Displacement Method of Analysis: Moment Distribution

12

The moment-distribution method is a displacement method of analysis that is easy to apply once certain elastic constants have been determined. In this chapter we will first state the important definitions and concepts for moment distribution and then apply the method to solve problems involving statically indeterminate beams and frames. Application to multistory frames is discussed in the last part of the chapter.

12.1 General Principles and Definitions

The method of analyzing beams and frames using moment distribution was developed by Hardy Cross, in 1930. At the time this method was first published it attracted immediate attention, and it has been recognized as one of the most notable advances in structural analysis during the twentieth century.

As will be explained in detail later, moment distribution is a method of successive approximations that may be carried out to any desired degree of accuracy. Essentially, the method begins by assuming each joint of a structure is fixed. Then, by unlocking and locking each joint in succession, the internal moments at the joints are “distributed” and balanced until the joints have rotated to their final or nearly final positions. It will be found that this process of calculation is both repetitive and easy to apply. Before explaining the techniques of moment distribution, however, certain definitions and concepts must be presented.

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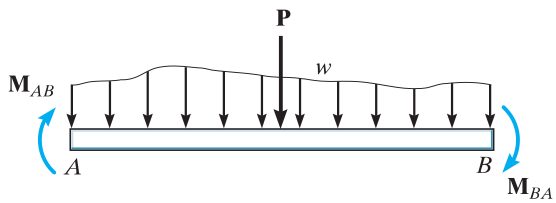


Fig. 12–1

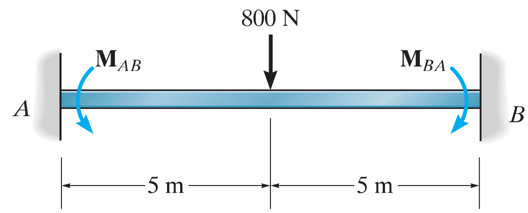


Fig. 12–2

Sign Convention. We will establish the same sign convention as that established for the slope-deflection equations: *Clockwise moments* that act *on the member* are considered *positive,* whereas *counterclockwise moments* are *negative,* Fig. 12–1.

Fixed-End Moments (FEMs). The moments at the "walls" or fixed joints of a loaded member are called *fixed-end moments.* These moments can be determined from the table given on the inside back cover, depending upon the type of loading on the member. For example, the beam loaded as shown in Fig. 12–2 has fixed-end moments of $\text{FEM} = PL/8 = 800(10)/8 = 1000\ \text{N}\cdot\text{m}$. Noting the action of these moments *on the beam* and applying our sign convention, it is seen that $M_{AB} = -1000\ \text{N}\cdot\text{m}$ and $M_{BA} = +1000\ \text{N}\cdot\text{m}$.

Member Stiffness Factor. Consider the beam in Fig. 12–3, which is pinned at one end and fixed at the other. Application of the moment **M** causes the end A to rotate through an angle θ_A. In Chapter 11 we related M to θ_A using the conjugate-beam method. This resulted in Eq. 11–1, that is, $M = (4EI/L)\,\theta_A$. The term in parentheses

$$K = \frac{4EI}{L} \qquad \text{Far End Fixed} \tag{12–1}$$

is referred to as the *stiffness factor* at A and can be defined as the amount of moment M required to rotate the end A of the beam $\theta_A = 1$ rad.

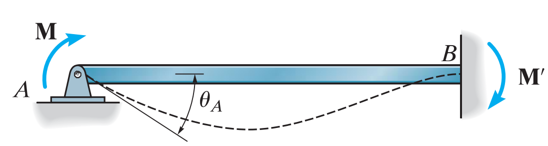


Fig. 12–3

Joint Stiffness Factor. If several members are fixed connected to a joint and each of their far ends is fixed, then by the principle of superposition, the *total stiffness factor* at the joint is the sum of the member stiffness factors at the joint, that is, $K_T = \Sigma K$. For example, consider the frame joint A in Fig. 12–4a. The numerical value of each member stiffness factor is determined from Eq. 12–1 and listed in the figure. Using these values, the total stiffness factor of joint A is $K_T = \Sigma K = 4000 + 5000 + 1000 = 10\,000$. This value represents the amount of moment needed to rotate the joint through an angle of 1 rad.

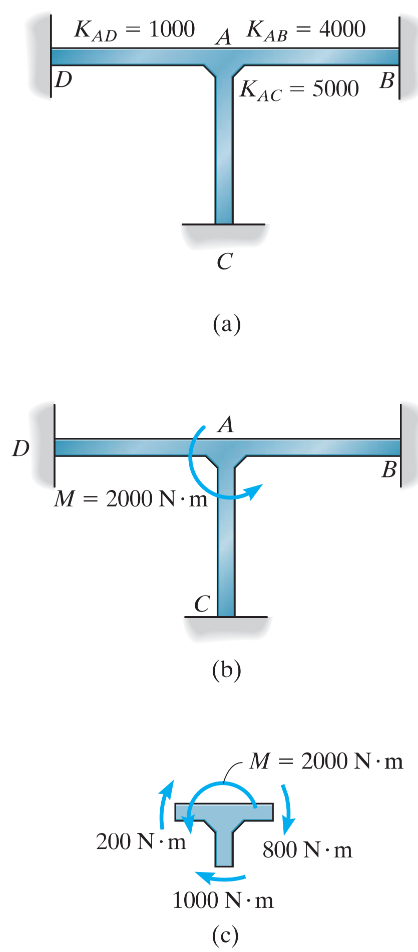


Fig. 12–4

Distribution Factor (DF). If a moment **M** is applied to a fixed connected joint, the connecting members will each supply a portion of the resisting moment necessary to satisfy moment equilibrium at the joint. That fraction of the total resisting moment supplied by the member is called the *distribution factor* (DF). To obtain its value, imagine the joint is fixed connected to n members. If an applied moment **M** causes the joint to rotate an amount θ, then each member i rotates by this same amount. If the stiffness factor of the ith member is K_i, then the moment contributed by the member is $M_i = K_i\theta$. Since equilibrium requires $M = M_1 + M_n = K_1\theta + K_n\theta = \theta\Sigma K_i$ then the distribution factor for the ith member is

$$\text{DF}_i = \frac{M_i}{M} = \frac{K_i\theta}{\theta\Sigma K_i}$$

Canceling the common term θ, it is seen that the distribution factor for a member is equal to the stiffness factor of the member divided by the total stiffness factor for the joint; that is, in general,

$$\boxed{\text{DF} = \frac{K}{\Sigma K}} \qquad (12\text{–}2)$$

For example, the distribution factors for members AB, AC, and AD at joint A in Fig. 12–4a are

$$\text{DF}_{AB} = 4000/10\,000 = 0.4$$
$$\text{DF}_{AC} = 5000/10\,000 = 0.5$$
$$\text{DF}_{AD} = 1000/10\,000 = 0.1$$

As a result, if $M = 2000$ N · m acts at joint A, Fig. 12–4b, the equilibrium moments exerted by the members on the joint, Fig. 12–4c, are

$$M_{AB} = 0.4(2000) = 800 \text{ N}\cdot\text{m}$$
$$M_{AC} = 0.5(2000) = 1000 \text{ N}\cdot\text{m}$$
$$M_{AD} = 0.1(2000) = 200 \text{ N}\cdot\text{m}$$

The statically indeterminate loading in bridge girders that are continuous over their piers can be determined using the method of moment distribution.

Member Relative-Stiffness Factor. Quite often a continuous beam or a frame will be made from the same material so its modulus of elasticity E will be the *same* for all the members. If this is the case, the common factor $4E$ in Eq. 12–1 will *cancel* from the numerator and denominator of Eq. 12–2 when the distribution factor for a joint is determined. Hence, it is *easier* just to determine the member's *relative-stiffness factor*

$$K_R = \frac{I}{L} \qquad (12\text{–}3)$$

Far End Fixed

and use this for the computations of the DF.

Carry-Over Factor. Consider again the beam in Fig. 12–3. It was shown in Chapter 11 that $M_{AB} = (4EI/L)\,\theta_A$ (Eq. 11–1) and $M_{BA} = (2EI/L)\,\theta_A$ (Eq. 11–2). Solving for θ_A and equating these equations we get $M_{BA} = M_{AB}/2$. In other words, the moment $\mathbf{M}$ at the pin induces a moment of $\mathbf{M}' = \frac{1}{2}\mathbf{M}$ at the wall. The carry-over factor represents the fraction of $\mathbf{M}$ that is "carried over" from the pin to the wall. Hence, in the case of a beam with *the far end fixed,* the carry-over factor is $+\frac{1}{2}$. The plus sign indicates both moments act in the same direction.

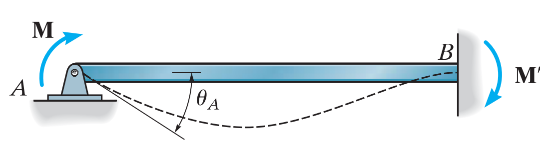


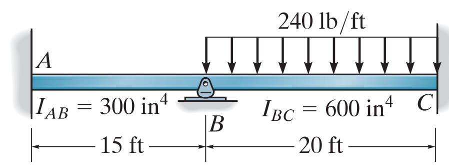
Fig. 12–3

12.2 Moment Distribution for Beams

Moment distribution is based on the principle of successively locking and unlocking the joints of a structure in order to allow the moments at the joints to be distributed and balanced. The best way to explain the method is by examples.

Consider the beam with a constant modulus of elasticity E and having the dimensions and loading shown in Fig. 12–5*a*. Before we begin, we must first determine the distribution factors at the two ends of each span. Using Eq. 12–1, $K = 4EI/L$, the stiffness factors on either side of B are

$$K_{BA} = \frac{4E(300)}{15} = 4E(20)\ \text{in}^4/\text{ft} \qquad K_{BC} = \frac{4E(600)}{20} = 4E(30)\ \text{in}^4/\text{ft}$$



(a)

Fig. 12–5

Thus, using Eq. 12–2, $\text{DF} = K/\Sigma K$, for the ends connected to joint B, we have

$$\text{DF}_{BA} = \frac{4E(20)}{4E(20) + 4E(30)} = 0.4$$

$$\text{DF}_{BC} = \frac{4E(30)}{4E(20) + 4E(30)} = 0.6$$

At the walls, joint A and joint C, the distribution factor depends on the member stiffness factor and the "stiffness factor" of the wall. Since in theory it would take an "infinite" size moment to rotate the wall one radian, the wall stiffness factor is infinite. Thus for joints A and C we have

$$\text{DF}_{AB} = \frac{4E(20)}{\infty + 4E(20)} = 0$$

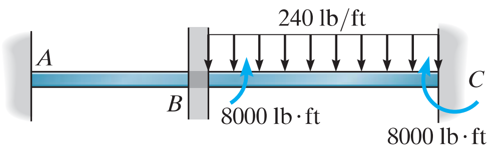
$$\text{DF}_{CB} = \frac{4E(30)}{\infty + 4E(30)} = 0$$

Note that the above results could also have been obtained if the relative stiffness factor $K_R = I/L$ (Eq. 12–3) had been used for the calculations. Furthermore, as long as a *consistent* set of units is used for the stiffness factor, the DF will always be dimensionless, and at a joint, except where it is located at a fixed wall, the sum of the DFs will always equal 1.

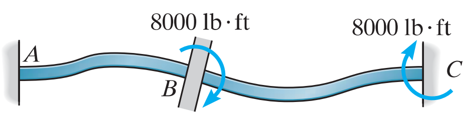
Having computed the DFs, we will now determine the FEMs. Only span BC is loaded, and using the table on the inside back cover for a uniform load, we have

$$(\text{FEM})_{BC} = -\frac{wL^2}{12} = -\frac{240(20)^2}{12} = -8000\ \text{lb}\cdot\text{ft}$$

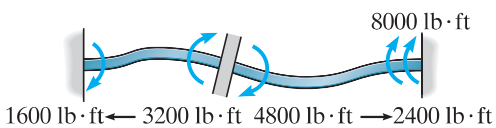
$$(\text{FEM})_{CB} = \frac{wL^2}{12} = \frac{240(20)^2}{12} = 8000\ \text{lb}\cdot\text{ft}$$



joint B held fixed
(b)



correction moment applied to joint B
(c)



moment at B distributed
(d)

Joint	A	B		C
Member	AB	BA	BC	CB
DF	0	0.4	0.6	0
FEM			−8000	8000
Dist,CO	1600 ←	3200	4800 →	2400
ΣM	1600	3200	−3200	10 400

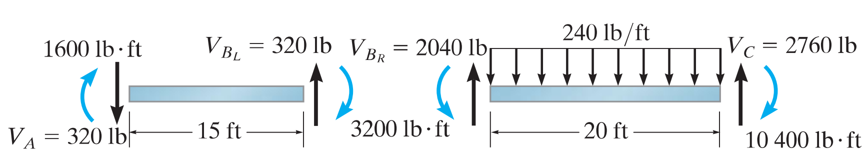
(e)

We begin by assuming joint B is fixed or locked. The fixed-end moment at B then holds span BC in this fixed or locked position as shown in Fig. 12–5*b*. This, of course, does not represent the actual equilibrium situation at B, since the moments on *each side* of this joint must be equal but opposite. To correct this, we will apply an equal, but opposite moment of 8000 lb · ft to the joint and allow the joint to rotate freely, Fig. 12–5*c*. As a result, portions of this moment are distributed in spans BC and BA in accordance with the DFs (or stiffness) of these spans at the joint. Specifically, the moment in BA is $0.4(8000) = 3200$ lb · ft and the moment in BC is $0.6(8000) = 4800$ lb · ft. Finally, due to the released rotation that takes place at B, these moments must be "carried over" since moments are developed at the far ends of the span. Using the carry-over factor of $+\frac{1}{2}$, the results are shown in Fig. 12–5*d*.

