ELECTRIC STRESS DUE TO H.V

Dr.-Ing. Getachew Biru

INTRODUCTION

- The electric field in the insulation material must be known under all possible conditions of operation, so that the insulation can be designed and chosen to have a dielectric strength greater than the applied field levels.

INTRODUCTION

- In high voltage engineering we are particularly interested in electric and magnetic fields.
- Electric fields are caused by the voltage difference between electrodes while magnetic fields are caused by currents flowing in conductors.
- The performance of electric insulating materials is adversely affected by excessive electric field magnitudes.

INTRODUCTION

- High AC currents cause time-varying magnetic fields that induce voltages in conducting loops.
- Similarly, high time-varying currents due to lightning may cause induced voltages. These overvoltages cause high electric fields in power system components that may cause failure of the insulation systems.

Basic Equations

- The work done on a charge when moved is E, is the electric potential. where dl is the path that the charge is moved, or

- Gauss theorem: for closed surface S

$$\iint \bar{E} \cdot d\bar{S} = \frac{q}{\varepsilon_0}$$

$$\bar{\nabla} \cdot \bar{D} = \rho$$

(Poisson's equation)

Basic Equations

- The potential at a point plays an important role in obtaining any information regarding the electrostatic field at that point. The electric field intensity can be obtained from the potential by gradient operation on the potential

$$E = -\nabla V \qquad (1.1)$$

- The electric flux density

$$D = \varepsilon E \qquad (1.2)$$

- The divergence of this flux density which is again a differentiation results in volume charge density.

$$\nabla \cdot D = \rho_v \qquad (1.3)$$

Basic Equations

- Writing $\varepsilon E = D$ in equation (1.3) we have

$$\nabla . \varepsilon E = \rho_v$$
$$-\nabla . \varepsilon . \nabla V = \rho_v$$
$$\varepsilon \nabla^2 V = -\rho_v$$
$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} \qquad (1.4)$$

- This is known as Poisson's equation. However, in most of the high voltage equipments, space charges are not present and hence $\rho_v = 0$ and hence equation (1.4) is written as

$$\nabla^2 V = 0$$

- is known as Laplace's equation

Basic Equations

- Here ∇ is a vector operator and is termed as del operator and expressed mathematically in Cartesian coordinates as

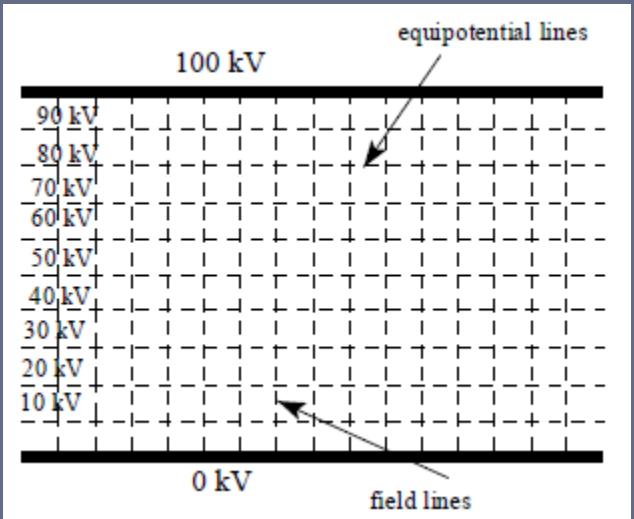
$$\nabla = \frac{\partial}{\partial x}\bar{a}_x + \frac{\partial}{\partial y}\bar{a}_y + \frac{\partial}{\partial z}\bar{a}_z \qquad (1.5)$$

- where a_x , a_y and a_z are unit vectors in the respective increasing directions.
- Hence Laplace's equation in Cartesian coordinates is given as

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Electric Field Stress

- Uniform fields: If a voltage difference exist between two parallel plate electrodes of large dimensions, the electrostatic field between the plates is uniform.



- Note that the equipotential lines are equidistant and that the field strength E is constant throughout the region. The field is therefore a uniform field.
- The lines perpendicular to the equipotential lines are known as field lines.
- The field lines indicate the direction that a positive charge would move if placed in the field

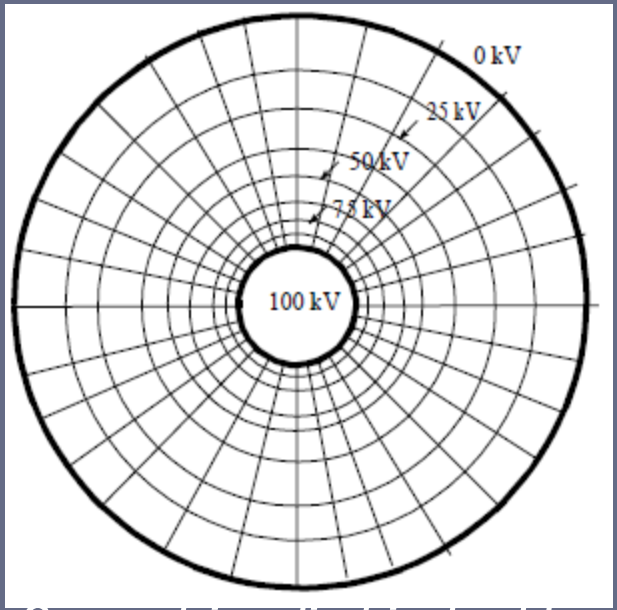
- In general, the electric field strength is given by:

$$E = \frac{V}{d}$$

V : voltage between plates
d : distance between the plates.

Electric Field Stress

- Non-uniform fields: the field strength is not constant throughout the region.



Concentric cylindrical cable

Note that the equipotential lines are much closer together near the inner conductor, resulting in higher field strength near the inner conductor. The field strength E is therefore not constant throughout the region and such a field is known as a non-uniform field.

