# 3. Qualitative modelling

- If values are best described by *qualitative adjectives*, membership functions indicating the particular quantitative descriptions of parameter or decision variable values can be used to quantify these qualitative descriptions.
- A particular mix of economic and environmental impacts may be more acceptable to some and less acceptable to others
- Qualitative adjectives examples: Large, small, pure, polluted, satisfactory, unsatisfactory, sufficient, insufficient, excellent, good, fair, poor, dry, hot, expensive, clean, high, low, etc.
- Qualitative, or so-called "fuzzy" statements convey information despite the imprecision of the italicized adjectives.

The quality of water is good for drinking.

 Membership functions of these uncertain or qualitative variables can be included in quantitative models.

### Membership functions (MF)

- Assume a set *A* = [18, 25].
- Any number *x* is either in or not in the set A.
- The statement "*x belongs to A*" is either *true or false*. Such *set A* is called a **crisp set**.
- If one is not able to say for certain whether or not any number x is in the set, then the set A could be referred to as fuzzy.
- The degree of truth attached to that statement is defined by a membership function.
- Membership functions range from 0 (completely false) to 1 (completely true)

#### MFs

- Consider the statement, "The water temperature should be suitable for swimming."
- Just what temperatures are suitable will depend on the persons asked.
- Thus shall be defined on the basis of the responses of many potential swimmers.
- It would be difficult for anyone to define precisely those temperatures that are suitable, if it is understood

that temperatures outside that range are **absolutely not suitable**.

 A function defining the interval or range of water temperatures suitable for swimming can be as shown



#### MFs

- The form or shape of a function depends on the individual subjective feelings of the "members" or individuals who are asked their opinions.
- To define this particular function, each individual *i* could be asked to define his or her comfortable water temperature interval  $(T_{min}, T_{max})$ .
- The degree of belonging value associated with any temperature value T equals the number of individuals who place that T within their range  $(T_{min}, T_{max})$ , divided by the total number of individual opinions obtained.
- It is the fraction of the total number of individuals that consider the water temperature *T* suitable for swimming. For this reason such functions are often called membership functions

### MF Implication for optimization

- Now suppose the water temperature applied to a swimming pool where the temperature could be regulated. The hotter the temperature the more it will cost.
- If we could quantify the economic benefits associated with various temperatures we could perform a benefit–cost analysis by maximizing the net benefits.
- Alternatively, we could maximize the fraction of people who consider the temperature good for swimming subject to a cost constraint using a membership function shown above in place of an economic benefit function.

### MF Implication for optimization

- Continuing with this example, assume you are asked to provide the desired temperature at a reasonable cost.
- Just what is reasonable can also be defined by another membership function, but this time the function applies to cost, not temperature.
- In this case one could consider there are in fact two objectives,

Suitable temperature and

Acceptable/Reasonable cost.

 A model that maximizes the minimum value of both membership functions is one approach for finding an acceptable policy for controlling the water temperature at this swimming pool.

- Consider the application of qualitative modeling to the three irrigation farm water allocation problem seen in chapter 2.
- The objective is to find the values of each allocation that maximizes the net benefits, NB(x).
- Maximize Net benefit Subject to constraints:

NB= 
$$p_1(12 - p_1) + p_2(20 - 1.5p_2) + p_3(28 - 2.5p_3) - 3(p_1)^{1.3} - 5(p_2)^{1.2} - 6(p_3)^{1.15}$$

•Where  $p_1 \le 0.4(x_1)^{0.9}$ ,  $p_2 \le 0.5(x_2)^{0.8}$  and  $p_3 \le 0.6(x_3)^{0.7}$ 

•Water-allocation restriction:  $x_1 + x_2 + x_3 = about 8$ 

• Remember that the optimal solution using GA was around  $x_1 = 0.7217$ ,  $x_2 = 2.9808$ ,  $x_3 = 4.2985$  and NB = 47.3721

- To create a qualitative equivalent for the model, the objective can be expressed as a membership function of the set of all possible objective values. The higher the objective value the greater the membership function value.
- Since membership functions range from 0 to 1, the objective needs to be scaled so that it also ranges from 0 to 1.
- The highest value of the objective occurs when there is sufficient water to maximize each firm's benefits; i.e. at the value where the derivative of the net benefit function of each firm equal zero.

