Chapter 7: Signal Design for Band-limited Channels

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Graduate Program

School of Electrical and Computer Engineering

Overview- Signal Design for Band-limited Signal

- Characterization of band-limited channels
- Signal design for band-limited channels
  - Criterion for No ISI - The Nyquist criterion
  - Controlled ISI
- Signal design with distortion

Characterization of Band-Limited Channels

- In the previous lectures, we saw transmission of digital information through *AWGN channels*
- Bandwidth constraints not considered on either
  - Communication system design or
  - Type of the information bearing signal
- We consider the problem of signal design when a channel is *band-limited* to some specified bandwidth of W Hz
- The channel may be modeled as a *linear filter* having an equivalent low-pass frequency response C(f) that is zero for $|f| > W$

Characterization of Band-Limited Channels…

- A linearly modulated signal in baseband is represented as

$$v(t) = \sum_n I_n g(t - nT)$$

- We consider the design of the signal pulse *g(t)* that efficiently utilizes the total available channel bandwidth W
- When the channel is *ideal* for $|f| \leq W$, a signal pulse can be designed that permits the transmission at symbol rates comparable to or exceeding the channel bandwidth
- For *non-ideal channels*, transmission at symbol rate equal to or exceeding W results in *intersymbol interference* (ISI) among adjacent channels
- For such channels *coding can shape the spectrum* of the transmitted signal to avoid ISI

Characterization of Band-Limited Channels…

- Consider telephone channels that are characterized as *band-limited linear filters*
  - Appropriate when FDM is used
- Filtering is still used on the analog signal prior to sampling and encoding
- A band-limited channel can be characterized as a *linear filter* having an equivalent low-pass frequency-response C(f) and an equivalent LP impulse response c(t)
- If a signal of the form

$$s(t) = Re[v(t) e^{j2\pi f_c t}]$$

Characterization of Band-Limited Channels …

- If transmitted over a band-pass channel, the received equivalent LP signal is given by

$$r_l(t) = \int_{-\infty}^{\infty} v(\tau)c(t-\tau)d\tau + z(t)$$

- Where z(t) denotes the additive noise
- Alternatively in frequency domain

$$R_l(f) = V(f)C(f)$$

- If the channel is band-limited to W Hz, then C(f) = 0 for $|f|$ >W
  - Frequency component of V(*f*) above $|f|$ =W will be filtered out by the channel
- Hence, need to limit the BW of transmitted signal to W Hz

Characterization of Band-Limited Channels …

- Within the bandwidth of the channel

$$C(f) = |C(f)| e^{j\theta(f)}$$

- Where |C(f)| is the *amplitude response* and $\theta(f)$ is the *phase response* characteristics
- Furthermore, the *envelope delay characteristic* is defined as

$$\tau(f) = -\frac{1}{2\pi}\frac{d\theta(f)}{df}$$

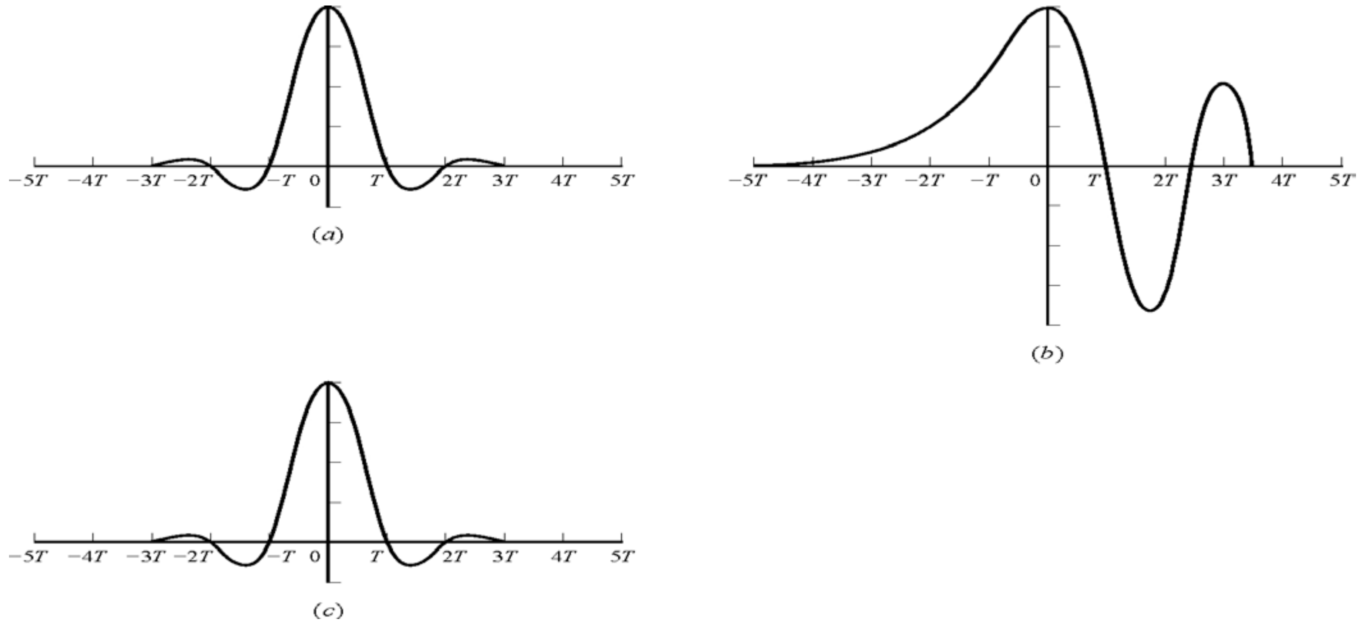
- A channel is said to be *non-distorting* or *ideal* if
  - $|C(f)|$ is constant for all $|f| \le W$ and
  - θ (f) is linear function of frequency or $\tau(f)$ is constant for all $|f| \le W$

