Statistical Digital Signal Processing

Chapter 3: Nonparametric Power Spectrum Estimation

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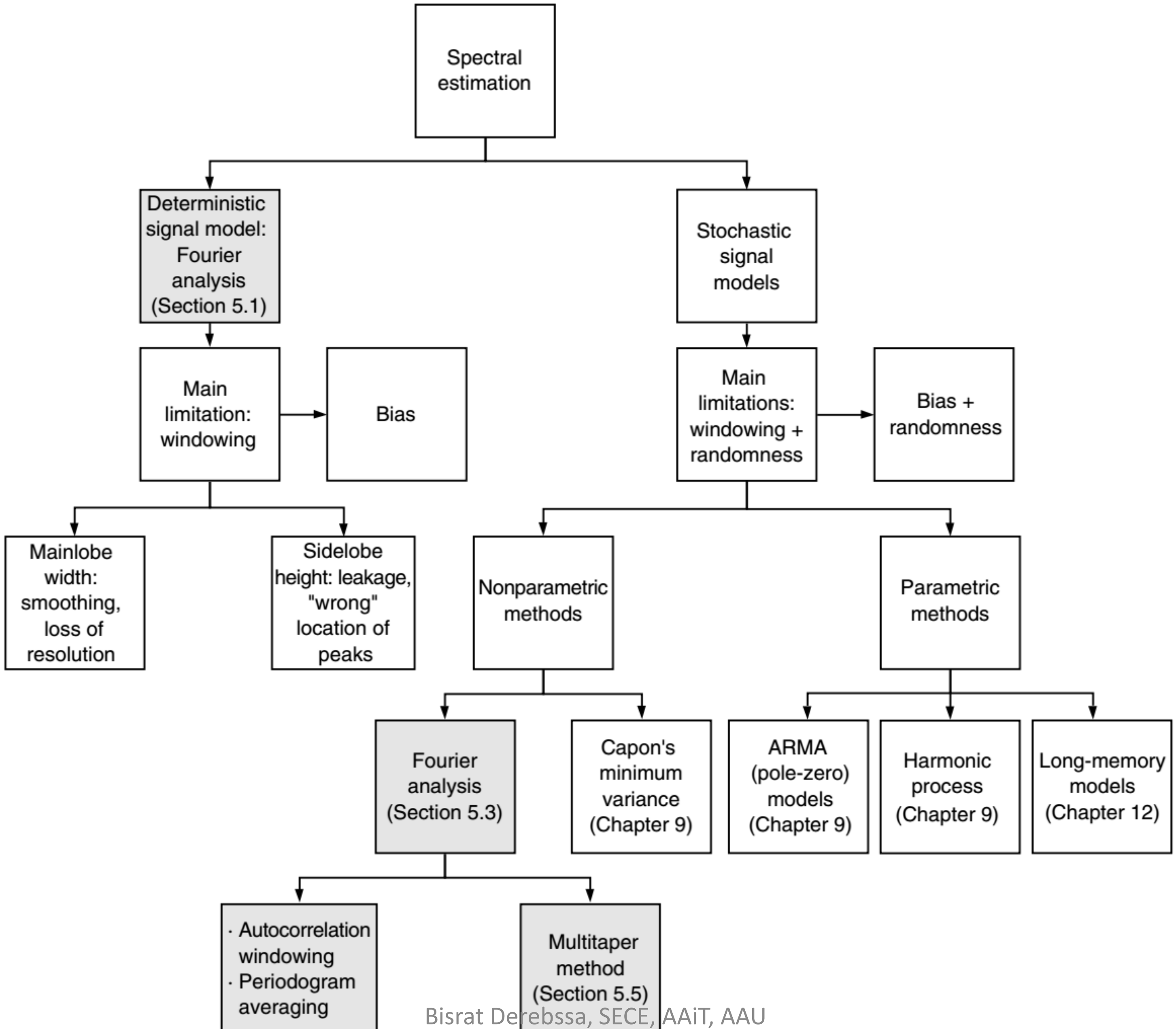
Introduction

- Frequency analysis is the representation of a signal as a superposition of sinusoidal components.
- In practical applications, where only a finite segment of a signal is available, we can only compute an approximation (estimate) of the spectrum of the adopted signal model.
- The power spectrum is the Fourier Transform of the autocorrelation sequence.

- The quality of the estimated spectrum depends on
  - How well the assumed signal model represents the data
  - What values we assign to the unavailable signal samples.
  - Which spectrum estimation method we use.
- Spectrum estimation in practical problems requires
  - sufficient a priori information,
  - understanding of the signal generation process,
  - knowledge of theoretical concepts, and
  - experience.

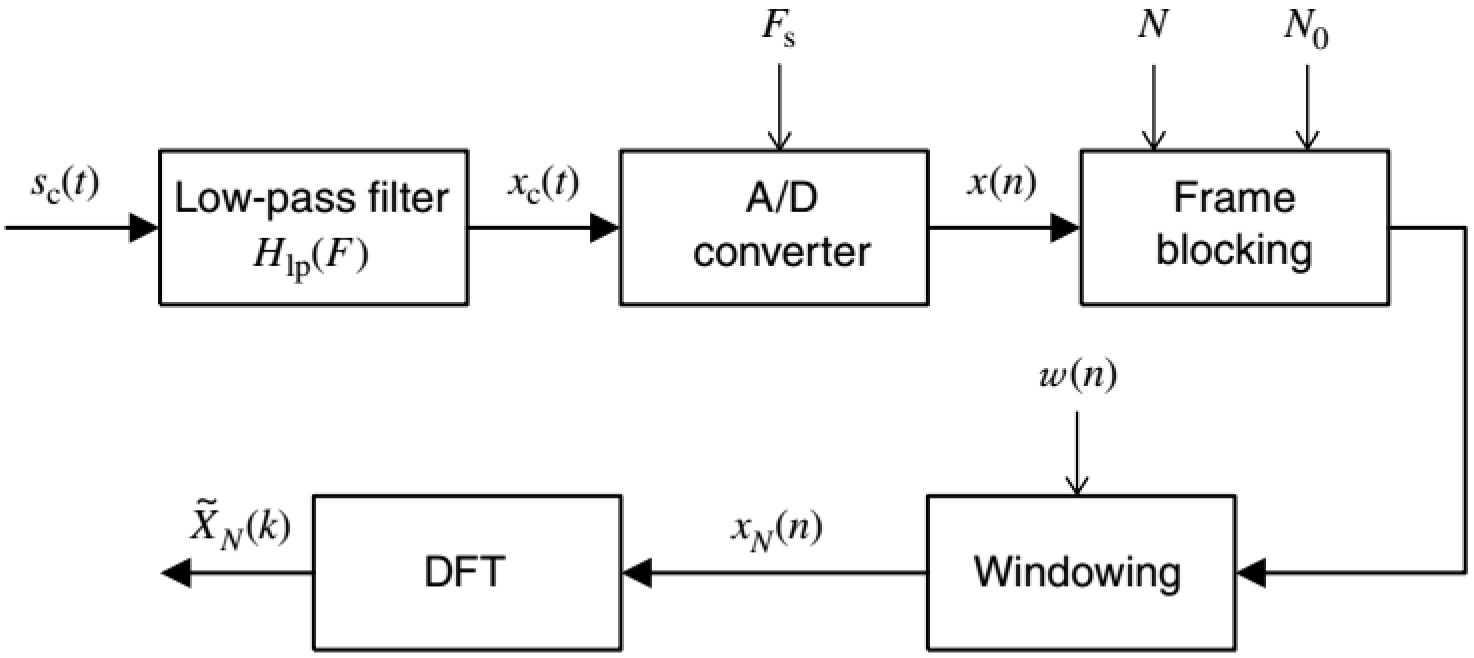
- In this chapter we discuss the most widely used correlation and spectrum estimation methods.
- We discuss only nonparametric techniques that do not assume a particular functional form, but allow the form of the estimator to be determined entirely by the data.
- These methods are based on the discrete Fourier transform of either the signal segment or its autocorrelation sequence.


Spectral estimation
Deterministic signal model: Fourier analysis (Section 5.1)
Stochastic signal models
Main limitation: windowing
Bias
Main limitations: windowing + randomness
Bias + randomness
Mainlobe width: smoothing, loss of resolution
Sidelobe height: leakage, "wrong" location of peaks
Nonparametric methods
Parametric methods
Fourier analysis (Section 5.3)
Capon's minimum variance (Chapter 9)
ARMA (pole-zero) models (Chapter 9)
Harmonic process (Chapter 9)
Long-memory models (Chapter 12)
· Autocorrelation windowing
· Periodogram averaging
Multitaper method (Section 5.5)

Spectral Analysis of Deterministic Signals

- Fourier series and Fourier transforms are used for deterministic signals.
  - These require knowledge of signal for -∞ to ∞.
- Rationale
  - Every realization of a stochastic process is a deterministic function.
  - Deterministic functions and sequences are used in many aspects of the study of stationary processes
  - Various spectra that can be defined for deterministic signals can be used to summarize important features of stationary processes.

DFT based Spectrum analysis



Effect of Sampling

- From sampling theorem

$$X(e^{j2\pi F/F_s}) = F_s \sum_{l=-\infty}^{\infty} X_c(F - lF_s)$$

- In order to avoid distortion due to overlapping replicas, Fs should be greater than the maximum frequency in x.

