

4 Pipelines and Pipe Network Hydraulics – I

The hydraulics of pipelines and pipe networks presented herein is limited to turbulent flow of water in closed conduits (pipes) flowing full. Closed conduits flowing partially full, such as storm sewers, are analyzed as open channel flow. Pipes are connected together in various configurations (called networks) to transport water from the supply to the user (customer).

When the pipes are connected in a series, the system is called a pipeline. When the pipes are connected with one or more closed loops, the system is called a network.

A pipe system can include many types of components: pipes of various lengths and diameters, valves to control flow rate and isolate parts of the system for repair, pumps and turbines to convert between hydraulic energy and mechanical, thermal, and electrical energy.

Flow in a pipe can either be steady or unsteady. Unsteady flow implies that the velocity is a function of time and will change on the scale of seconds. Steady flow implies that the velocity is constant, although it is often assumed to change slowly over time (sequence of steady states). Steady flow equations can be used to analyze a water distribution system where the demands on the system change hourly, pumps turn on and off, and storage tanks fill and drain as time proceeds.

4.1 Equations for Steady Flow

The hydraulics of steady flow in pipe systems is described by the continuity equation and the energy equation. Momentum is used when forces of flow in restraints (thrust blocks) are important. Head loss is the key term in the energy equation that relates frictional losses to flow in a pipe system.

4.2 Continuity Equation

Water is considered incompressible in steady flow in pipelines and pipe networks, and the conservation of mass reduces to the volumetric flow rate, Q ,

$$Q = AV \tag{1}$$

where A is the cross sectional of the pipe, and V is the mean section velocity. Typical units for discharge is liters per second (lps), gallons per minute (gpm), cubic meters per second (cms), cubic feet per second (cfs), and million gallons per day (mgd).

4.2.1 Continuity between two cross-sections

The continuity equation between two cross-sections of a pipe as depicted in Figure 9 is

$$A_1 V_1 = A_2 V_2 \quad (2)$$

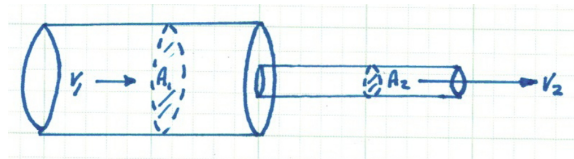


Figure 9: Continuity between cross-sections

4.2.2 Continuity at a junction (node)

Junctions (nodes) are where two or more pipes join together. A three-pipe junction node

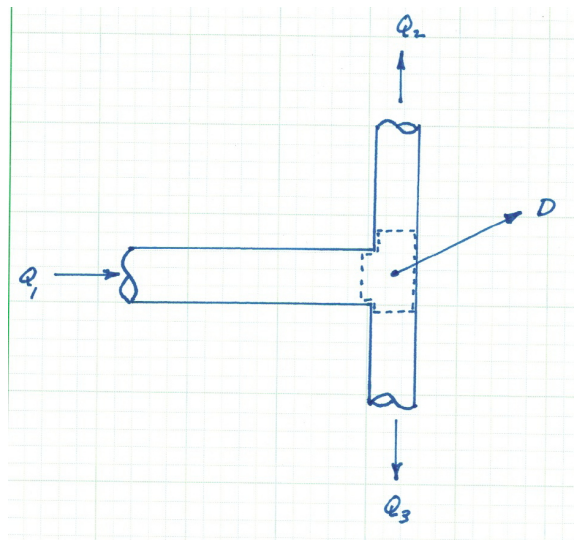


Figure 10: Continuity at a three-pipe junction with an external demand.

with constant external demand is shown in Figure 10. The continuity equation for the junction node is

$$Q_1 - Q_2 - Q_3 - D = 0 \quad (3)$$

In design analysis, all demands on the system are located at junctions (nodes), and the flow connecting junctions is assumed to be uniform. If a substantial demand is located between nodes, then an additional node is established at the demand location. Mays (2011), pages 66 – 68, provides a more extensive discussion of continuity for the interested reader.

4.3 Energy Equation

The design of a pressurized conduit involves selecting diameter and materials to satisfy hydraulic requirements at the lowest life-cycle cost. Most buyers of engineering services are “first-cost” buyers, so lowest life-cycle usually means lowest initial capital cost.

Equation 4 is the one-dimensional steady flow form of the energy equation typically applied for pressurized conduit hydraulics.

$$\frac{p_1}{\rho g} + \alpha_1 \frac{V_1^2}{2g} + z_1 + h_p = \frac{p_2}{\rho g} + \alpha_2 \frac{V_2^2}{2g} + z_2 + h_t + h_l \quad (4)$$

where $\frac{p}{\rho g}$ is the pressure head at a location, $\alpha \frac{V^2}{2g}$ is the velocity head at a location, z is the elevation, h_p is the added head from a pump, h_t is the added head extracted by a turbine, and h_l is the head loss between sections 1 and 2. Figure 11 is a sketch that illustrates the various components in Equation 4.

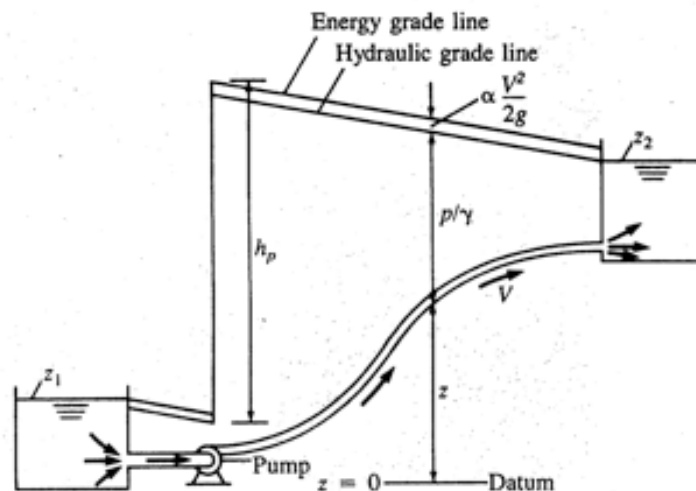


Figure 5-1 Definition sketch for terms in the energy equation

Figure 11: Definition sketch for energy equation

4.3.1 Velocity Head

The velocity in $\alpha \frac{V^2}{2g}$ is the mean section velocity and is the ratio of discharge to flow area. The kinetic energy correction coefficient is

$$\alpha = \frac{\int_A u^3 dA}{V^3 A} \quad (5)$$

where u is the point velocity in the cross section (usually measured relative to the centerline or the pipe wall; axial symmetry is assumed). Generally values of α are 2.0 if the flow is laminar, and approach unity (1.0) for turbulent flow. In most water distribution systems the flow is usually turbulent so α is assumed to be unity and the velocity head is simply $\frac{V^2}{2g}$.

4.3.2 Added Head — Pumps

The head supplied by a pump is related to the mechanical power supplied to the flow. Equation 6 is the relationship of mechanical power to added pump head.

$$\eta P = Q \rho g h_p \quad (6)$$

where the power supplied to the motor is P and the “wire-to-water” efficiency is η .

4.3.3 Added Head — Turbines

The head recovered by a turbine is also an “added head” but appears on the loss side of the equation. Equation 7 is the power that can be recovered by a turbine (again using the concept of “water-to-wire” efficiency is

$$P = \eta Q \rho g h_t \quad (7)$$

4.4 Hydraulic and Energy Grade Lines

The terms of Equation 4 have dimensions of length and this result provides a useful physical relationship. The sum of pressure and elevation head (static head) is a length that liquid would rise in a piezometer attached to the system at a particular location. The locus of all such points in a hydraulic system (visualize piezometers every few feet) is called the hydraulic grade line (HGL). In an open conduit, the water surface is the hydraulic grade line.

The total head includes the static head and the velocity head. The locus of these points is called the energy grade line (EGL). Engineers from time to time find the ability to sketch HGL and EGL useful to locate potential trouble spots in a hydraulic system.

Hints for drawing HGL and EGL sketches are listed as follows:

1. The EGL is above the HGL by a distance equal to the velocity head at a location. If the velocity is zero, or small, as in a reservoir, the HGL and EGL will coincide with the liquid surface (e.g. Figure 12).
2. Head loss requires the EGL to slope downward in the direction of flow. The only exception is when a pump adds energy (and pressure) to the flow. In this instance an abrupt rise in the EGL occurs from the upstream side (suction) of the pump to the downstream side (discharge).
3. When energy is removed from the system as in a turbine, the EGL and HGL will drop abruptly (e.g. Figure 12).
4. An expansion can be used to gradually convert velocity head to pressure head, a sudden expansion is much less efficient and wastes energy (e.g. Figure 12).

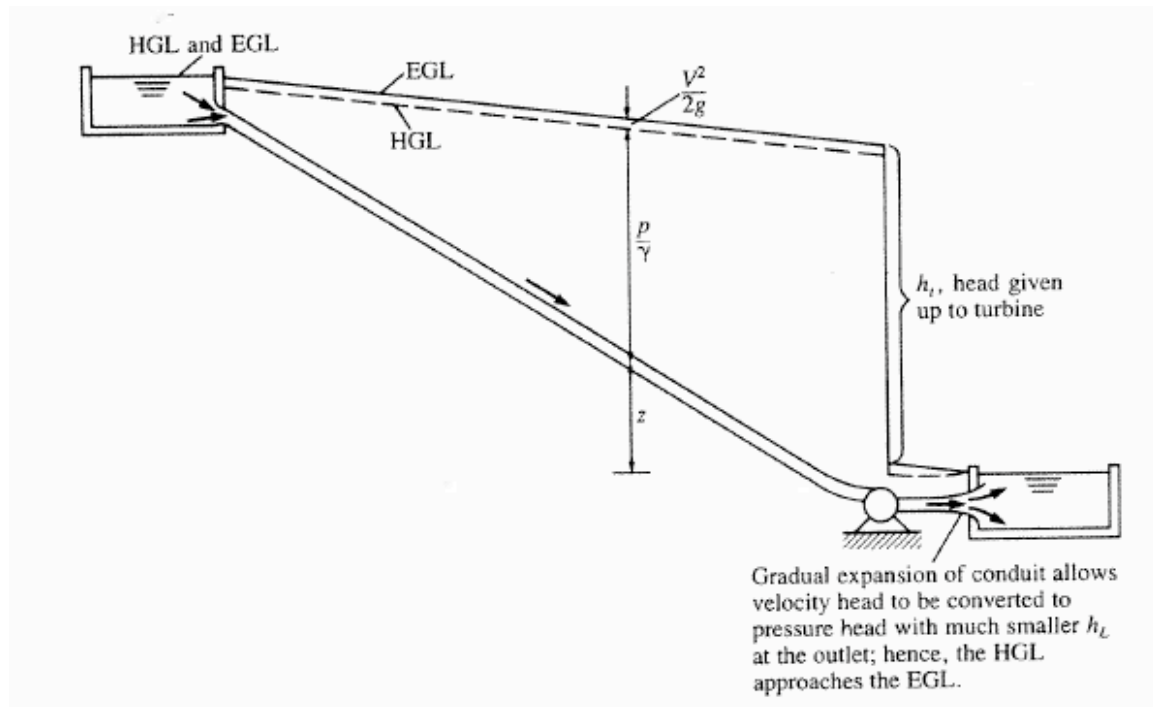


Figure 12: HGL and EGL and energy loss in a turbine system.