Characterization of Band-Limited Channels …

- If $|C(f)|$ is not constant for all $|f| \leq W$, the channel *distorts the transmitted signal in amplitude*
- If $\tau(f)$ is not constant for all $|f| \leq W$, the channel is said to *distort the signal V(f) in delay*
- The consequence of amplitude and phase distortion is to cause a *succession of pulses* transmitted at rates comparable to the bandwidth W to be smeared and become indistinguishable as well defined pulses at the receiver

Characterization of Band-Limited Channels …

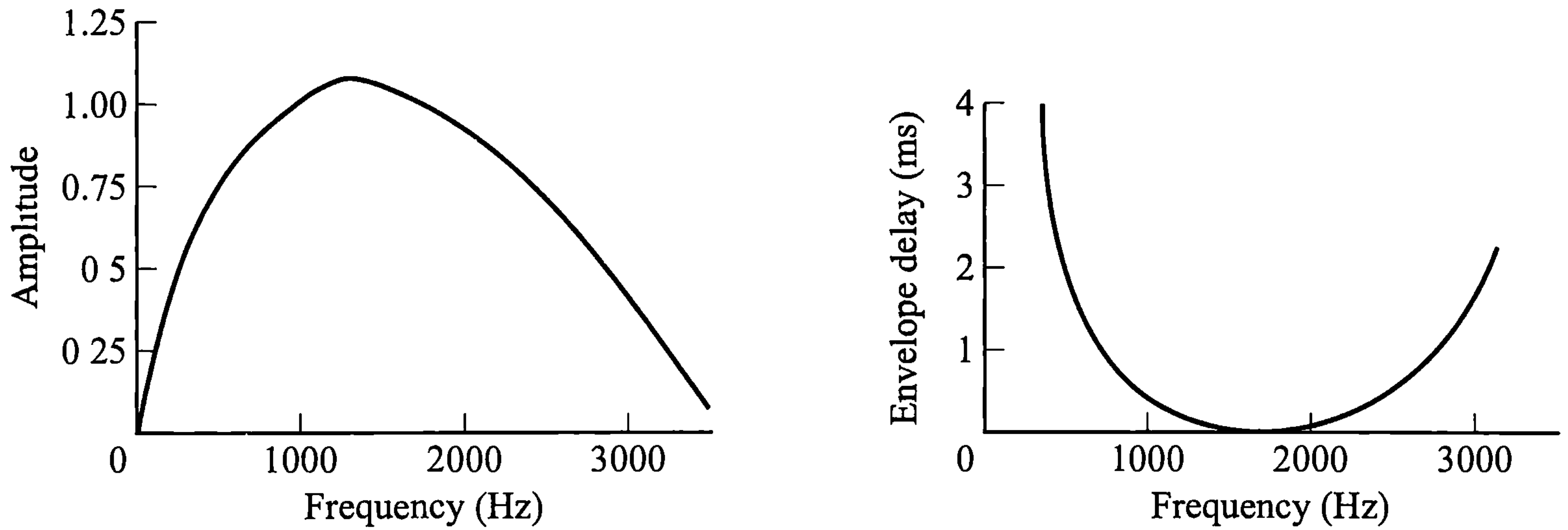
- Example of distortion on transmitted signal shape
- ISI will be severe for PAM signals if transmitted through a channel having a *linear envelope delay* characteristics
- Effect of channel can be corrected through *equalization*



Effect of channel distortion: **(a)** channel input; **(b)** channel output; **(c)** equalizer output

Characterization of Band-Limited Channels …

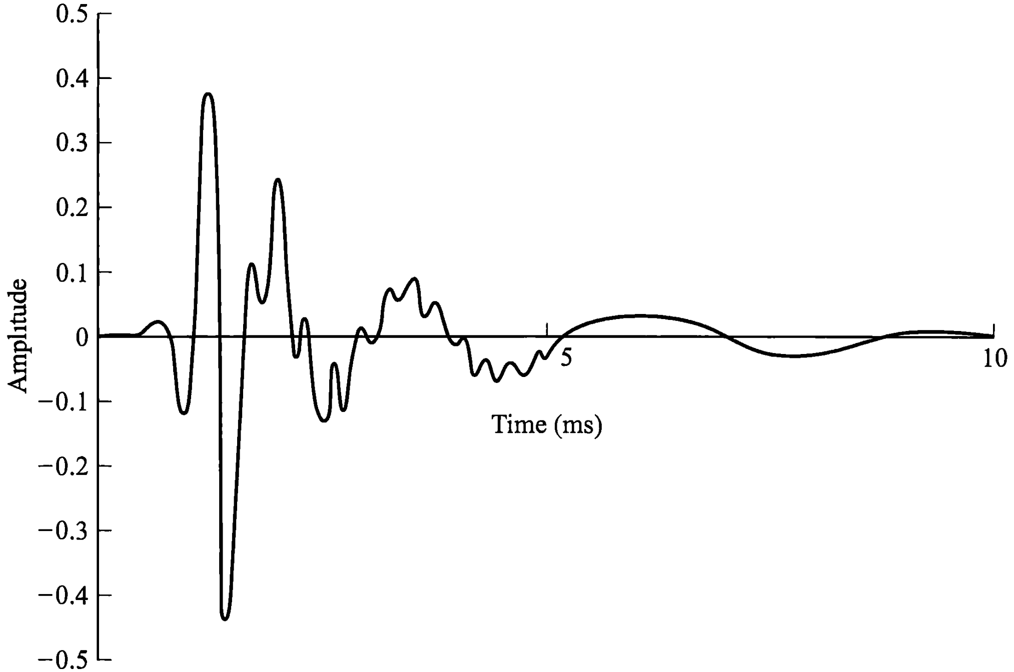
- Typical frequency response characteristics of the telephone channel are shown in the figure below
- Usable frequency band extends from about 0.3k to 3k Hz



