# Contents

**Preface** v

**1. REINFORCED CONCRETE STRUCTURES** **1**

**1.1 Introduction 1**

**1.2 Plain and Reinforced Concrete 4**

1.2.1 Plain Concrete 4
1.2.2 Reinforced Concrete 5

**1.3 Objectives of Structural Design 7**

**1.4 Reinforced Concrete Construction 8**

**1.5 Structural Systems 9**

**1.6 Reinforced Concrete Buildings 9**

1.6.1 Floor Systems 11
1.6.2 Vertical Framing System 17
1.6.3 Lateral Load Resisting Systems 19

**1.7 Structural Analysis and Design 21**

**1.8 Design Codes and Handbooks 22**

1.8.1 Purpose of Codes 22
1.8.2 Basic Code for Design 22
1.8.3 Loading Standards 23
1.8.4 Design Handbooks 23
1.8.5 Other Related Codes 23

**Review Questions 24**

**References 24**

**2. BASIC MATERIAL PROPERTIES** **25**

**2.1 Introduction 25**

2.1.1 Concrete Technology 25

**2.2 Cement 26**
2.2.1 Portland Cements 26
2.2.2 Other Cements 28
2.2.3 Tests on Cements 29
**2.3 Aggregate 29**
2.3.1 Aggregate Properties and Tests 30
2.3.2 Grading Requirements of Aggregate 31
**2.4 Water 32**
2.4.1 Water Content and Workability of Concrete 33
2.4.2 Water-Cement Ratio and Strength 34
2.4.3 Water for Curing 35
**2.5 Admixtures 36**
2.5.1 Types of Chemical Admixtures 37
2.5.2 Types of Mineral Admixtures 37
**2.6 Grade of Concrete 38**
2.6.1 Characteristic Strength 39
**2.7 Concrete Mix Design** 40
2.7.1 Nominal Mix Concrete 40
2.7.2 Design Mix Concrete 41
**2.8 Behaviour of Concrete under Uniaxial Compression 42**
2.8.1 Influence of Size of Test Specimen 43
2.8.2 Stress-Strain Curves 45
2.8.3 Modulus of Elasticity and Poisson's Ratio 46
2.8.4 Influence of Duration of Loading on Stress-Strain Curve 49
2.8.5 Maximum Compressive Stress of Concrete in Design Practice 50
**2.9 Behaviour of Concrete under Tension 50**
2.9.1 Modulus of Rupture 51
2.9.2 Splitting Tensile Strength 52
2.9.3 Stress-Strain Curve of Concrete in Tension 52
2.9.4 Shear Strength and Tensile Strength 53
**2.10 Behaviour of Concrete under Combined Stresses 53**
2.10.1 Biaxial State of Stress 53
2.10.2 Influence of Shear Stress 53
2.10.3 Behaviour under Triaxial Compression 55
**2.11 Creep of Concrete 55**
2.11.1 Time-Dependent Behaviour under Sustained Loading 55
2.11.2 Effects of Creep 56
2.11.3 Factors Influencing Creep 57
2.11.4 Creep Coefficient for Design 57
**2.12 Shrinkage and Temperature Effects in Concrete 57**
2.12.1 Shrinkage 57
2.12.2 Temperature Effects 59
**2.13 Durability of Concrete 59**
2.13.1 Environmental Exposure Conditions and Code Requirements 61
2.13.2 Permeability of Concrete 63

2.13.3 Chemical Attack on Concrete 63
2.13.4 Corrosion of Reinforcing Steel 65

**2.14 Reinforcing Steel 65**

2.14.1 Types, Sizes and Grades 66
2.14.2 Stress-Strain Curves 67

**2.15 List of Relevant Indian Standards 70**

**Review Questions 72**

**References 74**

**3. BASIC DESIGN CONCEPTS 77**

**3.1 Introduction 77**

3.1.1 Design Considerations 77
3.1.2 Design Philosophies 78

**3.2 Working Stress Method (WSM) 79**

**3.3 Ultimate Load Method (ULM) 80**

**3.4 Probabilistic Analysis and Design 80**

3.4.1 Uncertainties in Design 80
3.4.2 Classical Reliability Models 82
3.4.3 Reliability Analysis and Design 84
3.4.4 Levels of Reliability Methods 84

**3.5 Limit States Method (LSM) 85**

3.5.1 Limit States 85
3.5.2 Multiple Safety Factor Formats 85
3.5.3 Load and Resistance Factor Design Format 86
3.5.4 Partial Safety Factor Format 86

**3.6 Code Recommendations for Limit States Design 87**

3.6.1 Characteristic Strengths and Loads 87
3.6.2 Partial Safety Factors for Materials 88
3.6.3 Partial Safety Factors for Loads 88
3.6.4 Design Stress-Strain Curve for Concrete 89
3.6.5 Design Stress-Strain Curve for Reinforcing Steel 90

**Review Questions 93**

**References 94**

**4. BEHAVIOUR IN FLEXURE 95**

**4.1 Introduction 95**

4.1.1 Two Kinds of Problems: Analysis and Design 95
4.1.2 Bending Moments in Beams from Structural Analysis 96
4.1.3 From Bending Moment to Flexural Stresses 97

**4.2 Theory of Flexure for Homogeneous Materials 97**

4.2.1 Fundamental Assumption 97
4.2.2 Distribution of Stresses 97

4.2.3 Linear Elastic Material 99

**4.3 Linear Elastic Analysis of Composite Sections 99**

4.3.1 Distribution of Strains and Stresses 100
4.3.2 Concept of 'Transformed Section' 101

**4.4 Modular Ratio and Cracking Moment 101**

4.4.1 Modular Ratio in Reinforced Concrete 101
4.4.2 Transformed Area of Reinforcing Steel 102
4.4.3 Cracking Moment 103

**4.5 Flexural Behaviour of Reinforced Concrete 105**

4.5.1 Uncracked Phase 107
4.5.2 Linear Elastic Cracked Phase 107
4.5.3 Stages Leading to Limit State of Collapse 108

**4.6 Analysis at Service Loads (WSM) 112**

4.6.1 Stresses in Singly Reinforced Rectangular Sections 112
4.6.2 Permissible Stresses 115
4.6.3 Allowable Bending Moment 116
4.6.4 Analysis of Singly Reinforced Flanged Sections 122
4.6.5 Analysis of Doubly Reinforced Sections 128

**4.7 Analysis at Ultimate Limit State 134**

4.7.1 Assumptions in Analysis 134
4.7.2 Limiting Depth of Neutral Axis 135
4.7.3 Analysis of Singly Reinforced Rectangular Sections 137
4.7.4 Analysis of Singly Reinforced Flanged Sections 147
4.7.5 Analysis of Doubly Reinforced Sections 153
4.7.6 Balanced Doubly Reinforced Sections 159

**4.8 Analysis of Slabs as Rectangular Beams 160**

4.8.1 Transverse Moments in One Way Slabs 161

**Review Questions 163**

**Problems 164**

**References 167**

**5. DESIGN OF BEAMS AND ONE-WAY SLABS FOR FLEXURE 169**

**5.1 Introduction 169**

**5.2 Requirements of Flexural Reinforcement 170**

5.2.1 Concrete Cover 170
5.2.2 Spacing of Reinforcing Bars 172
5.2.3 Minimum and Maximum Areas of Flexural Reinforcement 174

**5.3 Requirements for Deflection Control 176**

5.3.1 Deflection Control by Limiting Span/Depth Ratios 176
5.3.2 Code Recommendations for Span/Effective Depth Ratios 177

**5.4 Guidelines for Selection of Member Sizes 179**

5.4.1 General Guidelines for Beam Sizes 179
5.4.2 General Guidelines for Slab Thicknesses 180

5.4.3 Deep Beams and Slender Beams 180

**5.5 Design of Singly Reinforced Rectangular Sections 181**

5.5.1 Fixing Dimensions of Rectangular Sections 182
5.5.2 Determining Area of Tension Steel 183
5.5.3 Design Check for Strength and Deflection Control 185

**5.6 Design of Continuous One-Way Slabs 189**

5.6.1 Simplified Structural Analysis — Use of Moment Coefficients 190
5.6.2 Design Procedure 192

**5.7 Design of Doubly Reinforced Rectangular Sections 197**

5.7.1 Design Formulas 197
5.7.2 Design Procedure for Given $M_u$ 199

**5.8 Design of Flanged Beam Sections 203**

5.8.1 Transverse Reinforcement in Flange 203
5.8.2 Design Procedure 204

**5.9 Curtailment of Flexural Tension Reinforcement 210**

5.9.1 Theoretical Bar Cut-off Points 210
5.9.2 Restrictions on Theoretical Bar Cut-off Points 212
5.9.3 Code Requirements 214
5.9.4 Bending of Bars 219

**Review Questions 221**

**Problems 222**

**References 223**

**6. DESIGN FOR SHEAR 225**

**6.1 Introduction 225**

**6.2 Shear Stresses in Homogeneous Rectangular Beams 226**

**6.3 Behaviour of Reinforced Concrete under Shear 228**

6.3.1 Modes of Cracking 228
6.3.2 Shear Transfer Mechanisms 230
6.3.3 Shear Failure Modes 232

**6.4 Nominal Shear Stress 234**

6.4.1 Members with Uniform Depth 234
6.4.2 Members with Varying Depth 234

**6.5 Critical Sections for Shear Design 236**

**6.6 Design Shear Strength without Shear Reinforcement 238**

6.6.1 Design Shear Strength of Concrete in Beams 238
6.6.2 Design Shear Strength of Concrete in Slabs 240
6.6.3 Influence of Axial Force on Design Shear Strength 241

**6.7 Design Shear Strength with Shear Reinforcement 242**

6.7.1 Types of Shear Reinforcement 242
6.7.2 Factors Contributing to Ultimate Shear Resistance 243
6.7.3 Limiting Ultimate Shear Resistance 244

6.7.4 Shear Resistance of Web Reinforcement 245
6.7.5 Influence of shear on longitudinal reinforcement 247
6.7.6 Minimum Stirrup Reinforcement 249

**6.8 Additional Comments on Shear Reinforcement Design 249**

**6.9 Interface Shear and Shear Friction 251**

6.9.1 Shear friction 251

6.9.2 Recommendation for Interface Shear Transfer 254

**6.10 Shear Connectors in Flexural Members 256**

6.10.1 Shear along Horizontal Planes 256

**6.11 Shear Design Examples – Conventional Method 257**

**Review Questions 263**

**Problems 264**

**References 266**

## 7. DESIGN FOR TORSION 267

**7.1 Introduction 267**

**7.2 Equilibrium Torsion and Compatibility Torsion 267**

7.2.1 Equilibrium Torsion 268
7.2.2 Compatibility Torsion 268
7.2.3 Estimation of Torsional Stiffness 270

**7.3 General Behaviour in Torsion 271**

7.3.1 Behaviour of Plain Concrete 271
7.3.2 Behaviour of Concrete with Torsional Reinforcement 273

**7.4 Design Strength in Torsion 274**

7.4.1 Design Torsional Strength without Torsion Reinforcement 274
7.4.2 Design Torsional Strength with Torsion Reinforcement 277
7.4.3 Design Strength in Torsion Combined with Flexure 280
7.4.4 Design Strength in Torsion Combined with Shear 282

**7.5 Analysis and Design Examples 284**

**Review Questions 291**

**Problems 292**

**References 294**

## 8. DESIGN FOR BOND 295

**8.1 Introduction 295**

8.1.1 Mechanisms of Bond Resistance 295
8.1.2 Bond Stress 296
8.1.3 Two Types of Bond 296

**8.2 Flexural Bond 297**

8.2.1 Effect of Flexural Cracking on Flexural Bond Stress 298

**8.3 Anchorage (Development) Bond 299**

8.3.1 Development Length 300

**8.4 Bond Failure and Bond Strength 301**

8.4.1 Bond Failure Mechanisms 301
8.4.2 Bond Tests 303
8.4.3 Factors Influencing Bond Strength 305

**8.5 Review of Code Requirements for Bond 305**

8.5.1 Flexural Bond 305
8.5.2 Development (Anchorage) Bond 306
8.5.3 Bends, Hooks and Mechanical Anchorages 306

**8.6 Splicing of Reinforcement 308**

8.6.1 Lap Splices 308
8.6.2 Welded Splices and Mechanical Connections 310

**8.7 Design Examples 311**

**Review Questions 314**

**Problems 315**

**References 316**

**9. ANALYSIS FOR DESIGN MOMENTS IN CONTINUOUS SYSTEMS 317**

**9.1 Introduction 317**

9.1.1 Approximations in Structural Analysis 317
9.1.2 Factored Moments from Elastic Analysis and Moment Redistribution 320

**9.2 Gravity Load Patterns for Maximum Design Moments 321**

9.2.1 Design Moments in Beams 322
9.2.2 Design Moments in Columns 323

**9.3 Simplified (Approximate) Methods of Analysis 324**

9.3.1 Moment Coefficients for Continuous Beams under Gravity Loads 324
9.3.2 Substitute Frame Method of Frame Analysis for Gravity Loads 324
9.3.3 Simplified Methods for Lateral Load Analysis 327

**9.4 Proportioning of Member Sizes for Preliminary Design 328**

**9.5 Estimation of Stiffnesses of Frame Elements 330**

**9.6 Adjustment of Design Moments at Beam-Column Junctions 331**

**9.7 Inelastic Analysis and Moment Redistribution 334**

9.7.1 Limit Analysis 334
9.7.2 Moment Redistribution 337
9.7.3 Code Recommendations for Moment Redistribution 341

**9.8 Design Examples 345**

**Review Questions 353**

**Problems 353**

**References 355**

**10. SERVICEABILITY LIMIT STATES: DEFLECTION AND CRACKING 357**

**10.1 Introduction 357**

**10.2 Serviceability Limit States: Deflection 358**

10.2.1 Deflection Limits 358
10.2.2 Difficulties in Accurate Prediction of Deflections 359

**10.3 Short-Term Deflections 360**

10.3.1 Deflections by Elastic Theory 360
10.3.2 Effective Flexural Rigidity 361
10.3.3 Tension Stiffening Effect 362
10.3.4 Effective Second Moment of Area 364
10.3.5 Average $I_{eff}$ for Continuous Spans 366
10.3.6 Effective Curvature Formulation 368
10.3.7 Additional Short-Term Deflection due to Live Loads alone 373

**10.4 Long-Term Deflection 380**

10.4.1 Deflection Due to Differential Shrinkage 381
10.4.2 Deflection Due to Creep 384
10.4.3 Deflection Due to Temperature Effects 387
10.4.4 Checks on Total Deflection 388

**10.5 Serviceability Limit State: Cracking 391**

10.5.1 Cracking in Reinforced Concrete Members 391
10.5.2 Limits on Cracking 393
10.5.3 Factors Influencing Crackwidths 393
10.5.4 Estimation of Flexural Crackwidth 395
10.5.5 Estimation of Crackwidth under Direct and Eccentric Tension 405
10.5.6 Thermal and Shrinkage Cracking 409

**Review Questions 412**

**Problems 413**

**References 415**

**11. DESIGN OF TWO-WAY SLAB SYSTEMS 417**

**11.1 Introduction 417**

11.1.1 One-Way and Two-Way Actions of Slabs 417
11.1.2 Torsion in Two-Way Slabs 419
11.1.3 Difference Between Wall-Supported Slabs and Beam/Column Supported Slabs 420

**11.2 Design of Wall-Supported Two-Way Slabs 422**

11.2.1 Slab Thickness Based on Deflection Control Criterion 422
11.2.2 Methods of Analysis 422
11.2.3 Uniformly Loaded and Simply Supported Rectangular Slabs 423
11.2.4 Uniformly Loaded ‘Restrained’ Rectangular Slabs 427
11.2.5 Shear Forces in Uniformly Loaded Two-Way Slabs 435

11.2.6 Design of Circular, Triangular and other Slabs 448
11.2.7 Two-Way Slabs Subjected to Concentrated Loads 454

**11.3 Design of Beam-Supported Two-Way Slabs 454**

11.3.1 Behaviour of Beam-Supported Slabs 454
11.3.2 Use of Codal Moment Coefficients for Slabs Supported on Stiff Beams 454
11.3.3 Slabs Supported on Flexible Beams — Codal Limitations 456
11.3.4 The 'Equivalent Frame' Concept 456

**11.4 Design of Column-Supported Slabs (with/without Beams) under Gravity Loads 460**

11.4.1 Codal Procedures Based on the Equivalent Frame Concept 460
11.4.2 Proportioning of Slab Thickness, Drop Panel and Column Head 463
11.4.3 Transfer of Shear and Moments to Columns in Beamless Two-Way Slabs 467

**11.5 Direct Design Method 469**

11.5.1 Limitations 469
11.5.2 Total Design Moments for a Span 470
11.5.3 Longitudinal Distribution of Total Design Moment 470
11.5.4 Apportioning of Moments to Middle Strips, Column Strips and Beams 473
11.5.5 Loads on the Edge Beam 476
11.5.6 Torsion in the Edge Beam 476
11.5.7 Moments in Columns and Pattern Loading 478
11.5.8 Beam Shears in Two Way Slab Systems with Flexible Beams 480

**11.6 Equivalent Frame Method 481**

11.6.1 Equivalent Frame for Analysis 481
11.6.2 Slab-Beam Member 483
11.6.3 Loading Patterns 491
11.6.4 Design Moments in Slab-Beam Members 492
11.6.5 Design Moments in Columns and Torsion in Transverse Beam 494

**11.7 Reinforcement Details in Column-Supported Two-Way Slabs 494**

**11.8 Shear in Column-Supported Two-Way Slabs 497**

11.8.1 One-Way Shear or Beam Shear 497
11.8.2 Two-Way Shear or Punching Shear 499

**11.9 Design Examples of Column-Supported Two-Way Slabs 504**

**Review Questions 528**

**Problems 529**

**References 531**

**12. DESIGN OF STAIRCASES 533**

**12.1 Introduction 533**

**12.2 Types of Staircases 535**

12.2.1 Geometrical Configurations 535
12.2.2 Structural Classification 536

**12.3 Loads and Load Effects on Stair Slabs 540**

12.3.1 Dead Loads 541
12.3.2 Live Loads 541
12.3.3 Distribution of Gravity Loads in Special Cases 541
12.3.4 Load Effects in Isolated Tread Slabs 542
12.3.5 Load Effects in Waist Slabs 542
12.3.6 Load Effects in Tread-Riser Stairs 544

**12.4 Design Examples of Stair Slabs Spanning Transversely 547**

**12.5 Design Examples of Stair Slabs Spanning Longitudinally 552**

**Review Questions 562**

**Problems 562**

**References 564**

**13. DESIGN OF COMPRESSION MEMBERS 565**

**13.1 Introduction 565**

13.1.1 Classification of Columns Based on Type of Reinforcement 565
13.1.2 Classification of Columns Based on Type of Loading 566
13.1.3 Classification of Columns Based on Slenderness Ratios 568

**13.2 Estimation of Effective Length of a Column 569**

13.2.1 Definition of Effective Length 569
13.2.2 Effective Length Ratios for Idealised Boundary Conditions 570
13.2.3 Effective Length Ratios of Columns in Frames 573

**13.3 Code Requirements on Slenderness Limits, Minimum Eccentricities and Reinforcement 581**

13.3.1 Slenderness Limits 581
13.3.2 Minimum Eccentricities 582
13.3.3 Code Requirements on Reinforcement and Detailing 582

**13.4 Design of Short Columns under Axial Compression 586**

13.4.1 Conditions of Axial Loading 586
13.4.2 Behaviour under Service Loads 587
13.4.3 Behaviour under Ultimate Loads 588
13.4.4 Design Strength of Axially Loaded Short Columns 590

**13.5 Design of Short Columns under Compression with Uniaxial Bending 594**

13.5.1 Distribution of Strains at Ultimate Limit State 594
13.5.2 Modes of Failure in Eccentric Compression 596
13.5.3 Design Strength: Axial Load-Moment Interaction 597
13.5.4 Analysis for Design Strength 600
13.5.5 Use of Interaction Diagram as an Analysis Aid 610
13.5.6 Non-dimensional Interaction Diagrams as Design Aids 618

**13.6 Design of Short Columns under Axial Compression with Biaxial Bending 625**

13.6.1 Biaxial Eccentricities 625
13.6.2 Interaction Surface for a Biaxially Loaded Column 627
13.6.3 Code Procedure for Design of Biaxially Loaded Columns 629

**13.7 Design of Slender Columns 634**

13.7.1 Behaviour of Slender Columns 634
13.7.2 Second-Order Structural Analysis of Slender Column Structures 639
13.7.3 Code Procedures for Design of Slender Columns 639

**Review Questions 649**

**Problems 650**

**References 653**

**14. DESIGN OF FOOTINGS AND RETAINING WALLS 655**

**14.1 Introduction 655**

**14.2 Types of Footings 656**

14.2.1 Isolated Footings 658
14.2.2 Combined Footings 658
14.2.3 Wall Footings 659

**14.3 Soil Pressures under Isolated Footings 659**

14.3.1 Allowable Soil Pressure 659
14.3.2 Distribution of Base Pressure 660
14.3.3 Instability Problems: Overturning and Sliding 664

**14.4 General Design Considerations and Code Requirements 665**

14.4.1 Factored Soil Pressure at Ultimate Limit State 665
14.4.2 General Design Considerations 667
14.4.3 Thickness of Footing Base Slab 667
14.4.4 Design for Shear 667
14.4.5 Design for Flexure 669
14.4.6 Transfer of Forces at Column Base 671
14.4.7 Plain Concrete Footings 673

**14.5 Design Examples of Isolated and Wall Footings 674**

**14.6 Design of Combined Footings 692**

14.6.1 General 692
14.6.2 Distribution of Soil Pressure 693
14.6.3 Geometry of Two-Column Combined Footings 693
14.6.4 Design Considerations in Two-Column Footings 693

**14.7 Types of Retaining Walls and Their Behaviour 703**

**14.8 Earth Pressures and Stability Requirements 706**

14.8.1 Lateral Earth Pressures 706
14.8.2 Effect of Surcharge on a Level Backfill 708
14.8.3 Effect of Water in the Backfill 709
14.8.4 Stability Requirements 710
14.8.5 Soil Bearing Pressure Requirements 711

**14.9 Proportioning and Design of Cantilever and Counterfort Walls**

712
14.9.1 Position of Stem on Base Slab for Economical Design 712
14.9.2 Proportioning and Design of Elements of Cantilever Walls 714
14.9.3 Proportioning and Design of Elements of a Counterfort Wall 715

**Review Questions 745**

**Problems 746**

**References 747**

**15. GOOD DETAILING AND CONSTRUCTION PRACTICES 749**

**15.1 Introduction 749**
15.1.1 Serviceability Failures 750
15.1.2 Reasons for Building Failures 751
15.1.3 Structural Integrity 751

**15.2 Design and Detailing Practices 752**
15.2.1 Reinforcement Layout 753
15.2.2 Design Drawings 754
15.2.3 Construction Details at Connections and special situations 754
15.2.4 Beam and Column Joints (Rigid Frame Joints) 761
15.2.5 Construction Joints 763
15.2.6 Bar Supports and Cover 764
15.2.7 Deflection Control 765

**15.3 Materials and Construction Practices 765**

**15.4 Summary 767**

**Review Questions 768**

**References 769**

**16. SPECIAL PROVISIONS FOR EARTHQUAKE-RESISTANT DESIGN 771**

**16.1 Introduction 771**

**16.2 Importance of Ductility in Seismic Design 773**
16.2.1 Measures of Ductility 773
16.2.2 Energy Dissipation by Ductile Behaviour 775
16.2.3 Flexural Yielding in Frames and Walls 777

**16.3 Major Design Considerations 778**
16.3.1 General Design Objectives 778
16.3.2 Requirements of Stability and Stiffness 778
16.3.3 Materials 779
16.3.4 Foundations 780
16.3.5 Flexural Members in Ductile Frames 780
16.3.6 Columns and Frame Members Subject to Bending and Axial Load 785
16.3.7 Joints in Ductile Frames 788
16.3.8 Shear Walls (Flexural Walls) 788

16.3.9 Infill frames 790
16.3.10 Soft storey 791
16.3.11 Performance limit states 792

**16.4 Closure 792**

**Review Questions 792**

**References 793**

## 17. Selected Special Topics 795

**17.1 Design for Shear by Compression Field Theory 795**

17.1.1 Introduction 795
17.1.2 General Concepts 796
17.1.3 Stress-Strain Relationship for Diagonally Cracked Concrete 798
17.1.4 Analysis Based on Modified Compression Field Theory 799
17.1.5 Simplified Design Procedure using Modified Compression Field Theory 804
17.1.6 CSA Code Provisions for Shear Design by the Compression Field Theory 808
17.1.7 Combined Shear and Torsion 810

**17.2 Design Using Strut-and-Tie Model 811**

**17.3 Fire Resistance 822**

17.3.1 Introduction 822
17.3.2 Factors which influence Fire Resistance Ratings of RC Assemblies 823
17.3.3 Code Requirements 825

**Problems 826**

**Review Questions 826**

**References 827**

## APPENDIX A: ANALYSIS AND DESIGN AIDS 829

**Table A.1** ANALYSIS AIDS (WSM) for Singly Reinforced Rectangular Beam Sections
Values of $M_{all}/bd^2$ (MPa) for given Values of $p_t$
(a) M 20, M 25 Concrete Grades 830
(b) M 30, M 35 Concrete Grades 831

**Table A.2** ANALYSIS AIDS (LSM) for Singly Reinforced Rectangular Beam Sections
Values of $M_{uR}/bd^2$ (MPa) for given Values of $p_t$
(a) M 20, M 25 Concrete Grades 832
(b) M 30, M 35 Concrete Grades 835

**Table A.3** DESIGN AIDS (LSM) for Singly Reinforced Rectangular Beam Sections
Values of $p_t$ for given Values of $R \equiv M_u/bd^2$ (MPa)
(a) M 20, M 25 Concrete Grades 839
(b) M 30, M 35 Concrete Grades 843

**Table A.4** DESIGN AIDS (LSM) for Doubly Reinforced Rectangular Beam Sections
Values of $p_t$ and $p_c$ for given Values of $R \equiv M_u/bd^2$ (MPa) for
(a) Fe 415 Steel, M 20 Concrete 849
(b) Fe 415 Steel, M 25 Concrete 855

**Table A.5** Areas ($mm^2$) of Reinforcing Bar Groups 861

**Table A.6** Areas ($mm^2$/m) of Uniformly Spaced Bars 862

**APPENDIX B: GENERAL DATA FOR DEAD LOADS AND LIVE LOADS 863**

**Table B.1** DEAD LOADS — Unit Weights of Some Materials/Components 864

**Table B.2** LIVE LOADS on Floors 865

**Table B.3** LIVE LOADS on Roofs 865

**Table B.4** HORIZONTAL LIVE LOADS on Parapets/Balustrades 865

**Index 867**

# 1

# Reinforced Concrete Structures

## 1.1 INTRODUCTION

Traditionally, the study of **reinforced concrete design** begins directly with a chapter on materials, followed by chapters dealing with design. In this book, a departure is made from that convention. It is desirable for the student to have first an overview of the world of reinforced concrete structures, before plunging into the finer details of the subject.

Accordingly, this chapter gives a general introduction to reinforced concrete and its applications. It also explains the role of structural design in reinforced concrete construction, and outlines the various structural systems that are commonly adopted in buildings.

That concrete is a common structural material is, no doubt, well known. But, how common it is, and how much a part of our daily lives it plays, is perhaps not well known — or rather, not often realised. Structural concrete is used extensively in the construction of various kinds of buildings, stadia, auditoria, pavements, bridges, piers, breakwaters, berthing structures, dams, waterways, pipes, water tanks, swimming pools, cooling towers, bunkers and silos, chimneys, communication towers, tunnels, etc. It is the most commonly used construction material, consumed at a rate of approximately one ton for every living human being. "Man consumes no material except water in such tremendous quantities" (Ref. 1.1).

Pictures of some typical examples of reinforced concrete structures are shown in Figs 1.1–1.5. Perhaps, some day in the future, the reader may be called upon to design similar (if not, more exciting) structures! The student will do well to bear this goal in mind.

**Fig. 1.1** Ferrocement Boat — "the first known example of reinforced concrete" is a boat, patented in 1848 by *Joseph-Louis Lambot* [Ref. : *Ferrocement*, National Academy of Sciences, Washington D.C., Feb. 1973]; the boat shown here is a later version (1887) of the original design, presently preserved in the Brignoles Museum, France.

**Fig. 1.2** A modern reinforced concrete multi-storeyed building — one of the tallest in New Delhi (102 m) : *Jawahar Vyapar Bhavan* [Architects : *Raj Rewal and Kuldip Singh*, Project Consultants : *Engineers India Limited*].
*Structural concept* : joist floor supported on Vierendeel girders (arranged in a 'plug on' fashion), cantilevered from core walls.

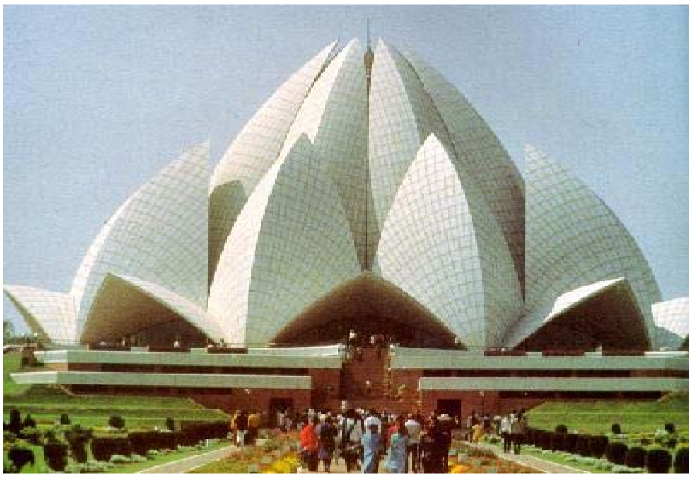


**Fig. 1.3** The Bahá'i House of Worship, New Delhi — a unique lotus-shaped reinforced concrete structure, with a complex shell geometry involving spheres, cylinders, torroids and cones [Architect : *Faríburz Sahbá*, Structural Consultants : *Flint & Neill*, Contractor : *Larsen & Toubro Ltd*.]

**Fig. 1.4** C N Tower — a communications tower at Toronto, Canada, rising to a height of 550 m, making it the tallest reinforced concrete tower in the world. (The picture also shows an elevator car which travels vertically along the shaft of the tower).

**Fig. 1.5** A reinforced concrete bow-string girder bridge spanning across the Bharathapuzha River at Kuttippuram, Kerala

## 1.2 PLAIN AND REINFORCED CONCRETE

### 1.2.1 Plain Concrete

Concrete may be defined [Ref. 1.2] as any solid mass made by the use of a cementing medium; the ingredients generally comprise sand, gravel, cement and water. That the mixing together of such disparate and discrete materials can result in a *solid mass* (of any desired shape), with well-defined properties, is a wonder in itself. Concrete has been in use as a building material for more than a hundred and fifty years. Its success and popularity may be largely attributed to (1) durability under hostile environments (including resistance to water), (2) ease with which it can be cast into a variety of shapes and sizes, and (3) its relative economy and easy availability. The main strength of concrete lies in its compression-bearing ability, which surpasses that of traditional materials like brick and stone masonry. Advances in *concrete technology*, during the past four decades in particular, have now made it possible to produce a wide range of concrete grades, varying in mass density (1200–2500 kg/m$^3$) and compressive strength (10 –100 MPa).

Concrete may be remarkably strong in compression, but it is equally remarkably weak in tension! [Fig. 1.6(a)]. Its tensile 'strength' is approximately one-tenth of its compressive 'strength'. Hence, the use of **plain concrete** as a structural material is limited to situations where significant tensile stresses and strains do not develop, as in hollow (or solid) block wall construction, small pedestals and 'mass concrete' applications (in dams, etc.).

### 1.2.2 Reinforced Concrete

Concrete would not have gained its present status as a principal building material, but for the invention of **reinforced concrete,** which is concrete with steel bars embedded in it. The idea of reinforcing concrete with steel has resulted in a new *composite* material, having the potential of resisting significant tensile stresses, which was hitherto impossible. Thus, the construction of load-bearing flexural members, such as beams and slabs, became viable with this new material. The steel bars (embedded in the tension zone of the concrete) compensate for the concrete's incapacity for tensile resistance, effectively taking up all the tension, without separating from the concrete [Fig. 1.6(b)]. The *bond* between steel and the surrounding concrete ensures *strain compatibility*, i.e., the strain at any point in the steel is equal to that in the adjoining concrete. Moreover, the reinforcing steel imparts *ductility* to a material that is otherwise brittle. In practical terms, this implies that if a properly reinforced beam were to fail in tension, then such a failure would, fortunately, be preceded by large deflections caused by the yielding of steel, thereby giving ample warning of the impending collapse [Fig.1.6(c)].

Tensile stresses occur either directly, as in direct tension or flexural tension, or indirectly, as in shear, which causes tension along diagonal planes ('diagonal tension'). Temperature and shrinkage effects may also induce tensile stresses. In all such cases, reinforcing steel is essential, and should be appropriately located, in a direction that cuts across the *principal tensile planes* (i.e., across potential tensile cracks). If insufficient steel is provided, cracks would develop and propagate, and could possibly lead to failure.

Reinforcing steel can also supplement concrete in bearing compressive forces, as in columns provided with longitudinal bars. These bars need to be confined by transverse steel ties [Fig. 1.6(d)], in order to maintain their positions and to prevent their lateral buckling. The lateral ties also serve to confine the concrete, thereby enhancing its compression load-bearing capacity.

As a result of extensive research on reinforced concrete over the past several decades in various countries, a stage has reached where it is now possible to predict the elastic and inelastic behaviour of this composite material with some confidence. No doubt, there exists some uncertainty in the prediction, but this is largely attributable to the variability in the strength of in-situ concrete (which, unlike steel, is not manufactured under closely controlled conditions). There are several factors which lead to this variability, some of which pertain to material properties (primarily of the aggregates), while others pertain to the actual making of concrete at site (mixing, placing, compacting and curing). This uncertainty can be taken care of, by providing an appropriate factor of safety in the design process. [The topic of structural safety in design is discussed in detail in Chapter 3].

The development of reliable design and construction techniques has enabled the construction of a wide variety of reinforced concrete structures all over the world: building frames (columns and beams), floor and roof slabs, foundations, bridge decks and piers, retaining walls, grandstands, water tanks, pipes, chimneys, bunkers and silos, folded plates and shells, etc.

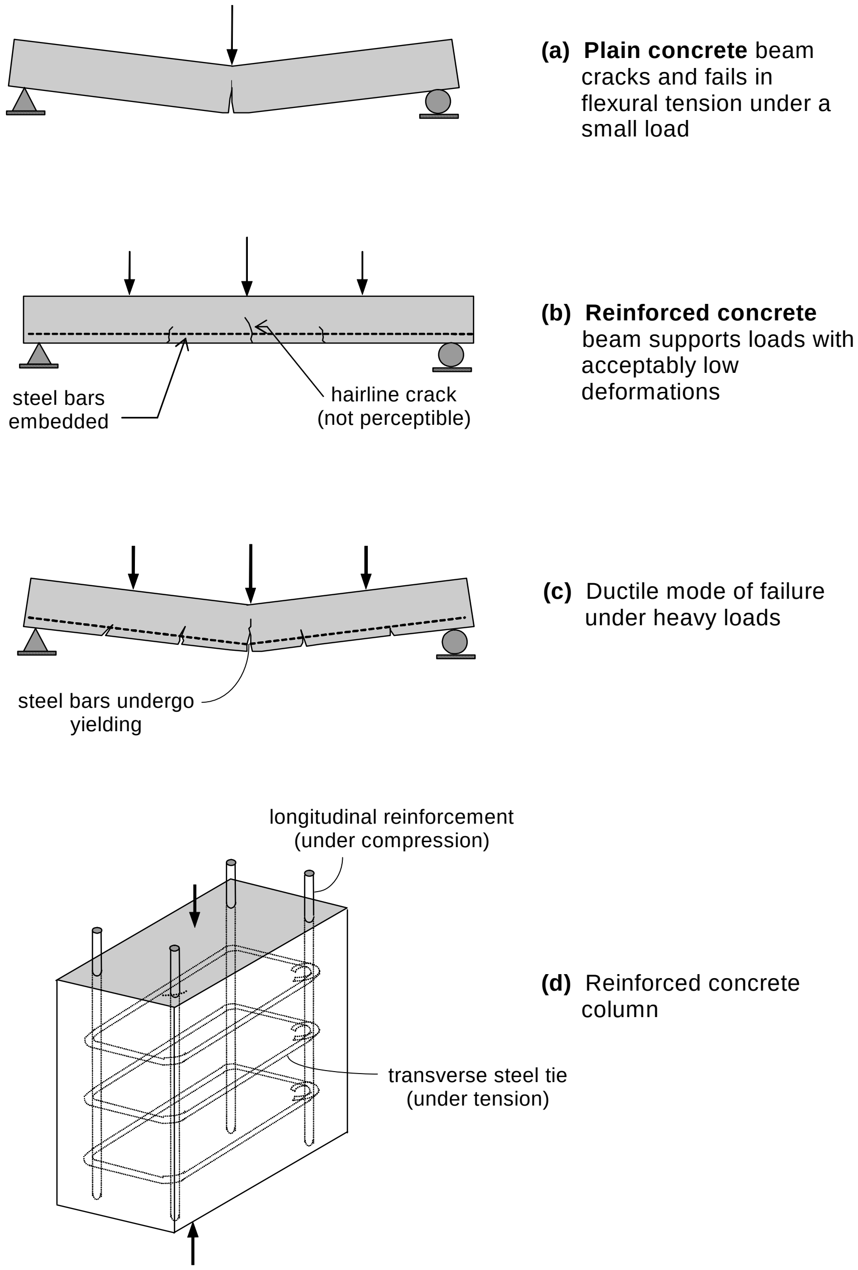



**Fig. 1.6** Contribution of steel bars in reinforced concrete

It is worth noting that, although these reinforced concrete structures appear to be completely different from one another, the actual principles underlying their design are the same. In the chapters to follow, the focus will be on these fundamental principles.

***Prestressed Concrete***: An introduction to reinforced concrete will not be complete without a mention of *prestressed concrete*, which is another ingenious invention that developed side-by-side with reinforced concrete. Prestressed concrete is high-strength concrete with high tensile steel wires embedded and tensioned, prior to the application of external loads. By this, the concrete can be pre-compressed to such a degree that, after the structure is loaded, there is practically no resultant tension developed in the beam. Prestressed concrete finds application in situations where long spans are encountered (as in bridges), or where cracks (even hairline) in concrete are not permitted (as in pressure vessels, pipes and water tanks), or where fatigue loading is encountered (as in railtrack sleepers), etc.

***Fibre-Reinforced Concrete and Ferrocement***: Recent developments in *concrete composites* have resulted in several new products that aim to improve the tensile strength of concrete, and to impart ductility. Among these, *fibre-reinforced concrete* and *ferrocement* constitute important developments. In the former, steel or glass fibres are incorporated in concrete at the time of mixing; in the latter, thin sections are formed by embedding multiple layers of steel wire mesh in cement mortar. Although ferrocement has gained popularity only in recent years, it represents one of the earliest applications of reinforced concrete to be experimented with [Fig. 1.1].

This book is concerned with reinforced concrete; hence, no further discussion on other concrete composites will be made.

## 1.3 OBJECTIVES OF STRUCTURAL DESIGN

The design of a structure must satisfy three basic requirements:

1) **Stability** to prevent overturning, sliding or buckling of the structure, or parts of it, under the action of loads;
2) **Strength** to resist safely the stresses induced by the loads in the various structural members; and
3) **Serviceability** to ensure satisfactory performance under *service load* conditions — which implies providing adequate ***stiffness*** and reinforcements to contain deflections, crack-widths and vibrations within acceptable limits, and also providing ***impermeability*** and ***durability*** (including corrosion-resistance), etc.

There are two other considerations that a sensible designer ought to bear in mind, viz., **economy** and **aesthetics**. One can always design a massive structure, which has more-than-adequate stability, strength and serviceability, but the ensuing cost of the structure may be exorbitant, and the end product, far from aesthetic.

In the words of Felix Candela [Ref. 1.3], the designer of a remarkably wide range of reinforced concrete shell structures,

> "... the architect has no weapons to fight against the scientific arguments of the engineer. A dialogue is impossible between two people who speak

> different languages. The result of the struggle is generally the same: science prevails, and the final design has generally lost the eventual charm and fitness of detail dreamed by the architect."

It is indeed a challenge, and a responsibility, for the structural designer to design a structure that is not only appropriate for the architecture, but also strikes the right balance between safety and economy [Ref. 1.4].

## 1.4 REINFORCED CONCRETE CONSTRUCTION

Reinforced concrete construction is not the outcome of structural design alone. It is a collaborative venture involving the client, the architect, the structural engineer, the construction engineer/project manager and the contractor. Other specialists may also have to be consulted, with regard to soil investigation, water supply, sanitation, fire protection, transportation, heating, ventilation, air-conditioning, acoustics, electrical services, etc. Typically, a construction project involves three phases viz. *planning*, *design* (including *analysis*) and *construction*.

***1. Planning Phase***: It is the job of the architect/planner to conceive and plan the architectural layout of the building, to suit the functional requirements of the client, with due regard to aesthetic, environmental and economic considerations. Structural feasibility is also an important consideration, and for this the structural designer has to be consulted.

***2. Design Phase***: Once the preliminary plans have been approved, the actual details of the project have to be worked out (on paper) by the various consultants. In the case of the structural engineer/consultant, the tasks involved are (i) selection of the most appropriate structural system and initial proportioning of members, (ii) estimation of loads on the structure, (iii) **structural analysis** for the determination of the stress resultants (member forces) and displacements induced by various load combinations, (iv) **structural design** of the actual proportions (member sizes, reinforcement details) and grades of materials required for *safety* and *serviceability* under the calculated member forces, and (v) submission of *working drawings* that are detailed enough to be stamped 'good for construction'.

***3. Construction Phase***: The plans and designs conceived on paper get translated into concrete (!) reality. A structure may be well-planned and well-designed, but it also has to be well-built, for, the proof of the pudding lies in the eating. And for this, the responsibility lies not only with the contractor who is entrusted with the execution, but also with the construction engineers who undertake supervision on behalf of the consultants. The work calls for proper management of various resources, viz. manpower, materials, machinery, money and time. It also requires familiarity with various construction techniques and specifications. In particular, expertise in concrete technology is essential, to ensure the proper mixing, handling, placing, compaction and curing of concrete. Management of contracts and following proper procedures, systems and documentation are also important aspects of the construction phase, especially in public works, however these are beyond the scope of this book.

During the construction phase, some redesign may also be required — in the event of unforeseen contingencies, such as complications in foundations, non-availability of specified materials, etc.

## 1.5 STRUCTURAL SYSTEMS

Any structure is made up of *structural elements* (load-carrying, such as beams and columns) and *non-structural elements* (such as partitions, false ceilings, doors). The structural elements, put together, constitute the 'structural system'. Its function is to resist effectively the action of gravitational and environmental loads, and to transmit the resulting forces to the supporting ground, without significantly disturbing the geometry, integrity and serviceability of the structure.

Most of the structural elements may be considered, from the viewpoint of simplified analysis, as *one-dimensional* (skeletal) elements (such as beams, columns, arches, truss elements) or *two-dimensional* elements (such as slabs, plates and shells). A few structural elements (such as shell-edge beam junctions, perforated shear walls) may require more rigorous analysis.

Consider, for example, a reinforced concrete overhead water tank structure [Fig. 1.7]. The structural system essentially comprises three subsystems, viz. the tank, the staging and the foundation, which are distinct from one another in the sense that they are generally designed, as well as constructed, in separate stages. The tank, in this example, is made up of a dome-shaped shell roof, a cylindrical side-wall (with stiffening ring beams at top and bottom), a flat circular base slab, and a main ring beam, which is supported by the columns of the staging. The staging comprises a three-dimensional framework of beams and columns, which are 'fixed' to the foundation. The foundation is a 'raft', comprising a slab in the shape of an annular ring, stiffened by a ring beam on top, and resting on firm soil below. The loads acting on the structure are due to *dead loads* (due to self-weight), *live loads* (due to water in the tank, maintenance on the roof), *wind loads* (acting on the exposed surface areas of the tank and staging), and *seismic loads* (due to earthquake induced ground excitation). The effect of the loads acting on the tank are transmitted to the staging through the main ring beam; the effect of the loads on the staging are, in turn, transmitted to the foundation, and ultimately, to the ground below.

## 1.6 REINFORCED CONCRETE BUILDINGS

The most common reinforced concrete construction is the *building* (planned for residential, institutional or commercial use). It is therefore instructive to look at its structural system and its *load transmission* mechanism in some detail. As the height of the building increases, lateral loads (due to wind and earthquake) make their presence felt increasingly; in fact, in very tall buildings, the choice of a structural system is dictated primarily by its relative economy in effectively resisting lateral loads (rather than gravity loads).

For convenience, we may separate the structural system into two load *transmission* mechanisms, viz. **gravity load resisting** and **lateral load resisting**, although, in effect, these two systems are complementary and interactive. As an integrated system, the structure must resist and transmit all the effects of gravity loads and lateral loads acting on it to the foundation and the ground below.

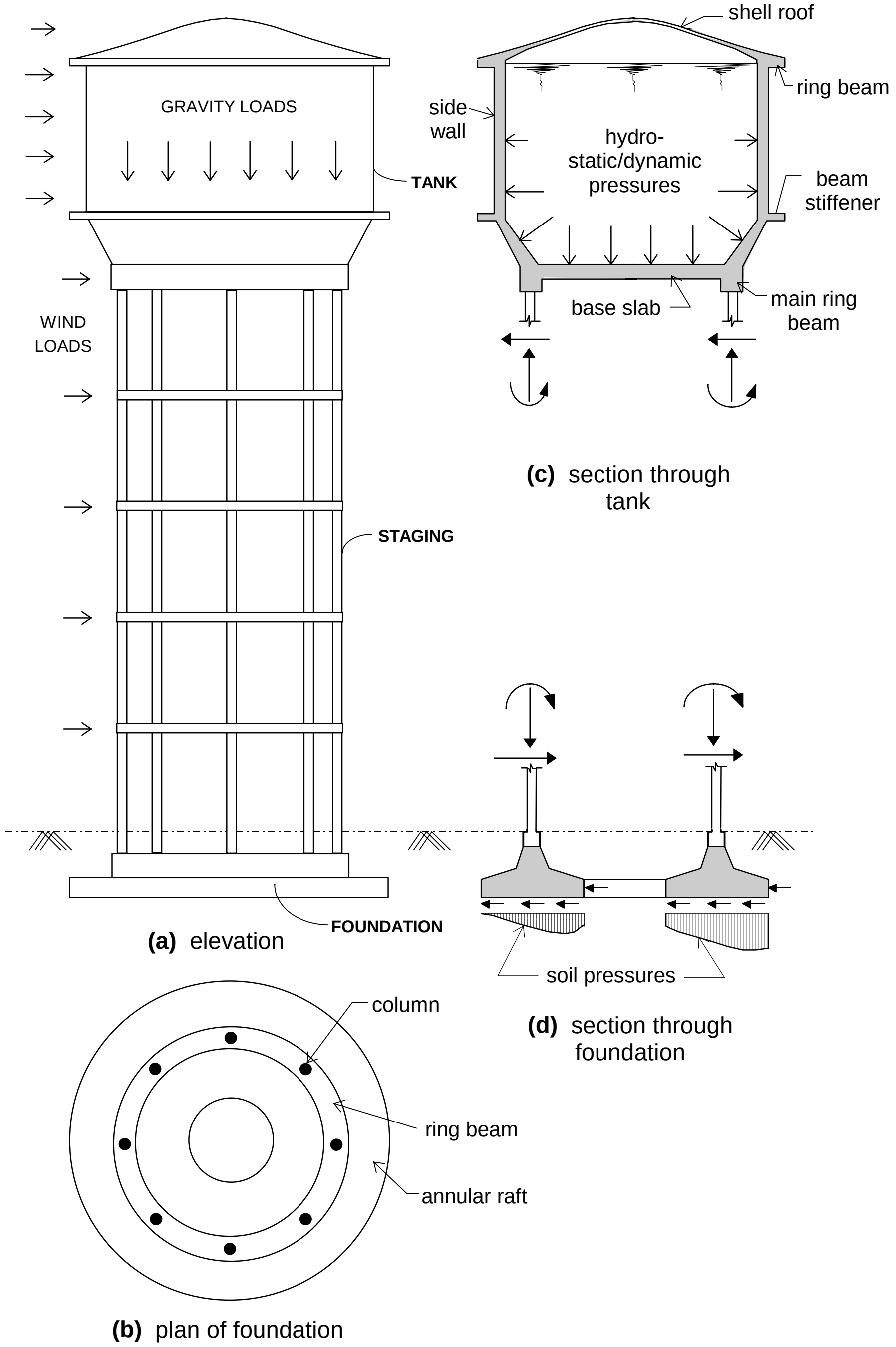



**Fig. 1.7** Structural system of an elevated water tank

Moreover, although the building is a three-dimensional structure, it is usually conceived, analysed and designed as an assemblage of two-dimensional (planar) sub-systems lying primarily in the horizontal and vertical planes (e.g., floors, roof, walls, plane frames, etc.), as indicated in Fig. 1.8. This division into a horizontal (floor) system and a vertical (framing) system is particularly convenient in studying the load resisting mechanisms in a building.

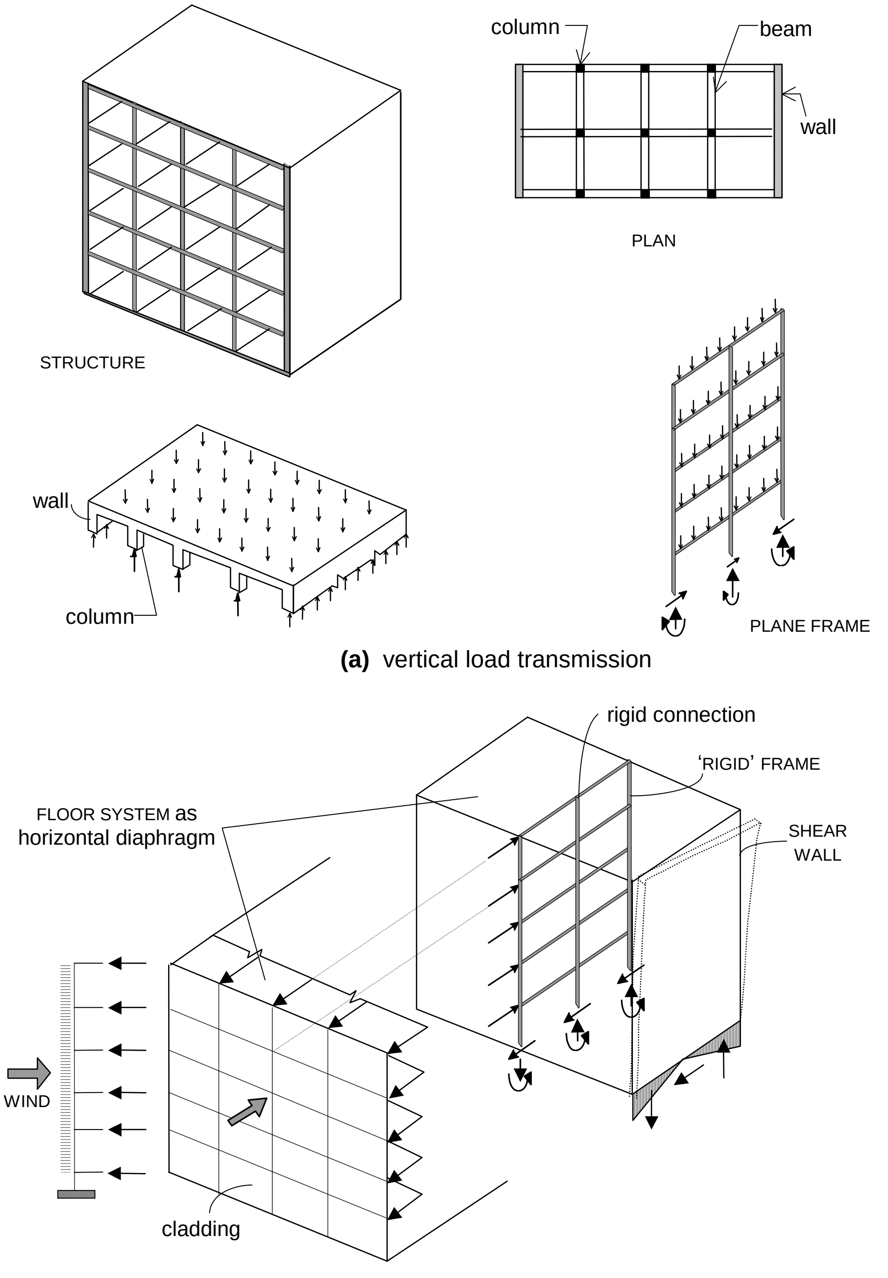
### 1.6.1 Floor Systems

The (horizontal) floor system resists the gravity loads (dead loads and live loads) acting on it and transmits these to the vertical framing system. In this process, the floor system is subjected primarily to flexure and transverse shear, whereas the vertical frame elements are generally subjected to axial compression, often coupled with flexure and shear [Fig. 1.8a]. The floor also serves as a horizontal diaphragm connecting together and stiffening the various vertical frame elements. Under the action of lateral loads, the floor diaphragm behaves rigidly (owing to its high in-plane flexural stiffness), and effectively distributes the lateral load effects to the various vertical frame elements and shear walls [Fig. 1.8b]. In cast-in-situ reinforced concrete construction, the floor system usually consists of one of the following:

#### *Wall-Supported Slab System*

In this system, the floor slabs, generally 100-200 mm thick with spans ranging from 3 m to 7.5 m, are supported on *load-bearing walls* (masonry). This system is mainly adopted in low-rise buildings. The slab panels are usually rectangular in shape, and can be supported in a number of ways.

When the slab is supported only on two opposite sides [Fig. 1.9(a)], the slab bends in one direction only; hence, it is called a *one-way slab*. When the slab is supported on all four sides, and the plan dimensions of length and breadth are comparable to each other [Fig. 1.9(c)], the slab bends in two directions (along the length and along the breadth); hence, it is called a *two-way slab*. However, if the plan is a long rectangle (length greater than about twice the width), the bending along the longitudinal direction is negligible in comparison with that along the transverse (short-span) direction, and the resulting slab action is effectively one-way [Fig. 1.9(b)]. If the wall extends above the floor level [Fig. 1.9(d)], the slab is no more simply supported; the partial fixity at the support introduces hogging moments in the slab. Furthermore, twisting moments are also introduced at the corners that are restrained (not free to lift up) — as established by the classical theory of plates. Generally, slabs are cast in panels that are continuous over several wall supports, and are called *one-way continuous* [Fig. 1.9(e)] or *two-way continuous* slabs, depending on whether the bending is predominantly along one direction or two directions. Hogging moments are induced in the slab in the region adjacent to the continuous support.




**(a)** vertical load transmission

**(b)** lateral load transmission

**Fig. 1.8** Load transmission mechanisms

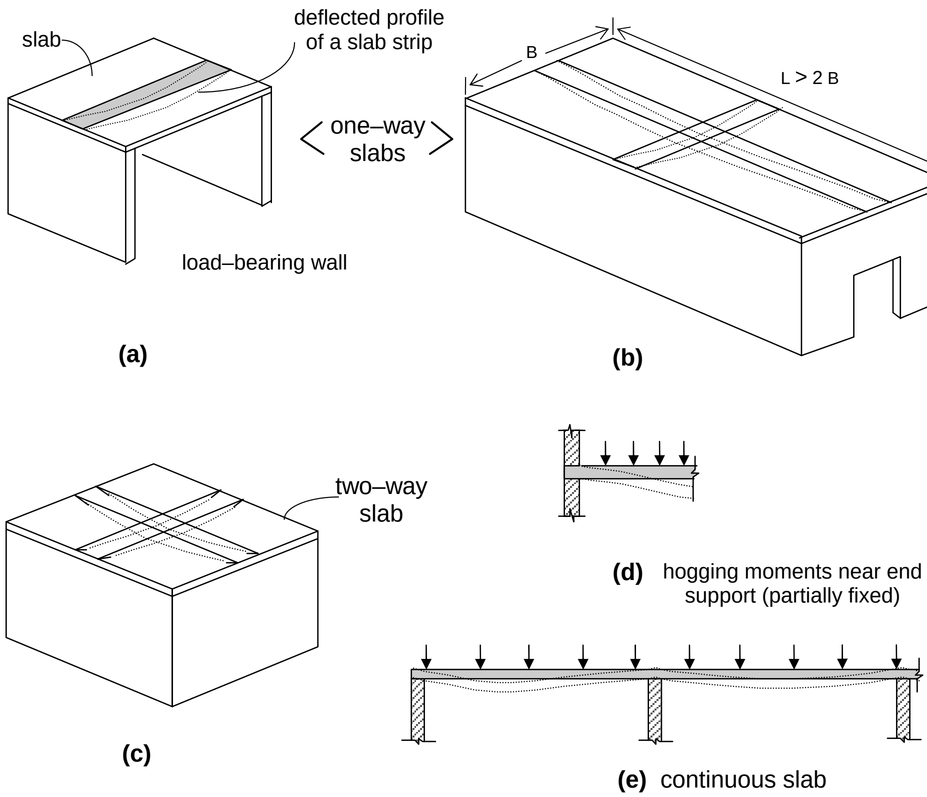



**Fig. 1.9** Wall-supported slab systems

### ***Beam-Supported Slab System***

This system is similar to the wall-supported slab system, except that the floor slabs are supported on beams (instead of walls). The beams are cast monolithically with the slabs in a grid pattern [Fig. 1.10(a)], with spans ranging from 3 m to 7.5 m. This system is commonly adopted in high-rise building construction, and also in low-rise *framed* structures. The gravity loads acting on the slabs are transmitted to the columns through the network of beams. The beams which are directly connected to the columns (forming the vertical frames) are called *primary* beams (or *girders*); whereas, the beams which are supported, not by columns, but by other (primary) beams, are called *secondary* beams [Figs 1.10(b),(c)].

If the beams are very stiff, the beam deflections are negligible, and the slab supports become relatively unyielding, similar to wall supports; the action may be either two-way or one-way [Fig. 1.10(b),(c)], depending on the panel dimensions. However, if the beams are relatively flexible, the beam deflections are no longer negligible and will influence the slab behaviour. When a large number of two-way secondary beams are involved (typically in a ‘grid floor’ with a large column-free space) [Fig. 1.10(d)], the slabs do not really 'rest' on the beams; the slab-beam system as a whole acts integrally in supporting the gravity loads.

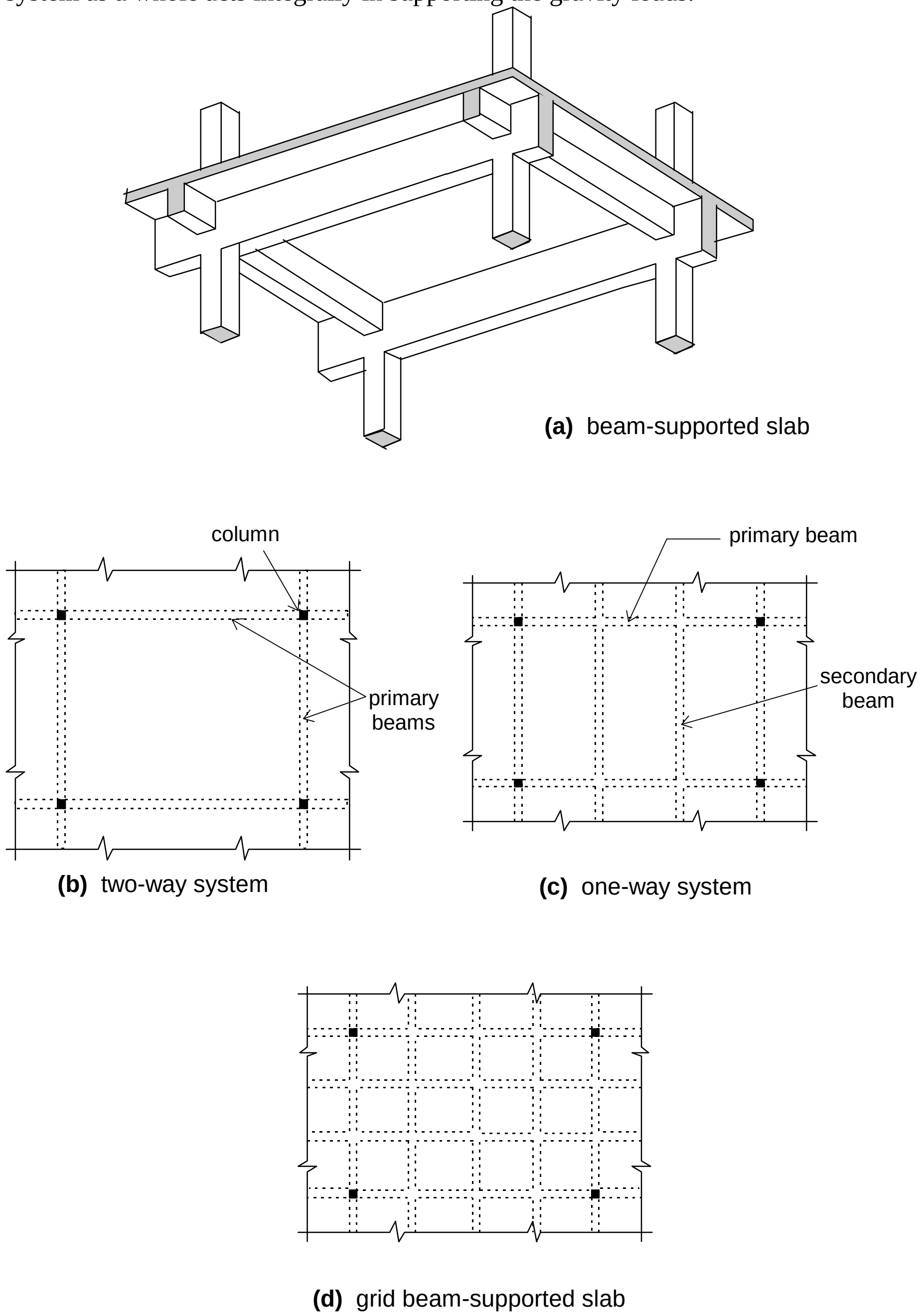



**Fig. 1.10** Beam-supported slab systems

### *Ribbed Slab System*

This is a special type of 'grid floor' slab-beam system, in which the 'slab', called *topping*, is very thin (50-100 mm) and the 'beams', called *ribs*, are very slender and closely spaced (less than 1.5 m apart). The ribs have a thickness of not less than 65 mm and a depth that is three-to-four times the thickness. The ribs may be designed in one-way or two-way patterns [Fig. 1.11(a),(b)], and are generally cast-in-situ, although precast construction is also possible.

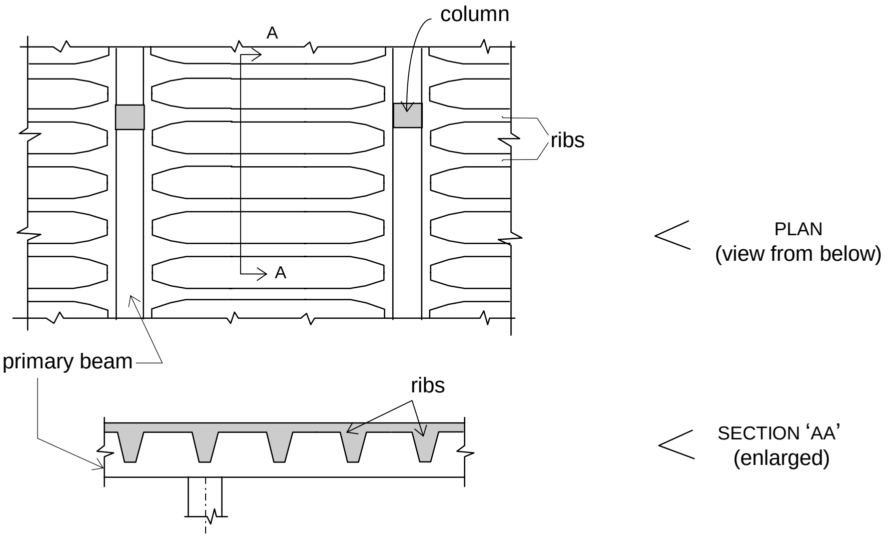



**Fig. 1.11(a)** One-way ribbed slab system

Two-way ribbed slabs are sometimes called *waffle slabs*. Along the outer edges, the ribbed slab system is generally supported on stiff edge beams or walls. In wall-supported systems, the thickness of the rib resting on the wall is usually increased to match the wall thickness for improved bearing. Waffle slabs, used in large-span construction, may rest directly on columns; in this case, the slab is made solid in the neighbourhood of the column.

### *Flat Plate System*

Here, the floor slab is supported directly on the columns, without the presence of stiffening beams, except at the periphery [Fig. 1.12]. It has a uniform thickness of about 125-250 mm for spans of 4.5-6 m. Its load carrying capacity is restricted by the limited shear strength and hogging moment capacity at the column supports. Because it is relatively thin and has a flat under-surface, it is called a *flat plate*, and certainly has much architectural appeal. It is used in the developed countries at locations (in apartments and hotels) where floor loads are low, spans are not large, and plane soffits serve as ceilings. However, it is yet to gain popularity in India — perhaps, because it is too daring a concept?

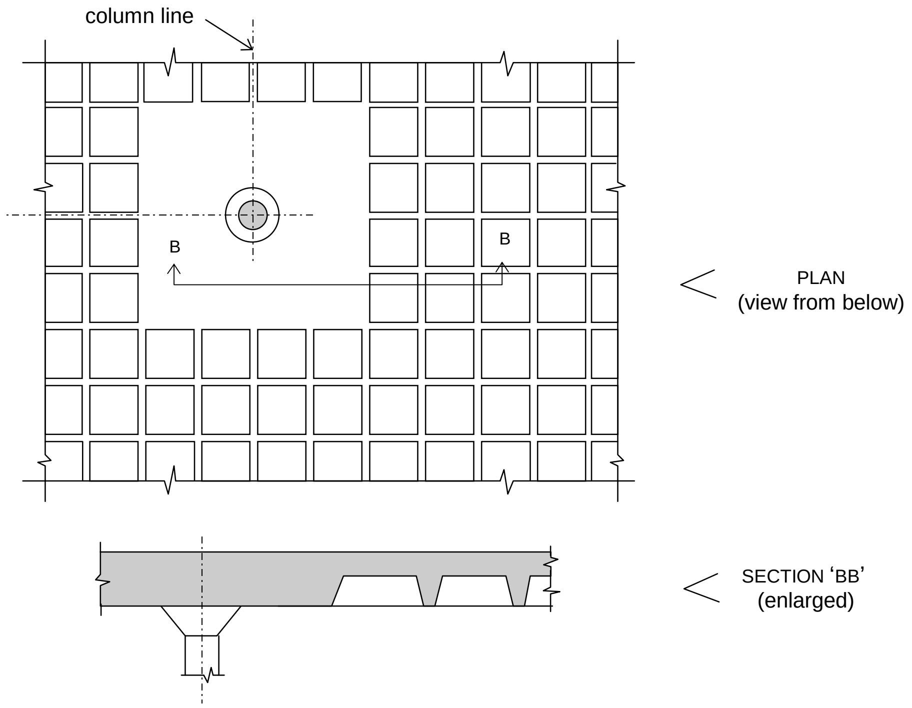



**Fig. 1.11(b)** Two-way ribbed (waffle) slab system

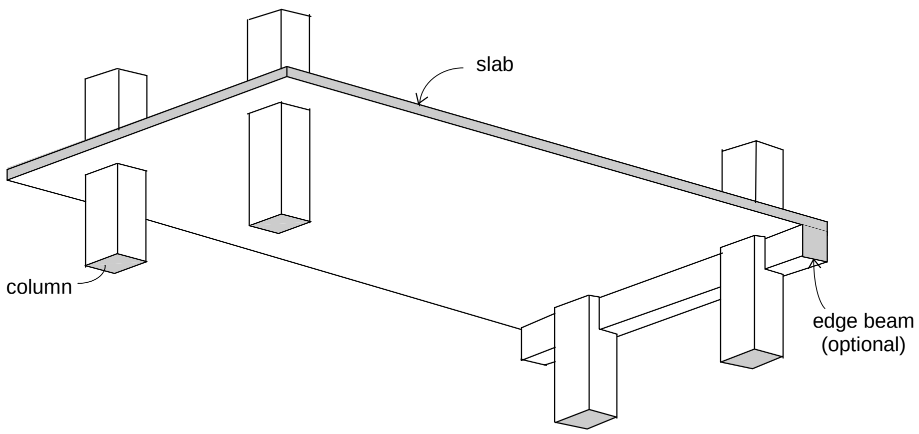



**Fig. 1.12** Flat plate system

### *Flat Slab System*

This is a more acceptable concept to many designers [Fig. 1.13]. It is adopted in some office buildings. The flat slabs are plates that are stiffened near the column supports by means of 'drop panels' and/or 'column capitals' (which are generally concealed under 'drop ceilings'). Compared to the flat plate system, the flat slab system is suitable for higher loads and larger spans, because of its enhanced capacity in resisting shear and hogging moments near the supports. The slab thickness varies from 125 mm to 300 mm for spans of 4-9 m. Among the various floor systems, the flat slab system is the one with the highest dead load per unit area.

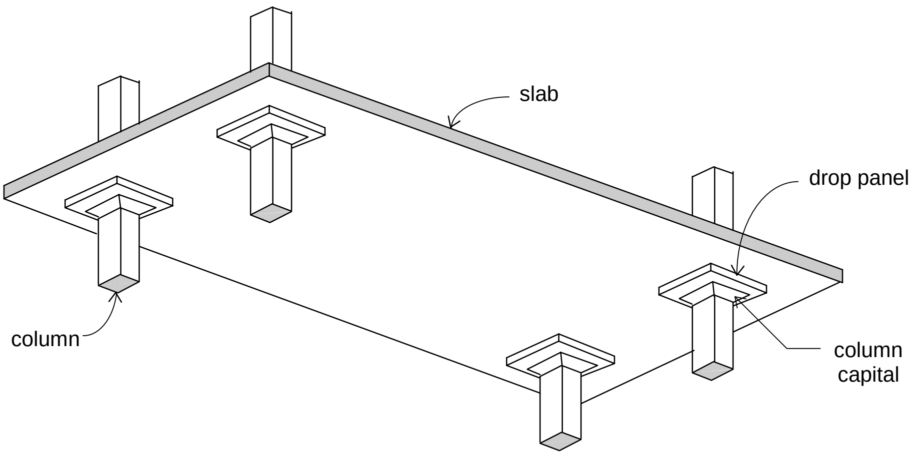



**Fig. 1.13** Flat slab system

### 1.6.2 Vertical Framing System

As mentioned earlier, the vertical framing system resists the gravity loads and lateral loads from the floor system and transmits these effects to the foundation and ground below. The framing system is made up of a three-dimensional framework of beams and columns. For convenience, we may divide the framework into separate *plane frames* in the transverse and longitudinal directions of the building.

In cast-in-situ reinforced concrete construction, the vertical framing system usually comprises the following:

### *Columns*

These are skeletal structural elements, whose cross-sectional shapes may be rectangular, square, circular, L-shaped, etc. — often as specified by the architect. The size of the column section is dictated, from a structural viewpoint, by its height and the loads acting on it — which, in turn, depend on the type of floor system, spacing of columns, number of storeys, etc. The column is generally designed to resist axial compression combined with (biaxial) bending moments that are induced by 'frame action' under gravity and lateral loads. These load effects are more pronounced in the lower storeys of tall buildings; hence, high strength concrete (up to 50 MPa) with high reinforcement area (up to 6 percent of the concrete area) is frequently adopted in such cases, to minimise the column size. In some situations, the column height between floor slabs may be excessive (more than one storey height); in such cases, it is structurally desirable to reduce the unsupported length of the column by providing appropriate *tie beams*; otherwise, the columns should be properly designed as *slender columns*.

### *Walls*

These are vertical elements, made of masonry or reinforced concrete. They are called *bearing walls* if their main structural function is to support gravity loads, and are referred to as *shear walls* if they are mainly required to resist lateral loads due to wind and earthquake. The thickness of reinforced concrete bearing walls varies from 125 mm to 200 mm; however, shear walls may be considerably thicker in the lower storeys of tall buildings. The walls around the lift cores of a building often serve as shear walls.

### *Transfer Girders*

In some buildings, the architectural planning is such that large column-free spaces are required in the lower floors — for banquet/convention halls (in hotels), lobbies, parking areas, etc.

In such cases, the vertical load-bearing elements (columns, bearing walls) of the upper floors are not allowed to continue downwards, through the lower floors, to the foundations below. This problem can be resolved by providing a very heavy beam, called *transfer girder*, whose depth may extend over one full storey [Fig. 1.14]. The upper-storey columns terminate above the transfer girder, and transmit their loads, through the beam action of the girder, to the main columns that support the girder from below.

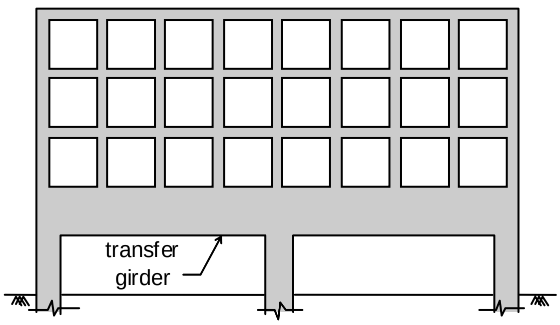



**Fig. 1.14** Use of transfer girder

### *Suspenders*

These are vertical elements used to suspend floor systems such as the cantilevered upper storeys of a multi-storeyed building from a central reinforced concrete core [Fig. 1.15]. Structural steel is often found to be better suited for use as suspenders (also called *hangers*), because the force to be resisted is direct tension; moreover, steel hangers take up very little of the floor space. The loads from the suspenders may be transmitted to the reinforced concrete core by means of large cantilevered beams, cross-braced trusses or Vierendeel girders [also refer Fig. 1.2].

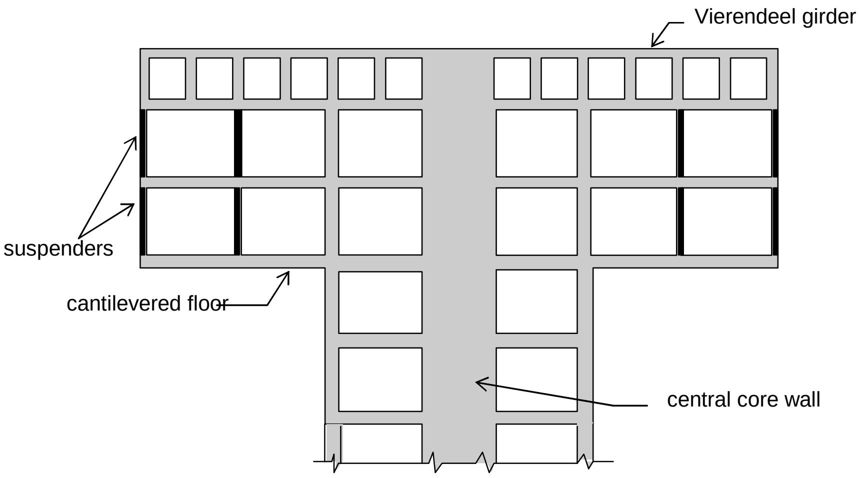



**Fig. 1.15** Use of suspenders

It may be noted that the vertical elements in the bow-string girder of Fig. 1.5 also act as suspenders, transmitting the loads of the bridge deck to the arches spanning between the piers.

## 1.6.3 Lateral Load Resisting Systems

As mentioned earlier, the horizontal and vertical sub-systems of a structural system interact and jointly resist both gravity loads and lateral loads. Lateral load effects (due to wind and earthquake) predominate in tall buildings, and govern the selection of the structural system.

Lateral load resisting systems of reinforced concrete buildings generally consist of one of the following:

### *Frames*

These are generally composed of columns and beams [Fig. 1.8(b) and 1.16(a)]. Their ability to resist lateral loads is entirely due to the rigidities of the beam-column connections and the moment-resisting capacities of the individual members. They are often (albeit mistakenly) called 'rigid frames', because the ends of the various members framing into a joint are 'rigidly' connected in such a way as to ensure that they all undergo the same rotation under the action of loads. In the case of the 'flat plate' or 'flat slab' system, a certain width of the slab, near the column and along the column line, takes the place of the beam in 'frame action'. Frames are used as the sole lateral load resisting system in buildings with up to 15 to 20 storeys [Fig. 1.16(e)].

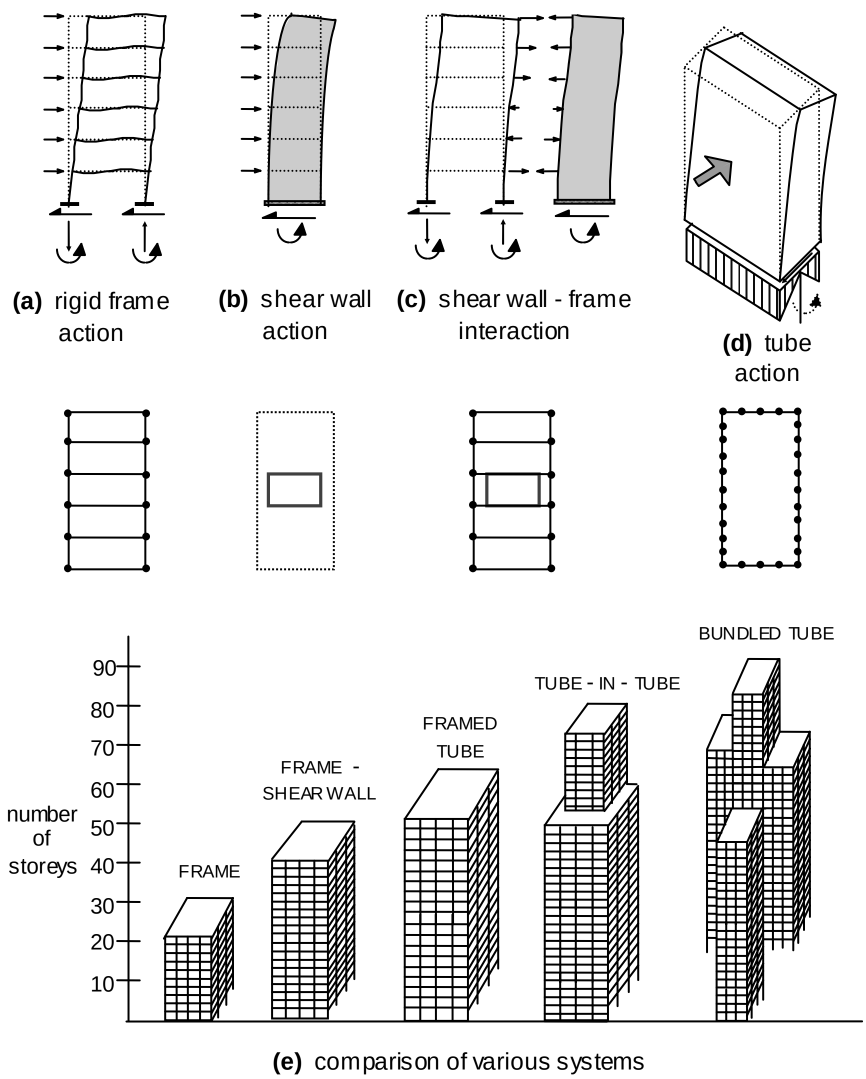



**Fig. 1.16** Lateral load resisting systems

### *Shear Walls*

These are solid walls, which usually extend over the full height of the building. They are commonly located at the lift/staircase core regions. Shear walls are also frequently placed along the transverse direction of a building, either as exterior (facade) walls or as interior walls. The walls are very stiff, having considerable depth in the direction of lateral loads [Fig. 1.16(b)]; they resist loads by bending like vertical cantilevers, fixed at the base.

The various walls and co-existing frames in a building are linked at the different floor levels by means of the floor system, which distributes the lateral loads to these different systems appropriately. The interaction between the shear walls and the frames is structurally advantageous in that the walls restrain the frame deformations in the lower storeys, while the frames restrain the wall deformations in the upper storeys [Fig. 1.16(c)]. Frame-shear wall systems are generally considered in buildings up to about 40 storeys, as indicated in Fig. 1.16(e) [Ref. 1.5].

### *Tubes*

These are systems in which closely-spaced columns are located along the periphery of a building. Deep spandrel beams, located on the exterior surface of the building, interconnect these columns. The entire system behaves like a perforated box or **framed tube** with a high flexural rigidity against lateral loads [Fig. 1.16(d)]. When the (outer) framed tube is combined with an 'inner tube' (or a central shear core), the system is called a **tube-in-tube**. When the sectional plan of the building comprises several perforated tubular *cells*, the system is called a **bundled tube** or 'multi-cell framed tube'. Tubular systems are effective up to 80 storeys, as indicated in Fig. 1.16(e). Widely adopted in the big cities of developed countries, these sky-scraping systems are on the verge of making an appearance in the metros of India.

## 1.7 STRUCTURAL ANALYSIS AND DESIGN

It is convenient to separate the work of a structural designer into **analysis** and **design,** although a rigid separation is neither possible nor desirable. When a student undergoes separate courses on **structural analysis** and **structural design**, it is essential that he realises the nature of their mutual relationship.

The purpose of analysis is to determine the stress resultants and displacements in the various members of a structure under any loading (static or dynamic). The purpose of design is to provide adequate member sizes, reinforcement and connection details, so as to enable the structure to withstand safely the calculated load effects. In order to perform analysis, the proportions of the various structural elements should be known in advance; for this, a preliminary design is generally required. Thus, in practice, analysis and design are interactive processes.

This book is confined to reinforced concrete **design**. It covers the basic principles of designing structural members for flexure, shear, torsion and axial compression — with applications to beams, slabs, staircases, columns, footings and retaining walls. Applications to special structures, such as bridges, chimneys, water tanks and silos are not covered here, although the basic principles of design remain the same.

Furthermore, the various methods of analysis of structures [Ref. 1.6-1.9] clearly lie outside the scope of this book. However, some approximations in analysis, as permitted by design codes, are discussed in some of the chapters to follow.

### Exposure to Construction Practices

In reinforced concrete structures, **construction practices** are as important as the **design**. Indeed, for a correct understanding of design as well as the Code provisions, some exposure to concrete laboratory work and to actual reinforced concrete construction work in the field is required.

> Frequently, major concrete structures are constructed right in the college campus (or nearby). Students (and teachers!) should take full advantage of these opportunities to visit the sites and supplement and reinforce the theory they learn in the class room. Learning can emerge from both good practice and bad practice!

## 1.8 DESIGN CODES AND HANDBOOKS

### 1.8.1 Purpose of Codes

National building codes have been formulated in different countries to lay down guidelines for the design and construction of structures. The codes have evolved from the collective wisdom of expert structural engineers, gained over the years. These codes are periodically revised to bring them in line with current research, and often, current trends.

The codes serve at least four distinct functions. Firstly, they ensure adequate structural safety, by specifying certain essential minimum requirements for design. Secondly, they render the task of the designer relatively simple; often, the results of sophisticated analyses are made available in the form of a simple formula or chart. Thirdly, the codes ensure a measure of consistency among different designers. Finally, they have some legal validity, in that they protect the structural designer from any liability due to structural failures that are caused by inadequate supervision and/or faulty material and construction.

> The codes are not meant to serve as a substitute for basic understanding and engineering judgement. The student is, therefore, forewarned that he will make a poor designer if he succumbs to the unfortunate (and all-too-common) habit of blindly following the codes. On the contrary, in order to improve his understanding, he must learn to question the code provisions — as, indeed, he must, nearly everything in life!

### 1.8.2 Basic Code for Design

The design procedures, described in this book, conform to the following Indian code for reinforced concrete design, published by the Bureau of Indian Standards, New Delhi:

**IS 456 : 2000** — Plain and reinforced concrete – Code of practice (fourth revision)

This code shall henceforth be referred to as 'the Code' in the chapters to follow. References have also been made to other national codes, such as ACI 318, BS 8110, CSA CAN3-A23.3 and Eurocode, wherever relevant.

### 1.8.3 Loading Standards

The loads to be considered for structural design are specified in the following loading standards:

**IS 875 (Parts 1-5) : 1987** — Code of practice for design loads (other than earthquake) for buildings and structures (second revision)

**Part 1** : Dead loads
**Part 2** : Imposed (live) loads
**Part 3** : Wind loads
**Part 4** : Snow loads
**Part 5** : Special loads and load combinations

**IS 1893 : 2002** — Criteria for earthquake resistant design of structures (fourth revision).

### 1.8.4 Design Handbooks

The Bureau of Indian Standards has also published the following handbooks, which serve as useful supplements to the 1978 version of the Code. Although the handbooks need to be updated to bring them in line with the recently revised (2000 version) of the Code, many of the provisions continue to be valid (especially with regard to structural design provisions).

**SP 16 : 1980** — Design Aids (for Reinforced Concrete) to IS 456 : 1978
**SP 24 : 1983** — Explanatory Handbook on IS 456 : 1978
**SP 34 : 1987** — Handbook on Concrete Reinforcement and Detailing
**SP 23 : 1982** — Design of Concrete Mixes

### 1.8.5 Other Related Codes

There are several other codes that the designer may need to refer to. The codes dealing with material specifications and testing are listed at the end of Chapter 2. Chapter 16 of this book deals with special design provisions related to earthquake-resistant design of reinforced concrete structures. The code related to this topic is:

**IS 13920 : 1993** — Ductile detailing of reinforced concrete structures subjected to seismic forces.

Other codes dealing with the design of special structures, such as liquid-retaining structures, bridges, folded plates and shells, chimneys, bunkers and silos, are not covered in this book, the scope of which is limited to basic reinforced concrete design.

## REVIEW QUESTIONS

1.1 What reasons do you ascribe to concrete gaining the status of the most widely used construction material?

1.2 The occurrence of *flexural tension* in reinforced concrete is well known. Cite practical examples where tension occurs in other forms in reinforced concrete.

1.3 What is the role of *transverse steel ties* [Fig. 1.6(d)] in reinforced concrete columns?

1.4 A reinforced concrete canopy slab, designed as a cantilever, is under construction. Prior to the removal of the formwork, doubts are expressed about the safety of the structure. It is proposed to prop up the free edge of the cantilever with a beam supported on pillars. Comment on this proposal.

1.5 What are the main objectives of structural design?

1.6 List the steps involved in the process of structural design.

1.7 Distinguish between structural *design* and structural *analysis*.

1.8 Consider a typical reinforced concrete building in your institution. Identify the various structural elements in the structural system of the building, and briefly explain how the loads are transmitted to the supporting ground.

1.9 Consider a symmetrical portal frame ABCD with the columns (AB and CD) 4 m high, fixed at the base points A and D. The beam BC has a span of 6 m and supports a uniformly distributed load of 100 kN. From structural analysis, it is found that at each fixed base support, the reactions developed are 50 kN (vertical), 30 kN (horizontal) and 40 kN m (moment). With the help of freebody, bending moment, shear force and axial force diagrams, determine the stress resultants in the design of the beam BC and the column AB (or CD).

1.10 Enumerate the various types of gravity load bearing systems and lateral load resisting systems used in reinforced concrete buildings.

## REFERENCES

1.1 Mehta, P.K. and Monteiro, P.J.M., *Concrete: Microstructure, Properties and Materials*, Indian edition, Indian Concrete Institute, Chennai, 1997.

1.2 Neville, A.M. and Brooks, J.J., *Concrete Technology*, ELBS edition, Longman, London, 1990.

1.3 Faber, C., *Candela, The Shell Builder*, Architectural Press, London, 1960.

1.4 Salvadori, M. and Heller, M., *Structure in Architecture*, Prentice-Hall International, 1986.

1.5 Fintel, M., *Handbook of Concrete Engineering*, Second edition, (Van Nostrand Co., New York), C.B.S. Publishers and Distributors, Delhi, 1986.

1.6 Wang, C.K., *Intermediate Structural Analysis*, McGraw-Hill International edition, 1983.

1.7 Weaver, W. and Gere, J.M., *Matrix Analysis of Framed Structures*, Second edition, Van Nostrand Co., New York, 1980.

1.8 Clough, R.W. and Penzien, J., *Dynamics of Structures*, Second edition, McGraw-Hill International edition, 1993.

1.9 Taranath, B.S., *Structural Analysis and Design of Tall Buildings*, McGraw-Hill International edition, 1988.

# 2

# Basic Material Properties

## 2.1 INTRODUCTION

In order to learn to design reinforced concrete structures, it is desirable to begin with an understanding of the basic materials, viz. **concrete** (including its ingredients) and **reinforcing steel**. Accordingly, this chapter describes briefly some of the important properties of these basic materials.

Much of this chapter is devoted to concrete rather than steel, because the designer (as well as the builder) needs to know more about concrete, which, unlike steel, is not manufactured in factories under controlled conditions. Concrete is generally prepared at the site itself, although *precast* concrete is also used in some cases.

### 2.1.1 Concrete Technology

The making of 'good' concrete is decidedly not an easy job. This is clear from the all-too-common 'bad' concrete. Both good and bad concrete can be prepared from exactly the same constituents: cement, aggregate, water (and, sometimes, admixtures). It is the mix proportions, the 'know-how' and the 'do-how' that makes the difference.

Good concrete is one that has the desired qualities of strength, impermeability, durability, etc., in the *hardened state*. To achieve this, the concrete has to be 'satisfactory' in the *fresh state* (which includes mixing, handling, placing, compacting and curing). Broadly, this means that the *mix* must be of the right proportions, and must be *cohesive* enough to be *transported* and *placed* without *segregation* by the means available, and its *consistency* must be such that it is *workable* and can be *compacted* by the means that are actually available for the job.

A competent concrete technologist will be able to get a fair idea of the nature and properties of hardened concrete by observation and a few simple tests on the fresh concrete. If found unsatisfactory, suitable remedial measures can and should be adopted without having to wait until the concrete hardens, by which time it is too late to effect corrections.

'Concrete technology' is a complete subject in itself, and the reader is advised to consult standard textbooks on the subject [Ref. 2.1, 2.2, 2.3] for a detailed study. In the following sections, some salient features of the making of concrete (covering both ingredients and process) are discussed, followed by a detailed description of the properties of hardened concrete and reinforcing steel.

## 2.2 CEMENT

Cement may be described as a material with adhesive and cohesive properties that make it capable of bonding mineral fragments ('aggregates') into a compact whole [Ref. 2.1]. In this process, it imparts *strength* and *durability* to the hardened mass called concrete. The cements used in the making of concrete are called *hydraulic cements* — so named, because they have the property of reacting chemically with water in an exothermic (heat-generating) process called *hydration* that results in water-resistant products[†]. The products of hydration form a viscous *cement paste*, which coats the aggregate surfaces and fills some of the void spaces between the aggregate pieces. The cement paste loses consistency ('stiffens') on account of gradual loss of 'free water', adsorption and evaporation, and subsequently 'sets', transforming the mixture into a solid mass. If the *consistency* of the cement paste is either excessively 'harsh' or excessively 'wet', there is a danger of *segregation*, i.e., the aggregate tends to separate out of the mix; this will adversely affect the quality of the hardened concrete and result in a 'honeycomb' appearance. The freshly set cement paste gains strength with time ('hardens'), on account of progressive filling of the void spaces in the paste with the reaction products, also resulting in a decrease in porosity and permeability.

There is a common misconception regarding the role of cement in concrete. Many people (including some civil engineers) assume that it is desirable to put in as much cement as possible in a concrete mix — provided, of course, cost is not a constraint. This is simply not true. The use of excessive cement results in cracking of concrete (due to the *heat of hydration* generated and due to *plastic shrinkage* of the cement paste), and leads to increased long-term effects of *creep* and *drying shrinkage* of hardened concrete, resulting in undesirable large deflections and cracking.

### 2.2.1 Portland Cements

The most common type of hydraulic cement used in the manufacture of concrete is known as *Portland cement*, which is available in various forms.

Portland cement was first patented in England in 1824, and was so named because its grey colour resembled a limestone (quarried in Dorset) called 'Portland stone'. Portland cement is made by burning together, to about 1400°C, an intimate mixture (in the form of a slurry) of limestone (or chalk) with alumina-, silica- and iron oxide-

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[†] Cements derived from calcination of gypsum or limestone are 'non-hydraulic' because their products of hydration are not resistant to water; however, the addition of pozzolanic materials can render gypsum and lime cements 'hydraulic' [Ref. 2.2].

bearing materials (such as clay or shale), and grinding the resulting 'clinker' into a fine powder, after cooling and adding a little gypsum. The cement contains four major compounds, viz., tricalcium silicate ($C_3S$), dicalcium silicate ($C_2S$), tricalcium aluminate ($C_3A$) and tetracalcium aluminoferrite ($C_4AF$). By altering the relative proportions of these major compounds, and including appropriate *additives*, different types of Portland cement, with different properties, can be made. For instance, increased proportions of $C_3S$ and $C_3A$ contribute to high early strength; on the contrary, an increased proportion of $C_2S$ retards the early development of strength (and generates less heat of hydration), but enhances ultimate strength [Ref. 2.2]. Adjusting the fineness of cement can also control these properties.

The use of any one of the following types of Portland cement is permitted by the Code (IS 456 : 2000):

***Ordinary Portland Cement*** (OPC) — presently available in three different 'grades' (denoting compressive strength), viz. C33, C43 and C53, conforming to IS 269 : 1989, IS 8112 : 1989 and IS 12269 : 1987 respectively. The numbers 33, 43 and 53 correspond to the 28-day (characteristic⁼) compressive strengths of cement, as obtained from standard tests on cement-sand mortar specimens. These are most commonly used in general concrete construction, where there is no special durability requirement (such as exposure to 'sulphate attack').

***Rapid Hardening Portland Cement*** (RHPC) — conforming to IS 8041 : 1990, is similar to OPC, except that it has more $C_3S$ and less $C_2S$, and it is ground more finely. It is used in situations where a rapid development of strength is desired (e.g., when formwork is to be removed early for reuse).

***Portland Slag Cement*** (PSC) — conforming to IS 455 : 1989, is made by inter-grinding Portland cement clinker and granulated *blast furnace slag* (which is a waste product in the manufacture of pig iron). It has fairly high sulphate resistance, rendering it suitable for use in environments exposed to sulphates (in the soil or in ground water).

***Portland Pozzolana Cements*** (PPC) — *flyash based* or *calcined clay based,* conforming respectively to Parts 1 and 2 of IS 1489 : 1991, involves the addition of 'pozzolana' (flyash or calcined clay) — a mineral additive containing silica; the pozzolana is generally cheaper than the cement it replaces. These cements hydrate and gain strength relatively slowly, and therefore require *curing* over a comparatively longer period. They are suitable in situations (such as mass concreting) where a low rate of heat of hydration is desired.

***Hydrophobic Portland Cement*** (HPC) — conforming to IS 8043 : 1991, is obtained by inter-grinding Portland cement with 0.1–0.4 percent of oleic acid or

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⁼ The term 'characteristic strength' is defined in Section 2.6.1. Higher grade OPC is now widely available in India, and is achieved in cement manufacture by increased proportion of lime (which enhances $C_3S$) and increased fineness (up to 325 kg/m$^2$). The higher the grade of cement, the quicker will be the strength gain of the concrete mixture. However, in the long run, the strength development curves more or less converge for the various grades of cement.

stearic acid. The 'hydrophobic' (water-resistant) property is due to the formation of a water-repellent film around each particle of cement. During the mixing of concrete, this film is broken, thereby making it possible for normal hydration to take place. Although its early strength is low, this cement is suitable in situations where cement bags are required to be stored for a prolonged period under unfavourable conditions, because it deteriorates very little.

***Low Heat Portland Cement*** (LHPC) — conforming to IS 12600 : 1989, is Portland cement with relatively lower contents of the more rapidly hydrating compounds, $C_3S$ and $C_3A$. The process of hydration is slow (as with PPC), and the consequent rate of heat generation is also low. This is desirable in mass concreting of gravity dams; as otherwise, the excessive heat of hydration can result in serious cracking. However, because of the slower rate of strength gain, adequate precaution should be taken in their use such as with regard to removal of formwork, etc.

***Sulphate Resisting Portland Cement*** (SRPC) — conforming to IS 12330 : 1988, is Portland cement with a very low $C_3A$ content and ground finer than OPC. This cement is 'sulphate-resistant' because the disintegration of concrete, caused by the reaction of $C_3A$ in hardened cement with a sulphate salt from outside is inhibited. SRPC is therefore ideally suited for use in concrete structures located in soils where sulphates are present. However, recent research indicates that the use of SRPC is not beneficial in environments where chlorides are present.

***Portland White Cement*** (PWC) — conforming to IS 269 : 1989, is Portland cement made from raw materials of low iron content, the clinker for which is fired by a reducing flame. Special precautions are required during the grinding of the clinker to avoid contamination. The addition of pigments to a white cement concrete mix makes it possible to produce concrete with pastel colours. White cement is far more expensive, compared to OPC, and is used mainly for architectural purposes — in floor and wall finishes, swimming pool surfaces, etc.

The Code permits the use of combinations of Portland cements with mineral admixtures, provided they meet the desired performance standards. The term 'blended cements' is now gaining popularity; it refers to cements obtained by combination with various pozzolanic admixtures such as *flyash* (of proper quality) and *ground granulated blast furnace slag*.

### 2.2.2 Other Cements

The Code also permits the use of the following special cements 'under special circumstances' — mainly prevention of chemical attack. However, the use of these cements should be done judiciously and with special care.

***High Alumina Cement*** (HAC) or *aluminous cement* — conforming to IS 6452: 1989, is very different in its composition from Portland cements. The raw materials used for its manufacture consist of 'bauxite' (which is a clay with high alumina content) and limestone (or chalk). It has good resistance against attack by sulphates and some dilute acids, and is particularly recommended in marine environments; it also shows a very high rate of strength development.

***Supersulphated Cement*** (SC) — conforming to IS 6909 : 1990, is made by intergrinding a mixture of 80–85 percent of granulated blast furnace slag with 10-15 percent of dead-burnt gypsum and about 5 percent Portland cement clinker. It is highly resistant to sea-water, and can withstand high concentrations of sulphates found in soil or ground water; it is also resistant to peaty acids and oils.

### 2.2.3 Tests on Cements

Testing of cement quality is very important in the production of quality concrete. The quality of cement is determined on the basis of its conformity to the performance characteristics given in the respective IS specification for the cement. Any special features or such other performance characteristics claimed/indicated by manufacturers alongside the "Statutory Quality Marking" or otherwise have no relation with characteristics guaranteed by the Quality Marking as relevant to that cement. Consumers should go by the characteristics given in the corresponding IS specification or seek expert advise (Cl. 5.1.3 of the Code).

Tests are performed in accordance with IS 269 : 1976 and IS 4031 : 1988 to assess the following:

- ***chemical composition*** — analysis to determine the composition of various oxides (of calcium, silica, aluminium, iron, magnesium and sulphur) present in the cement and to ensure that impurities are within the prescribed limits;
- ***fineness*** — a measure of the size of the cement particles, in terms of *specific surface* (i.e., surface area per unit mass); increased fineness enhances the rate of hydration, and hence, also strength development[†];
- ***normal consistency*** — determination of the quantity of water to be mixed to produce 'standard paste';
- ***initial*** and ***final setting times*** — measures of the rate of solidification of standard cement paste (using a 'Vicat needle'); the 'initial setting time' indicates the time when the paste becomes unworkable (to be not less than 30-45 min usually for OPC), whereas the 'final setting time' refers to the time to reach a state of complete solidification (to be not greater than 375-600 min for OPC);
- ***soundness*** — a quality which indicates that the cement paste, once it has set, does not undergo appreciable change in volume (causing concrete to crack); and
- ***strength*** — measured in terms of the stress at failure of hardened cement-sand mortar specimens, subject to compression and tension tests.

## 2.3 AGGREGATE

Since aggregate occupies about three-quarters of the volume of concrete, it contributes significantly to the structural performance of concrete, especially *strength*, *durability* and *volume stability*.

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[†] Modern cements are considerably finer than their predecessors, on account of improved grinding technology; accordingly, these cements also turn out to be stronger in the early stages. However, the heat of hydration released is also higher.

Aggregate is formed from natural sources by the process of weathering and abrasion, or by artificially crushing a larger parent (rock) mass. Other types of aggregates may be used for *plain concrete members* (Code Cl. 5.3.1), however, as far as possible, preference shall be given to natural aggregates. Aggregate is generally categorised into **fine aggregate** (particle size between 0.075 mm and 4.75 mm) and **coarse aggregate** (particle size larger than 4.75 mm), as described in IS 383 : 1970.

Sand, taken from river beds and pits, is normally used as fine aggregate, after it is cleaned and rendered free from silt, clay and other impurities; stone (quarry) dust is sometimes used as a partial replacement for sand.

Gravel and crushed rock are normally used as coarse aggregate. The maximum size of coarse aggregate to be used in reinforced concrete work depends on the thickness of the structural member and the space available around the reinforcing bars. Generally, a maximum *nominal*‡ size of 20 mm is found to be satisfactory in RC structural elements. However, in cases where the member happens to be very thin, the Code (Cl. 5.3.3) specifies that the size should be restricted to one-fourth of the minimum thickness of the member. In the case of heavily reinforced members, it should be restricted to 5 mm less than the minimum clear spacing between bars or minimum *cover* to reinforcement, whichever is smaller. In such situations, the maximum nominal size is frequently taken as 10 mm. In situations where there is no restriction to the flow of concrete, as in most plain concrete work, there is no such restriction on the maximum aggregate size. It is common to use aggregate up to 40 mm nominal size in the *base concrete* underneath foundations. The Code (Cl. 5.3.3) even permits the use of 'plums' above 160 mm in certain cases of mass concreting up to a maximum limit of 20 percent by volume of concrete. *Plums* are large random-shaped stones dropped into freshly-placed mass concrete to economise on the concrete; such mass concrete is sometimes called 'Cyclopean concrete' [Ref. 2.8].

Mention may also be made of a special type of aggregate, known as lightweight aggregate, which (although not used for reinforced concrete work) is sometimes used to manufacture 'lightweight concrete' masonry blocks, which have low unit weight and good thermal insulation and fire resistance properties. Lightweight aggregate may be obtained from natural sources (such as diatomite, pumice, etc.) or artificially, in the form of 'sintered fly ash' or 'bloated clay' (conforming to IS 9142 : 1979).

### 2.3.1 Aggregate Properties and Tests

A number of tests have been described in IS 2386 (Parts 1 - 8) to assess the quality of the aggregate, in terms of the following physical and mechanical properties:

- ***particle size, shape** and **surface texture***: 'size' and 'shape' influence *strength*; 'shape' and 'texture' influence *bond* (between the aggregate and the cement paste) — for instance, it is found that *angular* and somewhat *porous* aggregates are conducive to good bond;
- ***geological classification***: based on the mineral type of the parent rock;

‡ The term 'nominal' (commonly used in reinforced concrete design practice) refers to the *expected* value of any parameter, such as dimension and material strength. The *actual* value may be somewhat different, depending on admissible tolerances.

- ***specific gravity*** and ***bulk density***: of aggregate particle and aggregate whole respectively;
- ***moisture content, water absorption*** and ***bulking of sand*** : the moisture present in aggregate or the moisture that may be absorbed by the aggregate, as the case may be, must be accounted for in the *water content* of the concrete mix; moreover, the presence of water films in between sand particles results in an increase in volume (*bulking of sand*) that must be accounted for in case *volume batching* is employed in mix preparation;
- ***strength***: resistance to compression, measured in terms of the *aggregate crushing value*;
- ***toughness***: resistance to impact, measured in terms of the *aggregate impact value*;
- ***hardness***: resistance to wear, measured in terms of the *aggregate abrasion value*;
- ***soundness***: which indicates whether or not the aggregate undergoes appreciable volume changes due to alternate thermal changes, wetting and drying, freezing and thawing; and
- ***deleterious constituents***: such as iron pyrites, coal, mica, clay, silt, salt and organic impurities, which can adversely affect the hydration of cement, the bond with cement paste, the strength and the durability of hardened concrete.

### 2.3.2 Grading Requirements of Aggregate

'Grading' is the particle size distribution of aggregate; it is measured by *sieve analysis* [IS 2386 (Part 1) : 1963], and is generally described by means of a *grading curve*, which depicts the 'cumulative percentage passing' against the standard IS sieve sizes.

The grading (as well as the type and size) of aggregate is a major factor which influences the *workability* of fresh concrete, and its consequent degree of *compaction*. This is of extreme importance with regard to the quality of hardened concrete, because incomplete compaction results in *voids*, thereby lowering the *density* of the concrete and preventing it from attaining its full compressive strength capability [Fig. 2.1]; furthermore, the impermeability and durability characteristics get adversely affected. It is seen from Fig. 2.1 that as little as 5 percent of voids can lower the strength by as much as 32 percent.

From an economic viewpoint, it may appear desirable to aim for maximum density by a proper grading of aggregate alone — with the smaller particles fitting, as much as possible, into the voids of the larger particles in the dry state, thereby limiting the use of the (more expensive) cement paste to filling in the voids in the fine aggregate. Unfortunately, such a concrete mix is prone to be 'harsh' and unworkable. Moreover, it is very likely to *segregate*, with the coarser particles separating out or settling more than the finer particles.

Evidently, the cement paste must be in sufficient quantity to be able to coat properly all the aggregate surfaces, to achieve the required workability, and to ensure that the particle sizes are distributed as homogeneously as possible without segregation. The presence of more 'fines' (sand and cement) in a mix is found to improve both workability and resistance to segregation, because the fines tend to 'lubricate' the larger particles, and also fill into their voids as *mortar*. However, too much of fine aggregate in a mix is considered to be undesirable, because the durability and impermeability of the hardened concrete may be adversely affected.

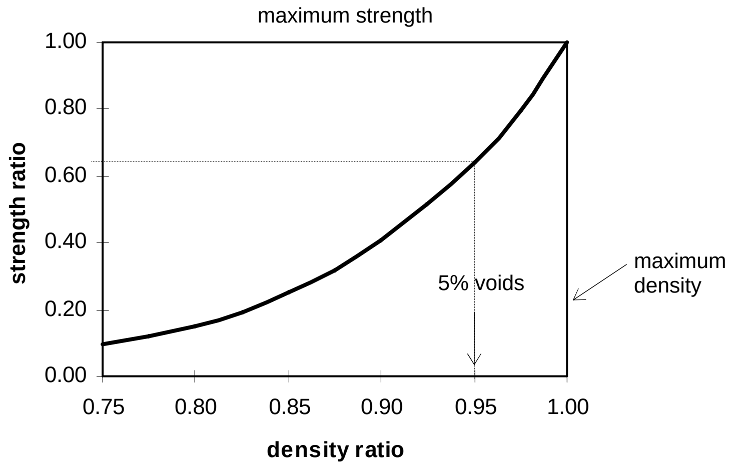


**Fig. 2.1** Relation between density ratio and strength ratio [Ref. 2.1]

On account of these and other interacting factors, it is difficult to arrive at a unique 'ideal' grading in *mix design*. In practice, grading limits are recommended in codes and specifications, which are found *to produce a strong and workable concrete* [Ref. 2.3].

## 2.4 WATER

Water has a significant role to play in the making of concrete — in *mixing* of fresh concrete and in *curing* of hardened concrete. In order to ensure proper strength development and durability of concrete, it is necessary that the water used for mixing and curing is free from impurities such as oils, acids, alkalis, salts, sugar and organic materials.

Water that is fit for human consumption (i.e., *potable water*) is generally considered to be suitable for concreting. However, when the potability of the water is suspect, it is advisable to perform a chemical analysis of the water, in accordance with IS 3025 (Parts 17–32). The pH value of the water should not be less than 6. The concentrations of solids in water should be within certain 'permissible limits' that are specified in the Code (Cl. 5.4). In particular, the content of sulphates (as $SO_3$) is limited to 400 mg/l, while that of chlorides is restricted to 500 mg/l in reinforced concrete (and 2000 mg/l in plain concrete)[†]. Sea water is particularly

[†] Steel reinforcing bars embedded in concrete are highly prone to corrosion in the presence of chlorides (as explained in Section 2.13.3); hence, the Code imposes a stricter control on the chloride control in reinforced concrete, compared to plain concrete.

unsuitable for mixing or curing of concrete. The Code also recommends testing for initial setting time of cement paste (as per IS 4031 (Part 5) : 1988) and compressive strength of concrete cubes (as per IS 516 : 1959), when there is doubt regarding the suitability of the water for proper strength development of concrete.

### 2.4.1 Water Content and Workability of Concrete

The water in a concrete mix is required not only for hydration with cement, but also for *workability*. 'Workability' may be defined as *'that property of the freshly mixed concrete (or mortar) which determines the ease and homogeneity with which it can be mixed, placed, compacted and finished'* [Ref. 2.8]. The main factor that influences workability is, in fact, the water content (in the absence of admixtures), as the 'inter-particle lubrication' is enhanced by the mere addition of water. The amount of water required for lubrication depends on the aggregate type, texture and grading: finer particles require more water to wet their larger specific surface; angular aggregates require more water than rounded ones of the same size; aggregates with greater porosity consume more water from the mix.

Water content in a mix is also related to the fineness of cement — the finer the cement, the greater the need for water — for hydration as well as for workability.

It may be recalled that workability is required to facilitate *full placement* in the formwork (even in areas of restricted access) and *full compaction*, minimising the voids in concrete. If a mix is too dry, bubbles of entrapped air create voids, and there is danger of *segregation* [refer section 2.3.2]. The addition of water provides for better cohesion of the mix and better compaction, and causes the air bubbles to get expelled. However, there is a danger in adding too much water, because it would be water, rather than cement paste, that takes the place of the air bubbles. This water evaporates subsequently, leaving behind voids. Hence, even if the fresh concrete were to be 'fully compacted', voids may still be present in the hardened concrete, adversely affecting its strength, impermeability, etc. Moreover, there is the danger of segregation of 'grout' (cement plus water) in a very wet mix. The excess water tends to rise to the surface of such a mix, as the solid constituents settle downwards; this is called *bleeding*.

The 'optimum' water content in a mix is that at which the *sum of volumes* of entrapped air and of entrapped water is a minimum, and for which the density achievable (by the method of compaction employed) is a maximum [Ref. 2.1].

The Code recommends that the workability of concrete should be controlled by the direct measurement of water content in the mix. For this, workability should be checked at frequent intervals, by one of the standard tests (*slump, compacting factor*, or *vee-bee*), described in IS 1199 : 1959. The Code (Cl. 7.1) also recommends certain ranges of slump, compacting factor and vee-bee time that are considered desirable for various 'degrees of workability' (very low, low, medium, high) and placing conditions.

For the purpose of *mix design* [refer section 2.7], the water content is usually taken in the range 180–200 lit/m$^3$ (unless admixtures are used). If the aggregate is wet, then this should be appropriately accounted for, by measuring the moisture content in the aggregate [refer Cl. 10.2 of the Code].

### 2.4.2 Water-Cement Ratio and Strength

As mentioned earlier, the addition of water in a concrete mix improves workability. However, the water should not be much in excess of that required for hydration. The **water-cement ratio,** defined as the ratio of the *mass* of 'free water' (i.e., excluding that absorbed by the aggregate) to that of cement in a mix, is the major factor that controls the strength and many other properties of concrete. In practice, this ratio lies generally in the range of 0.35 to 0.65, although the purely chemical requirement (for the purpose of complete hydration of cement) is only about 0.25.

It is seen that the compressive strength of hardened concrete is inversely proportional to the water-cement ratio, provided the mix is of workable consistency; this is the so-called **Abrams' law**. A reduction in the water-cement ratio generally results in an increased quality of concrete, in terms of density, strength, impermeability, reduced shrinkage and creep, etc.

In *mix design* (refer Section 2.7.2), the water-cement ratio is selected on the basis of the desired 28-day compressive strength of concrete *and* the 28-day† compressive strength of the cement to be used. For this purpose, appropriate design charts may be made use of [Ref. 2.4 and IS 10262 : 1982]. A simple chart (in which 'strength' is non-dimensionalised) developed for this purpose [Ref. 2.5] is shown in Fig. 2.2.

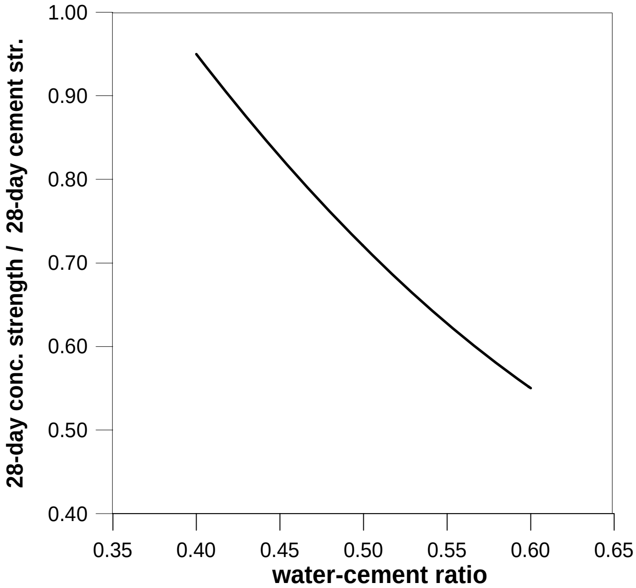


**Fig. 2.2** Relation between water-cement ratio and compressive strength [Ref. 2.5]

†The earlier practice of specifying 7-day strength is discarded, as it is found that some types of cement (such as RHPC) gain early high strength, but the strength at 28-days is no different from that of other cements (such as PPC) which gain strength relatively slowly. Most cements and concretes attain a major part of their long-term strength in about 28 days.

It is found that water-cement ratios of 0.4, 0.5 and 0.6 are expected to produce respectively 28-day concrete strengths that are about 0.95, 0.72 and 0.55 times the 28-day strength of the cement used.

### 2.4.3 Water for Curing

The water in a concrete mix takes one of the following three forms, as a consequence of hydration [Ref. 2.3]:

1. ***combined water*** — which is chemically combined with the products of hydration; it is non-evaporable;
2. ***gel water*** — which is held physically or adsorbed on the surface area of the 'cement gel' (solid hydrates located in tiny, impermeable 'gel pores'); and
3. ***capillary water*** — which partially occupies the 'capillary pores' that constitute the space in the cement paste remaining after accounting for the volumes of cement gel and unhydrated cement; this water is easily evaporated.

If the hardened cement paste is only partly hydrated (as is usually the case, soon after casting), the capillary pores tend to become interconnected; this results in low strength, increased permeability and increased vulnerability of the concrete to chemical attack. All these problems can be overcome, to a large extent, if the degree of hydration is sufficiently high for the capillary pore system to become 'segmented' through partial blocking by the newly developed cement gel.

**Curing** is the name given to procedures that are employed for actively promoting the hydration of cement in a suitable environment during the early stages of hardening of concrete. The Code (Cl. 13.5) defines it as "the process of preventing the loss of moisture from the concrete while maintaining a satisfactory temperature regime". Curing is essential for producing 'good' concrete that has the desired strength, impermeability and durability, and is of particular importance in situations where the water-cement ratio is low, or the cement has a high rate of strength development or if the pozzolanic content is high.

*Moist curing* aims to keep the concrete as nearly saturated as possible at normal temperature — by continually spraying water, or by 'ponding', or by covering the concrete with a layer of any kind of 'sacking' which is kept wet. The ingress of curing water into the capillary pores stimulates hydration. This process, in fact, goes on, even after active curing has stopped, by absorption of the moisture in the atmosphere. The period of curing should be as long as conveniently possible in practice. The Code specifies the duration as "at least seven days from the date of placing of concrete in case of OPC'' under normal weather conditions, and at least ten days when dry and hot weather conditions are encountered. When mineral admixtures or blended cements are used, the recommended minimum period is 10 days, which should preferably be extended to 14 days or more.

Moist curing improves the concrete strength very rapidly in the first few days; subsequently, the gain in strength becomes less and less, as shown in Fig. 2.3. The figure also shows the drastic loss in strength if moist curing is avoided altogether.

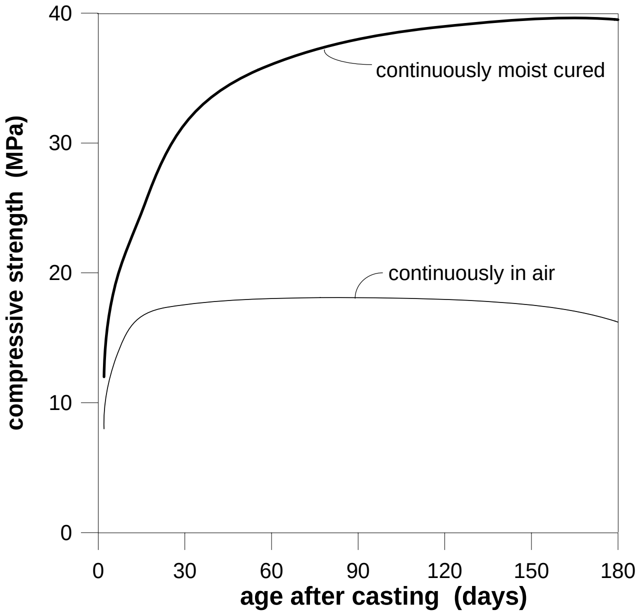


**Fig. 2.3** Influence of moist curing on concrete strength [Ref. 2.6]

Increase in temperature is found to enhance the rate of hydration and the consequent rate of gain of strength. However, the early application of high temperature is found to reduce the long-term strength of concrete [Ref. 2.3]. Hence, it is desirable to take appropriate steps to reduce the temperature of fresh concrete when concreting is done in hot weather.

In some cases, as in the manufacture of prefabricated components, a high early strength is desired, to facilitate handling and transfer of the concrete products soon after casting. In such cases, methods of *accelerated curing* such as 'steam curing' or the more advanced 'autoclave curing' are resorted to. In *steam curing*, steam at atmospheric pressure takes the place of water for curing.

In special cases, *membrane curing* may be resorted to, in lieu of moist curing, by applying either special compounds (usually sprayed on the surface) or impermeable membranes (such as polyethylene sheeting) to all exposed concrete surfaces immediately after the setting of concrete, to prevent the evaporation of water.

## 2.5 ADMIXTURES

Admixtures are additives that are introduced in a concrete mix to modify the properties of concrete in its fresh and hardened states. Some guidelines for admixtures are given in IS 9103 : 1999. A large number of proprietary products are currently available; their desirable effects are advertised in the market. These, as well as possible undesirable effects, need to be examined scientifically, before they are advocated [Ref. 2.7]. The Code (Cl. 5.5.3) recommends, "the workability, compressive strength and the slump loss of concrete with and without the use of admixtures shall be established during the trial mixes before the use of admixtures". Also, the use of admixtures should not impair durability and increase the risk of corrosion to reinforcement.

Admixtures are either 'chemical' (liquid) or 'mineral' (fine granular) in form. They are now being increasingly used in concrete production, particularly when there is an emphasis on either 'high strength' or 'high performance' (durability). The use of chemical admixtures is inevitable in the production of *ready-mixed concrete*, which involves transportation over large distances of fresh concrete that is manufactured under controlled conditions at a batching plant.

### 2.5.1 Types of Chemical Admixtures

Some of the more important chemical admixtures are briefly described here:

***Accelerators***: chemicals (notably, calcium chloride) to accelerate the hardening or the development of early strength of concrete; these are generally used when urgent repairs are undertaken, or while concreting in cold weather;

***Retarders***: chemicals (including sugar) to retard the setting of concrete, and thereby also to reduce the generation of heat; these are generally used in hot weather concreting and in *ready-mixed concrete*;

***Water-reducers*** (or *plasticizers*): chemicals to improve plasticity in the fresh concrete; these are mainly used for achieving higher strength by reducing the water-cement ratio; or for improving workability (for a given water-cement ratio) to facilitate placement of concrete in locations that are not easily accessible;

***Superplasticizers*** (or *high-range water-reducers*): chemicals that have higher dosage levels and are supposedly superior to conventional water-reducers; they are used for the same purposes as water-reducers, viz. to produce high-strength concrete or to produce 'flowing' concrete;

***Air-entraining agents***: organic compounds (such as animal/vegetable fats and oils, wood resins) which introduce discrete and microscopic air bubble cavities that occupy up to 5 percent of the volume of concrete; these are mainly used for protecting concrete from damage due to alternate freezing and thawing;

***Bonding admixtures***: polymer emulsions (latexes) to improve the adherence of fresh concrete to (old) hardened concrete; they are ideally suited for repair work.

### 2.5.2 Types of Mineral Admixtures

**Mineral admixtures** are used either as partial replacement of cement or in combination with cement, at the time of mixing, in order to modify the properties of concrete or achieve economy. Some of the more important mineral admixtures are described briefly here.

***Pozzolanas*** are materials containing amorphous silica, which, in finely divided form and in the presence of water, chemically react with calcium hydroxide at ordinary temperatures to form compounds possessing cementitious properties; the Code (Cl. 5.2) permits their use, provided uniform blending with cement is ensured.

Many of the pozzolanas (especially fly ash) are industrial 'waste products' whose disposal raise environmental concerns; their use in concrete making, hence, is commendable. These include:

* *fly ash*: ash precipitated electrostatically or mechanically from exhaust gases in coal-fired power plants, conforming to Grade 1 of IS 3812;
* *ground granulated blast-furnace slag*, conforming to IS 12089, has good pozzolanic properties, and produces concrete with improved resistance to chemical attack;
* *silica fume* (or *micro silica*), obtained as a by-product of the silicon industry, is found to be not only pozzolanic in character but also capable of producing very dense concrete, and is finding increasing use in the production of high-strength and high-performance concrete;
* *rice husk ash*: produced by burning rice husk at controlled temperatures;
* *metakaoline*: obtained by calcination of kaolinitic clay(a natural pozzolana), followed by grinding;

***Gas-forming admixtures***: *powdered zinc, powdered aluminium* and *hydrogen peroxide*, which generate gas bubbles in a sand-cement matrix; they are used in the manufacture of lightweight *aerated concrete* — which, although not suitable for heavy load-bearing purposes, can be used for its high thermal insulation properties.

## 2.6 GRADE OF CONCRETE

The desired properties of concrete are its compressive strength, tensile strength, shear strength, bond strength, density, impermeability, durability, etc. Among these, the property that can be easily tested, and is perhaps the most valuable (from the viewpoint of structural design) is the *compressive strength*. This is measured by standard tests on concrete cube (or cylinder) specimens. Many of the other important properties of concrete can be inferred from the compressive strength, using correlations that have been experimentally established.

The quality or *grade* of concrete is designated in terms of a number, which denotes its *characteristic* compressive strength (of 150 mm cubes at 28-days), expressed in MPa (or, equivalently, N/mm$^2$). The number is usually preceded by the letter 'M', which refers to 'mix'. Thus, for example, M 20 grade concrete denotes a concrete whose mix is so designed as to generate a **characteristic strength** of 20 MPa; the meaning of this term is explained in the next section.

In the recent revision of the Code, the selection of the minimum grade of concrete is dictated by considerations of durability, and is related to the kind of environment that the structure is exposed to [refer Table 5 of the Code]. The minimum grade of concrete in reinforced concrete work has been upgraded from M 15 to M 20 in the recent code revision†. However, this is applicable only under 'mild' exposure

† It may be noted that the traditional 'nominal mix' of 1:2:4 (cement : sand : coarse aggregate, by weight), which used to conform approximately to M 15 grade of concrete (using OPC of C 33 grade), is presently found to yield higher grades (M 20 and higher), with the modern use of C 43 and C 53 grades of cement, which are now commonly available in the market.

conditions. An exposure condition is considered 'mild' when the concrete surface is protected against weather or aggressive conditions and is not situated in a coastal area. Under more adverse environmental exposure[‡] conditions, higher grades of concrete are called for. For 'moderate', 'severe', 'very severe' and 'extreme' exposure conditions, the minimum grades prescribed are M 25, M 30, M 35 and M 40 respectively, for reinforced concrete work [Cl. 6.1.2 of the Code]. It should be noted that the higher grades specified here are dictated, not by the need for higher compressive strength, but by the need for improved durability [refer Section 2.13]. The need is for 'high performance' concrete, and it is only incidental that this high performance (obtained, for example, by reducing the water-cement ratio and adding mineral admixtures such as silica fume) is correlated with high strength. In practice, although M20 is the minimum grade specified for reinforced concrete, it is prudent to adopt a higher grade.

However, there are specific applications that may call for the grade of concrete to be decided on the basis of considerations of strength, rather than durability. For example, the use of high strength is desirable in the columns of very tall buildings, in order to reduce their cross-sectional dimensions; this is desirable even under 'mild' environmental exposure. Similarly, high strength concrete is required in prestressed concrete construction [refer IS 1343 : 1980]. The definition of the term 'high strength' has been changing over the years, with technological advancements resulting in the development of higher strengths. The present Code (in its recent revision) describes grades of concrete above M 60 as 'high strength concrete'. Concrete grades in the range M 25 to M 55 are described as 'standard strength concrete', and grades in the range M 10 to M 20 are termed 'ordinary concrete' [refer Table 2 of the Code].

### 2.6.1 Characteristic Strength

Concrete is a material whose strength is subject to considerable *variability*. Cube specimens that are taken from the same mix give different values of compressive strength in laboratory tests. This may be attributed largely to the non-homogeneous nature of concrete. The variability in the strength evidently depends on the degree of quality control [Fig. 2.4]. Statistically, it is measured in terms of either the 'standard deviation' ($\sigma$) or the coefficient of variation (*cov*), which is the ratio of the standard deviation to the mean strength ($f_{cm}$).

Experimental studies have revealed that the probability distribution of concrete strength (for a given mix, as determined by compression tests on a large number of specimens) is approximately 'normal' (Gaussian) [Ref. 2.9]. The coefficient of variation is generally in the range of 0.01 to 0.02; it is expected to reduce with increasing grade of concrete, in view of the need for increased quality control.

In view of the significant variability in the compressive strength, it is necessary to ensure that the designer has a reasonable assurance of a certain minimum strength of concrete. This is provided by the Code by defining a *characteristic strength*, which is applicable to any material (concrete or steel):

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[‡] The different types of exposure are described in detail in Section 2.13.1

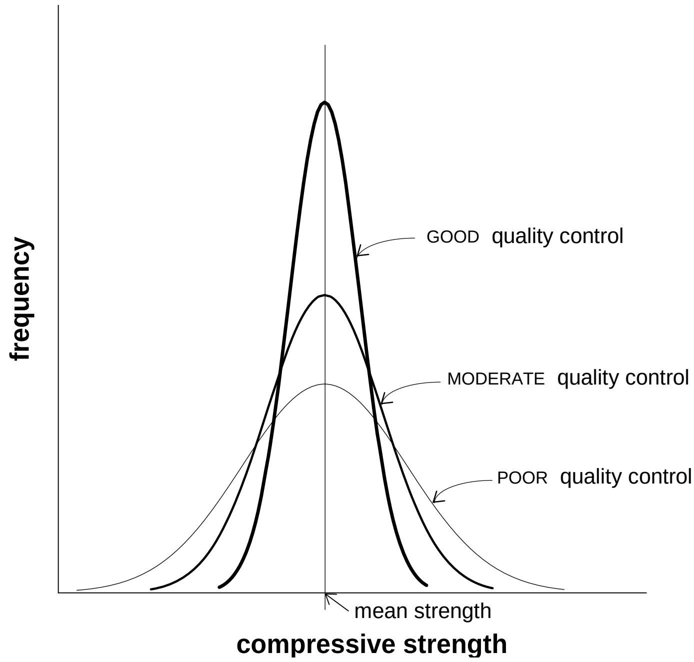



**Fig. 2.4** Influence of quality control on the frequency distribution of concrete strength

"**Characteristic strength** is defined as the strength of material below which not more than 5 percent of the test results are expected to fall"

Accordingly, the mean strength of the concrete $f_{cm}$ (as obtained from 28-day compression tests) has to be significantly greater than the 5 percentile characteristic strength $f_{ck}$ that is specified by the designer [Fig. 2.5].

## 2.7 CONCRETE MIX DESIGN

The design of a concrete mix for a specified *grade* involves the economical selection of the relative proportions (and type) of cement, fine aggregate, coarse aggregate and water (and admixtures, if any). Although compliance with respect to 'characteristic strength' is the main criterion for acceptance, it is implicit that the concrete must also have the desired *workability* in the fresh state, and *impermeability* and *durability* in the hardened state.

### 2.7.1 Nominal Mix Concrete

Concrete mix design is an involved process that calls for some expertise from the construction engineer/contractor. This is not often available. Traditionally, mixes were specified in terms of fixed ratios of cement : sand : coarse aggregate (by mass preferably, or by volume) such as 1 : 2 : 4, 1 : 1.5 : 3, etc. — which are rather crude and incorrect translations of concrete grades M 15, M 20, etc.

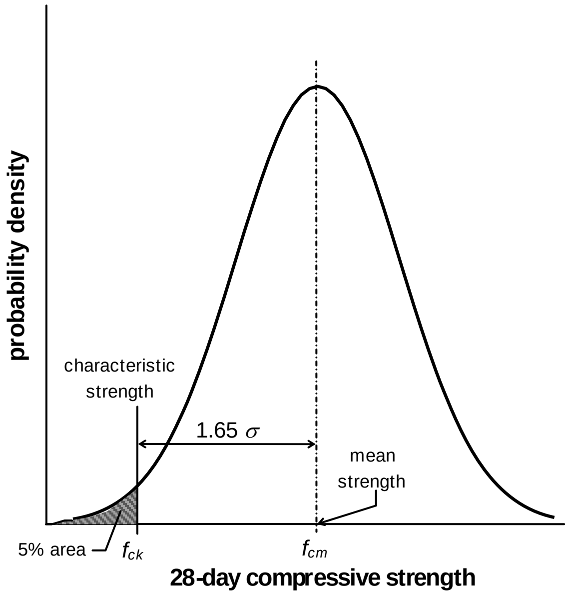



**Fig. 2.5** Idealised normal distribution of concrete strength

The Code (Table 9) attempts to provide more realistic ‘nominal mix’ proportions for M 5, M 7.5, M 10, M 15 and M 20 grades of concrete, in terms of the total mass of aggregate, proportion of fine aggregate to coarse aggregate and volume of water to be used per 50 kg mass of cement (i.e., one bag of cement). Such *nominal mix concrete* is permitted in ‘ordinary concrete construction’, which does not call for concrete grades higher than M 20.

However, the Code clearly highlights (Cl. 9.1.1) that *design mix concrete*, based on the principles of ‘mix design’, is definitely preferred to ‘nominal mix concrete’. In practice, it is found that design mix concrete not only yields concrete of the desired quality, but also often works out to be more economical than nominal mix concrete.

### 2.7.2 Design Mix Concrete

Several methods of mix design have been evolved over the years in different countries, and have become codified — such as the ACI practice [Ref. 2.10–2.12], the British practice [Ref. 2.13], etc. In India, recommendations for mix design are given in IS 10262: 1982 and SP 23: 1982 [Ref. 2.4]. These are merely ‘recommendations’; in practice, any proven method of design may be adopted. All that matters finally is that the designed mix must meet the desired requirements in the fresh and hardened states. The steps involved in the Indian Standard recommendations for mix design are summarised as follows:

1. Determine the *mean target strength* $f_{cm}$ from the desired 'characteristic strength' $f_{ck}$ [Fig. 2.5]:

$$f_{cm} = f_{ck} + 1.65\sigma \tag{2.1}$$

where the standard deviation $\sigma$ depends on the quality control, which may be assumed for design in the first instance, are listed in Table 8 of the Code. As test results of samples are available, actual calculated value is to be used].

2. Determine the *water-cement ratio*, based on the 28-day strength of cement and the mean target strength of concrete, using appropriate charts (such as Fig. 2.2); this ratio should not exceed the limits specified in Table 5 of the Code (for durability considerations).
3. Determine the *water content* $V_w$ based on workability requirements, and select the ratio of fine aggregate to coarse aggregate (by mass), based on the type and grading of the aggregate; the former is generally in the range of 180–200 lit/m$^3$ (unless admixtures are employed), and the latter is generally 1:2 or in the range of 1:1½ to 1:2½.
4. Calculate the *cement content* $M_c$ (in kg/m$^3$) by dividing the water content by the water-cement ratio, and ensure that the cement content is not less than that specified in the Code [Tables 4 and 5] for durability considerations. [Note that the Code (Cl.8.2.4.2) cautions against the use of cement content (not including fly ash and ground granulated blast furnace slag) in excess of 450 kg/m$^3$ in order to control shrinkage and thermal cracks]. Also, calculate the masses of fine aggregate $M_{fa}$ and coarse aggregate $M_{ca}$ based on the 'absolute volume principle':

$$\frac{M_c}{\rho_c} + \frac{M_{fa}}{\rho_{fa}} + \frac{M_{ca}}{\rho_{ca}} + V_w + V_v = 1.0 \tag{2.2}$$

where $\rho_c, \rho_{fa}, \rho_{ca}$ denote the mass densities of cement, fine aggregate and coarse aggregate respectively, and $V_v$ denotes the volume of voids (approx. 2 percent) per cubic metre of concrete.

5. Determine the weight of ingredients per batch, based on the capacity of the concrete mixer.

## 2.8 BEHAVIOUR OF CONCRETE UNDER UNIAXIAL COMPRESSION

The strength of concrete under *uniaxial compression* is determined by loading 'standard test cubes' (150 mm size) to failure in a compression testing machine, as per IS 516 : 1959. The test specimens are generally tested 28 days after casting (and continuous curing). The loading is *strain-controlled* and generally applied at a uniform strain rate of 0.001 mm/mm per minute in a standard test. The maximum stress attained during the loading process is referred to as the *cube strength* of concrete. As discussed in section 2.6.1, the cube strength is subject to variability; its characteristic (5-percentile) and mean values are denoted by $f_{ck}$ and $f_{cm}$ respectively.

In some countries (such as USA), 'standard test *cylinders*' (150 mm diameter and 300 mm high) are used instead of *cubes*. The *cylinder strength* is found to be invariably lower than the 'cube strength' for the same quality of concrete; its nominal value, termed as 'specified cylinder strength' by the ACI code [Ref. 2.21], is denoted by $f_c'$.

It should be noted that among the various properties of concrete, the one that is actually measured in practice most often is the compressive strength. The measured value of compressive strength can be correlated to many other important properties such as tensile strength, shear strength, modulus of elasticity, etc. (as discussed in the sections to follow).

### 2.8.1 Influence of Size of Test Specimen

It has been observed that the *height/width ratio* and the *cross-sectional dimensions* of the test specimen have a pronounced effect on the compressive strength (maximum stress level) obtained from the uniaxial compression test. These effects are illustrated in Fig. 2.6 for cylinder specimens.

The standard test cylinder has a diameter of 150 mm and a height-diameter ratio equal to 2.0. With reference to this 'standard', it is seen that, maintaining the same diameter of 150 mm, the strength increases by about 80 percent as the height/diameter ratio is reduced from 2.0 to 0.5 [Fig. 2.6(a)]; also, maintaining the same height/diameter ratio of 2.0, the strength drops by about 17 percent as the diameter is increased from 150 mm to 900 mm [Fig. 2.6(b)]. Although the real reasons for this behaviour are not known with certainty, some plausible explanations that have been proposed are discussed below.

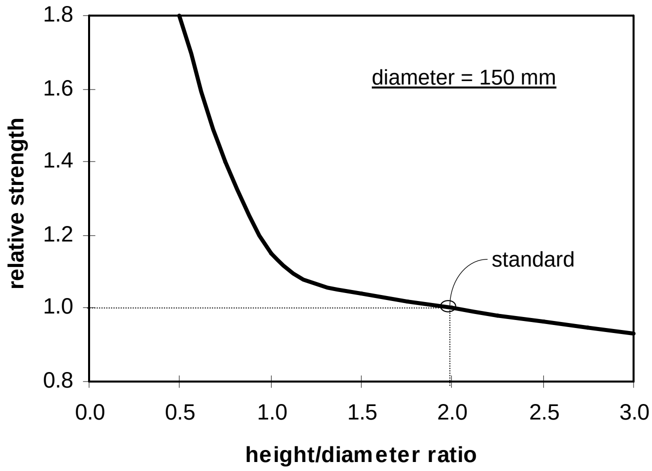
Firstly, a proper measure of uniaxial compressive stress can be obtained (in terms of load divided by cross-sectional area) only if the stress is *uniformly distributed* across the cross-section of the longitudinally loaded test specimen. Such a state of stress can be expected only at some distance away from the top and bottom surfaces where the loading is applied (*St. Venant's principle*) — which is possible only if the height/width ratio of the specimen is sufficiently large.

Secondly, *uniaxial* compression implies that the specimen is not subject to lateral loading or lateral restraint. However, in practice, lateral restraint, known as *platen restraint*, is bound to manifest owing to the friction between the end surfaces of the concrete specimen and the adjacent steel platens of the testing machine. This introduces radial (inward) shear forces at the top and bottom surfaces, resulting in restraint against free lateral displacements.

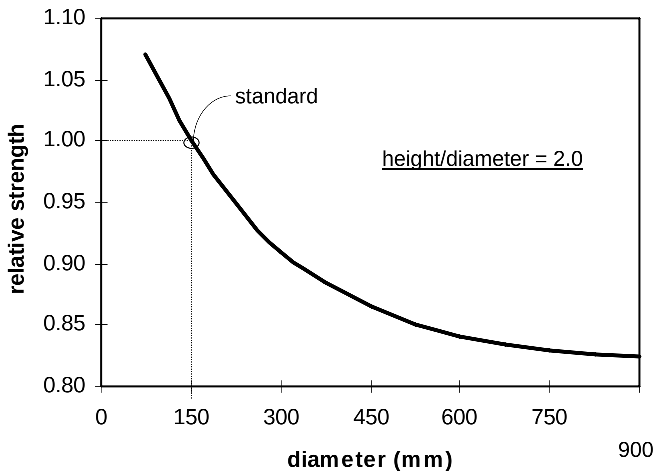
The effect of this lateral restraint is to enhance the *compressive strength* (maximum stress prior to failure) in the longitudinal direction; this effect dies down with increasing distance from the platen restraint. Thus, the value of the compressive strength depends on the *height/width ratio* of the specimen; the greater this ratio, the less the strength, because the less is the beneficial influence of the lateral restraint at the (weakest) section, located near the mid-height of the specimen.

The reduction in compressive strength with increasing size, while maintaining the same height/width ratio [Fig. 2.6(b)], is attributed to *size effect* — a phenomenon which requires a *fracture mechanics* background for understanding.

From the above, it also follows that the 'standard test *cube*' (which has a height/width ratio of 1.0) would register a compressive strength that is higher than that of the 'standard test *cylinder*' (with a height/diameter ratio of 2.0), made of the same concrete, and that the cylinder strength is closer to the true uniaxial compressive strength of concrete. The cube strength is found to be approximately 1.25 times the cylinder strength [Ref. 2.3], whereby $f_c' \approx 0.8 f_{cm}$. For design purposes, the cube strength that is relied upon by the Code is the 'characteristic strength' $f_{ck}$.




**(a)**




**(b)**

**Fig. 2.6** Influence of (a) height/diameter ratio and (b) diameter on cylinder strength [Ref. 2.3, 2.14]

Accordingly, the relation between the cube strength and the cylinder strength takes the following form:

$$f_c' \approx 0.8 f_{ck} \tag{2.3}$$

### 2.8.2 Stress-Strain Curves

Typical stress-strain curves of concrete (of various grades), obtained from standard uniaxial compression tests, are shown in Fig. 2.7. The curves are somewhat linear in the very initial phase of loading; the non-linearity begins to gain significance when the stress level exceeds about one-third to one-half of the maximum. The maximum stress is reached at a strain approximately equal to 0.002; beyond this point, an increase in strain is accompanied by a decrease in stress. For the usual range of concrete strengths, the strain at failure is in the range of 0.003 to 0.005.

The higher the concrete grade, the steeper is the initial portion of the stress-strain curve, the sharper the peak of the curve, and the less the failure strain. For low-strength concrete, the curve has a relatively flat top, and a high failure strain.

When the stress level reaches 70–90 percent of the maximum, internal cracks are initiated in the mortar throughout the concrete mass, roughly parallel to the direction of the applied loading [Ref. 2.15]. The concrete tends to expand laterally, and longitudinal cracks become visible when the lateral strain (due to the *Poisson effect*) exceeds the limiting tensile strain of concrete (0.0001—0.0002). The cracks generally occur at the aggregate-mortar interface. As a result of the associated larger lateral extensions, the apparent Poisson's ratio increases sharply [Ref. 2.16].

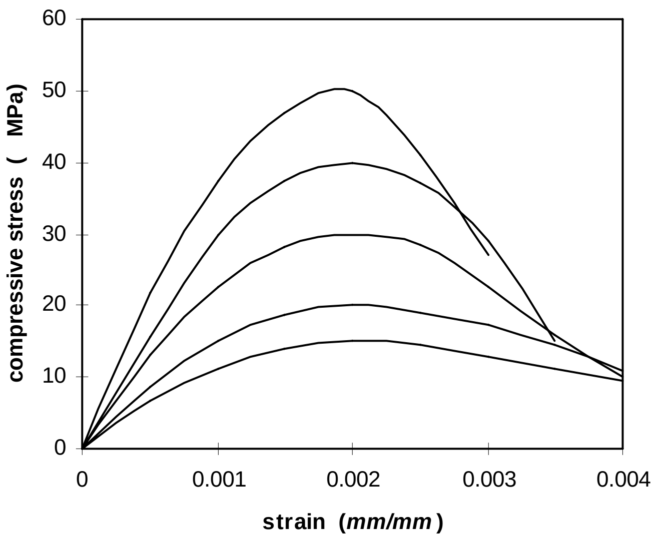



**Fig. 2.7** Typical stress-strain curves of concrete in compression

The descending branch of the stress-strain curve can be fully traced only if the strain-controlled application of the load is properly achieved. For this, the testing machine must be sufficiently rigid‡ (i.e., it must have a very high value of load per unit deformation); otherwise, the concrete is likely to fail abruptly (sometimes, explosively) almost immediately after the maximum stress is reached. The fall in stress with increasing strain is a phenomenon which is not clearly understood; it is associated with extensive *micro-cracking* in the mortar, and is sometimes called *softening* of concrete [Ref. 2.17].

Experimental studies [Ref. 2.17, 2.18] have also confirmed that the stress-strain relation for the compression zone of a reinforced concrete *flexural member* is nearly identical to that obtained for uniaxial compression. For the purpose of design of reinforced concrete flexural members, various simplified stress-strain curves have been adopted by different codes.

### 2.8.3 Modulus of Elasticity and Poisson's Ratio

Concrete is not really an *elastic* material, i.e., it does not fully recover its original dimensions upon unloading. It is not only *non-elastic*; it is also *non-linear* (i.e., the stress-strain curve is nonlinear). Hence, the conventional 'elastic *constants*' (*modulus of elasticity* and *Poisson's ratio*) are not strictly applicable to a material like concrete. Nevertheless, these find place in design practice, because, despite their obvious limitations when related to concrete, they are material properties that have to

‡ Alternatively, a *screw-type* loading mechanism may be used.

be necessarily considered in the conventional *linear elastic analysis* of reinforced concrete structures.

### Modulus of Elasticity

The *Young's modulus of elasticity* is a constant, defined as the ratio, within the *linear elastic* range, of axial stress to axial strain, under uniaxial loading. In the case of concrete under uniaxial compression, it has some validity in the very initial portion of the stress-strain curve, which is practically linear [Fig 2.8]; that is, when the loading is of low intensity, and of very short duration. If the loading is sustained for a relatively long duration, inelastic *creep* effects come into play, even at relatively low stress levels [refer Section 2.11]. Besides, non-linearities are also likely to be introduced on account of creep and shrinkage.

The *initial tangent modulus* [Fig 2.8] is, therefore, sometimes considered to be a measure of the **dynamic modulus** of elasticity of concrete [Ref. 2.3]; it finds application in some cases of cyclic loading (wind- or earthquake-induced), where long-term effects are negligible. However, even in such cases, the non-elastic behaviour of concrete manifests, particularly if high intensity cyclic loads are involved; in such cases, a pronounced *hysterisis effect* is observed, with each cycle of loading producing incremental permanent deformation [Ref. 2.18].

In the usual problems of structural analysis, based on *linear static analysis*, it is the **static modulus** of elasticity that needs to be considered. It may be noted that when the loads on a structure (such as dead loads) are of long duration, the long-term effects of creep reduce the effective modulus of elasticity significantly. Although it is difficult to separate the *long-term strains* induced by creep (and shrinkage) from the *short-term 'elastic' strains*, this is usually done at a conceptual level, for convenience. Accordingly, while estimating the deflection of a reinforced concrete beam, the total deflection is assumed to be a sum of an 'instantaneous' elastic deflection (caused by the loads) and the 'long-term' deflections induced by creep and shrinkage [refer Chapter 10]. The *short-term static modulus of elasticity* ($E_c$) is used in computing the 'instantaneous' elastic deflection.

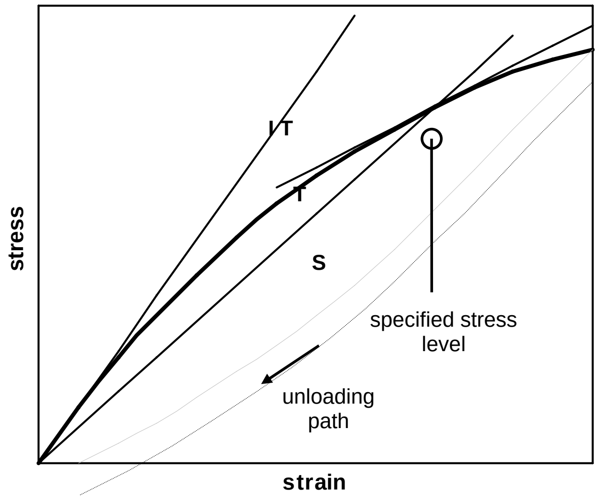



**Fig. 2.8** Various descriptions of modulus of elasticity of concrete: ( **I T** ≡ initial tangent, **T** ≡ tangent, **S** ≡ secant )

Various descriptions of $E_c$ are possible, such as *initial tangent modulus*, *tangent modulus* (at a specified stress level), *secant modulus* (at a specified stress level), etc. — as shown in Fig. 2.8. Among these, the **secant modulus** at a stress of about one-third the cube strength of concrete is generally found acceptable in representing an average value of $E_c$ under service load conditions (static loading) [Ref. 2.3].

The Code (Cl. 6.2.3.1) gives the following empirical expression for the static modulus $E_c$ (in *MPa* units) in terms of the characteristic cube strength $f_{ck}$ (in MPa units):

$$E_c = 5000\sqrt{f_{ck}} \tag{2.4}$$

It may be noted that the earlier version of IS 456 had recommended $E_c = 5700 \sqrt{f_{ck}}$, which is found to over-estimate the elastic modulus.

The ACI code [Ref. 2.21] gives an alternative formula† for $E_c$ in terms of the specified cylinder strength $f_c'$ and the mass density of concrete $\rho_c$ (in kg/m$^3$):

$$E_c = 0.0427\sqrt{\rho_c{}^3 f_c'} \tag{2.4a}$$

Considering $\rho_c$ = 2400 kg/m$^3$ for normal-weight concrete and applying Eq. 2.3, the above expression reduces to $E_c \approx 4500\sqrt{f_{ck}}$, which gives values of $E_c$ that are about 10 percent less than those given by the present IS Code formula [Eq. 2.4].

† The original formula in the ACI code, expressed in FPS units, is converted to SI units.

From a design viewpoint, the use of a lower value of $E_c$ will result in a more conservative (larger) estimate of the short-term elastic deflection of a flexural member.

### *Poisson's Ratio*

This is another elastic constant, defined as *the ratio of the lateral strain to the longitudinal strain*, under uniform axial stress. When a concrete prism is subjected to a uniaxial compression test, the longitudinal compressive strains are accompanied by lateral tensile strains. The prism as a whole also undergoes a volume change, which can be measured in terms of *volumetric strain*.

Typical observed variations of longitudinal, lateral and volumetric strains are depicted in Fig. 2.9 [Ref. 2.16]. It is seen that at a stress equal to about 80 percent of the compressive strength, there is a point of inflection on the volumetric strain curve. As the stress is increased beyond this point, the rate of volume reduction decreases; soon thereafter, the volume stops decreasing, and in fact, *starts increasing*. It is believed that this inflection point coincides with the initiation of major micro-cracking in the concrete, leading to large lateral extensions. Poisson's ratio appears to be essentially constant for stresses below the inflection point. At higher stresses, the apparent Poisson's ratio begins to increase sharply.

Widely varying values of Poisson's ratio have been obtained — in the range of 0.10 to 0.30. A value of about 0.2 is usually considered for design.

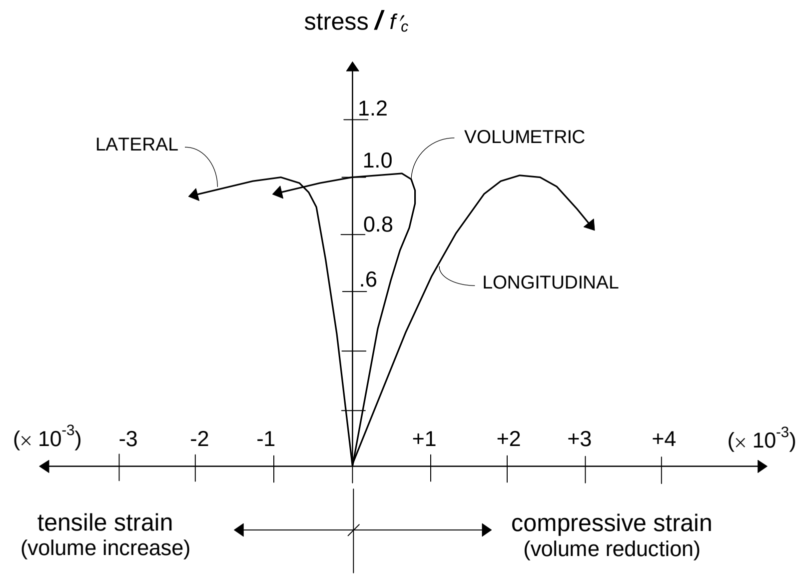



**Fig. 2.9** Strains in a concrete prism under uniaxial compression [Ref. 2.16]

### 2.8.4 Influence of Duration of Loading on Stress-Strain Curve

The standard compression test is usually completed in less than 10 minutes, the loading being gradually applied at a uniform strain rate of 0.001 mm/mm per minute. When the load is applied at a faster strain rate (which occurs, for instance, when an impact load is suddenly applied), it is found that both the modulus of elasticity and the strength of concrete increase, although the failure strain decreases [Ref. 2.19, 2.20].

On the other hand, when the load is applied at a slow strain rate, such that the duration of loading is increased from 10 minutes to as much as *one year* or more, there is a slight reduction in compressive strength, accompanied by a decrease in the modulus of elasticity and a significant increase in the failure strain, as depicted in Fig. 2.10; the stress-strain curve also becomes relatively flat after the maximum stress is reached.

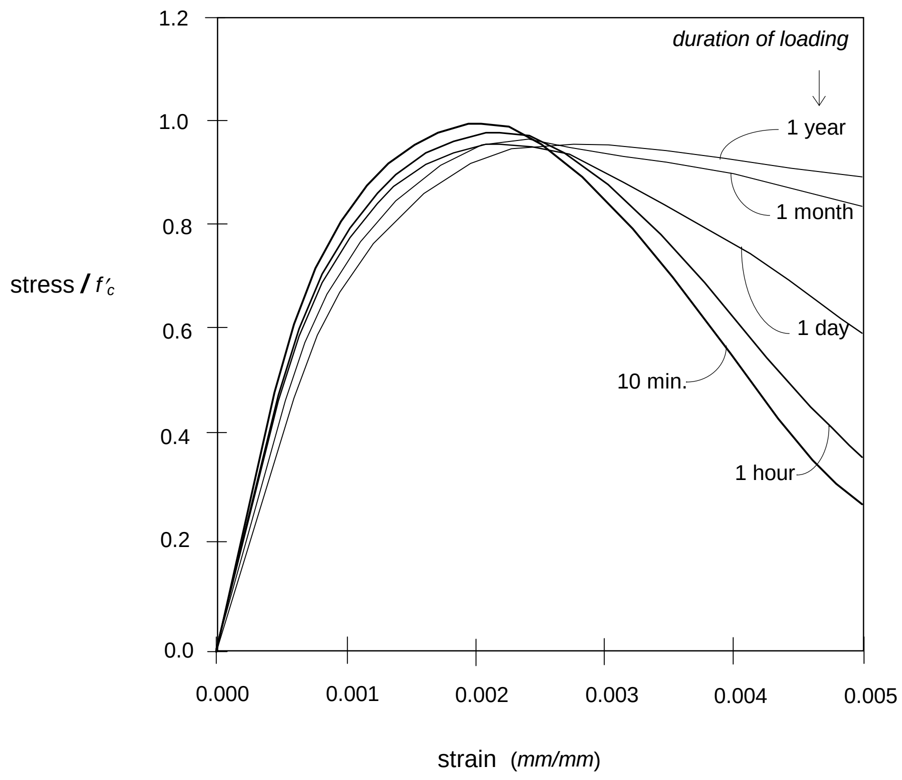



**Fig. 2.10** Influence of duration of loading (strain-controlled) on the stress-strain curve of concrete [Ref. 2.20]

It has also been reported [Ref. 2.20] that long-term *sustained loading* at a constant stress level results not only in *creep strains* [refer Section 2.11], but also in a reduced compressive strength of concrete.

### 2.8.5 Maximum Compressive Stress of Concrete in Design Practice

The compressive strength of concrete in an actual concrete structure cannot be expected to be exactly the same as that obtained from a standard uniaxial compression test for the same quality of concrete. There are many factors responsible for this difference in strength, mainly, the effects of duration of loading, size of the member (*size effect*) and the strain gradient.

The value of the maximum compressive stress (strength) of concrete is generally taken as 0.85 times the 'specified cylinder strength' ( $f_c'$ ), for the design of reinforced concrete structural members (compression members as well as flexural members) [Ref. 2.17, 2.20]. This works out approximately [Eq. 2.3] to 0.67 times the 'characteristic cube strength' ( $f_{ck}$ ) — as adopted by the Code. The Code also limits the failure strain of concrete to 0.002 under direct compression and 0.0035 under flexure.

When the predominant loading that governs the design of a structure is *short-term* rather than *sustained* (as in tall reinforced concrete chimneys subject to wind loading), it may be too conservative to limit the compressive strength to 0.85 $f_c'$ (or 0.67 $f_{ck}$ ); in such cases, it appears reasonable to adopt a suitably higher compressive strength [Ref. 2.22, 2.9].

When the occurrence of permanent sustained loads on a structure is delayed, then, instead of a reduction in compressive strength, some increase in strength (and in the quality of concrete, in general) can be expected due to the tendency of freshly hardened concrete to gain in strength with age, beyond 28 days. . This occurs due to the process of continued hydration of cement in hardened concrete, by absorption of moisture from the atmosphere; this is particularly effective in a humid environment.

The earlier version of the Code allowed an increase in the estimation of the *characteristic strength* of concrete when a member (such as a foundation or lower-storey column of a tall building) receives its full design load more than a month after casting. A maximum of 20 percent increase in $f_{ck}$ was allowed if the operation of the full load is delayed by one year or more. However, it is now recognised that such a significant increase in strength may not be realised in many cases, particularly involving the use of high-grade cement (with increased fineness), which has high early strength development. Consequently, the values of age factors have been deleted in the present version of the Code (Cl. 6.2.1), which stipulates, "the design should be based on the 28 days characteristic strength of concrete unless there is evidence to justify a higher strength".

The use of age factors (based on actual investigations) can assist in assessing the actual behaviour of a distressed structure, but should generally not be taken advantage of in design.

## 2.9 BEHAVIOUR OF CONCRETE UNDER TENSION

Concrete is not normally designed to resist direct tension. However, tensile stresses do develop in concrete members as a result of flexure, shrinkage and temperature changes. Principal tensile stresses may also result from multi-axial states of stress.

Often cracking in concrete is a result of the tensile strength (or limiting tensile strain) being exceeded. As pure shear causes tension on diagonal planes, knowledge of the direct tensile strength of concrete is useful for estimating the shear strength of beams with unreinforced webs, etc. Also, a knowledge of the flexural tensile strength of concrete is necessary for estimation of the 'moment at first crack'[†], required for the computation of deflections and crackwidths in flexural members.

As pointed out earlier, concrete is very weak in tension, the direct tensile strength being only about 7 to 15 percent of the compressive strength [Ref. 2.6]. It is difficult to perform a *direct* tension test on a concrete specimen, as it requires a purely axial tensile force to be applied, free of any misalignment and secondary stress in the specimen at the grips of the testing machine. Hence, *indirect* tension tests are resorted to, usually the *flexure test* or the *cylinder splitting test*.

### 2.9.1 Modulus of Rupture

In the *flexure test* most commonly employed [refer IS 516 : 1959], a 'standard' plain concrete beam of a square or rectangular cross-section is simply supported and subjected to third-points loading until failure. Assuming a linear stress distribution across the cross-section, the theoretical maximum tensile stress reached in the extreme fibre is termed the *modulus of rupture* ( $f_{cr}$ ). It is obtained by applying the flexure formula:

$$f_{cr} = \frac{M}{Z} \tag{2.5}$$

where $M$ is the bending moment causing failure, and $Z$ is the section modulus.

However, the actual stress distribution is not really linear, and the modulus of rupture so computed is found to be greater than the direct tensile strength by as much as 60–100 percent [Ref. 2.6]. Nevertheless, $f_{cr}$ is the appropriate tensile strength to be considered in the evaluation of the *cracking moment ($M_{cr}$)* of a beam by the flexure formula, as the same assumptions are involved in its calculation.

The Code (Cl. 6.2.2) suggests the following empirical formula for estimating $f_{cr}$ :

$$f_{cr} = 0.7\sqrt{f_{ck}} \tag{2.6}$$

where $f_{cr}$ and $f_{ck}$ are in MPa units.

The corresponding formula suggested by the ACI Code [Ref. 2.21] is:

$$f_{cr} = 0.623\sqrt{f_c'} \tag{2.6a}$$

From a design viewpoint, the use of a lower value of $f_{cr}$ results in a more conservative (lower) estimate of the 'cracking moment'.

### 2.9.2 Splitting Tensile Strength

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[†] Refer Chapter 4 for computation of *cracking moment $M_{cr}$*

The *cylinder splitting test* is the easiest to perform and gives more uniform results compared to other tension tests. In this test [refer IS 5816 : 1999], a 'standard' plain concrete cylinder (of the same type as used for the compression test) is loaded in compression *on its side* along a diametral plane. Failure occurs by the splitting of the cylinder along the loaded plane [Fig. 2.11]. In an elastic homogeneous cylinder, this loading produces a nearly uniform tensile stress across the loaded plane as shown in Fig. 2.11(c).

From theory of elasticity concepts, the following formula for the evaluation of the *splitting tensile strength* $f_{ct}$ is obtained:

$$f_{ct} = \frac{2P}{\pi \, d \, L} \tag{2.7}$$

where *P* is the maximum applied load, *d* is the diameter and *L* the length of the cylinder.

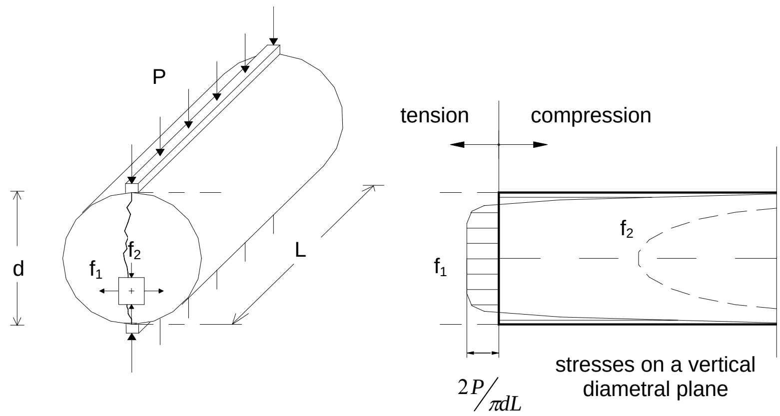


**Fig. 2.11** Cylinder splitting test for tensile strength

It has been found that for normal density concrete the splitting strength is about two-thirds of the modulus of rupture [Ref. 2.23]. (The Code does not provide an empirical formula for estimating $f_{ct}$ as it does for $f_{cr}$ ).

### 2.9.3 Stress-Strain Curve of Concrete in Tension

Concrete has a low failure strain in uniaxial tension. It is found to be in the range of 0.0001 to 0.0002. The stress-strain curve in tension is generally approximated as a straight line from the origin to the failure point. The modulus of elasticity in tension is taken to be the same as that in compression. As the tensile strength of concrete is very low, and often ignored in design, the tensile stress-strain relation is of little practical value.

### 2.9.4 Shear Strength and Tensile Strength

Concrete is rarely subjected to conditions of pure shear; hence, the strength of concrete in *pure shear* is of little practical relevance in design. Moreover, a state of pure shear is accompanied by principal tensile stresses of equal magnitude on a diagonal plane, and since the tensile strength of concrete is less than its shear strength, failure invariably occurs in tension. This, incidentally, makes it difficult to experimentally determine the resistance of concrete to pure shearing stresses. A reliable assessment of the shear strength can be obtained only from tests under combined stresses. On the basis of such studies, the strength of concrete in pure shear has been reported to be in the range of 10–20 percent of its compressive strength [Ref. 2.14]. In normal design practice, the shear strength of concrete is governed by its tensile strength, because of the associated principal tensile (diagonal tension) stresses and the need to control cracking of concrete.

## 2.10 BEHAVIOUR OF CONCRETE UNDER COMBINED STRESSES

Structural members are usually subjected to various combinations of axial forces, bending moments, transverse shear forces and twisting moments. The resulting three-dimensional state of stress acting at any point on an element may be transformed into an equivalent set of three normal stresses (*principal stresses*) acting in three orthogonal directions. When one of these three principal stresses is zero, the state of stress is termed *biaxial*. The failure strength of materials under combined stresses is normally defined by appropriate failure criteria. However, as yet, there is no universally accepted criterion for describing the failure of concrete.

### 2.10.1 Biaxial State of Stress

Concrete subjected to a biaxial state of stress has been studied extensively due to its relative simplicity in comparison with the triaxial case, and because of its common occurrence in flexural members, plates and thin shells. Figure 2.13 shows the general shape of the biaxial strength envelopes for concrete, obtained experimentally [Ref. 2.16, 2.24], along with proposed approximations.

It is found that the strength of concrete in *biaxial compression* is greater than in *uniaxial compression* by up to 27 percent. The *biaxial tensile strength* is nearly equal to its *uniaxial tensile strength*. However, in the region of combined compression and tension, the compressive strength decreases nearly linearly with an increase in the accompanying tensile stress. Observed failure modes suggest that tensile strains are of vital importance in the failure criteria and failure mechanism of concrete for both uniaxial and biaxial states of stress [Ref. 2.24].

### 2.10.2 Influence of Shear Stress

Normal stresses are accompanied by shear stresses on planes other than the *principal planes*. For a prediction of the strength of concrete in a general biaxial state of stress, *Mohr's theory of failure* is sometimes used. A more accurate (experiment based) failure envelope for the case of direct stress (compression or tension) in one direction, combined with shear stress, is shown in Fig. 2.13 [Ref. 2.25].

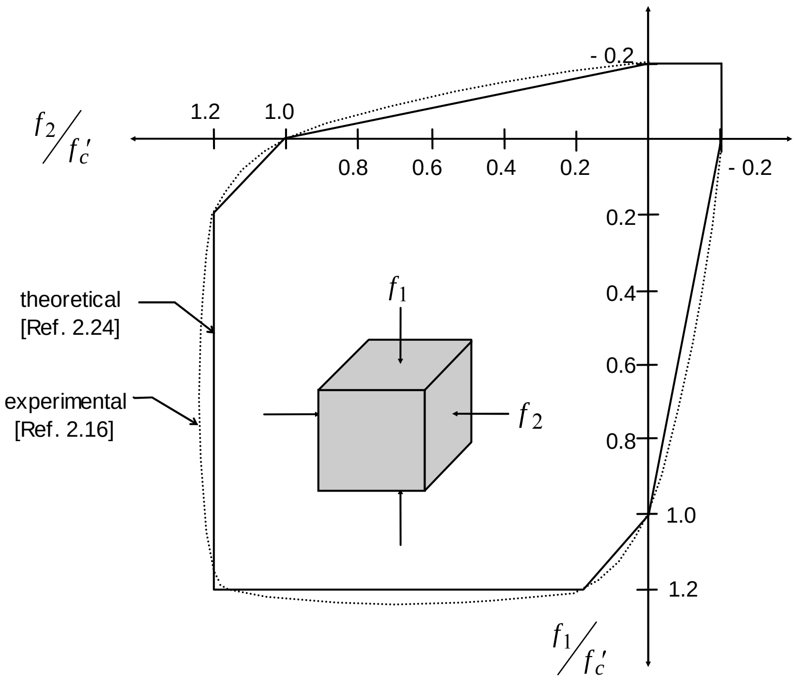


**Fig. 2.12** Failure stress envelope — biaxial stress [Ref. 2.16, 2.24]

It is seen that the compressive strength (as well as the tensile strength) of concrete is reduced by the presence of shear stress. Also, the shear strength of concrete is enhanced by the application of direct compression (except in the extreme case of very high compression), whereas it is (expectedly) reduced by the application of direct tension.

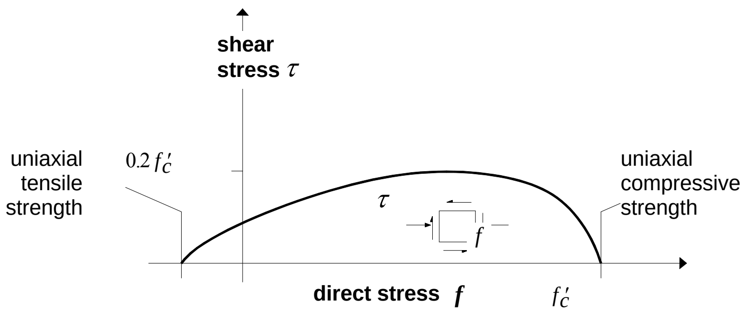


**Fig. 2.13** Failure stress envelope — direct stress combined with shear stress [Ref. 2.25]

### 2.10.3 Behaviour Under Triaxial Compression

When concrete is subject to compression in three orthogonal directions, its *strength* and *ductility* are greatly enhanced [Ref. 2.26, 2.27]. This effect is attributed to the all-round *confinement* of concrete, which reduces significantly the tendency for internal cracking and volume increase just prior to failure.

#### *Effect of confinement*

The benefit derived from confinement of concrete is advantageously made use of in reinforced concrete columns, by providing transverse reinforcement in the form of steel hoops and spirals [Fig. 1.6(c)]. It is found that continuous circular spirals are particularly effective in substantially increasing the ductility, and to some extent, the compressive strength of concrete; square or rectangular ties are less effective [Ref. 2.28]. The yielding of the confining steel contributes to increased ductility (ability to undergo large deformations prior to failure). Provision of ductility is of particular importance in the design and *detailing* of reinforced concrete structures subject to *seismic loads* (especially at the beam-column junctions), since it enables the material to enter into a plastic phase, imparting additional strength to the structure by means of redistribution of stresses [for details, refer Chapter 16].

## 2.11 CREEP OF CONCRETE

### 2.11.1 Time-Dependent Behaviour under Sustained Loading

As mentioned earlier, when concrete is subject to sustained compressive loading, its deformation keeps increasing with time, even though the stress level is not altered. The time-dependent component[†] of the total strain is termed **creep**. The time-dependent behaviour of the total strain in concrete (considering both 'instantaneous' strain and creep strain) is depicted in Fig. 2.14.

The *instantaneous strain* is that which is assumed to occur 'instantaneously' on application of the loading. This may have both 'elastic' and 'inelastic' components, depending on the stress level [Fig. 2.6]. In practice, as the stress level under service loads is relatively low, the inelastic component is negligible. If the stress is maintained at a constant level, the strain will continue to increase with time (as indicated by the solid line in the curve in Fig. 2.14), although at a progressively decreasing rate. The increase in strain at any time is termed the *creep strain*. This is sometimes expressed in terms of the *creep coefficient* ($C_t$), defined as the ratio of the creep strain at time $t$ to the instantaneous strain ('initial elastic strain'). The maximum value of $C_t$ is called the *ultimate creep coefficient* (designated as $\theta$ by the Code); its value is found to vary widely in the range 1.3 to 4.2 [Ref. 2.29].

If the sustained load is removed at any time, the strain follows the curve shown by the dashed line in Fig. 2.14. There is an instantaneous recovery of strain by an amount equal to the elastic strain (to the extent permitted by the prevailing modulus of elasticity) due to the load removed at this age. This is followed by a gradual decrease in strain, which is termed as *creep recovery*.

[†] excluding strains introduced by shrinkage and temperature variations.

### 2.11.2 Effects of Creep

The exact mechanism of creep in concrete is still not fully understood. It is generally attributed to internal movement of adsorbed water, viscous flow or sliding between the gel particles, moisture loss and the growth in micro-cracks.

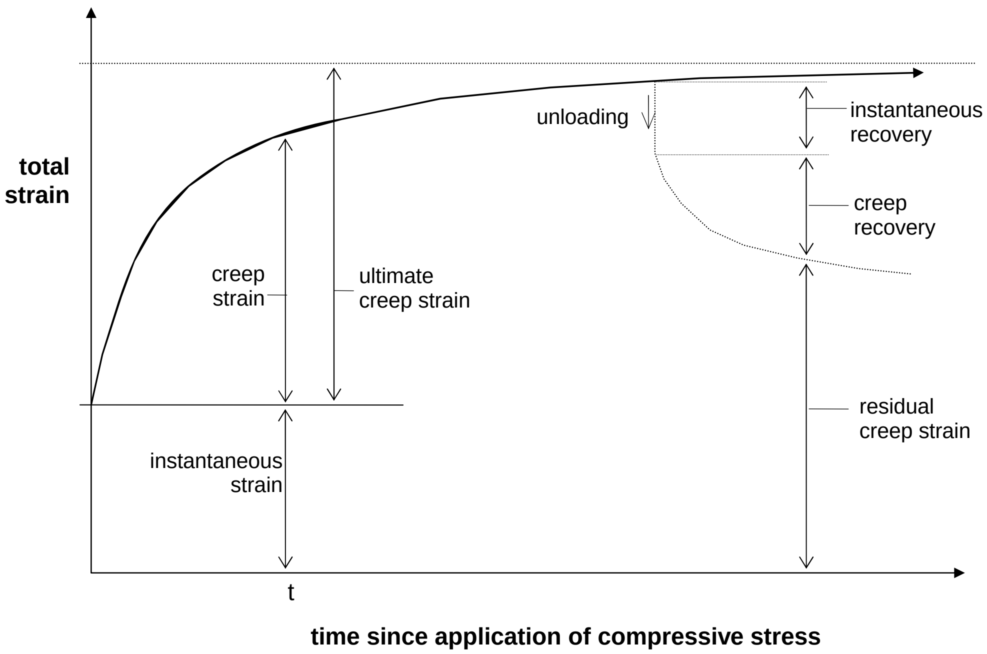


**Fig. 2.14** Typical strain-time curve for concrete in uniaxial compression

Creep of concrete results in the following detrimental results in reinforced concrete structures:

⇒ increased deflection of beams and slabs;
⇒ increased deflection of slender columns (possibly leading to buckling);
⇒ gradual transfer of load from concrete to reinforcing steel in compression members;
⇒ loss of prestress in prestressed concrete;

However, some effects of creep may even be beneficial — such as reduction of stresses induced by non-uniform or restrained shrinkage, resulting in a reduction of cracking [Ref. 2.3]. Also, in cases of stresses induced by imposed deformations (as with settlement of supports), creep effects tend to reduce the stresses.

### 2.11.3 Factors Influencing Creep

There are a number of independent and interacting factors related to the material properties and composition, curing and environmental conditions, and loading conditions that influence the magnitude of creep [Ref. 2.29]. In general, creep *increases* when:

* cement content is high;
* water-cement ratio is high;
* aggregate content is low;
* air entrainment is high;
* relative humidity is low;
* temperature (causing moisture loss) is high;
* size / thickness of the member is small;
* loading occurs at an early age; and
* loading is sustained over a long period.

### 2.11.4 Creep Coefficient for Design

Several empirical methods, such as the ACI method [Ref. 2.29] and the CEB-FIP method [Ref. 2.30], have been developed to arrive at a reasonable estimate of the creep coefficient for design purposes. In the absence of data related to the factors influencing creep, the Code (Cl. 6.2.5.1) recommends the use of the *ultimate creep coefficient* ($\theta$) — with values equal to 2.2, 1.6 and 1.1, for ages of loading equal to 7 days, 28 days and one year respectively.

Within the range of service loads, creep may be assumed to be proportional to the applied stress. This assumption facilitates the estimation of total deflection (initial plus creep deflection) of flexural members by the usual linear elastic analysis with a reduced elastic modulus. The Code (Cl. C 4.1) terms this reduced modulus as *effective modulus of elasticity* ($E_{ce}$), which can be expressed† in terms of the short-term elastic modulus ($E_c$) and the ultimate creep coefficient ($\theta$) as follows:

$$E_{ce} = \frac{E_c}{1+\theta} \tag{2.8}$$

Details of computation of long-term deflections of reinforced concrete beams due to creep are covered in Chapter 10.

## 2.12 SHRINKAGE AND TEMPERATURE EFFECTS IN CONCRETE

### 2.12.1 Shrinkage

Concrete shrinks in the hardened state due to loss of moisture by evaporation; the consequent reduction in volume is termed *drying shrinkage* (often, simply *shrinkage*). Like creep, shrinkage introduces time-dependent strains in concrete [Fig. 2.15].

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† For a more detailed explanation, refer Chapter 10 (Fig. 10.12).

Shrinkage and creep are not independent phenomena. However, for convenience, it is normal practice to treat their effects as separate, independent and additive. All the factors related to constituent material properties, composition of mix, curing and environmental conditions, member size and age that affect creep also affect shrinkage.

However, unlike creep, shrinkage strains are independent of the stress conditions in the concrete. Also, shrinkage is reversible to a great extent, i.e., alternating dry and wet conditions will cause alternating volume changes in concrete.

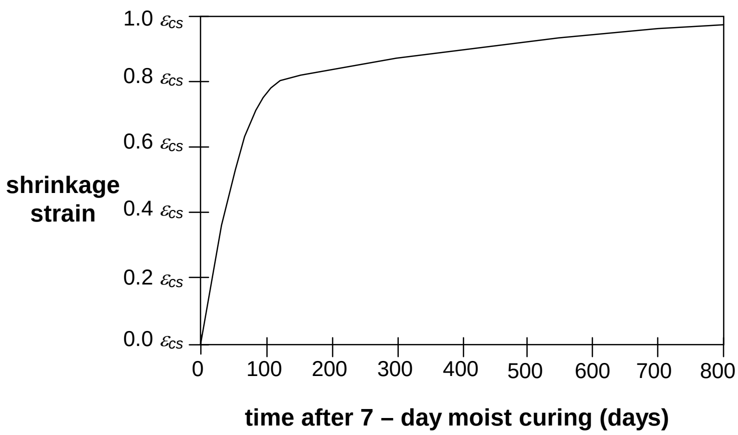


**Fig. 2.15** Typical variation of shrinkage with time

When shrinkage is restrained, as it often is in concrete structures, tensile stresses develop, and, if excessive, may lead to cracking. Similarly, a differential shrinkage, due to a moisture or thermal gradient, or due to a differential restraint to shrinkage (caused, for example, by unsymmetrically placed reinforcement in a beam) will result in internal stresses, curvature and deflections. Shrinkage, like creep, also leads to a loss of prestress in prestressed concrete structures.

Since the primary cause of shrinkage is moisture loss from the cement paste phase of the concrete, it can be minimised by keeping the unit water content in the mix as low as possible and the total aggregate content as high as possible.

### *Shrinkage Strain for Design*

Shrinkage is usually expressed as a linear strain (mm/mm). Empirical methods [Ref. 2.29, 2.30] are available for the estimation of the shrinkage strains for the purpose of design. Wide variations in the value of the *ultimate shrinkage strain* ($\varepsilon_{cs}$) — up to 0.001 mm/mm — have been reported. In the absence of reliable data, the Code (Cl. 6.2.4.1) recommends the use of an ultimate shrinkage strain value of 0.0003 mm/mm; this appears to be rather low, in comparison with ACI recommendation [Ref. 2.29] of an average value of 0.0008 mm/mm for moist-cured concrete.

Details of computation of long-term deflections of reinforced concrete beams due to shrinkage are covered in Chapter 10.

### 2.12.2 Temperature Effects

Concrete expands with a temperature rise and contracts as the temperature drops; thermal contraction, in fact, produces effects similar to shrinkage.

As a consequence of seasonal variations in temperature, internal stresses are induced in structures (which are statically indeterminate), owing to restrictions in free movements. In order to limit the development of temperature stresses in reinforced concrete buildings with large plan dimensions, it is desirable to provide suitable *expansion joints* at appropriate locations — particularly where there are marked changes in plan dimensions [refer Cl. 27 of the Code].

Temperature stresses also develop on account of differential temperature (thermal gradient), as in roof slabs (particularly of air-conditioned rooms) exposed to the sun, or in chimneys which release hot gases. In the design of many structures (such as reinforced concrete chimneys and cooling towers), 'temperature loads' need to be specially considered in the design.

In general, it is good design practice to provide some *nominal reinforcement* (close to the surface) in concrete at locations where cracks can potentially develop, due to the effects of temperature and shrinkage. This is particularly desirable in the case of large exposed surfaces of concrete (such as web faces of large-size beams) which are otherwise unreinforced.

#### *Coefficient of Thermal Expansion*

For the purpose of design, the coefficient of thermal expansion of concrete is required. This is found to depend on various factors, such as the types of cement and aggregate, relative humidity, member size, etc. The Code (Cl. 6.2.6) recommends values ranging from $6\times10^{-6}$ mm/mm per ºC (for concrete with calcareous aggregate) to $12\times10^{-6}$ mm/mm per ºC (for concrete with siliceous aggregate). However, for the design of special structures such as water tanks, chimneys, bins and silos, a value of $11\times10^{-6}$ mm/mm per ºC is recommended [Ref. 2.33]. This is very close to the coefficient of thermal expansion of steel (which is about $12\times10^{-6}$ mm/mm per ºC), so that there is little likelihood of any differential thermal expansion and associated relative movements between the steel and surrounding concrete.

## 2.13 DURABILITY OF CONCRETE

If concrete is to serve the purpose for which it is designed during its intended lifetime, it has to be *durable*. Unfortunately, many reinforced concrete structures built in the past (particularly, the not-too-distant past) in adverse environments have shown signs of increased structural distress, mainly[†] due to **chemical attack**, causing

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[†] Other factors include 'abrasive' actions on concrete surfaces (caused, for example, by machinery and metal tyres) and 'freezing and thawing' actions.

deterioration of concrete and corrosion of reinforcing steel. Loss of durability results in a reduced life of the structure. In an attempt to give increased importance to durability considerations, the recent revision of the Code has strengthened the provisions pertaining to durability, by shifting the guidelines from the Appendix (of the earlier Code) to the main body of the Code (Cl. 8), and by enhancing their scope and impact. These changes are in line with other national codes, such as BS 8100 and ACI 318.

Loss of durability in concrete structures is essentially attributable to two classes of factors, viz., external factors and internal factors. The external factors pertain to the type of environment to which the concrete is exposed, whereas the internal factors pertain to characteristics inherent to the built concrete. Primary among the internal factors is the relative **permeability** of the concrete, as chemical attack can occur only if harmful chemicals can ingress into the concrete. Chemical attack is caused by the ingress of water, oxygen, carbon dioxide, chlorides, sulphates, and other harmful chemicals (borne by surrounding ground or sea water, soil or humid atmosphere). It can also occur due to the presence of deleterious constituents (such as chlorides, sulphates and alkali-reactive aggregate) in the original concrete mix. Concrete members that are relatively thin or have inadequate cover to reinforcement are particularly vulnerable. Lack of good drainage of water to avoid standing pools and rundown of water along exposed surfaces, and cracks in concrete also lead to ingress of water and deterioration of concrete. Impermeability is governed by the constituents and workmanship used in making concrete. Despite the remarkable advances in concrete technology, regrettably, workmanship remains very poor in many construction sites in India, especially of smaller size projects.

Durability in concrete can be realised if the various internal factors are suitably accounted for (or modified), during the design and construction stages, to ensure that the concrete has the desired resistance to the anticipated external factors. Otherwise, the task of repairing and rehabilitating concrete that has been damaged (for want of proper design and quality of construction) can prove to be difficult and expensive.

The most effective ways of providing for increased durability of concrete against chemical attack in a known adverse environment are by:

- reducing permeability by
  - providing high grade of concrete
  - using adequate cement content
  - using well-graded, dense aggregate
  - using low water-cement ratio
  - using appropriate admixtures (including silica fume)
  - achieving maximum compaction
  - achieving effective curing
  - using appropriate surface coatings and impermeable membranes
  - avoiding sharp corners and locations where compaction is difficult
  - taking care while designing to minimise possible cracks
- providing direct protection to embedded steel against corrosion by
  - providing adequate clear cover
  - using appropriate corrosion-resistant or coated steel
  - using sophisticated techniques such as cathodic protection

- providing appropriate type of cement having the desired chemical resistance to sulphates and/or chlorides
- controlling the chloride and sulphate contents in the concrete mix constituents (within the limits specified in Cl. 8.2.5 of the Code)
- avoiding the use of alkali-reactive aggregate
- providing air-entraining admixtures when resistance against freezing and thawing is required
- providing adequate thickness of members
- providing adequate reinforcement designed to contain crack-widths within acceptable limits
- providing adequate drainage on concrete surfaces to avoid water retention (e.g., 'ponding' in roof slabs)

### 2.13.1 Environmental Exposure Conditions and Code Requirements

The Code (Cl. 8.2.2.1) identifies five categories of 'environmental exposure conditions', viz., **'mild'**, **'moderate'**, **'severe'**, **'very severe'** and **'extreme'**, in increasing degree of severity. The purpose of this categorisation is mainly to provide a basis for enforcing certain minimum requirements aimed at providing the desired performance related to the severity of exposure. These requirements, having implications in both design and construction of reinforced concrete work, pertain to:

- minimum grade of concrete (varying from M 20 to M 40)
- minimum clear cover to reinforcement (20 mm to 75 mm)
- minimum cement content (300 to 360 kg/m$^3$ for 20mm size aggregates)
- maximum water-cement ratio (0.55 to 0.40)
- acceptable limits of surface width of cracks (0.1 mm to 0.3 mm)

The descriptions of the five categories of environmental exposure, as well as the corresponding specifications for the minimum grade of concrete, 'nominal cover'[†] (minimum clear cover to reinforcement), minimum cement content and maximum free water-cement ratio, for reinforced concrete work, are summarised in Table 2.1[‡]. These specifications incorporated in the revised Code constitute perhaps the most significant changes in the Code, having tremendous practical (and economic) implications. These recommendations have been long overdue, and are in line with international practice.

It may be noted that in the same structure, different members may be subject to different categories of exposure. For example, a reinforced concrete building located in a port city (such as Chennai or Mumbai) would be exposed to a coastal environment, which qualifies to be categorised as 'severe' (or 'very severe', in case it is very near the beach, exposed to sea water spray). However, for concrete members located well inside the building (excepting foundations), sheltered from direct rain and aggressive atmospheric environment, the exposure category may be lowered by one level of severity; i.e., from 'severe' to 'moderate' (or 'very severe' to 'severe').

[†] Additional cover requirements pertaining to fire resistance are given in Table 16A of the Code.
[‡] The requirements for plain concrete (given in Table 5 of the Code) are not shown here.

**Table 2.1** Exposure conditions and requirements for RC work with normal aggregate (20 mm nominal size)

| **Exposure Category** | **Description** | **Min. grade** | **Min. Cover (mm)** | **Min. Cement ($kg/m^3$)** | **Max. free w/c ratio** |
|---|---|---|---|---|---|
| Mild | Protected against weather or aggressive conditions, except if located in coastal area | M 20 | 20* | 300 | 0.55 |
| Moderate | Sheltered from severe rain or freezing whilst wet, or<br>Exposed to condensation and rain, or<br>Continuously under water, or<br>In contact with or buried under non-aggressive soil or ground water, or<br>Sheltered from saturated ‘salt air’ in coastal area | M 25 | 30 | 300 | 0.50 |
| Severe | Exposed to severe rain, alternate wetting and drying or occasional freezing whilst wet or severe condensation, or<br>Completely immersed in sea water, or<br>Exposed to coastal environment | M 30 | 45** | 320 | 0.45 |
| Very Severe | Exposed to sea water spray, corrosive fumes or severe freezing whilst wet, or<br>In contact with or buried under aggressive sub-soil or ground water | M 35 | 50 | 340 | 0.45 |
| Extreme | Members in tidal zone, or<br>Members in direct contact with liquid/solid aggressive chemicals | M 40 | 75 | 360 | 0.40 |

* can be reduced to 15 mm, if the bar diameter is less than 12 mm.
** can be reduced by 5mm, if M 35 or higher grade is used.

Accordingly, corresponding to the 'severe' category, the roof‡ slab must be (at least) of M 30 grade concrete and its reinforcement should have a minimum clear cover of 45 mm. These values may be compared to M 15 grade and 15 mm cover hitherto adopted in design practice, as per IS 456 (1978). The increase in capital investment on account of the substantial increase in slab thickness and enhanced grade of concrete may appear to be drastic, but should be weighed against the significant gain in terms of prolonged maintenance-free life of the structure.

### 2.13.2 Permeability of Concrete

As mentioned earlier, reducing the permeability of concrete is perhaps the most effective way of enhancing durability. *Impermeability* is also a major *serviceability* requirement — particularly in water tanks, sewage tanks, gas purifiers, pipes and pressure vessels. In ordinary construction, roof slabs need to be impermeable against the ingress of rain water.

Permeability of concrete is directly related to the *porosity* of the cement paste, the distribution of *capillary pores* and the presence of *micro-cracks* (induced by shrinkage effects, tensile stresses, etc.). The main factors influencing capillary porosity are the *water-cement ratio* and the *degree of hydration*. The use of a **low water-cement ratio, adequate cement** and **effective curing** contribute significantly to reduced permeability. The steps to be taken to reduce permeability were listed in the previous Section. In addition, it is essential for the concrete to be dense; this requires the use of **well-graded, dense aggregate** and **good compaction**. For given aggregates, the cement content should be sufficient to provide adequate workability with a low water-cement ratio so that concrete can be completely compacted with the means available. The use of appropriate chemical admixtures (such as superplasticisers) can facilitate working with a reduced water-cement ratio, and the use of mineral admixtures such as silica fume can contribute to making a dense concrete with reduced porosity.

Provision of appropriate tested surface coatings and impermeable membranes also provide additional protection in extreme situations.

### 2.13.3 Chemical Attack on Concrete

The main sources of chemical attack, causing deterioration of concrete are sulphates, sea water (containing chlorides, sulphates, etc.), acids and alkali-aggregate reaction.

#### *Sulphate Attack*

Sulphates present in the soil or in ground (or sea) water attack hardened concrete that is relatively permeable. Sulphates of sodium and potassium, and magnesium in particular, react with calcium hydroxide and $C_3A$ to form calcium sulphate ('gypsum') and calcium sulphoaluminate ('ettringite') — which occupy a greater volume than the compounds they replace [Ref. 2.3]. This leads to expansion and disruption (cracking or disintegration) of hardened concrete. Sulphate-attacked

‡ In the case of intermediate floor slabs, the grade of concrete can be reduced to M 25 and the clear cover to 30 mm, corresponding to 'moderate' exposure.

concrete has a characteristic whitish appearance ('efflorescence' — due to leaching of calcium hydroxide), and is prone to cracking and spalling of concrete.

Resistance to sulphate attack can be improved by the use of special cements such as PSC, SRPC, HAC and SC [refer Section 2.2.1], and by reducing the permeability of concrete. Recommendations regarding the choice of type of cement, minimum cement content and maximum water-cement ratio, for exposure to different concentrations of sulphates (expressed as $SO_3$) in soil and ground water are given in Table 4 of the Code. The Code recommends the use of Portland slag cement (PSC) with slag content more than 50 percent, and in cases of extreme sulphate concentration, recommends the use of supersulphated cement (SC)† and sulphate resistant cement (SRC). The specified minimum cement content is in the range 280 to 400 kg/m$^3$ and the maximum free water-cement ratio is in the range 0.55 to 0.40 respectively for increasing concentrations of sulphates.

### Sea Water Attack

Sea water contains chlorides in addition to sulphates — the combination of which results in a gradual increase in porosity and a consequent decrease in strength. The same measures used to prevent sulphate attack are applicable here. However, the use of sulphate resistant cement (SRC) is not recommended. The Code recommends the use of OPC with $C_3A$ content in the range 5–8 percent and the use of blast furnace slag cement (PSC). In particular, *low permeability* is highly desirable. The inclusion of silica fume admixture can contribute to the making of the densest possible concrete. The use of soft or porous aggregate should be avoided. Concrete shall be at least M30 Grade in case of reinforced concrete. The use of a higher cement content (of at least 350 kg/m$^3$) above the low-tide water level, along with a lower water-cement ratio (of about 0.40) is recommended, owing to the extreme severity of exposure in this region, where construction joints should also be avoided [Ref. 2.30]. Adequate *cover to reinforcement* and other measures to prevent corrosion are of special importance to prevent chloride attack on reinforcement.

### Alkali-Aggregate Reaction

Some aggregates containing reactive silica are prone to reaction with alkalis ($Na_2O$ and $K_2O$) in the cement paste. The reaction, however, is possible only in the presence of a high moisture content within the concrete. The reaction eventually leads to expansion, cracking and disruption of concrete, although the occurrence of damage may be delayed, sometimes by five years or so [Ref. 2.3]. Care must be taken to avoid the use of such aggregates for concreting. Also, the use of low alkali OPC and inclusion of pozzolana are recommended by the Code (Cl. 8.2.5.4). The Code also recommends measures to reduce the degree of saturation of concrete during service by the use of impermeable membranes in situations where the possibility of alkali-aggregate reaction is suspected.

† However, use of supersulphated cement in situations where the ambient temperature exceeds 40 °C is discouraged.

### 2.13.4 Corrosion of Reinforcing Steel

The mechanism of corrosion of reinforcing steel, embedded in concrete, is attributed to electrochemical action. Differences in the electrochemical potential on the steel surface results in the formation of anodic and cathodic regions, which are connected by some salt solution acting as an electrolyte.

Steel in freshly cast concrete is generally free from corrosion because of the formation of a thin protective film of iron oxide due to the strongly alkaline environment produced by the hydration of cement. This passive protection is broken when the pH value of the regions adjoining the steel falls below about 9. This can occur either by *carbonation* (reaction of carbon dioxide in the atmosphere with the alkalis in the cement paste) or by the ingress of *soluble chlorides*. The extent of carbonation penetration or chloride penetration depends, to a great extent, on the *permeability* of concrete in the *cover* region. The increased cover stipulated in the recent code revision will contribute to increased protection against corrosion, provided, of course, the cover concrete is of good quality (low permeability). However, it may be noted that increased cover also contributes to increased flexural crack-widths [Ref. 2.31], and it is necessary to contain the cracking by suitable reinforcement design and detailing [refer Chapter 10].

The electrochemical process of corrosion takes place in the presence of the electrolytic solution and water and oxygen. The consequent formation and accumulation of rust can result in a significant increase in the volume of steel and a loss of strength; the swelling pressures cause cracking and spalling of concrete, thereby allowing further ingress of carbonation or chloride penetration. Unless remedial measures are quickly adopted, corrosion is likely to propagate and lead eventually to structural failure. *Cathodic protection* is the most effective (although expensive) way of arresting corrosion [Ref. 2.32].

Prevention is easier (and less costly) than cure. If it is known in advance that the structure is to be located in an adverse environment, the designer should aim for structural durability at the design stage itself, by adopting suitable measures such as:

- ⇒ *control crack widths* in reinforced concrete by suitable design [refer Chapter 10], or by resorting to partial or full *prestressing*;
- ⇒ provide *increased cover* to reinforcement [refer Cl. 26.4.2 of the Code];
- ⇒ ensure *low permeability* by specifying optimum cement content, minimum water-cement ratio, proper compaction and curing;
- ⇒ specify the use of special *corrosion-resistant steel* or *fusion bonded epoxy steel;*
- ⇒ use of special cements;
- ⇒ use of cathodic protection.

## 2.14 REINFORCING STEEL

As explained earlier (Section 1.2), concrete is reinforced with steel primarily to make up for concrete's incapacity for tensile resistance. Steel embedded in concrete, called reinforcing steel, can effectively take up the tension that is induced due to flexural tension, direct tension, 'diagonal tension' or environmental effects. Reinforcing steel also imparts ductility to a material that is otherwise brittle. Furthermore, steel is stronger than concrete in compression also; hence, concrete can be advantageously reinforced with steel for bearing compressive stresses as well, as is commonly done in columns.

### 2.14.1 Types, Sizes and Grades

Reinforcing steel is generally provided in the form of bars, wires or welded wire fabric.

Reinforcing bars (referred to as *rebars*) are available in *nominal* diameters† ranging from 5 mm to 50 mm, and may be *plain* or *deformed*. In the case of the latter, 'deformations', in the form of lugs or protrusions, are provided on the surface to enhance the *bond* between steel and concrete, and to mechanically inhibit the longitudinal movement of the bar relative to the concrete around it. The bars that are most commonly used are **high strength deformed bars** (generally *cold-twisted*), conforming to IS 1786 : 1985, and having a 'specified yield strength' of 415 MPa. Deformed bars of a higher specified strength of 500 MPa are also used in special cases. Plain **mild steel** bars are less commonly used in reinforced concrete, because they possess less strength (250 MPa yield strength) and cost approximately the same as high-strength deformed bars; however, they are used in practice in situations where nominal reinforcement is called for. Low strength steel is also preferred in special situations where deflections and crackwidths need to be controlled [refer Chapter 10], or where high ductility is required, as in earthquake-resistant design [refer Chapter 16]. The Code also permits the use of *medium tensile steel* (which has a higher strength than mild steel); but this is rarely used in practice. The requirements of both mild steel and medium tensile steel are covered in IS 432 (Part 1) : 1982.

For the purpose of reinforced concrete design, the Code grades reinforcing steel in terms of the 'specified yield strength'. Three grades have been specified, viz. Fe 250, Fe 415, and Fe 500, conforming to specified yield strengths of 250 MPa, 415 MPa and 500 MPa respectively. The specified yield strength normally refers to a guaranteed minimum. The actual yield strength of the steel is usually somewhat **higher** than the specified value. The Code (Cl. 36.1) specifies that the 'specified yield strength' may be treated as the *characteristic strength* of reinforcing steel. In some cases (e.g., in ductile, earthquake resistant design – see Ch. 16) it is **undesirable** to have a yield strength much higher than that considered in design.

**Hard-drawn steel wire fabrics**, conforming to IS 1566 : 1982, are sometimes used in thin slabs and in some precast products (such as pipes). Their sizes are specified in terms of mesh size and wire diameter.

**Rolled steel sections**, conforming to Grade A of IS 2062 : 1999, are also permitted by the Code in *composite construction*; however, this is strictly not in the purview of reinforced concrete design, and hence is not discussed in this book.

† The bar sizes (nominal diameters in mm) presently available in India are — 5, 6, 8, 10, 12, 16, 18, 20, 22, 25, 28, 32, 36, 40, 45 and 50.

### *Rust on reinforcement*

Rust on reinforcing steel (prior to concreting) is not an uncommon sight at construction sites. Loose mill scale, loose rust, oil, mud, etc. are considered harmful for the bond with concrete, and should be removed before fixing of reinforcement [Cl. 5.6.1 of the Code]. However, research has shown that a normal amount of rust on deformed bars and wire fabric is perhaps not undesirable, because it increases the bond with concrete [Ref. 2.33]; however, the rust should not be excessive as to violate the specified tolerances on the size of the reinforcement.

## 2.14.2 Stress-Strain Curves

The stress-strain curve of reinforcing steel is obtained by performing a standard *tension test* [refer IS 1608 : 1995]. Typical stress-strain curves for the three grades of steel are depicted in Fig. 2.16.

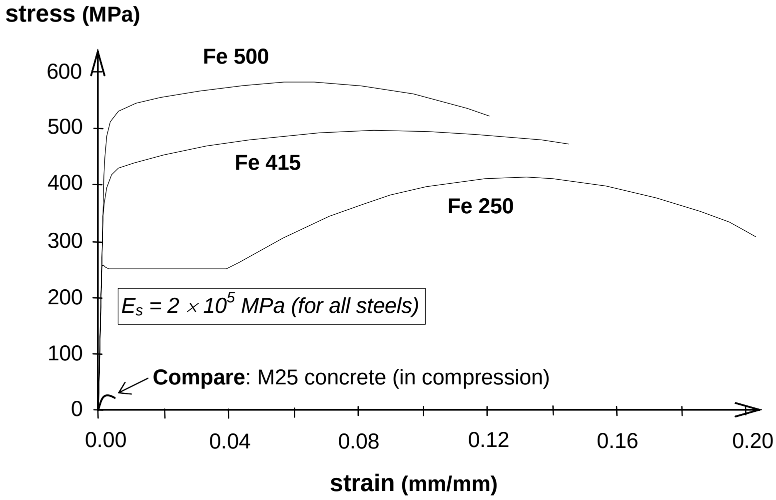


**Fig. 2.16** Typical stress-strain curves for reinforcing steels†

For all grades, there is an initial linear elastic portion with constant slope, which gives a *modulus of elasticity* ($E_s$) that is practically the same for all grades. The Code (Cl. 5.6.3) specifies that the value of $E_s$ to be considered in design is $2\times10^5$ MPa (N/mm$^2$).

The stress-strain curve of mild steel (*hot rolled*) is characterised by an initial linearly elastic part that is followed by an *yield plateau* (where the strain increases at almost constant stress), followed in turn by a *strain hardening range* in which the

† For comparison, the stress-strain curve for M25 concrete is also plotted to the same scale. Note the enormous difference in ultimate strength and strain and elastic modulus between concrete and steel. For steel, the stress-strain curve in compression is identical to the one in tension (provided buckling is restrained). In contrast, for concrete, the tensile strength and strain are only a small fraction of the corresponding values in compression.

stress once again increases with increasing strain (although at a decreasing rate) until the peak stress (*tensile strength*) is reached. Finally, there is a descending branch wherein the *nominal stress* (load divided by original area) decreases until *fracture* occurs. (The *actual* stress, in terms of load divided by the current reduced area, will, however, show an increasing trend). For Fe 250 grade steel, the ultimate tensile strength is specified as 412 MPa, and the minimum percentage elongation (on a specified gauge length) is 20–22 percent. [refer IS 432 (Part 1) : 1982].

The process of *cold-working* involves stretching and twisting of mild steel, beyond the yield plateau, and subsequently releasing the load, as indicated by the line *BCD* in Fig. 2.17. This steel on reloading will follow the path *DEF*. It can be seen that by unloading after the yield stress has been exceeded and reloading, a hysteresis loop is formed. The unloading and reloading curves *BCD* and *DEF* are initially very nearly parallel to the original elastic loading line *OA*. The hysteresis loop shown in Fig. 2.17 is greatly exaggerated and, except under continuous reversed cyclic loading, the unloading and reloading paths in this region may be assumed to be overlapping, elastic and parallel to line *OA*. Thus, upon reloading, the steel follows a linear elastic path (with the same modulus of elasticity $E_s$ as the original mild steel) up to the point where the unloading started — the new raised 'yield point'. (The point of yielding is **not likely to be well-defined** if the point of unloading lies beyond the yield plateau.) Thereafter, the material enters into the *strain hardening* range, following the path indicated by the curve *FGH*, which is virtually a continuation of the curve *OAB* [Fig. 2.17]. It can also be seen that cold-working results in a residual strain in the steel, represented by *OD* in Fig. 2.17.

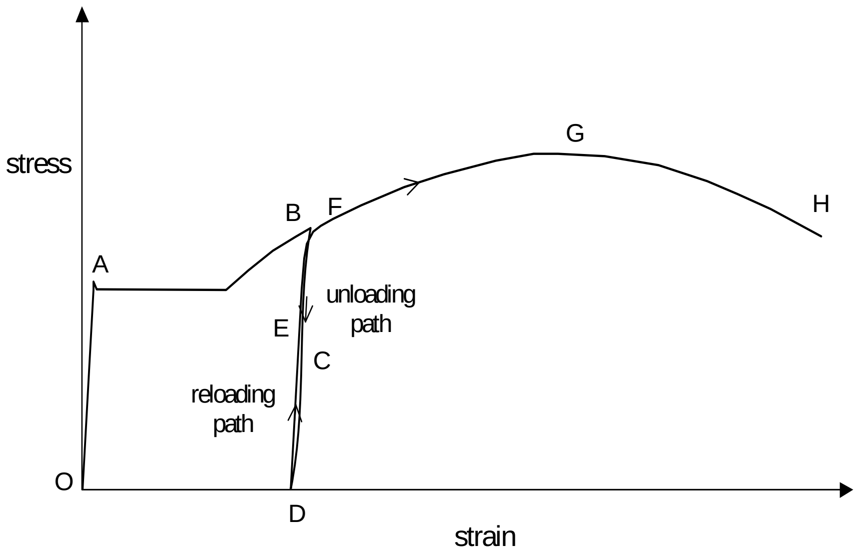


**Fig. 2.17** Effect of cold-working on mild steel bars [Ref. 2.34]

It should be noted that although the process of cold-working effectively increases the 'yield strength' of the steel, it also reduces the ductility in the material. Higher yield strengths can be achieved by suitably selecting the point of unloading in the strain hardening range, and by using higher grades of mild steel. It also follows that increasing yield strength through cold-working results in a decreased margin between yield strength and ultimate strength.

For Fe 415 grade steel, the ultimate tensile strength is expected to be 15 percent more than the yield strength, with a percentage elongation of 14.5 percent, whereas for Fe 500 grade steel, the ultimate tensile strength is expected to be only 10 percent more than the yield strength, with a percentage elongation of 12 percent [refer IS 1786 : 1985]. For design purposes, the increase in strength beyond the 'yield point' (due to strain-hardening) is generally ignored. Most design codes recommend the use of an ideal *elasto-plastic* stress-strain curve (with an initial linearly elastic line up to yield, followed by a line at constant stress, denoting the post-yielding behaviour).

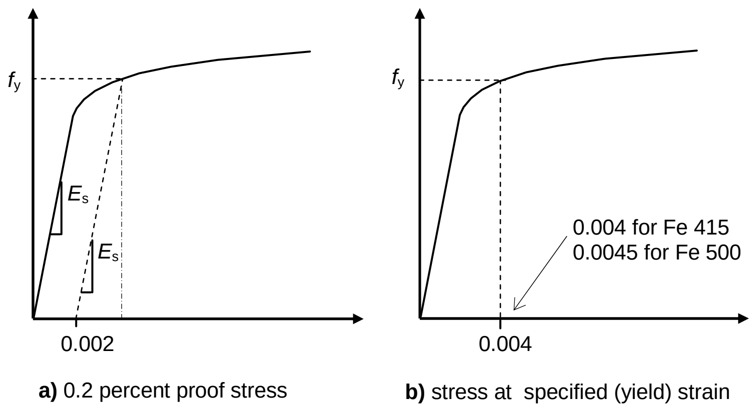


**Fig. 2.18** Definition of yield strength – high strength steel

In the absence of a definite yield point, the 0.2 percent 'proof stress'† is generally taken as the yield strength [Fig. 2.18(a)]. It is also admissible to take the yield strength as the stress corresponding to a specified strain (0.004 and 0.0045 for Fe 415 and Fe 500 grades respectively; see Cl. 8.2 of IS 1786: 1985) as shown in Fig. 2.18(b)‡. **A note of caution** will be appropriate here. For mild steel such as grade Fe 250 [see Fig. 2.16], there is a sharp and pronounced yielding. In a tension test on such steel conducted on a relatively 'stiff' testing machine, because of the sudden relaxation at yielding, the pointer of the load dial of the machine 'hesitates' or 'drops back'. Hence, the load corresponding to the 'drop of pointer' can be taken as the

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† the stress at which a non-proportional elongation equal to 0.2 percent of the original gauge length takes place [IS 1786 : 1985 & 1608 : 1995]; stress level, which on unloading, results in a residual strain of 0.002.

‡ these strains are indeed the sum of 0.002 and $f_y/E_s$ for the respective grades [see Fig. 2.18(b)].

yield load with reasonable accuracy. However, in cold-worked and high strength steels whose yielding is more gradual and the ‘yield point’ is not well-defined, the ‘drop-of-pointer’ method should not be used to determine the yield load and yield strength. In such cases, yield strength should be determined from the measured stress-strain diagram as the 0.2 percent ‘proof stress’[†].

The stress-strain behaviour of steel in compression is identical to that in tension. However, if the steel is stressed into the inelastic range in uniform tension, unloaded, and then subjected to uniform compression in the opposite direction, it is found that the stress-strain curve becomes nonlinear at a stress much lower than the initial yield strength [Fig. 2.19]. This is referred to as the ‘*Bauschinger effect’* [Ref. 2.28]. In this case, the hysteresis loop is also more pronounced. In inelastic deformation processes involving continual reversal of stress (such as metal working, high intensity reversed seismic loading, etc), the Bauschinger effect is very important and cannot be ignored. In other cases, where there is in general no more than one stress reversal, the Bauschinger effect can safely be neglected.

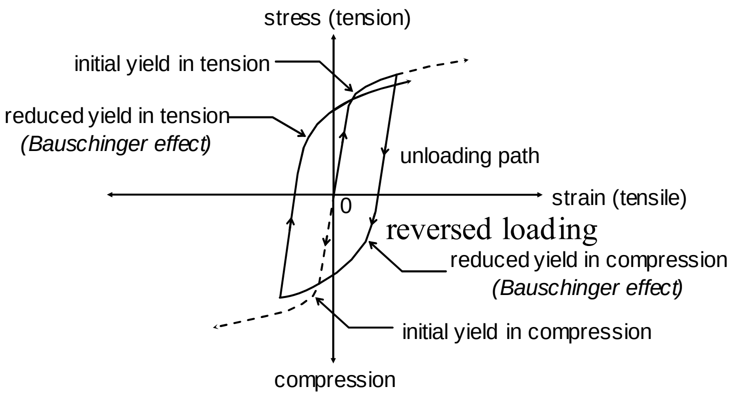


**Fig. 2.19** Bauschinger effect and hysteresis

The stress-strain curve for steel under compression is generally taken to be the same as under tension.

## 2.15 LIST OF RELEVANT INDIAN STANDARDS

All the codes on *material specifications* that have been referred to in this chapter are listed as follows:

[†] It has come to the notice of the authors that some educational institutions conducting commercial tests have been adopting the ‘drop-of-pointer’ method even for high strength reinforcing steel specimens. As a result, yield strengths far greater than the real values have been erroneously reported. Hence, this note of caution.

### *Standards on Cement*

**IS 269 : 1989** — Specification for 33 Grade ordinary Portland cement (fourth revision);

**IS 8112 : 1989** — Specification for 43 Grade ordinary Portland cement (first revision);

**IS 12269 : 1987** — Specification for 53 Grade ordinary Portland cement;

**IS 8041 : 1990** — Specification for rapid hardening Portland cement (second revision);

**IS 455 : 1989** — Specification for Portland slag cement (fourth revision);

**IS 1489 : 1991** — Specification for Portland pozzolana cement
**Part I** : Flyash based (third revision);
**Part II** : Calcined clay based (third revision);

**IS 8043 : 1991** — Specification for hydrophobic Portland cement (second revision);

**IS 12600 : 1989** — Specification for low heat Portland cement;

**IS 12330 : 1988** — Specification for sulphate resisting Portland cement;

**IS 8042 : 1978** — Specification for Portland white cement (first revision);

**IS 8043 : 1991** — Specification for hydrophobic Portland white cement (second revision);

**IS 6452 : 1989** — Specification for high alumina cement for structural use (first revision);

**IS 6909 : 1990** — Specification for supersulphated cement (first revision);

**IS 4031 : 1988** — Methods of physical tests for hydraulic cement;

### *Standards on Aggregate, Water and Admixtures*

**IS 383 : 1970** — Specification for coarse and fine aggregates from natural sources for concrete (second revision);

**IS 9142 : 1979** — Specification for artificial lightweight aggregates for concrete masonry units;

**IS 2386 (Parts 1–8**) — Methods of tests for aggregate for concrete;

**IS 3025 (Parts 17–32**) — Methods of sampling and test (physical and chemical) for water and waste water;

**IS 9103 : 1999** — Specification for admixtures for concrete (first revision);

**IS 3812 : 1981** — Specification for flyash for use as pozzolana and admixture (first revision);

**IS 1344 : 1981** — Specification for calcined clay pozzolana (second revision);

### *Standards on Concrete*

**IS 10262 : 1982** — Recommended guidelines for concrete mix design;

**IS 7861 (Part 1) : 1975** — Code of Practice for extreme weather concreting: Part 1 — Recommended practice for hot weather concreting;

**IS 4926 : 1976** — Ready-mixed concrete (first revision);

**IS 1199 : 1959** — Methods of sampling and analysis of concrete;

**IS 516 : 1959** — Methods of tests for strength of concrete;

**IS 5816 : 1999** — Method of test for splitting tensile strength of concrete cylinders (first revision);

**IS 3370 (Part 1) : 1965** — Code of Practice for the storage of liquids: Part 1 — General

**IS 1343 : 1980** — Code of Practice for Prestressed Concrete (first revision);

### *Standards on Reinforcing Steel*

**IS 432 (Part 1) : 1982** — Specification for mild steel and medium tensile steel bars for concrete reinforcement (third revision);

**IS 1786 : 1985** — Specification for high strength deformed steel bars for concrete reinforcement (third revision);

**IS 1566 : 1982** — Specification for hard-drawn steel wire fabric for concrete reinforcement (second revision);

**IS 2062 : 1999** — Steel for general structural purposes- Specification (Fifth revision);

**IS 1608 : 1995** — Mechanical testing of Metals – Tensile testing (second revision).

## REVIEW QUESTIONS

2.1 What are the types of cement that are suitable for (a) *mass concreting*, (b) resistance to *sulphate attack*?

2.2 How can the development of strength and heat of hydration be controlled in cement manufacture?

2.3 Can the use of excessive cement in concrete be harmful?

2.4 What do the terms *stiffening*, *setting* and *hardening* mean, with reference to cement paste?

2.5 What is the basis for deciding the maximum size of coarse aggregate in concrete work?

2.6 What is meant by *segregation* of concrete? Under what circumstances does it take place?

2.7 What is meant by *workability* of concrete, and how is it measured?

2.8 Discuss the role of water in producing ‘good’ concrete.

2.9 Mention the different types of ‘admixtures’ and their applications.

2.10 (a) Define *characteristic strength*. (b) Determine the 'mean target strength' required for the mix design of M25 concrete, assuming moderate quality control.

2.11 Enumerate the steps involved in the Indian Standard method of *mix design*.

2.12 Why is the *cube strength* different from the *cylinder strength* for the same grade of concrete?

2.13 Can concrete be assumed to be a *linear elastic* material? Discuss.

2.14 Distinguish between *static modulus* and *dynamic modulus* of elasticity of concrete.

2.15 Discuss the variations of longitudinal, lateral and volumetric strains that are observable in a typical uniaxial compression test on a concrete prism.

2.16 Why does the Code limit the compressive strength of concrete in structural design to 0.67 $f_{ck}$, and not $f_{ck}$ ?

2.17 Is the *modulus of rupture* of concrete equal to its *direct tensile strength*? Discuss.

2.18 The standard flexure test makes use of a 'third-point loading'. Is this necessary? Can a single point load at midspan be used as an alternative?

2.19 Why is it not possible to determine the *shear strength* of concrete by subjecting it to a state of pure shear?

2.20 What is the advantage of confinement of concrete? Give suitable examples to illustrate your point.

2.21 What does 'creep of concrete' mean? Is creep harmful or beneficial?

2.22 How is it that the deflection of a simply supported reinforced concrete beam increases due to shrinkage of concrete?

2.23 Consider a simple portal frame (with fixed base) made of reinforced concrete. Sketch the approximate shape of the deflection curve caused by (a) a uniform shrinkage strain, (b) a uniform temperature rise.

2.24 Consider the temperature gradient across the shell thickness of a reinforced concrete chimney (with tubular cross-section). Where would you provide reinforcing steel to resist tensile stresses due to the effect of temperature alone (caused by the emission of hot gases): close to the outer circumference or close to the inner circumference? Justify your answer.

2.25 How would you define 'durable concrete'? Discuss the ways of ensuring durability.

2.26 Cite two examples each for the five categories of 'environmental exposure' described in the Code.

2.27 Describe the main factors that affect the *permeability* of concrete.

2.28 Discuss briefly the factors that lead to corrosion of reinforcing steel.

2.29 What steps can a designer adopt at the design stage to ensure the *durability* of a reinforced concrete *offshore structure*?

2.30 What is meant by *strain hardening* of steel? How is it related to the *grade* of reinforcing steel?

2.31 What is meant by cold-working of mild steel? How does it affect the structural properties of the steel?

2.32 What is *Bauschinger effect*? Where is it relevant?

## REFERENCES

2.1 Neville, A.M., *Properties of Concrete*, Second edition, Pitman Publishing Co., London, 1973.

2.2 Mehta, P.K. and Monteiro, P.J.M., *Concrete: Microstructure, Properties and Materials*, Indian edition, Indian Concrete Institute, Chennai, 1997.

2.3 Neville, A.M. and Brooks, J.J., *Concrete Technology*, ELBS edition, Longman, London, 1990.

2.4 — *Design of Concrete Mixes*, Special Publication SP:23, Bureau of Indian Standards, New Delhi, 1982.

2.5 Rao, P.S. and Aravindan, P.K., *Concrete Mix Design Practice — Need for a Fresh Approach*, Indian Concrete Journal, May 1990, pp 234–237

2.6 Price, W.H., *Factors Influencing Concrete Strength*, Journal ACI, Vol. 47, Feb. 1951, pp 417–432.

2.7 — *Guide for Use of Admixtures in Concrete*, ACI Committee Report 212.2 R-81, Am. Conc. Inst., Detroit, Michigan, USA, 1981.

2.8 — *Cement and Concrete Terminology*, ACI (Committee 116) Special Publication SP-19, Am. Conc. Inst., Detroit, Michigan, USA, 1967.

2.9 Ellingwood, B. and Galambos, T.V. and MacGregor, J G. and Cornell, C.A., *Development of a Probability Based Load Criterion for American National Standard A58*, Special Publication No.°577, National Bureau of Standards, Washington D.C., 1980 .

2.10 — *Standard Practice for Selecting Proportions for Normal, Heavyweight and Mass Concrete*, ACI Standard 211.1–81, Am. Conc. Inst., Detroit, Michigan, USA, 1981.

2.11 — *Standard Practice for Selecting Proportions for Structural Lightweight Concrete*, ACI Standard 211.2–81, Am. Conc. Inst., Detroit, Michigan, USA, 1981.

2.12 — *Standard Practice for Selecting Proportions for No-Slump Concrete*, ACI Standard 211.3–75 (revised), Am. Conc. Inst., Detroit, Michigan, USA, 1980.

2.13 Teychenne, D.C., Franklin, R.E., Erntroy, H.C., *Design of Normal Concrete Mixes*, Dept. of Environment, Her Majesty's Stationary Office, London, 1975.

2.14 Kesler, C.E., *Hardened Concrete Strength*, 'Tests and Properties of Concrete', ASTM Special Testing Publication No. 169–A, Am. Soc. for Testing and Materials, 1966, pp 144–159.

2.15 Hsu, T.T.C. et al, *Microcracking of Plain Concrete and the Shape of the Stress-Strain Curve*, Journal ACI, Vol. 60, Feb. 1963, pp 209–223.

2.16 Kupfer, H., Hilsdorf, H.K. and Rüsch, H., *Behaviour of Concrete Under Biaxial Stresses*, Journal ACI, Vol. 66, Aug. 1969, pp 655–666.

2.17 Hognestad, E., Hanson, N.W. and McHenry, D., *Concrete Stress Distribution in Ultimate Strength Design*, Journal ACI, Vol. 52, Dec. 1955, pp 455–479.

2.18 Sinha, B.P., Gerstle, K.H. and Tulin, L.G., *Stress-Strain Relationships for Concrete Under Cyclic Loading*, Journal ACI, Vol. 61, Feb. 1964, pp 195–211.

2.19 ACI Committee 439, *Effect of Steel Strength and Reinforcement Ratio on the Mode of Failure and Strain Energy Capacity of R.C. Beams*, Journal ACI, Vol. 66, March 1969, pp 165–173.

2.20 Rüsch, H, *Researches Towards a General Flexural Theory for Structural Concrete*, Journal ACI, Vol. 57, July 1960, pp 1–28.

2.21 — *Building Code Requirements for Reinforced Concrete*, ACI Standard 318–89, Am. Conc. Inst., Detroit, Michigan, USA, 1989.

2.22 Rao, P.S. and Menon, D., *Ultimate Strength of Tubular R C Tower Sections Under Wind Loading*, Indian Concrete Journal, Feb. 1995, pp 117–123.

2.23 Wright, P.J.F., *Comments on an Indirect Tensile Test on Concrete Cylinders*, Magazine of Concrete Research, No. 20, 1955, p. 87.

2.24 Tasuji, M.E., Slate, F.O. and Nilson, A.H., *Stress-Strain Response and Fracture of Concrete in Biaxial Loading*, Journal ACI, Vol. 75, July 1978, pp 306–312.

2.25 Bresler, B. and Pister, K.S., *Strength of Concrete Under Combined Stresses*, Journal ACI, Vol. 55, Sept. 1958, pp 321–345.

2.26 Richart, F.E., Brandtzaeg, A. and Brown, R.L., *A Study of the Failure of Concrete Under Combined Stresses*, Univ. of Illinois Engineering Experimental Station, Bulletin No. 185, 1928.

2.27 Gerstle, K.H. et al,. *Strength of Concrete Under Multi-axial Stress States*, ACI Publication SP–55, Am. Conc. Inst., Detroit, Michigan, USA, 1978, pp 103–131.

2.28 Park, R. and Paulay, T., *Reinforced Concrete Structures*, John Wiley & Sons, Inc., New York, 1975.

2.29 ACI Committee 209, *Prediction of Creep, Shrinkage and Temperature Effects in Concrete Structures*, SP–27, Am. Conc. Inst., Detroit, Michigan, USA, 1971, pp 51–93.

2.30 CEB-FIP, *International Recommendations for the Design and Construction of Concrete Structures*, Comité Européen du Béton-Fédération Internationale de la Précontrainte, Paris, 1970.

2.31 Gouthaman, A. and Menon, D., *Increased Cover Specifications in IS 456 (2000) – Crack-width Implications in RC Slabs*, Indian Concrete Journal, Sept. 2001, pp 581–586.

2.32 ACI Committee 201, *Guide to Durable Concrete*, Journal ACI, Vol. 74, 1977, pp 573–609.

2.33 — *Explanatory Handbook on Indian Standard Code of Practice for Plain and Reinforced Concrete (IS 456:1978)*, Special Publication SP:24, Bureau of Indian Standards, New Delhi, 1983.

2.34 Purushothaman, P., *Reinforced Concrete Structural Elements — Behaviour, Analysis and Design,* Tata McGraw Hill Publication Co. Ltd., New Delhi, 1984.

# 3

# Basic Design Concepts

## 3.1 INTRODUCTION

Having gained a general overview of reinforced concrete structures (Chapter 1) and an understanding of the basic material properties (Chapter 2), it is time to get into the actual details of the design process. This chapter introduces the basic concepts relating to performance criteria in reinforced concrete design.

### 3.1.1 Design Considerations

The aim of structural design is to design a structure so that it fulfils its intended purpose during its intended lifetime with adequate *safety* (in terms of strength, stability and structural integrity), adequate *serviceability* (in terms of stiffness, durability, etc.) and *economy*.

**Safety** implies that the likelihood of (partial or total) collapse of the structure is acceptably low not only under the normal expected loads (*service loads*), but also under abnormal but probable overloads (such as due to earthquake or extreme wind). Collapse may occur due to various possibilities such as exceeding the load–bearing capacity, overturning, sliding, buckling, fatigue fracture, etc. Another related aspect of safety is *structural integrity* (see Section 15.1.3). The objective here is to minimise the likelihood of *progressive collapse*.

**Serviceability** implies satisfactory performance of the structure under *service loads*, without discomfort to the user due to excessive *deflection*, *cracking*, *vibration*, etc. Other considerations that come under the purview of serviceability are *durability*, *impermeability*, *acoustic* and *thermal insulation*, etc. A design that adequately satisfies the 'safety' requirement need not necessarily satisfy the 'serviceability' requirement. For example, a thin reinforced concrete slab can be made safe against collapse (by suitable reinforcement); but if it is too thin, it is likely to result in excessive deflections, crack-widths and permeability (leakage), and the exposed steel becomes vulnerable to corrosion (thereby affecting durability).

Increasing the design *margins of safety* can enhance safety and serviceability; but this increases the cost of the structure. In considering overall **economy**, the increased cost associated with increased safety margins should be weighed against the potential losses that could result from any damage.

### 3.1.2 Design Philosophies

Over the years, various design philosophies have evolved in different parts of the world, with regard to reinforced concrete design. A 'design philosophy' is built up on a few fundamental premises (assumptions), and is reflective of a way of thinking.

The earliest codified design philosophy is the **working stress method** of design (WSM). Close to a hundred years old, this traditional method of design, based on *linear elastic theory*, is still surviving in some countries (including India), although it is now sidelined by the modern *limit states design* philosophy. In the recent (2000) revision of the Code (IS 456), the provisions relating to the WSM design procedure have been relegated from the main text of the Code to an Annexure (Annex B) "so as to give greater emphasis to limit state design" (as stated in the 'Foreword').

Historically, the design procedure to follow the WSM was the **ultimate load method** of design (ULM), which was developed in the 1950s. Based on the (ultimate) strength of reinforced concrete at *ultimate loads*, it evolved and gradually gained acceptance. This method was introduced as an alternative to WSM in the ACI code in 1956 and the British Code in 1957, and subsequently in the Indian Code (IS 456) in 1964.

Probabilistic concepts of design developed over the years[†] and received a major impetus from the mid-1960s onwards. The philosophy was based on the theory that the various uncertainties in design could be handled more rationally in the mathematical framework of probability theory. The risk involved in the design was quantified in terms of a *probability of failure*. Such probabilistic methods came to be known as **reliability-based methods**. However, there was little acceptance for this theory in professional practice, mainly because the theory appeared to be complicated and intractable (mathematically and numerically).

In order to gain code acceptance, the probabilistic 'reliability-based' approach had to be simplified and reduced to a deterministic format involving *multiple (partial) safety factors* (rather than *probability of failure*). The European Committee for Concrete (CEB) and the International Federation for Prestressing (FIP) were among the earliest to introduce the philosophy of ***limit states method*** (LSM) of design, which is reliability-based in concept [Ref. 3.2]. Based on the CEB-FIP recommendations, LSM was introduced in the British Code CP 110 (1973) [now BS 8110 (1997)], and the Indian Code IS 456 (1978). In the United States, LSM was introduced in a slightly different format (*strength design* and *serviceability design*) in the ACI 318−71 (now ACI 318-95).

Thus, the past several decades have witnessed an evolution in design philosophy — from the traditional 'working stress method', through the 'ultimate load method', to the modern 'limit states method' of design.

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[†] For a detailed history, consult Ref. 3.1.

## 3.2 WORKING STRESS METHOD (WSM)

This was the traditional method of design not only for reinforced concrete, but also for structural steel and timber design. The conceptual basis of WSM is simple. The method basically assumes that the structural material behaves in a *linear elastic* manner, and that adequate safety can be ensured by suitably restricting the stresses in the material induced by the expected 'working loads' (service loads) on the structure. As the specified *permissible* ('allowable') stresses are kept well below the material strength (i.e., in the initial phase of the stress-strain curve), the assumption of linear elastic behaviour is considered justifiable. The ratio of the strength of the material to the permissible stress is often referred to as the *factor of safety*.

The stresses under the applied loads are analysed by applying the methods of 'strength of materials' such as the simple bending theory. In order to apply such methods to a composite material like reinforced concrete, *strain compatibility* (due to bond) is assumed, whereby the strain in the reinforcing steel is assumed to be equal to that in the adjoining concrete to which it is bonded. Furthermore, as the stresses in concrete and steel are assumed to be linearly related to their respective strains, it follows that the stress in steel is linearly related to that in the adjoining concrete by a constant factor (called the *modular ratio*), defined as the ratio of the modulus of elasticity of steel to that of concrete.

However, the main assumption of linear elastic behaviour and the tacit assumption that the stresses under working loads can be kept within the 'permissible stresses' are not found to be realistic. Many factors are responsible for this — such as the long-term effects of creep and shrinkage, the effects of stress concentrations, and other secondary effects. All such effects result in significant local increases in and *redistribution* of the calculated stresses[†]. Moreover, WSM does not provide a realistic measure of the actual factor of safety underlying a design. WSM also fails to discriminate between different types of loads that act simultaneously, but have different degrees of uncertainty. This can, at times, result in very unconservative designs, particularly when two different loads (say, dead loads and wind loads) have counteracting effects [Ref. 3.4].

Nevertheless, in defence against these and other shortcomings levelled against WSM, it may be stated that most structures designed in accordance with WSM have been generally performing satisfactorily for many years. The design usually results in relatively large sections of structural members (compared to ULM and LSM), thereby resulting in better serviceability performance (less deflections, crack-widths, etc.) under the usual working loads. The method is also notable for its essential simplicity — in concept, as well as application.

It may also be noted that although WSM has been superseded by the limit states method (LSM) in the design code for general RC structures (IS 456), it continues to

[†] For example, in the case of reinforced concrete columns subjected to sustained service loads, it is found that redistribution of stresses takes place with time to such an extent that the 'permissible' stress in reinforcing steel will not only be exceeded, but the stress is even likely to reach yield stress — thereby upsetting the assumptions and calculations of WSM based on a constant modular ratio [Ref. 3.3].

be the accepted method of design in India for certain special structures such as RC bridges (IRC 21), water tanks (IS 3370) and chimneys (IS 4998).

## 3.3 ULTIMATE LOAD METHOD (ULM)

With the growing realisation of the shortcomings of WSM in reinforced concrete design, and with increased understanding of the behaviour of reinforced concrete at *ultimate loads*, the ultimate load method of design (ULM) evolved in the 1950s and became an alternative to WSM. This method is sometimes also referred to as the *load factor method* or the *ultimate strength method*.

In this method, the stress condition at the state of impending *collapse* of the structure is analysed, and the non-linear stress–strain curves of concrete and steel are made use of. The concept of 'modular ratio' and its associated problems are avoided entirely in this method. The safety measure in the design is introduced by an appropriate choice of the *load factor*, defined as the ratio of the ultimate load (design load) to the working load. The ultimate load method makes it possible for different types of loads to be assigned different *load factors* under combined loading conditions, thereby overcoming the related shortcoming of WSM.

This method generally results in more slender sections, and often more economical designs of beams and columns (compared to WSM), particularly when high strength reinforcing steel and concrete are used.

However, the satisfactory 'strength' performance at *ultimate loads* does not guarantee satisfactory 'serviceability' performance at the normal *service loads*. The designs sometimes result in excessive deflections and crack-widths under service loads, owing to the slender sections resulting from the use of high strength reinforcing steel and concrete.

Moreover, the use of the non-linear stress-strain behaviour for the design of sections becomes truly meaningful only if appropriate *non-linear limit analysis* is performed on the structure. Unfortunately, such a structural analysis is generally not performed on reinforced concrete structures (except in the *yield line theory* for slabs), owing to the difficulties in predicting the behaviour of 'plastic hinges' in reinforced concrete. Commonly, the distribution of stress resultants at ultimate load is taken as the distribution at service loads, magnified by the load factor(s); in other words, analysis is still based on linear elastic theory. This is clearly in error, because significant inelastic behaviour and *redistribution* of stress resultants takes place, as the loading is increased from service loads to ultimate loads.

## 3.4 PROBABILISTIC ANALYSIS AND DESIGN

### 3.4.1 Uncertainties in Design

Safety margins are provided in design to safeguard against the risk of failure (*collapse* or *unserviceability*). In the traditional methods of design, these safety margins were assigned (in terms of 'permissible stresses' in WSM and 'load factors' in ULM) primarily on the basis of engineering judgement. Structures designed according to these traditional methods were found, in general, to be free from failure. However, the scientific basis underlying the provision of safety margins in design has been questioned time and again. As a result of persistent efforts over the past several decades in various fields of engineering, the science of reliability-based design evolved with the objective of providing a rational solution to the problem of ‘adequate safety’.

The main variables in design calculations that are subject to varying degrees of uncertainty and randomness are the *loads* [Fig. 3.1, for example], *material properties* [Fig. 3.2, for example] and *dimensions*. Further, there are idealisations and simplifying assumptions used in the theories of structural analysis and design. There are also several other variable and often unforeseen factors that influence the prediction of strength and serviceability — such as construction methods, workmanship and quality control, intended service life of the structure, possible future change of use, frequency of loading, etc.

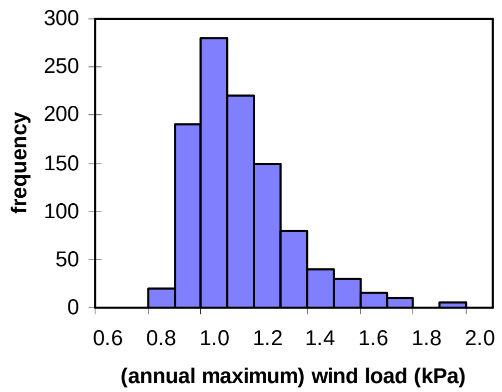


**Fig. 3.1** Typical example of frequency distribution of wind loads on a structure

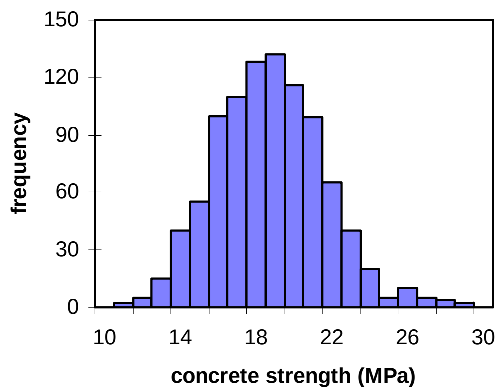


**Fig. 3.2** Typical example of frequency distribution of concrete strength

Thus, the problem facing the designer is to design economically on the basis of:

> "…prediction through imperfect mathematical theories of the performance of structural systems constructed by fallible humans from material with variable properties, when these systems are subjected to an unpredictable natural environment" [Ref. 3.5]

It should be evident that any realistic, rational and quantitative representation of safety must be based on *statistical* and *probabilistic* analysis. [Recent attempts include the application of *fuzzy logic* also.]

### 3.4.2 Classical Reliability Models

In this section, a simple introduction to reliability–based design is given. Two simple 'classical' models are considered here — one for 'strength design' and the other for 'serviceability design'.

#### *Strength Design Model*

The (lifetime maximum) *load effect* $\boldsymbol{S}$ on a structure and the ultimate *resistance* $\boldsymbol{R}$ of the structure (both expressed in terms of a stress resultant such as bending moment at a critical section) are treated as *random variables* whose respective probability density functions $f_S(S)$ and $f_R(R)$ are known [Fig. 3.3]. It is also assumed that $\boldsymbol{S}$ and $\boldsymbol{R}$ are statistically independent, which is approximately true for cases of normal static loading.

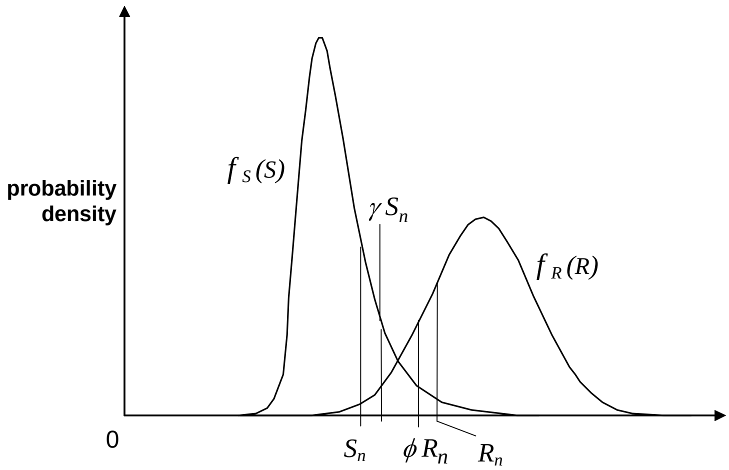


**Fig. 3.3** Classical reliability model for strength design

If $\boldsymbol{S} < \boldsymbol{R}$, the structure is expected to be safe, and if $\boldsymbol{S} > \boldsymbol{R}$, the structure is expected to fail. It is evident from Fig. 3.3 that there is always a probability, however small, that failure may occur due to the exceeding of the load-bearing capacity of the structure (or structural element under consideration).

The *probability of failure* $P_f$ may be calculated as follows :

$$P_f = Prob\left[\{R < S\} \cap \{0 < S < \infty\}\right]$$

$$\Rightarrow \quad P_f = \int_0^{\infty} f_S(S) \left[\int_0^{S} f_R(R)\ dR\right] dS \tag{3.1}$$

The desired margin(s) of safety (expressed in terms of one or more *factors of safety*) can thus be related to the desired ('target') probability of failure $P_f$ .

### *Serviceability Design Model*

Here, the variable to be considered is a *serviceability* parameter $\Delta$ (representing deflection, crack-width, etc.). Failure is considered to occur when the specified limit (maximum allowable limit of serviceability) $\Delta_{all}$ is exceeded [Fig. 3.4]. It may be noted that unlike the previous model, here the limit defining failure is deterministic, and not probabilistic.

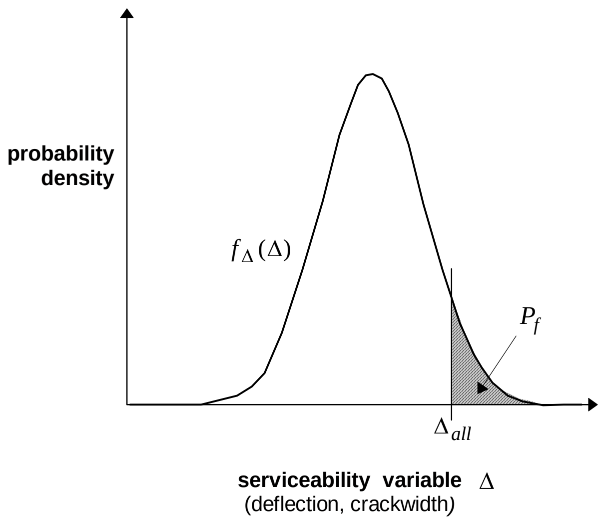


**Fig. 3.4** Classical reliability model for serviceability design

Accordingly, in this case, the probability of failure $P_f$ may be obtained as follows :

$$P_f = \int_{\Delta_{all}}^{\infty} f_\Delta(\Delta)\ d\Delta \tag{3.2}$$

where $f_\Delta$ denotes the probability density function of $\Delta$. Here also, the probability of failure can be restricted to a 'target' value, by suitably selecting the safety margin in the design.

### 3.4.3 Reliability Analysis and Design

From the discussions in the preceding section, it follows that a rational and quantitative solution to the problem of 'adequate safety' can be obtained by quantifying the acceptable risk in terms of target probability of failure or *target reliability*. ['Reliability' is expressed as the complement of the probability of failure, i.e., equal to $(1-P_f)$.]

Evaluating the probability of failure $P_f$ (or the reliability) underlying a given structure is termed *reliability analysis*, whereas designing a structure to meet the target reliability is termed *reliability design* [Ref. 3.6].

However, in practice, there are considerable difficulties involved in reliability analysis and design. Firstly, the problem becomes complicated when a large number of load and resistance 'basic variables' are involved, as is usually the case. The integral [Eq. 3.1] becomes multi–dimensional and quite formidable to solve (even with sophisticated techniques such as *simulation* with 'variance reduction'). Secondly, it is difficult to obtain statistical data regarding the joint probability distribution of the multiple variables. Thirdly, 'target reliabilities' are hard to define, since losses associated with failures are influenced by economic, social and moral considerations, which are difficult to quantify. Fourthly, it is now recognized that 'human error' is a major factor causing failure, and this is difficult to express probabilistically [Ref. 3.1].

### 3.4.4 Levels of Reliability Methods

There exist a number of *levels* of reliability analysis. These are differentiated by the extent of probabilistic information that is used [Ref. 3.1].

A full-scale probabilistic analysis (of the type discussed in Section 3.4.3) is generally described as a *Level III reliability method*. It is highly advanced, mathematically difficult, and generally used at a research level. It is clearly unsuitable for general use in practice.

The problem can be simplified by limiting the probability information of the basic variables to their 'second moment statistics' (i.e., *mean* and *variance*). Such a method is called a *Level II reliability method*. It evaluates the risk underlying a structural design in terms of a *reliability index* $\beta$ (in lieu of the 'probability of failure' $P_f$ used in Level III method). However, even such a 'simplified method' is unsuitable for day-to-day use in a design office, as it requires the application of optimisation techniques for the determination of $\beta$.

For code use, the method must be as simple as possible — using deterministic rather than probabilistic data. Such a method is called a *Level I reliability method*. The 'multiple safety factor' format of *limit states design* comes under this category. Even the traditional methods of design (WSM, ULM) belong to this category as they make use of deterministic measures of safety such as ‘permissible stresses’ and ‘load factors’.

## 3.5 LIMIT STATES METHOD (LSM)

The philosophy of the limit states method of design (LSM) represents a definite advancement over the traditional design philosophies. Unlike WSM, which based calculations on service load conditions alone, and unlike ULM, which based calculations on ultimate load conditions alone, LSM aims for a comprehensive and rational solution to the design problem, by considering *safety* at ultimate loads and *serviceability* at working loads.

The LSM philosophy uses a multiple safety factor format which attempts to provide *adequate safety* at ultimate loads as well as *adequate serviceability* at service loads, by considering all possible ‘limit states’ (defined in the next section). The selection of the various multiple safety factors is supposed to have a sound probabilistic basis, involving the separate consideration of different kinds of failure, types of materials and types of loads. In this sense, LSM is more than a mere extension of WSM and ULM. It represents a new ‘paradigm’ — a modern philosophy.

### 3.5.1 Limit States

A ***limit state*** is a state of impending *failure*, beyond which a structure ceases to perform its intended function satisfactorily, in terms of either *safety* or *serviceability*; i.e., it either *collapses* or becomes *unserviceable*.

There are two types of limit states :

1. ***Ultimate limit states*** (or ‘limit states of collapse’), which deal with strength, overturning, sliding, buckling, fatigue fracture, etc.
2. ***Serviceability limit states***, which deal with discomfort to occupancy and/or malfunction, caused by excessive deflection, crack-width, vibration, leakage, etc., and also loss of durability, etc.

### 3.5.2 Multiple Safety Factor Formats

The objective of limit states design is to ensure that the probability of any limit state being reached is acceptably low. This is made possible by specifying appropriate *multiple safety factors* for each limit state (Level I reliability). Of course, in order to be meaningful, the specified values of the safety factors should result (more-or-less) in a ‘target reliability’. Evidently, this requires a proper reliability study to be done by the code–making authorities.

Most national codes introduced multiple safety factors in limit states design in the 1970s — primarily based on experience, tradition and engineering judgement [Ref. 3.7]. Subsequently, codes have been engaged in the process of *code calibration* — to determine the range of the reliability index $\beta$ (or its equivalent probability of failure $P_f$) underlying the specified safety factors for different practical situations [Ref. 3.1, 3.6, 3.8]. With every code revision, conscious attempts are made to specify more rational reliability–based safety factors, in order to achieve practical designs that are *satisfactory* and *consistent* in terms of the degree of safety, reliability and economy.

### 3.5.3 Load and Resistance Factor Design Format

Of the many multiple safety factor formats in vogue, perhaps the simplest to understand is the Load and Resistance Factor Design (LRFD) format, which is adopted by the ACI Code [Ref. 3.5, 3.8, 3.9]. Applying the LRFD concept to the classical reliability model [Fig. 3.5], adequate safety requires the following condition to be satisfied :

$$\textit{Design Resistance}\ \left(\phi\ R_n\right)\ \geq\ \textit{Design Load effect}\ \left(\gamma\ S_n\right) \tag{3.3}$$

where $R_n$ and $S_n$ denote the nominal or characteristic values of *resistance* ***R*** and *load effect* ***S*** respectively; $\phi$ and $\gamma$ denote the ***resistance factor*** and ***load factor*** respectively. The resistance factor $\phi$ accounts for 'under-strength', i.e., possible shortfall in the computed 'nominal' resistance, owing to uncertainties related to material strengths, dimensions, theoretical assumptions, etc., and accordingly, it is less than unity. On the contrary, the load factor $\gamma$ , which accounts for 'overloading' and the uncertainties associated with $S_n$, is generally greater than unity.

Eq. 3.3 may be rearranged as

$$S_n\ <\ \frac{R_n}{\gamma/\phi} \tag{3.3 a}$$

which is representative of the safety concept underlying WSM, $\gamma/\phi$ here denoting the 'factor of safety' applied to the material strength, in order to arrive at the *permissible stress* for design.

Alternatively, Eq. 3.3 may be rearranged as

$$R_n\ \geq\ \left(\gamma/\phi\right)\ S_n \tag{3.3 b}$$

which is representative of the safety concept underlying ULM, $\gamma/\phi$ here denoting the so-called 'load factor' (ULM terminology) applied to the load in order to arrive at the *ultimate load* for design.

### 3.5.4 Partial Safety Factor Format

The multiple safety factor format recommended by CEB−FIP [Ref. 3.2], and adopted by the Code, is the so-called *partial safety factor* format, which may be expressed as follows:

$$R_d\ \geq\ S_d \tag{3.4}$$

where $R_d$ is the *design resistance* computed using the reduced material strengths $0.67\,f_{ck}/\gamma_c$ and $f_y/\gamma_s$ , involving two separate *partial (material) safety factors*

$\gamma_c$ (for concrete) and $\gamma_s$ (for steel), in lieu of a single overall resistance factor $\phi$, and $S_d$ is the *design load effect* computed for the enhanced loads $\left(\gamma_D.DL,\ \gamma_L.LL,\ \gamma_Q.QL,...\right)$ involving separate *partial load factors* $\gamma_D$ (for dead load), $\gamma_L$ (for live load), $\gamma_Q$ (for wind *or*[†] earthquake load). The other terms involved are the *nominal* compressive strength of concrete 0.67 $f_{ck}$ (refer Section 2.8.5) and the *nominal* yield strength of steel $f_y$ on the side of the resistance, and the *nominal* load effects DL, LL and QL representing dead loads, live loads and wind/earthquake loads respectively.

It may be noted that, whereas the multiplication factor $\phi$ is generally less than unity, the dividing factors $\gamma_c$ and $\gamma_s$ are greater than unity — giving the same effect. All the *load factors* are generally greater than unity, because over-estimation usually results in improved safety. However, one notable exception to this rule is the dead load factor $\gamma_D$ which is taken as 0.8 or 0.9 while considering stability against overturning or sliding, or while considering reversal of stresses when dead loads are combined with wind/earthquake loads; in such cases, under-estimating the counteracting effects of dead load results in greater safety.

One other effect to be considered in the selection of load factors is the reduced probability of different types of loads (DL, LL, QL) acting simultaneously at their peak values. Thus, it is usual to reduce the load factors when three or more types of loads are considered acting concurrently; this is referred to sometimes as the 'load combination effect'.

## 3.6 CODE RECOMMENDATIONS FOR LIMIT STATES DESIGN

The salient features of LSM, as prescribed by the Code, are covered here. Details of the design procedure for various limit states of collapse and serviceability are covered in subsequent chapters.

### 3.6.1 Characteristic Strengths and Loads

The general definition of the **characteristic strength** of a material (concrete or steel) was given in Section 2.6.1. It corresponds to the 5 percentile strength value. In the case of reinforcing steel, it refers to the 'specified yield stress' as mentioned in Section 2.14.1.

The **characteristic load** is defined as the load that *"has a 95 percent probability of not being exceeded during the life of the structure"* (Cl. 36.2 of the Code). However, in the absence of statistical data regarding loads, the *nominal* values specified for dead, live and wind loads are to be taken from IS 875 (Parts 1–3) : 1987 and the values for 'seismic loads' (earthquake loads) from IS 1893 : 2002.

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[†] Either wind *or* earthquake loads is considered at a time. The probability of the joint occurrence of an earthquake as well as an extreme wind is considered to be negligible.

### 3.6.2 Partial Safety Factors for Materials

The **design strength** of concrete or reinforcing steel is obtained by dividing the **characteristic strength** by the appropriate **partial safety factor**. In the case of concrete, while $f_{ck}$ is the characteristic *cube* strength, the characteristic strength of concrete in the *actual structure* is taken as $0.67 f_{ck}$ [refer Section 2.8.5], and hence the *design strength* of concrete is $0.67 f_{ck} / \gamma_c$ .

For *ultimate limit states*, the Code specifies $\gamma_c$ = 1.5 and $\gamma_s$ = 1.15. A higher partial safety factor has been assigned to concrete, compared to reinforcing steel, evidently because of the higher variability associated with it.

For *serviceability limit states*, $\gamma_c = \gamma_s =$ 1.0. A safety factor of unity is appropriate here, because the interest is in estimating the *actual* deflections and crack-widths under the service loads, and not 'safe' (conservative) values.

### 3.6.3 Partial Safety Factors for Loads

Three different load combinations have been specified (Cl. 36.4.1 of Code) involving the combined effects of dead loads (DL), 'imposed' or live loads (LL) and wind/earthquake loads (QL).

The code recommends the following weighted combinations for estimating the *ultimate load effect* (UL) and the *serviceability load effect* (SL):

***Ultimate limit states :***

- UL = 1.5 (DL + LL)
- UL = 1.5 (DL + QL) or (0.9DL + 1.5 QL)
- UL = 1.2 (DL + LL + QL)

The reduced load factor of 1.2 in the third combination above recognises the reduced probability of all the three loads acting together at their possible peak values.

For the purpose of structural design, the design resistance (using the material partial safety factors) should be greater than or equal to the maximum load effect that arises from the above load combinations [Eq. 3.4].

The Code makes a departure from the usual convention (adopted by other codes) of assigning a *lower* load factor for DL in comparison with the more variable LL — apparently on the grounds that it is more convenient in practice to deal with all *gravity* loads (dead plus live) together [Ref. 3.10]. Besides, applying an average load factor of 1.5 to the combined gravity loads is, in general, likely to have about the same effect as, say, (1.4 DL + 1.6 LL) — which is adopted by BS 8110.

***Serviceability limit states :***

- SL = 1.0 (DL + LL)
- SL = 1.0 (DL + QL)
- SL = (1.0 DL) + (0.8 LL) + (0.8 QL)

The use of a partial load factor of unity, in general, is appropriate, because it implies *service load* conditions — which is required for 'serviceability design'.

However, when live loads and wind loads are combined, it is improbable that both will reach their characteristic values simultaneously; hence a lower load factor is assigned to LL and QL in the third combination, to account for the reduced probability of joint occurrence.

### *Other Load Combinations*

Other loads to be considered, but not listed in the Code (Table 18) include the effects of creep, shrinkage, temperature variation and support settlement. Although a load factor of unity may be appropriate while considering these load effects (in addition to gravity loads) for 'serviceability limit states', the following combinations are recommended [Ref. 3.10] for 'ultimate limit states' :

- UL = 0.75 (1.4 DL + 1.4 TL + 1.7 LL)
- UL = 1.4 (DL + TL)

where TL represents the structural effects due to temperature variation, creep, shrinkage or support settlement.

## 3.6.4 Design Stress-Strain Curve for Concrete

The c*haracteristic* and *design* stress–strain curves specified by the Code for concrete in ***flexural compression*** are depicted in Fig. 3.5. The maximum stress in the 'characteristic' curve is restricted to 0.67 $f_{ck}$ for reasons explained in Section 2.8.5. The curve consists of a parabola in the initial region up to a strain of 0.002 (where the slope becomes zero), and a straight line thereafter, at a constant stress level of 0.67 $f_{ck}$ up to an *ultimate strain* of 0.0035.

For the purpose of *limit states design*, the appropriate partial safety factor $\gamma_c$ has to be applied, and $\gamma_c$ is equal to 1.5 for the consideration of ultimate limit states [refer Section 3.6.2]. Thus, the 'design curve' is obtained by simply scaling down the ordinates of the characteristic curve — dividing by $\gamma_c$ [Fig. 3.5]. Accordingly, the maximum *design stress* becomes equal to 0.447 $f_{ck}$, and the formula for the design compressive stress $f_c$ corresponding to any strain $\varepsilon \leq$ 0.0035 is given by :

$$f_c = \begin{cases} 0.447 f_{ck}\left[2\left(\frac{\varepsilon}{0.002}\right) - \left(\frac{\varepsilon}{0.002}\right)^2\right] & \text{for } \varepsilon < 0.002 \\ 0.447 f_{ck} & \text{for } 0.002 \leq \varepsilon \leq 0.0035 \end{cases} \quad (3.5)$$

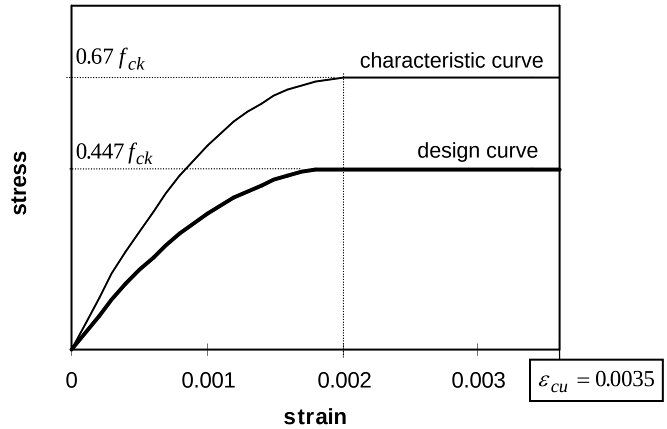



**Fig. 3.5** Characteristic and design stress-strain curves for concrete in flexural compression

When concrete is subjected to ***uniform compression***, as in the case of a concentrically loaded short column, the *ultimate strain* is limited to 0.002, and the corresponding *maximum design stress* is 0.447 $f_{ck}$. The stress–strain curve has no relevance in the limit state of collapse by (pure) compression of concrete, and hence is not given by the Code.

When concrete is subject to ***axial compression combined with flexure***, the ultimate strain is limited to a value between 0.002 and 0.0035, depending on the location of the neutral axis [refer Chapter 13]. The maximum design stress level remains unchanged at 0.447 $f_{ck}$.

### 3.6.5 Design Stress-Strain Curve for Reinforcing Steel

The *characteristic* and *design* stress–strain curves specified by the Code for various grades of reinforcing steel (in tension or compression) are shown in Figs. 3.6 and 3.7. The design yield strength $f_{yd}$ is obtained by dividing the specified yield strength $f_y$ by the partial safety factor $\gamma_s$ = 1.15 (for ultimate limit states); accordingly, $f_{yd} = 0.870\, f_y$. In the case of mild steel (Fe 250), which has a well–defined yield point, the behaviour is assumed to be perfectly linear-elastic up to a design stress level of 0.87 $f_y$ and a corresponding design yield strain $\varepsilon_y$ = 0.87 $f_y / E_s$; for larger strains, the design stress level remains constant at 0.87 $f_y$ [Fig. 3.6].

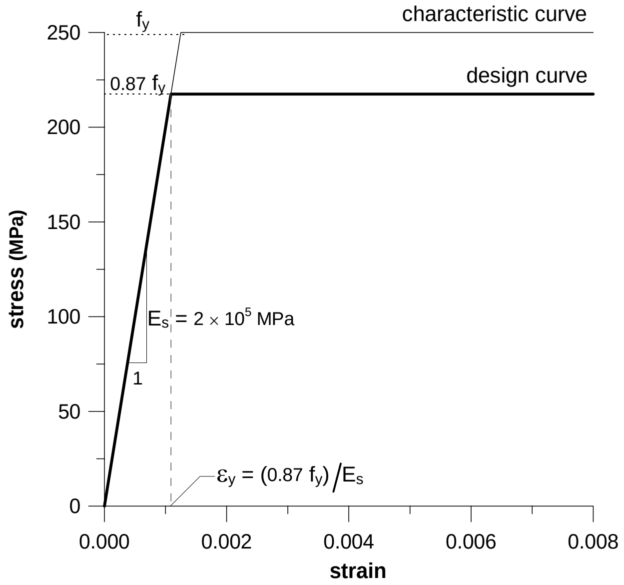



**Fig. 3.6** Characteristic and design stress-strain curves for **Fe 250** grade mild steel

In the case of cold-worked bars (Fe 415 and Fe 500), which have no specific yield point, the transition from linear elastic behaviour to nonlinear behaviour is assumed to occur at a stress level equal to 0.8 times $f_y$ in the *characteristic* curve and 0.8 times $f_{yd}$ in the *design* curve. The full design yield strength 0.87 $f_y$ is assumed to correspond to a 'proof strain' of 0.002; i.e, the design yield strain $\varepsilon_y$ is to be taken as 0.87 $f_y / E_s$ + 0.002, as shown in Fig. 3.7. The non-linear region is approximated as piecewise linear segments. The coordinates of the salient points of the design stress-strain curve for Fe 415 and Fe 500 are listed in Tables 3.1 and 3.2. The design stress, corresponding to any given strain, can be obtained by linear interpolation from Table 3.2.

It may be noted that whereas in the case of concrete [Fig. 3.5], the partial safety factor $\gamma_c$ is applicable at *all* stress levels, in the case of reinforcing steel, the partial safety factor $\gamma_s$ is applicable only for the inelastic region [Figs 3.6, 3.7]. This may be ascribed to the fact that in the case of concrete, the stress-strain curve (including the short–term modulus of elasticity) is directly affected by changes in the compressive strength of concrete [refer Fig. 2.7]; however, this is not the case for steel whose modulus of elasticity is independent of the variations in the yield strength.

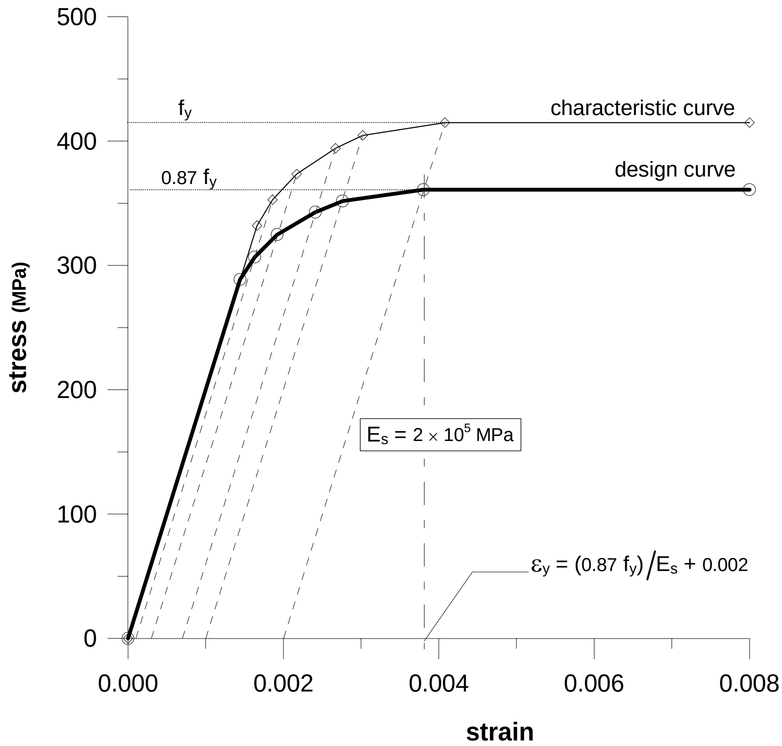



**Fig. 3.7** Characteristic and design stress-strain curves for **Fe 415** grade cold-worked steel†

**Table 3.1** Salient points on the stress-strain curve for cold-worked steels

| **Design Stress** | .80 $f_{yd}$ | .85 $f_{yd}$ | .90 $f_{yd}$ | .95 $f_{yd}$ | .975 $f_{yd}$ | 1.0 $f_{yd}$ |
|---|---|---|---|---|---|---|
| **Inelastic Strain** | 0.0000 | 0.0001 | 0.0003 | 0.0007 | 0.0010 | 0.0020 |

**Table 3.2** Design stresses at specified strains for (a) **Fe 415** and (b) **Fe 500**

| **Fe 415** | |
|---|---|
| **Strain** | **Stress** *(MPa)* |
| 0.000 00 | 0.0 |

| **Fe 500** | |
|---|---|
| **Strain** | **Stress** *(MPa)* |
| 0.000 00 | 0.0 |

† The characteristic and design stress-strain curves for Fe 500 grade cold-worked steel is similar; in this case, $f_y$ shall be taken as 500 MPa (instead of 415 MPa).

| | | | | |
|---|---|---|---|---|
| 0.001 44 | 288.7 | | 0.001 74 | 347.8 |
| 0.001 63 | 306.7 | | 0.001 95 | 369.6 |
| 0.001 92 | 324.8 | | 0.002 26 | 391.3 |
| 0.002 41 | 342.8 | | 0.002 77 | 413.0 |
| 0.002 76 | 351.8 | | 0.003 12 | 423.9 |
| $\geq$0.003 80 | 360.9 | | $\geq$0.004 17 | 434.8 |

## REVIEW QUESTIONS

3.1 Discuss the merits and demerits of the traditional methods of design (working stress method, ultimate load method).

3.2 How does *creep* of concrete affect the *modular ratio*?

3.3 What is the difference between *deterministic* design and *probabilistic* design?

3.4 Show that an alternative expression for the *probability of failure* for the classical reliability problem of safety [refer Eq. 3.1] can be obtained as:

$$P_f = 1 - \int_0^{\infty} f_R(R) \left[ \int_0^S f_S(S)\ dS \right] dR$$

3.5 Why is it difficult to undertake a fully probabilistic design of a structure in practice?

3.6 What is meant by *limit state*? Discuss the different 'limit states' to be considered in reinforced concrete design.

3.7 The maximum moments at a section in a reinforced concrete beam (end section) are obtained (for different independent service load conditions) from structural analysis, as –50 kNm, –80 kNm, $\pm$ 120 kNm and $\pm$ 180 kNm under dead loads, live loads, wind loads and earthquake loads respectively. Determine the *ultimate design* moments ('negative' as well as 'positive') to be considered (as per the Code) for the limit state of collapse (flexure).

3.8 Explain the basis for the selection of partial load and safety factors by the Code for 'serviceability limit states'.

3.9 Why is the partial safety factor for concrete ($\gamma_c$) greater than that for reinforcing steel ($\gamma_s$) in the consideration of ultimate limit states?

3.10 Why is it that $\gamma_c$ is applicable at all stress levels whereas $\gamma_s$ is applicable only near the 'yield stress' level?

3.11 Is the use of a single (overall) *resistance factor* $\phi$ more suitable than two separate *partial safety factors* $\gamma_c$ and $\gamma_s$? Give your views.

3.12 What is meant by *calibration* of codes? Why is it necessary?

3.13 Determine the *design stress* levels (at ultimate limit states) in (a) Fe 250, (b) Fe 415 and (c) Fe 500 grades of steel, corresponding to tensile strains of (i) 0.001, (ii) 0.002, (iii) 0.003 and (iv) 0.004.

## REFERENCES

3.1 Madsen, H.O., Krenk, S. and Lind, N.C., *Methods of Structural Safety*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1986.

3.2 CEB-FIP, *International Recommendations for the Design and Construction of Concrete Structures*, Comité Européen du Béton-Fédération Internationale de la Précontrainte, Paris, 1970.

3.3 Park, R. and Paulay, T., *Reinforced Concrete Structures*, John Wiley & Sons, Inc., New York, 1975.

3.4 Rao, P.S. and Menon, D., *Safety Considerations in Wind-Resistant Design of R C Chimneys*, Journal of Instn. of Engineers (India) — Civil Engg Div., Vol. 76, Part 4, pp 242–248.

3.5 Cornell, C.A., *A Probability-Based Structural Code*, Journal ACI, Vol. 66, Dec. 1969, pp 974–985.

3.6 Ranganathan, R., *Reliability Analysis and Design of Structures*, Tata McGraw-Hill Publication Co. Ltd., New Delhi, 1990.

3.7 Ellingwood, B., *Reliability Basis for Load and Resistance Factors for R C Design*, NBS Building Science Series 110, National Bureau of Standards, Washington D.C., 1978.

3.8 Ellingwood, B. and Galambos, T.V. and MacGregor, J G. and Cornell, C.A., *Development of a Probability Based Load Criterion for American National Standard A58*, Special Publication No.577, National Bureau of Standards, Washington D.C., 1980.

3.9 Ravindra, M.K. and Galambos, T.V., *Load and Resistance Factor Design for Steel*, ASCE Journal of Struct. Div., Vol. 104, Sept. 1978, pp 1337–1353.

3.10 — *Explanatory Handbook on Indian Standard Code of Practice for Plain and Reinforced Concrete (IS 456:1978)*, Special Publication SP:24, Bureau of Indian Standards, New Delhi, 1983.

# 4

# Behaviour in Flexure

## 4.1 INTRODUCTION

*Flexure* or *bending* is commonly encountered in structural elements such as *beams* and *slabs* (as well as *plates* and *shells*) which are transversely loaded. Flexure also occurs in *columns* and *walls* that are subjected to eccentric loading, lateral pressures and/or lateral displacements.

'Flexure' usually occurs in combination with *transverse shear*, and sometimes in combination with other structural actions, such as *axial compression* (or *tension*) and *torsion*. For convenience, the behaviour of reinforced concrete in flexure alone is considered in this chapter. The effects of shear, torsion and axial force are considered separately in subsequent chapters, as also their combined effects.

Furthermore, the actual *design* of reinforced concrete elements (in flexure) is deferred to the next chapter (Chapter 5). The scope of this chapter is limited to discussing the overall behaviour of reinforced concrete in flexure, and includes the *analysis* of beam sections.

### 4.1.1 Two Kinds of Problems : Analysis and Design

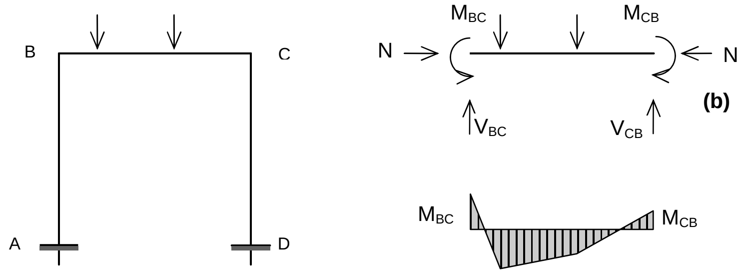
There are two kinds of problems commonly encountered in structural design practice. In the first kind, termed *analysis* (or 'review') problems, the complete cross-sectional dimensions (including details of reinforcing steel), as well as the material properties of the member are known. It is desired to compute (1) the stresses in the materials (or deflections, crack-widths, etc.) under given loads or (2) the allowable or ultimate bending moments (loads) that the member can resist. The 'analysis' referred to here (with regard to *stresses* in a given beam *section*) should not be confused with 'structural analysis', which refers to the determination of *stress resultants* (i.e., bending moments, shear forces, etc.) in an entire *structure* (such as a frame) or a structural element (such as a beam).

The second type of problem involves *design.* In this case, the load effects (stress resultants) are known from structural analysis, and it is required to select appropriate grades of materials and to arrive at the required member dimensions and reinforcement details. It is evident that there are many possible solutions to a design problem, whereas the solution to an analysis problem is unique. As mentioned earlier, the topic of 'design for flexure' is dealt with in the next chapter.

### 4.1.2 Bending Moments in Beams from Structural Analysis

The beam is a very commonly used structural element. It may exist independently, or may form a component of a structural framework (as in 'grids' and 'rigid frames'). In all such cases, the beam is treated as a *one-dimensional* (line) element (with known material and geometric properties) for the purpose of structural analysis.

Commonly, at any point in the beam line element, the stress resultants to be determined from structural analysis are the *bending moment* (*M*) and the transverse *shear force* (*V*). A *twisting moment* (*T*) may also exist in some situations (e.g., when the loading is eccentric to the 'shear centre' axis). Frequently, an *axial force* (*N*) — tensile or compressive — exists in combination with *M* and *V*, as shown in Fig. 4.1. The effect of the axial force may be neglected if the normal stresses induced by it are relatively low (as is often the case in *beams*, unlike *columns*). Occasionally, a beam may be subject to *biaxial bending*, involving bending moments and transverse shear forces in two orthogonal planes (as when a beam is laterally loaded, both vertically and horizontally).




**(a)** line diagram of a portal frame

**(b)** freebody of beam

**(c)** bending moment diagram in beam

**Fig. 4.1** Structural analysis to determine bending moments in a beam

In the present context, it is assumed that the distribution of bending moments along the length of the beam is known from structural analysis [Fig. 4.1 (c)]. The effect of uniaxial bending moment (*M*) alone at a given beam section is investigated in this chapter.

### 4.1.3 From Bending Moment to Flexural Stresses

Although in structural analysis, it suffices to treat the beam as a one-dimensional element, the fact is that the beam is actually three-dimensional. Each point on the one-dimensional beam element is representative of a cross-section (usually rectangular) on the actual three-dimensional beam, and the so−called 'bending moment' (stress-resultant) at any section of the beam manifests in the form of *normal stresses* (compressive and tensile) in concrete and reinforcing steel.

In the sections to follow in this chapter, the relationship between the bending moment (*M*) and the flexural (normal) stresses in concrete and steel (at various stages of loading) is described.

## 4.2 THEORY OF FLEXURE FOR HOMOGENEOUS MATERIALS

Reinforced concrete is a *composite* material. Before developing a theory of flexure for such a material, it is instructive to review the conventional 'theory of flexure' which was originally developed for a *homogeneous* material (such as structural steel). [It is presumed that the reader is familiar with the *simple theory of bending/flexure* and only a brief recapitulation is presented here].

### 4.2.1 Fundamental Assumption

The fundamental assumption in flexural theory is that plane cross-sections (taken normal to the longitudinal axis of the beam) remain *plane* even after the beam bends. This assumption is generally found to be valid for beams of usual proportions† [Ref. 4.1, 4.2].

For initially straight members, the assumption implies that the distribution of *normal strains* across the beam cross-section is *linear* [Fig. 4.2 (a), (b)]. That is, the normal strain at any points in the beam section is proportional to its distance from the *neutral* axis. For the case of a 'sagging' (designated 'positive' in this book) moment, as indicated in Fig. 4.2, the top 'fibres' (above the neutral axis) are subjected to compression and the bottom 'fibres' (below the neutral axis) to tension, with the maximum strains occurring at the most extreme (top/bottom) surfaces. Of course, if the moment is 'hogging' ('negative'), as in the case of a cantilever, the top fibres will be in tension and the bottom fibres in compression.

### 4.2.2 Distribution of Stresses

The normal stress induced by flexure (*flexural stress*) at any point such as *a* in Fig. 4.2(b) depends on the normal strain $\varepsilon_a$ at that level. This stress $f_a$ corresponding to strain $\varepsilon_a$ is determined from the stress-strain relation for the material [Fig. 4.2(d)]. Thus, corresponding to each strain level (such as $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$) in Fig. 4.2(b), the corresponding stress level ( $f_1, f_2, f_3$ ), indicated in Fig. 4.2(c) is obtainable from the stress-strain curve [Fig. 4.2(d)]. As the distribution of strains is *linear* (with strains increasing in the *y*-direction from a zero

† In 'deep beams' (i.e., beams whose depths are comparable to their spans), this assumption is not valid as significant *warping* of the cross-section occurs, on account of shear deformations.

value at the neutral axis) and as the stress-strain curve is plotted on a linear scale (with strains increasing along the *x*-axis), it follows that the distribution of stresses in the beam section will be identical to that in the stress-strain curve, but turned through 90 degrees.

The location of the *neutral axis* (NA) and the magnitude of the *flexural stresses* are obtained by solving two simple equations of static equilibrium:

$$C = T \tag{4.1}$$

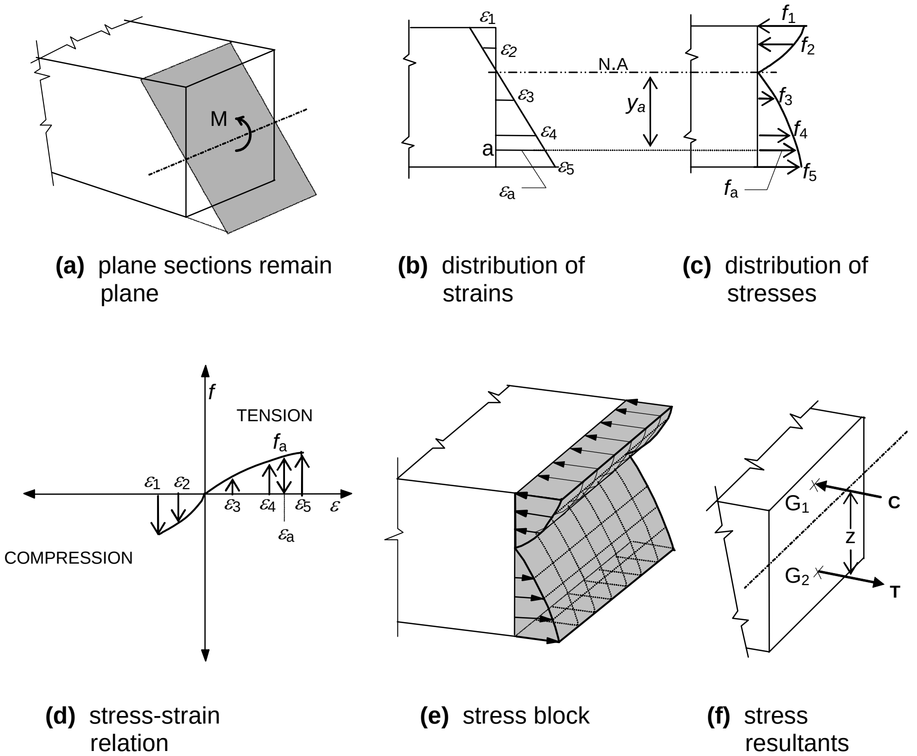
$$M = C\,z \tag{4.2a}$$

or

$$M = T\,z \tag{4.2b}$$

where $C$ and $T$ denote the resultant compression and tension respectively [refer 'stress block' in Fig. 4.2(e)], and $z$ is the lever arm of the 'couple'. In Fig. 4.2(f), $z$ is shown as the distance $G_1G_2$, where $G_1$ and $G_2$ denote the points of intersection of the lines of action of $C$ and $T$ with the cross-sectional surface.

It may be noted that Eq. 4.1 and 4.2 are valid whether or not the stress-strain behaviour of the material is *linear* and/or *elastic*; the material could just as well be *nonlinear* and *inelastic* in its behaviour.




**(a)** plane sections remain plane

**(b)** distribution of strains

**(c)** distribution of stresses

**(d)** stress-strain relation

**(e)** stress block

**(f)** stress resultants

**Fig. 4.2** Homogeneous section under flexure

### 4.2.3 Linear Elastic Material

If the homogeneous material of the beam follows Hooke's law (that is, it is *linearly elastic*), the stress-strain distribution will be linear, the constant of proportionality $E = f/\varepsilon$ being the *Young's modulus of elasticity*.

For such a material, evidently, the distribution of stresses across the cross-section of the beam will be linear [Fig. 4.3]. Incidentally, this 'straight-line distribution' of stresses is also valid for a linear *inelastic* material. However, traditionally, a linear stress-strain relation has been associated with elastic behaviour, and frequently the latter is implied to mean the former as well.

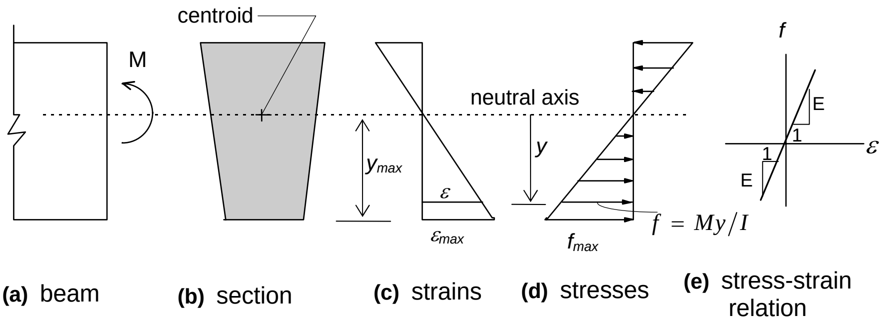



**Fig. 4.3** Linear elastic stress distribution in flexure

Assuming that $E$ has the same value in both tension and compression, it can be easily shown by applying the methods of 'strength of materials' [Ref. 4.3] that the *neutral axis passes through the centroid of the section* [Fig. 4.3], and

$$f = My/I \tag{4.3}$$

$$M = f_{\max} I/y_{\max} \tag{4.4}$$

where

$f \equiv$ flexural stress at any point at a distance $y$ from the neutral axis,
$f_{max} \equiv$ flexural stress in the extreme fibre ($y = y_{max}$),
$I \equiv$ second moment of area about the neutral axis, and
$M \equiv$ bending moment at the section.

## 4.3 LINEAR ELASTIC ANALYSIS OF COMPOSITE SECTIONS

The analysis described in Section 4.2.3 [including Eq. 4.3, 4.4] is not directly applicable to *nonhomogeneous* materials like reinforced concrete. For 'composite' materials, made of two (or more) different (linear elastic) materials [Fig. 4.4(a)], the theory has to be suitably modified.

### 4.3.1 Distribution of Strains and Stresses

The fundamental assumption in flexural theory (refer Section 4.2.1) that initially *plane cross-sections remain plane*, while subject to bending, is valid — provided the two materials are *bonded* together to act as an integral unit, without any 'slip' at their interface.

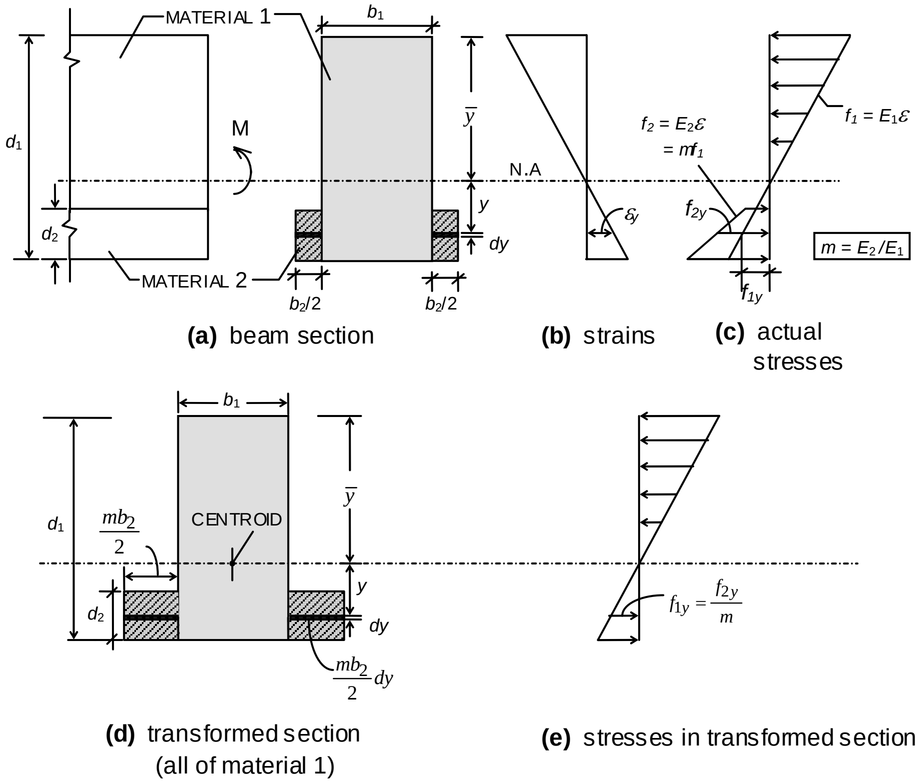



**Fig. 4.4** Concept of 'transformed section'

Accordingly, the strain variation in the section will be linear [Fig. 4.4(b)]. When two different (but bonded) materials are located at the same distance $y$ from the 'neutral axis', both materials will have exactly the same strain $\varepsilon_y$. The corresponding stresses will be $f_{1y} = E_1\varepsilon_y$ in the case of 'material 1' and $f_{2y} = E_2\varepsilon_y$ in the case of 'material 2', where $E_1$ and $E_2$ represent the elastic moduli of materials 1 and 2 respectively [Fig. 4.4(c)].

The stress $f_{2y}$ in the 'material 2' can be expressed in terms of the corresponding stress $f_{1y}$ in the 'material 1' (at points located at the same distance $y$ from the neutral axis) as follows:

$$f_{2y} = mf_{1y} \tag{4.5}$$

$$m = E_2/E_1 \tag{4.6}$$

The ratio of the two moduli of elasticity, $m$, is called the *modular ratio*.

### 4.3.2 Concept of 'Transformed Section'

The concept of 'modular ratio' makes it possible, for the purpose of analysis, to transform the composite section into an equivalent homogeneous section made up entirely of one material (say, 'material 1'). Evidently, this transformation must not alter the magnitude, direction and line of action of the resultant *forces* in the 'material 2' due to the flexural stresses $f_{2y}$.

Considering the resultant force $dF_2$ in an infinitesimal element of 'material 2' having thickness $dy$ (and corresponding breadth $b_2$), located at a distance $y$ from the neutral axis [Fig. 4.4 (a),(c)],

$$dF_2 = f_{2y}(b_2 dy)$$

Substituting Eq. 4.5, $dF_2$ can be expressed in terms of $f_{1y}$ as follows:

$$dF_2 = mf_{1y}(b_2 dy) = f_{1y}(mb_2)dy \tag{4.7}$$

Eq. 4.7 indicates that 'material 2' may be transformed into an equivalent 'material 1' simply by multiplying the original breadth $b_2$ (dimension parallel to the neutral axis at the depth $y$) with the modular ratio $m$.

In the *transformed section* [Fig. 4.4 (d)], as the material is homogeneous (all of 'material 1') and 'linear elastic', the analysis can proceed exactly in the manner described in Section 4.2.3.

The use of the 'transformed section' concept may be limited to determining the *neutral axis* as the 'centroidal axis' of the transformed section. The stresses induced in the two materials due to a given moment can then be determined by applying the basic equations of static equilibrium [Eq. 4.1, 4.2].

Alternatively, the stresses can be computed with the 'transformed section' itself, by applying the flexure formula [Eq. 4.3]; in this case the second moment of area $I_g$ of the 'transformed section' has to be considered. The stresses thus computed with reference to 'material 1' can be converted to the equivalent stresses in 'material 2' by involving the 'modular ratio' concept [Eq. 4.5].

## 4.4 MODULAR RATIO AND CRACKING MOMENT

### 4.4.1 Modular Ratio in Reinforced Concrete

In the case of the *working stress* analysis of reinforced concrete sections, it is usual to transform the composite section into an equivalent *concrete* section. Accordingly, for reinforced concrete, the 'modular ratio' $m$ [Eq. 4.6] is defined as the ratio of the elastic modulus of steel to that of concrete.

As discussed earlier (Section 2.8.3, 2.11.1), the modulus of elasticity of concrete is *not a constant* (unlike that of steel). The 'short-term static modulus' $E_c$, given by Eq. 2.4, is not considered appropriate for determining the modular ratio $m$ because it ignores the long-term effects of *creep* under sustained loading. Partly taking this into account, the Code [(Cl. B-1.3(d)] suggests the following approximate formula for determining the modular ratio:

$$m = 280/3\sigma_{cbc} \tag{4.8}$$

implying that $m\,\sigma_{cbc}$ is a constant† [Ref. 4.9]:

$$m\sigma_{cbc} = 280/3 \tag{4.9}$$

where $\sigma_{cbc}$ is the *permissible* compressive stress of concrete in bending (refer Table 21 of the Code). Values of $\sigma_{cbc}$ (in MPa units) and $m$ for different grades of concrete are listed in Table 4.1.

**Table 4.1** Values of $\sigma_{cbc}$ and *m* for different concrete grades

| Concrete Grade | $\sigma_{cbc}$ (MPa) | Modular ratio 'm' |
|---|---|---|
| M 15 | 5.0 | 18.67 |
| M 20 | 7.0 | 13.33 |
| M 25 | 8.5 | 10.98 |
| M 30 | 10.0 | 9.33 |
| M 35 | 11.5 | 8.11 |
| M 40 | 13.0 | 7.18 |
| M 45 | 14.5 | 6.44 |
| M 50 | 16.0 | 5.83 |

### 4.4.2 Transformed Area of Reinforcing Steel

***Tension Steel***

Applying the concept of 'transformed section', the area of tension reinforcement steel $A_{st}$ is transformed into equivalent concrete area as $mA_{st}$. This transformation is valid in reinforced concrete not only for flexural members but also for members subjected to *direct tension* [refer Cl. B-2.1.1 of the Code].

The stress in the tension steel, $f_{st}$ , is obtained from the corresponding stress $f_{cs}$ in the equivalent 'transformed' concrete (at the level of the steel) as $f_{st} = mf_{cs}$.

† This concept is used subsequently in some derivations. Hence, for consistency in calculations, the value of *m* (given by Eq. 4.8) should *not* be rounded off to an integer (as done in the traditional WSM).

### *Compression Steel*

When reinforcing steel is provided in compression in reinforced concrete beams or columns, the modular ratio to be considered for transformation is generally greater than that used for tension steel [Eq. 4.9]. This is because the long-term effects of creep and shrinkage of concrete, as well as the nonlinearity at higher stresses, result in much larger compressive strains in the compression steel than those indicated by the linear elastic theory using the normally specified value of $m$. Accordingly, the Code recommends that the transformed area of compression steel $A_{sc}$ be taken as $1.5mA_{sc}$, rather than $mA_{sc}$.

The stress in the compression steel, $f_{sc}$ , is obtained from the corresponding stress $f_{csc}$ in the equivalent 'transformed' concrete (at the level of the compression steel) as $f_{sc} = 1.5mf_{csc}$

It may be noted that, while considering the area of *concrete* (under compression) in the transformed section, the *net* area $A_c$, i.e., gross area $A_g$ minus $A_{sc}$ (making allowance for the concrete area displaced by the steel area) should be considered.

## 4.4.3 Cracking Moment

Concrete in the extreme tension fibre of a beam section is expected to crack (for the first time) when the stress reaches the value of the *modulus of rupture* $f_{cr}$ [refer Section 2.9.1]. At this stage, the maximum strains in compression and tension are of a low order. Hence, assuming a linear stress-strain relation for concrete in both tension and compression, with same elastic modulus, the following formula is obtained [applying Eq. 4.4] for the 'moment at first crack' or *cracking moment* $M_{cr}$:

$$M_{cr} = f_{cr} \frac{I_T}{y_t} \tag{4.10}$$

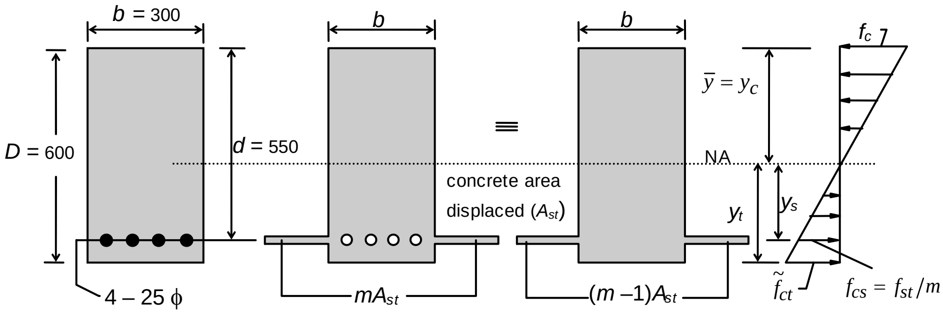
where $y_t$ is the distance between the neutral axis and the extreme tension fibre, and $I_T$ is the second moment of area of the transformed reinforced concrete section with reference to the NA.

If the contribution of the transformed area of reinforcing steel is not significant, an approximate value $M_{cr}$ is obtainable by considering the 'gross (concrete) section', i.e., treating the beam section as a *plain concrete* section.

If the beam is very lightly loaded (or designed to be crack-free), the maximum applied bending moment may be less than $M_{cr}$. In such a case of 'uncracked section', the concrete and steel both participate in resisting tension. The computation of stresses for such a situation is described in Example 4.1.

**EXAMPLE 4.1**

A reinforced concrete beam of rectangular section has the cross-sectional dimensions shown in Fig. 4.5(a). Assuming M 20 grade concrete and Fe 415 grade steel, compute (i) the *cracking moment* and (ii) the stresses due to an applied moment of 50 kNm.




**(a)** beam section **(b)** transformed section **(c)** stresses

**Fig. 4.5** Example 4.1 — 'Uncracked section'

**SOLUTION**

Material Properties: For M 20 concrete,

- modular ratio $m$ = 13.33 [Eq. 4.8 or Table 4.1]
- modulus of rupture $f_{cr} = 0.7\sqrt{20} = 3.13$ MPa [Eq. 2.6]

Approximate *Cracking Moment* (assuming *gross* section):

- Section modulus $Z = \dfrac{bD^2}{6} = \dfrac{300 \times 600^2}{6} = 18 \times 10^6\,\text{mm}^3$

  $\Rightarrow$ Cracking moment

  $M_{cr} \approx f_{cr} Z = 3.13\,\text{N/mm}^2 \times 18 \times 10^6\,\text{mm}^3 = 56.34 \times 10^6\,\text{Nmm}$ †

  = **56.3 kNm**.

Transformed Section Properties:

- Area of tension steel $A_{st} = 4 \times \pi (25)^2/4 = 1963\text{mm}^2$

  The transformed area $A_T$ comprises the concrete area $A_g - A_{st}$ plus the transformed steel area $mA_{st}$. It is convenient to take this as the sum of the gross concrete area $A_g$ and the additional contribution due to steel as $(m-1)A_{st}$ [Fig. 4.5(b)]:

  $$A_T = bD + (m-1)A_{st}$$

- *Depth of neutral axis* $\bar{y}$:

  Equating moments of areas of the transformed section about the top edge,

  $A_T \bar{y} = (bD)\left(D/2\right) + (m-1)\, A_{st}(d)$

† It is reasonable and adequate to include only three significant figures for *final* results in calculations. This practice is followed in this book.

$$\bar{y} \;=\; \frac{(300\times 600)(300)+(13.33-1)(1963)(550)}{(300\times 600)+(12.33\times 1963)} = 329.6 \text{ mm}$$

$\Rightarrow$ distance from NA to extreme compression fibre $y_c = 329.6$ mm ,

distance from NA to extreme tension fibre $y_t = 600 - 329.6 = 270.4$ mm

distance from NA to reinforcing steel $y_s = 550 - 329.6 = 220.4$ mm

- *Transformed second moment of area:*

$$I_T \;=\; by_c^3/3 \;+\; by_t^3/3 \;+\; (m-1)A_{st}(y_s)^2$$

$$= \; 300\times(329.6^3+270.4^3)/3+(12.33\times 1963)(220.4)^2 \;=\; 6.733\times 10^9\,\text{mm}^4$$

**(i)** ***Cracking Moment*** $M_{cr} \;=\; f_{cr}\,\dfrac{I_T}{y_t} = 3.13\times\dfrac{6.733\times 10^9}{270.4}$

$= \; 77.94\times 10^6$ Nmm $=$ **77.9 kNm.**

[Note that the error in the estimate of $M_{cr}$ by the use of the gross section (56.3 kNm) is 28% (underestimated).]

**(ii)** ***Stresses*** *due to applied moment M= 50* kNm:

(As $M < M_{cr}$, the assumption of ‘uncracked section’ is valid.)

- *Maximum Compressive Stress in Concrete*:

$$f_c = \frac{My_c}{I_T} = \frac{(50\times 10^6)\times 329.6}{6.733\times 10^9} = \textbf{2.45 MPa.}$$

- *Maximum Tensile Stress in Concrete:*

$$\tilde{f}_{ct} = \frac{My_t}{I_T} = f_c\left(\frac{y_t}{y_c}\right) \Rightarrow \tilde{f}_{ct} = 2.45\times\left(\frac{270.4}{329.6}\right) = \textbf{2.01 MPa}\;(< f_{cr} = 3.13\text{ MPa})$$

- *Tensile Stress in Steel*:

From the stress distribution diagram [Fig. 4.5(c)],

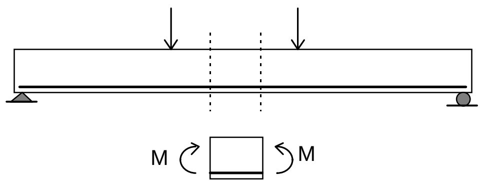
$$f_{st} \;=\; m\,f_{cs}$$

where $f_{cs} = f_c\left(\dfrac{y_s}{y_c}\right) = \tilde{f}_{ct}\left(\dfrac{y_s}{y_t}\right)$

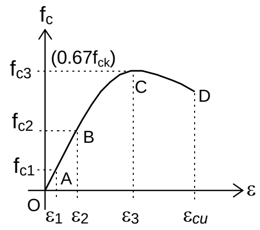
$$\Rightarrow \quad f_{st} = 13.33\times 2.45\times\left(\frac{220.4}{329.6}\right) = \textbf{26.6 MPa.}$$

## 4.5 FLEXURAL BEHAVIOUR OF REINFORCED CONCRETE

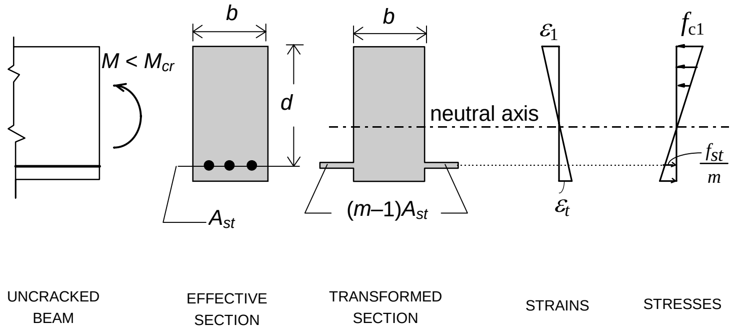
The general behaviour of reinforced concrete beam sections under flexure is discussed in detail here. The behaviour of the section at various stages of loading is described — from the initial *uncracked phase* to the final (ultimate) condition at *collapse* (due to the flexural resistance capacity of the section being exceeded). For convenience, it is assumed that the beam section is rectangular and that only tension reinforcing steel is provided [Fig. 4.6].