5. In a pipe where the pressure is zero, the HGL is coincident with the water surface in

the system because $p/\gamma = 0$ at these points. This hint is useful for finding the HGL in a system such as the outlet of a pipe where liquid discharges to the atmosphere, or at the upstream end where the pressure is zero in a reservoir.

6. Steady flow in a pipe of constant cross section and material properties (roughness) requires the head loss per unit length be some constant; thus the slope of the EGL and HGL will be constant along that section of pipe.
7. When flow passes to a pipe of a different cross section or material property, the velocity will change and the distance between the EGL and HGL will change. Furthermore, the slope of the EGL will change because the head loss per unit length will be larger in the conduit with the greater velocity.
8. If the HGL falls below the elevation of the pipe, then the pressure head is negative, as depicted in Figure 13.

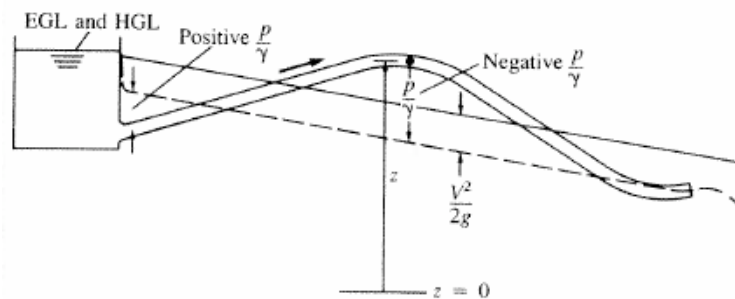


Figure 13: HGL and EGL in a negative pressure system.

If the negative head of the water is less than the vapor pressure head of the water (about -34 feet at STP), cavitation will occur. Cavitation is undesirable in engineered systems⁶, causing increased head loss, structural damage from vibrations and pitting of the conduit wall. Similarly undesirable is when the pressure decreases to the vapor pressure and stays that low a large vapor cavity forms. When the liquid rejoins with a system pressure rise a huge dynamic pressure is formed as the vapor cavity collapses (water hammer) and can rupture the pipe. If the pipe is thin walled, the negative pressure can cause the pipe itself to collapse (pinch). The engineer needs to be cautious in specifying sustained or high negative pressure in a system⁷.

⁶Except a sonicator, which uses cavitation to impart energy into a sample — however such devices are not common in civil engineering processes.

⁷There will be times when negative pressure is intentional, such as in a high-vacuum system, the warning about negative pressure is for routine pipe systems.

5 Pipeline and Pipe Network Hydraulics – II

Need a section introduction.

5.1 Head Loss Models – Darcy-Weisbach and Moody Chart

Head loss in straight pipes is caused by resistance of the pipe wall on the liquid. In laminar flow ($Re_d < 2000$) the head loss is explained entirely by liquid viscosity, and the loss is proportional to the mean section velocity. In turbulent flow the loss is caused by the dissipation of the turbulent component of kinetic energy, and the loss is a super-linear function of velocity as well as the pipe roughness.

The Darcy-Weisbach head loss model is used in civil engineering practice (as well as other disciplines). The model relates fluid properties (viscosity, density), flow and geometric properties (velocity, area), and pipe material properties (roughness, diameter) to a friction factor that is then applied in a quadratic drag formula. Equation 8 is the Darcy-Weisbach formula for pipe loss. These relationships for pipes are typically summarized by the friction factor concept and the Moody chart.

$$h_f = f \frac{L V^2}{D 2g} \quad (8)$$

where h_f is the head loss for a length of pipe L , D is the diameter, V is the mean section velocity, g is the constant of gravitational acceleration.

Equation 9 is the Darcy-Weisbach head loss formula in its often more convenient discharge form.

$$h_f = 8fL \frac{Q^2}{\pi^2 g D^5} \quad (9)$$

The friction factor f is a function of Reynolds number Re_d and pipe roughness ratio $\frac{\epsilon}{D}$. The relation of f , Re_d , and $\frac{\epsilon}{D}$ is plotted in Figure 14 which is called the Moody chart (or Moody-Stanton diagram)(CITE SOURCE). The diagram is a graphical representation of the analysis of thousands of experiments over the last century that established the functional relations represented by the various curves on the chart.

Typical problems using the Moody chart fall into three categories.

1. Estimate the head loss in a pipe h_f given the discharge, diameter, and material properties.

2. Estimate the discharge in a pipe Q given the available head loss, diameter, and material properties.
3. Estimate the diameter of a pipe D given the available head loss, discharge, and material properties.

These three types of problems are the subject of the next three examples.

| Material | e (ft) | e (mm) |
|----------------------------------|------------|---------|
| Riveted steel | 0.003–0.03 | 0.9–9.0 |
| Concrete | 0.001–0.01 | 0.3–3.0 |
| Cast iron | 0.00085 | 0.25 |
| Galvanized iron | 0.0005 | 0.15 |
| Commercial steel or wrought iron | 0.00015 | 0.046 |
| Drawn tubing | 0.000005 | 0.0015 |

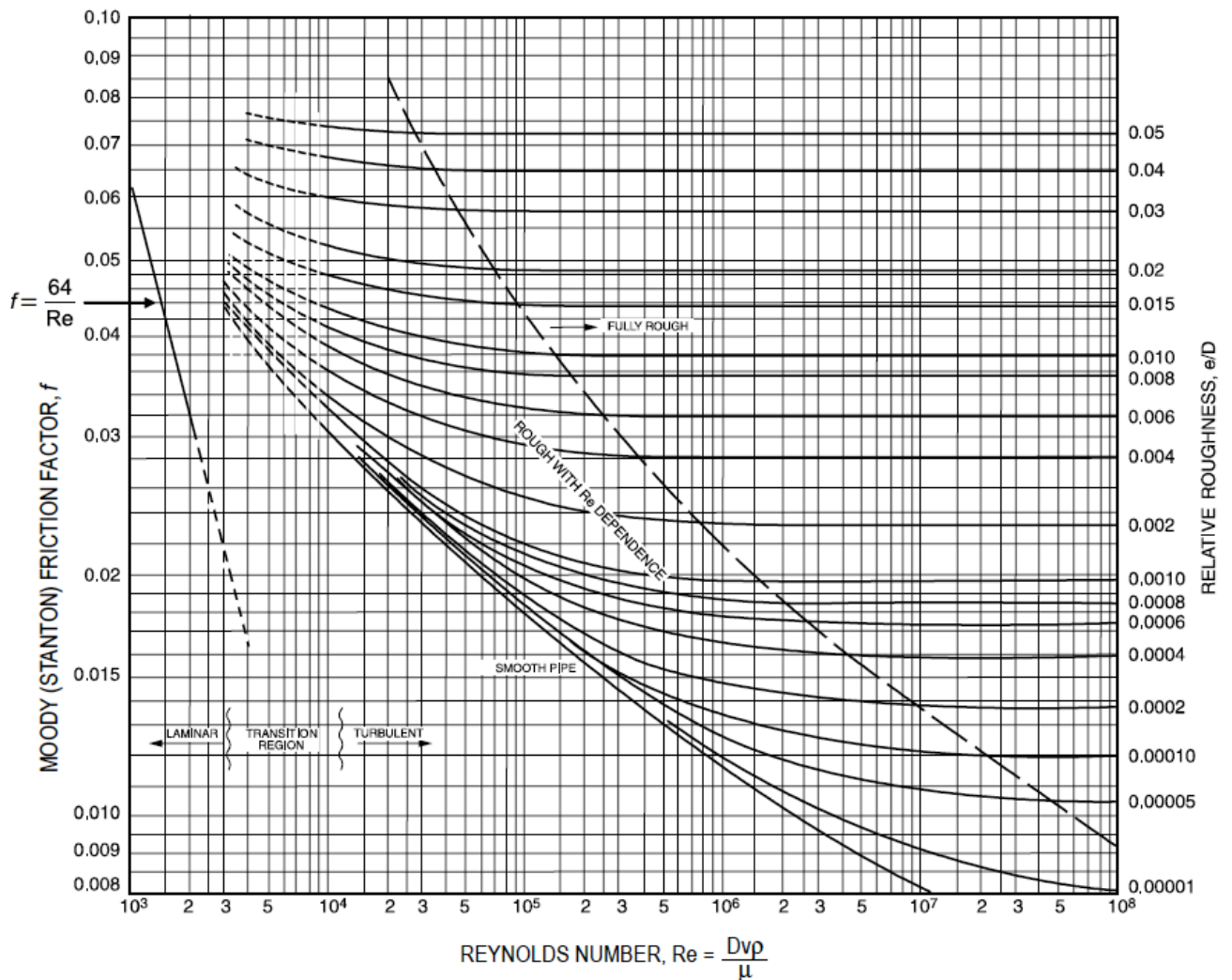


Figure 14: Moody-Stanton Diagram (from CITE NCEES).

Find h_f given Q , D , ϵ This kind of problem is relatively straightforward. The engineer computes Re_d from the discharge Q and the pipe diameter D . Then computes the roughness ratio from the tabulated ϵ for the pipe material. The friction factor, f , is then recovered directly from the Moody chart.

Example Oil with specific gravity 0.9, viscosity 0.00003 ft²/sec flows in a 2000-foot long, 6-inch diameter, cast-iron pipe at a flow rate of 1.0 cubic-feet-per-second. The pipe slopes upward at an angle of 5° in the direction of flow. Estimate the head loss in the pipe. Estimate the pressure drop in the pipe.

Solution Figure 15 is a sketch of the situation.

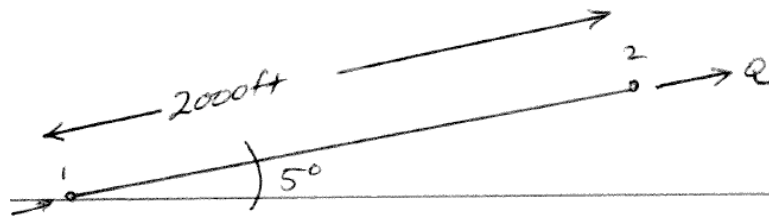


Figure 15: Sketch for Example

Equation 11 is the energy equation for the situation. The velocity terms are absent because they are equal, the added pump head and removed turbine head are absent because these devices are absent. All that remains is the pressure, elevation, and head loss terms.

$$\frac{p_1}{\rho g} + z_1 = \frac{p_2}{\rho g} + z_2 + h_l \quad (10)$$

Rearranging equation to isolate the head loss will be of value when we try to find the pressure drop.

$$\left(\frac{p_1}{\rho g} - \frac{p_2}{\rho g}\right) + (z_1 - z_2) = h_l \quad (11)$$

The first term in parenthesis is the pressure drop (rise), and the second term is the elevation drop (rise).

