Groundwater Hydraulics

Chapter 2 – Groundwater motion

CENG 6606

AAU

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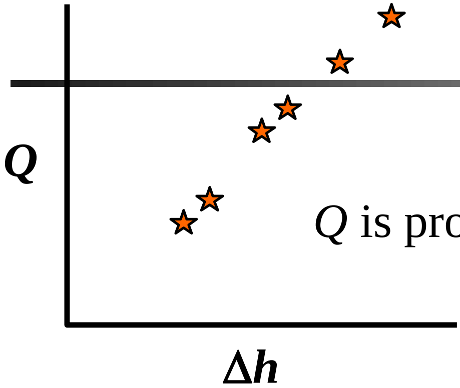
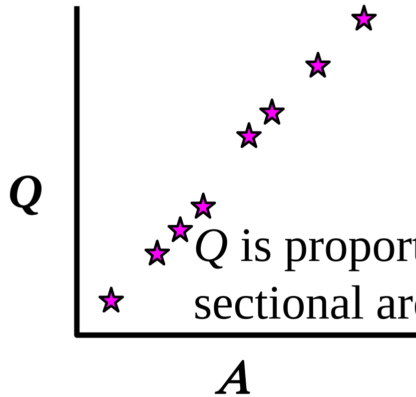
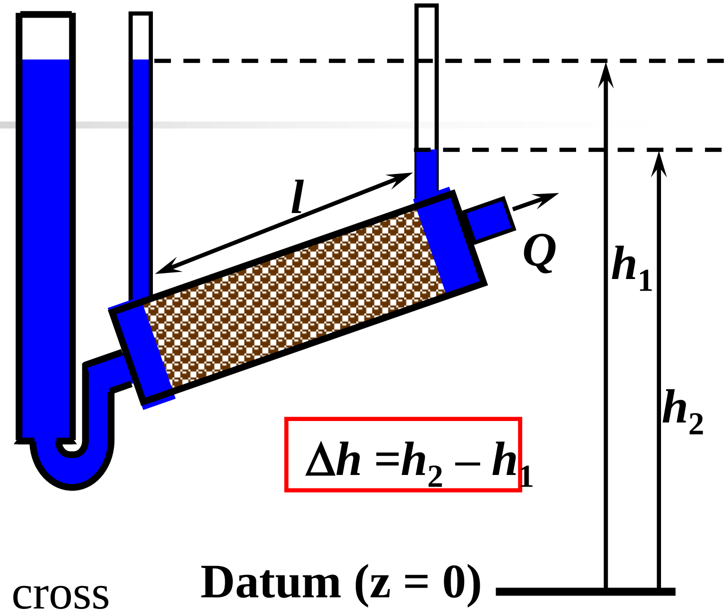
Darcy's Law

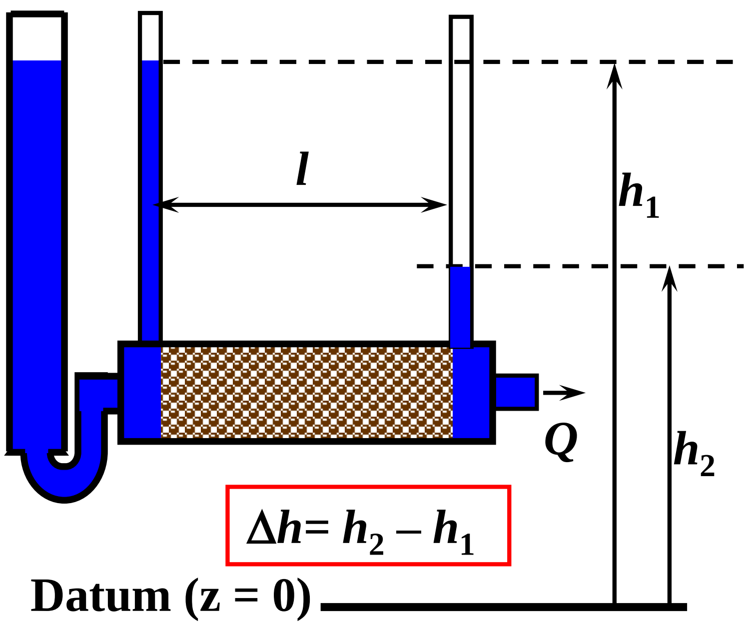


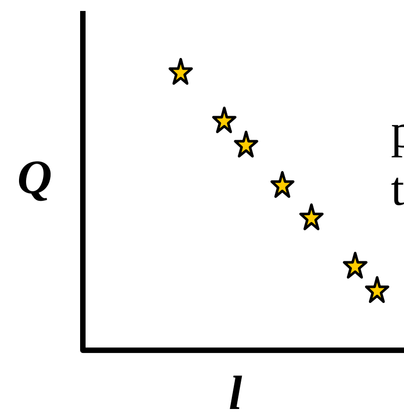
Q is proportional to Δh



Q is proportional to cross sectional area, A_{xs}







Q is inversely proportional to sample length, l

Darcy's Law

- Combine and insert a constant of proportionality

$$Q = -KA_{xc}\,[\Delta h/l]$$

- A_{xc} = sample cross-sectional area [m^2]
  - *Perpendicular* to flow direction
- K = hydraulic conductivity [m/s]
- $\Delta h/l$ = hydraulic gradient [-]
  - l measured along the flow direction
- Sometimes written as $Q/A_{xc} = q = -K[\Delta h/l]$
  - Where q = specific discharge a.k.a. "Darcy velocity"
- Hydraulic gradient often written as a differential, dh/dl

Effect of Geologic Material and Fluid Property

- $Q = -KA_{xc}[dh/dl]$
- Re-run experiments with different geologic materials
  - e.g., grain size
- General relationship still holds – but –
- Need a new constant of proportionality (K)
  - ***K is a property of the porous material***

- Re-run experiments with a different fluid
  - e.g. petroleum, trichloroethylene, ethanol
- General relationship still holds – but –
- Need a new constant of proportionality (K)
  - ***K is a property of the fluid***

Intrinsic Permeability - K_i

- Separate the effects of the fluid and the porous medium
- K = (porous medium property) x (fluid property)
- Porous medium property:
- k_i = intrinsic permeability
  - Essentially a function of pore opening size
  - Think of it as a 'friction' term
- $K = k_i$ x fluid driving force/fluid resisting force
- Fluid driving force = specific weight
- Fluid resisting force = dynamic viscosity
  - $K = K_i\,[\rho g/\mu]$
- $Q = -KA_{xc}\frac{dh}{dl} \rightarrow K = \frac{Qdl}{A_{xc}dh}$

Where:

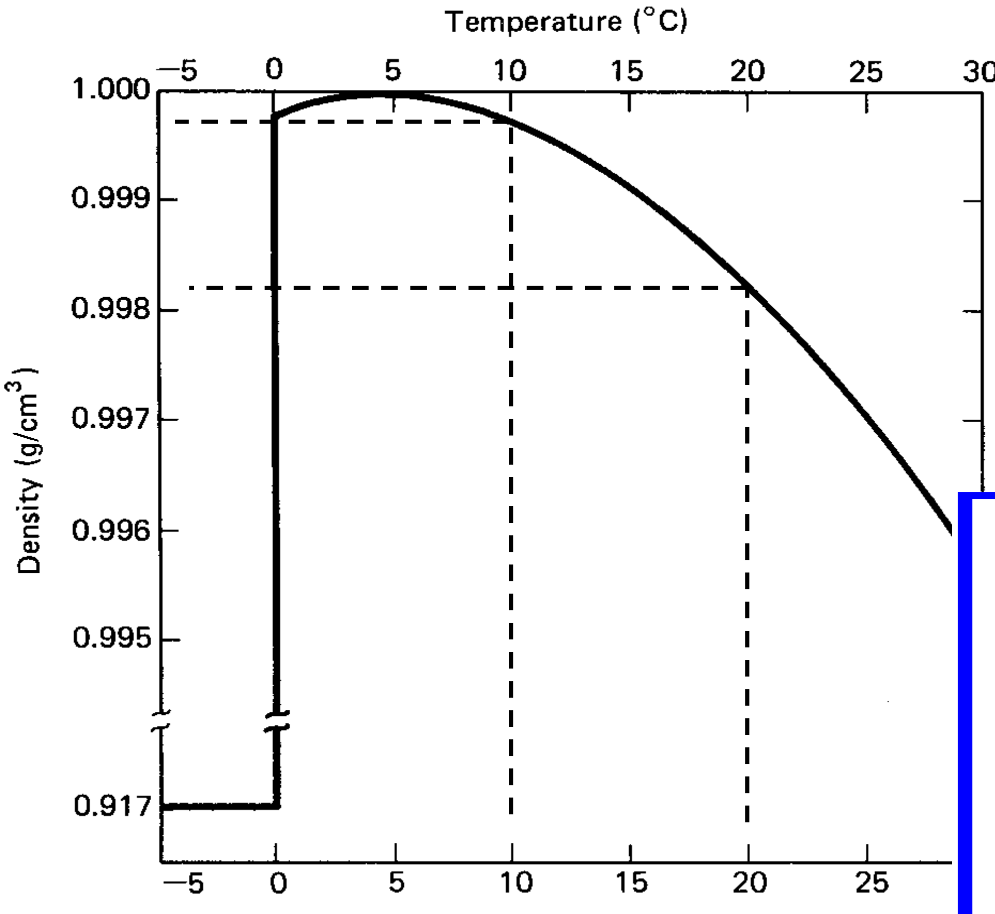
- dh/dl is dimensionless
- $Q \equiv [m^3\ s^{-1}]$
- $A_{xc} \equiv [m^2]$
- Therefore,
- $K \equiv \left[\frac{m^3}{s}\right]\left[\frac{1}{m^2}\right]\left[\frac{m}{m}\right] \equiv \left[\frac{m}{s}\right]$

Intrinsic Permeability - Units

- Write as: $Q = -K_i(\rho g/\mu)A_{xc}[dh/dl]$
- Solve for $k_i = \frac{Q\mu}{A_{xc}\rho g}\frac{dl}{dh}$
- Where: Q = [m^3/s]; A_{xc} =[m^2]; ρ = [kg/m^3]; g = [m/s^2]; μ = [kg/m/s]
- Therefore, $k_i = \left[\frac{m^3}{s}\right]\left[\frac{kg}{m\ s}\right]\left[\frac{1}{m^2}\right]\left[\frac{m^3}{kg}\right]\left[\frac{s^2}{m}\right]\left[\frac{m}{m}\right] = [m^2]$
- **Magnitude of Intrinsic Permeability**
- ($\rho g/\mu$) is a large number: For water at 15 ^{o}C,
  - $\rho g/\mu$ = 999.1 x 9.81/0.0011 = 8,910,155 [1/(m-s)]
- If K = 1 m/s then, $k_i = K/(\rho g/\mu)$ = 1.12x10^{-7} m^2
- Therefore we usually use a smaller unit –
  - 1 Darcy = 9.87x10^{-9} cm^2
- This course: typically use *hydraulic* conductivity (K)
- Contaminant transport, petroleum geology: k_i is important

Effect of Temperature

- Density and viscosity (ρ and μ) for water are a function of temperature
- ***K*** is therefore a function of temperature, but
- $\boldsymbol{K_i}$ is NOT a function of temperature
  - More fundamental unit controlling flow
- Lab standards are run at 60 ^{o}F (15.6 ^{o}C)
  - For most of the remainder of the course, we will assume that temperature is 15.6 ^{o}C
- So, how does ***K*** vary as a function of temperature??



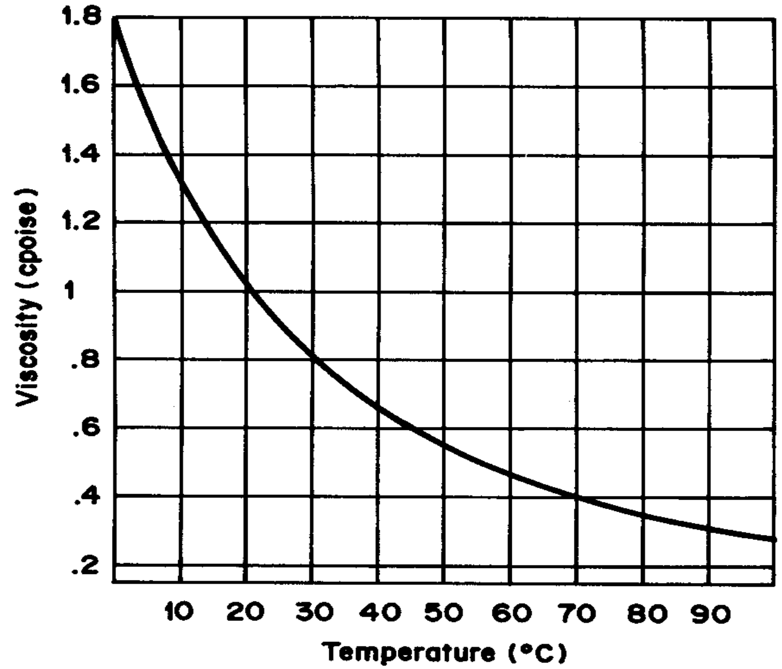
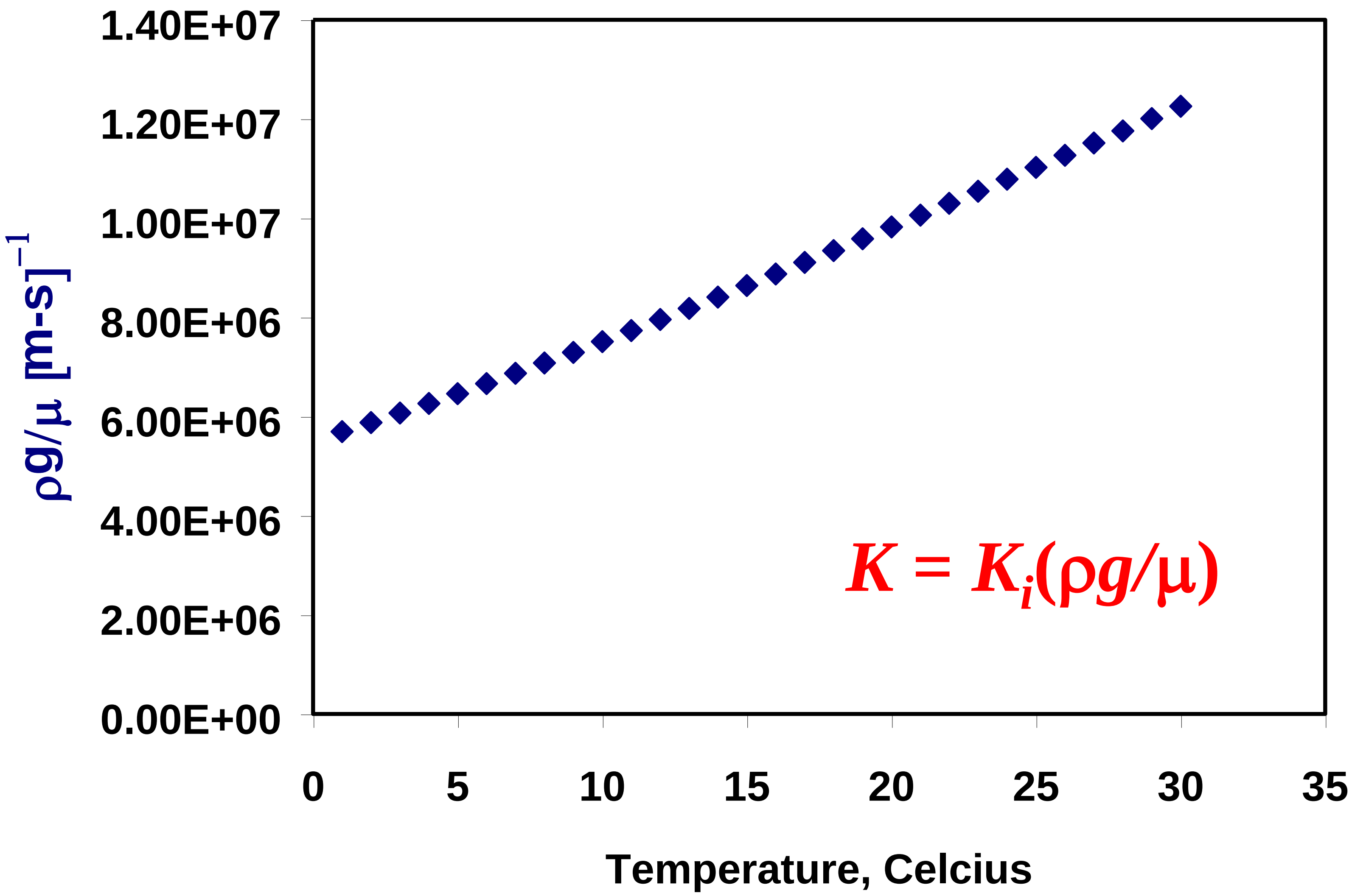


Fig. 3-2. Relationship between viscosity and temperature of water.

$$K = K_i(\rho g/\mu)$$

Effect of *T* on *K*



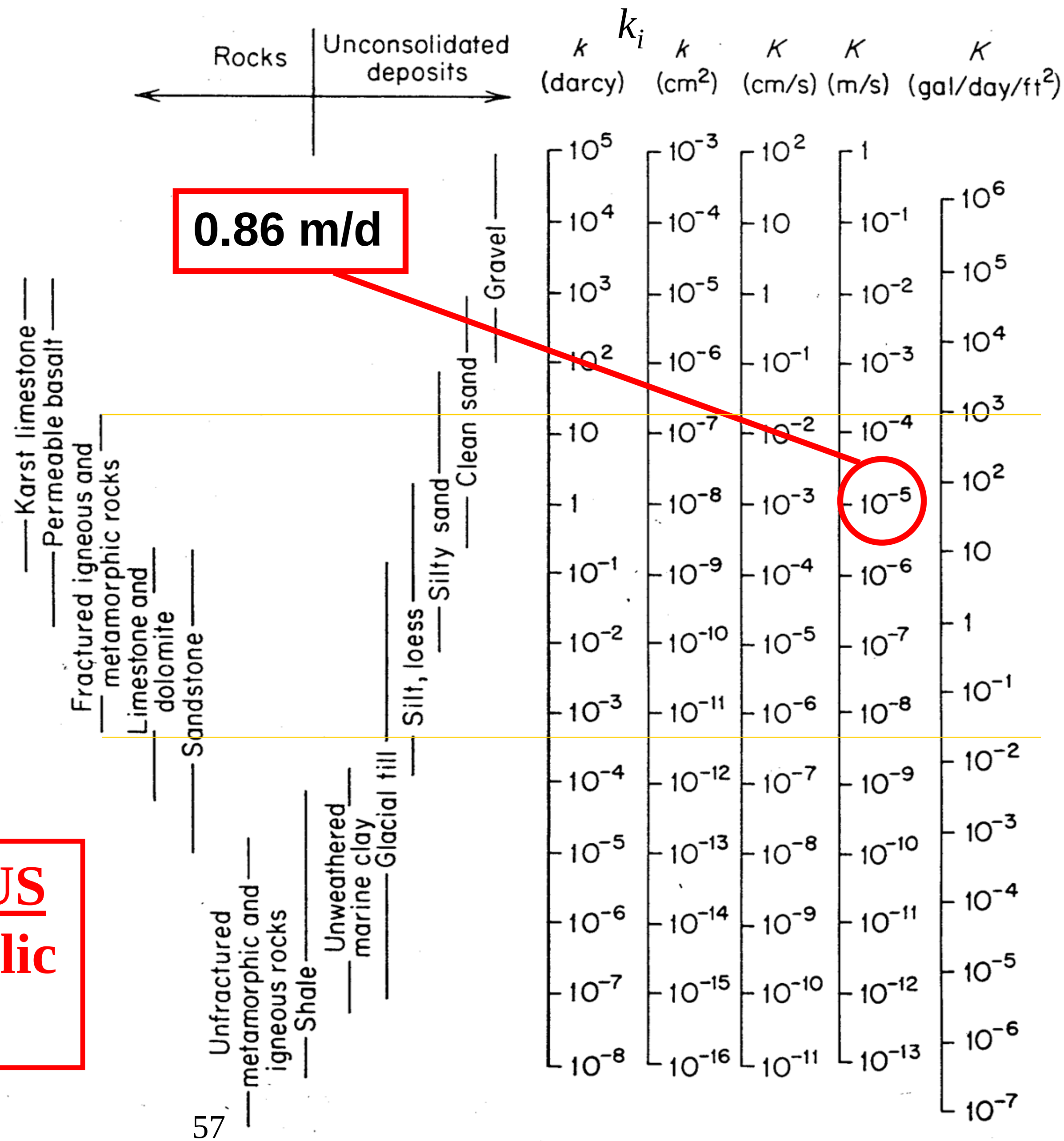
K_i in Rocks

- Primary openings
  - Formed as the rock forms - e.g. the initial porosity of the pre-cementation sediments
  - K_i in sedimentary rocks is the K_i of the sediments from which they form
  - Crystalline rocks (igneous, metamorphic): Low primary permeability (possible exception: some igneous rocks with interconnected pores)
- Secondary openings (after the rock formed)
  - Fractures
  - Dissolution
    - Along fractures, bedding planes
    - Important for chemically precipitated rocks - limestone, dolostone, gypsum, halite
  - Weathering

Estimating *K*

(1) past experience

Table 2.2 Range of Values of Hydraulic Conductivity and Permeability



Note the <u>ENORMOUS</u> variability in Hydraulic Conductivity!!!

