Theories of Failure

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Summary

- Maximum principal stress theory
- Maximum principal strain theory
- Maximum strain energy theory
- Distortion energy theory
- Maximum shear stress theory
- Octahedral stress theory

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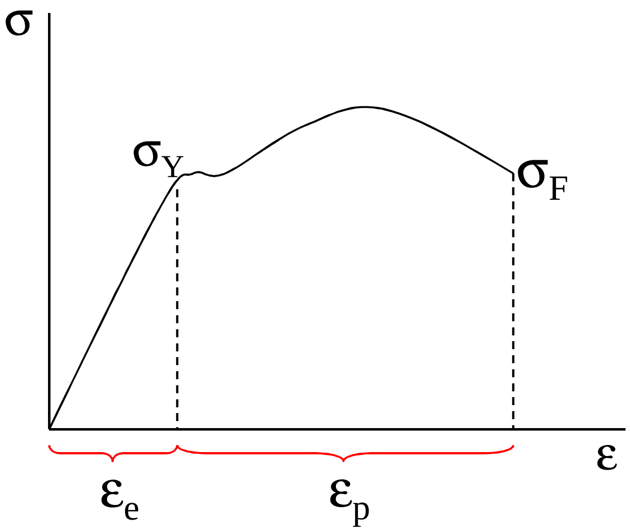
Introduction

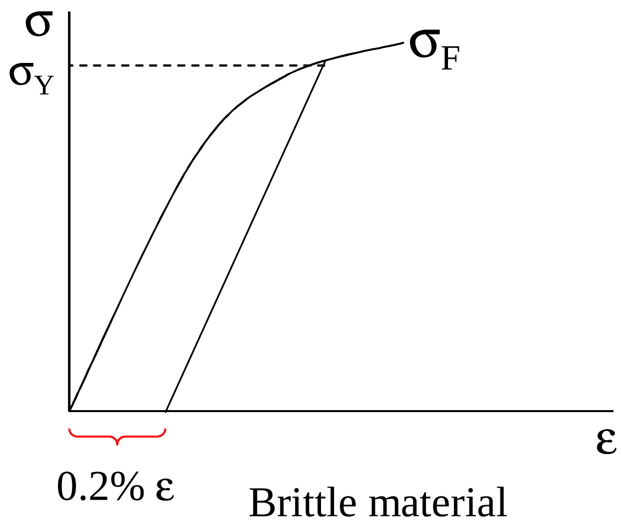
- Failure occurs when material starts exhibiting inelastic behavior
- Brittle and ductile materials – different modes of failures – mode of failure – depends on loading
- Ductile materials – exhibit yielding – plastic deformation before failure
- Yield stress – material property
- Brittle materials – no yielding – sudden failure
- Factor of safety (FS)

Introduction

- Ductile and brittle materials



Ductile material



Brittle material

Well – defined yield point in ductile materials – FS on yielding

No yield point in brittle materials sudden failure – FS on failure load

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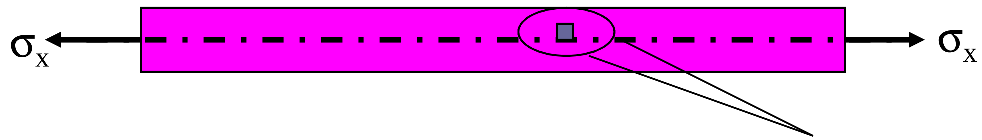
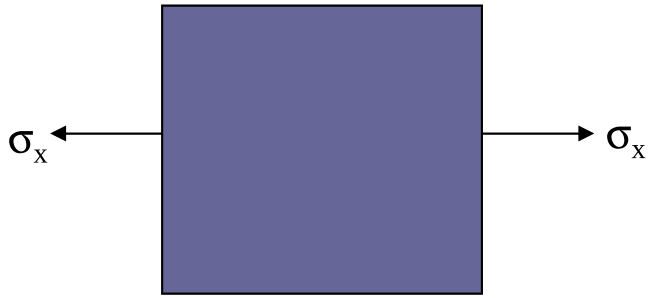
Introduction

- Stress developed in the material < yield stress
- Simple axial load



If $\sigma_x = \sigma_Y$ => yielding starts – failure



Yielding is governed by single stress component, σ_x

Similarly in pure shear – only shear stress.

If $\tau_{max} = \tau_Y$ => Yielding in shear

Multi-axial stress state ??

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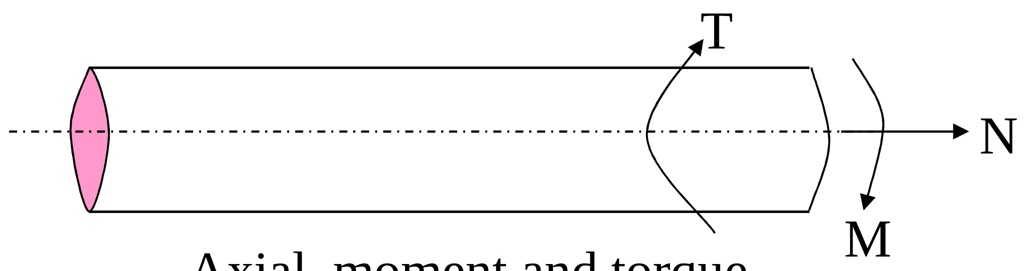
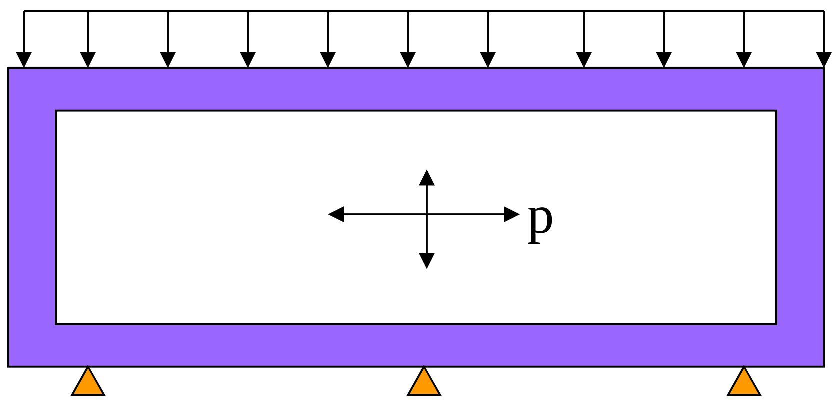
Introduction

- Various types of loads acting at the same time



Axial, moment and torque



Internal pressure and external UDL

Introduction

- Multiaxial stress state – six stress components – one representative value
- Define effective / equivalent stress – combination of components of multiaxial stress state
- Equivalents stress reaching a limiting value – property of material – yielding occurs – Yield criteria
- Yield criteria define conditions under which yielding occurs
- Single yield criteria – doesn't cater for all materials
- Selection of yield criteria
- Material yielding depends on rate of loading – static & dynamic

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Introduction

- Yield criteria expressed in terms of quantities like stress state, strain state, strain energy etc.
- Yield function => $f(\sigma_{ij}, Y)$, σ_{ij} = stress state
- If $f(\sigma_{ij}, Y)<0$ => No yielding takes place – no failure of the material
- If $f(\sigma_{ij}, Y) = 0$ – starts yielding – onset of yield

  If $f(\sigma_{ij}, Y) > 0$ - ??
- Yield function developed by combining stress components into a single quantity – effective stress => σ_e

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Introduction

- Equivalent stress depends on stress state and yield criteria – not a property
- Compare σ_e with yield stress of material
- Yield surface – graphical representation of yield function, $f(\sigma_{ij}, Y) = 0$
- Yield surface is plotted in principal stress space – Haiagh – Westergaard stress space
- Yield surface – closed curve

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Parameters in uniaxial tension

R&DE (Engineers), DRDO

- Maximum principal stress

Applied stress => Y

$\sigma_1 = Y, \sigma_2 = 0, \sigma_3 = 0$

- Maximum shear stress

$$\tau_{max} = \left|\frac{\sigma_1 - \sigma_3}{2}\right| = \frac{Y}{2}$$

- Maximum principal strain

$\sigma_1 = Y, \sigma_2 = 0, \sigma_3 = 0$

$$\varepsilon_Y = \frac{\sigma_1}{E} - \frac{\upsilon}{E}(\sigma_2 + \sigma_3) = \frac{Y}{E}$$

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Parameters in uniaxial tension

- Total strain energy density

Linear elastic material $$U = \frac{1}{2} Y \varepsilon_Y = \frac{1}{2} \frac{Y^2}{E}$$

- Distortional energy

$$[\sigma] = \begin{bmatrix} Y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} Y-p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix} + \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}$$

First invariant = 0 for deviatoric part => p = Y/3

$$U = U_D + U_V$$

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Parameters in uniaxial tension

Volumetric strain energy density, $U_V = p^2/2K$

$$U_V = \frac{p^2}{2K} = \frac{Y^2}{18K} = \frac{(1-2\upsilon)}{6E}Y^2$$

$$U_D = U - U_V$$

$$U_D = \frac{Y^2}{2E} - \frac{(1-2\upsilon)Y^2}{6E} = \frac{Y^2}{6E}(3-1+2\upsilon) = \frac{Y^2}{3E}(1+\upsilon)$$

$$U_D = \frac{Y^2}{6G}$$

Similarly for pure shear also

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Failure theories

- Failure mode –
  - Mild steel (M. S) subjected to pure tension
  - M. S subjected to pure torsion
  - Cast iron subjected to pure tension
  - Cast iron subjected to pure torsion
- Theories of failure
  - Max. principal stress theory – Rankine
  - Max. principal strain theory – St. Venants
  - Max. strain energy – Beltrami
  - Distortional energy – von Mises
  - Max. shear stress theory – Tresca
  - Octahedral shear stress theory

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Max. principal stress theory