Effect of Windowing

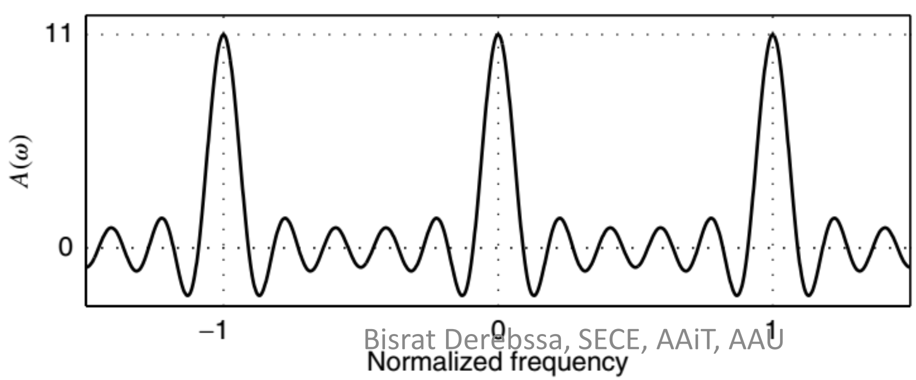
- Spectrum is estimated from finite duration segment.

Determine the spectrum of a signal $x(n)$, $-\infty < n < \infty$, from its values in a finite interval $0 \le n \le N-1$, that is, from a finite-duration segment.

- Approaches to deal with unknown data:
  - Periodic extension
  - Windowing
  - Extrapolation

- As long as the DFT length is greater than or equal to the data length, the DFT is periodic sample of the DTFT.
- It is possible to obtain the DTFT from the DFT by using the sinc function.
  - Impractical
- Error made by using linear interpolation can be made smaller by zero padding.
  - Does not increase resolution.

- Windowing in the time domain is convolving the spectrum with the DFT of the window.
- Ideal window = impulse spectrum.
- Practical windows have to problems
  - Loss of resolution due to **main lobe width**
  - Leakage due to **side lobe amplitude**



Commonly used windows

- Reading assignment.

Comparison of properties of commonly used windows. Each window is assumed to be of length N.

Window type	Peak sidelobe level (dB)	Approximate mainlobe width	Exact mainlode width
Rectangular	-13	$\frac{4\pi}{N-1}$	$\frac{1.81\pi}{N-1}$
Hanning	-32	$\frac{8\pi}{N-1}$	$\frac{5.01\pi}{N-1}$
Hamming	-43	$\frac{8\pi}{N-1}$	$\frac{6.27\pi}{N-1}$
Kaiser	$-A$	—	$\frac{A-8}{2.285N-1}$
Dolph-Chebyshev	$-A$	—	$\cos^{-1}\left[\left(\cosh\frac{\cosh^{-1}10^{A/20}}{N-1}\right)^{-1}\right]$

Autocorrelation estimation of stationary random signals

- Most widely used estimator is

$$\hat{r}_x(l) \triangleq \begin{cases} \frac{1}{N}\sum_{n=0}^{N-l-1} x(n+l)x^*(n) & 0 \le l \le N-1 \\ \hat{r}_x^*(-l) & -(N-1) \le l < 0 \\ 0 & \text{elsewhere} \end{cases}$$

N>50

|l|≤N/4

- Biased: $E\{\hat{r}_x(l)\} = \frac{1}{N} r_x(l) r_w(l)$
  - For Bartlett window

$$E\{\hat{r}_x(l)\} = \frac{1}{N} r_x(l) w_B(n) = r_x(l)\left(1 - \frac{|l|}{N}\right) w_R(n)$$

- Consistency:

$$\text{cov}\{\hat{r}_x(l_1), \hat{r}_x(l_2)\} \simeq \frac{1}{N}\sum_{l=-\infty}^{\infty} [r_x(l) r_x(l+l_2-l_1) + r_x(l+l_2) r_x(l-l_1)]$$

Power Spectrum Estimation of Stationary Random Signals

- The power spectral density of a zero-mean stationary stochastic process is

$$R_x(e^{j\omega}) \triangleq \sum_{l=-\infty}^{\infty} r_x(l)e^{-j\omega l}$$

- Goal: an estimate that faithfully characterizes the power density using a finite segment realization.
- Challenges:
  - Availability of only finite amount of data
  - Complexity of problem for infinite data.

Periodogram

- The periodogram is defined as

$$\hat{R}_x(e^{j\omega}) \triangleq \frac{1}{N}\left|\sum_{n=0}^{N-1} v(n)e^{-j\omega n}\right|^2 = \frac{1}{N}|V(e^{j\omega})|^2$$

$$v(n) = x(n)w(n) \qquad 0 \le n \le N-1$$

- The window may or may not be rectangular.
- It can be expressed in terms of autocorrelation estimate.

$$\hat{R}_x(e^{j\omega}) = \sum_{l=-(N-1)}^{N-1} \hat{r}_v(l)\, e^{-j\omega l}$$

- The periodogram is a natural estimate of the power spectrum.

Performance of Periodogram

- Mean of Periodogram

$$E\{\hat{R}_x(e^{j\omega})\} = \sum_{l=-(N-1)}^{N-1} E\{\hat{r}_v(l)\}e^{-j\omega l} = \frac{1}{N}\sum_{l=-(N-1)}^{N-1} r_x(l)r_w(l)e^{-j\omega l}$$

  - The periodogram is a biased estimator.
  - In frequency domain

$$E\{\hat{R}_x(e^{j\omega})\} = \frac{1}{2\pi N}\int_{-\pi}^{\pi} R_x(e^{j\theta})R_w(e^{j(\omega-\theta)})\, d\theta$$

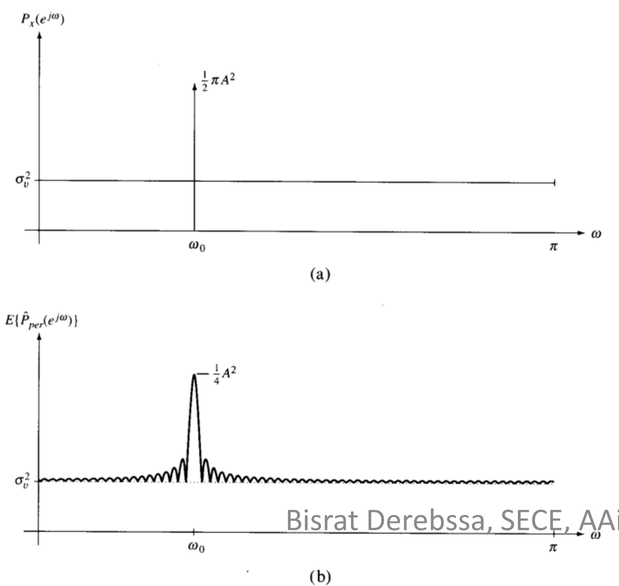
$$R_w(e^{j\omega}) = |W(e^{j\omega})|^2$$

  - For rectangular window

$$E\{\hat{R}_x(e^{j\omega})\} = \sum_{l=-(N-1)}^{N-1}\left(1 - \frac{|l|}{N}\right) r_x(l)e^{-j\omega l}$$

  - Modified periodogram: N changed to N-l.

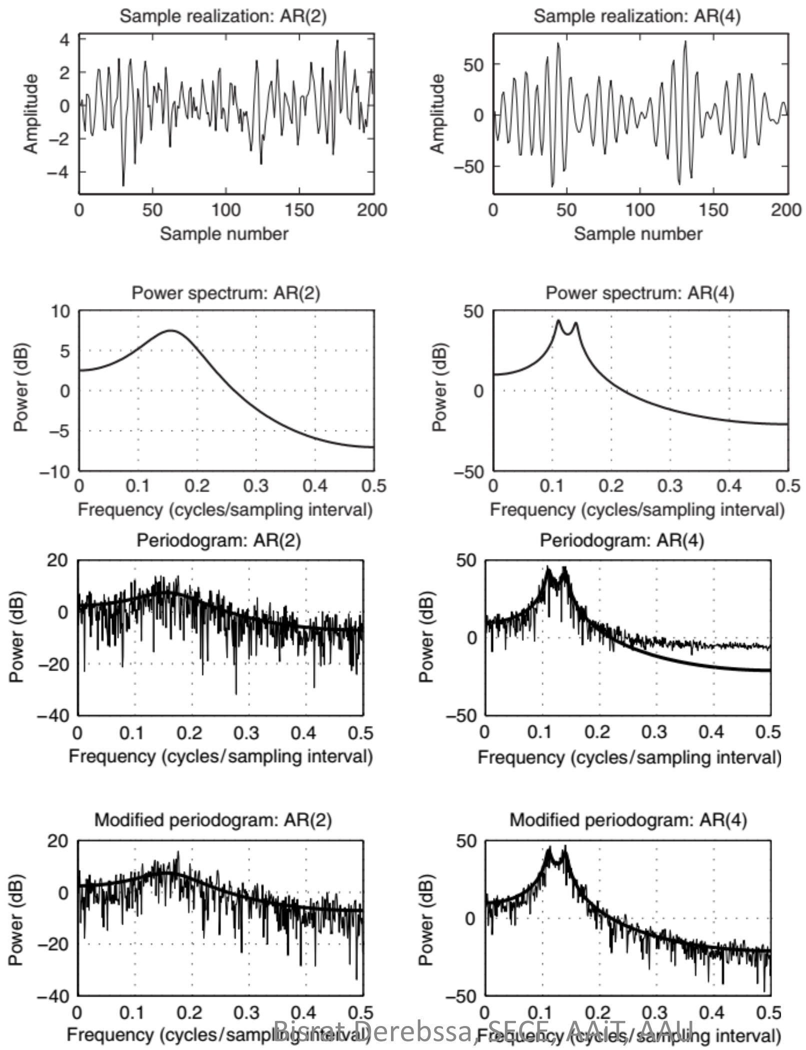
- The expected value of the periodogram is the convolution of the power spectrum with the Fourier transform of the window.
  - The main lobe width and the sidelobe amplitude of the window affect the resolution and variance.



