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Chapter Three
Rainfall-Runoff Analysis

- **Linear System Theory and Rainfall-Runoff Analysis**
  - Unit hydrograph theory
    - From Stream flow Data
    - Synthetically
    - "Fitted" Distributions
  - Instantaneous unit hydrograph (IUH)
  - IUH analysis methods:
    - S-Hydrograph
    - Conceptual model
    - Fitting Harmonic analysis Fourier transforms
    - Theoretically from Laplace transforms
- **River and Reservoir Flood Routing**
  - Flood Routing
  - Reservoir flood routing methods:
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  - Multiple reach Muskingum method
  - Nonlinear Muskingum method:

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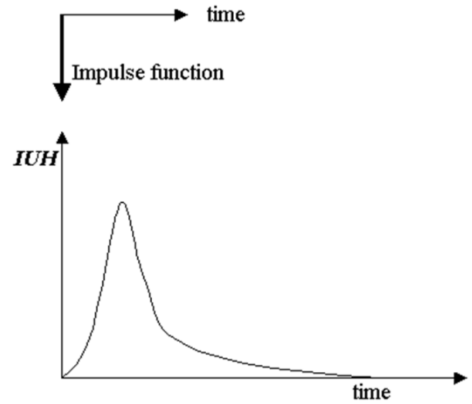
Instantaneous Unit Hydrograph (IUH)

- UH always named with it time of Duration of rainfall excess
- With the decease of time of duration the peak and the shape of UH shift towards the left axis
- If the duration of the rainfall excess becomes infinitesimally small with unit rainfall excess, which uniformly and instantaneously spread over the catchment , the resulting DRH is called **Instantaneous Unit Hydrograph (IUH)**
- In other words Instantaneous unit hydrograph is the direct runoff hydrograph resulted from an **Impulse function** rainfall, i.e., one unit of effective rainfall at a time instance.



Instantaneous Unit Hydrograph (IUH)

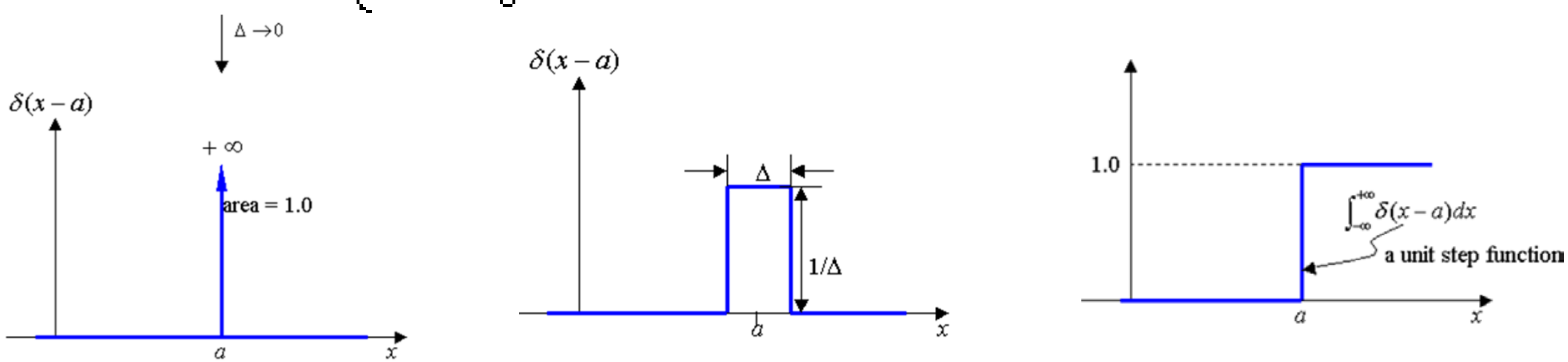
Definition of the *Unit Impulse function*

Let $\phi(x)$ be an arbitrary function that is continuous at $x = x_0$, then

$$\int_{-\infty}^{+\infty} \phi(x)\delta(x - x_0)dx = \phi(x_0)$$

The impulse function $\delta(x)$ can be interpreted as a function with infinite amplitude, area of unity, and zero duration.

$$\delta(x - x_0) = \begin{cases} 0 & x_0 \neq a \\ 1 & x_0 = a \end{cases}$$



Instantaneous Unit Hydrograph (IUH)

