Chapter 1

Complex Variable

1.1 Introduction

The theory of functions of a complex variable, also called for brevity complex variables or complex analysis, is one of the beautiful as well as useful branches of mathematics that investigates functions of complex numbers.

Although originating in an atmosphere of mystery, suspicion and distrust, as evidenced by the terms imaginary and complex present in the literature, it was finally placed on a sound foundation in the 19th century through the efforts of Cauchy, Riemann, Weierstrass, Gauss, and other great mathematicians. It is useful in many branches of mathematics, including number theory and applied mathematics; as well as in physics, hydrodynamics, thermodynamics, and electrical engineering. In modern times, it has become very popular through a new boost from complex dynamics and the pictures of fractals produced by iterating holomorphic functions.

1.2 Complex Numbers

A *complex number* z can be defines as an ordered pair (x, y) of real numbers x, y

$$z \equiv (x, y) \tag{1.1}$$

we call x the *real part* of z and y the *imaginary part* of z and write

$$\text{Re } z \equiv x, \quad \text{Im } z \equiv y \tag{1.2}$$

Equality, addition and multiplication on complex numbers $z_1 = (x_1, y_1)$ and

$z_2 = (x_2, y_2)$ can be defined as

$$z_1 = z_1 \quad \text{iff} \quad x_1 = x_2 \quad \text{and} \quad y_1 = y_2 \tag{1.3}$$
$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2) \tag{1.4}$$
$$z_1 z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1) \tag{1.5}$$

1.2.1 Representation in the Form $z = x + iy$

A complex number whose imaginary part is zero is of the form $(x, 0)$. For such numbers:

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0)$$
$$(x_1, 0)(x_2, 0) = (x_1 x_2, 0)$$

as for real numbers. We identify $(x, 0)$ with the real number x and hence the complex number system is an *extension* of the real number system.

We denote the complex number $(0, 1)$ by i or j

$$i \equiv (0, 1) \tag{1.6}$$

We observe that $i^2 = (0, 1)(0, 1) = (-1, 0) = -1$. Hence

$$\boxed{i^2 = -1} \tag{1.7}$$

Furthermore, for every real y, we have

$$iy = (0, 1)(y, 0) = (0, y) \tag{1.8}$$

Combining with $x = (x, 0)$, we obtain

$$(x, y) = (x, 0) + (0, y) \tag{1.9}$$

we can therefore write every complex number $z = (x, y)$ as

$$\boxed{z = x + iy} \tag{1.10}$$

Let $z = x + iy$, then $x - iy$ is called the *complex conjugate* of z and is denoted by $\bar{z}$ or z^*.

1.2.2 Complex Plane and Polar Form

Since a complex number $x + iy$ can be considered as an ordered pair (x, y), we can represent such number by points in an xy plane called the *complex plane* or *Argand diagram.*

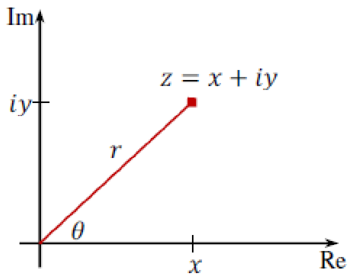

Figure 1.1: Complex plane.

From Figure 1.1, $x = r\cos\theta, y = r\sin\theta$ so that we have the *polar form* of the complex number as

$$z = x + iy = r(\cos\theta + i\sin\theta) \tag{1.11}$$

r is the amplitude, absolute value or *modulus* of z

$$|z| = r = \sqrt{x^2 + y^2} \tag{1.12}$$

θ is the *argument* of z

$$\theta = \arg z = \tan^{-1}\frac{y}{x} \tag{1.13}$$

For $z = 0$, $\arg z$ is not defined. For $z \neq 0$ it is determined only up to integer multiples of 2π. The value of θ that lies in the interval $-\pi < \theta \leq \pi$ is called the *principal value* of the argument of $z(\neq 0)$ and is denoted by Arg z. Thus by definition,

$$\theta = \text{Arg } z, \quad -\pi < \text{Arg } z \leq \pi \tag{1.14}$$

Example 1.1 Express $1 + i$ in polar form and determine the principal value of the argument. ◀