Average amplitude and delay characteristics of medium-range telephone channel

Characterization of Band-Limited Channels …

- The *impulse response* shown in the figure below extends over 10 ms band with 2500 symbols/s
- The ISI might extend over 20-30 symbols



Impulse response of average channel with amplitude and delay shown in the previous slide

Characterization of Band-Limited Channels …

- In addition to linear distortion, other impairments include:
  - Non-linear distortion
  - Frequency off-set
  - Phase jitter
  - Impulse and thermal noise
- *Non-linear distortion* – Arises from non linearities in amplifiers and companders
  - Effect is small but difficult to correct
- *Frequency off-set* – Approximately less than 5 Hz
  - Effect is severe in high speed transmission systems and synchronous coherent detection
- *Phase jitter* – Low index frequency modulation with the low-frequency harmonics of the power line(50 Hz)

Overview

- Characterization of band-limited channels
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Signal Design for Band-Limited Channels

- Low-pass equivalent of transmitted signals for different modulation techniques have the common form

$$v(t) = \sum_n I_n g(t - nT)$$

- Where
  - $\{I_n\}$ – Discrete information bearing sequence of symbols
  - $g(t)$ – Signal pulse assumed having a band-limited frequency response G(f) such that G(f)=0 for $|f| > W$

- And the channel characteristic is $C(f) = 0$ for $|f| > W$
- The received signal is then given by

$$r_l(t) = \sum_n I_n h(t - nT) + z(t)$$

Signal Design for Band-Limited Channels

- Where $$h(t) = \int_{-\infty}^{\infty} g(\tau)c(t-\tau)\,d\tau$$
- And *z(t)* is the AWGN
- Assume the received signal is passed through a matched filter with frequency response described by H*(*f*)
- The output of the receiving filter is

$$y(t) = \sum_{n=0}^{\infty} I_n x(t-nT) + z'(t)$$

- Where
  - *x(t)* is the pulse representing the response of the receiving filter to the input *h(t)*
  - *z' (t)* is the response of the receiving filter to the noise *z(t)*

Signal Design for Band-Limited Channels

- The output of the filter is sampled at the rate of 1/T samples/second and the sampled output is given by

$$y(kT+\tau_0) \equiv y_k = \sum_n I_n x(kT-nT+\tau_0) + z'(kT+\tau_0)$$

- Or equivalently

$$y_k = \sum_n I_n x_{k-n} + z'_k \; ; \quad k = 0, 1, 2, \ldots\ldots\ldots$$

- Where τ_0 is the transmission delay

Signal Design for Band-Limited Channels…

- The sample values can alternatively be expressed as

$$y_k = x_0\left(I_k + \frac{1}{x_0}\sum_{\substack{n=0\\n\neq k}}^{\infty} I_n x_{k-n}\right) + z'_k\,; \quad k = 0,1,2,.....$$

- Where x_0 is an arbitrary scale factor which can be set to unity for convenience such that

$$y_k = I_k + \sum_{\substack{n=0\\n\neq k}}^{\infty} I_n x_{k-n} + z'_k\,; \quad k = 0,1,2,...$$

- The term I_k represents the desired information symbol at the k^{th} sampling instant
- And the term $\sum_{\substack{n=0\\n\neq k}}^{\infty} I_n x_{k-n} + z'_k$ represents the ISI and AWGN variable at the same sampling instant

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Criterion for No ISI - The Nyquist criterion

- Assume $C(f)=1$ for $|f| \le W$ (channel response)
- The pulse x(t) is such that the frequency response is $X(f)=|G(f)|^2$ where

$$x(t) = \int_{-W}^{W} X(f) e^{j2\pi ft} df$$

- We want to determine the spectral properties of the pulse x(t) and hence the transmitted pulse g(t) that results in no ISI

Criterion for No ISI - The Nyquist criterion …

- For $$y_k = I_k + \sum_{\substack{n=0 \\ n \neq k}}^{\infty} I_n x_{k-n} + z'_k \ ; \quad k = 0,1,2$$
- The condition for No ISI is given by

$$x(t = kT) = x(kT) = x_k = \begin{cases} 1 & \mathbf{k = 0} \\ 0 & k \neq 0 \end{cases}$$

- The Nyquist pulse-shaping criterion or the Nyquist condition for zero ISI is stated as follows:
- *A necessary & sufficient condition for x(t) to satisfy the above condition is that the Fourier transform of x(nT)↔X(f) should satisfy*

$$\sum_{m=-\infty}^{\infty} X(f + \frac{m}{T}) = T$$

Criterion for No ISI - The Nyquist criterion …

- Proof: Since X(*f*) is the Fourier transform of x(t), in general it is given by