It cab be shown that

$$\int_{-\infty}^{x} \delta(\xi - x_0)d\xi = u(x - x_0)$$

or, equivalently

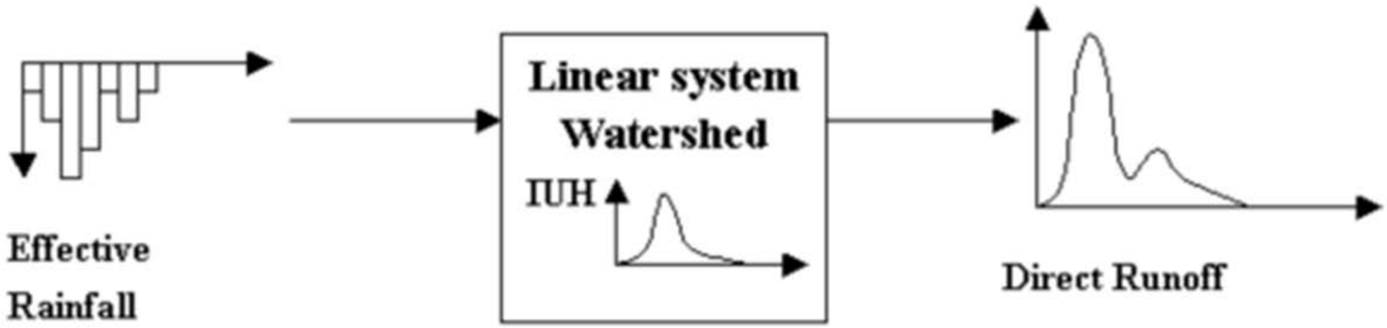
$$\frac{du(x)}{dx} = \delta(x)$$

where $u(x)$ is the unit-step function defined by

$$u(x) = \begin{cases} 1 & 0 < x \\ 0 & x < 0 \end{cases}$$

Instantaneous Unit Hydrograph (IUH)

- The time base of the *IUH* is the time of concentration of the watershed.
- The ordinate of the *IUH* at time *t*, *IUH*(t), is the system's response at time *t*.
- Consider a watershed as a linear system and the effective rainfall and direct runoff are respectively the input and output of this system.



$$Q(t) = \int_0^t i(\tau)u(t-\tau)d\tau$$

$i(t)$: effective rainfall, $u(t)$: instantaneous unit hydrograph, $Q(t)$: direct runoff.

Derivation of IUH

IUH can be derived by either of the following approaches

- s- Hydrograph
- Conceptual Models
  - Clark's Model
  - Nash Model
- Fitting Harmonic Series
- Theoretically from Laplace transform function

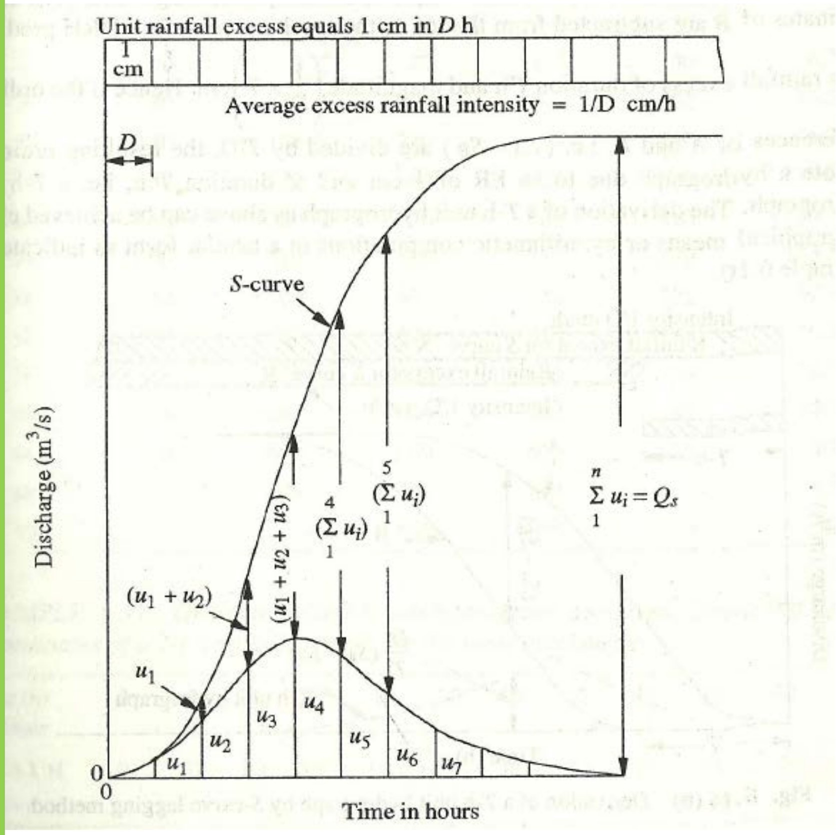
IUH from S-Hydrograph

S-Hydrograph

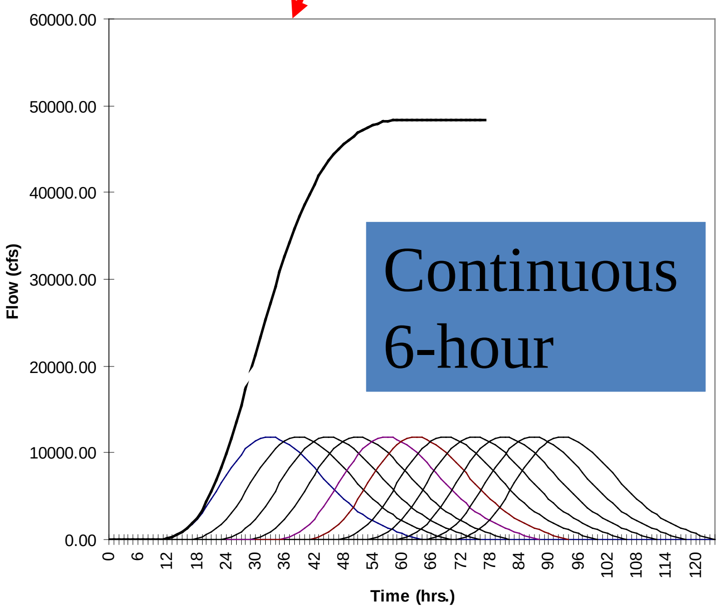
- S-hydrograph is a hydrograph produce by a continuous effective rainfall at a constant rate for an infinite period
- It is a curve obtained by summation of an infinite series of D-hr unit hydrographs spaced D-hr apart.
- The S-hydrograph method use for the conversion of an X- hour unit hydrograph into a Y-hour unit hydrograph, regardless of the ratio between X and Y.
- The procedure consists of the following step
  - Determine the X hour S-hydrograph. Accumulating the unit hydrograph ordinates at intervals equal to X, thus deriving the X-hour S-hydrograph.
  - Lag the X -hour S-hydrograph by a time interval equal to Y hours.
  - Subtract ordinates of these two S-hydrographs.
  - Multiply the resulting hydrograph ordinates by X/Y to obtain the Y-hour unit hydrograph.
  - The volume under X-hour and Y-hour unit hydrograph is the same. If Tb is the time base of the X-hour unit hydrograph, the time base of the Y-hour unit hydrograph is Tb - X + Y.