If there is another sinusoidal peak near the main lobe width, the two will overlap and appear as 1.

Example: AR(2) and AR(4)

$$\mathbf{a}_2 = [1 \;\; -0.75 \; 0.5]^T \qquad \mathbf{a}_4 = [1 \;\; -2.7607 \; 3.8106 \;\; -2.6535 \; 0.9238]^T$$

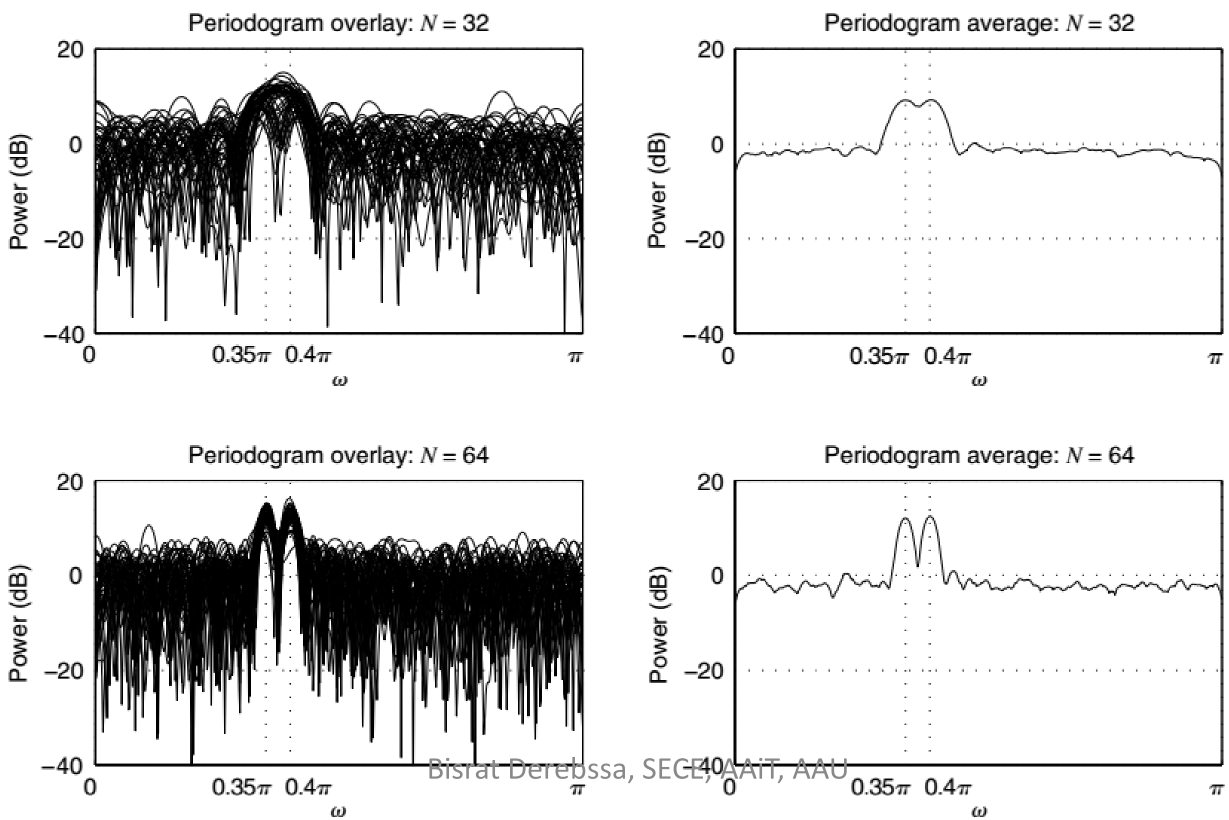


Example: Two sinusoidal at 0.35π and 0.4 π

$$x(n) = \cos(0.35\pi n + \phi_1) + \cos(0.4\pi n + \phi_2) + v(n)$$

- If we use a rectangular window,

$$N - 1 > \frac{1.81\pi}{0.4\pi - 0.35\pi} \qquad \text{or} \qquad N > 37$$



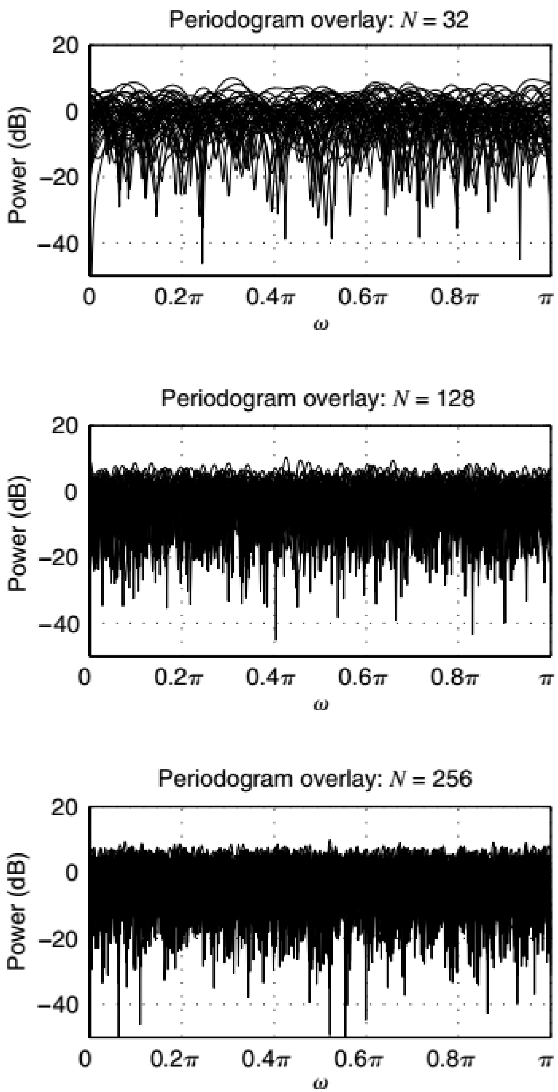
Variance of Periodogram

- For white Gaussian noise, the variance can be approximated by

$$\mathrm{var}\{\hat{R}_x(e^{j\omega})\} \simeq \begin{cases} R_x^2(e^{j\omega}) & 0 < \omega < \pi \\ 2R_x^2(e^{j\omega}) & \omega = 0, \pi \end{cases}$$

- This shows that the variance is independent of the record length.
- As it does not approach zero as N increases, the periodogram is not a consistent estimator.

Example: Periodogram of a white noise



Covariance of Periodogram

- The covariance of the periodogram is given as

$$\text{cov}\{\hat{R}_x(e^{j\omega_1}), \hat{R}_x(e^{j\omega_2})\} \simeq R_x(e^{j\omega_1})R_x(e^{j\omega_2}) \left(\left\{ \frac{\sin[(\omega_1+\omega_2)N/2]}{N \sin[(\omega_1+\omega_2)/2]} \right\}^2 + \left\{ \frac{\sin[(\omega_1-\omega_2)N/2]}{N \sin[(\omega_1-\omega_2)/2]} \right\}^2 \right)$$

- For $\omega_1 = 2\pi k_1/N$ and $\omega_2 = 2\pi k_2/N$

$$\text{cov}\{\hat{R}_x(e^{j\omega_1})\hat{R}_x(e^{j\omega_2})\} \simeq 0$$

- Thus, values of the periodogram spaced in frequency by integer multiples of $2\pi/N$ are approximately uncorrelated.

- In summary
  - The periodogram is a very poor estimator of the power spectrum.
  - It is a biased estimator.
  - It is an inconsistent estimator.
- The main problem is its variance and its erratic behavior.
  - The variance does not decrease with increasing N.

Modified Periodogram

- Instead of using a rectangular window, it uses other windows
  - Hanning, Hamming, Bartlett, …

$$\hat{P}_{per}(e^{j\omega}) = \frac{1}{N}|X_N(e^{j\omega})|^2 = \frac{1}{N}\left|\sum_{n=-\infty}^{\infty} x(n)w_R(n)e^{-jn\omega}\right|^2$$

- Its mean is

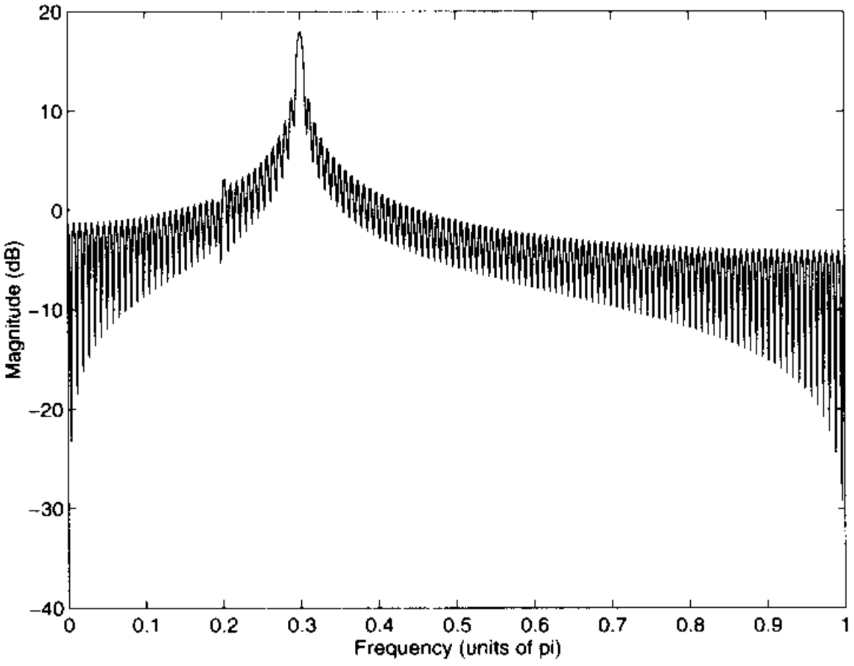
$$E\left\{\hat{P}_{per}(e^{j\omega})\right\} = \frac{1}{2\pi N}P_x(e^{j\omega}) * |W_R(e^{j\omega})|^2$$

  - Window property affects resolution.

