Chapter 3: Characterization of Communication Signals and Systems

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Graduate Program
School of Electrical and Computer Engineering

Goals of the Chapter

- Signals can be categorized in a number of ways
- Will cover characterization of signals and systems encountered in digital communication systems
- Focus points
  - Characterization of ***bandpass*** signals and systems
  - Vector space representation of signals
  - Representation of digitally modulated signals

Overview

- Signals and systems
- Stationary stochastic process
- Signal space

Communication Signals & Systems Characterization

- Binary bits from the source encoder mapped into signal waveforms, mostly after channel encoding
  - In channel encoding redundancy is added in a controlled manner for error correction at the receiver
- Example: Binary modulation: $0 \rightarrow s_1(t)$ and $1 \rightarrow s_2(t)$
- *b* bits at a time mapped using $M = 2^b$ waveforms $s_i(t)$, $i = 0,1,2,\ldots M-1$
  - I.e., one waveform for each of the 2^b possible bit sequences
  - M-ary modulation for $M > 2$
- Question: What should be the characteristics of these waveforms and how do we describe and use them?
  - Various forms of digitally modulated signals will be introduced along with their spectral and other characteristics

Bandpass Signals and Systems

- Channels have limited bandwidth centered about the carrier (DSB) or adjacent to the carrier (SSB)
- Narrowband Definition: BW<< f_c, carrier frequency
- Reduce all bandpass signals and channels to equivalent lowpass signals and channels
- Without any loss of generality, *it makes the analysis independent of the carrier frequency*
- We consider that s(t) has frequency content around a narrowband in the vicinity of the frequency, f_c

Bandpass Signals and Systems

- Construct an analytic signal that contains only the positive frequencies

$$S_+(t) = S(t) + j\hat{S}(t) \qquad OR \qquad S_+(f) = 2U(f)S(f)$$

$$Where \;\; \hat{S}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{s(\tau)}{t-\tau}d\tau, \;\; Hilbert\ transform\ of\ s(\tau)$$

- The analytic signal $S_+(t)$ is a bandpass signal
- Equivalent lowpass representation can be obtained by frequency translation

$$S_l(t) = S_+(t)\,e^{-j2\pi f_c t} = \left[S(t) + j\hat{S}(t)\right]e^{-j2\pi f_c t} \qquad OR$$

$$S(t) + j\hat{S}(t) = S_l(t)e^{j2\pi f_c t}$$

Bandpass Signals and Systems …

- Since $S_l(t)$ is in general complex, it may be expressed as

$$S_l(t) = x(t) + jy(t) \quad then$$

$$S(t) + j\hat{S}(t) = \left(x(t) + jy(t)\right)\left(\cos 2\pi f_c t + j\sin 2\pi f_c t\right)$$
$$= \left(x(t)\cos 2\pi f_c t - y(t)\sin 2\pi f_c t\right) +$$
$$j\left(x(t)\sin 2\pi f_c t + y(t)\cos 2\pi f_c t\right)$$

- Equating the real and imaginary parts

$$S(t) = x(t)\cos 2\pi f_c t - y(t)\sin 2\pi f_c t$$
$$\hat{S}(t) = x(t)\sin 2\pi f_c t + y(t)\cos 2\pi f_c t \qquad \textbf{(1)}$$

- *S(t)* is the desired form of the bandpass signal
- x(t) and y(t) are amplitude modulations impressed on carriers cos $2\pi f_c t$ and $\sin 2\pi f_c t$, which are in phase quadrature

Bandpass Signals and Systems …

- Alternatively,

$$\boxed{S(t)=Re\left\{\left[\left(x(t)+jy(t)\right)\right]e^{j2\pi f_c t}\right\}=Re\left[S_l(t)e^{j2\pi f_c t}\right]} \qquad (2)$$