IUH from S-Hydrograph

S-Hydrograph

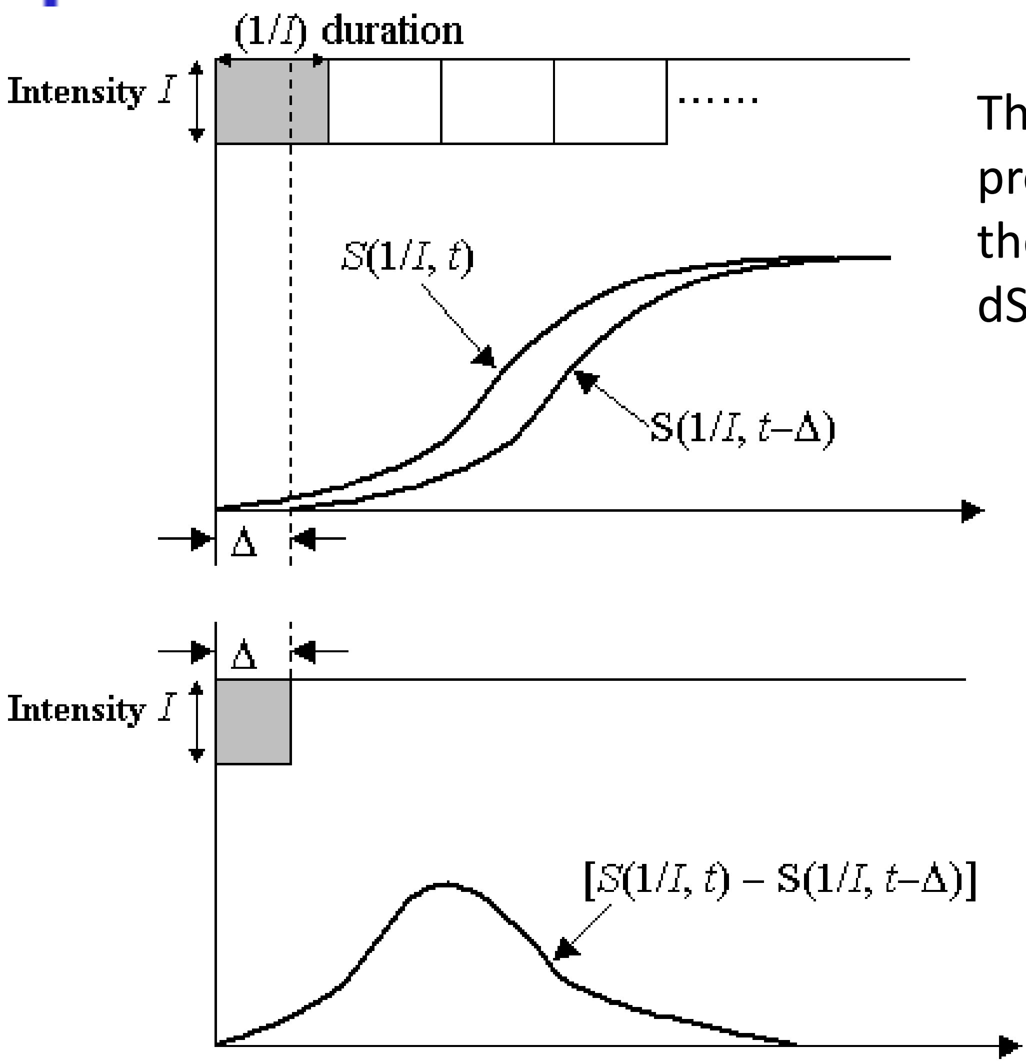


S-Curve: You get this by adding the ordinates of multiple 6 hr UHs below



IUH from S-Hydrograph

Relationship between IUH and the S curve



The ordinate of IUH(t) is proportional to the slope of the S-curve at time t, i.e. dS/dt.

$$UH(\Delta,t) = [S(1/I,t) - S(1/I,t-\Delta)] \cdot \frac{1}{I} \cdot \frac{1}{\Delta}$$