- Maximum principal stress reaches tensile yield stress (Y)
- For a given stress state, calculate principle stresses, σ_1, σ_2 and σ_3
- Yield function

$$f = \max\left(|\sigma_1|, |\sigma_2|, |\sigma_3|\right) - Y$$

$$\text{If,} \quad f < 0 \quad \text{no yielding}$$

$$f = 0 \quad \text{onset of yielding}$$

$$f > 0 \quad \text{not defined}$$

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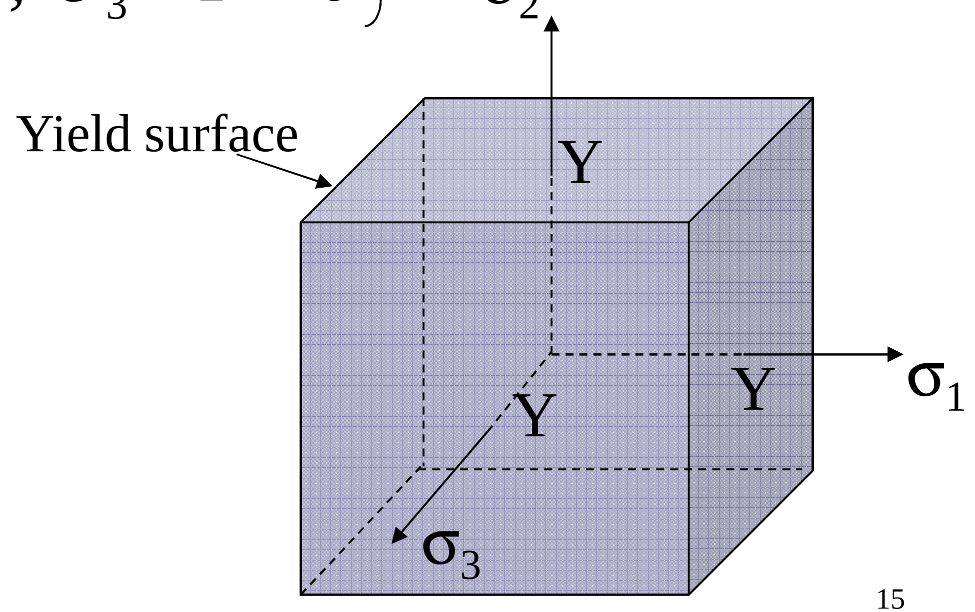
Max. principal stress theory

- Yield surface –

$$\sigma_1 = \pm Y \Rightarrow \sigma_1 + Y = 0,\ \sigma_1 - Y = 0$$
$$\sigma_2 = \pm Y \Rightarrow \sigma_2 + Y = 0,\ \sigma_2 - Y = 0$$
$$\sigma_3 = \pm Y \Rightarrow \sigma_3 + Y = 0,\ \sigma_3 - Y = 0$$

Represent six surfaces



Yield strength – same in tension and compression

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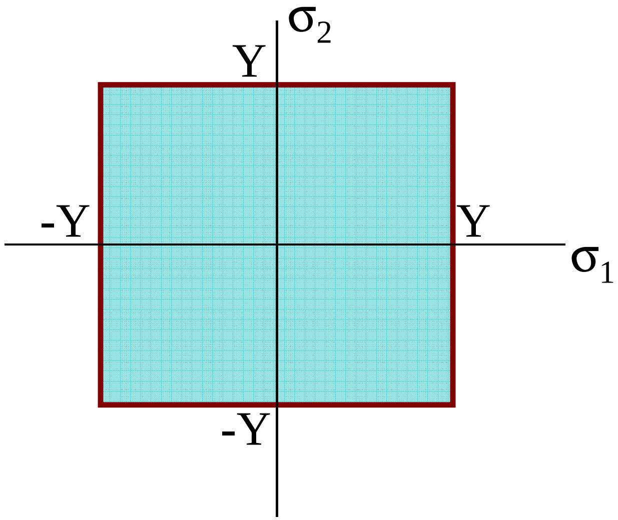
Max. principal stress theory

- In 2D case, $\sigma_3 = 0$ – equations become

$$\sigma_1 = \pm Y \Rightarrow \sigma_1 + Y = 0,\ \sigma_1 - Y = 0$$
$$\sigma_2 = \pm Y \Rightarrow \sigma_2 + Y = 0,\ \sigma_2 - Y = 0$$



Closed curve

Stress state inside – elastic, outside => Yielding

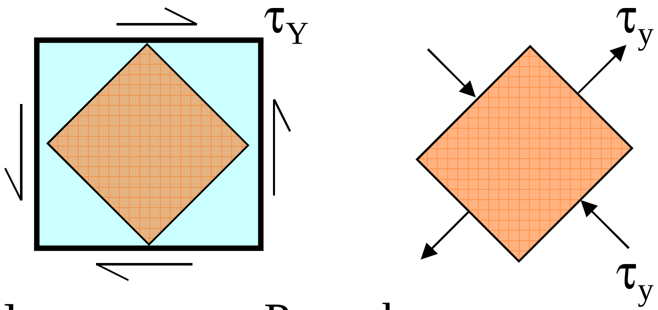
Pure shear test => $\sigma_1 = +\tau_Y$, $\sigma_2 = -\tau_Y$

For tension => $\sigma_1 = +\sigma_Y$

From the above => $\sigma_Y = \tau_Y$

Experimental results – Yield stress in shear is less than yield stress in tension

Predicts well, if all principal stresses are tensile



Pure shear

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Max. principal strain theory

- "Failure occurs at a point in a body when the maximum strain at that point exceeds the value of the maximum strain in a uniaxial test of the material at yield point"
- 'Y' – yield stress in uniaxial tension, yield strain, $\varepsilon_y = Y/E$
- The maximum strain developed in the body due to external loading should be less than this
- Principal stresses => σ_1, σ_2 and σ_3 strains corresponding to these stress => ε_1, ε_2 and ε_3

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Max. principal strain theory

Strains corresponding to principal stresses -

$$\varepsilon_1 = \frac{\sigma_1}{E} - \frac{\upsilon}{E}(\sigma_2 + \sigma_3)$$

$$\varepsilon_2 = \frac{\sigma_2}{E} - \frac{\upsilon}{E}(\sigma_1 + \sigma_3)$$

$$\varepsilon_3 = \frac{\sigma_3}{E} - \frac{\upsilon}{E}(\sigma_2 + \sigma_1)$$

Maximum of this should be less than ε_y

For onset of yielding

$$|\varepsilon_1| = \frac{Y}{E} => \sigma_1 - \upsilon(\sigma_2 + \sigma_2) = \pm Y$$

$$|\varepsilon_2| = \frac{Y}{E} => \sigma_2 - \upsilon(\sigma_3 + \sigma_1) = \pm Y$$

$$|\varepsilon_3| = \frac{Y}{E} => \sigma_3 - \upsilon(\sigma_1 + \sigma_2) = \pm Y$$

There are six equations – each equation represents a plane

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Max. principal strain theory

- Yield function

$$f = \max_{i \neq j \neq k} \left| \sigma_i - \upsilon\sigma_j - \upsilon\sigma_k \right| - Y, \quad i, j, k = 1, 2, 3$$

$$f = \sigma_e - Y$$

$$\sigma_e = \max_{i \neq j \neq k} \left| \sigma_i - \upsilon\sigma_j - \upsilon\sigma_k \right|$$

- For 2D case

$$\left| \sigma_1 - \upsilon\sigma_2 \right| = Y \Rightarrow \sigma_1 - \upsilon\sigma_2 = \pm Y$$

$$\left| \sigma_2 - \upsilon\sigma_1 \right| = Y \Rightarrow \sigma_2 - \upsilon\sigma_1 = \pm Y$$

There are four equations, each equation represents a straight line in 2D stress space

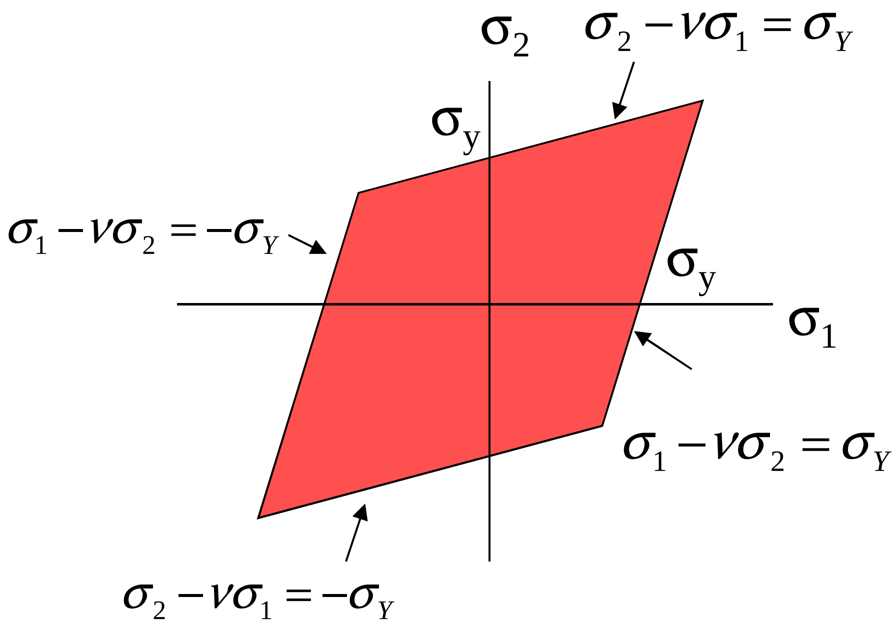
Max. principal strain theory

Equations –

$$\sigma_1 - \upsilon\sigma_2 = Y,\ \sigma_1 - \upsilon\sigma_2 = -Y$$

$$\sigma_2 - \upsilon\sigma_1 = Y,\ \sigma_2 - \upsilon\sigma_1 = -Y$$

Plotting in stress space



Failure – equivalent stress falls outside yield surface

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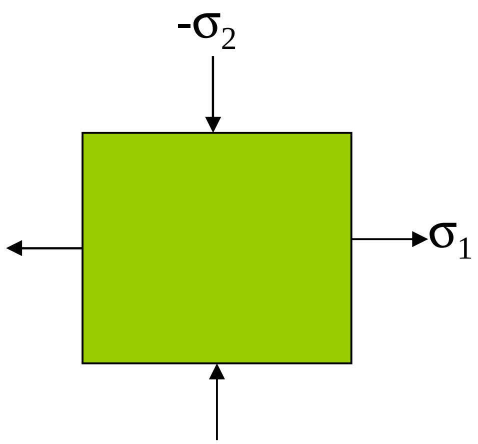
Max. principal strain theory

- Biaxial loading

For onset of yielding –

$$Y = \sigma_1 - \upsilon\sigma_2 = \sigma(1+\upsilon)$$
$$Y = \sigma(1+\upsilon)$$



$$\sigma_1 = |\sigma_2| = \sigma$$

Maximum principal stress theory –

$$Y = \sigma$$

Max. principal strain theory predicts smaller value of stress than max. principal stress theory

Conservative design

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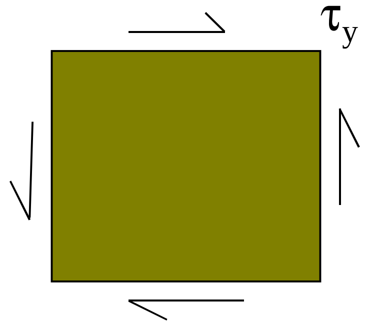
Max. principal strain theory