Example: two sinusoids

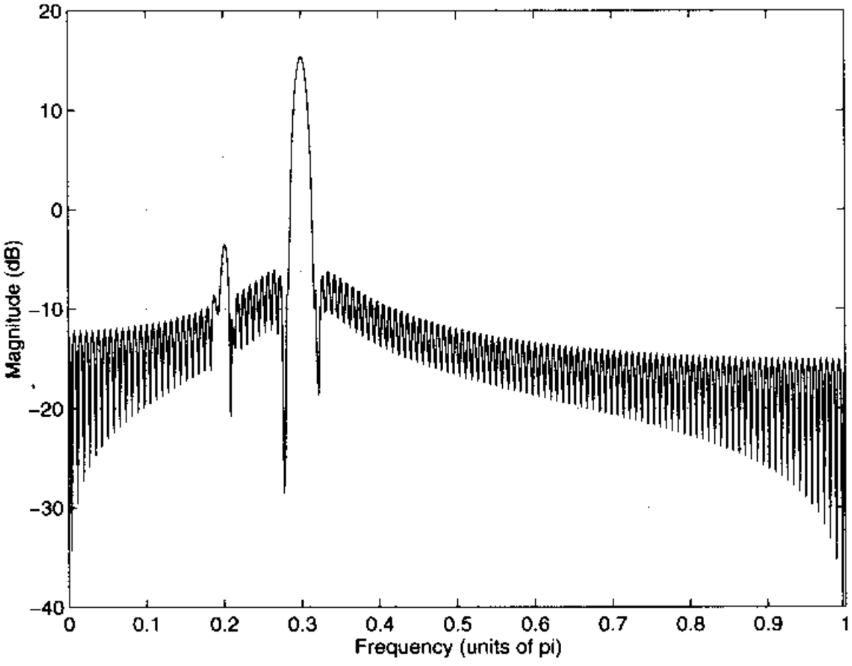
$$x(n) = 0.1\sin(n\omega_1 + \phi_1) + \sin(n\omega_2 + \phi_2) + v(n)$$

Average Periodogram



(a)

Average Modified Periodogram



(b)

- Variance

$$\mathrm{Var}\left\{\hat{P}_M(e^{j\omega})\right\} \approx P_x^2(e^{j\omega})$$

  – It is not a consistent estimator.

Blackman-Tukey Method

- Smoothing the periodogram using a moving average filter to reduce the variance.

$$\hat{R}_x^{(\mathrm{PS})}(e^{j\omega_k}) \triangleq \frac{1}{2M+1}\sum_{j=-M}^{M}\hat{R}_x(e^{j\omega_{k-j}}) \triangleq \sum_{j=-M}^{M} W(e^{j\omega_j})\hat{R}_x(e^{j\omega_{k-j}})$$

- Since samples of the periodogram are uncorrelated

$$\mathrm{var}\{\hat{R}_x^{(\mathrm{PS})}(e^{j\omega_k})\} \simeq \frac{1}{2M+1}\,\mathrm{var}\{\hat{R}_x(e^{j\omega_k})\}$$

- This can be done in the time domain via

$$\hat{R}_x^{(\mathrm{PS})}(e^{j\omega}) = \sum_{l=-(L-1)}^{L-1} \hat{r}_x(l) w_a(l) e^{-j\omega l}$$

$$w_a(l) = \frac{\sin(l\Delta\omega/2)}{\pi l} \qquad -\infty < l < \infty$$

- To avoid ripple $\quad w_a(l) = 0 \text{ for } |l| > L \leq \bar{N}$

- The Blackman-Tukey approach has three steps
  1. Estimate the autocorrelation sequence from the unwindowed data.
  2. Window the obtained autocorrelation samples.
  3. Compute the DTFT of the windowed autocorrelation.
- The resolution is determined by the duration of the correlation window

$$\Delta\omega \approx (2\pi / N)(2M + 1)$$

- For large L, its mean can be approximated as

$$E\{\hat{R}_x^{(\mathrm{PS})}(e^{j\omega})\} \simeq R_x(e^{j\omega}) \frac{1}{2\pi} \int_{-\pi}^{\pi} W_a(e^{j(\omega-\theta)})\, \mathrm{d}\theta$$

  - Asymptotically unbiased if

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} W_a(e^{j\omega})\, \mathrm{d}\omega = w_a(0) = 1$$

- Variance,

$$\mathrm{var}\{\hat{R}_x^{(\mathrm{PS})}(e^{j\omega})\} \simeq \frac{E_w}{N} R_x^2(e^{j\omega}) \qquad 0 < \omega < \pi$$

$$E_w = \frac{1}{2\pi} \int_{-\pi}^{\pi} W_a^2(e^{j\omega})\, \mathrm{d}\omega = \sum_{l=-(L-1)}^{L-1} w_a^2(l)$$

- Tradeoff between resolution and variance
  - To decrease variance E_w must be small
    - L<<N
  - In terms of resolution, the correlation window w_a should have narrow main lobe width
    - Large L
- Both can only be achieved for sufficiently large N.

- Confidence interval
  - Since they are not unbiased and consistent, need to know whether the spectral details are real or due to fluctuations of the PSD estimate.
  - (1-α)x100 percent confidence interval

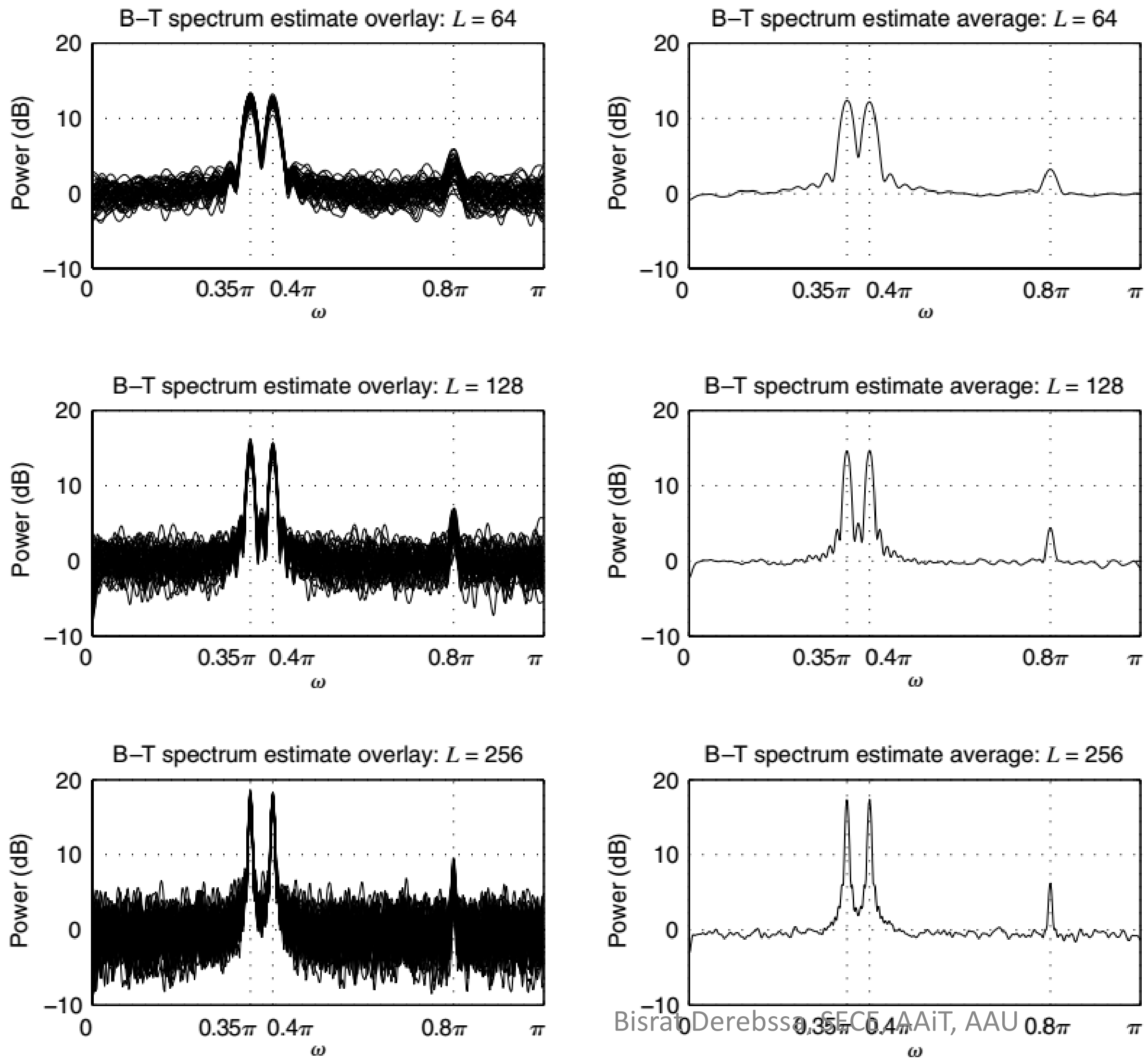
$$\left(10\log \hat{R}_x^{(\mathrm{PS})}(e^{j\omega}) - 10\log \frac{\chi_\nu^2(1-\alpha/2)}{\nu},\ 10\log \hat{R}_x^{(\mathrm{PS})}(e^{j\omega}) + 10\log \frac{\nu}{\chi_\nu^2(\alpha/2)}\right)$$

where

$$\nu = \frac{2N}{\displaystyle\sum_{l=-(L-1)}^{L} w_a^2(l)}$$

Example: three sinusoidal estimation

$$x(n) = \cos(0.35\pi n + \phi_1) + \cos(0.4\pi n + \phi_2) + 0.25\cos(0.8\pi n + \phi_3) + v(n)$$