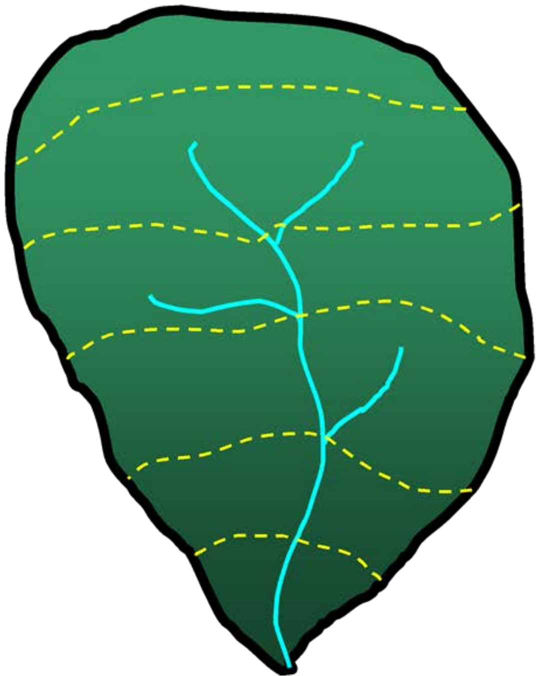
$$= \frac{1}{I} \frac{[S(1/I,t) - S(1/I,t-\Delta)]}{\Delta}$$

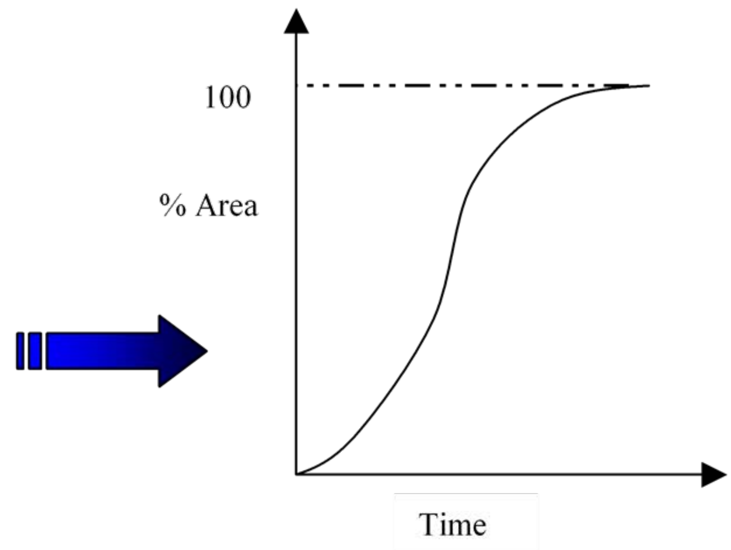
As $\Delta \to 0$,

$$IUH(t) = \frac{1}{I} \cdot \frac{dS(1/I,t)}{dt}$$

Conceptual Models : Clark's Model

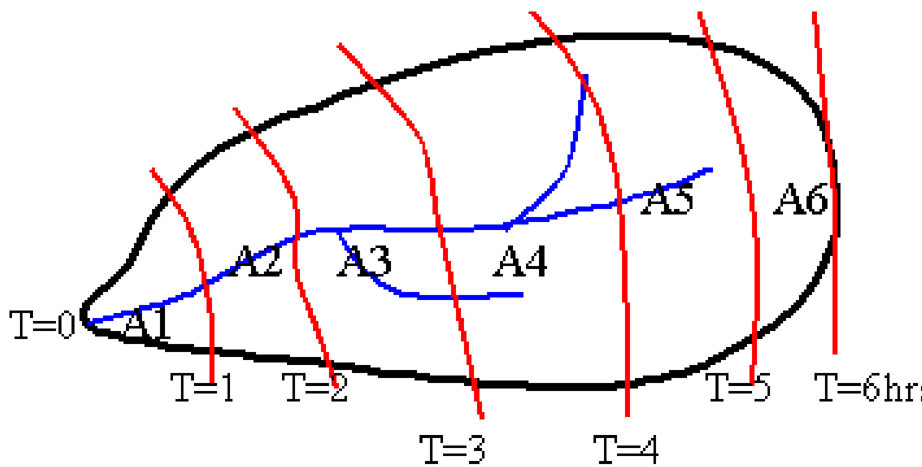
- Clark (1945) proposed an IUH model due to instantaneous rainfall excess over a drainage area by considering a linear channel in series with a linear reservoir.
- Clark's model developed by Time-Area concepts
- it tries to understand the flow transfer function based on the travel time of area.