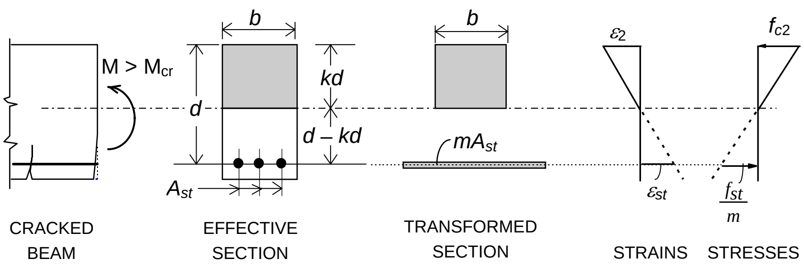

**(a)** beam with loading



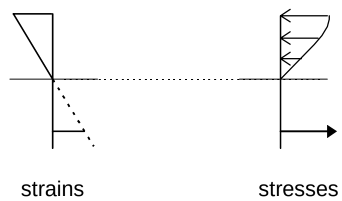

**(b)** concrete stress-strain curve




**(c)** uncracked phase




**(d)** cracked phase (linear stress distribution)




**(e)** cracked phase (nonlinear stress distribution)

**Fig. 4.6** Behaviour of reinforced concrete beam under increasing moment

### 4.5.1 Uncracked Phase

Consider a simply supported beam subjected to gradually increasing load [Fig. 4.6(a)]. In the early stages of loading, the applied moment (at any section) is less than the cracking moment $M_{cr}$ and the maximum tensile stress $\tilde{f}_{ct}$ in the concrete is less than its flexural tensile strength $f_{cr}$. This phase is the *uncracked* phase, wherein the *entire* section is effective in resisting the moment and is under stress. The distribution of strains and stresses are as indicated in Fig. 4.6(c). The calculation of stresses for a given moment is as shown in Example 4.1; similarly, the 'allowable moment' for given 'permissible stresses' can be computed.

The uncracked phase reaches its limit when the applied moment $M$ becomes equal to the cracking moment $M_{cr}$. In the concrete stress-strain curve shown in Fig. 4.6(b), the uncracked phase falls within the initial linear portion OA.

### 4.5.2 Linear Elastic Cracked Phase

As the applied moment exceeds $M_{cr}$, the maximum tensile stress in concrete exceeds the *flexural tensile strength* of concrete and the section begins to *crack* on the tension side. The cracks are initiated in the bottom (tensile) fibres of the beam, and with increasing loading, widen and propagate gradually towards the neutral axis [Fig. 4.6(d)]. As the cracked portion of the concrete is now rendered ineffective in resisting tensile stresses, the *effective* concrete section is reduced. The tension resisted by the concrete just prior to cracking is transferred to the reinforcing steel at the cracked section of the beam. For any further increase in the applied moment, the tension component has to be contributed solely by the reinforcing steel. With the sudden increase in tension in the steel, there is the associated increase in tensile strain in the steel bars at the cracked section. This relatively large increase in tensile strain at the level of the steel results in an upward shift of the neutral axis and an increase in curvature at the cracked section.

Because of the tensile cracking of concrete at very low stresses, it is generally assumed in flexural computations that concrete has no tensile resistance, and that:

> "…all tensile stresses are taken up by reinforcement and none by concrete, except as otherwise specifically permitted" [Cl. B-1.3(b) of the Code].

On this basis, the effective *cracked section* is shown in Fig. 4.6(d). The flexural strength of concrete in the tension zone below the neutral axis is neglected altogether. It is true that, during the first-time loading, a small part of the concrete below and close to the neutral axis (where the tensile strains are less than that corresponding to $f_{cr}$) will remain uncracked and effective. However, the magnitude of the resulting tensile force and the internal moment due to it are negligibly small. Moreover, if the loading is done on a previously loaded beam, it is possible that prior overloading may have caused the tensile cracks to penetrate high enough to effectively eliminate this little contribution from the tensile strength of concrete. Hence, the assumption that concrete resists no flexural tensile stress is satisfactory and realistic.

It is obvious that, in order to maximise the effectiveness of the reinforcing bars in resisting flexure, they should be positioned as distant as possible from the neutral axis — provided the requirements of minimum cover and spacing of bars are satisfied [refer Section 5.2].

It may be noted that the concrete on the tension side is not quite useless. It serves the important functions of holding the reinforcing bars in place, of resisting shear and torsion, of enhancing the flexural stiffness of the beam and thereby reducing deflections and of providing protection to the steel against corrosion and fire.

It may also be noted that cracks cannot be eliminated altogether in reinforced concrete flexural members under the normal range of applied loads. However, by proper design for *serviceability limit state* [Chapter 10], cracks can be *controlled* so that there will be several well-distributed fine hairline cracks rather than a few wide cracks. Hairline cracks (which are barely perceptible) neither affect the external appearance of the beam nor affect the corrosion protection of the reinforcing steel, and hence are acceptable in normal situations.

Finally, it may be noted that the stresses under service loads are usually in the 'cracked section' phase and within the *linear elastic* range. Hence, such an analysis (refer Section 4.3), involving the use of the 'modular ratio' concept, is called for in investigating the *limit states of serviceability* (calculation of deflections and crack-widths) as well as in the traditional *working stress method* of design (refer Section 3.2). The assumption of linear elastic behaviour is acceptable for beams with tension reinforcement, as long as the calculated maximum stress in concrete (under flexural compression) is less than about one-third of the cube strength [see the nearly linear part OAB of the stress-strain curve in Fig. 4.6(b)] and the steel stress is within the elastic limit (which is usually the case). However, when compression reinforcement is introduced, the modular ratio for the compression steel has to be suitably modified, as explained in Section 4.4.2.

Expressions for the stresses and the moment of resistance (based on 'permissible stresses') of reinforced concrete sections, using the linear elastic stress distribution and the concept of cracked-transformed sections, are derived in Section 4.6.

### 4.5.3 Stages Leading to Limit State of Collapse

As the applied moment on the beam section is increased beyond the 'linear elastic cracked phase', the concrete strains and stresses enter the nonlinear range BCD in Fig. 4.6(b). For example, if the strain in the extreme compression fibre reaches a value of $\varepsilon_3$ (equal to 0.002, according to the Code), corresponding to the maximum stress level 0.67 $f_{ck}$, the compressive stress distribution in the cracked section (above the neutral axis) will take the shape of the curve OBC in Fig. 4.6(b), as shown in Fig. 4.6(e). This occurs because the 'fundamental assumption' of a linear strain distribution holds good at all stages of loading, as validated experimentally [Ref. 4.2, 4.4].

The behaviour of the beam in the nonlinear phase depends on the amount of reinforcing steel provided.

The reinforcing steel can sustain very high tensile strains, due to the ductile behaviour of steel, following 'yielding'; the ultimate strain can be in the range of

0.12 to 0.20. However, the concrete can accommodate compressive strains which are much lower in comparison; the 'ultimate compressive strain' $\varepsilon_{cu}$ is in the range of 0.003 to 0.0045. As will be seen later, the final collapse of a normal beam at the ultimate limit state is caused inevitably by the crushing of concrete in compression, regardless of whether the tension steel has yielded or not. If the tension steel yields at the ultimate limit state, the beam is said to be *under-reinforced*; otherwise, if the steel does not yield, the beam is said to be *over-reinforced*. The terms 'under-' and 'over-' are used with reference to a benchmark condition called the 'balanced' section. If the area of tension steel provided at a beam section is less than that required for the *balanced* section condition, the beam is *under-reinforced*; otherwise, if the steel area is in excess, the beam is *over-reinforced*.

### *Balanced Section*

A 'balanced section' is one in which the area of tension steel is such that at the *ultimate limit state*, the two limiting conditions are reached *simultaneously*; viz., the compressive strain in the extreme fibre of the concrete reaches the ultimate strain $\varepsilon_{cu}$, and the tensile strain at the level of the centroid of the steel reaches the 'yield strain' $\varepsilon_y$. The failure of such a section, termed '*balanced failure*', is expected to occur by the simultaneous initiation of crushing of concrete and yielding of steel.

### *Under-Reinforced Section*

An 'under-reinforced section' is one in which the area of tension steel is such that as the ultimate limit state is approached, the yield strain $\varepsilon_y$ is reached in the steel before the ultimate compressive strain is reached in the extreme fibre of the concrete. When the reinforcement strain reaches $\varepsilon_y$ (and the stress reaches the yield strength $f_y$), the corresponding maximum concrete strain is less than $\varepsilon_{cu}$ — as depicted in 'stage 1' of Fig. 4.7. The equilibrium conditions are given by $C = T = A_{st} f_y$ and $M = Tz_1$.

A slight increase in the load (moment) at this stage causes the steel to *yield* and elongate significantly, without any significant increase in stress. The marked increase in tensile strain causes the neutral axis to shift upwards, thus tending to reduce the area of the concrete under compression. As the total tension $T$ remains essentially constant at $A_{st} f_y$, the compressive stresses (and hence, the strains) have to increase in order to maintain equilibrium ($C = T$). This situation is represented by 'stage 2' in Fig. 4.7. The corresponding moment of resistance is given by $M = Tz_2$, and represents a marginal increase over the moment at 'stage 1' owing to the slight increase in the lever arm — from $z_1$ to $z_2$. [Conversely, one can also see that any increase in load (and moment) beyond the first yield of steel requires (with $T$ constant) an increase in lever arm and hence a rise in the neutral axis level].

This process is accompanied by wider and deeper tensile cracks and increased beam curvatures and deflections, due to the relatively rapid increase in the tensile strain. The process continues until the maximum strain in concrete reaches the ultimate compressive strain of concrete $\varepsilon_{cu}$ ('stage 3'), resulting in the crushing of concrete in the limited compression zone.

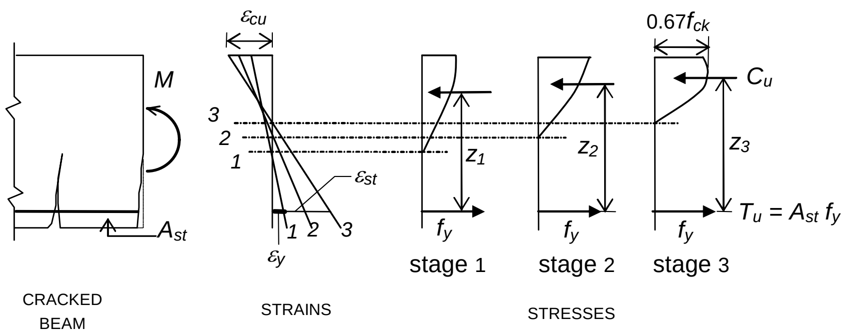



**Fig. 4.7** Behaviour of under-reinforced section (tension failure)

It is to be noted that the increase in the moment of resistance between 'stage 1' and 'stage 3' is marginal, being attributable solely to the slight increase in the lever arm $z$. However, there is a substantial increase in curvature, deflection, and width as well as spread of cracking during this process.

As indicated in Fig. 4.8, the curvature $\varphi$ (rotation per unit length) can be conveniently measured from the linear strain distribution as:

$$\varphi = \frac{\varepsilon_c + \varepsilon_{st}}{d} \tag{4.11}$$

where $\varepsilon_c$ is the compressive strain in the extreme concrete fibre, $\varepsilon_{st}$ is the strain at the centroid of the tension steel, and $d$ is the *effective depth* of the beam section.

*Effective depth* of a beam is defined as 'the distance between the *centroid* of the area of tension reinforcement and the maximum compression fibre' (Cl. 23.0 of the Code). Reinforcing bars are usually provided in multiple numbers, and sometimes in multiple layers, due to size and spacing constraints. In flexural computations, it is generally assumed that the entire steel area resisting tension is located at the centroid of the bar group, and that all the bars carry the same stress — corresponding to the centroid level (i.e., at the effective depth).

The failure of an under-reinforced beam is termed as *tension failure* — so called because the primary cause of failure is the yielding in tension of the steel. The onset of failure is gradual, giving ample prior *warning* of the impending collapse by way of increased curvatures, deflections and cracking. Hence, such a mode of failure is highly preferred in design practice. The actual collapse, although triggered by the yielding of steel, occurs by means of the eventual crushing of concrete in compression ('secondary compression failure'). A sketch of the moment-curvature relation for an under-reinforced beam is shown in Fig. 4.8(a). The large increase in curvature (rotation per unit length), prior to collapse, is indicative of a typical *ductile* mode of failure.

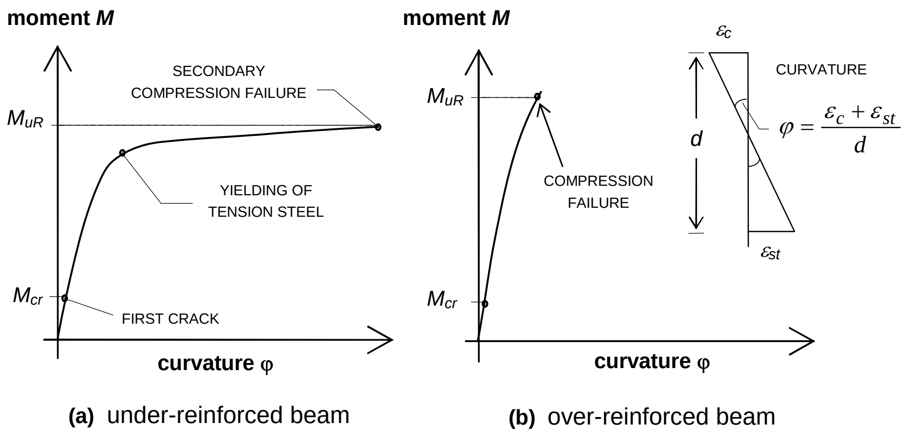



**Fig. 4.8** Moment-curvature relations

### *Over-Reinforced Section*

An 'over-reinforced' section is one in which the area of tension steel is such that at the ultimate limit state, the ultimate compressive strain in concrete is reached, however the tensile strain in the reinforcing steel is less than the yield strain $\varepsilon_y$ [Fig. 4.9].

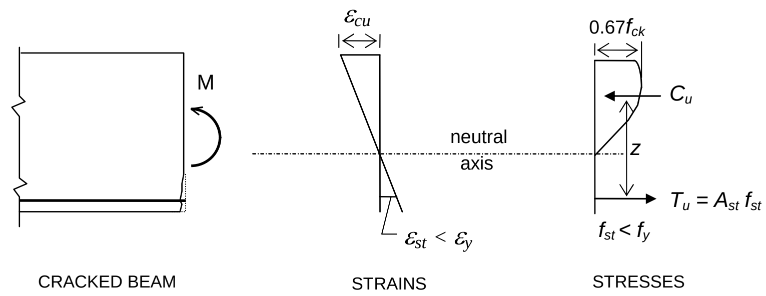



**Fig. 4.9** Behaviour of over-reinforced section (compression failure)

The concrete fails in compression before the steel reaches its yield point. Hence, this type of failure is termed *compression failure*. The failure occurs (often, explosively) without warning.

In this case, the tension steel remains in the elastic range up to collapse. As the limit state of collapse is approached, the tensile stress in steel increases proportionately with the tensile strain, whereas the compressive stress in concrete does not increase proportionately with the compressive strain, because it is in the nonlinear range. Hence, in order to maintain equilibrium ($C=T$), the area of concrete under compression has to increase; this is enabled by a lowering of the neutral axis. The strains across the section remain relatively low. Consequently, the curvatures [Fig. 4.8(b)], deflections and crack-widths – all the 'distress' signals – also remain relatively low in sharp contrast with the behaviour of the under-reinforced section at failure. Because the failure is sudden (without any signs of warning) and the deflections and curvatures remain low right up to failure, this type of failure is termed a *brittle* failure. For this reason, over-reinforced flexural members are not permitted by the Code.

It should be noted that one other type of failure is possible, although extremely rare in practice. This is failure by *fracture* of the reinforcing steel, which can happen with extremely low amounts of reinforcements and under dynamic loading.

## 4.6 ANALYSIS AT SERVICE LOADS (WSM)

Sections designed for *ultimate* limit states (under factored loads) must be checked for *serviceability* (deflection, crack-width, etc.) under the expected 'service loads', as mentioned earlier [refer Section 3.5]. The details of the calculations of deflections and crack-widths are covered in Chapter 10. These calculations require the computation of stresses under service loads. Moreover, these calculations form part of the working stress method of design (WSM). The **basic assumptions** involved in the analysis at the service load stage (Cl. B-1.3 of the Code) are summarised here. (These assumptions have already been explained earlier.)

a) Plane sections normal to the beam axis remain plane after bending, i.e., in an initially straight beam, strain varies linearly over the depth of the section.
b) All tensile stresses are resisted by the reinforcement, and none by concrete (except in the uncracked phase).
c) Stresses are linearly proportional to strains — for both concrete and steel.
d) The modular ratio, $m = E_s/E_c$, has the value, $280/(3\sigma_{cbc})$ [Eq. 4.8].

The expressions for stresses under service loads are derived here, using the linear elastic theory and the cracked-transformed section concept [refer Section 4.3]. Further, the expressions for 'allowable moment of resistance', based on WSM, are also derived.

The simple rectangular cross-section with tension reinforcement alone ('singly reinforced section') is studied first. Subsequently, 'flanged beams' and beams with compression reinforcement ('doubly reinforced') are dealt with.

### 4.6.1 Stresses in Singly Reinforced Rectangular Sections

Fig. 4.10(a) shows a 'singly reinforced' rectangular section of a beam, subjected to a specified (load) moment $M$ (assumed sagging). For this beam section, the corresponding 'cracked-transformed section' is shown in Fig. 4.10(c). The concrete on the tension side of the neutral axis is neglected. The neutral axis (NA) is located by the line passing through the 'centroid' of transformed section, and perpendicular to the plane of bending.

Expressing the depth of the neutral axis (from the extreme compression fibres) as a fraction $k$ of the *effective depth* $d$, and equating the moments of the compression and tension areas about the NA,

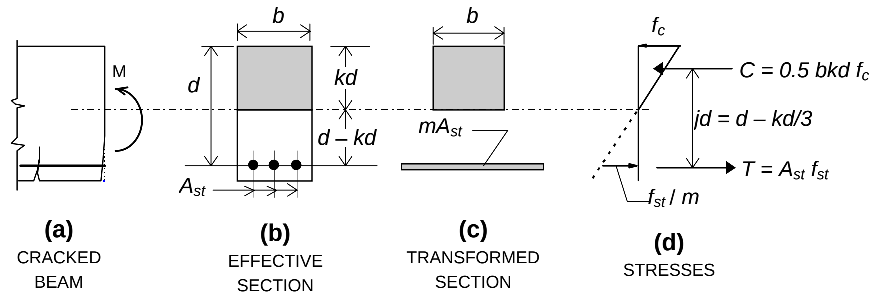



**Fig. 4.10** Concept of 'cracked-transformed section'

$$\frac{b(kd)^2}{2} = mA_{st}(d - kd) \tag{4.12}$$

This quadratic equation can be easily solved to determine $kd$. Of the two roots, only one is acceptable, namely $0 < k < 1$. The resulting expression for $k$ can be obtained as

$$k = \sqrt{2\rho m + (\rho m)^2} - \rho m \tag{4.13}$$

where ρ is termed the *reinforcement ratio*, given by:

$$\rho \equiv \frac{A_{st}}{bd} \tag{4.14}$$

The *second moment of area* of the cracked–transformed section, $I_{cr}$, is given by:

$$I_{cr} = \frac{b(kd)^3}{3} + mA_{st}(d - kd)^2 \tag{4.15}$$

Knowing the neutral axis location and the second moment of area, the stresses in the concrete (and steel) in the composite section [Fig. 4.10(d)] due to the applied moment $M$ may be computed from the flexure formula $f = My/I$, as explained in Section 4.3 (and Example 4.1). The same results could be obtained more simply and directly, by considering the static equilibrium of resultant forces and moments [Eq. 4.1 and 4.2]. Referring to Fig. 4.10,

$$C = \frac{bkd}{2} f_c \tag{4.16}$$

$$T = A_{st} f_{st} \tag{4.17}$$

$$j = (1 - k/3) \tag{4.18}$$

$$M = C\ jd = T\ jd$$

from which
$$f_c = \frac{M}{0.5b(kd)(jd)} \tag{4.19}$$

and
$$f_{st} = \frac{M}{A_{st}\ jd} \tag{4.20}$$

It is to be noted that if two points in the stress distribution diagram are known (such as $k$ and $f_c$), then the stress at any level can be computed using similar triangles. Thus, $f_{st}$ may be alternatively obtained as:

$$f_{st} = mf_c\frac{d-kd}{kd} = mf_c\left(\frac{1-k}{k}\right) \tag{4.21}$$

**EXAMPLE 4.2**

Consider the same beam section [Fig. 4.11] of Example 4.1. Assuming M 20 grade concrete and Fe 415 grade steel, compute the stresses in concrete and steel under a service load moment of 140 kNm.

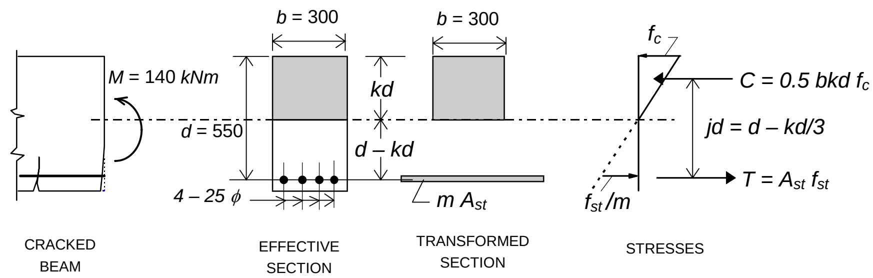



**Fig. 4.11** Example 4.2 — 'cracked section'

**SOLUTION**

From Example 4.1, it is seen that $M_{cr}$ = 77.9 kNm. For the present problem $M$ = 140 kNm > $M_{cr}$. Hence, the section would have 'cracked'. The cracked–transformed section is shown in Fig. 4.11.

<u>Transformed Section Properties:</u>

- modular ratio $m$ = 13.33 (for M 20 concrete)
- Transformed steel area = $mA_{st}$
  = 13.33×1963 = **26167 mm²**
- Equating moments of areas about the neutral (centroidal) axis,

$$\frac{300(kd)^2}{2} = 26167(550 - kd)$$

Solving, $kd$ = 234.6 mm.

$\Rightarrow$ neutral axis depth factor $k$ = 234.6/550 = 0.4265.

$\Rightarrow$ lever arm $jd = d - kd/3$ = 471.8 mm.

Stresses:

- *Maximum Concrete Stress*:

Taking moments about the tension steel centroid,

$$M = 0.5 f_c b(kd)(jd)$$

$$\Rightarrow f_c = \frac{140 \times 10^6}{0.5 \times 300 \times 234.6 \times 471.8} = \mathbf{8.43\ MPa}.$$

- *Tensile Stress in Steel*:

Taking moments about the line of action of $C$,

$$M = f_{st} A_{st} (jd)$$

$$\Rightarrow f_{st} = \frac{140 \times 10^6}{1963 \times 471.8} = \mathbf{151\ MPa}.$$

Alternatively, considering the linear stress distribution [Fig. 4.10]:

$$f_{st} = 13.33 \times 8.43 \times \frac{1 - 0.4265}{0.4265} = \mathbf{151\ MPa}.$$

[Note: The maximum concrete stress $f_c$ exceeds the permissible stress $\sigma_{cbc}$ = 7.0 $MPa$ for M 20 concrete [refer Section 4.6.2]; hence the beam is not 'safe' according to WSM provision of the Code although the steel stress is within allowable limit.]

- Note that the same results can be obtained by the 'flexure formula' $f = My/I_{cr}$, where

$$I_{cr} = \frac{300 \times (234.6)^3}{3} + 26167(550 - 234.6)^2 = 3.894 \times 10^9 \,\text{mm}^4$$

Accordingly, $$f_c = \frac{My_c}{I_{cr}} = \frac{\left(140 \times 10^6\right) \times 234.6}{3.894 \times 10^9} = \mathbf{8.43\ MPa}$$

$$f_{st} = m \frac{M(d - kd)}{I_{cr}} = \frac{13.33 \times \left(140 \times 10^6\right) \times (550 - 234.6)}{3.894 \times 10^9} = \mathbf{151\ MPa} \text{ (as before)}.$$

### 4.6.2 Permissible Stresses

In the traditional working stress method, analysis requires the designer to verify that the calculated stresses [Eq. 4.19 and 4.20] under service loads are within '*permissible limits*'. The 'permissible stress' in concrete under flexural compression (denoted as $\sigma_{cbc}$ by the Code) is as given in Table 4.1.

The 'permissible stress' in tension steel $\sigma_{st}$ (specified in Table 22 of the Code) takes values of 140 MPa, 230 MPa and 275 MPa for Fe 250, Fe 415 and Fe 500 grades respectively. However, for Fe 250 grade, the permissible stress is *reduced* to 130 MPa if the bar diameter exceeds 20*mm*.

In the case of reinforcing steel under compression in flexural members, the permissible stress $\sigma_{sc}$ is limited to *the calculated compressive stress in the surrounding concrete multiplied by* 1.5 *times the modular ratio or* $\tilde{\sigma}_{sc}$ (maximum permissible compressive stress in steel given in Table 22 of the Code), *whichever is lower*†. The specified values of $\tilde{\sigma}_{sc}$ are 130 MPa, 190 MPa and 190 MPa for Fe 250, Fe 415 and Fe 500 grades respectively.

### 4.6.3 Allowable Bending Moment

When it is desired to compute the 'allowable bending moment' capacity of a beam of known cross-section, in accordance with WSM, the procedure to be adopted is very similar to that given in Section 4.6.1 and Example 4.2. Here, the stresses in concrete and steel ($f_c$ and $f_{st}$) are taken as their respective 'permissible stresses' ($\sigma_{cbc}$ and $\sigma_{st}$) as specified in Section 4.6.2.

Considering the moment with reference to the tension steel [Fig. 4.10],

$$M_{all} = A_{st}\sigma_{st}\, jd \tag{4.22a}$$

Considering the moment with reference to the compression in concrete [Fig. 4.10],

$$M_{all} = 0.5\,\sigma_{cbc}\, b(kd)(jd) \tag{4.22b}$$

In a given beam section, the permissible stresses in both steel and concrete may not be reached simultaneously. Hence, the lower of the two moments computed by Eq. 4.22a and 4.22b will give the correct permissible moment, and the corresponding stress (either $f_{st}$ or $f_c$) will be the one to reach the permissible limit.

Alternatively, with the knowledge of certain constants (discussed in the subsection to follow), it is possible to predict whether it is the steel or the concrete that controls $M_{all}$.

#### *'Balanced (WSM)' section constants*

In *the working stress method*, the 'balanced' section is one in which both tensile steel stress $f_{st}$ and maximum compressive stress $f_c$ simultaneously reach their allowable limits $\sigma_{st}$ and $\sigma_{cbc}$ respectively [Fig. 4.12] under *service loads*. The corresponding area of steel $A_{st}$ is denoted as $A_{st,b}$; the percentage reinforcement $p_t \equiv 100A_{st,b}/\,bd$ is denoted as $p_{t,b}$; the neutral axis depth factor is denoted as $k_b$; the lever arm depth factor is denoted as $j_b$; and the allowable moment of the section is denoted as $M_{wb}$.

For such a case, from the linear distribution of stresses [Fig. 4.12(c)] in the transformed-cracked section [Eq. 4.21], it follows that:

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† Generally, the value of *1.5m* times the calculated compressive stress is lower than $\tilde{\sigma}_{sc}$, and hence controls.

$$\sigma_{st} = m\sigma_{cbc} \times \frac{1-k_b}{k_b}$$

The product $m\sigma_{cbc}$ is a constant, equal to 280/ 3, according to the Code [Eq. 4.9], whereby the above equation can be solved to give:

$$k_b = \frac{280}{280+3\sigma_{st}} \tag{4.23}$$

where $\sigma_{st}$ is in MPa units.

Thus, it is seen that the neutral axis depth factor ($k_b$) of a 'balanced (WSM)' section depends only on the permissible tensile stress in steel.

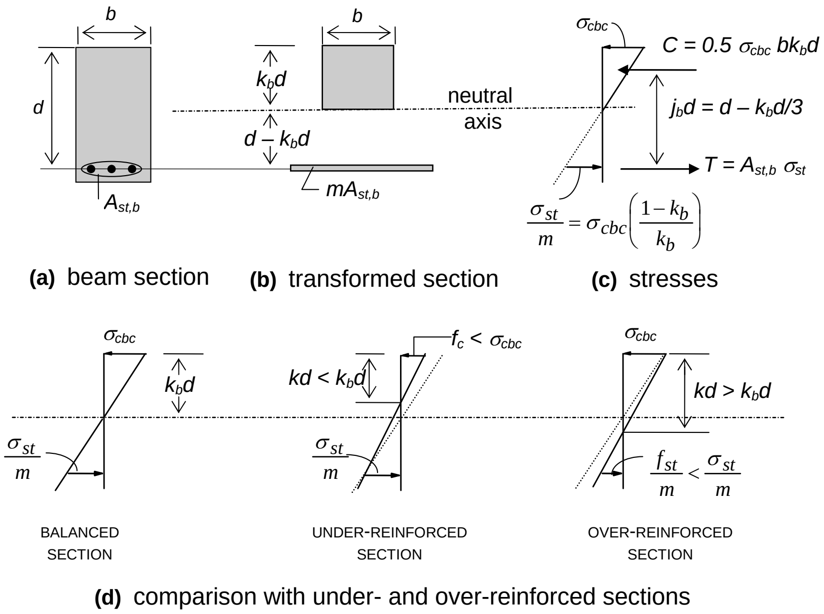


**Fig. 4.12** 'Balanced (WSM)' section

Further, considering the equilibrium of forces $C=T$, it follows that

$$0.5\sigma_{cbc}b(k_bd) = A_{st,b}\sigma_{st}$$

$$\Rightarrow \frac{100A_{st,b}}{bd} \equiv p_{t,b} = 50k_b \times \frac{\sigma_{cbc}}{\sigma_{st}} \tag{4.24}$$

Finally, considering moment equilibrium,

$$M_{wb} = C\ (j_bd) = 0.5\sigma_{cbc}b\ (k_bd)\left(1-\frac{k_b}{3}\right)\ d$$

$$\Rightarrow \frac{M_{wb}}{\sigma_{cbc} bd^2} \equiv Q_b = 0.5k_b\left(1-\frac{k_b}{3}\right) \tag{4.25}$$

It may be noted that Eq. 4.23 to 4.25 are expressed in nondimensional form. The quantities on the right-hand sides of the equations reduce to constants (for given material grades). Some typical values of the constants $k_b$, $p_{t,b}$ and $Q_b$ are listed in Table 4.2.

**Table 4.2** Constants for the balanced 'WSM' section

| **Steel Grade** | | **Fe 250** ($\phi$ > **20** mm) | **Fe 250** ($\phi \leq$ **20** mm) | **Fe 415** | **Fe 500** |
|---|---|---|---|---|---|
| $k_b$ | | 0.4179 | 0.4000 | 0.2887 | 0.2534 |
| $p_{t,b}$ | **M 20** | 1.1251 | 1.0000 | 0.4394 | 0.3225 |
| | **M 25** | 1.3662 | 1.2143 | 0.5334 | 0.3916 |
| | **M 30** | 1.6073 | 1.4286 | 0.6276 | 0.4607 |
| | **M 35** | 1.8484 | 1.6429 | 0.7218 | 0.5298 |
| | **M 40** | 2.0895 | 1.8571 | 0.8159 | 0.5989 |
| | **M 45** | 2.3306 | 2.0714 | 0.9100 | 0.6681 |
| | **M 50** | 2.5717 | 2.2857 | 1.0042 | 0.7372 |
| $Q_b \equiv M_{wb} / \sigma_{cbc} bd^2$ | | 0.1798 | 0.1733 | 0.1304 | 0.1160 |

### *'Under-Reinforced (WSM)' Sections*

According to the traditional WSM terminology, a given section is said to be 'under-reinforced (WSM)' if its area of tension steel is less than that corresponding to balanced conditions (i.e., $A_{st,b}$); so that the tensile stress in steel reaches the allowable limit before the maximum compressive stress in concrete reaches its allowable limit, and the allowable moment capacity is limited by the stress in steel, and not by the stress in concrete [i.e., $f_{st} = \sigma_{st}$ *and* $f_c < \sigma_{cbc}$].

With the help of the 'balanced (WSM)' section constants, it is evidently possible to predict whether a given section is 'under-reinforced (WSM)' or not. If the section is 'under-reinforced (WSM)', $k < k_b$ [Fig. 4.12(d)] and $p_t < p_{t,b}$ ; both conditions are equivalent, and either one may be checked.

Accordingly, the allowable moment is given by:

$$M_{all} = \left(A_{st}\ \sigma_{st}\right) \times \left(d - kd/3\right) \qquad \text{for } k < k_b \tag{4.26}$$

The corresponding maximum compressive stress in concrete is obtained by applying the condition of force equilibrium ($C=T$), (or from the stress distribution diagram knowing $\sigma_{st}$ and $k$).

$$0.5 f_c bkd = A_{st} \sigma_{st}$$

$$\Rightarrow f_c = \frac{2 A_{st} \sigma_{st}}{bkd} = \frac{p_t \sigma_{st}}{50k} \tag{4.27}$$

### *'Over-Reinforced (WSM)' Sections*

According to the traditional WSM terminology, a given section is said to be 'over-reinforced (WSM)' if its area of tension steel is more than $A_{st,b}$, so that allowable limiting stress is reached in the concrete before the steel stress reaches the limiting value, and the allowable moment capacity is limited by the stress in concrete, and not by the stress in steel [i.e., $f_c = \sigma_{cbc}$ *and* $f_{st} < \sigma_{st}$ ]

Evidently, for such a section, $k > k_b$ [Fig. 4.12(d)] and $p_t > p_{t,b}$ . The allowable moment is given by

$$M_{all} = \left(0.5\sigma_{cbc} bkd\right)\left(d - kd/3\right) \qquad \text{for } k > k_b \tag{4.28}$$

The corresponding stress in the tension steel is obtained (by applying the condition $T=C$ ) as:

$$f_{st} = \frac{0.5\sigma_{cbc} bkd}{A_{st}} = \frac{50\sigma_{cbc} k}{p_t} \tag{4.29}$$

These definitions of 'balanced (WSM)', 'under-reinforced (WSM)' and 'over-reinforced (WSM)', with reference to **service load** conditions, should not be confused with the definitions for balanced / under-reinforced / over-reinforced given earlier (in Section 4.5.3) with reference to the **ultimate limit state**. In order to distinguish the WSM descriptions, the term 'WSM' should be preferably attached (in parenthesis) — as done consistently in this chapter.

### *Variation of $M_{all}$ with $p_t$*

On the basis of Eq. 4.13, Eq. 4.26 and Eq. 4.28, the values of $M_{all}$ / ($bd^2$), in MPa units, corresponding to increasing values of $p_t$ have been computed and plotted in Fig. 4.13 — for two commonly used concrete grades (M 20, M 25), each combined with two steel grades (Fe 250[†], Fe 415).

Each curve in Fig. 4.13 is characterised by two distinctive portions: the initial segment (thick line), which is practically linear, conforms to *under-reinforced (WSM)* sections; this is followed by a non-linear segment (thin line), which conforms to *over-reinforced* (*WSM)* sections. The kink in each curve, marking the transition from 'under-reinforced (WSM)' to 'over-reinforced (WSM)', evidently corresponds to the *balanced (WSM)* section.

From the trends depicted in Fig. 4.13, it follows that (expectedly), the allowable moment capacity increases with increase in tensile reinforcement area. In fact, for 'under-reinforced (WSM)' sections, $M_{all}$ increases rapidly and nearly proportionately

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[†] Assuming bar diameter $\phi \leq 20$ mm.

with $p_t$ for $p_t < p_{t,b}$. Further, although the rate of gain in $M_{all}$ increases with the use of higher strength steel, the 'balanced' section limit is reached at a lower percentage of steel.

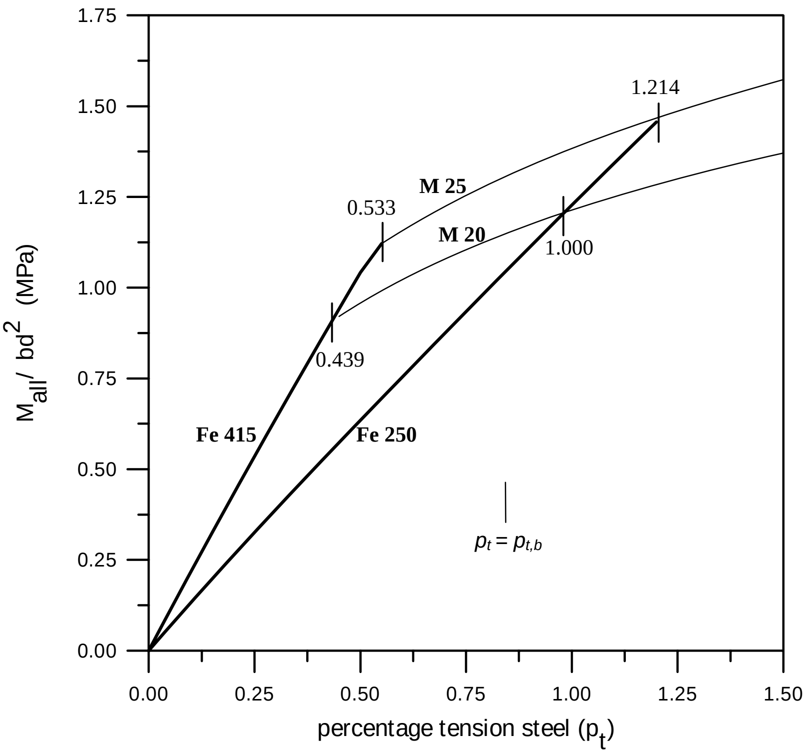


**Fig. 4.13** Variation of $M_{all}/bd^2$ with $p_t$ for different grades of concrete and steel

Improving the grade of concrete has practically no effect on $M_{all}$ (for a given $p_t$) for under-reinforced (WSM) sections, except that the 'balanced section' limit is raised.

For $p_t > p_{t,b}$ (i.e., for 'over-reinforced (WSM)' sections), the rate of gain in allowable moment capacity with increase in tensile reinforcement area drops off rapidly. This is so, because the allowable limit of stress is reached in concrete in *compression,* and, unless the compression capacity is suitably enhanced†, there is not much to gain in boosting the flexural tensile capacity of the beam section — either by adding more tension steel area or by improving the grade of steel.

For this reason, 'over-reinforced (WSM)' beams are considered to be highly uneconomical in the traditional WSM method of design.

† by improving the grade of concrete and/or providing compression reinforcement ('doubly reinforced' section).

Finally, it may be noted that 'over-reinforced (WSM)' sections often may turn out to be *under-reinforced* with reference to the *ultimate limit state* (leading to *ductile* failure), except when the percentage of tension steel is very high. For example, when M 20 concrete and Fe 415 steel are used in a beam section, if the tension steel area exceeds 0.439 percent, by the working stress method the section is 'over-reinforced (WSM)', — but it is *under-reinforced* in the ultimate limit sense (up to tension steel area of 0.961 percent) [refer Section 4.7].

### *Analysis Aids*

The variation of $M_{all}/bd^2$ with $p_t$ for different grades of concrete and steel (depicted in Fig. 4.13) is expressed in tabular form and presented in Tables A.1(a) and (b) in Appendix A of this book. These Tables serve as useful *analysis aids*, enabling the rapid determination of $M_{all}$ for any given singly reinforced rectangular beam section. The use of these Tables is demonstrated in Example 4.3.

**EXAMPLE 4.3**

Consider the same beam section [Fig. 4.11 of Examples 4.1 and 4.2]. Assuming M 20 grade concrete and Fe 415 grade steel, determine the *allowable* bending moment, and the stresses in concrete and steel corresponding to this moment.

**SOLUTION**

- Given: $\sigma_{cbc}$ = 7.0 MPa, $\sigma_{st}$ = 230 MPa, $m$ = 13.33, $A_{st}$ = 1963 mm$^2$,
  $b$ = 300 mm, $d$ = 550 mm.
- The transformed section properties [Fig. 4.11(b)] have already been worked out in Example 4.2. Accordingly, $kd$ = 234.6 mm $\Rightarrow$ $k$ = 0.4265.
  The neutral axis depth factor $k_b$ is a constant [Eq. 4.23].
  For Fe 415 steel ($\sigma_{st}$ = 230 MPa), $k_b = \dfrac{280}{280+3(230)} = 0.2887$

Stresses:

As $k > k_b$, the section is 'over-reinforced (WSM)'. [Alternatively, $p_t = \dfrac{100\times1963}{300\times550}$ = 1.190. $p_{t,b}$ = 0.440 for M 20 concrete with Fe 415 steel. As $p_t > p_{t,b},$ the section is 'over-reinforced (WSM)']. Accordingly, the concrete stress controls, and $f_c = \sigma_{cbc}$ = **7.0 MPa** (for M 20 concrete).

- Applying $T = C$,

$$f_{st} = \frac{0.5\sigma_{cbc}\ b(kd)}{A_{st}} = \frac{0.5\times7.0\times300\times234.6}{1963} = \mathbf{125\ MPa}$$

$$\left(< \sigma_{st} = 230\ \text{MPa}\right).$$

Alternatively, considering the linear stress distribution,

$$f_{st} = mf_c \frac{(1-k)}{k} = 13.33 \times 7.0 \left( \frac{1-0.4265}{0.4265} \right) = \mathbf{125\ MPa} \text{ (as before).}$$

Allowable bending moment:

- Taking moments of forces about the tension steel centroid,

  $$M_{all} = (0.5\sigma_{cbc} bkd)(d - kd/3) = (0.5 \times 7.0 \times 300 \times 234.6)(550 - 234.6/3)$$
  $$= 116.2 \times 10^6 \text{ Nmm} = \mathbf{116\ kNm}.$$

- [Alternatively, using the *analysis aids* given in Table A.1(a), for $p_t$ = 1.190 and M 20 concrete with Fe 415 steel, $\frac{M_{all}}{bd^2} = 1.28$ MPa

  $\Rightarrow M_{all}$ = 1.28 × 300 × 550$^2$ = 116.2 × 10$^6$ Nmm = **116 kNm** (as before)]

### 4.6.4 Analysis of Singly Reinforced Flanged Sections

In the previous discussions, beams of *rectangular section* (which are most common) and with tensile steel alone ('singly reinforced') were considered, for the sake of simplicity. The procedure of analysis is similar for other cross-sectional shapes.

Frequently, rectangular sections of beams are coupled with flanges — on top or bottom. If the flanges are located in the compression zone, they become effective (partly or wholly) in adding significantly to the area of the concrete in compression. However, if the flanges are located in the tension zone, the concrete in the flanges becomes ineffective in cracked section analysis.

#### *T–beams and L–beams*

Beams having effectively *T-sections* and *L-sections* (called *T*-beams and *L*-beams) are commonly encountered in beam-supported slab floor systems [refer Figs. 1.10, 4.14]. In such situations, a portion of the slab acts integrally with the beam and bends in the longitudinal direction of the beam. This slab portion is called the *flange* of the T- or L-beam. The beam portion below the flange is often termed the *web*, although, technically, the web is the full rectangular portion of the beam other than the overhanging parts of the flange. Indeed, in shear calculations, the web is interpreted in this manner.

When the flange is relatively wide, the flexural compressive stress is not uniform over its width. The stress varies from a maximum in the web region to progressively lower values at points farther away from the web†. In order to operate within the framework of the theory of flexure, which assumes a *uniform* stress distribution across the *width* of the section, it is necessary to define a *reduced effective flange*.

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† The term 'shear lag' is sometimes used to explain this behaviour. The longitudinal stresses at the junction of the web and flange are transmitted through in-plane shear to the flange regions. The resulting shear deformations in the flange are maximum at the junction and reduce progressively at regions farther away from the web. Such 'shear lag' behaviour can be easily visualised in the case of a rectangular piece of sponge that is compressed in the middle.

The 'effective width of flange' may be defined as the width of a hypothetical flange that resists in-plane compressive stresses of uniform magnitude equal to the peak stress in the original wide flange, such that the value of the resultant longitudinal compressive force is the same (Fig. 4.14).

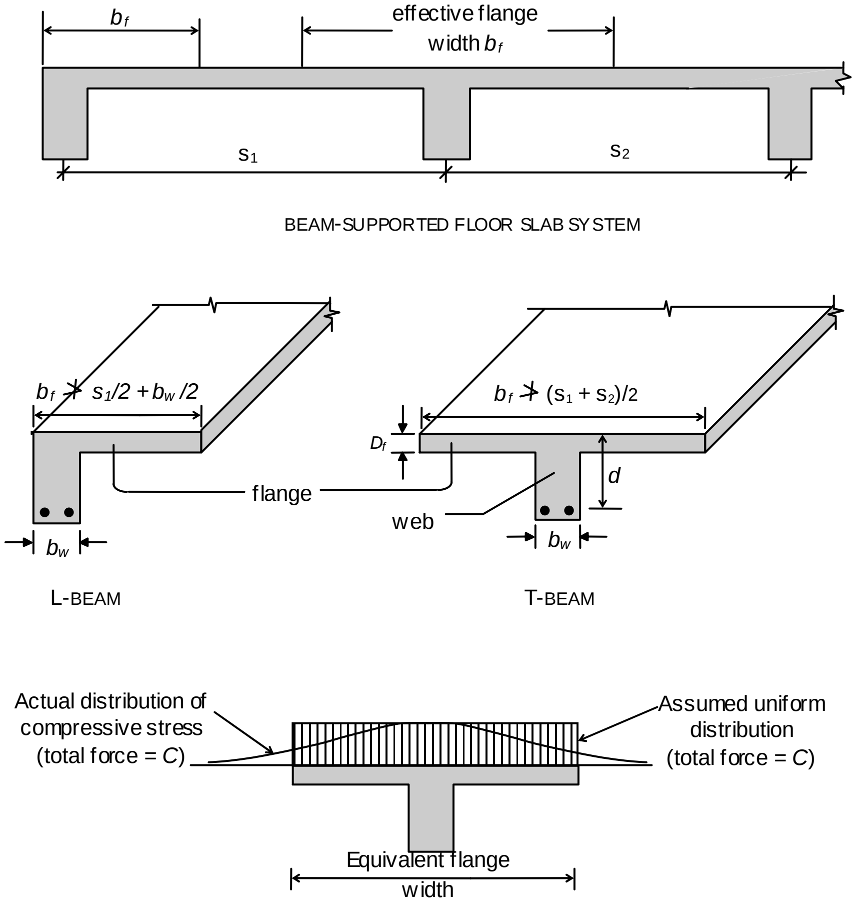


**Fig. 4.14** T-beams and L-beams in beam-supported floor slab systems

The effective flange width is found to increase with increased span, increased web width and increased flange thickness. It also depends on the type of loading [†](concentrated, distributed, etc.) and the support conditions (simply supported, continuous, etc.). Approximate formulae for estimating the 'effective width of flange' $b_f$ (Cl. 23.1.2 of Code) are given as follows:

[†] For example, it is seen that the equivalent flange width is less when a concentrated load is applied at the midspan of a simply supported beam, compared to the case when the same load is applied as a uniformly distributed load.

$$b_f = \begin{cases} l_0/6 + b_w + 6D_f & \text{for T-beams} \\ l_0/12 + b_w + 3D_f & \text{for L-beams} \end{cases} \tag{4.30a}$$

where $b_w$ is the breadth of the web, $D_f$ is the thickness of the flange [Fig 4.14], and $l_0$ is the "distance between points of zero moments in the beam" (which may be assumed as 0.7 times the effective span in continuous beams and frames). Obviously, $b_f$ cannot extend beyond the slab portion tributary to a beam, i.e., the actual width of slab available. Hence, the calculated $b_f$ should be restricted to a value that does not exceed $(s_1+s_2)/2$ in the case of T–beams, and $s_1/2 + b_w/2$ in the case of L–beams, where the spans $s_1$ and $s_2$ of the slab are as marked in Fig. 4.14.

In some situations, *isolated* T–beams and L–beams are encountered, i.e., the slab is discontinuous at the sides, as in a footbridge or a 'stringer beam' of a staircase. In such cases, the Code [Cl. 23.1.2(c)] recommends the use of the following formula to estimate the 'effective width of flange' $b_f$:

$$b_f = \begin{cases} \dfrac{l_0}{l_0/b + 4} + b_w & \text{for isolated T-beams} \\ \dfrac{0.5 l_0}{l_0/b + 4} + b_w & \text{for isolated L-beams} \end{cases} \tag{4.30b}$$

where $b$ denotes the *actual* width of flange; evidently, the calculated value of $b_f$ should not exceed $b$.

### *Analysis of T–beams and L–beams*

The neutral axis may lie either within the flange [Fig. 4.15(b)] or in the web of the flanged beam [Fig. 4.15(c)]. In the former case ($kd \leq D_f$), as all the concrete on the tension side of the neutral axis is assumed ineffective in flexural computations, the flanged beam may just as well be treated as a *rectangular* beam having a width $b_f$ and an *effective depth* $d$. The analytical procedures described in Sections 4.6.1 and 4.6.3, therefore, are identically applicable here, the only difference being that $b_f$ is to be used in lieu of $b$.

In the case $kd > D_f$ , the area of concrete in compression spreads into the web region of the beam [Fig. 4.15(c)]. The exact location of the neutral axis (i.e., $kd$) is determined by equating moments of areas of the cracked-transformed section in tension and compression [Eq. 4.31] and solving for $kd$:

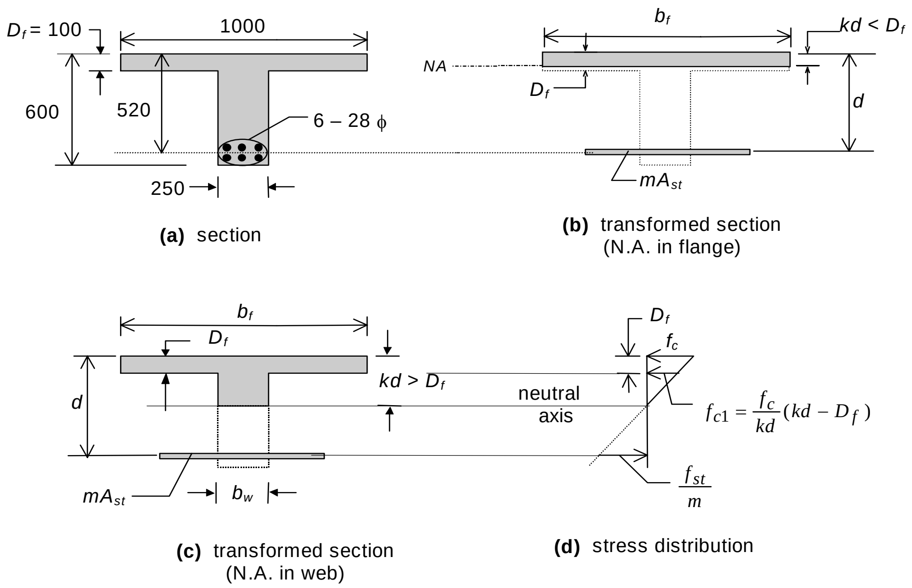


**Fig. 4.15** Example 4.4 — Cracked section analysis (WSM) of a T-beam

$$\left(b_f - b_w\right)D_f\left(kd - D_f/2\right) + b_w\,(kd)^2/2 \;=\; mA_{st}(d - kd) \tag{4.31}$$

This is valid only if the resulting $kd$ exceeds $D_f$.

With reference to the stress distribution shown in Fig. 4.15(d), the area of the concrete in compression can be conveniently obtained by considering the difference between the two rectangles $b_f \times kd$ and $\left(b_f - b_w\right)\times\left(kd - D_f\right)$. Accordingly, considering equilibrium of forces *(C = T)*,

$$\frac{1}{2} f_c\; b_f\,(kd) - \frac{1}{2} f_{c1}\left(b_f - b_w\right)\left(kd - D_f\right) \;=\; A_{st}\, f_{st} \tag{4.32}$$

where
$$f_{c1} \;=\; f_c\left(\frac{kd - D_f}{kd}\right) \tag{4.33}$$

Also, taking moments of forces about the centroid of tension steel,

$$M \;=\; \frac{1}{2} f_c\; b_f\,(kd)(d - kd/3) - \frac{1}{2} f_{c1}\left(b_f - b_w\right)\left(kd - D_f\right)\left\{d - D_f - \left(kd - D_f\right)/3\right\} \tag{4.34}$$

If the problem is one of determining the stresses $f_c$ and $f_{st}$ for a given moment $M$, then $f_c$ may be determined first by solving Eq. 4.34 (after substituting Eq. 4.33) and $f_{st}$ can then be determined either by solving Eq. 4.32, or by considering similar triangles in the stress distribution diagram (Eq. 4.21). On the other hand, if the problem is one of determining the allowable moment capacity ($M_{all}$) of the section, then it should first be verified whether the section is 'under–reinforced (WSM)' or 'over–reinforced (WSM)' — by comparing the neutral axis depth factor $k$ with $k_b$ (given by Eq. 4.23). If $k < k_b$, the section is 'under–reinforced (WSM),' whereby $f_{st} = \sigma_{st}$. The corresponding value of $f_c$ can be calculated using the stress distribution diagram. On the other hand, if $k > k_b$, the section is 'over-reinforced (WSM)', whereby $f_c = \sigma_{cbc}$. Using the appropriate value of $f_c$ in Eq. 4.34, $M_{all}$ can be determined.

### EXAMPLE 4.4

An isolated T-beam, having a span of 6 m and cross sectional dimensions shown in Fig. 4.15(a), is subjected to a service load moment of 200 kNm. Compute the maximum stresses in concrete and steel, assuming M 20 concrete and Fe 250 steel.

### SOLUTION

- It must be verified first whether the actual flange width $b$ = 1000 mm is fully effective or not. Applying Eq. 4.30(b) for isolated T-beams with $l_0$ = 6000 mm

$$b_f = \frac{l_0}{l_0/b + 4} + b_w = \frac{6000}{6+4} + 250 = \mathbf{850\ mm} < (b = 1000\text{ mm}).$$

- modular ratio (for M 20 concrete) $m$ =13.33.
- $A_{st} = 6 \times \pi(28^2)/4$ = 3695 mm$^2$, $d$ = 520 mm, $b_w$ = 250 mm, $D_f$ = 100 mm.

<u>Neutral axis depth:</u>

- First assuming $kd \le D_f$ [Fig. 4.15(b)], and equating moments of compression and (transformed) tension areas about the neutral axis,

$$b_f \times (kd)^2/2 = mA_{st}(d - kd) \Rightarrow 850 \times (kd)^2/2 = 13.33 \times 3695 \times (520 - kd)$$

  Solving, $kd$ = 194.2 mm.
  As this is greater than $D_f$ = 100 mm, the assumption $kd \le D_f$ is incorrect.

- For $kd > D_f$, the neutral axis is located in the web [Fig. 4.15(c)],

$$(850-250)(100)(kd-50) + 250\,(kd)^2/2 = 13.33 \times 3695 \times (520 - kd)$$

  Solving, $kd$ = **210.9 mm**.

<u>Stresses</u>:

- Relating the compressive stress $f_{c1}$ at the flange bottom to $f_c$,

$$f_{c1} = f_c\left(\frac{210.9-100}{210.9}\right) = 0.526\, f_c$$

- Compressive force $C = 0.5\, f_c\, b_f\,(kd) - 0.5\, f_{c1}(b_f - b_w)(kd - D_f)$

$$= 0.5 f_c[(850 \times 210.9) - 0.526 \times (850 - 250)(210.9 - 100)]$$
$$= 0.5 f_c[(179324) - (35022)]\text{ N.}$$

- Taking moments of forces about the tension steel centroid,

$$200\times10^6 = 0.5 f_c\left[(179324)(520-210.9/3)-(35022)\left(520-100-\frac{210.9-100}{3}\right)\right]$$

  Solving, $f_c$ = **5.95 MPa**.

  (which, incidentally, is less than the permissible stress $\sigma_{cbc}$ = 7.0 MPa for M 20 concrete).

- Now applying $C = T$,

$$0.5\times5.95\times[179324-35022] = 3695 f_{st}$$

  $\Rightarrow f_{st}$ = **116 MPa**.

- Alternatively, from the stress distribution diagram [Fig. 4.15(d)]

$$f_{st} = mf_c\left(\frac{d-kd}{kd}\right) \Rightarrow f_{st} = \frac{13.33\times5.95(520-210.9)}{210.9} = \textbf{116 MPa} \text{ (as before)}$$

  (which, incidentally, is less than the permissible stress $\sigma_{st}$ = 130 MPa )

### EXAMPLE 4.5

For the T-beam problem in Example 4.4, determine the *allowable moment capacity*.

### SOLUTION

- From the previous Example, the neutral axis depth factor

$$k = 210.9/520 = 0.4057.$$

- For a 'balanced section', as per Eq. 4.23,

$$k_b = \frac{280}{280+(3\times130)} = 0.4179.$$

- As $k < k_b$, the section is 'under-reinforced (WSM)'.

  Accordingly, $f_{st} = \sigma_{st}$ = 130 MPa (for Fe 250 steel, $\phi$ >20mm);

$$f_c = \left(\frac{kd}{d-kd}\right)\left(\frac{f_{st}}{m}\right) = \left(\frac{210.9}{520-210.9}\right)\left(\frac{130}{13.33}\right) = 6.66 \textbf{ MPa}$$

$$f_{c1} = 0.526 f_c = 3.5 \textbf{ MPa}$$

  substituting in Eq. 4.34,

- $$M_{all} = \frac{1}{2}\times6.66\times850\times210.9\times(520-210.9/3)$$
$$-\frac{1}{2}\times3.5\times(850-250)\times(210.9-100)\times\left[520-100-\left(\frac{210.9-100}{3}\right)\right]$$

  $= 223.9\times10^6$ Nmm = **223.9 kNm**.

- *Note*:

  The answer could have been easily obtained using the result of Example 4.4, and making use of the *linear elastic* assumption underlying WSM. A compressive stress $f_c$ = 5.95 MPa results from a moment $M$ = 200 kNm.

Hence, the allowable stress $f_c$ = 6.66 MPa corresponds to a moment $M_{all} = 200 \times \left(\frac{6.66}{5.95}\right)$ = 223.9 kNm (as before).

### 4.6.5 Analysis of Doubly Reinforced Sections

When *compression reinforcement* is provided in addition to *tension* reinforcement in beams, such beams are termed *doubly reinforced* beams. *Hanger bars* of nominal diameter, used for the purpose of holding stirrups, do not normally qualify as compression reinforcement — unless the area of such bars is significant (greater than 0.2 percent).

In the discussions related to Fig. 4.13, it was shown that merely providing tension steel in excess of that required for the 'balanced section' ($p_{t,b}$) is not an effective way of improving the allowable moment capacity of the section, because the increase in the beam's capacity to carry flexural tension (with $f_{st} = \sigma_{st}$) is not matched by a corresponding increase in its capacity to carry flexural compression. One of the ways of solving this problem is by providing compression steel.

It may further be recalled (refer Section 4.6.2) that the *permissible stress* in compression steel ($\sigma_{sc}$) is generally restricted to 1.5 $m$ times the stress in the adjoining concrete ($f_{csc}$). Accordingly, the 'transformed section' takes the configuration shown in Fig. 4.16(b) for rectangular sections. For convenience, the concrete area under compression (i.e., above the NA) is treated as the 'gross' concrete area, i.e., disregarding the area displaced by the steel embedded therein. The concrete area displaced by the embedded compression bars (area $A_{sc}$) is accounted for by taking the 'effective' transformed area of steel as $(1.5m–1)A_{sc}$. The compression steel is generally kept as close to the face of the extreme compression concrete fibre as permitted by considerations of minimum cover, in order to maximise its effectiveness. The distance between the centroid of the compression steel and the extreme compression fibre in concrete is usually denoted by *d'*.

#### *Rectangular Sections*

Referring to the transformed section shown in Fig. 4.16(b), the neutral axis is determined by solving the following equation (considering moments of areas about the NA),

$$\frac{b(kd)^2}{2} + (1.5m - 1)A_{sc}(kd - d') \quad = \quad mA_{st}(d - kd) \tag{4.35}$$

The force equilibrium equation is given by:

$$C_c + C_s = T \tag{4.36}$$

where $C_c$ and $C_s$ denote the net compressive forces in concrete and steel respectively:

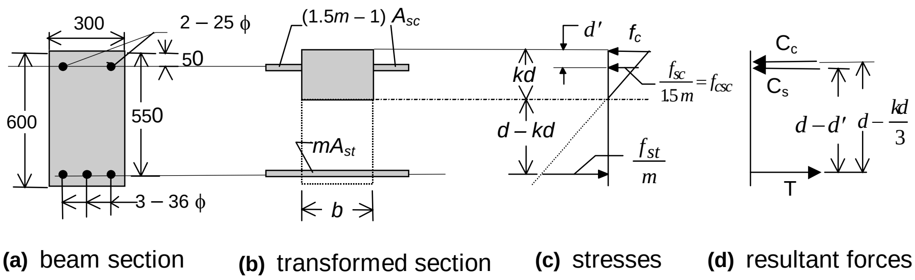


**Fig. 4.16** Cracked section analysis of a doubly-reinforced beam

$$C_c = 0.5 f_c b(kd) \tag{4.37a}$$

$$C_s = (1.5mf_{csc})A_{sc} - f_{csc}A_{sc}$$

$$\Rightarrow C_s = (1.5m - 1)A_{sc} f_{csc} \tag{4.37b}$$

where $$f_{csc} = f_c \left(\frac{kd - d'}{kd}\right) \tag{4.38}$$

and $$T = A_{st} f_{st} \quad \text{(as before)}$$

Further, taking moments of forces about the centroid of the tension steel,

$$M = C_c(d - kd/3) + C_s(d - d') \tag{4.39}$$

### *Flanged Sections*

If the neutral axis falls inside the flange ($kd \le D_f$), then the section can be effectively treated as a rectangular section, $b_f \times d$, as discussed earlier.

However, if $kd > D_f$, the equation to determine the neutral axis (in lieu of Eq. 4.35) is as follows:

$$(b_f - b_w)D_f(kd - D_f/2) + b_w(kd)^2/2 + (1.5m - 1)A_{sc}(kd - d') = mA_{st}(d - kd) \tag{4.40}$$

In the force equilibrium equation [Eq. 4.36], the net compressive force in concrete is determined (as before) by considering the difference between the rectangular $b_f \times kd$ and $(b_f - b_w) \times (kd - D_f)$ — whereby,

$$C_c = 0.5 f_c \left[ b_f(kd) - (b_f - b_w)\frac{(kd - D_f)^2}{kd} \right] \tag{4.41}$$

Finally, the moment equilibrium equation [Eq. 4.39] takes the following form:

$$M = 0.5 f_c \left[ b_f (kd)(d - kd/3) \; - \; (b_f - b_w)\frac{(kd - D_f)^2}{kd}\left(d - D_f - \frac{kd - D_f}{3}\right)\right] + (1.5m - 1)A_{sc}\, f_c\, \frac{(kd - d')}{kd}(d - d') \quad (4.42)$$

It may be noted that although Eq. 4.42 appears rather lengthy, it can be derived easily from first principles.

**EXAMPLE 4.6**

The cross–sectional dimensions of a doubly reinforced beam are shown in Fig. 4.16(a). Determine the stresses in concrete and steel corresponding to a service load moment of 175 kNm. Further, determine the allowable moment on the beam section. Assume M 20 concrete and Fe 250 steel

**SOLUTION**

- Given: $b$ = 300 mm, $d$ = 550 mm, $d'$ = 50 mm, $\sigma_{cbc}$ = 7.0 MPa, $\sigma_{st}$ = 130 MPa ($\phi > 20$ mm)

Transformed section properties [Fig. 4.16(b)]:

- modular ratio $m$ = 13.33 (for M 20 concrete)

  $$A_{st} = 3 \times \frac{\pi \times (36)^2}{4} = 3054 \text{ mm}^2; \quad A_{sc} = 2 \times \frac{\pi \times (25)^2}{4} = 982 \text{ mm}^2$$
- Transformed tension steel area = $mA_{st}$ = 40709 mm$^2$
- Transformed compression steel area = $(1.5m - 1)A_{sc}$ = 18653 mm$^2$

Neutral axis depth:

- Considering moments of areas about the neutral axis,

  $$\frac{300(kd)^2}{2} + 18653(kd - 50) = 40709(550 - kd)$$

  Solving, $kd$ = 243.3 mm.

Stresses due to $M$ = 175 kNm:

- Considering the linear stress distribution [Fig. 4.16(c)],

  $$f_{csc} = f_c\left(\frac{243.3 - 50}{243.3}\right) = 0.7945 f_c.$$

  $$C_c = 0.5 \times f_c \times 300 \times 243.3 = 36495 f_c$$

  $$C_s = 18653 \times 0.7945 \times f_c = 14819\, f_c$$
- Taking moments about the tension steel centroid,

  $$M = C_c(d - kd/3) + C_s(d - d')$$

  $$\Rightarrow 175 \times 10^6 = f_c[36495 \times (550 - 243.3/3) + 14819 \times (550 - 50)]$$

  $\Rightarrow f_c$ = **7.136 MPa**.

(which, incidentally, exceeds $\sigma_{cbc}$ = 7.0 MPa)

- *Compressive stress in steel*:
  $f_{sc} = 1.5 \; mf_{csc} = 1.5 \times 13.33 \times (0.7945 \times 7.136)$ = **113 MPa**.
- *Tensile stress in steel*:
  $f_{st} \; = \; mf_c \left(\frac{d - kd}{kd}\right) = 13.33 \times 7.136 \times \left(\frac{550 - 243.3}{243.3}\right)$ = **120 MPa**

  [Alternatively, $C_c + C_s = T \Rightarrow \; f_{st} = \frac{7.136(36495 + 14819)}{3054}$

  = 120 MPa (as before).

<u>Allowable bending moment</u>:

- For a 'balanced (WSM) section', $k_b = \frac{280}{280 + 3(130)}$ = 0.4179.

  For the given section, $k = 243.3/550 = 0.4424 > k_b = 0.4179$ .

  Hence the section is 'over-reinforced (WSM)',

  whereby $f_c = \sigma_{cbc}$ = 7.0 MPa.

  $M_{all} = 7.0 \; [36495 \times (550 - 243.3/3) + 14819 \times (500)]$

  $= 171 \times 10^6$ Nmm = **171 kNm**.

  [Alternatively, $M_{all} = 175 \times \left(\frac{7.0}{7.136}\right)$ = 171 kNm]

## EXAMPLE 4.7

In the previous Example, it is seen that the service load moment of 175 kNm exceeds the allowable moment (equal to 171 MPa). If the problem were a *design* problem (instead of an *analysis* problem), how is it possible to arrive at the appropriate values of $A_{st}$ and $A_{sc}$ (without changing the size of the section and the grades of concrete and steel) so that the allowable moment is raised to 175 kNm ?

### SOLUTION

- As explained earlier (with reference to Fig. 4.13), 'over–reinforced (WSM)' sections are uneconomical. This is true not only for 'singly reinforced' sections, but also 'doubly reinforced' sections. The neutral axis depth $kd$ should be ideally restricted to that corresponding to the balanced section ($k_b d$ = 0.4179 × 550 = 229.8 mm). Accordingly, applying Eq. 4.35,

  $300 \times (229.8)^2 / 2 + (1.5 \times 13.33 - 1) A_{sc} (229.8 - 50)$ = $13.33 A_{st} (550 - 229.8)$

  $\Rightarrow A_{st} = (0.8 A_{sc} + 1856) \, mm^2$

  $A_{sc}$ is to be determined from Eq. 4.39 for $M$ = 175 kNm and $f_c = \sigma_{cbc}$ = 7.0 MPa,

  $175 \times 10^6 = (0.5 \times 7.0 \times 300 \times 229.8) \times (550 - 229.8/3)$

  $+ (1.5 \times 13.33 - 1) A_{sc} \times \; 7.0 \times \frac{(229.8 - 50) \times (550 - 50)}{229.8}$

$\Rightarrow A_{sc}$ = **1168mm²**

whereby $A_{st} = 0.8 \times 1168 + 1856 =$ **2791mm²**

- **Alternative Solution**:

Let

$$A_{st} \;=\; A_{st1} + A_{st2} \tag{4.43}$$

where, $A_{st1}$ corresponds to the area required for a *singly reinforced* 'balanced (WSM)' section†, and $A_{st2}$ corresponds to the additional tension steel (with $f_{st} = \sigma_{st}$) required to resist the moment $M - M_{wb}$, in combination with the compression steel $A_{sc}$ whose stress is given by 1.5$m$ times $f_{csc}$, where

$$f_{\mathrm{csc}} \;=\; \sigma_{cbc}\left(1 - \frac{d'}{k_b d}\right) \tag{4.44}$$

[In doubly reinforced sections, the stress 1.5$mf_{csc}$ is generally less than the maximum limit $\tilde{\sigma}_{sc}$ given in Section 4.6.2.]

Accordingly,

$$A_{st1} \;=\; \frac{M_{wb}}{\sigma_{st} d\left(1 - k_b/3\right)} \tag{4.45}$$

$$A_{st2} \;=\; \frac{M - M_{wb}}{\sigma_{st}\left(d - d'\right)} \tag{4.46}$$

$$A_{sc} \;=\; \frac{M - M_{wb}}{(1.5m - 1)\, f_{\mathrm{csc}}\left(d - d'\right)} \tag{4.47}$$

The formula for $M_{wb}$ is obtainable by Eq. 4.25. For the present problem, applying the various formulae with $k_b$ = 0.418, $b = 300$ mm, $d$ = 550 mm, $m$ = 13.33, $\sigma_{st}$ = 130 MPa, $\sigma_{cbc}$ = 7.0 MPa, and $M = 175 \times 10^6$ Nmm ,

$$M_{wb} = 0.5 \times 7.0 \times 300 \times (0.4179 \times 550) \times (1 - 0.4179/3) 550 \qquad \text{[Eq. 4.25]}$$

$$= 114.27 \times 10^{6} \text{ Nmm .}$$

$$A_{st1} = \frac{114.27 \times 10^6}{130 \times 550 (1 - 0.4179/3)} = 1857 \text{ mm}^2 \qquad \text{[Eq. 4.45]}$$

$$A_{st2} = \frac{(175.0 - 114.27) \times 10^6}{130 (550 - 50)} = 935 \text{ mm}^2 \qquad \text{[Eq. 4.46]}$$

$$\therefore A_{st} = 1857 + 935 = \mathbf{2792mm^2} \qquad \text{[Eq. 4.43]}$$

$$f_{\mathrm{csc}} = 7.0 \times \left(1 - \frac{50}{0.4179 \times 550}\right) = 5.477 \text{ MPa} \qquad \text{[Eq. 4.44]}$$

$$\Rightarrow A_{sc} = \frac{(175.0 - 114.27) \times 10^6}{(1.5 \times 13.33 - 1)(5.477)(550 - 50)} = \mathbf{1168mm^2} \qquad \text{[Eq. 4.47]}$$

† Note that allowance has to be made for the area $A_{sc}$ displaced by the concrete; this is done in the calculation for the required area of the compression steel [Eq. 4.47].

## EXAMPLE 4.8

Consider the T-beam section of Example 4.4 with additional compression reinforcement of 3–28ϕ bars of Fe 250 grade and $d' = 50$ mm. Determine the allowable moment capacity.

**SOLUTION**

- Given: $b_f$ = 850 mm, $D_f$ = 100 mm, $b_w$ = 250 mm, $d$ = 520 mm, $A_{st}$ = 3695 mm$^2$, $m$ = 13.33 (for M 20 concrete), $d' = 50$ mm,

$$A_{sc} = \frac{3 \times \pi \times (28)^2}{4} = 1847 \text{ mm}^2$$

Transformed tension steel area = $mA_{st}$ = 49254 mm$^2$
Transformed compression steel area = $(1.5\,m - 1)A_{sc}$ = 35084 mm$^2$

- Assuming first $kd \le D_f$, and $kd > d'$, and solving Eq. 4.35 with $b = b_f$,

$kd$ = 173.2 mm.

As $kd > D_f$, the assumption $kd \le D_f$, is incorrect.

- Now solving Eq. 4.40 for $kd \ge D_f$, $kd$ = 181.8 mm $\Rightarrow$ $k$ = 181.8/ 520 = 0.3496

For a 'balanced (WSM)' section with $\sigma_{st}$ = 130 MPa [Eq. 4.23], $k_b = 0.4179$

- As $k = 0.3496 < k_b$, the section is 'under-reinforced (WSM)' whereby $f_{st} = \sigma_{st}$ = 130 MPa. Considering the linear stress distribution [Eq. 4.32(b)],

$$f_c = \frac{130}{13.33} \times \frac{181.8}{520 - 181.8} = 5.24 \text{ MPa } \left(< \sigma_{cbc} = 7.0 \text{ MPa}\right)$$

- Substituting in Eq. 4.42,

$$M_{all} = \mathbf{226\ kNm}$$

Two points may be noted with regard to *working stress design* of doubly-reinforced sections:

1) There is no advantage in using high strength steel as compression reinforcement as the permissible stress† is relatively low and unrelated to the grade of steel.
2) In order to resist a very high moment, a large area of compression steel is called for; the $A_{sc}$ required may even exceed $A_{st}$

These are serious shortcomings of WSM, which result in uneconomical designs. Owing to these and other reasons explained in Chapter 3, WSM is no longer employed in practical designs — being replaced by the more rational *limit states design*.

## 4.7 ANALYSIS AT ULTIMATE LIMIT STATE

Whereas the previous section (Section 4.6) dealt with the 'analysis at *service loads*', the present section deals with the 'analysis at *ultimate loads*'. The former is based on

† refer Section 4.6.2

the *working stress method* (WSM) and is also applicable to the analysis of 'serviceability limit states', whereas the latter is based on the 'ultimate limit state' of the *limit states method* (LSM).

Having studied analysis at '*service loads*' in some detail (including the solution to a number of Example problems), it is likely that the student may get somewhat confused while undertaking the task of analysing the same problems at '*ultimate loads*'. The important question that is likely to disturb the student is — why go through this process of analysing at service loads as well as ultimate loads? The answer to this question was given in Chapter 3, where it was explained that a structure has to be both **safe** (at various *ultimate* limit states) and **serviceable** (at various *serviceability* limit states). At ultimate limit states, the loads are those corresponding to impending failure of structure, whereas at serviceability limit states, the loads and stresses are those applicable in the day-to-day service of the structure. This section investigates the 'safety' of flexural members (of given design) at the *ultimate limit state in flexure*. The previous section discussed the calculation of flexural stresses under service loads required for serviceability analysis (described in Chapter 10), and also the calculation of 'allowable bending moment' based on the WSM concept of *permissible stresses*. The latter was included to enable the student to gain a first-hand understanding of the traditional (and, earlier much-used) working stress method — which retains a place in the Code, albeit as an Appendix, and is sometimes used in the design of special structures such as water tanks and road bridges.

Therefore, the student will do well to keep in perspective the background of the present section, dealing with the *analysis* at the 'ultimate limit state in flexure'. The expressions derived here will find use again in the next chapter (Chapter 5), which deals with the *design* of reinforced concrete beams at the ultimate limit state in flexure.

In this section, the Code procedure for analysis is discussed. The calculations are based on the idealised stress-strain curves for concrete and steel, as specified by the Code. Moreover, the *design* stress-strain curves (involving partial safety factors $\gamma_c, \gamma_s$) are used, as explained in Section 3.6.

### 4.7.1 Assumptions in Analysis

The behaviour of reinforced concrete beam sections at ultimate loads has been explained in detail in Section 4.5.3. The basic assumptions involved in the analysis at the ultimate limit state of flexure (Cl. 38.1 of the Code) are listed here [see also Fig. 4.17]. (Most of these assumptions have already been explained earlier.)

a) Plane sections normal to the beam axis remain plane after bending, i.e., in an initially straight beam, strain varies linearly over the depth of the section.

b) The maximum compressive strain in concrete (at the outermost fibre) $\varepsilon_{cu}$ shall be taken as 0.0035 [Fig. 4.17(b)]. This is so, because regardless of whether the beam is *under–reinforced* or *over-reinforced*, collapse invariably occurs by the crushing of concrete (as explained in Section 4.5.3).

c) The *design stress–strain curve* of concrete in flexural compression (recommended by the Code) is as depicted in Fig. 3.5. [The Code also permits the use of any other shape of the stress–strain curve *which results in substantial agreement with the results of tests.*] The partial safety factor $\gamma_c$ = 1.5 is to be considered.

d) The tensile strength of the concrete is ignored.

e) The *design stress–strain curves* for mild steel and cold-worked bars are as depicted in Fig. 3.6 and Fig. 3.7 respectively. The partial safety factor $\gamma_s$ = 1.15 is to be considered.

f) The strain $\varepsilon_{st}$ in the tension reinforcement (at its centroid) at the ultimate limit state shall not be less than $\varepsilon_{st}^*$ [Fig. 4.17], defined as:

$$\varepsilon_{st}^* \equiv (0.87\, f_y / E_s) + 0.002 \tag{4.48}$$

This is equivalent to defining the yield stress $f_y$ of steel as the stress corresponding to 0.002 strain offset (0.2% proof stress) — regardless of whether the steel has a well-defined yield point or not. The yield strain corresponding to $f_y$ is then given by $0.002 + f_y / E_s$. Introducing the partial safety factor $\gamma_s = 1.15$ to allow for the variability in the steel strength, the *design*† *yield strength,* $f_{yd} = f_y / 1.15 = 0.87 f_y$ and using this in lieu of $f_y$, the yield strain $\varepsilon_y$ [refer Fig. 3.7] is given by:

$$\varepsilon_y = 0.002 + (0.87\, f_y / E_s) \equiv \varepsilon_{st}^*$$

In the case of mild steel, which has a well-defined yield point ($\varepsilon_y = 0.87\, f_y / E_s$, as shown in Fig. 3.6), the requirement (f) cited above may appear to be conservative. However, the Code specifies a uniform criterion [Eq. 4.48] for *all* grades of steel. The intention here is to ensure that 'yielding' of the tension steel takes place at the ultimate limit state, so that the consequent failure is *ductile* in nature, providing ample warning of the impending collapse.

### 4.7.2 Limiting Depth of Neutral Axis

Based on the assumption given above, an expression for the depth of the neutral axis at the ultimate limit state, $x_u$, can be easily obtained from the strain diagram in Fig. 4.17(b). Considering similar triangles,

$$\frac{x_u}{d} = \frac{0.0035}{0.0035 + \varepsilon_{st}} \tag{4.49}$$

† It is interesting to note that the use of the *design* yield stress $f_{yd}$ = 0.87 $f_y$, instead of the characteristic yield stress $f_y$, results in a slightly *lesser* (and hence, less conservative!) value of the yield strain $\varepsilon_y$ [refer Figs 3.6, 3.7].

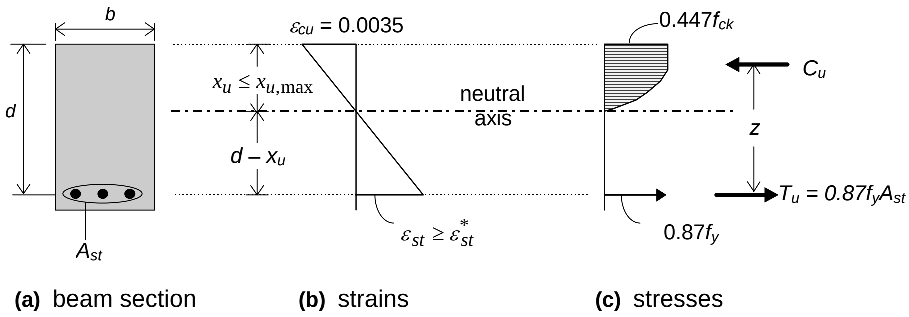


**Fig. 4.17** Behaviour of singly reinforced rectangular section at ultimate limit state in flexure

According to the Code [requirement (f) in Section 4.7.1], $\varepsilon_{st} \ge \varepsilon^*_{st}$, implying that there is a limiting (maximum) value of the neutral axis depth $x_{u,max}$ corresponding to $\varepsilon_{st} = \varepsilon^*_{st}$. This is obtained by substituting the expression for $\varepsilon^*_{st}$ [Eq. 4.48] in Eq. 4.49:

$$\frac{x_{u,\max}}{d} = \frac{0.0035}{0.0055 + 0.87 f_y / E_s} \tag{4.50}$$

The values of $x_{u,\max}/d$ for different grades of steel, obtained by applying Eq. 4.50, are listed in Table 4.3. It may be noted that the constants given in Table 4.3 are applicable to all cross–sectional shapes, and remain valid for doubly reinforced sections as well.

**Table 4.3** Limiting depth of neutral axis for different grades of steel

| **Steel Grade** | **Fe 250** | **Fe 415** | **Fe 500** |
|---|---|---|---|
| $x_{u,max}/d$ | 0.5313 | 0.4791 | 0.4560 |

The limiting depth of neutral axis $x_{u,max}$ corresponds to the so-called *balanced* section, i.e., a section that is expected to result in a 'balanced' failure at the ultimate limit state in flexure [refer Section 4.5.3]. If the neutral axis depth $x_u$ is less than $x_{u,max}$, then the section is *under-reinforced* (resulting in a 'tension' failure); whereas if $x_u$ exceeds $x_{u,max}$, it is *over-reinforced* (resulting in a 'compression' failure).

### 4.7.3 Analysis of Singly Reinforced Rectangular Sections

Analysis of a given reinforced concrete section at the ultimate limit state of flexure implies the determination of the *ultimate moment of resistance* $M_{uR}$ of the section. This is easily obtained from the couple resulting from the flexural stresses [Fig. 4.17(c)]:

$$M_{uR} = C_u \cdot z = T_u \cdot z \tag{4.51}$$

where $C_u$ and $T_u$ are the resultant (ultimate) forces in compression and tension respectively, and $z$ is the lever arm.

$$T_u = f_{st} A_{st} \tag{4.52}$$

where

$$f_{st} = 0.87 f_y \qquad \text{for} \quad x_u \le x_{u,\max}$$

and the line of action of $T_u$ corresponds to the level of the centroid of the tension steel.

#### *Concrete Stress Block in Compression*

In order to determine the magnitude of $C_u$ and its line of action, it is necessary to analyse the *concrete stress block* in compression. As ultimate failure of a reinforced concrete beam in flexure occurs by the crushing of concrete, for both under- and over-reinforced beams, the shape of the compressive stress distribution ('stress block') at failure will be, in both cases, as shown in Fig. 4.18. [also refer assumptions (b) and (c) in Section 4.7.1]. The value of $C_u$ can be computed knowing that the compressive stress in concrete is uniform at 0.447 $f_{ck}$ for a depth of $3x_u$ / 7, and below this it varies parabolically over a depth of $4x_u$ / 7 to zero at the neutral axis [Fig. 4.18].

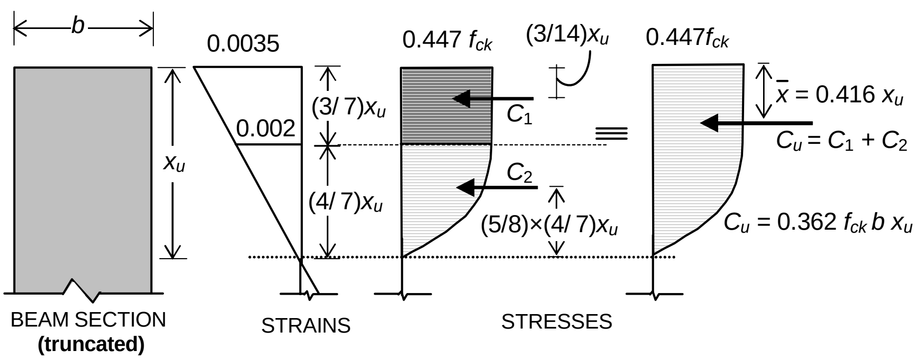


**Fig. 4.18** Concrete stress-block parameters in compression

For a rectangular section of width $b$,

$$C_u = 0.447\, f_{ck}\, b \left[ \frac{3x_u}{7} + \left( \frac{2}{3} \times \frac{4x_u}{7} \right) \right]$$

$$\Rightarrow C_u = 0.362\, f_{ck}\, b\, x_u \qquad (4.53)$$

Also, the line of action of $C_u$ is determined by the centroid of the stress block, located at a distance $\bar{x}$ from the concrete fibres subjected to the maximum compressive strain. Accordingly, considering moments of compressive forces $C_u$, $C_1$ and $C_2$ [Fig. 4.18] about the maximum compressive strain location,

$$\left(0.362\, f_{ck} b x_u\right) \times \bar{x} = \left(0.447\, f_{ck} b x_u\right) \left[ \left(\frac{3}{7}\right)\left(\frac{1.5x_u}{7}\right) + \left(\frac{2}{3} \times \frac{4}{7}\right)\left(x_u - \frac{5}{8} \times \frac{4x_u}{7}\right) \right]$$

Solving,

$$\bar{x} = 0.416\, x_u \qquad (4.54)$$

### *Depth of Neutral Axis*

For any given section, the depth of the neutral axis should be such that $C_u = T_u$, satisfying equilibrium of forces. Equating $C_u = T_u$, with expressions for $C_u$ and $T_u$, given by Eq. 4.53 and Eq. 4.52 respectively:

$$x_u = \frac{0.87 f_y A_{st}}{0.362 f_{ck} b}, \qquad \text{valid only if resulting } x_u \le x_{u,max} \qquad (4.55)$$

For the condition $x_u > x_{u,max}$ , $\varepsilon_{st} < \varepsilon^*_{st}$ [Fig. 4.17], implying that, at the ultimate limit state, the steel would not have 'yielded' (as per the proof stress definition† for $f_y$) and the steel stress cannot be taken as $f_y/\gamma_s = 0.87 f_y$. Hence Eq. 4.52 and therefore Eq. 4.55 are not applicable. When the steel has not yielded, the true location of the neutral axis is obtained by a trial-and-error method, called *strain compatibility method*, involving the following steps:

1) Assume a suitable initial (trial) value of $x_u$
2) Determine $\varepsilon_{st}$ by considering *strain compatibility* [Eq. 4.49]:

$$\varepsilon_{st} = 0.0035\,(d/x_u - 1) \qquad (4.56)$$

3) Determine the *design stress* $f_{st}$ corresponding to $\varepsilon_{st}$ using the design stress-strain curve [Fig. 3.7, Table 3.2].
4) Derive the value of $x_u$ corresponding to $f_{st}$ by considering $T_u = f_{st} A_{st}$ and applying the force equilibrium condition $C_u = T_u$, whereby

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† In the case of low grade mild steel (Fe 250), which has a sharply defined yield point, the steel would have yielded and reached $f_y$ even at a strain slightly lower than $\varepsilon^*_{st}$ . In such cases, one may find that $f_{st} = 0.87 f_y$ even for values of $x_u$ slightly in excess of $x_{u,max}$.

$$x_u = \frac{f_{st} A_{st}}{0.362 f_{ck} b} \tag{4.57}$$

5) Compare this value of $x_u$ with the value used in step (1). If the difference between the two values is acceptably small, accept the value given by step (4); otherwise, repeat steps (2) to (5) with an improved (say, average) value of $x_u$, until convergence.

### *Ultimate Moment of Resistance*

The *ultimate moment of resistance* $M_{uR}$ of a given beam section is obtainable from Eq. 4.51. The lever arm $z$, for the case of the singly reinforced rectangular section [Fig. 4.17(d), Fig. 4.18] is given by

$$z = d - 0.416 x_u \tag{4.58}$$

Accordingly, in terms of the concrete compressive strength,

$$M_{uR} = 0.362 f_{ck} b x_u (d - 0.416 x_u) \qquad \text{for all } x_u \tag{4.59}$$

Alternatively, in terms of the steel tensile stress,

$$M_{uR} = f_{st} A_{st} \left(d - 0.416 x_u\right) \qquad \text{for all } x_u \tag{4.60}$$

$$\text{with } f_{st} = 0.87 f_y \quad \text{for} \quad x_u \le x_{u,max}$$

### *Limiting Moment of Resistance*

The *limiting moment of resistance* $M_{u,lim}$ of a given (singly reinforced, rectangular) section, according to the Code (Cl. G–1.1), corresponds to the condition $x_u = x_{u,max}$, defined by Eq. 4.50. From Eq. 4.59, it follows that:

$$M_{u,\text{lim}} = 0.362 f_{ck} b x_{u,\text{max}} \left(d - 0.416 x_{u,\text{max}}\right) \tag{4.61}$$

$$\Rightarrow \frac{M_{u,\text{lim}}}{f_{ck} b d^2} \equiv K = 0.362 \left(\frac{x_{u,\text{max}}}{d}\right)\left(1 - 0.416 \frac{x_{u,\text{max}}}{d}\right) \tag{4.61a}$$

The values of the non-dimensional parameter $K$ for different grades of steel [refer Table 4.3] are obtained as 0.1498, 0.1389 and 0.1338 for Fe 250, Fe 415 and Fe 500 respectively.

### *Limiting Percentage Tensile Steel*

Corresponding to the limiting moment of resistance $M_{u,lim}$, there is a limiting percentage tensile steel $p_{t,lim} = 100 \times A_{st,lim}/bd$. An expression for $p_{t,lim}$ is obtainable from Eq. 4.55 with $x_u = x_{u,\text{max}}$:

$$\frac{x_{u,max}}{d} = \frac{0.87 f_y}{0.362 f_{ck}} \times \frac{p_{t,lim}}{100}$$

$$\Rightarrow \quad p_{t,\lim} \quad = \quad 41.61 \left(\frac{f_{ck}}{f_y}\right)\left(\frac{x_{u,\max}}{d}\right) \tag{4.62}$$

The values of $p_{t,lim}$ and $M_{u,lim}/bd^2$ (in MPa units) for, different combinations of steel and concrete grades are listed in Table 4.4. These values correspond to the so-called 'balanced' section [refer Section 4.5.3] for a singly reinforced rectangular section.

**Table 4.4** Limiting values of $p_{t,lim}$ and $M_{u,lim}/bd^2$ for singly reinforced rectangular beam sections for various grades of steel and concrete.

**(a)** $p_{t,lim}$ values

| | **M 20** | **M 25** | **M 30** | **M 35** | **M 40** | **M 45** | **M 50** |
|---|---|---|---|---|---|---|---|
| **Fe 250** | 1.769 | 2.211 | 2.653 | 3.095 | 3.537 | 3.979 | 4.421 |
| **Fe 415** | 0.961 | 1.201 | 1.441 | 1.681 | 1.921 | 2.162 | 2.402 |
| **Fe 500** | 0.759 | 0.949 | 1.138 | 1.328 | 1.518 | 1.708 | 1.897 |

**(b)** $M_{u,lim}/bd^2$ values (MPa)

| | **M 20** | **M 25** | **M 30** | **M 35** | **M 40** | **M 45** | **M 50** |
|---|---|---|---|---|---|---|---|
| **Fe 250** | 2.996 | 3.746 | 4.495 | 5.244 | 5.993 | 6.742 | 7.491 |
| **Fe 415** | 2.777 | 3.472 | 4.166 | 4.860 | 5.555 | 6.249 | 6.943 |
| **Fe 500** | 2.675 | 3.444 | 4.013 | 4.682 | 5.350 | 6.019 | 6.688 |

### *Safety at Ultimate Limit State in Flexure*

The bending moment expected at a beam section at the *ultimate limit state* due to the *factored loads* is called the *factored moment* $M_u$. For the consideration of various combinations of loads (dead loads, live loads, wind loads, etc.), appropriate load factors should be applied to the specified 'characteristic' loads (as explained in Chapter 3), and the factored moment $M_u$ is determined by structural analysis.

The beam section will be considered to be 'safe', according to the Code, if its ultimate moment of resistance $M_{uR}$ is greater than or equal to the factored moment $M_u$. In other words, for such a design, the *probability of failure* is acceptably low. It is also the intention of the Code to ensure that at ultimate failure in flexure, the type of failure should be a *tension (ductile) failure* — as explained earlier. For this reason, the Code requires the designer to ensure that $x_u \leq x_{u,\max}$ [Table 4.3], whereby it follows that, *for a singly reinforced rectangular section*, the tensile reinforcement percentage $p_t$ should not exceed $p_{t,\lim}$ and the ultimate moment of resistance $M_{uR}$ should not exceed $M_{u,\lim}$ [Table 4.4]. The Code (Cl. G–1.1d) clearly states:

"If $x_u/d$ is greater than the limiting value, the section shall be redesigned."

The topic of *design* is covered in detail in Chapter 5. The present chapter deals with analysis — and, in *analysis*, it is not unlikely to encounter beam sections (already constructed) in which $p_t > p_{t,lim}$, whereby $x_u > x_{u,max}$ and $M_{uR} > M_{u,\lim}$. Evidently, in such 'over-reinforced' sections, the *strength* requirement may be satisfied, but not the *ductility*† requirement. The question arises: are such sections acceptable ? The answer, in general, would be in the negative, except in certain special situations where the section itself is not 'critical' in terms of ductility, and will not lead to a *brittle failure* of the structure under the given *factored loads*. In such exceptional cases, where $M_{uR} > M_u$, and inelastic flexural response‡ is never expected to occur under the given *factored loads*, over-reinforced sections cannot be strictly objected to.

It may be noted that the exact determination of $M_{uR}$ of an over-reinforced section generally involves considerable computational effort, as explained in the next section. An approximate (but conservative) estimate of the ultimate moment capacity of such a section is given by the limiting moment of resistance, $M_{u,lim}$, which can be easily computed.

### *Variation of $M_{uR}$ with $p_t$ (for singly reinforced rectangular sections)*

$p_t \leq p_{t,lim}$

For $x_u \leq x_{u,\max}$. it is possible to arrive at a simple closed–form expression for the ultimate moment of resistance of a given section with a specified $p_t \leq p_{t,\lim}$. First, expressing $A_{st}$ in terms of $p_t$ :

$$A_{st} = \frac{p_t bd}{100} \tag{4.63}$$

and then substituting in Eq. 4.55,

$$x_u = \frac{0.87}{0.362}\frac{f_y}{f_{ck}}\frac{p_t d}{100} \tag{4.64}$$

Further substituting Eq. 4.63 and Eq. 4.64 in Eq. 4.60,

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† The ductility requirement may be partly satisfied in the case of mild steel (Fe 250), even if $x_u$ slightly exceeds $x_{u,max}$; this is explained later with reference to Fig. 4.19. [See also footnote on p 136.]

‡ for details on 'plastic hinge' formation at the ultimate limit state, refer Chapter 9.

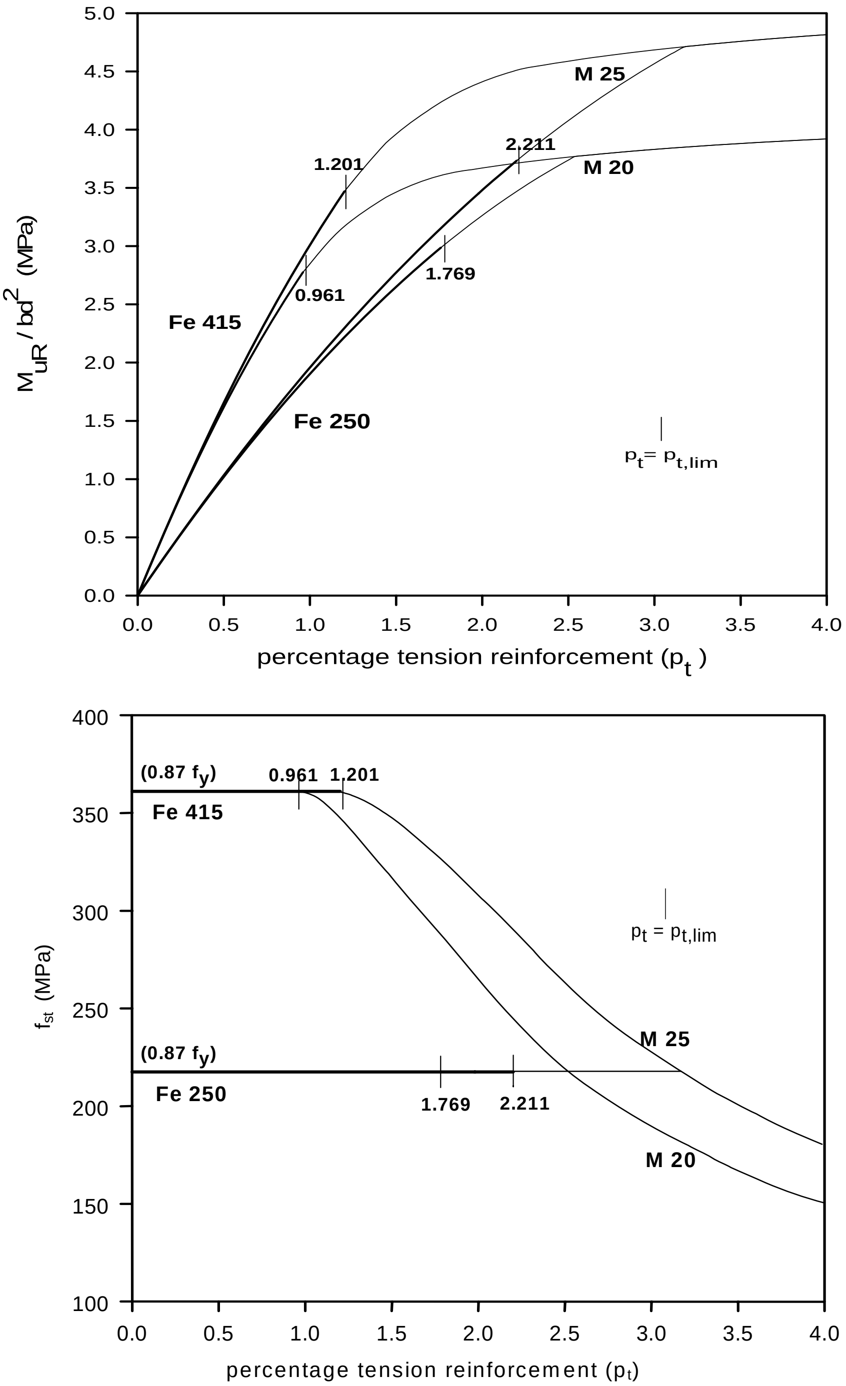


**Fig. 4.19** Variation of (a) $M_{uR}/bd^2$ and (b) $f_{st}$ with $p_t$

$$M_{uR} = 0.87 f_y \frac{p_t}{100} bd^2 \left(1 - \frac{0.416 \times 0.87}{0.362} \frac{f_y}{f_{ck}} \frac{p_t}{100}\right)$$

$$\Rightarrow \frac{M_{uR}}{bd^2} = 0.87 f_y \frac{p_t}{100}\left(1 - 1.000 \frac{f_y}{f_{ck}} \frac{p_t}{100}\right) \qquad \text{for } p_t \le p_{t,\lim} \qquad (4.65)$$

### $p_t > p_{t,lim}$

For $p_t > p_{t,lim}$ †, $x_u > x_{u,max}$, whereby the *design stress* in the tension steel takes a value $f_{st}$ which in general is not a constant, and depends on the value of $x_u$ [Eq. 4.56].

To determine $f_{st}$, the (trial-and-error) *strain compatibility method* (described earlier) has to be employed. The final expression, comparable to Eq. 4.65, takes the following form:

$$\frac{M_{uR}}{bd^2} = f_{st} \frac{p_t}{100}\left(1 - \frac{0.416}{0.362} \frac{f_{st}}{f_{ck}} \frac{p_t}{100}\right) \qquad (4.66)$$

where $f_{st} \le 0.87 f_y$ has to satisfy the force equilibrium condition [Eq. 4.57], and the strain $\varepsilon_{st}$ corresponding to $f_{st}$ [Fig. 3.6, 3.7] must satisfy the strain compatibility condition [Eq. 4.56]. For convenience, Eq. 4.57 is re-arranged as follows:

$$\frac{x_u}{d} = \frac{f_{st}}{0.362 f_{ck}} \frac{p_t}{100} \qquad (4.67)$$

The steps involved in the 'strain-compatibility method for determining $M_{uR}/bd^2$ for a given $p_t$, are as follows:

1) Assume an initial (trial) value of $x_u/d$: say, $x_{u,\max}/d$;
2) Determine $\varepsilon_{st}$ using Eq. 4.56;
3) Determine $f_{st}$ from $\varepsilon_{st}$ using the *design* stress-strain curves [Fig. 3.6, 3.7];
4) Calculate the new value of $x_u/d$ using Eq. 4.67;
5) Compare the new value of $x_u/d$ with the old value. If the difference is within acceptable tolerance, proceed to step (6); otherwise repeat steps (2) to (5) until convergence is attained.
6) Apply Eq. 4.66 and determine $M_{uR}/bd^2$.

A quick solution can be obtained by means of a computer program. The relationship between $M_{uR}/bd^2$ (expressed in MPa units) and $p_t$ is plotted in Fig. 4.19(a) for two typical grades of steel (Fe 250 and Fe 415) combined with the commonly used grades of concrete (M 20 and M 25). The corresponding relationship between stress $f_{st}$ (at the ultimate limit state) and $p_t$ is depicted in

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† This condition $p_t > p_{t,\lim}$ is not permitted in *design*. Its only relevance is in *analysis*.

Fig. 4.19(b). The relatively thick lines represent the under-reinforced condition $p_t \le p_{t,lim}$ (i.e., $M_{uR} \le M_{uR,lim}$), whereas the thin lines denote the over-reinforced condition $p_t > p_{t,lim}$ (i.e., $M_{uR} > M_{uR,lim}$), and the transition points are marked by thin vertical lines.

It can be seen that these curves in Fig. 4.19(a) (for the ultimate limit state) bear resemblance with the corresponding curves in Fig. 4.13 (for the service load state).

The gain in $M_{uR}$ with increase in $p_t$ follows a nearly linear relationship almost up to the 'balanced' point. Also the gain in $M_{uR}$ with higher grades of concrete is marginal for low values of $p_t$ and becomes pronounced only when $p_t$ exceeds $p_{t,lim}$.

It may be noted from Fig. 4.19(b) that with the steel percentage limited to $p_{t,lim}$, as ultimate moment of resistance $M_{uR}$ is reached, the steel would have already 'yielded' ($f_{st}$ = $0.87f_y$) and gone into the domain of large inelastic strains, thus ensuring a ductile response. For $p_t > p_{t,lim}$, the tension steel would not have 'yielded' at the ultimate limit state, with the definition of steel strain at balanced condition as $\varepsilon_{st}^* = 0.002 + (0.87 f_y)/E_s$ [refer Section 4.7.1]. However, in the case of Fe 250 steel [Fig. 3.6], it can be seen that 'yielding' will actually take place even with a steel strain less than $\varepsilon_{st}^*$.

Hence, in the case of Fe 250 steel, the actual 'under-reinforced' (ductile failure) behaviour extends to some $p_t$ values greater than $p_{t,lim}$ [Fig. 4.19]. However, as explained earlier, the use of $p_t$ values in this range is not permitted by the Code for *design* purpose.

It can also be seen that the gain in $M_{uR}$ with $p_t$ falls off significantly, and somewhat exponentially, beyond the point where $f_{st}$ drops below 0.87 $f_y$. Beyond the 'balanced' point, there is a stage when the ultimate moment capacity is dictated entirely by the compressive strength of concrete, and hence does not depend on the grade of steel; in this range, $\varepsilon_{st} << \varepsilon_y$, whereby the steel stress is given by $f_{st} = E_s \varepsilon_{st}$, regardless of the grade of steel, and the same $T_u = A_{st} f_{st}$ is obtained whether Fe 250 or Fe 415 steel is used. This is indicated by the merging together of the thin lines (for a given concrete grade, and for Fe 250 and Fe 415 steel grades) in Fig. 4.19(a).

It is thus evident that over-reinforced sections are undesirable not only from the Code perspective of the loss in ductility, but also from the practical viewpoint of economy.

### *Analysis Aids*

The variation of $M_u/bd^2$ with $p_t$ for different grades of concrete and steel (depicted in Fig. 4.19) is expressed in tabular form and presented in Tables A.2(a), (b) in Appendix A of this book. As with analysis by WSM [Tables A.1(a), (b)], these Tables serve as useful *analysis aids*. They enable rapid determination of the ultimate moment capacity of any given singly reinforced rectangular beam section. The use of these Tables is demonstrated in Example 4.11.

### EXAMPLE 4.9

Determine the neutral axis depth $x_u$ (at the ultimate limit state) for the beam section in Example 4.2.

**SOLUTION**

- Given: $b$ = 300 mm, $d$ = 550 mm, $A_{st}$ = 1963 mm$^2$, $f_y$ = 415 MPa , $f_{ck}$ = 20 MPa
- For Fe 415 steel, $x_{u,\max}/d$ = 0.479 [Table 4.3 or Eq. 4.50]

  $\Rightarrow x_{u,\max} = 0.479 \times 550 = 263.5$ mm
- Assuming $x_u \le x_{u,\max}$ and applying the force equilibrium condition $C_u = T_u$

  $x_u = \dfrac{0.87 \times 415 \times 1963}{0.362 \times 20 \times 300}$ = 326.3 mm (> $x_{u,\max}$ = 263.5 mm).
- As $x_u > x_{u,\max}$, steel would not have 'yielded'; accordingly, the 'strain compatibility method' is adopted to obtain the correct value of $x_u$.

  *First Cycle* :

  1) assume $x_u \approx (264 + 326)/2$ = 295 mm;
  2) strain compatibility $\Rightarrow \varepsilon_{st} = 0.0035\left(\dfrac{550}{295} - 1\right)$ = 0.00303;
  3) $\Rightarrow f_{st}$ = 351.8 + (360.9 – 351.8) $\left(\dfrac{303-276}{380-276}\right)$ = 354.2 MPa [refer Table 3.2, for Fe 415]
  4) $C_u = T_u \Rightarrow x_u = f_{st} \times \left(\dfrac{1963}{0.362 \times 20 \times 300}\right)$ = 354.2 × (0.9038) = 320.1 mm.

  *Second Cycle* :

  1) assume $x_u \approx (320 + 295)/2$ = 308 mm
  2) $\Rightarrow \varepsilon_{st} = 0.0035\left(\dfrac{550}{308} - 1\right)$ = 0.00275;
  3) $\Rightarrow$ [Table 3.2] $f_{st}$ = 351.5 MPa;
  4) $x_u = 351.5 \times (0.9038)$ = 317.7 mm.

  *Third Cycle* :

  1) assume $x_u \approx$ (318 + 308)/2 = 313 mm;
  2) $\Rightarrow \varepsilon_{st} = 0.0035\left(\dfrac{550}{313} - 1\right)$ = 0.00265;
  3) $\Rightarrow f_{st} = 342.8 + (351.8 - 342.8) \times \left(\dfrac{265-241}{276-241}\right)$ = 349.0MPa.

4) $x_u = 349.0 \times (0.9038)$ = 315.4 mm.
The final value of $x_u$ may be taken as: $x_u$ = **315 mm**.

### EXAMPLE 4.10

Repeat the problem in Example 4.9, considering Fe 250 grade steel in lieu of Fe 415.

**SOLUTION**

- Given: $b$ = 300 mm, $d$ = 550 mm, $A_{st}$ = 1963 mm$^2$, $f_y$ = 250 MPa , $f_{ck}$ = 20 MPa
- For Fe 250 steel, $x_{u,\max}/d$ = 0.5313 [Table 4.3, Eq. 4.50]

  $\Rightarrow x_{u,\max} = 0.5313 \times 550$ =292.2 mm.
- Assuming $x_u \le x_{u,\max}$ , and applying the force equilibrium condition

  $$x_u = \frac{0.87 \times 250 \times 1963}{0.362 \times 20 \times 300} = 196.6 \text{ mm} < x_{u,\max} = 292.2 \text{ mm.}$$

  Therefore, $x_u$ =**196.6 mm.**

### EXAMPLE 4.11

Determine the ultimate moments of resistance for the beam sections in (a) Example 4.9 and (b) Example 4.10.

**SOLUTION**

**(a)**

- Given: $b$ = 300 mm, $d$ = 550 mm, $A_{st}$ = 1963 mm$^2$, $f_y$ = 415 MPa , $f_{ck}$ = 20 MPa.
- $x_u$ = 315 mm > $x_{u,\max}$ = 263.5 mm (from Example 4.9).
- Taking moments about the tension steel centroid,

  $$M_{uR} = 0.362 f_{ck} b x_u (d - 0.416 x_u) = 0.362 \times 20 \times 300 \times 315 \times (550 - 0.416 \times 315)$$
  $$= 286.6 \times 10^6 \text{ Nmm} = \mathbf{287\ kNm}.$$

- Note that $M_{uR}$ can also be calculated in terms of the steel tensile stress $f_{st}$, which is less than $0.87f_y$, as $x_u > x_{u,max}$. From the last cycle of iteration in Example 4.9, the value of $f_{st}$ is obtained as 349 MPa.

  $\Rightarrow M_{uR} = f_{st} A_{st} (d - 0.416 x_u)$ = **287 kNm** (as before).

  Evidently, it is easier to evaluate $M_{uR}$ in terms of the concrete strength.

**Alternative (using analysis aids)**

$$p_t = \frac{100 \times 1963}{300 \times 550} = 1.190$$

Referring to Table A.2(a) — for M 20 concrete and Fe 415 steel, for $p_t$ = 1.190,

$$\frac{M_{uR}}{bd^2} = (3.145 + 3.170)/2 = 3.158 \text{ MPa}$$

$\Rightarrow M_{uR} = 3.158 \times 300 \times 550^2 = 286.6 \times 10^6$ Nmm

= **287 kNm** (exactly as obtained earlier).

**(b)**

- Given: $b$ = 300 mm, $d$ = 550 mm, $A_{st}$ = 1963 mm$^2$, $f_y$ = 250 MPa , $f_{ck}$ = 20 MPa
- $x_u$ = 196.6 mm < $x_{u,\max}$ = 292.2 mm (from Example 4.10)
- Taking moments about the tension steel centroid,

$$M_{uR} = 0.362 \times 20 \times 300 \times 196.6 \times (550 - 0.416 \times 196.6)$$
$$= 199.9 \times 10^6 \text{ Nmm} = \mathbf{200\ kNm}.$$

- Alternatively, as $x_u < x_{u,max}$ , it follows that $f_{st} = 0.87 f_y$, and

$$M_{uR} = 0.87 f_y A_{st} (d - 0.416 x_u)$$
$$= 0.87 \times 250 \times 1963 \times (550 - 0.416 \times 196.6)$$
$$= 199.6 \times 10^6 \text{ Nmm} = \mathbf{200\ kNm}.$$

**Alternative (using analysis aids)**

$p_t$ = 1.190 (as in the previous case).

Referring to Table A.2(a) — for M 20 concrete and Fe 250 steel,

$$\frac{M_{uR}}{bd^2} = (2.188 + 2.219)/2 = 2.204 \text{ MPa}$$

$\Rightarrow M_{uR} = 2.204 \times 300 \times 550^2 = 200.0 \times 10^6$ Nmm

= **200 kNm** (exactly as obtained earlier).

### 4.7.4 Analysis of Singly Reinforced Flanged Sections

Flanged beams (T-beams and L–beams) were introduced in Section 4.6.4, where the analysis at *service loads* was discussed. The present section deals with the analysis of these beam sections at the *ultimate limit state.*

The procedure for analysing flanged beams at ultimate loads depends on whether the neutral axis is located in the flange region [Fig. 4.20(a)] or in the web region [Fig. 4.20(b)].

If the neutral axis lies within the flange (i.e., $x_u \le D_f$ ), then — as in the analysis at service loads [refer Section 4.6.4] — all the concrete on the tension side of the neutral axis is assumed ineffective, and the T-section may be analysed as a rectangular section of width $b_f$ and effective depth $d$ [Fig. 4.20(a)]. Accordingly, Eq. 4.55 and Eq. 4.59 are applicable with $b$ replaced by $b_f$ .

If the neutral axis lies in the web region (i.e., $x_u > D_f$ ), then the compressive stress is carried by the concrete in the flange and a portion of the web, as shown in Fig. 4.20(b). It is convenient to consider the contributions to the resultant compressive force $C_u$ , from the *web*† portion ($b_w \times x_u$) and the *flange* portion (width

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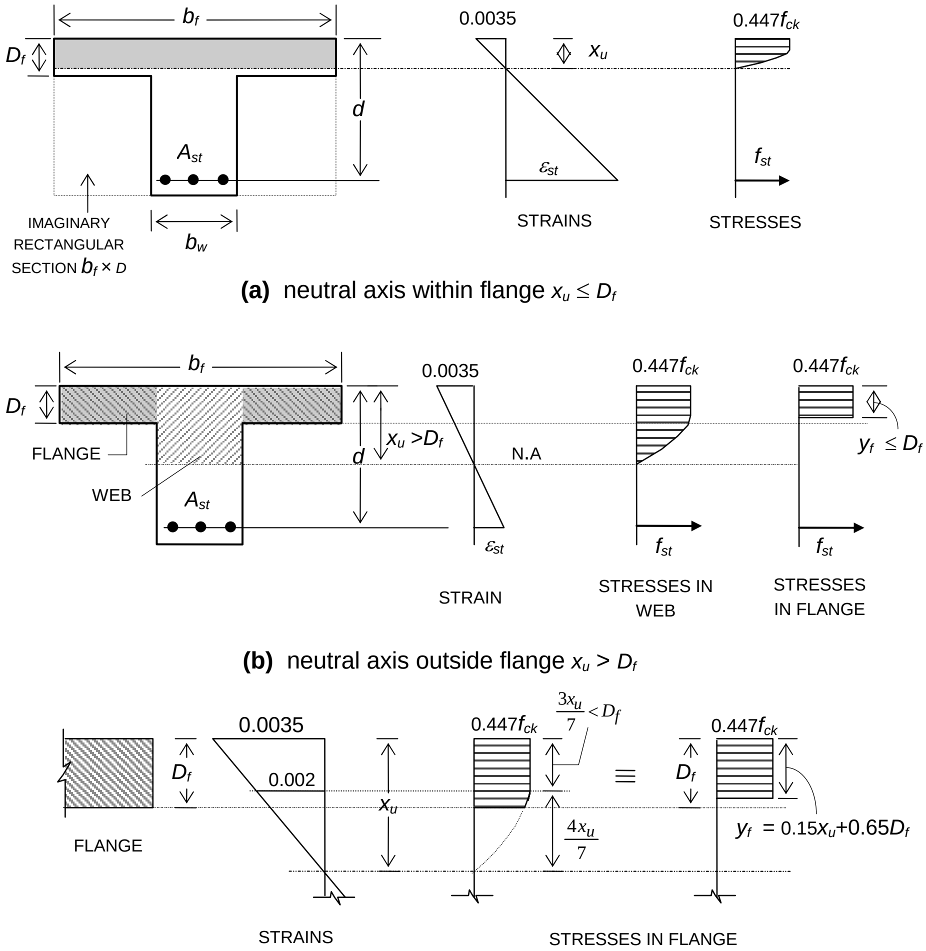
† In the computation of $C_{uw}$, the 'web' is construed to comprise the portion of the flanged beam (under compression) other than the overhanging parts of the flange.

$b_f - b_w$) separately, and to sum up these effects. Estimating the compressive force $C_{uw}$ in the 'web' and its moment contribution $M_{uw}$ is easy, as the full stress block is operative:

$$C_{uw} = 0.362 f_{ck} b_w x_u \tag{4.68a}$$

$$M_{uw} = C_{uw}\left(d - 0.416\, x_u\right) \tag{4.68b}$$

However, estimating the compressive force $C_{uf}$ in the flange is rendered difficult by the fact that the stress block for the flange portions may comprise a rectangular area plus a truncated parabolic area [Fig. 4.20(b)]. A general expression for the total area of the stress block operative in the flange, as well as an expression for the centroidal location of the stress block, is evidently not convenient to derive for such a case. However, when the stress block over the flange depth contains only a rectangular area (having a uniform stress $0.447 f_{ck}$), which occurs when $3x_u / 7 \geq D_f$ ,




**(a)** neutral axis within flange $x_u \le D_f$

**(b)** neutral axis outside flange $x_u > D_f$

**(c)** concept of equivalent flange thickness $y_f$ for ($D_f < x_u < 7/3\ D_f$)

**Fig. 4.20** Behaviour of flanged beam section at ultimate limit state

an expression for $C_{uf}$ and its moment contribution $M_{uf}$ can easily be formulated. For the case, $1 < x_u/D_f < 7/3$, an *equivalent* rectangular stress block (of area $0.447f_{ck}\,y_f$) can be conceived, for convenience, with an equivalent depth $y_f \le D_f$, as shown in Fig. 4.20(c). The expression for $y_f$ given in the Code (Cl. G – 2.2.1) is necessarily an approximation, because it cannot satisfy the *two* conditions of 'equivalence', in terms of area† of stress block as well as centroidal location. A general expression for $y_f$ may be specified for any $x_u > D_f$:

† It may be noted that the equivalence in terms of area is approximately satisfied at the limiting conditions $x_u / D_f = 1$, and exactly satisfied at $x_u / D_f = 7/3$.

$$y_f = \begin{cases} 0.15x_u + 0.65D_f & \text{for } 1 < x_u/D_f < 7/3 \\ D_f & \text{for } x_u/D_f \geq 7/3 \end{cases} \tag{4.69}$$

The expressions for $C_{uf}$ and $M_{uf}$ are accordingly obtained as:

$$C_{uf} = 0.447 f_{ck}\left(b_f - b_w\right)y_f \quad \text{for } x_u > D_f \tag{4.70a}$$

$$M_{uf} = C_{uf}\left(d - y_f/2\right) \tag{4.70b}$$

The location of the neutral axis is fixed by the force equilibrium condition (with $y_f$ expressed in terms of $x_u$ [Eq. 4.70]).

$$C_{uw} + C_{uf} = f_{st}A_{st} \tag{4.71}$$

where $f_{st}$ = 0.87 $f_y$ for $x_u \leq x_{u,max}$. Where $x_u > x_{u,max}$, the strain compatibility method has to be employed to determine $x_u$.

Substituting Eq. 4.68a and Eq. 4.70a in Eq. 4.71, and solving for $x_u$,

$$x_u = \frac{f_{st}A_{st} - 0.447 f_{ck}\left(b_f - b_w\right)y_f}{0.362 f_{ck} b_w} \quad \text{for } x_u > D_f \tag{4.72}$$

The final expression for the ultimate moment of resistance $M_{uR}$ is obtained as:

$$M_{uR} = M_{uw} + M_{uf} \tag{4.73}$$

$$\Rightarrow M_{uR} = 0.362 f_{ck} b_w x_u\left(d - 0.416x_u\right) + 0.447 f_{ck}\left(b_f - b_w\right)y_f\left(d - y_f/2\right) \tag{4.74}$$

### *Limiting Moment of Resistance*

The limiting moment of resistance $M_{u,lim}$ is obtained for the condition $x_u = x_{u,max}$, where $x_{u,max}$ takes the values of 0.531$d$, 0.479$d$ and 0.456$d$ for Fe 250, Fe 415 and Fe 500 grades of tensile steel reinforcement [refer Table 4.3]. The condition $x_u/D_f \geq 7/3$ in Eq. 4.69, for the typical case of Fe 415, works out, for $x_u = x_{u,max}$, as $0.479d/D_f \geq 7/3$, i.e., $D_f/d \leq 0.205$. The Code (Cl. G–2.2) suggests a simplified condition of $D_f/d \leq 0.2$ for *all* grades of steel — to represent the condition $x_u/D_f \geq 7/3$.

Eq. 4.74 and Eq. 4.69 take the following forms:

$$M_{u,\lim} = 0.362 f_{ck} b_w x_{u,\max}\left(d - 0.416 x_{u,\max}\right) + 0.447 f_{ck}\left(b_f - b_w\right) y_f\left(d - y_f/2\right) \quad \text{for } x_{u,max} > D_f \tag{4.75}$$

where

$$y_f = \begin{cases} 0.15x_{u,\max} + 0.65D_f & \text{for } D_f/d > 0.2 \\ D_f & \text{for } D_f/d \le 0.2 \end{cases} \quad (4.76)$$

The advantage of using Eq. 4.76 in lieu of the more exact Eq. 4.69 (with $x_u = x_{u,max}$) is that the estimation of $y_f$ is made somewhat simpler. Of course, for $x_{u,max} \le D_f$ (i.e., neutral axis within the flange),

$$M_{u,\lim} = 0.362 f_{ck} b_f x_{u,\max}\left(d - 0.416 x_{u,\max}\right) \quad \text{for } x_{u,max} \le D_f \quad (4.77)$$

As mentioned earlier, when it is found by *analysis* of a given T-section that $x_u > x_{u,max}$ , then the strain compatibility method has to be applied. As an approximate and conservative estimate, $M_{uR}$ may be taken as $M_{u,lim}$, given by Eq. 4.76 / 4.77. From the point of view of *design* (to be discussed in Chapter 5), $M_{u,lim}$ provides a measure of the ultimate moment capacity that can be expected from a T-section of given proportions. If the section has to be designed for a *factored moment* $M_u > M_{u,lim}$ , then this calls for the provision of compression reinforcement in addition to extra tension reinforcement.

**EXAMPLE 4.12**

Determine the ultimate moment of resistance for the T–section in Example 4.4

**SOLUTION**

- Given: $b_f$ = 850 mm, $D_f$ = 100 mm, $b_w$ = 250 mm, $d$ = 520 mm, $A_{st}$ = 3695 mm$^2$, $f_y$ = 250 MPa and $f_{ck}$ = 20 MPa
- $x_{u,\max}/d$ = 0.531 for Fe 250 $\Rightarrow x_{u,\max} = 0.531 \times 520$ = 276.1 mm.
- First assuming $x_u \le D_f$ and $x_u \le x_{u,\max}$, and considering force equilibrium

  $$C_u = T_u \quad \Rightarrow \quad 0.362 f_{ck} b_f x_u = 0.87 f_y A_{st}$$

  $$\Rightarrow x_u = \frac{0.87 \times 250 \times 3695}{0.362 \times 20 \times 850} = = 130.6 \text{ mm} > D_f = 100 \text{ mm.}$$

  Hence, this calculated value of $x_u$ is not correct, as $x_u > D_f$.
- As $x_u > D_f$, the compression in the 'web' is given by

  $$C_{uw} = 0.362 f_{ck} b_w x_u$$
  $$= 0.362 \times 20 \times 250 \times x_u = = (1810\, x_u) \text{ N}$$
- Assuming $x_u \ge \frac{7}{3} D_f$ = 233.3 mm, the compression in the 'flange' is given by

  $$C_{uf} = 0.447 f_{ck}\left(b_f - b_w\right) D_f$$
  $$= 0.447 \times 20 \times (850 - 250) \times 100 = 536400 \text{ N.}$$
- Also assuming $x_u \le x_{u,\max}$ = 276.1 mm,

  $$T_u = 0.87 \times 250 \times 3695 = 803662\ N.$$

- Applying the force equilibrium condition $\left(C_{uw} + C_{uf} = T_u\right)$,

  $1810\,x_u + 536400 = 803662. \Rightarrow x_u = 147.7$ mm $< \frac{7}{3}D_f = 233.3$ mm.

  Hence, this calculated value of $x_u$ is also not correct.

- As $D_f < x_u < \frac{7}{3}D_f$, the depth $y_f\ (\le D_f)$ of the equivalent concrete stress block is obtained as:

  $y_f = 0.15x_u + 0.65D_f = \left(0.15x_u + 65\right)$ mm.

  $\Rightarrow C_{uf} = 536400 \times \left(\frac{y_f}{D_f}\right) = \left(804.6x_u + 348660\right)$ N.

- $C_{uw} + C_{uf} = T_u$

  $\Rightarrow 1810x_u + \left(804.6x_u + 348660\right) = 803662.$

  $\Rightarrow x_u = \mathbf{174.0mm} < x_{u,\max}$; hence, the assumption $f_{st} = 0.87\,f_y$ is OK.

  $\Rightarrow y_f = (0.15 \times 174.0) + 65.0 = 91.1$ mm

- Taking moments of $C_{uw}$ and $C_{uf}$ about the centroid of tension steel,

  $$M_{uR} = C_{uw}\left(d - 0.416x_u\right) + C_{uf}\left(d - y_f/2\right)$$

  $$= \left(1810 \times 174.0\right) \times \left(520 - 0.416 \times 174.0\right) + \left(804.6 \times 174.0 + 348660\right) \times \left(520 - 91.1/2\right)$$

  $$= 372.8 \times 10^6\ Nmm = \mathbf{373\ kNm}.$$

**EXAMPLE 4.13**

Repeat the T–section problem in Example 4.12, considering 8 - 28ϕ bars instead of 6 - 28ϕ bars.

**SOLUTION**

- Given: $b_f = 850$ mm, $D_f = 100$ mm, $b_w = 250$ mm, $d = 520$ mm, $f_y = 250$ MPa and $f_{ck} = 20$ MPa, $A_{st} = 8 \times \frac{\pi}{4} \times (28)^2 = 4926\ \text{mm}^2$

- $x_{u,\max} = 276.1$ mm (as in Example 4.12)

- First assuming $x_u \le D_f$ and $x_u \le x_{u,\max}$,

  $$x_u = \frac{0.87 \times 250 \times 4926}{0.362 \times 20 \times 850} = 174.1\text{mm} > D_f = 100 \text{ mm.}$$

  Hence this calculated value of $x_u$ is not correct.

- As $x_u > D_f$,

  $C_{uw} = 0.362 \times 20 \times 250x_u$

  $= (1810\,x_u)$ N.

- Assuming $x_u \ge \frac{7}{3}D_f = 233.3$ mm,

$C_{uf} = 0.447 \times 20 \times (850 - 250) \times 100$
$= 536400$ N.

- Further assuming $x_u \leq x_{u,\max}$ = 276.1 mm,
  $T_u = 0.87 \times 250 \times 4926$ = 1071405 N.
- Applying the force equilibrium condition $\left(C_{uw} + C_{uf} = T_u\right)$,
  $x_u = \dfrac{1071405 - 536400}{1810}$ = **295.6** mm
  which implies $x_u > \frac{7}{3} D_f$ = 233.3 mm, but not $x_u \leq x_{u,\max}$ = 276.1 mm.

**Exact Solution (considering *strain compatibility*)**

- Corresponding to $x_u$ = 295.6 mm, $\varepsilon_{st} = 0.0035(520/295.6 - 1)$ = 0.00266 [Eq. 4.56]
  which is clearly greater than the strain at yield for Fe 250,
  i.e., $0.87 \times 250/(2.0 \times 10^5)$ = 0.00109.
  Hence, the design steel stress is indeed $f_{st}$ = $0.87 f_y$, and the so calculated $x_u$ = 295.6 mm is the correct depth of the neutral axis† .
  Accordingly,
  $$\begin{aligned} M_{uR} &= C_{uw}(d - 0.416 x_u) + C_{uf}(d - D_f/2) \\ &= (1810 \times 295.6) \times (520 - 0.416 \times 295.6) + 536400 \times (520 - 50) \\ &= 464.5 \times 10^6 \text{ Nmm} = \mathbf{465\ kNm} > M_{u,lim} \end{aligned}$$

This is the correct estimate of the ultimate moment capacity of the section; as the steel strain is beyond the yield strain a limited amount of ductile behaviour can also be expected. However, as per the Code, this will not qualify as an admissible under-reinforced section since $x_u > x_{u,max}$. [Note that if the $\varepsilon_{st}$ computed had turned out to be less than $\varepsilon_y$, $f_{st} < 0.87 f_y$ and a trial-and-error procedure has to be resorted to.]

**Approximate Solution**

- An approximate and conservative solution for $M_{uR}$ can be obtained by limiting $x_u$ to $x_{u,\max}$ = 276.1 mm, and taking moments of $C_{uw}$ and $C_{uf}$ about the centroid of the tension steel (Note that, following the Code procedure, $D_f/d = 100/520 = 0.192 < 0.2 \Rightarrow y_f = D_f$ = 100 mm [Eq. 4.76] ). . Accordingly,

$$\begin{aligned} M_{uR} \approx M_{u,lim} &= C_{uw}\left(d - 0.416 x_{u,max}\right) + C_{uf}\left(d - D_f/2\right) \\ &= (1810 \times 276.1) \times (520 - 0.416 \times 276.1) + 536400 \times (520 - 50) \end{aligned}$$

---

† This is a case where, being Fe 250 grade steel with a sharp yield point, the strain at first yield, $\varepsilon_y = f_y/E_s$, is lower than the strain for the 'balanced' condition $\varepsilon^*_{st}$ specified by the Code. Hence, even though $x_u > x_{u,max}$, the steel has yielded. See also footnote on p. 136 and Fig. 4.19(b).

$$= 454.6 \times 10^{6}\ Nmm \ = \mathbf{455\ kNm}.$$

### 4.7.5 Analysis of Doubly Reinforced Sections

Doubly reinforced beam sections (i.e., sections with compression steel as well as tension steel) were introduced in Section 4.6.5, where the analysis at *service loads* was discussed. The present section deals with the analysis of these beam sections (rectangular) at the *ultimate limit state.*

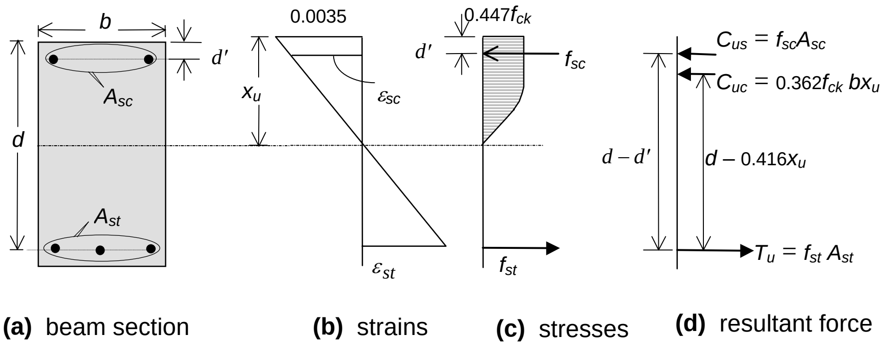



**Fig. 4.21** Behaviour of doubly reinforced rectangular section at ultimate limit state

The distributions of stresses and strains in a 'doubly reinforced' rectangular section [Fig. 4.21] are similar to those obtained in a 'singly reinforced' section [Fig. 4.17], except that there is a stress $f_{sc}$ in the compression steel (area $A_{sc}$) which also needs to be accounted for. This stress $f_{sc}$ may or may not reach the design yield stress 0.87$f_y$, depending on the strain $\varepsilon_{sc}$ in the compression steel. An expression for $\varepsilon_{sc}$ can be easily obtained from strain compatibility [Fig. 4.21(b)]:

$$\varepsilon_{sc} = 0.0035 \times \left(1 - d'/x_u\right) \tag{4.78}$$

where $d'$ is the distance between the centroid of the compression steel and the extreme compression fibre in the concrete. In practice, the ratio $d'/d$ is found to vary in the range 0.05 to 0.20. It can be shown that the compression steel will, in most cases, attain the design yield stress ($f_{sc}$ = 0.87$f_y$) in the case of Fe 250 grade steel, but is generally unlikely to do so in the case of Fe 415 and Fe 500 (because of their higher strains at yield). Values of the stress $f_{sc}$ (corresponding to $x_u = x_{u.max}$) for various grades of steel and ratios of $d'/d$ are listed in Table 4.5.

**Table 4.5** Value of $f_{sc}$ (in MPa units) at $x_u = x_{u,max}$ — for various $d'/d$ ratios and different grades of compression steel

| Grade of steel | $d'/d$ | | | |
|---|---|---|---|---|
| | **0.05** | **0.10** | **0.15** | **0.20** |
| **Fe 250** | 217.5 | 217.5 | 217.5 | 217.5 |
| **Fe 415** | 355.1 | 351.9 | 342.4 | 329.2 |
| **Fe 500** | 423.9 | 411.3 | 395.1 | 370.3 |

Applying the condition of force equilibrium [Fig. 4.21(d)]

$$C_{uc} + C_{us} = T_u \tag{4.79}$$

where $C_{uc}$ and $C_{us}$ denote, respectively, the resultant compressive forces in the concrete and the compression steel. For convenience, the full area of the concrete under compression ( $b \times x_u$) is assumed to be effective in estimating $C_{uc}$. The force in concrete area displaced by steel (equal to $A_{sc}$, stressed to a level that is exactly or nearly equal to 0.447 $f_{ck}$, the stress in concrete) already included in $C_{uc}$, is accounted for in the estimation of $C_{us}$ as follows:

$$C_{uc} = 0.362 f_{ck} b x_u \tag{4.80a}$$

$$C_{us} = \left(f_{sc} - 0.447 f_{ck}\right) A_{sc} \tag{4.80b}$$

$T_u = f_{st} A_{st}$, where $f_{st} = 0.87 f_y$ if $x_u \le x_{u,max}$. Accordingly, the depth of the neutral axis $x_u$ is obtainable from Eq. 4.79 as:

$$x_u = \frac{f_{st} A_{st} - \left(f_{sc} - 0.447 f_{ck}\right) A_{sc}}{0.362 f_{ck} b} \tag{4.81}$$

This equation provides a closed-form solution to $x_u$ only if $f_{st} = 0.87 f_y$ and $f_{sc} = 0.87 f_y$; otherwise, $f_{st}$ and $f_{sc}$ will depend on $x_u$. Initially the values of $f_{sc}$ and $f_{st}$ may be taken as $0.87 f_y$, and then revised, if necessary, employing the *strain compability* method. This is demonstrated in Example 4.15.

Having determined $f_{sc}$ and $x_u$ , the ultimate moment of resistance can be calculated by considering moments of $C_{uc}$ and $C_{us}$ about the centroid of the tension steel [Fig. 4.21(d)] as follows:

$$M_{uR} = C_{uc}\left(d - 0.416 x_u\right) + C_{us}\left(d - d'\right) \tag{4.82}$$

### *Limiting Moment of Resistance*

The 'limiting' value of $M_{uR}$, obtained for the condition $x_u = x_{u,max}$, is given by the following expression :

$$M_{u,\lim} = 0.362 f_{ck} b x_{u,\max}\left(d - 0.416 x_{u,\max}\right) + \left(f_{sc} - 0.447 f_{ck}\right) A_{sc}\left(d - d'\right) \tag{4.83}$$

where the value of $f_{sc}$ depends on $\varepsilon_{sc}$ (obtainable from Eq. 4.78 and Table 3.2). For convenience, the values of the stress $f_{sc}$ (corresponding to $x_u = x_{u,max}$) for various grades of steel and ratios of $d'/d$ are listed in Table 4.5. Linear interpolation may be used to determine $f_{sc}$ for any value of $d'/d$ other than the tabulated constants.

### EXAMPLE 4.14

Determine the ultimate moment of resistance of the doubly reinforced beam section of Example 4.6.

**SOLUTION**

- Given : $b$ = 300 mm, $d$ = 550 mm, $A_{st}$ = 3054 mm$^2$, $f_y$ = 250 MPa and $f_{ck}$ = 20 MPa, $d'$ = 50 mm, $A_{sc}$ = 982 mm$^2$
- $x_{u,max}/d = 0.531$ for Fe 250 $\Rightarrow$ $x_{u,max} = 0.531 \times 550$ = 292.1 mm.
- Assuming $f_{sc} = f_{st} = 0.87\, f_y$, and considering force equilibrium :

  $C_{uc} + C_{us} = T_u$, with
  $C_{uc} = 0.362 \times 20 \times 300 \times x_u = (2172 x_u)$ N
  $C_{us} = (0.87 \times 250 - 0.447 \times 20) \times 982$ =204806 N
  $T_u = 0.87 \times 250 \times 3054$ = 664 245$N$
  $\Rightarrow 2172\, x_u + 204806 = 664\,245$
  $\Rightarrow$ $x_u$ = **211.5 mm** < $x_{u,max}$ = 292.1 mm.
  Hence, the assumption $f_{st} = 0.87 f_y$ is justified.
- Also, $\varepsilon_{sc} = 0.0035(1 - 50/211.5) = 0.00267 > \varepsilon_y = \dfrac{0.87 \times 250}{2 \times 10^5} = 0.00109$

  $\Rightarrow$ $f_{sc} = 0.87\, f_y$ is also justified.
- *Ultimate moment of resistance*

  $M_{uR} = C_{uc}(d - 0.416 x_u) + C_{us} \times (d - d')$
  $= (2172 \times 211.5) \times (550 - 0.416 \times 211.5) + 204806 \times (550 - 50)$
  $= 314.6 \times 10^6$ Nmm = **315 kNm**.

### EXAMPLE 4.15

Repeat the problem in Example 4.14, considering Fe 415 instead of Fe 250.

**SOLUTION**

- Given : $b$ = 300 mm, $d$ = 550 mm, $A_{st}$ = 3054 mm$^2$, $f_y$ = 415 MPa and $f_{ck}$ = 20 MPa, $d'$ = 50 mm, $A_{sc}$ = 982 mm$^2$
- $x_{u,\max}/d$ = 0.479 for Fe 415 $\Rightarrow$ $x_{umax}$ = 0.479 ×550 = 263.5 mm
- Assuming $f_{sc} = f_{st} = 0.87 \times f_y$, and considering force equilibrium [Eq. 4.81],

$$x_u = \frac{(0.87 \times 415 \times 3054) - (0.87 \times 415 - 0.447 \times 20) \times 982}{0.362 \times 20 \times 300}$$

= 348.5 mm > $x_{u,\max} = 263.5$ mm.

Evidently, the section is over-reinforced.

**Exact Solution (considering *strain compatibility*) :**

- Applying [Eq. 4.81] : $x_u = \dfrac{f_{st} \times 3054 - (f_{sc} - 0.447 \times 20) \times 982}{0.362 \times 20 \times 300}$

  $\Rightarrow x_u = (3054 f_{st} - 982 f_{sc} + 8779)/2172.$

  *First Cycle* :

  1) Evidently, 263.5 mm < $x_u$ < 348.5 mm.
  2) Assume $x_u \approx \frac{1}{2}$ (263.5 + 348.5) = 306 mm.
  3) [Eq. 4.78] $\Rightarrow$ $\varepsilon_{sc}$ = $0.0035(1-50/306)$ = 0.00293
  4) [Eq. 4.56] $\Rightarrow$ $\varepsilon_{st}$ = $0.0035(550/306-1)$ = 0.00279
  5) [Table 3.2] $\Rightarrow$ $f_{sc} = 351.8 + (360.9 - 351.8) \times (293 - 276)/(380 - 276)$
     = 353.3 MPa
     and $f_{st} = 351.8 + (360.9 - 351.8) \times (279 - 276)/(380 - 276)$
     = 352.1 MPa
  6) $\Rightarrow x_u = (3054 \times 352.1 - 982 \times 353.3 + 8779)/2172$ = 339.3 mm.

  *Second Cycle* :

  1) Assume $x_u \approx \frac{1}{2}(306 + 339)$ = 323 mm
  2) [Eq. 4.78] $\Rightarrow$ $\varepsilon_{sc}$ = 0.00296
  3) [Eq. 4.56] $\Rightarrow$ $\varepsilon_{st}$ = 0.00246
  4) [Table 3.2] $\Rightarrow$ $f_{sc}$ =353.5 MPa (converged, insensitive to changes in $x_u$)
     and $f_{st}$ = 344.1 MPa
  5) $\Rightarrow x_u = (3054 \times 344.1 - 982 \times 353.5 + 8779)/2172$ = 328.0 mm.

  *Third Cycle* :

  1) $x_u \approx \frac{1}{2}$ (323 + 328) = 325.5 mm
  2) [Eq. 4.56] $\Rightarrow$ $\varepsilon_{st}$ = 0.00241
  3) [Table 3.2] $\Rightarrow$ $f_{st}$ = 342.8 MPa
  4) $\Rightarrow x_u = (3054 \times 342.8 - 982 \times 353.5 + 8779)/2172$ = 326.2 mm (converged).

- Taking $x_u$ = 326 mm, and applying Eq. 4.82,

  $M_{uR}$ = (0.362×20×300×326)×(550–0.416×326) +

(353.5–0.447×20)×982× (550–50)

= $462.6\times10^6$ Nmm = **463 kNm**

(Note : this moment is associated with *brittle* failure).

**Approximate Solution**

- As an approximate and conservative estimate, limiting $x_u$ to $x_{u,max}$ = 263.5 mm,
  $\varepsilon_{sc} = 0.0035(1 - 50/263.5) = 0.00284$
  $\Rightarrow f_{sc}$ = 352.5 MPa [Table 3.2].
  [This value is alternatively obtainable from Table 4.5 for $d'/d = 0.09$ and Fe 415.]
  Accordingly, limiting the ultimate moment of resistance $M_{uR}$ to the 'limiting moment' $M_{u,lim}$ [Eq. 4.83],

  $M_{u,lim}$ = 0.362 × 20 × 300 × 263.5 × (550–0.416 × 263.5)
  + (352.5–0.447 × 20) × 982 × (550–50) = $420.7\times10^6$ Nmm = **421 kNm**.

**EXAMPLE 4.16**

Determine the ultimate moment of resistance of the doubly reinforced section shown in Fig. 4.22. Assume M 20 concrete and Fe 415 steel.

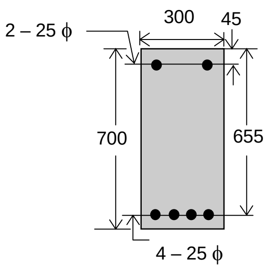



**Fig. 4.22** Example 4.16

**SOLUTION**

- Given : $b$ = 300 mm, $d$ = 655 mm, $d'$ = 45 mm, $f_y$ = 415 MPa and $f_{ck}$ = 20 MPa
  $A_{sc} = \pi(25)^2/4 \times 2$ = 491 × 2 =982 mm², $A_{st}$ = 491 × 4 = 1964 mm²
- $x_{u,\max}/d$ = 0.479 for Fe 415 $\Rightarrow$ $x_{u,max}$ = 0.479 × 655 = 313.7 mm
- Assuming (for a first approximation) $f_{sc} = f_{st} = 0.87f_y$ ,
  $C_{uc}$ = 0.362 × 20 × 300 × $x_u$ = $(2172x_u)$N
  $C_{us}$ = (0.87 × 415 – 0.447 × 20) × 982 = 345772$N$
  $T_u$ = 0.87 × 415 × 1964 = 709102$N$
- Considering force equilibrium : $C_{uc} + C_{us} = T_u$,

$2172x_u + 345772 = 709102$
$\Rightarrow\ x_u = 167.3$ mm < $x_{u,max} = 313.7$ mm

- Evidently, the assumption $f_{st} = 0.87f_y$ is justified
Further, $\varepsilon_{sc} = 0.0035(1 - 45/167.3) = 0.00256$
For Fe 415, $\varepsilon_y = \dfrac{0.87 \times 415}{2 \times 10^5} + 0.002 = 0.0038$
As $\varepsilon_{sc} < \varepsilon_y$, the assumption $f_{sc} = 0.87f_y$ is not justified, whereby the calculated value of $C_{us}$ (and hence of $x_u = 167.3$ mm) is also not correct. The correct value has to be obtained iteratively using *strain compatibility*.

*First cycle* :

- Assuming $\varepsilon_{sc} = 0.00256$,
$f_{sc} = 342.8 + (351.8 - 342.8) \times \dfrac{256 - 241}{276 - 241} = 346.7$ MPa
$\Rightarrow\ C_{us} = (346.7 - 0.447 \times 20) \times 982 = 331680$ N
$C_{uc} + C_{us} = T_u \Rightarrow\ x_u = \dfrac{709102 - 331680}{2172} = 173.8$ mm
$\Rightarrow\ \varepsilon_{sc} = 0.0035(1 - 45/173.8) = 0.00259 \approx 0.00256$ (calculated earlier)

*Second cycle* :

- Assuming $\varepsilon_{sc} = 0.00259$,
$f_{sc} = 342.8 + (351.8 - 342.8) \times \dfrac{259 - 241}{276 - 241} = 347.4$ MPa
$\Rightarrow\ C_{us} = (347.4 - 0.447 \times 20) \times 982 = 332368$ N
$\Rightarrow\ x_u = \dfrac{709102 - 332368}{2172} = 173.4$ mm (converged)
- Taking $x_u = 173.4$ mm,
$M_{uR} = C_{uc}(\mathrm{d} - 0.416x_u) + C_{us}(d - d')$
$= (2172 \times 173.4)(655 - 0.416 \times 173.4) + 332368\,(655 - 45)$
$= 422.3 \times 10^6$ Nmm = **422 kNm**

### 4.7.6 Balanced Doubly Reinforced Sections

As explained earlier, 'over-reinforced' sections are undesirable, both from the Code viewpoint of lack of ductile failure, as well as the practical viewpoint of loss of economy. Hence, from a *design* viewpoint it is necessary to restrict the depth of the neutral axis to the limit prescribed by the Code [Eq. 4.50].

In a *singly reinforced* rectangular beam, the requirement $x_u \le x_{u,max}$ can be ensured by limiting the tension reinforcement percentage $p_t \le p_{t,lim}$ where $p_{t,lim}$ (given by Eq. 4.62) corresponds to the 'balanced' condition $x_u = x_{u,max}$. If $p_t \le p_{t,lim}$, and yet compression reinforcement is provided (i.e., the beam is 'doubly reinforced'), then evidently the condition $x_u < x_{u,max}$ is satisfied.

If the section is *doubly reinforced* with $p_t > p_{t,lim}$ and $p_c > 0$, then the requirement $x_u \leq x_{u,max}$, can be ensured by restricting $p_t - p_{t,lim}$ to a value commensurate with the percentage compression reinforcement ($p_c = 100A_{sc}/bd$) provided. Alternatively, this can be ensured by providing adequate compression steel ($p_c$) for a given $p_t > p_{t,lim}$.

It is convenient to visualise $p_t$ as comprising a component $p_{t,lim}$ [Eq. 4.62] and another component ($p_t - p_{t,lim}$); the tensile force in the former is visualised as being balanced by the compressive force in the concrete $C_{uc} = 0.362\, f_{ck}\, b\, x_{u,max}$, and in the latter by the compressive force in the compression steel $C_{us}$ alone. Accordingly, considering force equilibrium in the latter parts, and denoting the value of $p_c$ for 'balanced' section a $p_c^*$:

$$0.87 f_y \frac{(p_t - p_{t,\lim})bd}{100} = (f_{sc} - 0.447 f_{ck}) \frac{p_c^*}{100} bd$$

$$\Rightarrow p_c^* = \frac{0.870 f_y}{f_{sc} - 0.447 f_{ck}} \left(p_t - p_{t,lim}\right) \qquad \text{for } x_u = x_{u,max} \quad (4.84)$$

where $f_{sc}$ is obtainable from Table 4.5.

It also follows that if the actual $p_c$ provided in a beam section exceeds $p_c^*$ (given by Eq. 4.84), then $x_u < x_{u,max}$, and hence the beam is 'under-reinforced'. On the other hand, if $p_c < p_c^*$, then the beam is 'over-reinforced'. For example, in the beam of Example 4.14, $p_c^*$ works out to 0.547, whereas the $p_c$ provided is 0.595 > $p_c^*$; hence, the beam is 'under-reinforced'. However, in Example 4.15, $p_c^*$ works out to 0.914 while $p_c$ provided remains at 0.595; hence, $p_c < p_c^*$ and the beam is 'over-reinforced'.

In the case of a 'balanced' section an expression for $M_{uR} = M_{u,lim-DR}$ can be derived in terms of the percentage tensile steel ($p_t$) as follows:

$$M_{u,lim-DR} = (0.87 f_y) \left[ \frac{p_{t,\lim} bd}{100} (d - 0.416 x_{u,\max}) + \frac{(p_t - p_{t,\lim})bd}{100} (d - d') \right]$$

$$\Rightarrow \frac{M_{u,\lim-DR}}{bd^2} = (0.87 f_y) \left[ \frac{p_{t,\lim}}{100} \left(1 - 0.416 \frac{x_{u,\max}}{d}\right) + \frac{(p_t - p_{t,\lim})}{100} \left(1 - d'/d\right) \right] \quad (4.85)$$

where the additional subscript 'DR' (for *doubly reinforced* section) is inserted — to avoid confusion with the $M_{u,lim}$ defined earlier for the *singly reinforced* section. The corresponding percentage compression steel $p_c = p_c^*$ is as given by Eq. 4.84.

## 4.8 ANALYSIS OF SLABS AS RECTANGULAR BEAMS

Slabs under flexure behave in much the same way as beams. A slab of uniform thickness subject to a bending moment uniformly distributed over its width [Fig. 4.23] may be treated as a wide shallow beam for the purpose of analysis and design.

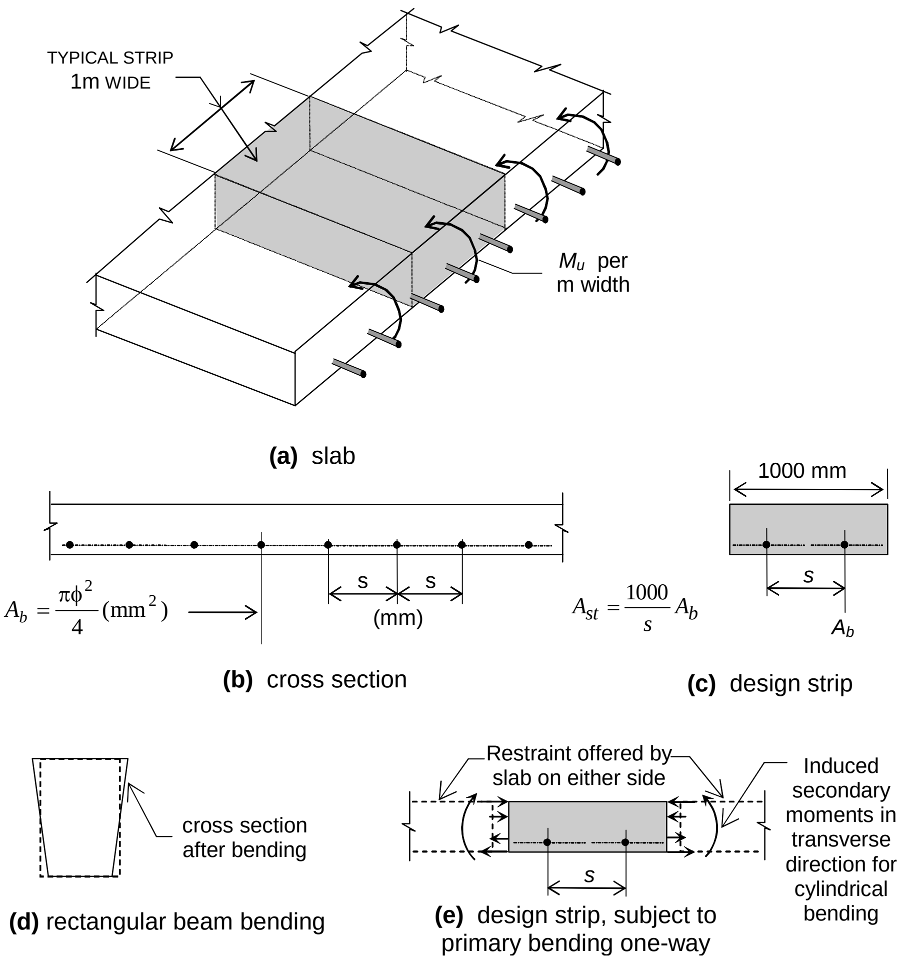



**Fig. 4.23** Analysis of slabs

In such slabs, the reinforcing bars are usually spaced uniformly over the width of the slab. For convenience, computations are generally based on a typical one-*metre* wide strip of the slab considered as a beam [Fig. 4.23(c)], i.e., with $b = 1000$ mm. The loads are generally uniformly distributed[†] and expressed in units of $kN/m^2$.

If $s$ is the centre-to-centre spacing of bars in mm, then the number of bars in the 1-metre wide strip is given by $1000/s$. Accordingly, denoting $A_b$ as the cross-sectional area of one bar (equal to $\pi\phi^2/4$), the area of tensile steel ($A_{st}$), expressed in units of $mm^2/m$, is given by

$$A_{st} = 1000\ A_b / s \tag{4.86}$$

In practice, reinforced concrete slabs are generally under-reinforced and singly reinforced. In the example to follow, the analysis of a typical slab is undertaken to determine the moment resisting capacity at *working* loads as well as at the *ultimate limit state.*

### 4.8.1 Transverse Moments in One-way Slabs

Although a one-metre wide strip of the slab is considered as a beam of width $b = 1000$ mm for the analysis/design for flexural strength, there is a difference which the student will do well to bear in mind. As a beam bends (sags), the portion of the section above the neutral axis is under compression and hence subjected to a lateral expansion due to the Poisson effect. Similarly, the part below the NA is subjected to a lateral contraction. Hence, after bending, the cross section will strictly not be rectangular, but nearly[‡] trapezoidal, as shown (greatly exaggerated) in Fig. 4.23(d). In the case of a one-way slab, for a design strip such as shown in Fig. 4.23(c, e), such lateral displacements (and hence strains) are prevented by the remainder of the slab on either side (except at the two edges). In other words, in order for the rectangular section to remain rectangular even after bending (as a slice of a long cylindrically bent surface, with no transverse curvature, should be), the remainder of the slab restrains the lateral displacements and strains, by inducing lateral stresses on the design strip as shown in Fig. 4.23(e). This is known as the 'plain strain' condition [Ref. 4.1]. These lateral stresses give rise to secondary moments in the transverse direction as shown in Fig. 4.23(e).

Hence, even a one-way slab will need ('secondary') reinforcements in the transverse direction to resist these secondary moments. Furthermore, bending moments in the transverse direction are generated locally when the slab is subject to concentrated loads. Also, shrinkage and temperature effects introduce secondary stresses which require transverse reinforcement.

[†] When concentrated loads act on a one-way slab, the simplified procedure given in Cl. 24.3.2 of the Code may be adopted.

[‡] To be exact, just as the beam undergoes a 'sagging' curvature along the span, there will be a 'hogging' ('anticlastic') curvature in the transverse direction. Thus the top surface will be curved rather than straight [see Ref. 4.1].

### EXAMPLE 4.17

Determine (a) the allowable moment (at service loads) and (b) the ultimate moment of resistance of a 150 mm thick slab, reinforced with 10 mm $\phi$ bars at 200 mm spacing located at an effective depth of 125 mm. Assume M 20 concrete and Fe 415 steel.

### SOLUTION

- Given : $d$ = 125 mm, $f_y$ =415 MPa and $f_{ck}$ = 20 MPa, and

$$A_{st} = \frac{1000\left(\pi \times 10^2/4\right)}{200} = 393 \text{ mm}^2\text{/m.}$$

**a)** ***Analysis at working loads*** **:**

For M 20 concrete, $\sigma_{cbc}$ = 7.0 MPa and $m$ = 13.33.

For Fe 415 steel, $\sigma_{st}$ = 230 MPa and $k_b = \dfrac{280}{280 + 3\sigma_{st}}$ = 0.289.

The neutral axis depth $kd$ is obtained by considering moments of areas in the transformed–cracked section [Eq. 4.12], and considering $b$ = 1000 mm

$1000 \times (kd)^2/2$ = 13.33 × 393 × (125 – $kd$)

Solving, $kd$ = 31.33 mm < $k_b d$ = 0.289 × 125 = 36.1 mm

Hence, the section is 'under–reinforced (WSM)'.

$\Rightarrow$ $f_{st} = \sigma_{st}$ = 230 MPa

$$M_{all} = \sigma_{st} A_{st}\left(d - kd/3\right)$$

= 230 × 393 × (125 – 31.33/3)

= 10.35 × $10^6$ N mm/m = **10.4 kNm/ m**.

**b)** ***Analysis at ultimate limit state***

For Fe 415 steel, $x_{u,max} = 0.479 \times 125$ = 59.9 mm

- Assuming $x_u \le x_{u,max}$, and considering $C_u = T_u$ ,

$$x_u = \frac{0.87 f_y A_{st}}{0.362 f_{ck} b} = \frac{0.87 \times 415 \times 393}{0.362 \times 20 \times 1000} = 19.60 \text{ mm} < x_{u,max}$$

Accordingly,

$$M_{uR} = 0.362 f_{ck}\ b x_u\left(d - 0.416 x_u\right)$$

$= 0.362 \times 20 \times 1000 \times 19.60 \times \left(125 - 0.416 \times 19.60\right)$

= 16.58 × $10^6$ N mm/m = **16.6 kNm/m**

- Alternatively,

$$p_t = \frac{100 \times 393}{1000 \times 125} = 0.314 < p_{t,lim} = 0.961 \text{ [Table 4.4]}$$

Applying Eq. 4.65, or using analysis aids [Table A.2(a)],

$$\frac{M_{uR}}{bd^2} = 0.87 \times 415 \times \frac{0.314}{100} \times \left(1 - \frac{415}{20} \times \frac{0.314}{100}\right) = 1.060 \text{ MPa}$$

$$\Rightarrow M_{uR} = 1.060 \times 1000 \times 125^2 = 16.56 \times 10^6 \text{ Nmm/m} = \mathbf{16.6\ kNm/m}.$$

## REVIEW QUESTIONS

4.1 What is the fundamental assumption in flexural theory? Is it valid at the ultimate state?

4.2 Explain the concept of 'transformed section', as applied to the analysis of reinforced concrete beams under service loads.

4.3 Why does the Code specify an effectively higher modular ratio for compression reinforcement, as compared to tension reinforcement?

4.4 Justify the assumption that concrete resists no flexural tensile stress in reinforced concrete beams.

4.5 Describe the moment-curvature relationship for reinforced concrete beams. What are the possible modes of failure?

4.6 The term 'balanced section' is used in both *working stress method* (WSM) and *limit state method* (LSM). Discuss the difference in meaning.

4.7 Why is it undesirable to design over-reinforced sections in (a) WSM, (b) LSM?

4.8 The concept of locating the *neutral axis* as a *centroidal axis* (in a reinforced concrete beam section under flexure) is applied in WSM, but not in LSM. Why?

4.9 Why is it uneconomical to use high strength steel as compression reinforcement in design by WSM?

4.10 Justify the Code specification for the limiting neutral axis depth in LSM.

4.11 "The ultimate moment of resistance of a singly reinforced beam section can be calculated either in terms of the concrete compressive strength or the steel tensile strength". Is this statement justified in all cases?

4.12 Compute and plot the ratio $\frac{M_{uR}}{M_{all}}$ for a given singly reinforced beam section for values of $p_t$ in the range 0.0 to 2.0, considering combinations of (i) M 20 and Fe 250 and (ii) M 25 and Fe 415. (Refer Figs 4.13 and 4.19). Comment on the graphs generated, in terms of the safety underlying beam sections that are designed in accordance with WSM.

4.13 Define "effective flange width".

4.14 What are the various factors that influence the effective flange width in a T-beam? To what extent are these factors accommodated in the empirical formula given in the Code?

4.15 Is it correct to model the interior beams in a continuous beam-supported slab system as T-beams for determining their flexural strength at *all* sections?

4.16 Discuss the variation of the ultimate moment of resistance of a singly reinforced beam of given rectangular cross-section and material properties with the area of tension steel.

4.17 Explain how the neutral axis is located in T-beam sections (at the ultimate limit state), given that it lies outside the flange.

4.18 Given percentages of tension steel ($p_t$) and compression steel ($p_c$) of a doubly reinforced section, how is it possible to decide whether the beam is under-reinforced or over-reinforced (at the ultimate limit state)?

4.19 Show that the procedure for analysing the flexural strength of reinforced concrete slabs is similar to that of beams.

4.20 What are the significant differences between the behaviour in bending of a beam of rectangular section and a strip of a very wide one-way slab?

4.21 Why is it necessary to provide transverse reinforcement in a one-way slab?

4.22 "A reinforced concrete beam can be considered to be *safe* in flexure if its ultimate moment of resistance (as per Code) at any section exceeds the factored moment due to the loads at that section". Explain the meaning of *safety* as implied in this statement. Does the Code call for any additional requirement to be satisfied for 'safety'?

4.23 If a balanced singly reinforced beam section is experimentally tested to failure, what is the ratio of actual moment capacity to predicted capacity (as per Code) likely to be? (Hint: to estimate actual strength, no safety factors should be applied; also, there is no effect of sustained loading).

## PROBLEMS

4.1 A beam has a rectangular section as shown in Fig. 4.24. Assuming M 20 concrete and Fe 250 steel,

(a) compute the stresses in concrete and steel under a service load moment of 125 kNm. Check the calculations using the flexure formula.

[*Ans.* : 4.84 MPa; 499.0 MPa]

(b) determine the *allowable moment* capacity of the section under service loads. Also determine the corresponding stresses induced in concrete and steel.

[*Ans.* : 164 kNm; 6.35 MPa; 130 MPa]

4.2 Determine the *allowable moment* capacity of the beam section [Fig. 4.24] of Problem 4.1, as well as the corresponding stresses in concrete and steel (under service loads), considering

(i) M 20 concrete and Fe 415 steel;

[*Ans.* : 181 kNm; 7.00 MPa; 143 MPa]

(ii) M 25 concrete and Fe 250 steel.

[*Ans.* : 164 kNm; 6.35 MPa; 130 MPa]

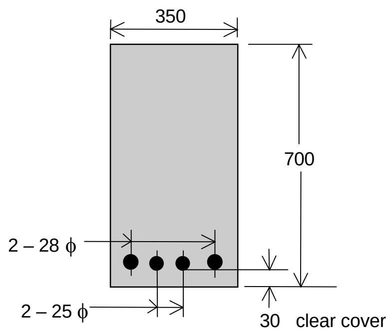



**Fig. 4.24** Problems 4.1 – 4.3

4.3 Determine the *ultimate* moment of resistance of the beam section [Fig. 4.24] of Problem 4.1, considering

(i) M 20 concrete and Fe 250 steel;

[*Ans.* : 278 kNm]

(ii) M 20 concrete and Fe 415 steel;

[*Ans.* : 420 kNm]

(iii) M 25 concrete and Fe 250 steel;

[*Ans.* : 285 kNm]

(iv) M 25 concrete and Fe 415 steel.

[*Ans.* : 440 kNm]

Compare the various results, and state whether or not, in each case, the beam section complies with the Code requirements for flexure.

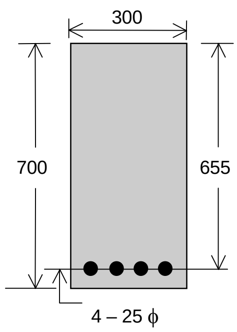



**Fig. 4.25** Problems 4.4 – 4.5

4.4 A beam carries a uniformly distributed service load (including self-weight) of 38 kN/m on a simply supported span of 7.0 m. The cross-section of the beam is shown in Fig. 4.25. Assuming M 20 concrete and Fe 415 steel, compute

(a) the stresses developed in concrete and steel at applied service loads;

[*Ans.* : 10.4 MPa; 209 MPa]

(b) the *allowable* service load (in kN/m) that the beam can carry (as per the Code). [*Ans.* : 25.5 kN/m]

4.5 Determine the ultimate moment of resistance of the beam section [Fig. 4.25] of Problem 4.4. Hence, compute the effective *load factor* (i.e., ultimate load/service load), considering the service load of 38 kN/m cited in Problem 4.4.

[*Ans.* : 366 kNm; 1.57]

4.6 The cross-sectional dimensions of a T- beam are given in Fig. 4.26. Assuming M 20 concrete and Fe 415 steel, compute :

(a) the stresses in concrete and steel under a service load moment of 150 kNm;

[*Ans.* : 4.30 MPa; 92.9 MPa]

(b) the *allowable* moment capacity of the section at service loads.

[*Ans.* : 244 kNm]

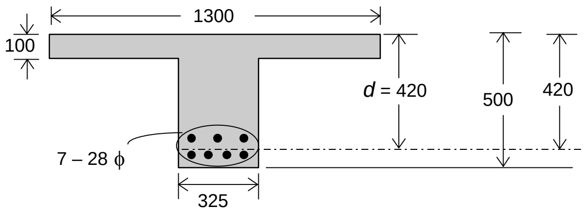



**Fig. 4.26** Problems 4.6 – 4.7

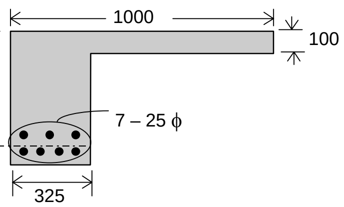



**Fig. 4.27** Problems 4.8 – 4.9

4.7 Determine the ultimate moment of resistance of the T - beam section [Fig. 4.26] of Problem 4.6.

[*Ans.* : 509 kNm]

4.8 Assuming M 25 concrete and Fe 415 steel, compute the ultimate moment of resistance of the L - beam section shown in Fig. 4.27.

[*Ans.* : 447 kNm]

4.9 Determine the ultimate moment of resistance of the L- section [Fig. 4.27] of Problem 4.8, considering Fe 250 grade steel (in lieu of Fe 415).

[*Ans.* : 288 kNm]

4.10 A doubly reinforced beam section is shown in Fig. 4.28. Assuming M 20 concrete and Fe 415 steel, compute

(a) the stresses in concrete and steel under a service load moment of 125 kNm;

[*Ans.* : 11.7 MPa; 170 MPa; 218 MPa]

(b) the allowable service load moment capacity of section.

[*Ans.* : 74.6 kNm]

4.11 Determine the ultimate moment of resistance of the beam section [Fig. 4.28] of Problem 4.10.

[*Ans.* : 201 kNm]

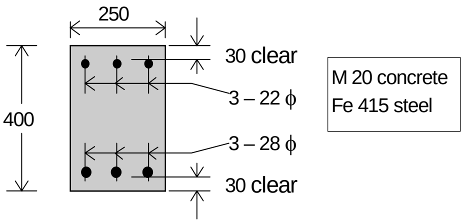



**Fig. 4.28** Problems 4.10 – 4.12

4.12 Repeat Problem 4.11, considering the compression bars to comprise 3 – 20 ϕ (instead of 3 – 22 ϕ, as shown in Fig. 4.28).

[*Ans.* : 196 kNm]

4.13 Determine (a) the allowable moment (at service loads) and (b) the ultimate moment of resistance of a 100 mm thick slab, reinforced with 8 mm ϕ bars at 200 mm spacing located at an effective depth of 75 mm. Assume M 20 concrete and Fe 415 steel.

[*Ans.* : (a) 4.21 kN/m;
(b) 6.33 kN/m]

4.14 A simply supported *one–way* slab has an effective span of 3.5 metres. It is 150 mm thick, and is reinforced with 10 mm ϕ bars @ 200 mm spacing located at an effective depth of 125 mm. Assuming M 20 concrete and Fe 415 steel, determine the superimposed service load (in kN/$m^2$) that the slab can safely carry (i) according to WSM , and (ii) according to LSM (assuming a load factor of 1.5).

[*Ans.* : (i) 3.01 kN/$m^2$;
(ii) 3.46 kN/$m^2$]

## REFERENCES

4.1 Timoshenko, S.P. and Goodier, J.N., *Theory of Elasticity*, Second edition, McGraw-Hill, 1951.

4.2 Hognestad, E., Hanson, N.W. and McHenry,D., *Concrete Stress Distribution in Ultimate Strength Design*, Journal ACI, Vol. 52, Dec. 1955, pp 455–479

4.3 Popov, E.P., *Mechanics of Solids*, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1976.

4.4 Rüsch, H, *Researches Towards a General Flexural Theory for Structural Concrete*, Journal ACI, Vol. 57, July 1960, pp 1–28.

4.5 — *Explanatory Handbook on Indian Standard Code of Practice for Plain and Reinforced Concrete (IS 456:1978)*, Special Publication SP:24, Bureau of Indian Standards, New Delhi, 1983.

# 5

# Design of Beams and One-Way Slabs for Flexure

## 5.1 INTRODUCTION

In the previous chapter, the behaviour of reinforced concrete beams (and one-way slabs) was explained, and procedures given for the **analysis** of sections. Analysis of beam sections may involve calculations of (1) stresses under known service load moments, (2) allowable service load moments $M_{all}$ (working stress method) and (3) ultimate moment of resistance $M_{uR}$ (limit states method). It may be noted that the results of the analysis of a *given* beam section are unique, being dictated solely by the conditions of equilibrium of forces and compatibility of strains. On the basis of these computations, it is possible to decide whether or not the beam is 'safe' under known moments.

The **design** problem is somewhat the reverse of the *analysis* problem. The external loads (or load effects), material properties and the skeletal dimensions of the beam are given, and it is required to arrive at suitable cross-sectional dimensions and details of the reinforcing steel, which would give adequate *safety* and *serviceability*. In designing for flexure, the distribution of bending moments along the length of the beam must be known from structural analysis. For this, the initial cross-sectional dimensions have to be assumed in order to estimate dead loads; this is also required for the analysis of indeterminate structures (such as continuous beams). The adequacy of the assumed dimensions should be verified and suitable changes made, if required.

Unlike the *analysis* problem, the *design* problem does not have a unique solution because the flexural strength of a section is dependent on its width and effective depth, and on the area of reinforcement; and there are several combinations of these which would give the required strength. Different designers may come up with different solutions, all of which may meet the desired requirements.

Of course, it is possible to conceive of an 'optimal' solution — by formulating the problem as an *optimisation* problem[†] [Ref. 5.1]. However, such a mathematically intensive procedure is not commonly adopted, nor warranted in standard design practice.

A complete design of a beam involves considerations of safety under the *ultimate limit states* in flexure, shear, torsion and bond, as well as considerations of the *serviceability limit states* of deflection, crack-width, durability etc.

The present chapter focuses on the considerations of safety under the ultimate limit state of flexure alone. Considerations of other limit states are covered in detail in subsequent chapters. However, some acceptance criteria under serviceability limit states are introduced indirectly, such as by specifying limiting span/depth ratios (for deflection control) and clear cover to reinforcement (for durability).

The traditional *working stress method* (WSM) of design is not considered here, as it is no longer used in practice, being superseded by the *limit states method* (LSM) of design. WSM is no longer recommended by most international Codes on reinforced concrete design. In the recent (2000) revision of the Code (IS 456), the provisions relating to the WSM design procedure, as an alternative to LSM, have been relegated from the main text of the Code to an Annexure (Annex B). It is generally not used in practice — except in the design of liquid retaining structures and bridges. However, most modern codes insist on LSM even for such structures, where the serviceability limit state of cracking is a major criterion; crack-widths are best *directly* controlled (as in LSM), rather than *indirectly* by permissible stresses in steel and concrete (as in WSM).

Prior to taking up problems related to design in flexure, it is necessary to have first an understanding of the requirements related to the placing of flexural reinforcement, control of deflection, as well as other guidelines for the selection of member sizes. These are discussed in the following sections.

## 5.2 REQUIREMENTS OF FLEXURAL REINFORCEMENT

### 5.2.1 Concrete Cover

*Clear cover* is the distance measured from the exposed concrete surface (without plaster and other finishes) to the nearest surface of the reinforcing bar. The Code (Cl. 26.4.1) defines the term *nominal cover* as "the design depth of concrete cover to all steel reinforcements, including links". This cover is required to protect the reinforcing bars from corrosion and fire, and also to give the reinforcing bars sufficient embedment to enable them to be stressed without 'slipping' (losing bond with the concrete). As mentioned earlier, the recent revision in the Code with its emphasis on increased durability, has incorporated increased cover requirements, based on the severity of the environmental exposure conditions (refer Table 2.1).

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[†] i.e., suitably defining an 'objective function' and various 'constraints', involving economy, deflection control, crack control, ductility requirements, headroom limitations, reinforcement percentage limits, etc., and a set of variables such as width, depth, material properties, area of reinforcement, etc.

The 'nominal cover' to meet *durability* requirements, depending on exposure condition, are summarised in Table 5.1.

As corrosion of reinforcing bars is a common and serious occurrence, it is advisable to specify liberal clear cover in general (particularly in excessively wet and humid environments, and in coastal areas). It may be noted that in actual construction, the clear cover obtained may be (and often is) less than the specified clear cover; however, this should be within the tolerance allowed and appropriate allowance should be made for such errors in construction. In this context, it is important to note the revised tolerance specified in IS 456 (2000), according to which the maximum deviation in clear cover from the value specified by the designer are "+10 mm and –0 mm†".

**Table 5.1** Nominal cover requirements based on exposure conditions

| Exposure Condition | Minimum Grade | Nominal Cover (mm) | Allowance permitted |
|---|---|---|---|
| Mild | M 20 | 20 | Can be reduced by 5mm for main rebars less than 12mm dia |
| Moderate | M 25 | 30 | . |
| Severe | M 30 | 45 | } Can be reduced by 5mm if concrete |
| Very severe | M 35 | 50 | } grade is M35 or higher |
| Extreme | M 40 | 75 | . |

The clause in the earlier version of the Code, limiting the maximum clear cover in any construction to 75 mm has, for some reason, been eliminated in the revised code. The general message underlying the revised recommendations in the code pertaining to clear cover seems to be: "the more the cover, the more durable the concrete". Unfortunately, the code does not also convey the message that the provision of very large covers (100 mm or more) is undesirable, and can be counter-productive, causing increased crack-widths, particularly in flexural members (such as slabs and beams). Large crack-widths (greater than 0.3 mm) permit the ingress of moisture and chemical attack to the concrete, resulting in possible corrosion to reinforcement and deterioration of concrete. There is little use in providing increased cover to reinforcement, if that cover is cracked, and the likelihood of cracking increases with increased cover. It is therefore necessary to impose an upper limit to clear cover (usually 75 mm), and to enforce the checking for the limit state of cracking when large covers are provided (Ref. 5.2).

It may be noted that in the earlier version of the Code, the clear cover requirements were based on the type of structural element (for example, 15 mm in slabs, 25 mm in beams, 40 mm in columns, etc.). The clear cover specifications are now made applicable for all types of structural elements. However, certain minimum clear cover requirements have been specified in Cl. 26.4.2.1 of the Code for columns

† i.e., no reduction in clear cover is permitted; an increase in clear cover up to 10 mm above the specified nominal cover is allowed.

(for longitudinal bars, 40 mm in general) and in Cl. 26.4.2.2 for footings (50 mm in general). These are discussed in Chapters 13 and 14.

In addition, the Code has introduced nominal cover requirements, based on *fire resistance* (in terms of hours) required. These provisions have been apparently borrowed from BS 8110. They are described in Cl. 26.4.3 of the Code. In general, for a nominal 1 hour fire resistance, the nominal cover specified is 20 mm for beams and slabs, and 40 mm for columns. Larger cover is required only if the structural element under consideration has to be specially designed for fire resistance.

### 5.2.2 Spacing of Reinforcing Bars

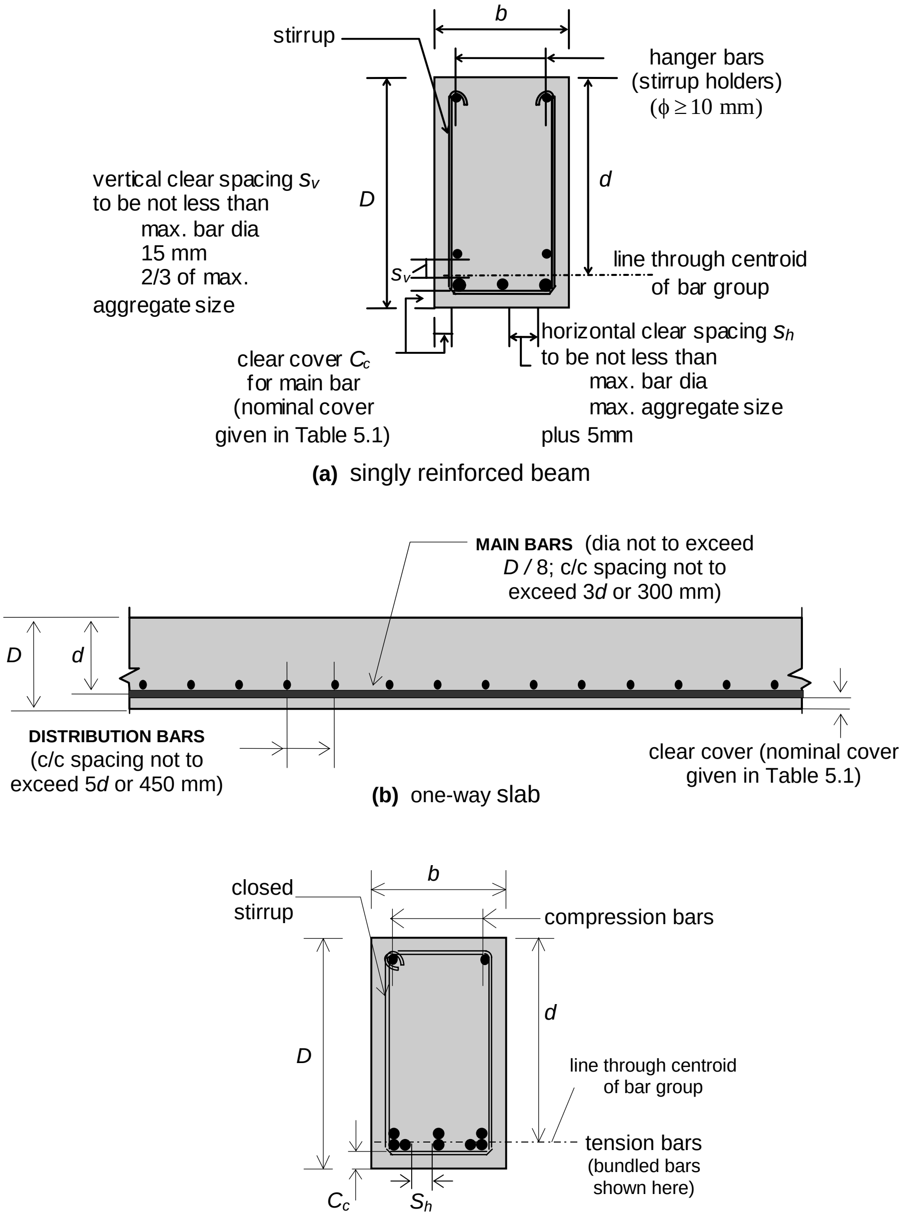
The Code specifies minimum and maximum limits for the spacing between parallel reinforcing bars in a layer. The *minimum* limits are necessary to ensure that the concrete can be placed easily in between and around the bars during the placement of fresh concrete. The *maximum* limits are specified for bars in tension for the purpose of controlling crack-widths and improving bond.

The minimum spacing limits can be met without difficulty in slabs in general, because of the large widths available and the relatively low percentage of flexural reinforcement required. However, in the case of beams, which have limited widths and are required to accommodate relatively large areas of flexural reinforcement, the minimum spacing requirements can sometimes govern the selection of the widths of the beams. If all the reinforcing bars cannot be accommodated in a single layer with the necessary clearance between the bars and the clear cover on the two sides [Fig. 5.1], the options are:

- to increase the beam width;
- to place the bars in two or more layers, properly separated [Fig. 5.1(a)]; and
- to bundle groups of parallel bars (two, three or four bars in each bundle) [Fig. 5.1(c)].

While fixing the overall size of the beam or the thickness of the slab, it is desirable to use multiples of 5 mm for slabs and 50 mm (or 25 mm) for beams. This will be convenient in the construction of the formwork. The requirements for placement of flexural reinforcement are described in Cl. 26.3 of the Code. The salient features of these specifications are summarised in Fig. 5.1. The student is advised to read the relevant clauses in the Code, while studying Fig. 5.1. The requirements for singly reinforced beams, slabs and doubly reinforced beams are depicted in parts (a), (b) and (c) respectively of Fig. 5.1.

Stirrups provided in beams serve as transverse shear reinforcement [refer Chapter 6]. In singly reinforced beams, they may be provided as U-shaped stirrups, with two hanger bars at top [Fig. 5.1(a)]. However, it is more common to provide fully closed rectangular stirrups [Fig. 5.1(c)], for both singly and doubly reinforced sections; this is mandatory in the latter case for the effective functioning of the compression steel. Stirrups required for resisting torsion must also be of the closed form [refer Chapter 7].




**(a)** singly reinforced beam

**(b)** one-way slab

**(c)** doubly reinforced beam

**(d)** **Fig. 5.1** Code requirements for flexural reinforcement placement

In addition to the requirements indicated in Fig. 5.1, the Code specifies limits to the maximum spacing of tension reinforcing bars for crack control [refer Table 15 of the Code]. It may be noted that, for a given area of tension reinforcement, providing several small-diameter bars (that are well distributed in one or more layers in the extreme tension zone) is more effective in controlling cracks and improving bond than providing fewer bars of larger diameter. For this reason, the Code (Cl. 26.5.2.2 & 26.3.3b) limits the maximum diameter of reinforcing bars in slabs to one-eighth of the total thickness of the slab, and the maximum spacing of such main bars to $3d$ or 300 mm (whichever is less) [Fig. 5.1(b)]. However, it may be noted that when large cover is provided, more stringent bar spacing may be required to achieve the desired crack control [Ref. 5.2].

Furthermore, in relatively deep flexural members, a substantial portion of the web will be in tension. Tension reinforcement properly distributed will, no doubt, control the crack width at its level; however, wider cracks may develop higher up in the web. Moreover, as explained in Section 2.12, cracking can occur in large unreinforced exposed faces of concrete on account of shrinkage and temperature variations. In order to control such cracks, as well as to improve resistance against lateral buckling of the *web* [Ref. 5.3], the Code (Cl. 26.5.1.3) requires *side face reinforcement* to be provided along the two faces of beams with overall depth exceeding 750 mm:

> "the total area of such reinforcement shall be not less than 0.1 percent of the web area and shall be distributed *equally* on two faces at a spacing not exceeding 300 mm or *web* thickness whichever is less"

### 5.2.3 Minimum and Maximum Areas† of Flexural Reinforcement

A minimum area of tension reinforcing steel is required in flexural members not only to resist possible load effects, but also to control cracking in concrete due to shrinkage and temperature variations.

#### *Minimum Flexural Reinforcement in Beams*

In the case of beams, the Code (Cl. 26.5.1.1) prescribes the following:

$$\frac{(A_{st})_{\min}}{bd} = \frac{0.85}{f_y} \tag{5.1}$$

which gives $(p_t)_{\min} \equiv \dfrac{100(A_{st})_{\min}}{bd}$ values equal to 0.340, 0.205 and 0.170 for Fe 250, Fe 415 and Fe 500 grades of steel respectively. In the case of flanged beams, the width of the web $b_w$ should be considered in lieu of $b$.

It can be shown that the $(A_{st})_{min}$ given by Eq. 5.1 results in an ultimate moment of resistance that is approximately equal to the 'cracking moment' of an identical plain concrete section. Thus, the minimum reinforcement requirement ensures that a sudden failure is avoided at $M = M_{cr}$.

#### *Minimum Flexural Reinforcement in Slabs*

† The limits specified here (as per IS 456) are applicable to reinforced concrete flexural members in general. However, for earthquake-resistant design ('ductile detailing'), different limits are applicable; this is described in Chapter 16.

As specified in Cl. 26.5.2, the minimum reinforcement $(A_{st})_{min}$ in either direction in slabs is given by

$$(A_{st})_{\min} = \begin{cases} 0.0015A_g & \text{for Fe 250} \\ 0.0012A_g & \text{for Fe 415} \end{cases} \tag{5.2}$$

where $A_g$ denotes the *gross* area of the section ($b \times D$).

In the design of one-way slabs, this minimum reinforcement is also to be provided for the *secondary* (or *distributor*) *reinforcement* (refer Section 4.8.1) along the direction perpendicular to the main reinforcement†, with the spacing of such bars not exceeding $5d$ or 450 mm (whichever is less) [Fig. 5.1(b)]. It may be noted that in the case of slabs, sudden failure due to an overload is less likely owing to better lateral distribution of the load effects. Hence, the minimum steel requirements of slabs are based on considerations of shrinkage and temperature effects alone, and not on strength. Accordingly, the specified value of $(p_t)_{min}$ is somewhat smaller in the case of slabs, compared to beams. However, for exposure conditions where crack control is of special importance, reinforcement in excess of that given by Eq. 5.2 should be provided.

### *Maximum Flexural Reinforcement in Beams*

Providing excessive reinforcement‡ in beams can result in congestion (particularly at beam-column junctions), thereby adversely affecting the proper placement and compaction of concrete. For this reason, the Code (Cl. 26.5.1) restricts the area of tension reinforcement ($A_{st}$) as well as compression reinforcement ($A_{sc}$) in beams to a maximum value of 0.04 $bD$. If both $A_{sc}$ and $A_{st}$ are provided at their maximum limits, the total area ($A_{sc} + A_{st}$) of steel would be equal to 8 percent of the gross area of the beam section; this is rather excessive. It is recommended that such high reinforcement areas should be generally avoided by suitable design measures. These include:

- increasing the beam size (especially depth);
- improving the grades of concrete and steel.

## 5.3 REQUIREMENTS FOR DEFLECTION CONTROL

Excessive deflections in slabs and beams are generally undesirable as they cause psychological discomfort to the occupants of the building, and also lead to excessive crack-widths and subsequent loss of durability and ponding in roof slabs.

The selection of cross-sectional sizes of flexural members (thicknesses of slabs, in particular) is often governed by the need to control deflections under service loads. For a given loading and span, the deflection in a reinforced concrete beam or slab is inversely proportional to its *flexural rigidity*. It is also dependent on factors related

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† Note that the direction of the secondary reinforcement need not be the same as that of the long span. This case is encountered, for example, in a slab supported on opposite edges, with the actual span dimension being larger than the transverse dimension.

‡ Heavy reinforcement may be designed in doubly reinforced beam sections and in flanged beam sections, without resulting in *over-reinforced* sections.

to long–term effects of creep and shrinkage [refer Sections 2.11, 2.12]. From the point of view of design, it is the ratio of the maximum deflection to the span that is of concern, and that needs to be limited. The Code (Cl. 23.2a) specifies a limit of span / 250 to the final deflection due to *all* loads (including long-term effects of temperature, creep and shrinkage). Additional limits are also specified in Cl. 23.2(b) of the Code — to prevent damage to partitions and finishes [refer Chapter 10 for details].

The explicit computation of maximum deflection can be rather laborious and made difficult by the need to specify a number of parameters (such as creep coefficient and shrinkage strain as well as actual service loads), which are not known with precision at the design stage. For convenience in design, and as an alternative to the actual calculation of deflection, the Code recommends certain span/effective depth ($l/d$) ratios which are expected to satisfy the requirements of deflection control $(\Delta/l < 1/250)$. Nevertheless, explicit calculations of deflections (refer Chapter 10) become necessary under the following situations [Ref. 5.3]:

- when the specified $l/d$ limits cannot be satisfied;
- when the loading on the structure is abnormal; and
- when stringent deflection control is required.

### 5.3.1 Deflection Control by Limiting Span/Depth Ratios

For a rectangular beam, made of a linearly elastic material, the ratio of the maximum elastic deflection to the span ($\Delta/l$) will be a constant if the span /overall depth ratio ($l/D$) is kept constant. This can be proved as follows for the case of a simply supported rectangular beam, subjected to a uniformly distributed load $w$ per unit length:

$$\Delta = \frac{5}{384}\frac{wl^4}{EI} \tag{5.3a}$$

$$M_{\max} = \frac{wl^2}{8}$$

$$\Rightarrow \quad w = (M_{\max})\frac{8}{l^2} = (\sigma Z)\frac{8}{l^2} = \left(\sigma\frac{bD^2}{6}\right)\frac{8}{l^2} \tag{5.3b}$$

where $\sigma$ is the bending stress at service loads and $Z = bD^2/6$ is the section modulus.

Substituting Eq. 5.3(b) and $I = bD^3/12$ in Eq. 5.3(a), it can be shown that

$$\left(\frac{\Delta}{l}\right) = \text{constant} \times \left(\frac{l}{D}\right) \tag{5.4}$$

where, in the present case of a simply supported beam with uniformly distributed loading, the 'constant' works out to $5\sigma/24E$.

Eq. 5.4 is generalised, and holds good for all types of loading and boundary conditions (with appropriately different constants). It is thus seen that, by limiting the $l/D$ ratio, deflection (in terms of $\Delta/l$) can be controlled.

Eq. 5.4 is not directly applicable in the case of reinforced concrete, because it is not a linearly elastic material and the parameters $\sigma$, $Z$ and $E$ are not constants, being dependent on such factors as the state of cracking, the percentage of reinforcement, as well as the long–term effects of creep and shrinkage. The Code however adopts this concept, with suitable approximations, and prescribes limiting $l/d$ ratios for the purpose of deflection control.

### 5.3.2 Code Recommendations for Span/Effective Depth Ratios

For prismatic beams of rectangular sections and slabs of uniform thicknesses and *spans*† *up to* 10 m, the limiting $l/d$ ratios are specified by the Code (Cl. 23.2.1) as:

$$(l/d)_{\max} = (l/d)_{basic} \times k_t \times k_c \tag{5.5}$$

$$\text{where} \quad (l/d)_{basic} = \begin{cases} 7 \text{ for cantilever spans} \\ 20 \text{ for simply supported spans} \\ 26 \text{ for continuous spans} \end{cases}$$

and the *modification factors* $k_t$ (which varies with $p_t$ and $f_{st}$) and $k_c$ (which varies with $p_c$) are as given in Fig. 4 and Fig. 5 of the Code (based on Ref. 5.4). Alternatively, the values of these 'modification factors' can be obtained from Tables 5.2 and 5.3 which are based on the figures given in the Code [Ref. 5.4].

It can be seen from Table 5.2 that the values of $k_t$ increase with the use of lower percentages and lower service load stress levels‡ of tension reinforcement. This is attributable to the fact that, under given service loads, lower values of $p_t$ and $f_{st}$ are indicative of larger beam (or slab) cross-sections, resulting in higher flexural rigidity, and hence lesser deflections. Alternatively, for given cross-sections, lower values of $p_t$ and $f_{st}$ are indicative of lower design loads and lower strains distributed across the cross-section, and hence lower curvatures and lesser deflections. The use of *mild steel* bars (Fe 250), with relatively low allowable stress levels, is particularly effective in reducing deflections; the values of $k_t$ are invariably greater than unity — even at high $p_t$ values. The calculation of the stress in the tension steel $f_{st}$ should ideally be worked out considering the transformed cracked section properties. However, for convenience, the Code permits an approximate calculation of $f_{st}$, given as follows:

$$f_{st} = 0.58\, f_y \, \frac{\text{Area of steel required}}{\text{Area of steel provided}}$$

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† The Code (Cl. 22.2) uses the term *effective span*, defined as the clear span plus effective depth, or centre-to-centre support distance, whichever is smaller.

‡ In the earlier version of the Code, the modification factor $k_t$ was a function of the characteristic (yield) strength, $f_y$; this has been now corrected in the 2000 revision of the Code.

It should be noted that, in the estimation of $k_t$ for the control of deflections in continuous beams and slabs, the value of $p_t$ and $f_{st}$ should be calculated at the *midspan region*; however, in the case of cantilevers, $p_t$ and $f_{st}$ should be calculated at the support [Ref. 5.3].

**Table 5.2** Modification factor $k_t$ for different values of $p_t$ and $f_{st}$
[Ref. Fig.4 of IS 456 : 2000]

| $p_t \equiv \frac{100A_{st}}{bd}$ | $f_{st}$ (MPa) | | | | |
|---|---|---|---|---|---|
| | **120** | **145** | **190** | **240** | **290** |
| 0.2 | - | - | - | 1.68 | 1.40 |
| 0.3 | - | - | 1.89 | 1.47 | 1.23 |
| 0.4 | - | - | 1.68 | 1.34 | 1.13 |
| 0.5 | - | 1.95 | 1.53 | 1.23 | 1.04 |
| 0.6 | - | 1.79 | 1.43 | 1.17 | 0.98 |
| 0.8 | 1.78 | 1.57 | 1.29 | 1.06 | 0.90 |
| 1.0 | 1.60 | 1.42 | 1.20 | 0.99 | 0.85 |
| 1.2 | 1.50 | 1.33 | 1.12 | 0.95 | 0.81 |
| 1.4 | 1.41 | 1.26 | 1.08 | 0.92 | 0.78 |
| 1.6 | 1.34 | 1.21 | 1.03 | 0.88 | 0.75 |
| 1.8 | 1.30 | 1.17 | 0.99 | 0.85 | 0.72 |
| 2.0 | 1.23 | 1.12 | 0.97 | 0.83 | 0.71 |
| 2.2 | 1.20 | 1.10 | 0.94 | 0.82 | 0.70 |
| 2.4 | 1.18 | 1.06 | 0.91 | 0.81 | 0.69 |
| 2.6 | 1.15 | 1.04 | 0.90 | 0.80 | 0.68 |
| 2.8 | 1.13 | 1.02 | 0.89 | 0.79 | 0.67 |
| 3.0 | 1.11 | 1.00 | 0.88 | 0.78 | 0.67 |

From Table 5.3, it can be seen that the provision of compression steel can significantly contribute towards reducing deflections. For example, the modification factor $k_c$ takes values of 1.25 and 1.50 for values of $p_c$ equal to 1 percent and 3 percent respectively — for all grades of compression steel. This beneficial effect of compression reinforcement is attributable to its contribution in reducing differential *shrinkage* strains across the reinforced concrete section [refer Section 2.12 and Chapter 10], thereby reducing long–term shrinkage deflections.

In the case of *flanged beams*, the Code (Cl. 23.2.1e) recommends that the values of $p_t$ and $p_c$ considered in estimating the modification factors should be based on an area of section equal to $b_f\, d$, and that the calculated $(l/d)_{\max}$ [Eq. 5.5] should be further modified by a 'reduction factor' which depends on $b_w / b_f$ (as given in Fig. 6 of the Code). However, this code procedure has been found to give anomalous results — as reported in the Explanatory Handbook to the Code [Ref. 5.3]. Hence, it is recommended that, for the purpose of using Eq. 5.5, the overhanging portions of the flanges be ignored, and that the beam be treated as a rectangular beam with width $b_w$ and effective depth $d$; this will give conservative results.

**Table 5.3** Modification factor $k_c$ for different values of $p_c$ [Ref. Fig. 5 of IS 456 : 2000]

| $p_c \equiv \frac{100A_{sc}}{bd}$ | $k_c$ |
|---|---|
| 0.00 | 1.00 |
| 0.25 | 1.08 |
| 0.50 | 1.14 |
| 0.75 | 1.20 |
| 1.00 | 1.25 |
| 1.25 | 1.29 |
| 1.50 | 1.33 |
| 1.75 | 1.36 |
| 2.00 | 1.39 |
| 2.25 | 1.42 |
| 2.50 | 1.45 |
| 2.75 | 1.48 |
| 3.00 | 1.50 |

## 5.4 GUIDELINES FOR SELECTION OF MEMBER SIZES

As explained in Sections 5.2 and 5.3, the selection of flexural member sizes (from a structural viewpoint) is often dictated by serviceability criteria (need to control deflections and crack-widths) as well as requirements related to the placement of reinforcement. However there are other structural, economic and architectural considerations that come into play in the design of reinforced concrete beams.

### 5.4.1 General Guidelines for Beam Sizes

The design problem does not have a unique solution (Section 5.1). Many choices of beam sizes are feasible in any given design situation. In general, for the purpose of designing for flexure, it is economical to opt for singly reinforced sections with moderate percentage tension reinforcement ($p_t \approx$ 0.5 to 0.8 times $p_{t,lim}$).

Given a choice between increasing either the width of a beam or its depth, it is always advantageous to resort to the latter. This results not only in improved moment resisting capacity, but also in improved flexural stiffness, and hence, less deflections, curvatures and crack-widths. However, very deep beams are generally not desirable, as they result in a loss of headroom or an overall increase in the building height. In general, the recommended ratio of overall depth ($D$) to width ($b$) in rectangular beam sections is in the range of 1.5 to 2. It may be higher (up to 3 or even more) for beams carrying very heavy loads. The width and depth of beams are also governed by the shear force on the section [refer Chapter 6]. Often, architectural considerations dictate the sizes of beams. If these are too restrictive, then the desired strength of the beam in flexure can be provided by making it 'doubly reinforced' and/or by providing high strength concrete and steel. In the case of beam-supported slab systems which are cast integrally, the beams can be advantageously modelled as 'flanged beams', as explained earlier.

In the case of building frames, the width of beams should, in general, be less than or equal to the lateral dimension of the columns into which they frame. Beam widths of 200 mm, 250 mm and 300 mm are common in practice world-wide. Where the beam is required to support a masonry wall, the width of the beam is often made such that its sides are flush with the finished surfaces of the wall; thus, beam widths of 230 mm are also encountered in practice in India. In design practice, the overall depths of beams are often fixed in relation to their spans. Span to *overall depth* ratios of 10 to 16 are generally found to be economical in the case of simply supported and continuous beams. However, in the case of cantilevers, lower ratios are adopted, and the beams are generally tapered in depth along their lengths, for economy. Such traditional heuristic methods of fixing the depth of beams are generally satisfactory from the viewpoint of deflection control — for the normal range of loads.

From practical considerations, it is desirable to limit the number of different beam sizes in the same structure to a few standard modular sizes, as this will greatly convenience the construction of formwork, and permit reusability of forms.

### 5.4.2 General Guidelines for Slab Thicknesses

In the case of *slabs*, whose thicknesses are very small in comparison with the depths of beams, the limiting span /depth ratios of Eq. 5.5 will generally govern the proportioning. In practice, Fe 415 grade steel is most commonly used, and for such steel, a $p_t$ value of about 0.4 – 0.5 percent may be assumed for preliminary proportioning. This gives a $k_t$ value of about 1.25 [Table 5.2]; accordingly, the required *effective depth* (for preliminary design) works out to about span/25 for simply supported slabs and about span/32 for continuous slabs.

In order to determine the thickness of the slab, the clear cover (based on exposure, refer Table 5.1) plus half the bar diameter of the main reinforcement (usually along the shorter span) have to be added to the effective depth, as indicated in Fig. 5.1(b). The calculated value of the thickness should be rounded off to the nearest multiple of 5 mm or 10 mm.

### 5.4.3 Deep Beams and Slender Beams

In certain extreme situations, the designer may be called upon to deal with very low span /depth ratios. In such cases, where the depth of the beam becomes comparable to its span, the beam is referred to as a *deep beam*[†]. It calls for special design requirements, which are covered in Cl. 29 of the Code.

In other situations, *slender beams* may be encountered. When the length of a beam is excessive in comparison with its cross–sectional dimensions (particularly its width $b$), there is a possibility of *instability* due to slenderness — in particular, lateral buckling in the compression zone. The Code (Cl. 23.3) specifies certain 'slenderness limits' to ensure *lateral stability*. The clear distance between lateral restraints should not exceed $60b$ or $250\,b^2/d$, whichever is less, in the case of simply supported and

[†] By definition, a 'deep beam' is one whose $l/D$ ratio is less than 2.0 for a simply supported beam, and 2.5 for a continuous beam [refer Cl. 29.1 of the Code].

continuous beams. For a cantilever, the distance from the free end to the edge of the support should not exceed 25$b$ or 100 $b^2/d$, whichever is less.

## 5.5 DESIGN OF SINGLY REINFORCED RECTANGULAR SECTIONS

The design problem is generally one of determining the cross-sectional dimensions of a beam, viz. $b$ and $D$ (including $d$), and the area of tension steel $A_{st}$ required to resist a known factored moment $M_u$. The material properties $f_{ck}$ and $f_y$ are generally prescribed/selected on the basis of exposure conditions, availability and economy. For normal applications, Fe 415 grade steel is used, and either M 20 or M 25 grade concrete is used (for exposures rated 'severe', 'very severe' and 'extreme', the minimum concrete grades specified are M 30, M 35 and M 40 respectively, as shown in Table 5.1). As explained earlier in Section 4.7.3, for under-reinforced sections, the influence of $f_{ck}$ on the ultimate moment of resistance $M_{uR}$ is relatively small; hence, the use of high strength concrete is not beneficial from the point of economy, although it is desirable from the point of durability.

The basic requirement for *safety* at the 'ultimate limit state of flexure' is that the factored moment $M_u$ should not exceed the ultimate moment of resistance $M_{uR}$, and that the failure at the limit state should be *ductile*. Accordingly, the *design equation* for flexure is given by:

$$M_u \leq M_{uR} \qquad \text{with } x_u \leq x_{u,\max}$$

This implies that, for singly reinforced beam sections, Eq. 4.65 is applicable, with $M_u = M_{uR}$:

$$\frac{M_u}{bd^2} \equiv R = 0.87 f_y \left(\frac{p_t}{100}\right)\left[1 - \frac{f_y}{f_{ck}}\left(\frac{p_t}{100}\right)\right] \qquad \text{for } p_t \leq p_{t,\lim} \tag{5.6}$$

For any chosen value of $p_t$, the constant $R \equiv \frac{M_u}{bd^2}$ (in MPa units) is determined from Eq. 5.6 (or, alternatively from the *analysis aids* given in Table A.2). The limiting percentage tension reinforcement $p_{t,lim}$ and the corresponding $R_{\lim} \equiv \frac{M_{u,\lim}}{bd^2}$ are constants given by Eq. 4.62 and 4.61(a):

$$p_{t,\lim} = \begin{cases} 0.0884 f_{ck} & \text{for Fe 250} \\ 0.0480 f_{ck} & \text{for Fe 415} \\ 0.0379 f_{ck} & \text{for Fe 500} \end{cases} \tag{5.7}$$

$$R_{\lim} = \frac{M_{u,\lim}}{bd^2} = \begin{cases} 0.1498 f_{ck} & \text{for Fe 250} \\ 0.1389 f_{ck} & \text{for Fe 415} \\ 0.1338 f_{ck} & \text{for Fe 500} \end{cases} \tag{5.8}$$

### 5.5.1 Fixing Dimensions of Rectangular Section

Obviously, there are several combinations of $p_t$, $b$ and $d$ (or $D$) which can satisfy Eq. 5.6. However, the problem is simplified if the values of $b$ and $D$ are either given (by architectural considerations) or arrived at on some logical basis.

In the case of slabs, $b$ is taken as 1000 mm (as explained in Section 4.8) and $d$ is governed by the limiting $l/d$ ratios for deflection control (refer Section 5.3.2). As suggested in Section 5.4.2, a trial value of $d$ may be assumed as approximately $l/25$ for simply supported spans, $l/32$ for continuous spans and $l/8$ for cantilevers. The overall depth $D$ may be taken as $d$ plus effective cover. The effective cover will be the sum of the clear cover, the diameter of the stirrup and half the bar diameter (in the case of a single layer of tension reinforcement). Assuming a stirrup diameter of 10mm and a bar diameter of 20mm, the effective cover will be in the range of 40 – 95 mm, depending on the exposure condition[†].

In the case of beams, it is generally found economical to adopt under–reinforced sections with $p_t < p_{t,lim}$. The value of $b$ may be suitably fixed as 200 mm, 250 mm, 300 mm, etc., and the value of $d$ corresponding to any $R \le R_{lim}$ is given by:

$$d = \sqrt{\frac{M_u}{Rb}} \tag{5.9}$$

where $M_u$ is the factored moment[‡] (in N mm) and $R$ is given by Eq. 5.6 for the chosen value of $p_t$. The minimum value of $d$ corresponding to the limiting case $p_t = p_{t,lim}$ is obtained by substituting $R = R_{lim}$ (given by Eq. 5.8). It is desirable to adopt a value of $d$ which is larger than $d_{min}$ in order to obtain an under-reinforced section. The overall depth of the beam may be taken as $D > d_{min}$ + effective cover, and should be expressed in rounded figures (for ease in formwork construction). Multiples of 50 mm (or 25 mm) are generally adopted in practice. However, as explained earlier, the resulting $D/b$ ratio should neither be excessive nor too small; ideally, it should be in the range 1.5 to 2.0. If the resulting $D/b$ ratio is unacceptable and needs to be modified, this can be achieved by suitably modifying $b$, recalculating $d$ (using Eq. 5.9) and fixing $D$.

Having fixed the rounded-off value of $D$, the correct value of the effective depth $d$ can be obtained (assuming that the reinforcing bars can be accommodated in one layer) as follows:

$$d = D - (\text{clear cover}) - \phi_{\text{tie}} - \frac{\phi}{2}$$

---

[†] effective cover (in mm) may be taken as 40, 50, 65, 70 and 95 respectively for mild, moderate, severe, very severe and extreme conditions of exposure.

[‡] This will include the contribution of the self-weight of the flexural member. A conservative estimate of the size of the member may be made at the initial stage, for calculating self–weight. The unit weight of concrete should be taken as 25 kN/m$^3$ [Cl. 19.2.1 of Code; see also Appendix B.1 of this book].

If the bars need to be accommodated in two or more layers, the values of $D$ and $d$ should be fixed accordingly [refer Fig. 5.1].

### 5.5.2 Determining Area of Tension Steel

At this stage of the design process, $b$ and $d$ are known, and it is desired to determine the required $A_{st}$ so that the section has an ultimate moment of resistance $M_{uR}$ equal to the factored moment $M_u$.

From Eq. 4.60, considering $M_u = M_{uR}$ and $x_u < x_{u,max}$, it follows that:

$$(A_{st})_{reqd} = \frac{M_u}{0.87 f_y d \ (1 - 0.416 x_u / d)} \tag{5.10}$$

where $x_u/d$ is obtained by solving Eq. 4.59:

$$M_u = 0.362 f_{ck} bd^2 \left(\frac{x_u}{d}\right)\left[1 - 0.416\left(\frac{x_u}{d}\right)\right]$$

$$\Rightarrow 0.416\left(\frac{x_u}{d}\right)^2 - \left(\frac{x_u}{d}\right) + \frac{M_u}{0.362 f_{ck} bd^2} = 0$$

which is a quadratic equation, whose solution gives:

$$\frac{x_u}{d} = 1.202\left[1 - \sqrt{1 - 4.597R/f_{ck}}\right] \tag{5.11}$$

where, as mentioned earlier, $R \equiv M_u / bd^2$ .

It is possible to calculate $(A_{st})_{reqd}$ directly, without having to determine $x_u/d$. By re-arranging Eq 5.6,

$$\frac{f_y}{f_{ck}}\left(\frac{p_t}{100}\right)^2 - \left(\frac{p_t}{100}\right) + R/0.87f_y = 0$$

which is a quadratic equation, whose solution gives:

$$\frac{p_t}{100} \equiv \frac{(A_{st})_{reqd}}{bd} = \frac{f_{ck}}{2f_y}\left[1 - \sqrt{1 - 4.598R/f_{ck}}\right] \tag{5.12}$$

The above formula provides a convenient and direct estimate of the area of tension reinforcement in singly reinforced rectangular sections.

#### *Alternative: Use of Design Aids*

In practice, this is the most widely used method. Expressing the relationship between $R \equiv M_u / bd^2$ and $p_t$ [Eq. 5.12] in the form of charts or tables for various combinations of $f_y$ and $f_{ck}$ is relatively simple. These are available in design handbooks such as SP : 16 [Ref. 5.5]. The tabular format is generally more convenient to deal with than the Chart.

Accordingly, Tables A.3(a) and A.3(b) have been developed (based on Eq. 5.12) for M 20, M 25, M 30 and M 35 grades of concrete, each Table covering the three grades of steel [Fe 250, Fe 415 and Fe 500]; these Tables are placed in Appendix A of this book. For a given value of $R$, and specified values of $f_y$ and $f_{ck}$, the desired value of $p_t$ can be read off (using linear interpolation for intermediate values).

### *Converting Area of Steel to Bars*

The calculated area of steel $(A_{st})_{reqd}$ has to be expressed in terms of bars of specified nominal diameter $\phi$ and number (or spacing). Familiarity with the standard bar areas $(A_b = \pi\phi^2/4)$ [Table 5.4] renders this task easy.

**Table 5.4** Standard bar areas ( $A_b = \pi\phi^2/4$ ) and mass per metre (kg/m)

| **$\phi$ (mm)** | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 25 | 28 | 32 | 36 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **$A_b$ (mm$^2$)** | 28.3 | 50.3 | 78.5 | 113 | 154 | 201 | 254 | 314 | 380 | 491 | 616 | 804 | 1018 |
| Mass kg/m | .222 | .395 | .616 | .887 | 1.21 | 1.58 | 1.99 | 2.46 | 2.98 | 3.85 | 4.84 | 6.31 | 7.99 |

For a chosen bar diameter $\phi$, the number of bars required to provide the area of tension steel $A_{st}$ is given by $A_{st}/A_b$, taken as a whole number. Alternatively, for a chosen number of bars, the appropriate bar diameter can be worked out. In some cases, it may be economical to select a combination of two different bar diameters (close to each other) in order to arrive at an area of steel as close as possible to the $A_{st}$ calculated. As explained earlier, in the case of slab, the area of steel is expressed in terms of centre-to-centre spacing of bars, given by

$$s_{reqd} = 1000 A_b / (A_{st})_{reqd}$$

The actual spacing provided should be rounded off to the nearest *lower* multiple of 5 mm or 10 mm.

For convenience, Tables A.5 and A.6 (provided in Appendix A) may be referred to — for a quick selection of bar diameter and number/spacing of bars. The values of bar areas given in Table 5.4 are also obtainable from Table A.5. Table 5.4 also gives the mass per metre length of the bars which may be useful in cost estimation.

The arrangement of bars finally proposed must comply with the Code requirements for placement of flexural reinforcement described in Section 5.2.

### 5.5.3 Design Check for Strength and Deflection Control

The actual $A_{st}$ and $d$ provided should be worked out, and it should be ensured that the consequent $p_t$ is less than $p_{t,lim}$ (for ductile failure at the ultimate limit state). It is good practice to calculate the actual $M_{uR}$ of the section designed (using Eq. 4.65 or 4.66), and thereby ensure that the actual $M_{uR} \geq M_u$.

A check on the adequacy of the depth provided for deflection control is also called for in flexural members. In the case of beams, the limiting ($l/d$) ratio given by Eq. 5.5 is generally more-than-adequately satisfied by singly reinforced sections. However, in the case of slabs, the criteria for deflection control are generally critical. In anticipation of this, it is necessary to adopt a suitable value of $d$ at the initial stage of the design itself, as explained in Section 5.5.1.

The section should be suitably redesigned if it is found to be inadequate.

**EXAMPLE 5.1**

A rectangular reinforced concrete beam, located inside a building in a coastal town, is simply supported on two masonry walls 230 mm thick and 6m apart (centre-to-centre). The beam has to carry, in addition to its own weight, a distributed live load of 10 kN/m and a dead load of 5 kN/m. Design the beam section for maximum moment at midspan. Assume Fe 415 steel.

**SOLUTION**

The beam is located inside the building, although in a coastal area, and thereby protected against weather, and not directly exposed to 'coastal environment'[†]. Hence, according to the Code (Table 3), the exposure condition may be taken as 'moderate'. The corresponding grade of concrete may be taken as M 25 and the clear cover as 30 mm. This cover will be adequate for normal fire resistance requirement also.

***Determining $M_u$ for design***

- Assume a trial cross-section $b$ = 250 mm, and $D$ = 600 mm (span/10).
  Let $d = D - 50 = 550$ mm.
  $\therefore$ Effective span (Cl. 22.2 of Code)
  $$l = \begin{cases} 6.0\text{ m} & \text{(distance between supports)} \\ (6.0-0.23)+0.55 = 6.32 & \text{(clear span} + d) \end{cases}$$
  Taking the lesser value (as per Code), $l = 6.0$ m
- Distributed load due to self-weight
  $\Delta w_{DL} = 25\text{ kN/m}^3 \times 0.25\text{ m} \times 0.6\text{ m} = 3.75$ kN/m
  $\therefore\ w_{DL} = 5.0 + 3.75 = 8.75$ kN/m, $w_{LL} = 10.0$ kN/m (given)
- $\therefore$ Factored load (as per Code):
  $$w_u = 1.5\,(w_{DL} + w_{LL}) = 1.5\,(8.75+10.0) = 28.1\text{ kN/m}$$

[†] Had the beam been located in the roof, the exposure condition would be 'severe'. Further, if the structure is located at the seafront (subject to sea water spray), the exposure condition would be 'very severe', according to the Code.

- $\Rightarrow$ Factored Moment (maximum at midspan)
  $M_u = w_u l^2/8 = 28.1 \times 6.0^2/8$ = **126 kNm.**

***Fixing up b, d and D***

- For Fe 415 steel, $M_{u,lim} = 0.1389 f_{ck} bd^2$ [Eq 5.8]
- For M 25 concrete,

$$f_{ck} = 25 \text{ MPa} \Rightarrow R_{\lim} \equiv \frac{M_{u,\lim}}{bd^2} = 0.1389 \times 25 = 3.472 \text{ MPa}$$

- Assuming $b$ = 250 mm, for a singly reinforced section, the minimum value of $d$, corresponding to $x_u = x_{u,max}$ is given by

$$d_{\min} = \sqrt{\frac{M_u}{R_{\lim} b}} = \sqrt{\frac{126 \times 10^6}{3.472 \times 250}} = 381 \text{ mm.}$$

- Adopt $D$ = 450 mm[†]. Assuming 25 $\phi$ bars, 8$\phi$ stirrups and clear cover of 30 mm, (note that specified cover is required for the stirrups as well),
  $d$ = 450 – 30 – 8 – 25/ 2 = **399mm**

***Determining $(A_{st})_{reqd}$***

- $$R \equiv \frac{M_u}{bd^2} = \frac{126 \times 10^6}{250 \times 399^2} = 3.166 \text{ MPa}$$

$$R = 0.87 f_y \left(\frac{p_t}{100}\right)\left[1 - \frac{f_y}{f_{ck}} \times \frac{p_t}{100}\right]$$

  Solving this quadratic equation in terms of $p_t$ [Eq. 5.12],

- $$\frac{p_t}{100} \equiv \frac{A_{st}}{bd} = \frac{25}{2(415)}\left[1 - \sqrt{1 - \frac{4.598 \times 3.166}{25}}\right] = 1.065 \times 10^{-2}$$

$$\Rightarrow \quad (A_{st})_{reqd} = (1.065 \times 10^{-2}) \times 250 \times 399 = 1062 \text{ mm}^2$$

- [Alternatively, using 'design aids' [Table A.3(a)], for $\dfrac{M_u}{bd^2}$ = 3.166 MPa, M 25 concrete and Fe 415 steel, $p_t$ = 1.065 — which gives the same result].

***Detailing***

- Using 3 bars in one layer, 3 × ($\pi \phi^2$ / 4) =1062 $\Rightarrow$ $\phi_{reqd}$ = 21.2 mm.

  Provide **1 -25** $\phi$ bar and **2 -20** $\phi$ bars, for which $A_{st}$ = 491+2(314) = 1119 > 1062.

  The placement of bars [Fig. 5.2] complies with the clearances specified by the Code.

***Design Checks***

- (a) *For strength in flexure*
  Actual $d$ = 450 – 30 – 8 – 25/2 = 399 mm.

$$\Rightarrow p_t = \frac{100 \times 1119}{250 \times 399} = 1.121 \quad < p_{t,lim} = 1.201.$$

---

[†] The resulting $D/b$ ratio is 1.8, which is satisfactory.

$$\Rightarrow M_{uR} = 0.87 \times 415 \times \left(\frac{1.121}{100}\right)\left[1 - \frac{415}{25}\left(\frac{1.121}{100}\right)\right] \times 250 \times 399^2$$

$=131.1 \times 10^6$ Nmm $>$ $M_u$ = 126 kNm — Hence, safe.

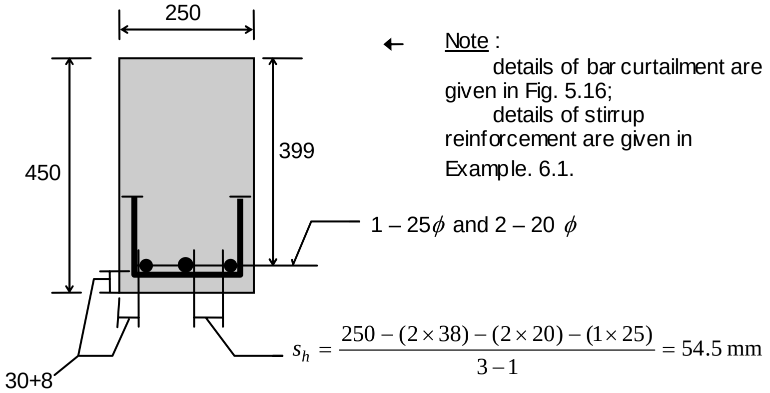



**Fig. 5.2** Singly reinforced beam design — Example 5.1

[*Note*: As the actual depth provided (399 mm) is greater than the calculation value ($d$ = 381 mm), and as the $A_{st}$ provided (1119 mm$^2$) is also greater than the required value $(A_{st})_{reqd} = (1.065 \times 10^{-2}) \times 250 \times 399$ = 1062 mm$^2$, it is evident (without the need for further proof) that the section is safe in flexure.]

- (b) *For deflection control*:

For $p_t$ = 1.121, and

$$f_s = 0.58 f_y \left(\frac{A_{st(req)}}{A_{st(provided)}}\right) = 0.58 \times 415 \times \left(\frac{1062}{1119}\right) = 228.4$$

$k_t$ = 1.014 (from Fig. 4 of Code or Table 5.2),

and, as $p_c$ = 0 (singly reinforced beam), $k_c$ = 1

$$\Rightarrow \text{[Eq 5.5]: } (l/d)_{\max} = 20 \times 1.014 \times 1 = 20.28$$

$$(l/d)_{provided} = 6000/399 = 15.04 < (l/d)_{\max} \quad \text{— Hence, OK.}$$

**EXAMPLE 5.2**

Design a one–way slab, with a clear span of 4.0 m, simply supported on 230 mm thick masonry walls, and subjected to a live load of 4 kN/m$^2$ and a surface finish of 1 kN/m$^2$. Assume Fe 415 steel. Assume the beam is subjected to moderate exposure conditions.

**SOLUTION**

***Determining $M_u$***

- Assume an effective depth $d \approx \frac{4000}{25}$ = 160 mm and an overall depth $D = 160 + 40 = 200$ mm

$\therefore$ Effective span $l = \begin{cases} 4000 + 230 = 4230 \text{ mm} & \text{(c / c distance)} \\ 4000 + 160 = 4160 \text{ mm} \end{cases}$

Taking the lesser values (as per Code), $l$ = 4.16 m.

- Distributed load due to self-weight, $\Delta w_{DL} = 25 \text{ kN/m}^3 \times 0.2 \text{ m} = 5.0 \text{ kN/m}^2$

  $$\therefore\ w_{DL} = 5.0 + 1.0 = 6.0 \text{ kN/m}^2\ ;\ w_{LL} = 4.0 \text{ kN/m}^2 \text{ (given)}$$

- $\therefore$ Factored load (as per Code) : $w_u = 1.5(w_{DL} + w_{LL})$

  $= 1.5\ (6.0+4.0) = 15.0$ kN/m$^2$

- $\Rightarrow$ Factored Moment (maximum at midspan)

  $$M_u = w_u l^2 / 8 = 15.0 \times 4.16^2 / 8 = \mathbf{32.4\ kNm/m}.$$

***Determining $A_{st}$ (main bars)***

- $R \equiv \dfrac{M_u}{bd^2} = \dfrac{32.4 \times 10^6}{10^3 \times 160^2}$ = 1.267 MPa.
- For moderate exposure conditions, considering M25 grade concrete, $f_{ck}$ = 25 MPa and applying Eq. 5.12,

  $\Rightarrow \dfrac{p_t}{100} \equiv \dfrac{A_{st}}{bd} = \dfrac{25}{2(415)} \left[1 - \sqrt{1 - \dfrac{4.598 \times 1.267}{25}}\right] = 0.374 \times 10^{-2}$

- $\Rightarrow (A_{st})_{reqd}$ = $(0.374 \times 10^{-2}) \times 1000 \times 160$ = 599 mm$^2$/m. [Alternatively, using 'design aids' [Table A.3(a)], the same result is obtained].
- Spacing of bars $s = 1000 A_b / A_{st}$

  Assuming 10 $\phi$ bars $(A_b = \pi \times 10^2 / 4 = 78.5 \text{ mm}^2)$,

  spacing $s_{reqd} = \dfrac{1000 \times 78.5}{599}$ = 131 mm

  [Alternatively, this can be obtained from Table A.6].
- Maximum spacing limits: 3$d$ = 3 × 160 = 480 mm or 300 mm (whichever less)

  $\therefore$ Provide 10 $\phi$ @ 125 mm c/c for main reinforcement.

***Distribution bars***

(to be provided at right angles, in plan, to the main reinforcement — refer Section 5.2.3)

- $(A_{st})_{dist} = 0.0012 \times 1000 \times 200$ = 240 mm$^2$/m.

  Assuming 8 $\phi$ bars $(A_b = \pi \times 8^2 / 4 = 50.3 \text{ mm}^2)$,

  spacing $s_{reqd} = \dfrac{1000 \times 50}{240}$

  = 208 mm.

  Maximum spacing limit: 5$d$ = 5 × 160 = 800 mm or 450 mm (whichever less)

  $\therefore$ Provide 8 $\phi$ @ 200 mm c/c for distribution bars.

***Strength check***

- Providing a clear cover of 30 mm, $d$ = 200–30–10/2 = 165 mm.

$A_{st} = \frac{1000 \times 78.5}{125}$ = 628 mm$^2$/m

$\Rightarrow \; p_t = \frac{100 \times 628}{1000 \times 165}$ = 0.380

$< p_{t,lim}$ = 0.72.

As the actual depth provided (165 mm) is greater than the calculation value (160 mm), and the steel area provided is also greater than the calculation value, it is evident that the section is safe in flexure.

***Deflection control check***

- For $p_t$ = 0.380 and $f_s = 0.58 \times 415 \times \frac{610}{628} = 234$ N/mm$^2$,

  $k_t$ = 1.40 [Fig. 3 of Code or Table 5.2]

  $\Rightarrow (l/d)_{\max} = 20 \times 1.40 = 28.0$

  $(l/d)_{provided} = 4160/165 = 25.2 < 28.0$

  — Hence, OK.

***Detailing***

The complete detailing of the slab† is indicated in Fig. 5.3; this meets the Code requirements [refer Section 5.5]. Alternate bars of the main reinforcement are bent up (cranked) near the supports at a distance of 0.1 $l$ from the support (Cl. D-1.6 of Code) — in order to resist any flexural tension that may possibly arise on account of partial fixity at the support [refer Fig. 1.9(d)].

## 5.6 DESIGN OF CONTINUOUS ONE–WAY SLABS

In wall–supported and beam–supported slab floor systems (with stiff beams) [refer Section 1.6.1], the slab panels are generally continuous over several supporting walls/beams. When the bending is predominantly in one–direction [Fig. 1.9(b),(e)], the slab is called a *one–way continuous* slab system.

### 5.6.1 Simplified Structural Analysis — Use of Moment Coefficients

In order to determine the distribution of bending moments under the design loads (dead loads plus live loads), *structural analysis* has to be performed. For convenience, a strip of 1 *metre* width [Fig. 5.4(a)] is considered (i.e., $b$ = 1000 mm) for analysis and design. As the live loads (unlike the dead loads) are not expected to act all the time, various arrangements of live load have to be considered [refer Cl. 22.4.1 of the Code] in order to determine the maximum load effects; this is discussed in detail in Chapter 9. The (linear elastic) analysis may be done by methods such as the 'moment distribution method'.

† Plans are generally drawn to a scale of 1:50 or 1:100, and section details to a scale of 1:10 or 1:20.

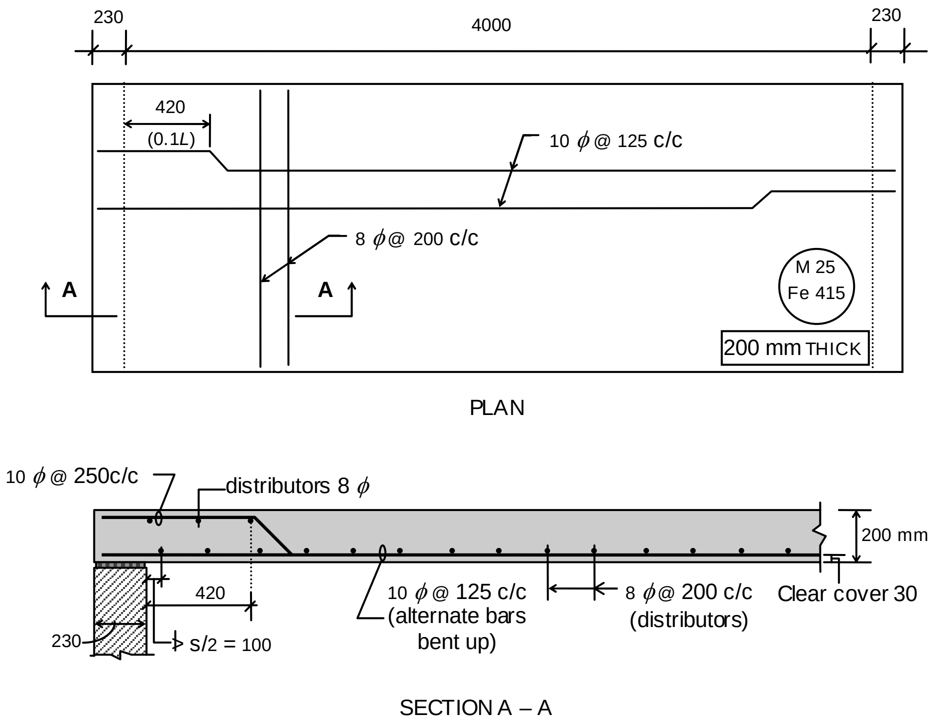
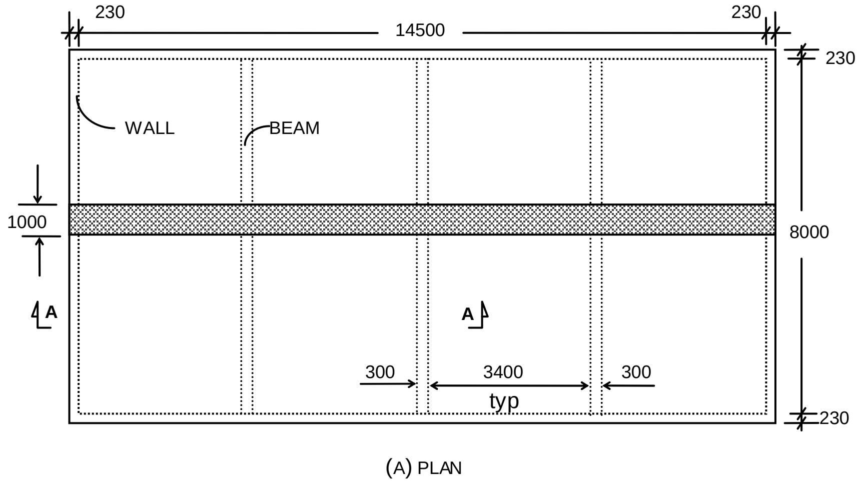



**Fig. 5.3** Details of a one-way slab — Example 5.2

For convenience, the Code (Cl. 22.5) lists *moment coefficients* (as well as *shear coefficients*‡) that are close to the 'exact' values of the maximum load effects obtainable from rigorous analyses on an infinite number of equal spans on point supports [refer 5.3]. The moment coefficients [Table 12 of the Code] are depicted in Fig. 5.4(b). These are applicable to cases of (uniformly loaded) one-way continuous slabs and (secondary) continuous beams with at least three spans "*which do not differ by more than 15 percent of the longest*". In the case of two adjacent spans which are either unequal or unequally loaded, for the negative moment at the support, the average of the two values may be taken. The 'shear coefficients' given by the Code are not shown here, as slabs do not generally have to be checked for shear, the shear stresses being kept in check by the adequate depths provided for deflection control [refer Chapter 6].

‡ The 'shear coefficients' [Table 13 of the Code] are required in the design of continuous secondary beams.



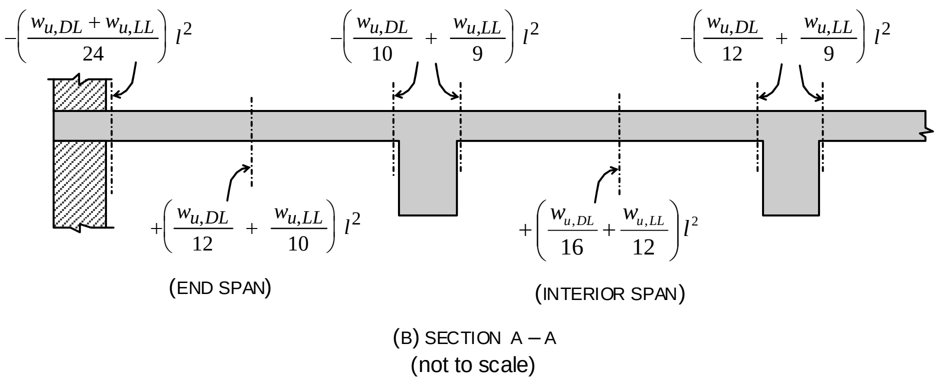




**Fig. 5.4** A continuous one-way slab floor system — Example 5.3

In any span, the maximum sagging ('positive') moment is assumed to be located at the midspan location, and the maximum hogging ('negative') moment at the face of the support (wall / beam). The magnitude of the moment due to the factored dead load $w_{u,DL}$ (per unit length) is obtained by multiplying $w_{u,DL}$ with the relevant moment coefficient and the square of the *effective span l*. Similarly, the moment due to the factored live load $w_{u,LL}$ is obtained.

### *Effective Span*

For continuous spans, the *effective span* (length) depends on the relative width of the support [vide Cl. 22.2 (b) of the Code]. If the width of the support exceeds 1/12 of the clear span or 600 mm, whichever is less, the effective span should be taken as the clear span — except for the end span (whose one end is discontinuous) for which the effective span should be taken as the clear span plus $d/2$ or clear span plus half the width of the discontinuous support, whichever is less. Otherwise, it should be taken as the clear span plus effective depth or centre-to-centre distance between supports, whichever is less (as for simply supported spans).

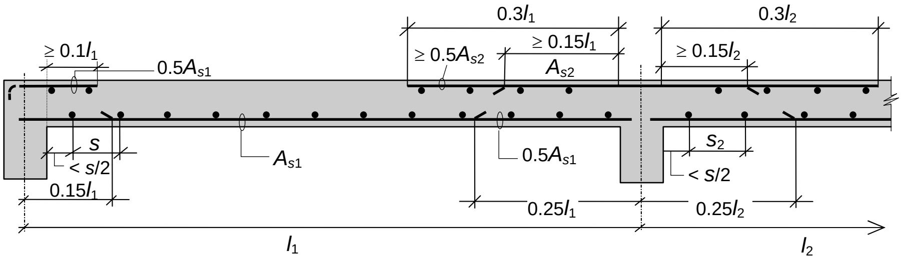
It should be noted that the moment coefficients have been derived, assuming *unyielding* supports. Hence, the use of these coefficients is justified only if the supports are walls or beams that are adequately rigid.

### 5.6.2 Design Procedure

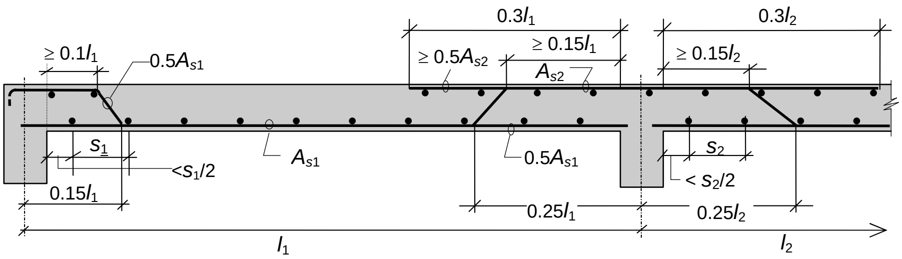
The factored moment $M_u$ at any section is obtained by detailed analysis (or the use of moment coefficients where appropriate), with the load factors applied to DL and LL. The thickness of the slab is usually governed by limiting $l/d$ ratios (for deflection control). In this regard, the *end span* (whose one end is discontinuous) is more critical than the *interior span*. As the midspan moment in the *end span* is significantly larger than that in the *interior span* [Fig. 5.4(b)], the end span will require a larger area of tensile steel, and will govern the thickness based on $(l/d)_{max}$ [Eq. 5.5]. Often, the same thickness is provided for the interior spans also — unless there are a large number of interior spans involved, whereby a separate and lesser thickness may be specified for the interior spans, in the interest of economy.

The required $A_{st}$ for the calculated $M_u$ at the different midspan and support sections should then be determined — by applying Eq. 5.12 or 'design aids' (SP : 16 or Tables A.3(a), (b) given in this book).

The 'positive' and 'negative' moment reinforcement required in the midspan and support regions may be provided in one of two alternative ways, as shown in Fig. 5.5(a) and (b). In the first method, separate reinforcement is detailed for the positive moments and the negative moments [Fig. 5.5(a)]. Alternatively, in the second method, the top ('negative moment') reinforcement over a support region may be provided by bending up alternate bars of the bottom ('positive moment') reinforcement from either side of the support, with additional bars provided (at top), if required.




x = 1059




**(b)** using bent-up bars

**Fig. 5.5** Arrangement of main reinforcement in one-way continuous slabs [Ref. 5.6]

The calculated spacings (required theoretically) in different spans/support regions are *not* directly provided as such (including rounding off to the nearest lower multiple of 5 mm or 10 mm). In practice, it is found desirable to correlate the spacings requirements at the different locations of the continuous slab, and to provide either the same spacing $s$ or a fraction/multiple of it ($s$/4, $s$/2, $2s$ etc.) in all spans/supports, so that placement (including bending up over support become convenient. *Detailing* of *bar cut–off* (curtailment), bending and extensions are discussed in Section 5.9. This involves detailed calculations (involving the bending moment envelope) which are not necessary in the present case, as a simplified analysis (using moment coefficients) is adopted. Accordingly, the details shown in Fig. 5.5, based on the recommendations of SP : 34 [Ref. 5.5], may be adopted for such continuous slabs.

The proposed design should be checked for adequacy in terms of deflection control. Appropriate distribution bars should also be provided, as required by the Code.

### EXAMPLE 5.3

The plan of a floor slab system, covering an area 8.0 m × 14.5 m (clear spans) is shown in Fig. 5.4(a). The slab rests on a 230 mm thick masonry wall all around. For economy, the span of the slab is reduced by providing three (equally spaced) intermediate beams along the 8.0 m direction, as shown. The specified floor loading consists of a live load of 4 kN/m$^2$, and a dead load (due to floor finish, partitions etc.) of 1.5 kN/m$^2$ in addition to the self–weight. Assuming Fe 415 steel, design and detail the floor slab. Assume the beam is subjected to moderate exposure conditions.

**SOLUTION**

- Assuming each beam to be 300 mm wide, the clear spacing between beams is equal to (14.5 – 0.3×3)/4 = 3.4 m. Each slab panel (with clear spans 3.4 m × 8.0 m) has an L/B ratio greater than 2.0, and hence may be treated as *one-way* (*continuous*) [refer Section 1.6.1].

***Determining Values of $M_u$ at Critical Sections***

Consider a 1 m wide design strip [Fig. 5.4(a)].

- *Thickness of slab*:
  Assume a uniform thickness for both *end span* and *interior span*. The 'end span', which is critical, is discontinuous on one edge and continuous at the other.
  Accordingly, assuming $(l/d)_{max} \approx \frac{20+26}{2} \times 1.34$ † = 30,

  $$d_{\min} \approx 3500/30 = 117 \text{ mm (for an assumed effective span of } l = 3.5\text{m)}$$

  Assume overall depth D ≈ 117+35 ≈ 160 mm for all spans and $d$ = 125 mm.
- *Effective length l*: As the beam width (300 mm) exceeds 3400/12 = 283 mm, $l$ =3400 mm (clear span) for the interior span — as per Cl. 22.2(b) of Code.
  For the *end span*, $l = 3400 + d/2$ = 3400+125/2 = 3463 mm.
- *Distributed load due to self-weight*:
  $\Delta w_{DL}$ = 25 kN/m$^3$ × 0.16 = 4.0 kN/m$^2$
  $\therefore w_{DL}$ = 4.0+1.5 = 5.5 kN/$m^2$; $w_{LL}$ = 4.0 kN/m$^2$ (given)
- $\therefore$ Factored loads $\begin{cases} w_{u,DL} = 5.5 \times 1.5 = 8.25 \text{ kN/m}^2 \\ w_{u,LL} = 4.0 \times 1.5 = 6.00 \text{ kN/m}^2 \end{cases}$
- *Factored Moments at critical sections*:
  As the spans are almost equal, uniformly loaded and more than three in number, the simplified analysis using *moment coefficients* [Table 12 of Code] can be applied [Fig. 5.4(b)].

  For end span ($l$ = 3.463 m),

$$M_u = \begin{cases} -\left(\dfrac{w_{u,DL} + w_{u,LL}}{24}\right) l^2 = -7.12 \text{ kNm/m} & \text{at end support} \\ +\left(\dfrac{w_{u,DL}}{12} + \dfrac{w_{u,LL}}{10}\right) l^2 = +15.44 \text{ kNm/m} & \text{at midspan} \\ -\left(\dfrac{w_{u,DL}}{10} + \dfrac{w_{u,LL}}{9}\right) l^2 = -17.89 \text{ kNm/m} & \text{at interior support} \end{cases}$$

† modification factor $k_t$ corresponding to $p_t \approx 0.4$ and $f_s$= 240 N/mm$^2$ [refer Section 5.4.2].

For interior span ($l$ = 3.400 m),

$$M_u = \begin{cases} -\left(\dfrac{w_{u,DL}}{10} + \dfrac{w_{u,LL}}{9}\right)l^2 = -17.24 \text{ kNm/m} & \text{at first interior support} \\ +\left(\dfrac{w_{u,DL}}{16} + \dfrac{w_{u,LL}}{12}\right)l^2 = +11.74 \text{ kNm/m} & \text{at midspan} \\ -\left(\dfrac{w_{u,DL}}{12} + \dfrac{w_{u,LL}}{9}\right)l^2 = -15.65 \text{ kNm/m} & \text{at interior support} \end{cases}$$

At the first interior support, an average value of $M_u$ should be considered:

$$M_u = -(17.89+17.24)/2 = -17.6 \text{ kNm/m}$$

***Determining $A_{st}$***

- For the maximum moment, $M_u = -17.6$ kNm/m at the first interior support,

$$R \equiv \frac{M_u}{bd^2} = \frac{17.6\times10^6}{1000\times125^2} = 1.126 \text{ MPa}$$

  Applying Eq. 5.12, or using design aids (Table A.3(a) SP 16), for M 25 concrete (since the slab is subjected to moderate exposure conditions) and Fe 415 steel,

$$\frac{p_t}{100} = \frac{25}{2(415)}\left[1-\sqrt{1-(4.598\times1.126/25)}\right] = 0.33\times10^{-2}$$

$$(p_t)_{reqd} = 0.33 \Rightarrow (A_{st})_{reqd} = \frac{0.33}{100}\times1000\times125 = 413 \text{ mm}^2/\text{m}$$

  Assuming 10 $\phi$ bars ($A_b$ = 78.5 mm$^2$), spacing reqd = $\dfrac{1000\times78.5}{413}$ = 190 mm.

  Alternatively, for 8 $\phi$ bars ($A_b$ = 50.3 mm$^2$), spacing reqd = $\dfrac{1000\times50.3}{413}$ = 122 mm

  [*Note*: maximum spacing allowed = 3 × 125 = 375 mm (< 450 mm)].

- For convenience, the results for all $(A_{st})_{reqd}$ at the various sections are tabulated in Table 5.5, after performing appropriate calculations (as shown for $M_u = 17.6$ kNm/m). The Table also shows the details of the actual steel provided (assuming the arrangement of bars shown in Fig. 5.5(a)).

- Distribution bars: $(A_{st})_{min} = 0.0012bD$ = 192 mm$^2$/m. Provide 8 mm $\phi$ @ 250c/c

***Deflection control check***

- Maximum midspan steel in the end span:

  $(A_{st})_{provided}$: 8 $\phi$ @ 110 $c/c$ = $\dfrac{1000\times50.3}{110}$ = 457 mm$^2$/m.

  Providing a clear cover of 30 mm, $d$ = 160–30–8/2 = 126 mm.

$$\Rightarrow p_t = \frac{100\times457}{1000\times126} = 0.363,\quad f_s = 0.58\times415\times\frac{360}{457} = 189.6 \text{ N/mm}^2$$

$$\Rightarrow k_t = 1.76 \text{ [Table 5.2]}$$

$\Rightarrow (l/d)_{max} = \frac{1}{2}(20+26)\times 1.76 = 40.4$

$(l/d)_{provided} = 3463/126 = 27.5 < 40.4 \Rightarrow$ OK

Evidently, the limiting $(l/d)$ ratio will be satisfied by the interior span as well.

**Table 5.5** Calculation of $A_{st}$ at critical locations of a one-way continuous slab system — Example 5.3

| Location | End Span | | | Interior Span | |
|---|---|---|---|---|---|
| | end support | midspan | first int.support | midspan | int.support |
| $M_u$ (kNm/m) | –7.1 | +15.4 | –17.6 | +11.7 | –15.6 |
| $M_u/bd^2$ (MPa ) | 0.456 | 0.988 | 1.126 | 0.751 | 1.002 |
| $(p_t)_{reqd}$ | 0.129 | 0.287 | 0.33 | 0.215 | 0.292 |
| $(A_{st})_{reqd}$ (mm$^2$/m) | 162 | 360 | 413 | 270 | 365 |
| $(A_{st})_{min}$ (mm$^2$/m) | $0.0012bD = 0.0012\times10^3\times160 = 192$ | | | | |
| Reqd. Spacing (mm) using | | | | | |
| a) 10 $\phi$ | 408 | 218 | 190 | 285 | 215 |
| b) 8 $\phi$ | 261 | 139 | 121 | 186 | 137 |
| Max.Spacing (mm) | $3\times d = 3\times126 = 378$ (> 300) | | | | |
| Spacing of bars provided | (top) | (bottom) | (top) | (bottom) | (top) |
| 10 $\phi$ | — | — | — | — | — |
| 8 $\phi$ | 220 | 110 | 110 | 150 | 110 |

***Detailing*** : the sectional details of the design are shown in Fig. 5.6.

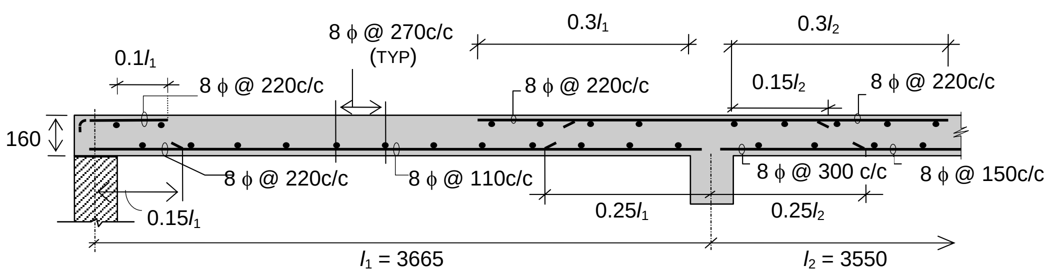



**Fig. 5.6** Details of one-way continuous slab — Example 5.3

## 5.7 DESIGN OF DOUBLY REINFORCED RECTANGULAR SECTIONS

As explained earlier, doubly reinforced sections are generally resorted to in situations where the cross-sectional dimensions of the beam are restricted (by architectural or other considerations) and where singly reinforced sections (with $p_t = p_{t,lim}$) are not adequate in terms of moment-resisting capacity. Doubly reinforced beams are also used in situations where reversal of moments is likely (as in multi-storeyed frames subjected to lateral loads). The presence of compression reinforcement reduces long-term deflections due to shrinkage (refer Section 5.3.2). All compression reinforcement must be enclosed by *closed* stirrups [Fig. 5.1(c)], in order to prevent their possible buckling and to provide some ductility by confinement of concrete.

### 5.7.1 Design Formulas

As the dimensions of the beam section already are fixed, the design problem is one of determining the areas of reinforcement required in tension ($A_{st}$) and compression ($A_{sc}$).

As explained in Section 4.7.5, from the design point of view, it is necessary to limit the neutral axis depth $x_u$ to $x_{u,max}$. This can be done conveniently by considering $x_u = x_{u,max}$ and resolving the factored moment $M_u = M_{uR}$ into two components [Fig. 5.7]:

$$M_u = M_{uR} = M_{u,\lim} + \Delta M_u \tag{5.13}$$

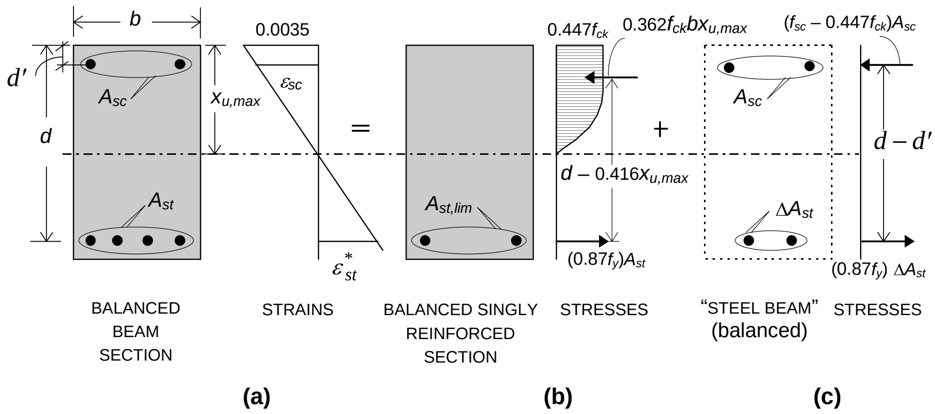



**Fig. 5.7** Concept underlying the design of a 'balanced' doubly reinforced section

where $M_{u,lim}$ is the limiting moment capacity of a singly reinforced 'balanced' section (without compression steel) [Fig. 5.7(b)] given by Eq. 5.8, and $\Delta M_u$ is the additional moment capacity desired from the compression steel $A_{sc}$ and the corresponding additional tension steel $\Delta A_{st}$ — which may be visualised as the flanges of an equivalent 'steel beam' [Fig. 5.7(c)]. As the contribution of concrete in compression is entirely accounted for in $M_{u,lim}$, it does not contribute to $\Delta M_u$. The distribution of strains, given by the condition, [Fig. 5.7(a)] is identical for both the components, and the corresponding distributions of stresses are as shown in Fig. 5.7(b) and (c).

If $p_t$ denotes the total percentage of tension steel required for a 'balanced' doubly reinforced section, then corresponding to Eq. 5.13, it can also be resolved into two components.

$$p_t = p_{t,lim} + \Delta p_t \tag{5.14a}$$

$$\text{or} \quad A_{st} = A_{st,lim} + \Delta A_{st} \tag{5.14b}$$

where $A_{st,lim} = \frac{p_{t,lim}}{100} bd$ or (given by Eq 5.7) is the tension steel corresponding to $M_{u,lim}$ and $\Delta p_t = 100(\Delta A_{st})/bd$ is that corresponding to $\Delta M_u$. Evidently, the moment $\Delta M_u$ is obtained from a couple comprising a (compressive) force $(f_{sc} - 0.447 f_{ck})A_{sc}$ and an equal and opposite (tensile) force $(0.87 f_y \, \Delta A_{st})$, with a lever arm $(d - d')$. The stress $f_{sc}$ in the compression steel (at the ultimate limit state) depends on the strain $\varepsilon_{sc}$ (given by Eq. 4.78) which is controlled by the linear strain distribution with the neutral axis located at $x = x_{u,max}$ [Fig. 5.7(a)]. Values of $f_{sc}$ for different grades of steel and typical $d'/d$ ratios are listed in Table 4.5. (It may be noted from Table 4.5 that the full design yield stress is attained only in the case of mild steel.) Based on the above, following formulas are obtainable:

$$\Delta A_{st} = \frac{M_u - M_{u,lim}}{0.87 f_y (d - d')} \tag{5.15a}$$

$$\frac{\Delta p_t}{100} = \frac{R - R_{lim}}{0.87 f_y (1 - d'/d)} \tag{5.15b}$$

$$p_t = p_{t,lim} + \frac{100(R - R_{lim})}{0.87 f_y (1 - d'/d)} \tag{5.16}$$

where $R \equiv M_u \big/ bd^2$ and $R_{lim} \equiv M_{u,lim} \big/ bd^2$

$$A_{sc} = \frac{(0.87 f_y)(\Delta A_{st})}{f_{sc} - 0.447 f_{ck}} \tag{5.17a}$$

$$\text{or} \quad p_c = \frac{0.87 f_y (p_t - p_{t,lim})}{f_{sc} - 0.447 f_{ck}} \tag{5.17b}$$

Applying Eq. 5.16 and Eq. 5.17b, design aids can be generated to give values of $p_t$ and $p_c$ for a given $M_u/(bd^2)$ — for various combinations of concrete and steel grades and different $d'/d$ ratios. These have been developed in Table A.4 (placed in Appendix A of this book) for the commonly used combination of M 20 and M 25 grades of concrete with Fe 415 steel. Four typical ratios of $d'/d$ (viz., 0.05, 0.10, 0.15, and 0.20) are covered in Table A.4. The Design Handbook, SP 16 [Ref. 5.5], gives such Tables for other combinations of concrete and steel grades.

### 5.7.2 Design Procedure for Given $M_u$

#### *Determining $A_{st}$*

For a given rectangular section (with given $b$, $d$, and given $f_{ck}$, $f_y$ ), the limiting moment capacity for a singly reinforced section ($M_{u,lim}$) should be first determined, using Eq. 5.8. If $M_{u,lim}$ is greater than or equal to the factored moment $M_u$, the section should be designed as a singly reinforced section — as described in Section 5.5.2. Otherwise (for $M_u > M_{u,lim}$), the section should be designed as a doubly reinforced section.

The value of $p_{t,lim}$ is determined from Eq. 5.7, and the values of $\Delta A_{st}$ from Eq. 5.15(a) assuming a suitable value for $d'$. The total $(A_{st})_{reqd}$ is then obtained from Eq. 5.14(b). The bars should be suitably selected such that the $A_{st}$ actually provided is as close as possible in magnitude to, but not less than $(A_{st})_{reqd}$. Tables A.5 and A.6 may be used for this purpose.

If the placement of bars results in a new value of the effective depth $d$ which is significantly different from the original value assumed, $M_{u,lim}$, $A_{st,lim}$ and $(A_{st})_{reqd}$ should be recalculated at this stage itself.

#### *Determining $A_{sc}$*

Using the value of $\Delta A_{st}$ *actually provided*, the value of $(A_{sc})_{reqd}$ may be calculated from Eq. 5.17a. The value of $f_{sc}$ can be obtained from the value of $\varepsilon_{sc}$ [Fig. 5.7(a) or Eq. 4.78] and the stress-strain relation [Table 3.2], or alternatively from Table 4.5 by interpolating for the calculated value of $d'/d$. The compression bars should be suitably selected such that the $A_{sc}$ provided is as close as possible (but not less than $(A_{sc})_{reqd}$). Such a design procedure will result invariably in an adequately 'safe' design with $M_{uR} \geq M_u$ and $x_u \leq x_{u,max}$; however, a design check for strength is always desirable.

#### *Alternative: Using Design Aids*

Design aids (Table A.4, SP:16) may be used to determine the $(p_t)_{reqd}$ and $(p_c)_{reqd}$ for the calculated value of $M_u/bd^2$; accordingly, $A_{st}$ and $A_{sc}$ are suitably provided. This is the most commonly adopted method in practice. However, it should be noted that if the $A_{st}$ actually provided is well in excess of the $(A_{st})_{reqd}$, there is a possibility of ending up with an *over-reinforced section* (with $x_u > x_{u,max}$). In order to avoid such a situation, (which is undesirable, and also not permitted by the Code), a correspondingly higher value of $A_{sc}$ should by provided [Eq. 5.17a] such that the resulting $p_c > p_c^*$ (given by Eq. 4.84).

A design check for safety in flexural strength is desirable — by properly analysing the designed section. It may be noted, however, that designers in practice, who are habituated with the use of design aids, often neglect to do this. It is often assumed that the strength design requirements are automatically satisfied if the values of $p_t$ and $p_c$ provided are in excess of that given in the Tables. This is not always true — as demonstrated in Example 5.4.

### Check for Deflection Control

The limiting ($l/d$) ratio for deflection control [Eq. 5.5] is generally satisfied by doubly reinforced beams, on account of the modification factor ($k_c$) for the compression steel [Table 5.3]. However, in the case of relatively shallow beams, a check for deflection control becomes necessary.

**EXAMPLE 5.4**

Design the flexural reinforcement for the beam in Example 5.1, given that its size is limited to 250 mm × 400 mm, and that it has to carry, in addition to the loads already mentioned, a concentrated dead load of 30 kN placed at the midspan point. Assume that the beam is subjected to moderate exposure conditions.

**SOLUTION**

***Determining $M_u$ for design***

Given $b$ = 250 mm, $D$ = 400 mm, $f_{ck}$ = 25 MPa , $f_y$ = 415 MPa
Let $d$ = D–50 = 350 mm.
$\Rightarrow$ Effective span $l$ = 6.0 m (as in Example 5.1).

- Loads: $w_{DL} = 5.0 \text{ kN/m} + \Delta w_{DL}$ (self-weight), $W_{DL} = 30$ kN at midspan.
  Due to self-weight, $\Delta w_{DL}$ = 25×0.25×0.4 = 2.5 kN/m, $W_{LL}$ = 10.0 kN/m
  Factored Load: $w_u$ = (5.0 + 2.5 + 10.0) × 1.5 = 26.25 kN/m
  and $W_u$ = 30 ×1.5 = 45 kN (at midspan)
- $\therefore$ Factored Moment (maximum at midspan):
  $M_u = 26.25 \times 6.0^2/8 + 45 \times 6.0/4$ = 185.6 kN$m$ = **186 kNm**.

***Singly reinforced or doubly reinforced section?***

- For $M_{u,lim} = 0.1389 f_{ck} bd^2 = 0.1389 \times 25 \times 250 \times 350^2$ (for Fe 415 with M 25)
  = 106.3 × $10^6$ Nmm = 106 kNm
  As $M_u > M_{u,\lim}$, the section has to be doubly reinforced, with $p_t > p_{t,\lim}$, where
  $$p_{t,lim} = 41.61\left(\frac{f_{ck}}{f_y}\right)\left(\frac{x_{u,\max}}{d}\right) = 1.201 \text{ for Fe 415 with M 25.}$$

***Determining $A_{st}$***

- Considering a 'balanced section' $(x_u = x_{u,\max})$,
  $A_{st} = A_{st,lim} + \Delta A_{st}$
  where $A_{st,\lim} = \frac{1.201}{100} \times 250 \times 350 = 1051 \text{ mm}^2$
- Assuming 20 mm $\phi$ bars for compression steel,

$d' \approx 48$ mm (30 mm clear cover + 8 mm stirrup + $\phi$/2)

$$(\Delta A_{st})_{reqd} = \frac{(186-106)\times10^6}{0.87\times415\times(350-48)} = 734 \text{ mm}^2$$

$$\therefore (A_{st})_{reqd} = 1051 + 734 = 1785 \text{ mm}^2$$

Using 3 bars, $\phi_{reqd} = \sqrt{\dfrac{1777/3}{\pi/4}} = 27.5$ mm

<u>Provide 3 nos 28 mm</u> $\phi$ [$A_{st}$ = 3 × 616 = 1848 mm$^2$].

- Actual $d$ (assuming 30 mm clear cover and 8 mm stirrups):
  $d$ = 400-(30+8+28/2) = 348 mm < 350 mm assumed earlier
  Revising the above calculations with $d$ = 348 mm,
  $M_{u,\lim}$ = 105 kNm, $A_{st,\lim}$ = 1.201×250×348/100 = 1045 mm$^2$,
  $(\Delta A_{st})_{reqd}$ = 748 mm$^2$, $(A_{st})_{reqd}$ = 1045+748 = 1793 mm$^2$,
  $\therefore$ Actual $(\Delta A_{st})_{provided}$ = 1848 – 1045 = 803 mm$^2$

***Determining $A_{sc}$***

- Assuming $x_u = x_{u,max}$, for $d'/d$ = 48/348 = 0.138, from Table 4.5,

$$f_{sc} = 351.9 - (351.9 - 342.4)\times\frac{0.138-0.100}{0.15-0.10} = 344.7 \text{ MPa}$$

[Alternatively, applying $x_{u,\max}/d$ = 0.479 in Eq. 4.78,

$\varepsilon_{sc} = 0.0035(1-0.138/0.479) = 0.00249$

$\Rightarrow f_{sc}$ = 344.7 MPa (from Table 3.2 or design stress-strain curve)]

$$(A_{sc})_{reqd} = \frac{0.87 f_y (\Delta A_{st})}{f_{sc} - 0.447 f_{ck}} = \frac{(0.87\times415)\times803}{344.7-(0.447\times25)} = 869 \text{ mm}^2$$

Using 2 bars, $\phi_{reqd} = \sqrt{\dfrac{869/2}{\pi/4}} = 23.5$ mm

Using 3 bars, $\phi_{reqd} = \sqrt{\dfrac{869/3}{\pi/4}} = 19.2$ mm

Provide 3 nos 20 mm $\phi$ [$A_{sc}$ = 3 × 314 = 942 mm$^2$ > 869 mm$^2$].

The proposed section is shown in Fig. 5.8.

[As an exercise in analysis, the student may verify that this section satisfies the design conditions: $M_{uR} \geq M_u$ and $x_u \leq x_{u,\max}$.]

**Alternative method: using design aids**

Assuming $d$ = 350 mm, $d'$ = 50 mm,

$$\frac{M_u}{bd^2} = \frac{186\times10^6}{250\times350^2} = 6.073 \text{ MPa}$$

Referring to Table A.4b (M 25 concrete and Fe 415 steel),
for $d'/d$ = 50/350 = 0.143 and $M_u/bd^2$ = 6.073 MPa, by linear interpolation,

$p_t \approx 2.042 \Rightarrow (A_{st})_{reqd} = \frac{2.042}{100} \times 250 \times 350 = 1787 \text{ mm}^2$

$p_c \approx 0.913 \Rightarrow (A_{sc})_{reqd} = \frac{0.913}{100} \times 250 \times 350 = 799 \text{ mm}^2$

Provide 3–28 $\phi$ for tension steel [$A_{st}$ = 3×616 = 1848 mm$^2$ > 1787]
and 4–16 $\phi$ for compression steel [$A_{sc}$ = 4×201 = 804 mm$^2$ > 799].

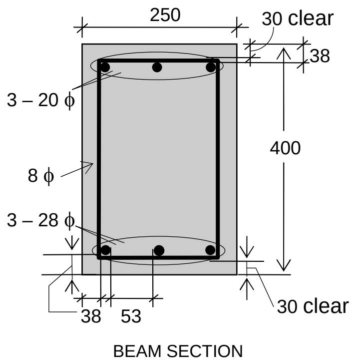



**Fig. 5.8** Doubly reinforced section design — Example 5.4

***Design check***

- To ensure $x_u \le x_{u,\max}$, it suffices to establish $p_c \ge p_c^*$ [Eq. 4.84]
  Actual $d$ provided: $d$ = 400 – 30 – 8 – 28/2 = 348 mm;
  $d' = 30 + 16/2$ +8= 46 mm
  For $d'/d$ = 46/348 = 0.132, $f_{sc}$ = 345.8 MPa [Table 4.5].
  [Alternatively, $\varepsilon_{sc} = 0.0035(1 - 0.132/0.479) = 0.00253$
  $\Rightarrow f_{sc}$ = 345.8 MPa (Table 3.2).]
  Actual $p_t$ provided: $p_t$ = 100×1848/ (250×348) = 2.124
  Actual $p_c$ provided: $p_c$ = 100×804/ (250×348) = 0.924

$$p_c^* = \frac{0.87 f_y (p_t - p_{t,lim})}{f_{sc} - 0.447 f_{ck}} = \frac{0.87 \times 415 \times (2.124 - 1.201)}{345.8 - (0.447 \times 25)} = 0.996$$

- As $p_c$ is slightly less than $p_c^*$, the section is slightly over-reinforced. [This can also be verified by applying Eq. 4.81, which gives
  $x_u/d$ = 0.505 > $x_{u,\max}/d$ = 0.479.]

***Revised design***

To ensure ductile failure,

- $A_{sc} > \frac{p_c^*}{100} bd = \frac{0.996}{100} \times 250 \times 348 = 867 \text{ mm}^2$

Provide 3–20$\phi$ for compression steel [ $A_{sc}$ = 3×314 = 942 mm$^2$ > 867 — as shown in Fig. 5.8]: $p_c$ = 100 × 942/(250 × 348) = 1.083 > $p_c^*$ $\Rightarrow$ OK.

***Check for deflection control***

$p_t$ = 2.124 and $f_{st}$ = 0.58×415×1787/1848 = 233 MPa

$\Rightarrow$ $k_t$ = 0.842 [Table 5.2 or Fig. 4 of Code]

$p_c$ = 1.083 $\Rightarrow$ $k_c$ = 1.263 [Table 5.3 or Fig. 5 of Code]

- Applying Eq. 5.5,
  $(l/d)_{max}$ = 20 × 0.842 × 1.263 = 21.27
  $(l/d)_{provided}$ = 6000/348 = 17.24 < 21.27 — Hence OK.

## 5.8 DESIGN OF FLANGED BEAM SECTIONS

T-beams and L-beams were introduced in Section 4.6.4. The integral† connection between the slab and the beam in cast in-situ construction makes the two act integrally, so that some portion of the slab functions as a flange of the beam. It should be noted that the flange is effective only when it is on the compression side, i.e., when the beam is in a 'sagging' mode of flexure (not 'hogging') with the slab on top. Alternatively, if the beam is 'upturned' (inverted T-beam) and it is subjected to 'hogging' moments (as in a cantilever), the T-beam action is effective, as the flange is under compression.

Ideal flanged beam action occurs when the flange dimensions are relatively small while the beam is deep — as in the case of closely spaced long-span bridge girders in a T-beam bridge. The beam is invariably heavily reinforced in such cases.

### 5.8.1 Transverse Reinforcement in Flange

The integral action between the flange and the web is usually ensured by the transverse bars in the slab and the stirrups in the beam. In the case of isolated flanged beams (as in spandrel beams of staircases), the detailing of reinforcement depicted in Fig. 5.9(a) may be adopted. The overhanging portions of the slab should be designed as cantilevers and the reinforcement provided accordingly.

Adequate transverse reinforcement must be provided near the top of the flange. Such reinforcement is usually present in the form of negative moment reinforcement in the continuous slabs which span across and form the flanges of the T-beams. When this is not the case (as in slabs where the main bars run *parallel* to the beam), the Code (Cl. 23.1.1b) specifies that transverse reinforcement should be provided in the flange of the T-beam (or L-beam) as shown in Fig. 5.9(b). The area of such steel should be not less than 60 percent of the main area of steel provided at the midspan of the slab, and should extend on either side of the beam to a distance not less than one-fourth of the span of the beam.

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† Where the slab and beam are not cast monolithically, flanged beam action cannot be assumed, unless special shear connectors are provided at the interface between beam and the slab.

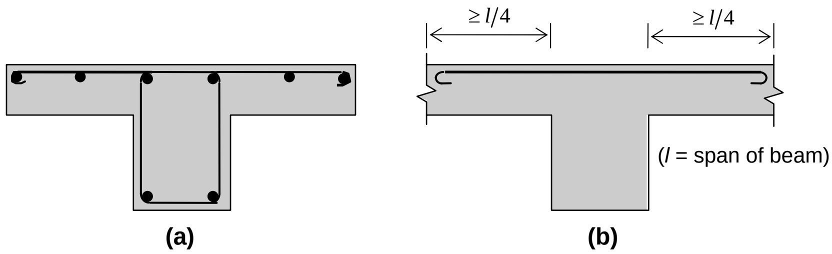



**Fig. 5.9** Detailing of flanged beams to ensure integral action of slab and beam.

### 5.8.2 Design Procedure

In the case of a *continuous* flanged beam, the negative moment at the face of the support generally exceeds the maximum positive moment (at or near the midspan), and hence governs the proportioning of the beam cross-section. In such cases of negative moment, if the slab is located on top of the beam (as is usually the case), the flange is under flexural tension and hence the concrete in the flange is rendered ineffective. The beam section at the support is therefore to be designed as a rectangular section for the factored negative moment† . Towards the midspan of the beam, however, the beam behaves as a proper flanged beam (with the flange under flexural compression). As the width of the web $b_w$ and the overall depth $D$ are already fixed from design considerations at the support, all that remains to be determined is the area of reinforcing steel; the *effective width of flange* is determined as suggested by the Code [Eq. 4.30].

In simply supported flanged beams, however, the web dimensions must also be designed (if not otherwise specified). The width of the web is generally fixed as 250 mm, 300 mm, 350 mm (as for a rectangular section), and the overall depth assumed to be approximately span/13 to span/16. An approximate estimate of the area of tension steel $A_{st}$ can be obtained as follows:

$$(A_{st})_{reqd} = \frac{M_u}{0.87 f_y z} \tag{5.18}$$

where the lever arm $z$ may be taken approximately as $0.9d$ or $(d - D_f/2)$, whichever is larger. If convenient, the reinforcement should be accommodated in one layer — although, often this may not be possible. When the tension steel is provided in more than one layer, the effective depth gets reduced.

The determination of the actual reinforcement in a flanged beam depends on the location of the neutral axis $x_u$, which, of course, should be limited to $x_{u,max}$. If $M_u$ exceeds $M_{u,lim}$ for a singly reinforced flanged section, the depth of the section should be suitably increased; otherwise, a doubly reinforced section is to be designed.

† In such cases it is desirable to distribute the tension steel at the top of the web across the effective width of the flange, to protect the integral flange from cracking — as recommended by the ACI Code. Alternatively, additional reinforcement may be provided in the flange region for this purpose.

### *Neutral Axis within Flange* ($x_u \le D_f$):

This is, by far, the most common situation encountered in building design. Because of the very large compressive concrete area contributed by the flange in T-beams and L-beams of usual proportions, the neutral axis lies within the flange ($x_u \le D_f$), whereby the section behaves like a rectangular section having width $b_f$ and effective depth $d$.

A simple way of first checking $x_u \le D_f$ is by verifying $M_u \le (M_{uR})_{x_u=D_f}$ where $(M_{uR})_{x_u=D_f}$ is the limiting ultimate moment of resistance for the condition $x_u = D_f$ and is given by

$$(M_{uR})_{x_u=D_f} = 0.362 f_{ck} b_f D_f \; (d - 0.416 D_f) \tag{5.19}$$

It may be noted that the above equation is meaningful only if $x_{u,max} > D_f$. In rare situations involving very thick flanges and relatively shallow beams, $x_{u,max}$ may be less than $D_f$. In such cases, $M_{u,lim}$ is obtained by substituting $x_{u,max}$ in place of $D_f$ in Eq. 5.19.

### *Neutral Axis within Web* ($x_u > D_f$):

When $M_u > (M_{uR})_{x_u=D_f}$, it follows that $x_u > D_f$. The accurate determination of $x_u$ can be somewhat laborious†. As explained in Chapter 4, the contributions of the compressive forces $C_{uw}$ and $C_{uf}$ in the 'web' and 'flange' may be accounted for separately as follows:

$$M_u = C_{uw}(d - 0.416 x_u) + C_{uf}(d - y_f/2) \tag{5.20}$$

where,

$$C_{uw} = 0.362 f_{ck} b_w x_u \tag{5.21}$$

$$C_{uf} = 0.447 f_{ck} (b_f - b_w) y_f \tag{5.22}$$

and the equivalent flange thickness $y_f$ is equal to or less than $D_f$ depending on whether $x_u$ exceeds $7D_f/3$ or not.

For $x_{u,max} \ge 7D_f/3$, the value of the ultimate moment of resistance $(M_{uR})_{x_u=7D_f/3}$ corresponding to $x_u = 7D_f/3$ and $y_f = D_f$ may be first computed. If the factored moment $M_u \ge (M_{uR})_{x_u=7D_f/3}$, it follows that $x_u > 7D_f/3$ and $y_f = D_f$. Otherwise, $D_f < x_u < 7D_f/3$ for $(M_{uR})_{x_u=D_f} < M_u < (M_{uR})_{x_u=7D_f/3}$ and

$$y_f = 0.15 x_u + 0.65 D_f \tag{5.23}$$

† As an alternative to this procedure, a design based on the approximate estimate of $A_{st}$ [Eq. 5.18] may be assumed, and the resulting section *analysed* to determine $M_{uR}$. The design becomes acceptable if $M_{uR} \ge M_u$ and $x_u \le x_{u,max}$.

Inserting the appropriate value — $D_f$ or the expression for $y_f$ (given by Eq. 5.23) — in Eq. 5.20, the resulting quadratic equation (in terms of the unknown $x_u$) can be solved to yield the correct value of $x_u$. Corresponding to this value of $x_u$, the values of $C_{uw}$ and $C_{uf}$ can be computed [Eq. 5.21, 5.22] and the required $A_{st}$ obtained by solving the force equilibrium equation.

$$T_u = 0.87 f_y A_{st} = C_{uw} + C_{uf}$$

$$\Rightarrow (A_{st})_{reqd} = \frac{C_{uw} + C_{uf}}{0.87 f_y} \qquad (5.24)$$

### EXAMPLE 5.5

Design the interior beam in the floor system in Example 5.3 [Fig. 5.4(a)]. Assume that the beam is subjected to moderate exposure conditions. Use Fe 415 steel.

#### SOLUTION

- The slab is one-way, spanning between the beams, which are simply supported and hence behave as T-beams [ $l_0$ = 8230 mm, $D_f$ = 160 mm, $b_w$ = 300 mm].

  *Effective flange width* (Cl 23.1.2 Code):

  $b_f = l_0/6 + b_w + 6D_f$ [Eq. 4.30]

  $= 8230/6 + 300 + (6 \times 160)$ = 2632 mm,

  which is acceptable as it is less than $b_w$ + clear span of slab (300+3400 = 3700).
- Assume overall depth D $\approx l/15$ = 550 mm.

  $\Rightarrow$ effective depth d $\approx$ 500 mm.

  $\Rightarrow$ effective span $l$ = 8.0 + 0.23 m = 8.23 m (less than 8.0 + 0.5 = 8.5 m).

***Determining $M_u$ for design***

- Distributed loads from slab (refer Example 5.3):

  $w_{DL}$ = 5.5 kN/m$^2$×3.7 m = 20.35 kN/m

  $w_{LL}$ = 4.0 kN/m$^2$×3.7 m = 14.8 kN/m

  Additional dead load due to self weight of *web:*

  $\Delta w_{DL}$ = 25×0.3×(0.55–0.16) = 2.93 kN/m
- $\therefore$ Factored load $w_u$ = 1.5×(20.35 + 14.8 + 2.93) = 57.12 kN/m.
- $\Rightarrow$ Factored moment (maximum at midspan)

  $M_u = w_u l^2/8$ = 57.12×8.23$^2$/8 = 484 kNm

***Determining approximate $A_{st}$***

- Assuming a lever arm $z$ equal to the larger of 0.9$d$ = 450 mm and $d - D_f/2$ = 420 mm, i.e., $z \approx$ 450 mm,

  $$(A_{st})_{reqd} = \frac{M_u}{0.87 f_y z} \approx \frac{484 \times 10^6}{0.87 \times 415 \times 450} = 2979 \text{ mm}^2$$

- Providing 3 bars, $\phi_{reqd} = \sqrt{\dfrac{2979/3}{\pi/4}}$ = 35.5 mm

  or, providing 4 bars, $\phi_{reqd} = \sqrt{\dfrac{2979/4}{\pi/4}}$ = 30.8 mm

- [Alternatively, this is obtainable from Table A.6.]

  It may be observed that the bars (either 3–36 $\phi$ or 4–32 $\phi$) can be accommodated in one layer, given $b_w$ = 300 mm. Assuming 32 mm $\phi$ bars and 8 mm $\phi$ stirrups,

  $\Rightarrow$ Actual $d$ = 550–32–8–32/2 = 494 mm

  (clear cover shall not be less than the diameter of the bar)

***Determining actual $A_{st}$***

$x_{u,max}$ = 0.479×494 = 237 mm > $D_f$ = 160 mm.

- Assuming the neutral axis to be located at $x_u = D_f$

  $(M_{uR})_{x_u = D_f}$ = 0.362×25×2632×160× (494–0.416×160) [for M 25 concrete]

  = 1629×10$^6$ Nmm > $M_u$ = 484 kNm

  Hence, the neutral axis is located definitely within the flange ($x_u < D_f$).

- Accordingly, designing the T-section as a singly reinforced rectangular section with $b = b_f$ = 2632 mm and $d$ = 494 mm,

  $$R \equiv \frac{M_u}{bd^2} = \frac{484\times10^6}{2632\times494^2} = 0.753 \text{ MPa}$$

  $$\frac{(p_t)_{reqd}}{100} = \frac{25}{2\times415}\left[1-\sqrt{1-\frac{4.598\times0.753}{25}}\right] = 0.216\times10^{-2}$$

  $$\Rightarrow (A_{st})_{reqd} = \frac{0.216}{100}\times2632\times494 = 2808 \text{ mm}^2$$

  (which incidentally is about 6 percent less than the approximate value calculated earlier).

- Provide 2–32 $\phi$ plus 2–28 $\phi$ bars

  [$A_{st}$ = (2×804) + (2×616) = 2840 mm$^2$ > 2808 mm$^2$].

  The cross-section of the beam, showing the location of bars, is depicted in Fig. 5.10.

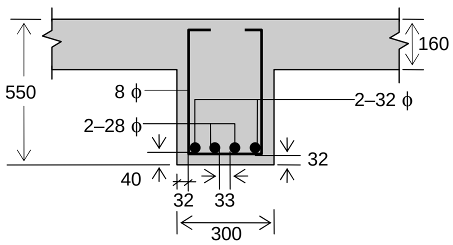



**Fig. 5.10** T-beam of Example 5.5

***Check for Deflection Control***

- Ignoring the contribution of flanges (conservative) [refer Section 5.3.2],

$p_t = \frac{100 \times 2840}{300 \times 494} = 1.92$; $f_s = 0.58 \times 415 \times \frac{2808}{2840} = 238$ MPa

$\Rightarrow k_t = 0.844$ [Table 5.2]

$\Rightarrow (l/d)_{max} = 20 \times 0.844 \times 1 = 16.88$ [Eq. 5.5]

$(l/d)_{provided} = 8230/494 = 16.66 < 16.88$ — Hence, OK.

## EXAMPLE 5.6

A continuous T-beam has the cross-sectional dimensions shown in Fig. 5.11(a). The web dimensions have been determined from the consideration of negative moment at support and shear strength requirements. The span is 10 m and the design moment at midspan under factored loads is 800 kNm. Determine the flexural reinforcement requirement at midspan. Consider Fe 415 steel. Assume that the beam is subjected to moderate exposure conditions.

### SOLUTION

***Determining approximate $A_{st}$***

- *Effective flange width $b_f$*

Actual flange width provided = 1500 mm; $D_f$ = 100 mm, $b_w$ = 300 mm.

Maximum width permitted = (0.7 × 10000)/6 + 300 + (6 × 100) = 2067 mm

> 1500 mm.

$\therefore b_f$ = 1500 mm

- Assuming $d \approx 650$ mm and a lever arm $z$ equal to the larger of $0.9d$ = 585 mm and $d - D_f/2$ = 600 mm, i.e., $z \approx 600$ mm,

$$(A_{st})_{reqd} \approx \frac{800 \times 10^6}{0.87 \times 415 \times 600} = 3693 \text{ mm}^2$$

- Providing 4 bars, $\phi_{reqd} = \sqrt{\frac{3693/4}{\pi/4}}$ = 34.3 mm, i.e., 36 mm.

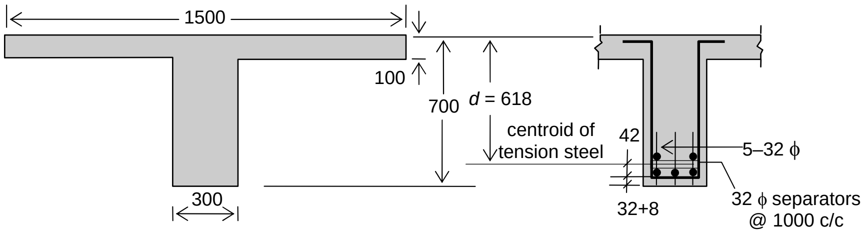
  As 4–36ϕ bars cannot be accommodated in one layer within the width $b_w$ = 300 mm, two layers are required.
  Assuming a reduced $d \approx 625$ mm, $z \approx 625 - 100/2 = 575$ mm.

  $\Rightarrow (A_{st})_{reqd} \approx 3693 \times \frac{600}{575} = 3854 \text{ mm}^2$.

- Provide 5–32ϕ [$A_{st} = 804 \times 5 = 4020 \text{ mm}^2$] with 3 bars in the lower layer plus 2 bars in the upper layer, with a clear vertical separation of 32 mm — as shown in Fig. 5.11(b). Assuming 8 mm stirrups and a clear 32 mm cover to *stirrups*,

  $\Rightarrow d = 700 - 32 - 8 - \frac{1}{5}[(3 \times 16) + 2 \times (32 + 32 + 16)]$

  $= 700 - 40 - 41.6 = 618$ mm




**(a)** given section **(b)** proposed reinforcement

**Fig. 5.11** T-beam of Example 5.6

***Determining actual $A_{st}$***

$x_{u,max} = 0.479 \times 618 = 296$ mm

- As $x_{u,max} > D_f = 100$ mm, the condition $x_u = D_f$ satisfies $x_u \leq x_{u,max}$:
- Assuming M 25 concrete, $f_{ck} = 25$ MPa

  $(M_{uR})_{x_u=D_f} = 0.362 \times 25 \times 1500 \times 100 \times (618 - 0.416 \times 100)$

  $= 782.5 \times 10^6 \text{ Nmm} < M_u = 800 \text{ kNm}$

  $\Rightarrow x_u > D_f$ and $M_u = C_{uw}(d - 0.416\, x_u) + C_{uf}\,(d - y_f/2)$,

  where $C_{uw} = 0.362 f_{ck}\, b_w\, x_u = 0.362 \times 25 \times 300 x_u = (2715 x_u)$ N

  and $C_{uf} = 0.447 f_{ck}\,(b_f - b_w) y_f = 0.447 \times 25 \times (1500 - 300) y_f = (13410 y_f)$ N
- Considering $x_u = 7D_f/3 = 233$ mm ( $< x_{u,max} = 296$ mm), $y_f = D_f = 100$ mm

  $\Rightarrow (M_{uR})_{x_u=7D_f/3} = (2715 \times 233)(618 - 0.416 \times 233) +$

  $(13410 \times 100) \times (618 - 100/2)$

  $= 1091.3 \times 10^6 \text{ Nmm} > M_u = 800 \text{ kNm}.$
- Evidently, $D_f < x_u < \frac{7}{3} D_f$ , for which $y_f = 0.15 x_u + 0.65 D_f$

  $\Rightarrow C_{uf} = 13410(0.15 x_u + 0.65 \times 100) = (2011.5 x_u + 871650)$ N.

  $M_u = 800 \times 10^6 = (2715 x_u)(618 - 0.416 x_u)$

  $+ (2011.5 x_u + 871650) \times [618 - (0.15 x_u + 65)/2]$

  $= -1280.3 x_u^2 + 2790229.5\, x_u + 510.35 \times 10^6$

  Solving this quadratic equation,

  $\boldsymbol{x_u = 109.3}$ **mm** $< x_{u,max} = 296$ mm

  $\Rightarrow y_f = 0.15 x_u + 65 = 81.4$ mm
- Applying $T_u = 0.87 f_y A_{st} = C_{uw} + C_{uf}$,

  $(A_{st})_{reqd} = \dfrac{(2715 \times 109.3) + (13410 \times 81.4)}{0.87 \times 415}$ = **3845 mm²**

The reinforcement (5–32 ϕ; $A_{st} = 4020$ mm², based on the approximate estimate of $A_{st}$ [Fig. 5.11(b)] is evidently adequate and appropriate.

## 5.9 CURTAILMENT OF FLEXURAL TENSION REINFORCEMENT

In simply supported beams, the maximum (positive) bending moment occurs at or near the midspan, and the beam section is accordingly designed. Similarly, in continuous spans, the cross-section at the face of the support is designed for the maximum negative moment, and the cross-section at the midspan region is designed for the maximum positive moment. Although the bending moment progressively decreases away from these critical sections, the same overall dimensions of the beam are usually maintained throughout the length of the beam — mainly for convenience in formwork construction.

In order to achieve economy in the design, it is desirable to progressively curtail ('cut-off') the flexural tension reinforcement, commensurate with the decrease in bending moment. However, there are several other factors to be considered in arriving at the actual bar cut-off points — such as unexpected shifts in maximum moments, development length requirements, influence on shear strength and development of diagonal tension cracks due to the effects of discontinuity. Accordingly, the Code (Cl. 26.2.3) has listed out a number of requirements that need to be considered for the curtailment of flexural reinforcement.

### 5.9.1 Theoretical Bar Cut-off Points

The 'theoretical cut-off point' for a bar in a flexural member is that point beyond which it is (theoretically) no longer needed to resist the design moment.

In a prismatic beam (with constant $b$, $d$) the required area of tension reinforcement varies nearly linearly with the bending moment. This was indicated in Fig. 4.19, and can further be demonstrated for a uniformly loaded and simply supported beam [Fig. 5.12a] as follows.

Let $A_{st}$ be the tension steel area required at the section of maximum factored moment $M_{u,max}$ [Fig. 5.12(b), (c)], and let $A_{st1}$ be the tension steel area required at a section where the factored moment decreases to $M_{u1}$. Evidently,

$$A_{st} = \frac{M_{u,max}}{0.87 f_y z} \text{ and } A_{st1} = \frac{M_{u1}}{0.87 f_y z_1}$$

$$\Rightarrow \frac{A_{st1}}{A_{st}} = \frac{M_{u1}}{M_{u,max}} \times \frac{z}{z_1} \approx \frac{M_{u1}}{M_{u,max}}, \text{ as } \frac{z}{z_1} \approx 1.$$

Actually, the lever arm ratio $z/z_1$ decreases slightly below unity at sections further removed from the critical section, as the area of steel is reduced [Fig. 5.12(c)]. Accordingly, the approximation $A_{st1} \approx (M_{u1}/M_{u,max})A_{st}$ is acceptable as it results in slightly conservative estimates of $A_{st1}$. Based on this, it can be seen that at a section where the moment is, say, 60 percent of $M_{u,max}$, the reinforcement area required is only 60 percent of the designed area $A_{st}$, and the remaining 40 percent may be 'cut off' — as far as the flexural requirement is concerned.

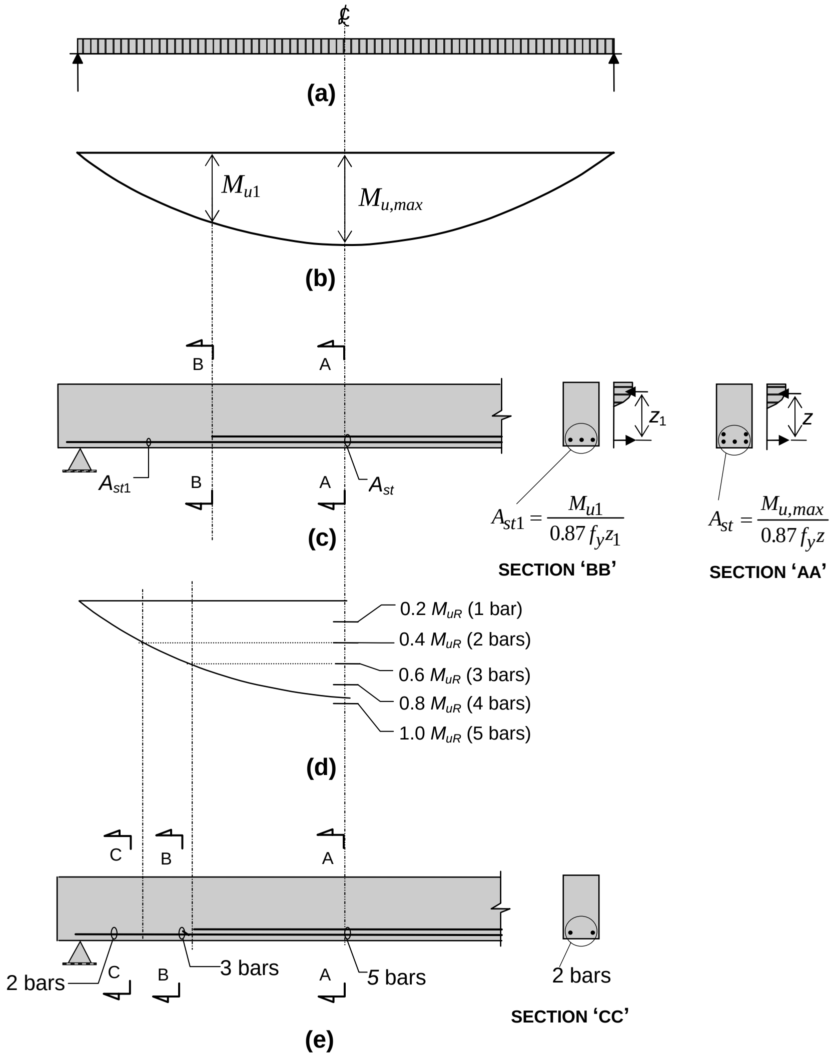



**Fig. 5.12** Illustration of theoretical bar cut-off points

As bars are available only in discrete sizes and can be provided only in full numbers, the actual reinforcement provided in practice at the critical section is often slightly greater than the calculated area $A_{st}$. Also, bars to be cut off are selected in terms of numbers rather than percentage of areas. Hence, for detailing of bar cut-off, it is appropriate to consider the strength contributed by each bar in terms of the critical section's ultimate moment resisting capacity. If there are $n$ bars provided at the critical section, and if the bars are all of the same diameter, then the strength per bar is $M_{uR}/n$, where $M_{uR}$ is the *actual* ultimate moment of resistance at the critical section.

Thus, for example, as shown in Fig. 5.12(d),(e), the 'theoretical cut-off point' for the first two bars (of the group of 5 bars) occurs at a section where the factored moment is equal to $(n–2)M_{uR}/n$. Similarly, the theoretical cut-off point for the third bar occurs at the section where the factored moment is equal to $(n–3)M_{uR}/n$, and so on.

In determining the theoretical bar cut-off points in this manner, the factored bending moment diagram must represent the possible maximum at each section, i.e., the *moment envelope* must be considered [Fig. 5.13].

This is of particular significance where moving loads are involved and in continuous spans where the loading patterns (of live loads) for the maximum negative moment at supports and for the maximum positive moment in the span are different [refer Chapter 9]. In a continuous span, the *point of zero moment*† for the negative moment envelope (marked $P_0^-$ in Fig. 5.13) is often different from that of the positive moment envelope, (marked $P_0^+$ in Fig. 5.13).

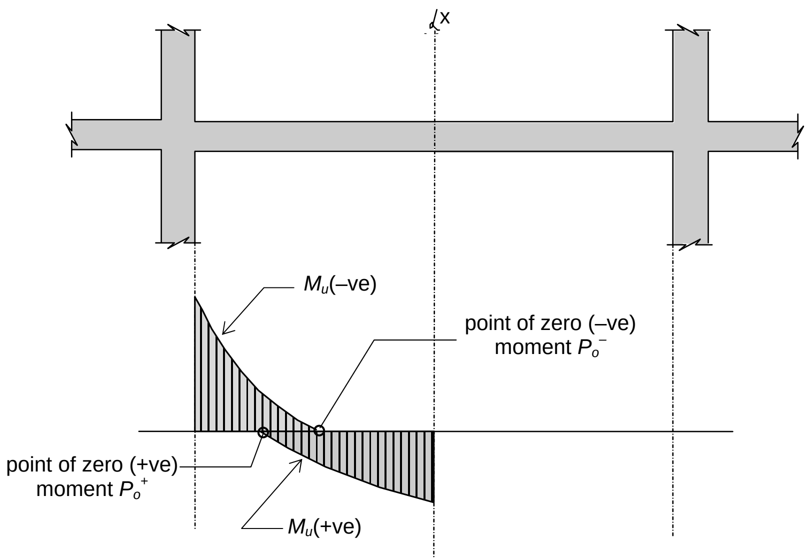



**Fig. 5.13** Moment envelope for continuous spans

### 5.9.2 Restrictions on Theoretical Bar Cut-off Points

As mentioned earlier, the *actual* bar cut-off points differ from the 'theoretical' ones for a number of reasons, some of which are described below:

- **Unexpected shifts in design moments**: The theoretical bending moment diagrams represent idealisations or 'best estimates'; these are subject to some

† It is not appropriate to use the term *point of inflection* here, as the reference is to a bending moment *envelope*. A point of inflection occurs where there is a change in sign of curvature (and hence, of bending moment) under a given loading.

variability on account of the assumptions and approximations involved in the calculation of load and load effects, yielding of supports, etc.