Estimating K, k_i

(2) **Empirical Relations to Grain Size**

Hazen's Formula

$$K = C\,(d_{10})^2$$

- Where K is hydraulic conductivity in cm/sec
- d_{10} is the effective grain size
  - 10%of the soil by weight is finer in grain size, 90% is coarser
- C is an empirical coefficient

K is proportional to the *square of grain size*

Soil Type	C
Very fine sand, poorly sorted	40-80
Fine sand with appreciable fines	40-80
Medium sand, well sorted	80-120
Coarse sand, poorly sorted	80-120
Coarse sand, well sorted, clean	120-150

- $k_i = C_0 (D_{50})^2 e^{-c\sigma_\Phi}$
  - Where k_i = intrinsic permeability (in Darcie); C_0 = an empirical constant ~760 Darcy/mm; c = an empirical constant ~ 1.31; D_{50} = median diameter of sediment (in mm); σ_φ = standard deviation of grain size in φ units

Estimating *K*, k_i

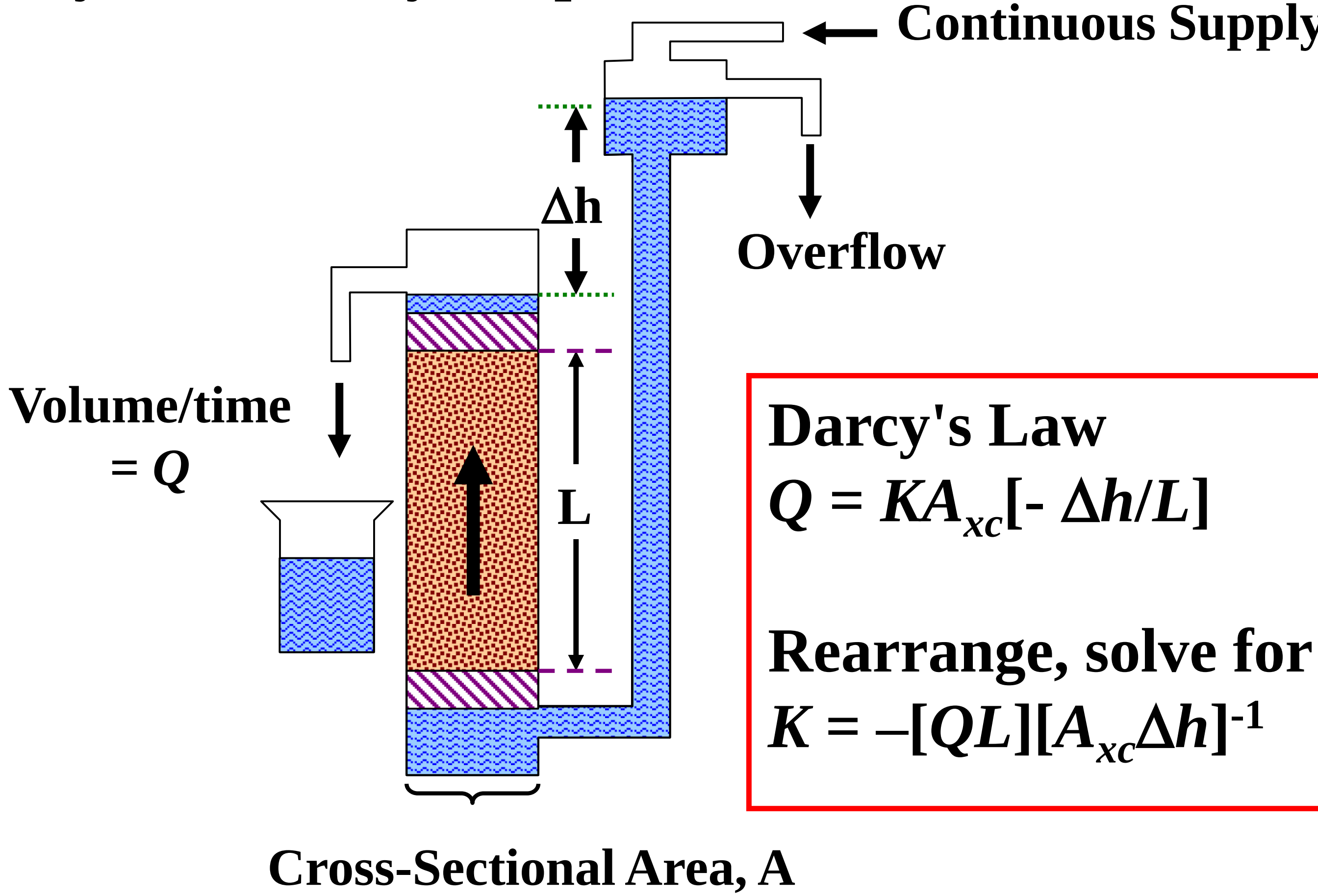
(2) **Empirical Relations to Grain Size**

- Kozeney (1927):
  - $K_i = Cn^3/S^{*2}$
  - Where K_i = intrinsic permeability (in darcies);
  - C = an empirical constant ~0.5, 0.562, & 0.597 for circular, square, and equilateral triangular pore openings;
  - n = porosity;
  - S^* = specific surface - interstitial surface areas of pores per unit bulk volume of the medium.
- Kozeny-Carmen Bear (1972)
  - $K = \frac{\rho_w g}{\mu} * \frac{n^3}{(1-n)^2}\left(\frac{d_m^2}{180}\right)$
  - Where K = hydraulic conductivity; ρ_w = fluid density; μ = fluid viscosity; g = gravitational constant; d_m = any representative grain size; n = porosity

Measuring K

(3) Constant Head Permeameter

- **Basically, redo Darcy's experiments:**



Darcy's Law

$$Q = KA_{xc}[-\Delta h/L]$$

Rearrange, solve for *K*

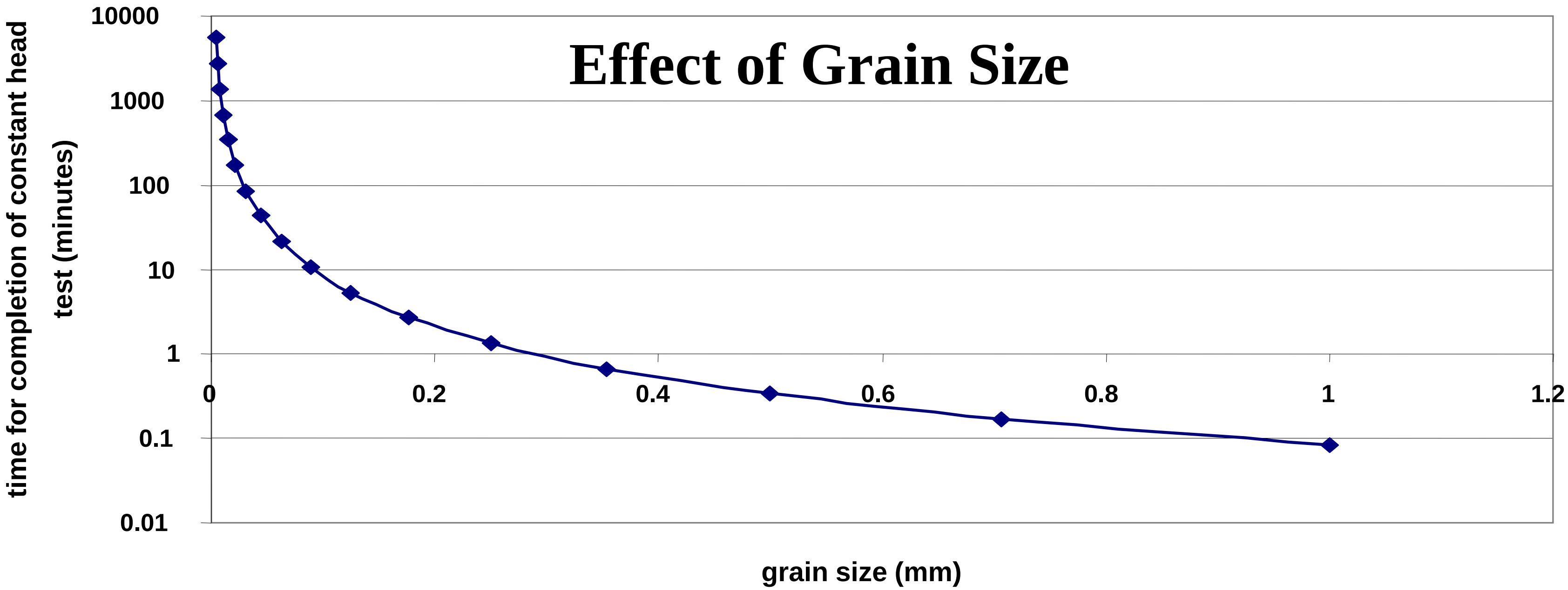
$$K = -[QL][A_{xc}\Delta h]^{-1}$$

Constant Head Permeameter Test Protocols

- Keep Δh at reasonable field conditions
  - ($< 1/2\ L$)
- Be certain that no air is trapped in the sample
  - Air bubbles will act as impermeable lenses
  - Fill slowly from the bottom to force air upwards
  - De-gas water
- Design test, so all parameters can be measured accurately
- Design test, so it can be conducted in a reasonable amount of time
- Good for relatively coarse grained material with relatively high hydraulic conductivity

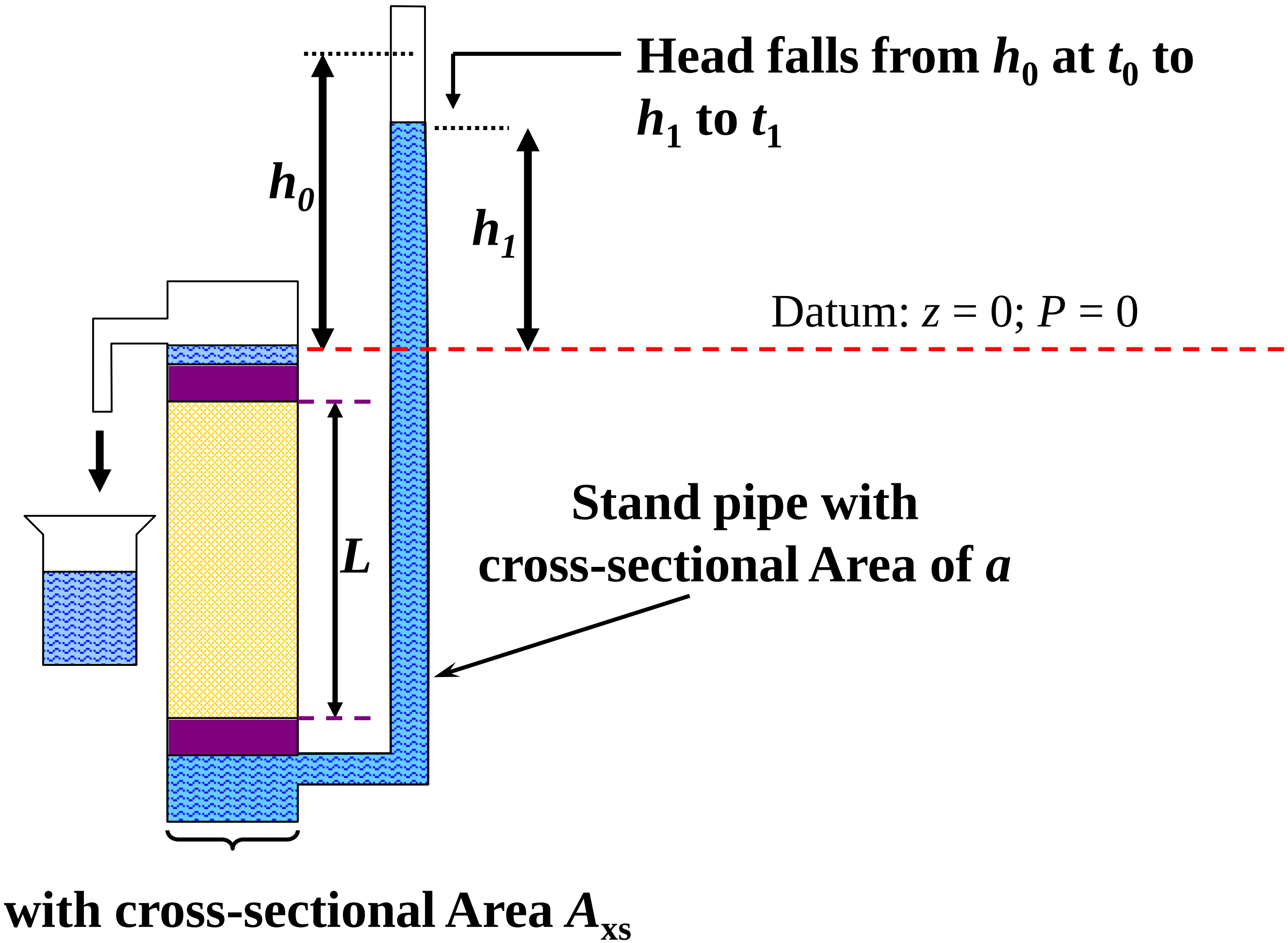
Constant Head Permeameter Test Design

- $K = -\frac{QL}{A\Delta h} = -\frac{{Vol}/{Time} \times L}{A \times \Delta h} \rightarrow Time = -\frac{L \times Vol}{K \times A \times \Delta h}$
  - Trial Design: L = 10 cm long; A_{xs} = 5 cm^2 ; Head difference (Δh) = – 5 cm; $K = 10^{-1}$ cm/sec(~ coarse sand);Volume collected = 100 ml;
  - Time = 10x100/(0.10x5x5) = 400 s



Measuring K

(3) Falling Head Permeameter



Falling Head Permeameter Analysis

- Apply to fine grained soils
  - Constant head permeameter test inaccurate, lengthy
- Mass balance – standpipe
  - $Q = \frac{dV}{dt} = a\frac{dh}{dt}$
- Darcy's Law – sample
  - $Q = -KA_{xs}\frac{h}{L}$
- Set Q equal
  - $a\frac{dh}{dt} = -KA_{xs}\frac{h}{L}$

Set datum at outlet

Therefore, $h_{outlet} = 0$ and

$\Delta h = h_{outlet} - h = -h$

At $t = t_o$

$\Delta h = h_{outlet} - h_o = -h_o$

At $t = t_1$

$\Delta h = h_{outlet} - h_1 = -h_1$

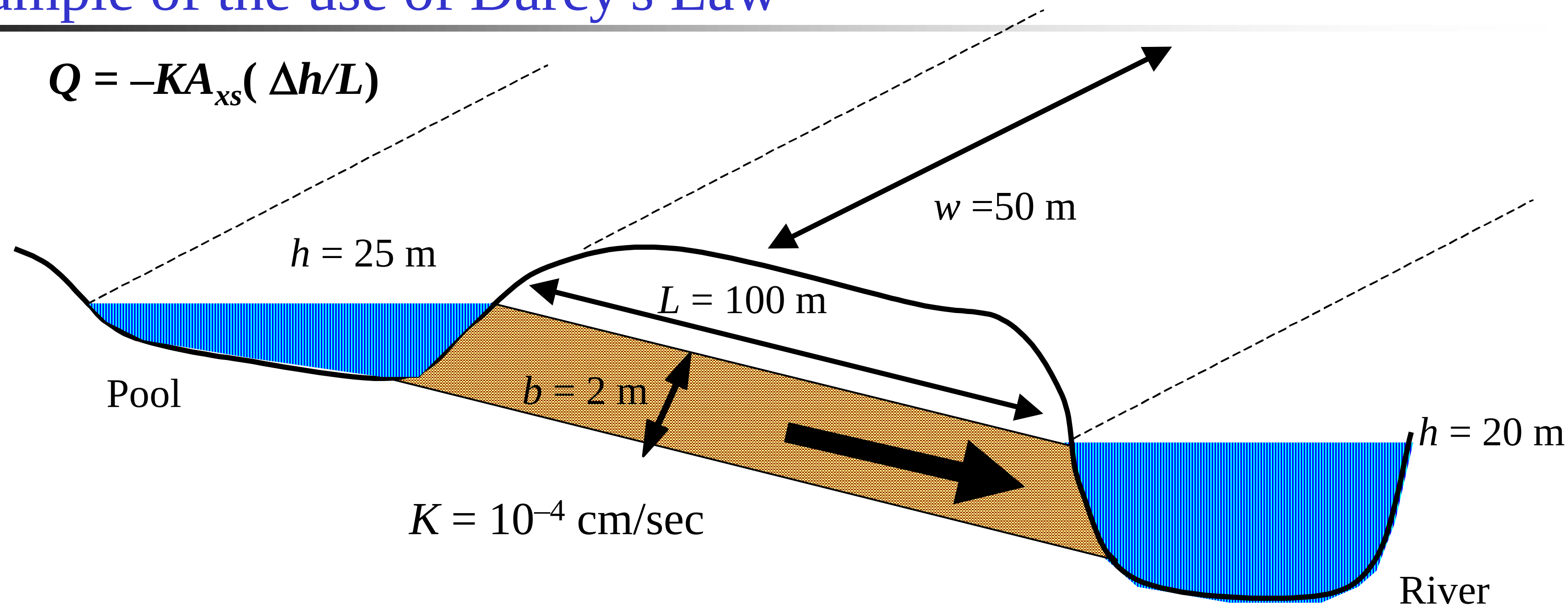
Falling Head Permeameter Analysis

- Combine mass balance and Darcy's Law
  - $a\frac{dh}{dt} = -KA_{xs}\frac{h}{L}$
- Separate variables and integrate
  - $-\int_{h_o}^{h_1}\frac{dh}{h} = \frac{KA_{xs}}{aL}\int_{t_o}^{t_1}dt$
  - $\ln\frac{h_o}{h_1} = \frac{KA_{xs}(t_1 - t_o)}{aL}$
  - $K = \frac{aL}{A_{xs}(t_1 - t_o)}\ln\frac{h_o}{h_1}$

Falling Head Permeameter Test Design

- Solve for time = $t_1 - t_o = \frac{aL}{KA_{xs}} \ln \frac{h_o}{h_1}$
- Trial Design:
  - L = 10 cm
  - A_{xs} = 10 cm^2
  - Stand pipe a = 0.5 cm^2
  - h_o = 20 cm; h_1 = 19 cm
  - K = 10^{-3} cm/sec (~ fine sand with silt)
- Time $= t_1 - t_o = \frac{0.5\times10}{0.001\times10} \ln \frac{20}{19} = 25.6$ s

Example of the use of Darcy's Law



- How much water is flowing from the pool into the river per second over a 50 m stretch?
- Δh = –5 m (head decreases in the direction of flow)
- l = 100 m; $\Delta h/l = -0.05$
- $A_{xc} = b \times w$ = 2 m x 50 m = 100 m^2
- $K = 10^{-4}$ cm/sec
- $Q = -10^{-4}$ cm/sec x 100 m^2 x 100^2 cm^2/m^2 x (–0.05) = 5 cm^3/s

PERPENDICULAR to the direction of flow!