- Pure shear

Principal stresses corresponding to shear yield stress



$$\sigma_1 = +\tau_y, \ \sigma_2 = -\tau_y$$

For onset of yielding – max. principal strain theory

$$Y = \tau_y + \upsilon\, \tau_y = \tau_y\,(1 + \upsilon)$$

Relation between yield stress in tension and shear

$$\tau_y = Y/\,(1 + \upsilon) \text{ for } \upsilon = 0.25$$

$\boldsymbol{\tau_y = 0.8Y}$ Not supported by experiments

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Strain energy theory

- "Failure at any point in a body subjected to a state of stress begins only when the energy density absorbed at that point is equal to the energy density absorbed by the material when subjected to elastic limit in a uniaxial stress state"
- In uniaxial stress (yielding)

$\sigma = E\varepsilon$ => Hooke's law

Strain energy density,

$$U = \int \sigma_{ij} \, d\varepsilon_{ij} \Rightarrow U = \int_0^{\varepsilon_y} \sigma \, d\varepsilon$$

$$U = \frac{1}{2}\frac{Y^2}{E}$$

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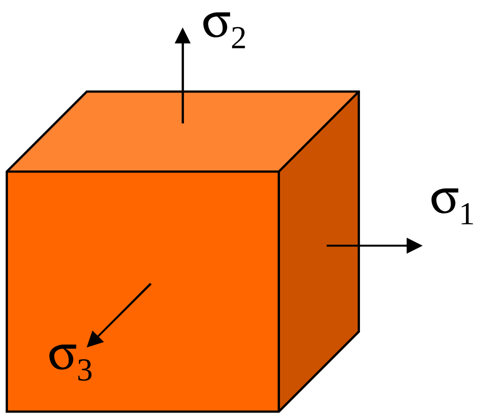
Strain energy theory

- Body subjected to external loads => principal stresses



Strain energy associated with principal stresses

$$U = \frac{1}{2}(\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3)$$

$$\varepsilon_1 = \frac{\sigma_1}{E} - \frac{\upsilon}{E}(\sigma_2 + \sigma_3)$$

$$\varepsilon_2 = \frac{\sigma_2}{E} - \frac{\upsilon}{E}(\sigma_3 + \sigma_1)$$

$$\varepsilon_3 = \frac{\sigma_2}{E} - \frac{\upsilon}{E}(\sigma_1 + \sigma_2)$$

$$U = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\upsilon(\sigma_1\sigma_2 + \sigma_3\sigma_1 + \sigma_2\sigma_3)\right]$$

For onset of yielding,

$$\frac{Y^2}{2E} = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\upsilon(\sigma_1\sigma_2 + \sigma_3\sigma_1 + \sigma_2\sigma_3)\right]$$

Strain energy theory

- Yield function –

$$f = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \upsilon(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) - Y^2$$

$$f = \sigma_e^2 - Y^2$$

$$\text{Equivalent stress} \Rightarrow \sigma_e^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \upsilon(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)$$

$$\text{Yielding} \Rightarrow f = 0, \quad \text{safe } f < 0$$

- For 2D stress state => $\sigma_3 = 0$ – Yield function becomes

$$f = \sigma_1^2 + \sigma_2^2 - \upsilon\sigma_1\sigma_2 - Y^2$$

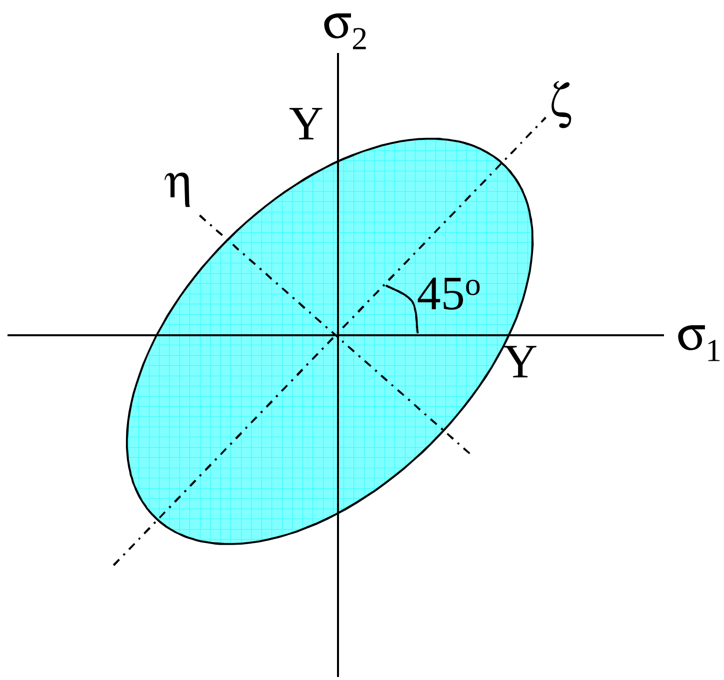
For onset of yielding => f = 0 $\quad \sigma_1^2 + \sigma_2^2 - \upsilon\sigma_1\sigma_2 - Y^2 = 0$

Plotting this in principal stress space

Strain energy theory

Rearrange the terms –

$$\left(\frac{\sigma_1}{Y}\right)^2+\left(\frac{\sigma_2}{Y}\right)^3-2\upsilon\left(\frac{\sigma_1}{Y}\frac{\sigma_2}{Y}\right)=1$$



This represents an ellipse – Transform to ζ-η csys

$$\sigma_1=\zeta\cos 45-\eta\sin 45=\frac{1}{\sqrt{2}}(\zeta-\eta)$$

$$\sigma_2=\zeta\sin 45+\eta\cos 45=\frac{1}{\sqrt{2}}(\zeta+\eta)$$

Equivalent stress inside – no failure

Substitute these in the above expression

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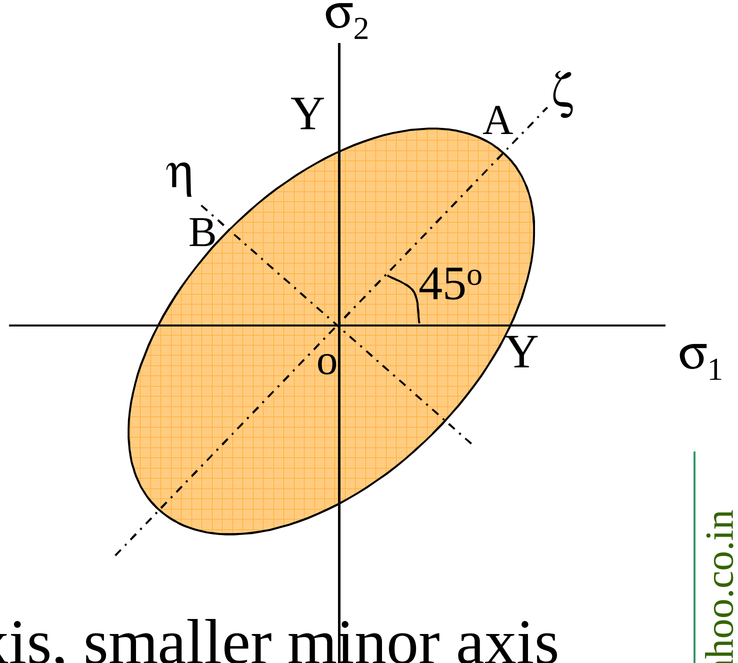
Strain energy theory

Simplifying, $$\frac{\zeta^2}{\frac{Y^2}{(1-\upsilon)}} + \frac{\eta^2}{\frac{Y^2}{(1+\upsilon)}} = 1 \Rightarrow \frac{\zeta^2}{a^2} + \frac{\eta^2}{b^2} = 1$$

Semi major axis – OA => $a = \frac{Y}{\sqrt{(1-\upsilon)}}$

Semi minor axis – OB => $b = \frac{Y}{\sqrt{(1+\upsilon)}}$



Higher Poisson ratio – bigger major axis, smaller minor axis

If $\upsilon = 0$ => circle of radius 'Y'

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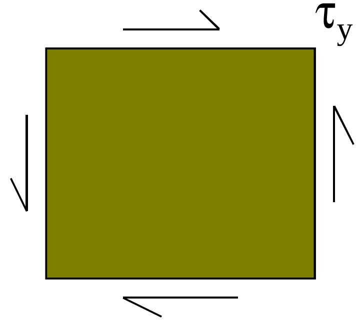
Strain energy theory

- Pure shear

Principal stresses corresponding to shear yield stress



$\sigma_1 = +\tau_y$, $\sigma_2 = -\tau_y$

$$\varepsilon_1 = \frac{\tau_y}{E}(1+\upsilon),\ \varepsilon_2 = -\frac{\tau_y}{E}(1+\upsilon)$$

Strain energy, $U_\tau = \frac{(1+\upsilon)}{2E} 2\tau_y^2 = \frac{1}{2E} Y^2 \Rightarrow Y = \sqrt{2(1+\upsilon)}\tau_y$

$$\tau_y = 0.632\ Y$$

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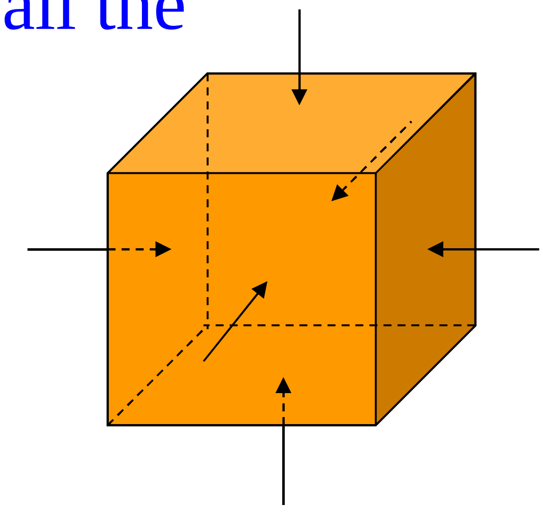
Distortional energy theory (von-Mises)

- Hydrostatic loading
  - applying uniform stress from all the directions on a body
  - Large amount of strain energy can be stored
  - Experimentally verified
  - Pressures beyond yield stress – no failure of material
  - Hydrostatic loading – change in size – volume



Pressure ‘p’ applied from all sides

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von-Mises theory

- Energy associated with volumetric change – volumetric strain energy
- Volumetric strain energy – no failure of material
- Strain energy causing material failure – distortion energy – associated with shear – First invariant of deviatoric stress = 0
- For a given stress state estimate distortion energy – this should be less than distortion energy due to uniaxial tensile – safe