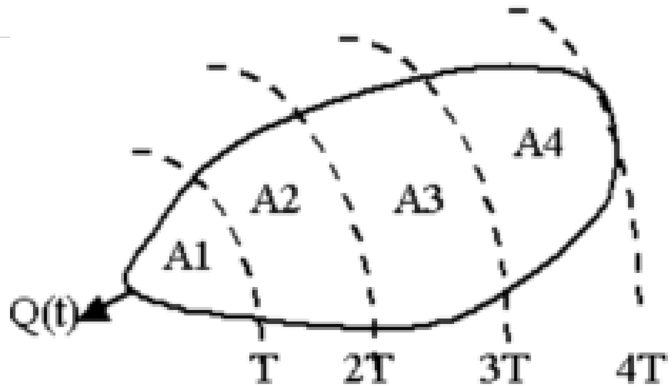


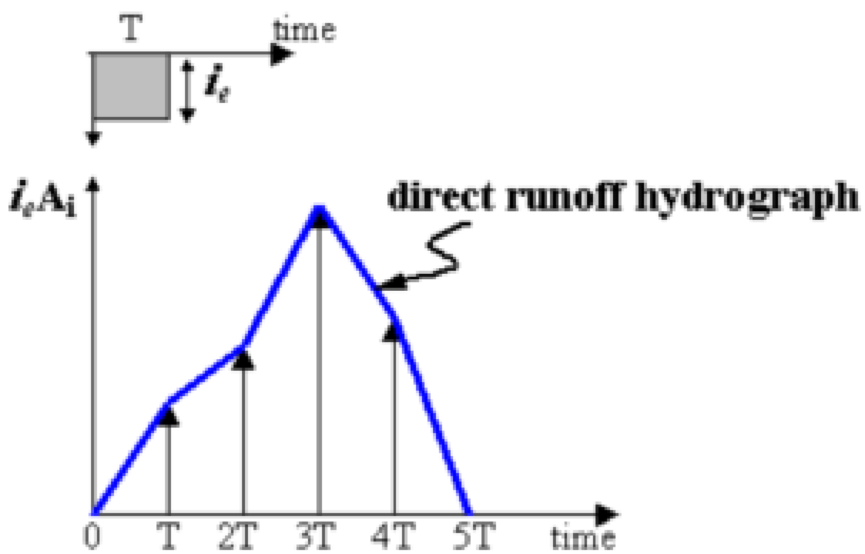
Conceptual Models : Clark's IUH (time-area method)

- The concept of isochrones
  - Isochrones are lines of equal travel time. Any point on a given isochrone takes the same time to reach the basin outlet. Therefore, for the following basin isochrone map and assuming constant and uniform effective rainfall, discharge at the basin outlet can be decomposed into individual contributing areas and rainfalls.



Conceptual Models : Clark's IUH (time-area method)





If one unit of rainfall excess, which has a constant intensity within $0 \le t \le T$ and duration T, falls over the watershed, the resulting direct runoff hydrograph is the unit hydrograph **UH(T,t)** and can be determined by the isochrones, as shown in the figure.

Remark: The *storage effect* of the watershed is neglected in the above description.

Conceptual Models : Clark's IUH (time-area method)

Contributing area and contributing rainfall

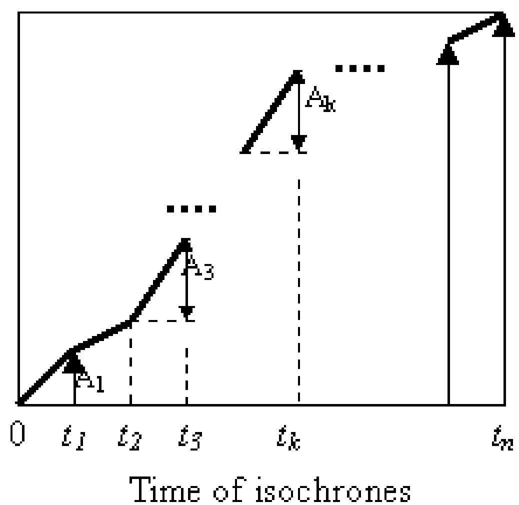
Duration of effective rainfall = 3 hrs		
Discharge	Contributing area	Contributing rainfall, P(x,t)
Q(0)	Outlet point, x=0	P(0,0)
Q(1)	A1 (x=0 → x=1)	P(0,1) → P(1,0)
Q(2)	A1, A2	P(0,2) → P(1, 1) → P(2, 0)
Q(3)	A1, A2, A3	P(0, 3) → P(1, 2) → P(2, 1) → P(3, 0)
Q(4)	A2, A3, A4	P(1, 3) → P(2, 2) → P(3, 1) → P(4, 0)
Q(5)	A3, A4, A5	P(2, 3) → P(3, 2) → P(4, 1) → P(5, 0)
Q(6)	A4, A5, A6	P(3, 3) → P(4, 2) → P(5, 1) → P(6, 0)
Q(7)	A5, A6,	P(4, 3) → P(5, 2) → P(6, 1)
Q(8)	A6	P(5, 3) → P(6, 2)
Q(9)	0	P(6, 3)
Remarks	x=*i*: location on the *i*-th isochrone, P(x, t): rainfall falls at location x at time t.	

Conceptual Models : Clark's IUH (time-area method)

Using the basin isochrone map, the cumulative contributing area curve can be developed. The derivatives or differences of this curve constitute the instantaneous unit hydrograph **IUH(t)**.

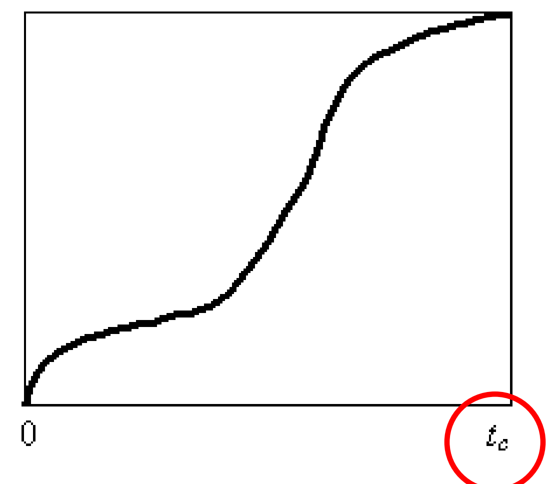
Cumulative contributing area

$$C(t_k) = \sum_{i=0}^{k} A(t_i)$$



Cumulative contributing area

$$C(t) = \int_{\tau=0}^{t} A(\tau)\,d\tau$$



$$IUH(t) = \frac{dC(t)}{dt}$$

Time base of the *IUH* is the time of concentration of the watershed.