Making use of

$$e^w = 1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \cdots \tag{1.15}$$

and putting $w = i\theta$, we have

$$\begin{aligned} e^{i\theta} &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + -\frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\ &= \cos\theta + i\sin\theta \end{aligned}$$

hence a complex number $z = x + iy$ can be represented exponentially as

$$\boxed{z = x + iy = re^{i\theta}} \tag{1.16}$$

Exercise 1.1 Verify the following identities

1. $\text{Re } z = \dfrac{z + \bar{z}}{2}, \quad \text{Im } z = \dfrac{z - \bar{z}}{2i}$
2. $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$
3. $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2, \quad \overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$
4. $\overline{z_1 + z_2 + \cdots + z_n} = \bar{z}_1 + \bar{z}_2 + \cdots + \bar{z}_n$ (use mathematical induction)
5. $\overline{z_1 z_2 \cdots z_n} = \bar{z}_1 \bar{z}_2 \cdots \bar{z}_n$
6. $z\bar{z} = |z|^2$
7. $|z_1 + z_2| \le |z_1| + |z_2|$ (triangle inequality)
8. $|z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n|$ (use mathematical induction)
9. $\left|\dfrac{z_1 + z_2}{z_3 + z_4}\right| \le \dfrac{|z_1| + |z_2|}{||z_3| - |z_4||}$ (for $|z_3| \ne |z_4|$)
10. Let $a_0, a_1, \ldots, a_n$ ($n \ge 1$) denote real numbers, show that
$$\overline{a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n} = a_0 + a_1 \bar{z} + a_2 \bar{z}^2 + \cdots + a_n \bar{z}^n$$

Exercise 1.2 Using the polar or complex form, show that

1. $z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$ and hence
$$|z_1 z_2| = |z_1||z_2|, \quad \arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$
2. $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$ and hence
$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \quad \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$
3. $z^n = r^n(\cos n\theta + i\sin n\theta)$ for any integer n.
Setting $|z| = r = 1$, yields de Moivre's relation $z^n = \cos n\theta + i\sin n\theta$
4. Use de Moivre's relation to show that
   (a) $\cos 2\theta = 2\cos^2\theta - 1, \quad \sin 2\theta = 2\sin\theta\cos\theta$
   (b) $\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$
   (c) $\sin^6\theta = \frac{1}{32}(10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta)$ ◄

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1.2.3 Roots of a Complex Number

It can be shown that there are q and only q distinct values (roots) of $(\cos\theta + i\sin\theta)^{\frac{1}{q}}$, q being positive integer.

Example 1.2 Prove the above statement and show that the q roots of $z = x + iy$ are

$$\begin{aligned}\sqrt[q]{z} &= [r(\cos\theta + i\sin\theta)]^{\frac{1}{q}} \\ &= r^{\frac{1}{q}}\left[\cos\left(\frac{2n\pi+\theta}{q}\right) + i\sin\left(\frac{2n\pi+\theta}{q}\right)\right], \quad n = 0, 1, \ldots, q-1.\end{aligned}$$

Example 1.3 Find $(1)^{\frac{1}{3}}$.

Example 1.4 Solve $z^6 = -1$.

Exercise 1.3

1. Find all the roots of
   (a) $(1+i)^{\frac{1}{3}}$
   (b) $(\frac{1}{2} + i\frac{\sqrt{3}}{2})^{\frac{3}{4}}$
   (c) $i^{\frac{1}{3}}$
   (d) $(-i)^{\frac{1}{6}}$
   (e) $32^{\frac{1}{5}}$
2. Solve the equations
   (a) $x^7 + x^4 + x^3 + 1 = 0$
   (b) $(x-1)^4 + x^4 = 0$
   (c) $(1+x)^3 = i(1-x)^3$
   (d) $(1+x)^n = (1-x)^n$, n- integer. ◀

1.3 Functions of a Complex Variable

If to each of a set of complex numbers z there corresponds one or more values of a variable w, then w is called a *function of the complex variable* z, written $w = f(z)$.