Other Ways to Express Flow

Flow per Unit Width

- What is the flow through the aquifer per unit width (per cm)?
  - $Q = -KA_{xs}(\Delta h/l)$ $A_{xs} = b \times w$
  - $Q = -K(b \times w) \times (\Delta h/l)$ divide both sides by w
  - $Q/w = -Kb(\Delta h/l)$
- $Q/w = -10^{-4}$ cm/sec x 2 m x 100 cm/m x (–.05)=

  = 0.001 [$cm^3\ s^{-1}\ cm^{-1}$]

Other Ways to Express Flow

Flow per Unit Width per Unit Gradient

- What is the flow through the aquifer per unit width (per cm) per unit hydraulic gradient?
  - This is a measure often used to compare aquifers.
  - $Q = -K(b \times w)(\Delta h/l)$ divide both sides by w
  - $Q/w = -Kb(\Delta h/l)$ divide both sides by $(\Delta h/l)$
  - $(Q/w)/(\Delta h/l) = -Kb$
- $(Q/w)/(\Delta h/l) = -10^{-4}$ cm/sec x 2 m x 100 cm/m =

  $= 0.02\ [\mathrm{cm}^2\ \mathrm{s}^{-1}]$

Transmissivity (T)

- The rate at which water is transmitted through a unit width of aquifer under a unit hydraulic gradient
- Our last calculation (flux per unit width per unit hydraulic gradient)
- A common unit in hydrogeology
  - $Q = -KA_{xs}(\Delta h/l)$ $\quad\quad$ $A_{xs} = b \times w$
  - $Q = -K(b \times w)(\Delta h/l)$ $\quad$ divide both sides by w
  - $Q/w = -K(b\ )(\Delta h/l)$ $\quad$ divide both sides by $-\Delta h/l$
  - $Q/w/(-\Delta h/l) = Kb$

$$T = Kb$$

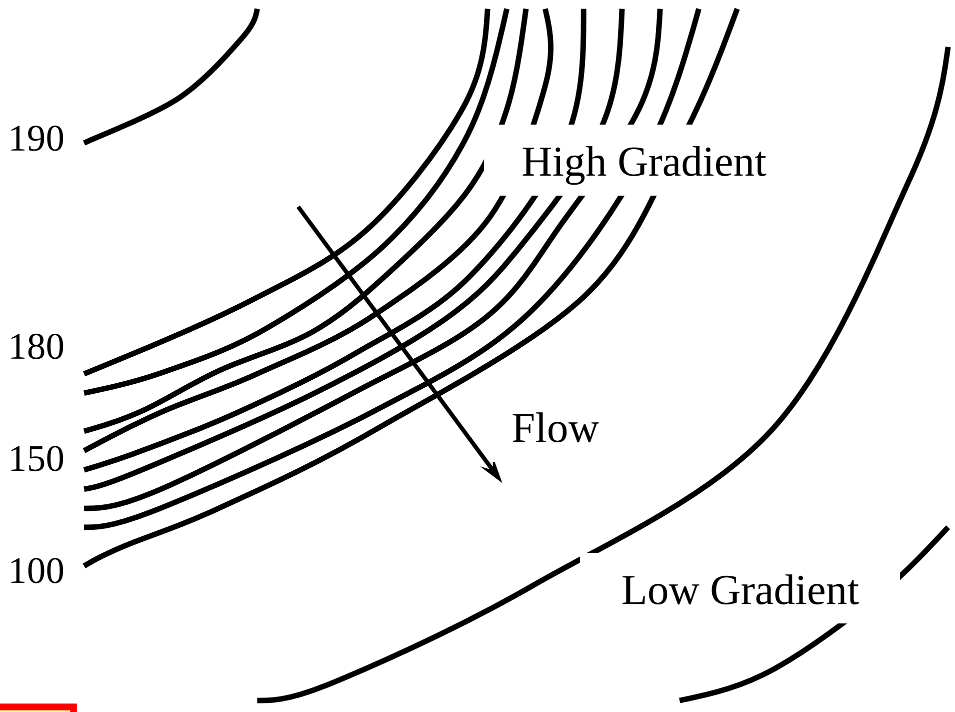
- K is the hydraulic conductivity
- b is the aquifer thickness

Transmissivity (T)

- For confined aquifers, b is the aquifer thickness (may vary in space)
- For unconfined aquifers, b is not well defined, since it can also change with position and through time. Use b as the saturated thickness
- Alternate way of expressing Darcy's Law
  - $Q = -KA_{xs}(\Delta h/l)$
  - $Q = -K(b \; x \; w)(\Delta h/l)$
  - $Q = -Tw(\Delta h/l)$
    - w is the aquifer width (horizontal dimension perpendicular to flow)
- Units: (volume/time)/length, Eg. gallons/day/foot or
- Units: length^2/time [m^2/s]

Gradients in Hydraulic Head

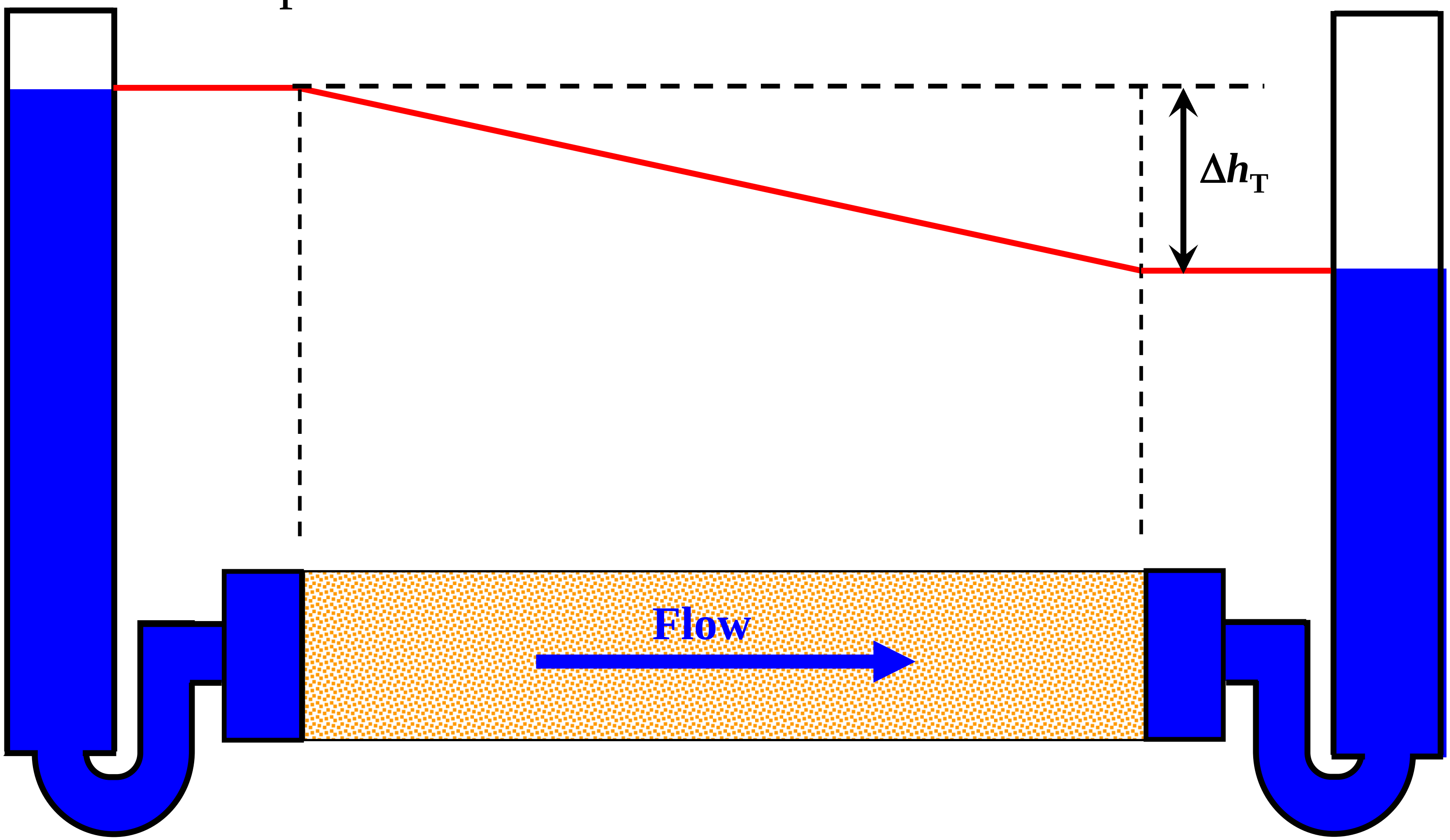
- We measure gradients in head using piezometers
- We can map these as shown
- We often observe changes in head gradient
- What aquifer properties can cause changes in these gradients?



Hydraulic gradient = Δ*h*/*l*

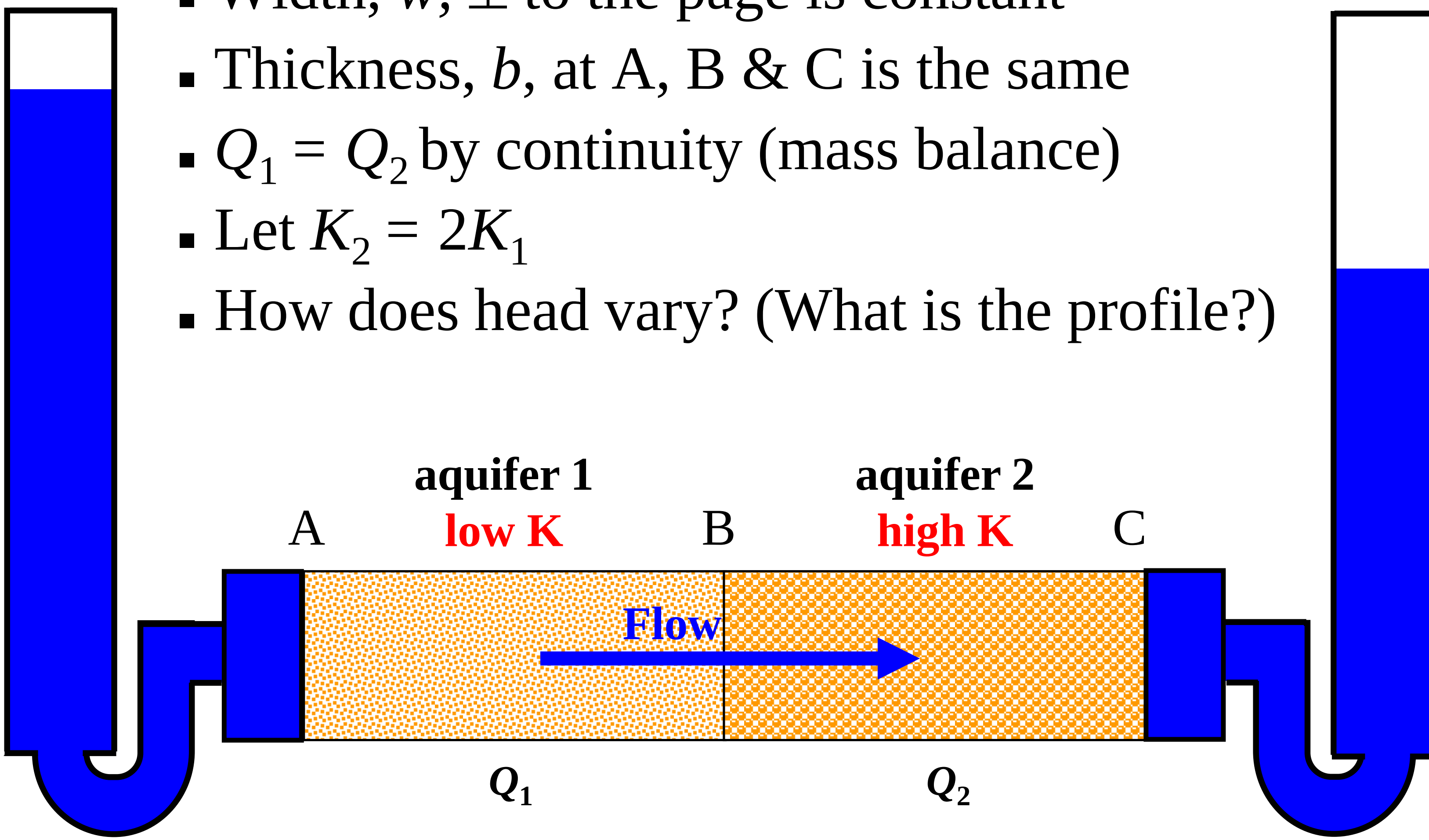
Head Gradient

- Head profile for homogeneous material
- Slope is constant



Effect of *K* on Change in Head Gradient

- Length, L, from A to B and B to C is same
- Width, w, ⊥ to the page is constant
- Thickness, b, at A, B & C is the same
- $Q_1 = Q_2$ by continuity (mass balance)
- Let $K_2 = 2K_1$
- How does head vary? (What is the profile?)



Head Profile (Effect of K)

- By continuity, $Q_1 = Q_2$
- Write Darcy's Law

$$-K_1 A_1 (\Delta h/l)_1 = -K_2 A_2 (\Delta h/l)_2$$

- Cancel like terms, A, l
- Substitute $K_2 = 2K_1$

$$K_1 \Delta h_1 = K_2 \Delta h_2 = 2K_1 \Delta h_2$$

- Cancel K_1; therefore,

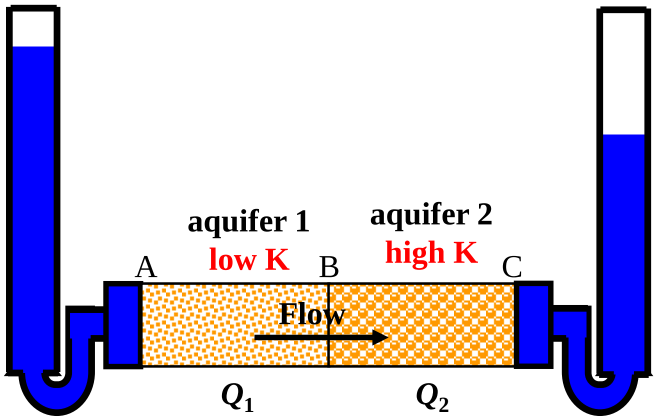
$$\Delta h_1 = 2 \Delta h_2$$

- Determine Δh_1 and Δh_2

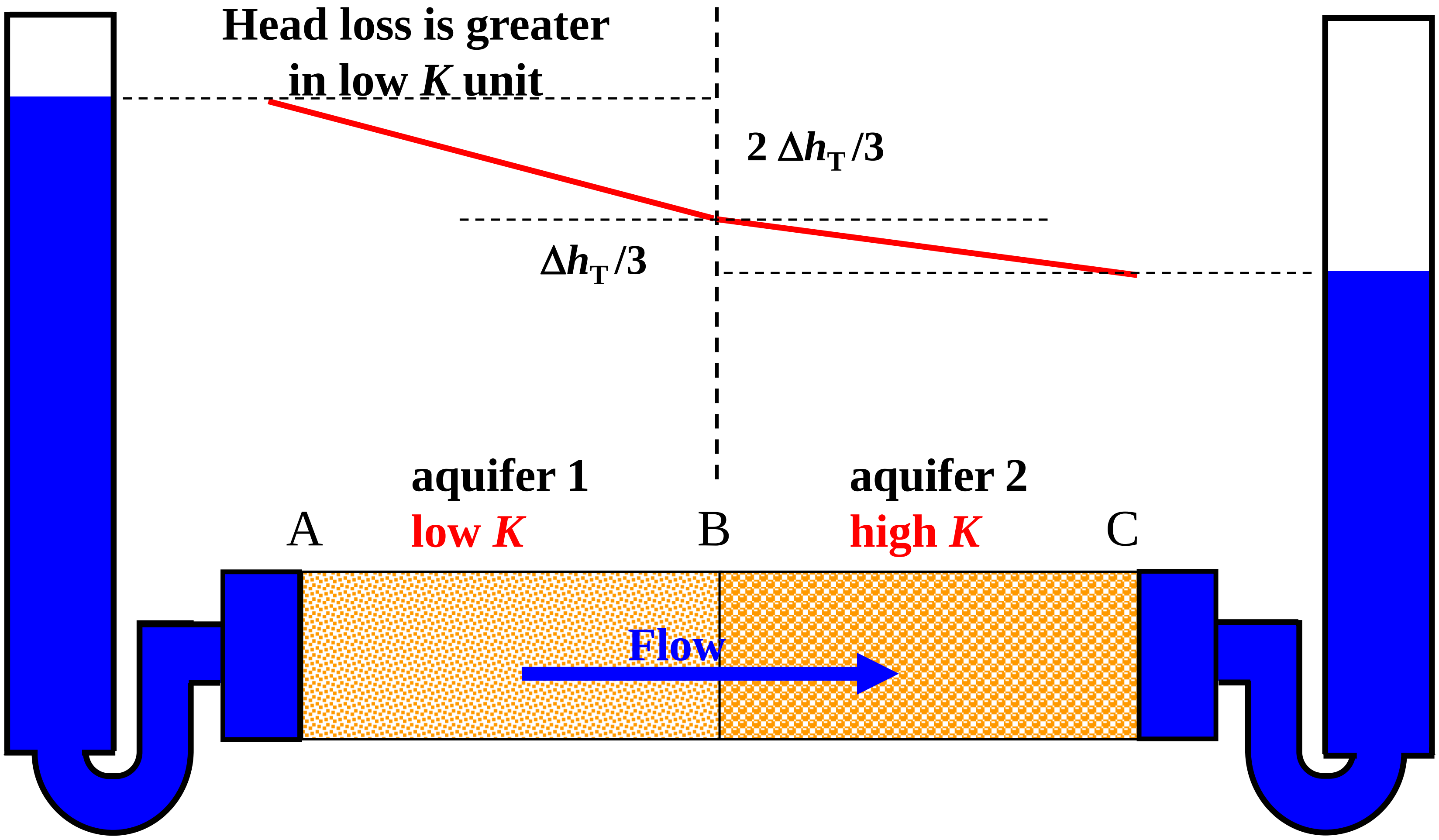
$$\Delta h_T = \Delta h_1 + \Delta h_2 = 2 \Delta h_2 + \Delta h_2 = 3 \Delta h_2$$

$$\Delta h_2 = \Delta h_T /3$$

$$\Delta h_1 = 2 \Delta h_T /3$$

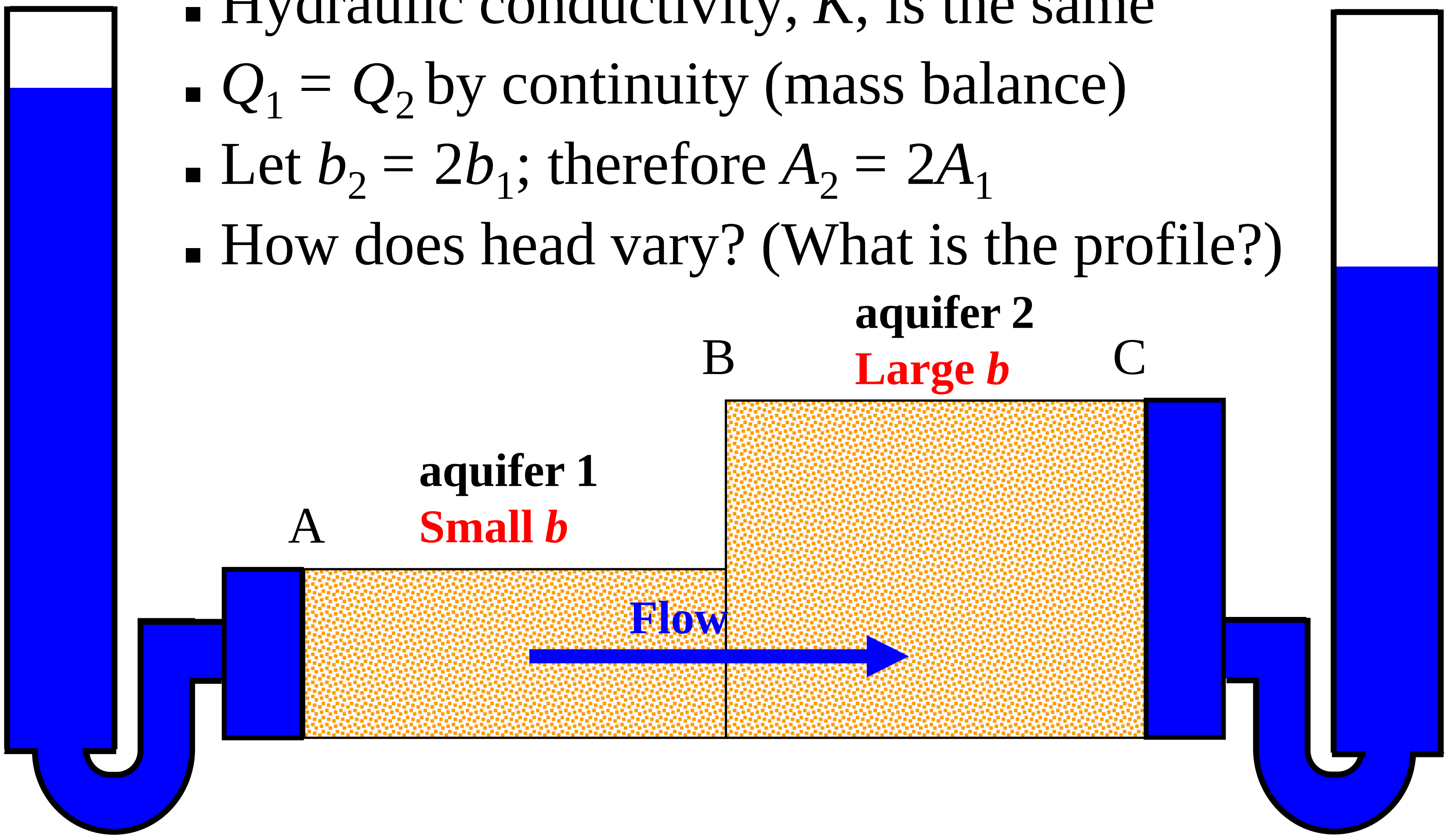


Change in *K* can cause Change in Head Gradient



Effect of *b* on Change in Head Gradient

- Length, L, from A to B and B to C is same
- Width, w, ⊥ to the page is constant
- Hydraulic conductivity, K, is the same
- $Q_1 = Q_2$ by continuity (mass balance)
- Let $b_2 = 2b_1$; therefore $A_2 = 2A_1$
- How does head vary? (What is the profile?)