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

- At the sampling instant t=nT, this becomes

$$x(nT) = \int_{-\infty}^{\infty} X(f) e^{j2\pi fnT} df$$

- Breaking up the integral into parts covering the infinite range of 1/T, we obtain

$$x(nT) = \sum_{m=-\infty}^{\infty} \int_{\frac{2m-1}{2T}}^{\frac{2m+1}{2T}} X(f) e^{j2\pi fnT} df = \sum_{m=-\infty}^{\infty} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} X(f+\frac{m}{T}) e^{j2\pi fnT} df$$

$$= \int_{-\infty}^{\infty} \left[\sum_{m=-\infty}^{\infty} X(f+\frac{m}{T}) \right] e^{j2\pi fnT} df = \int_{-\infty}^{\infty} B(f) e^{j2\pi fnT} df$$

Criterion for No ISI - The Nyquist criterion …

- Where $B(f) = \sum_{m=-\infty}^{\infty} X(f + \frac{m}{T})$
- B(f) is periodic with period 1/T and therefore can be expanded in terms of its Fourier series coefficients $\{b_n\}$ as

$$B(f) = \sum_{n} b_n \, e^{j2\pi nfT}$$

- Where $b_n = T \int B(f) e^{-j2\pi nfT} df$
- Note that $b_n = Tx(-nT)$
- Therefore $b_n = \begin{cases} T & n = 0 \\ 0 & n \neq 0 \end{cases}$

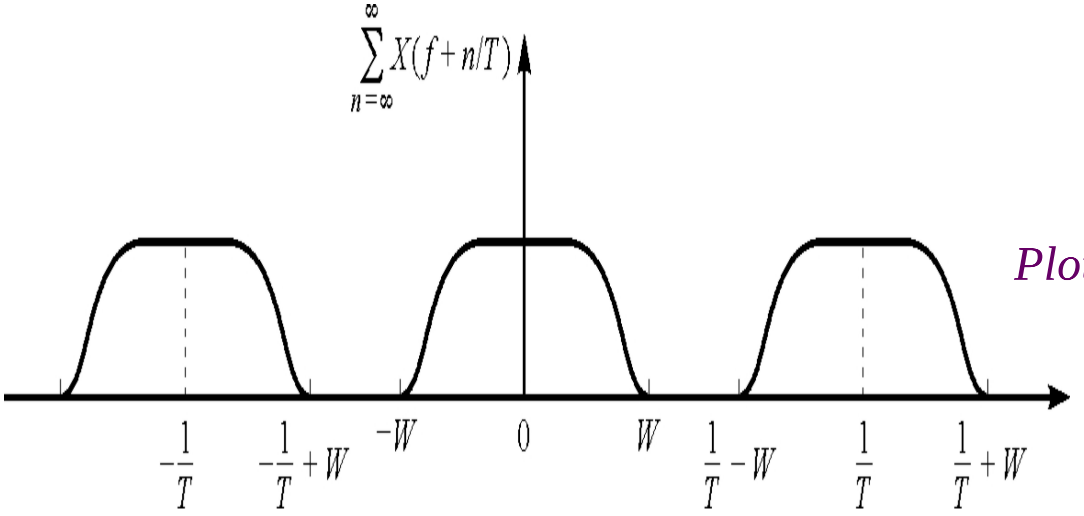
Criterion for No ISI - The Nyquist criterion …

- Which is the necessary condition that upon substituting in the expression for B(f) yields

$$B(f) = T \quad \text{or equivalently} \quad B(f) = \sum X(f + \frac{m}{T}) = T$$

Criterion for No ISI - The Nyquist criterion …

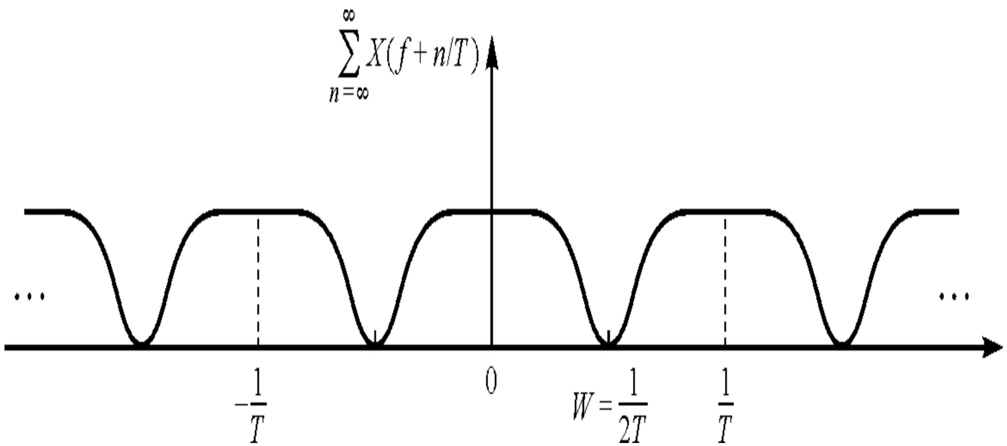
- Consider the different situations in terms of the relative values of T with respect to the signal bandwidth
    1. When T < 1/2W or 1/T > 2W, the bands of B(*f*) are *non-overlapping* and hence, *no choice of X(f) make B(f)=T*



Plot of B(f) for the case T < 1/2W

Criterion for No ISI - The Nyquist criterion …

2. When T = 1/2w, this gives the smallest value of T for which the ISI becomes zero



Plot of *B*(*f*) for the case *T* = 1/2*W*

- Here, $X(f)=\begin{cases} T & |f|<W \\ 0 & otherwise \end{cases}$ which corresponds to the pulse $x(t)=\frac{\sin(\pi t/T)}{\pi t/T}=sinc(\pi t/T)$

