Failure Criteria

Introduction

- **Failure:** Every material has certain strength, expressed in terms of stress or strain, beyond which it fractures or fails to carry the load.
- **Failure Criterion:** A criterion used to hypothesize the failure.
- **Failure Theory:** A Theory behind a failure criterion.

Why Need Failure Theories?

(a) To design structural components and calculate margin of safety.

(b) To guide in materials development.

(c) To determine weak and strong directions.

Introduction

- Cannot perform tests for all combinations of 3D loading

  $\rightarrow$ *so we need yield criterion to generalize from small number of tests.*

- What are the ideal properties of a 3D yield criterion?

$$f(\sigma_{ij}, Y) < 0 \quad \textit{Elastic behavior}$$

$$f(\sigma_{ij}, Y) = 0 \quad \textit{onset of Inelastic behavior}$$

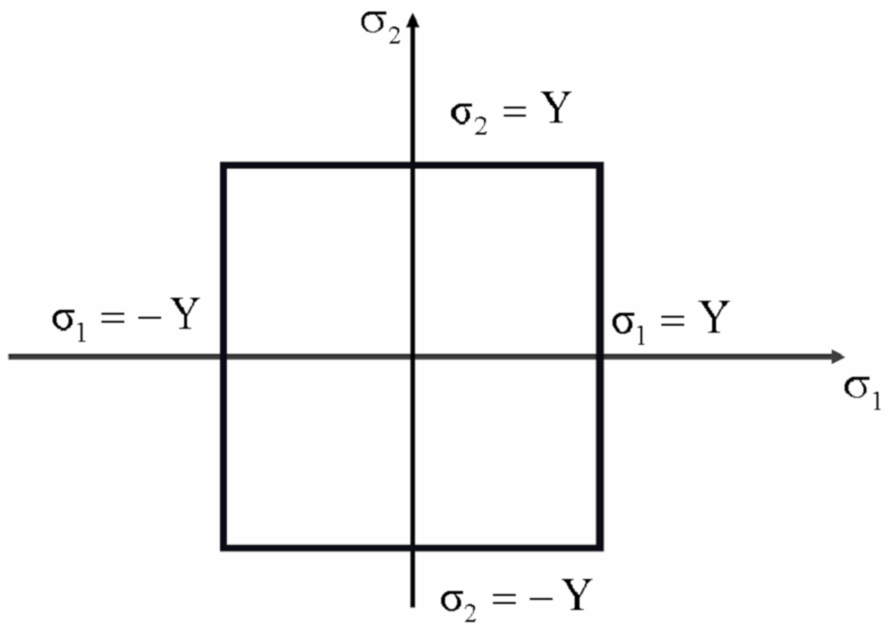
- We usually visualize yield criterion by a surface in principal stress space.

  $\rightarrow$ We also calculate “effective” stress to compare with yield stress.

Maximum principal stress criterion William Rankine (1820-1872)

- Applicable to brittle materials (mostly in tension).

$$f = \max(|\sigma_1|, |\sigma_2|, |\sigma_3|) = Y$$



Maximum Principal Stress Yield Surface

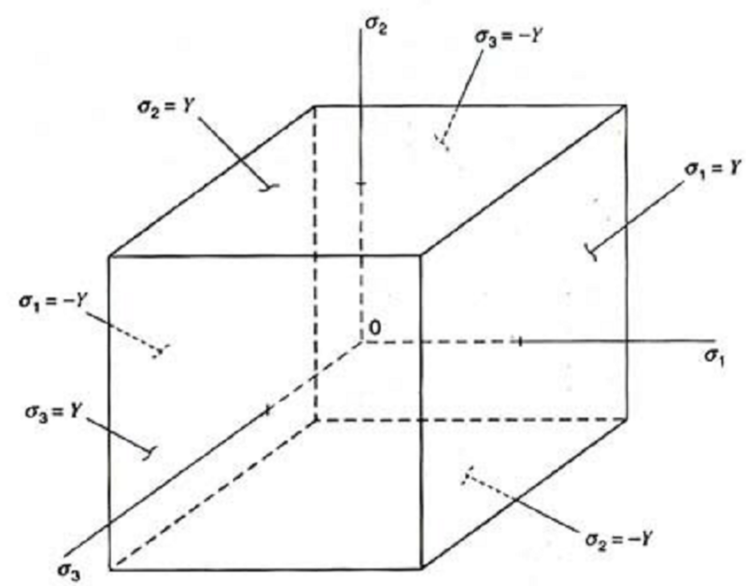


Figure 4.7 Maximum principal stress yield surface.

