CHAPTER 2

2. Slope Deflection Method

2.1. Introduction

The slope-deflection method uses displacements as unknowns and is referred to as a displacement method. The basic idea of the slope deflection method is to write the moments for each node of the members in terms of the deflections and rotations.

An important characteristic of the slope-deflection method is that it does not become increasingly complicated to apply as the number of unknowns in the problem increases. In the slope-deflection method the individual equations are relatively easy to construct regardless of the number of unknowns. In this method, the relationship is established between moments at the ends of the members and the corresponding rotations and displacements.

The basic assumptions are:

- axial deformation is neglected

  (axial deformation is considered for reasonable and highly sensitive structures)
- shear deformation is neglected
- deformation due to bending moment and rotation is considered

Degree of Freedom (DOF)

When a structure is loaded, it deforms into a unique shape that can be specified provided we know the displacement of specific joints on a structure. The displacements are angular and relative linear displacements. These displacements are called degree of freedom, DOF.

In the figure shown below, node A can rotate, node B is restrained from rotation and there is a relative settlement between the supports. Thus, θ_A and Δ are called DOFs.

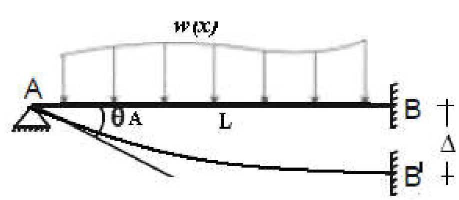


Fig. 2.1

Fixed-End Moments (FEM):

When all of the joints of a structure are clamped to prevent any joint rotation, the external loads produce certain moments at the ends of the members to which they are applied. These moments are referred to as fixed-end moments. These could be due to in-span loads, temperature variation and/or relative displacement between the ends of a member.

Fixed-end moments for several common types of loading conditions may be found tabulated in different structural analysis books for convenient reference.

2.2. Sign Convention

The moments at the end of a member are assumed to be positive when they tend to rotate the member clockwise about the joint. This implies that the resisting moment of the joint would be counter-clockwise. Accordingly, under gravity loading condition the fixed-end moment at the left end is assumed as counter-clockwise (−ve) and at the right end as clockwise (+ve).

2.3. Analysis of beams

2.3.1. Derivation of Slope-Deflection Equation

For the derivation of slope deflection equation, a typical member AB of a continuous beam shown below is considered.

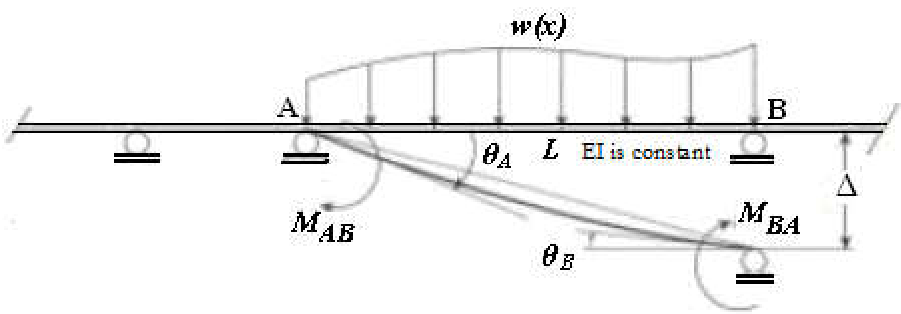


Fig 2.2

The slope-deflection equations can be obtained by superposing the moments induced at each support due to the applied load (FEM) and each of the displacements (θ_A, θ_B and Δ).

I. Contribution of the applied load

Moments due to the applied load, FEMs, ($M^F{}_{AB}$ and $M^F{}_{BA}$) are obtained from Tables.

