Chapter 8: Communication Through Band-limited Channels

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Signals Through Band-Limited Channels

- So far, we saw the design of modulator and demodulator filters for band-limited channels
  - Assumption: Channel characteristic is known a priori
- However, in most communication systems, the frequency response of the channel is
  - Either not sufficiently known
  - Or it varies with time
- E.g., telephone channels, wireless channels, …

Signals Through Band-Limited Channels

- We now consider the problem of receiver design taking:
  - The presence of channel distortion and
  - Under the condition that the channel is AWGN and its characteristics are not known a priori
- The channel distortion results in inter-symbol interference

Signals Through Band-Limited Channels …

- The strategy employed to solve the ISI problem is to design a receiver that
  - Compensates for the distortion introduced by the channel
  - Or reduce the ISI in the received signal – ***Equalization***

- Equalization methods
  - Based on maximum likelihood (ML) sequence detection criterion that is optimum from error probability point of view
  - Based on the use of linear filters with adjustable coefficients
  - Decision-feedback equalization: Uses previously detected symbols to suppress the ISI in the symbol being detected

Digital Signals over Channels with Distortion

- Consider the following transmitted signal

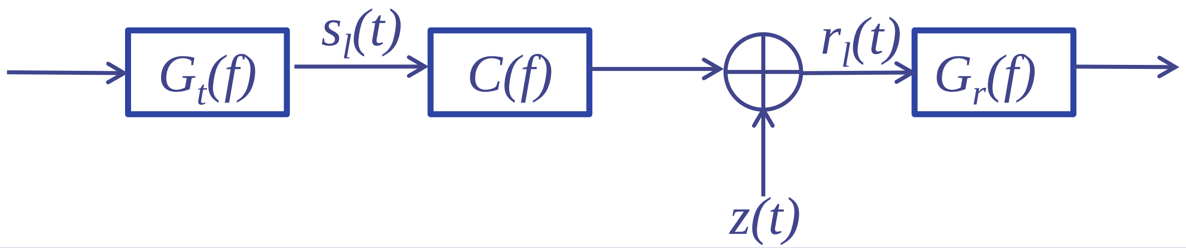
$$S_l(t) = \sum_{n=-\infty}^{+\infty} I_n g_t(t-nT)$$

- Passed through the channel, the equivalent lowpass of the received is given by

$$r_l(t) = \sum_{n=-\infty}^{+\infty} I_n h(t-nT) + z(t)$$

- where we define

$$h(t) = g_t(t) * c(t) \quad \text{and} \quad x(t) = g_t(t) * c(t) * g_r(t)$$



Digital Signals Over Channels with Distortion …

- Note that the optimum demodulator can be realized as:
  - A filter matched to h(t)
  - Followed by a sampler operating at the symbol rate 1/T and
  - A subsequent processing algorithm for estimating the information sequence $\{I_n\}$ from the sample values
- Let us next see an optimum Maximum-Likelihood receiver

Optimum Maximum-likelihood Receiver

- The received signal may be expanded in the series

$$r_l(t) = \lim_{N\to\infty} \sum_{k=1}^{N} r_k \ f_k(t)$$

- Where
  - $\{f_k(t)\}$ is a complete set of orthonormal functions
  - $\{r_k\}$ are the observable random variables obtained by projecting $r_l(t)$ onto the set $\{f_k(t)\}$

Optimum Maximum-likelihood Receiver …

- It can be shown that

$$r_k = \sum_n I_n h_{kn} + z_k$$

- Where
  - h_{kn} is the value obtained by projecting $h(t-nT)$ onto the set $\{ f_k(t) \}$
  - z_k is the value obtained by projecting $z_k(t)$ onto $f_k(t)$

- The sequence of $\{z_k\}$ is Gaussian with zero-mean and covariance

$$\frac{1}{2} E\{z_k^* z_m\} = N_0 \delta_{km}$$

Optimum Maximum-likelihood Receiver …

- The joint probability density function of the random variable $\mathbf{r}_N \equiv \{r_1, r_2,\ldots\ldots r_N\}$ conditional on the transmitted sequence $\mathbf{I}_P \equiv \{I_1, I_2,\ldots\ldots I_P\}$ where P ≤ N is

$$p(r_N | I_p) = \left(\frac{1}{2\pi N_0}\right)^N exp\left(-\frac{1}{2N_0}\sum_{k=1}^{N}\left|r_k - \sum_n I_n h_{kn}\right|^2\right)$$

