CE 6504
Finite Elements Method in Structures
(Part 1)

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Course Outline

1. **Introduction**
2. **Preliminaries**
3. **1D (2-Node) Line Elements**
   Bar, Truss, Beam-elements, Shape functions
4. **2D Elements**
   Plane Stress and Plane Strain Problems
5. **3D Elements**
   Tetrahedral, Hexahedral Elements
6. **Plate Bending & Shells**
7. **Further Issues**
   Modeling, Errors, Non-linearity

References

Finite Element Analysis (Text)
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By: K.-J. Bathe
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O.C. Zienkiewicz
An Introduction to the Finite Element Method
By: J. N. Reddy

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Department of Aerospace Engineering Sciences
University of Colorado at Boulder
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Advanced Finite Element Methods
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I Introduction

What is FEM?

Finite element method is a numerical method that generates approximate solutions to engineering problems which are usually expressed in terms of differential equations.

Used for stress analysis, heat transfer, fluid flow, electromagnetic etc.

What is FEM?

- Use of several materials within the same structure,
- complicated or discontinuous geometry,
- complicated loading, etc,

➔makes the closed form (analytical) solution of structural problems very difficult.

One resorts to a numerical solution, the best of which is the FEM.

What is FEM?

Structure is partitioned into FINITE ELEMENTS – that are joined to each other at limited number of NODES

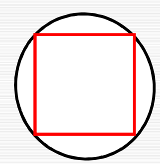
Behavior of an individual element can be described with a simple set of equations

Assembling the element equations, to a large set, is supposed to describe the behavior of the whole structure.

Discretization Example

Find the circumference of a circle with a unit diameter – find the value of π.

Approximation with that of regular polygons:



Discretization Example

Solution:

Let *n* be the number of sides of the inscribed or circumscribing polygon.

i) Inscribed polygon

Perimeter p = *n* sin (180/*n*)

ii) Circumscribing polygon

Perimeter p = *n* tan (180/*n*)

Discretization Example

Estimated vs. exact value of π = **3.1415926536**

No. of sides	Inscribed polygon	Error
3	2.5980762114	0.5435164422
4	2.8284271247	0.3131655288
8	3.0614674589	0.0801251947
16	3.1214451523	0.0201475013
32	3.1365484905	0.0050441630
64	3.1403311570	0.0012614966
128	3.1412772509	0.0003154027
1000	3.1415874859	0.0000051677
10000	3.1415926019	0.0000000517
100000	3.1415926531	0.0000000005
1000000	3.1415926536	0.0000000000

Brief History

A formal mathematical theory for the FEM started some 60 years ago

The steps in FEA are very similar to the method of the *direct stiffness method* in matrix structural analysis

- The term *finite element* was first used by Clough in 1960.
- The first book on the FEM by Zienkiewicz and Cheung was published in 1967.
- In the late 1960s and early 1970s, the FEM was applied to a wide variety of engineering problems.

Brief History

- **The Pioneers** – 1950 to 1962; Clough, Turner, Argyris, etc.; thought structural elements as a device to transmit forces (“force transducer”).
- **The Golden Age** – 1962–1972; Zienkiewicz, Cheung, Martin, Carey etc.; thought discrete elements approximate continuum models (displacement formulation).

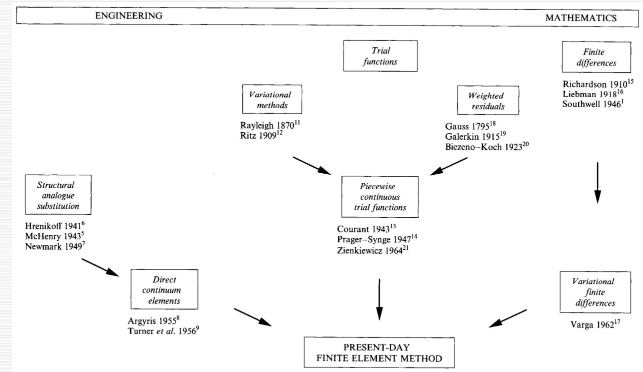
Brief History

- **Consolidation** – 1972 to mid 1980s; Hughes, Bathe Argyris, etc.; variational method, mixed formulation, error estimation.
- **Back to Basics** – early 1980s to the present; Elements are kept simple but should provide answers of engineering accuracy with relatively coarse meshes.

Brief History

- The 1970s ➔ advances in mathematical treatments, including the development of new elements, and convergence studies.
- Most commercial FEM software packages originated in the 1970s and 1980s.
- The FEM is one of the most important developments in computational methods to occur in the 20th century.

Brief History



Proprietary Software

- ANSYS
- MSC/NASTRAN
- ABAQUS
- ADINA
- ALGOR
- NISA
- COSMOS/M
- STARDYNE
- IMAGES-3D

Typical Steps in FEA Process

1. Discretize and Select Element Type
2. Select a Displacement Function
3. Define Strain/Displacement and Stress/Strain Relationships
4. Derive Element Stiffness Matrix & Eqs.
5. Assemble Equations and Introduce B.C.'s
6. Solve for the Unknown Degrees of Freedom
7. Solve for Element Stresses and Strains
8. Interpret the Results

Common FEA Procedure for Structures

<u>0. Idealization</u>

The given structure needs to be idealized based on engineering judgment. Identify the governing equation.

<u>1. Discretization</u>

The continuum system is disassembled into a number of small and manageable parts (finite elements).

Common FEA Procedure for Structures

<u>2 – 4. Derivation of Element Equations</u>

Derive the relationship between the unknown and given parameters at the nodes of the element.

$$\{f\}^e = [k]^e \{u\}^e$$

<u>5a. Assembly</u>

Assembling the global stiffness matrix from the element stiffness matrices based on compatibility of displacements and equilibrium of forces.

For example:

Common FEA Procedure for Structures

$$U_i = u_2^1 = u_1^2$$

$$V_i = v_2^1 = v_1^2$$

➔ Displacement of a node is always the same for the adjoining elements and for the whole structure.

Common FEA Procedure for Structures

$$F_x^i = fx_2^1 + fx_1^2$$

$$F_y^i = fy_2^1 + fy_1^2$$

➔ The sum of the forces on each element of a particular node must balance the external force at that node.

For the whole structure, this process results in the master stiffness equation:

$$\left[K'\right]\left\{U'\right\} = \left\{F'\right\}$$

Common FEA Procedure for Structures

5b. Introduce Boundary Conditions

After applying prescribed nodal displacements (and known external forces) to the master stiffness equation, the resulting equation becomes the modified master stiffness equation:

$$\left[K\right]\left\{U\right\} = \left\{F\right\}$$

Common FEA Procedure for Structures

6. Solve for the primary unknowns

$$\left\{U\right\} = \left[K\right]^{-1}\left\{F\right\}$$

7/8. Compute other values of interest

Secondary unknowns are determined using the known nodal displacements. Result interpretation.

Types of FEA in Structures

1. Linear analysis: small deflection and elastic material properties.
2. Non-linear analysis:

- Material non-linearity: small deflection and non-linear material properties.
- Geometric non-linearity: large deflection and elastic material properties.
- Both material and geometric non-linearity.

