Chapter 3: Characterization of Communication Signals and Systems

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Spectral Characteristics of Digitally Modulated Signals

- Generally, the available channel bandwidth is ***limited***
- In the selection of the modulation methods, we
  - Need to determine the spectral content of digitally modulated signals
  - Helps to take the effect of the BW constraint into account
- A digitally modulated signal is a stochastic process since the information sequences are random
- Need to determined power spectral density (PSD) of these processes
- From PSD one can find the channel bandwidth required to transmit the information-bearing signals

Spectral Characteristics of Digitally …

- Considering linearly modulated band-pass signal given by

$$s(t) = Re\left[v(t)\, e^{j2\pi f_c t}\right]$$

- Where $\nu(t)$ is the equivalent low-pass signal
- Autocorrelation function of s(t) is

$$\phi_{ss}(\tau) = Re\left[\phi_{vv}(\tau)e^{j2\pi f_c \tau}\right]$$

- And its Fourier transform yields the desired expression for the power density spectrum $\Phi_{ss}(f)$ as

$$\Phi_{ss}(f) = \frac{1}{2}\left[\Phi_{vv}(f - f_c) + \Phi_{vv}(-f - f_c)\right]$$

- Where $\Phi_{vv}(f)$ is the power density spectrum of $\nu(t)$

Spectral Characteristics of Digitally …

- To obtain the spectral characteristics of the bandpass signal *s(t)*, it suffices to determine the autocorrelation function and power spectral density of the equivalent low-pass signal $\nu(t)$

- Consider a *linear digital modulation* method for which $\nu(t)$ is represented in the general form

$$\boxed{v(t) = \sum_{n=-\infty}^{n=\infty} I_n \, g(t-nT)}$$

- Where:
  - $\{I_n\}$ represents the sequence of symbols resulting from mapping k-bit blocks into corresponding points
  - $1/T = R/k$ symbols/s is the transmission rate

Spectral Characteristics of Digitally …

- The autocorrelation function of ν(t) is

$$\boxed{\begin{aligned}\phi_{vv}(t+\tau,t) &= \frac{1}{2}E\left[v^*(t)v(t+\tau)\right] \\ &= \frac{1}{2}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}E\left[I_n^* I_m\right]g^*(t-nT)g(t+\tau-mT)\end{aligned}}$$

- Assuming the sequence of information symbols $\{I_n\}$ is wide-sense stationary with mean μ_i and autocorrelation function

$$\boxed{\phi_{ii}(m) = \frac{1}{2}E\left[I_n^* I_{n+m}\right]}$$

- Then the autocorrelation of ν(t) will be

$$\boxed{\begin{aligned}\phi_{vv}(t+\tau,t) &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\phi_{ii}(m-n)g^*(t-nT)g(t+\tau-mT) \\ &= \sum_{m=-\infty}^{\infty}\phi_{ii}(m)\sum_{n=-\infty}^{\infty}g^*(t-nT)g(t+\tau-nT-mT)\end{aligned}}$$

Spectral Characteristics of Digitally …

- The 2nd summation is periodic with T
- Thus the autocorrelation function is also periodic, i.e.,

$$\boxed{\phi_{vv}(t+\tau+T, t+T) = \phi_{vv}(t+\tau; t);}$$

- Further the mean of $\nu(t)$

$$\boxed{E[v(t)] = \mu_i \sum g(t-nT)}$$ is also periodic with period T

- $\nu(t)$ is a stochastic process having a periodic mean and autocorrelation function
  - *Called* a ***cyclostationary*** or a ***periodically stationary*** process
- To compute the power spectral density of $\phi_{\nu\nu}(t+\tau,t)$, we must eliminate the dependence on the t variable by averaging over a single period

Spectral Characteristics of Digitally …

$$\bar{\phi}_{vv}(\tau)=\frac{1}{T}\int_{-T/2}^{T/2}\phi_{vv}(t+\tau;t)\,dt$$
$$=\sum_{m=-\infty}^{\infty}\phi_{ii}(m)\sum_{n=-\infty}^{\infty}\frac{1}{T}\int_{-T/2}^{T/2}g^{*}(t-nT)g(t+\tau-nT-mT)dt$$
$$=\sum_{m=-\infty}^{\infty}\phi_{ii}(m)\sum_{n=-\infty}^{\infty}\frac{1}{T}\int_{-T/2-nT}^{T/2+nT}g^{*}(t-nT)g(t+\tau-mT)dt$$