- As the number N becomes large, the logarithm of $p(r_N | I_p)$ will be proportional to the metrics $PM(I_p)$, defined as follows

$$PM(I_p) = -\int_{-\infty}^{\infty}\left|r_l(t) - \sum_n I_n h(t-nT)\right|^2 dt$$

$$= -\int_{-\infty}^{\infty}|r_l(t)|^2 dt + 2\,Re\sum_n\left[I_n^* \int_{-\infty}^{\infty} r_l(t)h^*(t-nT)dt\right] + \sum_n\sum_m I_n^* I_m \int_{-\infty}^{\infty} h^*(t-nT)h(t-mT)\,dt$$

Optimum Maximum-likelihood Receiver …

- The maximum likelihood estimate of the symbols $I_1, I_2, \ldots\ldots I_P$ are those that maximize this quantity
- However, since the integral of the square of the magnitude of $r_l(t)$ is common to all metrics it may be discarded
- The other integral involving $r_l(t)$ gives rise to the variables

$$y_n \equiv y(nT) = \int_{-\infty}^{\infty} r_l(t)\, h^*(t - nT)\, dt$$

- These variables can be generated by passing $r(t)$ through a filter matched to $h(t)$ and sampling the output at the symbol rate 1/T

Optimum Maximum-likelihood Receiver …

- The samples $\{y_n\}$ form a set of sufficient statistics for the computation of PM(I_p) or, equivalently, of the correlation metrics

$$CM(I_p) = 2\,Re\left(\sum_n I_n^* y_n\right) - \sum_n \sum_m I_n^* I_m x_{n-m}$$

- Where by definition $x(t)$ is the response of the matched filter to $h(t)$ and is given by

$$x_n \equiv x(nT) = \int_{-\infty}^{\infty} h^*(t)\, h(t+nT)\, dt$$

- $x(t)$ represents the output of a filter having an impulse response $h^*(-t)$ and an excitation $h(t)$
- Hence, $x(nt)$ represents the autocorrelation of $h(t)$ sampled periodically at the rate of 1/T

Optimum Maximum-likelihood Receiver …

- Substituting for $r_l(t)$

$$y_k = \sum_n I_n x_{k-n} + v_k$$

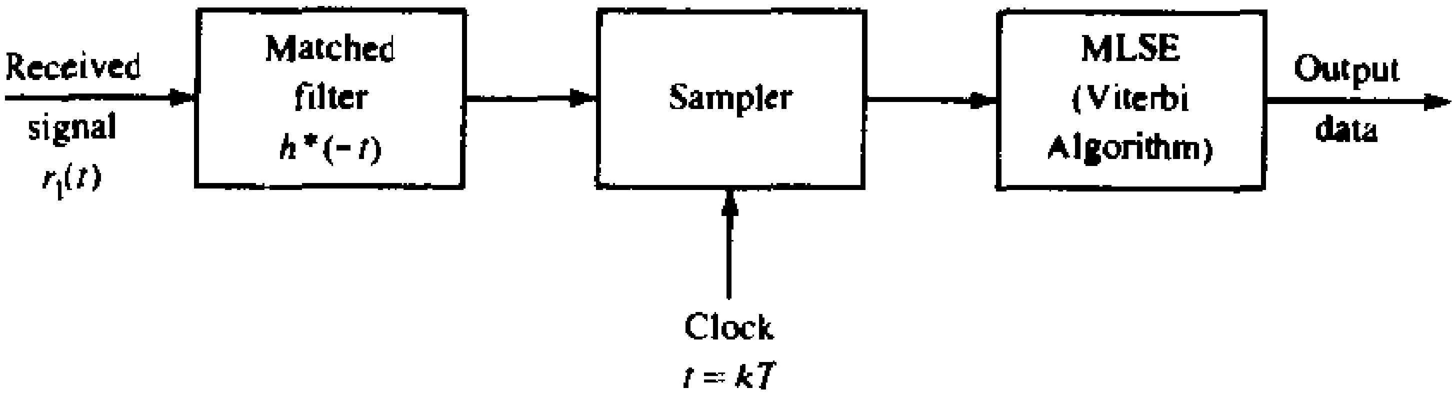
- Where $v_k = \int z(t) h^*(t-kT)dt$

- Note that y_k is corrupted by ISI
- We can reasonably assume that the ISI effects cover a length L ≤ |n|
- The output of the modulator can be considered as the output of a finite state machine
  - I.e., the channel output with ISI may be represented by a trellis diagram
  - The ML estimator of the information is the most probable path through the trellis give the received sequence y_n