The head loss is evaluated using the Darcy-Weisbach head loss model. First we compute Re_d

$$Re_d = \frac{V D}{\nu} = \frac{4 (1 cfs) (0.5 ft)}{\pi (0.5 ft)^2 (0.00003 sq.ft/sec)} \approx 84,822 \quad (12)$$

The Reynolds number is greater than 10,000 therefore we conclude the flow is turbulent.

The roughness height is $\epsilon=0.00085$ from the table on the Moody chart (in this document), so the roughness ratio is

$$\frac{\epsilon}{D} = \frac{0.00085}{0.5} \approx 0.0017 \tag{13}$$

| Material | e (ft) | e (mm) |
|----------------------------------|------------|---------|
| Riveted steel | 0.003–0.03 | 0.9–9.0 |
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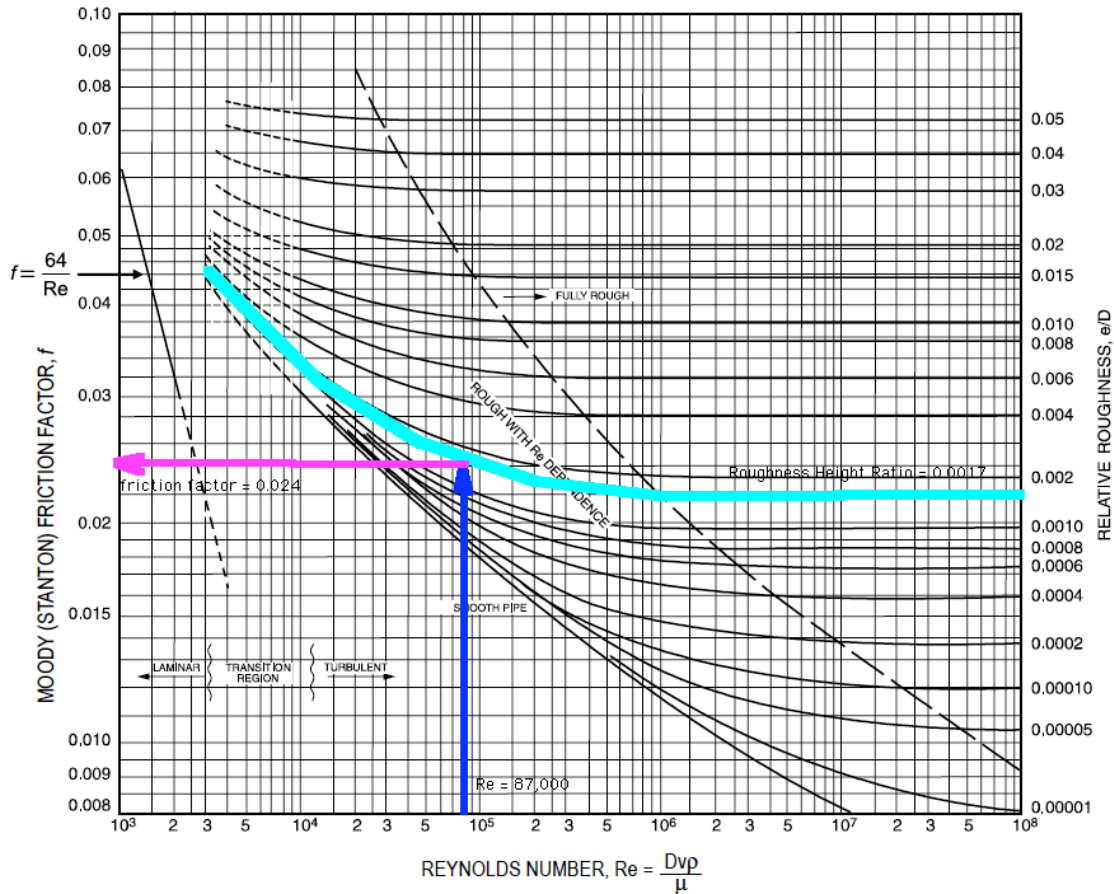


Figure 16: Moody-Stanton Diagram annotated with Example 1 components.

Figure 16 is the Moody chart with the roughness ratio shown as the light blue (cyan) curve,

the Reynolds number as the black line, and the recovered friction factor ($f=0.024$) from the magenta line.

To complete the analysis, we then use the Darcy-Weisbach equation for estimate the head loss as

$$h_l = 8fL \frac{Q^2}{\pi^2 g D^5} = 8(0.024)(2000ft) \frac{(1cfs)^2}{\pi^2 (32.2ft/s^2)(0.5ft)^5} \approx 38.6ft \quad (14)$$

Now to compute the pressure drop, we simply account for the elevation change and what remains must be pressure. First the change in elevation is about $2000 \sin 5^\circ \approx 175ft$. The change in pressure is therefore

$$\Delta p = \rho g (h_l + (z_2 - z_1)) = (38.6ft + 175ft)(62.4)(0.9) \approx 11,999lb/ft^2 = 83psi \quad (15)$$

Thus the oil pressure must be at least 83 psi greater at the lower elevation than the upper elevation for the oil to flow up the pipe. **Find Q given h_f , D , ϵ** This type of problem requires iteration if using the Moody chart, but it usually converges in two attempts. The alternative is to use one of the explicit equations in a later section. The engineer computes the Then computes the roughness ratio from the tabulated ϵ for the pipe material. Then makes a guess as to the flow rate, uses that guess to compute a Reynolds number, and look up a friction factor and compute the head loss. If the computed value is too large, then the flow rate is reduced; if too small then increased. The trial-and-error process is greatly helped by using a spreadsheet to record the various computations.

Example An 80-foot horizontal, 1/2-inch diameter wrought iron pipe has an observed head loss of 40 feet. Estimate the discharge in the pipe.

Solution Apply Darcy-Weisbach directly — the pipe is horizontal so the energy equation is quite boring,

$$h_l = \frac{\Delta p}{\rho g} = 40ft \quad (16)$$

First compute the roughness height ratio — it will be needed to look up friction factors.

$$\frac{\epsilon}{D} = \frac{0.00015ft}{0.5i/12ft} \approx 0.0036 \quad (17)$$

Then construct a table of computations as shown in Table 2. Increase (decrease) the flow rate until the computed head loss is about the same as the required head loss. The moody chart is used the same way as in the previous example. The engineer will need to exercise some judgement of when to stop, because as one gets close to the specified head loss, the ability to read changes in f diminishes.

Table 1: Computation table for Estimating Q from head loss and material properties.

| Q_{guess} | Re_d | f | h_l_{guess} |
|-------------|--------------------|-------|----------------|
| 0.001 | 2.83×10^3 | 0.036 | ≈ 0.57 |
| 0.005 | 1.41×10^4 | 0.032 | ≈ 12.7 |
| 0.008 | 2.26×10^4 | 0.031 | ≈ 31.6 |
| 0.009 | 2.25×10^4 | 0.030 | ≈ 37.9 |

The result in this example is that the pipe discharge is about 0.009 cfs. **Find D given Q , h_f , ϵ** The last kind of case is finding (design) the diameter required to move a specified volume of water given a specified working head and pipe material. In this kind of problem, the engineer must guess a diameter, then compute Re_d based on the flow rate and the diameter. The engineer must also compute $\frac{\epsilon}{D}$. Then the engineer enters the Moody chart with these two values (guesses), reads the friction factor and computed the head loss. If the head loss from the guess is too large, then the diameter is increased, too small then decreased. Upon convergence, the next larger commercially available diameter is selected for practical application.

Example An 600 foot wrought-iron pipe is to carry water at 20°C at a discharge of 3 CFS. The pipe drops 60 feet in the direction of flow and the desired pressure drop is 6 feet of head. What diameter pipe will function under these conditions?

Solution Figure 17 is a sketch of the situation.

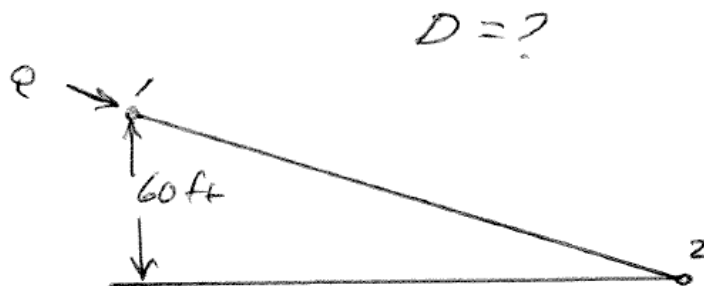


Figure 17: Sketch for Example

The energy equation for this situation is

$$\left(\frac{p_1}{\rho g} - \frac{p_2}{\rho g}\right) + (z_1 - z_2) = h_l \quad (18)$$

The elevation change is given as 60 feet and the pressure drop is given as 6 feet (that is the pressure is greater at location 1 by 6 feet than location 2). Thus the Darcy-Weisbach head

loss equation is

$$h_l = 8fL \frac{Q^2}{\pi^2 g D^5} \approx 66ft \quad (19)$$

As in the prior example, a computation table is useful. The sixth column is a computational trick to make a hand calculations faster. The term $\frac{h_l}{f}$ is evaluated by taking the Darcy-Weisbach equation and dividing out the friction factor (i.e. $\frac{h_l}{f} = 8L \frac{Q^2}{\pi^2 g D^5}$)

Table 2: Computation table for Estimating D from head loss, discharge and material properties.

| D_{guess} | Re_d | $\frac{\epsilon}{D}$ | f | $\frac{h_l}{f}$ | h_l |
|-------------|--------------------|----------------------|-------|--------------------|----------------|
| 0.25 | 1.41×10^6 | 0.0006 | 0.018 | 1.39×10^5 | ≈ 2500 |
| 0.50 | 7.06×10^5 | 0.0003 | 0.016 | 4.35×10^3 | ≈ 69.6 |
| 0.51 | 6.92×10^5 | 0.00029 | 0.016 | 3.94×10^3 | ≈ 63.0 |

The result after three tries is that the diameter is between 6-7 inches, commercially available 7 inch iron pipe exists⁸, so this size could be specified in such a situation.

5.2 Head Loss Models – Hazen-Williams

The Hazen-Williams head loss model is another common model in civil engineering practice. It is an empirical model **valid only for water**, and is not for use with other liquids. It appears in the Texas Administrative Code and has been used for water distribution systems design for at least a century.

Equation 20 is the formula that relates pipe velocity to head loss.

$$V = 1.381 C_h R^{0.63} S^{0.54} \quad (20)$$

where, V is the mean section velocity in feet per second, C_h is the Hazen-Williams loss coefficient (tabulated), R is the hydraulic radius in feet, and $S = h_f/L$ is the slope of the energy grade line (head loss per unit length)

If Equation 20 is rearranged, a formal head loss equation results, as in Equation 21 for circular pipes of diameter D in feet.

$$h_f = 3.02 L D^{-1.167} \left(\frac{V}{C_h} \right)^{1.85} \quad (21)$$

⁸7-inch pipe is uncommon, but is reported on NPS tables; so the pipe is commercially available somewhere.