Criterion for No ISI - The Nyquist criterion …

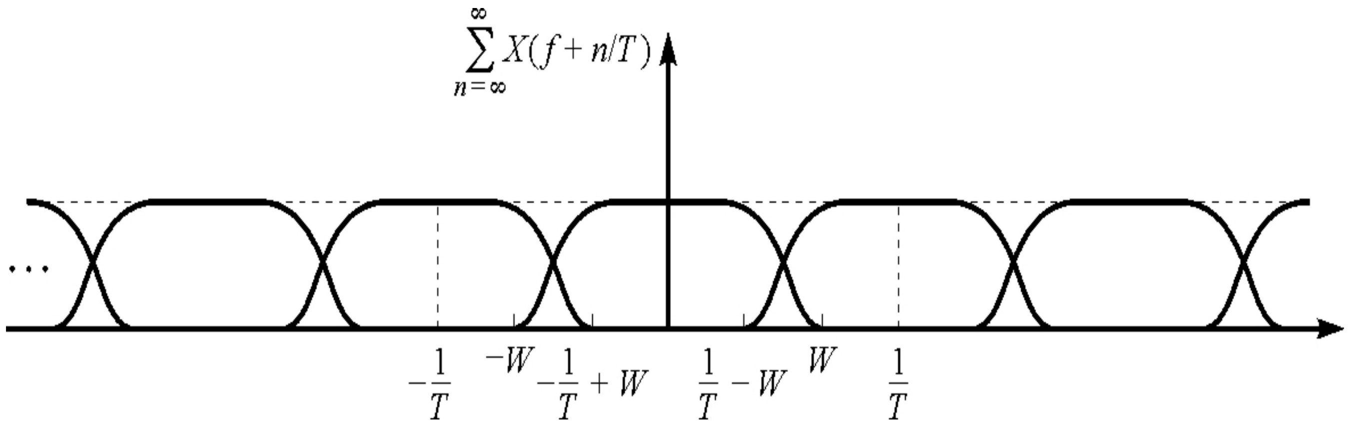
- Note that this function is *non-causal* & hence *non-realizable*
- To make it causal we use the *delayed version* of it, i.e.,

$$sinc\left[\frac{\pi(t-t_0)}{T}\right]$$

- Such that *x(t)* becomes essentially zero for t < 0
- Note that this function decays at the rate of 1/t

Criterion for No ISI - The Nyquist criterion …

3. When T > 1/2w, there are numerous choices of X(*f*) so that B(*f*) = T



Plot of B(f) for the case T > 1/2W

Criterion for No ISI - The Nyquist criterion …

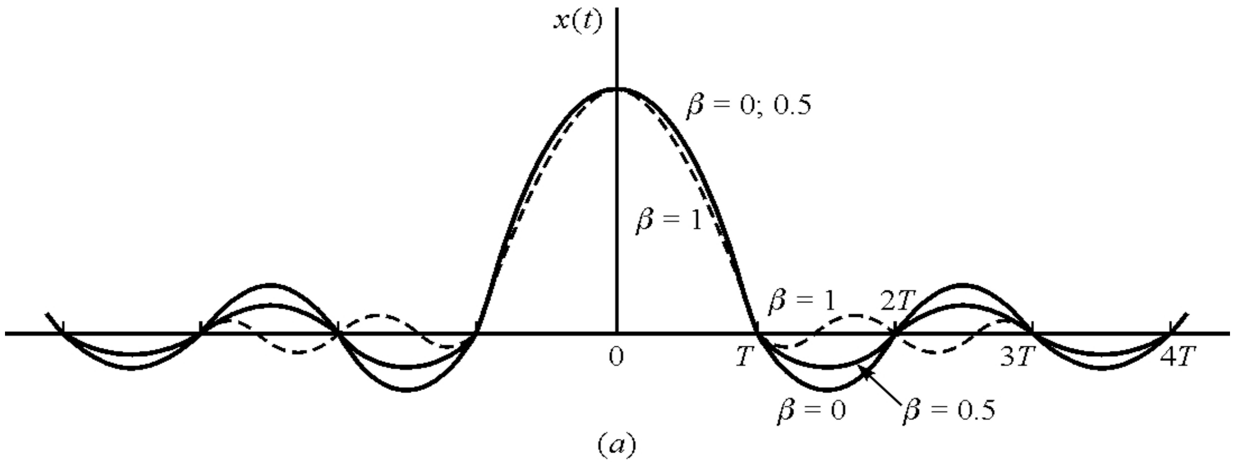
- For T > 1/2W, the *raised cosine function* has the desired spectral properties

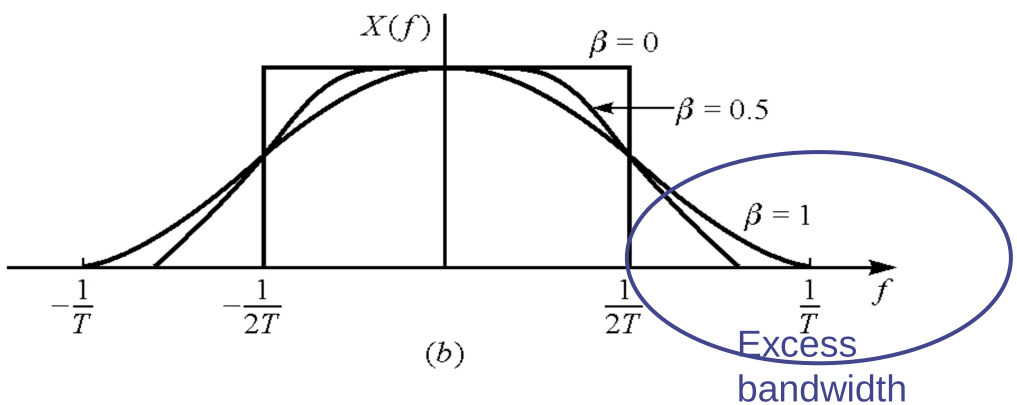
$$X_{rc}(f) = \begin{cases} T & 0 \le |f| \le \frac{1-\beta}{2T} \\ \frac{T}{2}\left(1 + \cos\left[\frac{\pi T}{\beta}\left(|f| - \frac{1-\beta}{2T}\right)\right]\right): & \frac{1-\beta}{2T} \le |f| \le \frac{1+\beta}{2T} \\ 0 & otherwise \end{cases}$$