It can be shown that the electric field strength E a distance *r* from the center line of the configuration is given by:

$$E_r = \frac{V}{r\,ln(b/a)}$$

V: voltage between the electrodes
b: outer radius
a: inner radius
r: distance from center line

Electric Field Stress

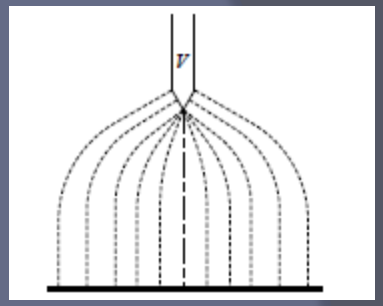
- In high voltage applications the dielectric strength of insulating materials are stressed by electric field.
- The electric stress to which an insulating material is subjected to is numerically equal to the voltage gradient and is equal to the electric field intensity.

$$\vec{E} = -\nabla V = -\nabla \Phi$$

- where Φ is the applied voltage and ∇ is the gradient.
- The dielectric strength of an insulating material can be defined as the maximum dielectric stress which the material can withstand.

Electric Field Stress

- The field efficiency factor η is defined as
- $\eta = \frac{E_{av}}{E_{max}}$



- E_{av} is the average value of electric field E_{max} is the maximum value of the field. This value indicates how much the level of non-uniformity of the field is.
- The average value of the field strength of an air gap is defined by equation:
- $E_{av} = \frac{V}{d}$
- V is applied voltage and d is the thickness of the insulation between the electrodes.

Field Calculation

Consider a single core cable with the following data.

r = radius of core (m)

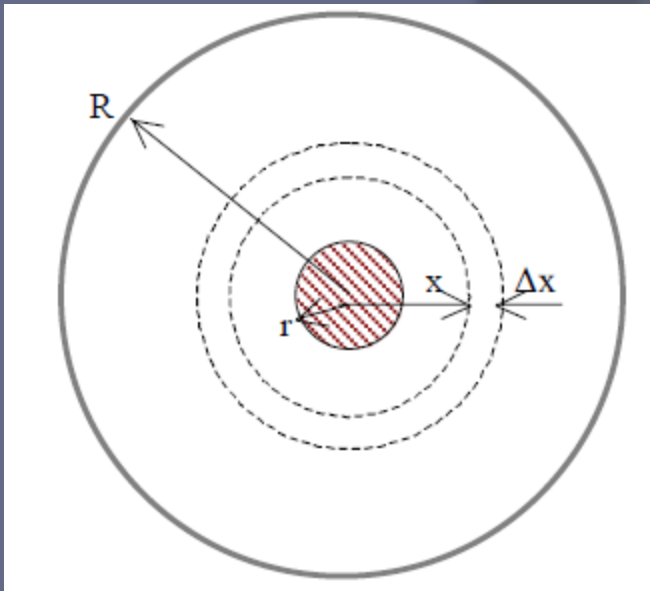
R = radius of earthed sheath (m)

q = charge/unit length of cable (C/m)

D = electric flux density (C/m²)

ε_0 = permittivity of free space

$= 1/(4\pi \times 9 \times 10^9)$ F/m



Field Calculation

Consider a single core cable (figure 5.9) with the following data.

r = radius of core (m)

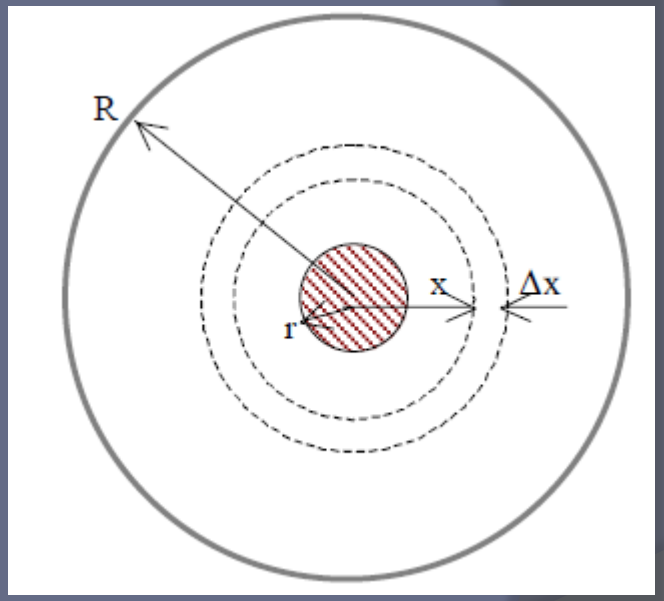
R = radius of earthed sheath (m)

Q = charge (C)

D = electric flux density (C/m^2)

ε_0 = permittivity of free space

$= 1/(4\pi \text{ x } 9 \text{ x } 10^9)$ F/m



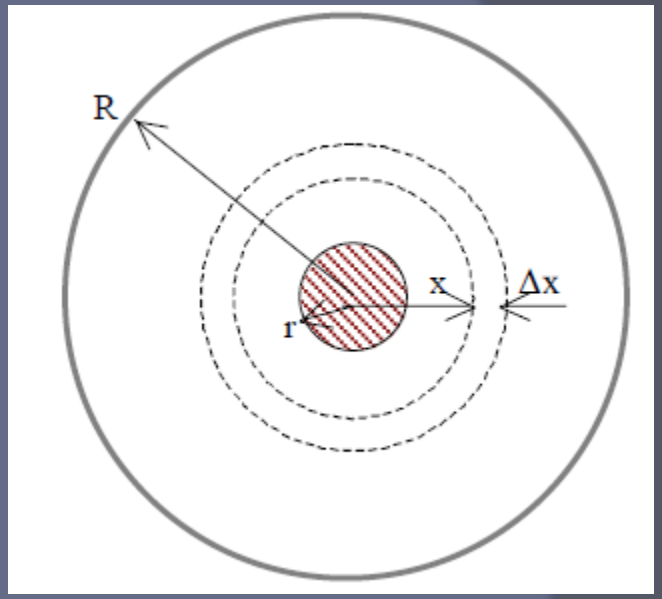
D(x) $2 * \pi *$x*l=Q

$$D(x) = \frac{Q}{2*\pi*x}$$

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$$

Field Calculation

Consider a single core cable (figure 5.9) with the following data.