This example indicates the basic steps necessary when distributing moments at a joint: Determine the unbalanced moment acting at the initially "locked" joint, unlock the joint and apply an equal but opposite unbalanced moment to correct the equilibrium, distribute the moment among the connecting spans, and carry the moment in each span over to its other end. The steps are usually presented in tabular form as indicated in Fig. 12–5*e*. Here the notation Dist, CO indicates a line where moments are distributed, then carried over. In this particular case only *one cycle* of moment distribution is necessary, since the wall supports at A and C "absorb" the moments and no further joints have to be balanced or unlocked to satisfy joint equilibrium. Once distributed in this manner, the moments at each joint are summed, yielding the final results shown on the bottom line of the table in Fig. 12–5*e*. Notice that joint B is now in equilibrium. Since M_{BC} is negative, this moment is applied to span BC in a counterclockwise sense as shown on free-body diagrams of the beam spans in Fig. 12–5*f*. With the end moments known, the end shears have been computed from the equations of equilibrium applied to each of these spans.

Consider now the same beam, except the support at C is a rocker, Fig. 12–6*a*. In this case only *one member* is at joint C, so the distribution factor for member CB at joint C is

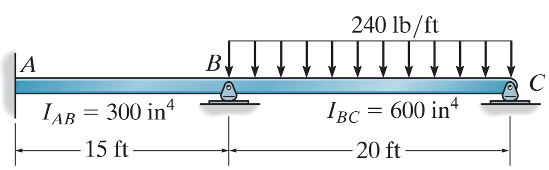
$$\text{DF}_{CB} = \frac{4E(30)}{4E(30)} = 1$$



(f)

Fig. 12–5

The other distribution factors and the FEMs are the same as computed previously. They are listed on lines 1 and 2 of the table in Fig. 12–6*b*. Initially, we will assume joints *B* and *C* are locked. We begin by unlocking joint *C* and placing an equilibrating moment of $-8000 \text{ lb}\cdot\text{ft}$ at the joint. The entire moment is distributed in member *CB* since $(1)(-8000) \text{ lb}\cdot\text{ft} = -8000 \text{ lb}\cdot\text{ft}$. The arrow on line 3 indicates that $\frac{1}{2}(-8000) \text{ lb}\cdot\text{ft} = -4000 \text{ lb}\cdot\text{ft}$ is carried over to joint *B* since joint *C* has been allowed to rotate freely. Joint *C* is now *relocked*. Since the total moment at *C* is *balanced,* a line is placed under the $-8000\text{-lb}\cdot\text{ft}$ moment. We will now consider the unbalanced $-12\,000\text{-lb}\cdot\text{ft}$ moment at joint *B*. Here for equilibrium, a $+12\,000\text{-lb}\cdot\text{ft}$ moment is applied to *B* and this joint is unlocked such that portions of the moment are distributed into *BA* and *BC*, that is, $(0.4)(12\,000) = 4800 \text{ lb}\cdot\text{ft}$ and $(0.6)(12\,000) = 7200 \text{ lb}\cdot\text{ft}$ as shown on line 4. Also note that $+\frac{1}{2}$ of these moments must be carried over to the fixed wall *A* and roller *C* since joint *B* has rotated. Joint *B* is now *relocked.* Again joint *C* is unlocked and the unbalanced moment at the roller is distributed as was done previously. The results are on line 5. Successively locking and unlocking joints *B* and *C* will essentially diminish the size of the moment to be balanced until it becomes negligible compared with the original moments, line 14. Each of the steps on lines 3 through 14 should be thoroughly understood. Summing the moments, the final results are shown on line 15, where it is seen that the final moments now satisfy joint equilibrium.



(a)

Joint	*A*	*B*		*C*	
Member	*AB*	*BA*	*BC*	*CB*	
DF	0	0.4	0.6	1	1
FEM			−8000	8000	2
			−4000 ←	−8000	3
	2400 ←	4800	7200 →	3600	4
			−1800 ←	−3600	5
	360 ←	720	1080 →	540	6
			−270 ←	−540	7
	54 ←	108	162 →	81	8
			−40.5 ←	−81	9
	8.1 ←	16.2	24.3 →	12.2	10
			−6.1 ←	−12.2	11
	1.2 ←	2.4	3.6 →	1.8	12
			−0.9 ←	−1.8	13
		0.4	0.5		14
ΣM	2823.3	5647.0	−5647.0	0	15

(b)

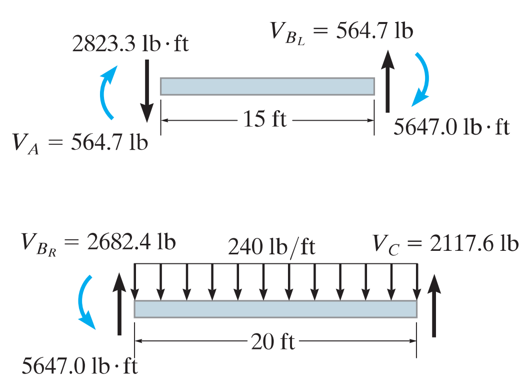
Fig. 12–6

Rather than applying the moment distribution process successively to each joint, as illustrated here, it is also possible to apply it to all joints at the *same time*. This scheme is shown in the table in Fig. 12–6*c*. In this case, we start by fixing all the joints and then balancing and distributing the fixed-end moments at both joints B and C, line 3. Unlocking joints B and C simultaneously (joint A is always fixed), the moments are then carried over to the end of each span, line 4. Again the joints are relocked, and the moments are balanced and distributed, line 5. Unlocking the joints once again allows the moments to be carried over, as shown in line 6. Continuing, we obtain the final results, as before, listed on line 24. By comparison, this method gives a slower convergence to the answer than does the previous method; however, in many cases this method will be more efficient to apply, and for this reason we will use it in the examples that follow. Finally, using the results in either Fig. 12–6*b* or 12–6*c*, the free-body diagrams of each beam span are drawn as shown in Fig. 12–6*d*.

Although several steps were involved in obtaining the final results here, the work required is rather methodical since it requires application of a series of arithmetical steps, rather than solving a set of equations as in the slope deflection method. It should be noted, however, that the

Joint	*A*	*B*		*C*	
Member	*AB*	*BA*	*BC*	*CB*	
DF	0	0.4	0.6	1	1
FEM			−8000	8000	2
Dist.		3200	4800	−8000	3
CO	1600		−4000	2400	4
Dist.		1600	2400	−2400	5
CO	800		−1200	1200	6
Dist.		480	720	−1200	7
CO	240		−600	360	8
Dist.		240	360	−360	9
CO	120		−180	180	10
Dist.		72	108	−180	11
CO	36		−90	54	12
Dist.		36	54	−54	13
CO	18		−27	27	14
Dist.		10.8	16.2	−27	15
CO	5.4		−13.5	8.1	16
Dist.		5.4	8.1	−8.1	17
CO	2.7		−4.05	4.05	18
Dist.		1.62	2.43	−4.05	19
CO	0.81		−2.02	1.22	20
Dist.		0.80	1.22	−1.22	21
CO	0.40		−0.61	0.61	22
Dist.		0.24	0.37	−0.61	23
ΣM	2823	5647	−5647	0	24

(c)



(d)

Fig. 12–6

fundamental process of moment distribution follows the same procedure as any displacement method. There the process is to establish load-displacement relations at each joint and then satisfy joint equilibrium requirements by determining the correct angular displacement for the joint (compatibility). Here, however, the equilibrium and compatibility of rotation at the joint is satisfied *directly,* using a "moment balance" process that incorporates the load-deflection relations (stiffness factors). Further simplification for using moment distribution is possible, and this will be discussed in the next section.

Procedure for Analysis

The following procedure provides a general method for determining the end moments on beam spans using moment distribution.

Distribution Factors and Fixed-End Moments

The joints on the beam should be identified and the stiffness factors for each span at the joints should be calculated. Using these values the distribution factors can be determined from $\text{DF} = K/\Sigma K$. Remember that $\text{DF} = 0$ for a fixed end and $\text{DF} = 1$ for an *end* pin or roller support.

The fixed-end moments for each loaded span are determined using the table given on the inside back cover. Positive FEMs act clockwise on the span and negative FEMs act counterclockwise. For convenience, these values can be recorded in tabular form, similar to that shown in Fig. 12–6*c*.

Moment Distribution Process

Assume that all joints at which the moments in the connecting spans must be determined are *initially locked*. Then:

1. Determine the moment that is needed to put each joint in equilibrium.

2. Release or "unlock" the joints and distribute the counterbalancing moments into the connecting span at each joint.

3. Carry these moments in each span over to its other end by multiplying each moment by the carry-over factor $+\frac{1}{2}$.

By repeating this cycle of locking and unlocking the joints, it will be found that the moment corrections will diminish since the beam tends to achieve its final deflected shape. When a small enough value for the corrections is obtained, the process of cycling should be stopped with no "carry-over" of the last moments. Each column of FEMs, distributed moments, and carry-over moments should then be added. If this is done correctly, moment equilibrium at the joints will be achieved.

EXAMPLE 12.1

Determine the internal moments at each support of the beam shown in Fig. 12–7*a*. *EI* is constant.

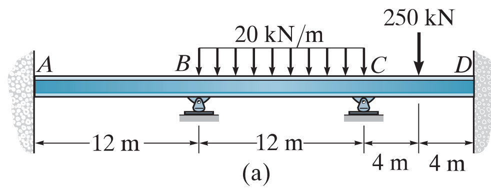


Fig. 12–7

SOLUTION

The distribution factors at each joint must be computed first.* The stiffness factors for the members are

$$K_{AB} = \frac{4EI}{12} \qquad K_{BC} = \frac{4EI}{12} \qquad K_{CD} = \frac{4EI}{8}$$

Therefore,

$$\text{DF}_{AB} = \text{DF}_{DC} = 0 \qquad \text{DF}_{BA} = \text{DF}_{BC} = \frac{4EI/12}{4EI/12 + 4EI/12} = 0.5$$

$$\text{DF}_{CB} = \frac{4EI/12}{4EI/12 + 4EI/8} = 0.4 \quad \text{DF}_{CD} = \frac{4EI/12}{4EI/12 + 4EI/8} = 0.6$$

The fixed-end moments are

$$(\text{FEM})_{BC} = -\frac{wL^2}{12} = \frac{-20(12)^2}{12} = -240 \text{ kN}\cdot\text{m} \qquad (\text{FEM})_{CB} = \frac{wL^2}{12} = \frac{20(12)^2}{12} = 240 \text{ kN}\cdot\text{m}$$

$$(\text{FEM})_{CD} = -\frac{PL}{8} = \frac{-250(8)}{8} = -250 \text{ kN}\cdot\text{m} \qquad (\text{FEM})_{DC} = \frac{PL}{8} = \frac{250(8)}{8} = 250 \text{ kN}\cdot\text{m}$$

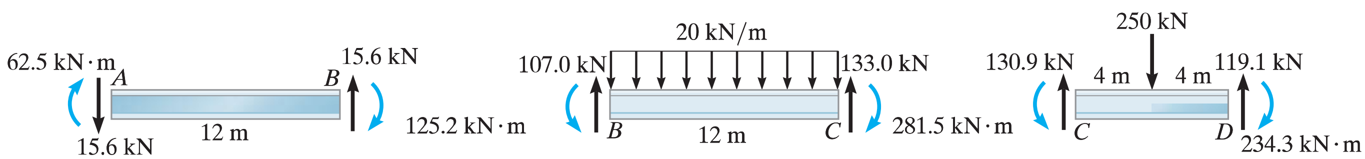
Starting with the FEMs, line 4, Fig. 12–7*b*, the moments at joints *B* and *C* are distributed *simultaneously*, line 5. These moments are then carried over *simultaneously* to the respective ends of each span, line 6. The resulting moments are again simultaneously distributed and carried over, lines 7 and 8. The process is continued until the resulting moments are diminished an appropriate amount, line 13. The resulting moments are found by summation, line 14.

Placing the moments on each beam span and applying the equations of equilibrium yields the end shears shown in Fig. 12–7*c* and the bending-moment diagram for the entire beam, Fig. 12–7*d*.

