CENG 6108 Construction Economics

Time Value of Money and Cash Flow Analysis

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1. Introduction
2. Interest and Interest Rates
3. Effective and Nominal Interest Rates
4. Continuous Compounding
5. Cash Flow Diagrams

Time Value of Money

- Importance of Time
  - Money can have different values at different times.
  - $10,000 now is worth more than $10,000 a year from now even if there is no inflation, as it can earn money during the interval by:
    - Doing business with it (Productive Use of Money) or
    - Investing it in a bank and earn interest on it.
  - Thus, interest rate is the more identifiable and accepted measure of the earning power of money.
  - Interest is thus usually accepted as the time value of money and indication of its earning power
  - However, the buying power of money is related to inflation, a concept closely related to interest rate.

Time Value of Money

- If we assume
  - that money can always be invested in the bank (or some other reliable source) now to gain a return with interest later
  - that as rational actors, we never make an investment which, we know, offers less money than we could get in the bank
- Then
  - Money in the ***present*** can be thought as of "*equal worth*" to a larger amount of money in the future
  - money in the ***future*** can be thought of as having an equal worth to a lesser "*present value*" of money

Interest Rate

- The rate of interest i is the percentage of the money we pay for its use over a time period.
  - The interest rate is referred to by different names such as rent, cost of money, and value of money.
  - In investment terminology, it is also called the minimum acceptable rate of return or MARR
- To compare the value of money at different points in time, we need to use an acceptable interest rate.
- The interest rate will depend on the:
  - The position in time that the money is needed,
  - The length of time it is required. For short periods, it is assumed that the economy is stable and the risk is predictable.
  - Availability of money in the financial market.

Interest

- Money has a Time Value
  - Money earns money over time

- Example
  - Deposit Br 1,000,000 today in a saving account at interest rate of 10%

Original Amount + Interest Amount = Total Amount

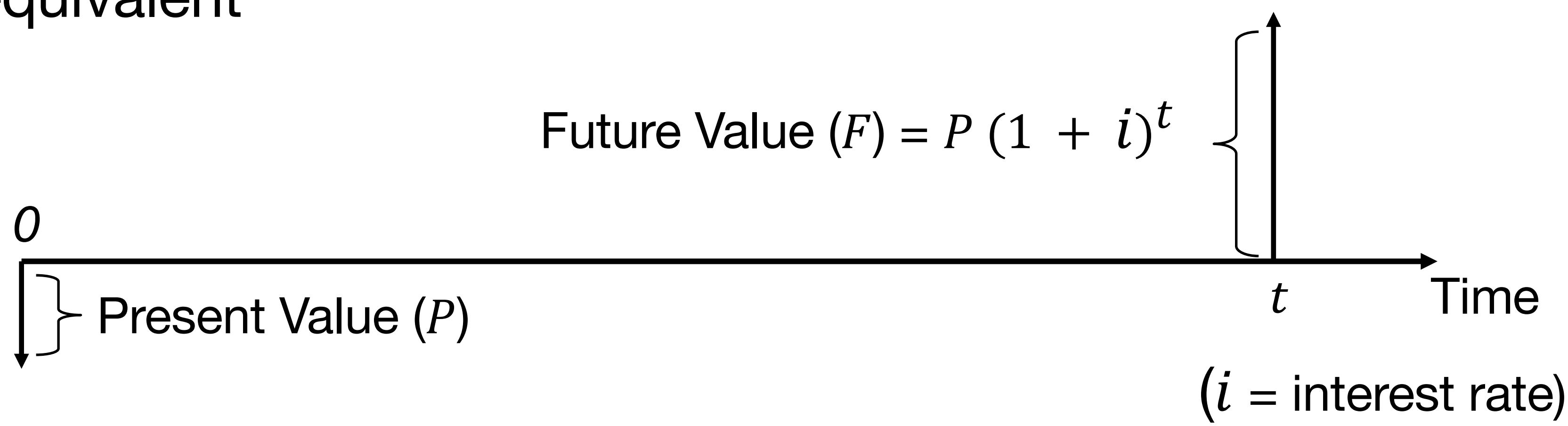
$ 1,000,000 → 10% → Earns $ 100,000 → After 1Year → $ 1,100,000