A function is *single-valued* if for each value of z there corresponds only one value of w; otherwise it is *multiple-valued* or *many-valued*. In general we can write

$$w = f(z) = u(x, y) + iv(x, y), \tag{1.17}$$

where $u(x, y)$ and $v(x, y)$ are real-valued functions. A function which is multiple-valued can be considered as a collection of single-valued functions.

Exercise 1.4 Express each function in the form $u(x, y) + iv(x, y)$

1. z^3
2. $\dfrac{1}{1-z}$
3. e^{3z}
4. $\ln z$ ◀

1.3.1 Limits and Continuity

Definitions of limit and continuity for functions of a complex variable are analogous to those of a real variable. Thus

$$\lim_{z \to z_0} f(z) = w_0 \tag{1.18}$$

if given any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(z) - w_0| < \epsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta. \tag{1.19}$$

Similarly $f(z)$ is *continuous* at z_0 if

$$\lim_{z \to z_0} f(z) = z_0 \tag{1.20}$$

1.3.2 Derivatives

Let $f(z)$ be a single-valued function of the variable z, the *derivative* of $f(z)$ is defined as

$$f'(z) \equiv \frac{d}{dz} f(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \tag{1.21}$$

provided the limit exits independent of the manner in which $\Delta z \to 0$.

If the limit (1.21) exits for $z = z_0$, then $f(z)$ is called *analytic* at z_0. If the limit exits for all z in a region $\mathcal{R}$, then $f(z)$ is called *analytic in* $\mathcal{R}$. In order to be analytic, $f(z)$ must be single-valued and continuous. The converse, however, is not necessarily true.

Differentiation rules of a real-valued functions can be similarly carried to complex functions.

Example 1.5 Show that $f(z) = z^2$ is analytic.

Example 1.6 Show that $f(z) = \bar{z}$ is not analytic anywhere. ◀

1.3.3 The Cauchy-Riemann Relations

The necessary and sufficient conditions for the function $f(z)$

$$w = f(z) = u(x, y) + iv(x, y)$$

to be analytic in a region $\mathcal{R}$, are

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \quad \text{are continuous functions af } x \text{ and } y \text{ in } \mathcal{R} \tag{1.22}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{1.23}$$

The conditions in (1.23) are known as the Cauchy-Riemann (C-R) relations.

Example 1.7 Prove the above statement.

Example 1.8

1. Show that the real and imaginary parts u and v of an analytic function satisfy the Laplace's equation, viz.,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

   (Hence u and v are known as *harmonic functions*.)

2. Prove that the function $\sinh z$ is analytic and find its derivative. ◀

Note that when the function $f(z)$ is known to be analytic, it can be differentiated in the ordinary way as if it is a real variable. Thus

$$\begin{aligned} f(z) &= z^2 \quad \Rightarrow \quad f'(z) = 2z \\ f(z) &= \sin z \quad \Rightarrow \quad f'(z) = \cos z, \text{ etc.} \end{aligned}$$

Exercise 1.5

1. Show that the following functions are non-analytic

   (a) $|z|^2$

(b) $z - \bar{z}$

(c) $2x + ixy^2$

(d) $e^x e^{-iy}$

(e) $z|z|$

2. Show that $f'(z)$ exits everywhere

(a) $f(z) = iz + 2$

(b) $f(z) = e^{-x}e^{-iy}$

(c) $f(z) = z^3$

(d) $f(z) = \cos x \cosh y - i \sin x \sinh y$

3. Are the following functions analytic?

(a) $f(z) = z^4$

(b) $f(z) = i|z|^4$

(c) $f(z) = i/z$

(d) $f(z) = \dfrac{1}{z-2}, \quad z \neq 2$

(e) $f(z) = \arg z$

(f) $f(z) = (1+i)z^2$

(g) $f(z) = \text{Re } z/\text{Im } z$

(h) $f(z) = (1+i)(x+y)^2$

(i) $f(z) = \ln|z| + i\text{Arg } z$

Exercise 1.6

(a) Determine an analytic function whose imaginary part is $2x(1-y)$.

