Problem Formulation and Solution Strategies

Review of field Equations

- Basic field equations for linear isotropic elasticity.

$\mapsto$ Strain displacement relations

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

$\mapsto$ Compatibility relations

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0$$

$\mapsto$ Equilibrium Equations

$$\sigma_{ij,j} + F_i = 0$$

$\mapsto$ Elastic constitutive law(Hooke's law)

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}$$

$$e_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

15 Equations

15 unknowns
3 displacements
6 strain components
6 stress components

This general system of equations is of such complexity that solutions by the analytical methods are essentially Impossible and further simplification is required to solve problems of interest.

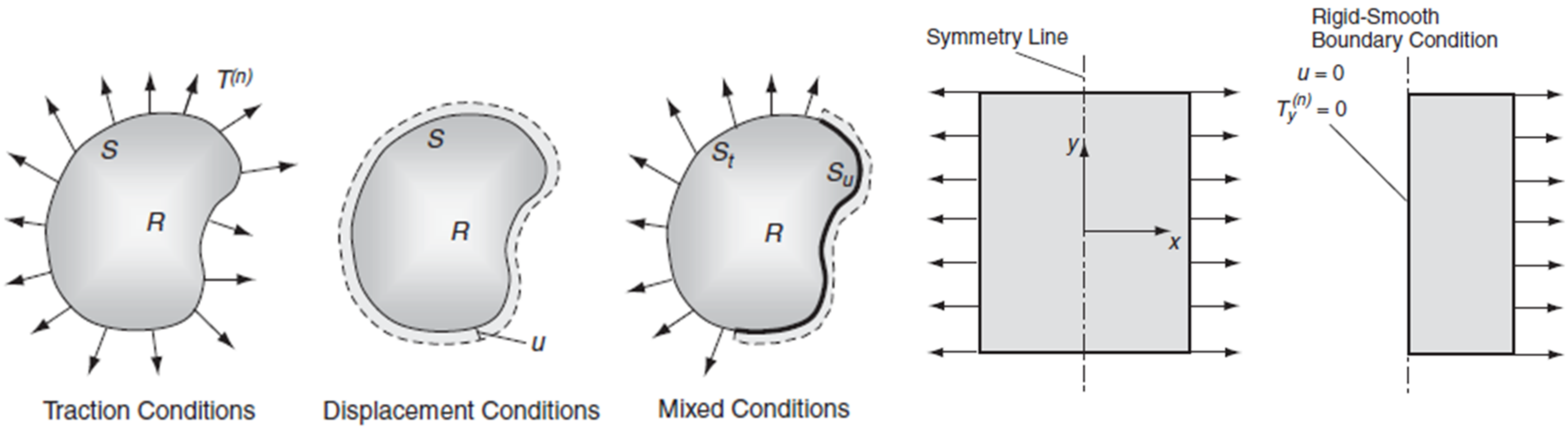
Boundary Conditions

- Similar to other field equations in engineering science the solution of the above system requires boundary conditions on the body under study.

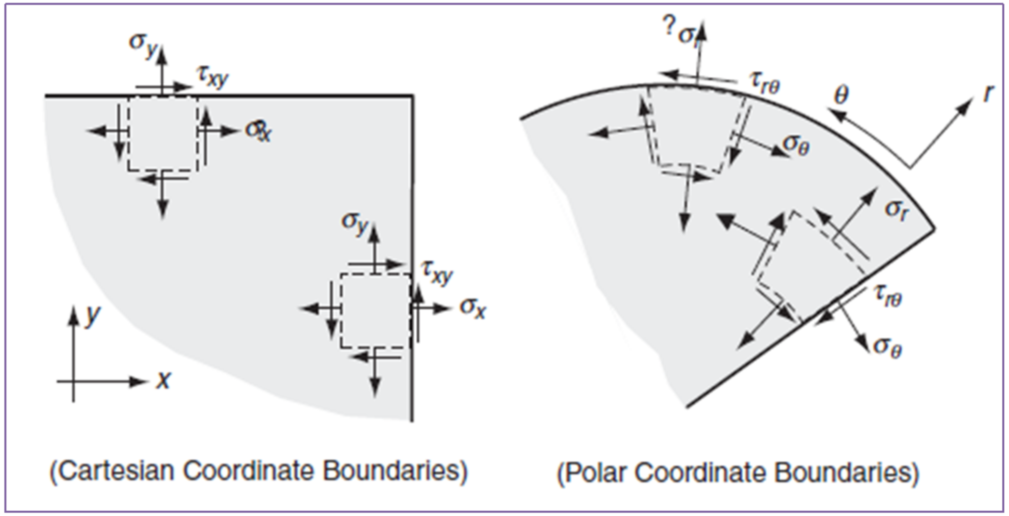
- Common types of boundary conditions for elasticity applications