Function of Interest Rate & Time

$ 1,000,000 at present (P) = $ 1,100,000 at future time (F)

Present and Future Values

- Because we can flexibly switch from one such value to another without cost, we can view these values as equivalent



- Given a reliable source offering annual return i (i.e., interest), we can shift without additional costs between cash *P* at time *0* and $P(1+i)^t$ at time t

Major Elements & Symbols

i = Interests rate per interest period

N = Number of interest periods

P = Equivalent value of amount of money at time zero (Present Worth)

F = Equivalent value of amount of money at time N (Future Worth)

A = An End-Of-Period payment or receipt in a uniform series (Annuity)

Methods of Calculating Interest (i)

- Simple Interest
  - Considers the interest to be earned on only the Principal amount during each interest period (the interest earned during each period does not earn additional interest in the remaining periods)

For one period $$I = i\,P$$

e.g.) $P = \$100, i = 10\%$

$$I = 0.1 * 100 = \$10$$

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For N periods $$I = (i\,P)\,N$$

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If $N = 5$,
$I = 0.1 * 100 * 5 = \$50$

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If $N = 5$,
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$$\vdots$$

$i\,N$ is a linear function of time

F @ end of N Periods

$$F = P + I = P(1 + i\,N)$$

$$F = 100 + 50 = \$150$$

Methods of Calculating Interest (i)

- Compound Interest
  - The interest in each period is based on the total amount owed at the end of the previous period *(the original principal plus the accumulated interest)*

$100

If you deposit ($ $\boldsymbol{P}$) at interest rate i

10%

What you would have at the end of period one

$10

$$I_1 = i\,P$$

$$F_1 = P + i\,P = P(1 + i)$$

$110

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$110

$$I_2 = i\,F_1$$

$$\$110 * 0.1$$

$$F_2 = F_1(1 + i)$$

$$\$110 * (1 + 0.1)$$

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$$F_1 = P + i\,P = P(1 + i)$$

$110

$110*(1+0.1)

$110*0.1 $I_2 = i\,F_1$

$121*0.1 $I_3 = i\,F_2$

$$F_2 = F_1(1 + i)$$

$$F_3 = F_2(1 + i)$$

…

…

$121*(1+0.1)