- A third possible form of representation of *S(t)* can be

$$\boxed{\begin{aligned}&S_l(t)=a(t)e^{j\theta(t)} \quad Where\\&a(t)=\sqrt{(x^2(t)+y^2(t))} \quad and \quad \theta(t)=tan^{-1}\frac{y(t)}{x(t)}\\&S(t)=Re\left(S_l(t)\,e^{j2\pi f_c t}\right)=a(t)cos\left(2\pi f_c t+\theta(t)\right)\end{aligned}} \qquad (3)$$

- *a*(t) and *θ*(t) are the envelope and the phase angle of *S(t),* respectively

Bandpass Signals and Systems …

- The signal can also be expressed in frequency domain through its Fourier transform

$$\boxed{S(f)=\int_{-\infty}^{\infty} S(t)e^{-j2\pi ft}dt=\int_{-\infty}^{\infty}\mathrm{Re}\left(S_l(t)e^{j2\pi f_c t}\right)e^{-j2\pi ft}dt}$$

- And using $\boxed{\mathrm{Re}(Z)=\frac{1}{2}\left(Z+Z^*\right)}$

- We get

$$\boxed{\begin{aligned} S(f)&=\frac{1}{2}\int_{-\infty}^{\infty}\left[S_l(t)e^{j2\pi f_c t}+S_l^*(t)e^{-j2\pi f_c t}\right]e^{-j2\pi ft}dt \\ &=\frac{1}{2}\left[S_l(f-f_c)+S_l^*(-f-f_c)\right]; \end{aligned}}$$

- Where $S_l(f)$ is the transform of $S_l(t)$

Bandpass Signals and Systems …

- The energy in the signal *S(t)* is given by

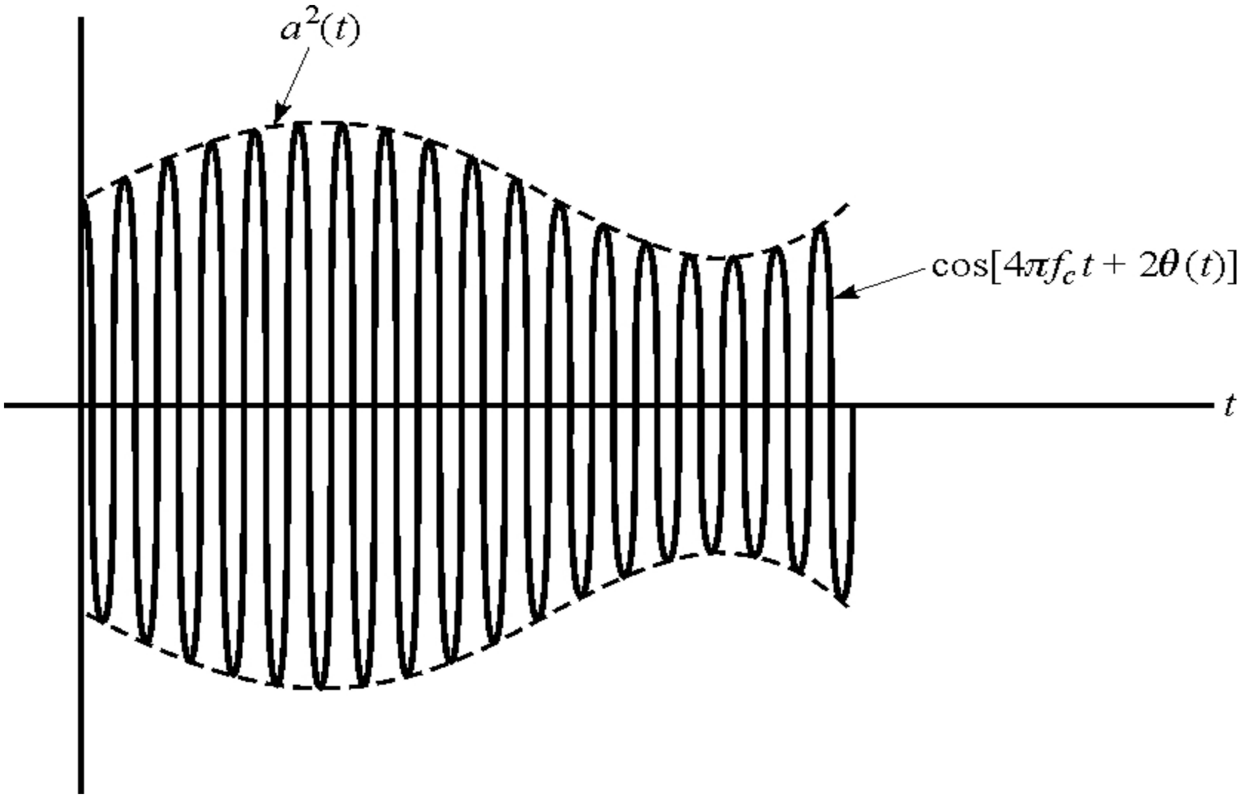
$$\varepsilon = \int_{-\infty}^{\infty} S^2(t)\,dt = \int_{-\infty}^{\infty} \left\{Re\left[S_l(t)e^{i2\pi f_c t}\right]\right\}^2 dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} |S_l(t)|^2\,dt + \frac{1}{2}\int_{-\infty}^{\infty} |S_l(t)|^2 \cos(4\pi f t + 2\theta(t))\,dt$$