Head Profile (Effect of *b*)

- By continuity, $Q_1 = Q_2$
- Write Darcy's Law

$$-K_1 A_1 (\Delta h/l)_1 = -K_2 A_2 (\Delta h/l)_2$$

- Cancel like terms, substitute $A_2 = 2A_1$

$$A_1 \Delta h_1 = A_2 \Delta h_2 = 2A_1 \Delta h_2$$

- Therefore,

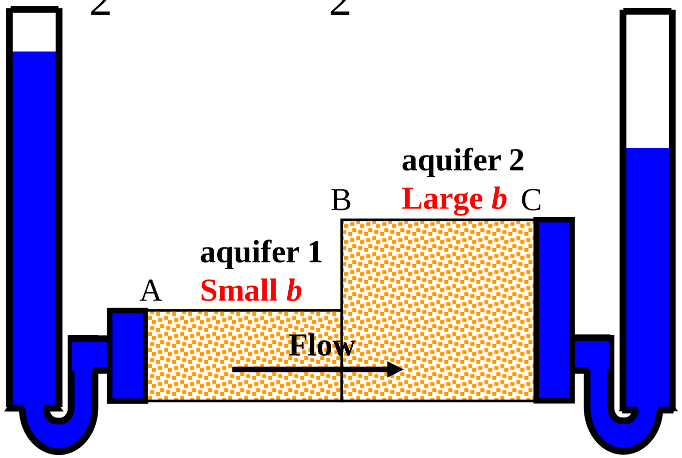
$$\Delta h_1 = 2 \Delta h_2$$

- Determine dh_1 and dh_2

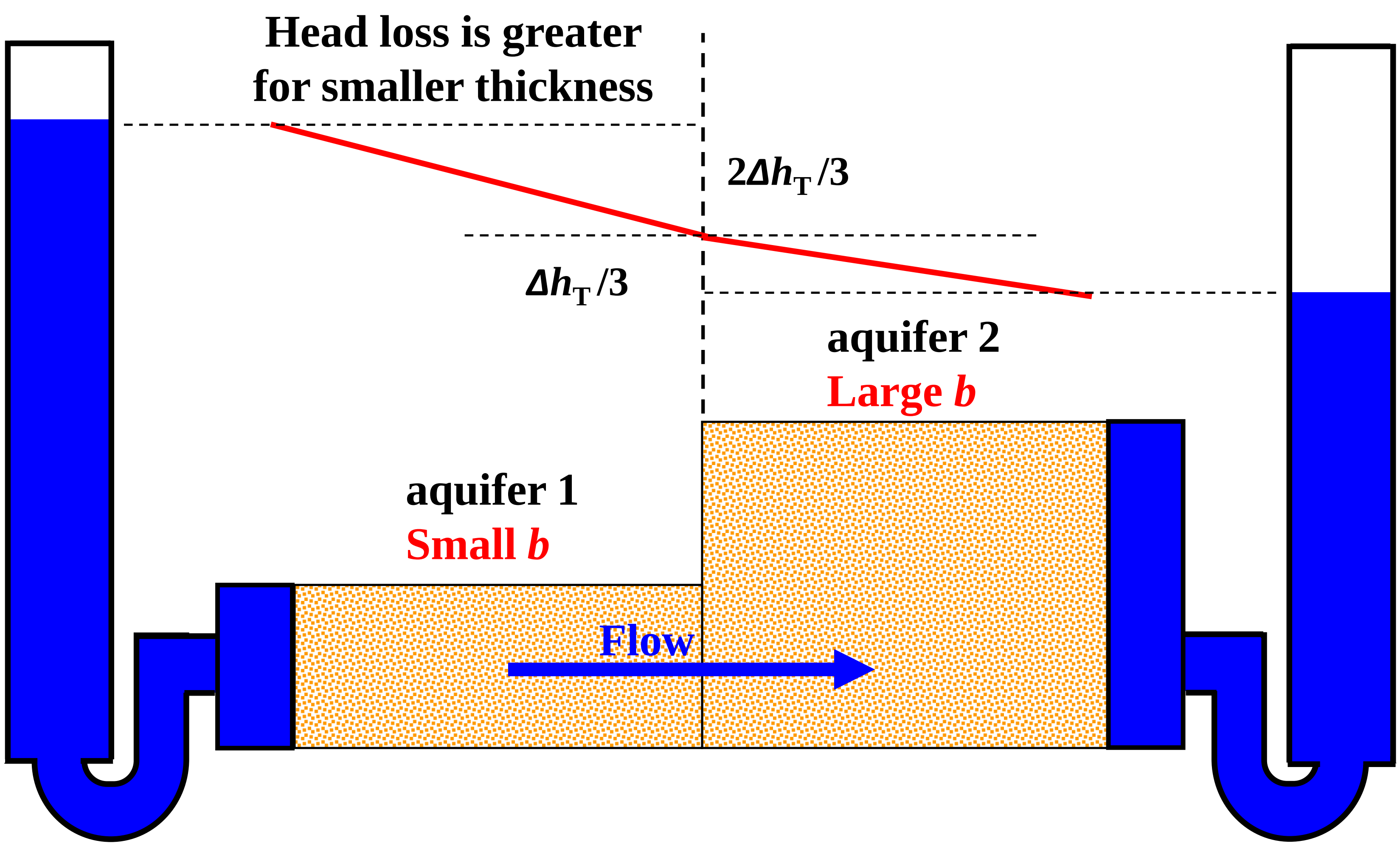
$$\Delta h_T = \Delta h_1 + \Delta h_2 = 2 \Delta h_2 + \Delta h_2 = 3 \Delta h_2$$

$$\Delta h_2 = \Delta h_T /3$$

$$\Delta h_1 = 2 \Delta h_T /3$$

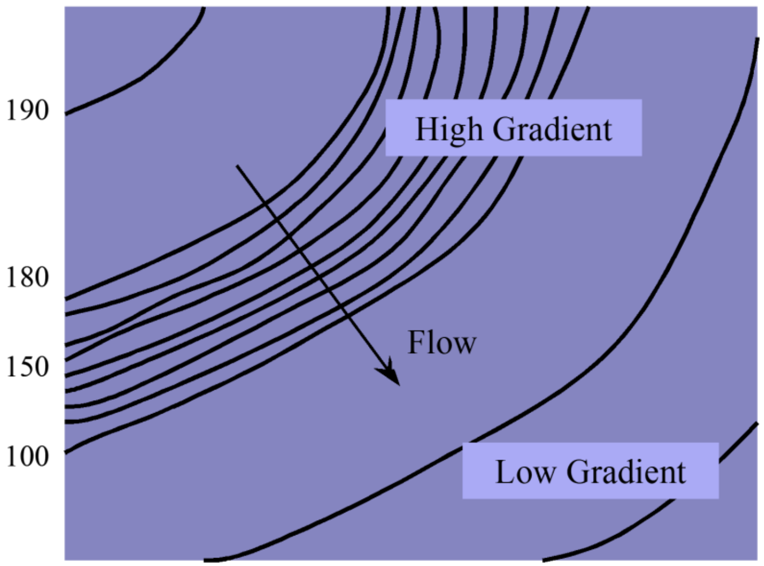


Changes in *b* can cause Changes in Head Gradient

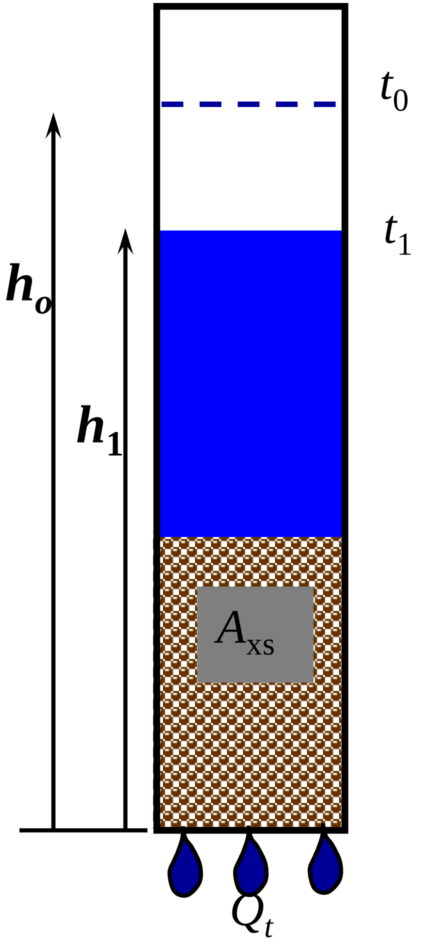


Head Distribution Reflects Transmissivity, *T*, not hydraulic conductivity *K*

- Groundwater computer models calculate the distribution of hydraulic head, try to match measured and calculated head
- Note that the head distribution reflects *T*, not *K*
- You can't determine *K* and *b* separately from head distribution (or hydraulic gradient) measurements. Must know one to calculate the other



How Fast is Groundwater Moving?



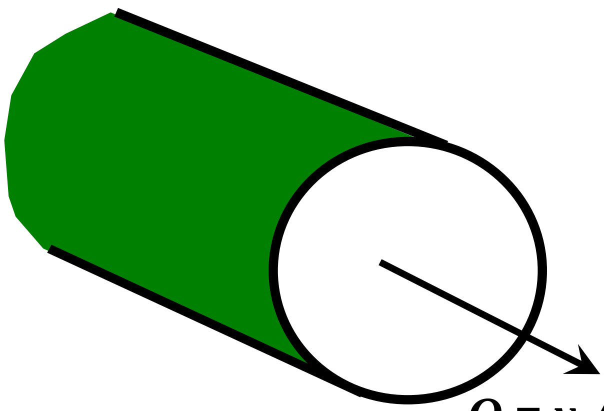
- Consider Darcy's experiment with a vertical sample
  - $Q_t = -KA_{xs}(h_t/L)$ Divide through by A_{xs}:
  - $Q_t/A_{xs} = -K(h_t/L) = q$ [m/s]
  - q = Specific Discharge (Darcy velocity)
  - $Q/A_{xs} = A_{xs}(h_o - h_1)/(t_1 - t_0)/A_{xs}$
  - $q = Q/A_{xs} = (h_o - h_1)/(t_1 - t_0)$

Specific Discharge – Darcy Velocity

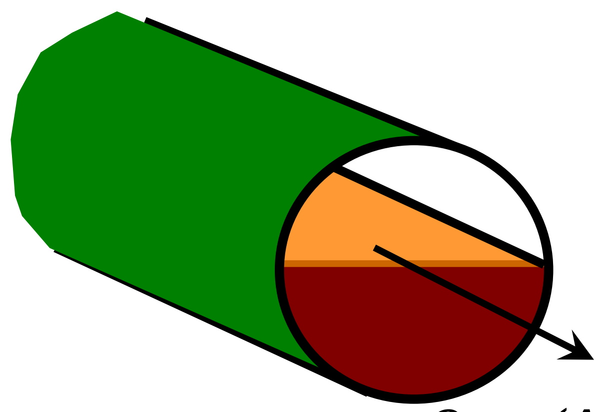
- Darcy Velocity is the velocity of water in the standpipe above the sample, not in the sample
- Specific discharge is an *apparent* velocity
  - Does not occur *in* porous media
- Also called an *approach* velocity
- It is the velocity of the water, IF the aquifer had been an open conduit
  - "Empty bed" velocity

How Fast is Groundwater Moving?

- How is groundwater velocity in the porous medium related to specific discharge?
- Consider a pipe carrying water under pressure
- If a pipe became half clogged, but the flow through the pipe was kept constant, the velocity would double.



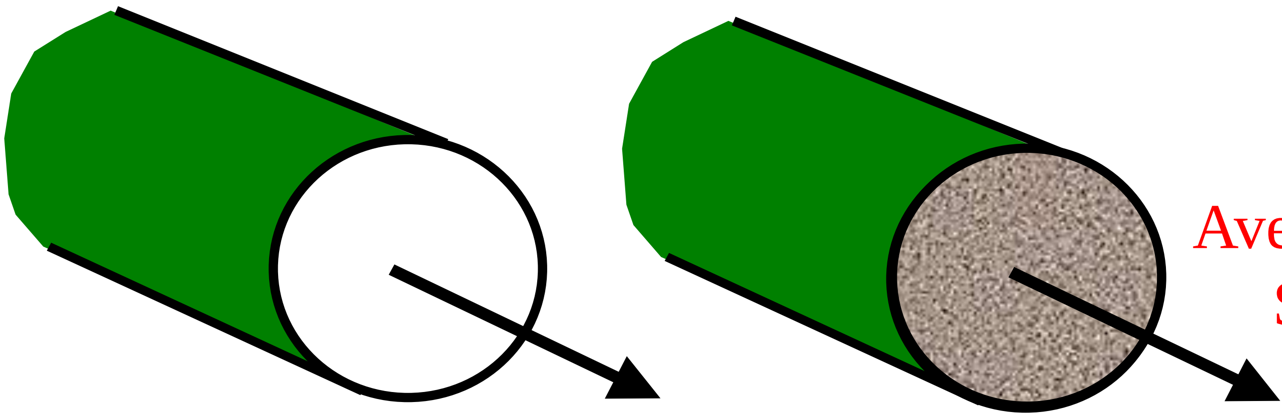
$Q = v_1 A_{xs}$
$v_1 = Q/A_{xs}$



$Q = v_2(A_{xs}/2)$
$v_2 = 2Q/A_{xs}$

Effect of Porosity on Velocity

- Similarly, if the pipe was filled with sand having a porosity of 50%, only half the area is available for flow
  - If the flow through the pipe was kept constant, the velocity would double
- The area available for flow is therefore $n_e A_{xs}$
- Groundwater velocity $v = Q/A_{flow} = Q/n_e A_{xs} = q/n_e$

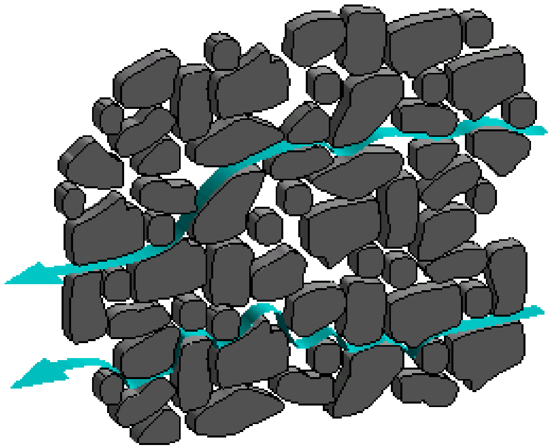


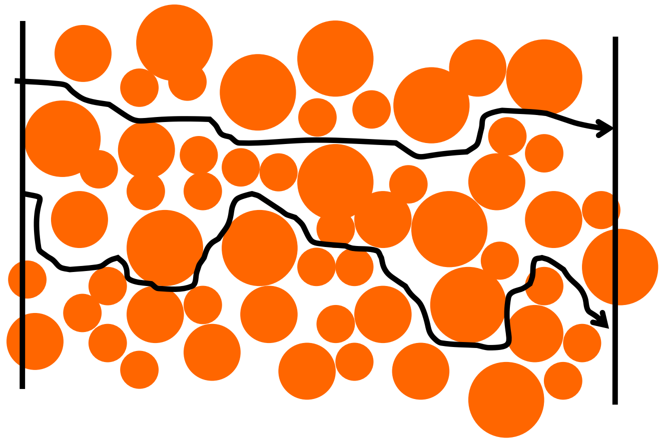
Average Linear Velocity Vs Microscopic Scale

Average linear velocity

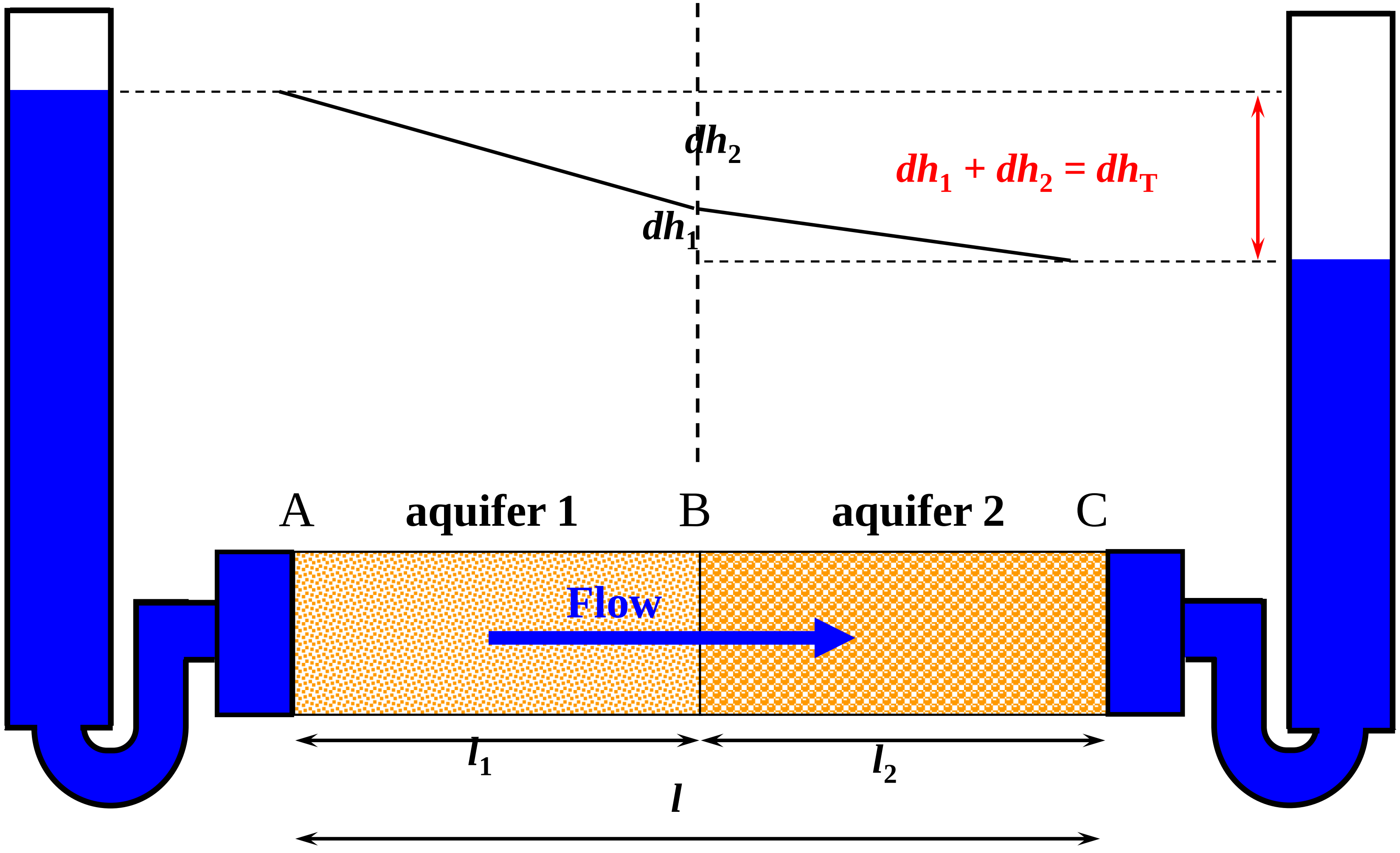
$$v = q/n_e = -K\,(dh/L)/n_e$$

- Pores have different sizes – velocity will differ in different size pores
- Water flowing near the pore walls will be slowed by viscosity, flow near the center of the pore throat will move fastest
- Flow paths are of different lengths, and some must split and branch around grains
- Actual *v* will vary about the mean

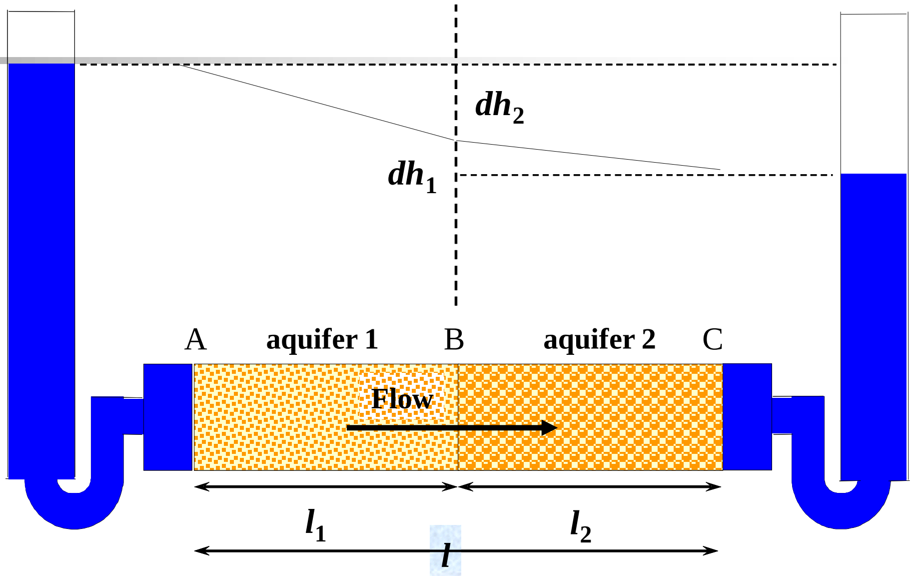




Flow Across Layers – Effective *K*



- Continuity: $Q_1 = Q_2$
- Head: $dh_1 + dh_2 = dh_T$
- Flow path: $l_1 + l_2 = l$
- Darcy's Law – solve for K_{eff}



$$Q = K_{eff} A \frac{dh_T}{l} = K_{eff} A \frac{dh_1 + dh_2}{l_1 + l_2}$$

$$K_{eff} = \frac{Q(l_1 + l_2)}{A(dh_1 + dh_2)}$$

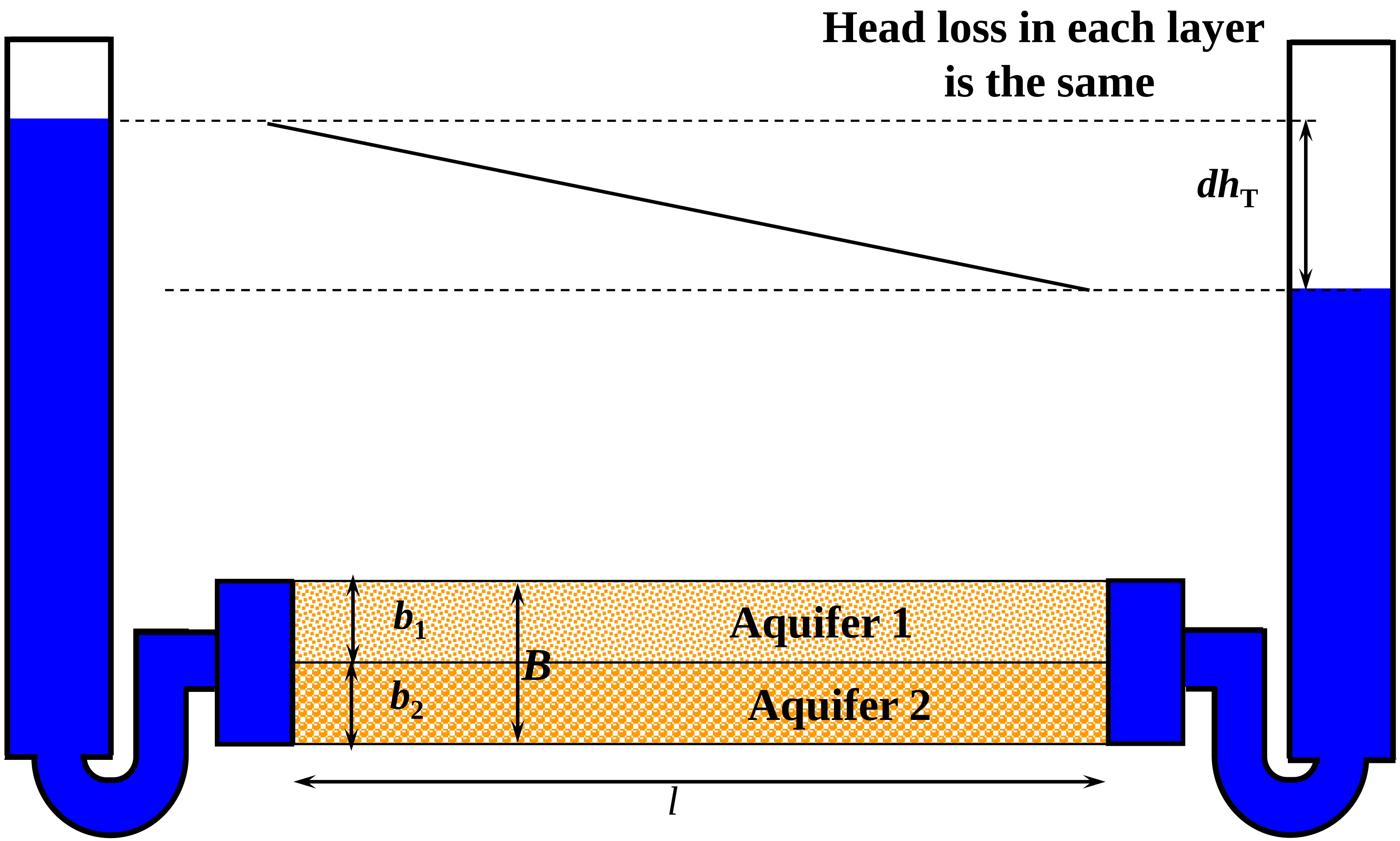
- **Darcy's Law – solve for dh_1 and dh_2**