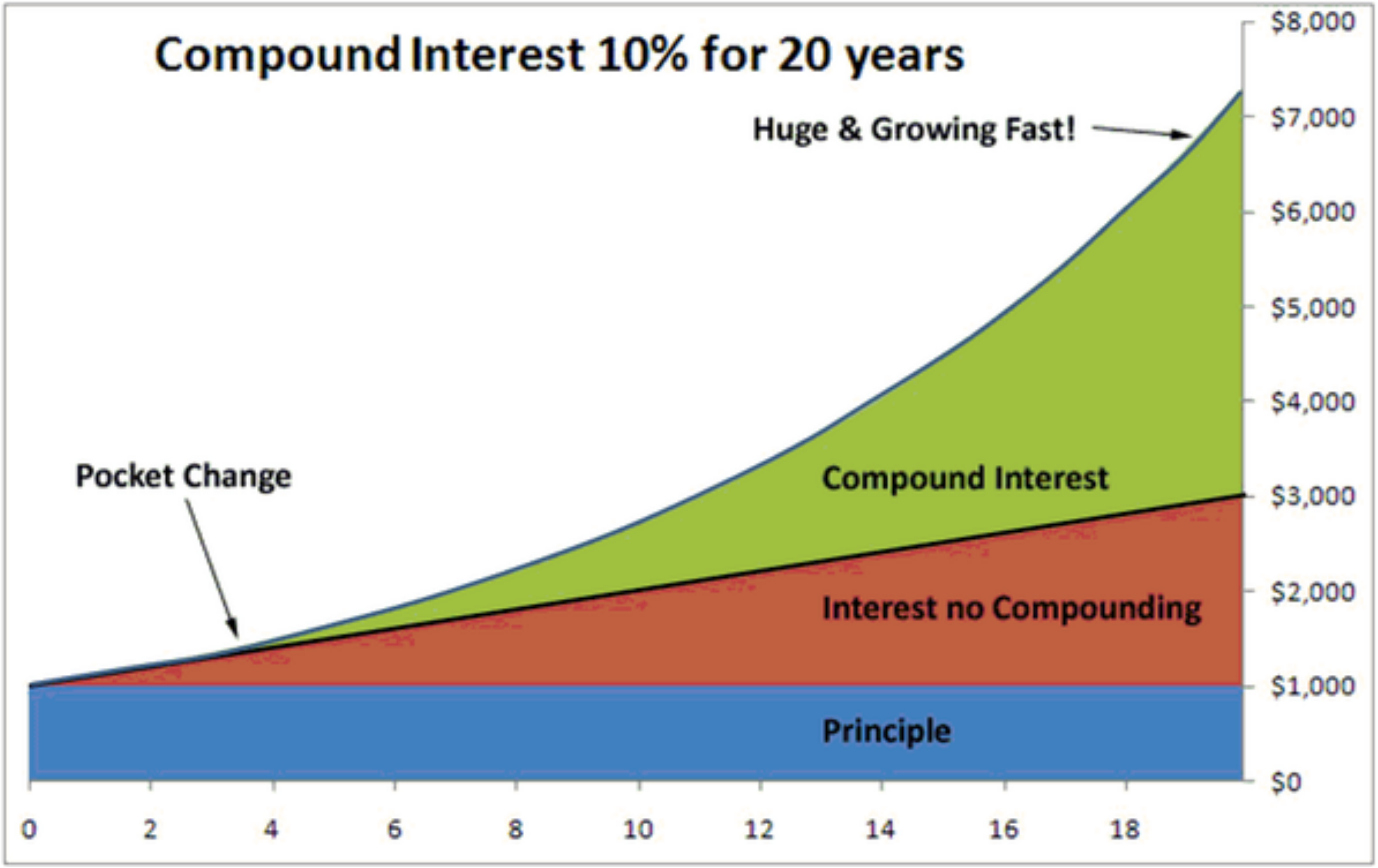
Methods of Calculating Interest (i)

- Compound Interest

End of Period	(A) Amount Owed	(B) Interest for Next Period	(C) = (A) + (B) Amount Owed for Next Period	
0	P	Pi	$P+Pi$	$= P(1+i)$
1	$P(1+i)$	$P(1+i)\,i$	$P(1+i) + P(1+i)i$	$= P(1+i)^2$
2	$P(1+i)^2$	$P(1+i)^2\,i$	$P(1+i)^2 + P(1+i)^2\,i$	$= P(1+i)^3$
3	$P(1+i)^3$	$P(1+i)^3\,i$	$P(1+i)^3 + P(1+i)^3\,i$	$= P(1+i)^4$
.				
.				
n-1	$P(1+i)^{n-1}$	$P(1+i)^{n-1}\,i$	$P(1+i)^{n-1} + P(1+i)^{n-1}\,I$	$= P(1+i)^n$
n	$P(1+i)^n$			

- Compound interest (I_c): $I_c = P(1+i)^n - P$

Methods of Calculating Interest (i)



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Interest Rates

- Interest rates are stated for some period, like a year, while the computation of interest is based on shorter compounding sub-periods such as months.
  - 12% per year interest compounded yearly? Or 1% per month interest compounded monthly?

- Nominal Interest Rate:
  - Is the conventional method of stating the annual interest rate.
  - It is calculated by multiplying the interest rate per compounding period by the number of compounding periods per year.
  - Suppose the nominal interest rate is r and the time period is divided into m equal sub-periods, then the interest rate for each sub-period is: $i_s = r/m$
  - Example: Nominal interest rate of 18% per year, compounded monthly, implies: $i_{s,monthly} = \frac{r}{m} = \frac{18\%}{12} = 1.5\%$

