Chapter 4

Finite Difference Methods

4.1 Introduction

Many of the basic numerical solution schemes for partial differential equations can be fit into two broad themes. The first, to be developed in the present chapter, are the *finite difference methods*, and the second category are the *finite element methods*, which will be the topic of the next chapter.

The finite-difference methods (FDM) are numerical methods for approximating the solutions to differential equations using *finite difference equations* to approximate *derivatives*.

The finite difference approximations for derivatives are one of the simplest and of the oldest methods to solve differential equations. It was already known by L. Euler (1707-1783), in one dimension of space and was probably extended to dimension two by C. Runge (1856-1927). The FDM was first developed by A. Thom in the 1920s under the title “the method of square” to solve nonlinear hydrodynamic equations.

The finite difference techniques are based upon the approximations that permit replacing differential equations by finite difference equations. These finite difference approximations are algebraic in form, and the solutions are related to grid points (Fig. 4.1). Thus, a finite difference solution basically involves three steps:

1. Dividing the solution into grids of nodes.
2. Approximating the given differential equation by finite difference equivalence that relates the solutions to grid points.
3. Solving the difference equations subject to the prescribed boundary conditions and/or initial conditions.

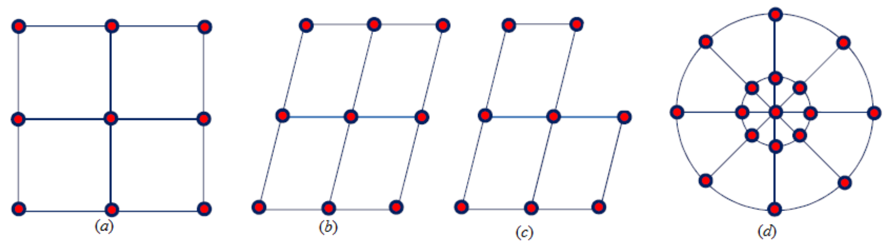


Figure 4.1: Common two-dimensional grids: (a) rectangular, (b) skew, (c) triangular, and (d) circular grids.

4.2 Finite Difference Schemes

Given a function $f(x)$ shown in Fig. 4.2, we can approximate its derivative, slope or tangent at P by the slope of the arcs PB, PA, or AB, for obtaining the *forward-difference*, *backward-difference*, and *central-difference* formulas respectively.

- **Forward-difference formula**:

$$f'(x_o) \approx \frac{f(x_o + \Delta x) - f(x_o)}{\Delta x} \tag{4.1}$$

- **Backward-difference formula**:

$$f'(x_o) \approx \frac{f(x_o) - f(x_o - \Delta x)}{\Delta x} \tag{4.2}$$

- **Central-difference formula**:

$$f'(x_o) \approx \frac{f(x_o + \Delta x) - f(x_o - \Delta x)}{2\Delta x} \tag{4.3}$$

The approach used for obtaining the above finite difference equations is Taylor's series:

$$f(x_o + \Delta x) = f(x_o) + \Delta x f'(x_o) + \frac{(\Delta x)^2}{2!} f''(x_o) + \frac{(\Delta x)^3}{3!} f'''(x_o) + \mathcal{O}(\Delta x)^4 \tag{4.4}$$

$$f(x_o - \Delta x) = f(x_o) - \Delta x f'(x_o) + \frac{(\Delta x)^2}{2!} f''(x_o) - \frac{(\Delta x)^3}{3!} f'''(x_o) + \mathcal{O}(\Delta x)^4 \tag{4.5}$$

where $\mathcal{O}(\Delta x)^4$ is the error introduced by truncating the series.

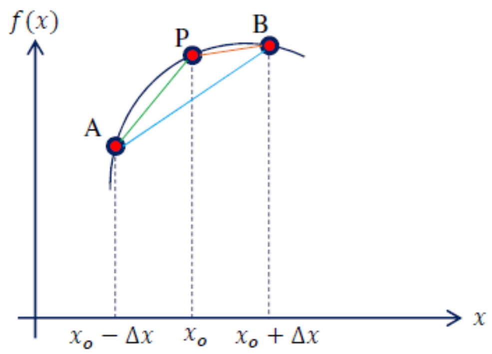


Figure 4.2: Estimates for the derivative of $f(x)$ at P using forward, backward, and central differences.