- **Development length requirements**: The stress at the end of a bar is zero; it builds up gradually along its length through bond with the surrounding concrete [refer Chapter 8]. In order to develop the full design stress ($0.87f_y$) in the bar at a section, a minimum *development length* $L_d$ is required on either side of the section. Some typical values of $L_d/\phi$ (in accordance with the Code — Cl. 26.2.1) are listed in Table 5.6.

**Table 5.6** $L_d / \phi$ values for fully stressed bars in tension*

| Grade of Steel | Grade of concrete | | | | |
|---|---|---|---|---|---|
| | **M 20** | **M 25** | **M 30** | **M 35** | **M 40 and above** |
| **Fe 250** | 45 | 39 | 36 | 32 | 29 |
| **Fe 415** | 47 | 40 | 38 | 33 | 30 |
| **Fe 500** | 57 | 49 | 45 | 40 | 36 |

*for bars in *compression*, multiply these values of $L_d / \phi$ by 0.8.

For example, if Fe 415 grade steel and M 20 concrete are used, $L_d$ should be taken as 47 times the bar diameter $\phi$. If the bar is subjected to a stress that is less than $0.87f_y$, then the required 'development length' is proportionately less. No bar should be terminated abruptly at any section, without extending it by the required development length.

- **Development of premature diagonal tension cracks**: Cutting off bars in the tension zone lowers substantially the shear strength (and ductility) of beams [Ref. 5.7]. The discontinuity at the cut end of the bar introduces stress concentration which can cause premature flexural cracks that may further develop into diagonal tension cracks — particularly if the shear stress at this section is relatively high [refer Chapter 6].

Such a diagonal tension crack in a flexural member without shear reinforcement is shown in Fig. 5.14. The equilibrium of forces† indicated on the freebody in Fig. 5.14(c) shows that the tensile force in the reinforcement at section '*b-b*' (located approximately *d* beyond the theoretical cut-off point at section '*a-a*') depends on the moment $M_{u1}$ at section '*a-a*'. Thus, the area of steel required at section '*a-a*' must extend up to section '*b-b*'.

† The forces due to 'aggregate interlock' and 'dowel action' in the reinforcing bars [refer Chapter 6] are neglected here.

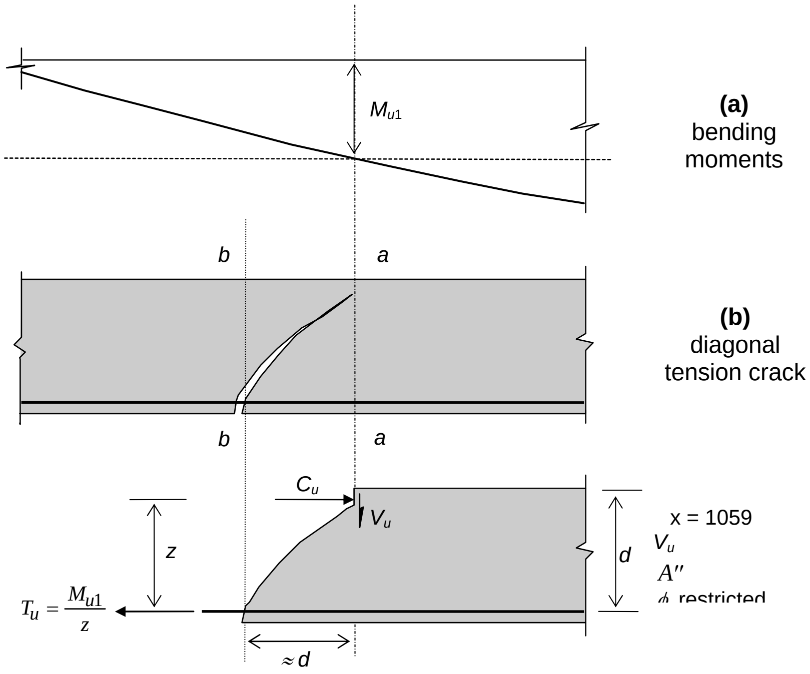



**Fig. 5.14** Influence of diagonal tension crack on tension steel stress

### 5.9.3 Code Requirements

In view of the various considerations involved in the curtailment of flexural reinforcement, the Code (Cl. 26.2.3) has specified certain requirements.

#### *Shear Strength Requirements for Curtailment*

To safeguard against the development of diagonal tension cracks, the tension steel should not be terminated unless *any one* of the following three requirements is satisfied:

1. The shear at the cut-off point does not exceed two-thirds of the shear resisting capacity of the section.
2. Excess stirrups are provided over a distance of 0.75$d$ from the cut-off point having an area $A_{sv}$ and a spacing $s_v$ such that:

$$A_{sv} \geq \frac{0.4 b_w s_v}{f_y} \quad ; \quad s_v \leq \frac{d}{8\beta_b}$$

   where $\beta_b$ is the ratio of the area of bars cut off to the total area of the bars at the section.

3. For 36 mm $\phi$ and smaller bars, the continuing bars provide at least twice the area required for flexure at the cut-off point *and* the shear does not exceed three-fourth of the shear resisting capacity of the section.

### *End Extension of Bars*

The extension of bars beyond the theoretical cut-off points, denoted $L_a$, should not be less than the effective depth $d$, as indicated in Fig. 5.15. The Code specifies that it should also not be less than 12 times the bar diameter $\phi$.

$$L_a \geq d \qquad \text{and} \qquad L_a \geq 12\phi$$

This requirement of 'end extension' should be satisfied by the *curtailed* bars (i.e., assuming there are additional continuing bars) of both positive moment reinforcement and negative moment reinforcement [Fig. 5.15].

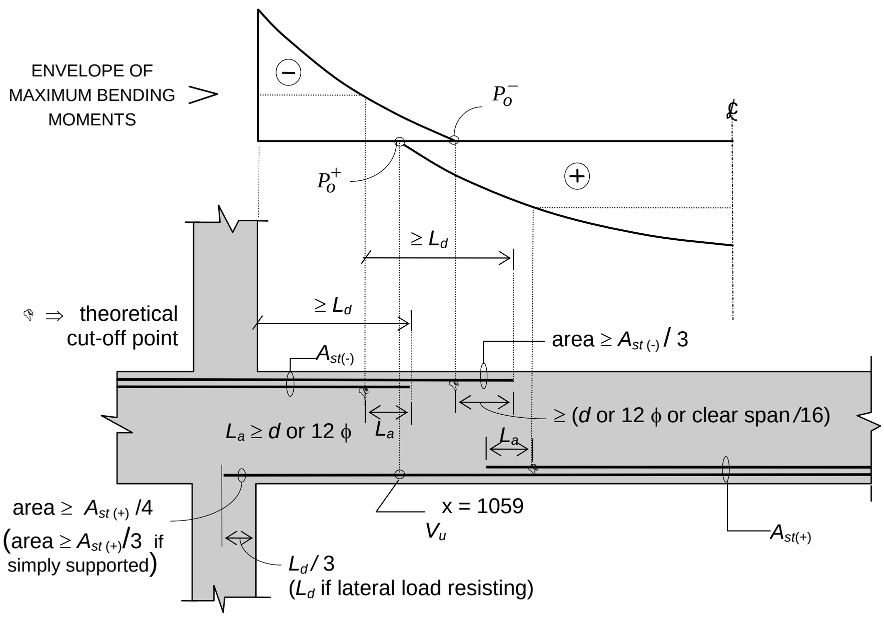



**Fig. 5.15** Code requirements (Cl. 25.2.3) for curtailment of tension reinforcement

For the continuing bars which are to be *finally terminated*, different requirements are applicable, regarding continuation beyond the point of zero moment (where they are no longer required theoretically). In the case of negative moment reinforcement, at least one-third of the total steel provided at the face of the support must extend beyond the *point of zero moment* $(P_o^-)$ for a distance not less than $d$, $12\phi$ or 1/16 times the clear span. In the case of positive moment reinforcement, at least one-third of the total steel in simply supported members and one-fourth the total steel in continuous members should be extended straight into the support by a distance not less than $L_d/3$ [Fig. 5.15]. If the support width is inadequate to provide this embedment, the bars should be suitably anchored by bending/hooks.

### *Development Length Requirements*

As mentioned earlier, no bar should be terminated without providing the required development length $L_d$ [Table 5.6] on either side of the point of maximum design stress ($0.87f_y$). At the supports at exterior columns, the bars may be bent (standard 90 degree bend) to anchor them suitably and thus provide the required $L_d$ — for the negative moment reinforcement.

In the case of positive moment reinforcement of beams (in frames) that constitute part of a lateral load resisting system, the Code requires that such steel should also be anchored into the support by a length $L_d$ beyond the face of the support.

Every point of stress in a bar requires a corresponding 'development length' that is directly proportional to the bar diameter $\phi$ as well as the stress level at the point. The values of $L_d$ generally specified [Table 5.6] correspond to the 'fully stressed' condition ($f_{st} = 0.87f_y$) which occurs at a 'critical section'. However, providing a length $L_d$ on either side of the critical section does not necessarily ensure that adequate embedment is provided at *all* stressed points along the length of the bar — except when the variation of bar stress is linear (which occurs only when the bending moment falls off linearly). In particular, the 'positive' moment regions of beams with distributed loading require special consideration, as in such regions the moment diagram is nonlinear (and convex), whereas the bar stress development over the length $L_d$ is assumed to be linear.

This is illustrated with the aid of Fig. 5.16, which shows a typical nonlinear variation of bending moment near a simple support of a beam. For the purposes of illustration, it is assumed that the theoretical cut-off point of a group of bars (marked 'a') is at point A, located at a distance equal to $L_d$ of the continuing group of bars (marked 'b') from the point of zero moment (support C). At A, bars 'b' are fully stressed ($f_{st} = 0.87f_y$) and possess an ultimate moment of resistance $M_{uR}$ equal to the factored moment $M_A$. As these bars are terminated at C, it follows that the bar stress will decrease linearly (assuming uniform bond stress distribution) from $0.87f_y$ at A to zero at C, and hence the moment resisting capacity will also drop linearly from $M_A$ to zero, as depicted by the dotted line A″C′ in Fig. 5.16. Evidently, this means that the ultimate moment capacity at any intermediate point (such as B at, say, $0.4L_d$ from C) in the region AC will generally fall short of the factored moment $M_B$, as shown in Fig. 5.16.

Clearly, this indicates the need for an adequate extension ($L_o$) of the tension bars beyond the point of zero moment (either at a simple support or at a point of inflection). How much to extend, of course, depends on how nonlinear the variation of the factored moment ($M_u$) is, in the region CA near the point of zero moment. A measure of this rate of change of the factored moment is given by the factored shear force $V_u$ (= $dM_u/dx$) at the point of zero moment, C [Fig. 5.17].

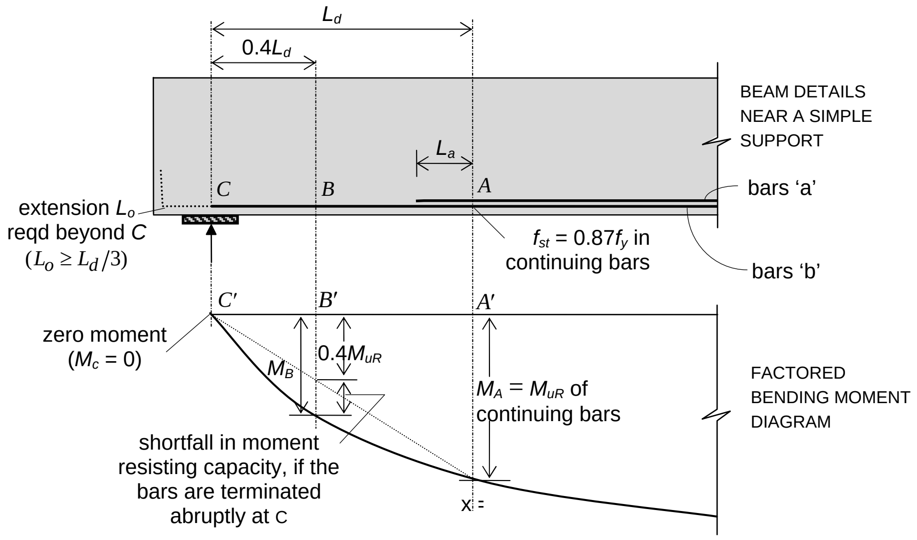



**Fig. 5.16** Need for extending bars beyond point of zero moment

It is seen that adequate flexural strength can invariably be ensured by providing a full development length $L_d$ to a point D beyond a critical section at B, located at a distance $M_{uR}/V_u$ from the section of zero moment (C). Here $M_{uR}$ denotes the ultimate moment of resistance due to the continuing bars and $V_u$ denotes the factored shear force at the section of zero moment, as depicted in Fig. 5.17.

With a full development length $L_d$ provided beyond point B, the *moment resistance diagram* will vary from zero at the bar end *D* to $M_{uR}$ at section B, and will lie completely outside the factored moment diagram [as shown in Fig. 5.17(a)]; this ensures that $M_{uR} > M_u$ all along the segment CA. In the case of a simple support where the reaction confines the ends of the reinforcement, the Code (Cl. 26.2.3.3c) permits an increase in $M_{uR}/V_u$ by 30 percent.

If the anchorage length beyond the zero moment location is denoted as $L_o$, then the Code requirement may be expressed as:

$$\left(M_{uR}/V_u\right)^* + L_o \;\;\geq\;\; L_d \tag{5.25}$$

where $(M_{uR}/V_u)^* = 1.3M_{uR}/V_u$ at a simple support with a confining reaction, and $M_{uR}/V_u$ otherwise.

At a point of inflection, the anchorage length $L_o$ is limited to the effective depth $d$ or $12\phi$, whichever is greater. At a simple support although the entire embedment length beyond the centre of support (including the equivalent anchorage value of any hook[†] or mechanical anchorage) qualifies as a proper measure of $L_o$, there may be practical limitations. The most effective way of satisfying Eq. 5.25 is by controlling the bar diameter $\phi$, thereby reducing $L_d$.

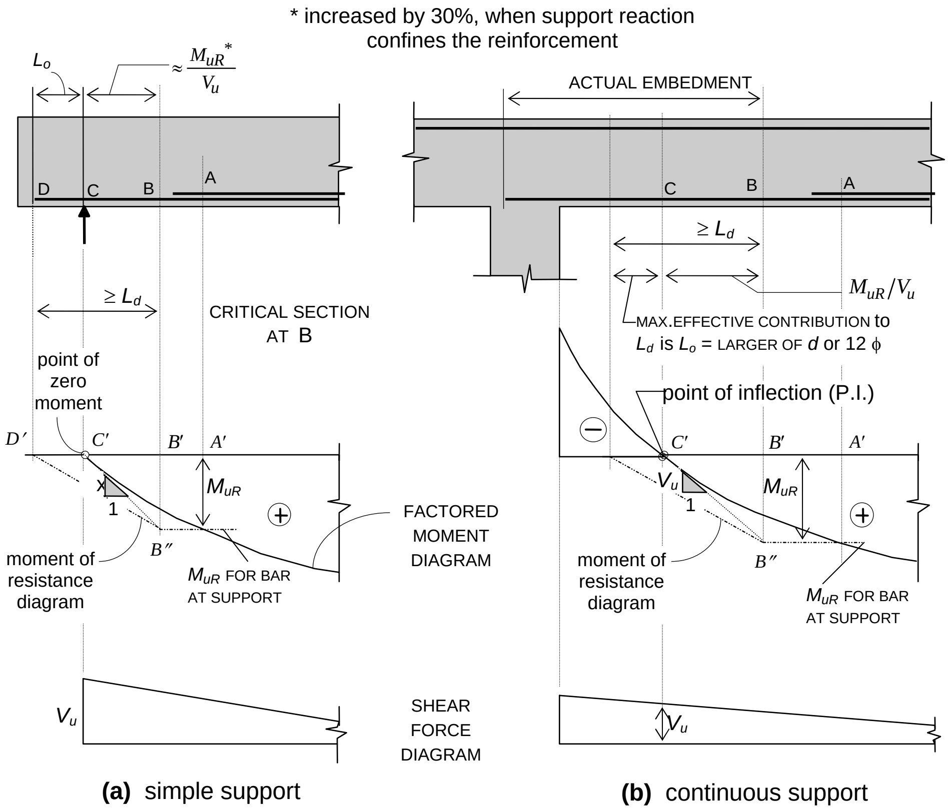



**Fig. 5.17** Code requirement (Cl. 25.2.3c) for limiting bar size of positive moment requirement at zero moment location

The Code (Cl. 26.2.3.3c) explicitly states that the purpose of the requirement given by Eq. 5.25 is to limit the bar diameter of positive tension reinforcement at a zero moment location. It is apparent that the objective is to ensure adequate development length and moment resisting capacity at locations such as depicted in Fig. 5.17.

[†] The anchorage value of a standard U-type hook should be taken as $16\phi$. If bends are provided, an anchorage value of $4\phi$ should be taken for each $45^o$ bend, subject to a maximum of $16\phi$ [Cl. 26.2.2.1b of the Code].

### *Curtailment of Bundled Bars*

Where *bundled bars* (provided as tension reinforcement) are to be curtailed, the individual bars in a bundle must be terminated at different points that are at least 40 diameters apart. This, however, is not applicable for bundles terminating at a support.

It is desirable to curtail first the bars in a bundle that are closer to the neutral axis; this is also desirable when bars (not bundled) are provided in multiple layers.

The 'development length' for each bar in a bundle should be taken as $1.1L_d$, $1.2L_d$, and $1.33L_d$ for 2-bar bundles, 3-bar bundles and 4-bar bundles respectively, where $L_d$ is the development length for an individual bar.

### 5.9.4 Bending of Bars

As an alternative to curtailment, *bending* ('cranking') of bars may be resorted to. In continuous beams, some of the bars (usually, not more than two at a time) may be bent over (at intervals, if large numbers are involved) from the bottom side of the beam to the top side, and continued over the support to form part of the negative moment reinforcement. Such a system is shown in Fig. 5.5(b) for one-way continuous slabs. The bars are usually bent at an angle of 45°, although angles up to 60° are also resorted to in practice for relatively deep beams. The bent portion of the bar contributes towards increased shear strength of the beam section by resisting diagonal tensile strength and restraining the spread of diagonal tension cracks [refer Chapter 6]. This contrasts with the adverse effect of bar curtailment on shear strength.

The discontinuity effects discussed in Section 5.9.2 are, therefore, less severe for bars which are bent, in comparison with bars which are cut off. Hence a requirement that the *bend points* be extended beyond the 'theoretical bar cut-off points' by the end extension distance ($L_a \geq d$ or $12\phi$) may be too conservative. It has been suggested [Ref. 5.6] that for bar extension purpose, a bent bar may be considered effective up to a section where the bar crosses the mid-depth of the beam; this would reduce the extension required by $d/2$. However, for bar extension, the Code does not distinguish between cut-off bars and bent bars.

**EXAMPLE 5.7**

Design a suitable longitudinal arrangement of the tension reinforcement (including bar cut-off) for the simply supported beam of Example 5.1.

**SOLUTION**

- Given: From Example 5.1, $A_{st}$ = 1119 mm$^2$ (1-25 $\phi$ and 2–20 $\phi$), $b$ = 250 mm, $d$ = 399 mm, $f_y$ = 415 MPa, $f_{ck}$ = 25 MPa, factored load $w_u$ = 28.1 kN/m, span = 6.0 m [Fig. 5.18(a)], $M_{u,max}$ = 126 kN*m*, $(M_{uR})_{3bars}$ = 131 kN*m*.
- The middle bar (1-25 $\phi$) may be curtailed. Let the theoretical cut-off point be at a distance $x$ from the support. The value of $x$ can be obtained by solving the moment equilibrium equation:

$$M_x = V_u x - w_u x^2/2$$

where $M_x$ is the ultimate moment of resistance of the two continuing 20 $\phi$ bars:

$$M_x \approx (2\times314/1119)\times131 \text{ kN}m = 73.5 \text{ kN}m. \text{ [Fig. 5.18(b)].}$$

$V_u = w_u l/2 = 28.1\times3.0 = 84.3$ kN.

$\Rightarrow 84.3x - 28.1\, x^2/2 = 73.5$

$\Rightarrow x^2 - 6x + 5.231 = 0$

Solving, $x = 1.059$ m = 1059 mm.

- ***End extension for curtailed bar***

  Bar extension beyond theoretical cut-off point is given by $d$ = 399 mm, which is greater than 12 $\phi$ = 240 mm.

  ∴ Actual point of termination = 1059 – 399 = 660 mm from the centre of support [Fig. 5.18(c)].

- ***Check bar size limitation at support***

$\left(1.3\dfrac{M_{uR}}{V_u} + L_o\right)$ must exceed $L_d$ [Eq. 5.25].

For Fe 415 steel and M 25 concrete, $L_d$ = 40×25 = 1000 mm [Table 5.6].

For the continuing two bars, $M_{uR}$ = 73.5 kNm

$\Rightarrow \left(1.3\dfrac{M_{uR}}{V_u} + L_o\right) = 1.3\times\dfrac{73.5\times10^3}{84.3} + L_o$ = (1133 mm + $L_o$ )

  which exceeds $L_d$ = 1000 mm, regardless of $L_o$. Hence, the bar diameter $\phi$ = 20 mm is acceptable.

- ***Extension over support for continuing bars***

  At the simple support the two bars must extend beyond the face of the support by a distance not less than: $L_d$ /3 = 1000/3 = 333 mm

  which is not possible over a support width of 230 mm (with end cover of 30 mm minimum) — unless the bar is bent upwards. Accordingly, provide standard 90° bend with 4 $\phi$ extension, having a total anchorage value of 12 $\phi$ = 240 mm, as shown in Fig. 5.18(c).

  ∴ Embedment length provided = (230 – 30 – 5×20) + 240 = 340 mm > $L_d$ /3

- ***Development length requirements***

  1) For the curtailed bar, the critical section is at midspan; the length provided (3000 – 660 = 2340 mm) is well in excess of $L_d$ = 940 mm.

  2) For the continuing bars, the requirement $M_u^*/V_u + L_o \geq L_d$ has been satisfied.

  — Hence, OK.

- ***Shear strength requirements for curtailment***

This is covered in Example 6.1 of Chapter 6.

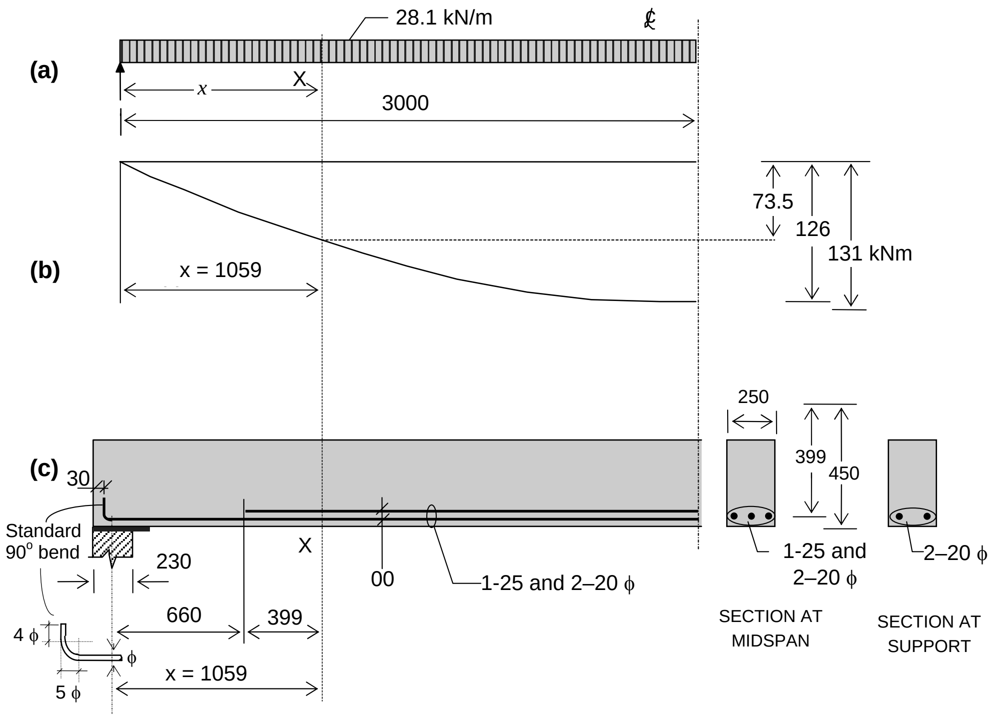



**Fig. 5.18** Example 5.7 — Curtailment of bar

## REVIEW QUESTIONS

5.1 Why does the Code impose minimum and maximum limits with regard to (a) spacing, (b) percentage area of flexural reinforcement?

5.2 What are the advantages and disadvantages of providing large clear cover to reinforcement in flexural members?

5.3 Show that deflection control in normal flexural members can be achieved by limiting span/effective depth ratios.

5.4 Explain the dependence of span/effective depth ratios (for deflection control, as per Code) on the percentage tension and compression reinforcement, as well as the grade of tension steel.

5.5 Under what circumstances are doubly reinforced beams resorted to?

5.6 Reinforced concrete slabs are generally singly reinforced. Why not doubly reinforced?

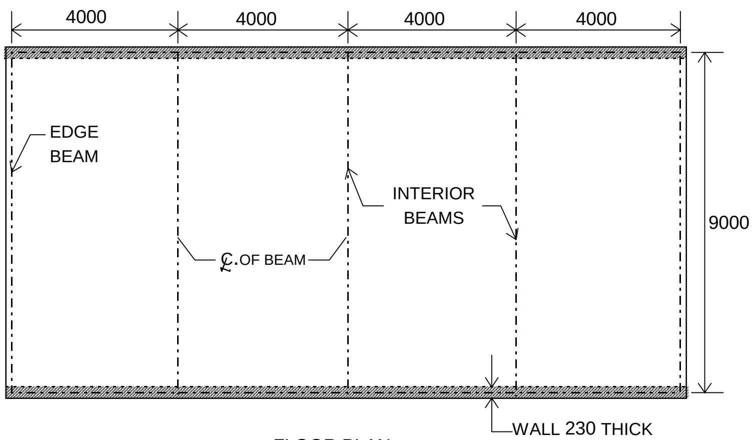
5.7 A designer provides areas of tension and compression reinforcement (in a doubly reinforced beam) that result in percentage $p_t$ and $p_c$ in *excess* of the values obtained from *design tables* (corresponding to a given $M_u/bd^2$). Is it guaranteed that the design will meet all the Code requirements?

5.8 Discuss the proportioning of sections in T-beam design.

5.9 What is a ‘theoretical bar cut-off point’? Why does the Code disallow curtailment of flexural tension reinforcement at this point?
5.10 Discuss the influence of diagonal tension cracks on the tension steel stress in a flexural member.
5.11 Discuss the Code requirement related to ensuring adequate development length in the bars near the zero moment location.

## PROBLEMS

5.1 A rectangular beam of span 7 m (centre-to-centre of supports), resting on 300 mm wide simple supports, is to carry a uniformly distributed dead load (excluding self-weight) of 15 kN/m and a live load of 20 kN/m. Using Fe 415 steel, design the beam section at midspan, based on first principles. Check the adequacy of the section for strength, using design aids. Also perform a check for deflection control. Assume that the beam is subjected to moderate exposure conditions.
5.2 Design a suitable arrangement of the tension reinforcement (including bar cut-off) for the beam in Problem 5.1
5.3 Design the beam section in Problem 5.1, given that the overall beam depth is restricted to 550 mm.
5.4 Design a one-way slab, with a clear span of 5.0m, simply supported on 230 mm thick masonry walls, and subjected to a live load of 3 kN/m$^2$ and a surface finish load of 1 kN/m$^2$, using Fe 415 steel. Assume that the beam is subjected to (a) mild exposure and (b) very severe exposure, and compare the results.
5.5 Repeat Problem 5.4, considering Fe 250 steel in lieu of Fe 415 steel.
5.6 The floor plan of a building is shown in Fig. 5.19. The specified floor loading consists of a live load of 5.0 kN/m$^2$ and a dead load of 2.5 kN/m$^2$ (excluding self-weight). Design the slab thickness and reinforcement area required at the various critical sections, using Fe 415 steel. Assume that the beam is subjected to moderate exposure conditions.



FLOOR PLAN

**Fig. 5.19** Floor system — Problems 5.6 – 5.8

5.7 Design the interior beam of the floor system in Fig. 5.19, considering the beam to be simply supported.

5.8 Design the edge beam (L-beam) of the floor system in Fig. 5.19, with the width of the web equal to 250 mm. Assume the beam to be simply supported and neglect the effect of torsion.

5.9 A T-beam of 8*m* clear span, simply supported on wall supports 230 mm wide, is subjected to a dead load of 20 kN/m (including self-weight) and a live load of 25 kN/m. The overall size of the beam is given in Fig. 5.20. Design the beam for tension reinforcement and detail the bar cut-off. Assume that 50 percent of bars are to be cut off. Use Fe 415 steel. Assume moderate exposure conditions.

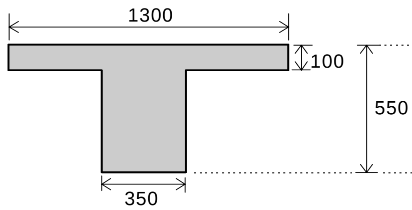



**Fig. 5.20** Problem 5.9

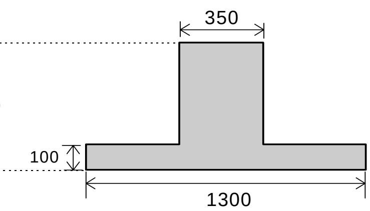



**Fig. 5.21** Problem 5.10

5.10 The section of a cantilever (inverted T-beam) is shown in Fig. 5.21. The cantilever has a clear span of 4*m* and carries a total distributed load of 25 kN/m (including self-weight) and a concentrated load of 50 kN at the free end. Design and detail the tension reinforcement, considering the cantilever to be supported from a 600 mm wide column. Assume that the beam is subjected to severe exposure conditions.

5.11 Work out the bar cut-off details for the beam designed in Example 5.5.

## REFERENCES

5.1 Arora, J.S., *Introduction to Optimum Design*, McGraw-Hill International edition, 1989.

5.2 Gouthaman, A. and Menon, D., *Increased Cover Specifications in IS 456 (2000) – Crack-width Implications in RC Slabs*, Indian Concrete Journal, Sept. 2001, pp 581–586

5.3 — *Explanatory Handbook on Indian Standard Code of Practice for Plain and Reinforced Concrete (IS 456:1978)*, Special Publication SP:24, Bureau of Indian Standards, New Delhi, 1983.

5.4 Beeby, A.W., *Modified Proposals for Controlling Deflection by Means of Ratios of Span to Effective Depth*, Cement and Concrete Association, Publ. No. 42.456, London, April 1971.

5.5 — *Design Aids (for Reinforced Concrete) to IS 456 : 1978*, Special Publication SP:16, Bureau of Indian Standards, New Delhi, 1980.

5.6 — *Handbook on Concrete Reinforcement and Detailing*, Special Publication SP 34, Bureau of Indian Standards, New Delhi, 1987.

5.7 ASCE-ACI Committee 426, *The Shear Strength of Reinforced Concrete Members*, ASCE Journal, Struct. Div., Vol. 99, June 1973, pp 1091–1187.

# 6

# Design for Shear

## 6.1 INTRODUCTION

As mentioned earlier (in Section 4.1), bending moments are generally accompanied by transverse shear forces, and sometimes by axial forces and torsion as well. The ultimate limit state considered in Chapters 4 and 5 dealt with flexure (bending) alone. This chapter deals with the ultimate limit state in *flexural shear*, i.e., shear associated with a varying bending moment. Commonly, flexural shear is simply referred to as 'shear'. The method described in this Chapter (for convenience, referred to as the *Conventional Method* or the "*Simplified Method*") is intended to be used for the design *of* "*flexural regions*" of members; ie. beams, columns, or walls (or portions of the member as are) designed by the conventional theory of flexure, in which the assumption that "*plane sections remain plane*" is reasonably valid. In this method, the transverse reinforcement is designed for the shear, while the longitudinal reinforcement is designed for the combined effects of flexure and axial (compressive only) load. The effects of shear on the longitudinal reinforcement (Section 5.9.2) are taken care of by bar detailing requirements. In the case of slabs, this type of shear is sometimes referred to as *one–way shear* — as distinct from *two–way shear* ('punching shear'), which is associated with the possibility of punching through a relatively thin slab by a concentrated column load (refer Chapter 11). Another type of shear that needs consideration is *torsional shear* (due to torsion), which, when it occurs, generally does so in combination with flexural shear; this is covered in Chapter 7.

A more general method for shear and torsion design, based on the so-called *Compression Field Theory*, is presented in Chapter 17. In this method, the member is designed for the combined effects of flexure, shear, axial (compressive or tensile) load and torsion.

A method based on the 'Strut-and-Tie Model' is also presented in Chapter 17. This method is particularly suitable for the design of regions where the assumption that "plane sections remain plane" is not applicable. Such regions include deep beams, parts of members with a deep shear span, brackets and corbels, pile caps, regions with abrupt changes in cross-section (web openings in beams and articulations in girders) and regions near discontinuities.

Interface shear transfer and the shear-friction procedure are also described in this Chapter (Section 6.9). This is applicable for situations involving the possibility of shear failure in the form of sliding along a plane of weakness.

Failure of a reinforced concrete beam in flexural shear often may not lead to an immediate *collapse* by itself. However, it can significantly reduce flexural strength (moment–bearing capacity) as well as ductility. Hence, the state of (impending) shear failure is treated by the Code as an *ultimate limit state* (i.e., limit state of collapse) for design purposes.

The behaviour of reinforced concrete under shear (flexural shear alone or in combination with torsion and axial forces) is very complex — mainly because of its non-homogeneity, presence of cracks and reinforcement, and the nonlinearity in its material response. The current understanding of and design procedures for shear effects are, to a large measure, based on the results of extensive tests and simplifying assumptions, rather than on an exact and universally accepted theory.

## 6.2 SHEAR STRESSES IN HOMOGENEOUS RECTANGULAR BEAMS

In order to gain an insight into the causes of flexural shear failure in reinforced concrete, the stress distribution in a homogeneous elastic beam of rectangular section is reviewed here. In such a beam, loaded as shown in Fig. 6.1(a), any transverse section (marked 'XX'), in general, is subjected to a bending moment $M$ and a transverse shear force $V$.

From basic mechanics of materials [Ref. 6.1], it is known that the *flexural* (*normal*) *stress* $f_x$ and the *shear stress* $\tau$ at any point in the section, located at a distance $y$ from the neutral axis, are given by:

$$f_x = \frac{M\,y}{I}$$

$$\tau = \frac{VQ}{Ib} \tag{6.1}$$

where $I$ is the second moment of area of the section about the neutral axis, $Q$ the first moment of area about the neutral axis of the portion of the section above the layer at distance $y$ from the NA, and $b$ is the width of the beam at the layer at which $\tau$ is calculated. The distributions of $f_x$ and $\tau$ are depicted in Fig. 6.1(b). It may be noted that the variation of shear stress is parabolic, with a maximum value at the neutral axis and zero values at the top and bottom of the section.

Considering an element at a distance $y$ from the NA [Fig. 6.1(c)], and neglecting any possible vertical normal stress $f_y$ caused by the surface loads, the combined flexural and shear stresses can be resolved into equivalent *principal stresses* $f_1$ and $f_2$ acting on orthogonal planes, inclined at an angle $\alpha$ to the beam axis (as shown):

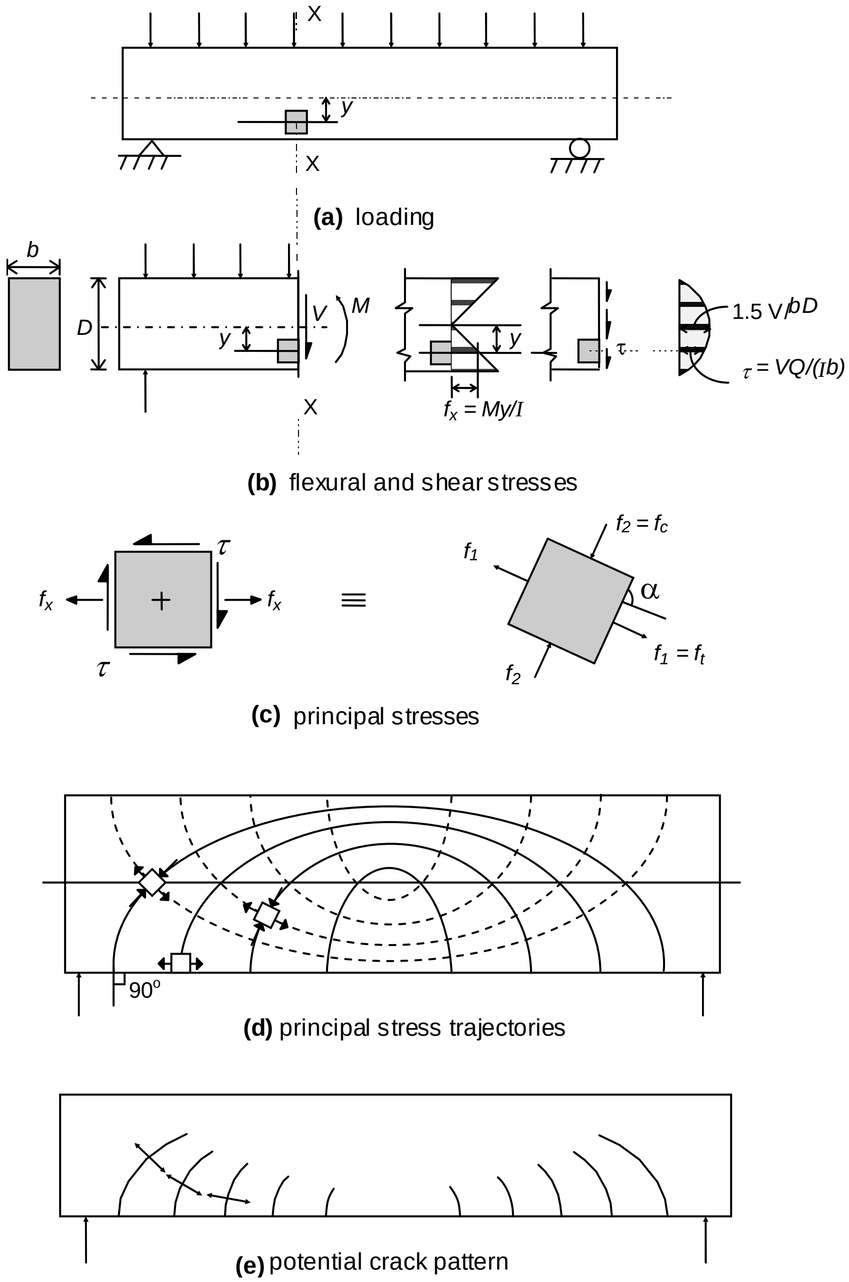



**Fig. 6.1** Stress distribution in homogeneous beams of rectangular section

$$f_{1,2} = \frac{1}{2} f_x \pm \sqrt{\left(\frac{1}{2} f_x\right)^2 + \tau^2} \qquad (6.2)$$

$$\tan\ 2\alpha = \frac{2\tau}{f_x} \qquad (6.3)$$

In general, the stress $f_1$ is tensile (say = $f_t$) and $f_2$ is compressive (say = $f_c$). The relative magnitudes of $f_t$ and $f_c$ and their directions depend on the relative values of $f_x$ and $\tau$ [Eq. 6.2, 6.3]. In particular, at the top and bottom fibres where shear stress $\tau$ is zero, it follows from Eq. 6.3 that $\alpha$ = 0, indicating that one of the principal stresses is in a direction parallel to the surface, and the other perpendicular to it, the latter being zero in the present case. Thus, along the top face, the nonzero stress parallel to the beam axis is $f_c$, and along the bottom face, it is $f_t$. On the other hand, a condition of 'pure shear' occurs for elements located at the neutral axis (where $\tau$ is maximum and $f_x = 0$), whereby $f_t = f_c = \tau_{max}$ and $\alpha = 45^o$. The stress pattern is indicated in Fig. 6.1(d), which depicts the *principal stress trajectories*† in the beam.

In a material like concrete which is weak in tension, tensile cracks would develop in a direction that is perpendicular to that of the principal tensile stress. Thus the compressive stress trajectories [firm lines in Fig. 6.1(d)] indicate *potential* crack patterns (depending on the magnitude of the tensile stress), as shown in Fig. 6.1(e). It should be noted, however, that once a crack develops, the stress distributions depicted here are no longer valid in that region, as the effective section gets altered and the above equations are no longer valid.

## 6.3 BEHAVIOUR OF REINFORCED CONCRETE UNDER SHEAR

### 6.3.1 Modes of Cracking

In reinforced concrete beams of usual proportions, subjected to relatively high flexural stresses $f_x$ and low shear stresses $\tau$, the maximum principal tensile stress is invariably given by the flexural stress $f_{x,max}$ in the outer fibre (bottom face of the beam in Fig. 6.1) at the peak moment locations; the resulting cracks are termed *flexural cracks* [Fig. 6.2(a)]. These are controlled by the tension bars. On the other hand, in short–span beams which are relatively deep and have thin webs (as in *I*-sections) and are subjected to high shear stresses $\tau$ (due to heavy concentrated loads) and relatively low flexural stresses $f_x$, it is likely that the maximum principal tensile stress is located at the neutral axis level at an inclination $\alpha = 45^o$ (to the longitudinal axis of the beam); the resulting cracks (which generally occur near the supports, where shear force is maximum) are termed *web shear cracks* or *diagonal tension cracks* [Fig. 6.2(b)].

† 'Principal stress trajectories' are a set of orthogonal curves whose tangent/normal at any regular point indicate the directions of the principal stresses at that point. The firm lines indicate directions of compressive stress, and the broken lines indicate directions of tensile stress.

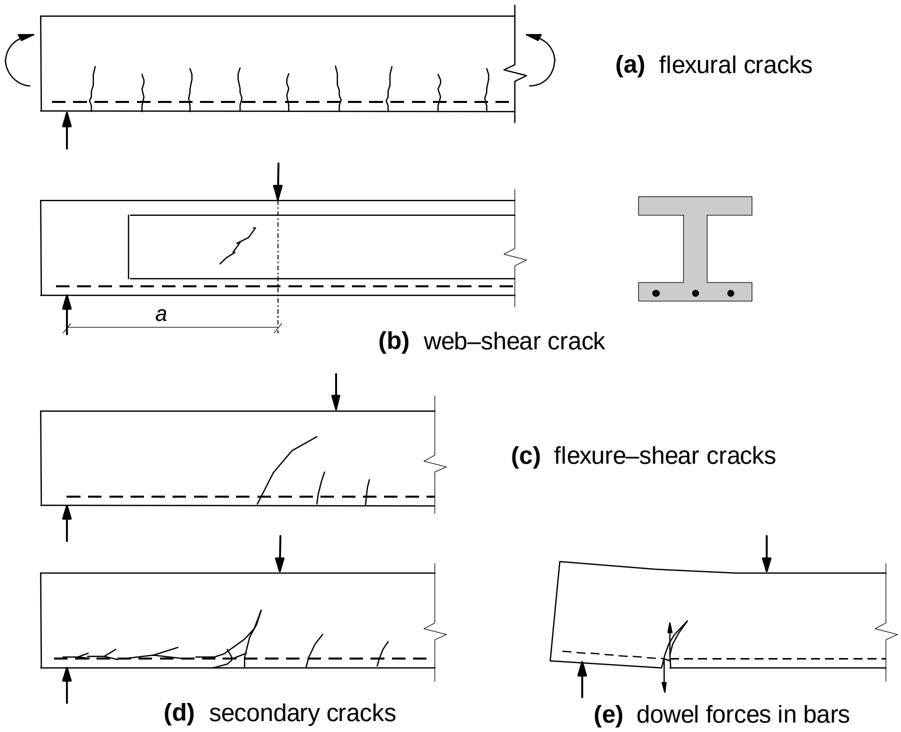



**Fig. 6.2** Modes of cracking

In general, in a beam under flexure and shear, a biaxial state of combined tension and compression exists at various points, as shown in Fig. 6.1. As explained in Section 2.10.2, the presence of shear stress reduces the *strength* of concrete in compression as well as tension. Accordingly, the tensile strength of the concrete in a reinforced concrete beam subjected to flexural shear will be less than the uniaxial tensile strength of concrete. The so-called 'diagonal tension cracks' can be expected to occur in reinforced concrete beams in general, and appropriate *shear reinforcement* is required to prevent the propagation of these cracks. When a 'flexural crack' occurs in combination with a 'diagonal tension crack' (as is usually the case), the crack is sometimes referred to as a *flexure-shear crack* [Fig. 6.2(c)]. In such a case, it is the flexural crack that usually forms first, and due to the increased shear stresses at the tip of the crack, this flexural crack extends into a diagonal tension crack.

Sometimes, the inclined crack propagates along the tension reinforcement towards the support [Fig. 6.2(d)]. Such cracks are referred to as *secondary cracks* or *splitting cracks*. These are attributed to the wedging action of the tension bar deformations and to the transverse 'dowel forces' introduced by the tension bars functioning as *dowels* across the crack, resisting relative transverse displacements between the two segments of the beam (*dowel action*) [Fig. 6.2(e)].

### 6.3.2 Shear Transfer Mechanisms

There are several mechanisms by which shear is transmitted between two adjacent planes in a reinforced concrete beam. The prominent among these are identified in Fig. 6.3, which shows the freebody of one segment of a reinforced concrete beam separated by a flexure-shear crack.

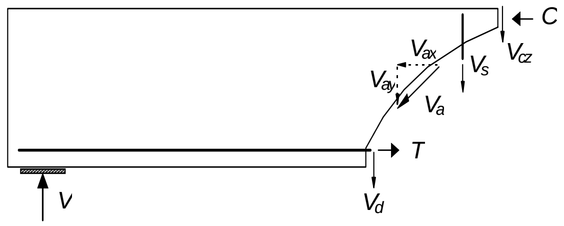



**Fig. 6.3** Internal forces acting at a flexural-shear crack

The transverse (external) shear force is denoted as $V$ (and has a maximum value near the support, equal to the support reaction). It is resisted by various mechanisms, the major ones [Fig. 6.3] being:

1. shear resistance $V_{cz}$ of the uncracked portion of concrete;
2. vertical component $V_{ay}$ of the 'interface shear' (*aggregate interlock*) force $V_a$;
3. dowel force $V_d$ in the tension reinforcement (due to dowel action); and
4. shear resistance $V_s$ carried by the shear (transverse) reinforcement, if any.

The interface shear $V_a$ is a tangential force transmitted along the inclined plane of the crack, resulting from the friction against relative slip between the interlocking surfaces of the crack. Its contribution can be significant, if the crack-width is limited. The dowel force $V_d$ comes from 'dowel action' [Fig. 6.2(e)], as explained earlier.

The equilibrium of vertical forces in Fig. 6.3 results in the relation:

$$V = V_{cz} + V_{ay} + V_d + V_s \tag{6.4}$$

The relative contribution of the various mechanisms depends on the loading stage, the extent of cracking and the material and geometric properties of the beam. Prior to flexural cracking, the applied shear is resisted almost entirely by the uncracked concrete $(V \approx V_{cz})$. At the commencement of flexural cracking, there is a redistribution of stresses, and some interface shear $V_a$ and dowel action $V_d$ develop. At the stage of diagonal tension cracking, the shear reinforcement (hitherto practically unstressed) that intercepts the crack undergoes a sudden increase in tensile strain and stress. All the four major mechanisms are effective at this stage. The subsequent behaviour, including the failure mode and the ultimate strength in shear, depends on how the mechanisms of shear transfer break down and how successfully the shear resisting forces are redistributed.

The presence of increased longitudinal reinforcement in the flexural tension zone not only contributes to enhanced dowel action ($V_d$), but also serves to control the propagation of flexural cracks and contributes to increasing the depth of the neutral axis, and thereby the depth of the uncracked concrete in compression; this enhances the contributions of $V_a$ and $V_{cz}$. Thus, the higher the percentage tension reinforcement, the greater the shear resistance in the concrete – up to a limit.

### *Beams without Shear Reinforcement*

In beams without shear reinforcement, the component $V_s$ is absent altogether. Moreover, in the absence of *stirrups* enclosing the longitudinal bars, there is little restraint against splitting failure, and the dowel force $V_d$ is small. Furthermore, the crack propagation is unrestrained, and hence, fairly rapid, resulting in a fall in the aggregate interface force $V_a$ and also a reduction in the area of the uncracked concrete (in the limited compression zone) which contributes to $V_{cz}$. However, in relatively deep beams, *tied–arch action* [Fig. 6.4(b)] may develop following inclined cracking, thereby transferring part of the load to the support, and so reducing the effective shear force at the section.

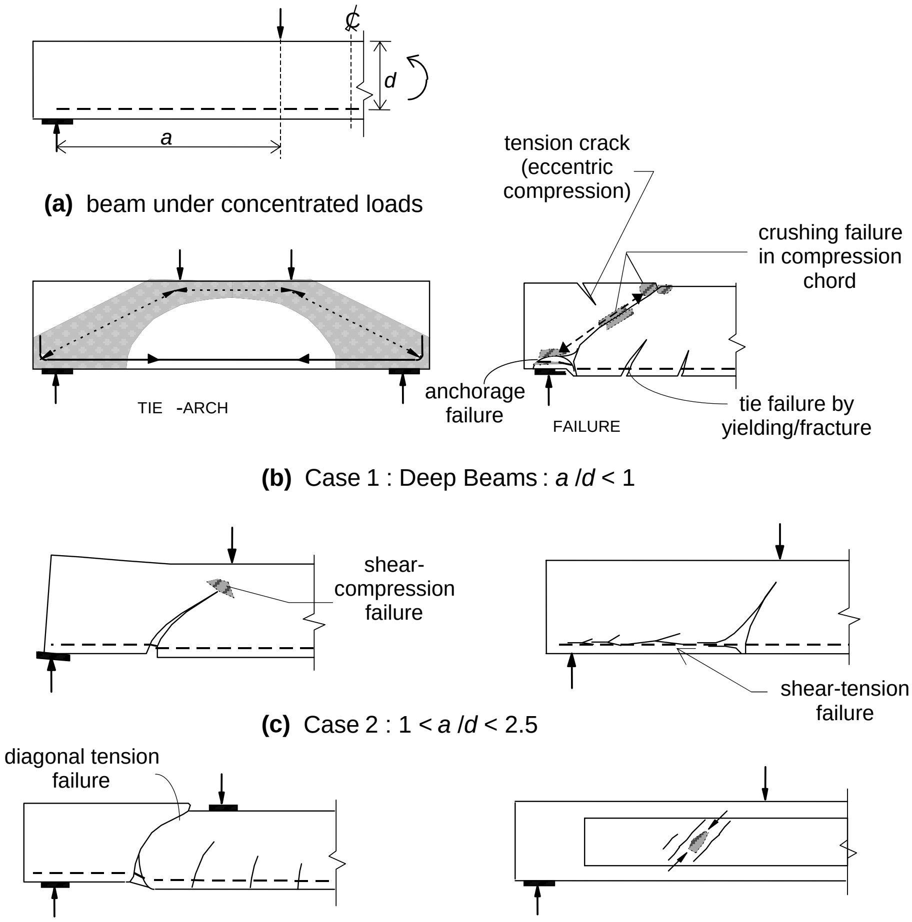
Thus, in beams without shear reinforcement, the breakdown of any of the shear transfer mechanisms may cause immediate failure, as there is little scope for redistribution. Further, owing to the uncertainties associated with all the above effects, it is difficult to predict precisely the behaviour and the strength beyond the stage of diagonal cracking in beams without shear reinforcement.

As seen in Chapter 5, design for flexure is done so as to ensure a ductile flexural failure. The objective of shear design is to avoid premature brittle shear failures, such as those displayed by beams without web reinforcement, before the attainment of the full flexural strength. Members should be designed so that the shear capacity is high enough to ensure a ductile flexural failure.

### *Beams with Shear Reinforcement*

In beams with moderate amounts of shear reinforcement, shear resistance continues to increase even after inclined cracking, until the shear reinforcement yields in tension, and the force $V_s$ cannot exceed its ultimate value $V_{us}$. Any additional shear $V$ has to be resisted by increments in $V_{cz}$, $V_d$ and/or $V_{ay}$. With progressively widening crack-width (now accelerated by the yielding of shear reinforcement), $V_{ay}$ decreases (instead of increasing), thereby forcing $V_{cz}$ and $V_d$ to increase at a faster rate until either a splitting (dowel) failure occurs or the concrete in the compression zone gets crushed under the combined effects of flexural compressive stress and shear stress.

Owing to the pronounced yielding of the shear reinforcement, the failure of shear reinforced beams is gradual and ductile in nature — unlike beams without shear reinforcement, whose failure in shear is sudden and brittle in nature. However, if excessive shear reinforcement is provided, it is likely that the 'shear-compression' mode of failure [see next section] will occur first, and this is undesirable, as such a failure will occur suddenly, without warning.




**(a)** beam under concentrated loads

**(b)** Case 1 : Deep Beams : $a/d < 1$

**(c)** Case 2 : $1 < a/d < 2.5$

**(d)** Case 3 : $2.5 < a/d < 6$

**(e)** web-crushing failure

**Fig. 6.4** Typical shear failure modes

### 6.3.3 Shear Failure Modes

As explained earlier (with reference to Eq. 6.2), the magnitude and direction of the maximum principal tensile stress, and hence the development and growth of inclined cracks are influenced by the relative magnitudes of the flexural stress $f_x$ and the shear stress $\tau$. As an approximation, stresses $f_x$ and $\tau$ can be considered proportional to $M/(bd^2)$ and $V/(bd)$ respectively, where $M$ and $V$ are the applied bending moment and shear force respectively at the beam section under consideration, $b$ is the width and $d$ the effective depth.
Accordingly,

$$\frac{f_x}{\tau} = \frac{F_1 M/(bd^2)}{F_2 V/(bd)} = F_3 \frac{M}{Vd} \tag{6.5}$$

where $F_1$, $F_2$, $F_3$ are constants of proportionality. For beams subjected to concentrated loads [Fig. 6.4(a)], the ratio $M/V$ at the critical section subjected to the *maximum V* works out exactly to the distance *a*, called *shear span*, between the support and the load. In such a case, the ratio $M/(Vd)$ becomes equal to $a/d$, the shear span-depth ratio, whereby Eq. 6.5 reduces to

$$\frac{f_x}{\tau} \propto \frac{a}{d} \tag{6.6}$$

It can be seen that the dimensionless ratio $a/d$ (or $M/(Vd)$) provides a measure of the relative magnitudes of the flexural stress and the shear stress, and hence enables the prediction of the mode of failure of the beam in flexural shear [Ref. 6.2, 6.3]. The prediction is based on considerable experimental evidence involving simply supported beams of rectangular cross-section subjected to symmetrical two-point loading.

### *Case* 1*: a/d* < 1

In very deep beams ($a/d < 1$) without web reinforcement, inclined cracking transforms the beam into a *tied-arch* [Fig. 6.4(b)]. The tied–arch may fail either by a breakdown of its tension element, viz. the longitudinal reinforcement (by yielding, fracture or failure of anchorage) or a breakdown of its compression chord (crushing of concrete), as shown in Fig. 6.4(b).

### *Case* 2*:* 1 < *a/d* < 2.5

In relatively short beams with $a/d$ in the range of 1 to 2.5, the failure is initiated by an inclined crack — usually a *flexure-shear crack*. The actual failure may take place either by (1) crushing of the reduced concrete section above the tip of the crack under combined shear and compression, termed *shear-compression* failure or (2) secondary cracking along the tension reinforcement, termed *shear-tension* failure. Both these types of failure usually occur before the flexural strength (full moment-resisting capacity) of the beam is attained.

However, when the loads and reactions applied on the top and bottom surfaces of the beam are so located as to induce a vertical compressive stress in concrete between the load and the reaction, the shear strength may be increased significantly — requiring very heavy loads to cause inclined cracking.

### Case 3: 2.5 < *a/d* < 6

Normal beams have $a/d$ ratios in excess of 2.5. Such beams may fail either in shear or in flexure. The limiting $a/d$ ratio above which flexural failure is certain depends on the tension steel area as well as strength of concrete and steel; generally, it is in the neighbourhood of 6.

For beams with $a/d$ ratios in the range 2.5 to 6, flexural tension cracks develop early. Failure in shear occurs by the propagation of inclined flexural-shear cracks. As mentioned earlier, if shear (web) reinforcement is not provided, the cracks extend rapidly to the top of the beam; the failure occurs suddenly and is termed *diagonal tension* failure [Fig. 6.4(d)]. Addition of web reinforcement enhances the shear strength considerably. Loads can be carried until failure occurs in a *shear-tension* mode (yielding of the shear reinforcement) or in a *shear-compression* mode[†], or in a *flexural* mode.

#### *Web–Crushing Failure*

In addition to the modes described above, thin-webbed members (such as *I*–beams with web reinforcement) may fail by the crushing of concrete in the web portion between the inclined cracks under diagonal compression forces [Fig. 6.4(e)].

## 6.4 NOMINAL SHEAR STRESS

The concept of *average shear stress* $\tau_{av}$ in a beam section is used in mechanics of materials, with reference to a homogeneous elastic material [Fig. 6.1]. It is defined as

$$\tau_{av} = \frac{\text{shear force}}{\text{area of cross section}}$$

For simplicity, this parameter is used as a *measure* of the shear stresses in a reinforced concrete beam section as well.

### 6.4.1 Members with Uniform Depth

For *prismatic* members of rectangular (or flanged) sections, the Code (Cl. 40.1) uses the term *nominal shear stress* $\tau_v$, defined at the ultimate limit state, as follows:

$$\tau_v = \frac{V_u}{bd} \qquad (6.7)$$

where $V_u$ is the *factored* shear force at the section under consideration, $b$ is the width of the beam (taken as the web width $b_w$ in flanged beams), and $d$ the effective depth of the section.

It should be noted that $\tau_v$ is merely a parameter intended to aid design and to control shear stresses in reinforced concrete; it does not actually represent the true average shear stress (whose distribution is quite complex in reinforced concrete).

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[†] If web steel is excessive, it may not yield; instead concrete in diagonal compression gets crushed.

### 6.4.2 Members with Varying Depth

In the case of members with varying depth [Fig. 6.5], the nominal shear stress, defined by Eq. 6.7, needs to be modified, to account for the contribution of the vertical component of the flexural tensile force $T_u$ which is inclined at an angle $\beta$ to the longitudinal direction.

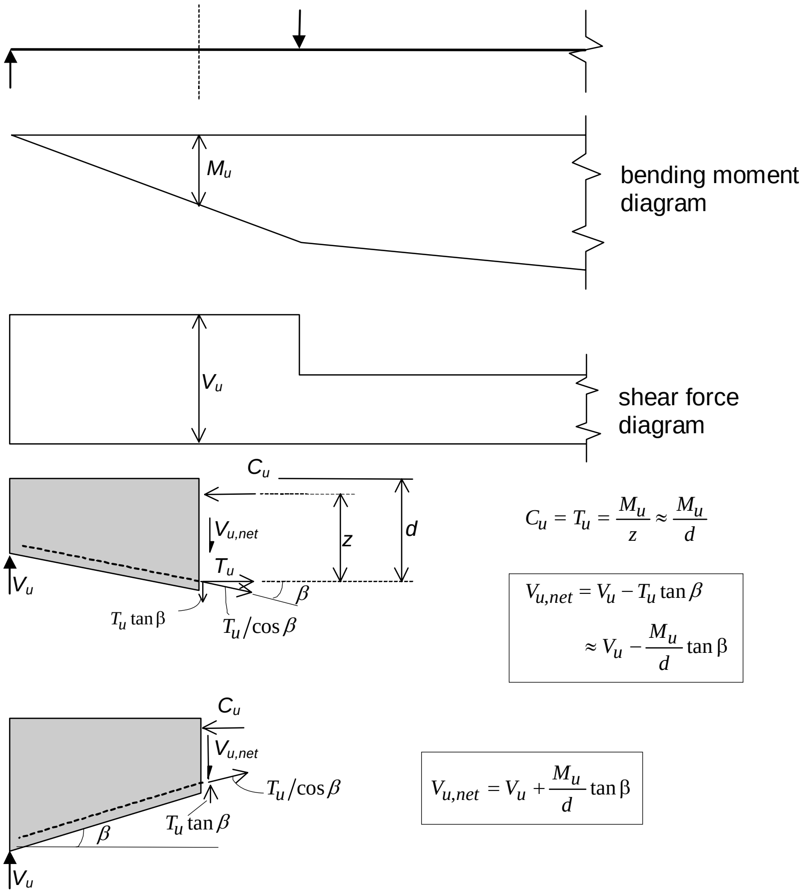


**Fig. 6.5** Design shear force in beams of variable depth

Assuming the horizontal component of $T_u$ as $M_u/z \approx M_u/d$, it can be seen from Fig. 6.5 that the *net shear force* $V_{u,net}$ for which the section should be designed is:

$$V_{u,net} = V_u \pm \frac{M_u}{d}\tan\beta \tag{6.8}$$

Accordingly, the *nominal shear stress* (Cl. 40.1.1 of the Code), defined with respect to $V_{u,net}$, is obtained as

$$\tau_v = \frac{V_u \pm \frac{M_u}{d}\tan\beta}{bd} \tag{6.9}$$

where $V_u$ and $M_u$ are the applied factored shear force and bending moment at the section under consideration. The negative sign in Eq. 6.8, 6.9 applies where $M_u$ *increases* in the same direction as the depth increases and the positive sign applies where $M_u$ decreases in this direction, as shown clearly in Fig. 6.5.

A similar adjustment to the shear $V_u$ and the nominal shear stress $\tau_v$ is called for when the flexural compression $C_u$ is inclined to the longitudinal axis of the beam, i.e., the compression face is sloping. Such a situation is encountered in tapered base slabs of footings [refer Chapter 14]. It can be shown that Eq. 6.9 holds good in this case also.

It may be noted that when the depth increases in the same direction as the bending moment (as is usually the case in cantilever beams), there is an advantage to be gained, in terms of reduced shear stress, by the application of Eq. 6.9 rather than Eq. 6.7. In such a case, the use of the simpler Eq. 6.7 for nominal shear stress $\tau_v$ (sometimes adopted in practice, for convenience) will give conservative results. However, the use of Eq. 6.9 becomes mandatory when the effect of the vertical component of $T_u$ is unfavourable, i.e., when the depth decreases with increasing moment.

## 6.5 CRITICAL SECTIONS FOR SHEAR DESIGN

In designing for flexural shear, the *critical sections* to be investigated first (for calculating the nominal shear stress $\tau_v$) are the ones where the shear force is maximum and/or the cross-sectional area is minimal.

The maximum shear force usually occurs in a flexural member at the face of the support, and progressively reduces with increasing distance from the support. When concentrated loads are involved, the shear force remains high in the span between the support and the first concentrated load.

When a support reaction introduces transverse compression in the end region of the member, the shear strength of this region is enhanced, and inclined cracks do not develop near the face of the support (which is usually the location of maximum shear). In such a case, the Code (Cl. 22.6.2.1) allows a section located at a distance $d$ (effective depth) from the face of the support to be treated as the *critical section* [Fig. 6.6(a)]. The beam segment between this critical section and the face of the support need be designed only for the shear force at the critical section. As the shear force at this critical section will be less than (or equal to) the value at the face of the support, the Code recommendation will usually result in a more favourable (less) value of $\tau_v$ than otherwise. This is of particular significance in base slabs of footing where flexural (one-way) shear is a major design consideration [refer Chapter 14].

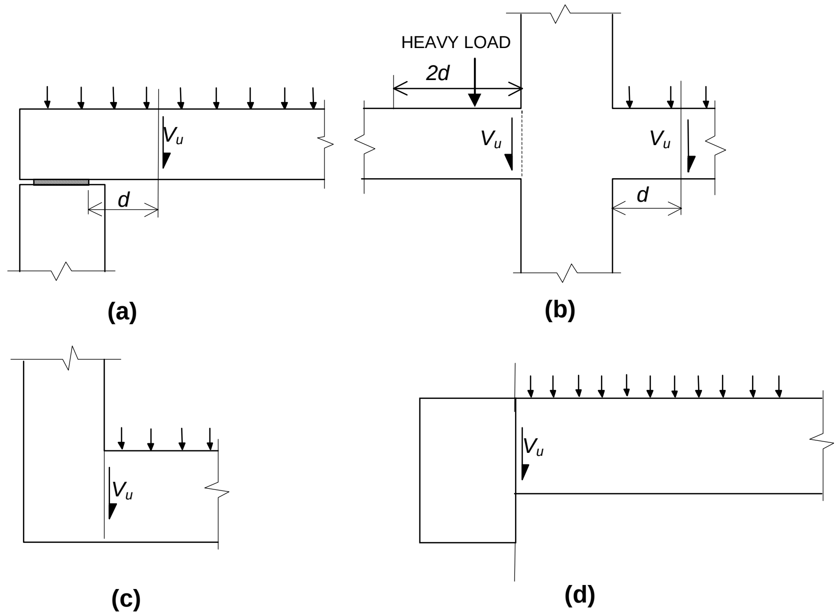


**Fig. 6.6** Critical sections for shear at support

However, when a heavy concentrated load is introduced within the distance 2$d$ from the face of the support, then the face of the support becomes the critical section [Fig. 6.6(b)], as inclined cracks can develop within this region if the shear strength is exceeded. In such cases, closely spaced stirrups should be designed and provided in the region between the concentrated load and the support face.

Also, when the favourable effect of transverse compression from the reaction is absent — as in a suspended beam [Fig. 6.6(c)], or a beam (or bracket) connected to the side of another supporting beam [Fig. 6.6(d)] — the critical section for shear should be taken at the face of the support.

In the latter case [Fig. 6.6(d)], special shear reinforcement detailing is called for — to ensure that effective shear transfer takes place between the *supported* beam (or bracket in some situation) and the *supporting* beam. It is recommended [Ref. 6.8] that full depth stirrups should be designed in both the supported member and the supporting member in the vicinity of the interface for 'hanging up' a portion of the interface shear, equal to $V_u\,(1 - h_b/D)$, the dimensions $h_b$ and $D$ being as shown in Fig. 6.7. The shear reinforcement (stirrups) so designed must be accommodated in the effective regions indicated in Fig. 6.7 [refer Section 6.8 for design procedure].

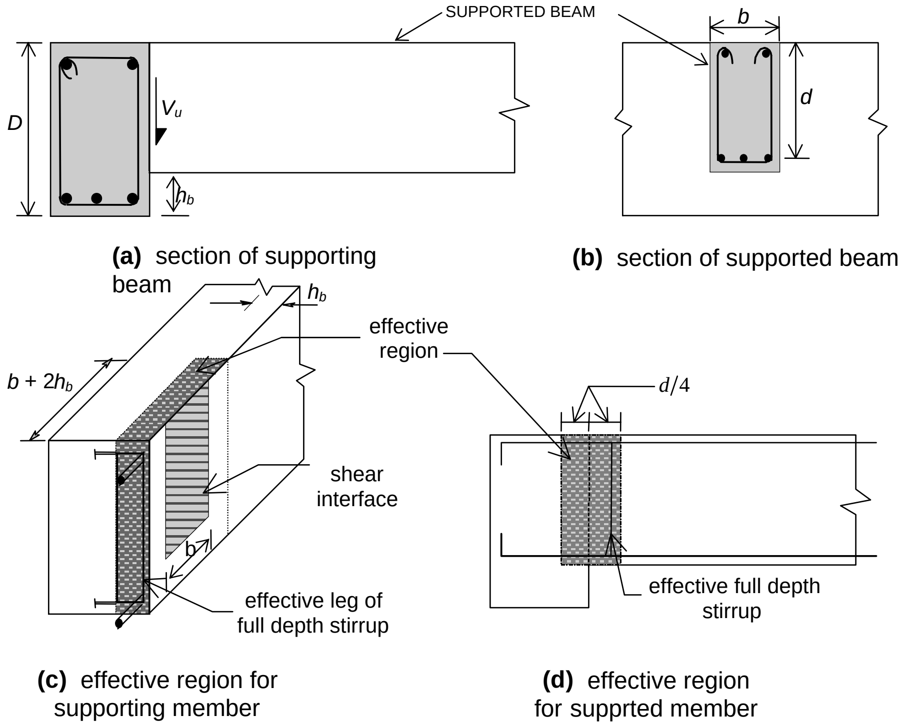


**Fig. 6.7** Detailing of stirrups for interface shear at indirect support

## 6.6 DESIGN SHEAR STRENGTH WITHOUT SHEAR REINFORCEMENT

### 6.6.1 Design Shear Strength of Concrete in Beams

As explained earlier in Section 6.3.2, the margin of strength beyond diagonal cracking is subject to considerable fluctuation on account of various factors, and hence is ignored for design purposes [Ref. 6.4, 6.5]. Accordingly, the (average) *design shear strength* $\tau_c$ of concrete in reinforced concrete beams without shear reinforcement is limited to the value of the nominal shear stress $\tau_v$ corresponding to the load at which the first inclined crack develops; some partial factor of safety ($\approx 1.2$) is also introduced.

The magnitude of the design shear strength $\tau_c$ depends on various factors (refer Section 6.3.2) that are related to the grade of concrete ($f_{ck}$) and the percentage tension steel $p_t = 100A_{st}/(bd)$. The values of $\tau_c$ given in the Code (Table 19) are based on the following empirical formula [Ref. 6.5, 6.6]:

$$\tau_c = 0.85\sqrt{(0.8 f_{ck})}(\sqrt{1+5\beta} - 1)\big/(6\beta) \tag{6.10}$$

**Table 6.1** Design Shear Strength of Concrete $\tau_c$ (MPa)

| $p_t = \frac{100A_{st}}{bd}$ | Concrete grade | | | | |
|---|---|---|---|---|---|
| | M 20 | M 25 | M 30 | M 35 | M 40 and above |
| 0.2 | 0.326 | 0.331 | 0.334 | 0.337 | 0.339 |
| 0.3 | 0.388 | 0.395 | 0.400 | 0.403 | 0.407 |
| 0.4 | 0.437 | 0.446 | 0.452 | 0.457 | 0.461 |
| 0.5 | 0.478 | 0.489 | 0.497 | 0.503 | 0.508 |
| 0.6 | 0.514 | 0.526 | 0.536 | 0.543 | 0.549 |
| 0.7 | 0.546 | 0.559 | 0.570 | 0.578 | 0.585 |
| 0.8 | 0.574 | 0.589 | 0.601 | 0.611 | 0.618 |
| 0.9 | 0.599 | 0.616 | 0.630 | 0.640 | 0.649 |
| 1.0 | 0.623 | 0.641 | 0.656 | 0.667 | 0.677 |
| 1.1 | 0.644 | 0.664 | 0.680 | 0.692 | 0.703 |
| 1.2 | 0.664 | 0.686 | 0.703 | 0.716 | 0.727 |
| 1.3 | 0.682 | 0.706 | 0.724 | 0.738 | 0.750 |
| 1.4 | 0.700 | 0.725 | 0.744 | 0.759 | 0.771 |
| 1.5 | 0.716 | 0.742 | 0.762 | 0.779 | 0.792 |
| 1.6 | 0.731 | 0.759 | 0.780 | 0.797 | 0.811 |
| 1.7 | 0.746 | 0.775 | 0.797 | 0.815 | 0.830 |
| 1.8 | 0.760 | 0.790 | 0.813 | 0.832 | 0.847 |
| 1.9 | 0.773 | 0.804 | 0.828 | 0.848 | 0.864 |
| 2.0 | 0.785 | 0.818 | 0.843 | 0.864 | 0.880 |
| 2.1 | 0.797 | 0.831 | 0.857 | 0.878 | 0.896 |
| 2.2 | 0.808 | 0.843 | 0.871 | 0.893 | 0.911 |
| 2.3 | 0.819 | 0.855 | 0.883 | 0.906 | 0.925 |
| 2.4 | 0.821 | 0.867 | 0.896 | 0.919 | 0.939 |
| 2.5 | 0.821 | 0.878 | 0.908 | 0.932 | 0.952 |
| 2.6 | 0.821 | 0.888 | 0.919 | 0.944 | 0.965 |
| 2.7 | 0.821 | 0.899 | 0.930 | 0.956 | 0.978 |
| 2.8 | 0.821 | 0.909 | 0.941 | 0.968 | 0.990 |
| 2.9 | 0.821 | 0.918 | 0.952 | 0.979 | 1.001 |
| 3.0 | 0.821 | 0.918 | 0.962 | 0.989 | 1.013 |
| 3.1 | 0.821 | 0.918 | 0.971 | 1.000 | 1.024 |
| 3.2 | 0.821 | 0.918 | 0.981 | 1.010 | 1.034 |
| 3.3 | 0.821 | 0.918 | 0.990 | 1.020 | 1.045 |
| 3.4 | 0.821 | 0.918 | 0.999 | 1.029 | 1.055 |
| 3.5 | 0.821 | 0.918 | 1.006 | 1.039 | 1.065 |
| 3.6 | 0.821 | 0.918 | 1.006 | 1.048 | 1.074 |
| 3.7 | 0.821 | 0.918 | 1.006 | 1.056 | 1.084 |
| 3.8 | 0.821 | 0.918 | 1.006 | 1.065 | 1.093 |
| 3.9 | 0.821 | 0.918 | 1.006 | 1.073 | 1.102 |
| 4.0 | 0.821 | 0.918 | 1.006 | 1.081 | 1.110 |
| 4.1 | 0.821 | 0.918 | 1.006 | 1.087 | 1.119 |
| 4.2 | 0.821 | 0.918 | 1.006 | 1.087 | 1.127 |
| 4.3 | 0.821 | 0.918 | 1.006 | 1.087 | 1.135 |
| 4.4 | 0.821 | 0.918 | 1.006 | 1.087 | 1.143 |
| 4.5 | 0.821 | 0.918 | 1.006 | 1.087 | 1.151 |
| 4.6 | 0.821 | 0.918 | 1.006 | 1.087 | 1.158 |
| 4.7 | 0.821 | 0.918 | 1.006 | 1.087 | 1.162 |

where $$\beta \equiv \begin{cases} (0.8 f_{ck})/(6.89 p_t) \\ 1 \end{cases} \text{ whichever is greater} \tag{6.10a}$$

Typical values of $\tau_c$ are listed in Table 6.1 for different values of $f_{ck}$ and $p_t$.

It may be observed that, for a given $f_{ck}$, there is a value of $p_t$ (corresponding to $\beta = 1$ in Eq. 6.10a), beyond which $\tau_c$ remains constant, implying that the beneficial effects due to dowel action, control of crack propagation and increased depth of uncracked concrete in compression, cannot increase indefinitely with increasing $p_t$.

Further, it may be noted that the use of the values of $\tau_c$ listed in Table 6.1 (based on Eq. 6.10) for a given value of $p_t$ are applicable at bar cut-off regions, only if the detailing requirements are adequately satisfied (refer Section 5.9.3). Where bars are proposed to be curtailed at locations where the shear requirements are not otherwise satisfied, it is necessary to provide additional stirrups locally near the cut-off points (thereby satisfying Cl. 26.2.3.2 of the Code).

As explained in Section 6.5 [Fig. 6.6(a)], the shear strength of concrete is enhanced in regions close to the support (located 2*d* away from the face of the support), when the support reaction introduces transverse compression. It is seen that a substantial portion of the load is transmitted to the support directly through strut action, rather than through flexural shear. A recent revision in the Code allows for enhancement of shear strength of concrete $\tau_c$ in this region, provided the flexural tension reinforcement is extended beyond this region and well anchored. The Code (Cl. 40.5) permits an increase in $\tau_c$ at any section located at a distance $a_v$ (less than 2*d*) from the face of the support by a factor $(2d)/a_v$. However, this increase should be used with caution, as it is implied that as the critical section approaches the face of the support, the shear strength will increase asymptotically, which is not realistic. The authors suggest that the shear strength in concrete for sections within a distance *d* from the face of the support should be limited to $2\tau_c$.

### 6.6.2 Design Shear Strength of Concrete in Slabs

Experimental studies [Ref. 6.2–6.4] have shown that slabs and shallow beams fail at loads corresponding to a nominal stress that is higher than that applicable for beams of usual proportion. Moreover, the thinner the slab, the greater is the increase in shear strength. In recognition of this, the Code (Cl. 40.2.1.1) suggests an increased shear strength, equal to $k\tau_c$ for 'solid slabs' (i.e., not including *ribbed slabs*), the multiplication factor $k$ having a value in the range 1.0 to 1.3, expressed as follows:

$$k = \begin{cases} 1.3 & \text{for } D \le 150 \text{ mm} \\ 1.6 - 0.002D & \text{for } 150 < D < 300 \text{ mm} \\ 1.0 & \text{for } D \ge 300 \text{ mm} \end{cases} \tag{6.11}$$

where $D$ is the overall depth of the slab in mm.

It should be noted that these provisions for design shear strength are applicable only for considerations of *flexural shear* (or 'one-way shear'). For *flat slabs* and column footings, *punching shear* ('two-way shear') has to be considered, which involve different considerations of shear strength [refer Chapter 11].

In general, slabs subjected to normal distributed loads satisfy the requirement $\tau_v < k\tau_c$, and hence do not need shear reinforcement. This is mainly attributable to the fact that the thickness of the slab (controlled by limiting deflection criteria) is usually adequate in terms of shear capacity. This is demonstrated in Example 6.1.

### 6.6.3 Influence of Axial Force on Design Shear Strength

In Section 2.10.2, it was indicated that the actual shear strength of concrete is generally improved in the presence of uniaxial compression and weakened in the presence of uniaxial tension.

As explained earlier (in Section 6.6.1) the design shear strength is based on a safe estimate of the limiting nominal stress at which the first inclined crack develops. The effect of an axial *compressive* force is to delay the formation of both flexural and inclined cracks, and also to decrease the angle of inclination $\alpha$ of the inclined cracks to the longitudinal axis [Ref. 6.5]. Likewise, an axial *tensile* force is expected to do exactly the reverse, i.e., it will decrease the shear strength, accelerate the process of cracking and increase the angle $\alpha$ of the inclined cracks.

Accordingly, the Code (Cl. 40.2.2) specifies that the design shear strength in the presence of *axial compression* should be taken as $\delta\tau_c$, the multiplying factor $\delta$ being defined as:

$$\delta = \begin{cases} 1 + \dfrac{3P_u}{A_g f_{ck}} \\ 1.5 \end{cases} \quad \text{whichever is less} \tag{6.12}$$

where $P_u$ is the factored compressive force (in *N*), $A_g$ is the gross area of the section (in mm$^2$) and $f_{ck}$ is the characteristic strength of concrete (in MPa).

Although the Code does not explicitly mention the case of *axial tension*, it is evident that some reduction in design shear strength is called for in such a case. The following simplified expression for $\delta$, based on the ACI Code [Ref. 6.7], may be used:

$$\delta = 1 + \frac{P_u}{3.45A_g} \quad \text{for } P_u < 0 \tag{6.13}$$

where $P_u$ is the factored axial tension (in N), with a negative sign.

Alternatively, when axial tension is also present, design for shear may be done based on the Compression Field Theory or the Strut-and-Tie Model, described in Chapter 17.

## 6.7 DESIGN SHEAR STRENGTH WITH SHEAR REINFORCEMENT

### 6.7.1 Types of Shear Reinforcement

*Shear reinforcement*, also known as *web reinforcement* may consist of any one of the following systems (Cl. 40.4 of the Code)

- stirrups perpendicular to the beam axis;
- stirrups inclined (at 45° or more) to the beam axis; and
- longitudinal bars bent-up (usually, not more than two at a time) at 45° to 60° to the beam axis, combined with stirrups.

By far, the most common type of shear reinforcement is the *two-legged stirrup*, comprising a closed or open loop, with its ends anchored properly around longitudinal bars/stirrup holders (to develop the yield strength in tension). It is placed perpendicular to the member axis ('vertical† stirrup'), and may or may not be combined with bent-up bars, as shown in Fig. 6.8.

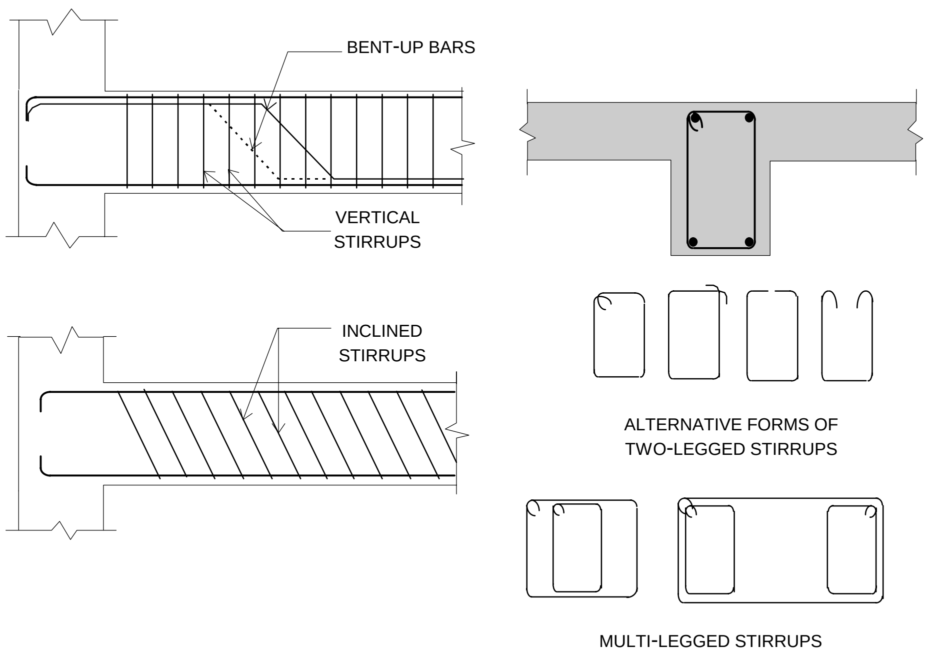


**Fig. 6.8** Types of shear reinforcement

† The term vertical is commonly used (as in the Code), with the assumption that the beam axis is horizontal (as is commonly the case); the term is also (although inappropriate) used in the case of beams inclined to the horizontal.

The direction of bending up of the tension bar (or the direction of the inclined stirrup) should be such that it intercepts the potential inclined (diagonal tension) crack, nearly at right angles, thereby most effectively restraining the opening up and propagation of crack. The bent-up bar must be properly anchored; in order to be effective, a full 'development length' $L_d$ is required beyond the midpoint of the inclined portion of the bar.

The stirrup (particularly, the inclined stirrup) is considered to be most effective in enhancing the overall shear resistance of the beam, because in addition to contributing in much the same way as the bent–up bar, it contributes significantly towards improved *dowel action* of the longitudinal tension bars, by restraining the latter from undergoing transverse (dowel) displacements. Strictly, simple U-shaped ('open') stirrups with the free ends anchored properly in the compression zone by hooks suffice as shear reinforcement. However, 'closed' loops are called for in resisting torsion and confining the compression reinforcement (when provided, in doubly reinforced beams). It is desirable to locate the hook in the closed stirrup in the compression zone, rather than the tension zone, for improved anchorage and to avoid crack initiation.

The shear resistance of bent-up bars cannot be fully relied upon, unless stirrups are also provided, to ensure adequate development of dowel action of the longitudinal bars. The Code (Cl. 40.4) specifies that

> "Where bent-up bars are provided, their contribution towards shear resistance shall not be more than half that of the total shear reinforcement"

### 6.7.2 Factors Contributing to Ultimate Shear Resistance

If $V_{uc}$ and $V_{us}$ denote respectively the ultimate shear resistance of the concrete and the shear reinforcement, then the total *ultimate shear resistance* $V_{uR}$ at any section of the beams is given by

$$V_{uR} = V_{uc} + V_{us} \tag{6.14}$$

In Eq. 6.14, the shear resistance $V_{uc}$ of concrete is made up of all the components $V_{cz}$, $V_{ay}$ and $V_d$ of Eq. 6.4. Although the relative magnitudes of these components of $V_{uc}$ vary with the stage of loading and the state of cracking (refer Section 6.3.2), their aggregate value $V_{uc}$ is assumed to be constant, and obtainable from the design strength of concrete $\tau_c$ as

$$V_{uc} = \tau_c bd \tag{6.15}$$

This follows from the concept of $\tau_c$ as the 'safe' limiting value of the nominal shear stress $\tau_v$ (given by Eq. 6.7) of concrete *without* shear reinforcement. In beams *with* shear reinforcement, it is found from actual measurements of strain (in the shear reinforcement) that, prior to the formation of diagonal tension cracks, there is practically no tensile stress developed in the shear reinforcement [Ref. 6.2]. Accordingly, $V_{uc}$ denotes the shear resistance at the stage of initiation of diagonal cracking in flexural members, regardless of whether or not shear reinforcement is provided.

As explained earlier in Section 6.3.2, all the four shear transfer mechanisms ($V_{cz}$, $V_{ay}$, $V_d$, $V_s$) become operational, following the development of inclined cracks. The shear reinforcement contributes significantly to the overall shear resistance by increasing or maintaining the individual components $V_{cz}$, $V_{ay}$ and $V_d$ of Eq. 6.4, in addition to directly contributing by means of the tension $V_s$ in the legs of the stirrup and the bent-up portions of the bent-up bars (where provided).

For simplicity, it is assumed that the contribution $V_{cz} + V_{ay} + V_d$ remains practically unchanged following the stage of inclined cracking. Hence, $V_{uc}$ (calculated using Eq. 6.15) is assumed to represent the shear resistance of the concrete at the ultimate limit state in beams with shear reinforcement as well. Further, as explained in Section 6.3.2, once the shear reinforcement starts yielding in tension, its shear resisting capacity remains practically constant as $V_{us}$. Expressions for $V_{us}$ are derived in Section 6.7.4.

From a design viewpoint, suitable shear reinforcement has to be designed if the factored shear $V_u$ exceeds $V_{uc}$ (i.e., $\tau_v$ exceeds $\tau_c$), and the shear resistance required from the web reinforcement is given by

$$V_{us} \geq V_u - V_{uc} = (\tau_v - \tau_c)\, bd \tag{6.16}$$

### 6.7.3 Limiting Ultimate Shear Resistance

As explained earlier, the yielding of the shear reinforcement at the ultimate limit state is essential to ensure a *ductile* failure (with ample warning). However, such a failure will not occur if the shear reinforcement provided is excessive. If the total cross-sectional area $A_{sv}$ of the stirrup legs and the bent-up bars exceeds a certain limit, it is likely that the section becomes stronger in diagonal tension compared to diagonal compression. Hence, a *shear-compression* failure [Fig. 6.4(c)] may occur even before the shear reinforcement has yielded (and thus realised its full potential). Such a situation is undesirable due to the *brittle* nature of the failure; moreover, it turns out to be uneconomical, in much the same way as over-reinforced beams.

In order to prevent such shear-compression failures and to ensure yielding of the shear reinforcement at the ultimate limit state, the Code (Cl. 40.2.3) has indirectly imposed a limit on the resistance $V_{us}$, by limiting the ultimate shear resistance $V_{uR}$:

$$V_{uR,\lim} = \tau_{c,\max} bd \tag{6.17}$$

where $\tau_{c,\max} (\approx 0.62\sqrt{f_{ck}})$ is given values (in MPa) of 2.4, 2.8, 3.1, 3.4, 3.7 and 3.9 for concrete grades M 15, M 20, M 25, M 30, M 35 and M 40 respectively (vide Table 20 of the Code).

Thus, if the calculated nominal shear stress $\tau_v\ (= V_u/bd)$ at a beam section exceeds the limit $\tau_{c,\max}$ (or the factored shear force $V_u$ exceeds $V_{uR,lim}$), the design should be suitably revised, either by improving the grade of concrete (thereby, raising $\tau_{c,\max}$) or increasing the dimensions of the beam (thereby, lowering $\tau_v$). The increase in $\tau_{c,\max}$ with the compressive strength of concrete follows logically from the fact that the shear strength in diagonal compression gets enhanced.

In the case of solid slabs, the Code (Cl. 40.2.3.1) specifies that $\tau_v$ should not exceed $0.5\,\tau_{c,\max}$ (i.e., 1.2 MPa for M 15, 1.4 MPa for M 20, 1.55 MPa for M 25, 1.7 MPa for M 30, 1.85 MPa for M 35 and 1.95 MPa for M 40 concrete).

### 6.7.4 Shear Resistance of Web Reinforcement

Traditionally, the action of web reinforcement in reinforced concrete beams has been explained with the aid of the *truss analogy*, the simplest form of which is shown in Fig. 6.9. This design model was first enunciated by Ritter in 1899. In this model, a reinforced concrete beam with inclined cracks is replaced with a pin-jointed truss, whose compression chord represents the concrete compression zone at the top, and whose tension chord at the bottom represents the longitudinal tension reinforcement. Further, the tension web members (shown vertical in Fig. 6.9a) represent the stirrups, and the diagonal web members represent the concrete in compression between the inclined cracks. (The truss model is akin to the Strut-and-Tie model).

In this model, the compression diagonals do not have to go from top of one stirrup to the bottom of the next. In reality, rather than having discrete diagonal compressive struts, there is a continuous field of diagonal compression contributing to shear resistance.

The truss model is a helpful tool in visualising the forces in the stirrups (under tension) and the concrete (under diagonal compression), and in providing a basis for simplified design concepts and methods. However, this model does not recognise fully the actual action of the web reinforcement and its effect on the various types of shear transfer mechanisms identified in Fig. 6.3.

Fig. 6.9(b) shows one segment of the beam separated by a diagonal tension crack. This is an idealization of Fig. 6.9(a), wherein the diagonal crack is assumed to be straight, inclined at an angle $\theta$ to the beam axis and extends over the full depth of the beam. The general case of inclined stirrups is considered in the freebody in Fig. 6.9(b); only the forces in the web reinforcement that contribute to the resistance $V_{us}$ are shown.

The inclined stirrups are assumed to be placed at an angle $\alpha$ (not less than 45° in design practice) with the beam axis, and spaced $s_v$ apart along the beam axis. If $A_{sv}$ is the total cross-sectional area of one stirrup (considering all the legs intercepting the inclined crack) and $0.87f_y$ is the design yield stress in it (assuming yielding at the ultimate limit state), then the total shear resistance of all the stirrups intercepting the crack is given by:

$$V_{us} = (\textit{vertical component of tension per stirrup}) \times (\textit{number of stirrups})$$

$$\Rightarrow V_{us} = (0.87\, f_y A_{sv} \sin\alpha) \times d(\cot\theta + \cot\alpha)/s_v$$

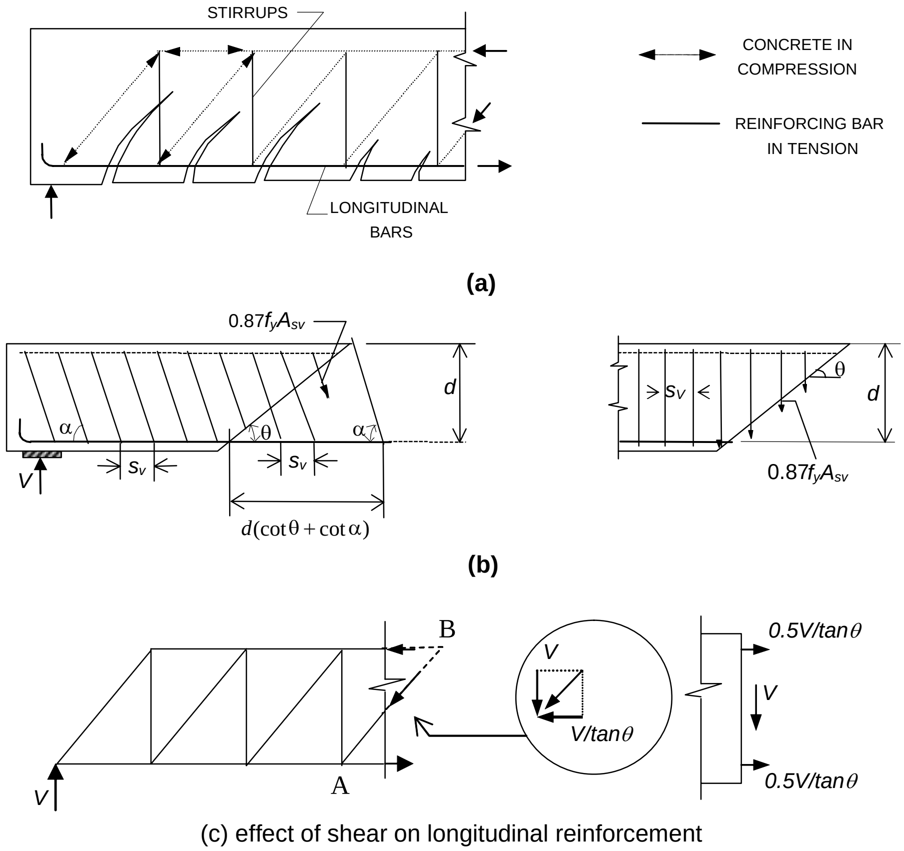


**Fig. 6.9** Classical truss analogy for action of web reinforcement

Assuming, for convenience, that the crack is located at $\theta$ = 45°, the above relation simplifies to

$$V_{us} = 0.87\, f_y A_{sv}\,(d/s_v)(\sin\alpha + \cos\alpha) \tag{6.18}$$

The case of 'vertical stirrups' may be considered as a special case with $\alpha$ = 90°. Hence, for vertical stirrups, the shear resistance $V_{us}$ is obtained from Eq. 6.18 as

$$V_{us} = 0.87\, f_y A_{sv}\, d/s_v \tag{6.19}$$

The shear resistance of *bent-up bars* may also be obtained from Eq. 6.18 — when a series of single or parallel bent-up bars are provided at regular intervals in the manner of inclined stirrups. However, when a *single* bar or a single group of parallel bars are bent-up at the same location at an angle $\alpha$ , the vertical component of the total tension in these bars is given[†] by

$$V_{us} = 0.87 f_y A_{sv} \sin\alpha \qquad (6.20)$$

### 6.7.5 Influence of Shear on Longitudinal Reinforcement

The truss analogy illustrates an important effect pertaining to influence of shear on the tension in the longitudinal reinforcement. Usually, the tension steel area requirement at any section is governed by the bending moment in the beam at that section. However, when the beam is cracked (especially, at ultimate loads), there will be a change in the calculated tensile stress. The presence of a diagonal crack will alter the tensile stress in the longitudinal steel, as observed earlier in the context of curtailment of bars (refer Fig. 5.14). This is also clear in the truss analogy, as revealed by the section of the truss shown in Fig. 6.9(c). By applying the 'method of sections', we observe that the compressive force in the top chord will be less than the tensile force in the bottom chord of the truss in any given panel (owing to the presence of the diagonals) and this difference will be equal to the horizontal component of the force in the diagonal. Whereas the force in the top chord (compression in concrete) is governed by the bending moment at A, the force in the bottom chord (tension reinforcement) is governed by the bending moment at B, which is higher than that at A. Thus, the presence of a diagonal tension crack due to shear results in an increase in the tension in the longitudinal reinforcement. The increased tension is given approximately by half of the horizontal component of the force in the diagonal strut in Fig. 6.9(c)[‡]; i.e., equal to $0.5V/tan\theta$. This influence of shear in enhancing the longitudinal reinforcement requirement was not realised till the 1950s. Even now, this is not directly reflected in the I.S. Code provisions as a specified additional amount of longitudinal reinforcement required for shear. Instead, it is accounted for indirectly by provisions for extension of flexural reinforcements (Cl. 26.2.3 – see also Sections 5.9.2 and 5.9.3). The Code (Cl. 26.2.3.1) requires that the flexural tension reinforcement be extended for a distance of $d$ or $12\phi$, whichever is greater, beyond the location required for flexure alone. Here $\phi$ is the nominal diameter of the longitudinal bar concerned, and the provision is applicable for locations other than at the supports of simple spans and at the free ends of cantilevers with concentrated loads. This provision is equivalent to the outward shifting of the design moment diagram by a distance of $d$ or $12\phi$ (Fig. 6.10a).

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[†] It may be noted that Eq. 6.20 is applicable only in the limited region where the bar is bent up.
[‡] The horizontal component of the force in the diagonal strut, equal to $V/tan\theta$ is assumed to be balanced equally by forces in the top and bottom chords. This, incidentally, also implies that there will be a reduction in the longitudinal compression in the top chord, equal to $0.5\ V/tan\theta$.

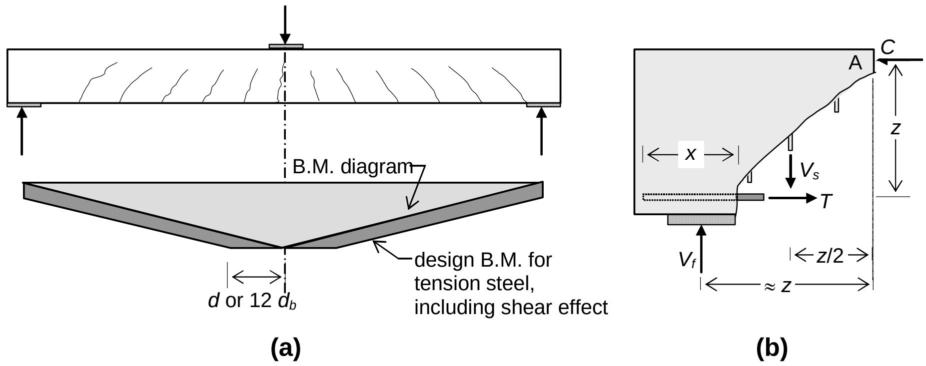


**Fig. 6.10** Design bending moment for tension steel, including shear effect

At simple supports and near free ends of cantilevers, the flexural tension reinforcement should be capable of resisting a tensile force of $V_f - 0.5\ V_s$ at the inside edge of the bearing area, where $V_s$ is the factored shear resistance provided by the shear reinforcement in this location. This expression can be derived from equilibrium considerations of the forces in the free body diagram of the support region, separated by a diagonal crack [Fig. 6.10b]. Taking moments about A and neglecting small quantities of second order,

$$V_f z \approx T z + V_s\ z/2$$

$$\Rightarrow T = V_f - 0.5 V_s \tag{6.21}$$

If the actual straight embedment length available at the support, $x$, is less than the development length, $L_d$, the stress that can be developed in the bar at the critical section at the inside edge of the bearing area (Fig. 6.10b) may be taken as:

$$f_s = \phi_s f_y \left(x/L_d\right)$$

Alternatively, the embedment length required to develop the stress $f_s$ in the bar can be computed as:

$$x = \frac{f_s}{\phi_s f_y} L_d$$

### *Code Recommendations*

Eq. 6.18–6.20 are given in the Code under Cl. 40.4. Further, the Code limits the maximum value of $f_y$ to 415 MPa, as higher strength reinforcement may be rendered brittle at the sharp bends of the web reinforcement; also, a shear compression failure could precede the yielding of the high strength steel.

The Code (Cl. 26.5.1.5) also limits the value of the spacing $s_v$ to 0.75 $d$ for 'vertical' stirrups and $d$ for inclined stirrups with $\alpha$ = 45°. This is done to ensure that every potential diagonal crack is intercepted by at least one stirrup. Further, the Code specifies that "*in no case shall the spacing exceed* 300 mm".

The overall shear resistance $V_{uR}$ is given by Eq. 6.14. For the purpose of *design* for a given factored shear force $V_u$, the web reinforcement is to be designed for a design shear force of ($V_u - \tau_c bd$ ), provided $\tau_v \leq \tau_{c,\max}$ (i.e., $V_u < V_{uR,lim}$).

From the viewpoint of *analysis* of a given reinforced concrete beam, the capacity $V_{us}$ may be determined from Eq. 6.18 – 6.20, assuming that the steel has yielded. However, the total shear resistance $V_{uR}$, given by Eq. 6.14, should be limited to $V_{uR,lim}$ given by Eq. 6.17.

### 6.7.6 Minimum Stirrup Reinforcement

The Code (Cl. 26.5.1.6) specifies a minimum shear reinforcement to be provided in the form of stirrups in all beams where the calculated nominal shear stress $\tau_v$ exceeds 0.5 $\tau_c$ :

$$\frac{A_{sv}}{bs_v} \geq \frac{0.4}{0.87 f_y} \quad \text{when } \tau_v > 0.5\ \tau_c \tag{6.22}$$

$$\Rightarrow\ s_v < \frac{2.175 f_y A_{sv}}{b} \tag{6.23}$$

The maximum spacing of stirrups should also comply with the requirements described earlier. For normal 'vertical' stirrups, the requirement is

$$s_v \leq \begin{cases} 0.75d \\ 300 \text{ mm} \end{cases} \quad \text{whichever is less} \tag{6.24}$$

The Code objective in recommending such minimum shear reinforcement is to prevent the sudden formation of an inclined crack in an unreinforced (or very lightly reinforced) web, possibly leading to an abrupt failure. Further, the provision of nominal web reinforcement restrains the growth of inclined shear cracks, improves the dowel action of the longitudinal tension bars, introduces ductility in shear and provides a warning of the impending failure.

## 6.8 ADDITIONAL COMMENTS ON SHEAR REINFORCEMENT DESIGN

- *Bent-up bars* generally give lower shear strength and often result in wider cracks than stirrups. Hence, unless there is a series of such bars bent up at relatively close spacings (as is possible in long-span bridge girders), there is not much economy resulting from considering their shear strength contribution. In normal situations, where there are only a few isolated bent-up bars scattered widely along the span, their shear strength contribution (not available at all sections) is ignored. Accordingly, the stirrups are designed to carry the full excess shear, given by:

$$V_{us} = (\tau_v - \tau_c)bd \tag{6.25}$$

- *Inclined stirrups* are most effective in reducing the width of the inclined cracks, and are desirable when full depth transverse cracks are likely (as in beams with high axial tension). However, such reinforcement may be rendered entirely ineffective if the direction of the shear force is reversed (as under seismic loads[†]).

- '*Vertical*' *stirrups* are the ones most commonly employed in practice. It should be noted that the use of closely spaced stirrups of smaller diameter gives better crack control than stirrups of larger diameter placed relatively far apart. The diameter is usually 8 mm, 10 mm or 12 mm. Where heavy shear reinforcement is called for, multiple-legged stirrups should be employed (as often required in the beams of slab-beam footings).
- For $n$-legged stirrups of diameter $\phi_s$ (where $n$ = 2, 4, 6),

$$A_{sv} = n\pi\phi_s^2/4 \tag{6.26}$$

The required spacing $s_v$ of 'vertical stirrups' for a selected diameter $\phi_s$ is given by applying Eq. 6.19, as:

$$s_v \leq \frac{0.87 f_y A_{sv}}{V_{us}/d} \tag{6.27}$$

where (from Eq. 6.25),

$$\frac{V_{us}}{d} = (\tau_v - \tau_c)b \tag{6.28}$$

It can also be seen from Eq. 6.19 and Eq. 6.26, that for a given arrangement of vertical stirrups (with specified $n$, $\phi_s$, $s_v$), the shear resistance in terms of $V_{us}/d$ is a constant (in $N$/mm units) given by

$$\frac{V_{us}}{d} = \frac{0.87 f_y A_{sv}}{s_v} \tag{6.29}$$

Accordingly, suitable design aids can be prepared expressing the above equation, as done in Table 62 of SP : 16 [Ref. 6.9] — to enable a quick design of vertical stirrups, for a specified $V_{us}/d$.

- The stirrup bar diameter is usually kept the same for the entire span of the beam. Theoretically, the required spacing of stirrups will vary continuously along the length of the beam owing to the variation in the shear force $V_u$. However, stirrups are usually arranged with the spacing kept uniform over portions of the span — satisfying the requirements of shear strength [Eq. 6.27] and maximum spacing [Eq. 6.23, 6.24]. The first stirrup should be placed at not more than one-half spacing ($s_v$ / 2) from the face of the support[†]. Also, a longitudinal bar (at least a 'hanger bar' of nominal diameter) must be located at every bend[‡] in a

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[†] Special provisions for shear reinforcement design, under earthquake loading, are covered in Chapter 16.
[†] See Section 6.5 and Fig. 6.7 regarding special case involving shear transfer.
[‡] The stirrup is tied to the longitudinal bar using 'binding wire'.

stirrup. The ends of the stirrup enclosing the longitudinal bars should satisfy *anchorage* requirements (discussed in Chapter 8).

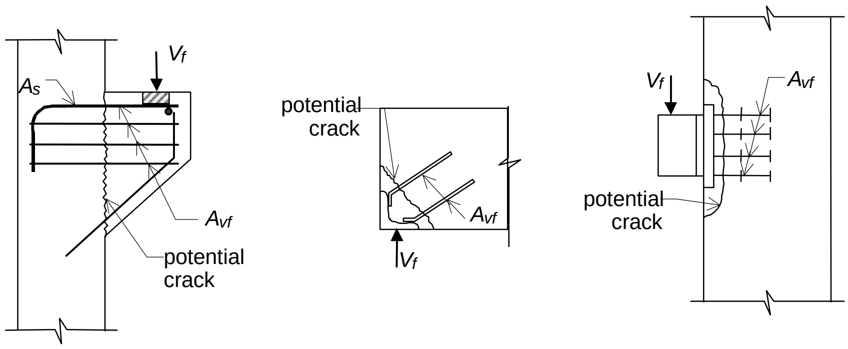
- Although the Code does not call for shear reinforcement in portions of beams where $\tau_v < \tau_c/2$, it is good design practice to provide minimum (nominal) stirrups [Eq. 6.23] in this region — to improve ductility and to restrain inclined cracks in the event of accidental overloading.
- The factored shear force $V_u$ to be considered for design at any section must take into account possible variations in the arrangement of live loads. The construction of *shear envelope* for this purpose is demonstrated in Examples 6.1 and 6.3.
- Termination of flexural reinforcement in the tension zone can lower the shear strength of beams (refer Section 5.9). Hence, such sections may also be critical and have to be checked for shear; if necessary, additional stirrups should be provided over a distance of 0.75$d$ from the cut-off point to satisfy the Code requirement (Cl. 26.2.3.2). This is demonstrated in Examples 6.1 and 6.3.
- When reversal of stresses occurs, as in the case of earthquake loading or reversed wind direction, the shear strength of the (previously cracked) concrete cannot be relied upon. In such cases, the stirrups should be designed to take the *entire shear*. Moreover, the stirrups should necessarily be in the form of closed loops placed perpendicular to the member axis. [The details of earthquake-resistant design for shear are described in Chapter 14.]

## 6.9 INTERFACE SHEAR AND SHEAR FRICTION

### 6.9.1 Shear-Friction

There are situations where shear has to be transferred across a defined plane of weakness, nearly parallel to the shear force and along which slip could occur (Fig. 6.11). Examples are planes of existing or potential cracks, interface between dissimilar materials, interfaces between elements such as webs and flanges, and interface between concrete placed at different times. In such cases, possible failure involves sliding along the plane of weakness rather than diagonal tension. Therefore it would be appropriate to consider shear resistance developed along such planes in the form of resistance to the tendency to slip. The *shear-friction* concept is a method to do this.

When two bodies are in contact with a normal reaction, $R$, across the surface of contact, the frictional resistance, $F$, acting tangential to this surface and resisting relative slip is known to be $F = \mu R$, where $\mu$ is the coefficient of friction (Fig. 6.12a). Figure 6.12(b) shows an idealised cracked concrete specimen loaded in shear. In such a specimen, a clamping force between the two faces of the crack can be induced by providing reinforcement (shear-friction reinforcement, $A_{vf}$) perpendicular to the crack surface. Any slip between the two faces of the rough irregular crack causes the faces to ride upon each other, which opens up the crack (Fig. 6.12c). This in turn induces tensile forces in the reinforcement, which ultimately yields (Fig. 6.12d). If the area of reinforcement is $A_{vf}$ and yield stress $f_y$, at ultimate, the clamping force between the two faces is $R = A_{vf} f_y$, and the frictional resistance is $\Sigma F = A_{vf} f_y \mu$..




**(a)** Corbel **(b)** Precast beam seat **(c)** Column face plate

**Fig. 6.11** Typical cases where shear friction is applicable (adapted from Ref. 6.10)

In reality, the actual resistance to shear, $V_r$, is composed of this frictional force ($\Sigma F$), the resistance to shearing off of the protrusions on the irregular surface of the crack, the dowel force developed in the transverse reinforcement, and when there are no cracks developed yet, the cohesion between the two parts as well. The *nominal* or *characteristic* (i.e. without safety factors) shear resistance, $V_{sn}$, due to the friction between the crack faces, is given by Eq. 6.30. Other less simple methods of calculation have been proposed (Refs. 6.11, 6.12) which result in predictions of shear transfer resistance in substantial agreement with comprehensive test results.

For shear-friction reinforcement placed perpendicular to the shear plane,

$$V_{sn} = A_{vf}\, f_y\, \mu \tag{6.30}$$

where, $V_{sn}$ = nominal shear resistance due to the assumed friction part alone contributed by reinforcement stress

$A_{vf}$ = area of shear-friction reinforcement, placed normal to the plane of possible slip

$\mu$ = coefficient of friction.

Shear-friction reinforcement may also be placed at an angle $\alpha_f$ to the shear plane, such that the shear force produces tension in the shear-friction reinforcement, as shown in Fig. 6.13(a), (b) (i.e., $\alpha_f \le 90°$). As the shear-friction reinforcement yields, the tensile force in the reinforcement is $A_{vf} f_y$, which has a component parallel to the shear plane of $A_{vf} f_y \cos\alpha_f$, and a component normal to the plane equal to $A_{vf} f_y \sin \alpha_f$. The latter produces the clamping force. The total force resisting shear is then obtained as $A_{vf} f_y \cos \alpha_f + \mu . A_{vf} f_y \sin \alpha_f$, and the nominal shear resistance is given by:

$$V_{sn} = A_{vf} f_y (\cos \alpha_f + \mu \sin \alpha_f) \tag{6.31}$$

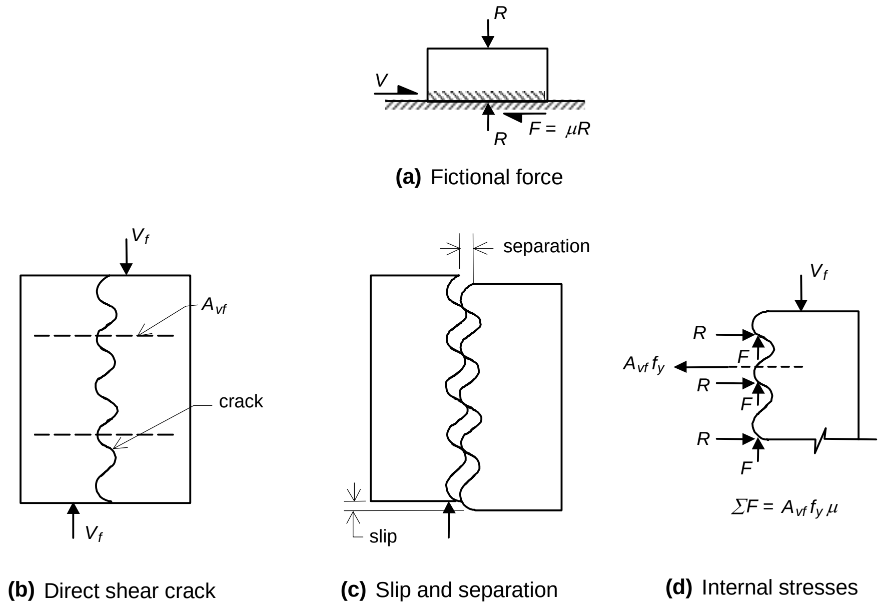



**Fig. 6.12** Shear-friction analogy

If the area of concrete section at the interface resisting shear transfer is $A_{cv}$, the nominal shear resistance per unit area can be expressed (from Eq. 6.31) as:

$$v_{sn} = \rho_v f_y \cos\alpha_f + \mu(\rho_v f_y \sin\alpha_f) \tag{6.31a}$$

where $\rho_v = A_{vf}/A_{cv}$, and $v_{sn}$ = nominal shear resistance due to the transverse reinforcement.

If there is a load, $N$, normal to the interface, this will either increase or decrease the effective normal pressure across the interface, and correspondingly affect the shear resistance associated with shear-friction, depending on whether it is compressive or tensile (Fig. 6.13d). Taking $N$ positive if compressive, the effective normal pressure $R$ across the interface will then be:

$$R = A_{vf} f_y \sin\alpha_f + N$$

Reinforcement inclined at an angle $\alpha_f > 90°$ is ineffective in resisting interface shear, because, as relative slip between the two parts occurs and the reinforcement deforms, the effect is to separate the two parts farther rather than to introduce any clamping forces (Fig. 6.13c). Hence reinforcement with $\alpha_f \leq 90°$ (i.e., placed such that shear force produces tension in the bar) only is effective as shear-friction reinforcement. [Indeed, this type of inclined bars will be more effective than bars perpendicular to the interface as tensile strains are initiated in the former sooner and more effectively than in the latter].

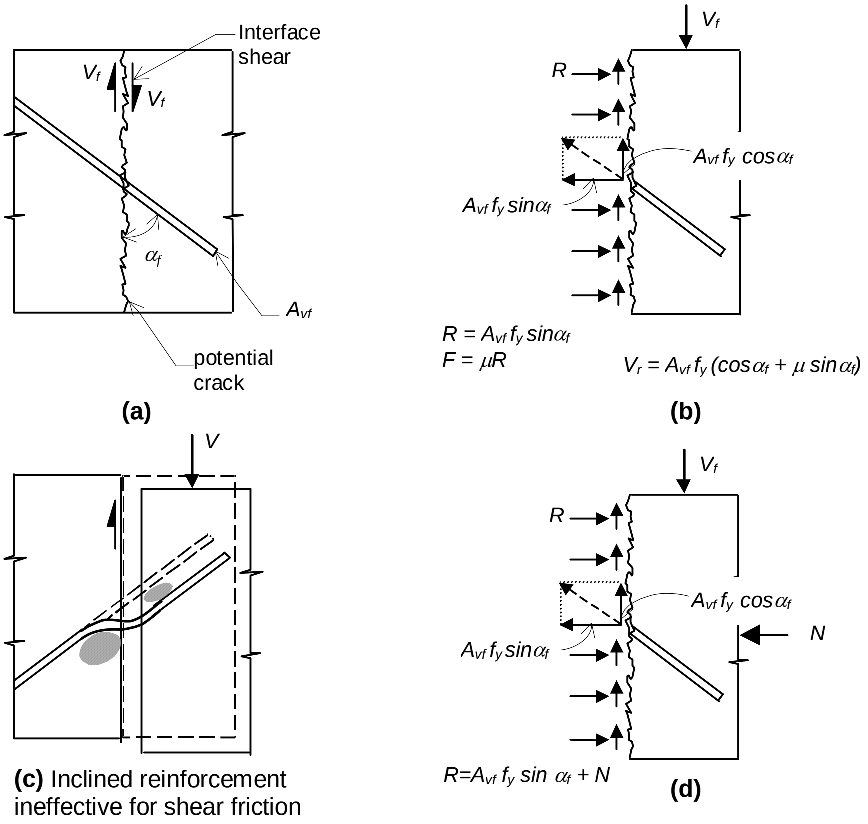



**Fig. 6.13** Inclined shear-friction reinforcement

If allowance is made for the shear strength contribution due to the cohesion between the two parts across the interface, the nominal shear resistance (for the general case of inclined shear-friction reinforcement and normal force *N*) can be obtained as:

$$v_{rn} = c + \mu(\rho_v f_y \sin\alpha_f + N / A_g) + \rho_v f_y \cos\alpha_f \tag{6.32}$$

where $c$ = stress due to cohesion

$N$ = load across shear plane (positive if compressive and negative if tensile)

### 6.9.2 Recommendation for Interface Shear Transfer

The Code IS 456 : 2000 does not give any guidance related to shear friction concepts. The Canadian standard CSA A23.3 recommends the following formula for determining the factored interface shear resistance, $v_r$, based on the shear-friction concept:

$$v_r = \lambda\phi_c (c + \mu\sigma) + \phi_s \rho_v f_y \cos\alpha_f \tag{6.33}$$

where, $\sigma = \rho_v f_y \sin\alpha_f + N/A_g$ (6.34)

$A_{cv}$ ≡ area of concrete section resisting shear
$A_g$ ≡ gross area of section transferring $N$
$A_{vf}$ ≡ area of shear-friction reinforcement
$c$ ≡ resistance due to cohesion
$f_y$ ≡ yield stress of shear-friction reinforcement
$N$ ≡ unfactored permanent compressive load perpendicular to the shear plane
$v_r$ ≡ $V_r/A_{cv}$ = factored shear stress resistance
$\alpha_f$ ≡ inclination of shear-friction reinforcement with shear plane
$\rho_v$ ≡ $A_{vf}/A_{cv}$ = ratio of shear-friction reinforcement
$\mu$ ≡ coefficient of friction
$\phi_c$, $\phi_s$ ≡ material resistance factors for concrete and steel reinforcement and
$\lambda$ ≡ factor to account for low density concrete

The *material resistance factors, $\phi_c$ and $\phi_s$,* applied to the material strengths as multipliers, used in the Canadian Code format correspond to the *inverse* of the *partial safety factors for materials* (see Sections 3.5.4 and 3.6.2) used in the IS Code format; and have values of 0.60 and 0.85 respectively. These compare well with corresponding values of 1/1.5 = 0.67 and 1/1.15 = 0.87 used in IS 456 for concrete and steel. The design Eq. 6.33 is obtained from the nominal strength Eq. 6.32, by introducing the safety factors $\phi_c$ *and* $\phi_s$ for concrete and steel and, in addition, a density factor $\lambda$ to allow for low density concrete (which has lower shear strength) when used. Recommended values for $\lambda$ are 1.00 for normal density concrete, 0.85 for structural semi-low density concrete and 0.75 for structural low density concrete. The CSA Code recommends the following values for $c$ and $\mu$:.

**Table 6.2** Values of c and μ to be used with Eq. 6.33

| Case | Concrete placed against: | $c$ (MPa) | $\mu$ |
|---|---|---|---|
| 1 | Hardened concrete | 0.25 | 0.60 |
| 2 | Hardened concrete, clean and intentionally roughened | 0.50 | 1.00 |
| 3 | Monolithic construction | 1.00 | 1.40 |
| 4 | As-rolled structural steel and anchored by headed studs or reinforcing bars | 0.00 | 0.60 |

An upper limit on the first term of Eq. 6.33 is specified, equal to $0.25\phi_c f_c' \leq 7.0\ \phi_c$ MPa, to avoid failure of concrete by crushing (here $f_c'$ is the cylinder strength of concrete).

Any direct tension, $N_f$, across the shear plane must be provided for by additional reinforcement having an area equal to $N_f/(\phi_s f_y)$. Such tensile forces may be caused by restraint of deformations due to temperature change, creep and shrinkage, etc. Although there is a beneficial effect of a permanently occurring net compressive force across the shear plane that reduces the amount of shear-friction reinforcement required, it is prudent to ignore this effect. When there is a bending moment acting on the shear plane, the flexural tensile and compressive forces balance each other, and the ultimate compressive force across the plane (which induces the frictional resistance) is equal to $A_s f_y$. Hence, the flexural reinforcement area, $A_s$, can be included in the area $A_{vf}$ for computing $V_r$. When there is no bending moment acting on the shear plane, the shear-friction reinforcement is best distributed uniformly along the shear plane in order to minimise crack widths. When a bending moment also exists, most of the shear-friction reinforcement is placed closer to the tension face to provide the required effective depth. Since it is assumed that the shear-friction reinforcement yields at the ultimate strength, it must be anchored on both sides of the shear plane so as to develop the specified yield strength in tension.

Equation 6.32 can be adapted to the IS code format by introducing the corresponding partial safety factors. Thus introducing the factors given in Section 3.6.2, the interface shear resistance may be taken as:

$$v_r = 0.447\,(c + \mu\sigma) + 0.87\rho_v\, f_y \cos\alpha_f \tag{6.34}$$

where, $\sigma = \rho_v\, f_y \sin\alpha_f + N/A_g$

Values for c and $\mu$ given in Table 6.2 may be adopted.

## 6.10 SHEAR CONNECTORS IN FLEXURAL MEMBERS

### 6.10.1 Shear along Horizontal Planes

The shear stress distribution in a homogeneous elastic beam was discussed in Section 6.2 and presented in Fig. 6.1. Just as a vertical section is subjected to shear stresses as shown in Fig. 6.1 (b), every horizontal plane in the beam is also subjected to shear stresses, as shown in the top and bottom faces of the element depicted in Fig. 6.1(c). At times, a beam is made up of two dissimilar materials, such as a rolled steel joist with a concrete compression flange as shown in Fig. 6.13(a). Similarly, under special circumstances, a concrete beam may be cast in two steps (such as a precast part and a cast-in-situ part, or a slab cast over a prestressed concrete beam) with a horizontal layer forming the interface between the concrete cast at different times (Fig. 6.13(b)). In such a situation, if the beam is to act as a single composite integral flexural member with the entire cross section acting integrally (rather than as two separate beams, one sitting on top of the other with a discontinuity along the plane of contact), provision has to be made to transmit the horizontal shear across the interface and prevent relative slippage between the parts above and below. In a composite beam such as shown in Fig. 6.13(a), this is achieved by providing *shear connectors* in the form of studs, channel shapes, etc. welded on top of the steel beam as shown in Fig. 6.13(c). In the case of beams with concrete-to-concrete interface, as in Fig. 6.13 (b), providing full depth stirrups together with the bond and friction along the interface can provide shear connection (Fig. 6.13(d)).

Sometimes, attempts are made to strengthen a flexural member in distress by increasing the depth by casting another layer on top. Such a layer will not be effective unless positive and effective steps are taken to have shear transfer across the interface. Cleaning and chipping the surface of the older member, application of bonding materials, etc. will not be effective except for very small localised areas.

The engineer should be wary of such dubious steps for the rehabilitation of distressed members.

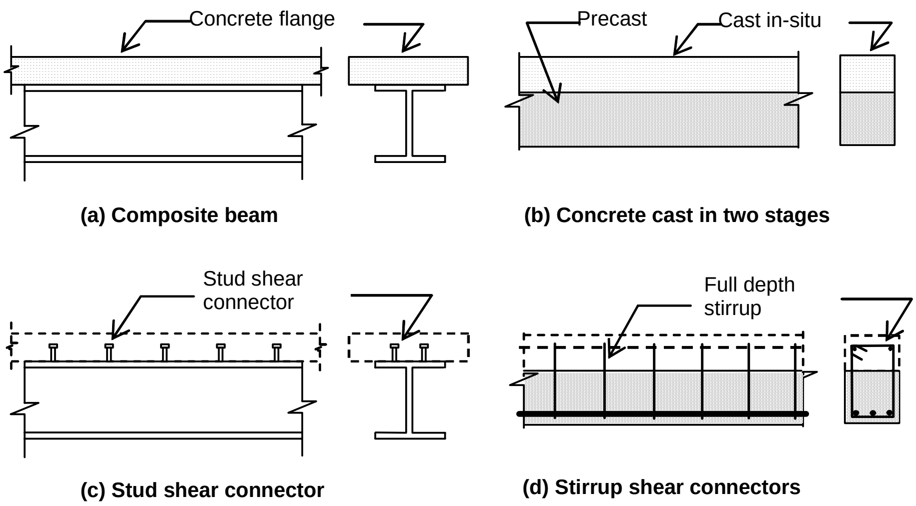



**Fig. 6.13** Shear Connectors

## 6.11 SHEAR DESIGN EXAMPLES – CONVENTIONAL METHOD

### EXAMPLE 6.1

The simply supported beam in Example 5.1 (and Example 5.7) is provided with web reinforcement of 8 mm plain bar U-stirrups at a uniform spacing of 200 mm, as shown in Fig. 6.14(a). Check the adequacy of the shear design. If necessary, revise the design.

**SOLUTION**

- ***Factored loads*** *[refer Example 5.1]*:

  Dead Load $w_{u,DL} = 1.5 \times 8.75 = 13.1$ kN/m
  Live Load $w_{u,LL} = 1.5 \times 10.0 = 15.0$ kN/m

- ***Factored shear force envelope***

  The placement of live load[†] giving the maximum shear force ($V_u$) at any section X on the left-half of the span is shown in Fig. 6.14(b). (The dead load act on the entire span). Although the resulting shape of the shear envelope is curvilinear, it is adequate and conservative to consider the shear force envelope to vary as a

[†] This follows from the influence line diagram for shear force at the section X [refer any basic text on structural analysis].

straight line between the maximum values computed for the support and for the midspan.

$$\text{At support, } V_u = (13.1 + 15.0) \times 6.0/2 = 84.3 \text{ kN}$$

$$\text{At midspan}^{\dagger}\text{, } V_u = 0 + (15.0 \times 3.0)/4 = 11.25 \text{ kN}$$

The factored shear force diagram is shown in Fig. 6.14(c).

- ***Factored shear force at critical section near support***

  The critical section for shear is at a distance $d$ = 399 mm from the face of support, i.e., 230/2 + 399 = 514 mm from the centre of support [Fig. 6.14(a)].
  The factored shear force at this section is obtainable from the shear force envelope [Fig. 6.14(c)]:

$$V_u = 11.25 + (84.3 - 11.25) \times \frac{3000 - 514}{3000} = 71.8 \text{ kN}$$

- ***Check adequacy of section***

  Nominal shear stress $\tau_v = \dfrac{V_u}{bd} = \dfrac{71.8 \times 10^3}{250 \times 399}$ = 0.72 MPa < $\tau_{c,max}$ = 3.1 MPa (for M 25 concrete). Hence the size of the section is adequate.

- ***Design shear resistance at critical section***

  At the critical section, $A_{st}$ (due to 2 – 20 $\phi$) = 314 × 2 = 628 mm$^2$

  $p_t = \dfrac{100 \times 628}{250 \times 399}$ = 0.63

  $\Rightarrow$ Design shear strength of concrete (from Eq. 6.10 or Table 6.1, for M 25 grade): $\tau_c$ = 0.536 MPa < $\tau_v$ = 0.72 MPa

$$\Rightarrow V_{uc} = 0.536 \times 250 \times 399 = 53466 \text{ N} = 53.47 \text{ kN}$$

  Shear resistance of 'vertical' stirrups (8 $\phi$ @ 200 c/c, Fe 250 grade):

$$A_{sv} = 2 \times 50.3 = 100.6 \text{ mm}^2,\ s_v = 200 \text{ mm}$$

$$\begin{aligned} V_{us} &= 0.87 f_y A_{sv}\, d/s_v \\ &= 0.87 \times 250 \times 100.6 \times 399/200 = 43651 \text{ N} \\ &= 43.65 \text{ kN} \end{aligned}$$

  $\therefore$ Total shear resistance at critical section:
  $V_{uR} = V_{uc} + V_{us}$ = 53.47 + 43.65 = 97.12 kN > $V_u$ = 71.8 kN
  Hence, the section is safe in shear.

- ***Check minimum stirrup requirements (maximum spacing)***

  Eq. 6.22: $(s_v)_{max} = 2.175 f_y A_{sv} / b$

$$\begin{aligned} &= 2.175 \times 250 \times 100.6/250 \\ &= 219 \text{ mm} \end{aligned}$$

$$(s_v)_{provided} = 200 \text{ mm} < (s_v)_{max} \qquad \Rightarrow \text{OK}$$

---

† The live load is placed only on one-half of the span for maximum shear force at the midspan section [refer Fig. 6.14(b)].

Further, applying Eq. 6.23,

$$s_v \le \begin{cases} 0.75d = 0.75 \times 399 = 299mm \\ 300mm \end{cases}$$

which are evidently satisfied by $s_v$ = 200 mm.

- ***Check shear strength at bar cut-off point***

The cut-off point is located at 660 mm from centre of support [Fig. 6.14(a)]. Factored shear at cut-off point [from Fig. 6.14(c)]:

$$V_u = 11.25 + (84.3 - 11.25) \times \frac{3000 - 660}{3000} = 68.23 \text{ kN}$$

Shear resistance of the section $V_{uR}$ = 97.12 kN

$$\frac{2}{3} \text{ shear resistance} = \frac{2}{3} \times 97.12 = 64.75 \text{ kN} < V_u = 68.23 \text{ kN}$$

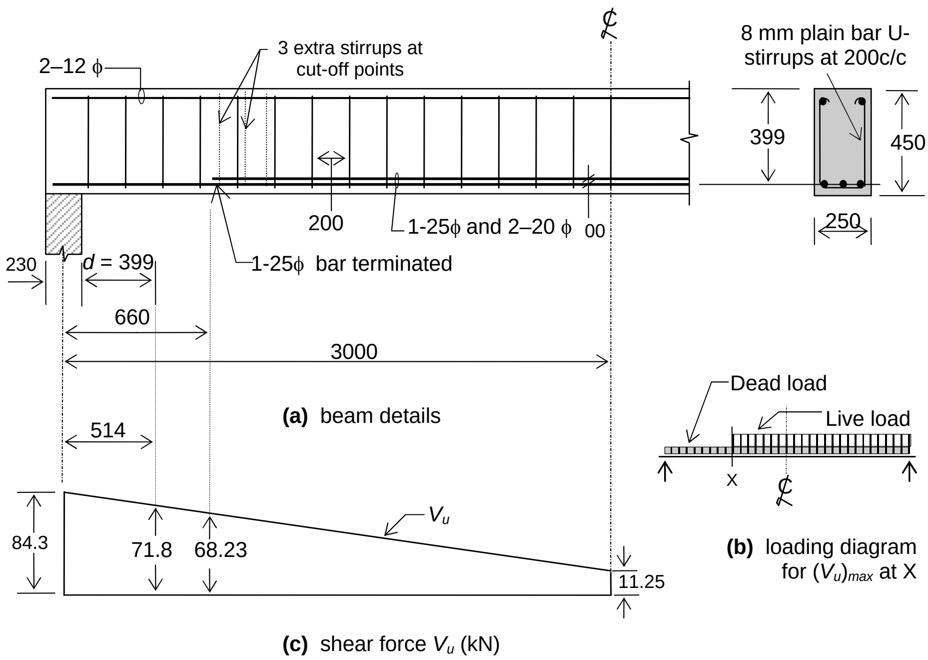



**Fig. 6.14** Example 6.1

As, shear strength requirements (Cl. 26.2.3.2(a) of the Code, Section 5.9.3) are NOT satisfied, additional stirrups must be provided over a distance of $0.75d$ = 299 mm ('along the terminated bar') with a spacing < $d/8\beta_b$ = 399/(8× 0.44) = 113 mm [since $\beta_b$ = 491/(491+628)) = 0.44, Cl. 26.2.3.2(b) of the Code]. This is achieved by adding three additional stirrups along the last portion of the cut-off bar, as shown in Fig. 6.14

$\Rightarrow$ spacing = 0.75 × 399/3 = 99.75 mm which is less than 113 mm

Excess stirrup area required = $\dfrac{0.4bs_v}{f_y} = \dfrac{0.4 \times 250 \times 83.3}{415}$ = 20.1 mm$^2$.

Although it suffices to provide 6 mm $\phi$ additional 2-legged stirrups ($A_{sv}$ = 56.6 mm$^2$), from a practical viewpoint, it is convenient to use the same 8 mm $\phi$ ($A_{sv}$ = 100.6 mm$^2$) for the additional stirrups.

## EXAMPLE 6.2

Slabs, in general, do not require shear reinforcement, as the depth provided (based on deflection criteria) is usually adequate to meet shear strength requirements. Verify this in the case of the one-way slab of Example 5.2.

### SOLUTION

- *Given*: Slab thickness = 200 mm, Effective depth = 165 mm, $f_{ck}$ = 25 MPa (from Example 5.1)

  $A_{st}$ provided at support (10 $\phi$ @ 250 c/c) = 314 mm$^2$/m

  $\Rightarrow p_t = \dfrac{100 \times 314}{1000 \times 165}$ = 0.19

  Factored load (DL + LL) $w_u$ = 15 kN/m$^2$

  simply supported span $l$= 4.165 m

- It is convenient to prove that the section has adequate shear strength at the support itself, which has the maximum factored shear, rather than at $d$ from the face of support:

  $$V_u = 15 \times 4.165/2 = 31.24 \text{ kN/m}$$

  $\Rightarrow$ Nominal shear stress $\tau_v = \dfrac{V_u}{bd} = \dfrac{31.24 \times 10^3}{10^3 \times 165}$ = 0.189 MPa

  Design shear strength (from Eq. 6.10): for M 25 concrete and $p_t$ = 0.19,

  $$\tau_c = 0.323 \text{ MPa}$$

  This value may further be enhanced by a multiplying factor

  $k$ = 1.6 – 0.002 × 200 = 1.2 [Eq. 6.11].

  $\Rightarrow k\tau_c$ = 1.2×0.323 = 0.39 MPa

  $$>> \tau_v = 0.189 \text{ MPa}$$

- As the section is safe at the support section itself (where shear is maximum), there is no need to confirm this at the 'critical' section located $d$ away from the face of support.

## EXAMPLE 6.3

Design the shear reinforcement for the beam in Example 5.4. Assume the curtailment of longitudinal bars as shown in Fig. 6.15(a). Assume Fe 415 grade steel for the shear reinforcement.

### SOLUTION

- ***Factored loads*** [refer Example 5.4]:

  Dead Load $w_{u,DL}$ = 7.5 kN/m × 1.5 = 11.25 kN/m
  Additional DL $W_{u,DL}$: 30 × 1.5 = 45.0 kN concentrated at midspan
  Live Load $w_{u,LL}$ = 10 kN/m × 1.5 = 15.0 kN/m

- ***Factored shear force envelope***
  As explained in the Example 6.1 it is adequate and conservative to consider the shear force envelope to vary as a straight line between the maximum values computed at the support and at the midspan.
  At support, $V_u = (11.25 + 15.0) \times 6.0/2 + 45.0/2$ = 101.25 kN
  At midspan, $V_u = 45.0/2 + (15.0 \times 3.0)/4$ = 33.75 kN

- ***Factored shear force at critical section***
  The critical section is $d$ = 348 mm from the face of support, i.e., 230/2+348 = 463 mm from the centre of support [Fig. 6.15(a)].
  $$V_u = 33.75 + (101.25 - 33.75) \times \frac{3000 - 463}{3000} = 90.8 \text{ kN}$$

- ***Check adequacy of section size***
  Nominal shear stress $\tau_v = \dfrac{90.8 \times 10^3}{250 \times 348}$ = 1.044 MPa
  which is less than $\tau_{c,\max}$ = 3.1 MPa (for M 25 concrete)
  Hence, the size of the section is adequate.

- ***Design shear strength of concrete***
  At the critical section, $A_{st}$ (due to 2–28 $\phi$) = 616 × 2 = 1232 mm$^2$
  $$\Rightarrow\ p_t = \frac{100 \times 1232}{250 \times 348} = 1.416$$
  $\Rightarrow$ Design shear strength of concrete (from Eq. 6.10 or Table 6.1, for M 25 concrete). $\tau_c$ = 0.728 MPa < $\tau_v$ = 1.019 MPa

- ***Design of 'vertical' stirrups***
  Shear to be resisted by stirrups $V_{us} = (\tau_v - \tau_c)bd$
  $$\Rightarrow\ \frac{V_{us}}{d} = (1.019 - 0.728) \times 250 = 72.75 \text{ N/mm}$$
  Assuming 2-legged closed stirrups of 8 mm dia,
  $A_{sv} = 2 \times 50.3$ = 100.6 mm$^2$

$$\Rightarrow \text{required spacing } s_v \le \frac{0.87 f_y A_{sv}}{V_{us}/d} = \frac{0.87 \times 415 \times 100.6}{72.75} = 499 \text{ mm}$$

Code requirements for maximum spacing:

$$s_v = \begin{cases} 2.175 f_y A_{sv} / b = 2.175 \times 415 \times 100.6 / 250 = 363 mm \\ 0.75d = 0.75 \times 348 = 261 mm \\ 300 mm \end{cases}$$

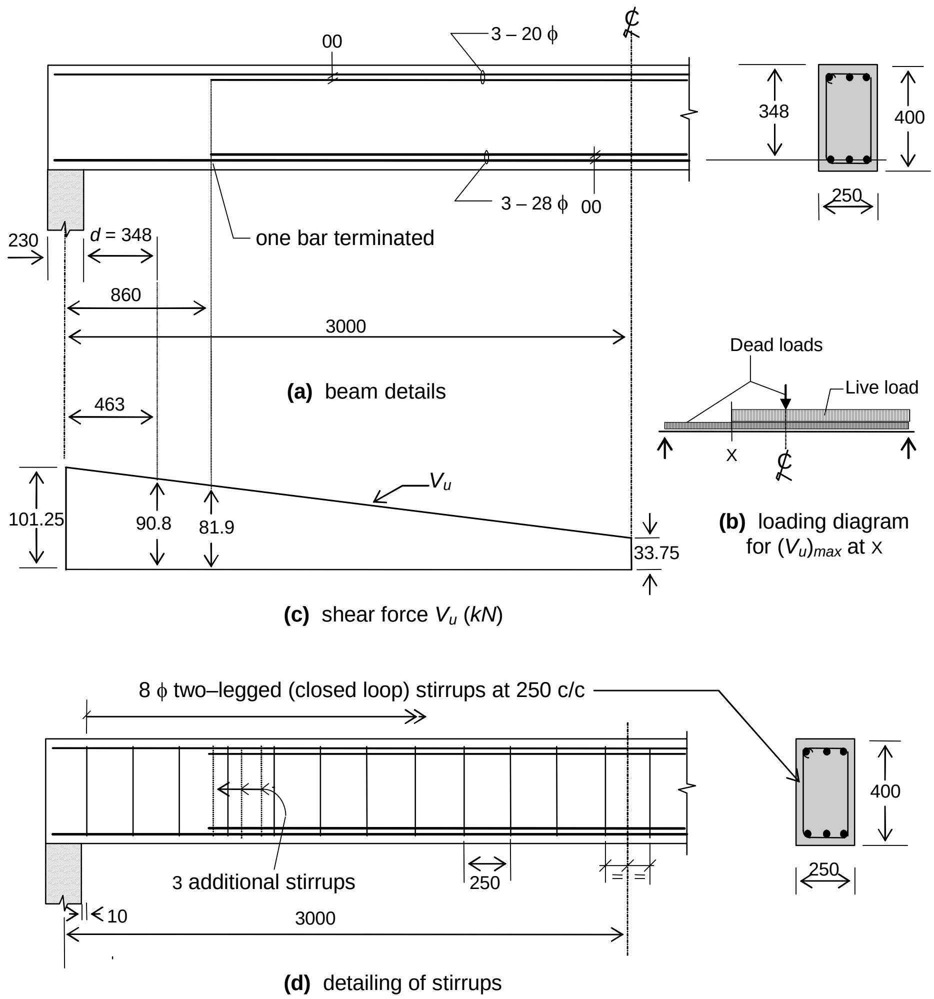



**Fig. 6.15** Example 6.3

Provide 8 $\phi$ two-legged closed stirrups at 250 mm c/c spacing.

[*Note*: The stirrups should be *closed* as the section is doubly reinforced].

- *Check shear strength at bar cut-off point*

  The cut-off point is located at 860 mm from the centre of support [Fig. 6.15(a)]. The factored shear force at this section [Fig. 6.15(b)] is given by:

  $$V_u = 33.75 + (101.25 - 33.75) \times \frac{(3000 - 860)}{3000} = 81.9 \text{ kN}$$

  Shear resistance $V_{uR} = \tau_c bd + \dfrac{0.87 f_y A_{sv} d}{s_v}$

  $= (0.728 \times 250 \times 348) + (0.87 \times 415 \times 100.6 \times 348) / 250$

  $= 113896$ N = 114 kN

  $\frac{2}{3}$ shear resistance = $\frac{2}{3} \times 114 = 76$ kN < 81.9 kN

  Hence, additional stirrups must be provided over a distance of $0.75d$ = 261 mm ('along the terminated bar') with a spacing < $d/8\beta_b$ = 348 / (8×1/3) = 130 mm [Cl. 26.2.3.2(b) of the Code].

  This is achieved by adding three additional stirrups along the last portion of the cut-off bar, as shown in Fig. 6.11(d).

  $\Rightarrow$ spacing = $250/3$ = 83.3 mm which is less than 133 mm

  Excess stirrup area required = $\dfrac{0.4bs_v}{f_y} = \dfrac{0.4 \times 250 \times 83.3}{415} = 20.1 \text{ mm}^2$.

  Although it suffices to provide 6 mm $\phi$ additional 2-legged stirrups ($A_{sv}$ = 56.6 mm$^2$), from a practical viewpoint, it is convenient to use the same 8 mm $\phi$ ($A_{sv}$ = 100.6 mm$^2$) for the additional stirrups.

## REVIEW QUESTIONS

6.1 Under what conditions is the traditional method of shear design inappropriate?

6.2 Under what situations do the following modes of cracking occur in reinforced concrete beams: (a) *flexural* cracks, (b) *diagonal tension* cracks, (c) *flexural-shear* cracks and (d) *splitting* cracks?

6.3 Describe the force components that participate in the shear transfer mechanism at a flexural-shear crack location in a reinforced concrete beam.

6.4 How does the *shear span* influence the mode of shear failure?

6.5 How is the computation of *nominal shear stress* for beams with variable depth different from that for prismatic beams?

6.6 Generally, the *critical section* for shear in a reinforced concrete beam is located at a distance $d$ (effective depth) away from the face of the support. Why? Under what circumstances is this not permitted?

6.7 Why is the design shear strength of concrete ($\tau_c$) related to the percentage tension steel $p_t$?

6.8 Reinforced concrete slabs are generally safe in shear and do not require shear reinforcement. Why?

6.9 How does the presence of an axial force (tension or compression) influence the shear strength of concrete?
6.10 Stirrups may be *open* or *closed*. When does it become mandatory to use *closed* stirrups?
6.11 Stirrups may be 'vertical' or inclined. When does it become mandatory to use vertical stirrups?
6.12 The shear resistance of bent-up bars cannot be counted upon, unless stirrups are also provided. Why?
6.13 Why is an upper limit $\tau_{c,\max}$ imposed on the shear strength of a reinforced concrete beam with shear reinforcement?
6.14 Explain the action of a reinforced concrete beam (with shear reinforcement) with the aid of the *truss analogy* model.
6.15 The provision of a *minimum* stirrup reinforcement is mandatory in all reinforced concrete beams. Why?
6.16 The site of curtailment of tension reinforcement in a reinforced concrete beam is considered a critical section for shear. Why?
6.17 In the traditional method of design for shear, how is the influence of shear on longitudinal reinforcement requirement taken care of?
6.18 What are shear connectors? Where are they needed? What are the different types used?
6.19 Explain the concept of interface shear and shear friction theory? Where are these relevant?
6.20 Relate interface shear and shear connectors.

## PROBLEMS

6.1 A simply supported beam of 6 m span (c/c), (shown in Fig. 6.16), is to carry a uniform dead load of 20 kN/m (including beam weight) and a uniform live load of 30 kNm. The width of the supporting wall is 230 mm. Assume M 25 concrete and Fe 415 steel.

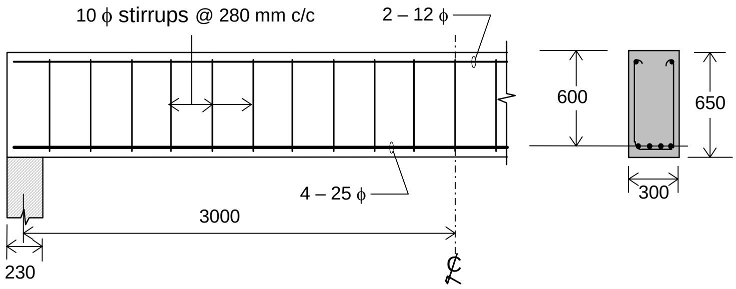



**Fig. 6.16** Problem 6.1

a) Determine the adequacy of the 10 mm $\phi$ U-stirrups as shear reinforcement.

[*Ans.*: adequate]

b) If the shear reinforcement is to be provided in the form of 10 $\phi$ stirrups inclined at 60° to the beam axis, determine the required spacing.

[*Ans.*: 450 mm]

c) If two of the tension reinforcement bars are terminated at 300 mm from the centre of the support, check the adequacy of shear strength at the bar cut-off point.

[*Ans.*: inadequate]

6.2 A simply supported T-beam of 9 m span (c/c) is subjected to a dead load (including self weight) of 20 kN/m and a live load of 25 kN/m. Details of the section and bar cut-offs are shown in Fig. 6.17. Design and detail the shear reinforcement using 'vertical' stirrups. Assume M 20 concrete and Fe 415 steel.

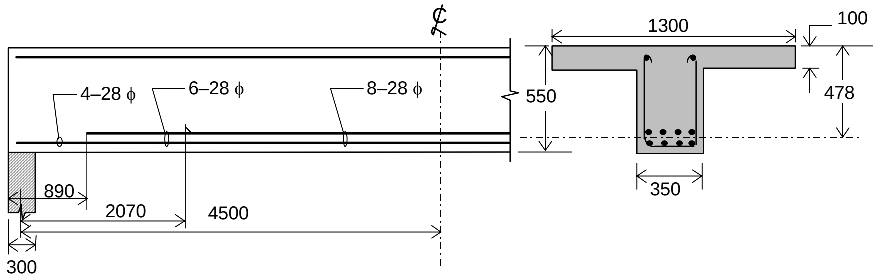



**Fig. 6.17** Problem 6.2

6.3 A simply supported beam (shown in Fig. 6.18) is subjected to a dead load (including self weight) of 20 kN/m and a live load of 20 kN/m. Design and detail the shear reinforcement using vertical stirrups. Use M 20 concrete and Fe 415 steel.

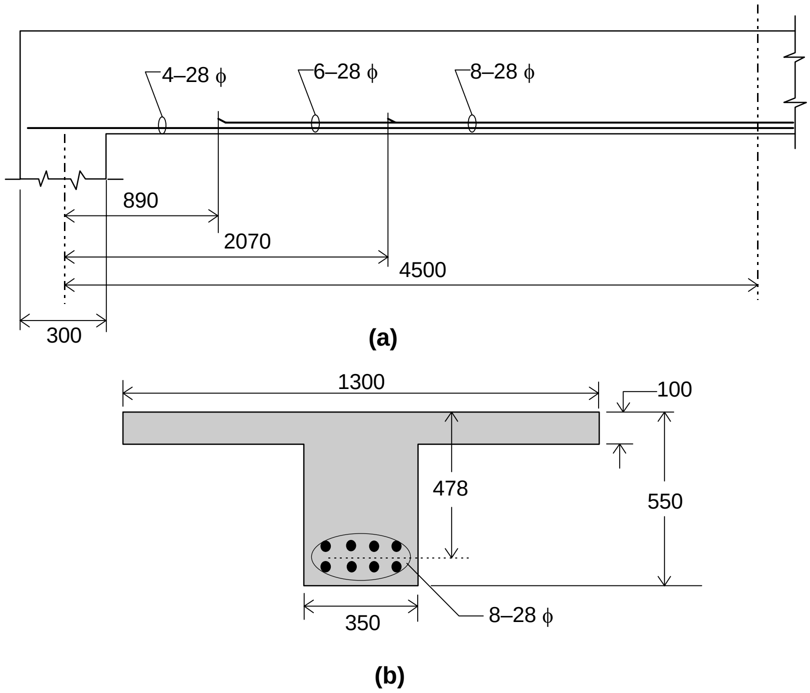


**Fig. 6.18** Problem 6.3

6.4 Figure 6.19 shows a uniformly loaded cantilever beam with the depth linearly tapered along the span. The dead load, including self-weight of the beam is 20 kN/m and the live load is 50 kN/m. Design the shear reinforcement using vertical stirrups. The bar cut-off details are as shown. Assume M 20 concrete and Fe 415 steel.

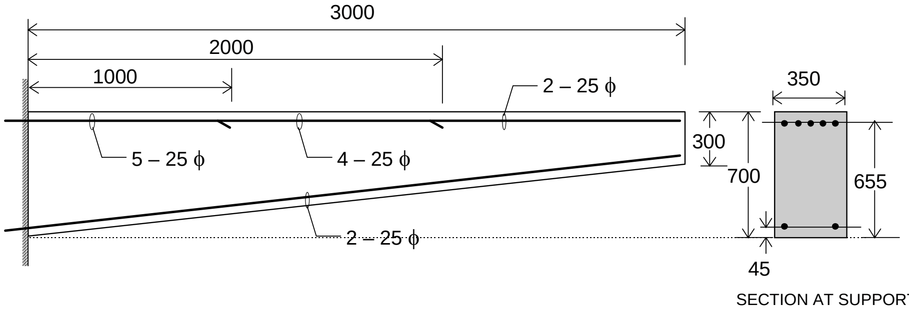



**Fig. 6.19** Problem 6.4

## REFERENCES


6.1 Popov, E.P., *Mechanics of Materials*, Second edition, Prentice Hall Inc., Englewood Cliffs, New Jersey, 1978.

6.2 ASCE–ACI Committee 426, *The Shear Strength of Reinforced Concrete Members*, ASCE Journal, Struct. Div., Vol. 99, June 1973, pp 1091–1187.

6.3 Bresler, B. and MacGregor, J.G., *Review of Concrete Beams Failing in Shear*, ASCE Journal, Struct. Div., Vol. 93, Feb. 1967, pp 343–372.

6.4 ACI–ASCE Committee 326 Report, *Shear and Diagonal Tension*, Journal ACI, Vol. 59, Jan., Feb., and Mar., 1962, pp 1–30, 277–334 and 352–396.

6.5 — *Explanatory Handbook on Indian Standard Code of Practice for Plain and Reinforced Concrete (IS 456:1978)*, Special Publication SP:24, Bureau of Indian Standards, New Delhi, 1983.

6.6 Rangan, B.V., *Diagonal Cracking Strengths in Shear of Reinforced Concrete Beams*, Civil Engg. Transactions, Instn of Engineers, Australia, Vol CE 14, No.1, 1972.

6.7 — *Building Code Requirements for Reinforced Concrete*, ACI Standard 318–89, Am. Conc. Inst., Detroit, Michigan, USA, 1989.

6.8 CSA Standard A23.3 – 94 — *Design of Concrete Structures*, Canadian Standards Association, Rexdale, Ontario, Canada, 1994.

6.9 — *Design Aids (for Reinforced Concrete) to IS 456 : 1978*, Special Publication SP:16, Bureau of Indian Standards, New Delhi, 1980.

6.10 *Commentary on Building Code Requirements for Structural Concrete* (ACI 318R-95), American Concrete Institute, Detroit, Michigan, 1995.

6.11 Mattock, A.H., *Shear Transfer in Concrete Having Reinforcement at an Angle to the Shear Plane, Shear in Reinforced Concrete*, SP-42, American Concrete Institute, Detroit, 1974, pp. 17-42.

6.12 Mattock, A.H., Li, W.K., Wang, T.C., *Shear Transfer in Lightweight Reinforced Concrete*, J. PCI, Vol. 21, No. 1, Jan.-Feb. 1976, pp. 20-39.

# 7

# Design for Torsion

## 7.1 INTRODUCTION

Torsion when encountered in reinforced concrete members, usually occurs in combination† with flexure and transverse shear. Torsion in its 'pure' form (generally associated with metal shafts) is rarely encountered in reinforced concrete.

The interactive behaviour of torsion with bending moment and flexural shear in reinforced concrete beams is fairly complex, owing to the non-homogeneous, nonlinear and composite nature of the material and the presence of cracks. For convenience in design, codes prescribe highly simplified design procedures, which reflect a judicious blend of theoretical considerations and experimental results.

These design procedures and their bases are described in this chapter, following a brief review of the general behaviour of reinforced concrete beams under torsion.

## 7.2 EQUILIBRIUM TORSION AND COMPATIBILITY TORSION

Torsion may be induced in a reinforced concrete member in various ways during the process of load transfer in a structural system. In reinforced concrete design, the terms 'equilibrium torsion' and 'compatibility torsion' are commonly used to refer to two different torsion-inducing situations‡.

In 'equilibrium torsion', the torsion is induced by an eccentric loading (with respect to the shear centre at any cross-section), and equilibrium conditions alone suffice in determining the twisting moments. In 'compatibility torsion', the torsion is induced by the need for the member to undergo an angle of twist to maintain deformation compatibility, and the resulting twisting moment depends on the torsional stiffness of the member.

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† In some (relatively rare) situations, axial force (tension or compression) may also be involved.

‡ It must be clearly understood that this is merely a matter of terminology, and that it does not imply, for instance, equilibrium conditions need not be satisfied in cases of 'compatibility torsion'!

There are some situations (such as circular beams supported on multiple columns) where both *equilibrium torsion* and *compatibility torsion* coexist.

### 7.2.1 Equilibrium Torsion

This is associated with twisting moments that are developed in a structural member to maintain static equilibrium with the external loads, and are independent of the torsional stiffness of the member. Such torsion must be necessarily considered in design (Code Cl. 41.1). The magnitude of the twisting moment does not depend on the torsional stiffness of the member, and is entirely determinable from statics alone. The member has to be designed for the full torsion, which is transmitted by the member to the supports. Moreover, the end(s) of the member should be suitably restrained to enable the member to resist effectively the torsion induced. Typically, *equilibrium torsion* is induced in beams supporting lateral overhanging projections, and is caused by the eccentricity in the loading [Fig. 7.1]. Such torsion is also induced in beams curved in plan and subjected to gravity loads, and in beams where the transverse loads are eccentric with respect to the *shear centre* of the cross-section.

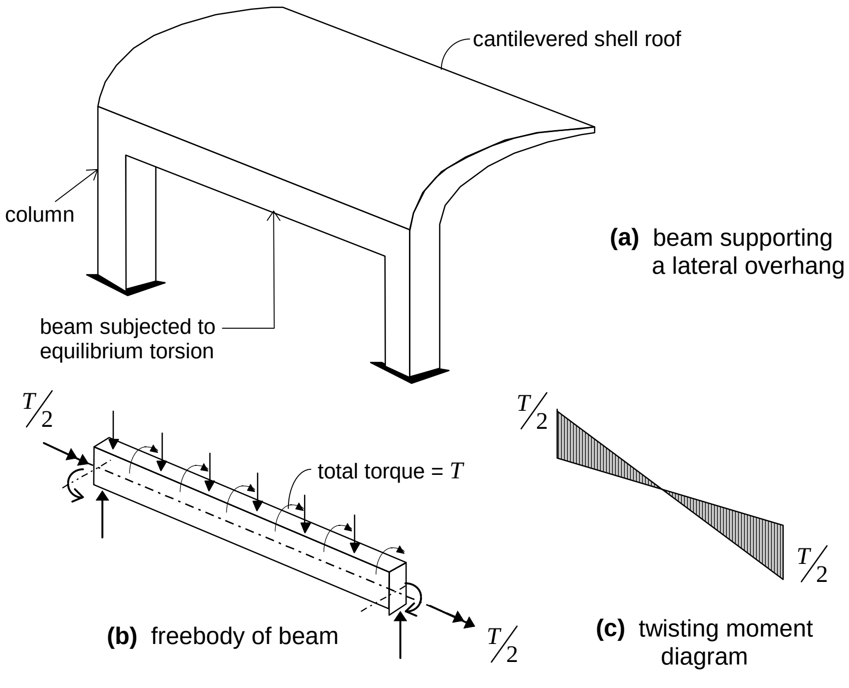



**Fig. 7.1** Example of 'equilibrium torsion'

### 7.2.2 Compatibility Torsion

This is the name given to the type of torsion induced in a structural member by rotations (twists) applied at one or more points along the length of the member. The twisting moments induced are directly dependent on the torsional stiffness of the member. These moments are generally statically indeterminate and their analysis necessarily involves (rotational) compatibility conditions; hence the name ‘compatibility torsion’. For example, in the floor beam system shown in Fig. 7.2, the flexure of the secondary beam *BD* results in a *rotation* $\theta_B$ at the end *B*. As the primary (spandrel) beam *ABC* is monolithically connected with the secondary beam *BD* at the joint *B*, deformation *compatibility* at *B* implies an *angle of twist*, equal to $\theta_B$, in the spandrel beam *ABC* at *B*. Corresponding to the angle $\theta_B$, a twisting moment will develop at *B* in beam *ABC*, and a bending moment will develop at the end *B* of beam *BD*. The bending moment will be equal to, and will act in a direction opposite to the twisting moment, in order to satisfy static equilibrium. The magnitude of $\theta_B$ and the twisting/bending moment at *B* depends on the torsional stiffness of beam *ABC* and the flexural stiffness of beam *BD*.

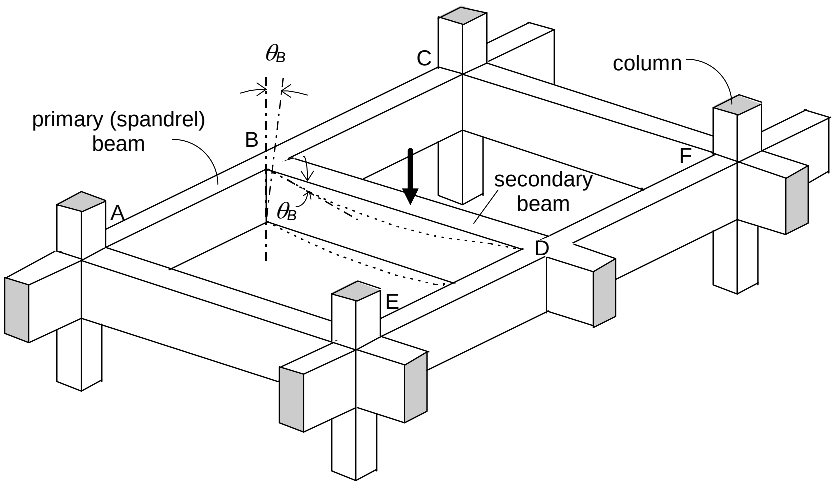



**Fig. 7.2** Example of ‘compatibility torsion’

In statically indeterminate structures (such as the grid floor system shown in Fig. 7.2), the torsional restraints are ‘redundant’, and releasing such redundant restraints will eliminate the compatibility torsion. Thus, the Code states:

> “…where torsion can be eliminated by releasing redundant restraints, no specific design for torsion is necessary, provided torsional stiffness is neglected in the calculation of internal forces.” [Cl. 41.1 of the Code].

The torsional stiffness of a reinforced concrete member is drastically reduced by torsional cracking. This results in a very large increase in the angle of twist (formation of a ‘torsional hinge’), and, in the case of ‘compatibility torsion’, a major reduction in the induced twisting moment.

With reference to Fig. 7.2, application of Code Cl. 41.1 implies providing a hinge-like connection (i.e., with no rotational restraint) at the end *B* (and *D*) of the beam *BD*; i.e., treating *BD* as a simply supported beam, and analysing it independent of

*ABC*. Alternatively, cognisance can be taken of the torsional hinge-like behaviour of the member ABC *after* torsional cracking and resulting release of flexural restraint offered by it to beam *BD* at end *B*. In this case, the grid system is analysed as a whole, but the value of the torsional stiffness of the member ABC is taken as zero† in the structural analysis for calculation of internal forces. Incidentally, this assumption helps in reducing the degree of static indeterminacy of the structure (typically, a grid floor), thereby simplifying the problem of structural analysis.

Of course, this simplification implies the acceptance of cracking and increased deformations in the torsional member. It also means that, during the first time loading, a twisting moment up to the *cracking torque* of the plain concrete section develops in the member, prior to torsional cracking. In order to control the subsequent cracking and to impart ductility to the member, it is necessary to provide a minimum torsional reinforcement, equal to that required to resist the 'cracking torque'. In fact, one of the intentions of the minimum stirrup reinforcement specified by the Code (Cl. 26.5.1.6) is to ensure some degree of control of torsional cracking of beams due to compatibility torsion.

If, however, the designer chooses to consider 'compatibility torsion' in analysis and design, then it is important that a realistic estimate of torsional stiffness is made for the purpose of structural analysis, and the required torsional reinforcement should be provided for the calculated twisting moment.

### 7.2.3 Estimation of Torsional Stiffness

Observed behaviour of reinforced concrete members under torsion [see also Section 7.3] shows that the torsional stiffness is little influenced by the amount of torsional reinforcement in the linear elastic phase, and may be taken as that of the plain concrete section. However, once torsional cracking occurs, there is a drastic reduction in the torsional stiffness. The post-cracking torsional stiffness is only a small fraction (less than 10 percent) of the pre-cracking stiffness, and depends on the amount of torsional reinforcement, provided in the form of closed stirrups and longitudinal bars. Heavy torsional reinforcement can, no doubt, increase the torsional resistance (strength) to a large extent, but this can be realised only at very large angles of twist (accompanied by very large cracks).

Hence, even with torsional reinforcement provided, in most practical situations, the maximum twisting moment in a reinforced concrete member under compatibility torsion is the value corresponding to the torsional cracking of the member. This 'cracking torque' is very nearly the same as the failure strength obtained for an identical *plain concrete* section.

In the usual linear elastic analysis of framed structures, the *torsional stiffness* $K_t$ (torque per unit twist $T/\theta$) of a beam of length $l$ is expressed as

$$K_t = \frac{GC}{l} \tag{7.1}$$

† For greater accuracy, this value may be treated as 10 percent of the uncracked torsional stiffness. The analysis of this indeterminate system will result in some flexural moment at end B in beam BD and twisting moments in beam ABC, which should be designed for.

where *GC* is the *torsional rigidity*, obtained as a product of the *shear modulus G* and the geometrical parameter† *C* of the section [Ref. 7.1]. It is recommended in the Explanatory Handbook to the Code [Ref. 7.2] that G may be taken as 0.4 times the modulus of elasticity of concrete $E_c$ (given by Eq. 2.4) and *C* may be taken as 0.5*K*, where *K* is the appropriate 'St. Venant torsional constant' calculated for the plain concrete section. For a rectangular section of size $b \times D$, with $b < D$,

$$K = \beta b^3 D \tag{7.2}$$

where $\beta$ is a constant which depends on the $D/b$ ratio, having values varying from 0.141 to 0.333 [Ref. 7.1]. Alternatively, the following formula [Ref. 7.1] may be used:

$$\beta = \left(1 - 0.63\frac{b}{D}\right) \Big/ 3 \tag{7.3}$$

For sections composed of rectangular elements (T-, L-, channel sections), the value of *K* (and hence, *C* and $K_t$) may be computed by summing up the individual values for each of the component rectangles, the splitting into component rectangles being so done as to maximise *K*.

## 7.3 GENERAL BEHAVIOUR IN TORSION

### 7.3.1 Behaviour of Plain Concrete

The theory of torsion (St. Venant torsion) of prismatic, homogeneous members having circular, non-circular and thin-walled cross-sections is described in detail in books on mechanics of materials [Ref. 7.1, 7.3]. It is seen that torsion induces shear stresses and causes warping of non-circular sections. For rectangular sections under elastic behaviour, the distribution of torsonal shear stress over the cross-section is as shown in Fig. 7.3.

The maximum torsional shear stress occurs at the middle of the wider face, and has a value given by

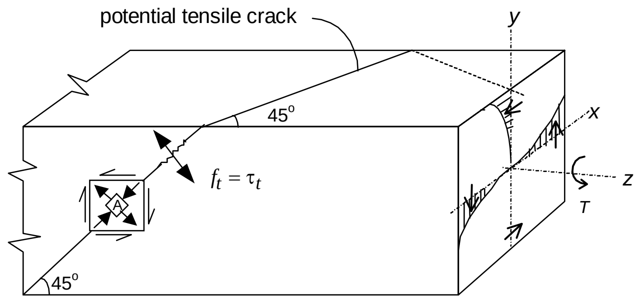
$$\tau_{t,\max} = \frac{T}{\alpha b^2 D} \tag{7.4}$$

where *T* is the twisting moment (torque), *b* and *D* are the cross-sectional dimensions (*b* being smaller), and $\alpha$ is a constant whose value depends on the $D/b$ ratio; $\alpha$ lies in the range 0.21 to 0.29 for $D/b$ varying from 1.0 to 5.0 respectively.

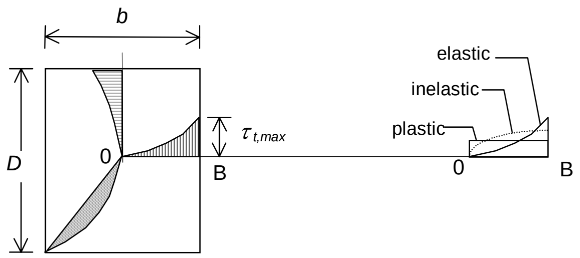
The state of *pure shear* develops direct tensile and compressive stresses along the diagonal directions, as shown in the element at A in Fig. 7.3(a). The principal tensile and compressive stress trajectories spiral around the beam in orthogonal directions at 45° to the beam axis. One such line (across which the principal tensile stress $f_t$ acts) is marked in Fig. 7.3(a); it is evidently a potential line of crack in the case of

† *C* is a property of the section having the same relationship to the torsional stiffness of a rectangular section as the polar moment of inertia has for a circular section.

concrete. Such a crack would develop in a concrete beam when the diagonal tensile stress reaches the tensile strength of concrete. Owing to the brittle nature of concrete under tension, the crack will rapidly penetrate inwards from the outer surface of the cross-section. This effectively destroys the torsional resistance, which is primarily contributed by the stresses in the outer fibres (that are the largest in magnitude and also have the greatest lever arm). Hence, in a plain concrete member, the diagonal torsional cracking in the outer fibres would lead, almost immediately, to a sudden failure of the entire section. Of course, as the point of failure is approached, some degree of plasticity is introduced, resulting in somewhat larger stresses in the interior fibres than what the elastic theory would indicate. Sometimes, for simplicity, the material is assumed to be rigid plastic with a uniform stress distribution over the entire cross-section [Fig. 7.3(c)].




**(a)** part section of beam




**(b)** torsional shear stress distributions

**(c)** degrees of plastic behaviour

**Fig. 7.3** Torsional shear stresses in a beam of rectangular section

The *cracking torque* $T_{cr}$ provides a measure of the ultimate torsional resistance (strength) of a plain concrete section. It is generally computed by equating the theoretical nominal maximum torsional shear stress $\tau_{t,\max}$ (which is a measure of the resulting diagonal tension) to the tensile strength of concrete. Various expressions for $T_{cr}$ have been derived based on (a) elastic theory [given by Eq. 7.3], (b) plastic theory, (c) skew bending theory and (d) equivalent tube analogy [Ref. 7.4, 7.5]. Each of these different expressions for $T_{cr}$ needs to be correlated experimentally with an appropriate measure of the tensile strength of concrete to be used with it. The Code has adopted the *design shear strength of concrete* $\tau_c$ [given by Table 6.1] as the measure of tensile strength, for convenience in combining the effects of torsional shear and flexural shear [refer Section 7.4.1].

A typical torque-twist relation for a plain concrete section is shown in Fig. 7.4(a) [Ref. 7.6]. The relationship is somewhat linear up to failure, which is sudden and brittle, and occurs immediately after the formation of the first torsional crack.

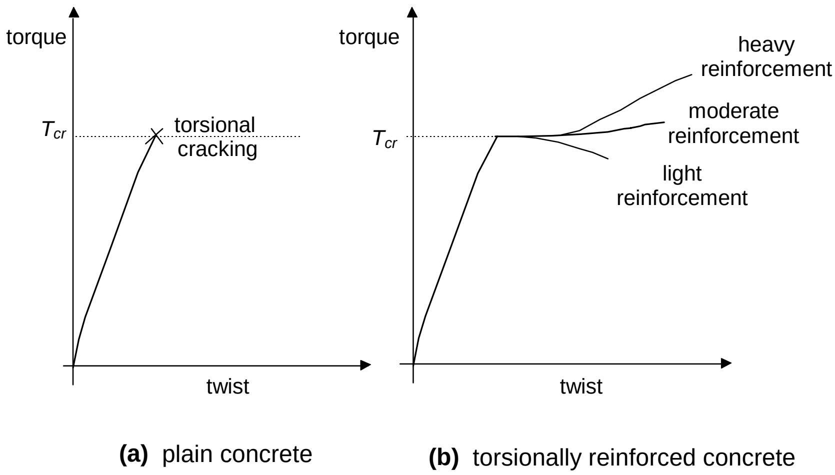



**Fig. 7.4** Typical torque-twist curves for concrete members in pure torsion

### 7.3.2 Behaviour of Concrete with Torsional Reinforcement

As mentioned earlier, the failure of a plain concrete member in torsion is caused by torsional cracking due to the diagonal tensile stresses. Hence, the ideal way of reinforcing the beam against torsion is by providing the steel in the form of a spiral along the direction of the principal tensile stresses. However, this is often impractical, and the usual form of torsional reinforcement consists of a combination of longitudinal and transverse reinforcement — the former in the form of bars distributed around the cross-section, close to the periphery, and the latter in the form of closed rectangular stirrups, placed perpendicular to the beam axis. It may be noted here that the longitudinal reinforcement (on the tension side) is also needed for flexure and the transverse reinforcement is needed for shear.

The torque-twist behaviour of torsionally reinforced concrete member is similar to that of plain concrete until the formation of the first torsional crack (corresponding to the cracking torque $T_{cr}$), as shown in Fig. 7.4(b) [Ref. 7.7]. The value of $T_{cr}$ is insensitive to the presence of torsional reinforcement, and is practically the same as for an identical plain concrete section. When cracking occurs, there is a large increase in twist under nearly constant torque, due to a drastic loss of torsional stiffness [Fig. 7.5]. Beyond this, however, the strength and behaviour depend on the amount of torsional reinforcement present in the beam.

For very small amounts of torsional reinforcement, no increase in torsional strength beyond $T_{cr}$ is possible, and failure occurs soon after the first crack, in a brittle manner. Increasing the torsional reinforcement will no doubt increase the ultimate torsional strength and the (ductile) failure is preceded by yielding of steel, but this can be realized only at very large angles of twist. However, the strength cannot be raised indefinitely with increasing torsional reinforcement as crushing of concrete in diagonal compression may precede, and thereby prevent, the yielding of the reinforcement in tension.

The torsional stiffness after cracking is primarily dependent on the amount of torsional reinforcement, and is usually in the range of 0 – 10 percent of the value prior to cracking.

## 7.4 DESIGN STRENGTH IN TORSION

### 7.4.1 Design Torsional Strength without Torsional Reinforcement

As already indicated, the strength of a torsionally reinforced member at torsional cracking $T_{cr}$ is practically the same as the failure strength of a plain concrete member under pure torsion. Although several methods have been developed to compute $T_{cr}$, the *plastic theory* approach is described here, as the Code recommendation can be explained on its basis.

#### *Cracking Torque*

As explained earlier, the idealised assumption that the unreinforced section is fully plasticised at the point of failure implies that the shear stress is constant throughout the section, having a magnitude $\tau_{t,\max}$ [Fig. 7.3(c)]. The resultant shears are obtainable from the shear flow diagram, as shown in Fig. 7.5.

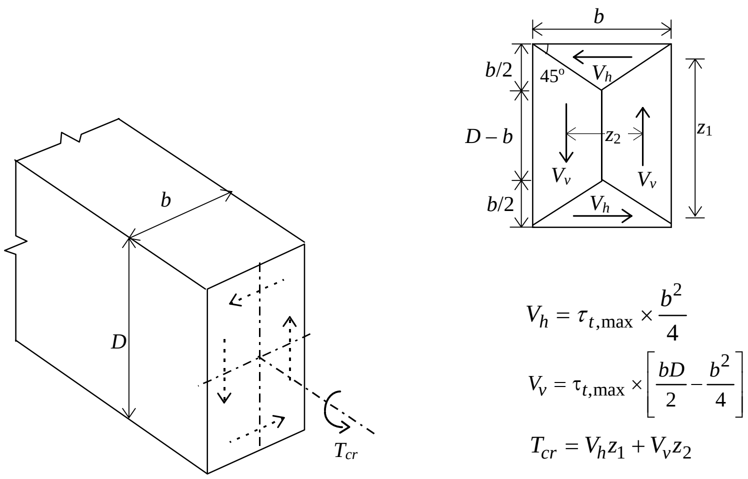
The resultant horizontal shear $V_h$ is simply obtained by multiplying the tributary area, triangular-shaped, (equal to $b^2/4$) by $\tau_{t,\max}$. Similarly, the resultant vertical shear $V_v$ is obtained by multiplying the trapezoidal tributary area $\left(\frac{bD}{2}-\frac{b^2}{4}\right)$ by $\tau_{t,\max}$. The two equal and opposite $V_h$ forces form a couple which has a lever arm $z_1 = D - b/3$. Similarly, the two equal and opposite $V_v$ force form another couple, with a lever arm $z_2$, which can be shown to be:

$$z_2 = 2 \times \frac{3D-b}{3(2D-b)} \times \frac{b}{2}$$

The summation of the two couple-moments gives the desired value of $T_{cr}$:

$$\begin{aligned} T_{cr} &= V_h z_1 + V_v z_2 \\ &= \tau_{t,\max}\left[\left(\frac{b^2}{4}\right)\left(D-\frac{b}{3}\right)+\left(\frac{bD}{2}-\frac{b^2}{4}\right)\left(\frac{b}{3}\times\frac{3D-b}{2D-b}\right)\right] \end{aligned}$$

$$\Rightarrow T_{cr} = \tau_{t,\max}\frac{b^2}{2}\left(D-b/3\right) \tag{7.5}$$




**(a)** **(b)**

**Fig. 7.5** Plastic theory to determine $T_{cr}$

The above relation can also be derived using the so-called 'sand heap' analogy. Evidently, the assumption of full plastification of the section is not justified for a material like concrete. Hence, a correction factor has to be applied, by either modifying the expression in Eq. 7.5, or by using a reduced value of $\tau_{t,\max}$ (which is otherwise equal to the tensile strength of concrete) — to conform with experimental results. Test results indicate an ultimate strength value of $\tau_{t,\max}$ (MPa units) of about $0.2\sqrt{f_{ck}}$ , to be used with Eq. 7.5.

### *Torsional Shear Stress*

As the torque-twist behaviour up to torsional cracking is approximately linear [Fig. 7.4], torsional shear stress $\tau_t$ corresponding to any factored torque, $T_u \leq T_{cr}$ may be obtained from Eq. 7.5 for a plain concrete rectangular section as

$$\tau_t = \frac{2T_u}{b^2(D - b/3)} \tag{7.6}$$

Extending this expression for $\tau_t$ to a reinforced concrete member with effective depth *d*, Eq. 7.6 reduces to the following form:

$$\tau_t \approx \frac{2T_u}{b^2 d} \times \frac{1}{constant}$$

where the constant, equal to $\left[(D/d) - (b/d)/3\right]$, takes values in the range 0.8 – 1.15 for most rectangular sections in practice. Considering an average value for this constant and further providing a correction factor for the assumption of full plastification of the section, the above expression reduces to the following simplified form, given in the Code:

$$\tau_t = \frac{1.6(T_u/b)}{bd} \tag{7.7}$$

where $T_u$ is the twisting moment acting on the section, $b$ is the width of the rectangular beam (or the width of the web of the flanged beam) and $d$ the effective depth.

It may be noted here that this expression [Eq. 7.7] for torsional stress $\tau_t$ has a form similar to that of the 'nominal' (flexural) shear stress $\tau_v = V_u/bd$ (given by Eq. 6.7). By comparison, it follows that $(1.6T_u/b)$ provides a measure of the equivalent 'torsional shear'.

### *Need for Torsional Reinforcement*

Torsional reinforcement has to be suitably designed when the torsional shear stress $\tau_t$ exceeds the shear strength $\tau_c$ of the plain concrete section. Where flexural shear $V_u$ occurs in combination with torsional shear (as is commonly the case), the combined shear stress (flexural plus torsional) has to be considered. For this purpose, the term *equivalent shear* $V_e$ is used by the Code (Cl. 41.3.1) to express the combined shear effects on a reinforced concrete beam, subject to flexural shear and torsional shear:

$$V_e = V_u + 1.6\frac{T_u}{b} \tag{7.8}$$

It may be noted that the shear† due to $V_u$ and $T_u$ are additive only on one side of the beam; they act in opposite directions on the other side.

The equivalent nominal shear stress, $\tau_{ve}$, is given by

$$\tau_{ve} = \frac{V_u + 1.6\,T_u/b}{bd} \tag{7.9}$$

If $\tau_{ve}$ exceeds $\tau_{c,\max}$ [refer Section 6.6], the section has to be suitably redesigned — by increasing the cross-sectional area (especially width) and/or improving the grade of concrete. If $\tau_{ve}$ is less than the design shear strength of concrete $\tau_c$ [refer Section 6.6], minimum stirrup reinforcement has to be provided, as explained in Section 6.7.5. If the value of $\tau_{ve}$ lies between $\tau_c$ and $\tau_{c,\max}$, suitable torsional reinforcement (both transverse and longitudinal) has to be designed for the combined effects of shear and torsion.

† Care must be taken to express $V_u$ and $T_u$ in consistent units, i.e., $V_u$ in N and $T_u$ in Nmm, $b$ in mm and $d$ in mm.

### 7.4.2 Design Torsional Strength with Torsional Reinforcement

Several theories have been proposed for the computation of the torsional strength of reinforced concrete members with torsional reinforcement — notably the *space-truss analogy* and the *skew bending* theory [Ref. 7.9 – 7.12].

#### *Space Truss Analogy*

The space truss analogy is essentially an extension of the *plane truss analogy* [Fig. 6.9] used to explain flexural shear resistance. The 'space-truss model' (illustrated in Fig. 7.6) is an idealisation of the effective portion of the beam, comprising the longitudinal and transverse torsional reinforcement and the surrounding layer of concrete. It is this 'thin-walled tube' which becomes fully effective at the post-torsional cracking phase. The truss is made up of the corner longitudinal bars as stringers, the closed stirrup legs as transverse ties, and the concrete between diagonal cracks as compression diagonals.

For a closed thin-walled tube, the shear flow $q$ (force per unit length) across the thickness of the tube [Ref. 7.1] is given by:

$$q = \frac{T_u}{2A_o} \tag{7.10}$$

where $A_o$ is the area enclosed by the centreline of the thickness. The proof for Eq. 7.10 is indicated in Fig. 7.6(c). For the *box section* under consideration,

$$A_o = b_1 d_1 \tag{7.11}$$

where $b_1$ and $d_1$ denote the centre-to-centre distances between the corner bars in the directions of the width and the depth respectively. Accordingly, substituting Eq. 7.11 in Eq. 7.10,

$$q = \frac{T_u}{2b_1 d_1} \tag{7.12}$$

Assuming torsional cracks (under pure torsion) at 45° to the longitudinal axis of the beam, and considering equilibrium of forces normal to section AB [Fig. 7.6(b)], the total force in each stirrup is given by $qs_v \tan 45^{\circ} = qs_v$ where $s_v$ is the spacing of the (vertical) stirrups. Further, assuming that the stirrup has yielded in tension at the ultimate limit state (with a design stress of $0.87f_y$), it follows from force equilibrium that

$$A_t(0.87 f_y) = qs_v \tag{7.13}$$

where $A_t$ is the cross-sectional area of the stirrup (equal to $A_{sv}/2$ for two legged stirrups). Substituting Eq. 7.12 in the above equation, the following expression is obtained for the ultimate strength $T_u = T_{uR}$ in torsion:

$$T_{uR} = 2A_t b_1 d_1 (0.87 f_y)/s_v \tag{7.14}$$

Further, assuming that the longitudinal steel (symmetrically placed with respect to the beam axis) has also yielded at the ultimate limit state, it follows from longitudinal force equilibrium [Fig. 7.6(a)] that:

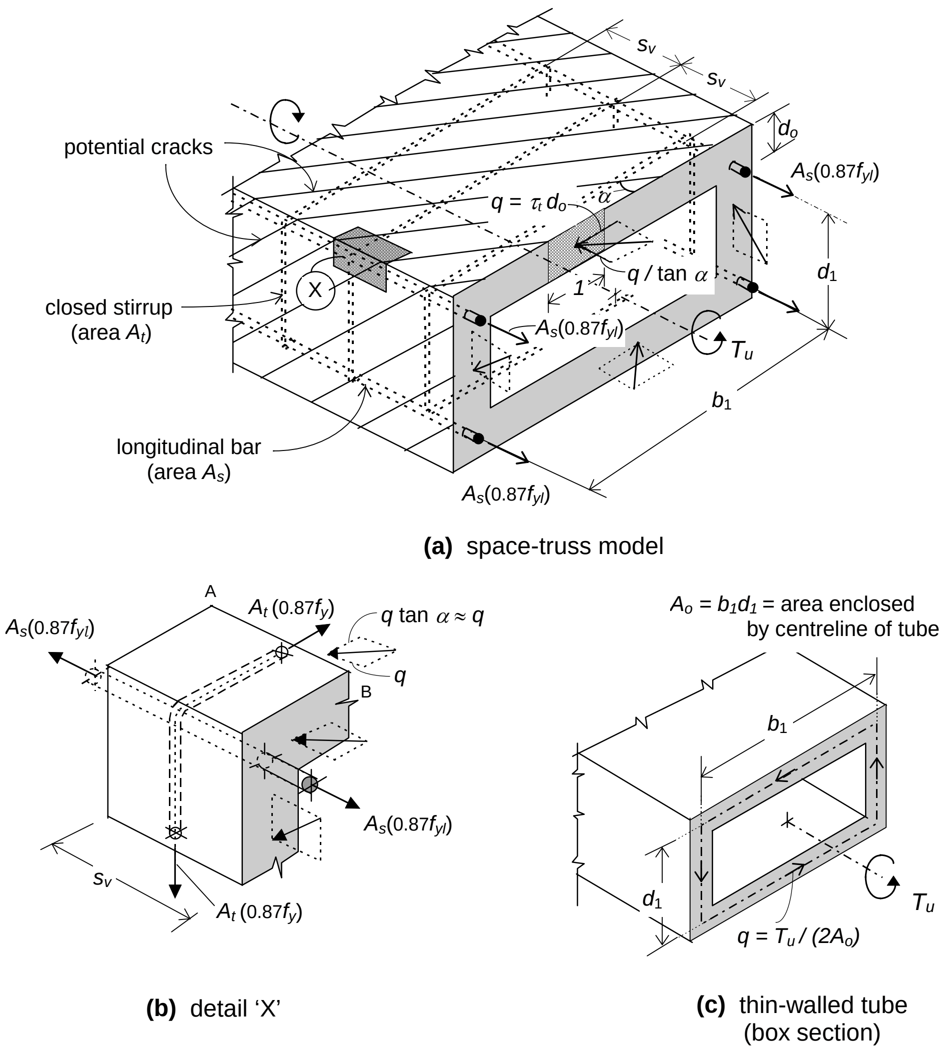



**Fig. 7.6** The idealised space-truss model

$$A_l\,(0.87\,f_{yl}) = \frac{q}{\tan 45^o} \times 2(b_1 + d_1) \tag{7.15}$$

where $A_l \equiv \Sigma A_s$ is the total area of the longitudinal steel and $f_{yl}$ its yield strength.

Substituting Eq. 7.12 in the above equation, the following expression is obtained for the ultimate strength $T_u = T_{uR}$ in torsion:

$$T_{uR} = A_l b_1 d_1\,(0.87\,f_{yl})/(b_1 + d_1) \tag{7.16}$$

The two alternative expressions for $T_{uR}$) viz. Eq. 7.14 and Eq. 7.16, will give identical results only if the following relation between the areas of longitudinal steel and transverse steel (as torsional reinforcement) is satisfied:

$$A_l = A_t \times \frac{2(b_1 + d_1)}{s_v} \times \frac{f_y}{f_{yl}} \tag{7.17}$$

If the relation given by the Eq. 7.17 is not satisfied, then $T_{uR}$ may be computed by combining Eq. 7.14 and Eq. 7.16 [Ref. 7.10], taking into account the areas of both transverse and longitudinal reinforcements:

$$T_{uR} = 2b_1 d_1 \sqrt{\left(\frac{A_t f_y}{s_v}\right)\left(\frac{A_l \times f_{yl}}{2(b_1 + d_1)}\right)} \times 0.87 \tag{7.18}$$

To ensure that the member does not fail suddenly in a brittle manner after the development of torsional cracks, the torsional strength of the cracked reinforced section must be at least equal to the cracking torque $T_{cr}$ (computed without considering any safety factor).

For any design torsional moment, $T_u$ (= $T_{uR}$ ), Eq. 7.14 and Eq. 7.17 give the required areas for transverse and longitudinal reinforcements respectively. These areas $A_t$ and $A_l$ calculated for the effect of torsion alone, should then be added to the corresponding reinforcement required to resist shear and flexure respectively. This procedure is adopted by many codes — *but not the IS Code*. The provisions in the IS Code are based on the *skew bending theory* for rectangular beams and not on the space truss analogy [Ref. 7.2].

### *Skew Bending Theory*

The post-cracking behaviour of reinforced concrete members may be alternatively studied on the basis of the *mechanism of failure,* rather than on the basis of stresses [Ref. 7.10, 7.13]. In the consideration of the failure mechanism, the *combined* action of torsion with flexure and shear has to be taken into account.

Three modes of failure have been identified for beams subjected to combined flexure and torsion [Fig. 7.7]. The action of torsion is to *skew* the failure surface (which is otherwise vertical under the action of flexure alone); the skewing is in the direction of the resultant moment-torsion vector. The most common type of failure is as shown in Mode 1 [Fig. 7.7(b)] — with bending predominating over torsion and the compression zone (shown shaded) remaining on top[†], albeit skewed ($\theta < 45^o$). This type of failure (sometimes called ‘modified bending failure’) will occur in wide beams, even if torsion is relatively high. However, if a beam with a *narrow* section ($D >> b$) is subject to predominant torsion, a Mode 2 type of failure [Fig. 7.7(c)] is likely, with the compression zone skewed to a side of the section; this type of failure is sometimes called a *lateral bending* failure. A third mode of failure — Mode 3 [Fig. 7.7(d)] — is possible when the compression zone occurs at the bottom and the area of the longitudinal top steel is much less than that of the bottom steel; this type of failure is sometimes called a *negative bending* failure.

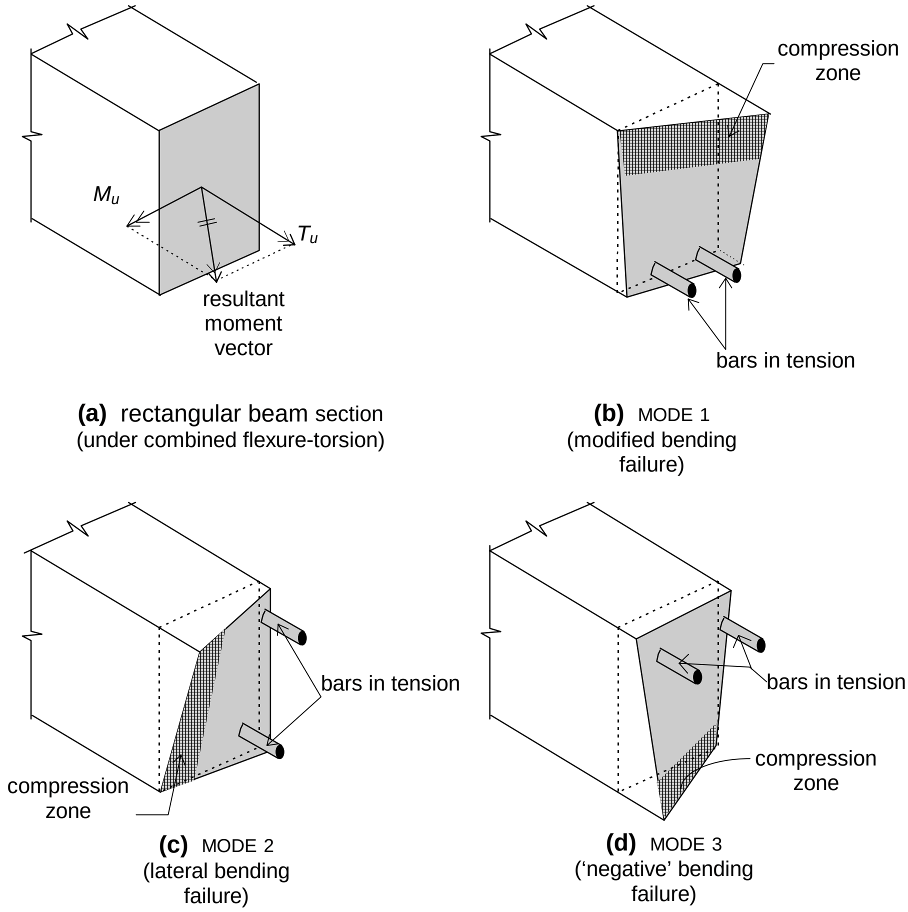



**Fig. 7.7** Failure Modes for combined flexure and torsion

In a beam with a square cross-section, with symmetrical longitudinal reinforcement, subjected to pure torsion, the three modes become identical.

Expressions for the ultimate strength in torsion have been derived for each of the three possible modes of failure. The interested reader is advised to refer to Ref. 7.13 (or Ref. 7.14) for the derivation of these expressions. It is customary to check for all the three modes and to choose the lowest value of the torsional strength.

It may be noted that the presence of shear may cause a beam to fail at a lower strength. The Code attempts to prevent the possibility of such shear type of failure by the concept of designing for *equivalent shear* [refer Section 7.4.4].

### 7.4.3 Design Strength in Torsion Combined with Flexure

The strength of a member subjected to combined torsion ($T_u$) and flexure ($M_u$) is best described in terms of the interaction of $T_u/T_{uR}$ with $M_u/M_{uR}$, where $T_{uR}$ and $M_{uR}$ denote respectively the strengths of the member under *pure torsion* and *pure flexure* respectively. The following parabolic interaction formulas [Ref. 7.15] have been proposed, based on experimental studies on rectangular reinforced beams:

$$\text{Mode 1 failure} \quad : \left(\frac{A_s'}{A_s}\right)\left(\frac{T_u}{T_{uR}}\right)^2 + \left(\frac{M_u}{M_{uR}}\right)^2 \le 1 \tag{7.19a}$$

$$\text{Mode 3 failure} \quad : \left(\frac{T_u}{T_{uR}}\right)^2 - \left(\frac{A_s}{A_s'}\right)\left(\frac{M_u}{M_{uR}}\right)^2 \le 1 \tag{7.19b}$$

where $A_s$ and $A_s'$ denote, respectively, the areas of longitudinal steel provided in the 'flexural tension zone' and 'flexural compression zone' of the rectangular beam section

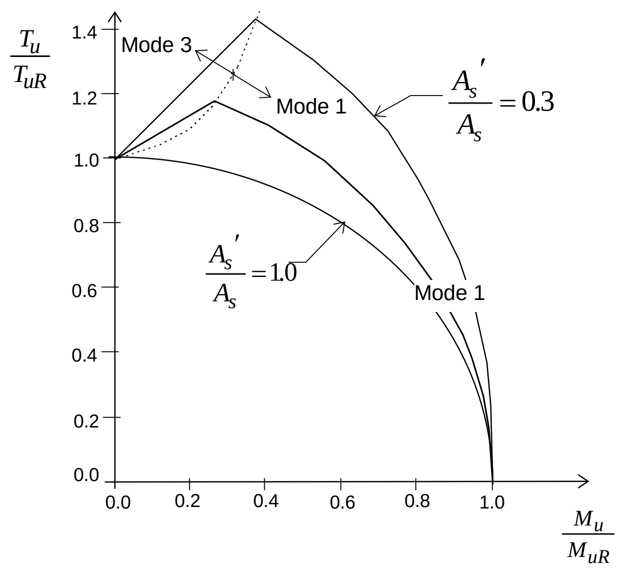



**Fig. 7.8** Torsion-Flexure Interaction

The torsion-flexure *interaction curves*, based on Eq. 7.19, are depicted in Fig. 7.8 for $A_s'/A_s$ in the range 0.3 to 1.0. Each curve represents a 'failure envelope', in the sense that any combination of $T_u/T_{uR}$ and $M_u/M_{uR}$ that falls outside the area bounded by the curve and the coordinate axes is 'unsafe'. In general, it is seen that the torsional strength ($T_u$) increases beyond the 'pure torsion' strength ($T_{uR}$) in the presence of bending moment ($M_u$) — provided $M_u/M_{uR}$ is low and $A_s'/A_s$ is also low. In such cases, failure may occur in Mode 3 (i.e., initiated by the yielding of the compression steel) at very low values of $M_u/M_{uR}$ [Fig. 7.8]. In general, however, a Mode 1 failure is likely to occur (i.e., initiated by the yielding of the 'tension' steel); this becomes inevitable when $A_s'$ is equal to $A_s$.

It may also be noted from Fig. 7.8 that the presence of torsion invariably brings down the flexural strength of the reinforced concrete member.

### *IS Code Provisions for Design of Longitudinal Reinforcement*

The Code (Cl. 41.4.2) recommends a simplified skew-bending based formulation [Ref. 7.15] for the design of longitudinal reinforcement to resist torsion combined with flexure in beams with rectangular sections. The torsional moment $T_u$ is converted into an effective bending moment $M_t$ defined† as follows:

$$M_t = T_u\,(1 + D/b)/1.7 \tag{7.20}$$

where $D$ is the overall depth and $b$ the width of the beam.

$M_t$, so calculated, is combined with the actual bending moment $M_u$ at the section, to give 'equivalent bending moments', $M_{e1}$ and $M_{e2}$:

$$M_{e1} = M_t + M_u \tag{7.21a}$$

$$M_{e2} = M_t - M_u \tag{7.21b}$$

The longitudinal reinforcement area $A_{st}$ is designed to resist the equivalent moment $M_{e1}$, and this steel is to be located in the 'flexural tension zone'. In addition, if $M_{e2} > 0$ (i.e., $M_t > M_u$), then a reinforcement area $A_{st}$' is to be designed to resist this equivalent moment, and this steel is to be located in the 'flexural compression zone'.

It follows from the above that in the limiting case of 'pure torsion' (i.e., with $M_u = 0$), *equal* longitudinal reinforcement is required at the top and bottom of the rectangular beam, each capable of resisting an equivalent bending moment equal to $M_t$.

### 7.4.4 Design Strength in Torsion Combined with Shear

Torsion-shear interaction curves have been proposed [Ref. 7.16], similar to torsion-flexure interaction curves. In general, the interaction between $T_u/T_{uR}$ and $V_u/V_{uR}$ takes the following form:

$$\left(\frac{T_u}{T_{uR}}\right)^{\alpha} + \left(\frac{V_u}{V_{uR}}\right)^{\alpha} \leq 1 \tag{7.22}$$

$T_u$ and $V_u$ are the given factored twisting moment and factored shear force respectively; $T_{uR}$ and $V_{uR}$ are the ultimate strengths in 'pure torsion' and 'flexural shear' (without torsion) respectively; $\alpha$ is a constant, for which values in the range 1 to 2 have been proposed [Fig. 7.9]. A value of $\alpha$ equal to unity results in a linear interaction and generally provides a conservative estimate.

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† This formula can alternatively be generated from the space truss analogy [Fig. 7.6(a)], by visualising the longitudinal tensile forces in the bars (located either at top or at bottom) as those required to resist an effective bending moment $M_t$ which can be shown to be equal to $T_u\,(1 + d_1/b_1)$; the Code has simplified this formula to the form given in Eq. 7.20.

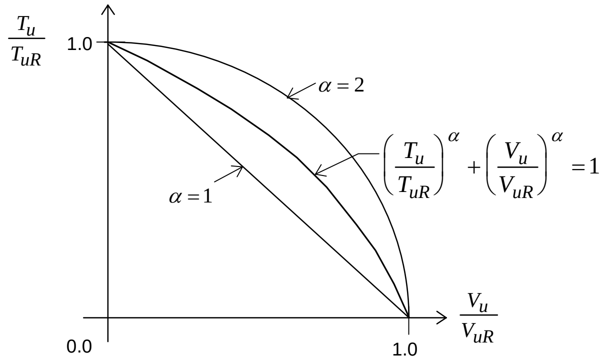



**Fig. 7.9** Torsion-shear interaction

### *IS Code Provisions for Design of Transverse Reinforcement*

The Code provisions (Cl. 41.4.3) for the design of transverse stirrup reinforcement (2-legged, closed) are based on the skew-bending theory and are aimed at resisting a Mode 2 failure [Fig. 7.7(b)], caused by a large torsion combined with a small flexural shear:

$$A_{sv} = \frac{T_u s_v}{b_1 d_1 (0.87 f_y)} + \frac{V_u s_v}{2.5 d_1 (0.87 f_y)} \qquad (7.23)$$

where $A_{sv} = 2A_t$ is the total area of two legs of the stirrup; $s_v$ is the centre-to-centre spacing of the stirrups; $b_1$ and $d_1$ are the centre-to-centre distances between the corner bars along the width and depth respectively; and $T_u$ and $V_u$ are the factored twisting moment and factored shear force acting at the section under consideration.

It may be observed that for the extreme case of strength in 'pure torsion' (i.e., with $V_u = 0$ and $T_u = T_{uR}$), Eq. 7.23 becomes exactly equivalent to Eq. 7.14, which was derived using the space-truss analogy.

In addition to Eq. 7.23, the Code (Cl. 41.4.3) specifies a minimum limit to the *total* area of transverse reinforcement:

$$A_{sv} \geq \frac{(\tau_{ve} - \tau_c)\, b s_v}{0.87 f_y} \qquad (7.24)$$

where $\tau_{ve}$ is the 'equivalent nominal shear stress' given by Eq. 7.7. The purpose of Eq. 7.24 is to provide adequate resistance against flexural shear failure, which is indicated in situations where $T_u$ is negligible in comparison with $V_u$. Indeed, for the extreme case of $T_u = 0$, Eq. 7.24 becomes exactly equivalent to Eq. 6.25, which was derived for flexural shear. It may be noted that the contribution of inclined stirrups and bent up bars can be included in the calculation of $A_{sv}$ in Eq. 7.24, but not Eq. 7.23.

### *Distribution of Torsional Reinforcement*

The Code (Cl. 26.5.1.7a) specifies maximum limits to the spacing $s_v$ of the stirrups provided as torsional reinforcement — to ensure the development of post-cracking torsional resistance, to control crack-widths and to control the fall in torsional stiffness on account of torsional cracks:

$$s_v \le \begin{cases} x_1 \\ (x_1 + y_1)/4 \\ 300 \text{ mm} \end{cases} \tag{7.25}$$

where $x_1$ and $y_1$ are, respectively, the short and long centre-to-centre dimensions of the rectangular closed stirrups. The spacing $s_v$ should satisfy *all* the limits given in Eq. 7.25.

The Code (Cl. 26.5.1.7b) also recommends that the *"longitudinal reinforcement shall be placed as close as is practicable to the corners of the cross-section, and in all cases, there shall be at least one longitudinal bar in each corner of the ties"*.

Further, if the torsional member has a cross-sectional dimension (usually, overall depth rather than width) that exceeds 450 mm, additional longitudinal bars are required to be provided as side face reinforcement, with an area not less than 0.1 percent of the web area. These bars are to be distributed equally on the two faces at a spacing not exceeding 300 mm or web thickness, whichever is less.

## 7.5 ANALYSIS AND DESIGN EXAMPLES

### EXAMPLE 7.1

A plain concrete beam (M 20 grade concrete) has a rectangular section, 300 mm wide and 500 mm deep (overall). Estimate the 'cracking torque'. Also determine the limiting torque beyond which torsional reinforcement is required (as per the Code), assuming $\tau_c$ = 0.3 MPa.

### SOLUTION

- ***Using the plastic theory formula*** [Eq. 7.5]**:**

  $$T_{cr} = \frac{1}{2}\tau_{t,\max} b^2 (D - b/3)$$

  where $b$ = 300 mm, $D$ = 500 mm.

  Assuming $\tau_{t,\max} \approx 0.2\sqrt{f_{ck}}$ = $0.2\sqrt{20}$ = 0.894 MPa.

  $$\Rightarrow T_{cr} = \frac{1}{2}(0.894) \times 300^2 \times (500 - 300/3)$$

  $= 16.09 \times 10^6$ Nmm = **16.1 kNm**.

- As per IS Code formulation, torsion has to be combined with shear for deciding whether or not torsional reinforcement is required.

  Torsional reinforcement is required if $\tau_v > \tau_c$, i.e., $\dfrac{V_u + 1.6\,T_u/b}{bd} > \tau_c$

  Assuming $V_u$ = 0, $d \approx 0.9D = 450 mm$ and $\tau_c$ = 0.3 MPa,

  $$\Rightarrow \frac{0 + 1.6(T_u \times 10^6/300)}{300 \times 450} > 0.30$$

  $\Rightarrow T_u > 7.59 \times 10^6$ Nmm = **7.6 kNm**.

### EXAMPLE 7.2

The beam of Example 7.1 is reinforced (using Fe 415 grade steel) as shown in Fig. 7.10(a). Determine the design torsional resistance of the beam under pure torsion. Assume moderate exposure condition.

### SOLUTION

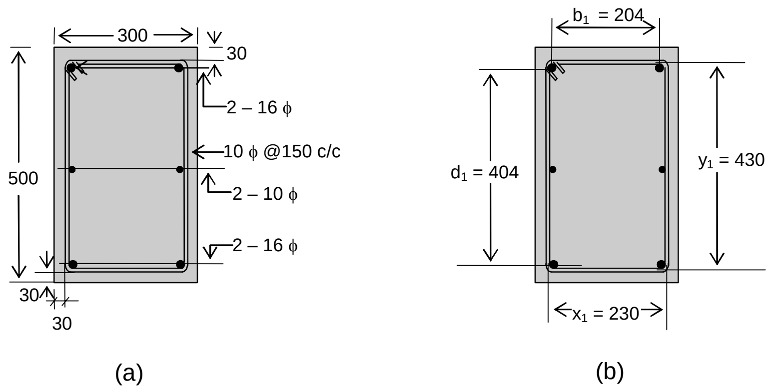



**Fig. 7.10** Example 7.2

- Given $b$ = 300 mm, $D$ = 500 mm, $f_{ck}$ = 20 MPa, $f_y = f_{yl}$ = 415 MPa,
  $A_l$ (due to 4 – 16 $\phi$ plus 2 – 10 $\phi$) = (201 × 4) + (78.5 × 2) = 961 mm$^2$.
  $A_t$ (10 $\phi$ stirrup) = 78.5 mm$^2$, $A_{sv} = 2A_t$ = 157 mm$^2$.
  $s_v$ = 150 mm
  $b_1$ = 300 – 30 × 2 – 10 × 2 – 16 = 204 mm
  $d_1$ = 500 – 30 × 2 – 10 × 2 – 16 = 404 mm [Fig. 7.10(b)].
- Applying the general space truss formulation, considering the contribution of both transverse and longitudinal reinforcements [Eq. 7.18]:

$$T_{uR} = 2b_1 d_1 (0.87 f_y)\sqrt{\left(\frac{A_t}{s_v}\right)\left(\frac{A_l}{2(b_1+d_1)}\right)}$$

$$= 2 \times 204 \times 404 \times (0.87 \times 415) \times \sqrt{\left(\frac{78.5}{150}\right)\left(\frac{961}{2(204+404)}\right)}$$

$$= 38.27 \times 10^6 \text{ Nmm} = \mathbf{38.3\ kNm}$$

  which is greater than $T_{cr}$ = 16.1 kNm (refer Example 7.1).
- Alternatively, using the IS Code formula, considering shear-torsion interaction [Eq. 7.23] with $V_u$ = 0, which corresponds to the space truss formulation considering the contribution of the transverse reinforcement alone [Eq. 7.14]:

$$T_{uR} = A_{sv} b_1 d_1 (0.87 f_y)/s_v$$

$$= 157 \times 204 \times 404 \times (0.87 \times 415)/150$$

$$= 31.1 \times 10^6 \text{ Nmm} = \mathbf{31.1\ kNm}.$$

- In the above formulation, we had tacitly assumed that the torsional strength is governed by shear considerations and not 'equivalent moment'. The reader may verify this assumption by checking the equivalent moment capacity due to the longitudinal reinforcement[†] using Eq.7.20.

**EXAMPLE 7.3**

A beam, framing between columns, has an effective span of 5.0 m and supports a cantilevered projection, 1 m wide [Fig. 7.11(a)] throughout its length. Assume that the cross-sectional details of the beam are exactly the same as in Example 7.2 [Fig. 7.10]. Determine the adequacy of the section (as per IS Code), assuming a total uniformly distributed load (DL+LL) of 5 kN/m$^2$ on the cantilever projection as shown. Assume fixity at the ends of the beam against torsion as well as flexure.

**SOLUTION**

### Structural Analysis

This is a problem involving equilibrium torsion, combined with flexure and shear.

- *Loads on beam* [Fig. 7.11(b)]:
  from projection: 5.0 kN/m$^2$ $\times$ 1*m* = 5.0 kN/m
  from self weight: 25.0 × 0.3 × 0.5 = 3.75 ”
  8.75 kN/m

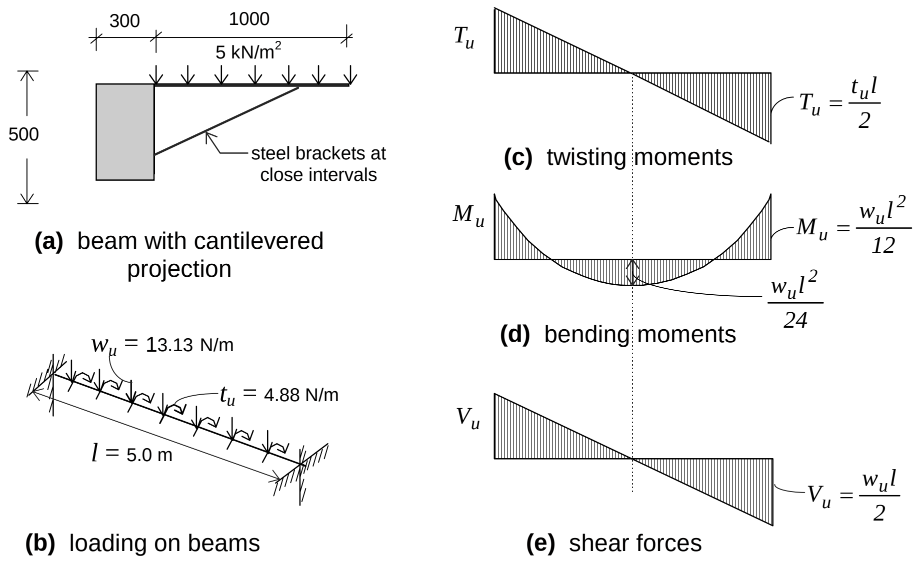



**Fig. 7.11** Example 7.3

[†] The corresponding value of $M_{uR}$ is 73.3 kNm, whereby $T_u$ = 46.7 kNm [Eq.7.20].

Assuming a load factor of 1.5,
Factored distributed load $w_u$ = 8.75 × 1.5 = 13.13 kN/m.

Eccentricity of cantilever load from beam centreline = 1.0/2 + 0.3/2 = 0.65 m.
∴ Factored distributed torque $t_u$ = (5.0 × 0.65) × 1.5 = 4.88 kNm/m

- *Stress resultants*:

  Max. twisting moment (at support) $T_u = \dfrac{t_u l}{2} = 4.88\left(\dfrac{5.0}{2}\right) = 12.20$ kNm.

  Max. bending moment (at support) $M_u = \dfrac{w_u l^2}{12} = 27.35$ kNm (hogging).

  Max. shear force (at support) $V_u = \dfrac{w_u l}{2} = 32.82$ kN.

  The distributions of twisting moment, bending moment and shear force are shown in Fig. 7.11(c), (d), and (e). Evidently, the critical section for checking the adequacy of the beam under the combined effects of $T_u$, $M_u$ and $V_u$ is at the support [at midspan, $T_u = 0$, $V_u = 0$].

**Need for torsional reinforcement**

- *Equivalent nominal shear stress* [Eq. 7.9]:

$$\tau_{ve} = \frac{V_u + 1.6\,T_u/b}{bd}$$

$$= \frac{(32.82\times10^3) + 1.6(12.20\times10^6)/300}{300\times462} = 0.706 \text{ MPa}$$

$$< \tau_{c\,\max} = 2.8 \text{ MPa}$$

- *Shear strength of concrete* [Eq. 6.10]:

$A_{st} = 402$ mm$^2$ $\Rightarrow p_t = \dfrac{402\times100}{300\times462} = 0.290 \Rightarrow \beta = 0.8\times20/(6.89\times0.290)$

$$= 8.01 > 1.0$$

$$\Rightarrow \tau_c = 0.85\sqrt{(0.8\times20)}\left(\sqrt{(1+5\times8.01)}-1\right)\Big/(6\times8.01) = 0.383 \text{ MPa}$$

- *As $\tau_{ve} > \tau_c$, torsional reinforcement is required.*

**Adequacy of longitudinal reinforcement**

- *Effective bending moment* $M_t = T_u\left(\dfrac{1+D/b}{1.7}\right)$ [Eq. 7.20]

$$= 12.20\left(\frac{1+500/300}{1.7}\right) = 19.14 \text{ kNm}$$

- *Equivalent bending moments* [Eq. 7.21]:

$M_{e1} = M_t + M_u = 19.14 + 27.35 = 46.49$ kNm (with flexural tension on top)
$M_{e2} = M_t - M_u < 0 \Rightarrow$ not to be considered.

- *Ultimate resisting moment of section with* $A_{st}$ = 402 mm$^2$ ($p_t$ = 0.290 at top)

$$M_{uR} = 0.87 f_y \frac{p_t}{100}\left(1 - \frac{f_y}{f_{ck}}\frac{p_t}{100}\right) bd^2 \qquad \text{[Eq. 4.65]}$$

$$= 0.870 \times 415 \times \frac{0.290}{100} \times \left(1 - \frac{415}{20} \times \frac{0.290}{100}\right) \times 300 \times 462^2$$

$$= 63.01 \times 10^6 \text{ Nmm } = 63.0 \text{ kNm}$$

- $M_{uR} > M_e$ = 46.49 kNm $\Rightarrow$ safe.

**Adequacy of side face reinforcement**

- As the depth (500 mm) exceeds 450 mm, additional $A_{st}$ = 0.001 $bD$ = 150 mm$^2$ is required at a spacing less than 300 mm, distributed equally on the two side faces. This has been provided in the form of 2 – 10 $\phi$ bars (157 mm$^2$) [Fig. 7.11(a)].

— Hence, OK.

**Adequacy of transverse reinforcement**

- Area of 2–legged 10 $\phi$ stirrups provided, $A_{sv}$ = 157 mm$^2$
  This should exceed the requirements given by Eq. 7.23 and Eq. 7.24
  $s_v$ = 150 mm, $b_1$ = 224 mm, $d_1$ = 424 mm [Fig. 7.10(a), (b)]
- [Eq. 7.23]: $(A_{sv})_{reqd} = (T_u/b_1 + V_u/2.5)\ (s_v/d_1(0.87 f_y))$

$$= (12.20 \times 10^6/224 + 32.82 \times 10^3/2.5) \times \frac{150}{424(0.87 \times 415)}$$

$$= 66.2 \text{ mm}^2 < 157 \text{ mm}^2 \text{ provided} \Rightarrow \text{ OK.}$$

- Minimum limit of area of transverse reinforcement [Eq. 7.24]:

$$(A_{sv})_{reqd} = \frac{(\tau_{ve} - \tau_c) b s_v}{0.87 f_y}$$

$$= \frac{(0.706 - 0.383) \times 300 \times 150}{0.87 \times 415}$$

$$= 40 \text{ mm}^2 < 157 \text{ mm}^2 \text{ provided} \Rightarrow \text{ OK.}$$

- Further, the spacing of stirrups provided ($s_v$ = 150 mm) should satisfy the requirements of Eq. 7.25:

$$(s_v)_{reqd} \le \begin{cases} x_1 = 224 + 16 + 10 = 250 \text{ mm} \\ (x_1 + y_1)/4 = (250 + 450)/4 = 175 \text{ mm} \\ 0.75d = 0.75 \times 462 = 346 \text{ mm} \end{cases}$$

$$(s_v)_{provided} = 150 \text{ mm} < (s_v)_{reqd} \Rightarrow \text{ OK.}$$

- Hence the section provided is adequate in all respects.

## EXAMPLE 7.4

Design the torsional reinforcement in a rectangular beam section, 350 mm wide and 750 mm deep, subjected to an ultimate twisting moment of 140 kNm, combined with an ultimate (hogging) bending moment of 200 kNm and an ultimate shear force of 110*kN*. Assume M 25 concrete, Fe 415 steel and *mild* exposure conditions.

### SOLUTION

- Given: $b$ = 350 mm, $D$ = 750 mm, $f_{ck}$ = 25 MPa, $f_y$ = 415 MPa, $T_u$ = 140 kNm, $M_u$ = 200 kNm, $V_u$ = 110*kN*.
  Minimum required cover to the stirrups is 20mm. Assuming 50 mm effective cover all around, $d$ = 700 mm.

**Design of longitudinal reinforcement**

- *Effective bending moment due to torsion*:
  $$M_t = T_u(1+D/b)/1.7$$
  $$= 140\times(1+750/350)/1.7 = 259 \text{ kNm}$$
- *Equivalent bending moments for design*:
  $$M_e = M_t \pm M_u$$
  $$= 259 \pm 200 = \begin{cases} 459 \text{ kNm} & \text{(flexural tension at top)} \\ 59 \text{ kNm} & \text{(flexural tension at bottom)} \end{cases}$$
- *Design of top steel*:
  $$R_1 \equiv \frac{M_{e1}}{bd^2} = \frac{459\times10^6}{350\times(700)^2} = 2.676 \text{ MPa}$$
  $$< \frac{M_{u,\lim}}{bd^2} = 0.1389 \times 25 = 3.472 \text{ MPa}$$
  $$\Rightarrow \text{[Eq. 5.12]: } \frac{p_t}{100} \equiv \frac{A_{st}}{bd} = \frac{f_{ck}}{2f_y}\left[1-\sqrt{4.598\,R_1/f_{ck}}\right]$$
  $$= \frac{25}{2(415)}\left[1-\sqrt{4.598\times2.676/25}\right]$$
  $$= 0.866 \times 10^{-2}$$
  [Note: The same result is obtainable directly using Design Aids — Table A.3(a) or SP : 16].
  $$\Rightarrow (A_{st})_{reqd} = 0.866\times10^{-2}\times350\times700 = 2122 \text{ mm}^2$$
  **Provide 2 – 28 ϕ + 2 – 25 ϕ** at top [$A_{st}$ = (616 × 2) + (491 × 2) = 2214 mm$^2$].
- *Design of bottom steel*:
  $$R_2 \equiv \frac{M_{e2}}{bd^2} = \frac{59\times10^6}{350\times(750)^2} = 0.344 \text{ MPa}$$
  $$\therefore \frac{(p_t)_{reqd}}{100} = \frac{25}{2(415)}\left[1-\sqrt{1-\frac{4.598\times0.344}{25}}\right] = 0.097\times10^{-2} \text{ (very low)}$$

Provide minimum reinforcement: $\dfrac{A_{st}}{bd} = \dfrac{0.85}{415} = 0.205 \times 10^{-2}$

$\Rightarrow (A_{st})_{reqd} = 0.205 \times 10^{-2} \times 350 \times 700 = 502 \text{ mm}^2$

**Provide 3 – 16 ϕ** ($A_{st} = 201 \times 3 = 603 \text{ mm}^2$) at bottom.

- *Side face reinforcement*:

  As $D > 450$ mm, side face reinforcement for torsion is required.
  $(A_{st})_{reqd} = 0.001bD = 0.001 \times 350 \times 750 = 263 \text{ mm}^2$
  **Provide 4 – 10 ϕ** ($A_{st} = 78.5 \times 4 = 452 \text{ mm}^2$), two bars on each side face. The (vertical) spacing between longitudinal bars will be less than 300 mm, as required by the Code. The designed cross-section is shown in Fig. 7.12.

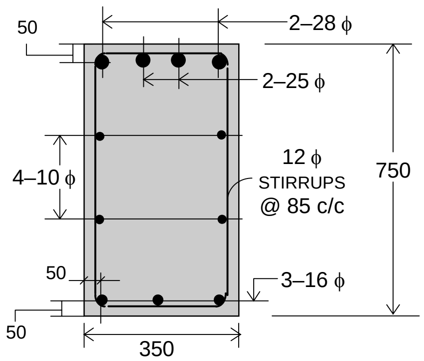



**Fig. 7.12** Example 7.4

**Design of transverse reinforcement**

- Equivalent nominal shear stress:

$$\tau_{ve} = \frac{V_u + 1.6\,T_u/b}{bd}$$

$$= \frac{(110 \times 10^3) + 1.6 \times (140 \times 10^6)/350}{350 \times 700} = 3.06 \text{ MPa}$$

$$< \tau_{c,\max} = 3.1 \text{ MPa (for M 25 concrete)}$$

- Shear strength of concrete [Eq. 6.10]:

  For $p_t = \dfrac{2214 \times 100}{350 \times 700} = 0.904$, $\beta = 3.211 \Rightarrow \tau_c = 0.618$ MPa (for M 25 concrete)

- As torsional shear is relatively high, Eq. 7.23 is likely to govern the design of stirrups (rather than Eq. 7.24).
  Assuming 10 ϕ 2-legged stirrups, $A_{sv} = 78.5 \times 2 = 157 \text{ mm}^2$.

$$\Rightarrow (s_v)_{reqd} = \frac{A_{sv}\, d_1 (0.87 f_y)}{T_u/b_1 + V_u/2.5}$$

  With 50 mm effective cover assumed all around, [Fig. 7.12].
  $d_1 = 750 - 50 \times 2 = 650$ mm

$b_1 = 350 - 50 \times 2 = 250$ mm

$$\Rightarrow (s_v)_{reqd} = \frac{157 \times 650 \times (0.87 \times 415)}{(140 \times 10^6 / 250) + (110 \times 10^3 / 2.5)} = 61.0 \text{ mm (low)}$$

Alternatively, providing 12 $\phi$ 2-legged stirrups, $A_{sv} = 113 \times 2 = 226$ mm$^2$

$$\Rightarrow (s_v)_{reqd} = 61.0 \times \frac{226}{157} = 87.8 \text{ mm}$$

Further, applying Eq. 7.24,

$$(s_v)_{reqd} = \frac{0.87 f_y A_{sv}}{(\tau_{ve} - \tau_e)b} = \frac{0.87 \times 415 \times 226}{(3.06 - 0.618) \times 350} = 95 \text{ mm}$$

- Minimum spacing requirements [Eq. 7.25]:

$$(s_v) \le \begin{cases} x_1 = 250 + 28 + 12 = 290 \text{ mm} \\ (x_1 + y_1)/4 = (290 + 650 + 34)/4 = 243 \text{ mm} \\ 300 \text{ mm} \end{cases}$$

**Provide 12 $\phi$ 2-legged stirrups at 85 mm c/c** [Fig. 7.12]

- Check cover:

With 50 mm effective cover, 12 $\phi$ stirrups and 28 $\phi$ longitudinal bars, clear cover to stirrups is: 50 – 12 – 28/2 = 24 mm, > 20 mm $\Rightarrow$ OK.

## REVIEW QUESTIONS

7.1 Explain, with examples, the difference between *equilibrium torsion* and *compatibility torsion*.

7.2 “Equilibrium torsion is associated with statically determinate structures, whereas compatibility torsion is associated with statically indeterminate structures”. Is this statement true? Comment.

7.3 Reinforced concrete columns are rarely subjected to torsion. Cite an example where this situation occurs, i.e., torsion exists in combination with axial compression, and perhaps also with flexure and shear.

7.4 How is *torsional stiffness* estimated for ‘compatibility torsion’?

7.5 (a) Estimate the torsional stiffness of a reinforced concrete beam element (of a frame), having a span $l$ = 6.0 m and a rectangular section with width $b$ = 200 mm and overall depth $D$ = 500 mm. Assume M 25 concrete.
(b) Compare the torsional stiffness with the flexural stiffness, $4EI / l$, for the same beam element.

7.5 In the case of a *circular* shaft subject to pure torsion, it is well known that the maximum torsional shear stress occurs at locations of maximum radius. If the member has a *rectangular* (instead of circular) cross-section, the corner points are the ones located furthest from the shaft axis. However, the torsional shear stress at these points is not the maximum; the stress, in fact, is zero! [refer Fig. 7.3(b)]. Why?

7.6 Discuss the torque-twist relationship for (a) *plain concrete* and (b) *reinforced concrete* members subjected to pure torsion.

7.7 Inclined stirrups and bent-up bars are considered suitable for shear reinforcement, but not torsional reinforcement. Why?

7.8 For a thin-walled tubular section of arbitrary shape (but uniform thickness) subjected to pure torsion $T_u$, the *shear flow* $q$ can be assumed to be constant (in the plastified state) at all points on the centreline of the thickness. Using this concept, derive the relationship [Eq. 7.9] between $q$ and $T_u$ in terms of $A_0$, the area enclosed by the centreline of the thickness.

7.9 Briefly explain the concept underlying the space truss analogy for estimating torsional strength of a reinforced concrete beam.

7.10 Briefly discuss the different modes of failure under combined flexure and torsion.

7.11 Briefly discuss torsion-shear interaction of reinforced concrete beams.

## PROBLEMS

7.1 Determine the design torsional resistance of the beam shown in Fig. 7.13 under pure torsion by (i) IS Code procedure, (ii) general space truss formulation. Assume M 25 concrete and Fe 415 steel.

*Ans.* (i) 48.6 kNm (ii) 62.2 kNm

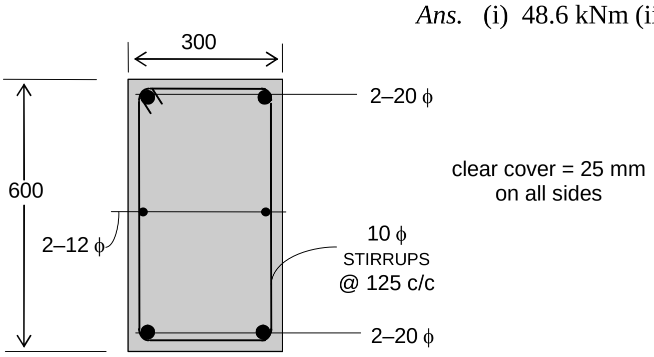



**Fig. 7.13** Problems 7.1, 7.2

7.2 For the beam section shown in Fig. 7.13, derive and hence plot suitable interaction relationships (satisfying IS Code requirements) between:

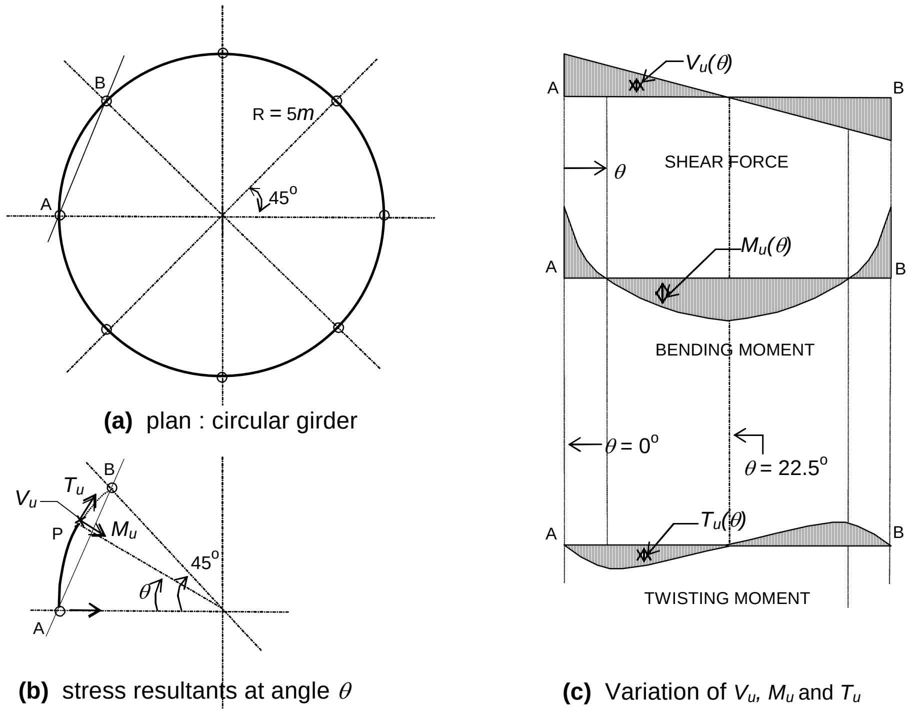
(a) torsion and bending ($T_u - M_u$)
(b) torsion and shear ($T_u - V_u$)

Plot $T_u$ on the $y$ - axis in both cases.

7.3 Consider the problem described in Example 7.3 as a *design* problem instead of an *analysis* problem. Consider a total distributed service load of 10 kN/m$^2$ (instead of 5 kN/m$^2$) on the 1 m wide cantilever projection [Fig. 7.11]. Design the reinforcement in the beam section (300 mm × 500 mm), assuming M 25 concrete, moderate exposure conditions and Fe 415 steel.

7.4 Design a rectangular beam section, 300 mm wide and 550 mm deep (overall), subjected to an ultimate twisting moment of 25 kNm, combined with an ultimate bending moment of 60 kNm and an ultimate shear force of 50 kN. Assume M 20 concrete, moderate exposure conditions and Fe 415 steel.

7.5 Repeat the Problem 7.4, considering an ultimate twisting moment of 50 kNm (instead of 25 kNm).




**(a)** plan : circular girder

**(b)** stress resultants at angle $\theta$

**(c)** Variation of $V_u$, $M_u$ and $T_u$

**Fig. 7.14** Problem 7.6

7.6 Consider a circular girder of radius $R = 5$ m with rectangular cross-section, 350 mm wide and 750 mm deep (overall), supported symmetrically on 8 pillars [Fig. 7.14(a)]. Design and detail one typical span (AB) of the girder, assuming M 20 concrete and Fe 415 steel. The total ultimate (uniformly distributed) load on span AB may be taken as $W_u$ = 1400 kN, inclusive of self-weight of the girder. Expressions for the bending moment ($M_u$), twisting moment ($T_u$) and shear force ($V_u$) at any location $\theta$ (in radians) [Fig. 7.14(b)] can be derived from first principles. For convenience, these expressions are summarised below:

$$M_u = (W_u R)[0.5 sin\theta + 1.20711 cos\theta - 1.27324]$$
$$T_u = (W_u R)\left[1.20711 sin\theta - 0.5 cos\theta - 4\theta/\pi + 0.5\right]$$
$$V_u = 0.5\ W_u\,(1 - 8\theta/\pi)$$

Assume moderate exposure conditions.

## REFERENCES


7.1 Timoshenko, S., and Goodier, J.N., *Theory of Elasticity*, Third edition, McGraw-Hill, New York, 1970.

7.2 — *Explanatory Handbook on Indian Standard Code of Practice for Plain and Reinforced Concrete (IS 456:1978)*, Special Publication SP:24, Bureau of Indian Standards, New Delhi, 1983.

7.3 Popov, E.P., *Mechanics of Materials*, Second edition, Prentice-Hall, Englewood Cliffs, New Jersey, 1978.

7.4 Hsu, T.T.C., *Plain Concrete Rectangular Sections*, Torsion of Structural Concrete, ACI Publication SP – 18, Am. Conc. Inst., Detroit, 1968, pp 207–238.

7.5 Collins, M.P., *The Torque-Twist Characteristics of Reinforced Concrete Beams*, Inelasticity and Nonlinearity in Structural Concrete, SM Study No. 8, University of Waterloo Press, Waterloo, 1972, pp 211–232.

7.6 Cowan, H.J., *Reinforced and Prestressed Concrete in Torsion*, Edward Arnold Ltd., London, 1965.

7.7 Hsu, T.T.C., *Behaviour of Reinforced Concrete Rectangular Members*, ACI Publication SP – 18, 'Torsion of Structural Concrete' Am. Conc. Inst., Detroit, 1968, pp 261–306.

7.8 —*Structural Use of Concrete : Part 2 : Code of Practice for Special Circumstances*, BS 8110 : Part 2 : 1985, British Standards Institution, 1985.

7.9 Hsu, T.T.C., *Ultimate Torque of Reinforced Rectangular Beams*, ASCE Journal, Struct. Div., Vol. 94, Feb. 1968, pp 485–510.

7.10 Lampert, P. and Collins, M.P., *Torsion, Bending and Confusion — An Attempt to Establish the Facts*, ACI Journal, Vol. 69, Aug. 1972, pp 500–504.

7.11 Mitchell, D. and Collins, M.P., *Diagonal Compression Field Theory — A Rational Method for Structural Concrete in Pure Torsion*, ACI Journal, Vol. 71, Aug. 1974, pp 396–408.

7.12 Collins, M.P., Walsh, P.F., Archer, F.E. and Hall A.S., *Ultimate Strength of Reinforced Concrete Beams Subjected to Combined Torsion and Bending*, ACI Publication SP – 18, 'Torsion of Structural Concrete', Am. Conc. Inst., March 1966.

7.13 Warner, R.F., Rangan, B.V. and Hall, A.S., *Reinforced Concrete*, Pitman, Australia, 1976.

7.14 Purushothaman, P., *Reinforced Concrete Structural Elements — Behaviour, Analysis and Design*, Tata McGraw-Hill Publ. Co. Ltd., New Delhi, 1984.

7.15 Iyengar, K.T.S. and Ram Parkash, N., *Recommendation for the Design of Reinforced Concrete Beams for Torsion, Bending and Shear*, Bridge and Structural Engineer, March 1974.

7.16 Mattock, A.H., *How to Design for Torsion*, ACI Publication SP – 18, 'Torsion of Structural Concrete', Am. Conc. Inst., March 1968.

# 8

# Design for Bond

## 8.1 INTRODUCTION

‘Bond’ in reinforced concrete refers to the adhesion between the reinforcing steel and the surrounding concrete. It is this *bond* which is responsible for the transfer of axial force from a reinforcing bar to the surrounding concrete, thereby providing *strain compatibility* and ‘composite action’ of concrete and steel [refer Section 1.2.2]. If this bond is inadequate, ‘slipping’ of the reinforcing bar will occur, destroying full ‘composite action’. Hence, the fundamental assumption of the theory of flexure, viz. plane sections remain plane even after bending, becomes valid in reinforced concrete only if the mechanism of bond is fully effective.

It is through the action of *bond resistance* that the axial stress (tensile or compressive) in a reinforcing bar can undergo variation from point to point along its length. This is required to accommodate the variation in bending moment along the length of the flexural member. Had the bond been absent, the stress at all points on a straight bar would be constant[†], as in a string or a straight cable.

### 8.1.1 Mechanisms of Bond Resistance

Bond resistance in reinforced concrete is achieved through the following mechanisms:

1. Chemical adhesion — due to a gum-like property in the products of hydration (formed during the making of concrete).
2. Frictional resistance — due to the surface roughness of the reinforcement and the grip exerted by the concrete shrinkage.
3. Mechanical interlock — due to the surface protrusions or ‘ribs’ (oriented transversely to the bar axis) provided in deformed bars.

[†] Such a situation is encountered in prestressed concrete — in *unbonded post-tensioned* members.

Evidently, the resistance due to 'mechanical interlock' (which is considerable) is not available when *plain* bars are used. For this reason, many foreign codes prohibit the use of plain bars in reinforced concrete — except for lateral spirals, and for stirrups and ties smaller than 10 mm in diameter. However, there is no such restriction, as yet, in the IS Code.

### 8.1.2 Bond Stress

Bond resistance is achieved by the development of tangential (shear) stress components along the interface (contact surface) between the reinforcing bar and the surrounding concrete. The stress so developed at the interface is called *bond stress*, and is expressed in terms of the tangential force per unit nominal surface area of the reinforcing bar.

### 8.1.3 Two Types of Bond

There are two types of loading situations which induce bond stresses, and accordingly 'bond' is characterised as:

1. *Flexural bond*;
2. *Anchorage bond* or *development bond*.

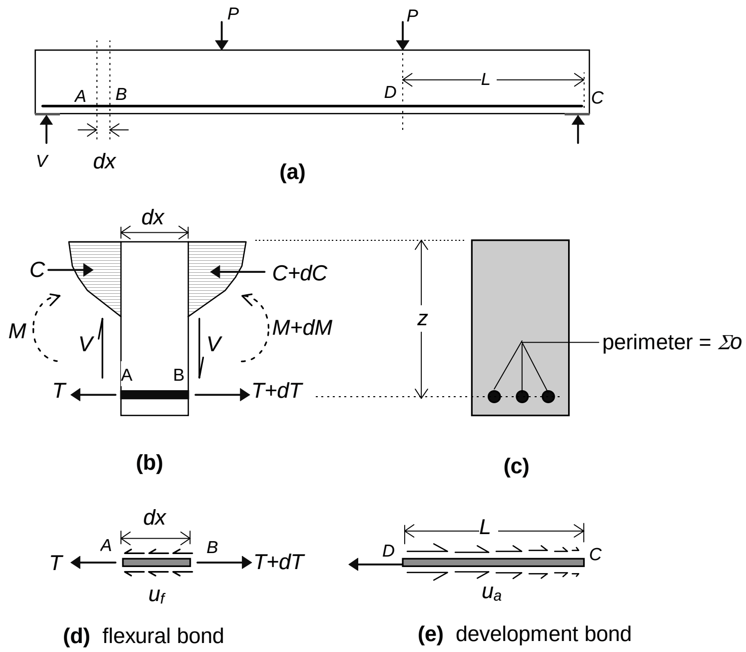



**Fig. 8.1** Bond stress in a beam

'Flexural bond' is that which arises in flexural members on account of shear or a variation in bending moment, which in turn causes a variation in axial tension along the length of a reinforcing bar [Fig. 8.1(d)]. Evidently, flexural bond is critical at points where the shear $(V = dM/dx)$ is significant.

'Anchorage bond' (or 'development bond') is that which arises over the length of anchorage provided for a bar or near the end (or cut-off point) of a reinforcing bar; this bond resists the 'pulling out' of the bar if it is in tension [Fig. 8.1(e)], or conversely, the 'pushing in' of the bar if it is in compression.

These two types of bond are discussed in detail in the sections to follow.

## 8.2 FLEXURAL BOND

As mentioned earlier, variation in tension along the length of a reinforcing bar, owing to varying bending moment, is made possible through *flexural bond.* The flexural stresses at two adjacent sections of a beam, $dx$ apart, subjected to a differential moment $dM$, is depicted in Fig. 8.1(b). With the usual assumptions made in flexural design, the differential tension $dT$ in the tension steel over the length $dx$ is given by

$$dT = \frac{dM}{z} \tag{8.1}$$

where $z$ is the *lever arm*.

This unbalanced bar force is transferred to the surrounding concrete by means of 'flexural bond' developed along the interface. Assuming the flexural (local) bond stress $u_f$ to be uniformly distributed over the interface in the elemental length $dx$, equilibrium of forces gives:

$$u_f (\sum o) dx = dT \tag{8.2}$$

where $\Sigma o$ is the total perimeter of the bars at the beam section under consideration [Fig. 8.1(c)].

From Eq. 8.2, it is evident that the bond stress is directly proportional to the change in the bar force. Combining Eq. 8.2 with Eq. 8.1, the following expression for the local bond stress $u_f$ is obtained:

$$u_f = \frac{dM/dx}{(\sum o) z} \tag{8.3a}$$

Alternatively, in terms of the transverse shear force at the section $V = dM/dx$,

$$u_f = \frac{V}{(\sum o) z} \tag{8.3b}$$

It follows that flexural bond stress is high at locations of high shear, and that this bond stress can be effectively reduced by providing an increased number of bars of smaller diameter bars (to give the same equivalent $A_{st}$).

It may be noted that the actual bond stress will be influenced by *flexural cracking, local slip, splitting* and other secondary effects — which are not accounted for in

Eq. 8.3. In particular, flexural cracking has a major influence in governing the magnitude and distribution of local bond stresses.

### 8.2.1 Effect of Flexural Cracking on Flexural Bond Stress

From Eq. 8.3(b), it appears that the flexural (local) bond stress $u_f$ has a variation that is similar to and governed by the variation of the transverse shear force *V*. In fact, it would appear that in regions of constant moment, where shear is zero, there would be no bond stress developed at all. However, this is not true. The tensile force *T* in the reinforcement varies between flexural crack locations, even in regions of constant moment, as indicated in Fig. 8.2. At the flexural crack location, the tension is carried by the reinforcement alone, whereas in between the cracks, concrete carries some tension and thereby partially relieves the tension in the steel bars. As local bond stress is proportional to the rate of change of bar force [Eq. 8.2], local bond stresses do develop in such situations.

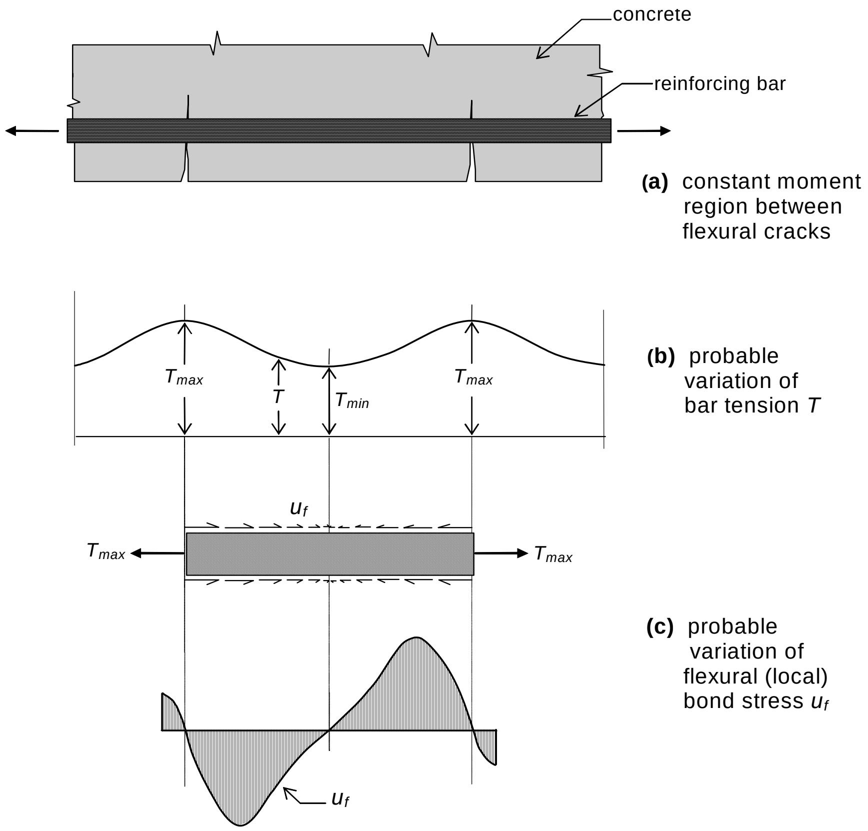



**Fig. 8.2** Effect of flexural cracks on flexural bond stress in constant moment region

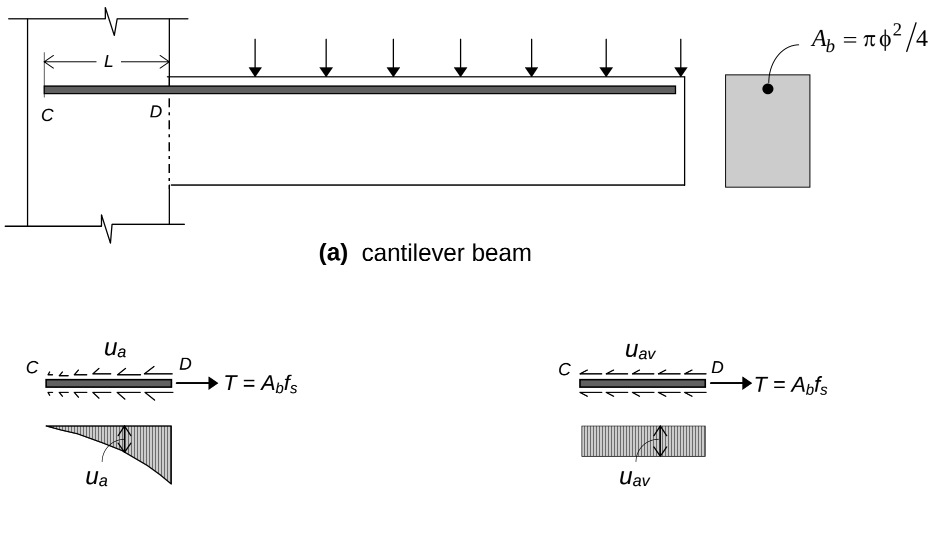
The bond stresses follow a distribution somewhat like that shown in Fig. 8.2(c), with the direction of the bond stress reversing between the cracks [Ref. 8.1]. The net bond force between the cracks will, of course, be zero in a region of constant moment. When the moment varies between the flexural cracks, the bond stress distribution will differ from that shown in Fig. 8.2(c), such that the net bond force is equal to the unbalanced tension in the bars between the cracks [Eq. 8.2].

Beam tests show that longitudinal splitting cracks tend to get initiated near the flexural crack locations where the local peak bond stresses can be high. The use of large diameter bars particularly renders the beam vulnerable to splitting and/or local slip.

Finally, it may be noted that flexural cracks are generally not present in the compression zone. For this reason, flexural bond is less critical in a compression bar, compared to a tension bar with an identical axial force.

## 8.3 ANCHORAGE (DEVELOPMENT) BOND

As mentioned earlier, *anchorage bond* or *development bond* is the bond developed near the extreme end (or cut-off point) of a bar subjected to tension (or compression). This situation is depicted in the cantilever beam of Fig. 8.3, where it is seen that the tensile stress in the bar segment varies from a maximum ($f_s$) at the continuous end $D$ to practically zero at the discontinuous end $C$.




**(b)** probable variation of anchorage bond stress $u_a$

**(c)** assumed uniform average bond stress $u_{ar}$

**Fig. 8.3** Anchorage bond stress

The bending moment, and hence the tensile stress $f_s$, are maximum at the section at *D*. Evidently, if a stress $f_s$ is to be developed in the bar at *D*, the bar should not be terminated at *D*, but has to be extended ('anchored') into the column by a certain length *CD*. At the discontinuous end *C* of the bar, the stress is zero. The difference in force between *C* and *D* is transferred to the surrounding concrete through anchorage bond. The probable variation of the *anchorage bond stress* $u_a$ is as shown in Fig. 8.3(b) — with a maximum value at *D* and zero at *C*. It may be noted that a similar (but not identical[†]) situation exists in the bar segment *CD* of the simply supported beam in Fig. 8.1(e).

An expression for an *average* bond stress $u_{av}$ can be derived by assuming a uniform bond stress distribution over the length *L* of the bar of diameter $\phi$ [Fig. 8.3(c)], and considering equilibrium of forces as given below:

$$(\pi\phi L)u_{av} = \left(\pi\,\phi^2/4\right)f_s \quad \Rightarrow u_{av} = \frac{\phi f_s}{4L} \tag{8.4}$$

This bond stress may be viewed as the average bond stress generated over a length *L* in order to *develop* a maximum tensile (or compressive) stress $f_s$ at a critical section; hence, this type of bond is referred to as 'development bond'. Alternatively, this bond may be viewed as that required to provide *anchorage* for a critically stressed bar; hence, it is also referred to as 'anchorage bond'.

### 8.3.1 Development Length

The term *development length* has already been introduced in Section 5.9.2, in relation to restrictions on theoretical bar cut-off points. The concept of 'development length' is explained in the Code as follows:

> "The calculated tension or compression in any bar at any section shall be developed on each side of the section by an appropriate development length or end anchorage or by a combination thereof" [Cl. 26.2].

The concept underlying 'development length' is that a certain minimum length of the bar is required on either side of a point of maximum steel stress, to prevent the bar from pulling out under tension (or pushing in, under compression). However, when the required bar embedment cannot be conveniently provided due to practical difficulties, *bends*, *hooks* and *mechanical anchorages* can be used to supplement with an equivalent embedment length [refer Section 8.5.3]. The term *anchorage length* is sometimes used in lieu of 'development length' in situations where the embedment portion of the bar is not subjected to any flexural bond [Fig. 8.3].

The expression given in the Code (Cl. 26.2.1) for 'development length' $\tilde{L}_d$ [‡] follows from Eq. 8.4:

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[†] It can be seen that in the case shown in Fig. 8.1(e), flexural bond coexists with anchorage bond, owing to variation of bending moment in the segment CD.

[‡] In this book, the notation $L_d$ is reserved for development length of fully stressed bars ($f_s = 0.87\,f_y$). For $f_s < 0.87\,f_y$, evidently $\tilde{L}_d = L_d\left(f_s/0.87f_y\right)$

$$\tilde{L}_d = \frac{\phi f_s}{4\tau_{bd}} \tag{8.5}$$

where $\tau_{bd}$ is the 'design bond stress', which is the permissible value of the average anchorage bond stress $u_a$. The values specified for $\tau_{bd}$ (Cl. 26.2.1.1 of the Code) are 1.2 MPa, 1.4 MPa, 1.5 MPa, 1.7 MPa and 1.9 MPa for concrete grades M 20, M 25, M 30, M 35 and M 40 and above respectively for *plain bars in tension*, with an increase of 60 percent for *deformed bars in tension*, and a further increase of 25 percent for *bars in compression*.

The development length requirements in terms of $L_d/\phi$ ratios for fully stressed bars ($f_s$ = 0.87 $f_y$) of various grades of steel in combination with various grades of concrete are listed in Table 5.6. It may be noted that when the area of steel $A_s$ actually provided is in excess of the area required (for $f_s$ = 0.87$f_y$), then the actual development length required $\tilde{L}_d$ may be proportionately reduced [Ref. 8.5]:

$$\tilde{L}_d = L_d \times \frac{(A_s)_{reqd}}{(A_s)_{provided}} \tag{8.6}$$

In the case of bundled bars, the Code specifies that the *"development length of each bar of bundled bars shall be that for the individual bar, increased by* 10 *percent for two bars in contact,* 20 *percent for three bars in contact and* 33 *percent for four bars in contact."* Such an increase in development length is warranted because of the reduction in anchorage bond caused by the reduced interface surface between the steel and the surrounding concrete.

## 8.4 BOND FAILURE AND BOND STRENGTH

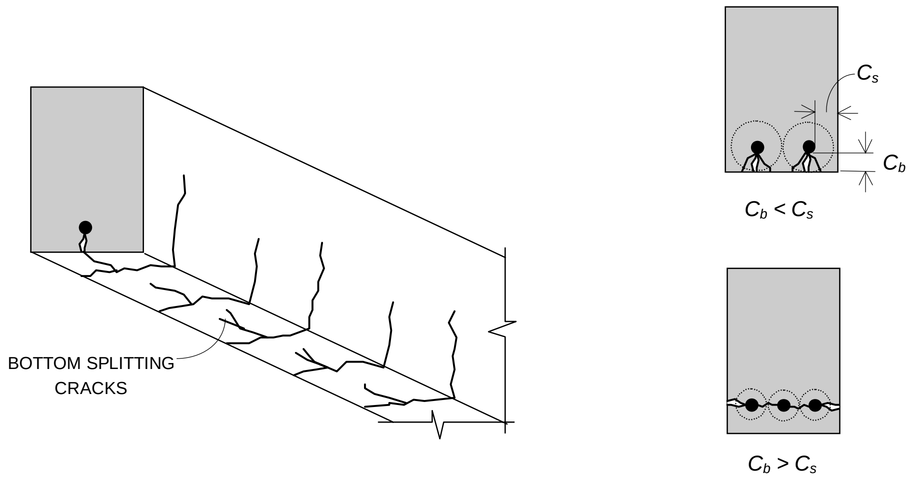
### 8.4.1 Bond Failure Mechanisms

The mechanisms that initiate bond failure may be any one or combination of the following:

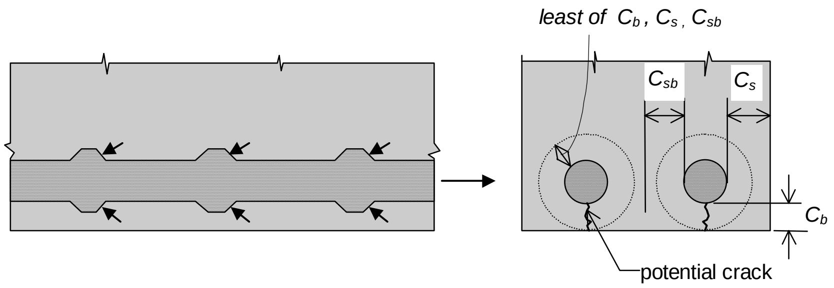
- break-up of adhesion between the bar and the concrete;
- longitudinal splitting of the concrete around the bar;
- crushing of the concrete in front of the bar ribs (in deformed bars); and
- shearing of the concrete keyed between the ribs along a cylindrical surface surrounding the ribs (in deformed bars).

The most common type of bond failure mechanism is the pulling loose of the reinforcement bar, following the *longitudinal splitting* of the concrete along the bar embedment [Fig. 8.4]. Occasionally, failure occurs with the bar pulling out of the concrete, leaving a circular hole without causing extensive splitting of the concrete. Such a failure may occur with plain smooth bars placed with large cover, and with very small diameter deformed bars (wires) having large concrete cover [Ref. 8.1]. However, with deformed bars and with the normal cover provided in ordinary beams, bond failure is usually a result of longitudinal splitting.

In the case of ribbed bars, the bearing pressure between the rib and the concrete is inclined to the bar axis [Fig. 8.4(b)]. This introduces radial forces in the concrete ('wedging action'), causing circumferential tensile stresses in the concrete surrounding the bar (similar to the stresses in a pipe subjected to internal pressure) and tending to split the concrete along the weakest plane [Ref. 8.2 – 8.4]. Splitting occurs along the thinnest surrounding concrete section, and the direction of the splitting crack ('bottom splitting' or 'side splitting') depends on the relative values of the bottom cover, side cover and bar spacing as shown in Fig. 8.4(b).




**(a)** bottom and side splitting cracks




**(b)** splitting forces with deformed

**Fig. 8.4** Typical bond splitting crack patterns

Splitting cracks usually appear on the surface as extensions of flexural or diagonal tension cracks in flexural members, beginning in regions of high local bond stress [Fig. 8.2]. With increased loads, these cracks propagate gradually along the length of embedment ('longitudinal splitting') with local splitting at regions of high local bond stress and associated redistribution of bond stresses. It is found that in a normal beam, local splitting can develop over 60 – 75 percent of the bar length without loss of average bond strength and without adversely affecting the load-carrying capacity of the beam [Ref. 8.1]. The presence of stirrups offers resistance to the propagation of continuous longitudinal splitting cracks [Fig. 8.5]. However, in beams without stirrups, the failure due to bond can occur early and suddenly, as the longitudinal split runs through to the end of the bar without the resistance offered by the stirrups.

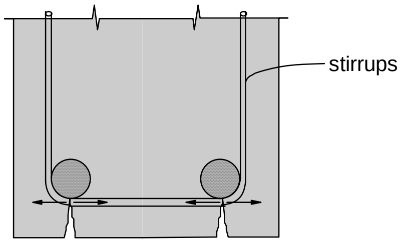



**Fig. 8.5** Stirrups resisting tensile forces due to bond

A simply supported beam can act as a two-hinged arch, and so carry substantial loads, even if the bond is destroyed over the length of the bar, provided the tension bars are suitably anchored at their ends [Fig. 8.6]. However, the deflections and crackwidths of such a beam may be excessive. The anchorage may be realised over an adequate length of embedment beyond the face of the support, and/or by bends and hooks or mechanical anchorages (welded plates, nuts and bolts, etc.).

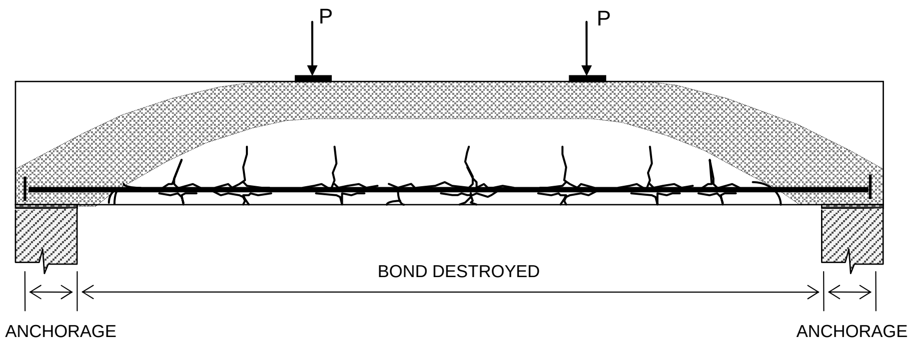
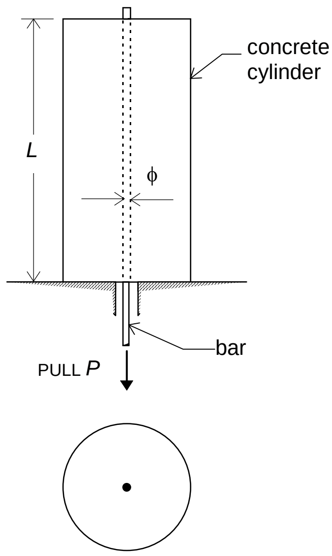



**Fig. 8.6** Tied-arch action with bar anchorage alone

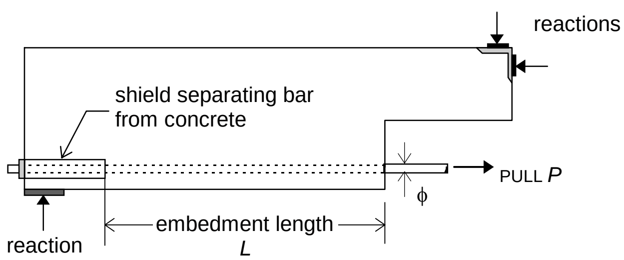
### 8.4.2 Bond Tests

Bond strength is usually ascertained by means of *pull out* tests or some sort of beam tests.

The typical 'pull out' test is shown schematically in Fig. 8.7(a). A bar embedded in a concrete cylinder or prism is pulled until failure occurs by splitting, excessive slip or pull out. The *nominal bond strength* is computed as $P/(\pi\phi L)$, where $P$ is the pull at failure, $\phi$ the bar diameter and $L$ the length of embedment. It may be noted, however, that factors such as cracking (flexural or diagonal tension) and dowel forces, which lower the bond resistance of a flexural member, are not present in a concentric pull out test. Moreover, the concrete in the test specimen is subjected to a state of compression (and not tension), and the friction at the bearing on the concrete offers some restraint against splitting. Hence, the bond conditions in a pull out test do not ideally represent those in a flexural member.




**(a)** 'pull out' test set up




**(b)** bond test on modified cantilever specimen

**Fig. 8.7** Bond tests

Of the several types of beam tests developed to simulate the actual bond conditions, one test set-up is shown in Fig. 8.7(b) [Ref. 8.3]. The bond strength measured from such a test, using the same expression [ $P/(\pi\phi L)$ ] as for a pull-out test, is bound to give a lesser (and more accurate) measure of the bond strength than the pull out strength. However, the pull out test is easier to perform, and for this reason, more commonly performed.

From the results of such bond tests, the 'design bond stress' (permissible average anchorage bond stress), $\tau_{bd}$ ,is arrived at — for various grades of concrete. Tests indicate that bond strength varies proportionately with $\sqrt{f_{ck}}/\phi$ for small diameter bars and $\sqrt{f_{ck}}$ for large diameter deformed bars [Ref. 8.1].

### 8.4.3 Factors Influencing Bond Strength

Bond strength is influenced by several factors, some of which have already been mentioned. In general, bond strength is enhanced when the following measures are adopted:

- deformed (ribbed) bars are used instead of *plain* bars;
- smaller bar diameters are used;
- higher grade of concrete (improved tensile strength) is used;
- increased cover is provided around each bar;
- increased length of embedment, bends and /or hooks are provided;
- mechanical anchorages are employed;
- stirrups with increased area, reduced spacing and/or higher grade of steel are used;
- termination of longitudinal reinforcement in tension zones is avoided;
- any measure that will increase the confinement of the concrete around the bar is employed.

Another factor which influences bond strength in a beam is the depth of fresh concrete below the bar during casting. Water and air inevitably rise towards the top of the concrete mass and tend to get trapped beneath the horizontal reinforcement, thereby weakening the bond at the underside of these bars. For this reason, codes specify a lower bond resistance for the top reinforcement in a beam.

## 8.5 REVIEW OF CODE REQUIREMENTS FOR BOND

### 8.5.1 Flexural Bond

Traditionally, design for bond required the consideration of both *flexural (local) bond stress* $u_f$ and *development* (*anchorage*) *bond stress* $u_{av}$. However, since the 1970s, there has been an increased awareness of the fact that the exact value of flexural bond stress cannot be accurately computed (using expressions like Eq. 8.3a or 8.3b) owing to the unpredictable and non-uniform distribution of the actual bond stress. In fact, it is found that localised bond failures can and do occur, despite the checks provided by Eq. 8.3. However, as explained earlier, these local failures do not impair the strength of the beam (in terms of ultimate load-carrying capacity) provided the bars are adequately anchored at their ends [Fig. 8.6]. Thus, the concept underlying limit state design for flexural bond has shifted from the control of local bond stresses (whose predicted values are unrealistic) to the development of required bar stresses through provision of adequate *anchorage* — at simple supports and at bar cut-off points.

The Code requirement for such a check on anchorage, in terms of *development length* plus *end anchorage* and the variation of tensile stress in the bar [Eq. 5.25], has already been discussed at length in Section 5.9.3, and illustrated with several examples.

### 8.5.2 Development (Anchorage) Bond

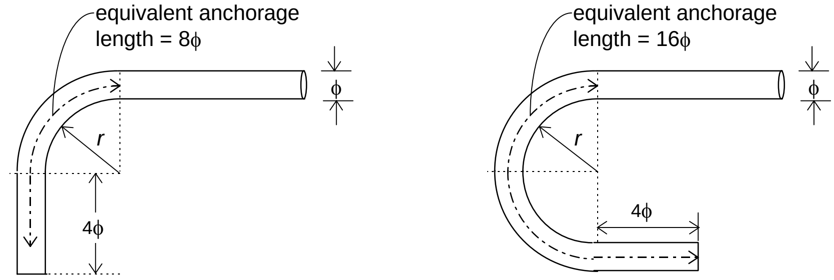
The computed stress at *every section* of a reinforcing bar (whether in tension or compression) must be developed on *both* sides of the section — by providing adequate 'development length', $\tilde{L}_d$ . Such development length is usually available near the midspan locations of normal beams (where sagging moments are generally maximum) and support locations of continuous beams (where hogging moments are generally maximum). Special checking is generally called for in the following instances only:

- in flexural members that have relatively short spans;
- at simple supports and points of inflection [refer Section 5.9.3];
- at points of bar cut-off [refer Section 5.9.3];
- at cantilever supports;
- at beam-column joints in lateral load resisting frames;
- for stirrups and transverse ties; and
- at *lap splices* [refer Section 8.6].

### 8.5.3 Bends, Hooks and Mechanical Anchorages

*Bends*, conforming to standards are frequently resorted to in order to provide *anchorage*, contributing to the requirements of development length of bars in tension or compression. The Code (Cl. 26.2.2.1) specifies that *"the anchorage value of a bend shall be taken as* 4 *times the diameter of the bar for each* 45° *bend, subject to a maximum of* 16 *times the diameter of the bar*".

Commonly a 'standard 90° bend' (anchorage value = $8\phi$) is adopted [Fig. 8.8(a)], including a minimum extension of $4\phi$. Any additional extension beyond the bend also qualifies to be included in development length calculations. However, for bars in compression (as in column bases), it is doubtful whether such extensions can meaningfully provide anchorage. The 90° bend itself is very effective in compression as it transfers part of the force by virtue of bearing stresses, and prevents the bar from punching through the concrete cover. When the bend is turned around 180° (anchorage value = $16\phi$) and extended beyond by $4\phi$, it is called a *standard U-type hook* [Fig. 8.8(b)]. The minimum (internal) turning radius ($r$ in Fig. 8.8) specified for a hook is $2\phi$ for plain mild steel bars and $4\phi$ for cold-worked deformed bars [Ref. 8.5]. Hooks are generally considered mandatory for *plain* bars in tension [refer Cl. 26.2.2.1a of the Code].




**(a)** standard 90° bend **(b)** standard U–type hook

**Fig. 8.8** Anchorage lengths of standard bends and hooks

In the case of stirrup (and transverse tie) reinforcement, the Code (Cl. 26.2.2.4b) specifies that *complete anchorage shall be deemed to have been provided* if any of the following specifications is satisfied:

- 90° bend around a bar of diameter not less than the stirrup diameter $\phi$, with an extension of at least 8 $\phi$;
- 135° bend with an extension of at least 6 $\phi$;
- 180° bend with an extension of at least 4 $\phi$.

It may be noted that bends and hooks introduce bearing stresses in the concrete that they bear against. To ensure that these bearing stresses are not excessive, the turning radius $r$ (in Fig. 8.8) should be sufficiently large. The Code (Cl. 26.2.2.5) recommends a check on the bearing stress $f_b$ inside any bend, calculated as follows:

$$f_b = \frac{F_{bt}}{r\phi} \tag{8.7}$$

where $F_{bt}$ is the design tensile force in the bar (or group of bars), $r$ the internal radius of the bend, and $\phi$ the bar diameter (or size of bar of equivalent area in case of a bundle). The calculated bearing stress should not exceed a limiting bearing stress, given by $\dfrac{1.5 f_{ck}}{1 + 2\phi/a}$, where $a$ is the centre-to-centre spacing between bars perpendicular to the bend, or, in the case of bars adjacent to the face of the member, the clear cover plus the bar diameter $\phi$. For fully stressed bars,

$$F_{bt} = 0.87 f_y \left(\frac{\pi\phi^2}{4}\right)$$

Accordingly, it can be shown that the limiting radius is given by

$$r \geq 0.456\phi \left(\frac{f_y}{f_{ck}}\right)\left(1 + \frac{2\phi}{a}\right) \tag{8.8}$$

Mechanical anchorages in the form of welded plates, nuts and bolts, etc. can be used, provided they are *capable of developing the strength of the bar without damage to concrete* (Cl. 26.2.2.3 of the Code). In general, the effectiveness of such devices must be ascertained through tests.

## 8.6 SPLICING OF REINFORCEMENT

Splices are required when bars placed short of their required length (due to non-availability of longer bars) need to be extended. Splices are also required when the bar diameter has to be changed along the length (as is sometimes done in columns). The purpose of 'splicing' is to transfer effectively the axial force from the terminating bar to the connecting (continuing) bar with the same line of action at the junction. This invariably introduces stress concentrations in the surrounding concrete. These effects should be minimised by:

- using proper splicing techniques;
- keeping the splice locations away from sections with high flexural/shear stresses; and
- staggering the locations of splicing in the individual bars of a group (as, typically in a column).

The Code recommends that

> "splices in flexural members should not be at sections where the bending moment is more than 50 percent of the moment of resistance; and not more than half the bars shall be spliced at a section" (Cl. 26.2.5).

When splicing in such situations becomes unavoidable, special precautions need to be employed, such as

- increasing the length of lap (in *lap splices* and *lap welding*);
- using spirals or closely-spaced stirrups around the length of the stirrups.

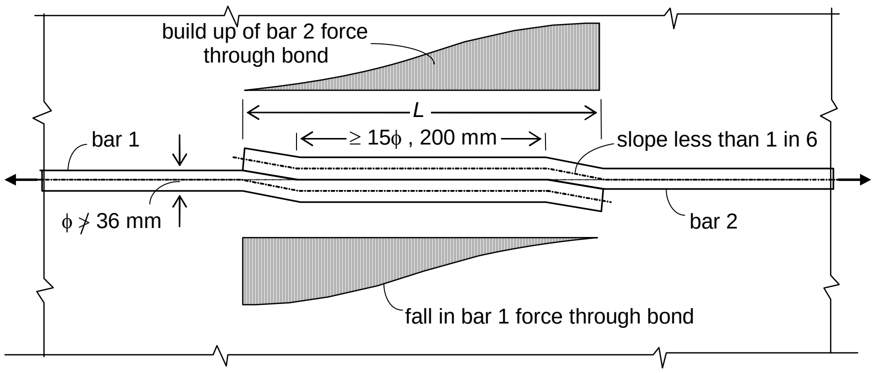
Splicing is generally done in one of the following three ways:

1. Lapping of bars (*lap splice*)
2. Welding of bars (*welded splice*)
3. Mechanical connection.

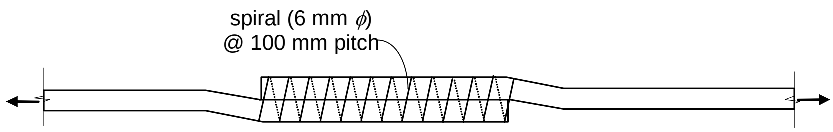
### 8.6.1 Lap Splices

*Lap splices* are achieved by overlapping the bars over a certain length, thereby enabling the transfer of axial force from the terminating bar to the connecting bar through the mechanism of anchorage (development) bond with the surrounding concrete — [Fig. 8.9(a)]. The cracking and splitting behaviour observed in *lap splice tests* are found to be similar to those in *anchorage bond tests* [Ref. 8.2].

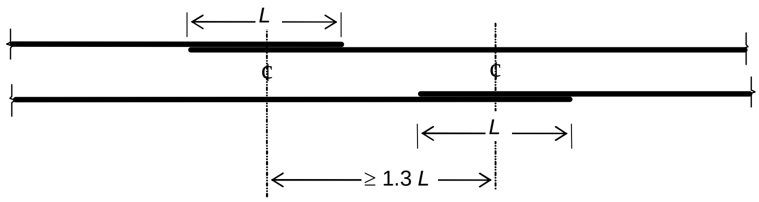
Lap splices are usually not permitted for very large diameter bars ($\phi > 36$ mm), for which *welded splices* are recommended. However, where welding is not practicable, the Code (Cl. 26.2.5.1a) permits lap splices with additional *spirals* around the lapped bars [Fig. 8.9(b)].




**(a)** lap splice action through development bond




**(b)** use of spirals in lap splices for large diameter bars




**(c)** staggered splicing of bars

**Fig. 8.9** Lap splices

It is desirable to bend the bars slightly (particularly large diameter bars) near the splice location in order to ensure a collinear transfer of force (without eccentricity), as shown in Fig. 8.9(a). The Code specifies that the straight length of the lap should not be less than $15\phi$ or 200 mm. As the force transfer is through development bond, the lap length should at least be equal to the *development length* $\tilde{L}_d$. The Code (Cl. 26.2.5.1c) specifies a lap length of $2\tilde{L}_d$ in situations where the member is subjected to direct tension. In no case should the lap length be less than $30\phi$ under flexural or direct tension and $24\phi$ under compression[†]. When bars of two different diameters are to be spliced, the lap length should be calculated on the basis of the smaller diameter. Splices in tension members shall be enclosed in spirals made of bars not less than 6 mm diameter with pitch not more than 10mm.

In the revised Code, some additional clauses have been incorporated (Cl. 26.2.5.1c) to account for the reduction in bond strength with regard to rebars located near the top region [refer Section 8.4.3[‡]]. When lapping of tension reinforcement is required at the top of a beam (usually near a continuous support location or a beam-column junction) and the clear cover is less than twice the diameter of the lapped bar, the lapped length should be increased by a factor of 1.4. If the rebar is required to turn around a corner (as in an exterior beam-column junction), the lapped length should be increased by a factor of 2.0. This factor may be limited to 1.4 in the case of corner bars when the clear cover on top is adequate but the side cover (to the vertical face) is less than twice the diameter of the lapped bar.

When more than one bar requires splicing, care must be taken to ensure that the splicing is staggered, with a minimum (centre-to-centre) separation of 1.3 times the lap length, as indicated in Fig. 8.9(c). It is also desirable to provide (extra) transverse ties (especially in columns), connecting the various longitudinal bars in the spliced region. In the case of *bundled bars*, the lap length should be calculated considering the increased $\tilde{L}_d$ [refer Section 8.3.1], and the individual splices within a bundle should be staggered.

### 8.6.2 Welded Splices and Mechanical Connections

*Welded splices* and *mechanical connections* are particularly suitable for large diameter bars. This results in reduced consumption of reinforcing steel. It is desirable to subject such splices to *tension tests* in order to ensure adequacy of strength [refer Cl. 12.4 of the Code]. Welding of cold-worked bars needs special precautions owing to the possibility of a loss in strength on account of welding heat [Ref. 8.6]. The Code (Cl. 26.2.5.2) recommends that the design strength of a welded splice should in general be limited to 80 percent of the design strength of the bar for tension splices.

*Butt welding* of bars is generally adopted in welded splices. The bars to be spliced should be of the same diameter. Additional two or three symmetrically positioned small diameter lap bars may also be provided (especially when the bars are subjected to tension) and fillet welded to the main bars. Even in the case of 'lap splices', *lap welding* (at intervals of $5\phi$) may be resorted to in order to reduce the lap length.

*End-bearing splices* are permitted by the Code (Cl. 26.2.5.3) for bars subject to compression. This involves *square cutting* the ends of the bars and welding the bar ends to suitable bearing plates that are embedded within the concrete cover.

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[†] Columns are frequently subjected to compression combined with bending. Hence, it may be prudent to calculate the lap length, assuming *flexural tension*, and *not* compression.

[‡] As explained in Section 8.4.3, this reduction in bond strength is strictly applicable for all situations, not only lapping. It may also be noted that the reduction in bond strength of top reinforcement may not be significant in shallow members.

## 8.7 DESIGN EXAMPLES

### EXAMPLE 8.1

Check the adequacy of the anchorage provided for the longitudinal bars in the cantilever beam shown in Fig. 8.10 and suggest appropriate modifications, if required. The beam is subjected to a uniformly distributed factored load of 100 kN (total, including self-weight). Assume M 20 concrete and Fe 415 steel, deformed bars.

### SOLUTION

**Preliminary check on anchorage length**

- Assuming the bars are fully stressed at the location of maximum moment (i.e., face of column support), full development length $L_d$ is required for anchorage of the bars inside the column, beyond this section.

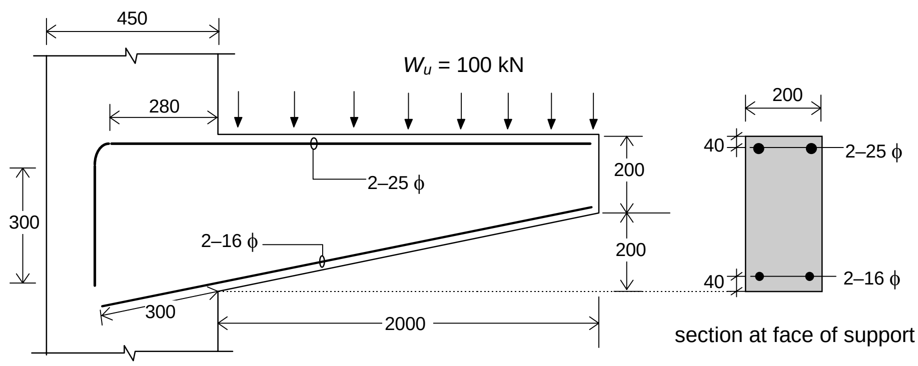



**Fig. 8.10** Example 8.1

- *For the tension bars* (2 – 25 $\phi$ at top), $L_d = \left(\frac{0.87 f_y}{4\tau_{bd}}\right)\phi = 47\,\phi$

  [This follows from $f_y$ = 415 MPa and $\tau_{bd}$ = 1.2 MPa × 1.6 for M 20 concrete with Fe 415 steel deformed bars; $L_d/\phi$ ratios are also listed in Table 5.4]

  $\Rightarrow L_d = 47 \times 25 = 1175$ mm.

  Actual anchorage length provided (including effect of the 90° bend and extension of bar beyond bend + 4ϕ minimum extension) = 280 + (4 × 25) + 300 = 680 mm

  $< L_d = 1175$ mm $\Rightarrow$ Not OK.

- *For the compression bars* (2 – 16 $\phi$ at bottom),

  $\tau_{bd}$ can be increased by 25 percent, whereby $L_d = (47\,\phi) \times 0.8$

  $\Rightarrow L_d = 47 \times 16 \times 0.8 = 602$ mm.

  Actual anchorage length provided = 300 mm.

  $< L_d = 602$ mm $\Rightarrow$ Not OK.

- Before providing increased anchorage length, it is desirable to verify whether the bars are fully stressed under the given loading, and to make more precise development length $\left(\tilde{L}_d\right)$ calculations [refer Eq. 8.6].

**Actual anchorage length required**

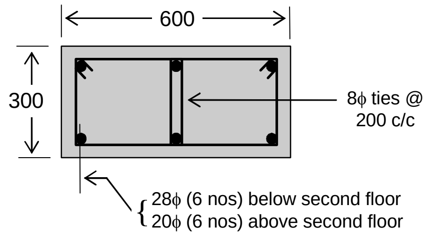
- *Maximum factored moment at the critical section* :
  $M_u = W_u \times L/2$ = 100 kN × (2.0 m)/2 = 100kNm
  $b$ = 200 mm, $d$ = 400 – 40 = 360 mm
  $\Rightarrow \dfrac{M_u}{bd^2} = \dfrac{100 \times 10^6}{200 \times 360^2}$ = 3.858 MPa > $M_{u,lim}$ = 0.1389 × 20
  = 2.778 MPa (for M 20)
  Hence, the section has to be doubly-reinforced.
- Using Design Aids [Table A.4 or SP : 16], with $d'/d$ = 40/360 = 0.11,
  $(p_t)_{reqd}$ = 1.30 $\Rightarrow$ $(A_{st})_{reqd}$ = (1.30/100) × (200 × 360) = 936 mm$^2$
  $(p_c)_{reqd}$ = 0.37 $\Rightarrow$ $(A_{sc})_{reqd}$ = (0.37/100) × (200 × 360) = 266 mm$^2$
  $(A_{st})_{provided}$ = 2 × 491 = 982 mm$^2$ > 932 mm$^2$
  $(A_{sc})_{provided}$ = 2 × 201 = 402 mm$^2$ > 266 mm$^2$
- Actual anchorage length required = $\tilde{L}_d = \dfrac{(A_s)_{reqd}}{(A_s)_{provided}} \times L_d$
  $\Rightarrow$ For the *tension* bars, $\tilde{L}_d = \dfrac{936}{982} \times 1175$ = 1120 mm > 680 mm provided.
  For the *compression* bars, $\tilde{L}_d = \dfrac{266}{402} \times 602$ = 398 mm > 300 mm provided.
  $\Rightarrow$ Not OK.

**Modifications proposed**

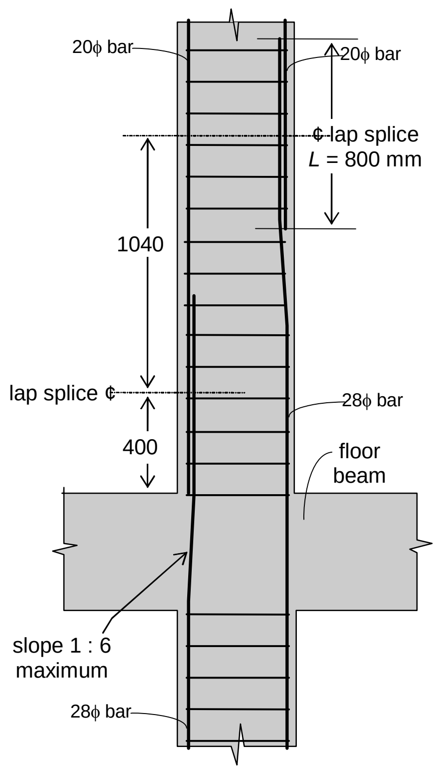
- It is desirable to reduce the anchorage length requirements by providing smaller diameter bars:
- For *tension* bars (at top), *provide* 3 – 20 mm $\phi$ (instead of 2 – 25 $\phi$):
  $\Rightarrow A_{st}$ = 3 × 314 = 942 mm$^2$ > 936 mm$^2$ reqd.
  $\tilde{L} \approx L_d$ = 47 × 20 = 940 mm > 680 mm provided
  $\Rightarrow$ <u>Extend the bars by 940 – 680 = 260 mm</u>.
- For *compression* bars (at bottom), *provide* 3 – 12 mm $\phi$ (instead of 2 – 16 $\phi$)
  $\Rightarrow A_{sc}$ = 3 × 113 = 339 mm$^2$ > 266 mm$^2$ reqd.
  $\tilde{L}_d = \dfrac{266}{339}$ × (47 × 12) × 0.8 = 354 mm > 300 mm provided.
  $\Rightarrow$ <u>Extend the bars by providing a standard 90$^o$ bend</u> (Additional anchorage obtained = 8 $\phi$ = 8 × 12 = 96 mm). — Hence OK.

## EXAMPLE 8.2

The plan of a ground floor column in a building is shown in Fig. 8.11(a). It is desired to reduce the longitudinal bar diameter from 28 mm to 20 mm above the second floor level. Design and detail a suitable *lap splice*. Assume M 25 concrete and Fe 415 steel.




**(a)** plan of column




**(b)** typical lap slice detail
(only two bars shown here)

**Fig. 8.11** Example 8.2

**SOLUTION**

- As the column is subjected to compression combined with flexure, some of the bars may be under tension. Hence, the lap length should be calculated, assuming tension, i.e., $L = L_d$ or 30 $\phi$, whichever is greater. Moreover, at the splice location, as the smaller diameter (20 mm $\phi$) bars are adequate in providing the derived strength, the lap length calculation should be based on the smaller diameter:

  $$L = \begin{cases} L_d = \dfrac{0.87 \times 415\phi}{4 \times (1.4 \times 1.25)} = 52\phi \quad \text{(greater)} \\ 30\phi \end{cases} \Rightarrow L = 40\phi$$

  $\Rightarrow L = 40 \times 20 = 800$ mm.

- *Staggered splicing*:
  As required by the Code (Cl. 25.2.5.1), the splicing of the bars should be ideally staggered with a minimum (centre-to-centre) separation of
  $1.3L = 1.3 \times 800 = 1040$ mm.
  The splice detail is shown in Fig. 8.11(b).

## REVIEW QUESTIONS

8.1 How is the assumption that *plane sections remain plane even after bending* related to 'bond' in reinforced concrete?

8.2 What are the mechanisms by which bond resistance is mobilised in reinforced concrete?

8.3 Explain clearly the difference between *flexural bond* and *development bond.*

8.4 There is no direct check on *flexural bond stress* in the present Code. Comment on this.

8.5 Define 'development length'. What is its significance?

8.6 Briefly describe the various bond failure mechanisms.

8.7 How is *bond strength* of concrete measured in the laboratory?

8.8 Enumerate the main factors that influence *bond strength*.

8.9 Can there be a difference in the bond resistance of identical bars placed at the *top* and *bottom* of a beam? If so, why? Does the current Code IS 456 recognise this in (i) development length, (ii) lap splice?

8.10 Briefly describe the situations where a check on *development bond* is called for.

8.11 What is the most effective way of reducing the development length requirement of bars in tension?

8.12 What is the criterion for deciding the minimum turning radius in a bend in a reinforcing bar?

8.13 Determine the minimum internal radius at a bend in a 20 mm $\phi$ bar of Fe 415 grade in concrete of grade M 20. Assume that the centre-to-centre spacing of bars normal to the bend is 100 mm.

8.14 What is the purpose of *splicing* of reinforcement? What are the different ways by which this can be achieved?

8.15 Why is a *welded splice* generally preferred to a *lap splice*?

8.16 How would you decide on the location of a splice in the tension reinforcement in a flexural member?

8.17 When there is a marked change in the direction of a bar, care must be taken to ensure that the resultant force in the bar at the bending point does not have a component that tends to break the concrete cover. Cite a suitable example and suggest detailing measures to solve such a problem.

## PROBLEMS

8.1 The outline of a typical (exterior) beam-column joint is shown in Fig. 8.12. The maximum factored moment in the beam at the face of the column is found to be 350 kNm (hogging) under gravity loads. Design the flexural reinforcement in the beam at this critical section, and determine the desired anchorage for the reinforcement. Mark the reinforcement and anchorage details in Fig. 8.12. Assume M 25 concrete and Fe 415 steel.

8.2 In the case of the beam of Problem 8.1 [Fig. 8.12], it is seen that under lateral (wind) loads combined with gravity loads, the maximum factored design moments are obtained as 350 kNm (hogging) or 150kNm (sagging). Does the earlier design (solution to Problem 8.1) need any modification? Detail the modification, if any.

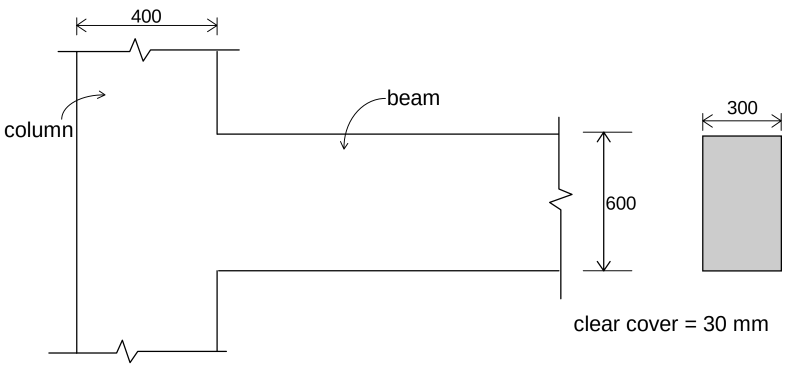



**Fig. 8.12** Problem 8.1

8.2 In a reinforced concrete tension member, a 16 mm $\phi$ bar has to be lap spliced with a 20 mm $\phi$ bar. Assuming M 20 concrete and Fe 415 steel, design a suitable *lap splice.*

8.3 The plan of a column (with 4–25ϕ bars plus 4–20ϕ bars) at a certain level in a multi-storeyed building is shown in Fig. 8.13. It is desired to reduce the longitudinal bar sizes from 25 mm to 20 mm, and from 20 mm to 16 mm, above the next floor level. Design and detail suitable lap splices. Assume M 25 concrete and Fe 415 steel.

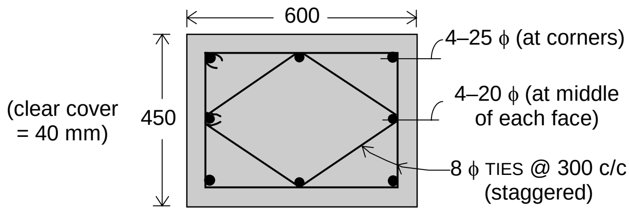



**Fig. 8.13** Problem 8.4

## REFERENCES


8.1 ACI Committee 408, *Bond Stress — The State of the Art*, Journal ACI, Vol.63, Oct.1966, pp1161–1190.

8.2 Orangun, C.O., Jirsa, J.O. and Breen, J.E., *A Re-evaluation of Test Data on Development Length and Splices*, Journal ACI, Vol.74, March 1977, pp114–122.

8.3 Kemp, E.L. and Wilhelm, W,J., *Investigation of the Parameters Influencing Bond Cracking*, Journal ACI, Vol.76, Jan.1979, pp 47–71.

8.4 Jimenez, R., White, R.N. and Gergely, P., *Bond and Dowel Capacities of Reinforced Concrete*, Journal ACI, Vol. 76, Jan.1979, pp 73–92.

8.5 — *Handbook on Concrete Reinforcement and Detailing*, Special Publication SP 34, Bureau of Indian Standards, New Delhi, 1987.

8.6 — *Recommendations for Welding Cold Worked Bars for Reinforced Concrete Construction* (first revision), IS 9417 : 1989, Bureau of Indian Standards, New Delhi, 1989.

# 9

# Analysis for Design Moments in Continuous Systems

## 9.1 INTRODUCTION

The behaviour and design of flexural members subjected to given bending moments were covered in Chapters 4 and 5. In the design procedure, it was assumed implicitly that *the distribution of bending moments along the length of the beam is known from structural analysis* (as stated in Section 4.1.2). It is not within the scope of this book to describe detailed and exact methods of analysis of indeterminate structures. However, there are many approximations, assumptions and simplified procedures permitted by codes, which assist in the analysis of indeterminate structures and these are discussed here. In particular, this chapter deals with the following aspects:

- gravity load patterns for maximum moments;
- simplified (approximate) methods of analysis;
- proportioning of member sizes for preliminary design;
- estimation of stiffnesses of frame elements;
- adjustments in calculated moments at beam-column junctions; and
- inelastic analysis and moment redistribution.

### 9.1.1 Approximations in Structural Analysis

In a typical reinforced concrete building [refer Section 1.6], the structural system is quite complex. The structure is three-dimensional, comprising floor slabs, beams, columns and footings, which are monolithically connected and act integrally to resist gravity loads (dead loads, live loads) and lateral loads (wind loads, seismic loads). The nature of the load transfer mechanisms [Fig. 1.8] and the various load combinations (as well as the associated load factors) have already been discussed in Sections 1.6 and 3.6 respectively.

### *Gravity Load Patterns*

From the viewpoint of designing for the limit state in flexure, what is essentially required is the distribution of the maximum ('positive' as well as 'negative') bending moments (*moment envelope*) under the 'worst' combination of *factored loads*. This is a problem of structural analysis, and for this purpose, the Code (Cl. 22.1) recommends that *all structures may be analysed by the linear elastic theory to calculate internal actions produced by design loads*. In order to simplify the analysis, the effects of *gravity loads* and *lateral loads* may be considered separately and their results superimposed†, after applying appropriate weighting factors (load factors), to give the design *factored moments* (as well as shear forces, axial forces and twisting moments). In the case of *live loads*, special loading patterns have to be considered so that the loads are so placed as to produce the worst effects at the design section considered. These loading patterns, as well as the related Code simplifications, are discussed in Section 9.2.

### *Simplified Methods of Frame Analysis*

In order to simplify the analysis, the three-dimensional framed structure is generally divided into a series of independent parallel *plane frames* along the column lines in the longitudinal and transverse directions of the building, as shown in Fig. 9.1. To analyse the gravity load effects, these plane frames may further be simplified into continuous beams or partial frames. The Code also permits the use of certain moment and shear coefficients for continuous beams, which directly give the design moments and shear forces. Simplified methods are also suggested in the Code for analysing plane frames subjected to gravity loads as well as lateral‡ loads. These methods are described in Section 9.3.

Unless the structure is very unsymmetrical or very tall or of major importance, the simplified methods of analysis are usually adequate. In such cases, rigorous analyses can be avoided. From a practical design viewpoint, it should be appreciated that when considerable uncertainties exist with regard to the loads and material properties (inputs of structural analysis), the very concept of an $n^{\text{th}}$ order accuracy in the computation of design moments is questionable. A good designer is, therefore, one who makes sensible decisions regarding the modelling and analysis of the structural system, depending on its behaviour and complexity. With the increasing availability of digital computers as well as software packages (finite element method based), the modern trend is to let the computer do all the analyses as a 'black box', with as much rigour as is available with the software. However, the results of such analyses are highly dependent on the input data fed into the computer, and errors in the input — conceptual or numerical — may lead to disastrous results. The approximate (manual)

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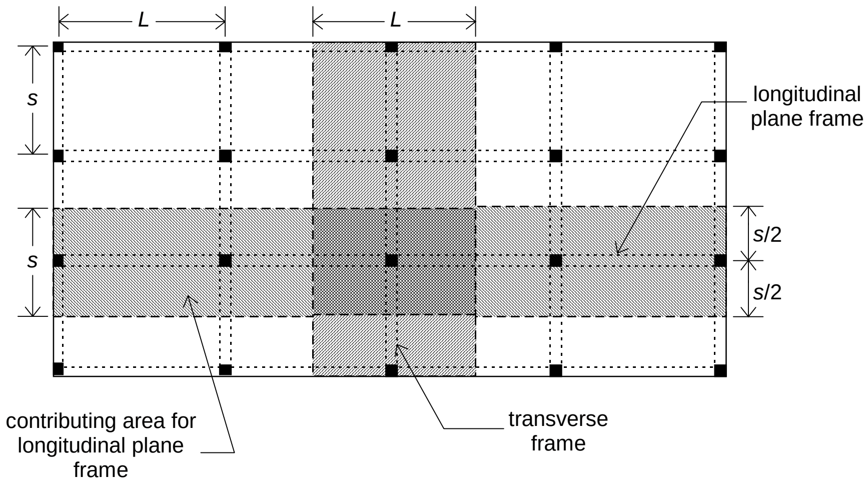
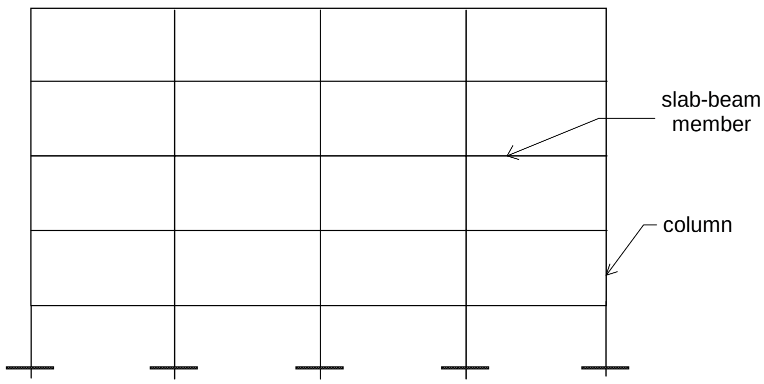
† The *principle of superposition* is strictly applicable only in a 'first order' analysis, where the structure is assumed to behave in a linear elastic manner. In cases where deflections are significant, a 'second order' analysis (involving the so-called *P–Δ effect*) may be called for; in such cases, superposition is not valid.

‡ Wind load or earthquake loads are dynamic loads, which require *dynamic analysis*. However, the Code permits the use of conventional static analysis in most cases, wherein the loads are treated as equivalent static loads.

methods of analyses prove to be highly useful in this context, as they provide a rough check on the detailed analyses. Moreover, these approximate methods are useful in proportioning members in the preliminary design stage; this is discussed in Section 9.4.




**(a)** floor plan




**(b)** longitudinal plane frame

**Fig. 9.1** Building frame idealised as a series of independent plane frames

### *Stiffness of Frame Elements*

The sizes of the various members must be known prior to structural analysis, not only to enable the calculation of dead loads, but also to fix the *relative stiffnesses* of the various members of the indeterminate structure. As mentioned earlier, *cracking* is unavoidable in reinforced concrete structures, and the presence of cracks complicates the determination of the stiffness (flexural, torsional or axial) of a reinforced concrete member. Accordingly, certain simplified assumptions are called for in the estimation of stiffnesses — even for the 'rigorous' methods of analysis; these are discussed in Section 9.5.