Optimum Maximum-likelihood Receiver …

- It can be shown that the correlation matrix can be computed recursively in the Viterbi algorithm according to the relation

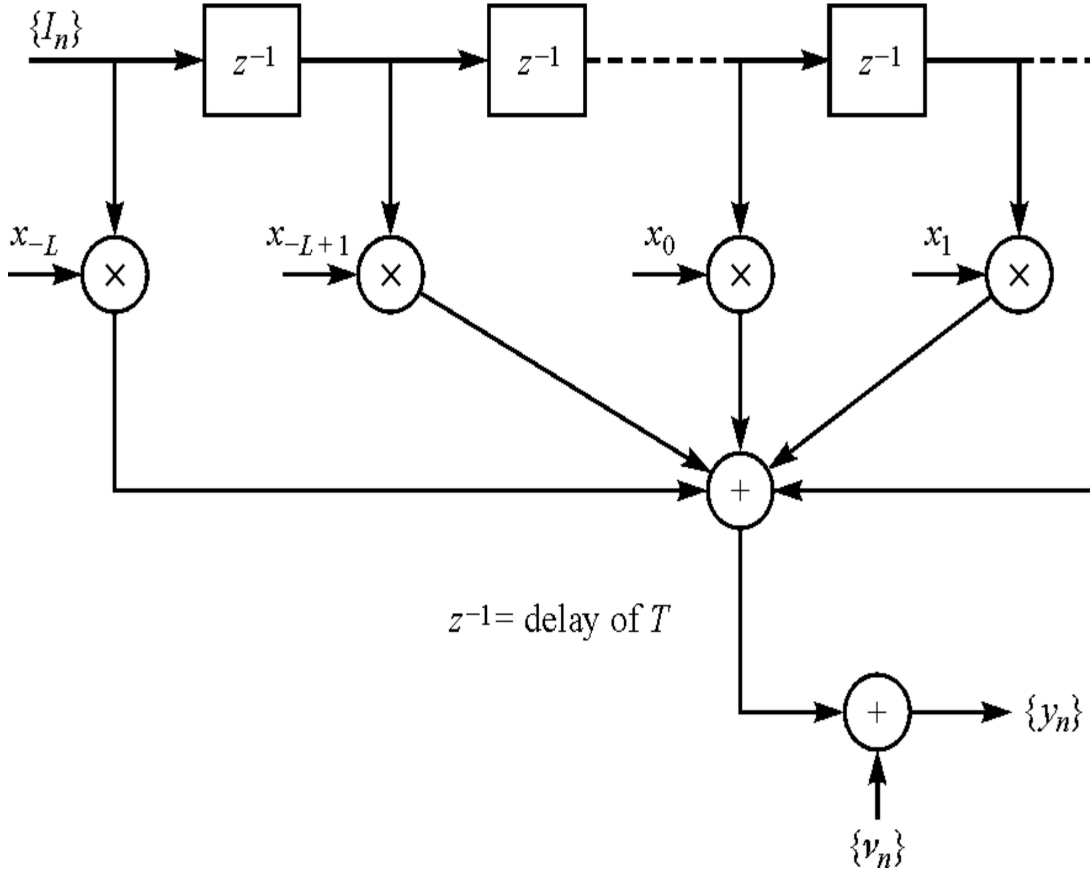
$$CM_n(I_n) = CM_{n-1}(I_{n-1}) + Re\left[I_n^* \left(2y_n - x_0 I_n - 2\sum_{m=1}^{L} x_m I_{n-m} \right) \right]$$