Advantages of Finite Element Analysis

- *Models Bodies of Complex Shape*
- *Can Handle General Loading/Boundary Conditions*
- *Models Bodies Composed of Composite Materials*
- *Model is Easily Refined for Improved Accuracy by Varying Element Size and Type (Approximation Scheme)*
- *Time Dependent and Dynamic Effects Can Be Included*
- *Can Handle a Variety Nonlinear Effects*

Common Sources of Error in FEA

- *Domain Approximation*
- *Element Interpolation/Approximation*
- *Numerical Integration Errors*
  (Including Spatial and Time Integration)
- *Computer Errors (Round-Off, Etc.,)*

Measures of Accuracy in FEA

<u>Accuracy</u>

Error = |(Exact Solution)-(FEM Solution)|

<u>Convergence</u>

Limit of Error as:

Number of Elements (*h-convergence*)
or
Approximation Order (*p-convergence*)

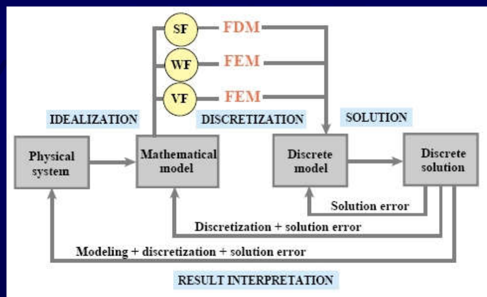
Increases

Ideally, Error → 0 as Number of Elements or Approximation Order is Higher

Numerical Methods

- Several approaches can be used to transform the physical formulation of the problem to its finite element discrete analogue.
- Ritz/ Galerkin methods – the physical formulation of the problem is known as a differential equation.
- Variational formulation – the physical problem can be formulated as minimization of a functional.

Variational Method cont.



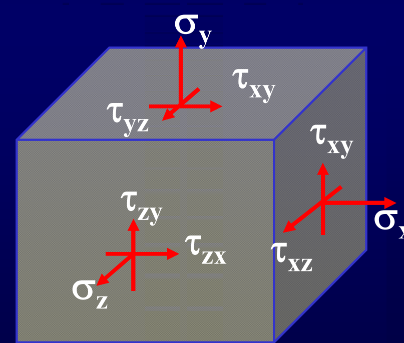
- A **mathematical model** is a set of mathematical statements which attempts to describe a given physical system.

Variational Method cont.

- **Strong Form (SF)**: A system of *ordinary or partial differential equations* in space and/or time, complemented by appropriate boundary conditions.
- **Weak Form (WF)**: A *weighted integral equation* that "relaxes" the strong form into a domain-averaging statement.
- **Variational Form (VF)**: A *functional* whose stationary conditions generate the weak and strong forms.

Elasticity

- 3D Stress block



Elasticity cont.

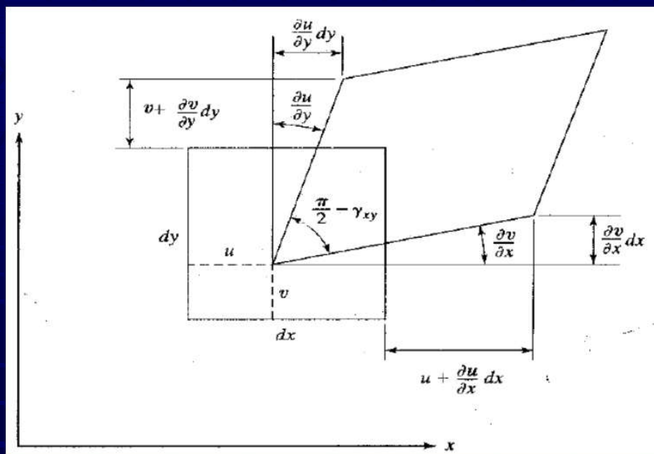
- Stress Equilibrium Equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X_b = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + Y_b = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z_b = 0$$

Elasticity cont.



Elasticity cont.

- Strain – Displacement

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

(u,v,w) are the x, y and z components of displacement

Elasticity cont.

- 3D Stress – Strain Relationships

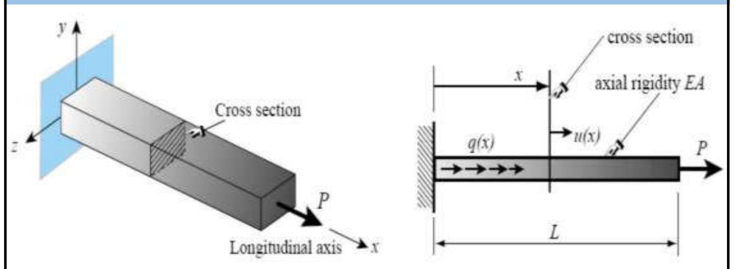
$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = [D] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

Bar Elements

(1) The *longitudinal dimension* or *axial dimension* is much larger that the *transverse dimension(s)*. The intersection of a plane normal to the longitudinal dimension and the bar defines the *cross sections*.

(2) The bar resists an internal axial force along its longitudinal dimension.

Bar Elements cont.



Bar Elements cont.

Quantity	Meaning
x	Longitudinal bar axis*
$(.)'$	$d(.)/dx$
$u(x)$	Axial displacement
$q(x)$	Distributed axial force, given per unit of bar length
L	Total bar length
E	Elastic modulus
A	Cross section area; may vary with x
EA	Axial rigidity
$e = du/dx = u'$	Infinitesimal axial strain
$\sigma = Ee = Eu'$	Axial stress
$p = A\sigma = EAe = EAu'$	Internal axial force
P	Prescribed end load

Bar Elements cont.

- It must be in equilibrium.
- It must satisfy the elastic stress–strain law (Hooke's law)
- The displacement field must be compatible.
- It must satisfy the strain–displacement equation.

Governing Equation

The governing differential equation of the bar element is given by

$$\frac{d}{dx}\left(AE\frac{du}{dx}\right) + q = 0$$

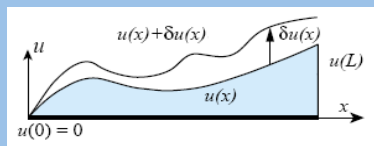
boundary conditions

$$u\big|_{x=0} = 0$$

$$\left(AE\frac{du}{dx}\right)\bigg|_{x=L} = P$$

Approximate Solution

- Admissible displacement function is <u>continuous</u> over the length and satisfies any <u>boundary condition</u>*:*



- **Principles of Minimum Potential Energy —** *Of all kinematically admissible displacement equations, those corresponding to equilibrium extermize the TPE. If the extremum is a minimum, the equilibrium state is stable.*

Kinematically admissible Displacement Functions

those that satisfy the single-valued nature of displacements (compatibility) and the boundary conditions

Usually Polynomials

Continuous within element.

Inter-element compatibility. Prevent overlap or gaps.

Allow for rigid body displacement and constant strain.