Interest Rates

- Effective Interest Rate:
  - Is the actual but not usually stated interest rate.
  - It is determined by converting a given interest rate with an arbitrary compounding period (normally less than a year) to an equivalent interest rate with a one-year compounding period.
  - The compound interest every sub-period will be:

$$F = P(1 + i_s)^m$$

  - The effective interest rate, i_e, that yields the same future amount F at the end of the full period from the present amount P:

$$P(1 + i_s)^m = P(1 + i_e)$$

$$(1 + i_s)^m = (1 + i_e)$$

$$i_e = (1 + i_s)^m - 1$$

  - To convert from a nominal rate r to an annual effective rate:

$$i_e = \left(1 + \frac{r}{m}\right)^m - 1$$

Interest Rates

- Example A
- A local bank announces that a deposit over $1,000 will receive a monthly interest of 0.5%. If you leave $10,000 in this account, how much would you have at the end of one year?
- $F\ = P(1+i)^n = 10{,}000 * (1+0.005)^{12} = 10{,}000 * (1.062) = 10{,}620$
- This means that over a one-year period, $620 has been added to our money, which is the same as a 6.2% annual interest rate.
- We can see that this is not 12 times the monthly interest rate of 0.5% which is 6%. The difference between the 6.2% and 6% rates is the result of compounding monthly rather than annually.

Interest Rates

- Example B
- The annual interest rate is 6%, and the interest is compounded quarterly. What is the quarterly nominal interest rate? What is the effective annual interest rate if compounded quarterly and monthly?
- Nominal Interest Rate:
- $i_{s,quaterly} = \frac{r}{m} = \frac{6\%}{4} = 1.5\%$ $\quad$ $i_{s,monthly} = \frac{r}{m} = \frac{6\%}{12} = 0.5\%$
- Effective Interest Rate, quarterly:
- $i_e = (1 + i_{s,Q})^m - 1 = (1 + 0.015)^4 - 1 = 0.0613 = 6.13\%$
- Effective Interest Rate, monthly:
- $i_e = (1 + i_{s,M})^m - 1 = (1 + 0.005)^{12} - 1 = 0.0617 = 6.17\%$

Interest Rates

- Example C
- A Credit Card company charges a nominal 24% interest on overdue accounts, compounded daily? What is the effective interest rate?
- Assuming 365 days per year:
- $i_{s,daily} = \frac{r}{m} = \frac{0.24}{365} = 0.0006575$
- The effective interest rate (per year):
- $i_e = \left(1 + i_{s,D}\right)^m - 1 = (1 + 0.0006575\ )^{365} - 1 = 0.271 = 27.1\%$