- β is called the *roll-off factor* and has values in the range $0 \le \beta \le 1$

Criterion for No ISI - The Nyquist criterion …

- Plots of this function with β as a parameter are shown below, both in the frequency and time domains





Pulses having a raised cosine spectrum

Criterion for No ISI - The Nyquist criterion …

- The bandwidth occupied by the signal beyond the Nyquist frequency 1/2T is called *excess bandwidth* and is usually expressed as a percentage of the Nyquist frequency
- Thus for β = 0.5, the excess bandwidth is 50%; while it will be 100% for β = 1
- The pulse having the raised cosine spectrum is given by

$$x(t) = \frac{\sin(\pi t/T)}{\pi t/T}\,\frac{\cos(\pi\beta t/T)}{1-4\beta^2 t^2/T^2}$$

$$= \operatorname{sinc}(\pi t/T)\,\frac{\cos(\pi\beta t/T)}{1-4\beta^2 t^2/T^2}$$

Criterion for No ISI - The Nyquist criterion …

- $x(t)$ is normalized such that $x(0) = 1$
- Note that for β=0 the pulse reduces to the sinc function and the symbol rate is $1/T = 2w$
- When β=1, the symbol rate is $1/T = w$

- The tails of $x(t)$ decay as $1/t^3$ for $\beta > 0$ and mistiming error in sampling leads to a series of ISI components that converge to a finite value
- Also, since the raised cosine spectrum is smooth, it is possible to *design practical filters* for both the transmitter and receiver that approximate the desired frequency response

Criterion for No ISI - The Nyquist criterion …

- In the case where the channel is ideal C(*f*) = 1 for |*f* | ≤ w

$$X_{rc} = G_T(f)G_R(f)$$

- In the case where the receiver filter is matched to the transmitter filter

$$X_{rc} = \left|G_T\right|^2 \quad since \quad G_R(f) = G_T^*(f)$$

$$G_T(f) = \sqrt{\left|X_{rc}\right|}\, e^{-j2\pi f t_0}$$

- Where t_0 is some nominal delay that is required to insure physical realizability of the filter

Overview

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Controlled ISI

- We have seen that for *zero ISI* $1/T < 2W$ - the Nyquist rate
- Relaxing the condition of zero ISI, we can achieve a transmission rate of 2W symbols/s
- This can be achieved through allowing a controlled amount of ISI
- Allow a *non-zero value of ISI at one time* instant by permitting *one additional non-zero* value in the samples $\{x(nT)\}$

Controlled ISI …

- Since the ISI we introduce is deterministic we can take this into account at the receiver
- One possible special case specified by the samples

$$x\left(\frac{n}{2w}\right) = x(nT) = \begin{cases} 1 & (n = 0, 1) \\ 0 & (otherwise) \end{cases}$$

- Or alternatively, $b_n = \begin{cases} T & (n = 0, -1) \\ 0 & (otherwise) \end{cases}$

- Which yields

$$B(f) = T + Te^{-j2\pi fT}$$

Controlled ISI …

- Note: This relationship is impossible to satisfy for T < 1/2W
- However, for T =1/2 W, we obtain

$$X(f) = \begin{cases} \frac{1}{2W}(1+e^{-j\pi f/W}) & (|f|<W) \\ 0 & (otherwise) \end{cases}$$