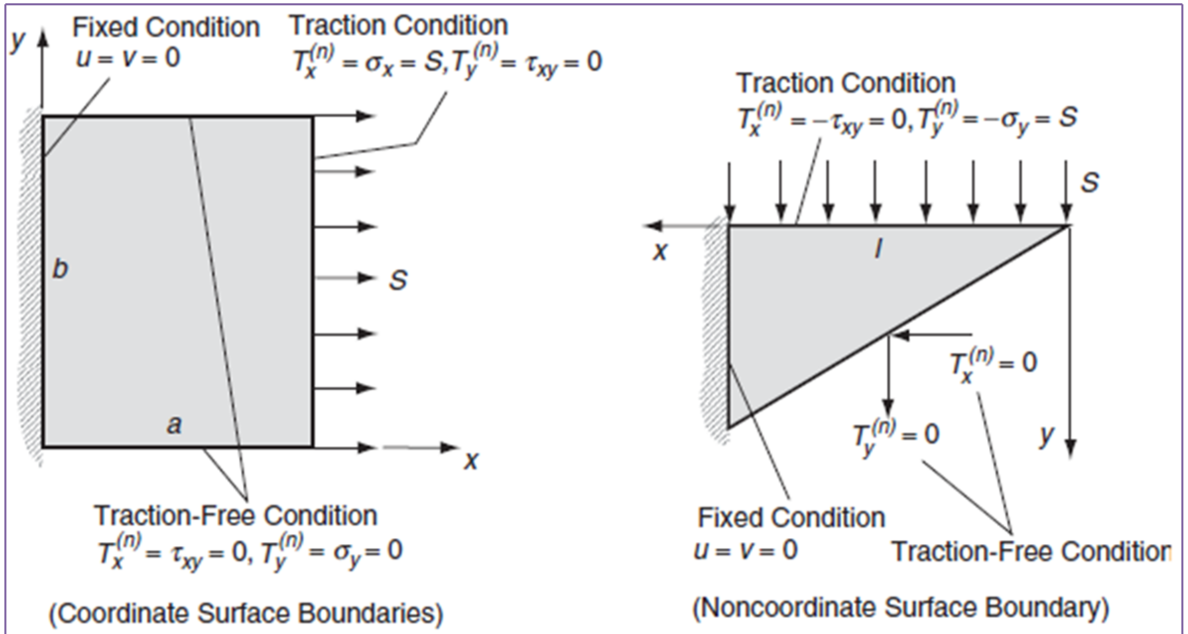
  How the body is supported → *displacements*