$$Q = K_1 A \frac{dh_1}{l_1} \qquad dh_1 = \frac{Ql_1}{AK_1}$$

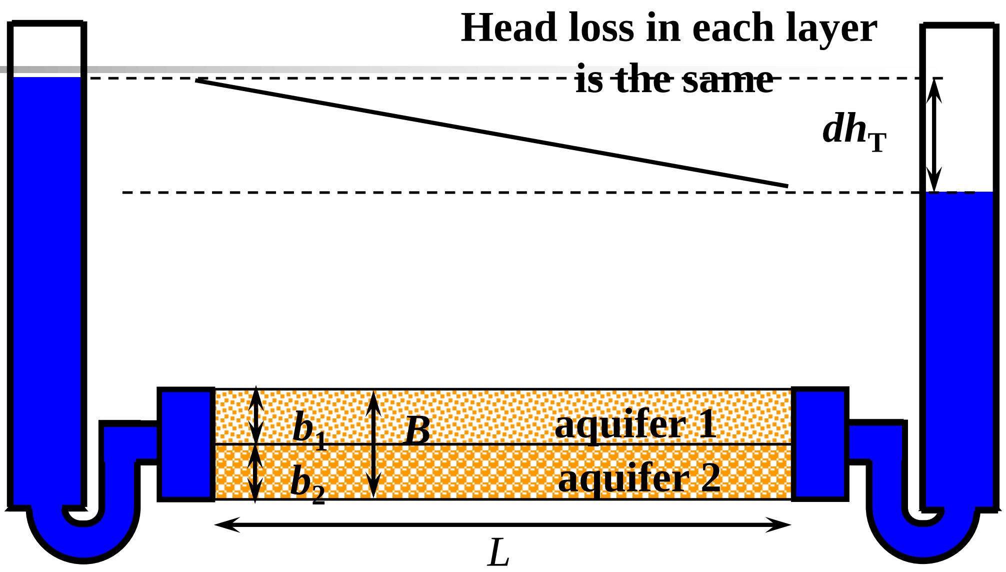
- **Substitute**

$$K_{eff} = \frac{(l_1 + l_2)}{\left[\frac{l_1}{K_1} + \frac{l_2}{K_2}\right]} = \frac{l}{\left[\frac{l_1}{K_1} + \frac{l_2}{K_2}\right]}$$

Flow Along Layers – Effective *K*



- Continuity: $Q_1 + Q_2 = Q_T$
- Head: $dh_1 = dh_2 = dh_T$
- Flow area: $b_1 w + b_2 w = A$
- Darcy's Law – solve for K_{eff}



$$Q_T = K_{eff}(b_1 + b_2)w\frac{dh_T}{L} \qquad K_{eff} = \frac{Q_T L}{(b_1 + b_2)w\,dh_T}$$

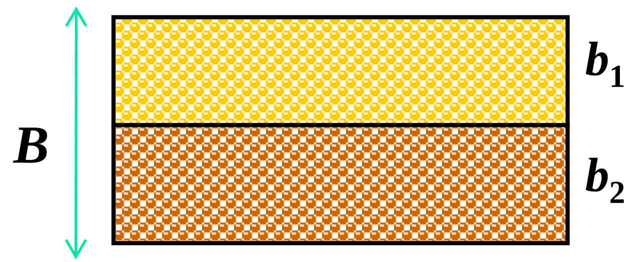
- **Darcy's Law – solve for Q_1**

$$Q_1 = K_1 b_1 w \frac{\Delta h_T}{l}$$

- **Substitute $Q_1 + Q_2 = Q_T$**

$$K_{eff} = \frac{K_1 b_1 + K_1 b_1}{(b_1 + b_2)} = \frac{K_1 b_1 + K_2 b_2}{B}$$

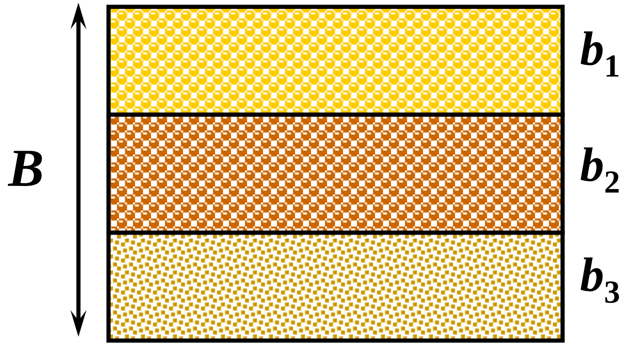
Vertical vs Horizontal *K*



- Vertical flow – across layers
- Horizontal flow – along layers
- Example
  - $K_1 = 1$ and $K_2 = 100$ m/d
  - $b_1 = 2$ and $b_2 = 2$ m
- Find K_{eff} for horizontal and vertical flow
- For vertical K (Flow across layers)
  - $K_{eff} = \frac{(b_1+b_2)}{\left[\frac{b_1}{K_1}+\frac{b_2}{K_2}\right]} = \frac{4}{\left[\frac{2}{1}+\frac{2}{100}\right]} = 1.98$ [m/d]
- For horizontal K (Flow along Layers)
  - $K_{eff} = \frac{K_1 b_1 + K_1 b_1}{(b_1+b_2)} = \frac{1\times2+100\times2}{4} = 50.5$ [m/d]

Vertical vs Horizontal *K*

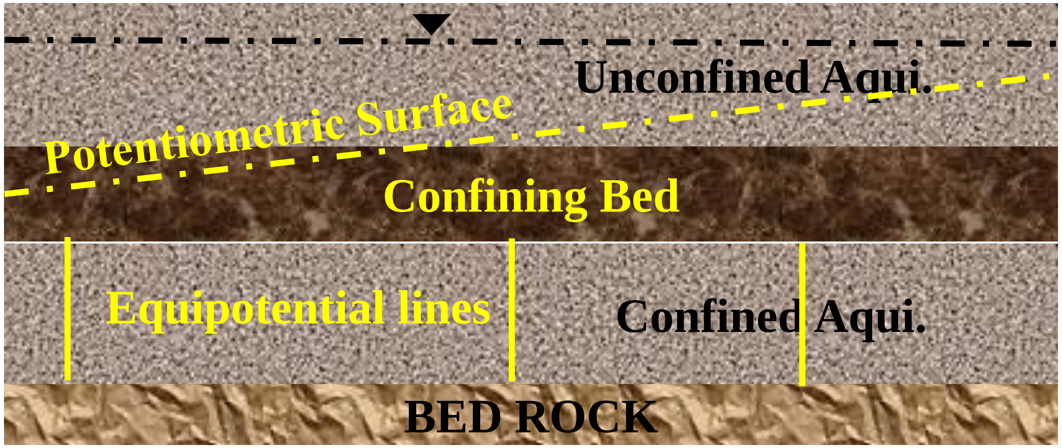
- Vertical effective conductivity is dominated by the layer having the lowest *K*
- Horizontal effective conductivity is dominated by the high *K* layer
- Horizontal effective *K* is much larger than the vertical effective *K*



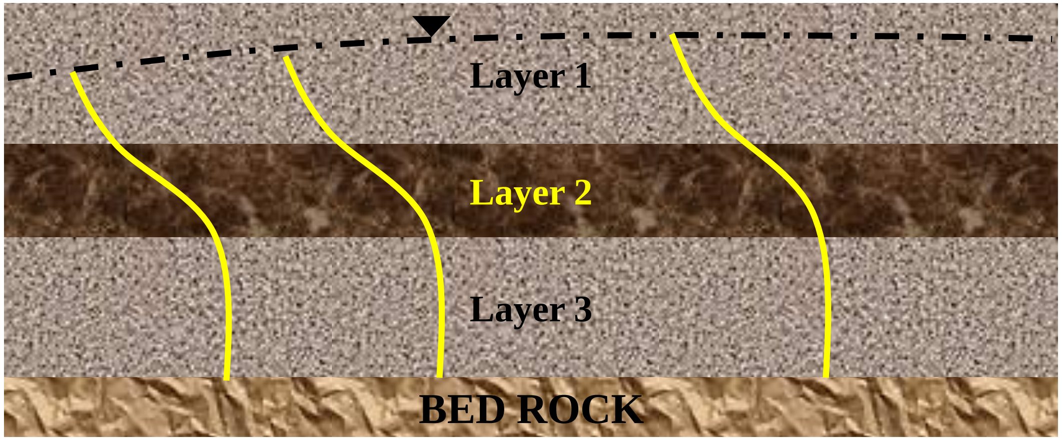
2. Governing equation

- Two conceptual views of groundwater:
  - Aquifer system view point
  - Flow system view point
- The aquifer view point:
  - Is based on the concept of confined and unconfined aquifers.
  - Is especially suited to analysis of flow to pumping wells
  - Is the basis for many analytical solutions including those of Theim, Theis and Jacob.
  - The groundwater flow assumed to be strictly horizontal through aquifers and strictly vertical through confining beds.
  - Is used to simulate two dimensional horizontal flow.
- In the flow system view point equipotential lines pass through all geologic units, both aquifers and confining beds.

The geologic system



The aquifer
System view point



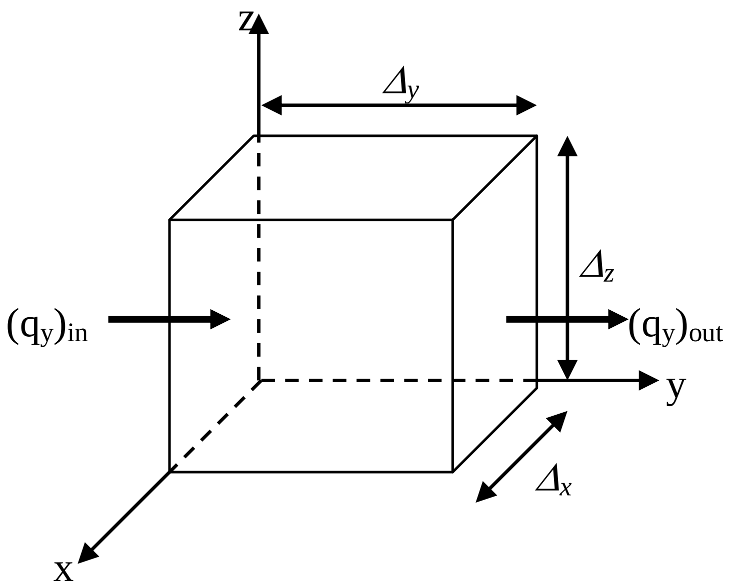
The flow system
view point

Modified from Anderson and Woessner, 1992

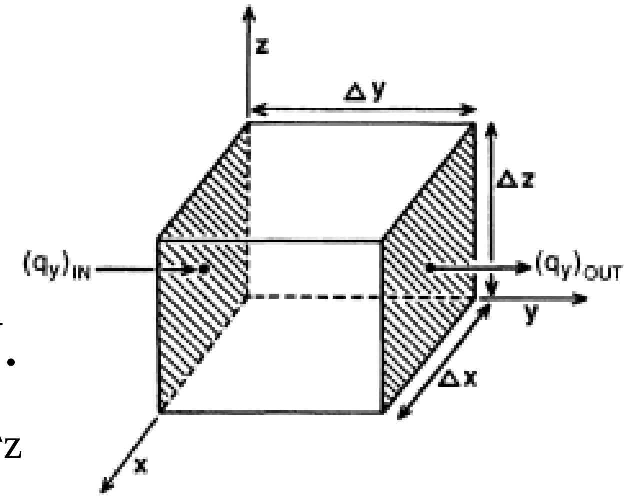
- The governing equation for water flow in saturated medium can be obtained by combining a special form of Darcy’s law (derived from the water phase momentum balance) and the continuity equation written for the water phase.
- The derivation is traditionally done by referring to a cube of porous material (Figure 1) that is large enough to be representative of the properties of the porous medium and yet small enough so that the change of head within the volume is relatively small (Anderson and Woessner, 1992).

Groundwater Flow Equation

- **Figure 1** Representative elementary volume used in the derivation
- The cube in Figure 1 is called the representative elementary volume (REV). Its volume is equal to $\Delta_x\Delta_y\Delta_z$. The flow of water through the REV is expressed in terms of the discharge rate (q), whose magnitude in the three coordinates will be q_x, q_y, and q_z.



- The water balance equation (conservation of mass) states that:
  - Mass Out – Mass In = Change of the Mass in storage
- Consider flow along the y-axis of the REV. Influx to REV occurs through the face $\Delta_x\Delta_z$ and is equal to $(q_y)_{in}$. Flux out is $(q_y)_{out}$.



The volumetric flow rate along y-axis is: $(q_{y,out} - q_{y,in})\Delta_x\Delta_z$

This can also be written as: $\frac{(q_{y,out} - q_{y,in})}{\Delta_y}\Delta_x\Delta_y\Delta_z$

Dropping the *'in'* and *'out'* subscripts, the change in flow rate through the REV along the y-axis is: $\frac{\partial q_y}{\partial y}\Delta_x\Delta_y\Delta_z$

- Similar expression can be written for the change in flow rate along the *x*- and *z*- axes. The total change in flow rate is equal to the rate of change in storage:

$$\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}\right)\Delta_x\Delta_y\Delta_z = Rate\ of\ \ Change\ in\ storage \qquad 1$$

- The existence of sink (e.g. a pumping well) or source of water (e.g. injection well or some other source of recharge) within the REV is undeniable. The **volumetric inflow rate** of such sources is represented by $R^*\Delta_x\Delta_y\Delta_z$. Here the R^* is defined to be intrinsically positive when it is a source of water; therefore it is added to the right hand side of Eq. 1. Therefore Eq. 1 becomes:

$$\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}-R^*\right)\Delta_x\Delta_y\Delta_z = Rate\ of\ \ Change\ in\ storage \qquad 2$$

- The change in storage is represented by specific storage (S_S). It is defined as the volume of water released from storage per unit change in head (h) per unit volume of aquifer (Anderson and Woessner, 1992) i.e.

$$S_S = -\frac{\Delta V}{\Delta h \Delta_x \Delta_y \Delta_z}$$

- The sign convention is that the ΔV is intrinsically positive when the Δh is negative, in other words, water is released from the REV when head decreases.
- The rate of change in storage in REV will be:

$$\frac{\Delta V}{\Delta t} = -S_S \frac{\Delta h}{\Delta t} \Delta_x \Delta_y \Delta_z \qquad 3$$

- Combining Eq. 2 and Eq. 3:

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} = -S_S \frac{\partial h}{\partial t} + R^* \qquad 4$$

- Darcy law is used to set the relationship between *q* and *h*. Darcy law in three dimension is written as (Anderson and Woessner, 1992):

$$q_x = -K_x \frac{\partial h}{\partial x} \qquad q_y = -K_y \frac{\partial h}{\partial y} \qquad q_z = -K_z \frac{\partial h}{\partial z}$$

- Substituting these expressions in Eq. A.4 the desired groundwater flow equation is formulated:

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial h}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial h}{\partial z}\right) = S_S \frac{\partial h}{\partial t} - R^*$$

- Where K_x, K_y, and K_z are components of the hydraulic conductivity.

- In the above derivation it is assumed that K_x, K_y, and K_z are collinear to the x, y- and z- axes.
- If the geology is such that it is not possible to align the principal direction of the hydraulic conductivity tensor with the rectilinear coordinate system, a modified form of equation that utilizes the hydraulic conductivity tensor is required.

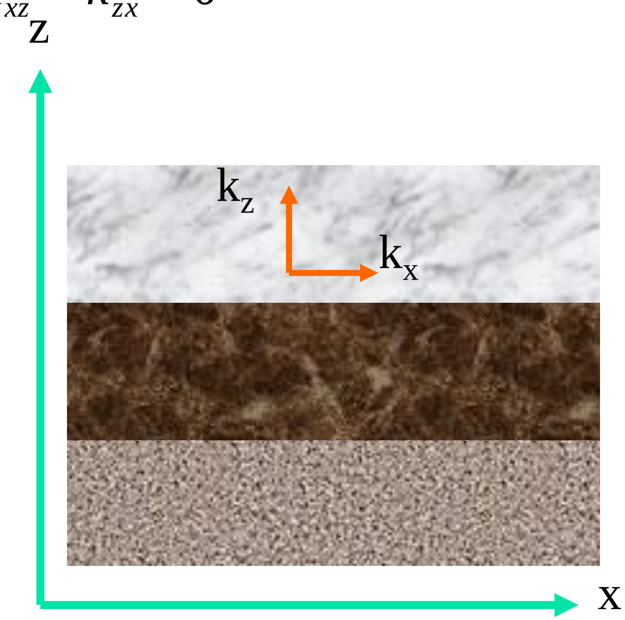
$$K = \begin{bmatrix} K_{xx} & K_{xy} & K_{xz} \\ K_{yx} & K_{yy} & K_{yz} \\ K_{zx} & K_{zy} & K_{zz} \end{bmatrix}$$

- By using a global coordinate system for the entire problem domain and a local coordinate system for each REV in the grid, the off diagonal terms in the hydraulic conductivity tensor could have zero value (Anderson and Woessner, 1992).

$$K = \begin{bmatrix} K_x & 0 & 0 \\ 0 & K_y & 0 \\ 0 & 0 & K_z \end{bmatrix}$$

$$\frac{\partial}{\partial x}\left(k_{xx}\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial z}\left(k_{zz}\frac{\partial h}{\partial z}\right)=S_S\frac{\partial h}{\partial t}-R^*$$

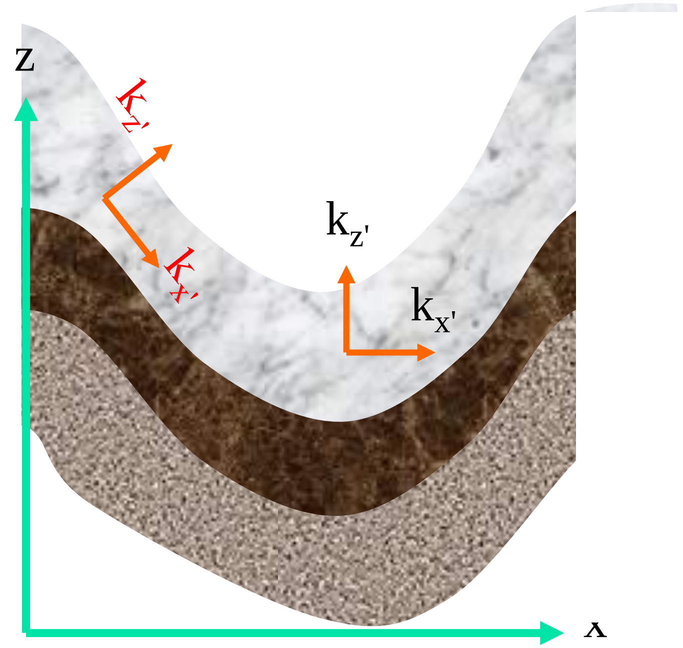
$$k_{xz}=k_{zx}=0$$



- The x-z coordinate system is aligned with the principal directions of the hydraulic conductivity tensor.

$$\frac{\partial}{\partial x}\left(k_{xx}\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial x}\left(k_{xz}\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial z}\left(k_{zz}\frac{\partial h}{\partial z}\right)+$$

$$\frac{\partial}{\partial z}\left(k_{zx}\frac{\partial h}{\partial z}\right)=S_S\frac{\partial h}{\partial t}-R^*;\quad k_{xz}\neq 0; k_{zx}\neq 0$$



A global coordinate system (x-z) is defined. Local coordinates (x'-z') are aligned with the principal directions of the local hydraulic conductivity tensor.

3. Initial and boundary conditions

- For a well posed boundary value problem: (i) A solution must exist, (ii) The solution must be unique and (iii) The solution must be stable, in the sense that sufficiently small variations in the given data should lead to arbitrary small changes in the solution
- Initial and boundary conditions are needed for a unique solution of the groundwater flow equations (second-order partial differential equations) for a specific flow domain of interest
- **Initial conditions:** specification of the distribution of the state variable (hydraulic head for the groundwater flow equation) at some initial time, usually at t = 0.
- For example $h = h(x, y, z, 0) = f(x, y, z)$ in D
- in which $f(x,y,z)$ is a known function, D is the flow domain.