(b) Determine an analytic function whose real part is $e^{2x}(x\cos 2y - y\sin 2y)$.

(c) Find p such that the function $f(z) = r^2\cos 2\theta + ir^2\sin p\theta$ is analytic.

(d) Prove that there is no analytic function whose imaginary part is $x^2 - 2y$.

(e) Show that the function $u = e^{-2xy}\sin(x^2 - y^2)$ is harmonic.

(f) Show that the function $f(z) = \sqrt{|xy|}$ is not regular at the origin, although all the R-C relations are satisfied. ◀

1.4 Power Series

Power series in complex variable is a natural extension to that of real-variables

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1.24}$$

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where the a_n are in general complex numbers.

The ratio test for real series can be employed to investigate the absolute convergence of complex power series.

The series (1.24) is absolutely convergent if

$$\lim_{n\to\infty} \frac{|a_{n+1}||z|^{n+1}}{|a_n||z|^n} = \lim_{n\to\infty} \frac{|a_{n+1}||z|}{|a_n|} < 1 \tag{1.25}$$

and the radius of convergence of the series is given by

$$\frac{1}{R} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \tag{1.26}$$

Alternatively, the series (1.24) is absolutely convergent if $|z| < R$ and divergent if $|z| > R$.

The cases $R = 0$ and $R = \infty$ correspond respectively to convergence at the origin only and convergence everywhere. For R finite the convergence occurs in a restricted part of the z-plane. For a power series about a general point z_0, the circle of convergence is of course on that point.

Example 1.9 Find the part of the z-plane for which the following series are convergent

1. $\displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!}$
2. $\displaystyle\sum_{n=0}^{\infty} n!z^n$
3. $\displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n}$ ◀

It can be shown that the power series $\sum_{n=0}^{\infty} a_n z^n$ has a sum that is an analytic function of z inside its circle of convergence.

Exercise 1.7 Prove the above statement.

Exercise 1.8 Find the part of the z-plane for which the following series are convergent

1. $\sum_{n=0}^{\infty} \frac{n!}{(n+1)^n} z^n$
2. $\sum_{n=0}^{\infty} \frac{n}{n^2+1} z^n$
3. $\sum_{n=0}^{\infty} \frac{n^2}{3^n} z^n$ ◀

1.5 Some Elementary Functions

Elementary functions can be defined through power series.

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad (z = x + iy) \tag{1.27}$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \tag{1.28}$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \tag{1.29}$$

$$\tan z = \frac{\sin z}{\cos z} \tag{1.30}$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \tag{1.31}$$

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \tag{1.32}$$

$$\tanh z = \frac{\sinh z}{\cosh z} \tag{1.33}$$

Exercise 1.9 Verify the following identities

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}), \quad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}),$$
$$\cosh z = \frac{1}{2}(e^{z} + e^{-z}), \quad \sinh z = \frac{1}{2}(e^{z} - e^{-z})$$

Exercise 1.10 Find the part of the z-plane for which the above functions are convergent.