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von-Mises theory

- Given stress state referred to principal co-ordinate system –

$$[\sigma]=\begin{bmatrix}\sigma_1 & 0 & 0\\ 0 & \sigma_2 & 0\\ 0 & 0 & \sigma_3\end{bmatrix}=\begin{bmatrix}\sigma_1-p & 0 & 0\\ 0 & \sigma_2-p & 0\\ 0 & 0 & \sigma_3-p\end{bmatrix}+\begin{bmatrix}p & 0 & 0\\ 0 & p & 0\\ 0 & 0 & p\end{bmatrix}$$

$$\text{First invariant,} \quad J_1=0$$

$$(\sigma_1-p)+(\sigma_2-p)+(\sigma_3-p)=0$$

$$=> p=\frac{1}{3}\sigma_{ii}$$

Principal strains => $\varepsilon_1, \varepsilon_2, \varepsilon_3$

Volumetric strain => $\varepsilon_V = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$

von-Mises theory

This gives –

$$\varepsilon_V = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \frac{1}{E}\left\{(\sigma_1 + \sigma_2 + \sigma_3) - 2\upsilon(\sigma_1 + \sigma_2 + \sigma_3)\right\}$$

$$\varepsilon_V = \frac{(1-2\upsilon)}{E}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{3(1-2\upsilon)}{E}p$$

$$\text{Volumetric strain energy, } U_V = \frac{1}{2}p\varepsilon_V$$

$$U_V = \frac{1}{2}p\frac{3(1-2\upsilon)}{E}p = \frac{3(1-2\upsilon)}{2E}p^2 = \frac{(1-2\upsilon)}{6E}(\sigma_1 + \sigma_2 + \sigma_3)^2$$

$$U = \text{strain energy due to principal stresses \& strains}$$

$$U = \frac{1}{2E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) - \frac{\upsilon}{E}(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)$$

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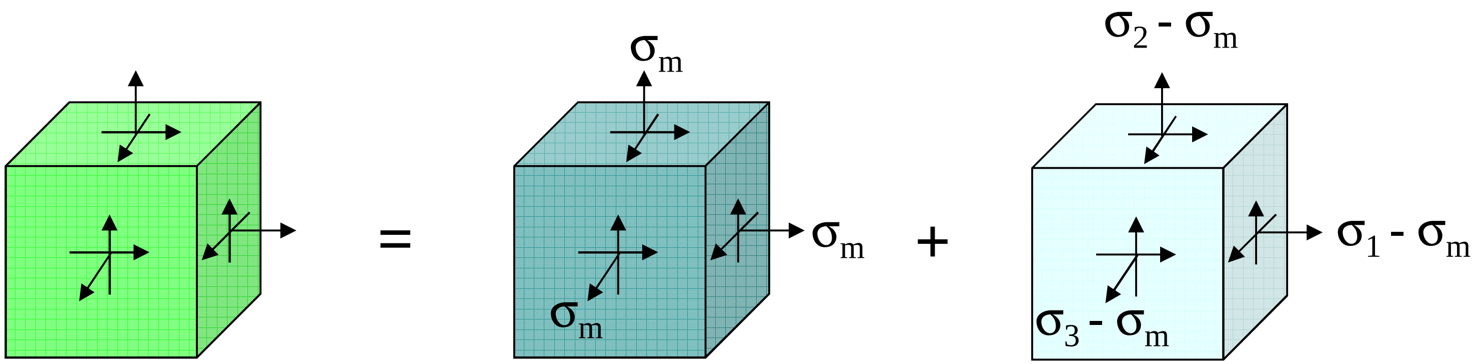
von-Mises theory

- Distortional energy –



$$U_D = U - U_V$$

$$U_D = \frac{1}{2E}\left[\left(\sigma_1^2+\sigma_2^2+\sigma_3^2\right)-2\upsilon\left(\sigma_2\sigma_1+\sigma_2\sigma_3+\sigma_3\sigma_1\right)\right]-\left(\frac{1-2\upsilon}{6E}\right)\left(\sigma_1+\sigma_2+\sigma_3\right)^2$$

Simplifying this

$$U_D = \frac{1}{12G}\left[\left(\sigma_1-\sigma_2\right)^2+\left(\sigma_2-\sigma_3\right)^2+\left(\sigma_3-\sigma_1\right)^2\right]$$

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von-Mises theory

- Compare this with distortion in uniaxial tensile stress

$$U_D == \frac{Y^2}{6G} = \frac{1}{12G}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]$$

$$=> 2Y^2 = (\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2$$

Yield function,

$$f = \sigma_e^2 - Y^2$$

Equivalent stress, $\sigma_e^2 = \frac{1}{2}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]$

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von-Mises theory

Principal stresses of deviatoric shear stress, S_{ii}

$$[\sigma]=\begin{bmatrix}\sigma_1 & 0 & 0\\ 0 & \sigma_2 & 0\\ 0 & 0 & \sigma_3\end{bmatrix}=\begin{bmatrix}\sigma_1-p & 0 & 0\\ 0 & \sigma_2-p & 0\\ 0 & 0 & \sigma_3-p\end{bmatrix}+\begin{bmatrix}p & 0 & 0\\ 0 & p & 0\\ 0 & 0 & p\end{bmatrix}$$

$$[\sigma]=\begin{bmatrix}S_1 & 0 & 0\\ 0 & S_2 & 0\\ 0 & 0 & S_3\end{bmatrix}+\begin{bmatrix}p & 0 & 0\\ 0 & p & 0\\ 0 & 0 & p\end{bmatrix}$$

$$S_{ii} = \sigma_{ii} - p \Rightarrow \sigma_{ii} = S_{ii} + p$$

$$2Y^2 = (\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2$$

$$2Y^2 = ((S_1+p)-(S_2+p))^2+((S_2+p)-(S_3+p))^2+((S_3+p)-(S_1+p))^2$$

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von-Mises theory

Simplifying this expression –

$$2Y^2 = (S_1 - S_2)^2 + (S_2 - S_3)^2 + (S_3 - S_1)^2$$

Hydrostatic pressure does not appear in the expression

- von-Mises criteria has square terms – result independent of signs of individual stress components
- Von-Mises equivalent stress => +ve stress

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von-Mises theory

R&DE (Engineers), DRDO

- 2D stress state => $\sigma_3 = 0$

Yield function, $f = \sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 - Y^2$

Onset of yielding, $\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = Y^2$

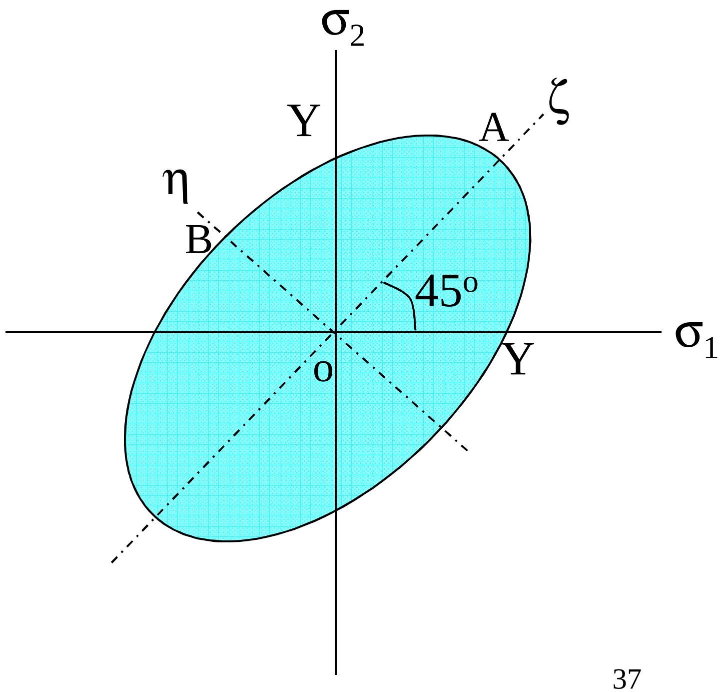
Re-arrange the terms –

$$\left(\frac{\sigma_1}{Y}\right)^2 + \left(\frac{\sigma_2}{Y}\right)^2 - \left(\frac{\sigma_1\sigma_2}{Y^2}\right) = 1$$

This represents an ellipse

Semi-major axis, $OA = \sqrt{2}Y$

Semi-minor axis, $OB = \sqrt{\frac{2}{3}}Y$



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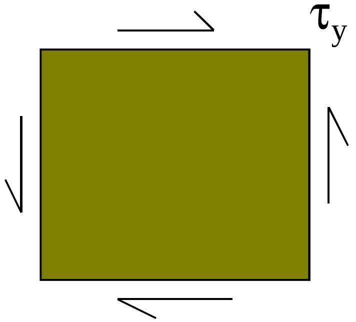
von-Mises theory

- Pure shear –

Principal stresses corresponding to shear yield stress



$\sigma_1 = +\tau_y$, $\sigma_2 = -\tau_y$

$$Y^2 = \sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = 3\tau_y^2 \Rightarrow \tau_y = 0.577Y$$

Shear yield = 0.577 * Tensile yield

Suitable for ductile materials

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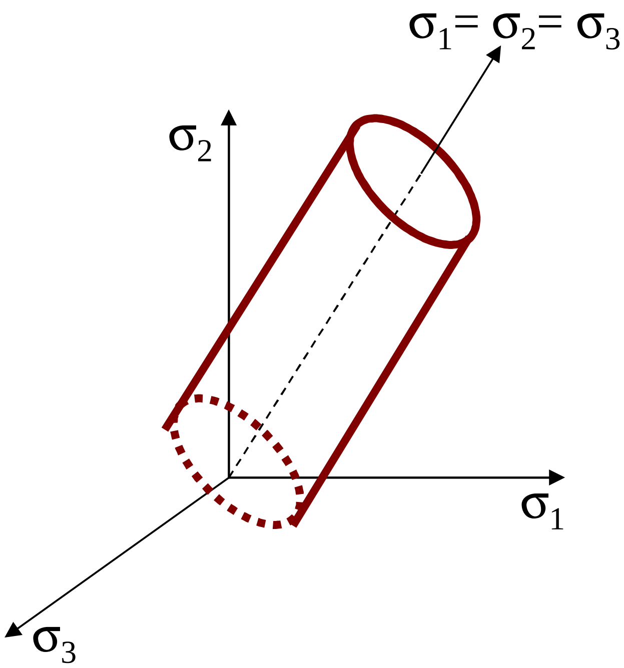
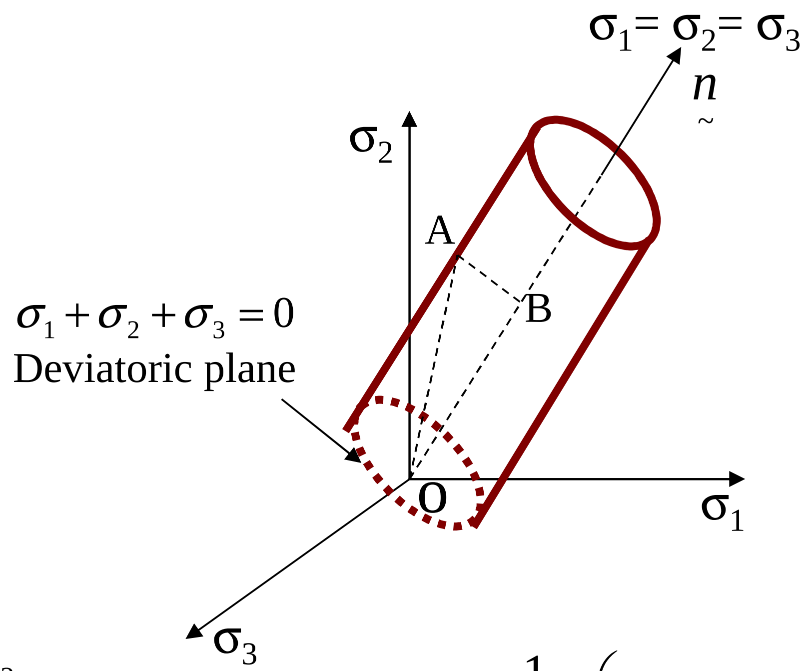
von-Mises theory