- **Boundary conditions:** specification of the interaction between the flow domain and its surrounding environment, which is a mathematical representation of the physical reality
  - Known water fluxes
  - Known values of state variables, such as hydraulic head, that the external domain imposes on the flow regime
- Different initial and boundary conditions result in different solutions
- Three mathematical boundary conditions:
  1. Dirichlet
  2. Neumann
  3. Cauchy

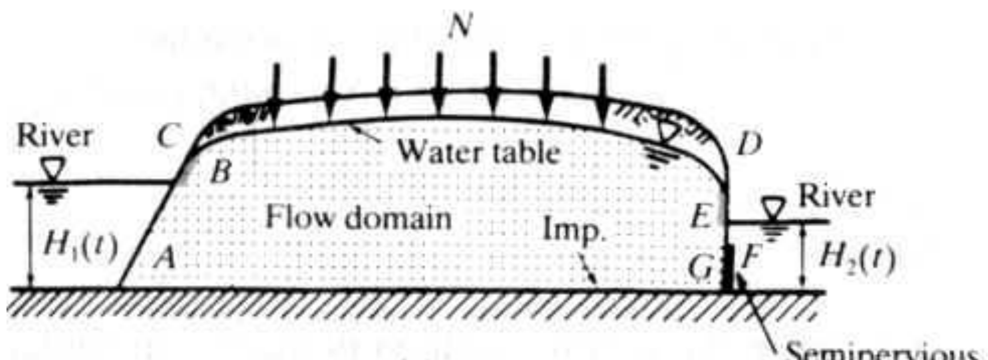
- Three mathematical boundary conditions
  - **Dirichlet condition** (boundary condition of the first kind): the fluid pressure (or hydraulic head) is specified as a known function of space and time.
    - This occurs whenever the porous medium flow domain is in contact with a body of open water (AB, EG surfaces)

$$p(\mathbf{x},t) = f(\mathbf{x},t) \quad \text{on B}$$

$$h(\mathbf{x},t) = g(\mathbf{x},t) \quad \text{on B}$$

***f* and *g* are two known functions**

  - Special case : Equipotential boundary



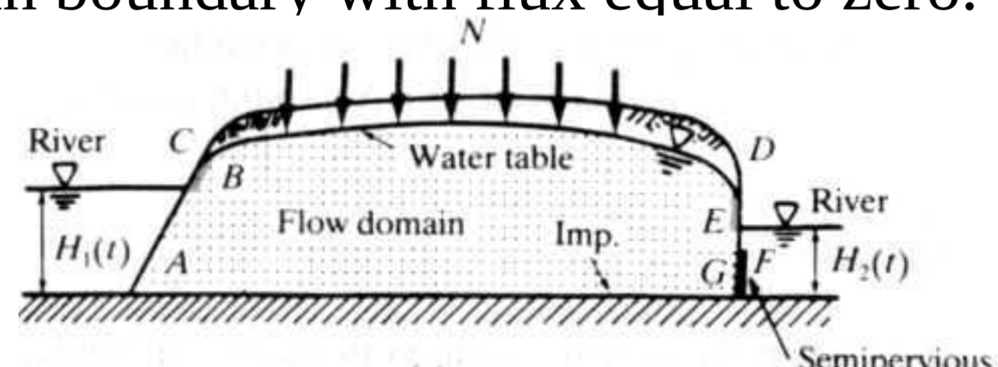
(a)

- **Neumann condition** (boundary condition of the second kind): the pressure gradient (or hydraulic gradient), or a linear combination of their components, is specified as a known function of space and time on the boundary.

$$q_r \cdot \upsilon = m(x,t) \; on \; B$$

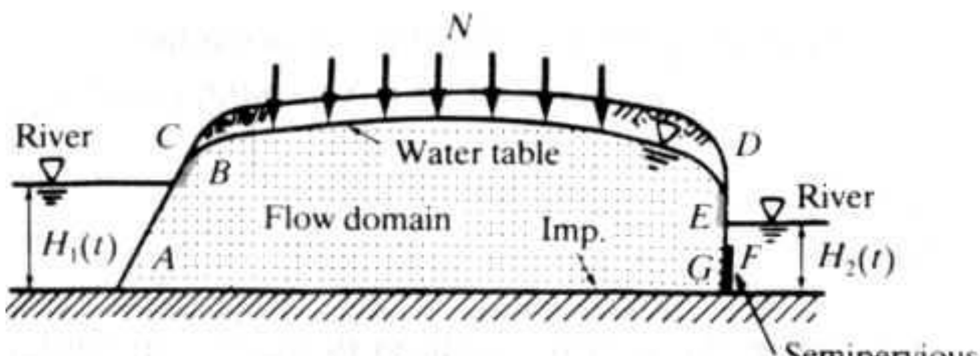
m(x, t) is a known function

- This occurs when constant flux (discharge) is seen across a certain portion of the boundary (BE).
- Thus an impervious boundary (Boundary along AG) is Neumann boundary with flux equal to zero.



(a)

- **Cauchy,** Mixed boundary condition, boundary condition of the third kind) : the condition which specifies the information on the relationship between the state variable and its derivatives
- This occurs when the porous medium domain is in contact with a body of water continuum (or another porous medium domain) through a relatively thin semi pervious layer separating the two domains (e.g., FG in the bottom figure)



(a)

Analytical Method Example:

- **The ends A and B of a soil column, 200 cm long, have head at 0 cm and 40 cm until steady state prevails. If the head of the ends are changed to 0 cm. Find the head distribution in the soil column at any time t. Take Ss as 10^{-3} and K as 10^{-5} cm/s.**

$$Ss\frac{\partial h}{\partial t} = K\frac{\partial^2 h}{\partial x^2} \Rightarrow \frac{\partial h}{\partial t} = \frac{10^{-5}}{10^{-3}}\frac{\partial^2 h}{\partial x^2}$$

$$\Rightarrow \frac{\partial h}{\partial t} = 0.01\frac{\partial^2 h}{\partial x^2}$$

For steady state:

$$\frac{\partial h}{\partial t} = 0.01\frac{\partial^2 h}{\partial x^2} = 0 \Rightarrow \frac{\partial^2 h}{\partial x^2} = 0$$

$$\text{Upon Integration}: h(x) = c_1 x + c_2$$

$$h(0) = c_1 \times 0 + c_2 = 0 \Rightarrow c_2 = 0cm$$

$$h(x) = c_1 x$$

$$h(200) = c_1 \times 200 = 40 \Rightarrow c_1 = 0.2$$

$$h(x) = 0.2x$$

Since the head at A and B are suddenly changed we gain transient state whose initial condition could be described by the above equation.

$$\frac{\partial h}{\partial t} = 0.01 \frac{\partial^2 h}{\partial x^2}$$

Let $h = TX$ *is the solution*

$$\frac{\partial h}{\partial t} = \frac{dT}{dt} X ; \frac{\partial h}{\partial x} = T \frac{dX}{dx}$$

$$\Rightarrow \frac{\partial^2 h}{\partial x^2} = T \frac{d^2 X}{dx^2}$$

Substituting :

$$\frac{dT}{dt} X = 0.01 T \frac{d^2 X}{dx^2}$$

$$\frac{dT}{Tdt} = 0.01 \frac{d^2 X}{Xdx^2}$$

$$\frac{dT}{Tdt} = -c^2 \Rightarrow \frac{dT}{T} = -c^2 dt$$

$$\ln \frac{T}{c_1} = -c^2 t \Rightarrow T = c_1 e^{-c^2 t}$$

$$0.01 \frac{d^2 X}{Xdx^2} = -c^2 \Rightarrow \frac{d^2 X}{dx^2} + 100 c^2 X = 0$$

Solving this ODE

$$X = c_2 \sin(10cx) + c_3 \cos(10cx)$$

thus $h = TX$

$$\Rightarrow h = c_1 e^{-c^2 t} \left(c_2 \sin(10cx) + c_3 \cos(10cx) \right)$$

$$h = e^{-c^2 t} \left(C_1 \sin(10cx) + C_2 \cos(10cx) \right)$$

$$h(0,t) = e^{-c^2 t} \left(C_1 \times 0 + C_2 \times 1 \right) = 0 \Rightarrow C_2 = 0$$

$$\Rightarrow h(x,t) = C_1 e^{-c^2 t} \sin(10cx)$$

$$h(200,t) = C_1 e^{-c^2 t} \sin(10c \times 200) = 0$$

$$\sin(2000c) = 0 = \sin(n\pi) \Rightarrow c = \frac{n\pi}{2000}$$

The general solution would be

$$h(x,t)=\sum_{n=0}^{\infty} b_n e^{-\left(\frac{n\pi}{2000}\right)^2 t}\sin\left(\frac{n\pi x}{200}\right);\ at\ t=0\ h=0.2x \Rightarrow x=\sum_{n=0}^{\infty} 5b_n\sin\left(\frac{n\pi x}{200}\right)$$

$$\Rightarrow 5b_n=\frac{2}{200}\int_0^{200} x\sin\left(\frac{n\pi x}{200}\right)dx\ \ Note: Fourier\ \sin e\ series$$

$$5b_n=\frac{2}{200}\left[x\frac{200}{n\pi}\left(-\cos\left(\frac{n\pi x}{200}\right)\right)-\left(\frac{200}{n\pi}\right)^2\sin\left(\frac{n\pi x}{200}\right)\right]_0^{200}$$

$$5b_n=(-1)^{n+1}\frac{400}{n\pi}\Rightarrow b_n=(-1)^{n+1}\frac{80}{n\pi}$$

$$h(x,t)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{80}{n\pi}e^{-\left(\frac{n\pi}{2000}\right)^2 t}\sin\left(\frac{n\pi x}{200}\right)$$

$$h(x,t)=\frac{80}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-\left(\frac{n\pi}{2000}\right)^2 t}\sin\left(\frac{n\pi x}{200}\right)$$

4. Dupuit assumption

- The boundary of an unconfined aquifer (z) is indeed the solution (h) that needs to be determined.
- Dupuit assumptions: First developed by Dupuit (1863) and then advanced by Forchheimer (1930), or called Dupuit-Forchheimer theory
  - From observations, the slope of phreatic surface (water table) is very small (commonly 1/1000)
  - Two assumptions
    - The hydraulic gradient is equal to the slope of the free surface and is invariant with depth
    - The equipotential lines are vertical, i.e., the flow lines are horizontal, i.e.,

$$\frac{\partial p}{\partial z} = -\rho g$$

$$q_s = -K\frac{dh}{ds}$$

$$= -K\frac{dz}{ds}$$

$$= -K\sin\theta$$

$$\approx -K\tan\theta$$

$$= -K\frac{dh}{dx}$$

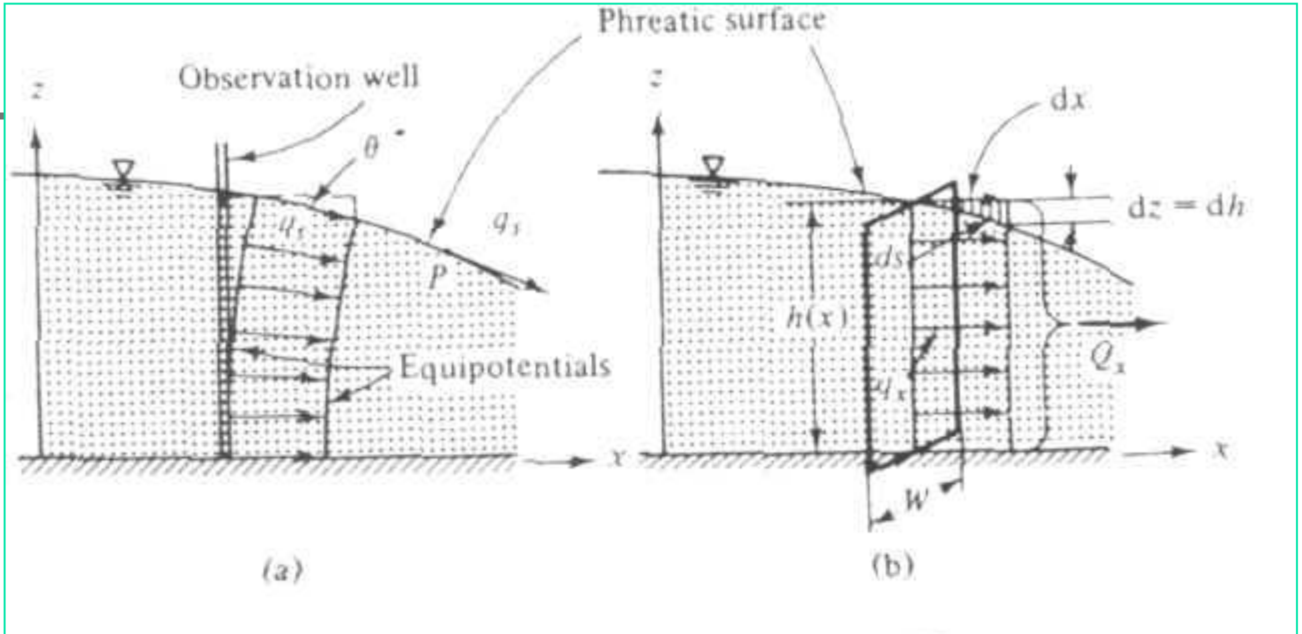


Fig. 2.6. The Dupuit assumption (Bear, 1979).

a. The real flow field with non-vertical equipotential lines near the water table

b. The flow field obtained by the Dupuit assumption, i.e., vertical equipotential lines

For small θ, sin θ can be replaced by tan θ, then

$$q_s \equiv q_x = -K\tan\theta = -K\frac{dh}{dx} \qquad (\text{for } h = h(x))$$

(A) For horizontal bottom and 3-D steady-state, free surface flows

$h(x,y,z) \rightarrow h(x,y)$ because the assumption of vertical equipotential lines (or horizontal flows)

$$\left.\begin{aligned} q_x &= -K\frac{\partial h}{\partial x} \\ q_y &= -K\frac{\partial h}{\partial y} \end{aligned}\right\} \Rightarrow \mathbf{q} = -K\nabla' h \quad or \quad \left.\begin{aligned} Q_x &= -KWh\frac{\partial h}{\partial x} \\ Q_y &= -KWh\frac{\partial h}{\partial y} \end{aligned}\right\} \Rightarrow \mathbf{Q} = -KWh\nabla' h \text{ or } \mathbf{Q}/W = -Kh\nabla' h$$

in which $\nabla' = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j}$ (W is the width of the unconfined aquifer)

$\left(\text{Note that for 1-D flows, } h(x,y) \rightarrow h(x) \text{ and } \nabla' \text{ is replaced by } \frac{d}{dx}\right)$

$\left(\text{Note that } \left[\frac{\mathbf{Q}}{W}\right] = \left[\frac{Q_x}{W}\right] = \left[\frac{Q_y}{W}\right] = \frac{L^2}{T} = \text{discharge per unit width}\right)$

Example : two-dimensional steady-state flow without accretion

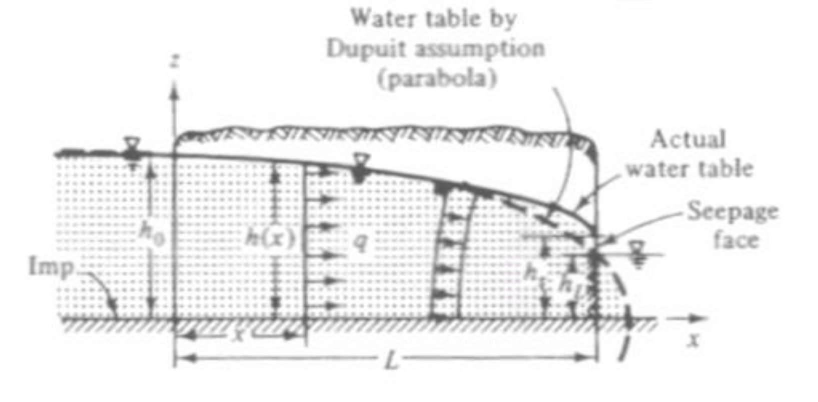


Fig. 2.7. Steady unconfined flow between two reservoirs (Bear, 1979).

(After Bear and Verruijt, 1987)

$$Q_x = -Kh\frac{dh}{dx} = \text{constant}$$

$$\Rightarrow Q_x dx = -Khdh$$

$$\Rightarrow Q_x \int_0^L dx = -K\int_{h_0}^{h_L} hdh = K\frac{\left(h_0^2 - h_L^2\right)}{2}$$

$$\Rightarrow Q_x = \frac{K\left(h_0^2 - h_L^2\right)}{2L} \qquad (1)$$

(Dupuit equation)

(Q_x = flow per unit width)

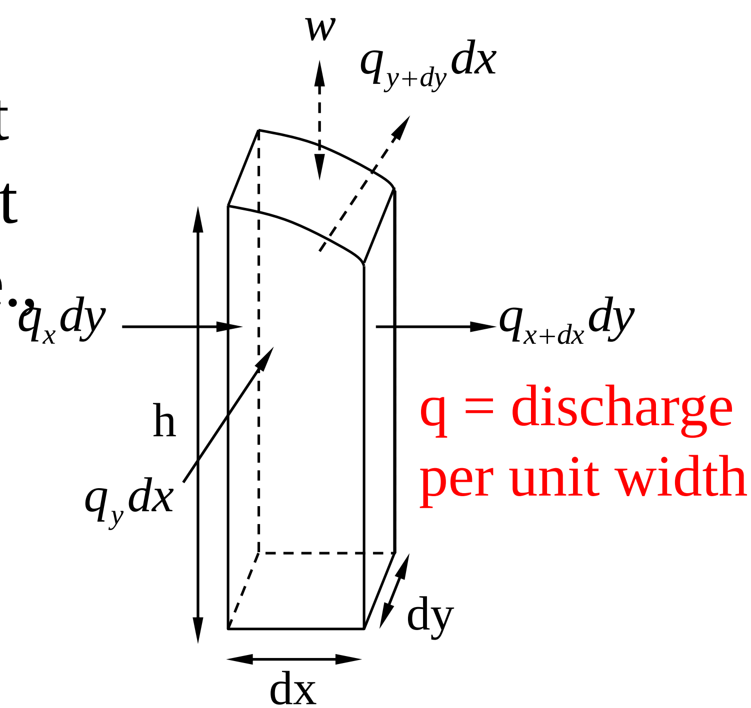
Example : three-dimensional steady-state flow with accretion

Mass in – mass out = Δ M = 0 (steady state)

Under Dupuit assumptions : $h(x, y, z) \rightarrow h(x, y)$

w [L/T] : rate of water into or out of the unconfined aquifer per unit area of the unconfined aquifer i.e., $w > 0$ for infiltration, $w < 0$ for evaporation

(Adapted from Fetter, 1994)



(impervious, horizontal bottom)

$$\rho q_x dy\Delta t - \rho q_{x+dx} dy\Delta t + \rho q_y dx\Delta t - \rho q_{y+dy}\Delta t + \underbrace{\rho w dxdy\Delta t}_{\text{sink/source}} = 0$$

$$\Rightarrow \rho\Delta t dy\left[\underbrace{-Kh\left(\frac{\partial h}{\partial x}\right)_x}_{q_x} + \underbrace{K\left(h\frac{\partial h}{\partial x}\right)_{x+dx}}_{q_{x+\Delta x}} \right] + \rho\Delta t dx\left[\underbrace{-Kh\left(\frac{\partial h}{\partial y}\right)_y}_{q_y} + \underbrace{K\left(h\frac{\partial h}{\partial y}\right)_{y+dy}}_{q_{y+\Delta y}} \right]$$

$$+\rho w dxdy\Delta t = 0$$

$$\Rightarrow \rho\Delta t dy\left[K\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right)dx\right] + \rho\Delta t dx\left[K\frac{\partial}{\partial y}\left(h\frac{\partial h}{\partial y}\right)dy\right] + \rho w dxdy\Delta t = 0$$

$$\Rightarrow K\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right)dxdy + K\frac{\partial}{\partial y}\left(h\frac{\partial h}{\partial y}\right)dxdy + wdxdy = 0$$

$$\Rightarrow K\left(\frac{\partial^2 h^2}{\partial x^2} + \frac{\partial^2 h^2}{\partial y^2}\right) = -2w \qquad (2)$$

Furthermore, for one-dimensional flows, Eq(2) reduces to

$$K\frac{d^2}{dx^2}\left(h^2\right) = -2w$$

BC's : $h(x=0) = h_1$, $h(x=L) = h_2$

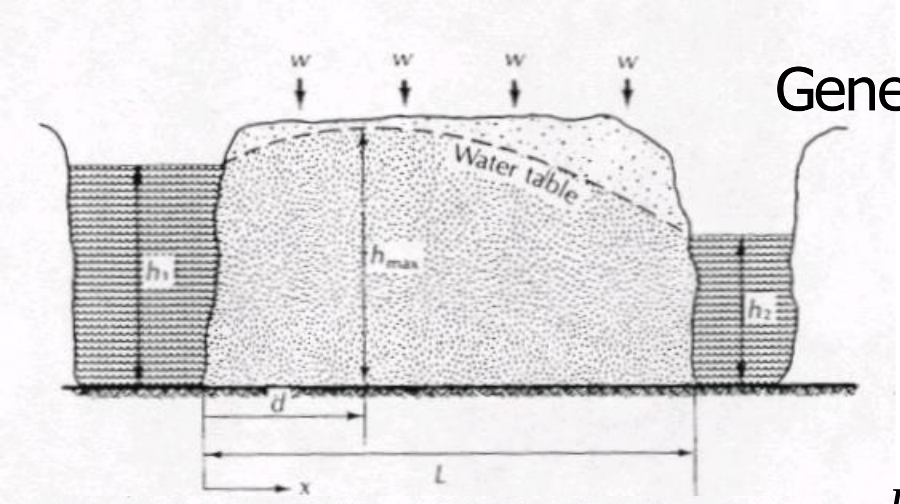


FIGURE 5.19 Unconfined flow, which is subject to infiltration or evaporation.