Exercise 1.11 Show that Euler's formula is valid in complex:

$$e^{iz} = \cos z + i \sin z$$

Exercise 1.12 Verify the following identities

1. $\cos z = \cos x \cosh y - i \sin x \sinh y$
2. $\sin z = \sin x \cosh y + i \cos x \sinh y$
3. $|\cos z|^2 = \cos^2 x + \sinh^2 y$
4. $|\sin z|^2 = \sin^2 x + \sinh^2 y$
5. $\cos^2 z + \sin^2 z = 1$
6. $\cosh iz = \cos z, \quad \sinh iz = i \sin z$
7. $\cos iz = \cosh z, \quad \sin iz = i \sinh z$
8. $\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$
9. $\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$
10. $\cosh^2 z - \sinh^2 z = 1$
11. $\cosh^2 z + \sinh^2 z = \cosh 2z$
12. $\cosh z = \cosh x \cos y + i \sinh x \sin y$
13. $\sinh z = \sinh x \cos y + i \cosh x \sin y$
14. $\cos \bar{z} = \overline{\cos z}, \quad \sin \bar{z} = \overline{\sin z}$
15. $\left(\dfrac{ia-1}{ia+1}\right)^{ib} = \exp(-2b \cot^{-1} a)$ where a and b are real
16. (a) $\displaystyle\sum_{n=0}^{N-1} \cos nx = \frac{\sin(Nx/2)}{\sin x/2} \cos(N-1)\frac{x}{2},$

    (b) $\displaystyle\sum_{n=0}^{N-1} \sin nx = \frac{\sin(Nx/2)}{\sin x/2} \sin(N-1)\frac{x}{2}$
17. For $-1 < p < 1$ show that

    (a) $\displaystyle\sum_{n=0}^{\infty} p^n \cos nx = \frac{1 - p\cos x}{1 - 2p\cos x + p^2}$

    (b) $\displaystyle\sum_{n=0}^{\infty} p^n \sin nx = \frac{p \sin x}{1 - 2p\cos x + p^2}$

Exercise 1.13 Show that

1. $|\sinh y| \leq |\sin z| \leq \cosh y$
2. $|\sinh y| \leq |\cos z| \leq \cosh y$
3. $|\cosh z| \leq \cosh x$
4. $|\sin x| \leq |\sin z|, \quad |\cos x| \leq |\cos z|$
5. $|z-1| < |\sqrt{z^2-1}| < |z+1|, \quad$ for Re $(z) > 0$
6. $\sin \bar{z}$ and $\cos \bar{z}$ are non-analytic functions of z.

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7. $(\text{sech } z)' = -\text{sech } z \tanh z, \quad (\text{csch } z)' = -\text{csch } z \coth z$

Exercise 1.14 Find all the roots of

1. $\cosh z = \frac{1}{2}$ *ans.* $(2n \pm \frac{1}{3})\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$
2. $\sinh z = i$ *ans.* $(2n + \frac{1}{2})\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$
3. $\cos z = 2$ *ans.* $2n\pi + i\cosh^{-1} 2 \quad (n = 0, \pm 1, \pm 2, \ldots)$

Exercise 1.15

1. Prove that the usual quadratic formula solves the quadratic equation

$$az^2 + bz + c = 0, \qquad (a \neq 0)$$

where a, b and c are complex numbers. Specifically, by completing the square on the left-hand side, prove that the roots of the equation are

$$z = \frac{-b + (b^2 - 4ac)^{1/2}}{2a}$$

where the two roots are to be considered when $b^2 - 4ac \neq 0$.

2. Use the above result to show the roots of

$$z^2 + 2z + (1 - i) = 0$$

are $\left(-1 + \frac{1}{\sqrt{2}}\right) + \frac{i}{\sqrt{2}}$, and $\left(-1 - \frac{1}{\sqrt{2}}\right) - \frac{i}{\sqrt{2}}$ ◀

1.5.1 Logarithms

The natural logarithm of $z = x + iy$ denoted by $\ln z$ is defined as the inverse of the exponential function; i.e., $w = \ln z$ is defined for $z \neq 0$ by the relation

$$e^w = z$$

If $z = re^{i\theta}$, then

$$\boxed{\ln z = \ln r + i\theta}, \qquad (r = |z|, \theta = \arg z) \tag{1.34}$$