$$D(x)= \frac{Q}{2*\pi*x*l}$$

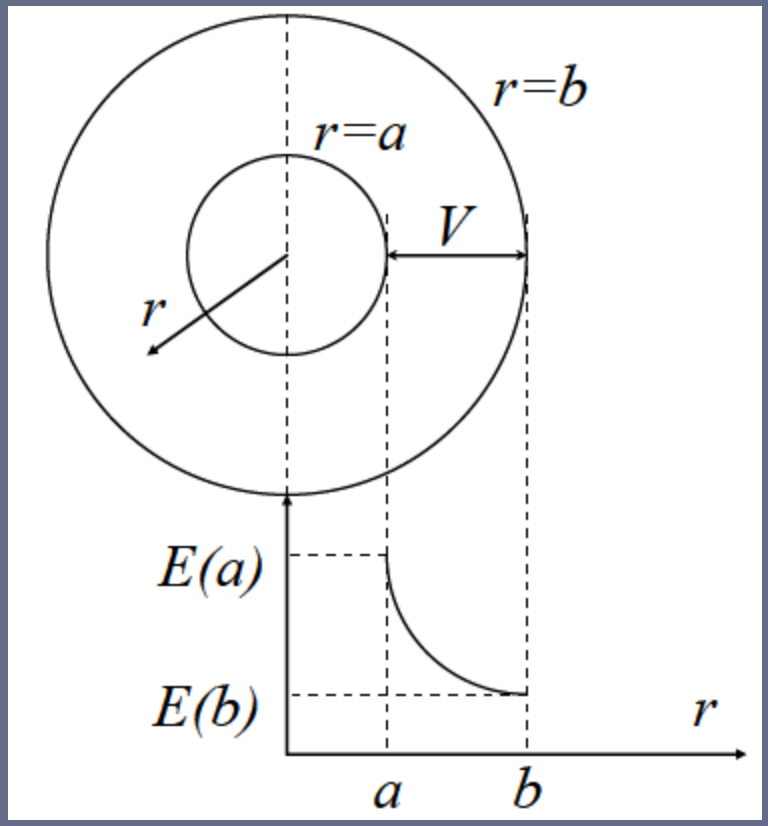
$$E(x)= \frac{Q}{2*\pi*\varepsilon*x*l}$$

$$V=\int_r^R E(x)\,dx = \int_r^R \frac{Q}{2*\pi*\varepsilon*x*l} = \frac{Q}{2*\pi*\varepsilon*l}\ln\left(\frac{R}{r}\right)$$

$$E(x)= \frac{V}{x\,\ln\left(\frac{R}{r}\right)}$$

Field Calculation

Single core cable



Highest stress occurs at surface of inner conductor.
In HV cables, this is taken as the *operating stress*, which is the determining factor in the design.
The field factor is very low, indicating the dielectric material is not effectively utilized.

Field Calculation

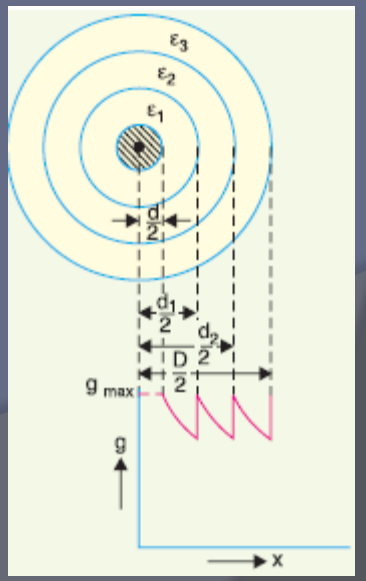
The process of achieving uniform electrostatic stress in the dielectric of cables is known as grading of cables.

- Capacitance Grading: The process of achieving uniformity in the dielectric stress by using layers of different dielectrics is known as capacitance grading.

$$\frac{1}{\varepsilon_1 d} = \frac{1}{\varepsilon_2 d_1} = \frac{1\dagger}{\varepsilon_3 d_2}$$

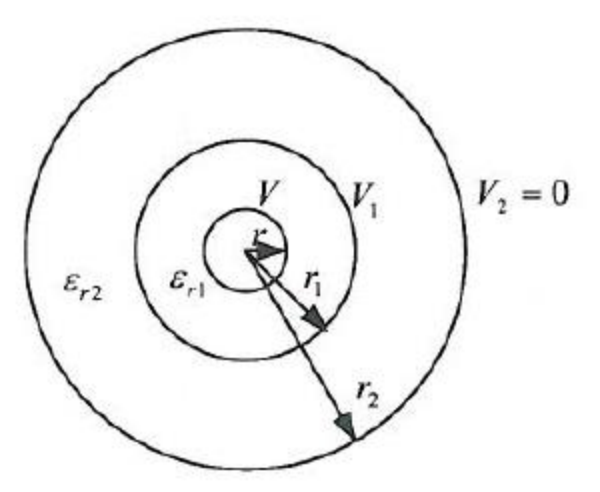
$$\varepsilon_1 d = \varepsilon_2 d_1 = \varepsilon_3 d_2$$

$$V = \frac{g_{1max}}{2} d \log_e \frac{d_1}{d} + \frac{g_{2max}}{2} d_1 \log_e \frac{d_2}{d_1} + \frac{g_{3max}}{2} d_2 \log_e \frac{D}{d_2}$$



Field Calculation

Exercise: A single core, lead covered capacitance graded cable is to be designed for 60 kV peak voltage to ground. The conductor radius is 15 mm and there are two concentric layers of insulation materials with relative permittivities of 5 and 3 respectively and maximum permissible electric stress is 4 and 3 kV/mm (Peak), respectively. Calculate the minimum inner radius (r_1 and r_2) for the two mediums.