General solution : $h^2(x) = -\frac{w}{K}x^2 + c_1 x + c_2$

$$h(x=0) = h_1 \Rightarrow c_2 = h_1^2$$

$$h(x=L) = h_2 \Rightarrow c_1 = \left(\frac{h_2^2 - h_1^2}{L}\right) + \frac{wL}{K}$$

$$h^2(x) = h_1^2 - \frac{\left(h_1^2 - h_2^2\right)x}{L} + \frac{w}{K}(L-x)x$$

(After Fetter, 1994)

Hence

$$h(x) = \sqrt{h_1^2 - \frac{\left(h_1^2 - h_2^2\right)x}{L} + \frac{w}{K}(L-x)x} \quad (3)$$

From Eq. (3)

$$h(x)=\sqrt{h_1^2-\frac{\left(h_1^2-h_2^2\right)x}{L}+\frac{w}{K}(L-x)x}=\sqrt{-ax^2+bx+c}=\sqrt{T}$$

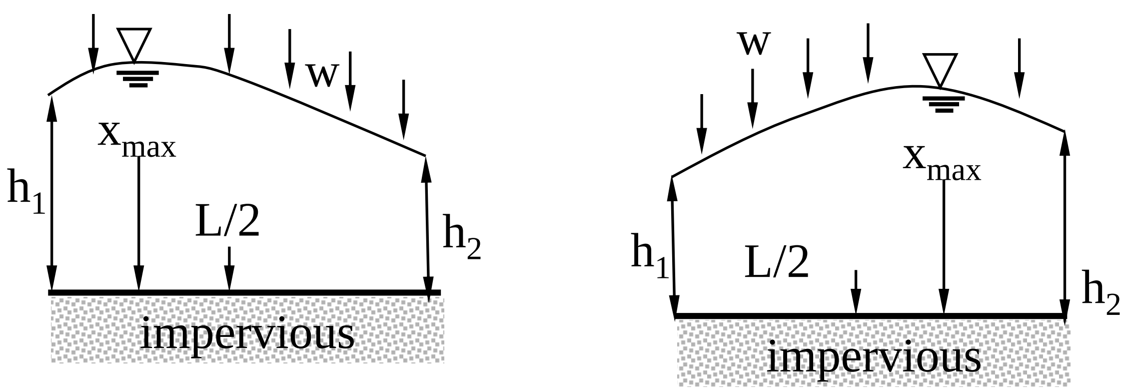
$$a=\frac{w}{K}>0\ ,\ b=\frac{\left(h_2^2-h_1^2\right)}{L}+\frac{wL}{K},\ c=h_1^2>0$$

$$\Rightarrow \begin{array}{l} \dfrac{dh}{dx}=\dfrac{-2ax+b}{2\sqrt{T}},\ x=\dfrac{b}{2a}=\dfrac{L}{2}+\dfrac{K\left(h_2^2-h_1^2\right)}{2wL}\ \text{ where }\ \dfrac{dh}{dx}=0 \\ \dfrac{d^2h}{dx^2}=\dfrac{-4ac-b^2}{4\sqrt{T^3}}<0 \end{array}$$

- Hence, the water table surface is a hyperbola with maximum elevation occurs at

$$x_{max} = \frac{L}{2} + \frac{K\left(h_2^2 - h_1^2\right)}{2wL}, \quad h_{max} = \sqrt{\frac{\left(h_1^2 + h_2^2\right)}{2} + \frac{wL^2}{4K} + \frac{K\left(h_2^2 - h_1^2\right)^2}{4wL^2}}$$

- The location of maximum h occurs to the left of the midpoint if $h_2 < h_1$, or to the right if $h_2 > h_1$.

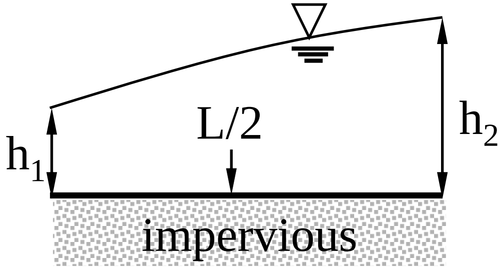


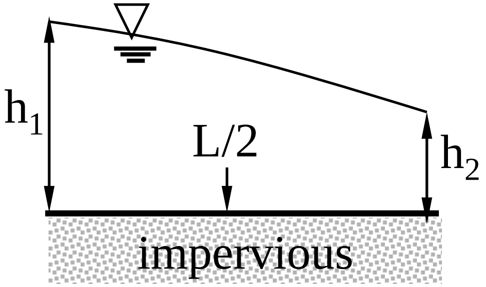
From Eq(3) and if w = 0 then

$$h(x)=\sqrt{h_1^2-\frac{\left(h_1^2-h_2^2\right)x}{L}}=\sqrt{ax+b}=\sqrt{T},\ a=\frac{h_2^2-h_1^2}{L}\ ,\ b=h_1^2>0$$

$$\Rightarrow \begin{aligned} \frac{dh}{dx}&=\frac{a}{2\sqrt{ax+b}} \\ \frac{d^2h}{dx^2}&=-\frac{a^2}{4\left(ax+b\right)^{3/2}}<0 \end{aligned}$$

Hence, the water table surface is a parabola with a positive slope when $h_2 > h_1$, or a negative slope when $h_2 < h_1$



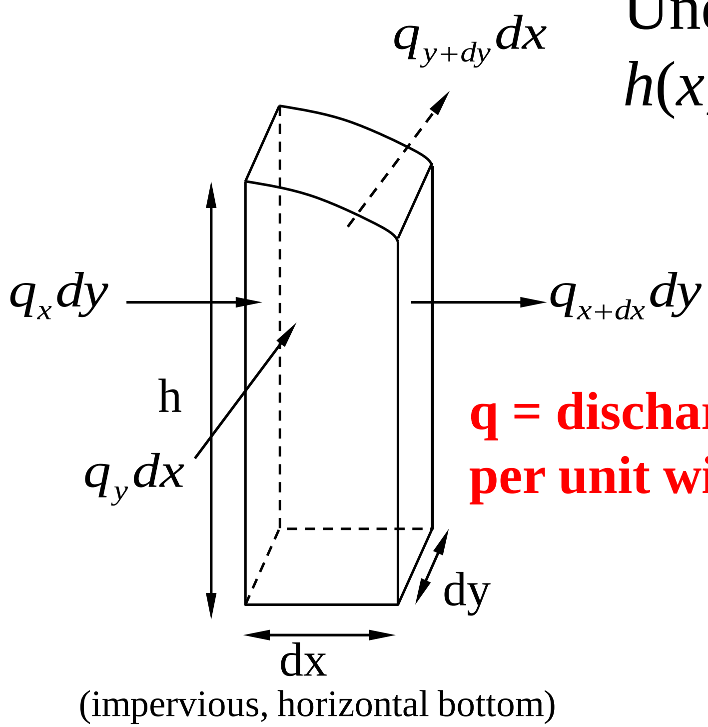


- Transient 2-D unconfined flows

For incompressible fluids and homogeneous and isotropic aquifers

Mass in – mass out = Δ M

Under Dupuit assumptions: $h(x,y,z) \rightarrow h(x,y)$



q = discharge per unit width

$$\rho q_x dy\Delta t - \rho q_{x+dx} dy\Delta t + \rho q_y dx\Delta t - \rho q_{y+dy}\Delta t = \rho S_y dxdy\Delta h$$

$$\Rightarrow \rho\Delta t dy\left[\underbrace{-Kh\left(\frac{\partial h}{\partial x}\right)_x}_{q_x} + \underbrace{K\left(h\frac{\partial h}{\partial x}\right)_{x+dx}}_{q_{x+\Delta x}}\right]$$

$$S_y = \text{specific yield} \equiv \frac{\Delta V_w}{A\Delta h}$$

$$+\rho\Delta t dx\left[\underbrace{-Kh\left(\frac{\partial h}{\partial y}\right)_y}_{q_y} + \underbrace{K\left(h\frac{\partial h}{\partial y}\right)_{y+dy}}_{q_{y+\Delta y}}\right] = \rho S_y dxdy\Delta h$$

$$\Rightarrow K\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right)dxdy + K\frac{\partial}{\partial y}\left(h\frac{\partial h}{\partial y}\right)dxdy = S_y dxdy\frac{\partial h}{\partial t}$$

$$\Rightarrow \frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial h}{\partial y}\right) = \frac{S_y}{K}\frac{\partial h}{\partial t} \quad \text{(Boussinesq equation)} \qquad (4)$$

- Boussinesq equation is a non-linear PDE, which can not be solved analytically except under some idealized conditions
- Approximations: Drawdown in the aquifer is small, i.e., h ≈ b (averaged thickness assumed to be constant over the aquifer)

From (4):

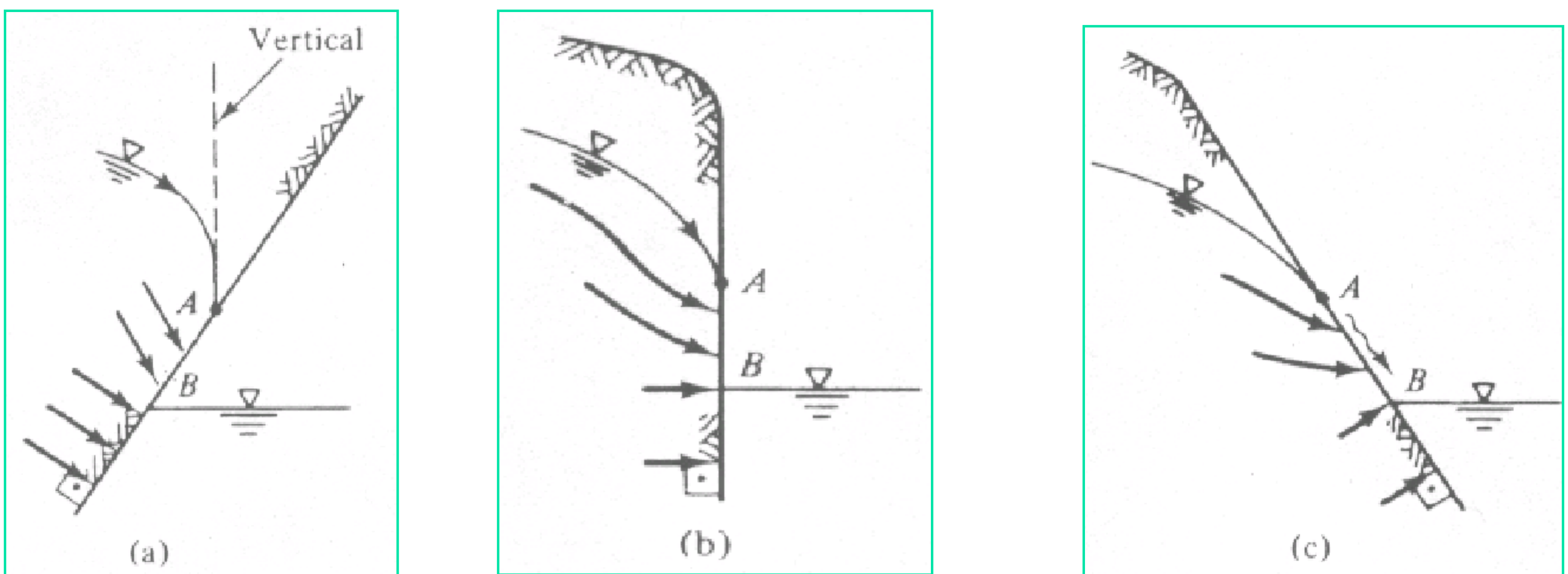
$$\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial h}{\partial y}\right)\approx\frac{\partial}{\partial x}\left(b\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(b\frac{\partial h}{\partial y}\right)=\frac{S_y}{K}\frac{\partial h}{\partial t}$$

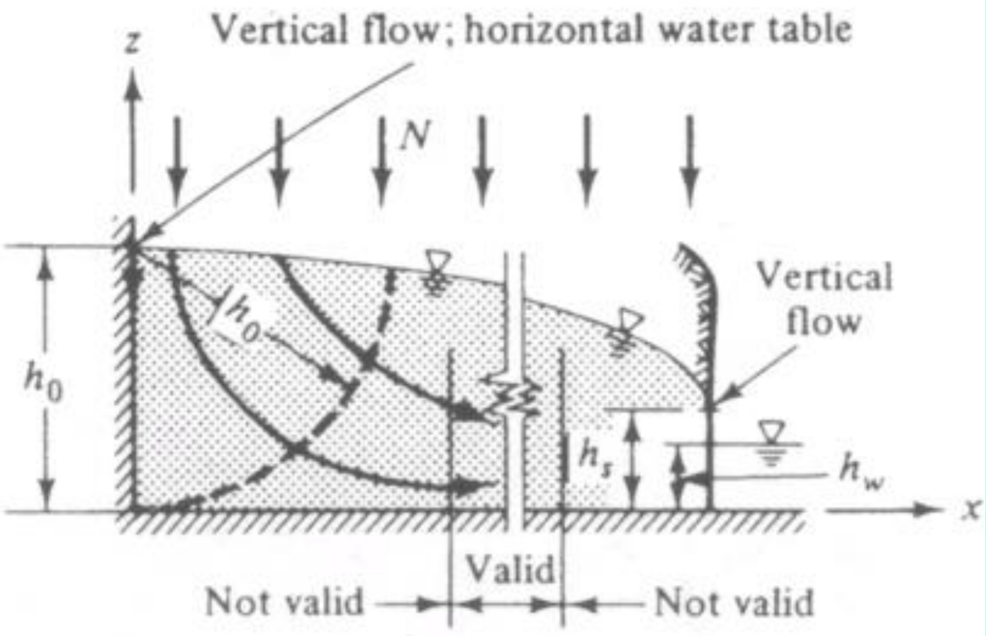
$$\Rightarrow \frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}=\frac{S_y}{Kb}\frac{\partial h}{\partial t} \qquad (5)$$

(Note that (5) is similar to the 2-D flow in a confined aquifer, except that S, Storativity of a confined aquifer, is used instead of S_y)

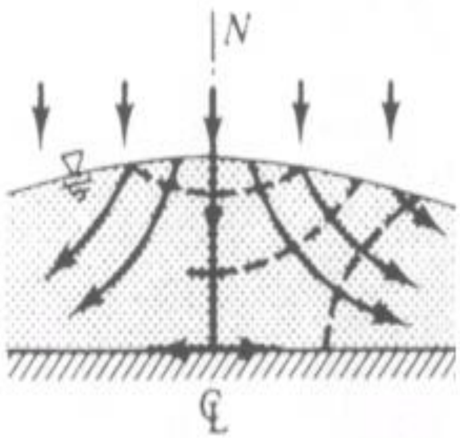
- Conditions when Dupuit assumption does not work
- Vertical flow is not negligible (Vertical impervious boundary; Crest of water table (or water divide); Seepage face
- Rule of thumb (Bear and Verruijt, 1987): Dupuit assumption is valid at distances from the downstream end larger than twice the average height of the flow domain. However, discharge calculated from Dupuit assumption is a satisfactory estimation for most cases

Examples of seepage face

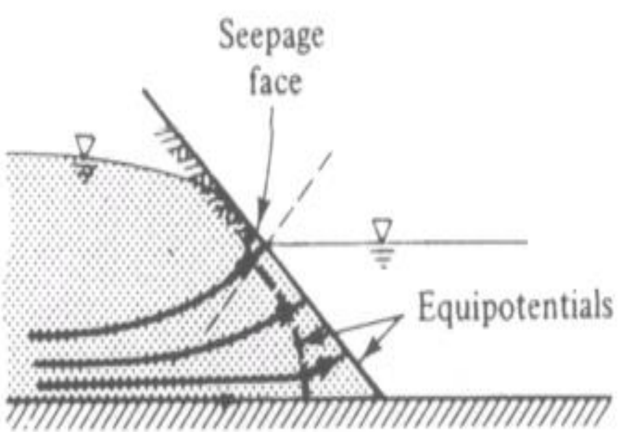




(a)



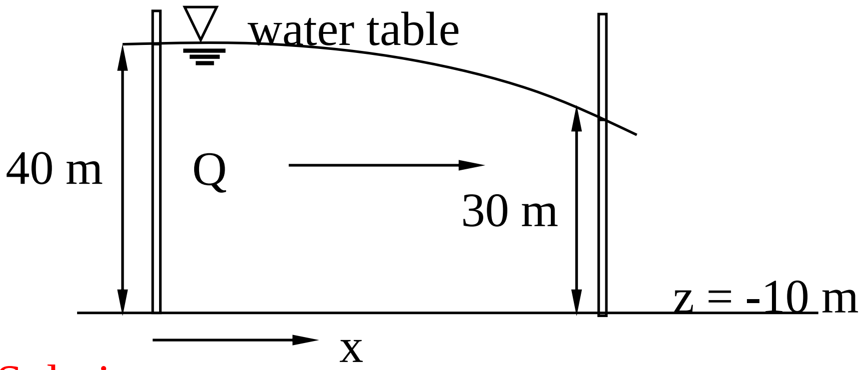
(b)



(c)

- Examples where Dupuit assumption is not valid

Example : (Problem 2.14, Bear and Verruijt, 1987)



(a) Determine Q if K = 18 m/d

(b) Repeat (a) if K = 30 m/d from x = 0 to x = 800 m, and K = 10 m/d for the remaining 400 m.

Solution :

(a) Because the flow field is steady-state, Q is a constant. Hence

$$Q = -Kh\frac{dh}{dx} = -\frac{K}{2}\frac{dh^2}{dx} = -\frac{18}{2}\frac{40^2 - 30^2}{1200} = 5.25\, m^2/d$$

(b) The hydraulic head at x = 800 must be continuous. Furthermore, Q is a constant because the flow field is steady-state. Hence

$$Q = \left(-Kh\frac{dh}{dx}\right)_1 = \left(-Kh\frac{dh}{dx}\right)_2 = -\frac{30}{2}\frac{h^2 - 40^2}{800} = -\frac{10}{2}\frac{30^2 - h^2}{400}$$

$$\Rightarrow h = 36.33\, m$$

$$\Rightarrow Q = 5.25\, m^2/d$$

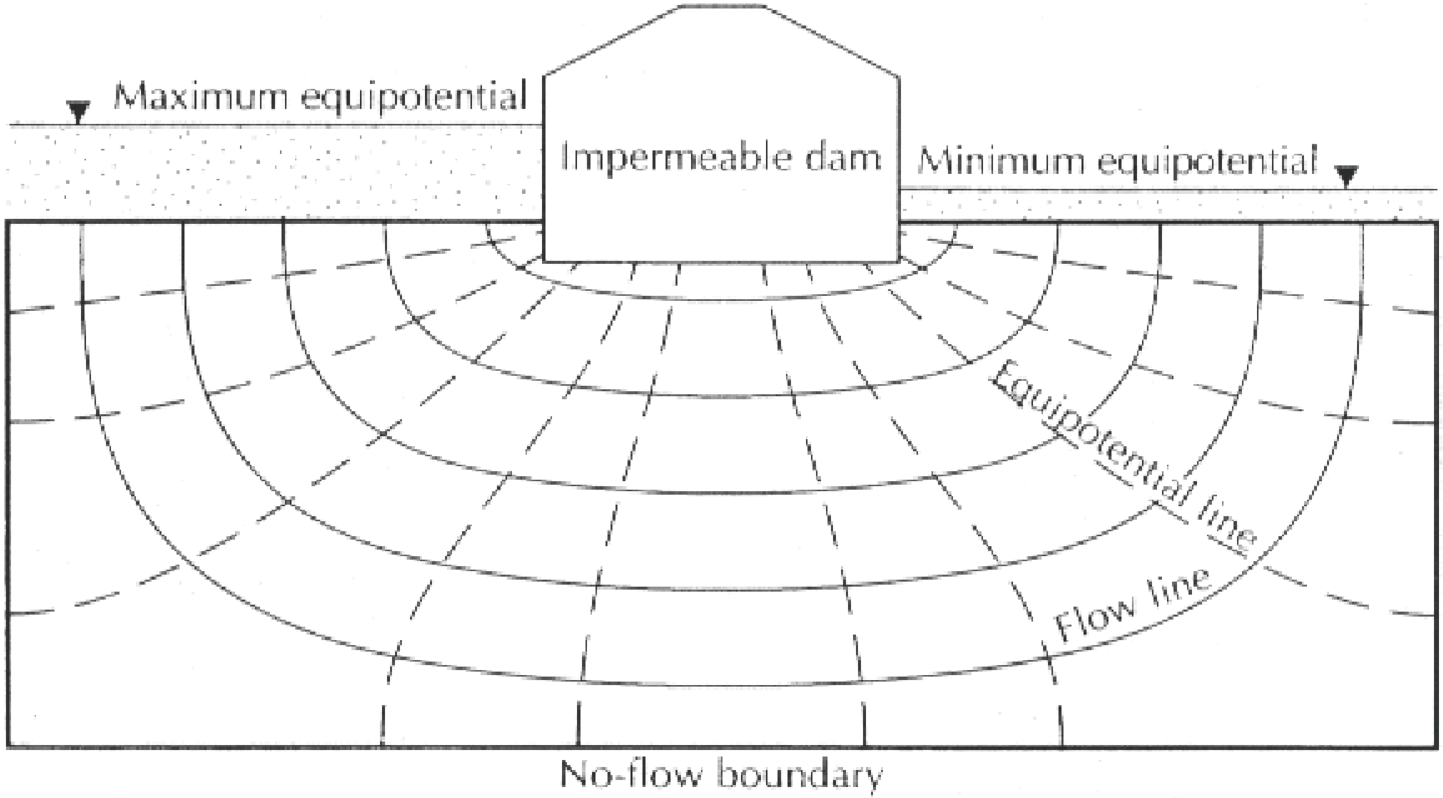
5. Flow net

- The 2D steady state Groundwater flow equation in isotropic and homogeneous porous medium can be expressed by Laplace's Equation:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$$

- Graphically, the equation can be represented by two sets of curves known as '*Equipotential line*' and '*flow lines*', that intersect at right angles. The combined representation of two sets of lines is called a flow net. With the help of a flow net, the groundwater flow problems can be analyzed.