Since $\arg z$ is determined only up to integer multiples of 2π, the complex natural logarithm $\ln z$ $(z \neq 0)$ is *infinitely many-valued.*

The *principal value* of $\ln z$ is defined by

$$\text{Ln } z = \ln |z| + i\text{Arg } z \tag{1.35}$$

This logarithmic function is a *single-valued* function. So,

$$\ln z = \text{Ln } z \pm 2n\pi i, \qquad (n = 0, 1, 2, 3, \ldots) \tag{1.36}$$

Exercise 1.16 Show that (for $n = 0, 1, 2, 3, \ldots$)

1. $\ln 1 = 0, \pm 2\pi i, \pm 4\pi i, \ldots;$ $\quad$ $\text{Ln } 1 = 0$
2. $\ln(-1) = \pm\pi i, \pm 3\pi i, \ldots;$ $\quad$ $\text{Ln } (-1) = \pi i$
3. $\ln(3-4i) = 1.609 - 0.927i \pm 2n\pi i;$ $\quad$ $\text{Ln } (3-4i) = 1.609 - 0.927i$
4. $\ln i = \pi i/2, -3\pi i/2, 5\pi i/2, \ldots;$ $\quad$ $\text{Ln } i = \pi i/2$

Exercise 1.17 Show that

1. $\text{Ln } (-ei) = 1 - \frac{1}{2}\pi i$
2. $\text{Ln } (1-i) = \frac{1}{2}\text{Ln } 2 - \frac{1}{4}\pi i$
3. $\text{Ln } [(1+i)^2] = 2\text{Ln } (1+i)$ but $\text{Ln } [(-1+i)^2] \neq 2\text{Ln } (-1+i)$
4. $\ln(z_1 z_2) = \ln z_1 + \ln z_2,$ $\quad$ $\ln(z_1/z_2) = \ln z_1 - \ln z_2$ ◀

1.5.2 General Power

General power of a complex number z is defined by

$$z^c = e^{c \ln z} \qquad (c \text{ complex}, \ z \neq 0) \tag{1.37}$$

Since $\ln z$ is infinitely many-valued, z^c will, in general, be multi-valued. The particular value

$$z^c = e^{c\text{Ln } z} \tag{1.38}$$

is called the *principal value* of z^c.

Example 1.10 Evaluate: i^i; $\quad (1+i)^{2-i}$

Exercise 1.18 Evaluate

1. $i^{1/2}$
2. $(2i)^{2i}$
3. 3^{4-i}
4. $(5-2i)^{3+\pi i}$
5. $(-5)^{2-4i}$

Exercise 1.19 Solve for z

1. $\ln z = -\frac{1}{2}\pi i$
2. $\ln z = e - \pi i$
3. $\ln z = -2 - \frac{3}{2}i$

Exercise 1.20 The inverse of sine $w = \sin^{-1} z$ is defined such that $\sin w = z$ and so on. Using $\sin w = \frac{1}{2i}(e^{iw} - e^{-iw})$ and similar relations show that

1. $\sin^{-1} z = -i \ln(iz + \sqrt{1 - z^2})$
2. $\cos^{-1} z = -i \ln(z + \sqrt{z^2 - 1})$
3. $\cosh^{-1} z = \ln(z + \sqrt{z^2 - 1})$
4. $\sinh^{-1} z = \ln(z + \sqrt{z^2 + 1})$
5. $\tan^{-1} z = \dfrac{i}{2} \ln \dfrac{i + z}{i - z}$
6. $\tanh^{-1} z = \dfrac{1}{2} \ln \dfrac{1 + z}{1 - z}$ ◀