More useful is the head loss in discharge form for circular pipes with discharge Q is cubic feet per second.

$$h_f = 3.02 L D^{-1.167} \left(\frac{4Q}{\pi D^2 C_h} \right)^{1.85} \quad (22)$$

The loss coefficients are tabulated in many places and are usually based on material type (cement, steel, etc.) and an estimate of age. Table 3 is one such tabulation⁹, and is a rather extensive list. The model is intended for pipes in sizes from 2-inches to 6-feet; outside these ranges the model may not be applicable.

Table 3: Hazen-Williams Coefficients for Different Materials.

| Material | C_h | Material | C_h |
|---|-----------|---|-----------|
| ABS - Acrylonite Butadiene Styrene | 130 | Aluminum | 130 - 150 |
| Asbestos Cement | 140 | Asphalt Lining | 130 - 140 |
| Brass | 130 - 140 | Brick sewer | 90 - 100 |
| Cast-Iron - new unlined (CIP) | 130 | Cast-Iron 10 years old | 107 - 113 |
| Cast-Iron 20 years old | 89 - 100 | Cast-Iron 30 years old | 75 - 90 |
| Cast-Iron 40 years old | 64-83 | Cast-Iron, asphalt coated | 100 |
| Cast-Iron, cement lined | 140 | Cast-Iron, bituminous lined | 140 |
| Cast-Iron, wrought plain | 100 | Cast-Iron, seal-coated | 120 |
| Cement lining | 130 - 140 | Concrete | 100 - 140 |
| Concrete lined, steel forms | 140 | Concrete lined, wooden forms | 120 |
| Concrete, old | 100 - 110 | Copper | 130 - 140 |
| Corrugated Metal | 60 | Ductile Iron Pipe (DIP) | 140 |
| Ductile Iron, cement lined | 120 | Fiber | 140 |
| Fiber Glass Pipe - FRP | 150 | Galvanized iron | 120 |
| Glass | 130 | Lead | 130 - 140 |
| Metal Pipes - Very to extremely smooth | 130 - 140 | Plastic | 130 - 150 |
| Polyethylene, PE, PEH | 140 | Polyvinyl chloride, PVC, CPVC | 150 |
| Smooth Pipes | 140 | Steel new unlined | 140 - 150 |
| Steel, corrugated | 60 | Steel, welded and seamless | 100 |
| Steel, interior riveted, no projecting rivets | 110 | Steel, projecting girth and horizontal rivets | 100 |
| Steel, vitrified, spiral-riveted | 90 - 110 | Steel, welded and seamless | 100 |
| Tin | 130 | Vitrified Clay | 110 |
| Wrought iron, plain | 100 | Wooden or Masonry Pipe - Smooth | 120 |
| Wood Stave | 110 - 120 | | |

Example Estimate the head loss in a 72-inch, 10,000-foot steel pipe carrying water at 200 CFS using the Hazen-Williams formula.

Solution Using Table 3 an estimate of the C_h is 100. Next substitute into the HW formula as

$$h_f = 3.02 (10,000 ft) (6 ft)^{-1.167} \left(\frac{4(200 cfs)}{\pi (6 ft)^2 100} \right)^{1.85} \approx 28 ft \quad (23)$$

⁹Adapted from http://www.engineeringtoolbox.com/hazen-williams-coefficients-d_798.html.

5.3 Head Loss Models – Manning’s

5.3.1 Non-circular conduits

Hydraulic radius concept.

5.4 Head Loss Models – Explicit Equations

An alternative to the Moody chart are explicit equations based on regression models of the data represented by the Moody chart. These equations are useful in programming solutions to common pipe-flow design problems.

5.4.1 Friction Factor Formulas

Equation 24 is the formula for friction factor for laminar flow ($Re_d < 2000$).

$$f = \frac{64}{Re_d} \quad (24)$$

Equation 25 is the formula for friction factor for transitional to fully rough flow for the ranges of Re_d and roughness ratio shown (CITE SOURCE).

$$f = \frac{0.25}{[\log_{10}(\frac{k_e}{3.7D} + \frac{5.74}{Re_d^{0.9}})]^2} \quad (25)$$

valid for : $4,000 < Re_d < 100,000,000$ and $0.00001 < \frac{\epsilon}{D} < 0.002$.

5.4.2 Discharge Formula

Combining Equation 25 and Equation 9, then rearranging to isolate the discharge Q produces an explicit formula for estimating discharge from head loss and material properties. Equation 26 is such an explicit formula (CITE SOURCE).

$$Q = -2.22D^{5/2} \times \sqrt{gh_f/L} \times [\log_{10}(\frac{\epsilon}{3.7D} + \frac{1.78\nu}{D^{3/2}\sqrt{gh_f/L}})] \quad (26)$$

5.4.3 Diameter Formula

A formula for diameter is also available. Equation 27 is a formula to estimate the required pipe diameter for a particular discharge, head loss, and roughness (CITE SOURCE).

$$D = 0.66[\epsilon^{1.25} \times \left(\frac{LQ^2}{gh_f}\right)^{4.75} + \nu Q^{9.4} \times \left(\frac{L}{gh_f}\right)^{5.2}]^{0.04} \quad (27)$$

5.5 Head Losses in Bends and Fittings — (aka Minor Losses)

In addition to head loss in the conduit, other losses are created by inlets, outlets, transitions, and other connections in the system. In fact such losses can be used to measure discharge (think of the orifice plate in the fluids laboratory). The fittings create additional turbulence that generates heat and produces the head loss.

Equation 28 is the typical loss model

$$h_{minor} = K \frac{V^2}{2g} \quad (28)$$

where K is called a minor loss coefficient, and is tabulated (e.g. Table 4) for various kinds of fittings.

Table 4: Minor Loss Coefficients for Different Fittings

| Fitting Type | K |
|---------------------------------|------|
| Tee, Flanged, Line Flow | 0.2 |
| Tee, Threaded, Line Flow | 0.9 |
| Tee, Flanged, Branched Flow | 1.0 |
| Tee, Threaded, Branch Flow | 2.0 |
| Union, Threaded | 0.08 |
| Elbow, Flanged Regular 90° | 0.3 |
| Elbow, Threaded Regular 90° | 1.5 |
| Elbow, Threaded Regular 45° | 0.4 |
| Elbow, Flanged Long Radius 90° | 0.2 |
| Elbow, Threaded Long Radius 90° | 0.7 |
| Elbow, Flanged Long Radius 45° | 0.2 |
| Return Bend, Flanged 180° | 0.2 |
| Return Bend, Threaded 180° | 1.5 |
| Globe Valve, Fully Open | 10 |
| Angle Valve, Fully Open | 2 |
| Gate Valve, Fully Open | 0.15 |
| Gate Valve, 1/4 Closed | 0.26 |
| Gate Valve, 1/2 Closed | 2.1 |
| Gate Valve, 3/4 Closed | 17 |
| Swing Check Valve, Forward Flow | 2 |
| Ball Valve, Fully Open | 0.05 |
| Ball Valve, 1/3 Closed | 5.5 |
| Ball Valve, 2/3 Closed | 200 |
| Diaphragm Valve, Open | 2.3 |
| Diaphragm Valve, Half Open | 4.3 |
| Diaphragm Valve, 1/4 Open | 21 |
| Water meter | 7 |

The use is straightforward, and multiple fittings are summed in the loss term in the energy equation.

Example What is the pressure drop across a valve with nominal diameter of 8 cm, a loss coefficient of 3.2, and a flow rate of $0.04 \text{ m}^3/\text{sec}$?

Solution First write the minor loss equation, solve for head loss.

$$h_l = K \frac{V^2}{2g} = (3.2) \frac{\left(\frac{4 \cdot 0.04}{\pi(0.08)^2}\right)^2}{2(9.8)} \approx 10.3m \quad (29)$$

Then convert the head loss into a pressure drop from

$$\rho g * h_l = \Delta p = 9800N/m^3 * 10.3m = 101,321Pa \approx 101kPa \quad (30)$$

5.6 Exercise Set

1. A 5-foot diameter, enamel coated, steel pipe carries 60°F water at a discharge of 295 cubic-feet per second (cfs). Using the Moody chart, estimate the head loss in a 10,000 foot length of this pipe.
2. Using the Moody chart, and the energy equation, estimate the diameter of a cast-iron pipe needed to carry 60°F water at a discharge of 10 cubic-feet per second (cfs) between two reservoirs 2 miles apart. The elevation difference between the water surfaces in the two reservoirs is 20 feet.
3. Points A and B are 3 miles apart along a 24-inch ductile iron pipe carrying 50°F water. Point A is 30 feet higher in elevation than point B. The pressure at point B is 20 pounds per square-inch (psi) greater than the pressure at point A. Using the Moody chart, and the energy equation, determine the direction and magnitude of flow (i.e. discharge and direction).
4. (Writing Intensive Problem) **Build** and **document**¹⁰ a spreadsheet tool that implements the explicit formulas. Use the tool to repeat problems 1-3 and use these in your documentation as tutorial examples for other users of your spreadsheet tool.

¹⁰Use the spreadsheet report format

6 Pipeline System Hydraulics – II

6.1 Conduit Systems

Conduit systems include pipes in series (which can already be analyzed methods presented herein) as well as branching pipes, parallel pipes, manifolds, and networks.