- Equipotential line: A line on which values of hydraulic head are the same.
  - Potential of groundwater $\phi = \nabla h$ = mechanical energy (pressure energy + elevation energy) per unit mass of groundwater. Equipotential lines are always perpendicular to the direction of ∇h, no matter the isotropy of the medium
- Flow line (Fetter, 1994): An imaginary line that traces the path that a particle of groundwater would follow as it flows through an aquifer.
  - Flow lines will cross equipotential lines at right angles in an isotropic aquifer
  - Flow lines will cross the equipotential lines at an angle dictated by the degree of anisotropy and the orientation of ∇h to the hydraulic conductivity tensor ellipsoid
  - Flow lines are parallel to ∇h in isotropic media but not in anisotropic media



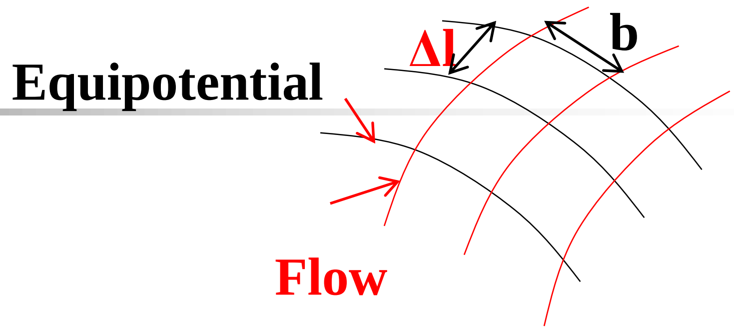
$$Q = \frac{KmH}{n}$$

Q : flow per unit width $[L^2/T]$

K : homogeneous/isotropic hydraulic conductivity [L/T]

m : # of stream tubes (flow tubes, i.e., area between two adjacent flow lines)

n : # of divisions of head in the flow net



- Darcy's equation: $v = ki$; where k is hydraulic conductivity (m/s) and i ($= \Delta h/\Delta l$) is hydraulic gradient. The seepage flow q, through a cross sectional area A is computed as; $q = vA = kiA$.
- In the flow net case: for a single net $A = bX1 = b$; $q = kb\Delta h/\Delta l$, but $\Delta h = H/N_d$ where N_d is the number of equipotential drops; and H is the head difference between the initial and end section along the groundwater flow direction.
- The total discharge per unit width $Q = N_f(q) = N_f kbH/(N_d \Delta l)$; however if the flow net is drawn so that $b \approx \Delta l$, $Q = kHN_f/N_d$
  - Where N_f is the number of flow tubes.

Boundary conditions Vs flow lines

- Boundary conditions vs flow lines and equipotential lines
- **No-flow boundary** (Neumann): Adjacent flow lines are parallel to this boundary, and equipotential lines are perpendicular to this boundary
- **Constant-head boundary** (Dirichlet): This boundary represents an equipotential line and adjacent equipotential lines are parallel to this boundary. Flow lines will intersect the constant-head boundary at right angles
- **Water-table boundary**: the water table, in general, is neither a flow line nor an equipotential line. It is a line where head is known. If Dupuit assumption is valid, equipotential lines are vertical and flow lines are horizontal. If there is recharge or discharge across the water table, flow lines will be at an oblique angle to the water table.

Three BC's vs flow lines and equipotential lines

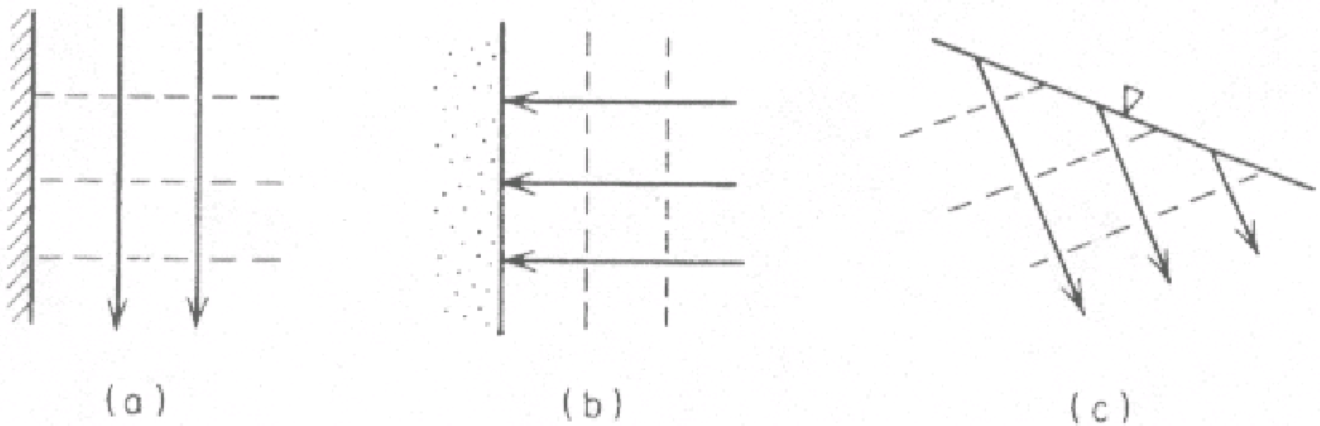


Figure 5.1 Groundwater flow in the vicinity of (a) an impermeable boundary, (b) a constant-head boundary, and (c) a water-table boundary.

(After Freeze and Cherry, 1979)

Flow nets for anisotropic media

- For isotropic soil the flow net is orthogonal; however the flow net in case of anisotropic soil is not orthogonal. Thus the two dimensional seepage flow equation is not a Laplace equation.
- As the permeability is different in the two directions. For example in horizontally stratified aquifers, the horizontal permeability is usually greater than the vertical. Thus the seepage flow equation in an isotropic soils will be:

$$k_x \frac{\partial^2 h}{\partial x^2} + k_y \frac{\partial^2 h}{\partial y^2} = 0$$

- However this equation can be modified to work as Laplace equation as:

$$\frac{k_x}{k_y}\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}=0 \; let \; x_t = x\sqrt{k_y/k_x};$$

$$\frac{\partial^2 h}{\partial {x_t}^2}+\frac{\partial^2 h}{\partial y^2}=0$$

- For example if $k_x = 4k_y$; $x_t = x/2$; The section of the medium is transformed by halving the horizontal dimension. Draw the flow net for the transformed section then transfer the flow net back to the original section.

Steps:

1. Transform the coordinates according to a specific scaling
2. Construct a flow net for the transformed, isotropic medium
3. Invert the scaling ratio

The total discharge per unit width:

$Q = N_f kbH/(N_d \Delta l)$;

Where $k = (k_x k_y)^{1/2}$

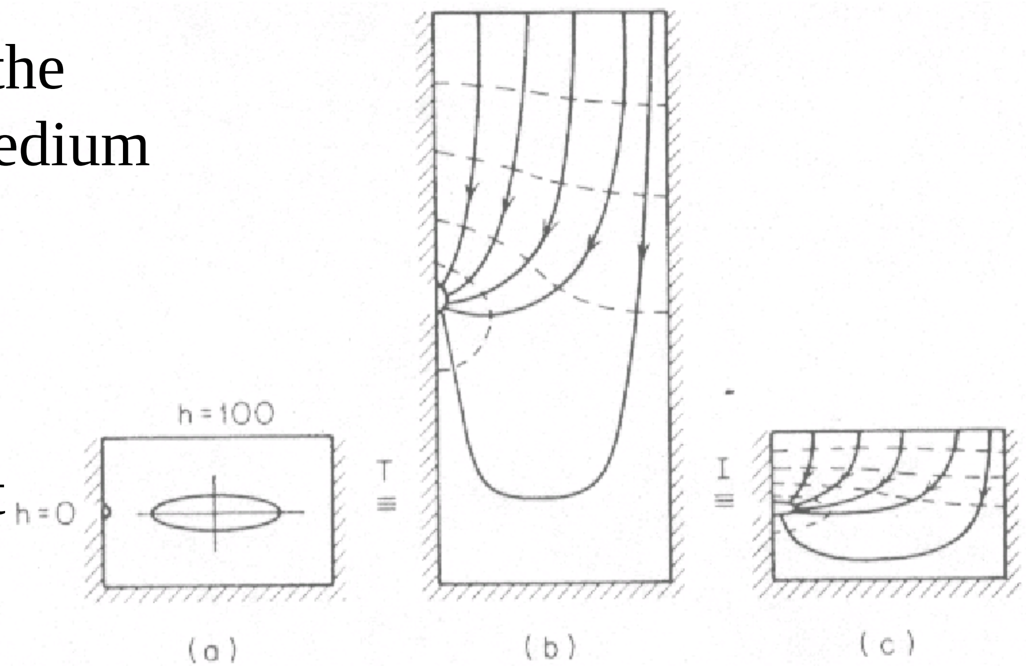
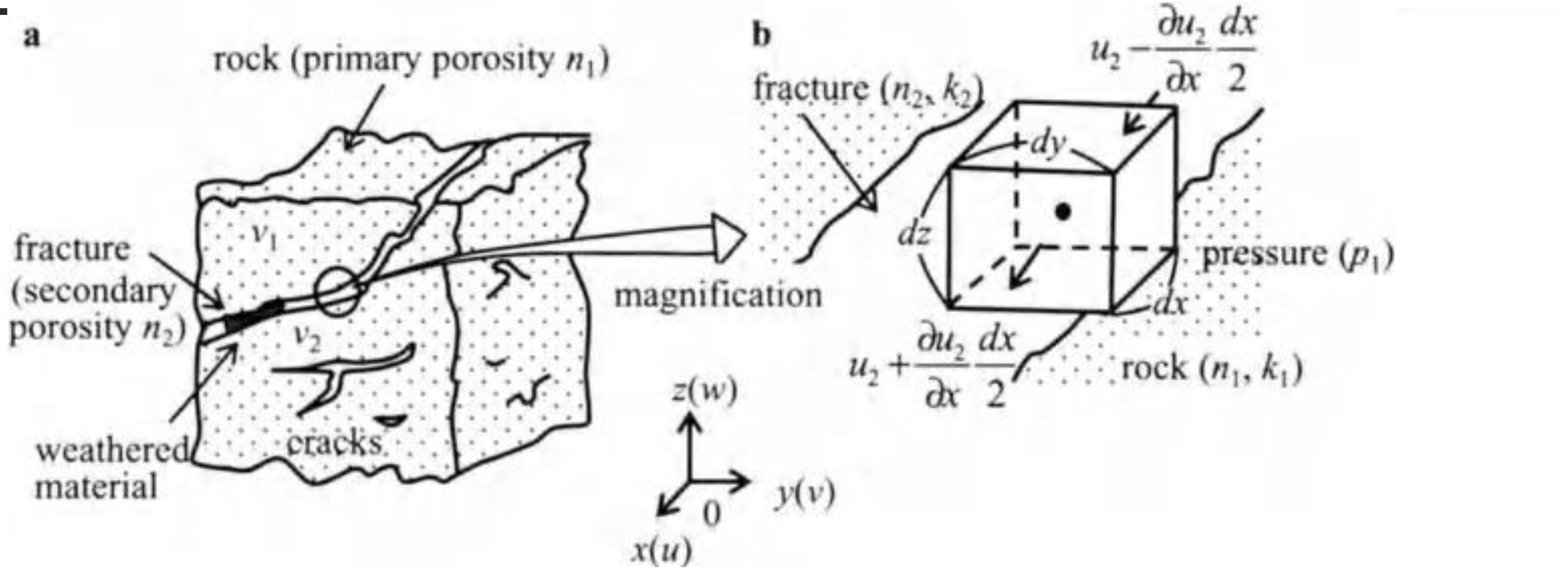
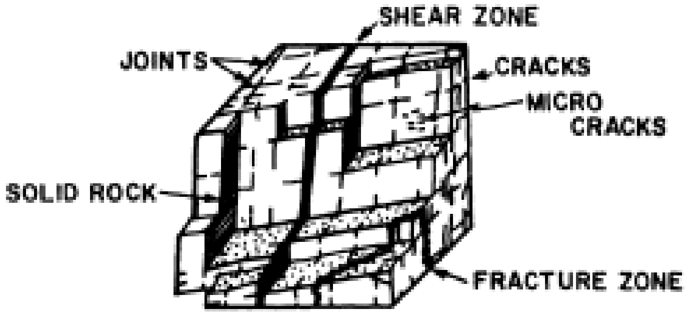


Figure 5.8 (a) Flow problem in a homogeneous anisotropic region with $\sqrt{K_x}/\sqrt{K_z} = 4$. (b) Flow net in the transformed isotropic section. (c) Flow net in the actual anisotropic section. *T*, transformation; *I*, inversion.

6. Approaches to groundwater flow analysis in fractured aquifers



- A fractured medium consists of solid rock with some primary porosity cut by a system of cracks, microcracks, joints, fracture zones, and shear zones that create secondary porosity

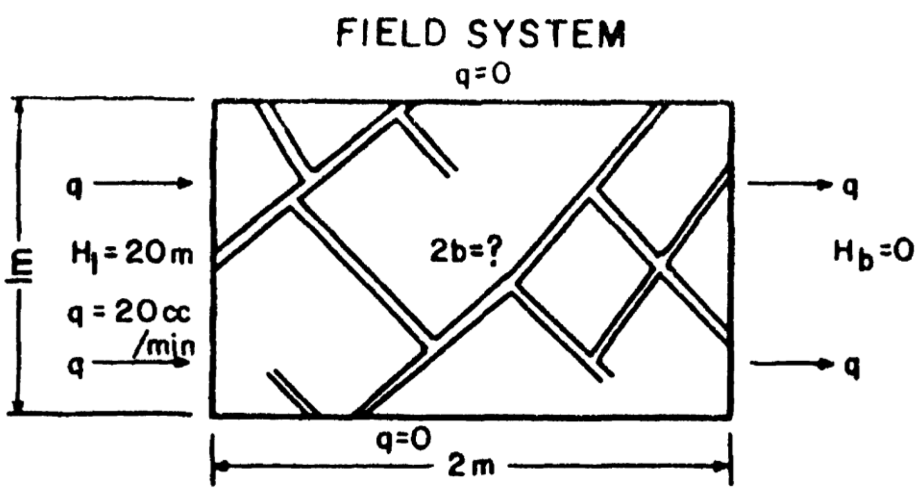


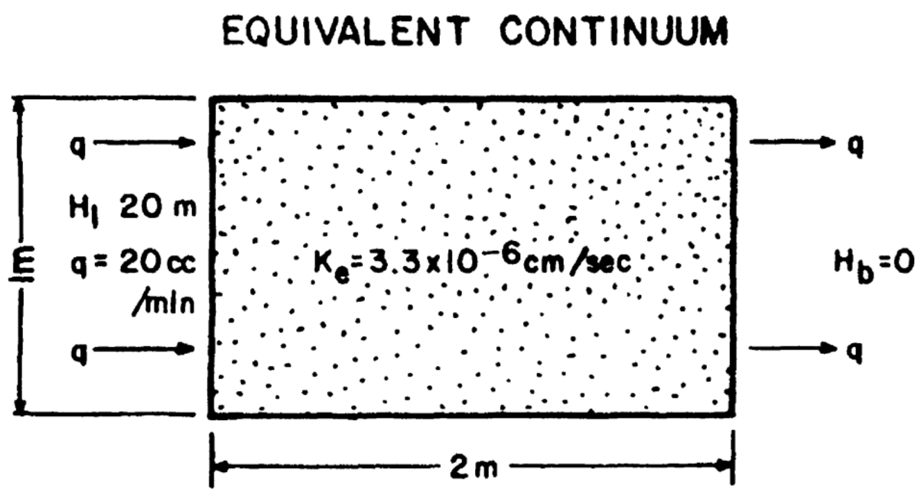
- Fractured systems are typically modeled using one or more of the following conceptual models:

1. **Equivalent porous medium (EPM)**: This model Replaces the primary and secondary hydrogeologic parameters with a continuous porous medium having so-called equivalent or effective hydraulic properties. EPM assumes that the fractured material can be treated as a continuum.
2. **Discrete fractures (DF)**: Flow through a single fracture may be idealized as occurring between two parallel plates with a uniform separation equal to the fracture aperture (2b). Typically applied to fractured media with low primary permeability such as crystalline rocks.
3. **Dual porosity (DP):** If the rock matrix containing the fracture network has significant primary permeability, a DP model may be used.

Equivalent porous Medium (EPM)

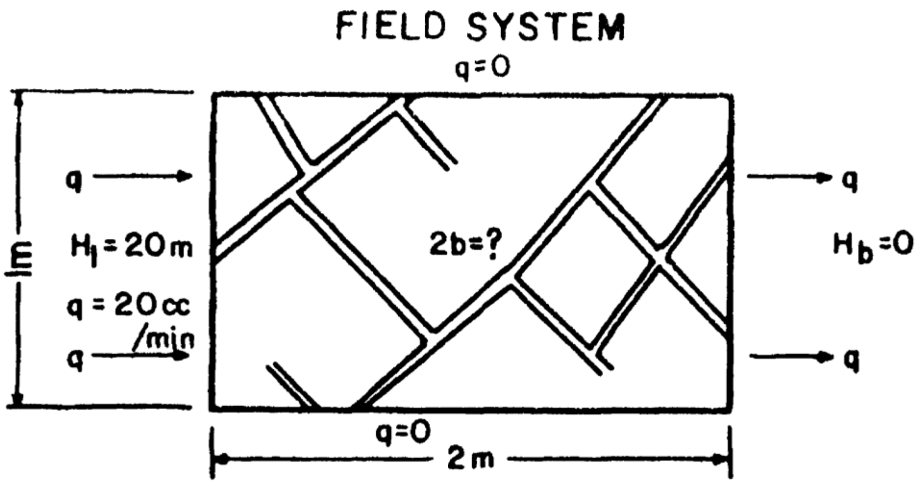
- The difficulty in applying the EPM approach arises in determining the appropriate size of the REV needed to define equivalent hydraulic properties.
- When fractures are few and far between and the unfractured block hydraulic conductivity is low, the EPM method may not be appropriate even with a large REV.
- EPM approach may adequately represent the behavior of a regional flow system, but poorly reproduces local conditions.

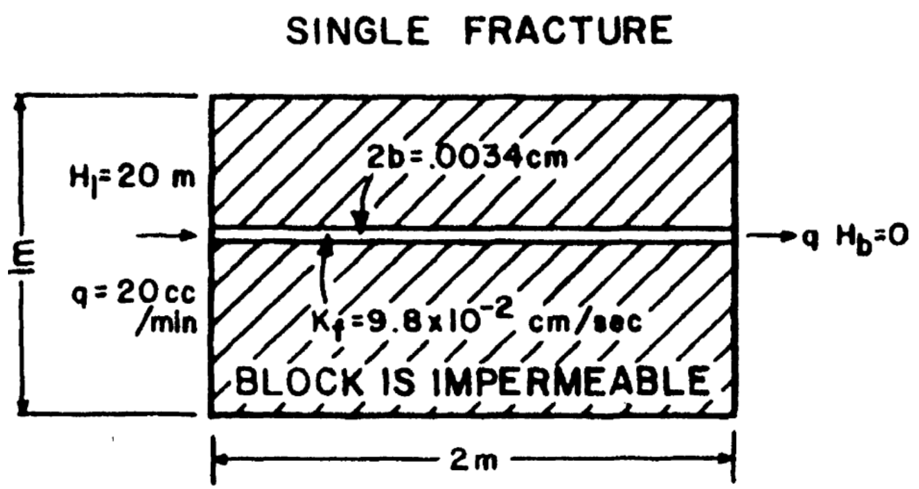




Discrete Fracture (DF) approach

- The flow rate (Q_f): $Q_f = 2bwK_f(dh/dl)$, where w is the width, K_f $(= \rho g(2b)^2/(12\mu))$ the hydraulic conductivity of the fracture, h is hydraulic head, and *1* is the length over which the hydraulic gradient is measured, ρ is fluid density, μ is viscosity, and g is gravity. Note that Q_f is proportional to the $(2b)^3$.
- Use of a cubic model (above equation) requires a description of the fracture network, including fracture apertures and geometry. These data are extremely difficult to collect or estimate.





- Further complications in using a DF model arise when fracture widths are less than 10 μm and when portions of the fracture surfaces touch or are rough. Under these conditions the cubic law for flow through a fracture may not be valid.
- Furthermore, increases in stress with depth and with decreases in pore pressure (e.g., from dewatering) cause a decrease in fracture aperture. Hence, the relative orientation of the fractures and the stress field in relation to the groundwater flow field must also be considered.
- Models based on the DF approach are computationally intensive. To date, applications have been mainly to research problems.

Dual Porosity (DP)

- In this conceptual model, flow through the fractures is accompanied by exchange of water and solute to and from the surrounding porous rock matrix.
- Obviously, the fracture network as well as the properties of the porous blocks must be described prior to modeling.