• $x_1 = 10.18$  (NB<sub>1</sub> = 14.552),  $x_2 = 13.56$  (NB<sub>2</sub> = 29.611), and  $x_3 = 14.54$  (NB<sub>3</sub> = 42.476).

 This unconstrained solution would result in a total benefit of 86.639.

- The objective membership function,  $m_G$ , can be expressed by •  $m_G = \begin{bmatrix} p_1(12 - p_1) + p_2(20 - 1.5p_2) + p_3(28 - 2.5p_3) \\ -3(p_1)^{1.3} - 5(p_2)^{1.2} - 6(p_3)^{1.15} \end{bmatrix} / 86.639$
- The goal of maximizing objective function is changed to that of maximizing the degree of reaching the objective target.
- The optimization problem becomes:

•*Maximize*  $m_G(x)$ 

•Subject to:  $x_1 + x_2 + x_3 \le about 8$ 

- The total amount of resources available to be allocated is limited to "*about 8*" which is a qualitative constraint.
- The membership function describing the above constraint can be defined by

 $m_{\mathcal{C}}(x) = 1, if x_1 + x_2 + x_3 \le 7$   $m_{\mathcal{C}}(x) = [9 - (x_1 + x_2 + x_3)]/2,$   $if 7 \le (x_1 + x_2 + x_3) \le 9$  $m_{\mathcal{C}}(x) = 0, if x_1 + x_2 + x_3 \ge 9$ 



- The qualitative optimization problem becomes: maximize the minimum  $(m_G(x), m_C(x))$
- subject to

$$m_{G} = \begin{bmatrix} p_{1}(12 - p_{1}) + p_{2}(20 - 1.5p_{2}) + p_{3}(28 - 2.5p_{3}) \\ -3(p_{1})^{1.3} - 5(p_{2})^{1.2} - 6(p_{3})^{1.15} \end{bmatrix} / 86.639$$
  
$$m_{C}(x) = \begin{bmatrix} 9 - (x_{1} + x_{2} + x_{3}) \end{bmatrix} / 2$$

This yields (using excel solver)

 $x_1 = 0.84044$ ,  $x_2 = 2.50404$ ,  $x_3 = 4.57079$  and Q = 7.91526

 $-m_G(x) = m_C(x) = 0.542$ , and

- The total net benefit, is TB(X) = 46.990.
- Compare this with the GA solution of x<sub>1</sub> = 0.7217, x<sub>2</sub> = 2.9808, x<sub>3</sub>= 4.2985 and NB = 47.3721.

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#### Example: Qualitative Water Quality

- Consider the stream pollution problem where the stream receives waste,  $W_i$  from sources located at sites i = 1 and 2.
- Without some waste treatment at these sites, the pollutant concentrations at *sites 2* and *3* will exceed the maximum desired concentration. The problem is to find the fraction of wastewater removal,  $x_i$ , at *sites i* = 1 and 2 required to meet the quality standards at *sites 2* and *3* at a minimum total cost.
- The data used for the problem shown in next slide
- The crisp model for this problem is: Minimize  $C_1x_1 + C_2x_2$ , Subject to
- Water quality constraint at site 2:

 $[P_1Q_1 + W_1(1 - x_1)]\sigma_{12}/Q_2 \le P_2^{max}$ 

■ $[32 \times 20 + 250,000(1 - x_1)/86.4] 0.25/12 \le 20 \rightarrow x_1 \ge 0.78$ 

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	meter	Unit	Value	Remark
	$Q_1$	m <sup>3</sup> /s	10	Flow just u/s of site 1
	$Q_2$	m <sup>3</sup> /s	12	Flow just u/s of site 2
Flow	$\overline{Q_3}$	m <sup>3</sup> /s	13	Flow at park
	$\mathbf{W}_{1}$	Kg/day	250,000	Pollutant mass produced at site 1
Waste	$W_2$	Kg/day	80,000	Pollutant mass produced at site 2
	$P_1$	mg/l	32	Concentration Just upstream of site 1
Pollutant	-			Maximum Allowable concentration upstream
Concentr	$P_2$	mg/l	20	of site 2
ation	$P_3$	mg/l	20	Maximum Allowable concentration at site 3
	$\sigma_{12}$	_	0.25	Fraction of site 1 pollutant mass at site 2
Decay	$\sigma_{13}$	_	0.15	Fraction of site 1 pollutant mass at site 3
Fraction	$\sigma_{23}$	_	0.6	Fraction of site 2 pollutant mass at site 3