- In the above expression, the integral can be interpreted as the time autocorrelation of the function *g(t)*

$$\phi_{gg}(\tau)=\int_{-\infty}^{\infty}g^{*}(t)\,g(t+\tau)\,dt$$

- So that

$$\bar{\phi}_{vv}(\tau)=\frac{1}{T}\sum_{m=-\infty}^{\infty}\phi_{ii}(m)\,\phi_{gg}(\tau-mT)$$

Spectral Characteristics of Digitally …

- The average power spectral density of ν(t) is the Fourier transform of the average of its autocorrelation which may be expressed as

$$\boxed{\Phi_{vv}(f) = \frac{1}{T}\left|G(f)\right|^2 \Phi_{ii}(f)}$$

- Where G(f) is the Fourier transform of g(t) and

$$\boxed{\Phi_{ii}(f) = \sum_{m=-\infty}^{\infty} \phi_{ii}(m) e^{-j2\pi fmT}}$$

- Note that the power spectral density depends on the spectral characteristic of the pulse *g(t)* and the information sequence $\{I_n\}$

Spectral Characteristics of Digitally …

- Or, the spectral characteristics of $v(t)$ can be controlled by the choice of *g(t)* and the correlation characteristics of the information sequence
- Note also that $\Phi_{ii}(f)$ is related to the autocorrelation $\phi_{ii}(m)$ in the form of an ***exponential Fourier series*** with $\phi_{ii}(m)$ as the Fourier coefficient such that

$$\boxed{\phi_{ii}(m) = T \int_{-1/2T}^{1/2T} \Phi_{ii}(f) e^{j2\pi fmT}\, df}$$

- Consider the information symbols in the sequence are real and mutually uncorrelated such that

$$\boxed{\phi_{ii}(m) = \begin{cases} \sigma_i^2 + \mu_i^2 & (m = 0) \\ \mu_i^2 & (m \neq 0) \end{cases}}$$

Spectral Characteristics of Digitally …

- Where σ_i^2 is the variance of the information sequence; then

$$\Phi_{ii}(f) = \sigma_i^2 + \mu_i^2 \sum_{m=-\infty}^{\infty} e^{-j2\pi fmT}$$

- Which is periodic with period 1/T
- The above can be viewed as the exponential Fourier series of a periodic train of impulses each with an area of 1/T; i.e,

$$\Phi_{ii}(f) = \sigma_i^2 + \frac{\mu_i^2}{T} \sum_{m=-\infty}^{\infty} \delta(f - \frac{m}{T})$$