Discrete-time Model for a Channel with ISI

- We note that:
  - Transmitter sends discrete-lime symbols at a rate of 1/T
  - Sampled output of the matched filter at the receiver is also a discrete-time signal with samples occurring at a rate of 1/T
- An *equivalent discrete-time transversal filter* having tap gain coefficients $\{x_k\}$ can be used to represent the cascade of
  - Transmitter filter with impulse response $g(t)$
  - Channel with impulse response $c(t)$
  - Matched filter at the receiver with impulse response $h^*(-t)$ and
  - The sampler
- Input of the filter is the sequence of information symbols $\{I_k\}$ and its output is the discrete-time sequence $\{y_k\}$

Discrete-time Model for a Channel with ISI …



Equivalent discrete-time model of channel with ISI

Discrete-time Model for a Channel with ISI

- The output of the receiver filter is given by

$$y(t) = r_l(t) * g_r^*(-t) = \sum_{n=-\infty}^{\infty} I_n\, x(t - nT) + z(t) * g_r^*(t)$$

- The received signal sample is obtained as

$$y_k = y(KT) = \sum_{n=-\infty}^{\infty} I_n x_{n-k} + v_k$$

- The optimum receiver is matched to *h(t)*, that is $g_r(t) = h^*(-t)$

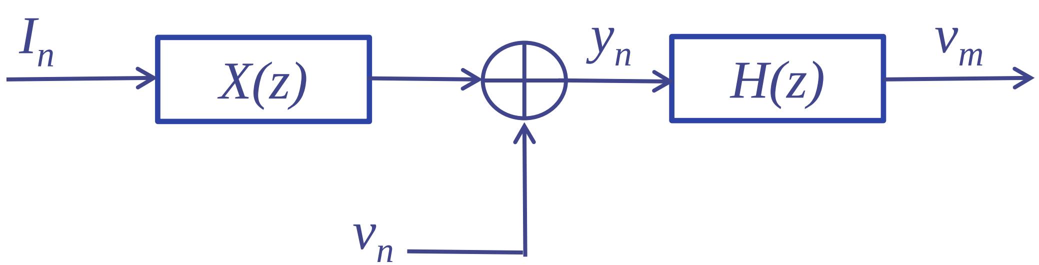
$$x_n = \int_{-\infty}^{\infty} h(t) h^*(t+nT)\,dt; \quad v_k = \int_{-\infty}^{\infty} z(t) h^*(t+nT)\,dt$$

Discrete-time Model for a Channel with ISI …

- The autocorrelation of the noise samples at the correlator output is given by

$$\frac{1}{2}E(v_k^* v_j) = \begin{cases} N_0 x_{j-k} & (|k-j|) \le L \\ 0 & otherwise \end{cases}$$

- which is the correlated noise sequence at the receiver filter output
- This complicates the detection process by introducing distortion!
- To solve the problem we use a *whitening filter*



Discrete-time Model for a Channel with ISI …

- Let *X(z)* denote the two sided Z transform of the sampled autocorrelation function $\{x_k\}$

$$X(z) = \sum_{k=-L}^{L} x_k z^{-k}$$

- Since $x_k = x_{-k}^*$, it follows that $X(z) = X^*(1/z^*)$ and the 2L roots of *X(z)* have symmetry such that if ρ is a root, $1/\rho^*$ is also a root
- Hence, *X(z)* can be factored and expressed us

$$X(z) = F(z)F^*(z^{-1})$$

- Where *F(z)* is a polynomial of degree L and roots $\rho_1, \ldots\ldots, \rho_L$ and $F^*(z^{-1})$ also has degree L and roots $1/\rho_1^*, \ldots\ldots, 1/\rho_L^*$