6.1.1 Branching Pipes

Figure 18 is a sketch of a branched system. The three reservoirs are connected to a common junction (D) called a node. There are four unknowns, V_{AD} , V_{BD} , V_{DC} , and p_D/γ . The

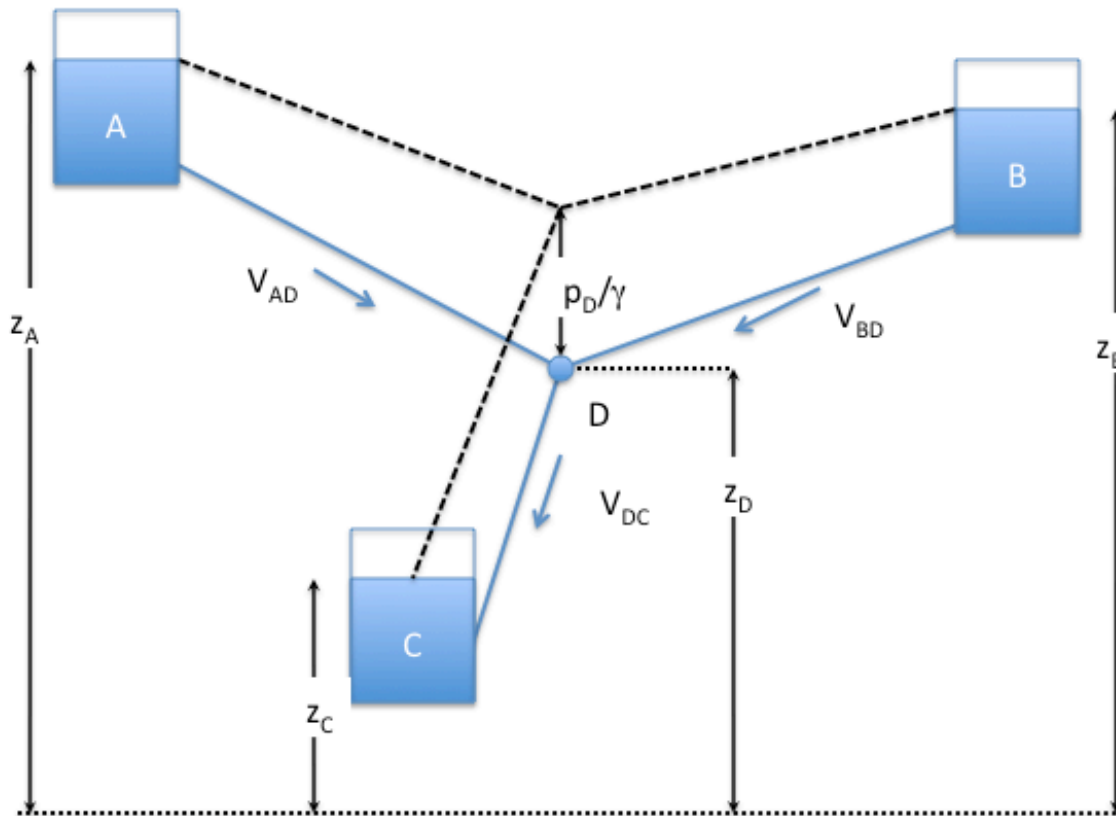


Figure 18: Branched pipe system

typical challenge is to find the **flow** (e.g. V_A) in each of the three pipes and the pressure at the junction. The fundamental relationships are energy loss in the branches, and continuity of flow at the node.

First energy along each pipe:

$$z_A = z_D + \frac{p_D}{\gamma} + f_{AD} \frac{L_{AD}}{D_{AD}} \frac{V_{AD}^2}{2g} \quad (31)$$

$$z_B = z_D + \frac{p_D}{\gamma} + f_{BD} \frac{L_{BD}}{D_{BD}} \frac{V_{BD}^2}{2g} \quad (32)$$

$$z_D + \frac{p_D}{\gamma} = z_C + f_{DC} \frac{L_{DC}}{D_{DC}} \frac{V_{DC}^2}{2g} \quad (33)$$

These three energy equations contain the four unknowns. For a unique solution to exist, we need a fourth independent equation — in this instance, conservation of mass at the node provides the fourth relation as

$$A_{AD}V_{AD} + A_{BD}V_{BD} = A_{DC}V_{DC} \quad (34)$$

Now we have 4 equations in 4 unknowns; 3 equations happen to be nonlinear in an unknown and if one is solving by hand this is not a problem. If we use a computer spreadsheet, we need to construct the quadratic term as $V|V|$, this "vee times absolute vee" maintains the sign required to keep track of flow direction.

Example

Consider the branched system shown in Figure 19. In this example the friction factor is assumed constant for simplicity, but in practice would vary during the solution computations.

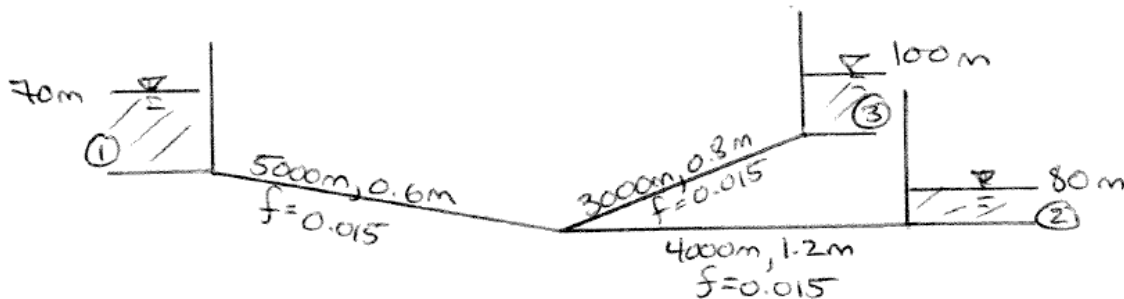


Figure 19: Branched pipe system

The hydraulics question is what is the discharge in each pipe and what is the total head at the junction (notice we don't know the junction elevation in this example — if the elevation were specified, we could also find the pressure head).

Solution First populate the four equations with the appropriate numerical values.

$$70 = h_D + (0.015) \frac{5000 V_{AD} |V_{AD}|}{0.6 \cdot 2(9.8)} \quad (35)$$

$$100 = h_D + (0.015) \frac{3000 V_{BD} |V_{BD}|}{0.8 \cdot 2(9.8)} \quad (36)$$

$$-80 = -h_D + (0.015) \frac{4000 V_{DC} |V_{DC}|}{1.2 \cdot 2(9.8)} \quad (37)$$

$$\frac{\pi(0.6)^2}{4} V_{AD} + \frac{\pi(0.8)^2}{4} V_{BD} = \frac{\pi(1.2)^2}{4} V_{DC} \quad (38)$$

Next, compute all the constants, and organize the 4 equations into a system of simultaneous equations

$$\begin{array}{rcccccl} h_D & 6.377V_{AD}|V_{AD}| & 0 & 0 & = & 70 \\ h_D & 0 & 2.869V_{BD}|V_{BD}| & 0 & = & 100 \\ h_D & 0 & 0 & -2.551V_{DC}|V_{DC}| & = & 80 \\ 0 & 0.2827V_{AD} & 0.5026V_{BD} & -1.1309V_{DC} & = & 0 \end{array} \quad (39)$$

Because the system is non-linear we need to employ some tricks. First, it is a quadratic system so if a solution exists, Excel solver should be able to find one.

First a more conventional vector-matrix structure

$$\begin{pmatrix} 1 & 6.377|V_{AD}| & 0 & 0 \\ 1 & 0 & 2.869|V_{BD}| & 0 \\ 1 & 0 & 0 & -2.551|V_{DC}| \\ 0 & 0.2827 & 0.5026 & -1.1309 \end{pmatrix} \cdot \begin{pmatrix} h_D \\ V_{AD} \\ V_{BD} \\ V_{DC} \end{pmatrix} - \begin{pmatrix} 70 \\ 100 \\ 80 \\ 0 \end{pmatrix} = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \end{pmatrix} \quad (40)$$

Now we can use solver to find a solution. We notice that the right-hand-side is a vector of “errors”. If we square these values and sum them ($\vec{\epsilon}^T \vec{\epsilon}$), then minimize this sum of squared errors by changing h_D, V_{AD}, V_{BD} , and V_{DC} , the result should be the solution (if the error is small enough).

Figure 20 is a screen capture of an Excel worksheet before starting the solution algorithm. Notice the column labeled **X** contains an initial guess for the unknowns.

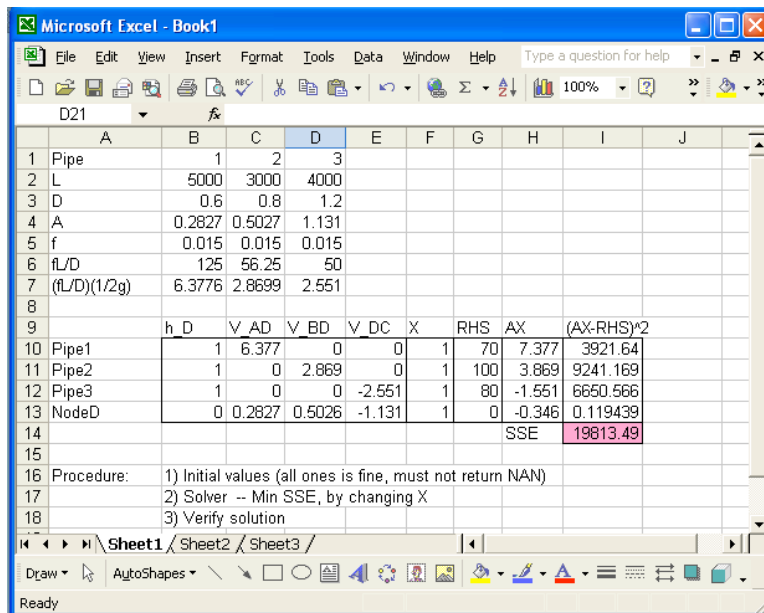


Figure 20: Branched pipe system — Excel solution set-up

After building the spreadsheet and preparing the columns, then the engineer must select solver to minimize the error vector (the last column in the worksheet). The SOLVER add-in does the job nicely, GOAL SEEK might work, but provides less control of precision and stopping criterion.

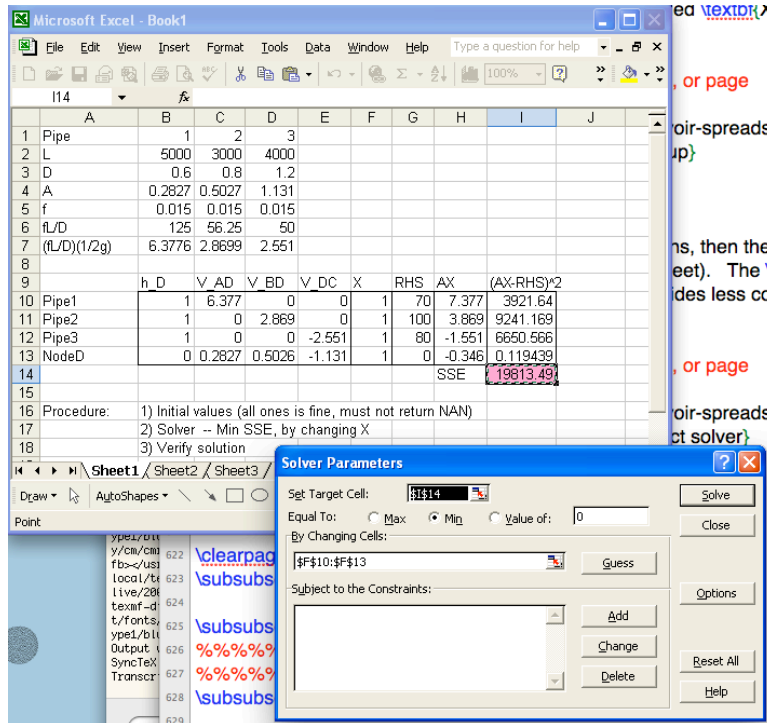


Figure 21: Branched pipe system — Excel solution select solver

Figure 21 is the solver set-up for this example. When the engineer selects SOLVE then the program will attempt to minimize the error by changing values of the **X** vector. The example does NOT use program defaults, but instead reduces the tolerance and precision values (in the Options section) and chooses the conjugate-gradient direction finding method and centered differences.