- And substituting this in the expression for $\Phi_{vv}(f)$

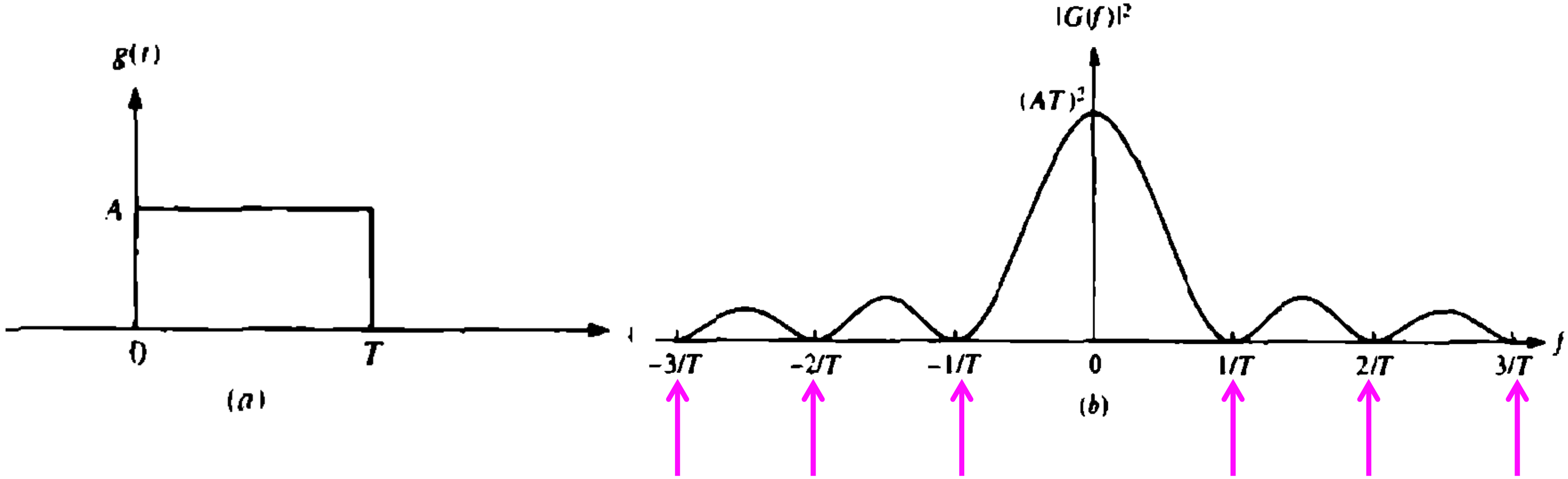
$$\Phi_{vv}(f) = \frac{\sigma_i^2}{T} \left|G(f)\right|^2 + \frac{\mu_i^2}{T} \sum \left|G\left(\frac{m}{T}\right)\right|^2 \delta\left(f - \frac{m}{T}\right)$$

Spectral Characteristics of Digitally …

- The first term is a continuous spectrum and its shape depends on the spectral characteristics of signal pulse *g(t)*
- The second expression contains discrete frequency components spaced 1/T apart in frequency
  - Each spectral line has power proportional to $|G(f)|^2$ evaluated at $f=m/T$
- If the information sequences have zero mean the discrete frequency components will vanish, i.e., $\mu_i = 0$
- This property is most desirable for digital modulation and can be achieved when the information sequences are ***equally likely*** & ***symmetrically positioned*** in a complex plane

Spectral Characteristics of Digitally …

- Example 1: Consider g(t) to be a rectangular pulse as shown in the figure below with Fourier transform $|G(f)|$



- Rectangular pulse and its energy density spectrum

$$\boxed{G(f) = AT\frac{\sin \pi f T}{\pi f T}} \quad \text{and} \quad \boxed{|G(f)|^2 = (AT)^2\left(\frac{\sin \pi f T}{\pi f T}\right)^2}$$