Furthermore, there are approximations involved in specifying the stiffnesses at the beam-column junctions of frames [Fig. 9.6]. The errors arising from such approximations, and the consequent adjustments required in the design moments at the beam-column junctions are discussed in Section 9.6.

### 9.1.2 Factored Moments from Elastic Analysis and Moment Redistribution

As mentioned earlier, the maximum load effects (moments, shear forces, etc.) are generally determined (vide Cl. 22.1 of the Code) on the basis of *elastic* analyses of the structure under service loads (*characteristic loads*). The *factored moments* (design moments) are then obtained by multiplying the service load moments by the specified *load factors*, and combining the results of different load combinations. This is equivalent to considering elastic moment distributions under the *factored loads.*

There is an apparent inconsistency in determining the design moments based on an *elastic analysis*, while doing the design based on a *limit state design* procedure. The *structural analysis* is based on linear elastic theory, whereas the *structural design* is based on inelastic section behaviour. It should be noted, however, that there is no real inconsistency if the moment-curvature ($M$-$\varphi$) relationship remains *linear* even under ultimate loads. As explained earlier in Section 4.5.3, the moment-curvature relationship is practically linear up to the point of yielding of the tension steel in under-reinforced sections[†]. If under the factored loads, no significant yielding takes place at any section in the structure, the bending moment distribution at the ultimate limit state will indeed be the same as that obtained from a linear elastic analysis under factored loads. In other words, the structure continues to behave more-or-less in a linear elastic manner even under the factored loads, provided no significant yielding of the reinforcing steel takes place. As the design moments at various critical sections, determined by superimposing different combined factored load effects, are greater than (or at best equal to) the bending moments due to any one loading pattern, most sections (excepting a few) would not have reached their ultimate moment capacities and will be well within the linear phase.

Also, as explained earlier, the main advantage underlying under-reinforced sections is that they exhibit *ductile* behaviour, due to the ability of the sections to undergo large curvatures at nearly constant moment after the yielding of steel [Fig. 4.8(a)]. This ductile behaviour enables the structure to enter into an inelastic phase, wherein the sections which have reached their ultimate moment capacities undergo rotations (under constant moment). This causes additional load effects to be

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[†] Actually, the linear $M$–$\varphi$ relationship is valid only up to the point of yielding of the tension steel in under-reinforced sections [Fig. 4.8(a)]. However, the moment of resistance at this point is practically equal to $M_{uR}$, which is finally attained only after a significant increase in curvature.

borne by less stressed sections — a phenomenon which is described as *redistribution* of moments (or, in general, stresses). This capacity for *moment redistribution* can be advantageously made use of in many cases, resulting in designing for ultimate moments that are less than the peak factored moments obtained from elastic analysis. The Code (Cl. 37.1.1) permits a limited redistribution of moments, provided adequate ductility is ensured at the critical sections. This is discussed in the concluding section (Section 9.7) of this chapter.

## 9.2 GRAVITY LOAD PATTERNS FOR MAXIMUM DESIGN MOMENTS

Gravity loads comprise *dead loads* and *live loads* — to be estimated in accordance with Parts 1 and 2 respectively of IS 875 : 1987 [also refer Appendix B]. Whereas dead loads, by their inherent nature, act at all times, live loads occur randomly — both temporally and spatially. In order to determine the maximum ('positive' as well as 'negative') moments that can occur at any section in a continuous beam or frame, it is first necessary to identify the spans to be loaded with live loads so as to create the 'worst' (most extreme) effects.

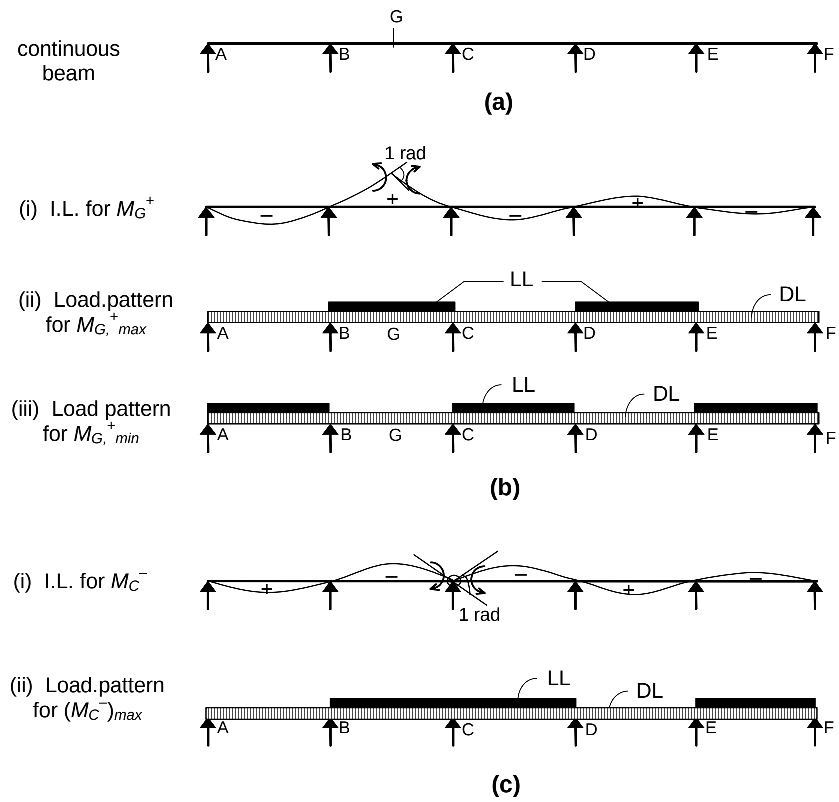
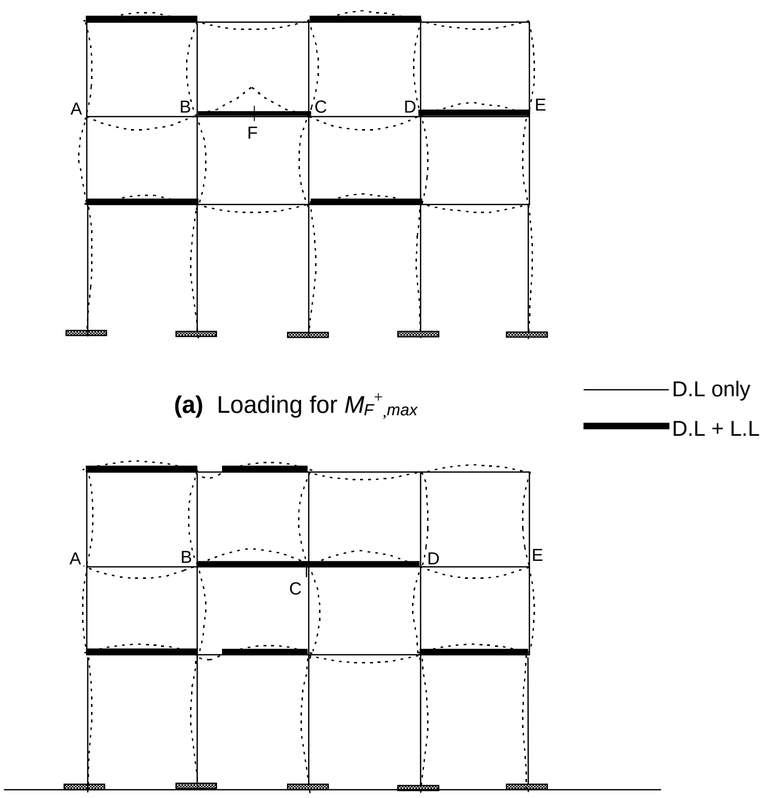



**Fig. 9.2** Influence lines and gravity load patterns for a continuous beam

This can be conveniently done by sketching, qualitatively, the shape of the *influence line* for the bending moment at the section under consideration, using the *Müller-Breslau Principle* [Ref. 9.1]. A fictitious hinge is first inserted at the section under consideration, and a rotation introduced therein in a direction corresponding to the moment desired. The resulting deflected shape, corresponding to a unit value of the imposed rotation, gives the desired influence line. The influence lines for the bending moments at two sections (one in the midspan region and another near the support) in a continuous beam, and at two similar sections in a plane frame, are shown in Fig. 9.2 and Fig. 9.3 respectively.




**(a)** Loading for $M_{F}^{+}{}_{,max}$

**(b)** Loading for $M_{C}^{-}{}_{,max}$

**Fig. 9.3** Influence lines and gravity load patterns for a plane frame

### 9.2.1 Design Moments in Beams

Based on the shapes of the influence lines in Figs 9.2 and 9.3, the following conclusions may be drawn regarding the positioning of live loads for obtaining the critical design moments in continuous beams and plane frames:

(1) The maximum 'positive' moment in a span occurs when live loads are placed on that span and every other alternate span [Fig. 9.2b(ii)]; in the case of a plane frame, this arrangement corresponds to a 'checkerboard' pattern [Fig. 9.3(a)].

(2) The minimum 'positive' moment in a span (which may turn out to be a maximum 'negative' moment in some cases†) occurs when it is *not* loaded with live loads, and when live loads are placed on adjoining spans, as well as alternate spans further away [Fig. 9.2b(iii)].

(3) The maximum 'negative' moment at a support section [marked 'C' in Fig. 9.2(c) and Fig. 9.3(b)] occurs when live loads are placed on the span (BC) in which the support section is located as well as the adjoining span CD, and also on every alternate span thereafter, as shown. [Note that in Fig. 9.3(b), the influence line is partially positive in the beams above and below beam BC, and strictly, live loads should not be placed on the small 'positive' portions in these spans. However, this is ignored generally and the full live load is placed on these spans, as the error involved is small.]

(4) The influence of loads on spans far removed from the sections under consideration is relatively small. (This forms the basis of the substitute frame method, to be discussed in Section 9.3).

### *Code Recommendations*

The Code recommendations for 'arrangement of imposed load' (Cl. 22.4.1) in continuous beams (and one-way slabs) and frames are in conformity with the conclusions (1) – (3) cited above. It may be noted, however, that with respect to the live load pattern required to estimate the maximum moment at a support section, the Code does not call for live loads to be placed on alternate spans [refer span EF in Fig. 9.2(c)] in addition to the placement on the two spans adjacent to an interior support [this is justified by conclusion (4) above].

Furthermore, in the case of frames in which the *design live load does not exceed three-fourths of the design dead load*, the Code (Cl. 22.4.1b) permits the designer to ignore altogether the problem of analysing different live load patterns. In such cases, it suffices to perform a single frame analysis for gravity loading — with full design dead load plus live load on *all* the spans. It may be noted that this major concession is permitted only for frames, and *not* for continuous beam and one-way slabs. Also, it should be noted that *redistribution of moments* (refer Section 9.8) cannot be applied in this case [Ref. 9.2].

## 9.2.2 Design Moments in Columns

Bending moments in columns, unlike beams, need to be studied in association with co-existing axial compressive forces. The interaction between axial compressive strength and flexural strength of a given column section is such that its ultimate moment resisting capacity $M_{uR}$ is nonlinearly dependent on the factored axial load $P_u$

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† For example, in a lightly loaded short span, flanked on either side by heavily loaded long spans.

[refer Chapter 13]. Strictly, this calls for an investigation of all gravity load patterns that result in all possible combinations of $P_u$ and $M_u$. Strictly, the combinations should include $P_u$, $M_{ux}$ and $M_{uy}$ — considering the biaxial bending moments that occur simultaneously from the longitudinal and transverse frames connected to the same column [Fig. 9.1].

However, it is generally accepted [Ref. 9.3] that considerations may be limited to gravity load patterns that result in (a) maximum eccentricity $e = M_u/P_u$ and (b) maximum $P_u$. The former is generally obtainable from the checkerboard patterns of loading (which are, at any rate, required to determine the maximum span moments in beams). The latter is obtained by loading *all* the panels on *all* the floors above the storey under consideration. It may be noted here that the Loading Code [IS 875 : 1987 (Part 2)] permits some reduction in live load values to account for the low probability of simultaneous occurrence of full live loads on all the floor slab areas in all the floors above. Incidentally, the live load arrangement on the floor below have some influence on the bending moment to be considered in combination with the maximum $P_u$. For this purpose, it is recommended [Ref. 9.3] that it is sufficient to consider live loads on a single span (the one that is longer and more heavily loaded) adjoining the lower end of the column member under consideration.

## 9.3 SIMPLIFIED (APPROXIMATE) METHODS OF ANALYSIS

### 9.3.1 Moment Coefficients for Continuous Beams Under Gravity Loads

The use of *moment coefficients* (in lieu of rigorous analyses of various gravity load patterns) was introduced in Section 5.6.1, with reference to continuous one-way slabs. These 'moment coefficients' as well as related 'shear coefficients' (Cl. 22.5.1 of the Code) are listed in Table 9.1. The use of the moment coefficients has already been demonstrated in Example 5.3.

It may be noted that the use of the 'moment coefficients, as well as the related 'shear coefficients', for continuous beams and one-way slabs is subject to the following conditions (Cl. 22.5.1 of the Code):

- the continuous spans should be at least three in number;
- the supports should be fairly rigid and should not themselves deflect;
- all the spans should have the same cross-sections;
- the effective length of each span should be more-or-less the same, and at any rate should not differ by more than 15 percent of the largest effective span;
- the loading on all spans should be 'substantially uniformly distributed'; and
- no redistribution of moments is permitted.

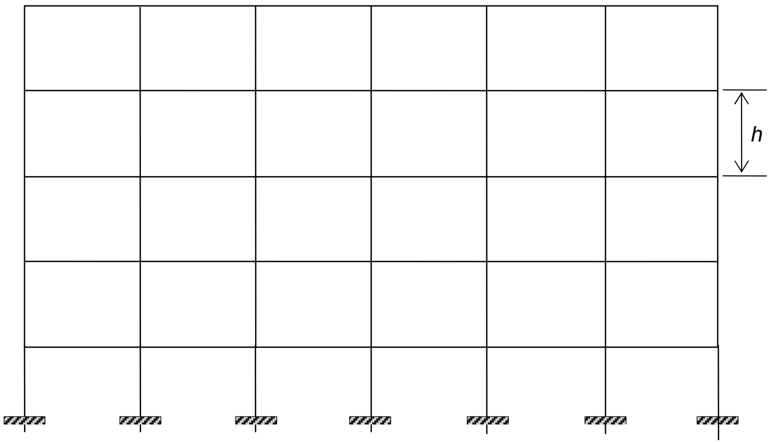
### 9.3.2 Substitute Frame Method of Frame Analysis for Gravity Loads

As explained earlier, the skeleton of a typical framed building is a three-dimensional 'rigid frame', which may be resolved, for the purpose of analysis, into a series of independent parallel 'plane frames' along the column lines in the longitudinal and transverse directions of the building. Each of these plane frames [Fig. 9.4(a)] in turn, has to be analysed separately for a number of gravity load patterns as well as lateral loads.

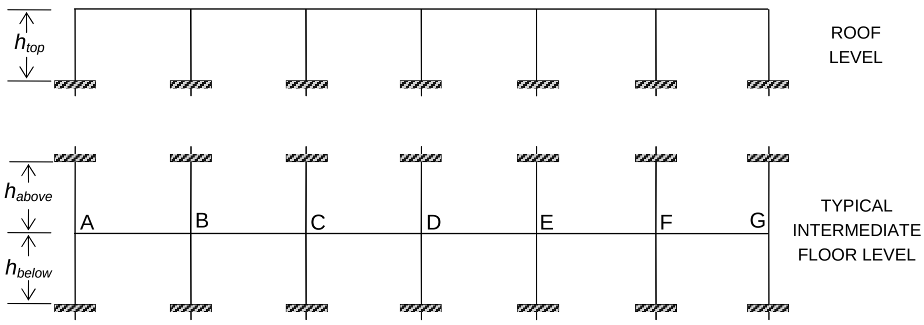
**Table 9.1** Factored moments and shears in continuous beams using Code coefficients (Cl. 22.5 of the Code)

| | |
|---|---|
| (a) **'POSITIVE' MOMENTS**: | |
| 1. End Spans | $+\left(\frac{w_{u,DL}}{12}+\frac{w_{u,LL}}{10}\right)l^2$ |
| 2. Interior Spans | $+\left(\frac{w_{u,DL}}{16}+\frac{w_{u,LL}}{12}\right)l^2$ |
| (b) **'NEGATIVE' MOMENTS**: | |
| 1. End support (if partially restrained) | $-\left(\frac{w_{u,DL}}{24}+\frac{w_{u,LL}}{24}\right)l^2$ |
| 2. First interior support | $-\left(\frac{w_{u,DL}}{10}+\frac{w_{u,LL}}{9}\right)l^2$ |
| 3. Other interior supports | $-\left(\frac{w_{u,DL}}{12}+\frac{w_{u,LL}}{9}\right)l^2$ |
| (c) **SHEAR FORCES**: | |
| 1. End support | |
| • *unrestrained* | $(0.5w_{u,DL}+0.5w_{u,LL})\,l$ |
| • *partially restrained* | $(0.4w_{u,DL}+0.45w_{u,LL})\,l$ |
| 2. First interior support | |
| • *exterior face* | $(0.6w_{u,DL}+0.6w_{u,LL})\,l$ |
| • *interior face* | $(0.55w_{u,DL}+0.6w_{u,LL})\,l$ |
| 3. Other interior supports | $(0.5w_{u,DL}+0.6w_{u,LL})\,l$ |

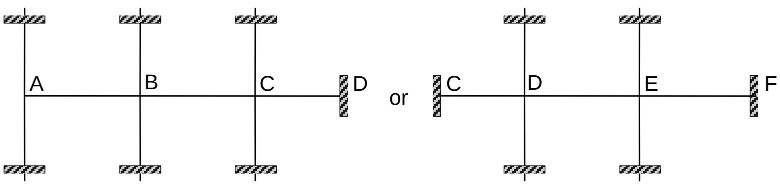
This is essentially a problem of structural analysis of indeterminate structures, and various techniques are available for this purpose [Ref. 9.4, 9.5]. For the detailed analysis of large frames, with high indeterminacy, computer-based matrix methods are ideally suitable [Ref. 9.5]. A large number of established 'finite element method' based software packages (such as ANSYS, NASTRAN, NISA, SAP) are available in the market and are increasingly being used by designers worldwide. However, the good old manual methods such as the *Moment Distribution Method* are still in vogue, and are ideally suited for analysing small *non-sway* frames under gravity loading. Convenient tabular arrangements for performing moment distribution (up to four cycles) are given in Ref. 9.6 and 9.7.




**(a)** typical plane frame for analysis




**(b)** substitute frames




**(c)** substitute frames — truncated

**Fig. 9.4** 'Substitute frames' for gravity load analysis

In the cases of frames that are substantially unsymmetrical or unsymmetrically loaded, and are otherwise not braced against *sway*, the effects of sway also have to be considered. For such analyses, the use of the *Moment Distribution Method* can be cumbersome, as it requires more than one distribution table to be constructed (and later combined); *Kani's Method* is better suited for such analyses, as it manages with a single distribution table.

In cases where the effects of sway are negligible, it is found that the problem of frame analysis can be considerably simplified by resolving the frame into partial frames ('substitute frames'), as indicated in Fig. 9.4 [Ref. 9.2]. This simplification is justified on the grounds that the bending moment or shear force at a particular section (of a beam or column) is not influenced significantly by gravity loads on spans far removed from the section under consideration [refer conclusion (4) in Section 9.2.1]. Accordingly, an entire floor, comprising the beams and columns (above and below) may be isolated, with the far ends of the columns above and below considered 'fixed' [Fig. 9.4(b)]. This frame can be conveniently analysed for various gravity load patterns by the moment distribution method; two cycles of moment distribution are generally adequate [Ref. 9.8, 9.2]. This method of 'substitute frame analysis' is permitted by the Code (Cl. 22.4.2). If there are a large number of bays in the substitute frame, then a further simplification is possible by truncating the beams appropriately, and treating the truncated end as 'fixed' [Fig. 9.4(c)]. In determining the support moment, the beam member may be assumed as fixed at any support two panels distant, provided the beam continues beyond that point. In analysing for maximum and minimum 'positive' moments in spans, the far ends of adjacent spans may similarly be considered as fixed.

### 9.3.3 Simplified Methods for Lateral Load Analysis

Multi-storeyed buildings have to be designed to resist the effects of lateral loads due to wind or earthquake, in combination with gravity loads. The lateral load transfer mechanism of a framed building has been explained briefly in Chapter 1 (refer Fig. 1.8). It is seen that the lateral loads are effectively resisted by the various plane frames aligned in the direction of the loads. The wind loads and seismic loads are to be estimated in accordance with IS 875:1987 (Part 3) and IS 1893:2002 respectively. They are assumed to act at the floor levels and are appropriately distributed among the various resisting frames (in proportion to the relative translational stiffness[†]). Although these loads are essentially dynamic in nature, the loading Codes prescribe

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[†] A measure of the translational stiffness of a plane frame in a multi-storey building is approximately given by $E(\Sigma I_c)/h$ where $\Sigma I_c$ denotes the sum of moments of inertia of the columns (with appropriate axis of bending) that form part of the frame. However, the presence of an infill masonry wall can substantially increase the stiffness (up to two times or more), and this should also be accounted for. Also, if the lowermost storey of the building is free of infill walls (on account of ground floor parking), the adverse effects of reduction in mass and stiffness in this critical storey should be specially accounted for in seismic-resistant design, as advocated in IS 1893 (2002).

equivalent static loads to facilitate static analysis in lieu of the more rigorous dynamic analysis [Ref. 9.9]. However, if the building is very tall and/or unsymmetrically proportioned, rigorous dynamic analysis is called for. Otherwise, in most ordinary situations, simplified static analysis is permitted by the Code (Cl. 22.4.3). In fact, the Explanatory Handbook to the Code [Ref. 9.2] states:

> "Considerable uncertainty prevails regarding the magnitude as well as the distribution of wind and earthquake forces. Therefore, it will be sufficient, in most cases, to use an approximate method which gives an accuracy which is greater than that of the load data and other assumptions".

The simple methods of lateral load analysis in vogue are the *Portal Method* and the *Cantilever Method* [Ref. 9.10], wherein the static indeterminacy in the frame is eliminated by making reasonable assumptions. The Portal Method, in particular, is very simple and easy to apply. It is based on the following assumptions:

- The inflection points of all columns and beams are located at their respective middle points.
- For any given storey, the 'storey shear' (which is equal and opposite to the sum of all lateral loads acting above the storey) is apportioned among the various columns in such a way that each interior column carries twice the shear that is carried by each exterior column.

The limitation of the Portal Method (as well as the Cantilever Method) is that it does not account for the relative stiffnesses of the various beams and columns. An improved method of analysis, which takes into account these relative stiffnesses (in locating the points of inflection and apportioning the storey shear to the various columns) is the so-called *Factor Method*, developed by Wilbur [Ref. 9.10]. This method works out to be fairly accurate, although it requires more computational effort.

## 9.4 PROPORTIONING OF MEMBER SIZES FOR PRELIMINARY DESIGN

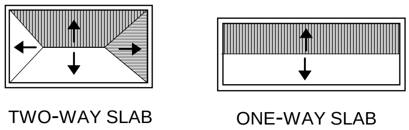
As mentioned earlier, it is necessary to estimate the cross-sectional dimensions of beams and columns prior to frame analysis and subsequent design — in order to

- assess the dead loads due to self-weight;
- determine the various member stiffnesses for analysis.

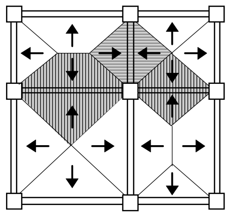
This requires a preliminary design, whereby the design values of bending moments, shear forces and axial forces in the various members may be approximately computed.

### *Gravity Load Effects*

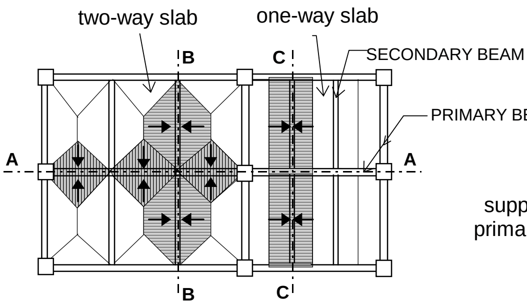
The apportioning of gravity loads to the various secondary/primary beams and columns may be done by considering the *tributary areas* shown in Fig. 9.5. These areas are based on the assumption that the (uniformly distributed) gravity loads in any panel are divided among the supporting beams by lines midway between the lines of support, the load in each area is transferred to the adjacent support. In the general case of a 'two-way' rectangular slab panel, this implies a triangular shaped tributary area on each of the two short span beams and a trapezoidal area on each of the two long span beams, as shown in Figs 9.5 (a), (b) [refer Cl. 24.5 of the Code]†. When the slab is 'one-way', the trapezoidal tributary area on each long span beam may be approximated (conservatively) as a simple rectangle, as indicated.

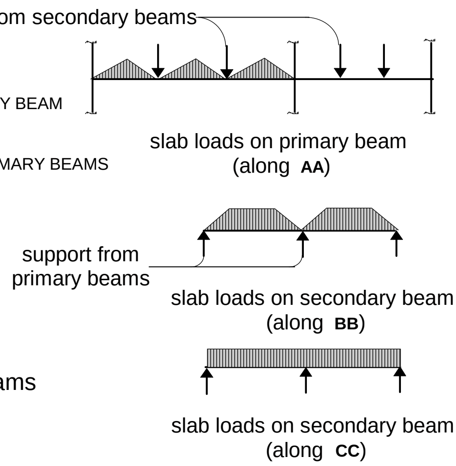



**(a)** slabs

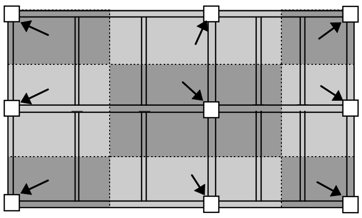


**(b)** twoway slabs with primary beams




**(c)** slabs with primary and secondary beams



**(d)** column tributary areas

**Fig. 9.5** Tributary areas for beams and columns

† The lines forming the two sides of the triangle may be assumed to be at 45° to the base.

When the floor system is made up of a combination of primary beams and secondary[‡] beams, the tributary areas may be formed in a similar fashion [Figs. 9.5(c), (d)]. One-way slabs are assumed to transfer the loads only on to the longer supporting sides [Fig. 9.5(d)]. The loads from the 'secondary beams' are transferred as concentrated loads to the supporting 'primary beams' [Fig. 9.5(c)]. However, for preliminary calculations, a uniformly distributed load may be assumed on each primary beam, with the tributary areas appropriately taken.

The axial loads on the columns at each floor level are obtained from the tributary areas of the primary beams, the load on each primary beam being shared equally by the two supporting columns. The tributary areas for the columns are indicated in Fig. 9.5(d). In addition to the gravity loads transferred from the floor slabs, loads from masonry walls (wherever applicable) as well as self-weight of the beams/columns should be considered.

For a preliminary design, the influence of different possible live load patterns may be ignored, and all panels may be assumed to be fully loaded with dead loads plus live loads. Substitute frame analysis may be done to determine the bending moments and shear forces in the primary beams and columns; the secondary beams may be analysed as continuous beams using moment coefficients (except when they are discontinuous at both ends). A crude estimate of the maximum design moment in a beam may be taken as $\pm W_u l/10$, where $W_u$ is the total factored load on the beam and $l$ its span; similarly the design shear force may be approximately taken as $W_u/2$.

Sizes of interior columns are primarily determined by the axial loads coming from the tributary areas of all the floors above the floor under consideration. However, exterior columns are subjected to significant bending moments (on account of unbalanced beam end moments) in addition to axial forces. Accordingly, these columns must be designed for the combined effect of axial compression and bending moment; this can be conveniently done using appropriate interaction curves (design aids), as explained in Chapter 13.

Finally, it should be noted that if the frames are unsymmetrical, the additional moments induced by sway should also be accounted for.

### *Lateral Load Effects*

In the case of tall buildings, the effects of wind or earthquake moments are likely to influence the design bending moments in primary beams and column, especially in the lower floors. These load effects may, for the purpose of preliminary design, be determined approximately using the Portal Method described in Section 9.3.3. The design (factored) moments are obtained by combining the effects of gravity loads with lateral loads, using the specified partial load factors. Earthquake-resistant design calls for special design and detailing, as discussed in detail in Chapter 16.

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[‡] 'Primary' beams (or *girders*) are those which frame into the columns, whereas 'secondary' beams are those which are supported by the primary beams — as explained in Section 1.6.1 [refer Fig. 1.10]. In the load transfer scheme shown in Fig. 9.5(c), it is assumed that the primary beams offer 'rigid' supports to the secondary beams.

## 9.5 ESTIMATION OF STIFFNESSES OF FRAME ELEMENTS

A typical building frame — even a 'substitute frame' — is a statically indeterminate structure. To enable its analysis, whether by approximate methods or rigorous methods, it is necessary to know the stiffnesses (flexural, torsional, axial) of the various members that constitute the frame. Frequently, it is only the *flexural stiffness* that needs to be known. Axial stiffness is generally high, resulting in negligible axial deformations. Furthermore, as explained in Chapter 7, the torsional stiffness of a reinforced concrete member is drastically reduced following torsional cracking and hence can be ignored altogether. Recommendations for computing torsional stiffness, wherever required, are given in Section 7.2.3.

The 'flexural stiffness' of a beam element (with the far end 'fixed') is given by $4EI/l$, where *EI* is the 'flexural rigidity', obtained as a product of the modulus of elasticity *E* and the second moment of area *I*, and $l$ is the length of the member. As the analysis is generally a 'linear static' analysis, the appropriate value of *E* is given by the static modulus of elasticity $E_c$, defined in Section 2.8.3. The problem lies with specifying the value of *I*, which must ideally reflect the degree of cracking, the amount of reinforcement and the participation of flanges (in beam-slab members).

The Code (Cl. 22.3.1) permits the calculation of flexural stiffness based on the 'gross' concrete section, the 'uncracked-transformed' section or the 'cracked-transformed' section [refer Chapter 4]; however, the *same* basis is to be applied to all the elements of the frame to be analysed. This is reasonable, because what really matters is the *relative* stiffness, and not the absolute stiffness. The most common (and simplest) procedure is to consider the 'gross' section (i.e., ignoring both the amount of reinforcement and the degree of cracking) for calculating the second moment of area. An alternative procedure, which better reflects the higher degree of flexural cracking in beams relative to columns, is to use $I_{gross}$ for columns and 0.5 $I_{gross}$ for beam stems [Ref. 9.11].

In slab-beam systems, the presence of the flange enhances the stiffness of the beam. However, the flanged beam action (with effective width $b_f$ as described in Section 4.6.4) is not fully effective when the flanges are subjected to flexural tension, as in the regions of 'negative' moment. Some designers ignore the contribution of the flanges altogether (mainly for convenience) and treat the beam section as being rectangular. An improved procedure, suggested in Ref. 9.8, is to use twice the moment of inertia of the gross web section ( $I = 2 \times b_w D^3/12$ ). This corresponds to an effective flange width $b_f \approx 6b_w$ with $D_f/D \approx 0.2$ to 0.4. The use of such a procedure eliminates the explicit consideration of the flanges; it gives reasonable results and is simple to apply, and hence is recommended by the authors of this book.

## 9.6 ADJUSTMENT OF DESIGN MOMENTS AT BEAM-COLUMN JUNCTIONS

In frame analysis, *centreline* dimensions of beams and columns are generally used to define the geometry of the frame 'line diagram' [Fig. 9.6(a)] (refer Cl. 22.2c of the Code). In the analysis, it is tacitly assumed that the specified flexural stiffness of any member (beam or column) is valid even at the ends of the member [i.e., right up to the point of intersection of the centre lines, including the zone where the column and beam merge, as shown in Fig. 9.6(b)]. It is also assumed tacitly that the restraint offered by the column against vertical deflection of the beam is limited to a single point, corresponding to the centre of the beam-column junction. This results in beam deflections at points located between the centreline and the face of the column [see point 'B' in Fig. 9.6(c)] — which is obviously not possible in reality.

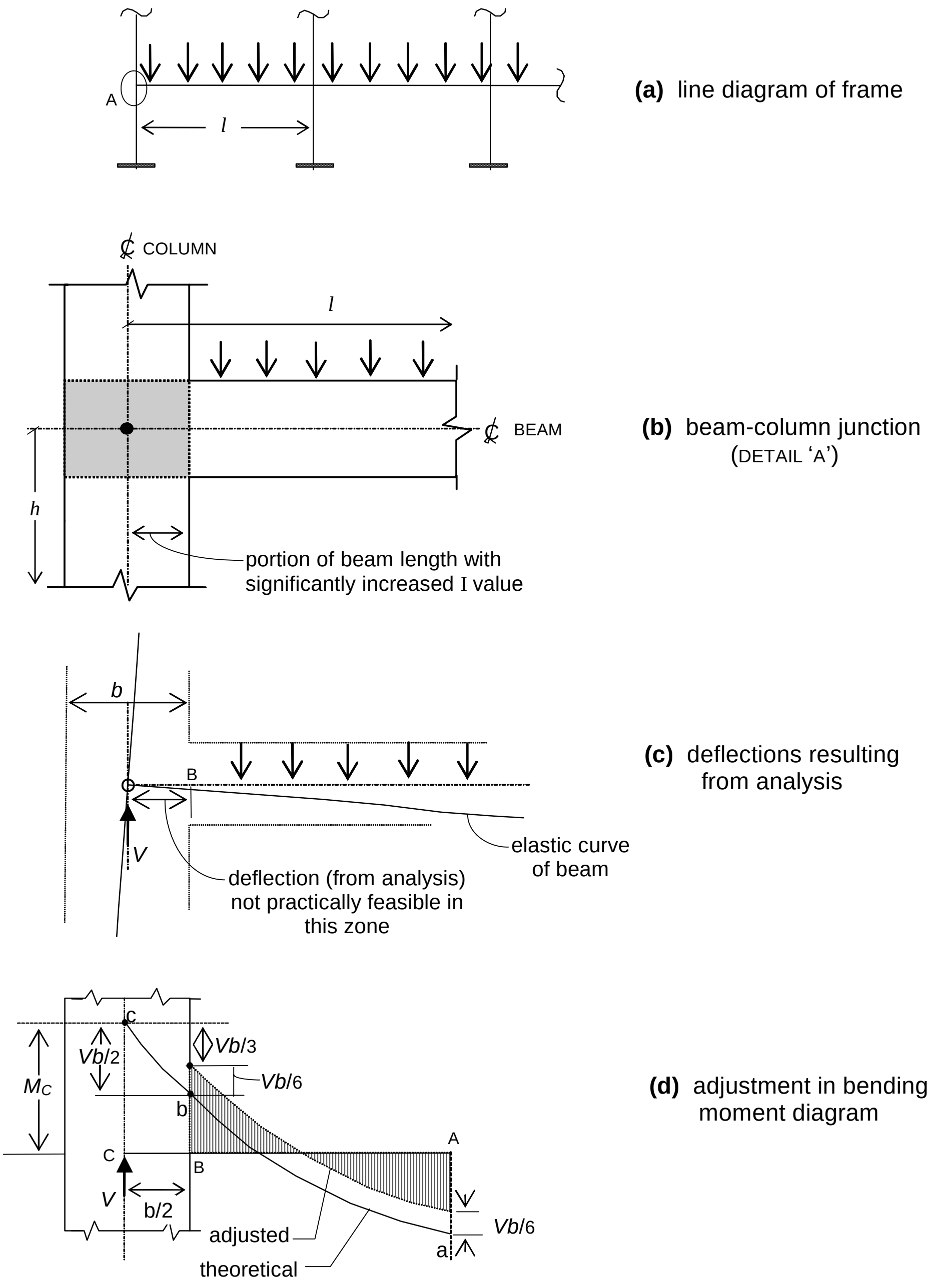



**Fig. 9.6** Assumptions implicit in frame analysis at beam-column junctions

The neglect of the increased stiffness and restraint of the beam within the column width results in a slight under-estimation of the 'negative' moment at the beam end and a corresponding over-estimation of the 'positive' moment in the span, under gravity loads. It has been shown that, in the interest of greater accuracy, the beam bending moment diagram (from frame analysis) may be adjusted[†] [Fig. 9.6(d)] by an upward shift by $Vb/6$, where $V$ is the shear force at the column centreline and $b$ the width of the column support [Ref. 9.8]. This results in an adjusted 'negative' moment at the face of the column support, equal to $M_c - Vb/3$, where $M_c$ is the theoretically computed moment at the column centreline. The slight reduction (equal to $Vb/6$) in the design 'positive' moment at midspan may be ignored; this is conservative and satisfactory.

The Code (Cl. 22.6.1) permits flexural members (in monolithic construction) to be designed for moments computed at the *faces* of the supports, as the effective depth (and hence, flexural resistance) of the flexural member (beam or column) is greatly enhanced in the region where the beam merges with the column. In the case of beams, this often results in a much lower moment than the computed 'negative' moment at the support centreline, owing to the steep variation in the bending moment diagram of the support region [Fig. 9.6(d)]. However, in the case of columns, the moment gradient is not so significant, and so there is little to gain in taking the moment at the beam face, rather than at the beam centreline [Fig. 9.7].

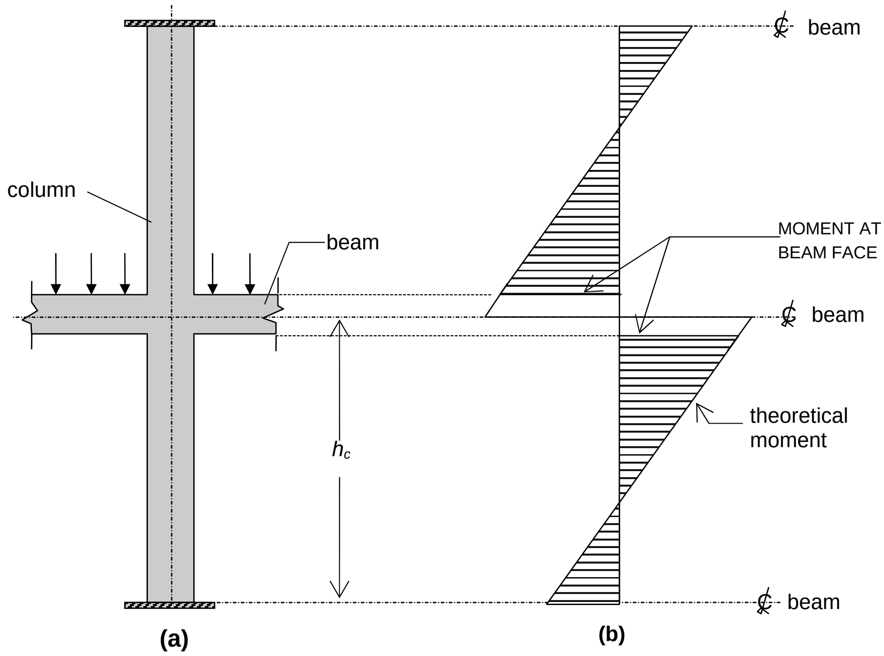



**Fig. 9.7** Column moments from frame analysis

Also, the adjustment in the bending moment diagram due to the increased stiffness of column at the beam-column junction is generally negligible. Hence, the column

[†] This is based on the assumption that the stiffness of the beam is infinite within the column width $b$, where it is monolithic with the column.

moment for design may be taken as that obtained from frame analysis at the beam centreline.

It may be noted that, in practice, most designers do not bother to consider the adjustment in the bending moment diagram, indicated in Fig. 9.6(d); the design moment at the face of the column supports is simply taken as ($M_c - Vb/2$), rather than as ($M_c - Vb/3$). This is also justifiable, if consideration is given to the possibility of *redistribution of moments*, as explained in Section 9.7.

## 9.7 INELASTIC ANALYSIS AND MOMENT REDISTRIBUTION

### 9.7.1 Limit Analysis

Reinforced concrete structures are generally analysed by the conventional *elastic theory* (refer Cl. 22.1 of the Code). In flexural members, this is tantamount to assuming a linear moment-curvature relationship, even under *factored loads*. For under-reinforced sections, this assumption is approximately true [refer Fig. 4.8(a)], provided the reinforcing steel has not yielded at any section. Once yielding takes place (at any section), the behaviour of a statically indeterminate structure enters an inelastic phase, and linear elastic structural analysis is strictly no longer valid.

For a proper determination of the distribution of bending moments for loading beyond the yielding stage at any section, *inelastic* analysis is called for. This is generally referred to as *limit analysis*, when applied to reinforced concrete framed structures [Ref. 9.12–9.17], and 'plastic analysis' when applied to steel structures. In the special case of reinforced concrete slabs, the inelastic analysis usually employed is the 'yield line analysis' due to Johansen [Ref. 9.18]. The assumption generally made in limit analysis is that the moment-curvature relation is an idealised bilinear elasto-plastic relation [Fig. 9.8]. This has validity only if the section is adequately under-reinforced and the reinforcing steel has a well-defined yield plateau. The ultimate moment of resistance ($M_{uR}$) of such sections, with specified area of steel, can be easily assessed, as described in Chapter 4.

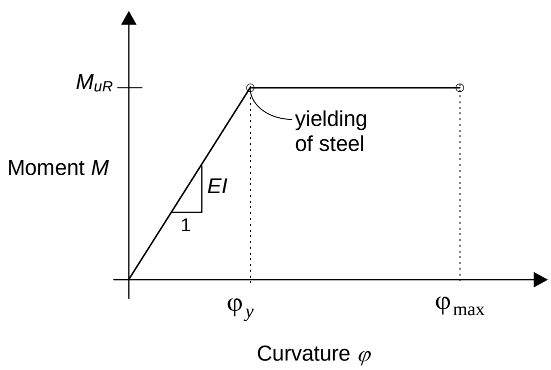



**Fig. 9.8** Idealised moment-curvature relation

***Plastic Hinge Formation***

With the idealised $M - \varphi$ relation, the ultimate moment of resistance ($M_{uR}$) is assumed to have been reached at a 'critical' section in a flexural member with the yielding of the tension steel [Fig. 9.8]. On further straining (increase in curvature: $\varphi > \varphi_y$), the moment at the section cannot increase. However, the section 'yields', and the curvature continues to increase under a constant moment ($M = M_{uR}$). In general (with bending moment varying along the length of the member), the zone of 'yielding' spreads over a small region in the immediate neighbourhood of the section under consideration, permitting continued rotation, as though a 'hinge' is present at the section, but one that continues to resist a fixed moment $M_{uR}$. A *plastic hinge* is said to have formed at the section. If the structure is statically indeterminate, it is still stable after the formation of a plastic hinge, and for *further loading*, it behaves as a modified structure with a hinge at the plastic hinge location (and one less degree of indeterminacy). It can continue to carry additional loading (with formation of additional plastic hinges) until the limit state of collapse is reached on account of one of the following reasons:

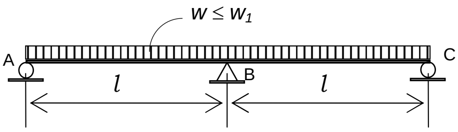
- formation of sufficient number of plastic hinges, to convert the structure (or a part of it) into a 'mechanism';
- limitation in ductile behaviour (i.e., curvature $\varphi$ reaching the ultimate value $\varphi_{max}$, or, in other words a plastic hinge reaching its ultimate rotation capacity) at any one plastic hinge location, resulting in local crushing of concrete at that section.

### *Example of Limit Analysis*

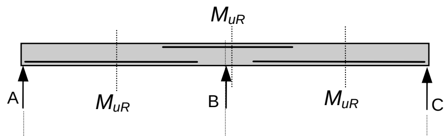
A simple application of limit analysis is demonstrated here, with reference to a two-span continuous beam subjected to an increasing uniformly distributed load $w$ per unit length [Fig. 9.9(a)]. For convenience, it may be assumed that the beam has uniform flexural strength ($M_{uR}$) at all sections [Fig. 9.9(b)]. The limit of the linear elastic behaviour of the structure is reached at a load $w = w_1$, corresponding to which the maximum moment (occurring at the continuous support) becomes equal to $M_{uR}$ [Fig. 9.9(c)], i.e.,

$$M_{uR} = \frac{w_1 l^2}{8} \Rightarrow w_1 = 8M_{uR}/l^2$$

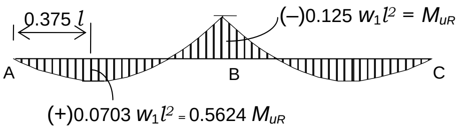
At this load, a plastic hinge will form at the continuous support [Fig. 9.9(d)]. However, the maximum moment in the span is only 0.5624 $M_{uR}$. How much additional load the beam can take will now depend on the plastic rotation capacity of this 'plastic hinge'. For any additional loading, the beam behaves as a two-span beam with a hinge at support B (i.e., two simply supported spans) and the span moment alone increases [Fig. 9.9(d),(f)] while the support moment remains constant at $M_{uR}$. Assuming that the support section is sufficiently under-reinforced such that it will not break down prior to the formation of the next plastic hinge(s), this phase of behaviour will continue until the peak moment in the span reaches $M_{uR}$ [Fig. 9.9(e),(f)]. Analysis of the structure for this condition [Fig. 9.9(e)] indicates that this corresponds to a maximum span moment given by:



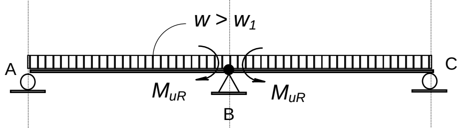

**(a)** loading on beam



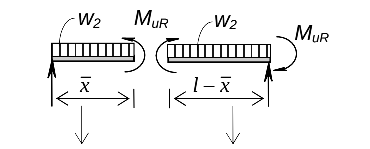

**(b)** flexural detailing of beam




**(c)** bending moments up to limit of elastic behaviour (phase I)



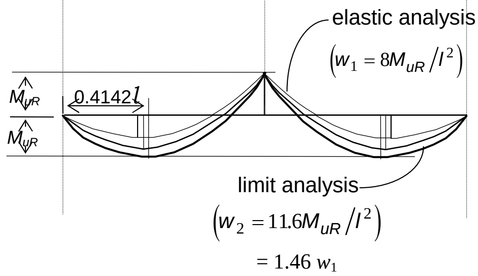

**(d)** plastic hinge at continuous support



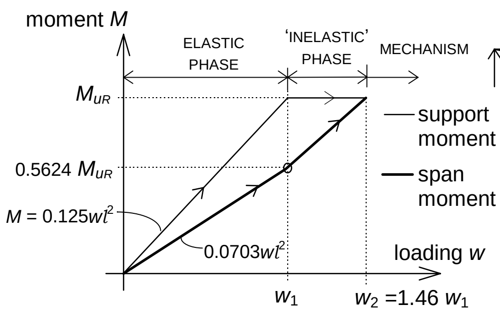

$$M_{uR} = \frac{w_2 \bar{x}^2}{2} \qquad 2M_{uR} = \frac{w_2 (l - \bar{x})^2}{2}$$

$$Solving,\ \bar{x} = 0.4142l \text{ and } M_{uR} = \frac{w_u l^2}{11.656}$$

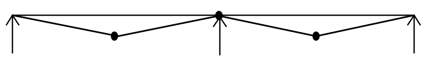
**(e)** limit analysis ("equilibrium method")




**(f)** bending moment distribution(s) in the 'inelastic' phase (phase II)




**(g)** variation of support/span moment with loading



**(h)** hinged mechanism at ultimate load $w = w_2$

**Fig. 9.9** Limit analysis of a two-span continuous beam

$$M_{uR} = \frac{w_2 l^2}{11.656} \Rightarrow w_2 = 11.656 M_{uR}/l^2 = 1.46 w_1$$

As the load $w_2$ is reached, two additional plastic hinges† are formed in the two spans at the peak moment locations, and the structure is transformed into an unstable hinged mechanism which can deflect with no increase in load [Fig. 9.9(h)]. Obviously, $w_2$ is the ultimate (collapse) load of the structure, even allowing for inelastic behaviour.

This indicates that the beam is capable of carrying additional loads up to 46 percent beyond the limit of elastic behaviour, thanks to the ductile behaviour of the beam section at the continuous support.

The bending moment distributions in the inelastic phase are indicated in Fig. 9.9(f). It is seen in this example that, a 'redistribution of moments' takes place, with the support moment remaining constant at $M_{uR}$ while the span moments continue to increase until they too reach $M_{uR}$. The variation of support moment and maximum span moment with increasing loading is shown in Fig. 9.9(g). The gain in moments is linear in the 'elastic phase' ($w < w_1$), and corresponding to the formation of the first plastic hinge (at $w = w_1$), there is a discontinuity in each of the two $M$–$w$ curves.

It may be noted that the so-called 'limit analysis' is essentially a superposition of a series of 'elastic analyses', the inelasticity being confined to the plastic hinge locations. The structure gets modified — with the introduction of successive plastic hinges, and each so-called 'inelastic' phase of the analysis is in fact 'elastic'. This is reflected in the piece-wise linear segments of the *M–w* relationship in Fig. 9.9(h).

In deriving the expression for $w_2$, allowing full moment redistribution on to the spans, it was assumed above that the plastic hinge at the support section will continue to yield (rotate) without breakdown. If the rotation capacity‡ of the plastic hinge at B gets exhausted prior to the span moment reaching $M_{uR}$, the 'inelastic' phase will get terminated at a stage $w_1 < w < w_2$. If the plastic hinge possesses adequate ductility, then the maximum collapse load is reached at $w_u = w_2$, corresponding to the formation of a 'mechanism'.

### 9.7.2 Moment Redistribution

As seen in the previous section, the distribution of bending moments in a continuous beam (or frame) gets modified significantly in the inelastic phase. The term *moment redistribution* is generally used to refer to the transfer of moments to the less stressed sections as sections of peak moments yield on their ultimate capacity being reached (as witnessed in the example above). From a design viewpoint, this behaviour can be taken advantage of by attempting to effect a redistributed bending moment diagram

† In this example, two plastic hinges will form simultaneously — one in each span, due to symmetry in the geometry as well as the loading.

‡ For a detailed calculation of plastic rotations, the reader is advised to consult Ref. 9.15.

which achieves a reduction in the maximum moment levels (and a corresponding increase in the lower moments at other locations).

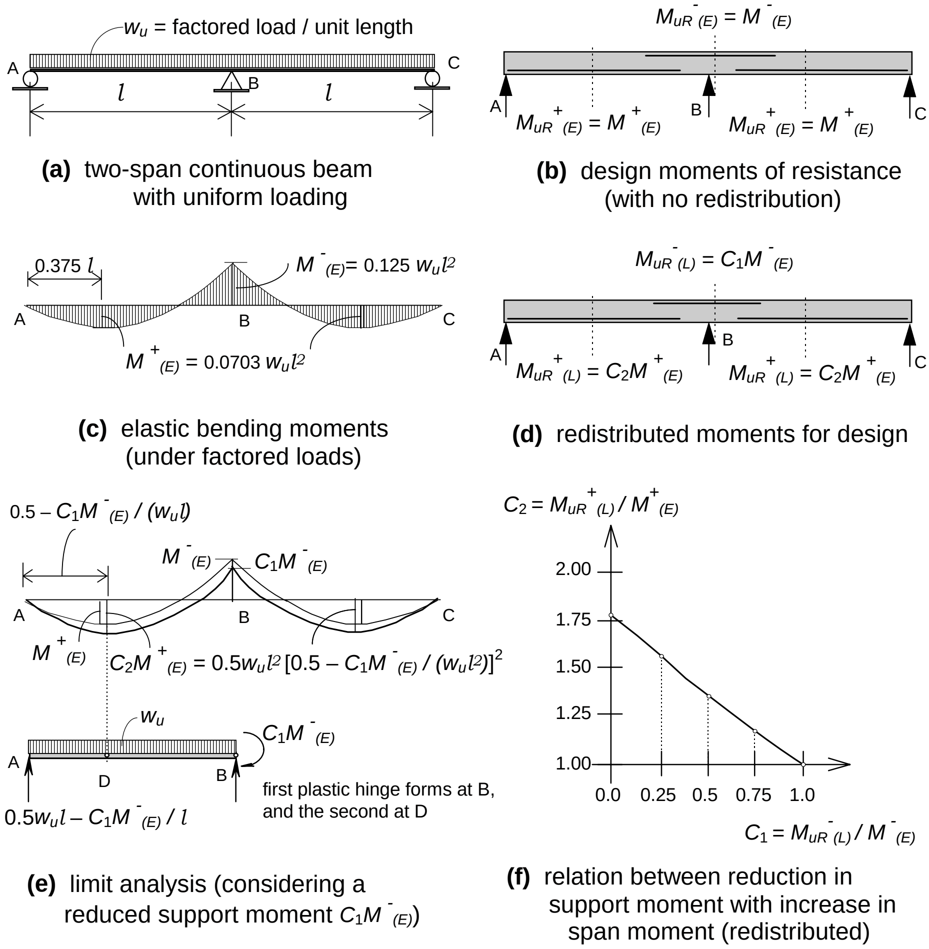



**Fig. 9.10** Moment redistribution in a two-span continuous beam

Such an adjustment in the moment diagram often leads to the design of a more economical structure with better balanced proportions, and less congestion of reinforcement at the critical sections.

Considering the earlier example of the two-span continuous beam [Fig. 9.10(a)], as a *design* problem (rather than an *analysis* problem), it may be seen that the designer has several alternative factored moment diagrams to choose from, depending on the amount of redistribution to be considered. If the design [Fig. 9.10(b)] is to be based on a purely elastic moment distribution (without considering any redistribution) then the bending moment diagram to be considered is as shown in Fig. 9.10(c), and the corresponding design support moment $M^-_{uR(E)}$ and span moment $M^+_{uR(E)}$ are obtained as:

$$
\begin{aligned}
M^-_{uR(E)} &= M^-_{(E)} \equiv \frac{1}{8} w_u l^2 \\
M^+_{uR(E)} &= M^+_{(E)} \equiv \frac{9}{128} w_u l^2 \quad \left(= \frac{9}{16} M^-_{(E)}\right)
\end{aligned}
\tag{9.1}
$$

where $M^-_{(E)}$ and $M^+_{(E)}$ denote the support and span moments in the elastic solution; the subscript ($E$) here represents *elastic* analysis†.

### *Reduction in Peak 'Negative' Moments*

The relatively high support moment $M_{(E)}^-$ may call for a large section (if singly reinforced); alternatively, for a given limited cross-section, large amounts of reinforcement may be required. Therefore, in such situations, it is desirable to reduce the design moment at the support to a value, say $C_1 M_{(E)}^-$ (where the factor $C_1$ has a value less than unity), and to correspondingly increase the span (positive) moments which are otherwise relatively low. The percentage reduction in the design support moment is given by:

$$\delta M = (1 - C_1) \times 100 \tag{9.2}$$

Consequent to a reduction in the support moment from $M_{(E)}^-$ to $C_1$ $M_{(E)}^-$, there is an increase in the design ('positive') moment in the span region from $M_{(E)}^+$ to $C_2$ $M_{(E)}^+$, where the factor $C_2$ obviously is greater than unity. Accordingly, as indicated in Fig. 9.9(d),

$$M^-_{uR(L)} = C_1 M^-_{(E)} \tag{9.3a}$$

$$M^+_{uR(L)} = C_2 M^+_{(E)} \tag{9.3b}$$

where the subscript ($L$) represents *limit* analysis. The factor $C_2$ (indicating the increase in the elastic span moment $M_{(E)}^+$) depends on the factor $C_1$ The factor $C_1$ is fixed (based on the percentage reduction desired), and the factor $C_2$ has to be determined for design — by considering 'limit analysis' [Fig. 9.10(e)]. It can be shown easily, by applying static equilibrium, that:

$$C_2 M^+_{(E)} = \frac{w_u l^2}{2} \left[ \frac{1}{2} - \left( \frac{C_1 M^-_{(E)}}{w_u l^2} \right) \right]^2 \tag{9.4}$$

Introducing Eq. 9.1 in Eq. 9.4, the following quadratic relationship between the constants $C_2$ and $C_1$ can be established:

$$C_2 = \frac{64}{9} \left( \frac{1}{2} - \frac{C_1}{8} \right)^2 \tag{9.5}$$

---

† In this example, it is tacitly assumed that the gravity loads indicated in Fig. 9.10 are entirely due to permanent dead loads, and that there are no live loads.

This relation is depicted graphically in Fig. 9.10(f). It is seen that, for instance, a 25 percent reduction in the elastic support moment ($M_{(E)}^{-}$) results in a 17.3 percent increase in the span moment ($M_{(E)}^{+}$) and a 50 percent reduction in $M_{(E)}^{-}$ results in a 36.1 percent increase in $M_{(E)}^{+}$. However, it should be noted that a large amount of moment redistribution requires a correspondingly large amount of plastic rotation of the plastic hinge (at the support, in this example) — which is often not practically feasible. If the desired ductility is not available, a premature failure is likely (due to crushing of the concrete in the compression zone at the plastic hinge forming region) at a load that is less than the factored load $w_u$.

> For the desired moment redistribution to take place, the plastic hinges that develop must have the required rotation capacities to 'hold on' without inducing premature failure.

Through proper design and detailing, it may be possible to muster the ductility required for significant amounts of moment redistribution. However, excessive moment redistribution can be undesirable if it results in plastic hinge formation at low loads (less than the service loads), and the consequent crack-widths and deflections are likely to violate *serviceability* requirements. Codes generally attempt to preclude such a situation by ensuring that plastic hinges are not allowed to form under normal service loads. In general, codes allow only a limited amount of redistribution in reinforced concrete structures.

### *Reduction in Peak 'Positive' Moments*

Moment redistribution[†] may also be advantageously applied to situations where 'positive' moments are relatively high and need to be reduced — for greater economy and less congestion of reinforcement.

For instance, with reference to the earlier example of the two-span continuous beam, if part of the total factored load $w_u$ is due to live load $w_{u,LL}$, then the arrangement of loads for maximum span moment is as shown in Fig. 9.11(a).

[†] It may be noted that 'redistribution' merely refers to a transfer of load effects from heavily stressed locations to less heavily (or lightly) stressed locations, regardless of whether the peak moments are 'positive' or 'negative'.

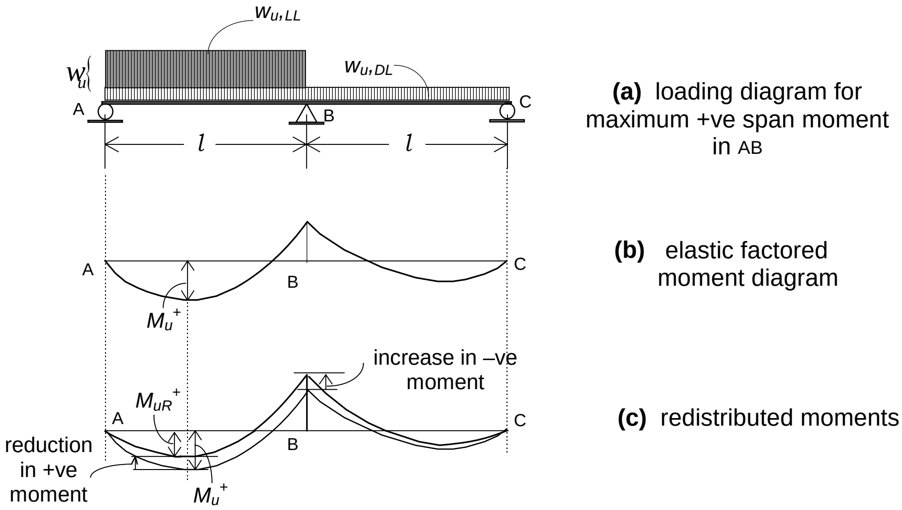



**Fig. 9.11** Moment redistribution: reduction in peak positive moment

The maximum span moment from elastic analysis [Fig. 9.11(b)] can be redistributed by allowing the first plastic hinge to form in the span region. The reduction in span moment is accompanied by a corresponding increase in the support moment [Fig. 9.11(c)] — to maintain equilibrium at the limit state. Such a redistribution may be desirable if the elastic span moment is relatively high, as would be the case if the live load component in the loading is high.

### 9.7.3 Code Recommendations for Moment Redistribution

The Code (Cl. 37.1.1) permits the designer to select the envelope of redistributed factored moment diagrams for design, in lieu of the envelope of elastic factored moments, provided the following conditions are satisfied:

1. **Limit Equilibrium:** The redistributed moments must be in a state of static equilibrium with the factored loads at the limit state.
2. **Serviceability**: The ultimate moment of resistance ($M_{uR}$) at any section should not be less than 70 percent of the factored moment ($M_{u,max}$) at that section, as obtained from the elastic moment envelope (considering all loading combinations). In other words, the flexural strength at any section should not be less than that given by the elastic factored moment envelope, scaled by a factor of 0.7:

$$M_{uR} \geq 0.7\left(M_{u,\max}\right)_{elastic} \quad \text{at all sections} \qquad (9.6)$$

   This restriction is aimed at ensuring that plastic hinge formation does not take place under normal service loads, and even if it does take place, the yielding of the steel will not be so significant as to result in excessive crack-widths and deflections. It is mentioned in the Explanatory Handbook to the Code that the value of 70% is arrived at as the ratio of service loads to ultimate loads with respect to load combinations involving a uniform load factor of 1.5, as 1/1.5 = 0.67 ≅ 0.7.

3. **Low Demand for High Plastic Hinge Rotation Capacities:** The reduction in the elastic factored moment ('negative' or 'positive') at any section due to a particular combination of factored loads should not exceed 30 percent of the absolute maximum factored moment ($M_{u,max}$), as obtained from the envelope of factored elastic moments (considering all loading combinations). Although the basis for this clause in the Code (Cl. 37.1.1.c) is different from the previous clause, which is based on the idea of preventing the formation of plastic hinges at service loads, for the case of gravity loading, in effect, this clause is no different. However, in the design of lateral load resisting frames (with number of storeys exceeding four), the Code (Cl. 37.1.1.e) imposes an additional over-riding restriction. The reduction in the elastic factored moment is restricted to 10 percent of $M_{u,max}$. Thus,

$$(M_u)_{elastic} - M_{uR} \le \begin{cases} 0.3(M_{u,\max})_{elastic} & \textit{in general} \\ 0.1(M_{u,\max})_{elastic} & \textit{for lateral load-resisting frames only} \end{cases} \tag{9.7}$$

This restriction is intended to ensure that the ductility requirements at the plastic hinge locations are not excessive.

4. **Adequate Plastic Hinge Rotation Capacity**: The design of the critical section (plastic hinge location) should be such that it is sufficiently under-reinforced, with a *low* neutral axis depth factor ($x_u/d$), satisfying:

$$\frac{x_u}{d} \le 0.6 - \frac{|\delta M|}{100} \tag{9.8}$$

where $|\delta M|$ denotes the percentage reduction in the maximum factored elastic moment $(M_{u,max})_{elastic}$ at the section:

$$\delta M \equiv \left[\frac{(M_{u,\max})_{elastic} - M_{uR}}{(M_{u,\max})_{elastic}}\right] \times 100 \tag{9.9}$$

In practice, it is sometimes more convenient to express Eq. 9.8 alternatively as:

$$\frac{\delta M}{100} \le 0.6 - \frac{x_u}{d} \tag{9.10}$$

For singly reinforced rectangular beam sections, the expression for $x_u/d$ is given by Eq. 5.11, which is repeated here for convenience, with $M_{uR} = M_u$,

$$\frac{x_u}{d} = 1.202\left[1 - \sqrt{1 - 4.597R/f_{ck}}\right] \tag{9.11}$$

### *Moment Redistribution in Beams*

Low values of $x_u/d$ (and, thus large values of $\delta M$) are generally not possible in beams without resorting to very large sections, which may be uneconomical. However, even with the extreme case of a *balanced* section (with $x_u = x_{u,max}$), it can be shown, by applying Eq. 9.10 and Eq. 4.50 (or Table 4.3), that

$$\delta M < \begin{cases} 6.9 & \text{for } Fe250 \\ 12.1 & \text{for } Fe415 \\ 14.4 & \text{for } Fe500 \end{cases} \quad \text{with } x_u = x_{u,max} \tag{9.12}$$

Thus, it is seen that a limited moment distribution (for example, up to 12.1 percent in the case of Fe 415 steel) is possible, even with the *limiting* neutral axis depth permitted for design [refer Chapter 4].

### Inelastic Analysis of Slabs

As discussed earlier (in Chapter 5), the thicknesses of reinforced concrete slabs are generally governed by deflection control criteria, with the result that the sections are invariably under-reinforced, with low $x_u/d$ values. Hence, significant inelastic action is possible in such cases.

It may be noted, however, that, in the case of one way continuous slabs, (and continuous beams), no moment redistribution is permitted by the Code (Cl. 22.5.1) if the analysis is based on the use of the Code moment coefficients [Table 12 of the Code]. This is so, because such coefficients are only approximations, and minor errors are assumed to be accommodated through the inherent capacity for moment redistribution in the structure [Ref. 9.2].

In the case of *two-way* slab systems, which are statically indeterminate, detailed inelastic analysis (*yield line analysis*) is often resorted to [Fig. 9.12], and, in fact, the moment coefficients given in the Code (Table 26) for two-way rectangular slabs with various possible edge conditions are based on such analyses [refer Chapter 11].

'Yield line analysis' is the equivalent for a two-dimensional flexural member (plate or slab) of the limit analysis of a one-dimensional member (continuous beam), explained in Section 9.7.1. It is based on the elastic-plastic $M$–$\varphi$ relation [Fig. 9.8], according to which, as the moment at a section reaches $M_{uR}$, a plastic hinge is formed, and therefore rotation takes place at constant moment. In slabs, peak moments occur along *lines* (such as 'negative' moments along support lines and 'positive' moments along lines near the midspan), and hence the yielding (plastic hinge formation) occurs along lines ("yield lines"), and not at sections, as in beams. In a skeletal structure (continuous beam, grid, plane frame, space frame), the ultimate (collapse) load is reached when sufficient number of plastic hinges are formed to transform the structure into a mechanism. In a similar way, the ultimate load is reached in plates when sufficient number of yield lines are formed to transform the slab into a series of plate segments[†] connected by 'yield lines', resulting in mechanism type behaviour.

As in the case of limit analysis of beams and frames, it is assumed in 'yield line analysis' [Fig. 9.12] that the plastic hinges which form (along the 'yield lines')

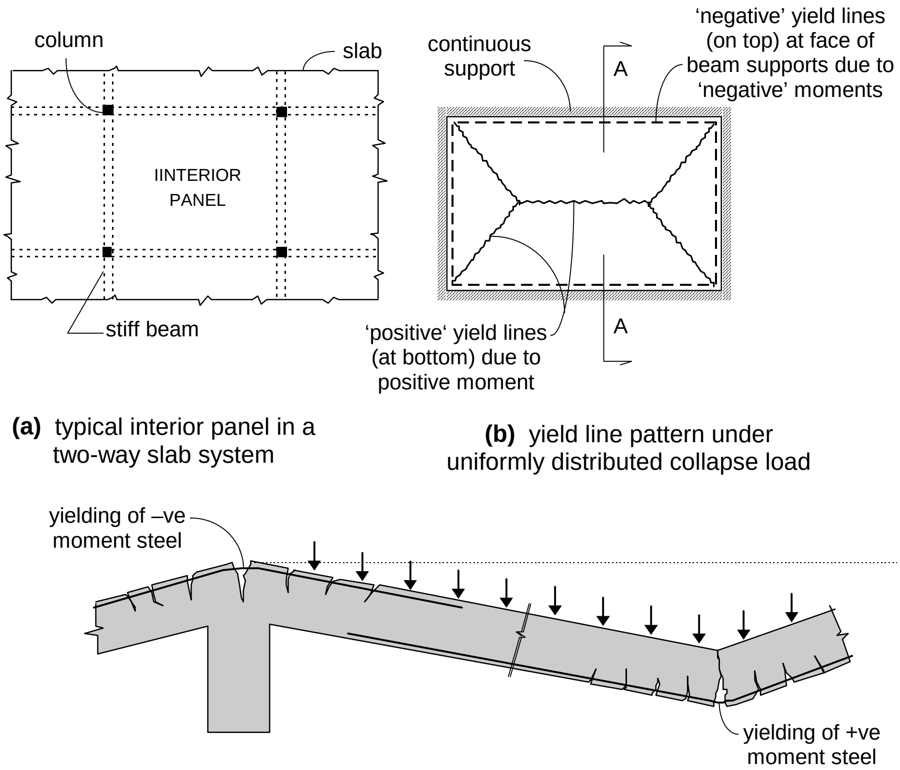
[†] Such plate segment can freely rotate about the 'yield line', in much the same way as a door can rotate about a line hinge.

possess adequate plastic rotation capacities to 'hold on' till a complete set of yield lines are formed, leading to a mechanism type of collapse. This is justifiable in view of the relatively low $x_u/d$ values in slabs in general. Applications of yield line analysis are discussed further in Chapter 11. For a more comprehensive study, reference may be made to Refs. 9.18–9.22.

### *Moment Redistribution in Columns*

Reduction of moments on account of moment redistribution is generally not applied to columns, which are essentially *compression* members that are also subjected to bending (due to frame action). In general, the neutral axis location[‡] at the limit state is such that the Code requirements [Eq. 9.8] cannot be satisfied by a column section — unless the column is very lightly loaded axially and the eccentricity in loading is very large. Furthermore, in the case of a typical beam-column joint in a reinforced concrete building, it is desirable that the formation of the plastic hinge occurs in the beam, rather than in the column, because the subsequent collapse is likely to be less catastrophic. This is particularly necessary in earthquake-resistant design [refer Chapter 16].

[‡] When the loading on the column is not very eccentric, the neutral axis will lie *outside* the cross-section.




**(a)** typical interior panel in a two-way slab system

**(b)** yield line pattern under uniformly distributed collapse load

**(c)** deflected shape at collapse (section A–A)

**Fig. 9.12** Concept underlying yield line analysis of slabs

It may be noted that when moment redistribution is applied to frames with the objective of reducing the peak moments in *beams*, this will also result in changes in the elastic factored moments in columns. These changes in column moments may be ignored in design, if the redistribution results in a reduction in column moments (which is usually the case). However, if the redistribution results in an increase in column moments, then the column section must necessarily be designed for the increased moments.

### *Limit Analysis with Torsional Hinges*

The basis for the flexural plastic hinge formation, moment redistribution and Limit Analysis is the moment-curvature relation of under-reinforced beams, which can be idealised as a bilinear elastic-plastic relation [Fig. 9.8]. Interestingly, a beam with adequate torsional reinforcements has a similar bilinear elastic-plastic torque-twist relation [see Fig. 7.4(b)]. In such beams, the formation of a *plastic hinge in torsion* and subsequent redistribution of torque/moments can occur [Ref. 9.23]. Limit Analysis can also be extended to structures with members subjected to significant torsion, such as transversely loaded grid structures (or bridges), where the collapse mechanisms may involve torsional hinges as well [Ref. 9.24]. Recognising this, some codes [Ref. 9.3] permit the limiting of the maximum design torque in spandrel beams to $0.67T_{cr}$. Here $T_{cr}$ is the cracking torque of the spandrel beam, and $0.67T_{cr}$ represents a torque corresponding to a 'plastic torsional hinge' formation and consequent cracking and reduction in torsional stiffness.

## 9.8 DESIGN EXAMPLES

### EXAMPLE 9.1

Analyse a three-span continuous beam (with equal spans $l$), subjected to a uniformly distributed load $w$ per unit length, to determine the critical 'positive' moments $M_1$ (in the end span) and $M_2$ (in the interior span), as well as the 'negative' moment $M_3$ at the continuous support [Fig. 9.13(a)]. Assume that the dead load ($w_D$) and live load ($w_L$) components of the total load ($w$) are equal. Also assume all spans to have the same cross-section. Compare the moment coefficients obtained by

a) elastic analysis considering total load $w$ on all spans† ;
b) elastic analysis considering 'pattern loading';
c) Code recommendations for moment coefficients.

**SOLUTION**

**(a) Elastic analysis considering total load (w) on all spans**

By taking advantage of the symmetry in the structural geometry and loading, the analysis can be easily performed by considering a simple one-cycle moment distribution, as shown in Fig. 9.13(a). The results indicate:

$$\textit{span moments} \quad \begin{cases} M_1 = +\ 0.0800wl^2 \\ M_2 = +\ 0.0250wl^2 \end{cases}$$

$$\textit{support moment} \quad M_3 = -0.1000wl^2$$

**(b) Elastic analysis considering 'pattern loading'**

Here, too, the advantage of symmetry of the structure can be availed of for analysing the maximum span moments due to live loads appropriately arranged, as shown in Fig. 9.13(b) and (c). The results of uniform dead loads on *all* spans is obtainable form Fig. 9.13(a), by considering $w_D$ in lieu of $w$. By superimposing the effects of live load and dead load contributions separately, and considering $w_L = w_D = 0.5w$, the final results (critical moments) may easily be obtained, as shown in Fig. 9.14(a),(b).

$$M_1 = +\ 0.0903wl^2$$

$$M_2 = +\ 0.0500wl^2 \ \text{ and } \ -0.0125\,wl^2$$

† Note that such an analysis is included here only for the purpose of comparison; such analysis is not permitted by the Code for design purposes.

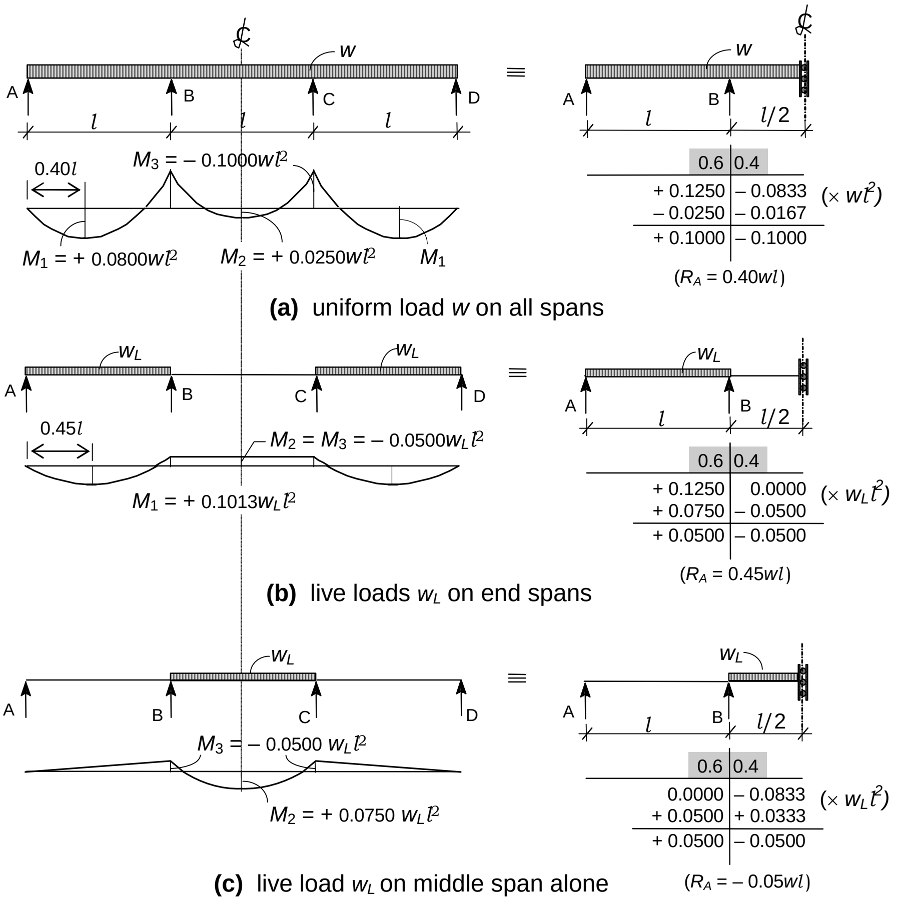



**Fig. 9.13** Example 9.1 — analysis by moment distribution method

The loading arrangement for maximum 'negative' moment $M_3$ at the continuous support is shown in Fig. 9.14(c). The corresponding moment distribution table is shown in Fig. 9.14(c), from which it follows that

$$M_3 = -\ 0.1082wl^2$$

**(c) Use of Code moment coefficients**

The results are easily obtainable from Table 9.1 of this chapter (Cl. 22.5):

$$\textit{span}\text{ moments}\ \begin{cases} M_1 = +(0.5w)l^2\left(\dfrac{1}{12}+\dfrac{1}{10}\right) = +0.0917wl^2 \\ M_2 = +(0.5w)l^2\left(\dfrac{1}{16}+\dfrac{1}{12}\right) = +0.0729wl^2 \end{cases}$$

$$\text{support moments}\ \ M_3 = -\ (0.5w)l^2\left(\frac{1}{10}+\frac{1}{9}\right) = -0.1056wl^2$$

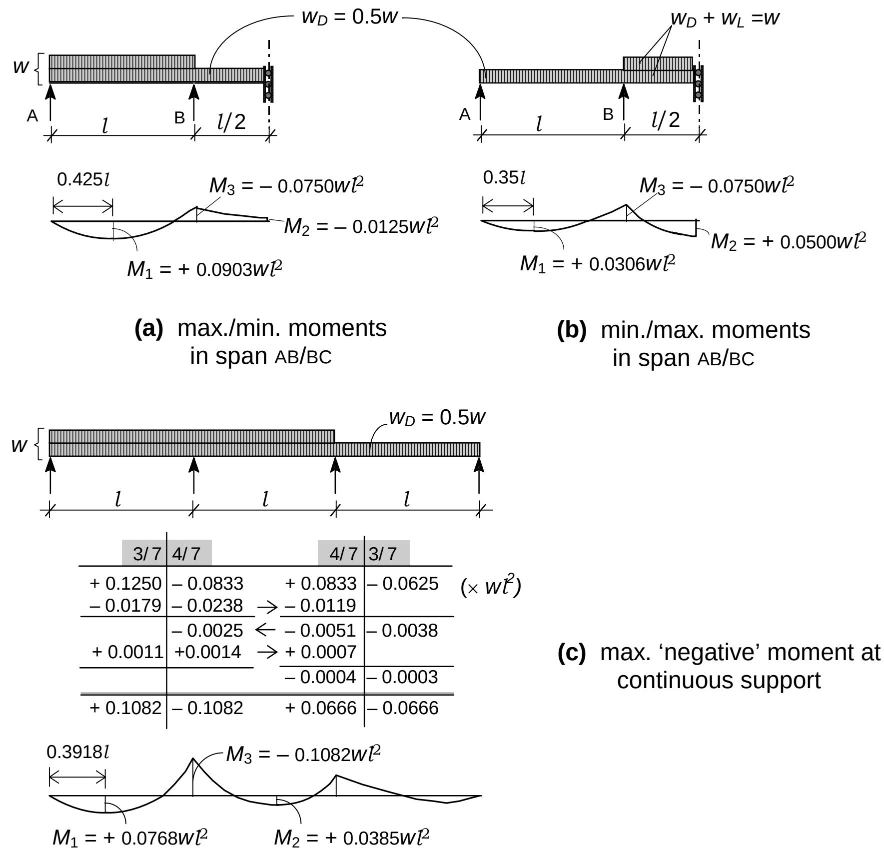



**Fig. 9.14** Example 9.1 — analysis considering $w_D = w_L = 0.5w$

## Comparison of results

The results of moment coefficients obtained by the three methods are summarised in Table 9.2 as follows:

**Table 9.2** Example 9.1 — Moment coefficients by various methods

| Method | Span moments | | Support moment |
|---|---|---|---|
| | $M_1$ | $M_2$ | $M_3$ |
| (a) Total load on all spans | + 0.0800 | + 0.0250 | – 0.1000 |
| (b) Pattern loading | + 0.0903 | + 0.0500<br>– 0.0125 | – 0.1082 |
| (c) Code coefficients | + 0.0917 | + 0.0729 | – 0.1056 |

**Comments**

It is evident that the 'exact' analysis corresponds to case (b), viz. consideration of 'pattern loading'. With reference to these results, for the particular problem analysed, it follows that:

- the simplified consideration of total loading on all spans [case (a)] results in a 50 percent under-estimation in the 'positive' midspan moment ($M_2$) in the interior span; $M_1$ and $M_3$ are also under-estimated, but marginally;
- the Code coefficient method over-estimates the midspan 'positive' moment $M_3$ in the interior span by as much as 45.8 percent and the moment $M_1$ in the end span by 1.5 percent;
- in general, the relatively crude method of considering total loads on all spans results in an unconservative design, whereas the use of Code moment coefficients results in a relatively conservative† design.

## EXAMPLE 9.2

**(a)** Based on the elastic factored moment envelope obtainable from Example 9.1, design the flexural reinforcement in the three-span continuous beam of Example 9.1, given the following data:

$w$ = 30 kN/m ($w_D$ = 15 kN/m, $w_L$ = 15 kN/m); $l$ = 8.0 m
Assume a partial load factor of 1.5 for both dead loads and live loads (as per IS Code). Use M 20 concrete and Fe 415 steel.

**(b)** Redesign the three-span continuous beam by applying moment redistribution (to the extent permitted by the Code).

### SOLUTION

**(a)**

- Factored load $w_u = 1.5 \times 30 = 45$ kN/m

  $\Rightarrow w_u l^2 = 45 \times (8.0)^2 = 2880$ kNm
- The elastic factored moment envelope, based on the results of Example 9.1, is shown in Fig. 9.15(a). The critical design moments are:

  $$\textit{span moments} \begin{cases} M_{u1} = +\ 0.0903 w_u l^2 = +\ 260.1 \text{ kNm (end span)} \\ M_{u2} = +\ 0.0500 w_u l^2 = +\ 144.0 \text{ kNm (interior span)} \end{cases}$$

  [Also note that $M_{u2,\,min} = -\,0.0125\, w_u l^2$]

  *support moment* $M_{u3} = -\ 0.1082\ w_u l^2 = -\,311.7$ kNm

---

† Note that although the moment at the continuous support is slightly under-estimated (by 2.4 percent), this difference can be easily accommodated by moment redistribution. Also, the *minimum* span moments in the *central* span are hogging in nature (requiring steel to be designed at top); however, $M_2 = -\,0.0125wl^2$ is relatively small and is likely to fall within the flexural strength of the nominal top steel provided.

**Proportioning of beam section***:*

- Assume a beam width $b$ = 300 mm. Considering the maximum design moment of 311.7 kN/m, for an under-reinforced section,

effective depth $d = \sqrt{\dfrac{M_u}{Rb}}$

where $R_{lim}$ = 0.1389$f_{ck}$ = 0.1389 × 20 = 2.778 MPa

$\Rightarrow d_{\min} = \sqrt{\dfrac{311.7 \times 10^6}{2.778 \times 300}}$ = 611.6 mm

- Assume overall depth $D$ = 700 mm and $d \approx 655$ mm (for an economical design)

**Design of flexural reinforcement**

- $\dfrac{p_t}{100} \equiv \dfrac{A_{st}}{bd} = \dfrac{f_{ck}}{2 f_y}\left[1 - \sqrt{1 - 4.598\, R/f_{ck}}\right]$

where $R \equiv \dfrac{M_u}{bd^2}$

- Considering $f_{ck}$ = 20 MPa, $f_y$ = 415 MPa, $b$ = 300 mm, $d$ = 655 mm, the following results are obtained:
  1. for $M_{u1}$ = + 260.1 kNm, $R$ = 2.021 MPa $\Rightarrow p_t$ = 0.647
     $\Rightarrow (A_{st})_{reqd}$ = 1271 mm$^2$
     <u>Provide 2–25ϕ + 1–20ϕ</u> at bottom in the end span [$A_{st}$ = 1296 mm$^2$ > 1271]
  2. for $M_{u2}$ = + 144.0 kNm, $R$ = 1.119 MPa $\Rightarrow p_t$ = 0.333
     $\Rightarrow (A_{st})_{reqd}$ = 655 mm$^2$
     <u>Provide 2–16ϕ + 1–20ϕ</u> at bottom in the central span [$A_{st}$ = 716 mm$^2$ > 654]
     $M_{u2} = -\,0.0125\, w_u l^2$ = 36 kNm is accommodated by the nominal top steel (2–16ϕ bars) provided[†] [see Fig. 9.15(b)].
  3. for $M_{u3}$ = – 311.7 kNm, $R$ = 2.422 MPa $\Rightarrow p_t$ = 0.805
     $\Rightarrow (A_{st})_{reqd}$ = 1583 mm$^2$
     <u>Provide 2–28 ϕ + 2–16 ϕ</u> at top [$A_{st}$ = 1634 mm$^2$ > 1583] up to, say 0.3$l$ on the end span side, and 0.4$l$ on the central span side of the continuous support; beyond this, the 2–16 ϕ bars may be extended over the span regions as nominal top steel.

**(b)**

- By applying 'moment redistribution', the maximum 'negative' moment at the continuous support can be reduced. The amount of reduction possible depends on the plastic hinge rotation capacity at the section. [Eq. 9.10 has to be satisfied].
- The maximum reduction in moment permitted by the Code is 30 percent, corresponding to which, the design moment at the continuous support is given by:

$\tilde{M}_{u3} = 0.7\, M_{u3} = 0.7 \times -\,0.1082\ \, w_u l^2$

---

[†] The flexural strength due to the 2–16ϕ bars, in fact, works out to 91kNm*;* this may be verified.

$= -0.07574\, w_u l^2$
$= -218.2$ kNm

- Assuming $b$ = 300 mm and $d$ = 655 mm (as before),
- [Eq. 9.11]: $\dfrac{x_u}{d} = 1.202\left[1-\sqrt{1-\dfrac{4.598\times 218.2\times 10^6}{20\times 300\times 655^2}}\right]$

  $= 0.263$

  which satisfies the Code requirement [Eq. 9.8]:

  $\dfrac{x_u}{d} \le 0.6 - \dfrac{30}{100} = 0.30$ (for 30% reduction in $M_{u3}$).

  Hence, the desired plastic rotation capacity is ensured.

**Bending moment envelope after redistribution**

- In the elastic analyses for maximum/minimum span moments [Fig. 9.14(a)], the support moment was found to be equal to $0.0750 w_u l^2$, which is less than the design support moment (after redistribution), $\tilde{M}_{u3} = 0.07574 w_u l^2$. Hence, no plastic hinge will form at the continuous support under these loading conditions (alternate spans loaded with live load). Accordingly, the bending moment distributions shown in [Fig. 9.14(a),(b)] do not get altered, as no redistribution takes place.
- For the loading pattern shown in Fig. 9.14(c), the possibility of redistribution has been recognised by reducing the design flexural strength at the continuous support from the elastic solution value of $-0.1082\, w_u l^2$ to $\tilde{M}_{u3} = -0.07574\, w_u l^2$. What remains to be done is to calculate the revised span moments and locations of points of inflection, corresponding to the lowering of the support moment, i.e., redistribution.
- By performing an analysis of the continuous beam with a plastic hinge at the continuous support, (with $\tilde{M}_{u3} = -0.07574\, w_u l^2$) [Fig. 9. 15], the maximum 'positive' moments in the end span and central span are obtained as:

  *end span*: $\tilde{M}_{u1} = +0.0900\, w_u l^2 < (M_{u1})_{elastic} = +0.0903\, w_u l^2$

  *central span*: $\tilde{M}_{u2} = +0.0497\, w_u l^2 < (M_{u2})_{elastic} = +0.0500\, w_u l^2$

  In fact, the bending moment diagrams obtained, after redistribution [Fig. 9.15(c)], for spans AB and BC, are very much similar to those obtained earlier in Fig. 9.14(a), (b).
- Thus, it is seen that, in the case of the end span as well as the central span, the design moments remain governed by the loading conditions given in Fig. 9.14(a) and (b), with $\tilde{M}_{u1} = +0.0903\, w_u l^2 = +260.1$ kNm, and $\tilde{M}_{u2} = +0.0500\, w_u l^2 = +144.0$ kNm. The corresponding 'positive' moment envelopes are shown in Fig. 9.16(a).

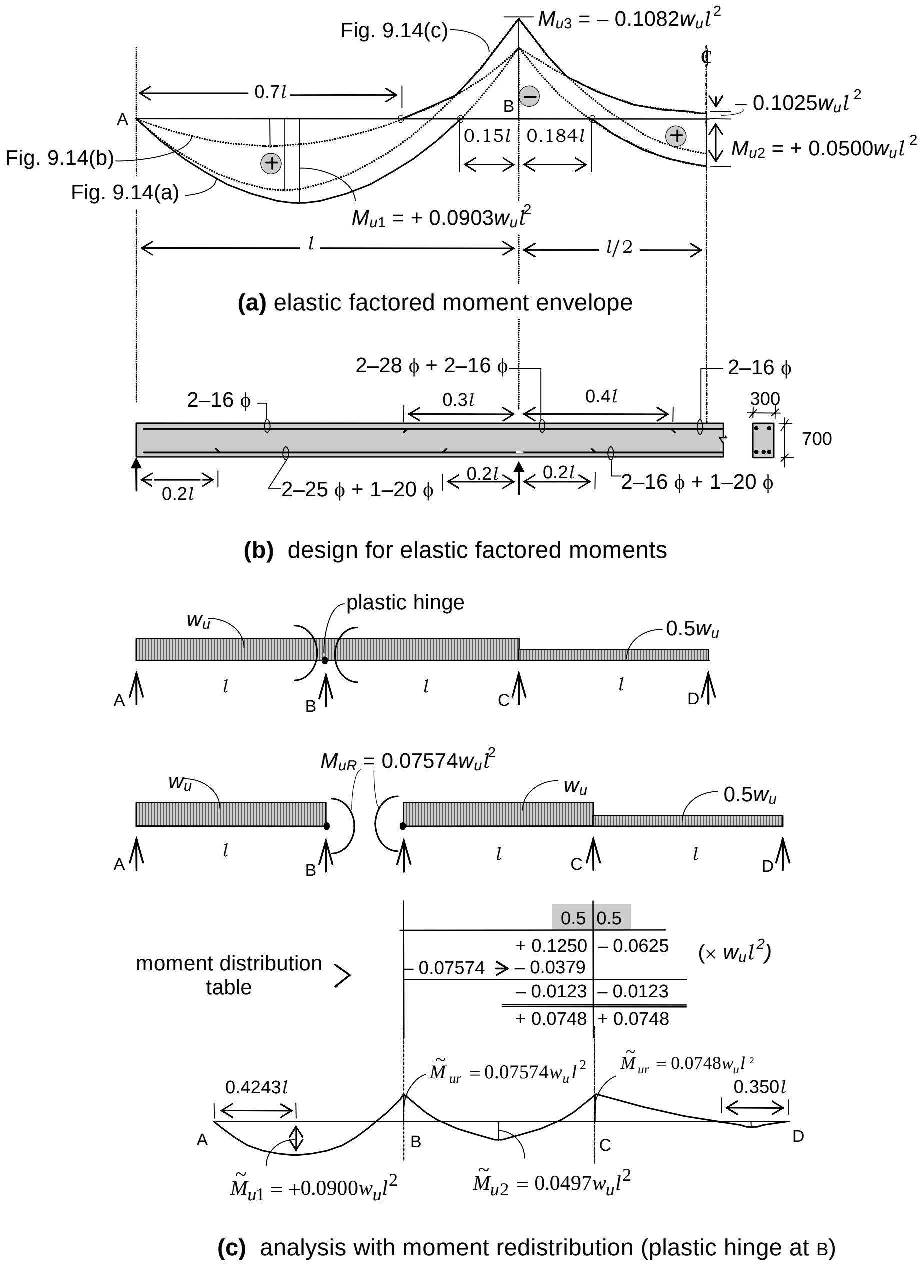



**Fig. 9.15** Example 9.2

- The moment envelope is obtained by combining the bending moment diagrams of Figs. 9.14(a), 9.14(b) and 9.15(c); the diagram of 9.15(c) is seen to practically merge with Fig. 9.14(a),(b). This is depicted in Fig. 9.16(a). In combining these diagrams, the outermost lines yield the moment envelope.

**Design of flexural reinforcement**

- Considering $f_{ck}$ = 20 MPa, $f_y$ = 415 MPa, $b$ = 300 mm, $d$ = 655 mm (as before),

  for $\tilde{M}_{u1}$ = + 260.1 kNm, (which is identical to Part(a) of this Example).

  Provide 2–25 ϕ + 1–20 ϕ at bottom in the end span ($A_{st}$ = 1296 mm$^2$ >1256).

  *Note*: There is no increase in the reinforcement provided [refer Fig. 9.15(b)] on account of redistribution.

  1) for $\tilde{M}_{u2}$ = + 144.0 kNm, (which is identical to Part(a) of this Example)

     Provide 2–16 ϕ + 1–20 ϕ at bottom in the central span (exactly as before).

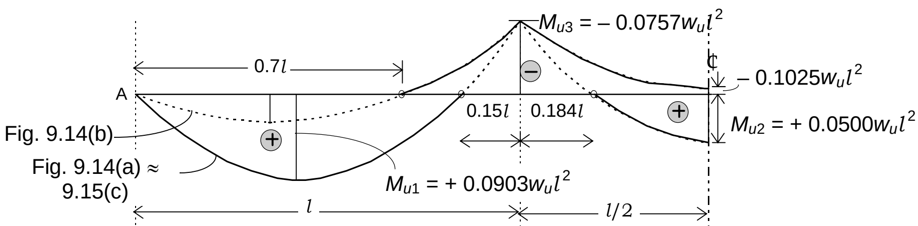
  2) for $\tilde{M}_{u3}$ = + 218.2 kNm, $R \equiv \dfrac{M_u}{bd^2}$ = 1.695 MPa

     $\Rightarrow p_t = 0.527 \Rightarrow (A_{st})_{reqd} = 1036$ mm$^2$

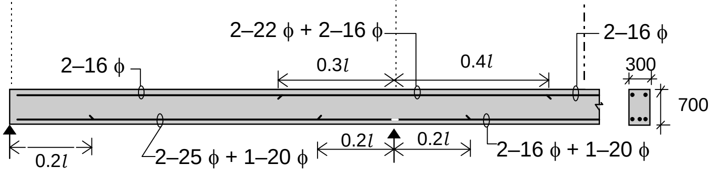
     Provide 2–22 ϕ + 2–16 ϕ at top ($A_{st}$ = 1162 mm$^2$ >1036), with the 2–22 ϕ bars curtailed exactly as before.

     *Note*: This results in some savings, compared to the earlier design [Fig. 9.15(b)], which required 2–28 ϕ + 2–16 ϕ.

- The detailing is shown in Fig. 9.16(b).




**(a)** design moment envelope (after redistribution)




**(b)** design for redistributed moments

**Fig. 9.16** Example 9.2 (contd.)

## REVIEW QUESTIONS

9.1 Comment on the apparent inconsistency in combining *elastic analysis* of structures with *design* at the *ultimate limit state*.
9.2 Explain how the critical live load patterns in a plane frame can be obtained by the application of the Müller-Breslau Principle.
9.3 Under what circumstances does the Code permit the neglect of 'pattern loading' for the purpose of arriving at the critical design moments in a framed structure?
9.4 Why does the Code disallow *moment redistribution* when bending moments in continuous beams are based on the Code moment coefficients?
9.5 When is it inappropriate to apply the *substitute frame method* for multi-storeyed buildings?
9.6 Justify the use of approximate methods of frame analysis for multi-storeyed buildings.
9.7 What are the basic assumptions underlying the approximate methods of lateral load analysis of multi-storeyed frames?
9.8 What are the problems associated with specifying the flexural stiffnesses of reinforced concrete frame members for the purpose of structural analysis? How may these problems be resolved?
9.9 "The bending moment diagram obtained from frame analysis needs adjustment in order to obtain the design moments at *beam-column junctions*". Discuss this statement.
9.10 What is meant by 'moment redistribution' and what are its implications in design?
9.11 Explain the bases underlying the various limitations imposed by the Code with regard to moment redistribution.
9.12 Can moment redistribution be applied to reduce bending moments in columns? Explain.
9.13 Can moment redistribution be applied to reduce bending moment in beams with doubly reinforced sections? Explain.
9.14 What is meant by a 'torsional plastic hinge'? Cite practical situations where such hinges are encountered.

## PROBLEMS

9.1 Repeat the problem given in Example 9.1, considering the live load component ($w_L$) to constitute 80 percent of the total load ($w$).

*Ans.: (a)* $M_1 = +0.0800 w_u l^2$; $M_2 = +0.0250 w_u l^2$; $M_3 = -0.1000 w_u l^2$.
*(b)* $M_1 = +0.0968 w_u l^2$; $M_2 = +0.0650 w_u l^2$; $M_3 = -0.1143 w_u l^2$.
*(c)* $M_1 = +0.0968 w_u l^2$; $M_2 = +0.0750 w_u l^2$; $M_3 = -0.1089 w_u l^2$.

9.2 Using the results of Problem 9.1, design the three-span continuous beam, considering 25 percent reduction in the maximum factored elastic 'negative' moment, with moment redistribution. Assume $w$ = 30 kN/m, $l$ = 8 m, M 20 concrete and Fe 415 steel.
9.3 Consider the symmetric portal frame in Fig. 9.17. From an elastic analysis under factored loads, the hogging moment at B and C is obtained as $M_B = M_C$ = 100 kNm.

(a) Determine the ratio $M_{uR}^{-} : M_{uR}^{+}$, as required by elastic analysis, where $M_{uR}^{-}$ and $M_{uR}^{+}$ denote respectively the design hogging and sagging moment capacities for the beam BC. In order to make this ratio 1:1, determine the percentage redistribution required. Draw the redistributed bending moment diagram for the portal frame, indicating the values at the critical locations.
(b) Show that the above redistribution is allowable, given that the beam is 200 mm wide, and has an effective depth of 450 mm. Assume M20 concrete and Fe 415 steel. Also determine the area of tension steel $A_{st}$ (mm$^2$) required at the support/midspan sections.

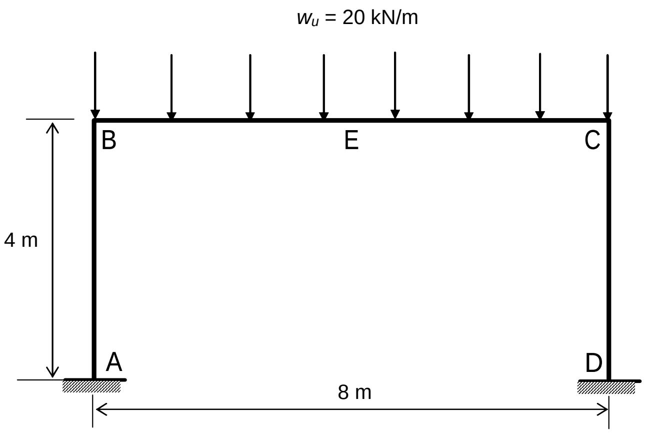



**Fig. 9.17** Problem 9.3

9.4 Analyse the design moments and sketch the moment envelope for the beam members in the *substitute frame* shown in Fig. 9.18. Assume suitable dimensions for the frame members. The beams are integrally connected to a floor slab 150 mm thick.

9.5 Applying appropriate moment redistribution (to the results of Problem 9.3), design the flexural reinforcement in the beams AB and BC in Fig. 9.18. Assume M 20 concrete and Fe 415 steel.

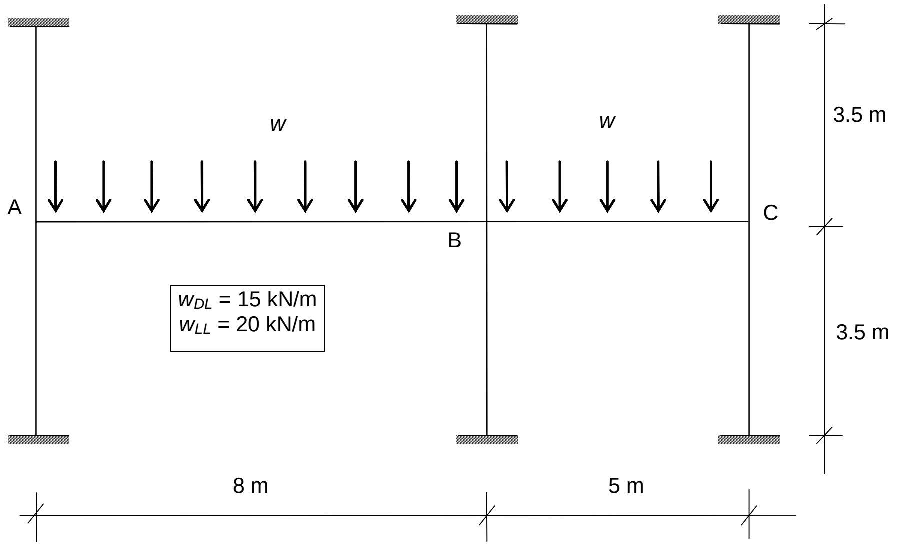



**Fig. 9.18** Problems 9.4, 9.5

## REFERENCES

9.1 Hsieh, Y.Y., *Elementary Theory of Structures*, Prentice-Hall, Inc., New Jersey, 1970.

9.2 — *Explanatory Handbook on Indian Standard Code of Practice for Plain and Reinforced Concrete (IS 456:1978)*, Special Publication SP:24, Bureau of Indian Standards, New Delhi, 1983.

9.3 CSA Standard CAN – A23.3 – M84 — *Design of Concrete Structures for Buildings*, Canadian Standards Association, Rexdale, Ontario, 1984.

9.4 Wang, C.K., *Intermediate Structural Analysis*, McGraw-Hill International edition, 1983.

9.5 Weaver, W. and Gere, J.M., *Matrix Analysis of Framed Structures*, Second edition, Van Nostrand Co., New York, 1980.

9.6 — *The Applications of Moment Distribution*, The Concrete Association of India, Publication of The Associated Cement Companies Limited, Bombay, 1978.

9.7 — *Frames and Continuous Structures — Analysis by Moment Distribution*, Concrete Information IS 210.01D, Portland Cement Association, Skokie, Illinois, 1980.

9.8 — *Continuity in Concrete Building Frames*, Fourth edition, Portland Cement Association Chicago, Illinois, 1959.

9.9 Clough, R.W. and Penzien, J., *Dynamics of Structures*, Second edition, McGraw-Hill International edition, 1993.

9.10 Norris, C.H., Wilbur, J.B. and Utku, S., Elementary Structural Analysis, Third edition, McGraw-Hill International edition,1977.

9.11 ACI Committee 318, *Commentary on Building Code Requirements for Reinforced Concrete* (ACI 318–89), Am. Conc. Institute, Detroit, Michigan, 1989.
9.12 ACI–ASCE Committee 428, *Limit Design of Reinforced Concrete Beams and Frames — Addendum*, (supplement to bibliographies published in ACI Journal in Nov. 1961 and Dec. 1962), Journal ACI, Vol. 60, Oct. 1963, pp 1471–1473.
9.13 Cohn, M. Z., *Limit-Design Solutions for Concrete Structures*, ASCE Journal, Struct. Div., Vol. 83, Feb. 1967, pp 37–57.
9.14 Lazaro, A.L. and Richards, R., *Full-Range Analysis of Concrete Frames*, ASCE Journal, Struct. Div., Vol. 99, August 1973, pp1761–1783.
9.15 Baker, A.L.L., *Ultimate Load Theory Applied to the Design of Reinforced and Prestressed Concrete Frames*, Concrete Publications Ltd., London, 1956.
9.16 Colin M. A., *Limit Design of Reinforced Concrete Frames*, ASCE Journal, Struct. Div., Vol.94, Oct. 1968, M. 2647–2483.
9.17 Furlong, R.W., *Design of Concrete Frames by Assigned Limit Moments*, Journal ACI, Vol.67, Apr. 1970, pp 341–353.
9.18 Johansen, K.W., *Yield Line Theory*, (translated from the Danish), Cement and Concrete Association, London, 1962.
9.19 Wood, R.H., *Plastic and Elastic Design of Slabs and Plates*, The Ronald Press Company, New York, 1961.
9.20 Jones, L.L. and Wood, R.H., *Yield Line Analysis of Slabs*, Thames and Hudson, Chatto and Windus, London, 1967.
9.21 Shukla, S. N., *Handbook for Design of Slabs by Yield line and Strip Methods*, Structural Engineering Research Centre, Roorkee, India, 1973.
9.22 Purushothaman, P., *Reinforced Concrete Structural Elements — Behaviour, Analysis and Design*, Tata McGraw Hill Publ. Co. Ltd., New Delhi, 1984.
9.23 Unnikrishna Pillai, S. and Bhargavan, K.K., *Bending Moment-Torque Redistribution in Reinforced Concrete Frames*, Proc., Conference on Tall Buildings, Kuala Lampur, Dec. 1974, pp 4.66–4.75.
9.24 Pillai, S.U. and Lash, S.D., *Ultimate Strength of Reinforced Concrete Grid and Slab Bridges*, First International Symposium: Concrete Bridge Design, ACI Publication SP–23, Am. Conc. Institute, Detroit, 1969, pp 361–384.

# 10

# Serviceability Limit States: Deflection and Cracking

## 10.1 INTRODUCTION

According to the design philosophy of the *limit states method* (refer Section 3.5), there are two distinct classes of limit states to be considered: *ultimate limit states* and *serviceability limit states*. Whereas the former deals with *safety* in terms of strength, overturning, sliding, buckling, fatigue fracture, etc., the latter deals with *serviceability* in terms of deflection, cracking, vibration, durability, etc. The aim of structural design by the LSM philosophy is to ensure both 'safety' and 'serviceability', so that the structure performs its intended function satisfactorily.

In the chapters considered hitherto, the focus has been on *ultimate* limit states (or 'limit states of collapse') dealing with *strength*† (flexure, shear, torsion and bond). The importance of controlling *deflection* and *cracking* has been briefly covered in Chapter 5, in terms of proper detailing of reinforcement (Section 5.2) and limiting span/effective depth ratios (Section 5.3) in beams and slabs. In fact, the Code (Cl. 42 and 43) does not require the designer to perform any explicit check on deflection or crack-width for *all normal cases*, provided the Code recommendations for limiting *l*/*d* ratios (for deflection control) and spacing of flexural reinforcement (for crack control) are complied with. However, the minimum depths indicated by these *l*/*d* ratios may not be adequate for certain sequences of shoring during construction or large live to dead load ratios.

It is well recognised that, in modern practice, structural 'failures' are all-too-common in terms of serviceability, and are relatively rare in terms of safety. In particular, it is the serviceability limit state of *durability* that calls for particular attention, indeed all over the world. The problem of inadequate durability is linked

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† The ultimate limit state of strength in *compression*, not considered so far, is discussed at length in Chapter 13. The ultimate limit states in overturning and sliding are considered in Chapter 14, in connection with the design of foundations.

not only to such factors as improper making of concrete, chemical attack from the environment and corrosion of reinforcement [refer Section 2.13], but also to inadequate cover to reinforcement [refer Section 5.2.1], improper detailing and inadequate sizes for structural members, resulting in excessive deflections and crack-widths (and consequent loss of durability). Some of these problems as well as their solutions are addressed in Chapter 15 of this book.

Adoption of limit states design and higher grades of concrete and steel in modern reinforced concrete design has led to overall thinner member sections and higher stress levels at service loads. These, in turn, have resulted in larger deflections, crack-widths, vibrations, etc., in such structures, compared to earlier ones designed by more the conservative working stress design and using mild steel and lower grades of concrete. Hence, the need for serviceability checks has assumed greater importance in present-day design.

The scope of the present chapter is limited to describing methods of explicitly calculating deflections and crack-widths in flexural members for the purpose of checking the serviceability limit states of deflections and cracking. This is required especially when the limiting $l/d$ ratios of the Code are not complied with, when the specified load is abnormally high, and in special structures where limits to deflection and crack-width are of particular importance.

## 10.2 SERVICEABILITY LIMIT STATES: DEFLECTION

### 10.2.1 Deflection Limits

Various factors are involved in prescribing limits to deflection in flexural members, such as:

- aesthetic/psychological discomfort;
- crack-width limitation (limiting deflection is an indirect way of limiting crack-widths);
- effect on attached structural and non-structural elements;
- ponding in (roof) slabs.

The selection of a limit to deflection depends on the given situation, and this selection is somewhat arbitrary [Ref. 10.1].

The Code (Cl. 23.2) prescribes the following two limits for flexural members in general:

| | |
|---|---|
| 1. span/250 | — the *final* deflection due to *all* loads (including long-term effects of creep and shrinkage); |
| 2. span/350 or 20 mm (whichever is less) | — the deflection (including long-term effects of creep and shrinkage) that occur *after* the construction of partitions and finishes†. |

† This involves loads applied after this stage, which in general, comprise live loads.

The first limit is based on considerations of crack control and aesthetic/psychological discomfort to occupants, and the second limit is aimed at preventing damage to partitions and finishes [Ref. 10.2]. These limits constitute broad guidelines and may be exceeded in situations where the deflections are considered to not adversely affect the appearance or efficiency of the structure.

It may be noted that the prescribed Code limits are concerned only with deflections that occur under *service loads*; hence, the partial load factor to be applied on the *characteristic* load should be taken as unity in general [refer Section 3.6.3]. Furthermore, the Code (Cl. 36.4.2.2) specifies that the modulus of elasticity and other properties to be considered in deflection calculations should be based on the characteristic strength of concrete and steel.

### 10.2.2 Difficulties in Accurate Prediction of Deflections

Accurate prediction of deflections in reinforced concrete members is difficult because of the following factors:

- uncertainties in predicting the *flexural rigidity* (*EI*) of the member, which is influenced by:
    - ∗ varying degrees of tensile cracking of concrete;
    - ∗ varying amounts of flexural reinforcement;
    - ∗ variations in modulus of elasticity of concrete; and
    - ∗ variations in modulus of rupture of concrete;
- uncertainties regarding time-dependent and environmental effects which influence shrinkage and creep;
- inelastic flexural behaviour of members; and
- inherent high variability in measured deflections, even under carefully controlled laboratory conditions.

In view of the above uncertainties, approximations and simplifications are essential in deflection calculations. The calculations are considered in two parts: (i) *immediate* or short-term deflection occurring on application of the load, and (ii) additional *long-term* deflection, resulting mostly from differential shrinkage and creep under sustained loading. For calculating the 'immediate' deflection, the loading to be considered is the full load (dead plus live). However, for calculating the long-term deflection due to creep, only the 'permanent' load (dead load plus the sustained part of live load) is to be considered. Despite these simplifications and assumptions, the calculations are quite lengthy and may give the impression of being sophisticated and rigorous — which is rather illusory, in view of the random nature of deflection and its high variability. Nevertheless, in the absence of more precise information, these deflection calculations become necessary, and may be regarded as providing representative values that serve the purpose of comparison with empirically set deflection limits.

## 10.3 SHORT-TERM DEFLECTIONS

### 10.3.1 Deflections by Elastic Theory

Short-term deflections, due to the applied *service* loads, are generally based on the assumption of linear elastic behaviour, and for this purpose, reinforced concrete is treated as a homogeneous material [refer Section 4.2]. Expressions for the maximum elastic deflection $\Delta$ of a homogeneous beam of effective span $l$ and flexural rigidity $EI$ (for any loading and support conditions) can be derived using the standard methods of structural analysis, and are available for several standard cases in handbooks [Ref. 10.3, 10.4]. Typically, they take the form:

$$\Delta = k_w \frac{Wl^3}{EI} = k_m \frac{Ml^2}{EI} \qquad (10.1)$$

where $W$ is the total load on the span, $M$ the maximum moment, and $k_w$ and $k_m$ are constants which depend on the load distribution, conditions of end restraint and variation in the flexural rigidity $EI$ (if any). For the standard case of a simply supported beam of uniform section, subjected to a uniformly distributed load, $k_w$ = 5/384 and $k_m$ = 5/48, as shown in Fig. 10.1(a). If the same beam is subjected instead to an end moment $M$ alone, the midspan deflection $\Delta_m$ is given by $k_m$ = 1/16 in Eq. 10.1 [Fig. 10.1(b)].

Generally, the expression $\Delta$ in Eq. 10.1 refers to the *midspan deflection* ($\Delta_m$), which is usually very close to the maximum value. For example, in the case of a 'propped cantilever' with a uniformly distributed load, $\Delta_m$ is within 3.5 percent of the maximum deflection. This is found to be true even when the beam or loading is unsymmetrical about the midspan location, provided the beam is supported at both ends (i.e., not free at one end).

A standard case frequently encountered in design is that of a continuous beam of uniform section, subject to a uniformly distributed load [Fig. 10.1(c)]. As the support moments ($M_1$, $M_2$) are usually known from structural analysis of the statically indeterminate structure, it would be convenient to express the midspan deflection $\Delta_m$ in terms of these two moments as well as the midspan moment $M_m = M_0 - (M_1 + M_2)/2$. Applying the principle of superposition, and making use of the results of Fig. 10.1(a) and Fig. 10.1(b), an expression for the midspan deflection $\Delta_m$ may be derived as follows:

$$\Delta_m = \frac{5}{48}\frac{M_o l^2}{EI} - \frac{M_1 l^2}{16EI} - \frac{M_2 l^2}{16EI}$$

$$= \frac{5l^2}{48EI}\left[M_o - (3/5)(M_1 + M_2)\right]$$

where $M_o = Wl/8$

Substituting $M_m = M_o - (M_1 + M_2)/2$, and eliminating $M_o$,

$$\Delta_m = \frac{5l^2}{48EI}\left[M_m - (M_1 + M_2)/10\right] \qquad (10.2)$$

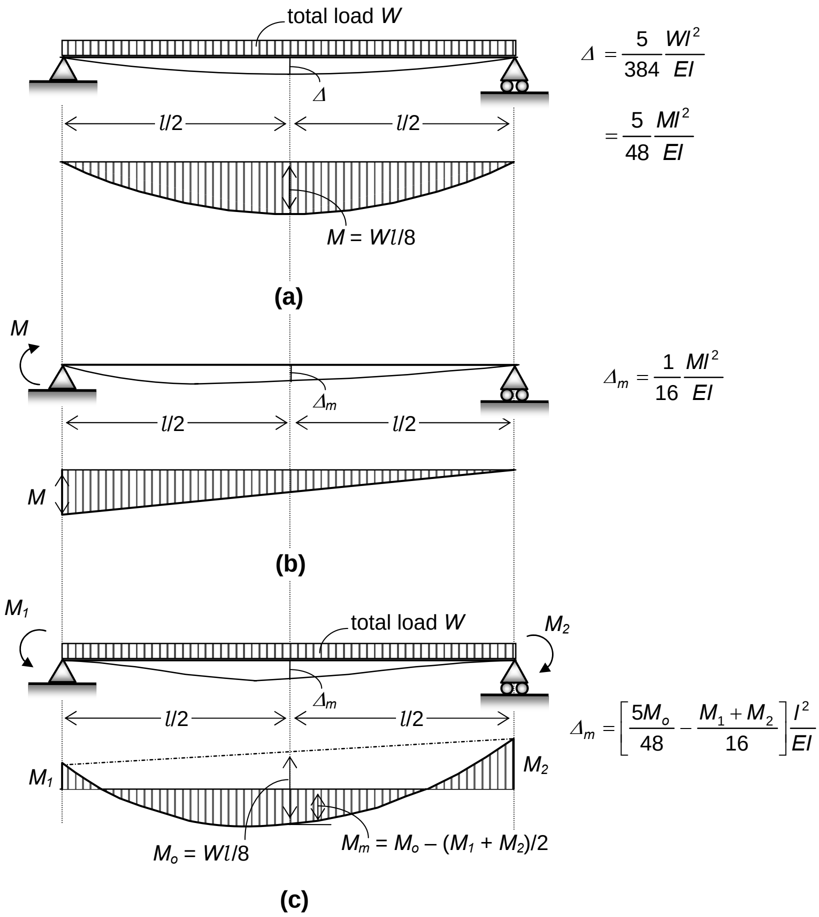


**Fig. 10.1** Midspan deflections of homogeneous beams by elastic theory

Alternatively, eliminating ($M_1 + M_2$),

$$\Delta_m = \frac{5l^2}{48EI}\left[1.2M_m - 0.2M_o\right] \tag{10.3}$$

Similar expressions can be worked out for concentrated loadings [Ref. 10.5].

### 10.3.2 Effective Flexural Rigidity

For the purpose of calculating *short-term deflections* in reinforced concrete flexural members, expressions such as Eq. 10.1 – 10.3, based on elastic theory, may be made use of. An important parameter that needs to be considered in these calculations is the flexural rigidity $EI$, which is the product of the modulus of elasticity of concrete[†] $E = E_c$, and the second moment of area, $I$, of the cross-section. As discussed in Chapter 2 (Section 2.8), the modulus of elasticity of concrete depends on factors such as concrete quality, age, stress level and rate or duration of applied load. However, for short-term loading up to service load levels, the Code expression [Eq. 2.4] for the short-term static modulus of elasticity $\left(E_c = 5000\sqrt{f_{ck}}\right)$ is satisfactory.

The second moment of area, $I$, to be considered in the deflection calculations is influenced by percentage of reinforcement as well as the extent of flexural cracking, which in turn depends on the applied bending moment and the *modulus of rupture* $f_{cr}$ of concrete.

### 10.3.3 Tension Stiffening Effect

During the first time loading of a reinforced concrete beam, the portions of the beam where the applied moment ($M$) is less than the *cracking moment* ($M_{cr}$) will remain uncracked and have the second moment of area ($I_T$) corresponding to the gross transformed section [refer Section 4.4.3 and Example 4.1]. Where the moment exceeds $M_{cr}$, the concrete in tension is expected to fail at the outer tension fibres and the cracks propagate inward (towards the neutral axis). The average spacing between cracks reduces and the average crack-width increases with increase in moment $M$ beyond $M_{cr}$. In a beam segment subject to a constant moment $M > M_{cr}$, theoretically, the entire segment should be fully cracked on the tension side of the neutral axis. But, in practice, it is seen that this does not happen, and in fact, the flexural cracks are dispersed randomly such that there are significant portions in between the cracks, which remain uncracked, as shown in Fig. 10.2(a).

The concrete in between the cracks resists some tension, and this is reflected by a reduction in tensile strain in the reinforcement [Fig. 10.2(b)], a lowering of the neutral axis [Fig. 10.2(a)], a fluctuation in the bond stress[†] [see also Fig. 8.2] as well as a reduction in curvature [Fig. 10.2(c)] — with reference to these parameters calculated at the crack location.

The tensile strain in the steel midway between the cracks (at the section marked '2' in Fig. 10.2) may be as low as 60 percent of the strain at the crack location (marked '1' in the Fig. 10.2) — at service load levels. Of course, at higher load levels, increased cracking occurs, and the difference between the two strains gets reduced — and eventually gets practically eliminated as ultimate load conditions are approached (in an under-reinforced beam).

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[†] The material called 'reinforced concrete' is essentially concrete, as the embedded reinforcement comprises only a very small fraction of the volume of the reinforced concrete member.

[†] As explained in Chapter 8, [Fig. 8.2], the bond stress is zero at every crack location and also midway between cracks (in the region of constant moment); elsewhere the bond stress varies nonlinearly.

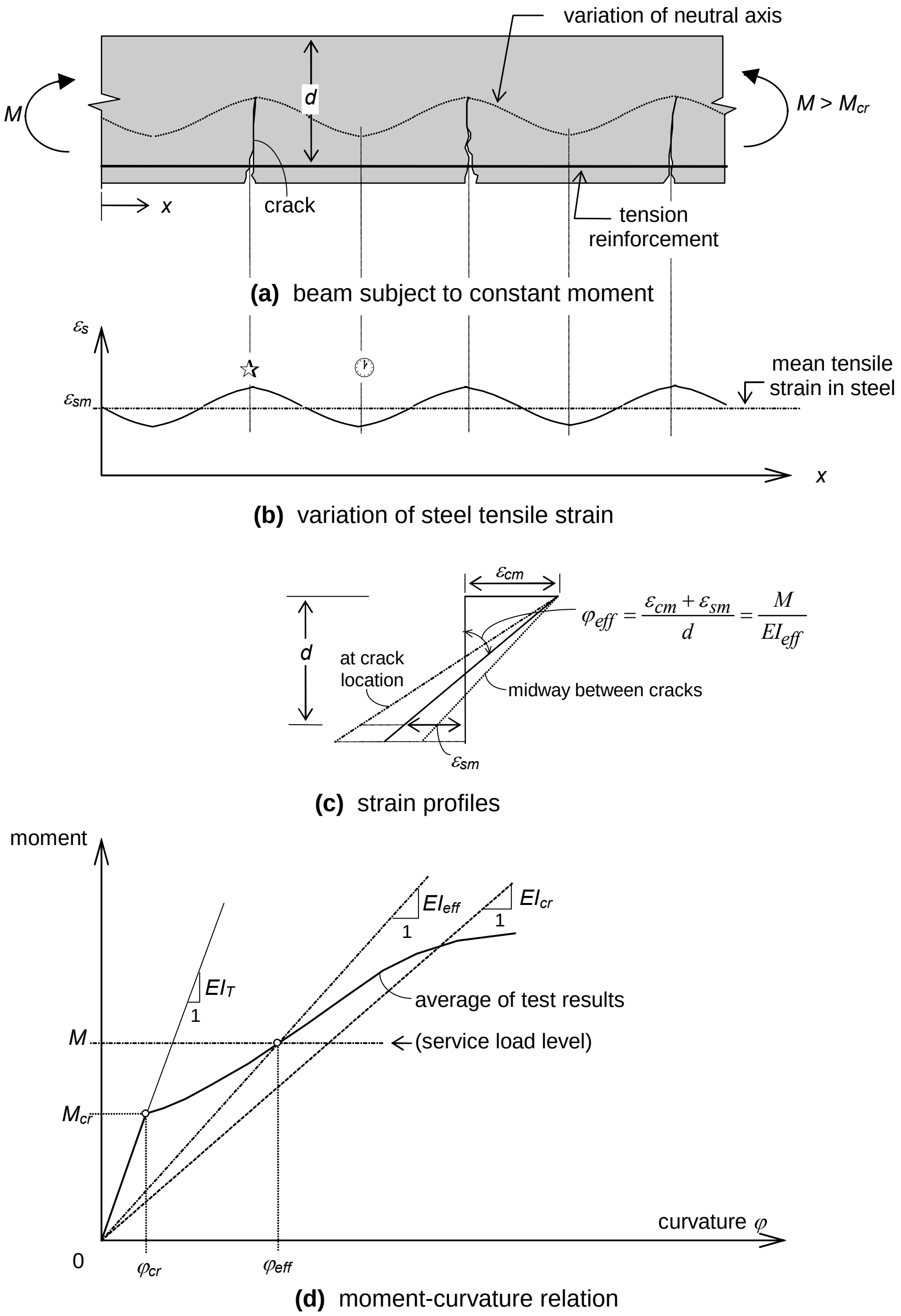


**Fig. 10.2** Effective flexural rigidity of a beam subject to constant moment

An *effective curvature* $\varphi_{eff}$ may be defined for the beam segment, as being representative of the mean curvature of the segment, under the action of a constant moment $M$. For this purpose, the strain profile to be considered may be reasonably based on the *mean strain* profile [Fig. 10.2(c)], rather than the strain profile at the crack location (which is obviously higher):

$$\varphi_{eff} = \frac{\varepsilon_{cm} + \varepsilon_{sm}}{d} = \frac{M}{EI_{eff}} \tag{10.4}$$

where $\varepsilon_{cm}$ and $\varepsilon_{sm}$ are the mean strains in the extreme compression fibre in concrete and tension steel respectively, $d$ is the effective depth, and $EI_{eff}$ is the effective flexural rigidity of the section.

'Flexural rigidity' $EI$ is obtainable as the slope (secant modulus) of the moment-curvature relationship, which in turn can be established from an average of a number of test results [Ref. 10.5 – 10.11]. As shown in Fig. 10.2(d), this may be obtained variously as:

- $EI_T$ — based on the 'uncracked-transformed' section;
- $EI_{gr}$ — based on the 'gross' (uncracked) section, i.e., ignoring the presence of steel;
- 
- $EI_{eff}$ — based on the 'effective' section;
- $EI_{cr}$ — based on the 'cracked-transformed' section [refer Eq. 4.15].

Evidently, $EI_T$ represents the true flexural rigidity for $M < M_{cr}$, and $EI_{eff}$ represents the true flexural rigidity for $M > M_{cr}$. Whereas $EI_T$ is a constant and a property of the beam section, $EI_{eff}$ depends on the load level (applied moment). It follows that:

$$EI_T > EI_{gr} > EI_{eff} > EI_{cr}$$

Thus, determining the flexural rigidity (stiffness) on the basis of the uncracked section results in an under-estimation of the actual deflection of a reinforced concrete beam under service loads; whereas doing so on the basis of the (fully) cracked section results in an over-estimation of the actual deflection.

> The increase in stiffness over the 'cracked section' stiffness, on account of the ability of concrete (in between cracks) to resist tension, is referred to as the *tension stiffening* effect.

### 10.3.4 Effective Second Moment of Area Formulation

Various empirical expressions for the 'effective second moment of area' $I_{eff}$ (for calculating short-term deflections in simply supported beams) have been proposed [Ref. 10.5 – 10.11] and incorporated in different codes. Some of these formulations are based on assumed transition moment-curvature relations [Ref. 10.5 – 10.8], whereas the others [Ref. 10.9 – 10.11] are based on assumed transition of strains/stresses in the region between cracks (and involve stress-strain relations and equilibrium of forces). The expression given in the Indian Code (IS 456: 2000, Cl. C–2.1) is based on an earlier version of the British Code, which assumes an idealised trilinear moment-curvature relation [line OABCD in Fig. 10.3]. The initial uncracked stiffness $EI_{gr}$ and the cracked stiffness (at ultimate load) $EI_{cr}$ are represented by the slopes of lines OA and OC respectively. Any intermediate stiffness ($EI_{eff}$ corresponding to line OB) can be interpolated by defining the slope of the intermediate line ABC in the region $M_{cr} < M < M_u$. The slope of this line (which commences with the cracking moment point A) is approximated as $0.85EI_{cr}$.

It follows that:

$$EI_{eff} = \frac{M}{\varphi_{cr} + (M - M_{cr})/(0.85EI_{cr})} \qquad \text{for } M_{cr} \le M \le M_u$$

where $\varphi_{cr} = M_{cr}/EI_{gr}$ and $M_{cr} = f_{cr}I_{gr}/y_t$ [Note that this expression for $M_{cr}$ is similar to Eq. 4.10, except that $I_{gr}$† is used instead of $I_T$]. Simplifying the above expression, it can be shown that

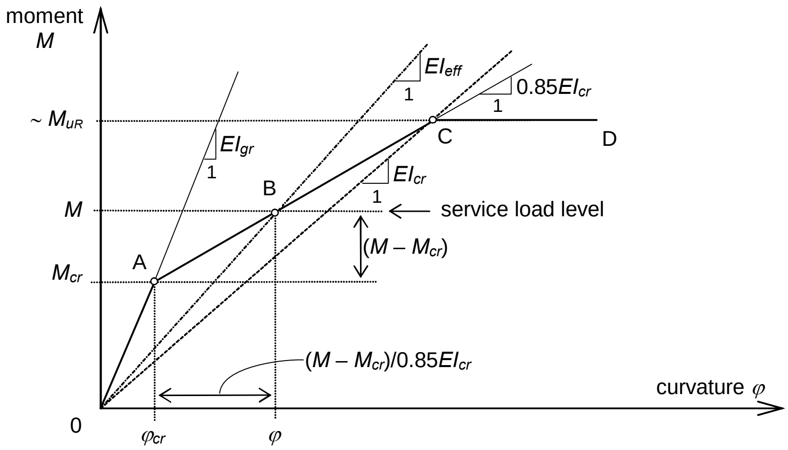


**Fig. 10.3** Idealised trilinear moment-curvature relation

$$I_{eff} = \frac{I_{cr}}{1.2 - (M_{cr}/M)\eta} \qquad \text{with } I_{cr} \le I_{eff} \le I_{gr} \tag{10.5a}$$

where $$\eta \equiv 1.2 - I_{cr}/I_{gr} \tag{10.5b}$$

The IS code formula for the effective moment of inertia is identical to Eq. 10.5a, except that the non-dimensional parameter $\eta$ in the equation takes a more complicated form than the one given by Eq. 10.5b:

$$\eta \equiv \frac{z}{d}\left(1 - \frac{x}{d}\right)\frac{b_w}{b} \tag{10.6}$$

† The use of the *gross transformed* section (i.e., ignoring the contribution of steel), instead of the *uncracked-transformed* section, is done for convenience.

where $x \equiv kd$ is the depth of the neutral axis, $z \equiv jd$ the lever arm, $b_w$ the 'breadth of the web', and $b$ the 'breadth of the compression face'. Sometimes, the calculations will yield values of $I_{\text{eff}}$ that may exceed $I_{\text{gr}}$ or be less than $I_{\text{cr}}$. In such cases, the bounds on $I_{\text{eff}}$, as indicated in Eq. 10.5a, should be applied.

An alternative expression for $I_{eff}$, widely used in North American practice (ACI code [Ref. 10.13], Canadian code [Ref. 10.14]), is due to Branson [Ref. 10.5] and takes the form:

$$I_{eff} = \left(\frac{M_{cr}}{M}\right)^3 I_{gr} + \left[1 - \left(\frac{M_{cr}}{M}\right)^3\right] I_{cr} \qquad \text{for } M > M_{cr} \qquad (10.7)$$

Eq. 10.7 gives a value of $I_{eff}$ which is effectively a weighted average of $I_{gr}$ and $I_{cr}$.

Comparison with test results indicates reasonable agreement (within the range of ± 20 percent !) between the deflections measured and those computed using Eq. 10.5a or Eq. 10.7. It may be noted that for the general case of a non-uniform bending moment diagram, $I_{eff}$ varies along the span, and the consequent calculation of maximum deflection can be very difficult. However, Eq. 10.5a and Eq. 10.7 have been developed with the intention of generating a single value of $I_{eff}$ (for the entire beam, assumed to be prismatic) in association with the *maximum* moment ($M$) on the beam, which is assumed to be bent in single curvature. Thus, Eq. 10.5 (or 10.7) can be used to give an average value† of $I_{eff}$ for simply supported spans and cantilever spans, but not continuous spans. It may also be noted that these expressions cannot be applied in the case of bending combined with axial force, or in the case of two-way slab bending.

### 10.3.5 Average $I_{eff}$ for Continuous Spans

In the case of continuous spans, the sense of curvature at midspan is different from that near the support; the former is (generally) 'sagging', and the latter 'hogging'. In beam-slab construction, the flanged section properties are substantially different under 'positive' and 'negative' moments; in the case of the former, the flange is effective, being under compression, but in the case of the latter, the flange is under tension [refer Sections 4.6.4, 4.7.4]. Even in the case of beams with rectangular cross-sections, there are differences in reinforcement ratios and differences in the influence of cracking in the two regions. Hence, a weighted average of the $I_{eff}$ values at mid-span and support regions is generally recommended.

A simple expression for a weighted average $I_{eff,\,av}$, recommended in Ref. 10.5 and 10.15, is as follows:

$$I_{eff,av} = 0.7 I_{eff,m} + 0.15(I_{eff,\,1} + I_{eff,\,2}) \qquad (10.8a)$$

(for beams with both ends continuous)

$$I_{eff,av} = 0.85 I_{eff,m} + 0.15 I_{eff,cont} \qquad (10.8b)$$

(for beams with one end continuous)

† Assuming a uniform $I_{eff}$ (calculated with respect to the *maximum* span moment) will generally result in a conservative estimate of deflection.

where the subscript *m* denotes the midspan location, and the subscripts *cont*, 1, 2 denote the continuous end location(s).

Typically, in a continuous span, the gravity loading pattern which produces the maximum deflection (as well as maximum 'positive' moment) is different from that which produces the maximum 'negative' moment at the supports [refer Chapter 9]. In order to account for the more widespread cracking near the support regions under these larger 'negative' moments, it becomes necessary to consider the maximum 'negative' moment at the support region for evaluating $I_{eff,1}$, $I_{eff,2}$ or $I_{eff,cont}$ using Eq. 10.5a or Eq. 10.7. Similarly, the maximum 'positive' moment should be considered for evaluating $I_{eff,m}$. Of course, once the $I_{eff,av}$ has been evaluated this way (using Eq. 10.8), the deflection calculation (using Eq. 10.3) should be based on values of $M_m$, $M_1$ and $M_2$, corresponding to that loading diagram which causes maximum deflection (in the span region).

It may also be noted that, generally, in continuous spans, $I_{eff}$ values at the continuous ends have a much smaller effect† on the deflections than $I_{eff,m}$, and reasonable predictions of deflection can be obtained by using $I_{eff,m}$ alone, instead of a weighted average [Ref. 10.15]. However, when there is a significant variation in flexural rigidity (as in flanged beams), or when the negative moment at either continuous end is relatively large, the use of a weighted average such as Eq. 10.8 is recommended.

The weighted average expression given in the Code (Cl. C–2.1) for continuous beams is somewhat complicated and takes the form (similar to Eq. 10.5a):

$$I_{eff,av} = \frac{I_{cr,av}}{1.2 - (M_{cr,av}/M)\ \eta} \qquad \text{with } I_{cr,av} \le I_{eff,av} \le I_{gr,av} \qquad (10.9a)$$

where $I_{cr,av}$, $I_{gr,av}$ and $M_{cr,av}$ are to be computed as weighted averages using the following generalised expression:

$$X_{av} = k_1(X_1 + X_2)/2 + (1 - k_1)X_m \qquad (10.9b)$$

where the subscripts 1 and 2 denote the two continuous support locations and *m* denotes the midspan location; $k_1$ is a weighting factor which lies between 0 and 1, and depends on the ratio of the sum of the corresponding fixed end moment $(M_{F1} + M_{F2})$ — as given in Table 21 of the Code. The expression for $\eta$ in Eq. 10.9(a) is the same as that given in Eq. 10.6. It is not clear from either the Code or the Explanatory Handbook to the Code [Ref. 10.2] whether, in the case of continuous beams, the value of the applied moment *M* (in Eq. 10.9a) and the values of the neutral axis depth *x* and lever arm *z* (in Eq. 10.6) are to be based on the midspan location or the support location, or as a weighted average. [It appears logical that when an expression such as Eq. 10.2 is used for calculation of $\Delta$, the *M* should be the moment at midspan.] In view of these uncertainties and complications,

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† This can be easily observed from the *conjugate beam method*, where the span moment in the conjugate beam (which gives $\Delta$) is governed predominantly by the $M/EI$ values (which is the loading on the conjugate beam) in the midspan regions than in the end regions.

the Code procedure for evaluating $I_{eff,\,av}$ in its present form, is not generally used in design practice.

### 10.3.6 Effective Curvature Formulation

In the latest British code, BS: 8110 [Ref. 10.12], the use of Eq. 10.5a (involving the concept of $I_{eff}$) is dispensed with, and the formulation is now based on an assumed distributions of strains and stresses that attempt to account for the tension-stiffening effect more directly. From the assumed distribution of strains, the effective curvature is directly obtained [Fig. 10.4], and the deflections may be computed from the effective curvatures. It may be noted that if the beam section under consideration is subject to a low bending moment, it is likely to behave as an uncracked section, and this possibility [Fig. 10.4a] should also be investigated. It is stated in the British code that the curvature at any section should be taken (conservatively) as the larger of the values obtained by considering the section as (a) uncracked and (b) cracked [Fig. 10.4]. This clause is intended to ensure that even if the applied bending moment is less than the cracking moment, it is possible that the section may behave as a cracked section due to some prior heavier loading (or cracking due to temperature and shrinkage effects)†. It is also mentioned that the calculation of effective curvature is to be done at the mid-span for simply supported beams and at the support section for cantilevers, and the appropriate relationship between elastic deflection and curvature used for calculating the maximum deflection [Eq. 10.1].

It may be noted that the distribution of strains [Fig. 10.4] is linear for both uncracked and cracked sections (being based on the fundamental assumption of plane sections remaining plane after bending), and the formula for effective curvature, applicable for both sections, may be easily derived using strain compatibility considerations. The formula may be expressed in terms of either the concrete compressive stress $f_c$:

$$\varphi_{eff} = \frac{\varepsilon_c}{x} = \frac{f_c}{xE_c} \tag{10.10a}$$

or, in terms of the mean tensile stress in steel:

$$\varphi_{eff} = \frac{\varepsilon_{sm}}{(d-x)} = \frac{f_{sm}}{(d-x)E_s} \tag{10.10b}$$

where

$f_c \equiv$ compressive stress in concrete at the extreme compression fibre
$f_{sm} \equiv$ mean tensile stress in steel
$f_{cts} \equiv$ allowable tensile stress in concrete at the level of the tension steel, to be taken appropriately.

The value of $f_{cts}$ is to be limited to 1.0 MPa in the case of the uncracked section [Fig. 10.4b], and the corresponding stress in steel is given by $f_{sm} = m\, f_{cts}$. It may be noted that the corresponding stress in the extreme tension fibre in concrete, given by $f_{cts}.(D-x)/(d-x)$, will be considerably less than the modulus of rupture ($0.7\sqrt{f_{ck}}$).

† Refer Example 10.1

This is done in order to ensure that when the applied moment is less than but close to the 'cracking moment' $M_{cr}$, it would be more appropriate (and conservative) to treat the section as a cracked section for the purpose of estimating deflections. The magnitude of cracking moment, by this formulation reduces to:

$$\tilde{M}_{cr} = \left(f_{cts}\frac{0.5D}{d-0.5D}\right)\left(\frac{I_{gr}}{0.5D}\right) \tag{10.11}$$

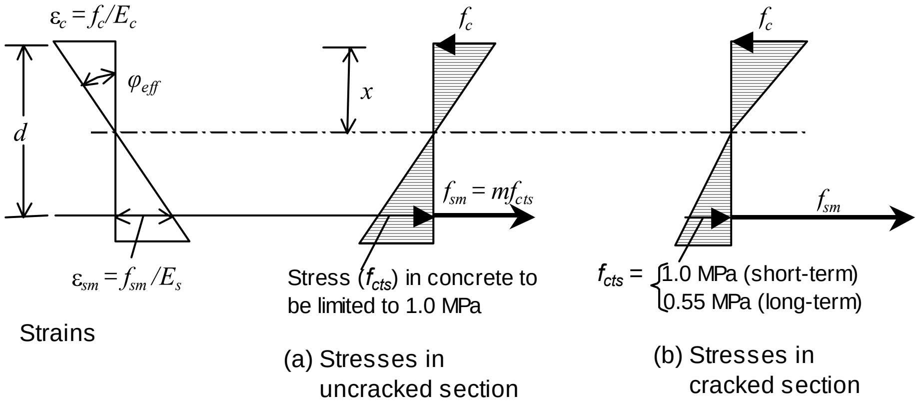


**Fig. 10.4** Distribution of strains and stresses (BS 8110) for serviceability calculations, including tension stiffening effect

In the case of the cracked section, the maximum contribution of concrete in tension is considered by taking $f_{cts}$ = 1.0 MPa, and considering the short-term elastic modulus $E_c$. However, for the purpose of calculating long-term deflections due to creep, it is recommended that a lower value, $f_{cts}$ = 0.55 MPa, should be taken (as indicated in Fig. 10.4b), and the effective modulus of elasticity (including creep coefficient) $E_{ce}$ should be considered. It may be noted that in both the cracked and uncracked cases, the total tensile force resisted by concrete below the neutral axis may be obtained by assuming a triangular stress block.

For the uncracked section, the curvature can be obtained directly, using Eq. 10.4 with $I_{eff}$=$I_{gr}$ (which ignores the contribution of the steel, and effectively assumes $x$ = 0.5$D$). However, for the cracked section, it is necessary to apply Eq. 10.10, which involves the neutral axis depth $x$ and either the concrete compressive stress $f_c$ [Eq. 10.10a] or the mean tensile stress in steel $f_{sm}$ [Eq. 10.10b]. This cannot be directly determined by means of a closed-form solution, and an iterative (trial-and-error) procedure is required. The following force and moment equilibrium equations need to be satisfied:

$$0.5bxf_c = f_{sm}A_{st} + 0.5b(D-x)f_{ct} \tag{10.12}$$

$$M = A_{st}f_{sm}\left(d-\frac{x}{3}\right)+\frac{1}{3}bD\,f_{ct}\left(D-x\right) \tag{10.13}$$

where $$f_{ct} = \left(\frac{D-x}{d-x}\right) \times f_{cts} \tag{10.14}$$

Using strain compatibility relations [Fig. 10.4(a)],

$$f_c = \frac{x}{d-x}\frac{E_c}{E_s} f_{sm} \tag{10.15}$$

whereby $$x = \frac{d}{1 + f_{sm}/(mf_c)} \tag{10.15a}$$

One may begin by assuming a trial value of the neutral axis depth $x$ ($\approx d/3$). Next, trial values of $f_{ct}$ and $f_{sm}$ are calculated by solving Eq. 10.14 and 10.13 respectively. Using these values, $f_c$ is calculated solving Eq. 10.12, and an improved value of $x$ can now be obtained from Eq. 15a. For the next trial, an average of this value and the initial trial value of $x$ may be considered, and the procedure repeated. Usually, convergence in the trial-and-error procedure can achieved within two or three iterations, as demonstrated in Example 10.2.

**EXAMPLE 10.1**

For the one-way (simply supported) slab system designed in Example 5.2, compute the maximum short-term deflection due to dead loads plus live loads. Solve (a) using the concept of $I_{eff}$ specified in IS 456 (2000), and (b) using the concept of effective curvature given in BS 8110 (1997).

**SOLUTION**

- Given: $l$ = 4.16 m, $D$ = 200 mm, $A_{st}$ = 628 mm$^2$/m, ($p_t$ = 0.380), $d$ = 165 mm, $f_{ck}$ = 25 MPa, $f_y$ = 415 MPa, $w_{DL}$ = 6.0 kN/m$^2$ $w_{LL}$ = 4.0 kN/m$^2$, [refer Example 5.2].
- Formula for maximum short-term deflection (at midspan):

$$\Delta = \frac{5}{384}\frac{wl^4}{EI_{eff}} = \frac{5}{48}\frac{Ml^2}{EI_{eff}}$$

- *Maximum moment* at midspan (under service loads — dead plus live):

  $M = (6.0 + 4.0) \times 4.16^2/8 = 21.63$ kNm per m width
- *Short-term modulus of elasticity*:

$E = E_c = 5000\sqrt{f_{ck}}$

$= 5000\sqrt{25} = 25000$ MPa

**(a) <u>SOLUTION AS PER IS 456 (2000)</u>**

- $M_{cr} = f_{cr} I_{gr}/(0.5D)$

  where $f_{cr} = 0.7\sqrt{f_{ck}}$ (as per Code)

  $= 0.7\sqrt{25} = 3.5$ MPa

$\Rightarrow M_{cr}$ = 3.5 × 6.667 × $10^8$ / (0.5 × 200)
= 23.33 × $10^6$ Nmm/m = 23.33 kNm/m
> $M$ = 21.63 kNm/m (implying that the section is likely to be uncracked).
$\Rightarrow I_{eff} = I_{gr}$
$I_{eff} = I_{gr} = bD^3/12$ = 1000 × $(200)^3$/12 = 6.667 × $10^8$ mm$^4$

Maximum short-term deflection (uncracked section)

$$\Delta = \frac{5}{48}\frac{Ml^2}{EI_{eff}} = \frac{5}{48} \times \frac{21.63\times10^6(\text{Nmm})\times(4160)^2(\text{mm})^2}{25000(\text{N}/\text{mm}^2)\times(6.667\times10^8)(\text{mm}^4)}$$
= 2.34 mm ($l$ / 1778)

**Note**: As the applied moment of 21.63 kNm/m is close to the cracking moment of 23.33 kNm/m, it may be unconservative to calculate deflections based on the uncracked section. It would be more prudent (as suggested in BS 8110) to limit the use of the uncracked section to applied moments that are less than $M_{cr}$ [refer Eq. 10.10 for $\tilde{M}_{cr}$]. The IS Code, however, does not give any recommendations in this regard. A recent study indicates that the use of the BS Code estimate of $\tilde{M}_{cr}$, when coupled with the IS Code procedure, results in very large deflection estimates. The authors suggest that, for the application of the IS Code procedure, the value of $\tilde{M}_{cr}$ may be taken as approximately 0.7 $M_{cr}$. In the present problem, the reduced cracking moment works out to:

$\tilde{M}_{cr}$ = 0.7 × 23.33 kNm/m = 16.33 kNm/m.

(The use of Eq. 10.11 results in a more conservative estimate of $\tilde{M}_{cr}$ = 10.26 kNm/m).
This value (16.33 kNm/m) is considerably less than the applied moment of 21.63 kNm/m.
Hence, the section should be treated as a cracked section for calculation of short-term deflection. For meaningful results in the estimation of $I_{eff}$, $M_{cr}$ should be replaced by $\tilde{M}_{cr}$ in Eq. 10.5a.

**Effective second moment of area**

- $I_{eff} = \dfrac{I_{cr}}{1.2-(\tilde{M}_{cr}/M)\eta}$

  $\eta = j(1-k)(b_w/b)$ as per IS 456 formulation
- $I_{cr} = b(kd)^3/3 + m\,A_{st}(d-kd)^2$ [see Fig. 10.5]

The neutral axis (NA) depth $kd$ is obtainable by considering moments of areas of the cracked transformed section about the NA:

- $b\times(kd)^2/2 = m\,A_{st}\,(d-kd)$

where $m^{\ddagger} = E_s/E_c = 2 \times 10^5/25000 = 8.0$
$\Rightarrow 1000 \times (kd)^2/2 = (8.0) \times 628\ (165 - kd)$
Solving, $kd = 36.00$ mm $\Rightarrow k = 36.00/165 = 0.2182$
[Note: $k$ can be directly obtained from Eq. 4.13].

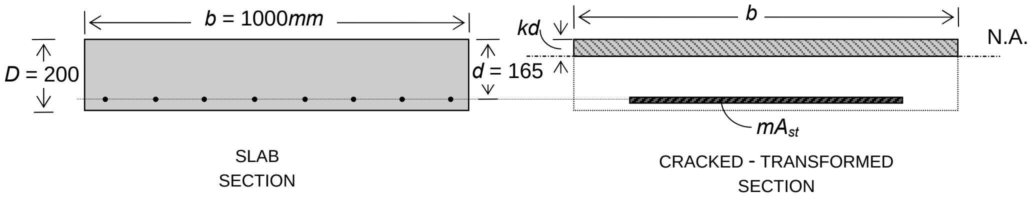


**Fig. 10.5** Example 10.1

$\Rightarrow I_{cr} = 1000(36.0)^3/3 + (8.0 \times 628) \times (165 - 36.0)^2$
$= 0.9916 \times 10^8\ \text{mm}^4\ (= 0.1487\, I_{gr})$
$\eta = j(1-k)(b_w/b)$
$= (1 - k/3)\ (1 - k)\ (1) = (1 - 0.2182/3)\ (1 - 0.2182)$
$= 0.7249$

$$I_{eff} = \frac{I_{cr}}{1.2 - (\tilde{M}_{cr}/M)\eta} = \frac{(0.9916 \times 10^8)}{1.2 - (16.33/21.63) \times 0.7249} = 1.5192 \times 10^8\ \text{mm}^2$$

(which lies between $I_{cr}$ and $I_{gr}$) and is equal to $0.2279\, I_{gr}$).

**Maximum short-term deflection (cracked section)**

$$\Delta = \frac{5}{48}\frac{Ml^2}{EI_{eff}} = \frac{5}{48} \times \frac{21.63 \times 10^6\,(\text{Nmm}) \times (4160)^2\,(\text{mm})^2}{25000(\text{N/mm}^2) \times (1.5192 \times 10^8)(\text{mm}^4)}$$
$$= \mathbf{10.26\ mm}\ (= l/405)$$

[Note: The deflection calculated on the basis of 'cracked section' (10.26 mm) is considerably larger than the value calculated on the basis of 'uncracked section' (2.34 mm). This is because $I_{eff} = 0.228\, I_{gr}$].

**(b) <u>ALTERNATIVE SOLUTION AS PER BS: 8110-1997)</u>**

The effective curvature should be calculated assuming that the section is (i) uncracked and (ii) cracked, and the higher value is to be taken.

<u>(i) Uncracked section</u>

The curvature is given by

‡ Note: for short-term deflection calculations, $E_c$ should be taken as the short-term modulus of elasticity. Hence, the empirical expression for *modular ratio* given by the Code for flexural design ($m = 280/3\sigma_{cbc}$) should not be used here.

$$\varphi_{eff} = \frac{M}{E_c I_{gr}} = \frac{21.63\times10^6\,(\text{Nmm})}{25000(\text{N/mm}^2)\times(6.667\times10^8)(\text{mm}^4)} = \mathbf{1.298\times10^{-6}\ /\ mm}$$

(ii) Cracked section

The equations 10.11 – 10.14 have to be satisfied.

Trial 1: Assume $x \approx d / 3 = 165/3 = 55$ mm.

$$\Rightarrow f_{ct} = \left(\frac{200-x}{165-x}\right)\times1.0 = 1.318 \text{ MPa}$$

$$\Rightarrow f_{sm} = \left(21.63\times10^6 - \frac{1000\times200\times f_{ct}\times(200-x)}{3}\right)\bigg/ 628\times\left(165-\frac{x}{3}\right) = 96.5\text{MPa}$$

$$\Rightarrow f_c = \frac{f_{sm}A_{st} + 0.5b(D-x)f_{ct}}{0.5bx} = 5.68 \text{ MPa}$$

$$\Rightarrow x = \frac{165}{1+f_{sm}/(8.0f_c)} = 52.8 \text{ mm}$$

Trial 2: Assuming an average value $x \approx (55 + 52.8)/2 = 53.9$ mm, and repeating the procedure,

$\Rightarrow f_{ct} = 1.315$ MPa

$\Rightarrow f_{sm} = 95.5$ MPa

$\Rightarrow f_c = 5.79$ MPa

$\Rightarrow$ **x = 53.9 mm**, which indicates convergence.

The effective curvature of the cracked section may now be calculated using either Eq.10.10a:

$$\varphi_{eff} = \frac{f_c}{xE_c} = \frac{5.79}{53.9\times25000} = 4.297\times10^{-6} \text{ per mm}$$

or Eq.10.10b:

$$\varphi_{eff} = \frac{f_{sm}}{(d-x)E_s} = \frac{95.5}{(165-53.9)(2\times10^5)} = \mathbf{4.298\times10^{-6}\ per\ mm}$$

The curvature due to the cracked section ($4.298 \times 10^{-6}$ per mm) is larger than the one due to uncracked section ($1.298 \times 10^{-6}$ per mm), and accordingly, considering this larger value, for the uniformly loaded beam,

$$\Delta = \frac{5}{48}\varphi_{eff}l^2 = \frac{5}{48}\times4.298\times10^{-6}\times(4160)^2 = \mathbf{7.75\ mm}$$

which is less than **10.26 mm** predicted as per IS 456 (cracked section).

### 10.3.7 Additional Short-Term Deflection Due to Live Loads Alone

As mentioned in Section 10.2.1, the check on deflection involves a separate check on deflection due to live loads (including long-term effects of creep and shrinkage) that occur after the construction of partitions and finishes. This requires the calculation of the short-term deflection due to live load alone.

Because of the variations in effective flexural rigidity with the applied moment [Fig. 10.2], the load-deflection behaviour of a reinforced concrete beam is non-linear [Fig. 10.6]; hence, the principle of superposition is not applicable in deflection calculations. Unlike the live loads, the dead loads act all the time. Hence, the immediate (short-term) deflection due to the live load part alone, $\Delta_L$, has to be obtained as the difference between the short-term deflection due to dead plus live loads, $\Delta_{D+L}$, and that due to dead load alone, $\Delta_D$:

$$\Delta_L = \Delta_{D+L} - \Delta_D \tag{10.16}$$

This is depicted in Fig. 10.6. In the calculation of $\Delta_D$ , for deciding whether to consider the section to be cracked or uncracked, it is prudent to compare the dead load moment $M_D$ with the reduced cracking moment $\tilde{M}_{cr}$, as discussed in Example 10.1(a). The section may be treated as uncracked only if $M_D$ is less than $\tilde{M}_{cr}$.

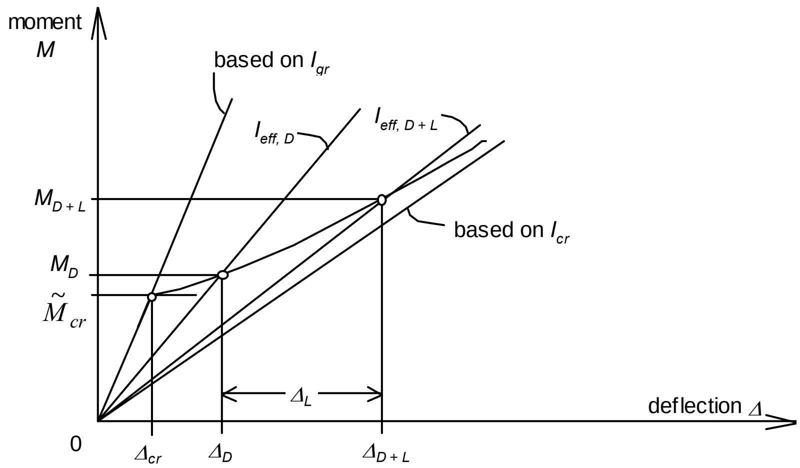


**Fig. 10.6** Short-term deflections due to dead loads and live loads

**EXAMPLE 10.2**

For the slab of Example 10.1, determine the short-term deflection due to live loads alone.

**SOLUTION**

- $\Delta_L = \Delta_{D+L} - \Delta_D$, where $\Delta_{D+L}$ = 10.26 mm as per IS 456 (from Example 10.1a)

**Short-term deflection due to dead loads alone**: $\Delta_D$

- $M_D = w_{DL}l^2/8 = 6.0\times 4.16^2/8$ = 12.98 kNm per m width
  $< \tilde{M}_{cr}$ = 16.33 kNm per m width [refer Example 10.1(a)].

Hence, the section should be treated as uncracked.

$\Rightarrow I_{eff,\,D} = I_{gr} = 6.667\times 10^8$ mm$^4$; [refer Example 10.1]

$E = E_c = 25000$ MPa

$$\Rightarrow \Delta_D = \frac{5}{48}\frac{M_D l^2}{EI_{eff}} = \frac{5}{48} \times \frac{(12.98\times10^6)\times(4160)^2}{25000\times(6.667\times10^8)}$$

$= 1.35$ mm

**Short-term deflection due to live loads alone**: $\Delta_L$

$\Delta_L = \Delta_{D+L} - \Delta_D$
$= 10.26 - 1.35$
$\mathbf{= 8.91\ mm}$

## EXAMPLE 10.3

Determine the maximum short-term deflection under dead loads and live loads for the doubly reinforced beam of Example 5.4. Also determine the short-term deflection due to live loads only.

### SOLUTION

- *Given*: $l$ = 6.0 m, $b$ = 250 mm, $D$ = 400 mm, $d$ = 348 mm, $d'$ = 48mm, $A_{st}$ = 1848 mm$^2$, $A_{sc}$ = 942.5 mm$^2$, $f_{ck}$ = 25 MPa, $f_y$ = 415 MPa, $w_{DL}$ = 7.5 kN/m (including self-weight) plus $W_{DL}$ = 30 kN at midspan, $w_{LL}$ = 10.0 kN/m [refer Example 5.4]

  The details of the beam loading and section are shown in Fig. 10.7.
- Formula for maximum short-term deflection (at midspan):

$$\Delta = \frac{5}{384}\frac{wl^4}{EI_{eff}} + \frac{1}{48}\frac{Wl^3}{EI_{eff}}$$

$$= \frac{5}{48}\frac{l^2}{EI_{eff}}[M_1 + 0.8M_2] \qquad \text{[refer Fig. 10.7(a), (b)]}$$

  where $M_1$ and $M_2$ are the midspan moments due to distributed loading and concentrated load respectively.
- *Maximum moments at midspan*

  i) due to $DL$ alone: $M_D = M_{1,D} + M_{2,D} = w_{DL}l^2/8 + W_{DL}l/4$
  $= (7.5 \times 6.0^2/8) + (30.0 \times 6.0/4)$
  $= 33.75 + 45.0 = 78.75$ kNm

  ii) due to $DL + LL$: $M_{D+L} = M_D + M_{1,L}$
  $= 78.75 + (10.0 \times 6.0^2/8)$
  $= 78.75 + 45.0 = 123.8$ kNm
- *Short-term modulus of elasticity*:

$E = E_c = 5000\sqrt{25} = 25000$ MPa
- *Gross section properties*:

  $I_{gr} = bD^3/12 = 250 \times 400^3/12 = 13.3333 \times 10^8$ mm$^4$

  $M_{cr} = f_{cr}I_{gr}/(0.5D)$

where $f_{cr} = 0.7\sqrt{20} = 3.13$ MPa
$\Rightarrow M_{cr} = 3.13 \times (13.3333 \times 10^8)/(0.5 \times 400)$
$= 20.87 \times 10^6$ Nmm $= 20.87$ kNm
$\tilde{M}_{cr} = 0.7\, M_{cr} = 0.7 \times 20.87 \times 10^6 = 14.61 \times 10^6$ Nmm $< M_D, M_{D+L}$
Hence, $I_{eff} < I_{gr}$

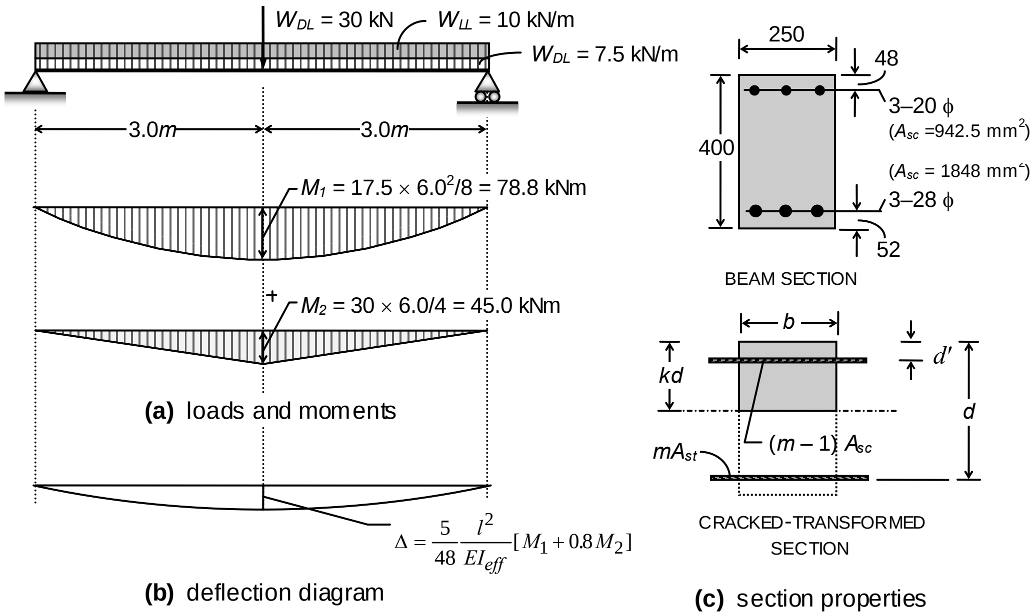


**Fig. 10.7** Example 10.3

**Effective second moment of area**

- $I_{eff} = \dfrac{I_{cr}}{1.2 - (\tilde{M}_{cr}/M)\eta}$;

  $\eta = j(1-k)(b_w/b)$ as per IS 456 formulation

  where $I_{cr} = b(kd)^3/3 + m\,A_{st}(d-kd)^2 + (m-1)A_{sc}(kd-d')^2$
- The NA depth, $kd$, is obtainable by considering moments of areas of the cracked-transformed section[†] about the NA (centroidal axis) [Fig. 10.7(b)]

  $b(kd)^2/2 + (m-1)A_{sc}(kd-d') = m\,A_{st}(d-kd)$

  where $m = E_s/E_c = 2 \times 10^5/25000 = 8$

[†] Note that the transformed area of compression steel is taken as $(m-1)A_{sc}$ and ***not*** $(1.5m-1)A_{sc}$ as considered in stress calculations [refer Section 4.6.5], because the increased modular ratio $1.5m$ (to account for creep effects) is not applicable in the context of short-term deflections.

$\Rightarrow 250(kd)^2/2 + (8 - 1)(942.5)(kd - 48) = (8 \times 1848)(348 - kd)$
$\Rightarrow 125(kd)^2 + 21381.5(kd) - 5461512 = 0$
Solving, $kd$ = 140.3 mm ($\Rightarrow k$ = 140.3/348 = 0.4032)
$\Rightarrow I_{cr} = 250(140.3)^3/3 + (8 \times 1848)(348 - 140.3)^2 + (7 \times 942.5)(140.3 - 48)^2$
$= 9.24117 \times 10^8$ mm$^4$
$\Rightarrow \eta = j(1-k)(b_w/b)$ = (1-0.4032/3)(1-0.4032)1.0 = 0.5166

- $M_D$ = 78.75 kNm $\Rightarrow I_{eff,D}/I_{cr}$ = [1.2 – (14.61/78.75)(0.5166)]$^{-1}$
  = 0.906< 1.0

  As $I_{eff}$ cannot be less than $I_{cr}$, and $I_{eff,\,D+L} \le I_{eff,\,D}$, it follows that
  $\mathbf{I_{eff,\,D} = I_{eff,\,D+L} = I_{cr} = 9.24117 \times 10^8}$ **mm$^4$**

**Maximum short-term deflection**

(i) due to dead loads plus live loads:

$$\Delta_{D+L} = \frac{5l^2}{48EI_{eff,D+L}}[M_{1_{D+L}} + 0.8\,M_{2_{D+L}}]$$
$$= \frac{5 \times (6000)^2 \times \{78.75 + (0.8 \times 45.0)\} \times 10^6}{48 \times 25000 \times (9.24117 \times 10^8)}$$
$$= 18.62 \text{ mm } (= l/322)$$

(ii) due to dead loads alone:

$$\Delta_D = \frac{5l^2}{48EI_{eff,D}}[M_1 + 0.8\,M_2]$$
$$= \frac{5 \times (6000)^2 \times [33.75 + 0.8 \times 45] \times 10^6}{48 \times 25000 \times (9.24117 \times 10^8)}$$
$$= 11.32 \text{ mm}$$

$\Delta_L = \Delta_{D+L} - \Delta_D$
= 18.62 – 11.32
= **7.3** mm

*Note*: The reader may compare the results obtained in this Example with other methods (BS 8110 and Branson's formula).

**EXAMPLE 10.4**

For the one-way continuous slab system designed in Example 5.3, compute the maximum midspan deflection in the end span due to dead loads and live loads. Also compute the deflection due to live loads only.

**SOLUTION**

- *Given*: $l$ = 3463 m, $D$ = 160 mm, $d$ = 127 mm, $A_{st}$ = 357 mm$^2$/m (at midspan), $A_{st}$ = 457 mm$^2$/m, and $A_{sc}$ = 178 mm$^2$/m, at first interior support (as shown in Fig. 10.8), $f_{ck}$ = 25 MPa, $f_y$ = 415 MPa, $w_{DL}$ = 5.25 kN/m$^2$ $w_{LL}$ = 4.0 kN/m$^2$. [Refer Fig. 10.8].

  $E_c = 5000\sqrt{25}$ = 25000 MPa

$m = E_s/E_c = 2 \times 10^5/25000 = 8$

$I_{gr} = 1000 \times 160^3/12 = 3.4133 \times 10^8 \text{ mm}^4$

$f_{cr} = 0.7\sqrt{25} = 3.5 \text{ MPa}$

$\Rightarrow M_{cr} = f_{cr}\, I_{gr}/(0.5D) = 3.5 \times (3.4133 \times 10^8)/(0.5 \times 160)$

$= 14.93 \times 10^6 \text{ Nmm} = 14.93 \text{ kNm per m width}$

$\tilde{M}_{cr} = 0.7M_{cr} = 10.45 \text{ kNm per m width}$

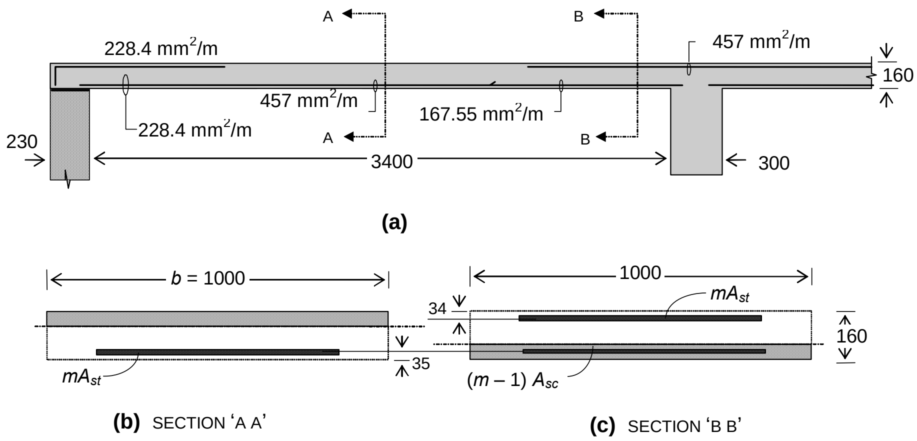


**Fig. 10.8** Example 10.4

- The moment coefficients prescribed by the Code (and used in Example 5.3) will be used here to determine the moments.

(a) *at midspan*: $M_{M,D} = w_{DL}l^2/12 = 5.5 \times 3.463^2/12$

$= 5.5 \text{ kNm per m width} < \tilde{M}_{cr} = 10.45 \text{ kNm}$

$M_{M,D+L} = M_{M,D} + w_{LL}l^2/10 = 5.5 + (4.0 \times 3.463^2/10)$

$= 10.3 \text{ kNm per m width} < \tilde{M}_{cr}$

(b) *at end support*: $M_{1,D} = w_{DL}l^2/24 = 5.5 \times 3.463^2/24$

$= 2.75 \text{ kNm} < \tilde{M}_{cr}$

$M_{1,D+L} = (w_{DL} + w_{LL})l^2/24 = 9.5 \times 3.463^2/24$

$= 4.75 \text{ kNm} < \tilde{M}_{cr}$

(c) *at first interior support*:

$M_{2,D} = w_{DL}l^2/10 = 5.5 \times 3.463^2/10$

$= 6.60 \text{ kNm per m width} < \tilde{M}_{cr}$

$M_{2,D+L} = M_{2,D} + w_{LL}l^2/9 = 6.60 + (4.0 \times 3.463^2/9)$

$= 11.93 \text{ kNm} > \tilde{M}_{cr}$

**Effective second moment of area**

(a) *at midspan*:

(i) due to *DL*: $M_{M,D} < \tilde{M}_{cr} \Rightarrow (I_{eff,\,m})_D = I_{gr} = 3.4133 \times 10^8$ mm$^4$

(ii) due to *DL* + *LL*: $M_{M,D+L} = 10.3$ kNm $< \tilde{M}_{cr}$

$\Rightarrow (I_{eff,\,m})_{D+L} = 3.4133 \times 10^8$ mm$^4$

(b) *at end support*:

$M_{1,D+L} < \tilde{M}_{cr} \;\; \Rightarrow (I_{eff,\,1})_D = (I_{eff,\,1})_{D+L} = I_{gr} = 3.4133 \times 10^8$ mm$^4$

(c) *at first interior support*:

(i) due to *DL*: $M_{2,D} = 6.6$ kNm $< \tilde{M}_{cr}$

(ii) due to *DL* + *LL*: $M_{2,D+L} = 11.93$ kNm $> \tilde{M}_{cr}$

The section is doubly reinforced [Fig. 10.8(c)]. Taking moments of areas of the cracked-transformed about the NA,

$1000 \times (kd)^2/2 + (7 \times 167.55)(kd - 35) = (8 \times 457) \times (126 - kd)$

$\Rightarrow 500(kd)^2 + 4828.85\,(kd) - 501705.8 = 0$

Solving, $kd = 27.2$ mm, k = 27.2 /126 =0.2158

$\Rightarrow I_{cr} = 1000 \times (27.2\,)^3/3 + (7 \times 167.55)\,(27.2 - 34)^2 + (8 \times 457)(126 - 27.2)^2$

$0.4245 \times 10^8$ mm$^4$

$\Rightarrow \eta = j(1-k)(b_w/b) = (1\text{-}0.2158/3)(1\text{-}0.2158)1.0 = 0.7278$

$$\Rightarrow \; I_{eff} = \frac{I_{cr}}{1.2-(\tilde{M}_{cr}/M)\eta} = \frac{0.4245\times10^8}{1.2-(10.45/11.93)0.7278} = 0.7547 \times 10^8$$

$M_{2,D} = 6.6$ kNm $\Rightarrow (I_{eff,\,2})_D = I_{gr} = 3.4133 \times 10^8$ mm$^4$

$M_{2,D+L} = 11.63$ kNm $\Rightarrow (I_{eff,\,2})_{D+L} = 0.7547 \times 10^8$ mm$^4 > I_{cr}$

- *Weighted average* [Fig. 10.8(a)]

  $I_{eff,\,av} = 0.7 I_{eff,\,m} + 0.15(I_{eff,1} + I_{eff,\,2})$

  (i) under *DL*: $(I_{eff,\,av})_D = 0.7 \times (3.4133 \times 10^8) + 0.15\,(3.4133 + 3.4133) \times 10^8$
  $= 3.4133 \times 10^8$ mm$^4$

  (ii) under *DL* + *LL*: $(I_{eff,\,av})_{D+L} = 0.7 \times (3.4133 \times 10^8)$
  $+ 0.15\,(3.4133 + 0.7547) \times 10^8 = 3.0145 \times 10^8$ mm$^4$

**Short-term deflection**

- $\Delta \approx \Delta_m = \dfrac{5l^2}{48EI_{eff}}\,[1.2M_m - 0.2M_o]$ (Eq. 10.3)

  where $M_o = wl^2/8$

  (i) *due to DL*, $M_m = 5.5$ kNm, $M_o = 5.5 \times 3.463^2/8 = 8.245$ kNm

$$\Rightarrow \; \Delta_D = \frac{5\times(3463)^2\times(1.2\times5.5-0.2\times8.245)\times10^6}{48\times25000\times(3.4133\times10^8)} = 0.73 \text{ mm}$$

  (ii) *due to DL* + *LL*, $M_m = 10.3$ kNm , $M_o = 9.5 \times 3.463^2/8 = 14.24$ kNm

$$\Rightarrow \; \Delta_{D+L} = \frac{5\times(3463)^2\times(1.2\times10.3-0.2\times14.24)\times10^6}{48\times25000\times(3.0145\times10^8)} = 1.58 \text{ mm } (= l/2192)$$

  (iii) *due to LL* alone: $\Delta_L = \Delta_{D+L} - \Delta_D$
  $= 1.58 - 0.73$

$$= \mathbf{0.85\ mm}$$

## 10.4 LONG-TERM DEFLECTION

The deflection of a reinforced concrete flexural member increases with time, mainly due to:

- differential shrinkage or temperature variation (causing differential strains across the cross-section, resulting in curvature);
- creep under sustained loading; and
- temperature effects in statically indeterminate frames [Fig. 10.9].

The factors affecting shrinkage and creep are related to the environment, making of concrete and loading history; these have been described in detail in Sections 2.11 and 2.12. It may be noted that, unlike creep strains, shrinkage and temperature strains are independent of the stress considerations in the concrete. Furthermore, shrinkage and temperature effects are reversible to a large extent, unlike creep effects.

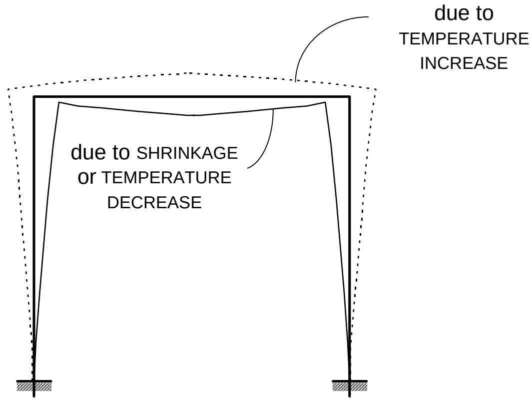


**Fig. 10.9** Deflections in a statically indeterminate frame due to temperature effects or shrinkage

The combined long-term deflection due to shrinkage, creep and temperature effects may be **as large as two to three times the short-term deflection** due to dead and live loads. It may also be noted that, whereas excessive short-term deflections can be effectively countered by *cambering*, this is not generally done in the case of long-term deflections. Providing a *camber* to a reinforced concrete flexural member implies casting the member in such a configuration that, following the removal of the formwork, the member deflects (under dead loads) into a horizontal position. This is commonly done in the case of cantilever beams and slabs.

The fact that the observed deflections of a structure soon after construction are well within limits is no guarantee that this will remain so in the future. In most cases, it is the long-term deflection that gradually builds up over a period of time (2–5 years or even more), and if the total deflection exceeds the acceptable limit, a serviceability failure will occur. Designers will do well to bear this in mind.

Additional factors which can contribute to increased long-term deflection (not considered here) include formation of new cracks, widening of earlier cracks, and effects of repeated load cycles.

### 10.4.1 Deflection Due to Differential Shrinkage

In an unrestrained reinforced concrete member, drying shrinkage of concrete results in shortening of the member. However, the reinforcing steel embedded in the concrete resists this shortening to some extent, with the result that compressive stress is developed in the steel, and tensile stress is developed in the concrete. When the reinforcement is placed symmetrically in the cross section, shrinkage does not result in any curvature of the member — except in statically indeterminate frame elements, where shrinkage results in an overall change in geometry of the entire frame, and has an effect similar to temperature [Fig. 10.9].

When the reinforcement is unsymmetrically placed in the cross-section (as is usually the case in flexural members), differential strains are induced across the cross section — with the locations with less (or no) reinforcement shrinking more than the location with relatively high reinforcement. This is depicted in Fig. 10.10(a) for a simply supported beam (where the main bars are located at the bottom), and in Fig. 10.10(b) for a cantilever beam (where the main bars are on top). For convenience, it is assumed that the strain gradient is linear. The differential shrinkage causes a curvature ($\varphi_{sh}$) in the member, which is in the same direction as that due to flexure under loading. Thus the shrinkage deflection enhances the deflection due to loads.

#### *Code Expression for Shrinkage Curvature*

The shrinkage curvature $\varphi_{sh}$ (due to differential shrinkage) may be expressed in terms of the shrinkage strains $\varepsilon_{sh}$ (at the extreme concrete compression fibre) and $\varepsilon_{st}$ (at the level of the tension steel) [refer Fig. 10.10] as follows:

$$\varphi_{sh} = \frac{\varepsilon_{sh} - \varepsilon_{st}}{d} \tag{10.17}$$

where $d$ is the effective depth.

$$\Rightarrow \varphi_{sh} = \left(\frac{\varepsilon_{sh}}{d}\right)\left[1 - \varepsilon_{st}/\varepsilon_{sh}\right]$$

$$\Rightarrow \varphi_{sh} = \left(\frac{\varepsilon_{sh}}{D}\right) k$$

where $$k = \frac{1 - \varepsilon_{st}/\varepsilon_{sh}}{d/D}$$

The parameter, $k$, evidently depends, amongst other things, on the extent of asymmetry in the reinforcement provided in the cross-section. The Code (Cl. C–3.1) suggests the following expression for shrinkage curvature based on empirical fits with test data [Ref. 10.2]:

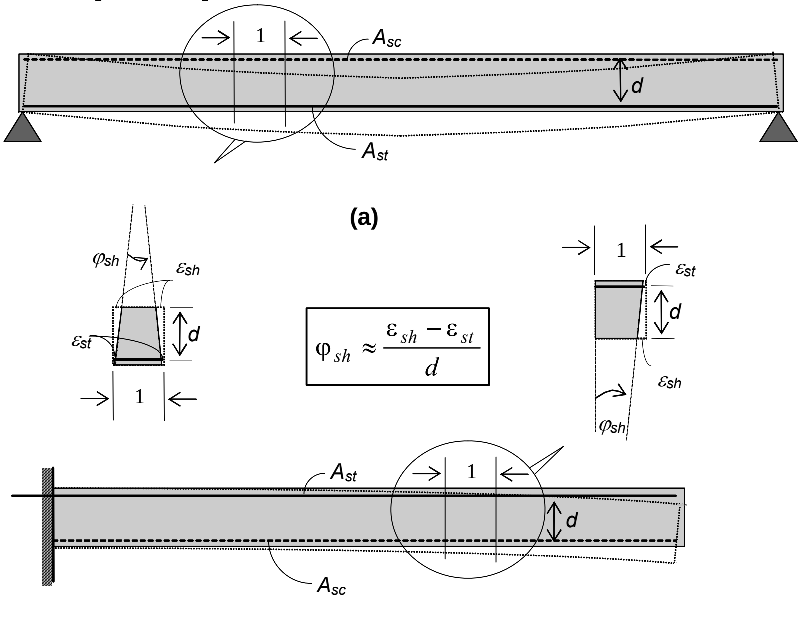


**Fig. 10.10** Curvature due to differential shrinkage

$$\varphi_{sh} = k_4 \frac{\varepsilon_{cs}}{D} \tag{10.18}$$

where, $\varepsilon_{cs}$ ≡ the ultimate shrinkage strain of concrete, and

$$k_4 = \begin{cases} 0.72(p_t - p_c)\big/\sqrt{p_t} \le 1.0 & \text{for } 0.25 \le (p_t - p_c) < 1.0 \\ 0.65(p_t - p_c)\big/\sqrt{p_t} \le 1.0 & \text{for } (p_t - p_c) \ge 1.0 \end{cases} \tag{10.19}$$

where $p_t \equiv 100A_{st}/(bd)$ and $p_c \equiv 100A_{sc}/(bd)$ denote the percentages of tension reinforcement and compression reinforcement respectively. When $p_t = p_c$ (i.e., the beam is symmetrically reinforced), $k_4 = 0$, and hence $\varphi_{sh} = 0$.

### *Relation Between Deflection and Shrinkage Curvature*

The deflection due to shrinkage in a reinforced concrete beam depends on the variation of shrinkage curvature $\varphi_{sh}$ along the span of the beam and the boundary conditions of the beam. For convenience, it may be assumed that the curvature $\varphi_{sh}$

remains constant in magnitude† and merely changes sign at points of inflection, which separate the beam into *sagging* ('positive' curvature) and *hogging* ('negative' curvature) segments. This is depicted in Fig. 10.11 for some typical boundary conditions.

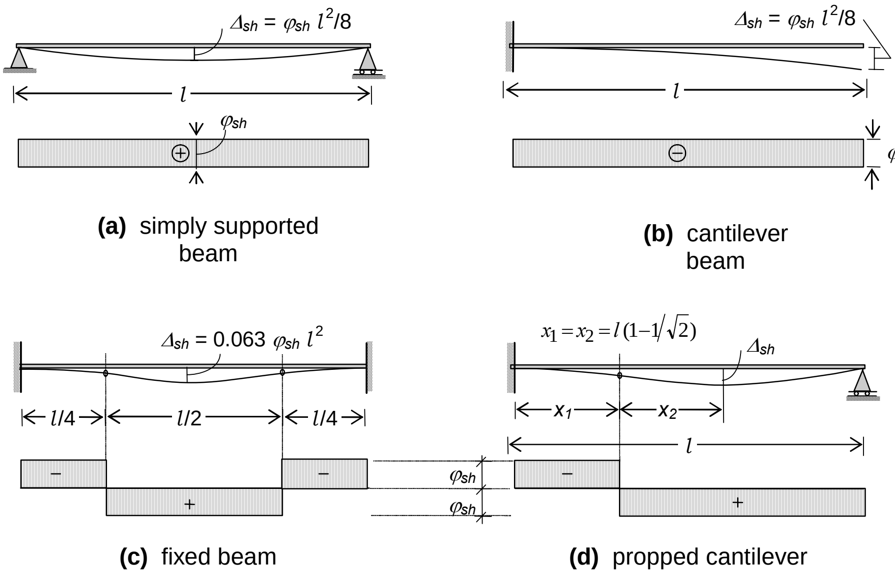


**Fig. 10.11** Relation between deflection and shrinkage curvature

The relationship between deflection $\Delta_{sh}$ and curvature $\varphi_{sh}$, for any set of boundary conditions [Fig. 10.11] can be established by any of the standard methods of structural analysis (such as the *conjugate beam* method or the *moment area* method). In these methods, the curvature $\varphi_{sh}$ takes the place of *M*/*EI*. Using these methods, the deflections can be obtained as:

$$\Delta_{sh} = \begin{cases} 0.125\varphi_{sh} l^2 & \text{for simply supported beams} \\ 0.500\varphi_{sh} l^2 & \text{for cantilever beams} \\ 0.063\varphi_{sh} l^2 & \text{for fixed beams} \\ 0.086\varphi_{sh} l^2 & \text{for propped cantilevers} \end{cases} \tag{10.20}$$

or, in general,

$$\Delta_{sh} = k_3 \varphi_{sh} l^2 \tag{10.21}$$

† In actual beams, the ratio $(p_t - p_c)/\sqrt{p_t}$ in Eq . 10.19, may not be uniform along the span. It is generally satisfactory to compute this ratio (and hence, $k_4$) at the midspan location in simply supported and continuous beams, and at the support section in cantilever beams.

where $l$ is the effective span of the beam.

The ratio $(\Delta_{sh}/\varphi_{sh} l^2)$ is denoted by the parameter $k_3$ in the Code (Cl. C–3.1) and the results obtained above are summarised in the Code. For convenience, the Code permits the use of the condition of 'fixity' for continuous ends, and thus the Code recommends values of 0.086 and 0.063 for $k_3$ for beams continuous at one end and both ends respectively.

### *Expression for Maximum Shrinkage Deflection*

The maximum shrinkage deflection $\Delta_{sh}$ in a reinforced concrete flexural member may be expressed (by combining Eq. 10.18 Eq. 10.21 follows:

$$\Delta_{sh} = k_3 k_4 \varepsilon_{cs} (l^2/D) \tag{10.22}$$

where $k_3$ is a constant which varies between 0.063 and 0.50, depending on the boundary conditions [Eq. 10.20]; $k_4$ is another constant which varies between 0.0 and 1.0, depending on the relative magnitudes of $p_t$ and $p_c$ [Eq. 10.19]; $\varepsilon_{cs}$ is the *ultimate shrinkage strain* of concrete [refer Section 2.12.1]; $D$ is the overall depth and $l$ the effective span of the flexural member.

The value of $\varepsilon_{cs}$ to be considered for calculating $\Delta_{sh}$ [Eq. 10.22] depends on various factors such as the constituents of concrete (especially water content at the time of mixing), size of the member, relative humidity and temperature. As explained in Section 2.12.1, the value of $\varepsilon_{cs}$ = 0.0003 mm/mm suggested by the Code (Cl. 6.2.4.1) in the absence of test data is rather low. Under hot and low-humidity conditions, and where high water content has been employed in the making of concrete, values of $\varepsilon_{cs}$ up to 0.0010 mm/mm have been reported. Hence, the choice of $\varepsilon_{cs}$ should be judiciously made. It may be noted that ACI Committee 435 [Ref. 10.17] and Branson [Ref. 10.5] have suggested the use of $\varepsilon_{cs}$ = 0.0004 mm/mm for routine deflection calculation, and higher values wherever required.

Other empirical methods of determining shrinkage deflection are described in Ref. 10.5 – 10.7, 10.10, 10.16 – 10.18.

### 10.4.2 Deflection Due to Creep

Under sustained loading, compressive strains in concrete keep increasing nonlinearly with time, owing to the phenomenon called *creep*. The variation of *creep strain* with time for concrete under uniaxial compression [refer Fig. 2.15] and the factors influencing creep in concrete have been described in Section 2.11. The *creep coefficient*, $C_t$, defined as the ratio of the creep strain, $\varepsilon_{cp}$, to the initial elastic strain ('instantaneous strain'), $\varepsilon_i$, provides a measure of creep in concrete at any given time. The maximum value of $C_t$, called the *ultimate creep coefficient* (designated as $\theta$ by the Code), is required for predicting the maximum deflection of a flexural member due to creep. In the absence of data related to the factors influencing creep, the Code (Cl. 6.2.5.1) recommends values of $\theta$ equal to 2.2, 1.6 and 1.1 for ages of loading equal to 7 days, 28 days and 1 year respectively.

In a flexural member, the distribution of creep strains across the depth at any cross-section is non-uniform, with a practically linear variation similar to that produced by the applied loading (i.e., bending moment). This linear variation of creep strains [Fig. 10.12(b)] results in a *creep curvature,* $\varphi_{cp}$, which is additive to the *initial elastic curvature,* $\varphi_i$, and is similar in effect to the *shrinkage curvature* $\varphi_{sh}$ described in Section 11.4.1. It may be noted that although the creep effect is primarily related to increased strains in *concrete* under *compression*, there is also a marginal increase in the tensile strain in the steel†, as indicated in Fig. 10.12.

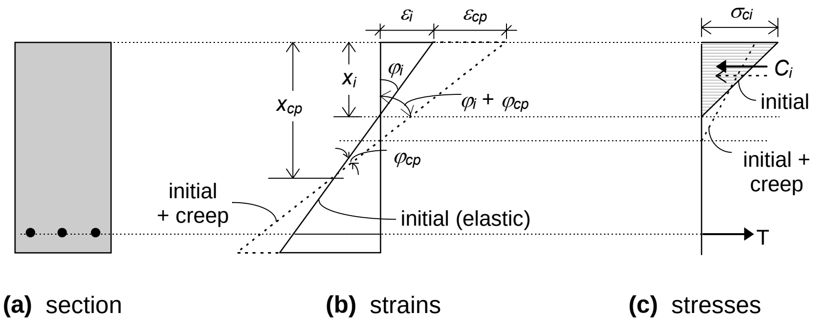


**Fig. 10.12** Creep curvature in a flexural member

### *Relation Between Creep Deflection and Initial Elastic Deflection*

Within the range of service loads, creep curvature $\varphi_{cp}$ may be assumed to be proportional to the initial elastic curvature $\varphi_i$. With reference to Fig. 10.12(b),

$$\frac{\varphi_{cp}}{\varphi_i} = \frac{\varepsilon_{cp}/x_{cp}}{\varepsilon_i/x_i} = k_r\, C_t \tag{10.23}$$

where $C_t \equiv \varepsilon_{cp}/\varepsilon_i$ is the creep coefficient, and $k_r \equiv x_i/x_{cp}$ is the ratio of the 'initial' neutral axis depth ($x_i$) to the neutral axis depth due to creep ($x_{cp}$). As $x_i < x_{cp}$, the coefficient $k_r$ is less than unity. For singly reinforced flexural members, $k_r \approx 0.85$ [Ref. 10.17]. The presence of compression reinforcement reduces $\varepsilon_{cp}$ and hence $\varphi_{cp}$. To allow for this reduced curvature and deflections, the following expression is recommended by ACI [Ref. 10.17]:

$$k_r = 0.85/(1 + 0.5p_c) \tag{10.24}$$

where $p_c \equiv 100A_{sc}/bd$ is the percentage of compression reinforcement.

The variation of creep curvature $\varphi_{cp}$ along the span of the flexural member may be assumed to be identical to the variation of $\varphi_i$. Hence, it follows that:

† Steel itself is not subjected to creep. However, due to creep in concrete (see Fig. 10.12), there is a slight increase in the depth of neutral axis, with a consequent reduction in the internal lever arm. Hence, to maintain static equilibrium with the applied moment at the section, there has to be a slight increase in the steel stress, and hence the steel strain.

$$\frac{\Delta_{cp}}{\Delta_i} = \frac{\varphi_{cp}}{\varphi_i} = k_r\, C_t$$

where $\Delta_i$ and $\Delta_{cp}$ denote respectively the maximum initial elastic deflection and the additional deflection due to creep. For estimating maximum (ultimate) deflection due to creep, the 'ultimate creep coefficient' $\theta$ should be used in lieu of $C_t$. Accordingly,

$$\Delta_{cp} = (k_r\, \theta)\, \Delta_i \tag{10.25}$$

where $\Delta_i$ is to be taken as the 'initial' elastic displacement due to the permanently applied loads (dead loads plus sustained part of live loads). It may be noted that although transient live loads are excluded in the computation of $\Delta_i$, the possibility of a reduced flexural stiffness on account of prior cracking due to such live loads should be considered. Hence, the calculation of $\Delta_i$ should be based on $I_{eff,\,D+L}$, and not on $I_{eff,D}$. It is in this respect that $\Delta_i$ differs from $\Delta_D$ [refer Fig. 10.5]. In case both live load moments and dead load moments have the same distribution along the span, it follows that:

$$\Delta_{cp} = \frac{w_{DL}}{w_{DL} + w_{LL}}\,\Delta_{D+L} \tag{10.26}$$

This is because both $\Delta_{cp}$ and $\Delta_{D+L}$ are computed with the same flexural rigidity, $EI_{eff,\,D+L}$ and hence they are proportional to the load intensity.

### *Code Procedure for Estimating Creep Deflection*

The procedure given in the Code (Cl. C – 4.1) for calculating creep deflection is different from the ACI recommendation [Eq. 10.27]. Instead of determining the creep deflection $\Delta_{cp}$ directly in terms of the initial elastic deflection $\Delta_i$ (due to the permanent loads), the Code procedure involves the explicit calculation of the deflection $\Delta_{i+cp}$ due to the permanent load plus creep, using an effective modulus of elasticity $E_{ce} = E_c/(1+\theta)$ [refer Section 2.11.4, Eq. 2.8]. The creep deflection $\Delta_{cp}$ is then obtained as the difference between $\Delta_{i+cp}$ and $\Delta_i$:

$$\Delta_{cp} = \Delta_{i+cp} - \Delta_i \tag{10.27}$$

where $\Delta_{i+cp}$ is calculated assuming $E_{ce}$ and the corresponding $I_{eff}$[†], whereas $\Delta_i$ is calculated assuming $E_c$ and $I_{eff,\,D+L}$.

The Code formula for the effective modulus of elasticity is based on the reasonable assumption that the total strain in concrete $\varepsilon_{i+cp}$ (i.e., initial elastic strain plus creep strain) is directly proportional to the stress $\sigma_i$ induced by the permanent loads [Fig. 10.13].

As

[†] The increased modular ratio $m = E_s/E_{ce}$ is generally quite high, with the result that the second moment of area of the corresponding cracked-transformed section (with steel area contributions $mA_{st}$, $mA_{sc}$) will also be high — but has to be limited to $I_{gr}$. In case the calculated $I_{cr}$ is less than $I_{gr}$, then $I_{eff}$ has to be calculated using Eq. 10.5a and considering the moment due to dead load plus live load.

$$\varepsilon_{i+cp} = \varepsilon_i + \varepsilon_{cp} = \varepsilon_i (1 + \theta)$$

it follows that

$$E_{ce} \equiv \frac{\sigma_i}{\varepsilon_{i+cp}} = \frac{\sigma_i}{\varepsilon_i}\frac{1}{1+\theta} = \frac{E_c}{1+\theta} \qquad (10.28)$$

where $E_c \equiv \sigma_i/\varepsilon_i$

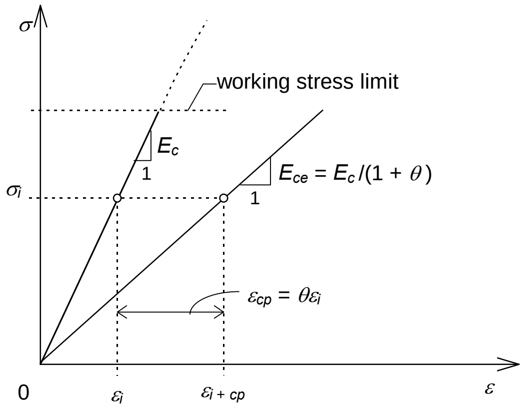


**Fig. 10.13** Effective modulus of elasticity under creep

The steps involved in the Code procedure for determining $\Delta_{cp}$ may be summarised [Ref. 10.2] as follows:

1. Compute $\Delta_{D+L}$ due to the characteristic dead plus live loads (considering $E_c$ and $I_{eff,\,D+L}$).
2. Compute $\Delta_i$ due to permanent (dead) loads alone, considering $E_c$ and $I_{eff,\,D+L}$.
3. Compute $\Delta_{i+cp}$ due to permanent loads plus creep, considering $E_{ce}$ and $I_{eff,\,D+L}$.
4. Calculate the creep deflection $\Delta_{cp} = \Delta_{i+cp} - \Delta_i$ [Eq. 10.27].

### 10.4.3 Deflection Due to Temperature Effects

As mentioned earlier with reference to Fig. 10.9, seasonal changes in temperature can introduce curvatures (and stresses) in statically indeterminate frames, that may be significant. The determination of bending moments due to temperature loading may be done by any of the standard methods of structural analysis of indeterminate frames. An appropriate value of the coefficient of thermal expansion should be considered — as explained in Section 2.12.2. The deflections in the various beam members may now be determined (using Eq. 10.2 or 10.3) in a manner identical to the determination of short-term deflection in continuous beams [refer Section 10.3.5]. The same procedure is applicable to deflections induced by overall shrinkage in statically indeterminate frames.

In addition to overall curvatures in an integrated structure, local deflections may be introduced in individual members (such as beams and slabs) owing to unsymmetrical reinforcement. The calculation of such deflections is similar to the calculation of deflections induced by differential shrinkage [refer Section 10.4.1]. In case of a known thermal gradient across the depth of a flexural member, a simple 'strength of materials' approach may be adopted for calculating deflections.

It may be noted, however, that in routine designs of reinforced concrete buildings, deflections due to temperature effects are rarely computed. This is largely because such deflections are not usually significant† and are reversible. However, the tensile stresses (and consequent cracks) induced by restraints against temperature changes can be significant, and these need to be taken care of by proper detailing of reinforcement — as explained in Section 2.12.2.

### 10.4.4 Checks on Total Deflection

After calculating the various components of short-term deflection ($\Delta_D$, $\Delta_{D+L}$) and long-term deflection ($\Delta_{sh}$, $\Delta_{cp}$, $\Delta_{temp}$), the acceptability of these deflections should be verified with reference to the deflection limits specified by the Code. As explained in Section 10.2.1, two checks on total deflection are called for:

$$\Delta_{D+L} + \Delta_{long\text{-}term} \le l/250 \tag{10.29a}$$

$$\Delta_L + \Delta_{long\text{-}term} \le \begin{cases} l/350 \\ 20\text{ mm} \end{cases} \quad \text{(whichever is less)} \tag{10.29b}$$

where

$$\Delta_{long\text{-}term} = \Delta_{sh} + \Delta_{cp} + \Delta_{temp} \tag{10.29c}$$

and $l$ is the effective span of the flexural member under considerations.

In case these limits are exceeded, it may be investigated if the excess can be contained by providing a *camber* (as explained earlier). Otherwise, the design of the member should be suitably revised — ideally, by enhancing the depth (and thereby the stiffness) of the member. If there are constraints on the cross-sectional dimensions, then the grade of concrete (and hence, $E_c$) should be increased suitably. Alternatively, measures may be taken to reduce the effective span and/or the loading on the member. Providing compression reinforcement will also reduce deflections, especially those due to shrinkage and creep.

#### EXAMPLE 10.5

In continuation with Example 10.1 and 10.2, determine the maximum long-term deflections due to shrinkage and creep, and hence apply the Code checks on the total deflection. Assume an ultimate shrinkage strain $\varepsilon_{cp}$ = 0.004 and an ultimate creep coefficient $\theta$ = 1.6.

#### SOLUTION

- From Examples 10.1 and 10.2,

† In special circumstances, these may be significant, and in such cases deflections due to temperature effects should be explicitly computed.

$l$ = 4160 mm, $D$ = 200 mm, $A_{st}$ = 628 mm$^2$/m ($p_t$ = 0.38), $d$ = 165 mm, $f_{ck}$ = 25 MPa, $f_y$ =415 MPa, $w_{DL}$ = 6.0 kN/m$^2$, $w_{LL}$ = 4.0 kN/m$^2$, $E_c$ = 25000 MPa, $\tilde{M}_{cr}$ = 16.33 kNm/m., $I_{cr}$ = 0.9916 × 10$^8$ mm$^4$, $I_{gr}$ = 6.667 × 10$^8$ mm$^4$, $M_{D+L}$ = 21.63 kNm/m, $I_{eff,\,D+L}$ = 1.5192 × 10$^8$ mm$^4$, $\Delta_{D+L}$ = 10.26 mm,

$M_D$ = 12.98 kNm/m, $I_{eff,\,D} = I_{gr}$, $\Delta_D$ = 1.35 mm, $\Delta_L$ = 8.91mm.

**Long-term deflection due to shrinkage**: $\Delta_{sh}$

- $\Delta_{sh} = k_3\, k_4\, \varepsilon_{cs}\, (l^2/D)$ [refer Eq. 10.22]
  where $k_3$ = 0.125 for simply supported end conditions,

  $k_4 = 0.72\sqrt{p_t} = 0.72\sqrt{0.38} = 0.444$ (< 1.0) [refer Eq. 10.19, $p_c$ = 0]

  $\varepsilon_{cs}$ = 0.0004 (given)

  $\Rightarrow \Delta_{sh}$ = (0.125)(0.444)(0.0004)(4160)$^2$/(200) = **1.92 mm**

**Long-term deflection due to creep**: $\Delta_{cp}$

- $E_{ce} = E_c/(1 + \theta)$
  = 25000/(1 + 1.6) = 9615.4 MPa

  $\Rightarrow m = E_s/E_{ce} = 2 \times 10^5/9615.4 = 20.8$

  $\Rightarrow \rho m = (0.38 \times 10^{-2}) \times 20.8 = 0.079$

  $\Rightarrow k = \sqrt{2(\rho m) + (\rho m)^2} - \rho m = 0.3263$

  $\Rightarrow kd = 0.3263 \times 165 = 53.84$ mm

  $\Rightarrow I_{cr} = 1000 \times (53.84)^3/3 + (20.8 \times 628) \times (165 - 53.84)^2$
  $= 2.1343 \times 10^8 \text{ mm}^4 < I_{gr} = 6.667\times 10^8 \text{ mm}^4$

  $\Rightarrow \eta = j(1-k)(b_w/b)$ = (1-0.3263/3)(1-0.3263)1.0 = 0.6

  $$\Rightarrow I_{eff} = \frac{I_{cr}}{1.2 - (\tilde{M}_{cr}/M_{D+L})\eta} = \frac{2.1343\times10^8}{1.2-(16.33/21.63)(0.6)}$$
  $$= 2.8571 \times 10^8 \text{ mm}^4 < I_{gr}$$

- $$\Delta_{i+cp} = \frac{5}{48}\frac{M_D l^2}{E_{ce} I_{eff,D+L}} = \frac{5}{48}\times\frac{12.98\times10^6\times(4160)^2}{9615.4\times(2.8571\times10^8)} = 8.52 \text{ mm}$$

- $$\Delta_i = \frac{w_{DL}}{w_{DL}+w_{LL}}\Delta_{D+L} = \frac{6}{10}\times 10.26 = 6.16 \text{ mm}$$

  $\therefore \Delta_{cp}{}^{\dagger} = \Delta_{i+cp} - \Delta_i$ = 8.52 – 6.16 = **2.36 mm**

**Checks on total deflection**

1. $\Delta_{D+L} + \Delta_{sh} + \Delta_{cp}$ = 10.26 + 1.92 + 2.36 = **14.54 mm**
   Allowable limit = $l$/250 = 4160/250 = 16.64 mm > 14.54 — OK.
2. $\Delta_L + \Delta_{sh} + \Delta_{cp}$ = 8.91 + 1.92 + 2.36 = **13.19 mm**
   Allowable limit = $l$/350 = 11.88 mm (which is less than 20 mm)
   < 13.19 mm, indicating a marginal excess

---

† Alternatively, applying the ACI method, $k_r$ = 0.85 for $p_c$ = 0 [Eq. 10.24] $\Rightarrow \Delta_{cp} = k_r\, \theta\, \Delta_i$ = 0.85 × 1.6 × 6.16 = 8.38 mm, which is much greater than $\Delta_{cp}$ = 2.36 mm obtained as per IS Code procedure.

#### EXAMPLE 10.6

In continuation with Example 10.3, determine the maximum long-term deflections due to shrinkage and creep, and hence apply the checks on the total deflection. Assume an ultimate shrinkage strain $\varepsilon_{cp}$ = 0.0004 and an ultimate creep coefficient $\theta$ = 1.6.

#### SOLUTION

- From Example 10.3,

$l$ = 6000 mm, $b$ = 250 mm, $D$ = 400 mm, $d$ = 348 mm, $d'$ = 48 mm, $A_{st}$ = 1848 mm$^2$ ($p_t$ = 2.124), $A_{sc}$ = 942.5 mm$^2$ ($p_c$ = 1.0833), $f_{ck}$ = 25 MPa, $f_y$ = 415 MPa, $w_{DL}$ = 7.5 kN/m, plus $W_{DL}$ = 30 kN, at midspan, $w_{LL}$ = 10.0 kN/m [refer Fig. 10.7], $E_c$ = 25000 MPa, $\tilde{M}_{cr}$ = 16.33 kNm, $I_{cr}$ = 9.24117 × 10$^8$ mm$^4$, $I_{gr}$ = 13.3333 × 10$^8$ mm$^4$, $M_D = M_1 + M_2$ = 23.75 + 45.0 = 78.75 kNm, $M_{D+L}$ = 123.8 kNm;

$I_{eff,\,D} = I_{eff,\,D+L} = I_{cr}$ = **9.24117**× 10$^8$ mm$^4$, $\Delta_{D+L}$ = 18.62 mm, $\Delta_D$ = 11.32 mm, $\Delta_L$ = 7.3 mm.

**Long-term deflection due to shrinkage:** $\Delta_{sh}$

- $\Delta_{sh} = k_3\, k_4\, \varepsilon_{cs}\, (l^2/D)$
  where $k_3$ = 0.125 for simply supported end conditions,
  $p_t - p_c$ = 2.124 –1.0833 = 1.0407 > 1.0
  $\Rightarrow k_4 = 0.65\ (p_t - p_c)\big/\sqrt{p_t} = 0.65\ (1.0407)\big/\sqrt{2.124}$ = 0.464 (< 1.0) [Eq. 10.19]
  $\varepsilon_{cs}$ = 0.0004 (given)
  $\Rightarrow \Delta_{sh}$ = (0.125)( 0.464)(0.0004)(6000)$^2$/(400) = **2.088 mm**

**Long-term deflection due to creep:** $\Delta_{cp}$

- $E_{ce} = E_c/(1 + \theta)$ = 25000/(1 + 1.6) = 9615.4 MPa
  $\Rightarrow m = E_s/E_{ce} = 2 \times 10^5/9804 = 20.8$
  Taking moments of areas of the cracked-transformed section about the NA [Fig. 10.7(b)],
  $250(kd)^2/2 + (20.8 - 1)(942.5)(kd - 48) = (20.8 \times 1848)(348 - kd)$
  $\Rightarrow 125(kd)^2 + 57099.9(kd) - 14272315 = 0$
- Solving, $kd$ = 179.45 mm
  $\Rightarrow I_{cr} = 250\ (179.45)^3/3 + (20.8 \times 1848)\ (348 - 179.45)^2 + (19.8 \times 942.5)\ (179.45 - 48)^2$
  $= 18.96 \times 10^8$ mm$^4$
  but cannot exceed $I_{gr} = 13.3333 \times 10^8$ mm$^4$
  $\Rightarrow I_{eff} = I_{gr} = 13.3333 \times 10^8$ mm$^4$

$$\Rightarrow \Delta_{i+cp} = \frac{5}{48}\frac{l^2}{E_{ce}I_{eff}} \times [M_{1D} + 0.8\, M_{2D}]$$

$$= \frac{5 \times (6000)^2 \times \{33.75 + (0.8 \times 45.0)\} \times 10^6}{48 \times 9615.4 \times (13.3333 \times 10^8)} = 20.4 \text{ mm}$$

$\Delta_i = \Delta_D$ (as $I_{eff,\,D} = I_{eff,\,D+L}$ — refer Example 10.3)

$= 11.32$ mm
$\Rightarrow \therefore\ \Delta_{cp}^{\dagger} = \Delta_{i+cp} - \Delta_i = 20.4 - 11.32 =$ **9.08 mm**

**Checks on total deflection**

1. $\Delta_{D+L} + \Delta_{sh} + \Delta_{cp} = 18.62 + 2.088 + 9.08 =$ **29.79 mm**
   Allowable limit = $l/250$ = 6000/250 = 24.0 mm < 29.79 mm. Hence, there is an excess of 5.79 mm beyond the permissible limit — which can be overcome by providing a camber of at least 6 mm at the midspan location. [Compensation by camber is usually limited to the dead load deflection. Here, $\Delta_D$ = 11.32 mm; hence OK. The camber can be as much as 10 mm.]
2. $\Delta_L + \Delta_{sh} + \Delta_{cp} = 7.3 + 2.088 + 9.08 =$ **18.47 mm**
   Allowable limit = $l/350$ = 17.14 mm (which is less than 20 mm)
   < 18.47 mm
   which represents only a marginal excess.

## 10.5 SERVICEABILITY LIMIT STATE: CRACKING

### 10.5.1 Cracking in Reinforced Concrete Members

Cracking of concrete will occur whenever the tensile strength (or the ultimate tensile strain) of concrete is exceeded. As concrete has relatively low tensile strength as well as low failure strain in tension, cracking is usually inevitable in normal reinforced concrete members. However, the degree of cracking (in terms of width and spacing of cracks) can be controlled through proper design. Cracking is considered undesirable, not only for obvious aesthetic reasons, but also because it adversely affects durability (in aggressive environments) and leads to corrosion of the embedded steel. Also, in some cases, such as liquid-retaining structures and pressure vessels, it can adversely affect the basic functional requirements (such as water-tightness in a water tank). Hence, it is important for the designer to have an understanding of the various causes of cracking, the allowable limits on crack-widths under different situations as well as the methods to achieve crack control. It may also be stated that there is presently widespread lack of awareness of these aspects related to the serviceability limit state of cracking. In the modern trend of adopting limit states design concepts, the emphasis has been on the limit of state of collapse, and often the limit state of serviceability with respect to cracking of concrete gets compromised, as is the case with deflection.

Cracking in reinforced concrete members is attributable to various causes [Ref. 10.18], particularly:

1. flexural tensile stress due to bending under applied loads [Fig. 10.14];
2. diagonal tension due to shear and/or torsion;

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† Alternatively, using the ACI formulation, $k_r = 0.85/(1 + 0.5 \times 1.0833) = 0.551$ [Eq. 10.24] $\Rightarrow$ $\Delta_{cp} = k_r\ \theta\ \Delta_i = 0.551 \times 1.6 \times 11.32 = 9.98$ mm, which is close to (but slighly greater than) $\Delta_{cp}$ = 9.08 mm obtained by the Code procedure.

3. direct tensile stress under applied loads (for example, hoop tension in a circular water tank);
4. lateral tensile strains accompanying high compressive stress/strain due to Poisson effect (as in a compression test) or due to heavy concentrated loads as in a split cylinder test;
5. restraint against volume changes due to shrinkage and temperature [Fig. 10.15], as well as due to creep and chemical effects; and
6. additional curvatures due to continuity effects, settlement of supports, etc.

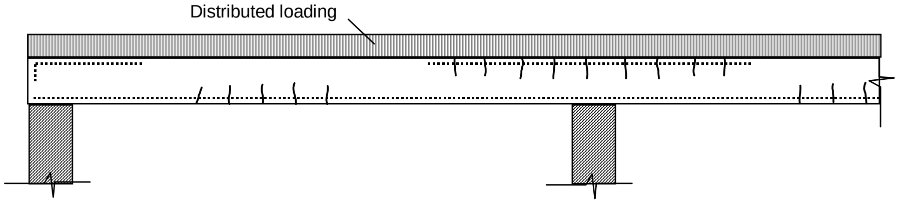


**Fig. 10.14** Typical flexural cracks in a continuous one-way slab due to gravity loading

Structural cracking in concrete occurs in tension, flexure or a combination of the two effects (eccentric tension). When this happens, splitting of the concrete occurs at the surface, penetrating inwards. Under direct tension, the crack generally runs through the thickness of the member (wall or slab), whereas under flexure, the crack is limited to the flexural tension zone. In all cases, the spacing of cracks as well as width of individual cracks depends not only on the magnitude of tensile force acting, but also on the reinforcement detailing, properties of concrete and thickness of section. It is observed that wide crack spacing is associated with relatively wide crack-widths, which is undesirable. Such cracking is often associated with low reinforcement percentages, wide spacing of bars and the use of high strength reinforcing steels, such as Fe 415 and Fe 500 (because the associated tensile strains at service load levels will be relatively high).

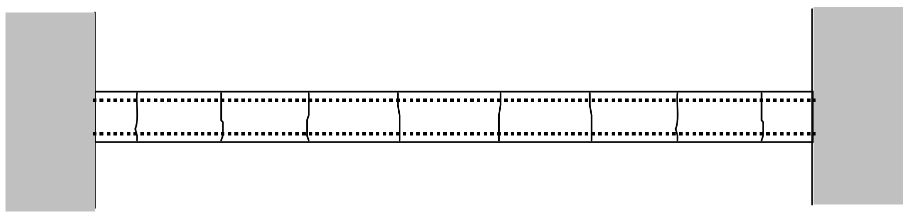

**Fig. 10.15** Cracking due to restrained shrinkage and temperature effects in a lightly loaded slab

Although cracking of concrete is inevitable, it is desirable to aim for a large number of well-distributed fine hairline cracks, rather than a few but wide cracks. This is the objective of limit state design for the serviceability limit state of cracking.

It should be noted that although through proper design (for example, providing increased thickness), it is possible to contain the tensile stress in concrete within acceptable limits, the occurrence of cracking due to restrained shrinkage and thermal effects and induced tensile strains cannot be altogether eliminated[†].

The discussion in this Chapter is primarily limited to the estimation of crack-widths due to applied bending moments and direct tensile forces. The problems associated with the prediction of crack-widths due to restrained deformation are also briefly discussed.

Cracking of concrete cannot be predicted with accuracy, because of inherent randomness in the nature of cracking. The width (and spacing) of cracks is subject to wide scatter, and methods of crack-width calculation only attempt to predict the most probable[‡] maximum crack-width as observed in laboratory tests. The primary objective of calculating crack width is to avoid gross errors in design, which might result in concentration and excessive width of cracks.

### 10.5.2 Limits on Cracking

The acceptable limits of cracking vary with the type of structure and its environment. The Code (Cl. 35.3.2) recommends a maximum limit of 0.3 mm on the assessed surface width of cracks for concrete structures subject to 'mild' exposure. This limit of 0.3 mm is based essentially on aesthetic considerations, but this limit is also considered to be adequate for the purpose of durability *when the member is completely protected against weather or aggressive conditions* [Ref. 10.2].

However, in the case of "members where cracking in the tensile zone is harmful either because they are exposed to the effects of weather or continuously exposed to moisture or in contact with soil or ground water" ('moderate' exposure category), the crack-width limit specified by the Code is 0.2mm. Under more aggressive environments (exposure categories: 'severe', 'very severe' and 'extreme'), a more stringent limit of 0.1mm is recommended.

For water-retaining structures, the British code BS 8007 (1987) [Ref.10.19] recommends a limiting surface crack-width of 0.2mm in general (deemed adequate for water-tightness) and a more stringent limit of 0.1mm when aesthetic appearance is of particular importance. It is believed that cracks less than 0.2mm heal autogenously, as water percolates through the crack and dissolves calcium salts in the cement, preventing subsequent leakage.

### 10.5.3 Factors Influencing Crack-widths

Crack-widths in RC members subject to flexure, direct tension or eccentric tension, are influenced by a large number of factors, many of which are inter-related. These include:

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[†] For this reason, even in liquid retaining structures, where proportioning is sometimes done so as to keep tensile stresses in concrete at levels below its tensile strength, some minimum reinforcement is specified by Codes. Also, the designer is required to design the reinforcement to resist fully the applied tension, ignoring the tension-resisting capacity of concrete.

[‡] Statistically, with 90 percent confidence limit, in general.

- tensile stress in the steel bars;
- thickness of concrete cover;
- diameter and spacing of bars;
- depth of member and location of neutral axis; and
- bond strength and tensile strength of concrete.

When a beam is subject to a uniform bending moment [Fig. 10.2] and when the limiting tensile strain of concrete is exceeded (at the weakest location), a flexural crack will form, and the concrete in the regions adjoining the crack will no longer be subject to tensile force. Due to the variability in tensile strength and ultimate tensile strain along the length of the beam, discrete cracks will develop at different stages of loading (within ±10 percent of the 'cracking moment') [Ref. 10.18]. Experimental studies indicate that these initial cracks (sometimes referred to as 'primary' cracks) are roughly uniformly spaced. As discussed earlier (with reference to Fig. 10.2), the concrete in-between the cracks resist some tension, and the tensile strain is maximum midway between the cracks. With increase in loading, additional cracks (called 'secondary cracks') will form somewhere midway between these cracks, when the limiting tensile strain capacity is exceeded. Both primary and secondary cracks will widen with increase in loading, and additional cracks may form (as the loading approaches the ultimate load), provided the bond between concrete and steel is capable of sustaining the development of significant tensile strain in the concrete (which can exceed the limiting strain capacity). A similar mechanism of development of primary and secondary cracks is found to occur in RC members subject to direct or eccentric tension [Ref. 10.18]. In all cases of applied loading, the width of the crack is found to be maximum at the surface of the member, reducing (tapering) to a near-zero value at the surface of the reinforcement[†]. Internal cracks (not visible from outside) are also likely to develop in the tension zone, with the width increasing at distances remote from the reinforcing bar.

Studies have shown that the **width of the crack** at a point depends primarily on the following three key factors:

1. the mean tensile strain ($\varepsilon_{sm}$) in the neighbouring reinforcement;
2. the distance ($a_{cr}$) to the nearest longitudinal bar that runs perpendicular to the crack; and
3. the distance to the neutral axis location (in the case of flexural cracks).

From the point of view of crack-width control, it is the surface crack-width that is of concern. The most obvious ways of minimising surface crack-widths at service loads are by (1) limiting the tensile stress in the steel (cracked section analysis), (2) minimising the spacing of reinforcing bars, and (3) providing bars as close as possible to the concrete surface in the tension zone (including side face reinforcement in relatively deep beams).

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[†] If, however, the member is very lightly reinforced, the crack-width may be significant at the surface of the reinforcement, and the bar may even yield at the crack location. This condition is, of course, highly undesirable under normal service loads, and can be avoided by providing appropriate minimum reinforcement.

It may be noted that increased cover results in increased crack-widths at the surface. On the other hand, increased cover is highly desirable from the point of view of durability and protection against corrosion of reinforcement. These two aspects appear to be contradictory [Ref. 10.2], but it is always desirable to provide the requisite concrete cover for durability, and to control the crack-width by adopting other measures such as increasing the depth of the flexural member, reducing the bar diameter and spacing, and maintaining low stress levels in the tension steel. The use of low grade steel (mild steel) as tension reinforcement is particularly desirable in flexural members (such as bridge girders) where both control of crack-width and prevention of reinforcement corrosion are of extreme importance.

In the earlier version of the Code, no explicit recommendations were given regarding procedures to calculate crack-widths. However, in the recent revision (2000) of the Code, a procedure is given in Annex F for the estimation of flexural crack-widths (in beams and one-way slabs). Unfortunately, the Code does not emphasise sufficiently on the need to estimate crack-widths, especially where relatively large cover is used. Instead, the Code has retained a rather outdated clause from the old Code (which was perhaps valid in earlier times when admissible minimum clear covers were 15mm in slabs and 25mm in beams). According to this clause (Cl. 43.1), explicit calculations of crack-widths are required only if the spacing of reinforcement exceeds the nominal requirements specified for beams and slabs [refer Section 5.2], regardless of the cover provided. To make matters worse, the clause in the earlier code, limiting the maximum clear cover in any construction to 75 mm has, for some reason, been eliminated. Of course. the highest figure given in Table 16 of the Code for the nominal cover is 75 mm, however, since this is the *minimum* required cover and there is no upper limit specified, a designer may be tempted to give a larger cover. However, as highlighted in a recent study [Ref. 10.21], increased cover implies increased crack-widths, particularly in flexural members.

> There is little use in providing increased cover to reinforcement (with the desired objective of providing increased durability against chemical attack and corrosion) if that cover is cracked. Such cracking (due to large covers) can be considerable, unless proper precautions are taken during design. Precautionary steps, under such circumstances, include the reduction of the tensile stress in the steel (by providing more reinforcement) and use of more closely spaced bars.

### 10.5.4 Estimation of Flexural Crack-width

The estimation of the probable maximum width of surface cracks in a flexural member is a fairly complex problem, and despite a fair amount of research in this field over the past four decades, the different equations evolved over the years predict values of crack-widths that are, in some cases, widely different. Different methods (with semi-empirical formulations) have been adopted by different international codes, and these too have undergone revisions over the years.

Experiments have shown that the average spacing $s_{av}$ of surface cracks is directly proportional to the distance $a_{cr}$ from the surface of the main reinforcing bar. Cracks are most likely to form on the surface in the tensile zone mid-way between two adjacent bars, as shown in Fig. 10.16. The value of $a_{cr}$, in terms of the bar spacing $s$, bar diameter $d_b$ and effective cover $d_c$ is given by:

$$a_{cr} = \sqrt{(s/2)^2 + {d_c}^2} - d_b/2 \tag{10.30}$$

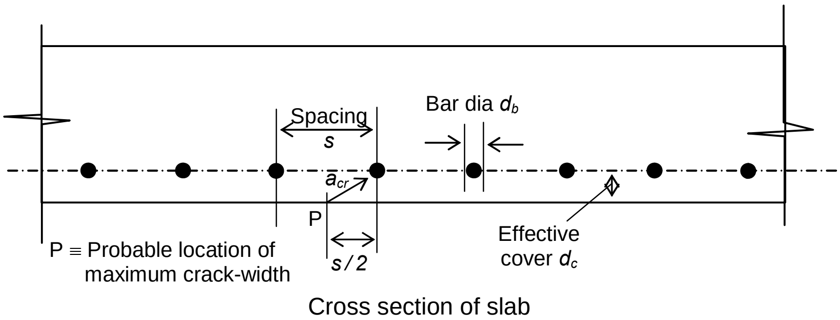


**Fig. 10.16** Geometrical parameters of relevance in determining flexural crack-width in a slab

In the case of a typical rectangular beam, the locations for determining maximum surface crack-widths are mainly at the soffit of the beam (when subject to sagging curvature), at distances where the value of $a_{cr}$ is likely to be maximum [points $P_1$ and $P_2$ in Fig. 10.17]. Also, points on the side of the beam section, mid-way between the neutral axis and the centreline of reinforcement, should be investigated [point $P_3$ in Fig. 10.17].

$$w_{cr} = \text{constant} \times a_{cr} \times \varepsilon_m \tag{10.31}$$

where the constant has a value of 3.33 for deformed bars and 4.0 for plain round bars, and the cracks are likely to occur at a mean spacing of $1.67a_{cr}$.

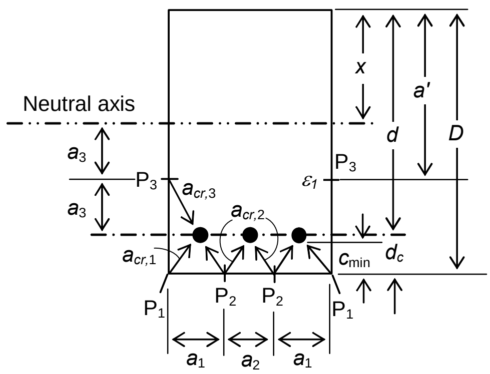


**Fig. 10.17** Critical locations for determining flexural crack-width in a typical beam section

Thus, it is seen from Eq. 10.31 that the flexural crack-width at a beam surface is directly proportional to (i) the distance to the nearest longitudinal bar (or the neutral axis for location such as $P_3$) and (ii) the mean strain in the concrete at the level under consideration.

### *IS 456 Formulation*

The IS code in Annex F describes a procedure[†] for predicting flexural crack-widths, which is apparently borrowed from the British Code [Ref. 10.12]. The expression for $w_{cr}$ takes the form shown in Eq. 10.31:

$$w_{cr} = \frac{3a_{cr}\varepsilon_m}{1+2\,(a_{cr} - C_{\min})/(D-x)} \tag{10.32}$$

where $a_{cr}$ and $\varepsilon_m$ are as defined earlier (Eq. 10.31), $D$ is the overall depth of the member, $C_{min}$ the minimum cover to the main longitudinal bar, and $x$ the neutral axis depth [refer Fig. 10.17]. The calculations of $\varepsilon_m$ and $x$ are crucial, and must account for the effect of 'tension stiffening'. Accordingly, the British Code BS 8110, on which Eq. 10.31 is based, recommends that they be based on the procedure described in Section 10.3.6 (which involves iterative calculations). It may be noted that the calculations should also accommodate the long-term effects of creep. This is done by taking the value of the allowable tensile stress in concrete at the level of the steel, $f_{cts}$, to be equal to 0.55 MPa [compared to the value 1.0 MPa for short-term effects - refer Fig. 10.4(b)], and by including considerations of creep in the modular ratio. It is also important to note that it is implicitly assumed that the strains in the steel are well within the elastic limit. The Code, accordingly, does not permit the application of Eq. 10.32 to situations where the tensile stress in the steel (at the crack location) exceeds 0.8 $f_y$. After calculating $x$ and $f_{sm}$, the average tensile stress in the reinforcement, the value of $\varepsilon_m$ at any point is easily obtained as:

$$\varepsilon_m = \frac{f_{sm}}{E_s}\,\frac{a'-x}{d-x} \tag{10.33}$$

where $a'$ refers to the distance from the extreme compression fibre to the point (such as $P_1$, $P_2$, $P_3$ etc) at which the surface crack-width is being calculated [refer Fig. 10.17].

The calculations, involving an iterative procedure to determine $x$ and $f_{sm}$, as described above, can be tedious, as shown in Example 10.7. The Code IS 456:2000 allows for an approximation for computing $\varepsilon_m$ which involves only the conventional 'cracked' section analysis using the modular ratio concept, whereby $x$ and the

[†] However, the manner in which this formulation has been presented in Annex F of the Code lacks clarity, because it is incomplete and the background information regarding the exact procedure to estimate the neutral axis depth, $x$ is not given. Also, it may be noted that while the British Code adopts the same formulation (for the calculation of $x$) for computations of both crack-widths and deflections (based on 'effective curvature'), the IS Code adopts a different formulation for deflection calculations (based on 'effective second moment of area'), which is borrowed from a much older version of the British Code.

'apparent' strain $\varepsilon_1$ at the location of interest are first calculated (assuming concrete to resist no tension), with

$$\varepsilon_1 = \frac{f_{st}}{E_s}\frac{a' - x}{d - x} \tag{10.34}$$

where $f_{st}$ is the tensile stress in the reinforcement (in MPa units) and $E_s$ (= $2\times10^5$ MPa) the modulus of elasticity of steel.

The effect of tension stiffening is now accounted for through a term $\varepsilon_2$, such that the desired mean tensile strain in concrete $\varepsilon_m$ is now obtained simply as:

$$\varepsilon_m = \varepsilon_1 - \varepsilon_2 \quad .$$

The strain $\varepsilon_2$ corresponds to a reduction in tensile stress in reinforcement. For computing this, the Code (Annex F) suggests a reduction in tensile force in reinforcement, equal to the force generated by the triangular distribution of tensile stresses in concrete given in Fig. 10.4(b). The tensile stress in concrete at steel level $f_{cts}$ is to be taken as 1.0 MPa (for short-term calculations) and 0.55 N/mm$^2$ (for inclusion of long-term effects) [refer Annex F of Code]. Accordingly, $\varepsilon_2$ works out to:

$$\varepsilon_2 = f_{cts} \times b\,(D - x)\,(a' - x)/\{2E_s\,A_{st}\,(d - x)\} \tag{10.35a}$$

The following empirical expression for $\varepsilon_2$ is given in the Code†:

$$\varepsilon_2 = \frac{b(D - x)(a' - x)}{3E_s A_{st}(d - x)} \tag{10.36}$$

where $b$ is the width of the section at the centroid of the tension steel, $A_{st}$ is the area of the tension steel, and $E_s = 2\times10^5$ MPa. If the value of $\varepsilon_2$, calculated by Eq. 10.36, exceeds $\varepsilon_1$ (i.e., $\varepsilon_m < 0$), then it should be construed that the section is uncracked.

Considering the crack-width at the surface remote from the compression face [refer Fig. 10.17],

$$a' = D$$

whereby Eq. 10.35 simplifies to

$$\varepsilon_m = \frac{1}{E_s}\frac{D - x}{d - x}\left[f_{st} - \frac{b(D - x)}{3A_{st}}\right] \tag{10.37}$$

where $f_{st}$ is the stress at the centroid of the tension steel (cracked section), expressed in MPa units.

### *Gergely Lutz Formula*

The practice in North America [Ref. 10.13, 10.14] is based on a formula due to Gergely and Lutz [Ref. 10.23]. In several European countries, the CEB–FIP procedure [Ref. 10.10] is favoured. Some recent studies [Refs 10.24, 10.25], which

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† Note that in the empirical Eq. 10.36, the multiplication factor 1/3 has the unit of stress (N/mm$^2$).

compare the predictions of the prevailing international Code methods with a very large number of experimental test results reported by Clark, Hognestad, Base and others, conclude that the best results are predicted by the Gergely and Lutz formula.

The formula for the maximum probable crack-width, in this case, is given by[†]:

$$w_{cr} = (11 \times 10^{-6})\ \sqrt[3]{d_c\,(A_e/n)}\left(\frac{D-x}{d-x}\right)f_{st} \qquad (10.38)$$

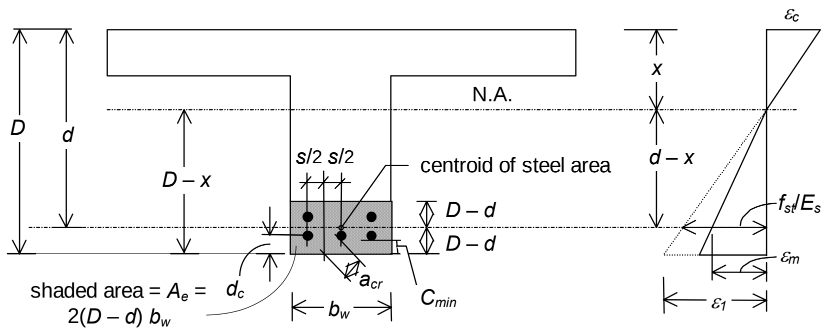
where

$d_c$ ≡ thickness of concrete cover measured from the extreme tension fibre to the centre of the nearest bar;

$A_e$ ≡ effective area of concrete in tension surrounding the main tension reinforcement, having the same centroid as the tension steel [$A_e = 2(D - d)b_w$ in Fig. 10.18(a)].

$n$ ≡ number of bars in tension; in case different diameters are used, $n$ shall be taken as the total steel area divided by the area of the largest bar diameter; and

$f_{st}$ ≡ stress (in MPa) at the centroid of the tension steel.




**(a)** beam section **(b)** strains

**Fig. 10.18** Parameters for crack-width calculation (Gergely-Lutz formula)

**EXAMPLE 10.7**

Determine the maximum probable crack-width for the one-way slab designed in Example 5.2 (and analysed for deflection in Example 10.1).

**SOLUTION**

- From Example 5.2 and Example 10.1,
  $D$ = 200 mm, $d$ = 165 mm, $C_{min}$ = 30 mm, $d_b$ = 10 mm $d_c$= 35 mm,

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[*] Note that for dimensional homogeneity in the empirical Eq. 10.38, the constant $11 \times 10^{-6}$, which is obtained from statistical analysis of the experimental data, evidently has the inverse unit of stress.

$s$ = 125 mm, $A_{st}$ = 628 mm$^2$/m (10 $\phi$ @ 125 mm c/c), $p_t$ = 0.3806,
$f_y$ = 415 MPa, $f_{ck}$ = 25 MPa, $f_{cts}$ = 0.55 MPa,
$M$ (at midspan, under service loads) = 21.63 kNm/m.

**a) Based on detailed procedure for calculating *x* and $\varepsilon_m$**

The procedure for calculating the neutral axis depth $x$ is identical to the one described in Example 10.1, except that allowable tensile stress in steel $f_{cts}$ needs to be altered, in order to account for the long-term effects due to creep.
$f_{cts}$ = 0.55 MPa [refer Fig. 10.4(b)].
Also, the modular ratio† $m$ may be taken as:
$m = 280/(3\ \sigma_{cbc}) = 280/(3 \times 8.5) = 11.0$

Trial 1: Assume $x \approx d / 3 = 165/3 = 55$ mm.

$$\Rightarrow f_{ct} = \left(\frac{200-x}{165-x}\right)\times 0.55 = 0.725 \text{ MPa}$$

$$\Rightarrow f_{sm} = \left(21.63\times10^6 - \frac{1000\times200\times f_{ct}\times(200-x)}{3}\right)\Big/ 628\times\left(165-\frac{x}{3}\right)$$

= 158.7 MPa

$$\Rightarrow f_c = \frac{f_{sm}A_{st} + 0.5b(D-x)f_{ct}}{0.5bx} = 5.54 \text{ MPa}$$

$$\Rightarrow x = \frac{165}{1+f_{sm}/(11 f_c)} = 45.8 \text{ mm}$$

Trial 2: Assuming an average value $x \approx (55 + 45.8)/2 = 50.4$and repeating the procedure,

$\Rightarrow f_{ct}$ = 0.718 MPa
$\Rightarrow f_{sm}$ = 155.5 MPa
$\Rightarrow f_c$ = 6.0 MPa
$\Rightarrow x$ = 49.2 mm.

Trial 3: Assuming an average value $x \approx (50.4 + 49.2)/2 = 49.8$ mm, and repeating the procedure,

$\Rightarrow f_{ct}$ = 0.717 MPa
$\Rightarrow f_{sm}$ = 155.0 MPa
$\Rightarrow f_c$ = 6.1 MPa
$\Rightarrow$ ***x* = 49.8 mm**, which indicates convergence.

- $w_{cr} = (3a_{cr}\ \varepsilon_m)/[1 + 2(a_{cr} - C_{min})/(D - x)]$

  $a_{cr} = \sqrt{(s/2)^2 + d_c^{\ 2}} - d_b/2 = \sqrt{(125/2)^2 + (35)^2} - 5 = 66.6$ mm

† According to BS 8110, the value of $m$ should be based on $E_s / E_{ce}$, considering $E_{ce}$ as half the short-term elastic modulus of concrete.

$$\varepsilon_m = \frac{f_{sm}}{E_s} \times \frac{D-x}{d-x} = \frac{155.0}{2\times 10^5} \times \frac{200-49.8}{165-49.8} = 0.00101$$

$w_{cr}$ = (3 × 66.6 × 0.00101)/[1 + 2(66.6 – 30)/(200– 49.8)]
= **0.136 mm**

**b) <u>Based on 'Approximate' procedure of IS 456 (Annex F)</u>**

This calls for the conventional cracked section analysis, with the modular ratio[†] $m$ taken as:

$m = 280/(3\ \sigma_{cbc}) = 280/(3 \times 8.5) = 10.98$

$k = \sqrt{2(\rho m) + (\rho m)^2} - (\rho m)$ where $\rho = p_t/100 = 0.3806\times 10^{-2}$

$\Rightarrow k = 0.2443 \Rightarrow x = kd = 0.2443 \times 165 = 40.3$ mm

$\Rightarrow I_{cr} = (1000)(40.3)^3/3 + (11.0 \times 628)(165 - 40.3)^2 = 1.2904\times 10^{-2}$ mm$^4$

$$\Rightarrow f_{st} = \frac{11.0\times(21.63\times 10^6)\times(165-40.3)}{1.2904\times 10^8} = 229.5 \text{ MPa}$$

(which is within 230 MPa, the allowable stress for FE 415 steel, as per working stress design)

$$\varepsilon_1 = \frac{f_{st}}{E_s}\times\frac{D-x}{d-x} = \frac{229.5}{2\times 10^5}\times\frac{200-40.3}{165-40.3} = 0.001470$$

$$\varepsilon_2 = \frac{b(D-x)(a'-x)}{3E_s A_{st}(d-x)} = \frac{1000(200-40.3)^2}{3(2\times 10^5)(628)(165-40.3)} = 0.000543$$

$\Rightarrow \varepsilon_m = \varepsilon_1 - \varepsilon_2 = 0.000927 > 0$

$\Rightarrow w_{cr} = (3a_{cr}\ \varepsilon_m)/[1 + 2(a_{cr} - C_{min})/(D - x)]$
= (3 × 66.6 × 0.000927) / [1 + 2(66.6 – 30)/(200 – 40.3)]
= **0.127 mm**

(which compares reasonably with the value of 0.136mm obtained by the 'exact' procedure).

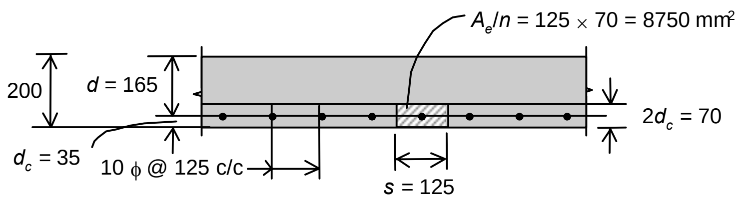



**Fig. 10.19** Example 10.7

[†] According to BS 8110, the value of $m$ should be based on $E_s / E_{ce}$, considering $E_{ce}$ as half the short-term elastic modulus of concrete.

**Crack-width calculation using Gergely & Lutz formula**

- $w_{cr} = (11 \times 10^{-6})\ \sqrt[3]{d_c\,(A_e/n)}\ \dfrac{D-x}{d-x}\ f_{st}$ [Eq. 10.38]

  where
  $d_c$ = 30 + 10/2 = 35 mm [refer Fig. 10.14]
  $A_e/n$ = effective concrete area in tension per bar
  = $s(2d_c)$ = 125 × 70 = 8750 mm$^2$

  $\Rightarrow w_{cr} = (11 \times 10^{-6})\ \sqrt[3]{35 \times 8750}\ \dfrac{200-40.3}{165-40.3} \times (229.5)$

  = **0.218 mm**

  (which is larger than the value of 0.13 mm obtained by the IS/BS Code formula).
  *Note*: It is clear that the maximum probable crack-width is less than 0.3 mm, and hence is acceptable.

**EXAMPLE 10.8**

Determine the maximum probable crack-width for the doubly reinforced beam designed in Example 5.4.

**SOLUTION**

- From Example 5.4,
  $D$ = 400 mm, b = 250 mm, $d$ = 348 mm, $C_{min}$ = 30 mm, $d'$ = 48 mm, $A_{st}$ = 1848 mm$^2$, (3 – 28 $\phi$), $A_{sc}$ = 942.5 mm$^2$ (3 – 20 $\phi$), $M_{D+L}$ (at midspan, under service loads) = 124 kNm, $f_{ck}$ = 25 MPa, $f_y$ = 415 MPa.
  The cross-sectional details are shown in Fig. 10.20.
- *Depth of neutral axis*: $x = kd$
  $m$ =280 / (3×8.5) = 11
  Taking moments of areas of the cracked-transformed section about the NA [refer Section 4.6.5],
  $b\,(x)^2/2 + (1.5m - 1)\,A_{sc}\,(x - d') = mA_{st}\,(d - x)$
  $\Rightarrow 250\,(x)^2/2 + (15.5 \times 942.5)\,(x - 48) = (11.0 \times 1848)\,(348 - x)$
  $\Rightarrow 125\,(x)^2 + 34936.75\,(x) - 7775364 = 0$
  Solving, $x$ = 146.14 mm

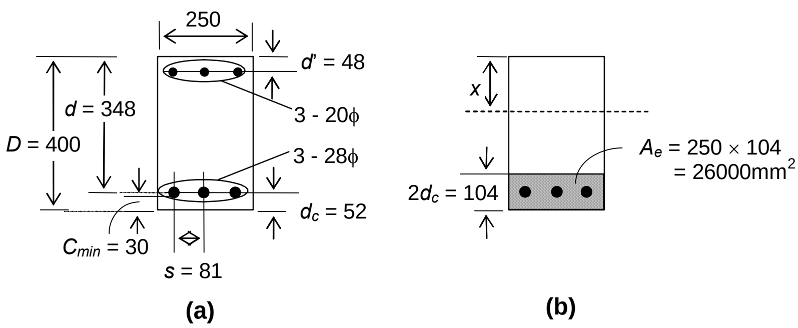



**Fig. 10.20** Example 10.8

- *Depth of neutral axis*: $x = kd$
  $m = 280 / (3\times8.5) = 11$
  Taking moments of areas of the cracked-transformed section about the NA [refer Section 4.6.5],
  $b\,(x)^2/2 + (1.5m - 1)\,A_{sc}\,(x - d') = mA_{st}\,(d - x)$
  $\Rightarrow 250\,(x)^2/2 + (23 \times 942.5)\,(x - 48) = (16 \times 1848)\,(348 - x)$
  $\Rightarrow 125\,(x)^2 + 51245.5\,(x) - 11330184 = 0$
  Solving, $x = 159.24$ mm
- *Tensile stress in steel under service loads*: $f_{st}$
  $$f_{st} = m\frac{M(d-x)}{I_{cr}}$$
  where $M = 124$ kNm per m width.
  $I_{cr} = 250\,(146.14)^3/3 + (11.0 \times 1848)\,(348 - 146.14)^2 + (15.5 \times 942.5) \times (146.14 - 48)^2$
  $= 12.29 \times 10^8$ mm$^4$
  $$\Rightarrow f_{st} = \frac{11\times(124\times10^6)\times(348-146.14)}{12.29\times10^8}$$
  $= 224.0$ MPa
  (which, incidentally, is slightly less than $\sigma_{st} = 230$ MPa allowed as per working stress design).

**Crack-width calculation using IS Code formula**

The approximate procedure is followed here. The reader may verify that the detailed procedure yields roughly the same results. Crack-widths are calculated at the three critical positions ($P_1$, $P_2$ and $P_3$) indicated in Fig. 10.17.

For position $P_1$

$$a_{cr} = \sqrt{(s/2)^2 + d_c^{\,2}} - d_b/2$$
$$= a_{cr} = \sqrt{(52)^2 + (52)^2} - 14 = 51.91 \text{ mm with } a' = D$$

- $$\varepsilon_m = \frac{1}{E_s}\frac{D-x}{d-x}\left[f_{st} - \frac{b(D-x)}{3A_{st}}\right] \quad \text{[Eq. 10.37]}$$
  $$= \frac{1}{2\times10^5}\;\frac{400-146.14}{348-146.14}\left[224.0 - \frac{250(400-146.14)}{3\times1848}\right]$$
  $= 1.336 \times 10^{-3}$
- $C_{min} = 30$ mm
- $w_{cr} = (3a_{cr}\,\varepsilon_m)/[1 + 2(a_{cr} - C_{min})/(D - x)]$
  $= (3 \times 51.91 \times 1.336 \times 10^{-3})/\,[1 + 2(51.91 - 30)/\,(400 - 146.14)]$
  = **0.177 mm**

For position $P_2$ (corner)

$$a_{cr} = \sqrt{(s/2)^2 + d_c^{\,2}} - d_b/2$$

$= a_{cr} = \sqrt{(52)^2 + (52)^2} - 14$ = 73.54 mm and $a'$=D

- $\varepsilon_m = \dfrac{1}{E_s}\dfrac{D-x}{d-x}\left[f_{st} - \dfrac{b(D-x)}{3A_{st}}\right]$ [Eq. 10.37]

  $= \dfrac{1}{2\times10^5}\ \dfrac{400-146.14}{348-146.14}\left[224.0\ -\ \dfrac{250(400-146.14)}{3\times1848}\right]$

  = 1.336 × $10^{-3}$
- $C_{min}$ = 30 mm
- $w_{cr} = (3a_{cr}\ \varepsilon_m)/[1 + 2(a_{cr} - C_{min})/(D - x)]$

  = (3 × 73.54 × 1.336 × $10^{-3}$)/ [1 + 2(73.54 – 30)/ (400 – 146.14)]

  = **0.219 mm**

<u>For position $P_3$ (side of beam)</u>

$C_e$ = 30 + 8 +14 = 52 mm

$s = (d - x)/2$ = 100.93 mm

- $a_{cr} = \sqrt{\left(\dfrac{d-x}{2}\right)^2 + (d_c)^2} - 14 = \sqrt{(100.93)^2 + (52)^2} - 14 =$ 99.54 mm
- $a'$= distance of point where crack-width is desired from the extreme compression fibre

  = 100.93 + 146.14 = 247.07 mm

$\varepsilon_1$ is the strain at the crack location considering a cracked section

$\varepsilon_1 = \dfrac{f_{st}}{2E_s} = \dfrac{224.0}{2\times2\times10^5}$ = 5.6 × $10^{-4}$

- $\varepsilon_m = \varepsilon_1 - \dfrac{b_t(D-x)(a'-x)}{3E_sA_{st}(d-x)} = 5.6\times10^{-4} - \dfrac{250(400-146.14)(247.07-146.14)}{3\times2\times10^5\times1848(348-146.14)}$

  = 5.314× $10^{-4}$
- $w_{cr} = (3a_{cr}\ \varepsilon_m)/[1 + 2(a_{cr} - C_{min})/(D - kd)]$

  = (3 × 99.54 × 5.313 × $10^{-4}$)/ [1 + 2(99.54 – 30)/ (400 – 146.14)]

  = **0.102 mm**

Hence, the maximum probable surface crack-width is **0.219 mm**., and this occurs at the bottom corner.

**Crack-width calculation using Gergely & Lutz formula**

- $d_c$ = 30 +8+ 28/2 = 52 mm [refer Fig. 10.15a]

  $A_e$ = 250 × (2 × 52) = 26000 $mm^2$ [as shown in Fig. 10.15(b)]

  $n$ = 3

<u>For positions $P_1$ and $P_2$</u>

- $w_{cr}$ = (11 × $10^{-6}$) $\sqrt[3]{d_c(A_e/n)}\,\dfrac{D-x}{d-x}\,f_{st}$

  = (11 × $10^{-6}$) $\sqrt[3]{52(26000/3)}\,\dfrac{400-146.14}{348-146.14}\times224.0$

  = **0.237 mm**

*Note*: The maximum probable crack-width is less than 0.3 mm, and hence is acceptable.

### 10.5.5 Estimation of Crack-width under Direct and Eccentric Tension

In some reinforced concrete members, such as the walls of water tanks, bins and silos, pressure vessels, suspenders and ties in arch, roof and bridge structures, it is axial (or membrane) tension that is the predominant structural action, and not flexure. In many instances in such members, direct or eccentric tension due to applied loading, may act in combination with restraint to volume changes caused by temperature and shrinkage [Ref. 10.18]. This can lead to significant cracking, which should be controlled in the interest of serviceability. In this Section, the discussion is limited to cracking caused by applied loading; thermal and shrinkage cracking are discussed in the next Section.

Cracking due to direct tension is of somewhat more serious concern than flexural cracking, because it causes a clear separation in the concrete, through the entire thickness of the member, as shown in Fig. 10.21. Control of such cracking is therefore of particular importance in liquid retaining structures and pressure vessels.

At the crack location, the reinforcement is required to resist the entire tension, and the width and spacing of cracks are governed primarily by the reinforcement detailing. If very low percentages of reinforcement are provided, the steel may yield and result in wide crack-widths. However, crack-widths can be significantly controlled by maintaining sufficient reinforcement at close spacing, with relatively low tensile stress and strain in the steel at service loads. As in the case of flexure, the concrete in between the cracks resists some tension ('tension stiffening' effect). The axial stiffness of the member can be greatly reduced by the presence of wide cracks, but this reduction is mitigated by the tension stiffening effect. Recommendations for assessing axial stiffness are given in Ref. 10.18. These may be used in the structural analysis of statically indeterminate reinforced concrete structures involving significant axial tension.

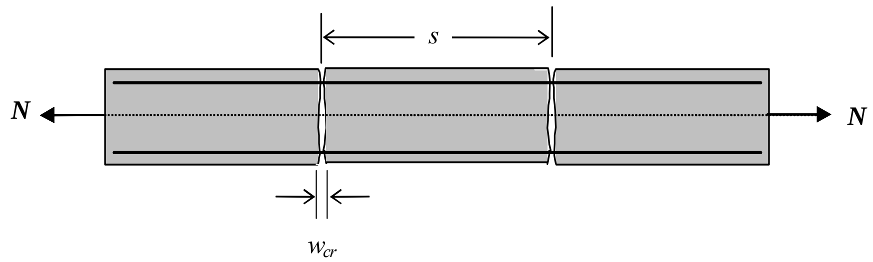
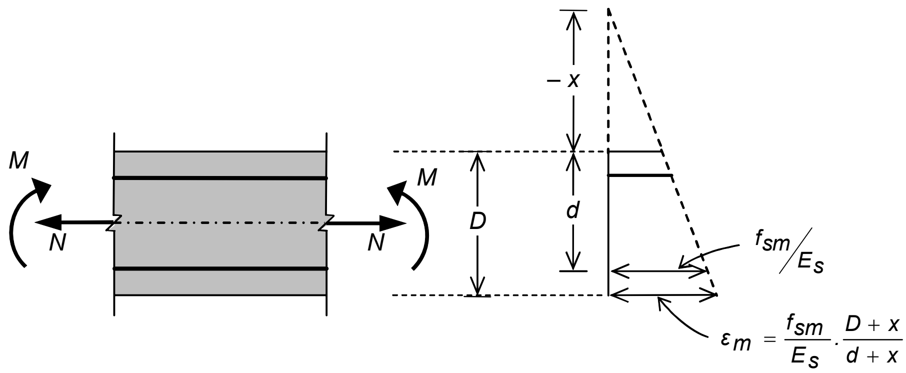



**Fig. 10.21** Cracking under direct tension

The IS Code does not give any recommendations for estimating crack-widths in members subject to direct or eccentric tension. However, such recommendations are given in the British codes BS 8110 [Ref. 10.12] and BS 8007 [Ref. 10.19], and these are discussed here. BS 8110 permits the extension of the procedure (approximate method) prescribed for crack-width prediction under pure flexure [Eq. 10.34–10.36] to situations of flexure combined with axial tension. When axial tension predominates, the entire section is likely to be under tension with the neutral axis lying outside the section, as shown in Fig. 10.22. In such cases, the neutral axis depth $x$ will have a negative value, and this must be taken special note of, while applying Eq. 10.34 and 10.36. In the extreme case of pure tension (i.e., $x \to -\infty$), it is suggested that it suffices to consider $x = -D$.




(a)

section subjected to direct tension *N* and bending moment *M*

(b)

distribution of mean tensile strains

**Fig. 10.21** Analysis for crack-width under eccentric tension

BS 8007 [Ref. 10.19] suggests an alternative empirical formula for the probable surface crack-width in members subject to direct tension:

$$w_{cr} = 3\, a_{cr}\, \varepsilon_m \tag{10.39}$$

which has the form given by Eq. 10.31, with the terms $a_{cr}$ and $\varepsilon_m$ as defined earlier. The constant, ‘3’ is based on a probability of exceedance of the calculated value of crack-width being equal to about 1 in 100. The expression for the mean tensile strain $\varepsilon_m$ is the same as given in Eq. 10.35:

$$\varepsilon_m = \varepsilon_1 - \varepsilon_2$$

where, as in the case of flexure, $\varepsilon_1$ is the ‘apparent strain’ (tensile strain in steel at the crack location) given by Eq. 10.34 and $\varepsilon_2$ is the reduction in strain due to the tension stiffening effect, which is given by:

$$\varepsilon_2 = \frac{f_{ct} A_{gr}}{A_{st} E_s} \tag{10.40}$$

where $A_{gr}$ is the gross area of the cross-section, $A_{st}$ the area of tension steel, and $f_{ct}$ is the allowable tensile stress in concrete. While the above formulas can be used to compute the crack width for an applied tensile force, conversely it can also be used to compute the admissible $f_{st}$ (and hence the tension admissible) for a permissible crack width. For such calculations, the allowable value of $f_{ct}$ is specified as 1.0 MPa if the permissible crack-width is 0.1 mm, and 0.67 MPa if the permissible crack-width is 0.2 mm. For other crack-widths, values of $f_{ct}$ have not been specified by BS 8007 (which is the British code for design of water retaining structures), and linear interpolation or extrapolation are not permitted.

Unlike the British codes (BS 8110 and BS 8007), the ACI codes (ACI 318 for general RC design and ACI 350 for liquid retaining structures) do not specifically recommend any procedure for the estimation of crack-widths under direct or eccentric tension. However, reference is made to the formula given in the ACI 224 Committee Report [Ref. 10.26], which is based on a formula by Broms and Lutz (similar to the one by Gergely and Lutz for flexure). In SI units, this equation takes the following form

$$w_{cr} = 0.02 f_{st} t_e \times 10^{-3} \qquad (10.41)$$

where $w_{cr}$ is the maximum probable surface crack-width (in mm), $f_{st}$ is the tensile stress (in MPa) in the steel at the crack, and $t_e$ is the 'effective concrete cover', which may be taken as:

$$t_e = d_c \sqrt{1 + \left( \frac{s}{4d_c} \right)^2} \qquad (10.42)$$

where $s$ is the spacing of bars, located at an effective cover $d_c$.

**EXAMPLE 10.9**

Consider a cylindrical water tank, whose wall is subject to hoop tension on account of hydrostatic pressure. If the wall is 250 mm thick, and reinforced with 12 mm dia bars @ 150 mm c/c, determine the allowable hoop tension (per unit width of wall) corresponding to a permissible crack-width of 0.2mm. Assume M 30 concrete and Fe 415 steel. Assume a clear cover of 40 mm.

**SOLUTION**

- $D$ = 250 mm, $C_{min}$ = 40 mm, $d_b$ = 12 mm, $s$ = 150 mm, $d_c$= 46 mm
  $A_{st}$ = (201×10$^3$) / 150 = 1340 mm$^2$/m

  The allowable hoop tension is given by $N = f_{st} A_{st}$, where the allowable value of $f_{st}$ is to be determined.

**BS 8007 formula**

$w_{cr} = 3\, a_{cr} (\varepsilon_1 - \varepsilon_2)$ = 0.2 mm
where $a_{cr} = \sqrt{(s/2)^2 + d_c^{\,2}} - d_b/2 = \sqrt{(150/2)^2 + (46)^2} - 6$ = 82.0 mm

$$\varepsilon_2 = \frac{f_{ct}A_{gr}}{E_s A_{st}} = \frac{(0.67)(1000\times 250)}{(2\times 10^5)(1340)} = 0.000625$$

$$\Rightarrow \varepsilon_1 = \frac{w_{cr}}{3a_{cr}} + \varepsilon_2 = \frac{0.2}{3(82.0)} + 0.000625 = 0.001438$$

$\Rightarrow f_{st} = E_s\,\varepsilon_1 = (2\times 10^5)(0.001438) = 287.6$ MPa

(which incidentally is higher than 230 MPa assumed in usual working stress design, but is well within the yield strength).

$\Rightarrow N = f_{st}A_{st}$ = (287.6 N/mm$^2$)(1340 mm$^2$/m) = 385,384 N = **385.4 kN**

**BS 8100 procedure**

$$w_{cr} = \frac{3a_{cr}\varepsilon_m}{1+2\,(a_{cr}-C_{\min})/(D-x)} = 0.2 \text{ mm}$$

$$\Rightarrow \varepsilon_m = 0.2\{1+2\,(a_{cr}-C_{\min})/(D-x)\}/3a_{cr}$$

where, $x = -D = -250$mm for pure tension [refer Section 10.5.5]

$d = 250 - 40 - 6 = 204$ mm

$$\Rightarrow \varepsilon_m = 0.2\{1+2\,(82-40)/(250+250)\}/(3\times 82) = 0.0009496$$

$\varepsilon_m = \varepsilon_1 - \varepsilon_2$

$$\varepsilon_2 = \frac{b(D-x)(a'-x)}{3E_s A_{st}(d-x)} = \frac{1000(250+250)^2}{3(2\times 10^5)(1340)(204+250)} = 0.0006849$$

$\varepsilon_1 = \varepsilon_m + \varepsilon_2 = 0.0016345$

$$\varepsilon_1 = \frac{f_{st}}{E_s}\frac{D-x}{d-x}$$

$\Rightarrow f_{st} = (2\times 10^5)(0.0016345)(204+250)/(500) = 296.8$ MPa

$N = f_{st}A_{st}$ = (296.8 N/mm$^2$)(1340 mm$^2$/m) = 397745 N = **397.7 kN**

(which compares very well with the solution obtained by the BS 8007 formula).

**ACI Committee 224 procedure**

The effective concrete cover is given by

$$t_e = d_c\sqrt{1+\left(\frac{s}{4d_c}\right)^2} = 46\sqrt{1+\left(\frac{150}{4\times 46}\right)^2} = 59.35\text{mm} \qquad \text{[Eq. 10.42]}$$

$$\Rightarrow w_{cr} = 0.02 f_{st} t_e \times 10^{-3} = 0.2 \text{ mm} \qquad \text{[Eq. 10.41]}$$

$\Rightarrow f_{st} = 168.5$ MPa

$\Rightarrow N = f_{st}A_{st}$ = (168.5 N/mm$^2$)(1340 mm$^2$/m) = 225790 N = **225.8 kN**

(which is a conservative estimate in comparison to the BS codes).

### 10.5.6 Thermal and Shrinkage Cracking

#### *Early Thermal Shrinkage*

During mixing of concrete, heat of hydration is generated, causing the temperature of concrete to rise. The temperature at casting may be significantly higher than the ambient temperature, but the peak hydration temperature (which may occur within three days or so after casting) can be considerably higher than the ambient temperature. This difference in temperature (between the peak and ambient temperatures) may vary from 10 to 50°C, depending on various factors, such as the volume of concrete cast, the volume of cement content, the type of cement used, type of formwork, etc[†]. The temperature in the middle region is often higher than at the surface (with a gradient of up to 20°C in very thick members). The concrete cools and attains the ambient temperature within seven days or so. This results in shortening of the hardened concrete, and such shortening is sometimes referred to as 'early thermal shrinkage'. The concrete at the surface is subject to tension (and the concrete in the middle region to compression) owing to the temperature gradient across the thickness. Additionally, the overall shortening is frequently restrained by the adjoining concrete, resulting in further development of tensile strains throughout the section. If the overall tensile strain (maximum at the surface regions) exceeds the ultimate tensile strain capacity of the hardening concrete, cracking is likely to occur. Such 'early thermal cracking' can be prevented by controlling the factors affecting the heat of hydration (using a low cement content, using pozzalanic admixtures, using ice to control temperature during mixing, etc.). Studies have also established that delaying the process of cooling, such as by delaying the formwork removal and insulating the concrete, are also advantageous [Ref. 10.27].

#### *Drying Shrinkage*

Accompanying the early thermal movement associated with the hardening of concrete, another factor which contributes to the shortening of concrete is *drying shrinkage* [refer Section 2.12.1]. However, unlike early thermal movements, drying shrinkage is a long-term phenomenon. It is best controlled by adopting as low a water-cement ratio as practicable, thereby keeping to a minimum the volume of moisture in the concrete that can evaporate. Cracking due to drying shrinkage occurs due to restraints that cause tensile stresses and strains in the concrete. The external restraints are due to fixity with adjoining members or friction against any surface in contact (such as the earth, in the case of footings supporting walls). There is also internal restraint caused by the embedded steel resisting the free shrinkage of concrete. (It may be noted that such internal restraint does not occur in the case of thermal shrinkage, because the coefficient of thermal expansion for concrete and steel are approximately the same.) If the external restraints are absent, then drying shrinkage can generate a system of self-equilibrating forces, with the concrete in

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[†] Temperatures increase with the use of rapid-hardening cement or higher grade cement (with greater fineness), with increased cement content, larger volumes cast (with larger thickness of section) and use of timber formwork (instead of steel).

tension and the embedded steel in compression (with longitudinal bond forces at the interface). If the percentage of steel provided is relatively low, cracking of concrete is likely to occur, and the likelihood of such cracking is enhanced in the presence of external restraints.

### *Other Thermal Effects*

There are two other factors that can contribute to thermal cracking. The first is due to seasonal variations in temperature, resulting in volume changes in the hardened concrete. When the season changes from summer to winter, thermal shrinkage occurs, and in the presence of restraints, this can result in cracking. The second factor is due to the differential temperature gradient caused by the action of the sun on the external surface of a slab or a wall, with the other surface relatively insulated ('solar radiation' effect). The presence of restraints restricts the free expansion of the external surface and induces flexural tension on the inner surface, causing possible cracking. The incidence of such cracking particularly needs to be controlled in structural elements such as walls of overhead water-tanks (whose liquid face is prone to cracking).

### *Methods of Crack Control and Crack-width Estimation*

Cracking due to thermal and shrinkage effects can be effectively controlled by provision of (i) adequate reinforcement and (ii) appropriate 'movement joints', such as contraction and expansion joints [Fig. 10.22]. The designer may choose to adopt closely spaced movement joints with low percentage of reinforcement, or widely spaced joints with a relatively high percentage of reinforcement. The choice is governed by factors such as the size of the structure, method of construction and economics [Ref. 10.20].

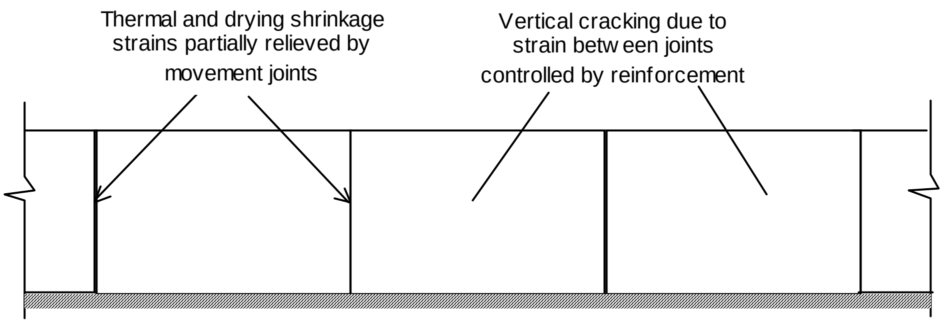


**Fig. 10.22** Crack control in a base-restrained wall

The methods of estimating crack-widths under flexure, direct tension and eccentric tension described earlier can be applied here, provided the magnitude of internal forces are known. Unfortunately, methods to estimate crack width and spacing due to thermal and drying shrinkage are fraught with various uncertainties. These pertain, not only to uncertainties associated with temperature and shrinkage parameters as well as degree of restraint, but more important, to uncertainties associated with the tensile forces induced in the structure. These are 'deformation-induced forces', which get partially relieved on account of cracking. It would be very conservative to estimate these forces by the usual elastic structural analysis (including finite element analysis), considering all sections to be uncracked, as the tensile forces generated are likely to be high. It would indeed be appropriate to assign appropriate 'effective' stiffnesses (axial and flexural), taking into account the effect of tension stiffening, while performing the structural analysis. However, this is a nonlinear problem, as the amount of tension stiffening depends on the magnitude of internal forces, and these may vary from location to location. The designer must exercise good engineering judgement and caution in estimating the design forces for a maximum permissible crack-width. If the reinforcement provided is inadequate and movement joints are altogether absent, the consequences of thermal and shrinkage cracking can be alarming [Fig. 10.23].

**Fig. 10.23** Typical thermal and shrinkage cracks in a large reservoir structure

In the case of structures such as water tanks, where crack-width control is of special importance, design codes attempt to achieve such control through guidelines relating to provision of movement joints and minimum reinforcement requirements [Ref. 10.20]. The absolute minimum reinforcement area $(A_{st})_{\min}$ is usually governed by the consideration that the steel should not yield when cracking occurs (due to the ultimate tensile strength of concrete being exceeded). Accordingly, considering force equilibrium,

$$(A_{st})_{\min} f_y = A_g f_{ct}$$

$$\Rightarrow (p_t)_{cr} \equiv 100 \frac{(A_{st})_{\min}}{A_g} = 100 \frac{f_{ct}}{f_y} \tag{10.43}$$

where $(p_t)_{cr}$ is called the 'critical percentage reinforcement', $f_{ct}$ and $f_y$ denote the limiting tensile strength of concrete and yield strength of steel respectively, and $A_g$ denotes the gross area of the cross-section under consideration. As control of cracking is critical during the early life of the concrete, it is recommended that the value of $f_{ct}$ should correspond to the 3-day tensile strength of concrete, which may be taken as 1.15 MPa for M 25 grade, 1.3 MPa for M 30 grade and 1.6 MPa for M 35 grade [Ref. 10.20].

Thus, for example, considering M 25 grade concrete, the critical percentage reinforcement works out to 0.28 in the case of Fe 415 grade steel and 0.46 in the case of Fe 250 steel. It is desirable to provide this minimum reinforcement in two equal layers (in each direction) with minimum cover (from durability point-of-view), and if the member (wall or slab) is very thick (thicker than 600 mm), the calculations may be based on a maximum thickness of 600 mm (i.e., considering only 300mm on either surface). However, additional reinforcement is called for if movement joints are not provided in large structures, or if high crack-width control is desired. A minimum value of 0.65% is suggested in Ref. 10.28 (referred to in ACI 350), based on considerations of drying shrinkage alone. For large continuous constructions, without expansion joints, this value may be as high as 1.0%, to include the effect of other temperature effects, in the absence of more rigorous analysis.

Finally, it may be mentioned that recent trends in concrete technology suggest that 'high performance concrete' with new materials such as high volume fly ash (HVFA) have the advantage of well-bonded micro-structure with relatively low potential for shrinkage and thermal cracking [Ref. 10.29]. Such materials, which are also 'sustainable' (low cement consumption), are likely to be used widely in the decades to come.

## REVIEW QUESTIONS

10.1 Explain the importance of serviceability limit states in the structural design of reinforced concrete flexural members.

10.2 Why is it necessary to limit deflections in reinforced concrete flexural members?

10.3 Why is it difficult to make an accurate prediction of (a) the total deflection, (b) the maximum crack-width in a reinforced concrete flexural member?

10.4 Distinguish between *short-term* deflection and *long-term* deflection.

10.5 What is meant by the *tension stiffening effect* in reinforced concrete members subject to flexure? Explain with suitable sketches.

10.6 How is the short-term deflection due to live loads alone estimated? What is its relevance?

10.7 Explain the difficulty in estimating the short-term deflection as per IS Code procedure when the applied moment at service loads is marginally less than the cracking moment (calculated using the modulus of rupture of concrete).

10.8 Explain briefly the BS Code procedure of estimating short-term deflections using the concept of 'effective curvature'.

10.9 How does shrinkage of concrete lead to deflections in reinforced concrete flexural members?

10.10 How does the magnitude of compression reinforcement affect deflections due to (a) shrinkage (b) creep?
10.11 Explain the Code procedure for calculating deflection due to creep.
10.12 Explain how temperature effects lead to deflections in reinforced concrete flexural members.
10.13 What are the different options available to a designer with regard to control of cracking in flexural members?
10.14 What are the major factors which influence crack-widths in flexural members?
10.15 The Code does not call for explicit checks on the serviceability limit states of deflection provided certain requirements are complied with in the design. What are these requirements?
10.16 Are the nominal detailing requirements of the Code adequate for ensuring crack-width control? Comment.
10.17 Discuss the issues involved in designing for achieving control over thermal and shrinkage cracking in large RC structures.

## PROBLEMS

10.1 The section of a cantilever beam, designed for a span of 4.0 m is shown in Fig. 10.24. The beam has been designed for a bending moment of 200 kNm (at the support) under service loads, of which 60 percent is due to permanent (dead) loads. The loading is uniformly distributed on the span. Assume M 20 concrete and Fe 415 steel.

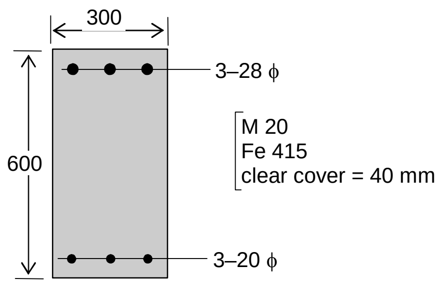


**Fig. 10.24** Example 10.1

(a) Calculate
(i) the maximum short-term elastic deflection;
(ii) the short-term deflection due to live loads alone;
(iii) the maximum deflection due to shrinkage, assuming $\varepsilon_{cs}$ = 0.0004;
(iv) the maximum deflection due to creep, assuming $\theta$ = 1.6.
(b) Check whether the beam satisfies the deflection limits specified by the Code.

10.2 A one-way slab has been designed for a simply supported span of 4.0 m with an overall depth of 170 mm and clear cover of 20 mm, using M 20 concrete and Fe 415 steel. The dead loads are taken as 5.0 kN/m$^2$ and the live loads as 2.0 kN/m$^2$. The longitudinal bars are designed as 10 mm $\phi$ @ 150 c/c. Verify the adequacy of the thickness provided,

(i) applying the limiting span/effective depth ratio;

(ii) actual calculation of total deflections.

10.3 For the T-beam designed in Example 5.5, calculate the following:

(i) short-term deflection due to service loads;

(ii) incremental short-term deflection due to live loads;

(iii) long-term deflection due to shrinkage;

(iv) long-term deflection due to creep.

Hence, verify whether the design satisfies the Code limits on deflection.

10.4 Repeat Problem 10.3 for the continuous T-beam designed in Example 5.6. Assume that 60 percent of the loading is due to dead (permanent) loads. Also assume that the moments at the two supports are equal to the midspan moment of 533.3 kNm (under service loads) and that the reinforcement provided at the support section is as shown in Fig. 10.25.

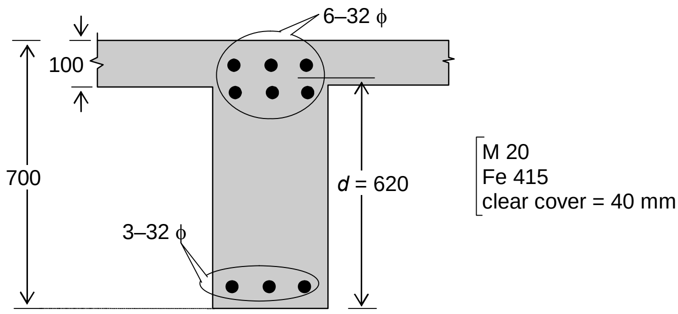


**Fig. 10.25** Example 10.4 — support section

10.5 Determine the maximum probable crack-width for the cantilever beam of Example 10.1

10.6 Determine the maximum probable crack-width for the one-way slab of Example 10.2.

10.7 Determine the maximum probable crack-width for the T-beam of Example 10.3.

10.8 A 150 mm thick wall of a cylindrical water tank is subject to a direct tensile force of 270 kN/m due to hydrostatic loading. Determine the required spacing of 12 mm dia bars in order to achieve crack-width control of 0.1 mm. Assume a clear cover of 40mm, M 30 concrete and Fe 415 steel.

## REFERENCES

10.1 ACI Committee 435, (Subcommittee 1), *Allowable Deflections*, Journal ACI, Vol. 65, No. 6, June 1968, pp 433–444.

10.2 — *Explanatory Handbook on Indian Standard Code of Practice for Plain and Reinforced Concrete (IS 456:1978)*, Special Publication SP:24, Bureau of Indian Standards, New Delhi, 1983.

10.3 Hsieh, Y.Y., *Elementary Theory of Structures*, Prentice Hall, Inc., New Jersey, 1970.

10.4 Roark, R.J., and Young, W.C., *Formulas for Stress and Strain*, Fifth edition, McGraw-Hill Book Co., New York, 1975.

10.5 Branson, D.E., *Deformation of Concrete Structures*, McGraw-Hill Inc., New York, 1977.

10.6 Yu, W.W., and Winter, G., *Instantaneous and Long-time Deflections of Reinforced Concrete Beams under Working Loads*, Journal ACI, Vol. 57, No. 1, July 1960, pp 29–50.

10.7 ACI Committee 435, *Deflections of Reinforced Concrete Flexural Members*, Journal ACI, Vol. 63, No. 6, June 1966, pp 637–674.

10.8 Beeby, A.W., and Miles, J.R., *Proposals for the Control of Deflection in the New Unified Code*, Concrete, Vol. 3, March 1969, pp 101–110.

10.9 Rao, P.S. and Subrahmanyam, B.V., *Trisegmental Moment-Curvature Relations for Reinforced Concrete Members*, Journal ACI, May 1973, No. 70–39, pp 346–352.

10.10 CEB–FIP, *International Recommendations for the Design and Construction of Concrete Structures*, Comité Européen du Béton–Fédération Internationale de la Précontrainte, Paris, 1970.

10.11 Prakhya, G.K.V., and Morely, C.T., *Tension Stiffening and Moment-Curvature Relations of Reinforced Concrete Elements*, Journal ACI, Vol. 87, Sept.–Oct. 1990, pp 597–602.

10.12 — *Structural Use of Concrete : Part 2 : Code of Practice for Special Circumstances*, BS 8110 : Part 2 : 1985, British Standards Institution, 1985.

10.13 — *Commentary on Building Code Requirements for Reinforced Concrete* ACI 318–95, American Concrete Institute, Detroit, 1995.

10.14 CSA Standard CAN – A23.3 – M84 — *Design of Concrete Structures for Buildings*, Canadian Standards Association, Rexdale, Ontario, 1984.

10.15 ACI Committee 435, *Deflections of Continuous Concrete Beams*, Journal ACI, Vol. 70, No. 12, Dec. 1973, pp 781–787.

10.16 ACI Committee 209, *Prediction of Creep, Shrinkage and Temperature Effects in Concrete Structures*, SP–27, Am. Conc. Inst., Detroit, Michigan, USA, 1971, pp 51–93.

10.17 ACI Committee 435, *Proposed Revisions by Committee* 435 to ACI *Building Code and Commentary Provisions on Deflections*, Journal of ACI, Vol. 75, No.6, June 1978, pp 229–238.

10.18 ACI Committee 224, *Control of Cracking in Concrete Structures*, Journal ACI, Vol. 69, No. 12, Dec. 1972, pp 717–753.

10.19 BS 8007 : 1987 — *Design of Concrete Structures for Retaining Aqueous Liquids*, British Standards Institution, London, 1987.
10.20 Anchor, R.D., *Design of Liquid Retaining Concrete Structures*, Second edition, Edward Arnold Publishers, London, 1992.
10.21 Gouthaman, A. and Menon, D., *Increased Cover Specifications in IS 456 (2000) – Crack-width Implications in RC Slabs*, Indian Concrete Journal, Sept. 2001, pp 581–586.
10.22 Mosley, W.H., Bungey, J.H. and Hulse, R., *Reinforced Concrete Design*, Fifth edition, Macmillan Press Ltd, London, 1999.
10.23 Gergely, P. and Lutz, L.A., *Maximum Crack Width in Reinforced Concrete Flexural Members*, Causes, Mechanism, and Control of Cracking in Concrete, (SP–20), American Concrete Institute, Detroit, 1968, pp 87–117.
10.24 Ganesan, N. and Shivananda, K.P., *Comparison of International Codes for the Prediction of Maximum Width of Cracks in Reinforced Concrete Flexural Members*, Indian Concrete Journal, November 1996, pp 635– 641.
10.25 Srinivas, N. and Menon, D., *Design Criteria for Crack Control in RC Liquid Retaining Structures – Need for a Revision of IS 3370 (Part II) 1965*, Indian Concrete Journal, Aug. 2001, pp 451–458.
10.26 ACI Committee 224, *Cracking of Concrete Members in Direct Tension*, Journal ACI, Vol. 83, No. 1, Jan-Feb. 1986, pp 3–13.
10.27 Fitzgibbon, M.E., *Continuous Casting of Concrete,* in *"New Concrete Technologies and Building Design",* ed. Neville, A.M., The Construction Press, 1979.
10.28 Vetter, C.P., *Stresses in Reinforced Concrete Due to Volume Changes*, Transactions, ASCE, Vol. 98, 1933, pp 1039–1053.
10.29 Mehta, P.K., *Concrete Durability: Critical Issues for the Future*, Concrete International, Vol. 19, No. 7, 1997, pp. 27–33.

# 11

# Design of Two-Way Slab Systems

## 11.1 INTRODUCTION

Slab panels that deform with significant curvatures in two orthogonal directions must be designed as *two-way slabs*, with the principal reinforcement placed in the two directions.

This chapter deals with the design of rectangular two-way slab systems [refer Section 1.6.1], and includes:

- slabs supported on walls (or rigid beams);
- slabs supported on flexible beams;
- slab supported directly on columns ('flat plates' and 'flat slabs').

### 11.1.1 One-Way and Two-Way Actions of Slabs

The design of *one-way* slabs (simply supported/continuous/cantilevered) has already been described in Chapter 5. It was pointed out that 'one-way' action may be assumed when the predominant mode of flexure is in *one* direction alone. Rectangular slabs which are supported only on two opposite sides by unyielding (wall) supports and are uniformly loaded (along the direction parallel to the supports) provide examples of ideal one-way action [Fig. 11.1(a)]. The initially plane surface of the slab deforms into a cylindrical surface[†], in which curvatures (and hence, primary bending moments) develop only in one direction. For the purpose of analysis and design, the slab may be divided into a parallel series of identical one-way beam-strips, with the *primary* reinforcement placed (with uniform spacing)

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[†] Within the elastic phase, when a plate bends into a *cylindrical* surface, although the curvature in the direction parallel to the axis of the cylinder is zero, the moment in this direction is not zero, but is equal to $\upsilon$ times the moment in the direction of the curvature, where $\upsilon$ is the Poisson's ratio of the material of the plate (see also Section 4.8).

along each strip [refer Section 4.8]. Some nominal *secondary* reinforcement should also be placed in the perpendicular direction (normal to the beam-strip span) — to take care of tensile stresses that arise due to the secondary moment in this direction caused by the Poisson effect [Ref. 11.1], any non-uniform (or concentrated) loading, and temperature and shrinkage effects.

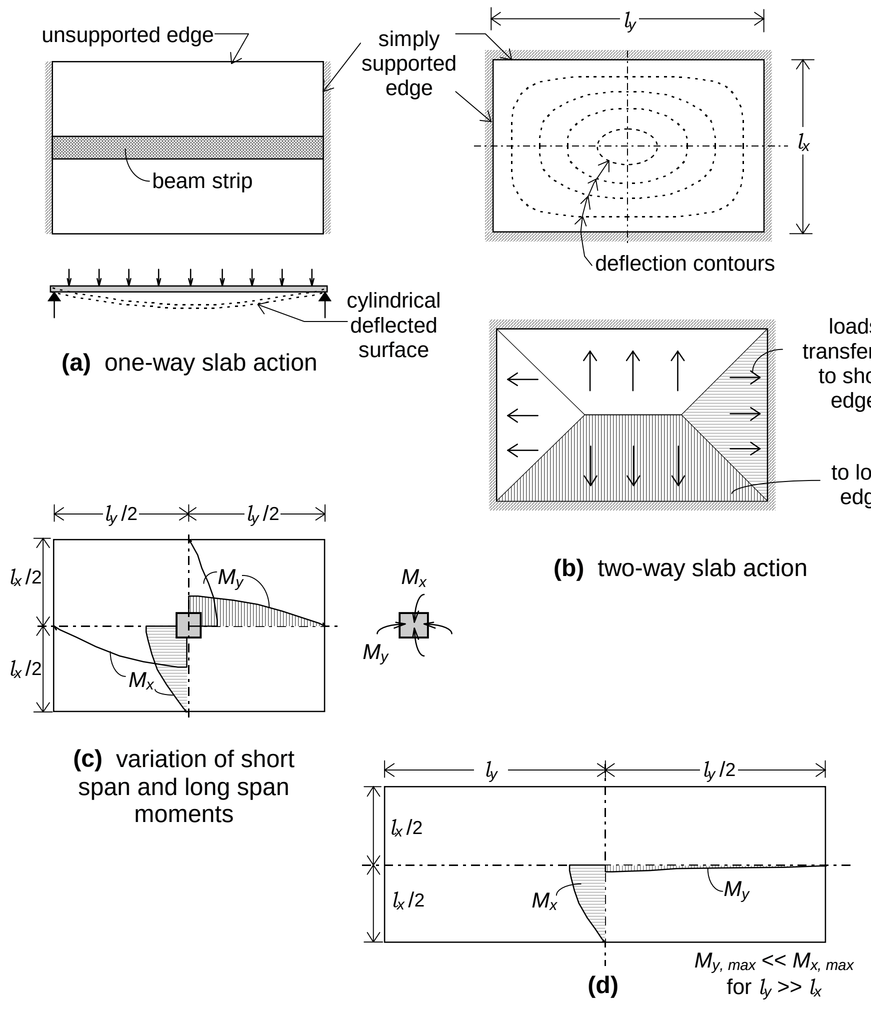


**Fig. 11.1** One-way and two-way slab actions

This primary one-way action [Fig. 11.1(a)] ceases to exist if either the support conditions or the loading conditions are altered. For example, if the uniformly loaded rectangular slab of Fig. 11.1(a) is supported on all four edges, then the deformed surface of the slab will be *doubly curved*, with the load effects transferred to all the four supporting edges [Fig. 11.1(b)]. Such action is called a *two-way* action, involving significant curvatures along two orthogonal directions. The typical variation of longitudinal and transverse bending moments is depicted in Fig. 11.1(c). The bending moments are expectedly maximum at the middle of the slab, and of the two principal moments ($M_x$, $M_y$) at the middle point, the one ($M_x$) along the short span ($l_x$) is invariably greater.

As the 'aspect ratio' $l_y / l_x$ (i.e., long span/short span) increases, the curvatures and moments along the long span progressively reduce, and more and more of the slab load is transferred to the two long supporting edges by bending in the short span direction. In such cases, the bending moments ($M_y$) are generally low in magnitude [Fig. 11.1(d)]. Hence, such long rectangular slabs ($l_y / l_x > 2$) may be approximated as one-way slabs, for convenience in analysis and design [refer Example 5.3].

The reinforcements in a two-way slab should ideally be oriented in the directions of the two principal moments (i.e., principal curvatures) at every point. However, this is not generally convenient in practice, and the bars are usually placed along the transverse and longitudinal directions[†] throughout the slab. Such slabs are said to be 'orthotropically reinforced'; they are said to be 'isotropically reinforced' in case the reinforcements are such that the moment of resistance per unit width of slab is the same in both directions at the point considered.

### 11.1.2 Torsion in Two-Way Slabs

In general, twisting moments develop in addition to bending moments in a two-way slab element — except when the element is oriented along the principal curvatures. These twisting moments can become significant at points along the slab diagonals [Fig. 11.2(a)]. The variations of principal moments ($M_1$, $M_2$) along and across a diagonal of a simply supported and uniformly loaded square slab of homogeneous material, as obtained from an elastic analysis [Ref. 11.1], are depicted in Fig. 11.2(b). It is seen that the principal moment $M_1$ (along the diagonal) is 'negative' (hogging) at locations close to the corner, and the reactions developed at the supports in the corner region will be downward in nature. If such downward reactions cannot be developed at the supports, the corners will lift up [Fig. 11.2(d)].

In practice, however, corners are usually prevented from lifting up (by wall loads from above, or by monolithic edge beams framing into columns); such slabs are said to be *torsionally restrained*. In such cases, the corners have to be suitably reinforced at top, (for the moment $M_1$ with reinforcement placed parallel to the diagonal) and also at bottom (for the moment $M_2$ with reinforcement placed perpendicular to the diagonal); otherwise cracks are liable to form at the corner, as shown in Fig. 11.2(c).

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[†] It may be noted that an orthogonally placed set of reinforcing bars in a slab is capable of generating flexural strength in any direction.

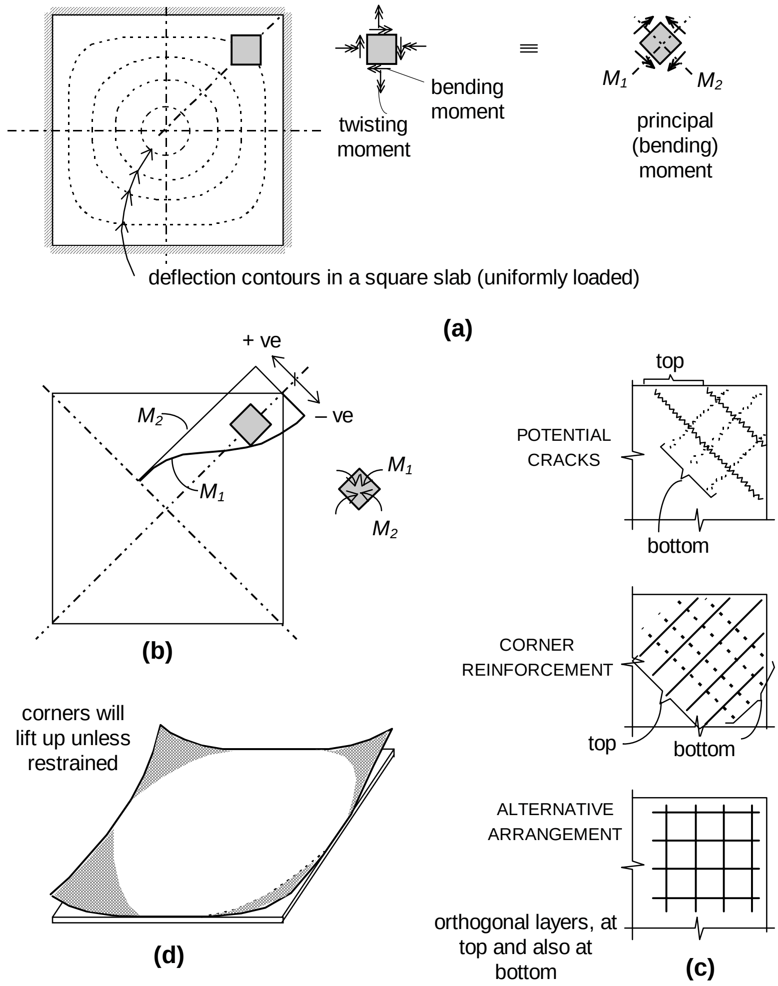



**Fig. 11.2** Torsion effects in a two-way slab

### 11.1.3 Difference between Wall-Supported Slabs and Beam/Column Supported Slabs

The distributed load $w$ on a typical two-way slab is transmitted partly ($w_x$) along the short span to the long edge supports, and partly ($w_y$) along the long span to the short edge supports. In *wall-supported* panels, these portions ($w_x$, $w_y$) of the load are transmitted by the respective wall supports directly to their foundations (or other supports) vertically below, as shown in Fig. 11.3(a). On the contrary, when the edge supports comprise beams spanning between columns, the portion of the load transmitted by the slab in any one direction is in turn transmitted by the beam in the perpendicular direction to the two supporting columns, as shown in Fig. 11.3(b) (i).

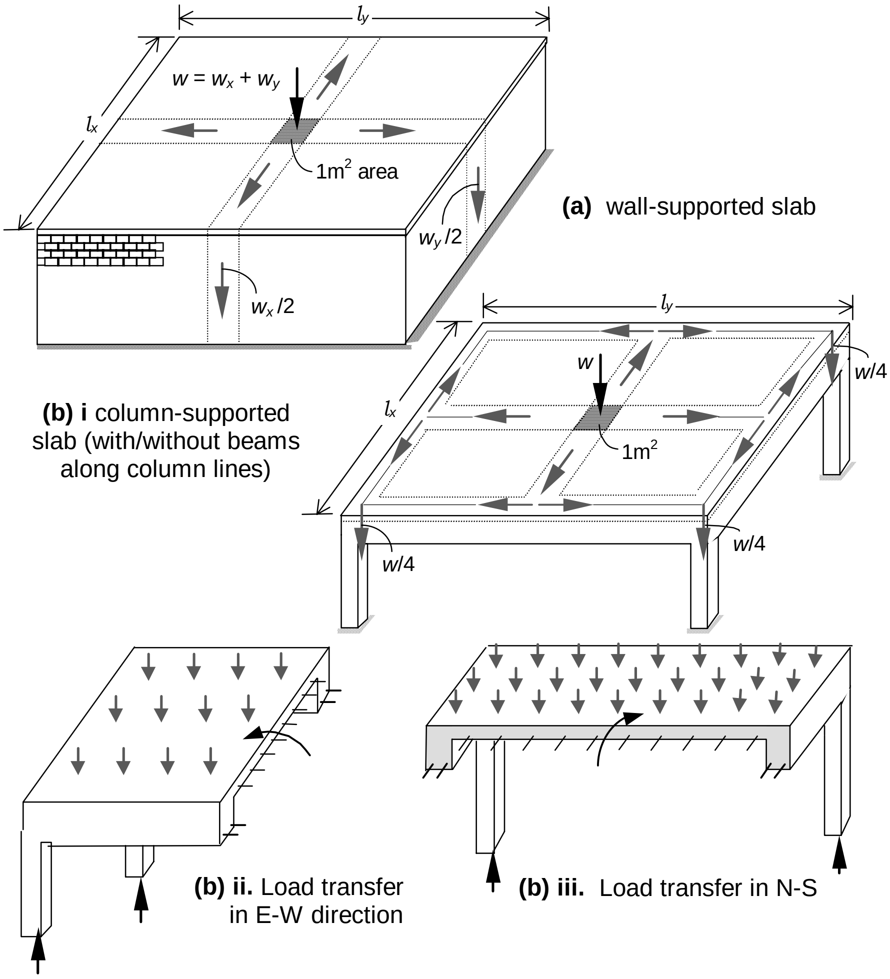



**Fig. 11.3** Load transfer in wall-supported and column-supported slabs

In general, in *column-supported* slabs, with or without beams along the column lines, 100 percent of the slab load has to be transmitted by the floor system in *both directions* (transverse and longitudinal) towards the columns (Fig. 11.3 (b) ii & iii). In such cases, the entire floor system and the columns act integrally in a two-way frame action. The analysis of such systems is described in Section 11.4.

It may be noted, however, that if beams are provided along the column lines, and if these beams are sufficiently rigid, then the analysis and design of the slab part can be considered separately and treated in same manner as wall-supported slabs.

The design of wall-supported two-way slabs is covered in Section 11.2. Subsequent sections in this chapter (Sections 11.3 to 11.9) deal with the design of two-way slabs supported on columns, with or without beams along the column lines.

## 11.2 DESIGN OF WALL-SUPPORTED TWO-WAY SLABS

The design considerations of wall-supported two-way slabs are similar to those pertaining to one-way slabs [refer Chapter 5]. The thickness of the slab is generally based on deflection control criteria, and the reinforcements in the two orthogonal directions are designed to resist the calculated maximum bending moments in the respective directions at the critical sections. [Additional reinforcement may be required at the corners of two-way slabs in some cases, as explained later]. The slab thickness should be sufficient against shear, although shear is usually not a problem in two-way slabs subjected to uniformly distributed loads.

### 11.2.1 Slab Thickness Based on Deflection Control Criterion

The initial proportioning of the slab thickness may be done by adopting the same guidelines regarding span/effective depth ratios, as applicable in the case of one-way slabs [refer Chapter 5]. The effective span in the *short span* direction should be considered for this purpose. However, the percentage tension reinforcement requirement in the short span direction for a two-way slab is likely to be less than that required for a one-way slab with the same effective span. Hence, the modification factor $k_t$ to be considered for two-way slabs may be taken to be higher than that recommended for one-way slabs [refer Section 5.4.2]. A value of $k_t \approx 1.5$ may be considered for preliminary design. The adequacy of the effective depth provided should be verified subsequently, based on the actual $p_t$ provided.

For the special case of two-way slabs with spans up to 3.5 m and live loads not exceeding 3.0 kN/m$^2$, the Code(Cl. 24.1, Note 2) permits the slab thickness (overall depth $D$) to be calculated directly as follows, without the need for subsequent checks on deflection control:

(i) using mild steel (Fe 250 grade),

$$D \geq \begin{cases} l_x/35 & \text{for simply supported slabs} \\ l_x/40 & \text{for continuous slabs} \end{cases} \tag{11.1a}$$

(ii) using Fe 415 grade steel,

$$D \geq \begin{cases} l_x/28 & \text{for simply supported slabs} \\ l_x/32 & \text{for continuous slabs} \end{cases} \tag{11.1b}$$

### 11.2.2 Methods of Analysis

Two-way slabs are highly statically indeterminate†. They may be visualised as being comprised of intersecting, closely-spaced grid beam-strips which are subject to flexure, torsion and shear. Owing to the high static indeterminacy, rigorous solutions are not generally available. The available solutions, based on the classical theory of

† It should be noted that on account of the multiple load paths possible, two-way slabs are capable of considerable stress redistribution. Hence, the reinforcements in such slabs can be designed in many different ways.

plates [Ref. 11.1], for such standard problems as simply supported and uniformly loaded two-way slabs (due to Navier and Levy), need modification to accommodate the differences observed experimentally on account of the non-homogeneous and nonlinear behaviour of concrete. Such solutions have been proposed by Westergaard [Ref. 11.2, 11.3] and others [Ref. 11.4] in the form of convenient moment coefficients, which have been widely used by codes all over the world. Another approximate method, very elementary in approach, is the so-called Rankine-Grashoff method. The main feature of this method is that it simplifies a highly indeterminate problem to an equivalent simple determinate one. This method, as well as its modified version due to Marcus [Ref. 11.5, 11.6] have also been widely in use during the past five decades. Modern computer-based methods include the *finite difference method* [Ref. 11.5] and the *finite element method* [Ref. 11.7]. Other methods, which are particularly suited for limit state design, and are relatively simple, are *inelastic* methods based on *yield line analysis* [Ref. 11.8 – 11.10].

According to the Code (Cl. 24.4), two-way slabs may be designed *by any acceptable theory*. In the case of uniformly loaded two-way rectangular slabs, the Code suggests design procedures for

- simply supported slabs whose corners are not restrained from lifting up [Cl. D–2 of the Code].;
- 'torsionally restrained' slabs, whose corners are restrained from lifting up and whose edges may be continuous or discontinuous [Cl. D–1 of the Code].

### 11.2.3 Uniformly Loaded and Simply Supported Rectangular Slabs

The moment coefficients prescribed in the Code (Cl. D–2) to estimate the maximum moments (per unit width) in the short span and long span directions are based on the Rankine-Grashoff theory. According to this theory, the slab can be divided into a series of orthogonal crossing unit (beam) strips, and the load can be apportioned to the short span and long span strips such that the deflections $\delta$ of the two *middle strips* is the same at their intersection [Fig. 11.4].