- Plot yield function in 3D principal stress space





$$f = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 - 2Y^2 = 0$$

Cylinder, with hydrostatic stress as axis

Axis makes equal DCs with all axes

$$\underset{\sim}{n} = \frac{1}{\sqrt{3}}\left(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}\right)$$

$$\underset{\sim}{OA} = \sigma_1 \underset{\sim}{i} + \sigma_2 \underset{\sim}{j} + \sigma_3 \underset{\sim}{k}$$

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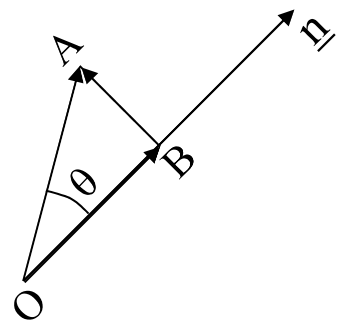
von-Mises theory

- Projection of OA on hydrostatic axis



$$\underset{\sim}{OA}.\underset{\sim}{n} = |OA||n|\cos\theta \Rightarrow OA\ \cos\theta = \frac{\underset{\sim}{OA}.\underset{\sim}{n}}{|n|} = OB$$

$$OB = \frac{\left(\sigma_1 \underset{\sim}{i} + \sigma_2 \underset{\sim}{j} + \sigma_3 \underset{\sim}{k}\right).\left(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}\right)}{\sqrt{3}}$$

$$OB = \frac{1}{\sqrt{3}}(\sigma_1 + \sigma_2 + \sigma_3)$$

$$\Rightarrow \underset{\sim}{OB} = OB\ \underset{\sim}{n} = \frac{1}{\sqrt{3}}(\sigma_1 + \sigma_2 + \sigma_3)\frac{1}{\sqrt{3}}\left(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}\right)$$

$$\underset{\sim}{OB} = \frac{(\sigma_1 + \sigma_2 + \sigma_3)}{3}\left(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}\right) = p\left(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}\right)$$

$$\underset{\sim}{OA} = \underset{\sim}{OB} + \underset{\sim}{BA}$$

$$\underset{\sim}{BA} = \underset{\sim}{OA} - \underset{\sim}{OB}$$

BA = r = radius of cylinder

von-Mises theory

- Radius of cylinder

$$\underset{\sim}{BA} = \underset{\sim}{R} = \underset{\sim}{OA} - \underset{\sim}{OB} = \left(\sigma_1 \underset{\sim}{i} + \sigma_2 \underset{\sim}{j} + \sigma_3 \underset{\sim}{k}\right) - p\left(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}\right)$$

$$\underset{\sim}{R} = \left(\frac{2\sigma_1 - \sigma_2 - \sigma_3}{3}\right)\underset{\sim}{i} + \left(\frac{2\sigma_2 - \sigma_3 - \sigma_1}{3}\right)\underset{\sim}{j} + \left(\frac{2\sigma_3 - \sigma_1 - \sigma_2}{3}\right)\underset{\sim}{k}$$

$$\underset{\sim}{R} = S_1 \underset{\sim}{i} + S_2 \underset{\sim}{j} + S_3 \underset{\sim}{k}$$

$$\text{Radius} \Rightarrow \quad \text{R} = \sqrt{S_1^2 + S_2^2 + S_3^2}$$

$$\text{First invariant of deviatoris stress tensor, } J_1 = 0 \Rightarrow S_1 + S_2 + S_3 = 0$$

$$(S_1 + S_2 + S_3)^2 = 0 = S_1^2 + S_2^2 + S_3^2 = -2(S_1S_2 + S_2S_3 + S_3S_1)$$

$$\text{Yield criteria} \Rightarrow (S_1 - S_2)^2 + (S_2 - S_3)^2 + (S_3 - S_1)^2 = 2Y^2$$

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von-Mises theory

Yield criteria,

$$Y^2 = S_1^2 + S_2^2 + S_3^2 - S_1S_2 - S_2S_3 - S_3S_1$$

Use, $S_1^2 + S_2^2 + S_3^2 = -2(S_1S_2 + S_2S_3 + S_3S_1)$

$$Y^2 = S_1^2 + S_2^2 + S_3^2 + \frac{1}{2}\left(S_1^2 + S_2^2 + S_3^2\right)$$

$$Y^2 = \frac{3}{2}\left(S_1^2 + S_2^2 + S_3^2\right) = \frac{3}{2}R^2$$

$$Y = \sqrt{\frac{3}{2}}R$$

Yielding depends on deviatoric stresses
Hydrostatic stress has no role in yielding

von-Mises theory

- Second invariant of deviatoric stress

$$[S]=\begin{bmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{bmatrix} \Rightarrow J_2 = \begin{vmatrix} S_1 & 0 \\ 0 & S_2 \end{vmatrix} + \begin{vmatrix} S_2 & 0 \\ 0 & S_3 \end{vmatrix} + \begin{vmatrix} S_1 & 0 \\ 0 & S_3 \end{vmatrix}$$

$$J_2 = S_1S_2 + S_2S_3 + S_3S_1$$

$$S_1S_2 + S_2S_3 + S_3S_1 = -\frac{1}{2}\left(S_1^2 + S_2^2 + S_3^2\right)$$

$$J_2 = -\frac{1}{2}\left(S_1^2 + S_2^2 + S_3^2\right) \Rightarrow |J_2| = \frac{1}{2}\left(S_1^2 + S_2^2 + S_3^2\right) = \frac{R^2}{2}$$

$$Y^2 = \frac{3}{2}R^2 \Rightarrow Y^2 = 3|J_2|$$

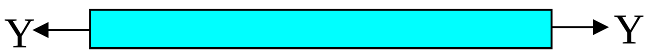
Redfining yield function => $f = 3|J_2| - Y^2$

J_2 Materials

Max. shear stress theory (Tresca)

- "Yielding begins when the maximum shear stress at a point equals the maximum shear stress at yield in a uniaxial tension"

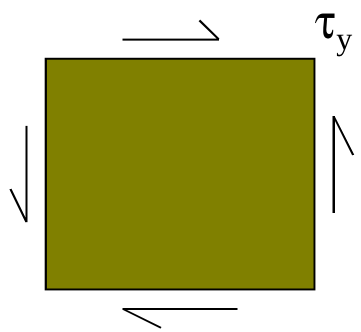
$$\tau_{\max} = K_T = \left|\frac{\sigma_1 - \sigma_2}{2}\right| \Rightarrow \tau_{\max} = \frac{Y}{2} = K_T$$



If maximum shear stress < Y/2 => No failure occurs

For pure shear, $\sigma_1 = +\tau_y$, $\sigma_2 = -\tau_y$

$$\tau_{\max} = K_T = \left|\frac{\sigma_1 - \sigma_2}{2}\right| \Rightarrow \tau_{\max} = \tau_y = K_T$$



Shear yield = 0.5 Tensile yield

Tresca theory

- In 3D stress state – principal stresses => σ_1, σ_2 and σ_3
- Maximum shear stress

$$\max.\left\{\left|\frac{\sigma_1-\sigma_2}{2}\right|,\ \left|\frac{\sigma_2-\sigma_3}{2}\right|,\ \left|\frac{\sigma_3-\sigma_1}{2}\right|\right\}$$

- Yield function

$$f=\max.\left\{\left|\frac{\sigma_1-\sigma_2}{2}\right|,\ \left|\frac{\sigma_2-\sigma_3}{2}\right|,\ \left|\frac{\sigma_3-\sigma_1}{2}\right|\right\}-K_T\left(=\frac{Y}{2}\right)$$

$$f<0 \quad \Rightarrow \text{ No yielding}$$

$$f=0 \quad \Rightarrow \text{Onset of yielding}$$

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Tresca theory

R&DE (Engineers), DRDO

- Following equations are obtained

$$\left|\frac{\sigma_1 - \sigma_2}{2}\right| = K_T \;\Rightarrow\; \frac{\sigma_1 - \sigma_2}{2} = \pm K_T$$

$$f_1(\sigma_1, \sigma_2) = \sigma_1 - \sigma_2 - 2K_T; \qquad f_2(\sigma_1, \sigma_2) = \sigma_1 - \sigma_2 + 2K_T$$

$$\left|\frac{\sigma_2 - \sigma_3}{2}\right| = K_T \;\Rightarrow\; \frac{\sigma_2 - \sigma_3}{2} = \pm K_T$$

$$f_3(\sigma_2, \sigma_3) = \sigma_2 - \sigma_3 - 2K_T; \qquad f_4(\sigma_2, \sigma_3) = \sigma_2 - \sigma_3 + 2K_T$$

$$\left|\frac{\sigma_3 - \sigma_1}{2}\right| = K_T \;\Rightarrow\; \frac{\sigma_3 - \sigma_1}{2} = \pm K_T$$

$$f_5(\sigma_3, \sigma_1) = \sigma_3 - \sigma_1 - 2K_T; \qquad f_6(\sigma_3, \sigma_1) = \sigma_3 - \sigma_1 + 2K_T$$