Discrete-time Model for a Channel with ISI …

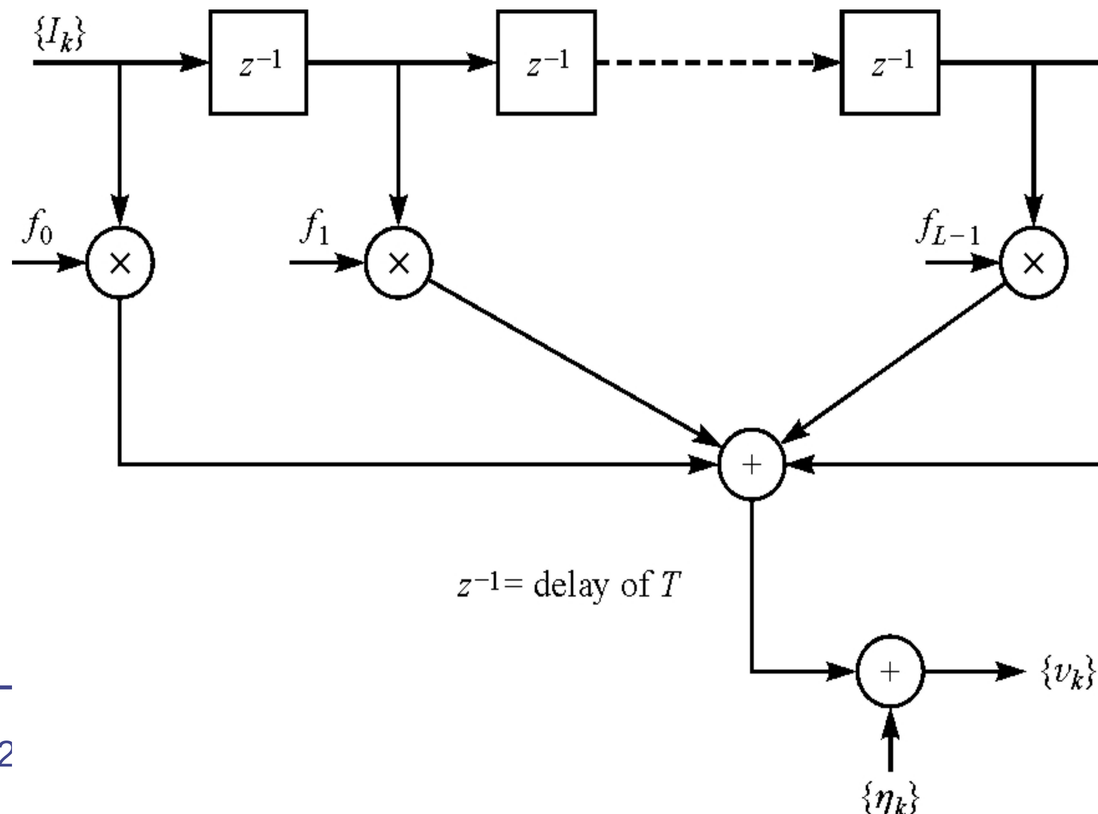
- We can now choose a *whitening filter* as $H(z)=1/F^*(z^{-1})$ having all roots inside the unit circle resulting in a physically realizable & stable recursive discrete-time filter
- The passage of the sequence $\{y_k\}$ through the whitening filter results in an output $\{v_k\}$

$$v_k = \sum_{n=0}^{L} f_n I_{k-n} + \eta_k$$

- $\{\eta_k\}$ is a zero-mean white Gaussian noise sequence with variance σ^2_n
- $\{f_k\}$ is a set of tap coefficients of an *equivalent discrete-time transversal filters* with transfer functions $F(z)$

Discrete-time Model for a Channel with ISI …

- The cascade of *g(t), c(t), $h^*(-t)$,* the sampler, and the whitening filter $1/F(z^{-1})$ is represented as an equivalent discrete-time transversal filter with $\{f_n\}$ as its tap coefficients
- The additive noise sequence $\{\eta_k\}$ is white Gaussian noise sequence having zero mean and variance N_o



Discrete-time Model for a Channel with ISI …

- Example: Consider that the
  - Transmitter signal pulse *g(t)* has duration T and unit energy
  - The received signal pulse is *h(t) = g(t) + ag(t-T)*
  - Determine the equivalent discrete-time white noise filter model
- The sampled autocorrelation function is given by