Maximum principal strain criterion Adhémar Jean Claude Barréde Saint-Venant (1797 - 1886)

- Has the advantage that strains are often easier to measure than stresses
- Assumes that ε_1 is the largest principal strain

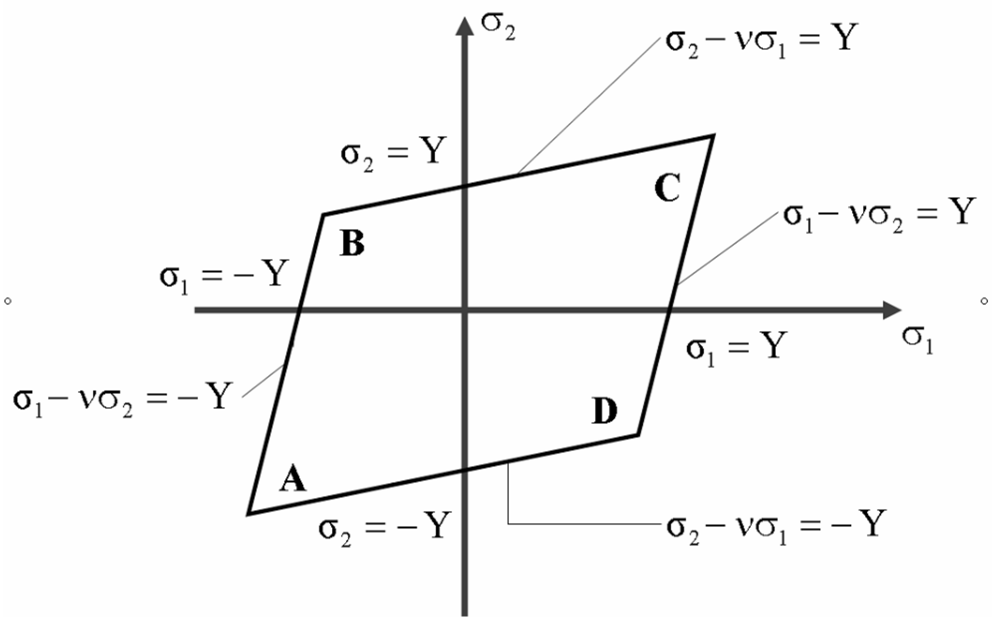
$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2 - \nu\sigma_3)$$

$$f_1 = \left|(\sigma_1 - \nu\sigma_2 - \nu\sigma_3)\right| - Y$$

$$\sigma_1 - \nu\sigma_2 - \nu\sigma_3 = \pm Y$$

$$\sigma_e = \max_{i \neq j \neq k}\left|\sigma_i - \nu\sigma_j - \nu\sigma_k\right|$$

$$\rightarrow f = \sigma_e - Y$$



Anti-optimization for selecting test conditions

- What test will give us maximum difference between maximum principal stress and maximum principal strain criteria?

  $\rightarrow$ *Obviously,*

$$\sigma_1 = \sigma_2 = \sigma_3 = \sigma$$
$$\textit{with}\ \mathbf{max}\,\textit{imum pricipal stress};\ \sigma_e = \sigma$$
$$\textit{with}\ \mathbf{max}\,\textit{imum pricipal strain};\ \sigma_e = \sigma(1-2\nu)$$

$\rightarrow$ *Alternatively,*

$$\sigma_1 = -\sigma_2 = -\sigma_3 = \sigma$$
$$\textit{with}\ \mathbf{max}\,\textit{imum pricipal stress};\ \sigma_e = \sigma$$
$$\textit{with}\ \mathbf{max}\,\textit{imum pricipal strain};\ \sigma_e = \sigma(1+2\nu)$$

- What is bad about these test conditions?

Strain energy density criterion (Eugenio Beltrami 1835 -1900)

- The theory assumes that the failure occurs when the total strain energy for a complex state of stress system is equal to that at the yield point of a tensile test.

$$U_o = \frac{1}{2E}[\sigma_1{}^2 + \sigma_2{}^2 + \sigma_3{}^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)]$$

$$Uniaxial\ Test = \frac{\sigma_1{}^2}{2E} = \frac{Y^2}{2E}$$

It may be noted that this theory gives fair and good results for ductile materials.