Welch-Bartlett Method

- Subdivide the existing record of length N into K smaller segments.

$$x_i(n) = x(iD + n)w(n)$$

- The spectrum estimate is obtained by averaging the K periodograms.

$$\hat{R}_x^{(\mathrm{PA})}(e^{j\omega}) \triangleq \frac{1}{K}\sum_{i=0}^{K-1} \hat{R}_{x,i}(e^{j\omega}) = \frac{1}{KL}\sum_{i=0}^{K-1} |X_i(e^{j\omega})|^2$$

- Assuming that $r_x(l)$ is very small for $|l|>L$, the signal segments are uncorrelated.

- Mean of estimation

$$E\{\hat{R}_x^{(\mathrm{PA})}(e^{j\omega})\} = \frac{1}{K}\sum_{i=0}^{K-1} E\{\hat{R}_{x,i}(e^{j\omega})\} = E\{\hat{R}_x(e^{j\omega})\}$$

  - Asymptotically unbiased if $R_x(e^{jw})$ is unbiased.

- Variance

$$\mathrm{var}\{\hat{R}_x^{(\mathrm{PA})}(e^{j\omega})\} = \frac{1}{K}\mathrm{var}\{\hat{R}_x(e^{j\omega})\}$$

  - Assuming segments are independent

$$\mathrm{var}\{\hat{R}_x^{(\mathrm{PA})}(e^{j\omega})\} \simeq \frac{1}{K} R_x^2(e^{j\omega})$$

  - Asymptotically consistent estimator.

- If N is fixed and $N = KL$,
  - Increasing K to reduce the variance (or equivalently obtain a smoother estimate) results in a decrease in L, that is, a reduction in resolution (or equivalently an increase in bias).

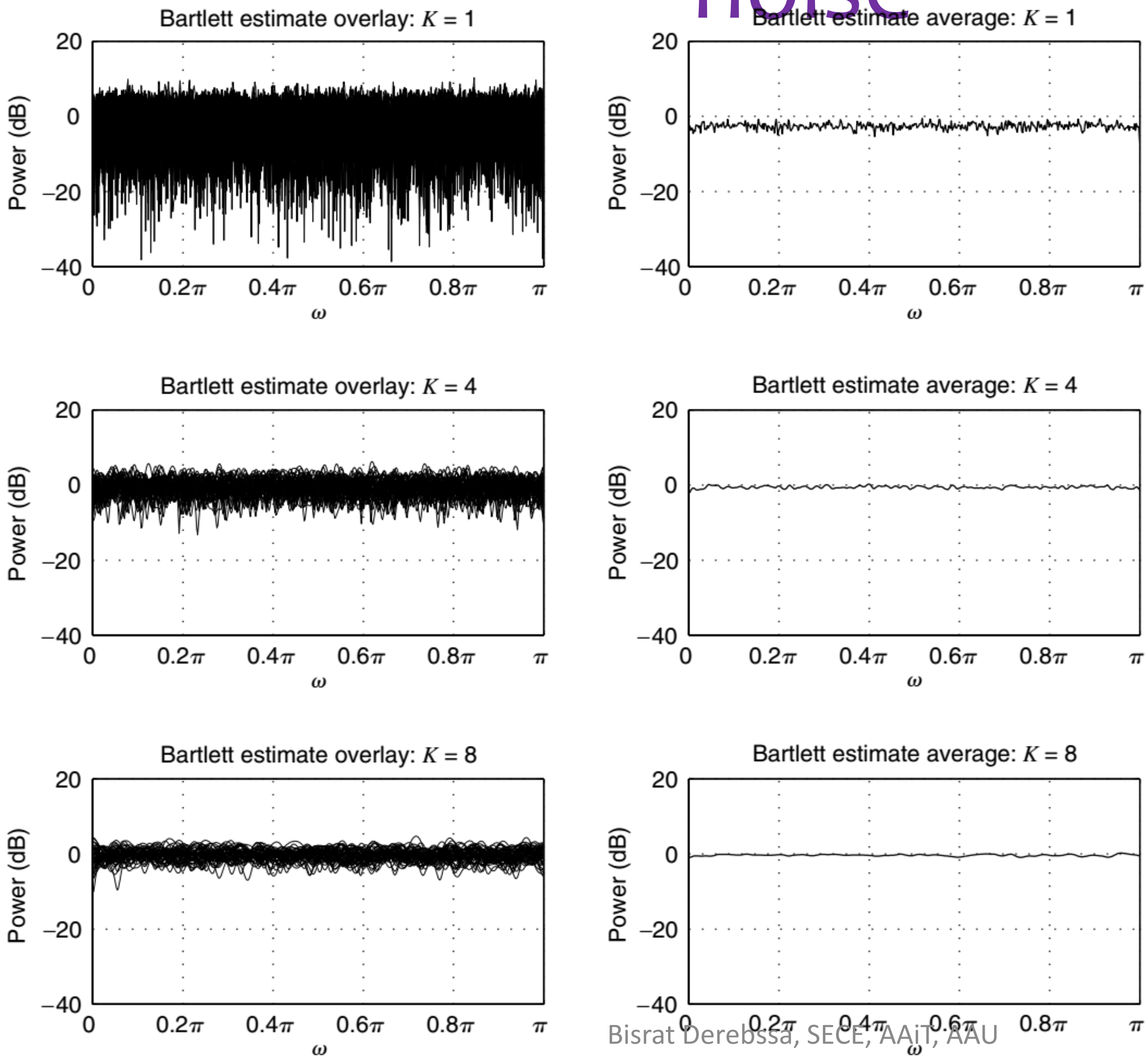
- The segments can be chosen to overlap.
- Welch showed that overlapping the segment by 50% reduces the variance by a factor of 2.
- More overlap does not result in additional reduction.

- Confidence interval
  - (1-α)x100 percent confidence interval

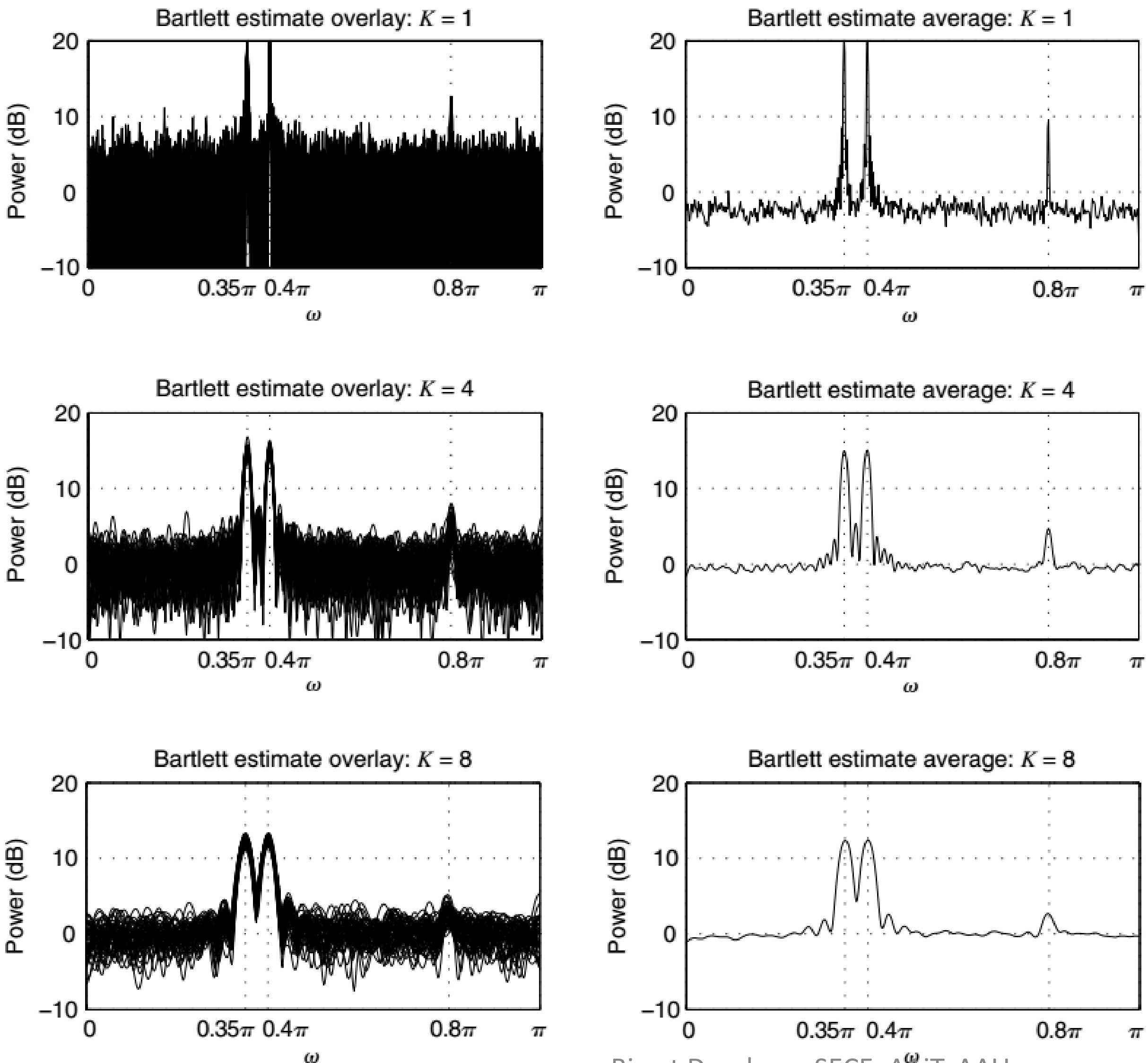
$$\left(10\log \hat{R}_x^{(\mathrm{PA})}(e^{j\omega}) - 10\log \frac{\chi^2_{2K}(1-\alpha/2)}{2K},\ 10\log \hat{R}_x^{(\mathrm{PA})}(e^{j\omega}) + 10\log \frac{2K}{\chi^2_{2K}(\alpha/2)}\right)$$