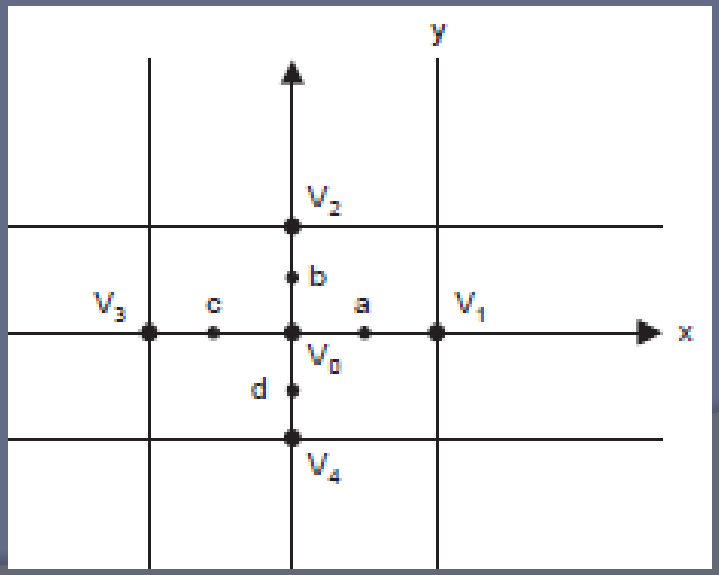


Numerical Solutions: Finite Differences

- Analytical solutions exist only for simple geometries and linear material behavior.
- For complex geometry, one has to use the numerical methods to solve field problems.

Numerical Solutions: Finite Differences

- Let us assume that voltage variations is a two dimensional problem i.e. it varies in x-y plane and it does not vary along z-coordinate and let us divide the interior of a cross section of the region where the potential distribution is required into squares of length h on a side as shown in Fig.



Numerical Solutions: Finite Differences

- Assuming the region to be charge free

$$\nabla \cdot D = 0 \quad \text{or} \quad \nabla \cdot E = 0$$

- and for a two-dimensional situation

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0$$

- From Lap. Eq (1).

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Numerical Solutions: Finite Differences

Approximate values for these partial derivatives may be obtained in terms of the assumed values (Here V_0 is to be obtained when V_1, V_2, V_3 and V_4 are known in Fig. 1.

$$\left.\frac{\partial V}{\partial x}\right|_a = \frac{V_1 - V_0}{h} \quad \text{and} \quad \left.\frac{\partial V}{\partial x}\right|_c = \frac{V_0 - V_3}{h}$$

From the gradients

$$\left.\frac{\partial^2 V}{\partial x^2}\right|_0 = \frac{\left.\frac{\partial V}{\partial x}\right|_a - \left.\frac{\partial V}{\partial x}\right|_c}{h} = \frac{V_1 - V_0 - V_0 + V_3}{h^2}$$

Similarly $$\left.\frac{\partial^2 V}{\partial y^2}\right|_0 = \frac{V_2 - V_0 - V_0 + V_4}{h^2}$$

Numerical Solutions: Finite Differences

Substituting in equation (1) we have

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{V_1 + V_2 + V_3 + V_4 - 4V_0}{h^2} = 0$$

$$V_0 = \frac{1}{4}(V_1 + V_2 + V_3 + V_4)$$

As mentioned earlier the potentials at four corners of the square are, these correspond to boundary potentials which are known a priori.

The process is repeated over the entire region until the difference in values is less than a pre-specified value.

Numerical Solutions: Finite Differences

Example: The Poisson's equation

$$\frac{\partial^2 V}{\partial x^2} + V = 0$$

- Over a one-dim. Region with $0 \leq x \leq 1$ and boundary conditions V(0)=0 and V(1)=1 is to be solved.

- The region is divided into n equal segments (elements), with (n+1) grid points. Let us assume n=4
- The grid points are denoted by i=0, 1, 2, 3..
- At the grid points x=i*h, where h is the spacing between two adjacent grid points.

Numerical Solutions: Finite Differences

Example:

- We can approximate the differential equation with central-difference formula
- $\frac{\partial}{\partial x}\frac{\partial V}{\partial x} = \frac{\partial}{\partial x}\left[\frac{V_{i+1} - Vi}{h}\right] = \frac{1}{h}\left\{\left[\frac{V_{i+1} - V_i}{h}\right] - \left[\frac{V_i - Vi_{-1}}{h}\right]\right\}$
- $= \frac{1}{h^2}\left[V_{i+1} - 2 * V_i + V_{i-1}\right]$
- $16\left[V_{i+1} - 2 * V_i + V_{i-1}\right] + V_i = o$

- From this the system of equation is established for i=1, 2,3 with h=1/4
- A matrix of 3X 3 is formulated to solve the three unknowns.

Numerical Solutions: Finite Element

The finite element analysis of any problem involves basically four steps:

- discretizing the solution region into a finite number of sub-regions or elements,
- deriving governing equations for a typical element,
- assembling of all elements in the solution region, and
- solving the system of equations obtained.