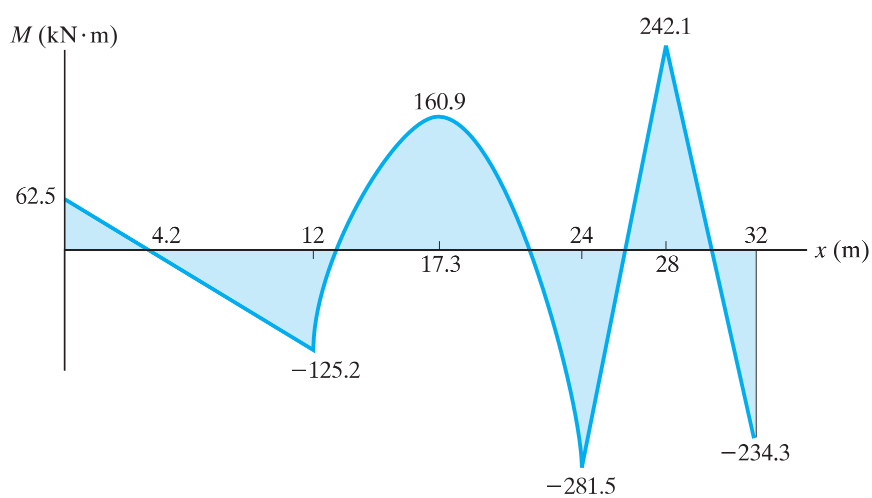
*Here we have used the stiffness factor $4EI/L$; however, the relative stiffness factor I/L could also have been used.

Joint	A	B		C		D	1
Member	AB	BA	BC	CB	CD	DC	2
DF	0	0.5	0.5	0.4	0.6	0	3
FEM			−240	240	−250	250	4
Dist.		120	120	4	6		5
CO	60		2	60		3	6
Dist.		−1	−1	−24	−36		7
CO	−0.5		−12	−0.5		−18	8
Dist.		6	6	0.2	0.3		9
CO	3		0.1	3		0.2	10
Dist.		−0.05	−0.05	−1.2	−1.8		11
CO	−0.02		−0.6	−0.02		−0.9	12
Dist.		0.3	0.3	0.01	0.01		13
ΣM	62.5	125.2	−125.2	281.5	−281.5	234.3	14

(b)



(c)



(d)

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EXAMPLE 12.2

Determine the internal moment at each support of the beam shown in Fig. 12–8*a*. The moment of inertia of each span is indicated.

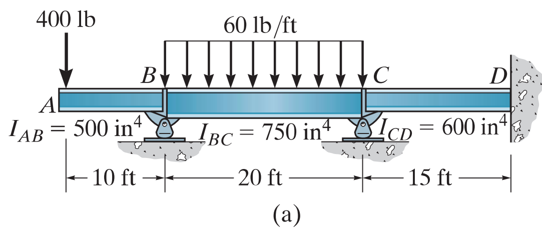


Fig. 12–8

SOLUTION

In this problem a moment does not get distributed in the overhanging span AB, and so the distribution factor $(\text{DF})_{BA} = 0$. The stiffness of span BC is based on $4EI/L$ since the pin rocker is not at the far end of the beam. The stiffness factors, distribution factors, and fixed-end moments are computed as follows:

$$K_{BC} = \frac{4E(750)}{20} = 150E \qquad K_{CD} = \frac{4E(600)}{15} = 160E$$

$$\text{DF}_{BC} = 1 - (\text{DF})_{BA} = 1 - 0 = 1$$

$$\text{DF}_{CB} = \frac{150E}{150E + 160E} = 0.484$$

$$\text{DF}_{CD} = \frac{160E}{150E + 160E} = 0.516$$

$$\text{DF}_{DC} = \frac{160E}{\infty + 160E} = 0$$

Due to the overhang,

$$(\text{FEM})_{BA} = 400 \text{ lb}(10 \text{ ft}) = 4000 \text{ lb} \cdot \text{ft}$$

$$(\text{FEM})_{BC} = -\frac{wL^2}{12} = -\frac{60(20)^2}{12} = -2000 \text{ lb} \cdot \text{ft}$$

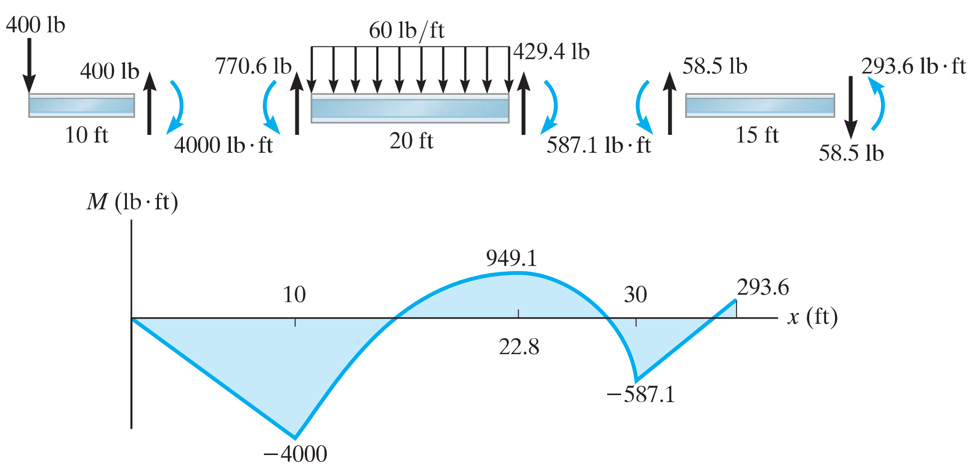
$$(\text{FEM})_{CB} = \frac{wL^2}{12} = \frac{60(20)^2}{12} = 2000 \text{ lb} \cdot \text{ft}$$

These values are listed on the fourth line of the table, Fig. 12–8*b*. The overhanging span requires the internal moment to the left of B to be $+4000$ lb · ft. Balancing at joint B requires an internal moment of -4000 lb · ft to the right of B. As shown on the fifth line of the table -2000 lb · ft is added to BC in order to satisfy this condition. The distribution and carry-over operations proceed in the usual manner as indicated.

Since the internal moments are known, the moment diagram for the beam can be constructed (Fig. 12–8*c*).

Joint	*B*		*C*		*D*
Member		*BC*	*CB*	*CD*	*DC*
DF	0	1	0.484	0.516	0
FEM	4000	−2000	2000		
Dist.		−2000	−968	−1032	
CO		−484	−1000		−516
Dist.		484	484	516	
CO		242	242		258
Dist.		−242	−117.1	−124.9	
CO		−58.6	−121		−62.4
Dist.		58.6	58.6	62.4	
CO		29.3	29.3		31.2
Dist.		−29.3	−14.2	−15.1	
CO		−7.1	−14.6		−7.6
Dist.		7.1	7.1	7.6	
CO		3.5	3.5		3.8
Dist.		−3.5	−1.7	−1.8	
CO		−0.8	−1.8		−0.9
Dist.		0.8	0.9	0.9	
CO		0.4	0.4		0.4
Dist.		−0.4	−0.2	−0.2	
CO		−0.1	−0.2		−0.1
Dist.		0.1	0.1	0.1	
ΣM	4000	−4000	587.1	−587.1	−293.6

(b)



(c)

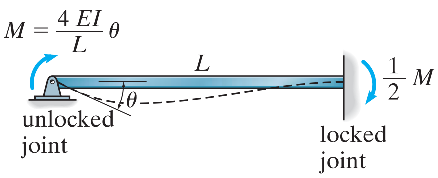


Fig. 12–9

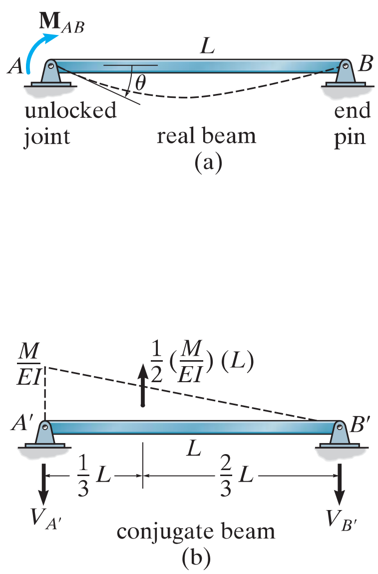


Fig. 12–10

12.3 Stiffness-Factor Modifications

In the previous examples of moment distribution we have considered each beam span to be constrained by a fixed support (locked joint) at its far end when distributing and carrying over the moments. For this reason we have computed the stiffness factors, distribution factors, and the carry-over factors based on the case shown in Fig. 12–9. Here, of course, the stiffness factor is $K = 4EI/L$ (Eq. 12–1), and the carry-over factor is $+\frac{1}{2}$.

In some cases it is possible to modify the stiffness factor of a particular beam span and thereby simplify the process of moment distribution. Three cases where this frequently occurs in practice will now be considered.

Member Pin Supported at Far End. Many indeterminate beams have their far end span supported by an end pin (or roller) as in the case of joint B in Fig. 12–10a. Here the applied moment **M** rotates the end A by an amount θ. To determine θ, the shear in the conjugate beam at A' must be determined, Fig. 12–10b. We have

$$\circlearrowleft + \Sigma M_{B'} = 0; \qquad V'_A(L) - \frac{1}{2}\left(\frac{M}{EI}\right)L\left(\frac{2}{3}L\right) = 0$$

$$V'_A = \theta = \frac{ML}{3EI}$$

or

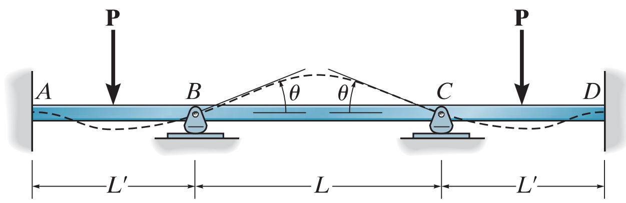
$$M = \frac{3EI}{L}\theta$$

Thus, the stiffness factor for this beam is

$$K = \frac{3EI}{L} \qquad (12\text{–}4)$$

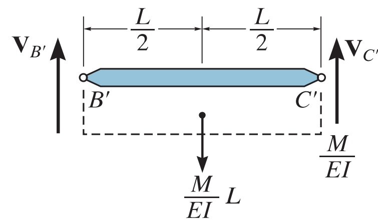
Far End Pinned or Roller Supported

Also, note that *the carry-over factor is zero,* since the pin at B does not support a moment. By comparison, then, *if the far end was fixed supported, the stiffness factor* $K = 4EI/L$ *would have to be modified by* $\frac{3}{4}$ *to model the case of having the far end pin supported*. If this modification is considered, the moment distribution process is simplified since the end pin does *not* have to be unlocked–locked successively when distributing the moments. Also, since the end span is pinned, the fixed-end moments for the span are computed using the values in the right column of the table on the inside back cover. Example 12–4 illustrates how to apply these simplifications.



real beam

(a)



conjugate beam

(b)

Fig. 12–11

Symmetric Beam and Loading. If a beam is symmetric with respect to both its loading and geometry, the bending-moment diagram for the beam will also be symmetric. As a result, a modification of the stiffness factor for the center span can be made, so that moments in the beam only have to be distributed through joints lying on either half of the beam. To develop the appropriate stiffness-factor modification, consider the beam shown in Fig. 12–11*a*. Due to the symmetry, the internal moments at B and C are equal. Assuming this value to be **M**, the conjugate beam for span BC is shown in Fig. 12–11*b*. The slope θ at each end is therefore

$$\circlearrowright + \Sigma M_{C'} = 0; \qquad -V_{B'}(L) + \frac{M}{EI}(L)\left(\frac{L}{2}\right) = 0$$

$$V_{B'} = \theta = \frac{ML}{2EI}$$

or

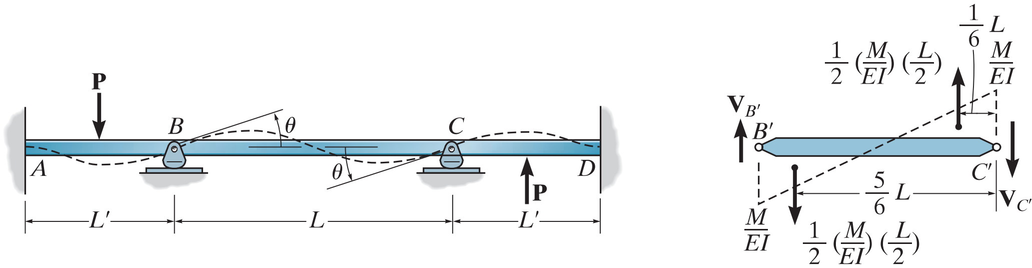
$$M = \frac{2EI}{L}\theta$$

The stiffness factor for the center span is therefore

$$K = \frac{2EI}{L} \qquad (12\text{–}5)$$

Symmetric Beam and Loading

Thus, moments for only half the beam can be distributed provided the stiffness factor for the center span is computed using Eq. 12–5. *By comparison, the center span's stiffness factor will be one half that usually determined using* $K = 4EI/L$.



real beam

(a)

conjugate beam

(b)

Fig. 12–12

Symmetric Beam with Antisymmetric Loading. If a symmetric beam is subjected to antisymmetric loading, the resulting moment diagram will be antisymmetric. As in the previous case, we can modify the stiffness factor of the center span so that only one half of the beam has to be considered for the moment-distribution analysis. Consider the beam in Fig. 12–12*a*. The conjugate beam for its center span *BC* is shown in Fig. 12–12*b*. Due to the antisymmetric loading, the internal moment at *B* is equal, but opposite to that at *C*. Assuming this value to be **M**, the slope θ at each end is determined as follows:

$$\circlearrowleft + \Sigma M_{C'} = 0; \quad -V_{B'}(L) + \frac{1}{2}\left(\frac{M}{EI}\right)\left(\frac{L}{2}\right)\left(\frac{5L}{6}\right) - \frac{1}{2}\left(\frac{M}{EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{6}\right) = 0$$

$$V_{B'} = \theta = \frac{ML}{6EI}$$

or

$$M = \frac{6EI}{L}\theta$$

The stiffness factor for the center span is, therefore,

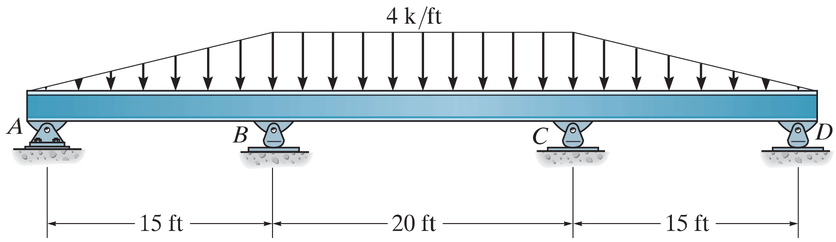
$$K = \frac{6EI}{L} \qquad (12\text{–}6)$$

Symmetric Beam with Antisymmetric Loading

Thus, when the stiffness factor for the beam's center span is computed using Eq. 12–6, the moments in only half the beam have to be distributed. *Here the stiffness factor is one and a half times as large as that determined using* $K = 4EI/L$.