- Since in the second term $|S_l(t)|$ varies very slowly its contribution may be neglected and the energy of the signal given by the first element of the sum only

$$\boldsymbol{\varepsilon} \approx \frac{1}{2}\int_{-\infty}^{\infty} |S_l(t)|^2\,dt$$

Bandpass Signals and Systems …



Linear Bandpass System

- Linear bandpass systems are characterized by the impulse response *h(t)* or by the frequency response H(*f*), which is the Fourier transform of *h(t)*
- Note that for real *h(t)*, $H^*(-f) = H(f)$
- Define:

$$\boxed{H_l(f - f_c) = \begin{cases} H(f) & f > 0 \\ 0 & f < 0 \end{cases}}$$

- Then:

$$\boxed{\begin{array}{l} H_l^*(-f - f_c) = \begin{cases} 0 & f > 0 \\ H(f) & f < 0 \end{cases} \\ \therefore \quad H(f) = H_l(f - f_c) + H_l^*(-f - f_c) \end{array}}$$

Linear Bandpass System

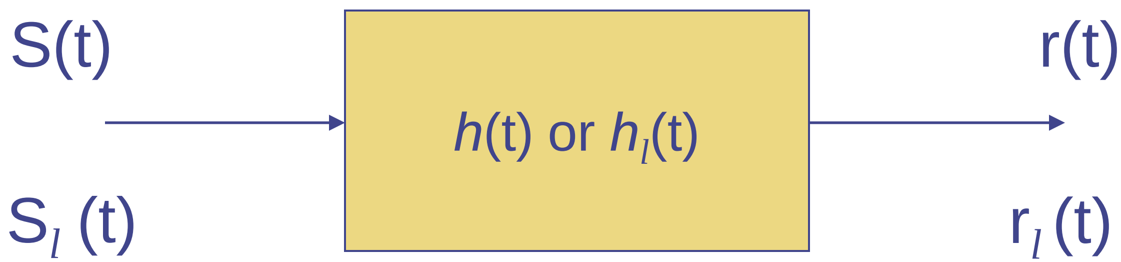
- Or in time domain using the inverse transform

$$\boxed{\begin{aligned} h(t) &= h_l(t)e^{j2\pi f_c t} + h_l^*(t)e^{-j2\pi f_c t} \\ &= 2\,Re\left[h_l(t)e^{j2\pi f_c t}\right] \end{aligned}}$$

- Where $h_l(t)$ and $H_l(f)$ are Fourier transform pairs and are in general complex valued functions that characterize the equivalent lowpass system

Response of Bandpass System to Bandpass Signal

- Assume an input signal S(t) is a narrowband bandpass (BP) signal and the system is also narrowband BP



$$\boxed{r(t) = Re(r_l(t)e^{j2\pi f_c t})}$$

- Where $\boxed{r(t) = \int_{-\infty}^{\infty} S(\tau)h(t-\tau)dt \quad or \quad R(f) = S(f)H(f)}$

$$\boxed{R(f) = \frac{1}{2}[S_l(f-f_c) - S_l^*(-f-f_c)]\,[H_l(f-f_c) - H_l^*(-f-f_c)]}$$