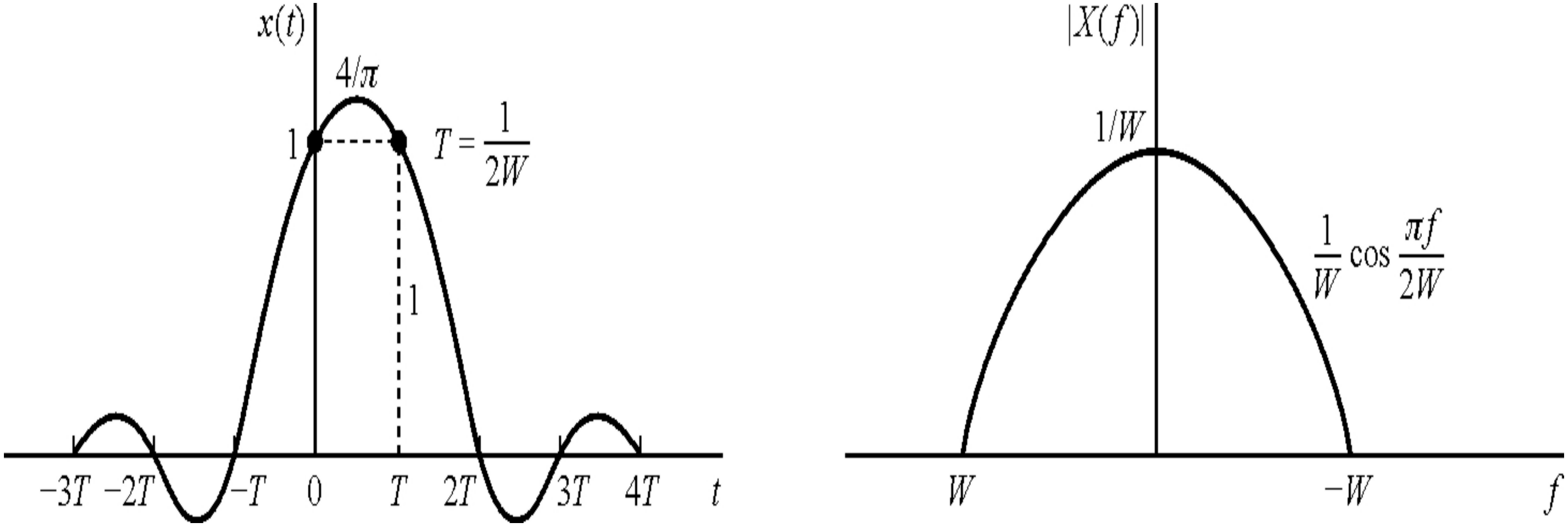
$$= \begin{cases} \frac{1}{W} e^{-j\pi f/2W} \cos\frac{\pi f}{2W} & (|f|<W) \\ 0 & (otherwise) \end{cases}$$

- Thus

$$x(t) = sinc(2\pi Wt) + sin[2\pi(Wt - \frac{1}{2})]$$

Controlled ISI …

- This pulse is called a *duobinary pulse* and illustrated below



Time-domain and frequency-domain characteristics of a duobinary signal

- Note that the *spectrum decays to zero* smoothly, which means that the corresponding filter is physically realizable to achieve a symbol rate of 2W

Controlled ISI …

- Another special case that leads to approximately realizable filter at both the transmitter and receiver is specified by

$$x\left(\frac{n}{2W}\right) = x(nT) = \begin{cases} 1 & (n=-1) \\ -1 & (n=1) \\ 0 & (otherwise) \end{cases}$$

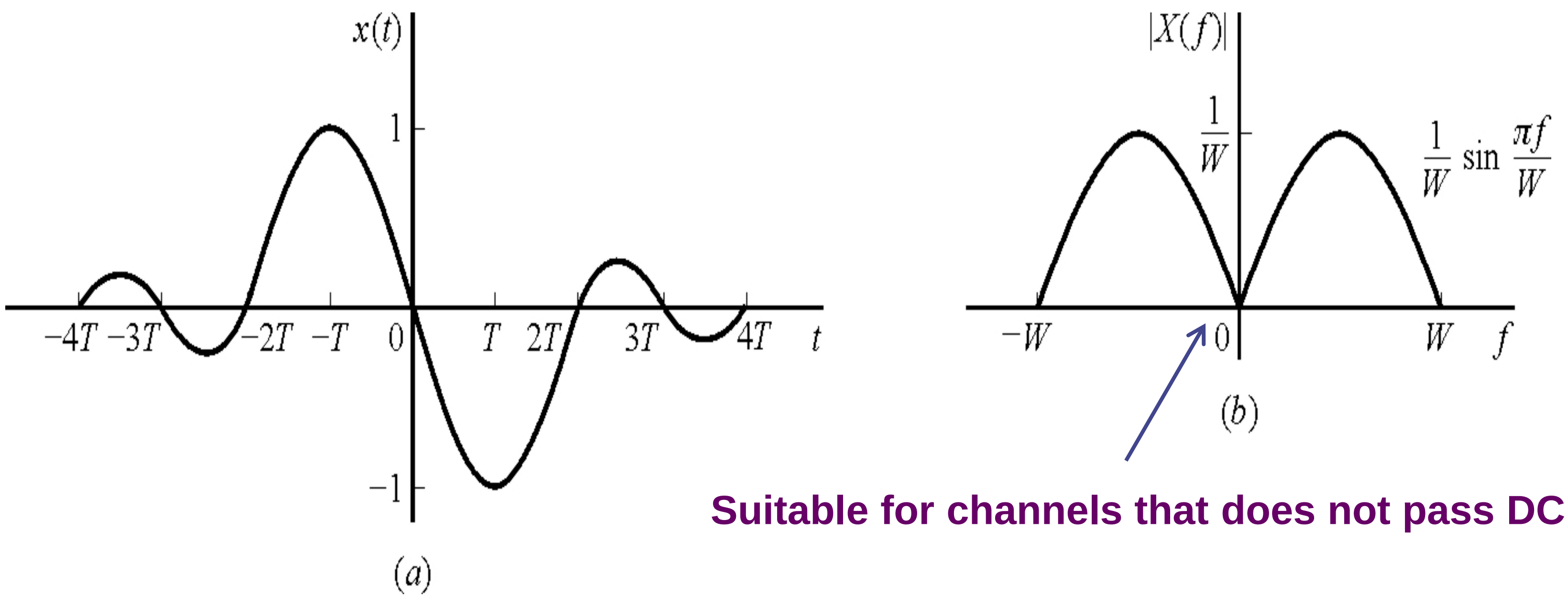
- Whose corresponding pulse is given by

$$x(t) = sinc\left(\frac{\pi(t+T)}{T}\right) - sinc\left(\frac{\pi(t-T)}{T}\right)$$