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Tresca theory

- Redefining yield function as,

$$f(\sigma_1,\sigma_2,\sigma_3) = f_1(\sigma_1,\sigma_2).f_2(\sigma_1,\sigma_2).f_3(\sigma_2,\sigma_3).f_4(\sigma_2,\sigma_3).f_1(\sigma_1,\sigma_2)$$

$$f(\sigma_1,\sigma_2,\sigma_3) = (\sigma_1-\sigma_2-2K_T)(\sigma_1-\sigma_2+2K_T)$$
$$(\sigma_2-\sigma_3-2K_T)(\sigma_2-\sigma_3+2K_T)$$
$$(\sigma_3-\sigma_1-2K_T)(\sigma_3-\sigma_1+2K_T)$$

Each function represents a plane in 3D principal stress space

$$f(\sigma_1,\sigma_2,\sigma_3) = \left((\sigma_1-\sigma_2)^2-4K_T^2\right)\left((\sigma_2-\sigma_3)^2-4K_T^2\right)\left((\sigma_3-\sigma_1)^2-4K_T^2\right)$$

No effect of hydrostatic pressure in Tresca criteria

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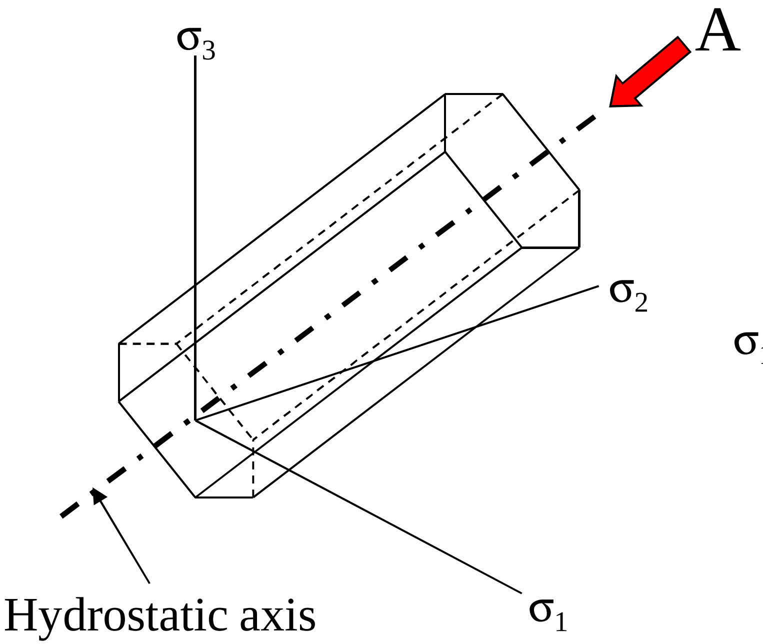
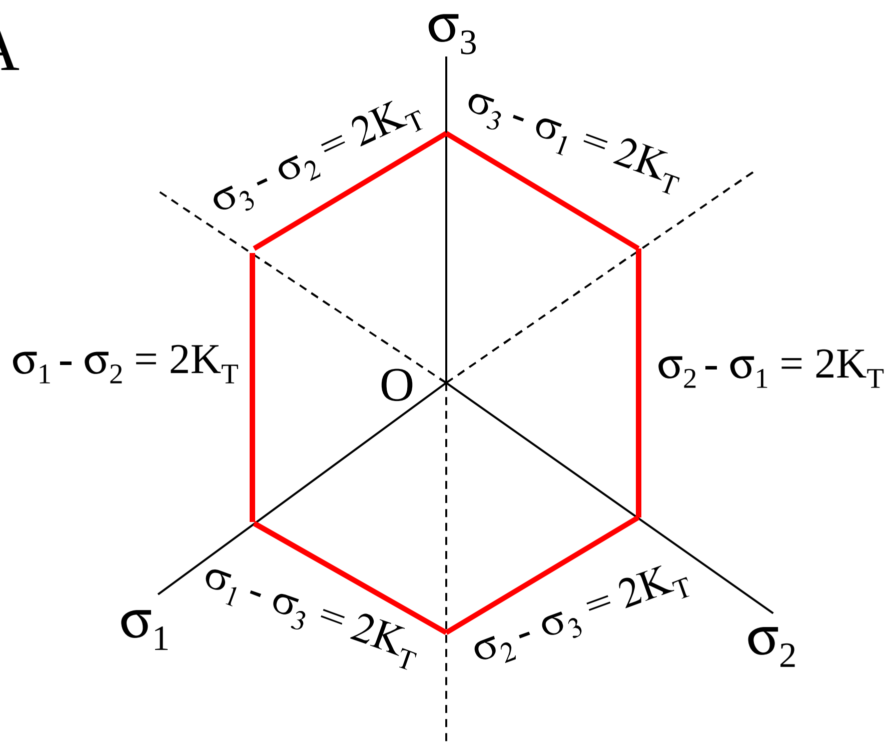
Tresca theory

- Yield function in principal stress space



Tresca yield surface



View 'A' – along hydrostatic axis

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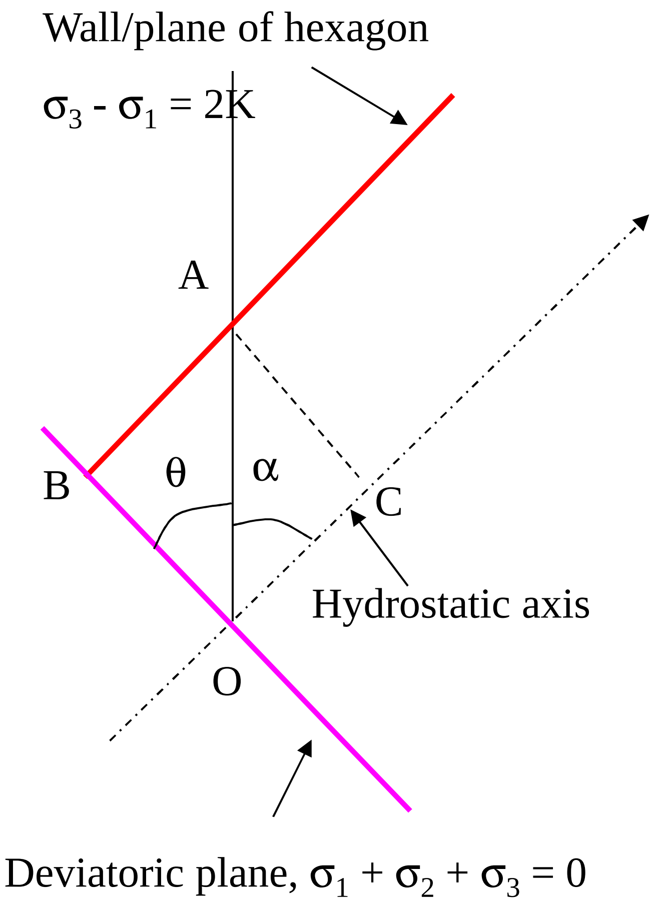
Tresca theory

- Yield surface intersects principal axes at $2K_T$



$$\text{Hydrostati c axis} \Rightarrow \sigma_1 = \sigma_2 = \sigma_3$$

$$\cos\alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = 54.73^0$$

$$\alpha + \theta = 90^0 \Rightarrow \theta = 35.26^0$$

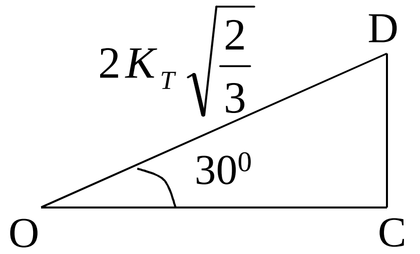
$$OA = 2K_T$$

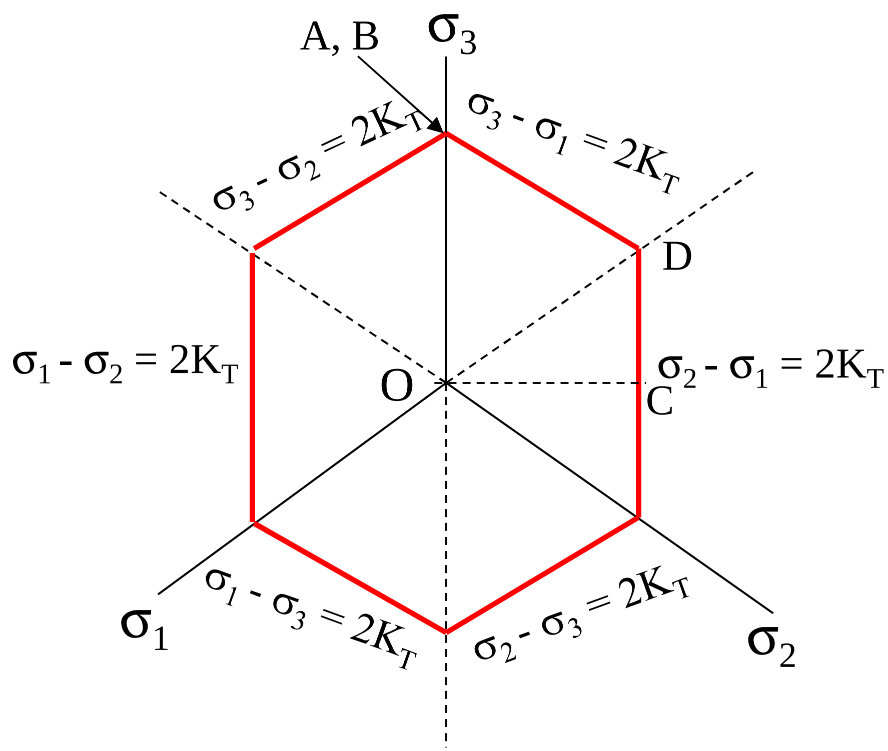
$$OB = OA\cos\theta = 2K_T\sqrt{\frac{2}{3}}$$