Response of Bandpass System to Bandpass Signal

- For narrowband signal and narrowband impulse response

$$\boxed{\begin{aligned}&S_l(f - f_c) \approx 0 \quad and \quad H_l(f - f) = 0 \quad for\ f < 0\\ &\text{Thus} \quad S_l(f - f_c)H_l^*(-f - f_c) = S_l^*(-f - f)H_l(f - f) = 0\end{aligned}}$$

$$\boxed{\begin{aligned}R(f) &\approx \frac{1}{2}[s_l(f - f_c)\,H_l(f - f_c) + S_l^*(f - f_c)H_l^*(-f - f_c)]\\ &= \frac{1}{2}[R_l(f - f_c) + R_l^*(-f - f_c)]\end{aligned}}$$

- Where

$$\boxed{R_l(f) = S_l(f)H_l(f)}$$

- is output spectrum of the LPF system excited by LP signal

Overview

- Signals and systems
- Stationary stochastic process
- Signal space

Bandpass Stationary Stochastic Processes

- Suppose n(t) is a sample function of a wide sense stationary (WSS) stochastic process with zero mean and power spectral density $\Phi_{nn}(f)$
  - $\Phi_{nn}(f)$ is assumed zero outside an interval Δf centered around $\pm f_c$
- n(t) is narrowband process if $\Delta f \ll f_c$
- n(t) may be represented by any of the following three forms

$$\boxed{\begin{aligned} n(t) &= a(t)\cos\left(2\pi f_c t + \theta(t)\right) \\ &= x(t)\cos 2\pi f_c t - y(t)\sin 2\pi f_c t \\ &= Re\left(z(t)e^{-j2\pi f_c t}\right) \end{aligned}}$$

- Where

$$\boxed{\begin{aligned} & z(t) = x(t) + jy(t) \\ & \theta(t) = \tan^{-1}\frac{y(t)}{x(t)} \text{ and } a(t) = \sqrt{x^2(t) + y^2(t)} \end{aligned}}$$

Bandpass Stationary Stochastic Processes …

- Since *n(t)* is zero-mean, *x(t)* and *y(t)* are also zero mean
- Furthermore, from the stationarity of *n(t)* follows

$$\boxed{\begin{aligned}&\phi_{xx}(\tau)=\phi_{yy}(\tau); \qquad \phi_{xy}(\tau)=-\phi_{yx}(\tau) \quad and\\&\phi_{nn}(\tau)=\phi_{xx}(\tau)\cos 2\pi f_c\tau-\phi_{yx}(\tau)\sin 2\pi f_c\tau\end{aligned}}$$

- Which is identical in form with the expression for n(t)
- The autocorrelation of the equivalent lowpass process

$$\boxed{\begin{aligned}&z(t)=x(t)+jy(t) \text{ is}\\&\phi_{zz}(\tau)=\frac{1}{2}E\left\{z^*(t)z(t+\tau)\right\}=\phi_{xx}(\tau)+j\phi_{yx}(\tau)\end{aligned}}$$

- Which is the autocorrelation of the complex evelope
- Finally, $\boxed{\phi_{nn}(\tau)=\mathrm{Re}\left(\phi_{zz}(\tau)e^{j2\pi f_c\tau}\right)}$

Bandpass Stationary Stochastic Processes …

- Thus $\Phi_{nn}(\tau)$ of the bandpass stochastic process is uniquely determined from the autocorrelation function $\Phi_{ZZ}(\tau)$ of the equivalent lowpass process $z(t)$ and carrier frequency f_c
- Note that

$$\boxed{\begin{aligned}\phi_{nn}(f) &= \int_{-\infty}^{\infty} \left[\mathrm{Re}\left(\phi_{zz}(\tau) e^{j2\pi f_c \tau}\right)\right] e^{-j2\pi f \tau} d\tau \\ &= \frac{1}{2}\left[\phi_{zz}(f-f_c) + \phi_{zz}(-f-f_c)\right]\end{aligned}}$$