  How the body is loaded → *Tractions*



Boundary Conditions





Fundamental boundary value problems in theory of elasticity

- **Problem 1: Traction problem**
  - *Determine the distribution of displacements, strains, and stresses in the interior of an elastic body in equilibrium when body forces are given and the distribution of tractions is prescribed over the surface of the body:*

$$T_i^{(n)}(x_i^{(s)}) = f_i(x_i^{(s)})$$

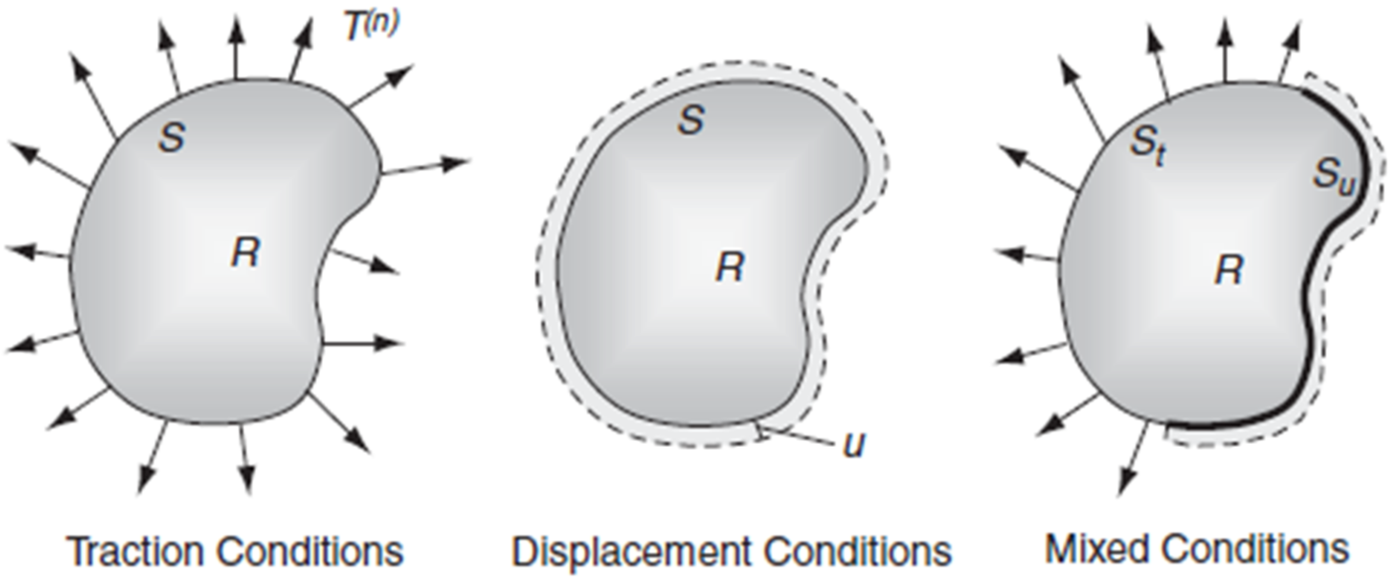
where $x_i^{(s)}$ denotes boundary points and $f_i(x_i^{(s)})$ are the prescribed traction values.

- **Problem 2: Displacement problem**
  - *Determine the distribution of displacements, strains, and stresses in the interior of an elastic body in equilibrium when body forces are given and the distribution of displacements is prescribed over the surface of the body:*

$$u_i^{(n)}(x_i^{(s)}) = g_i(x_i^{(s)})$$

where $x_i^{(s)}$ denotes boundary points and $g_i(x_i^{(s)})$ are the prescribed displacement values.

- Problem 3: Mixed problem
  - *Determine the distribution of displacements, strains, and stresses in the interior of an elastic body in equilibrium when body forces are given and the distribution of tractions is prescribed as per (p1) over the surface S_t and the distribution of displacements is prescribed as per (p2) over the surface S_u of the body.*



Stress formulation

- Using Hooke's Law and eliminating the strains in the compatibility relations and incorporating equilibrium:

$$\sigma_{ij,kk} + \frac{1}{1+v}\sigma_{kk,ij} = \frac{1}{1+v}\sigma_{mm,kk}\delta_{ij} - F_{i,j} - F_{j,i}$$