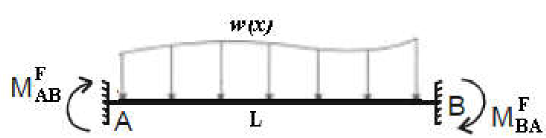
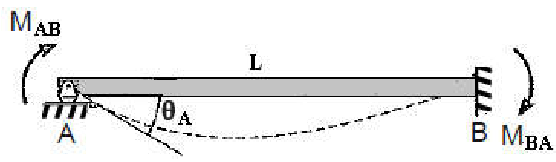


Fig. 2.3

II. Contribution of rotation at A, θA

In order to develop the method, consider the beam shown in Fig 2.4(a) below. By the conjugate beam theorem, the deflection of the real beam at A and B is zero and therefore the sum of moments at nodes A' and B' of the conjugate beam must be zero.



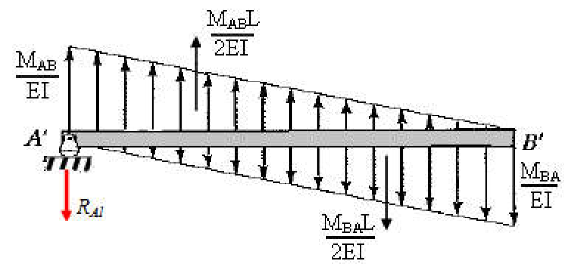


Fig.2.4 (a) real beam Fig. 2.4 (b) conjugate beam

For the conjugate beam,

$$\sum M_{A'} = \frac{M_{BA}L}{2EI}(2L/3) + \frac{M_{AB}L}{2EI}(L/3) = 0 \tag{2.1}$$

$$\sum M_{B'} = R_{A1}L + \frac{M_{BA}L}{2EI}(L/3) - \frac{M_{AB}L}{2EI}(2L/3) = 0 \tag{2.2}$$

Based on the conjugate beam theorem, the shear force of the conjugate beam is the rotation of the corresponding actual beam. Therefore, $R_{A1}=\theta_A$. From Eqs. (2.1) and (2.2) the following relations are obtained:

$$M_{AB} = \frac{4EI}{L}\theta_A \quad \text{and} \quad M_{BA} = \frac{2EI}{L}\theta_A \rightarrow \qquad M_{BA} = \frac{1}{2}M_{AB} \tag{2.3}$$

The contribution of rotation at A (θ_A) on the end moment at A is twice as compared with the end moment at B.

III. Contribution of rotation at B, θ_B

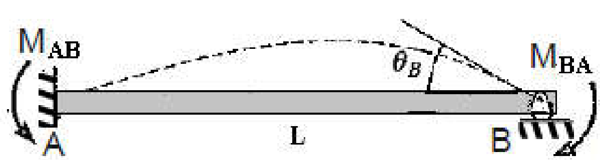


Fig. 2.5

In a similar manner, the end moments due to rotation of node B are:

$$M_{AB} = \frac{2EI}{L}\theta_B \quad \text{and} \quad M_{BA} = \frac{4EI}{L}\theta_B \tag{2.4}$$

From the above results, it is observed that the end-moment at A has twice the contribution from the rotation at A compared with the rotation at B. More generally, the contribution from a rotation at the "near" end is twice that of the rotation at the "far" end. The constant (1/2) is called the carry-over factor.

IV. Contribution of relative linear displacement, Δ

If the far node B of the member is displaced relative to A, so that the chord rotates clockwise (the member experiences negative rotations) and yet both ends do not rotate, then equal but opposite moment and shear reactions are developed in the member.

Using the conjugate beam method, the moment of the conjugate beam is the displacement of the real beam and hence end B' of conjugate beam shown in Fig. 2.6(b) must have subjected to a moment of magnitude Δ.