EXAMPLE 12.3

Determine the internal moments at the supports for the beam shown in Fig. 12–13*a*. *EI* is constant.



(a)

Fig. 12–13

SOLUTION

By inspection, the beam and loading are symmetrical. Thus, we will apply $K = 2EI/L$ to compute the stiffness factor of the center span BC and therefore use only the left half of the beam for the analysis. The analysis can be shortened even further by using $K = 3EI/L$ for computing the stiffness factor of segment AB since the far end A is pinned. Furthermore, the distribution of moment at A can be skipped by using the FEM for a triangular loading on a span with one end fixed and the other pinned. Thus,

$$K_{AB} = \frac{3EI}{15} \quad \text{(using Eq. 12–4)}$$

$$K_{BC} = \frac{2EI}{20} \quad \text{(using Eq. 12–5)}$$

$$\text{DF}_{AB} = \frac{3EI/15}{3EI/15} = 1$$

$$\text{DF}_{BA} = \frac{3EI/15}{3EI/15 + 2EI/20} = 0.667$$

$$\text{DF}_{BC} = \frac{2EI/20}{3EI/15 + 2EI/20} = 0.333$$

$$(\text{FEM})_{BA} = \frac{wL^2}{15} = \frac{4(15)^2}{15} = 60 \text{ k}\cdot\text{ft}$$

$$(\text{FEM})_{BC} = -\frac{wL^2}{12} = -\frac{4(20)^2}{12} = -133.3 \text{ k}\cdot\text{ft}$$

Joint	A	B	
Member	AB	BA	BC
DF	1	0.667	0.333
FEM Dist.		60 48.9	−133.3 24.4
ΣM	0	108.9	−108.9

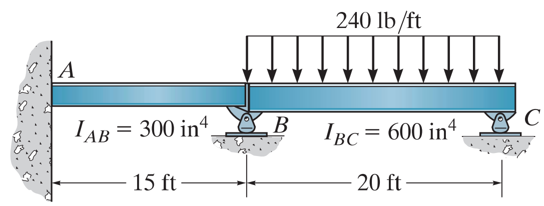
(b)

These data are listed in the table in Fig. 12–13*b*. Computing the stiffness factors as shown above considerably reduces the analysis, since only joint B must be balanced and carry-overs to joints A and C are not necessary. Obviously, joint C is subjected to the same internal moment of 108.9 k · ft.

12

EXAMPLE 12.4

Determine the internal moments at the supports of the beam shown in Fig. 12–14*a*. The moment of inertia of the two spans is shown in the figure.



(a)

Fig. 12–14

SOLUTION

Since the beam is roller supported at its far end C, the stiffness of span BC will be computed on the basis of $K = 3EI/L$. We have

$$K_{AB} = \frac{4EI}{L} = \frac{4E(300)}{15} = 80E$$

$$K_{BC} = \frac{3EI}{L} = \frac{3E(600)}{20} = 90E$$

Thus,

$$\text{DF}_{AB} = \frac{80E}{\infty + 80E} = 0$$

$$\text{DF}_{BA} = \frac{80E}{80E + 90E} = 0.4706$$

$$\text{DF}_{BC} = \frac{90E}{80E + 90E} = 0.5294$$

$$\text{DF}_{CB} = \frac{90E}{90E} = 1$$

Further simplification of the distribution method for this problem is possible by realizing that a *single* fixed-end moment for the end span BC can be used. Using the right-hand column of the table on the inside back cover for a uniformly loaded span having one side fixed, the other pinned, we have

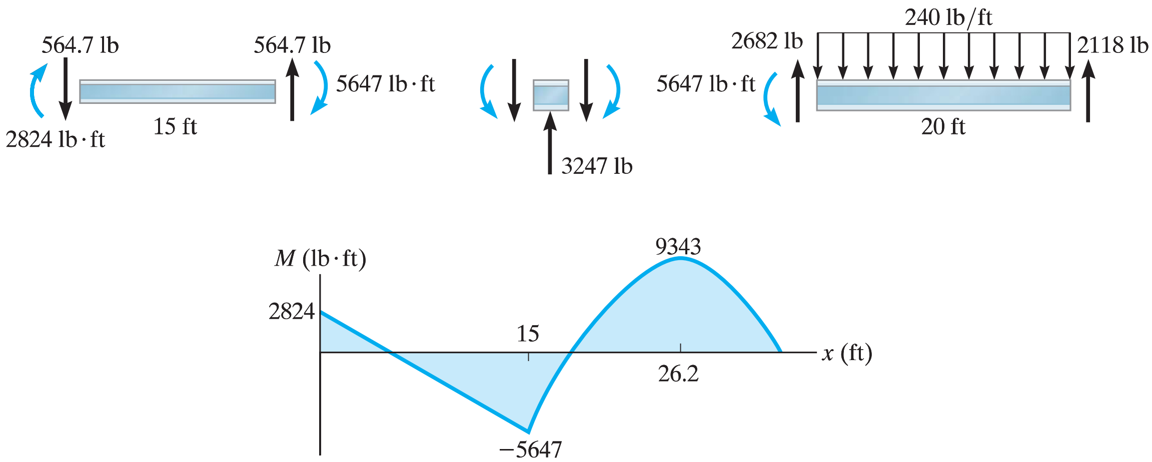
$$(\text{FEM})_{BC} = -\frac{wL^2}{8} = \frac{-240(20)^2}{8} = -12\,000\text{ lb}\cdot\text{ft}$$

The foregoing data are entered into the table in Fig. 12–14*b* and the moment distribution is carried out. By comparison with Fig. 12–6*b*, this method considerably simplifies the distribution.

Using the results, the beam's end shears and moment diagrams are shown in Fig. 12–14*c*.

Joint	*A*	*B*		*C*
Member	*AB*	*BA*	*BC*	*CB*
DF	0	0.4706	0.5294	1
FEM Dist.		 5647.2	−12 000 6352.8	
CO	2823.6			
Σ*M*	2823.6	5647.2	−5647.2	0

(b)

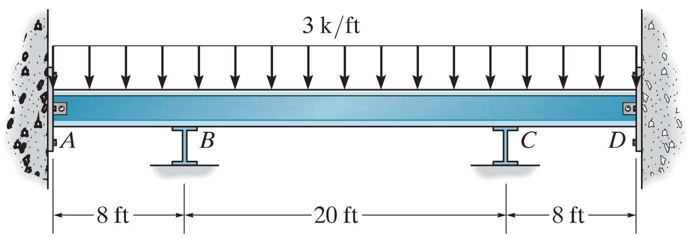


(c)

12

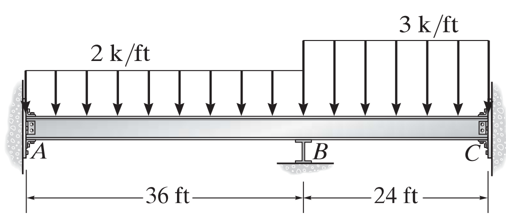
PROBLEMS

12–1. Determine the moments at B and C. EI is constant. Assume B and C are rollers and A and D are pinned.



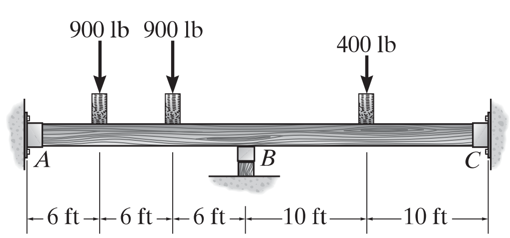
Prob. 12–1

12–2. Determine the moments at A, B, and C. Assume the support at B is a roller and A and C are fixed. EI is constant.



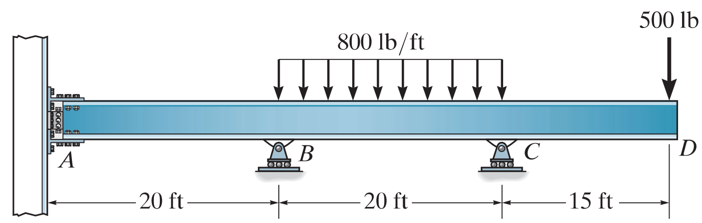
Prob. 12–2

12–3. Determine the moments at A, B, and C, then draw the moment diagram. Assume the support at B is a roller and A and C are fixed. EI is constant.



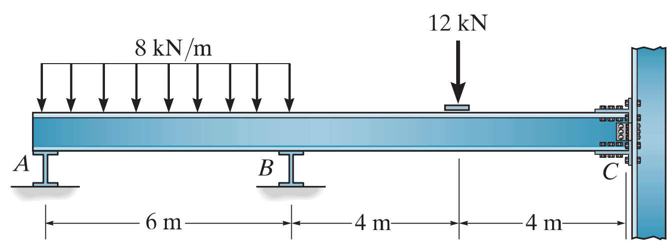
Prob. 12–3

***12–4.** Determine the reactions at the supports and then draw the moment diagram. Assume A is fixed. EI is constant.



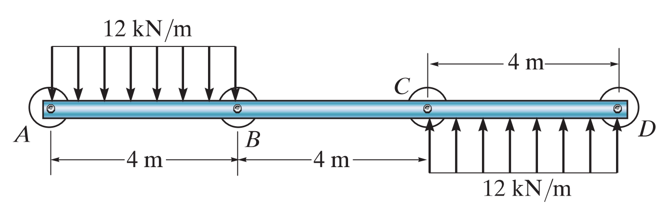
Prob. 12–4

12–5. Determine the moments at B and C, then draw the moment diagram for the beam. Assume C is a fixed support. EI is constant.



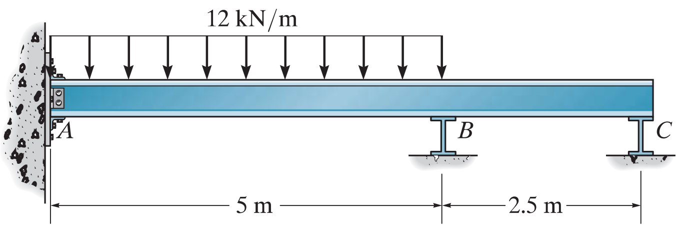
Prob. 12–5

12–6. Determine the moments at B and C, then draw the moment diagram for the beam. All connections are pins. Assume the horizontal reactions are zero. EI is constant.



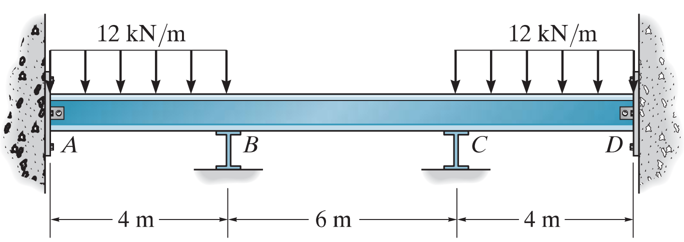
Prob. 12–6

12–7. Determine the reactions at the supports. Assume A is fixed and B and C are rollers that can either push or pull on the beam. EI is constant.



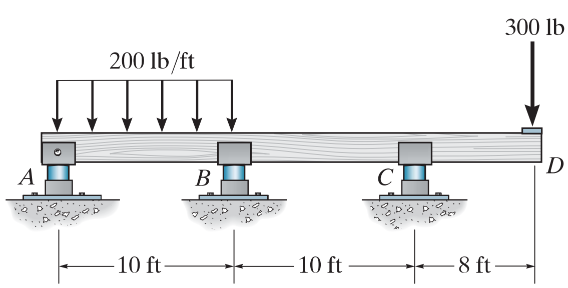
Prob. 12–7

***12–8.** Determine the moments at B and C, then draw the moment diagram for the beam. Assume the supports at B and C are rollers and A and D are pins. EI is constant.