Finally, when the solver is completed, we recover the solution by multiplying the results by pipe areas to obtain the flows. Figure 22 is the solution for this example (the arithmetic was added at the bottom after the program converged to the near-zero "sum-squared-error").

| | A | B | C | D | E | F | G | H | I | J | K |
|----|--------------|---|-----------------|-----------------|-----------------|-------|-----|-------|------------|---|---|
| 4 | A | 0.2827 | 0.5027 | 1.131 | | | | | | | |
| 5 | f | 0.015 | 0.015 | 0.015 | | | | | | | |
| 6 | fL/D | 125 | 56.25 | 50 | | | | | | | |
| 7 | (fL/D)(1/2g) | 6.3776 | 2.8699 | 2.551 | | | | | | | |
| 8 | | | | | | | | | | | |
| 9 | | h _D | V _{AD} | V _{BD} | V _{DC} | X | RHS | AX | (AX-RHS)*2 | | |
| 10 | Pipe1 | 1 | 8.5962 | 0 | 0 | 81.6 | 70 | 70 | 3.47E-24 | | |
| 11 | Pipe2 | 1 | 0 | 7.2681 | 0 | -1.35 | 100 | 100 | 2.64E-22 | | |
| 12 | Pipe3 | 1 | 0 | 0 | -2.012 | 2.53 | 80 | 80 | 2.96E-20 | | |
| 13 | NodeD | 0 | 0.2827 | 0.5026 | -1.131 | 0.79 | 0 | 6E-10 | 4.2E-19 | | |
| 14 | | | | | | | | SSE | 4.5E-19 | | |
| 15 | | | | | | | | | | | |
| 16 | Procedure: | 1) Initial values (all ones is fine, must not return NAN) | | | | | | | | | |
| 17 | | 2) Solver -- Min SSE, by changing X | | | | | | | | | |
| 18 | | 3) Verify solution | | | | | | | | | |
| 19 | | | | | | | | | | | |
| 20 | Answers: | h _D = | 81.588 | | | | | | | | |
| 21 | | Q _{AD} = | -0.381 | | | | | | | | |
| 22 | | Q _{BD} = | 1.2734 | | | | | | | | |
| 23 | | Q _{DC} = | 0.8922 | | | | | | | | |
| 24 | | | | | | | | | | | |
| 25 | | | | | | | | | | | |
| 26 | | | | | | | | | | | |

Figure 22: Branched pipe system — Excel solution

The results are:

1. $Q_{AD} = -0.381\text{m}^3/\text{s}$, from Node D to Reservoir A (opposite the original assumed direction).
2. $Q_{BD} = 1.273\text{m}^3/\text{s}$, from Reservoir B to the node (the assumed flow direction in this pipe).
3. $Q_{DC} = 0.89\text{m}^3/\text{s}$ from the Node D to reservoir C (the assumed direction in this pipe).
4. $h_D = 81.6\text{m}$.

An alternate solution approach is to guess values of h_D and use the energy equations to find resulting flows in each pipe. Then adjust the guess until continuity is satisfied. Lastly, the reader might (correctly) conclude that we could have built the spreadsheet to directly solve by the Newton-Raphson method where the Jacobian is computed from the coefficient matrix (and some calculus) and an update formula is directly built into the spreadsheet. This approach (Newton-Raphson) is outlined in the section on networks and is the method employed in the EPA-NET computer program.

6.1.2 Parallel Pipes

Parallel pipes is an extension of the branching system where two pipes branch at one location and rejoin at another. In these systems there are two nodes and two (or more pipes) that join at each node. The approach is quite similar with the head loss in either branch being equal (or more conventionally, head loss around a closed loop is zero), and continuity is satisfied at each node. An example is used to illustrate the parallel system analysis.

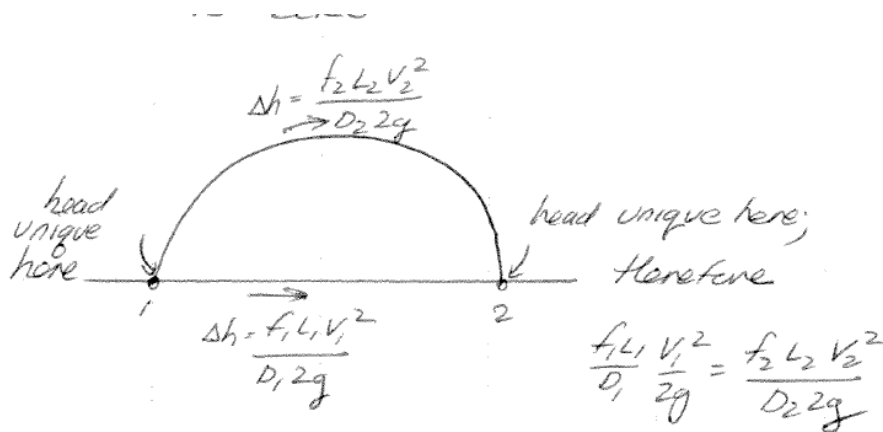


Figure 23: Parallel pipe system — notice that the head loss in either pipe must be equal for unique head at each node. Flows must sum to zero at each node.

Example

Figure 24 is a parallel pipe system with two pipes that together convey 20 cubic-feet-per-second of water.

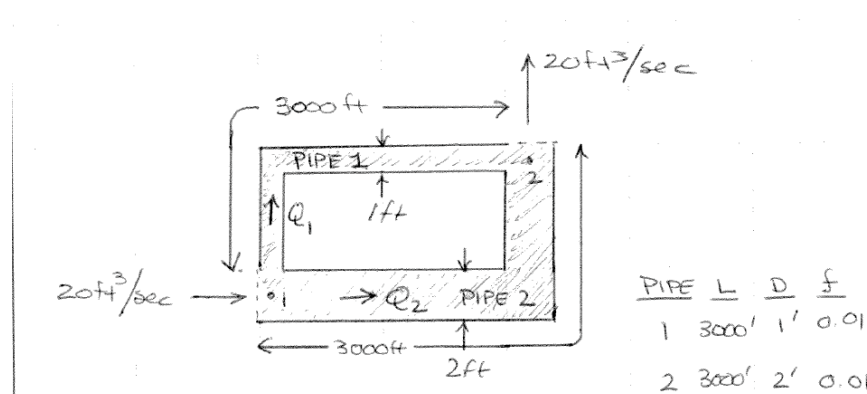


Figure 24: Parallel pipe system. Find Q_1 and Q_2 .

Pipe 1 is a 3000-foot long, 12-inch diameter pipe with constant friction factor of 0.01. Pipe 2

is a 3000-foot long, 24-inch diameter pipe, also with friction factor 0.01. Find the discharge in each pipe.

Solution

This example is really a simple network. First we assume flow directions (shown in the sketch). Next we will write the node equations.

$$\begin{aligned} P_1 : +20 - Q_1 - Q_2 - Q_f &= 0 \\ P_2 : -20 + Q_1 + Q_2 + 0 &= 0 \end{aligned} \quad (41)$$

The fictitious pipe (Q_f) will be explained shortly — at the solution its flow value will be forced to zero.

Next we write a head loss equation around the loop, the sign convention is that if the loop traverse direction (CCW or CW) is the same as the assumed flow direction, then the head loss is positive. If the loop traverse direction and the flow direction are opposite, then the head loss is negative. Figure 25 illustrates this convention (and uses CCW as the traverse direction).

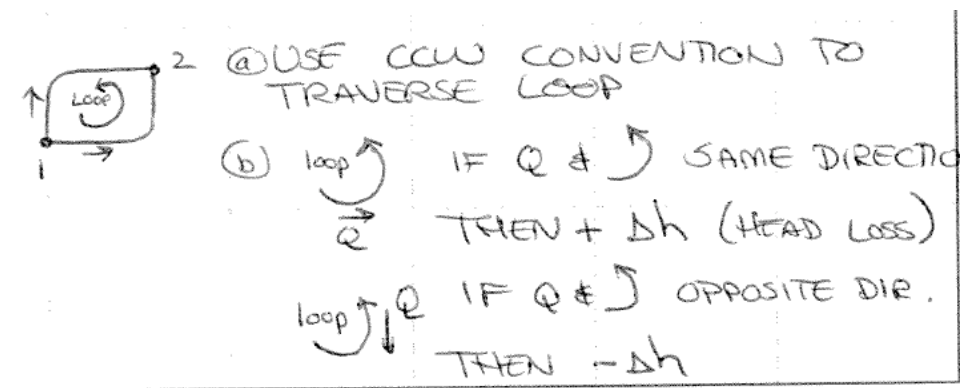


Figure 25: Sign convention in loop traverse

Using this convention we then write the loop equation(s).

$$L_1 : +0 - \frac{8fL_1}{\pi^2gD_1^5}Q_1|Q_1| + \frac{8fL_2}{\pi^2gD_2^5}Q_2|Q_2| + 0 = 0 \quad (42)$$

Next we combine the node and loop equations into a system of equations (as before) and build a solution algorithm.

$$\begin{pmatrix} -1 & -1 & 1 \\ 1 & 1 & 0 \\ -\frac{8fL_1}{\pi^2gD_1^5}|Q_1| & \frac{8fL_2}{\pi^2gD_2^5}|Q_2| & 0 \end{pmatrix} \cdot \begin{pmatrix} Q_1 \\ Q_2 \\ Q_f \end{pmatrix} = \begin{pmatrix} -20 \\ 20 \\ 0 \end{pmatrix} \quad (43)$$

We now have a matrix-vector form of three non-linear simultaneous equations in three unknowns (with the added knowledge that one of the unknowns must be darned close to zero at the answer!). We could solve the same way as before, but instead, will directly build a Newton-Raphson solution.

To do that, we need to recall some calculus. Recall Newton's method for non-linear root finding:

$$x^{k+1} = x^k - \frac{f(x^k)}{\left. \frac{df}{dx} \right|_{x^k}} \quad (44)$$

This update equation solves $f(x) = 0$ if the initial guess is adequate and the derivative is non-zero. Our problem is the same except we need a multi-dimensional extension of Newton's method.

Our function is $\mathbf{G}(\mathbf{Q}) = \mathbf{0}$ and the multi-dimensional "derivative" is the Jacobian of \mathbf{G} evaluated at the current (k) guess.