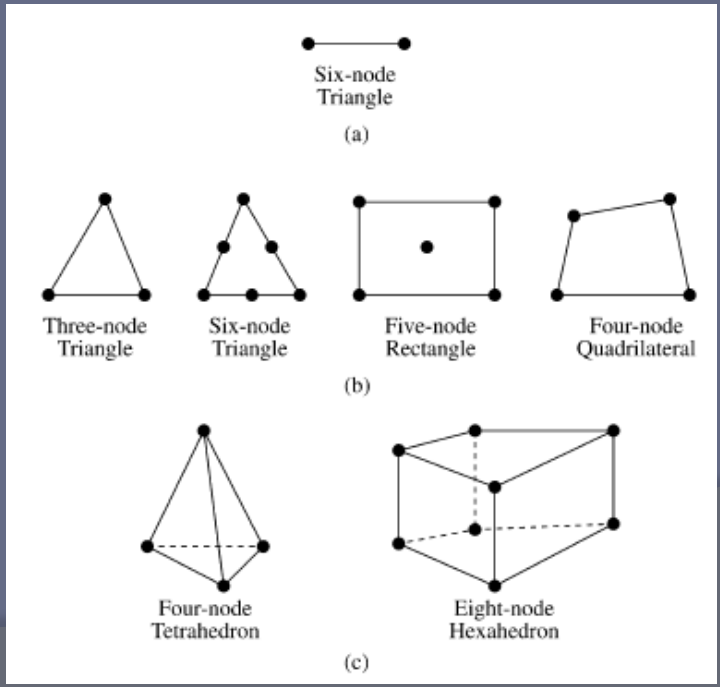
Numerical Solutions: Finite Element

The finite element analysis of any problem involves basically four steps:

- discretizing the solution region into a finite number of sub-regions or elements,
- deriving governing equations for a typical element,
- assembling of all elements in the solution region, and
- solving the system of equations obtained.

Numerical Solutions: Finite Element

Discretization of the continuum involves dividing up the solution region into subdomains, called finite elements. The Figure, shows some typical elements for one-,two-, and three-dimensional problems.



Numerical Solutions: Finite Element

- As an application of FEM to electrostatic problems, let us apply the four steps mentioned above to solve Laplace's equation, $\nabla^2 V = 0$

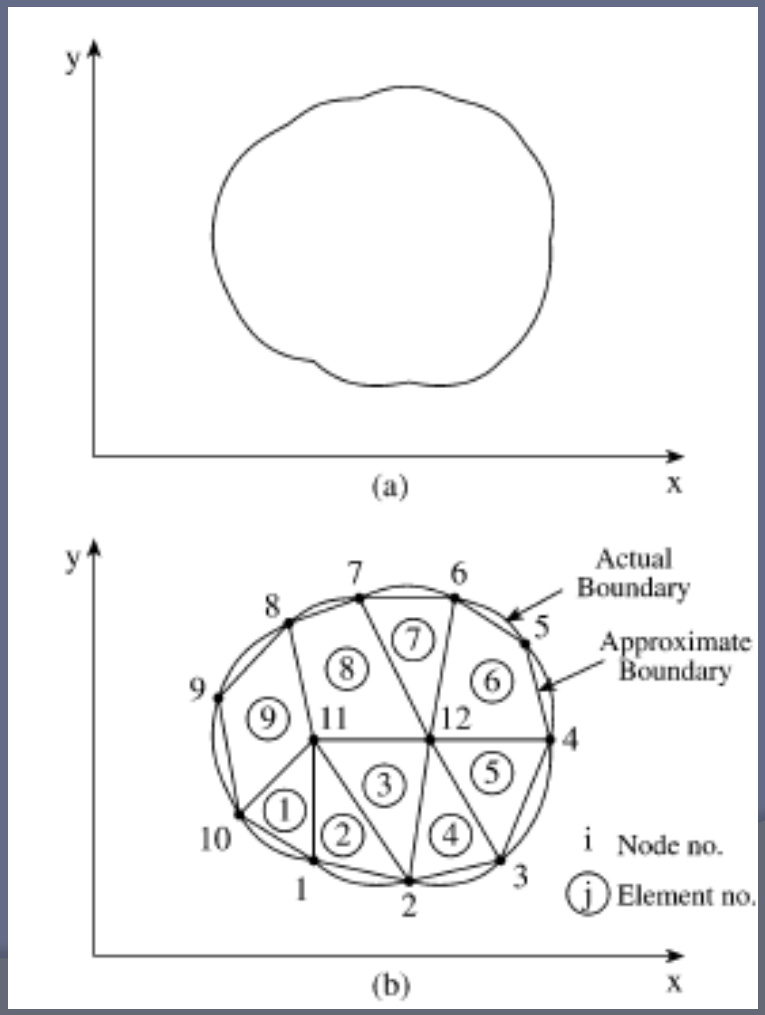
Numerical Solutions: Finite Element

Finite Element Discretization

- To find the potential distribution V (x, y) for the two-dimensional solution region shown in Fig, we divide the region into a number of finite elements as illustrated in Fig. 6.2(b). In Fig. 6.2(b), the solution region is subdivided into nine nonoverlapping *finite elements* 6, 8, and 9 are four-node quadrilaterals, while other elements are three-node triangles. In practical situations.

Numerical Solutions: Finite Element

Finite Element Discretization



Numerical Solutions: Finite Element

Finite Element Discretization

- We seek an approximation for the potential V_e within an element e and then interrelate the potential distribution in various elements such that the potential is continuous across interelement boundaries. The approximate solution for the whole region is

$$V(x, y) \simeq \sum_{e=1}^{N} V_e(x, y),$$

- where N is the number of elements into which the solution region is divided.