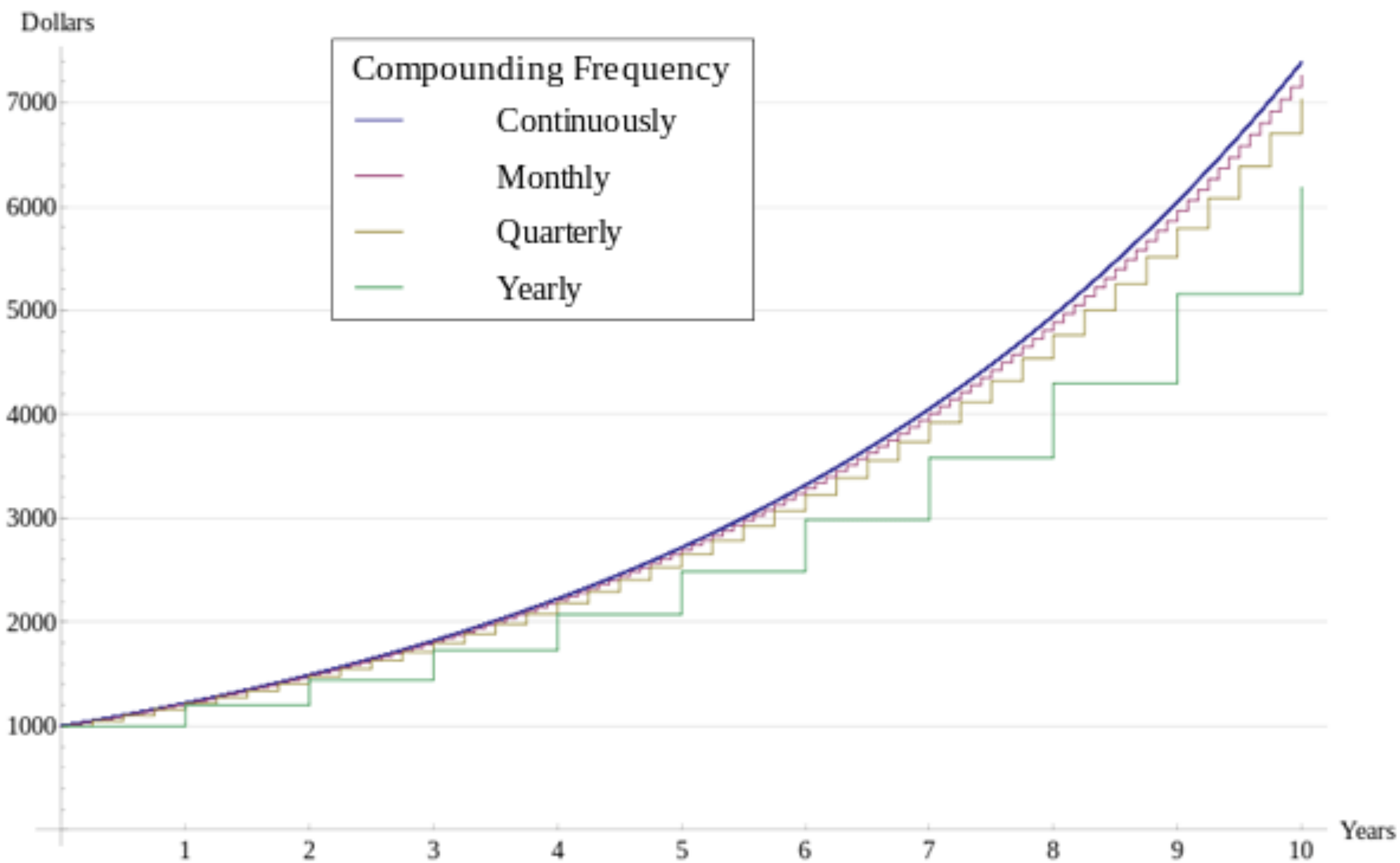
Interest Rates

- ## Continuous Compounding:
  - Compounding can be done yearly, quarterly, monthly, daily, or even on smaller time sub-periods.
  - If the period is made infinitesimally small, the interest will be compounded continuously.
  - Continuously compounding equation:

$$i_e = \lim_{m \to \infty} \left(1 + \frac{r}{m}\right)^m - 1 \quad \rightarrow \quad \lim_{m \to \infty} \left(1 + \frac{r}{m}\right)^m = e^r \quad \rightarrow \quad i_e = e^r - 1$$

- ## Example D
- An investment in a new stock is expected to return a nominal interest rate of 40%, compounded continuously. What is the effective interest rate earned by this stock?
- $i_e = e^r - 1 = e^{0.4} - 1 = 1.492 - 1 = 0.492 \ or \ 49.2\%$

Methods of Calculating Interest (i)



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Cash Flow Diagram

- The graphic presentation of the costs and benefits over the time.
- This is the time profile of all the costs and benefits. It is a presentation of what costs have to be incurred and what benefits are received at all points in time.
- The following conventions are used:
  - The horizontal axis represents time
  - The vertical axis represents costs and benefits
  - Costs are shown by downward arrows
  - Benefits are shown by upward arrows
  - All the benefits and/or costs incurred during a period are assumed to have been incurred at the end of that period. Since the period is normally a year, this is called the ''end of the year" rule.

Cash Flow Diagram

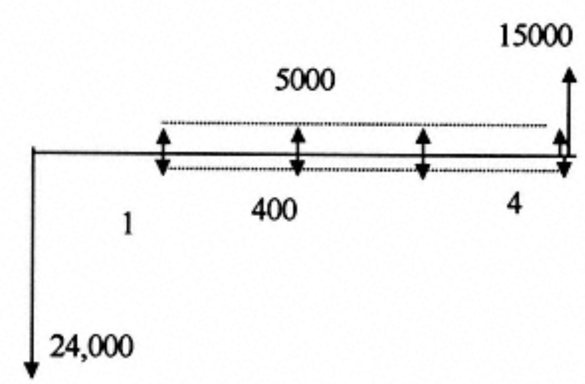
- Example C
- A car leasing company buys a car from a wholesaler for $24,000 and leases it to a customer for four years at $5,000 per year. Since the maintenance is not included in the lease, the leasing company has to spend $400 per year in servicing the car. At the end of the four years, the leasing company takes back the car and sells it to a secondhand car dealer for $15,000.