If torsion between the interconnecting strips and the influence of adjoining strips on either side are ignored[†], each of the two strips along the centrelines can be considered to be simply supported and subjected to uniformly distributed loads $w_x$ (on the short span strip) and $w_y$ (on the long span strip) [Fig. 11.4]. Hence, the mid-point deflection $\delta$ is easily obtained as:

$$\delta = \frac{5}{384}\frac{w_x l_x^4}{EI} = \frac{5}{384}\frac{w_y l_y^4}{EI} \tag{11.2}$$

where it is assumed that the second moment of area, $I$, is the same for both strips. A simple relation between $w_x$ and $w_y$ is obtainable from Eq. 11.2:

$$w_x = w_y (l_y / l_x)^4 \tag{11.3}$$

Also,

[†] It should be noted that these assumptions lead to a high degree of approximation.

$$w_x + w_y = w \tag{11.4}$$

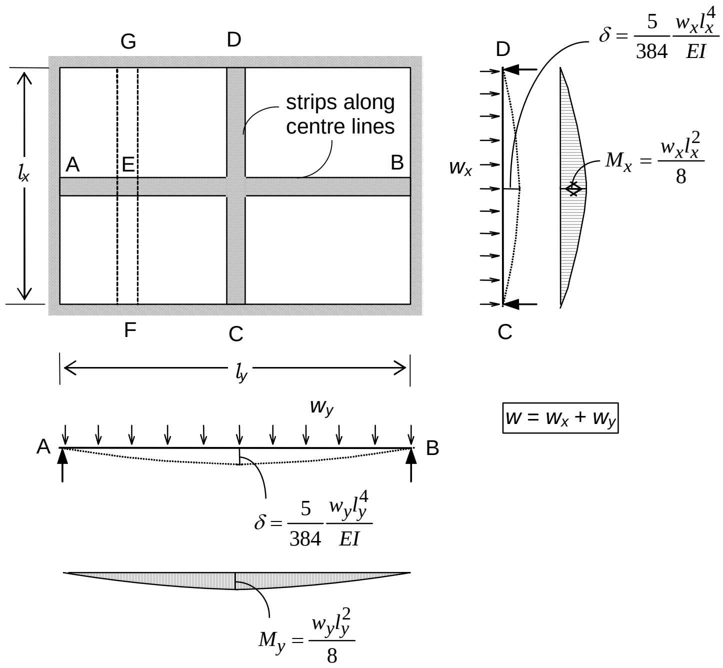



**Fig. 11.4** Concept underlying Rankine-Grashoff theory

$$\Rightarrow \left.\begin{aligned} w_x &= w\left[\frac{r^4}{1+r^4}\right] \\ w_y &= w\left[\frac{1}{1+r^4}\right] \end{aligned}\right\} \tag{11.5}$$

where $r \equiv l_y/l_x$

The maximum short span moment $M_x$ (per unit width) and maximum long span moment $M_y$ (per unit width) are easily obtained as:

$$\left.\begin{aligned} M_x &= w_x l_x^2/8 \\ M_y &= w_y l_y^2/8 \end{aligned}\right\} \tag{11.6}$$

Substituting Eq. 11.5 in Eq. 11.6, the following expressions (in the format presented in the Code) are obtained:

$$\left.\begin{aligned} M_x &= \alpha_x \; w l_x^2 \\ M_y &= \alpha_y \; w l_x^2 \end{aligned}\right\} \tag{11.7}$$

where the 'moment coefficients' $\alpha_x$ and $\alpha_y$ are given by

$$\left.\begin{aligned} \alpha_x &= \frac{1}{8}\left[\frac{r^4}{1+r^4}\right] \\ \alpha_y &= \frac{1}{8}\left[\frac{r^2}{1+r^4}\right] \end{aligned}\right\} \tag{11.8}$$

It is important to note that both $M_x$ and $M_y$ are given (in Eq. 11.7) in terms of the short span $l_x$.

Values of $\alpha_x$ and $\alpha_y$ for different aspect ratios $r \equiv l_y / l_x$ are listed in Table 11.1 (also given in Table 27 of the Code). The variations of these coefficients with the aspect ratio, $r$, are also depicted in Fig. 11.5.

**Table 11.1** Rankine-Grashoff moment coefficients for simply supported, uniformly loaded rectangular slabs (with corners torsionally unrestrained)

| $l_y / l_x$ | **1.0** | **1.1** | **1.2** | **1.3** | **1.4** |
|---|---|---|---|---|---|
| $\alpha_x$ | 0.0625 | 0.0743 | 0.0843 | 0.0926 | 0.0992 |
| $\alpha_y$ | 0.0625 | 0.0614 | 0.0586 | 0.0548 | 0.0506 |
| $l_y / l_x$ | **1.5** | **1.75** | **2.0** | **2.5** | **3.0** |
| $\alpha_x$ | 0.1044 | 0.1130 | 0.1176 | 0.1219 | 0.1235 |
| $\alpha_y$ | 0.0464 | 0.0369 | 0.0294 | 0.0195 | 0.0137 |

As expected, the short span moment coefficient progressively increases and the long span moment coefficient $\alpha_y$ progressively decreases as the aspect ratio $r$ increases. In the case of a square slab ($l_y/l_x$ = 1), $w_x = w_y = w/2$, and $M_x = M_y$ $= \frac{1}{2}(wl_x^2/8) = 0.0625\,wl_x^2$. For high values of $l_y / l_x$, $\alpha_x$ approaches the 'one-way' value of 1/8 = 0.125 and $\alpha_y$ becomes negligible.

Also shown in Fig. 11.5, by means of thinner lines, are the variations of $\alpha_x$ and $\alpha_y$ with $r$ for the case of a rectangular slab with corners torsionally restrained (using Code moment coefficients). This is discussed in Section 11.2.4. It is evident that the corner restraints lead to a reduction in moment coefficients. With regard to the long span moment coefficient, the Code recommends a constant value of $\alpha_y \equiv 0.056$ for all aspect ratios [refer Fig. 11.5].

It should be noted that the moments predicted by the Rankine-Grashoff theory are somewhat conservative because the effect of the restraint along the sides of the strips offered by the rest of the slab through torsion, transverse shear and moment are ignored. For example, the slope $\theta$ in the elastic curve of the longitudinal strip AB at E (Fig. 11.4) is also the angle of twist at E of the transverse strip GF. Owing to the torsional stiffness of the strip GF, a twisting moment will develop at E in the transverse strip GF ('compatibility torsion' — as explained in Chapter 7). The twisting moments thus developed in the various transverse strips (such as GF, CD) at the middle, due to the integral action with the longitudinal middle strip AB, effectively reduce the flexure in AB. Conversely, the bending in the transverse middle strip CD is reduced by torsion in the interconnecting longitudinal strips (parallel to AB). Thus, the treatment of strips such as AB and CD as *discrete strips* free of interaction from the rest of the slab results in conservative estimates of the design bending moments.

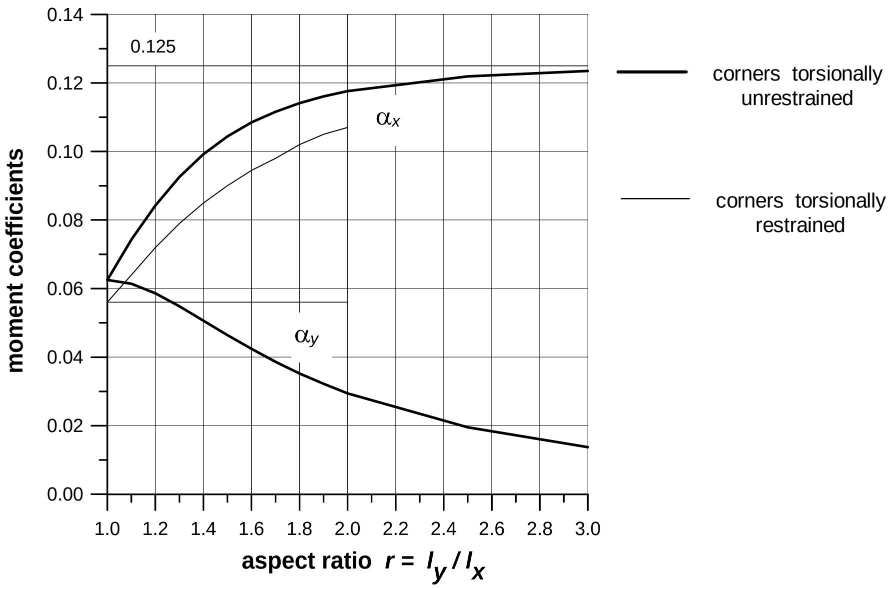



**Fig. 11.5** Variation of IS Code moment coefficients $\alpha_x$, $\alpha_y$ with $l_y/l_x$ for simply supported and uniformly loaded rectangular slabs

### *Detailing of Reinforcement*

The flexural reinforcements in the two directions are provided to resist the maximum bending moments $M_{ux} = \alpha_x\, w_u\, l_x^2$ (in the short span) and $M_{uy} = \alpha_y\, w_u\, l_x^2$ (in the long span). The steel requirements at the midspan locations in strips distant from the middle strip progressively reduce with the distance from the middle strip. However, the usual design practice is to provide bars that are uniformly spaced† throughout the span (in both directions), with a flexural resistance that is not less than the calculated maximum ultimate bending moment ($M_{ux}$ or $M_{uy}$).

Furthermore, considering any particular strip (transverse or longitudinal), the bending moment varies from a maximum value at the midspan to zero at either support [Fig. 11.4]. Hence, it is possible to curtail the bars in accordance with the Code provisions explained in Section 5.9. For the special case of simply supported

† The spacing of reinforcement should not exceed 3*d* or 300 mm (whichever is smaller).

two-way slabs (torsionally unrestrained), the Code (Cl. D–2.1.1) suggests a simplified procedure for reinforcement curtailment. According to this procedure, up to 50 percent of the bars may be terminated within a distance of 0.1*l* from the support, while the remaining bars must extend fully into the supports.

If the slab is truly *simply supported* at the edges, there is no possibility of ‘negative’ moments developing near the supports, due to partial fixity. However, it is good design practice to always safeguard against the possibility of partial fixity. As explained with reference to the design of one-way slabs [refer Chapter 5] this can be achieved either by bending up alternate bars [Fig. 5.3, 5.5(b)], or by providing separate top steel, with area equal to 0.5 times that provided at bottom at midspan, with an extension of 0.1*l* from the face of the support [Fig. 5.5(a)]. [The recent trend is to do away with bent up bars and instead to opt for separate layers at top and bottom. This type of detailing is illustrated in Example 11.1 [Fig. 11.14].

### 11.2.4 Uniformly Loaded ‘Restrained’ Rectangular Slabs

The Code (Cl. D–1) uses the term *restrained slabs* to refer to slabs whose corners are prevented from lifting and contain suitable reinforcement to resist torsion [Ref. 11.11]. All the four edges of the rectangular ‘restrained’ slab are assumed to be supported (tied down) rigidly against vertical translation, and the edges may be either continuous/fixed or discontinuous. Accordingly, nine different configurations of restrained rectangular slab panels are possible (as shown in Fig. 11.6), depending on the number of discontinuous edges (zero, one, two, three or four) and also depending on whether the discontinuous edge is ‘short’ or ‘long’. Panel type À corresponds to the slab with all four edges continuous/fixed, and panel type È corresponds to the slab with all four edges simply supported [Fig. 11.6]. [Incidentally, there will be several mare cases if combinations involving *free* (unsupported) edges are also considered].

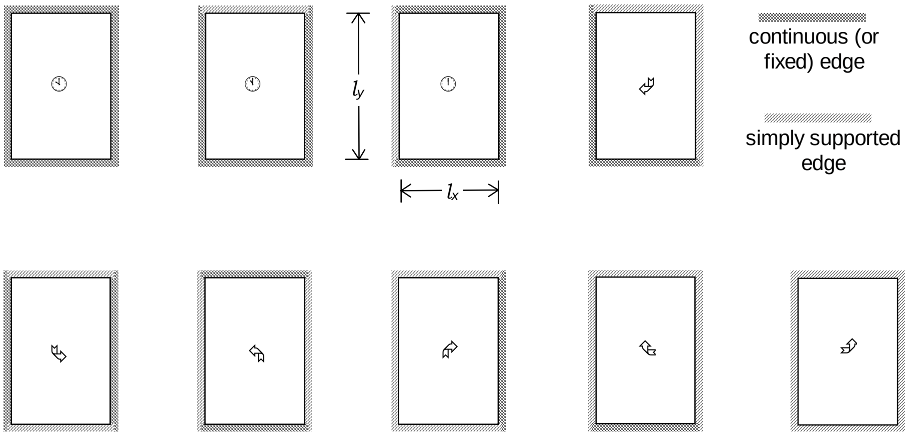



**Fig. 11.6** Nine different types of ‘restrained’ rectangular slab panels

The torsional restraint at the corner calls for the provision of special corner reinforcement, as explained earlier [refer Fig. 11.2(c)]. The corner restraints have the beneficial effect of reducing the deflections and curvatures in the middle of the slab. Expressions for design moment coefficients for uniformly loaded two-way 'restrained' rectangular slabs with fixed or simply supported edge conditions, based on the classical theory of plates, are available [Ref. 11.1, 11.5, 11.6]. Approximate solutions based on the Rankine-Grashoff theory are also available. Modifications to these solutions were proposed by Marcus ('Marcus correction'), whereby the moment coefficients are reduced to account for the effects of torsional restraint at the corners, as well as torsional resistance of the transverse and longitudinal unit strips [Ref. 11.5]. For example, the 'Marcus correction' in the case of simply supported slabs, results in a reduction in the design moment $M_x$ by about 42 percent for $l_y/l_x$ =1.0 (square slab) and 9 percent for $l_y/l_x$ = 3.0, when compared to the slab with corners free to lift up [Table 11.1]; these results compare favourably with the rigorous solutions from elastic theory [Ref. 11.5].

However, the moment coefficients recommended in the Code (Cl. D–1) are based on *inelastic analysis* (yield line analysis) [Ref. 11.12, 11.13], rather than elastic theory. This analysis is based on the following assumptions:

- the bottom steel in either direction is uniformly distributed over the 'middle strip' which spreads over 75 percent of the span;
- the 'edge strip' lies on either side of the middle strip, and has a width equal to $l_x/8$ or $l_y/8$ [Fig. 11.7];
- top steel is provided in the edge strip adjoining a continuous edge (and at right angles to the edge) such that the corresponding flexural strength (ultimate 'negative' moment capacity) is 4/3 times the corresponding ultimate 'positive' moment capacity due to the bottom steel provided in the middle strip in the direction under consideration;
- the corner reinforcement provided is sufficient to prevent the formation of 'corner levers', i.e., forking of diagonal yield lines near the corners.

The resulting moment coefficients $\alpha_x^+$, $\alpha_y^+$ for 'positive' moments at midspans in the short span and long span directions respectively, and the coefficients $\alpha_x^-$, $\alpha_y^-$ for 'negative' moments at the continuous edge(s) in the two directions, for the nine different sets of boundary conditions [Fig. 11.6] are listed in Table 26 of the Code. The design factored moment is obtained as

$$M_u = \alpha\, w_u l_x^2 \tag{11.9}$$

where $w_u$ is the uniformly distributed factored load and $l_x$ the effective short span and $\alpha$ is the appropriate moment coefficient.

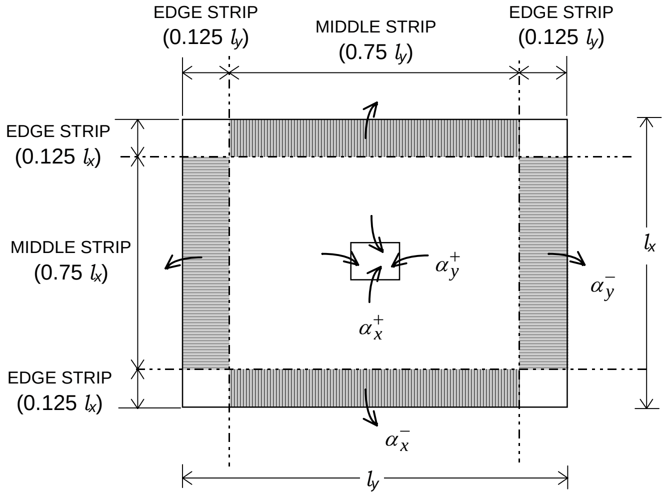



**Fig. 11.7** Basis for Code moment coefficients for 'restrained' two-way slabs

### *'Positive' moment coefficients $\alpha_x^+$, $\alpha_y^+$*

The variations of the short span 'positive' moment coefficient $\alpha_x^+$, with $l_y/l_x$ is plotted for the nine types of two-way slabs in Fig. 11.8. In all cases, there is a marked increase in $\alpha_x^+$ as $l_y/l_x$ increases from 1.0 to 2.0. The Code recommends a constant value of $\alpha_y^+$ for all values of $l_y/l_x$. The value of $\alpha_y^+$ is obtainable from the following formula [Ref. 11.13]:

$$\alpha_y^+ = (24 + 2n_d + 1.5n_d^2)\big/1000 \tag{11.10}$$

where $n_d$ denotes the number of discontinuous edges. Corresponding to $n_d$ = 0, 1, 2, 3 and 4, the values of $\alpha_y^+$ are obtained as 0.0240, 0.0275, 0.0340, 0.0435 and 0.0560 respectively[†].

An expression for $\alpha_x^+$ may be obtained in terms of $\alpha_y^+$ and $r \equiv l_y/l_x$ from yield line analysis [Ref. 11.12, 11.13] as follows:

$$\alpha_x^+ = \frac{2}{9}\left[\frac{3 - \sqrt{18\alpha_y^+}\,(C_{s1} + C_{s2})\big/r}{(C_{l1} + C_{l2})^2}\right] \tag{11.11}$$

where

$$C = \begin{cases} 1 & \text{for a discontinuous edge} \\ \sqrt{7/3} & \text{for continuous edge} \end{cases} \tag{11.12}$$

and the subscripts $s$ and $l$ denote 'short edge' and 'long edge' respectively, while the additional subscripts '1' and '2' represent the two edges in either direction. Thus, for

---

[†] In Table 26 of the Code, the specified values of $\alpha_y$ are 0.024, 0.028, 0.035, 0.043, and 0.056 — corresponding to $n_d$ = 0, 1, 2, 3 and 4 respectively.

example, for the slab panel of type '4' ("two adjacent edges discontinuous"), $C_{s1} + C_{s2} = Cl_1 + Cl_2 = 1 + \sqrt{7/3} = 2.5275$. For such a case, $n_d = 2$; hence, applying Eq. 11.10, $\alpha_y^+ = 0.0340$. Further, applying Eq. 11.11,

$$\alpha_x^+ = \frac{2}{9}\left[\frac{3 - \sqrt{18 \times 0.034}(2.5275)/r}{(2.5275)^2}\right] = \frac{2}{9}\left[\frac{3 - 1.9773/r}{6.3884}\right]$$

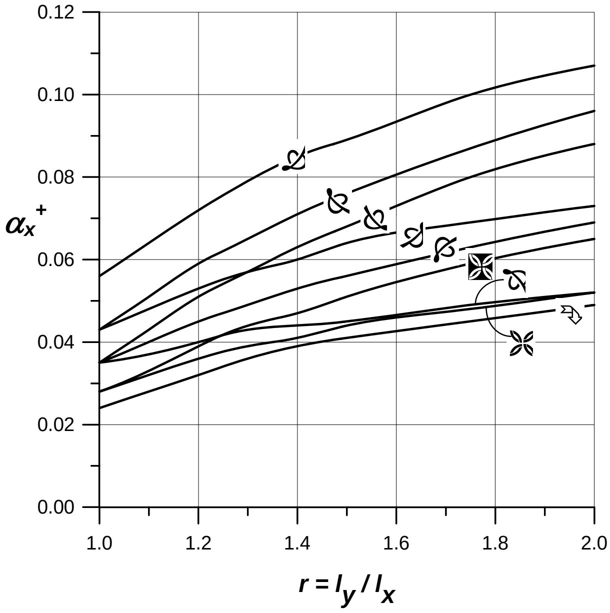



**Fig. 11.8** Variations in short span 'positive' moment coefficients with $l_y/l_x$ in 'restrained' two-way slabs

This results in values of $\alpha_x^+$ varying from 0.0356 (for $r$ = 1.0) to 0.0700 (for $r$ = 2.0); the corresponding values given in Table 26 of Code are 0.035 (for $r$ = 1.0) and 0.069(for $r$ = 2.0).

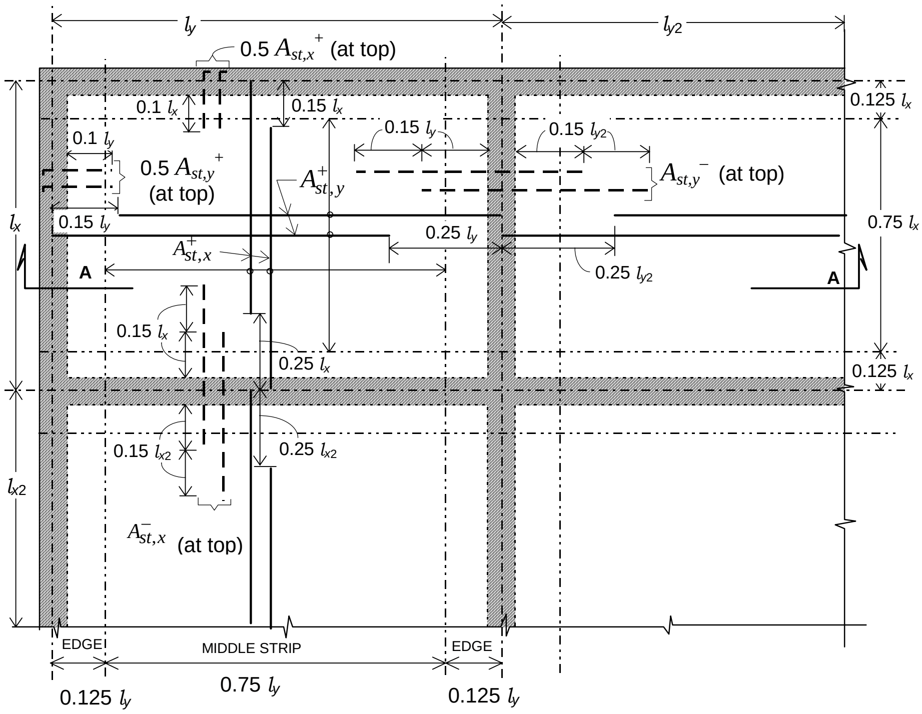
Similarly, values of $\alpha_x^+$ and $\alpha_y^+$ can be easily obtained for any value of $r \equiv l_y/l_x$ in the range [1.0, 2.0] and given set of boundary conditions. If the value of $l_y/l_x$ exceeds 2.0, the Code (Cl. D–1.11) recommends that the slab should be treated as *one-way* [refer Chapter 5]; the provision of the *secondary reinforcement* in the long span direction is expected to take care of the nominal bending moments that may arise in this direction.

### *'Negative' moment coefficients $\alpha_x^-$, $\alpha_y^-$*

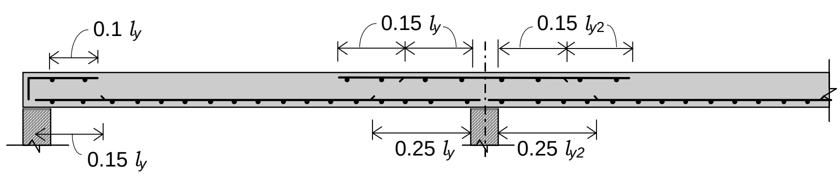
As explained earlier, the Code moment coefficients have been derived (using yield line analysis) with the basic assumption that the ultimate moment of resistance (for 'negative' moment) at a continuous support is 4/3 times the 'positive' moment capacity in the midspan region. Of course, at a discontinuous support, the 'negative' moment developed is zero[‡]. Accordingly,

$$\alpha^{-} = \begin{cases} 0 & \text{at a discontinuous support} \\ \dfrac{4}{3}\alpha^{+} & \text{at a continuous support} \end{cases} \tag{11.13}$$

### *Detailing of Flexural Reinforcement*




**(a)** PLAN




**(b)** SECTION 'AA'

**Fig. 11.9** Detailing of flexural reinforcement in two-way 'restrained' rectangular slabs[†] (excluding corner reinforcement)

[‡] However, as explained earlier, the possibility of partial restraint must be considered at the time of detailing.

- The bottom steel for the design moments (per unit width) $M_{ux}^{+} = \alpha_x^{+} w_u l_x^2$ and $M_{uy}^{+} = \alpha_y^{+} w_u l_x^2$ should be uniformly distributed across the 'middle strips' in the short span and long span directions respectively. The Code (Cl. D–1.4) recommends that these bars should extend to within 0.25*l* of a continuous edge or 0.15*l* of a discontinuous edge. It is recommended [Ref. 11.14] that alternate bars (bottom steel) should extend fully into the support, as shown in Fig. 11.9.
- The top steel calculated for the design moments $M_{ux}^{-} = \alpha_x^{-} w_u l_x^2$ and $M_{uy}^{-} = \alpha_y^{-} w_u l_x^2$ at continuous supports should be uniformly distributed across the 'edge strips' in the long span and short span directions respectively. The Code (Cl. D–1.5) recommends that at least 50 percent of these bars should extend to a distance of 0.3*l* from the face of the continuous support, on either side. The remaining bars may be curtailed at a distance of 0.15*l* from the face of the continuous support, as shown in Fig. 11.9[†] .
- To safeguard against possible 'negative' moments at a discontinuous edge due to partial fixity, the Code (Cl. D–1.6) recommends that top steel with area equal to 50 percent of that of the bottom steel at mid-span (in the same direction) should be provided, extending over a length of 0.1*l*, as shown in Fig. 11.9.
- In the edge strip, distribution bars parallel to that edge (conforming to the minimum requirements specified in Section 5.2) should be provided — at top and bottom — to tie up with the main bars [Fig. 11.9].

### *Detailing of Torsional Reinforcement at Corners*

Torsional reinforcement is required at the corners of rectangular slab panels whose edges are discontinuous. This can conveniently be provided in the form of a mesh (or grid pattern) at top and bottom. The bars can be made U-shaped (wherever convenient) and provided in the two orthogonal directions as shown in Fig. 11.10 [Ref. 11.14]. The Code (Cl. D–1.8) recommends that the mesh should extend beyond the edge[‡] over a distance not less than one-fifth of the shorter span ($l_x$). The total area of steel to be provided in *each* of the four layers should be not less than:

- 0.75 $A_{st,x}^{+}$ if both edges meeting at the corner are discontinuous;
- 0.375 $A_{st,x}^{+}$ if one edge is continuous and the other discontinuous.

Here, $A_{st,x}^{+}$ is the area of steel required for the maximum midspan moment in the slab.

It may be noted that if both edges meeting at a corner are continuous, torsional reinforcement is not called for at the corner [refer Cl. D–1.10]. This is indicated in Fig. 11.10. [However, this area will have some reinforcement provided anyway,

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[†] For convenience in estimating lengths of bars and locations of bar cut-off points, $l_x$ and $l_y$ may be taken as the spans, measured centre-to-centre of supports.

[†] In Fig. 11.9, the top bars and bottom bars are shown as being separate. Alternatively, the bottom bars can be bent up to form the top steel, as shown in Fig. 5.5.

[‡] Here, the term 'edge' refers to the face of the support.

because of the 'negative' moment reinforcements over supports in the middle strips and the distributor reinforcements in the edge strips.]

### *Design 'Negative' Moments at Continuous Supports*

In a wall-supported continuous slab system, each rectangular slab panel is analysed separately (for design moments) using the Code moment coefficients. The 'negative' moments ($M_1$, $M_2$) calculated for two panels sharing a common continuous edge may not be equal [Fig. 11.11] due to one or more of the following reasons:

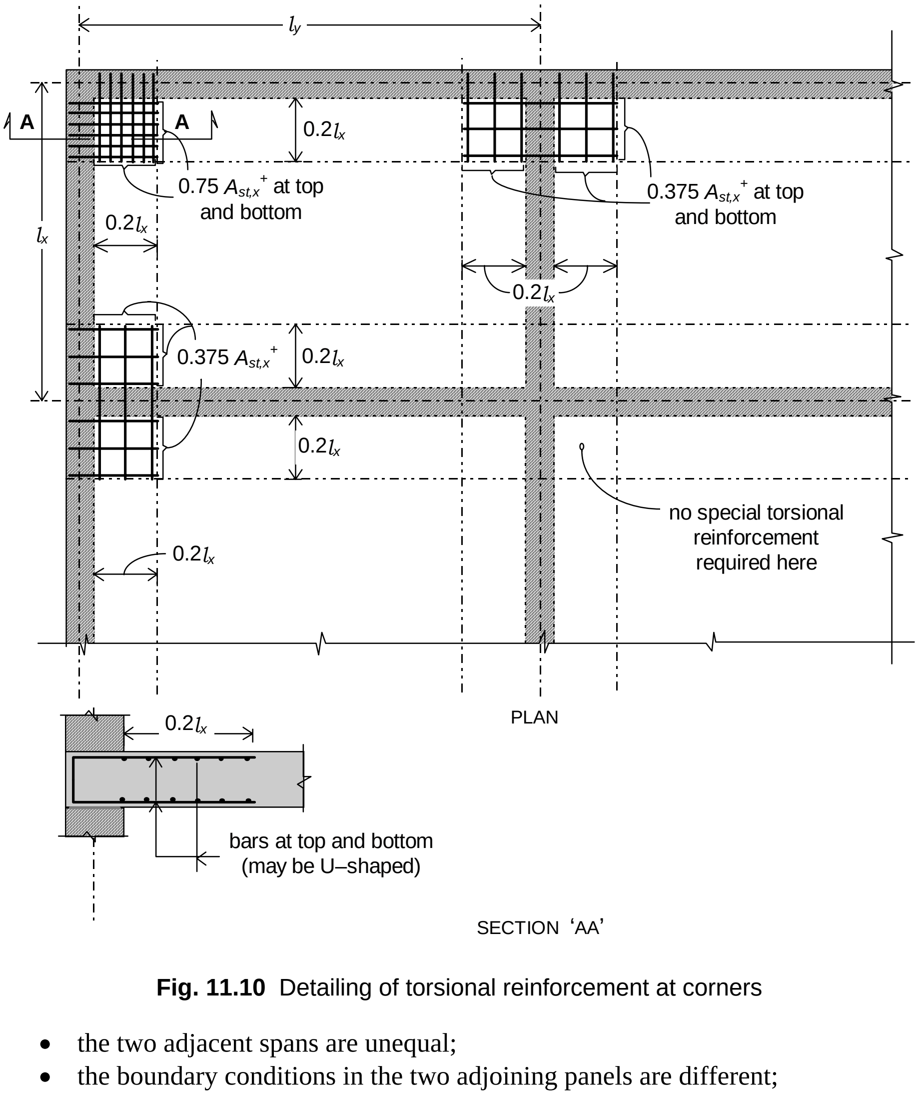


**Fig. 11.10** Detailing of torsional reinforcement at corners

- the two adjacent spans are unequal;
- the boundary conditions in the two adjoining panels are different;
- the loading on one panel is different from that in the other panel.

Since the Code moment coefficients are based on *inelastic* analysis, with a fixed ratio (4/3) of 'negative' to 'positive' moment capacities, no redistribution of moments is permissible†. Hence, it is logical to take the *larger* factored moment ($M_1$ in Fig. 11.11) as the design 'negative' moment at the continuous edge.

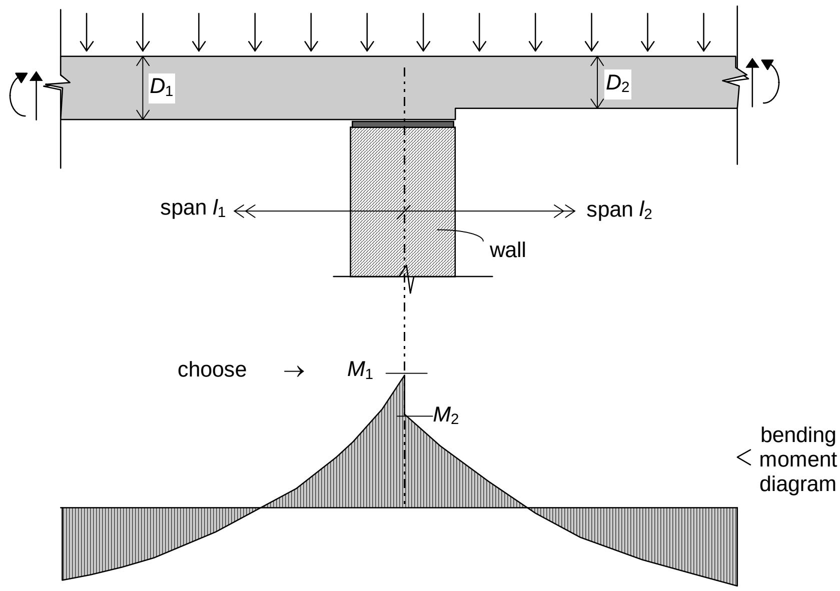


**Fig. 11.11** Design 'negative' moment at a continuous support

### *Influence of Pattern Loading*

In the case of continuous slab systems — whether one-way or two-way — the influence of variability in live loads must be considered. The concept of 'pattern loading' was introduced in Section 9.7, with reference to elastic analysis of multistoreyed frames, and also in Section 5.6, with reference to continuous beams and one-way slabs. In the case of two-way slabs, the 'checkerboard pattern' of loading [Fig. 11.12(a)] generally results in the maximum 'positive' moments in slabs, and the 'strip pattern' of loading [Fig. 11.12(b)] results in the maximum 'negative' moments in slabs.

As explained earlier, the Code moment coefficients for 'restrained' slab panels are based on inelastic analysis, and not elastic analysis. Each panel is analysed separately for its worst ('collapse') loading, and hence the concept of pattern loading is not relevant here.

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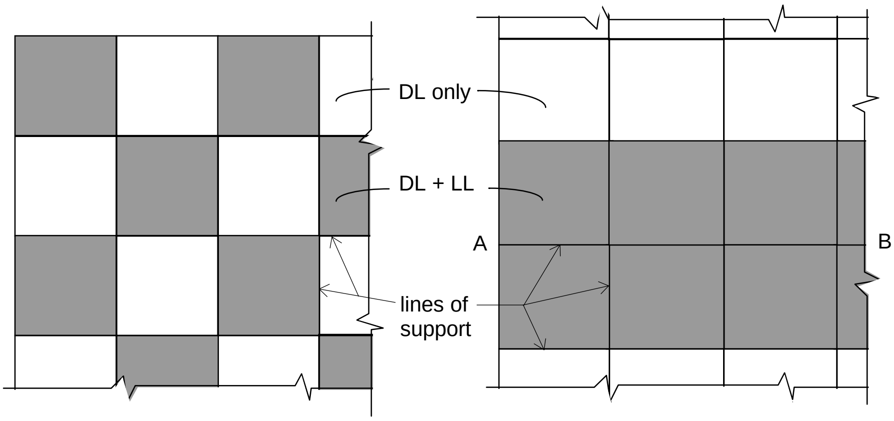
† It may be noted that the Code recommends the use of the same moment coefficients for design by the *working stress method*. In a WSM context (i.e., under service loads), it may be argued that the design moments should be obtained through some kind of moment distribution procedure [Ref. 11.5]. However, this is not meaningful in design by LSM. Moreover, the basis of the Code moment coefficients is *inelastic* analysis, and not elastic analysis.




**(a)** 'checkerboard loading' pattern for maximum 'positive' moments (in panels with LL)

**(b)** 'strip loading' pattern for maximum 'negative' moments along panel edges/support lines (along AB)

**Fig. 11.12** Pattern loadings for maximum slab moments

### *Redistribution of Moments*

Bending moments in continuous systems, based on elastic analysis, can be redistributed, as explained in Chapter 9, for more economical distribution of reinforcement. However, when the Code moment coefficients for 'restrained' slabs are used, moment redistribution is prohibited [refer Cl. D–1.3], as the coefficients are based on inelastic analysis.

### 11.2.5 Shear Forces in Uniformly Loaded Two-Way Slabs

Shear is generally not a governing design consideration in wall-supported reinforced concrete slabs subject to uniformly distributed loads. This was explained earlier [refer Example 6.2] with reference to one-way slabs. With two-way action, the magnitude of shear stresses are likely to be even lesser than with one-way action.

The distribution of shear forces at the various edges of a two-way slab is complicated in general. However, the Code (Cl. 24.5) recommends a simple distribution of loads on the supporting edges (as explained earlier in Section 9.4), according to which, the distribution of load on the short edge is triangular, and the distribution of load on the long edge is trapezoidal, with the lines demarcating the contributing areas at 45 degrees to the boundaries [Fig. 11.13]. The critical section for shear[†] is to be considered *d* away from the face of the support.

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[†] This type of shear is called 'one-way shear' or 'beam shear', which is distinct from 'two-way shear' ('punching shear') applicable for slabs supported on columns [see Section 11.8].

As shown in Fig. 11.13, the maximum factored shear force per unit length, $V_u$, is obtained as:

$$V_u = w_u (0.5 l_{xn} - d) \tag{11.14}$$

where $l_{xn}$ is the *clear span* in the short span direction.

The corresponding nominal shear stress $\tau_v = V_u/bd$ (with $b$ = 1000 mm) should be less than the design shear strength of concrete for slabs, $k\tau_c$ [refer Section 6.6.2]. An average effective depth $d = (d_x + d_y)/2$ may be considered in the calculations.

For a more accurate estimation of load distribution in slabs with different boundary conditions, reference may be made to Ref. 11.16.

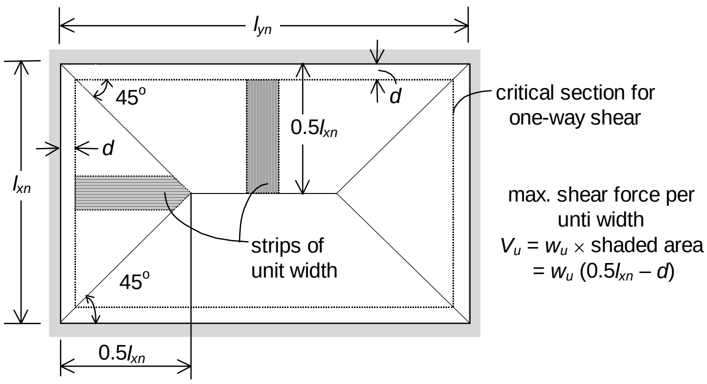



**Fig. 11.13** Assumed distribution of loads on the edges of a rectangular slab, uniformly loaded

## EXAMPLE 11.1

Design a simply supported slab to cover a room with internal dimensions 4.0 m × 5.0 m and 230 mm thick brick walls all around. Assume a live load of 3 kN/m$^2$ and a finish load of 1 kN/m$^2$. Use M 20 concrete and Fe 415 steel. Assume that the slab corners are free to lift up. Assume *mild* exposure conditions.

### SOLUTION

- Effective short span ≈ 4150 mm

  Assume an effective depth $d \approx \dfrac{4150}{20 \times 1.5}$ = 138 mm

  With a clear cover of 20 mm and say, 10 $\phi$ bars, overall thickness of slab $D \approx 138 + 20 + 5 = 163$ mm.

  ⇒ Provide $D$ = 165 mm

  ⇒ $d_x = 165 - 20 - 5 = 140$ mm

  $d_y = 140 - 10 = 130$ mm

$\Rightarrow$ Effective spans $\begin{cases} l_x = 4000 + 140 = 4140 \text{ mm} \\ l_y = 5000 + 130 = 5130 \text{ mm} \end{cases} \Rightarrow r \equiv \dfrac{l_y}{l_x} = \dfrac{5130}{4140} = 1.239$

[Note that effective span is taken as (clear span + $d$), as this is less than centre-to-centre span (between supports)].

**Loads on slab:**

(i) self weight @ 25 kN/$m^3$ × 0.165$m$ = 4.13 kN/m²
(ii) finishes (given) = 1.0 ”
(iii) live loads (given) = 3.0 ”
$w$ = 8.13 kN/m²

$\Rightarrow$ Factored load $w_u = 8.13 \times 1.5 = 12.20$ kN/m²

**Design Moments** (for strips at midspan, 1 m wide in each direction)

- As the slab corners are torsionally unrestrained, the Rankine-Grashoff method [Cl. D–2 of Code] may be applied:
  *short span*: $M_{ux} = \alpha_x w_u l_x^2$
  *long span*: $M_{uy} = \alpha_y w_u l_x^2$
  where

  $$\alpha_x = \frac{1}{8}\left[\frac{r^4}{1+r^4}\right] = \frac{1}{8}\left[\frac{1.239^4}{1+1.239^4}\right] = 0.0878$$

  and $$\alpha_y = \frac{1}{8}\left[\frac{r^2}{1+r^4}\right] = \frac{1}{8}\left[\frac{1.239^2}{1+1.239^4}\right] = 0.0571$$

  $\Rightarrow M_{ux} = 0.0878 \times 12.20 \times 4.140^2 = 18.36$ kNm/m
  $M_{uy} = 0.0571 \times 12.20 \times 4.140^2 = 11.94$ kNm/m

**Design of Reinforcement**

- $R_x \equiv \dfrac{M_{ux}}{bd_x^2} = \dfrac{18.36 \times 10^6}{10^3 \times 140^2} = 0.9367$ MPa
- $R_y \equiv \dfrac{M_{uy}}{bd_y^2} = \dfrac{11.94 \times 10^6}{10^3 \times 130^2} = 0.7065$ MPa

  $$\frac{(p_t)_{reqd}}{100} \equiv \frac{(A_{st})_{reqd}}{bd} = \frac{f_{ck}}{2f_y}\left[1 - \sqrt{1 - 4.589R/f_{ck}}\right]$$

  $$\Rightarrow \frac{(p_t)_{x,reqd}}{100} = \frac{20}{2 \times 415}\left[1 - \sqrt{1 - (4.598 \times 0.9367)/20}\right] = 0.275 \times 10^{-2}$$

  $\Rightarrow (A_{st})_{x,\,reqd} = (0.275 \times 10^{-2}) \times 1000 \times 140 = 385$ mm²/m

  $\Rightarrow$ required spacing of 10 $\phi$ bars = $\dfrac{1000 \times 78.5}{385} = 204$ mm

- Similarly, $\dfrac{(p_t)_{y,reqd}}{100} = \dfrac{20}{2 \times 415}\left[1 - \sqrt{1 - (4.598 \times 0.7065)/20}\right] = 0.204 \times 10^{-2}$

  $\Rightarrow (A_{st})_{y,\,reqd} = (0.204 \times 10^{-2}) \times 1000 \times 130 = 265.7$ mm²/m

$\Rightarrow$ required spacing of 10 $\phi$ bars = $\dfrac{1000 \times 78.5}{265.7}$ = 295 mm

Maximum spacing for primary reinforcement = 3$d$ or 300 mm

$$= \begin{cases} 3 \times 140 = 420 \text{ mm (short span)} \\ 3 \times 130 = 390 \text{ mm (long span)} \end{cases}$$

- Provide $\begin{cases} 10\,\phi\ @\ 200\ c/c \text{ (short span)} & \Rightarrow A_{st,x} = 392.5 \text{ mm}^2/\text{m} \\ 10\,\phi\ @\ 290\ c/c \text{ (long span)} & \Rightarrow A_{st,y} = 270.7 \text{ mm}^2/\text{m} \end{cases}$

  The detailing is shown in Fig. 11.14.

**Check for deflection control**

- $p_{t,x} = \dfrac{392.5}{10^3 \times 140} \times 100 = 0.280$
- $f_s$ = 0.58 × 415 × 385/392.5 = 236 MPa

  $\Rightarrow$ modification factor $k_t$ = 1.5 (from Table 5.2 or Fig. 3 of Code)

  $\Rightarrow (l/d)_{max}$ = 20 × 1.5 = 30

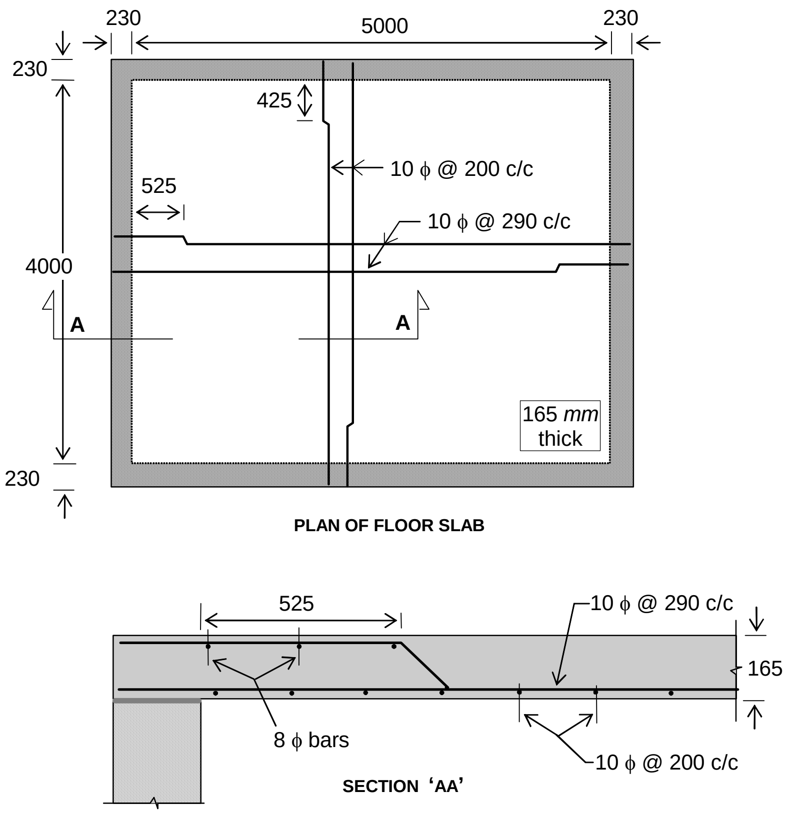



**Fig. 11.14** Example 11.1

$(l/d)_{provided} = \dfrac{4140}{140} = 29.6 < 30$ — OK.

**Check for shear**†

- average effective depth $d$ = (140 + 130)/2 = 135 mm
  $V_u = w_u(0.5l_{xn} - d)$ = 12.20 (0.5 × 4.0 – 0.135) = 22.75 kN/m
  $\Rightarrow \tau_v = 22.75 \times 10^3/(1000 \times 135) = 0.169$ MPa
- $p_t = 0.28 \Rightarrow \tau_c = 0.376$ MPa $\Rightarrow k\tau_c > \tau_v$ — Hence, OK.

**EXAMPLE 11.2**

Repeat Example 11.1, assuming that the slab corners are prevented from lifting up.

**SOLUTION**

- [Refer Example 11.1]: Assume $D$ = 160 mm (which is 5 mm less than the previous case)
  Assuming 8 φ bars ⇒ $d_x$ = 160 – 20 – 4 = 136 mm, $d_y$ = 136 – 8 = 128 mm

$$\Rightarrow \begin{cases} l_x = 4000 + 136 = 4136 \text{ mm} \\ l_y = 5000 + 128 = 5128 \text{ mm} \end{cases} \Rightarrow \frac{l_y}{l_x} = 1.240$$

**Loads on slab**: (same as in Example 11.1)

- Factored load $w_u$ = 12.20 kN/m$^2$

**Design Moments** (for middle strips, 1 m width in each direction).

As the slab corners are to be designed as torsionally restrained, the moment coefficients given in Table 26 of the Code (Cl. D–1) may be applied‡ for $l_y/l_x$ = 1.240:

- *Short span*: $\alpha_x = 0.072 + (0.079 - 0.072) \times \dfrac{1.240 - 1.2}{1.3 - 1.2} = 0.0748$

  $\Rightarrow M_{ux} = \alpha_x w_u l_x^2$
  $= 0.0748 \times 12.20 \times 4.136^2 = 15.61$ kNm/m
  (which, incidentally, is about 15 percent less than the value of 18.36 kNm/m obtained in Example 11.1)
- *Long span*: $\alpha_y = 0.056$
  $\Rightarrow M_{uy} = \alpha_y w_u l_x^2$
  $= 0.056 \times 12.20 \times 4.136^2 = 11.69$ kNm/m
  (which is comparable to the earlier value of 11.94 kNm/m)

**Design of reinforcement**

- $R_x \equiv \dfrac{M_{ux}}{bd_x^2} = \dfrac{15.61 \times 10^6}{10^3 \times 136^2} = 0.844$ MPa

† As explained earlier, a check on shear is not really called for in uniformly loaded, wall-supported two-way slabs. This is evident from the results of this example.
‡ Alternatively, Eq. 11.10, 11.11 may be applied.

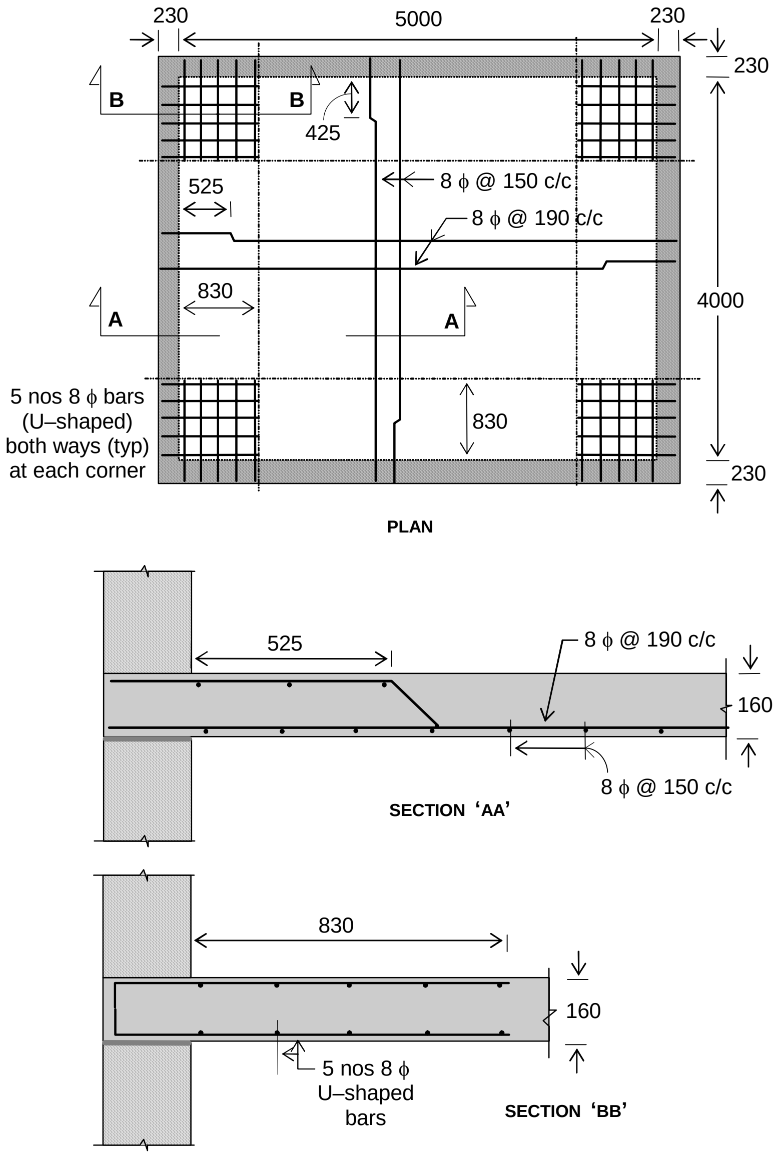



**Fig. 11.15** Example 11.2

$$R_y \equiv \frac{M_{uy}}{bd_y^2} = \frac{11.69\times10^6}{10^3\times128^2} = 0.714 \text{ MPa}$$

$$\Rightarrow \frac{(p_t)_{x,reqd}}{100} = \frac{20}{2\times415}\left[1-\sqrt{1-4.589\times0.844/20}\right] = 0.2465\times10^{-2}$$

$\Rightarrow (A_{st})_{x,\,reqd} = (0.246 \times 10^{-2}) \times 1000 \times 136 = 334\ \text{mm}^2/\text{m}$

$\Rightarrow$ Required spacing of 8 $\phi$ bars = $\dfrac{1000 \times 50.3}{334}$ = 150.7 mm

Maximum spacing permitted = 3 × 136 = 408 mm, but < 300 mm.

$$\Rightarrow \frac{(p_t)_{y,reqd}}{100} = \frac{20}{2 \times 415}\left[1 - \sqrt{1 - 4.589 \times 0.714/20}\right] = 0.206 \times 10^{-2}$$

$\Rightarrow (A_{st})_{y,\,reqd} = (0.206 \times 10^{-2}) \times 1000 \times 128 = 264\ \text{mm}^2/\text{m}$

$\Rightarrow$ Required spacing of 8 $\phi$ bars = $\dfrac{1000 \times 50.3}{264}$ = 191 mm

Maximum spacing permitted = 3 × 128 = 384 mm, but < 300 mm

- Provide $\begin{cases} 8\,\phi\ @\,150\ c/c & \text{(short span)} \\ 8\,\phi\ @\,190\ c/c & \text{(long span)} \end{cases}$

  The detailing is shown in Fig. 11.15.

**Check for deflection control**

- $p_{t,x}$ = 0.2465
- $f_s$ = 0.58 × 415 × 334/335 = 240 MPa
  $\Rightarrow$ modification factor $k_t$ = 1.55 (from Table 5.2 or Fig. 3 of Code)
- $\Rightarrow (l/d)_{max}$ = 20 × 1.55 = 31

  $(l/d)_{provided} = \dfrac{4136}{136}$ = 30.4 < 31 — Hence, OK.

**Corner Reinforcement**

As the slab is designed as 'torsionally restrained' at the corners, corner reinforcement has to be provided [vide Cl. D–1.8 of the Code] over a distance $l_x/5$ = 830 mm in both directions in meshes at top and bottom (four layers), each layer comprising 0.75 $A_{st,\,x}$.

$\Rightarrow$ spacing of 8 $\phi$ bars = $\dfrac{150}{0.75}$ = 200 c/c

Provide 8 $\phi$ @ 200 c/c both ways at top and bottom at each **corner** over an area 830 mm × 830 mm, i.e., 5 bars U-shaped in two directions, as shown in Fig. 11.15.

**EXAMPLE 11.3**

The floor slab system of a two-storeyed building is shown in Fig. 11.16. The slab system is supported on load-bearing masonry walls, 230 mm thick, as shown. Assuming a floor finish load of 1.0 kN/m$^2$ and a live load of 4.0 kN/m$^2$, design and detail the multipanel slab system. Use M 20 concrete and Fe 415 steel. Assume *mild* exposure conditions.

**SOLUTION**

- The slab system [Fig. 11.6] has two axes of symmetry passing through the centre, owing to which the number of different slab panels to be designed is four:

| panel | clear spans | boundary conditions | type [refer Fig. 11.6] |
|---|---|---|---|
| $S_1$ | 4.0 m × 5.0 m | one short edge discontinuous | Á |
| $S_2$ | 3.0 m × 5.0 m | one long edge discontinuous | Â |
| $S_3$ | 3.0 m × 5.0 m | two adjacent edges discont. | Ã |
| $S_4$ | 4.0 m × 5.0 m | all four edges continuous | À |

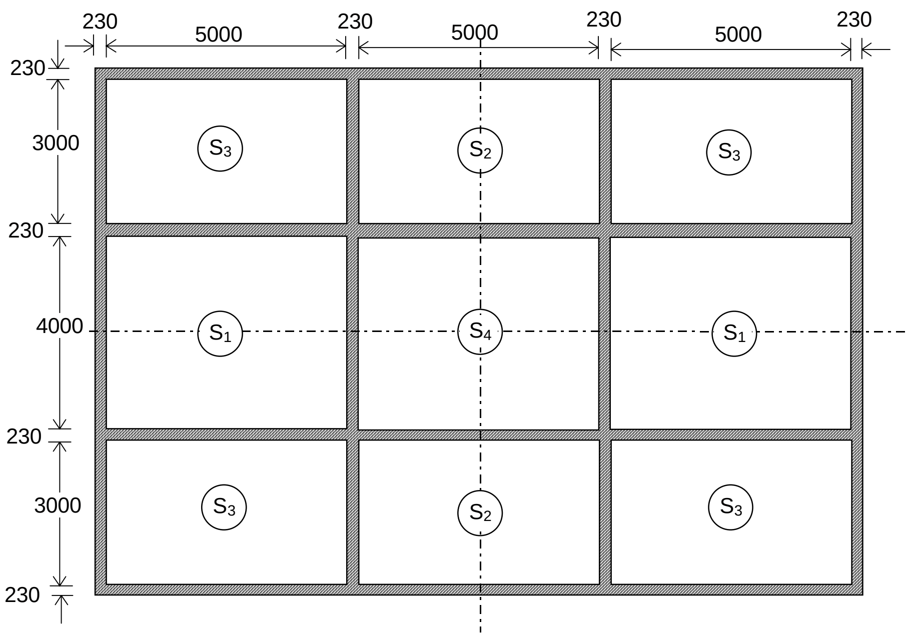



**Fig. 11.16** Floor slab system — Example 11.3

**Slab thicknesses**: based on deflection control criteria

- ***Panels $S_1$ and $S_4$***: 4.0 m × 5.0 m clear spans

  $l_x \approx 4000 + 150 = 4150$ mm

  $\Rightarrow d_x = \approx \dfrac{4150}{26 \times 1.5} = 107$ mm

  Assuming a clear cover of 20 mm and 8 $\phi$ bars,

  $D \approx 107 + 20 + 8/2 = 131$ mm

  Provide $\boldsymbol{D}$ **= 135 mm**

$\Rightarrow$ Effective depths $\begin{cases} d_x = 135 - 20 - 4 = 111 \text{ mm} \\ d_y = 111 - 8 \qquad = 103 \text{ mm} \end{cases}$

$\Rightarrow$ effective spans[†] $\begin{cases} l_x = 4000 + 111 = 4111 \text{ mm} \\ l_y = 5000 + 103 = 5103 \text{ mm} \end{cases}$

$\Rightarrow l_y/l_x$ = 5103/4111 = 1.241

- ***Panels $S_2$ and $S_3$***: 3.0 m × 5.0 m clear spans
  $\Rightarrow l_x \approx$ 3000 + 100 = 3100 mm
  $\Rightarrow d_x = \approx$ 3100/(23 × 1.5) = 90 mm [The continuity effect is only partial in the case of panel $S_3$. Hence, it is appropriate to consider a basic $l/d$ ratio which is an average of the simply supported and continuous cases, i.e., (20 + 26)/2 = 23.]
  Assuming a clear cover of 20 mm and 8 $\phi$ bars, $D \approx$ 90 + 20 + 8/2 = 114 mm
  Provide ***D* = 115 mm**

  $\Rightarrow$ effective depths $\begin{cases} d_x = 115 - 20 - 4 = 91 \text{ mm} \\ d_y = 91 - 8 \qquad = 83 \text{ mm} \end{cases}$

  $\Rightarrow$ effective spans $\begin{cases} l_x = 3000 + 91 = 3091 \text{ mm} \\ l_y = 5000 + 83 = 5083 \text{ mm} \end{cases}$

  $\Rightarrow l_y/l_x$ = 5083/3091 = 1.644

**Loading on slabs**

- self-weight of slab $\begin{cases} @\ 25\, kN/m^3 \times 0.135 m = 3.375 \text{ kN/m}^2 & \text{for } S_1, S_4 \\ @\ 25 \ \times \ 0.115 \ = 2.875 \text{ kN/m}^2 & \text{for } S_2, S_3 \end{cases}$

finishes @ 1.0 kN/$m^2$
live loads @ 4.0 kN/$m^2$

- $\Rightarrow$ Factored load $w_u$ = $\begin{cases} 1.5 \times (3.375 + 1.0 + 4.0) = 12.56 \text{ kN/m}^2 & \text{for } S_1, S_2 \\ 1.5 \times (2.875 + 1.0 + 4.0) = 11.81 \text{ kN/m}^2 & \text{for } S_3, S_4 \end{cases}$

**Design Moments** (using Code moment coefficients for 'restrained' slabs)

- Referring to Table 26 of the Code, or alternatively applying Eq. 11.10 – 11.12, the following moment coefficients $\alpha_x^+$, $\alpha_y^+$ (for 'positive' moments in the middle strip) in the different panels are obtained as:

$$\alpha_x^+, \alpha_y^+ = \begin{cases} 0.037, & 0.028 \quad \text{panel } S_1 \\ 0.056, & 0.028 \quad \text{"} \quad S_2 \\ 0.060, & 0.035 \quad \text{"} \quad S_3 \\ 0.034, & 0.024 \quad \text{"} \quad S_4 \end{cases}$$

[†] The width of the continuous support (230 mm) is less than 1/12 of the clear span (4000/12 = 333 mm); hence, the effective span is to be taken as (clear span + $d$) or (centre-to-centre distance between supports), whichever is less [refer Cl. 22.2 of Code].

- The coefficients for the 'negative' moments in the various continuous edge strips are easily obtained as $\alpha^{-} = 4/3\alpha^{+}$
- The corresponding design (factored) moments $M_u = \alpha\, w_u l_x^2$ in the various panels are accordingly obtained as follows:

| panel | | $S_1$ | $S_2$ | $S_3$ | $S_4$ |
|---|---|---|---|---|---|
| load $w_u$ (kN/m$^2$) | | 12.56 | 11.81 | 11.81 | 12.56 |
| span $l_x$ (m) | | 4.111 | 3.091 | 3.091 | 4.111 |
| short span moments | $M_{ux}^{+}$ | 7.85 | 6.32 | 6.77 | 7.22 |
| | $M_{ux}^{-}$ | 10.47 | 8.43 | 9.03 | 9.62 |
| long span moments (kNm/m) | $M_{uy}^{+}$ | 5.94 | 3.16 | 3.95 | 5.09 |
| | $M_{uy}^{-}$ | 7.92 | 4.21 | 5.27 | 6.79 |

**Design 'negative' moments at common supports**

- The 'negative' moments at the continuous edges, as obtained from the Code coefficients, are unequal — as shown in Fig. 11.17(a). In all such cases, the design 'negative' moment is taken as the *larger* of the two values obtained from either sides of the support. The design moments so obtained are shown in Fig. 11.17.

**Flexural reinforcement requirements**

- $\dfrac{(p_t)_{reqd}}{100} \equiv \dfrac{(A_{st})_{reqd}}{bd} = \dfrac{f_{ck}}{2f_y}\left[1-\sqrt{1-4.598M_u/(f_{ck}bd^2)}\right]$

  where $f_{ck}$ = 20 MPa, $f_y$ = 415 MPa, $b$ = 1000 mm

**Panel $S_1$**: $d_x$ = 111 mm, $d_y$ = 103 mm

- *short span*: $M_{ux}^{+}$ = 7.85 kNm/m ⇒ $(A_{st})_{reqd}$ = 203.6 mm$^2$/m

  ⇒ reqd spacing of 8 φ bars = 247 mm

  $M_{ux}^{-}$ = 10.47 kNm/m ⇒ $(A_{st})_{reqd}$ = 275.5 mm$^2$/m

  ⇒ reqd spacing of 8 φ bars = 183 mm
- *long span*: $M_{uy}^{+}$ = 5.94 kNm/m ⇒ $(A_{st})_{reqd}$ = 165.2 mm$^2$/m

  ⇒ reqd spacing of 8 φ bars = 304 mm— to be limited to 300 mm

  $M_{uy}^{-}$ = 7.92 kNm/m ⇒ $(A_{st})_{reqd}$ = 223 mm$^2$/m

  ⇒ reqd spacing of 8 φ bars = 225 mm

**Panel $S_2$**: $d_x$ = 91 mm, $d_y$ = 83 mm

- *short span*: $M_{ux}^{+}$ = 6.32 kNm/m ⇒ $(A_{st})_{reqd}$ = 201.6 mm$^2$/m

  ⇒ reqd spacing of 8 φ bars = 249 mm

  $M_{ux}^{-}$ = 9.62 kNm/m ⇒ $(A_{st})_{reqd}$ = 315.5 mm$^2$/m

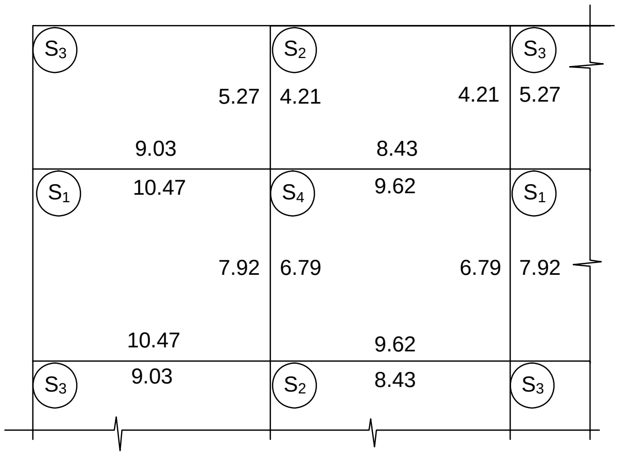
  ⇒ reqd spacing of 8 φ bars = 159 mm

- *long span*: $M_{uy}^{+} = 3.16$ kNm/m $\Rightarrow (A_{st})_{reqd} = 108.4$ mm$^2$/m

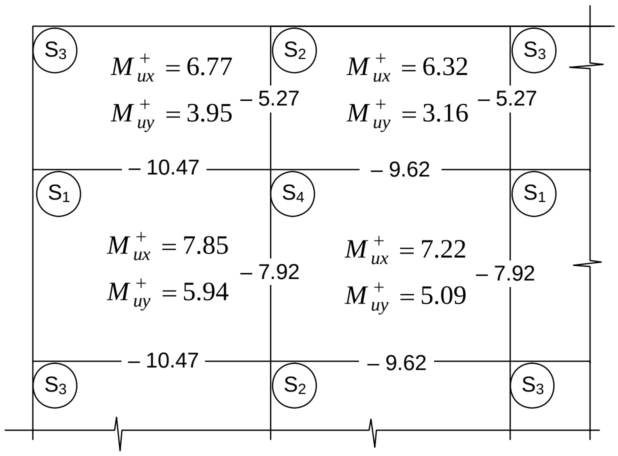
  $\Rightarrow$ reqd spacing of 8 $\phi$ bars = 464 mm — to be limited to $3d_y$ = 249 mm

  $M_{uy}^{-} = 5.27$ kNm/m $\Rightarrow (A_{st})_{reqd} = 184.4$ mm$^2$/m

  $\Rightarrow$ reqd spacing of 8 $\phi$ bars = 272 mm — to be limited to $3d_y$ = 249 mm




**(a)** 'negative' moments (kNm/m) at continuous edges for each panel




**(b)** final design moments (kNm/m)

**Fig. 11.17** Design moments — Example 11.3

**Panel $S_3$**: $d_x$ = 91 mm, $d_y$ = 83 mm

- *short span*: $M_{ux}^{+}$ = 6.77 kNm/m ⇒ $(A_{st})_{reqd}$ = 216.8 mm$^2$/m
  ⇒ reqd spacing of 8 ϕ bars = 232 mm
  $M_{ux}^{-}$ = 10.47 kNm/m ⇒ $(A_{st})_{reqd}$ = 346 mm$^2$/m
  ⇒ reqd spacing of 8 ϕ bars = 145 mm
- *long span*: $M_{uy}^{+}$ = 3.95 kNm/m ⇒ $(A_{st})_{reqd}$ = 136.5 mm$^2$/m
  ⇒ reqd spacing of 8 ϕ bars = 368 mm — to be limited to $3d_y$ = 249 mm
  $M_{uy}^{-}$ =5.27 kNm/m ⇒ $(A_{st})_{reqd}$ = 184.4 mm$^2$/m
  ⇒ reqd spacing of 8 ϕ bars = 272 mm — to be limited to $3d_y$ = 249 mm

**Panel $S_4$**: $d_x$ = 111 mm, $d_y$ = 103 mm

- *short span*: $M_{ux}^{+}$ = 7.22 kNm/m ⇒ $(A_{st})_{reqd}$ = 186.7 mm$^2$/m
  ⇒ reqd spacing of 8 ϕ bars = 269 mm
  $M_{ux}^{-}$ = 9.62 kNm/m ⇒ $(A_{st})_{reqd}$ = 251.9 mm$^2$/m
  ⇒ reqd spacing of 8 ϕ bars = 199 mm
- *long span*: $M_{uy}^{+}$ = 5.09 kNm/m ⇒ $(A_{st})_{reqd}$ = 140.9 mm$^2$/m
  ⇒ reqd spacing of 8 ϕ bars = 357 mm — to be limited to $3d_y$ or 300 mm
  $M_{uy}^{-}$ = 7.92 kNm/m ⇒ $(A_{st})_{reqd}$ = 223 mm$^2$/m
  ⇒ reqd spacing of 8 ϕ bars = 225 mm

## Detailing of Reinforcement

- Based on the requirements of reinforcement calculated above, the detailing of flexural reinforcement in the various middle strips and edge strips is shown in Fig. 11.18. For practical convenience, only two different bar spacings (220 mm and 150 mm) are adopted (except for slab S3 for $M_{ux}^{-}$, for which a spacing of 145 mm is used). The detailing is in conformity with the requirements specified in Cl. D–1 of the Code, and satisfies the requirements of minimum spacing.
- Nominal top steel (50 percent of bottom steel) is provided at the discontinuous edges — against possible 'negative' moments due to partial fixity.

***Torsional reinforcement at corners***

- As required by the Code, the reinforcement is provided in the form of a mesh, extending over a distance of $0.2l_x$ beyond the face of the supporting wall. The bars are provided as U-shaped (i.e., with the mesh extending over top and bottom).
- At the extreme corner of the slab system, required spacing of 8 mm ϕ bars = 4/3 × 220 = 293 mm — over a distance of 0.2 × 3103 = 620 mm.

⇒ Provide 3 nos 8 mm $\phi$ U-shaped bars in both directions at the extreme corner of the slab system — over a distance 620 mm × 620 mm.

- The area of steel required at the other corners, where torsional reinforcement is required, is half the above requirement — however, provide 3 nos 8 mm $\phi$ at top and bottom. The size of the mesh is 620 mm × 620 mm at the junction of $S_2$ and $S_3$, where one edge of the corner is discontinuous.
- The size of the mesh is 0.2 × 4111 ≈ 820 mm at the junction of $S_1$ and $S_3$. Provide 3 nos 8 mm ϕ bars at top and bottom.

  The detailing is shown in Fig. 11.18.

  [*Note*: The slab panels satisfy the limiting *l* /*d* ratios for deflection control; this may be verified.]

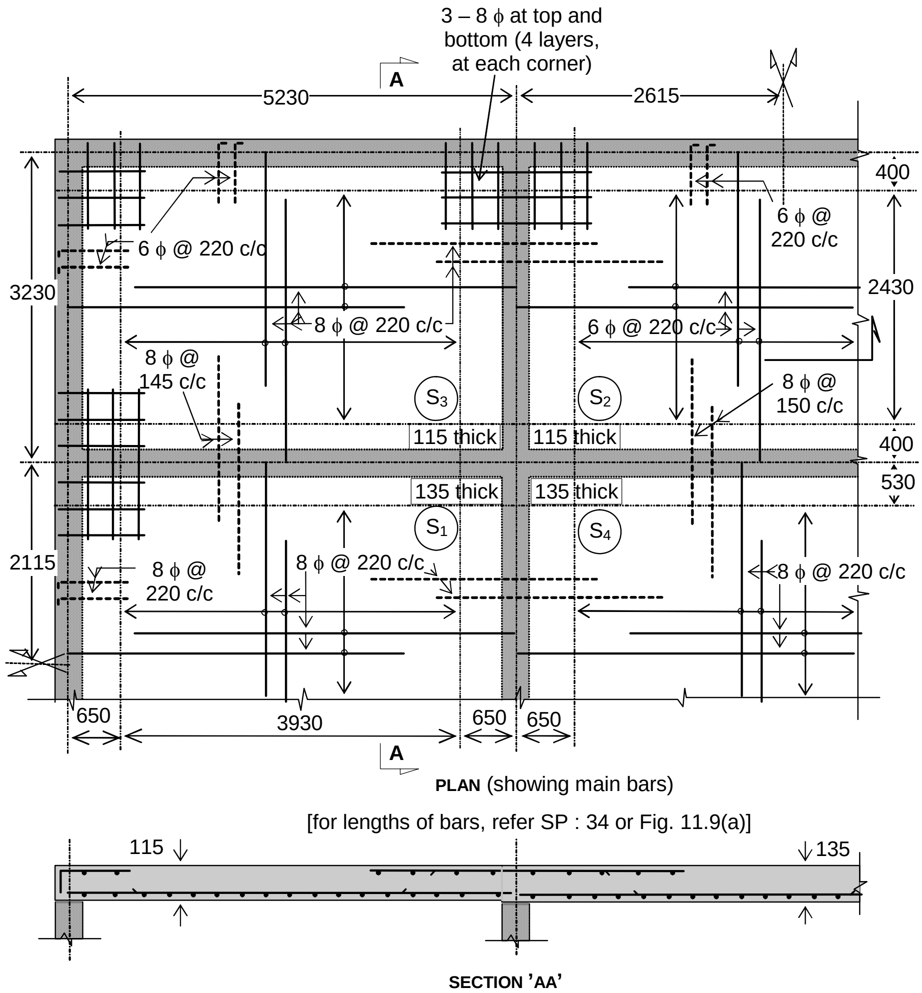



**Fig. 11.18** Detailing of multi panel slab system — Example 11.3

### 11.2.6 Design of Circular, Triangular and Other Slabs

Non-rectangular slabs, with shapes such as circular, triangular and trapezoidal, are sometimes encountered in structural design practice. Rectangular slabs, supported on three edges or two adjacent edges, are also met with in practice. Classical solutions, based on the elastic theory, are available in the case of rectangular and circular plates, uniformly loaded [Ref. 11.1]. Nonrectangular slabs are sometimes designed by considering the largest circle that can be inscribed within the boundaries of the slabs, and treating these slabs as equivalent circular slabs [Ref. 11.16]. More accurate analyses of stresses in nonrectangular and other slabs are obtainable from the computer-based finite difference method [Ref. 11.6] and finite element method [Ref. 11.7]. Yield line analyses provide simple and useful solutions for slabs of all possible shapes and boundary conditions [Ref. 11.8 – 11.10].

Some of the standard solutions for a few typical cases are given here (without derivation). In all these cases, slabs are assumed to be subjected to uniformly distributed loads *w* (per unit area).

***Circular Slabs, Simply Supported*** [Fig. 11.19]

**Elastic theory**

- Moment in radial direction $M_r = \dfrac{w}{16}[(3+\nu)(a^2 - r^2)]$ (11.15a)
- Moment in circumferential direction $M_\theta = \dfrac{w}{16}[a^2(3+\nu) - r^2(1+3\nu)]$ (11.15b)

  where $a \equiv$ radius of the circular slab;
  $r \equiv$ radius where moment is determined ($0 \le r \le a$);
  $\nu \equiv$ Poisson's ratio — may be taken as zero in the case of reinforced concrete.

- Maximum moments (at centre): $M_{r,\max} = M_{\theta,\max} = 3wa^2/16$ (11.15c)
- The two-way reinforcement may be provided by means of an orthogonal mesh with isotropic reinforcement [Fig. 11.19(b)]; Providing radial plus circumferential reinforcement [Fig. 11.19(c)] is also (theoretically) a solution; however, this is not convenient in practice, as the radial bars need to be specially welded at the centre.

**Yield line theory** (assuming isotropic reinforcement)

- Collapse load $w_u = 6M_{uR}/a^2$
  $\Rightarrow$ design moment $M_u = w_u a^2/6$ (11.16)
  (which is less than the elastic theory solution: $M_u = w_u a^2/5.333$)

***Circular Slabs, Fixed at the Edges***

**Elastic theory**

- $M_r = \dfrac{w}{16}[a^2(1+\nu) - r^2(3+\nu)]$ (11.17a)

- $M_\theta = \dfrac{w}{16}[a^2(1+\nu) - r^2(1+3\nu)]$ (11.17b)

  where the notations are exactly as mentioned earlier [Fig. 11.19]

  *Design moments* (*assuming* $\upsilon = 0$):

$$M^+_{r,\max} = M^+_{\theta,\max} = wa^2/16 \text{ (‘positive’ at centre)} \qquad (11.17c)$$

$$M^-_{r,\max} = (-)\, wa^2/8\,;\; M^-_{\theta,\max} = 0 \text{ (at edges)} \qquad (11.17d)$$

  [Top steel is required near the supports, in the radial direction].

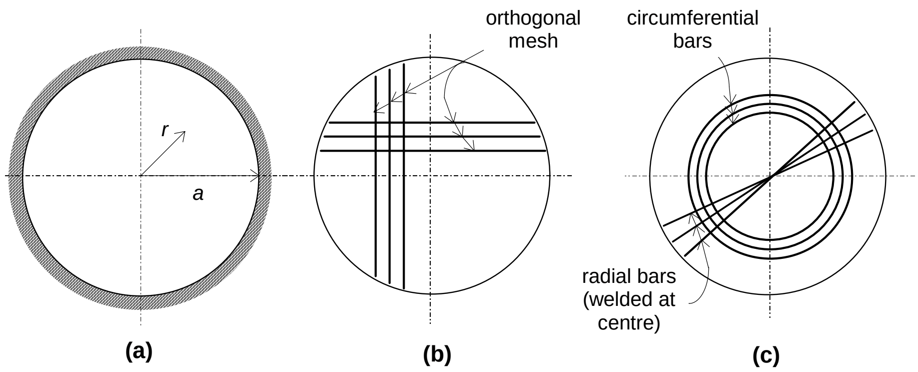



**Fig. 11.19** Circular slabs, simply supported and isotropically reinforced

**Yield line theory** (assuming isotropic reinforcement)

- Collapse load $w_u = 6(M^+_{uR} + M^-_{uR})/a^2$

  As the elastic moment at the support is twice that at the centre, it is desirable to provide $M^+_{uR} : M^-_{uR}$ in the ratio 1 : 2

  Accordingly, for design, $M_u^+ = (+)\, wa^2/18$ (11.18a)

$$M_u^- = (-)\, wa^2/9 \qquad (11.18b)$$

***Equilateral Triangular Slabs, Simply Supported*** [Fig. 11.20(a)]

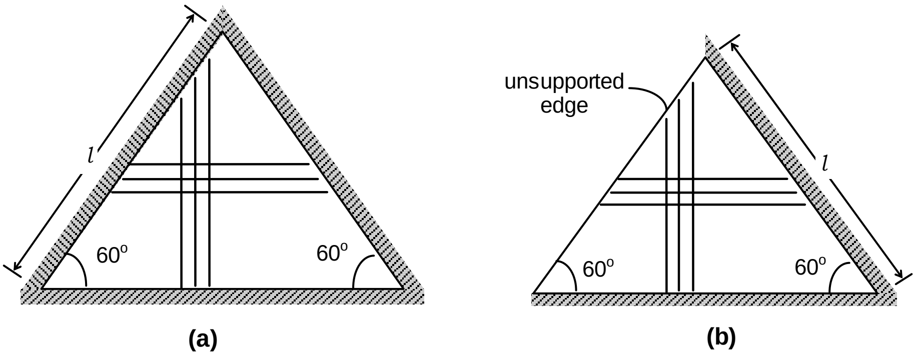



**Fig. 11.20** Equilateral triangular slabs, isotropically reinforced

**Yield line theory** (assuming isotropic reinforcement)

- Collapse load $w_u = 72\ M_{uR}/l^2$
  where $l$ is the length of one side
  $\Rightarrow$ design moment (at midspan) $M_u = w_u l^2/72$ (11.19)

### *Equilateral Triangular Slabs, Two Edges Simply Supported and One Edge Free* [Fig. 11.20(b)]

**Yield line theory** (assuming isotropic reinforcement)

- Collapse load $w_u = 24\ M_{uR}/l^2$
  where $l$ is the length of one side
  $\Rightarrow$ design moment (at midspan) $M_u = w_u l^2/24$ (11.20)

### *Rectangular Slabs, Three Edges Simply Supported and One Edge Free* [Fig. 11.21(a)]

**Yield line theory**

Let $\mu \equiv M_{uR,Y}/M_{uR,X}$ and $R \equiv l_Y/l_X$ ; the $X$- and $Y$- directions are as indicated in Fig. 11.21(a).

- Design moments:

$$M_{uX} = \begin{cases} \dfrac{w_u R^2 l_X^2}{24\mu}\left\{\sqrt{4+9\mu/R^2}-2\right\} \\ \dfrac{w_u l_X^2}{24}\left\{\sqrt{3+\dfrac{\mu}{4R^2}}-\dfrac{\sqrt{\mu}}{2R}\right\}^2 \end{cases} \text{(whichever is greater)} \qquad (11.21)$$

- By suitably selecting $\mu$ (which can even be taken as unity), design moments $M_{uX}$ and $M_{uY} = \mu\ M_{uX}$ can be determined.

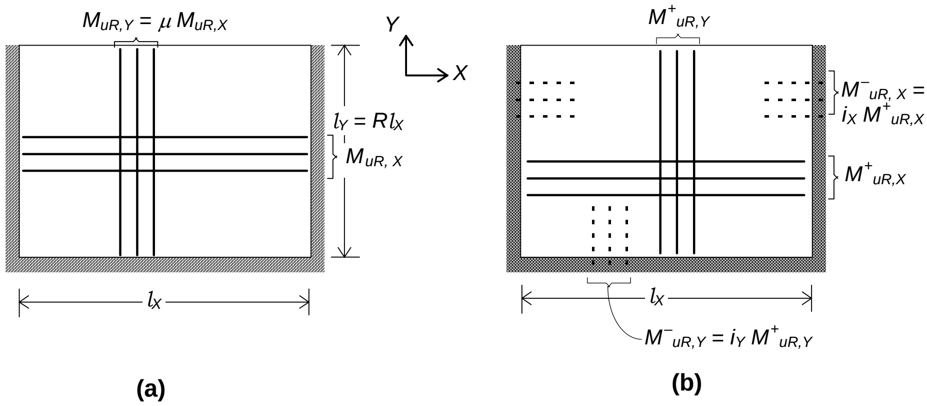



**Fig. 11.21** Rectangular slabs, supported on three edges

### ***Rectangular Slabs, Three Edges Fixed and One Edge Free*** [Fig. 11.21(b)]

**Yield line theory**

Let $\mu \equiv M_{uR,Y} / M_{uR,X}$

$i_X \equiv M^{-}_{uR,X} / M^{+}_{uR,X}$

$i_Y \equiv M^{-}_{uR,Y} / M^{+}_{uR,Y}$

- Design moments:
$$M^{+}_{uY} = \begin{cases} \dfrac{w_u l_Y^2 (3-\alpha_1)}{6(i_Y + \alpha_1)} \\ \dfrac{w_u l_Y^2 \ \alpha_2^2}{6(1+i_Y)} \end{cases} \text{(whichever is greater)} \qquad (11.22)$$

where

$$\alpha_1 \equiv 4\left(\sqrt{4+3K_1} - 2\right) \Big/ K_1$$

$$K_1 \equiv \frac{\mu(3+i_Y)}{R^2(1+i_X)}$$

$$\alpha_2 \equiv \left(\sqrt{1+3K_2} - 1\right) \Big/ K_2$$

$$K_2 \equiv \frac{4(1+i_X)}{\mu(1+i_Y)}$$

- By suitably selecting $\mu$, $i_X$ and $i_Y$, all the design moments can be determined. This is illustrated in Example 11.5.

### ***Rectangular Slabs, Two Adjacent Edges Fixed and the Other Two Free*** [Fig. 11.22]

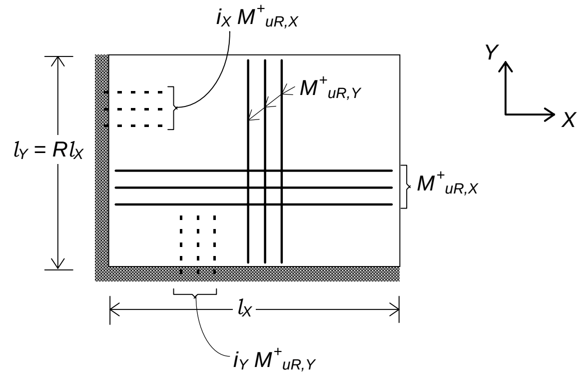



**Fig. 11.22** Rectangular slab, two edges fixed and the other two free

**Yield line theory**

- Using the same notations as in the previous case [refer Fig. 11.22], design moment:

$$M^+_{uX} = \begin{cases} \dfrac{w_u l_X^2}{6} \Big/ \left\{ \dfrac{1+i_X}{\alpha_3^2} + \dfrac{\mu}{R^2} \right\} \\ \dfrac{w_u l_X^2}{6} \Big/ \left\{ i_X + i_Y \mu / R^2 \right\} \end{cases} \text{(whichever is greater)} \qquad (11.23)$$

where $\alpha_3 \equiv \left(\sqrt{1+3K_1} - 1\right) \Big/ K_1$

- By suitably selecting $\mu$, $i_X$ and $i_Y$, all the design moments can be determined.

**EXAMPLE 11.4**

Design a circular slab of 3.5 m diameter to cover an underground sump. The slab is simply supported at the periphery by a wall 200 mm thick. Assume a finish load of 1.0 kN/m$^2$ and live loads of 4.0 kN/m$^2$. Use M 20 concrete and Fe 415 steel. Assume *mild* exposure conditions.

**SOLUTION**

- Clear span = 3500 – (200 × 2) = 3100 mm
  Assuming a slab thickness of 100 mm, with 20 mm clear cover (*mild* exposure condition) and 8 mm $\phi$ bars (in an orthogonal mesh),
  average effective depth $d$ = 100 – 20 – 8 = 72 mm
  $\Rightarrow$ effective span (diameter) = 3100 + 72 = 3172 mm
  $\Rightarrow$ effective radius $a$ = 3172/2 = 1586 mm
- *Loads*: (i) self weight @ 25 kN/m$^3$ × 0.10 m = 2.5 kN/m$^2$
  (ii) finishes = 1.0 ”
  (iii) live loads = 4.0 ”
  $w$ = 7.5 kN/m$^2$

$\Rightarrow$ Factored load $w_u = 7.5 \times 1.5 = 11.25$ kN/m$^2$

- *Design moments* (assuming yield line theory with isotopic reinforcement)

  $M_u = w_u a^2/6 = 11.25 \times 1.586^2/6 = 4.72$ kNm/m

  $$\Rightarrow R \equiv \frac{M_u}{bd^2} = \frac{4.72\times10^6}{10^3\times72^2} = 0.910 \text{ MPa}$$

  $$\frac{p_t}{100} \equiv \frac{20}{2\times415}\left[1-\sqrt{1-(4.598\times0.910/20)}\right] = 0.267\times10^{-2}$$

  $\Rightarrow A_{st,\,reqd} = (0.267 \times 10^{-2}) \times 10^3 \times 72 = 192$ mm$^2$/m

  $\Rightarrow$ required spacing of 8 mm $\phi$ bar $= 50.3 \times 10^3/192$
  $= 262$ mm

  Maximum spacing allowed $= 3d = 3 \times 72 = 216$ mm

- Provide **8 mm $\phi$ 210 c/c both ways** at bottom, as shown in Fig. 11.23.

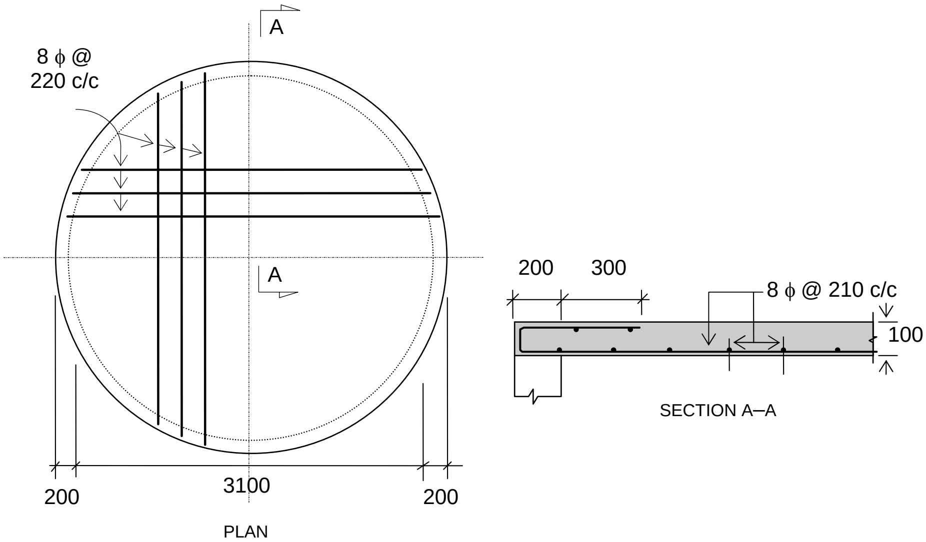



**Fig. 11.23** Circular slab – Example 11.4

**EXAMPLE 11.5**

Determine the design moments in a square slab (4 m × 4 m), with three edges continuous and one edge free, subject to a uniformly distributed factored load $w_u = 10.0$ kN/m$^2$. Assume the slab to be isotropically reinforced. Also assume that the 'negative' moment capacity at the continuous support to be equal to that at midspan in either direction.

**SOLUTION**

- Applying the yield line theory solution [Eq. 11.22] with $l_X = l_Y = 4.0$ m, $R = 1$, $\mu = i_X = i_Y = 1$,

$$K_1 = \frac{\mu(3+i_Y)}{R^2(1+i_X)} = 2$$

$$K_2 = \frac{4(1+i_X)}{\mu(1+i_Y)} = 4$$

$$\Rightarrow \alpha_1 = 4\left(\sqrt{4+3K_1}-2\right)\Big/K_1 = 2.325$$

$$\alpha_2 = \left(\sqrt{1+3K_2}-1\right)\Big/K_2 = 0.651$$

$$\Rightarrow M_{uY}^{+} = \begin{cases} \dfrac{w_u l_Y^2 (3-\alpha_1)}{6(i_Y+\alpha_1)} & = 0.0338\ w_u l_Y^2 \\ \dfrac{w_u l_Y^2\ \alpha_2^2}{6(1+i_Y)} & = 0.0353\ w_u l_Y^2 \quad (greater) \end{cases}$$

$$\Rightarrow M_{uY}^{+} = M_{uX}^{+} = M_{uY}^{-} = M_{uX}^{-} = 0.0353 \times 10.0 \times 4.0^2$$
$$= 5.65 \text{ kNm/m}$$

### 11.2.7 Two-Way Slabs Subjected to Concentrated Loads

Two-way slabs subjected to concentrated loads (such as wheel loads) are frequently encountered in the design of bridge slabs. Analyses by elastic theory have been developed by Westergaard and Pigeaud. However, the theory is mathematically complex, and recourse has to be made to design charts developed for this purpose. The reader may refer to Handbooks such as Ref. 11.16 for such design charts.

## 11.3 DESIGN OF BEAM-SUPPORTED TWO-WAY SLABS

### 11.3.1 Behaviour of Beam-Supported Slabs

Slabs supported on beams [refer Fig. 1.8] behave differently, when compared to slabs supported on walls, because of the influence of the following three factors:

1. deflections in the supporting beams;
2. torsion in the supporting beams;
3. displacements (primarily rotations) in the supporting columns.

Deflections in the supporting beams become significant when they have relatively large span/depth ratios and their end connections are relatively flexible. These deflections, caused by relatively low flexural stiffnesses of the beams, get enhanced with time, due to the long-term effects of creep and shrinkage. The support flexibility significantly influences the magnitudes and distributions of bending moments and shear forces in slabs, with the result that the ‘moment coefficients’ applicable for wall-supported slabs [refer Section 11.2] are not applicable for slabs supported on **flexible** beams.

The monolithic construction of beam and slab results in the twisting of the beam along with the bending of the slab. This results in torsion in the beam, the magnitude of the torsion being equal to the unbalanced moment (if any) in the slabs at the slab-beam junction. The magnitude of the torsion depends, among other factors, on the torsional stiffness of the beam. In general, it is observed that with increase in beam torsional stiffness, there is a consequent decrease in 'positive' moments in interior slab panels, but a significant increase in the 'negative' moment at the discontinuous edge of the exterior panel.

The columns also influence the behaviour of the beam-supported slabs, because they form part of an integral slab-beam-column system which can sway and bend in a variety of ways.

### 11.3.2 Use of Code Moment Coefficients for Slabs Supported on Stiff Beams

When the supporting beams (and columns) are relatively rigid, the slabs may be *assumed* to be supported on *undeflecting supports*. The torsion in the beams may also be neglected for convenience. This results in conservative estimates of moments in the slabs, except at the discontinuous edge of the exterior panel (as explained earlier). The latter problem may be resolved by providing suitable top steel at the discontinuous edge of the slab — as required by the detailing requirements of the Code (to account for 'negative' moments due to partial fixity). Furthermore, the minimum stirrup requirements prescribed for beams by the Code provide some measure of torsional strength to the supporting beams. With these assumptions, slabs supported on stiff beams may be designed using the Code moment coefficients of wall-supported slabs [refer Section 11.2].

The Code does not provide any specific recommendations for the procedure for designing beam-supported two-way slabs. Generally, in Indian practice, the design is done by treating continuous beam-supported slabs as identical to continuous wall-supported slabs, for which moment coefficients are readily available in the Code (Table 12 of the Code for one-way slabs, and Table 26 of the Code for two-way slabs). However, this is justifiable only if the supporting beams are adequately stiff.

The Code limitation on beam deflection (Cl. 23.2) does provide some indirect control on the flexural stiffness of the supporting beam. However, the designer must take precautions to ensure that the supporting beams are adequately stiff, in order to justify the application of the Code moment coefficients for the slab design.

A simple guideline for selecting the overall depth of a beam $D_b$ (to ensure that it is 'adequately stiff'), based on Swedish regulations is given in Ref. 11.17:

$$D_b \geq \begin{cases} 2.5D_s & for \;\; r \leq 1.5 \\ 2.5rD_s & for \;\; r > 1.5 \end{cases} \tag{11.24a}$$

where $D_s$ is the thickness of the slab and $r \equiv l_y/l_x$.

Thus, for example, a 100 mm thick slab requires the supporting beams to be at least 250 mm deep for a square panel, and at least 500 mm deep for a rectangular panel with $l_y/l_x = 2.0$.

An alternative guideline, given by the Canadian code [Ref. 11.18], which explicitly accounts for the width of the beam, $b$ and its clear span $l$, is as follows:

$$D_b \geq D_s (2\, l/b)^{1/3} = 1.26 D_s (l/b)^{1/3} \tag{11.24b}$$

Considering a square panel with clear spans of 2.8 m and a slab 100 mm thick, the required beam depth for a width of 250 mm works out to 282 mm (which is slightly higher than 250 mm, obtained earlier). Furthermore, if the panel is rectangular, with clear spans 2.8 m × 5.6 m, and if the slab thickness remains as 100 mm, the required depth of the long span beam (assuming $b$ = 250 mm) is 355 mm, while that of the short span beam is 282 mm.

By treating the beam supports as wall supports, the slab system can be effectively isolated from the integral slab-beam-column system, for design purposes. The reactions due to the gravity loads on the slab are transferred to the supporting stiff beams, as explained in Section 9.4 [refer Fig. 9.5]. If the supporting beams are 'secondary' [refer Fig. 1.10(c)], then these 'secondary' beams may, in turn, be isolated and assumed to be supported on the 'primary' beams, for design purposes. The primary beams and columns constitute a continuous skeletal framework, which can be separated into plane frames (longitudinal and transverse), and can be designed to resist gravity loads as well as lateral loads [refer Chapter 9].

### 11.3.3 Slabs Supported on Flexible Beams — Code Limitations

As an alternative to the idealised assumption as continuous slabs supported on walls, the Code (Cl. 24.3) suggests that slabs monolithically connected with beams may be analysed *as members of a continuous framework with the supports, taking into account the stiffness of such supports*. This suggestion becomes significant in situations where the supporting beams are not adequately stiff.

It may be noted that the ACI Code had made such a treatment mandatory for *all* (flexible) beam-supported two-way slabs, as far back as in 1971, and had altogether dispensed with the use of moment coefficients for such slabs. Another significant change introduced in the 1971 version of the ACI Code was the unification of the design methods for all slabs supported on columns — with and without beams, including flat slabs. This is considered to be an advancement as it ensures that all types of slabs have approximately the same *reliability* (or risk of failure[†]) [Ref. 11.6]. Some other codes, such as the Canadian code [Ref. 11.18], have also incorporated these changes, but retain the moment coefficient method as an alternative for slabs supported on *walls or stiff beams*.

In the IS Code, such a procedure, based on the concept of 'equivalent frame', is prescribed for *flat slabs*. This method [Cl. 31 of the Code] follows closely the extensive research undertaken in this area, since 1956, at the University of Illinois, USA [Ref. 11.11]. However, unlike the ACI and Canadian codes, the IS Code is yet to extend the 'equivalent frame' concept of analysis to beam-supported slabs.

The problem of designing slabs on flexible beams has therefore not yet been satisfactorily addressed by the IS Code. This information is also not generally available in standard Indian books on reinforced concrete design.

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[†] It is reported that the application of the 'moment coefficient' procedure to beam-supported slabs results in more conservative designs, with the result that such slabs turn out to be significantly stronger than beamless (flat) slabs, given the same gravity loads and material grades [Ref. 11.6].

In the sections to follow, procedures for analysis and design of beam-supported slabs are described — in line with the by-now-well-established ACI concept of unified procedures for all slabs, supported on columns, with or without beams.

### 11.3.4 The 'Equivalent Frame' Concept

As mentioned in Section 11.1.3, in the case of beam-supported two-way slabs, 100 percent of the gravity loads on the slabs are transmitted to the supporting columns, in both longitudinal and transverse directions (see Fig. 11.3(b)). The mechanism of load transfer from slab to columns is achieved by flexure, shear and torsion in the various elements. The slab-beam-column system behaves integrally as a three-dimensional system, with the involvement of all the floors of the building, to resist not only gravity loads, but also lateral loads. However, a rigorous three-dimensional analysis of the structure is complex, and not warranted except in very exceptional structures.

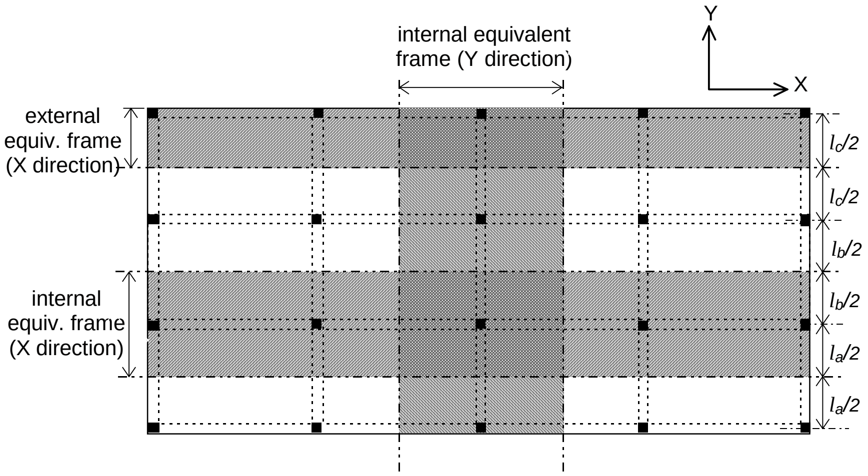
Conventionally, when stiff beams are provided along column lines, the slab design is separated from the design of beams (and columns), as in the case of wall-supported slabs. The remaining part of the structure, comprising a three-dimensional skeletal framework of beams and columns, is separated for convenience, into (two-dimensional) plane frames in the longitudinal and transverse directions of the building. As the integrally cast slab also contributes to the strength and stiffness of the beams, the beam members are considered as flanged beams (T-beams, L-beams), with *portions of the slab* acting as the flanges of these beams; this concept was explained in Chapter 9. However, when the beams are flexible or absent, it is not appropriate to separate the slab design from the beam design.

In using the concept of a plane frame comprising columns and slab-beam members at various floor levels, fundamentally, the slab-beam member should consist of the *entire floor member* (slab and beam, if any) *tributary* to a line of columns forming the frame. This is illustrated in Fig. 11.24(a) and (b), which show how a building structure may be considered as a series of 'equivalent (plane) frames', each consisting of a row of columns and the portion of the *floor system tributary to it*. The part of the floor bound by the panel centrelines, on either side of the columns, forms the slab-beam member in this plane frame. Such 'equivalent frames' must be considered in both longitudinal and transverse directions, to ensure that load transfer takes place in both directions [Fig. 11.24(a)].

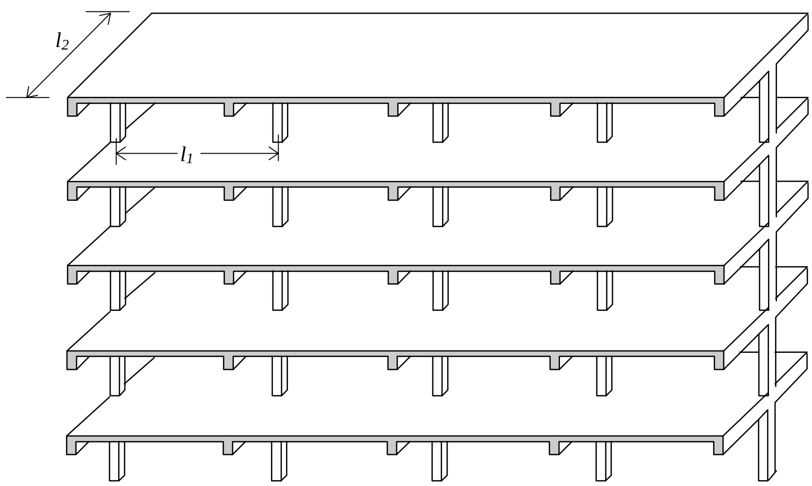
The equivalent frames can now be analysed under both gravity loads and lateral loads using the procedures mentioned in Chapter 9. The primary difference between the frame in Fig. 9.1(b) and the one in Fig. 11.24(b) lies in the width of the slab-beam member and the nature of its connections with the columns. Whereas in the conventional skeletal frame, the full beam is integral with the column, and the rotational restraint offered by the column at the joint is for the entire beam (with both beam and column undergoing the same rotation at the joint), in the 'equivalent frame', the column connection is only over part of the slab-beam member width, and hence the flexural restraint offered by the column to the slab-beam member is only partial. Thus, the rotation of the slab-beam member along a transverse section at the column support will vary, and will be equal to the column rotation only in the immediate vicinity of the column. This, in turn, results in torsion in the portion of the slab transverse to the span and passing through the column (i.e., a cross-beam running over the column).

In the elastic analysis of the plane frame in Fig. 9.1(b), it was shown (in Section 9.3) that several approximations can be made, subject to certain limitations. Similar approximations can also be made in the present case.

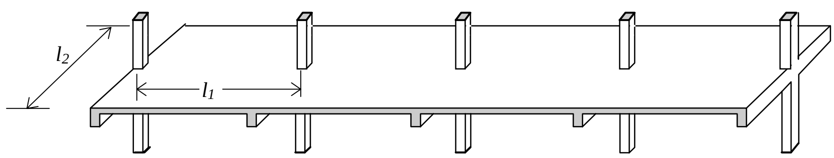
For example, for the purpose of gravity load analysis, it is possible to simplify the analysis by applying the concept of *substitute frames*. Accordingly, instead of analysing the full 'equivalent frame' [Fig. 11.24(b)], it suffices to analyse separate *partial* frames [Fig. 11.24(c)], comprising each floor (or roof), along with the columns located immediately above and below. The columns are assumed to be fixed at their far ends [refer Cl. 24.3.1 of the Code]. Such substitute frame analysis is permissible provided the frame geometry (and loading) is relatively symmetrical, so that no significant *sway* occurs in the actual frame.




**(a)** Floor Plan — definition of *equivalent frame*




**(b)** typical internal equivalent frame (X - direction)




**(c)** substitute internal equivalent frame (X - direction)

**Fig. 11.24** The ‘equivalent frame’ concept

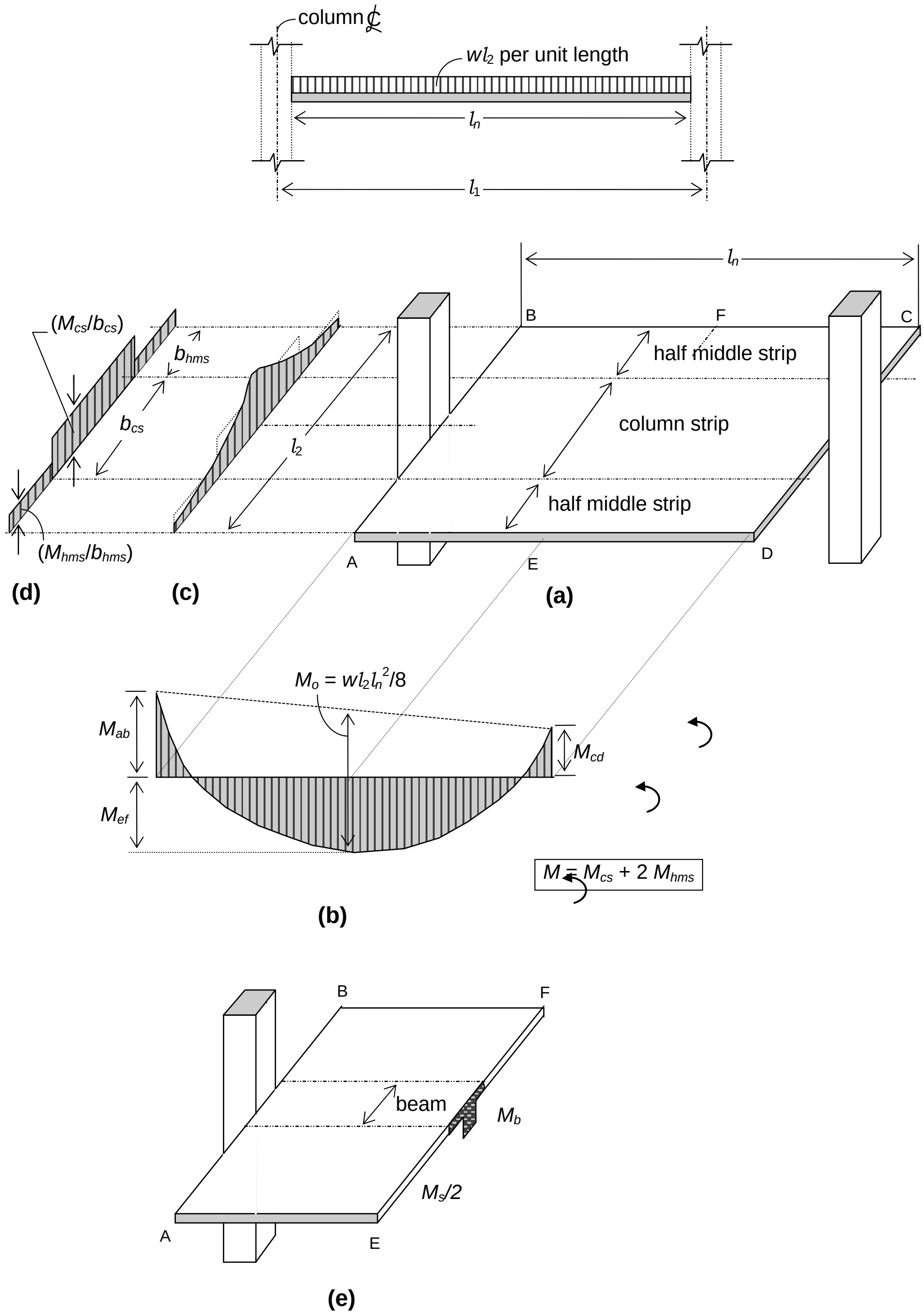



**Fig. 11.25** Moment variations in a two-way slab panel

### *Variations of Moments in a Two-Way Slab Panel*

Although the horizontal member in the 'equivalent partial frame' in Fig. 11.24(c) is modelled as a very wide beam (i.e., slab with or without beam along the column line), it is actually supported on a very limited width. Hence, the outer portions of the member are less stiff than the part along the column line, and the distribution of moment across the width of the member is not uniform — unlike the beam in the conventional plane frame [Fig. 9.1(b)]. The probable variations of moments in a typical panel of the 'equivalent frame' are shown in Fig. 11.25.

The variation of bending moment in the floor member along the span, under gravity loads is sketched in Fig. 11.25(b). Such a variation — with 'negative moments' near the supports and 'positive moments' in the neighbourhood of the midspan — is typical in any beam subject to uniformly distributed loads. In Fig. 11.25(b), $M_{ab}$ denotes the total 'negative' moment in the slab-beam member along the support line AB (extending over the full width of the panel), and $M_{ef}$ denotes the total 'positive moment' along the middle line EF of the panel.

These moments are distributed across the width of the panel nonuniformly, as sketched in Fig. 11.25(c). The actual variation along AB or EF (marked by the solid line in Fig. 11.25(c) depends on several factors, such as the span ratio $l_2/l_1$, relative stiffness of beam (if any) along column lines, torsional stiffness of transverse beams (if any), etc. The actual moment variation is very difficult to predict exactly, and hence suitable approximations need to be made. This is generally achieved by dividing the slab panel into a *column strip* (along the column line) and two *half-middle strips* [Fig. 11.25(a)], and by suitably apportioning the total moment ($M_{ab}$ or $M_{ef}$) to these strips with the assumption that the moment within each strip is uniform. This is indicated by the broken lines in Fig. 11.25(c), and is also clearly shown in Fig. 11.25(d).

When beams are provided along the column line, the beam portion is relatively stiffer than the slab and resists a major share of the moment at the section. In this case, the moment has to be apportioned between the beam part and the slab part of the slab-beam member as indicated in Fig. 11.25(e).

The calculations involved in the design procedure are given in the next section, which follows the unified procedure of design for all types of column-supported slabs — with or without beams (i.e., including flat slabs).

## 11.4 DESIGN OF COLUMN-SUPPORTED SLABS (WITH / WITHOUT BEAMS) UNDER GRAVITY LOADS

### 11.4.1 Code Procedures Based on the Equivalent Frame Concept

Two-way slabs supported on columns include *flat plates* [Fig. 1.12], *flat slabs* [Fig. 1.13], *waffle* (*ribbed*) *slabs* [Fig. 1.11], and solid slabs with beams along the column lines [Fig. 1.10(b)]. Such slabs may be designed by any procedure which satisfies the basic conditions of equilibrium and geometrical compatibility, and the Code requirements of strength and serviceability. Specific design procedures have been laid out in the Code (Cl. 31) for the design of 'flat slabs', which are defined, according to the Code (Cl. 31.1) as follows:

The term 'flat slab' means a reinforced concrete slab with or without drops, supported generally without beams, by columns with or without flared column heads. A flat slab may be a solid slab or may have recesses formed on the soffit so that the soffit comprises a series of ribs in two directions.

The above definition is very broad and encompasses the various possible column-supported two-way slabs mentioned earlier. Flat slabs may have an edge beam, which helps in stiffening the discontinuous edge, increasing the shear capacity at the critical exterior column supports and in supporting exterior walls, cladding etc. Furthermore, they have a favourable effect on the minimum thickness requirement for the slab (see Section 11.4.2). As mentioned earlier, the Code procedure is based on the elastic analysis of 'equivalent frames' [Fig. 11.24] under gravity loads, and follows closely the 1977 version of the ACI Code [Ref. 11.11]. However, unlike the unified ACI Code procedure, there is no elaboration in the IS Code (Cl. 31) for the particular case of two-way slabs with beams along column lines as in Fig. 1.10(b).

The design procedures described hereinafter† will not only cover the provisions in the prevailing IS Code, but also include provisions in other international codes [Ref. 11.18, 11.19] to cover the case of two-way rectangular slabs supported on flexible beams. These Code procedures are an outcome of detailed analyses of results of extensive tests, comparison with theoretical results based on the theory of plates, and design practices employed successfully in the past. The interested reader may also refer to the background material for the Code procedures presented in Refs. 11.20 - 11.22.

The following two methods are recommended by the Code (Cl. 31.3) for determining the bending moments in the slab panel: either method is acceptable (provided the relevant conditions are satisfied):

1. Direct Design Method
2. Equivalent Frame Method

These methods are applicable only to two-way rectangular slabs (not one-way slabs), and, in the case of the Direct Design Method, the recommendations apply to the gravity loading condition alone (and not to the lateral loading condition).

Both methods are based on the 'equivalent frame concept' (described in Section 11.3.4). The slab *panel* is defined (Cl. 31.1.1c of the Code) as that part of the slab bounded on each of its four sides by the column centrelines. Each slab panel is divided into *column strips* and *middle strips* [Fig. 11.26]. A 'column strip' is defined (Cl. 31.1.1a of the Code) as a design strip having a width equal to the lesser of $0.25l_1$ or $0.25l_2$ on each side of the column centreline, and includes within this width any drop panel or beam (along the column line). Here, $l_1$ and $l_2$‡ are the two spans of the rectangular panel, measured centre-to-centre of the column supports.

† Other design procedures based on yield line analysis and finite element analysis are also acceptable, provided they satisfy all the requirements mentioned earlier.

‡ In general, subscript 1 identifies parameters in the direction where moments are being determined, and subscript 2 relates to the perpendicular direction.

The 'middle strip' is defined (Cl. 31.1.1b) as a design strip bound on each of its sides by the column strip.

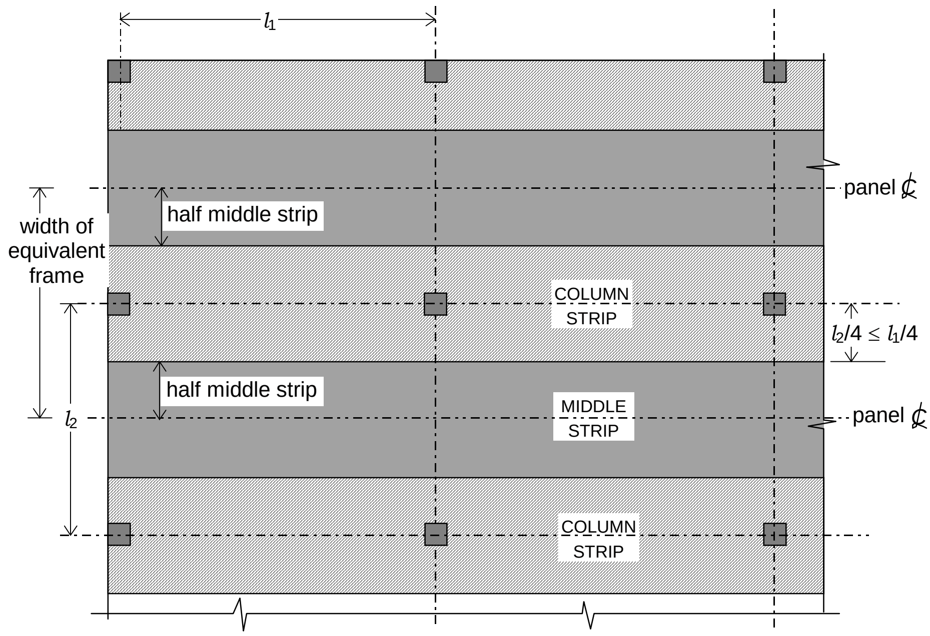



**Fig. 11.26** Column strip and middle strip in a slab panel

While considering an 'equivalent frame' along a column line, the slab width $l_2$ consists of two half middle strips flanking one column strip, as shown in Fig. 11.25(a) and Fig. 11.26. When monolithic beams are provided along the column lines, the *effective* (flanged) *beam sections* (which form part of the column strip) are considered to include a portion of the slab on either side of the beam, extending by a distance equal to the projection of the beam above or below the slab (whichever is greater), but not exceeding four times the slab thickness, as shown in Fig. 11.27 [Ref. 11.18]. In cases where the beam stem is very short, the T-beam may be assumed to have a width equal to that of the column support [Fig. 11.27(c)].

The Direct Design Method (described in Section 11.5) and the Equivalent Frame Method (described in Section 11.6) for gravity load analysis differ essentially in the manner of determining the distribution of bending moments along the span in the slab-beam member [Fig. 11.25(b)]. The former uses moment coefficients (similar in concept to the simplified Code procedure for continuous beams and one-way slabs – see Sections 5.6.1 and 9.3), whereas the latter requires an elastic partial frame analysis. The procedure for apportioning the factored moments between the middle strip and the column strip (or between the slab and the beam when beams are present along the column line) is identical for both design methods.

Both methods require the values of several relative stiffness parameters in order to obtain the longitudinal and transverse distribution of factored moments in the design strips. For this purpose, as well as for determining the dead loads on the slab, it is necessary to assume, initially, the gross section dimensions of the floor system (and the columns). These dimensions may need to be modified subsequently, and the analysis and design may therefore need to be suitably revised.

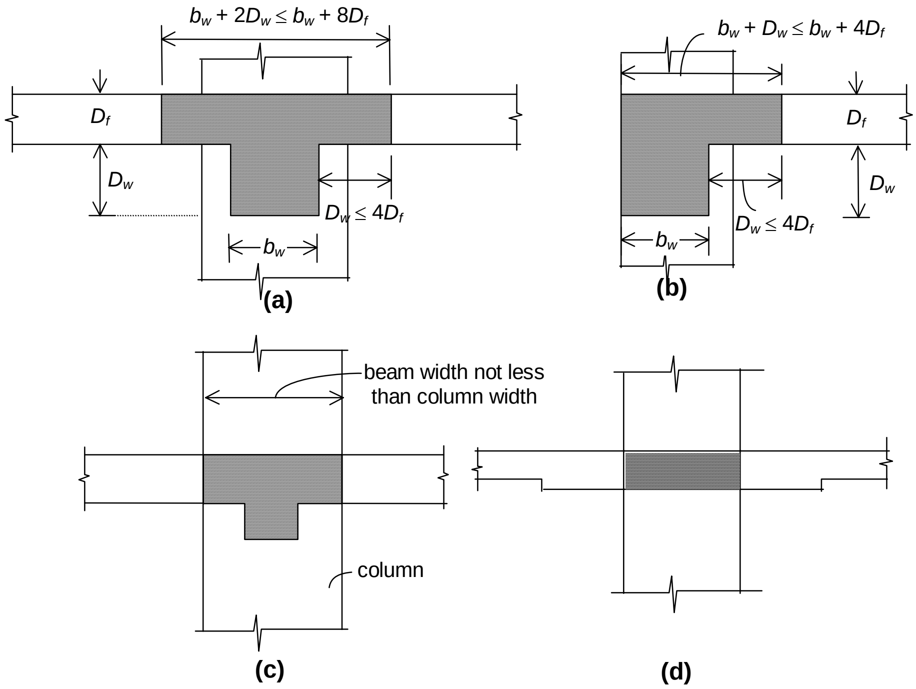



**Fig. 11.27** Definition of beam section

### 11.4.2 Proportioning of Slab Thickness, Drop Panel and Column Head

#### *Slab Thickness*

The thickness of the slab is generally governed by deflection control criteria. [Shear is also an important design criterion — especially in *flat plates* (slabs without beams and drop panels) and at exterior column supports]. The calculation of deflections of two-way slab systems is quite complex, and recourse is often made to empirical rules which limit maximum span/depth ratios as indirect measures of deflection control. For this purpose, the Code (Cl. 31.2.1) recommends the same *l*/*d* ratios prescribed (in Cl. 23.2, also refer Section 5.3.2) for flexural members in general, with the following important differences:

- the *longer* span should be considered[‡] (unlike the case of slabs supported on walls or stiff beams, where the *shorter* span is considered);
- for the purpose of calculating the modification factor $k_t$ [Table 5.2] for tension reinforcement, an *average* percentage of steel across the whole width of panel should be considered [Ref. 11.11];
- When *drop panels* conforming to Cl. 31.2.2 are *not* provided around the column supports, in flat slabs the calculated $l/d$ ratios should be further reduced by a factor of 0.9;
- the minimum thickness of the flat slab should be 125 mm.

### *Slab Thickness Recommended by other Codes*

Other empirical equations for maximum span/depth ratios have been established, based on the results of extensive tests on floor slabs, and have been supported by past experience with such construction under normal values of uniform loading [Ref. 11.18, 11.19]. Thus Ref. 11.8 recommends equations 11.25 to 11.26a for the minimum overall thickness of slabs necessary for the control of deflections. If these minimum thickness requirements are satisfied, deflections need not be computed. These equations are also applicable for two-way slabs supported on stiff beams. However, these thicknesses may not be the most economical in all cases, and may even be inadequate for slabs with large live to dead load ratios. In the calculation of span/depth ratio, the *clear span* $l_n$ in the *longer* direction and the overall depth (thickness) $D$ are to be considered.

For flat plates and slabs with column capitals, the minimum overall thickness of slab is:

$$D \geq [l_n\,(0.6 + f_y / 1000)] / 30 \tag{11.25}$$

However, discontinuous edges shall be provided with an edge beam with stiffness ratio, $\alpha_b$, of not less than 0.8, failing which the thickness given by Eq. 11.25 shall be increased by 10 per cent.

For slabs with drop panels, the minimum thickness of slab is:

$$D \geq [l_n\,(0.6 + f_y / 1000)] / [30\{1+(2x_d/l_n)(D_d\text{-}D)/D\}] \tag{11.26}$$

where $x_d / (l_n/2)$ is the smaller of the values determined in the two directions, and $x_d$ is not greater than $l_n/4$, and $(D_d\text{-}D)$ is not larger than $D$.

For slabs with beams between all supports, the minimum thickness of slab is:

$$D \geq [l_n\,(0.6 + f_y / 1000)] / \{30 + 4\beta\alpha_{bm}\} \tag{11.26a}$$

where $\alpha_{bm}$ is not greater than 2.0. This limit is to ensure that with heavy beams all around the panel, the slab thickness does not become too thin. In the above equations,

$D_d \equiv$ overall thickness of drop panel, mm;
$x_d \equiv$ dimension from face of column to edge of drop panel, mm;
$f_y \equiv$ characteristic yield strength of steel (in MPa);

[‡] These IS Code provisions apply to ‘Flat Slabs’ as defined in Cl. 31.1. However, it is not clear from the Code whether they apply to slabs with flexible beams between all supports as such slabs are not specifically covered by the Code. Ref. 11.8 does cover such slabs also.

$\beta \equiv$ (clear long span)/(clear short span);
$\alpha_{bm} \equiv$ average value of $\alpha_b$ for all beams on edges of slab panel;
$\alpha_b \equiv$ 'beam stiffness parameter', defined as the ratio of the flexural stiffness of the beam section to that of a width of slab bounded laterally by the centreline of the adjacent panel (if any) on each side of the beam.

Referring to Fig. 11.27,

$$\alpha_b = \frac{E_c I_b}{E_c I_s} \Rightarrow \alpha_b = \frac{I_b}{I_s} \tag{11.27}$$

where $I_b$ is the second moment of area with respect to the centroidal axis of the *gross* flanged section of the beam [shaded area in Fig. 11.27(a), (b), (c) and (d)] and $I_s = lD^3/12$ is the second moment of area of the slab. The minimum thickness for flat slabs obtained from Eq. 11.25 are given in Table 11.2

**Table 11.2** Minimum thicknesses for two-way slabs without beams between interior column supports (Eq. 11.25)

| Steel Grade | Without Drop Panels* | |
|---|---|---|
| | no edge beam | with edge beam** |
| Fe 250 | $l_n$/32.1 | $l_n$/35.3 |
| Fe 415 | $l_n$/26.9 | $l_n$/29.6 |

* Thickness to be not less than 125 mm (as per Code).
** Edge beam must satisfy $\alpha \geq 0.80$.

### *Drop Panels*

The 'drop panel' is formed by local thickening of the slab in the neighbourhood of the supporting column. Drop panels (or simply, *drops*) are provided mainly for the purpose of reducing shear stresses around the column supports. They also help in reducing the steel requirement for 'negative' moments at the column supports. [Also refer Section 11.7 for calculation of reinforcement at drop panels].

The Code (Cl. 31.2.2) recommends that drops should be rectangular in plan, and have a length in each direction not less than one-third† of the panel length in that direction. For exterior panels, the length, measured perpendicular to the discontinuous edge from the column centreline should be taken as one-half of the corresponding width of drop for the interior panel [Fig. 11.28(a)].

The Code does not specify a minimum thickness requirement for the drop panel. It is, however, recommended [Ref. 11.18, 11.19] that the projection below the slab should not be less than one-fourth the slab thickness, and preferably not less than 100 mm [Fig. 11.28(b)].

† This may be interpreted as one-sixth of the centre-to-centre dimension to columns on either side of the centre of the column under consideration, as depicted in Fig. 11.28(a).

### *Column Capital*

The 'column capital' (or *column head*), provided at the top of a column, is intended primarily to increase the capacity of the slab to resist *punching shear* [see Section 11.8.2]. The flaring of the column at top is generally done such that the plan geometry at the column head is similar to that of the column.

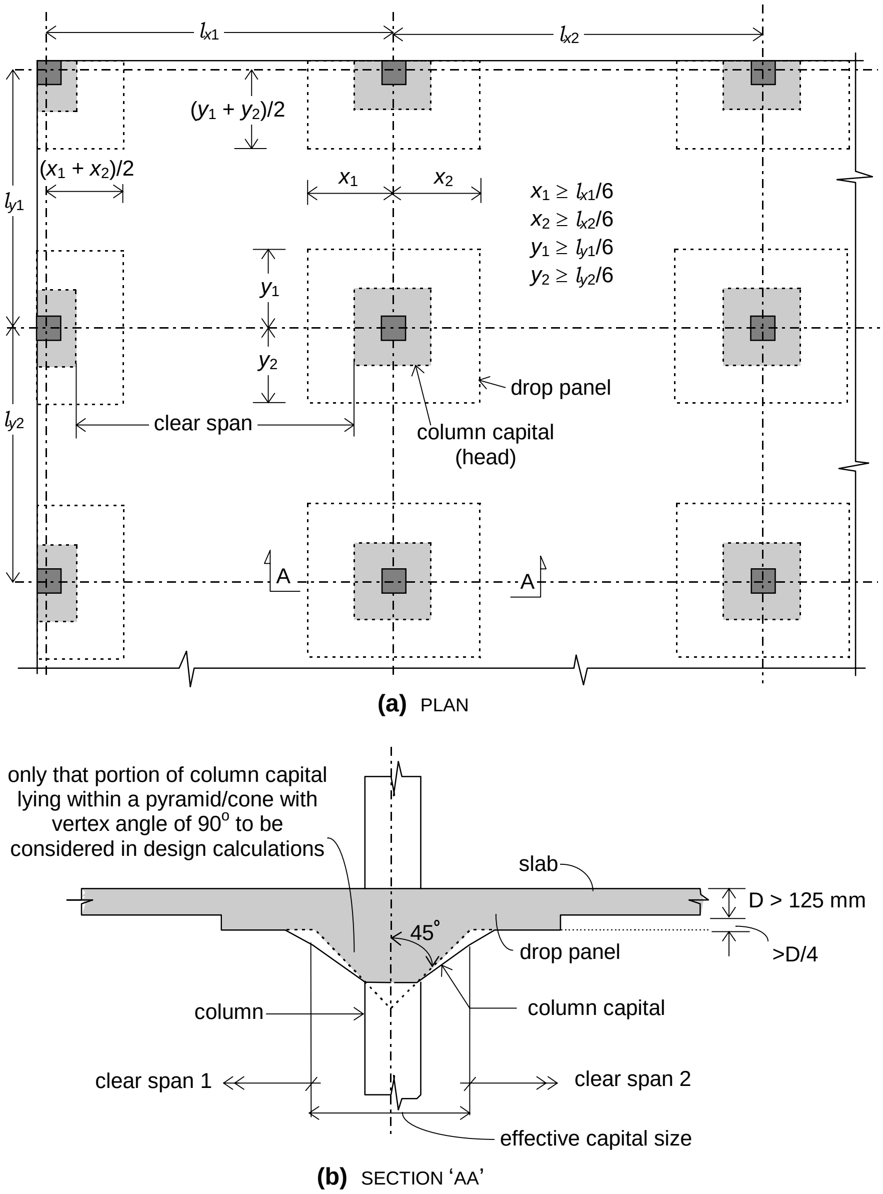



**Fig. 11.28** Drop panel and column capital

The Code (Cl. 31.2.3) restricts the structurally useful portion of the column capital to that portion which lies within the largest (inverted) pyramid or right circular cone which has a vertex angle of 90 degrees, and can be included entirely within the outlines of the column and the column head [Fig. 11.28(b)]. This is based on the assumption of a 45 degree failure plane, outside of which enlargements of the support are considered ineffective in transferring shear to the column [Ref. 11.11].

In the Direct Design Method, the calculation of bending moments is based on a clear span $l_n$, measured face-to-face of the supports (including column capitals, if any) *but not less than* 0.65 *times the panel span* in the direction under consideration [Cl. 31.4.2.2. of the Code]. When the column (support) width in the direction of span exceeds 0.35 $l_1$, (to be more precise, $0.175l_1$ on either side of the column centreline), the critical section for calculating the factored 'negative' moment should be taken at a distance not greater than $0.175l_1$ from the centre of the column (Cl.31.5.3).

### 11.4.3 TRANSFER OF SHEAR AND MOMENTS TO COLUMNS IN BEAMLESS TWO-WAY SLABS

Shear forces and bending moments have to be transferred between the floor system and the supporting columns. In slabs without beams along column lines, this needs special considerations.

The design moments in the slabs are computed by frame analysis in the case of the Equivalent Frame Method, and by empirical equations in the case of the Direct Design Method. At any column support, the total unbalanced moment must be resisted by the columns above and below in proportion to their relative stiffnesses [Fig. 11.29(a)].

In slabs without beams along the column line, the transfer of the unbalanced moment from the slab to the column takes place partly through direct flexural stresses, and partly through development of non-uniform shear stresses around the column head. A part ($M_{ub}$) of the unbalanced moment $M_u$ can be considered to the transferred by flexure and the balance ($M_{uv}$) through eccentricity of shear forces, as shown in Fig. 11.29(b) and (c). The Code recommendation (Cl. 31.3.3) for the apportioning of $M_{ub}$ and $M_{uv}$ is based on a study described in Ref. 11.23:

$$M_{ub} = \gamma \, M_u \tag{11.28a}$$

$$M_{uv} = (1-\gamma)M_u \tag{11.28b}$$

where

$$\gamma = \frac{1}{1+\frac{2}{3}\sqrt{\frac{c_1+d}{c_2+d}}} \tag{11.29}$$

Here, $c_1$ and $c_2$ are the dimensions of the equivalent rectangular column, capital or bracket, measured in the direction moments are being determined and in the transverse direction, respectively, and $d$ is the effective depth of the slab at the critical section for shear [refer Section 11.8.2]. For square and round columns, $c_1 = c_2$, and $\gamma = 0.6$.

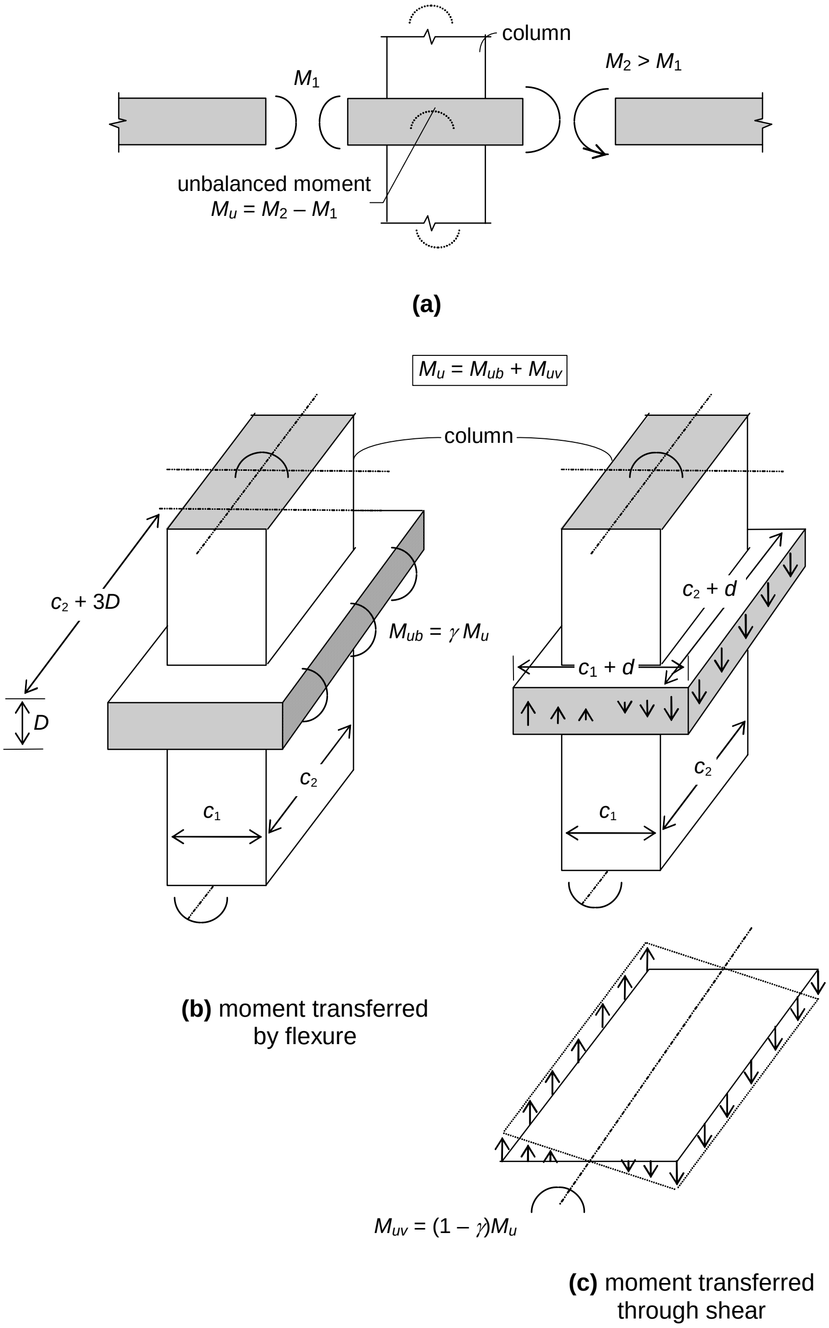



**Fig. 11.29** Transfer of unbalanced moment from slab to column

The width of the slab considered effective in resisting the moment $M_{ub}$ is taken as the width between lines a distance 1.5 times slab/drop thickness on either side of the column or column capital [Fig. 11.29(b)], and hence this strip should have adequate reinforcement to resist this moment. The detailing of reinforcement for moment transfer, particularly at the exterior column where the unbalanced moment is usually the largest, is critical for the safety as well as the performance of flat slabs without edge beams.

The critical section considered for moment transfer by eccentricity of shear is at a distance $d/2$ from the periphery of the column or column capital [Fig. 11.29(c)]. The shear stresses introduced because of the moment transfer, (assumed to vary linearly about the centroid of the critical section), should be added to the shear stresses due to the vertical support reaction [refer Section 11.8.2].

## 11.5 DIRECT DESIGN METHOD

### 11.5.1 Limitations

The Direct Design Method (DDM) is a simplified procedure of determining the 'negative' and 'positive' design moments (under gravity loads) at critical sections in the slab (slab-beam member), using empirical moment coefficients. In order to ensure that these design moments are not significantly different from those obtained by an elastic analysis, the Code (Cl. 31.4.1) specifies that the following conditions must be satisfied by the two-way slab systems for the application of DDM.

1. There must be at least three continuous spans in each direction.
2. Each panel must be rectangular, with the long to short span ratio not exceeding 2.0.
3. The columns must not be offset by more than 10 percent of the span (in the direction of offset) from either axis between centrelines of successive columns†.
4. The successive span lengths (centre-to-centre of supports), in each direction, must not differ by more than one-third of the longer span.
5. The factored live load must not exceed three times the factored dead load (otherwise, moments produced by pattern loading would be more severe than those calculated by DDM).

There is an additional limitation prescribed by other codes [Ref. 11.18, 11.19], with regard to the application of this method to slab panels supported on flexible beams on all sides. The following condition, relating the relative stiffnesses of the beams in the two perpendicular directions, needs to be satisfied in such cases:

$$0.2 \le \frac{\alpha_{b1} l_2^2}{\alpha_{b2} l_1^2} \le 5.0 \qquad (11.30)$$

where $\alpha_b$ is the *beam stiffness parameter* (defined by Eq. 11.27), and the subscripts 1 and 2 refer to the direction moments are being determined, and transverse to it,

† If the column offsets result in variation in spans in the transverse direction, the adjacent transverse spans should be averaged while carrying out the analysis [Cl. 31.4.2.4].

respectively. Ref. 1.18 also limits the live load/dead load ratio to ≤ 2.0. Furthermore, the loads are assumed to be gravity loads, uniformly distributed over the entire panel.

### 11.5.2 Total Design Moment for a Span

In any given span, $l_1$, the *total* (*factored*) *design moment* $M_o$ for the span, is expressed as [refer Cl. 31.4.2.2 of the Code]:

$$M_o = w_u \, l_2 \, l_n^2 / 8 \tag{11.31}$$

where $w_u \equiv$ factored load per unit area of the slab;
$l_n \equiv$ clear span in the direction of $M_o$, measured face-to-face of columns[§], capitals, brackets or walls, *but not less than* $0.65 l_1$[‡];
$l_1 \equiv$ length of span in the direction of $M_o$; and
$l_2 \equiv$ length of span transverse to $l_1$.

With reference to Fig. 11.25(b), it can be seen that considering statics, the absolute sum of the 'positive' and average 'negative' design moments in the slab panel must not be less than $M_o$. The expression for $M_o$ [Eq. 11.31] is obtained as the maximum midspan static moment in an equivalent simply supported span $l_n$, subjected to a uniformly distributed total load $W = w_u(l_2 l_n)$, where $l_2 l_n$ is the effective panel area on which $w_u$ acts. When drop panels are used, the contribution of the additional dead load (due to local thickening at drops) should be suitably accounted for; this is illustrated in Example 11.7.

### 11.5.3 Longitudinal Distribution of Total Design Moment

The typical variation of moments (under gravity loads) in the slab, along the span, has already been introduced in Fig. 11.25(b). The Code recommendation (Cl. 31.4.3.3) for the distribution of the calculated 'total design moment', $M_o$, between critical 'negative' moment sections (at the face of equivalent rectangular supports) and 'positive' moment sections (at or near midspan) is as depicted in Fig. 11.30.

- *Interior span*:
  - ∗ 'negative' design moment $M_o^- = 0.65 M_o$ (11.32a)
  - ∗ 'positive' design moment $M_o^+ = 0.35 M_o$ (11.32b)
- *Exterior span*:
  - ∗ 'negative' design moment at *exterior* support $M_{o,ext}^- = (0.65/q) M_o$ (11.33a)

---

[§] Circular / nonrectangular column supports are to be treated as equivalent square/rectangular supports having the same area (Cl. 31.4.2.3 of the Code).

[‡] This condition is imposed in order to prevent undue reduction in the design moment when the columns are long and narrow in cross-section or have large brackets or capitals [Ref. 11.11].

- ∗ 'negative' design moment at *interior* support $M^{-}_{o,\text{int}} = (0.75 - 0.10/q)M_o$ (11.33b)
- ∗ 'positive' design moment $M^{+}_{o} = (0.63 - 0.28/q)M_o$ (11.33c)

where $q \equiv 1 + (1/\alpha_c)$ (11.34)

$\alpha_c \equiv \left(\sum K_c\right)/K_{sb}$ (11.35)

$\sum K_c$ ≡ sum of flexural stiffnesses of columns meeting at the exterior joint; and

$K_{sb}$ ≡ flexural stiffness of the slab (or slab-beam member) in the direction moments are calculated (i.e., along span $l_1$), at the exterior joint.

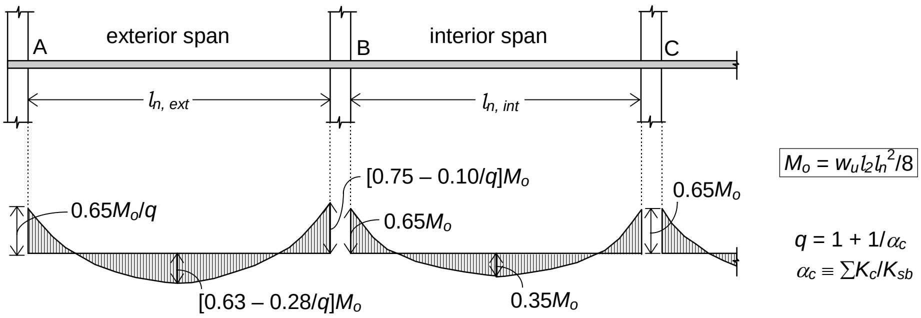



**Fig. 11.30** Distribution of $M_o$ into 'negative' and 'positive' design moments (longitudinal)

> At interior supports such as B in Fig. 11.30, the 'negative' moment sections must be designed to resist the *larger* of the two design 'negative' moments (on either side of the support), unless an analysis is made to distribute the unbalanced moment in accordance with the relative stiffnesses of the elements framing into the support (Cl. 31.4.3.5 of the Code). [However, the moments may be modified as given below]

The Code (Cl. 31.4.3.4) permits a limited readjustment in the apportioned design moments — but by no more than 10 percent, because of the approximations and limitations inherent in the Direct Design Method — provided the total design moment $M_o$ for the panel is not less than the value given by Eq. 11.31. No additional redistribution of moments is permitted.

For the purpose of calculating the flexural stiffness ($K_{sb}$) of the slab-beam member and that of the column ($K_c$), it is permitted [Ref. 11.11] to assume that the members are prismatic (i.e., having uniform cross-section throughout their lengths). This is done purely for convenience, and is in keeping with the simplifications underlying DDM. This assumption implies that the contributions of drop panels, column capitals and brackets may be neglected. Furthermore, the increase in the second moment of area of the slab-beam member, between the column centreline and column face, may be neglected. With these simplifications, the flexural stiffness $K$ (of the column or slab-beam member) is simply obtained as:

$$K = 4E_c I/l \tag{11.36}$$

where $E_c \equiv$ short-term modulus of elasticity of concrete (for the grade applicable to the element);
$I \equiv$ second moment of area (considering the *gross* section† ); and
$l \equiv$ appropriate centre-to-centre span.

The Code (under Cl. 31) covers only flat slabs, and as such Eq. 11.32 to 11.35 are not specifically meant for use in the case of two-way slabs with flexible beams between column supports. However, other codes [Ref. 11.18] do permit the use of DDM for the latter case mentioned above. For two-way slabs with beams, for interior spans, Eq. 11.32(a), (b) are applicable. For exterior spans, the distribution of $M_o$ for slabs with beams may be made according to the factors given in Table 11.3 – case (2).

**Table 11.3** Moment factors for end span [Ref. 11.18]

| Case | (1) | (2) | (3) | (4) |
|---|---|---|---|---|
| | **Exterior edge not restrained** | **Slab with beams between all supports** | **Slab without beams between interior supports** | **Exterior edge fully restrained** |
| Interior negative factored moment | 0.75 | 0.70 | 0.70 | 0.65 |
| Positive factored moment | 0.66 | 0.59 | 0.52 | 0.35 |
| Exterior negative factored moment | 0 | 0.16 | 0.26 | 0.65 |

It may be noted that when the slab is stiffened with beams along the column lines, the calculation of the second moment of area $I$ must logically include the contribution of the portion of the beam projecting below (or above) the slab.

The calculations related to flexural stiffness are required for the purpose of determining the parameter $\alpha_c$ [Eq. 11.35], required for the moment factors in the end span [Eq. 11.33]. An alternative and simpler scheme for end spans (with moment coefficients independent of $\alpha_c$), suggested in Ref. 11.18, is shown in Table 11.3. [As already indicated above, this Table also covers the case of end spans of two-way slabs with beams between all columns.]

† In the case of the slab-beam member, the width of the section should be taken as the full panel width, $l_2$.

### 11.5.4 Apportioning of Moments to Middle Strips, Column Strips and Beams

As mentioned earlier [Fig. 11.25(c), (d)], the calculated design 'positive' and 'negative' slab moments in the panel in the longitudinal direction have to be apportioned transversely to the design strips of the panel (*column strip* (*cs*) and *half middle strips* (*hms*) in flat slabs, and the *beam* part and the *slab* part when there are beams) at all critical sections. This procedure of transverse distribution of moments is common for both Direct Design Method and Equivalent Frame Method..

#### *IS Code Recommendations*

The Code recommendations for flat slabs (Cl. 31.5.5) in this regard are based mainly on studies reported in Ref. 11.24:

- '*Negative*' *moment at exterior support*:
  - ∗ *column strip*: $$M^{-}_{cs,ext} = \begin{cases} 1.00 M^{-}_{o,ext} & \textit{if column / wall width} < 0.75 l_2 \\ (b_{cs}/l_2) M^{-}_{o,ext} & \textit{otherwise} \end{cases} \qquad (11.37a)$$
  - ∗ *half middle strip*: $$M^{-}_{hms,ext} = \begin{cases} 0 & \textit{if col. width} < 0.75 l_2 \\ 0.5(1 - b_{cs}/l_2) M^{-}_{o,ext} & \textit{otherwise} \end{cases} \qquad (11.37b)$$

where $b_{cs}$ is the width of the column strip [Fig. 11.25].

- '*Negative*' *moment at interior support*:
  - ∗ *column strip*: $M^{-}_{cs,\text{int}} = 0.75\, M^{-}_{o,\text{int}}$ (11.38a)
  - ∗ *half middle strip*: $M^{-}_{hms,\text{int}} = 0.125\, M^{-}_{o,\text{int}}$ (11.38b)
- '*Positive*' *moment for all spans:*
  - ∗ *column strip*: $M^{+}_{cs} = 0.60\, M^{+}_{o}$ (11.39a)
  - ∗ *half middle strip*: $M^{+}_{hms} = 0.20\, M^{+}_{o}$ (11.39b)

The transverse distribution of moments for a typical exterior slab panel is depicted in Fig. 11.31. The moments indicated are assumed to be uniformly distributed across the width of the respective design strips.

In the case of a panel with a discontinuous edge in the direction of $M_o$ (span $l_1$), such as in the *external* equivalent frame shown in Fig. 11.24(a), the design of the half-column strip adjoining and parallel to the discontinuous edge, as well as the middle strip in the panel, depends on whether a marginal beam (with depth > $1.5D_s$) or wall is supporting the slab at the edge. If such a stiffening of the edge exists, the bending moments in the half-column strip should be taken as one-quarter of that for the first interior column strip, and the moments in the middle strip as twice that assigned to the half-middle strip corresponding to the first row of interior columns (Cl. 31.3.2b and 31.5.5.4c of the Code).

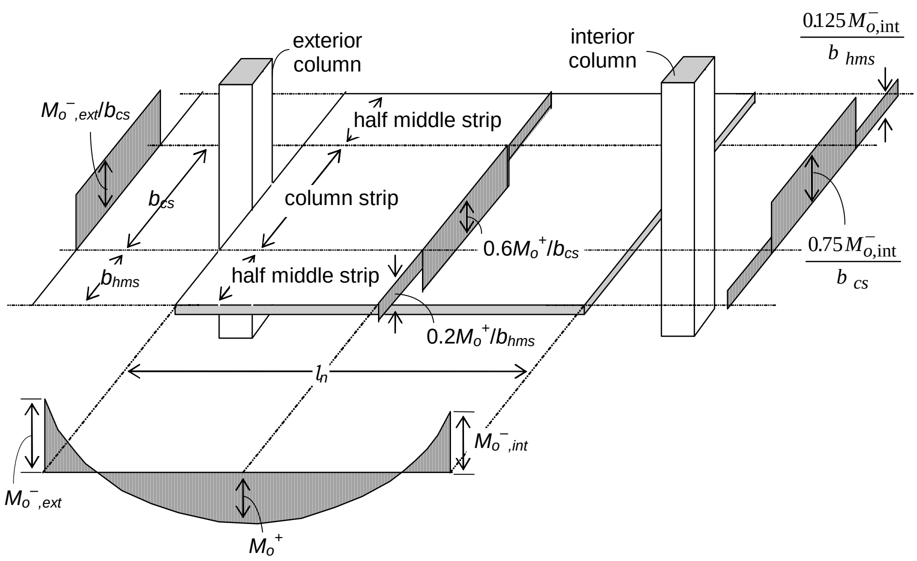



**Fig. 11.31** Transverse distribution of bending moments in a typical exterior panel

## *Canadian Code Recommendations*

A more simplified scheme for transverse distribution of moment, which accounts for the ability of slabs to redistribute moments, is given in the Canadian code [Ref. 11.18]. Slabs are highly statically indeterminate and usually greatly under-reinforced. This leads to the formation of yield lines along sections of peak moment, effecting considerable moment redistribution to sections of lesser moment (see Section 9.7 and Fig. 9.12). This inherent ability of the slab gives the designer considerable leeway in adjusting the moment field and designing the reinforcement accordingly, subject to static equilibrium conditions and requirements of strength and serviceability being met. Reflecting this flexibility, for slabs without beams, the Canadian code gives a *range of values* for the column strip share of moment, from which the designer can *choose* an appropriate value; the balance is apportioned to the middle strip. For slabs with beams, the distribution is between the beam part and the slab part, the proportions being dependent on the beam stiffness ratio $\alpha_{b1}$ and the span ratio $l_2/l_1$. This procedure is applicable to both Direct Design Method and Equivalent Frame Method. These provisions are summarised below:

***(a) Regular slabs without beams***

This includes flat plates and slabs with drop panels and/or column capitals, which may or may not have edge beams along the discontinuous edges.

*(i) Column strip moments:*

The column strips are designed to resist the total negative or positive moments at the critical sections (given in Fig. 11.30 for DDM) multiplied by an appropriate factor within the following ranges:

- Negative moment at interior column - 0.6 to 1.00
- Negative moment at exterior column - 1.00
- Positive moment in all spans - 0.50 to 0.70

Furthermore, at interior columns, at least one-third of the reinforcement for the total negative moment shall be located in a band with a width extending a distance of 1.5*D* from the sides of the column. Similarly, reinforcement for the total negative moment at exterior column is placed within such a bandwidth. This is to facilitate the transfer of unbalanced moment to the column by flexure.

*(ii) Middle strip moments:*
At all critical sections, the portion of the negative and positive moments not resisted by the column strip is assigned proportionately to the two half middle strips on either side of the column strip. A full middle strip in a panel has moments assigned to its two halves from the equivalent frames on either side (Fig. 11.32). The middle strip adjacent to and parallel with an edge supported by a wall must be designed for twice the moment assigned to its interior half portion forming part of the equivalent frame along the first row of interior supports (Fig. 11.32).

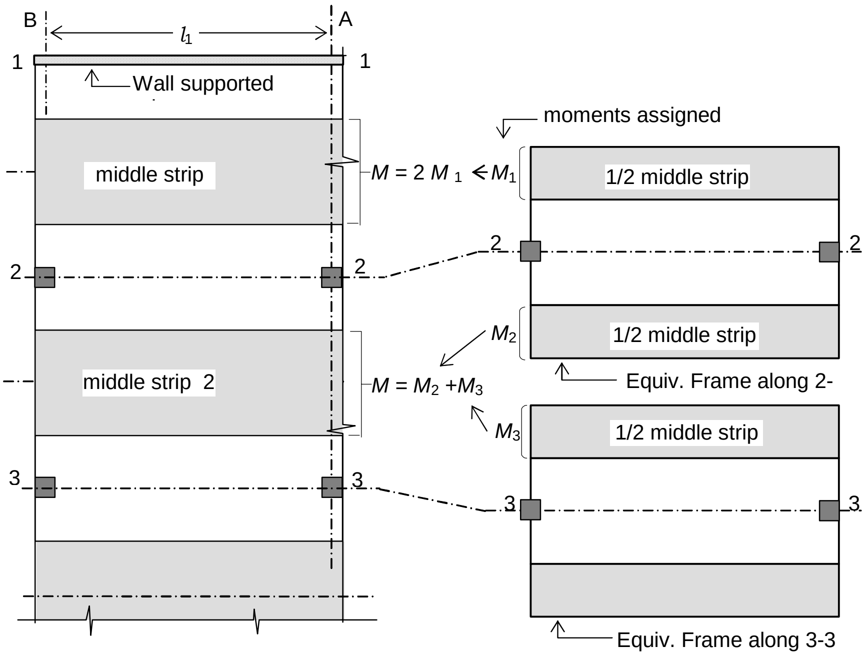



**Fig. 11.32** Factored moments in middle strips

***(b) Regular slabs with beams between all supports***
In this case, the slab-beam member is divided into the beam part (see Figs.11.25e and 11.27) and the slab part which is the portion of the member outside the beam part.

*(i) Moments in beams:*
The beam shall be designed to resist the following fractions of the total negative and positive moments at the critical sections (given in Fig. 11.30 for DDM):

- Negative moment at interior columns and positive moment in all spans -

$$\alpha_{b1}/[1+(l_2/l_1)^2] \qquad (11.40)$$

- Negative moment at an exterior column – 100 percent

Here $\alpha_{b1}$ is not taken larger than 1.0. The beam must also resist moments due to loads directly applied on it and not considered in slab design such as weight of walls and the beam rib. The negative moment reinforcement at exterior support must be placed within a band of width extending a distance 1.5$D$ past the sides of the column or the side of the beam web, whichever is larger.

*(ii) Moments in slabs:*

The portion of the negative and positive moments not resisted by the beam is assigned to the slab parts outside the beam. The slab reinforcement for the negative moment at interior supports is uniformly distributed over the width of the slab. Positive moment reinforcements may also be distributed uniformly.

### 11.5.5 Loads on the Edge Beam

The IS Code (Cl. 31.3.2) describes a procedure for the design of 'marginal beams' (edge beams) having an overall depth greater than 1.5 times the slab thickness or walls supporting a flat slab. According to this procedure, the slab portion in the 'half-column strip' adjacent to the edge beam (or wall) should be designed to resist one-quarter of the design moment assigned to the 'first interior column strip'. The edge beam or wall should be designed to carry the loads acting directly on it (if any) plus a uniformly distributed load equal to one-quarter of the total load on the slab panel [also refer Section 11.5.8].

### 11.5.6 Torsion in Edge Beam

Although the IS Code does not offer any specific recommendation for torsion in the transverse beam at the exterior edge, it is evident that some of the 'negative' moment at the exterior edge of the panel ( $M_{o,ext}^{-}$ ) will be transferred to the column by torsion in the edge beam [Fig. 11.33], and the balance will be transferred to the column through flexure at the column-slab connection.

The edge beam, therefore, has to be designed as a *spandrel beam* subjected to a torsional moment distributed along its length (in addition to the bending moments and shear force due to the loads indicated in Section 11.5.5). For this purpose, it may be assumed, conservatively, that the entire 'negative' design moment at the exterior support, $M_{o,ext}^{-}$ is uniformly distributed over the width of the design strip, $l_2$, as shown in Fig. 11.33. This results in a linear variation of twisting moment in the edge beam, with a zero value at the midspan (panel centreline) and a maximum value at the face of the column, as illustrated.

If the maximum torque ($T_{max}$) exceeds the cracking torque ($T_{cr}$) of the edge (spandrel) beam, torsional cracking will occur; there will be a reduction in the torsional stiffness and a consequent redistribution of moments[†] [Ref. 11.29] resulting in a relaxation in the induced torque. For this reason, if $T_{max}$ exceeds 0.67$T_{cr}$, Ref. 11.18 permits the use of a maximum factored torque of 0.67$T_{cr}$, provided a corresponding readjustment is made to the 'positive' moment in the span. If this is

[†] refer Section 9.7.3 ('torsional plastic hinge').

done, $M^{-}_{o,ext}$ gets restricted to $2 \times 0.67\ T_{cr}\ l_2 / (l_2 - c_2)$, and the 'positive' moment in the span has to be correspondingly adjusted to maintain the value of $M_o$ [Eq. 11.31].

The design of the edge beam for torsion should conform to the requirements described in Chapter 7.

Some torsion can also be expected to occur in the transverse beams at the interior columns, due to unbalanced moments in the panels on the two sides of these beams. However, such torsion is generally negligible (except in exceptional cases), and hence not considered in design.

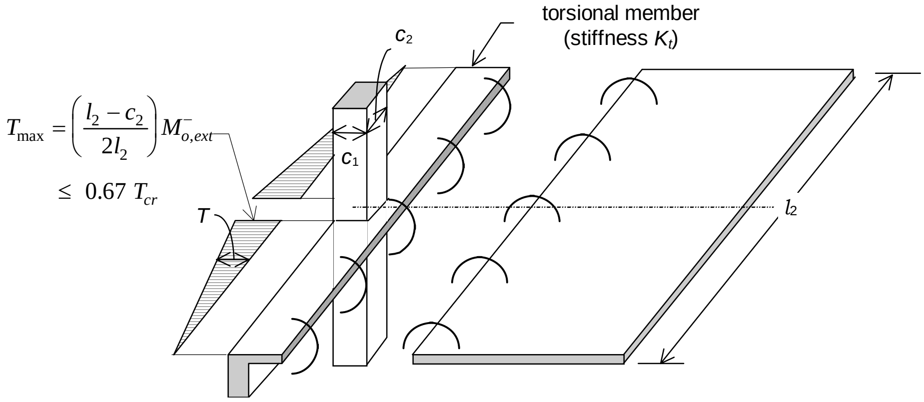



**Fig. 11.33** Torsion in edge beam

### *Torsional member and stiffness*

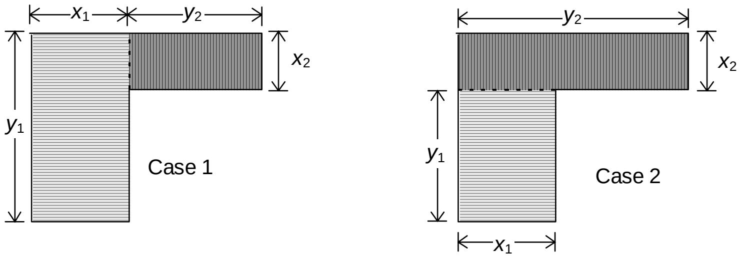
The transverse torsional member (at the edge as in Fig. 11.33 or over an interior column) is assumed to have a cross section consisting of the larger of: (i) a portion of the slab having a width equal to that of the column, bracket, or capital measured in the direction $l_1$ plus that part of the transverse beam (if any) above and below the slab; and (ii) the beam section as defined in Fig. 11.27. For the calculation of the torsional stiffness of this member, needed for the Equivalent Frame Method, the torsional property C of the section (refer Section 7.2.3) is needed.

The computation of an exact value of $C$ for a flanged section being very difficult, it suffices to obtain an approximate value for $C$ by subdividing the section into rectangles such that the summation of the $C$ values of the component rectangles results in a maximum value [Fig. 11.34]. This is done by subdividing in such a way as to minimise the length of the common boundaries.

The expression for $C$ [refer Section 7.2.3] is accordingly obtained as:

$$C = \sum \left(1 - 0.63\frac{x}{y}\right)\frac{x^3 y}{3} \tag{11.41}$$

where $x$ and $y$ are the short and long dimensions of the rectangular part [Fig. 11.34].




Adopt larger $C$ computed for cases 1 and 2

**Fig. 11.34** Computation of torsional property $C$ for a flanged edge beam

### 11.5.7 Moments in Columns and Pattern Loading

As explained in Section 11.4.3, columns (and walls) built monolithically with the slab must be designed to resist the unbalanced moments transferred from the slab. The total moment transferred to the *exterior* column is the same as the 'negative' design moment ( $M_{o,ext}^{-}$ ) at the exterior support, computed by the factors in Fig. 11.30 and Table 11.3 for DDM and obtained by frame analysis in EFM. The negative moments at faces of supports computed by these factors for the DDM correspond to the action of full factored live load plus dead load, whereas the maximum unbalanced moment at the *interior* support would occur under pattern loading. Hence, in the case of the *interior* column, the Code (Cl. 31.4.5.2) recommends the use of the following empirical expression† for the total unbalanced (factored) moment to be resisted by the column:

$$M_u = \frac{0.08\,[(w_{u,DL} + 0.5 w_{u,LL}) l_2 l_n^2 - w'_{u,DL} l'_2 (l'_n)^2]}{1 + 1/\alpha_c} \tag{11.42}$$

where $\alpha_c \equiv (\Sigma K_c / \Sigma K_s)$

$w_{u,DL}$, $w_{u,LL}$ ≡ design (factored) dead and live loads per unit area on the longer span;

$w'_{u,DL}$ ≡ design dead load per unit area on the shorter span;

$l_2$ , $l'_2$ ≡ lengths transverse to the direction of $M_u$ in the long span and short span respectively;

$l_n$ , $l'_n$ ≡ lengths of clear spans (measured face to face of supports) in the direction of $M_u$, in the long span and short span respectively;

$\alpha_c$ ≡ relative column stiffness parameter; and

$K_c$ and $K_s$ are flexural stiffnesses of column and slab respectively.

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† This expression for $M_u$ has been derived for the case of two adjacent unequal spans, with full dead load plus half live load on the long span and dead load alone on the short span, thereby (partially) accounting for the effects of pattern loading [Ref. 11.11].

The total unbalanced moment transferred to the column (exterior or interior) should be distributed to the columns above and below the floor under consideration, in direct proportion to their relative stiffnesses.

Furthermore, in the case of beamless slabs, as explained in Section 11.4.3, a fraction, $M_{ub} = \gamma M_u$, of the unbalanced moment is transferred by flexure of a width of slab, and the balance, $M_{uv} = M_u - M_{ub}$, is transferred by shear. The design of the slab should account for the resulting flexural and shear stresses.

### *Effects of Pattern Loading on Slab Moments*

The moments at critical sections in the slab, computed in DDM using the factors in Fig. 11.30 also correspond to the application of full factored load (dead load plus live load) on all spans. Due to 'pattern loading' (i.e., occurrence of dead load on all spans and live load only on certain critical spans), it is possible that the 'positive' bending moments could exceed the calculated values for full loading (on all spans), in an extreme case, by as much as 100 percent [Ref. 11.11]. However, there is no likelihood of a significant increase in the calculated 'negative' moment at the support, because the loading pattern for maximum moment for such a case requires full factored loads to be considered on both spans adjoining the support.

If the relative stiffness of the columns, measured by the parameter $\alpha_c$ (defined by Eq. 11.35), is high, such excess of the maximum 'positive' moment under pattern loading (over that calculated with full loading on all spans) is low. The Code (Cl. 31.4.6) prescribes that if $\alpha_c$ is *not* less than a specified value $\alpha_{c,\,min}$, then the possible increase in the design 'positive' moment may be ignored. If, however, the columns lack the desired minimum relative stiffness, i.e., $\alpha_c < \alpha_{c,\,min}$, then the calculated 'positive' design moments should be increased by a factor, $\delta_s$, defined as follows:

$$\delta_s = 1 + \frac{2 - (w_{DL}/w_{LL})}{4 + (w_{DL}/w_{LL})}\left(1 - \frac{\alpha_c}{\alpha_{c,\min}}\right) \tag{11.43}$$

where $w_{DL}$ and $w_{LL}$ denote respectively the characteristic (unfactored) dead load and live load per unit area.

The value of $\alpha_{c,\,min}$ depends on the $w_{DL}/w_{LL}$ load ratio, the $l_2/l_1$ span ratio, as well as the beam stiffness parameter $\alpha_b$ (defined by Eq. 11.27). The values of $\alpha_{c,\,min}$, listed in Table 17 of the Code are for Flat Slabs (i.e. without beams) for which $\alpha_b = 0$; and these are same as the values given as the first set in column 3 of Table 11.4. More comprehensive values for $\alpha_{c,\,min}$, covering values of $\alpha_b$ other than zero, as given in the 1984 revision of the Canadian Code are included in Table 11.4. For intermediate values of $w_{DL}/w_{LL}$, $l_2/l_1$ and $\alpha_b$, linear interpolation may be resorted to. The more recent (1994) revision of the Canadian Code [Ref. 11.18] restricts the use of DDM to cases with $w_{LL}/w_{DL} \leq 2$ and dispenses with the factor $\delta_s$.

**Table 11.4** Values of $\alpha_{c,\,min}$

| $w_{DL}/w_{LL}$ | $l_2/l_1$ | Beam stiffness parameter $\alpha_{b1}$ | | | | |
|---|---|---|---|---|---|---|
| | | **0** | **0.5** | **1.0** | **2.0** | **4.0** |
| 2.0 | 0.5 – 2.0 | 0 | 0 | 0 | 0 | 0 |
| 1.0 | 0.5 | 0.6 | 0 | 0 | 0 | 0 |
| | 0.8 | 0.7 | 0 | 0 | 0 | 0 |
| | 1.0 | 0.7 | 0.1 | 0 | 0 | 0 |
| | 1.25 | 0.8 | 0.4 | 0 | 0 | 0 |
| | 2.0 | 1.2 | 0.5 | 0.2 | 0 | 0 |
| 0.5 | 0.5 | 1.3 | 0.3 | 0 | 0 | 0 |
| | 0.8 | 1.5 | 0.5 | 0.2 | 0 | 0 |
| | 1.0 | 1.6 | 0.6 | 0.2 | 0 | 0 |
| | 1.25 | 1.9 | 1.0 | 0.5 | 0 | 0 |
| | 2.0 | 4.9 | 1.6 | 0.8 | 0.3 | 0 |
| 0.33 | 0.5 | 1.8 | 0.5 | 0.1 | 0 | 0 |
| | 0.8 | 2.0 | 0.9 | 0.3 | 0 | 0 |
| | 1.0 | 2.3 | 0.9 | 0.4 | 0 | 0 |
| | 1.25 | 2.8 | 1.5 | 0.8 | 0.2 | 0 |
| | 2.0 | 13.0 | 2.6 | 1.2 | 0.5 | 0.3 |

### 11.5.8 Beam Shears in Two-way Slab System with Flexible Beams

Shear in two-way slabs without flexible beams along column lines is discussed in Section 11.8. When there are beams, in addition to designing the slab to resist the shear force in it, the beam must also be designed to resist the shear it is subjected to. As mentioned earlier, the design of two-way slabs supported on *flexible* beams is not adequately covered in the Code, and hence there are no specific recommendations for determining the design shear forces in such beams.

In general, the Code (Cl. 24.5) suggests that the design shear in *beams supporting solid slabs spanning in two directions at right angles and supporting uniformly distributed loads* may be computed as that caused by loads in tributary areas bounded by 45$^{o}$ lines drawn from the corners of the panels, as explained in Chapter 9 [Fig. 9.5]. However, such an idealisation is meaningful only if the supporting beams can be considered to be "adequately stiff", which, as explained earlier, is indicated by the condition $\alpha_{b1}\, l_2/l_1 \geq 1.0$. In the other extreme, when there are no beams, $\alpha_{b1}\, l_2/l_1 = 0$, the 'beam' carries no load and the full shear in the panel is transmitted by the slab to the column in two-way action [refer Section 11.8]. For the case of *flexible* beams, with $0 < \alpha_{b1}\, l_2/l_1 < 1$, the beams framing into the columns transmit only part of the shear, and the balance of the shear in the panel is assumed to be transmitted by the slab to the column. A simple means of evaluating the shear component in the beam in such a case is by applying linear interpolation between the above two extreme conditions of 'adequate stiffness' ($\alpha_{b1} l_2/l_1 \geq 1.0$) and zero stiffness ($\alpha_{b1} l_2/l_1 = 0$), as recommended in Ref. 11.18. Accordingly,

$$V_{flexible\ beam} = (\alpha_{b1} l_2/l_1) V_{stiff\ beam} \qquad for \quad 0 < \alpha_{b1} l_2/l_1 < 1 \qquad (11.44)$$

This means that, in such cases, the beams framing into the column transmit only part of the shear from panels to the column, and the balance shear is transmitted by the slab directly to the columns in two-way shear. In such cases, the total shear strength of the slab-beam-column connection has to be checked to ensure that resistance to the full shear occurring on a panel is provided. This involves the checking of the shear strength of the slab-beam part around the column perimeter as in the case of flat slabs (Section 11.8.2).

It may be noted that, in addition to the shear due to slab loads, beams must also resist shears due to factored loads applied directly on the beams.

The application of the Direct Design Method is illustrated in Example 11.6.

## 11.6 EQUIVALENT FRAME METHOD

The 'equivalent frame method' (EFM) of design (also called Elastic Frame Method) of two-way beam-supported slabs, flat slabs, flat plates and waffle slabs is a more general (and more rigorous) method than DDM, and is not subject to the limitations of DDM [refer Section 11.5.1]. Furthermore, under lateral loads, recourse has to be taken to design by EFM.

The 'equivalent frame' concept has already been introduced in Section 11.3.4. Such a concept simplifies the analysis of a three-dimensional reinforced concrete building by subdividing it into a series of two-dimensional (plane) frames ('equivalent frames') centred on column lines in longitudinal as well as transverse directions [Fig. 11.24]. The 'equivalent frame method' differs from DDM in the determination of the total 'negative' and 'positive' design moments in the slab panels — for the condition of gravity loading. However, the apportioning of the moments to 'column strips' and 'middle strips' (or to beam and slab) across a panel [refer Section 11.5.4] is common to both methods.

### 11.6.1 Equivalent Frame for Analysis

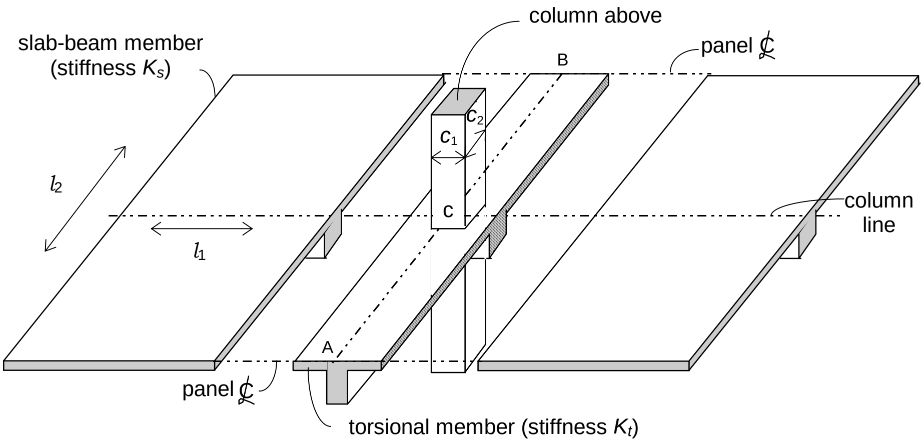
The bending moments and shear forces in an 'equivalent frame' are obtained in EFM by an elastic analysis[†]. Such an analysis should generally be performed on the entire plane frame [Cl. 31.5.1(a) of the Code]. However, if the frame is subjected to gravity loading alone, and if the frame geometry and loading are not so unsymmetrical as to cause significant 'sway' (lateral drift) of the frame, each floor may be analysed separately, considering the appropriate 'substitute frame', with the columns attached to the floor assumed fixed at their far ends [Fig. 11.24(c)]. A further simplification may be made for the purpose of determining the design moment at a given support or

[†] It is now possible to do such analysis of the entire frame by methods such as the *Finite Element Method*. The successive levels of simplifications and approximations given below are for use when such computer–based methods are not resorted to, or are found unnecessary.

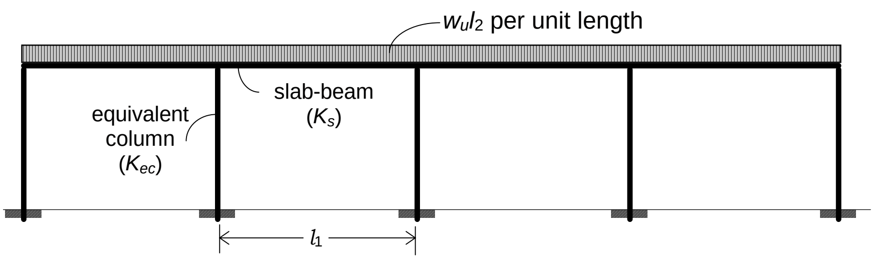
span in the slab-beam member, by assuming the slab-beam member to be fixed at any support two panels distant, provided the slab is continuous beyond this point [refer Cl. 31.5.1(b) of the Code].

The load transfer system in the 'equivalent frame' involves three distinct interconnected elements [Fig. 11.35(a)]:

- the slab-beam members (along span $l_1$);
- the columns (or walls); and
- the torsional members, transverse to the frame (along span $l_2$) and along the column lines.




**(a)** elements of equivalent frame at a connection




**(b)** equivalent frame for analysis

**Fig. 11.35** Equivalent frame method

In conventional plane frames, the torsional members are absent, and the skeletal frame comprises only beams and columns. However, in the case of the 'equivalent frame', the wide slab-beam member is supported at its support section only over part of its width by the column, and the remaining (and generally substantial) portion is supported by the transverse torsional member, which provides only elastic (flexible) restraint (spring support) — both rotationally (in a twisting mode) and translationally (in a vertical direction). In other words, significantly, the flexural restraint to the slab-beam member (horizontal member) at the support section is less (i.e., the member is more *flexible*), than in the case of a beam-column connection in a conventional plane frame. In effect, it is as though the vertical supporting member (column) has less flexural stiffness than it really has.

The nonuniform translational restraint to the slab-beam member along a transverse line at the column is ignored in frame analysis, its effect being accounted for in the apportioning of the design 'positive' and 'negative' moments over the panel width to column strip and middle strips. However, the increased flexibility of the slab-to-column connection has to be accounted for in some manner in the assessment of the relative stiffnesses of the various members of the 'equivalent frame'. Although the IS Code provisions do not include any specific suggestion as to how this can be done, the ACI Code (on which the EFM procedure of the IS Code is based) recommends the concept of an 'equivalent column' with stiffness $K_{ec}$ which can be used to replace the actual columns (above and below the floor at any joint) as well as the torsional member at the column line under consideration [Fig. 11.35(b)].

Thus, for the purpose of gravity load analysis, the *substitute frame* to be analysed by EFM can be modelled as a simple multi-bay, single storeyed portal frame, comprising only horizontal slab-beam members and vertical 'equivalent columns' [Fig. 11.35(b)].

The calculation of stiffnesses of the slab-beam members and the equivalent columns are to be based on their respective *gross* concrete sections [Cl. 31.5.1(c) of the Code]. Details of the calculation procedure are discussed in the next section.

### 11.6.2 Slab-Beam Member

The slab-beam member in an interior frame is bounded laterally by the centreline of the panel on each side of the column line, thus comprising a column strip plus two half-middle strips. For an exterior frame, the slab-beam member extends laterally from the edge to the centreline of the adjacent panel [Fig. 11.24(a)]. The slab-beam comprises the slab, drop panel (if provided) and beam(s) (if provided).

The cross-section of the slab-beam member varies along its span, on account of provision of drop panels (if provided) and the increased cross-section within the bounds of the supporting column; the consequent variation of second moment of area along the span must be accounted for in the frame analysis by EFM [Cl. 31.5.1(d)[†] of the Code]. In order to account for the enhancement in the second moment of area of the slab-beam member in the region between the column face and the column centreline, a magnification factor of $(1 - c_2/l_2)^2$ is recommended [Ref. 11.18, 11.19]. The variation of the second moment of area of the slab-beam member in a flat slab (with drop panels) is shown in Fig. 11.36(a), (b), (c).

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[†] With reference to waffle slabs ('recessed' or 'coffered') which are made solid in the region of the columns [Fig. 1.11(b)], the Code suggests that the stiffening effect may be ignored provided the solid part of the slab does not extend more than $0.15l_e$ into the span measurement from the centreline of the columns.

The calculation of the stiffness factors, carry-over factors and fixed-end moments of the slab-beam member (required for conventional frame analysis‡ ) are dependent on the variation of the second moment of area along the span. Such factors have been tabulated for common geometric and loading configurations in various design handbooks [Ref. 11.11, 11.25, 11.26]. Factors for two typical cases are listed in Tables 11.5 and 11.6. The use of such tables is demonstrated in Example 11.7.

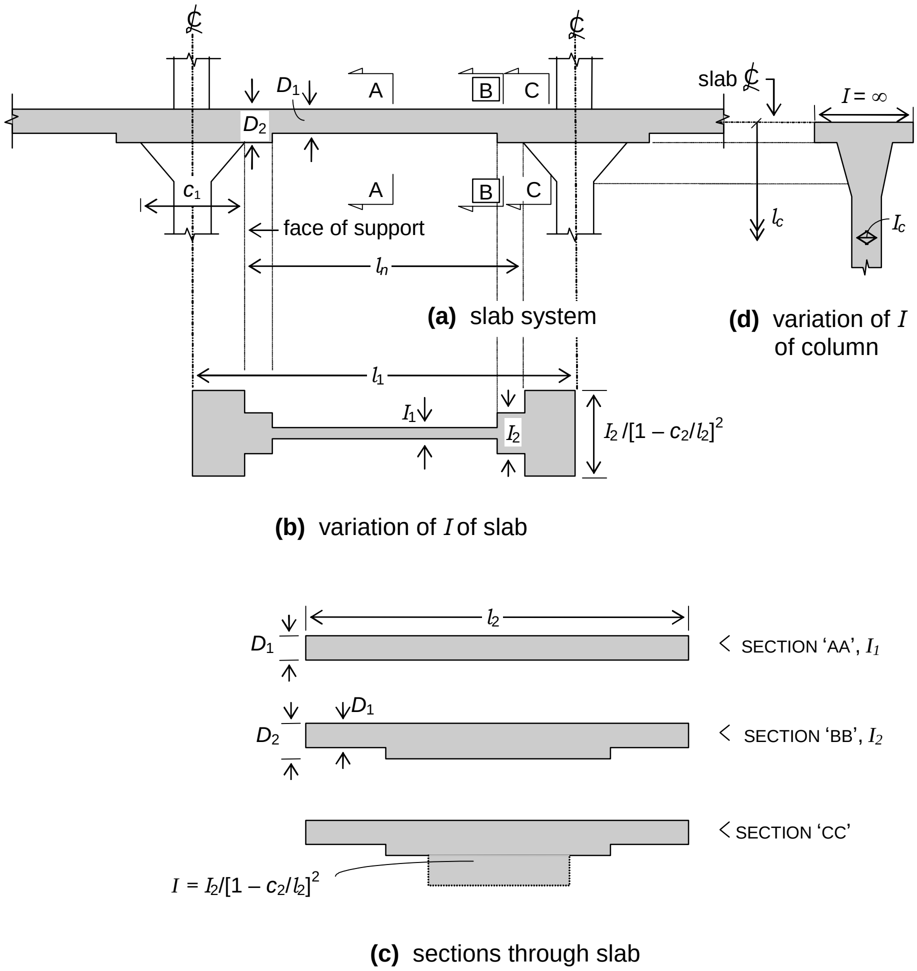
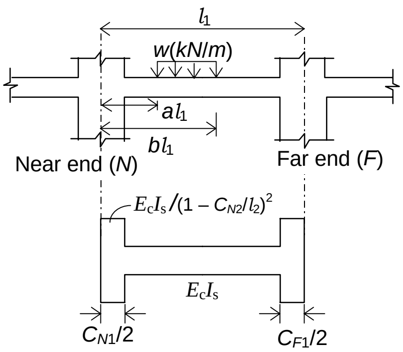
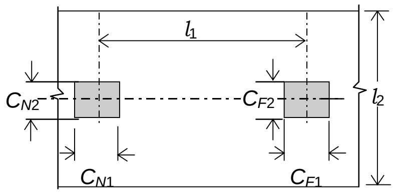



**Fig. 11.36** Variation of second moment of area along member axis

‡ However, these factors are not required in modern computer-based analyses using the finite element method; nodes are introduced at the locations where the second moment of area changes.

**Table 11.5** Moment distribution constants for slab-beam elements




$$FEM_{NF} = \sum_{i=1}^{n} m_{NFi}\, w_i\, l_1^2$$

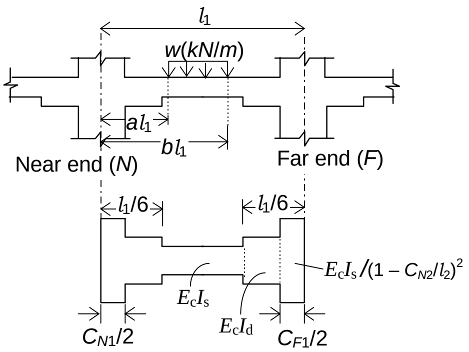
$K_{NF} = k_{NF}\, E_c I_s / l_1$

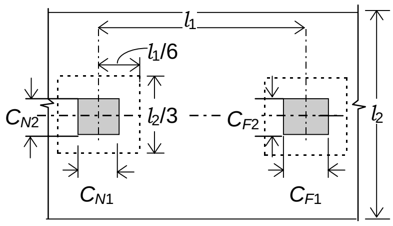
| $C_{N1}/l_1$ | $C_{N2}/l_2$ | Stiff-ness factor $K_{NF}$ | Carry over factor $C_{NF}$ | Unif. load f.e.m. coeff. ($m_{NF}$) | Fixed end moment coeff. ($m_{NF}$) for (b – a) = 0.2 | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | a = 0.0 | a = 0.2 | a = 0.4 | a = 0.6 | a = 0.8 |
| $C_{F1} = C_{N1}$ ; $C_{F2} = C_{N2}$ | | | | | | | | | |
| 0.00 | -- | 4.00 | 0.50 | 0.0833 | 0.0151 | 0.0287 | 0.0247 | 0.0127 | 0.00226 |
| | 0.00 | 4.00 | 0.50 | 0.0833 | 0.0151 | 0.0287 | 0.0247 | 0.0127 | 0.00226 |
| | 0.10 | 4.18 | 0.51 | 0.0847 | 0.0154 | 0.0293 | 0.0251 | 0.0126 | 0.00214 |
| 0.10 | 0.20 | 4.36 | 0.52 | 0.0860 | 0.0158 | 0.0300 | 0.0255 | 0.0126 | 0.00201 |
| | 0.30 | 4.53 | 0.54 | 0.0872 | 0.0161 | 0.0301 | 0.0259 | 0.0125 | 0.00188 |
| | 0.40 | 4.70 | 0.55 | 0.0882 | 0.0165 | 0.0314 | 0.0262 | 0.0124 | 0.00174 |
| | 0.00 | 4.00 | 0.50 | 0.0833 | 0.0151 | 0.0287 | 0.0247 | 0.0127 | 0.00226 |
| | 0.10 | 4.35 | 0.52 | 0.0857 | 0.0155 | 0.0299 | 0.0254 | 0.0127 | 0.00213 |
| 0.20 | 0.20 | 4.72 | 0.54 | 0.0880 | 0.0161 | 0.0311 | 0.0262 | 0.0126 | 0.00197 |
| | 0.30 | 5.11 | 0.56 | 0.0901 | 0.0166 | 0.0324 | 0.0269 | 0.0125 | 0.00178 |
| | 0.40 | 5.51 | 0.58 | 0.0921 | 0.0171 | 0.0366 | 0.0276 | 0.0123 | 0.00156 |
| | 0.00 | 4.00 | 0.50 | 0.0833 | 0.0151 | 0.0287 | 0.0247 | 0.0127 | 0.00226 |
| | 0.10 | 4.49 | 0.53 | 0.0863 | 0.0155 | 0.0301 | 0.0257 | 0.0128 | 0.00219 |
| 0.30 | 0.20 | 5.05 | 0.56 | 0.0893 | 0.0160 | 0.0317 | 0.0267 | 0.0128 | 0.00207 |
| | 0.30 | 5.69 | 0.59 | 0.0923 | 0.0165 | 0.0334 | 0.0278 | 0.0127 | 0.00190 |
| | 0.40 | 6.41 | 0.61 | 0.0951 | 0.0171 | 0.0352 | 0.0287 | 0.0124 | 0.00167 |
| | 0.00 | 4.00 | 0.50 | 0.0833 | 0.0151 | 0.0287 | 0.0247 | 0.0127 | 0.00226 |
| | 0.10 | 4.61 | 0.53 | 0.0866 | 0.0154 | 0.0302 | 0.0259 | 0.0129 | 0.00225 |
| 0.40 | 0.20 | 5.35 | 0.56 | 0.0901 | 0.0158 | 0.0318 | 0.0271 | 0.0131 | 0.00221 |
| | 0.30 | 6.25 | 0.60 | 0.0936 | 0.0162 | 0.0337 | 0.0284 | 0.0131 | 0.00211 |
| | 0.40 | 7.37 | 0.64 | 0.0971 | 0.0168 | 0.0359 | 0.0297 | 0.0128 | 0.00195 |

**Table 11.5** (contd.)

| $C_{N1}/l_1$ | $C_{N2}/l_2$ | Stiff-ness factor $K_{NF}$ | Carry over factor $C_{NF}$ | Unif. load f.e.m. coeff. ($m_{NF}$) | Fixed end moment coeff. ($m_{NF}$) for (b – a) = 0.2 | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | a = 0.0 | a = 0.2 | a = 0.4 | a = 0.6 | a = 0.8 |
| $C_{F1} = 0.5C_{N1}$ ; $C_{F2} = 0.5C_{N2}$ | | | | | | | | | |
| 0.00 | -- | 4.00 | 0.50 | 0.0833 | 0.0151 | 0.0287 | 0.0247 | 0.0127 | 0.0023 |
| 0.10 | 0.00 | 4.00 | 0.50 | 0.0833 | 0.0151 | 0.0287 | 0.0247 | 0.0127 | 0.0023 |
| | 0.10 | 4.16 | 0.51 | 0.0857 | 0.0155 | 0.0296 | 0.0254 | 0.0130 | 0.0023 |
| | 0.20 | 4.31 | 0.52 | 0.0879 | 0.0158 | 0.0304 | 0.0261 | 0.0133 | 0.0023 |
| | 0.30 | 4.45 | 0.54 | 0.0900 | 0.0162 | 0.0312 | 0.0267 | 0.0135 | 0.0023 |
| | 0.40 | 4.58 | 0.54 | 0.0918 | 0.0165 | 0.0319 | 0.0273 | 0.0138 | 0.0023 |
| 0.20 | 0.00 | 4.00 | 0.50 | 0.0833 | 0.0151 | 0.0287 | 0.0247 | 0.0127 | 0.0023 |
| | 0.10 | 4.30 | 0.52 | 0.0872 | 0.0156 | 0.0301 | 0.0259 | 0.0132 | 0.0023 |
| | 0.20 | 4.61 | 0.55 | 0.0912 | 0.0161 | 0.0317 | 0.0272 | 0.0138 | 0.0023 |
| | 0.30 | 4.92 | 0.57 | 0.0951 | 0.0167 | 0.0332 | 0.0285 | 0.0143 | 0.0024 |
| | 0.40 | 5.23 | 0.58 | 0.0989 | 0.0172 | 0.0347 | 0.0298 | 0.0148 | 0.0024 |
| 0.30 | 0.00 | 4.00 | 0.50 | 0.0833 | 0.0151 | 0.0287 | 0.0247 | 0.0127 | 0.0023 |
| | 0.10 | 4.43 | 0.53 | 0.0881 | 0.0156 | 0.0305 | 0.0263 | 0.0134 | 0.0023 |
| | 0.20 | 4.89 | 0.56 | 0.0932 | 0.0161 | 0.0324 | 0.0281 | 0.0142 | 0.0024 |
| | 0.30 | 5.40 | 0.59 | 0.0986 | 0.0167 | 0.0345 | 0.0300 | 0.0150 | 0.0024 |
| | 0.40 | 5.93 | 0.62 | 0.1042 | 0.0173 | 0.0367 | 0.0320 | 0.0158 | 0.00235 |
| 0.40 | 0.00 | 4.00 | 0.50 | 0.0833 | 0.0151 | 0.0287 | 0.0247 | 0.0127 | 0.0023 |
| | 0.10 | 4.54 | 0.54 | 0.0884 | 0.0155 | 0.0305 | 0.0265 | 0.0135 | 0.0024 |
| | 0.20 | 5.16 | 0.57 | 0.0941 | 0.0159 | 0.0326 | 0.0286 | 0.0145 | 0.0025 |
| | 0.30 | 5.87 | 0.61 | 0.1005 | 0.0165 | 0.0350 | 0.0310 | 0.0155 | 0.0025 |
| | 0.40 | 6.67 | 0.64 | 0.1076 | 0.0170 | 0.0377 | 0.0336 | 0.0166 | 0.0026 |
| $C_{F1} = 2C_{N1}$ ; $C_{F2} = 2C_{N2}$ | | | | | | | | | |
| 0.00 | -- | 4.00 | 0.50 | 0.0833 | 0.0151 | 0.0287 | 0.0247 | 0.0127 | 0.0023 |
| 0.10 | 0.00 | 4.00 | 0.50 | 0.0833 | 0.0150 | 0.0287 | 0.0247 | 0.0127 | 0.0023 |
| | 0.10 | 4.27 | 0.51 | 0.0517 | 0.0153 | 0.0289 | 0.0241 | 0.0116 | 0.0018 |
| | 0.20 | 4.56 | 0.52 | 0.0798 | 0.0156 | 0.0290 | 0.0234 | 0.0103 | 0.0013 |
| 0.20 | 0.00 | 4.00 | 0.50 | 0.0833 | 0.0151 | 0.0287 | 0.0247 | 0.0127 | 0.0023 |
| | 0.10 | 4.49 | 0.51 | 0.0819 | 0.0154 | 0.0291 | 0.0240 | 0.0114 | 0.0019 |
| | 0.20 | 5.11 | 0.53 | 0.0789 | 0.0158 | 0.0293 | 0.0228 | 0.0096 | 0.0014 |

**Table 11.6** Moment distribution constants for slab-beam elements, drop thickness = 0.50*D*




$$FEM_{NF} = \sum_{i=1}^{n} m_{NFi}\, w_i\, l_1^2$$

$$K_{NF} = k_{NF}\, E_c I_s / l_1$$

| $C_{N1}/l_1$ | $C_{N2}/l_2$ | Stiff-ness factor $K_{NF}$ | Carry over factor $C_{NF}$ | Unif. load f.e.m. coeff. ($m_{NF}$) | Fixed end Moment coeff. ($m_{NF}$) for (b – a) = 0.2 | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | a = 0.0 | a = 0.2 | a = 0.4 | a = 0.6 | a = 0.8 |
| ***$C_{F1} = C_{N1}$ ; $C_{F2} = C_{N2}$*** | | | | | | | | | |
| 0.00 | -- | 5.84 | 0.59 | 0.0926 | 0.0164 | 0.0335 | 0.0279 | 0.0128 | 0.0020 |
| 0.10 | 0.00 | 5.84 | 0.59 | 0.0926 | 0.0164 | 0.0335 | 0.0279 | 0.0128 | 0.0020 |
| | 0.10 | 6.04 | 0.60 | 0.0936 | 0.0167 | 0.0341 | 0.0282 | 0.0126 | 0.0018 |
| | 0.20 | 6.24 | 0.61 | 0.0940 | 0.0170 | 0.0347 | 0.0285 | 0.0125 | 0.0017 |
| | 0.30 | 6.43 | 0.61 | 0.0952 | 0.0173 | 0.0353 | 0.0287 | 0.0123 | 0.0016 |
| 0.20 | 0.00 | 5.84 | 0.59 | 0.0926 | 0.0164 | 0.0335 | 0.0279 | 0.0128 | 0.0020 |
| | 0.10 | 6.22 | 0.61 | 0.0942 | 0.0168 | 0.0346 | 0.0285 | 0.0126 | 0.0018 |
| | 0.20 | 6.62 | 0.62 | 0.0957 | 0.0172 | 0.0356 | 0.0290 | 0.0123 | 0.0016 |
| | 0.30 | 7.01 | 0.64 | 0.0971 | 0.0177 | 0.0366 | 0.0294 | 0.0120 | 0.0014 |
| 0.30 | 0.00 | 5.84 | 0.59 | 0.0926 | 0.0164 | 0.0355 | 0.0279 | 0.0128 | 0.0020 |
| | 0.10 | 6.37 | 0.61 | 0.0947 | 0.0168 | 0.0348 | 0.0287 | 0.0126 | 0.0018 |
| | 0.20 | 6.95 | 0.63 | 0.0967 | 0.0172 | 0.0362 | 0.0294 | 0.0123 | 0.0016 |
| | 0.30 | 7.57 | 0.65 | 0.0986 | 0.0177 | 0.0375 | 0.0300 | 0.0119 | 0.0014 |
| ***$C_{F1} = 0.5C_{N1}$ ; $C_{F2} = 0.5C_{N2}$*** | | | | | | | | | |
| 0.00 | -- | 5.84 | 0.59 | 0.0926 | 0.0164 | 0.0335 | 0.0279 | 0.0128 | 0.0020 |
| 0.10 | 0.00 | 5.84 | 0.59 | 0.0926 | 0.0164 | 0.0335 | 0.0279 | 0.0128 | 0.0020 |
| | 0.10 | 6.00 | 0.60 | 0.0945 | 0.0167 | 0.0343 | 0.0285 | 0.0130 | 0.0020 |
| | 0.20 | 6.16 | 0.60 | 0.0962 | 0.0170 | 0.0350 | 0.0291 | 0.0132 | 0.0020 |
| 0.20 | 0.00 | 5.84 | 0.59 | 0.0926 | 0.0164 | 0.0335 | 0.0279 | 0.0128 | 0.0020 |
| | 0.10 | 6.15 | 0.60 | 0.0957 | 0.0169 | 0.0348 | 0.0290 | 0.0131 | 0.0020 |
| | 0.20 | 6.47 | 0.62 | 0.0987 | 0.0173 | 0.0360 | 0.0300 | 0.0134 | 0.0020 |
| ***$C_{F1} = 2C_{N1}$ ; $C_{F2} = 2C_{N2}$*** | | | | | | | | | |
| 0.00 | -- | 5.84 | 0.59 | 0.0926 | 0.0164 | 0.0335 | 0.0279 | 0.0128 | 0.0020 |
| 0.10 | 0.00 | 5.84 | 0.59 | 0.0926 | 0.0164 | 0.0335 | 0.0279 | 0.0128 | 0.0020 |
| | 0.10 | 6.17 | 0.60 | 0.0907 | 0.0166 | 0.0337 | 0.0273 | 0.0116 | 0.0015 |

### *Equivalent columns*

As mentioned earlier, the actual columns above and below, and the torsional member are replaced by an equivalent column of stiffness $K_{ec}$. The cross-section to be considered for the torsional member is that of the flanged section of the transverse beam defined earlier in Fig. 11.27, and in the absence of a beam along the column line, it may be limited to the portion of the slab having a width equal to that of the column, bracket or capital measured in the direction of the span $l_1$ [Ref. 11.18].

The concept of an 'equivalent column' is introduced to account for the *increased flexibility* (reduced flexural restraint) of the connection of the slab-beam member to its support (see Section 11.6.1), because of its connection to the column, for most of its width, through a torsional member. This is effected by taking the equivalent (or effective) *flexibility* (inverse of *stiffness*) of the connection as equal to the sum of the flexibilities of the actual columns and the torsional member. The stiffness, $K_{ec}$, of the equivalent column is thus obtained from:

$$\frac{1}{K_{ec}} = \frac{1}{\sum K_c} + \frac{1}{K_t} \tag{11.45a}$$

$$\Rightarrow \quad K_{ec} = \frac{K_t \sum K_c}{K_t + \sum K_c} \tag{11.45b}$$

Evidently, $K_{ec} \le \Sigma K_c$; i.e., the effect of the flexibility of the torsional member is to reduce the rotational restraint offered to the slab-beam member at the support section. The condition $K_{ec} = \Sigma K_c$ (implying that the rotation along the entire length AB in Fig. 11.35(a) is the same) can be assumed to occur only if the torsional stiffness of the transverse torsional member is infinite or if the column is in fact a wall with a width extending over the full width of the slab-beam member. On the other hand, if $\Sigma K_c = \propto$, $K_{ec} = K_t$, implying that although the column is infinitely stiff, the slab undergoes the same rotation as that of the torsional member along the length AB in Fig. 11.35(a) except for the width at the column location.

Another interesting result of $K_{ec} = 0$ is obtained for the hypothetical case of a torsional member with $K_t = 0$, or for the case of a slab simply supported on a masonry wall ($K_c = 0$) throughout the length AB. In the former case, the slab is flexurally unrestrained along the length AB (except for the width $c_2$ at the column location), and in the latter case, the slab is flexurally unrestrained throughout the entire length AB, and in both cases $K_{ec}$ should naturally be zero.

For the purpose of computing the torsional stiffness of the transverse member, the following approximate expression [Ref. 11.18] is recommended:

$$K_t = \sum \frac{9E_c C}{l_2(1 - c_2/l_2)^3} \tag{11.46}$$

where

$E_c \equiv$ modulus of elasticity of concrete (in the torsional member);

$C \equiv$ torsional property of the cross-section[†] [refer Eq. 11.41]; and

$c_2 \equiv$ width of the equivalent rectangular column, capital or bracket, measured transverse to the direction in which moments are being determined.

It should be noted that the concept of the 'equivalent column stiffness', $K_{ec}$, explained here, is applicable for gravity load analysis, and not for lateral load analysis.

It may be noted that the elastic frame analogy may also be used for the lateral load analysis of this type of unbraced frames, comprising column-supported slab system. However, the Code does not give any guidance on assigning member stiffnesses in such situations. For gravity load analysis, the reduced restraint to the slab-beam member at the column support is accounted for by reducing the effective column stiffness, as explained above. By analogy, for lateral load analysis, the effective stiffness of the slab-beam member has to be reduced. However, it should be realised that, due to the large flexural stiffness of the floor slab in its own plane, the *sway* for all columns in a storey will, in most cases, be very nearly equal. Hence, the columns share the lateral load very nearly in proportion to their stiffnesses; and the column shears determine the column moments at the connection. For this reason, the effective slab-beam member (flexural) stiffness does not usually affect the column moments under lateral load significantly [Ref. 11.29]. However, if the *lateral drift* (sway) is to be computed, a realistic assessment of the effective slab-beam member stiffness is required. Some studies [such as Ref. 11.29] have indicated that calculations with an effective width for the slab-beam member, $b_e = c_2 + 2D$, gives satisfactory drift predictions.

For the calculation of the column stiffness, $K_c$, the variation of the second moment of area along the height of the actual column should be taken into account. The height of the column $l_c$, is measured centre-to-centre of floors, and the second moment of area of the column section for the portion integral with the slab-beam member should be taken as infinite [Fig. 11.36(d)]. The change in cross-section on account of the column capital and drop panel (wherever provided) should also be considered in computing the stiffness and carry-over factors. To facilitate calculations, these factors have been evaluated for common configurations in various design handbooks [Ref. 11.11, 11.25, 11.26], and are indicated in Table 11.7.

### *Method of Analysis*

The gravity load analysis of the equivalent (substitute) frame, with the appropriate stiffnesses for the slab-beam members and equivalent columns, may be done by any method of structural analysis. For manual calculations, the *moment distribution method* is particularly suitable, as illustrated in Example 11.7.

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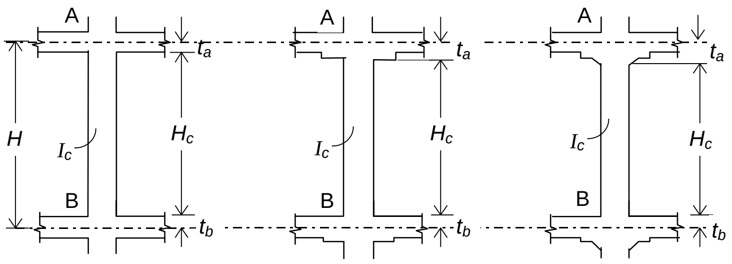
[†] For calculation purposes, the torsional member is assumed to have a uniform section throughout its length. In flat slabs, this implies that the provision of drop panels (if any) is ignored.

**Table 11.7** Stiffness and carry-over factors for columns




$K_c = k(EI_c)/H$

For values of $k_{BA}$ and $C_{BA}$, read $(t_a/t_b)$ as $(t_b/t_a)$

| $t_a/t_b$ | $H/H_c$ | 1.05 | 1.10 | 1.15 | 1.20 | 1.25 | 1.30 | 1.35 | 1.40 | 1.45 | 1.50 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | $K_{AB}$ | 4.20 | 4.40 | 4.60 | 4.80 | 5.00 | 5.20 | 5.40 | 5.60 | 5.80 | 6.00 |
| | $C_{AB}$ | 0.57 | 0.65 | 0.73 | 0.80 | 0.87 | 0.95 | 1.03 | 1.10 | 1.17 | 1.25 |
| 0.2 | $K_{AB}$ | 4.31 | 4.62 | 4.95 | 5.30 | 5.65 | 6.02 | 6.40 | 6.79 | 7.20 | 7.62 |
| | $C_{AB}$ | 0.56 | 0.62 | 0.68 | 0.74 | 0.80 | 0.85 | 0.91 | 0.96 | 1.01 | 1.07 |
| 0.4 | $K_{AB}$ | 4.38 | 4.79 | 5.22 | 5.67 | 6.15 | 6.65 | 7.18 | 7.74 | 8.32 | 8.94 |
| | $C_{AB}$ | 0.55 | 0.60 | 0.65 | 0.70 | 0.74 | 0.79 | 0.83 | 0.87 | 0.91 | 0.94 |
| 0.6 | $K_{AB}$ | 4.44 | 4.91 | 5.42 | 5.96 | 6.54 | 7.15 | 7.81 | 8.50 | 9.23 | 10.01 |
| | $C_{AB}$ | 0.55 | 0.59 | 0.63 | 0.67 | 0.70 | 0.74 | 0.77 | 0.80 | 0.83 | 0.85 |
| 0.8 | $K_{AB}$ | 4.49 | 5.01 | 5.58 | 6.19 | 6.85 | 7.56 | 8.31 | 9.12 | 9.98 | 10.89 |
| | $C_{AB}$ | 0.54 | 0.58 | 0.61 | 0.64 | 0.67 | 0.70 | 0.72 | 0.75 | 0.77 | 0.79 |
| 1.0 | $K_{AB}$ | 4.52 | 5.09 | 5.71 | 6.38 | 7.11 | 7.89 | 8.73 | 9.63 | 10.60 | 11.62 |
| | $C_{AB}$ | 0.54 | 0.57 | 0.60 | 0.62 | 0.65 | 0.67 | 0.69 | 0.71 | 0.73 | 0.74 |
| 1.2 | $K_{AB}$ | 4.55 | 5.16 | 5.82 | 6.54 | 7.32 | 8.17 | 9.08 | 10.07 | 11.12 | 12.25 |
| | $C_{AB}$ | 0.53 | 0.56 | 0.59 | 0.61 | 0.63 | 0.65 | 0.66 | 0.68 | 0.69 | 0.70 |
| 1.4 | $K_{AB}$ | 4.58 | 5.21 | 5.91 | 6.68 | 7.51 | 8.41 | 9.38 | 10.43 | 11.57 | 12.78 |
| | $C_{AB}$ | 0.53 | 0.55 | 0.58 | 0.60 | 0.61 | 0.63 | 0.64 | 0.65 | 0.66 | 0.67 |
| 1.6 | $K_{AB}$ | 4.60 | 5.26 | 5.99 | 6.79 | 7.66 | 8.61 | 9.64 | 10.75 | 11.95 | 13.24 |
| | $C_{AB}$ | 0.53 | 0.55 | 0.57 | 0.59 | 0.60 | 0.61 | 0.62 | 0.63 | 0.64 | 0.65 |
| 1.8 | $K_{AB}$ | 4.62 | 5.30 | 6.06 | 6.89 | 7.80 | 8.79 | 9.87 | 11.03 | 12.29 | 13.65 |
| | $C_{AB}$ | 0.52 | 0.55 | 0.56 | 0.58 | 0.59 | 0.60 | 0.61 | 0.61 | 0.62 | 0.63 |
| 2.0 | $K_{AB}$ | 4.63 | 5.34 | 6.12 | 6.98 | 7.92 | 8.94 | 10.06 | 11.27 | 12.59 | 14.00 |
| | $C_{AB}$ | 0.52 | 0.54 | 0.56 | 0.57 | 0.58 | 0.59 | 0.59 | 0.60 | 0.60 | 0.61 |
| 2.2 | $K_{AB}$ | 4.65 | 5.37 | 6.17 | 7.05 | 8.02 | 9.08 | 10.24 | 11.49 | 12.85 | 14.31 |
| | $C_{AB}$ | 0.52 | 0.54 | 0.55 | 0.56 | 0.57 | 0.58 | 0.58 | 0.59 | 0.59 | 0.59 |
| 2.4 | $K_{AB}$ | 4.66 | 5.40 | 6.22 | 7.12 | 8.11 | 9.20 | 10.39 | 11.68 | 13.08 | 14.60 |
| | $C_{AB}$ | 0.52 | 0.53 | 0.55 | 0.56 | 0.56 | 0.57 | 0.57 | 0.58 | 0.58 | 0.58 |
| 2.6 | $K_{AB}$ | 4.67 | 5.42 | 6.26 | 7.18 | 8.20 | 9.31 | 10.53 | 11.86 | 13.29 | 14.85 |
| | $C_{AB}$ | 0.52 | 0.53 | 0.54 | 0.55 | 0.56 | 0.56 | 0.56 | 0.57 | 0.57 | 0.57 |
| 2.8 | $K_{AB}$ | 4.68 | 5.44 | 6.29 | 7.23 | 8.27 | 9.41 | 10.66 | 12.01 | 13.48 | 15.07 |
| | $C_{AB}$ | 0.52 | 0.53 | 0.54 | 0.55 | 0.55 | 0.55 | 0.56 | 0.56 | 0.56 | 0.56 |
| 3.0 | $K_{AB}$ | 4.69 | 5.46 | 6.33 | 7.28 | 8.34 | 9.50 | 10.77 | 12.15 | 13.65 | 15.28 |
| | $C_{AB}$ | 0.52 | 0.53 | 0.54 | 0.54 | 0.55 | 0.55 | 0.55 | 0.55 | 0.55 | 0.55 |
| 3.2 | $K_{AB}$ | 4.70 | 5.48 | 6.36 | 7.33 | 8.40 | 9.58 | 10.87 | 12.28 | 13.81 | 15.47 |
| | $C_{AB}$ | 0.52 | 0.53 | 0.53 | 0.54 | 0.54 | 0.54 | 0.54 | 0.54 | 0.54 | 0.54 |
| 3.4 | $K_{AB}$ | 4.71 | 5.50 | 6.38 | 7.37 | 8.46 | 9.65 | 10.97 | 12.40 | 13.95 | 15.64 |
| | $C_{AB}$ | 0.51 | 0.52 | 0.53 | 0.53 | 0.54 | 0.54 | 0.54 | 0.53 | 0.53 | 0.53 |

**Table 11.7** (contd.)

| $t_a/t_b$ | $H/H_c$ | 1.05 | 1.10 | 1.15 | 1.20 | 1.25 | 1.30 | 1.35 | 1.40 | 1.45 | 1.50 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 3.6 | $K_{AB}$ | 4.71 | 5.51 | 6.41 | 7.41 | 8.51 | 9.72 | 11.05 | 12.51 | 14.09 | 15.80 |
| | $C_{AB}$ | 0.51 | 0.52 | 0.53 | 0.53 | 0.53 | 0.53 | 0.53 | 0.53 | 0.53 | 0.52 |
| 3.8 | $K_{AB}$ | 4.72 | 5.53 | 6.43 | 7.44 | 8.56 | 9.78 | 11.13 | 12.60 | 14.21 | 15.95 |
| | $C_{AB}$ | 0.51 | 0.52 | 0.53 | 0.53 | 0.53 | 0.53 | 0.53 | 0.52 | 0.52 | 0.52 |
| 4.0 | $K_{AB}$ | 4.72 | 5.54 | 6.45 | 7.47 | 8.60 | 9.84 | 11.21 | 12.70 | 14.32 | 16.08 |
| | $C_{AB}$ | 0.51 | 0.52 | 0.52 | 0.53 | 0.53 | 0.52 | 0.52 | 0.52 | 0.52 | 0.51 |
| 4.2 | $K_{AB}$ | 4.73 | 5.55 | 6.47 | 7.50 | 8.64 | 9.90 | 11.27 | 12.78 | 14.42 | 16.02 |
| | $C_{AB}$ | 0.51 | 0.52 | 0.52 | 0.52 | 0.52 | 0.52 | 0.52 | 0.51 | 0.51 | 0.51 |
| 4.4 | $K_{AB}$ | 4.73 | 5.56 | 6.49 | 7.53 | 8.68 | 9.95 | 11.34 | 12.86 | 14.52 | 16.32 |
| | $C_{AB}$ | 0.51 | 0.52 | 0.52 | 0.52 | 0.52 | 0.52 | 0.51 | 0.51 | 0.51 | 0.50 |
| 4.6 | $K_{AB}$ | 4.74 | 5.57 | 6.51 | 7.55 | 8.71 | 9.99 | 11.40 | 12.93 | 14.61 | 16.43 |
| | $C_{AB}$ | 0.51 | 0.52 | 0.52 | 0.52 | 0.52 | 0.52 | 0.51 | 0.51 | 0.50 | 0.50 |
| 4.8 | $K_{AB}$ | 4.74 | 5.58 | 6.53 | 7.58 | 8.75 | 10.03 | 11.45 | 13.00 | 14.69 | 16.53 |
| | $C_{AB}$ | 0.51 | 0.52 | 0.52 | 0.52 | 0.52 | 0.51 | 0.51 | 0.50 | 0.50 | 0.49 |
| 5.0 | $K_{AB}$ | 4.75 | 5.59 | 6.54 | 7.60 | 8.78 | 10.07 | 11.50 | 13.07 | 14.77 | 16.62 |
| | $C_{AB}$ | 0.51 | 0.51 | 0.52 | 0.52 | 0.51 | 0.51 | 0.51 | 0.50 | 0.49 | 0.49 |
| 6.0 | $K_{AB}$ | 4.76 | 5.63 | 6.60 | 7.69 | 8.90 | 10.24 | 11.72 | 13.33 | 15.10 | 17.02 |
| | $C_{AB}$ | 0.51 | 0.51 | 0.51 | 0.51 | 0.50 | 0.50 | 0.49 | 0.49 | 0.48 | 0.47 |
| 7.0 | $K_{AB}$ | 4.78 | 5.66 | 6.65 | 7.76 | 9.00 | 10.37 | 11.88 | 13.54 | 15.35 | 17.32 |
| | $C_{AB}$ | 0.51 | 0.51 | 0.51 | 0.50 | 0.50 | 0.49 | 0.48 | 0.48 | 0.47 | 0.46 |
| 8.0 | $K_{AB}$ | 4.78 | 5.68 | 6.69 | 7.82 | 9.07 | 10.47 | 12.01 | 13.70 | 15.54 | 17.56 |
| | $C_{AB}$ | 0.51 | 0.51 | 0.50 | 0.50 | 0.49 | 0.49 | 0.48 | 0.47 | 0.46 | 0.45 |
| 9.0 | $K_{AB}$ | 4.79 | 5.69 | 6.71 | 7.86 | 9.13 | 10.55 | 12.11 | 13.83 | 15.70 | 17.74 |
| | $C_{AB}$ | 0.50 | 0.50 | 0.50 | 0.50 | 0.49 | 0.48 | 0.47 | 0.46 | 0.45 | 0.45 |
| 10.0 | $K_{AB}$ | 4.80 | 5.71 | 6.74 | 7.89 | 9.18 | 10.61 | 12.19 | 13.93 | 15.83 | 17.90 |
| | $C_{AB}$ | 0.50 | 0.50 | 0.50 | 0.49 | 0.48 | 0.48 | 0.47 | 0.46 | 0.45 | 0.44 |

### 11.6.3 Loading Patterns

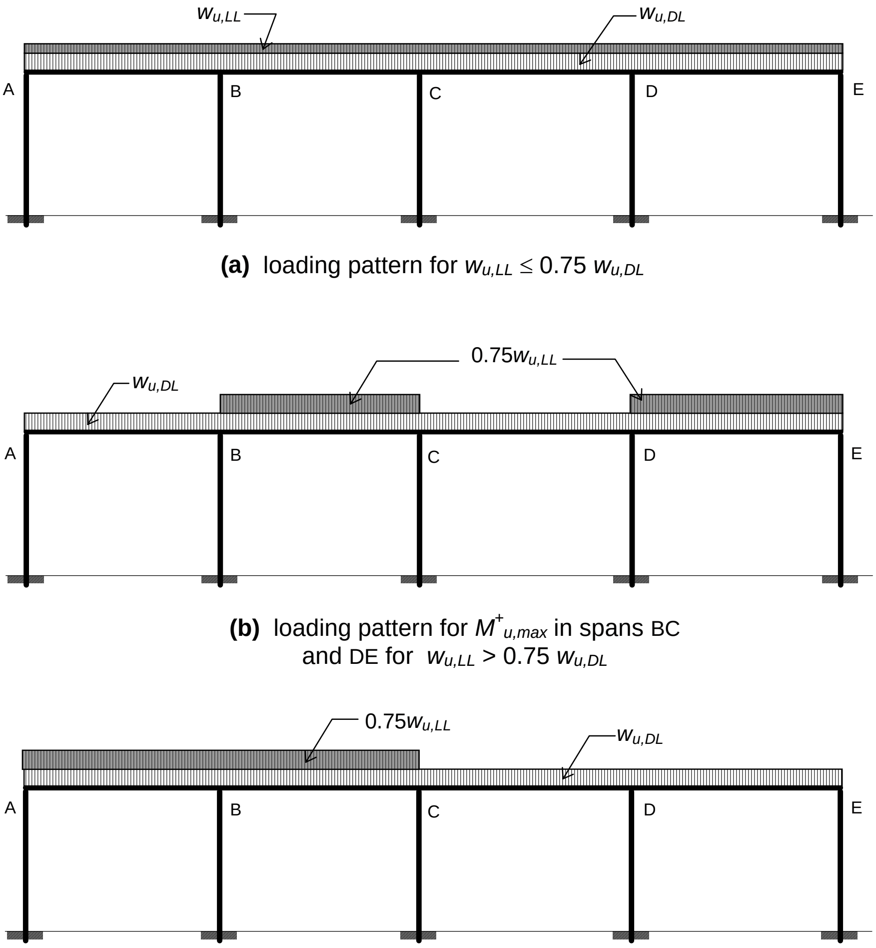
The gravity load analysis of the equivalent (substitute) frame, with the appropriate stiffnesses for the slab-beam members and equivalent columns, may be done by any method of structural analysis. For manual calculations, the *moment distribution method* is particularly suitable, as illustrated in Example 11.7.

When specific gravity load patterns are indicated[†] , the equivalent frame should be analysed for those patterns [Cl. 31.5.2.1 of the Code]. When the live load is variable, but does not exceed three-fourths of the dead load, or the nature of the live load is such that all the panels will be loaded simultaneously the Code (Cl. 31.5.2.2) recommends a single loading case of full factored loads (dead plus live) on all spans for analysis for design moments in the slab [Fig. 11.37(a)].

For larger live load/dead load ratios ($w_{LL}/w_{DL} > 0.75$), it is acceptable to design only for three-fourths of the full live load on alternate spans for maximum 'positive' moments in spans, and on adjacent spans for maximum 'negative' moments at supports, as indicated in Fig. 11.37(b) and (c) respectively [Cl. 31.5.2.3 of the Code].

[†] For example, water tank bases are often two-way slabs (with or without beams) supported on columns. The loading due to water should be expected to act on all the panels (bounded within the tank walls) simultaneously [Ref. 11.11].

However, in no case must the design moments be taken as less than those occurring with full factored loads (dead plus live) on all spans [Fig. 11.37(a)]. The use of only three-fourths of the full design live load for maximum moment loading patterns accounts for possible moment redistribution in the frame [Ref. 11.11].




**(a)** loading pattern for $w_{u,LL} \leq 0.75\ w_{u,DL}$

**(b)** loading pattern for $M^{+}_{u,max}$ in spans BC and DE for $w_{u,LL} > 0.75\ w_{u,DL}$

**(c)** loading pattern for $M^{-}_{u,max}$ at support B for $w_{u,LL} > 0.75\ w_{u,DL}$

**Fig. 11.37** Gravity loading patterns for equivalent frame analysis

### 11.6.4 Design Moments in Slab-Beam Members

The results of the equivalent frame analysis using centreline dimensions gives 'negative' moments at the centreline of the supports, from which the design 'negative' moments at the critical sections have to be deduced, as shown in

Fig. 11.38. The Code (Cl. 31.5.3.1) recommends that the critical section of the slab-beam member for 'negative' moment at an interior support is to be taken at the face of the support (column, capital or bracket), but in no case at a distance greater than $0.175l_1$ from the centre of the column. At an exterior support with a capital or bracket, the critical section is to be taken [Cl. 31.5.3.2 of the Code] at a distance from the face of the column not greater than one-half of the projection of the bracket or capital beyond the face of the column‡. These Code specifications are illustrated in Fig. 11.38.

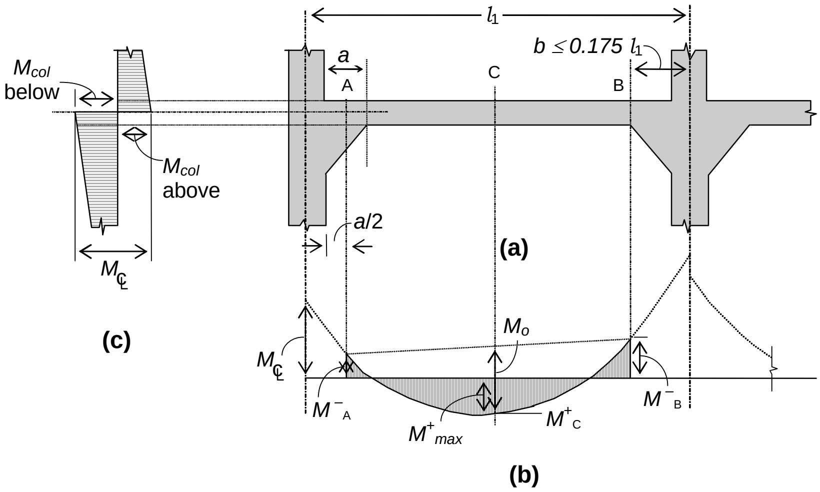



**Fig. 11.38** Critical sections for 'negative' design moments in the slab-beam member and column

The design 'positive' (maximum) moment in the beam-slab member should be obtained from the zero shear location, which, in general need not correspond to the midspan location [Fig. 11.38(b)].

To maintain consistency in design requirements, the Code (Cl. 31.5.4) provides that in a two-way slab system which meets the limitations of DDM (given in Section 11.5.1), but is analysed by EFM, the design moments obtained from equivalent frame analysis, if found excessive, may be reduced in such proportion that the numerical sum of the 'positive' moment at midspan and the average 'negative' moment used in design need not exceed the value $M_o$ obtained from Eq. 11.31 [Fig. 11.38(b)].

The apportioning of the design 'positive' and 'negative' moments in the transverse direction to the column strip and half-middle strips in the case of slabs without beams, and to the beam and slab in the case of slabs with beams between all supports, should be done as explained in Section 11.5.4.

‡ Circular or regular polygon shaped supports are to be treated as equivalent square supports having the same cross–sectional area [Cl. 31.5.3.3 of the Code].

### 11.6.5 Design Moments in Columns and Torsion in Transverse Beam

The design moments in the 'equivalent columns' are obtained from equivalent frame analysis. The moment (due to gravity loads) is usually significant only at the exterior support. This moment has to be distributed to the actual columns located above and below the floor in proportion to their relative flexural stiffnesses [Fig. 11.38(c)].

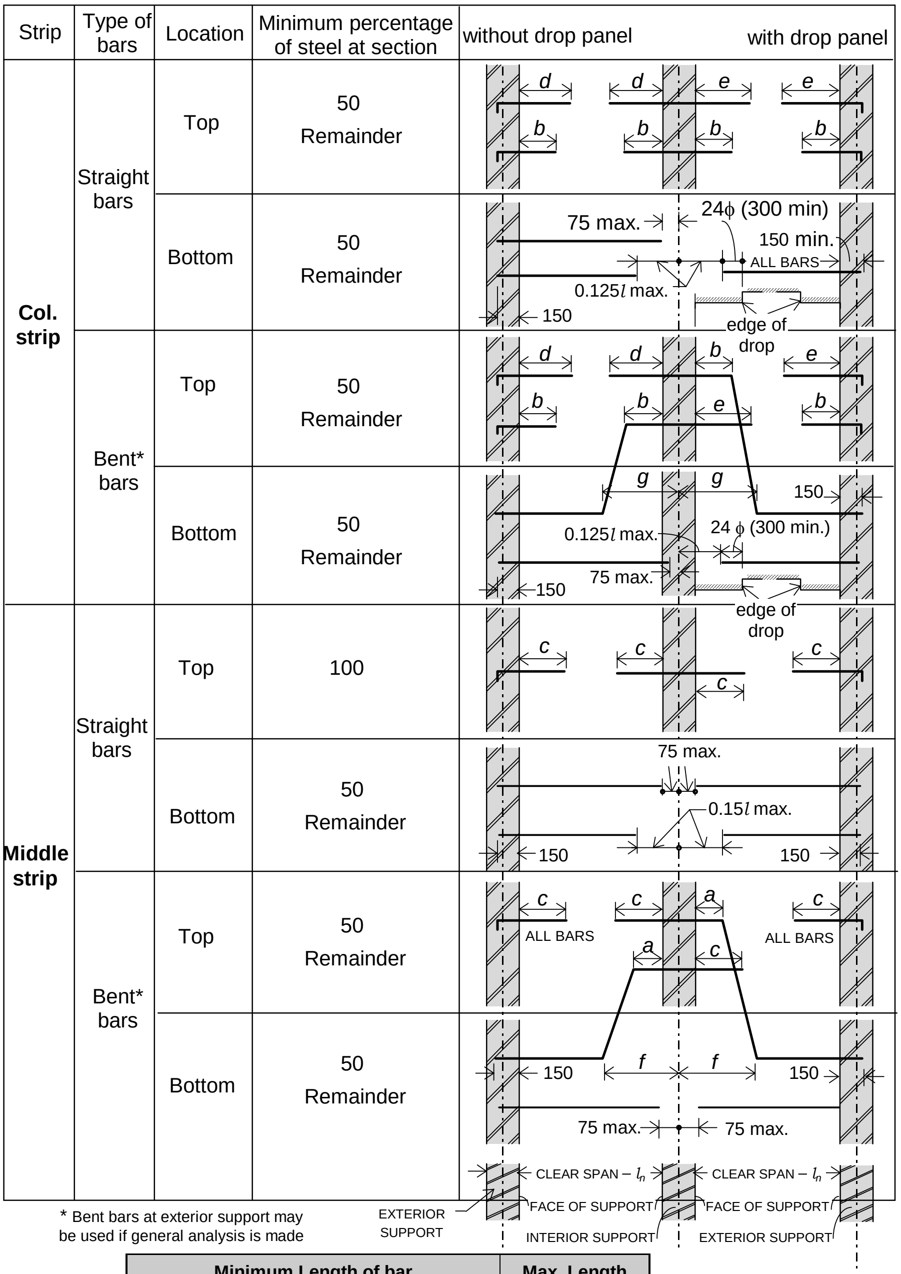
At an exterior support, the negative moment in the slab part of the slab-beam member acts as a twisting moment in the torsional member [see Section 11.5.6]. As indicated in Fig. 11.33 and conservatively, the twisting moment may be assumed to be uniformly distributed along the length of the torsional member (with a maximum value at the face of the column support. The twisting moment will generally be significant only in the presence of an edge beam†, and the edge beam has to be suitably designed to resist this moment, using the design principles explained in Chapter 7. Transverse beams at interior supports will also be subject to some torsion due to the unbalanced negative moments in the slabs on either side, but the magnitude of this will usually be very small.

## 11.7 REINFORCEMENT DETAILS IN COLUMN-SUPPORTED TWO-WAY SLABS

- When slabs are provided with drop panels, the slab thickness to be considered for calculation of area of 'negative' reinforcement at the support should be limited to the total thickness of the drop panel or the thickness of the slab plus one-fourth the distance between the edge of the drop and the edge of the column capital, whichever is smaller [Cl. 31.7.2 of the Code]. This limitation is intended to discourage the use of excessively thick drop panels solely for the purpose of reducing 'negative' reinforcement area [Ref. 11.11].
- The flexural reinforcement requirements, calculated for the design 'positive' and 'negative' moments at the critical sections, should not be less than the minimum specified for shrinkage and temperature stresses [Cl. 26.5.2.1 and Cl. 26.3.3(b) of the Code]. Furthermore, the Code (Cl. 31.7.1) limits the spacing of bars at critical sections in *flat slabs* to a maximum of twice the slab thickness. This limitation is intended to ensure *slab action*, reduce cracking, and to provide for the distribution of concentrated loads.
- Considerable uncertainties are generally associated with the 'degree of fixity' along the exterior edge of the slab system. The degree of flexural restraint depends on the torsional stiffness of the edge beam (if provided) and interaction with an exterior wall (if provided).

---

† In the absence of a beam, the torsional stiffness $K_t$ will be very low, which means that the slab is nearly unrestrained flexurally at the edges. Hence the negative moment in the slab part and consequently the twisting moment in the edge portion of the slab (which acts as a torsional member) will be negligible.




| | Minimum Length of bar | | | | | Max. Length | |
|---|---|---|---|---|---|---|---|
| **Mark** | ***a*** | ***b*** | ***c*** | ***d*** | ***e*** | ***f*** | ***g*** |
| **Length** | $0.14l_n$ | $0.20l_n$ | $0.22l_n$ | $0.30l_n$ | $0.33l_n$ | $0.20l_n$ | $0.24l_n$ |

**Fig. 11.39** Minimum lengths of reinforcements in beamless two-way slabs

It is therefore desirable to provide 'negative' moment reinforcement at the discontinuous edge, as required for wall-supported slabs [Cl. D–1.6 of the Code], and to provide proper anchorage for the same. Furthermore, all 'positive' reinforcement perpendicular to the discontinuous edge should be extended to the slab edge, and embedded for a length, straight or hooked, of at least 150 mm [Cl. 31.7.4 of the Code].

- For two-way systems supported on relatively stiff beams ($\alpha_{b1} l_2/l_1 > 1.0$), it is necessary to provide the special corner reinforcement at exterior corners, as in the case of wall-supported 'restrained' slabs [refer Cl. D–1 of the Code, and Section 11.2.4].
- The location of bar cut-off or bend points must be based on the moment envelopes (obtainable in the Equivalent Frame Method) and the requirements of development length and bar extensions described in Section 5.9.
- In the case of flat slabs and flat plates, the Code (Cl. 31.7.3) prescribes specific bend point locations and minimum extensions for reinforcement. These recommendations are based on ACI code recommendations and have been incorporated in other international codes [Ref. 11.18, 11.19]. They are depicted in Fig. 11.39. Under lateral loads, (combined with gravity loads), the actual lengths of reinforcement should be worked out by equivalent frame analysis, but must not be less than those prescribed in Fig. 11.39.
- It is seen that punching shear failure (discussed in Section 11.8.2) may lead to the tearing out of the top steel over the support section from the top surface of the slab [Fig. 11.40(a)], resulting in a complete 'punch through' at the support. Such a complete failure of one support will result in the slab load in this area getting transferred to the neighbouring support, overloading it, which may, in turn, cause its failure and thus set off a progressive type of collapse. In order to prevent such a collapse, it is recommended that at slab supports, adequate *bottom* steel should be provided[†], such that it passes through the columns in both span directions and has sufficient anchorage [Fig. 11.40(b)]. With this, even after a punching shear failure, the slab will be hanging by these reinforcements from the column head. The minimum area of such bottom reinforcement in each direction, $A_{sb}$, is prescribed in Ref. 11.18 as:

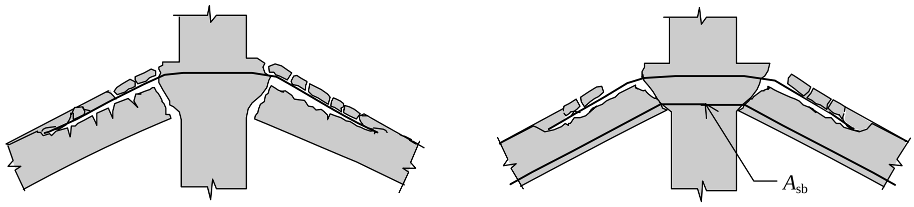
$$A_{sb} = \frac{0.5\, w\, l_2\, l_n}{0.87 f_y} \qquad (11.47)$$

  where $w$ is the total (dead plus live) characteristic load per unit area, but not less than $2w_{DL}$. At least two bars should be provided in each direction, and the bars should be effectively lap spliced, preferably outside the reaction area, with a minimum *lap length* of 2 $L_d$ [refer Chapter 8]. At discontinuous edges, proper anchorage should be provided into the support by means of bends, hooks, etc., so as to develop the full design yield stress at the face of the support on the slab side.

---

[†] Such reinforcement is, however, not called for if punching shear reinforcement is provided in the slabs [refer Section 11.8].

- When openings are provided in flat slabs and flat plates, the requirements of Cl. 31.8 of the Code should be satisfied. In particular, the requirement for the total amount of reinforcement for the slab without opening should be maintained, such that the equivalent of the reinforcement interrupted should be added on all sides of the openings.




**(a)** Tearing out of top reinforcement after shear failure

**(b)** Bottom reinforcement for hanging up slab

**Fig. 11.40** Minimum bottom steel required to pass through column

## 11.8 SHEAR IN COLUMN-SUPPORTED TWO-WAY SLABS

There are two types of shear to be considered in the design of two-way slabs supported on columns (with or without beams along column lines):

1. *one-way shear* or *beam shear*, and
2. *two-way shear* or *punching shear*.

Considerations of one-way shear predominate when beams are provided along the column lines; in fact, there is no need to check for two-way shear when the beams provided are relatively stiff. On the other hand, two-way shear considerations predominate in the case of beamless slabs (flat slabs and flat plates). However, both one-way shear and two-way shear need to be checked in two-way slabs supported on flexible beams[†] (see also Section 11.5.8) .

### 11.8.1 One-Way Shear or Beam Shear

The critical section for one-way shear in column-supported slabs is located at a distance *d* from the face of the support (column, capital or bracket), as shown in section 1–1 in Fig. 11.41. The slab acts as a wide beam supported on, and spanning between, the columns (and hence, the name *beam shear*, i.e., shear as in the case of beams). The shear stress may be computed for the full slab width, $l_2$, or for a typical strip of slab one metre wide, shown shaded in Fig. 11.41. Assuming the shear to be zero at midspan[‡], the factored one-way shear force per unit length $V_{u1}$ is given (similar to Eq. 11.14) as:

$$V_{u1} = w_u(0.5l_n - d) \tag{11.48}$$

[†] As mentioned earlier, the Code does not adequately cover provisions related to slabs supported on flexible beams. In the general category of flat slabs, the Code (Cl. 31.6) confines its attention to two-way shear alone.

In the case of a slab with drop panels [Fig. 11.41(b)], a second section where one-way shear may be critical is at section 2–2, at a distance $d_2$ from the edge of the drop panel, where $d_2$ is the effective depth of the slab outside the drop.

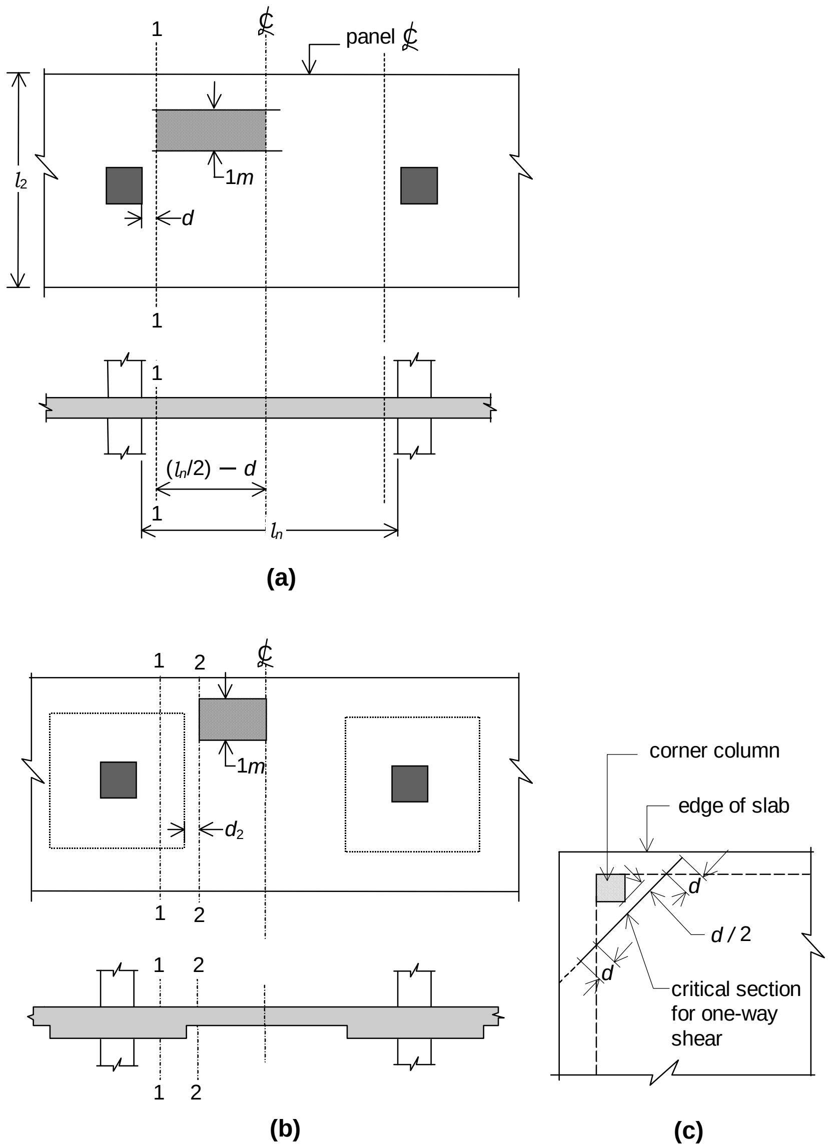



**Fig. 11.41** Critical sections for one-way shear

The slab thickness must be adequate to ensure that the shear resistance in one-way action (equal to $\tau_c d$ per unit length) is not less than the factored one-way shear $V_{u1}$. However, generally, deflection control criteria are more critical in governing the slab thickness [see Example 11.1], and slabs are mostly safe in one-way shear.

In the vicinity of a corner column, the critical section of the slab for one-way shear is taken along a straight line having a minimum length and located no farther than $d/2$ from the corner column. In case the slab cantilevers beyond the face of the corner column, the critical section may be extended into the cantilevered portion by a length not exceeding $d$, [Fig. 11.41(c)].

### 11.8.2 Two-Way Shear or Punching Shear

When a large concentrated load is applied on a small slab area‡, there is a possibility of a 'punch through' type of shear failure. A similar situation arises in flat plates and flat slabs supported on columns and subjected to gravity loading and consequent two-way bending. The reaction to the loading on the slab is concentrated on a relatively small area, and if the thickness of the slab is not adequate in this region, shear failure can occur by punching through of the reaction area along a truncated cone or pyramid, with the failure surface sloping outwards in all directions from the perimeter of the loaded (reaction) area, as shown in Fig. 11.42(a). The shear associated with this type of failure is termed *two-way shear* or *punching shear*.

Extensive research related to punching shear [Ref. 11.27] indicates that the critical section governing the ultimate shear strength in two-way action of slabs (and footings) is along the perimeter of the loaded area. Furthermore, for square columns and loaded areas, it is found that the ultimate shear stress at this section is a function of two parameters, viz., $\sqrt{f_{ck}}$ and the ratio of the side of the square loaded area to the effective depth of the slab. The shear strength can be made relatively independent of the second parameter by considering a critical section for punching shear at a distance $d/2$ beyond the edge of the loaded area [Fig. 11.41(b)]. An expression for the design shear strength $\tau_{c2}$ (in two-way shear), based on Ref. 11.27 and 11.28, is given [Cl. 31.6.3.1 of the Code] as:

$$\tau_{c2} = k_s\left(0.25\sqrt{f_{ck}}\right) \tag{11.49}$$

where $$k_s \equiv 0.5 + \beta_c \le 1.0 \tag{11.49a}$$

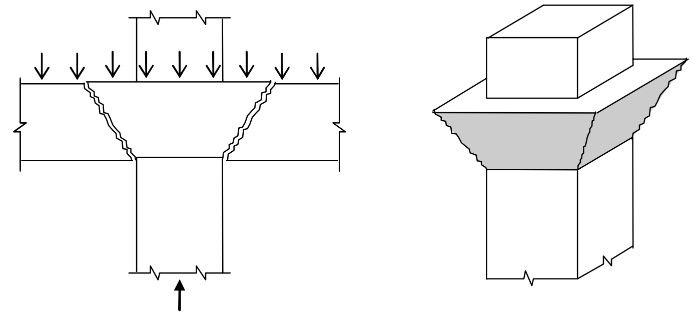
and $\beta_c$† is the ratio of the short side to the long side of the column or capital. The corresponding ultimate shear resistance, $V_{c2}$, is given by

$$V_{c2} = \tau_c b_o d \tag{11.50}$$

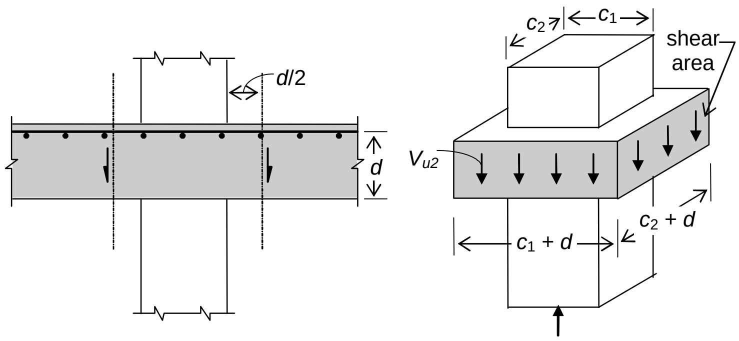
where $b_o$ is the perimeter of the critical section, equal to $2(c_1 + c_2 + 2d)$ for the column, as indicated in Fig. 11.42(b).

‡ Such a situation is encountered in a footing supporting a column [see Chapter 14].

† Tests have shown that the shear strength reduces with increasing rectangularity of the loaded area [Ref. 11.28].



**(a)** punching shear failure




**(b)** assumed critical section

**Fig. 11.42** Shear stresses in slabs due to punching shear

In general, the factored shear force, $V_{u2}$, causing punching shear, may be computed as the net upward column reaction minus the downward load within the area of the slab enclosed by the perimeter of the critical section.

In the Equivalent Frame Method, the column reaction can be obtained from frame analysis. In the Direct Design Method, and for preliminary design purposes, $V_{u2}$ may be computed as the total design load acting on the shaded area shown in Fig. 11.43.

For computing $V_{u2}$ (and $V_{c2}$), the critical section to be considered should be at a distance $d/2$ from the periphery of the column/capital/drop panel, perpendicular to the plane of the slab [Cl. 31.6.1 of the Code], and having a plan shape geometrically similar to the column section, as shown in Fig. 11.43(a), (b). Here, $d$ is to be taken as the effective depth at the section under consideration. For column sections (or loaded areas) with re-entrant corners, a section, no closer than $d/2$ from column face and having the least perimeter, may be taken as the critical section [Fig. 11.43(d) and (e)].

When openings in the slab are located within a distance of ten times the slab thickness from a concentrated load or reaction area, or within a column strip in a flat slab, the portion of the periphery of the critical section which is enclosed by radial projections of the openings to the centroid of the loaded area must be considered ineffective in computing the shear stress [Cl. 31.6.1.2 of the Code]. Some examples of the effective portions of critical sections for slabs with openings are shown in Fig. 11.44.

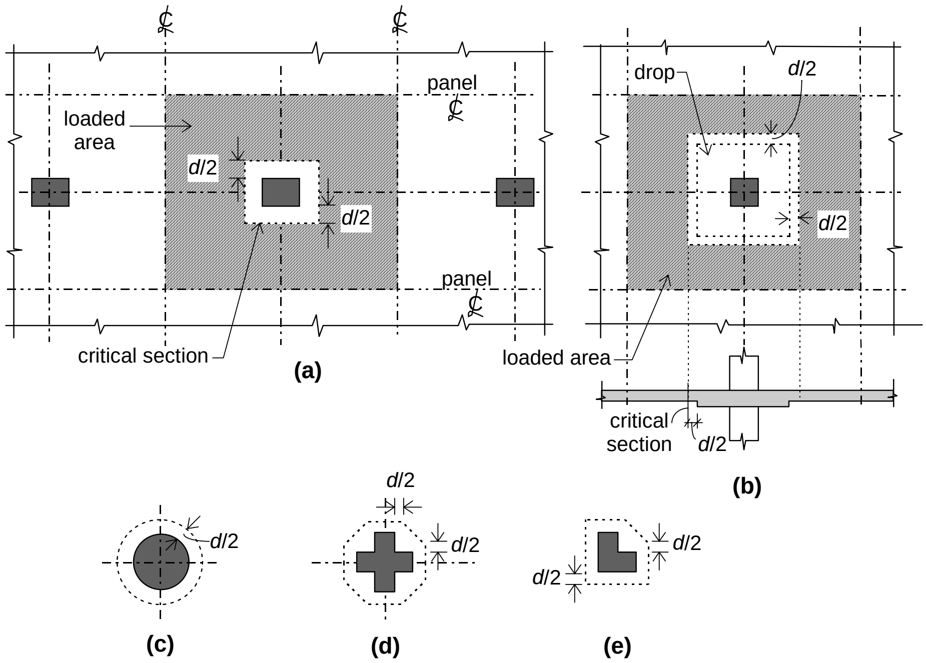



**Fig. 11.43** Critical sections and loading for punching shear

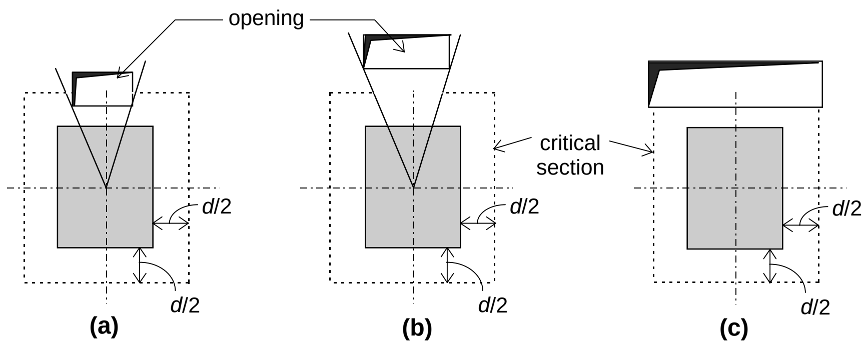



**Fig. 11.44** Effective perimeter for punching shear calculations in slabs with openings

The ultimate shear stress induced by the factored punching shear force $V_{u2}$ at the critical section around a column must be combined with the shear stress due to the transfer of part of the unbalanced slab moment ($M_{uv}$) to the column through shear [refer Section 11.4.3 and Fig. 11.29(c)]. For the purpose of computing shear stresses at the critical section due to $M_{uv}$, the Code (Cl. 31.6.2.2) recommends that the shear stresses may be assumed to vary linearly about the centroid of the critical section.

Accordingly, combining the effects of both $V_{u2}$ and $M_{uv}$, the shear stress distribution is as shown in Fig. 11.45, and the maximum shear stress $\tau_{v2}$ (two-way) may be expressed as:

$$\tau_{v2} = \frac{V_{u2}}{b_o d} + \frac{M_{uv}\, c}{J_c} \tag{11.51}$$

where

$b_o \equiv$ perimeter of the critical section;

$J_c \equiv$ property of the critical section analogous to the polar moment of inertia; and

$c \equiv$ distance of the point under consideration on the face of critical section to the centroidal axisof the critical section.

Expressions for $J_c$ and $c$ (for maximum shear) are indicated in Fig. 11.45 for two typical cases. For other cases, reference may be made to design handbooks such as Ref. 11.25.

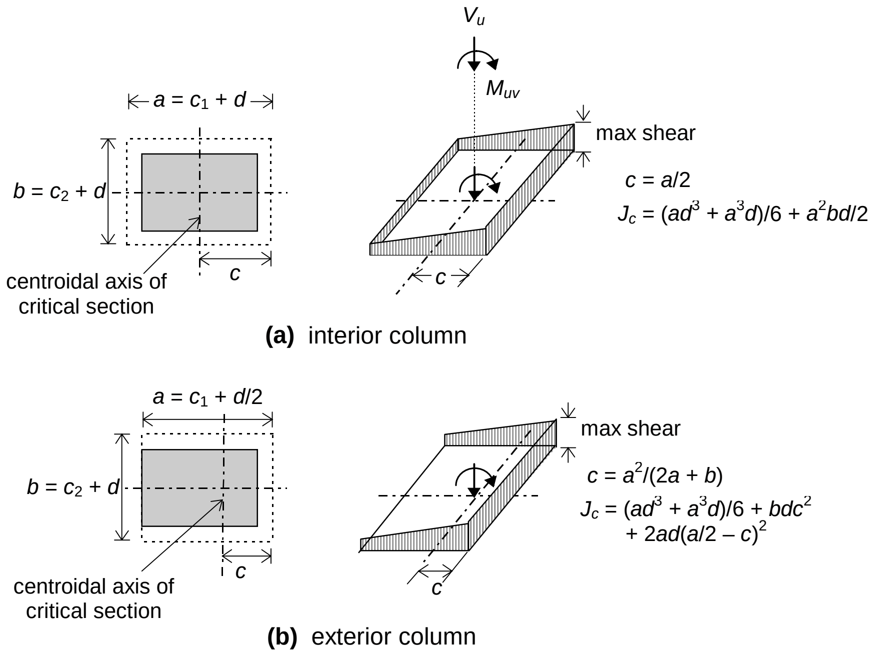



**Fig. 11.45** Combined shear due to punching and transfer of unbalanced moment from slab to column

If the calculated factored shear stress $\tau_{v2}$ exceeds the design shear strength $\tau_{c2}$ (given by Eq. 11.49), but not $1.5\,\tau_{c2}$, appropriate shear reinforcement must be provided along the perimeter of the column. The total cross-sectional area $A_{sv}$ of all the stirrup legs in the perimeter is calculated using the following expression [refer Cl. 31.6.3.2 and Ref. 11.11]:

$$A_{sv} = \frac{(\tau_{v2} - 0.5\tau_{c2})b_o d}{0.87 f_y} \tag{11.52}$$

Some typical types of shear reinforcement, recommended in Ref. 11.11 and Ref. 11.14, are shown in Fig. 11.46.

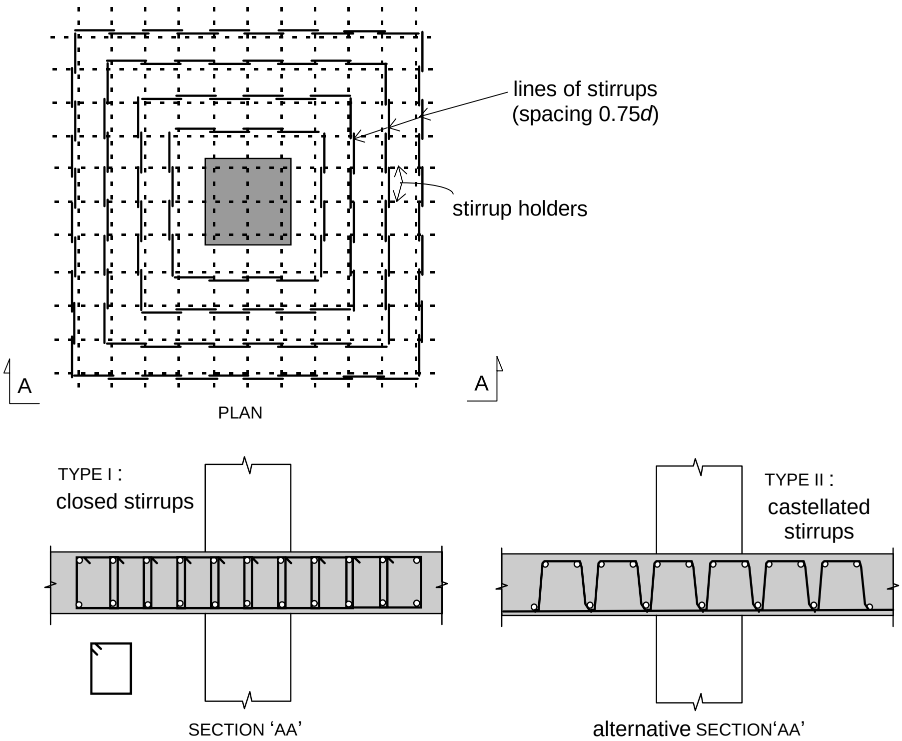



**Fig. 11.46** Reinforcement for punching shear

Stirrups may be *closed* or *castellated* and must pass around one row of tension steel running perpendicular to the stirrups at each face of the relevant section. If the value of $\tau_{v2}$ exceeds $1.5\,\tau_{v2}$, the slab thickness should be suitably increased. Alternatively, reinforcement may be made up of 'shearhead reinforcement', consisting of structural steel I-section or channel section embedded within the slab, and designed in accordance with the ACI Code provision [Ref. 11.19]. Generally, when shear reinforcement is provided, the critical section for punching shear gets shifted farther from the column. Hence, the Code (Cl. 31.6.3.2) requires that the shear stresses should be investigated at successive sections (at intervals of 0.75*d*, as per Ref. 11.11) more distant from the column, and shear reinforcement should be provided up to a section where the shear stress does not exceed 0.5 $\tau_{v2}$ .

It is recommended that the spacing of stirrups should not exceed 0.75*d* and must be continued to a distance *d* beyond the section at which the shear stress is within allowable limits [Ref. 11.11, 11.29].

The design of such shear reinforcement is demonstrated in Example 11.7

## 11.9 DESIGN EXAMPLES OF COLUMN-SUPPORTED TWO-WAY SLABS

Two examples are presented here for the design of two-way slabs supported on columns (with or without beams) by the unified approach using the equivalent frame concept. In the first example to follow (Example 11.6), the Direct Design Method is applied, and in the next example (Example 11.7), the Equivalent Frame Method is applied.

### EXAMPLE 11.6: DIRECT DESIGN METHOD

The plan of a two-way floor slab system, with beams along the column lines, is shown in Fig. 11.47. Based on preliminary estimates, the columns are of size 400 mm × 400 mm and the beams are of size 400 mm × 550 mm. The floor-to-floor height is 3.5 m. Assume a live load of 5.0 kN/m$^2$ and a finish load of 1.0 kN/m$^2$. Determine the design moments and reinforcement requirements in the various strips in the E–W direction for an edge panel and an interior panel (marked $S_1$ and $S_2$ respectively in Fig. 11.47), using the Direct Design Method. Assume M 20 concrete and Fe 415 steel.

#### SOLUTION

The moments in panels $S_1$ and $S_2$ in Fig. 11.47(a) (in the E–W direction) can be determined by DDM, by considering an 'equivalent frame' along column line 2–2, which is isolated and shown in Fig. 11.47(b).

**1. Check limitations of DDM**

1. There are three continuous spans in each direction.
2. The panels are rectangular with long span/short span ratio = 7.5/6.0
   = 1.25 < 2.0.
3. There are no offset columns.
4. There is no difference in successive span lengths.
5. Assuming the slab to be 180 mm thick, $w_{DL}$ = 1.0 + (25 × 0.18) = 5.5 kN/m$^2$

$\Rightarrow w_{u,DL} = 5.5 \times 1.5 = 8.25$ kN/m$^2$

$w_{u,LL} = 5.0 \times 1.5 = \underline{7.50}$ ”

$w_u = 15.75$ kN/m$^2$

$w_{u,LL}/w_{u,DL} = 7.50/8.25 = 0.91 < 3.0$ — OK

[The more severe condition in Ref. 11.8 is also satisfied as the ratio is < 2.0, and the loads are uniformly distributed gravity loads.]

6. Relative stiffnesses of beams: $\frac{\alpha_{b1} l_2^2}{\alpha_{b2} l_1^2}$

   where $\alpha_{b1} = I_{b1}/I_{s1}$ , $\alpha_{b2} = I_{b2}/I_{s2}$

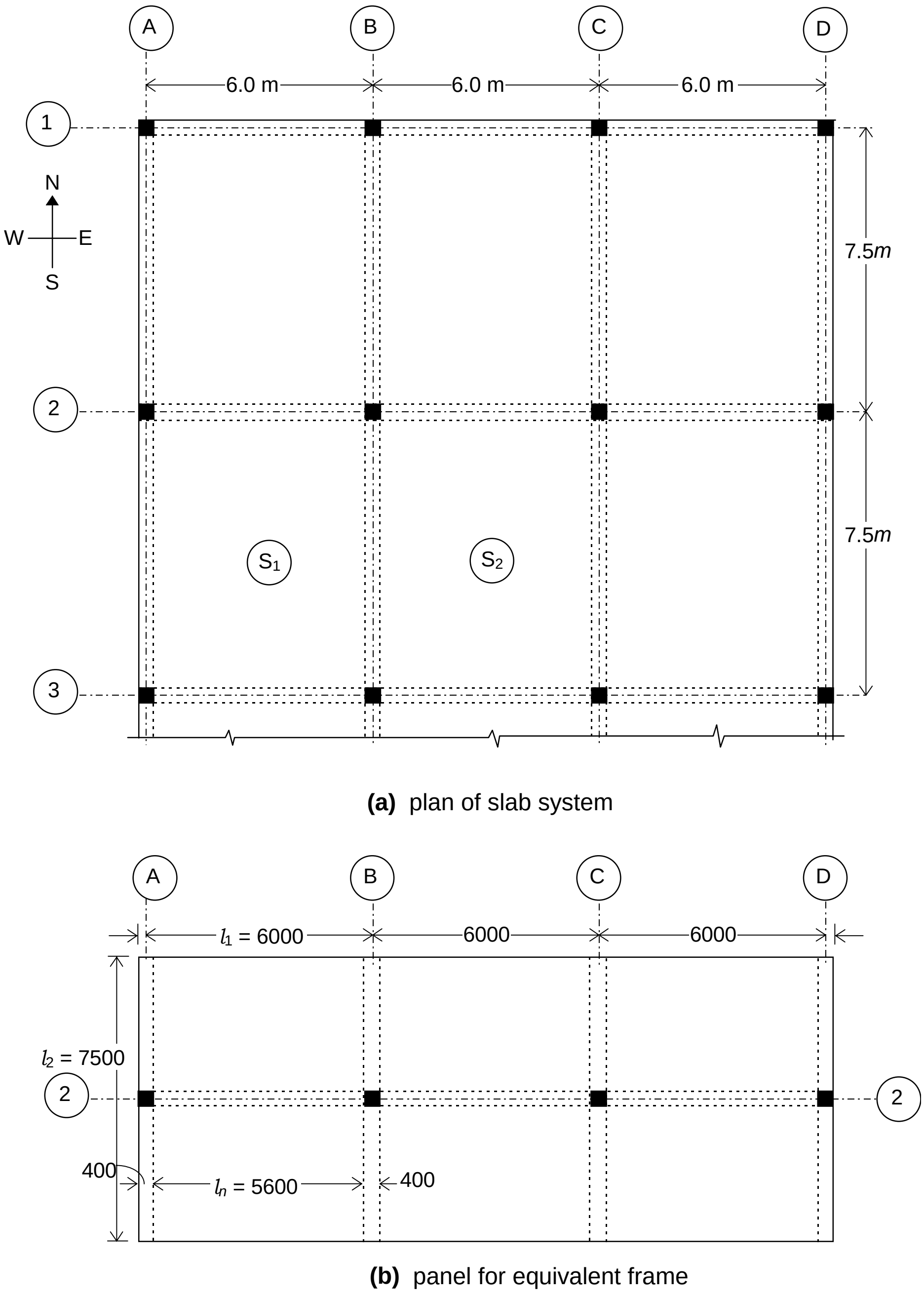



**Fig. 11.47** Example 11.6

As the beam stem dimensions are the same along all edges of the panel, it may be assumed that $I_{b1} \approx I_{b2}$

$$\Rightarrow \frac{\alpha_{b1}}{\alpha_{b2}} \approx \frac{I_{s2}}{I_{s1}} = \frac{l_1}{l_2}$$

$$\Rightarrow \frac{\alpha_{b1} l_2^2}{\alpha_{b2} l_1^2} \approx \frac{l_1}{l_2} \times \frac{l_2^2}{l_1^2} = \frac{l_2}{l_1} = \frac{7.5}{6} = 1.25$$

which lies between 0.2 and 5.0, thereby satisfying Eq. 11.30.

- Hence, all limitations are satisfied, and DDM is applicable.

**2. Slab thickness for deflection control**

- The critical panel to be considered is the exterior panel $S_1$.
- Applying the Canadian Code formula [Eq. 11.26a]:

$$D \geq [l_n (0.6 + f_y / 1000)] / \{30 + 4\beta\alpha_{bm}\}$$

where

$l_n$ = 7100 mm (longer clear span)

$\beta$ = longer clear span/shorter clear span = 7100/5600 = 1.268

$f_y$ = 415 MPa

$\alpha_{bm} \equiv$ average value of $\alpha_b$ for all beams on the edges of panel $S_1$.

The beam and slab sections for computing $\alpha_b = I_b/I_s$ along the four edges (along with values of $I_b$ and $I_s$) are depicted in Fig. 11.48(a), (b), (c). [Note that the flanged beam section corresponds to Fig. 11.27].

As indicated in Fig. 11.48,

$$\alpha_b = \begin{cases} 4.807 & \text{for transverse beams at exterior support} \\ 2.999 & \text{for transverse beams at interior support} \\ 2.399 & \text{for longitudinal beams} \end{cases}$$

$\Rightarrow \alpha_{bm} = \{4.807 + 2.999 + (2.399 \times 2)\}/4 = 3.151$

But $\alpha_{bm}$ is not greater than 2.0

$\Rightarrow D \geq [7100(0.6+415/1000)]/ \{30 + 4 \times 1.268 \times 2.0\} = 179.5$ mm

[Note that the parameter, $\alpha_{b1} l_2/l_1$, is greater than 1.0 for all the four beams. Hence, the supporting beams can be considered to be 'adequately stiff', whereby the limiting $l/d$ ratios prescribed by the IS Code can be applied, with $l$ taken as the effective *short* span.]

- A slab thickness of **180 mm** is therefore adequate.

**3. Total (static) design moment**

As calculated earlier†,

$w_u$ = 15.75 kN/m$^2$

$\Rightarrow$ Total design (factored) moment in the E–W direction in an interior equivalent frame:

---

† Note: the self-weight of the beam stem as well as other additional loads applied directly on the beam are to be accounted for separately in the design of the beam; hence, these are not considered here in the slab design.

$$M_o = w_u\ l_2\ l_n^2 / 8$$
$$= 15.75 \times 7.5 \times 5.6^2/8$$
$$= \mathbf{463\ kNm}$$

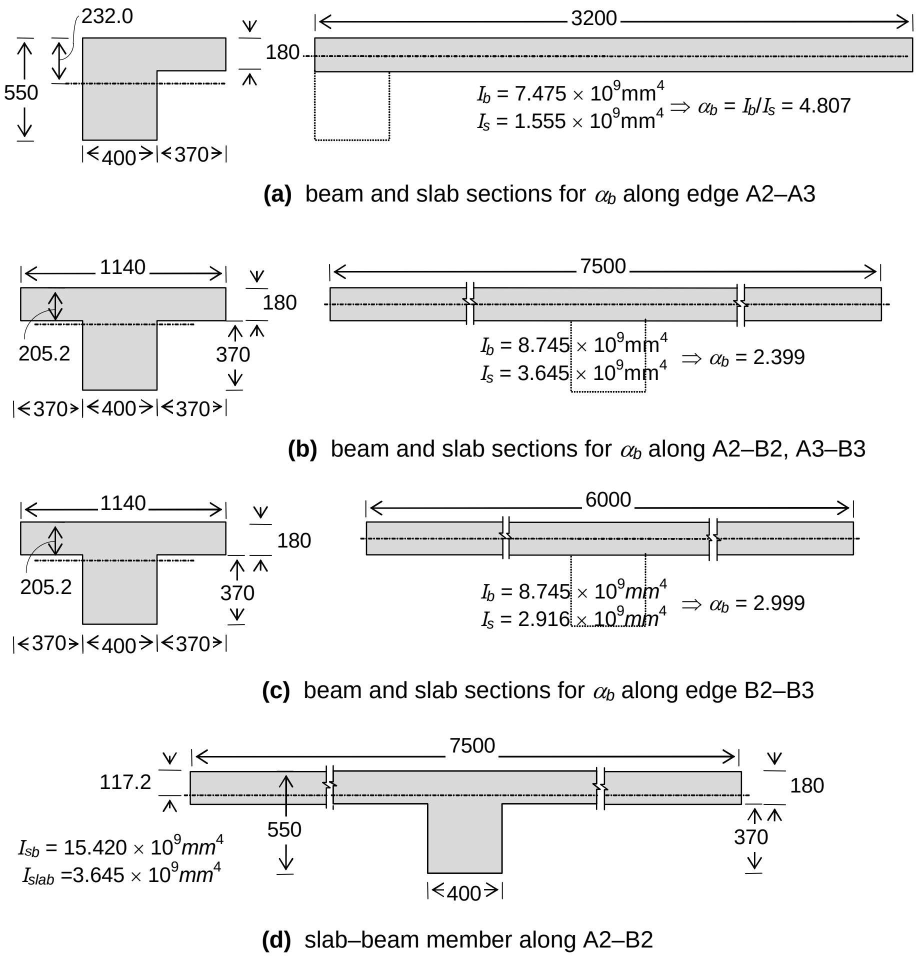



**Fig. 11.48** Sections of beams and slabs — Example 11.6

### 4. Longitudinal distribution of $M_o$

***Interior Span*** – Panel $S_2$ [Eq. 11.32a, b]

- 'negative' design moment $M_o^-$ = 0.65 × 463 = 301 kNm
- 'positive' design moment $M_o^+$ = 0.35 × 463 = 162 kNm

***Exterior Span*** – Panel $S_1$ [Eq. 11.33a, b, c]

$$\alpha_c = \frac{\sum K_c}{K_{sb}} = \frac{(4EI_c/h_c)\times 2}{4EI_{sb}/l_1} = \frac{2I_c l_1}{I_{sb}h_c}$$

where $I_c = (400)^4/12 = 2.133 \times 10^9$ mm$^4$; $h_c$ = 3500 mm (given)

$I_{sb} = 15.420 \times 10^9$ mm [see Fig. 11.48(d)]; $l_1$ = 6000 mm

$$\Rightarrow \alpha_c = \frac{2\times(2.133\times10^9)\times6000}{(15.420\times10^9)\times3500} = 0.4743$$

$\Rightarrow q = 1 + 1/\alpha_c = 1 + (1/0.4743) = 3.108$ [Eq. 11.34]

- 'negative' design moment at exterior support $M^-_{o,ext} = (0.65/q)M_o$
  $= 0.209 \times 463$
  $= 96.8$ kNm
- 'negative' design moment at interior support $M^-_{o,\text{int}} = (0.75 - 0.10/q)M_o$
  $= 0.718 \times 463$
  $= 332$ kNm
- 'positive' design moment $M^+_o = (0.63 - 0.28/q)M_o = 0.540 \times 463 = 250$ kNm

[Alternatively, following the Canadian code recommendations, the coefficients given in Table 11.3, case (2) may be applied. The corresponding factors there are 0.16, 0.70 and 0.59 respectively, which compare well with the factors above.]

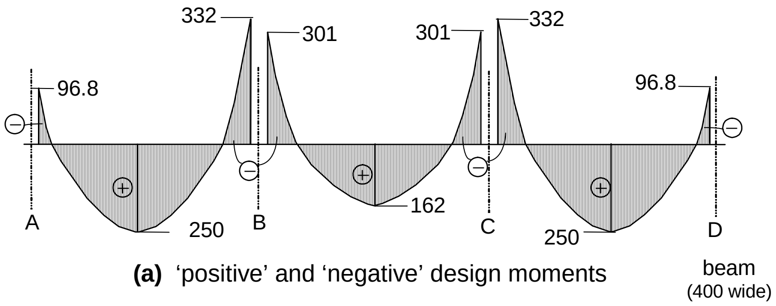
- *Check for effects of pattern loading*:
  Corresponding to $w_{DL}/w_{LL} = 5.5/5.0 = 1.1$, $l_2/l_1 = 7.5/6.0 = 1.25$, and $\alpha_{b1} = 2.40$, referring to Table 11.4,

  $$\alpha_{c,\,min} = 0$$

  Actual $\alpha_c = 0.4743 > 0$; hence, the column stiffness is adequate, and so there is no need to modify the 'positive' design moments to account for the effects of pattern loading.
- The longitudinal distribution of moments (as per IS Code) is shown in Fig. 11.49(a). [The 10 percent modification of bending moments permitted by the Code is not considered here.]

**5. Transverse distribution of moments in design strips**

- At each critical section, part of the design moment is assigned to the beam and the balance to the slab portion. The fraction of the positive moment in all spans and negative moment at interior columns to be resisted by the beam is given by Eq. 11.40, with $\alpha_{b1}$ taken not larger than 1.0 (in this case $\alpha_{b1}$ = 2.40), as:
  $1.0/[1 + (7.5/6)^2] = 0.39 = 39$ percent
  Hence, the moment share of the slab is 61 percent.
  At the exterior support, the beam must be designed for 100 percent of the exterior negative moment.
- These distributed moments in the beam part and the slab part are depicted in Fig. 11.49(b). The beam width is 1140 mm (Fig. 11.48c), and each slab part on either side of beam is 3180 mm wide.



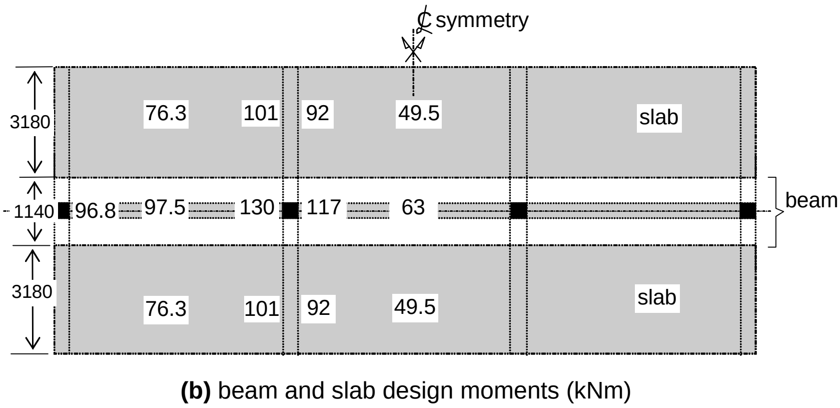




**Fig. 11.49** Design moments in various elements in E–W direction — Example 11.6

- The factored design moments (in the E–W direction) in the panels $S_1$ and $S_2$ of the slab are summarised in Fig. 11.50(a). The moments, expressed in kNm per m width, are also indicated in parenthesis in Fig. 11.50(a). As the moments on either side of the common continuous support (shared by $S_1$ and $S_2$) are unequal, for slab design purposes, it is sufficient and conservative to design for the higher moment — in this case, occurring in panel $S_1$. [Note that this difference could have been brought down and a more economical design obtained by resorting to the modification of moments by up to 10 percent as permitted by Code, Cl.31.4.3.4].

**6. Flexural reinforcements in slab and beam**

- The calculations of reinforcement requirements in the slab panels $S_1$ and $S_2$ are summarised in Table 11.8. For the slab panel, the shorter span is in the E-W direction and hence the reinforcement in this direction is placed at the outer layer. The effective depth, $d$, used in the calculations is based on the assumption of 10 $\phi$ bars with 20 mm clear cover.

$$\Rightarrow d = 180 - 20 - 10/2 = 155 \text{ mm}$$

The largest moment per metre width in the slab is 31.8 kNm at the first interior support. If a reinforcement ratio of $p_t$ = 0.48 is selected (which corresponds to about 0.5 $p_{t,lim}$ ), $R$ = 1.560 MPa (Table A.2(a)), and the required effective depth is:

$$D = \sqrt{[31.8 \times 10^6/(1.560 \times 1000)]} = 143 \text{ mm}$$

The effective depth provided, 155 mm, is more than this and the slab will be under-reinforced throughout. In this case, deflection control dictated the slab thickness.

The effective depth of the beam, assuming 30-mm clear cover, 8 $\phi$ stirrups and 20 $\phi$ main bars is 502 mm. The beam width is taken as 400 mm. It is assumed that there are no additional loads acting directly on the beams [to be more exact, the weight of beam rib should be taken as a directly applied load, but this being negligible, is not done here]. For the largest moment of 130 kNm at the first interior support, $R$ = 1.29 for which required $p_t$ = 0.389 which is well below the limiting value. Beam reinforcements are indicated in Table 11.8. The negative moment reinforcement in the beam at the top must be spread over a width of 400 + 1.5 $D_s$ = 940 mm (i.e. beam rib + width of slab of 270 mm on either side).

- The percentage tension steel requirement, $(p_t)_{reqd}$, is calculated for the respective $R \equiv M_u/bd^2$ values, using Eq. 5.12 or Table A.3(a) — for M 20 concrete and Fe 415 steel.

**Table 11.8** Slab reinforcement requirements — Example 11.6

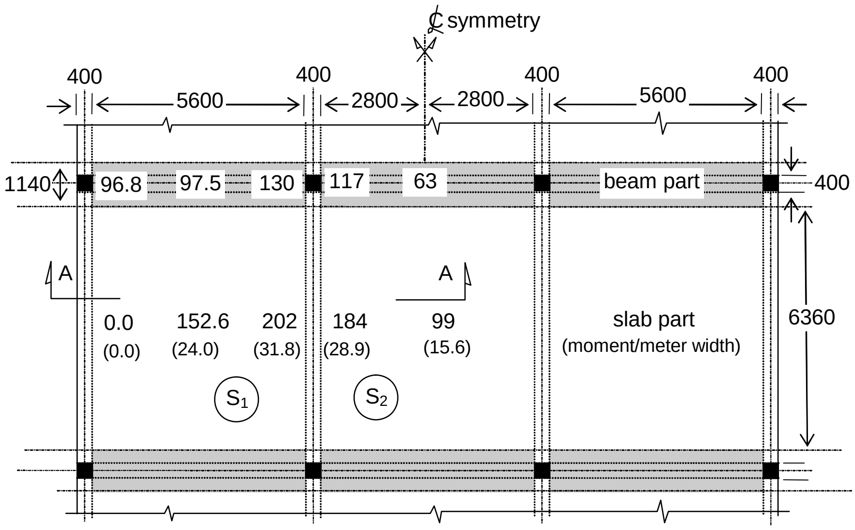
| Location | Panel $S_1$ | | | Panel $S_2$ |
|---|---|---|---|---|
| | exterior support | midspan | interior support | midspan |
| **1. beam portion** | | | | |
| $M_u$ (kNm) | (–) 96.8 | (+) 97.5 | (–) 130 | (+) 63 |
| $R \equiv \frac{M_u}{bd^2}$ (MPa) | 0.960 | 0.967 | 1.290 | 0.625 |
| $(p_t)_{reqd}$ , ($\times 10^{-2}$) | 0.283 | 0.285 | 0.389 | 0.180 |
| $(A_{st})_{reqd}$ ( mm$^2$) | 567 | 572 | 780 | 361 |
| $(A_{st})_{min}$ | - | - | - | 411* |
| No. of bars | 2 - 20 $\phi$ | 2 – 20 $\phi$ | 2-20 $\phi$ + 1-16$\phi$ | 2 – 18 $\phi$ |
| **2. slab** | | | | |
| $M_u$ (kNm/m) | 0.0 | (+) 24.0 | (–) 31.8 | (+) 15.6 |
| $R$ (MPa) | - | 0.999 | 1.324 | 0.649 |
| $(p_t)_{reqd}$ | * | 0.295 | 0.400 | 0.187 |
| $(A_{st})_{reqd}$ ( mm$^2$/m) | 216* | 457 | 620 | 290 |

* *Note*: $(A_{st})_{min}$ governs

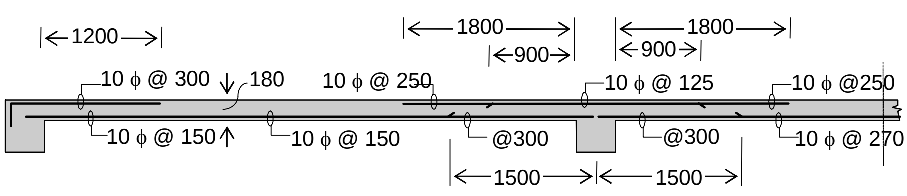
- *Minimum reinforcement*:
  For beam (Cl. 26.5.1.1), $(A_{st})_{min}$ = 400 × 502 × 0.85/415 = 411 mm$^2$
  For slab (Cl. 26.5.2.1), $(A_{st})_{min}$ = 0.0012 $bD$ = 0.0012 × 1000 × 180
  = 216 mm$^2$/m

*Maximum spacing of bars*: $3d = 3 \times 155 = 465$ mm, to be limited to 300 mm

- Spacing of bars
  - ∗ corresponding to $(A_{st})_{min} = 216$ mm$^2$/*m*, using 10 $\phi$ bars, spacing reqd = 1000 × 78.5/216 = 363 mm, minimum 300 mm controls.
  - ∗ corresponding to $A_{st} = 620$ mm$^2$/m, at interior support, spacing of 10 $\phi$ bars reqd = 1000 × 78.5/620 = 127 mm (at top)
  - ∗ corresponding to $A_{st} = 457$ mm$^2$/m, at midspan of panel $S_1$, spacing of 10 $\phi$ bars reqd = 1000 × 78.5/457 = 172 mm (at bottom)
  - ∗ corresponding to $A_{st} = 290$ mm$^2$/m, at midspan of panel $S_2$, spacing of 10 $\phi$ bars reqd = 1000 × 78.5/290 = 270 mm (at bottom)
- There is no negative moment assigned to the slab part at the exterior support. However, the slab at this part has to be provided with the minimum reinforcement. Although the design moment is zero, in order to take care of possible negative moments due to partial fixity, it would be appropriate (Cl. D-1.6) to provide top reinforcement equal to 50 percent of that provided at mid-span, extending 0.1 *l* into the span.




**(a)** factored design moments in beam part and slab part




SECTION 'AA' (through slab)

**(b)** reinforcement details

**Fig. 11.50** Design moments and reinforcement details in panels $S_1$ and $S_2$ in E–W direction — Example 11.6

- Furthermore, all positive moment reinforcement perpendicular to the discontinuous edge should be extended to the slab edge and embedded (Cl. 31.7.4).
- With no moment assigned to the slab at the exterior support, the torsion on the edge beam transmitted by the slab is also zero. In actuality, some torque will be introduced, but this will be low. Therefore, the torsional stresses in the edge beam may be neglected. Similar stresses at interior supports will be even lesser.
- The actual spacings of 10 ϕ bars provided in panels $S_1$ and $S_2$ are indicated in Fig. 11.50(b).
- As the supporting beams are 'adequately stiff' (αb1l2/l1 > 1.0), torsional reinforcement has to be provided at the corners with discontinuous edges in panel $S_1$ — as in the case of wall-supported slabs. The area requirement ($0.375\,A_{st}^{+}$) works out to $0.375 \times 457 = 172$ mm$^2$/m, at top and bottom over a square of size $0.2 \times 6.0 = 1.2m$. Provide 10 ϕ bars at 300-mm spacing.

**7. Transfer of moments to columns**

Since beams are provided along column lines, transfer of unbalanced moment to the supporting columns is not critical in this example. The largest unbalanced moment occurs at the exterior support and the beam is designed to resist this in full. The requirement in Cl. 31.3.3 for moment transfer between slab and column through flexure is of primary concern in flat slabs.

**8. Shear in slab and beams**

Since all beams in this example have the parameter $\alpha_{b1}l_2/l_1$ greater than 1.0, they are adequately stiff. Hence they are proportioned to resist the full shear from the loads on the respective tributary areas (Fig. 11.51) and no part of this need be assigned to the slab to be resisted by two-way action (Section 11.5.8). In beam-supported slabs, the slab shear is essentially that associated with one-way action. For one-way shear in slab, considering a one metre wide strip, the distribution of loads as shown in Fig. 11.51 may be assumed, with the critical section located $d$ away from the face of the beam. The factored shear force $V_u$ is given by:

$$V_u = w_u(0.5l_n - d)$$

$$= 15.75 \times (0.5 \times 5.6 - 0.155) = 41.66 \text{ kN}$$

$$\Rightarrow \tau_v = 41.66 \times 10^3/(10^3 \times 155) = 0.269 \text{ MPa}$$

shear strength of concrete in one-way shear = $k\,\tau_c$

where, corresponding to M 20 concrete and $p_t = \dfrac{1000 \times 78.5 \times 100}{125 \times 1000 \times 155} = 0.405$,

$$\tau_c = 0.44 \text{ MPa, and } k > 1.0$$

$$\Rightarrow k\,\tau_c > \tau_v \quad \text{— Hence, OK.}$$

*Note*: As the beams in this example are adequately stiff, the slab system may be alternatively designed by the (much simpler) method of moment coefficients prescribed by the Code for wall-supported slabs.

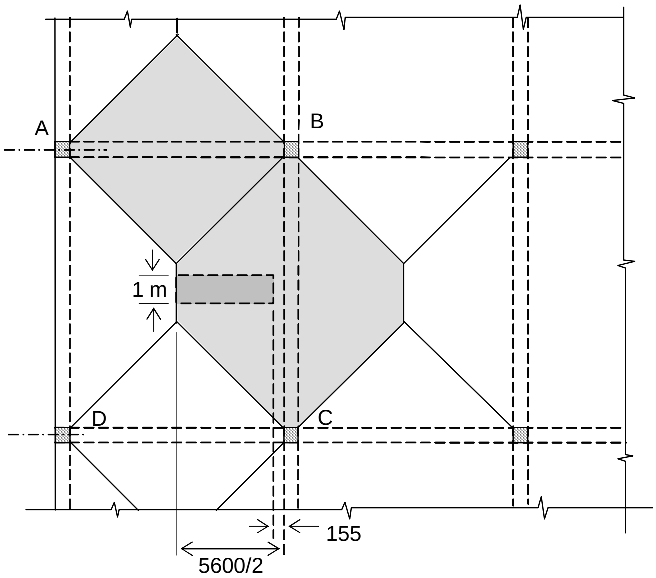



**Fig. 11.51** Shear in slab and beams

### EXAMPLE 11.7: EQUIVALENT FRAME METHOD

The floor plan of a flat slab floor for an office building is shown in Fig. 11.52(a). The floor-too-floor height may be taken as 3.3 m. The columns are of size 500 mm × 500 mm. Assume live loads of 2.5 kN/m$^2$ and superimposed dead loads of 2.7 kN/m$^2$ (comprising finishes: 1.0 kN/m$^2$; partitions: 1.0 kN/m$^2$; false ceiling: 0.2 kN/m$^2$, mechanical and electrical fittings: 0.5 kN/m$^2$). Analyse a typical interior bay in the E–W direction, shown shaded in Fig. 11.52(a), using the *equivalent frame method*. Compute the reinforcement requirements in the various design strips of the slab, and check shear stresses. Assume M 20 concrete and Fe 415 steel. Exposure condition is *mild*.

**SOLUTION**

**1. Slab thickness for deflection control**

The same thickness will be provided for the floor slab in all panels. At discontinuous edges, edge beams with a stiffness ratio $\alpha \geq 0.8$ will be provided so that no increase in slab thickness is required for the exterior panels.

For slabs with drops conforming to the requirements, the Code (Cl. 31.2.1) recommends span to effective depth ratios given in Cl. 23.2, with the longer span considered. For continuous spans, the ratio recommended is 26. For two-way slabs, as already mentioned in Section 11.2.1, the modification factor (Fig. 4 of

Code) may be taken as 1.5. Considering the larger span, $l_n$ = 7500 mm. With these, the effective depth required is:

$$d \geq 7500 / (1.5 \times 26) = 192 \text{ mm}$$

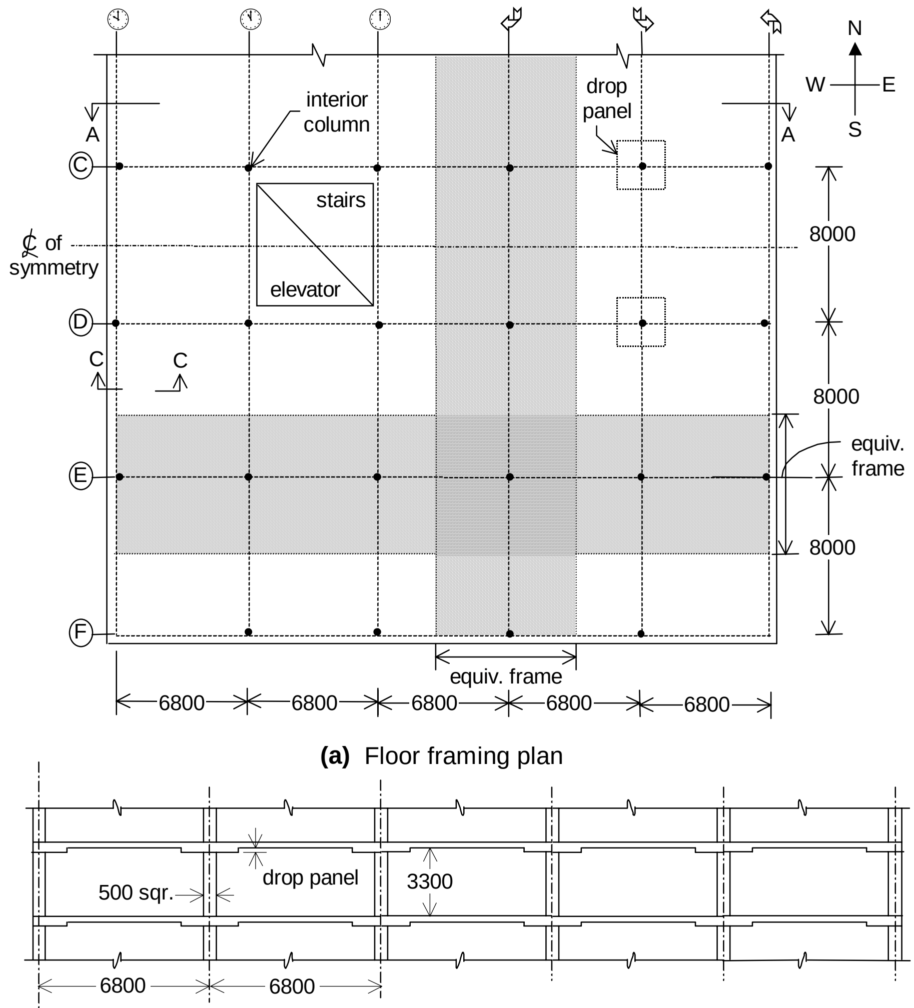



**Fig. 11.52** Example 11.7

An alternative is to use Eq. 11.26. Assume drop panels extending 1.5 m in each direction from column centre (i.e. 3m × 3m panels) and projecting by half the slab thickness below the slab. In Eq. 11.26,

$l_n$ = larger span = 8000-500 =7500 mm

$x_d$ = distance from face of column to edge of drop panel

$= 1500 - 250 = 1250$ mm $< l_n/4$, OK

Of the two directions, the smaller value of $x_d/(l_n/2) = 2 \times 1250/7500 = 1/3$
$(D_d - D)/D = 0.5$
With these, Eq. 11.26 gives

$$D \geq [7500\,(0.6 + 415/1000)] / [30\{1+(1/3)(0.5\}] = 217.5\text{mm}$$

A slab thickness of 200 mm is selected. This being marginally less than the value calculated above, strictly, a deflection computation is required to check that it is within specified limits.

**2. Drop panels**

As specified by the Code (Cl. 31.2.2), the minimum extension of the drop panel in each direction from the centre of the column is 1/6 centre-to-centre span, i.e., 6.8/6 = 1.133 m in the E–W direction, and 8.0/6 = 1.333 m in the N–S direction. As assumed earlier, their extension is taken as 1.5 m, resulting in a drop panel size of 3 m × 3 m. The thickness of the drop panel, projecting below the slab is taken as 100 mm (which is greater than $D/4$). The overall thickness of the drop panel is 300 mm; the entire thickness is effective in calculations of 'negative' moment reinforcement, as this is less than $D$ + 0.25 (distance between face of column and edge of drop) = 200 + 0.25(1250) = 512.5 mm [Cl. 31.7.2 of the Code].