Total Potential Energy (TPE)

$$\pi_p = U - W$$

Strain Energy (Internal work)

$$dU = \sigma_x (\Delta x)(\Delta y)(\Delta z) d\varepsilon_x$$

$$dU = \sigma_x d\varepsilon_x dV$$

$$U = \iiint_V \left\{ \int_0^{\varepsilon_x} \sigma_x d\varepsilon_x \right\} dV$$

$$U = \frac{1}{2} \iiint_V \sigma_x \varepsilon_x dV$$

Total Potential Energy

External work of loads

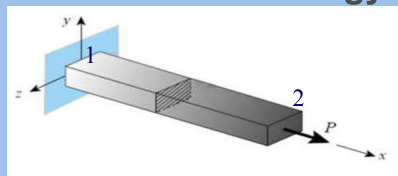
$$\pi_p = U - W$$

$$W = \iiint_V \hat{X}_b \, \hat{u} \, dV + \iint_S \hat{T}_x \, u \, dS + \sum_{i=1}^{M} \hat{f}_i \, \hat{d}_i$$

TPE

$$\pi_p = \frac{1}{2} \int_L \varepsilon_x (AE) \varepsilon_x dx - W$$

Total Potential Energy



- Strain energy $\frac{1}{2}\int_L \sigma^T \varepsilon \, Adx$
- External energy Pu_2
- Total potential energy $\Pi = \frac{1}{2}\int_L \sigma^T \varepsilon \, Adx - Pu_2$

Ritz-Method

- For continua, the total potential energy, π_P, can be used to finding an approximate solution. The Rayleigh-Ritz method involves the construction of an assumed displacement field [u, v, w]:

$$u = \sum a_i \phi_i(x,y,z) \quad i = 1 \; to \; l$$

$$v = \sum a_j \phi_j(x,y,z) \quad j = l+1 \; to \; m$$

$$w = \sum a_k \phi_k(x,y,z) \quad k = m+1 \; to \; n \qquad n > m > l$$

Ritz-Method

Using the Ritz-method, approximate displacement function is obtained by:

Assume arbitrary displacement

$$\Phi = a_1 f_1 + a_2 f_2 + \ldots + a_n f_n$$

Introduce this into the TPE functional

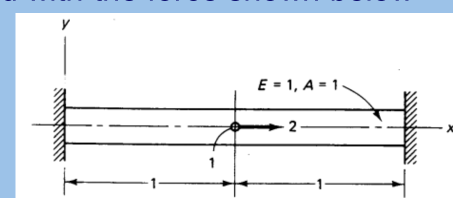
Performing differentiation and integration to obtain a function

Minimizing the resulting function

$$\frac{d\Pi}{da_i} = 0 \quad for \quad i = 1, 2, \ldots, \; n$$

Ritz-Method

Consider the linear elastic one-dimensional rod with the force shown below



- The potential energy of this system is:

$$\Pi = \tfrac{1}{2}\int_0^2 EA\left(\frac{du}{dx}\right)^2 dx - 2u_1$$

Ritz-Method

Consider the polynomial function:

$$u = a_1 + a_2 x + a_3 x^2$$

To be kinematically admissible u must satisfy the boundary conditions u = 0 at both (x = 0) and (x = 2)

$$a_1 = 0 \quad \& \quad 2a_2 + 4a_3 = 0$$

Thus:

$$u = a_3(-2x + x^2)$$

$$\frac{du}{dx} = a_3 2(x-1)$$

Ritz-Method

- TPE of this system becomes:

$$\Pi = \tfrac{4}{3} a_3^2 + 2a_3$$

- Minimizing the TPE:

$$\frac{\partial \Pi}{\partial a_3} = \frac{8}{3} a_3 + 2 = 0$$

$$\Rightarrow a_3 = -\tfrac{3}{4}$$

- Thus, an approximate u is given by:

$$u(x) = 0.75\left(2x - x^2\right)$$

- Rayleigh-Ritz method assumes trial functions over entire structure

Galerkin-Method

For the one-dimensional rod considered in the pervious example, the governing equation is:

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) = 0$$

The Galerkin method aims at setting the residual relative to a weighting function Wi, to zero. The weighting functions, Wi, are chosen from the basis functions used for constructing *û (approximate displacement function)*.

Galerkin-Method

Using the Galerkin-method, approximate displacement function is obtained by:

The governing DE is written in residual form

$$RES = \frac{d}{dx}\left(AE\frac{du}{dx}\right)$$

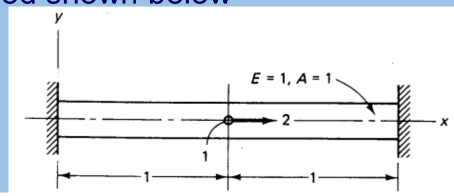
Multiply this ***RES*** by weight function *f* and integrate and equate to zero

Perform differentiation and integration to obtain the approximate u

$$\int_L f \; R \; dx = 0$$

Galerkin-Method

Consider the rod shown below



$$\frac{d}{dx}EA\frac{du}{dx}=0 \qquad u=0 \text{ at } x=0 \qquad u=0 \text{ at } x=2$$

- Multiplying by Φ (weighting function) and integrating gives (by parts):

$$-\int_0^2 EA\frac{du}{dx}\frac{d\phi}{dx}dx+\left(\phi EA\frac{du}{dx}\right)_0^1+\left(\phi EA\frac{du}{dx}\right)_1^2=0$$

Galerkin-Method

The function Φ is zero at (x = 0) and (x = 2) and EA(du/dx) is the force in the rod, which equals 2 at (x = 1). Thus:

$$-\int_0^2 EA\frac{du}{dx}\frac{d\phi}{dx}dx+2\phi_1=0$$

- Using the same polynomial function for u and Φ and if u_1 and Φ_1 are the values at (x = 1):

$$u=\left(2x-x^2\right)u_1 \qquad \phi=\left(2x-x^2\right)\phi_1$$

- *Setting these and E=A=1 in the integral:*

$$\phi_1\left[-u_1\int_0^2 (2-2x)^2dx+2\right]=0$$

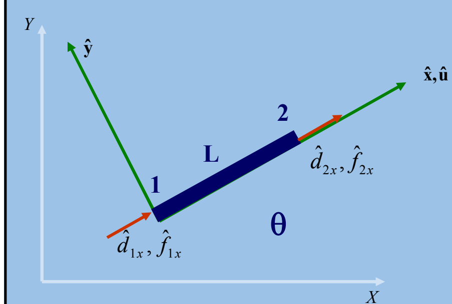
Galerkin-Method

Equating the value in the bracket to zero and performing the integral:

$$u_1=\tfrac{3}{4} \qquad u=0.75\left(2x-x^2\right)$$

- In elasticity problems Galerkin's method turns out to be the principle of virtual work.