  - ${x_{2k}}^2$ is a chi-squared distribution with 2k degrees of freedom.

Example: Periodogram of a white noise



Example: three sinusoidal estimation



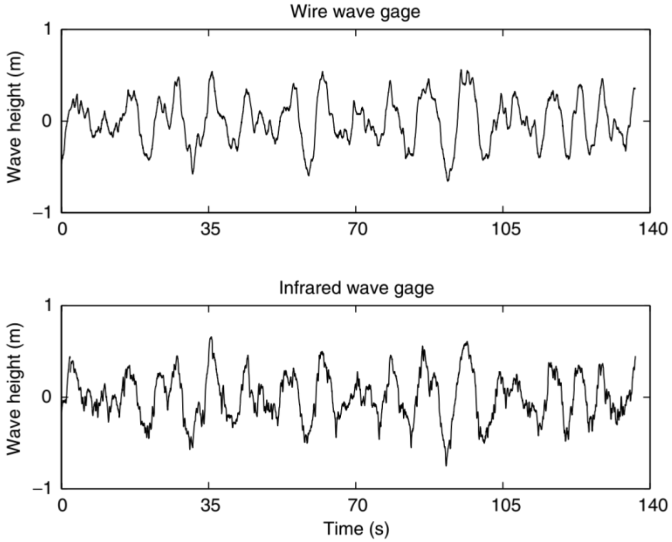
Reading Assignment

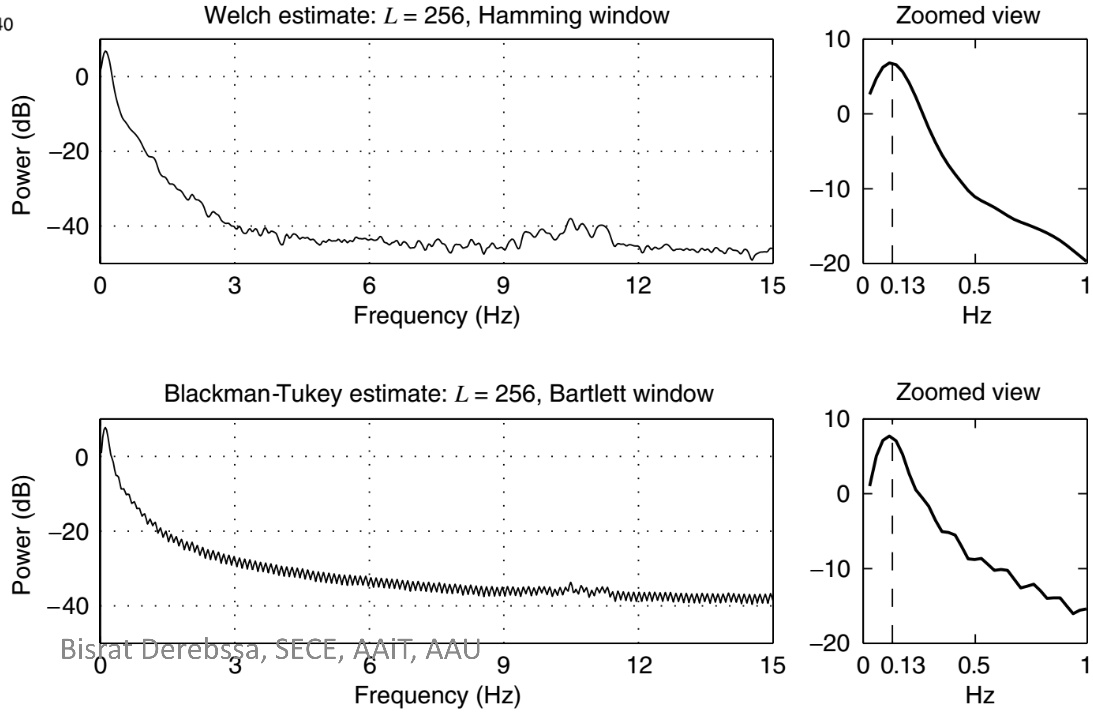
- Practical computation of Blackman-Tukey and Welch-Bartlett.
  - Manolakis: pp. 226-227 and 231-232

Comparison of PSD estimation methods.

	Periodogram $\hat{R}_x(e^{j\omega})$	Single-periodogram smoothing (Blackman-Tukey): $\hat{R}_x^{(PS)}(e^{j\omega})$	Multiple-periodogram averaging (Bartlett-Welch): $\hat{R}_x^{(PA)}(e^{j\omega})$	Multitaper (Thomson): $\hat{R}_x^{(MT)}(e^{j\omega})$
Description of the method	Compute DFT of data record	Compute DFT of windowed autocorrelation estimate (see Figure 5.17)	Split record into K segments and average their modified periodograms (see Figure 5.20)	Window data record using K orthonormal tapers and average their periodograms (see Figure 5.30)
Basic idea	Natural estimator of $R_x(e^{j\omega})$; the error $\lvert r_x(l) - \hat{r}_x(l)\rvert$ is large for large $\lvert l\rvert$	Local smoothing of $\hat{R}_x(e^{j\omega})$ by weighting $\hat{r}_x(l)$ with a lag window $w_a(l)$	Overlap data records to create more segments; window segments to reduce bias; average periodograms to reduce variance	For properly designed orthogonal tapers, periodograms are independent at each frequency. Hence averaging reduces variance
Bias	Severe for small N; negligible for large N	Asymptotically unbiased	Asymptotically unbiased	Negligible for properly designed tapers
Resolution	$\propto \frac{1}{N}$	$\propto \frac{1}{L}$, L is maximum lag	$\propto \frac{1}{L}$ L is segment length	$\propto \frac{1}{N}$
Variance	Unacceptable: about $R_x^2(e^{j\omega})$ for all N	$R_x^2(e^{j\omega}) \times \frac{E_w}{N}$	$\frac{R_x^2(e^{j\omega})}{K}$ K is number of segments	$\frac{R_x^2(e^{j\omega})}{K}$ K is number of tapers

Analysis of ocean wave data





Reading Assignment

- Comparison between these four methods
  - Hayes: pp. 424-426

Joint Signal Analysis

- Concerned with the analysis and computation of the correlation and associated spectral estimation of jointly stationary RP.
- Let x(n) and y(n) be two zero-mean, jointly stationary random process with PSD of $R_x(e^{jw})$ and $R_y(e^{jw})$.
- The joint cross-power spectral density is

$$R_{xy}(e^{j\omega}) = \sum_{l=-\infty}^{\infty} r_{xy}(l) e^{-j\omega l}$$

- $r_{xy}(l)$ is the cross-correlation sequence.

Estimation of Cross Power Spectrum

- Let $\{x(n),y(n)\}_0^{N-1}$ be the available data for estimation.
- By using the periodogram, the estimator is

$$\hat{R}_{xy}(e^{j\omega}) \triangleq \sum_{l=-(N-1)}^{N-1} \hat{r}_{xy}(l)e^{-j\omega l}$$

$$\hat{r}_{xy}(l) = \begin{cases} \frac{1}{N}\sum_{n=0}^{N-l-1} x(n+l)y^*(n) & 0 \le l \le N-1 \\ \frac{1}{N}\sum_{n=0}^{N+l-1} x(n)y^*(n-l) & -(N-1) \le l \le -1 \\ 0 & l \le -N \text{ or } l \ge N \end{cases}$$

- It can also be written as

$$\hat{R}_{xy}(e^{j\omega}) = \frac{1}{N}\left[\sum_{n=0}^{N-1} x(n)e^{-j\omega n}\right]\left[\sum_{n=0}^{N-1} y(n)e^{-j\omega n}\right]^*$$

- Alternatively,

$$|\hat{R}_{xy}(e^{j\omega})|^2 = \left(\frac{1}{N}\right)^2 \left|\sum_{n=0}^{N-1} x(n)e^{-j\omega n}\right|^2 \left|\sum_{n=0}^{N-1} y(n)e^{-j\omega n}\right|^2 = \hat{R}_x(e^{j\omega})\hat{R}_y(e^{j\omega})$$

- This implies a magnitude-squared coherence function is always 1.
  - This is due to the fact that the frequency-domain correlation coefficient at each frequency is estimated by using only a single pair of observations from the two signals.