**3. Edge beam**

The edge beam must have a beam stiffness parameter $\alpha_b \geq 0.8$ (to have a favourable effect on the minimum thickness requirement). Assuming a beam of 250 mm width and 450 mm depth [Fig. 11.53(a), (b)], the flanged section [Fig. 11.53(b)] has a second moment of area which can be shown to be

$$I_b = 2.714 \times 10^9 \text{ mm}^4$$

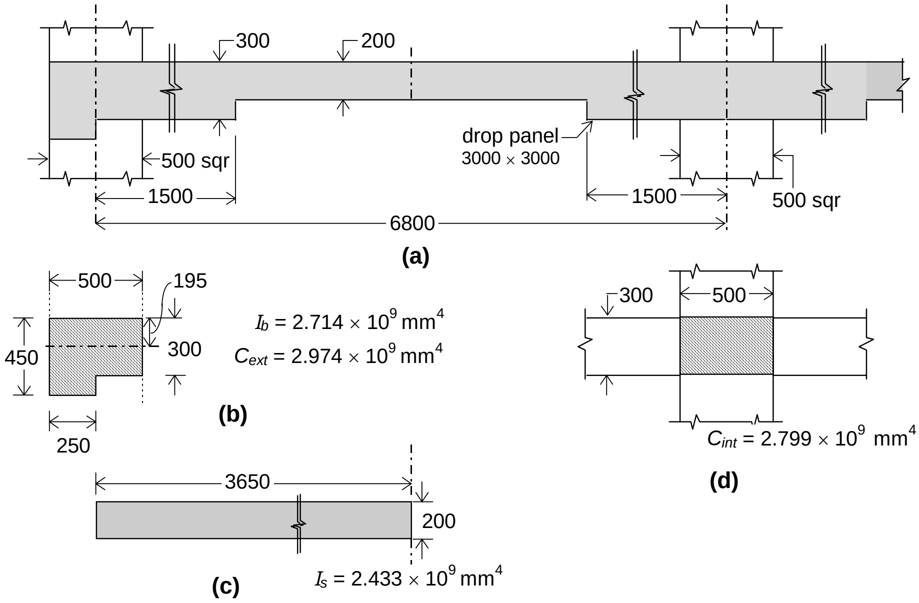



**Fig. 11.53** Slab dimensions and sectional properties

- For the edge beam along the long edge, the associated slab width is 6800/2 + 250 = 3650 mm [Fig. 14.53(c)], and the second moment of area of the slab is

$$I_s = 3650 \times 200^3/12 = 2.433 \times 10^9 \text{ mm}^4$$

$$\Rightarrow \alpha_b = I_b/I_s = 2.714/2.433 = 1.115 > 0.8 \text{ — OK}$$

- For the edge beam along the short edge, the associated slab width is 8000/2 + 250 = 4250 mm, and

$$I_s = 4250 \times 200^3/12 = 2.833 \times 10^9 \text{ mm}^4$$

$$\Rightarrow \alpha_b = I_b/I_s = 2.714/2.833 = 0.958 > 0.8 \qquad \text{— OK.}$$

**4. Effective depths**

- Reinforcement will be placed in the outer layer for the bars in the N–S direction (in order to resist the larger moments in this direction), and in the inner layer for the bars in the E–W direction. Assuming 16 ϕ bars with a clear cover of 20 mm, the effective depths are obtained as:

  $d_{N\text{–}S\ slab}$ = 200 – 20 – 16/2 = 172 mm
  $d_{N\text{–}S\ drop}$ = 300 – 20 – 16/2 = 272 mm
  $d_{E\text{–}W\ slab}$ = 172 – 16 = 156 mm
  $d_{E\text{–}W\ drop}$ = 272 – 16 = 256 mm

**5. Factored loads**

| | | |
|---|---|---|
| (i) | self-weight of slab @ 25 × 0.2 | = 5.0 kN/m$^2$ |
| (ii) | superimposed dead load | = 2.7 ” |
| (iii) | live load | = 2.5 ” |
| | | 10.2 kN/m$^2$ |

$\Rightarrow$ factored load on slab $w_u$ = 10.2 × 1.5 = 15.3 kN/m$^2$

- additional factored load in drop panel
  = (25 × 0.1) × 1.5 = 3.75 kN/m$^2$

**6. Relative stiffness parameters of equivalent frame**

**a. *Column stiffness*,** $K_c$ [refer Fig. 11.54]

$I_c = (500)^4/12 = 5.208 \times 10^4$ mm$^4$

$H/H_c = 3.3/3.0 = 1.1$

$t_a/t_b = 200/100 = 2.0$, $t_b/t_a = 0.5$

Referring to Table 11.7, the stiffness and carry-over factors are:

- for column below, $H/H_c = 1.1$, $t_a/t_b = 2$
  $\Rightarrow K_{AB} = 5.34$ , $K_c = E_c I_c/H = 5.34 \times E_c \times (5.208 \times 10^9)/3300 = (8.427 \times 10^6)E_c$
  $C_{AB} = 0.54$
- for column above, $H/H_c = 1.1$, $t_a/t_b = 0.5$
  $\Rightarrow K_{AB} = 4.85$, $K_c = 4.85 \times E_c \times (5.208 \times 10^9)/3300 = (7.654 \times 10^6)\ E_c$
  $C_{BA} = 0.595$

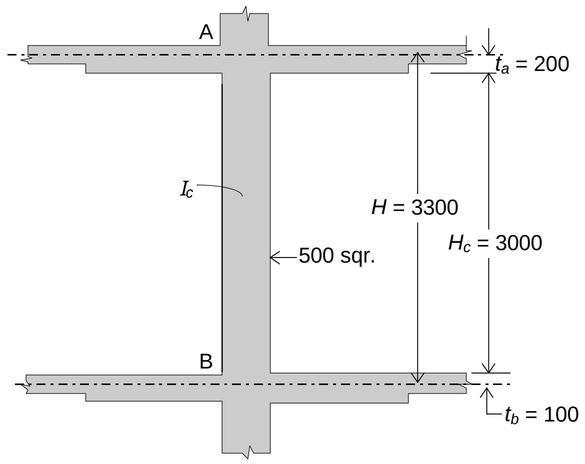



**Fig. 11.54** Column properties — Example 11.7

**b. *Torsional member stiffness*,** $K_t$

- For exterior columns, the long span edge beam [Fig. 11.53(b)] acts as the torsional member, having a torsional constant

$$C_{ext} = (1 - 0.63 \times 150/250) \times (150^3 \times 250)/3 + (1 - 0.63 \times 300/500) \times (300^3 \times 500)/3$$

$$= 2.974 \times 10^9 \text{ mm}^4$$

$$\Rightarrow K_{t,ext} = \sum \frac{9E_cC_{ext}}{l_2(1-c_2/l_2)^3} \quad \text{[Eq. 11.46]}$$

$$= \frac{2\times 9E_c \times (2.974\times 10^9)}{8000(1-500/8000)^3} = (8.121 \times 10^6)E_c$$

- For interior columns, the cross-section of the torsional member is as shown in Fig. 11.53(d), having a torsional constant

  $C_{int}$ = (1 – 0.63 × 300/500) × ($300^3$ × 500)/3 = 2.799 × $10^9$ $mm^4$

  $$\Rightarrow K_{t,int} = \frac{2\times 9E_c \times (2.799\times 10^9)}{8000(1-500/8000)^3} = (7.643 \times 10^6)E_c$$

**c. *Equivalent column stiffnesses***, $K_{ec}$

$$K_{ec} = \frac{\sum K_c}{1+\left(\sum K_c/K_t\right)} \quad \text{[Eq. 11.45(b)]}$$

where $\sum K_c$ = (8.427 + 7.654) × $10^6 E_c$ = (16.081 × $10^6$) $E_c$

- For external 'equivalent column',

  $\sum K_c/K_{t,\,ext}$ = 16.081/8.121 = 1.980

  $\Rightarrow K_{ec,ext}$ = (16.081 × $10^6$)$E_c$/(1 + 1.980) = (5.396 × $10^6$) $E_c$
- For internal 'equivalent column',

  $\sum K_c/K_{t,\,int}$ = 16.081/7.643 = 2.104

  $\Rightarrow K_{ec,int}$ = (16.081 × $10^6$)$E_c$/(1 + 2.104) = (5.181 × $10^6$) $E_c$

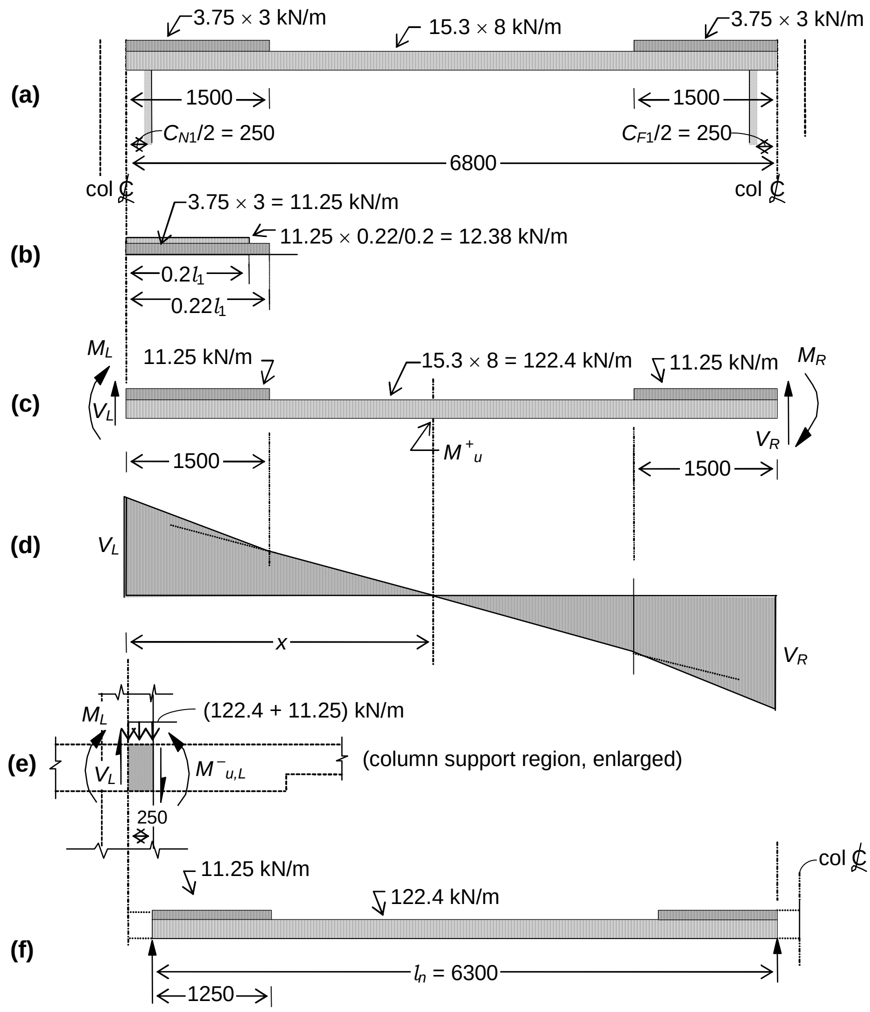



**Fig. 11.55** Loading details on slab-beam member — Example 11.7

**d.** ***Slab stiffnesses $K_s$ and fixed-end moment coefficients***

[Although Table 11.6 is strictly applicable for a drop extension of $l_1/6 = 1.13$ m, the same is used in this example for a drop extension of 1.5 m.]

- Referring to Fig. 11.53(a) for the slab geometry and Fig. 11.55(a) for the loading, and to Table 11.6 for the various stiffness and moment coefficients, with

  $C_{F1} = C_{N1}$; $C_{F2} = C_{N2}$

  $C_{N1}/l_1 = 0.5/6.8 = 0.074$, $C_{N2}/l_2 = 0.5/8.0 = 0.063$

$\Rightarrow$ (interpolating from Table 11.6 for these values),

- $*$ stiffness factor $K_{NF} = 5.933$

- carry-over factor $C_{NF}$ = 0.595
- FEM coefficient $m_{NF1}$ = 0.0931 for uniform load

- For the additional load due to the drop panel, Table 11.6 gives moment coefficients for partial loadings of $0.2l_1$, with $b - a = 0.2$. The drop panel loading, 3.75 kN/m$^2$ × 3 m = 11.25 kN/m applies over a length of 1.5 m = $0.22l_1$ from the column centrelines. An equivalent load acting over a length of $0.2l_1$, equal to 11.25 × 0.22/0.20 = 12.38 kN/m, may be considered to facilitate the use of Table 11.6 [see Fig. 11.55(b)].
  Interpolating from Table 11.6, for $C_{N1}/l_1 = 0.074$ and $C_{N2}/l_2 = 0.063$,
  - for drop panel at near end ($a$ = 0), $m_{NF2}$ = 0.01654
  - for drop panel at far end ($a$ = 0.8), $m_{NF3}$ = 0.00191
- *Slab stiffness* $K_s = K_{NF}\, E_c\, I_s / l_1$
  where $I_s$ is the second moment of area of the slab section beyond the drop:
  $I_s = 8000 \times 200^3/12 = 5.333 \times 10^9$ mm$^4$
  $\Rightarrow K_s = 5.933 \times E_c \times (5.333 \times 10^9)/6800$
  $= (4.653 \times 10^6)\, E_c$

**7. Equivalent Frame Analysis**

- The properties of the ‘equivalent frame’ for analysis by the moment distribution method is indicated in Table 11.9. As the specified live load (2.5 kN/m$^2$) is less than three-fourths of the dead load (7.7 kN/m$^2$), it suffices to consider a single loading pattern, comprising full factored dead plus live loads on all spans. The factored loads on each slab-beam member are indicated in Fig. 11.55(a).
- The slab fixed-end moments are:

$$FEM_{NF} = \sum m_{NF}\, w\, l_1^2$$
$$= \{(0.0931 \times 122.4) + (0.01654 + 0.00191) \times 12.38\} \times 6.8^2$$
$$= 537 \text{ kNm}$$

- The moment distribution factors are calculated based on the relative stiffnesses, and are indicated (for the slab-beam members), along with the carry-over factors in Table 11.9, which also shows the moment distribution procedure.
- With reference to the freebody diagram of a typical slab-beam member shown in Fig. 11.56(c), the following expression for maximum shear forces $V_L$ and $V_R$ may be derived, in terms of the moments $M_L$ and $M_R$, considering static equilibrium:

$$V_L = \frac{122.4 \times 6.8}{2} + (11.25 \times 1.5) - \frac{M_L + M_R}{6.8}$$

  $\Rightarrow V_L = [433.0 - (M_L + M_R)/6.8]$ kN
  and $V_R = [433.0 + (M_L + M_R)/6.8]$ kN
- The location of maximum ‘positive’ moment is given by the location of zero shear, marked $x$ from the left support [Fig. 11.56(d)]

$$x = \left(\frac{V_L - (11.25 \times 1.5)}{122.4}\right)$$

- The corresponding maximum ‘positive’ moment is given by:

$$M_u^+ = [M_L + V_L x - (11.25 \times 1.5)(x - 0.75) - 122.4 \times x^2/2] \text{ kNm}$$

**Table 11.9** Equivalent frame analysis (moment distribution method) — Example 11.7

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $K_s \times 10^6\ E_c$ = | 4.653 | | 4.653 | | 4.653 | | | | | |
| $K_{ec} \times 10^6\ E_c$ = | 5.396 | | 5.181 | | 5.181 | | | | | |
| *DF* | 0.463 | 0.321 | 0.321 | 0.321 | 0.321 | 0.321 | 0.321 | 0.321 | 0.321 | 0.463 |
| *COF* | 0.595 | | 0.595 | | 0.595 | | 0.595 | | 0.595 | |
| *FEM* | – 537 | 537 | – 537 | 537 | – 537 | 537 | – 537 | 537 | – 537 | 537 |
| *Bal* | 249 | | | | | | | | | – 249 |
| *C.O.* | | 148 | | | | | | | – 148 | |
| *Bal.* | | – 48 | – 48 | | | | | 48 | 48 | |
| *C.O.* | – 28 | | | – 28 | | | 28 | | | 28 |
| *Bal.* | 13 | | | 9 | 9 | – 9 | – 9 | | | – 13 |
| *C.O.* | | 8 | 5 | | – 5 | 5 | | – 5 | – 8 | |
| *Bal.* | | – 4 | – 4 | 2 | 2 | 2 | – 2 | 4 | 4 | |
| *C.O.* | – 2 | | 1 | – 2 | – 1 | 1 | 2 | – 1 | | 2 |
| | 1 | -- | -- | 1 | 1 | – 1 | – 1 | -- | -- | – 1 |
| $\sum$ = $M_L$, $M_R$ (kNm) | – 304 | 641 | – 583 | 519 | – 531 | 531 | – 519 | 583 | – 641 | 304 |
| $V_L$, $V_R$ (kN) | 383 | 483 | 442 | 424 | 423 | 433 | | | | |
| $x$ (m) | 2.991 | | 3.473 | | 3.400 | | | | | |
| $M_u^+$ (kNm) | 256 | | 168 | | 189 | | | | | |
| $M_{u,L}^-$, $M_{u,R}^-$ (kNm) | – 212 | 524 | – 477 | 417 | – 427 | 427 | | | | |
| $\overline{M}$ (kNm) | 614 | | 615 | | 616 | | | | | |
| $M_o$ (kNm) | 616 | | 616 | | 616 | | | | | |

- The critical sections for the 'negative' design moments are at the column faces; the moments at the left end ( $M_{u,L}^-$ ) and right end ( $M_{u,R}^-$ ) are, accordingly given by [Fig. 11.56(e)].

$$M_{u,L}^+ = [M_L + (V_L \times 0.25) - (122.4 + 11.25) \times (0.25)^2/2] \text{ kNm}$$
$$M_{u,R}^+ = [M_R - (V_R \times 0.25) + (122.4 + 11.25) \times (0.25)^2/2] \text{ kNm}$$

- The values of $V_L$, $V_R$, $x$, $M_u^+$, $M_{u,L}^-$ and $M_{u,R}^+$ have been tabulated, using the above formulas for the various spans in Table 11.9[†] .
- As the slab satisfies the limitations for the Direct Design Method, the sum of the 'positive' moment at midspan and average negative moment, $\overline{M}$ , need not exceed $M_o$, the maximum static moment on the simply supported span $l_n$ = 6.3 m [Fig. 11.56(f)]:

$$M_o = (122.4 \times 6.3^2/8) + (11.25 \times 1.25^2/2) = 616 \text{ kNm}$$

[†] These are indicated only for the first three spans, as the six-bay frame is symmetric with respect to its middle.

It is seen that $\overline{M} \leq M_o$ for all the three spans [refer Table 11.9]; hence the proportional reduction in the design moments (for $\overline{M} > M_o$), permitted by the Code, is not applicable here.

**8. Transverse distribution of moments**

The transverse distribution specified by Code Cl. 31.5.5 is given by Eq. 11.37 to 11.39.

Accordingly:

$\Rightarrow M^-_{cs,ext} = 1.00\ M^-_{o,ext}$ [Eq. 11.37a]

$\Rightarrow M^-_{hms,ext} = 0$

- *'Negative' moment at interior support*

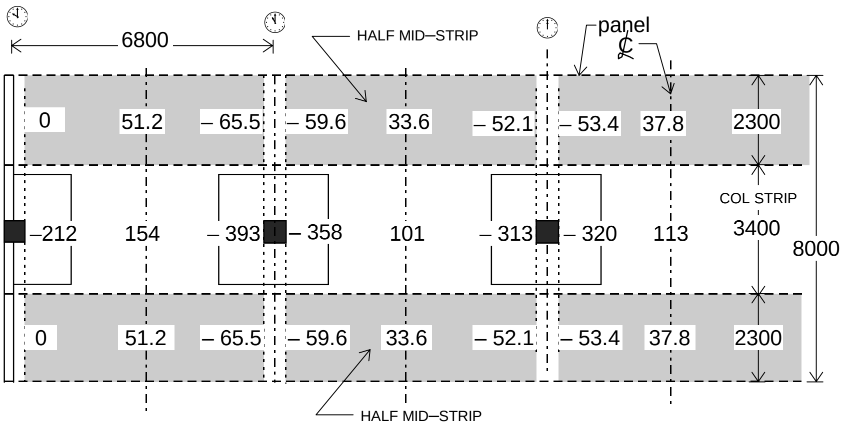
  $M^-_{cs,\text{int}} = 0.75\ M^-_{o,\text{int}}$ [Eq. 11.38a]

  $\Rightarrow M^-_{hms,\text{int}} = 0.125\ M^-_{o,\text{int}}$

- *'Positive' moment*

  $M^+_{cs} = 0.6\ M^+_o$ [Eq. 11.39a]

  $\Rightarrow M^+_{hms} = 0.2\ M^+_o$




| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Factored Moment | – 212 | 256 | – 524 | – 477 | 168 | – 417 | – 427 | 189 | *kNm* |
| Column Strip | 100 | 60 | 75 | 75 | 60 | 75 | 75 | 60 | percent |
| Half Mid–Strip | 0 | 20 | 12.5 | 12.5 | 20 | 12.5 | 12.5 | 20 | percent |

**Fig. 11.56** Factored moments in column and middle strips in E–W direction — Example 11.7

The distributed moments in the column strips and half-middle strips in the various panels (in the E–W direction) are indicated in Fig. 11.56‡.

**9. Column moments**

The total unbalanced slab moments at the various supports are transmitted to the respective columns. At each support, the unbalanced slab moment is shared by the column above and the column below in proportion to their relative stiffnesses.

- Fraction of moment in column above = $\dfrac{7.654}{8.427+7.654}$ = 0.476
  with a carry-over factor = 0.595 (determined earlier)
- Fraction of moment in column below = 1.0 – 0.476 = 0.524
  with a carry-over factor = 0.54

The unbalanced slab moments are obtainable from Table 11.9:

- at exterior column 1: 304 kNm
- at interior column 2: 641 – 583 = 58 kNm
- at interior column 3: 531 – 519 = 12 kNm

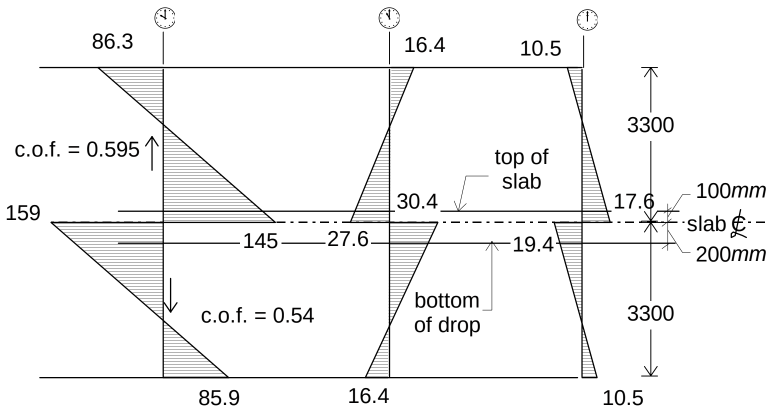



**Fig. 11.57** Column moments (kNm) — Example 11.7

However, at interior column locations, the unbalanced moment should not be less than that given by Eq. 11.42:

$$M_u = 0.08\left[(w_{u,DL} + 0.5w_{u,LL})l_2 l_n^2 - w'_{u,DL} l'_2 (l'_n)^2\right] \Big/ \left(1 + 1/\alpha_c\right)$$

where $w_{u,DL} = w'_{u,DL}$ = 7.7 × 1.5 = 11.55 kN/m$^2$ (neglecting drop panel)

$w_{u,LL}$ = 2.5 × 1.5 = 3.75 kN/m$^2$

$l_2 = l'_2$ = 8.0 m, $l_n = l'_n$ = 6.3 m

‡ This distribution is nearly identical to one that will be obtained by following Ref. 11.18 and the median values of the *range* of factors given in Section 11.5.4 (sub-section (a)) for the Canadian Code.

$$\alpha_c = \sum K_c / k_{sb} = \frac{(8.427 + 7.654) \times 10^6 E_c}{(4.653 \times 10^6) E_c} = 3.456$$

$$\Rightarrow M_u = 0.08 \left[ (11.55 + 0.5 \times 3.75)(8.0 \times 6.3^2) - (11.55 \times 8.0 \times 6.3^2) \right] \Big/ \left(1 + 1/3.456\right)$$

$$= 0.08 \times 461.7 = 37 \text{ kNm}$$

Accordingly, the unbalanced moment at the interior column 3 should be taken as 37 kNm (and not 12 kNm).

The distribution of unbalanced moments to the various columns, above and below, including carry-over effects (using the coefficients derived) are depicted in Fig. 11.57.

**10. Flexural reinforcement**

The requirement of flexural (tension) reinforcement at all critical sections is computed in Table 11.10 for the column strip (3400 mm wide for ‘positive’ moments, and 3000 mm† wide for ‘negative’ moments), and in Table 11.11 for the middle strip (4600 mm wide for ‘positive’ moments and 5000 mm wide for ‘negative’ moments). 16 mm diameter bars have been used and the Code restrictions of minimum reinforcement and maximum spacing have been adopted.

**Table 11.10** Design of reinforcement in column strip, E–W direction — Example 11.7

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Moment in col. strip, $M_u$ (kNm) | 212 | 154 | 393 | 358 | 101 | 313 | 320 | 113 |
| Width of strip, or drop $b$ (mm) | 3000 | 3400 | 3000 | 3000 | 3400 | 3000 | 3000 | 3400 |
| Effective depth $d$ (mm) | 256 | 156 | 256 | 256 | 156 | 256 | 256 | 156 |
| $R \equiv M_u\ (bd^2)$ (MPa) | 1.078 | 1.861 | 1.999 | -- | 1.221 | -- | 1.628 | 1.366 |
| $p$ [Table A.3(a)] | 0.319 | 0.587 | 0.638 | -- | 0.366 | -- | 0.504 | 0.414 |
| $A_{st} = (p/100)bd$ (mm$^2$) | 2456 | 3113 | 4900 | -- | 1941 | -- | 3871 | 2196 |
| Number of 16 $\phi$ bars | 12 | 16 | 25 | -- | 10 | -- | 201 | 11 |
| Spacing (mm) | 250 | 212 | 120 | -- | 340 | -- | 150 | 309 |

It may be noted that, where the unbalanced slab moment is significant (at the exterior support only, in this Example), adequate reinforcement should be provided over a distance $c_2 + 3D_{drop}$ = 1400 mm, centred about the column line, to permit the

† The effective width of the column strip at supports (for ‘negative’ moments) is restricted to that of the drop panel, for convenience.

transfer by flexure from the slab to the column the portion $M_{ub}$ of the unbalanced moment.

$M_{ub} = 0.6 \times 304 = 182$ kNm [Eq. 11.28a]

$$\Rightarrow R \equiv \frac{M_{ub}}{bd^2} = \frac{182 \times 10^6}{1400 \times 256^2} = 1.984 \text{ MPa}$$

$\Rightarrow p_t = 0.632$ [Table A.3(a) or Eq. 5.12, for M 20 and Fe 415]

$\Rightarrow A_{st} = (0.632/100) \times (1400 \times 256) = 2265 \text{ mm}^2$

$\Rightarrow$ No. of 16 $\phi$ bars required = 2265/201 = 12

Thus, at the exterior support, 12 nos of 16 $\phi$ bars must be distributed over a width of 1400 mm centred over the column. The remaining outer portions of the drop panel, with width equal to (3000 – 1400)/2 = 800 mm may be provided with two bars each, to limit the spacing to < 2*D*. The column strip 'negative' moment reinforcement at the exterior support, adds up to 12 + (2 $\times$ 2) = 16 bars, compared to the 12 bars indicated in Table 11.10.

**11. Check on one-way shear stress**

There are two critical sections to be considered:

(i) $d$ = 256 mm from the face of column; and

(ii) $d$ = 156 mm from the edge of drop panel.

**Table 11.11** Design of reinforcement in middle strip, E–W direction — Example 11.7

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Moment in col. strip, $M_u$ (kNm) | 0 | 102 | 131 | 119 | 67.2 | 104 | 107 | 75 |
| Width of strip 8000 – col. strip $b_s$ (mm) | 5000 | 4600 | 5000 | 5000 | 4600 | 5000 | 5000 | 4600 |
| Effective depth $d$ (mm) | 156 | 156 | 156 | 156 | 156 | 156 | 156 | 156 |
| $R \equiv M_u/(bd^2)$ (MPa) | -- | 0.911 | 1.077 | -- | 0.600 | -- | 0.879 | 0.679 |
| $p$ [Table A.3(a)] | 0.002* | 0.267 | 0.319 | -- | 0.172 | -- | 0.257 | 0.196 |
| $A_{st} = (p/100)bd$ (mm$^2$) | 1200 | 1917 | 2488 | -- | 1234 | -- | 2005 | 1406 |
| Number of 16 $\phi$ bars (No.) | 6 | 10 | 13 | -- | 7 | -- | 10 | 7 |
| Spacing (mm) | 833† | 460† | 385† | -- | 657† | -- | 500† | 657† |
| Number of 16 $\phi$ bars to limit spacing < 2*D* | 12 | 12 | 13 | -- | 12 | -- | 12 | |

* minimum reinforcement ($A_{st} = 0.0012\, bD$) governs.

† maximum allowable spacing governs.

From the equivalent frame analysis [Table 11.9], the maximum shear force, equal to 483 kN, is found to be at the exterior face of first interior support. Referring to Fig. 11.56(d), with $x$ = 2.991 m and $V_R$ = 483 kN, and to Fig. 11.58,

**(i) at *d* = 256 mm from face of column**, i.e., 506 mm from centre of column,

$$V_{u1} = 483 - (122.4 + 11.25) \times 0.506 = 415 \text{ kN}$$

Assuming the shear force to be uniformly distributed over the width of the panel (8.0*m*), and considering a 1 m strip of the slab outside the drop panel (where $d$ = 156 mm),

$$\tau_{v1} = \frac{(415 \times 10^3)/8.0}{1000 \times 156} = 0.333 \text{ MPa}$$

which is less than $k\,\tau_c$ given by the Code (Cl. 40.2.1) for $p_t$ = 0.32 and M 20 concrete.

**(ii) at *d* = 156 mm from the edge of the drop panel,** the shear force and hence the shear stress, will be less than that calculated above.
This will obviously be safe.

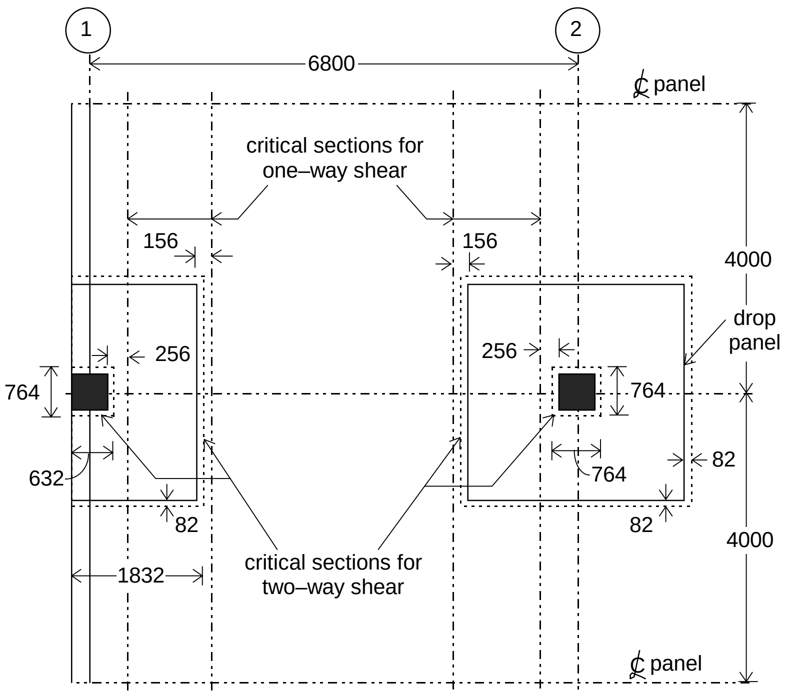



**Fig. 11.58** Critical sections for one-way and two-way shear — Example 11.7

**12. Check on two-way (punching) shear stress**

The maximum column reaction and unbalanced slab moment (in an interior column) occurs at column 2 (or 5). From Table 11.9, the vertical reaction is obtained as the sum of the shears on either side as:

$$R = 483 + 442 = 925 \text{ kN}$$

**a.** ***Interior column***

The effective depth in punching shear calculations may be taken as the average of the effective depths in the E–W and N–S directions. The critical sections are

(i) $d/2 = (256 + 272)/4 = 132$ mm from the column face all around; and

(ii) $d/2 = (156 + 172)/4 = 82$ mm from the edge of the drop panel,

as indicated in Fig. 11.58

i) For the critical section at $d/2$ from the column face, the punching shear is [Fig. 11.55]

$$V_{u2} = 925 - (15.3 + 3.75) \times 0.764^2 = 914 \text{ kN}$$

and the perimeter is

$$b_o = 4 \times 764 = 3056, \text{ with } d = 264 \text{ mm}$$

The moment transferred by shear is 40 percent of the unbalanced moment $M_u = 58$ kNm at column 2 [Eq. 11.28b]

$$M_{uv} = 0.4 \times 58 = 23.2 \text{ kNm}$$

and the parameters $c$ and $J_c$ in Eq. 11.51 [Fig. 11.45(a)] are given by:

$$c = (c_1 + d)/2 = 764/2 = 382 \text{ mm}$$

$$J_c = \frac{(c_1 + d)\, d^3}{6} + \frac{(c_1 + d)^3\, d}{6} + \frac{(c_1 + d)^2 (c_2 + d)\, d}{2}$$

$$= \frac{764 \times 264^3}{6} + \frac{764^3 \times 264}{6} + \frac{764^3 \times 264}{2}$$

$$= 80.83 \times 10^9 \text{ mm}^4$$

Applying Eq. 11.51,

$$\tau_{v2} = \frac{V_{u2}}{b_o d} + \frac{M_{uv} c}{J_c}$$

$$= \frac{914 \times 10^3}{3056 \times 264} + \frac{(23.2 \times 10^6) \times 382}{80.83 \times 10^9}$$

$$= 1.133 + 0.110 = 1.243 \text{ MPa}$$

$$\tau_{c2} = k_s\left(0.25\sqrt{f_{ck}}\right) \quad \text{[Eq. 11.49]}$$

$$= 1.0 \times 0.25\sqrt{20} = 1.118 \text{ MPa}^\dagger$$

---

† Note that if the concrete grade is improved to M 25, $\tau_{c2}$ = 1.25 MPa > $\tau_{v2}$, and no shear reinforcement is required.

The applied shear stress $\tau_{v2}$, therefore, marginally exceeds the shear strength $\tau_{c2}$. Note, however that $\tau_{v2} < 1.5\,\tau_{c2}$. Hence, shear reinforcement is required, with a total cross-sectional area:

$$A_{sv} = \frac{\left(\tau_{v2} - 0.5\tau_{c2}\right)b_o d}{0.87 f_y} \quad \text{[Eq. 11.52]}$$

$$= \frac{(1.243 - 0.5 \times 1.118)(3056 \times 264)}{(0.87 \times 415)} = 1526 \text{ mm}^2$$

- Using 8 mm ϕ stirrups [Type I arrangement, Fig. 11.46], number of vertical legs required on each side (of the square of side 764 mm) = 1528/(50.3 × 4) = 8. This can be achieved by providing 8-legged 8 mm ϕ stirrups comprising 4 nos 2-legged closed stirrups [Fig. 11.46] on each side. Nominal 10 mm ϕ holder bars may be provided at the stirrup corners if regular top steel and bottom steel are not otherwise available. This arrangement of stirrups should be provided at a spacing of not more than 0.75*d* = 0.75 × 264 = 198 mm ≈ 200 mm and should be continued to a distance *d* = 264 mm beyond the section where the shear stress does not exceed $0.5\,\tau_{c2}$.
- Checking at a section four spacings further removed from the first critical section, and now ignoring the marginal shear stress due to the unbalanced moment,

$$V_{u2} = 925 - (15.3 + 3.75) \times (0.764 + 0.2 \times 4)^2 = 878 kN$$

$$b_o = [764 + (200 \times 4)] \times 4 = 6256 \text{ mm},$$

$$\Rightarrow \tau_{v2} = \frac{V_{u2}}{b_o d} = \frac{878 \times 10^3}{6256 \times 264} = 0.532 \text{ MPa}$$

$$< 0.5\,\tau_{c2} = 0.5 \times 1.118 = 0.559 \text{ MPa} \quad \text{— OK.}$$

Thus, it is necessary to provide 4 + 1 = 5 spacings of stirrups on all sides @ 200*m*m c/c — within the drop panel, with the first line of stirrups located *d*/2 = 132 mm away from the column face.

(ii) For the critical section *d*/2 = 82 mm from the edge of the drop panel,

$$V_{u2} = 925 - (15.3 \times 3.164^2) - (3.75 \times 3^2) = 738 \text{ kN}$$

$$b_o = 3164 \times 4 = 12656 \text{ mm}, \; d = 164 \text{ mm}$$

$$\Rightarrow \tau_{v2} = \frac{738 \times 10^3}{12656 \times 164} = 0.356 \text{ MPa}$$

$$<< \tau_{c2} = 1.118 \text{ MPa} \quad \text{— OK.}$$

**b. *Exterior column***

Owing to the presence of the edge beam, part of the shear will be transmitted by this beam to the column by beam action. However, to simplify calculations, the shear stress is conservatively computed by neglecting the contributions of the edge beam.

The reaction at the exterior column = 383 kN [Table 11.9]

i) For the critical section, $d/2$ = 132 mm from the column face [Fig. 11.58].

$$V_{u2} = 383 - (15.3 + 3.75) \times (0.764 \times 0.632) = 374 \text{ kN}$$

$$M_{uv} = 0.4 \times 304 = 122 \text{ kNm}$$

$$b_o = (632 \times 2) + 764 = 2028 \text{ mm}, \; d = 264 \text{ mm}$$

Referring to Fig. 11.45(b),

$$c = \frac{(c_1 + d/2)^2}{2c_1 + c_2 + 2d} = \frac{(632)^2}{(2 \times 500) + 500 + (2 \times 264)} = 197 \text{ mm}$$

$$J_c = [(c_1 + d/2)\, d^3 + (c_1 + d/2)^3\, d\,]/6 + (c_2 + d)\, dc^2 + 2(c_1 + d/2)(d) \times [(c_1 + d/2)/2 - c]^2$$

$$= [(632)(264)^3 + (632)^3(264)]/6 + (764)(264)(197)^2 + 2(632)(264) \times [632/2 - 197]^2$$

$$= 25.60 \times 10^9 \text{ mm}^4$$

$$\Rightarrow \; \tau_{v2} = \frac{V_{u2}}{b_o d} + \frac{M_{uv} c}{J_c} = \frac{374 \times 10^3}{2028 \times 264} + \frac{(122 \times 10^6) \times 197}{25.60 \times 10^9}$$

$$= 0.699 + 0.939 = 1.638 \text{ MPa}$$

which is greater than $\tau_{c2}$ = 1.118 MPa but less than 1.5 $\tau_{c2}$ = 1.677 MPa.

Hence, shear reinforcement is required to be designed; this may be done as shown in the case of the interior column.

(ii) For the critical section at $d/2$ = 82 mm from the edge of the drop panel [Fig. 11.58],

$$V_{u2} = 383 - (15.3 \times 1.832 \times 3.164) - (3.75 \times 3.0 \times 1.75) = 275 \text{ kN}$$

$$b_o = (2 \times 1832) + 3164 = 6828 \text{ mm}, \; d = 164 \text{ mm}$$

$$\Rightarrow \; \tau_{v2} = \frac{275 \times 10^3}{6828 \times 164} = 0.246 \text{ MPa}$$

$$<< \tau_{c2} = 1.118 \text{ MPa} \quad \text{— OK.}$$

## REVIEW QUESTIONS

11.1 Explain clearly the difference in the behaviour of one-way slabs and two-way slabs.

11.2 Explain the need for corner reinforcement in two-way rectangular slabs whose corners are prevented from lifting up.

11.3 Explain the difference in load transfer between wall-supported slabs and beam/column supported slabs.

11.4 What are the main considerations that generally govern the thickness of a two-way slab?

11.5 Explain the concept underlying the Rankine-Grashoff theory as applied to uniformly loaded and simply supported rectangular two-way slabs.

11.6 What are the assumptions underlying the Code moment coefficients for two-way 'restrained' slabs?

11.7 In the design of a multipanel two-way slab system by the use of the Code moment coefficients, it is found that the design 'negative' moments at continuous supports are often unbalanced. Why does this occur, and how may this problem be resolved?

11.8 How are two-way wall-supported slabs checked for shear?

11.9 How is two-way slab behaviour in a column-supported slab system, with beams along column lines, affected by the stiffnesses of the supporting beams?

11.10 Why is it inappropriate to apply the Code moment coefficients for two-way slabs, if the slabs are supported on flexible beams?

11.11 Explain briefly the 'equivalent frame' concept. Also sketch the variations of moments in a typical two-way slab panel supported on flexible beams.

11.12 Briefly describe the Direct Design Method and compare it with the Equivalent Frame Method.

11.13 What is the function of (i) the *drop panel* and (ii) the *column capital*, in *flat slab* design?

11.14 Explain how the 'unbalanced' moment is transferred from the slab to the column in flat slabs.

11.15 Discuss briefly how the effects of pattern loading can be included in (i) Direct Design Method (ii) Equivalent Frame Method.

11.16 In flat slabs, what are the parameters, which determine (i) the longitudinal distribution of moments and (ii), the transverse distribution of moments in a panel?

11.17 In the transverse distribution of moments at critical sections, the column strip is given a larger share than the middle strips. Why?

11.18 Reference 11.8 gives some freedom to the designer in choosing the transverse distribution of moments at critical sections in the slab-beam member in two-way slab systems. How is this justified?

11.19 In flat slab floors, it is desirable to provide an edge beam along the discontinuous edges. Why?

11.20 What are the considerations in the design of edge beams in flat slab floors?

11.21 Explain how shear forces in beams are estimated in two-way slab systems supported on flexible beam.

11.22 Explain the concept of 'equivalent column' in the Equivalent Frame Method.

11.23 What is the difference between one-way shear and two-way shear in column-supported slab systems?

11.24 How is punching shear stress calculated in column-supported slab systems?

11.25 Suggest suitable reinforcement details for resistance against punching shear in flat slabs and flat plates.

11.26 In flat slabs, how can a total punch-through and a progressive collapse prevented?

## PROBLEMS

11.1 Design a simply supported slab to cover a hall with internal dimensions 4.0 m × 6.0 *m*. The slab is supported on masonry walls 230 mm thick. Assume a live load of 3 kN/m$^2$ and a finish load of 1 kN/m$^2$. Use M 20 concrete and Fe 415 steel. Assume that the slab corners are free to lift up.

11.2 Repeat Problem 11.1, considering the slab corners to be prevented from lifting up.

11.3 Repeat Problem 11.1, considering the slab to be an internal panel which is part of a multipanel slab system.

11.4 Design the multipanel floor slab system shown in Fig. 11.59, assuming a live load of 4.0 kN/m$^2$ and a floor finish load of 1.0 kN/m$^2$. The slab is supported on 230 mm thick masonry walls, as shown. Use M 20 concrete and Fe 415 steel.

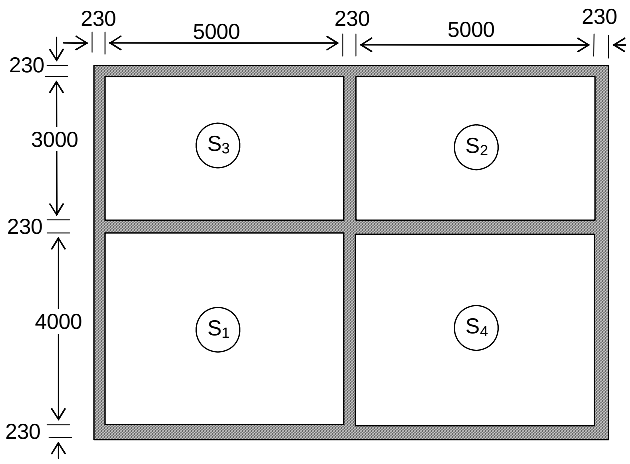



**Fig. 11.59 :** Problem 11.4

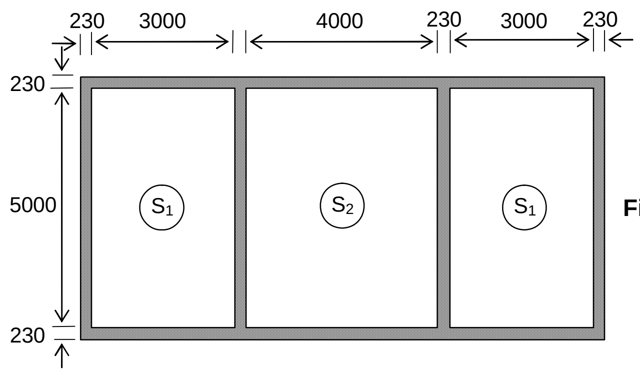



**Fig. 11.60 :** Problem 11.5

11.5 Design the multipanel floor slab system shown in Fig. 11.60, assuming a live load of 4.0 kN/m$^2$ and a floor finish load of 1.0 kN/m$^2$. The slab is supported on 230 mm thick masonry walls, as shown. Use M 20 concrete and Fe 415 steel.

11.6 Design the slab panels $S_1$ and $S_2$ in the multipanel floor system of Example 11.6, [Fig. 11.47] using the Code moment coefficients, assuming that the supporting beams are adequately stiff. Compare the results with those obtained earlier. (in Example 11.6).

11.7 Design a circular slab of 4.0 m diameter (overall), simply supported at the periphery, by a masonry wall 230 mm thick. Assume a live load of 5.0 kN/m$^2$ and a finish load of 1.0 kN/m$^2$. Use M 20 concrete and Fe 415 steel.

11.8 In continuation with Example 11.6, apply the Direct Design Method to design the corner panel in the slab system [Fig. 11.47].

11.9 Repeat Example 11.6, considering the slab system as a *flat slab* system, with suitable beams only along the exterior edges and with suitable drop panels.

11.10 Repeat Example 11.7, applying the Direct Design Method, instead of the Equivalent Frame Method.

11.11 Repeat Example 11.6, applying Equivalent Frame Method instead of the Direct Design Method.

11.12 Redesign the interior equivalent frame in Example 11.7, assuming that beams 300 mm wide and 500 mm deep (overall) are provided along the column lines, and no drop panels are used.

## REFERENCES


11.1 Timoshenko, S., and Woinosky-Krieger, S., *Theory of Plates and Shells*, McGraw-Hill Book Co., New York, 1959.

11.2 Westergaard, H.M. and Slater, W.A., *Moments and Stresses in Slabs*, Proc. ACI, Vol.17, 1921, pp 415–538.

11.3 Westergaard H.M., *Formulas for the Design of Rectangular Floor Slabs and the Supporting Girders*, Proc. ACI, Vol. 22, 1926, pp 26–46.

11.4 Di Stasio, J. and Van Buren, M.P., *Slabs supported on Four Sides*, Proc. ACI Vol. 32, pp 350–364.

11.5 Purushothaman, P., *Reinforced Concrete Structural Elements — Behaviour, Analysis and Design*, Tata McGraw Hill Publ. Co. Ltd., New Delhi, 1984.

11.6 Park, R. and Gamble, W.L., *Reinforced Concrete Slabs*, John Wiley & Sons, New York, 1980.

11.7 Zienkiewicz, O.C., *The Finite Element Method in Engineering Sciences*, Second edition, McGraw-Hill Book Co., London, 1971.

11.8 Johansen, K.W., *Yield Line Theory*, (translated from the Danish), Cement and Concrete Association, London, 1962.

11.9 Jones, L.L. and Wood, R.H., *Yield Line Analysis of Slabs*, Thames and Hudson, Chatto and Windus, London, 1967.

11.10 Shukla, S. N., *Handbook for Design of Slabs by Yield Line and Strip Methods*, Structural Engineering Research Centre, Roorkee, India, 1973.

11.11 — *Explanatory Handbook on Indian Standard Code of Practice for Plain and Reinforced Concrete (IS 456:1978)*, Special Publication SP:24, Bureau of Indian Standards, New Delhi, 1983.

11.12 Tayler, R., Hayes, B. and Bhai, M., *Coefficients for Design of Slabs by Yield Line Theory*, Concrete, Vol. 13, no.5, 1969.

11.13 —*Structural Use of Concrete: Part 1: Code of Practice for Design and Construction*, BS 8110 : Part 1 : 1985, British Standards Institution, 1985.

11.14 — *Handbook on Concrete Reinforcement and Detailing*, Special Publication SP 34, Bureau of Indian Standards, New Delhi, 1987.

11.15 Haan, J., *Structural Analysis of Beams and Slabs*, Sir Isaac Pitman and Sons, London, 1966.
11.16 Reynolds, C.E. and Steedman, J.C., *Reinforced Concrete Designer's Handbook*, Rupa & Co., Delhi, 1981.
11.17 Regan, P.E. and Yu, C.W., *Limit State Design for Structural Concrete*, Chatto and Windus, London, 1973.
11.18 CSA Standard A23.3 – 94 — *Design of Concrete Structures*, Canadian Standards Association, Rexdale, Ontario, 1994.
11.19 — *Building Code Requirements for Reinforced Concrete*, ACI Standard 318–89, Am. Conc. Inst., Detroit, Michigan, USA, 1989.
11.20 Sozen, M.A. and Siess, C.P., *Investigation of Multiple-Panel Reinforced Concrete Floor Slabs: Design Methods — Their Evolution and Comparison*, Journal ACI, Proc. Vol. 60, No.8, Aug. 1963, pp 999–1028.
11.21 Corley, W.G., and Jirsa, J.O., *Equivalent Frame Analysis for Slab Design*, Journal ACI, Proc. Vol. 67, No. 11, Nov. 1970, pp 875–884.
11.22 Gamble, W.L., *Moments in Beam Supported Slabs*, Journal ACI, Proc. Vol. 69, No. 3, Mar. 1972, pp 149–157.
11.23 Hanson, N.W., and Hanson, J.M., *Shear and Moment Transfer Between Concrete Slabs and Columns*, Journal, PCA Research and Development Laboratories, Vol. 10, No. 1, Jan. 1968, pp 2–16.
11.24 Gamble, W.L., Sozen, M.A., and Seiss, C.P., *Tests on a Two-Way Reinforced Concrete Floor Slab*, Proc. ASCE, Journal of Struct. Div., Vol. 95, No. ST6, June 1969, pp 1073–96.
11.25 *Concrete Design Handbook*, Canadian Portland Cement Association, Ottawa, Canada, 1985.
11.26 *Commentary on Building Code Requirements for Reinforced Concrete* (ACI 318–89), American Concrete Institute, Detroit, Michigan, 1989.
11.27 ACI–ASCE Committee 326, *Shear and Diagonal Tension*, Journal ACI, Proc. Vol. 59, Nos.1 – 3, Jan – March 1962.
11.28 ACI–ASCE Committee 426, *The Shear Strength of Reinforced Concrete Members*, Proc. ASCE, Vol. 100, No. ST 8, Aug. 1974, pp 1543–1591.
11.29 Pillai, S.U., Kirk, W., and Scavuzzo, L., *Shear Reinforcement at Slab-Column Connections in a Reinforced Concrete Flat Plate Structure*, Journal ACI, Proc. Vol. 79, No. 1, Jan.–Feb. 1982, pp 36–42.

# 12

# Design of Staircases

## 12.1 INTRODUCTION

The design of staircases is generally included in a first course on reinforced concrete design, and for this reason, this topic is included in this book. It should be noted that, from a structural viewpoint, the staircase merely comprises slab/beam elements, whose basic principles of design have already been dealt with in the previous chapters.

Functionally, the staircase is an important component of a building, and often the only means of access between the various floors in the building. It consists of a *flight* of steps, usually with one or more intermediate *landings* (horizontal slab platforms) provided between the floor levels. The horizontal top portion[†] of a step (where the foot rests) is termed *tread* and the vertical projection of the step (i.e., the vertical distance between two neighbouring steps) is called *riser* [Fig. 12.1]. Values of 300 mm and 150 mm are ideally assigned to the tread and riser respectively — particularly in public buildings. However, lower values of tread (up to 250 mm) combined with higher values of riser (up to 190 mm) are resorted to in residential and factory buildings. The *width* of the stair is generally around 1.1 – 1.6*m*, and in any case, should normally not be less than 850 mm; large stair widths are encountered in entrances to public buildings. The horizontal projection (plan) of an inclined flight of steps, between the first and last risers, is termed *going*. A typical flight of steps consists of two landings and one going, as depicted in Fig. 12.1(a). Generally, risers in a flight should not exceed about 12 in number. The steps in the flight can be designed in a number of ways: with *waist slab*, with *tread-riser* arrangement (without waist slab) or with *isolated tread slabs* — as shown in Fig. 12.1(b), (c), (d) respectively.

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[†] The tread is sometimes projected outwards to provide more space; this projection is termed *nosing*. Frequently, the nosing is provided in the finish over the concrete tread.

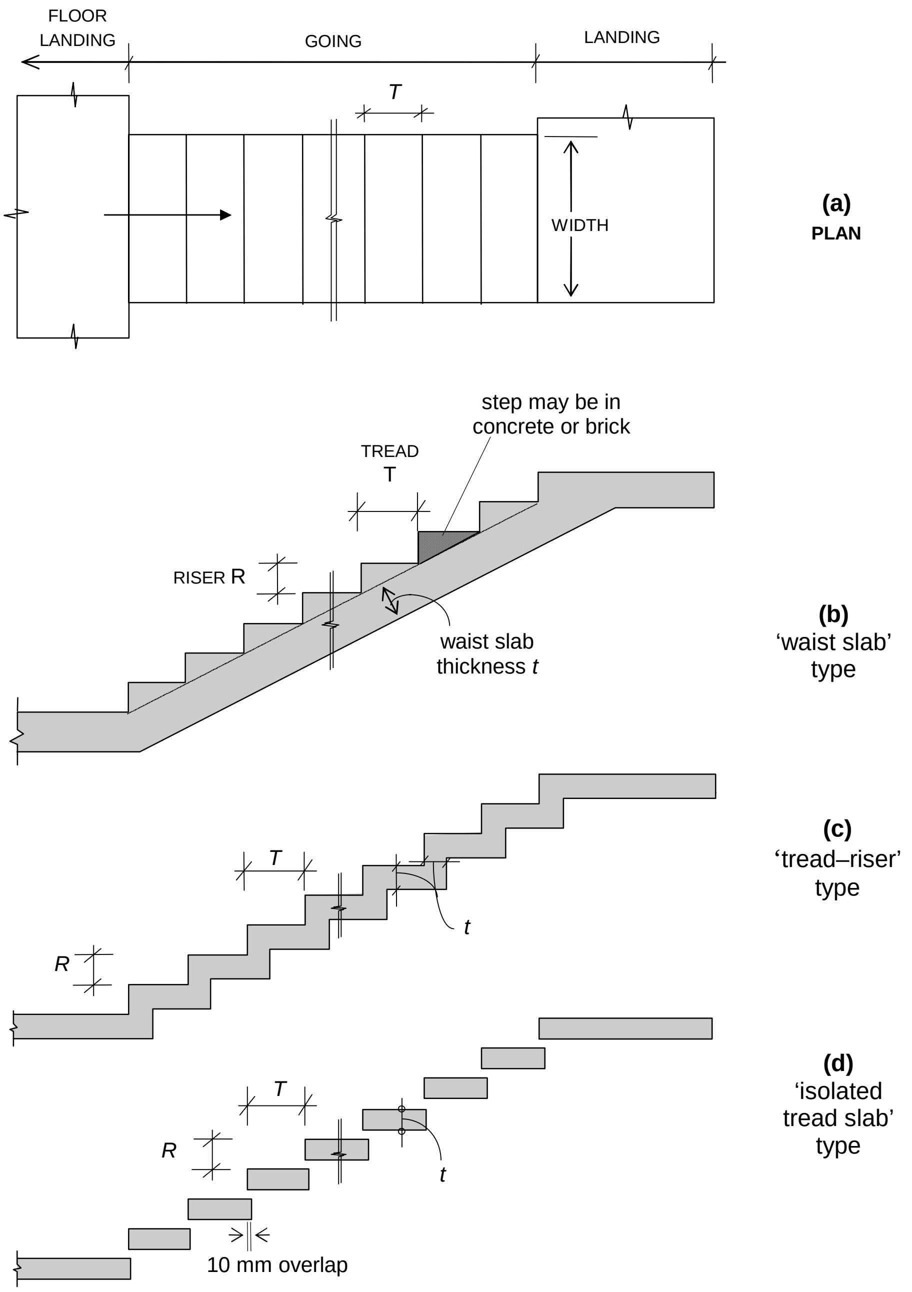


**Fig. 12.1** A typical flight in a staircase

## 12.2 TYPES OF STAIRCASES

### 12.2.1 Geometrical Configurations

A wide variety of staircases are met with in practice. Some of the more common geometrical configurations are depicted in Fig. 12.2. These include:

- straight stairs (with or without intermediate landing) [Fig. 12.2(a)]
- quarter-turn stairs [Fig. 12.2(b)]
- dog-legged stairs [Fig. 12.2(c)]
- open well stairs [Fig. 12.2(d)]
- spiral stairs [Fig. 12.2(e)]
- helicoidal stairs [Fig. 12.2(f)]

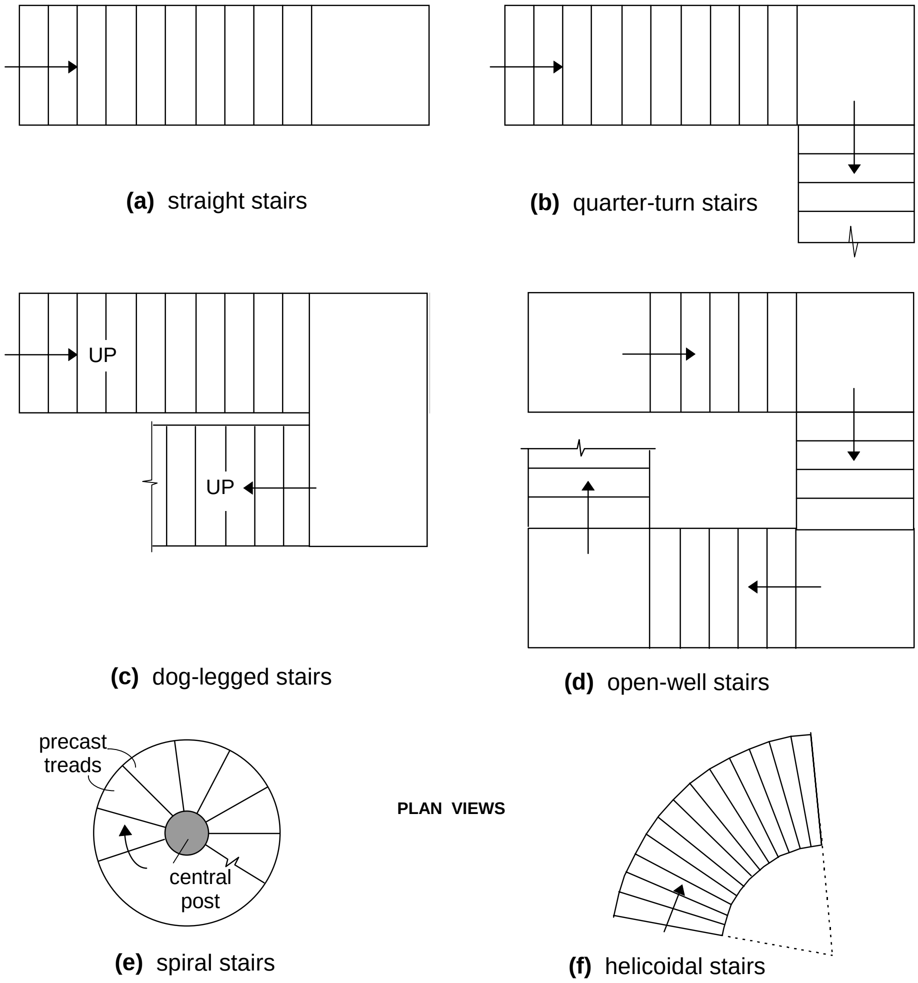


**Fig. 12.2** Common geometrical configurations of stairs

The architectural considerations involved in the selection and location of staircases include accessibility, function, comfort, lighting, ventilation, aesthetics, etc. Structural feasibility also has a major role in deciding the proportioning of the slab thickness (dimension '*t*' in Fig. 12.1) which contributes much to the aesthetic appeal of a staircase. Perhaps the most daring of all staircases is the *free-standing staircase* which is supported entirely at the base, and behaves essentially as a cantilever in space [Ref. 12.1, 12.2, 12.7]. The helicoidal staircase (or ramp), with intermediate supports also presents a challenging problem of structural analysis [Ref. 12.3, 12.4, 12.7]. These problems, however, are not discussed in the present chapter, the scope of which is limited to the simple geometrical configurations.

### 12.2.2 Structural Classification

Structurally, staircases may be classified largely into two categories, depending on the predominant direction in which the slab component of the stair undergoes flexure:

1. stair slab spanning transversely (stair widthwise);
2. stair slab spanning longitudinally (along the incline).

#### *Stair Slab Spanning Transversely*

This category generally includes:

1. slab cantilevered from a spandrel beam or wall [Fig. 12.3(a)];
2. slab doubly cantilevered from a central spine beam [Fig. 12.3(b)];
3. slab supported between two stringer beams or walls [Fig. 12.3(c)].

The slab component of the stair (whether comprising an isolated tread slab, a tread-riser unit or a waist slab [Fig. 12.1]) is supported on its side(s) or cantilevers laterally from a central support [Fig. 12.1(b)]. The slab supports gravity loads by bending essentially in a *transverse vertical plane*, with the span along the *width* of the stair.

In the case of the cantilevered slabs [Fig. 12.3(a), (b)], it is economical to provide isolated treads[‡] (without risers), as indicated in Fig. 12.1(d). However, the tread-riser type of arrangement [Fig. 12.1(c)] and the waist slab type [Fig. 12.1(b)] are also sometimes employed in practice, as cantilevers. The spandrel beam is subjected to torsion ('equilibrium torsion'), in addition to flexure and shear.

When the slab is supported at the two sides by means of 'stringer beams' or masonry walls [Fig. 12.3(c)], it may be designed as simply supported, but reinforcement at the top should be provided near the supports to resist the 'negative' moments that may arise on account of possible partial fixity.

It may be noted that, although the stair slab spans *transversely*, the supporting spandrel/spine/stringer beams span *longitudinally* along the incline of the stair,

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[‡] The isolated tread slabs are often *precast*. The treads may form part of a straight flight or a curved flight [Fig. 12.2(e), (f)]. For example, in a reinforced concrete chimney or tower, cantilevered tread slabs are fixed to the circular shaft and arranged in a helicoidal pattern.

framing into supporting columns. The design of the beam is not discussed in this chapter, as all the design principles have already been covered in earlier chapters.

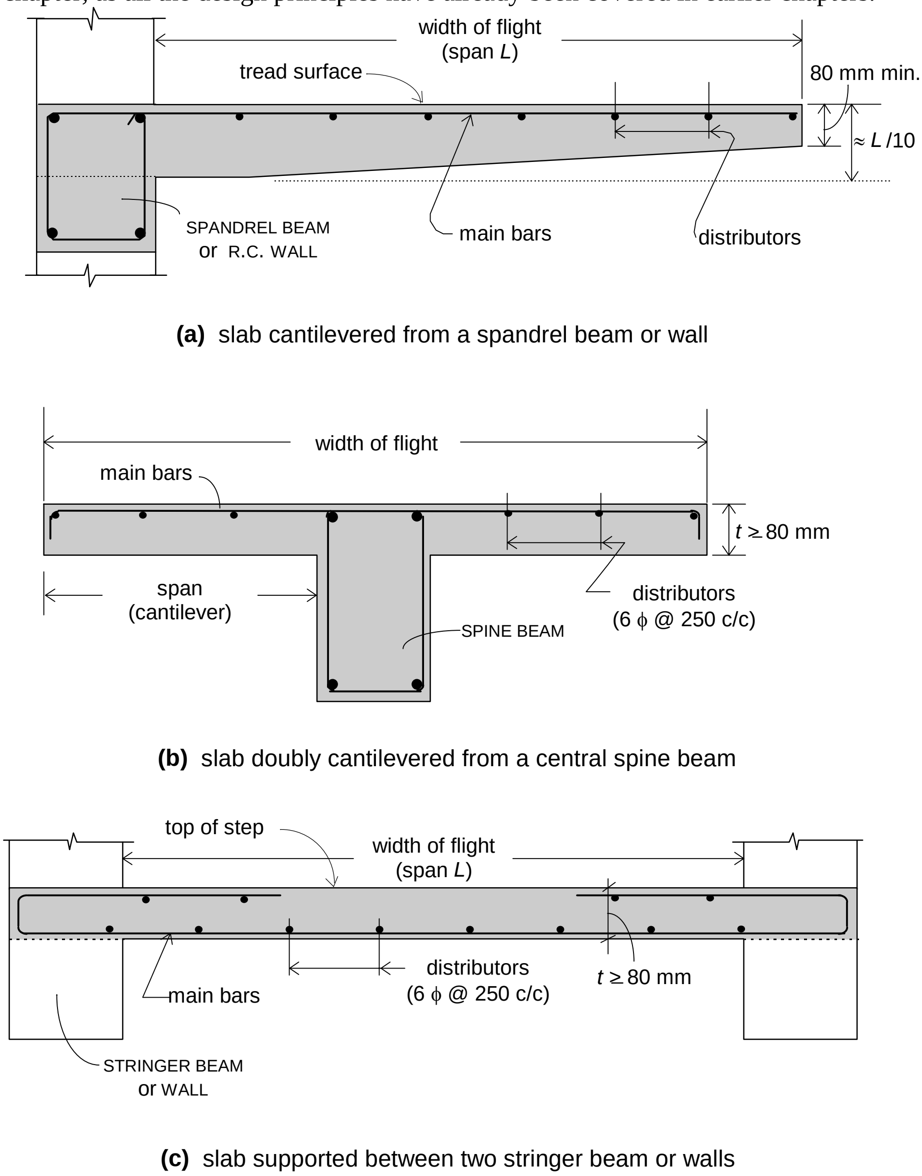


**Fig. 12.3** Typical examples of stair slabs spanning transversely†

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† The figures depict transverse sections of the stairs.

When the slab is doubly cantilevered from a central (spine) beam [Fig. 12.3(b)], it is essential to ensure, by proper detailing, that the slab does not separate from the beam when loaded on one side only. This can be done by anchoring the slab reinforcement into the beam, so that the same reinforcement acts as a stirrup in the beam, as shown in Fig. 12.3(b). Alternative arrangements are possible; however, it should be ensured that the beam stirrups are 'closed', to provide the desired torsional resistance. When the slab units are precast, suitable mechanical connections have to be provided between the beam and the slab units.

### *Stair Slab Spanning Longitudinally*

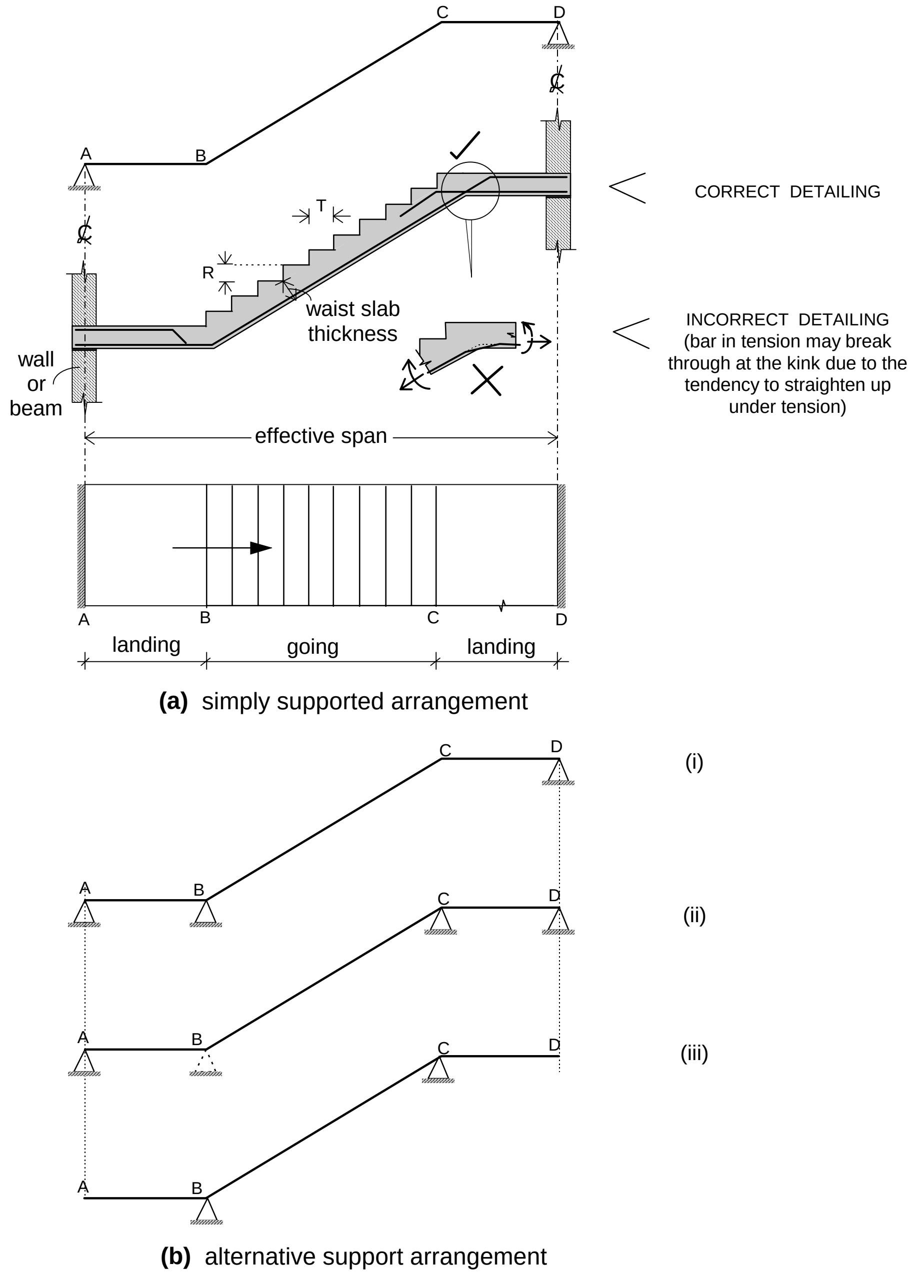
In this case, the supports to the stair slab are provided parallel to the riser at two or more locations, causing the slab to bend longitudinally between the supports, as shown in Fig. 12.4. It may be noted that longitudinal bending can occur in configurations other than the straight stair configuration (shown in Fig. 12.4), such as quarter-turn stairs, dog-legged stairs, open well stairs and helicoidal stairs [Fig. 12.2].

The slab arrangement may either be the conventional 'waist slab' type [Fig. 12.1(b)] or the 'tread-riser' type [Fig. 12.1(c)]. The slab thickness depends on the 'effective span', which should be taken as the centre-to-centre distance between the beam/wall supports, according to the Code (Cl. 33.1a, c). Fig. 12.4(a) shows a simple arrangement with simple supports at the far ends of the two landings. However, such an arrangement can result in large slab thicknesses for relatively long spans (4*m* or more). In such cases, it is economical to reduce the span, and hence the slab thickness, by providing additional intermediate supports — at locations 'B' and 'C', as shown in Fig. 12.4(b); this will induce 'negative' moments near the supports, requiring steel at the top in these regions. It is sometimes economical to provide supports at B and C alone and to treat the landings (AB, CD) as overhangs — as depicted in the 'balanced cantilever' design shown in Fig. 12.4(b)(iii).

In certain situations, beam or wall supports may not be available parallel to the riser at the landing. Instead, the flight is supported between the landings, which span transversely, parallel to the risers, as shown in Fig. 12.5(a). In such cases, the Code(Cl. 33.1b) specifies that the effective span for the flight (spanning longitudinally) should be taken as *the going of the stairs plus at each end either half the width of the landing or one metre*, *whichever is smaller*, as depicted in Fig. 12.5(a). Recent research [Ref. 12.5, 12.6, 12.8] indicates that 'negative' moments (of magnitudes comparable to the 'positive' span moments) develop at the junction where the inclined waist slab meet the landing slabs, and it is desirable to detail the slab accordingly.

Another case frequently encountered in residential and office buildings is that of the landings supported on three sides, as shown in Fig. 12.5(b). This case has not been explicitly covered by the Code. The ACI Code and BS Code also do not have any special provision as yet for this condition. However, recent studies (based on experiments as well as finite element analysis) reveal that the flight essentially spans between the landing-going junctions, with hogging moments developing at these junctions. It is recommended that an economical and conservative design can be achieved by designing for a 'positive' moment of $wl^2/8$ for the going (at midspan) and a 'negative' moment of $wl^2/8$ at the junction of landing and the going[†]. Here, $w$ is the distributed gravity load acting on the going and $l$ is the length of the going (projected on a horizontal plane) [Ref. 12.5, 12.6, 12.8].

[†] More detailed expressions for design moments in dog-legged and open-well stairs (with waist slab or with tread-riser) are described in Ref. 12.8.

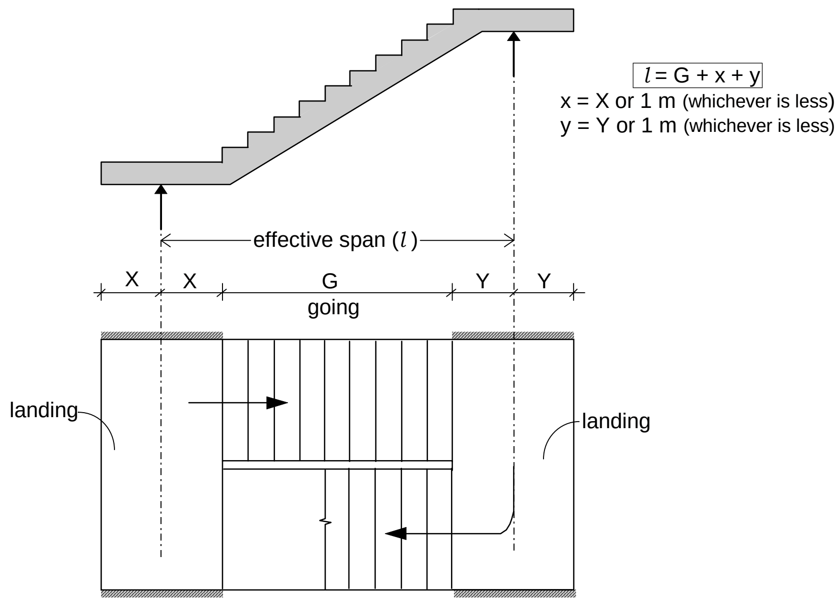



**(a)** simply supported arrangement

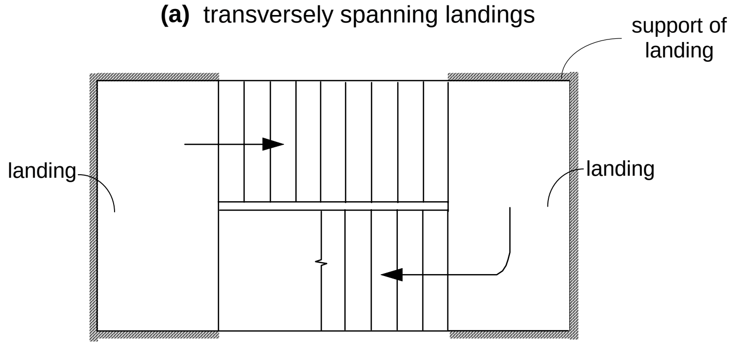


**(b)** alternative support arrangement

**Fig. 12.4** Typical examples of stair slabs spanning longitudinally




**(a)** transversely spanning landings




**(b)** landings supported on three edges

**Fig. 12.5** Special support conditions for longitudinally spanning stair slabs

## 12.3 LOADS AND LOAD EFFECTS ON STAIR SLABS

Stair slabs are usually designed to resist gravity loads[†] , comprising *dead loads* and *live loads*.

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[†] In the case of cantilevered tread slabs, the effects of *seismic* loads should also be investigated. The vertical vibrations induced by earthquakes may induce flexural stresses of considerable magnitude. It is desirable to provide bottom steel in the cantilever slabs (near the support locations) to counter the possibility of reversal of stresses.

### 12.3.1 Dead Loads

The components of the dead load to be considered comprise:

- self-weight of stair slab (tread/tread-riser slab/waist slab);
- self-weight of step (in case of 'waist slab' type stairs);
- self-weight of tread finish (usually 0.5 – 1.0 kN/m$^2$)

The unit weight of reinforced concrete for the slab and step may be taken as 25kN/m$^3$ as specified in the Code (Cl. 19.2.1).

### 12.3.2 Live Loads

Live loads are generally assumed to act as uniformly distributed loads on the horizontal projection of the flight, i.e., on the 'going'. The Loading Code [IS 875 : 1987 (Part II)] recommends a uniformly distributed load of 5.0 kN/m$^2$ in general, on the going, as well as the landing. However, in buildings (such as residences) where the specified floor live loads do not exceed 2.0 kN/m$^2$, and the staircases are not liable to be overcrowded, the Loading Code recommends a lower live load of 3.0 kN/m$^2$ [Fig. 12.6(a)].

$$w_{LL} = \begin{cases} 5.0 \ \ \text{kN/m}^2 & \text{in general} \\ 3.0 \ \ \text{kN/m}^2 & \text{when overcrowding is unlikely} \end{cases}$$

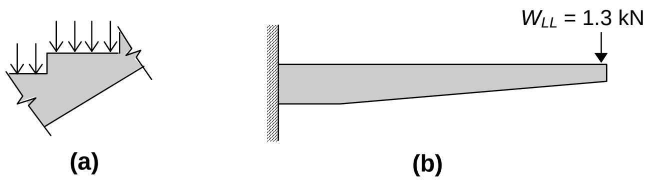



**Fig. 12.6** Code specifications for live loads on stair slabs

Further, in the case of *structurally independent cantilever steps*, the Loading Code requires the tread slab to be capable of safely resisting a concentrated live load of 1.3 kN applied to the free end of each cantilevered tread [Fig. 12.6(b)].

It may be noted that the specified live loads are *characteristic* loads; these loads as well as the characteristic dead loads should be multiplied by the appropriate *load factors* in order to provide the *factored loads* required for 'limit state design'.

### 12.3.3 Distribution of Gravity Loads in Special Cases

The Code (Cl. 33.2) specifies the following:

- When a staircase takes a right-angled turn [Fig. 12.2 b, d], fifty percent of the gravity loads on the areas (usually landings) common to the two flights (at right angles) may be assumed to act in each direction.
- When a longitudinally spanning flight (or landing) is embedded at least 110 mm into a side wall, then some marginal 'two-way' action can be expected. In such cases, the longitudinally acting component of the gravity

load can be assumed to act on a reduced width of flight; a reduction of 150 mm is permitted by the Code. Furthermore, *the effective width of the section can be increased by* 75 mm *for purposes of design*. In other words, if the width of the flight be $W$ (in mm) then the load may be assumed to act over a reduced width ($W$–150) mm and the effective width resisting flexure may be taken as ($W$+75) mm.

### 12.3.4 Load Effects in Isolated Tread Slabs

As mentioned earlier, isolated tread slabs [Fig. 12.1(d)] are invariably associated with stair slabs spanning transversely. The tread slabs are structurally independent and are designed as simple one-way slabs.

If the tread slab is simply supported, the thickness required is generally minimal (for stair widths less than 2 m). A slab thickness of 80 mm is usually provided, with minimum reinforcement (comprising at least 3nos 8 mm $\phi$ bars). The distribution bars may be of 6 mm $\phi$, with a nominal spacing of 250 mm. It suffices to use Fe 250 grade steel in such cases, as the steel requirement is minimal.

In the case of cantilevered tread slabs [Fig. 12.3a,b], the slab thickness may be taken as at least one-tenth of the effective cantilever span. For large spans, it is economical to taper the slab thickness to a minimum value of 80 mm at the free end, as shown in Fig. 12.3(a). The design of a cantilevered tread slab is demonstrated in Example 12.1.

### 12.3.5 Load Effects in Waist Slabs

In the 'waist slab' type staircase, the longitudinal axis of the flight is inclined to the horizontal and the steps form a series of triangles on top of the waist slab [Fig. 12.1(b), 12.7(a)]. The steps† are usually treated as non-structural elements and it is the waist slab which is designed to resist the load effects on the stairs. Some nominal reinforcement is provided in the step (if made in concrete) — mainly to protect the nosing from cracking [Fig. 12.7(a)].

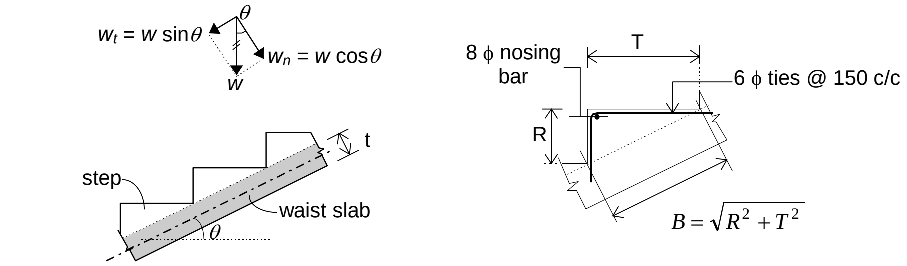
The vertical acting gravity loads $w$ may be resolved into two orthogonal components, as shown in Fig. 12.7(a). The component $w_n = w\cos\theta$ acts normal to the waist slab and the component $w_t = w\sin\theta$ acts tangential to the waist slab. The manner in which these load components are resisted by the waist slab depends on whether the slab spans transversely or longitudinally.

#### *Waist Slab Spanning Transversely*

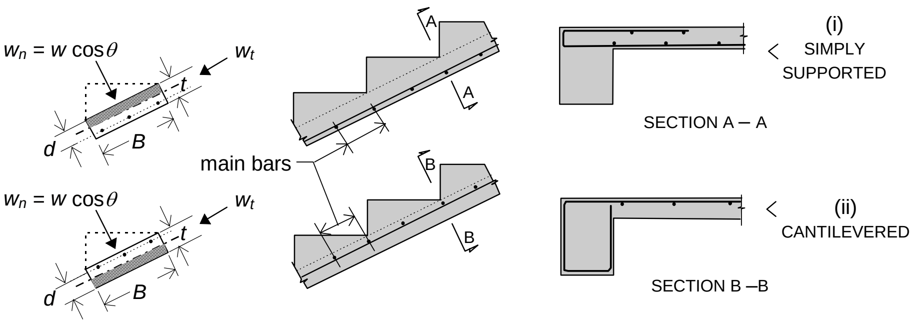
In this case, the normal load component $w_n$ causes the waist slab to bend in transverse planes normal to the sloping surface of the slab. The loading direction, cross sectional dimensions, neutral axis position, compression zone, main reinforcement and effective depth for a design strip of slab having a width $B$ corresponding to one tread (one step) are sketched in Fig. 12.7(b)(i),(ii) for the simply supported and

† The steps are sometimes constructed using brickwork.

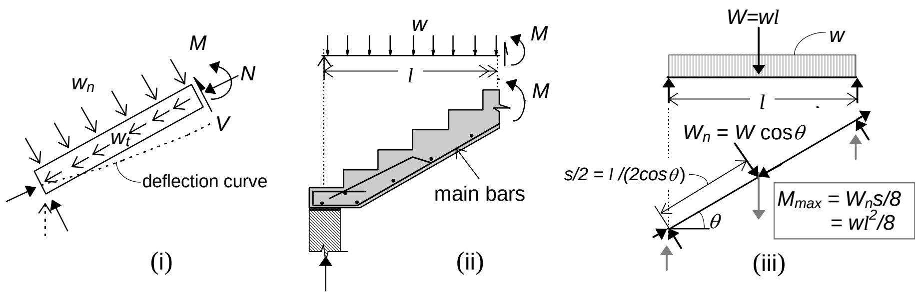
cantilever cases respectively, As can be seen, the main bars[†] are provided transversely, either at the bottom or top, depending on whether the slab is simply supported or cantilevered.




**(a)** waist slab–steps arrangement




**(b)** transversely spanning waist slabs




**(c)** longitudinally spanning waist slabs

**Fig. 12.7** Load effects and detailing in waist slabs

[†] It is desirable to provide a bar in line with every point where the step meets the waist slab, as the effective depth is minimum at this location, if the step is considered to behave integrally with the waist slab. This is shown in Fig. 12.7(b).

The tangential load component $w_t$ causes the waist slab to bend in its own plane. However, as the slab is extremely deep in this plane, the flexural stresses so induced are of a small order, and do not call for any particular design. Distributor bars are provided in the longitudinal direction. The proportioning of the waist slab thickness is as described earlier (in Section 12.3.4) for cantilevered treads.

#### *Waist Slab Spanning Longitudinally*

In this case, the slab thickness $t$ may be taken as approximately $l/20$ for simply supported end conditions and $l/25$ for continuous end conditions. The normal load component $w_n$ causes flexure in vertical planes containing the span direction (parallel to the longitudinal axis of the slab), and the tangential load component $w_t$ causes axial compression (of low order) in the slab [Fig. 12.7(c)(i)]. The main bars are placed longitudinally, and designed for the bending moments induced in the vertical planes along the slab span. These moments may be conveniently computed by considering the entire vertical load $w$ acting on the projected horizontal span (going), rather than considering the normal load component $w_n$ acting on the inclined span $s$ [Fig. 12.7(c)(iii)]. The distributor bars are provided in the transverse directions.

Care must be taken to ensure proper detailing of the longitudinal bars at the junction of the flight and landing slab. The bottom bars in the waist slab should not be continued to the bottom of the upper landing slab at the reentrant corner, but extended to the top of the landing slab. This is to prevent the bars (in tension) from breaking out at reentrant corners, as shown in the detail in Fig. 12.4(a).

### 12.3.6 Load Effects in Tread-Riser Stairs

In the tread-riser type of arrangement [Fig. 12.1(c)], the 'slab'[‡] is repeatedly folded, and behaves essentially like a 'folded plate'. A rigorous analysis of such a structure is difficult and laborious. However, the analysis can be rendered simple by means of certain idealisations, and designs based on such simplified analysis are found to work well in practice. These simple design methods are described here.

#### *Tread-Riser Units Spanning Transversely*

In this case, the assumption made is that each tread-riser unit, comprising the 'riser slab' and one-half of each 'tread slab' on either side [Fig. 12.8(b)], can be assumed to behave independently as a beam with a Z-section. Such an assumption is made in one-way slab design (where design is done for a standard strip of unit width), and indeed, slabs spanning transversely are basically one-way slabs, designed for uniformly distributed gravity loads.

This 'tread-riser' unit behaves essentially as a flanged beam which is transversely loaded. The overall depth of the beam is given by $(R + t)$, where $R$ is the riser and $t$ the thickness of the 'slab' [Fig. 12.8(a)].

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[‡] Sometimes, this type of stair is referred to as a 'slabless' stair, referring to the absence of a continuous waist slab.

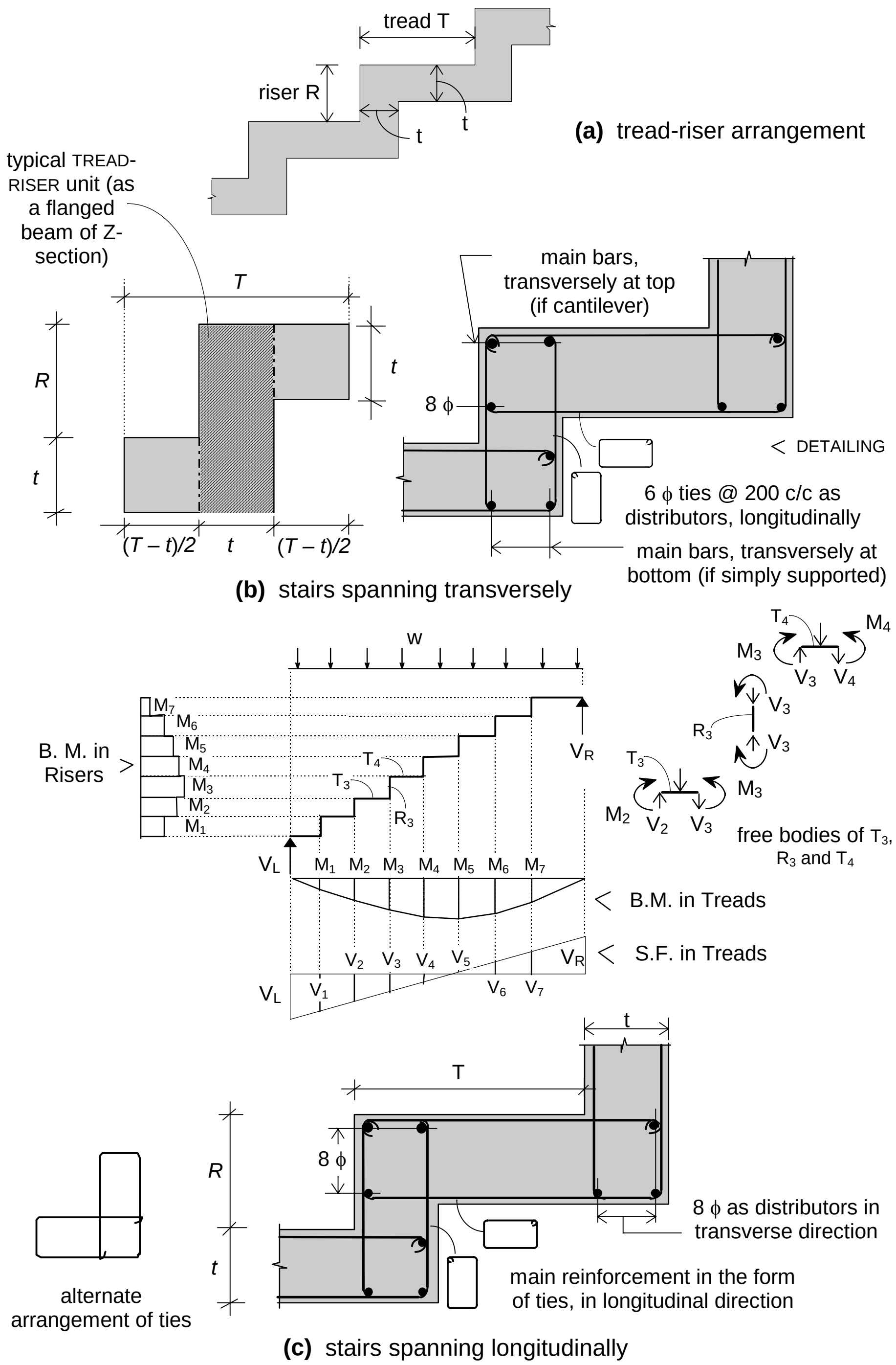



**Fig. 12.8** Load effects and detailing in tread-riser units

In most cases of tread-riser units spanning transversely, the bending moments are low, and it generally suffices to provide a nominal slab thickness $t$ = 100 mm. For convenience in calculations, the 'flange' portions of the beam may be ignored and the rectangular portion of the riser slab alone may be considered. It will be found that the reinforcement required is nominal. The detailing of the tread-riser slab may be done as indicated in Fig. 12.8(b). The nominal distributor bars (generally 6 $\phi$ @ 200 c/c) may be provided in the form of ties (stirrups) — in both riser slab and tread slab — as shown. The main bars are concentrated in the 'riser slab' portion, and may be located at the top or bottom, depending on whether the slab is cantilevered or simply supported. At every bend in the ties where there are no main bars, a nominal 8 mm $\phi$ bar should be provided. The clear cover to the main bars should be as required for normal slabs.

### *Tread-Riser Units Spanning Longitudinally*

In this case, the bending moments to be considered occur in the longitudinal direction, in the 'riser slab' as well as the 'tread slab'. The overall behaviour of the inter-connected tread-riser units, including calculation of bending moments, is similar to longitudinally spanning *waist slabs*. The variation of bending moment along the span is as for a horizontal slab having the projected horizontal span with the entire vertical load acting on it [Fig. 12.8(c)]. As depicted in the freebody diagram in Fig. 12.8(c), each 'tread slab' is subjected to a bending moment (which varies slightly along the tread) combined with a shear force, whereas each 'riser slab' is subjected to a bending moment (which is constant for a given riser) combined with an axial force (which may be compressive or tensile[†] ). It is assumed that the connection between the 'riser slab' and the adjoining 'tread slab' is a 'rigid joint'. For all practical purposes, it suffices to design both tread slabs and riser slabs for flexure alone, as the shear stresses in tread slabs and axial stresses in riser slabs are relatively low. The slab thickness $t$ may be kept the same for both tread slab and riser slab, and may be taken as about span/25 for simply supported stairs and span/30 for continuous stairs.

It is generally accepted that the tread-riser arrangement has considerable aesthetic appeal, and in this sense, it is superior to the conventional 'waist slab' type of staircase. However, this aesthetic appeal of the tread-riser staircase is lost if the slab thickness $t$ is excessive — especially if it exceeds the riser $R$. For this reason, it becomes necessary to work out a suitable support scheme for the tread-riser staircase, which results in a relatively low effective span — generally not exceeding about 3.5 m.

The reinforcement detailing, shown in Fig. 12.8(c), is similar to that shown in Fig. 12.8(b) — except that the main bars (ideally in the form of closed loops, as shown) lie in the longitudinal direction, while the distributors (generally 8 $\phi$) are located transversely. The closed loop arrangement of the main bars (in the tread slab as well as the riser slab) serves to provide the required development length. Furthermore, this arrangement provides reinforcement at top, required to resist

[†] The axial force is generally tensile in the risers located in the upper half of the flight; this tension is resisted by the closed ties provided as main reinforcement [Fig. 12.8(c)].

negative moments near the supports which are likely to be partially restrained. Also, the closed loop arrangement enhances both the shear- and axial force-resisting capacities, as well as ductility of the slabs. The diameter and/or spacing of the main bars in the tread-riser units may be suitably varied along the span (to conform to the bending moment diagram), in order to achieve an economical design.

## 12.4 DESIGN EXAMPLES OF STAIR SLABS SPANNING TRANSVERSELY

### EXAMPLE 12.1

A straight staircase is made of structurally independent tread slabs, cantilevered from a reinforced concrete wall. Given that the riser is 150 mm, tread is 300 mm, and width of flight is 1.5 m, design a typical tread slab. Apply the live loads specified in the IS Loading Code for stairs liable to be overcrowded. Use M 20 concrete and Fe 250 steel. Assume *mild* exposure conditions.

**SOLUTION**

- Given: $R$ = 150 mm, $T$ = 300 mm, $W$ = 1.5 m
  $\Rightarrow$ effective span $l$ = 1.5 m
  It is desirable to make the actual width of the tread slab, $B$, about 10 mm more than the effective tread, $T$, so that there is a marginal overlap between adjacent tread slabs [see Fig. 12.1(d)]. $\Rightarrow$ $B$ = 310 mm
- Assume a slab thickness at the fixed support, $t \approx \frac{l}{10}$ = 150 mm. The slab thickness may be kept constant for a distance of, say, 300 mm, from the support, and tapered to a minimum thickness of 80 mm, as shown in Fig. 12.9.
- *Dead Loads*:
  (i) self weight of tread slab $\approx 25\text{ kN/m}^3 \times (0.15^{\dagger} \times 0.31)\text{ m}^2$ = 1.162 kN/m
  (ii) finishes $\approx 0.6\text{ kN/m}^2 \times 0.31\text{ m}$ = 0.186 kN/m
  1.348 kN/m
  $\Rightarrow$ $w_{u,DL}$ = 1.348 × 1.5 = 2.022 kN/m
- *Live Loads*:
  Alternative I: $w_{u,LL} = (5.0\text{ kN/m}^2 \times 0.3\text{ m}) \times 1.5 = 2.250$ kN/m
  Alternative II: $W_{u,LL}$ = 1.3kN × 1.5 = 1.95 kN (at free end)
- *Design Moment*:
  At fixed end, $M_{u,DL} = 2.022 \times 1.5^2/2$ = 2.27 kNm

[†] The actual slab thickness varies along the slab; however, it is convenient and conservative to assume a uniform thickness equal to 150 mm for the purpose of calculating dead load and bending moment.

$$M_{u,LL} = \begin{cases} 2.250 \times 1.5^2/2 & = 2.53 \text{ kNm} \\ 1.95 \times 1.5 & = 2.93 \text{ kNm (more critical)} \end{cases}$$

$\Rightarrow M_u = 2.27 + 2.93 = 5.20$ kNm

- *Design of Main Bars*:
  Assuming a clear cover of 20 mm (mild exposure) and a bar diameter of 10 mm, effective depth $d$ = 150 – 20 – 10/2 = 125 mm.

  $$R \equiv \frac{M_u}{bd^2} = \frac{5.20 \times 10^6}{310 \times 125^2} = 1.0735 \text{ MPa}$$

  $$\Rightarrow \frac{p_t}{100} \equiv \frac{A_{st}}{bd} = \frac{f_{ck}}{2 f_y}\left[1 - \sqrt{1 - 4.598 R/f_{ck}}\right]$$

  $= 0.528 \times 10^{-2}$ (for $f_{ck}$ = 20 MPa and $f_y$ = 250 MPa)
  [Alternatively, this is obtainable from design aids Table A.3(a)]
  $\Rightarrow (A_{st})_{reqd} = (0.528 \times 10^{-2}) \times 310 \times 125 = 205 \text{ mm}^2$
  **Provide 3–10 ϕ bars** [$A_{st} = 78.5 \times 3 = 235.5 \text{ mm}^2 > 205$].

- Anchorage length required: $L_d = \dfrac{(0.87 \times 250) \times 10}{4 \times 1.2}$ = 453 mm [Eq. 8.5]

- *Distributors*
  $(A_{st})_{min} = 0.0015bt$ (for Fe 250 bars, Cl. 26.5.2.1)
  $= 0.0015 \times 10^3 \times 150 = 225 \text{ mm}^2$/m (assuming uniform slab thickness)

  Required Spacing of 8 ϕ bars = $\dfrac{50.3 \times 10^3}{225}$ = 223 mm

  **Provide 8 ϕ distributors @ 220c/c**

- *Check for shear*‡
  Design (factored) shear force at support:
  $V_u = (2.022 + 2.250) \times 1.5 = 5.72$ kN

  $$\Rightarrow \tau_v = \frac{V_u}{bd} = \frac{5720}{310 \times 127} = 0.145 \text{ MPa}$$

  $\tau_c = (0.47 \times 1.3)\, MPa$ [vide Cl. 40.2.1.1 of the Code].
  $\Rightarrow \tau_v << \tau_c$ — Hence, safe.

‡ This is generally not required, as shear stresses are invariably of low magnitude. A check for deflection control is also not called for in the case of well-proportioned slabs — especially since the major load component (live load) is a transient load.

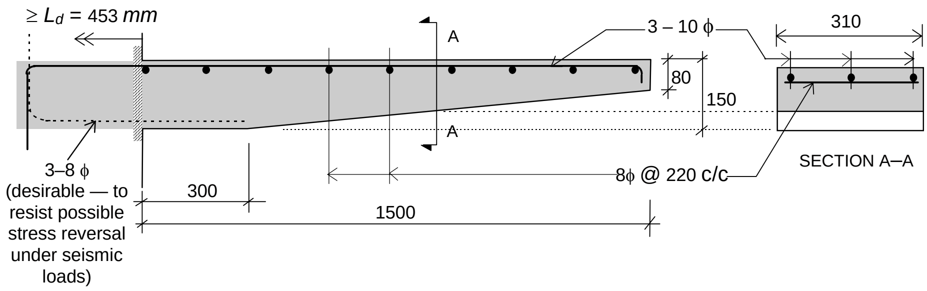



**Fig. 12.9** Example 12.1

- *Detailing*
  The detailing of the tread slab is shown in Fig. 12.9.

**EXAMPLE 12.2**

Repeat Example 12.1, considering a tread-riser arrangement spanning transversely.

**SOLUTION**

- Given (as in previous Example):
  $R$ = 150 mm, $T$ = 300 mm, $l$ = 1.5 m.
- Assume a nominal slab thickness $t$ = 100 mm.
  Effective depth (assuming 20 mm cover, 10 ϕ bars and 8 ϕ ties)
  $d$ = (150 + 100) – 20 – 8 – 10/2 = 217 mm.
- *Load on a typical 'tread-riser' unit* [Fig. 12.10(a)]:
  *Dead Loads*:
  (1) self-weight @ 25kN/m$^3$ × (0.3 + 0.15) m × 0.1 m = 1.125 kN/m
  (2) finishes @ 0.6 kN/m$^2$ × 0.3 m = 0.180 ”
  = 1.305 kN/m
  ⇒ Factored dead load $w_{u,DL}$ = 1.305 × 1.5 = 1.958 kN/m
  *Live Loads:*
  Alternative I: $w_{u,LL}$ = 1.5 × (5.0 kN/m$^2$ × 0.3 m) = 2.250 kN/m
  Alternative II: $W_{u,LL}$ = 1.5 × 1.3kN = 1.95 kN (at free end)
- *Design Moment*
  Consideration of concentrated live load at the free end [see Ex. 12.1] will result in a slightly larger design moment of
  $$M_u = 1.958 \times 1.5^2 / 2 + 1.95 \times 1.5 = 5.13 \text{ kNm}$$
  This moment is resisted by the flanged section shown in Fig. 12.10(a). Ignoring the contribution of the flanges (for convenience), and considering a rectangular section with $b$ = 100 mm, $d$ = 217 mm,
  $$R \equiv \frac{M_u}{bd^2} = \frac{5.13 \times 10^6}{100 \times 217^2} = 1.089 \text{ MPa}$$
- *Design of main bars*

$$\Rightarrow \frac{p_t}{100} \equiv \frac{A_{st}}{bd} = \frac{20}{2 \times 250}\left[1-\sqrt{1-4.598 \times 1.089/20}\right]$$
$$= 0.537 \times 10^{-2}$$
$$(A_{st})_{reqd} = (0.537 \times 10^{-2}) \times (100 \times 217) = 117 \text{ mm}^2$$

**Provide 2–10 ϕ bars on top** ($A_{st}$ = 78.5 × 2 = 157 mm$^2$ > 117)
Anchorage length = 453 mm (as in previous Example)

- *Distributors*
  $(A_{st})_{min}$ = 0.0015$bt$ (for Fe 250 bars)
  = 0.0015 × 1000 × 100 = 225 mm$^2$/m
  **Provide 8 ϕ @ 220c/c distributors** (as in Example 12.1). These distributors are provided in the form of closed loops, with a bar (8ϕ minimum) provided transversely at each bend, as shown in Fig. 12.10(b).

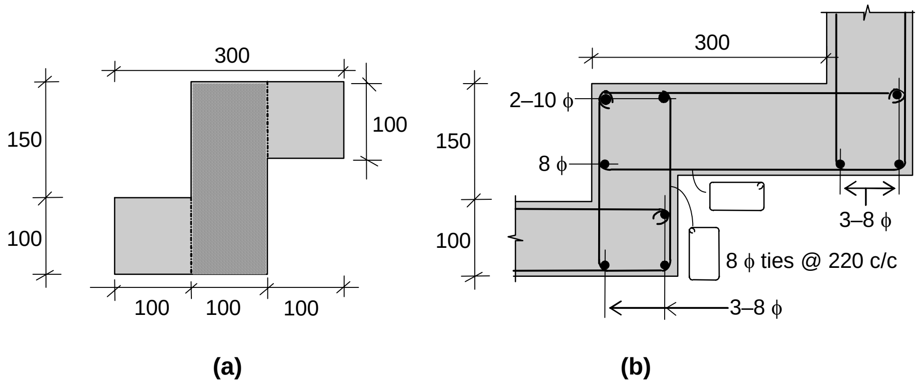



**Fig. 12.10** Example 12.2

### EXAMPLE 12.3

Design a 'waist slab' type staircase comprising a straight flight of steps, supported between two stringer beams along the two sides. Assume an effective span of 1.5 m, a riser of 150 mm and a tread of 270 mm. Assume a live load of 3.0kN/m$^2$. Use M 20 concrete and Fe 250 steel. Assume *mild* exposure conditions.

### SOLUTION

- Given $R$ = 150 mm, $T$ = 270 mm, $l$ = 1.5 m
  $\Rightarrow \sqrt{R^2+T^2}$ = 309 mm
- Assume a nominal waist slab thickness $t$ = 80 mm [Fig. 12.11(a)]. Further, assuming the flexural resistance to be provided entirely by the waist slab, with 20 mm clear cover (mild exposure) and 10 ϕ bars,
  effective depth $d$ = 80 – 20 – 10/2 = 55 mm.
- *Loads acting vertically over each tread width*:

(1) self-weight of slab @ 25 kN/m$^3$ × (0.080 × 0.309) m$^2$ = 0.618 kN/m

(2) self-weight of step @ 25 kN/m$^3$ × $\left(\frac{1}{2}\times 0.15\times 0.27\right)$ m = 0.506 ”

(3) finishes @ 0.6 kN/m$^2$ × 0.27 m = 0.162 ”

(4) live loads @ 3.0 kN/m$^2$ × 0.27 m = 0.810 ”

$w$ = 2.096 kN/m

Factored load causing flexure in the transverse[‡] direction [Fig. 12.11(a)]:

$$(w\times 1.5)\cos\theta = (2.096\times 1.5)\times\left(\frac{270}{309}\right) = 2.747 \text{ kN/m}$$

⇒ Distributed factored load <u>per *m* width</u> along inclined slab

$$= \frac{2.747}{0.309} = 8.89 \text{kN/m}^2$$

- *Design of main bars* (spanning transversely)

Maximum moment at midspan:

$$M_u = \frac{8.89\times 1.5^2}{8} = 2.50 \text{ kNm/m}$$

$$R \equiv \frac{M_u}{bd^2} = \frac{2.50\times 10^6}{1000\times 55^2} = 0.826 \text{ MPa}$$

$$\Rightarrow \frac{p_t}{100} \equiv \frac{A_{st}}{bd} = \frac{f_{ck}}{2f_y}\left[1-\sqrt{1-4.598R/f_{ck}}\right] = 0.4\times 10^{-2}$$

$$\Rightarrow (A_{st})_{reqd} = (0.4\times 10^{-2})\times 10^3\times 55 = 220 \text{ mm}^2/\text{m}$$

⇒ Required spacing of 10 ϕ bars = $\frac{78.5\times 10^3}{220}$ = 357 mm

Required spacing of 8 ϕ bars = $\frac{50.0\times 10^3}{220}$ = 227 mm

(Minimum spacing = 3$d$ = 3 × 55 = 165 mm)

**Provide 8 ϕ bars @ 309/2 = 155 mm c/c,** as shown in Fig. 12.11(b).

[‡] The load component $w_t = w\sin\theta$ acting tangentially in the longitudinal direction (i.e., in the plane of the waist slab) results in very low flexural stresses owing to the large depth of the waist slab in its own plane; hence, this is ignored.

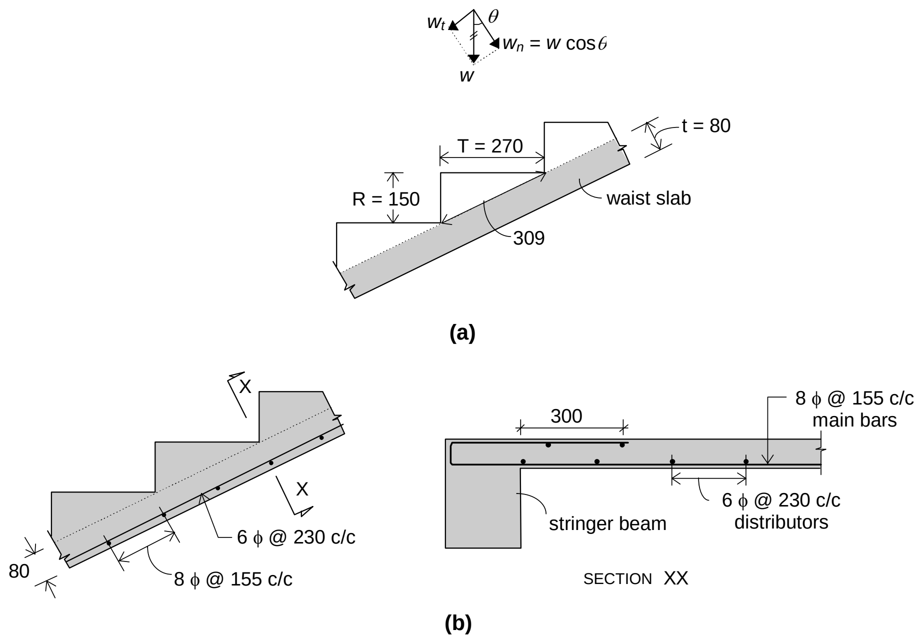



**Fig. 12.11** Example 12.3

- *Distributors* (spanning longitudinally)
  $(A_{st})_{min}$ = 0.0015 *bt* (for Fe 250 bars)
  = 0.0015 × 1000 × 80 = 120 mm$^2$/m

  Spacing of 6 ϕ bars = $\dfrac{28.3\times10^3}{120}$ = 235 mm

  **Provide 6 ϕ distributors @ 230c/c**, as shown in Fig. 12.11(b)

## 12.5 DESIGN EXAMPLES OF STAIR SLABS SPANNING LONGITUDINALLY

### EXAMPLE 12.4

Design the staircase slab, shown in Fig. 12.12(a). The stairs are simply supported on beams provided at the first riser and at the edge of the upper landing. Assume a finish load of 0.8 kN/m2 and a live load of 5.0 kN/m2. Use M 20 concrete and Fe 415 steel. Assume *mild* exposure conditions.

### SOLUTION

- Given: $R$ = 150 mm, $T$ = 300 mm $\Rightarrow \sqrt{R^2+T^2}$ = 335.4 mm
  Effective span = c/c distance between supports = 4.5 m [Fig. 12.12(a)]
- Assume a waist slab thickness $\approx l/20$ = 4500/20 = 225 mm, say 230 mm

  Assuming 20 mm clear cover and 12 ϕ main bars, effective depth $d$ = 230 – 20 – 12/2 = 204 mm.

- *Loads on going* [Ref. Fig. 12.12(b)] on projected plan area:
  (1) self-weight of waist slab @ 25 kN/m$^3$ × (0.230 × 335.4/300)$m$ = 6.43 kN/m$^2$
  (2) self-weight of steps @ 25 kN/m$^3$ × (0.5 × 0.15) m = 1.88 ”
  (3) finishes (given) = 0.80 ”
  (4) live load (given) = 5.00 ”
  14.11 kN/m$^2$

$\Rightarrow$ Factored load = 14.11 × 1.5 = 21.17 kN/m$^2$

- *Loads on landing*
  (1) self-weight of slab @ 25 × 0.23 = 5.75 kN/m$^2$
  (2) finishes @ 0.80 ”
  (3) live loads @ 5.00 ”
  11.55 kN/m$^2$

  $\Rightarrow$ Factored load = 11.55 × 1.5 = 17.33 kN/m$^2$

- *Design Moment* [refer Fig. 12.12(c)], considering 1$m$ wide strip of waist slab:

  Reaction $R_1 = \left(21.17 \times 3.45 \times \dfrac{4.5 - 1.725}{4.5}\right) + \left(17.33 \times 1.05 \times \dfrac{0.525}{4.5}\right)$ = 47.16 kN/m

  Max. factored moment occurs at the section of zero shear, located at
  $x$ = 47.16/21.17 = 2.228 m from the left support.
  $\Rightarrow M_u$ = (47.16 × 2.228) – (21.17 × 2.228$^2$/2) = 52.53 kNm/m

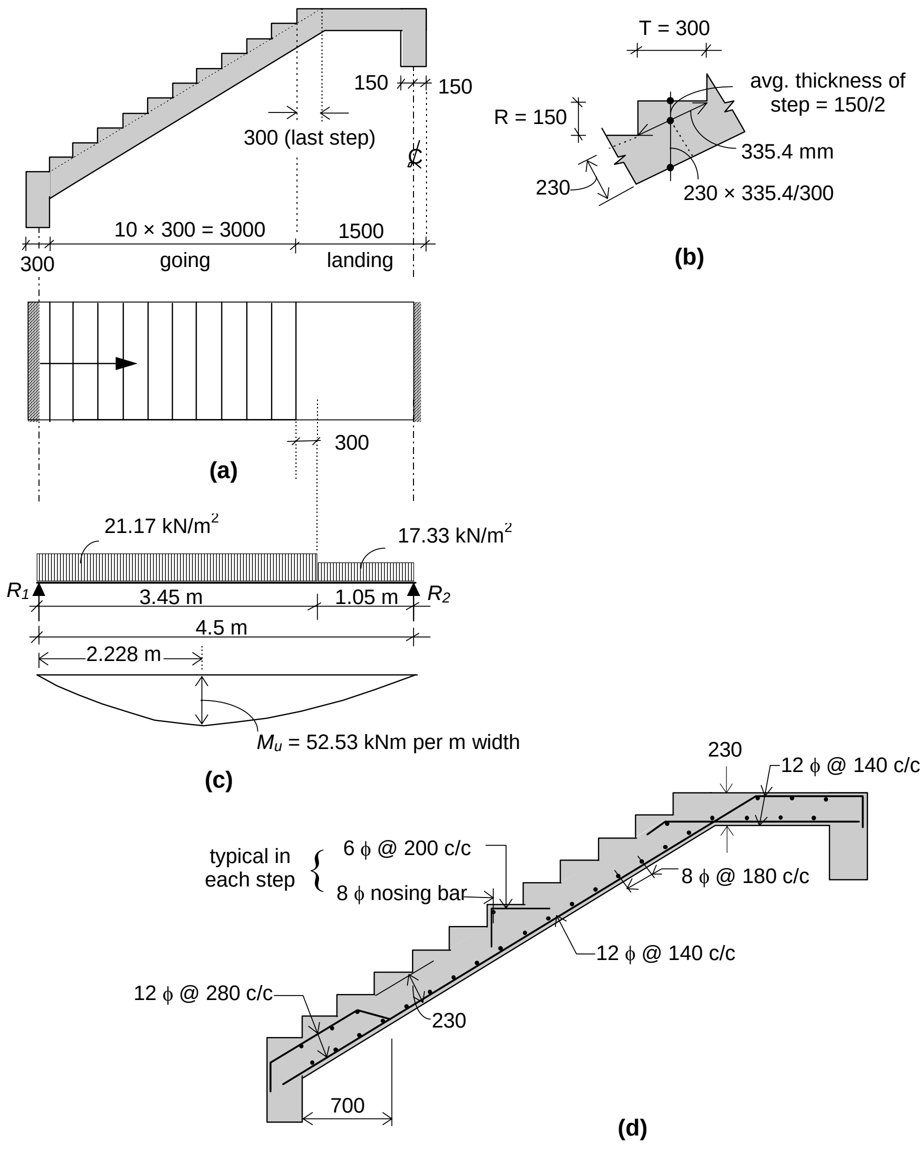



**Fig. 12.12** Example 12.4

- *Main reinforcement*

$$R \equiv \frac{M_u}{bd^2} = \frac{52.53 \times 10^6}{10^3 \times 204^2} = 1.262 \text{ MPa}$$

Assuming $f_{ck}$ = 20 MPa, $f_y$ = 415 MPa,

$$\Rightarrow \frac{p_t}{100} \equiv \frac{A_{st}}{bd} = \frac{f_{ck}}{2f_y}\left[1 - \sqrt{1 - 4.598R/f_{ck}}\right] = 0.379 \times 10^{-2}$$

[This may also be obtained from design aids: Table A.3(a)].

$\Rightarrow \ (A_{st})_{reqd} = (0.379 \times 10^{-2}) \times 10^3 \times 204 = 774 \,\text{mm}^2/\text{m}$

Required spacing of 12 ϕ bars = $\dfrac{113 \times 10^3}{774}$ = 146 mm

**Provide 12 ϕ @ 140c/c**

- *Distributors*

  $(A_{st})_{reqd}$ = 0.0012 $bt$ (for Fe 415 bars)

  = 0.0012 × $10^3$ × 230 = 276 mm$^2$/m

  Assuming 8 ϕ bars, spacing reqd = $\dfrac{50.3 \times 1000}{276}$ = 182 mm

  **Provide 8 ϕ @ 180 c/c distributors**

  The detailing of bars is shown in Fig. 12.12(d).

- *Check for shear*‡ (check at $d$ = 204 mm from support)

  $V_u$ = 47.16 – (21.17 × 0.204) = 42.8 kN/$m$

  $\tau_v = \dfrac{42.8 \times 10^3}{10^3 \times 204}$ = 0.21 MPa << $\tau_c$ = 0.42 × 1.19 = 0.499 MPa

  [refer Cl. 40.2.1.1 of the Code]. Hence, safe.

### EXAMPLE 12.5

Design a (‘waist slab’ type) dog-legged staircase for an office building, given the following data:

- height between floor = 3.2 m;
- riser = 160 mm, tread = 270 mm;
- width of flight = landing width = 1.25 m
- live load = 5.0 kN/m$^2$
- finishes load = 0.6 kN/m$^2$

Assume the stairs to be supported on 230 mm thick masonry walls at the outer edges of the landing, parallel to the risers [Fig. 12.13(a)]. Use M 20 concrete and Fe 415 steel. Assume *mild* exposure conditions.

### SOLUTION

- Given: $R$ = 160 mm, $T$ = 270 mm $\Rightarrow \sqrt{R^2 + T^2}$ = 314 mm

  Effective span = c/c distance between supports = 5.16 m [Fig. 12.13(a)].

- Assume a waist slab thickness $\approx l/20$ = 5160/20 = 258 $\rightarrow$ 260 mm.

  Assuming 20 mm clear cover (*mild* exposure) and 12 ϕ main bars, effective depth $d$ = 260 – 20 – 12/2 = 234 mm.

  The slab thickness in the landing regions may be taken as 200 mm, as the bending moments are relatively low here.

‡ As observed earlier in Example 12.1, the slab (if well-proportioned) is invariably safe in shear, and does not require shear reinforcement. Also, as explained earlier, a check for deflection control is not called for here.

- *Loads on going* [Ref. 12.13(b)] on projected plan area:

| | | | |
|---|---|---|---|
| (1) | self-weight of waist slab @ 25 × 0.26 × 314/270 | | = 7.56 kN/m$^2$ |
| (2) | self-weight of steps @ 25 × $\left(\frac{1}{2}\times 0.16\right)$ | | = 2.00 ” |
| (3) | finishes | (given) | = 0.60 ” |
| (4) | live load | (given) | = 5.00 ” |
| | | | 15.16 kN/m$^2$ |

  ⇒ Factored load = 15.16 × 1.5 = 22.74 kN/m$^2$
- *Loads on landing*

| | | |
|---|---|---|
| (1) | self-weight of slab | @ 25 × 0.20 = 5.00 kN/m$^2$ |
| (2) | finishes | @ 0.6 ” |
| (3) | live loads | @ 5.0 ” |
| | | 10.60 kN/$m^2$ |

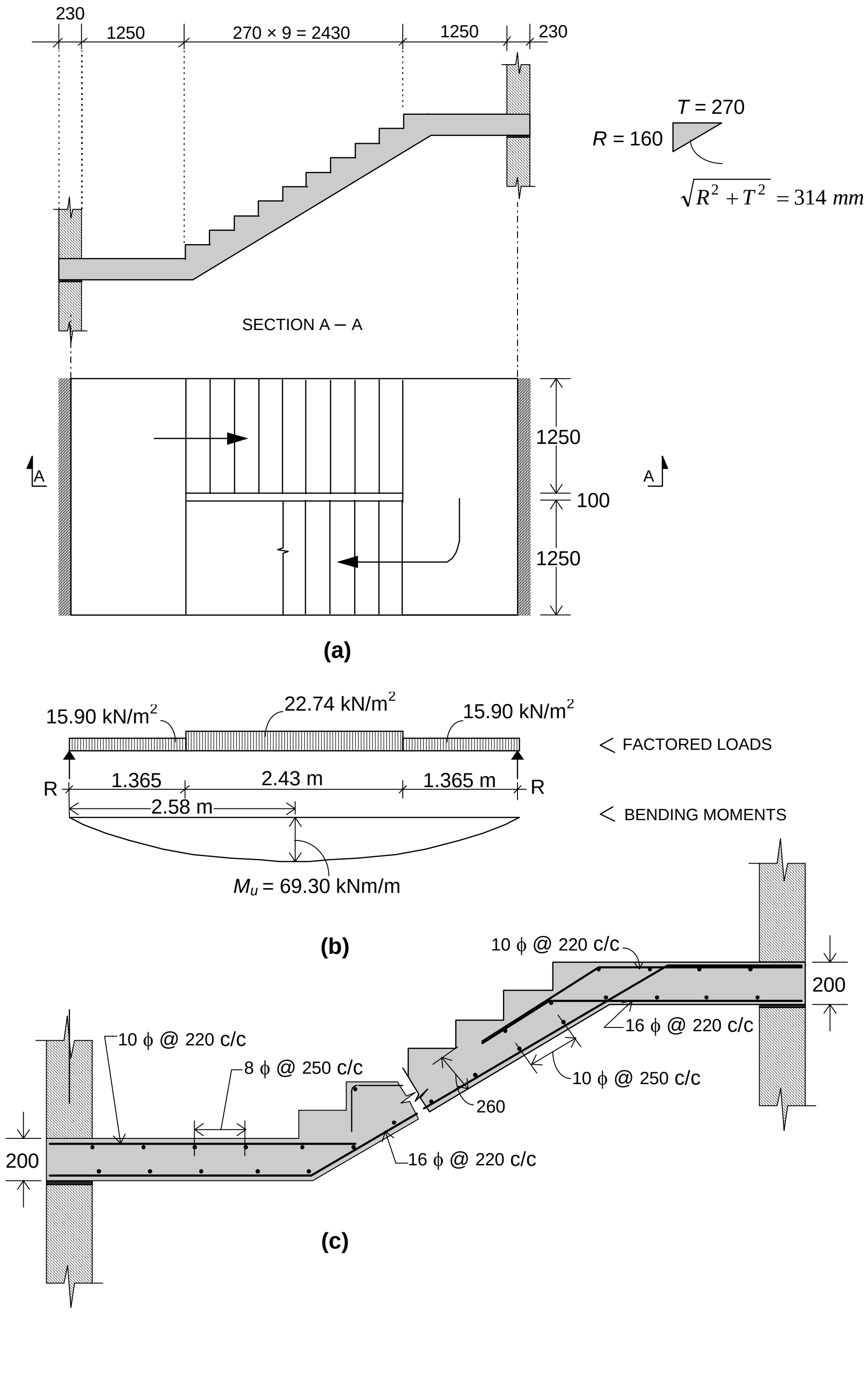
  ⇒ Factored load = 10.60 × 1.5 = 15.90 kN/m$^2$

- *Design Moment* [refer Fig. 12.13(b)]
  Reaction $R = (15.90\times 1.365) + (22.74\times 2.43)/2$ = 49.33 kN/m
  Maximum moment at midspan:
  $M_u$ = (49.33 × 2.58) – (15.90 × 1.365) × (2.58 – 1.365/2) – (22.74) × (2.58 – 1.365)$^2$/2
  = 69.30 kNm/m

- *Main reinforcement*
  $$R \equiv \frac{M_u}{bd^2} = \frac{69.30\times 10^6}{10^3\times 234^2} = 1.265 \text{ MPa}$$
  Assuming $f_{ck}$ = 20 MPa, $f_y$ = 415 MPa,
  $$\frac{p_t}{100} \equiv \frac{A_{st}}{bd} = \frac{20}{2\times 415}\left[1-\sqrt{1-4.598\times 1.265/20}\right] = 0.381\times 10^{-2}$$
  [This may also be obtained from design aids Table 3(a)].
  $\Rightarrow (A_{st})_{reqd} = (0.381\times 10^{-2})\times 10^3\times 234 = 892 \text{ mm}^2/\text{m}$

  Required spacing of 12 ϕ bars = $\frac{113\times 10^3}{892}$ = 127 mm

  Required spacing of 16 ϕ bars = $\frac{201\times 10^3}{892}$ = 225 mm (to be reduced slightly to account for reduced effective depth)
  **Provide 16 ϕ @ 220c/c**
- *Distributors*
  $(A_{st})_{reqd}$ = 0.0012 *bt* (for Fe 415 bars)
  = 0.0012 × 10$^3$ × 260 = 312 mm$^2$/m
  spacing 10 ϕ bars = $78.5\times 10^3$ / 312 = 251 mm
  **Provide 10 ϕ @ 250c/c as distributors**



230
1250
270 × 9 = 2430
1250
230
T = 270
R = 160
$\sqrt{R^2 + T^2} = 314\ mm$
SECTION A – A
1250
100
1250
A
A
(a)
15.90 kN/m²
22.74 kN/m²
15.90 kN/m²
FACTORED LOADS
R
1.365
2.43 m
1.365 m
R
2.58 m
BENDING MOMENTS
$M_u$ = 69.30 kNm/m
(b)
10 φ @ 220 c/c
200
16 φ @ 220 c/c
10 φ @ 220 c/c
8 φ @ 250 c/c
10 φ @ 250 c/c
260
200
16 φ @ 220 c/c
(c)

**Fig. 12.13** Example 12.5

The detailing of bars for the first flight is shown in Fig. 12.13(c). Some nominal reinforcement (10 $\phi$ @ 220c/c) is provided in the landing slabs near the support at top to resist possible 'negative' moments on account of partial fixity; 8 $\phi$ @ 250 c/c distributors are also provided.

### EXAMPLE 12.6

Repeat the problem of the dog-legged staircase in Example 12.5, considering the landings to be supported only on two edges perpendicular to the risers [Fig. 12.14(a)].

### SOLUTION

- The prevailing IS Code recommendations are adopted here for determination of the design moments†.
- Given: $R$ = 160 mm, $T$ = 270 mm $\Rightarrow \sqrt{R^2 + T^2}$ = 314 mm
  As the flight is supported on the landings (whose length is less than 2.0 m), the effective span (as per Code) is given by the c/c distance between landings.
  $l = 2.43 + 1.25 = 3.68$ m
- Assume a waist slab thickness $\approx$ 3680/20 = 184 $\rightarrow$ 185 mm.
  Let thickness of the landing slabs also be 185 mm
  Assuming 20 mm cover and 12 $\phi$ bars, $d$ = 185 – 20 – 12/2 = 159 mm
- *Loads on going* [Ref. 12.14(b)] on projected plan area:

| | | | |
|---|---|---|---|
| (1) | self-weight of waist slab @ 25 × 0.185 × 314/270 | | = 5.38 kN/m$^2$ |
| (2) | self-weight of steps @ 25 × $\left(\frac{1}{2} \times 0.16\right)$ | | = 2.00 ” |
| (3) | finishes | (given) | = 0.60 ” |
| (4) | live load | (given) | = 5.00 ” |
| | | | 12.98 kN/m$^2$ |

  $\Rightarrow$ Factored load = 12.98 × 1.5 = 19.47 kN/m$^2$
- *Loads on landing*

| | | |
|---|---|---|
| (1) | self-weight of slab @ 25 × 0.185 | = 4.63 kN/m$^2$ |
| (2) | finishes | @ 0.60 ” |
| (3) | live loads | @ 5.00 ” |
| | | 10.23 kN/m$^2$ |

  $\Rightarrow$ Factored load = 10.23 × 1.5 = 15.35 kN/m$^2$
  50% of this load may be assumed to be acting longitudinally,
  i.e., 15.35 ×1/2 = 7.68 kN/m$^2$ [Fig. 12.14(b)].
  **Design of waist slab** [refer Fig. 12.14(b)]

---

† As explained earlier, this will result in a conservative estimate of sagging moments (and consequently, thicker waist slab) and does not address the development of hogging moments at the going-landing junctions. More rational and economical design procedures are described in Ref. 12.6 and 12.8.

Reaction on landing $R = (7.68 \times 0.625) + (19.47 \times 2.43/2)$ = 28.46 kN/m

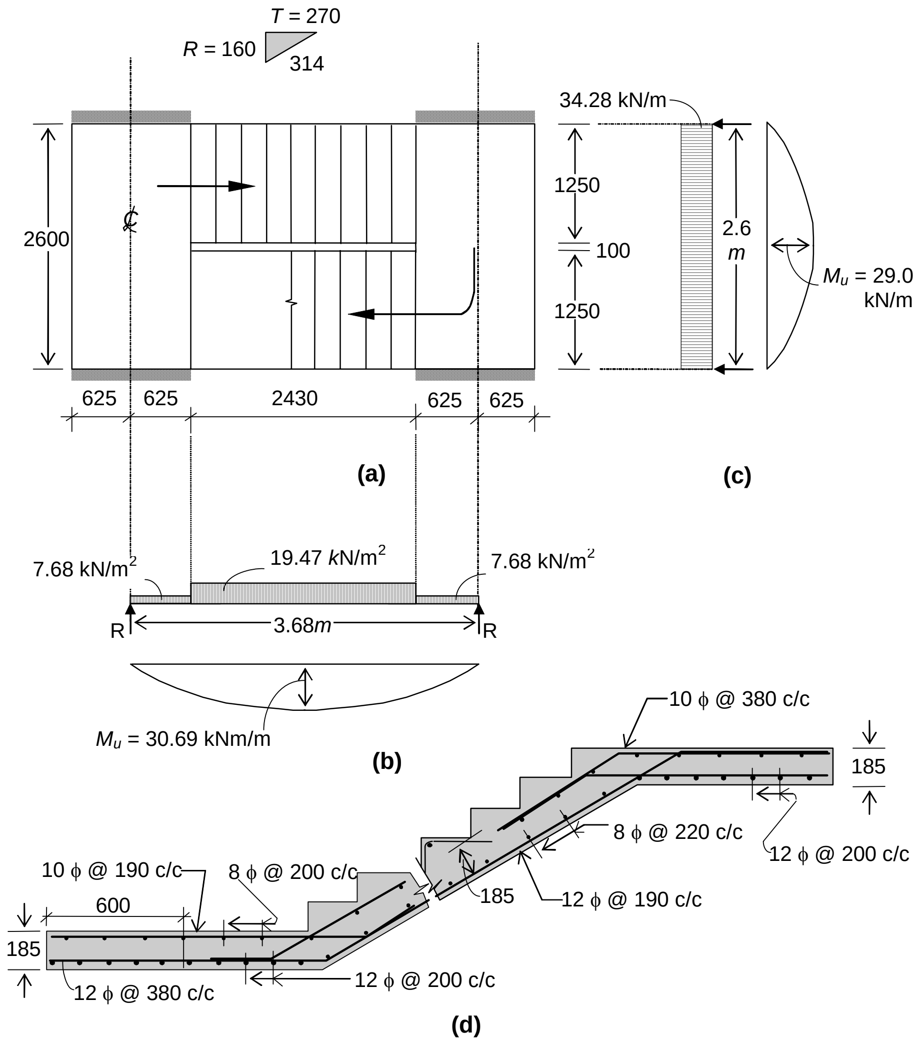



**Fig. 12.14** Example 12.6

- *Design Moment at midspan*:
  $M_u$ = (28.46 × 3.68/2) – (7.68 × 0.625) × (1.84 – 0.625/2) – 19.47 × $1.215^2$/2
  = 30.69 kN/*m*
- *Main reinforcement*

  $$\Rightarrow R \equiv \frac{M_u}{bd^2} = \frac{30.69 \times 10^6}{10^3 \times 159^2} = 1.214 \text{ MPa}$$

  Assuming M 20 concrete and Fe 415 steel,

$$\frac{p_t}{100} \equiv \frac{A_{st}}{bd} = \frac{20}{2 \times 415}\left[1-\sqrt{1-4.598 \times 1.214/20}\right] = 0.364 \times 10^{-2}$$

$$\Rightarrow (A_{st})_{reqd} = (0.364 \times 10^{-2}) \times 10^3 \times 159 = 579 \text{ mm}^2/\text{m}$$

Required spacing of 12 $\phi$ bars = $\frac{113 \times 10^3}{579}$ = 195 mm.

**Provide 12 $\phi$ @ 190c/c** main bars in the waist slab; these bars are continued into the landing slab, as shown in Fig. 12.14(c). Nominal top steel 10 $\phi$ @ 190c/c is also provided at top at the junction of the waist slab with the landing slab to resist possible 'negative' moments.

- *Distributors*:

$(A_{st})_{min}$ = 0.0012 × 1000 × 185 = 222 mm$^2$/m

Required spacing 8 $\phi$ bars = $\frac{50.3 \times 10^3}{222}$ = 226 mm

**Provide 8 $\phi$ @ 220c/c distributors** in the waist slab.

**Design of landing slabs** [refer Fig. 12.14(c)].

The entire loading on the staircase is transmitted to the supporting edges by the bending of the landing slab in a direction parallel to the risers.

- *Loads* (assumed to be uniformly distributed):

(considering the full width of landing of 1.25 m)

(i) directly on landing: 15.35 × 1.25 = 19.19 kN/m

(ii) from going: 19.47 × 2.43/2 = 23.66 "

42.85 kN/m

$\Rightarrow$ Loading on 1 m wide strip = 42.85/1.25 = 34.28 kN/m

Effective span = 2.60 m

- *Design Moment* (at midspan):

$M_u$ =34.28 × 2.60$^2$/8 = 29.0 kNm/m

$$\Rightarrow \frac{M_u}{bd^2} = \frac{29.0 \times 10^6}{10^3 \times 159^2} = 1.147 \text{ MPa}$$

$$\frac{p_t}{100} \equiv \frac{A_{st}}{bd} = \frac{20}{2 \times 415}\left[1-\sqrt{1-4.598 \times 1.147/20}\right] = 0.342 \times 10^{-2}$$

$$\Rightarrow (A_{st})_{reqd} = (0.342 \times 10^{-2}) \times 10^3 \times 159 = 544 \text{ mm}^2/\text{m}$$

Required spacing of 12 $\phi$ bars = $\frac{113 \times 10^3}{544}$ = 207 mm.

**Provide 12 $\phi$ @ 200 c/c** at bottom in a direction parallel to the risers.

- The detailing of the staircase (one typical flight) is depicted in Fig. 12.14(d). Note that the bars from the waist slab are kept above the main bars of the landing slab so that the desired maximum effective depth is obtained for the main bars in the landing slab. This arrangement is *essential* all the more because the waist slab is supported by the landing, and to facilitate effective load transfer, the waist slab bars must be placed *above* the main bars in the landing. Nominal bars 8 ϕ @ 200 c/c are also provided at top in the landing slabs.

### EXAMPLE 12.7

Repeat Example 12.6, considering a 'tread-riser' type of staircase, instead of a 'waist slab' type.

### SOLUTION

- Given: (as in Example 12.6) $R$ = 160 mm, $T$ = 270 mm $\Rightarrow \sqrt{R^2+T^2}$ = 314 mm
  Effective span of the flight [Fig. 12.14(a)]:
  $$l = 2.43 + 1.25 = 3.68 \text{ m}$$
  Assume thickness of tread slab = thickness of riser slab $\approx l/25$ = 147 $\rightarrow$ 145 mm.
  Assuming 20 mm cover and 12 ϕ bars, $d$ = 145 – 20 – 12/2 = 119 mm
- *Loads on going* [Ref. 12.15(a)] on projected plan area:

| | | |
|---|---|---|
| (1) | self-weight of tread-riser slab @ 25×(0.16+0.27) × 0.145/0.27 | = 5.77 kN/m$^2$ |
| (2) | finishes | = 0.60 ” |
| (3) | live load | = 5.00 ” |
| | | 11.37 kN/m$^2$ |

  $\Rightarrow$ Factored load = 11.37 × 1.5 = 17.06 kN/m$^2$
- *Loads on landing* (assume 175 mm thick)

| | | |
|---|---|---|
| (1) | self-weight of slab @ 25 × 0.175 | = 4.38 kN/m$^2$ |
| (2) | finishes | = 0.60 ” |
| (3) | live loads | = 5.00 ” |
| | | 9.98 kN/m$^2$ |

  $\Rightarrow$ Factored load = 9.98 × 1.5 = 14.97 kN/m$^2$
  50% of this load may be assumed to be acting longitudinally, as in Example 12.6; i.e., 14.97 ×1/2 = 7.49 kN/$m^2$ [refer Fig. 12.15(a)]

**Design of tread-riser unit**

Reaction on landing $R = (7.49 \times 0.625) + (17.06 \times 2.43)/2$ = 25.41 kN/m

- *Design Moment at midspan*:
  $M_u$ = (25.41 × 3.68/2) – (7.49 × 0.625) × (1.84 – 0.625/2) – 17.06 × 1.215$^2$/2 = 27.01 kNm/m

$$\Rightarrow R \equiv \frac{M_u}{bd^2} = \frac{27.01\times10^6}{10^3\times119^2} = 1.907 \text{ MPa}$$

Assuming M 20 concrete and Fe 415 steel,

$$\frac{p_t}{100} \equiv \frac{A_{st}}{bd} = \frac{20}{2\times415}\left[1-\sqrt{1-4.598\times1.907/20}\right] = 0.604\times10^{-2}$$

$$\Rightarrow (A_{st})_{reqd} = (0.604\times10^{-2})\times10^3\times119 = 719 \text{ mm}^2/\text{m}$$

Required spacing of 12 ϕ bars = $\dfrac{113\times10^3}{719}$ = 157 mm.

**Provide 12 ϕ @ 150 c/c** in the form of closed ties [Fig. 12.15(b)], as explained earlier in Section 12.3.5 [Fig. 12.8(c)].
*Distributors:* provide an 8 ϕ bar transversely at each bend.

### Design of landing slabs

- The entire loading on the staircase is transmitted by flexure of the landing slabs in a direction parallel to the risers.

  Effective span = 2.60 m; effective depth = 175 – 20 – 12/2 = 149 mm

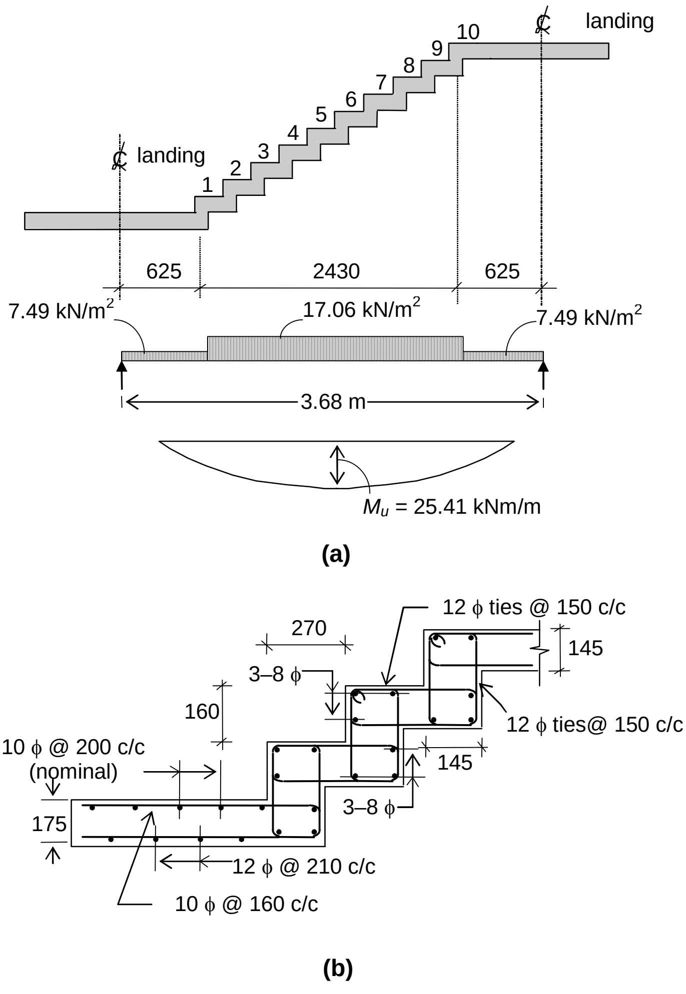



**Fig. 12.15** Example 12.7

- *Loads* (assumed to be uniformly distributed) — as in Example 12.6
  (i) directly on landing @14.97 kN/m$^2$

(ii) from going @ 17.06 kN/m$^2$ × 2.43 m / 2 $\qquad$ 16.58 kN/m$^2$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ 31.55 kN/m$^2$

- *Design Moment* (at midspan):

  $M_u$ = 31.55 × 2.60$^2$/8 = 26.66 kN/m

  $$\Rightarrow \frac{M_u}{bd^2} = \frac{26.66 \times 10^6}{10^3 \times 149^2} = 1.201 \text{ MPa}$$

  $$\Rightarrow \frac{p_t}{100} \equiv \frac{A_{st}}{bd} = \frac{20}{2 \times 415}\left[1 - \sqrt{1 - 4.598 \times 1.201/20}\right] = 0.359 \times 10^{-2}$$

  $$\Rightarrow (A_{st})_{reqd} = (0.359 \times 10^{-2}) \times 10^3 \times 149 = 536 \text{ mm}^2/\text{m}$$

  Required spacing of 12 ϕ bars = $\dfrac{113 \times 10^3}{536}$ = 211 mm

  **Provide 12 ϕ @ 210c/c** at bottom in a direction parallel to the risers. In the perpendicular direction, provide nominal bars 10 ϕ @ 160c/c. The detailing of the bars is shown in Fig. 12.15(b).

## REVIEW QUESTIONS

12.1 Describe the common geometrical configurations of staircases.

12.2 Explain the basic difference in structural behaviour between 'stair slabs spanning transversely' and 'stair slabs spanning longitudinally'.

12.3 The gravity loading on a 'waist slab' type flight can be resolved into components normal to the flight and tangential to the flight. Describe their load effects on the waist slab if it is (i) spanning transversely, (ii) spanning longitudinally.

12.4 In the case of 'tread-riser' type stairs spanning longitudinally, discuss the load effects produced by gravity loading.

12.5 Sketch the appropriate detailing of longitudinal bars in longitudinally spanning 'waist slab' type stairs at the junction of the flight and (i) lower landing slab, (ii) upper landing slab. Is there any special requirement at re-entrant corners?

12.6 What is meant by "stair slabs supported on landings"? Explain the Code recommendations for the *effective span* of the stair slab in such cases.

## PROBLEMS

12.1 A straight staircase is made of structurally independent tread slabs, with riser 160 mm, tread 280 mm, and width 1600 mm, cantilevered from a reinforced concrete wall. Design a typical tread slab, assuming M 20 concrete and Fe 415 steel. Apply the live loads specified in the IS Loading Code for stairs liable to be overcrowded. Assume mild exposure conditions

12.2 Repeat Problem 12.1, considering a tread-riser arrangement.

12.3 Repeat Problem 12.1, considering the isolated tread slabs to be supported on two stringer beams, each 250 mm wide. The clear spacing between the beams is 1600 mm.

12.4 Repeat Problem 12.3, considering a 'waist slab' type arrangement.

12.5 Design a dog-legged staircase ('waist slab' type) for an office building, assuming a floor-to-floor height of 3.0*m*, a flight width of 1.2*m*, and a landing width of 1.25*m*. Assume the stairs to be supported on 230 mm thick masonry walls at the edges of the landing, parallel to the risers. Use M 20 concrete and Fe 415 steel. Assume live loads of 5.0 $kN/m^2$ and mild exposure conditions.

12.6 Repeat Problem 12.5, considering each of the landings to be supported only on two edges perpendicular to the risers.

12.7 Repeat Problem 12.6, considering a 'tread-riser' type of staircase, instead of a 'waist slab' type.

12.8 Design a single flight straight staircase, with 11 risers, each 160 mm, and with the tread 280 mm, and upper and lower landings of 1250 mm width each. The edges of the two landings are simply supported on two masonry walls, 230 mm thick. Design a 'waist slab' type stair, assuming M 20 concrete and Fe 415 steel. Apply the live loads specified in the IS Loading Code for stairs liable to be overcrowded. Assume mild exposure conditions.

12.9 Repeat Problem 12.8, considering a 'tread-riser' type of staircase, instead of a 'waist slab' type.

12.10 Design and detail a typical intermediate flight (shown in section 'AA') of the 'open-well' staircase, details of which are shown in Fig. 12.16. Use M 20 concrete and Fe 415 steel and assume live loads of 5.0 $kN/m^2$. Assume mild exposure conditions.

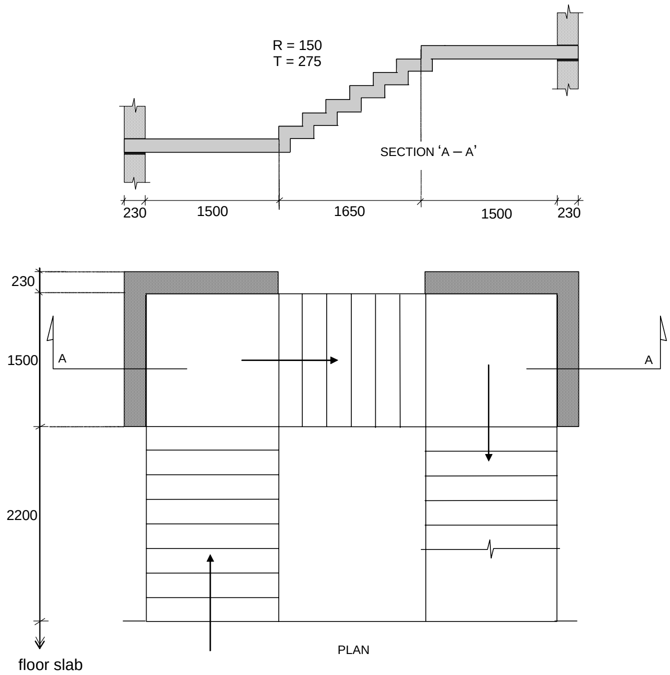



**Fig. 12.16** Problem 12.8

## REFERENCES


12.1 Gould, P.L., *Analysis and Design of a Cantilever Staircase*, ACI Journal, Vol. 60, July 1963, pp 881–889.

12.2 Solanki, H.T., *Free Standing Stairs with Slabless Tread-Risers*, Journal ASCE, Structural Div., Vol. 101, August 1975, pp 1733–1738.

12.3 Bergman, V.R., *Helicoidal Staircases of Reinforced Concrete*, ACI Journal, Vol. 53, October 1956, pp 403–412.

12.4 Scordelis, A.C., *Internal Forces in Uniformly Distributed Loaded Helicoidal Girders*, ACI Journal, Vol. 56, April 1960, pp 1013–1026.

12.5 Ahmed, I., Muqtadir, A. and Ahmed, S., *A Design Basis for Stair Slabs Supported at Landing Level*, J. Structural Engg., ASCE, Vol. 121(7), July 1995, pp 1051–1057.

12.6 Ahmed, I., Muqtadir A., and Ahmed, S., *Design Provisions for Stair Slabs in the Bangladesh Building Code*, J. Structural Engg., ASCE, Vol. 122(3), March 1996, pp 262–266.

12.7 Bangash, M. Y. H, and Bangash T., *Staircases: Structural Analysis and Design*, A. A. Balkema, Rotterdam, 1999.

12.8 Siva Raman, K. V., *R.C. Stairs Supported on Landing Edges – Elastic Behaviour, Analysis & Design*, M.S. Thesis, Dept. of Civil Engg., Indian Institute of Technology Madras, September 2000.

# 13

# Design of Compression Members

## 13.1 INTRODUCTION

A 'compression member' is a structural element which is subjected (predominantly) to axial compressive forces. Compression members are most commonly encountered in reinforced concrete buildings as *columns* (and sometimes as reinforced concrete *walls*), forming part of the 'vertical framing system' [refer Section 1.6.2]. Other types of compression members include truss members ('struts'), inclined members and rigid frame members.

The 'column' is representative of all types of compression members, and hence, sometimes, the terms 'column' and 'compression member' are used interchangeably. The Code (Cl. 25.1.1) defines the column as a *compression member*, the 'effective length[†]' of which exceeds three times the least lateral dimension. The term 'pedestal' is used to describe a vertical compression member whose 'effective length' is less than three times its least lateral dimension [Cl. 26.5.3.1(h) of the Code].

### 13.1.1 Classification of Columns Based on Type of Reinforcement

Reinforced concrete columns may be classified into the following three types based on the type of reinforcement provided:

1) Tied columns : where the main longitudinal bars are enclosed within closely spaced *lateral ties* [Fig. 13.1(a)];
2) Spiral columns : where the main longitudinal bars are enclosed within closely spaced and continuously wound spiral reinforcement [Fig. 13.1(b)];

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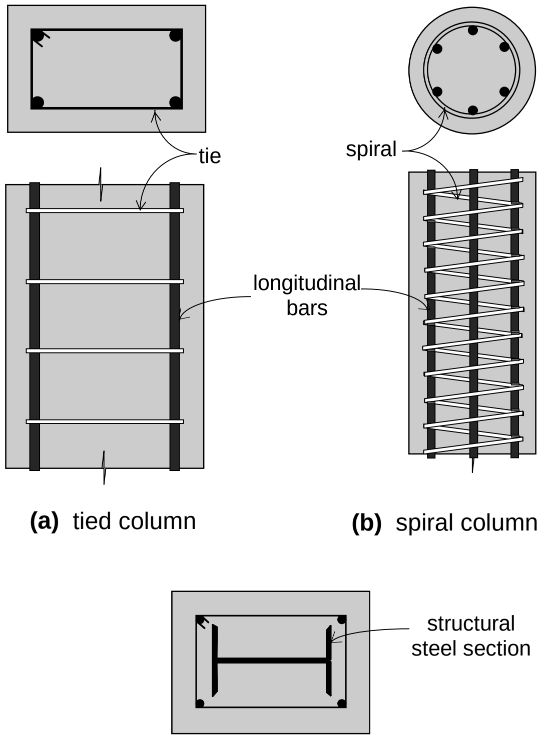
[†] For the definition of 'effective length', refer Section 13.2.

3) Composite columns : where the reinforcement is in the form of structural steel sections or pipes, with or without longitudinal bars [Fig. 13.1(c)].




**(a)** tied column **(b)** spiral column

**(c)** composite column

**Fig. 13.1** Types of columns — tied, spiral and composite

This chapter primarily deals with tied columns and spiral columns, which are the most commonly used types in reinforced concrete construction. Among these two, tied columns are more common, being applicable to all cross-sectional shapes (square, rectangle, T-, L-, cross, etc.). Spiral columns are used mainly for columns that are circular in shape, and also for square and octagonal sections.

### 13.1.2 Classification of Columns Based on Type of Loading

Columns may be classified into the following three types, based on the nature of loading:

1. Columns with axial loading (applied concentrically) [Fig. 13. 2(a)];
2. Columns with uniaxial eccentric loading [Fig. 13. 2(b)];
3. Columns with biaxial eccentric loading [Fig. 13. 2(c)].

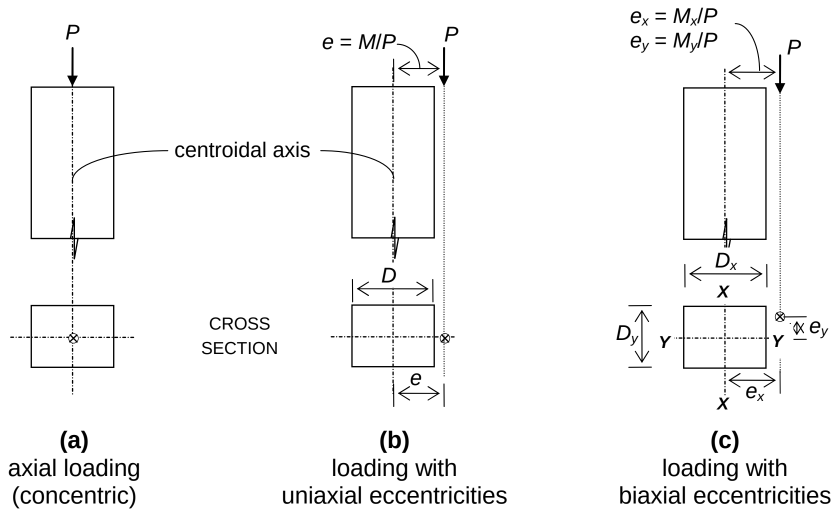



**Fig. 13.2** Different loading situations in columns

The occurrence of 'pure' axial compression in a column (due to concentric loads) is relatively rare. Generally, flexure (and, sometimes, shear[‡]) accompanies axial compression — due to 'rigid frame' action, lateral loading and/or actual (or even, unintended/accidental) eccentricities in loading. The combination of axial compression ($P$) with bending moment ($M$) at any column section is statically equivalent to a system consisting of the load $P$ applied with an eccentricity $e = M/P$ with respect to the longitudinal centroidal axis of the column section. In a more general loading situation, bending moments ($M_x$ and $M_y$) are applied simultaneously on the axially loaded column in two perpendicular directions — about the major axis (XX) and minor axis (YY) of the column section. This results in biaxial eccentricities $e_x = M_x/P$ and $e_y = M_y/P$, as shown in [Fig. 13.2(c)].

Columns in reinforced concrete framed buildings, in general, fall into the third category, viz. columns with biaxial eccentricities. The biaxial eccentricities are particularly significant in the case of the columns located in the building corners. In the case of columns located in the interior of symmetrical, simple buildings, these eccentricities under gravity loads are generally of a low order (in comparison with the lateral dimensions of the column), and hence are sometimes neglected in design calculations. In such cases, the columns are assumed to fall in the first category, viz. columns with axial loading. The Code, however, ensures that the design of such columns is sufficiently conservative to enable them to be capable of resisting nominal eccentricities in loading [refer Section 13.3.2].

Frequently, the eccentricity about one axis is negligible, whereas the eccentricity about the other axis is significant. This situation is encountered in the exterior

[‡] Considerations of shear in columns are usually neglected because the shear stresses are generally low, and the shear resistance is high on account of the presence of axial compression and the presence of lateral reinforcement.

columns of interior frames in a reinforced concrete building, under gravity loads. Under lateral loads (wind or seismic), indeed *all* columns (external as well as internal) in multi-storeyed buildings are subjected to significant uniaxial† bending moments. Such columns fall into the second category, viz. columns with uniaxial eccentricity.

### 13.1.3 Classification of Columns Based on Slenderness Ratios

Columns (i.e., compression members) may be classified into the following two types, depending on whether *slenderness effects* are considered insignificant or significant:

1. *Short* columns;
2. *Slender* (or *long*) columns.

'Slenderness' is a geometrical property of a compression member which is related to the ratio of its 'effective length' to its lateral dimension. This ratio, called *slenderness ratio*, also provides a measure of the vulnerability to failure of the column by elastic instability (buckling) — in the plane in which the slenderness ratio is computed. Columns with low slenderness ratios, i.e., relatively short and stocky columns, invariably fail under ultimate loads with the material (concrete, steel) reaching its ultimate strength, and not by buckling. On the other hand, columns with very high slenderness ratios are in danger of buckling (accompanied with large lateral deflection) under relatively low compressive loads, and thereby failing suddenly. Design codes attempt to preclude such failure by specifying 'slenderness limits' to columns [refer Section 13.3.1].

There is another important consequence of slenderness of a column subjected to eccentric compression. When a column is subjected to flexure combined with axial compression, the action of the axial compression in the displaced geometry of the column introduces 'secondary moments' — commonly referred to as the $P-\Delta$ effect — which is ignored in the usual 'first-order' structural analysis. These secondary moments become increasingly significant with increasing column slenderness. On the other hand, the secondary moments are negligible in columns with low slenderness ratios; such columns are called *short columns*. Design codes provide guidelines, in terms of slenderness ratios, in drawing the line between 'short columns' (wherein secondary moments can be ignored) and 'slender (or long) columns' (wherein secondary moments must be explicitly considered).

According to the IS Code (Cl. 25.1.2), a compression member may be classified as a 'short column' if its slenderness ratios with respect to the 'major principal axis' ($l_{ex}/D_x$) as well as the 'minor principal axis' ($l_{ey}/D_y$) are *both* less than 12‡ ; otherwise, it should be treated as 'slender column'. Here $l_{ex}$ and $D_x$ denote the effective length and lateral dimension ('depth') respectively for buckling *in the plane* passing through the longitudinal centroidal axis and normal to the major principal axis; i.e. causing

† Lateral loads, with their maximum design values, are generally assumed to operate only in one direction *at a time*. The action of lateral loads (especially seismic) in a diagonal direction (inducing biaxial bending in columns) may also have to be investigated in some cases.

‡ In the British Code, this value is specified as 15 for 'braced columns' and 10 for 'unbraced columns' [Ref. 13.2].

buckling *about the major axis* [refer Fig. 13.2(c)]; likewise, $l_{ey}$ and $D_y$ refer to the minor principal axis.

Such a definition, is, however, not suitable for non-rectangular and non-circular sections — where the slenderness ratio is better expressed in terms of the *radius of gyration*† $r$ (as in steel columns), rather than the lateral dimension $D$. In such cases, reference may be made to the ACI Code [Ref. 13.1], which recommends that the dividing line between short columns and slender columns be taken as $l_e/r$ equal to 34 for 'braced columns' and 22 for 'unbraced columns' [refer Section 13.2.3 for definitions of braced/unbraced columns]. A more precise definition of this demarcating slenderness ratio, in terms of the magnitudes and directions of the applied primary moments (at the column ends) is given in Section 13.7.1.

The design of slender columns is described in Section 13.7. The design of short columns subject to axial compression, uniaxially eccentric compression and biaxially eccentric compression are described in Sections 13.4, 13.5 and 13.6 respectively. Code requirements relating to slenderness limits, minimum eccentricities and reinforcement are explained in Section 13.3.

The 'effective length' of a column ($l_{ex}$, $l_{ey}$) is an important parameter in its design. Methods of estimating the effective length are described in the next section.

## 13.2 ESTIMATION OF EFFECTIVE LENGTH OF A COLUMN

### 13.2.1 Definition of Effective Length

The *effective length* of a column in a given plane may be defined as the distance between the points of inflection‡ in the buckled configuration of the column in that plane. The effective length depends on the *unsupported length* $l$ (i.e., distance between lateral connections, or actual length in case of a cantilever) and the boundary conditions at the column ends introduced by connecting beams and other framing members. An expression for $l_e$ may be obtained as

$$l_e = k\,l \tag{13.1}$$

where $k$ is the *effective length ratio* (i.e., the ratio of effective length to the unsupported length — also known as *effective length factor*) whose value depends on the degrees of rotational and translation restraints at the column ends.

#### *Unsupported Length*

The Code (Cl. 25.1.3) defines the 'unsupported length' $l$ of a column explicitly for various types of constructions. In conventional framed construction, $l$ is to be taken as *the clear distance between the floor and the shallower beam framing into the columns in each direction at the next higher floor level*. By this, it is implied that

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† For a rectangular section, $r \cong 0.3D$; for a circular section, $r = 0.25D$.

‡ When there exists relative translation of the ends of the column member, the points of inflection (zero moment) may not lie within the member. In such cases, they may be located by extending the deflection curve beyond the column end(s) and by applying conditions of symmetry, as shown in [Fig. 13.4(a), (b)].

when a column is framed in any direction by beams of different depths on either side, then the unsupported length (with respect to buckling about a perpendicular axis) shall be considered, conservatively, with reference to the shallower beam. It may be noted that the unsupported length in one direction may be different from that in the perpendicular direction. For a rectangular column section (width $D_y \times$ depth $D_x$), we may use the terms, $l_{ex} = k_x\, l_x$ and $l_{ey} = k_y\, l_y$ to denote the effective lengths referring to buckling about the major and minor axes respectively, where $l_x$ and $l_y$ denote the corresponding unsupported lengths and $k_x$ and $k_y$ denote the corresponding effective length factors. These concepts are made clear in Fig. 13.2a, and further illustrated in Examples 13.1 and 13.2.

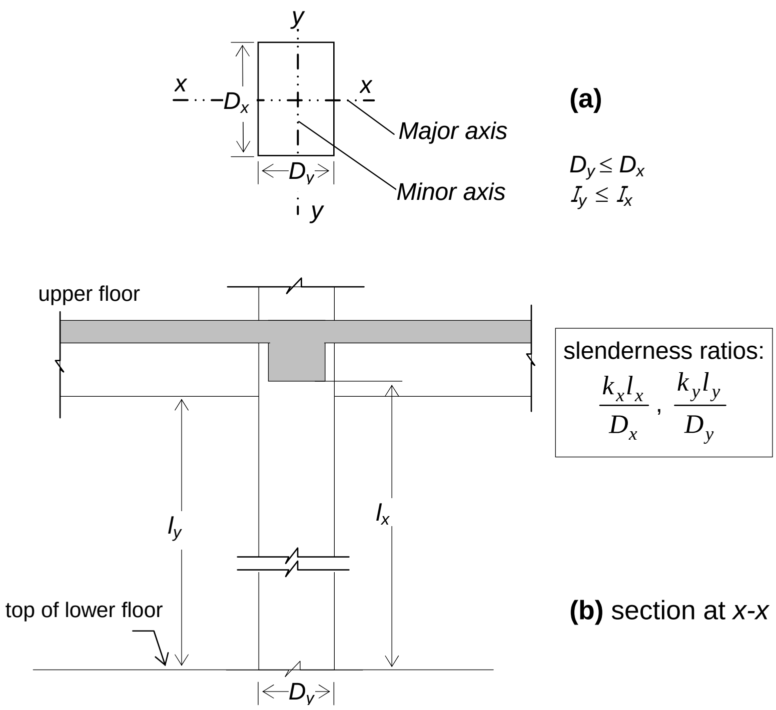
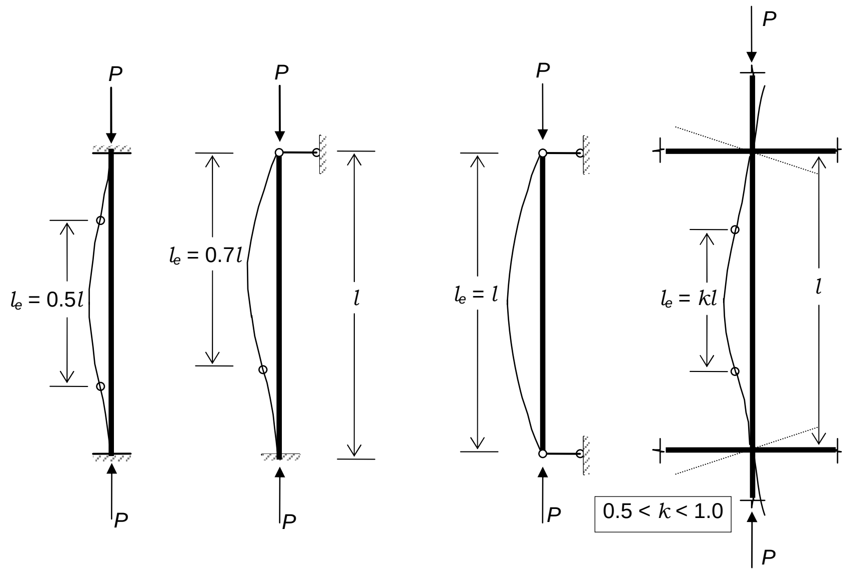



**Fig. 13.2a** Definitions of unsupported and effective lengths in a rectangular column

In the case of 'flat slab construction', the unsupported length $l$ is to be taken as *the clear distance between the floor and the lower extremity of the capital, the drop panel or slab, whichever is the least.*

### 13.2.2 Effective Length Ratios for Idealised Boundary Conditions

When relative transverse displacement between the upper and lower ends of a column is prevented, the frame is said to be *braced* (against sideway). In such cases, the effective length ratio $k$ varies between 0.5 and 1.0, as shown in Fig. 13.3. The extreme value $k$ = 0.5 corresponds to 100 percent rotational fixity at both column ends [Fig. 13.3(a)] (i.e., when the connecting floor beams have infinite flexural stiffness), and the other extreme value $k$ = 1.0 corresponds to zero rotational fixity at both column ends ('pinned') [Fig. 13.3(c)] (i.e., when the beams have zero flexural stiffness). When one end is fully 'fixed' and the other 'pinned', $k$ = 0.7 [Fig. 13(b)].




**(a)** both ends rotationally fixed

**(b)** one end rotationally fixed, the other free

**(c)** both ends rotationally free

**(d)** both ends partially restrained (rotational)

**Fig. 13.3** Effective lengths of columns *braced* against sideway

When relative transverse displacement between the upper and lower ends of a column is not prevented, the frame is said to be *unbraced* (against sideway). In such cases, the effective length ratio $k$ varies between 1.0 and infinity, as shown in Fig. 13.4. The lower limit $k$ = 1.0 corresponds to 100 percent rotational fixity at both column ends [Fig. 13.4(a)], and the upper theoretical $k = \propto$ corresponds to zero rotational fixity at both column ends, i.e. a column pinned at both ends and permitted to sway (unstable) [Fig. 13.4(c)]. When one end is fully 'fixed' and the other 'free', the column acts like a vertical cantilever in the buckled mode, corresponding to which $k$ = 2 [Fig. 13.4(b)].

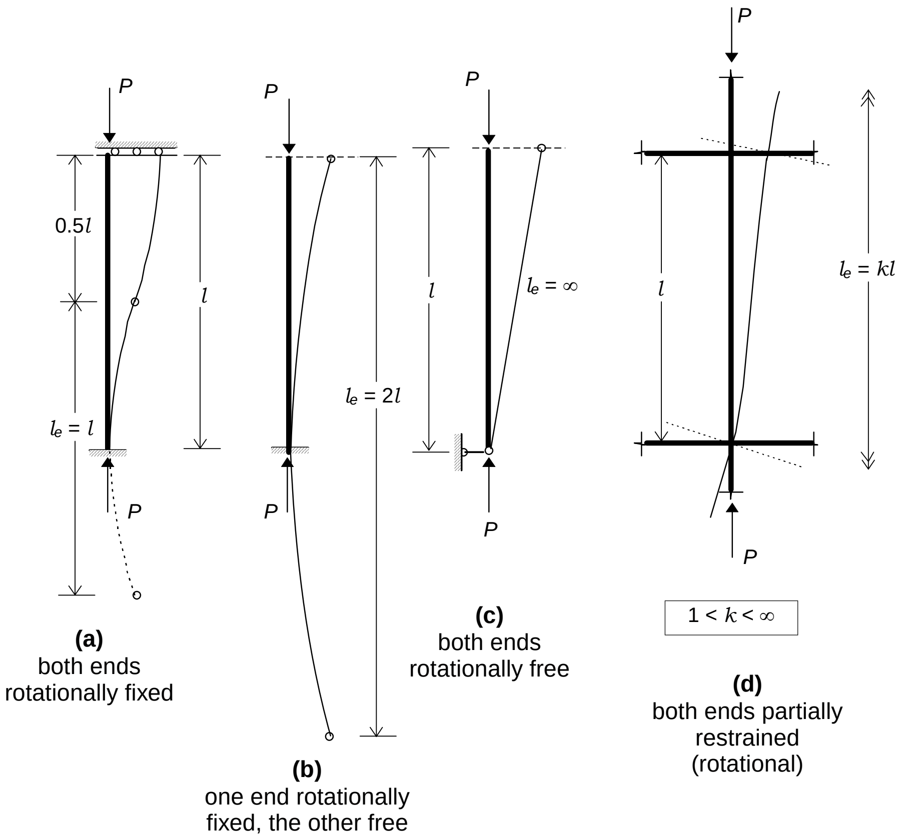



**Fig. 13.4** Effective lengths of columns *unbraced* against sideway

### *Code Recommendations for Idealised Boundary Conditions*

Although, in design practice, it is convenient to assume the idealised boundary conditions of either zero or full restraint (rotational and translational) at a column end, the fact is that such idealisations cannot generally be realised in actual structures. For this reason, the Code (Cl. E–1), while permitting these idealisations, recommends the use of 'effective length ratios' $k = l_e/l$ that are generally more conservative than those obtained from theoretical considerations [Fig. 13.3, 13.4].

These recommended values of $k$ are as follows:

1. *columns braced against sideway*:
   a) both ends 'fixed' rotationally [Fig. 13.3(a)] : 0.65 (instead of 0.5)
   b) one end 'fixed' and the other 'pinned'[Fig. 13.3(b)] : 0.80 (instead of 0.7)
   c) both ends 'free' rotationally ('pinned') [Fig. 13.3(c)] : 1.00
2. *columns unbraced against sideway*:

| | |
|---|---|
| a) both ends 'fixed' rotationally [Fig. 13.4(a)] | : 1.20 (instead of 1.0) |
| b) one end 'fixed' and the other 'partially fixed' | : 1.50 |
| c) one end 'fixed' and the other free [Fig. 13.4(b)] | : 2.00 |

The most common case encountered in framed buildings is the one involving partial rotational fixity at both ends of the column. For such a case in a 'braced' frame, $k = 0.85$ may be assumed for preliminary designs — conforming to an average of cases 1(a) and 1(c) indicated above. If the frame cannot be considered to be fully braced[†] , it is desirable to assume a more conservative estimate — say $k = 1.0$ or more. However, if the frame is clearly 'unbraced', it is necessary to ascertain the effective length ratio more accurately, as described in the next section.

### 13.2.3 Effective Length Ratios of Columns in Frames

The rotational restraint at a column end in a building frame is governed by the flexural stiffnesses of the members framing into it, relative to the flexural stiffness of the column itself. Hence, it is possible to arrive at measures of the 'degree of fixity' at column ends, and thereby arrive at a more realistic estimate of the effective length ratio of a column than the estimates given in Section 13.2.2 (for idealised boundary conditions).

For this purpose, different methods have been recommended by different codes. The IS Code (Cl. E–1) recommendations are based on design charts proposed by Wood [Ref. 13.3]. The British Code [Ref. 13.2] and the Commentary to the ACI Code [Ref. 13.1] recommend the use of certain simplified formulas, which are particularly suitable for computer-aided design. Other methods, including the use of certain 'alignment charts' [Ref. 13.4, 13.5], have also been proposed. All of these methods provide two different sets of charts/formulas: one set for columns 'braced' against sideway, and the other set for 'unbraced' columns. This is a shortcoming in these methods, because columns in actual frames are rarely completely 'braced' (prevented from side-sway) and rarely completely 'unbraced', and the difference between the two estimates of effective length ratio can be considerable. A recent study [Ref. 13.22] shows how this problem can be resolved using fuzzy logic concepts, which incorporates the concept of 'partial bracing'. This aspect of 'partial bracing' may be more accurately accounted for by means of a proper second-order analysis of the entire frame is required [Ref. 13.6]; however, this is computationally too difficult for routine design problems.

#### *Deciding Whether a Column is Braced or Unbraced*

An approximate way of deciding whether a column is 'braced' or 'unbraced' is given in the ACI Code commentary. For this purpose, the 'stability index' Q of a storey in a multi-storeyed building is defined as:

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[†] Generally, the assumption of a fully braced frame can be safely made if there are special bracing elements in a building such as shear walls, shear trusses, etc. [Ref. 13.7] (which are designed to resist practically all the lateral loads on the frame). Even otherwise, if a rigid frame possesses sufficient inherent translational stiffness, and especially if there are in-fill masonry walls, it may be considered to be braced — at least partially, if not fully.

$$Q = \frac{\sum P_u}{h_s} \times \frac{\Delta_u}{H_u} \qquad (13.2)$$

where $\sum P_u \equiv$ sum of axial loads on all columns in the storey;
$h_s \equiv$ height of the storey;
$\Delta_u \equiv$ elastic first-order lateral deflection of the storey;
$H_u \equiv$ total lateral force acting on the storey.

It can be shown [Ref. 13.8] that, in the absence of bracing elements, the 'lateral flexibility' measure of the storey $\Delta_u/H_u$ (storey drift per unit storey shear) may be taken (for a typical intermediate storey) as:

$$\frac{\Delta_u}{H_u} = \frac{h_s^{\,2}}{12E_{c,col}\sum(I_c/h_s)} + \frac{h_s^{\,2}}{12E_{c,beam}\sum(I_b/l_b)} \qquad (13.3)$$

where $\sum I_c \equiv$ sum of second moments of areas of all columns in the storey in the plane under consideration;
$\sum I_b/l_b \equiv$ sum of ratios of second moment of area to span of all floor members in the storey in the plane under consideration;
$E_c \equiv$ modulus of elasticity of concrete.

Eq. 13.2 is derived by assuming points of inflection at the mid-heights of all columns and midspan locations of all beams, and by applying the *unit load method* to an isolated storey [Ref. 13.8]. If special bracing elements such as shear walls, shear trusses and infill walls are present, then their effect will be to reduce $\Delta_u/H_u$ significantly.

The recommendation given in the ACI Code is that the storey can be considered to be *braced* [†] only if the stability index $Q < 0.05$.

The application of this concept is demonstrated in Example 13.1.

### *Use of Code Charts*

Charts are given in Fig. 26 and Fig. 27 of the Code for determining the effective length ratios of braced columns and unbraced columns respectively, in terms of coefficients $\beta_1$ and $\beta_2$ which represent the degrees of rotational freedom at the top and bottom ends of the column [Ref. 13.7]:

$$\beta = \frac{\sum_{jt} I_c/h_s}{\sum_{jt} I_c/h_s + \sum_{jt} 0.5(I_b/l_b)} \quad \text{for braced columns} \qquad (13.4a)$$

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[†] It is found that when $Q < 0.05$, the second-order moments due to the 'lateral drift effect' will be less than 5 percent of the first-order moments [refer Section 13.7]. It may also be noted that in the earlier versions of ACI 318, the limiting value of $Q$ was specified as 0.04. Code IS 456:2000 (Annex E) gives limiting value for $Q$ as 0.04.

$$\beta = \frac{\sum_{jt} I_c/h_s}{\sum_{jt} I_c/h_s + \sum_{jt} 1.5(I_b/l_b)} \quad \text{for unbraced columns} \qquad (13.4b)$$

where the notation *jt* denotes that the summation is to be done for the members framing into the top joint (in case of $\beta_1$) or the bottom joint (in case of $\beta_2$). The increased beam stiffness for unbraced columns [Eq. 13.4b], compared to braced columns [Eq. 13.4a], is attributable to the fact that in the case of the latter, the (braced) columns are bent in *single curvature*, whereas in the case of the former, the (unbraced) columns are bent in *double curvature*.

The limiting values $\beta$ = 0 and $\beta$ = 1 represent 'fully fixed' and 'fully hinged' conditions respectively.

The use of these Code charts (not reproduced in this book) is demonstrated in Examples 13.1 and 13.2.

### Use of Formulas

The following formulas, given in BS 8110 [Ref. 13.2] and the Commentary to the ACI Code [Ref. 13.1], provide useful estimates of the effective length ratio $k$:

$$k = \begin{cases} 0.7 + 0.05\,(\alpha_1 + \alpha_2) \\ 0.85 + 0.05\ \alpha_{\min} \end{cases} \quad \text{whichever is less, for \textbf{braced} columns} \qquad (13.5a)$$

$$k = \begin{cases} 1.0 + 0.15\,(\alpha_1 + \alpha_2) \\ 2.0 + 0.30\ \alpha_{\min} \end{cases} \quad \text{whichever is less, for \textbf{unbraced} columns} \qquad (13.5b)$$

where
$$\alpha_1 \equiv \frac{\sum_{jt} I_c/h_s}{\sum_{jt} (I_b/l_b)_{top}} \quad \text{and} \quad \alpha_2 \equiv \frac{\sum_{jt} I_c/h_s}{\sum_{jt} (I_b/l_b)_{bottom}} \qquad (13.5c)$$

For a fully fixed condition, $\alpha$ = 0 may be considered, and for a 'hinged' condition, $\alpha$ = 10 may be considered.

The application of these formulas is demonstrated in Examples 13.1 and 13.2.

#### EXAMPLE 13.1

The framing plan of a multi-storeyed building is shown in Fig. 13.5(a). Assume that all the columns have a size 300 mm × 400 mm; the longitudinal beams (global *X*-direction) have a size 250 mm × 600 mm and the transverse beams (global *Y*-direction) have a size 250 mm × 400 mm as shown. The storey height $h_s$ = 3.5 m. For a column in a typical lower floor of the building, determine the effective lengths $l_{ex}$ and $l_{ey}$ with respect to the local *x*- and *y*- axes (major and minor), as shown in Fig. 13.5(b).

For the purpose of estimating the total axial loads on the columns in the storey, assume a total distributed load of 35 kN/m$^2$ from all the floors above (combined). Also assume M 25 grade concrete for the columns and M 20 grade concrete for the beams.

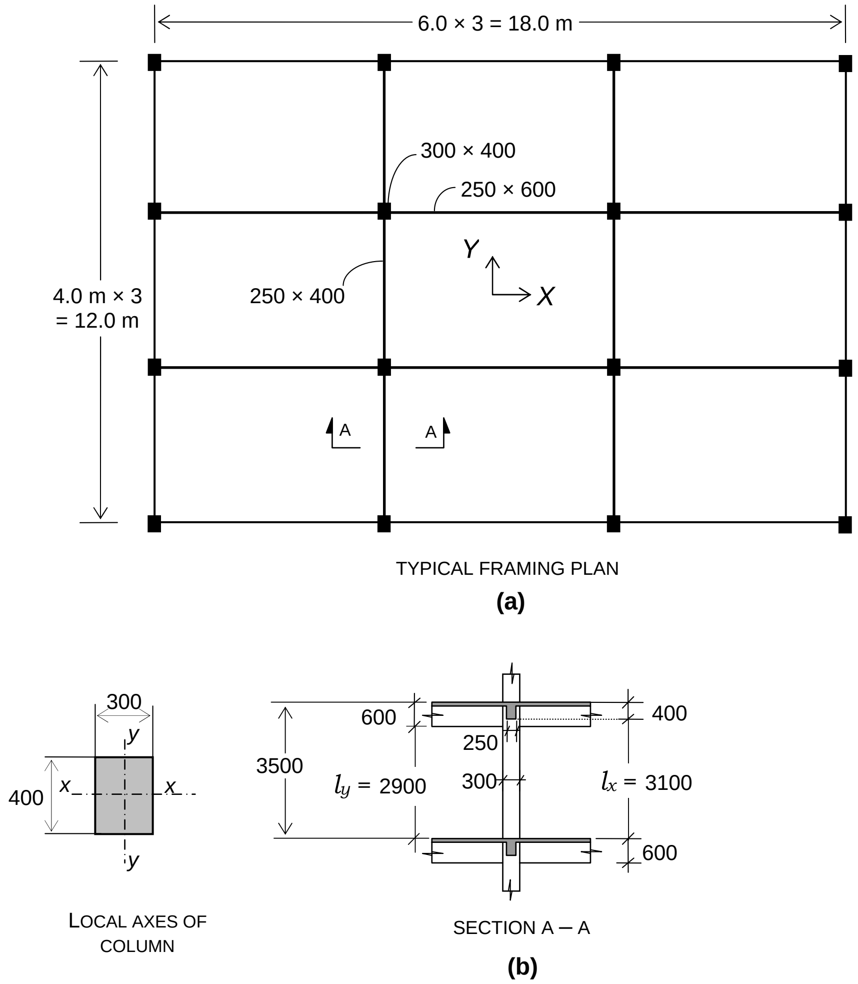


**Fig. 13.5** Example 13.1

## SOLUTION

*Unsupported lengths* of column

$l_y$ = 3500 – 600 = 2900 mm (for buckling about $y$-axis)

$l_x$ = 3500 – 400 = 3100 mm (for buckling about $x$-axis)

*Relative stiffness measures* of columns and beams

- Columns: 16nos, 300 mm × 400 mm, $h_s$ = 3500 mm

  $\Rightarrow \sum I_c/h_s = 16 \times \dfrac{400 \times 300^3/12}{3500} = 4114 \times 10^3\,\text{mm}^3$ (for sway in the global $X$ – direction), and

$\Rightarrow \sum I_c/h_s = 16 \times \dfrac{300 \times 400^3/12}{3500} = 7314 \times 10^3\,\text{mm}^3$ (for sway in the global $Y$-direction).

- Longitudinal Beams: 12nos, 250 mm × 600 mm, $l_b$ = 6000 mm

  $\Rightarrow (\sum I_b/l_b)_X = 12 \times \dfrac{(250)\times(600)^3/12}{6000}$ = 9000 × 10$^3$ mm$^3$

- Transverse Beams: 12nos, 250 mm × 400 mm, $l_b$ = 4000 mm

  $\Rightarrow (\sum I_b/l_b)_Y = 12 \times \dfrac{(250)\times(400)^3/12}{4000}$ = 4000 × 10$^3$ mm$^3$

**Columns Braced or Unbraced ?**

*Lateral Flexibility measures* of the storey: $(\Delta_u/H_u)_X$ and $(\Delta_u/H_u)_Y$

Ignoring the contribution of in-fill walls [Eq. 13.3]:

- $$\frac{\Delta_u}{H_u} = h_s^2\left[\left\{12E_{c,col} \times \sum\left(I_c/h_s\right)\right\}^{-1} + \left\{12E_{c,beam} \times \sum\left(I_b/l_b\right)\right\}^{-1}\right]$$

  where $E_c = 5000\sqrt{f_{ck}}$ (as per Code Cl. 6.2.3.1)

  For columns, $f_{ck}$ = 25 MPa $\Rightarrow E_{c,col} = 5000\sqrt{25}$ = 25000 MPa

  For beams, $f_{ck}$ = 20 MPa $\Rightarrow E_{c,beam} = 5000\sqrt{20}$ = 22361 MPa

- Longitudinal direction (global $X$-direction):

  $$\left(\frac{\Delta_u}{H_u}\right)_X = 3500^2\,[(12 \times 25000 \times 4114 \times 10^3)^{-1} + (12 \times 22361 \times 9000 \times 10^3)^{-1}]$$

  $= 1.4998 \times 10^{-5}$ mm/N

- Transverse direction (global $Y$-direction)::

  $$\left(\frac{\Delta_u}{H_u}\right)_Y = 3500^2[(12 \times 25000 \times 7314 \times 10^3)^{-1} + (12 \times 22361 \times 4000 \times 10^3)^{-1}]$$

  $= 1.6996 \times 10^{-5}$ mm/N

*Stability Index Q*

- Total axial load on all columns ≈ 35 kN/m$^2$ × (12.0 m × 18.0 m) = 7560 kN

  $Q = \dfrac{\sum P_u}{h_s} \times \dfrac{\Delta_u}{H_u}$ [Eq. 13.2]

  Longitudinal direction: $Q_X = \dfrac{7560 \times 10^3}{3500} \times 1.4998 \times 10^{-5}$ = 0.0324

  Transverse direction: $Q_Y = \dfrac{7560 \times 10^3}{3500} \times 1.6996 \times 10^{-5}$ = 0.0367

- As $Q_X$ = 0.0324 < 0.05, the storey can be considered 'braced' in the longitudinal direction.
  As $Q_Y$ = 0.0367 < 0.05, the storey can be considered 'braced' in the transverse direction.
  Hence, the columns in the storey may be assumed to be 'braced' in both directions. (Note that $Q_X$ and $Q_Y$ are less than the IS Code limit of 0.04 as well)

**Effective Lengths by IS Code charts**

$$\beta_1 = \beta_2 = \frac{\sum_{jt} I_c/h_s}{\sum_{jt} I_c/h_s + \sum_{jt} 0.5(I_b/l_b)} \qquad \text{[Eq. 13.4a]}$$

- *Buckling with respect to minor (local y-) axis*:

  $$\sum_{jt}(I_c/h_s) = \frac{400\times 300^3/12}{3500}\times 2 = 514\times 10^3\,\text{mm}^3$$

  $$\sum_{jt}(I_b/l_b) = \frac{250\times(600)^3/12}{6000}\times 2 = 1500\times 10^3\,\text{mm}^3$$

  $$\beta_1 = \beta_2 = \frac{514}{514+0.5(1500)} = 0.407$$

  Referring to Fig. 26 of the Code, $k_y$ = 0.64
  $\Rightarrow l_{ey} = l_y \times k_y =$ 0.64 × 2900 = **1856 mm**

- *Buckling with respect to major (local x-) axis*:

  $$\sum_{jt}(I_c/h_s) = \frac{300\times 400^3/12}{3500}\times 2 = 914\times 10^3\,\text{mm}^3$$

  $$\sum_{jt}(I_b/l_b) = \frac{250\times(400)^3/12}{4000}\times 2 = 667\times 10^3\,\text{mm}^3$$

  $$\beta_1 = \beta_2 = \frac{914}{914+0.5(667)} = 0.733$$

  Referring to Fig. 26 of the Code, $k_x$ = 0.82
  $\Rightarrow l_{ex} = l_x \times k_x =$ 0.82 × 3100 = = **2542 mm**

**Alternative: Effective Lengths by Formulas** [Eq. 13.5a]

$$\alpha_1 = \alpha_2 = \frac{\sum_{jt} I_c/h_s}{\sum_{jt}(I_b/l_b)}$$

- *Buckling with respect to minor (local y-) axis*: $\alpha_1 = \alpha_2 = \dfrac{514}{1500} = 0.3427$

$$\Rightarrow k_y = \begin{cases} 0.7 + 0.05(2 \times 0.3427) = 0.7343 & \text{(lesser)} \\ 0.85 + 0.05(0.3427) = 0.8671 \end{cases}$$

$\Rightarrow l_{ey} = 0.7343 \times 2900 =$ **2129 mm**

- <u>*Buckling with respect to major (local x-) axis*</u>: $\alpha_1 = \alpha_2 = \dfrac{914}{667} = 1.3703$

$$\Rightarrow k_x = \begin{cases} 0.7 + 0.05(2 \times 1.3703) = 0.8370 & \text{(lesser)} \\ 0.85 + 0.05(1.3703) = 0.9185 \end{cases}$$

$\Rightarrow l_{ex} = 0.8370 \times 3100 =$ **2595 mm**

- *Note* 1: The use of formulas gives effective lengths that are generally within ± 8 percent of the corresponding values obtained from the Code charts.

- *Note* 2: Alternatively, the designer may assume idealised boundary conditions — braced columns with partial rotational fixity at top and bottom.
  Assuming a value $k = 0.85$ (as explained in Section 13.2.2)
  $\Rightarrow l_{ey} = 0.85 \times 2900 = 2465$ mm, and $l_{ex} = 0.85 \times 3100 = 2635$ mm
  This results in a slightly conservative estimate of effective length.

- *Note* 3: A realistic assessment of effective length is called for in the case of *slender* columns. In the present case, as $l_{ey}/D_y$ *and* $l_{ex}/D_x$ are approximately 7, and well below 12, the column is definitely a *short column*, and there is no real need for a rigorous calculation of effective length.

### EXAMPLE 13.2

Repeat the problem in Example 13.1, considering a column size of 250 mm × 250 mm (instead of 300 mm × 400 mm).

### SOLUTION

- *Unsupported lengths* of column
  $l_y = 2900$ mm and $l_x = 3100$ mm (as in Example 13.1)
- *Relative stiffness measures* of columns and beams
  Columns: $\sum I_c/h_s = 16 \times (250)^4/(12 \times 3500) = 1488 \times 10^3$ mm$^3$
  Longitudinal Beams: $\left(\sum I_b/l_b\right)_X = 9000 \times 10^3$ mm$^3$ (as in Example 13.1)
  Transverse Beams: $\left(\sum I_b/l_b\right)_Y = 4000 \times 10^3$ mm$^3$ (as in Example 13.1)
- *Lateral Flexibility measures* of the storey:

$$\frac{\Delta_u}{H_u} = h_s^2\left[\left\{12E_{c,col} \times \sum\left(I_c/h_s\right)\right\}^{-1} + \left\{12E_{c,beam} \times \sum\left(I_b/l_b\right)\right\}^{-1}\right]$$

Substituting $E_{c,col} = 25000$ MPa, $E_{c,beam} = 22361$ MPa (as in Example 13.1) and $h_s = 3500$ mm, and values of relative stiffness measures, Longitudinal direction (global *X*-direction): $\left(\dfrac{\Delta_u}{H_u}\right)_X$ = $3.2514 \times 10^{-5}$ mm/N

Transverse direction (global *Y*-direction): $\left(\dfrac{\Delta_u}{H_u}\right)_Y$ = $3.8855 \times 10^{-5}$ mm/N

[Comparing these values with those obtained in Example 13.1, it is seen that the reduction in column size results in a drastic increase (more than double) in the lateral flexibility of the storey].

- *Stability Index Q*

  $P_u$ = 7560 kN (as in Example 13.1)

  $$Q_X = \frac{7560 \times 10^3}{3500} \times 3.2514 \times 10^{-5} = 0.0702 > 0.05$$

  $$Q_Y = \frac{7560 \times 10^3}{3500} \times 3.8855 \times 10^{-5} = 0.0839 > 0.05$$

  Hence, the columns in the storey should be considered as 'unbraced' in both directions.

**Effective Lengths by IS Code charts**

$$\beta_1 = \beta_2 = \frac{\sum_{jt} I_c/h_s}{\sum_{jt} I_c/h_s + \sum_{jt} 1.5(I_b/l_b)} \qquad \text{[Eq. 13.4 (b)]}$$

- <u>*Buckling with respect to minor (local y-) axis*</u>:

  $$\sum_{jt}(I_c/h_s) = \frac{250^4/12}{3500} \times 2 = 186 \times 10^3 \text{ mm}^3 \;;\; \sum_{jt}(I_b/l_b) = 1500 \times 10^3 \text{ mm}^3$$

  (as in Example 13.1)

  $$\beta_1 = \beta_2 = \frac{186}{186 + 1.5(1500)} = 0.076$$

  Referring to Fig. 27 of the Code, $k_y$ = 1.04

  $\Rightarrow l_{ey} = l_y \times k_y =$ 1.04 × 2900 = **3016 mm**

- <u>*Buckling with respect to major (local x-) axis*</u>:

  $$\sum_{jt}(I_c/h_s) = 186 \times 10^3 \text{ mm}^3 \;;\; \sum_{jt}(I_b/l_b) = 667 \times 10^3 \text{ mm}^3 \text{ (as in Example 13.1)}$$

  $$\beta_1 = \beta_2 = \frac{186}{186 + 1.5(667)} = 0.157$$

  Referring to Fig. 27 of the Code, $k_x$ = 1.09

  $\Rightarrow l_{ex} = l_x \times k_x =$ 1.09 × 3100 = **3379 mm**

**Alternative: Effective Lengths by Formulas** [Eq. 13.5a]

$$\alpha_1 = \alpha_2 = \frac{\sum_{jt} I_c / h_s}{\sum_{jt} (I_b / l_b)}$$

- *Buckling with respect to minor (local y-) axis*: $\alpha_1 = \alpha_2 = \dfrac{186}{1500} = 0.124$

  $\Rightarrow k_y = \begin{cases} 1.0 + 0.15(2 \times 0.124) = 1.0372 & \text{(lesser)} \\ 2.0 + 0.30(0.124) = 2.0372 \end{cases}$

  $\Rightarrow l_{ey}$ = 1.037 × 2900 = **3007 mm**

- *Buckling with respect to major (local x-) axis*: $\alpha_1 = \alpha_2 = \dfrac{186}{667} = 0.2789$

  $\Rightarrow k_x = \begin{cases} 1.0 + 0.15(2 \times 0.2789) = 1.084 & \text{(lesser)} \\ 2.0 + 0.30(0.2789) = 2.084 \end{cases}$

  $\Rightarrow l_{ex}$ = 1.084 × 3100 = **3360 mm**

- *Note* 1: The effective lengths predicted by the two different methods are fairly close.

- *Note* 2: Considering the effective lengths given by the Code charts, the slenderness ratios of the column are obtained as follows:

  $l_{ey}/D_y$ = 3016/250 = 12.1.
  $l_{ex}/D_x$ = 3379/250 = 13.5;

  The column should be designed as a 'slender column'.

## 13.3 CODE REQUIREMENTS ON SLENDERNESS LIMITS, MINIMUM ECCENTRICITIES AND REINFORCEMENT

### 13.3.1 Slenderness Limits

Slenderness effects in columns effectively result in reduced strength, on account of the additional 'secondary' moments introduced [refer Section 13.7]. In the case of very slender columns, failure may occur suddenly under small loads due to instability ('elastic buckling'), rather than due to material failure. The Code attempts to prevent this type of failure (due to instability) by specifying certain 'slenderness limits' in the proportioning of columns.

The Code (Cl. 25.3.1) specifies that the ratio of the unsupported length ($l$) to the least lateral dimension ($d$) of a column should not exceed† a value of 60:

$$l/d \leq 60 \qquad (13.6)$$

Furthermore, in case one end of a column is free (i.e., cantilevered column) in any given plane, the Code (Cl. 25.3.2) specifies that

† In the case of 'unbraced' columns, it is desirable to adopt a more stringent limit — say, $l/d < 40$.

$$l \leq 100b^2/D \tag{13.7}$$

where $D$ is the depth of the cross-section measured in the plane of the cantilever and $b$ is the width (in the perpendicular direction).

### 13.3.2 Minimum Eccentricities

As explained in Section 13.1.2, the general case of loading on a compression member is one comprising axial compression combined with biaxial bending. This loading condition is represented by a state of *biaxial eccentric compression*, wherein the axial load $P$ acts eccentric to the longitudinal centroidal axis of the column cross-section, with eccentricities $e_x$ and $e_y$ with respect to the major and minor principal axes [Fig. 13.2(c)].

Very often, eccentricities not explicitly arising out of structural analysis calculations act on the column due to various reasons, such as:

- lateral loads not considered in design;
- live load placements not considered in design;
- accidental lateral/eccentric loads;
- errors in construction (such as misalignments); and
- slenderness effects underestimated in design.

For this reason, the Code (Cl. 25.4) requires every column to be designed for a minimum eccentricity $e_{min}$ (in any plane) equal to the unsupported length/500 plus lateral dimension/30, subject to a minimum of 20 mm. For a column with a rectangular section [Fig. 13.2], this implies:

$$e_{x,\min} = \begin{cases} l/500 + D_x/30 \\ 20 \text{ mm} \end{cases} \text{(whichever is greater)} \tag{13.8a}$$

$$e_{y,\min} = \begin{cases} l/500 + D_y/30 \\ 20 \text{ mm} \end{cases} \text{(whichever is greater)} \tag{13.8b}$$

For non-rectangular and non-circular cross-sectional shapes, it is recommended [Ref. 13.7] that, for any given plane,

$$e_{\min} = \begin{cases} l_e/300 \\ 20 \text{ mm} \end{cases} \quad \text{(whichever is greater)} \tag{13.8c}$$

where $l_e$ is the *effective length* of the column in the plane considered.

### 13.3.3 Code Requirements on Reinforcement and Detailing

#### ***Longitudinal Reinforcement*** (refer Cl. 26.5.3.1 of the Code)

- *Minimum Reinforcement*: The longitudinal bars must, in general, have a cross-sectional area not less than 0.8 percent of the gross area of the column section. Such a minimum limit is specified by the Code:

* ∗ to ensure nominal flexural resistance under unforeseen eccentricities in loading; and
* ∗ to prevent the yielding of the bars due to creep[†] and shrinkage effects, which result in a transfer of load from the concrete to the steel.

In very large-sized columns (where the large size is dictated, for instance, by architectural considerations, and not strength) under axial compression, the limit of 0.8 percent of gross area may result in excessive reinforcement. In such cases, the Code allows some concession by permitting the minimum area of steel to be calculated as 0.8 percent of *the area of concrete required to resist the direct stress, and not the actual* (*gross*) *area.*

However, in the case of *pedestals* (i.e., compression members with $l_e/D < 3$) which are designed as plain concrete columns, the minimum requirement of longitudinal bars may be taken as 0.15 percent of the gross area of cross-section.

In the case of reinforced concrete walls, the Code (Cl. 32.5) has introduced detailed provisions regarding minimum reinforcement requirements for vertical (and horizontal) steel. The vertical reinforcement should not be less than 0.15 percent of the gross area in general. This may be reduced to 0.12 percent if welded wire fabric or deformed bars (Fe 415 / Fe 500 grade steel) is used, provided the bar diameter does not exceed 16 mm. This reinforcement should be placed in two layers if the wall is more than 200mm thick. In all cases, the bar spacing should not exceed three times the wall thickness or 450 mm, whichever is less.

- *Maximum Reinforcement*: The maximum cross-sectional area of longitudinal bars should not exceed 6 percent of the gross area of the column section. However, a reduced maximum limit of 4 percent is recommended in general in the interest of better placement and compaction of concrete — and, in particular, at lapped splice locations.

  In tall buildings, columns located in the lowermost storeys generally carry heavy reinforcement (~ 4 percent). The bars are progressively curtailed in stages at higher levels.

- *Minimum diameter / number of bars and their location*: Longitudinal bars in columns (and pedestals) should not be less than 12 mm in diameter and should not be spaced more than 300 mm apart (centre-to-centre) along the periphery of the column[‡] [Fig. 13.6(a)]. At least 4 bars (one at each corner) should be provided in a column with rectangular cross-section, and at least 6 bars (equally spaced near the periphery) in a circular column. In 'spiral columns' (including noncircular shapes), the longitudinal bars should be placed in contact with the

---

[†] Creep effects can be quite pronounced in compression members under sustained service loads [refer Section 13.4.2]. The consequent increase in steel stress (due to creep strain) is found to be relatively high at very low reinforcement percentages: hence, the minimum limit of 0.8 percent is prescribed [Ref. 13.7].

[‡] In the case of reinforced concrete walls, the Code (Cl. 32.5b) recommends a maximum spacing of three times the wall thickness or 450 mm, whichever is smaller.

spiral reinforcement, and equidistant around its inner circumference [Fig. 13.6(b)]. In columns with T-, L-, or other cross-sectional shapes, at least one bar should be located at each corner or apex [Fig. 13.6(c)].
Longitudinal bars are usually located close to the periphery (for better flexural resistance), but may be placed in the interior of the column when eccentricities in loading are minimal. When a large number of bars need to be accommodated, they may be *bundled*, or, alternatively, *grouped*, as shown in [Fig. 13.6(d)].

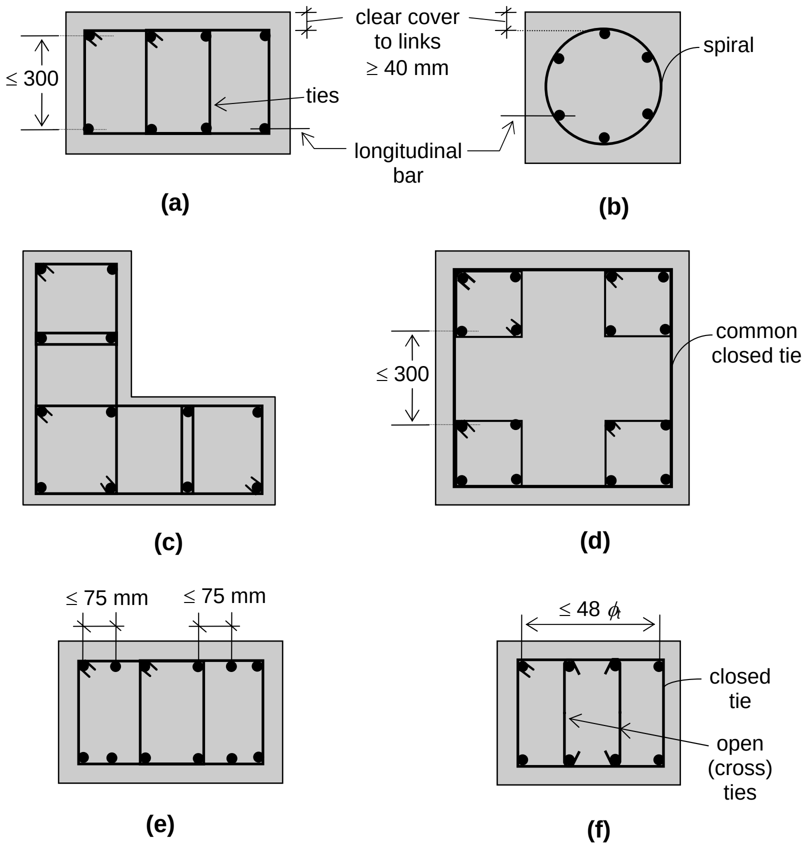


**Fig. 13.6** Some Code recommendations for detailing in columns

- *Cover to reinforcement*: A minimum clear cover of 40 mm or bar diameter (whichever is greater), to the column ties is recommended by the Code (Cl. 26.4.2.1) for columns in general; a reduced clear cover of 25 mm is permitted in small-sized columns ($D \leq 200$ mm and whose reinforcing bars do not exceed 12mm) and a minimum clear cover of 15 mm (or bar diameter, whichever is greater) is specified for walls. However, in aggressive environments, it is desirable, in the interest of durability, to provide increased cover [Table 5.1] — but preferably not greater than 75 mm.

***Transverse Reinforcement*** (refer Cl. 26.5.3.2 of the Code)

- *General*: All longitudinal reinforcement in a compression member must be enclosed within transverse reinforcement, comprising either *lateral ties* (with internal angles not exceeding 135°) or *spirals*. This is required:
  - to prevent the premature buckling of individual bars;
  - to confine the concrete in the 'core', thus improving ductility and strength;
  - to hold the longitudinal bars in position during construction; and
  - to provide resistance against shear and torsion, if required.
- *Lateral Ties*: The arrangement of lateral ties should be effective in fulfilling the above requirements. They should provide adequate lateral support to each longitudinal bar, thereby preventing the outward movement of the bar. The diameter of the tie $\phi_t$ is governed by requirements of stiffness, rather than strength, and so is independent of the grade of steel [Ref. 13.7]. The *pitch* $s_t$ (centre-to-centre spacing along the longitudinal axis of the column) of the ties should be small enough to reduce adequately the unsupported length (and hence, slenderness ratio) of each longitudinal bar. The Code recommendations (based on Ref. 13.9) are as follows:

$$\text{tie diameter}^{\dagger} \quad \phi_t \geq \begin{cases} \phi_{long,\max}/4 \\ 6\,\text{mm} \end{cases} \tag{13.9}$$

$$\text{tie spacing}^{\ddagger} \quad s_t \leq \begin{cases} D \\ 16\phi_{long,\min} \\ 300\,\text{mm} \end{cases} \tag{13.10}$$

where $\phi_{long}$ denotes the diameter of longitudinal bar to be tied and $D$ denotes the least lateral dimension of the column.

Ideally, the tie must turn around (and thereby provide full lateral restraint to) every longitudinal bar that it encloses — particularly the corner bars. When the spacing of longitudinal bars is less than 75 mm, lateral support need only be provided for the corner and alternate bars [Fig. 13.6(e)]. The straight portion of a closed tie (between the corner bars) is not really effective if it is large, as it tends to bulge outwards when the concrete core is subjected to compression [Ref. 13.10]. For this reason, supplementary cross ties are required for effective confinement of the concrete. If the longitudinal bars spaced at a distance not exceeding $48\phi_t$ are effectively tied in two directions, then the additional longitudinal bars in between these bars need be tied only in one direction by *open* ties [Fig. 13.6(f)].

† In earlier version of Code, minimum tie diameter was specified as 5 mm, instead of 6 mm.

‡ In earlier version of Code, maximum tie spacing was specified as $48\times\phi_t$ instead of 300 mm

The ends of every tie (whether closed or open) should be properly anchored. In the case of grouping of longitudinal bars at the corners of a large-sized column [Fig. 13.6(d)], each group should be separately tied together, along with a single common closed tie for all groups. The diameter and pitch of this common tie should be computed [Eq. 13.9 and 13.10] by treating each bar group as a single longitudinal bar (of equivalent area); however the diameter of the common tie need not exceed 20 mm.

Finally, it should be noted that when multiple ties are provided in a column (which is usually the case, except in small-sized columns), the locations of these different ties should preferably be staggered along the longitudinal axis of the column. Extra ties should be provided at lapped splice locations in the longitudinal reinforcement — especially at the bends.

- *Spirals*: Helical reinforcement provides very good confinement to the concrete in the 'core' and enhances significantly the ductility of the column at ultimate loads. The diameter and pitch of the spiral may be computed as in the case of ties [Eq. 13.9, 13.10] — except when the column is designed to carry a 5 percent overload (as permitted by the Code), in which case

$$\text{pitch } s_t < \begin{cases} 75 \text{ mm} \\ \textit{core diameter}/6 \end{cases} \tag{13.11a}$$

$$\text{and } s_t > \begin{cases} 25 \text{ mm} \\ 3\phi_t \end{cases} \tag{13.11b}$$

The ends of the spiral should be anchored properly by providing one and a half extra turns.

## 13.4 DESIGN OF SHORT COLUMNS UNDER AXIAL COMPRESSION

### 13.4.1 Conditions of Axial Loading

Axial loading on a compression member may be defined as loading that produces a uniform (compressive) strain distribution across the cross-section. If the column is symmetrically reinforced, as shown in Fig. 13.7(a), the line of action of the load $P_o$ must coincide with the longitudinal centroidal axis of the column section, in order to produce a uniform strain distribution. Equilibrium conditions require the resultant compressive force $C_c + C_s$ in the section to be equal to and act opposite to and through the point of application of the external load $P_o$ [Fig. 13.7(a)]. If $f_{cc}$ and $f_{sc}$ denote respectively the stresses in the concrete and the longitudinal steel, corresponding to the uniform compressive strain $\varepsilon_c$, then it follows that

$$\begin{aligned} P_o &= C_c + C_s \\ &= f_{cc}A_c + f_{sc}A_{sc} \\ \Rightarrow P_o &= f_{cc}A_g + (f_{sc} - f_{cc})A_{sc} \end{aligned} \tag{13.12}$$

where $A_g$ = gross area of cross-section = $A_c + A_{sc}$ ;

$A_{sc}$ = total area of longitudinal reinforcement = $\sum A_{si}$ ;

$A_c$ = net area of concrete in the section = $A_g - A_{sc}$

When the section is unsymmetrically reinforced and subject to axial load conditions [Fig. 13.7(b)], Eq. 13.12 remains valid. However, the line of action of the applied load $P_o$ must now be eccentric to the geometric centroidal axis. This eccentricity $e_c$ is easily obtained from moment equilibrium conditions as

$$e_c = \frac{\left(f_{sc} - f_{cc}\right)\sum A_{si}\,x_i}{P_o} \tag{13.12a}$$

where $x_i$ = distance measured [+ve as indicated in Fig. 13.7(b)] of the i[th] row of reinforcement of area $A_{si}$ with reference to the geometrical centroid.

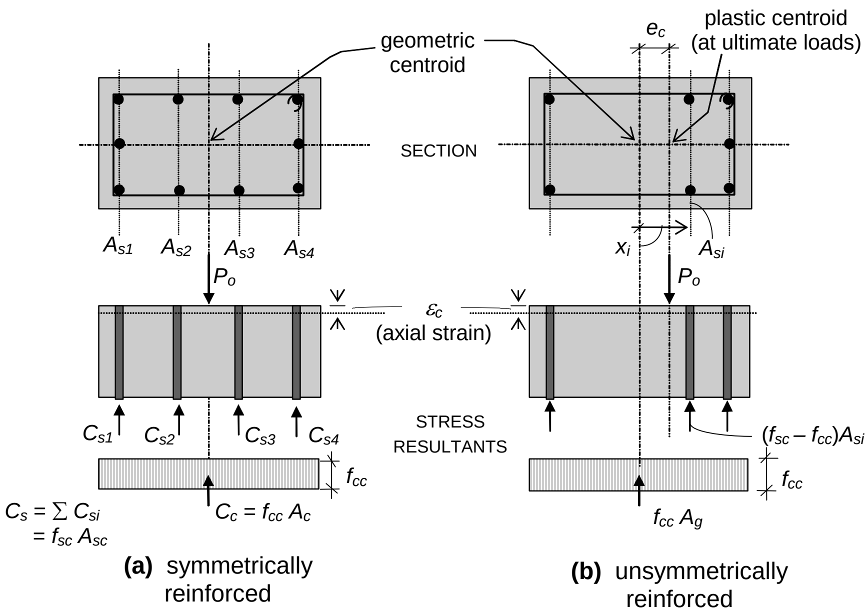


**Fig. 13.7** Axial loading on short columns

At the limit state of collapse, the section would have 'plastified', and for this reason, the axis through which $P_o$ must act for axial load conditions is termed the *plastic centroidal* axis. Under symmetrical reinforcement conditions of axial loading, $e_c = 0$, and the plastic centroid coincides with the geometric centroid.

### 13.4.2 Behaviour Under Service Loads

Until the early 1950s, reinforced concrete columns were invariably designed using the elastic theory, which involves the concepts of *permissible stresses*, *modular ratio* and *transformed section* [refer Sections 3.2, 4.6]. It was realized subsequently, from extensive experimental investigations, that there can be no fixed ratio of steel stress

$f_{sc}$ to concrete stress $f_{cc}$, even under axial loading conditions [Ref. 13.10]. The ratio of these stresses depended on:

1. the amount of creep, which is influenced by the history of sustained loading and numerous factors related to the quality of concrete [refer Section 2.11].
2. the amount of shrinkage, which in turn depended on the age of concrete, method of curing, environmental conditions and several other factors related to the quality of concrete [refer Section 2.12]

In general, the strain in the cross-section $\varepsilon_c$ increases with age on account of creep and shrinkage, with a consequent redistribution of stresses in concrete and steel, such that the load shared by the concrete is partially transferred to the steel. Consequently, it becomes difficult to predict the stresses $f_{cc}$ and $f_{sc}$ (in Eq. 13.12) under service loads.

According to conventional working stress method of design, substituting the *permissible stresses* $\sigma_{cc}$ and $\sigma_{sc}$ in lieu of $f_{sc}$ and $f_{cc}$ respectively, the design equation is obtained from Eq. 13.12 as

$$P_o = \sigma_{cc} A_g + (\sigma_{sc} - \sigma_{cc}) A_{sc} \tag{13.13}$$

where $\sigma_{sc}$ is taken approximately as $1.5m\sigma_{cc}$ (as in doubly reinforced beams — refer Section 4.6). However, this assumption renders the steel stress $\sigma_{sc}$ independent of the grade of steel, and results in unrealistic and uneconomical designs. The Code (B-2.2) in its provision for working stress design, attempts to somewhat remedy this situation by recommending

$$\sigma_{sc} = \begin{cases} 130 \text{ MPa} & \text{for Fe 250 steel} \\ 190 \text{ MPa} & \text{for Fe 415, Fe 500 steels} \end{cases} \tag{13.14a}$$

The allowable stresses in concrete ($\sigma_{cc}$) under direct compression are specified as

$$\sigma_{cc} = \begin{cases} 4.0 \text{ MPa} & \text{for M 15} \\ 5.0 \text{ MPa} & \text{for M 20} \\ 6.0 \text{ MPa} & \text{for M 25} \\ 8.0 \text{ MPa} & \text{for M 30} \\ 9.0 \text{ MPa} & \text{for M 35} \end{cases} \tag{13.14b}$$

However, most codes of other countries have dispensed with the *working stress method* (WSM) of design altogether, with the advent of the *ultimate load method* (ULM) of design initially, and the more rational *limit states method* (LSM) of design subsequently (since the 1980s). Indeed, in the revised Indian Code too, priority is given to the LSM design procedure and the WSM relegated to an Annex.

### 13.4.3 Behaviour Under Ultimate Loads

Unlike service load conditions, the behaviour of an axially compressed short column is fairly predictable under ultimate load conditions. It is found that the ultimate strength of the column is relatively independent of its age and history of loading. As axial loading is increased, axial shortening of the column increases linearly up to about 80 percent of the ultimate load $P_{uo}$ (path OA in Fig. 13.8); this behaviour is found to be independent of the type of transverse reinforcement [Ref. 13.10]. However, beyond the ultimate load (point B in Fig. 13.8), the behaviour depends on the type and amount of transverse reinforcement.

### *Tied columns*

Generally, the longitudinal steel would have reached 'yield' conditions at the ultimate load level $P_{uo}$ [point B in Fig. 13.8] — regardless of whether transverse reinforcement is provided or not†. However, in the absence of transverse reinforcement (or with widely spaced lateral ties), failure will be sudden and brittle, caused by crushing and shearing of the concrete (as in a plain concrete cylinder test — refer Section 2.8) and accompanied by the buckling of longitudinal bars. In the case of tied columns, some marginal ductility [paths BC, BD in Fig. 13.8] can be introduced by providing closely spaced lateral ties which undergo yielding in tension prior to collapse of the columns. The descent in the load-axial shortening curve is attributable to 'softening' and micro-cracking in the concrete.

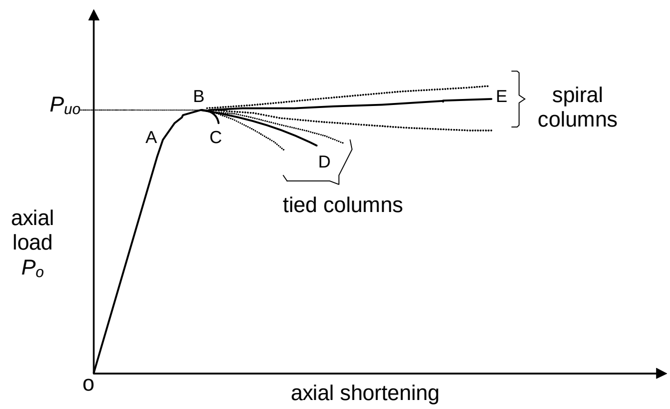


**Fig. 13.8** Behaviour of axially loaded tied and spiral columns

### *Spiral Columns*

It is with the *spiral column* that substantial ductility is achieved prior to the collapse of the column [path BE in Fig. 13.8]. It is found that, approximately at load level $P_{uo}$ [point B in Fig. 13.8], the outer shell of the concrete (covering the spiral) spalls off; but the concrete in the 'core', laterally confined by the helical reinforcement, continues to carry load. Collapse ultimately takes place when the spiral reinforcement yields in tension. The load carrying capacity after the spalling can exceed $P_{uo}$ provided the amount of spiral reinforcement is such that the load capacity

† It may be noted, however, that, for design purposes, the Code limits the ultimate strain in concrete to 0.002, as a conservative measure. Corresponding to this strain, yield conditions will not be attained in the case of Fe 415 and Fe 500 grades of steel [refer Fig. 3.6, 3.7].

contributed by it more than makes up for the loss in load capacity due to spalling of the concrete shell. Based on experimental findings [Ref. 13.10], the Code (Cl. 39.4) permits a 5 percent increase in the estimation of strength beyond $P_{uo}$, provided the following requirement is satisfied by the spiral reinforcement:

$$\rho_s \geq 0.36\left(\frac{A_g}{A_{core}} - 1\right)\frac{f_{ck}}{f_{sy}} \qquad (13.15)$$

where $\rho_s \equiv \dfrac{\text{volume of spiral reinforcement}}{\text{volume of core}}$ per unit length of column;

$A_{core} \equiv$ total area of concrete core, measured outer-to-outer of the spirals;

$A_g \equiv$ gross area of cross-section;

$f_{sy} \equiv$ characteristic (yield) strength of spiral, limited to 415 MPa.

However, it may be observed that the 5 percent increase in strength will be realised only after extensive cracking of concrete and will be accompanied by large deformations. It is stated in the Explanatory Handbook to the Code [Ref. 13.7] that *the permitted increase in design capacity of such columns is because the failure will be gradual and ductile* and not because the Code intends *to make use of increase in capacity beyond the spalling load*.

In addition to increased ductility and warning prior to collapse, spiral columns exhibit increased toughness (resistance to impact loading) and are particularly effective under dynamic loading conditions (such as seismic loading).

### 13.4.4 Design Strength of Axially Loaded Short Columns

The maximum compressive strain in concrete under axial loading at the limit state of collapse in compression is specified as $\varepsilon_c$ = 0.002 by the Code (Cl. 39.1a). Corresponding to this (somewhat conservative) limiting strain of 0.002, the design stress in the concrete is $0.67f_{ck}/1.5 = 0.447f_{ck}$ (refer Fig. 3.5), and the design stress in steel is $0.87f_y$ in the case of Fe 250 (refer Fig. 3.6) and $0.790f_y$ and $0.746f_y$ in the case of Fe 415 and Fe 500 respectively (refer Fig. 3.7 and Table 3.2).

Accordingly, under 'pure' axial loading conditions, the design strength of a short column is obtainable from Eq. 13.12 as:

$$P_{uo} = 0.447 f_{ck} A_g + (f_{sc} - 0.447 f_{ck}) A_{sc} \qquad (13.16)$$

$$\text{with } f_{sc} = \begin{cases} 0.870\ f_y & \text{for Fe 250} \\ 0.790\ f_y & \text{for Fe 415} \\ 0.746\ f_y & \text{for Fe 500} \end{cases} \qquad (13.16a)$$

However, as explained in Section 13.3.2, the Code requires all columns to be designed for 'minimum eccentricities' in loading. Hence, Eq. 13.16 cannot be directly applied. Nevertheless, where the calculated minimum eccentricity (in any plane) does not exceed 0.05 times the lateral dimension (in the plane considered), the

Code (Cl. 39.3) permits the use of the following simplified formula, obtained by reducing $P_{uo}$ (from Eq. 13.16) by approximately 10 percent[†] [Ref. 13.7]:

$$\tilde{P}_{uo} = 0.4 f_{ck} A_g + (0.67 f_y - 0.4 f_{ck}) A_{sc} \qquad (13.17)$$

where $\tilde{P}_{uo}$ denotes the design strength in uniaxial compression permitted by the Code (including the effect of minimum eccentricities). It is found that the use of Eq. 13.17 results in a conservative design, compared to the rigorous design involving axial compression and biaxial bending with the minimum eccentricities.

As mentioned earlier, the Code (Cl. 39.4) permits the load capacity be enhanced by 5 percent when spiral reinforcement is provided, conforming to Eq. 13.15.

### EXAMPLE 13.3

Design the reinforcement in a column of size 450 mm × 600 mm, subject to an axial load of 2000 kN under service dead and live loads. The column has an unsupported length of 3.0*m* and is braced against sideway in both directions. Use M 20 concrete and Fe 415 steel.

### SOLUTION

*Short Column or Slender Column* ?

- Given: $lx = ly$ = 3000 mm, $D_y$ = 450 mm, $D_x$ = 600 mm

  slenderness ratios $\begin{cases} l_{ex}/D_x = k_x\, l_x/D_x = k_x \times 3000/600 = 5k_x \\ l_{ey}/D_y = k_y\, l_y/D_y = k_y \times 3000/450 = 6.67k_y \end{cases}$

  As the column is braced against sideway in both directions, effective length ratios $k_x$ and $k_y$ are both less than unity, and hence the two slenderness ratios are both less than 12.
- Hence, the column may be designed as a *short column*.

*Minimum Eccentricities* [Eq. 13.8]

$$e_{x,\min} = \frac{3000}{500} + \frac{600}{30} = 26.0 \text{ mm } (> 20.0 \text{ mm})$$

$$e_{y,\min} = \frac{3000}{500} + \frac{450}{30} = 21.0 \text{ mm } (> 20.0 \text{ mm})$$

- As $0.05D_x = 0.05 \times 600 = 30.0$ mm $> e_{x,min} = 26.0$ mm
  and $0.05D_y = 0.05 \times 450 = 22.5$ mm $> e_{y,min} = 21.0$ mm,
  the Code formula for axially loaded short columns can be used.

*Factored Load*

- $P_u$ = *service load* × *partial load factor*
  = 2000 × 1.5 = 3000 kN

[†] The reduction works out as 10 percent with respect to Fe 415/Fe 500 grades of steel. However, with respect to Fe 250 steel, the reduction in $f_{sc}$ is as high as 30 percent, while the reduction in $f_{cc}$ is 10 percent.

*Design of Longitudinal Reinforcement*

- $P_u = 0.4 f_{ck} A_g + (0.67 f_y - 0.4 f_{ck}) A_{sc}$ [Eq. 13.17]

  $\Rightarrow 3000 \times 10^3 = 0.4 \times 20 \times (450 \times 600) + (0.67 \times 415 – 0.4 \times 20)A_{sc}$
  $= 2160 \times 10^3 + 270.05 A_{sc}$
  $\Rightarrow A_{sc} = (3000–2160) \times 10^3/270.05 = 3111 \text{ mm}^2$
- In view of the column dimensions (450 mm, 600 mm), it is necessary to place intermediate bars, in addition to the 4 corner bars:
  Provide **4–25 ϕ at corners** : $4 \times 491 = 1964 \text{ mm}^2$
  and **4–20 ϕ additional**: $4 \times 314 = 1256 \text{ mm}^2$
  $\Rightarrow A_{sc} = 3220 \text{ mm}^2 > 3111 \text{ mm}^2$

  $\Rightarrow p = (100 \times 3220) / (450 \times 600) = 1.192 > 0.8$ (minimum reinf.) — OK.

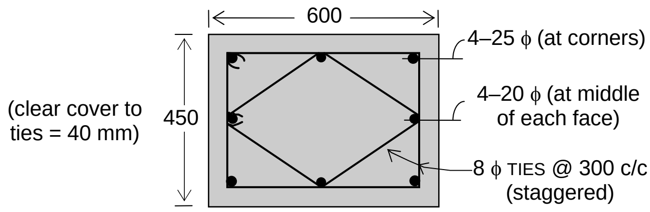


**Fig. 13.9** Example 13.3

*Lateral Ties*

- Tie diameter $\phi_t > \begin{cases} 25/4 \\ 6 \text{ mm} \end{cases}$ : provide 8 mm dia;

  Tie spacing $s_t < \begin{cases} 450 \text{ mm} \\ 16 \times 20 = 320 \text{ mm} \\ 300 \text{ mm} \end{cases}$ : provide 300 mm.

  $\therefore$ Provide **8 ϕ ties @ 300 c/c**

The detailing of reinforcement is shown in Fig. 13.9

### EXAMPLE 13.4

Design the reinforcement in a spiral column of 400 mm diameter subjected to a factored load of 1500 kN. The column has an unsupported length of 3.4 m and is braced against sideway. Use M 25 concrete and Fe 415 steel.

### SOLUTION

*Short Column or Slender Column* ?

- Given: $l$ = 3400 mm, $D$ = 400 mm $\Rightarrow$ slenderness ratio = $l_e/D \leq 3400/400 = 8.5$ (as column is braced).
- As $l_e/D < 12$, the column may be designed as a *short column*.

*Minimum eccentricity*

- $e_{\min} = \dfrac{3400}{500} + \dfrac{400}{30} = 20.1$ mm (> 20.0 mm)

  As $0.05D = 20.0$ mm $\approx e_{min}$, the Code formula for axially compressed short columns may be used.

*Factored Load*

- $P_u = 1500$ kN (given)

  $= 1.05\,[0.4 f_{ck}\,A_g + (0.67\,f_y - 0.4\,f_{ck})\,A_{sc}]$

  for spiral columns (appropriately reinforced)

*Design of longitudinal reinforcement*:

- $\dfrac{1500 \times 10^3}{1.05} = 0.4 \times 25 \times \dfrac{\pi \times 400^2}{4} + (0.67 \times 415 - 0.4 \times 25)A_{sc}$

  $\Rightarrow 1428.6 \times 10^3 = 1256.6 \times 10^3 + 268.05\,A_{sc}$

  $\Rightarrow A_{sc} = (1428.6 - 1256.6) \times 10^3/268.05$

  $= 642$ mm$^2$ (equal to 0.51% of gross area).

- $A_{sc,min}$ at 0.8% of $A_g$

  $= \dfrac{0.8}{100} \times \dfrac{\pi \times 400^2}{4} = 1005$ mm$^2$

  **Provide 6 nos 16 $\phi$**: $A_{sc} = 201 \times 6 = 1206$ mm$^2$ > 1005 mm$^2$.

*Design of Spiral reinforcement*

- Assuming a clear cover of 40 mm over spirals,
  Core diameter = 400 – (40 × 2) = 320 mm
- Assuming a bar diameter of 6 mm and pitch $s_t$

  $\rho_s \equiv \dfrac{\text{Volume of spiral reinforcement}}{\text{Volume of core}}$ per unit length of column

  $= \dfrac{(\pi \times 6^2/4) \times \pi \times (320-6)/s_t}{\pi \times 320^2/4} = \dfrac{0.3468}{s_t}$

- As per the Code requirement (Eq. 13.15, Cl. 39.4.1 of Code)

  $\rho_s \geq 0.36\left(\dfrac{A_g}{A_{core}} - 1\right)\left(\dfrac{f_{ck}}{f_{sy}}\right)$

  Assuming for the spiral reinforcement, $f_{sy} = 415$ MPa

  $\Rightarrow \dfrac{0.3468}{s_t} \geq 0.36\left(\dfrac{\pi \times 400^2/4}{\pi \times 320^2/4} - 1\right)\left(\dfrac{25}{415}\right)$

  $\Rightarrow s_t \leq 28.4$ mm

- Code restrictions on pitch (Eq. 13.11, Cl. 26.5.3.2d of Code)

  $s_t < \begin{cases} 75 \text{ mm} \\ \text{core dia}/6 = 53.3 \text{ mm} \end{cases}$

  $s_t > \begin{cases} 25 \text{ mm} \\ 3\phi_t = 18 \text{ mm} \end{cases}$

**Provide 6 ϕ spiral @ 28 mm c/c pitch**

- The detailing of reinforcement is shown in Fig. 13.10.

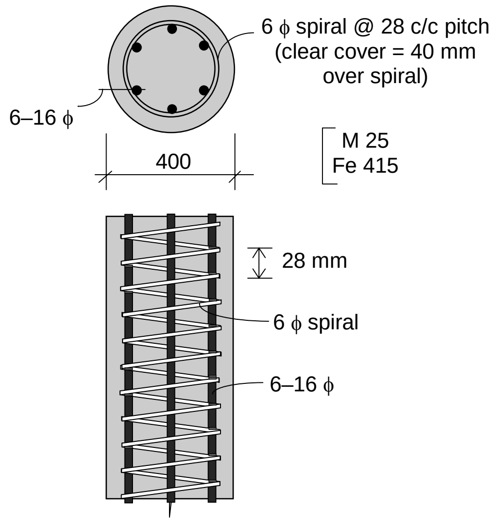


**Fig. 13.10** Example 13.4

## 13.5 DESIGN OF SHORT COLUMNS UNDER COMPRESSION WITH UNIAXIAL BENDING

This section deals with the behaviour and design of short compression members subject to axial compression combined with uniaxial bending, i.e., bending with respect to either the major axis or minor axis (but not both). As explained in Section 13.1.2, this loading condition is statically equivalent to a condition of *uniaxial eccentric compression* wherein the factored axial load $P_u$ is applied at an eccentricity $e = M_u/P_u$ with respect to the centroidal axis, $M_u$ being the factored bending moment.

The traditional 'working stress method' of design is not covered in this section, not only because of the fact that it has become obsolete, but also because the Code (Cl. B 4.3) makes it mandatory that designs for eccentric compression by WSM, based on 'cracked section' analysis† *should be further checked for their strength under ultimate load conditions to ensure the desired margin of safety*. This condition effectively makes WSM redundant, as it suffices to design in accordance with LSM.

### 13.5.1 Distribution of Strains at Ultimate Limit State

A special limiting case of uniaxial eccentric compression is the condition of zero eccentricity ($e$ = 0, i.e., $M_u$ = 0) which corresponds to the axial loading condition, discussed in Section 13.4. Corresponding to this condition, the strain across the

† 'Uncracked section' analysis is permitted by the Code (Cl. 46.1) when the eccentricity in loading is so small that the resulting flexural tension, if any, can be borne by the concrete.

column section is uniform and limited to $\varepsilon_{cu}$ = 0.002 at the *limit state of collapse in compression* (as per the Code).

The other limiting case of uniaxial eccentric compression corresponds to infinite eccentricity ($e = \infty$, i.e., $P_u = 0$), which is equivalent to a condition of 'pure' flexure, discussed in Chapter 4. Corresponding to this condition, the strains are linearly distributed across the section with a 'neutral axis' (NA) located somewhere within the section, and with tensile strains on one side of the NA and compressive strains on the other side. Under ultimate load conditions, i.e., at the *limit state of collapse in flexure*, the strain in the highly compressed edge of the column is specified by the Code as $\varepsilon_{cu}$ = 0.0035 [refer Section 4.7].

In the general case of uniaxial eccentric compression ($M_u \neq 0$, $P_u \neq 0$), it follows that $0 \leq e < \infty$, and for such a condition, the strain profile is non-uniform and assumed to be linearly varying across the section, with the maximum strain in the highly compressed edge, $\varepsilon_{cu}$, having a value between 0.002 and 0.0035 at the ultimate limit state. This is depicted in the Fig. 13.11.

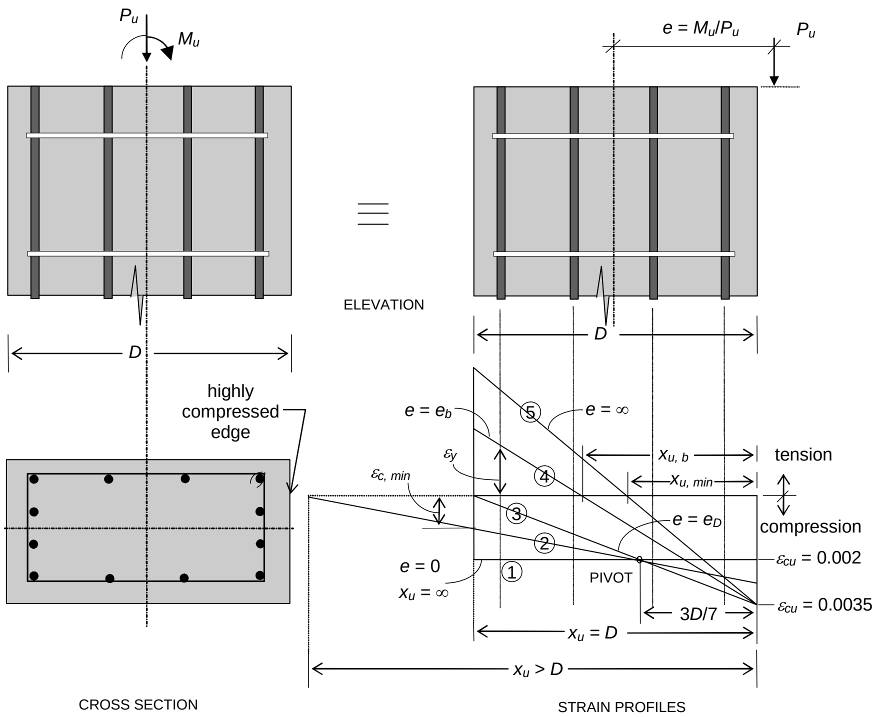


**Fig. 13.11** Possible strain profiles under ultimate limit state in eccentric compression

It may be noted that all the assumptions made in the analysis of the ultimate limit state in flexure [refer Section 4.7] — excluding the one related to the minimum tensile strain $\varepsilon_{st}^*$ at the centroid of the tension steel — are also applicable in the case of eccentric compression [refer Cl. 39.1 of the Code]. In fact, the assumption of a linear distribution of strains [Fig. 13.11] follows directly from the basic assumption that *plane sections before bending remains plane after bending*; this has been validated experimentally. In the case of eccentric compression, however, the 'depth' of the NA (with reference to the 'highly compressed edge') can vary from a minimum value $x_{u,min}$ (corresponding to $e = \infty$) to the maximum value $x_u = \infty$ (i.e., no neutral axis(!), corresponding to $e = 0$).

The Code (Cl. 39.1) permits $\varepsilon_{cu}$ = 0.0035 to be considered in cases where the loading eccentricity (i.e., $M_u/P_u$) is sufficiently high as to induce some tensile strain in the column section. The limiting condition for this occurs when the resulting neutral axis coincides with the edge farthest removed from the highly compressed edge, i.e., $x_u = D$, corresponding to which $e = e_{x_u = D} \equiv e_D$, as indicated in Fig. 13.11.

When the loading eccentricity is relatively low, such that the entire section is subjected to (non-uniform) compression and the NA lies outside the section ($x_u > D$), the Code (Cl. 39.1b) limits the strain in the highly compressed edge to a value between 0.002 and 0.0035 as follows:

$$\varepsilon_{cu} = 0.0035 - 0.75\ \varepsilon_{c,\,min} \quad \text{for } x_u \geq D \tag{13.18}$$

where $\varepsilon_{c,\,min}$ is the strain in the *least compressed edge*, as shown in Fig. 13.11. It can be seen that Eq. 13.18 satisfies the limiting strain conditions $\varepsilon_{cu}$ = 0.0035 (corresponding to $\varepsilon_{c,\,min} = 0$; i.e, $x_u$ = D or $e = e_D$) and $\varepsilon_{cu}$ = 0.002 (corresponding to $\varepsilon_{c,\,min}$ = 0.002; i.e, $x_u = \infty$ or $e = 0$). The point of intersection of these two limiting strain profiles (corresponding to $e = 0$ and $e = e_D$) occurs at a distance of $3D/7$ from the 'highly compressed edge', and in fact, this point acts like a 'pivot' for strain profiles. It serves as a common point through which all strain profiles (with $x_u \geq D$) pass, as indicated in Fig. 13.11. Using similar triangles, it can be shown that:

$$\varepsilon_{cu} = 0.002\left[1 + \frac{3D/7}{x_u - 3D/7}\right] \quad \text{for } x_u \geq D \tag{13.18a}$$

### 13.5.2 Modes of Failure in Eccentric Compression

Although the term *limit state of collapse in compression* is generally used by the Code (Cl. 39) to describe the 'ultimate limit state' of compression members (whether axially loaded or eccentrically loaded), the actual failure need not necessarily occur in compression. This is because an eccentrically loaded column section is subjected to an axial compression ($P_u$) as well as a bending moment ($M_u$).

The mode of failure depends on the eccentricity of loading; i.e., the relative magnitudes of $P_u$ and $M_u$. If the eccentricity $e = M_u/P_u$ is relatively small, the axial compression behaviour predominates, and the consequent failure is termed *compression failure*. On the other hand, if the eccentricity is relatively large, the flexural behaviour predominates, and the consequent failure is termed *tension failure*. In fact, depending on the exact magnitude of the loading eccentricity $e$, it is possible to predict whether a 'compression failure' or a 'tension failure' will take place.

### *Balanced Failure*

In between 'compression failure' and 'tension failure', there exists a critical failure condition, termed 'balanced failure'. This failure condition refers to that ultimate limit state wherein the yielding of the outermost row of longitudinal steel on the tension side and the attainment of the maximum compressive strain in concrete $\varepsilon_{cu}$ = 0.0035 at the highly compressed edge of the column occur simultaneously. In other words, both crushing of concrete (in the highly compressed edge) and yielding of steel (in the outermost tension steel) occur simultaneously. In this context, for design purpose, the 'yield strain' $\varepsilon_y$ is defined simply as that corresponding to the conventional definition of 'yield point' in the design stress-strain curve for steel [refer Fig. 3.6, 3.7], i.e.,

$$\varepsilon_y = \begin{cases} 0.87 f_y / E_s & \text{for Fe 250} \\ 0.87 f_y / E_s + 0.002 & \text{for Fe 415/Fe 500} \end{cases} \tag{13.19}$$

The 'balanced strain profile' is depicted, along with other strain profiles in Fig. 13.11. The corresponding eccentricity in loading is denoted $e_b \equiv e_{x_u = x_{u,b}}$; i.e., the eccentricity which results in a 'balanced' neutral axis depth $x_u = x_{u,\,b}$. Evidently, $e_D < e_b < \infty$, where, as explained earlier with reference to Fig. 13.11, $e_D$ corresponds to a neutral axis depth $x_u = D$ and $e = \infty$ corresponds to a minimum neutral axis depth $x = x_{u,\,min}$ (when $P_u = 0$).

### *Compression Failure*

When the loading eccentricity is less than that corresponding to the 'balanced failure' condition, i.e., when $e < e_b$, 'yielding' of longitudinal steel in tension does not take place, and failure occurs at the ultimate limit state by crushing of concrete at the highly compressed edge. The compression reinforcement may or may not yield, depending on the grade of steel and its proximity to the highly compressed edge.

### *Tension Failure*

When the loading eccentricity is greater than that corresponding to the 'balanced failure' condition, i.e., when $e > e_b$, failure will be initiated by the yielding of the tension steel. The outermost longitudinal bars in the tension side of the neutral axis first undergo yielding and successive inner rows (if provided), on the tension side of the neutral axis, may also yield in tension with increasing strain. Eventually, collapse occurs when the concrete at the highly compressed edge gets crushed.

## 13.5.3 Design Strength: Axial Load - Moment Interaction

The design strength of an eccentrically loaded short column depends on the eccentricity of loading. For uniaxial eccentricity, $e$, the design strength (or resistance) has two components: an axial compression component, $P_{uR}$, and a corresponding uniaxial moment component, $M_{uR} = P_{uR}\, e$.

As seen in Section 13.5.1, there exists a unique strain profile (and neutral axis location) at the ultimate limit state, corresponding to a given eccentricity of loading

[Fig. 13.11]. Corresponding to this distribution of strains ('strain compatibility'), the stresses in concrete and steel, and hence, their respective resultant forces[†] $C_c$ and $C_s$, can be determined. Applying the condition of static equilibrium, it follows that the two design strength components are easily obtainable as:

$$P_{uR} = C_c + C_s \tag{13.20}$$

and $$M_{uR} = M_c + M_s \tag{13.21}$$

where $M_c$ and $M_s$ denote the resultant moments due to $C_c$ and $C_s$ respectively, with respect to the centroidal axis (principal axis under consideration).

From the nature of the equilibrium equations [Eq. 13.20, 13.21], it may be observed that, for a given location of the neutral axis ($x_u/D$), the design strength values $P_{uR}$ and $M_{uR}$ can be directly determined, and the eccentricity $e = M_{uR}/P_{uR}$ resulting in such a NA location can be deduced. However, given an arbitrary value of $e$, it is possible to arrive at the design strength ($P_{uR}$ or $M_{uR} = P_{uR}\, e$) using Eq. 13.20, only after first locating the neutral axis — which can be achieved by considering moments of forces $C_c$ and $C_s$ about the eccentric line of action of $P_{uR}$, and applying static equilibrium. Unfortunately, the expressions for $C_c$ and $C_s$ in terms of $x_u$ (derived in Section 13.5.4) are such that, in general, it will not be possible to obtain a closed-form solution for $x_u$ in terms of $e$. The relationship is highly nonlinear, requiring a trial-and-error solution.

### *Interaction Curve*

The 'interaction curve' is a complete graphical representation of the design strength of a uniaxially eccentrically loaded column of given proportions. Each point on the curve corresponds to the design strength values of $P_{uR}$ and $M_{uR}$ associated with a specific eccentricity ($e$) of loading. That is to say, if load $P$ is applied on a short column with an eccentricity $e$, and if this load is gradually increased till the ultimate limit state (defined by the Code) is reached, and that ultimate load at failure is given by $P_u = P_{uR}$ and the corresponding moment by $M_u = M_{uR} = P_{uR}\, e$, then the coordinates ($M_{uR}$, $P_{uR}$)[†] form a unique point on the interaction diagram (such as point '2' in Fig. 13.12). The interaction curve defines the different ($M_{uR}$, $P_{uR}$) combinations for all possible eccentricities of loading $0 \le e < \infty$. For design purposes, the calculations of $M_{uR}$ and $P_{uR}$ are based on the *design* stress-strain curves (including the partial safety factors), and the resulting interaction curve is sometimes referred to as the *design interaction curve* (which is different from the *characteristic* interaction curve).

Using the design interaction curve for a given column section, it is possible to make a quick judgement as to whether or not the section is 'safe' under a specified *factored* load effect combination ($P_u$, $M_u$). If the point given by the coordinates ($M_u$, $P_u$) falls within the design interaction curve, the column is 'safe'; otherwise, it is not.

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[†] Some of the longitudinal steel may be subjected to tension, rather than compression. The term $C_s$ here denotes the net force (assumed positive if compressive) considering all the bars in the section.

[†] It is customary to use the $x$ - axis for $M_u$ values and the $y$ - axis for $P_u$ values.

In other words, the design interaction curve serves as a *failure envelope*. Of course, it must be appreciated that by the term 'safe', all that is implied is that the risk of failure is deemed by the Code to be acceptably low. It does *not* follow (as some designers are inclined to believe), that if the point ($M_u$, $P_u$) falls outside the failure envelope, the column *will* fail!

### Salient Points on the Interaction Curve

The salient points, marked 1 to 5 on the interaction curve [Fig. 13.12] correspond to the failure strain profiles, marked 1 to 5 in Fig. 13.11:

- The point 1 in Fig. 13.12 corresponds to the condition of axial loading with $e = 0$. For this case of 'pure' axial compression, $M_{uR} = 0$ and $P_{uR}$ is denoted as $P_{uo}$ (given by Eq. 13.16).

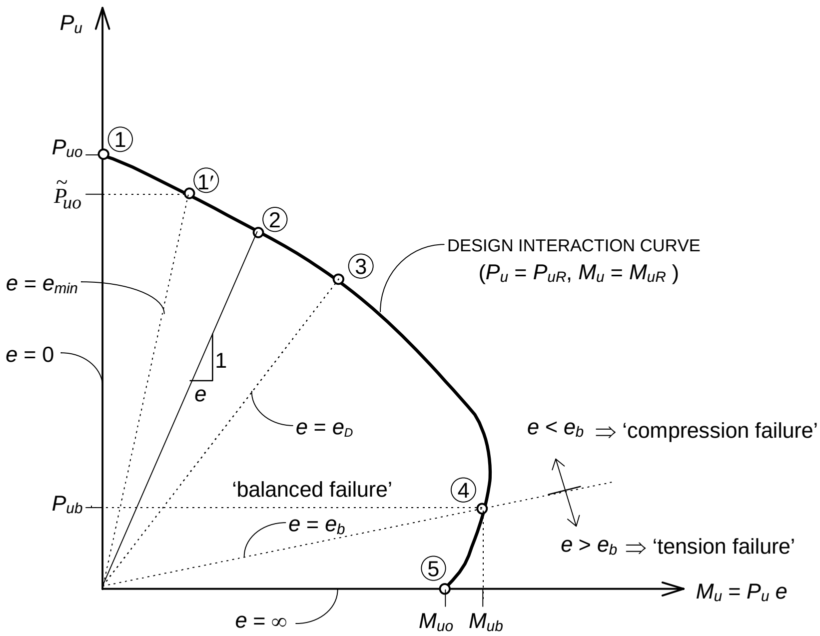


**Fig. 13.12** Typical $P_u$ — $M_u$ interaction diagram

- The point 1′ in Fig. 13.12 corresponds to the condition of axial loading with the mandatory minimum eccentricity $e_{min}$ [prescribed by the Code (Cl. 25.4 and 39.3)]. The corresponding ultimate resistance is approximately given by $\tilde{P}_{uo}$ (Eq. 13.17).
- The point 3 in Fig. 13.12 corresponds to the condition $x_u = D$ [refer Fig. 13.11], i.e., $e = e_D$. For $e < e_D$, the entire section is under compression and the neutral axis is located outside the section ($x_u > D$), with $0.002 < \varepsilon_{cu} < 0.0035$. For $e > e_D$, the NA is located within the section ($x_u < D$) and $\varepsilon_{cu} = 0.0035$ at the 'highly

compressed edge' [Fig. 13.11]. Point 2 represents a *general* case, with the neutral axis outside the section ($e < e_D$ ).

- The point 4 in Fig. 13.12 corresponds to the *balanced failure* condition, with $e = e_b$ and $x_u = x_{u,\,b}$ [refer Fig. 13.11]. The design strength values for this 'balanced failure' condition are denoted as $P_{ub}$ and $M_{ub}$. For $P_{uR} < P_{ub}$ (i.e., $e > e_b$), the mode of failure is called *tension failure*, as explained earlier. It may be noted that $M_{ub}$ is close to the maximum‡ value of ultimate moment of resistance that the given section is capable of, and this value is higher than the ultimate moment resisting capacity $M_{uo}$ under 'pure' flexure conditions [point 5 in Fig. 13.12].
- The point 5 in Fig. 13.12 corresponds to a 'pure' bending condition ($e = \infty$, $P_{uR} = 0$); the resulting ultimate moment of resistance is denoted $M_{uo}$ and the corresponding NA depth takes on a minimum value $x_{u,\,min}$.

### 13.5.4 Analysis for Design Strength

In this section, the detailed calculations for determining the *design strength* of a uniaxially eccentrically loaded column with a rectangular cross-section ($b \times D$) is described in detail. The notation $D$ denotes the 'depth' of the rectangular section in the plane of bending, i.e., either $D_x$ or $D_y$, depending on whether bending occurs with respect to the major axis or minor axis, and the notation $b$ denotes the 'breadth' (width) of the section (in the perpendicular direction). The basic procedure for other cross-sectional shapes (including circular sections) is similar, and this is demonstrated in Example 13.8 for an H-shaped section. This procedure can also be extended to large tubular towers (such as chimneys), albeit with some modifications [Ref. 13.11].

As explained in Section 13.5.3, the design strength of an eccentrically loaded column is not a unique value, but comprises infinite sets of values of $P_{uR}$ and $M_{uR}$ (corresponding to $0 < e < \infty$) — all of which are describable by means of a single curve, termed the *design interaction curve* [Fig. 13.12]. It was also pointed out that the analysis for design strength basically entails two conditions: strain compatibility [Fig. 13.11] and equilibrium [Eq. 13.20, 13.21].

The distribution of strains in the rectangular column section and the corresponding (compressive) stresses in concrete are depicted in Fig. 13.13. Two different cases need to be distinguished. In the first case [Fig. 13.13(a)], the loading eccentricity is relatively high [e > $e_D$ in Fig. 13.11], such that the neutral axis is located inside the column section ($x_u \leq D$). In the second case [Fig. 13.13(b)], the loading eccentricity is relatively low [e < $e_D$ in Fig. 13.11], such that the NA is located outside the section. In both cases, the force/moment equilibrium equations, described by Eq. 13.20 and 13.21, remain valid; however, the formulas for $C_c$, $C_s$, $M_c$ and $M_s$ involve parameters that have different expressions for the two cases.

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‡ $M_{uo}$ corresponds to the ultimate moment of resistance of an under–reinforced beam section. The presence of some axial compression delays the yielding of the tension steel (and hence, the development of the ultimate limit state), thereby enhancing the moment resisting capacity beyond $M_{uo}$. However, the presence of axial compression also enhances the compressive stress in concrete, and the gain in $M_{uR}$ due to delayed yielding of tension steel becomes offset by the loss in $M_{uR}$ due to hastening of the compression failure condition, when $P_{uR}$ exceeds $P_{ub}$.

Generalised expressions for the resultant force in concrete ($C_c$) as well as its moment ($M_c$) with respect to the centroidal axis of bending may be derived as follows, based on Fig. 13.13:

$$C_c = a\, f_{ck}\, bD \tag{13.22}$$

$$M_c = C_c (D/2 - \bar{x}) \tag{13.23}$$

where $a \equiv$ stress block area factor

$\bar{x} \equiv$ distance between highly compressed edge and the line of action of $C_c$ (i.e., centroid of stress block area)

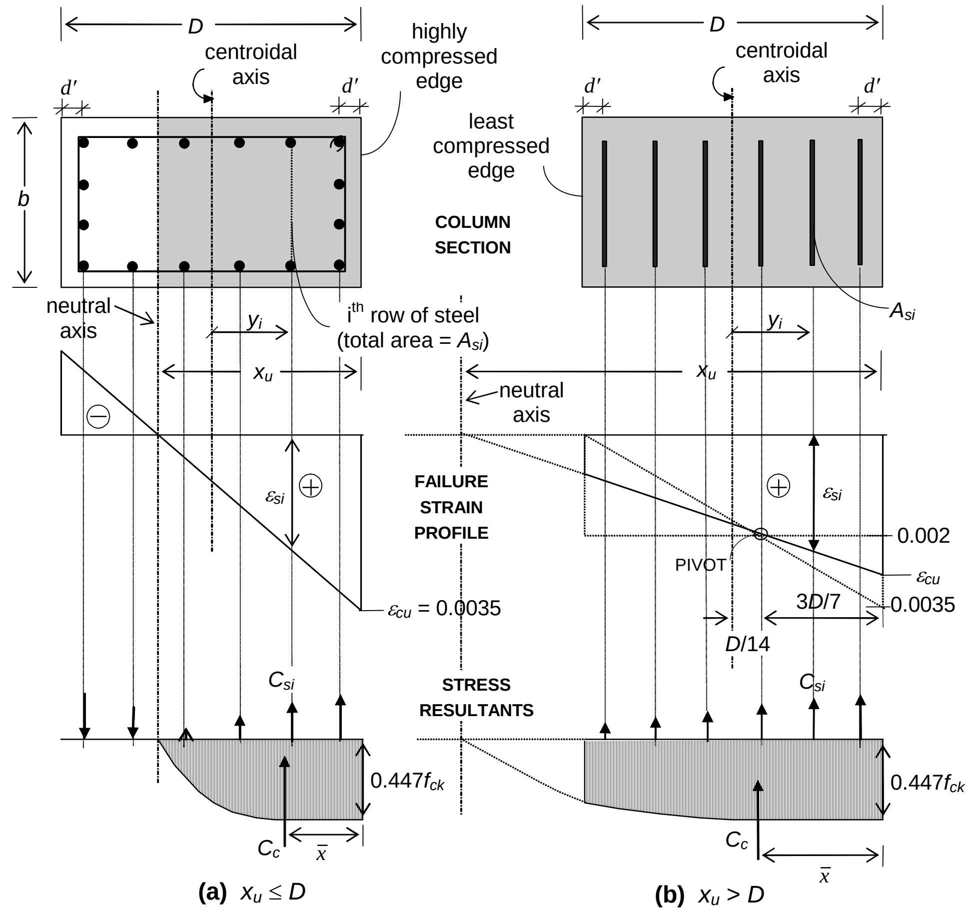



**Fig. 13.13** Analysis of design strength of a rectangular section under eccentric compression

By means of simple integration, it is possible to derive expression for $a$ and $\bar{x}$ for the case (a): $x_u \le D$ [refer Section 4.7] and for the case (b): $x_u > D$ [Ref. 13.12]:

$$a = \begin{cases} 0.362\, x_u / D & \text{for } x_u \le D \\ 0.447(1 - 4g/21) & \text{for } x_u > D \end{cases} \tag{13.24}$$

$$\bar{x} = \begin{cases} 0.416 x_u & \text{for } x_u \le D \\ (0.5 - 8g/49)\{D/(1 - 4g/21)\} & \text{for } x_u > D \end{cases} \tag{13.25}$$

where $$g = 16 \big/ (7x_u/D - 3)^2 \tag{13.26}$$

Similarly, the expressions for the resultant force in the steel ($C_s$) as well as its moment ($M_s$) with respect to the centroidal axis of bending is easily obtained as:

$$C_s = \sum_{i=1}^{n} (f_{si} - f_{ci}) A_{si} \tag{13.27}$$

$$M_s = \sum_{i=1}^{n} (f_{si} - f_{ci}) A_{si}\, y_i \tag{13.28}$$

where

$A_{si}$ ≡ area of steel in the i$^{th}$ row (of $n$ rows) [refer Fig. 13.13];

$y_i$ ≡ distance of i$^{th}$ row of steel from the centroidal axis, measured positive in the direction towards the highly compressed edge;

$f_{si}$ ≡ design stress in the i$^{th}$ row (corresponding to the strain $\varepsilon_{si}$) obtainable from design stress-strain curves for steel;

$\varepsilon_{si}$ ≡ strain in the i$^{th}$ row, obtainable from strain compatibility conditions ($\varepsilon_{si}$ and $f_{si}$ are assumed to be positive if compressive, and negative if tensile);

$f_{ci}$ ≡ design compressive stress level in concrete, corresponding to the strain $\varepsilon_{ci} = \varepsilon_{si}$ adjoining the i$^{th}$ row of steel, obtainable from the design stress-strain curve for concrete [Fig. 3.5] [Note: $f_{ci} = 0$ if the strain is tensile]:

$$f_{ci} = \begin{cases} 0 & \text{if } \varepsilon_{si} \le 0 \\ 0.447 f_{ck} & \text{if } \varepsilon_{si} \ge 0.002 \\ 0.447 f_{ck} \left[ 2(\varepsilon_{si}/0.002) - (\varepsilon_{si}/0.002)^2 \right] & \text{otherwise} \end{cases} \tag{13.29}$$

Also, from Fig. 13.13, it can be observed (applying similar triangles) that:

$$\varepsilon_{si} = \begin{cases} 0.0035 \left[ \dfrac{x_u - D/2 + y_i}{x_u} \right] & \text{for } x_u \le D \\ 0.002 \left[ 1 + \dfrac{y_i - D/14}{x_u - 3D/7} \right] & \text{for } x_u > D \end{cases} \tag{13.30}$$

It should be noted that, in the case of *spiral columns*, the Code permits an enhancement in the design strength (both $P_{uR}$ and $M_{uR}$) by 5 percent — for short columns only.

### EXAMPLE 13.5

For the column section shown in Fig. 13.14(a), determine the design strength components corresponding to the condition of 'balanced failure'. Assume M 25 concrete and Fe 415 steel. Consider loading eccentricity with respect to the *major axis* alone. Assume 8 $\phi$ ties and 40 mm clear cover.

### SOLUTION

- *Given*: $b$ = 300 mm, $D$ = 500 mm, $f_{ck}$ = 25 MPa, $f_y$ = 415 MPa,

$A_{s1} = A_{s2} = A_{s3} = 2 \times 491$ mm$^2$ = 982 mm$^2$, [as shown in Fig. 13.14(b)],
$y_1$ = (–)189.5 mm, $y_2$ = 0 mm, $y_3$ = (+)189.5 mm with reference to centroidal axis

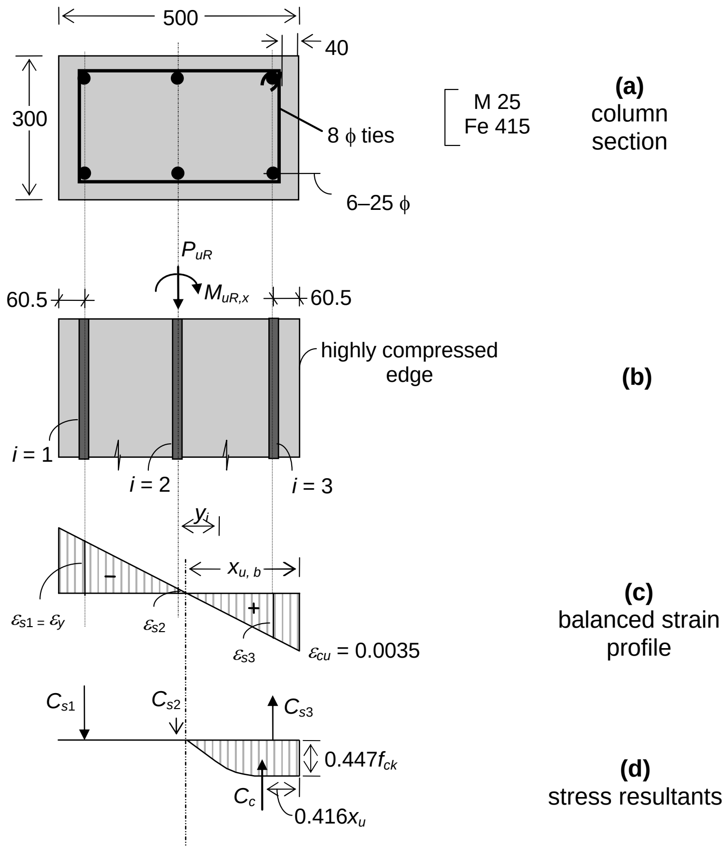



**Fig. 13.14** Example 13.5

- *Neutral axis depth $x_{u,b}$*

The 'balanced strain' profile is as shown in Fig. 13.14(c).

For Fe 415 steel, $\varepsilon_y = \dfrac{0.87 \times 415}{2 \times 10^5} + 0.002 = 0.003805$

Considering similar triangles [Fig. 13.14(c)],

$$x_{u,b} = \frac{0.0035\times(500-60.5)}{0.0035+0.003805} = 210.6 \text{ mm } (< D/2 = 250 \text{ mm})$$

- *Strains in steel* (3 rows): Considering Fig. 13.14(b), (c),

  $\varepsilon_{s1} = (–)\varepsilon_y = – 0.003805$ (tensile)

  $$\varepsilon_{s2} = (–)0.0035 \times \frac{250-210.6}{210.6} = – 0.000655 \text{ (tensile)}$$

  $$\varepsilon_{s3} = (+)0.0035 \times \frac{210.6-60.5}{210.6} = + 0.002495 \text{ (compression)} > 0.002$$

- *Design stresses in steel* (3 rows): Referring to the design stress-strain curve for Fe 415 [Fig. 3.7, Table 3.2],

  $f_{s1} = (–)0.87f_y = –360.9$ MPa

  $f_{s2} = E_s\ \varepsilon_{s2} = (2 \times 10^5) \times (–)0.000581 = –131$ MPa

  $$f_{s3} = +[342.8 + \left(\frac{249.5-241}{276-241}\right) \times (351.8 – 342.8)] = +345 \text{ MPa}$$

- *Design strength component in axial compression*: $P_{ub,x}$

  $P_{ub,x} = C_c + C_s$ [refer Fig. 13.14(d)]

  $C_c = 0.362\ f_{ck}\ b\ x_{u,b} = 0.362 \times 25 \times 300 \times 210.6 = 571779$ N

  $$C_s = \sum_{i=1}^{3} C_{si} = \sum (f_{si} - f_{ci})A_{si} \quad \text{[Eq. 13.27]}$$

  $= [(–360.9) + (–131) + (345 – 0.447 \times 25)] \times 982$

  $= –155230N$ (tensile)

  $\Rightarrow P_{ub,x} = (571.8– 155.2)$ kN = **416.6 kN**

- *Design strength component in flexure*: $M_{ub,x}$

  $M_{ub,x} = M_c + M_s$

  where, considering moments of stress resultants about the centroidal axis,

  $M_c = C_c\ (0.5D – 0.416x_u)$

  $= 571779 \times (250 – 0.416 \times 210.6) = 92.85 \times 10^6$ Nmm

  $M_s = \sum C_{si} y_i$

  $= [(–360.9)(–189.5 ) + (–131)(0) + (345 – 0.447 \times 25)(189.5 )] \times 982$

  $= 129.3 \times 10^6$Nmm

  $\Rightarrow M_{ub,x} = (92.85 + 129.3)$ kNm = **222.15 kNm**

  [Note: $e_{b,x} = \dfrac{M_{ub,x}}{P_{ub,x}} = \dfrac{222.15\times10^3}{416.6} = 533.2$ mm $\Rightarrow (e/D)_x = 1.066$ and corresponding $x_{u,b}/D = 210.6/500 = 0.4212$. This implies that *tension failure* occurs only if $(e/D)_x > 1.066$ or $x_u/D < 0.4212$].

**EXAMPLE 13.6**

For the column section shown in Fig. 13.14(a), determine the design strength components corresponding to a neutral axis location given by $x_u/D$ =1.2. Consider loading eccentricity with respect to the *major axis* alone.

**SOLUTION**

- *Given*: data as in Example 13.5 [Figs. 13.14(a), (b)].
- *Neutral axis depth* $x_u$ = 1.2 × 500 = 600 mm
  As the NA falls outside the section, the entire section is under compression, and the corresponding failure strain diagram is as shown in Fig. 13.13(b).

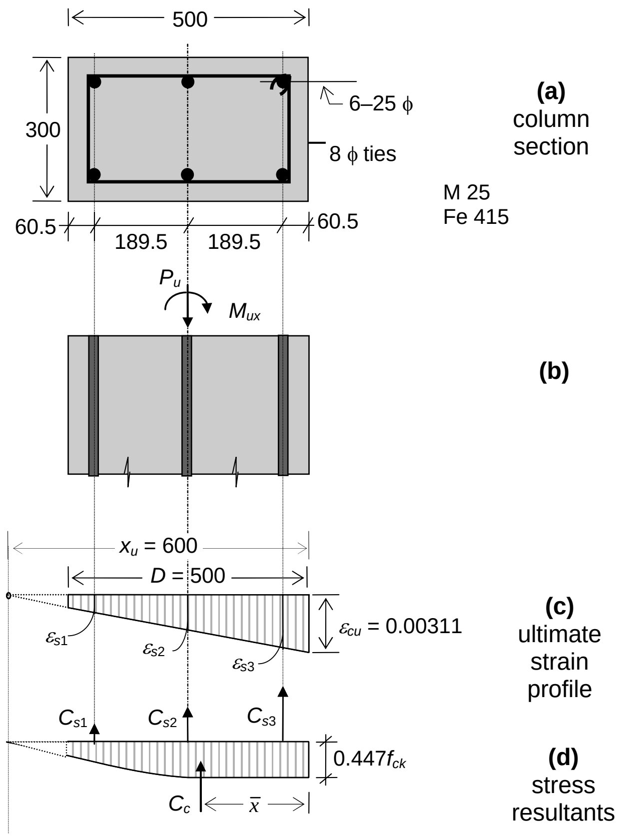



**Fig. 13.15** Example 13.6

$$\varepsilon_{cu} = 0.002\left[1+\frac{3D/7}{x_u-3D/7}\right] = 0.002\left[1+\frac{1500/7}{600-1500/7}\right] = 0.003111$$

- *Strains in steel*

  From the distribution of failure strains shown in Fig. 13.15(a), by applying similar triangles,

  $\varepsilon_{s1}$ = (+)0.003111 × 160.5/600 = + 0.000832 < 0.002

  $\varepsilon_{s2}$ = (+)0.003111 × 350/600 = + 0.001815 < 0.002

  $\varepsilon_{s3}$ = (+)0.003111 × (600 – 60.5)/600 = + 0.002797 > 0.002

  [Note that these values can alternatively be obtained by applying the generalised formula given by Eq. 13.30]

- *Design stresses in steel*

  Referring to the design stress-strain curve for Fe 415 [Fig. 3.7, Table 3.2],

  $f_{s1} = (2 \times 10^5) \times (+)0.000832 = +\ 166.4$ MPa

  $f_{s2} = +\ [306.7 + \dfrac{181.5-163}{192-163} \times (324.8 - 306.7)] = +\ 318.2$ MPa

  $f_{s3} = +\ [351.8 + \dfrac{279.7-276}{380-276} \times (360.9 - 351.8)] = +\ 352.1$ MPa

- *Design strength component in axial compression*: $P_{uR}$

  $P_{uR} = C_c + C_s$ [refer Fig. 13.15(b)]

  The properties of the truncated stress block have to be considered [Eq. 13.24 – 13.26] as $x_u > D$:

  $g = 16/(7x_u/D - 3)^2$ [Eq. 13.26]

  $= 16/(7 \times 1.2 - 3)^2 = 0.5487$

  $a = 0.447 \times (1 - 4g/21)$ [Eq. 13.24]

  $= 0.447 \times 0.8955 = 0.4003$

  $\therefore\ C_c = a\, f_{ck}\, b\, D = 0.4003 \times 25 \times 300 \times 500 = 1501125$ N

  $$C_s = \sum_{i=1}^{3} (f_{si} - f_{ci})A_{si}$$

  $= [(166.4 - 0.295^{\dagger} \times 25) + (318.2 - 0.443^{\ddagger} \times 25) + (352.5 - 0.447 \times 25)] \times 982$

  $= 792940N$

  $\Rightarrow P_{uR} = (1501.1 + 792.9)$ kN = **2294 kN**

- *Design strength component in flexure*: $(M_{uR})_x$

  $(M_{uR})_x = M_c + M_s$

  where, considering moments of resultant forces about the centroidal axis,

  $M_c = C_c\,(0.5D - \bar{x})$

  $\bar{x} = (0.5 - 8g/49)\{D/(1 - 4g/21)\}$ [Eq. 13.25]

  $= (0.5 - 8 \times 0.5487/49)(500/0.8955)$

  $= 229.2$ mm

  $\Rightarrow M_c = 1501125\,(0.5 \times 500 - 229.2) = 31.22 \times 10^6$ Nmm

---

† Applying Eq. 13.29 with $\varepsilon_{s1}$ = + 0.000832, $f_{c1} = 0.447 f_{ck} \times 0.6589 = 0.295\, f_{ck}$.

‡ Applying Eq. 13.29 with $\varepsilon_{s2}$ = + 0.001815, $f_{c2} = 0.447 f_{ck} \times 0.9914 = 0.443\, f_{ck}$.

$$M_s = \sum_{i=1}^{3} (f_{si} - f_{ci}) A_{si}\, y_i$$

$= [(166.4 - 0.295 \times 25)(-189.5) + 0 + (352.5 - 0.447 \times 25)(+189.5)] \times 982$

$= 33.92 \times 10^6$ Nmm

$\Rightarrow (M_{uR})_x = (31.22 + 33.92)$ kNm = **65.1 kNm**

[Note that $e_x = \dfrac{(M_{uR})_x}{P_{uR}} = \dfrac{65.1 \times 10^3}{2294}$ = 28.4 mm $\Rightarrow (e/D)_x = 0.057$,

corresponding to $x_u/D = 1.2$]

## EXAMPLE 13.7

For the column section shown in Fig. 13.14(a), determine the design strength components and corresponding eccentricity of loading with respect to the *minor axis* alone, for the limiting condition of 'no tension' in the section.

### SOLUTION

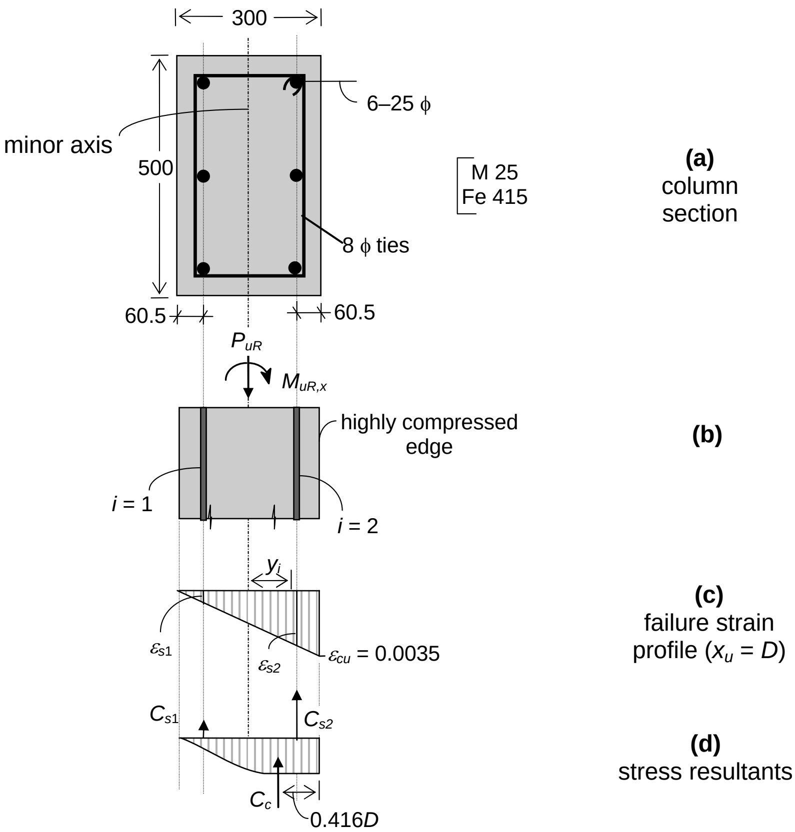



**Fig. 13.16** Example 13.6

- For bending about the *minor axis*, the column details are as shown in Fig. 13.16(a). In this case, $b$ = 500 mm, $D$ = 300 mm. There are only two rows of steel.
  $A_{s1} = A_{s2} = 3 \times 491\ \text{mm}^2 = 1473\ \text{mm}^2$
  $y_1$ = (–)89.5 mm and $y_2$ = (+)89.5 mm with reference to the centroidal axis.
- *Neutral axis depth*
  $x_u$ = 300 mm, as shown in Fig. 13.16(c) for the limiting strain profile for 'no tension' (i.e., $x_u = D$)
- *Strains in steel* (2 rows): Considering Fig. 13.16(c),

  $$\varepsilon_{s1} = (+)0.0035 \times \frac{60.5}{300} = +\,0.000706 \qquad < 0.002$$

  $$\varepsilon_{s2} = (+)0.0035 \times \frac{300-60.5}{300} = +\,0.00279 \qquad > 0.002$$
- *Design stresses in steel* (2 rows):
  Referring to the design stress-strain curve for Fe 415 [Fig. 3.7, Table 3.2],
  $f_{s1} = (2 \times 10^5) \times (+)0.000706 = +141.2$ MPa

  $$f_{s2} = +[351.8 + \frac{279-276}{380-276} \times (360.9 - 351.8)] = +352.0 \text{ MPa}$$
- *Design strength component in axial compression*: $P_{uR}$
  $P_{uR} = C_c + C_s$ [refer Fig. 13.16(d)]
  $C_c = 0.362 \times 25 \times 500 \times 300 = 1357500$ N
  $C_s = C_{s1} + C_{s2} = [(141.2 - 0.259^{\dagger} \times 25) + (352.0 - 0.447 \times 25)] \times 1473$
  $= 700485$ N
  $\Rightarrow P_{uR} = (1357.5 + 700.5)$ kN = **2058 kN**
- *Design strength component in flexure*: $M_{uR,\,y}$
  $M_{uR,\,y} = M_c + M_s$
  where, considering moments of stress resultants about the centroidal axis,
  $M_c = 1357500 \times (0.5 \times 300 - 0.416 \times 300) = 34.21 \times 10^6$ Nmm
  $M_s = C_{s1}\,y_1 + C_{s2}\,y_2$
  $= [(141.2 - 0.259 \times 25)(-89.5) + (352.0 - 0.447 \times 25) \times (+89.5)] \times 1473$
  $= 27.17 \times 10^6$ Nmm
  $\Rightarrow M_{uR,\,y} = (34.21 + 27.17)$ kNm = **61.38 kNm**
- *Eccentricity $e_{D,y}$ corresponding to 'no tension' limit*

  $$e_{D,y} = \frac{(M_{uR})_y}{P_{uR}} = \frac{61.38 \times 10^3}{2058} = 29.82 \text{ mm}$$

  [$\Rightarrow (e/D)_y = 29.82/300 = 0.0994$, implying that the entire section will be under compression at the ultimate limit state if $(e/D)_y < 0.0994$].

---

† Applying Eq. 13.29 with $\varepsilon_{s1} = +\,0.000706$, $f_{c1} = 0.447 f_{ck} \times 0.5814 = 0.259 f_{ck}$.

**EXAMPLE 13.8**

For the H-shaped column section shown in Fig. 13.17(a), determine the design strength components corresponding to a neutral axis location given by $x_u/D$ = 0.75. Consider loading eccentricity with respect to the *major axis* alone. Assume M 30 concrete and Fe 415 steel.

**SOLUTION**

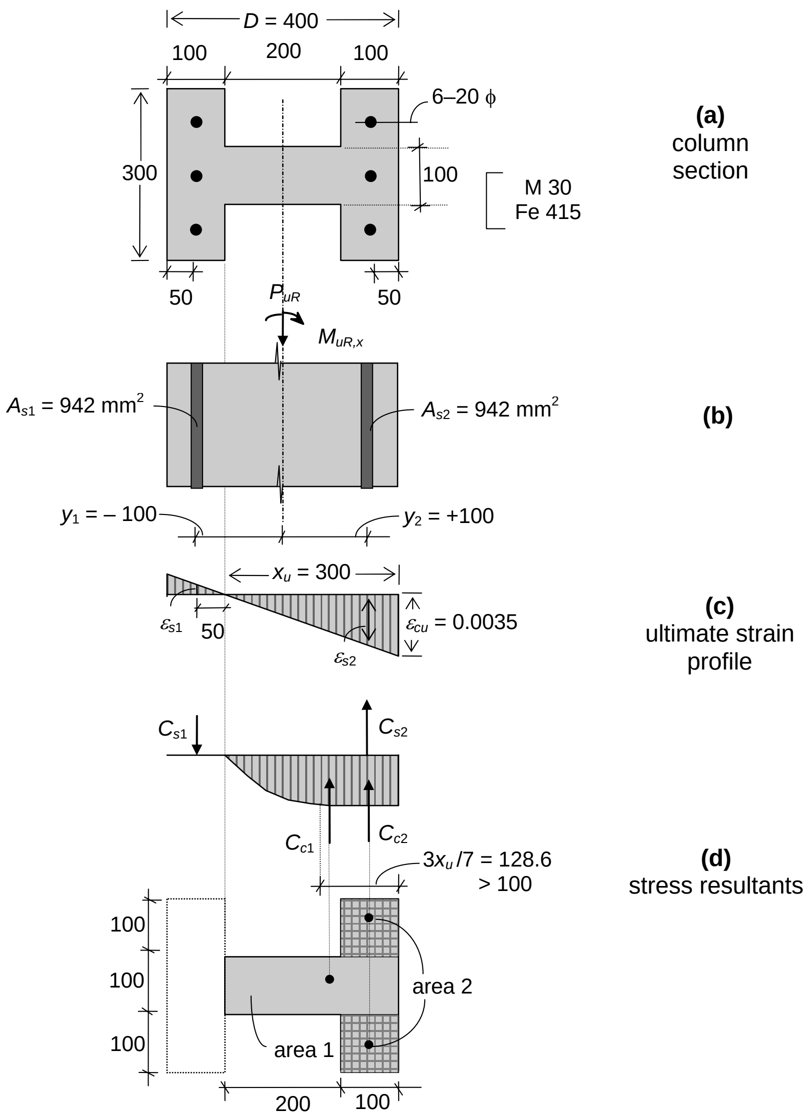



**Fig. 13.17** Example 13.8

- *Given:* data as indicated in Fig. 13.17(a), (b)
  $A_{s1} = A_{s2} = 314 \times 3 = 942 \text{ mm}^2$ ; $y_1 = -150$ mm, $y_2 = +150$ mm,
  $f_{ck} = 30$ MPa $f_y = 415$ MPa
  $D = 400 \text{ mm} \Rightarrow x_u = 0.75 \times 400 = 300$ mm
- *Strains in steel* (2 rows)
  Referring to strain profile shown in Fig. 13.17(c), $\varepsilon_{cu} = 0.0035$ (as $x_u < D$),
  $\varepsilon_{s1} = (-)0.0035 \times 50/300 = -\ 0.000583$ (tensile)
  $\varepsilon_{s2} = (+)0.0035 \times 250/300 = +\ 0.002917$ (compressive)
- *Design stresses in steel*
  Referring to the design stress-strain curve for Fe 415 [Fig. 3.7, Table 3.2],
  $f_{s1} = (2 \times 10^5) \times (-)0.000583 = -116.6$ MPa
  $$f_{s2} = +[351.8 + \frac{291.7 - 276}{380 - 276} \times (360.9 - 351.8)] = +353.2 \text{ MPa}$$
- *Design strength component in axial compression*: $P_{uR}$
  $P_{uR} = (C_{c1} + C_{c2}) + (C_{s1} + C_{s2})$, as shown in Fig. 13.17(d)
  $C_{c1} = 0.362 \times 30 \times 100 \times 300 = 325800$ N
  $C_{c2} = 0.447 \times 30 \times 200 \times 100 = 268200$ N
  $C_{s1} = -116.6 \times 942 = -109837$ N
  $C_{s2} = +(353.2 - 0.447 \times 30) \times 942 = +320082$ N
  $\Rightarrow P_{uR} = (325.80 + 268.20 - 109.84 + 320.08)$ kN = **804.2 kN**
- *Design strength component in flexure*: $M_{uR,\,x}$
  Referring to Fig. 13.17(d), taking moments of stress resultants about the centroidal axis of bending,
  $$\begin{aligned} M_{uR,\,x} &= C_{c1}(0.5D - 0.416x_u) + C_{c2}(0.5D - 50) + C_{s1}\, y_1 + C_{s2}\, y_2 \\ &= 325800(200 - 0.416 \times 300) + 268200\,(200 - 50) + (-109837)(-150) \\ &\qquad + (320082)(150) \\ &= (24.50 + 40.23 + 16.48 + 48.01) \times 10^6 \text{ Nmm} \\ &= \mathbf{129.2\ kNm} \end{aligned}$$
- *Corresponding eccentricity* $e_x = \dfrac{M_{uR},_x}{P_{uR}} = \dfrac{129.2 \times 10^3}{804.2} = 135.8$ mm

  [$\Rightarrow (e/D)_x = 135.8/400 = 0.339$, corresponding to $x_u/D = 0.75$].

### 13.5.5 USE OF INTERACTION DIAGRAM AS AN ANALYSIS AID

*Analysis* of the strength of a given column section basically implies determination of its design strength components $P_{uR}$ and $M_{uR}$ — with the objective of assessing the safety of the column section subjected to specified factored load effects $P_u$ and $M_u$ (i.e., either $M_{ux}$ or $M_{uy}$). It should be noted that $\{P_{uR}, M_{uR}\}$ denotes the *resistance* inherent in the column section, whereas $\{P_u, M_u\}$ denotes the *load effects* induced in the section by the action of external *factored* loads on the structure. The point $\{P_{uR}, M_{uR}\}$ lies on the design 'interaction curve', whereas the point $\{P_u, M_u\}$ is any point on the 'interaction diagram' (i.e., in the two-dimensional space bounded by the two coordinate axes). Various combinations of factored axial compression $P_u$ and factored uniaxial moment $M_u$ will act on the column section due to different factored loading patterns on the structure. As explained in Chapter 9, it generally suffices to consider two critical combinations of $P_u$ and $M_u$, viz. (i) maximum $P_u$ along with the corresponding $M_u$, and (ii) $P_u$ and $M_u$ corresponding to maximum eccentricity $e = M_u/P_u$. These critical load effects are obtainable from structural analyses of the structure (of which the column under consideration is a part) under different loading patterns (gravity loads, lateral loads) [refer Section 9.2].

Thus, in effect, the column strength analysis problem reduces to determining whether a given column section, subjected to given factored load effects $\{P_u, M_u\}$, is 'safe' or not. One way of checking this is by determining the design strength $\{P_{uR}, M_{uR}\}$, corresponding to the applied eccentricity $e = M_u/P_u$, and if $P_{uR} \geq P_u$ *and* $M_{uR} \geq M_u$, the column section can be considered safe†, according to the Code. An alternative method of checking safety is by assuming that the ultimate limit state has been reached under the factored load $P_u$, i.e., $P_{uR} = P_u$, and then comparing the corresponding ultimate moment of resistance $M_{uR}$ with the applied factored moment $M_u$; if $M_{uR} \geq M_u$, the column section is 'safe'.

Regardless of the criterion used for checking safety of a column section under factored load effects $\{P_u, M_u\}$, a trial-and-error type of procedure has to be adopted, if calculations are to be based on first principles. The basic stress resultants $C_c$ and $C_s$ (in Eq. 13.22(c) and 13.23) are expressed in terms of an unknown neutral axis depth ($x_u$), and the nature of this relationship is too complicated to enable a closed-form solution for $x_u$ — by solving a suitable equilibrium equation for a given $P_{uR}$ (Eq. 13.20) or a given eccentricity† $e$.

Indeed, as mentioned in the Code (Note to Cl. 39.5):

> "The design of a member subject to combined axial load and uniaxial bending will involve lengthy calculation by trial and error. In order to overcome these difficulties, interaction diagrams may be used".

Although the above statement refers to *design* (discussed in Section 13.5.6), it is equally valid in the case of *analysis*. If an *interaction diagram* [refer Fig. 13.12] is readily available (or can be constructed) for the given column section, then the analysis problem simply reduces to determining whether or not the point corresponding to factored load effects $\{P_u, M_u\}$ lies within the envelope of the interaction curve, as explained in Section 13.5.3. Furthermore, the design strength components $\{P_{uR}, M_{uR}\}$ can be easily read off from the interaction curve, corresponding to any given eccentricity $e$, or given $P_u$ (= $P_{uR}$).

---

† By the term 'safe', it is only implied that the risk of failure (measured in terms of *probability of failure*) is acceptably low [refer Chapter 3].

† For a given eccentricity $e$, Eq. 13.20 is not suitable as it involves an unknown $x_u$ as well as an unknown $P_{uR}$. In this case, it is best to construct a moment equilibrium equation with reference to the eccentric line of action of $P_{uR}$, thereby eliminating $P_{uR}$.

### *Construction of a Design Interaction Curve*

The coordinates of the 'design interaction curve', viz. $M_{uR}$ (on the $x$-axis) and $P_{uR}$ (on the $y$-axis), can be determined for any arbitrary neutral axis depth $x_u$, using Eqs. 13.20 and 13.21. The starting value of $x_u$ corresponding to $P_{uR}$ = 0, viz. $x_{u,min}$ [refer Fig. 13.11], poses some problem as it is unknown and has to be determined by solving Eq. 13.20 with $P_{uR}$ = 0 by trial-and-error. To begin with, a trial value $x_{u,min} \approx 0.15D$ can be assumed; this, in all probability, will result in a negative value of $P_{uR}$, which corresponds to a condition of *eccentric tension*. By incrementing $x_u/D$ suitably (in steps of 0.05 or less), the transition between $P_{uR} < 0$ and $P_{uR} > 0$ can be traced. (The 'exact' value of the ultimate moment of resistance $M_{uo}$ corresponding to 'pure bending' ($P_{uR}$ = 0), if required, can be obtained by repeated trial-and-error‡ , using very fine increments of $x_u/D$).

Having located (approximately) $x_{u,\,min}/D$, the coordinates of the design interaction curve can be obtained (using Eq. 13.20, 13.21) and tabulated for incremental values of $x_u/D$ — say, increments of 0.05. The process can be terminated when $P_{uR}$ exceeds the maximum limit $\tilde{P}_{uo}$ permitted by the Code [refer Eq. 13.17], and may be extended to $P_{uo}$ ($e$ = 0) [Eq. 13.16]. As the procedure is repetitive, this can be more conveniently done on a computer. The coordinates ($M_{uR}$, $P_{uR}$) of the design interaction curve can then be tabulated and/or plotted. It is useful to include the ratios $x_u/D$ and $e/D$ in the Table. The construction and use of a design interaction curve for a typical column section is demonstrated in Examples 13.9 and 13.11.

#### EXAMPLE 13.9

For a column section shown in Fig. 13.14(a), construct the *design interaction curve* for axial compression combined with uniaxial bending about the *major axis*. Hence, investigate the safety of the column section under the following factored load effects:

(i) $P_u$ = 2275 kN, $M_{ux}$ = 46.4 kNm (maximum axial compression);

(ii) $P_u$ = 1105 kN, $M_{ux}$ = 125 kNm (maximum eccentricity).

#### SOLUTION

- *Given*: $b$ = 300 mm, $D$ = 500 mm, $f_{ck}$ = 25 MPa, $f_y$ = 415 MPa, $A_{sc}$ = 2946 mm$^2$
  $A_{s1} = A_{s2} = A_{s3} = 2 \times 491$ mm$^2$ = 982 mm$^2$ [as in Example 13.5]
  $y_1$ = –189.5 mm, $y_2$ = 0 mm, $y_3$ = +189.5 mm
- *Maximum axial compression resistance*:

$$\tilde{P}_{uo} = 0.4f_{ck}\,bD + (0.67\,f_y - 0.4f_{ck})A_{sc} \qquad \text{[Eq. 13.17]}$$
$$= (0.4 \times 25 \times 300 \times 500) + (0.67 \times 415 - 0.4 \times 25) \times (2946)$$
$$= (1500\,000 + 789\,675)\text{ N} = 2290\text{ kN}$$

‡ The problem of determining $x_u/D$ for any given $P_{uR}$ can be more elegantly solved using a suitable numerical procedure [Ref. 13.13]. Thus, a computer program can be written and used to derive the interaction curve coordinates, using increments of $P_{uR}$, rather than $x_u/D$.

**Interaction Curve Coordinates**

The theoretical maximum axial compression (with $e = 0$) is given by Eq. 13.16:

$P_{uo} = 0.447 f_{ck}\, bD + (0.790 f_y - 0.447 f_{ck})\, A_{sc}$ — for Fe 415

$= (0.447 \times 25 \times 300 \times 500) + (0.79 \times 415 - 0.447 \times 25) \times (2946)$

$= (1676250 + 932925)$ N = 2609 kN.

The coordinates ($M_{uR,x}$, $P_{uR}$) are derived for $0 \le P_{uR} \le P_{uo}$, considering incremental values of $x_u/D$, using the equilibrium equations (Eq. 13.20, 13.21). The results, obtained by a computer program, are tabulated in Table 13.1(a). The reader may verify some of the solutions by simple manual calculations. The 'balanced failure' point ($P_{ub,x}$ = 416.6 kN, $M_{ub,x}$ = 222.15 kNm) obtained in Example 13.5 is a salient point on the interaction curve.

An alternative, and perhaps more elegant, computer-based procedure for determining the interaction curve coordinates, is by considering incremental values of $P_{uR}$ (instead of $x_u/D$). The 'bisection method' was employed here to determine $x_u/D$ (to an accuracy of $10^{-6}$) for a given $P_{uR}$. Thus, it becomes possible to accurately compute $x_{u,\,min}/D$ and $M_{uo}$, corresponding to $P_{uR} = 0$; the values are obtained as

$$x_{u,\,min}/D = 0.284$$
$$M_{uo,\,x} = 199.8 \text{ kNm}$$

The results obtained by this alternative procedure are tabulated in Table 13.1(b). The design interaction curve is plotted in Fig. 13.18.

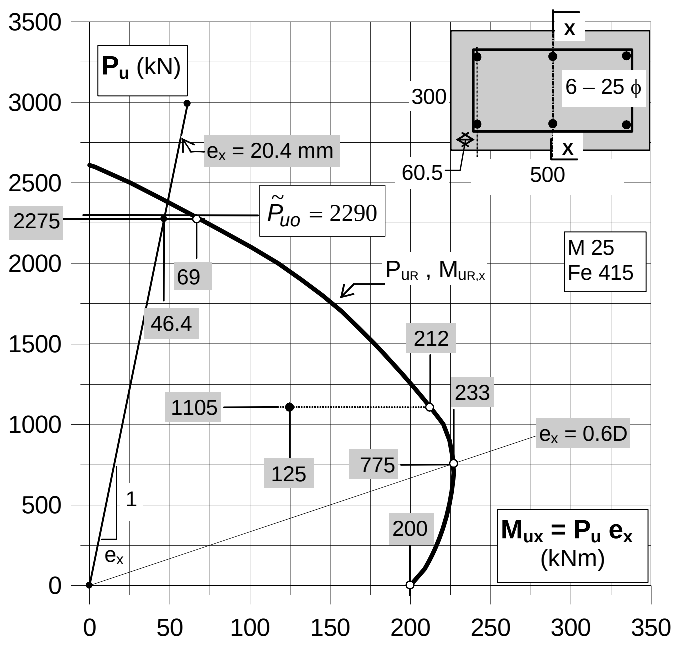



**Fig. 13.18** Example 13.9: Interaction diagram

**Table 13.1** Example 13.9 — Design interaction curve coordinates

(a)

| $x_u/D$ | $P_{uR}$ ( kN) | $M_{uR,x}$ ( kNm) |
|---|---|---|
| 0.28 | -9.1 | 198.9 |
| 0.30 | 33.0 | 203.1 |
| 0.34 | 125.0 | 210.6 |
| 0.38 | 268.6 | 216.9 |
| 0.42 | 412.2 | 222.0 |
| 0.46 | 545.3 | 225.1 |
| 0.50 | 667.8 | 226.9 |
| 0.54 | 785.9 | 226.4 |
| 0.58 | 900.5 | 224.4 |
| 0.62 | 1018.3 | 219.5 |
| 0.66 | 1162.4 | 207.9 |
| 0.70 | 1298.3 | 196.2 |
| 0.74 | 1425.6 | 184.7 |
| 0.78 | 1545.5 | 173.2 |
| 0.82 | 1659.0 | 161.6 |
| 0.86 | 1764.5 | 149.7 |
| 0.90 | 1857.6 | 137.7 |
| 0.94 | 1946.5 | 125.3 |
| 0.98 | 2030.5 | 112.5 |
| 1.02 | 2102.6 | 100.2 |
| 1.06 | 2157.7 | 90.1 |
| 1.10 | 2204.4 | 81.5 |
| 1.14 | 2244.2 | 74.2 |
| 1.18 | 2278.6 | 67.9 |
| 1.22 | 2308.6 | 62.4 |
| 1.26 | 2334.5 | 57.5 |
| 1.30 | 2357.2 | 53.1 |
| ∞ | 2609.0 | 0.0 |

(b)

| $P_{uR}$ ( kN) | $M_{uR,x}$ ( kNm) | $x_u/D$ |
|---|---|---|
| 0 | 199.8 | 0.284 |
| 100 | 208.7 | 0.330 |
| 200 | 214.2 | 0.362 |
| 300 | 218.1 | 0.388 |
| 400 | 221.6 | 0.416 |
| 500 | 224.1 | 0.446 |
| 600 | 226.1 | 0.477 |
| 700 | 227.0 | 0.511 |
| 800 | 226.3 | 0.545 |
| 900 | 224.4 | 0.580 |
| 1000 | 220.4 | 0.614 |
| 1100 | 213.0 | 0.642 |
| 1200 | 204.6 | 0.671 |
| 1300 | 196.0 | 0.700 |
| 1400 | 187.0 | 0.732 |
| 1500 | 177.6 | 0.765 |
| 1600 | 167.7 | 0.799 |
| 1700 | 157.1 | 0.835 |
| 1800 | 145.2 | 0.875 |
| 1900 | 132.0 | 0.919 |
| 2000 | 117.3 | 0.965 |
| 2100 | 100.7 | 1.018 |
| 2200 | 82.3 | 1.096 |
| 2300 | 64.0 | 1.208 |
| 2400 | 44.8 | 1.392 |
| 2500 | 25.2 | 1.755 |
| 2600 | 3.0 | 5.717 |
| 2609 | 0.0 | ∞ |

**Safety under given factored load effects**

(i) $P_u$ = 2275 kN, $M_{ux}$ = 46.4 kNm

From fig. 13.18, it can be seen that this point falls within the design interaction curve (failure envelope). Hence, the section is 'safe' under the given load effects.

Alternatively, corresponding to $P_{uR}$ = 2275 kN, $M_{uR,x} \approx$ 69 kNm (from Fig. 13.18). As this ultimate moment of resistance is greater than $M_{ux}$ = 46.4 kNm, the column section is safe.

Alternatively, $e_x = \dfrac{46.4 \times 10^6}{(2275 \times 10^3)}$ = 20.40 mm† (i.e., $(e/D)_x$ = 0.0408). The design strength corresponding to this loading eccentricity can be obtained from Fig. 13.18 as the intersection of the design interaction curve with a straight line passing through the origin and the given point. Accordingly,

† This roughly corresponds to the minimum uniaxial eccentricity (for short columns) specified by the Code.

$P_{uR}$ = 2402 kN > $P_u$ = 2275 kN
$M_{uR}$ = 49 kNm > $M_u$ = 46.4 kNm — Hence, safe.

(ii) $P_u$ = 1105 kN, $M_{ux}$ = 125 kNm
This point also falls within the design interaction curve [Fig. 13.18]. Hence, the section is 'safe'.
Alternatively, corresponding to $P_{uR}$ = 1105 kN (from Fig. 13.18),
$M_{uR,\,x}$ = 212 kNm > $M_{ux}$ = 125 kNm — Hence, safe.

### EXAMPLE 13.10

Using the design interaction curve obtained in Example 13.9, determine

(i) the maximum eccentricity $e_x$ with which a factored load $P_u$ = 1400 kN can be safely applied;

(ii) the design strength components corresponding to an eccentricity $e_x = 0.6D$.

### SOLUTION

(i) Considering the design interaction curve in Fig. 13.18, or Table 13.1(b) corresponding to $P_{uR} = P_u$ = 1400 kN, the design flexural strength is obtained as

$$M_{uR,\,x} = 187 \text{ kNm}$$

This is the maximum factored moment that can be applied on the column section, in combination with $P_u$ = 1400 kN. The corresponding eccentricity is given by

$$e_x = \frac{M_{ux}}{P_u} = \frac{187 \times 10^3}{1400} = \mathbf{133.6\ mm}$$

(ii) $e_x = 0.6D = 0.6 \times 500 = 300$ mm
Draw the radial line with $e_x$ = 0.3, and locate its intersection with the interaction curve. For this, consider a point with coordinates $P_u$ = 1000 kN and $M_{ux}$ = 1000 × 0.30 = 300 kNm. Passing a straight line from the origin to this point on the interaction diagram (extending this line if necessary), to intersect the design interaction curve [see Fig. 13.18], the design strength components are obtained as

$P_{uR}$ = **753 kN**
$M_{uR,\,x}$ = **226 kNm**

### EXAMPLE 13.11

For the column section shown in Fig. 13.14(a), construct the design interaction curve for axial compression combined with uniaxial bending about the *minor* axis. Hence, determine:

(i) the maximum eccentricity $e_y$ with which a factored load $P_u$ = 1400 kN can be safely applied;

(ii) the design strength components corresponding to an eccentricity $e_y$ = 180 mm.

**SOLUTION**

- The design interaction curve $P_{uR} - M_{uR,\,y}$ is constructed in a manner similar to the $P_{uR} - M_{uR,\,x}$ curve of Example 13.9. In the present case, the depth of the section is taken as 300 mm and the width as 500 mm. There are only two rows of reinforcement to be considered [see Example 13.7]:
  $A_{s1} = A_{s2} = 3 \times 491 = 1473\ \text{mm}^2$
  $y_1 = -89.5$ mm, and $y_2 = +89.5$ mm
  The coordinates of the interaction curve (along with values of $x_u/D$) are shown in Table 13.2 and plotted in Fig. 13.19.
- Some salient points on this interaction curve are:
  $\tilde{P}_{uo} = 2290$ kN, $P_{uo} = 2609$ kN (as in Example 13.9)
  $M_{uo,y} = 105.4$ kNm (as against $M_{uo,\,x} = 199.8$ kNm in Example 13.9)
  $P_{ub,y} = 650.0$kN, $M_{ub,y} = 145.2$ kNm

**Table 13.2** Example 13.11 — Design Interaction Curve Coordinates

| $P_{uR}$ ( kN) | $M_{uR,y}$ ( kNm) | $x_u/D$ |
|---|---|---|
| 0 | 105.0 | 0.250 |
| 100 | 113.8 | 0.274 |
| 200 | 122.4 | 0.300 |
| 300 | 130.7 | 0.329 |
| 400 | 138.2 | 0.369 |
| 500 | 143.0 | 0.420 |
| 600 | 145.3 | 0.471 |
| 700 | 145.0 | 0.518 |

| | | |
|---|---|---|
| 800 | 142.7 | 0.557 |
| 900 | 138.0 | 0.585 |
| 1000 | 132.8 | 0.611 |
| 1100 | 127.5 | 0.639 |
| 1200 | 122.0 | 0.669 |
| 1300 | 116.3 | 0.701 |
| 1400 | 110.4 | 0.734 |
| 1500 | 104.2 | 0.769 |
| 1600 | 97.7 | 0.806 |
| 1700 | 90.8 | 0.845 |
| 1800 | 83.4 | 0.886 |
| 1900 | 75.5 | 0.929 |
| 2000 | 66.8 | 0.973 |
| 2100 | 57.1 | 1.026 |
| 2200 | 46.4 | 1.103 |
| 2300 | 35.8 | 1.212 |
| 2400 | 25.1 | 1.384 |
| 2500 | 14.3 | 1.699 |
| 2600 | 1.7 | 5.713 |
| 2609 | 0.0 | $\infty$ |

(i) Corresponding to $P_{uR} = P_u = 1400$ kN, the design flexural strength is obtained from Table 13.2 or Fig. 13.19 as
$M_{uR,\,y} = 110.4$ kNm (compared to 187 kNm in Example 13.10)
This is maximum factored moment $M_{uy}$ that can be applied on the column section, in combination with $P_u = 1400$ kN. The corresponding eccentricity is given by

$$e_y = \frac{M_{uy}}{P_u} = \frac{110.4 \times 10^3}{1400} = \mathbf{78.86\ mm}$$

(ii) $e_y = 180$ mm
Considering a straight line with slope such that $P_u : M_{uy} = 1000$ kN : 180 kNm, i.e., a slope of 1:$e_y$, the desired strength components are obtained as the point of intersection of the design curve with this straight line [see Fig. 13.19]:

$P_{uR}$ = **789 kN**
$M_{uR,\,y}$ = **142 kNm**

[In this case, $e_y/D = 180/300 = 0.60$. The results may be compared with $P_{uR} = 775$ kN, $M_{uR,x} = 233$ kNm corresponding to $e_x/D = 0.6$, obtained in Example 13.10(ii)].