Bar Element



$$\sigma=E\varepsilon$$

$$\varepsilon=\frac{d\hat{u}}{d\hat{x}}$$

$$A\sigma_{\hat{x}}=T=A\sigma$$

$$\frac{d}{d\hat{x}}\left(AE\frac{d\hat{u}}{d\hat{x}}\right)=0$$

Assumptions

- The bar cannot resist shear forces.
- That is: $\hat{f}_{1\hat{y}} = \hat{f}_{2\hat{y}} = 0$
- Effects of transverse displacements are ignored.
- Hooke's law applies.
- That is:

$$\sigma_x = E\varepsilon_x$$

Select a Displacement Function

- Assume a linear function. $\hat{u}$

$$\hat{u} = a_1 + a_2\hat{x}$$

- No. of coefficients = No. of DOF
- Written in matrix form: $\hat{u} = \begin{bmatrix} 1 & \hat{x} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}$

Expressed as function of $\hat{d}_{1x}$ and $\hat{d}_{2x}$

$$\hat{u}(0) = \hat{d}_{1x} = a_1 + a_2(0) \Rightarrow \hat{d}_{1x} = a_1$$

$$\hat{u}(L) = \hat{d}_{2x} = a_1 + a_2(L) \Rightarrow \hat{d}_{2x} = \hat{d}_{1x} + a_2 L$$

$$\hat{u} = \left(\frac{\hat{d}_{2x} - \hat{d}_{1x}}{L}\right)\hat{x} + \hat{d}_{1x}$$

$$\hat{u} = \begin{bmatrix} 1 - \frac{\hat{x}}{L} & \frac{\hat{x}}{L} \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

$$\hat{u} = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

Where:

Shape functions: $N_1 = 1 - \frac{\hat{x}}{L}$ and $N_2 = \frac{\hat{x}}{L}$

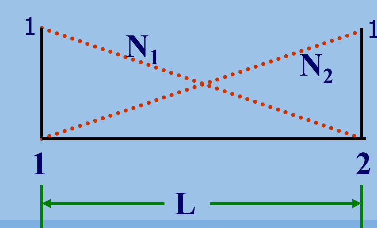
Shape Functions

N_1 and N_2 are called Shape Functions or Interpolation Functions. They express the shape of the assumed displacements.

$N_1 = 1$ $N_2 = 0$ at node 1

$N_1 = 0$ $N_2 = 1$ at node 2

$N_1 + N_2 = 1$ at any point



Define Strain/Displacement and Stress/Strain Relationships

$$\hat{u} = \left(\frac{\hat{d}_{2x} - \hat{d}_{1x}}{L}\right)\hat{x} + \hat{d}_{1x}$$

$$\varepsilon = \frac{d\hat{u}}{d\hat{x}} = \frac{\hat{d}_{2x} - \hat{d}_{1x}}{L}$$

$$\sigma = E\varepsilon = E\,\frac{\hat{d}_{2x} - \hat{d}_{1x}}{L}$$

$$f = A\sigma$$

$$\hat{f}_{1x} = AE\left(\frac{\hat{d}_{1x} - \hat{d}_{2x}}{L}\right)$$

$$\hat{f}_{2x} = AE\left(\frac{\hat{d}_{2x} - \hat{d}_{1x}}{L}\right)$$

Derive the Element Stiffness Matrix and Equations

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

$$\left[\hat{k}\right] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K] = \sum_{e=1}^{N}\left[\hat{k}^{(e)}\right]$$

Assemble Global Stiffness Matrix, apply BC and solve the Master stiffness equation for the unknown displacements:

$$\{F\} = \sum_{e=1}^{N}\left\{\hat{f}^{(e)}\right\}$$

$$[K]\{d\} = \{F\}$$

Potential Energy Approach

$$\pi_p = U - W$$

$$dU = \sigma_x\, d\varepsilon_x\, dV$$

$$U = \frac{1}{2}\iiint_V \sigma_x \varepsilon_x\, dV$$

$$W = -\iiint_V \hat{X}_b\, \hat{u}\, dV - \iint_S \hat{T}_x\, u\, dS - \sum_{i=1}^{M} \hat{f}_{ix}\, \hat{d}_{ix}$$

Potential Energy Approach

$$\pi_p = \frac{1}{2}\int_0^L A\sigma_x\, \varepsilon_x\, d\hat{x} - \hat{f}_{1x}\hat{d}_{1x} - \hat{f}_{2x}\hat{d}_{2x} - \iint_S \hat{u}\,\hat{T}_x\, dS - \iiint_V \hat{u}\ \hat{X}_b\, dv$$

$$\hat{u} = [N]\{\hat{d}\}$$

$$[N] = \begin{bmatrix} 1 - \frac{\hat{x}}{L} & \frac{\hat{x}}{L} \end{bmatrix}$$

$$\{\hat{d}\} = \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

$$\{\varepsilon_x\} = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}\{\hat{d}\}$$

$$\{\varepsilon_x\} = [B]\{\hat{d}\}$$

$$[B] = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}$$

Potential Energy Approach

$$\{\sigma_x\} = [D]\{\varepsilon_x\}$$

$$[D] = [E]$$

$$\{\sigma_x\} = [D][B]\{\hat{d}\}$$

[D] is the constitutive matrix (elasticity property matrix)

$$\pi_p = \frac{A}{2}\int_0^L \sigma_x\, \varepsilon_x\, d\hat{x} - \hat{f}_{1x}\hat{d}_{1x} - \hat{f}_{2x}\hat{d}_{2x} - \iint_S \hat{u}\,\hat{T}_x\, dS - \iiint_V \hat{u}\,\hat{X}_b\, dv$$

Potential Energy Approach

$$\pi_p = \frac{A}{2}\int_0^L \{\sigma_x\}^T \{\varepsilon_x\}\, d\hat{x} - \{\hat{d}\}^T\{P\} - \iint_S \{\hat{u}\}^T\{\hat{T}_x\}\, dS - \iiint_V \{\hat{u}\}^T\{\hat{X}_b\}\, dv$$

$$\pi_p = \frac{A}{2}\int_0^L \{\hat{d}\}^T [B]^T [D]^T [B]\{\hat{d}\}\, d\hat{x}$$

$$-\{\hat{d}\}^T\{P\} - \iint_S \{\hat{d}\}^T [N]^T \{\hat{T}_x\}\, dS - \iiint_V \{\hat{d}\}^T [N]^T \{\hat{X}_b\}\, dv$$

$$\pi_p = \frac{AL}{2}\{\hat{d}\}^T [B]^T [D]^T [B]\{\hat{d}\} - \{\hat{d}\}^T\{\hat{f}\}$$

$$\{\hat{f}\} = \{P\} + \iint_S [N]^T\{\hat{T}_x\}\, dS + \iiint_V [N]^T\{\hat{X}_b\}\, dv$$

Potential Energy Approach

$$\{U^*\} = \{\hat{d}\}^T [B]^T [D]^T [B]\{\hat{d}\}$$

$$\{U^*\} = \begin{bmatrix}\hat{d}_{1x} & \hat{d}_{2x}\end{bmatrix}\begin{Bmatrix}-\frac{1}{L}\\ \frac{1}{L}\end{Bmatrix}[E]\begin{bmatrix}-\frac{1}{L} & \frac{1}{L}\end{bmatrix}\begin{Bmatrix}\hat{d}_{1x}\\ \hat{d}_{2x}\end{Bmatrix}$$

$$U^* = \frac{E}{L^2}\left(\hat{d}_{1x}^2 - 2\hat{d}_{1x}\hat{d}_{2x} + \hat{d}_{2x}^2\right)$$

$$\{\hat{d}\}^T\{\hat{f}\} = \hat{d}_{1x}\hat{f}_{1x} + \hat{d}_{2x}\hat{f}_{2x}$$