Subtracting (4.5) from (4.4), we obtain

$$f(x_o + \Delta x) - f(x_o - \Delta x) = 2\Delta x f'(x_o) + \mathcal{O}(\Delta x)^3$$

which could be re-written as

$$f'(x_o) = \frac{f(x_o + \Delta x) - f(x_o - \Delta x)}{2\Delta x} + \mathcal{O}(\Delta x)^2$$

which is the central-difference formula. Note that the truncation error is is of order two for the central-difference.

The forward-difference and backward-difference formulas could be obtained by re-arranging (4.4) and (4.5) respectively, and we have

$$f'(x_o) = \frac{f(x_o + \Delta x) - f(x_o)}{\Delta x} + \mathcal{O}(\Delta x)$$

for forward-difference, and

$$f'(x_o) = \frac{f(x_o) - f(x_o - \Delta x)}{\Delta x} + \mathcal{O}(\Delta x)$$

for backward-difference. The truncation errors for forward- and backward-differences is of order one.

Upon adding (4.4) and (4.5)

$$f(x_o + \Delta x) + f(x_o - \Delta x) = 2f(x_o) + (\Delta x)^2 f''(x_0) + \mathcal{O}(\Delta x)^4$$

and rearranging, we obtain

$$f''(x_o) = \frac{f(x_o + \Delta x) - 2f(x_o) + f(x_o - \Delta x)}{(\Delta x)^2} + \mathcal{O}(\Delta x)^2 \qquad (4.6)$$

Higher order finite difference approximations can be obtained by taking more terms in Taylor series expansion.

To apply the difference method to find the solution of a function $\Phi(x,t)$, we divide the solution region in $x-t$ plane into equal rectangles or meshes of sides Δx and Δt. We let the coordinates (x,t) of a typical grid point or node be

$$x = i\Delta x, \quad i = 0, 1, 2, \ldots$$
$$t = j\Delta t, \quad j = 0, 1, 2, \ldots$$

and the value of Φ at P be

$$\Phi_P = \Phi(i\Delta x, j\Delta t) = \Phi(i,j)$$

With this notation, the central difference approximations of the derivatives of Φ at the (i,j)th node are (Fig. 4.3)

$$\Phi_x|_{i,j} \approx \frac{\Phi(i+1,j) - \Phi(i-1,j)}{2\Delta x} \tag{4.7}$$

$$\Phi_t|_{i,j} \approx \frac{\Phi(i,j+1) - \Phi(i,j-1)}{2\Delta t} \tag{4.8}$$

$$\Phi_{xx}|_{i,j} \approx \frac{\Phi(i+1,j) - 2\Phi(i,j) + \Phi(i-1,j)}{(\Delta x)^2} \tag{4.9}$$

$$\Phi_{tt}|_{i,j} \approx \frac{\Phi(i,j+1) - 2\Phi(i,j) + \Phi(i,j-1)}{(\Delta t)^2} \tag{4.10}$$

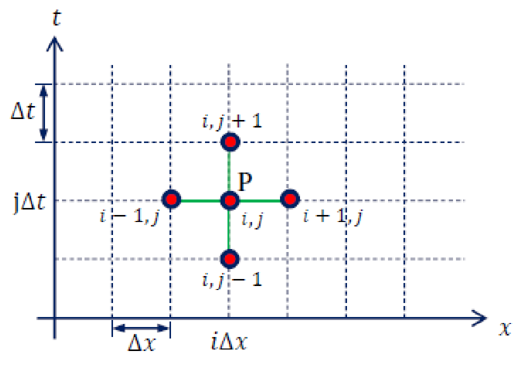
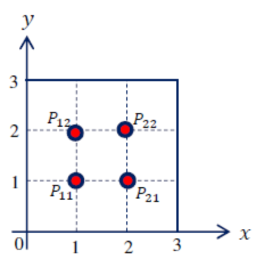


Figure 4.3: Finite difference mesh for two independent variable x and t.