- Welch approach
  - Subdivide the existing data records into K overlapping smaller segments of length L

$$x_i(n) = x(iD+n)w(n) \qquad 0 \le n \le L-1, 0 \le i \le K-1$$
$$y_i(n) = y(iD+n)w(n)$$

    - where the 50% overlap is chosen: D=L/2.
  - Obtain the cross-periodogram of each segment

$$\hat{R}_i(e^{j\omega}) = \frac{1}{L} X_i(e^{j\omega}) Y_i^*(e^{j\omega}) = \frac{1}{L}\left[\sum_{n=0}^{L-1} x_i(n) e^{-j\omega n}\right]\left[\sum_{n=0}^{L-1} y_i(n) e^{-j\omega n}\right]^*$$

  - Smooth the cross spectrum

$$\hat{R}_{xy}^{(\text{PA})}(e^{j\omega}) = \frac{1}{K}\sum_{i=0}^{K-1} \hat{R}_i(e^{j\omega}) = \frac{1}{KL}\sum_{i=0}^{K-1} X_i(e^{j\omega}) Y_i^*(e^{j\omega})$$

- Estimation of cospectra and quadrature spectra
  - The cospectra is the real part of the cross-power spectrum

$$\hat{C}_{xy}^{(\mathrm{PA})}(\omega) = \mathrm{Re}[\hat{R}_{xy}^{(\mathrm{PA})}(e^{j\omega})] = \mathrm{Re}\left[\frac{1}{KL}\sum_{i=0}^{K-1} X_i(e^{j\omega})Y_i^*(e^{j\omega})\right]$$

  - The quadrature spectra is the imaginary part

$$\hat{Q}_{xy}^{(\mathrm{PA})}(\omega) = -\mathrm{Im}[\hat{R}_{xy}^{(\mathrm{PA})}(e^{j\omega})] = -\mathrm{Im}\left[\frac{1}{KL}\sum_{i=0}^{K-1} X_i(e^{j\omega})Y_i^*(e^{j\omega})\right]$$

- Estimation of cross-amplitude and phase spectra
  - Cross-amplitude spectra is the magnitude of the cross-power spectrum

$$\hat{A}_{xy}^{(\mathrm{PA})}(\omega) = \sqrt{[\hat{C}_{xy}^{(\mathrm{PA})}(\omega)]^2 + [\hat{Q}_{xy}^{(\mathrm{PA})}(\omega)]^2}$$

  - Cross-amplitude spectra is the magnitude of the cross-power spectrum

$$\hat{\Phi}_{xy}^{(\mathrm{PA})}(\omega) = \tan^{-1}\{-\hat{Q}_{xy}^{(\mathrm{PA})}(\omega)/\hat{C}_{xy}^{(\mathrm{PA})}(\omega)\}$$

  - Coherency spectrum

$$|\hat{\mathcal{G}}_{xy}^{(\mathrm{PA})}(\omega)| = \frac{|\hat{R}_{xy}^{(\mathrm{PA})}(e^{j\omega})|}{\sqrt{\hat{R}_{x}^{(\mathrm{PA})}(e^{j\omega})\hat{R}_{y}^{(\mathrm{PA})}(e^{j\omega})}} = \left\{\frac{[\hat{C}_{xy}^{(\mathrm{PA})}(\omega)]^2 + [\hat{Q}_{xy}^{(\mathrm{PA})}(\omega)]^2}{\hat{R}_{x}^{(\mathrm{PA})}(e^{j\omega})\hat{R}_{y}^{(\mathrm{PA})}(e^{j\omega})}\right\}^{1/2}$$

Estimation of Frequency Response Functions

- When random processes x(n) and y(n) are the input and output of some physical system, the bivariate spectral estimation techniques can be used to estimate the system frequency response.
- Application
  - Characterization of a channel over which signals are transmitted.

- Since the cross-correlation of the output of an LTI system to WSS random process is given as

$$r_{yx}(l) = h(l) * r_x(l) \qquad R_{yx}(e^{j\omega}) = H(e^{j\omega}) R_x(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{R_{yx}(e^{j\omega})}{R_x(e^{j\omega})}$$

- Reading Assignment:
  - Estimation of frequency response examples
    - Manolakis: pp. 241-246

Multitaper power spectrum Estimation

- Rather than use a single rectangular data taper as in the periodogram estimate, **several data tapers** are used on the **same data** record to compute several modified periodograms.
- These modified periodograms are then averaged to produce the multitaper spectral estimate.
- The central premise of this multitaper approach is that if the data tapers are properly designed **orthogonal functions**, then the spectral estimates would be **independent** of each other at every frequency.
  - Averaging would reduce variance.

- The data tapers used are selected to be orthonormal to each other.

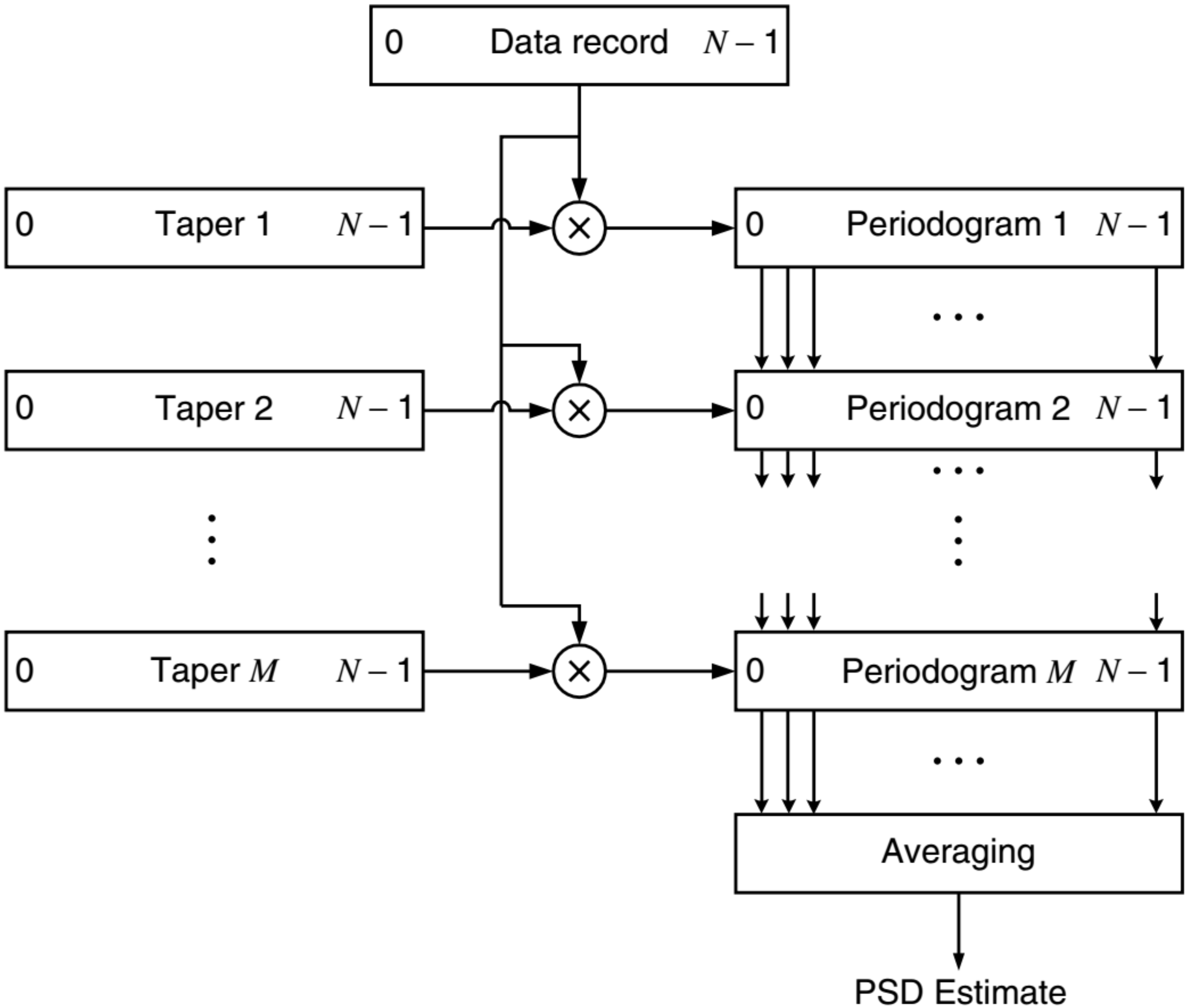
$$\sum_{n=0}^{N-1} w_k(n) w_l(n) = \begin{cases} 1 & k = l \\ 0 & k \neq l \end{cases}$$

- The periodogram estimator based on the kth taper window is

$$\hat{R}_{k,x}(e^{j\omega}) = \frac{1}{N} \left| \sum_{n=0}^{N-1} w_k(n) x(n) e^{-j\omega n} \right|^2$$

- The simple averaged multitaper estimator is

$$\hat{R}_x^{(\mathrm{MT})}(e^{j\omega}) = \frac{1}{K} \sum_{k=0}^{K-1} \hat{R}_{k,x}(e^{j\omega})$$