Constructing \mathbf{G} is straightforward, we just subtract the right hand side (the demand vector) from the model.

$$\mathbf{G}(\mathbf{Q}) = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 1 & 0 \\ -\frac{8fL_1}{\pi^2 g D_1^5} |Q_1| & \frac{8fL_2}{\pi^2 g D_2^5} |Q_2| & 0 \end{pmatrix} \bullet \begin{pmatrix} Q_1 \\ Q_2 \\ Q_f \end{pmatrix} - \begin{pmatrix} -20 \\ 20 \\ 0 \end{pmatrix} \quad (45)$$

The Jacobian is trickier. The Jacobian is the matrix of partial derivatives of \mathbf{G} , evaluated at the current guess.

$$\mathbf{J}(\mathbf{G})^k = \begin{pmatrix} \frac{\partial G_1}{\partial Q_1} & \frac{\partial G_1}{\partial Q_2} & \frac{\partial G_1}{\partial Q_f} \\ \frac{\partial G_2}{\partial Q_1} & \frac{\partial G_2}{\partial Q_2} & \frac{\partial G_2}{\partial Q_f} \\ \frac{\partial G_3}{\partial Q_1} & \frac{\partial G_3}{\partial Q_2} & \frac{\partial G_3}{\partial Q_f} \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 1 & 0 \\ -2\frac{8fL_1}{\pi^2 g D_1^5} |Q_1| & 2\frac{8fL_2}{\pi^2 g D_2^5} |Q_2| & 0 \end{pmatrix} \quad (46)$$

Now the update formula by analogy. The single variable derivative term is replaced by the vector matrix operation using \mathbf{G} and its Jacobian

$$-\frac{f(x^k)}{\left. \frac{df}{dx} \right|_{x^k}} \approx -\mathbf{J}[\mathbf{G}(\mathbf{Q}^k)]^{-1} \bullet \mathbf{G}(\mathbf{Q}^k) \quad (47)$$

Thus the update formula (updates each element of the \mathbf{Q} vector) becomes

$$\mathbf{Q}^{k+1} = \mathbf{Q}^k - \mathbf{J}[\mathbf{G}(\mathbf{Q}^k)]^{-1} \bullet \mathbf{G}(\mathbf{Q}^k) \quad (48)$$

In this case we stop when the function actually gets to zero or when the solution stops changing. The trick in setting up a spreadsheet is to be sure the update occurs after the

computations. The easiest way is to keep a current guess row in the data area and update that row at the end of each iteration. Spreadsheets calculate from Top to Bottom, Left to Right.

Figure 26 is a spreadsheet implementation of this example, before the updating begins.

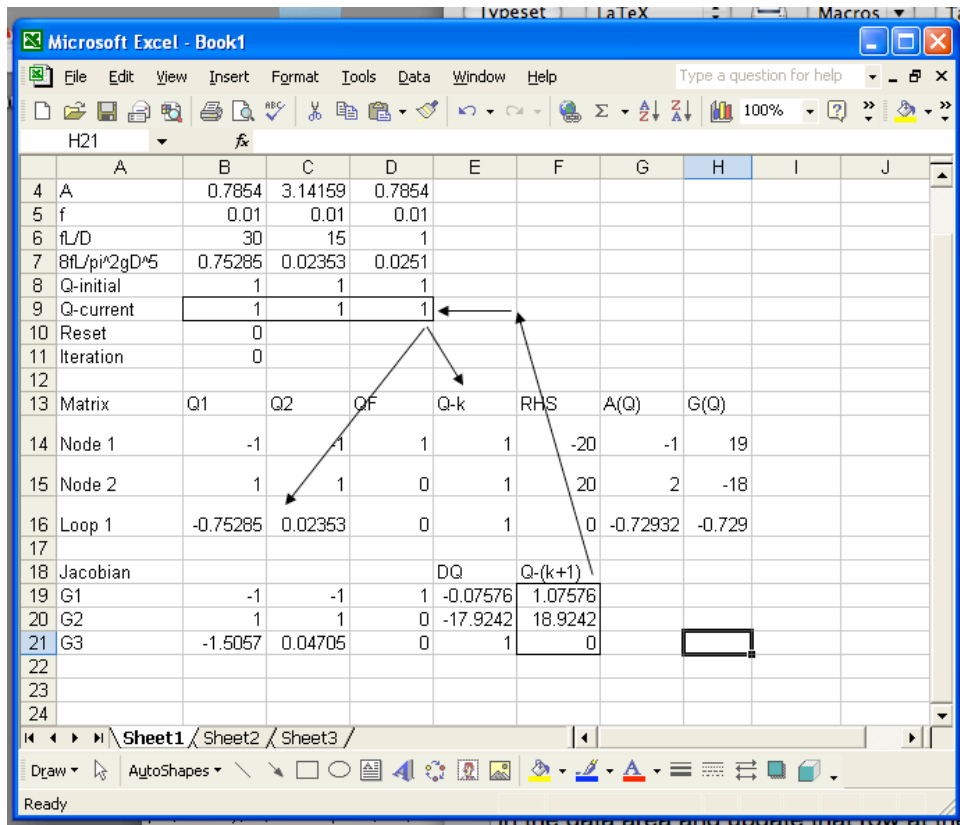


Figure 26: Simple network model using Newton-Raphson updates before iterations. Arrows indicate update path — notice the loopback is at bottom and right, so all rows are done before update. Also notice reset and iteration counter. These help control computations.

The function matrix is shown and the $G(Q)$ is computed in column H. The $A(Q)$ is simply a matrix multiplication from which the RHS is subtracted.

The Jacobian matrix is built from the function matrix. The column DQ is the ratio of the function and derivative at the current guess, and the lower right column $Q-(k+1)$ is the update following the multi-dimensional analog to Newton’s method.

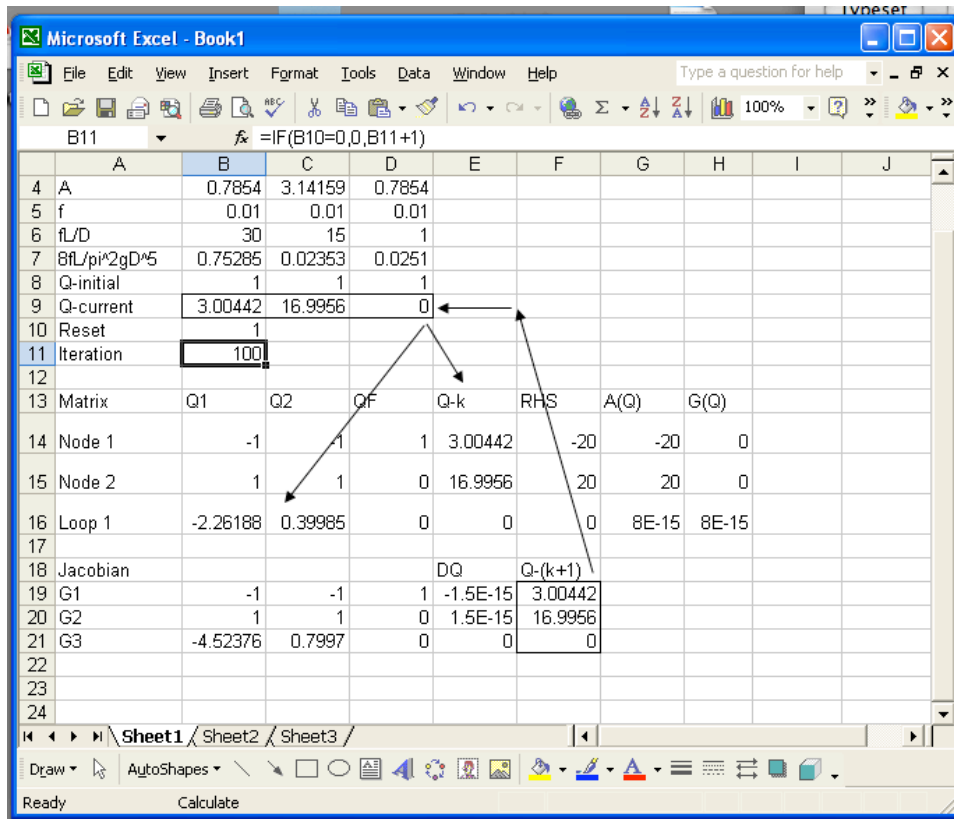


Figure 27: Simple network model using Newton-Raphson after 100 iterations. $G(Q)$ is for all practical purposes 0 , hence the problem is completed.

Figure 27 is the result of 100 iterations and represents the solution. In this example, the flow distribution is

1. $Q_1 = 3.00m^3/s$ in the assumed direction.
2. $Q_2 = 17.00m^3/s$ in the assumed direction.
3. $Q_f = 0$, which it should at the solution!
4. $\Delta h_{1 \rightarrow 2} = 6.79m$, regardless of path. The head loss around a loop must be zero, but just as important is the head loss between the two nodes.

6.1.3 Manifolds

FUTURE VERSION

6.1.4 Pipe Networks

Pipe networks, like single path pipelines, are analyzed for head losses in order to size pumps, determine demand management strategies, and ensure minimum pressures in the system. Conceptually the same principles are used for steady flow systems: conservation of mass and energy; with momentum used to determine head losses. The following brief description of the arithmetic behind such analyses and the algorithm common to most, if not all network analysis programs is accomplished using a simple example.

Keep in mind that the network is simply an extension of the branched pipe example.

Illustrative Example

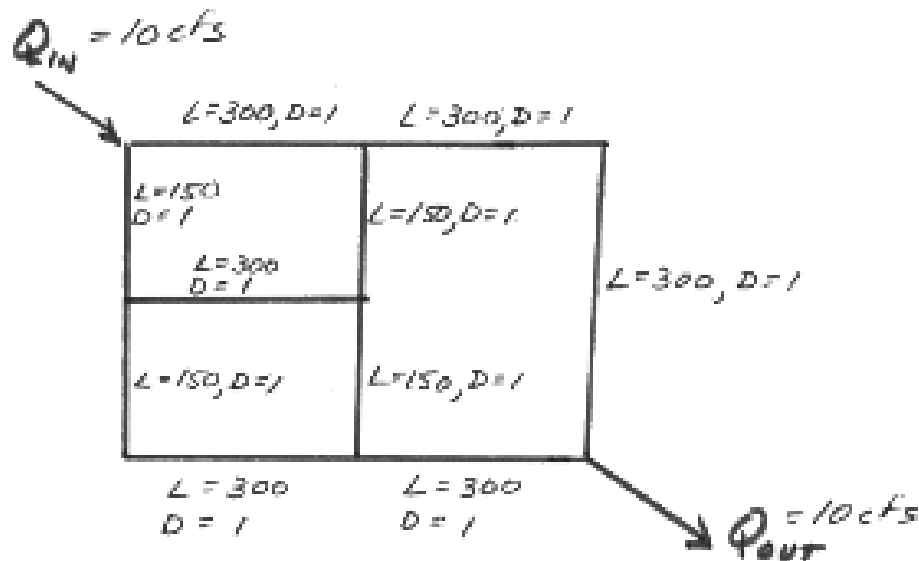


Figure 28: Pipe network for illustrative example with supply and demands identified. Pipe dimensions and diameters are also depicted.

Figure 28 is a sketch of the example problem that will be used. The network supply and demand node¹¹ are annotated with the discharge, in this case $10 \text{ ft}^3/\text{s}$. **Sketch the network** The first step in analysis is to sketch the network

Check geometry

A unique solution exists only if the sum of the node count and loop count is equal to the

¹¹A single demand is unusual, but helps with clarity in the example.