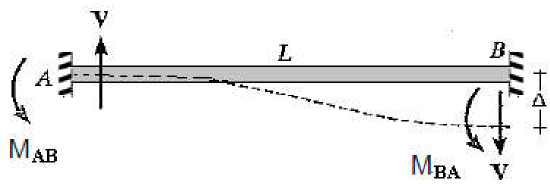


Fig. 2.6 (a)-real beam

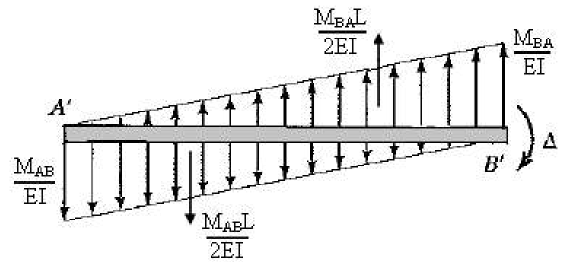


Fig.2.6 (b): conjugate beam

Fig.4b $\sum F_y = 0 \rightarrow M_{AB} = M_{BA}$ (2.5)

Fig.4b $\sum M_{B'} = \Delta + \frac{M_{BA}L}{2EI}(L/3) - \frac{M_{AB}L}{2EI}(2L/3) = 0$ (2.6)

Solve Eqs. (2.5) and (2.6) simultaneously,

$$M_{AB} = M_{BA} = -\frac{6EI\Delta}{L^2}$$

The total moment at each node is obtained by adding all the moments due to FEM, contributions from rotations at A and B and relative linear displacement.

$$M_{AB} = M^F_{AB} + \frac{2EI}{L}(2\theta_A + \theta_B - 3\psi) \qquad (2.7)$$

$$M_{BA} = M^F_{BA} + \frac{2EI}{L}(2\theta_B + \theta_A - 3\psi) \qquad (2.8)$$

In Eqs. (2.7) and (2.8) ψ is the span's cord angle =Δ/L. Theses equations, which express the moments at the ends of a member in terms of its end rotations and translations for a specified external loading, are called the slope-deflection equations.

From Equations (5a & 5b), we observe that the two slope-deflection equations have the same form and that either one of the equations can be obtained from the other simply by switching the subscripts A and B.

2.4. Beams with Hinged Ends

The slope-deflection equations derived previously (Eq.(2.7) or Eq. (2.8)) are based on the condition that the member is rigidly connected to joints at both ends, so that the member end rotations θ_A and θ_B are equal to the rotations of the adjacent joints. When one of the member's ends is connected to the adjacent joint by a hinged connection, the moment at the hinged end must be zero. The slope-deflection equations can be easily modified to reflect this condition. With reference to Fig.2.2, if the end B of member AB is hinged, then the moment at B must be zero. By substituting $M_{BA}=0$ into Eq. (2.7), we write

$$M_{AB} = M_{AB}^{F} + \frac{2EI}{L}(2\theta_A + \theta_B - 3\psi) \tag{2.9}$$

$$M_{BA} = M_{BA}^{F} + \frac{2EI}{L}(2\theta_B + \theta_A - 3\psi) = 0 \tag{2.10}$$

Solving for θ_B from Eq. (2.10) and substitute this value into Eq. (2.9), thus obtaining the modified slope-deflection equations for member AB with a hinge at end B:

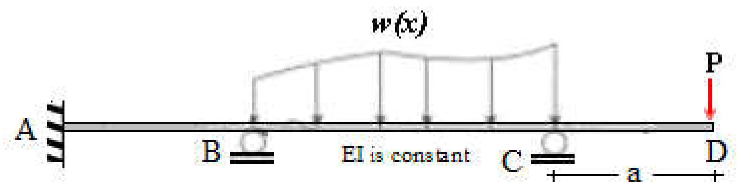
$$\begin{cases} M_{AB} = \dfrac{3EI}{L}(\theta_A - \psi) + \left(M_{AB}^{F} - \dfrac{M_{BA}^{F}}{2} \right) \\ \\ M_{BA} = 0 \end{cases} \tag{2.11}$$

Similarly, it can be shown that for a member AB with a hinge at end A, the rotation of the hinged end is given by:

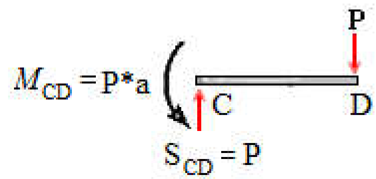
$$\begin{cases} M_{BA} = \dfrac{3EI}{L}(\theta_B - \psi) + \left(M_{BA}^{F} - \dfrac{M_{AB}^{F}}{2} \right) \\ \\ M_{AB} = 0 \end{cases} \tag{2.12}$$