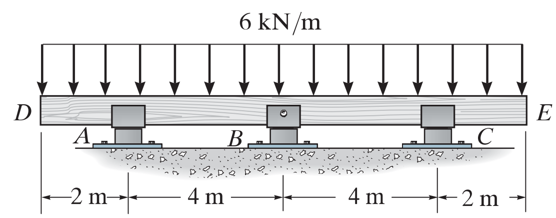
Prob. 12–8

12–9. Determine the moments at B and C, then draw the moment diagram for the beam. Assume the supports at B and C are rollers and A is a pin. EI is constant.



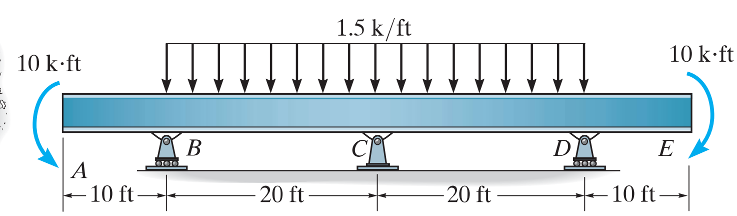
Prob. 12–9

12–10. Determine the moment at B, then draw the moment diagram for the beam. Assume the supports at A and C are rollers and B is a pin. EI is constant.



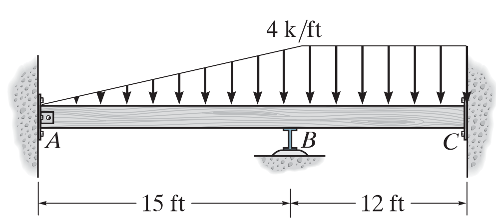
Prob. 12–10

12–11. Determine the moments at B, C, and D, then draw the moment diagram for the beam. EI is constant.



Prob. 12–11

***12–12.** Determine the moment at B, then draw the moment diagram for the beam. Assume the support at A is pinned, B is a roller and C is fixed. EI is constant.



Prob. 12–12

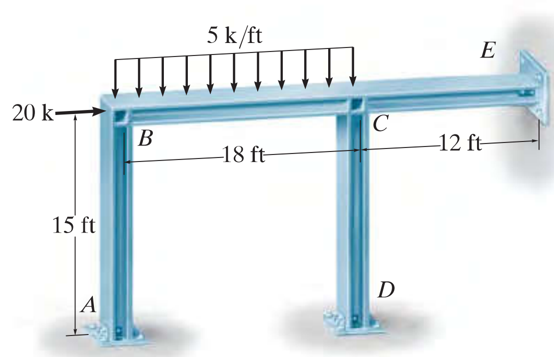
12

12.4 Moment Distribution for Frames: No Sidesway

Application of the moment-distribution method for frames having no sidesway follows the same procedure as that given for beams. To minimize the chance for errors, it is suggested that the analysis be arranged in a tabular form, as in the previous examples. Also, the distribution of moments can be shortened if the stiffness factor of a span can be modified as indicated in the previous section.

EXAMPLE 12.5

Determine the internal moments at the joints of the frame shown in Fig. 12–15*a*. There is a pin at *E* and *D* and a fixed support at *A*. *EI* is constant.



(a)

Joint	*A*	*B*		*C*			*D*	*E*
Member	*AB*	*BA*	*BC*	*CB*	*CD*	*CE*	*DC*	*EC*
DF	0	0.545	0.455	0.330	0.298	0.372	1	1
FEM			−135	135				
Dist.		73.6	61.4	−44.6	−40.2	−50.2		
CO	36.8		−22.3	30.7				
Dist.		12.2	10.1	−10.1	−9.1	−11.5		
CO	6.1		−5.1	5.1				
Dist.		2.8	2.3	−1.7	−1.5	−1.9		
CO	1.4		−0.8	1.2				
Dist.		0.4	0.4	−0.4	−0.4	−0.4		
CO	0.2		−0.2	0.2				
Dist.		0.1	0.1	−0.1	0.0	−0.1		
ΣM	44.5	89.1	−89.1	115	−51.2	−64.1		

(b)

Fig. 12–15

SOLUTION

By inspection, the pin at E will prevent the frame from sidesway. The stiffness factors of CD and CE can be computed using $K = 3EI/L$ since the far ends are pinned. Also, the 20-k load does not contribute a FEM since it is applied at joint B. Thus,

$$K_{AB} = \frac{4EI}{15} \qquad K_{BC} = \frac{4EI}{18} \qquad K_{CD} = \frac{3EI}{15} \qquad K_{CE} = \frac{3EI}{12}$$

$$\text{DF}_{AB} = 0$$

$$\text{DF}_{BA} = \frac{4EI/15}{4EI/15 + 4EI/18} = 0.545$$

$$\text{DF}_{BC} = 1 - 0.545 = 0.455$$

$$\text{DF}_{CB} = \frac{4EI/18}{4EI/18 + 3EI/15 + 3EI/12} = 0.330$$

$$\text{DF}_{CD} = \frac{3EI/15}{4EI/18 + 3EI/15 + 3EI/12} = 0.298$$

$$\text{DF}_{CE} = 1 - 0.330 - 0.298 = 0.372$$

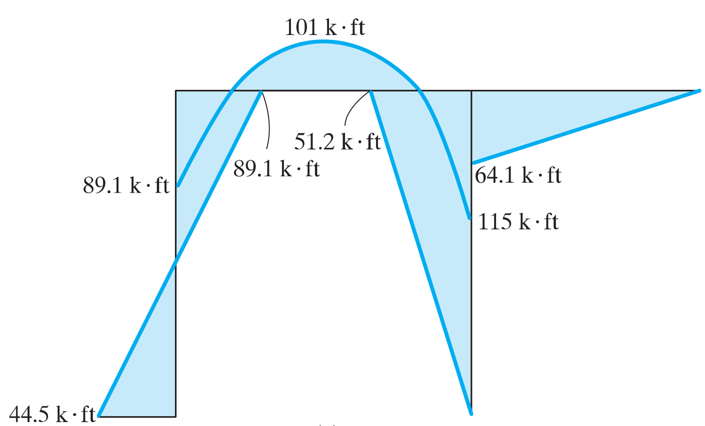
$$\text{DF}_{DC} = 1 \qquad \text{DF}_{EC} = 1$$

$$(\text{FEM})_{BC} = \frac{-wL^2}{12} = \frac{-5(18)^2}{12} = -135 \text{ k} \cdot \text{ft}$$

$$(\text{FEM})_{CB} = \frac{wL^2}{12} = \frac{5(18)^2}{12} = 135 \text{ k} \cdot \text{ft}$$

The data are shown in the table in Fig. 12–15*b*. Here the distribution of moments successively goes to joints B and C. The final moments are shown on the last line.

Using these data, the moment diagram for the frame is constructed in Fig. 12–15*c*.



(c)

12.5 Moment Distribution for Frames: Sidesway

It has been shown in Sec. 11–5 that frames that are nonsymmetrical or subjected to nonsymmetrical loadings have a tendency to sidesway. An example of one such case is shown in Fig. 12–16*a*. Here the applied loading **P** will create unequal moments at joints B and C such that the frame will deflect an amount Δ to the right. To determine this deflection and the internal moments at the joints using moment distribution, we will use the principle of superposition. In this regard, the frame in Fig. 12–16*b* is first considered held from sidesway by applying an artificial joint support at C. Moment distribution is applied and then by statics the restraining force **R** is determined. The equal, but opposite, restraining force is then applied to the frame, Fig. 12–16*c*, and the moments in the frame are calculated. One method for doing this last step requires first *assuming* a numerical value for one of the internal moments, say $\mathbf{M}'_{BA}$. Using moment distribution and statics, the deflection Δ' and external force $\mathbf{R}'$ corresponding to the assumed value of $\mathbf{M}'_{BA}$ can then be determined. Since linear elastic deformations occur, the force $\mathbf{R}'$ develops moments in the frame that are *proportional* to those developed by **R**. For example, if $\mathbf{M}'_{BA}$ and $\mathbf{R}'$ are known, the moment at B developed by **R** will be $M_{BA} = M'_{BA}(R/R')$. Addition of the joint moments for both cases, Fig. 12–16*b* and *c*, will yield the actual moments in the frame, Fig. 12–16*a*. Application of this technique is illustrated in Examples 12–6 through 12–8.

Multistory Frames. Quite often, multistory frameworks may have several *independent* joint displacements, and consequently the moment distribution analysis using the above techniques will involve more computation. Consider, for example, the two-story frame shown in Fig. 12–17*a*. This structure can have two independent joint displacements, since the sidesway Δ_1 of the first story is independent of any displacement

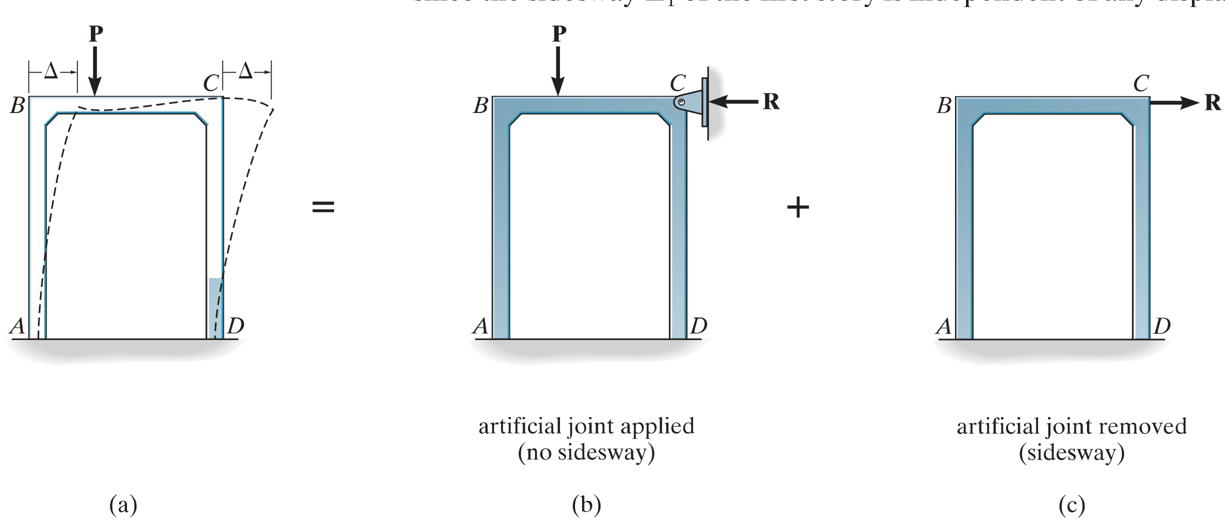


Fig. 12–16

Δ_2 of the second story. Unfortunately, these displacements are not known initially, so the analysis must proceed on the basis of superposition, in the same manner as discussed previously. In this case, two restraining forces $\mathbf{R_1}$ and $\mathbf{R_2}$ are applied, Fig. 12–17*b*, and the fixed-end moments are determined and distributed. Using the equations of equilibrium, the numerical values of $\mathbf{R_1}$ and $\mathbf{R_2}$ are then determined. Next, the restraint at the floor of the first story is removed and the floor is given a displacement Δ'. This displacement causes fixed-end moments (FEMs) in the frame, which can be assigned specific numerical values. By distributing these moments and using the equations of equilibrium, the associated numerical values of $\mathbf{R_1'}$ and $\mathbf{R_2'}$ can be determined. In a similar manner, the floor of the second story is then given a displacement Δ'', Fig. 12–17*d*. Assuming numerical values for the fixed-end moments, the moment distribution and equilibrium analysis will yield specific values of $\mathbf{R_1''}$ and $\mathbf{R_2''}$. Since the last two steps associated with Fig. 12–17*c* and *d* depend on *assumed* values of the FEMs, correction factors C' and C'' must be applied to the distributed moments. With reference to the restraining forces in Fig. 12–17*c* and 12–17*d*, we require equal but opposite application of $\mathbf{R_1}$ and $\mathbf{R_2}$ to the frame, such that

$$R_2 = -C'R_2' + C''R_2''$$
$$R_1 = +C'R_1' - C''R_1''$$

Simultaneous solution of these equations yields the values of C' and C''. These correction factors are then multiplied by the internal joint moments found from the moment distribution in Fig. 12–17*c* and 12–17*d*. The resultant moments are then found by adding these corrected moments to those obtained for the frame in Fig. 12–17*b*.

Other types of frames having independent joint displacements can be analyzed using this same procedure; however, it must be admitted that the foregoing method does require quite a bit of numerical calculation. Although some techniques have been developed to shorten the calculations, it is best to solve these types of problems on a computer, preferably using a matrix analysis. The techniques for doing this will be discussed in Chapter 16.