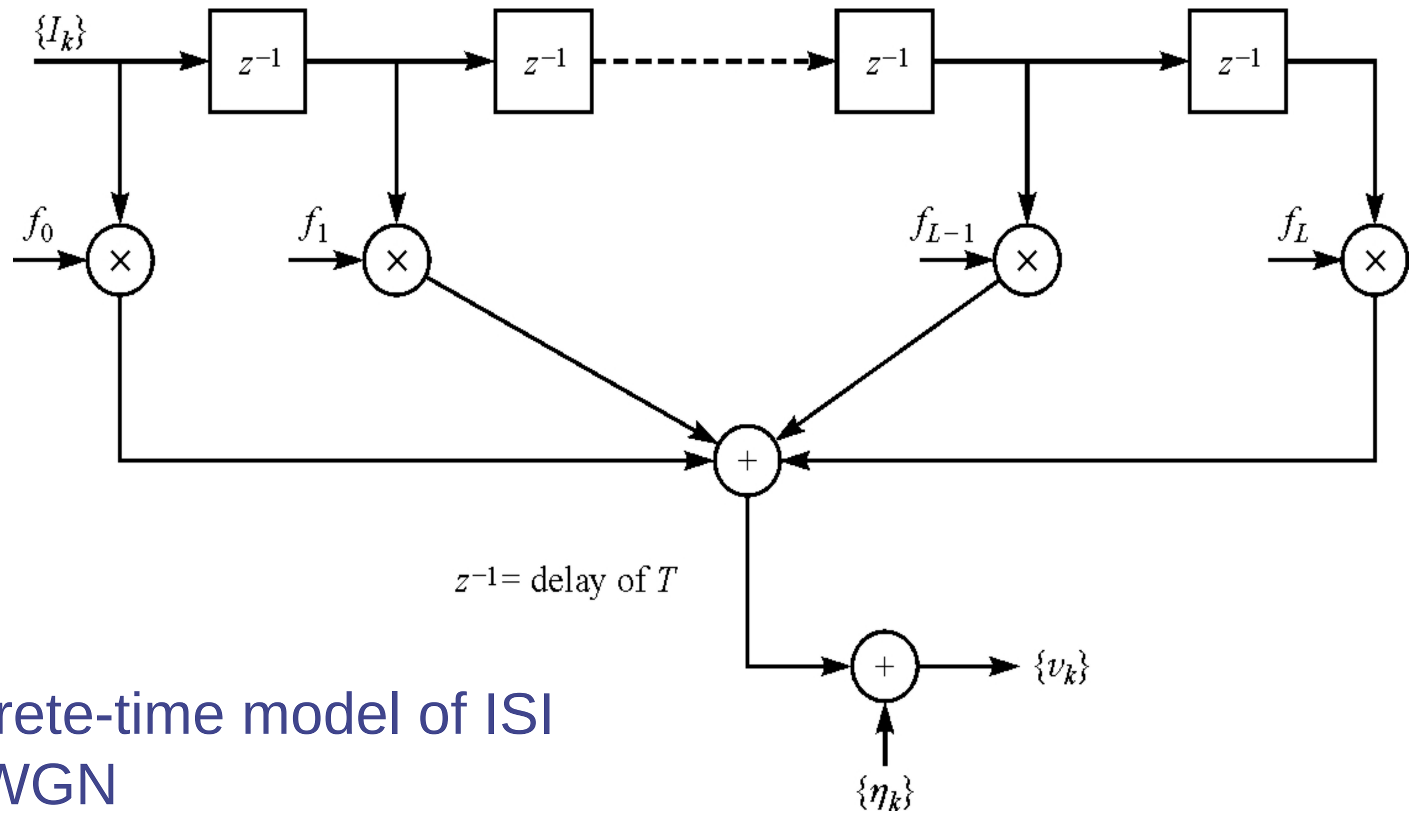
$$x_k = \begin{cases} a^* & (k=-1) \\ 1+|a|^2 & (k=0) \\ a & (k=1) \end{cases}$$

- The Z transform of x_k is

$$\begin{aligned} X(z) &= \sum_{k=-1}^{1} x_k z^{-k} \\ &= a^* z + (1+|a|^2) + a z^{-1} \\ &= (az^{-1}+1)(a^* z+1) \end{aligned}$$

Discrete-time Model for a Channel with ISI …

- Assuming that $|a| < 1$, we can choose $F(z)= az + 1$
- The two tap coefficients of the transversal filter are $f_0 = 1$ and $f_1=a$



Equivalent discrete-time model of ISI channel with AWGN

Assignments!

- Read the working principle of
  - Linear filters with adjustable coefficients
  - Decision-feedback equalization