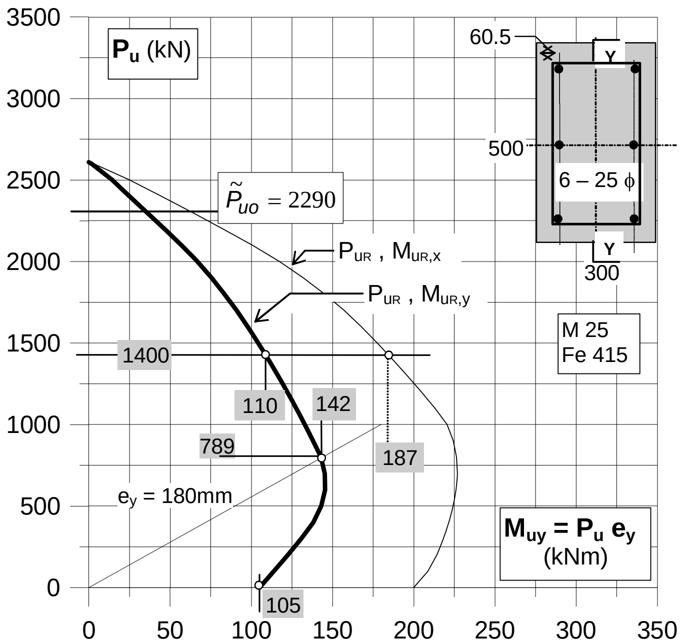



**Fig. 13.19** Example 13.11

### 13.5.6 Non-dimensional Interaction Diagrams as Design Aids

In a typical *design* situation, the column section has to be designed to resist certain critical combinations of factored axial compression ($P_u$) and factored bending moment ($M_u$) — as obtained from structural analyses. Commonly, the overall dimensions of the column are assumed†, and the grades of concrete and steel are specified; of course, these could be changed subsequently — in a redesign. The problem of 'design', therefore, reduces to the provision of longitudinal and transverse reinforcement. Designing of transverse reinforcement is relatively simple, as it is largely a matter of compliance with Code specifications related to bar diameter and spacing [refer Section 13.3.3]. Designing of longitudinal reinforcement can also be made simple by the use of interaction diagrams, as explained in Section 13.5.5. Indeed, as pointed out earlier, in the absence of such analysis/design aids, the problem is difficult to solve as it calls for repeated trial-and-error in order to locate the neutral axis.

The interaction diagrams discussed hitherto [such as Fig. 13.18 or Fig. 13.19] are applicable only for particular sections, details of which are indicated in the inset of each diagram. The application of the interaction diagram can be rendered more

† Indeed, these dimensions need to be assumed at the stage of structural analysis itself, if the column forms part of a statically indeterminate structure — as is usually the case.

versatile by making coordinates $P_u$ and $M_u$ independent of the cross-sectional dimensions — by defining suitable non-dimensional parameters $p_u$ and $m_u$:

$$p_u \equiv \frac{P_u}{f_{ck} bD} \tag{13.31}$$

$$m_u \equiv \frac{M_u}{f_{ck} bD^2} \tag{13.32}$$

The $p_u - m_u$ interaction diagram [Fig. 13.20] now becomes applicable to all sections geometrically similar to and having the same material (steel) properties as the one shown in the inset of the figure. The design interaction curve has coordinates $p_{uR}$ and $m_{uR}$ (corresponding to $P_{uR}$ and $M_{uR}$); expressions for $p_{uR}$ and $m_{uR}$ are obtainable from Eq. 13.20 – 13.30:

$$p_u \equiv \frac{P_u}{f_{ck} bD} = a + \sum_{i=1}^{n} (f_{si} - f_{ci}) \frac{k_i p}{100 f_{ck}} \tag{13.33}$$

$$m_u \equiv \frac{M_u}{f_{ck} bD^2} = a\,(0.5 - \bar{x}/D) + \sum_{i=1}^{n} (f_{si} - f_{ci}) \frac{k_i p}{100 f_{ck}} \frac{y_i}{D} \tag{13.34}$$

where expressions for the non-dimensional parameters $a$ and $\bar{x}/D$ are obtainable from Eq. 13.24 and 13.25; and $k_i$ is the fraction of the total percentage of reinforcement $p$ located at the $i^{th}$ row, where

$$p \equiv \frac{100 A_s}{bD} \tag{13.35}$$

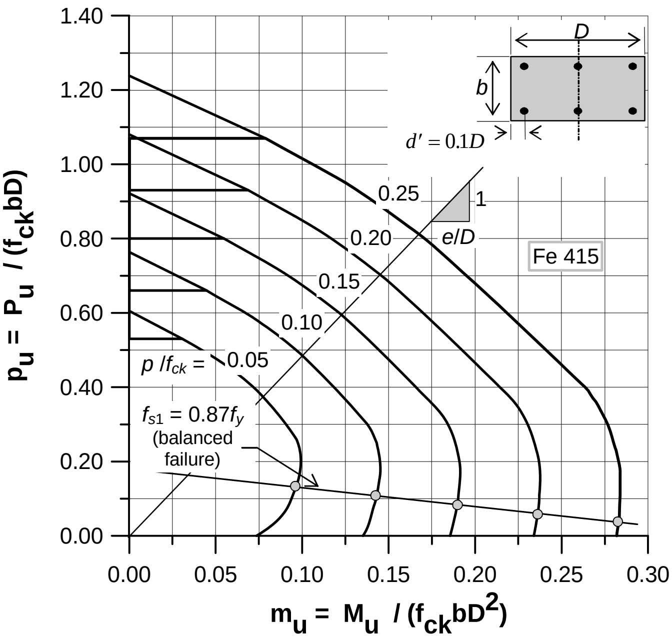



**Fig. 13.20** Typical non-dimensional interaction diagram

From the nature of Eq. 13.33 and 13.34, it follows that by selecting $p/f_{ck}$ as a parameter, the interaction diagram (valid for given arrangement of bars), can be rendered independent of the grade of concrete† . A family of non-dimensional design interaction curves ($p_{uR} - m_{uR}$) can thus be generated for a given arrangement of bars. For example, considering the arrangement of six bars in the rectangular column section of Example 13.5 [see Fig. 13.14(a), Fig. 13.18], with bending about the major axis, a family of interaction curves can be generated, as shown in Fig. 13.20 — for some typical values of $p/f_{ck}$ (0.05, 0.10, 0.15, 0.20, 0.25) and specific grade of steel, viz. Fe 415. In this particular arrangement of bars, it is assumed that all the bars are of equal diameter and symmetrically arranged, with one pair of bars located along the major centroidal axis. This is indicated in the inset shown at the top of the figure. The ratio of the effective cover $d'$ to the overall depth $D$ is another parameter of significance. A typical value $d'/D = 0.10$ is assumed in the interaction diagram shown in Fig. 13.20. In practice, $d'/D$ usually varies in the range 0.05 – 0.20, and $p/f_{ck}$ in the range 0.01 – 0.26. For a fairly exhaustive set of interaction diagrams (including different bar arrangements and grades of steel), reference may be made to the Design Handbook, SP : 16 [Ref. 13.12]

† Note that the term $f_{ci}$ in Eq. 13.33 and 13.34 is dependent on $f_{ck}$ if $\varepsilon_{si} > 0$ (compressive) [refer Eq. 13.29]. However, it is sufficiently accurate to consider $f_{ck}$ = 20 MPa or 25 MPa for this purpose and thereby make the curves applicable for all grades of concrete [Ref. 13.12].

It is seen that the different 'balanced failure' condition points on the various design interaction curves of the same 'family' all happen to fall on the same straight line — as indicated in Fig. 13.20.

It may also be noted that the non-dimensional design strength components $p_{uR}$ and $m_{uR}$ can be obtained for any given eccentricity $e$, as the point of intersection of the non-dimensional interaction curve with a straight line passing through the origin and having a slope given by :

$p_u : m_u = 1 : e/D$, as indicated in Fig. 13.20.

The non-dimensional interaction curves can be used to handle all types of design and analysis problems. In the design problem, the desired value of percentage reinforcement $p$ can be easily obtained from the family of interaction curves for a given $p_u$ and $m_u$. This is demonstrated in Examples 13.12 – 13.14.

### *Design Charts (for Uniaxial Eccentric Compression) in SP : 16*

The design Charts (non-dimensional interaction curves) given in the Design Handbook, SP : 16 [Ref. 13.12] cover the following three cases of symmetrically arranged reinforcement :

(a) rectangular sections with *reinforcement distributed equally on two sides* (Charts 27 – 38): the 'two sides' refer to the sides parallel to the axis of bending; there are no inner rows of bars, and each outer row has an area of $0.5A_s$ [Fig. 13.21(a)], this includes the simple 4–bar configuration;

(b) rectangular sections with *reinforcement distributed equally on four sides* (Charts 39 – 50): two outer rows (with area $0.3A_s$ each) and four inner rows (with area $0.1A_s$ each) have been considered in the calculations† ; however, the use of these Charts can be extended, without significant error, to cases of not less than two inner rows (with a minimum area $0.3A_s$ in each outer row), as shown in Fig. 13.21(b).

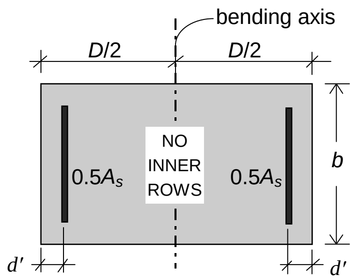
(c) *circular column sections* (Charts 51 – 62): the Charts are applicable for circular sections with at least six bars (of equal diameter) uniformly spaced circumferentially, as shown in Fig. 13.21(c).

Corresponding to each of the above three cases, there are as many as 12 Charts available — covering the 3 grades of steel (Fe 250, Fe 415, Fe 500), with 4 values of $d'/D$ ratio for each grade (viz., $d'/D$ = 0.05, .0.10, 0.15, 0.20). For intermediate values of $d'/D$, linear interpolation may be done. Each of the 12 Charts of SP : 16 covers a family of non-dimensional design interaction curves with $p/f_{ck}$ values ranging from 0.0 to 0.26.

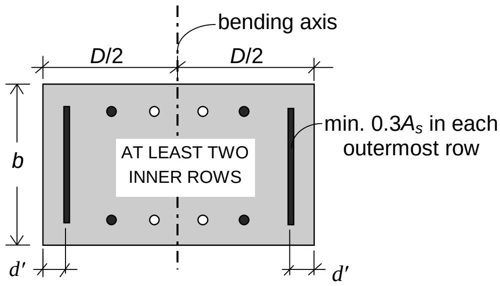
It may be noted that there are other types of symmetrical reinforcement arrangements such as the 6–bar arrangement of Fig. 13.14(a) which are not covered by SP : 16 [Fig. 13.21]. In such cases, the designer may make judicious

† If bars of equal diameter are used, this is equivalent to using 20 bars. While actually providing reinforcement to conform to $A_s$ computed using these Charts, some adjustments may be called for in practice. Providing a greater proportion of reinforcement (more than $0.3A_s$) on the outermost rows is on the safer side [refer Example 13.13].

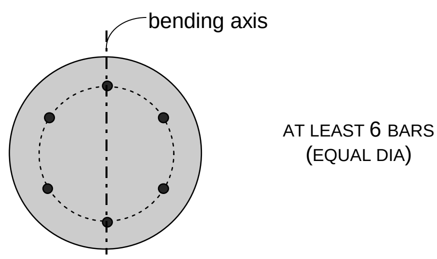
approximations if he still wishes to avail of the SP : 16 Charts. However, a proper course of action would be to construct the proper interaction diagram (as in Examples 13.9 and 13.11) for the section chosen, and thereby to verify the safety of the section; if required, the design should be suitably revised, to make it more economical.




**(a)** rectangular section with
"*reinforcement distributed equally on two sides*"




**(b)** rectangular section with
"*reinforcement distributed equally on all sides*"




**(c)** circular column section

**Fig. 13.21** Reinforcement arrangements for which SP : 16 Charts are applicable

There are other situations, encountered in practice, which are not amenable for the use of SP : 16 Charts. These include cases of:

- unsymmetrically arranged reinforcement in rectangular sections;
- non-rectangular and non-circular sections — such as L-shaped, T-shaped, H-shaped, cross shaped sections, etc.

In such cases, it becomes necessary to construct proper interaction diagrams in order to obtain accurate and reliable solutions.

### EXAMPLE 13.12

Using the design aids given in SP : 16, design the longitudinal reinforcement in a rectangular reinforced concrete column of size 300 mm × 600 mm subjected to a factored load of 1400 kN and a factored moment of 280 kNm with respect to the major axis. Assume M 20 concrete and Fe 415 steel.

### SOLUTION

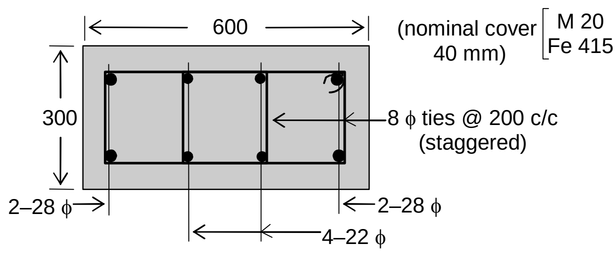



**Fig. 13.22** Example 13.12

- *Given*: $b$ = 300 mm, $D$ = 600 mm, $f_{ck}$ = 20 MPa, $f_y$ = 415 MPa, $P_u$ = 1400 kN, $M_{ux}$ = 280 kNm
- *Arrangement of bars*: as $D$ = 600 mm, the spacing between the corner bars will exceed 300 mm; hence inner rows of bars have to be provided to satisfy detailing requirements [refer Section 13.3.3]. Assuming two or more inner rows, the SP : 16 Charts for "equal reinforcement on four sides" can be made use of [Fig. 13.21(b)].
- Assuming an effective cover $d'$ = 60 mm,

  $\Rightarrow d'/D$ = 60/600 = 0.1

  $$p_u = \frac{P_u}{f_{ck}bD} = \frac{1400 \times 10^3}{20 \times 300 \times 600} = 0.389$$

  $$m_u = \frac{M_{ux}}{f_{ck}bD^2} = \frac{280 \times 10^6}{20 \times 300 \times 600^2} = 0.130$$

- Referring to Chart 44 ($d'/D$ = 0.10) of SP : 16, it can be observed that, the coordinates $p_u$ = 0.389, $m_u$ = 0.130 would lie on a design interaction curve with $p/f_{ck} \approx 0.11$

  $\Rightarrow p_{reqd}$ = 0.11 × 20 = 2.2

  $\Rightarrow A_{s,\,reqd}$ = 2.2 × 300 × 600/100 = **3960 mm²**

*Detailing of longitudinal reinforcement*

- The design chart used refers to the case of "equal reinforcement on four sides" [Fig. 13.21(b)],
  *Outermost rows*
  Minimum area required in each outermost row = 0.3 × 3960 = 1188 mm$^2$
  **Provide 2 – 28 ϕ**: area = 616 × 2 = 1232 mm$^2$ > 1188 mm$^2$
  *Inner rows*
  Total area required = 3960 – (1232 × 2) = 1496 mm$^2$
  **Provide 4 – 22 ϕ in two inner rows**: area = 380 × 4 = 1520 mm$^2$ > 1496 mm$^2$
- Total area provided = (1232 × 2) + 1520 = 3984 mm$^2$ > 3960 mm$^2$

(⇒ $p$ = 100 × 3984/(300 × 600) = 2.213)

Assuming 8mm ties, effective cover = 40+8+(28/2) = 62mm ≈ 60mm – OK

The detailing is shown in Fig. 13.22. Details of transverse reinforcement are also indicated in the figure. (It may be verified that this detailing adequately satisfies the Code requirements).

### EXAMPLE 13.13

Referring to the column section shown in Fig. 13.22, investigate the safety of the column section under uniaxial eccentric compression with respect to the *minor* axis, considering $P_u$ = 1400 kN and $M_{uy}$ = 200 kNm. If the section is unsafe, suggest suitable modifications to the reinforcement provided.

### SOLUTION

- *Given*: $b$ = 600 mm, $D$ = 300 mm, $f_{ck}$ = 20 MPa, $f_y$ = 415 MPa, $P_u$ = 1400 kN
  $M_{uy}$ = 200 kNm, $A_s$ = 3984 mm$^2$, $p$ = 2.213
  Effective cover $d'$ = 40 + 8 + 14 = 62 mm
- The arrangement of bars in this case conforms to "reinforcement distributed equally on two sides" [Fig. 13.22(a)].
  $d'/D$ = 62/300 = 0.207 ≈ 0.2
  $p/f_{ck}$ = 2.213/20 = 0.1106 ≅ 0.11

$$p_u = \frac{P_u}{f_{ck}bD} = \frac{1400 \times 10^3}{20 \times 600 \times 300} = 0.389$$

$$m_u = \frac{M_{ux}}{f_{ck}bD^2} = \frac{200 \times 10^6}{20 \times 600 \times 300^2} = 0.185$$

- Referring to Chart 34 ($d'/D$ = 0.2) of SP: 16, it can be seen that the point $P_u$ = 0.389, $m_u$ = 0.185 lies outside the design interaction curve envelope for $p/f_{ck}$ = 0.11 and $d'/D$ = 0.09. The value of $p/f_{ck}$ corresponding to $p_{uR}$ = 0.389 and $m_{uR}$ = 0.185 is given by:
  $(p/f_{ck})_{reqd}$ = 0.18 > $(p/f_{ck})_{provided}$ = 0.11. Hence, the given section is **unsafe**.
- *Revised Design*
  Corresponding to $(p/f_{ck})_{reqd}$ = 0.175,
  $p_{reqd}$ = 0.175 × 20 = 3.5

$\Rightarrow A_{s,\,reqd}$ = 3.5 × 600 × 300/100 = 6300 mm$^2$
(to be provided equally in two rows)
Provide 8 – 32 ϕ (instead of 4 – 28 ϕ + 4 – 22 ϕ)
$\Rightarrow A_{s,\,provided}$ = 804 × 8 = 6432 mm$^2$ > 6300 mm$^2$ ($p$ = 3.573)
The detailing for the revised design is shown in Fig. 13.23

**EXAMPLE 13.14**

Consider a spiral circular short column, with details as given in Fig. 13.10 (Example 13.4). Assess the safety of the column section, when subjected to a factored compressive load of 260 kN and a factored bending moment of 78 kNm. In case the section is found unsafe, redesign the reinforcement.

**SOLUTION**

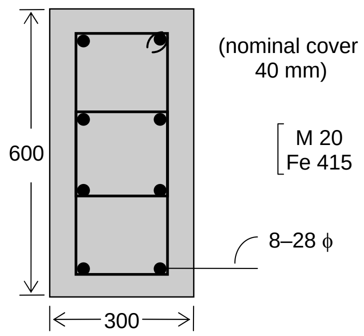



**Fig. 13.23** Example 13.13

- *Given*: diameter $D$ = 400 mm, $A_s$ = 1206 mm$^2$(6 – 16 ϕ), $f_{ck}$ = 25 MPa,
  $f_y$ = 415 MPa, $P_u$ = 260 kN, $M_u$ = 78 kNm
- As the spiral reinforcement satisfies the Code requirements (Cl. 39.4.1), as seen in Example 13.4, the design strength (i.e., $P_{uR}$, $M_{uR}$) may be enhanced by 5 percent, as permitted for short columns.
- Effective cover $d'$ = 40 + 6 + 16/2 = 54 mm

  $\Rightarrow d'/D$ = 54/400 = 0.135 (which lies between 0.10 and 0.15)
- percentage of reinforcement $p = \dfrac{100A_s}{\pi D^2/4} = \dfrac{100 \times 1206}{\pi \times 400^2/4}$ = 0.960

  $\Rightarrow p/f_{ck}$ = 0.960/25 = 0.0384
- *Eccentricity of loading* $e = \dfrac{M_u}{P_u} = \dfrac{78 \times 10^3}{260}$ = 300 mm

  $\Rightarrow e/D$ = 300/400 = 0.75
  $\Rightarrow p_{uR} : m_{uR}$ = 1 : 0.75, or more conveniently, 0.1 : 0.075
  Thus, the design strength components $p_{uR}$, $m_{uR}$ can be obtained as the point of intersection of the design interaction curve with a straight line joining the origin to a point with $P_u$ = 0.1, $m_u$ = 0.75 in the non-dimensional interaction diagram

- Referring to Chart 56 ( $d'/D$ = 0.10) and Chart 57 ( $d'/D$ = 0.15) of SP : 16, and interpolating results for $d'/D$ = 0.135 and $p/f_{ck}$ = 0.038, and enhancing the $p_{uR}$, $m_{uR}$ values by 5 percent ( as permitted by the Code for spiral columns, suitably reinforced),

  $p_{ur} \equiv P_{uR} / f_{ck} D^2 \cong 0.063 \times 1.05 = 0.0661$

  $m_{ur} \equiv M_{uR} / f_{ck} D^3 \cong 0.0515 \times 1.05 = 0.0541$

  $\Rightarrow P_{uR} = 0.0661 \times 25 \times 400^2 = 264.4 \times 10^3$ N = **264 kN**

  $M_{uR} = 0.0541 \times 25 \times 400^3 = 86.56 \times 10^6$ Nmm = **86.5 kNm**

  As $P_{uR}$ = **264** kN is greater than $P_u$ = 260 kN *and* $M_{uR}$ = **86.5** kNm is greater than $M_u$ = 78 kNm, the section can be considered to be safe, and hence there is no need for a redesign.

## 13.6 DESIGN OF SHORT COLUMNS UNDER AXIAL COMPRESSION WITH BIAXIAL BENDING

### 13.6.1 Biaxial Eccentricities

As mentioned in Section 13.3.2, all columns are (in a strict sense) to be treated as being subject to axial compression combined with *biaxial* bending, as the design must account for possible eccentricities in loading ($e_{min}$ at least) with respect to both major and minor principal axes of the column section. *Uniaxial loading* is an idealised approximation which can be made when the $e/D$ ratio with respect to one of the two principal axes can be considered to be negligible. Also, as mentioned in Section 13.4.1, if the $e/D$ ratios are negligible with respect to both principal axes, conditions of *axial loading* may be assumed, as a further approximation.

In the recent revision to the Code, it is clarified (Cl. 25.4) that "where biaxial bending is considered, it is sufficient to ensure that eccentricity exceeds the minimum about one axis at a time". This implies that if either one or both the factored bending moments $M_{ux}$ and $M_{uy}$ (obtained from analysis) is less than the corresponding value, calculated from minimum eccentricity considerations (Eq. 13.8), it suffices to ensure that at least one of the two minimum eccentricity conditions is satisfied. However, it also becomes necessary to check for the other biaxial bending condition wherein the minimum eccentricity in the other direction is also satisfied. In lieu of the above, of course, it will be sufficient and conservative to ensure that both minimum eccentricities are simultaneously satisfied in a single design check.

The factored moments $M_{ux}$ and $M_{uy}$ acting on a column section (with respect to bending about the major axis and minor axis respectively) can be resolved into a single resultant moment $M_u$ which acts about an axis inclined to the two principal axes [Fig. 13.24(a)]:

$$M_u = \sqrt{M_{ux}^2 + M_{uy}^2} \tag{13.36}$$

Alternatively, the resultant eccentricity $e = M_u/P_u$ may be obtained [refer Fig. 13.24(b)] as:

$$e = \sqrt{e_x^2 + e_y^2} \tag{13.37}$$

When the column section (including the reinforcement) is axisymmetric (with reference to the longitudinal axis) — as in a circular column — the resultant axis of bending is also a principal axis [Fig. 13.24(c)]. In such a situation, the case of biaxial bending simplifies into a case of uniaxial bending. The neutral axis, in this instance, will remain parallel to the resultant axis of bending.

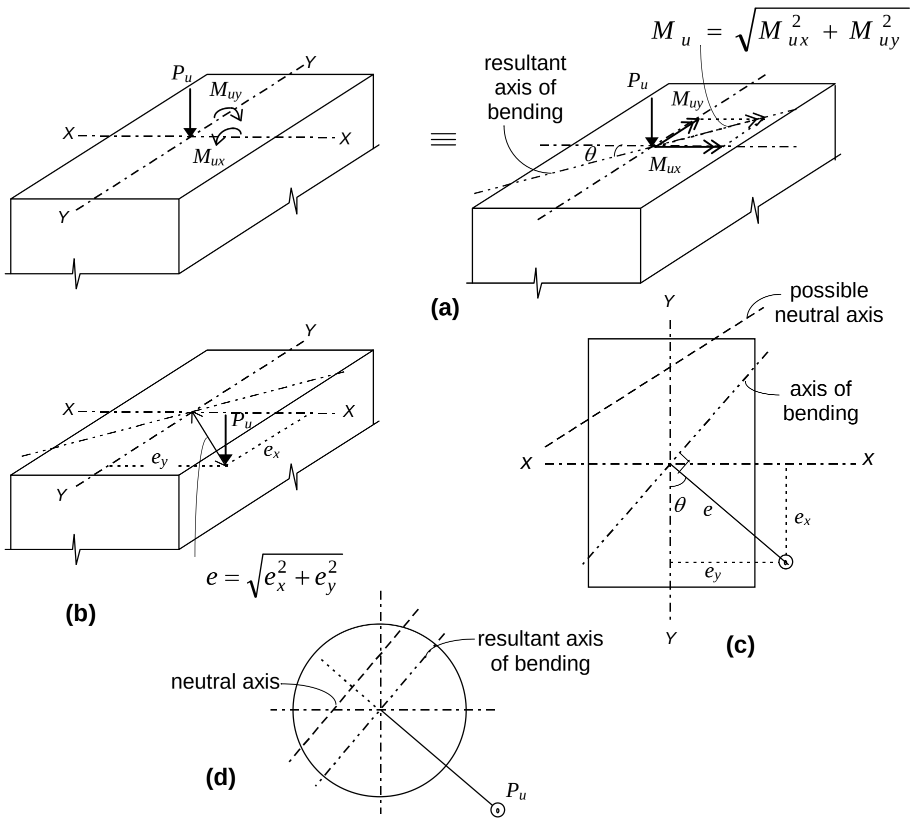



**Fig. 13.24** Resultant eccentricity of loading

However, in the more general case of non-axisymmetric reinforced concrete column sections, the neutral axis is generally not parallel to the resultant axis of bending [Fig. 13.24(d)]. In fact, the determination of the exact neutral axis location is a laborious process of trial and error. For a given neutral axis location, however, the failure strain distribution can be drawn (with the same assumptions as in the case of uniaxially eccentric compression) [Fig. 13.25].

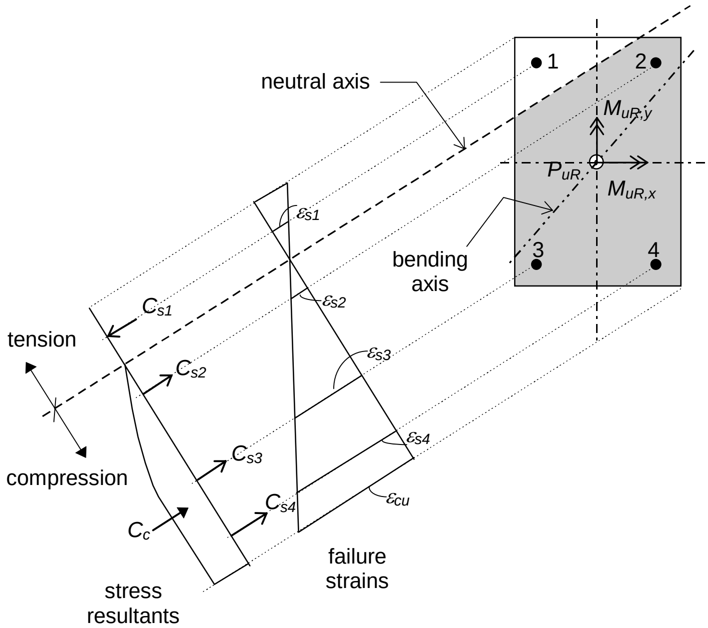



**Fig. 13.25** Analysis of design strength for a given location of neutral axis

### 13.6.2 Interaction Surface for a Biaxially Loaded Column

Various simplified procedures for the design of biaxially loaded columns have been proposed [Ref. 13.14, 13.15] and adopted by different design codes. Most of these simplified procedures are based on an approximation of the *interaction surface*, which may be visualised in a three-dimensional plot of $P_{uR} - M_{ux} - M_{uy}$ [Fig. 13.26].

The surface is generated as the envelope of a number of design interaction curves for different axes of bending. Each point on the interaction surface [Fig. 13.26] corresponds to values of $P_{uR}$, $M_{ux}$ and $M_{uy}$ obtained from the analysis of a chosen neutral axis location and orientation, such as the one in Fig. 13.25. The design interaction surface can be considered to be a *failure surface* in that the region bounded within this surface is a 'safe' region and any point ($P_u$, $M_{ux}$, $M_{uy}$) that lies outside the surface is 'unsafe'†.

The traces of the interaction surface on the *x-z* and *y-z* (vertical) planes correspond to the design interaction curves for uniaxial eccentricity with respect to the major and minor principal axes respectively. In order to avoid confusion, the notations used for the design flexural strength under uniaxial eccentricity and under biaxial eccentricities, the following notations shall be used in the context of biaxial loading of columns:

$M_{uR,\,x}$ ≡ design flexural strength with respect to *major* axis under *biaxial* loading
$M_{uR,\,y}$ ≡ design flexural strength with respect to *minor* axis under *biaxial* loading

† That is, the corresponding probability of failure is unacceptable, according to the Code.

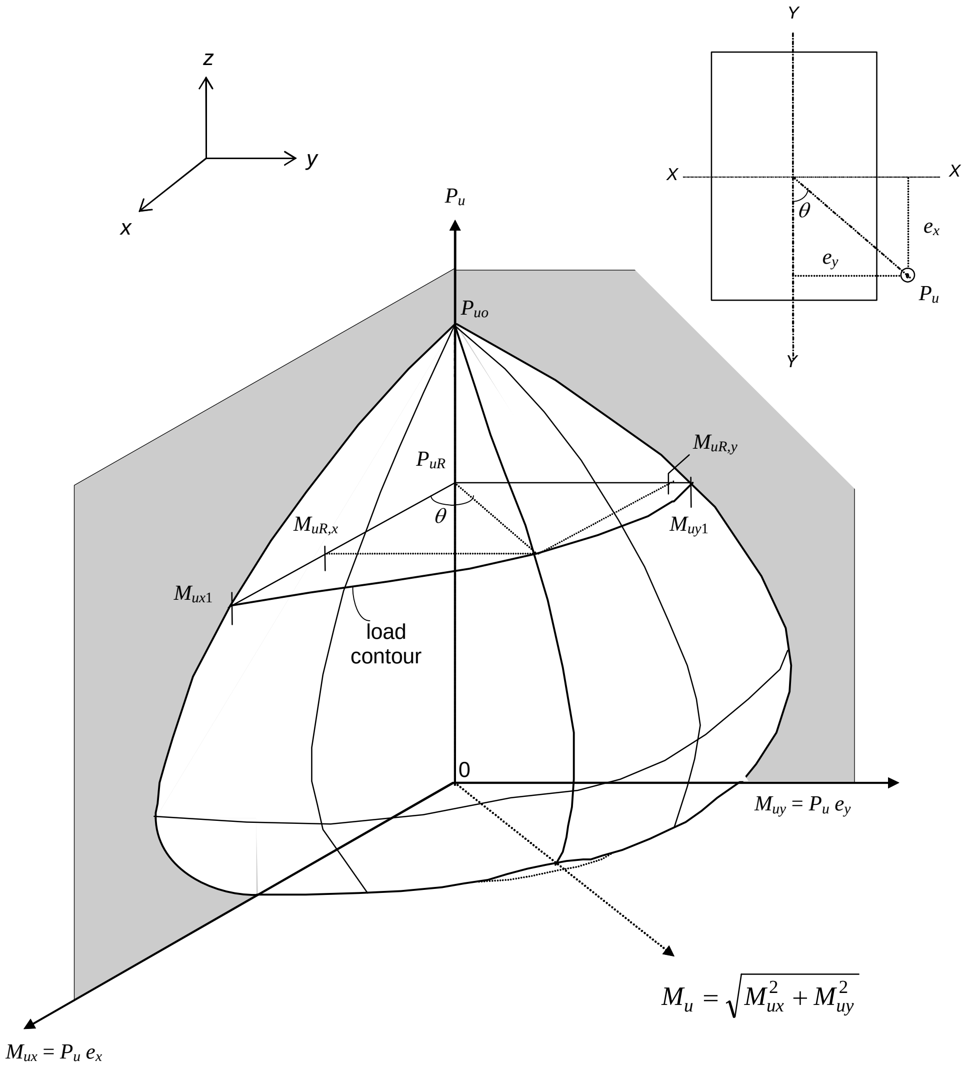



**Fig. 13.26** Interaction surface for a biaxially loaded column

$M_{ux1}$ ≡ design flexural strength with respect to *major* axis under *uniaxial* loading (i.e., $e_y = 0$)

$M_{uy1}$ ≡ design flexural strength with respect to *minor* axis under *uniaxial* loading (i.e., $e_x = 0$)

The notations and their respective meanings are depicted in Fig. 13.26, corresponding to an axial compression $P_u = P_{uR}$.

It is interesting to note (in Fig. 13.26) that the trace of the interaction surface on a horizontal plane (parallel to the *x*-*y* plane) at any load level $P_u$ is also an interaction curve — depicting the interaction between the biaxial bending capacities $M_{uR,x}$ and $M_{uR,y}$. Such an interaction curve is sometimes referred to as a *load contour*, as all the points on the curve pertain to a constant axial load level.

### 13.6.3 Code Procedure for Design of Biaxially Loaded Columns

The simplified method adopted by the Code (Cl. 39.6) is based on Bresler's formulation [Ref. 13.14] for the 'load contour' — whereby an approximate relationship between $M_{uR,x}$ and $M_{uR,y}$ (for a specified $P_u = P_{uR}$) is established. This relationship is conveniently expressed in a non-dimensional form as follows:

$$\left(\frac{M_{ux}}{M_{ux1}}\right)^{\alpha_n} + \left(\frac{M_{uy}}{M_{uy1}}\right)^{\alpha_n} \le 1 \tag{13.38}$$

where $M_{ux}$ and $M_{uy}$ denote the factored biaxial moments acting on the column, and (as explained earlier) $M_{ux1}$ and $M_{uy1}$ denote the uniaxial moment capacities with reference to the major and minor axes respectively, all under an accompanying axial load $P_u = P_{uR}$. It may be noted that $M_{ux}$, $M_{uy}$ (and $P_u$) are measures of the *load effects* due to *external* loading on the structure, whereas $M_{ux1}$, $M_{uy1}$ (and $P_{uR}$) are measures of the inherent *resistance* of the column section.

$\alpha_n$ in Eq. 13.38 is a constant which depends on the factored axial compression $P_u$ and which defines the shape of the 'load contour' [refer Fig. 13.27]. For low axial load levels, the load contour (in non-dimensional coordinates) is approximated as a straight line; accordingly $\alpha_n = 1$. For high axial load levels, the load contour is approximated as the quadrant of a circle; accordingly $\alpha_n = 2$. For moderate load levels, $\alpha_n$ takes a value between 1 and 2, as shown in Fig. 13.27(a). In order to quantitatively relate $\alpha_n$ with $P_u$, it is convenient to normalise $P_u$ with the maximum axial load capacity of the column (under 'pure compression'). This was denoted as $P_{uo}$ in Section 13.4.3, and defined by Eq. 13.16, with slightly different expressions for different grades of steel. In the context of biaxial loading, the Code (Cl. 39.6) uses the notation $P_{uz}$ (instead of $P_{uo}$), and suggests the following rounded-off version of Eq. 13.16, applicable for all grades of steel:

$$P_{uz} = 0.45 f_{ck} A_c + 0.75 f_y A_{sc},$$

$$\Rightarrow P_{uz} = 0.45 f_{ck} A_g + (0.75 f_y - 0.45 f_{ck}) A_{sc} \tag{13.39}$$

where $A_g$ denotes the gross area of the section and $A_{sc}$ the total area of steel in the section.

$\alpha_n = 1$ for $P_u/P_{uz} < 0.2$; $\alpha_n = 2$ for $P_u/P_{uz} > 0.8$; and $\alpha_n$ is assumed to vary linearly for values of $P_u/P_{uz}$ between 0.2 and 0.8 as shown in Fig. 13.27(b). Accordingly,

$$\alpha_n = \begin{cases} 1.0 & \text{for } P_u/P_{uz} < 0.2 \\ 2.0 & \text{for } P_u/P_{uz} > 0.8 \\ 0.667 + 1.667\, P_u/P_{uz} & \text{otherwise} \end{cases} \tag{13.40}$$

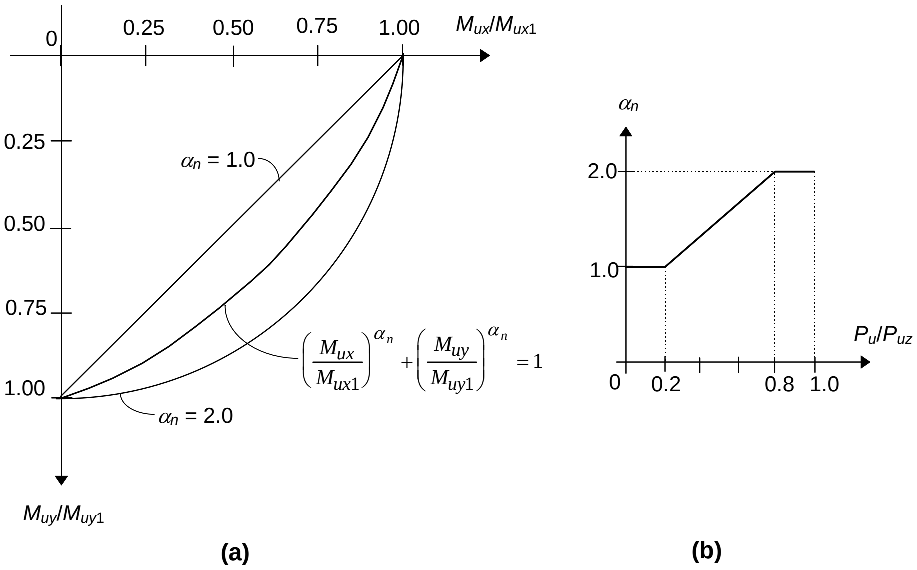



**Fig. 13.27** Approximation of load contour

A recent study [Ref. 13.23], based on rigorous analyses of several rectangular column sections with varying aspect ratios and reinforcement patterns has brought out that the above simplified formulation in the Code turns out to be conservative at very low axial load levels ($0.0 \le P_u/P_{uz} < 0.3$), and a little unconservative at higher axial load levels (levels ($0.5 < P_u/P_{uz} < 0.8$); improved expressions for $\alpha_n$ are proposed.

### *Code Procedure*

1. Given $P_u$, $M_{ux}$, $M_{uy}$, verify that the eccentricities $e_x = M_{ux}/P_u$ and $e_y = M_{uy}/P_u$ are not less than the corresponding minimum eccentricities (refer Section 13.6.1).
2. Assume a trial section for the column.
3. Determine $M_{ux1}$ and $M_{uy1}$, corresponding to the given $P_u$ (using appropriate design aids). Ensure that $M_{ux1}$ and $M_{uy1}$ are significantly greater than $M_{ux}$ and $M_{uy}$ respectively; otherwise, suitably redesign the section[†] .
4. Determine $P_{uz}$ [Eq. 13.39], and hence $\alpha_n$ [Eq. 13.40].
5. Check the adequacy of the section [Eq. 13.38]; if necessary, redesign the section and check again.

### *Selection of Trial Section*

Generally, in practice, the cross-sectional dimensions of the column are tentatively fixed in advance, and the structural analysis is performed on the basis of these dimensions. Indeed, the biaxial moments obtained from frame analyses (considering

[†] This is usually achieved by increasing the percentage of reinforcement and/or improving the grade of concrete; the dimensions may also be increased, if required.

various load combinations) are based on the assumed cross-sectional dimensions (required for stiffness calculations). Hence, in the selection of the 'trial section' for the design of biaxially loaded columns, it is only reinforcement details that need to be suitably assumed in practical situations.

One simple way of doing this is by designing the trial section for uniaxial eccentricity, considering a moment of approximately 15 percent[‡] in excess of the resultant moment, i.e.,

$$M_u \cong 1.15\sqrt{M_{ux}^2 + M_{uy}^2} \tag{13.41}$$

This bending moment should be considered to act with respect to the major principal axis if $M_{ux} \geq M_{uy}$; otherwise, it should be with respect to the minor principal axis. The reinforcement may be assumed to be distributed equally on all sides of the section.

**EXAMPLE 13.15**

A corner column (400 mm × 400 mm), located in the lowermost storey of a system of braced frames, is subjected to factored loads: $P_u$ = 1300 kN, $M_{ux}$ = 190 kNm and $M_{uy}$ = 110 kNm. The unsupported length of the column is 3.5*m*. Design the reinforcement in the column, assuming M 25 concrete and Fe 415 steel.

**SOLUTION**

- *Given*: $D_x = D_y$ = 400 mm, $l$ = 3500 mm, $P_u$ = 1300 kN, $M_{ux}$ = 190 kNm, $M_{uy}$ = 110 kNm, $f_{ck}$ = 25MPa, $f_y$ = 415MPa.

*Slenderness ratios*

- Assuming an effective length ratio of 0.85 for the braced column,

$l_{ex} = l_{ey} = 0.85 \times 3500 = 2975$ mm

$\Rightarrow l_{ex}/D_x = l_{ey}/D_y = 2975/400 = 7.44 < 12$

Hence the column may be designed as a *short column*.

*Check minimum eccentricities*

- Applied eccentricities: $e_x = 190 \times 10^3/1300 = 146$ mm
  $e_y = 110 \times 10^3/1300 = 84.6$ mm
  Minimum eccentricities as per Code [Eq. 13.8]:
  $e_{x,\,min} = e_{y,\,min} = 3500/500 + 400/30 = 20.3$ mm > 20 mm
  As the minimum eccentricities are less than the applied eccentricities, no modification to $M_{ux}$, $M_{uy}$ is called for.

*Trial section: Longitudinal reinforcement*

- Designing for uniaxial eccentricity with $P_u$ = 1300 kN and

$$M_u \approx 1.15\sqrt{M_{ux}^2 + M_{uy}^2}$$
$$= 1.15\sqrt{190^2 + 110^2} = 252 \text{ kNm}$$

- Assuming $d'$ = 60 mm,

[‡] Lower percentages (up to 5 percent) can be assumed if the axial loading level ($P_u/P_{uz}$) is relatively high.

$d'/D$ = 60/400 = 0.15

$$\frac{P_u}{f_{ck}bD} = \frac{1300 \times 10^3}{25 \times 400^2} = 0.325$$

$$\frac{M_u}{f_{ck}bD^2} = \frac{252 \times 10^6}{25 \times 400^3} = 0.157$$

- Referring to chart 45 of SP : 16 ("equal reinforcement on all sides"),
  $p/f_{ck} = 0.14$
  $\Rightarrow p_{reqd} = 0.14 \times 25 = 3.5$
  [Note: This relatively high percentage of steel is particularly acceptable for a column located in the lowermost storey of a tall building.]
  $\Rightarrow A_{s,\,reqd} = 3.5 \times 400^2/100 = 5600 \text{ mm}^2$
- **Provide 12 – 25 ϕ**: $A_s = 491 \times 12 = 5892 \text{ mm}^2 > 5600 \text{ mm}^2$. The arrangement of bars is shown in Fig. 13.28.

*Uniaxial moment capacities*: $M_{ux1}$, $M_{ux2}$ [Here, due to symmetry, $M_{ux1} = M_{ux2}$]

$P_u / f_{ck}bD$ = 0.325 (as calculated earlier)

- $p_{provided} = 5892 \times 100/400^2 = 3.68$
  $\Rightarrow p/f_{ck} = 3.68/25 = 0.147$
  $d' = 40 + 8 + 25/2 = 60.5$ mm (assuming a clear cover of 40 mm and 8 mm ties)
  $\Rightarrow d'/D = 60.5/400 = 0.151 \approx 0.15$
- Referring to Chart 45 ($d'/D$ = 0.15),

$$\frac{M_{ux1}}{f_{ck}bD^2} = 0.165$$

$\Rightarrow M_{ux1} = M_{uy1} = 0.165 \times 25 \times 400^3 = 264 \times 10^6$ Nmm
= 264 kNm

which is significantly greater than $M_{ux}$ = 190 kNm and $M_{uy}$ = 110 kNm

*Values of $P_{uz}$ and $\alpha_n$*

- $P_{uz} = 0.45f_{ck}\,A_g + (0.75f_y - 0.45f_{ck})A_{sc}$ [Eq. 13.40]
  $= (0.45 \times 25 \times 400^2) + (0.75 \times 415 - 0.45 \times 25) \times 5892$
  $= (1800 \times 10^3 + 1767.6 \times 10^3)N = 3568$ kN

$\Rightarrow P_u/P_{uz} = 1300/3568 = 0.364$ (which lies between 0.2 and 0.8)

$$\Rightarrow \alpha_n = 1.0 + \frac{0.364 - 0.2}{0.8 - 0.2}(2.0 - 1.0) = 1.273$$

[Alternatively, Eq. 13.40 may be used].

*Check safety under biaxial loading*

- $$\left(\frac{M_{ux}}{M_{ux1}}\right)^{\alpha_n} + \left(\frac{M_{uy}}{M_{uy1}}\right)^{\alpha_n} = \left(\frac{190}{264}\right)^{1.273} + \left(\frac{110}{264}\right)^{1.273}$$

$$= 0.658 + 0.328$$
$$= 0.986 < 1.0$$

Hence, the trial section is safe under the applied loading.

*Transverse reinforcement*

- The minimum diameter $\phi_t$ and maximum spacing $s_t$ of the lateral ties are specified by the Code [Eq. 13.9, 13.10]:

$$\phi_t > \begin{cases} 25/4 = 6.25\,\text{mm} \\ 6\,\text{mm} \end{cases}$$

$\Rightarrow$ Provide 8 $\phi$ ties

$$s_t < \begin{cases} D = 400\,\text{mm} \\ 16 \times 25 = 400\,\text{mm} \\ 300\,\text{mm} \end{cases}$$

**Provide 8 $\phi$ ties@ 300 c/c** as shown in Fig. 13.28.

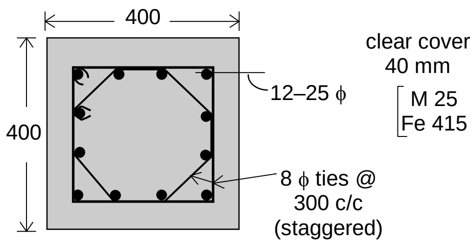



**Fig. 13.28** Example 13.15

### EXAMPLE 13.16

Verify the adequacy of the short column section Fig. 13.14(a) under the following load conditions:
$P_u$ = 1400 kN, $M_{ux}$ = 125 kNm, $M_{uy}$ = 75 kNm
The design interaction curves [Fig. 13.18, 13.19] derived earlier may be used for this purpose. Assume that the column is a 'short column'.

### SOLUTION

- *Given*: $D_x$ = 500 mm, $D_y$ = 300 mm, $A_s$ = 2946 mm$^2$ $M_{ux}$ = 125 kNm, $M_{uy}$ = 75 kNm, $f_{ck}$ = 25 MPa, $f_y$ = 415 MPa [refer Example 13.5].

*Applied eccentricities*

- $e_x = M_{ux}/P_u = 125 \times 10^3/1400 = 89.3$ mm $\Rightarrow e_x/D_x = 0.179$
  $e_y = M_{uy}/P_u = 75 \times 10^3/1400 = 53.6$ mm $\Rightarrow e_y/D_y = 0.179$
- These eccentricities for the short column are clearly not less than the minimum eccentricities specified by the Code.

*Uniaxial moment capacities*: $M_{ux1}$, $M_{uy1}$

- As determined in Example 13.10 and 13.11 [also see Fig. 13.19], corresponding to $P_u$ = 1400 kN,
  $M_{ux1}$ = 187 kNm
  $M_{uy1}$ = 110 kNm

*Values of $P_{uz}$ and $\alpha_n$*

- $P_{uz} = 0.45 f_{ck} A_g + (0.75 f_y - 0.45 f_{ck}) A_{sc}$
  $= (0.45 \times 25 \times 300 \times 500) + (0.75 \times 415 - 0.45 \times 25) \times 2946$
  $= (1687500 + 883800) N = 2571$ kN
- $\Rightarrow P_u/P_{uz} = 1400/2571 = 0.545$ (which lies between 0.2 and 0.8)
- $\Rightarrow \alpha_n = 1.0 + \dfrac{0.545 - 0.2}{0.8 - 0.2}(2.0 - 1.0) = 1.575$

*Check safety under biaxial bending*

- $$\left(\frac{M_{ux}}{M_{ux1}}\right)^{\alpha_n} + \left(\frac{M_{uy}}{M_{uy1}}\right)^{\alpha_n} = \left(\frac{125}{187}\right)^{1.575} + \left(\frac{75}{110}\right)^{1.575}$$
  $$= 0.530 + 0.547$$
  $$= 1.077 > 1.0$$

Hence, the given load is found to marginally exceed the safe limit prescribed by the Code (by 8%).

## 13.7 DESIGN OF SLENDER COLUMNS

### 13.7.1 Behaviour of Slender Columns

As discussed in Section 13.1.3, compression members are categorised as being either *short* or *slender* (*long*), depending on whether slenderness effects can be ignored or need special consideration. It is also explained in Section 13.1.3 that the *slenderness ratios* ($l_{ex}/D_x$, $l_{ey}/D_y$) provide a simple basis for deciding whether a column is short or 'slender'. The behaviour and design of short columns under axial, uniaxial eccentric and biaxial eccentric loading conditions have been extensively described in Sections 13.4, 13.5 and 13.6 respectively.

This section describes the behaviour of slender columns, and shows how this behaviour increasingly deviates from the short column behaviour with increasing slenderness ratios. To begin with, a simple example of a pin-ended column with an eccentrically applied load [Fig. 13.29(a)] is considered. The height $l$ between the pinned ends is the 'unsupported length', which, in this case, is also equal to the 'effective length' [refer Fig. 13.3(a)]. By considering different heights of the column, with the same cross-section, the effects of different slenderness ratios can be studied. Subjecting the column to a gradually increasing load $P$, applied at an eccentricity $e$ (with the undeflected longitudinal axis), the behaviour of the column can be observed until failure.

Due to the applied eccentricity $e$, 'primary moments' $M_{pr} = Pe$ are developed not only at the end sections of the column, but all along the height [Fig. 13.29(b)]. The bending of the column causes it to deflect laterally, thereby introducing additional displacement (load) dependent eccentricities. If the lateral deflection of the longitudinal axis is denoted as $\Delta$, then the total eccentricity is $e + \Delta$, and the total moment $M$ at any section is given by

$$M = P(e + \Delta) \tag{13.42a}$$

$$M = M_{pr} + P\Delta \tag{13.42b}$$

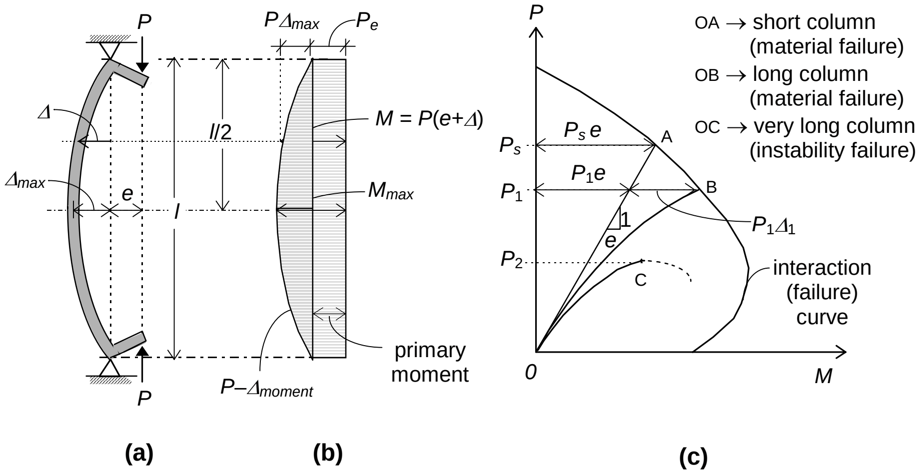



**Fig. 13.29** Behaviour of slender columns

where $P\Delta$ is the 'secondary moment' (also called '$P$–$\Delta$ moment'), which has a variation along the height of the column that is identical to that of $\Delta$ [refer Fig. 13.29(a), (b)]. The maximum value of $\Delta$ (i.e., $\Delta_{max}$), and hence the maximum value of the total moment $M_{max} = P(e + \Delta_{max})$ occurs at the mid-height section of the column.

It should be noted that the lateral deflection $\Delta_{max}$ is not only due to the curvature produced by the primary moment $M_{pr}$, but also due to the $P$–$\Delta$ moment. Hence, the variation of $M_{max}$ with $P$ is nonlinear, with $M_{max}$ increasing at a faster rate as $P$ increases. The axial thrust $P$ effectively reduces the flexural stiffness of the column ('beam column'), and, in the case of a very slender column, it may so happen that the flexural stiffness is effectively reduced to zero, resulting in an instability (buckling) failure. On the other hand, in the case of a very short column, the flexural stiffness is so high that the lateral deflection $\Delta$ is negligibly small† ; consequently, the $P$–$\Delta$ moment is negligible, and the primary moment $M_{pr}$ alone is of significance in such a case.

Fig. 13.29(c) shows the axial load-moment interaction curve (at the ultimate limit state) for the column *section*. This curve, therefore, represents the strength of the column. Also shown in Fig. 13.29(c) are three different loading paths OA, OB, OC that are possible (for different slenderness ratios) as the column in Fig. 13.29(a) is loaded to failure, with increasing $P$ (and hence, $M$) and constant eccentricity $e$. In the case of a very short column, $\Delta_{max} \approx 0$ (as explained earlier) and $M_{max} = Pe$. The resulting $P$ – $M$ path is linear, as indicated by the line OA in Fig. 13.29(c). The

† Theoretically, $\Delta_{max} = 0$ only if the effective length $l_e = 0$ or if $e = 0$ (pure axial loading). In practical 'short' columns, some lateral deflection is unavoidable, particularly at high eccentricities of loading. However, it is expected that the $P$–$\Delta$ moment in a short column will not exceed about 5 percent of the primary moment, and so may be neglected.

termination of this line at the point of intersection A with the interaction (failure) curve indicates the failure of the column at a load, say $P = P_s$ and a moment $M_{max} = P_s e$. The failure occurs by the crushing of concrete at the section of maximum moment.

Had the column been longer (and hence, 'slender'), with increasing load $P$, the deflection $\Delta_{max}$ is no longer negligible, and the moment $M_{max} = P(e + \Delta_{max})$ will vary nonlinearly with $P$, as indicated by the line OB in Fig. 13.29(c). Failure occurs at a load $P = P_1$ and a moment $M_{max} = P_1(e + \Delta_1)$; this is represented by the point B on the interaction curve. In this case, $P_1 e$ and $P_1 \Delta$, denote respectively the primary moment and secondary ($P$–$\Delta$) moment at failure. As shown in the figure, the secondary moment can become comparable to the primary moment in magnitude at the ultimate limit state. Furthermore, comparing the loading paths OA with OB, it follows that although the column section and the eccentricity in loading are identical in the two cases, the mere fact that one column is longer than the other can result in a reduction in the load-carrying capacity (as well as the primary moment resistance). In both cases, the final failure will be a material failure — either a 'compression failure' or a 'tension failure' — depending on which parts of the interaction curve the points A and B lie [refer Section 13.5.2]. Most columns in practical building frames are expected to have this type of failure at the ultimate limit state.

If the column in Fig. 13.29 is very long, the increase in lateral deflection $\Delta_{max}$ may be so excessive that the load-moment path corresponds to OC, with $dP/dM$ reaching zero at the point C. In this case, the column is so slender that it fails by instability (buckling) at a relatively low axial load $P_2$. This type of failure may occur in very slender columns in unbraced frames.

### *Braced Slender Columns: Member Stability Effect*

As explained in Section 13.2.3, a 'braced column' is one which is not subject to sidesway, i.e., there is no significant relative lateral displacement between the top and bottom ends of the column. The pin-jointed column of Fig. 13.29 is a simple example of a braced column. In general, the ends of a braced column (which forms part of a 'braced frame') are partially restrained against rotation (by the connecting beams). The primary moments $M_1$ and $M_2$ that are applied at the two ends of the column are determined from a 'first-order' structural analysis; i.e., analysis which assumes linear elastic behaviour, and neglects the influence of change in geometry of the frame due to deflections. The column may be bent in *single curvature* or *double curvature*, depending on the directions of $M_1$ and $M_2$ [Fig. 13.30]. The notations $M_1$ and $M_2$ generally refer to the smaller and larger column end moments, and the ratio $M_1/M_2$ is considered positive if the column is bent in single curvature, and negative if it is bent in double curvature.

If $M_1/M_2 = +1.0$, the column is bent in symmetrical single curvature, and the slenderness in the column will invariably result in an increased moment. However, in the more general case of unequal end moments ($M_1/M_2 \neq 1.0$), it is not necessary that slenderness will result in a peak moment in the column that is greater than the larger primary end moment $M_2$ — as indicated by the curves labelled "1" in Fig. 13.30. If, however, the column is very slender, and the consequent lateral deflection $\Delta_2$ is sufficiently high (curve '2' in Fig. 13.30), the total moment to be considered in design (i.e., including the additional moment $P\Delta_2$) may exceed $M_2$. This is less likely in columns bent in double curvature [Fig. 13.30(b)]. In fact, the chances of a given slenderness resulting in a peak design moment larger than $M_2$ fall off significantly as the ratio $M_1/M_2$ drops below about +0.5 and approaches the limit of –1.0. The possible amplification in bending moment (over the primary moment $M_2$) on account of lateral displacements (relative to the chord joining the column ends) is termed as *member stability effect*.

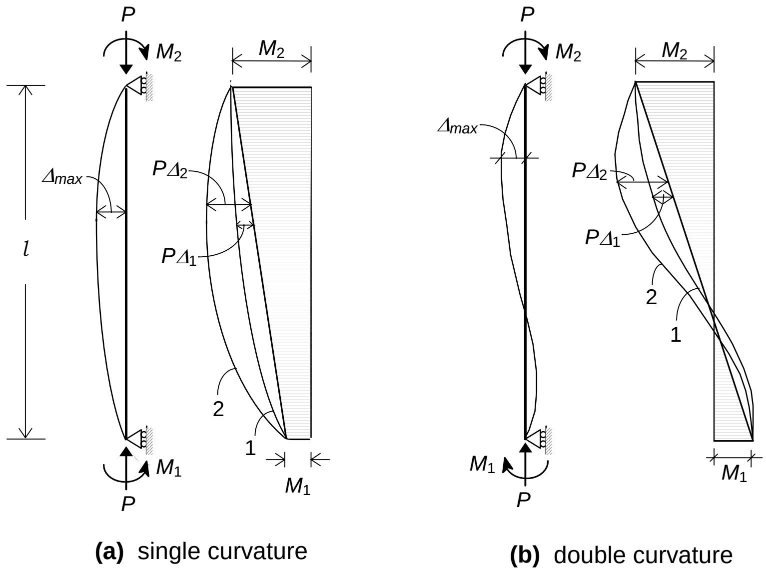



**Fig. 13.30** Braced columns: member stability effect

Thus, the criticality of slenderness effects is also dependent on the ratio $M_1/M_2$. The ACI Code [Ref. 13.1] recommends that slenderness effects may be ignored (i.e., the column may be designed as a 'short column') if, for a braced column,

$$l_e/r < 34 - 12\, M_1/M_2 \qquad (13.43)$$

where $l_e$ is the effective length and $r$ the radius of gyration. Thus, the slenderness ratio ($l_e/r$) limit for short columns lies in the range 22–34 in single curvature and 34–46 in double curvature. It is shown [Ref. 13.16] that this slenderness limit [Eq. 13.43] corresponds to effective lengths for which the ultimate axial load capacity, including 'member stability' effect, is at least 95 percent of the axial compressive strength of the cross-section.

### *Unbraced Slender Column: Lateral Drift Effect*

As explained in Section 13.2.3, an 'unbraced column' is one which is subject to sideway (or 'lateral drift'), i.e., there is significant lateral displacement between the top and bottom ends of the column. The lateral drift may occur due to the action of lateral loads, or due to gravity loads when the loading or the frame is asymmetric.

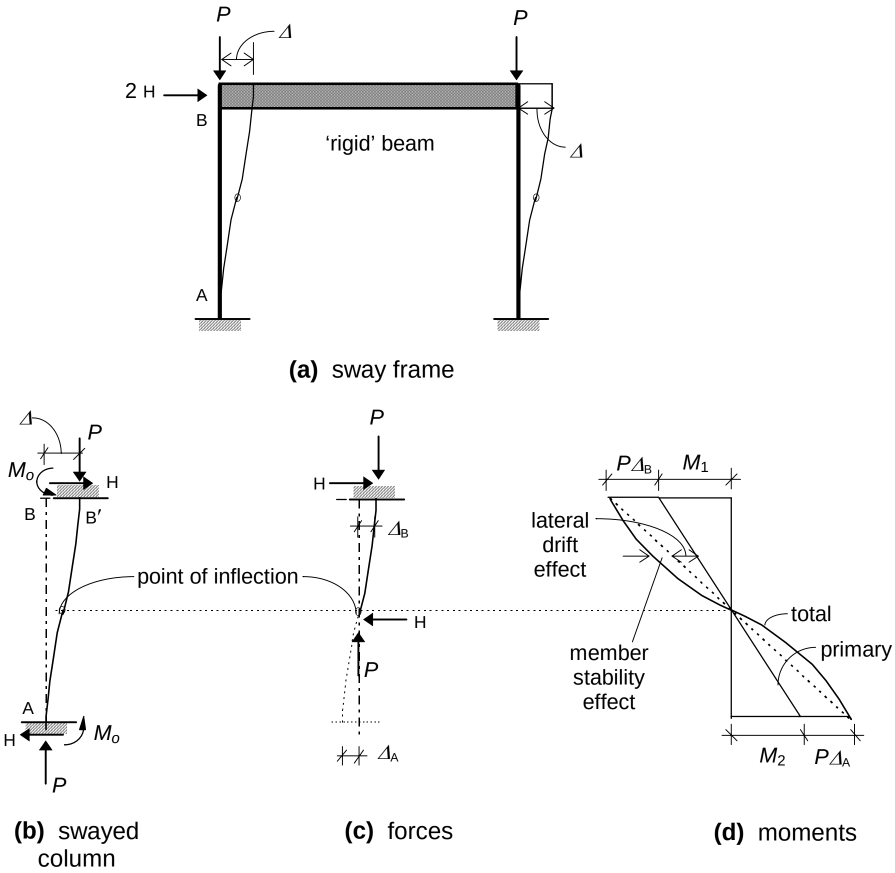



**Fig. 13.31** Unbraced columns: lateral drift effect

Considering the simple portal frame of Fig. 13.31(a) (in which the beam is assumed to be infinitely rigid, for convenience), the lateral drift (or sideway) of the column is the relative translational displacement $\Delta$ (= $\Delta_A$ + $\Delta_B$) between the ends of the column. The additional moments at the column ends caused by the action of the vertical load acting on the deflected configuration of the unbraced column is termed the *lateral drift effect*. In unbraced columns, the action of the primary moments ($M_1$, $M_2$) generally results in 'double curvature', which is further enhanced by the lateral drift effect. In addition, there is the 'member stability effect' (described earlier) on account of the lateral displacements at points along the length of the column relative to the chord joining the column ends [Fig. 13.31(d)]. However, generally, for unbraced columns, the moments at the column ends are maximum, and these are due to the primary moments enhanced by the lateral drift effect alone.

The moment amplification possible due to lateral drift effect in an unbraced column is generally much more than that due to member stability effect in a braced column. Further, as explained in Section 13.2.3, the effective length of an unbraced column is much more than that of a braced column with the same unsupported length. Hence, columns in unbraced frames are weaker than similar columns in braced frames.

### 13.7.2 Second-Order Structural Analysis of Slender Column Structures

The main problem with slender column design lies in determining the factored moments (including $P$–$\Delta$ effects) to be considered in design. In other words, the problem is essentially one of structural *analysis*, rather than structural *design*. The principles of designing a column section under a given factored axial compression $P_u$ and factored moments $M_{ux}$, $M_{uy}$ [described in Section 13.6] remain the same for both short columns and slender columns; the only difference is that $M_{ux}$ and $M_{uy}$ must include secondary moment components in slender column design, whereas these secondary moment components (being negligible) are ignored in short column design.

#### *Rigorous Analysis*

In general, the Code (Cl. 39.7) broadly recommends that when slender columns are involved in a reinforced concrete structure, a detailed 'second-order' structural analysis should be carried out to determine the bending moments and axial forces for which the slender columns are to be designed. Indeed, such a rigorous analysis is particularly desirable for slender columns in unbraced frames. Such analysis must take into account *all* slenderness effects, viz. the influence of column and frame deflections on moments, effects of axial loads and effects of sustained loads. Realistic moment-curvature relationships should be made use of. The details of procedures for second-order analysis lie outside the scope of this book; these details are presented in Ref. 13.17 – 13.19.

It should be noted that the *principle of superposition* is not valid in second-order analysis, and for this reason, the load effects due to different load combinations cannot be obtained by an algebraic summing up (with appropriate load factors); each load combination should be investigated separately. This requires substantial computational effort.

### 13.7.3 Code Procedures for Design of Slender Columns

In routine design practice, only first-order structural analysis (based on the linear elastic theory and undeflected frame geometry) is performed, as second-order analysis is computationally difficult and laborious. In recognition of this, the Code recommends highly simplified procedures for the design of slender columns, which either attempt to predict the increase in moments (over primary moments), or, equivalently, the reduction in strength, due to slenderness effects.

### *Strength Reduction Coefficient Method*

This is a highly simplified procedure, which is given in the Code for the working stress method of design [refer Section 13.4.3]. According to this procedure (B-3.3 of the Code) the *permissible stresses* in concrete and steel [Eq. 13.15, 13.16] are reduced by multiplication with a *strength reduction coefficient* $C_r$, given by:

$$C_r = 1.25 - \frac{l_e}{48d} \tag{13.44a}$$

where $d$ is the least lateral dimension of the column (or diameter of the core in a spiral column). Alternatively, *for more exact calculations*,

$$C_r = 1.25 - \frac{l_e}{160 r_{\min}} \tag{13.44b}$$

where $r_{min}$ is the least radius of gyration of the column. There is some ambiguity in Eq. 13.44(a), (b) regarding the plane in which the effective length $l_e$ is to be estimated. This can be resolved by considering $(l_e/d)_{max}$ in Eq. 13.44(a) and $(l_e/r)_{max}$ in Eq. 13.44(b) i.e., considering the maximum effective slenderness ratio of the column.

It is recommended in the Explanatory Handbook to the Code [Ref. 13.7] that instead of applying the strength reduction factor $C_r$ to the 'permissible stresses', this factor may be directly applied to the load-carrying capacity estimated for a corresponding short column. Furthermore, it may be noted that although this method has been prescribed for WSM, it can be extended to the limit state method (LSM) for the case of axial loading (without primary bending moments). This is demonstrated in Example 13.17.

### *Additional Moment Method*

The method prescribed by the Code (Cl. 39.7.1) for slender column design by the limit state method is the 'additional moment method'[†] , which is based on Ref. 13.20, 13.21. According to this method, every slender column should be designed for biaxial eccentricities which include the $P–\Delta$ moment ("additional moment") components $e_{ax} \equiv M_{ax}/P_u$ and $e_{ay} \equiv M_{ay}/P_u$ :

$$\tilde{M}_{ux} = P_u\,(e_x + e_{ax}) = M_{ux} + M_{ax} \tag{13.45a}$$

$$\tilde{M}_{uy} = P_u\,(e_y + e_{ay}) = M_{uy} + M_{ay} \tag{13.45b}$$

Here, $\tilde{M}_{ux}$ and $\tilde{M}_{uy}$ denote the total design moments; $M_{ux}$, $M_{uy}$ denote the *primary* factored moments[‡] (obtained from first-order structural analyses); and $M_{ax}$, $M_{ay}$ denote the *additional moments* with reference to bending about the major and minor axes respectively.

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[†] An alternative method called the 'moment magnification method' is adopted by the ACI and Canadian codes.

[‡] The primary moments should not be less than those corresponding to the minimum eccentricities spcified by the Code.

The essence of this method lies in a simple formulation for the determination of the *additional eccentricities* $e_{ax}$, $e_{ay}$. In the basic formulation, the $P$–$\Delta$ effect in a braced slender column with pin-joined ends [Fig. 13.29a] is considered. The 'additional eccentricity' $e_a$ is equal to $\Delta_{max}$ in Fig. 13.29(a), which is a function of the curvatures to which the column is subjected. If the maximum curvature (at mid-height) is denoted as $\varphi_{max}$, it can be shown [refer Fig. 13.32] that $\Delta_{max}$ lies between $\varphi_{max}l^2/12$ and $\varphi_{max}l^2/8$, the former limit corresponding to a linearly varying curvature (with zero at the pin joints and a maximum of $\varphi_{max}$ at midheight) and the latter corresponding to a constant curvature along the column height [Fig. 13.32(b),(c)].

Taking an average value,

$$e_a = \Delta_{max} \approx \varphi_{max}\, l^2/10 \tag{13.46}$$

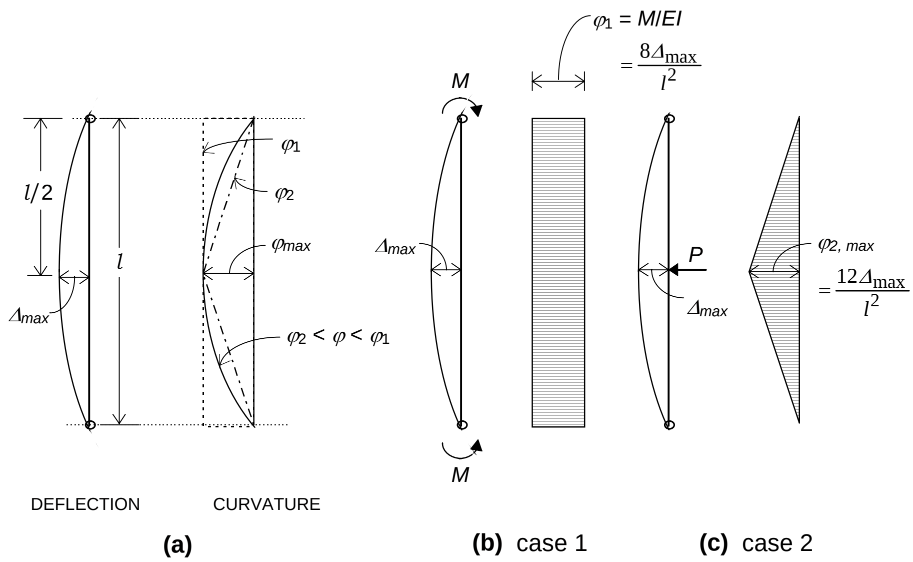


**Fig. 13.32** Relation between $\Delta_{max}$ and $\varphi_{max}$ in a pin-joined braced slender column

Failure of the column at the ultimate limit state is expected to occur at the section corresponding to $\varphi_{max}$. By making suitable assumptions, $\varphi_{max}$ can be expressed in terms of the failure strains $\varepsilon_{cu}$ and $\varepsilon_{st}$ in concrete (at the highly compressed edge) and steel (in the outermost row) respectively, as shown in Fig. 13.33. The values of $\varepsilon_{cu}$ and $\varepsilon_{st}$ evidently depend on the factored axial load $P_u$ (as explained in Section 13.5.1); this determines the location of the point of failure, marked B in the interaction curve in Fig. 13.29(c).

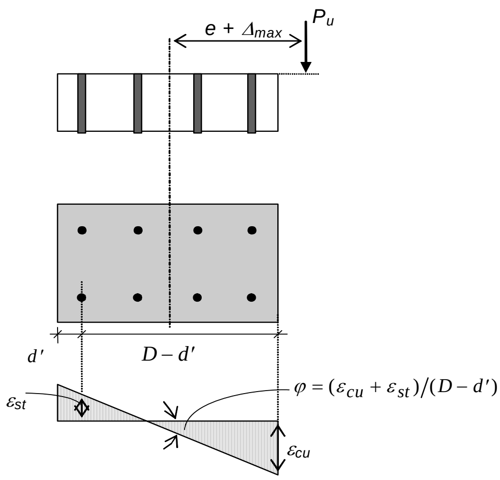


**Fig. 13.33** Determination of curvature from failure strain profile

Assuming that $\varepsilon_{cu}$ = 0.0035 and $\varepsilon_s$ = 0.002[†] , $d' \approx 0.1D$ and further assuming (rather conservatively) that the additional moment comprises about 80 percent of the total moment,

$$\varphi_{\max} \cong \frac{0.0035 + 0.002}{0.9D} \times 0.8 \approx \frac{1}{200D} \tag{13.47}$$

Combining Eq. 13.47 with Eq. 13.46, the following expression for the additional eccentricity ratio $e_a/D$ is obtained:

$$e_a/D \approx \frac{(l/D)^2}{2000} \tag{13.48}$$

Accordingly, the following expressions for additional moments $M_{ax}$, $M_{ay}$ [in Eq. 13.45a, b] are obtained, as given in the Code (Cl. 39.7.1):

$$M_{ax} = P_u\, e_{ax} = \frac{P_u D_x}{2000}\,(l_{ex}/D_x)^2 \tag{13.49a}$$

$$M_{ay} = P_u\, e_{ay} = \frac{P_u D_y}{2000}\,(l_{ey}/D_y)^2 \tag{13.49b}$$

where $l_{ex}$ and $l_{ey}$ denote the *effective lengths*, and $D_x$ and $D_y$ denote the depths of the rectangular column section with respect to bending about the major axis and minor axis respectively. It may be noted that the height $l$ in Eq. 13.48 has been replaced by

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[†] This approximately corresponding to the 'balanced failure' condition, whereby $\varepsilon_{st} = \varepsilon_y$ at the cracked section. For deflection calculations, the mean steel strain should be considered, including the effect of 'tension stiffening' (refer chapter 10).

the effective length $l_e$ in Eq. 13.49 to extend the application of the formulation to the various boundary conditions (other than the pinned-end condition) that occur in practical columns including unbraced columns. It is reported [Ref. 13.7] that the use of Eq. 13.49 has been validated with reference to a large number of experimental tests [Ref. 13.20]. It is seen from Eq. 13.48 and Eq. 13.49 that the $e_a/D$ ratio increases with the square of the slenderness ratio $l_e/D$; $e_a/D$ has a minimum value of 0.072 for $l_e/D$ =12 (transition between 'short column' and 'slender column') and a maximum value of 0.450 for $l_e/D$ = 30 (recommended limit for unbraced columns) and 1.800 for $l_e/D$ = 60 (braced column).

It should be noted that Eq. 13.49 relates to the 'additional moments' to be considered in addition to the maximum factored primary moments $M_{ux}$, $M_{uy}$ in a column. Under eccentric loading, these primary moments should not be less than those corresponding to the minimum eccentricities specified by the Code. Where a primary moment is not considered, i.e., taken as zero, (as under axial loading), it should be ensured that the corresponding additional moment is not less than that computed from considerations of minimum eccentricity. The derivation of Eq. 13.49 assumes that the column is braced and bent symmetrically in single curvature; some modification is required when the primary moments applied at the column ends are unequal and/or of different signs. Further, it is assumed that the axial load level corresponds approximately to the 'balanced failure' condition $P_u = P_{ub}$; Eq. 13.49 needs to be modified for other axial load levels. Hence, the Code recommends the following modifications to be incorporated with the use of Eq. 13.49 (and Eq. 13.45) for the design of slender columns in general:

- For $P_u > P_{ub}$, the failure mode is one of 'compression failure', and the corresponding $e/D$ ratio is low. At relatively high axial loads, the entire section may be under compression, suggesting low curvatures [Fig. 13.34]. Hence, the use of Eq. 13.49 in such situations can result in highly conservative results. The additional moments $M_{ax}$, $M_{ay}$ given by Eq. 13.49 may be reduced by multiplying factors (refer Cl. 39.7.1.1 of the Code) defined as:

$$k_{ax} = \frac{P_{uz} - P_u}{P_{uz} - P_{ub,x}} \quad \text{for } P_u > P_{ub,x} \tag{13.50a}$$

$$k_{ay} = \frac{P_{uz} - P_u}{P_{uz} - P_{ub,y}} \quad \text{for } P_u > P_{ub,y} \tag{13.50b}$$

  where $P_{uz}$ is the maximum 'pure compression' strength of the column and $P_{ub,x}$ and $P_{ub,y}$ correspond to the axial strength corresponding to balanced failure with respect to bending about the major axis and minor axis respectively. $P_{uz}$ is readily obtainable from Eq. 13.39 and $P_{ub}$ from the interaction curve (refer Fig. 13.20) corresponding to a design tensile stress of $f_{yd} = 0.87\ f_y$ in the outermost layer of steel.

  It can be seen that $k$ varies linearly from zero (for $P_u = P_{uz}$) to unity (for $P_u = P_{ub}$) and is a highly simplified formula. It should also be noted that Eq. 13.50 is not applicable for $P_u < P_{ub}$; i.e., $k_a = 1$ for $P_u < P_{ub}$.

- For *braced columns* subject to unequal primary moments $M_1$, $M_2$ at the two ends [Fig. 13.30(a)], the value of $M_u$ to be considered in the computation of the total moment $\tilde{M}_u$ in Eq. 13.45 may be taken as:

$$M_u = 0.4M_1 + 0.6M_2 \geq 0.4M_2 \qquad (13.51)$$

where $M_2$ is the higher column end moment. As mentioned earlier, with reference to Fig. 13.30, $M_1$ and $M_2$ are considered to be of opposite signs if the column is bent in double curvature. In the case of braced columns subject to double curvature, it is possible that the use of Eq. 13.51 in Eq. 13.45 may result in a total moment $\tilde{M}_u$ that is less than $M_2$; this obviously, cannot be allowed. Hence, a further condition needs to be imposed:

$$\tilde{M}_u \geq M_2 \qquad \text{for braced columns} \qquad (13.52)$$

- In the case of *unbraced columns*, the lateral drift effect (hitherto not considered) needs to be included [Fig. 13.31]. An approximate way of accounting for this is by assuming that the additional moment $M_a$ (given by Eq. 13.49†) acts at the column end where the maximum primary moment $M_2$ is operational. Hence, for design purposes, the total moment $\tilde{M}_u$ may be taken as:

$$\tilde{M}_u = M_2 + M_a \qquad \text{for unbraced columns} \qquad (13.53)$$

**EXAMPLE 13.17**

Determine the maximum factored axial load-carrying capacity of the column in Fig. 13.14(a), given that the column is 'braced' against sideway, and has an unsupported length of 7.0 m. Assume effective length ratios $k_x = k_y = 0.85$.

**SOLUTION**

- *Given*: (refer Example 13.5): $D_x$ = 500 mm, $D_y$ = 300 mm, $A_{sc}$ = 2946 mm$^2$, $f_{ck}$ = 25 MPa, $f_y$ = 415 MPa.
  Also, $l$ = 7000 mm, $k_x = k_y = 0.85$.

*Slenderness ratios:*

- $l_{ex} = 0.85 \times 7000 = 5950$ mm $\Rightarrow l_{ex}/D_x = 5950/500 = 11.90 < 12$
  $l_{ey} = 0.85 \times 7000 = 5950$ mm $\Rightarrow l_{ey}/D_y = 5950/300 = 19.83 > 12$
- Hence, the column has to be treated as a *slender* column.

**Strength reduction coefficient method**

- Extending the *strength reduction coefficient method* given in the Code (B-3.3) for WSM to LSM,

$$C_r = 1.25 - \frac{(l_e/D)_{\max}}{48}$$
$$= 1.25 - 19.83/48 = 0.837$$

† It is inadvisable to apply the reduction factor $k$ (given by Eq. 13.50) for unbraced columns.

- Considering *short column* behaviour (with dimensions satisfying Cl. 39.3 of the Code),

  $\tilde{P}_{uo} = 0.4f_{ck} A_g + (0.67f_y - 0.4f_{ck})A_s$

  $= 2290$ kN (as determined earlier in Example 13.9).
- Considering *slender column* behaviour,

  $(P_u)_{max} = C_r \tilde{P}_{uo}$

  $= 0.837\times 2290$

  $=$ **1917 kN**

*Note*: The safety of the column under this factored load, combined with minimum eccentricities, may now be verified by the *additional moment method* given in the Code for LSM.

**Additional moment method**

- *Minimum eccentricities*

  $$e_{x,min} = \frac{l}{500} + \frac{D_x}{30} = \frac{7000}{500} + \frac{500}{30} = 30.67 \text{ mm} > 20 \text{ mm}$$

  $$e_{y,min} = \frac{l}{500} + \frac{D_y}{30} = \frac{7000}{500} + \frac{300}{30} = 24.00 \text{ mm} > 20 \text{ mm}$$
- *Primary moments*

  As the column is under axial loading, $M_{ux} = M_{uy} = 0$. However, it must be ensured that the total moments $\tilde{M}_{ux}$, $\tilde{M}_{uy}$ should not be less than those due to corresponding minimum eccentricities.

*Additional moments*

- Without modification factors:

  $e_{ax} = D_x (l_{ex}/D_x)^2/2000 = 500\ (11.90)^2/2000 = 35.40$ mm

  $e_{ay} = D_y (l_{ey}/D_y)^2/2000 = 300\ (19.83)^2/2000 = 58.98$ mm
- modification factors $k_{ax}$, $k_{ay}$

  $$k_{ax} = \frac{P_{uz} - P_u}{P_{uz} - P_{ub,x}};\ k_{ay} = \frac{P_{uz} - P_u}{P_{uz} - P_{ub,y}}$$

  where

  $P_{uz} = 0.45f_{ck} A_g + (0.75f_y - 0.45f_{ck})A_s$

  $= (0.45 \times 25 \times 300 \times 500) + (0.75 \times 415 - 0.45 \times 25) \times 2946$

  $= (1687.5 \times 10^3 + 883.8 \times 10^3)N = 2571$ kN

  $P_{ub,x} = 445$ kN, $P_{ub,y} = 470$ kN (as determined in Examples 13.9, 13.11).

  $\Rightarrow k_{ax} = (2571 - 1935)/(2571 - 445) = 0.299$

  $k_{ay} = (2571 - 1935)/(2571 - 470) = 0.303$

  $M_{ax} = P_u\ (k_{ax}e_{ax}) = 1935 \times (0.299 \times 0.0354) = 20.5$ kNm

  $M_{ay} = P_u\ (k_{ay}e_{ay}) = 1935 \times (0.303 \times 0.05898) = 34.6$ kNm
- *Total Moments*

  $\tilde{M}_{ux} = M_{ux} + M_{ax} = 0.0 + 20.5 = 20.5$ kNm

  $< P_u\, e_{x,\,min} = 1935 \times 0.03067 =$ **59.3 kNm**

$\tilde{M}_{uy} = M_{uy} + M_{ay} = 0.0 + 34.6 = 34.6$ kNm

$< P_u e_{y,\,min} = 1935 \times 0.0240 =$ **46.4 kNm**

*Check safety under biaxial loading*

- Corresponding to $P_u = 1935$ kN,
  $M_{ux1} = 130$ kNm [refer Table 13.1, Fig. 13.18] > $\tilde{M}_{ux} = 59.3$ kNm
  $M_{uy1} = 75.7$ kNm [refer Table 13.2, Fig. 13.19] > $\tilde{M}_{uy} = 46.4$ kNm
- $P_u/P_{uz} = 1935/2571 = 0.753$ (which lies between 0.2 and 0.8)

$\Rightarrow \alpha_n = 1.0 + \dfrac{0.753-0.2}{0.8-0.2}(2.0-1.0) = 1.922$

- $$\left(\frac{\tilde{M}_{ux}}{M_{ax1}}\right)^{\alpha_n} + \left(\frac{\tilde{M}_{uy}}{M_{ay1}}\right)^{\alpha_n} = \left(\frac{59.3}{130}\right)^{1.922} + \left(\frac{46.4}{75.7}\right)^{1.922}$$
  $$= 0.221 + 0.390$$
  $$= 0.611 < 1.0 \qquad \text{— safe}$$
- Evidently, $P_u = 1935$ kN is a slightly conservative estimate of the axial load-carrying capacity of the slender column. This is as expected, owing to the conservatism built into the highly simplified 'strength reduction coefficient' method.

### EXAMPLE 13.18

Design the longitudinal reinforcement for a braced column, 300 mm × 400 mm, subject to a factored axial load of 1500 kN and factored moments of 60 kNm and 40 kNm with respect to the major axis and minor axis respectively at the top end. Assume that the column is bent in double curvature (in both directions) with the moments at the bottom end equal to 50 percent of the corresponding moments at top. Assume an unsupported length of 7.0 m and an effective length ratio of 0.85 in both directions. Use M 30 concrete and Fe 415 steel.

### SOLUTION

- *Given*: $D_x = 400$ mm, $D_y = 300$ mm, $P_u = 1500$ kN; $M_{ux} = 60$ kNm, $M_{uy} = 40$ kNm, at top; $M_{ux} = 30$ kNm, $M_{uy} = 20$ kNm, at bottom; $l = 7000$ mm, $k_x = k_y = 0.85$.

*Slenderness ratios*

- $l_{ex} = l_{ey} = 0.85 \times 7000 = 5950$ mm
  $\Rightarrow l_{ex}/D_x = 5950/400 = 14.88 > 12$
  $l_{ey}/D_y = 5950/300 = 19.83 > 12$
  Hence, the column should be designed as a *slender column*.

*Minimum eccentricities*

- $e_{x,min} = \dfrac{l}{500} + \dfrac{D_x}{30} = \dfrac{7000}{500} + \dfrac{400}{30} = 27.33$ mm > 20 mm
  $e_{y,min} = \dfrac{l}{500} + \dfrac{D_y}{30} = \dfrac{7000}{500} + \dfrac{300}{30} = 24.00$ mm > 20 mm

*Primary moments for design*

- As the column is braced and bent in double curvature [refer Eq. 13.51],
  $M_{ux} = (0.6 \times 60 - 0.4 \times 30) = 24$ kNm  $(\geq 0.4 \times 60 = 24$ kNm$)$
  $M_{uy} = (0.6 \times 40 - 0.4 \times 20) = 16$ kNm  $(\geq 0.4 \times 40 = 16$ kNm$)$
  $\Rightarrow$ corresponding (primary) eccentricities:
  $e_x = 24 \times 10^3/1500 = 16$ mm $< e_{x\,min} = 27.33$ mm
  $e_y = 16 \times 10^3/1500 = 10.67$ mm $< e_{y\,min} = 24.00$ mm
  The primary eccentricities should not be less than the minimum eccentricities.
- $\therefore$ Primary moments for design: $M_{ux} = 1500 \times (27.33 \times 10^{-3}) = 41.0$ kNm
  $M_{uy} = 1500 \times (24.00 \times 10^{-3}) = 36.0$ kNm

*Additional moments*

- Without modification factor, additional eccentricities
  $e_{ax} = D_x\,(l_{ex}/D_x)^2/2000$
  $= 400\,(14.88)^2/2000 = 44.28$ mm
  $e_{ay} = D_y\,(l_{ey}/D_y)^2/2000$
  $= 300\,(19.83)^2/2000 = 58.98$ mm
- Assuming modification factors $k_{ax} = k_{ay} \approx 0.5$ (to be verified later), additional moments:
  $M_{ax} = P_u\,(k_{ax}e_{ax}) = 1500(0.5 \times 0.04428)$
  $= 33.2$ kNm
  $M_{ay} = P_u\,(k_{ay}e_{ay}) = 1500(0.5 \times 0.05898) = 34.6$ kNm
  $= 44.2$ kNm

*Total factored moments*

$\tilde{M}_{ux} = M_{ux} + M_{ax} = 41.0 + 33.2 = 74.2$ kNm

$\tilde{M}_{uy} = M_{uy} + M_{ay} = 36.0 + 44.2 = 80.2$ kNm $> \tilde{M}_{ux} = 74.2$ kNm

*Trial section*

- Designing for a resultant uniaxial moment with respect to the *minor* axis,
  $M_u \approx 1.15\sqrt{74.2^2 + 80.2^2} = 126$ kNm
  combined with $P_u = 1500$ kNm
  $\Rightarrow P_u/f_{ck}bD = (1500 \times 10^3/(30 \times 400 \times 300) = 0.417$
  $M_u/f_{ck}bD^2 = 126 \times 10^6/(30 \times 400 \times 300^2) = 0.117$
- Assuming 25 $\phi$ for main bars, 8 $\phi$ ties and 40 mm clear cover, $d' = 60.5$ mm
  $\Rightarrow d'/D \approx 60.5/300 = 0.201 \approx 0.20$ with "equal reinforcement on all sides", and referring to Chart 46 of SP : 16,
  $p/f_{ck} = 0.13 \Rightarrow p_{reqd} = 0.13 \times 30 = 3.9$
  $\Rightarrow A_{s,reqd} = 3.9 \times 300 \times 400/100 = 4680$ mm$^2$
- Provide 8–28 $\phi$ (as shown in Fig. 13.34).
  [$A_s = 8 \times 616$ mm$^2$] 4928 > 4680
  $\Rightarrow p_{provided} = 4928 \times 100/(300 \times 400) = 4.107$
  $\Rightarrow p/f_{ck} = 4.107/30 = 0.137.$

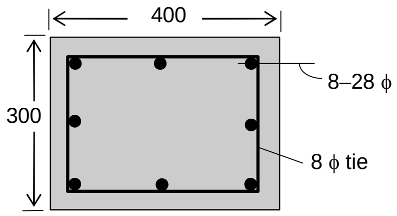



**Fig. 13.34** Example 13.18

*Check additional moments*

- Assuming a clear cover of 40 mm, $d' = 40 + 8 + 28/2 = 62$ mm
  $\Rightarrow d'/D_x = 0.155 \approx 0.15$ and $d'/D_y = 0.207 \approx 0.20$
- Referring to Charts 45 ($d'/D = 0.15$) and 46 ($d'/D = 0.20$) of SP :16, the ultimate loads $P_{ub,x}$ $P_{ub,y}$ at balanced failure can be determined by considering the stress level $f_{yd} = 0.87 f_y$ (marked on the interaction curves).
- Corresponding to $p/f_{ck} = 0.137$,
  for $d'/D_x = 0.15$, $P_{ub,x}/f_{ck}bD = 0.07 \Rightarrow P_{ub,x} = 252$ kN
  for $d'/D_y = 0.20$, $P_{ub,y}/f_{ck}bD = 0.03 \Rightarrow P_{ub,y} = 108$ kN
- $P_{uz} = 0.45 f_{ck} A_g + (0.75 f_y - 0.45 f_{ck}) A_s$
  $= (0.45 \times 30 \times 300 \times 400) + (0.75 \times 415 - 0.45 \times 30) \times 4928$
  $= (1620 \times 10^3 + 1467 \times 10^3) = 3087$ kN
- *Modification factors:*

$$k_{ax} = \frac{P_{uz} - P_u}{P_{uz} - P_{ub,x}} = \frac{3087 - 1500}{3087 - 252} = 0.559$$

$$k_{ay} = \frac{P_{uz} - P_u}{P_{uz} - P_{ub,y}} = \frac{3087 - 1500}{3087 - 108} = 0.533$$

- Hence, the assumed values $k_{ax} = k_{ay} = 0.5$ are fairly accurate. The actual (revised) total moments are obtained as:
  $\tilde{M}_{ux} = 41.0 + 1500(0.559 \times 0.04428) = 78.13$ kNm
  $\tilde{M}_{uy} = 36.0 + 1500(0.533 \times 0.05898) = 83.15$ kNm

*Check safety under biaxial bending*

- Referring to the design Charts in SP : 16, uniaxial moment capacities corresponding to
  $P_u/f_{ck}bD = 0.417$ and $p/f_{ck} = 0.137$
  are obtained as
  $M_{ux1}/f_{ck}bD^2 = 0.135$ (for $d'/D_x = 0.15$)
  $M_{uy1}/f_{ck}bD^2 = 0.115$ (for $d'/D_y = 0.20$)
  $\Rightarrow M_{ux1} = 0.135 \times 30 \times 300 \times 400^2 = 194.4 \times 10^6$ Nmm $= 194.4$ kNm
  $> \tilde{M}_{ux} = 78.1$ kNm

$M_{uy1}$ = 0.115 × 30 × 400 × 300$^2$ =124.2 × 10$^6$ Nmm = 124.2 kNm

$> \tilde{M}_{uy}$ = 83.1 kNm

- $P_u/P_{uz}$ = 1500/3087 = 0.486 (which lies between 0.2 and 0.8)

  $\Rightarrow \alpha_n = 1.0 + \dfrac{0.486-0.2}{0.8-0.2}(2.0 - 1.0) = 1.477$

- $\left(\dfrac{\tilde{M}_{ux}}{M_{ux1}}\right)^{\alpha_n} + \left(\dfrac{\tilde{M}_{uy}}{M_{uy1}}\right)^{\alpha_n} = \left(\dfrac{78.1}{194.4}\right)^{1.477} + \left(\dfrac{83.1}{124.2}\right)^{1.447}$

  $= 0.260 + 0.552$

  $= 0.812 < 1.0$ — Hence, safe

## REVIEW QUESTIONS

13.1 What is meant by *slenderness ratio* of a compression member and what are its implications?

13.2 Distinguish between (i) *unsupported length* and *effective length* of a compression member; (ii) *braced column* and *unbraced column*.

13.3 Why does the Code require all columns to be able to resist a *minimum eccentricity* of loading?

13.4 Why does the Code specify limits to the minimum and maximum reinforcement in columns?

13.5 A short column, 600 mm × 600 mm in section, is subject to a factored axial load of 1500 kN. Determine the *minimum* area of longitudinal steel to be provided, assuming M 20 concrete and Fe 415 steel.

13.6 Enumerate the functions of the transverse reinforcement in a reinforced concrete column.

13.7 Explain the limitations of the traditional working stress method with regard to the design of axially loaded reinforced concrete column.

13.8 Compare the behaviour of tied columns with spiral columns, subject to axial loading.

13.9 Sketch a typical axial load — moment interaction curve for a column and explain the salient points on it.

13.10 A column is subject to a uniaxially eccentric load which results in a point (on the interaction diagram) that lies (i) marginally outside (ii) marginally inside the envelope of the 'design interaction curve'. Comment on the safety of the column for the two situations.

13.11 Explain the reinforcement arrangement details underlying the design interaction curve given in SP : 16 for the condition "rectangular section with reinforcement distributed equally on four sides".

13.12 Briefly explain the difficulties in a rigorous analysis for the design strength components of a given rectangular column section under biaxial loading.

13.13 Explain the basis for the simplified Code procedure for analysing the design strength components of a biaxially loaded column with rectangular cross section.

13.14 What is the main difference, in terms of structural behaviour, between a 'short column' and a 'slender column'?

13.15 Distinguish between 'member stability effect' and 'lateral drift effect' in slender column behaviour.

13.16 In frame analysis, the columns are assumed to be fixed at their bases and the foundations have to be designed to resist the base moments as well as axial loads. In the case of slender columns located at the lowermost storey, is it necessary to include 'additional moments' (due to slenderness effect) while designing the foundations? (Hint: Does this depend on whether the frame is 'braced' or 'unbraced'?)

## PROBLEMS

13.1 A seven-storeyed building has a floor-to-floor height of 4*m* and a plan area of 18*m* × 30*m* with columns spaced at 6*m* intervals in the two directions. Assume that all columns have a size 400 mm × 400 mm with M 25 concrete, and all primary beams have a size 250 mm × 600 mm with M 20 concrete.

(a) Determine the *stability indices* of the structure in the transverse and longitudinal directions, considering the second storey. Assume a total distributed load of 50 kN/m$^2$ from all the floors above combined.

(b) Determine the *effective lengths* of a corner column in the second storey.

13.2 With reference to the short column section shown in Fig. 13.35, assuming axial loading conditions, determine the maximum service load that the column can be safely subjected to:

(i) according to the LSM provisions of the Code (assuming a load factor of 1.5)

(ii) according to the WSM provisions of the Code.

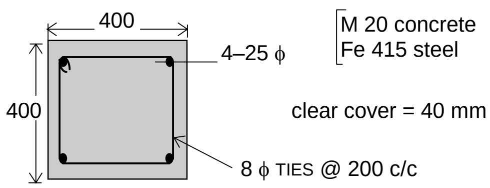
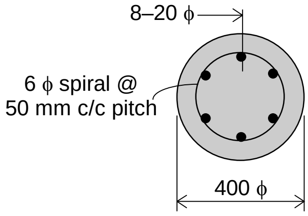



**Fig. 13.35** Problem 13.2

13.3 Repeat Problem 13.2 with reference to the column shown in Fig. 13.36. [*Hint*: The 5 percent increase in strength is allowed subject to certain conditions. Verify].



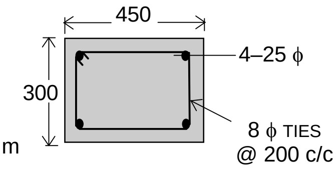




**Fig. 13.36** Problem 13.3 **Fig. 13.37** Problem 13.6

13.4 Design the reinforcement in a column of size 400 mm × 600 mm, subject to a factored axial load of 2500 kN. The column has an unsupported length of 3.0 m and is braced against sideway in both directions. Use M 20 concrete and Fe 415 steel.

13.5 Repeat Problem 13.4, considering a circular column of 400 mm diameter. Assume (i) lateral ties (ii) spiral reinforcement.

13.6 For the column section shown in Fig. 13.37, determine the design strength components corresponding to
(i) the condition of 'balanced failure';
(ii) $x_u/D = 0.55$;
(iii) $x_u/D = 1.1$.
Assume bending with respect to the major axis.

13.7 Repeat Problem 13.6, considering bending with respect to the minor axis.

13.8 Generate the design interaction curves for the column section in Fig. 13.37, considering uniaxial eccentricity with respect to (i) the major axis (ii) the minor axis. [It is convenient to achieve this with the help of a suitable computer program]. Verify with reference to the charts in SP : 16.

13.9 For the L - shaped section shown in Fig. 13.38, determine the design strength components corresponding to the neutral axis location shown in the Figure.

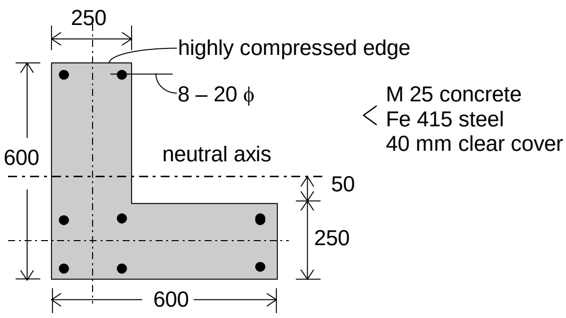



**Fig. 13.38** Problem 13.9

13.10 A short square column 300 mm × 300 mm is reinforced with 4 bars of 25 ϕ, placed with a clear cover of 45 mm. Assuming M 25 concrete and Fe 415 steel, determine

(i) the maximum eccentricity with which a factored load of 1250 kN can be safely applied;

(ii) the maximum factored load that can be applied at an eccentricity of 400 mm.

13.11 A short circular tied column 350 mm diameter is reinforced with 6 bars of 20 ϕ, placed with a clear cover of 40 mm. It is subject to a factored axial load of 1000 kN, combined with factored bending moments of 50 kNm each applied in two perpendicular directions. The concrete is of grade M 25 and the steel of grade Fe 415. Check the safety of the column. If the column is found to be unsafe, suggest suitable modification to the proposed reinforcement.

13.12 Design a short square column, with effective length 3.0*m*, capable of safely resisting the following factored load effects (under uniaxial eccentricity):

(i) $P_u$ = 1625 kN, $M_u$ = 75 kNm

(ii) $P_u$ =365 kN, $M_u$ = 198 kNm.

Assume M 25 concrete and Fe 415 steel.

13.13 Repeat Problem 13.12, considering a suitably proportioned *rectangular* section.

13.14 Repeat Problem 13.12, considering a circular column with spiral reinforcement.

13.15 Design the reinforcement for a column with $l_{ex} = l_{ey}$ = 3.5*m* and size 300 mm × 500 mm, subject to a factored axial load of 1250 kN with biaxial moments of 180 kNm, and 100 kNm with respect to the major axis and minor axis respectively (i.e., $M_{ux}$ = 180 kNm, $M_{uy}$ = 100 kNm). Assume M 25 concrete and Fe 415 steel.

13.16 Repeat Problem 13.15, considering $P_u$ = 1500 kN, $M_{ux}$ = 100 kNm, $M_{uy}$ = 80 kNm.

13.17 Consider a square column, 400 mm × 400 mm, with 4 – 25 ϕ bars at corners placed with a clear cover of 45 mm, and $l_{ex} = l_{ey} = 12D$, subject to axial loading conditions. Determine the maximum factored axial load $P_u$ that the column can safely carry considering:

(a) short column behaviour under axial loading, assuming $l = 12D$;

(b) short column behaviour under biaxial loading with minimum eccentricities;

(c) slender column behaviour (considering 'additional eccentricities' alone†).

Comment on the results obtained. Assume M 20 concrete and Fe 415 steel.

13.18 Design the reinforcement in a column of size 250 mm × 400 mm, with an unsupported length of 6.0 m, subject to a factored axial load of 1100 kN.

† That is, assuming zero primary moments.

Assume the column to be braced, and pinned at both ends in both directions. Assume M 25 concrete and Fe 415 steel, and design by

(i) strength reduction coefficient method;

(ii) additional moment method.

13.19 Repeat Problem 13.18(ii), considering biaxial moments $M_{ux} = M_{uy} = 100$ kNm in addition to $P_u = 1100$ kN.

## REFERENCES


13.1 — *Commentary on Building Code Requirements for Reinforced Concrete* ACI 318–95, American Concrete Institute, Detroit, 1995.

13.2 — *Structural Use of Concrete: Part* 1 *: Code of Practice for Design and Construction*, BS 8110 : Part 1 : 1997, British Standards Institution, London, 1997.

13.3 Wood, R.H., *Effective Lengths of Columns in Multi-Storey Buildings*, The Structural Engineer, Vol. 57, Nos 7–9 (3 parts), July, August and September 1974.

13.4 Kavanagh, T.C., *Effective Length of Framed Columns*, Transactions, ASCE, Vol. 127, Part II, 1962, pp 81–101.

13.5 Breen J.E., MacGregor, J.G., and Pfrang, E.O., *Determination of Effective Length Factors for Slender Concrete Columns*, Journal ACI, Vol. 69, No. 11, Nov.1972, pp 669–672.

13.6 Timoshenko, S.P. and Gere, J.M., *Theory of Elastic Stability*, Second edition, McGraw Hill International edition, 1963.

13.7 — *Explanatory Handbook on Indian Standard Code of Practice for Plain and Reinforced Concrete (IS 456:1978)*, Special Publication SP:24, Bureau of Indian Standards, New Delhi, 1983.

13.8 Taranath, B.S., *Structural Analysis and Design of Tall Buildings*, McGraw-Hill International edition, 1988.

13.9 Bresler, B. and Gilbert, P.H., *The Requirements for Reinforced Concrete Columns*, Journal ACI, Vol. 58, No. 5, November 1961, pp 555–570.

13.10 Park, R. and Paulay, T., *Reinforced Concrete Structures*, John Wiley & Sons, Inc., New York, 1975.

13.11 Rao, P.S and Menon, D., *Ultimate Strength of Tubular Reinforced Concrete Tower Sections Under Wind Loading*, Indian Concrete Journal, February 1995, pp 117–123.

13.12 — *Design Aids (for Reinforced Concrete) to IS 456* : *1978*, Special Publication SP:16, Bureau of Indian Standards, New Delhi, 1980.

13.13 Press, W.H., Flannery, B.P., Teukolsky, S.A. and Vetterling, W.T., *Numerical Recipes in C*, Cambridge University Press, Cambridge, 1988.

13.14 Bresler, B., *Design Criteria for Reinforced Concrete Columns Under Axial Load and Biaxial Bending*, Journal ACI, Vol. 57, 1960, pp 481–490.

13.15 Gouwens, A.J., *Biaxial Bending Simplified, Reinforced Concrete Columns*, ACI Special Publication SP–50, American Concrete Institute, Detroit, 1975, pp 223–261.

13.16 MacGregor, J.G., Breen, J.E., and Pfrang, E.O., *Design of Slender Concrete Columns*, Journal ACI, Vol. 67, No. 1, Jan. 1970, pp 6–28.

13.17 Wood, B.R., Beaulieu, D., and Adams, P.F., *Column Design by P–Delta Method*, ASCE Journal of Struct. Div., Vol. 102, Feb. 1976, pp 411–427.

13.18 Wood, B.R., Beaulieu, D., and Adams, P.F., *Further Aspects of Design by P–Delta Method*, ASCE Journal of Struct. Div., Vol. 102, Mar. 1976, pp 487–500.

13.19 MacGregor, J.G., and Hage, S.E., *Stability Analysis and Design of Concrete Columns*, ASCE Journal of Struct. Div., Vol. 10, Oct. 1977, pp 19–53.

13.20 Cranston, W.B., *Analysis and Design of Reinforced Concrete Columns*, Research Report 20, Cement and Concrete Assn., London, 1972.

13.21 CP 110 : 1972 — *The Structural Use of Concrete, Part I*: *Design Materials and Workmanship*, British Standards Institution, London, 1972.

13.22 Menon, D., *Fuzzy Logic Based Estimation of Effective Lengths of Columns in Partially Braced Multi-storey Frames*, Structural Engineering and Mechanics, Vol. 11, No.3, 2001, pp 287-299.

13.23 Srinivasan, S. and Menon, D., *RC Rectangular Column Sections Under Biaxial Eccentric Compression — An Improved Design Recommendation*, Journal of Structural Engg, SERC, Vol.29, No.4, Jan-Mar 2003.

# 14

# Design of Footings and Retaining Walls

## 14.1 INTRODUCTION

In a typical structure built on ground, that part of the structure which is located above ground is generally referred to as the *superstructure*, and the part which lies below ground is referred to as the *substructure* or the 'foundation structure' (or simply, foundation). The purpose of the foundation is to effectively support the superstructure by

1. transmitting the applied load effects (reactions in the form of vertical and horizontal forces and moments) to the soil below, without exceeding the 'safe bearing capacity' of the soil, and
2. ensuring that the *settlement* of the structure is within tolerable limits, and as nearly uniform[†] as possible.

Further, the foundation should provide adequate safety against possible *instability* due to *overturning* or *sliding* and/or possible *pullout*. Design against forces inducing overturning and sliding are of special importance in the design of *retaining walls*, whose very purpose is to provide lateral support to earthfill/embankment in order to retain the side of the earthfill in a vertical position. The choice of the type of foundation depends not only on the type of the superstructure and the magnitudes and types of reactions induced at the base of the superstructure, but also on the nature of the soil strata on top of which the substructure is to be founded. This comes under

[†] Non-uniform (differential) settlement of a structure generally results in significant stresses in the superstructure, which are usually not foreseen in design. In order to avoid this, it is necessary to ensure that the different footings in a building are proportioned in such a way as to result in soil pressures of nearly equal magnitude under their bases (under permanent loads). In those exceptional situations where differential settlements are unavoidable, it is necessary to consider this in the analysis of the structure itself; this will involve a trial-and-adjustment process, as the settlements are not known a priori.

the specialised domain of geotechnical engineering (soil mechanics), and for important structures and/or difficult soil conditions, the type of foundation to be used is based on a soil study by a geotechnical consultant. In the case of retaining walls, the choice of the type of wall is governed by the height of the earth to be retained and other site/soil conditions.

It is not the objective of this book to cover the designs of all the different types of foundations and retaining walls. Nor is it the objective of the Code on Reinforced Concrete Design (IS 456) to do this. The Code recommendations (Cl. 34) are confined to *the design of footings that support isolated columns or walls and rest directly on soil or on a group of piles* [Ref. 14.1]. This chapter is, accordingly, confined to the design of these simple types of footings† (including combined footings supporting two columns) as well as retaining walls (*cantilever* and *counterfort* walls). These simple types of footings [Fig. 14.1] are the most widely used types of foundation and are relatively cheap to build. The design of more complex types of foundations (continuous footings, raft foundations, pile foundations, wells and caissons, etc.) is clearly outside the scope of this book, and for this, reference may be made to books on foundation engineering [Ref. 14.2, 14.3] and related IS Codes [IS 2911 (Parts I–III), IS 2950, etc.]. The special codes related to the design of simple footings (discussed in this chapter) are IS 1904:1986 [Ref. 14.4] and IS 1080:1980 [Ref. 14.5].

Sections 14.2 – 14.6 deal with the types, behaviour and design of footings, while Sections 14.7 – 14.9 deal with the types, behaviour and design of retaining walls.

## 14.2 TYPES OF FOOTINGS

'Footings' belong to the category of *shallow foundations* (as opposed to *deep foundations* such as piles and caissons) and are used when soil of sufficient strength is available within a relatively short depth below the ground surface. Shallow foundations comprise not only *footings* (which support columns/walls, and have a limited area/width in plan) but also rafts which support multiple columns on a large plan area). The shallow foundation (footing or raft) has a large plan area in comparison with the cross-sectional area of the column(s) it supports because:

- the loads on the columns (axial thrust, bending moments‡) are resisted by concrete under compression and reinforcing steel under tension and/or compression, whereas these load effects are transmitted by the footing/raft to a relatively weak supporting soil by *bearing pressures* alone;
- the 'safe bearing capacity' of the soil is very low (100 – 400 kPa) in comparison with the *permissible* compressive stresses in concrete (5–15 MPa) and steel (130–190 MPa) in a column under service loads.

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† The design of pile caps is not included in this chapter.

‡ Shear forces are also induced in columns, which may result in significant horizontal forces at column bases, under lateral loads. These are resisted by friction between the underside of the footing and the soil below, and also by passive resistance of the soil adjoining the sides of the footing, and in some cases, by 'keys' cast integrally with the footing.

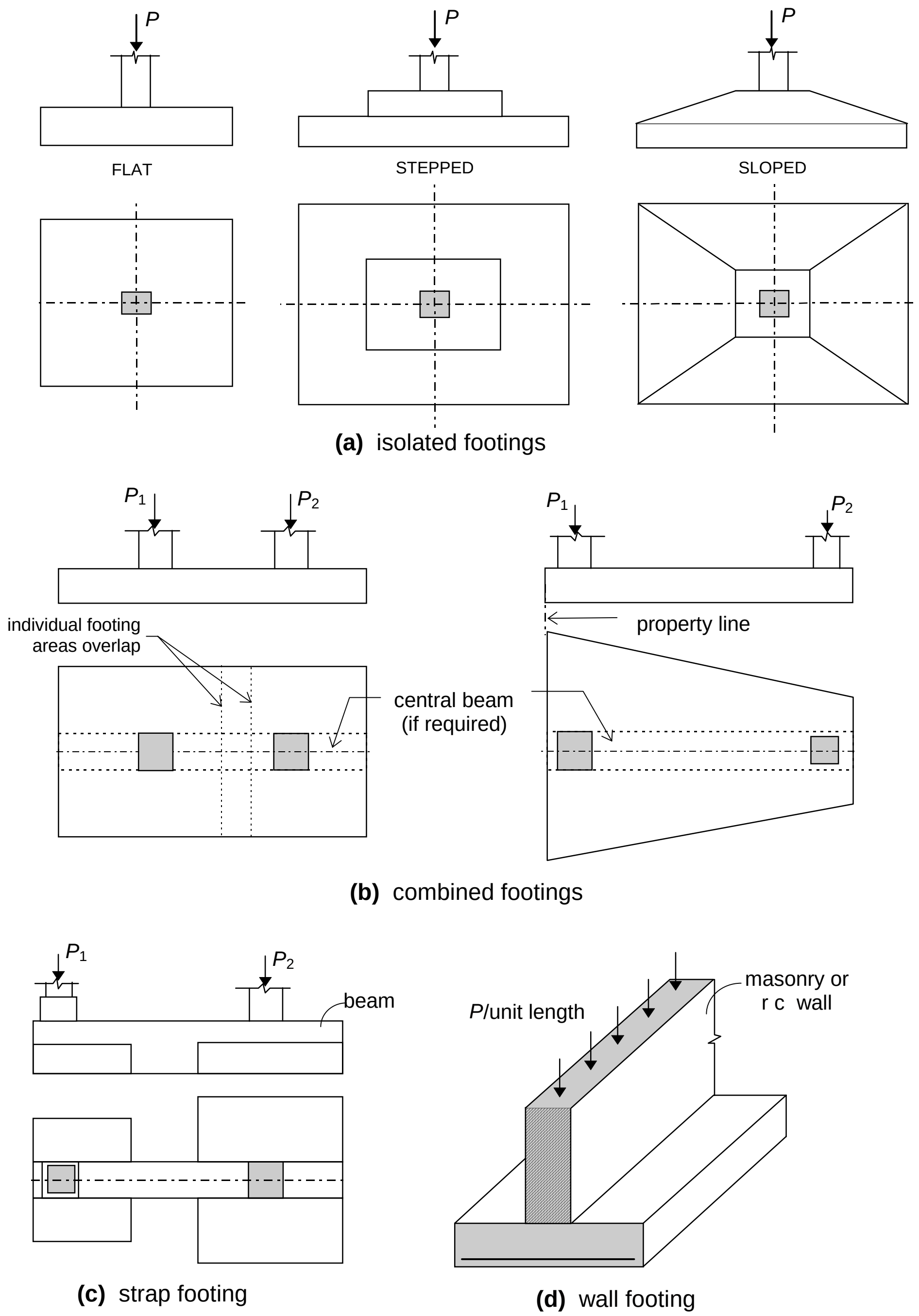


**Fig. 14.1** Types of footings

### 14.2.1 Isolated Footings

For ordinary structures located on reasonably firm soil, it usually suffices to provide a separate footing for every column. Such a footing is called an *isolated footing*. It is generally square or rectangular in plan; other shapes are resorted to under special circumstances. The footing basically comprises a thick slab which may be flat (of uniform thickness), stepped or sloped (on the upper surface), as shown in Fig. 14.1(a).

The soil bearing pressures from below tend to make the base slab of the footing bend upwards, somewhat into a saucer-like shape (cantilever action), and hence the footing needs to be suitably reinforced by a mesh provided at the bottom of the slab. However, in the exceptional case of very small and relatively thick footings, the structural action is likely to occur, not by bending of the footing slab, but by a lateral dispersion of the compressive stress at the base of the column; in such a case, it suffices to provide a plain concrete *pedestal footing* [refer Section 14.4.7].

The term 'pedestal' is also used to refer to that portion of a column below ground level where the cross-sectional dimensions are enlarged. The provision of a pedestal is optional, but is often resorted to by design engineers, as it results in reduced development length requirements for the column bars, reduced slenderness of the column[†] (especially when the founding depth is large), increased direct bearing area on the footing base slab, and reduced shear stresses and design moments. Pedestals are also used to support structural steel columns, the load transfer between the steel column and the concrete pedestal being achieved generally through gussetted steel base plates with 'holding down' bolts.

### 14.2.2 Combined Footings

In some cases it may be inconvenient to provide separate isolated footings for columns (or walls) on account of inadequate areas available in plan. This may occur when two or more columns (or walls) are located close to each other and/or if they are relatively heavily loaded and/or rest on soil with low safe bearing capacity, resulting in an overlap of areas if isolated footings are attempted.

In such cases, it is advantageous to provide a single *combined footing* [Fig. 14.1(b)] for the columns. Often, the term 'combined footing' is used when *two* columns are supported by a common footing, the term 'continuous strip footing' is used if the columns (three or more in number) are aligned in one direction alone, and the term 'raft foundation' ('mat foundation') is used when there is a grid of multiple columns[‡]. The combining of footings contributes to improved integral behaviour of the structure.

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[†] Tie beams are also sometimes provided (for this purpose), interconnecting different columns at the top of pedestal level (about 150 mm below ground level). Plinth beams also serve as tie beams.

[‡] The raft foundation consists of a thick slab which may be (i) of uniform thickness (flat plate), (ii) with locally thicker panels near column bases (flat slab), or (iii) with stiffening beams interconnecting the columns.

Fig. 14.1(b) also shows a two-column combined footing, in which there is a 'property line' which restricts the extension of the footing on one side. In this case, the non-availability of space near the exterior column is circumvented by combining the footing with that of an interior column. The width of the footing may be kept uniform or tapered, as shown. The trapezoidal shaped footing (with a larger width near the exterior column) is required when the exterior column is more heavily loaded than the interior column. Another option is a combined footing which is T-shaped. It is sometimes economical to provide a central beam interconnecting the column bases; this causes the base slab to bend transversely, while the beam alone bends longitudinally.

An alternative to the conventional combined footing is the *strap footing*, in which the columns are supported essentially on isolated footings, but interconnected with a beam, as shown in Fig. 14.1(c).

### 14.2.3 Wall Footings

Reinforced concrete footings are required to support reinforced concrete walls, and are also sometimes employed to support load-bearing masonry walls[†]. *Wall footings* distribute the load from the wall to a wider area, and are continuous throughout the length of the wall [Fig. 14.1(d)]. The footing slab bends essentially in the direction transverse to the wall (a 'one-way' slab), and hence is reinforced mainly in the transverse direction, with only distributors in the longitudinal direction.

## 14.3 SOIL PRESSURES UNDER ISOLATED FOOTINGS

### 14.3.1 Allowable Soil Pressure

The plan area of a footing base slab is selected so as to limit the maximum soil bearing pressure induced below the footing to within a safe limit. This safe limit to the soil pressure is determined using the principles of soil mechanics [Ref. 14.2, 14.3]. The main considerations in determining the allowable soil pressure, as well as fixing the depth of foundation, are (i) that the soil does not fail under the applied loads, and (ii) that the settlements, both overall and differential, are within the limits permissible for the structure. The safety factor, used in soil mechanics, lies in the range 2 – 6, and depends on the type of soil, and related uncertainties and approximations.

It should be noted that the value of the *safe soil bearing capacity* ('allowable soil pressure'), $q_a$, given to the structural designer by the geotechnical consultant[‡], is applicable for *service load* conditions, as $q_a$ includes the factor of safety. Hence, the calculation for the required area of a footing must be based on $q_a$ and the *service* load effects. The 'partial load factors' to be used for different load combinations (*DL*, *LL*,

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[†] It is more common to have stepped masonry (stone or brick) foundation for masonry walls.

[‡] The soil bearing capacity, according to soil mechanics theory, depends on the size of the footing, and this is to be accounted for (approximately) in the recommendation made in the Soil Report.

*WL*/*EL*) should, therefore, be those applicable for the *serviceability limit state* and *not* the 'ultimate limit state' [refer Section 3.6.3] when used in association with $q_a$.

Another point to be noted is that the prescribed allowable soil pressure $q_a$ at a given depth is generally the *gross* pressure, which includes the pressure due to the existing overburden (soil up to the founding depth), and not the *net* pressure (in excess of the existing overburden pressure). Hence, the total load to be considered in calculating the maximum soil pressure $q$ ($\leq q_a$) must include the weight of the footing itself and that of the backfill. Often, in preliminary calculations these weights are accounted for approximately as 10 – 15 percent of the axial load on the column; however, this assumption should be verified subsequently.

### 14.3.2 Distribution of Base Pressure

The distribution of the soil reaction acting at the base of the footing depends on the rigidity of the footing as well as the properties of the soil. The distribution of soil pressure is generally non-uniform. However, for convenience, a linear distribution of soil pressure is assumed in normal design practice.

#### *Concentrically Loaded Footings*

Thus, in a symmetrically loaded footing, where the resultant vertical (service) load $P + \Delta P$ (where $P$ is the load from the column and $\Delta P$ the weight of footing plus backfill) passes through the centroid of the footing, the soil pressure is assumed to be uniformly distributed [Fig. 14.2], and its magnitude $q$ is given by

$$q = \frac{P + \Delta P}{A} \tag{14.1}$$

where $A$ is the base area of the footing.

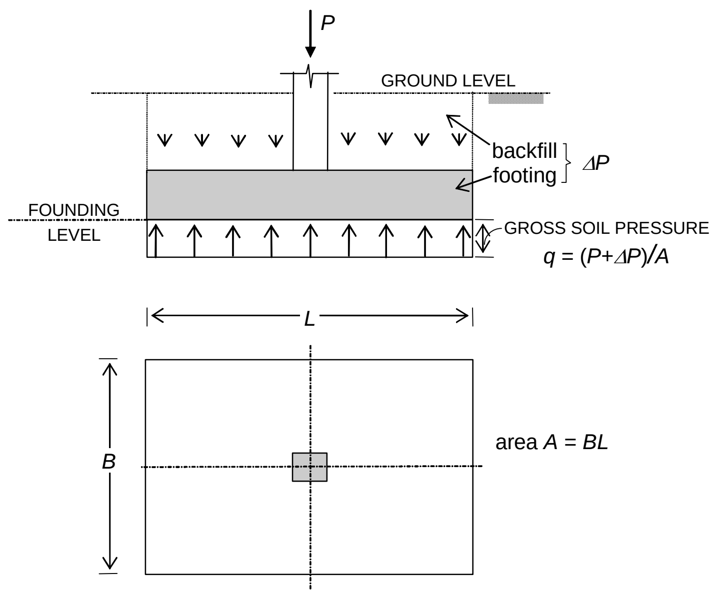

**Fig. 14.2** Assumed uniform base pressure distribution under concentric loading

Limiting $q$ to the allowable soil pressure $q_a$ will give the minimum required area of footing:

$$A_{reqd} = \frac{P + \Delta P}{q_a} \tag{14.1a}$$

### *Eccentrically Loaded Footings*

The load $P$ acting on a footing may act eccentrically with respect to the centroid of the footing base. This eccentricity $e$ may result from one or more of the following effects:

- the column transmitting a moment $M$ in addition to the vertical load [Fig. 14.3(a)];
- the column carrying a vertical load offset with respect to the centroid of the footing [Fig. 14.3(b)];
- the column (or pedestal) transmitting a lateral force located above the foundation level, in addition to the vertical load [Fig. 14.3(c)].

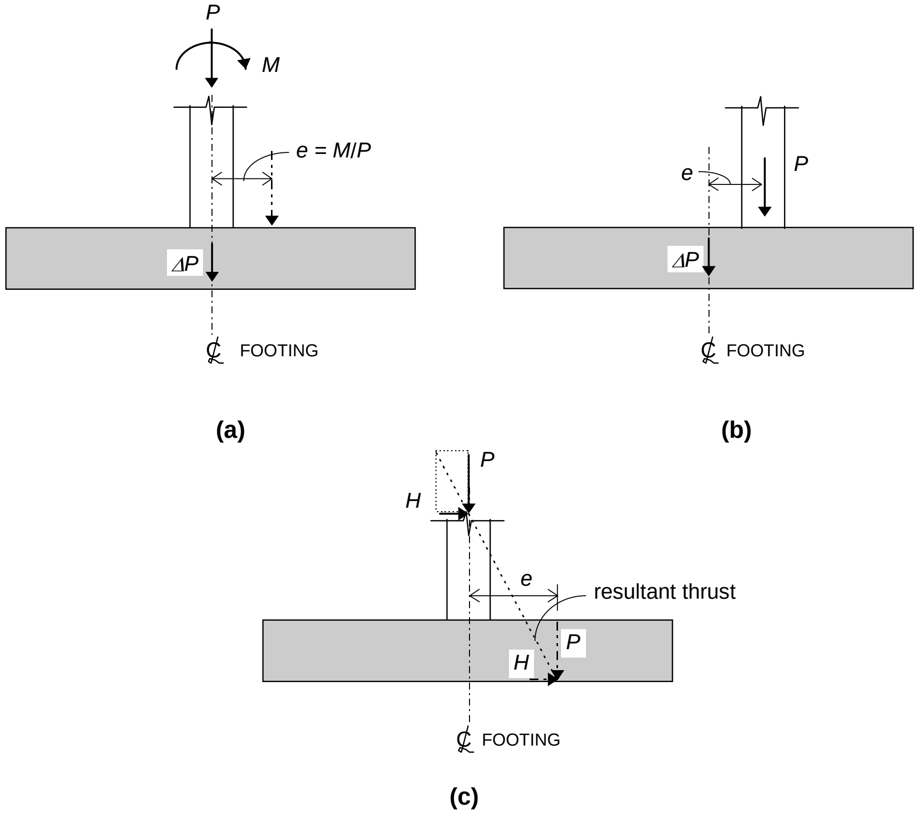

**Fig. 14.3** Eccentric loading on a footing

In a general case, biaxial eccentricities (i.e., eccentricities of loading with respect to both the principal centroidal axes of the footing) are possible — as in the case of a footing for the corner column of a building. However, the case of uniaxial eccentricity[†] is more commonly encountered in design practice.

For the purpose of determining the base pressures under eccentric loading, the footing is assumed to be rigid and the contact pressure distribution to be linear. The magnitude of the pressure distribution is determined from considerations of simple static equilibrium. Essentially this means that the centre of pressure (through which the resultant soil reaction $R$ acts) must be collinear with the resultant line of action of the eccentrically applied load $P + \Delta P$, with $R = P + \Delta P$ [Fig. 14.4].

For preliminary calculations, $\Delta P$, the weight of footing plus backfill, may be taken as 10–15 percent of $P$. Various possible linear base pressure distributions are depicted in Fig. 14.4 for the case of uniaxially eccentric loading on a rectangular footing.

***Case 1:*** $|e| \leq L/6$

If the resultant loading eccentricity $e = M/(P + \Delta P)$ lies within the "middle third" of the footing (i.e., $|e| \leq L/6$), it is seen that the entire contact area of the footing is subject to a (nonuniform) pressure which varies linearly from $q_{min}$ to $q_{max}$ [Fig. 14.4(a)]. These pressures are easily obtained by superposing the separate effects due to the direct load $(P + \Delta P)$ and the bending moment $M = (P + \Delta P)\,e$:

$$q_{\text{max,min}} = \frac{(P+\Delta P)}{A} \pm \frac{(P+\Delta P)e}{Z} \tag{14.2a}$$

with area $A = BL$ and section modulus $Z = BL^2/6$, where $L$ is the length of the footing in the direction of the eccentricity $e$, and $B$ the width of the footing. Accordingly,

$$q_{\text{max,min}} = \frac{(P+\Delta P)}{A}\left(1 \pm \frac{6e}{L}\right) \quad \text{for } |e| \leq L/6 \tag{14.2b}$$

In the limiting case of $|e| = L/6$, $q_{min} = 0$ and $q_{max} = 2(P + \Delta P)/A$, resulting in a triangular pressure distribution. The uniform pressure distribution $q = (P + \Delta P)/A$ [Eq. 14.1] is obtained as special case of Eq. 14.2b, with $e = 0$.

This limiting case of $|e| = L/6$, is valid only for uniaxial bending. In case of bi-axial bending, the limiting case shall be taken as

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[†] Eccentricities in loading can be quite significant in footings which support columns that form part of a lateral load resisting frame. However, as the lateral loads are generally assumed to act (with maximum values) in only one direction at a time, the problem is essentially one of uniaxial eccentricity . Eccentricities in both directions should be considered, but usually only one at a time.

$$\frac{e_x}{L_x/6}+\frac{e_y}{L_y/6}\leq 1 \tag{14.2c}$$

***Case 2:*** $|e| > L/6$

When the resultant eccentricity $e$ exceeds $L/6$, Eq. 14.2 becomes invalid because it will yield a negative value for $q_{min}$, implying a tensile force at the interface. However, such tension resisting capacity cannot be practically expected from soil† . Assuming a triangular pressure distribution (considering the soil under compression alone), and considering a collinear line of action of the resultant soil reaction $R$ with the eccentric load $P + \Delta P$, with $R = P + \Delta P$ (for static equilibrium), [Fig. 14.4(b)],

$$q_{\max}=\frac{2(P+\Delta P)}{BL'} \tag{14.3}$$

† In fact, it can be expected that the soil will tend to separate from the footing base, thereby offering no pressure whatsoever in the base regions farthest removed from $q_{max}$.

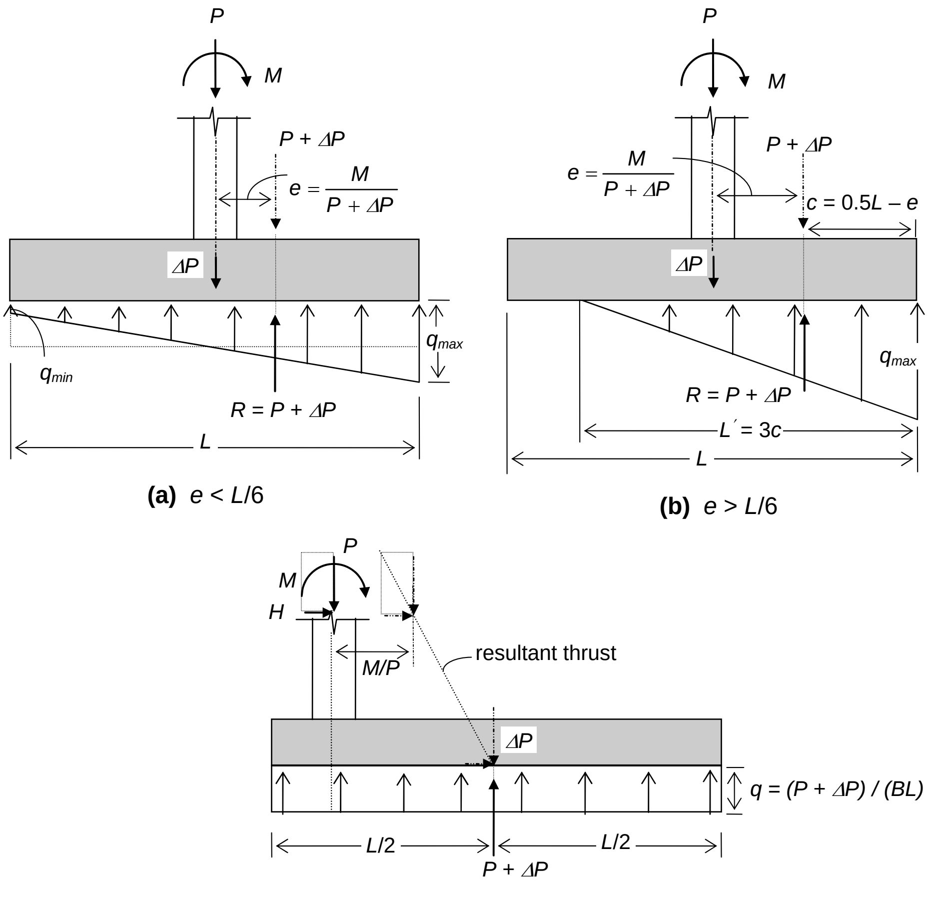


**Fig. 14.4** Assumed linear base pressure distributions under uniaxially eccentric loading on rectangular footings

$$\left.\begin{aligned} L' &= 3c \\ c &= 0.5L - e \end{aligned}\right\} \tag{14.4}$$

Thus, it is seen that the effective length of contact is reduced from $L$ to $L' = 3c$, and the maximum soil pressure $q_{max}$ is increased from $(P + \Delta P)/A$ to twice the load $(P + \Delta P)$ divided by the effective area $BL'$. In order to limit $q_{max}$ to the allowable bearing pressure $q_a$, and also to maximise the effective bearing area ratio $L'/L$, it may be necessary to design a footing with a large base area. Such footings are commonly encountered in industrial buildings where columns are relatively lightly loaded axially, but subject to high bending moments due to lateral wind loads or eccentric gantry crane loads.

It may be noted that highly nonuniform base pressures (especially under *sustained* eccentric loads) are undesirable as this can result in possible tilting of the footing.

Hence, proportioning of the footing base should be such as to make the contact pressure as uniform as possible.

### *Case 3: Eliminating Eccentricity in Loading*

Where the magnitude of eccentricity in loading is known with some degree of certainty and its direction is fixed, it is possible to arrive at an economical design solution by laterally shifting the footing base, relative to the column, such that the effective eccentricity in loading is reduced considerably, if not eliminated altogether. This is not only desirable from the viewpoint of economy but also desirable from the viewpoint of eliminating possible titling of the footing on account of non-uniform base pressure. The ideal situation of zero effective eccentricity is depicted in Fig. 14.4(c), where it is shown that by suitably offsetting the footing base so that the resultant line of thrust passes through the centroid of the footing, a uniform pressure distribution is obtainable, with $q = (P + \Delta P)/A$. However, some increase in bearing pressure should be considered in practice, to account for possible variations in the estimated $M/P$ ratio.

Indeed, such a design solution becomes impracticable when the $M/P$ ratio is highly uncertain in magnitude, and especially when the bending moment can be reversible (as under wind loads).

### 14.3.3 Instability Problems: Overturning and Sliding

When lateral loads act on a structure, adequate stability of the structure as a whole should be ensured at the foundation level — against the possibilities of *overturning* and *sliding*. Instability due to overturning may also occur due to eccentric loads, in footings for columns which support cantilevered beams/slabs.

The Code (Cl. 20) recommends a factor of safety of *not less than* 1.4 against both sliding and overturning[†] under the most adverse combination of the applied *characteristic* loads. In cases where dead loads contribute to improved safety, i.e., increased frictional resistance against sliding or increased restoring moment against overturning moment, only 0.9 times the characteristic dead load should be considered.

It may be noted that problems of overturning and sliding are relatively rare in reinforced concrete buildings, but are commonly encountered in such structures as retaining walls [refer Section 14.8], chimneys, industrial sheds, etc. The resistance against sliding is obtained by friction between the concrete footing base and the soil below, as well as the passive resistance of the soil in contact with the vertical faces of the footing. Improved resistance against sliding can be obtained by providing a local 'shear key' at the base of the footing, as is sometimes done in foundations for retaining walls. Such a 'shear key' serving as *construction joint*, may also be provided at the interface of the wall/column and the footing, thereby facilitating the

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[†] Against overturning, the Code (Cl. 20.1) permits a reduced minimum factor of safety of 1.2 if the overturning moment is entirely due to dead loads. However, it is advisable to apply a uniform minimum factor safety of 1.4 in all cases of loading.

transfer of horizontal shear forces (due to lateral loads) at the base of the wall/column.

The restoring moment, counterbalancing the overturning moment due to lateral/eccentric loads is generally derived from the weight of the footing plus backfill. In some cases, this may call for footings with large base area [refer Fig. 14.4(b)] and large depths of foundation. However, in cases where the overturning moment (not due to wind or earthquake) is not reversible, the problem can be more economically solved by suitably making the column/wall eccentric to the centre of the footing [refer Fig. 14.4(c)].

Another possibility, relatively rare in practice, is the case of *pullout* of a foundation supporting a tension member. Such a situation is encountered, for example, in an overhead tank (or silo) structure (supported on multiple columns), subjected to a very severe lateral wind load. Under minimal gravity load conditions (tank empty), the windward columns are likely to be under tension, with the result that the forces acting on these column foundations will tend to pull out the column-footing from the soil. The counteracting forces, comprising the self weight of the footing and the weight of the overburden, should be sufficiently large to prevent such a 'pullout'. If the tensile forces are excessive, it may be necessary to resort to *tension piles* for proper anchorage.

## 14.4 GENERAL DESIGN CONSIDERATIONS AND CODE REQUIREMENTS

### 14.4.1 Factored Soil Pressure at Ultimate Limit State

As mentioned earlier, the area of a footing is fixed on the basis of the allowable bearing pressure $q_a$ and the applied loads and moments under service load conditions (with *partial load factors* applicable for the 'serviceability limit state[†]'). Once the base area of the footing is determined, the subsequent structural design of the footing is done for the *factored loads*, using the partial load factors applicable for the 'ultimate limit state'. In order to compute the factored moments, shears, etc., acting at critical sections of the footing, a fictitious factored soil pressure $q_u$, corresponding to the factored loads, should be considered.

It may further be noted that the soil pressure which induces moments and shears in the footing base slab are due to the *net pressure* $q_{net}$, i.e., excluding the pressure induced by the weight $\Delta P$ of the footing and the backfill (assumed to be uniformly distributed). This net pressure is due to the concentrated load on the column (from the superstructure) and the moments at the base of the column (or pedestal), as shown in Fig. 14.5. Using *gross pressures* instead of *net pressures* will result in needlessly conservative designs. The 'factored net soil pressure' $q_u$ to be considered in the design of the footing at the limit state is obtainable from the factored loads on the column ($P_u$, $M_u$) as shown in Fig. 14.5(b).

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[†] As mentioned in Section 3.6.3, the partial load factor may be taken as unity in general — except for the load combination $DL + LL + WL/EL$, where a partial load factor of 0.8 is applicable for live loads ($LL$) and for wind loads ($WL$)/earthquake loads ($EL$).

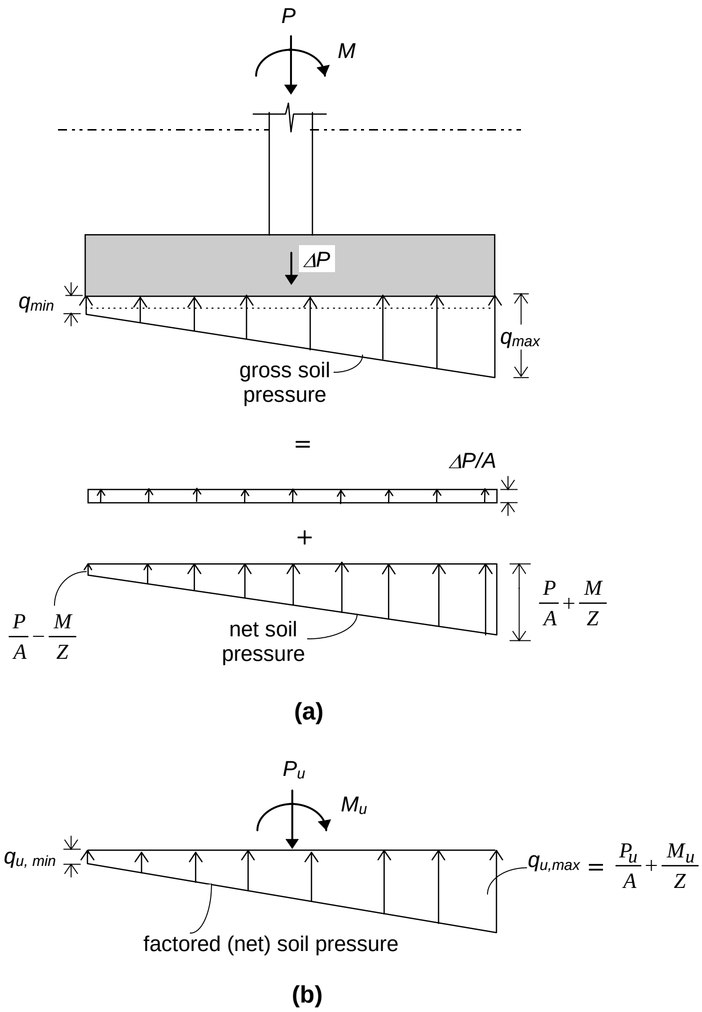


**Fig. 14.5** Net soil pressure causing stresses in a footing

### 14.4.2 General Design Considerations

The major design considerations in the structural design of a footing relate to *flexure*, *shear* (both one-way and two-way action), *bearing* and *bond* (development length). In these aspects, the design procedures are similar to those for beams and two-way slabs supported on columns. Additional considerations involve the transfer of force from the column/pedestal to the footing, and in cases where horizontal forces are involved, safety against sliding and overturning.

Deflection control is not a consideration in the design of footings which are buried underground (and hence not visible). However, control of crack-width and protection of reinforcement by adequate cover are important serviceability considerations, particularly in aggressive environments. It is considered sufficient to limit the crack-width to 0.3 mm in a majority of footings, and for this the general detailing requirements will serve the purpose of crack-width control [Ref. 14.1].

Although the minimum cover prescribed in the Code (Cl. 26.4.2.2) is 50 mm, it is desirable to provide a clear cover of 75 mm to the flexural reinforcement in all footings.

### 14.4.3 Thickness of Footing Base Slab

The thickness of a footing base slab is generally based on considerations of shear and flexure, which are critical near the column location. Generally, shear considerations predominate, and the thickness is based on shear criteria.

Except in the case of small footings, it is economical to vary the thickness from a minimum at the edge to a maximum near the face of the column, in keeping with the variations in bending moment and shear force. This may be achieved either by sloping the top face of the base slab or by providing a stepped footing.

In any case, the Code (Cl. 34.1.2) restricts the minimum thickness at the edge of the footing to 150 mm for footings in general (and to 300 mm in the case of pile caps). This is done to ensure that the footing has sufficient rigidity to provide the calculated bearing pressures. A 'levelling course' of lean concrete (about 100 mm thick) is usually provided below the footing base.

### 14.4.4 Design for Shear

The thickness (depth) of the footing base slab is most often dictated by the need to check shear stress, and for this reason, the design for shear usually precedes the design for flexure.

Both one-way shear and two-way shear ('punching shear') need to be considered in general [refer Cl. 34.2.4.1 of the Code]. However, in wall footings [Fig. 14.1(d)] and combined footings provided with a central beam [Fig. 14.1(b)], the base slab is subjected to one-way bending, and for this reason, need to be designed for one-way shear alone. The critical section for one-way shear is taken, as for beams, at a distance *d* (effective depth) from the face of the column/pedestal [Fig. 14.6(a)] or wall/beam [Fig. 14.6(d)]. The effective area resisting one-way shear [Fig. 14.6(a), (d)] may be rectangular or polygonal, depending on whether the footing is flat [Fig. 14.6(a)] or sloped [Fig. 14.6(c)].

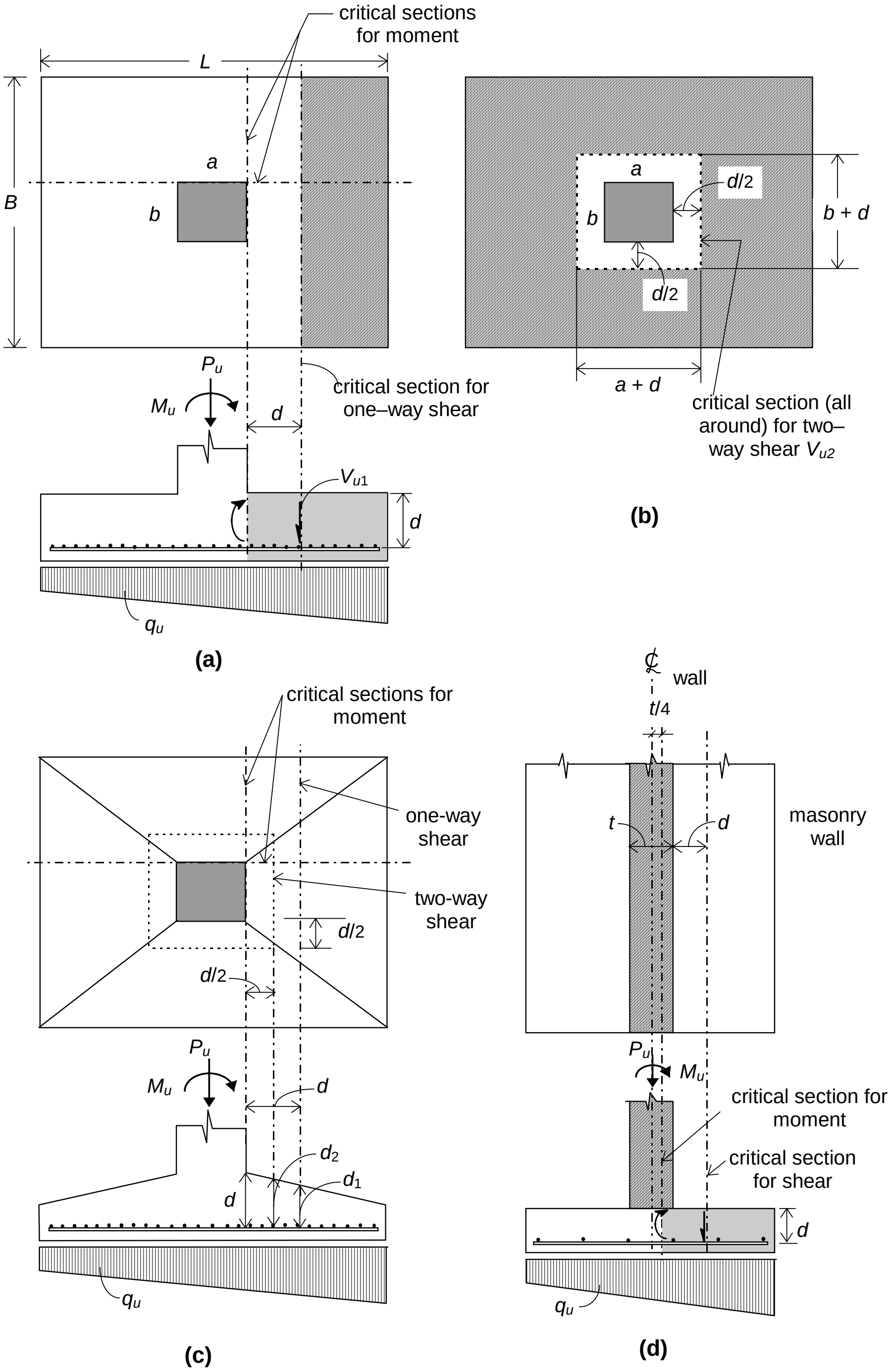


**Fig. 14.6** Critical sections for shear and moment

The behaviour of footings in two-way (punching) shear is identical to that of a two-way flat slab supported on columns, discussed in Chapter 11. The critical section for two-way shear is taken at a distance *d*/2 from the periphery of the column, as shown in Fig. 14.6(b), (c).

The design procedures for one-way and two-way shear are identical to those discussed in Chapters 6 and 11 respectively. However, shear reinforcement is generally avoided in footing slabs, and the factored shear force $V_u$ is kept below the factored shear resistance of the concrete $V_{uc}$[‡] by providing the necessary depth. Where, for some reason, there is a restriction on the depth of the footing base slab on account of which $V_u > V_{uc}$, appropriate shear reinforcement should be designed and provided, to resist the excess shear $V_u - V_{uc}$.

Finally, it may be noted that in the case of a column/pedestal with a circular or octagonal cross-section, the Code (Cl. 34.2.2) recommends that an equivalent square section should be considered, for the purpose of locating the critical sections for shear (and moment). The equivalent squares should be inscribed *within* the perimeter of the round or octagonal column or pedestal.

### 14.4.5 Design for Flexure

As mentioned earlier, the footing base slab bends upward into a saucer-like shape on account of the net soil pressure $q_u$ from below [Fig. 14.6(a)]. Based on extensive tests, it has been determined that the footing base slab may be designed against flexure by considering the bending moment at a critical section defined as a straight section passing through

- the face of a column, pedestal or wall for a footing supporting a concrete column, pedestal or wall [Fig. 14.6(a)];
- halfway between the face and centreline of the wall for a footing supporting masonry wall [Fig. 14.6(d)].

In one-way reinforced footings (such as wall footings), the flexural reinforcement (calculated for the moment at the critical section) is placed perpendicular to the wall at a uniform spacing. In the perpendicular direction (along the length of the wall), nominal *distributor* reinforcement should be provided — mainly to account for secondary moments due to Poisson effect and possible differential settlement, and also to take care of shrinkage and temperature effects.

In two-way reinforced square footings also, flexural reinforcement may be placed at a uniform spacing in both directions. In two-way reinforced rectangular footings, the reinforcement in the long direction is uniformly spaced across the full width of the footing, but in the short direction, the Code (Cl. 34.3.1c) requires a larger concentration of reinforcement to be provided within a central band width, equal to the width *B* of the footing:

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[‡] For the purpose of calculating the design shear strength $\tau_c$ of concrete, a nominal percentage of flexural tensile reinforcement ($p_t$ = 0.25) may be assumed (in preliminary calculations).

$$\text{Reinforcement in central band width} = A_{st,short} \times \frac{2}{\beta + 1} \tag{14.5}$$

where

$A_{st,short} \equiv$ total flexural reinforcement required in the short direction

and $\beta \equiv$ ratio of the longside ($L$) to the short side ($B$) of the footing.

This reinforcement is to be uniformly distributed within the central band width (equal to width $B$), and the remainder of the reinforcement distributed uniformly in the outer portions of the footing, as shown in Fig. 14.7. This is done to account (approximately) for the observed variation of the transverse bending moment along the length of the footing.

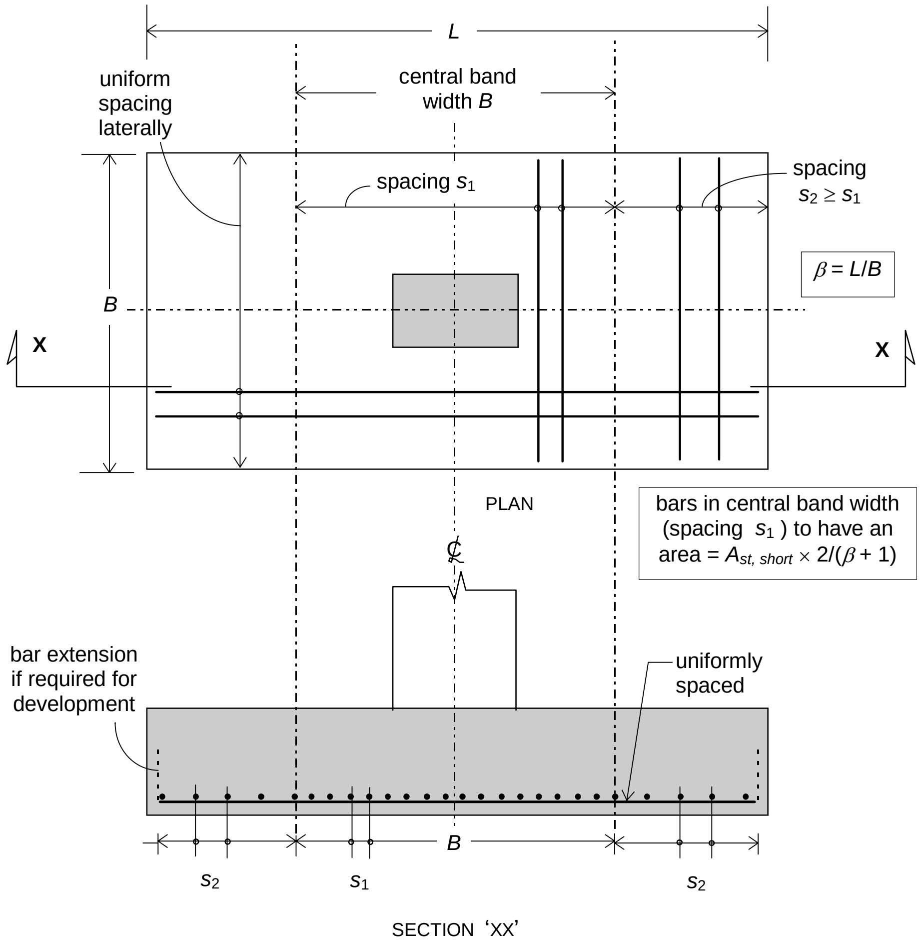



**Fig. 14.7** Detailing of flexural reinforcement in a rectangular footing with uniform thickness

These special detailing requirements are strictly intended for footings with uniform slab thickness. In the case of sloped footings, it usually suffices to provide uniformly distributed reinforcement in the short direction also, as the reduced bending moment in the outer portions is coupled with reduced effective depths in these regions. In the long direction also, the common practice is to provide uniformly spaced reinforcement throughout the width of the footing, despite the variations in depth.

In general, the percentage flexural reinforcement requirement in footing base slabs is low, owing to the relatively large thickness provided on shear considerations. At any rate, the reinforcement should not be less than the minimum prescribed for slabs [refer Chapter 5], unless the footing is designed as a plain concrete (pedestal) footing. Furthermore, the percentage reinforcement provided should be adequate to mobilise the required one-way shear strength in concrete.

It is advisable to select small bar diameters with small spacings, in order to reduce crackwidths and development length requirements.

Development length requirements for flexural reinforcement in a footing should be satisfied at the sections of maximum moment, and also at other sections where the depth is altered. Shortfall in required development length can be made up by bending up the bars near the edges of the footings. This may be required in footings with small plan dimensions.

Furthermore, the longitudinal reinforcement in the column/pedestal must also have the required development length, measured from the interface between the column/pedestal and the footing. When the column is subjected to compression alone (without the bars being subject to tension), it is possible to achieve a full transfer of forces from the column/pedestal to the footing by *bearing*, as discussed in the next section (Section 14.4.7). Where this is not possible, and the transfer of force is accomplished by reinforcement, such reinforcement must also have adequate development length on each side.

### 14.4.6 Transfer of Forces at Column Base

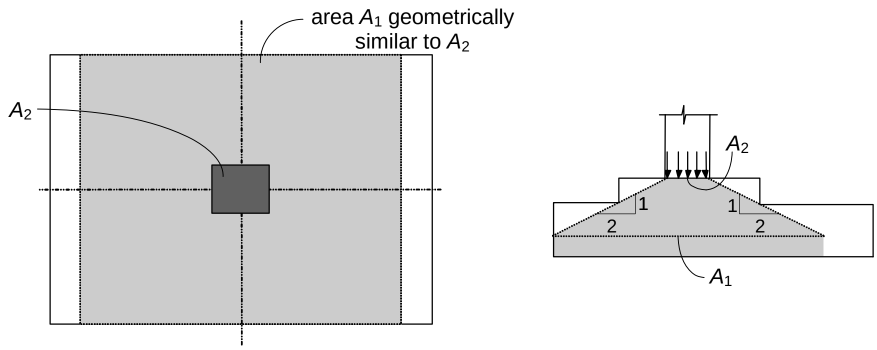
All forces (axial force, moment) acting at the base of the column† (or pedestal) must be transferred to the footing either by compression in concrete or by tension/compression in reinforcing steel. The force transfer achieved through compression in concrete at the interface is limited by the *bearing* resistance of concrete for *either* surface (i.e., supported surface or supporting surface). Under factored loads, the maximum bearing stress $f_{br,\,max}$ is limited by the Code (Cl. 34.4) to

$$f_{br,\max} = 0.45\, f_{ck} \sqrt{A_1/A_2} \tag{14.6}$$

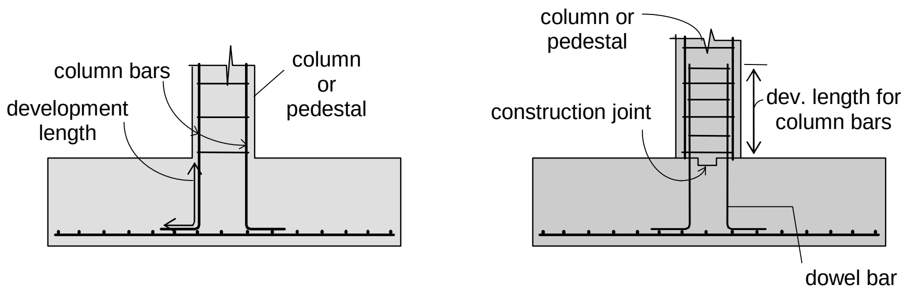
where $A_2$ is the loaded area at the column base, and $A_1$ the maximum area of the portion of the *supporting* surface that is geometrically similar to and concentric with the loaded area. In the case of stepped or sloping footings, the area $A_1$ is to be taken as that of the lower base of the largest frustrum of a pyramid (or cone) contained

† This is equally applicable in the case of force transfer from the column base to the pedestal (if provided) and from the pedestal base to the footing.

wholly within the footing (with area $A_2$ on top) with a side slope of 1 in 2, as shown in Fig. 14.8(a). The factor $\sqrt{A_1/A_2}$ in Eq. 14.6 allows for the increase in concrete strength in the bearing area in the footing due to confinement offered by the surrounding concrete. This factor $\sqrt{A_1/A_2}$ is limited to 2.0. A limitation on the bearing stress is imposed because very high axial compressive stresses give rise to transverse tensile strains which may lead to spalling, lateral splitting or bursting of concrete. This possibility, however, can be countered by providing suitable transverse and confinement reinforcement.




**(a)** definition of area $A_1$ (Cl. 34.4 of Code)




**(b)** reinforcement across column–footing interface

**Fig. 14.8** Transfer of forces at column base

It should be noted that $f_{br,\,max}$ may be governed by the bearing resistance of the concrete in the column at the interface (for which $\sqrt{A_1/A_2}$ is obviously unity), rather than that of the concrete in the footing (for which $1 < \sqrt{A_1/A_2} \le 2$). If the actual compressive stress exceeds $f_{br,max}$, then the excess force is transferred by reinforcement, dowels or mechanical connectors. For transferring a moment at the column base (involving *tension* in the reinforcement), it may be necessary to provide the same amount of reinforcement in the footing as in the column, although some relief in the compression reinforcement is obtainable on account of transfer through bearing. This may be achieved by either continuing the column/pedestal bars into the footing or by providing separate *dowel bars* across the interface as depicted in Fig. 14.8(b). The diameter of dowels should not exceed the diameter of the column bars by 3 mm. Furthermore, the reinforcement provided across the interface must comprise at least four bars, with a total area not less than 0.5 percent of the cross-sectional area of the supported column or pedestal [refer Cl. 34.4.3 of the Code].

Finally, it should be ensured that all reinforcement provided across the interface (whether by extension of column bars or dowels) must have the necessary development length in compression or tension, (as applicable) on both sides of the interface.

Where pedestals are provided, and full force transfer is possible at the interface of column and pedestal, no reinforcement is theoretically required in the pedestal. However, the Code (Cl. 26.5.3.1h) specifies that nominal longitudinal reinforcement (i.e., in a direction parallel to the column load) of not less than 0.15 percent of the cross-sectional area should be provided, for reasons similar to those pertaining to minimum reinforcement in columns [refer Section 13.3.3].

### 14.4.7 Plain Concrete Footings

When the column is relatively lightly loaded (without any bars in tension) and the base area requirement of a footing is relatively low, it may be economical to provide a simple plain concrete block as a footing. Such a footing is sometimes called a *pedestal footing*.

If the bearing stress at the column base under ultimate loads is less than $f_{br,max}$ (given by Eq. 14.6), the force transfer from the column base to the footing (pedestal) is achievable without the need for any reinforcement at the interface. Further, if the base area of the footing falls within a certain zone of dispersion of internal pressure in the footing, the entire force is transmitted to the footing base by compression[†] (*strut action*, as shown in Fig. 14.9a), and the soil pressure does not induce any bending in the footing. The (imaginary) struts are inclined to the vertical, and the horizontal component of the strut forces will necessarily call for some *tie action* ('strut and tie' concept – see Section 17.2), as shown in Fig. 14.9(b). To carry the tie forces and to avoid possible cracking of concrete due to the resulting tensile forces, it is necessary to provide some minimum reinforcement to serve as effective ties [Fig. 14.9(b)].

> Although, the Code does not specify the need for any reinforcement, a minimum reinforcement is necessary, not only for tie action, but also to provide resistance against temperature and shrinkage effects.

For the purpose of defining this zone of dispersion of internal pressure in the footing, thereby enabling the determination of the required thickness of the footing block, the Code (Cl. 34.1.3) defines an angle $\alpha$ between the plane through the bottom edge of the footing and the corresponding edge of the column at the interface [Fig. 14.9], such that

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[†] This is only a convenient idealisation; the actual state of stress is difficult to assess.

$$\tan\alpha \geq 0.9\sqrt{100 q_{\max}/f_{ck} + 1} \tag{14.7}$$

where $q_{max}$ is the maximum soil pressure under service loads, as defined earlier [Eq. 14.2a, Fig. 14.4a].

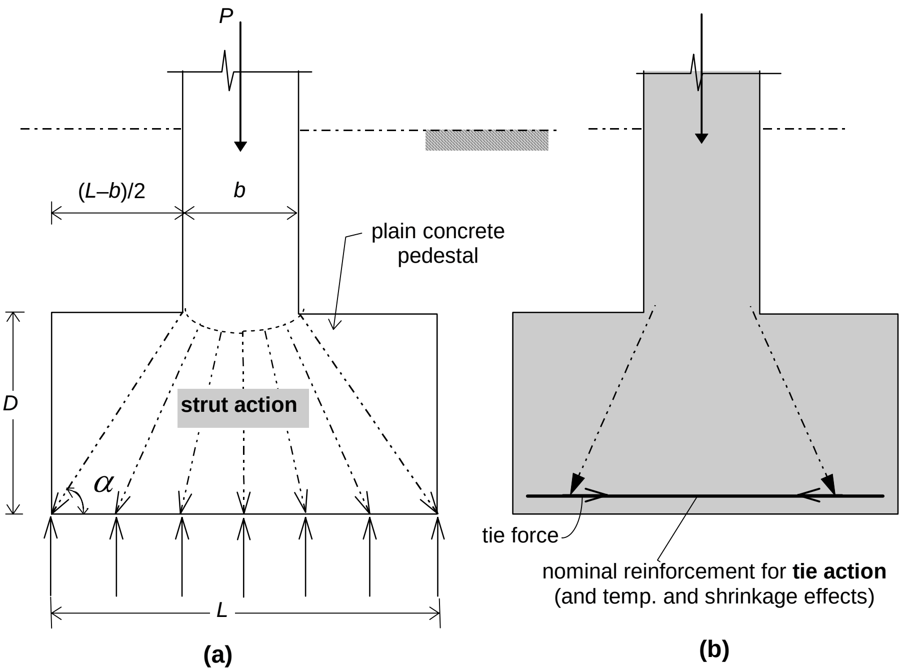


**Fig. 14.9** Plain concrete (pedestal) footing

An expression for the thickness $D$ of the footing block is obtainable [Fig. 14.9] as:

$$D = (L - b)(\tan\alpha)/2$$

$$\Rightarrow \quad D \geq 0.9\sqrt{100 q_{\max}/f_{ck} + 1}\,(L - b)/2 \tag{14.8}$$

where $b$ is the width of the column and the expression for $\tan\alpha$ in Eq. 14.7 governs the minimum thickness of the footing.

The design of a plain concrete footing is demonstrated in Example 14.1.

## 14.5 DESIGN EXAMPLES OF ISOLATED AND WALL FOOTINGS

### EXAMPLE 14.1: Design of a Plain Concrete Footing

Design a plain concrete footing for a column, 300 mm × 300 mm, carrying an axial load of 330 kN (under service loads, due to dead and live loads). Assume an allowable soil bearing pressure of 360 kN/m$^2$ at a depth of 1.0 m below ground. Assume M 20 concrete and Fe 415 steel.

**SOLUTION**

**Transfer of axial force at base of column**

- In order to provide a plain concrete block footing, full force transfer must be possible at the column base, without the need for reinforcement at the interface. That is, the factored axial load $P_u$ must be less than the limiting bearing resistance $F_{br}$.
- Assuming a load factor of 1.5, $P_u = 330 \times 1.5 = 495$ kN

  Limiting bearing stress $f_{br,\,max} = 0.45 f_{ck}\sqrt{A_1/A_2}$

  At the column-footing interface, $f_{br,\,max}$ will be governed by the column face in this case (and not the footing face), with $A_1 = A_2 = (300 \times 300)$ mm$^2$

  $\Rightarrow F_{br} = 0.45 \times 20 \times 300^2 = 810 \times 10^3$ N

  $> P_u = 495$ kN

Hence, full force transfer is possible without the need for reinforcement.

**Size of footing**

- Assuming the weight of footing + backfill to comprise 10 percent of the axial load, base area required = $\dfrac{330 \times 1.1}{360} = 1.01$ m$^2$

  Provide 1 m × 1 m footing, as shown in Fig. 4.10.

**Thickness of footing**

- $D = \left(\dfrac{1000 - 300}{2}\right)\tan\alpha$

  where $\tan\alpha \geq 0.9\sqrt{100 q_{\max}/f_{ck} + 1}$

  $q_{max} = 360$ kN/m$^2$ = 0.360 N/mm$^2$

  $f_{ck} = 20$ N/mm$^2$

  $\Rightarrow D \geq 350 \times 0.9\sqrt{100 \times 0.36/20 + 1}$

  $= 527$ mm

  Provide 530 mm.

  Hence, provide a concrete block 1000 × 1000 × 530 mm.
- Further, it is necessary to provide minimum reinforcement to provide for ‘tie action’, and to account for temperature and shrinkage effects:

  $A_{st,min} = 0.0012BD = 0.0012 \times 1000 \times 530 = 636$ mm$^2$

  Provide 6 – 12 mm $\phi$ bars ($A_{st}$ = 678 mm$^2$) both ways with a clear cover of 75 mm, as shown in Fig. 4.10. The spacing is within limits (< 5$d$ or 450 mm).

**Check gross base pressure**

- Assuming unit weight of concrete and soil as 24 kN/m$^3$ and 18 kN/m$^3$ respectively,

  actual gross soil pressure $q_{max} = \dfrac{330}{1.0 \times 1.0} + (24 \times 0.53) + (18 \times 0.47)$

$= 330.0 + 12.7 + 8.5$
$= 351.2 \text{ kN/m}^2$
$< q_a = 360 \text{ kN/m}^2$ — Hence, safe.

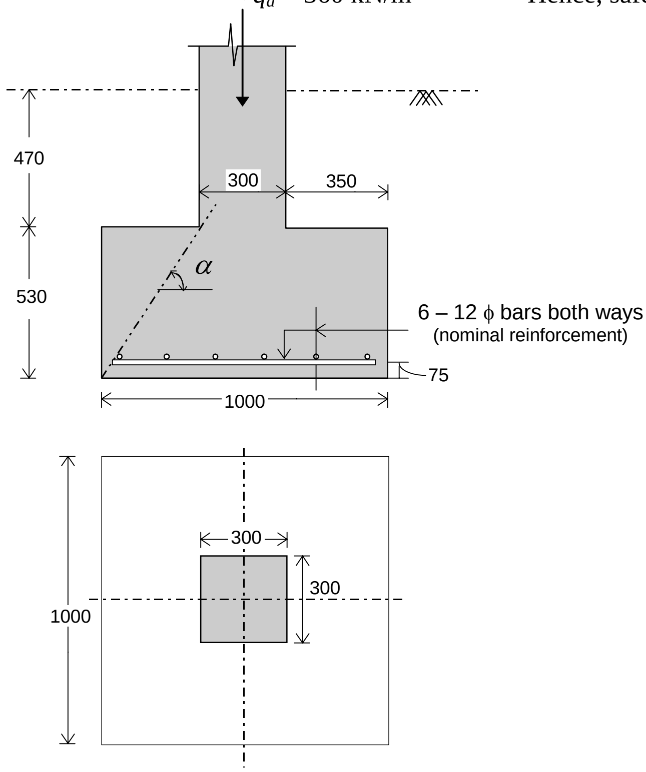


**Fig. 14.10** Example 14.1

#### EXAMPLE 14.2: Square Isolated Footing, Concentrically Loaded

Design an isolated footing for a square column, 450 mm × 450 mm, reinforced with 8–25 ϕ bars, and carrying a service load of 2300 kN. Assume soil with a safe bearing capacity of 300 kN/m² at a depth of 1.5 m below ground. Assume M 20 grade concrete and Fe 415 grade steel for the footing, and M 25 concrete and Fe 415 steel for the column.

#### SOLUTION

**Size of footing**

- *Given*: $P = 2300$ kN, $q_a = 300 \text{ kN/m}^2$ at $h = 1.5$ m

Assuming the weight of the footing + backfill to be 10 %[†] of the load $P$ = 2300 kN, base area required = $\dfrac{2300 \times 1.1}{300}$ = 8.43 m$^2$

$\Rightarrow$ Minimum size of square footing = $\sqrt{8.43}$ = 2.904 m

Assume a 3 m × 3 m footing base

**Thickness of footing slab based on shear**

- Net soil pressure at ultimate loads (assuming a load factor of 1.5)

$$q_u = \frac{2300 \times 1.5}{3.0 \times 3.0} = 383 \text{ kN/m}^2$$
$$= 0.383 \text{ N/mm}^2$$

**(a) One-way shear**

- The critical section is at a distance $d$ from the column face [refer Fig. 14.11].

$\Rightarrow$ Factored shear force $V_{u1} = 0.383 \times 3000 \times (1275 - d)$
$= (1464{,}975 - 1149d)$ N.

Assuming $\tau_c$ = 0.36 MPa (for M 20 concrete with, say, $p_t$ = 0.25) [refer Table 6.1 or Table 13 of the Code],

One-way shear resistance $V_{c1} = 0.36 \times 3000 \times d$
$= (1080d)$ N

$V_{u1} \leq V_{c1} \Rightarrow 1464975 - 1149d \leq 1080d$
$\Rightarrow d \geq 658$ mm

**(b) Two-way shear**

- The critical section is at $d$/2 from the periphery of the column [refer Fig. 14.11]

$\Rightarrow$ Factored shear force $V_{u2} = 0.383 \times [3000^2 - (450 + d)^2]$

Assuming $d$ = 658 mm (obtained earlier[†] )

$$V_{u2} = 2976.8 \times 10^3 N$$

Two-way shear resistance $V_{c2} = k_s\, \tau_c \times [4 \times (450 + d)\, d]$

where $k_s$ = 1.0 for a square column, and $\tau_c = 0.25\sqrt{20}$ = 1.118 MPa (refer Cl. 31.6.3.1 of the Code)

$\Rightarrow V_{c2} = 1.0 \times 1.118 \times 4d\,(450 + d)$
$= (2012.4d + 4.472d^2)$ N

$V_{u2} \leq V_{c2} \Rightarrow 2976.8 \times 10^3 \leq 2012.4d + 4.472d^2$

Solving, $d \geq 621$ mm

- Evidently, in this problem, one-way shear governs the thickness. Assuming a clear cover of 75 mm and 16 $\phi$ bars in both directions, with an average $d$ = 658 mm,

thickness $D \geq 658 + 75 + 16 = 749$ mm

---

[†] This assumption is verified subsequently.

[†] Actual effective depth provided will not be less than this value; hence, the use of this value in this context can only be on a slightly conservative side; such an assumption simplifies calculations.

Provide ***D* = 750 mm**. The effective depths in the two directions will differ by one bar diameter, which is not significant in relatively deep square footings. For the purpose of flexural reinforcement calculations, an average value of *d* may be assumed:

$\Rightarrow d = 750 - 75 - 16 =$ **659 mm**

- Assuming unit weights of concrete and soil as 24 kN/m$^3$ and 18 kN/m$^3$ respectively, actual gross pressure at footing base (under service loads)

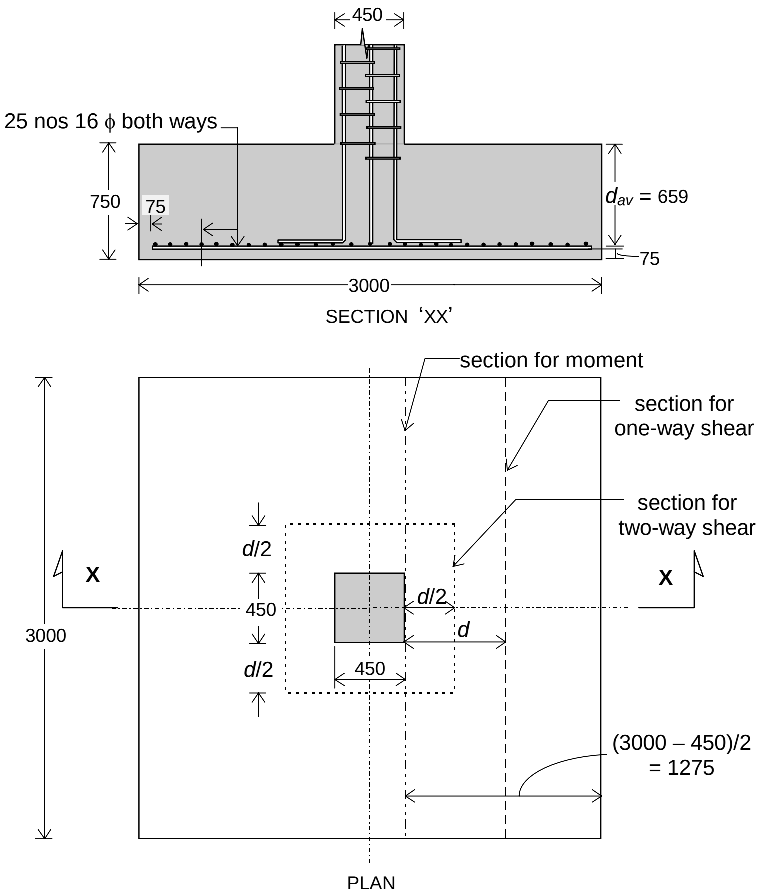


**Fig. 14.11** Example 14.2

$q = 2300/(3.0 \times 3.0) + (24 \times 0.75) + (18 \times 0.75) = 287$ kN/m$^2$
$< 300$ kN/m$^2$ — OK.

**Design of flexural reinforcement**

- Factored moment at column face (in either direction):
$M_u = 0.383 \times 3000 \times 1275^2/2 = 933.9 \times 10^6$ Nmm

$\Rightarrow R \equiv \dfrac{M_u}{Bd^2} = \dfrac{933.9\times10^6}{3000\times659^2}$ = 0.717 MPa

$\Rightarrow \dfrac{(p_t)_{reqd}}{100} = \dfrac{20}{2\times415}\left[1-\sqrt{1-4.598\times0.717/20}\right] = 0.207\times10^{-2}$

$A_{st,min} = 0.0012BD = 0.0012 \times 3000 \times 750 = 2700$ mm$^2$

$\Rightarrow p_{t,min} = 100 \times 2700/(3000 \times 659) = 0.137 < 0.207$

- However, this reinforcement is less than assumed for one-way shear design[‡], ($\tau_c$ = 0.36 MPa).

  for which $p_t$ = 0.25 (for M 20 concrete)

  $\Rightarrow A_{st,reqd} = 0.25 \times 3000 \times 659/100 = 4943$ mm$^2$

  Using 16 mm ϕ bars, number of bars required = 4943/201 = 25

  [corresponding spacing $s = \{3000 - (75 \times 2) - 16\}/(25 - 1) = 118$ mm — is acceptable.]

  **Provide 25 nos 16 ϕ bars** both ways as shown in Fig. 14.11

  Required development length $L_d = \dfrac{\phi(0.87 f_y)}{4\tau_{bd}}$ [refer Cl. 26.2.1 of Code]

  For M 20 concrete and Fe 415 steel, $L_d = \phi\,(0.87 \times 415)/(4 \times 1.2 \times 1.6) = 47.0\,\phi$

  For 16 ϕ bars in footing, $L_d = 47.0 \times 16 = 752$ mm

  Length available = 1275 – 75 = 1200 mm > 752 mm — Hence, OK.

**Transfer of force at column base**

- Factored compressive force at column base: $P_u = 2300 \times 1.5 = 3450$ kN

  Limiting bearing stress at column-footing interface, $f_{br,\,max} = 0.45 f_{ck}\sqrt{A_1/A_2}$ .

  (i) for column face, $f_{ck}$ = 25 MPa, $A_1 = A_2 = 450^2$ mm$^2$

  $\Rightarrow f_{br,\,max\text{ - }col} = 0.45 \times 25 \times 1 = 11.25$ MPa

  (ii) for footing face, $f_{ck}$ = 20 MPa, $A_1 = 3000^2$ mm$^2$, $A_2 = 450^2$ mm$^2$

  $\Rightarrow \sqrt{A_1/A_2}$ = 3000/450 = 6.67, limited to 2.0

  $\Rightarrow f_{br,\,max\text{-}\,ftg} = 0.45 \times 20 \times 2 = 18.0$ MPa

- Evidently, the column face governs, and $f_{br,\,max}$ = 11.25 MPa

  $\Rightarrow$ Limiting bearing resistance $F_{br} = 11.25 \times 450^2 = 2278.1 \times 10^3$ N

  $< P_u$ = 3450 kN

  $\Rightarrow$ Excess force (to be transferred by reinforcement):

  $\Delta P_u = 3450 - 2278 = 1172$ kN

  This may be transferred by reinforcement, dowels or mechanical connectors. In this case, it is convenient to extend the column bars into the footing, as shown in Fig. 14.11.

‡ Unless the footing dimensions are revised (to result in less shear stress), the reinforcement requirement here will be governed by shear strength requirements, and not flexural strength requirements. If the resulting $p_t$ is excessive, it may be more economical to revise the footing dimensions, providing larger plan area and less depth of footing. In a practical design, this should be investigated.

- Required development length of the 8–25 $\phi$ bars provided in the column, assuming a stress level equal to $(0.87f_y) \times (\Delta P_u/P_u)$, and M 20 concrete with Fe 415 steel (in compression)

  For fully stressed bars in compression (M 20, Fe 415): $L_d = \dfrac{\phi\,(0.87 \times 415)}{4(1.2 \times 1.6 \times 1.25)}$

  $= 37.6\ \phi$

  $\Rightarrow \tilde{L}_d = L_d \times \Delta P_u/P_u$

  $= 37.6 \times 25 \times 1172/3450$

  $= 319$ mm.

  Available vertical embedment length in footing ($d$ = 659 mm) > 319 mm.

  The bars are bent (with 90$^{o}$ standard bend) into the footing, and may rest directly on the top of the reinforcement layer in the footing, as shown in Fig. 14.11.

**Alternative Design**

- Providing a uniform thickness of 750 mm for the footing slab is rather uneconomical, as such a high thickness is required essentially near the face of the column (due to shear considerations); the effective depth requirement falls off with increasing distance from the critical section for one-way shear; theoretically, only a minimum thickness (150 mm, specified by the Code) need be provided at the edge of the footing.
- However, the slope provided at the top of the footing should preferably not exceed about 1 in 1.5 (i.e., 1 vertical : 1.5 horizontal), as a steeper slope will require the use of additional formwork on top (to prevent the concrete from sliding down).

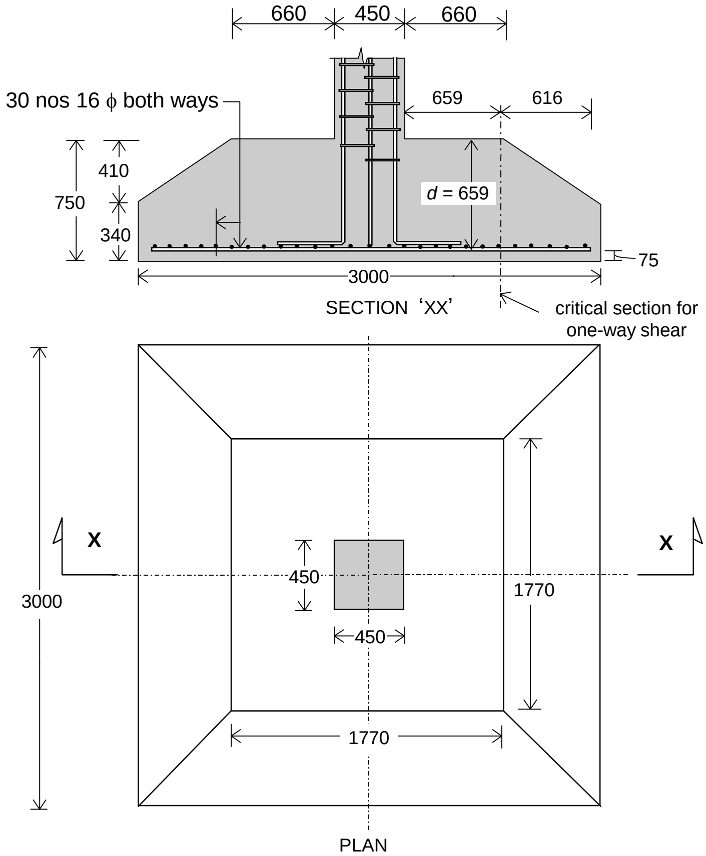


**Fig. 14.12** Example 14.2 — Alternative

- As the thickness of the footing near the column base is governed by shear (one-way shear, in this example) and the effective area available at the critical section is a truncated rectangle, the effective depth required is slightly larger than that for a flat footing.
- Assuming a thickness $D$ = 750 mm up to a distance of 660 mm (> $d$ = 659 mm) from the periphery of the column; and providing a slope of 1 in 1.5 over the remaining distance of 1275 – 660 = 615 mm on all four sides [Fig. 14.12], the edge thickness is obtained as 750 – 615/1.5 = 340 mm

  $\Rightarrow V_{u1} = 0.383 \times (1275 - 659) \times 3000 = 707784$ N

  $\Rightarrow \tau_{v1} = 707784 / (3000 \times 659 - 616 \times 410)^{\ddagger} = 0.410$ MPa

  Providing $p_t = 0.35 \Rightarrow \tau_c = 0.413$ MPa (for M 20 concrete) [refer Table 6.1]

  $\tau_c > \tau_v$ — Hence, OK.

  $\Rightarrow (A_{st})_{reqd} = (0.35/100) \times (3000 \times 659 - 616 \times 410) = 6036$ mm$^2$

‡ The area resisting the shear is polygonal in shape.

⇒ No. of 16 ϕ bars required = 6036/201 = 30 (as shown in Fig. 14.12).

- Other alternative designs are possible. These include (i) providing a proper sloped footing with a thickness varying linearly from a minimum at the edge to a maximum[†] at the *face* of the column, and (ii) providing a stepped footing. In the latter case, the section at the step location becomes a critical section at which one-way shear, flexural reinforcement and development length requirements need to be verified.

### EXAMPLE 14.3: Rectangular Isolated Footing, Concentrically Loaded

Redesign the footing for the column in Example 14.2, including a spatial restriction of 2.5 m on one of the plan dimensions of the footing.

**SOLUTION**

**Size of footing**

- As in Example 14.2, required base area = 8.43 m$^2$
  Width $B$ = 2.5 m, ⇒ length $L$ = 8.43/2.5 = 3.37 m
  ⇒ Provide a rectangular footing 3.4 m × 2.5 m.
  ⇒ Net factored soil pressure = 2300 × 1.5/(3.4 × 2.5) = 406 kN/m$^2$
  = 0.406 N/mm$^2$

**Thickness required for shear**

- An exact solution for the required depth for shear (one-way shear as well as two-way shear) may be obtained using the conditions $V_{u1} \le V_{c1}$ and $V_{u2} \le V_{c2}$, as done in Example 14.2. In this example, a trial-and-error procedure is used.
  Assuming an overall depth (thickness) of footing $D$ = 850 mm, with clear cover of 75 mm and 20 mm ϕ bars in the long direction (placed at bottom) and 16 mm ϕ bars in the short direction,
  effective depth (long span) $d_x$ = 850 – 75 – 20/2 = 765 mm
  effective depth (short span) $d_y$ = 850 – 75 – 20 – 16/2 = 747 mm
  Average $d$ for two-way shear calculations: $d$ = (765 + 747)/2 = 756 mm
- One-way shear at $d_x$ = 765 mm away from column face in the long direction:
  $V_{u1}$ = 0.406 × 2500 × (1475 – 765) = 720650 N
  ⇒ $\tau_{v1}$ = 720650/(2500 × 765) = 0.377 MPa
  For $\tau_c = \tau_{v1}$ = 0.377 MPa, $(p_t)_{reqd}$ = 0.28 [refer Table 6.1].
  [In the short span direction, $d_y$ = 747 mm and $V_{u1}$ = 0.406 × 3400 × (1025 – 747) = 383751 N ⇒ $\tau_{v1}$ =383751/(3400 × 747) = 0.151 MPa << $\tau_{c,\min}$ ; hence this is not critical].

[†] In this case, advantage may be availed of the reduction in one-way shear owing to the inclination in the compressive force [refer Section 6.4.2].

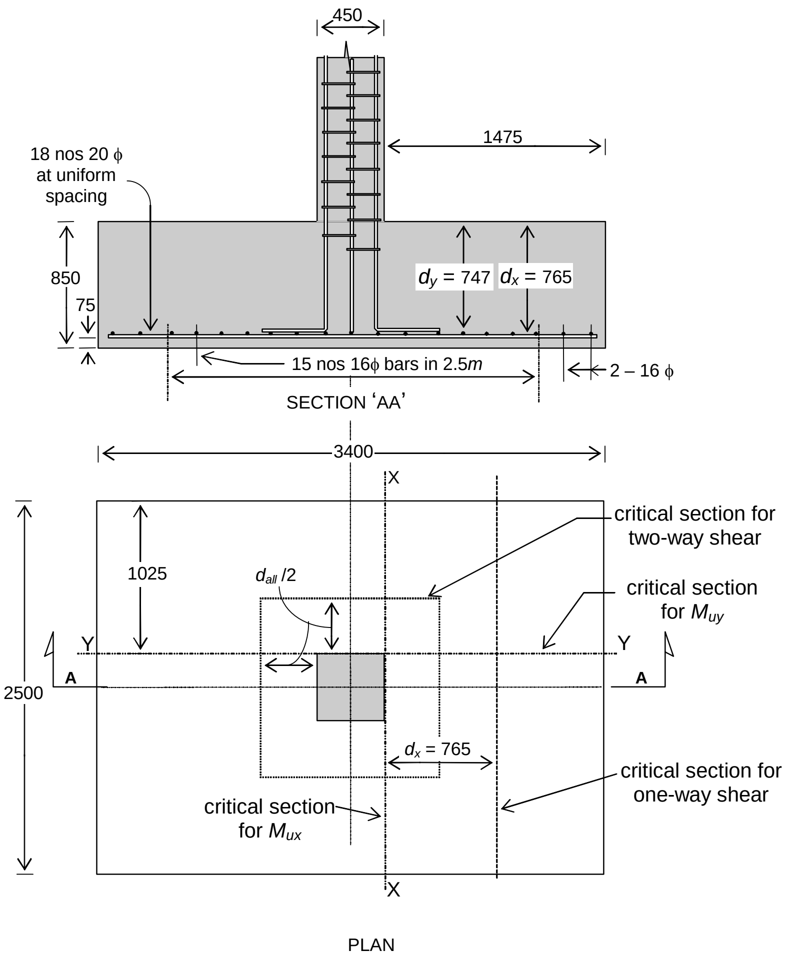


**Fig. 14.13** Example 14.3

- Two-way shear (at $d_{av}/2$ from column periphery):
  $V_{u2} = 0.406 \times [3400 \times 2500 - (450 + 756)^2] = 2860 \times 10^3$ N
  $\Rightarrow \tau_v = 2860 \times 10^3/\{(450 + 756) \times 4 \times 756\} = 0.784$ MPa
  $< k_s \tau_c = 1.0 \times 0.25\sqrt{20} = 1.118$ MPa (refer Cl. 31.6.3.1 of Code).
  —Hence, OK.

**Design of flexural reinforcement**

(a) *long direction* (section XX in Fig. 14.13)

- $M_{ux} = 0.406 \times 2500 \times 1475^2/2 = 1104.1 \times 10^6$ Nmm
  $$\Rightarrow R \equiv \frac{M_u}{Bd_x^{\,2}} = \frac{1104.1 \times 10^6}{2500 \times 765^2} = 0.755 \text{ MPa}$$

$\Rightarrow \dfrac{(p_t)_{reqd}}{100} = \dfrac{20}{2 \times 415}\left[1 - \sqrt{1 - 4.598 \times 0.755/20}\right] = 0.219 \times 10^{-2}$

- This is less than $p_t = 0.28$ required for one-way shear.
  $\Rightarrow (A_{st,x})_{reqd} = 0.28 \times 2500 \times 765/100 = 5355 \text{ mm}^2$
  Using 20 $\phi$ bars, number required = 5355/314 = 18
  $\Rightarrow$ Corresponding spacing $s = \{2500 - (75 \times 2) - 20\}/17 = 137$ mm — OK.
  Provide **18 nos 20 $\phi$** bars at uniform spacing in the long direction.
- Development length required: $L_d = 47.0\ \phi$ (for M 20 concrete, Fe 415 steel, as in previous Example)
  $= 47.0 \times 20 = 940$ mm
  Development length available = 1475 – 75 = 1400 mm > 940 mm — OK.

(b) *short direction* (section YY in Fig. 14.13)

- $M_{uy} = 0.406 \times 3400 \times 1025^2/2 = 725.1 \times 10^6$ Nmm
  $\Rightarrow R \equiv \dfrac{M_u}{Bd_y^{\,2}} = \dfrac{725.1 \times 10^6}{3400 \times 747^2} = 0.382$ MPa
  $\Rightarrow \dfrac{(p_t)_{reqd}}{100} = \dfrac{20}{2 \times 415}\left[1 - \sqrt{1 - 4.598 \times 0.382/20}\right] = 0.108 \times 10^{-2}$
- This is less than the minimum reinforcement required for slabs:
  $(A_{st})_{min} = 0.0012\ bD = 0.0012 \times 3400 \times 850 = 3468 \text{ mm}^2$
  Using 16 $\phi$ bars, number required = 3468/201 = 18
- $A_{st}$ to be provided within a central band width $B$ = 2500 mm is:
  $3468 \times \dfrac{2}{\beta + 1} = 3468 \times \dfrac{2}{(3.5/2.5 + 1)} = 2890 \text{ mm}^2$
  Using 16 $\phi$ bars, number required = 2890/201 = 15
- Provide **15 nos 16 $\phi$** bars at uniform spacing within the central band of width 2.5 m, and **2 nos 16 $\phi$** bars each in the two outer segments; making a total of 19 bars, as shown in Fig. 14.13. The spacings are within limits (3*d*, 300 mm).
- Required development length = 47.0 × 16 = 752 mm
  Development length available = 1025 – 75 = 950 mm > 752 mm — OK.

**Transfer of force at column base**

The calculations are identical to those given in Example 14.2 (except that for the footing face, $\sqrt{A_1/A_2}$ = 2500/450 = 5.56, limited to 2.0). The excess force of 1171.9 kN may be transferred across the column-footing interface by simply extending the column bars, as in the previous Example, and as indicated in Fig. 14.13.

**Alternative:** As in the previous Example, a sloped footing may be designed; this is likely to be more economical than a flat footing.

### EXAMPLE 14.4: Masonry Wall Footing

Design a reinforced concrete footing for a 230 mm thick masonry wall which supports a load (inclusive of self-weight) of 200 kN/m under service loads. Assume a safe soil bearing capacity of 150 kN/m$^2$ at a depth of 1 m below ground. Assume M 20 grade concrete and Fe 415 grade steel.

**SOLUTION**

**Size of footing**

- *Given:* $P$ = 200 kN/m, $q_a$ = 150 kN/m$^2$ at a depth of 1 m.
- Assuming the weight of the footing + backfill to constitute 10 percent of the applied load $P$, and considering a 1 m length of footing along the wall,

  required width of footing = $\dfrac{200 \times 1.1}{150}$ = 1.47 m.

  Provide 1.5 m wide footing.

**Thickness of footing based on shear considerations**

- Factored net soil pressure (assuming a load factor of 1.5) is:

  $q_u = \dfrac{200 \times 1.5}{1.5 \times 1.0}$ = 200 kN/m$^2$ = 0.200 N/mm$^2$
- The critical section for (one-way) shear is located at a distance $d$ away from the face of the wall

  $\Rightarrow V_u = 0.200 \times 1000\ [(1500 - 230)/2 - d]$

  $= (127000 - 200d)$ N
- Assuming nominal flexural reinforcement ($p_t$ = 0.25), $\tau_c$ = 0.36 MPa for M 20 concrete, the shear resistance of concrete is:

  $V_{uc} = 0.36 \times 1000 \times d = (360d)$ N.
- $V_u \le V_{uc} \Rightarrow 127000 - 200d \le 360d$

  $\Rightarrow d \ge 227$ mm
- Assuming a clear cover of 75 mm and 16 $\phi$ bars,

  thickness $D \ge 227 + 75 + 16/2 = 310$ mm

  Provide ***D* = 310 mm** upto a distance of 250 mm from the face of the wall. At the edge of the footing, a minimum thickness of 150 mm may be provided, and the thickness linearly tapered upto 310 mm, as shown in Fig. 14.14.

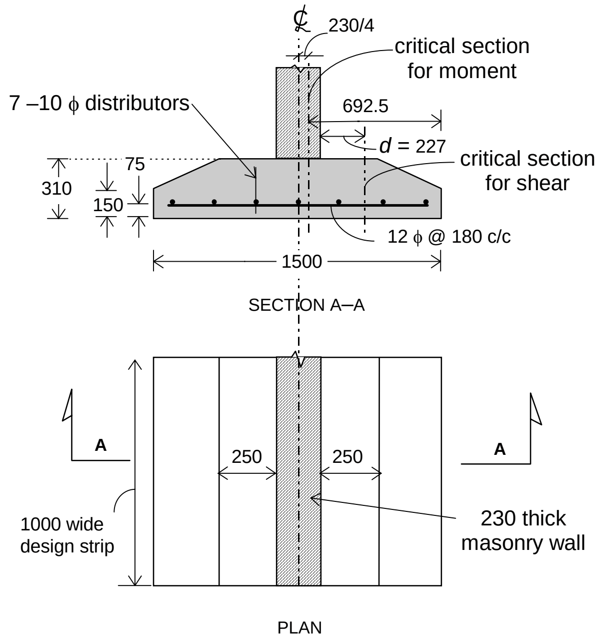


**Fig. 14.14** Example 14.4

**Design of flexural reinforcement**

- The critical section for maximum moment is located halfway between the centreline and edge of the wall, i.e., at a distance 1500/2 – 230/4 = 692.5 mm from the edge of the footing [refer Fig. 14.4]. Considering a 1 m long footing strip with $d$ = 227 mm,
- $\Rightarrow M_u = 0.200 \times 1000 \times 692.5^2/2 = 48.0$ kNm

  $\Rightarrow R \equiv \dfrac{M_u}{bd^2} = \dfrac{48.0 \times 10^6}{1000 \times 227^2} = 0.932$ MPa

  $\Rightarrow \dfrac{(p_t)}{100} = \dfrac{20}{2 \times 415}\left[1 - \sqrt{1 - 4.598 \times 0.932/20}\right] = 0.274 \times 10^{-2}$

  which is greater than the nominal value ($p_t$ = 0.25) assumed for $\tau_c$
- $(A_{st})_{reqd} = 0.274 \times 10^{-2} \times 1000 \times 227 = 622$ mm$^2$ per m length of footing.

  $\Rightarrow$ Spacing of 16 ϕ bars = $\dfrac{1000 \times 20}{622}$ = 323 mm

  Spacing of 12 ϕ bars = $\dfrac{1000 \times 113}{622}$ = 182 mm (< 3$d$ or 300 mm)

  **Provide 12 ϕ @ 180 c/c,** as shown in Fig. 14.14.
- Development length required = 47.0 ϕ (for M 20 concrete, Fe 415 steel)

$= 47.0 \times 12 = 564$ mm[†]

Length available = 692.5 – 75 = 617.5 mm > 564 mm — OK.

**Distributors**

Some nominal bars may be provided, to account for possible secondary stresses due to differential settlement.

Provide 10 ϕ distributors @ about 200 c/c (7 nos will be adequate) [Fig. 14.4].

**Transfer of force at wall base**

Assuming a load factor of 1.5, maximum bearing stress at wall/footing interface (loaded area is 230 mm wide)

$$f_{br} = \frac{200 \times 10^3 \times 1.5}{1000 \times 230} = 1.304 \text{ MPa}$$

which is relatively low and can be accommodated by the concrete [$f_{br,\,max} = 0.45\, f_{ck}\sqrt{A_1/A_2}$ ] in the footing face; the masonry must also be capable of providing this bearing resistance.

#### EXAMPLE 14.5: Isolated footing, eccentrically loaded

Design an isolated footing for a column, 300 mm × 500 mm, reinforced with 6–25 ϕ bars with Fe 415 steel and M 25 concrete [refer Fig. 13.14(a), Example 13.5], subject to a factored axial load $P_u$ = 1000 kN and a factored uniaxial moment $M_{ux}$ = 120 kNm (with respect to the major axis) at the column base. Assume that the moment is reversible. The safe soil bearing capacity may be taken as 200 kN/m$^2$ at a depth of 1.25 m. Assume M 20 concrete and Fe 415 steel for the footing.

#### SOLUTION

**Size of footing**

- *Given:* $P_u$ = 1000 kN, $M_{ux}$ = 120 kNm, $q_a$ = 200 kN/m$^2$ at a depth of 1.25 m.
- As the moment is reversible, the footing should be symmetric with respect to the column. Assuming the weight of the footing plus backfill to constitute about 15 percent of $P_u$, resultant eccentricity of loading at footing base,

  $$e = \frac{120 \times 10^3}{1000 \times 1.15}$$

  $= 104$ mm
- Assuming $e < L/6$ (i.e., $L > 6 \times 104 = 624$ mm)

  $$\frac{1000 \times 1.15}{BL} + \frac{120}{BL^2/6} \leq (200 \times 1.5^{\ddagger}) \text{ kN/m}^2$$

---

[†] This is required with reference to the section of maximum moment. Strictly, development length should also be checked at sections where the thickness is reduced. However, in this case, it can be seen that in the region of tapered thickness, the drop in bending moment (due to cantilever action) is steeper than the drop in effective depth, and hence there is no cause for concern.

[‡] Assuming an enhanced soil pressure under ultimate loads is equivalent to considering allowable pressures at the serviceability limit state. A load factor of 1.5 is assumed here.

$\Rightarrow 300\ BL^2 - 1150L - 720 \leq 0$

various combinations of width $B$ and length $L$ can satisfy the above equation.

Assuming $B = 1.0$ m $\Rightarrow L \geq 4.381$ m

$B = 1.5$ m $\Rightarrow L \geq 3.075$ m

$B = 2.0$ m $\Rightarrow L \geq 2.414$ m

- An economical proportion of the base slab is generally one in which the projection beyond the face of column (or pedestal) is approximately equal in both directions (for effective two-way behaviour, i.e., $(L - a)/2 \approx (B - b)/2$ [refer Fig. 14.7].
- Provide B = 2000 mm and L = 2450 mm; this gives projection of 850 mm (in the short direction) and 975 mm (in the long direction), as shown in Fig. 14.15.

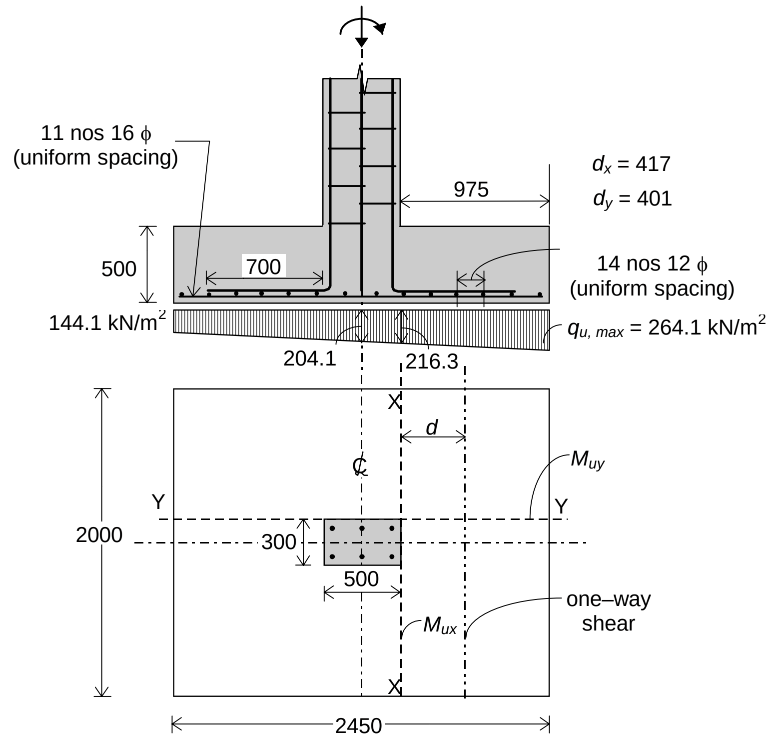


**Fig. 14.15** Example 14.5

**Thickness of footing based on shear**

- Factored (net) soil pressure $q_{u,max} = \dfrac{1000}{2.0 \times 2.45} + \dfrac{120 \times 6}{2.0 \times 2.45^2} = 204.1 + 60.0$

$$= 264.1 \text{ kN/m}^2$$

- $q_{u,min} = 204.1 - 60.0 = 144.1$ kN/m$^2$

($a$) *One-way shear*

- The critical section is located $d$ away from the column face, as shown in Fig. 14.5. The average pressure contributing to the factored one-way shear is†

  $q_u = 264.1 - 60.0 \times \{(975 - d)/2\}/1225$
  $= (240.2 + 0.02449d)$ kN/m$^2$
  $\approx 255$ kN/m$^2$ (assuming $d = 600$ mm conservatively)
  $= 0.255$ N/mm$^2$

  $\Rightarrow V_{u1} = 0.255 \times 2000 \times (975 - d)$
  $= (497250 - 510d)$ N
- Assuming $\tau_c = 0.36$ MPa (for M 20 concrete with nominal $p_t = 0.25$),

  $V_{uc} = 0.36 \times 2000 \times d = (720d)$ N
- $V_{u1} \leq V_{uc} \Rightarrow \quad 497250 - 510d \leq 720d$

  $\Rightarrow d \geq 404$ mm

($b$) *Two-way shear*

- The critical section is located $d/2$ from the periphery of the column all around. The average pressure contributing to the factored two-way shear is $q_u = 204.1$ kN/m$^2$ = 0.2041 N/mm$^2$

  $\Rightarrow V_{u2} = 0.2041\,[2000 \times 2450 - (300 + d)(500 + d)]$

  Assuming $d = 404$ mm (conservatively),

  $V_{u2} = 870 \times 10^3$ N
- For two-way shear resistance, limiting shear stress of concrete

  $\tau_{cz} = k_s\,(0.25\sqrt{f_{ck}}\,)$, where $k_s = 0.5 + 300/500$, but limited to 1.0.

  $\Rightarrow \tau_{cz} = 1.0 \times 0.25\sqrt{20} = 1.118$ MPa

  $\Rightarrow V_{uc} = 1.118 \times [(300 + d) + (500 + d)] \times 2 \times d$
  $= (1788.8d + 2.236d^2)$ N

  $d = 404$ mm $\Rightarrow V_{uc} = 1088$ kN $> V_{u2} = 870$ kN
- Hence, one-way shear governs the footing slab thickness and $d \geq 404$ mm. Assuming a clear cover of 75 mm and a bar diameter of 16 mm,

  $D \geq 404 + 75 + 16/2 = 487$ mm

  Provide $D = 500$ mm

  $\Rightarrow$ effective depth (long span) $d_x = 500 - 75 - 8 = 417$ mm
  effective depth (short span) $d_y = 417 - 16 = 401$ mm

**Check maximum soil pressure**

- Assuming unit weights of concrete and soil as 24 kN/m$^3$ and 18 kN/m$^3$ respectively, at the factored loads,

  $$q_{max\text{-}gross} = \frac{1000}{2.0 \times 2.45} + \{(24 \times 0.5) + 18 \times (1.25 - 0.5)\} \times 1.5 + \frac{120 \times 6}{2.0 \times 2.45^2}$$
  $$= 302 \text{ kN/m}^2 \approx 200 \times 1.5 \text{ kN/m}^2 \quad \text{— Hence, OK.}$$

---

† In this problem, the variation in the soil pressure over the length from the edge to the critical section is not very large. In such cases, it is sufficient and conservative to assume a uniform pressure equal to the maximum at the edge.

**Design of flexural reinforcement**

- The critical sections for moment are located at the faces of the column in both directions (XX and YY) as shown in Fig. 14.15.

(a) *long span*

- cantilever projection = 975 mm, width = 2000 mm, $d_x$ = 417 mm, $q_u$ = 0.2163 N/mm$^2$ at face of column, 0.2641 N/mm$^2$ at footing edge.
- $M_{ux} = (0.2163 \times 2000 \times 975^2/2) + (0.2641 - 0.2163) \times \frac{1}{2} \times 2000 \times 975^2 \times 2/3$

  $= (205.6 + 30.3) \times 10^6 = 236 \times 10^6$ Nmm

  $\Rightarrow R \equiv \dfrac{M_u}{b{d_x}^2} = \dfrac{236 \times 10^6}{2000 \times 417^2} = 0.679$ MPa

  $\Rightarrow \dfrac{(p_t)}{100} = \dfrac{20}{2 \times 415}\left[1 - \sqrt{1 - 4.598 \times 0.679/20}\right] = 0.197 \times 10^{-2}$

  $p_t$ assumed for one-way shear = 0.25 > 0.197

  $\Rightarrow (A_{st})_{reqd} = 0.25 \times 2000 \times 417/100 = 2085$ mm$^2$
- Using 16 $\phi$ bars, number required = 2085/201 = 11

  [corresponding spacing = (2000 – 75 × 2 – 16)/10 = 183 mm]

  **Provide 11 nos 16 $\phi$ bars** at uniform spacing in the long direction, as shown in Fig. 14.15.
- Development length required = 47.0 $\phi$ (for M 20 with Fe 415)

  = 47.0 × 16 = 752 mm

  < 900 mm available — OK.

(b) *short span*

- cantilever projection = 850 mm, width = 2450 mm, $d_y$ = 401 mm, $q_u$ varies along the section YY, with an average value of 0.2041 N/mm$^2$ at the middle. Considering a slightly greater value (mean of values at centre and footing edge), $q_u \approx (0.2041 + 0.2641)/2 = 0.2341$ N/mm$^2$
- $M_{uy} = 0.2341 \times 2450 \times 850^2/2) = 207.2 \times 10^6$ Nmm

  $\Rightarrow R \equiv \dfrac{M_u}{b{d_y}^2} = \dfrac{207.2 \times 10^6}{2450 \times 401^2} = 0.526$ MPa

  $\Rightarrow \dfrac{(p_t)}{100} = \dfrac{20}{2 \times 415}\left[1 - \sqrt{1 - 4.598 \times 0.526/20}\right] = 0.150 \times 10^{-2}$

  $\Rightarrow (A_{st})_{reqd} = 0.150 \times 10^{-2} \times 2450 \times 401 = 1474$ mm$^2$

  $(A_{st})_{min} = 0.0012 \times 2450 \times 500 = 1470$ mm$^2$ < 1474 mm$^2$
- Number of 12 $\phi$ bars required = 1474/113 = 14

  As the difference in dimensions between the two sides ($B$ = 2000 mm, $L$ = 2450 mm) is not significant, it suffices to provide these bars at a uniform spacing.
- **Provide 14 nos 12 $\phi$ bars** in the short direction at uniform spacing, as shown in Fig. 14.15.
- Development length required = 47.0 × 12 = 564 mm

  < 775 mm available — Hence, OK.

**Transfer of forces at column base**

- As some of the bars are in tension, no transfer of the tensile force is possible through bearing at the column-footing interface, and these bars may be extended into the footing.
- Required development length of 25 $\phi$ bars in tension = 47.0 × 25
  = 1175 mm
- Length available (including standard 90$^{o}$ bend on top of upper layer of footing reinforcement) = (500 – 75 – 16 – 12 – 25/2) + 8 × 25 = 584 mm. The balance, 1175 – 584 = 591 mm, can be made up by extending these bars into the footing beyond the bend. A total extension of 4 × 25 + 591 = 691 ≈ 700 mm needs to be provided beyond the bend point, as shown in Fig. 14.15. As the moment on the column is reversible, this embedment should be provided for all the column bars.
- Alternatively, a pedestal (with cross-sectional dimensions of, say, 450 mm × 750 mm) may be provided to the column below ground level (or 150 mm below GL), and the longitudinal bars in the pedestal designed to resist the factored axial load-moment combination; small diameter bars (say 16 mm $\phi$) may be selected, with the aim of reducing the development length requirements.

#### EXAMPLE 14.6: Isolated footing eccentrically loaded

Redesign the footing in Example 14.5 for a uniformly distributed base pressure, considering that the applied moment at the column base is entirely due to dead loads (and hence, irreversible).

#### SOLUTION

- *Given:* (as in the previous Example) $P_u$ = 1000 kN, $M_{ux}$ = 120 kNm,
  $q_a$ = 200 kN/m$^2$ at a depth of 1.25 m, $f_{ck}$ = 20 MPa, $f_y$ = 415 MPa

**Size of footing**

- Required eccentricity between column centroid and footing centroid = $M_u/P_u$
  $= \dfrac{120 \times 10^3}{1000}$ = 120 mm
- Assuming the weight of the footing + backfill to constitute 10 percent of $P_u$, and assuming a load factor of 1.5,
  base area required = $\dfrac{1000 \times 1.1}{200 \times 1.5}$ = 3.67 m$^2$
- For economical proportions, the cantilever projections (for flexural design) should be approximately equal in the two directions.
  ⇒ <u>Provide $L = B$ = 1.95 m</u> (Area = 3.80 m$^2$ > 3.67 m$^2$)
  With the column offset by 120 mm, this results in cantilever projections of 845 mm and 825 mm in the two directions, as shown in Fig. 14.16.

**Thickness of footing based on shear**

- Factored (net) soil prressure $q_u = \dfrac{1000}{1.95 \times 1.95}$ = 263.0kN/m$^2$
  = 0.263 N/mm$^2$

(a) *One-way shear*

- The critical section is located $d$ away from the column face [refer Fig. 14.16]
  $V_{u1} = 0.263 \times 1950 \times (845 - d) = (433358 - 512.8d)$ N
- Assuming $\tau_c = 0.36$ MPa (for M 20 concrete with nominal $p_t = 0.25$),
  $V_{uc} = 0.36 \times 1950 \times d = (702d)$ N
- $V_{u1} \le V_{uc} \Rightarrow 433358 - 512.8d \le 702d$
  $\Rightarrow d \ge 356.7$ mm

(b) *Two-way shear*

- The critical section is located $d/2$ from the column periphery all around.
  $V_{u2} = 0.263 \times [1950^2 - (300 + d)(500 + d)]$
- Assuming $d \ge 357$ mm, $V_{u2} \le 851976$ N

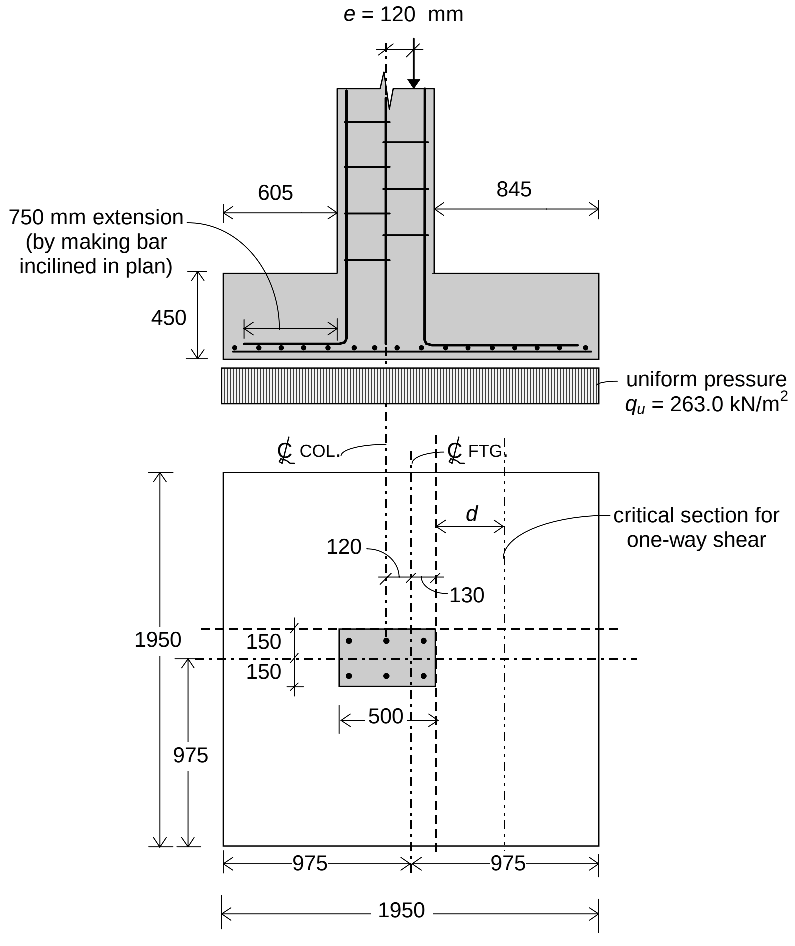


**Fig. 14.16** Example 14.6

- Two-way shear resistance (as in Example 14.5)
  $V_{uc} = 1.118 \times [(300 + d) + (500 + d)] \times 2 \times d$
  $d = 357$ mm $\Rightarrow V_{uc} = 1208.6$ kN $> V_{u2,max} = 852.0$ kN
  Hence, one-way shear governs the thickness. As a square footing is provided and the one-way shear requirement is equally applicable in both directions, the $d$ calculated may be taken as an average depth: $(d_x + d_y)/2$.
  Assuming 75 mm clear cover and 12 $\phi$ bars,
  $D \geq 357 + 75 + 12 = 444$ mm
- Provide $D = 450$ mm and consider the average effective depth,
  $d = 450 - 75 - 12 = 363$ mm while designing for flexure.

**Design of flexural reinforcement**

- Maximum cantilever projection = 845 mm (from face of column)
  $M_u = 0.263 \times 1950 \times 8452/2 = 183.1 \times 10^6$ kNm

  $$\Rightarrow R \equiv \frac{M_u}{Bd^2} = \frac{183.1\times10^6}{1950\times363^2} = 0.713 \text{ MPa}$$

  $$\Rightarrow \frac{p_t}{100} = \frac{20}{2\times415}\left[1-\sqrt{1-4.598\times0.713/20}\right] = 0.206 \times 10^{-2}$$

  $p_t = 0.25$ has been assumed for one-way shear strength
  Accordingly, $A_{st} = 0.25 \times 1950 \times 363/100 = 1770$ mm$^2$
  Number of 12 $\phi$ bars required = 1770/113 = 16
  [corresponding spacing = (1950 – 75 × 2 – 12)/15 = 119 mm — OK].
  Provide 16 nos 12 $\phi$ bars in both directions.
  Development length required = 47.0 $\phi$ = 47.0 × 12 = 564 mm — available [refer Fig. 14.16].

**Transfer of forces at column base**

- This is as explained in Example 14.5, with the difference that some of the bars are always under compression, requiring reduced development length. However, the bars in tension need an additional extension of 50 mm beyond the bend point, on account of the reduced footing thickness of 450 mm (as against 500 mm in Example 14.5). The total extension of 641 + 50 = 741 ≈ 750 mm requires reorienting the bars diagonally in plan for this length to be available.

## 14.6 DESIGN OF COMBINED FOOTINGS

### 14.6.1 General

As mentioned in Section 14.2.2, a footing supporting more than a single column or wall is called a *combined footing*, and when many columns (more than two) are involved, terms such as *continuous strip footing* (if columns are aligned in one direction only) and *raft foundation* or *mat foundation* are used. Multiple column foundations become necessary in soils having very low bearing capacities. However, even in soils having moderate or high 'safe bearing capacity' for the use of individual footings, combined footings become necessary sometimes — as when:

- columns are so closely spaced that isolated footings cannot be conveniently provided, as the estimated base areas tend to overlap;
- an exterior column located along the periphery of the building is so close to the property line that an isolated footing cannot be symmetrically placed without extending beyond the property line.

### 14.6.2 Distribution of Soil Pressure

As mentioned earlier (in Section 14.3.2), the prediction of the exact distribution of base pressure under a footing is difficult, as it depends on the rigidity of the footing as well as the properties of the soil. If this is difficult for an isolated footing, indeed, it is more so for a combined footing.

For a very rigid footing supported on an elastic soil base, a straight line pressure distribution is appropriate. Such an assumption is found to lead to satisfactory designs in the case of relatively rigid footings. However, for relatively flexible footings, such an assumption is not realistic; the problem is rather complex and involves consideration of *soil-structure interaction*.

In this section, only two-column combined footings are considered. The footing is assumed to be rigid and the soil response elastic, and hence a straight line distribution of soil pressure is assumed.

### 14.6.3 Geometry of Two-Column Combined Footings

Examples of two-column combined footings are shown in Fig. 14.17. The geometry of the footing base should preferably be so selected as to ensure that the centroid of the footing area coincides with the resultant of the column loads (including consideration of moments if any, at the column bases). This will result in a uniform distribution of soil pressure, which is desirable in order to avoid possible tilting of the footing (as mentioned earlier in Section 14.3.2).

The footing may be rectangular or trapezoidal in shape [Fig. 14.17], depending on the relative magnitudes of loads on the two columns which the footing supports. When the exterior column (which has the space limitation for an independent footing) carries the lighter load $\left(\bar{x} > s/2\right)$, a rectangular footing [Fig. 14.17(b)] or a trapezoidal footing (with a reduced width under the exterior column) as shown in Fig. 14.17(c) may be provided. On the other hand, when the exterior column carries the heavier load [ $\bar{x} < s/2$ in Fig. 14.17(e)], the wider end of the trapezoidal footing should be located under the exterior column.

### 14.6.4 Design Considerations in Two-Columns Footings

#### *Fixing Plan Dimensions*

As discussed earlier with reference to Fig. 14.17, the plan dimensions of the two-column combined footing may be selected to satisfy the following two requirements

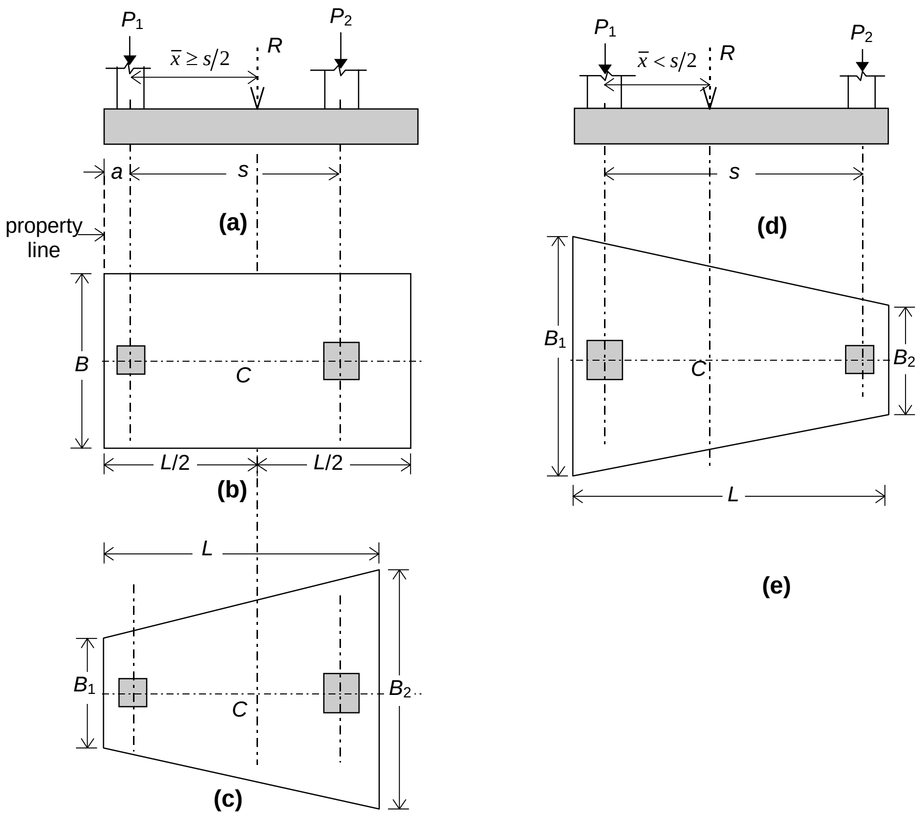


**Fig. 14.17** Geometry of two-column combined footings

1. Base area of footing $A$ = Total (service) load[†]/$q_a$.
2. The line of action of the resultant of the column loads must pass through the centroid of the footing.

In the case of a rectangular footing [Fig. 14.17(b)], the second requirement results in a length $L$ of the footing equal to $2(\bar{x}+a)$. The edge distance $a$ may be fixed with reference to a property line (as shown in Fig. 14.17); otherwise, it may be suitably assumed. Having fixed $L$, the footing width $B$ is obtained as $A/L$. In the case of a trapezoidal footing [Fig. 14.8(c), (e)], usually the length $L$ is selected first, and the dimensions $B_1$, $B_2$ adjusted to satisfy the two requirements cited above.

### *Load Transfer Mechanism*

As in the case of isolated footings, the factored net soil pressure $q_u$ is computed as the resultant factored load divided by the base area provided, and the pressure may be

[†] Including the weight of the footing plus backfill.

assumed to be uniformly distributed‡ [refer Fig. 14.18].

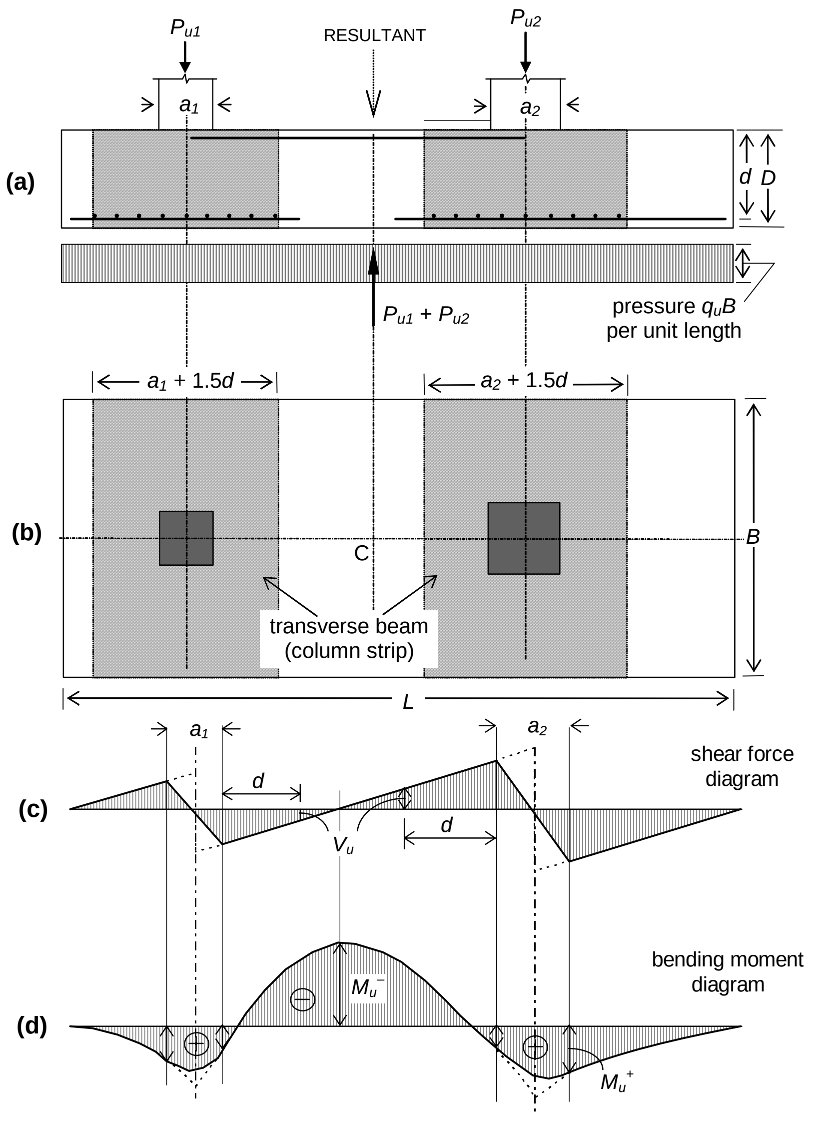


**Fig. 14.18** Assumed load transfer in two-column combined footing

‡ In cases where the ratio of the column loads $P_1 : P_2$ is subject to uncertainty, or when these loads are accompanied by moments which may be reversible, the line of action of the resultant load will not always match with the centroid of the footing, and the pressure distribution will be nonuniform. However, it is conservative to assume uniform distribution with maximum $q_u$.

The base slab of the combined footing is subject to two-way bending, and one-way as well as two-way shear (as in the case of isolated footing). In general, the width of the footing (*B*) is much less than the length (*L*), with the result that the flexural behaviour is predominantly one-way (i.e., in the longitudinal direction), and the two-way action (i.e., including transverse bending) is limited to the neighbourhood of the column locations.

For the purpose of structural design, a simplified (and usually conservative) load transfer mechanism may be assumed — as shown in Fig. 14.18. In this idealised model, the footing is treated as a uniformly loaded wide longitudinal beam (width *B*, length *L* and factored load $q_u B$ per unit length), supported on two column strips, which in turn act as transverse beams cantilevered from the columns. The width of each column strip may be taken approximately as the width of the column (*a*) plus 0.75*d* on either side of the column [Fig. 14.18(b)].

The thickness of the footing is generally governed by shear considerations, as in isolated footings. The critical sections for one-way shear are at a distance *d* from the column face [Fig. 14.18(c)], and at *d*/2 from each column periphery for two-way shear. The distribution of longitudinal shear forces and bending moments may be easily determined from statics, treating the footing slab as being simply supported† on the two column strips, with overhangs (if any) beyond each column strip, as shown in Fig. 14.18(c), (d).

The flexural reinforcement in the longitudinal direction is designed for the 'positive' moment at the face of the column and the maximum 'negative' moment between the columns; the reinforcement is placed at the bottom in the case of the former, and at top in the case of latter, as depicted in Fig. 14.18(a),(d). The flexural reinforcement in the transverse direction (in the column strip) is designed for the 'positive' moment at the section in line with the face of the column, considering the column strip as a beam with uniformly distributed factored loads (whose total magnitude is equal to the factored load on the column). This reinforcement is provided at the bottom, and located in a layer above the longitudinal reinforcement [refer Fig. 14.18(a)]. Nominal transverse reinforcement may be provided elsewhere (i.e., other than the column strips), to tie with the longitudinal reinforcement (wherever provided); these nominal bars, however, are not indicated in Fig. 14.18(a). Development length requirements should be satisfied by the flexural reinforcement provided.

The column strip (transverse beam) should also be checked for one-way shear at a distance, equal to the effective depth of the transverse reinforcement, from the face of the column/pedestal. The design of a two-column rectangular footing is illustrated in Example 14.7.

### *Beam-Slab Combined Footings*

If the case of relatively large footings, providing a uniform large thickness for the entire footing results in a somewhat expensive footing. In such a case, it may be

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† As the design section for both shear and moment are outside the column section, it suffices to assume the supports to be concentrated at the column centrelines; the corresponding shear force and bending moment distributions are shown by dashed lines in Fig. 14.18(c), (d).

more economical to design a *beam-slab* footing, in which the footing consists of a base slab stiffened by means of a central longitudinal beam (of sufficient depth), interconnecting the columns [Fig. 14.19].

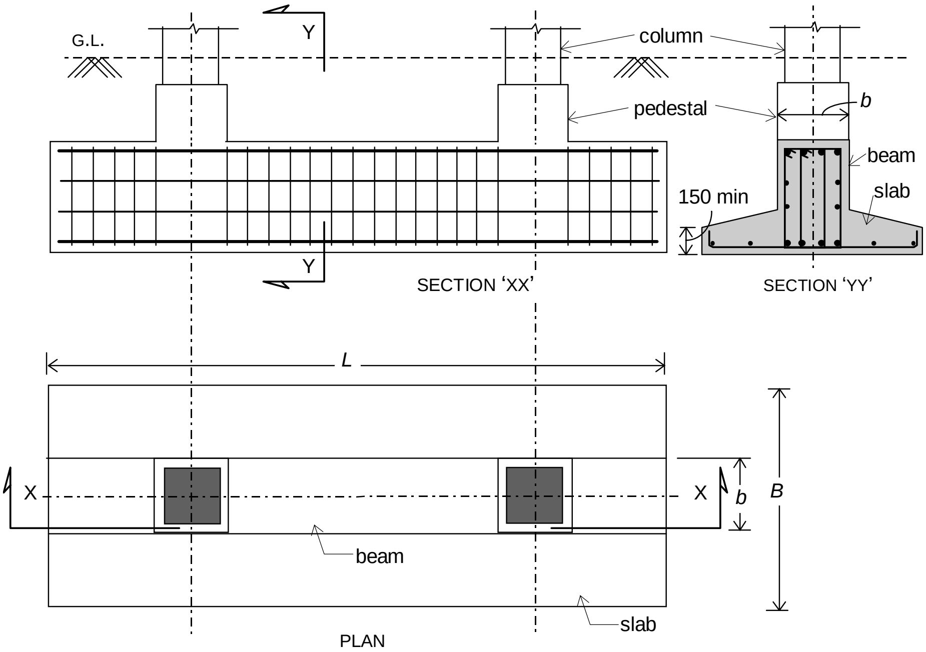


**Fig. 14.19** Beam-slab combined footing

The base slab behaves like a one-way slab, supported by the beam, and bends transversely under the uniform soil pressure acting from below. The loads transferred from the slab are resisted by the longitudinal beam. The size of the beam is generally governed by (one-way) shear at $d$ from the face of the column/pedestal. For effective load transfer, the width of the footing beam should be made equal to the column/pedestal width, and it is advantageous to provide a pedestal to the column. The high shear in the beam will usually call for heavy shear reinforcement, usually provided in the form of multi-legged stirrups [Fig. 14.19].

The base slab may be tapered (if the span $(B - b)/2$ is large), for economy. The thickness of the slab should be checked for one-way shear at $d$ (of slab) from the face of the beam. The flexural reinforcement in the slab is designed for the cantilever moment at the face of the beam, and provided at the bottom, as shown in Fig. 14.19. Two-way shear is not a design consideration in beam-slab footings. The top and bottom reinforcement in the beam should conform to the longitudinal bending moment diagram, and development length requirements should be satisfied.

**EXAMPLE 14.7**

Design a combined footing for two columns $C_1$ (400 mm × 400 mm with 4–25 $\phi$ bars) and $C_2$ (500 mm × 500 mm with 4–28 $\phi$ bars) supporting axial loads $P_1$ = 900 kN and $P_2$ = 1600 kN respectively (under service dead and live loads). The column $C_1$ is an exterior column whose exterior face is flush with the property line. The centre-to-centre distance between $C_1$ and $C_2$ is 4.5 m. The allowable soil pressure at the base of the footing, 1.5 m below ground level, is 240 kN/m$^2$. Assume steel of grade Fe 415 in columns as well as footing, and concrete of M 30 grade in columns and M 20 grade in footing.

**SOLUTION**

**Footing base dimensions**

- Assuming the weight of the combined footing plus backfill to constitute 15 percent of the column loads,

  $$A_{reqd} = \frac{P_1 + P_2 + \Delta P}{q_a} = \frac{(900 + 1600) \times 1.15}{240} = 11.98 \text{ m}^2$$
- In order to obtain a uniform soil pressure distribution, the line of action of the resultant load must pass through the centroid of the footing. Let the footing centroid be located at a distance $\bar{x}$ from the centre of $C_1$ [refer Fig. 14.20(a)]:
- Assuming a load factor of 1.5, the factored column loads are:

  $P_{u1}$ = 900 × 1.5 = 1350 kN; $P_{u2}$ = 1600 × 1.5 = 2400 kN $\Rightarrow$ $P_{u1} + P_{u2}$ = 3750 kN
- spacing between columns $s$ = 4500 mm

  $$\Rightarrow \bar{x} = \frac{P_{u2}\, s}{P_{u1+} + P_{u2}} = \frac{2400 \times 4500}{3750} = 2880 \text{ mm}$$
- As $\bar{x} > s/2$ = 2250 mm, a *rectangular* footing may be provided, with length

  $L$ = 2(2880 + 200) = 6160 mm

  **Provide *L* = 6.16 m**

  $\Rightarrow$ width required $B \geq A/L$ = 11.98/6.16 = 1.95 m

  **Provide *B* = 2.00 m**

**Stress resultants in longitudinal direction**

- Treating the footing as a wide beam ($B$ = 2000 mm) in the longitudinal direction, the uniformly distributed load (acting upward) is given by

  $q_u B = (P_{u1} + P_{u2})/L$ = 3750/6.16 = 608.8 kN/m [as shown in Fig. 14.20(b)].
- The distribution of shear force is shown in Fig. 14.20(c). The critical section for one-way shear is located at a distance $d$ from the (inside) face of $C_2$, and has a value

  $V_{u1} = 2400 - 608.8\,(1460 + 250 + d) \times 10^{-3} = (1359 - 0.6088d)$ kN
- The distribution of bending moment is shown in Fig. 14.20(d). The maximum 'positive' moment at the face of column $C_2$ is given by

  $M_u^+ = 608.8 \times (1.460 - 0.250)^2/2 = 446$ kNm
- The maximum 'negative' moment occurs at the location of zero shear, which is at a distance $x$ from the edge (near $C_1$) of the footing [Fig. 14.20(c)]:

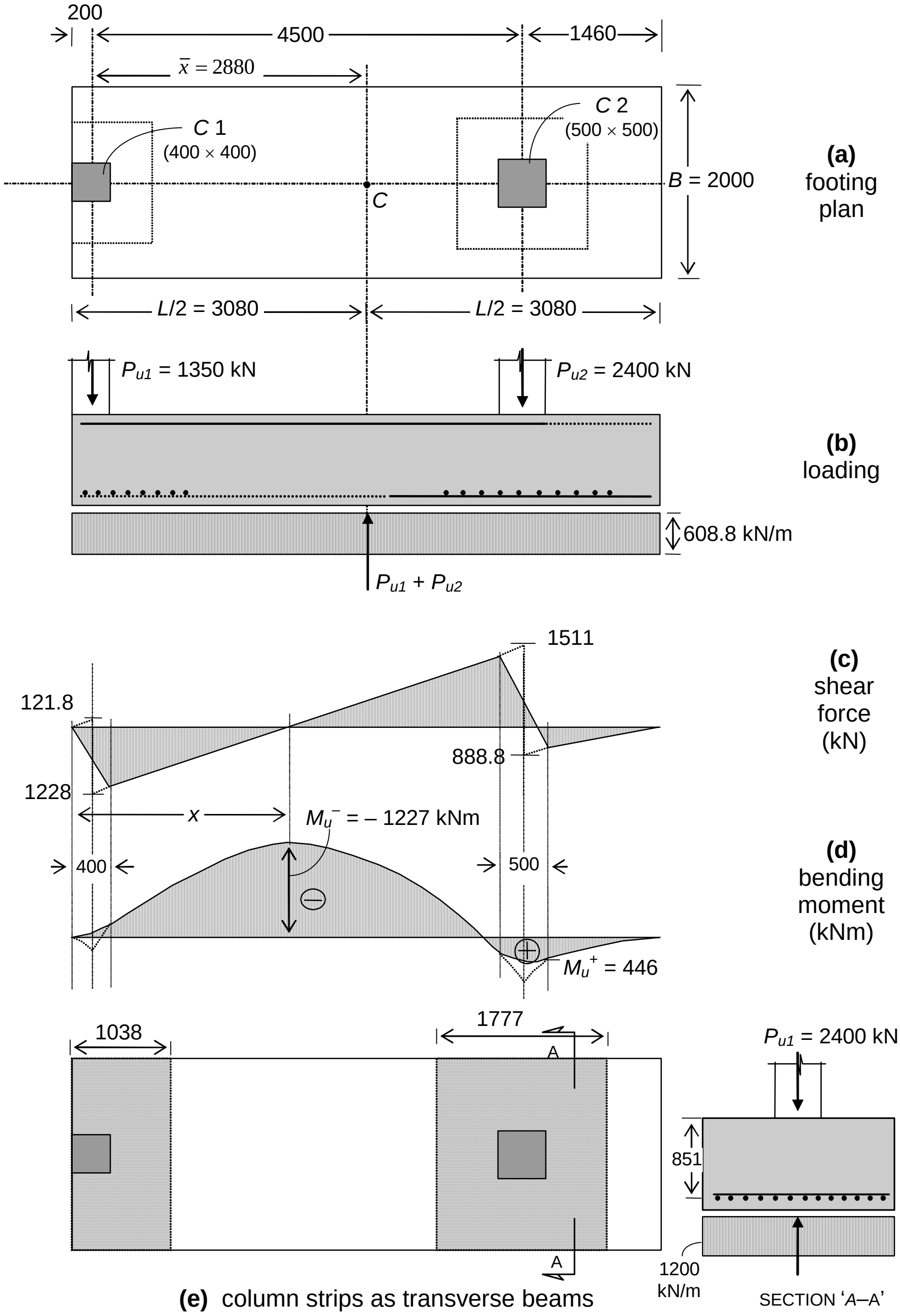


**Fig. 14.20** Example 14.7

$x = 1350/608.8 = 2.2175$ m

- $\Rightarrow M_u^- = 608.8 \times (2.2175)^2/2 - 1350 \times (2.2175 - 0.2) = (-)\ 1227$ kNm

**Thickness of footing based on shear**

(a) *One-way shear* (longitudinal): $V_{u1}$

- Assuming $\tau_c = 0.48$ MPa (for M 20 concrete, assuming $p_t = 0.50$)

  $V_{uc} = 0.48 \times 2000 \times d = (960d)\ N$
  $V_{u1} = V_{uc} \Rightarrow (1359 - 0.6088d) \times 10^3 \le 960d$
  $\Rightarrow d \ge 866$ mm

(b) *Two-way shear*

- The critical section is located $d/2$ from the periphery of columns $C_1$ and $C_2$ [Fig. 14.20(a)], and the factored soil pressure $q_u = (q_u B)/B = 608.8/2.0$
  $= 304.4$ kN/m$^2$.

  Assuming $d = 866$ mm,

  $$V_{u2} = \begin{cases} 1350 - 304.4(0.4 + 0.866)(0.4 + 0.866/2) = 1029 \text{ kN} & \text{at column } C_1 \\ 2400 - 304.4(0.5 + 0.866)^2 = 1832 \text{ kN} & \text{at column } C_2 \end{cases}$$

  Limiting two-way shear stress $\tau_{c2} = k_s\ (0.25\sqrt{f_{ck}}\ )$

  For square columns, $k_s = 1.0 \Rightarrow \tau_{c2} = 1.0 \times 0.25\sqrt{20} = 1.118$ MPa

  $$\Rightarrow V_{uc} = \begin{cases} 1.118 \times (1266 + 833 \times 2) \times (866) = 2839 \times 10^3 \text{ N} > 1029 \text{ kN} \\ 1.118 \times (1366 \times 4) \times (866) = 5290 \times 10^3 \text{ N} > 1832 \text{ kN} \end{cases}$$

  Hence, the depth is governed by considerations of one-way shear alone. Assuming an overall thickness ***D* = 950 mm** and 20 mm $\phi$ bars with a clear cover of 75 mm, effective depth $d = 950 - 75 - 20/2 = 865$ mm
  (very close to 866 mm required — OK)

- *Check base pressure*:
  Assuming unit weights of 24 kN/m$^3$ for concrete and 18 kN/m$^3$ for backfill, gross soil pressure under service loads
  $q = (900 + 1600)/(6.16 \times 2.0) + (24 \times 0.95) + (18 \times 0.55)$
  $= 235.6$ kN/m$^2 < q_a = 240$ kN/m$^2$ — OK.

**Design of longitudinal flexural reinforcement**

- Maximum 'negative' moment: $M_u^- = 1227$ kN/m

  $$\Rightarrow R \equiv \frac{M_u}{Bd^2} = \frac{1227 \times 10^6}{2000 \times 865^2} = 0.820 \text{ MPa}$$

  $$\Rightarrow \frac{p_t}{100} = \frac{20}{2 \times 415}\left[1 - \sqrt{1 - 4.598 \times 0820/20}\right] = 0.239 \times 10^{-2}$$

  $\Rightarrow p_t = 0.239 < 0.50$[†] required for one-way shear

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[†] Note: In general, it is not good practice (and often, not economical) to fix the flexural steel requirement based on shear strength requirements, if the steel requirement is excessive. However, in this situation, $p_t = 0.50$ cannot be considered to be excessive.

$\Rightarrow (A_{st})_{reqd} = 0.50 \times 2000 \times 865/100 = 8650 \text{ mm}^2$
$> (A_{st})_{min} = 0.0012BD$

Number of 20 mm $\phi$ bars required = 8650/314 = 28

[Corresponding spacing = (2000 – 75 × 2 – 20)/27 = 68 mm, which is low but acceptable.]

$\therefore$ **Provide 28 nos 20 mm $\phi$ bars at top** between the two columns as indicated in Fig. 14.21.

- Required development length (with M 20 concrete and Fe 415 bars) will be less than $L_d = 47.0 \times 20 = 940$ mm
  Adequate length is available on both sides of the peak moment section.
- Maximum 'positive' moment: $M_u^+ = 446$ kNm (at face of column $C_2$)

$$\Rightarrow R \equiv \frac{M_u}{Bd^2} = \frac{446 \times 10^6}{2000 \times 865^2} = 0.298 \text{ MPa}$$

$$\Rightarrow \frac{p_t}{100} = \frac{20}{2 \times 415}\left[1 - \sqrt{1 - 4.598 \times 0\,298/20}\right] = 0.084 \times 10^{-2} \text{ (low)}$$

$(A_{st})_{min} = 0.0012\ BD = 0.0012 \times 2000 \times 950 = 2280 \text{ mm}^2$

- Number of 16 mm $\phi$ bars required = 2280/201 = 12
  [Corresponding spacing = (2000 – 75 × 2 – 16)/11 = 167 mm — OK.]
  $\therefore$ **Provide 12nos 16 mm $\phi$ bars at bottom** as indicated in Fig. 14.21.
- Required development length = 47.0 × 16 = 752 mm, which is available on the side of the column $C_2$ close to the edge of the footing; by placing the bars symmetrically with respect to column $C_2$, the required length will be available on both sides of the section of maximum 'positive' moment.

**Design of column strips as transverse beams** [Fig. 14.20(e)].

(a) *Transverse beam under column* C1:

- Factored load per unit length of beam = 1350/2.0 = 675 kN/m
  Projection of beam beyond column face = (2000 – 400)/2 = 800 mm
- Maximum moment at column face:
  $M_u = 675 \times 0.80^2/2 = 216$ kNm
- Effective depth for transverse beam (16 mm $\phi$ bars placed above the 16 mm $\phi$ longitudinal bars): $d = 950 - 75 - 16 \times 1.5 = 851$ mm
  Width of beam = width of column + $0.75d$
  = 400 + 0.75 × 851 = 1038 mm

$$\Rightarrow R \equiv \frac{M_u}{Bd^2} = \frac{216 \times 10^6}{1038 \times 851^2} = 0.287 \text{ MPa (low)}$$

$\Rightarrow$ Provide minimum reinforcement: $A_{st} = 0.0012\ bD$

$\Rightarrow A_{st} = .0012 \times 1038 \times 950 = 1183 \text{ mm}^2$

- Number of 16 mm $\phi$ bars required = 1183/201 = 6
  [Corresponding spacing = (1038 – 75 – 16)/5 = 189 mm]
  Alternatively, no. of 12 mm $\phi$ bars required = 1183/113 = 11
  **Provide 11 nos 12 mm $\phi$ bars**

Required development length = 47.0 × 12 = 564 mm < (800 – 75) mm available — OK.

- There is no need to check one-way transverse shear in this case as the critical section (located at $d$ = 851 mm from column face) lies outside the footing.

(b) *Transverse beam under column* C2:

- Factored load per unit length = 2400/2.0 = 1200 kN/m
  Projection beyond column face = (2000 – 500)/2 = 750 mm
  $\Rightarrow$ Moment at column face = $1200 \times 0.75^2/2$ = 338 kNm
- Width of beam = 500 + 1.5 × 851 = 1777 mm
  $\Rightarrow R \equiv \dfrac{M_u}{Bd^2} = \dfrac{338 \times 10^6}{1777 \times 851^2}$ = 0.263 MPa (low)
  $\Rightarrow$ Provide$(A_{st})_{min}$ = 0.0012 × 1777 × 950 = 2026 mm$^2$
- Number of 12 mm $\phi$ bars required = 2026/113 = 18
  **Provide 18 nos 12 mm$\phi$ bars**
- Required development length = 47.0 × 12 = 564 mm is available beyond the column face.
- As in the previous case, check for one-way shear is not called for.

**Transfer of force at column base**

(a) *Column* C1:

- Limiting bearing stress at i) column face = $0.45 f_{ck}$ = 0.45 × 30 = 13.5 MPa
  ii) footing face = $0.45 f_{ck}\sqrt{A_1/A_2}$
  [As the column is located at the edge of the footing, $A_1 = A_2 = 400^2$ mm$^2$]
  = 0.45 × 20 × 1.0 = 9.0 MPa
  < 13.5 MPa

  $\Rightarrow$ Limiting bearing resistance at column-footing interface
  $F_{br} = 9.0 \times 400^2 = 1440 \times 10^3$ N > $P_{u1}$ = 1350 kN — OK.
- Hence, full force transfer can be achieved without the need for reinforcement across the interface. However, it is desirable to provide some nominal dowels (4 nos 20 mm $\phi$), as shown in Fig. 14.21.

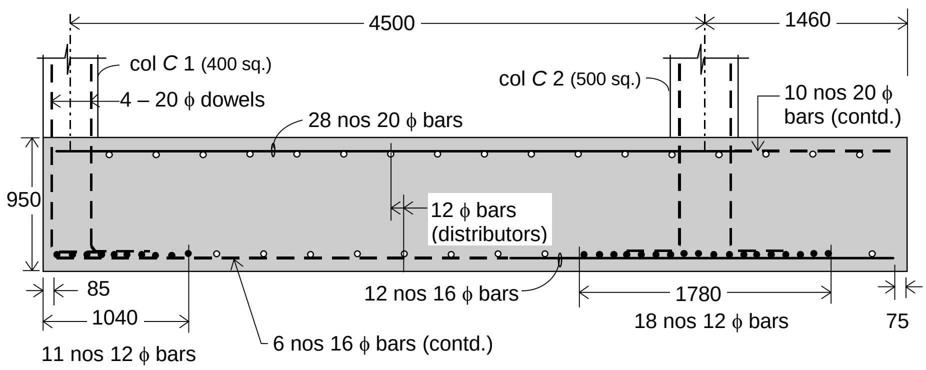


**Fig. 14.21** Details of reinforcement, Example 14.7

(b) *Column* C2:

- Limiting bearing stress at i) column face = $0.45 f_{ck}$ = 13.5 MPa (as before)

  ii) footing face = $0.45 f_{ck}\sqrt{A_1/A_2}$

  [$A_1 = 2000^2$ , $A_2 = 500^2$ mm$^2 \Rightarrow \sqrt{A_1/A_2}$ = 4.0, limited to 2.0]

  $= 0.45 \times 20 \times 2.0 = 18.0$ MPa

  $> 13.5$ MPa

  $\Rightarrow F_{br} = 13.5 \times 500^2 = 3375 \times 10^3$ kN $> P_{u2} = 2400$ kN.
- In this case also, full force transfer can be achieved without the need for reinforcement across the interface. However, it is desirable to provide some nominal dowels (4 nos 20 mm $\phi$) as shown in Fig. 14.21.

**Reinforcement details**

- The reinforcement details are indicated in Fig. 14.21. Some of the longitudinal bars at the bottom are shown (arbitrarily) extended across the full length of the footing in order to provide some nominal reinforcement in the large (otherwise unreinforced) area of concrete between the columns and also to tie up with the transverse bars under column C1. Nominal transverse reinforcement is also indicated at top between the columns, in order to tie up with the main longitudinal bars provided.

## 14.7 TYPES OF RETAINING WALLS AND THEIR BEHAVIOUR

As explained in Section 14.1, retaining walls are used to retain earth (or other material) in a vertical (or nearly vertical) position at locations where an abrupt change in ground level occurs. The wall, therefore, prevents the retained earth from assuming its natural angle of repose. This causes the retained earth to exert a lateral pressure on the wall, thereby tending to bend, overturn and slide the retaining wall structure. The wall, including its supporting footing, must therefore be suitably designed to be *stable* under the effects of the lateral earth pressure, and also to satisfy the usual requirements of strength and serviceability.

Retaining walls are usually of the following types:

**1. Gravity Wall** [Fig. 14.22(a)]

The 'gravity wall' provides stability by virtue of its own weight, and therefore, is rather massive in size. It is usually built in stone masonry, and occasionally in plain concrete. The thickness of the wall is also governed by the need to eliminate or limit the resulting tensile stress to its permissible limit (which is very low[†] in the case of concrete and masonry). Plain concrete gravity walls are not used for heights exceeding about 3 m, for obvious economic reasons.

[†] The 'middle third rule' is generally applied, wherein the wall thickness is made sufficiently large, to ensure that the resultant thrust at any cross-section falls within the 'middle third' region of the section.

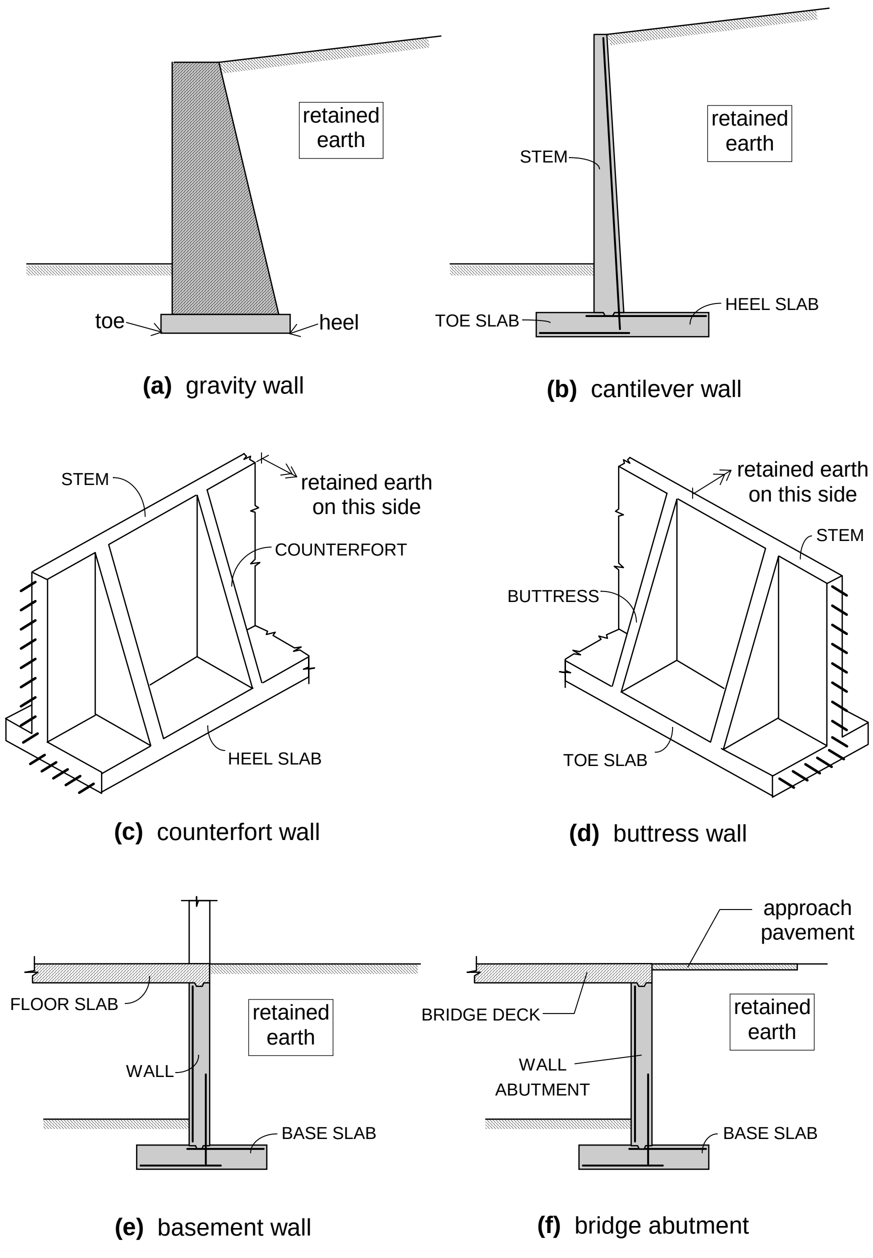


**Fig. 14.22** Types of retaining wall structures

2. **Cantilever Wall** [Fig. 14.22(b)]

   The 'cantilever wall' is the most common type of retaining structure and is generally economical for heights up to about 8 m. The structure consists of a vertical *stem*, and a *base slab*, made up of two distinct regions, viz. a *heel slab* and a *toe slab*. All three components behave as one-way cantilever slabs: the 'stem' acts as a vertical cantilever under the lateral earth pressure; the 'heel slab' acts as a (horizontal) cantilever under the action of the weight of the retained earth (minus soil pressure acting upwards from below); and the 'toe slab' also acts as a cantilever under the action of the resulting soil pressure (acting upward). The detailing of reinforcement (on the flexural tension faces) is accordingly as depicted in Fig. 14.22(b). The stability of the wall is maintained essentially by the weight of the earth on the heel slab plus the self weight of the structure.

3. **Counterfort Wall** [Fig. 14.22(c)]

   For large heights, in a cantilever retaining wall, the bending moments developed in the *stem*, *heel slab* and *toe slab* become very large and require large thicknesses. The bending moments (and hence stem/slab thicknesses) can be considerably reduced by introducing transverse supports, called *counterforts*, spaced at regular intervals of about one-third to one-half of the wall height), interconnecting the *stem*[‡] with the *heel slab*. The counterforts are concealed within the retained earth (on the rear side of the wall). Such a retaining wall structure is called the *counterfort wall*, and is economical for heights above (approx.) 7 m. The counterforts subdivide the vertical slab (stem) into rectangular panels and support them on two sides (suspender-style), and themselves behave essentially as vertical cantilever beams of T-section and varying depth. The *stem* and *heel slab* panels between the counterforts are now effectively 'fixed' on three sides (free at one edge), and for the stem the predominant direction of bending (and flexural reinforcement) is now horizontal (spanning between counterforts), rather than vertical (as in the cantilever wall).

4. **Buttress Wall** [Fig. 14.22(d)]

   The 'buttress wall' is similar to the 'counterfort wall', except that the transverse stem supports, called *buttresses*, are located in the front side, interconnecting the *stem* with the *toe slab* (and not with the *heel slab*, as with counterforts). Although buttresses are structurally more efficient (and more economical) than counterforts, the counterfort wall is generally preferred to the buttress wall as it provides free usable space (and better aesthetics) in front of the wall.

5. **Other Types of Walls**

   Retaining walls often form part of a bigger structure, in which case their structural behaviour depends on their interaction with the rest of the structure. For example, the exterior walls in the *basement* of a building [Fig. 14.22(e)] and

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[‡] The toe slab is also frequently interconnected with the stem (in the front side of the wall) by means of a 'front counterfort', whose height is limited by the ground level on the toe side, so that it is concealed and provides free usable space in front of the wall.

wall-type *bridge abutments* [Fig. 14.22(f)] act as retaining walls. In both these situations, the vertical stem is provided an additional horizontal restraint at the top, due to the slab† at the ground floor level (in the case of the basement wall) and due to the bridge deck (in the case of bridge abutment). The stem is accordingly designed as a beam, fixed at the base and simply supported or partially restrained at the top. The side walls of *box culverts* also act as retaining walls. In this case, the box culvert (with single/multiple cells) acts as a closed rigid frame, resisting the combined effects of lateral earth pressures, dead loads (due to self weight and earth above), as well as live loads due to highway traffic.

In the sections to follow, only the cantilever and counterfort retaining walls are discussed — with particular emphasis on the cantilever wall, which is the most common type of retaining wall structure.

## 14.8 EARTH PRESSURES AND STABILITY REQUIREMENTS

### 14.8.1 Lateral Earth Pressures

The lateral force due to earth pressure constitutes the main force acting on the retaining wall, tending to make it bend, slide and overturn. The determination of the magnitude and direction of the earth pressure is based on the principles of soil mechanics, and the reader may refer to standard texts in this specialised area (such as Ref. 14.2, 14.3, 14.8) for a detailed study.

In general, the behaviour of lateral earth pressure is analogous to that of a fluid, with the magnitude of the pressure $p$ increasing nearly linearly with increasing depth $z$ for moderate depths below the surface:

$$p = C\gamma_e z \tag{14.9}$$

where $\gamma_e$ is the unit weight of the earth and $C$ is a coefficient that depends on its physical properties, and also on whether the pressure is *active* or *passive*. 'Active pressure' ($p_a$) is that which the retained earth exerts on the wall as the earth moves in the same direction as the wall deflects. On the other hand, 'passive pressure' ($p_p$) is that which is developed as a resistance when the wall moves and presses against the earth (as on the toe side of the wall). The coefficient to be used in Eq. 14.9 is the *active pressure coefficient*, $C_a$, in the case of active pressure, and the *passive pressure coefficient*, $C_p$, in the case of passive pressure; the latter ($C_p$) is generally much higher than the former ($C_a$) for the same type of soil.

In the absence of more detailed information, the following expressions for $C_a$ and $C_p$, based on Rankine's theory [Ref. 14.2, 14.3], may be used for cohesionless soils and level backfills:

$$C_a = \frac{1-\sin\phi}{1+\sin\phi} \tag{14.10a}$$

† The slab is integrally connected to numerous beam–column frames, and the lateral restraint offered by it is due to the high *storey stiffness* at the lowermost storey.

$$C_p = \frac{1+\sin\phi}{1-\sin\phi} \tag{14.10b}$$

where $\phi$ is the *angle of shearing resistance* (or *angle of repose*). For a typical granular soil (such as sand), $\phi \approx 30^o$, corresponding to which, $C_a$ = 1/3 and $C_p$ = 3.0.

When the backfill is sloped† , as shown in Fig. 14.23, the expression [Eq. 14.10a] for $C_a$ should be modified as follows:

$$C_a = \left[\frac{\cos\theta - \sqrt{\cos^2\theta - \cos^2\phi}}{\cos\theta + \sqrt{\cos^2\theta - \cos^2\phi}}\right]\cos\theta \tag{14.11}$$

where $\theta$ is the angle of inclination of the backfill, i.e., the angle of its surface with respect to the horizontal [Fig. 14.23].

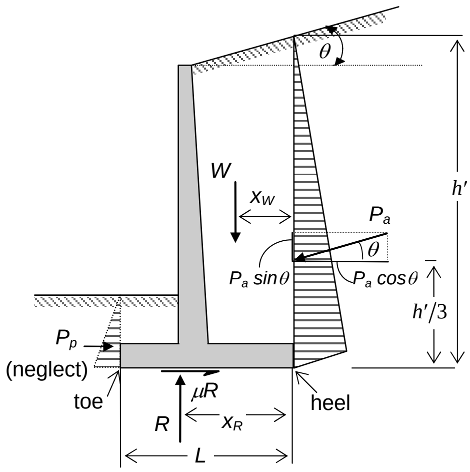


**Fig. 14.23** Forces acting on a cantilever retaining wall

The direction of the active pressure, $p_a$ [given by Eq. 14.9], is parallel to the surface of the backfill. The pressure has a maximum value at the heel, and is equal to $C_a\gamma_e h'$, where $h'$ is the height of the backfill, measured vertically above the heel [Fig. 14.23]. For the case of a level backfill, $\theta = 0$ and $h' = h$, and the direction of the lateral pressure is horizontal and normal to the vertical stem.

The force, $P_a$, exerted by the active earth pressure, due to a backfill of height $h'$ above the heel, is accordingly obtained from the triangular pressure distribution [Fig. 14.23] as

$$P_a = C_a\gamma_e (h')^2/2 \tag{14.12}$$

† Sometimes, the term 'inclined surcharge' is used to refer to a sloping backfill; the term 'surcharge' implies the additional height of the backfill above the level of the top of the wall [refer Section 14.8.2].

This force has units of kN per m length of the wall, and acts at a height $h'/3$ above the heel at an inclination $\theta$ with the horizontal.

The force, $P_p$, developed by passive pressure on the toe side of the retaining wall is generally small (due to the small height of earth†) and usually not included in the design calculations, as this is conservative.

### 14.8.2 Effect of Surcharge on a Level Backfill

Frequently, gravity loads act on a level backfill due to the construction of buildings and the movement of vehicles near the top of the retaining wall. These additional loads can be assumed to be static‡ and uniformly distributed on top of the backfill, for calculation purposes. This distributed load $w_s$ (kN/m$^2$) can be treated as statically equivalent to an additional (fictitious) height, $h_s = w_s/\gamma_e$, of soil backfill with unit weight $\gamma_e$. This additional height of backfill is called *surcharge*, and is expressed either in terms of height $h_s$, or in terms of the distributed load $w_s$ [Fig. 14.24].

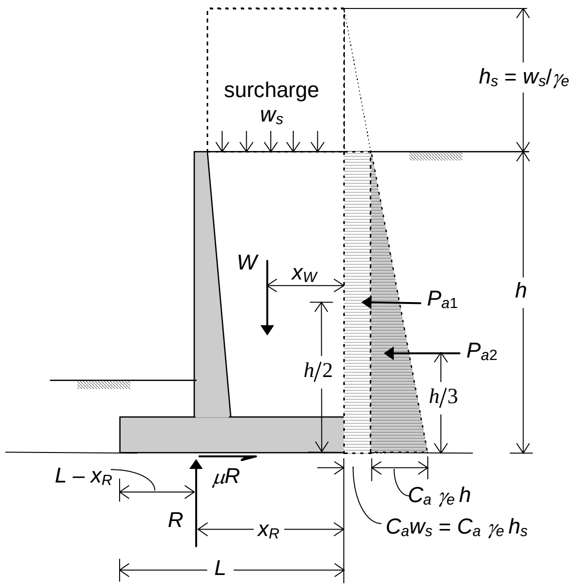


**Fig. 14.24** Effect of surcharge on a level backfill

The presence of the surcharge not only adds to the gravity loading acting on the heel slab, but also increases the lateral pressure on the wall by $C_a\gamma_e h_s = C_a w_s$. The resulting trapezoidal earth pressure distribution is made up of a rectangular pressure

† Strictly, for the full development of passive earth pressure, it is necessary that during the construction of the wall, there should be no disturbance to the soil against which the concrete in the toe slab is placed.

‡ In the case of vehicular traffic and other live loads, the equivalent loading should include a dynamic magnification factor.

distribution (of intensity $C_a w_s$), superimposed on the triangular pressure distribution due to the actual backfill, as shown in Fig. 14.24. The total force due to active pressure acting on the wall is accordingly given by

$$P_a = P_{a1} + P_{a2} \tag{14.13}$$

where

$$P_{a1} = C_a w_s\, h = C_a\, \gamma_e\, h_s\, h \tag{14.13a}$$

$$P_{a2} = C_a\, \gamma_e\, h^2/2 \tag{14.13b}$$

with the lines of action of $P_{a1}$ and $P_{a2}$ at $h/2$ and $h/3$ above the heel.

### 14.8.3 Effect of Water in the Backfill

When water accumulates in the backfill, it can raise the lateral pressure on the wall to very high levels. If the water in the backfill does not have an escape route, it will build up a hydrostatic pressure on the wall, causing it to behave like a dam. The resulting pressure† distributions are depicted in Fig. 14.25.

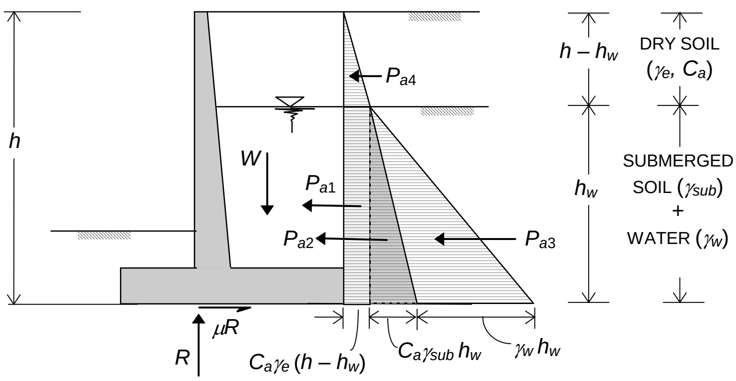


**Fig. 14.25** Effect of water in the backfill

The purpose of the retaining wall is to retain earth, and not water. Hence, submerged earth conditions should be avoided by providing and maintaining proper drainage facilities (including provision of *weep holes*). Failure to do so can result in the building up of enormous pressures, which, if not anticipated in the design, can result in serious failures. Such failures are not uncommon in practice; the designer must accept some responsibility for this.

† The presence of water does not significantly alter the shearing resistance of granular soils; hence the coefficient, $C_a$ is practically the same for both dry and submerged conditions.

### 14.8.4 Stability Requirements

The Code (Cl. 20) specifies that the factors of safety against overturning (Cl. 20.1) and sliding (Cl. 20.2) should not be less than 1.4. Furthermore (as explained in Section 14.3.3), as the stabilising forces are due to dead loads, the Code specifies that these stabilising forces should be factored by a value of 0.9 in calculating the factor of safety, *FS*. Accordingly,

$$FS = \frac{0.9 \times \text{(stabilising force or moment)}}{\text{destabilising force or moment}} \geq 1.4 \tag{14.14}$$

#### *Overturning*

If the retaining wall structure were to overturn, it would do so with the toe acting as the centre of rotation. In an overturning context, there is no upward reaction $R$ acting over the base width $L$. The expressions for the overturning moment $M_o$ and the stabilising (restoring) moment $M_r$ depend on the lateral earth pressure and the geometry of the retaining wall.

For the case of a sloping backfill [Fig. 14.23],

$$M_o = (P_a \cos\theta)(h'/3) = \left[C_a \gamma_e (h')^3/6\right]\cos\theta \tag{14.15}$$

$$M_r = W(L - x_w) + (P_a \sin\theta)L \tag{14.16}$$

where $W$ denotes the total weight of the reinforced concrete wall structure plus the retained earth resting on the footing† (heel slab), and $x_w$ is the distance of its line of action from the heel, as shown in Fig. 14.23.

For the case of a level backfill with surcharge [Fig. 14.24],

$$M_o = P_{a1}(h/2) + P_{a2}(h/3) \tag{14.17}$$

where $P_{a1}$ and $P_{a2}$ are as given by Eq. 14.13(a) and Eq. 14.13(b) respectively. The expression for $M_r$ is the same as that given by Eq. 14.16, but with $\theta = 0$.

The factor of safety required against overturning [Eq. 14.14] is obtained as

$$(FS)_{overturning} = \frac{0.9M_r}{M_o} \geq 1.4 \tag{14.18}$$

#### *Sliding*

The resistance against sliding is essentially provided by the friction between the base slab and the supporting soil, given by

$$F = \mu R \tag{14.19}$$

where $R = W$ is the resultant soil pressure acting on the footing base and $\mu$ is the coefficient of static friction between concrete and soil. [In a sloping backfill, R will

† The weight of the earthfill above the toe slab is usually (conservatively) ignored. Similarly, the passive earth pressure $P_p$ is also usually ignored.

also include the vertical component of earth pressure, $P_a \sin\theta$ (see Fig. 14.23)]. The value of $\mu$ varies between about 0.35 (for silt) to about 0.60 (for rough rock) [Ref. 14.2].

The factor of safety against sliding [Eq. 14.14] is obtained as

$$(FS)_{sliding} = \frac{0.9F}{P_a \cos\theta}, \text{ which should be} \geq 1.4 \tag{14.19a}$$

When active pressures are relatively high (as when surcharge is involved), it will be generally difficult to mobilise the required factor of safety against sliding, by considering frictional resistance below the footing alone [Eq. 14.19]. In such a situation, it is advantageous to use a *shear key* projecting below the footing base and extending throughout the length of the wall [Fig. 14.26]. When the concrete in the 'shear key' is placed in an unformed excavation (against undisturbed soil), it can be expected to develop considerable passive resistance. Different procedures have been proposed to estimate this passive resistance $P_p$ [Ref. 14.8, 14.9]. A simple and conservative estimate is obtained by considering the pressure developed over a region, $h_2 - h_1$, below the toe:

$$P_p = C_p \gamma_e \left(h_2^2 - h_1^2\right)/2 \tag{14.20}$$

where $h_1$ and $h_2$ are as indicated in Fig. 14.26. It may be noted that the overburden due to the top 0.3 m of earth below ground level is usually ignored in the calculation.

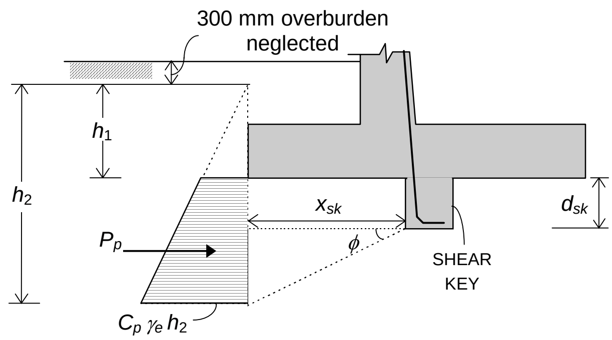


**Fig. 14.26** Passive resistance due to shear key

The shear key is best positioned at a distance $x_{sk}$ from the toe in such a way that the flexural reinforcement from the stem can be extended straight into the shear key near the toe.

### 14.8.5 Soil Bearing Pressure Requirements

The width $L$ of the base slab must be adequate to distribute the vertical reaction $R$ to the foundation soil without causing excessive settlement or rotation. As explained in Section 14.3, the required founding depth and the associated allowable pressure $q_a$ are usually prescribed by a geotechnical consultant on the basis of a soil study, and the control on vertical settlement is built into these recommendations. However, the designer must further ensure that tilting of the footing is also avoided by avoiding a highly non-uniform base pressure in weak soils.

## 14.9 PROPORTIONING AND DESIGN OF CANTILEVER AND COUNTERFORT WALLS

Prior to carrying out a detailed analysis and design of the retaining wall structure, it is necessary to assume preliminary dimensions of the various elements of the structure using certain approximations. Subsequently, these dimensions may be suitably revised, if so required by design considerations.

### 14.9.1 Position of Stem on Base Slab for Economical Design

An important consideration in the design of cantilever and counterfort walls is the position of the vertical stem on the base slab. It can be shown [Ref. 14.10] that an economical design of the retaining wall can be obtained by proportioning the base slab so as to align the vertical soil reaction $R$ at the base with the front face of the wall (stem). For this derivation, let us consider the typical case of a level backfill [Fig. 14.27]. The location of the resultant soil reaction, $R$, is dependent on the magnitude and location of the resultant vertical load, $W$, which in turn depends on the dimension $X$ (i.e., the length of heel slab, inclusive of the stem thickness). For convenience in the derivation, $X$ may be expressed as a fraction, $\alpha_x$, of the full width $L$ of the base slab ($X = \alpha_x L$). Assuming an average unit weight $\gamma_e$ for all material (earth plus concrete) behind the front face of the stem (rectangle *abcd*), and neglecting entirely the weight of concrete in the toe slab,

$$R = W = \gamma_e\, h\, X = \gamma_e\, h\, (\alpha_X L)$$

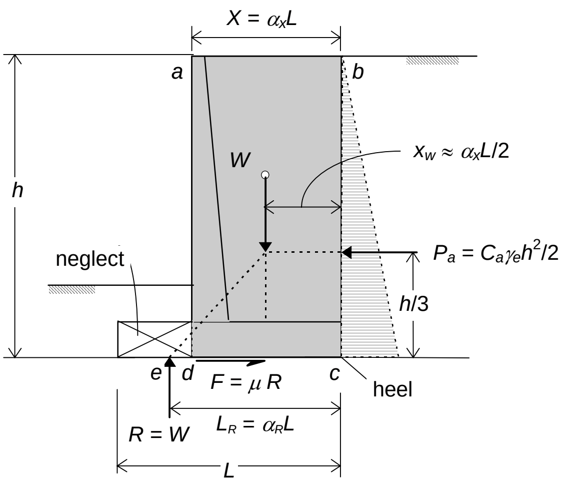


**Fig. 14.27** Proportioning of retaining wall

For a given location of $R$ corresponding to a *chosen value of X*, the toe projection of the base slab (and hence its total width, $L$) can be so selected by the designer as to give any desired distribution of base soil pressure. Thus, representing the distance, $L_R$, from the heel to $R$ as a fraction $\alpha_R$ of base width $L$, [Fig. 14.27], the base pressure will be uniform if $L$ is so selected as to make $\alpha_R = 0.5$. Similarly, for $\alpha_R = 2/3$, the base pressure distribution will be triangular. Thus, for any selected distribution of base pressure, $\alpha_R$ is a constant and the required base width $L = L_R/\alpha_R$.

Considering static equilibrium and taking moments about reaction point $e$, and assuming $X_w \approx \alpha_X L/2$,

$$W(\alpha_R L - \alpha_X L/2) = P_a\, h/3$$

$$\Rightarrow \gamma_e\, hL^2\,(\alpha_R\,\alpha_X - \alpha_X^2/2) = C_a \gamma_e\, h^3/6$$

$$\Rightarrow \frac{L}{h} = \sqrt{\frac{C_a/3}{2\alpha_R\alpha_X - \alpha_X^2}} \tag{14.21}$$

For economical proportioning for a given height of wall ($h$), the length of the base ($L$) must be minimum, i.e., $L/h$ should be minimum. From Eq. 14.21, this implies that $(2\,\alpha_R\,\alpha_X - \alpha_X^2)$ should be maximum. The location of $R$, and hence the base width for any selected pressure distribution, is dependent on the variable $X$, i.e., $\alpha_x$. For maximising $(2\alpha_R\,\alpha_x - \alpha^2_x)$,

$$\alpha_X = \alpha_R$$

$$\Rightarrow \alpha_R\, L = \alpha_X\, L = X$$

Hence, for an economical design, the soil pressure resultant should line up with the front face of the wall.

### *Width of Base*

Applying the above principle, an approximate expression for the minimum length of base slab for a given height of wall is obtained from Eq. 14.21 as:

$$\left(\frac{L}{h}\right)_{\min} \approx \frac{1}{\alpha_R}\sqrt{\frac{C_a}{3}}$$

$$\Rightarrow L_{\min} \approx (h/\alpha_R)\sqrt{C_a/3} \tag{14.22}$$

Alternatively, the minimum width of heel slab is given by:

$$X_{min} = \alpha_X\, L_{min} = \alpha_R\, L_{min} = h\sqrt{C_a/3} \tag{14.23}$$

The effect of surcharge or sloping backfill may be taken into account, approximately, by replacing $h$ with $h + h_s$, or $h'$, respectively.

Alternatively, and perhaps more conveniently, using the above principle, the heel slab width ($X$ in Fig. 14.27) may be obtained by equating moments of $W$ and $P_a$ about the point *d*. The required *L* can then be worked out based on the base pressure distribution desired.

It may be noted that the total height *h* of the retaining wall is the difference in elevation between the top of the wall and the bottom of base slab. The latter is based on geotechnical considerations (availability of firm soil) and is usually not less than 1 m below the ground level on the toe side of the wall.

After fixing up the trial width of the heel slab ( = X) for a given height of wall and backfill conditions, the dimension *L* may be fixed up. Initially, a triangular pressure distribution may be assumed, resulting in $L = \frac{3}{2}X$. Using other approximations (discussed in the next section) related to stem thickness and base slab thickness, a proper analysis‡ should be done to ascertain that

(1) the factor of safety against overturning is adequate;
(2) the allowable soil pressure, $q_a$, is not exceeded; and
(3) the factor of safety against sliding is adequate.

Condition (1) is generally satisfied; however, if it is not, the dimensions *L* and *X* may be suitably increased. If condition (2) is not satisfied, i.e., if $q_{max} > q_a$, the length *L* should be increased by suitably extending the length of the toe slab; the dimension *X* need not be changed. If condition (3) is not satisfied, which is usually the case, a suitable 'shear key' should be designed.

### 14.9.2 Proportioning and Design of Elements of Cantilever Walls

#### *Initial Thickness of Base Slab and Stem*

For preliminary calculations, the thickness of the base slab may be taken as about 8 percent of the height of the wall plus surcharge (if any); it should not be less than 300 mm. The base thickness of the vertical stem may be taken as slightly more than that of the base slab. For economy, the thickness may be tapered linearly to a minimum value (but not less than 150 mm) at the top of the wall; the front face of the stem is maintained vertical†. If the length of the heel slab and/or toe slab is excessive, it will be economical to provide a tapered slab.

With the above preliminary proportions, the stability check and determination of soil pressure (at the base) may be performed, and dimensions *L* and *X* of the base slab [Fig. 14.27] finalised. It may be noted that changes in thicknesses of base slab and stem, if required at the design stage, will be marginal and will not affect significantly either the stability analysis or the calculated (gross) soil pressures below the base slab.

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‡ In such an analysis, it will be seen that the actual vertical reaction *R* below the footing base will be close to, although rarely coincident with, the front face of the stem (as assumed initially).

† It is recommended that a batter of 1 : 50 be provided to the front face of the stem during construction, to offset the deflection of the stem or possible forward tilting of the structure [Ref. 14.10].

### *Design of Stem, Toe Slab and Heel Slab*

The three elements of the retaining wall, viz., *stem*, *toe slab* and *heel slab* have to be designed as cantilever slabs to resist the factored moments and shear forces. For this a load factor of 1.5 is to be used.

In the case of the toe slab, the net pressure is obtained by deducting the weight[§] of the concrete in the toe slab from the upward acting gross soil pressure. The net loading acts upward (as in the case of usual footings) and the flexural reinforcement has to be provided at the bottom of the toe slab. The critical section for moment is at the front face of the stem, while the critical section for shear is at a distance *d* from the face of the stem. A clear cover of 75 mm may be provided in base slabs.

In the case of the heel slab, the pressures acting downward, due to the weight of the retained earth (plus surcharge, if any), as well as the concrete in the heel slab, exceed the gross soil pressures acting upward. Hence, the net loading acts downward, and the flexural reinforcement has to be provided at the top of the heel slab. The critical section for moment is at the rear face of the stem base.

> The critical section for shear in the heel slab should be taken at the face of the support and not *d* away from it, because there is no compression introduced by the support reaction, and the probable inclined crack may extend ahead of the rear face of the stem [also refer Fig. 6.6(c)].

In the case of the stem (vertical cantilever), the critical section for shear may be taken *d* from the face of the support (top of base slab), while the critical section for moment should be taken at the face of the support. For the main bars in the stem, a clear cover of 50 mm may be provided. Usually, shear is not a critical design consideration in the stem (unlike the base slab). The flexural reinforcement is provided near the rear face of the stem, and may be curtailed in stages for economy [refer Example 14.9].

Temperature and shrinkage reinforcement ($A_{st,min}$ = 0.12 percent of gross area) should be provided transverse to the main reinforcement. Nominal vertical and horizontal reinforcement should also be provided near the front face which is exposed.

## 14.9.3 Proportioning and Design of Elements of a Counterfort Wall

### *Initial Thicknesses of Various Elements*

In a counterfort wall, counterforts are usually provided at a spacing of about one-third to one-half of the height of the wall. The triangular shaped counterforts are provided in the rear side of the wall, interconnecting the stem with the heel slab. Sometimes, small buttresses are provided in the front side below the ground level, interconnecting the toe slab with the lower portion of the stem.

The presence of counterforts enables the use of stem and base slab thicknesses that are much smaller than those normally required for a cantilever wall. For

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[§] The weight of the earthfill in this region is (conservatively) ignored.

preliminary calculations, the stem thickness and heel slab thickness may be taken as about 5 percent of the height of the wall, but not less than 300 mm. If the front buttress is provided, the thickness of the toe slab may also be taken as $0.05h$; otherwise, it may be taken as in the case of the cantilever wall ($0.08h$). The thickness of the counterforts may be taken as about 6 percent of the height of the wall at the base, but not less than 300 mm. The thickness may be reduced along the height of the wall.

With the above preliminary proportions, the stability check and determination of soil pressures (at the base) may be performed, and dimensions $L$ and $X$ of the base finalised, as in the case of the cantilever wall.

### *Design of Stem, Toe Slab and Heel Slab*

Each panel of the stem and heel slab, between two adjacent counterforts, may be designed as two-way slabs fixed on three sides, and free on the fourth side (free edge). These boundary conditions are also applicable to the toe slab, if buttresses are provided; otherwise the toe slab behaves as a horizontal cantilever, as in the case of the cantilever wall.

The loads acting on these elements are identical to those acting on the cantilever wall discussed earlier. For the stem, bending in the horizontal direction between counterforts[†] is generally more predominant than bending in the vertical direction. Near the counterforts, the main reinforcement will be located close to the rear face of the stem, whereas midway between counterforts, the reinforcement will be close to the outside face; the latter is indicated in Fig. 14.22(c). These two-ways slabs, subject to triangular/trapezoidal pressure distributions may be designed by the use of moment and shear coefficients (based on plate theory), available in various handbooks, and also in the IS Code for the design of liquid storage structures, viz., IS 3370 (Part 4) [Ref. 14.11]. Alternatively, the slabs may be designed by the yield line theory. An alterantive simplified method of analysis is demonstrated in Example 14.10.

### *Design of Counterforts*

The main counterforts should be firmly secured (by additional ties) to the heel slab, as well as to the vertical stem, as the loading applied on these two elements tend to separate them from the counterforts. In addition, the counterfort should be designed to resist the lateral (horizontal) force transmitted by the stem tributary to it. The counterfort is designed as a vertical cantilever, fixed at its base. As the stem acts integrally with the counterfort, the effective section resisting the cantilever moment is a flanged section, with the flange under compression. Hence, the counterforts may be designed as T-beams [refer Chapter 5] with the depth of section varying (linearly) from the top (free edge) to the bottom (fixed edge), and with the main reinforcement provided close to the sloping face. Since these bars are inclined (not parallel to the

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[†] An approximate and conservative estimate of this bending moment can be obtained by treating the slab as one-way continuous slab spanning the counterforts.

compression face), allowance has to be made for this in computing the area of steel required.

### EXAMPLE 14.8

Determine suitable dimensions of a cantilever retaining wall, which is required to support a bank of earth 4.0 m high above the ground level on the toe side of the wall. Consider the backfill surface to be inclined at an angle of 15° with the horizontal. Assume good soil for foundation at a depth of 1.25 m below the ground level with a safe bearing capacity of 160 kN/m². Further assume the backfill to comprise granular soil with a unit weight of 16 kN/m³ and an angle of shearing resistance of 30°. Assume the coefficient of friction between soil and concrete to be 0.5.

### SOLUTION

1. **Data given**: $h = 4.0 + 1.25 = 5.25$ m; $\mu = 0.5$
$\theta = 15^{\circ}$ $\gamma_e = 16$ kN/m³
$\phi = 30^{\circ}$ $q_a = 160$ kN/m²

- Earth pressure coefficients: $C_a = \left[\dfrac{\cos\theta - \sqrt{\cos^2\theta - \cos^2\phi}}{\cos\theta + \sqrt{\cos^2\theta - \cos^2\phi}}\right]\cos\theta = 0.373$

$$C_p = \frac{1+\sin\theta}{1-\sin\theta} = 3.0$$

2. **Preliminary proportions**

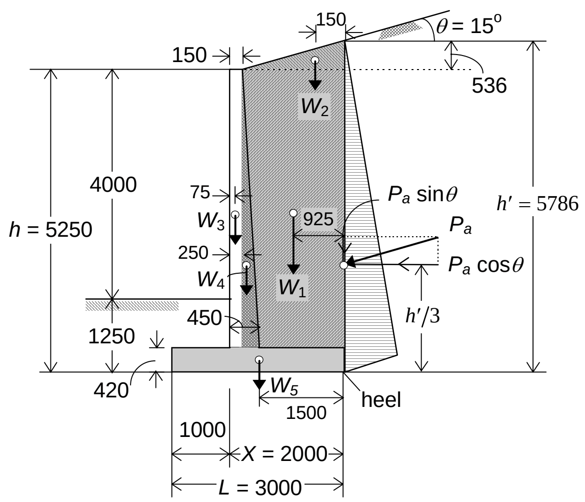
- Thickness of footing base slab $\approx 0.08h = 0.08 \times 5.25 = 0.42$ m
Assume a thickness of 420 mm.
- Assume a stem thickness of 450 mm at the base of the stem, tapering to a value of 150 mm at the top of the wall.
- For an economical proportioning of the length $L$ of the base slab, it will be assumed that the vertical reaction $R$ at the footing base is in line with the front face of the stem. For such a condition, (assuming the height above top of wall to be about 0.4 m), the length of the heel slab (inclusive of stem thickness) [Eq. 14.23]:
$$X \approx \left(\sqrt{C_a/3}\right)\, h' = \sqrt{0.373/3}\ (5.25 + 0.4) \approx 2.0 \text{ m}$$
- Assuming a triangular base pressure distribution,
$L = 1.5X = 3.0$ m
- The preliminary proportions are shown in Fig. 14.28(a).

3. **Stability against overturning**

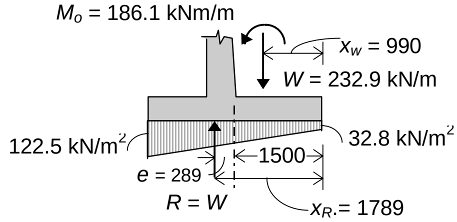
- Force due to active pressure: $P_a = C_a \gamma_e h'^2/2$
where $h' = h + X\tan\theta$ [Fig. 14.28(a)]
$= 5250 + 2000 \tan 15^{\circ} = 5786$ mm
$P_a = (0.373)(16)(5.786)^2/2 = 99.9$ kN (per m length of wall)
$\Rightarrow P_a \cos\theta = 99.9 \cos 15^{\circ} = 96.5$ kN
$P_a \sin\theta = 99.9 \sin 15^{\circ} = 25.9$ kN

- Overturning moment $M_o = (P_a \cos\theta)h'/3 = (96.5)(5.786/3) = 186.1$ kNm
- Line of action of resultant of vertical forces [Fig. 14.28(a)] with respect to the heel can be located by applying statics, considering 1 m length of the wall:

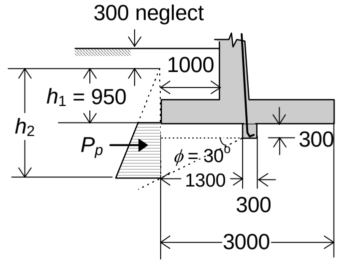
| force (kN) | | distance from heel (m) | moment (kNm) |
|---|---|---|---|
| $W_1 = (16)(1.85)(5.25 - 0.42)$ | = 143.0 | 0.925 | 132.3 |
| $W_2 = (16)(1.85)(0.5 \times 0.536)$ | = 7.9 | 0.617 | 4.9 |
| $W_3 = (25)(0.15)(5.25 - 0.42)$ | = 18.1 | 1.925 | 34.8 |
| $W_4 = (25 - 16)(4.83)(0.5 \times 0.30)$ | = 6.5 | 1.750 | 11.4 |
| $W_5 = (25)(3.0)(0.42)$ | = 31.5 | 1.500 | 47.2 |
| $P_a \sin\theta$ | = 25.9 | 0.000 | 0.0 |
| | $W$ = 232.9 | | $M_W$ = 230.6 kNm |




**(a)** forces on wall (with preliminary proportions)




**(b)** calculation of soil pressures




**(c)** design of shear key

**Fig. 14.28** Example 14.8

$\Rightarrow$ distance of resultant vertical force from heel

$x_W = M_W/W = 230.6/232.9 = 0.990$ m

- Stabilising moment (about toe):

  $M_r = W\,(L - x_W)$

  $= 232.9 \times (3.0 - 0.99)$

  $= 468.1$ kNm (per m length of wall)

  $\Rightarrow (FS)_{overturning} = \dfrac{0.9 M_r}{M_o} = \dfrac{0.9 \times 468.1}{186.1} = 2.26 > 1.40$ — OK

**4. Soil pressures at footing base** [refer Fig. 14.28(b)]

- resultant vertical reaction $R = W = 232.9$ kN (per m length of wall)
- distance of $R$ from heel: $L_R = (M_W + M_o)/R$

  $= (230.6 + 186.1)/232.9 = 1.789$ m†
- eccentricity $e = L_R - L/2 = 1.789 - 3.0/2 = 0.289$ m, $< L/6 = 0.5$
- Hence, the resultant lies within the middle third of the base, which is desirable

  $\dfrac{6e}{L} = \dfrac{6 \times 0.289}{3.0} = 0.578$

  $\Rightarrow q_{max} = \dfrac{R}{L}\left(1 + \dfrac{6e}{L}\right) = \dfrac{232.9}{3.0}(1 + 0.578)$

  $= 122.5$ kN/m$^2 < q_a$ — OK

  and $q_{min} = \dfrac{232.9}{3.0}(1 - 0.578) = 32.8$ kN/m$^2$ [refer Fig. 14.28(b)]

**5. Stability against sliding**

- Sliding force $= P_a \cos\theta = 96.5$ kN
- Resisting force (ignoring passive pressure on the toe side) $F = \mu R$

  $= 0.5 \times 232.9 = 116.4$ kN

  $\Rightarrow (FS)_{sliding} = \dfrac{0.9F}{P_a \cos\theta} = \dfrac{0.9 \times 116.4}{96.5} = 1.085 < 1.40$
- Hence, a *shear key* may be provided to mobilise the balance force through passive resistance.
- Assume a shear key 300 mm × 300 mm, at a distance of 1300 mm from toe as shown in Fig. 14.28(c). Distance $h_2 = 0.950 + 300 + 1.300 \tan 30^o = 2.001$ m

  $P_p = C_p \gamma_e (h^2{}_2 - h^2{}_1)/2 = 3 \times 16 \times (2.001^2 - 0.95^2)/2$

  $= 74.44$ kN

  $(F.S)_{sliding} = \dfrac{0.9(116.4 + 74.44)}{96.5} = 1.78 > 1.4$ — OK

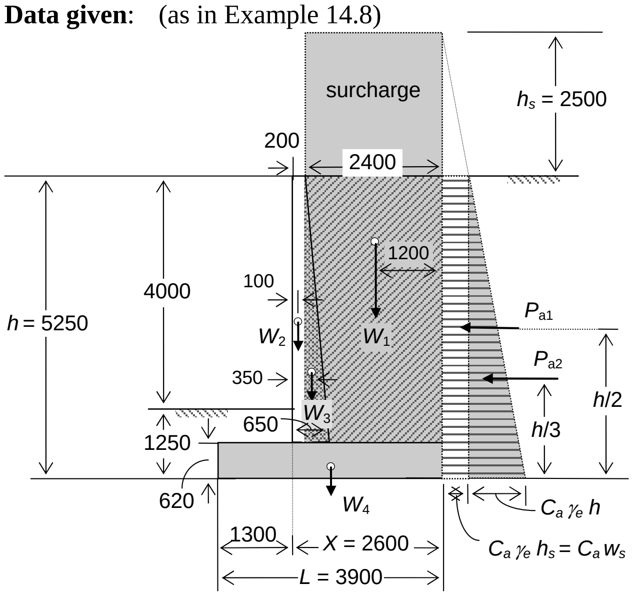
### EXAMPLE 14.9

Repeat the problem in Example 14.8, considering the backfill to be level, but subject to a surcharge pressure of 40 kN/m$^2$ (due to the construction of a building). Design the retaining wall structure, assuming M 20 and Fe 415 steel.