Exercise 4.1 Verify the following finite difference approximations for Φ_x and Φ_{xx} in the table shown below, where FD =Forward Difference, BD = Backward Difference, and CD = Central Difference.

Derivative	**FD Approximation**	**Type**	**Error**
Φ_x	$\dfrac{\Phi_{i+1}-\Phi_i}{\Delta x}$	FD	$\mathcal{O}(\Delta x)$
	$\dfrac{\Phi_i-\Phi_{i-1}}{\Delta x}$	BD	$\mathcal{O}(\Delta x)$
	$\dfrac{\Phi_{i+1}-\Phi_{i-1}}{\Delta x}$	CD	$\mathcal{O}(\Delta x)^2$
	$\dfrac{-\Phi_{i+2}+4\Phi_{i+1}-3\Phi_i}{2\Delta x}$	FD	$\mathcal{O}(\Delta x)^2$
	$\dfrac{3\Phi_i-4\Phi_{i-1}+\Phi_{i-2}}{2\Delta x}$	BD	$\mathcal{O}(\Delta x)^2$
	$\dfrac{-\Phi_{i+2}+8\Phi_{i+1}-8\Phi_{i-1}+\Phi_{i-2}}{12\Delta x}$	CD	$\mathcal{O}(\Delta x)^4$
Φ_{xx}	$\dfrac{\Phi_{i+2}-2\Phi_{i+1}+\Phi_i}{(\Delta x)^2}$	FD	$\mathcal{O}(\Delta x)^2$
	$\dfrac{\Phi_i-2\Phi_{i-1}+\Phi_{i-2}}{(\Delta x)^2}$	BD	$\mathcal{O}(\Delta x)^2$
	$\dfrac{\Phi_{i+1}-2\Phi_i+\Phi_{i-1}}{(\Delta x)^2}$	CD	$\mathcal{O}(\Delta x)^2$
	$\dfrac{-\Phi_{i+2}+16\Phi_{i+1}-30\Phi_i+16\Phi_{i-1}-\Phi_{i-2}}{(\Delta x)^2}$	CD	$\mathcal{O}(\Delta x)^4$

Example 4.1 **Elliptic type**: Using the grid shown in the figure, compute the potentials at the four interior points for the following boundary conditions. (Note that the electrostatic potential satisfies Laplace's equation, $\nabla^2 V = 0$)



1. $V(1,0) = -80, V(2,0) = 400$ and $V = 0$ on the three edges of the boundary.
2. $V(x,0) = x^4, V(3,y) = 81 - 54y^2 + y^4, V(x,3) = x^4 - 54x^2 + 81, V(0,y) = y^4$.

Exercise 4.2 **Parabolic type**: Solve the diffusion equation

$$\Phi_{xx} = \Phi_t, \quad 0 \le x \le 1$$

subject to the boundary conditions

$$\Phi(0,t) = 0 = \Phi(1,t), \quad t > 0$$

and initial condition

$$\Phi(x,0) = 100$$

using the explicit and implicit (Crank and Nicholson) methods. Compare your results with the analytic solution at some selected points.

Exercise 4.3 **Hyperbolic type**: Solve the wave equation

$$\Phi_{tt} = \Phi_{xx}, \quad 0 < x < 1, t \geq 1$$

subject to the boundary conditions

$$\Phi(0,t) = 0 = \Phi(1,t), \quad t > 0$$

and initial conditions

$$\Phi(x,0) = \sin \pi x, \quad 0 < x < 1$$
$$\Phi_t(x,0) = 0, \quad 0 < x < 1$$

Compare your result with the analytic solution at some selected points.

Exercise 4.4 The wave equation

$$u^2 \Phi_{xx} = \Phi_{tt}$$

can be finite differenced as

$$\Phi_{i,j+1} = 2(1-r)\Phi_{i,j} + r[\Phi_{i+1,j} + \Phi_{i-1,j}] - \Phi_{i,j-1}$$

where r is the *aspect ratio* given by

$$r = \left(\frac{u\Delta t}{\Delta x}\right)^2$$

Use the von Neumann approach to determine the stability condition.