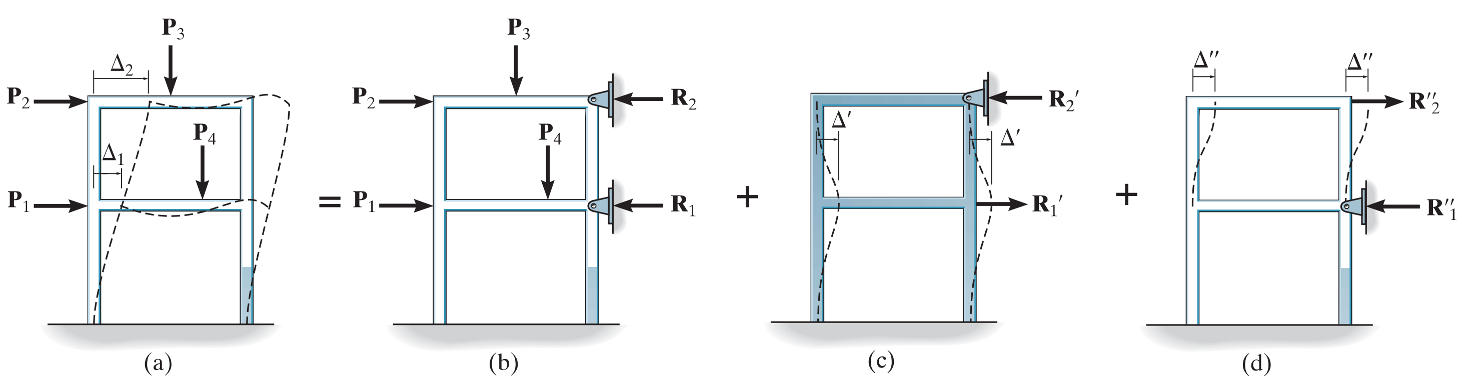
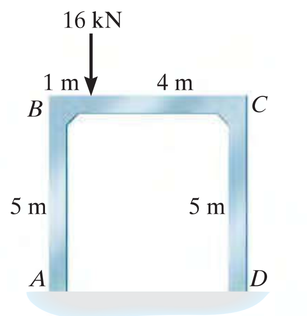


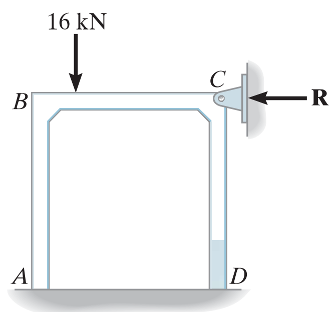
Fig. 12–17

EXAMPLE 12.6



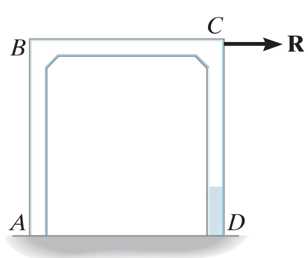
(a)

=



(b)

+



(c)

Fig. 12–18

Determine the moments at each joint of the frame shown in Fig. 12–18*a*. *EI* is constant.

SOLUTION

First we consider the frame held from sidesway as shown in Fig. 12–18*b*. We have

$$(\text{FEM})_{BC} = -\frac{16(4)^2(1)}{(5)^2} = -10.24 \text{ kN}\cdot\text{m}$$

$$(\text{FEM})_{CB} = \frac{16(1)^2(4)}{(5)^2} = 2.56 \text{ kN}\cdot\text{m}$$

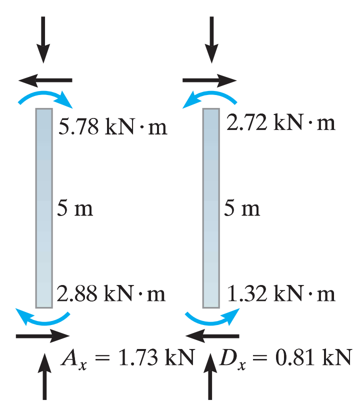
The stiffness factor of each span is computed on the basis of $4EI/L$ or by using the relative-stiffness factor I/L. The DFs and the moment distribution are shown in the table, Fig. 12–18*d*. Using these results, the equations of equilibrium are applied to the free-body diagrams of the columns in order to determine $\mathbf{A}_x$ and $\mathbf{D}_x$, Fig. 12–18*e*. From the free-body diagram of the entire frame (not shown) the joint restraint $\mathbf{R}$ in Fig. 12–18*b* has a magnitude of

$$\Sigma F_x = 0; \qquad R = 1.73 \text{ kN} - 0.81 \text{ kN} = 0.92 \text{ kN}$$

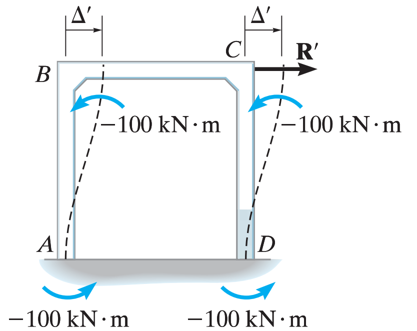
An equal but opposite value of $R = 0.92$ kN must now be applied to the frame at C and the internal moments computed, Fig. 12–18*c*. To solve the problem of computing these moments, we will assume a force $\mathbf{R}'$ is applied at C, causing the frame to deflect Δ' as shown in Fig. 12–18*f*. Here the joints at B and C are *temporarily restrained from rotating,* and as a result the fixed-end moments at the ends of the columns are determined from the formula for deflection found on the inside back cover, that is,

Joint	*A*	*B*		*C*		*D*
Member	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
DF	0	0.5	0.5	0.5	0.5	0
FEM			−10.24	2.56		
Dist.		5.12	5.12	−1.28	−1.28	
CO	2.56		−0.64	2.56		−0.64
Dist.		0.32	0.32	−1.28	−1.28	
CO	0.16		−0.64	0.16		−0.64
Dist.		0.32	0.32	−0.08	−0.08	
CO	0.16		−0.04	0.16		−0.04
Dist.		0.02	0.02	−0.08	−0.08	
ΣM	2.88	5.78	−5.78	2.72	−2.72	−1.32

(d)



(e)



(f)

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
DF	0	0.5	0.5	0.5	0.5	0
FEM	−100	−100			−100	−100
Dist.		50	50	50	50	
CO	25		25	25		25
Dist.		−12.5	−12.5	−12.5	−12.5	
CO	−6.25		−6.25	−6.25		−6.25
Dist.		3.125	3.125	3.125	3.125	
CO	1.56		1.56	1.56		1.56
Dist.		−0.78	−0.78	−0.78	−0.78	
CO	−0.39		−0.39	−0.39		−0.39
Dist.		0.195	0.195	0.195	0.195	
ΣM	−80.00	−60.00	60.00	60.00	−60.00	−80.00

(g)

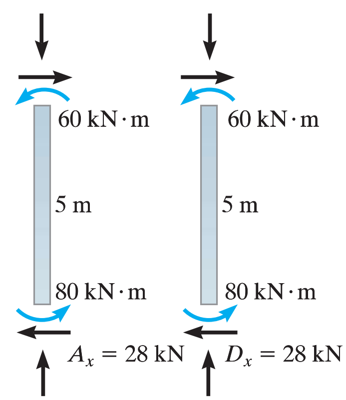
$$M = \frac{6EI\Delta}{L^2}$$

Since *both* B and C happen to be displaced the same amount Δ', and AB and DC have the *same* E, I, and L, the FEM in AB will be the *same* as that in DC. As shown in Fig. 12–18*f*, we will arbitrarily *assume* this fixed-end moment to be

$$(\text{FEM})_{AB} = (\text{FEM})_{BA} = (\text{FEM})_{CD} = (\text{FEM})_{DC} = -100\text{ kN}\cdot\text{m}$$

A *negative sign* is necessary since the moment must act *counterclockwise* on the column for deflection Δ' to the right. The value of $\mathbf{R}'$ associated with this −100 kN·m moment can now be determined. The moment distribution of the FEMs is shown in Fig. 12–18*g*. From equilibrium, the horizontal reactions at A and D are calculated, Fig. 12–18*h*. Thus, for the entire frame we require

$$\Sigma F_x = 0; \qquad R' = 28 + 28 = 56.0\text{ kN}$$



(h)

Hence, $R' = 56.0$ kN creates the moments tabulated in Fig. 12–18*g*. Corresponding moments caused by $R = 0.92$ kN can be determined by proportion. Therefore, the resultant moment in the frame, Fig. 12–18*a*, is equal to the *sum* of those calculated for the frame in Fig. 12–18*b* plus the proportionate amount of those for the frame in Fig. 12–18*c*. We have

$$M_{AB} = 2.88 + \tfrac{0.92}{56.0}(-80) = 1.57\text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$

$$M_{BA} = 5.78 + \tfrac{0.92}{56.0}(-60) = 4.79\text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$

$$M_{BC} = -5.78 + \tfrac{0.92}{56.0}(60) = -4.79\text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$

$$M_{CB} = 2.72 + \tfrac{0.92}{56.0}(60) = 3.71\text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$

$$M_{CD} = -2.72 + \tfrac{0.92}{56.0}(-60) = -3.71\text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$

$$M_{DC} = -1.32 + \tfrac{0.92}{56.0}(-80) = -2.63\text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$

12

EXAMPLE 12.7

Determine the moments at each joint of the frame shown in Fig. 12–19*a*. The moment of inertia of each member is indicated in the figure.

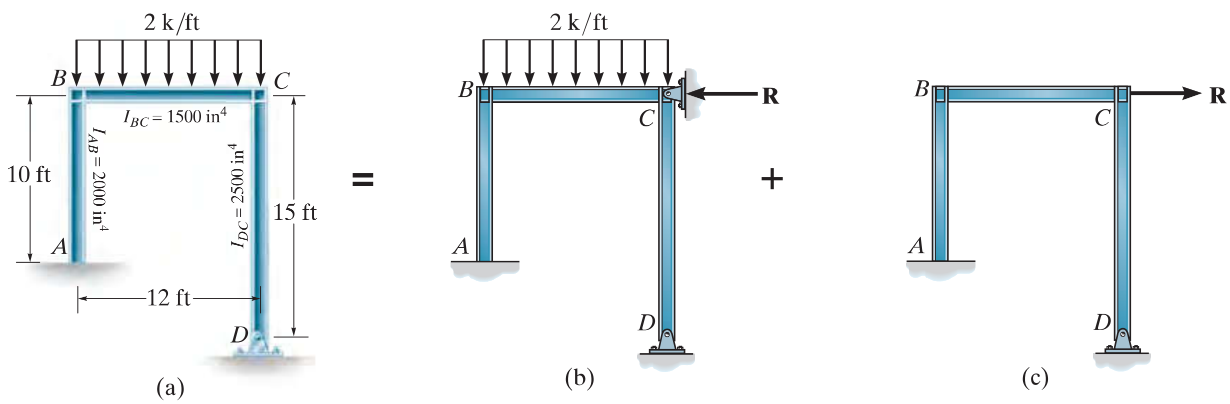


Fig. 12–19

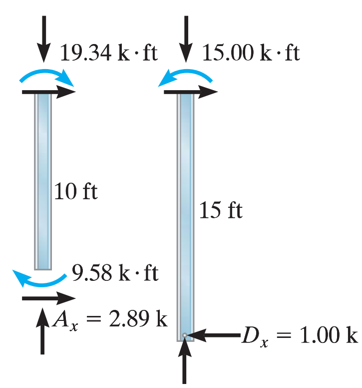
SOLUTION

The frame is first held from sidesway as shown in Fig. 12–19*b*. The internal moments are computed at the joints as indicated in Fig. 12–19*d*. Here the stiffness factor of *CD* was computed using $3EI/L$ since there is a pin at *D*. Calculation of the horizontal reactions at *A* and *D* is shown in Fig. 12–19*e*. Thus, for the entire frame,

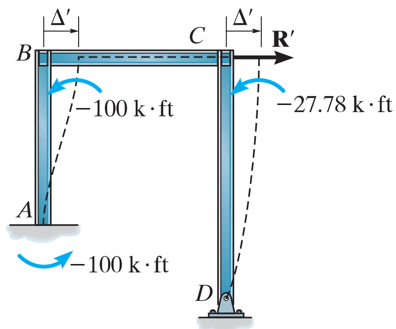
$$\Sigma F_x = 0; \qquad R = 2.89 - 1.00 = 1.89 \text{ k}$$

Joint	*A*	*B*		*C*		*D*
Member	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
DF	0	0.615	0.385	0.5	0.5	1
FEM			−24	24		
Dist.		14.76	9.24	−12	−12	
CO	7.38		−6	4.62		
Dist.		3.69	2.31	−2.31	−2.31	
CO	1.84		−1.16	1.16		
Dist.		0.713	0.447	−0.58	−0.58	
CO	0.357		−0.29	0.224		
Dist.		0.18	0.11	−0.11	−0.11	
ΣM	9.58	19.34	−19.34	15.00	−15.00	0

(d)



(e)



(f)

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
DF	0	0.615	0.385	0.5	0.5	1
FEM	−100	−100			−27.78	
Dist.		61.5	38.5	13.89	13.89	
CO	30.75		6.94	19.25		
Dist.		−4.27	−2.67	−9.625	−9.625	
CO	−2.14		−4.81	−1.34		
Dist.		2.96	1.85	0.67	0.67	
CO	1.48		0.33	0.92		
Dist.		−0.20	−0.13	−0.46	−0.46	
ΣM	−69.91	−40.01	40.01	23.31	−23.31	0