Conceptual Models : Clark's IUH (time-area method)

- If the effect of watershed storage is to be considered, the unit hydrograph described above is routed through a hypothetical linear reservoir with a storage coefficient *k* located at the watershed outlet.

- For a linear reservoir with storage coefficient *k*, we have

$$S(t) = kQ(t)$$

$S(t)$: reservoir storage at time t

$Q(t)$: reservoir outflow at time t.

Conceptual Models : Clark's IUH (time-area method)

- Consider the continuity of the hypothetical reservoir during a time interval Δt.

$$I(t) - Q(t) = dS(t)/dt = kdQ(t)/dt$$

$$\bar{I} - \bar{Q} = k(Q_2 - Q_1)/\Delta t$$

$$\bar{I} = (I_1 + I_2)/2, \quad \bar{Q} = (Q_1 + Q_2)/2$$

I_1, I_2 and Q_1, Q_2 : inflow and outflow at time t and $t+\Delta t$.

$$Q_2 = C_0 \bar{I} + C_1 Q_1$$

$$C_0 = \frac{2\Delta t}{2k + \Delta t}, \quad C_1 = \frac{2k - \Delta t}{2k + \Delta t}$$

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Conceptual Models : Clark's IUH (time-area method)

Example:

A drainage basin has the following characteristics: Area 110km2, time off concentration = 18hr, storage constant = 12hr, and inter-isochrone area distribution given as below

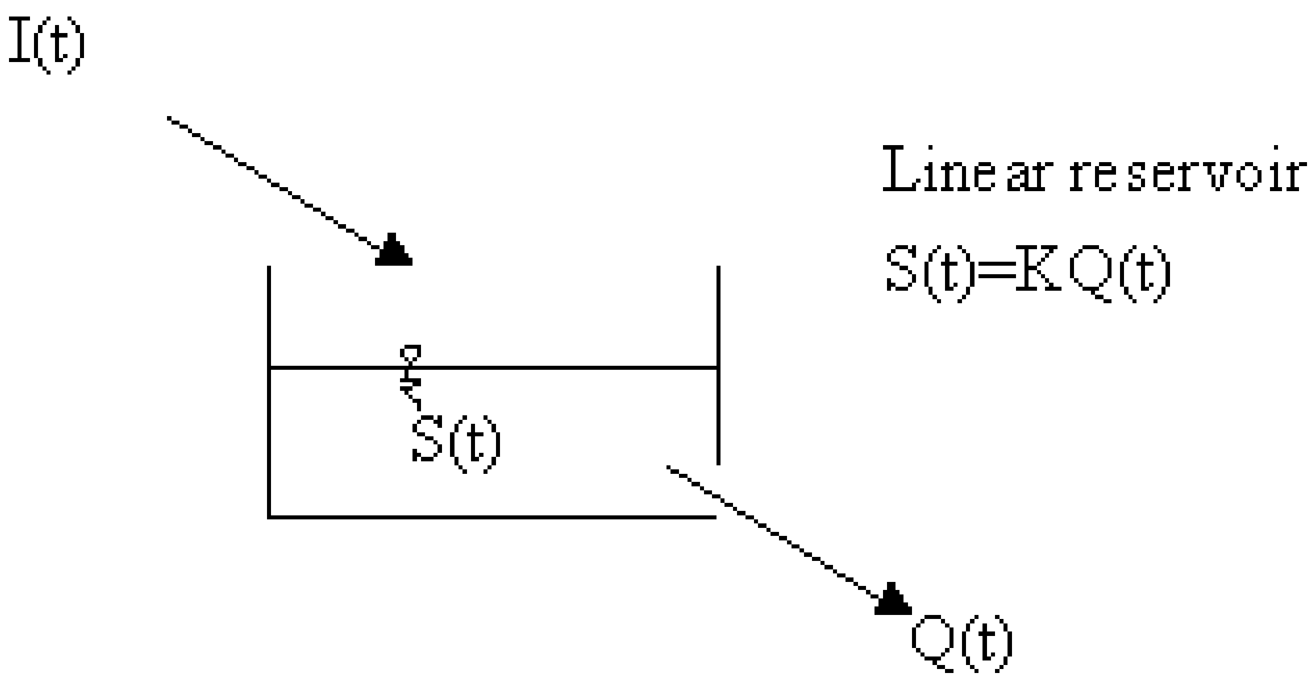
Time (hr)	0-2	2-4	4-6	6-8	8-10	10-12	12-14	14-16	16-18
Inter-Isochrone	100	200	800	1200	1800	1600	1000	400	100

Determine the IUH for this catchment

Nash's linear reservoir model

- Consider a linear reservoir which has the following characteristics:

$$S(t) = KQ(t)$$



Nash's linear reservoir model

From the continuity equation, it yields

$$I(t) = Q(t) + \frac{dS}{dt} = Q(t) + K\frac{dQ(t)}{dt}$$

$$\frac{dQ(t)}{dt} + \frac{Q(t)}{K} = \frac{I(t)}{K}$$

For constant inflow ($I(t) = I$),

$$\frac{dQ(t)}{dt} + \frac{Q(t) - I}{K} = 0$$

$$\therefore \frac{dQ(t)}{Q(t) - I} = -\frac{dt}{K}$$

$$\int_0^t \frac{dQ(\tau)}{Q(\tau) - I} = -\int_0^t \frac{d\tau}{K} + C$$

Nash's linear reservoir model

$$\int_0^t \frac{dQ(t)}{Q(t) - I} = -\int_0^t \frac{dt}{K} + c$$

$$\ln[Q(t) - I]\Big|_0^t = -\frac{t}{K} + c \Rightarrow \ln[Q(t) - I] - \ln[Q(0) - I] = -\frac{t}{K} + c$$