Numerical Solutions: Finite Element

Finite Element Discretization

- The most common form of approximation for V_e within an element is polynomial approximation, namely,

$$V_e(x, y) = a + bx + cy \quad (1)$$

- for a triangular element and for a quadrilateral element.

$$V_e(x, y) = a + bx + cy + dxy$$

Numerical Solutions: Finite Element

Finite Element Discretization

- We use triangular elements throughout our analysis in this chapter. Notice that our assumption of linear variation of potential within the triangular element as in Eq. is the same as assuming that the electric field is uniform within the element, i.e.

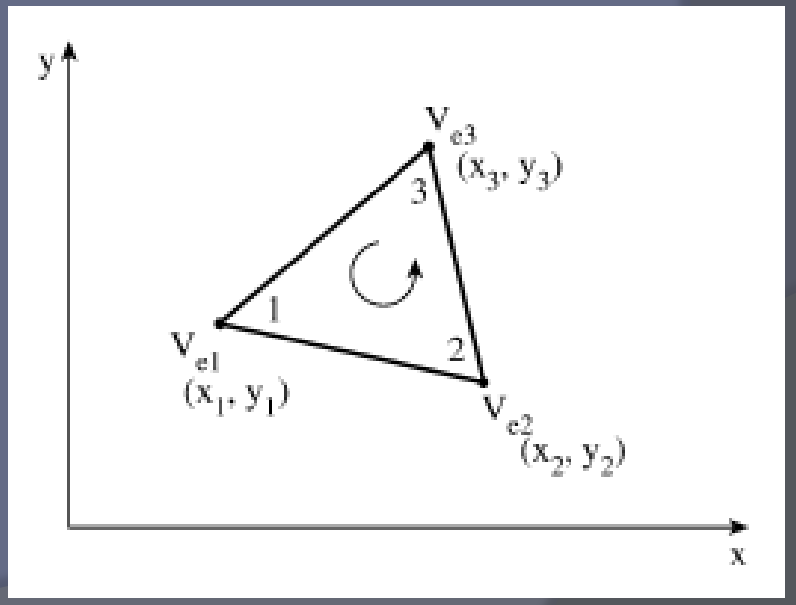
$$\mathbf{E}_e = -\nabla V_e = -\left(b\mathbf{a}_x + c\mathbf{a}_y\right)$$

Numerical Solutions: Finite Element

Element Governing Equations

Consider a typical triangular element shown in Fig. The potential V_{e1}, V_{e2}, and V_{e3} at nodes 1, 2, and 3, respectively, are obtained using Eq. (*), i.e.,

$$\begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad **$$



Numerical Solutions: Finite Element

Element Governing Equations

The coefficients *a*, *b* and *c* are determined from Eq. (**) as

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix}$$

Numerical Solutions: Finite Element

Element Governing Equations

Substituting this into Eq. (*) gives

$$V_e = [1\ x\ y]\frac{1}{2A}\begin{bmatrix}(x_2y_3 - x_3y_2) & (x_3y_1 - x_1y_3) & (x_1y_2 - x_2y_1)\\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2)\\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1)\end{bmatrix}\begin{bmatrix}V_{e1}\\ V_{e2}\\ V_{e3}\end{bmatrix}$$

Or

$$V_e = \sum_{i=1}^{3} \alpha_i(x, y)V_{ei} \qquad (3)$$

Note that Eq. gives the potential at any point (x, y) within the element provided that the potentials at the vertices are known. This is unlike finite difference analysis, where the potential is known at the grid points only.

Numerical Solutions: Finite Element

Element Governing Equations

Where

$$\alpha_1 = \frac{1}{2A}\left[(x_2y_3 - x_3y_2) + (y_2 - y_3)\,x + (x_3 - x_2)\,y\right],$$

$$\alpha_2 = \frac{1}{2A}\left[(x_3y_1 - x_1y_3) + (y_3 - y_1)\,x + (x_1 - x_3)\,y\right],$$

$$\alpha_3 = \frac{1}{2A}\left[(x_1y_2 - x_2y_1) + (y_1 - y_2)\,x + (x_2 - x_1)\,y\right],$$

where

$$2A = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = (x_1y_2 - x_2y_1) + (x_3y_1 - x_1y_3) + (x_2y_3 - x_3y_2)$$

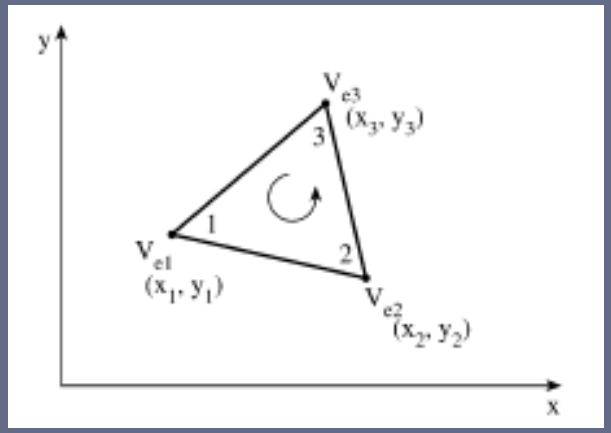
$$A = \frac{1}{2}\left[(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)\right]$$

A is the area of the element

Numerical Solutions: Finite Element

Element Governing Equations

The value of A is positive if the nodes are numbered counterclockwise (starting from any node) as shown by the arrow in Fig



Typical triangular element; local node numbering 1-2-3 must proceed counterclockwise as indicated by the arrow.