- For the case i = j, the above relation reduces to:

$$\sigma_{ii,kk} = -\frac{1+v}{1-v}F_{i,j}$$

- Substituting the above result back into the first equation:

$$\sigma_{ij,kk} + \frac{1}{1+v}\sigma_{kk,ij} = -\frac{v}{1-v}\delta_{ij}F_{k,k} - F_{i,j} - F_{j,i}$$

- The result is compatibility relations in terms of the stress and is commonly called Beltramin-Michell compatibility equations

continued

- For the case with no body forces:

$$(1+\nu)\nabla^2\sigma_x + \frac{\partial^2}{\partial x^2}(\sigma_x + \sigma_y + \sigma_z) = 0$$

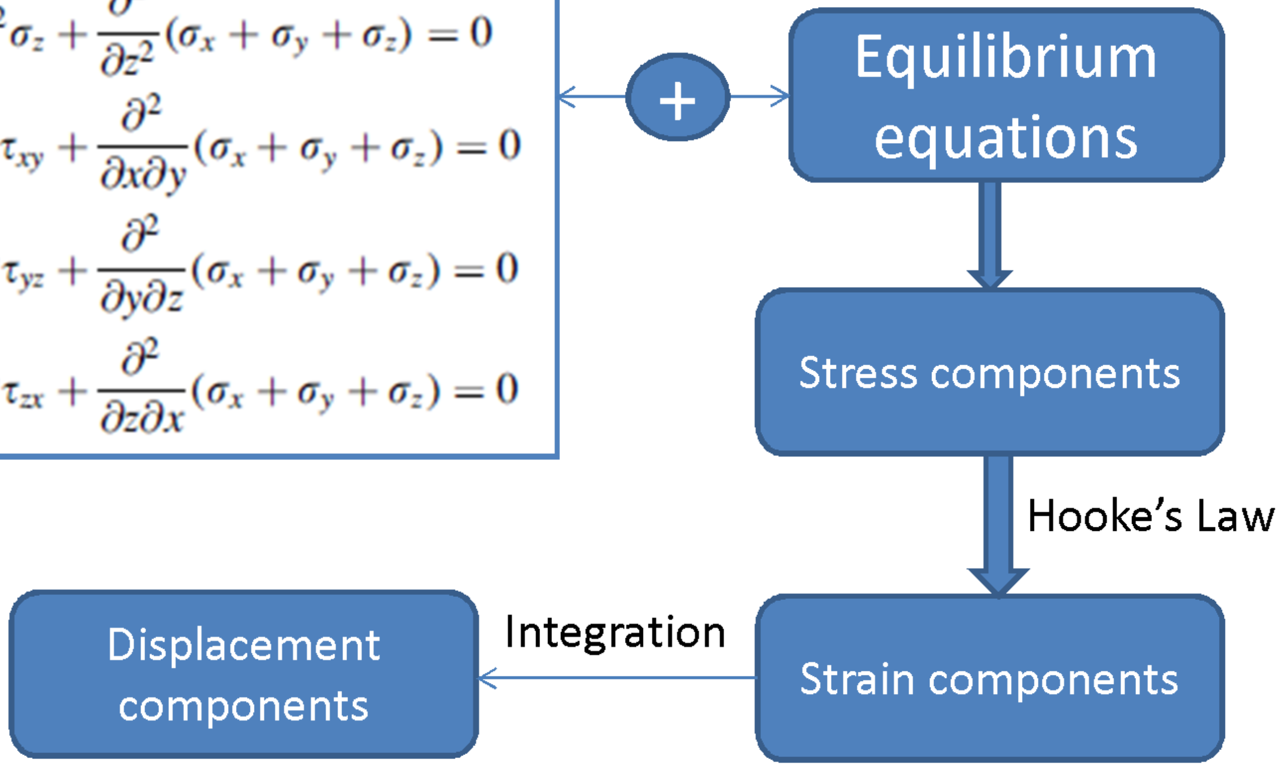
$$(1+\nu)\nabla^2\sigma_y + \frac{\partial^2}{\partial y^2}(\sigma_x + \sigma_y + \sigma_z) = 0$$