Potential Energy Approach

$$\frac{\partial \pi_p}{\partial \hat{d}_{1x}} = \frac{AL}{2}\left[\frac{E}{L^2}\left(2\hat{d}_{1x} - 2\hat{d}_{2x}\right)\right] - \hat{f}_{1x} = 0$$

$$\frac{\partial \pi_p}{\partial \hat{d}_{2x}} = \frac{AL}{2}\left[\frac{E}{L^2}\left(2\hat{d}_{2x} - 2\hat{d}_{1x}\right)\right] - \hat{f}_{2x} = 0$$

$$\Rightarrow \frac{AE}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}\hat{d}_{1x}\\ \hat{d}_{2x}\end{Bmatrix} = \begin{Bmatrix}\hat{f}_{1x}\\ \hat{f}_{2x}\end{Bmatrix}$$

$$[k] = \int_V [B]^T [D][B]\, dV \qquad [D]^T = [D]$$

Transformation Matrix (Local ⇔ Global)

Let

$C=\cos\theta$

$S=\sin\theta$

$$\begin{Bmatrix}\hat{d}_{1x}\\ \hat{d}_{1y}\\ \hat{d}_{2x}\\ \hat{d}_{2y}\end{Bmatrix}=\begin{bmatrix}C & S & 0 & 0\\ -S & C & 0 & 0\\ 0 & 0 & C & S\\ 0 & 0 & -S & C\end{bmatrix}\begin{Bmatrix}d_{1x}\\ d_{1y}\\ d_{2x}\\ d_{2y}\end{Bmatrix}$$

$$\{\hat{d}\}=[T]\{d\}$$

$$\{\hat{f}\}=[T]\{f\}$$

$$[T]=\begin{bmatrix}C & S & 0 & 0\\ -S & C & 0 & 0\\ 0 & 0 & C & S\\ 0 & 0 & -S & C\end{bmatrix}$$

Element stiffness equation in local coordinate:

$$\begin{Bmatrix}\hat{f}_{1x}\\ \hat{f}_{1y}\\ \hat{f}_{2x}\\ \hat{f}_{2y}\end{Bmatrix}=\frac{AE}{L}\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}d_{1x}\\ d_{1y}\\ d_{2x}\\ d_{2y}\end{Bmatrix}$$

$$[T]\{f\}=[\hat{k}][T]\{d\}$$

$$\{f\}=[T]^{-1}[\hat{k}][T]\{d\}$$

$$[T]^{-1}=[T]^{T}$$

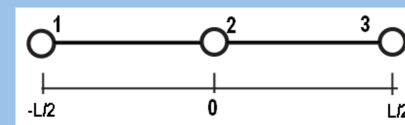
$$\{f\}=[T]^{T}[\hat{k}][T]\{d\}$$

Element stiffness matrix in the global coordinate:

$$[k]=[T]^{T}[\hat{k}][T]$$

Quadratic Bar Element

Obtain displacement function for the one-dimensional quadratic element with three nodes shown below.



- A quadratic displacement function:

$$\hat{u}=a_1+a_2\hat{x}+a_3\hat{x}^2$$

- No. of coefficients = No. of DOF
- Written in matrix form: $\hat{u}=\begin{bmatrix}1 & \hat{x} & \hat{x}^2\end{bmatrix}\begin{Bmatrix}a_1\\ a_2\\ a_3\end{Bmatrix}$

$$\hat{u}\big|_{\hat{x}=\frac{-L}{2}}=d_1 \qquad \hat{u}\big|_{\hat{x}=0}=d_2 \qquad \hat{u}\big|_{\hat{x}=\frac{L}{2}}=d_3 \quad\Rightarrow\quad \begin{aligned}d_1&=a_1-\frac{a_2L}{2}+\frac{a_3L^2}{4}\\ d_2&=a_1\\ d_3&=a_1+\frac{a_2L}{2}+\frac{a_3L^2}{4}\end{aligned}$$

$$\hat{u}=d_2+\frac{d_3-d_1}{L}\hat{x}+2\left(d_3-2d_2+d_1\right)\frac{\hat{x}^2}{L^2}$$

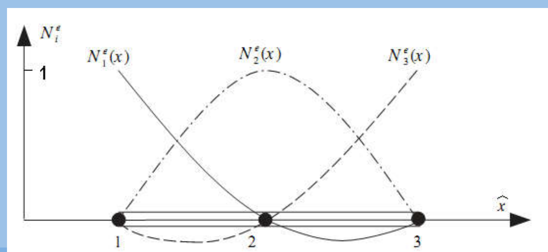
$$=\begin{bmatrix}-\frac{\hat{x}}{L}+\frac{2\hat{x}^2}{L^2} & 1-\frac{4\hat{x}^2}{L^2} & \frac{\hat{x}}{L}+\frac{2\hat{x}^2}{L^2}\end{bmatrix}\begin{Bmatrix}d_1\\ d_2\\ d_3\end{Bmatrix}$$

$$=\begin{bmatrix}N_1 & N_2 & N_3\end{bmatrix}\begin{Bmatrix}d_1\\ d_2\\ d_3\end{Bmatrix}$$

Quadratic Shape Functions

$N_1 = 1 \quad N_2 = N_3 = 0$ at node 1
$N_2 = 1 \quad N_1 = N_3 = 0$ at node 2
$N_3 = 1 \quad N_1 = N_2 = 0$ at node 3

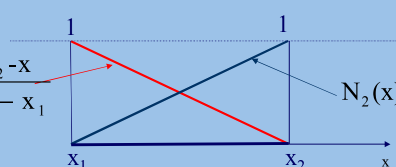
$N_1 + N_2 + N_3 = 1$ at any point



PROPERTIES OF THE SHAPE FUNCTIONS

The shape function at any node has a value of 1 at that node and a value of zero at ALL other nodes.

$$N_1(x) = \frac{x_2 - x}{x_2 - x_1}$$



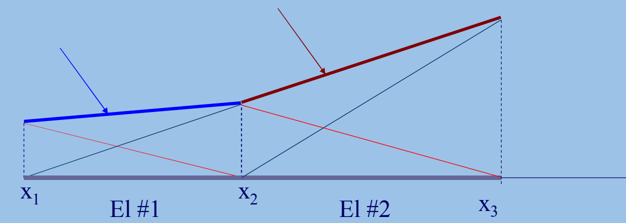
$$N_2(x) = \frac{x - x_1}{x_2 - x_1}$$

$$N_1(x) = \frac{x_2 - x}{x_2 - x_1}$$

Check $\Rightarrow N_1(x = x_1) = \frac{x_2 - x_1}{x_2 - x_1} = 1$

and $N_1(x = x_2) = \frac{x_2 - x_2}{x_2 - x_1} = 0$

Compatibility: The displacement approximation is **continuous across element boundaries**