Controlled ISI …

- And its spectrum is

$$X(f) = \begin{cases} \dfrac{1}{2W}\left(e^{j\pi\pi f/} - e^{-j\pi\pi f/}\right) = \dfrac{j}{W}\, sin\, \dfrac{\pi f}{W}; & |f| \le W \\ 0 & |f| > W \end{cases}$$



Time-domain and frequency-domain characteristics of a modified duobinary signal

Controlled ISI …

- In general, the class of band-limited signal having the form

$$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) sinc\left(2\pi W\left(t - \frac{n}{2W}\right)\right)$$

- And the corresponding spectra is

$$X(f) = \begin{cases} \dfrac{1}{2W} \displaystyle\sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) e^{-j\pi\pi nf/} & |f| \leq W \\ 0 & (otherwise) \end{cases}$$

- Are called *partial response signals*, when controlled ISI is purposely introduced by selecting two or more non-zero samples from {*x(n/2W)*}

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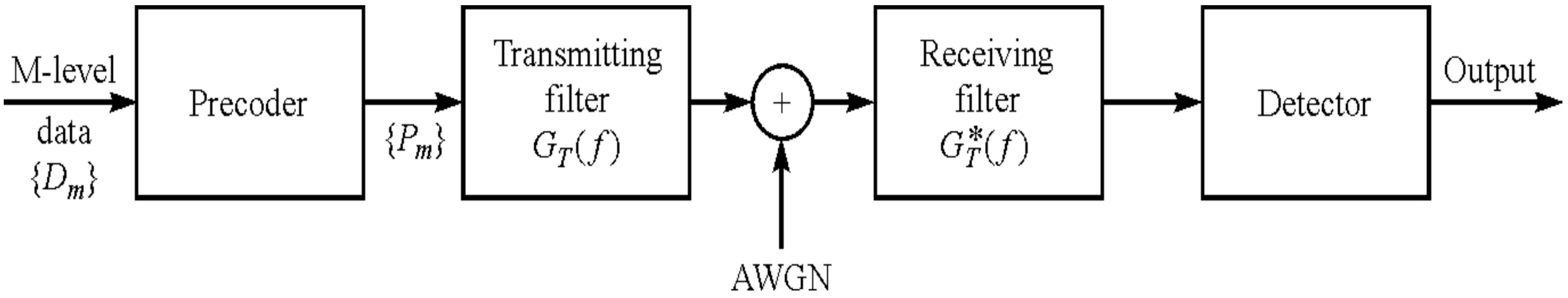
Signal Design with Distortion

- Assume channel frequency response $C(f)$ is known for $|f| \leq W$ and $C(f) = 0$ for $|f| \geq W$

- Criterion for optimization of filter responses $G_T(f)$ and $G_R(f)$ is the *maximization* of the SNR at the output of the demodulation filter



Block diagram of modulator and demodulator for partial-response signals

Controlled ISI …

$$G_T(f)C(f)G_R(f) = X_d(f)e^{-j2\pi f t_0} \quad for\ |f| \le W$$

- X_d (*f*) - Desired frequency response at the output and t_0 is the time delay necessary to ensure physical realizability
- X_d (*f*) may be selected to yield either zero ISI or controlled ISI at the sampling instants

- For zero ISI select X_d (*f*) = X_{rc} (*f*) – raised cosine spectrum with arbitrary roll-off factor β
- The noise output of the demodulator filter is

$$v(t) = \int_{-\infty}^{\infty} n(t-\tau)\, g_R\, d\tau$$

- Where n(t) is zero-mean Gaussian and $\phi_{vv}(f) = \phi_{nn}(f).|G_R(f)|^2$

Signal Design with Distortion …

- If we consider binary PAM transmission, the sampled output of the matched filter can be expressed as

$$y_m = I_m + v_m = \pm d + v_m$$

$$\sigma_v^2 = \int_{-\infty}^{\infty} \phi_{nn}(f) \left| G_R(f) \right|^2 df$$

- And the probability of error is

$$p_e = \frac{2}{\sqrt{2\pi}} \int_{\frac{d}{\sigma_v}} e^{-y^2} \, dy = Q\left(\sqrt{\frac{d^2}{\sigma^2}} \right)$$

Signal Design with Distortion …

- P_e is minimized by maximizing the SNR = $\dfrac{d^2}{\sigma^2}$
- But the average power is given as

$$P_{av} = \frac{E(I^2)}{T}\int_{-\infty}^{\infty} g_T^2(t)\,dt = \frac{d^2}{T}\int_{-\infty}^{\infty} g_T^2(t)\,dt$$

$$\frac{1}{d^2} = \frac{1}{P_{av}T}\int_{-\infty}^{\infty} \left|G_T(f)\right|^2 df$$

Signal Design with Distortion …

- Where $G_T(f)$ must be chosen to satisfy the zero ISI condition such that

$$|G_T(f)| = \begin{cases} \dfrac{|X_{rc}(f)|}{|C(f)||G_R(f)|} & |f| \le W \\ 0 & (otherwise) \end{cases}$$

$$\frac{1}{d^2} = \frac{1}{P_{av}T}\int_{-\infty}^{\infty}\frac{|X_{rc}(f)|^2}{|C(f)|^2|G_R(f)|^2}df$$

- The noise–to-signal ratio to be minimized with respect to $G_R(f)$ within the band of frequencies W

$$\frac{\sigma_v^2}{d^2} = \frac{1}{P_{av}T}\int_{-W}^{W}\phi_{nn}(f)|G_R(f)|^2\int_{-W}^{W}\frac{|X_{rc}(f)|^2}{|C(f)|^2|G_R(f)|^2}df$$