Numerical Solutions: Finite Element

Element Governing Equations

Note that Eq. gives the potential

$$V_e = \sum_{i=1}^{3} \alpha_i(x, y) V_{ei}$$

- at any point *(x, y)* within the element provided that the potentials at the vertices are known.
- Also note that α_i are linear interpolation functions. They are called the *element shape functions* and they have the following properties:

$$\alpha_i = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

$$\sum_{i=1}^{3} \alpha_i(x, y) = 1$$

Numerical Solutions: Finite Element

Element Governing Equations

The functional corresponding to Laplace's equation $\nabla^2 V=0$ is given by

$$W_e = \frac{1}{2}\int \epsilon\,|\mathbf{E}_e|^2\,dS = \frac{1}{2}\int \epsilon\,|\nabla V_e|^2\,dS \qquad (4)$$

- (Physically, the functional W_e is the energy per unit length associated with the element e.)

Numerical Solutions: Finite Element

Element Governing Equations

From Eq. (4),

$$\nabla V_e = \sum_{i=1}^{3} V_{ei} \nabla \alpha_i$$

Substituting Eq. (3) into (4) gives

$$W_e = \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \epsilon V_{ei} \left[\int \nabla \alpha_i \cdot \nabla \alpha_j \, dS \right] V_{ej}$$

If we define the term in brackets as

$$C_{ij}^{(e)} = \int \nabla \alpha_i \cdot \nabla \alpha_j \, dS \, ,$$

Numerical Solutions: Finite Element

Element Governing Equations

we may write the Eq. in matrix form as

$$W_e = \frac{1}{2}\epsilon [V_e]^t [C^{(e)}][V_e]$$

where the superscript *t* denotes the transpose of the matrix,

$$[V_e] = \begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix}$$

$$\left[C^{(e)}\right] = \begin{bmatrix} C_{11}^{(e)} & C_{12}^{(e)} & C_{13}^{(e)} \\ C_{21}^{(e)} & C_{22}^{(e)} & C_{23}^{(e)} \\ C_{31}^{(e)} & C_{32}^{(e)} & C_{33}^{(e)} \end{bmatrix}$$

Numerical Solutions: Finite Element

Element Governing Equations

The matrix [C(e)] is usually called the element coefficient matrix (or "stiffness matrix"). The element $C(e)_{ij}$ of the coefficient matrix may be regarded as the coupling between nodes i and j. For example

$$
\begin{aligned}
C_{12}^{(e)} &= \int \nabla\alpha_1 \cdot \nabla\alpha_2 \, dS \\
&= \frac{1}{4A^2} \left[(y_2 - y_3)(y_3 - y_1) + (x_3 - x_2)(x_1 - x_3) \right] \int dS \\
&= \frac{1}{4A} \left[(y_2 - y_3)(y_3 - y_1) + (x_3 - x_2)(x_1 - x_3) \right]
\end{aligned}
$$

Numerical Solutions: Finite Element

Element Governing Equations

Similarly

$$C_{13}^{(e)} = \frac{1}{4A}\left[(y_2 - y_3)(y_1 - y_2) + (x_3 - x_2)(x_2 - x_1)\right],$$

$$C_{23}^{(e)} = \frac{1}{4A}\left[(y_3 - y_1)(y_1 - y_2) + (x_1 - x_3)(x_2 - x_1)\right],$$

$$C_{11}^{(e)} = \frac{1}{4A}\left[(y_2 - y_3)^2 + (x_3 - x_2)^2\right],$$

$$C_{22}^{(e)} = \frac{1}{4A}\left[(y_3 - y_1)^2 + (x_1 - x_3)^2\right],$$

$$C_{33}^{(e)} = \frac{1}{4A}\left[(y_1 - y_2)^2 + (x_2 - x_1)^2\right]$$

$$C_{21}^{(e)} = C_{12}^{(e)}, \quad C_{31}^{(e)} = C_{13}^{(e)}, \quad C_{32}^{(e)} = C_{23}^{(e)}$$

Numerical Solutions: Finite Element

Assembling of All Elements

Having considered a typical element, the next step is to assemble all such elements in the solution region. The energy associated with the assemblage of elements is

$$W = \sum_{e=1}^{N} W_e = \frac{1}{2}\epsilon [V]^t [C][V]$$

$$[V] = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_n \end{bmatrix},$$

n is the number of nodes, *N* is the number of elements, and [*C*] is called the overall or *global coefficient matrix,* which is the assemblage of individual element coefficient matrices.

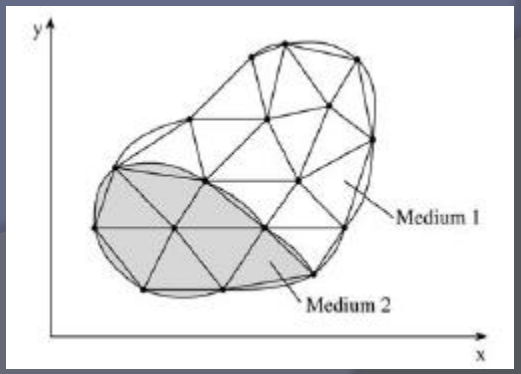
Numerical Solutions: Finite Element

Assembling of All Elements

- Notice that we have assumed that the whole solution region is homogeneous so that ε is constant. For an inhomogeneous solution region the region is discretized such that each finite element is homogeneous.
- For we may replace multiply the integrand in Eq.

by ε_r .

$$C_{ij}^{(e)} = \int \nabla \alpha_i \cdot \nabla \alpha_j \, dS \,,$$

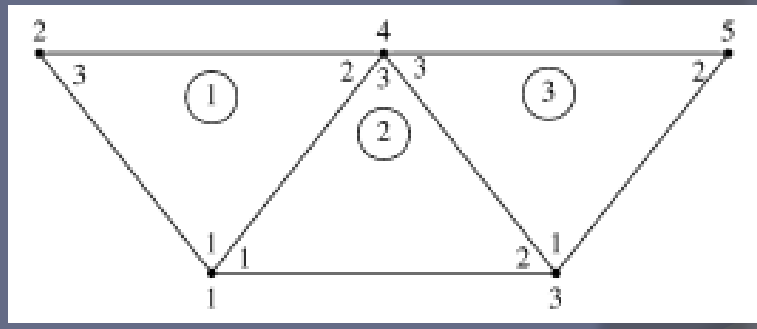


Numerical Solutions: Finite Element

Assembling of All Elements

- Assembly of three elements; i-j-k corresponds to local numbering (1-2-3) of the element

The numbering of nodes 1, 2, 3, 4, and 5 is called *global* numbering. The numbering *i-j -k* is called *local* numbering, and it corresponds with 1 - 2 - 3 of the element



Note that the local numbering must be in counterclockwise sequence starting from any node of the element.