- Criterion;

$$[\sigma_1{}^2 + \sigma_2{}^2 + \sigma_3{}^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)] - Y^2 = 0$$

- Effective stress;

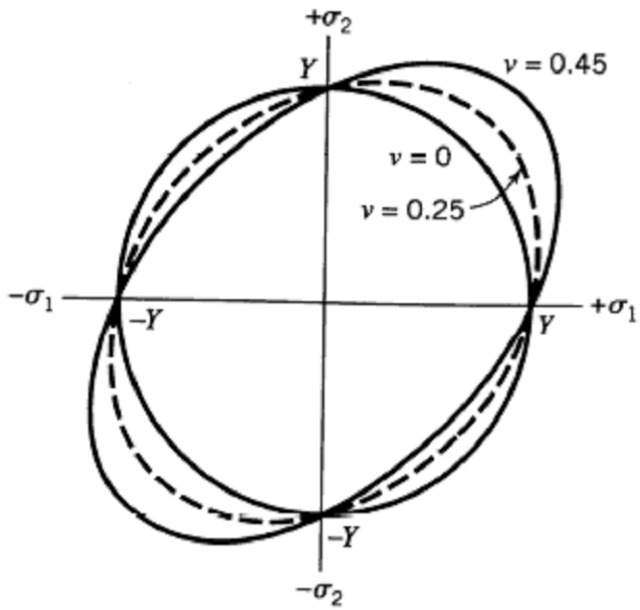
$$f = (\sigma_e)^2 - Y^2; \quad (\sigma_e)^2 = [\sigma_1{}^2 + \sigma_2{}^2 + \sigma_3{}^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)]$$

$\rightarrow$ *Extreme case,*

$$\sigma_1 = -\sigma_2 = -\sigma_3 = \sigma; \quad \sigma_e = \sigma\sqrt{3 + 2\nu}$$

Continued…

- For plane stress



Strain energy density yield surface for biaxial stress state

What is common to strain and energy criteria?

Maximum shear-stress criterion (Henri Eduard Tresca, 1814-1885)

- This theory states that the failure can be assumed to occur when the maximum shear stress in the complex stress system is equal to the value of maximum shear stress in simple tension.

$$\text{Uniaxial Test} \rightarrow \sigma_1 = Y; \sigma_2 = \sigma_3 = 0$$

$$\tau_{\max} = \frac{Y - 0}{2} = \frac{Y}{2}$$

Conservative for metals in shear

- Criterion;

$$f = \sigma_e - \frac{Y}{2}$$

- Effective stress;

$$\sigma_e = \tau_{\max} = \max(\tau_1, \tau_2, \tau_3)$$

$$\tau_1 = \frac{|\sigma_2 - \sigma_3|}{2}; \quad \tau_2 = \frac{|\sigma_1 - \sigma_3|}{2}; \quad \tau_3 = \frac{|\sigma_1 - \sigma_2|}{2}$$

→ *Extreme case,*

$$\sigma_1 = -\sigma_2 = -\sigma_3 = \sigma; \quad \sigma_e = \sigma$$

Distortional energy density criterion (Ludwig von Mises 1881-1973)

- This theory states that the failure occurs when the maximum shear strain energy component for the complex state of stress system is equal to that at the yield point in the tensile test.

$$U_o = U_V + U_D = \frac{1}{2E}[\sigma_1{}^2 + \sigma_2{}^2 + \sigma_3{}^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)]$$

$$U_V = \frac{\left(\sigma_1{}^2 + \sigma_2{}^2 + \sigma_3{}^2\right)}{18K}; \quad U_D = \frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2}{12G}$$

$$Uniaxial\ Test \rightarrow \sigma_1 = Y; \sigma_2 = \sigma_3 = 0$$

$$U_D = \frac{(Y-0)^2 + (0-0)^2 + (Y-0)^2}{12G} = \frac{Y^2}{6G}$$

- Effective stress;

$$\sigma_e = \sqrt{0.5[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2]}$$

- Criterion;