(g)

The opposite force is now applied to the frame as shown in Fig. 12–19*c*. As in the previous example, we will consider a force $\mathbf{R}'$ acting as shown in Fig. 12–19*f*. As a result, joints B and C are displaced by the same amount Δ'. The fixed-end moments for BA are computed from

$$(\text{FEM})_{AB} = (\text{FEM})_{BA} = -\frac{6EI\Delta}{L^2} = -\frac{6E(2000)\Delta'}{(10)^2}$$

However, from the table on the inside back cover, for CD we have

$$(\text{FEM})_{CD} = -\frac{3EI\Delta}{L^2} = -\frac{3E(2500)\Delta'}{(15)^2}$$

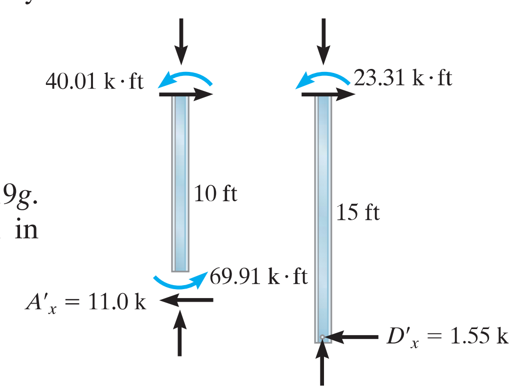
Assuming the FEM for AB is $-100\text{ k}\cdot\text{ft}$ as shown in Fig. 12–19*f*, the *corresponding* FEM at C, causing the *same* Δ', is found by comparison, i.e.,

$$\Delta' = -\frac{(-100)(10)^2}{6E(2000)} = -\frac{(\text{FEM})_{CD}(15)^2}{3E(2500)}$$

$$(\text{FEM})_{CD} = -27.78\text{ k}\cdot\text{ft}$$

Moment distribution for these FEMs is tabulated in Fig. 12–19*g*. Computation of the horizontal reactions at A and D is shown in Fig. 12–19*h*. Thus, for the entire frame,

$$\Sigma F_x = 0; \qquad R' = 11.0 + 1.55 = 12.55\text{ k}$$



(h)

The resultant moments in the frame are therefore

$$M_{AB} = 9.58 + \left(\tfrac{1.89}{12.55}\right)(-69.91) = -0.948\text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

$$M_{BA} = 19.34 + \left(\tfrac{1.89}{12.55}\right)(-40.01) = 13.3\text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

$$M_{BC} = -19.34 + \left(\tfrac{1.89}{12.55}\right)(40.01) = -13.3\text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

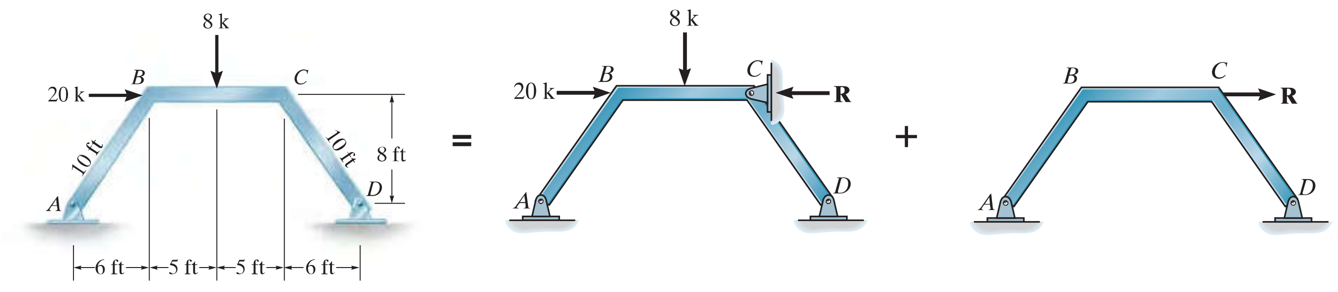
$$M_{CB} = 15.00 + \left(\tfrac{1.89}{12.55}\right)(23.31) = 18.5\text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

$$M_{CD} = -15.00 + \left(\tfrac{1.89}{12.55}\right)(-23.31) = -18.5\text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

12

EXAMPLE 12.8

Determine the moments at each joint of the frame shown in Fig. 12–20*a*. *EI* is constant.



(a) (b) (c)

Fig. 12–20

SOLUTION

First sidesway is prevented by the restraining force **R**, Fig. 12–20*b*. The FEMs for member *BC* are

$$(\text{FEM})_{BC} = -\frac{8(10)}{8} = -10 \text{ k}\cdot\text{ft} \qquad (\text{FEM})_{CB} = \frac{8(10)}{8} = 10 \text{ k}\cdot\text{ft}$$

Since spans *AB* and *DC* are pinned at their ends, the stiffness factor is computed using $3EI/L$. The moment distribution is shown in Fig. 12–20*d*.

Using these results, the *horizontal reactions* at *A* and *D* must be determined. This is done using an equilibrium analysis of *each member,* Fig. 12–20*e*. Summing moments about points *B* and *C* on each leg, we have

$$\curvearrowleft + \Sigma M_B = 0; \qquad -5.97 + A_x(8) - 4(6) = 0 \qquad A_x = 3.75 \text{ k}$$

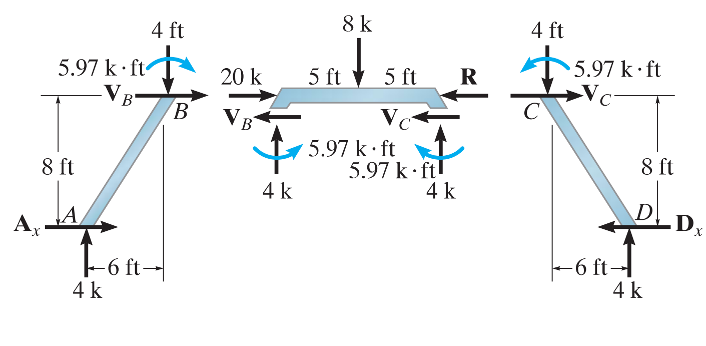
$$\curvearrowleft + \Sigma M_C = 0; \qquad 5.97 - D_x(8) + 4(6) = 0 \qquad D_x = 3.75 \text{ k}$$

Thus, for the entire frame,

$$\Sigma F_x = 0; \qquad R = 3.75 - 3.75 + 20 = 20 \text{ k}$$

Joint	A	B		C		D
Member	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
DF	1	0.429	0.571	0.571	0.429	1
FEM			−10	10		
Dist.		4.29	5.71	−5.71	−4.29	
CO			−2.86	2.86		
Dist.		1.23	1.63	−1.63	−1.23	
CO			−0.82	0.82		
Dist.		0.35	0.47	−0.47	−0.35	
CO			−0.24	0.24		
Dist.		0.10	0.13	−0.13	−0.10	
ΣM	0	5.97	−5.97	5.97	−5.97	0

(d)



(e)

The opposite force $\mathbf{R}$ is now applied to the frame as shown in Fig. 12–20*c*. In order to determine the internal moments developed by $\mathbf{R}$ we will first consider the force $\mathbf{R}'$ acting as shown in Fig. 12–20*f*. Here the dashed lines do not represent the distortion of the frame members; instead, they are constructed as straight lines extended to the final positions B' and C' of points B and C, respectively. Due to the symmetry of the frame, the displacement $BB' = CC' = \Delta'$. Furthermore, these displacements cause BC to rotate. The vertical distance between B' and C' is $1.2\Delta'$, as shown on the displacement diagram, Fig. 12–20*g*. Since each span undergoes end-point displacements that cause the spans to rotate, fixed-end moments are induced in the spans. These are: $(\text{FEM})_{BA} = (\text{FEM})_{CD} = -3EI\Delta'/(10)^2$, $(\text{FEM})_{BC} = (\text{FEM})_{CB} = 6EI(1.2\Delta')/(10)^2$.

Notice that for BA and CD the moments are *negative* since clockwise rotation of the span causes a *counterclockwise* FEM.

If we arbitrarily assign a value of $(\text{FEM})_{BA} = (\text{FEM})_{CD} = -100\ \text{k}\cdot\text{ft}$, then equating Δ' in the above formulas yields $(\text{FEM})_{BC} = (\text{FEM})_{CB} = 240\ \text{k}\cdot\text{ft}$. These moments are applied to the frame and distributed, Fig. 12–20*h*. Using these results, the equilibrium analysis is shown in Fig. 12–20*i*. For each leg, we have

$$\circlearrowleft + \Sigma M_B = 0; \quad -A'_x(8) + 29.36(6) + 146.80 = 0 \qquad A'_x = 40.37\ \text{k}$$

$$\circlearrowleft + \Sigma M_C = 0; \quad -D'_x(8) + 29.36(6) + 146.80 = 0 \qquad D'_x = 40.37\ \text{k}$$

Thus, for the entire frame,

$$\Sigma F_x = 0; \qquad R' = 40.37 + 40.37 = 80.74\ \text{k}$$

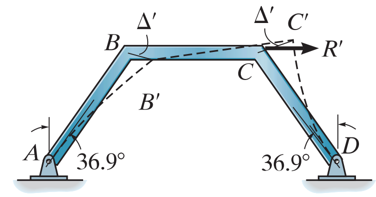
The resultant moments in the frame are therefore

$$M_{BA} = 5.97 + \left(\frac{20}{80.74}\right)(-146.80) = -30.4\ \text{k}\cdot\text{ft} \qquad \textit{Ans.}$$

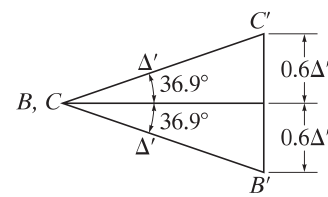
$$M_{BC} = -5.97 + \left(\frac{20}{80.74}\right)(146.80) = 30.4\ \text{k}\cdot\text{ft} \qquad \textit{Ans.}$$

$$M_{CB} = 5.97 + \left(\frac{20}{80.74}\right)(146.80) = 42.3\ \text{k}\cdot\text{ft} \qquad \textit{Ans.}$$

$$M_{CD} = -5.97 + \left(\frac{20}{80.74}\right)(-146.80) = -42.3\ \text{k}\cdot\text{ft} \qquad \textit{Ans.}$$



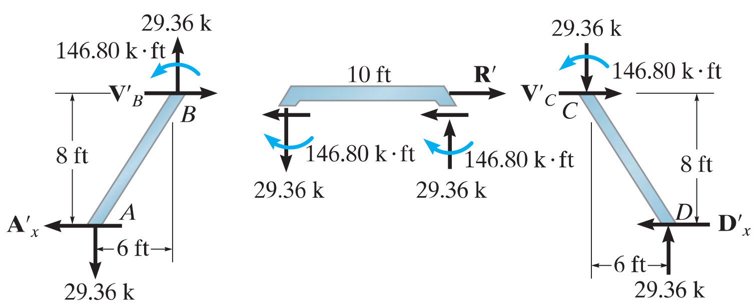
(f)



(g)

Joint	*A*	*B*		*C*		*D*
Member	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
DF	1	0.429	0.571	0.571	0.429	1
FEM		−100	240	240	−100	
Dist.		−60.06	−79.94	−79.94	−60.06	
CO			−39.97	−39.97		
Dist.		17.15	22.82	22.82	17.15	
CO			11.41	11.41		
Dist.		−4.89	−6.52	−6.52	−4.89	
CO			−3.26	−3.26		
Dist.		1.40	1.86	1.86	1.40	
CO			0.93	0.93		
Dist.		−0.40	−0.53	−0.53	−0.40	
ΣM	0	−146.80	146.80	146.80	−146.80	0

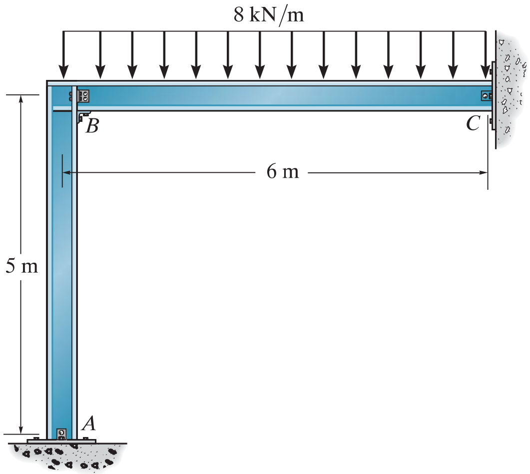
(h)



(i)

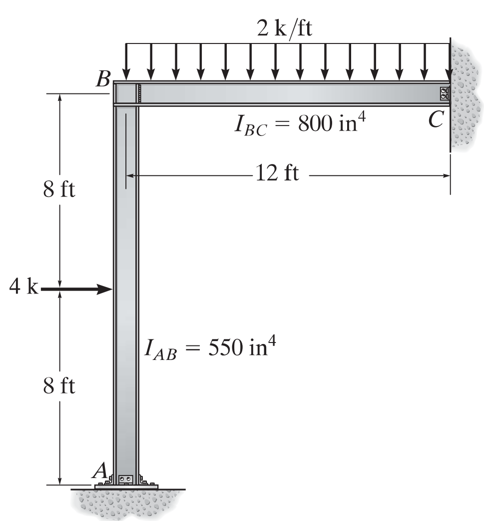
PROBLEMS

12–13. Determine the moment at *B*, then draw the moment diagram for each member of the frame. Assume the supports at *A* and *C* are pins. *EI* is constant.