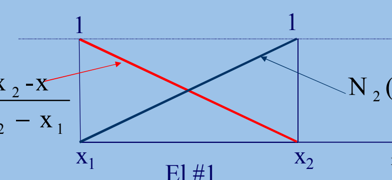


Hence the displacement approximation is continuous across elements

Writing shape functions (without deriving):

The **Kronecker delta property** (the shape function at any node has value of 1 at that node and a value of zero at all other nodes)

$$N_1(x) = \frac{x_2 - x}{x_2 - x_1}$$



$$N_2(x) = \frac{x - x_1}{x_2 - x_1}$$

Node at which N_1 is 0

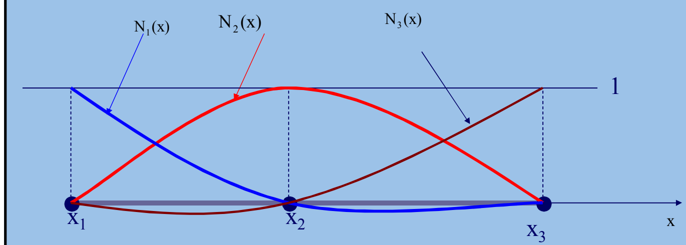
$$N_1(x) = \frac{(x_2 - x)}{(x_2 - x_1)}$$

Notice that the length of the element = $x_2 - x_1$

$$N_2(x) = \frac{(x_1 - x)}{(x_1 - x_2)} = \frac{(x - x_1)}{(x_2 - x_1)}$$

The denominator is the numerator evaluated at the node itself

3-Node bar element: varying **quadratically** inside the bar



$$N_1(x)=\frac{(x_2-x)(x_3-x)}{(x_2-x_1)(x_3-x_1)}$$

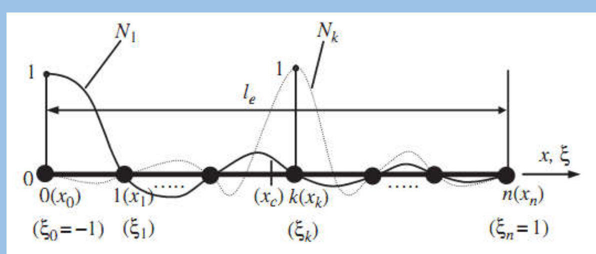
$$N_2(x)=\frac{(x_1-x)(x_3-x)}{(x_1-x_2)(x_3-x_2)}$$

$$N_3(x)=\frac{(x_1-x)(x_2-x)}{(x_1-x_3)(x_2-x_3)}$$

$$u(x)=N_1(x)d_{1x}+N_2(x)d_{2x}+N_3(x)d_{3x}$$

This is a **quadratic finite element** in 1D and it has three nodes and three associated shape functions per element.

Writing shape functions for higher DOFs:



$$N_k(x)=\prod_{\substack{m=0\\ m\neq k}}^{n}\frac{x-x_m}{x_k-x_m}=\frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

STRAIN and STRESS WITHIN EACH ELEMENT

From equation (1), the displacement within each element

$$\Phi(x)=\underline{N}\ \underline{d}$$

The **strain** in the bar $\varepsilon=\dfrac{d\Phi}{dx}$

Hence

$$\varepsilon=\left[\frac{d\underline{N}}{dx}\right]\underline{d}=\underline{B}\ \underline{d} \qquad (2)$$

The matrix $\underline{B}$ is known as the "**strain-displacement matrix**"

$$\underline{B}=\left[\frac{d\underline{N}}{dx}\right]$$

For a linear finite element

$$\underline{N}=\left[N_1(x)\quad N_2(x)\right]=\left[\frac{x_2-x}{x_2-x_1}\quad\frac{x-x_1}{x_2-x_1}\right]$$

Hence

$$B=\left[\frac{-1}{x_2-x_1}\quad\frac{1}{x_2-x_1}\right]=\frac{1}{x_2-x_1}\left[-1\quad 1\right]$$

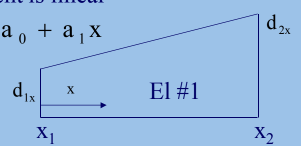
$$\varepsilon=\underline{B}\ \underline{d}=\left[\frac{-1}{x_2-x_1}\quad\frac{1}{x_2-x_1}\right]\begin{Bmatrix}d_{1x}\\ d_{2x}\end{Bmatrix}$$

$$=\frac{d_{2x}-d_{1x}}{x_2-x_1}$$

For a linear bar element, strain is a **constant** within the element.

Displacement is linear

$\Phi(x) = a_0 + a_1 x$



$$\varepsilon = \frac{d_{2x} - d_{1x}}{x_2 - x_1}$$

Strain is constant

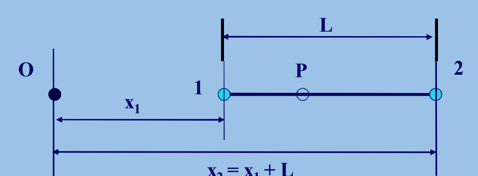
The **stress** in the bar $\sigma = E\,\varepsilon = E\,\frac{du}{dx}$

Inside the element, the approximate stress is $\sigma = E\,\underline{B}\,\underline{d}$

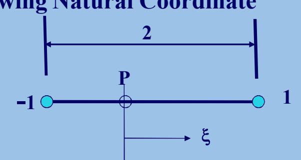
The **stiffness matrix**

$$[k] = \int_V [B]^T E\,[B]\,dV$$

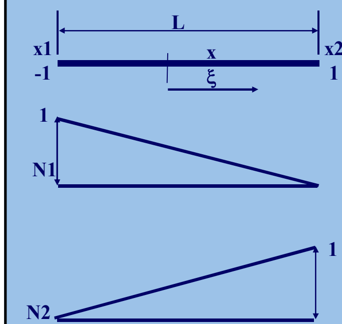
Natural Coordinates



Mapped on the following Natural Coordinate

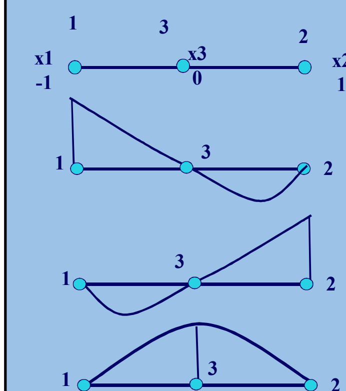


2 Node Linear Element



x	ξ
$N1 = \frac{x2 - x}{x2 - x1}$	$N1 = \frac{1 - \xi}{2}$
$N2 = \frac{x - x1}{x2 - x1}$	$N2 = \frac{\xi + 1}{2}$