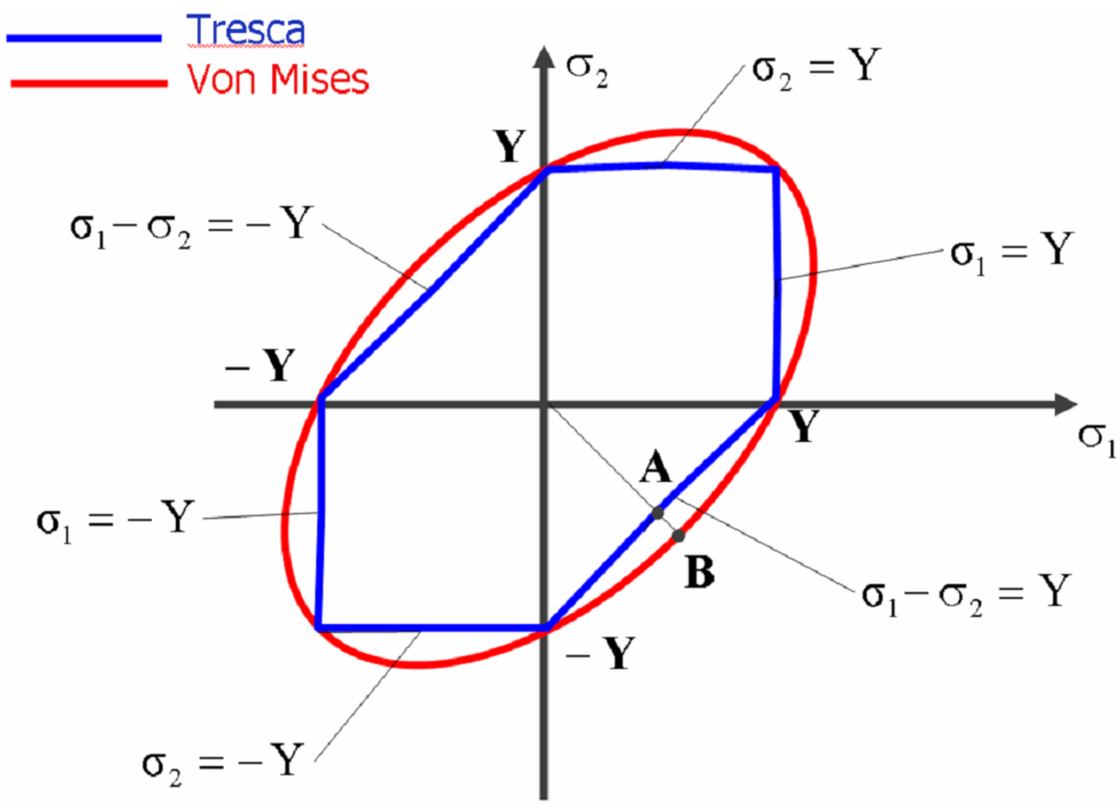
$$f = \sigma_e - Y$$

$\rightarrow$ *Extreme case,*

$$\sigma_1 = \sigma_2 = \sigma_3 = \sigma; \quad \sigma_e = 0$$

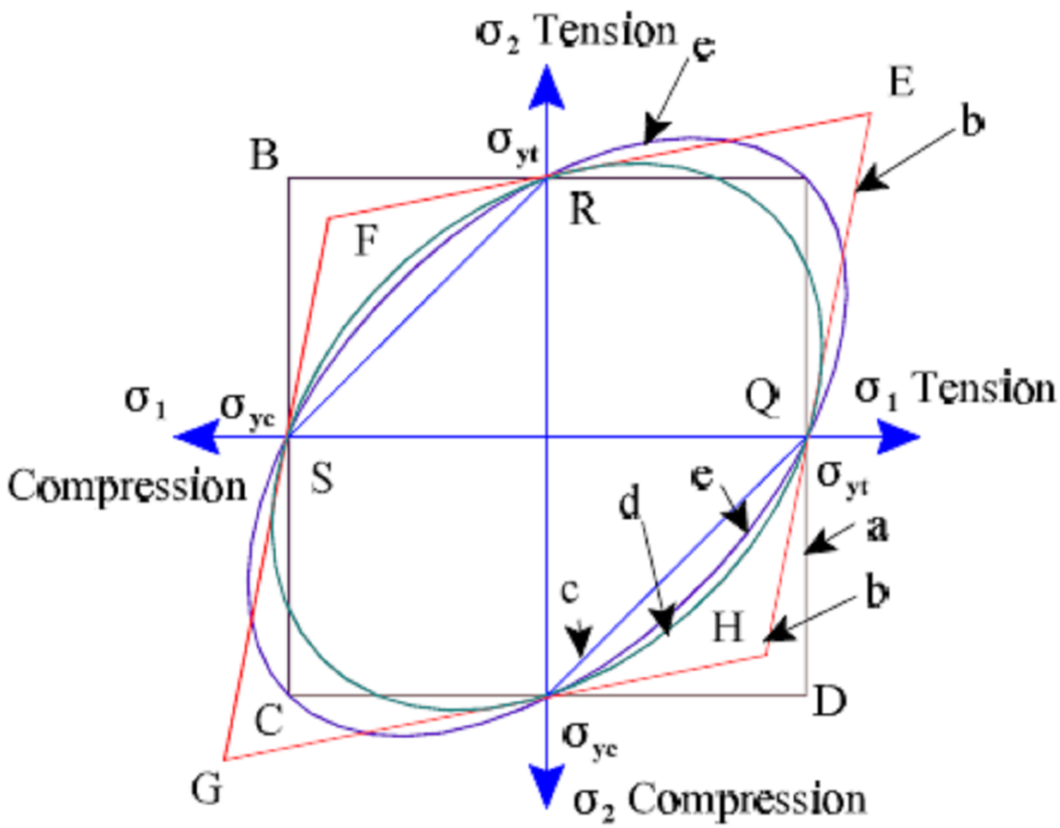
Continued…

- For plane stress



Where is the maximum difference?

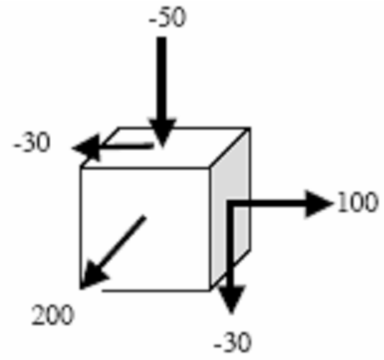
General



a = Maximum principle stress theory
b = Maximum principle strain theory
c = Maximum shear stress theory
d = Total strain energy theory
e = Shear strain energy theory

Example comparing failure criteria

Q. Stress analysis of a spacecraft structural member gives the state of stress as shown below. If the part is made from an alloy with Y = 500 Mpa, check yielding according to Rankine, Tresca and von Mises criteria. What is its safety factor for each criterion?



Maximum principal stress Criterion (Rankine)

$$\sigma = \begin{bmatrix} 200 & 0 & 0 \\ 0 & 100 & -30 \\ 0 & -30 & -50 \end{bmatrix} MPa$$

- The principal stresses are:

$$\sigma_1 = 200; \ \sigma_2 = 105.77; \ \sigma_3 = -55.77;$$

- The maximum principal stresses is 200MPa.
- Factor of safety:

$$FS = \frac{500}{200} = 2.5$$

Maximum shear stress Criterion (Tresca)

- Yield function:

$$f = \sigma_e - \frac{Y}{2}$$

- Maximum shear stress:

$$\sigma_e = \frac{|\sigma_1 - \sigma_3|}{2} = \frac{|200-(-55.77)|}{2} = 127.89 MPa;$$

- Shear stress for uniaxial tension.

$$Y = \frac{500}{2} = 250 \qquad f = 127.89 - 250 < 0$$

- Factor of safety:

$$FS = \frac{250}{127.89} = 1.95$$

Distortional Energy Density Criterion (von Mises)

- Yield function:

$$f = \sqrt{0.5[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2]} - Y$$

$$f = 224 - 500 < 0$$

- Factor of safety:

$$FS = \frac{500}{224} = 2.23$$

- Why are all the results so close?

$$\sigma_1 = 200;\ \sigma_2 = 105.77;\ \sigma_3 = -55.77;$$

Question: Under what conditions will we have maximum differences between Tresca and Von-Mises criteria? (consider both the case that Tresca is more conservative and the case that Von Mises is)

Mohr-Coulomb yield criterion (Charles Augustin de Coulomb, 1736-1806, Christian Otto Mohr, 1835-1918)

- Yield of materials like rock and concrete does change with hydrostatic pressure
- Yield function written in terms of cohesion *c* and internal friction angle ϕ as

$$f = \sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3)\sin\phi - 2c\cos\phi \qquad \sigma_1 \geq \sigma_2 \geq \sigma_3$$

- What would it look like for plane stress?

From test results

- For tension test

$$\sigma_1 = Y_T, \sigma_2 = \sigma_3 = 0, \quad f = 0 \quad \Rightarrow \quad Y_T = \frac{2\cos\phi}{1+\sin\phi}$$

- For compression test

$$\sigma_3 = -Y_C, \sigma_2 = \sigma_1 = 0, \; f = 0 \quad \Rightarrow \quad Y_C = \frac{2\cos\phi}{1-\sin\phi}$$

- Get

$$c = \frac{Y_T}{2}\sqrt{\frac{Y_T}{Y_C}} \qquad \phi = \frac{\pi}{2} - \tan^{-1}\left(\sqrt{\frac{Y_T}{Y_C}}\right)$$