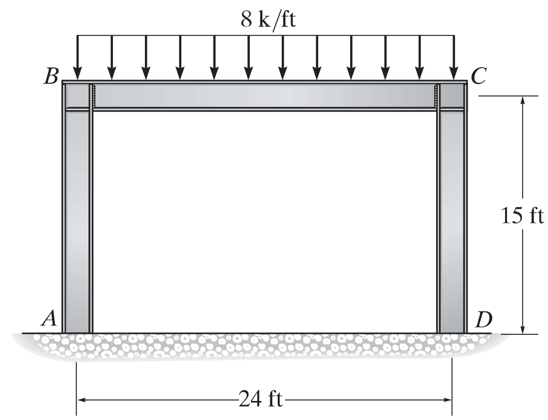
Prob. 12–13

12–14. Determine the moments at the ends of each member of the frame. Assume the joint at *B* is fixed, *C* is pinned, and *A* is fixed. The moment of inertia of each member is listed in the figure. $E = 29(10^3)$ ksi.



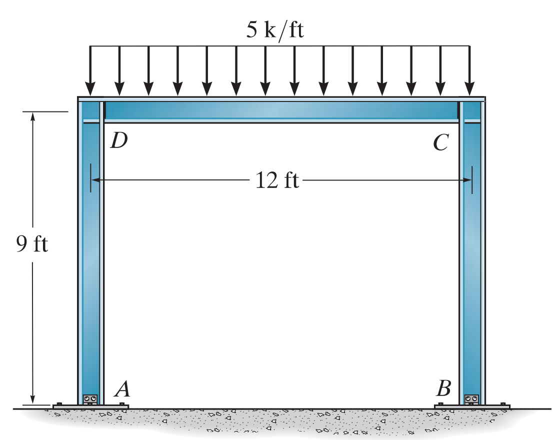
Prob. 12–14

12–15. Determine the reactions at *A* and *D*. Assume the supports at *A* and *D* are fixed and *B* and *C* are fixed connected. *EI* is constant.



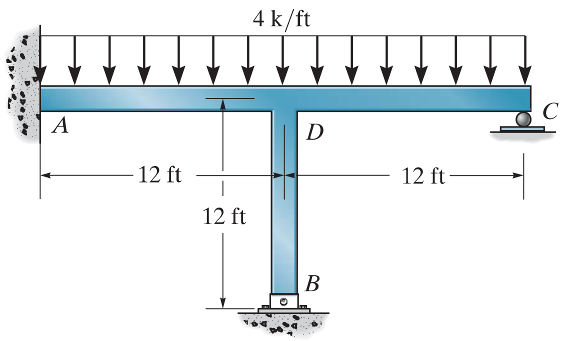
Prob. 12–15

***12–16.** Determine the moments at *D* and *C*, then draw the moment diagram for each member of the frame. Assume the supports at *A* and *B* are pins and *D* and *C* are fixed joints. *EI* is constant.



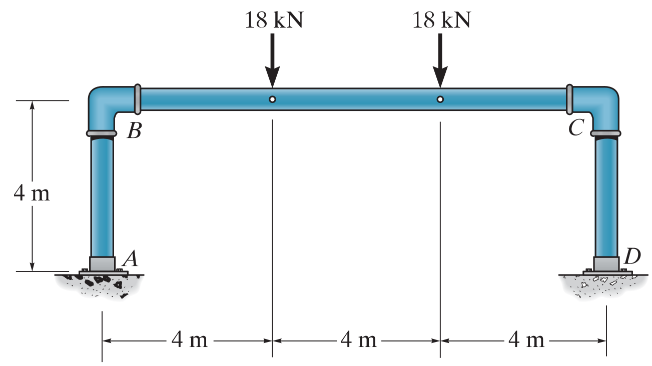
Prob. 12–16

12–17. Determine the moments at the fixed support A and joint D and then draw the moment diagram for the frame. Assume B is pinned.



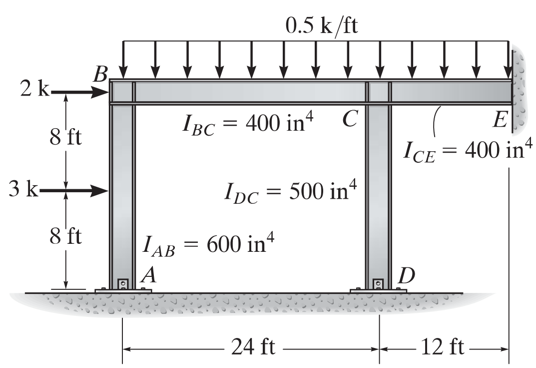
Prob. 12–17

12–19. The frame is made from pipe that is fixed connected. If it supports the loading shown, determine the moments developed at each of the joints. EI is constant.



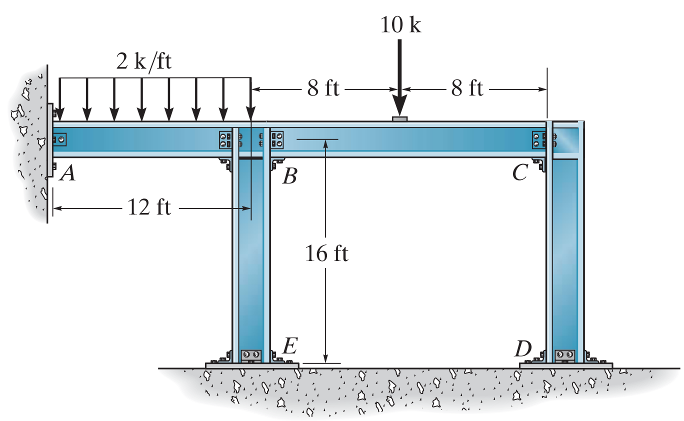
Prob. 12–19

12–18. Determine the moments at each joint of the frame, then draw the moment diagram for member BCE. Assume B, C, and E are fixed connected and A and D are pins. $E = 29(10^3)$ ksi.



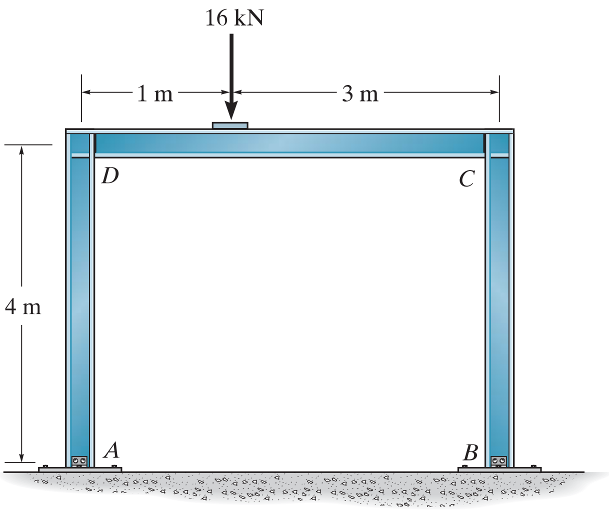
Prob. 12–18

***12–20.** Determine the moments at B and C, then draw the moment diagram for each member of the frame. Assume the supports at A, E, and D are fixed. EI is constant.



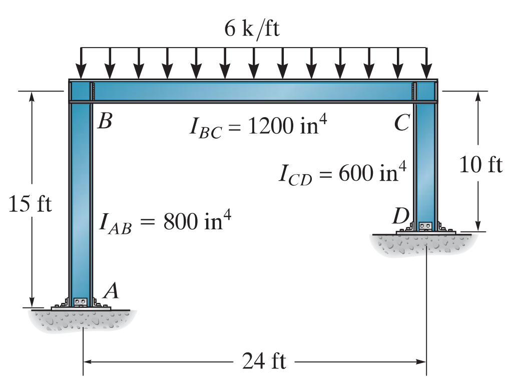
Prob. 12–20

12–21. Determine the moments at D and C, then draw the moment diagram for each member of the frame. Assume the supports at A and B are pins. EI is constant.



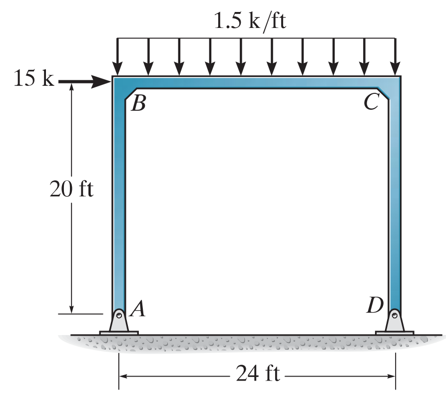
Prob. 12–21

12–22. Determine the moments acting at the ends of each member. Assume the supports at A and D are fixed. The moment of inertia of each member is indicated in the figure. $E = 29(10^3)$ ksi.



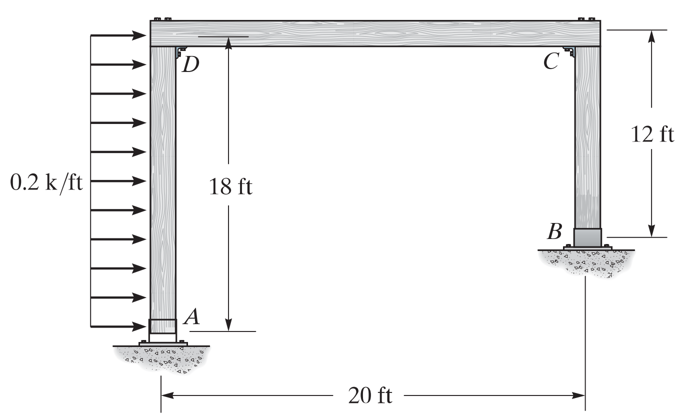
Prob. 12–22

12–23. Determine the moments acting at the ends of each member of the frame. EI is the constant.



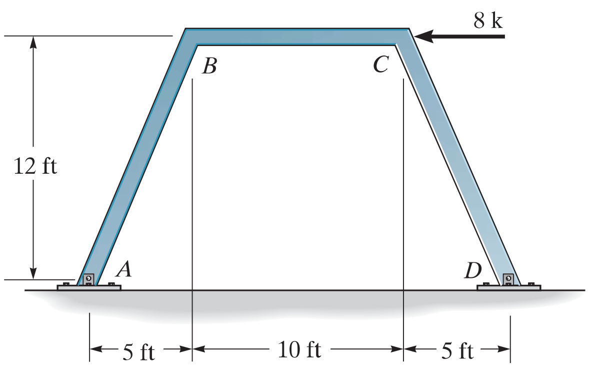
Prob. 12–23

***12–24.** Determine the moments acting at the ends of each member. Assume the joints are fixed connected and A and B are fixed supports. EI is constant.



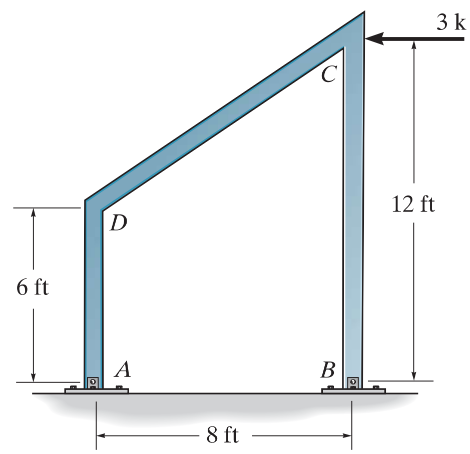
Prob. 12–24

12–25. Determine the moments at joints B and C, then draw the moment diagram for each member of the frame. The supports at A and D are pinned. EI is constant.



Prob. 12–25

12–26. Determine the moments at C and D, then draw the moment diagram for each member of the frame. Assume the supports at A and B are pins. EI is constant.



Prob. 12–26

CHAPTER REVIEW

Moment distribution is a method of successive approximations that can be carried out to any desired degree of accuracy. It initially requires locking all the joints of the structure. The equilibrium moment at each joint is then determined, the joints are unlocked and this moment is distributed onto each connecting member, and half its value is carried over to the other side of the span. This cycle of locking and unlocking the joints is repeated until the carry-over moments become acceptably small. The process then stops and the moment at each joint is the sum of the moments from each cycle of locking and unlocking.

The process of moment distribution is conveniently done in tabular form. Before starting, the fixed-end moment for each span must be calculated using the table on the inside back cover of the book. The distribution factors are found by dividing a member's stiffness by the total stiffness of the joint. For members having a far end fixed, use $K = 4EI/L$; for a far-end pinned or roller supported member, $K = 3EI/L$; for a symmetric span and loading, $K = 2EI/L$; and for an antisymmetric loading, $K = 6EI/L$. Remember that the distribution factor for a fixed end is DF $= 0$, and for a pin or roller-supported end, DF $= 1$.