3 Node Quadratic Element



$$N_1 = \frac{x_2 - x}{x_2 - x_1} \bullet \frac{x_3 - x}{x_3 - x_1} = \frac{1-\xi}{2} \bullet \left(\frac{-\xi}{1}\right) = \frac{1}{2}\left(-\xi + \xi^2\right)$$

$$N_2 = \frac{x_1 - x}{x_1 - x_2} \bullet \frac{x_3 - x}{x_3 - x_2} = \frac{-1-\xi}{-2} \bullet \frac{-\xi}{-1} = \frac{1}{2}\left(\xi + \xi^2\right)$$

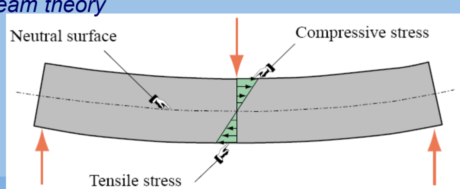
$$N_3 = \frac{x_1 - x}{x_1 - x_3} \bullet \frac{x_2 - x}{x_2 - x_3} = \frac{-1-\xi}{-1} \bullet \frac{1-\xi}{1} = \left(1 - \xi^2\right)$$

Beam Theory – Terminology

- A *general beam* is a bar-like member designed to resist a combination of loading actions such as biaxial bending, transverse shears, axial stretching or compression, and possibly torsion.
- If the beam is subject primarily to bending and axial forces, it is called a *beam-column*.
- A beam is *straight* if its longitudinal axis is straight. It is *prismatic* if its cross section is constant.
- A *spatial beam* supports transverse loads that can act on arbitrary directions along the cross section.
- A *plane beam* resists primarily transverse loading on a preferred longitudinal plane.

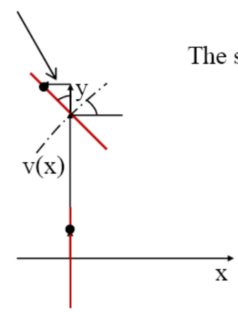
Mathematical Models

- One-dimensional mathematical models of structural beams are constructed on the basis of *beam theories*.
- The simplest and best known models for straight, prismatic beams are based on the:
  - *Bernoulli-Euler* theory (also called *classical beam theory* or *engineering beam theory)*
  - *Timoshenko beam theory*



Beam Theory – Kinematics

$$u = -y\frac{\partial v}{\partial x}$$

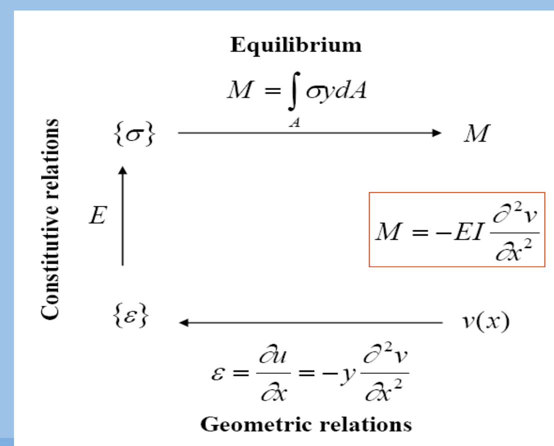


Assume that a straight line orthogonal to the middle line of the beam remains straight and orthogonal.

The segment is bent with the curvature κ

$$\varepsilon = \frac{\partial u}{\partial x} = -y\frac{\partial^2 v}{\partial x^2} = y\kappa$$

Beam Theory – Terminology



Bernoulli-Euler Model

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} -y\dfrac{\partial v(x)}{\partial x} \\ v(x) \end{bmatrix} = \begin{bmatrix} -yv' \\ v(x) \end{bmatrix} = \begin{bmatrix} -y\theta \\ v(x) \end{bmatrix}$$

$$e = \frac{\partial u}{\partial x} = -y\frac{\partial^2 v}{\partial x^2} = -y\frac{d^2 v}{dx^2} = -y\kappa$$

$$\sigma = Ee = -Ey\frac{d^2 v}{dx^2} = -Ey\kappa$$

$$M = EI\kappa$$

equilibrium equation $M'' = q$ (not used specifically in FEM)

Mathematical Model

Total Potential Energy of Beam Members

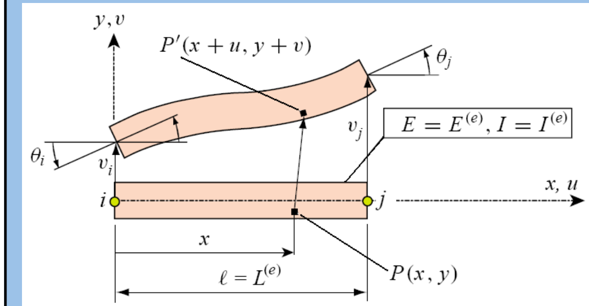
$$\Pi = U - W$$

$$U = \tfrac{1}{2}\int_V \sigma_{xx} e_{xx}\, dV = \tfrac{1}{2}\int_0^L M\kappa\, dx = \tfrac{1}{2}\int_0^L EI\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx$$

$$= \tfrac{1}{2}\int_0^L EI\kappa^2\, dx$$

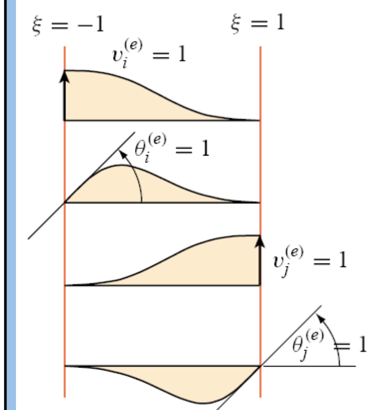
$$W = \int_0^L q\, v\, dx$$

Bernoulli-Euler Beam Theory



$$\mathbf{u}^{(e)} = \begin{bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{bmatrix}$$

Iso-P Shape Function



$$N_{vi}^{(e)}(\xi) = \tfrac{1}{4}(1-\xi)^2(2+\xi)$$

$$N_{\theta i}^{(e)}(\xi) = \tfrac{1}{8}\ell(1-\xi)^2(1+\xi)$$

$$N_{vj}^{(e)}(\xi) = \tfrac{1}{4}(1+\xi)^2(2-\xi)$$

$$N_{\theta j}^{(e)}(\xi) = -\tfrac{1}{8}\ell(1+\xi)^2(1-\xi)$$

Iso-P Shape Function

$$v^{(e)} = [\, N_{v_i}^{(e)} \quad N_{\theta_i}^{(e)} \quad N_{v_j}^{(e)} \quad N_{\theta_j}^{(e)} \,] \begin{bmatrix} v_i^{(e)} \\ \theta_i^{(e)} \\ v_j^{(e)} \\ \theta_j^{(e)} \end{bmatrix} = \mathbf{N}\mathbf{u}^{(e)}$$

$$\xi = \frac{2x}{\ell} - 1$$

$$N_{v_i}^{(e)} = \tfrac{1}{4}(1-\xi)^2(2+\xi), \qquad N_{\theta_i}^{(e)} = \tfrac{1}{8}\ell(1-\xi)^2(1+\xi),$$
$$N_{v_j}^{(e)} = \tfrac{1}{4}(1+\xi)^2(2-\xi), \qquad N_{\theta_j}^{(e)} = -\tfrac{1}{8}\ell(1+\xi)^2(1-\xi).$$