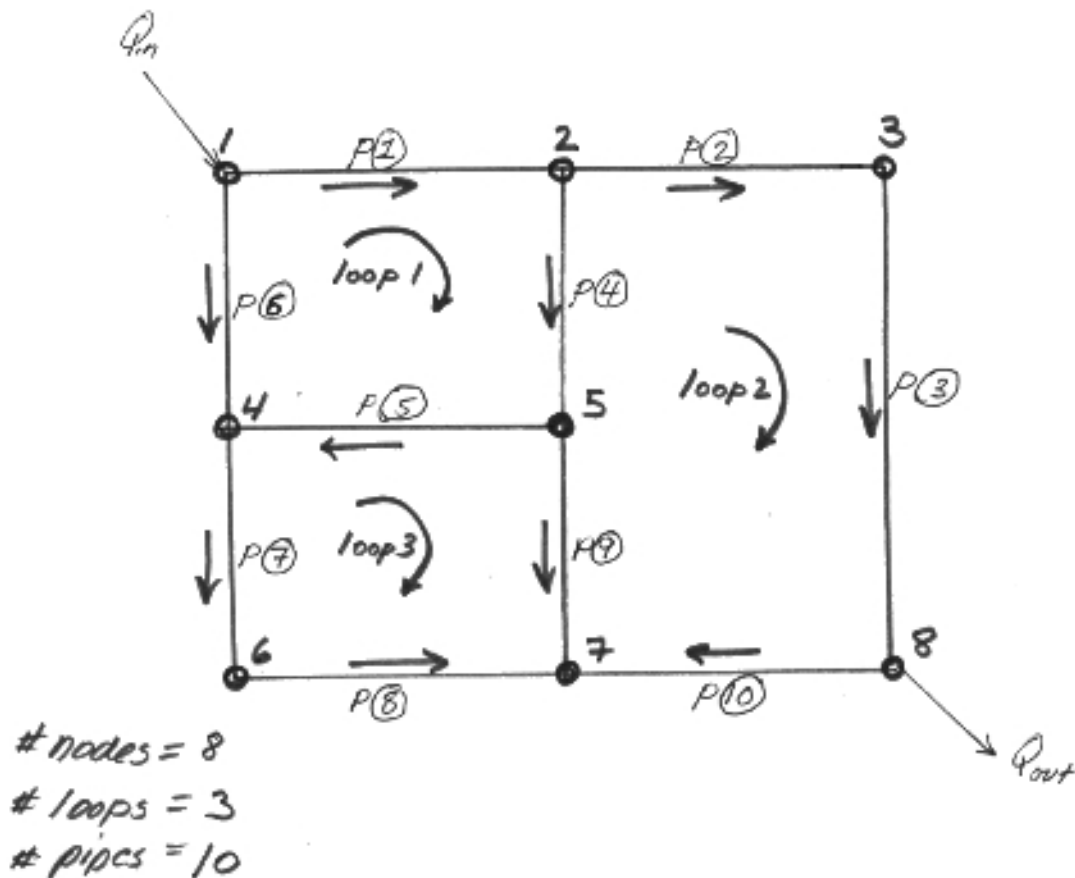


Figure 29: Pipe network for illustrative example with loops, pipes, and nodes labeled.

pipe count. In the current example this sum is 11, but the drawing shows only 10 pipes. A modeling trick is to add a fictitious pipe at either a supply or demand point to satisfy this geometric requirement. The particulars of this imaginary pipe are irrelevant as flow in this pipe should vanish at the solution.

Prepare f , K , Re Tables

The next step is to prepare tables for use in the head loss equations. In these notes, the Darcy-Weisbach formula is used for head loss, thus the relevant equations for any particular pipe are:

1. The head loss coefficient (just the constant part) to be multiplied by $|Q|Q$ to obtain loss for the pipe;

$$K = \frac{4\rho}{\pi^2 g D^5} \quad (49)$$

2. The Reynolds number coefficient, to be multiplied by Q to obtain the pipe Reynolds number for determination of friction factors.

$$\frac{Re}{Q} = \frac{8L}{\mu\pi D} \quad (50)$$

3. An the friction factor table (if variable factors are to be used); typically the Colebrook-White formula is used, but table look-up is also valid and fast. In this example fixed values will be used, so the Reynolds number component is superfluous.

The head loss in any pipe is $H_{loss} = fK|Q|Q$

Write the mass balance for each node and head loss for each loop

This step builds the equation system, the matrix below has two partitions, the upper partition corresponds to the nodal equations, and the lower partition to the loop equations. Notice that the lower partition will change value for any change in the discharges.

| | | | | | | | | | | |
|-----------|-----------|-----------|------------|------------|------------|------------|------------|--------------|--------------|----|
| -1 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 |
| 1 | -1 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 1 | 1 | -1 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | -1 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | -1 |
| $fK Q_1 $ | 0 | 0 | $fK Q_4 $ | $fK Q_5 $ | $-fK Q_6 $ | 0 | 0 | 0 | 0 | 0 |
| 0 | $fK Q_2 $ | $fK Q_3 $ | $-fK Q_4 $ | 0 | 0 | 0 | 0 | $-fK Q_9 $ | $fK Q_{10} $ | 0 |
| 0 | 0 | 0 | 0 | $-fK Q_5 $ | 0 | $-fK Q_7 $ | $-fK Q_8 $ | $fK Q_{10} $ | 0 | 0 |

The convention used here is that flow into a node is algebraically positive, while flow away from a node is negative. The sign convention in the loop partition is that if the loop traverse is the same direction as the assumed flow, then the sign convention is a positive loss, while a negative loss is computed for the opposite situation. The array above is a coefficient matrix dependent on discharge. This array is represented here as $\mathbf{A}(\mathbf{Q})$.

The discharges are written as the vector \mathbf{Q}

$$\begin{matrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 = \mathbf{Q} \\ Q_7 \\ Q_8 \\ Q_9 \\ Q_{10} \\ Q_{11} \end{matrix}$$

The demand is written as the vector \mathbf{D}

$$\begin{matrix} -10 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 = \mathbf{D} \\ 0 \\ 10 \\ 0 \\ 0 \\ 0 \end{matrix}$$

So the resulting system of equations¹² are

$$[\mathbf{A}(\mathbf{Q})] \cdot \mathbf{Q} = \mathbf{D} \quad (51)$$

Finding a Solution to the Network Equations

The network equations include the node equations¹³ and the loop equations¹⁴. The set of equations are solved simultaneously for pipe discharge, and then these results are used to determine system pressures or other nodal quantities. The system for a pipeline network happens to be a quadratic system of equations, therefore non-linear, and therefore some adaptation of linear solvers is used.

At the “correct” solution the following matrix-vector system is true.

$$[\mathbf{A}(\mathbf{Q})] \cdot \mathbf{Q} = \mathbf{D} \quad (52)$$

¹²Non-linear because of the $|Q|Q$ term.

¹³These equations are statements of conservation of mass.

¹⁴These equations are statements of conservation of energy and independence of path.

In this expression, \mathbf{A} is a function of \mathbf{Q} ; therefore the coefficient matrix is a function of the solution vector — a non-linear system. If the system were a univariate non-linear equation, we would proceed by subtracting the right-hand side from the left hand side and re-expressing the entire relationship as a function of the solution variable that is supposed to equal 0.

Newton-Raphson Method Theory

The Newton-Raphson method extends the classical Newton's method to vector valued functions of vector arguments. The first derivative of Newton's method is replaced by the Jacobian of the function, but otherwise the method is for all purposes identical to the univariate Newton's method.

Starting with the network equations, they are rewritten into a functional form suitable for a Newton's-type approach.

$$[\mathbf{A}(\mathbf{Q})] \cdot \mathbf{Q} - \mathbf{D} = \mathbf{f}(\mathbf{Q}) = \mathbf{0} \quad (53)$$

Recall from Newton's method that

$$x_{k+1} = x_k - \left(\frac{df}{dx} \Big|_{x_k} \right)^{-1} f(x_k) \quad (54)$$

thus the extension to the pipeline case is

$$\mathbf{Q}_{k+1} = \mathbf{Q}_k - [\mathbf{J}(\mathbf{Q}_k)]^{-1} \mathbf{f}(\mathbf{Q}_k) \quad (55)$$

where $\mathbf{J}(\mathbf{Q}_k)$ is the Jacobian of the coefficient matrix \mathbf{A} evaluated at \mathbf{Q}_k . Although a bit cluttered, here is the formula for a single update step, with the matrix, demand vector, and the solution vector in their proper places.

$$\mathbf{Q}_{k+1} = \mathbf{Q}_k - [\mathbf{J}(\mathbf{Q}_k)]^{-1} \{ [\mathbf{A}(\mathbf{Q}_k)] \cdot \mathbf{Q}_k - \mathbf{D} \} \quad (56)$$

The Jacobian of the pipeline model is a matrix with the following properties:

1. The partition of the matrix that corresponds to the node formulas is identical to the original coefficient matrix — it will be comprised of 0 or ± 1 in the same pattern at the equivalent partition of the \mathbf{A} matrix.
2. The partition of the matrix that corresponds to the loop formulas, will consist of values that are twice the values of the coefficients in the original coefficient matrix (at any supplied value of \mathbf{Q}_k).

In the current example the Jacobian would look like the following array (columns and rows are abbreviated to fit the page) :

$$\begin{array}{cccccc}
-1 & 0 & 0 & \dots & 0 & 0 \\
1 & -1 & 0 & \dots & 0 & 0 \\
\dots & & & & & \\
\dots & & & & & \\
0 & 0 & 1 & \dots & -1 & -1 \\
2fK|Q_1| & 0 & 0 & \dots & 0 & 0 \\
0 & 2fK|Q_2| & 2fK|Q_3| & \dots & 2fK|Q_{10}| & 0 \\
0 & 0 & 0 & \dots & 0 & 0
\end{array}$$

In this document, the pipeline solution is a true “Newton’s” method because analytical Jacobian values are used. If a numerical method to approximate the derivatives is used it would be called a quasi-Newton method.

As an algorithm, the engineer would supply a guess for \mathbf{Q}_k , compute the update value, use this just computed value as the new guess, and repeat the computation until the computed vector is relatively unchanging. Typically, even with a poor first guess the solution can be found in $\approx 2 \times \text{rank}(\mathbf{A})$

This method is the basis of nearly all network models (it is even used in groundwater hydraulics and surface water networks). The method with some effort can be extended to transient systems¹⁵.

Spreadsheet Example

Figure 30 is an image of a spreadsheet model to implement such calculations. The purpose here is to illustrate a bit of the layout, the spreadsheet itself required manual recalculation, and iteration counter, and reasonably intricate updating and linear algebra operations. All the pieces needed are part of a spreadsheet system and an actual spreadsheet will be illustrated in class.

¹⁵In a transient solver, there would be a set of iterations per time-step, hence the method is used over-and-over to evolve forward in time. In a transient case, analytical derivatives would be extremely desirable, but if geometry changes as in open channel cases, the programs usually sacrifice speed and use numerical approximation of the Jacobian at each time step.