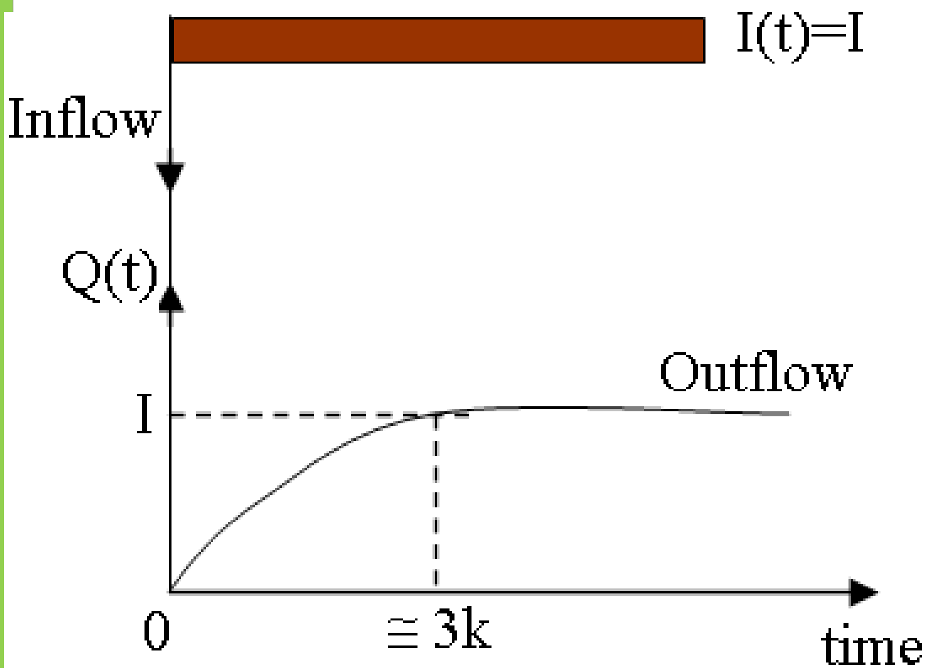
$$Q(0) = 0 \Rightarrow \ln[\frac{Q(t) - I}{-I}] = -\frac{t}{K} + c \Rightarrow \frac{Q(t) - I}{-I} = c' e^{-t/k}$$

$$Q(t) - I = -Ic' e^{-t/k}$$

$$t = 0 \Rightarrow Q(t) = 0 \therefore -I = -Ic', \therefore c' = 1$$

$$\therefore Q(t) = I - Ie^{-t/k} = I(1 - e^{-t/k})$$

Nash’s linear reservoir model



Discharge from a linear reservoir under a constant input I is Q(t)=I[1-exp(-t/K)] (an exponential model)

K: the storage caeff. of the reservoir.

$$Q(t) = I(1 - e^{-t/k})$$

Nash's linear reservoir model

Assume that I terminates at $t=t_0$

$$\begin{pmatrix} I(t) = I & 0 \le t \le t_0 \\ I(t) = 0 & t \ge t_0 \end{pmatrix}$$