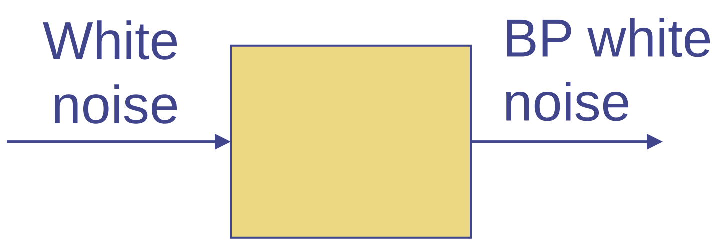
- Where $\Phi_{ZZ}(f)$ is the power spectrum of the lowpass process $z(t)$
- Since $\Phi_{ZZ}(\tau) = \Phi_{ZZ}^{*}(-\tau)$, it follows that $\Phi_{ZZ}(f)$ is ***real valued*** function of frequency

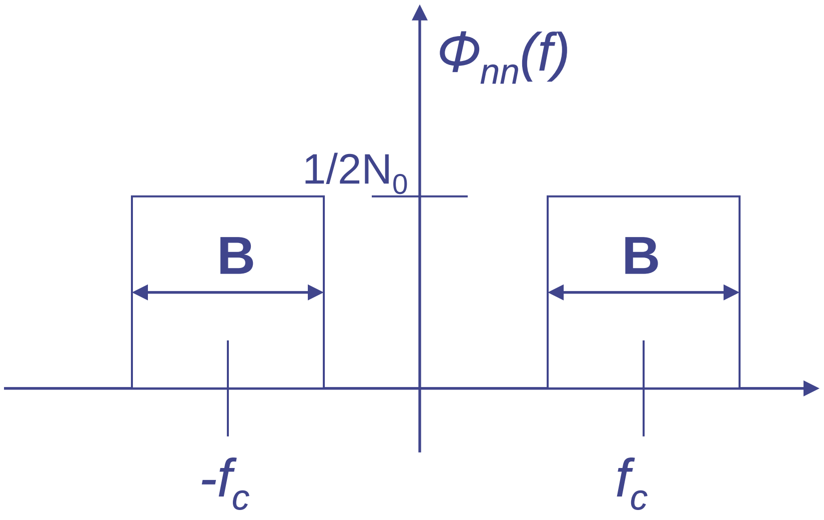
Bandpass Stationary Stochastic Processes …

- $\phi_{xy}(\tau)$ is an odd function of τ and $\phi_{xy}(0)=0$ and hence x(t) and y(t) are uncorrelated for $\tau = 0$
- If *n(t)* is Gaussian *x(t)* and *y(t)* are jointly Gaussian and for $\tau = 0$ are independent

Representation of White Noise

- White noise is wideband and cannot be represented in terms of quadrature components
- If the noise is assumed to have passed through an ideal bandpass filter, the output can be represented by quadrature components





Representation of White Noise …

- The equivalent lowpass noise z(t) has a power spectral density:

$$\phi_{zz}(f) = \begin{cases} N_0 & |f| \le \frac{1}{2}B \\ 0 & |f| > \frac{1}{2}B \end{cases}$$

- And autocorrelation function is:

$$\phi_{zz}(\tau) = N_0 \frac{\sin \pi B\tau}{\pi B\tau} \quad \text{which} \rightarrow \mathrm{N}_0\delta(\tau) \;\; \text{as B} \rightarrow \infty$$

$$\phi_{yx}(\tau) = 0 \text{ for all } \tau \text{ and } \phi_{zz}(\tau) = \phi_{xx}(\tau) = \phi_{yy}(\tau)$$

- i.e the quadrature components x(t) and y(t) are uncorrelated for all time shifts τ and the autocorrelation of z(t), x(t) and y(t) are all equal