$$(1+\nu)\nabla^2\sigma_z + \frac{\partial^2}{\partial z^2}(\sigma_x + \sigma_y + \sigma_z) = 0$$

$$(1+\nu)\nabla^2\tau_{xy} + \frac{\partial^2}{\partial x \partial y}(\sigma_x + \sigma_y + \sigma_z) = 0$$

$$(1+\nu)\nabla^2\tau_{yz} + \frac{\partial^2}{\partial y \partial z}(\sigma_x + \sigma_y + \sigma_z) = 0$$

$$(1+\nu)\nabla^2\tau_{zx} + \frac{\partial^2}{\partial z \partial x}(\sigma_x + \sigma_y + \sigma_z) = 0$$



Displacement formulation

- Using Hooke's Law and eliminating the strains in the compatibility relations and incorporating equilibrium:

$$\sigma_{ij} = \lambda u_{k,k}\delta_{ij} - \mu\left(u_{i,j} + u_{j,i}\right)$$

which can be expressed as six scalar expressions:

$$\sigma_x = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial u}{\partial x}$$

$$\sigma_y = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial v}{\partial y}$$

$$\sigma_z = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial w}{\partial z}$$

$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),\ \tau_{yz} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right),\ \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

Using these relations in equilibrium equations gives the result:

$$\mu u_{i,kk} + (\lambda + \mu)u_{i,ki} + F_i = 0$$

- The result is equilibrium equations in terms of the displacements and is referred to us Navier's or Lame's equations

Continued

- The system can be expressed in vector form as:

$$\mu\nabla^2 u-(\lambda+\mu)\nabla(\nabla.u)+F=0$$

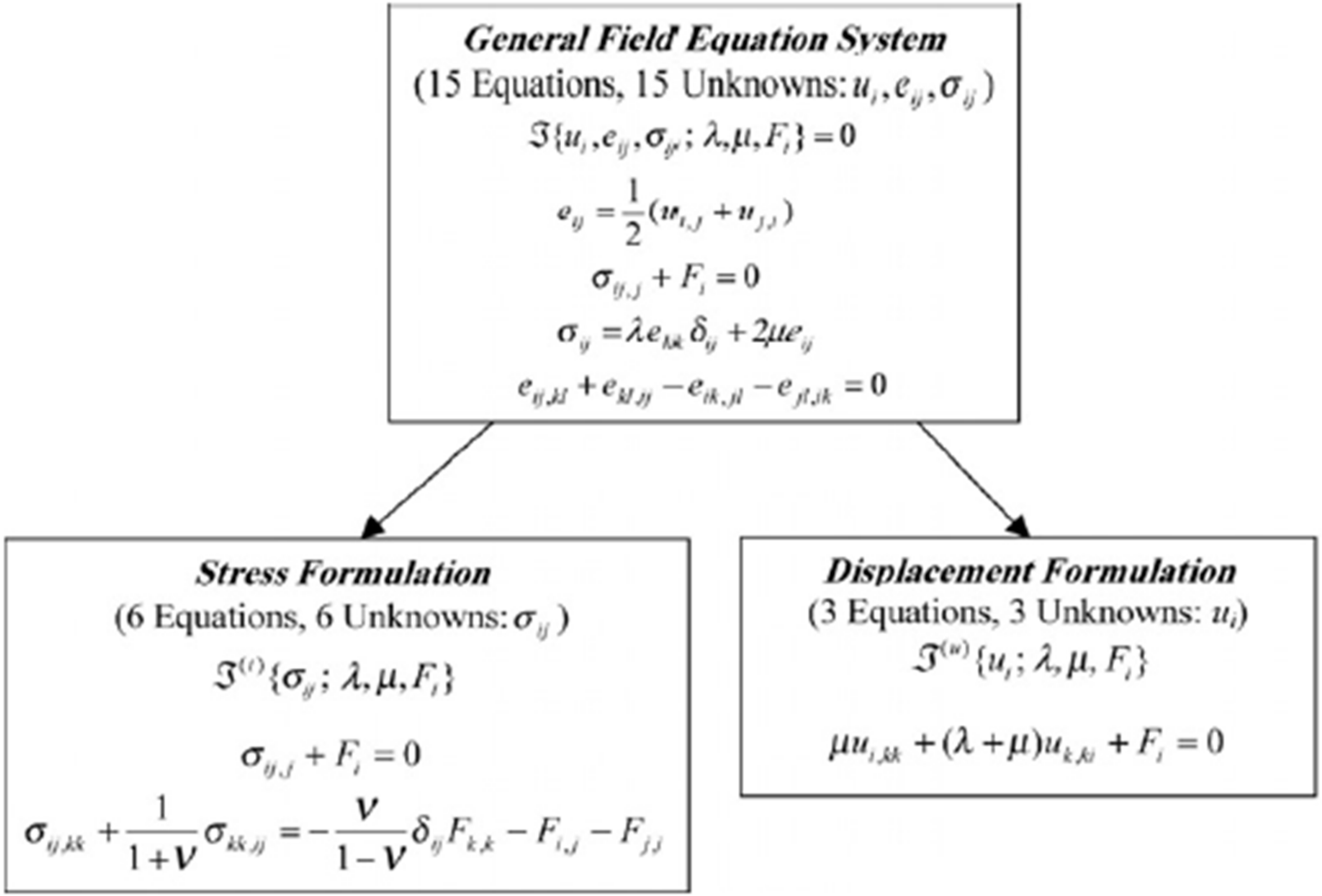
- In terms of three scalar equations:

$$\mu\nabla^2 u+(\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+F_x=0$$
$$\mu\nabla^2 v+(\lambda+\mu)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+F_y=0$$
$$\mu\nabla^2 w+(\lambda+\mu)\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+F_z=0$$

Where the Laplacian is given by:

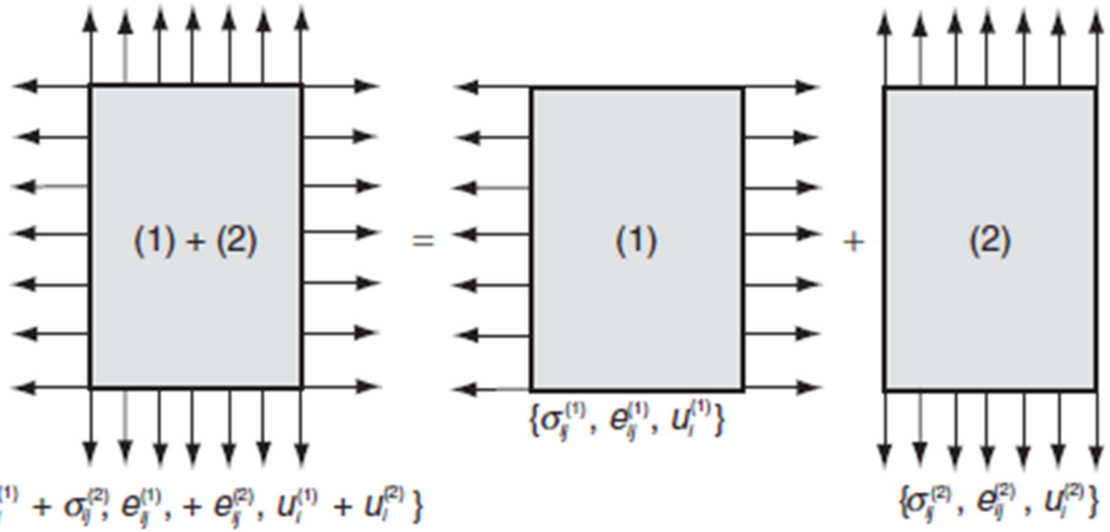
$$\nabla^2=\left(\frac{\partial^2}{\partial x^2}\right)+\left(\frac{\partial^2}{\partial y^2}\right)+\left(\frac{\partial^2}{\partial z^2}\right)$$