1.6 Conformal Transformations

We know that the real function $y = f(x)$ can be represented graphically by a curve in the xy-plane. Similarly the real function $z = f(x, y)$ by a surface in the three dimensional space. However, this method of graphical representation fails in the case of complex function because $w = f(z)$, i.e., $u + iv = f(x + iy)$ involves four real variables, two independent x, y and two dependent variables u, v. Thus a four dimensional region is required to represent it graphically in the cartesian fashion. As it is not possible, we choose, two complex planes and call them z-plane and w-plane. In the z-plane, we plot the point $z = x + iy$ and in the w-plane, we plot the corresponding points $w = u + iv$. Thus the function $w = f(z)$ defines a correspondence between the points of two plane. If the point z describes some curve C in the z-plane, the point w will move along a corresponding curve C' in the w-plane. The function $w = f(z)$ thus defines a *mapping* or *transformation* of the z-plane into the w-plane.

Example 1.11 Given $w = f(z) = z + (1 - i)$ determine the region D' of the w-plane corresponding to the rectangular region D in the z-plane bounded by $x = 0, y = 0, x = 1, y = 2$. ◀

Suppose two curves C_1, C_2 in the z-plane intersect at the point P and the corresponding curves C_1', C_2' in the w-plane intersect at P' under the transformation $w = f(z)$ (see Fig. 1.2). If the angle of intersection of the curves at P is the same as the angle of intersection of the curves at P', both in magnitude and sense, then the transformation is said to be *conformal* at P.

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Nov 2013.

The conditions under which the transformation $w = f(z)$ is conformal are given by the following theorem:

If $f(z)$ is analytic and $f'(z) \neq 0$ in a region $\mathcal{R}$ of the z-plane, then the mapping $w = f(z)$ is conformal at all points of $\mathcal{R}$.

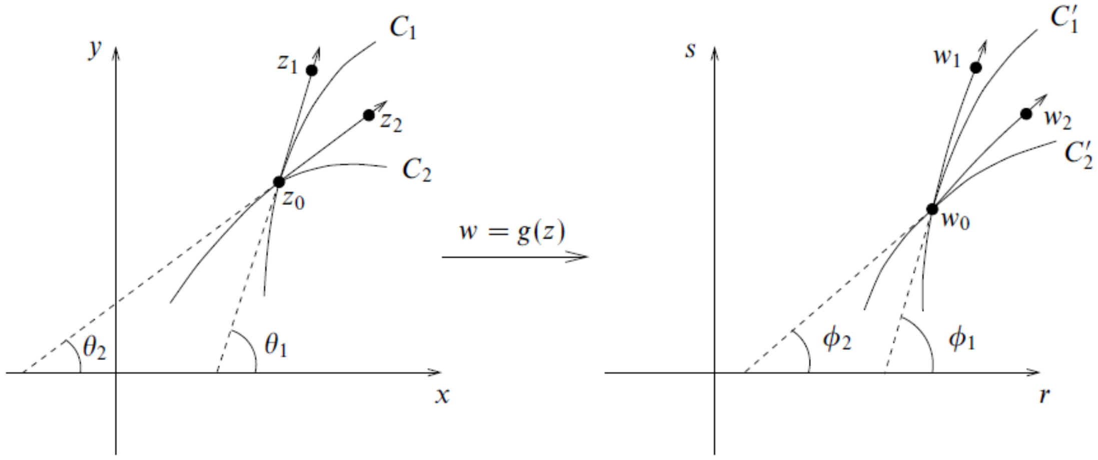

Figure 1.2: Two curves C_1 and C_2 in the z-plane, which are mapped onto C_1' and C_2' in the w-plane.

Exercise 1.21 Prove the above theorem.

Exercise 1.22 Show that under the conformal transformation $w = f(z)$, the lengths of the arcs through P are magnified in the ratio $\rho : 1$, where $\rho = |f'(z)|$. Thus an infinitesimal length in the z-plane is magnified by a factor $|f'(z)|$ in the w-plane and consequently infinitesimal ares in the z-plane are magnified by the factor $|f'(z)|^2$ in the w-plane. Also the tangent to the curve C at P is rotated through an angle $\phi = \text{Arg}\ [f'(z)]$ under the given transformation.