OB – projection of OA on deviatoric plane

Tresca theory

- Tresca hexagon





$$OC = OD\cos 30$$

$$\Rightarrow OC = 2K_T\sqrt{\frac{2}{3}}\,\frac{\sqrt{3}}{2}$$

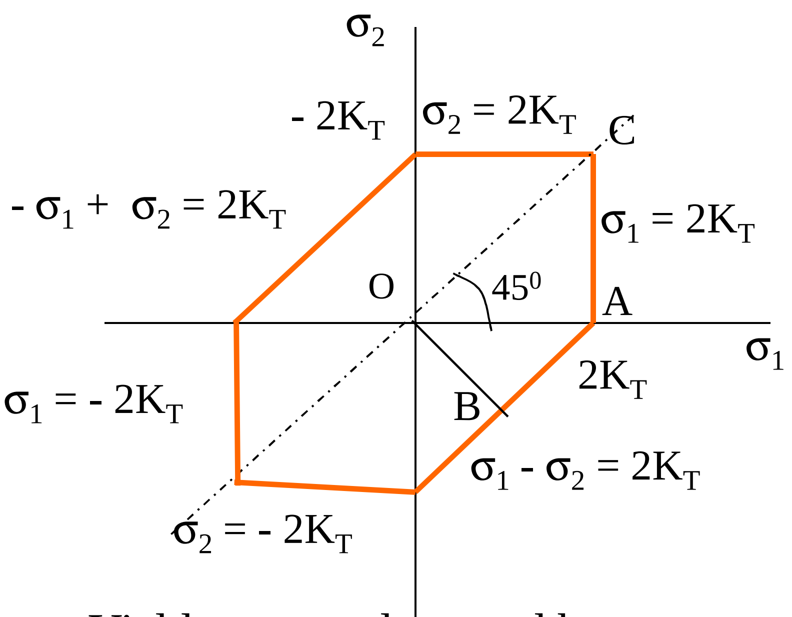
$$\Rightarrow OC = \sqrt{2}K_T$$

Tresca theory

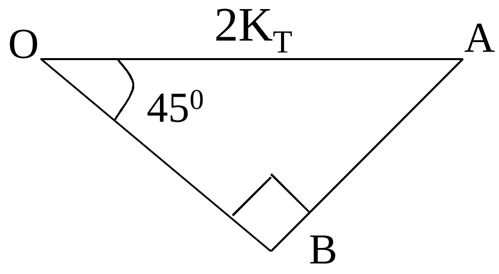
- 2D stress state - $\sigma_3 = 0$

$$\sigma_1 - \sigma_2 = \pm 2K_T$$
$$\sigma_2 = \pm 2K_T$$
$$\sigma_1 = \pm 2K_T$$

Each equation represents two lines in 2D stress space



Yield curve – elongated hexagon



$$OB = OA\cos 45 = 2K_T \frac{1}{\sqrt{2}} = \sqrt{2}K_T$$

$$OC = \frac{OA}{\cos 45} = 2\sqrt{2}K_T$$

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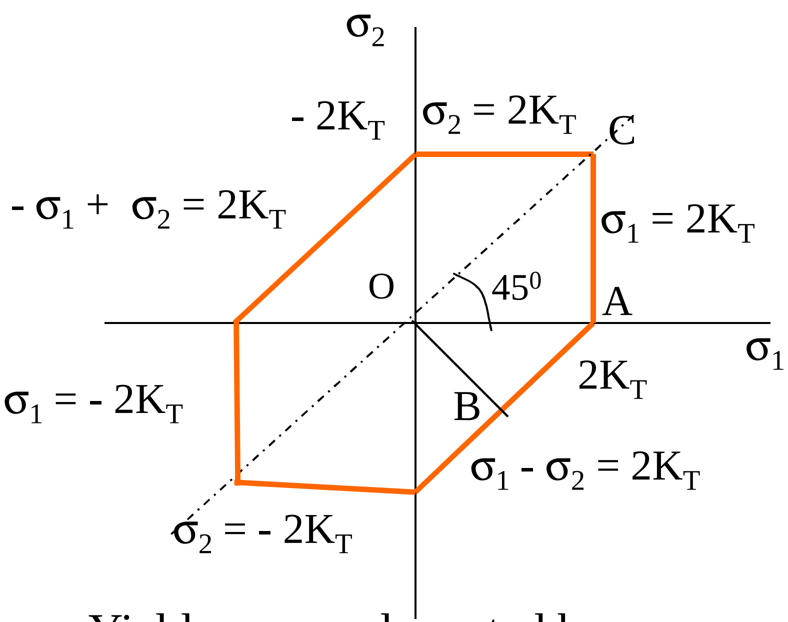
Tresca theory

- 2D stress state - $\sigma_3 = 0$

Each equation represents two lines in 2D stress space



Yield curve – elongated hexagon

$$\sigma_1 - \sigma_2 = \pm 2K_T$$

$$\sigma_2 = \pm 2K_T$$

$$\sigma_1 = \pm 2K_T$$

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von-Mises – Tresca theories

- Pure tension – $\sigma_1 = Y, \sigma_2 = 0, \sigma_3 = 0$

von-Mises criteria => $J_2 = K_M^2 = \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$

Tresca's criteria => $K_T = \max\left\{\left|\frac{\sigma_1 - \sigma_2}{2}\right|, \left|\frac{\sigma_2 - \sigma_3}{2}\right|, \left|\frac{\sigma_3 - \sigma_1}{2}\right|\right\}$

$$K_M = \frac{1}{\sqrt{3}} Y, \quad K_T = \frac{1}{2} Y$$

Pure shear => $\sigma_1 = +\tau_y, \sigma_2 = -\tau_y$ => $K_M = \tau_y = K_T$

$$K_M = \frac{1}{\sqrt{3}} Y = \tau_y, \quad K_T = \frac{1}{2} Y = \tau_y$$

$$\tau_y = 0.577Y \text{ (von – Mises)}, \quad \tau_y = 0.5Y \text{ (Tresca)}$$

von-Mises criteria predicts 15% higher shear stress than Tresca

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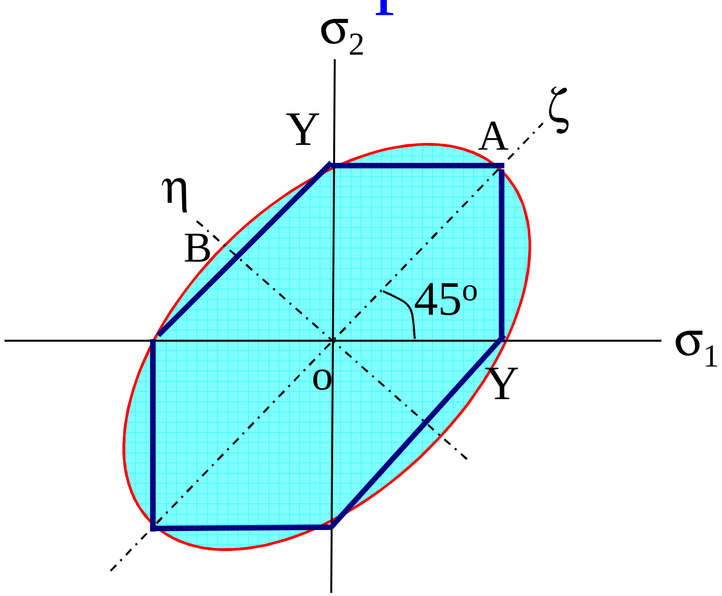
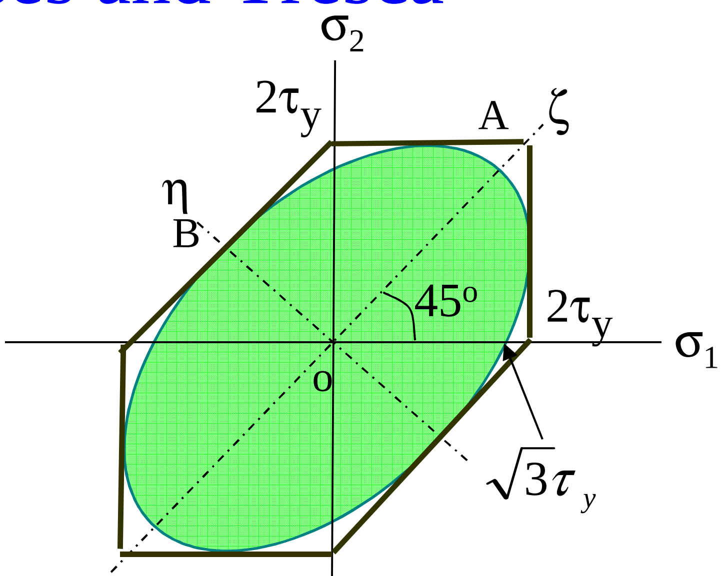
von-Mises – Tresca theories

- 2D stress space – von-Mises and Tresca





$$3\tau_y^2 = \sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2$$

$$Y^2 = \sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2$$

$$Y = \max.\left\{\left|\sigma_1 - \sigma_2\right|,\ \left|\sigma_1\right|,\ \left|\sigma_2\right|\right\}$$

$$\frac{\tau_y}{2} = \max.\left\{\left|\frac{\sigma_1 - \sigma_2}{2}\right|,\ \left|\frac{\sigma_1}{2}\right|,\ \left|\frac{\sigma_2}{2}\right|\right\}$$

Yielding in uniaxial tension

Tresca – conservative

Yielding in shear

von-Mises – conservative

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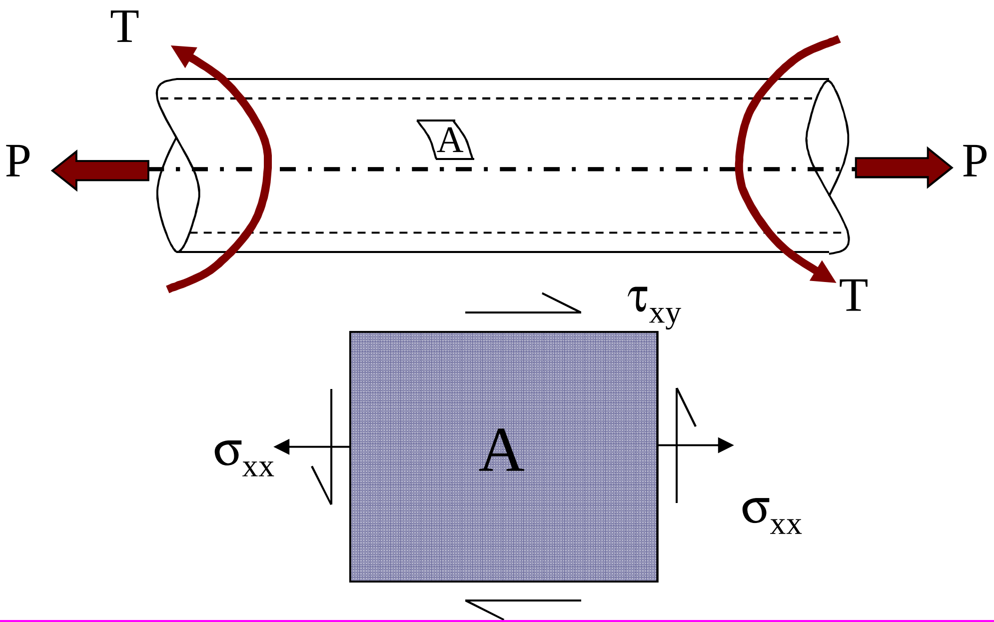
von-Mises – Tresca theories