For example, for element 3, the global numbering 3 - 5 - 4 corresponds with local numbering 1 - 2 - 3 of the element.

Numerical Solutions: Finite Element

Assembling of All Elements

- The global coefficient matrix is expected to have the form which is a 5 × 5 matrix since five nodes ($n = 5$) are involved.

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} \end{bmatrix}$$

Numerical Solutions: Finite Element

Assembling of All Elements

- Again, C_{ij} is the coupling between nodes *i* and *j* . We obtain C_{ij} by using the fact that the potential distribution must be continuous across inter-element boundaries.
- The contribution to the *i, j* position in [*C*] comes from all elements containing nodes *i* and *j*. For example, in the Fig., elements 1 and 2 have node 1 in common; hence

$$C_{11} = C_{11}^{(1)} + C_{11}^{(2)}$$

- Node 2 belongs to element 1 only; hence $C_{22} = C_{33}^{(1)}$

Numerical Solutions: Finite Element

Assembling of All Elements

- Node 4 belongs to elements 1, 2, and 3; consequently

$$C_{44} = C_{22}^{(1)} + C_{33}^{(2)} + C_{33}^{(3)}$$

- Nodes 1 and 4 belong simultaneously to elements 1 and 2; hence

$$C_{14} = C_{41} = C_{12}^{(1)} + C_{13}^{(2)}$$

- Since there is no coupling (or direct link) between nodes 2 and 3,

$$C_{23} = C_{32} = 0$$

Numerical Solutions: Finite Element

Assembling of All Elements

- Continuing in this manner, we obtain all the terms in the global coefficient matrix by inspection of the Fig. as

$$\begin{bmatrix} C_{11}^{(1)}+C_{11}^{(2)} & C_{13}^{(1)} & C_{12}^{(2)} & C_{12}^{(1)}+C_{13}^{(2)} & 0 \\ C_{31}^{(1)} & C_{33}^{(1)} & 0 & C_{32}^{(1)} & 0 \\ C_{21}^{(2)} & 0 & C_{22}^{(2)}+C_{11}^{(3)} & C_{23}^{(2)}+C_{13}^{(3)} & C_{12}^{(3)} \\ C_{21}^{(1)}+C_{31}^{(2)} & C_{23}^{(1)} & C_{32}^{(2)}+C_{31}^{(3)} & C_{22}^{(1)}+C_{33}^{(2)}+C_{33}^{(3)} & C_{32}^{(3)} \\ 0 & 0 & C_{21}^{(3)} & C_{23}^{(3)} & C_{22}^{(3)} \end{bmatrix}$$

Numerical Solutions: Finite Element

Assembling of All Elements

Note the following properties of the matrix [C]:

- It is symmetric ($C_{ij} = C_{ji}$) just as the element coefficient matrix.
- Since $C_{ij} = 0$ if no coupling exists between nodes i and j, it is expected that for a large number of elements [C] becomes sparse.
- It is singular.

Numerical Solutions: Finite Element

Solving the Resulting Equations

It can be shown that Laplace's equation is satisfied when the total energy in the solution region is minimum. Thus we require that the partial derivatives of W with respect to each nodal value of the potential be zero, i.e.,

$$\frac{\partial W}{\partial V_1} = \frac{\partial W}{\partial V_2} = \cdots = \frac{\partial W}{\partial V_n} = 0$$

$$\frac{\partial W}{\partial V_k} = 0, \qquad k = 1, 2, \ldots, n$$

Numerical Solutions: Finite Element

Solving the Resulting Equations

For example, to get $\partial W/\partial V1 = 0$ for the finite element mesh of the Fig., we substitute the coefficient matrix into Eq. $W = \sum_{e=1}^{N} W_e = \frac{1}{2}\epsilon[V]^t[C][V]$ and take the partial derivative of W with respect to V_1. We obtain

$$0 = \frac{\partial W}{\partial V_1} = 2V_1C_{11} + V_2C_{12} + V_3C_{13} + V_4C_{14} + V_5C_{15} + V_2C_{21} + V_3C_{31} + V_4C_{41} + V_5C_{51}$$

$$0 = V_1C_{11} + V_2C_{12} + V_3C_{13} + V_4C_{14} + V_5C_{15}$$

Numerical Solutions: Finite Element

Solving the Resulting Equations

In general, $\partial W/\partial V_k = 0$ leads to

$$0 = \sum_{i=1}^{n} V_i C_{ik}$$

- where n is the number of nodes in the mesh. By writing the Eq. for all nodes $k = 1, 2, \ldots, n$, we obtain a set of simultaneous equations from which the solution of $[V]^t = [V_1, V_2, \ldots, V_n]$ can be found.

Numerical Solutions: Finite Element

Solving the Resulting Equations

- Iteration Method: Suppose node 1 in the Fig., for example, is a free node

$$V_1 = -\frac{1}{C_{11}} \sum_{i=2}^{5} V_i C_{1i}$$

- Thus, in general, at node *k* in a mesh with *n* nodes

$$V_k = -\frac{1}{C_{kk}} \sum_{i=1, i \neq k}^{n} V_i C_{ki}$$