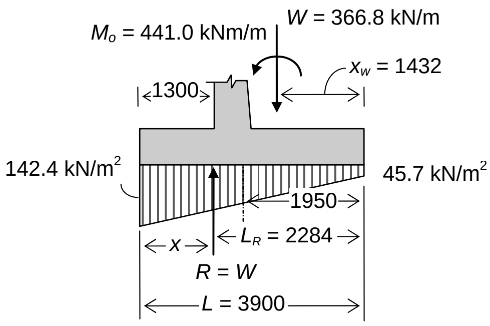
† Note that this value of $L_R$ is different from, although close to, the value of $X$ = 2.0 m asssumed in the initial proportioning.

**SOLUTION**

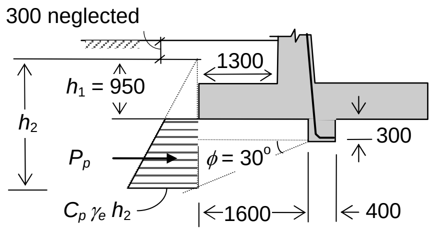
1. **Data given**: (as in Example 14.8)




**(a)** forces on wall (with preliminary proportions)




**(b)** calculation of soil pressures




**(c)** design of shear key

**Fig. 14.29** Example 14.9

$h$ = 4.0 + 1.25 = 5.25 m $\qquad\qquad \gamma_e$ = 16 kN/m$^3$
$\phi$ = 30$^o$ $\qquad\qquad q_a$ = 160 kN/m$^2$
$\mu$ = 0.5 $\qquad\qquad w_s$ = 40 kN/m$^2$

⇒ Equivalent height of earth as surcharge, $h_s = \dfrac{w_s}{\gamma_e} = \dfrac{40}{16}$ = 2.5 m

⇒ $h + h_s$ = 5.25 + 2.5 = 7.75 m

- Earth pressure coefficients: $C_a = \dfrac{1-\sin\phi}{1+\sin\phi}$ = 1/3

  $C_p = 1/C_a$ = 3.0

**2. Preliminary proportions**

- Thickness of footing base slab ≈ 0.08 ($h$ + $h_s$) = 0.08 × 7.75 = 0.620. Assume a thickness of 620 mm.
- Assume a stem thickness of 650 mm at the base of the stem, tapering to a value of 200 mm at the top of the wall.
- For an economical proportioning of the length $L$ of the base slab, it will be assumed that the vertical reaction $R$ at the footing base is in line with the front face of the stem. For such a condition, the length of the heel slab (inclusive of stem thickness)

$$X \approx \sqrt{C_a/3}(h+h_s) = \sqrt{(1/3)/3}(7.75) = 2.58 \text{ m}$$

  Let $X$ = 2.6 m.
- Assuming a triangular soil pressure distribution below the base,

$$L = 1.5X = 1.5 \times 2.6 = 3.9 \text{ m}$$

- The preliminary proportions are shown in Fig. 14.29(a)

**3. Stability against overturning**

- Forces due to active pressure (per m length of wall) [Fig. 14.29(a)]:
  $P_{a1} = C_a\, w_s\, h$ = (1/3)(40)(5.25) = 70.0 kN
  $P_{a2} = C_a\, \gamma_e\, h^2/2$ = (1/3)(16)(5.25)$^2$/2 = 73.5 kN
  ⇒ $P_a$ = 70.0 + 73.5 = 143.5 kN
- Overturning moment $M_o = P_{a1}\, h/2 + P_{a2}\, h/3$
  ⇒ $M_o$ = (70.0)(5.25/2) + (73.5)(5.25/3)
  = 312.4 kNm (per m length of wall)
- Line of action of resultant of vertical forces [Fig. 14.29(a)] with respect to the heel can be located by applying statics, considering 1 m length of the wall:

| force (kN) | | distance from heel (m) | moment (kNm) |
|---|---|---|---|
| $W_1$ = (16)(2.40)(7.75 – 0.62) | = 273.8 | 1.20 | 328.6 |
| $W_2$ = (25)(0.20)(4.63) | = 23.2 | 2.50 | 58.0 |
| $W_3$ = (25 – 16) (0.5 × 0.45) (4.63) | = 9.4 | 2.25 | 21.1 |
| $W_4$ = (25)(3.90)(0.62) | = 60.4 | 1.95 | 117.8 |
| | $W$ = 366.8 | | $M_W$ = 525.5 kNm |

$\Rightarrow$ distance of resultant vertical force from heel

$x_W = M_W/W = 525.5/366.8 = 1.432$ m

Referring to Fig. 14.29(b),

- Stabilising moment (about toe):

  $M_r = W(L - x_W)$

  $= 366.8 \times (3.9 - 1.432)$

  $= 905.3$ kNm (per m length of wall)

  $\Rightarrow (FS)_{overturning} = \dfrac{0.9M_r}{M_o} = \dfrac{0.9 \times 905.3}{312.4} = 2.61 > 1.40$ — OK

**4. Soil pressures at footing base** [refer Fig. 14.29(b)]

- resultant vertical reaction $R = W = 366.8$ kN (per m length of wall)
- distance of $R$ from heel: $L_R = (M_W + M_o)/R$

  $= (525.5 + 312.4)/366.8 = 2.284$ m[†]
- eccentricity $e = L_R - L/2 = 2.284 - 3.9/2 = 0.334$ m ($< L/6 = 0.65$)

  indicating that the resultant lies well inside the middle third of the base.

  $\Rightarrow \dfrac{6e}{L} = \dfrac{6 \times 0.334}{3.9} = 0.514$

  $\Rightarrow q_{max} = \dfrac{R}{L}\left(1 + \dfrac{6e}{L}\right) = \dfrac{366.8}{3.9}(1 + 0.514)$

  $= 142.4$ kN/m$^2$ $< q_a = 150$ kN/m$^2$ — OK.

  $\Rightarrow q_{min} = \dfrac{R}{L}\left(1 - \dfrac{6e}{L}\right) = \dfrac{366.8}{3.9}(1 - 0.514) = 45.7$ kN/m$^2$,

  as shown in Fig. 14.29(b).

**5. Stability against sliding**

- Sliding force = $P_a = 143.5$ kN (per m length of wall)
- Resisting force (ignoring passive pressure) $F = \mu R$

  $= 0.5 \times 366.8 = 183.4$ kN $> P_a$
- $(F.S)_{sliding} = \dfrac{0.9F}{P_a} = \dfrac{0.9 \times 183.4}{143.5} = 1.15 < 1.4$
- Hence, a *shear key* needs to be provided to generate the balance force through passive resistance.

  Required $P_p = 1.40 \times 143.5 - 0.9 \times 183.4 = 35.8$ kN (per m length of wall)

  Providing a shear key 300 mm × 400 mm at 1.6 m from toe [Fig. 14.29(c)],

  $h_2 = 0.95 + 0.3 + 1.6 \tan 30^\circ = 2.17$ m

  $P_p = 3 \times 16(2.17^2 - 0.95^2)/2 = 91.4$ kN

  $\Rightarrow (F.S)_{sliding} = \dfrac{0.9(183.4 + 91.4)}{143.5} = 1.72 > 1.4$ — OK

---

[†] Note that this value of $x_R$ is close to, but not equal to, the value of $X = 2.6$ m assumed in the initial proportioning.

**6. Design of toe slab**

- The loads considered for the design of the toe slab are as shown in Fig. 14.30(a). The net pressures, acting upward, are obtained by reducing the uniformly distributed self-weight of the toe slab from the gross pressures at the base.
  Self-weight loading = 25 × 0.62 = 15.5 kN/m$^2$
- The net upward pressure varies from 126.9 kN/m$^2$ to 94.7 kN/m$^2$, as shown in Fig. 14.30(b).
- Assuming a clear cover of 75 mm and 16 $\phi$ bars, $d$ = 620 – 75 – 8 = 537 mm
- Applying a load factor of 1.5, the design shear force (at $d$ = 537 mm from the front face of the stem) and the design moment at the face of the stem are given by:
  $V_u \approx 1.5(126.9 + 94.7)/2 \times (1.3 - 0.537) = 126.8$ kN/m
  $M_u = 1.5 \times [(94.7 \times 1.3^2/2) + (126.9 - 94.7) \times 0.5 \times 1.3^2 \times 2/3] = 147.2$ kNm/m
- Nominal shear stress $\tau_v = \dfrac{V_u}{bd} = \dfrac{126.8 \times 10^3}{10^3 \times 537}$ = 0.236 MPa
  For a $\tau_c$ = 0.24 MPa, the required $p_t$ = 0.10 with M 20 concrete [refer Eq. 6.1]

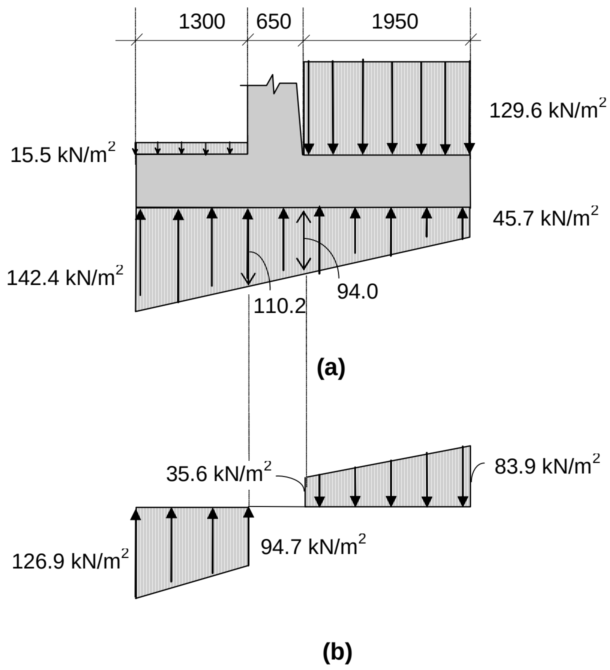



**Fig. 14.30** Net soil pressures acting on base slab

- $R \equiv \dfrac{M_u}{bd^2} = \dfrac{147.2 \times 10^6}{10^3 \times 537^2}$ = 0.510 MPa

$\Rightarrow \dfrac{(p_t)_{reqd}}{100} = \dfrac{20}{2\times 415}\left[1-\sqrt{1-4.598\times 0.510/20}\right] = 0.15\times 10^{-2}$, which is adequate for shear also

$\Rightarrow (A_{st})_{reqd} = (0.15\times 10^{-2})\times 10^3\times 537 = 806$ mm$^2$/m

- Using 16 $\phi$ bars, spacing required = $201\times 10^3/806 = 249$ mm

  **Provide 16 $\phi$ bars @ 240 c/c** at the bottom of the toe slab. The bars should extend by at least a distance $L_d = 47.0\times 16 = 752$ mm beyond the front face of the stem, on both sides. As the toe slab length is only 1.3 m overall, no curtailment of bars is resorted to here.

**7. Design of heel slab**

- The loads considered for the design of the heel slab are as shown in Fig. 14.30(a). The distributed loading acting downward on the heel slab is given by

  i) overburden + surcharge @ $16\times(7.75-0.62) = 114.1$ kN/m$^2$

  ii) heel slab @ $25\times 0.62$ = 15.5 ”

  $\Rightarrow w = 129.6$ kN/m$^2$

- The net pressure acts downwards, varying between 35.6 kN/m$^2$ and 83.9 kN/m$^2$ as shown in Fig. 14.30(b).
- Applying a load factor of 1.5, the design shear force and bending moment at the (rear) face of the stem are given by

  $V_u = 1.5(35.6+83.9)/2\times 1.95 = 174.8$ kN/m

  $M_u = 1.5\times[(35.6\times 1.95^2/2)+(83.9-35.6)\times 0.5\times 1.95^2\times 2/3] = 193.4$ kNm/m

- Assuming a clear cover of 75 mm and 16 $\phi$ bars, $d = 620-75-8 = 537$ mm
- Nominal shear stress $\tau_v = \dfrac{V_u}{bd} = \dfrac{174.8\times 10^3}{10^3\times 537}$

  $= 0.326$ MPa

  Corresponding $\tau_c = 0.33$, with M 20 concrete [refer Eq. 6.1],

  $(p_t)_{reqd} = 0.20$

- $R \equiv \dfrac{M_u}{bd^2} = \dfrac{193.4\times 10^6}{10^3\times 537^2}$

  $= 0.670$ MPa

  $\Rightarrow \dfrac{(p_t)_{reqd}}{100} = \dfrac{20}{2\times 415}\left[1-\sqrt{1-4.598\times 0.670/20}\right]$

  $= 0.193\times 10^{-2}$

  $< 0.20\times 10^{-2}$ required for shear

  $\Rightarrow (A_{st})_{reqd} = (0.20\times 10^{-2})\times 10^3\times 537$

  $= 1074$ mm$^2$/m

- Using 16 $\phi$ bars, spacing required = $201\times 10^3/1074 = 187$ mm

  **Provide 16 $\phi$ bars @ 180 c/c** at the top of the heel slab. The bars should extend by at least a distance $L_d = 47.0\times 16 = 752$ mm beyond the rear face of the stem, on both sides. The bars may be curtailed part way to the heel; however, since the length is relatively short, this is not resorted to in this example.

**8. Design of vertical stem**

- Height of cantilever above base $h$ = 5.250 – 0.62 = 4.63 m
- Assuming a clear cover of 50 mm and 20 $\phi$ bars,
  $d$ (at the base) = 650 – 50 – 10 = 590 mm
- Assuming a load factor of 1.5, maximum design moment
  $M_u = 1.5[C_a\, w_s\, h^2/2 + C_a\, \gamma_e\, h^3/6]$
  $= 1.5 \times (1/3)[40 \times 4.63^2/2 + 16 \times 4.63^3/6]$
  $= 346.7$ kNm/m
  $$\Rightarrow R \equiv \frac{M_u}{bd^2} = \frac{346.7\times10^6}{10^3\times590^2} = 1.00 \text{ MPa}$$
  $$\Rightarrow \frac{(p_t)_{reqd}}{100} = \frac{20}{2\times415}\left[1-\sqrt{1-4.598\times1.00/20}\right] = 0.295 \times 10^{-2}$$
  $$\Rightarrow (A_{st})_{reqd} = (0.295 \times 10^{-2}) \times 10^3 \times 590 = 1741 \text{ mm}^2\text{/m}$$
- Using 16 $\phi$ bars, spacing required = $\dfrac{201\times10^3}{1741}$ = 115 mm

  **Provide 16 $\phi$ @ 110 c/c**, bars extending into the 'shear key'. [This anchorage will be more than the minimum required: $L_d = 47.0 \times 16 = 752$ mm]

- *Check for shear at base*:

  Critical section is at $d$ = 0.59 m above base, i.e., at $z_s$ = 4.63 – 0.59 = 4.04 m below top edge. Shear force at critical section = $1.5\,[C_a\, w_s\, z_s + C_a\, \gamma_s\, z_s^2/2]$
  $= 1.5 \times (1/3)[40 \times 4.04 + 16 \times 4.04^2/2]$
  $= 146$ kN/m

  $$\tau_v = \frac{146\times10^3}{10^3\times590} = 0.248 \text{ MPa} < \tau_c \text{ for } p_t = 0.295$$

  — OK

  Note that since the shear stress is low and flexural reinforcement ratio also is low, the thickness of stem at base could be reduced for a more economical design.

- *Curtailment of bars*:

  The curtailment of the bars may be done in two stages (at one-third and two-third heights of the stem above the base) as shown in Fig. 14.31. It can be verified that the curtailment satisfies the Code requirements.

- *Temperature and Shrinkage reinforcement*

  Provide two-thirds of the (horizontal) bars near the front face (which is exposed to weather and the remaining one-third near the rear face. For the lowermost one-third height of the stem above base,

  $A_{st} = (0.0012 \times 10^3 \times 650) \times 2/3$
  $= 520$ mm²/m

- Using 8 $\phi$ bars, spacing required = $50.3 \times 10^3/520 = 97$ mm $\approx$ 100 mm. Provide 8 $\phi$ @ 100 c/c near front face and 8 $\phi$ @ 200 c/c near rear face in the lowermost one-third height of the wall; 8 ϕ @ 200 c/c near front face and 8 ϕ @ 400 c/c in the middle one-third height; and 8 ϕ @ 300 c/c near front face and 8 ϕ @ 600 c/c near the rear face in the top one-third height of the wall.

- Also provide nominal bars 10 ϕ bars @ 300 c/c vertically near the front face.
- The detailing is shown in Fig. 14.31.

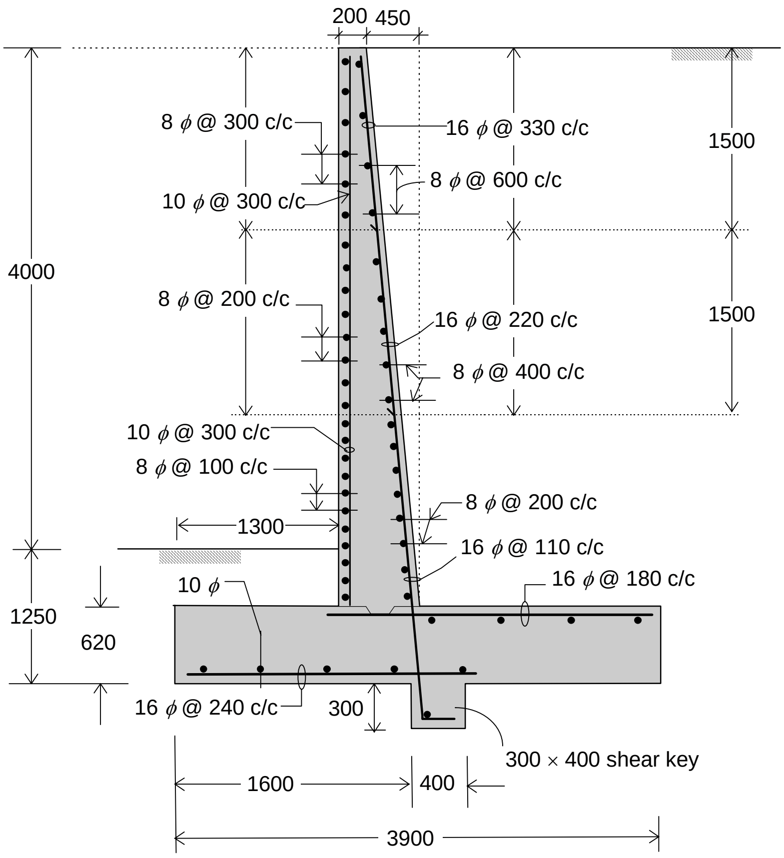



**Fig. 14.31** Detailing of cantilever wall — Example 14.9

#### EXAMPLE 14.10

Design a suitable **counterfort** retaining wall to support a level backfill, 7.5 m high above the ground level on the toe side. Assume good soil for foundation at a depth of 1.5 m below the ground level with a safe bearing capacity of 170 kN/m$^2$. Further assume the backfill to comprise granular soil with a unit weight of 16 kN/m$^3$ and an angle of shearing resistance of 30$^o$. Assume the coefficient of friction between soil and concrete to be 0.5. Use M 25 and Fe 415 steel.

**SOLUTION**

1. **Data given**: $h = 7.5 + 1.5 = 9.0$ m; $\mu = 0.5$
$\theta = 0^o$ $\gamma_e = 16$ kN/m$^3$
$\phi = 30^o$ $q_a = 170$ kN/m$^2$

- Earth pressure coefficients: $C_a = \dfrac{1 - \sin\theta}{1 + \sin\theta} = 0.333$

$$C_p = \frac{1 + \sin\theta}{1 - \sin\theta} = 3.0$$

2. **Preliminary proportions**

- The (triangular shaped) counterforts are provided on the rear (backfill) side of the wall, interconnecting the stem with the heel slab.

  Spacing of counterforts ≈ $\dfrac{1}{3}h$ to $\dfrac{1}{2}h$ = 3.0 m to 4.5 m

  Assume the counterforts are placed with a clear spacing of 3.0 m.
  Thickness of counterforts ≈ 0.05$h$ = 0.05 × 9.0 = 0.45 m. Assume a thickness of 500 mm.
- Thickness of heel slab ≈ 0.05$h$ = 0.05 × 9.0 = 0.45 m. Assume a thickness of 500 mm
- Assuming that the front buttresses are not provided,
  Thickness of toe slab ≈ 0.08$h$ = 0.08 × 9.0 = 0.72 m. Assume a thickness of 720 mm
- Thickness of stem slab ≈ 0.06$h$ = 0.06 × 9.0 = 0.54 m. Assume a stem thickness of 600 mm at the base of the stem, tapering to a value of 300 mm at the top of the wall.

- For an economical proportioning of the length $L$ of the base slab, it is assumed that the vertical reaction $R$ at the footing base is in line with the front face of the stem. For such a condition, (inclusive of stem thickness) [Eq. 14.23]:

$$X \approx \left(\sqrt{C_a/3}\right)\, h = \sqrt{0.333/3}\ (9.0) = 3.0 \text{ m}$$

- Assuming a triangular base pressure distribution,

  $L = 1.5X = 4.5$ m
- The preliminary proportions are shown in Fig. 14.32(a).

3. **Stability against overturning**

- Forces due to active pressure (per m length of wall) [Fig. 14.32(a)]:
  $P_a = C_a\, \gamma_e\, h^2/2 = (0.333)(16)(9.0)^2/2 = 216.0$ kN
- Overturning moment $M_o = P_a \times h/3$
  $\Rightarrow M_o = 216.0 \times (9.0/3)$
  $= 648.0$ kNm (per m length of wall)

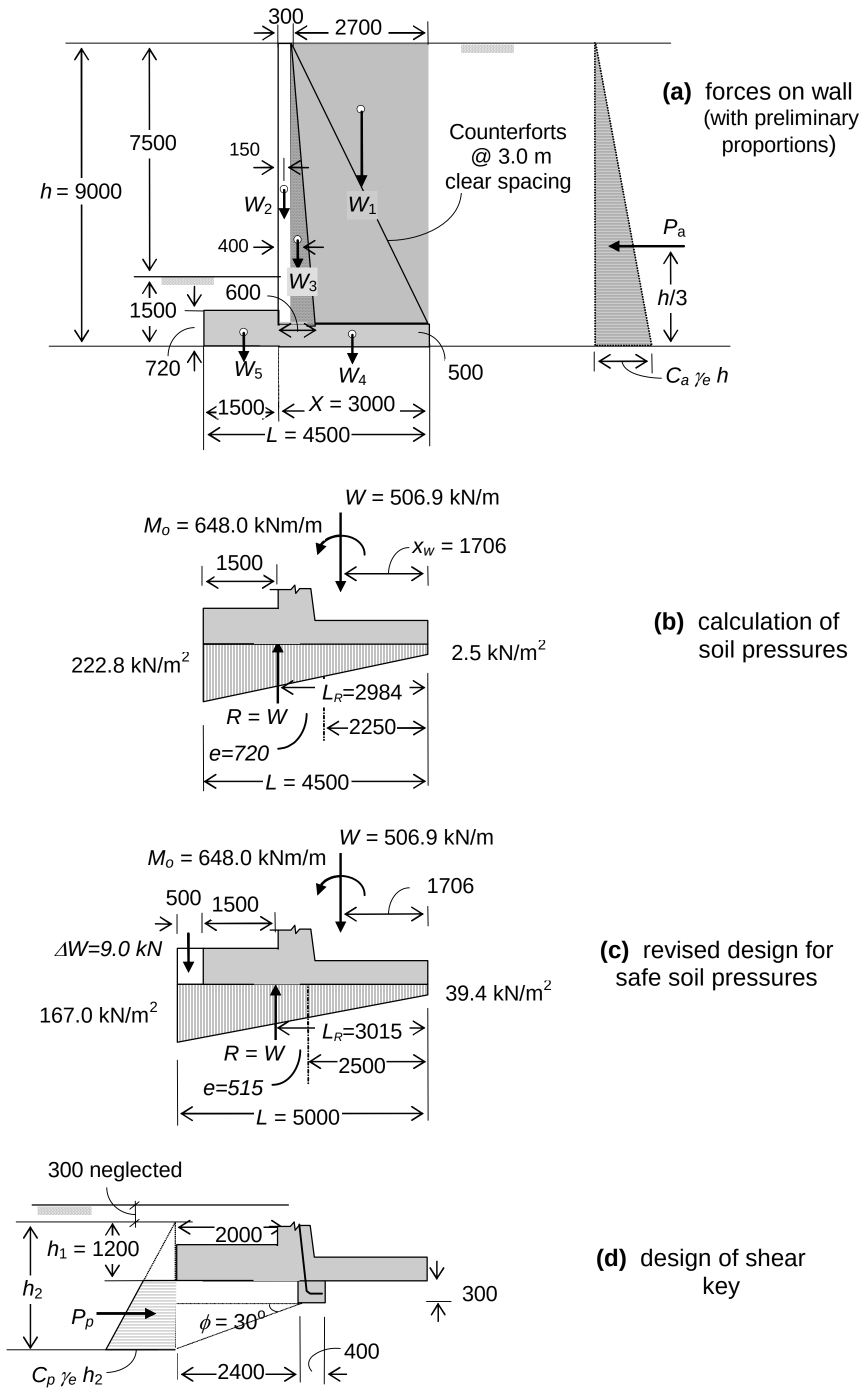



**Fig. 14.32** Example 14.10

- Line of action of resultant of vertical forces [Fig. 14.32(b)] with respect to the heel can be located by applying statics, considering 1 m length of the wall (the marginal additional weight due to counterfort is ignored).

| force (kN) | | distance from heel (m) | moment (kNm) |
|---|---|---|---|
| $W_1$ = (16)(2.7)(9.0 – 0. 5) | = 367.20 | 1.35 | 495.72 |
| $W_2$ = (25)(0.3)(9.0 – 0.5) | = 63.75 | 2.85 | 181.70 |
| $W_3$ = (25-16)(0.5)(0.3)(9.0 – 0.5) | = 11.48 | 2.60 | 29.85 |
| $W_4$ = (25)(3.0)(0.5) | = 37.50 | 1.50 | 56.30 |
| $W_5$ = (25)(1.5)(0.72) | = 27.00 | 3.75 | 101.25 |
| | $W$ = 506.9 | | $M_W$ = 864.8 |

$\Rightarrow$ distance of resultant vertical force from heel

$$x_W = M_W/W = 864.8 / 506.9 = 1.706 \text{ m}$$

Referring to Fig. 14.32(b),

- Stabilising moment (about toe):

$$M_r = W(L - x_W)$$
$$= 506.9 \times (4.5 - 1.706)$$
$$= 1416.4 \text{ kNm (per m length of wall)}$$

$$\Rightarrow (FS)_{overturning} = \frac{0.9M_r}{M_o} = \frac{0.9\times1416.4}{648.0} = 1.967 > 1.40 \qquad \text{— OK}$$

**4. Soil pressures at footing base** [refer Fig. 14.32(b)]

- resultant vertical reaction $R = W = 506.9$ kN (per m length of wall)
- distance of $R$ from heel: $L_R = (M_W + M_o)/R$

$$= (864.8 + 648.0) / 506.9 = 2.984 \text{ m}^{\dagger}$$

- eccentricity $e = L_R - L/2 = 2.984 - 4.5/2 = 0.734$ m $(< L/6 = 0.75)$
  indicating that the resultant lies well inside the middle third of the base.

$$\Rightarrow \frac{6e}{L} = \frac{6\times0.734}{4.5} = 0.978$$

$$\Rightarrow q_{min} = \frac{R}{L}\left(1-\frac{6e}{L}\right) = \frac{506.9}{4.5}(1 - 0.978) = 2.5 \text{ kN/m}^2 > 0 \qquad \text{— OK}$$

$$\Rightarrow q_{max} = \frac{R}{L}\left(1+\frac{6e}{L}\right) = \frac{506.9}{4.5}(1 + 0.978)$$

$$= 222.8 \text{ kN/m}^2 > q_a = 170 \text{ kN/m}^2$$ — **UNSAFE**.

Hence, the length of the base slab needs to be suitably increased on the toe side – say, by 500 mm.

---

† Note that this value of $x_R$ is very close to the value of $X$ = 3.0 m assumed in the initial proportioning.

Let $L$ = 5.0 m (as shown in Fig. 14.32c).
Additional weight due to 500 mm extension of toe slab

$\Delta W = 25 \times 0.5 \times 0.72 = 9.0$ kN
$\Rightarrow R = W + \Delta W = 506.9 + 9.0 = 515.9$ kN

Considering moments about the heel [Fig. 14.32 (c)]
$515.9\, L_R = 864.8 + (9.0)(5.0 - 0.25) + 648.0$
$\Rightarrow L_R = 3.015$ m

- Revised eccentricity $e = L_R - L/2 = 3.015 - 5.0/2$
  $= 0.515$ m (< $L/6 = 0.83$)

$$\Rightarrow \frac{6e}{L} = \frac{6 \times 0.515}{5.0} = 0.618$$

$$\Rightarrow q_{min} = \frac{R}{L}\left(1 - \frac{6e}{L}\right) = \frac{515.9}{5.0}(1 - 0.618) = 39.4 \text{ kN/m}^2 > 0 \qquad \text{— OK}$$

$$\Rightarrow q_{max} = \frac{R}{L}\left(1 + \frac{6e}{L}\right) = \frac{515.9}{5.0}(1 + 0.618) = 167.0 \text{ kN/m}^2$$

$$< q_a = 170 \text{ kN/m}^2 \qquad \text{— OK.}$$

as shown in Fig. 14.32c.

**5. Stability against sliding**

- Sliding force = $P_a$ = 216.0 kN (per m length of wall)
- Resisting force (ignoring passive pressure) $F = \mu R$
  $= 0.5 \times 515.9 = 257.9$ kN > $P_a$
- $$(F.S)_{sliding} = \frac{0.9F}{P_a} = \frac{0.9 \times 257.9}{216.0} = 1.075 < 1.4 \quad \text{— \textbf{UNSAFE}.}$$
- Hence, a *shear key* needs to be provided to generate the balance force through passive resistance.

Required $P_p = 1.4 \times 216.0 - 0.9 \times 257.9 = 70.3$ kN (per m length of wall)

Providing a shear key 400 mm × 300 mm at 2.4 m from toe [Fig. 14.32(d)],
$h_2 = 1.2 + 0.3 + 2.4 \tan 30^{\circ} = 2.89$ m
$P_p = 3 \times 16(2.89^2 - 1.2^2)/2 = 165.9$ kN

$$\Rightarrow (F.S)_{sliding} = \frac{0.9 \times (257.9 + 165.9)}{216.0} = 1.766 > 1.4 \qquad \text{— OK}$$

**6. Design of toe slab**

- The loads considered for the design of the toe slab are as shown in Fig. 14.33(a). The net pressures, acting upward, are obtained by reducing the uniformly distributed self-weight of the toe slab from the gross pressures at the base.

Self-weight loading = $25 \times 0.72 = 18.0$ kN/m$^2$

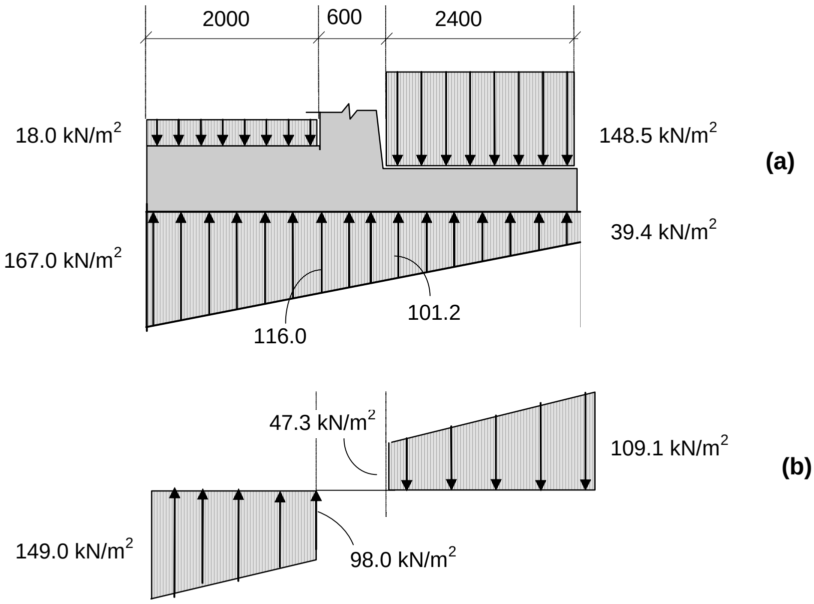



**Fig. 14.33** Net soil pressures acting on base slab

- The net upward pressure varies from 149.0 kN/m$^2$ to 97.9 kN/m$^2$, as shown in Fig. 14.33(b).
- Assuming a clear cover of 75 mm and 16 $\phi$ bars, $d$ = 720 – 75 – 8 = 637 mm
- Applying a load factor of 1.5, the design shear force (at $d$ = 637 mm from the front face of the stem) and the design moment at the face of the stem are given by:
  $V_u \approx 1.5 \times (149.0 + 97.9)/2 \times (2.0 - 0.637) = 252.4$ kN/m
  $M_u = 1.5 \times [(97.9 \times 2.0^2/2) + (149.0 - 97.9) \times 0.5 \times 2.0^2 \times 2/3] = 395.9$ kNm/m
- Nominal shear stress $\tau_v = \dfrac{V_u}{bd} = \dfrac{252.4 \times 10^3}{10^3 \times 637}$ = 0.396 MPa

  For a $\tau_c$ = 0.396 MPa, the required $p_t$ = 0.32 with M 25 concrete [refer Eq. 6.1]
- $R \equiv \dfrac{M_u}{bd^2} = \dfrac{395.9 \times 10^6}{10^3 \times 637^2}$ = 0.976 MPa

  $\Rightarrow \dfrac{(p_t)_{reqd}}{100} = \dfrac{25}{2 \times 415}\left[1 - \sqrt{1 - 4.598 \times 0.976/25}\right] = 0.284 \times 10^{-2}$

  $< 0.32 \times 10^{-2}$ required for shear

  $\Rightarrow (A_{st})_{reqd} = (0.32 \times 10^{-2}) \times 10^3 \times 637 = 2039$ mm$^2$/m

- Using 16 $\phi$ bars, spacing required = $201 \times 10^3/2039 = 98.6$ mm
- Using 20 $\phi$ bars, spacing required = $314 \times 10^3/2039 = 154$ mm

**Provide 20 $\phi$ bars @ 150 c/c** at the bottom of the toe slab. The bars should extend by at least a distance $L_d = 47.0 \times 20 = 940$ mm beyond the front face of the stem, on both sides.

Distribution steel:
Provide **10 $\phi$ bars @ 200 c/c** for the transverse reinforcement.

**7. Design of heel slab**

- The loads (net pressures) considered for the design of the heel slab are as shown in Fig. 14.33(a). The distributed loading acting downward on the heel slab is given by

  i) overburden @ $16 \times (9.0 - 0.5)$ = 136.0 kN/m$^2$
  ii) heel slab @ $25 \times 0.5$ = 12.5 ”
  $\Rightarrow w$ = 148.5 kN/m$^2$

  The net pressure acts downwards, varying between 47.3 kN/m$^2$ and 109.1 kN/m$^2$ as shown in Fig. 14.33(b).

- The counterforts are provided at a clear spacing of 3.0 m throughout the length of the wall [Fig. 14.32(a)]. Thus, each heel slab panel (2.4m $\times$ 3.0m) may be considered to be fixed (continuous) at three edges (counterfort locations and junction with stem) and free at the fourth edge. The moment coefficients given in IS 456 do not cater to this set of boundary conditions, and reference needs to be made to other handbooks. Alternatively, we may apply the formulas obtained from yield line theory (such as those given in Section 11.2.6).

  A common simplified design practice is to assume that some tributary (triangular) portion of the net load acting on the heel slab is transmitted through cantilever action [Fig. 14.34(a)], while much of the load (particularly near the free edge) is transmitted in the perpendicular direction through continuous beam action. The reinforcements in the remaining regions are judiciously apportioned. This procedure is followed here.

- **Design of heel slab for continuous beam action**

  Assuming a clear cover of 75 mm and 16 $\phi$ bars, $d = 500 - 75 - 8 = 417$ mm
  Consider a 1 m wide strip near the free edge of the heel (Fig.14.34b). The intensity of pressure at a distance of 1 m from the free edge is 83.4 kN/m$^2$. Hence, the average loading on the strip may be taken as (83.4 + 109.1)/2 = 96.25 kN/m$^2$. Applying a load factor of 1.5, $w_u = 1.5 \times 96.25 = 144.4$ kN/m$^2$. The effective span is given by $l = 3.0 + 0.417 = 3.417$ m

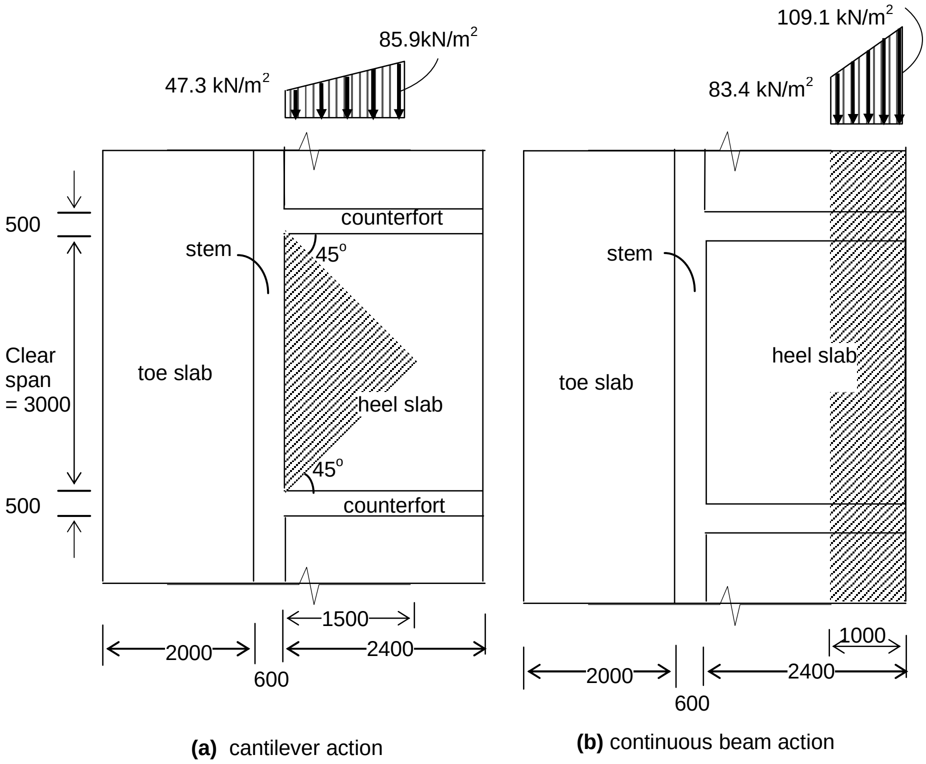



**Fig. 14.34** Loading considerations for simplified analysis of heel slab

Max. negative moment occurring in the heel slab at the counterfort location is given by

$M_{u,-ve} = w_u l^2/12 = 144.4 \times 3.417^2 /12 = 140.5$ kNm/m

Max. mid-span moment may be taken as

$M_{u,+ve} = w_u l^2/16 \approx 0.75 \times M_{u,-ve} = 105.4$ kNm/m

Design shear force

$V_u = w_u \times (\text{clear span} / 2 - d) = 144.4 \times (3.0/2 - 0.417) = 156.4$ kN/m

<u>Design of top reinforcement (for –ve moments) at the counterforts</u>

- Nominal shear stress $\tau_v = \dfrac{V_u}{bd} = \dfrac{156.4\times10^3}{10^3\times417}$ = 0.375 MPa

  For a $\tau_c$ = 0.375 MPa with M 25 concrete [refer Eq. 6.1], the required $p_t$ = 0.28.

- $R \equiv \dfrac{M_u}{bd^2} = \dfrac{140.5\times10^6}{10^3\times417^2}$ = 0.808 MPa

$\Rightarrow \dfrac{(p_t)_{reqd}}{100} = \dfrac{25}{2\times415}\left[1-\sqrt{1-4.598\times0.808/25}\right] = 0.233 \times 10^{-2} < p_t = 0.28$ required for shear (in the absence of stirrups).

$\Rightarrow (A_{st})_{reqd} = (0.28 \times 10^{-2}) \times 10^3 \times 417 = 1168$ mm$^2$/m (required at 1m from the free edge)

Using 16 ϕ bars, spacing required = $201 \times 10^3 / 1168 = 172$ mm

Using 12 ϕ bars, spacing required = $113 \times 10^3 / 1168 = 96$ mm

Minimum reinforcement for temperature and shrinkage:

Min. $A_{st} = \dfrac{0.12}{100}(1000)(500) = 600$ mm$^2$/m < 1168 mm$^2$/m – OK.

At a distance beyond 1m from the free edge, only minimum reinforcement need be provided:

Spacing of 12 ϕ bars required for min. reinf. = $113 \times 10^3 / 600 = 188$ mm

Provide **12 ϕ bars @ 180 c/c** at the top of the heel slab throughout, and introduce additional 12 ϕ bars in between two adjacent bars at the counterforts near the free edge over a distance of approx. 1m;

i.e., **Provide 5 additional 12 ϕ bars on top,** extending 1m from either side of the face of the counterfort.

<u>Design of bottom reinforcement (for +ve moment) at mid-span of heel slab</u>

- $R \approx 0.75 \times 0.808 = 0.606$ MPa

  $\Rightarrow \dfrac{(p_t)_{reqd}}{100} = \dfrac{25}{2\times415}\left[1-\sqrt{1-4.598\times0.606/25}\right] = 0.173 \times 10^{-2}$

  $\Rightarrow (A_{st})_{reqd} = (0.173 \times 10^{-2}) \times 10^3 \times 417 = 721$ mm$^2$/m > $(A_{st})_{min} = 600$ mm$^2$/m

  Spacing of 12 ϕ bars required = $113 \times 10^3 / 721 = 156$ mm

  Provide **12 ϕ bars @ 150 c/c** at the bottom of the heel slab throughout.

  <u>Distribution steel:</u>

  Provide **10 ϕ bars @ 200 c/c** for the transverse reinforcement.

- **Design of heel slab for cantilever action**

  Consider the triangular loading on the heel slab [Fig. 14.34(a)] to be carried by cantilever action with fixity at the face of the stem.

  The intensity of load at the face of the stem = 47.3 kN/m$^2$.

  The intensity of load at a distance of 1.5m from the face of the stem is 85.9 kN/m$^2$.

  Total B.M. due to loading on the triangular portion

  $$= \left(\frac{1}{2}\times3.0\times1.5\right)\times\left[47.3\times\frac{1.5}{3}+(85.9-47.3)\times\frac{1.5}{2\times3}\right] = 74.93 \text{ kNm}$$

  This moment is distributed non-uniformly across the width of 3.0m. For design purposes, the max. moment intensity (in the middle region) may be taken as two times the average value

$\Rightarrow \quad M_{max} = 2 \times (74.93 / 3.0)$
$= 49.95$ kNm/m
$d = 417 - 12 = 405$mm

Applying a load factor of 1.5,

$$R \equiv \frac{M_u}{bd^2} = \frac{1.5 \times 49.95 \times 10^6}{1000 \times 405^2} = 0.457 \text{ MPa}$$

$$\Rightarrow \frac{(p_t)_{reqd}}{100} = \frac{25}{2 \times 415}\left[1 - \sqrt{1 - 4.598 \times 0.457/25}\right] = 0.130 \times 10^{-2}$$

$\Rightarrow (A_{st})_{reqd} = (0.130 \times 10^{-2}) \times 10^3 \times 405 = 527$ mm$^2$/m $< (A_{st})_{min} = 600$ mm$^2$/m (for temperature and shrinkage)

Provide **12 ϕ bars @ 180 c/c** at the top of the heel slab throughout.

**8. Design of vertical stem**

The simplified analysis procedure adopted for the heel slab is used here for the vertical stem also. The cantilever action is limited to the bottom region only (triangular portion) with fixity at the junction of the stem with the base slab. Elsewhere, the stem is treated as a continuous beam spanning between the counterforts. The bending moments reduce along the height of the stem, owing to the reduction in the lateral pressures with increasing height.

Height of stem above base $h = 9.0 - 0.5 = 8.5$ m.

Intensity of earth pressure at the base of the stem is
$p_a = C_a \gamma_e h = (0.333)(16)(8.5)$
$= 45.33$ kN/m$^2$ (linearly varying to zero at the top)
Applying a load factor of 1.5, $w_u = 1.5 \times 45.33 = 68.0$ kN/m$^2$ at base.
Clear spacing between the counterforts = 3.0 m.

- **Design of stem for continuous beam action**

***At base***
Assuming a clear cover of 50 mm and 20 ϕ bars,
$d = 600 - 50 - 10 = 540$ mm and effective span, $l = 3.0 + 0.54 = 3.54$ m

Max. –ve moment occurring in the stem at the counterfort location is given by
$M_{u,-ve} = w_u l^2/12 = 68.0 \times 3.54^2 /12 = 71.0$ kNm/m
Max. mid-span moment may be taken as
$M_{u,+ve} = w_u l^2/16 \approx 0.75 \times M_{u,-ve} = 53.3$ kNm/m
Design shear force
$V_u = w_u \times (\text{clearspan}/2 - d) = 68.0 \times (3.0/2 - 0.54) = 65.3$ kN/m

Design of (rear face) reinforcement for –ve moments at the counterforts

$$\Rightarrow R \equiv \frac{M_u}{bd^2} = \frac{71.0 \times 10^6}{10^3 \times 540^2} = 0.244 \text{ MPa}$$

$\Rightarrow \dfrac{(p_t)_{reqd}}{100} = \dfrac{25}{2 \times 415}\left[1-\sqrt{1-4.598 \times 0.244/25}\right]$ = 0.068 × $10^{-2}$

$\Rightarrow (A_{st})_{reqd}$ = (0.068 × $10^{-2}$) × $10^3$ × 540 = 369 $mm^2$/m

Min. $A_{st} = \dfrac{0.12}{100}(1000)(600)$ = 720 $mm^2$/m > 369 $mm^2$/m

Check for shear at base

$\tau_v = \dfrac{65.3 \times 10^3}{10^3 \times 540}$ = 0.121 MPa < $\tau_c$ = 0.29 MPa (for minimum $p_t$ = 0.15) — OK

(Evidently, it is possible to reduce the thickness of the stem, for economy).

Design of (front face) reinforcement for +ve moments in the mid-span of stem

The minimum reinforcement requirement will govern the design on both faces, since $M_{u,+ve} < M_{u,-ve}$.

Using 12 ϕ bars, spacing required = 113 × 1000/ 720 = 156 mm
Provide **12 ϕ bars (horizontal) @ 150 c/c on both faces** of the stem (up to one-third height above base).

***At one-third height above base***
$d$ = 500 – 50 – 6 = 444 mm and effective span $l$ = 3.444m
$M_{u,-ve} = w_u l^2/12$ = (68.0 × 2/3) × $(3.444)^2$ /12 = 44.81 kNm/m

$\Rightarrow R \equiv \dfrac{M_u}{bd^2} = \dfrac{44.81 \times 10^6}{10^3 \times 444^2}$ = 0.227 MPa

$\Rightarrow \dfrac{(p_t)_{reqd}}{100} = \dfrac{25}{2 \times 415}\left[1-\sqrt{1-4.598 \times 0.227/25}\right]$ = 0.064 × $10^{-2}$

$\Rightarrow (A_{st})_{reqd}$ = (0.064 × $10^{-2}$) × $10^3$ × 444 = 282 $mm^2$/m

Min. $A_{st} = \dfrac{0.12}{100}(1000)(500)$ = 600 $mm^2$/m > 282 $mm^2$/m

Using 12 ϕ bars, spacing required = 113 × 1000/ 600 = 188 mm
Provide **12 ϕ bars (horizontal) @ 180 c/c on both faces** of the stem (in the middle one-third height).

***At two-thirds height above base***

Min. $A_{st} = \dfrac{0.12}{100}(1000)(400)$ = 480 $mm^2$/m

Using 10 ϕ bars, spacing required = 78.5 × 1000/ 480 = 163 mm
Using 12 ϕ bars, spacing required = 113 × 1000/ 480 = 235 mm
Provide **12 ϕ bars (horizontal) @ 230 c/c on both faces** of the stem (in the upper one-third height).

- **Design of stem for cantilever action**
  Consider the triangular loading on the stem [Fig. 14.35] to be carried by cantilever action about the face of the stem as follows:

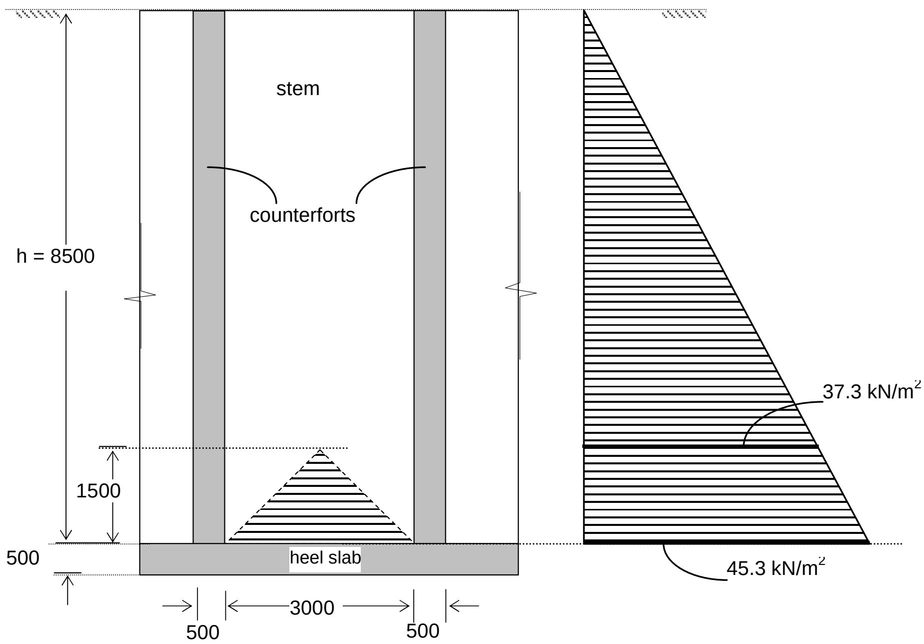



**Fig. 14.35** Loading considerations for simplified analysis of stem

The intensity of horizontal pressure at the base of the stem = 45.3 kN/m$^2$.
The intensity of horizontal pressure at a distance of 1.5 m from the base of the stem is 37.3 kN/m$^2$.

Total B.M. due to loading on the triangular portion

$$= \left(\frac{1}{2}\times 3.0\times 1.5\right)\times\left[37.3\times\frac{1.5}{2}+(45.3-37.3)\times\frac{1.5}{2\times 3}\right] = 67.5 \text{ kNm}$$

This moment is distributed non-uniformly across the width of 3.0m. For design purposes, the max. moment intensity (in the middle region) may be taken as two times the average value

$\Rightarrow \quad M_{max} = 2 \times (67.5 / 3.0)$
$= 45.0$ kNm/m

effective depth $d = 515 - 12 = 503$ mm

$$R \equiv \frac{M_u}{bd^2} = \frac{1.5\times 45.0\times 10^6}{1000\times 503^2} = 0.267 \text{ MPa}$$

$\Rightarrow \dfrac{(p_t)_{reqd}}{100} = \dfrac{25}{2\times 415}\left[1-\sqrt{1-4.598\times 0.267/25}\right] = 0.075\times 10^{-2}$ (required up to 1.5m height above base)

$\Rightarrow (A_{st})_{reqd} = (0.075 \times 10^{-2}) \times 10^3 \times 503 = 377$ mm$^2$/m < $(A_{st})_{min} = 720$ mm$^2$/m (for temperature and shrinkage)
The minimum reinforcement requirement will govern the design.
Provide **12 ϕ bars (vertical) @ 150 c/c on both faces** of the stem through out the height of the stem.

The reinforcement details for the stem, toe slab and heel slab are shown in Fig. 14.36

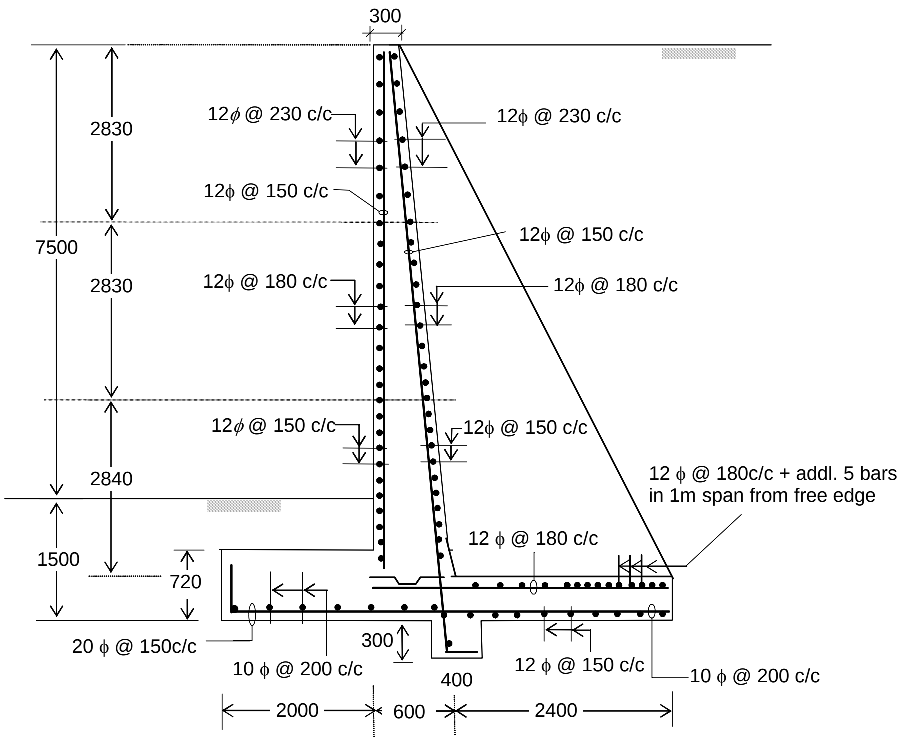



**Fig. 14.36** Reinforcement details of stem, toe slab and heel slab

## 9. Design of interior counterfort

The typical interior counterfort acts as a T beam of varying section cantilevering out of the base slab. The design should include:

- provision for beam action
- provision of horizontal ties against separation from stem
- provision of vertical ties against separation of base

- **Design of counterfort for T-beam action**

  The thickness of counterforts = 500 mm
  Clear spacing of counterforts = 3.0 m
  Thus, each counterfort receives earth pressure from a width of
  $l$= 3.0 + 0.5 = 3.5 m

  ***At base***

  The intensity of earth pressure at the base of the stem is

  $$p_a = C_a \gamma_e h = (0.333)(16)(8.5) = 45.33 \text{ kN/m}^2$$

  Applying a load factor of 1.5,

  $$M_u = 1.5 \times \left(\frac{1}{2}\times 45.33\times 8.5\right)\times 3.5\times\frac{8.5}{3} = 2866 \text{ kNm}$$

  $$V_u = 1.5 \times \left(\frac{1}{2}\times 45.33\times 8.5\right)\times 3.5 = 1012 \text{ kN}$$

  From Fig. 14.37,
  tan θ = 2700/8500 ⇒ θ = 17.6°
  and $D_{base}$ = 2400 × cosθ = 2287 mm
  Assuming a clear cover of 50 mm and 25 ϕ bars,
  $d$ = 2287 – 50 – 12.5 = 2224 mm

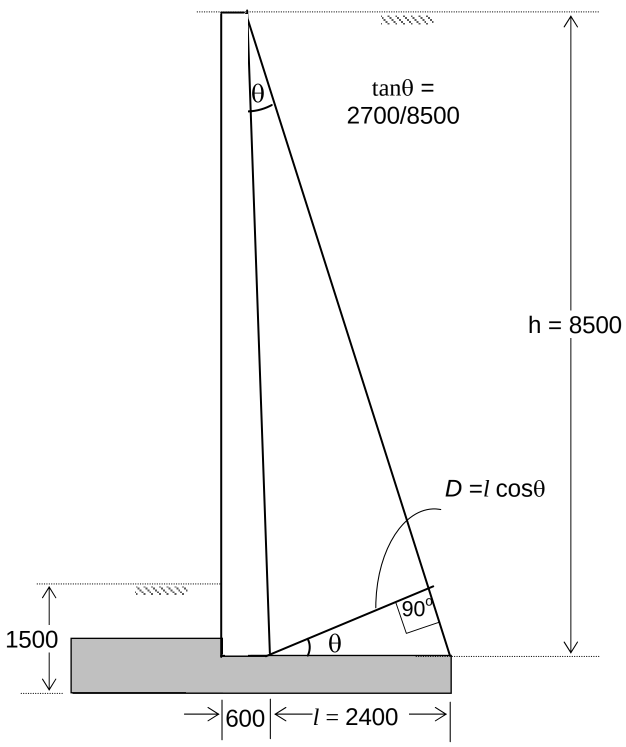


**Fig. 14.37** Depth consideration for analysis of counterfort

- *Effective flange width* (Cl 23.1.2 Code):

  $b_f = l_0/6 + b_w + 6D_f$ [Eq. 4.30]

  $= 8500/6 + 500 + (6 \times 600) = 5517$ mm,

  $b_f = b_w$ + clear span of slab

  = 500 + 3000 = 3500 mm

  Thus, $b_f$ = 3500 mm (least of the above two values)

  Approximate requirement of tension steel is given by assuming a lever arm $z$ to be the larger of 0.9$d$ = 2001mm and $d - D_f/2$ = 1924mm, i.e., 2001mm:

  $$(A_{st})_{\text{reqd}} = \frac{M_u}{0.87 f_y z} = \frac{2866 \times 10^6}{0.87(415)(2001)} = 3967 \text{ mm}^2$$

  No. of 25 ϕ bars required = $\frac{3967}{491} \approx 8$ bars (provide in two layers, with 25 ϕ spacer bars)

  $\Rightarrow d$ = 2287 – 50 – 25 – 12.5 = 2199 mm

  Assuming the neutral axis to be located at $x_u = D_f$,

  $M_{uR} = 0.362 \times 25 \times 3500 \times 600 \times (2199 - 0.416 \times 600) = 37048 \times 10^6$ Nmm

  $> M_u = 2866 \times 10^6$ Nmm

  This clearly indicates that the neutral axis lies within the flange.

  $$R \equiv \frac{M_u}{bd^2} = \frac{2866 \times 10^6}{3500 \times 2199^2} = 0.169 \text{ MPa}$$

  $$\Rightarrow \frac{(p_t)_{reqd}}{100} = \frac{25}{2 \times 415}\left[1 - \sqrt{1 - 4.598 \times 0.169/25}\right] = 0.047 \times 10^{-2}$$

  $\Rightarrow (A_{st})_{reqd} = (0.047 \times 10^{-2}) \times 3500 \times 2199 = 3639$ mm$^2$/m (which is close to the approximate value of 3967 mm$^2$ calculated)

  Minimum reinforcement in a beam is given by $\frac{A_s}{bd} = \frac{0.85}{f_y}$

  $\Rightarrow A_s = 0.85 \times 500 \times 2199 / 415 = 2252$ mm$^2$ < 3639 mm$^2$

  **Provide 8 nos 25 ϕ bars in two layers**, four bars in each layer with a 25 mm separation.

  ***Above one-third height from the base***

  The intensity of earth pressure at $h$ (= 8.5 × 2 / 3) = 5.67m from top is

  $p_a = C_a \gamma_e h = 45.33 \times 2/3 = 30.22$ kN/m$^2$

  Applying a load factor of 1.5,

  $$M_u = 1.5 \times \left(\frac{1}{2} \times 30.22 \times 5.67\right) \times 3.5 \times \frac{5.67}{3} = 850 \text{ kNm}$$

$$V_u = 1.5 \times \left(\frac{1}{2}\times 30.22\times 5.67\right)\times 3.5 = 450 \text{ kN}$$

$D_{h=5.67}$ = 2287 × 2 / 3 = 1525 mm
Assuming a clear cover of 50 mm and 25 φ bars,
$d$ = 1525 – 50 – 12.5 = 1462 mm
Approximate requirement of tension steel is given by assuming a lever arm $z$ to be the larger of 0.9$d$ = 1316mm and $d - D_f/2$ = 1212 mm, i.e., 1316 mm:

$$(A_{st})_{\text{reqd}} = \frac{M_u}{0.87 f_y z} = \frac{850\times 10^6}{0.87(415)(1316)} = 1789 \text{ mm}^2$$

No. of 25 φ bars required = $\frac{1789}{491} \approx 4$ bars

$\Rightarrow d$ = 1462 mm

Assuming the neutral axis to be located at $x_u = D_f$,
$M_{uR}$ = 0.362×25×3500×500 × (1462 – 0.416×500) = 19860 × 10$^6$ Nmm
> $M_u$ = 850 × 10$^6$ Nmm
This clearly indicates that the neutral axis lies within the flange.

$$R \equiv \frac{M_u}{bd^2} = \frac{850\times 10^6}{3500\times 1462^2} = 0.114 \text{ MPa}$$

$$\Rightarrow \frac{(p_t)_{reqd}}{100} = \frac{25}{2\times 415}\left[1-\sqrt{1-4.598\times 0.114/25}\right] = 0.032 \times 10^{-2}$$

$\Rightarrow (A_{st})_{reqd}$ = (0.032 × 10$^{-2}$) × 3500 × 1462 = 1638 mm$^2$/m (which is close to the approximate value of 1789 mm$^2$ calculated)

Minimum reinforcement in a beam is given by $\frac{A_s}{bd} = \frac{0.85}{f_y}$

$\Rightarrow A_s$ = 0.85 × 500 × 1462 / 415 = 1497 mm$^2$ < 1638 mm$^2$

**Curtail 4 nos 25 φ bars and extend 4 nos 25 φ bars (rear face)**.

In order to satisfy the minimum reinforcement criteria, **4 nos 25 φ bars** may be extended to the top of the counterfort, without any further curtailment.

- **Design of horizontal ties**

Horizontal tie (closed stirrup) reinforcement in the counterfort serves as shear reinforcement against flexural shear in the counterfort and also as ties resisting the separation of the stem from the counterfort due to the lateral pressure.

***At base***
Shear reinforcement requirement:

$$V_{u,net} = V_u - \frac{M_u}{d}\tan\theta$$

$$= 1012 - \frac{2866}{2.199}\tan(17.6) = 598.0 \text{ kN}$$

Nominal shear stress $\tau_v = \dfrac{V_{u,net}}{bd} = \dfrac{598 \times 10^3}{500 \times 2199} = 0.544$ MPa

$p_t = 100\, A_{st}/b = 100 \times (8{\times}491) / (500{\times}2199) = 0.357$

$\Rightarrow \tau_c = 0.416$ MPa

Hence, shear reinforcement is to be provided for a shear force of

$V_{us} = (\tau_v - \tau_c)\, bd = (0.544 - 0.416)\ 500 \times 2199$

$= 140.8 \times 10^3$ N

Assuming 10 $\phi$ 2-legged stirrups,

$A_{sv} = 2 \times 78.5 = 157$ mm$^2$

Required spacing = $s_v = \dfrac{0.87 f_y A_{sv} d}{V_{us}} = \dfrac{0.87 \times 415 \times 157 \times 2199}{140.8 \times 10^3} = 885$ mm

Max. spacing specified by Code = $\begin{cases} 0.75d \\ 300\,\text{mm} \end{cases}$ = 300mm (lesser value)

Tie connection requirement:

The tension resisted by the tie reinforcement is given by the lateral pressure on the wall multiplied by the tributary area. At the base ($p_a$ = 45.33 kN/m$^2$), the tensile force intensity is accordingly given by:

$T = 45.33$ kN/m$^2$ $\times$ 3.5m = 158.7 kN/m

Applying a load factor of 1.5, the total area of reinforcement required to resist this direct tension = $\dfrac{1.5 \times 158.7 \times 10^3}{0.87 \times 415}$ = 660 mm$^2$/m.

Spacing of 10 $\phi$ 2 legged stirrups required = $2 \times 78.5 \times 10^3 / 660 = 237$ mm

This tie reinforcement requirement governs (compared to shear reinforcement requirement).

**Provide 10 $\phi$ 2 legged stirrups @ 200 mm c/c in lower one-third region**

The tie reinforcement requirement will vary linearly along the height of the stem, as the lateral pressure variation is linear.

***At one-third height from the base***

$(A_{st})_{\text{reqd}} = (2/3) \times 660 = 440$ mm$^2$/m

Spacing of 8 $\phi$ 2 legged stirrups required = $2 \times 50.3 \times 10^3 / 440 = 228$ m

**Provide 8 $\phi$ 2 legged stirrups @ 200 mm c/c above one-third height.**

- **Design of vertical ties**

As in the case of the connection between the counterfort and the vertical stem, the connection between the counterfort and the heel slab must be designed to resist the tension arising out of the net downward pressures acting on the heel slab [Fig. 14.34b]. Considering a 1m strip from the free edge, the average downward pressure is (83.4 + 109.1)/2 = 96.25 kN/m$^2$, and hence the average tensile force intensity is:

$T$ = 96.25 kN/m$^2$ × 3.5m = 336.9 kN/m

Applying a load factor of 1.5, the total area of reinforcement required to resist this direct tension = $\dfrac{1.5 \times 336.9 \times 10^3}{0.87 \times 415}$ = 1400 mm$^2$/m.

Spacing of 10 $\phi$ 2 legged ties required = 2 × 78.5 × 10$^3$ / 1400 = 112 mm

Provide **10 $\phi$ 2 legged vertical ties @ 100 mm c/c** up to 1m from the free edge. The spacing may be increased to 150 mm beyond 1m, owing to the significant reduction in net pressure.

The counterfort reinforcement details are shown in Fig. 14.38 and Fig. 14.39.

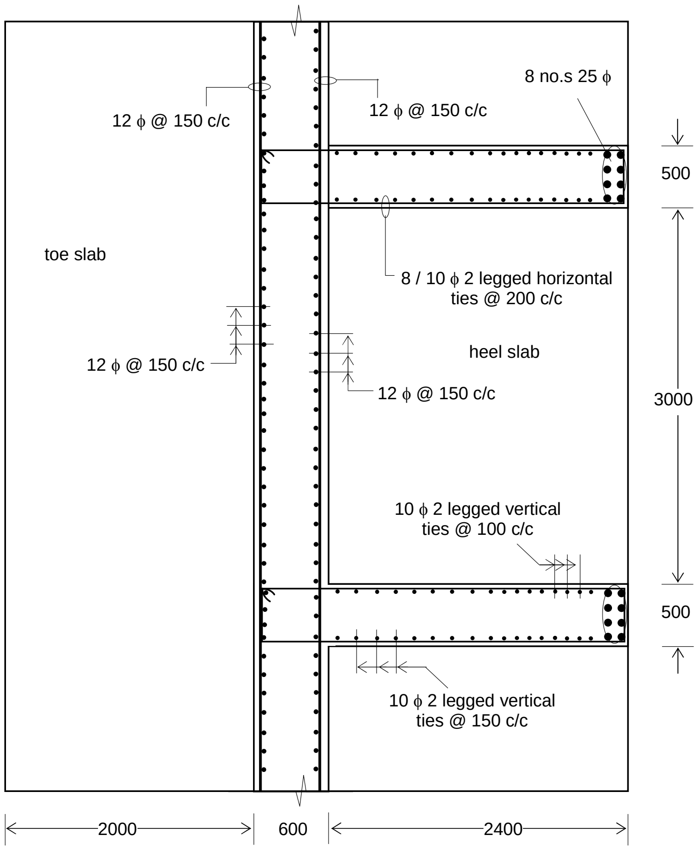



**Fig. 14.38** Reinforcement details of stem and counterfort

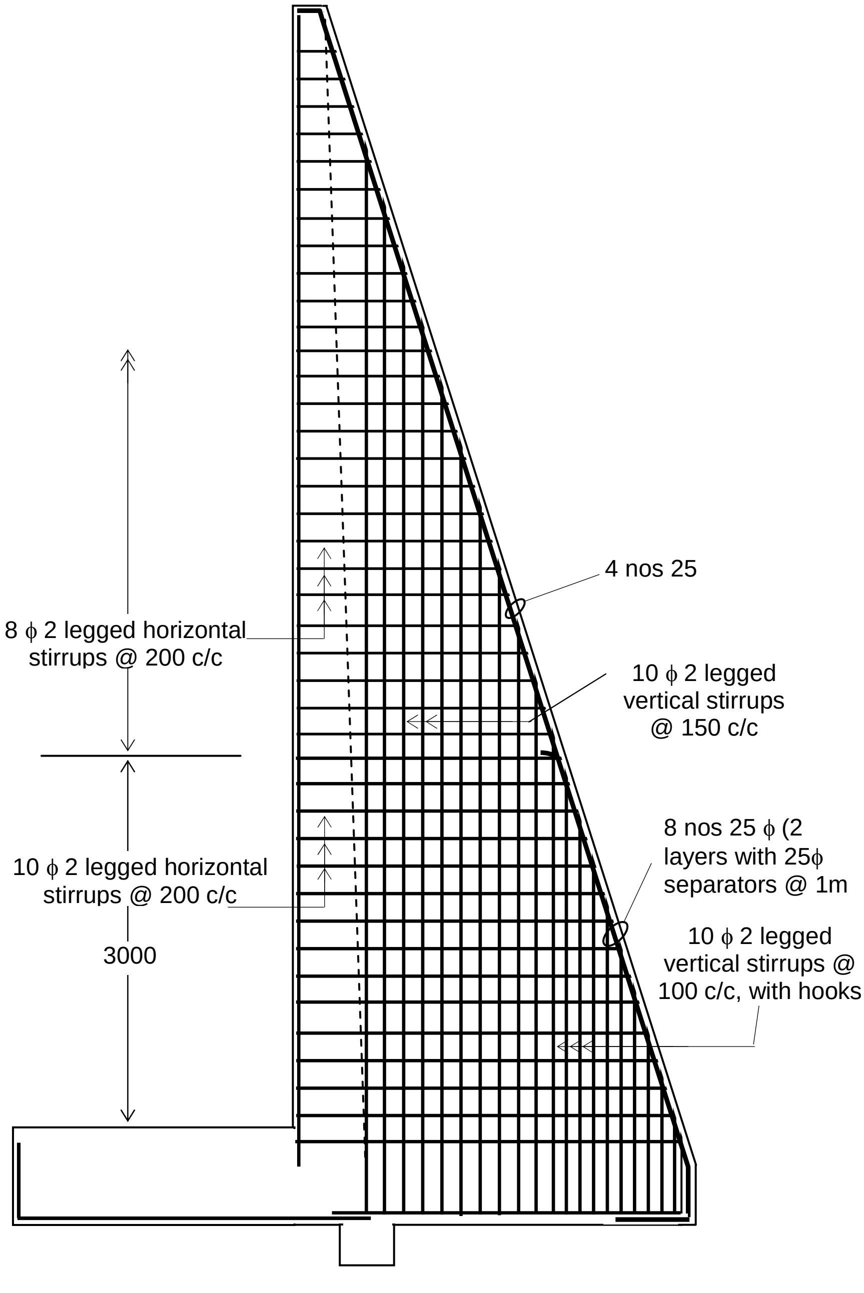



**Fig. 14.39** Section through counterfort showing counterfort reinforcement

## REVIEW QUESTIONS

14.1 What are the main requirements of a foundation system for a structure?

14.2 Why is it necessary to ensure, by proper proportioning of footings, that the bearing pressures underlying all the footings in a building are more-or-less of the same order of magnitude?

14.3 What are the situations in which *combined footings* are preferred to *isolated footings*?

14.4 Distinguish among the terms (i) *allowable* soil pressure (ii) *gross* soil pressure (iii) *net* soil pressure, (iv) *factored* soil pressure.

14.5 What is meant by *eccentric loading* on a footing, and under what circumstances does this occur?

14.6 Why is it desirable to eliminate eccentricity in loading on a footing, wherever possible, by means of proper proportioning?

14.7 From structural analyses, it is found that the following stress resultants develop at a column base under the action of *characteristic* loads:

(i) $P$ = 475 kN, $M$ = 35 kNm under *dead* loads;

(ii) $P$ = 380 kN, $M$ = 39 kNm under *live* loads;

(iii) H = ± 30 kN, $P$ = ±12 kN, $M$ = ± 41 kNm under *wind* loads.

Determine the combined loads to be considered in deciding the area of the footing to be located in a soil with an allowable soil pressure of 200 kN/m$^2$ at a depth of 1.5 m below ground level.

14.8 What are the advantages of providing pedestals to columns?

14.9 Briefly explain the conditions in which transfer of forces at the interface of column (or pedestal) and footing can be achieved without the aid of reinforcement.

14.10 Under what circumstances is a trapezoidal shape preferred to a rectangular shape for a two-column combined footing?

14.11 Describe briefly the load transfer mechanism in a two-column combined footing.

14.12 What is the purpose of a *retaining wall?* What are the different types of concrete retaining walls?

14.13 Distinguish between *active pressure* and *passive pressure* of earth, in relation to retaining wall structures?

14.14 What is meant by (a) *surcharge* (b) *inclined surcharge*?

14.15 Describe the effect of water in the backfill on the active earth pressure on a retaining wall.

14.16 What is the purpose of a *shear key?* Describe its action.

14.17 Briefly describe the behaviour of the various elements of a *cantilever* retaining wall.

14.18 Briefly describe the behaviour of the various elements of a *counterfort* retaining wall.

14.19 Where are the critical sections for shear located in the case of (a) the *toe slab* (b) the *heel slab* in the design of the base slab of a cantilever retaining wall?

## PROBLEMS

14.1 Design a plain concrete footing for a column, 400 mm × 400 mm, carrying an axial (service) load of 400 kN. Assume an allowable soil pressure of 350 kN/m$^2$ at a depth of 1.0 m below ground. Assume M 20 concrete and Fe 415 steel.

14.2 Design a square footing for a rectangular column 300 mm × 500 mm, reinforced with 6–25 ϕ bars, and carrying a service load of 1250 kN. Assume soil with an allowable pressure of 200 kN/m$^2$ at a depth of 1.25 m below ground. Assume Fe 415 grade steel for both column and footing, and M 20 grade concrete for the footing and M 25 grade concrete for the column.

14.3 Repeat Problem 14.2, considering a uniaxial moment (with respect to the major axis of the column) of 100 kNm (under service loads — dead plus live) in addition to the axial force of 1250 kN at the column base. Assume a suitable rectangular footing. Also assume that the moment is irreversible.

14.4 Design a square footing for a circular column, 500 mm in diameter, reinforced with 8–25 ϕ bars, and carrying an axial load of 2500 kN. Assume soil with a safe bearing capacity of 300 kN/m$^2$ at a depth of 1.5 m below ground. Assume Fe 415 grade steel for both column and footing, and M 20 grade concrete for the footing and M 30 grade concrete for the column.

14.5 Repeat Problem 14.4, considering a rectangular footing with a spatial restriction of 2.5 m on one of the plan dimensions.

14.6 Design a footing for a 250 mm thick reinforced concrete wall which supports a load (inclusive of self-weight) of 250 kN/m under service loads. Assume a safe soil bearing capacity of 180 kN/m$^2$ at a depth of 1 m below ground. Assume M 20 grade concrete and Fe 415 grade steel for both wall and footing. Assume the longitudinal reinforcement of the wall to comprise 0.25 percent of the gross cross-sectional area.

14.7 Repeat Problem 14.6, considering the wall to be made of masonry (instead of reinforced concrete).

14.8 Repeat Problem 14.6, considering a bending moment of 30 kNm/m (reversible) at the base of the wall, in addition to the axial load of 250 kN/m, under service loads.

14.9 Repeat the design of the two-column combined footing of Example 14.7, considering the property line to be located 500 mm away from the centre of column C1.

14.10 Repeat the design of the two-column combined footing of Example 14.7, considering a beam-slab footing, and assuming that the allowable soil pressure is 180 kN/$m^2$ (instead of 240 kN/$m^2$).

14.11 Design and detail the stem and base slab of the cantilever retaining wall of Example 14.8.

14.12 Design a cantilever wall to retain earth with a backfill sloped at $20^o$ to the horizontal. The top of the wall is 5.5 m above the ground level, and the foundation depth may be taken as 1.2 m below ground level, with a safe bearing capacity of 120 kN/$m^2$. Assume that the backfill has a unit weight of 17 kN/$m^2$ and an angle of shearing resistance of $35^o$. Further, assume a coefficient of friction between soil and concrete, $\mu = 0.55$. Use M 20 concrete and Fe 415 steel.

14.13 Repeat Problem 14.12, considering the backfill to be level, with a surcharge, equivalent to an additional 2.52 m of the backfill.

14.14 Suggest suitable proportions for a counterfort retaining wall to support difference in ground elevation of 9 m. The foundation depth may be taken as 1.5 m below ground level, with a safe bearing capacity of 160 kN/$m^2$. Assume a level backfill with a unit weight of 16 kN/$m^3$ and an angle of shearing resistance of $30^o$. Assume a coefficient of friction, $\mu = 0.5$, between soil and concrete. Check the stability of the wall.

14.15 Design and detail the various elements of the counterfort wall structure of Problem 14.14.

## REFERENCES

14.1 — *Explanatory Handbook on Indian Standard Code of Practice for Plain and Reinforced Concrete (IS 456:1978)*, Special Publication SP:24, Bureau of Indian Standards, New Delhi, 1983.

14.2 Bowles, J.E., *Foundations Analysis and Design*, Third edition., McGraw-Hill Book Co., New York, 1982.

14.3 Peck, R.R., Hanson, W.E., and Thornburn, T.H., *Foundation Engineering*, Second edition, John Wiley & Sons Inc., 1974.

14.4 — *Code of Practice for Structural Safety of Buildings*: *Shallow Foundations*, IS:1904 (third revision), Bureau of Indian Standards, New Delhi, 1986.

14.5 — *Code of Practice for Design and Construction of Simple Spread Foundations*, IS:1080 (First revision), Bureau of Indian Standards, New Delhi, 1980.

14.6 ACI Committee 336, *Suggested Design Procedures for Combined Footings and Mats*, Journal ACI, Vol. 63, No. 10, Oct. 1966, pp 1041–1057.

14.7 Kramisch, F. and Roberts, P., *Simplified Design of Combined Footings*, ASCE Journal, Soil Mechanics Div., Vol. 87, No.SM5, October 1961, pp 19–44.

14.8 Huntington, W.C., *Earth Pressures and Retaining Walls*, John Wiley, New York, 1968.

14.9 Fisher, G.P. and Mains, R.M., *Sliding Stability of Retaining Walls*, Civil Engineering, July 1952, pp 490.

14.10 Wang, C–K. and Salmon, C.G., *Reinforced Concrete Design*, Fourth edition, Harper & Row, New York, 1985.

14.11 — *Code of Practice for Concrete Structures for the Storage of Liquids*, Part 4: *Design Tables*, IS:3370 (Part 4) 904 (Third revision), Bureau of Indian Standards, New Delhi, 1967.

# 15

# Good Detailing and Construction Practices

## 15.1 INTRODUCTION

The objective of structural design and construction is to build *safe*, *serviceable*, *economical*, *durable* and *aesthetic* structures. Analysis and design, together, comprise only one of the phases in the process of a building construction. The elaborate computations involved in this phase become worthwhile only if the design is translated into a correspondingly high quality structure. This necessitates good detailing and construction practices.

In Chapter 3, it was explained that the primary aim of design by the Limit States Method (LSM) is to minimise the probability of failure to an acceptable low value. In this context, *failure* is defined as the attainment of a *limit state*. *Limit states* imply those conditions whereby a structure ceases to fulfil the functions for which it has been designed. The limit states include both *ultimate* limit states and *serviceability* limit states. Thus, the term, *failure*, in general, includes both *ultimate failure* — local or overall — (exceeding the load carrying capacity, instability and buckling, overturning, sliding, fatigue and fracture, and progressive type of collapse) under factored loads, and *serviceability failure* (unacceptable deflections, vibrations, cracking, inadequate durability, permanent deformation, leakage, wetting, spalling of concrete, etc.) under service loads.

It is rarely that buildings fail in a manner that can be classified as an ultimate limit state failure (collapse). On the other hand, it is much too common for comfort that buildings (especially those of more recent construction) perform unsatisfactorily in their day-to-day normal service; i.e., fail to meet serviceability criteria.

In the fifties and sixties, reinforced concrete buildings used to be designed by the working stress method (WSM) with relatively low permissible stresses (for example, 5 MPa for 1:2:4 nominal mix concrete and 140 MPa for mild steel). Moreover, many analysis and design methods used in those days employed approximations which 'erred on the safe side'. Modern structures are designed with higher strength materials, for higher stresses, and by the Limit States Method (including allowances for inelasticity and moment redistribution) with 'partial safety factors' lower than the 'factors of safety' inherent in the WSM as practised in the early sixties. Furthermore, the methods of analysis and design have become more sophisticated and accurate, and the conservatism in-built in approximate methods is no longer available. As a result, these modern structures are comparatively taller and have longer spans, more slender members and thinner slabs and walls, and are built at a faster pace. They are therefore, more flexible (in terms of deflections) and are more 'crack prone', as compared with the old structures, which used to be low in height, had thicker (stockier) members, were lightly stressed and were built at a slow pace. Thus, *the serviceability criteria assume far greater importance in modern structures.* It is in this context that this chapter is included in this book. It is meant to draw the attention of engineers involved in all the stages of planning, design, detailing, fabrication and construction to these important aspects related to the performance of buildings.

Detailing practices, construction practices, quality control in construction, building failures, causes and prevention of cracks/leakage in buildings, etc. are all large enough topics to write separate books and/or publish journals on each of them, as indeed have been done (Ref. 15.1 to 15.8). Nor are all these topics strictly within the scope of this book. As such, an attempt is made here only to draw attention to some of the major and most common causes of failure pertaining to the design (and construction) of reinforced concrete buildings. These are by no means exhaustive. For a more comprehensive coverage, reference may be made to Ref. 15.1–15.11.

In this context it is worthwhile to note that, in most cases, the cost of the structure itself forms only a small part of the total cost of a project. Further, the cost of the concrete and reinforcement forms only a fraction of the cost of the structure. Hence, the designer would do well to remember that:

> aiming for minimum quantities of concrete and reinforcement (based on strength criteria alone) may not lead to significant cost reduction, and may well result in unforeseen long-term cost or failure related to serviceability criteria.

### 15.1.1 Serviceability Failures

The commonly observed shortcomings in the context of serviceability requirements are:

- leakage from roofs and floors, particularly at construction / expansion joints, junctions, etc.;
- wetting of ceilings, walls (especially around toilet areas), leading to dampness, discoloration, growth of algae and moss on surfaces, growth of vegetation in fissures in walls and around drain pipes, sunshades, ledges, etc.
- cracking in slabs, beams, walls, etc.;
- poor drainage and ponding of water on roof slabs, sunshades, bathrooms, open staircase steps, etc.;
- corrosion of reinforcement and spalling of concrete;
- excessive deflections of slabs, beams, etc.

Many of the above effects are interactive. For example, large deflections of roof slabs can lead to ponding of rain water as well as cracking on the top side in negative moment regions, which in turn can lead to wetting and leakage, and also corrosion of reinforcement.

### 15.1.2 Reasons for Building Failures

There are many causes that could lead to the failure (*ultimate* and/or *serviceability*) of a structure. Some of these, which must be of concern to the design and construction engineers are listed below:

| ***Failure during construction or soon after*** | ***Failure a long time after construction*** |
|---|---|
| • Deficiency in design / shift from actual design; | • Failure of a primary load carrying member by accident; |
| • Poor detailing; | • Change in use (change of structural arrangement) or overloading. |
| • Poor quality materials; | • Unforeseen disasters like severe earthquake, bomb blast, etc. |
| • Poor quality construction. | • Deterioration arising out of poor quality materials, construction and/or lack of repair and maintenance. |
| • Poor formwork/scaffolding | • Exposure to adverse environment, not considered in original design |

### 15.1.3 Structural Integrity

Some causes for failures long after completion of construction are identified in the above section. Most of these causes such as accidents, overloading and disasters are not directly related to either the design or the construction. However, a related design consideration is the need for the structure to have *structural integrity*. A structure is said to have structural integrity if it is able to withstand localised damage or failure of a structural member, caused by any unforeseen or abnormal events (that may *reasonably be expected*) without spread of damage or collapse to a large part of the structure. In other words, the failure of one element should not lead to a *progressive collapse* or *incremental collapse* of the rest of the structure.

A typical example of a progressive type of collapse is the failure of a flat plate structure originating from the punching shear failure at one slab-column connection (in the absence of special reinforcement for structural integrity at such connections; refer Section 11.7, Fig. 11.40). Punching shear failure at one column can lead to increased shear and moment at an adjacent column connection, causing it to fail. Such progressive failure of slab-column joints may lead to the slab falling on to the slab below, causing it to fail as well, and so on vertically down the building. Examples of such progressive collapse are reviewed in Refs. 15.12 and 15.13.

In design, consideration should be given to the integrity of the overall structural system to minimise the likelihood of progressive collapse. This involves a careful selection of the structural system, understanding its behaviour under load and possible failure modes and ensuring a robust and stable design with sufficient *redundancy* and *alternative load paths*. Most continuously reinforced cast-in-place concrete structures designed and detailed in accordance with codes will generally possess a satisfactory level of structural integrity. Special provisions for structural integrity may be required for two-way flat slabs at slab-column connections, precast concrete structures, unusual structural systems, and structures exposed to severe loads such as vehicle impact, fire accident or explosion.

## 15.2 DESIGN AND DETAILING PRACTICES

It is very rare that a building fails as a result of a major design flaw. The design codes are fairly conservative, if not up to date, as far as reinforced concrete design is concerned. The engineering curriculum is also reasonably up to date in this regard. Moreover, highly sophisticated and accurate softwares are now available for computer aided analysis, and also for design and drafting. However, in using these softwares, a word of caution is appropriate. The outputs of these programs are only as good as the inputs are! Hence they should only be used by persons who have a full understanding of what the program does, as well as of the properties of materials, structural behaviour, failure modes, structural system employed, overall deformation patterns, compatibility conditions, load transmission paths to supports, and possible weak links, if any. Furthermore, it should be possible to identify the critical and primary load carrying elements, and to perform an approximate manual check on the stress resultants and design resistance of such members, in order to avoid gross errors. For example, the use of appropriate moment coefficients applied to a simplified substitute frame will yield a rough estimate of moments [refer Chapter 9]. Similarly, an approximate estimation of the ultimate moment of resistance of a beam/slab section can be obtained as $(0.87\, f_y\, A_{st})(0.9d)$.

As mentioned earlier, the analysis of structures for stress resultants and the design of individual elements (critical sections of slabs, beams and columns) for maximum load effects (bending moment, shear and torsion, and axial force) are done, in general, fairly competently. However, the attention given to the combining of these elements together to form the whole structure is generally found wanting both in the engineering curriculum and in many design offices. This includes the attention given to such important details as: termination, extending and bending of bars; anchorage and development; stirrup anchorage; splices; construction details at connections (slab-beam, beam-column, rigid frame corners, etc.); provision of continuity/discontinuity at junctions of members; construction sequencing and reinforcement placement to suit; deflection calculations (including long-term deflections) and control; crack control; special cases such as upturned (inverted) beams, edge and spandrel beams, cantilevered members; cover, bar support and reinforcement protection; durability; recognition of and allowance for long-term effects of creep, shrinkage and temperature; details of control/construction/expansion joints; structures needing special procedures (tanks, chimneys, etc.).

The practical work in concrete Laboratory Courses in most universities is also found wanting. Experiments and assignments are mostly limited to the standard tests on cement, aggregate and hardened concrete. Applied problems aimed at the understanding of concrete technology (influence of various parameters, and concrete mix design) and fabrication and testing of reinforced concrete elements (design-fabricate-cast-cure-test-analyse type assignments) are seldom included in the laboratory courses.

### 15.2.1 Reinforcement Layout

Heavy live loads and lack of regularity in framing (widely differing adjoining spans, variation in column sizes and spacings, fluctuations in relative stiffnesses, etc.) can move the zone of contraflexure considerably, thereby affecting the termination, extension and bending of bars considerably. Such lack of regularity could necessitate continuing of top bars over the full length of a short span located between two long ones (common in school buildings, hotels, etc. with narrow central corridors), or may make it impracticable to bend any bar at all, or conversely, may make it possible to bend up considerably more than half the bottom bars in a heavy girder, anchoring and terminating them in a compression zone, or may necessitate provision of stirrups for the full length of a member. Admittedly, considerably improved design skill is required in delineating the reinforcement throughout the length of members than in selecting top and bottom bars for maximum moments.

Concrete being weak in tension, reinforcement is provided to take care of all tensions envisioned by the designer, whether direct/flexural (main reinforcement) or diagonal (stirrups). In addition, suitable reinforcement must be provided across any potential crack. In particular, attention should be paid to locations at which tensile stresses, not ordinarily calculated, exist — such as due to shrinkage, settlement, temperature and stress concentration effects. Cracking due to thermal and shrinkage movements can be reduced by making provision in the design and construction of structures for unrestrained movement of parts, wherever feasible, by introducing movement joints (expansion joints, control joints and slip joints). Where provision of movement joints is not structurally feasible (as in rigid frames, shell roofs), thermal stresses have to be taken into account at the structural design itself. Even where joints for movement are provided, some amount of restraint to movement due to bond and friction is unavoidable. In cases where the design reinforcement is only in one direction (as in one-way slabs, cantilevered slabs, etc.) cracks could develop across the perpendicular direction due to contraction and shrinkage in that direction, and it is necessary to provide some reinforcement (variously called as ‘distributor reinforcement’ or ‘temperature reinforcement’) in the direction perpendicular to the main reinforcement. In case of members exposed to the sun (sunshades, fins, canopies, balconies, roof slab without adequate insulation cover, etc.), the “minimum” reinforcement specified by the Code should be increased by 50 to 100 percent, depending upon the severity of exposure, size of member and local conditions [Ref. 15.5]. Reference may also be made to Section 5.2.2, with regard to the need for *side face reinforcement* to control cracking in large unreinforced exposed faces of concrete members.

### 15.2.2 Design Drawings

Much time and expense can be saved, and costly mistakes avoided, if simple, clear and complete drawings are prepared. More than serving as a graphic delineation of a structure, a drawing acts as a definite order to workmen to perform certain operations in a specified manner. The drawing also serves as a record of some of the important assumptions made in the design (which, for example, can reveal whether or not future expansion of the structure is possible at a later date). Engineering drawings prepared by the designer should specify grades of concrete and steel, live load, dimensions, reinforcement, lap lengths, concrete cover, and all other information needed for detailing the reinforcement, building the forms, placing the reinforcement and placing the concrete. The designer has full data on the assumptions made, the computations, moment diagrams and the whole philosophy of structural design, and it is his responsibility to define the design requirements by way of anchorages, laps, bends, splices and similar details. Indeed, it is only he who can supply such information to the detailers, fabricators and construction personnel. Hence, design drawings must be complete to the extent that every bit of information regarding the size and arrangement of concrete members, and the size, positioning and detailing of reinforcing bars is completely covered either by a drawing, description, diagram, note, rule, or reference to a standard manual.

Notes and statements should be clear and unambiguous. A note “16 bars 2 ways 2 faces” is highly ambiguous, as it could mean 16 bars each way each face (total 64) or 4 bars each way each face (total 16), or indeed almost anything in between. Instead, the note should have been made explicit, giving the number *each way* in *each face*. Similarly, descriptions are best given in the imperative: “Do this”, “Bend these bars”, rather than “This may be done”, or “These bars may be bent”. Graphical representations (true-scale elevations, sections, etc.) are preferable to complicated notes and descriptions, to show precisely what is wanted; they are also, to a considerable extent, self-checking.

While, with uniform loads and equal spans, it may be satisfactory to bend bars at the quarter-point and to extend them to the $3/10^{th}$ point as in Fig. 5.5 (following standard practices/manuals for such cases), this is not safe as a general practice for all cases. Hence, the drawing should make it clear, by way of a separate section, indicating the departure from standard practice.

### 15.2.3 Construction Details at Connections and Special Situations

Frequently, locations and members needing special detailing considerations are encountered. These include locations of abrupt changes in section size and other sudden discontinuities, edge beams, inverted beams, odd-shaped/sized members, connections, etc. A few such cases are described below. In many such instances, a useful procedure for design and/or detailing is to adopt the *strut-and-tie model* [see Section 17.2].

### *Offset Columns*

When a column in a particular storey is smaller than the one below, some of the vertical bars from below may have to be offset to come within the column above, or dowel splices must be used [Fig. 15.1(a)].

The slope of the inclined portion should not exceed 1 in 6. Where column verticals are offset, additional ties shall be provided and placed close to the point of bend in order to carry the transverse force generated due to the change of direction at the bend. When the offset between column faces exceeds 75 mm, the vertical bars in the column below shall be terminated at the floor slab, and splicing of column bars by dowels may be necessary [Fig. 15.1(b)]. Dowels may also be necessary when the placing of part of the structure is delayed, and also between various units of structures (such as footings and columns). Dowels should, in general, be of the same size and grade as the bars joined, and should be of sufficient length to splice with the main bars.

When column bars are spliced, additional ties shall be provided at and near the ends of spliced bars, to provide confinement to the highly stressed concrete in the regions of the bar ends.

### *Members with a Break in Direction*

Whenever there is a change of direction in a main reinforcing bar, a resultant radial force is generated at the location of the kink, as shown in Fig. 15.2(a). If the radial force acts outwards, as is the case in Fig. 15.2(a), this force tends to push out the cover concrete causing splitting. Moreover, as a straight length (such as AC) is shorter than the bent length (ABC) of the bar, the spalling will lead to a relieving of the bar stress resulting in lowering of the resistance of the section, and possible failure. When the angular change is small (say < $15^{o}$) the radial force resultant ($R$) is small and can be carried and transferred to the compression zone by providing adequate number of stirrups at the location of the kink and on either side at close spacing as shown [Fig. 15.2(b)].

When the angular change is larger, the reinforcement from either side should be continued straight and anchored to develop the full design stress [see Fig. 15.3]. Note that in Fig. 15.3 the bar A, which cannot get a straight length of $L_d$ beyond the location of the kink in the beam face is continued on the compression face and anchored there, so that no outward splitting force is developed due to the bend in this bar. Examples are junction of stairs and landing[‡] , inside corners of rigid frames [Fig. 15.4(a)] and where the soffit of beam forms an angle as in a gable bent [Fig. 15.4(b)].

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[‡] This detail is indicated in Fig. 12.4(a) of Chapter 12.

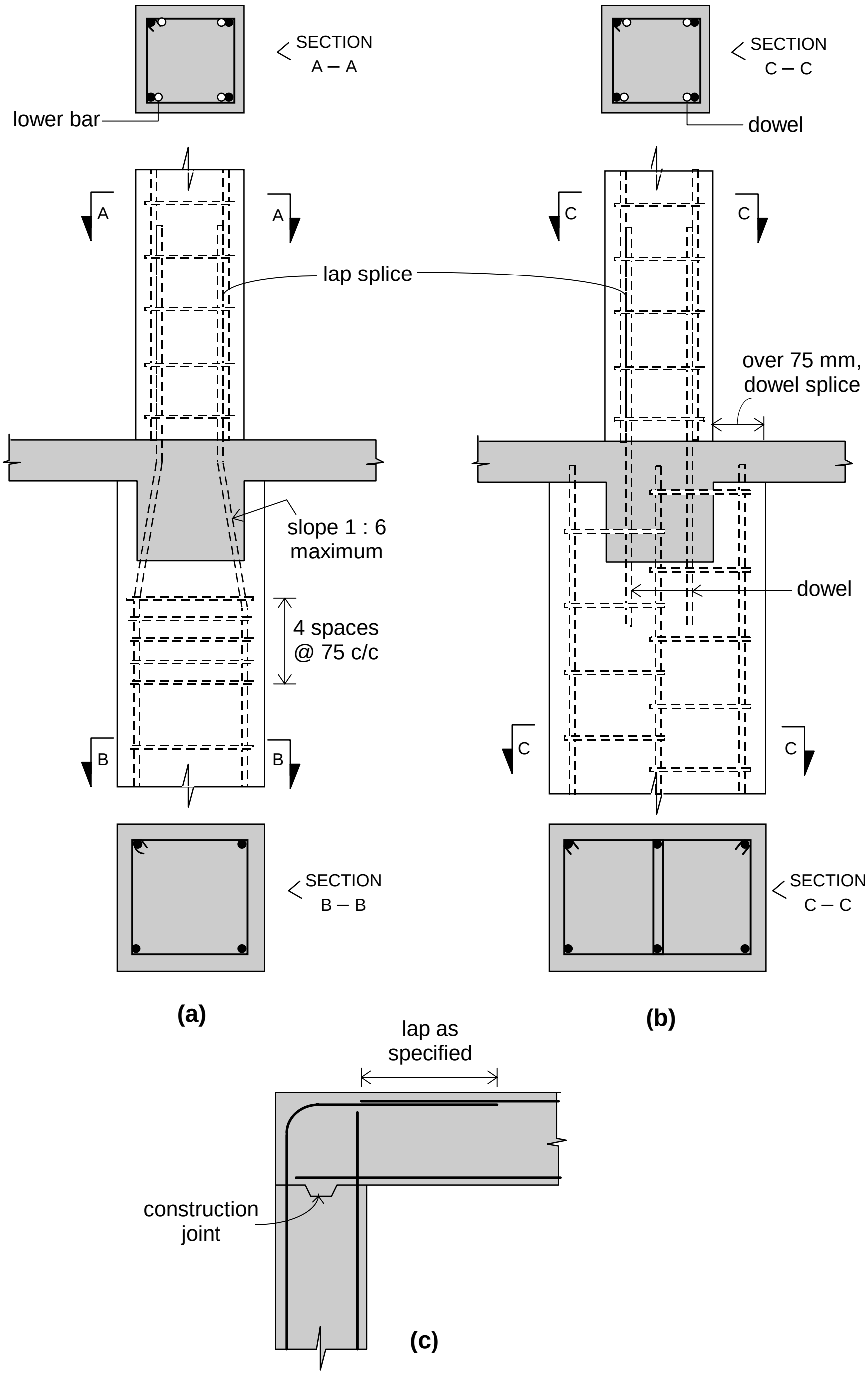



**Fig. 15.1** Some construction details at connections

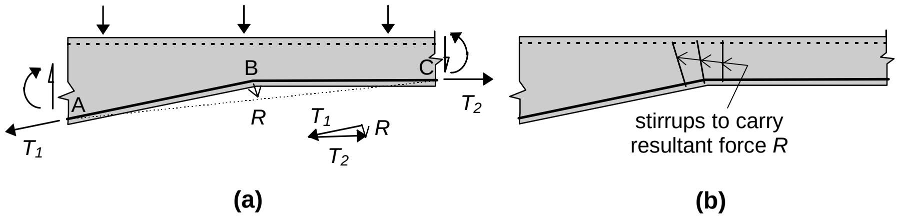



**Fig. 15.2** Member with a change in direction in flexural stresses (small angle)

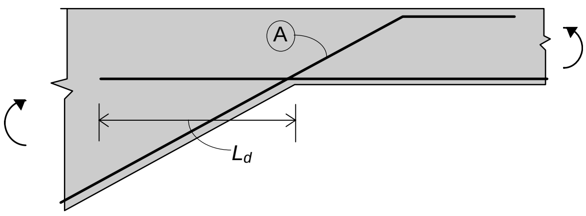



**Fig. 15.3** Large directional change in flexural stresses

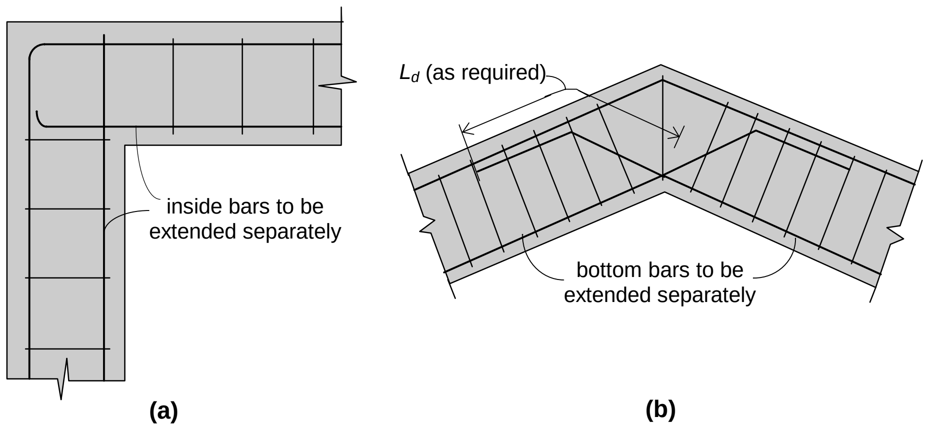
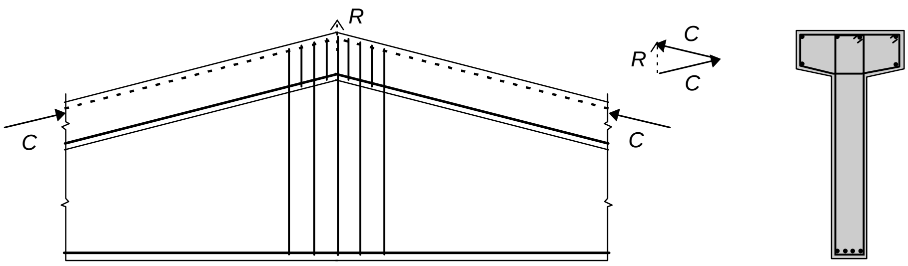



**Fig. 15.4** Reentrant corners with tension bars

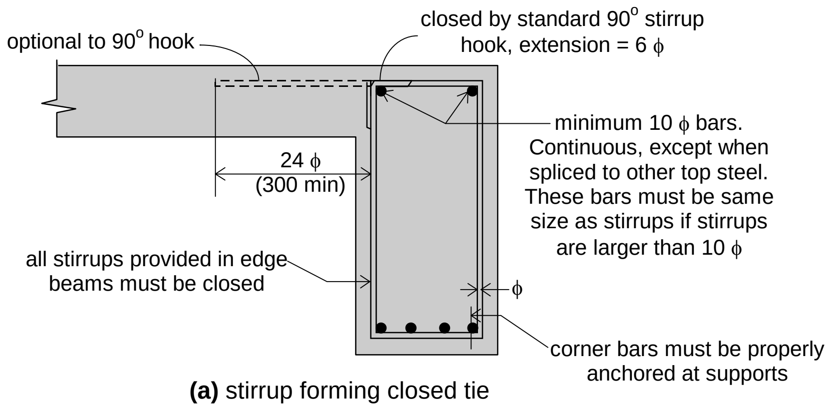
A similar situation exists when the internal compression force changes direction in such a way that the resultant force acts outward [Fig. 15.5(a)]. In the example shown, a breaking away of the flange can be prevented by transverse reinforcement tying the flange to the web of the beam [Fig. 15.5(b)].

Construction and bar placing details of the corner connection of a rigid frame are shown in Fig 15.1(c). In detailing such connections, care must be taken particularly in providing full continuity around as large a uniform radius as possible in splicing the top bars from the girder to the outside bars in the column. Rigid corner connections of beams to columns often require closed stirrups or ties around the bend [see also Section 15.2.4 on Rigid Frame Joints]. The designer must provide complete information showing the radius of bend, location and dimensions of the lap splices (or other type of splices) used and stirrup details.

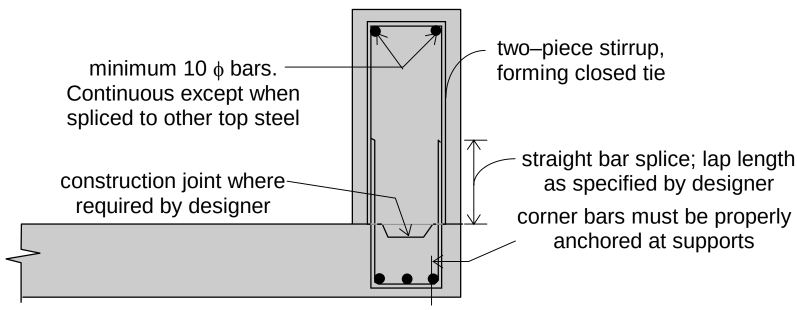



**(a)** **(b)**

**Fig. 15.5** Change in direction of compressive stresses




**(a)** stirrup forming closed tie




**(b)** two-piece stirrup forming closed tie

**Fig. 15.6** Typical edge and spandrel beam details

### *Edge Beams*

In edge and spandrel beams, stirrups must be of the closed type and at least one longitudinal bar should be located at each corner of the beam section. Typical details are shown, for normal and inverted edge/spandrel beams in Fig. 15.6(a) and (b) respectively. For easier placing of the longitudinal bars in an inverted beam, two-piece closed stirrups can also be used as shown in Fig. 15.6(b).

### *Corners of Walls*

In concrete walls, horizontal reinforcement may be required to resist moment, shear or temperature and shrinkage effects. All such bars in both faces of wall must be sufficiently extended past a corner or intersection to satisfy development requirements. Typical details are shown in Fig. 15.7 for resistance against moment (inward and outward), with the reinforcement from the appropriate faces anchored.

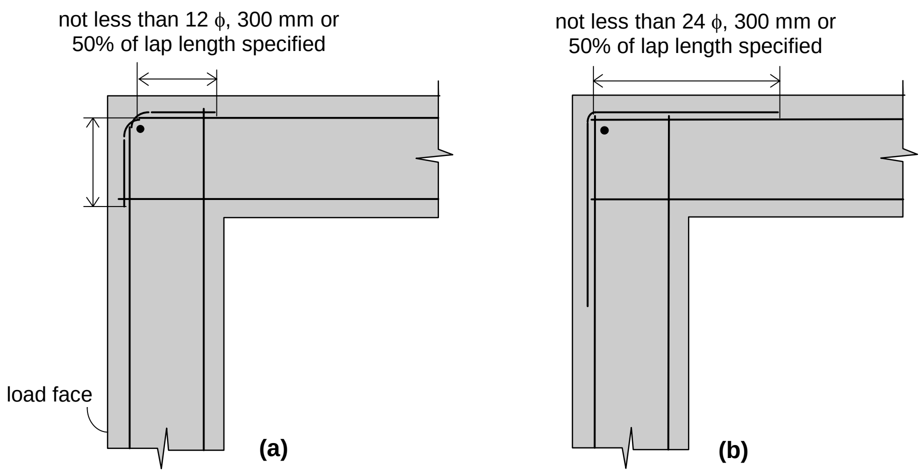



**Fig. 15.7** Typical corner details in walls

### *Special Conditions*

For special or unusual conditions, adequate details should be shown for proper placing of reinforcement, as the average steel setter cannot be expected to understand engineering principles. Examples are cantilevers and continuous footings, in which the reinforcement is in the opposite side from the one to which the steel setter is accustomed.

There are situations where the embedment length available for end anchorage of bars is insufficient to develop the design stress in the bar through bond. Examples include corbels, deep beams, small size footings, precast beams, etc. In such cases, special devices such as welded cross bars, end plates [Fig. 15.8] must be provided.

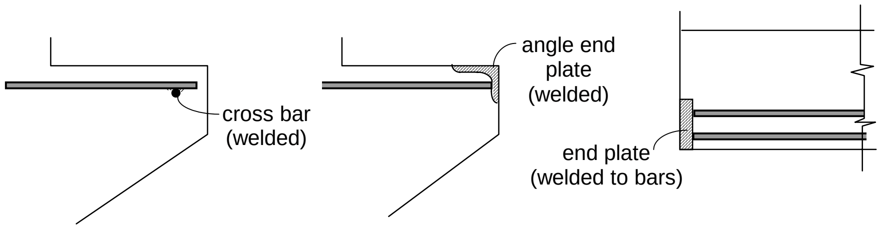



**Fig. 15.8** Special anchorage devices

### *Intersection of Members*

Congestion of steel should be avoided at locations where members intersect, such as intersection of (secondary) beams with girder (primary beam) and girders with column. In the interior beam-column joint, generally there is overcrowding of the negative (top) reinforcement in the beam if they are all placed within the beam width [Fig. 15.9(a)]. This usually interferes with proper placing and compaction of the concrete at the joint. The bond developed in these top reinforcement also is relatively inferior. The spreading of the top reinforcement into the adjoining slab, preferably using smaller diameter bars, [Fig. 15.9(b)] has been shown to reduce the crack-widths in these beams considerably [Ref. 15.12]. This has the added benefit that the effective depth is slightly increased and the placement and compaction of concrete is facilitated better.

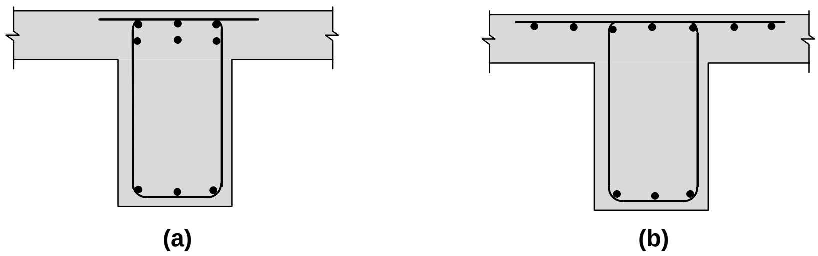



**Fig. 15.9** Negative moment reinforcement at beam-column joint

At the intersection of a beam and girder, the beam bars should be placed at a different elevation than those in the girder so as to avoid interference. The relative positions of the bars must be in accordance with the load transfer order assumed in design. Thus the beam reinforcement must come over the girder reinforcement at the intersection [Fig. 15.10]. In addition, adequate hanging up bars (suspender stirrups) should be provided in both members in the joint zone as given in Section 6.10 [see also Fig. 6.15]. Similarly, at slab-beam junctions, the bottom and top bars of the slab must be draped over the bottom and top bars in beam respectively. When slabs frame flush with the bottom of inverted beams or hanger walls, special stirrup hanger reinforcement shall be provided [see also Section 6.5 and Figs 6.7 and 15.4].

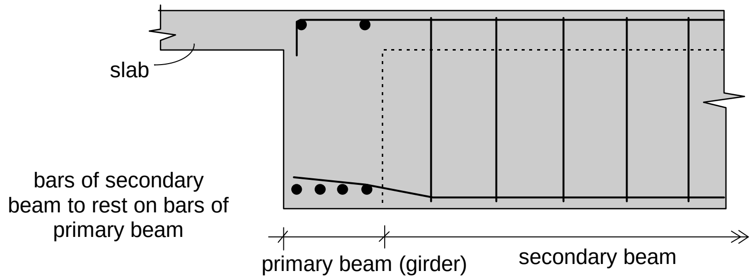



**Fig. 15.10** Bars of secondary beam to be placed over bars of primary beam, with suspender stirrups enclosing primary beam bars

### 15.2.4 Beam and Column Joints (Rigid Frame Joints)

Joints of beams and columns in rigid-jointed frames are critical locations requiring careful design and detailing. Such joints should have adequate strength to enable the development, at the joint face, of the full design strengths (and plastic hinges with adequate ductility, if required by design) in the members framing into each joint under the most adverse loading pattern, without distress in the joint itself. This type of rigid connection occurs in rigid jointed multistorey frames, portal frames, box culverts, at the base of cantilever retaining walls, etc.

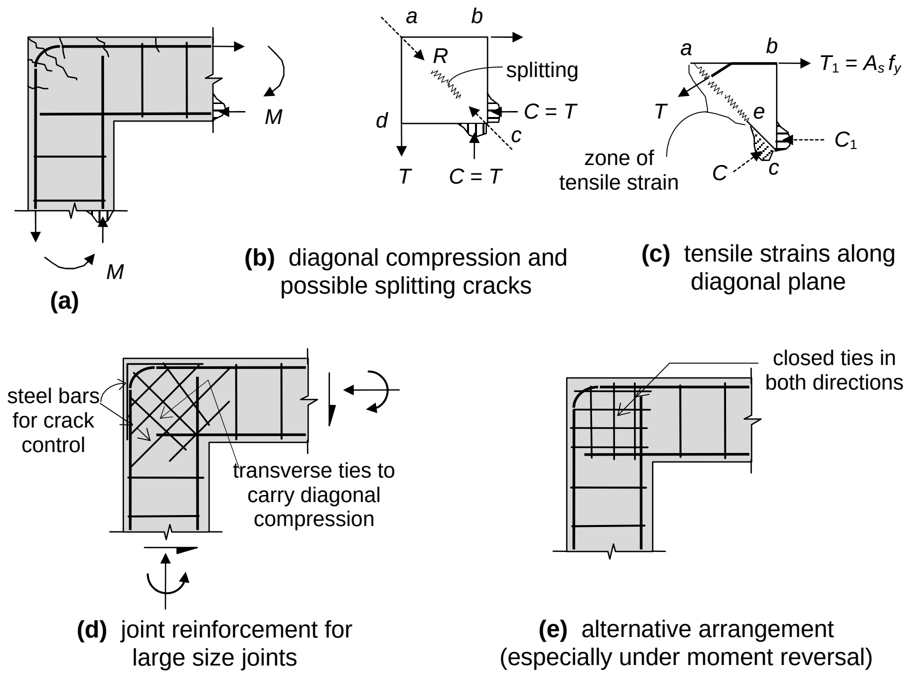



**Fig. 15.11** Knee joint subjected to ‘closing’ moment

A simple example of a rigid beam-column joint, for which the design and detailing considerations can be explained with relative ease, is the corner joint of a portal frame, usually called a "knee joint" [Fig. 15.11]. The joint, to be rigid, must have full continuity between the two members. The main flexural reinforcement in the joint zone undergoes a change of direction, and as a result transverse forces are developed, as in the cases considered in Fig. 15.11. There can be three types of stress patterns to which the joint zone is subjected to, depending on the nature of loading on the structure itself, namely:

(a) where the moment at the joint tends to 'close' the knee i.e., hogging moment causing tension in the outer fibres [Fig. 15.11],

(b) where the moment tends to "open" the knee [Fig. 15.12] and

(c) where the moment is subject to reversal as in the case of seismic loading.

The nature of induced forces in the joint zone in case (a) is shown in Fig. 15.11(b) and (c). The diagonal resultant thrust tends to develop splitting cracks along the diagonal *ac*. A significant portion (ae) of the diagonal *ac* will be under tensile stress and liable to crack, as shown in Fig. 15.11(c). It may be noted that the joint is usually subjected to axial forces and shearing forces, in addition to the bending moment. For satisfactory performance in this case, the outer tension bars should be continuous around the corner. The inner bars are in compression; however, the concrete alone may be adequate to carry the compressive forces here. These bars are better continued straight, as shown, rather than being made continuous by bending around near the re-entrant corner. In case these bars are also accounted as contributing part of the required compressive force, they should be continued straight and anchored to develop the design stress at the joint faces along corner *c*. Furthermore, the diagonal compression along *ac* and the possible diagonal cracking along *ae* should be countered. When the members are of small size (as in the case of slab-wall joint or the corner of a small size lightly reinforced and thin walled box culvert), no special provisions may be needed to carry the diagonal compression and tension along diagonals *ac* and *bd*. However, in large size or heavier reinforced members, the diagonal compression may be resisted, and the diagonal cracking controlled, by secondary reinforcements placed along diagonal directions as shown in Fig. 15.11(d). An alternative arrangement is to place these reinforcement orthogonally as in Fig. 15.11(e). This arrangement is particularly suited for cases of moment reversals.

The case of a knee joint subjected to 'opening' moment is shown in Fig. 15.12. In general this loading case (moment tending to open the knee joint) is more critical than the case of moment tending to 'close' the knee joint [case (a) discussed in Fig.15.11]. The nature of stresses in the joint and potential crack locations are shown in Fig. 15.12(b) and (c). The need to provide reinforcements along diagonal *ac* to carry the resultant tension in this direction and parallel to diagonal *bd* closer to the interior corner *c*, to control flexural cracking are self evident. The concrete near the corner *a* is stress-free and is likely to spall off, being pushed out by the resultant thrust along *ca*, and nominal reinforcement is required to control such cracking.

Suggested reinforcing details [Ref. 15.11] for large size joints of type (b) are shown in Fig. 15.12(d).

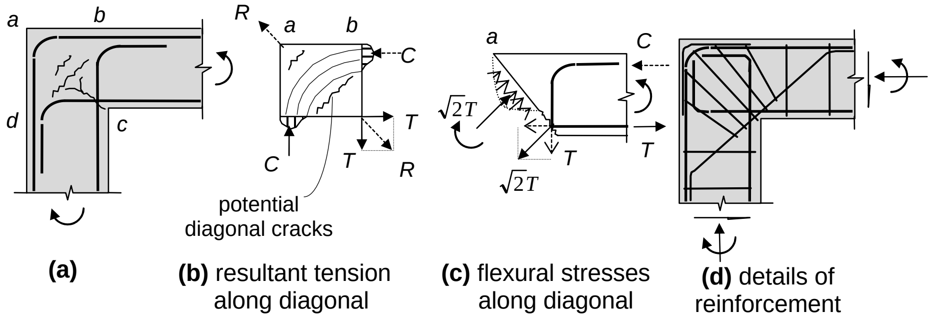



**Fig. 15.12** Knee joint subjected to 'opening' moment

When the moment is subject to reversals, the concrete in the joint zone is likely to crack along both diagonals and significant amounts of secondary reinforcements are required along both diagonals. For this situation it is more convenient to provide an orthogonal mesh of reinforcement (horizontal and vertical) in the joint zone in the form of closed ties. These will resist the horizontal and vertical components of the tensile forces along the diagonal. A model for computing the area of the horizontal and vertical secondary steel (stirrups) required is suggested in Ref. 15.11.

When the joint is subject to high intensity reversed loading for several cycles, the concrete in the joint is likely to be cracked along both principal directions, and it is recommended that the resistance offered by the concrete should not be taken into account.

In multi-storey building frames the joint behaviour is more complex as up to four beams may be framing into a joint with columns above and below at an interior joint. When both beams framing into a joint from opposite directions reach their ultimate capacity and bend in a reverse curvature mode, the diagonal compressive and tensile stresses induced in the joint panel may be very high. Moreover, the beam/column reinforcement may have to develop full anchorage within the joint zone, which may be difficult if the concrete is severely cracked, parallel to both diagonal directions. The joint shear may also be twice as high as that in an exterior joint with beam only on one side. The diagonal compressive stresses and potential diagonal cracking would require an orthogonal mesh of well anchored horizontal and vertical reinforcement ties in the joint region. Furthermore, the concrete in the joint core should be laterally confined. Effective lateral confinement would require cross ties between the legs of the orthogonal ties to prevent the lateral bulging of the concrete in the core.

### 15.2.5 Construction Joints

It is desirable to indicate at least some of the more obvious construction joints so that all the trades are working along the same line. This also facilitates the designer to indicate shouldered joints [such as in rigid frame bents — see Fig. 15.1(c)], reinforced to take moment, shear and thrust. The lapping and splicing of bars should be illustrated clearly, and not just schematically (indicating the amount of steel required at a few points).

### 15.2.6 Bar Supports and Cover

It is essential to have the reinforcing steel accurately located in the forms and firmly held in place before and during the placing of concrete by means of supports and spacers. Such supports should be adequate to prevent displacement during the course of construction and to keep the bars at the proper distance (cover) from the forms. In countries such as USA and Canada, standard bar supports in the form of individual or continuous metal bar chairs (plain, galvanised, or plastic protected) are commercially available and usually specified, although precast concrete blocks are also used. However, in India, while the recommended practice is to use precast mortar blocks (of thickness equal to the specified cover) for bottom bars and individual (locally fabricated) metal chairs for top/bent bars, actual site practices vary considerably. It is fairly common to see bottom bars being supported by just slipping in pieces of coarse aggregate between the bar and the formwork. Needless to say, these angular aggregate pieces, precariously poised between the formwork and the round steel bars, slip out easily during placing of concrete (if not earlier). The result is that the bar often comes to rest on the formwork with little or no cover. When the formwork is removed, it is a sorry sight to see the bars exposed from underneath slabs and beams at several places! This gets covered up in plaster soon enough to give an appearance that all is well (well, at least for the time being!). However, it is not long before the poorly protected reinforcing bars get corroded, and in this process increase in volume, setting up internal bursting stresses in the concrete. In course of time, this causes first cracks in line with the reinforcement, and later spalling of the concrete dislodging the plaster and whatever little concrete cover there is, thereby fully exposing the corroded bars. The seriousness of the resulting damage is obvious to all.

In the case of top bars, locally made individual high chairs are used. However, often these are few and far between, and at times are too flexible [Fig. 15.13]. In such cases, with the unskilled workmen frequently stepping on the top bars, the chairs may get bent (sag) or bent bars may get turned sideways, in either case resulting in a reduced effective depth (particularly so in slabs) in the negative moment regions. A reduced depth in this high moment region is likely to lead to undesirable cracking on the top face of slabs near the support (continuous) regions. This is one reason for the wetting visible underneath roof slabs where they join supporting beams. The multiple and cumulative effects of reduced effective depth at support on deflections, cracking, wetting, leakage, and corrosion of reinforcement can be easily understood.

*Inadequate concrete cover for steel bars is a very commonly observed construction error in India*. Much more attention needs to be paid for providing adequate cover and proper bar supports. Significantly, the recent (2000) revision of the Code has enhanced the cover requirements for reinforcements, including links.

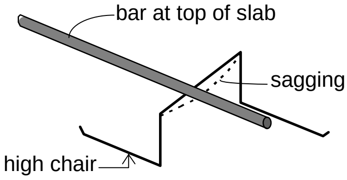



**Fig. 15.13** Inadequate rigidity of a steel chair

### 15.2.7 Deflection Control

In Section 15.1, it was explained that modern designs result in relatively slender members with associated larger deflections. Creep and shrinkage causes deflections to increase with age, and *such increases may be as much as 2 to 3 times the initial elastic deflections*. The possible adverse effects of large deflections on a continuous roof slab is schematically shown in Fig. 15.14.

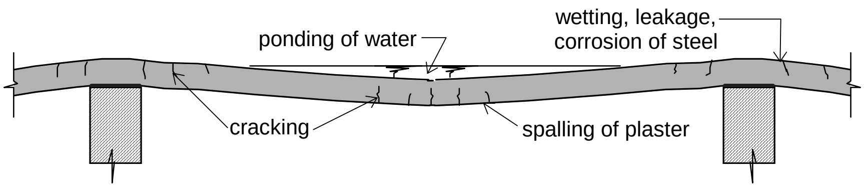



**Fig. 15.14** Adverse effects of large deflections on a continuous roof slab

Beam/slab deflections can also cause cracking in other elements (such as walls) supported by it. This points to the need for deflection calculations and control. While for normal slabs and beams it is enough to control deflections indirectly by limiting span/depth ratios [see Section 5.3], for members which are heavily loaded or exposed to adverse environmental conditions, it is necessary to calculate initial and long-term deflections and to ensure that these are within limits. In large spans where significant deflections can be anticipated, it is desirable to provide initial upward camber in floor slab/beam so as to offset deflection, especially in roof slabs and cantilevered spans.

## 15.3 MATERIALS AND CONSTRUCTION PRACTICES

Strict adherence to codes and specifications, use of good quality materials, engagement of trained and skilled masons and labour, and good workmanship under strict, honest and competent supervision are prerequisites for avoiding malfunction and failures, and for ensuring high quality structures. Regular materials testing, selection of competent contractors, supervisors and construction engineers as well as periodic inspection by regulatory agencies are needed for good quality construction.

Poor quality of construction materials is a real problem in India. Periodic testing of building materials is seldom resorted to in most construction sites. Adulteration of cement has been reported from the sites of even major hydro-electric projects undertaken by large public sector organisations. Reinforcing steel is being supplied by many rerolling mills in the small scale sector, which do not have in-house testing and quality control facilities, with the result that there is little guarantee about the strength, ductility, uniformity, and dimensional tolerances of such bars. All these underscore the need for periodic testing and quality control of materials used for construction.

The major reason for poor performance of reinforced concrete structures is the poor quality of construction. This is apparent if one realises that with the same quantity of cement and aggregate, both very poor quality and very high quality concrete can be obtained, depending on the water-cement ratio used and degree of control exercised. With reduction in w/c ratio, nearly all the engineering properties of concrete (strength, modulus of elasticity, durability, reduced shrinkage and creep, impermeability, etc.) improve. Reduction in creep and shrinkage also results in reduced long-term curvatures and deflections due to them, and reduced shrinkage cracks in concrete. Yet, most masons routinely use excess water in the mix to save time and labour on compaction and screeding; and this could be a major contributing factor to every one of the serviceability failures listed in Section 15.1. It being *the most important single factor* influencing concrete quality, the quantity of water used in the mix should be the minimum, consistent with requirements of laying and proper compaction. To enable the use of a dry mix (low w/c ratio) and yet get good compaction, *all structural concrete should be compacted using vibrators*. There are other requirements for good quality concrete like grading and cleanliness of aggregates, weight-batching, thorough mixing, adequate curing, etc., and for more details on these and other such requirements, reference may be made to books on *concrete technology* [Ref. 2.1–2.6].

In this context, it is worth noting that *leakage through concrete slabs is a very frequent problem in many parts of the country*. Such leakages, if restricted to small areas, can be repaired. However, repairs of a poorly built roof slab with extensive leakage spread over large areas is virtually impossible[†]. Therefore, *it is prudent and it pays to do the initial construction meticulously*.

Yet another common mistake seen usually in sites of small projects relates to *too flimsy and inadequately supported formwork.* Such forms deflect and vibrate as workmen move about over it. Deflection and vibration during periods of placing, setting and early curing of concrete can result in cracks developing in the concrete. Formwork supports are also frequently seen to be infirm, unstable or yielding.

Other common constructions errors include *inadequate curing, improper levels and slopes, inadequate drainage arrangements, poor bonding between hardened and*

[†] Note: Various manufactures and proprietry agencies advertise various methods such as tar felting, special plastering with chemical/adhesive cements, etc., for repairing leaky slabs. These may work for small areas and for a short period for larger areas, but do not offer a permanent solution.

*freshly laid concrete at construction points, poorly constructed expansion joints, superficial filling of holes cut for plumbing/electrification fixtures, etc.*

Reference has been made here to deterioration arising out of poor quality materials and construction. Associated with this is the need for timely repair and maintenance. Apart from regular and routine inspection and maintenance, every concrete structure should undergo a special inspection and associated *special repairs* once in 10–15 years.

## 15.4 SUMMARY

In this chapter, an attempt has been made to draw attention to some of the essential precautions to be taken and to a few of the very common mistakes with regard to the design, detailing and construction of concrete structures. It would be desirable for designers to develop a list of "do's and don'ts" based on codes and specifications, manuals such as Ref. 10.1–10.6, and their own experiences and observations. A partial list of such guidelines is given below:

1. It should be ensured (to the extent possible) that the materials specified can be readily obtained in the size, length and grade required. The quality of materials should be ensured by regular materials testing.
2. Apart from strength and durability considerations, the specification for concrete should also be decided so as to obtain minimum of drying shrinkage and creep.
3. Concrete mix design should aim at obtaining durable concrete of required strength through proper grading of aggregates, control of w/c ratio, thorough mixing, proper compaction and adequate curing.
4. The quantity of water used in concrete should be the minimum practicable, consistent with requirements for proper placement and compaction.
5. Consistent with requirements of economy, the integrity of the structure should be improved by building in structural redundancy in the framing system, reinforcement placement, etc.
6. Flexural members (slabs and beams) should have adequate stiffness so as to limit deflections.
7. In members liable to undergo large deflections, upward camber may be provided to offset the deflections.
8. Reinforcement design and detailing should take care of all tensions in concrete, whether direct, flexural, diagonal or due to shrinkage and temperature.
9. Suitable reinforcement should be provided across all potential cracks in concrete, whatever the cause.
10. To minimise shrinkage and temperature stresses, wherever feasible, provision may be made for unrestricted movements of parts, by introducing control/expansion/slip joints.
11. Concrete slabs in exposed situations, such as sunshades, balconies, canopies, open verandas, etc. should be provided with adequate quantities of temperature reinforcement in order to prevent cracks due to shrinkage and contraction
12. Adequate development length and/or anchorage should be provided so that the computed stress at every section of a reinforcing bar is fully developed on both sides.

13. It should be ensured that hooked and bent bars can be placed conveniently and have adequate concrete protection.
14. Congestion of bars should be avoided at points where members intersect, and it should be ensured that all the reinforcement required can be properly placed.
15. When a member has a break in its direction so that the reinforcement in tension tends to separate from the body of concrete, special anchorage should be provided and properly detailed.
16. When slabs frame flush with bottom of inverted beams or when a load is applied to the side of a member through brackets, ledges or cross beams, special stirrup hanger reinforcement should be provided.
17. Liberal concrete cover for reinforcement should be provided in general, and particularly in humid, wet or aggressive environments. The desired cover should be ensured in actual construction with proper cover blocks / bar supports.
18. Complete and accurate dimensions should be specified in engineering drawings. The notations must be unambiguous and not liable to be misinterpreted.
19. The lengths of laps, points of bend and extension of bars should be specified clearly in drawings.
20. Details of corners, intersection of members, control and construction / expansion joints and similar special locations should be drawn.
21. For special and unusual conditions, adequate details should be shown in drawings so as to ensure proper placing of reinforcement [Examples: cantilevers, continuous footings, hinged base of rigid frames, etc.]
22. Proper bar supports should be provided to ensure that the reinforcing bars are accurately and firmly held in place before and during concreting, and thus the required concrete cover and effective depths are obtained.
23. Compaction of concrete should be done using vibrators (wherever feasible), enabling the use of low w/c ratio and ensuring better strength and durability.
24. Formwork should be built firmly and with rigid supports and without gaps and holes through which the cement paste can escape.
25. Curing of the concrete should be done for durations as recommended by the Code, and should be terminated gradually to prevent quick drying.
26. In case of members which are liable to large deflections (Examples: cantilever beams and slabs), the removal of centering should be delayed as much as possible so that the concrete attains sufficient strength.
27. Adequate slopes for roofs, bathroom floors, etc., should be provided to ensure quick drainage.

## REVIEW QUESTIONS

15.1 List the areas in which the basic Reinforced Concrete Design course content in your university is deficient.

15.2 List ten most common construction mistakes in the order of their importance.

15.3 What are the factors in concrete-making that influence creep and shrinkage of the concrete ?

15.4 What are the precautions to be taken (a) at the design stage, (b) at the detailing stage and (c) at the construction stage for ensuring a high quality structure ?

15.5 What are the types of *serviceability failures* that can occur ?

15.6 Describe a 'serviceability failure' that you have observed in a structure you are familiar with, and analyse its causes and effects and suggest remedial/repair measures.

15.7 Explain the concept of 'structural integrity'.

15.8 In a three-hour long training programme to be given to masons engaged in construction of concrete structures, list out the important topics that you would include.

15.9 Make a literature survey and write a 'state-of-the-art' report on
(a) Building failure studies;
(b) Deflection calculations and control in concrete structures;
(c) Cracking and control in concrete structures; and
(d) Leakage and control in concrete buildings.

## REFERENCES

15.1 — *Handbook on Concrete Reinforcement and Detailing*, Special Publication SP 34, Bureau of Indian Standards, New Delhi, 1987

15.2 — *Concrete Materials and Methods of Concrete Construction,* CSA Standards A 23.1–94, Canadian Standards Association, Rexdale (Toronto), Canada, 1994.

15.3 — *Structural Failures: Modes, Causes and Responsibilities*, American Society of Civil Engineers, New York, 1973.

15.4 Feld, Jacob, *Construction Failures*, John Wiley & Sons, New York, 1968.

15.5 — *Handbook on Causes and Prevention of Cracks in Buildings*, Special Publications SP 25, Bureau of Indian Standards, New Delhi, 1984.

15.6 — *Formwork Striking Times — Criteria*, *Prediction and Methods of Assessment*, CIRIA Report, American Society of Civil Engineers, New York, 1996.

15.7 Hoover, C.A. and Greene, M.R. (editors), *Construction Quality*, *Education and Seismic Safety*, Earthquake Engineering Research Institute, Oakland, California, 1996.

15.8 Grant, E.L. and Leavenworth, R.S., *Statistical Quality Control*, Sixth edition, McGraw-Hill Book Co., New York, 1988.

15.9 Levin, R.I., Rubin, D.S., Stinson, J.P. and Garden Jr., E.S., *Quantitative Approaches to Management*, Eighth edition, McGraw-Hill Book Co., 1992.

15.10 Waestlund, G., *Use of High-Strength Steel in Reinforced Concrete*, Journal ACI, Vol. 30, No. 12, June 1959, pp 1237–1250.

15.11 Park, R. and Paulay, T., *Reinforced Concrete Structures*, John Wiley & Sons, Inc., New York, 1975.

15.12 Allen, D.E. and Shriever, W.R., *Progressive Collapse, Abnormal Loads and Building Codes*, in 'Structural Failure: Modes, Causes, Responsibilities', American Society of Civil Engineers, 1973, pp 21–47.

15.13 Taylor, D.A., *Progressive Collapse*, Canadian Journal of Civil Engineering, Vol. 2, No. 4 ,Dec. 1975, pp 517–529.

# 16

# Special Provisions for Earthquake-Resistant Design

## 16.1 INTRODUCTION

During an earthquake, ground motions occur in a random fashion, both horizontally and vertically, in all directions radiating from the epicentre. The ground accelerations cause structures to vibrate and induce inertial forces on them. Hence, structures in such locations need to be suitably designed and detailed to ensure stability, strength and serviceability with acceptable levels of safety under seismic effects. The resultant inertial force at any floor level[†] depends on the mass at the floor level and also the height above the foundation. The inertial forces usually follow a parabolic distribution in regular multi-storey buildings, with maximum values at the top floor levels. In regions of high seismic intensity, it is desirable to minimise the weights at various floor levels, especially the roofs and upper storeys. Also, it is desirable to avoid discontinuities in mass or stiffness in plan or elevation. Torsional effects should particularly be accounted for in buildings with asymmetry in plan[‡]. The codes published by the Bureau of Indian Standards, which specify minimum design requirements for earthquake-resistant design, are listed as Refs. 16.1–16.3. These requirements take into consideration the characteristics and probability of occurrence of earthquakes, the characteristics of the structure and the foundation, and the amount of damage that is considered tolerable. References 16.4–16.7 give details of code provisions in some seismic regions of the world.

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[†] Sufficient number of modes of vibration have to be considered in the 'response spectrum' analysis, as prescribed in IS 1893 (2002), and a suitable mode combination scheme (such as 'SRSS' or 'CQC') has to be employed.

[‡] Torsional effects should be considered when the eccentricity between the 'centre of mass' and 'centre of stiffness' at any floor level is significant (more than 5% of the floor plan dimension).

The criteria adopted by codes for fixing the level of the design seismic loading are generally as follows [Ref. 16.7]:

- structures should be able to resist *minor* earthquakes without damage;
- structures should be able to resist *moderate* earthquakes without significant structural damage, but with some nonstructural damage; and
- structures should be able to resist *major* earthquakes without collapse, but with some structural and nonstructural damage.

The magnitude of the forces induced in a structure due to a given ground acceleration (or given intensity of earthquake) will depend, amongst other things, on the mass of the structure, the material and type of construction, and the *damping*, *ductility* and *energy dissipation capacity* of the structure. By enhancing ductility and energy dissipation capacity in the structure, the induced seismic forces can be reduced, and a more economical structure obtained, or alternatively, the probability of collapse reduced. Buildings with lateral load resisting system comprising (i) a *ductile moment-resisting space frame* or (ii) a dual system consisting of ductile moment resisting space frame and ductile flexural (shear) wall, qualify for very low seismic induced forces.

> **Ductility** may be broadly defined as the ability of a structure or member to undergo inelastic deformations beyond the initial yield deformation with no decrease in the load resistance.

Since reinforced concrete is relatively less ductile in compression and shear, dissipation of seismic energy is best achieved by *flexural yielding*. A frame of continuous construction, comprising flexural members, columns and their connections, designed and detailed to accommodate reversible lateral displacements after the formation of plastic hinges (without decrease in strength), is known as a *ductile moment-resisting frame*. Similarly, *shear walls* (more appropriately called *flexural walls*), are reinforced concrete structural walls cantilevering vertically from the foundation, and designed and detailed to be ductile and to resist seismic forces and to dissipate energy through flexural yielding at one or more plastic hinges.

Modern codes [Ref. 16.1†, 16.4, 16.5] provide for reduction of seismic forces through provision of special ductility requirements. Details for achieving ductility in reinforced concrete structures are given in IS 13920 [Ref. 16.3]. Methods of determining design seismic forces, either in the form of equivalent static lateral loading or through proper dynamic analysis, lie outside the scope of this chapter; the reader may refer to the codes, handbooks and other texts [Ref. 16.8–16.10] for this purpose.

This chapter explains the major code provisions, particularly those given in Ref. 16.3, dealing with designing and detailing for ductility in moment-resisting frames and shear walls. Such provisions are mandatory for structures located in

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† In the recent revision of IS 1893 (2002), the procedure recommended is to first calculate the actual force that may be experienced by the structure during the 'probable maximum earthquake', if it were to remain elastic. Then, the effect of ductile deformation and energy dissipation is accounted for by means of a 'response reduction factor'.

relatively high intensity *seismic zones* (zones III, IV and V), specified in IS 1893 : 2002 [Ref. 16.1].

## 16.2 IMPORTANCE OF DUCTILITY IN SEISMIC DESIGN

### 16.2.1 Measures of Ductility

A general qualitative definition of ductility was given in the preceding section. A quantitative measure of ductility has to be with reference to a load-deformation response. A *ductile* response would be reflected in the deformation increasing at nearly constant load such as was shown in Fig. 9.8. Then, the ratio of the ultimate deformation to the deformation at the beginning of the horizontal path (or, at first 'yield') can give a measure of ductility. However, each choice of deformation (*strain*, *rotation*, *curvature*, or *deflection*) may give a different value for the ductility measure.

#### *Curvature Ductility*

For an under-reinforced beam section in flexure, the moment-curvature ($M$–$\varphi$) relation is typically as shown in Fig. 16.1(a). Based on the idealised $M$–$\varphi$ behaviour, *curvature ductility*, $\mu$, may be defined as the ratio $\varphi_u/\varphi_y$, where $\varphi_y$ is the curvature at first yield (idealised), and $\varphi_u$ the maximum (ultimate) curvature at the section:

$$\mu \equiv \frac{\varphi_u}{\varphi_y} \tag{16.1}$$

Indeed, IS 13920 [Ref. 16.3] defines *curvature ductility* as the ratio of curvature at the ultimate strength to the curvature at first yield of tension steel in the section.

The value of $\mu$ is a property of the beam *cross section*, and can be computed easily using the principles described in Chapter 4. It can be shown that 'curvature ductility', $\mu$, of a section *increases* with:

- decrease in the percentage tension steel ($p_t$);
- increase in the percentage compression steel ($p_c$);
- decrease in the tensile strength of steel;
- increase in the compressive strength of concrete† ;
- increase in the compression flange area in flanged beams; and
- increase in the transverse (shear) reinforcement.
- increase in the confinement of concrete and compression reinforcement (by closely spaced hoops, spirals, etc.).

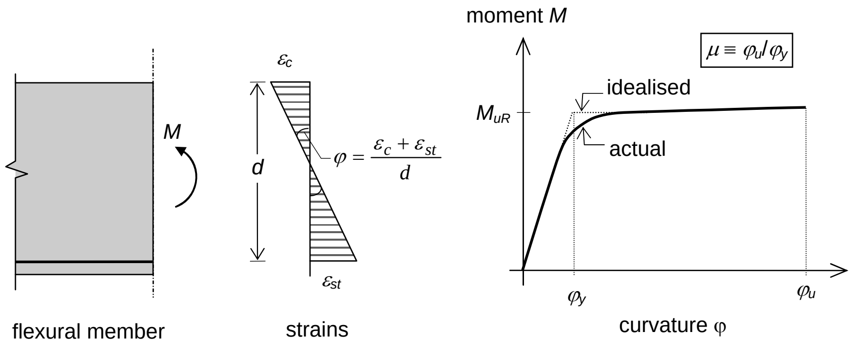
A curvature ductility of at least 5 is considered to be adequate for reinforced concrete [Ref. 16.2].

#### *Different Measures of Ductility*

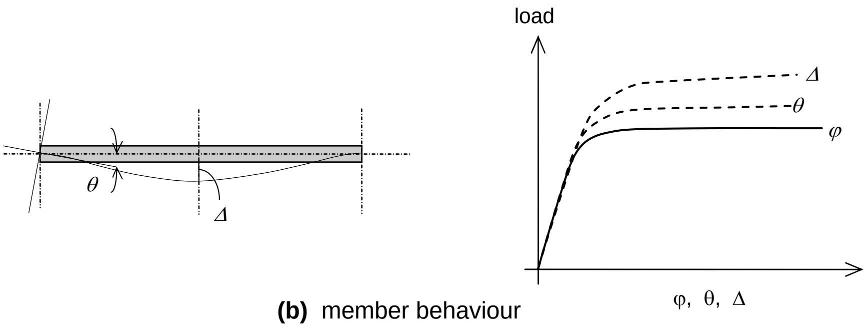
In the case of a beam member [Fig. 16.1(b)], it is more difficult to define a unique ductility ratio, as it could be in terms of the curvature ($\varphi$) at a particular section, or

† However, very high grades of concrete are undesirable, as the they have lower ultimate compressive strains [refer Fig. 2.7].

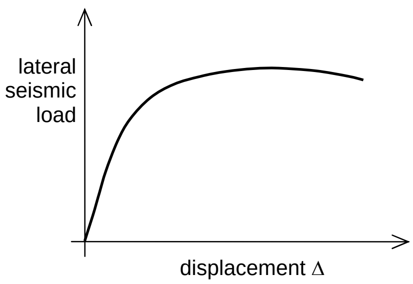
the rotation ($\theta$) at a joint, or the displacement ($\Delta$) at a selected point. The ductility ratios
obtained by the three methods will differ. Furthermore, the rotations and displacements will depend on several factors, such as the span dimensions, shape of the moment diagram, type of support restraints, etc.




**(a)** curvature ductility for an under-reinforced section




**(b)** member behaviour




**(c)** structure behaviour

**Fig. 16.1** Measures of ductility

The problem becomes more complex when it comes to defining a ductility measure for an entire structure. In general, a reinforced concrete ductile structure will have a load-displacement response as shown schematically in Fig. 16.1(c). This ductility is achieved by ensuring ductile member section responses (as indicated by the $M$–$\varphi$ relation in Fig. 16.1(a)), so that an adequate member of *plastic hinges* [refer Section 9.7] would develop at appropriate locations under extreme lateral seismic forces.

### 16.2.2 Energy Dissipation by Ductile Behaviour

Under seismic forces, structures are subject to several cycles of reversed cyclic loading. If the structure, modelled (for simplicity) as a single degree-of-freedom system, were to behave in a *linear elastic* manner under reversed cyclic loading, it will exhibit a linear load-displacement behaviour as shown in Fig. 16.2(a). The shaded area under the curve denotes the potential energy stored in the structure at the maximum displacement position; this gets released and converted to kinetic energy as the structure returns to its zero-load position.

However, if the structure responds in an elastoplastic (ductile) manner, developing fully plastic behaviour at a load level $F'$ (less than $F$ shown in Fig. 16.2a), then the load-displacement behaviour is as shown in Fig. 16.2(b). In this case, the maximum deflection $\Delta'$ is greater than that ($\Delta$) obtained in Fig. 16.2(a) for elastic behaviour.

Furthermore, when the structure returns to its zero-load position, the actual energy which gets converted to kinetic energy is limited to the triangular area *cde* in Fig. 16.2(b). The remainder of the input energy (given by area *abcd*) gets *dissipated*† by the plastic hinge. In summary, under seismic loadings, for a given energy input, elastoplastic response differs from elastic response in the following ways:

- the energy gets dissipated;
- the induced force is less; and
- the maximum deflection is more.

> Thus, while ductility helps in reducing induced forces and in dissipating some of the input energy, it also demands larger deformations to be accommodated by the structure.

It may be noted that the actual behaviour of reinforced concrete is different from the idealised behaviour shown in Fig. 16.2(b). As indicated schematically in Fig. 16.2(c), the hysterisis behaviour of reinforced concrete is characterised by 'rounding' and 'pinching' of the loops, which is associated with the *Bauschinger effect*‡ in steel, and *stiffness degradation* in concrete (due to repeated opening and closing of cracks and bar slip at anchorage zones) [Ref. 16.11]. This results in the areas within successive loops becoming smaller.

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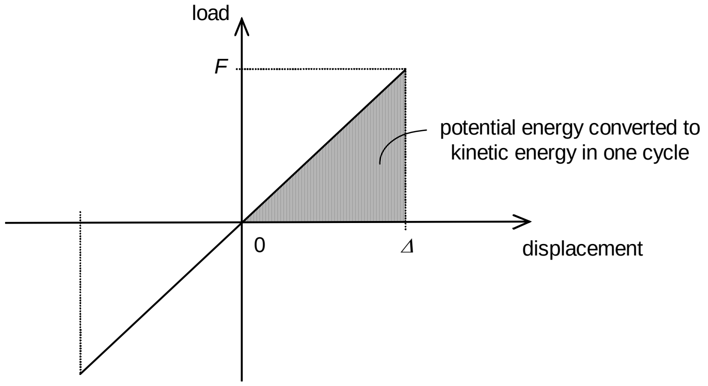
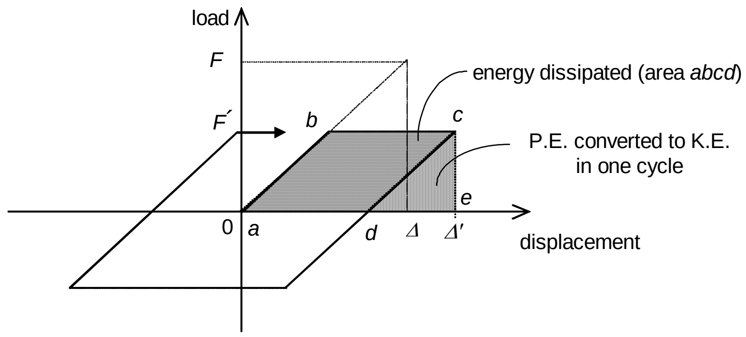
† gets converted into heat and other forms of nonrecoverable energy.

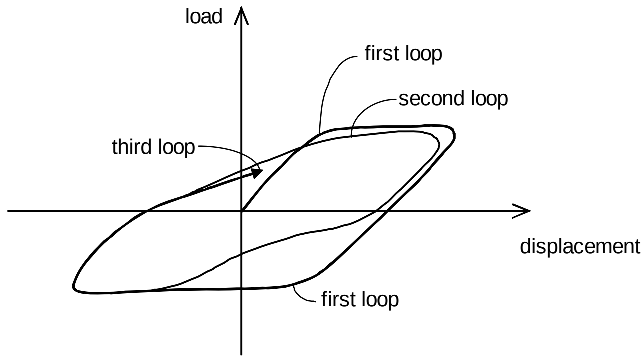
‡ When reinforcing steel is subjected to reversed cyclic loading, it is found that the yield strength obtained in the reloading or reversed direction is substantially less than the initial yield strength; this is known as the *Bauschinger effect* [refer Fig 2.19].




**(a)** elastic response




**(b)** elastoplastic response




**(c)** hysteris behaviour of reinforced concrete

**Fig. 16.2** Load-displacement behaviour under reversed cyclic loading

### 16.2.3 Flexural Yielding in Frames and Walls

As reinforced concrete is relatively less ductile in compression and shear, dissipation of seismic energy is best achieved by flexural yielding. Hence, weakness in compression and shear, in relation to flexure, should be avoided.

In a structure composed of ductile moment-resisting frames and/or shear (flexural) walls, the desired inelastic (ductile) response is developed by the formation of plastic hinges (flexural yielding) in the members, as shown in Fig. 16.3.

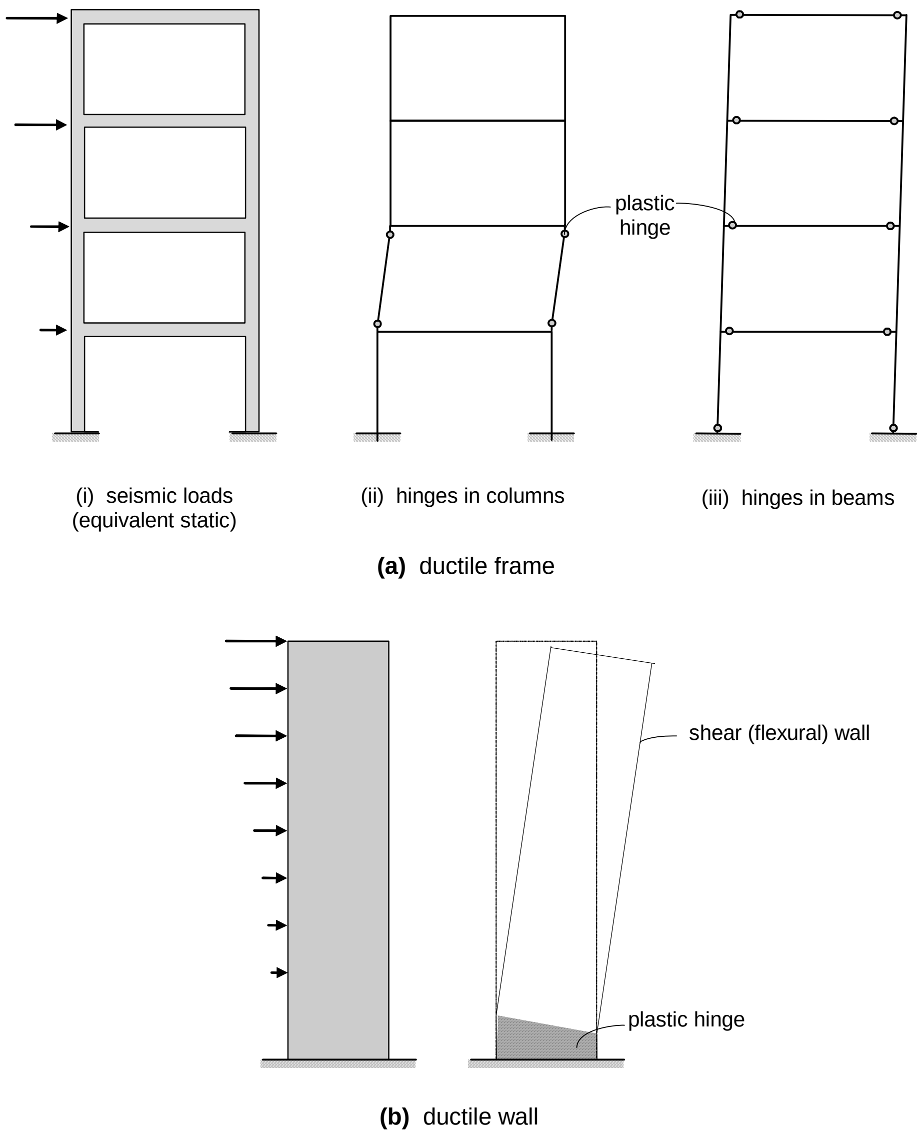



**Fig. 16.3** Formation of plastic hinges in a ductile structure

In the case of ductile frames, plastic hinges may form in the beams or in the columns, as shown in Fig. 16.3(a). It is desirable to design the frame such that the plastic hinges form in the beams [Fig. 16.3(a)(iii)], and not the columns, because:

- plastic hinges in beams have larger rotation capacities than in columns;
- mechanisms involving beam hinges have larger energy-absorptive capacity on account of the larger number of beam hinges (with large rotation capacities) possible;
- eventual collapse of a beam generally results in a localised failure, whereas collapse of a column may lead to a 'global' failure; and
- columns are more difficult to straighten and repair than beams, in the event of residual deformation and damage.

## 16.3 MAJOR DESIGN CONSIDERATIONS

### 16.3.1 General Design Objectives

The objective of the special design and detailing provisions in IS 13920 [Ref. 16.3] is to ensure adequate toughness and ductility (with ability to undergo large inelastic reversible deformations) for individual members such as beams, columns and walls and their connections, and to prevent other non-ductile types of failure.

> In order to maintain overall ductile behaviour of the structure with minimal damage, it becomes necessary to achieve, *in relative terms*, combinations of
> - strong foundations and weak superstructure;
> - members stronger in shear than in flexure; and
> - strong columns, and beams with little over-strength.

Some of the main design considerations in providing ductility include:

- using a low tensile steel ratio (with relatively low grade steel) and/or using compression steel;
- providing adequate stirrups to ensure that shear failure does not precede flexural failure;
- confining concrete and compression steel by closely spaced hoops or spirals; and
- proper detailing with regard to connections, anchorage, splicing, minimum reinforcement, etc.

Furthermore, continuity in construction and redundancy in structural framing are desirable for the development of more inelastic response, and thereby more moment redistribution and energy dissipation at several plastic hinges. Earthquake is often followed by fire and hence fire resistance should also be a major consideration in building construction in seismic regions.

### 16.3.2 Requirements of Stability and Stiffness

Under a severe earthquake, it is expected that in a structure designed to resist seismic forces in a ductile manner, large lateral deformations and oscillations will be induced, resulting in the development of reversible plastic hinges at various locations in the ductile frames and walls. The structural system should be so designed as to ensure that the formation of plastic hinges at suitable locations may, at worst, result in the failure of individual elements, but will not lead to instability or progressive collapse. This calls for building-in redundancy into the structural system. Redundancy assists in the development of alternative load paths, thereby helping redistribution of forces, dissipation of energy and avoidance of progressive collapse.

In addition to the requirements of stability and strength to resist seismic forces, the structure must have sufficient *stiffness* to limit the lateral deflection or *drift*. Ref. 16.4 suggests that the anticipated drift due to seismic forces may be taken as three times the lateral deflection obtained from the usual elastic analysis under equivalent factored static loads. [This factor is intended to account for the effects of material and geometric nonlinearities, as well as additional amplification due to dynamic effects]. The inter-storey drift is to be limited to 0.004 times the storey height (under the specified seismic forces) as per IS code [Ref. 16.1]. The effect of drift on the vertical load-carrying capacity of the lateral load resisting system should also be taken into account in the analysis.

### 16.3.3 Materials

#### *Reinforcing Steel*

As mentioned earlier, ductility calls for the use of relatively low grades of steel. Lower grade steel has clearly defined and longer yield plateau, and hence the plastic hinges formed will have larger rotation capacities, leading to greater energy dissipation. Similarly, locations of potential plastic hinges *should not have too much over*-strength, i.e., strength more than the required design strength. Over-strength will result in the section not yielding, as intended, at the expected lateral load levels. This may result in adjoining elements and/or foundations being subjected to loads larger than the design loads, with consequent damage. In other words, the *actual yield strength* of the steel used *should not be markedly higher* than the yield strength specified and used in design computations. Furthermore, yield strength, far in excess of that specified, may lead to excessive shear and bond stresses, as the plastic moment is developed. Another point to note is that, the lower the grade of steel, the higher is the ratio of the *ultimate* tensile strength ($f_u$) to the *yield* strength ($f_y$) [refer Section 2.14.2]. A high ratio of $f_u/f_y$ is desirable, as it results in an increased length of plastic hinge (along the member axis), and thereby an increased plastic rotation capacity.

For these reasons, mild steel (Fe 250) is best suited for use as flexural reinforcement in earthquake-resistant design. However, its use will necessitate larger sections of flexural members. Hence, the code [Ref. 16.3] permits the use of the higher grade Fe 415 (which is most commonly used in practice), but prohibits the use of grades higher than Fe 415.

### *Concrete*

With regard to the grade of concrete, the code [Ref. 16.3] limits the minimum grade of concrete to M 20 (*for all buildings which are more than* 3 *storeys in height*). It may be noted that very high strength concrete is also undesirable because higher compressive strength is associated with lower ultimate compressive strain ($\varepsilon_{cu}$) [refer Section 2.8.2, Fig. 2.7] — which adversely affects ductility. Likewise, low density concrete is undesirable because of its relatively poor performance under reversed cyclic loading. The ACI and Canadian codes [Ref. 16.4, 16.6] limits the maximum cylinder strength of low density concrete for use in earthquake-resistant design to 30 MPa.

### 16.3.4 Foundations

It is important to ensure that the foundation of a structure does not fail prior to the possible failure of the superstructure. As plastic deformations are permitted to occur at suitable locations in the superstructure under a severe earthquake, the maximum seismic forces transmitted to the foundation will be governed by the lateral loads at which *actual yielding* takes place in the structural elements transferring the lateral loads to the foundation. The ultimate moment, corresponding to 'actual yielding' at a section is obtained as its *characteristic* (nominal) moment capacity†, i.e., without applying partial safety factors (i.e., with $\gamma_c = \gamma_s = 1.0$).

The corresponding moments, shear forces and axial forces transferred from the frames and walls to the foundation system (under conditions of 'actual yielding') should be resisted by the foundation system with the usual margin of safety (i.e., with $\gamma_c = 1.5$ and $\gamma_s = 1.15$) in order to ensure a combination of a relatively stronger foundation and weaker superstructure. Although such a recommendation is yet to be incorporated in the IS codes [Ref. 16.1–16.3], it is in vogue in several international codes (such as Ref. 16.4, 16.7). Such a design concept is necessary to provide for ductile behaviour of the superstructure without serious damage to the foundation.

### 16.3.5 Flexural Members in Ductile Frames

The code recommendations [Ref. 16.3] for design and detailing of flexural members in earthquake-resistant design are as follows:

- To qualify as "flexural members", the factored axial stress under earthquake loading should not exceed 0.1 $f_{ck}$. Further, the overall depth $D$ should not exceed one-fourth of the clear span (to limit shear deformations) and the width $b$ should not be less than 200 mm, with a $b/D$ ratio of more than 0.3 (to avoid lateral instability and provide for improved torsional resistance).
- To ensure significant ductile behaviour even under reversals of displacements in the inelastic range, to avoid congestion of steel, and to limit the shear stresses in beams, the tensile reinforcement ratio $\rho_{max}$ is limited to 0.025 i.e., $p_{t,\,max} = 2.5$.

† Alternatively, this may be taken (conservatively) as 1.4 times the factored moment of resistance ($M_{uR}$) — as recommended by the code [Ref. 16.3], for estimating plastic moment capacities in the calculation of design shear forces.

- To avoid sudden brittle failure of a beam (when the cracking moment of the section is reached) a minimum reinforcement ratio, $\rho_{min} = 0.24\sqrt{f_{ck}} / f_y$, must be provided at both the top and bottom for the entire length of the member, with at least two bars placed at each face.
- Flexural members of lateral force resisting ductile frames are assumed to yield at the design earthquake. To ensure proper development of reversible plastic hinges near continuous supports (beam-column connections) where they usually develop in such members,
  - ∗ the 'positive' moment reinforcement at a joint face must not be less than half the 'negative' moment reinforcement at that joint face;
  - ∗ the top and bottom steel at any section along the length of the member should not be less than one-fourth of the 'negative' moment reinforcement at the joint face on either side;
  - ∗ both top and bottom bars must be taken through the column and made continuous wherever possible, in case of an interior joint. In other cases, they must be extended to the far face of the confined column core and provided an anchorage length of $L_d$ + 10 $\phi$, where $L_d$ is the development length of the bars (diameter $\phi$) in tension [Fig. 16.4]; and
  - ∗ Not more than 50 percent of the bars shall be spliced at one section. Because of the possibility of spalling of the concrete shell (cover) under large reversed strains, lap splices of flexural reinforcement are not permitted in and near possible plastic hinge locations. If welded splices or mechanical connections are used, it must be ensured that not more than 50 percent of the bars are spliced in the region of potential plastic hinging.
  - ∗ The provisions for redistribution of moments (See Section 9.7.3) shall be used only for vertical load moments and not for lateral load moments.

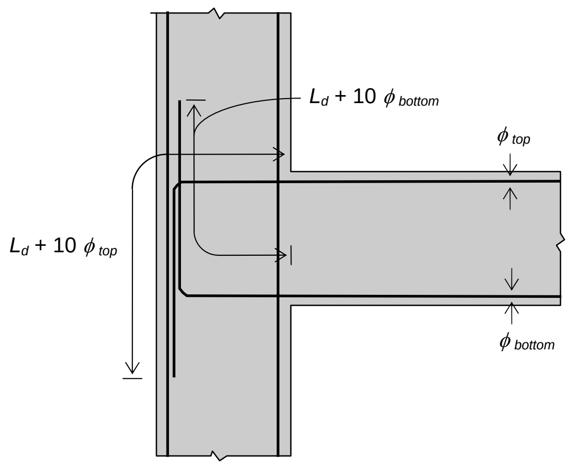



**Fig. 16.4** Anchorage of beam bars at an external joint

- When lap splices are provided (at regions other than plastic hinging regions), transverse reinforcement for confining concrete and to support longitudinal bars, in the form of closed stirrups or 'hoops' (with a 135$^o$ hook and 10 $\phi \geq$ 75 mm extension) should be provided over the entire splice length, at a spacing not exceeding 150 mm [Fig. 16.5].
- The bar extensions must provide for possible shifts in the inflection points, which may occur under the combined effects of gravity and seismic loadings.
- During an earthquake, a structure should be capable of undergoing extensive inelastic deformation (through ductile behaviour) without a significant loss in strength. Yielding softens the structure, which effectively increases its time period and reduces the earthquake force. Damping also increases significantly in the inelastic range of response and this further helps to improve the earthquake response. For these desirable effects to take place, it should be ensured that none of the brittle modes of failure (particularly, shear failure) should occur before ductile flexural failure. Hence, the shear design philosophy in an earthquake resistant structure differs significantly from that in an ordinary structure [Ref. 16.20, 16.24]. **Due to extensive cracking in the zones of high shear, it is desirable to completely ignore the shear strength of concrete ($\tau_c$) and to design the stirrups to resist the entire shear.**

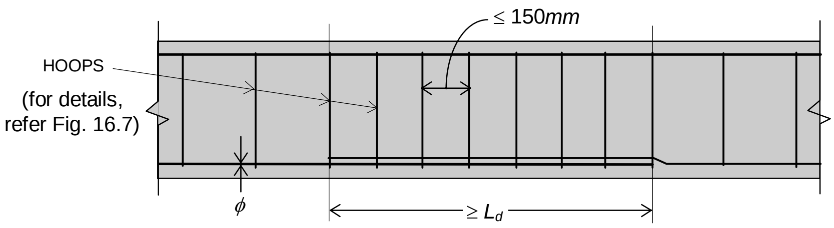



**Fig. 16.5** Lap splice in a flexural member

- In earthquake resistant structure, the design shear force will be the larger of
  1. Shear force as obtained from analysis for given load combinations, and
  2. Actual shear force likely to develop in a member after flexural failure has taken place.

  According to the code (Cl. 6.3.3; IS 13920: 1993), the web reinforcement in the form of vertical stirrups shall be provided so as to develop the vertical shear due to formation of the plastic hinges at both ends of the beam plus the factored gravity load on the span.
- To ensure that a shear failure does not precede the full development of plastic hinges in a beam, the design shear forces in the member should be suitably overestimated, considering plastic moment capacities† of 1.4 $M_{uR}$ at the beam ends, as shown in Fig. 16.6(b). The component shear force diagrams, including

† The factor of 1.4 specified by the code [Ref. 16.3] is intended to account for the condition of 'actual yielding' (involving *characteristic* values of material strengths) as well as increased tensile strength due to possible strain hardening, and also some margin of safety.

the effects of factored gravity loads and sway in either direction, are indicated in Fig. 16.6(c). The maximum design shear forces ($V_u$) at the support faces (left or right) are accordingly obtained as:

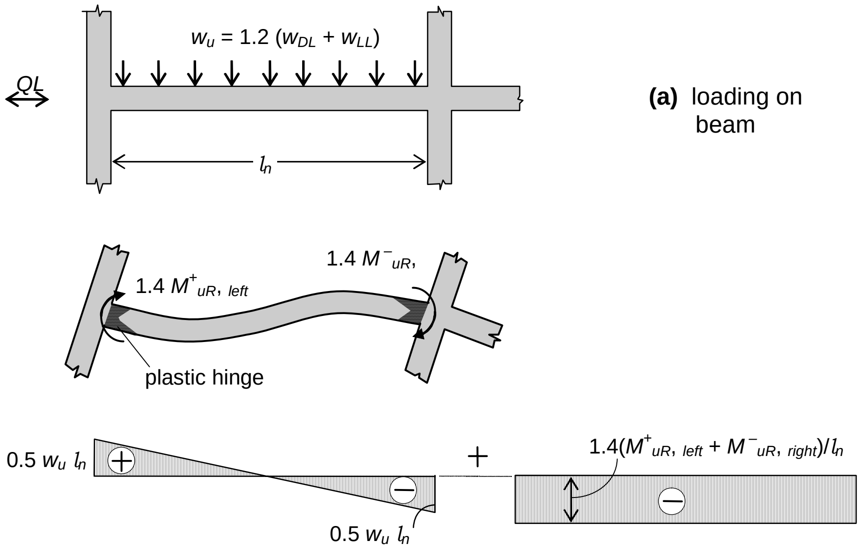
$$V_{u,left} = 0.5 w_u l_n + 1.4\left(M^{-}_{uR,left} + M^{+}_{uR,right}\right)/l_n \tag{16.2a}$$

$$V_{u,right} = 0.5 w_u l_n + 1.4\left(M^{+}_{uR,left} + M^{-}_{uR,right}\right)/l_n \tag{16.2b}$$

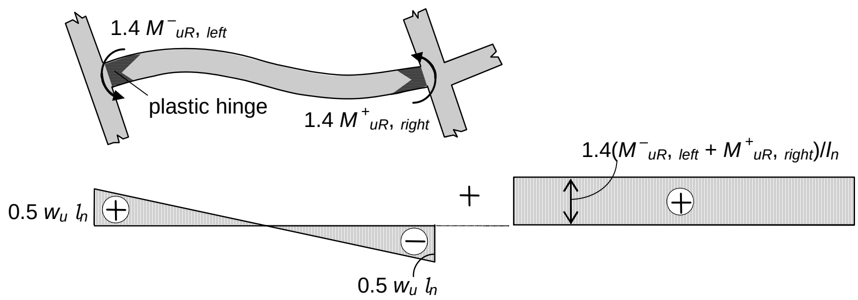
where $l_n$ is the clear span, and

$$w_u = 1.2\left(w_{DL} + w_{LL}\right) \tag{16.2c}$$

assuming that the gravity loads (dead loads $w_{DL}$ and live loads $w_{LL}$) are uniformly distributed.




**(b)** design shear forces in beam (sway to right)




**(c)** design shear forces in beam (sway to left)

**Fig. 16.6** Calculation of design shear forces in beams

- Because of the alternating direction of the shear force due to seismic effects, the direction of the associated diagonal tensile stress also alternates, as shown in Fig. 16.7(a). For this reason, *inclined bars* (which are effective only against shear in one direction) are not allowed as effective shear reinforcement.
- Web reinforcement for seismic design must be in the form of closed stirrups, called *hoops*, placed *perpendicular to the longitudinal reinforcement* and must be provided throughout the length of the member. These hoops should have a minimum diameter $\phi_s$ of 8 mm in beams with a clear span exceeding 5*m* (6mm in shorter beams). The free ends of the hoops should be bent at $135^o$ with a minimum bar extension of 10 $\phi_s$ (but not < 75 mm) [Fig. 16.7(b)], so that the ends are adequately anchored in the core of the concrete.

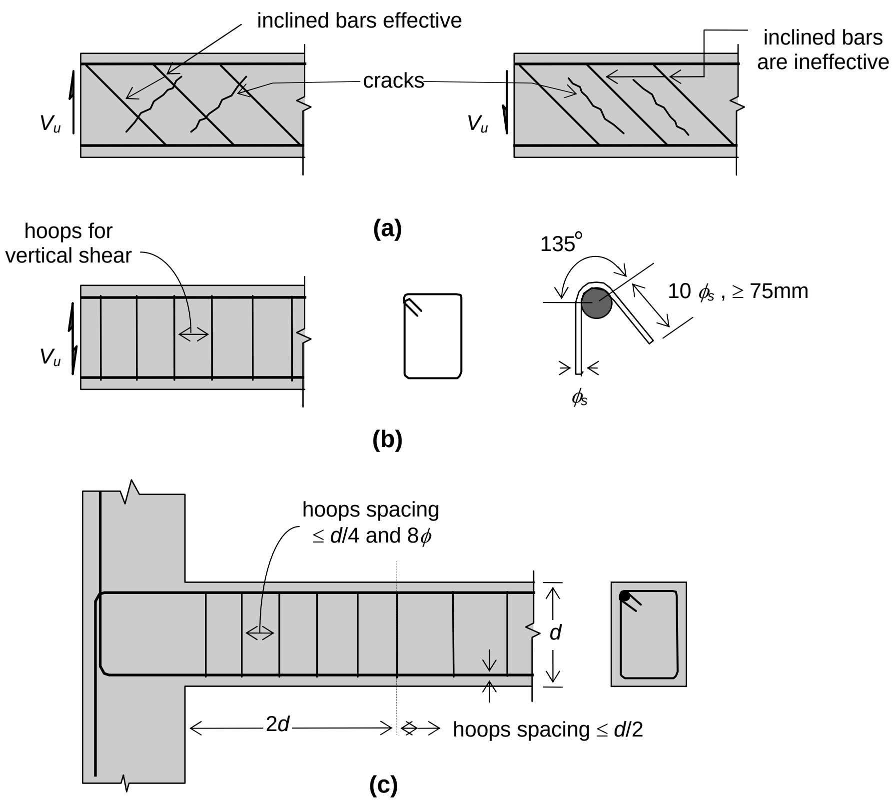



**Fig. 16.7** Type of web reinforcement for reversed shear condition

- The hoops serve the additional purposes of confining the concrete and preventing buckling of the longitudinal bars, particularly near the beam-column joints, where reversible plastic hinges are expected to develop and where the concrete cover is liable to spall off after a few cycles of inelastic rotations. The code [Ref. 16.3] specifies a closer spacing of hoops over a length equal to twice the effective depth (i.e., 2*d*) from the face of the column. The hoop spacing should not exceed *d*/4 or

8 times the diameter of the smallest longitudinal bar, with the first hoop located at a distance not exceeding 50 mm from the column face. Elsewhere, in the beam, the spacing of hoops should not exceed $d/2$, as shown in Fig. 16.7(c).

### 16.3.6 Columns and Frame Members Subject to Bending and Axial Load

- Members in this category are those having a factored axial stress which is greater than 0.1 $f_{ck}$ under the effect of seismic forces. Further, the minimum dimension of the member should not be less than 200 mm, with the ratio of the shortest cross-sectional dimension to the perpendicular dimension preferably not being less than 0.4. However, in frames having beams with centre to centre span exceeding 5 m or columns with unsupported length exceeding 4 m, the shortest dimension should not be less than 300 mm [Ref. 16.3].
- To ensure that the combined flexural resistance of the columns is greater than that of the beams at the beam-column joint (so that the plastic hinges form at the beam ends, rather than the column ends), it is necessary to design the column section for a suitably higher moment. Although the IS code [Ref. 16.3] does not make any specific recommendation in this regard, the ACI and Canadian codes [Ref. 16.4, 16.7] recommend that the sum of the *factored* moment resistances of the columns framing into the joint be at least 1.1 times the sum of the *characteristic* moment resistances (i.e., $\gamma_c = \gamma_s = 1.0$) of the beams[†] framing into the joint [Fig. 16.8(a)].
- Lap splices are not permitted near the ends of the column where spalling of the concrete shell is likely to occur. Lap splices (suitably designed as tension splices), however, are permitted in the central half of the member length. Hoops should be provided over the entire splice length at a spacing not exceeding 150 mm (centre-to-centre). Not more than 50 percent of the bars should be spliced at any section.
- The design shear force in a column should be taken as the larger of (1) the shear force due to the factored loads and (2) the shear force in the column due to the development of the plastic moments (suitably enhanced, as in Eq. 16.2) in the beams framing into the column, given approximately by [Ref. 16.3]:

$$V_u = 1.4\left(M_{uR,b1} + M_{uR,b2}\right)/h_{st} \tag{16.3}$$

  where $M_{uR,\,b1}$ and $M_{uR,\,b2}$ are the factored moments of resistance (of opposite sign) of beam ends '1' and '2' framing into the column from opposite faces, and $h_{st}$ is the storey height [refer Fig. 16.8(b)].
- Unless a larger amount of transverse reinforcement is required from shear strength considerations, **special confining reinforcement** should be provided as given below. Special confining reinforcement must be provided over a length $l_o$ from each joint face (high moment regions), and on both sides of any section, where flexural yielding may occur under seismic forces [Fig. 16.9(a)]. The

[†] The effects of the slab reinforcement within a distance of three times the slab thickness on either side of the beam should be included in calculating the beam moment capacity.

length $l_o$ should not be less than (a) the larger lateral dimension of the member at the section where yielding may occur, (b) 1/6 of the clear span (height) of the member, and (c) 450 mm. The spacing of hoops used as special confining reinforcement should not exceed 1/4 of the minimum member dimension, but need be less than 75 mm, or more than 100 mm. The area of cross-section ($A_{sh}$) of the bar to be used as special confining reinforcement should be taken as:

$$A_{sh} \geq \begin{cases} 0.09 s D_k \dfrac{f_{ck}}{f_y}\left[\dfrac{A_g}{A_k} - 1\right] & \text{for circular hoops / spiral} \\ 0.18 s D_h \dfrac{f_{ck}}{f_y}\left[\dfrac{A_g}{A_k} - 1\right] & \text{for rectangular hoops} \end{cases} \tag{16.4}$$

where

$s \equiv$ pitch of spiral or spacing of hoops;

$D_k \equiv$ diameter of core, measured to the outside of the spiral or hoop;

$D_h \equiv$ longer dimension of the rectangular hoop, measured to its outer face — not to exceed 300 mm;

$A_g \equiv$ gross area of the column section; and

$A_k \equiv$ area of the concrete core (contained within the outer dimension of the hoop/spiral).

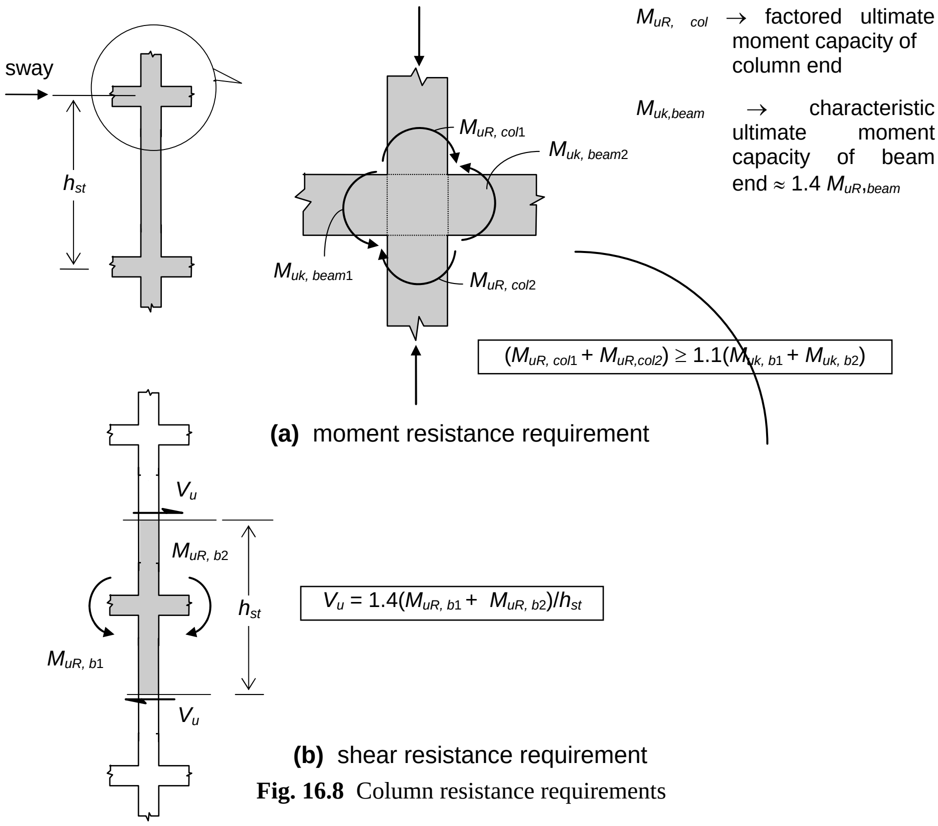
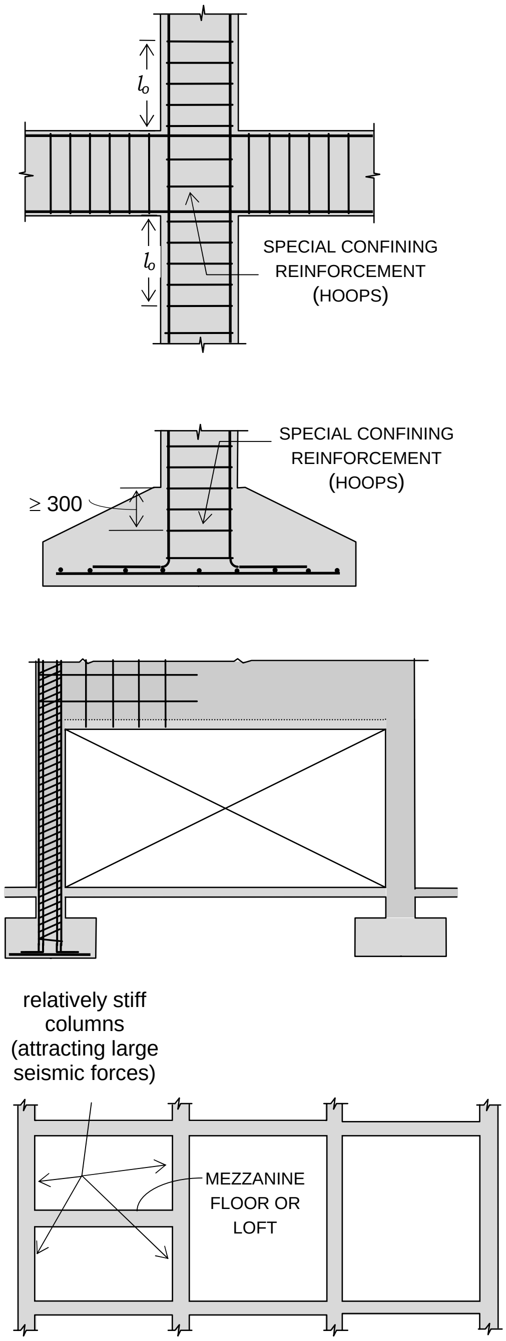



**Fig. 16.8** Column resistance requirements




**(a)** detailing of hoops in column (at and near joint)

**(b)** detailing at column-footing interface

**(c)** special confining reinforcement requirement for columns under discontinued walls

**(d)** columns with varying stiffness

**Fig. 16.9** Special confining reinforcement

- When a column terminates into a footing or mat, the 'special confining reinforcement' should extend at least 300 mm into the footing or mat, to account for possible development of plastic hinges at the base of a building [Fig. 16.9(b)]. Such detailing should also be provided in columns supporting discontinued stiff members (such as walls or trusses) for the full height of the column as shown in Fig. 16.9(c). Provision of special confining reinforcement over the full height of the column is also required in cases where there is a significant variation of stiffness along the height of the column — as when mezzanine floors/lofts are provided locally [Fig. 16.9(d)] or due to in-filled masonry walls not extending fully over the panel.

### 16.3.7 Joints in Ductile Frames

- Beam-column joints in ductile frames must have adequate shear strength and ductility to facilitate the development of large inelastic reversible rotations, in the event of a severe earthquake. Tests have indicated that the shear strength of joints is dependent primarily on the grade of concrete and is not sensitive to the amount of shear reinforcement [Ref. 16.4]; hence, it is desirable to use high strength concrete in the joint regions, and to achieve good compaction of this concrete [Ref. 16.12].
- The *special confining reinforcement* (hoops) provided near the column ends should be extended through the joint as well [see Fig. 16.9(a)]. However, when the joint is 'externally confined', this reinforcement may be reduced to one-half of that required at the end of the column, with the maximum spacing limited to 150 mm (Code Cl. 8.2). A joint is said to be 'externally confined' if beams frame into all the vertical faces of the joint, and if each beam width is at least three-fourths of the column width at the joint [Ref. 16.3 &16.12].
- Development length requirements of the flexural reinforcement within the joint [refer Fig. 16.4] are particularly important. The joint zone is an area of high concentration of beam bars, column bars and hoops. Extreme care is needed in detailing the reinforcement at the beam-column connection in order to provide for proper stress transfer, and to avoid congestion and placing difficulties for both reinforcement and concrete. Many structural failures under seismic loading can be traced to poor detailing of beam-column joints.
- The reinforcements detailed in this chapter pertain to monolithic concrete construction. In *precast* construction, subject to seismic loading, the most critical location is the beam-column connection. However, it has been shown that by careful detailing, ductile beam-column connections (having adequate strength, stiffness, ductility and energy-dissipating capacity) can be made in precast concrete construction as well [Ref. 16.19].

### 16.3.8 Shear Walls (Flexural Walls)

- Ductile 'shear walls' (more appropriately called *flexural walls*), which form part of the lateral load resisting system, are vertical members cantilevering vertically from the foundation, designed to resist lateral forces in its own plane, and are subjected to bending moment, shear and axial load. Unlike a beam, a wall is relatively thin and deep, and is subjected to substantial axial forces. The wall must be designed as an axially loaded beam, capable of forming reversible plastic hinges (usually at the base[†], as shown in Fig. 16.3(b)) with sufficient rotation capacity.

- The code [Ref. 16.3] recommends that the thickness of any part of the wall should preferably be not less than 150 mm. Walls that are thin are susceptible to instability (buckling) at regions of high compressive strain. Stability of the compression zones can be improved by local thickening of the wall or by providing flanges or cross walls (which is convenient at such locations as lift cores). Flanged walls also have higher bending resistance and ductility. The code [Ref. 16.3] restricts the effective flange width of flanged walls to (a) half the distance to an adjacent shear wall web, and (b) one-tenth of the total wall height.
- The wall should be reinforced with uniformly distributed reinforcement in both vertical and horizontal directions, with a minimum reinforcement ratio of 0.0025 of the gross section in each direction. The bar diameter should not exceed one-tenth the wall thickness, and the bar spacing in either direction should not exceed (a) 1/5 of the horizontal length of wall, (b) thrice the wall (web) thickness, and (c) 450 mm. The distributed reinforcement provides the shear resistance, controls the cracking, inhibits local breakdown in the event of severe cracking during an earthquake, and also resists shrinkage and temperature stresses. The vertical reinforcement, comprising both the distributed reinforcement and concentrated reinforcement near wall ends (see below), should be designed for the required flexural and axial load resistance.
- In walls which do not have flanges (‘boundary elements’), concentrated vertical reinforcement should be provided towards each end face of the wall, in addition to the uniformly distributed steel. A minimum of 4 nos 12 mm $\phi$ bars arranged in at least two layers should be provided near each end face of the wall [Ref. 16.3]. The concentrated vertical flexural reinforcement near the ends of the wall must be tied together by transverse ties, as in a column, to provide confinement of the concrete, and to ensure yielding without buckling of the compression bars when a plastic hinge is formed.
- Where the extreme fibre compressive stress in the wall exceeds 0.2 $f_{ck}$, *boundary elements* should be provided along the vertical boundaries of walls. These are portions along the wall edges that are strengthened by longitudinal and transverse reinforcement, and may have the same or larger thickness as that of the wall web.
- To prevent a premature brittle shear failure of the wall before the development of its full plastic resistance in bending, it is desirable to design the shear resistance of the wall for an overestimated shear force, as in the case of the column. Because of possible severe shear cracking under reversed cyclic loading, the shear carried by concrete in the plastic hinge region is neglected.
- For other details regarding the design of boundary elements, coupled shear walls, walls with openings, etc., reference should be made to Ref. 16.3.

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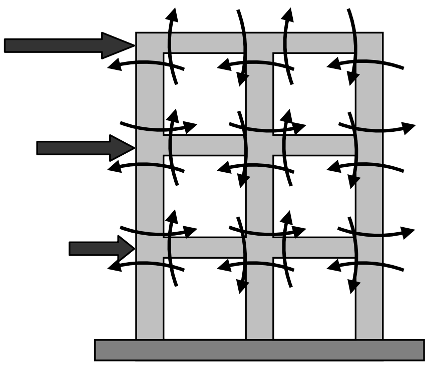
[†] Locations of abrupt changes in the strength and stiffness of the lateral load resisting system are also potential zones of flexural yielding in ductile walls.

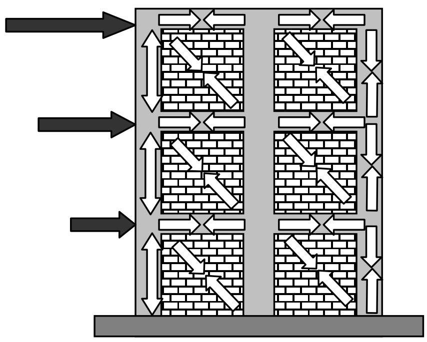
### 16.3.9 Infill Frames

Generally, in the analysis of multi-storey buildings, the contribution of masonry infill walls is ignored, and the frame analysis is based on the bare RC frame. The mass of the masonry infill is considered, but the stiffness and strength contributions of the masonry infill are neglected. However, the infill frame has some significant effects under lateral loading that merit consideration [Ref. 16.21, 16.22]:

- Infills alter the behaviour of buildings from predominantly frame action to predominantly shear action [Fig. 16.10]. Also, the infills are capable of resisting the applied lateral seismic forces through axial compression along the diagonal; there is no tensile resistance capability in the other diagonal, but the cracking induced in the masonry on account of this serves to dissipate energy.
- The neglect of infill contribution results in a significant under-estimation of the lateral stiffness of the structure, and thereby can result in an under-estimation of the seismic forces. Infills may also significantly modify the position of centre of rigidity and consequently can affect the behaviour in torsion.
- In regular multi-storey buildings, in general, the neglect of infill frame action results in a conservative estimation of bending moments in columns and beams (except when 'soft storey' is provided, as shown in Fig. 16.11).



**(a)** Bare frame: predominant frame action



**(b)** Infilled frame: predominant shear action

**Fig. 16.10** Behaviour of infill frame [Ref. 16.23]

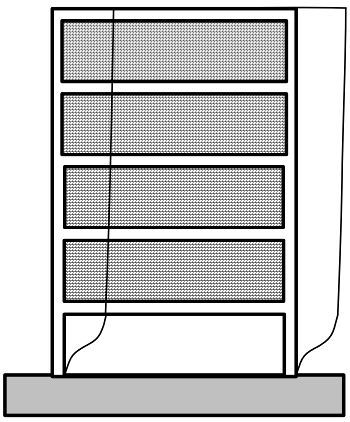
Thus, the bare RC frame of the building considered in design is inconsistent with actual behaviour. In general, however, the infills are expected to significantly reduce the demand on the RC frame members. Numerous cases are cited in the technical literature where brick walls acting together with RC elements have saved buildings from collapse during earthquake.

Several techniques have been proposed to evaluate the allowable horizontal force of an infilled frame subject to in-plane bending and axial force. The simplest procedure is to model the masonry infill by means of an equivalent compressive diagonal strut [Ref. 16.21]. At the tensile corners of the non-integral infill walls, separation of frame occurs from the infill at early stages. The panels are in contact with the frame only at the compression corners, and this contact is strengthened under increased loading, with high stress concentrations near the corners. The diagonal part of the infill acts as a compressive diagonal strut and is effective in resisting lateral loads. As the tensile corners are subjected to very small stress, the tensile diagonal region is not really effective in resisting lateral loads. Since the infills act as diagonal struts, an infill wall can be replaced by an equivalent strut in the analysis model.

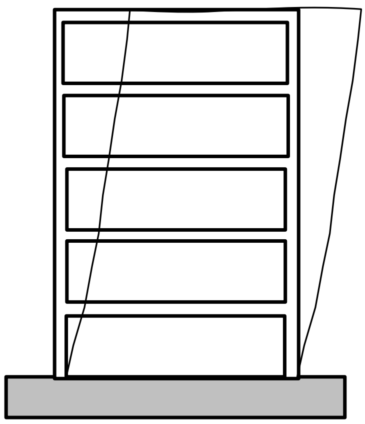
### 16.3.10 Soft Storey

The soft storey concept is related to a discontinuity in the stiffness of building. According to IS 1893: 2002 a soft storey is one in which the sum of the lateral translational stiffness is less than 70% of that in the storey above or less than 80% of the average lateral translational stiffness of three stories above. In modern multi-storey construction, such soft stories are commonly encountered in the ground storey. Owing to high cost of land and small sizes of plots, parking is often accommodated in the ground storey area of the building itself. Frame bays of the ground storey are not infilled with masonry walls, as in the case of upper stories. Usually, all panels are left open for parking. The sudden discontinuity in stiffness and mass at the lowermost storey (soft storey) leads to the following effects that make soft storey construction particularly dangerous.

- The stiffness discontinuity leads to severe stress concentrations at the soft storey corners, accompanied by large plastic deformations [Fig. 16.11(a)].
- Most of the deformation energy is dissipated by the soft storey columns, and this leads to major overstressing of these elements; onset of plastic hinges may transform the soft storey into a mechanism resulting in collapse. Such a collapse could also turn out to be more catastrophic.



(a) Open ground storey



(b) Bare frame

**Fig. 16.11** Lateral load responses of open ground storey frame (with infills above ground storey) and bare frame

For designing a 'soft storey' building, dynamic analysis should be carried out including the strength and stiffness effects of infills and inelastic deformation in the members. Alternatively, the code suggests the following design criteria based on a conventional earthquake analysis, neglecting the effect of infill walls in other stories[†]. The columns and beams of the soft storey are to be designed for 2.5 times the storey shears and moments calculated under seismic load (which ignores the infill frame effect).

### 16.3.11 Performance Limit States

In several countries, seismic design is in the process of fundamental change. One important reason for the change is that although code-designed buildings performed well (in countries such as USA) in recent earthquakes from a life-safety perspective, the level of damage to structures, economic loss due to non-usage of buildings and costs of repair were unexpectedly high. Conventional methods of seismic design have the objectives to provide for life safety (through strength and ductility) and damage control (through serviceability – drift limits). The design criteria are defined by limits on stresses and member forces calculated from prescribed levels of applied lateral shear force. Performance-based design philosophy involves design criteria that are expressed in terms of achieving stated performance objectives when the structure is subjected to stated levels of seismic hazard [Ref. 16.25]. The performance targets may be a level of stress, a load, a displacement, a limit state or a target damage state not to be exceeded. Required performance criteria for a seismic hazard are 'safety', 'restorability' and 'usability'. Safety refers to protection of human life. Restorability refers to structural integrity. Usability refers to function and habitability.

## 16.4 CLOSURE

The purpose of this chapter is to explain the background to the seismic design provisions of the IS code and related international codes. A detailed discussion of seismic analysis and design of reinforced concrete structures is beyond the scope of this book. Rapid advances are being made in this area, and recent publications (for example, Refs. 16.13-16.18) may be consulted for more details.

## REVIEW QUESTIONS

16.1 What are the objectives of earthquake-resistant design of reinforced concrete structures?

16.2 What is meant by *ductility*? Give a qualitative description and also describe briefly the qualitative measures of ductility in reinforced concrete.

16.3 What are the measures one can take for improving the ductility of a reinforced concrete structure?

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[†] The design criteria have been newly introduced in the recent (2002) revision of IS 1893. If these are applied to existing buildings, it will be seen that a majority of such buildings will be found deficient in terms of earthquake resistant design.

16.4 What are the advantages/disadvantages of elastoplastic behaviour over elastic behaviour in structures subjected to severe earthquakes?
16.5 What are the objectives behind the special detailing provisions in IS 13920?
16.6 Differentiate between the terms *strength, stiffness* and *stability* as applied to a reinforced concrete structure.
16.7 Why is it desirable to design for the formation of *plastic hinges* in beams rather than columns in earthquake-resistant design?
16.8 Is it desirable to have (a) high strength steel (b) high strength concrete in earthquake-resistant design of reinforced concrete structures? Justify your answers.
16.9 Suggest a design procedure for ensuring that the foundation is stronger than the superstructure in earthquake-resistant design.
16.10 What are the limits placed on tensile reinforcement ratios in beams in earthquake-resistant design? Why are such limits enforced?
16.11 How are the design shear forces estimated in the beams of ductile frames?
16.12 Why are inclined stirrups and bent-up bars unsuitable as shear reinforcement in earthquake-resistant design?
16.13 What is meant by *special confining reinforcement* in columns of ductile frames?
16.14 What are the design requirements of beam-column joints in earthquake-resistant design?
16.15 Explain the differences between an *ordinary wall* and a *shear wall* in a reinforced concrete tall building, with regard to function, loading and design.
16.16 What are the main design requirements of ductile shear (flexural) walls in earthquake-resistant design?
16.17 What is the effect of ignoring the contribution of masonry infill in the lateral load analysis of a multi-storey frame?
16.18 In what manner is the behaviour of a 'soft storey' construction likely to be different from a regular construction in the event of an earthquake?

## REFERENCES

16.1 IS 1893 (Part I): 2002 — *Criteria for Earthquake Design of Structures – Part I: General Provisions and Buildings* (Fifth revision), Bureau of Indian Standards, New Delhi, 2002.
16.2 IS 4326 : 1993 — *Code of Practice* for *Earthquake Resistant Design and Construction of Buildings*, Bureau of Indian Standards, New Delhi, 1993 (reaffirmed 1998).
16.3 IS 13920 : 1993 — *Ductile Detailing of Reinforced Concrete Structures Subjected to Seismic Forces — Code of Practice*, Bureau of Indian Standards, New Delhi, 1993.
16.4 ACI Standard 318-95, *Building Code Requirements for Structural Concrete* and *Commentary* (ACI 318R–95), Am. Conc. Institute, Detroit, Michigan, 1995.
16.5 *National Building Code of Canada 1995, Part 4*: *Structural Design*, National Research Council of Canada, Ottawa, 1995.

16.6 CSA Standard A23.3 – 94 — *Design of Concrete Structures*, Canadian Standards Association, Rexdale, Ontario, 1994.
16.7 *SEAOC*, *Recommended Lateral Force Requirements and Commentary*, Seismology Committee, Structural Engineer's Association of California, San Francisco, 1980.
16.8 — *Explanatory Handbook or Codes for Earthquake Engineering,* Special Publication SP 22, Bureau of Indian Standards, New Delhi, 1982.
16.9 Newmark, N.M. and Rosenbleuth, E., *Fundamentals of Earthquake Engineering,* Prentice-Hall, Englewood, Cliffs, N.J., 1971.
16.10 Clough, R.W. and Penzien, J., *Dynamics of Structures*, Second edition, McGraw-Hill International edition, 1993.
16.11 Park, R. and Paulay, T., *Reinforced Concrete Structures*, John Wiley & Sons, Inc., New York, 1975.
16.12 ACI–ASCE Committee 352, *Recommendations for Design of Beam-Column Joints in Monolithic Reinforced Concrete Structures*, (ACI 352 R–76, Reaffirmed 1981), Am. Conc. Institute, Detroit, 1976.
16.13 Arnold, C. and Reitherman, R., *Building Configuration and Seismic Design*, John Wiley & Sons, Inc., New York, 1982.
16.14 Applied Technology Council, *Tentative Provisions for the Development of Seismic Regulations for Buildings*, ATC 3–06, National Bureau of Standards, Special Publication 510, U.S. Government Printing Office, Washington, D.C., 1978.
16.15 Dowrick, D.J., *Earthquake Resistant Design*, John Wiley & Sons, Chichester, U.K., 1977.
16.16 — *Reinforced Concrete Structures in Seismic Zones*, ACI Publication SP–53, Am. Conc. Institute, Detroit, 1977.
16.17 — *Reinforced Concrete Structures Subjected to Wind and Earthquake Forces*, ACI Publication SP–63, Am.Conc. Institute, Detroit, 1980.
16.18 — *Earthquake Effects on Reinforced Concrete Structure*, ACI Publication SP–84, Am. Conc. Institute, Detroit, 1985.
16.19 Pillai S.U., and Kirk, D.W., *Ductile Beam-Column Connection in Precast Concrete*, ACI Journal, Vol. 78, Nov–Dec 1981, pp 480–487.
16.20 Murty C.V.R., and Jain S.K., *A Review of IS 1893-1984: Provisions on Seismic Design of Buildings,* The Indian Concrete Journal, November 1994, pp 619–629.
16.21 Scarlat A.S., *Approximate Methods in Structural Seismic Design*, E & FN Spon, London, 1996.
16.22 Mallick D.V., and Severn R.T., *The Behaviour of Infilled Frames,* Proceedings of Institute of Civil Engineers, Vol. 38, Dec 1967, pp 639–656.
16.23 — *Bhuj, India Earthquake of January 26, 2001: Reconnaissance Report,* Supplement A to Volume 18 Earthquake Spectra, July 2002.
16.24 Medhekar M.S., Gehad E.R., and Jain S.K., *Shear Reinforcement for Aseismic Design of Flexural Members*, The Indian Concrete Journal, June 1992, pp 319–324.
16.25 Ghobarah A., *Performance Based Design in Earthquake Engineering: State of Development*, Engineering Structures, Vol.23, 2001, pp 878-884

# 17

# Selected Special Topics

## 17.1 DESIGN FOR SHEAR BY COMPRESSION FIELD THEORY

### 17.1.1 Introduction

In the traditional method given in Chapter 6, the transverse reinforcement for shear is designed separately and added on to the reinforcement designed for flexure (with axial load if any), and for torsion. Influence of shear on longitudinal reinforcement requirements is taken care of by detailing provisions. This procedure, widely adopted in practice, does not explicitly account for the interaction among the various stress resultants (shear force, bending moment and axial force). Also, the calculations aim to satisfy equilibrium requirements, and do not account for the requirements of deformation compatibility. In the absence of shear, however, the combined effect of flexure and concurrent axial force is made in one step considering the deformation pattern ('plane section remaining plane'), stress and strain compatibility and equilibrium conditions (refer Chapter 13). Indeed, there have been attempts to design for flexure, shear and axial force taking all their effects together. The *Compression Field Theory* [Ref. 17.1] is an attempt in this direction. However, the mechanics involved are such that an exact solution is complex and intractable. Hence, recourse has been made to several simplifying assumptions, which are perhaps questionable.

The name "compression field theory" is based on the analogous problem of the post-buckling shear resistance of thin-webbed metal girders (plate girders). In such girders, following the buckling of the thin web due to diagonal compression caused by shear, the web cannot resist any more compression. Instead, the shear is resisted by a 'field of diagonal tension' [Fig. 17.1]. This approach is known as **tension field theory**. Similarly, in the case of concrete beams, after diagonal cracking, shear would not be resisted by diagonal tension, however a field of diagonal compression would still resist shear. This concept came to be called **compression field theory**.

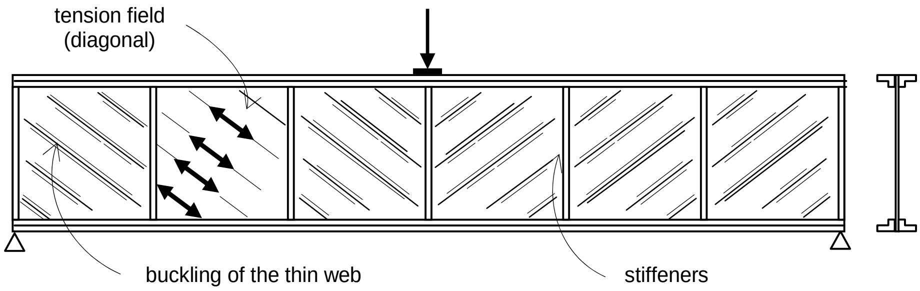



**Fig.17.1** Tension field in thin-webbed metal girder under shear

As in the case of the conventional method dealt with in Chapter 6, in its simplified version, the compression field theory also uses the truss analogy. However, while in the conventional method the inclination of the diagonal cracks is taken as 45°, here, the angle of inclination, θ, of the diagonal compressive stresses is considered variable. Also, the negative influence of diagonal tension cracking on the diagonal compressive strength of concrete [see Section 17.1.3] is accounted for. Moreover, the influence of shear on the design of longitudinal reinforcement is accounted for more directly.

### 17.1.2 General Concepts

In order to understand the complexity involved in an exact analysis for shear strength, consider the compressive stress trajectories in a beam subjected to a bending moment, *M*, an axial force, *N* (considered positive if tensile), and a shear force, *V*, as shown in Fig. 17.2. At any section 1-1, the magnitude and direction of the principal compressive stresses and principal compressive strains will vary over the depth of the section. At the bottom face, the inclination θ will be 90°, and at the top face θ will have a minimum value. The shear stress will also vary over the depth of section. On a small element such as at *A* at a depth *y*, the stresses, strains, and the corresponding Mohr's circles are as shown in Fig. 17.2(h) and (j). Concrete is assumed to have no tensile strength. In addition, the directions of principal stresses and principal strains are assumed to coincide.

For a correct analysis, at each point over the depth of the section, three parameters are required to be known/computed. These may be considered, for instance, as the principal strains $\varepsilon_1$ and $\varepsilon_2$ and the angle $\theta$. In addition, the stress-strain relationships for concrete and reinforcing steel are necessary. With these known, the principal stress, $f_2$, can be computed ($f_1 = 0$) and hence the normal stress, $f_{cx}$, and the tangential stress, $v$, at all points over the depth of the section. The stress in longitudinal steel, $f_s$, can be computed from the steel strain, $\varepsilon_{sx}$, assuming that reinforcing steel carries only axial forces. Thus, the distributions of axial and tangential stresses over the cross section can be obtained as shown in Fig. 17.2(e) and (d). By integrating these stresses (multiplied by the width of the cross-section) over the depth of section, the stress resultants *N*, *M* and *V* can be obtained [Fig. 17.2(f)]. The strain in the transverse direction, $\varepsilon_t$, determines the tensile stress, $f_v$, in the transverse shear reinforcement, and the tension in this reinforcement balances the transverse compressive stress in the concrete, $f_{cy}$, over the area tributary to it [Fig. 17.2(g)].

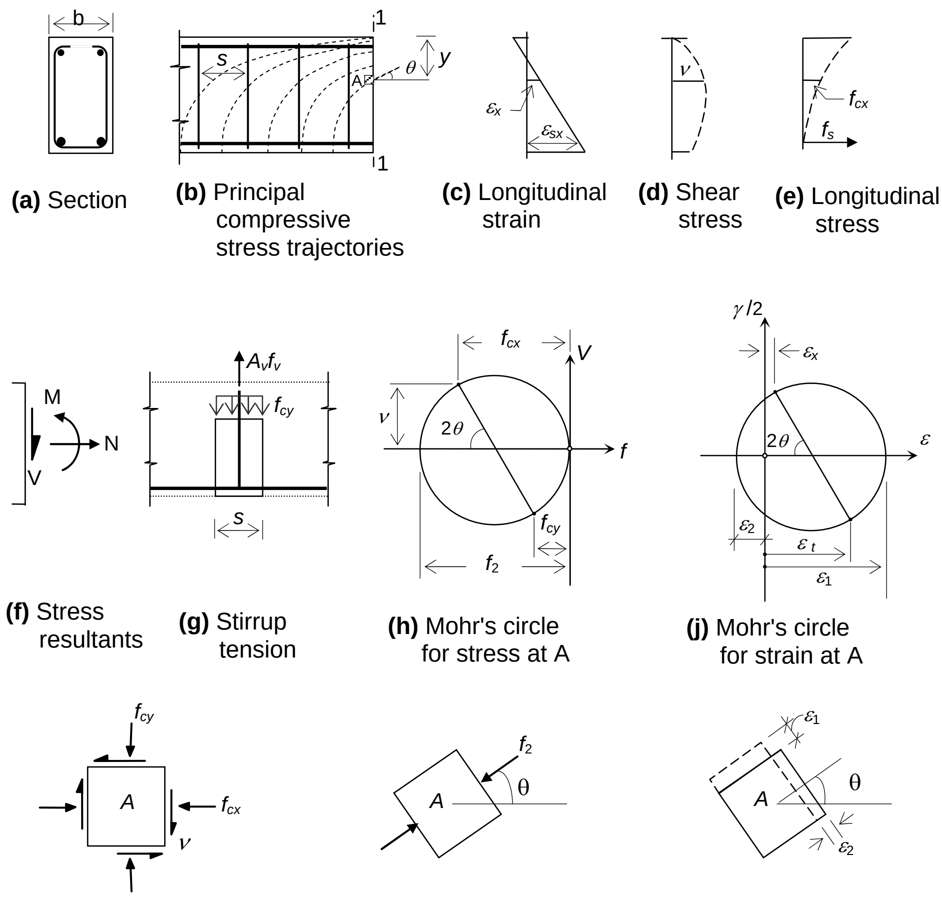



**Fig.17.2** Stress and strain under combined stress resultants *M*, *N*, and *V*

The strain distribution must be compatible with the geometry of deformation. Thus, with the usual assumption that plane sections of the beam remain plane (in shallow flexural members), the distributions of $\varepsilon_1$, $\varepsilon_2$ and $\theta$ must be such that the corresponding longitudinal strain, $\varepsilon_x$ varies linearly over the depth of the member, as shown in Fig. 17.2(c). Obviously, knowing/assuming, a priori, such distributions of $\varepsilon_1$, $\varepsilon_2$ and $\theta$ over the depth at all sections is a tall order!

Because of the large number of unknowns involved, a direct solution to the problem is not possible, and a trial and error procedure together with simplifying assumptions has to be used. Two parameters that may be assumed initially are the shear stress distribution and the longitudinal strain distribution. This gives $v$ and $\varepsilon_x$ at all points. Taking trial values of a third parameter also, such as $\varepsilon_1$, and by successive iterations to satisfy equilibrium, compatibility, and stress-strain relations, the appropriate values of $f_{cx}$ and $v$ at element $A$ can be found. Such a procedure to compute the shear strength of a given section subjected to moment $M$ and axial force $N$ is presented in Ref. 17.2. However, such procedures are lengthy and tedious and seldom resorted to in practice. Instead, approximate procedures are specified in Codes, for example the Canadian specifications CSA A23.3-94.

The procedure recommended in CSA A23.3-94 (Cl. 11.4) is based on the 'modified compression field theory' [Ref. 17.1, 17.3]. This is dealt with in Section 17.1.4.

### 17.1.3 Stress-Strain Relationship for Diagonally Cracked Concrete

For the evaluation of shear strength, the stress-strain relationship for steel and concrete must be known. The state of stress in element A is shown in Fig. 17.2(h). The maximum (principal) compressive stress in concrete, $f_2$, is inclined at an angle θ to the axis of the member. The maximum compressive strain along $f_2$ is $\varepsilon_2$, and the maximum tensile strain, $\varepsilon_1$, is at right angles to the direction of $f_2$. Because of the low tensile strength of concrete (which is neglected here), tensile cracks will develop early along the direction of $f_2$, and the concrete in between these cracks acts as the parallel compression diagonals in the truss analogy. Therefore, the concrete carrying the diagonal compressive stress has cracks parallel to the direction of compression as shown in Fig. 17.3(a).

Biaxially strained concrete, as in Fig. 17.3(a), with compression in one direction and a concurrent transverse tensile strain is weaker than concrete in uniaxial compression as in a cube or cylinder test [Fig. 17.3(b)], where the lateral strain is only due to the Poisson effect. Based on tests [Ref. 17.3] the maximum compressive strength, $f_{2,\max}$, of concrete in the presence of transverse tensile strain, $\varepsilon_1$, is given by:

$$\begin{aligned} f_{2,\max} &= f'_c/(0.8+170\varepsilon_1) \\ &\le f'_c \end{aligned} \tag{17.1}$$

where $f_{2,\max}$ = compressive strength of concrete in presence of transverse tensile strain $\varepsilon_1$

$f'_c$ = specified compressive (cylinder) strength of concrete

$\varepsilon_1$ = transverse tensile strain

Eqn. 17.1 gives $f_{2,\max}$, the *maximum strength* of concrete under transverse tensile strain $\varepsilon_1$. To compute the stress $f_2$ corresponding to a compressive strain $\varepsilon_2$ (concurrent with transverse tensile strain $\varepsilon_1$), a stress-strain relation for biaxially strained concrete [Fig. 17.3(a)] is necessary. For this, it may be assumed that the general shape of this stress-strain relation remains the same as for uniaxial compression. One such relationship proposed in Ref. 17.4 is given in Eqn. 17.2. Equation 17.2 is also shown in Fig. 17.3(c) where it is compared with the parabolic stress-strain diagram for uniaxial compression

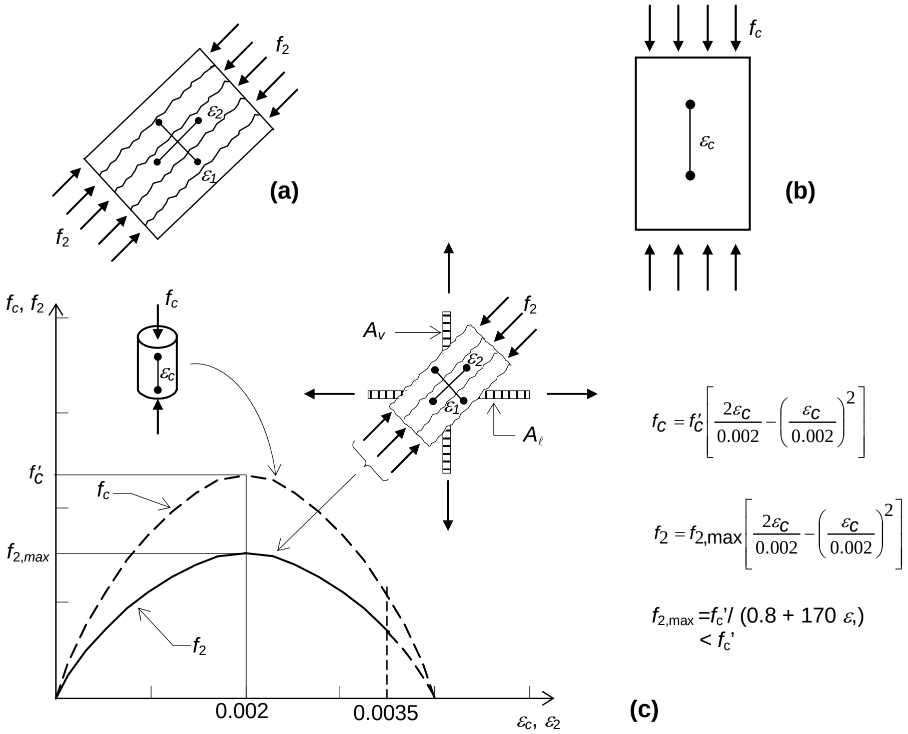



**Fig.17.3** Stress-strain relationship for diagonally cracked concrete

$$f_2 = f_{2,\max}\left[\frac{2\varepsilon_2}{0.002} - \left(\frac{\varepsilon_2}{0.002}\right)^2\right] \tag{17.2}$$

### 17.1.4 Analysis Based on Modified Compression Field Theory

#### (a) *Assumptions and Equations – Case of Pure Shear*

To begin with, the simple case of a symmetrically reinforced beam under pure shear is considered. The effects of bending moment and axial force are considered subsequently.

Prior to cracking, pure shear causes principal tensile and compressive stresses of equal magnitude along diagonal directions, inclined at $45^o$ to the beam axis. After diagonal tension cracks are formed, the corresponding tensile stress in concrete is reduced to zero at the cracks, while the concrete in-between cracks can still sustain tensile stresses. The early compression field theory neglected any contribution to the shear strength from such diagonal tensile stresses in the cracked concrete and assumed the tensile stress in concrete to be uniformly zero throughout and hence was found to be conservative. In contrast, the *modified compression field theory* accounts for the contribution of the diagonal tensile stresses in the cracked concrete. Such tensile stresses vary from zero at the cracks to a maximum value in between cracks and, for deriving equilibrium equations, an *average* value, $f_1$, can be used. This average stress, $f_1$, is less than the maximum tensile stress reached prior to diagonal cracking. Furthermore, the following simplifying assumptions are made in deriving the equations that follow:

(i) The shear stress, $v$, is uniformly distributed over the web, which has a width $b_w$ and depth $d_v$ (taken as the distance between the resultants of the tensile and compressive forces due to flexure), so that.

$$\nu = \frac{V}{b_w d_v} \tag{17.3}$$

(ii) Under uniform shear stress as above, and with symmetry, the longitudinal strain, $\varepsilon_x$, and the inclination, $\theta$, of the principal compressive stress remain constant over the depth $d_v$

(iii) The stress – strain relationship in compression for the diagonally cracked concrete is given by Eqns. 17.1 and 17.2.

With these assumptions, the internal forces, stress and strain distributions and the stress resultants at a section subjected to shear only (such as at a point of contraflexure) are as shown in Fig. 17.4. The Mohr's circles for stress and strain states at all points on the section are shown in Fig. 17.4 (viii) and (ix). From the Mohr's circle of stress,

$$v = \frac{f_1 + f_2}{2} \sin 2\theta$$

$$\Rightarrow \ f_2 = \frac{v}{\sin\theta\cos\theta} - f_1 \tag{17.4}$$

$$= \frac{V}{b_w d_v \sin\theta\cos\theta} - f_1 \tag{17.5}$$

The force in the transverse reinforcement balances the vertical components of the concrete stresses $f_1$ and $f_2$. Considering equilibrium of stirrup forces and vertical components of $f_1$ and $f_2$ acting over the concrete area tributary to a stirrup, as shown in Fig. 17.4,

$$A_v f_v = b_w s\,(f_2 \sin^2\theta - f_1 \cos^2\theta) \tag{17.6}$$

Substituting for $f_2$ from Eqn. 17.5,

$$\frac{A_v f_v}{b_w s} = \frac{V}{b_w d_v} \tan\theta - f_1 \tag{17.7}$$

and
$$V = \frac{A_v f_v d_v}{s} \cot\theta + f_1 b_w d_v \cot\theta \tag{17.8}$$

$$= V_s + V_c \tag{17.8a}$$

where
$$V_s = \frac{A_v f_v d_v}{s} \cot\theta \tag{17.9}$$

and
$$V_c = f_1 b_w d_v \cot\theta \tag{17.10}$$

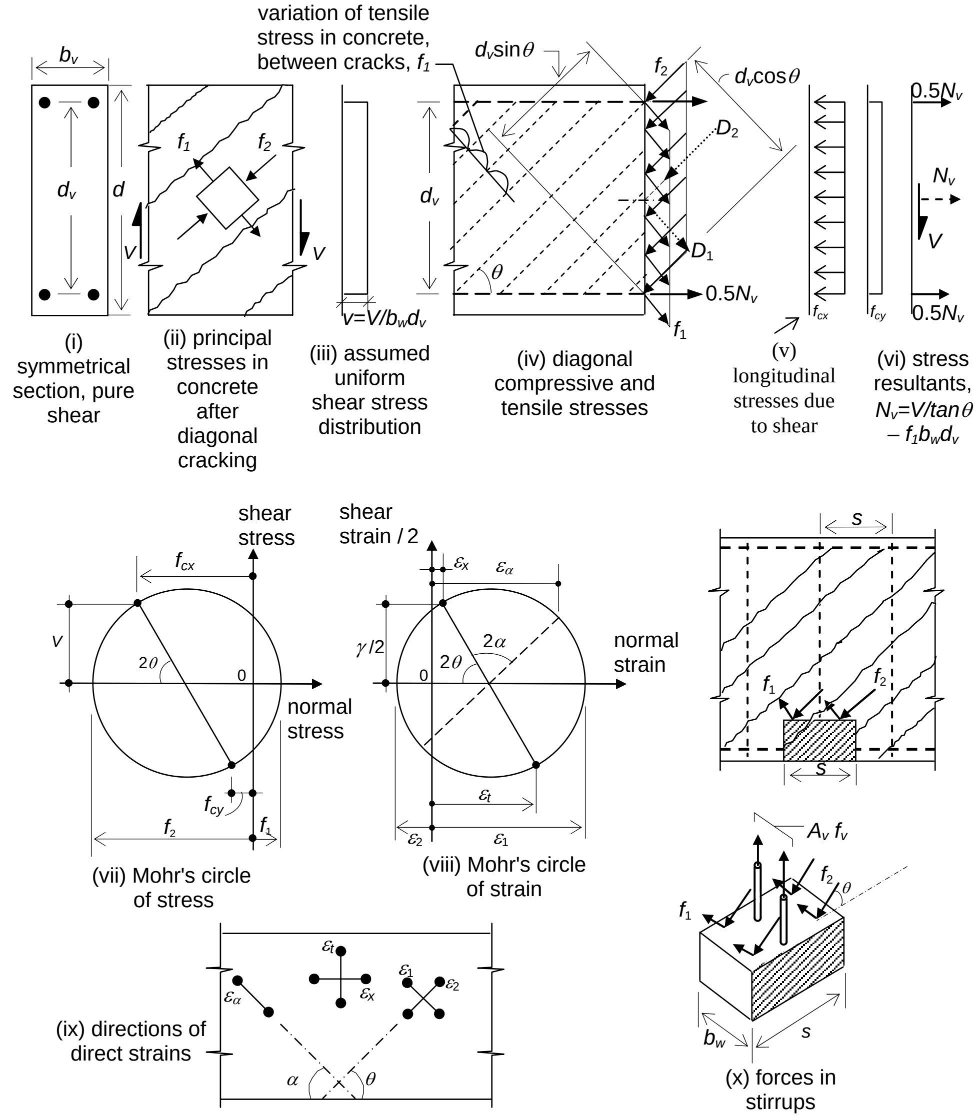



**Fig.17.4** Modified compression field theory–Analysis for shear force V

Equation 17.8 shows that the shear resistance consists of a part, $V_s$, contributed by the shear reinforcement, and a part, $V_c$, contributed by the tensile stresses in concrete. The part $V_c$ depends on the average tensile stress, $f_1$, in the diagonally cracked concrete. $V_s$ is the same as derived earlier in Section 6.7.4. (Strictly, the part $V_s$ includes a concrete contribution arising out of the diagonal compressive stress $f_2$ also. The ultimate shear strength in the conventional method [Eqn. 6.14] also has a concrete contribution, however its value is derived empirically based on a safe limiting value for the nominal shear stress in concrete).

The stresses $f_1$ and $f_2$ over a cross section [Fig. 17.4(iv) and (v)] resulting from shear $V$ has a net axial resultant, $N_v$, given by:

$$N_v = b_w\, d_v\, (f_2 \cos^2\theta - f_1 \sin^2\theta) \tag{17.11}$$

There has to be longitudinal reinforcement to resist this. With longitudinal reinforcement symmetrically placed at top and bottom, the tensile force in each of them will be 0.5 $N_v$. Thus, pure shear necessitates longitudinal reinforcements as well. If $A_{sx}$ is the *total* area of such reinforcement and $f_{sx}$ the tensile stress due to shear, $A_{sx} f_{sx} = N_v$, then substituting for $f_2$ from Eqn. 17.5 into Eqn. 17.11,

$$A_{sx}\, f_{sx} = N_v = V \cot\theta - f_1\, b_w\, d_v \tag{17.12}$$

In Eqns 17.8 and 17.12, $f_1$ is the *average* principal tensile stress carried by the concrete between diagonal cracks. Based on tests [Ref. 17.3], a relation between average tensile stress, $f_1$, and corresponding average tensile strain, $\varepsilon_1$, recommended in Ref. 17.1 is:

$$f_1 = E_c \varepsilon_1 \qquad \text{for} \qquad \varepsilon_1 \le \varepsilon_{cr} \tag{17.13}$$

$$f_1 = \frac{\alpha_1 \alpha_2\, f_{cr}}{1 + \sqrt{500\varepsilon_1}} \qquad \text{for} \qquad \varepsilon_1 > \varepsilon_{cr} \tag{17.14}$$

where $f_{cr}$ is the tensile stress at cracking and factors $\alpha_1$ and $\alpha_2$ account for the bond characteristics of the reinforcement and the type of loading.

There are several other considerations in choosing the appropriate value for $f_1$. In deriving the equations above, uniform *average stresses* and *strains* have been used. However, the tensile stress in concrete will be zero at the crack. There will be a corresponding local increase in the tensile stress in the transverse reinforcement, thereby providing the required tensile stress component across the crack interface. Once the stress in the transverse reinforcement (which is highest at the crack location) reaches yield, any increase in shear force can be resisted only by shear stresses, $v_{ci}$, transmitted along the crack interface [Fig. 17.5(b) and (d)]. The magnitude of the shear stress, $v_{ci}$ that can be transmitted between the two sides along the crack interface will depend primarily on the crack width, $w$, [Fig. 17.5(b)]. The crack width, $w$, in turn depends on the average tensile strain, $\varepsilon_1$, and the average spacing, $s_\theta$, of the diagonal cracks. Recommended limiting value of $v_{ci}$ to avoid slipping along cracks is [Ref. 17.1]:

$$v_{ci} = \frac{0.18\sqrt{f_c'}}{0.3 + \dfrac{24w}{a + 16}} \tag{17.15}$$

where, $a$ = the maximum size of aggregate and $w$ may be taken as the product of the average principal tensile strain and the average crack spacing so that:

$$w = \varepsilon_1\, s_\theta \tag{17.16}$$

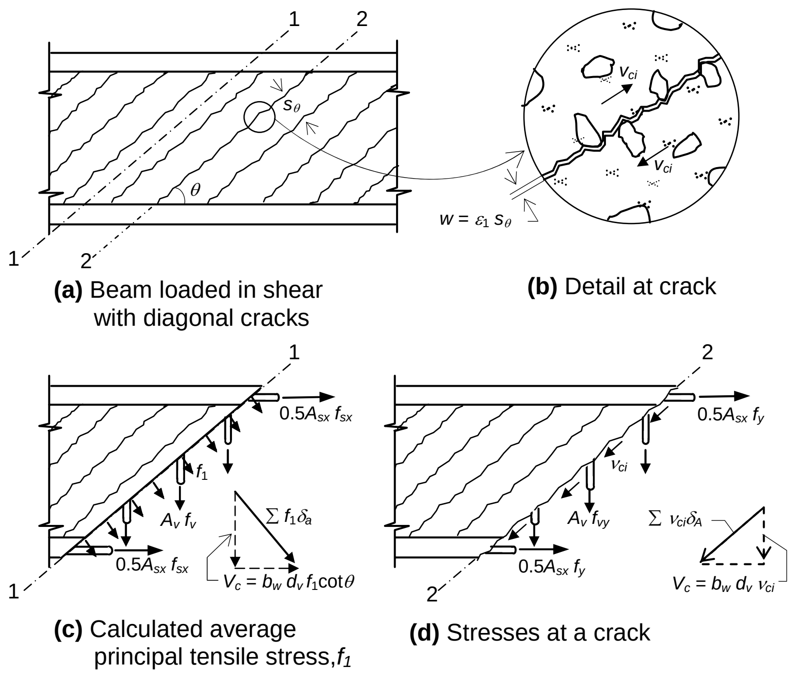



**Fig.17.5** Transmission of forces across diagonal cracks

The spacing of diagonal cracks, $s_\theta$, depends on the type, amount and distribution of the longitudinal and transverse reinforcements. Expressions for estimating $s_\theta$ are given in Ref. 17.1. In Fig. 17.5(c), the *average* tensile stress in concrete, $f_1$, is assumed to be developed midway between diagonal cracks. As one moves towards the crack, the concrete tensile stress decreases and the slack is taken up by increases in transverse reinforcement stress and/or the interface shear, $v_{ci}$. After yielding of transverse reinforcement at a crack, the limit on the stress $v_{ci}$ that can be mobilized can limit the shear capacity of the member.

Equations 17.8 and 17.12 give the shear strength and the axial tensile reinforcement requirements in the case of pure shear. If three parameters, such as $\varepsilon_1$, $\varepsilon_2$, and $\theta$ are known, $f_1$ , $f_2$, and steel stress can be computed from the respective strains and the stress – strain relations, and the shear strength determined. Stresses $f_1$, $f_2$ and $f_v$ must be within their limits. A trial and error procedure is presented in Ref. 17.1. In this, trial values are selected for $\varepsilon_1$, $\theta$, and $f_v$ and these are adjusted until equilibrium, compatibility, and limiting stress conditions are met.

### (b) *Shear with Bending Moment and Axial Force*

In practice, shear occurs in combination with bending moment and, often, axial force as well. Bending moment and/or axial tension increases the axial tensile strain, $\varepsilon_x$, reducing the shear strength. With bending moment, the longitudinal strain, $\varepsilon_x$, and the inclination $\theta$ of the principal compressive stress vary over the depth of the section. Hence, a detailed analysis of a section subjected to shear, moment and axial force is complex [Ref. 17.4]. Therefore, recourse is made to simplifications. One such procedure is to consider the stresses and strains at just one level in the beam depth and to calculate corresponding $\theta$, which is then considered applicable for the entire depth. Again, a trial and error process is required for a solution. The strain profile over the depth (linear variation – plane section theory) and the value of $\theta$ are adjusted so that the stress limits are not exceeded and the internal stresses are in equilibrium with given bending moment, $M$, and axial force, $N$.

### 17.1.5 Simplified Design Procedure using Modified Compression Field Theory

In general shear design involves checking the adequacy of the section, selected based on applied flexure and axial loads, and computing the required shear reinforcements (both transverse and additional longitudinal) to carry the applied shear. The nominal shear strength of the section can be expressed as [also Eqn. 17.8(a)]:

$$V = V_c + V_s \tag{17.17}$$

where $V_c$ is the part contributed by tensile stresses in the concrete and $V_s$ is the part contributed by the transverse reinforcement, given by Eqn. 17.9. Assuming that only the optimum amount of transverse reinforcement is used so that they yield as ultimate strength is reached, $f_v$ can be taken as the yield stress $f_y$, so that:

$$V_s = (A_v\ f_y\ d_v \cot\theta)\ /\ s \tag{17.18}$$

The part $V_c$ is given by Eqn. 17.10, which may be expressed as:

$$V_c = \beta\sqrt{f_c'}\ b_w d_v \tag{17.19}$$

where
$$\beta = f_1 \cot\theta / \sqrt{f_c'} \tag{17.20}$$

Computation of shear strength [Eqn. 17.17] now reduces to the determination of the appropriate values of $\theta$ and $\beta$ to be used in Eqns. 17.18 and 17.19. In Eqn. 17.20, substituting for $f_1$ from Eqn. 17.14, and assuming, $f_{cr}$ = $0.33\sqrt{f_c'}$ and taking the factor $\alpha_1\alpha_2$ as equal to unity,

$$\beta = 0.33\cot\theta/(1+\sqrt{500}\varepsilon_1) \tag{17.21}$$

If the transverse reinforcement has yielded at failure, the shear contribution $V_c$ has to be maintained at the diagonal cracks by the transverse component of the interface shear $v_{ci}$ [Fig. 17.5(a) and (b)]. Then:

$$V_c = f_1\ b_w\ d_v \cot\theta\ = v_{ci} b_w\ d_v \tag{17.22}$$

$$\Rightarrow f_1 = v_{ci} \tan\theta \tag{17.22a}$$

As discussed earlier, to avoid slipping along the diagonal cracks [Fig. 17.5b], $v_{ci}$ has to be within its limiting value [Eqn. 17.15]. Correspondingly, $f_1$ and hence $\beta$ have limiting values. Substituting the limiting value for $v_{ci}$ from Eqn. 17.15, the corresponding limit on $f_1$ is:

$$f_1 = \frac{0.18\sqrt{f_c'}}{0.3 + \dfrac{24w}{a+16}} \tan\theta \tag{17.23}$$

Substituting this value of $f_1$ in Eqn. 17.20 yields the limiting value for $\beta$ as:

$$\beta \le \frac{0.18}{0.3 + \dfrac{24w}{a+16}} \tag{17.24}$$

Thus the expressions for $\beta$ are:

$$\beta = \frac{f_1 \cot\theta}{\sqrt{f_c'}} = \frac{0.33\ \cot\theta}{1+\sqrt{500\varepsilon_1}} \le \frac{0.18}{0.3 + \dfrac{24w}{a+16}} \tag{17.25}$$

where, $w$ = $\varepsilon_1 s_\theta$ is the crack width,
$a$ = maximum size of aggregate,
$s_\theta$ = average spacing of diagonal cracks, and
$\varepsilon_1$ = average principal tensile strain in concrete.

It can be seen from Eqn. 17.25 that as $\varepsilon_1$ increases, $\beta$ and $V_c$ decreases. $\varepsilon_1$ will depend on magnitudes of $\varepsilon_x$, $\theta$ and $\varepsilon_2$ From the Mohr's circle of strain, the principal tensile strain, $\varepsilon_1$, may be expressed as:

$$\varepsilon_1 = \varepsilon_x + (\varepsilon_x + \varepsilon_2) \cot^2\theta \tag{17.26}$$

For diagonally cracked concrete, $\varepsilon_2$ is given by Eqn. 17.2 as:

$$\varepsilon_2 = 0.002\left(1 - \sqrt{1 - f_2 / f_{2,\max}}\right)$$

where $f_{2,\max} = \sqrt{f_c'} / (0.8 + 170\varepsilon_1)$

and $f_2$ may be taken conservatively (neglecting value of $f_1$ in Eqn. 17.4) as:

$$f_2 = v / (\sin\theta \cos\theta) = v(\tan\theta + \cot\theta)$$

Substituting these values in Eqn. 17.26,

$$\varepsilon_1 = \varepsilon_x + \cot^2\theta \left[\varepsilon_x + 0.002\left(1 - \sqrt{1 - \frac{V}{\sqrt{f_c'}}(\tan\theta + \cot\theta)(0.8 + 170\varepsilon_1)}\right)\right] \tag{17.27}$$

If $\varepsilon_x$, $v/f_c'$ and $\theta$ are known, $\varepsilon_1$ can be found and $\beta$ can be computed from Eqn. 17.25 assuming crack spacing $s_\theta$ and aggregate size $a$, if not known). An increase in the strain $\varepsilon_x$ results in a decrease in the shear strength. The axial strain $\varepsilon_x$ may be taken conservatively as the longitudinal strain in the flexural tension chord of the equivalent truss [Fig. 17.6]. Accordingly, at a section subjected to a bending moment $M$ and axial force $N$,

$$\varepsilon_x = \frac{0.5(N + V\cot\theta) + M / d_v}{E_s A_s} \tag{17.28}$$

The value of the parameter $v/ f_c'$ can be computed knowing the applied shear force $V$. For $\theta$, a trial-and-error approach is needed. The appropriate value of $\theta$ is chosen such that:

(i) $f_2$ does not exceed $f_{2,\max}$
(ii) strain in transverse reinforcement, $\varepsilon_v$, is at least equal to 0.002, and
(iii) the shear reinforcement is near minimum

On the above basis, Tables/graphs have been prepared giving values of $\theta$ and $\beta$ for different combinations of $v / f_c'$ and $\varepsilon_x$. For members containing at least a minimum amount of transverse reinforcement, for computing the $\beta$ values the average spacing of diagonal cracks is assumed as 300 mm and the aggregate size as about 19 mm.

The strain in the longitudinal reinforcement has its peak value at the crack location. Consider the stress patterns *for pure shear* given in Fig. 17.5. Equating the resultant horizontal force for the *average* stress condition along 1-1 (midway between cracks) shown at (c), and for the conditions at a diagonal crack along 2-2 shown at (d), if yielding of longitudinal bars at the crack is to be avoided,

$$A_{sx} f_y - v_{ci}\, b_w\, d_v \cot\theta \geq A_{sx} f_{sx} + b_w\, d_v\, f_1 \tag{17.29}$$

where $A_{sx}$ is the *total area* of longitudinal steel (both at top and bottom included). When the transverse reinforcement yields at failure, $v_{ci}$ is given by Eqn. 17.22. Further, for average stress conditions, $A_{sx} f_{sx}$ is given by Eqn. 17.12. Substituting these values in the above equation and simplifying yields:

$$A_{sx} f_y \geq V \cot\theta + f_1\, b_w\, d_v \cot^2\theta \tag{17.30}$$

Substituting $V_c = f_1\, b_w\, d_v \cot\theta$ from Eqn. 17.10

$$A_{sx} f_y \geq V \cot\theta + V_c \cot\theta$$

Since $V_c + V_s = V$,

$$A_{sx} f_y \geq (2V - V_s) \cot\theta$$

Considering the reinforcement on one side only,

$$A_s f_y \geq (V - 0.5V_s) \cot\theta$$

If stresses due to applied bending moment, $M$, and axial tension, $N$, are also included, to avoid yielding of the longitudinal reinforcement on the flexural tension side,

$$A_s f_y \geq \frac{M}{d_v} + 0.5N + (V - 0.5V_s) \cot\theta \tag{17.31}$$

For members without transverse reinforcements, the spacing of the diagonal cracks will be greater than the 300 mm assumed in the above case. For such cases also, tables have been prepared listing $\theta$ and $\beta$ values for various combinations of longitudinal strain $\varepsilon_x$ and a crack spacing parameter. In both cases, an over estimation of $\varepsilon_x$ will give more conservative predictions of the shear strength.

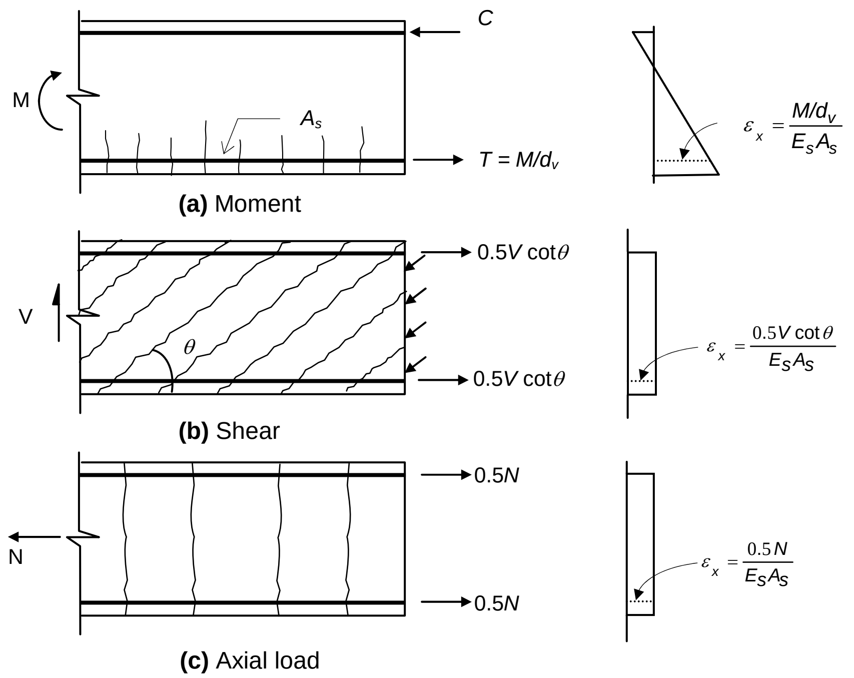



**Fig.17.6** Longitudinal strain at flexural tension steel level

***Summary***

This method aims to arrive at more rational solutions by considering such aspects as influence of cracking on compressive strength, the tensile strength of cracked concrete, variable angle of inclination of principal stresses, influence of shear on stresses in longitudinal reinforcements, strain compatibility, etc. At the same time, to make the procedure tractable and suitable for a code format, a series of simplifying assumptions are made. These include

- neglect of the redistribution of shear stress,
- assumption of uniform shear stress distribution over the depth,
- consideration of the stresses and strains at only one level in the cross section and applying the results to the entire section,
- taking the longitudinal strain at the flexural tension steel level,
- assumption of a constant crack spacing of 300 mm for all beams with shear reinforcement,
- assumption that the shear reinforcement yields, working out the equations for pure shear and accounting for effects of flexure and axial load by modifying longitudinal steel strain only.

Despite the above simplifying assumptions, a closed form solution is not possible and a trial-and-error approach involving complex equations, tables and charts are used. Even so, modification factors have to be applied in some cases, as with the constant 1.3 in Eqn. 17.33. Unfortunately, the objective for rigour is compromised by the need to introduce so many assumptions. Indeed, the traditional method uses far less and more justifiable assumptions and is supported by the sound concepts of the Strut-and-Tie model. In any case, IS Code does not specify Compression Field Theory as a method for design in shear. For these reasons, the authors do not recommend this as a standard method for design especially in the Indian context. However, the topic has been included as it is a relatively recent theoretical development. For the sake of completeness, the CSA Code provisions are given below and an example worked out later [Ex. 17.1].

### 17.1.6 CSA Code Provisions for Shear Design by the Compression Field Theory

IS 456:2000 does not include any provision based on compression field theory. One of the Codes which introduced the compression field theory for shear design early on is the Canadian Standards Association (CSA) Standard CSA A23.3-94: *Design of Concrete Structures*. The provisions in that Code are briefly discussed here.

CSA Standard uses the cylinder strength $f_c'$ as the specified concrete strength. To relate this to the cube strength, Eqn. 2.3 may be used. Moreover, this Code uses *material resistance factors* of $\phi_c = 0.6$ for concrete and $\phi_s = 0.85$ for reinforcing bars. These material resistance factors correspond to the inverse of the partial safety factors for materials, $\gamma$, referred to in Section 3.6.2. The factor $\lambda$ accounts for the effects of concrete density on tensile strength and other properties; [see also Section 6.9.2].

The procedure designated as "*general method*" for shear design in CSA A23.3–94 (Cl. 11.4) follows the simplified procedure described in Section 17.1.5. The load and resistance factors are also incorporated. The controlling design equation is:

$$V_{rg} = V_{cg} + V_{sg} \geq V_f \tag{17.32}$$

where, $V_{rg}$ is the factored shear resistance,
$V_{cg}$ is the factored shear resistance attributed to concrete,
$V_{sg}$ is the factored shear resistance provided by the shear reinforcement,
$V_f$ is the factored shear force at the section, and
$\lambda$ is the factor to account for low density concrete

$$V_{cg} = 1.3\lambda\phi_c\beta\sqrt{f_c'}\,b_w d_v \tag{17.33}$$

For stirrups perpendicular to beam axis:

$$V_{sg} = \frac{A_v\phi_s f_y d_v \cot\theta}{s} \tag{17.34}$$

For transverse reinforcement inclined at an angle $\alpha$ to the longitudinal axis,

$$V_{sg} = \frac{A_v\phi_s f_y d_v (\cot\theta + \cot\alpha)\sin\alpha}{s} \tag{17.35}$$

The factor 1.3 in Eqn. 17.33 compensates for the low value of $\phi_c$ and partially offsets the conservatism of this method. The expressions for $V_{sg}$ are the same as those derived on the basis of the truss model in Section 6.7.4 [Eqns. 6.18(a)]. To ensure that the transverse reinforcement will yield prior to the crushing of the concrete in the web in diagonal compression, $V_{rg}$ is limited to:

$$V_{rg} \le 0.25\phi_c f_c' b_w d_v \tag{17.36}$$

Tables and graphs are presented in the Code for determining values of $\beta$ and $\theta$ for sections with and without the minimum amount of transverse reinforcements. For sections *with transverse reinforcement*, the table is in terms of parameters $v_f/(\lambda\ \phi_c\ f_c')$, where $v_f$ is the factored shear stress, and $\varepsilon_x$, the longitudinal strain at the tension steel level. For evaluating these parameters,

$$v_f = V_v/(b_w d_v) \tag{17.37}$$

and

$$\varepsilon_x = [0.5(N_f + V_f \cot\theta) + M_f / d_v]/(E_s A_s) \le 0.002 \tag{17.38}$$

For sections without the minimum transverse reinforcement, the parameters to be used are the crack spacing parameter, $s_z$, determination of which is as per CSA Code Cl.11.4.7, and $\varepsilon_x$.

Longitudinal reinforcement is to be designed for the combined effects of flexure, axial load and shear. Accordingly, as in Eqn. 17.31, at all sections,

$$A_s f_y \ge M_f / d_v + 0.5N_f + (V_f - 0.5V_{sg})\cot\theta \tag{17.39}$$

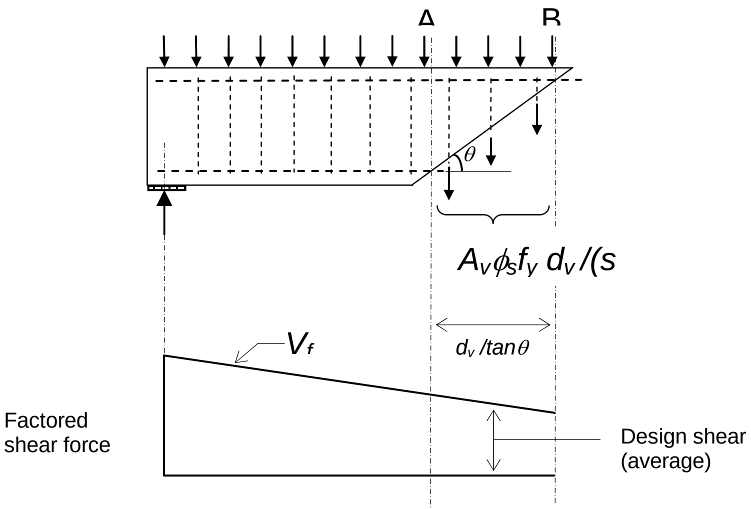



**Fig. 17.7** Design for average shear over length $d_v\cot\theta$

In the case of members not subjected to significant axial tension, the requirement of Eqn. 17.39 may be satisfied by extending the flexural tension reinforcement a distance of $d_v \cot\theta$ beyond the location needed for flexure alone [compare with Fig. 6.10(a)]. Similarly computation similar to development of Eqn. 6.21 [Fig. 6.10(b)] with diagonal crack at angle $\theta$ will show that at exterior direct bearing supports, the bottom longitudinal reinforcement should be capable of resisting a tensile force $T$ at the inside edge of the bearing area, given by:

$$T = (V_f - 0.5V_{sg})\cot\theta + 0.5N_f \tag{17.40}$$

Shear failure by yielding of transverse reinforcements involves the reinforcements over a length of about $d_v \cot\theta$, as can be seen in Fig. 17.7. Accordingly, the CSA Code (Cl. 11.4.8) permits the provision of transverse reinforcement over such lengths based on the average requirement for this length.

### 17.1.7 Combined Shear and Torsion

The shear stress due to torsion and the shear stress due to transverse shear are additive on one side of the cross section and they counteract on the opposite side. The transverse reinforcement is designed considering the side where the stresses are additive. The Code requires that the transverse reinforcement provided shall be at least equal to the sum of that required for the shear and the coexisting torsion. The equations presented in Section 17.1.6 above are used to compute the area of transverse reinforcement required for shear. Equations based on the *space truss analogy*, with cracks making an angle $\theta$ with the longitudinal axis, and assuming that cracked concrete carries no tension, are used for the computation of the transverse reinforcement required for torsion [see Section 7.4.2 and Fig. 7.6]. On this basis, the factored torsional resistance of the section, $T_{rg}$, is given by [see Eqns. 7.12 and 7.13]:

$$T = 2A_o \frac{\phi_s A_t f_y}{s} \cot\theta \tag{17.41}$$

Here $A_t$ is the area of one leg of closed transverse torsion reinforcement The area enclosed by shear flow path, $A_o$, is to be taken as 0.85 $A_{oh}$, where $A_{oh}$ is the area enclosed by centerline of exterior closed transverse torsion reinforcement. Angle $\theta$ is to be determined from tables and graphs given in the Code, as explained in Section 17.1.6. To determine $\theta$, the factored shear stress, $v_f$, and the longitudinal strain, $\varepsilon_x$, are required.

For thin walled tubular sections, the torsional shear is uniform over the thickness [Chapter 7]. Hence for box type sections, the factored shear stress due to combined shear, $V_f$, and torsion, $T_f$, is given by:

$$v_f = \frac{V_f}{b_w d_v} + \frac{T_f\, p_h}{A_{oh}^2} \tag{17.42}$$

For other cross sectional shapes, such as a rectangle, torsional shear stress at first (diagonal) cracking varies over the section from zero to a maximum, with the possibility for considerable redistribution. To allow for this, the factored shear stress is taken as:

$$v = \sqrt{\left(\frac{V_f}{b_w d_v}\right)^2 + \left(\frac{T_f\, p_h}{A_{oh}^2}\right)^2} \tag{17.43}$$

where $p_h$ is the perimeter of the centerline of the closed transverse torsion reinforcement. The longitudinal strain, $\varepsilon_x$, may be taken as 0.002, or alternatively computed from Eqn. 17.44 below:

$$\varepsilon_x = \frac{0.5N_f + 0.5\cot\theta\sqrt{V_f^2 + \left(\frac{0.9\,p_h}{2A_0}\right)^2} + \frac{M_f}{d_v}}{E_s A_s} \geq 0 \tag{17.44}$$

Similarly, allowing for torsion also, the longitudinal reinforcement is to be proportioned such that:

$$A_s \phi_s f_y \geq \frac{M_f}{d_f} + 0.5N_f + \cot\theta\sqrt{\left(V_f - 0.5V_{sg}\right)^2 + \left(\frac{0.45\,p_h T_f}{2A_0}\right)^2} \tag{17.45}$$

## 17.2 DESIGN USING STRUT-AND-TIE MODEL

The *strut-and-tie* concept was mentioned in Section 15.2.3. Reinforced concrete members or portions of them can be analyzed, designed and detailed by idealizing them as composed of a series of reinforcing steel tensile ties and concrete compressive struts, interconnected at nodes to form a truss capable of transmitting the loads to the supports. This *strut-and-tie* concept is depicted in Fig. 17.8. It is a very basic concept in structural design that, for transferring a system of loads to the supports, any stable skeletal framework such as a truss, grid, arch or catenary, compatible with the actual deformation pattern, may be delineated, and the members and their joints designed for the resulting forces thereon. The skeleton (or truss/arch/catenary) may be either explicit and externally visible, as in a real truss, or implicit and embedded within a member, as in the case of the truss analogy for shear design of concrete beams [Fig. 6.9] and the truss analogy for plate girder design.

For a given structure and loading, a number of different strut-and-tie arrangements are conceivable. Thus, for a *deep beam*, for instance, it is possible to conceive a triangulated truss, a tied arch or a strutted catenary, as depicted Fig. 17.9, for the purpose of modeling the skeletal load transfer scheme and for designing accordingly. The optimum design is the one that is economic, has stability, adequate reserve strength and ductility, and meets the serviceability conditions satisfactorily. Usually, the model involving the most direct load path to supports and relatively large angles between the strut and tie at nodes will be more efficient. This is a very simple and handy concept for design and detailing, particularly in regions with discontinuities, such as locations adjacent to supports, concentrated loads, or abrupt changes in cross section [refer Fig. 17.8]. Insightful designers have always used this concept for design in special situations, such as openings in webs of beams, corbels, end blocks in prestressed concrete beams, etc.

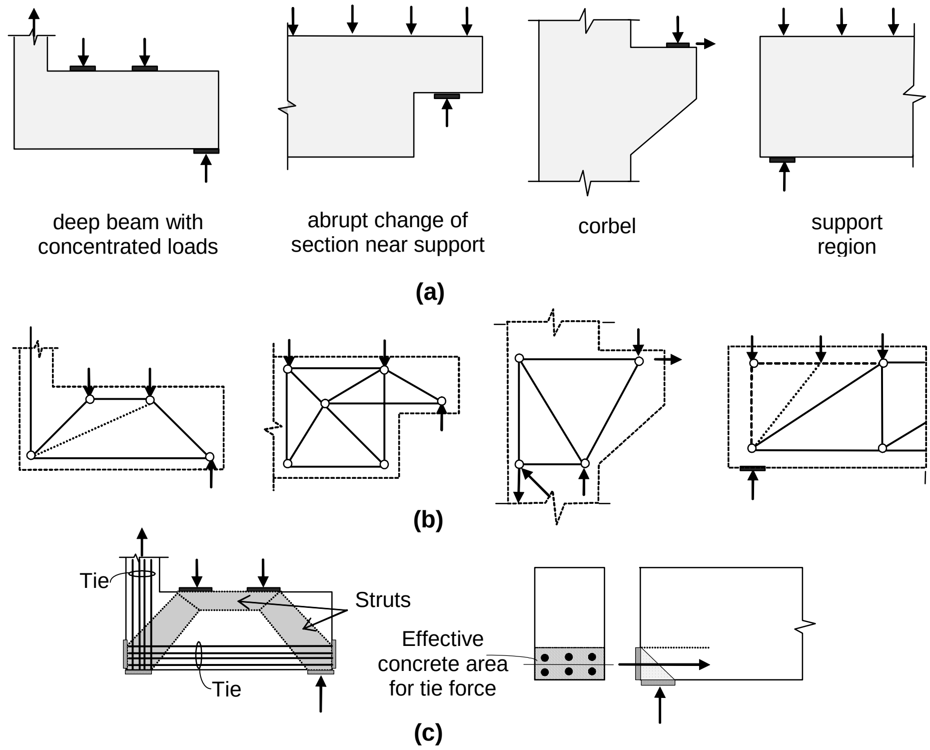



**Fig. 17.8** Examples of the *strut-and-tie* (truss) concept of load transfer

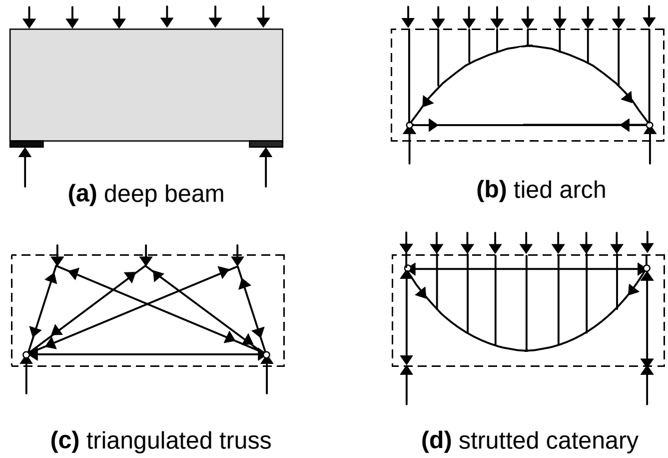



**Fig. 17.9** Strut-and-tie model — Alternative schemes of load transfer

Truss models for load transfer schemes for the cases shown in Fig. 17.8(a) are shown in Fig. 17.8(b). The forces in the truss members can be computed using methods of truss analysis. The truss members and their joints (node regions) must have adequate strength to carry and transmit these forces. In the case of a real truss, the identification of the member areas and joint details, and their design is fairly straight-forward. However, in the case of an implicit truss embedded in concrete such as the one shown in Fig. 18.8(b) and (c) and Fig. 17.10, the determination of appropriate member cross sectional areas and node dimensions is not so simple, especially for the compressive struts and nodes. Furthermore, the concrete in the struts may have tensile cracking parallel to its axis and the nodes may be under biaxial or triaxial states of stress. Although IS 456:2000 (Cl. 28) recommends the strut-and-tie model for design of corbels, no guidelines are given for determination of concrete strut and node dimensions and for the allowable stresses. Hence, the following general guidelines, which are based on CSA Standard A23.3-94, may be helpful here[†]:

1. The stress distribution in the cross section of a truss member may be assumed to be uniform, so that the member force will act along the member centerline.
2. All member forces, loads, and reactions meeting at a node must form a system of concurrent forces.
3. The node region is bounded by sections of the members meeting at the node and the load or reaction bearing area, if applicable.
4. The member section areas and the node dimensions should be adequate to carry the loads without exceeding the stress limits applicable.

[†] Note that in using these provisions and the associated limiting stresses borrowed from the Canadian Code, concrete strength is the cylinder strength and material resistance factors are to be used with characteristic strengths [see also Section 6.9.2]

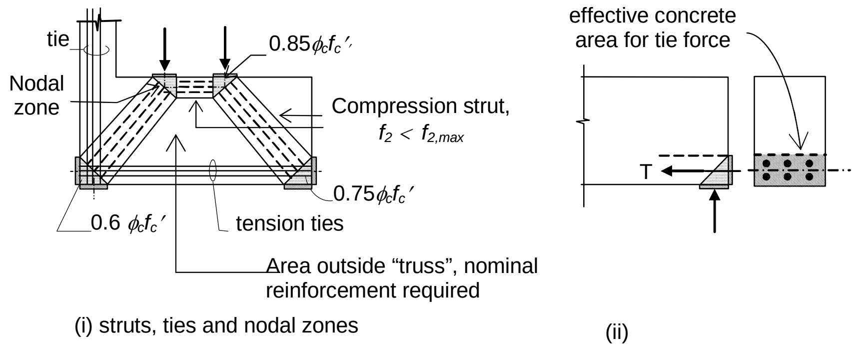
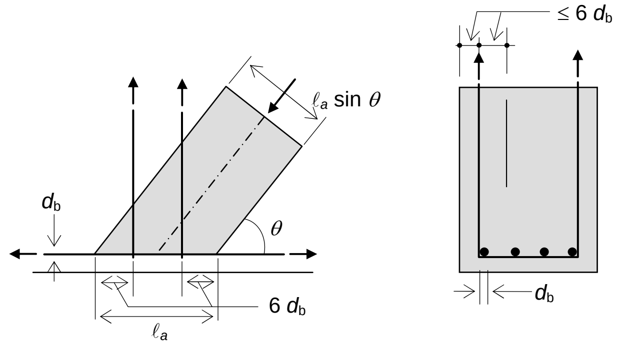



**Fig. 17.10** Identification of Strut-and-Tie members and nodes

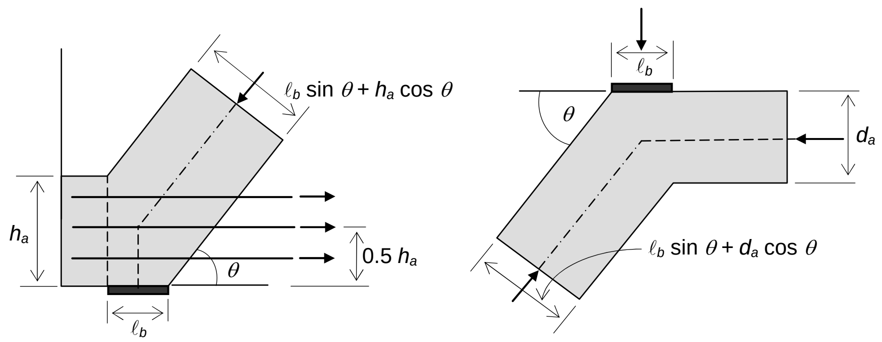
Flexural members may be designed for shear and torsion using the strut-and-tie model. Further, regions of members where the assumption "plane sections remain plane" is not applicable have to be proportioned for shear and torsion using the strut-and-tie model. Such regions include areas of static or geometric discontinuities, deep beams and corbels [Fig. 17.8].

The tension reinforcement forms the effective tension tie and hence its required area, $A_{st}$, is obtained as the tie force divided by $\phi_s f_y$. This reinforcement has to be so distributed as to give adequate dimensions for the nodes and joining compressive struts to carry their respective forces satisfactorily [Fig 17.10]. This reinforcement must be anchored by appropriate embedment lengths, hooks, or mechanical devices so that it is capable of developing the required stress at the inner edge of the node region [Chapter 8]. For straight bars, if the extension beyond the inner edge of the node region, $x$, is less than the development length, $l_d$, of the bar, the bar stress has to be limited to $f_y\,(x\,/l_d)$.

The cross sectional area of the strut has to be computed based on the guidelines given above and considering the concrete area available, as well as the anchorage conditions at the end of the strut. A few typical cases are depicted in Fig. 17.11.




**(a)** Strut anchored by reinforcement




**(b)** Strut anchored by bearing plate and reinforcement

**(c)** Strut anchored by bearing plate and strut

**Fig. 17.11** Compressive strut dimensions

The strut is likely to have tension cracks developed parallel to its axis [Fig. 17.12]. If so, the allowable compressive stress in the strut has to take this into account [Section 17.1.3]. If a tie reinforcement is crossing a strut [Fig. 17.10] and has a strain $\varepsilon_s = f_s / E_s$ along the tie, the principal strains in the concrete at this location, $\varepsilon_1$ and $\varepsilon_2$, must be compatible with $\varepsilon_s$. On this basis, and assuming that the maximum principal compressive strain, $\varepsilon_2$, in the direction of the strut equals 0.002, the following equations may be used for the limiting compressive stress in concrete strut, $f_{cu}$.

$$f_{cu} = \frac{f_c'}{0.8 + 170\varepsilon_1} \quad \le 0.85 f_c' \tag{17.46}$$

where,
$$\varepsilon_1 = \varepsilon_s + (\varepsilon_s + 0.002)\cot^2\theta_s \tag{17.47}$$

and $\theta_s$ is the smallest angle between the compressive strut and the adjoining tensile tie and $\varepsilon_s$ is the tensile strain in this tie.

If the compressive strut is reinforced for compression with bars having an area $A_{ss}$, placed parallel to the strut axis and detailed so as to develop its yield strength, $f_y$, the limiting strength of the strut is given by :

$$A_c \phi_c f_{cu} + A_{ss} \phi_s f_y \tag{17.48}$$

Concrete in the nodal zone is subjected to multidirectional compression, where the struts and ties of the truss meet. The allowable compressive stress in these regions are dependant on the degree of confinement and the adverse effects of tensile straining caused by anchoring of tension ties in this region, if any. The effective area of the node zone can be increased and the stresses in the region reduced by increasing the size of the bearing plates, by increasing the section dimensions of the compressive struts, and by increasing the effective anchorage area of tension ties. Unless special confining reinforcement is provided, the compressive stresses in the concrete in the node regions are limited to the maximum values given below:

(a) $0.85\phi_c f_c'$ in node regions bounded by compressive struts and bearing areas

(b) $0.75\phi_c f_c'$ in node regions anchoring a tension tie in only one direction; and

(c) $0.65\phi_c f_c'$ in node regions anchoring tension ties in more than one direction.

Examples of zones to which each of these limits apply are identified in Fig. 17.10(i). In most situations, since compressive stress in the compression strut is limited to a maximum of $f_{2,\max}$ [Eqn. 17.46], the compressive stress on the face of the nodal region bearing against a compression strut will be within safe limits. The stress limits in the node region may be considered satisfied if:

1. The bearing stress due to concentrated loads or reactions does not exceed the limits given above, and
2. The tie reinforcement is uniformly distributed over an effective area of concrete at least equal to the tie force divided by the stress limits given above.

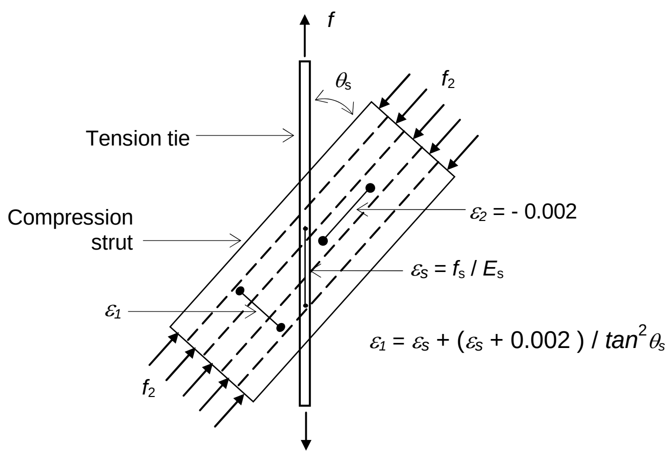



**Fig. 17.12** Strain conditions in concrete strut

The strut-and-tie portion identified and discussed so far leaves large areas of concrete in the member, outside the truss, unreinforced [see Fig. 17.10(i)]. In order to control crack widths and to impart some ductility to the member, the Code also requires placing of an orthogonal grid of reinforcing bars near each face. Such reinforcement should be not less than 0.002 times the gross concrete area in each direction, and should have a maximum spacing of 300 mm.

## EXAMPLE 17.1

Design the shear reinforcement for the T – beam shown in Fig. 17.13(a) using the General Method. Assume that the flexural reinforcement consists of 6 No. 30 bars (No. 30 bar has a nominal diameter of 29.9 mm and area of 700 mm$^2$), curtailed as shown in the figure. The shear force envelope may be assumed linear, varying from 257 kN at centre of support to 44.6 kN at mid-span as shown in Fig 17.13(b). Concrete has a cylinder strength of 20 MPa.

## SOLUTION†

**Shear force diagram**

The factored shear force envelope is shown in Fig. 17.13(b).

**Check adequacy of the section**

Near the support section, where the shear is maximum,

$d$ = 434 mm, $b_w$ = 300 mm, $d_v = 0.9\,d$ = 391 mm

Admissible maximum $V_{rg}$ is given by Eqn. 17.36

$$V_{rg,max} = 0.25\ \phi_c f_c'\ b_w\ d_v$$
$$= 0.25 \times 0.6 \times 20 \times 300 \times 391 \times 10^{-3}$$
$$= 351.9 \text{ kN}$$

The maximum shear at the critical section near the support, distant $d_v$ from the face of the support is

$$V_f = 44.6 + (257 - 44.6)\,(4 - 0.12 - 0.391) / 4$$
$$= 230 \text{ kN} \quad < \quad V_{rg,max} \qquad \text{OK}$$

† Note that the reference to Code clause in this example refers to CSA A23.3-94. The solution here is given only to demonstrate the procedure. Since the Tables for values of θ and β given in the Canadian Code are not reproduced here, the reader may not be able to adopt this method for design.

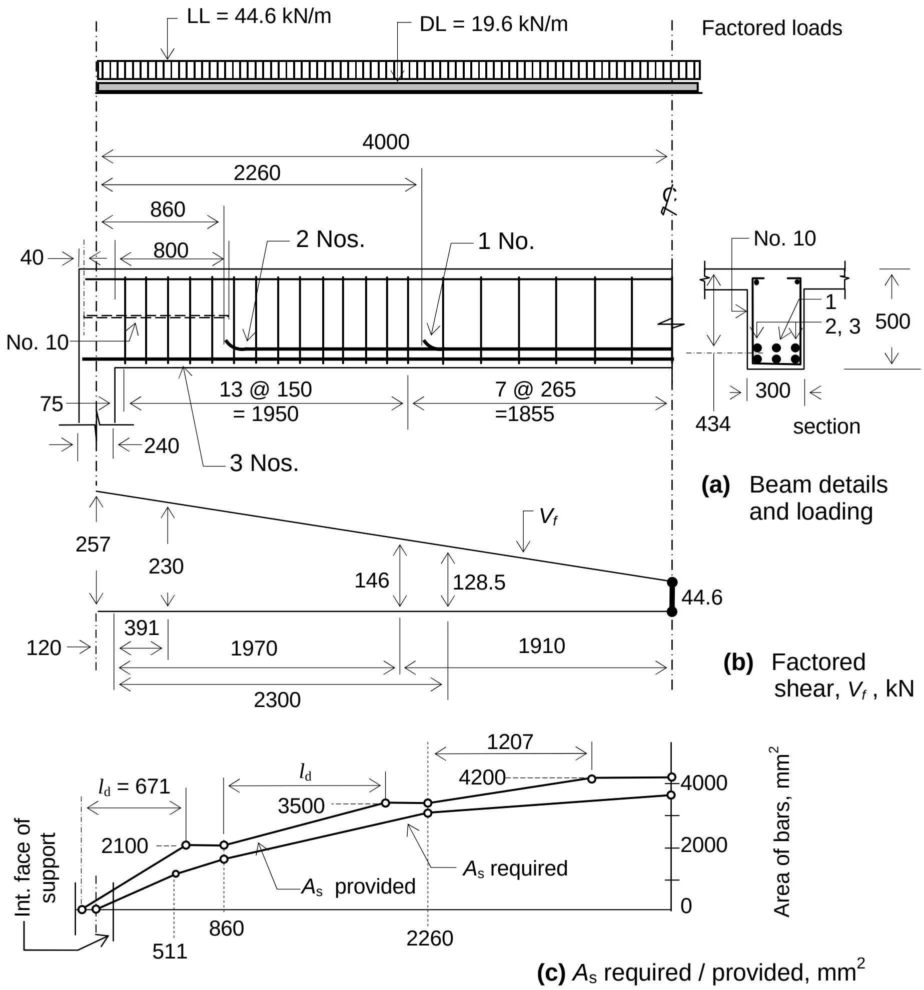



**Fig. 17.13** Example 17.1

**Different zones for shear reinforcement and spacing.**

As per **CSA A23.3-94**, shear reinforcement and spacing restrictions are:

(i) $V_f \le 0.5V_c$, No shear reinforcement required

(ii) $0.5V_c < V_f < 0.1\,\lambda\,\phi_c\,f_c'\,b_w\,d_v$, spacing limited to 600mm and 0.7$d$

(iii) $0.1\,\lambda\,\phi_c\,f_c'\,b_w\,d_v < V_f$, spacing limited to 300mm and 0.35$d$

- Calculation of $V_c$ [**CSA A23.3-94** Cl. 11.2.8.2]

$$V_c = 0.2\lambda\phi_c\sqrt{f_c'}b_w d_v$$

$$= 0.2 \times 1.0 \times 0.6 \times \sqrt{20} \times 300 \times 434 \times 10^{-3}$$

$= 69.87$ kN

Since the minimum $V_f$ = 44.6 kN is greater than $0.5V_c$ calculated above, shear reinforcement is required throughout the length of the beam.

- Spacing limitations

The normal spacing requirements [**CSA A23.3-94** Cl. 11.2.11(a) **-** 600mm and 0.7 *d*] applies where

$V_f < 0.1\, \lambda\, \phi_c\, f_c'\, b_w\, d_v$

$= 0.1 \times 1.0 \times 0.6 \times 20 \times 300 \times 406 \times 10^{-3}$

$= 146$ kN

The location where $V_f$ = 146 kN is given by

(146 – 44.6) × 4000 /(257 – 44.6) = 1910 mm from midspan, or a distance of 1970 mm from face of support.

Where $V_f$ > 146 kN, spacing limits are 300 mm and $0.35d$

**Design of stirrups**

- Critical section at $d_v$ = 391 mm from face (or 371+120 = 511mm from centre) of support, where $V_f$ = 230 kN.

The parameters $\theta$ and $\beta$ are to be determined from Table 11.1 of **CSA A23.3-94**. For this the factored shear stress is $v_f = V_f/(b_w\, d_v)$

$= 230 \times 10 /(300 \times 391)$

$= 1.96$ MPa

Factored shear stress ratio,

$v_f /(\lambda\, \phi_c\, f_c') = 1.96 / (1.0 \times 0.6 \times 20)$

$= 0.163$

Longitudinal strain is given by Eqn. 17.38,

$$\varepsilon_x = ( 0.5\, V_f \cot\theta + M_f /d_v ) / (E_s A_s),$$

where $M_f$ is the bending moment at the critical section at 391 mm from the support, corresponding to the load causing maximum shear at this section. However, for convenience and to be conservative, this moment is taken here as the moment with full load on the entire span. Accordingly,

$M_f = 64.2 \times (4 \times 0.511 - 0.511^2 /2 )$

$= 122.8$ kN·m

$\varepsilon_x = (0.5 \times 230 \times 10^3 \cot\theta + 122.8 \times 10^6 / 391) / (200\,000 \times 2100)$

$= ( 0.274 \cot\theta + 0.748 ) \times 10^{-3}$

Since $\theta$ is not yet known, a trial and error procedure is needed. It is conservative to overestimate $\varepsilon_x$. For $v_f /(\lambda\, \phi_c\, f_c') \le 0.200$ and $\varepsilon_x \le 0.001$, Table 11.1 of the Code **CSA A23.3-94** gives $\theta = 34.5^o$. With this value of $\theta$, $\varepsilon_x$ is calculated as:

$\varepsilon_x = ( 0.274 \cot 34.5^o + 0.748) \times 10^{-3}$

$= 1.147 \times 10^{-3}$

which is greater than the value 0.001 assumed. Choosing from the table the value for $\varepsilon_x \le 0.0015$, $\theta = 35^o$ and corresponding $\varepsilon_x = 0.001139 < 0.0015$ assumed and hence OK. For this, from Table 11.1, $\beta = 0.100$.

The factored shear resistance contributed by concrete is [Eqn. 17.33]

$V_{cg} = 1.3 \times 1.0 \times 0.6 \times 0.100 \times \sqrt{20} \times 300 \times 391 \times 10^{-3}$

$= 40.9$ kN

The factored shear resistance to be provided by stirrups is

$$V_{sg} = V_f - V_{cg}$$
$$= 230 – 40.9 = 189.1 \text{ kN}$$

Assuming No. 10 U stirrups placed perpendicular to beam axis, the required spacing is given by [Eqn. 17.34]

$$s = (\phi_s A_v f_y d_v \cot\theta) / V_{sg}$$
$$= (0.85 \times 200 \times 400 \times 391 \cot 35^o) / 189\,100$$
$$= 201 \text{ mm}$$

As $V_f$ > 146 kN, the limiting spacing is given by 300 mm or
$0.35\, d = 0.35 \times 434 = 152$ mm

Hence, the limiting spacing controls, and a spacing of 150 mm is selected. The shear force is 146 kN at a distance of 1970 mm from the face of support, and the limiting spacing is applicable upto this location. Therefore, provide the first stirrup at a distance of 75 mm from the face of the support, followed by 13 more stirrups at 150 mm, covering a total length of 2025 mm from the face of the support.

- Section at 2.3 m from face of support

The Code permits the design of stirrups for the average shear over a length of $d_v \cot\theta$. Here, $d_v \cot\theta$ is in the range of 391cot35° = 558 mm (although both $d_v$ and $\theta$ will vary slightly along the span). In practice, designing for every discrete lengths of $d_v \cot\theta$ is not warranted. In this example, the next section for design is taken at a distance of 2 m from face of support (which is approximately 558/2 mm from the location of the last stirrup designed.

The shear at 2.3 m from face (or 2.42 m from centre) of support is

$$V_f = 44.6 + (257 – 44.6) (4 –2.42) / 4$$
$$= 128.5 \text{ kN.}$$

The bending moment corresponding to this shear is,

$$M_f = 19.6 \times 4 \times 2.42 – 19.6 \times 2.42^2 / 2 + 44.6 \times 5.58^2 \times 2.42 /(8 \times 2)$$
$$= 342.4 \text{ kN·m.}$$

At 2.3 m from face of support, there are 5 No. 30 bars and effective depth is different from at support. Here, conservatively, the effective depth at mid-span, equal to 406 mm will be used for this section also. Corresponding
$d_v = 0.9 \times 406 = 365$ mm.

Factored shear stress ratio,

$$v_f /(\lambda\ \phi_c\ f_c') = 128.5 \times 10^3 /(300 \times 365 \times 1.0 \times 0.6 \times 20) = 0.098$$

Longitudinal strain is

$$\varepsilon_x = (0.5 \times 128.5 \times 10^3 \cot\theta + 342.4 \times 10^6 /365)/ (200\,000 \times 3500)$$
$$= (0.092 \cot\theta + 1.340) \times 10^3$$

For $v_f /(\lambda\ \phi_c\ f_c')$ < 0.100 and $\varepsilon_x$ < 0.0015, from Table 11.1 of **CSA A23.3-94**, $\theta$ = 38°. Corresponding to this, $\varepsilon_x$ = 1.458 × $10^{-3}$ which is less than the 0.0015

assumed and hence OK. Corresponding $\beta = 0.143$.
Shear strength due to concrete,

$$V_{cg} = 1.3 \times 1.0 \times 0.6 \times 0.143 \times \sqrt{20} \times 300 \times 365 \times 10^{-3}$$
$$= 54.62 \text{ kN}$$

Shear due to stirrups is 128.5 – 54.62 = 73.9 kN.

Spacing of stirrups is,

$$s = (0.85 \times 200 \times 400 \times 365 \cot 38^{o})/ 73900 = 430 \text{ mm}$$

The maximum spacing in this region is given by

600 mm or 0.7 $d = 0.7 \times 406 = 284$ mm.

Hence, the limiting spacing controls for the remaining portions of the beam. Here, from the last stirrup already provided earlier, 7 more stirrups may be provided at a uniform spacing of 265 mm, which results in the last stirrup being placed at mid-span. The arrangement of stirrups is shown in Fig. 17.13(a).

**Check adequacy of longitudinal reinforcements**

Out of the 6 – No. 30 bars at mid-span, one is terminated at a distance of 2260 mm and two more at a distance of 860 mm from the center of support, respectively. Allowing for the effects of shear, the required factored resistance of tension reinforcement is given by

$N_s = M_f / d_v + (V_f - 0.5\, V_{sg}) \cot\theta$

or, with the stress in steel fully developed to $f_y$, the required area of steel is given by $A_s = N_s / f_y$. The stress $f_y$ will be developed in a bar at a distance equal to the development length, $l_d$, from the free end.

The loading conditions for the maximum moment and the maximum shear force at a section are different, and the combination to be checked is the maximum $M_f$ and concurrently occurring $V_f$ and vice versa. However, in this example, the maximum moment $M_f$ and the shear $V_f$ determined from the shear envelope in Fig. 17.13(b) will be taken together for the checking. The section at a distance $d_v$ from the face of the support, the sections were bars are terminated and the mid-span section will be investigated. The calculations are shown in Table 17.1.

**Table 17.1**

| **Distance of section from support $^{C}/_{L}$ (m)** | **$M_f$ (kN·m)** | **$V_f$ (kN)** | **$d_v$ (mm)** | **$V_{sg}$ (kN)** | **$\theta$ (degrees)** | **Required $A_s$ (mm$^2$)** |
|---|---|---|---|---|---|---|
| 0.511 | 122.8 | 230 | 391 | 189.1 | 35 | 1269 |
| 0.860 | 197.1 | 211 | 391 | 168.9 | 37 | 1680 |
| 2.260 | 416.4 | 137 | 371 | 90.4 | 39 | 3089 |
| 4.000 | 513.6 | 44.6 | 365.4 | Nil | 43 | 3633 |

The terminated bars will be fully effective only at a distance $l_d$ from the free end. For No. 30 bottom bars in regions containing minimum stirrups, from Table 12.1 of **CSA A23.3-94**,

$$l_d = 0.45 k_1 k_2 k_3 k_4 f_y d_b / \sqrt{f_c'}$$

$$= 0.45 \times 1.0 \times 1.0 \times 1.0 \times 1.0 \times 400 \times 30 / \sqrt{20} = 1207 \text{ mm}$$

The required $A_s$ and the actual area provided are presented graphically in Fig. 17.13c. The longitudinal reinforcement provided is OK.

**Check adequacy of reinforcement at exterior support (CSA A23.3-94** Cl.11.4.9.4)

Tensile force to be resisted at the inside edge of bearing area is [Eqn. 17.40]

$$T = (V_f - 0.5 V_{sg}) \cot\theta = (230 - 0.5 \times 189.1) \cot 35^o = 193.4 \text{ kN}$$

If the bar is provided straight and the cover at the edge is 40 mm, the available development length up to the inside edge of the bearing area is = 240 – 40 = 200 mm.

The stress that can be developed at the inside edge is

$$f_s = (200 / 1207) \times 400 = 66.3 \text{ MPa.}$$

Hence, the force that can be resisted is $A_s f_s = 2100 \times 66.3 \times 10^{-3} = 139$ kN.

This is inadequate. Hence the bars may be provided with hooks so as to develop a stress of at least $193.4 \times 10^3 / 2100 = 92$ MPa.

Providing the bars continued over the support region with standard 90$^o$ hooks, the development length, $l_{dh}$, is given by [**CSA A23.3-94** Cl. 12.5]

$$l_{dh} = 100\, d_b / \sqrt{f_c'}$$

$$= 100 \times 30 / \sqrt{20} = 671 \text{ mm}$$

The stress developed in the bars at the inside edge of bearing area is

$400 \times (200 / 671) = 119$ MPa $> 92$ MPa, OK.

Note that in Fig. 17.13(c), the stresses in the bars are taken as fully developed over a length of 671 mm for the bars at the support and over a length of 1207 mm for the terminated bars.

## 17.3 FIRE RESISTANCE

### 17.3.1 Introduction

The purpose of this Section is to make the reader aware of the need to consider the fire resistance aspect and the related requirements of the applicable building codes, where these can be critical, while designing reinforced concrete structures. It is not the intent here to deal with the behaviour of structures in real fire situations. The objective is only to present some basic information, in order to aid the designer in considering fire hazards and certain fire protection features which could be kept in perspective while doing the structural design and detailing. Indian standard IS 1641 [Ref. 17.5] classifies buildings according to the use or the character of occupancy into 9 groups. Further, the city or area is to be demarcated into distinct Fire Zones (numbered 1 – 3) based on fire hazard inherent in the buildings and structures in the area, according to occupancy groups. The standard IS 1642 [Ref. 17.6] classifies the types of construction, according to fire resistance ratings, into four categories – Type 1 to Type 4. Restrictions are imposed on admissible Type of construction for new buildings erected in different Fire zones [Ref. 17.5]. The *fire resistance ratings* for structural and non-structural members for various Types of construction are specified in IS 1642 (Table 1, Ref. 17.6). Type 1 construction has the highest fire resistance rating and Type 4 the lowest among the four.

***Fire resistance*** generally refers to the property of a material or assembly to withstand fire or give protection from it. As applied to elements of buildings or a structure, it is characterized by the resistance to **flame penetration**, **heat transmission** and **failure**. ***Fire resistance rating*** is the *time in hours* or fraction thereof that a material or assembly of materials will withstand the passage of flame and transmission of heat when exposed to fire under specified conditions of test and performance criteria (or as determined by extension or interpretation of information so derived).

The fire resistance rating of building components is determined by standard fire resistance test. In such tests, a building assembly such as a portion of a floor, wall, roof or column is subjected to increasing temperatures that vary with time, approximating the conditions which would prevail during a fire within a moderate size compartment having a relatively small amount of ventilation. Floor and roof specimens are exposed to the fire from below, beams from the bottom and sides, walls from one side and columns from all sides. The end of the test is reached and the fire endurance of the specimen established when (i) specified rise in temperature of unexposed surface, (ii) flame penetration to the unexposed surface through cracks or fissures or (iii) failure to sustain specified load occurs. In general, fire resistance of concrete floor/roof assemblies and walls is governed by heat transmission (i.e. the temperature of the unexposed surface rises by/to a specified value), and columns and beams by failure to sustain the applied load, or beam reinforcement temperature rising to a limiting value. A standard fire resistance test is specified in IS 3809 [Ref. 17.7]. The designer has also the option to use ratings assigned to common assemblies and members in various codes such as Ref. 17.8.

Fire resistance of concrete elements depends upon details such as member size, cover to steel, reinforcement detailing and type of aggregate (normal weight or light weight).

### 17.3.2 Factors which influence Fire Resistance Ratings of RC Assemblies

***Type of Concrete and Aggregates***

Concrete is one of the most highly fire resistant structural material used in construction. However, the properties of concrete and reinforcing steel do change significantly at high temperatures caused by fire. For both materials, at high temperatures, the strength and modulus of elasticity are reduced, the coefficient of expansion increases, and creep and stress relaxations are considerably higher.

The compressive strength of concrete during fire exposure mainly depends upon the aggregate it contains. Concrete in which the coarse aggregate is limestone, calcareous gravel, sandstone, blast furnace slag or similar dense material containing not more than 30% quartz performs better during fire exposure, compared with concrete with coarse aggregate such as granite, quartzite, siliceous gravel or other dense materials containing more than 30% quartz. In general, low-density aggregate concretes exhibit better fire performance than natural stone aggregate concretes.

***Member Size and Detailing***

Other things being equal, the larger the thickness or overall cross-section of an assembly, the greater its fire resistance rating. In slab-like members, such as floors, roofs and walls, the rating can be improved by increasing the thickness. For beams and columns, larger cross sections suffer less average concrete strength loss than smaller ones for a given period of fire exposure, as the larger sections take longer to heat up. Where a plaster finish is used, the plaster thickness can be included in the member thickness for fire resistance requirements. In the case of sections containing cores or voids, an *equivalent thickness* of the slab must be used. For hollow-core concrete slabs and panels having a uniform thickness and cores of constant cross section throughout their length, the equivalent thickness to be used may be obtained by dividing the net cross sectional area of the slab or panel by its width.

***Reinforcing Steel and Cover***

The loss of strength at high temperatures will be slower for hot rolled reinforcing steel than for cold worked steel and prestressing tendons. Since the load carrying capacity of a member depends largely upon the tensile strength of the reinforcement in it, its fire resistance rating depends upon the type of reinforcing steel and the level of stress in the steel.

The rate at which heat reaches the reinforcement in a member and hence the loss of strength of the reinforcement is inversely proportional to the concrete cover provided. It should be noted that while plaster thickness, where provided, cannot be reckoned as part of cover for meeting the durability requirements (Table 16 of Code), it can be included in the cover thickness for meeting the fire resistance requirements (Table 16A of Code).

***Continuity and Restraint (structural system)***

This depends on (1) boundary conditions and (2) type of structural element.

- <u>Fire endurance of statically determinate beams and slabs</u>

Consider a simply supported reinforced concrete beam, subject to a uniformly distributed load. If the underside of the beam is exposed to fire, this side will expand more than the top, resulting in additional curvature and hence deflection of the beam. If the member is not restrained, no additional stress due to such deformation is likely to be induced (when the temperature gradient across the depth is linear). The compressive strength of concrete and tensile strength of steel will decrease with increase in temperature, and flexural collapse occurs at a 'critical temperature' when the material strength (usually that of steel) lowers to such an extent as to eliminate the margin of safety. The presence of end restraints in such a beam will delay the collapse, as it induces a compressive stress in the steel (and a membrane tension or catenary effect at large deflections); this beneficial effect, however, does not occur in the case of a cantilever or if the top surface of the simply supported member is subject to fire.

- Fire endurance of continuous beams and slabs

Structures that are continuous or otherwise statically indeterminate undergo changes in stresses when subjected to fire. Such changes in stress result from temperature induced deformations which may be restrained, or changes in strength of materials at high temperatures, or both. The continuous system may fail either by the critical temperature being reached in the reinforcement in the exposed side or by temperature-induced additional stress in the reinforcement in the unexposed side.

Consider a multi-span continuous beam, whose underside is exposed to fire. The bottom of the beam gets hotter than the top and tends to expand more than the top. This differential expansion, under restrained conditions, results in increase in hogging (negative) moments at the interior supports and decrease in sagging (positive) moments in the span regions [Ref. 17.9].

The cover specified by IS 456 for a continuous system for the same endurance rating is less than that for a simply supported system in view of the redundancy in the continuous system. The nominal cover for reinforcement to be provided also depends on the dimensions of the structural element, which govern the rate at which the temperature of steel increases.

***Fire Protection Measures***

Occasionally, members or assemblies may be provided with fire protection treatments such as plasterboard layers, lightweight aggregate gypsum plaster, fire resistant false ceiling, fire resistant finishes, sprayed fireproofing, etc. When used, contribution of such treatments to the fire resistance must be assessed separately based on test results with such materials.

### 17.3.3 Code Requirements

In the design of concrete structures, requirements for fire resistance are often overlooked. As already indicated, the two important design parameters are the member dimensions (especially thickness in the case of slab-like members) and cover to reinforcement. The thickness controls transmission of heat and the cover structural integrity.

The Code (Cl. 21) states that "a structure or structural element required to have fire resistance should be designed to possess an appropriate degree of resistance to flame penetration; heat transmission and failure". However, except for giving the minimum requirements of concrete cover and member dimensions for normal-weight aggregate concrete members to have the required fire resistance (Cl. 21.2), the Code does not cover any aspect of design to achieve the required fire resistance ratings. Minimum requirements of member dimensions and concrete cover for normal-weight aggregate concrete members so as to have the required fire resistance are given in Fig. 1 and Table 16A of the Code respectively.

It is emphasized that the nominal cover specified in the Code (Cl. 26.4.2) to meet durability requirements may not be sufficient to meet required fire resistance rating.

However, too large a cover (in excess of 40 mm for beams and 35 mm for slabs), especially over closely spaced grid type reinforcement layers, may lead to spalling of the cover concrete. In such cases, additional measures should be adopted to give protection against spalling (Code Cl. 21.3.1). Such are the cases falling below the bold line in Table 16A of Code. In column members, when the required cover exceeds 60mm, wire mesh reinforcement with 1.57 mm diameter wire and 100 mm openings may be incorporated midway in the concrete cover to retain the concrete in position.

## EXAMPLE 17.2

Determine the fire resistance rating of the floor slab designed in Example 5.2. Assume the concrete as normal-weight.

### SOLUTION

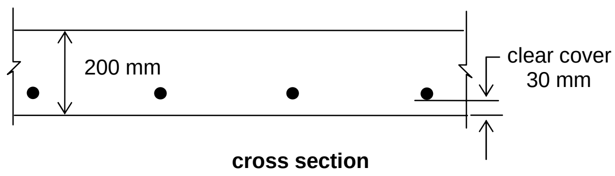



**Fig. 17.14** Example 17.2

The slab cross section is shown in Fig. 17.14

- Rating for transmission of heat:
  From the design, thickness of slab = 200 mm.
  Rating for transfer of heat, as per Fig.1 of Code (as $D$ > 170 mm) is 4 hours (+).
- Rating for structural integrity:
  Clear concrete cover = 30 mm. The slab is simply supported.
  Referring to Table 16A of Code, the cover is > 25 mm, but less than 35 mm.
  The rating for this cover is 1.5 hours (+).
  Hence, structural integrity governs, and the rating is 1.5 hours.

## EXAMPLE 17.3

Determine the fire resistance rating of the beam and slab floor designed in Example 11.6. Assume the concrete as normal-weight.

### SOLUTION

The relevant floor system dimensions are shown in Fig. 17.13

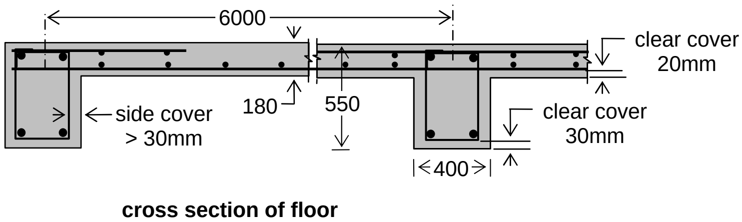



**Fig. 17.15** Example 17.3

- Fire resistance of the beam section
  The beam is continuous. For 30 mm cover, from Table 16A of Code, the fire resistance rating is 2 hours. Width of beam is 400 mm. Rating for this, from Fig.1 of Code is in excess of 4 hours. Hence the beam qualifies for a rating of 2 hours
- Fire resistance of the slab

  From Fig. 1 of Code, for 180 mm thick normal-weight aggregate concrete, rating to resist transfer of heat is in excess of 4 hours.
  From Table 16A, for a clear cover of 20 mm, for continuous slab, the rating is 1.5 hours. Hence, the structural rating controls for the slab and is 1.5 hours.
  Thus, the total floor assembly qualifies for a rating of 1.5 hours.

## PROBLEMS

See Chapters 6 and 7 for problems on shear and torsion design, to be solved using the General Method.

## REVIEW QUESTIONS

17.1 Explain *Tension Field Theory*. Where is it applicable?
17.2 How does the Compression Field Theory differ from the traditional method of design for shear?
17.3 How does the compressive stress-strain relationship of concrete with cracks parallel to the compression differ from that of uncracked concrete? What is the maximum strength of such cracked concrete?
17.4 Enumerate the assumptions used in the Modified Compression Field Theory and critically evaluate them.
17.5 Give examples where the Strut-and-Tie Model is most appropriate for design?
17.6 What are the design considerations in design using the Strut-and-Tie model?
17.7 In a test for fire resistance of a building component, what are the limiting endurance criteria?
17.8 What are the factors that influence fire resistance ratings of reinforced concrete assemblies?
17.9 How can the fire resistance of an already cast floor system improved?

## REFERENCES


17.1 Collins, M.P., Mitchell, D., P*restressed Concrete Structures,* Prentice Hall, Englewood cliffs, NJ, 1991, pp 338-411.

17.2 *Concrete Design Handbook,* Part II, Chapter 4, Canadian Portland Cement Association, Ottawa, 1985.

17.3 Vecchio, F.J. and Collins, M.P., *The Modified Compression-Field Theory for Reinforced Concrete Elements Subjected to Shear,* J. ACI, Vol. 83, March-April 1986, pp. 219-231.

17.4 Park, R. and Paulay, T., *Reinforced Concrete Structures*, John Wiley & Sons, Inc., New York, 1975, 769 pp.

17.5 — *Fire Safety of Buildings (General): General Principles of Fire Grading and Classification* IS:1641-1988, Bureau of Indian Standards, New Delhi, 1988, 5pp.

17.6 — *Fire Safety of Buildings (General): Details of Construction* IS:1642-1989, Bureau of Indian Standards, New Delhi, 1989, 15pp.

17.7 — *Fire Resistance Test of Structure* IS:3809-1979, Bureau of Indian Standards, New Delhi, 1979.

17.8 — *National Building Code of* Canada *1995: Appendix "D", Fire Performance Ratings*, National Research Council of Canada, Ottawa, 1995.

17.9 Gustaferro, A.H., *Fire Resistance,* Chapter 7 in *Handbook of Concrete Engineering* edited by Fintel, M., CBS Publishers, 1986, pp 212-226.