2.5. Structures with Cantilever Overhangs

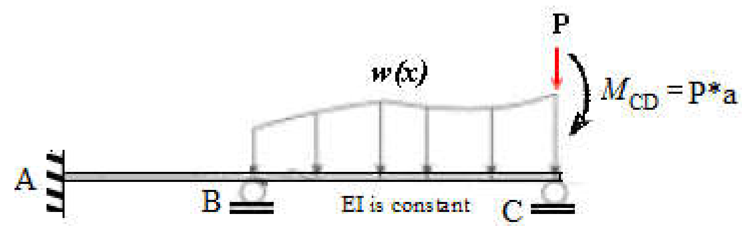
Consider a continuous beam with a cantilever overhang, as shown in Fig. 2.7(a). Since the cantilever portion CD of the beam is statically determinate in the sense that the shear and moment at its end C can be obtained by applying the equations of equilibrium (Fig. 2.7(b)), it is not necessary to include this portion in the analysis. Thus, for the purpose of analysis, the cantilever portion CD can be removed from the structure, provided that the moment and the force exerted by the cantilever on the remaining structure are included in the analysis. The indeterminate part AC of the structure, which needs to be analyzed, is shown in Fig. 2.7(c).



(a) Actual beam



(b) Cantilever portion



(c) Part to be analyzed

Fig 2.7 Continuous beam with a cantilever overhang

Generally, the procedure for the analysis of beams by the slope-deflection method can be summarized as follows:

- Identify the degrees of freedom of the structure. For continuous beams, the degrees of freedom consist of the unknown rotations of the joints.
- Compute fixed-end moments. For each member of the structure, evaluate the fixed-end moments due to the external loads
- In the case of support settlements, determine the rotations of the chords of members adjacent to the supports that settle ($\psi = \Delta/L$)
- Write slope-deflection equations. For each member, apply Eq. (2.7) to write two slope-deflection equations relating member end moments to the unknown rotations of the adjacent joints.
- Write equilibrium equations. For each joint that is free to rotate, $\sum M = 0$
- Determine the unknown joint rotations. Substitute the slope deflection equations into the equilibrium equations, and solve the resulting system of equations for the unknown joint rotations.

- Calculate member end moments by substituting the numerical values of joint rotations determined in step 6 into the slope-deflection equations.
- Compute member end shears. For each member,
  a) draw a free body diagram showing the external loads and end moments and
  b) apply the equations of equilibrium to calculate the shear forces at the ends of the member
- Determine support reactions by considering the equilibrium of the joints of the structure.
- To check the calculations of member end shears and support reactions, apply the equations of equilibrium to the free body of the entire structure. If the calculations have been carried out correctly, then the equilibrium equations should be satisfied.
- Draw shear and bending moment diagrams of the structure by using the beam sign convention.

2.6. Analysis of frames

The slope deflection equations may be applied to statically indeterminate frames with or without side sway. For frame analysis, the solution procedure is amended to include the identification of unknown joint displacements and to establish shear equilibrium equations in addition to moment equilibrium equations.

2.6.1. Frames without side sway

The analysis of frames using the slope-deflection method can also be carried out by applying the two governing equations of beams.

A side sway will not occur if

(a) the frame geometry and loading are symmetric, and

(b) side sway is prevented due to supports

For the frame shown below, the values of angular displacement at each joint are obtained from,

$$\begin{aligned} &\sum \mathrm{M_B} = 0 \rightarrow M_{BA} + M_{BC} = 0 \\ &\sum \mathrm{M_C} = 0 \rightarrow M_{CB} + M_{CD} = 0 \quad \text{and} \\ &\sum F_x = 0 \rightarrow H_A + H_D = 0 \end{aligned} \qquad (2.13)$$