Element Equations

$$\mathbf{B} = \frac{1}{\ell}\left[\, 6\frac{\xi}{\ell} \quad 3\xi - 1 \quad -6\frac{\xi}{\ell} \quad 3\xi + 1 \,\right]$$

$$\Pi^{(e)} = \tfrac{1}{2}\mathbf{u}^{(e)T}\mathbf{K}^{(e)}\mathbf{u}^{(e)} - \mathbf{u}^{(e)T}\mathbf{f}^{(e)}$$

$$\mathbf{K}^{(e)} = \int_0^\ell EI\ \mathbf{B}^T\mathbf{B}\, dx = \int_{-1}^{1} EI\ \mathbf{B}^T\mathbf{B}\ \tfrac{1}{2}\ell\, d\xi$$

$$\mathbf{f}^{(e)} = \int_0^\ell \mathbf{N}^T q\, dx = \int_{-1}^{1} \mathbf{N}^T q\ \tfrac{1}{2}\ell\, d\xi$$

Element Stiffness

$$\mathbf{K}^{(e)} = \frac{EI}{2\ell^3}\int_{-1}^{1} \begin{bmatrix} 36\xi^2 & 6\xi(3\xi-1)\ell & -36\xi^2 & 6\xi(3\xi+1)\ell \\ & (3\xi-1)^2\ell^2 & -6\xi(3\xi-1)\ell & (9\xi^2-1)\ell^2 \\ & & 36\xi^2 & -6\xi(3\xi+1)\ell \\ symm & & & (3\xi+1)^2\ell^2 \end{bmatrix} d\xi$$

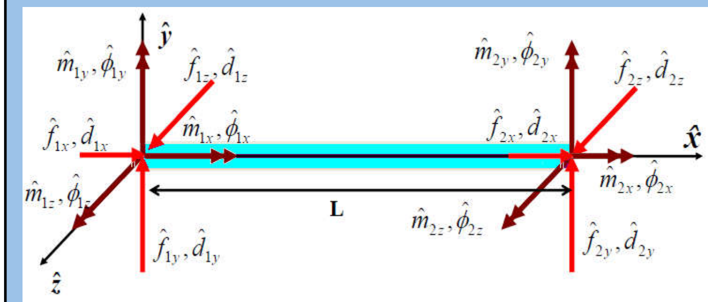
$$= \frac{EI}{\ell^3}\begin{bmatrix} 12 & 6\ell & -12 & 6\ell \\ & 4\ell^2 & -6\ell & 2\ell^2 \\ & & 12 & -6\ell \\ symm & & & 4\ell^2 \end{bmatrix}$$

Loading

$$\mathbf{f}^{(e)} = \tfrac{1}{2}q\,\ell\int_{-1}^{1} \mathbf{N}\, d\xi$$

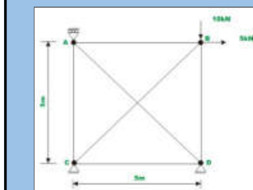
$$= \tfrac{1}{2}q\,\ell\int_{-1}^{1} \begin{bmatrix} \tfrac{1}{4}(1-\xi)^2(2+\xi) \\ \tfrac{1}{8}\ell(1-\xi)^2(1+\xi) \\ \tfrac{1}{4}(1+\xi)^2(2-\xi) \\ -\tfrac{1}{8}\ell(1+\xi)^2(1-\xi) \end{bmatrix} d\xi$$

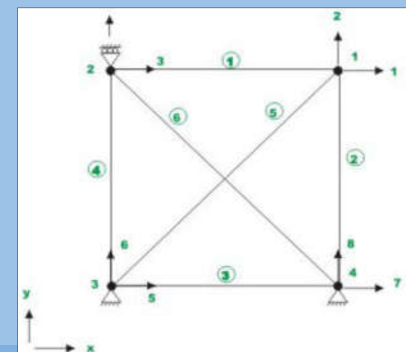
General 3D Frame Element



$$[K_{i-j}]=\begin{bmatrix}
\frac{EA_x}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{EA_x}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} & 0 & -\frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} \\
0 & 0 & \frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 & 0 & 0 & -\frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 \\
0 & 0 & 0 & \frac{GI_x}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{GI_x}{L} & 0 & 0 \\
0 & 0 & -\frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} \\
-\frac{EA_x}{L} & 0 & 0 & 0 & 0 & 0 & \frac{EA_x}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} \\
0 & 0 & -\frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 & 0 & 0 & \frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 \\
0 & 0 & 0 & -\frac{GI_x}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GI_x}{L} & 0 & 0 \\
0 & 0 & -\frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} & 0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L}
\end{bmatrix}$$

Example 1 – Analyse the plane truss





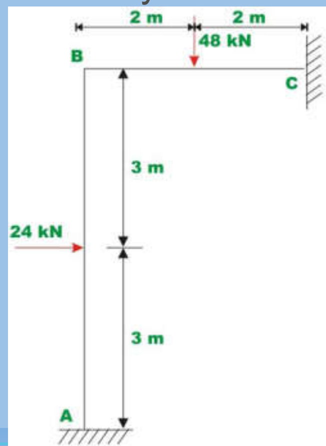
Example 1

$$u_1 = \frac{72.855}{AE}\ ;\ u_2 = \frac{-55.97}{AE};\ u_3 = \frac{53.825}{AE}$$

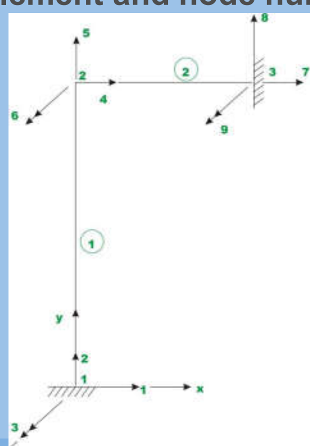
$$p_4 = -3.80 \quad \text{kN}\ ;\quad p_5 = -1.19 \quad \text{kN}\ ;\ p_6 = -1.19 \quad \text{kN}\ ;$$

$$p_7 = \ 3.80 \quad \text{kN}\ ; \qquad p_8 = 15.00 \quad \text{kN}$$

Example 2 – Analyze the Plane Frame



Example 2 – Element and node numbering



Example 2 – Boundary Condition & Solve

$$\begin{Bmatrix} u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} 6.28\times10^{-6} \\ -1.695\times10^{-5} \\ -0.13\times10^{-3} \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_7 \\ R_8 \\ R_9 \end{Bmatrix} = \begin{Bmatrix} -12 \\ 0 \\ 18 \\ 0 \\ 24 \\ -24 \end{Bmatrix} + \begin{Bmatrix} 0.57 \\ 22.59 \\ -1.14 \\ -12.57 \\ 1.40 \\ -1.92 \end{Bmatrix} = \begin{Bmatrix} -11.42 \\ 22.59 \\ 16.85 \\ -12.57 \\ 25.40 \\ -25.92 \end{Bmatrix}$$