• Water quality constraint at site 3:

 $\left[ \left[ P_1 Q_1 + W_1 (1 - x_1) \right] \sigma_{13} + W_2 (1 - x_2) \sigma_{23} \right] / Q_3 \le P_3^{max}$ 

$$\begin{bmatrix} [32 \times 20 + 250,000 \times (1 - x_1)/86.4] \times 0.15 \\ + 80,000(1 - x_2)/86.4 \times 0.6 \end{bmatrix} / 13 \le 20$$
  
$$\Rightarrow x_1 + 1.28x_2 \ge 1.79$$

• Restrictions on fractions of waste removal:

• $0 \le x_i \le 1$ , for sites i = 1 and 2

• Excel solver for this problem using linear programming will result in:

 $\mathbf{x}_1 = 0.779 \text{ and } \mathbf{x}_2 = 0.791$ 

Essentially 80% removal efficiency is expected.

• Compare this solution with that of the next qualitative model.

- The maximum allowable pollutant concentrations in the stream at *sites 2* and *3* were expressed as "*about 20 mg/l*"
- Obtaining opinions of individuals of what they consider to be "*about 20 mg/l*" a membership function  $(m_P)$  can be defined as shown below:



• Regardless of whether or not this is required to meet stream quality standards, the government environmental agency expects each polluter to install:

 Best available technology (BAT) or To carry out best management practices (BMP)

- Asking experts just what BAT or BMP means with respect to treatment efficiencies could result in a variety of answers.
- These responses were used to define membership function (m<sub>T</sub>) for each of the two firms in this example.

Membership function for best available treatment technology



$$m_T(x) = 0, if x_i \le 0.7$$
  

$$m_T(x) = 20x_i - 14,$$
  

$$if 0.7 \le x_i \le 0.75$$
  

$$m_T(x) = 1, if 0.75 \le x_i \le 0.8$$
  

$$m_T(x) = 9 - 10x_i,$$
  

$$if 0.8 \le x_i \le 0.9$$
  

$$m_T(x) = 0, if x_i \ge 0.9$$

- There is a third concern that has to do with equity. It is expected that no polluter should be required to treat at a much higher efficiency than any other polluter.
- A membership function  $(m_E)$  defining just what differences are acceptable or equitable could quantify this concern.



- Considering each of these membership functions as objectives, a number of fuzzy optimization models can be defined.
- One is to find the treatment efficiencies that maximize the minimum value of each of these membership functions.

•*Maximize*  $m = max \{min (m_P, m_T, m_E)\}$ 

- Considering each of these membership functions as objectives, a number of fuzzy optimization models can be defined.
- One is to find the treatment efficiencies that maximize the minimum value of each of these membership functions.

•*Maximize*  $m = max \{min (m_P, m_T, m_E)\}$ 

Solving this model yields the results shown below.

Variable	Value	Remark	Variable	Value	Remark
m	0.93		m <sub>p2</sub>	0.94	
<b>x</b> <sub>1</sub>	0.81		m <sub>p3</sub>	0.93	
<b>x</b> <sub>2</sub>	0.81		m <sub>T1</sub>	0.93	
P <sub>2</sub>	18.28		m <sub>T2</sub>	0.93	
P <sub>3</sub>	18.36		m <sub>E</sub>	1	

• This solution confirms the assumptions made when constructing the representations of the membership functions in the model.

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#### Exercise 3

- Consider the application of qualitative modeling to the three irrigation farm water allocation problem seen in GA section.
- The problem is to find the allocations of water to each farm that maximize the total benefits TB(X):

•*Maximize*  $TB(x) = (6x_1 - x_1^2) + (7x_2 - 1.5x_2^2) + (8x_3 - 0.5x_3^2)$ 

- These allocations cannot exceed the amount of water available, Q, less any that must remain in the river, R.
- Assuming the available flow for allocations is about 6 units
- The maximize equation is subject to the resource constraint:

 $\mathbf{x}_1 + x_2 + x_3 \le about \ 6 \ unit$