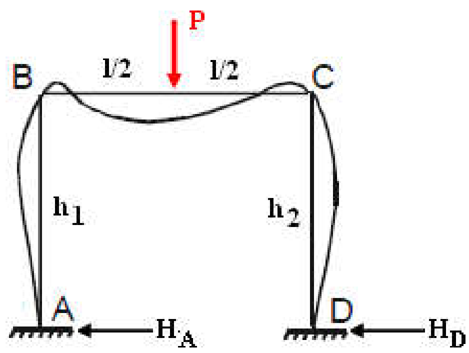


Fig. 2.7(a) Frame ABCD

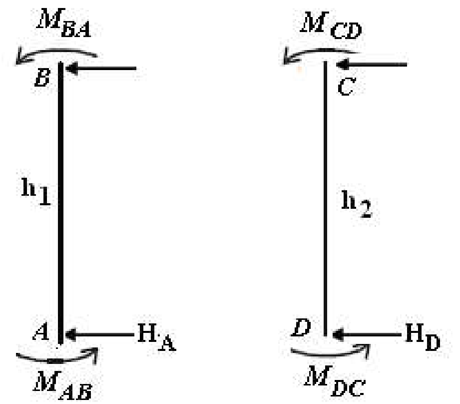


Fig. 2.7(b) FBD of members AB and DC

H_A and H_D are obtained by considering the FBDs of members AB and DC.

From FBD of member AB, $\sum M_B = 0 \rightarrow H_A = \dfrac{M_{BA} + M_{BC}}{h_1}$ and

From FBD of member CD, $\sum M_c = 0 \rightarrow H_D = \dfrac{M_{CD} + M_{CB}}{h_2}$

2.6.2. Frames with side sway

A side sway will occur if

(a) the frame geometry and loading are unsymmetrical, and

(b) side sway is not prevented due to supports.

For analysis of side sway frames using slope-deflection method, the contribution of Δ should be superimposed & it is necessary to consider the shear forces at the base of the columns, and the horizontal external load must be in equilibrium (force equilibrium equation) in addition to the equilibrium of joints.

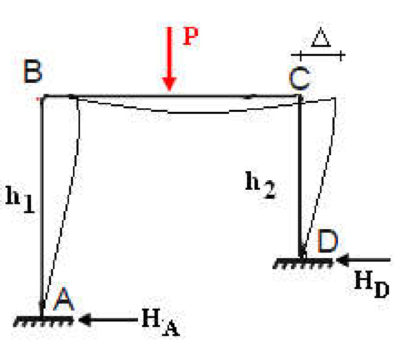


Fig. 2.8(a) Frame ABCD

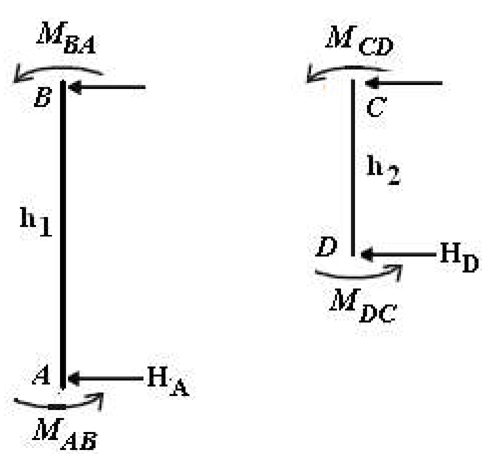


Fig. 2.8(b) FBD of members AB and DC

H_A and H_D are obtained by considering the FBDs of members AB and DC.

From FBD of member AB, $\sum M_B = 0 \rightarrow H_A = \dfrac{M_{BA} + M_{BC}}{h_1}$ and

From FBD of member CD, $\sum M_c = 0 \rightarrow H_D = \dfrac{M_{CD} + M_{CB}}{h_2}$

$$\sum M_B = 0 \rightarrow M_{BA} + M_{BC} = 0$$
$$\sum M_C = 0 \rightarrow M_{CB} + M_{CD} = 0 \quad \text{and}$$
$$\sum F_x = 0 \rightarrow H_A + H_D = 0$$