Overview

- Signals and systems
- Stationary stochastic process
- Signal space

Signal Space Concepts

- Analogous to space vectors, we represent a family of signals such as $\boxed{X = \{x_0(t), x_1(t), \ldots\ldots\ldots x_{M-1}\}}$ by a signal space over a given time interval

- A signal space is defined by its orthonormal basis

$$\boxed{\{f_0(t),\ f_1(t), \ldots\ldots\ldots\ldots\ldots f_{N-1}\}}$$

- The inner product of two signals is defined as

$$\boxed{\langle x_1(t), x_2(t)\rangle = \int_a^b x_1(t)\, x_2^*(t)\, dt}$$

- The norm of a signal is defined as

$$\boxed{\|x(t)\| = \left(\int_a^b |x(t)|^2\, dt\right)^{1/2}}$$

Signal Space Concepts …

- A set of signals are orthonormal iff they are orthogonal and their norms are each unity
- The set $\left\{ f_0(t),\ f_1(t),\ldots\ldots\ldots\ldots f_{N-1} \right\}$ is an orthonormal basis of the signal X iff

$$\forall x_j(t) \in X, \exists a_i \in \Re \,\middle|\, x_j(t) = \sum_{i=0}^{N-1} a_i f_i(t)$$

- Let *s(t)* be a deterministic, real valued signal with finite energy

$$E_s = \int_{-\infty}^{\infty} s^2(t)\,dt$$

- And let the set $\left\{ f_n(t) \right\},\quad n = 0, 1, 2, \ldots\ldots\ldots\ldots, N-1$ be an orthonormal set of signals or waveforms

Signal Space Concepts …

- We can approximate *s(t)* by

$$\hat{S}(t) = \sum_{n=0}^{N-1} s_n f_n(t), \quad where$$

$$s_n = \int_{-\infty}^{\infty} s(t) f_n(t), \quad n = 1,2,\ldots\ldots\ldots\ldots N-1$$

- The set of waveforms $\{f_n(t)\}, \quad n = 0,1,2,\ldots\ldots\ldots\ldots\ldots N-1$ is said to be complete iff the error energy is zero, i.e.,

$$\varepsilon_e = \int_{-\infty}^{\infty} \left(s(t) - \hat{s}(t) \right)^2 dt = E_s - \sum_{n=0}^{N-1} s_n^2 = 0$$