Schematic of Elasticity Field Equations



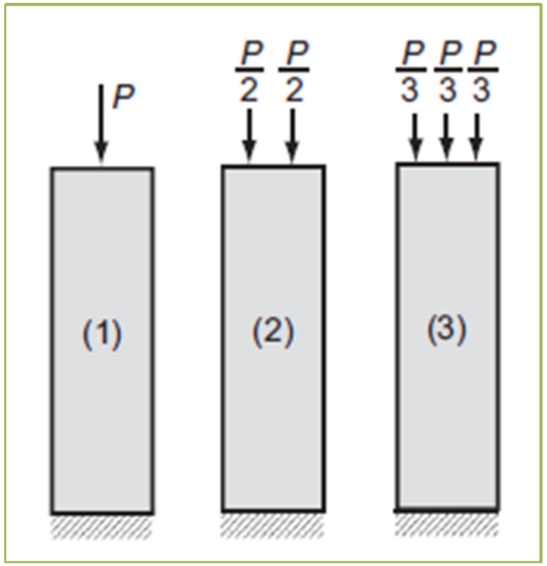
Principle of superposition

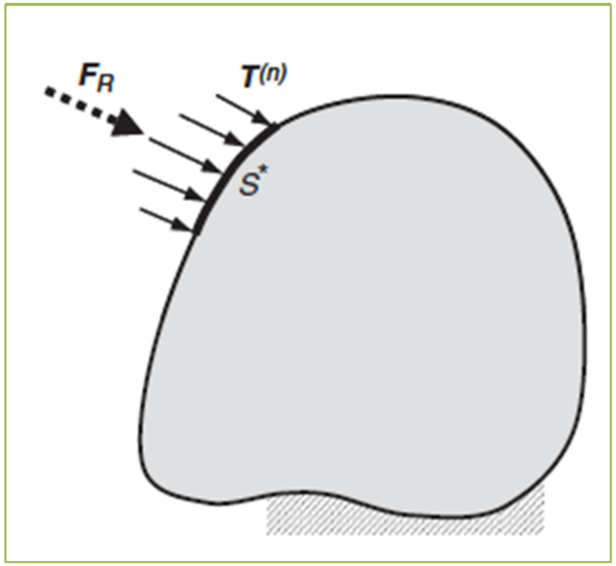
- For a given problem domain, if the state $\{\sigma_{ij}^{(1)}, e_{ij}^{(1)}, u_{ij}^{(1)}\}$ is a solution to the fundamental elasticity equations with prescribed body forces $F_i^{(1)}$ and surface tractions $T_i^{(1)}$, and the state $\{\sigma_{ij}^{(2)}, e_{ij}^{(2)}, u_{ij}^{(2)}\}$ is a solution to the fundamental equations with prescribed body forces $F_i^{(2)}$ and surface tractions $T_i^{(2)}$, then the state $\{\sigma_{ij}^{(1)} + \sigma_{ij}^{(2)}, e_{ij}^{(1)} + e_{ij}^{(2)}, u_{ij}^{(1)} + u_{ij}^{(2)}\}$ will be a solution to the problem with body forces $F_i^{(1)} + F_i^{(2)}$ and surface tractions $T_i^{(1)} + T_i^{(2)}$.



Saint-Venant's Principle

- The stress, strain and displacement fields caused by two different statically equivalent force distributions on parts of the body far away from the loading points are approximately the same.





- If we restrict our solution to points away from the boundary loading, Saint-Venant's principle allows us to change the given boundary conditions to a simpler statically equivalent statement and not affect the resulting solution. Such a simplification of boundary conditions greatly increases our chances of finding an analytical solution to the problem.

General Solution strategies