Cash Flow Diagram

- Example C
- Step 1:
  - Draw the horizontal axis to represent 1,2,3, and 4 years.
- Step 2:
  - At time zero, i.e., the beginning of year 1, the leasing company spends $24,000. Hence, at time zero, on the horizontal axis, a downward arrow represents this number.
- Step 3:
  - At the end of year 1, the company receives $5,000 from his customer. This is represented by an upward arrow at the end of year 1. The customer also spends $400 for maintaining the car; this is represented by a downward arrow.
  - The situations at years 2 and 3 are exactly the same as year 1 and are the presentations on the cash flow diagram exactly as for the first year.

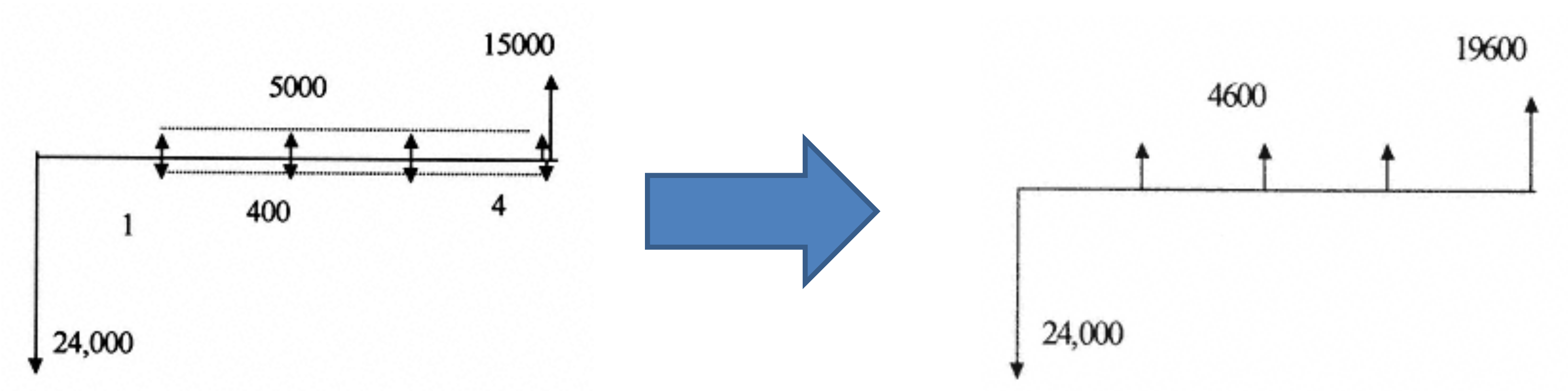
Cash Flow Diagram

- Step 4:
  - At the end of the fourth year, in addition to the income and the expenditure as in the previous years, the leasing company receives $15,000 by selling the car. This additional income is represented by an upward arrow.
  - The project ends at this time, so we have nothing else to insert in the cash flow diagram. We have represented all the costs and benefits in the cash flow.
  - At this point, it is a good idea to go back through the life of the project and make sure that nothing as expressed in the description of the project is left out.
  - Cash flow diagram