Signal Space Concepts …

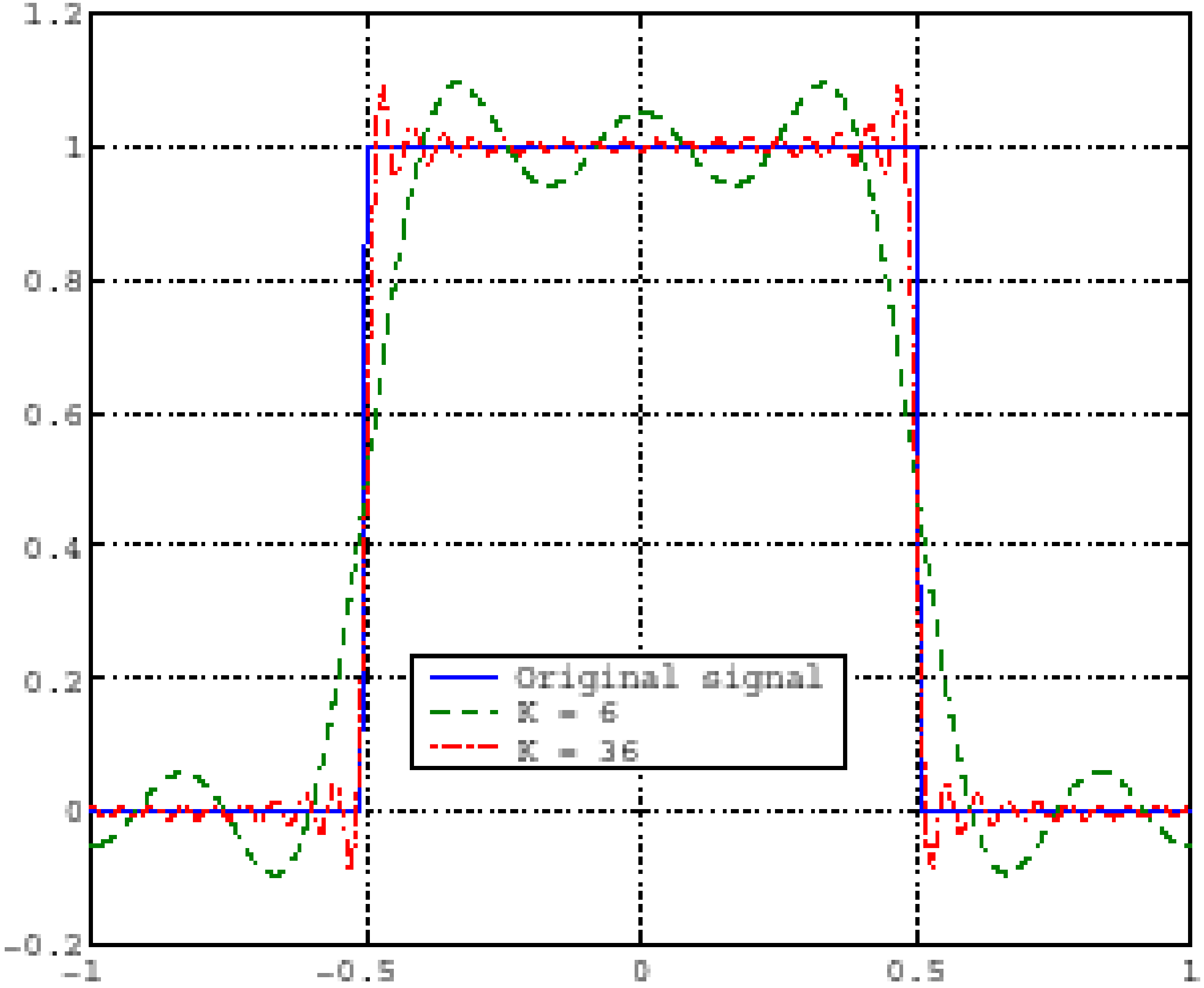
- Consider the signal x(t) given by

$$x(t) = \begin{cases} 1 & -0.5 \le t \le 0.5 \\ 0 & elsewhere \end{cases}$$

- We approximate x(t) by

$$x(t) = \sum_{n=0}^{K} x_n \cos n\pi\, \mathrm{t} \qquad -1 \le t \le 1$$

Signal Space Concepts …



Signal Space Concepts …

- In general, assume that we have a set of waveforms (signals)

$$\boxed{\left\{s_0(t), s_1(t), \ldots\ldots\ldots\ldots\ldots\ldots s_{M-1}(t)\right\}}$$