For $t > t_0 \Rightarrow Q(t) = -K\dfrac{dQ(t)}{dt}$

$$-\frac{dt}{K} = \frac{dQ(t)}{Q(t)}$$

$$-\frac{1}{K}\int_{t_0}^{t_0+\tau} dt + c = \ln Q(t)\Big|_{t_0}^{t_0+\tau}$$

$$-\frac{\tau}{K} + c = \ln\left[\frac{Q(t_0+\tau)}{Q(t_0)}\right]$$

$$c' e^{-\tau/K} = \frac{Q(t_0+\tau)}{Q(t_0)}$$

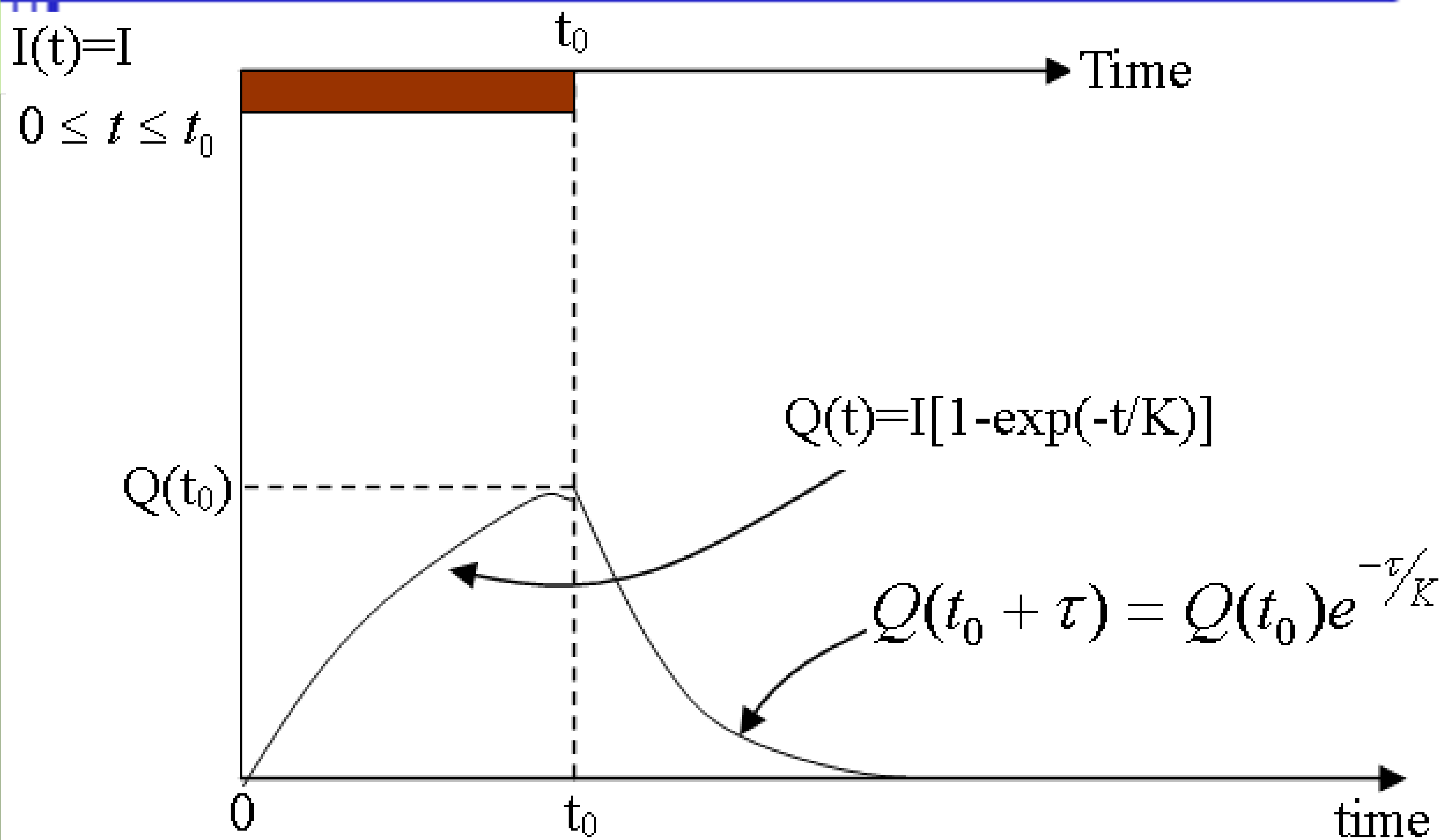
Nash’s linear reservoir model

At $(t = t_0, \tau = 0), c' = 1.0$

$$\therefore Q(t_0 + \tau) = Q(t_0)e^{-\tau/K}$$

The discharge after the input I ends is characterized by an exponential decay function

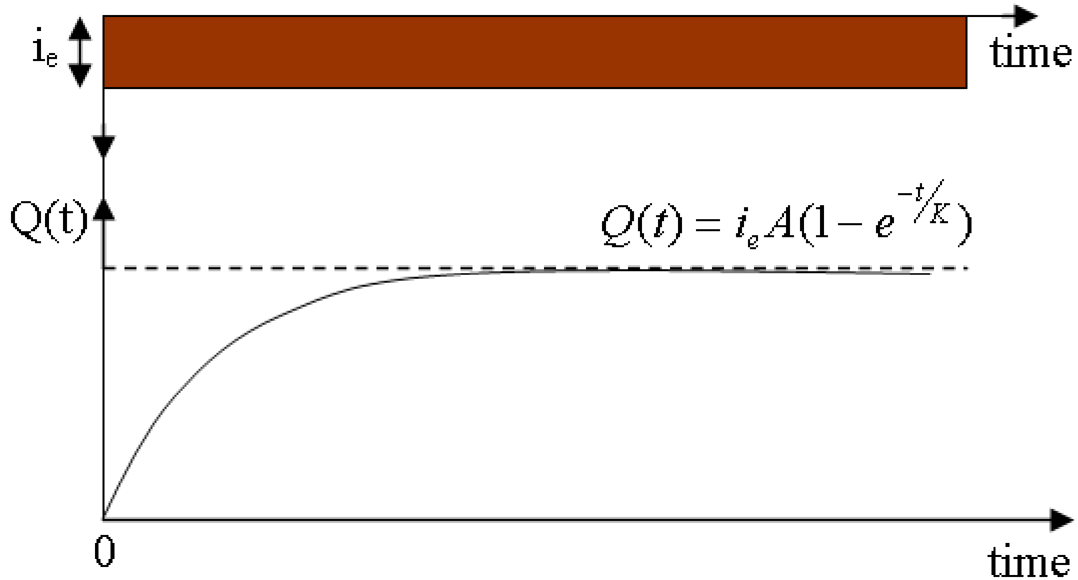
Nash's linear reservoir model



Summary of the Conceptual IUH model

Assumption: Linear Reservoir $S(t)=kQ(t)$

(i) Constant rainfall intensity i_e and infinite duration



The above curve of Q(t) can be considered as an S-curve, $S(1/i_e,t)$