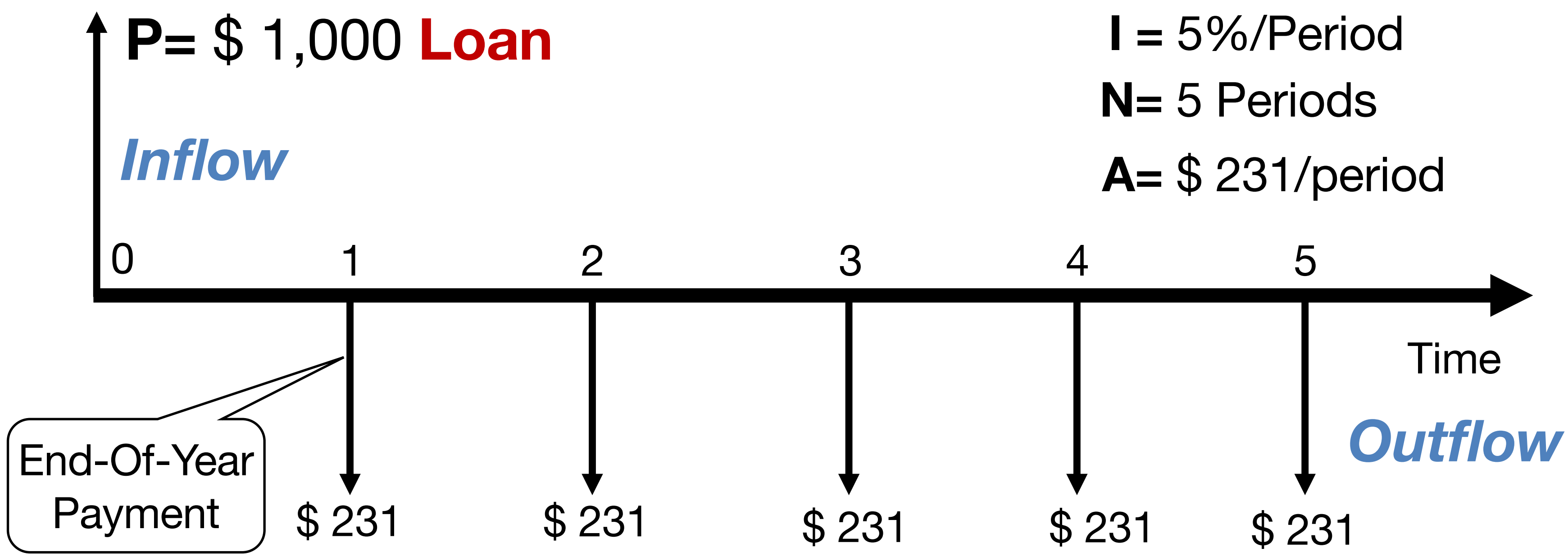


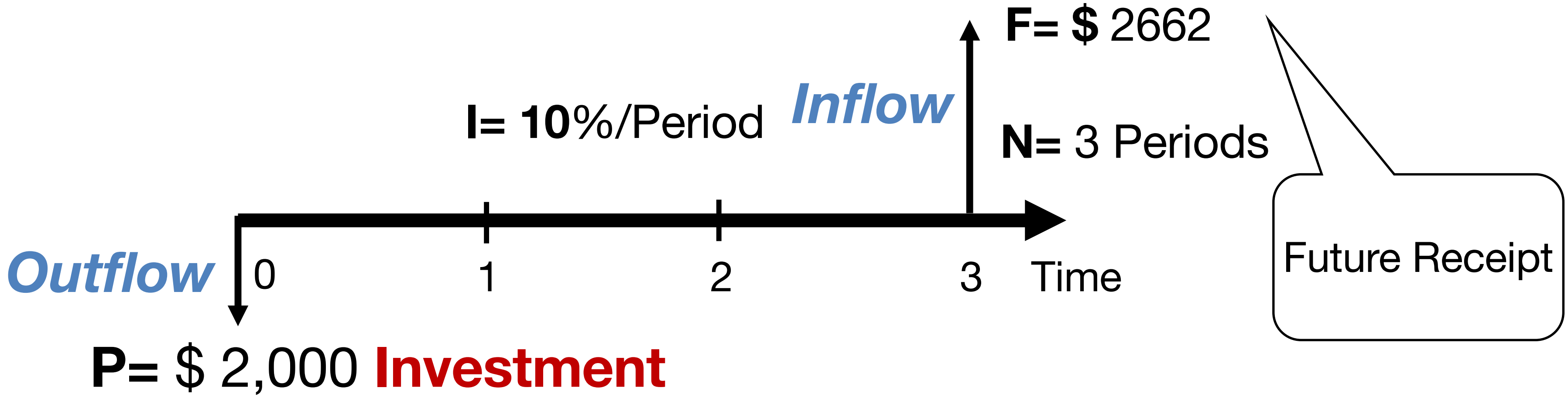
Cash Flow Diagram

- Netted cash flow diagram
  - After the cost and benefits of each year are deducted from each other:



Cash Flow Diagram: Example





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