- And we wish to construct a set of orthonormal waveforms

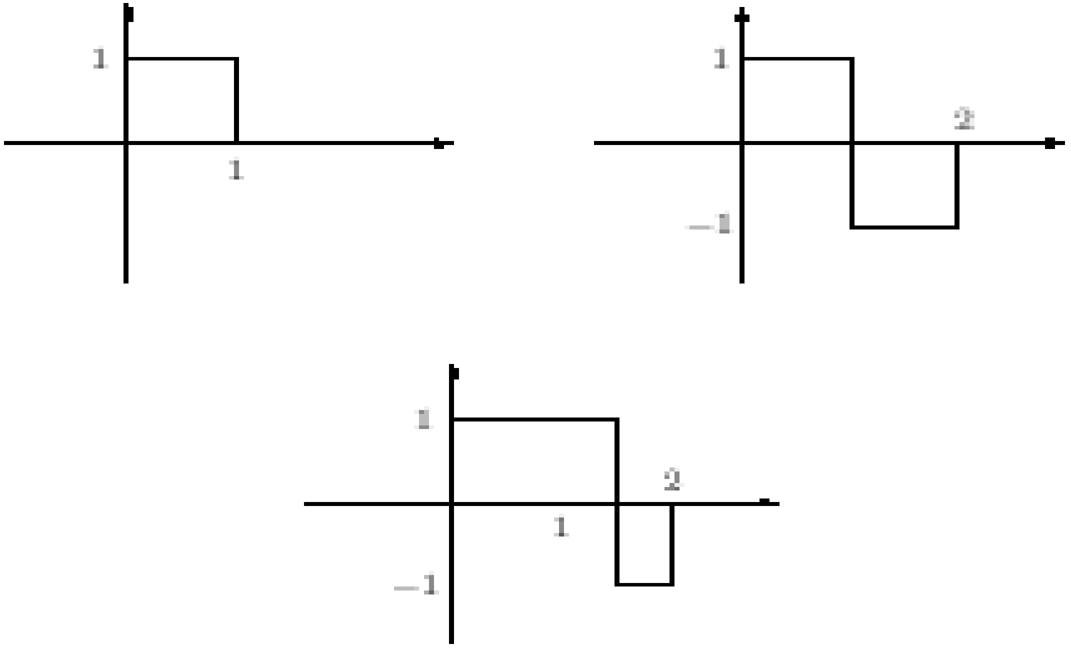
$$\boxed{\left\{\ f_0(t),\ \ f_1(t), \ldots\ldots\ldots\ldots\ldots f_{N-1}\ \right\}}$$

from the original signal set

- This can be done either:
  - Formally by using the Gram-Schmidt orthogonalization procedure (READ Page 163 of the text); or
  - In simple cases, by inspection

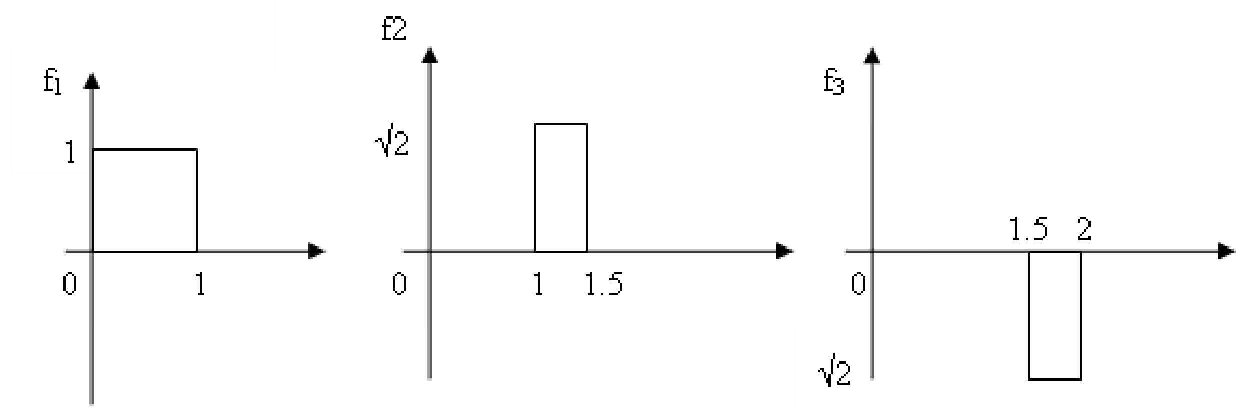
Signal Space Concepts …

- Example: Find an orthonormal basis for the following set of waveforms and determine the coordinates of each waveform in the signal space defined by the orthonormal basis functions



Signal Space Concepts …

- By inspection the following orthonormal function can be the basis for the representation of the three waveforms shown above



$$s_1(t) = f_1(t); \quad s_2(t) = f_1(t) - \frac{1}{\sqrt{2}} f_2(t) + \frac{1}{\sqrt{2}} f_3(t)$$

$$s_3(t) = f_1(t) + \frac{1}{\sqrt{2}} f_2(t) + \frac{1}{\sqrt{2}} f_3(t)$$

In vector form

$$\mathbf{S}_s = [1,0,0]; \quad \mathbf{S}_2 = \left[1, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]; \quad \mathbf{S}_3 = \left[1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$$