- Experiments by Taylor & Quinney**
- Thin walled tube subjected to axial and torsional loads



$$\sigma_1 = \frac{\sigma_{xx}}{2} + \sqrt{\left(\frac{\sigma_{xx}}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = \frac{\sigma_{xx}}{2} - \sqrt{\left(\frac{\sigma_{xx}}{2}\right)^2 + \tau_{xy}^2}$$

** Taylor and Quinney "Plastic deformation of metals", Phil. Trans. Roy. Soc.A230, 323-362, 1931

von-Mises – Tresca theories

- Tresca criteria

$$Y = |\sigma_1 - \sigma_2| = 2\sqrt{\left(\frac{\sigma_{xx}}{2}\right)^2 + \tau_{xy}^2} \Rightarrow Y^2 = \sigma_{xx}^2 + 4\tau_{xy}^2$$

$$\text{No yielding if,} \quad \left(\frac{\sigma_{xx}}{Y}\right)^2 + \left(\frac{\tau_{xy}}{Y/2}\right)^2 < 1$$

- von-Mises criteria

$$Y^2 = \sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2$$

$$Y^2 = \sigma_{xx}^2 + 3\tau_{xy}^2$$

$$\text{No yielding if,} \quad \left(\frac{\sigma_{xx}}{Y}\right)^2 + \left(\frac{\tau_{xy}}{Y/\sqrt{3}}\right)^2 < 1$$

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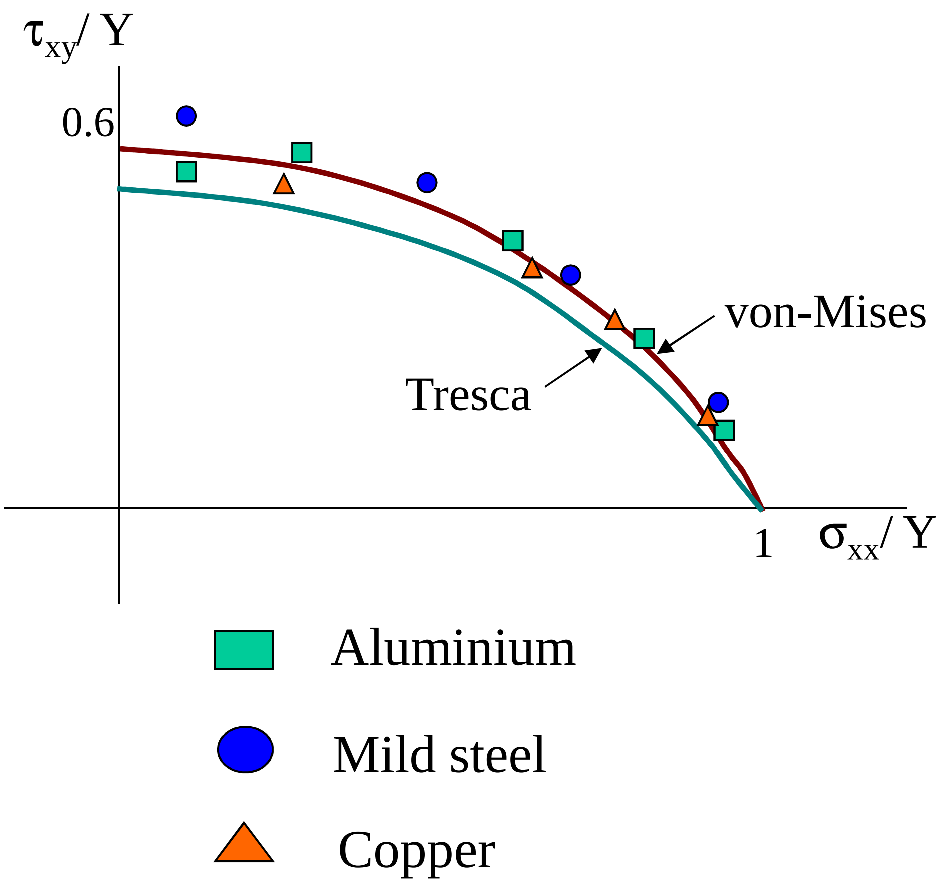
von-Mises – Tresca theories

- Plotting these two criteria –



Experimental data shows good agreement with von-Mises theory.

Tresca – conservative

von-Mises theory more accurate – generally used in design

Experiments show that for ductile materials yield in shear is 0.5 to 0.6 times of yield in tensile

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Octahedral shear stress theory

- Octahedral plane – makes equal angles with all principal stress axes – direction cosines same
- Shear stress acting on this plane – octahedral shear

$$\tau_{oct}^2 = \frac{1}{9}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

Body subjected to pure tension, $\sigma_1 = Y$, $\sigma_2 = \sigma_3 = 0$

$$\tau_{oct}^2 = \frac{2}{9} Y^2$$

$$2Y^2 = \left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

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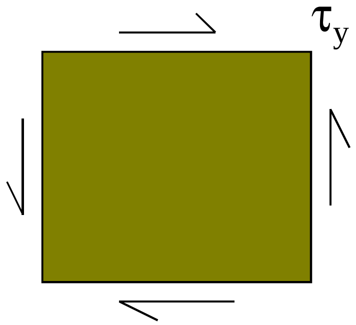
Octahedral shear stress theory

- Comparing this with von-Mises theory => both are same
- Pure shear



$$\sigma_1 = \tau_y,\ \sigma_2 = -\tau_y,\ \sigma_3 = 0$$

$$\tau_{oct}^2 = \frac{1}{9}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right] = \frac{6}{9}\tau_y^2$$

Same as von-Mises theory in pure shear

$$\tau_{oct}^2 = \frac{2}{3}\tau_y^2$$

$$6\tau_y^2 = \left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]$$

Octahedral shear stress theory => von-Mises theory

Tensile & shear yield strengths

- Each failure theory gives a relation between yielding in tension and shear ($\upsilon = 0.25$)

Failure Criteria

Theory	Loading: Uniaxial	Loading: Pure Shear	Relationship
Maximum principal stress	$\sigma_{max} = \sigma_{YP}$	$\sigma_{max} = \tau_{YP}$	$\tau_{YP} = \sigma_{YP}$
Maximum principal strain	$\varepsilon_{max} = \sigma_{YP} / E$	$\varepsilon_{max} = 5\tau_{YP} / 4E$	$\tau_{YP} = 0.8\ \sigma_{YP}$
Maximum octahedral shear stress	$\tau_{oct} = \sigma_{YP} \frac{\sqrt{2}}{3}$	$\tau_{oct} = \tau_{YP} \sqrt{\frac{2}{3}}$	$\tau_{YP} = 0.577\ \sigma_{YP}$
Maximum distortional energy density			$\tau_{YP} = 0.577\ \sigma_{YP}$
Maximum shear stress	$\tau_{max} = \sigma_{YP} / 2$	$\tau_{max} = \tau_{YP}$	$\tau_{YP} = 0.5\ \sigma_{YP}$

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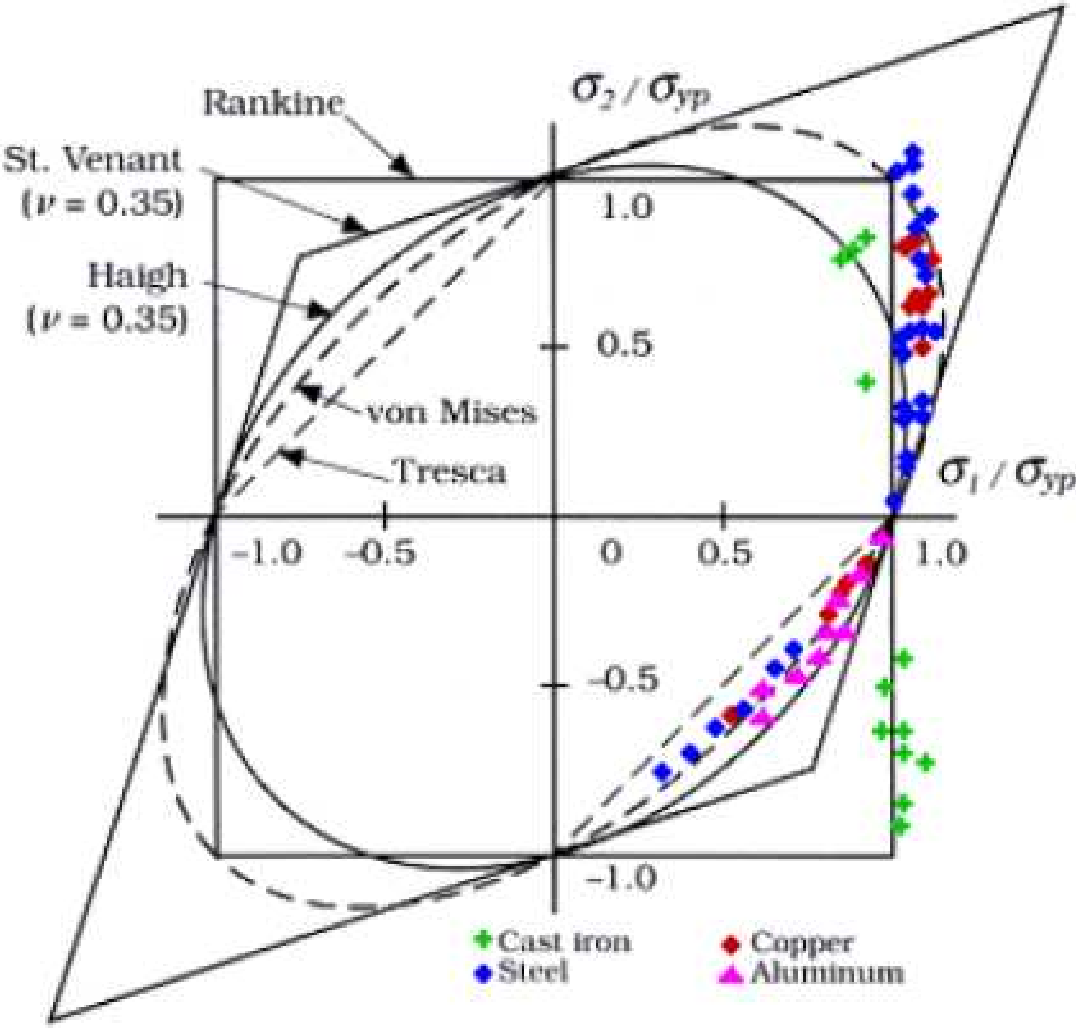
Failure theories in a nut shell



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