Spectral Characteristics of Digitally …

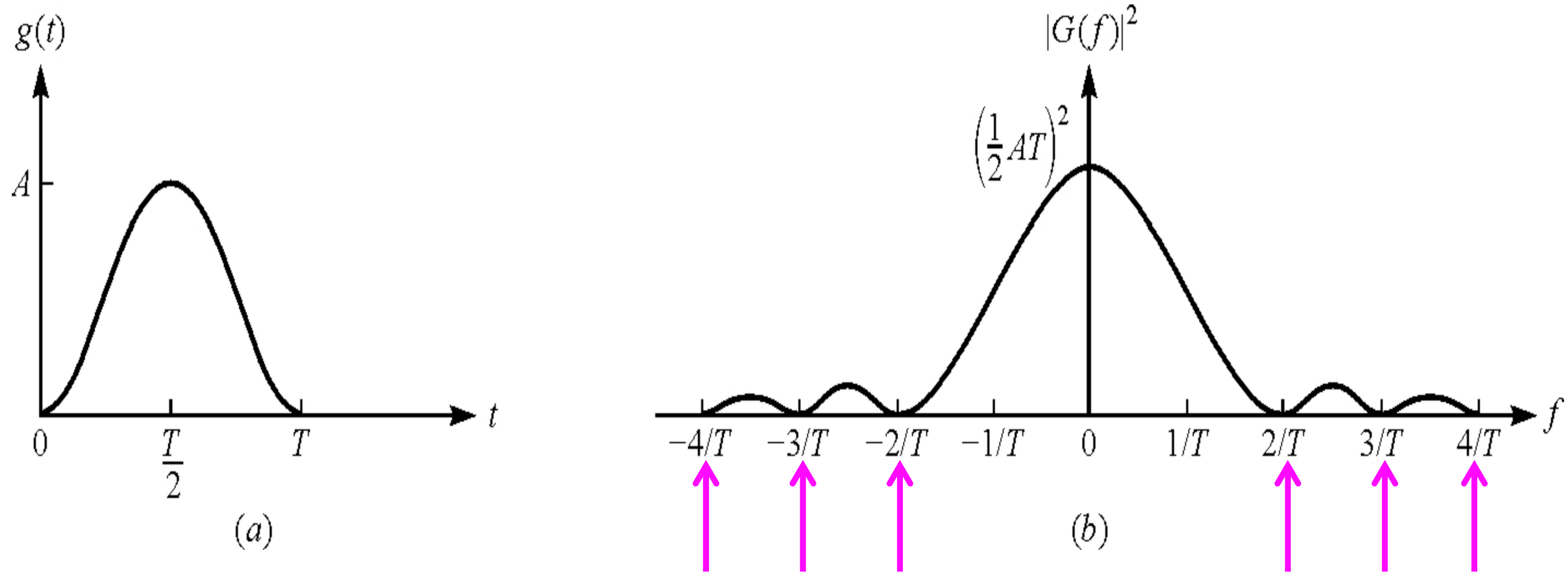
- It contains zeros at multiples of 1/T in frequency and it also decays inversely as the square of the frequency variable
- As a result, all but one of the discrete spectral components in $\Phi_{vv}(f)$ vanishes
- Thus, upon substitution for $|G(f)|$ from above, we get

$$\boxed{\Phi_{vv}(f) = \sigma_i^2 A^2 T \left(\frac{\sin \pi f T}{\pi f T} \right)^2 + A^2 \mu_i^2 \delta(f)}$$

Spectral Characteristics of Digitally …

- Example 2: Consider the case where g(t) is a raised cosine pulse

$$\boxed{g(t) = \frac{A}{2}\left[1 + \cos\frac{2\pi}{T}\left(t - \frac{T}{2}\right)\right] \qquad for \quad 0 \le t \le T}$$



Raised cosine pulse and its energy density spectrum

Spectral Characteristics of Digitally …

- Its Fourier transform is given as

$$\boxed{G(f) = \frac{AT}{2}\frac{\sin \pi f T}{\pi f T(1 - f^2T^2)}e^{-j\pi\pi f}}$$

- Note the spectrum has zeros at $f = n/T$; $n = \underline{+}2, \underline{+}3, \underline{+}4$…….
- Hence, all the spectral components, except those at zero and $f = \underline{+}1/T$ vanish
- Compared to that of the rectangular pulse, the spectrum has a broader main lobe but the tails decay inversely as f^6

Spectral Characteristics of Digitally …

- Spectrum can also be shaped by operations performed on the input information sequence
- Example 3: Consider a binary sequence $\{b_n\}$ from which we form the symbol

$$I_n = b_n + b_{n-1}$$

- Where the $\{b_n\}$ are assumed to be uncorrelated random variables, each having zero mean and unit variance
- The autocorrelation of the sequence $\{I_n\}$ is

$$\boxed{\begin{aligned}\phi_{ii}(m) &= E\left(I_n I_{n-m}\right) \\ &= \begin{cases} 2 & (m = 0) \\ 1 & (m = \pm 1) \\ 0 & (\text{Otherwise}) \end{cases}\end{aligned}}$$

Spectral Characteristics of Digitally …

- The power spectral density of the input sequence is

$$\boxed{\Phi_{ii}(f) = 2(1 + \cos 2\pi fT) = 4\cos^2 \pi fT}$$

- Then, the corresponding power spectral density of the (low-pass equivalent) modulated signal becomes

$$\boxed{\Phi_{vv}(f) = \frac{4}{T}\left|G(f)\right|^2 \cos^2 \pi fT}$$