0 Data record N – 1
0 Taper 1 N – 1
0 Periodogram 1 N – 1
0 Taper 2 N – 1
0 Periodogram 2 N – 1
0 Taper M N – 1
0 Periodogram M N – 1
Averaging
PSD Estimate

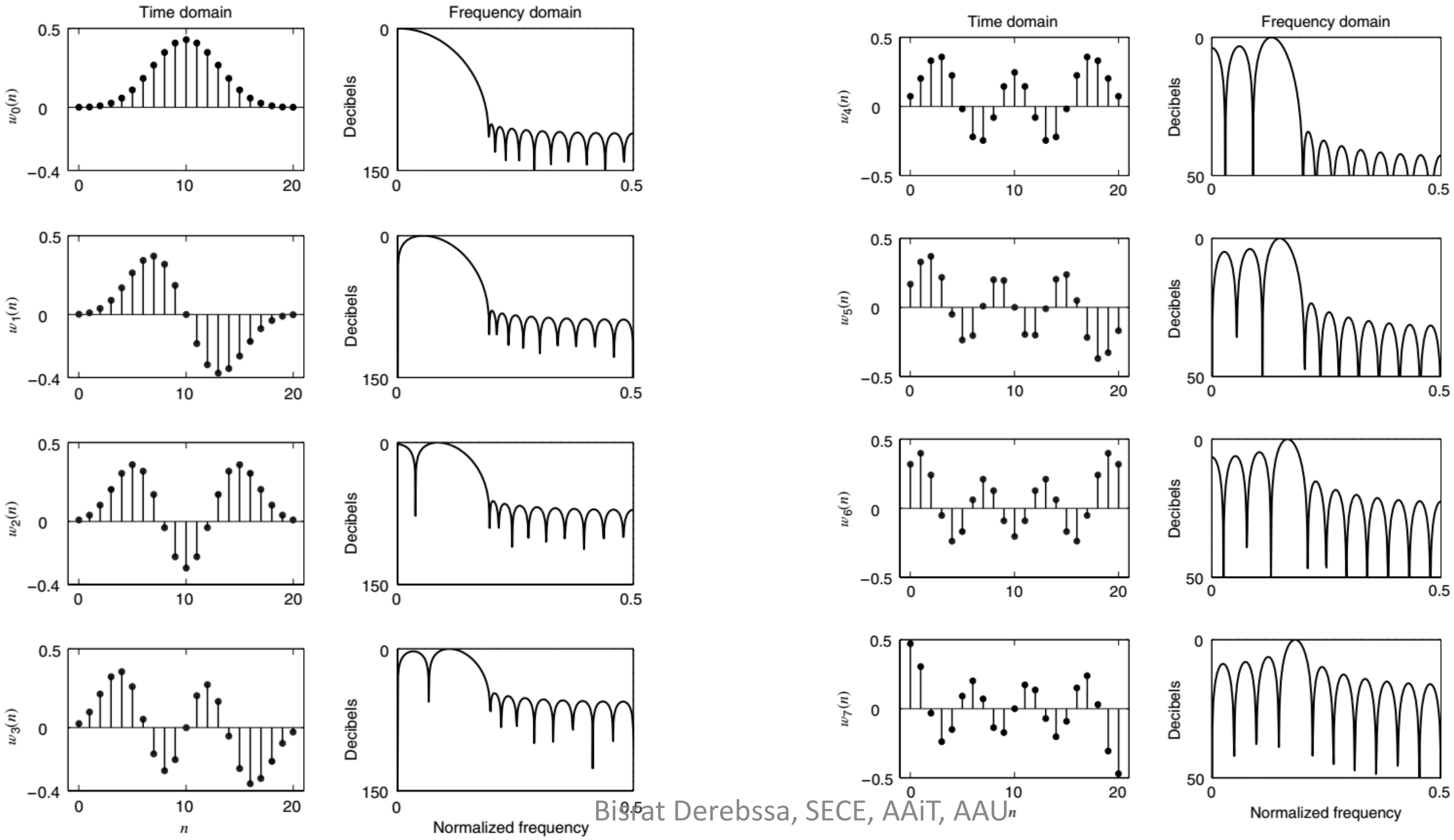
- The selection of the K orthonormal tapers determines the accuracy of the estimation.
  - Mean of estimation

$$E\{\hat{R}_x^{(\mathrm{MT})}(e^{j\omega})\} = \frac{1}{2\pi N}\int_{-\pi}^{\pi} R_x(e^{j\theta})\bar{R}_w(e^{j(\omega-\theta)})\,\mathrm{d}\theta$$

$$\bar{R}_w(e^{j\omega}) \triangleq \frac{1}{K}\sum_{k=0}^{K-1} |W_k(e^{j\omega})|^2$$

  - For good estimation
    - Each taper must have low sidelobe levels.
    - Each spectral estimate is uncorrelated.

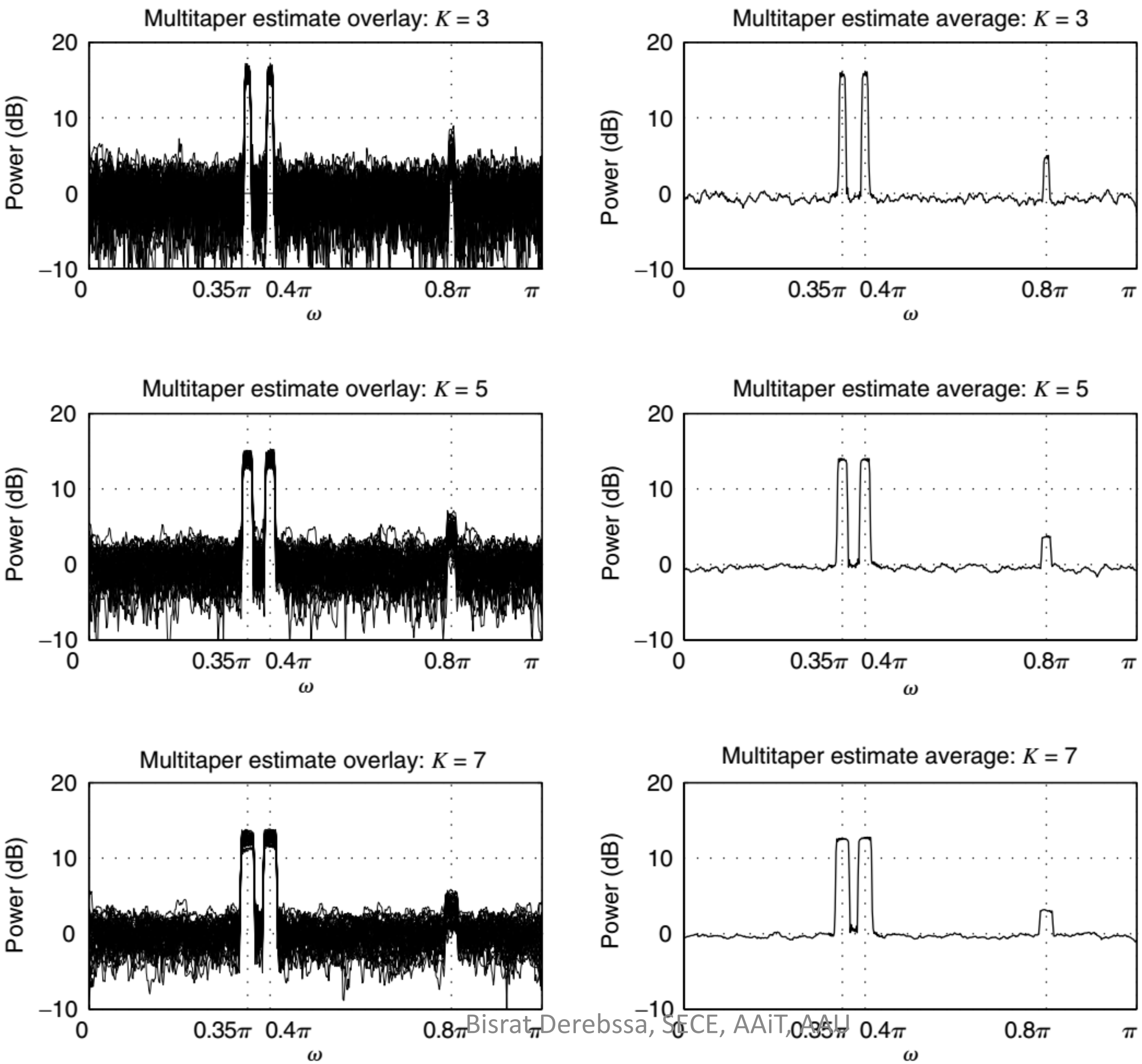
- One set is obtained by using the discrete prolate spheroidal sequences DPSS



- Harmonically related sinusoidal tapers

$$w_k(n) = \sqrt{\frac{2}{N+1}} \sin \frac{\pi (k+1)(n+1)}{N+1} \qquad n = 0, 1, \ldots, N-1$$

Example: three sinusoidal estimation



- Reading Assignment
  - Cross power spectrum estimation using the multitapering approach.
    - Manolakis: pp:252-254

Assignment 3

- Record 15 seconds of your speech in a noisy environment (use sampling frequency of 12 kHz).
- Take a 32.5 msec frame in the middle of the recorded speech (make sue there is a speech in the selected frame).
- Obtain the power spectrum by using
  - Periodogram,
  - Blackman-Tukey approach (L=32, 64, 128) and
  - Welch-Barttlet approach (K=1, 4, 8).
- Plot these and discuss what you observe.