Example 1.12 Given $w = z^2$, show that

1. the coefficient of magnification at $z = 1 + i$ is $2\sqrt{2}$.
2. the angle of rotation at $z = 1 + i$ is $\pi/4$. ◀

1.6.1 Some Standard Transformations

1. **Translation**: $w = z + c$
   Let $c = a + ib$, then $u = x + a, v = y + b$. Thus the transformation is

mere translation of the axes and preserves shape and size.

2. **Rotation and Magnification**: $w = cz$
Let $c = \rho e^{i\alpha}, z = re^{i\theta}$ and $w = Re^{i\phi}$. Then

$$\begin{aligned} Re^{i\phi} &= \rho r e^{i(\theta+\alpha)} \\ \Rightarrow R &= \rho r \quad \text{and} \quad \phi = \theta + \alpha \end{aligned}$$

Thus the transformation maps a point $P(r, \theta)$ in the z-plane into a point $P(\rho r, \theta + \alpha)$ in the w-plane.

3. **Inversion**: $w = 1/z$
Let $z = re^{i\theta}$ and $w = Re^{i\phi}$. Then

$$Re^{i\phi} = \frac{1}{r}e^{-i\theta} \Rightarrow \quad R = \frac{1}{r}, \phi = -\theta$$

Thus the transformation maps the point $P(r, \theta)$ in the z-plane into the point $P(\frac{1}{r}, -\theta)$ in the w-plane.

4. **Bilinear or Mobius transformation**: $w = \dfrac{az + b}{cz + d}$
For $ad - bc \neq 0$, the bilinear transformation is a combination of i) translation, ii) rotation and magnification, and iii) inversion. This can be seen by rewriting w as

$$w = \frac{a}{c} + \frac{bc - ad}{c^2} \frac{1}{z + \dfrac{d}{c}}$$

Exercise 1.23

1. Under $w = e^{i\pi/4}z$, determine the image of the region bounded by the lines $x = 0, y = 0$ and $x + y = 1$.
2. Find the image of the circle $|z| = 2$ under the transformation $w = z + 3 + 2i$.
3. Find the image of the following curves under the mapping $w = 1/z$
   (a) the line $y - x + 1 = 0$
   (b) the circle $|z - 3| = 5$.
4. Show that the map of the real axis of the z-plane on the w-plane by the transformation $w = \frac{1}{z+i}$ is a circle and find its center and radius.
5. Find and sketch the image of the following regions under the mapping $w = e^z$,
   (a) $|x| < 1, |y| < \pi/2$
   (b) $x < 1, |y| \leq \pi$
   (c) $0 \leq y \leq \pi/2$.

Murad Ridwan,
School of of Electrical & Computer Engineering
AAiT, Addis Ababa University.

Nov 2013.

6. Show that the transformation $w = z + \frac{a^2-b^2}{4z}$ transforms the circle $|z| = \frac{1}{2}(a+b)$ in the z-plane into an ellipse of semi-axes a, b in the w-plane.
7. Find the electrostatic potential $V(r, \theta)$ in the space $0 < r < 1, 0 < \theta < \pi/4$, bounded by the half planes $\theta = 0$ and $\theta = \pi/4$ and portion $0 \leq \theta \leq \pi/4$ of the cylindrical surface $r = 1$, when $V = 1$ on the planar surfaces and $V = 0$ on the cylindrical one. Verify that the solution obtained satisfies the boundary conditions. ◀

1.7 Singularities and Zeros of Complex Functions

A *singular point* of a function $f(z)$ is a value of z at which $f(z)$ fails to be analytic. If $f(z)$ is analytic everywhere in some region except at an interior point $z = a$, we call $z = a$ an *isolated singularity* of $f(z)$.

For example in $f(z) = \dfrac{1}{(z-3)^3}$, the point $z = 3$ is an isolated singularity of $f(z)$.

If $f(z) = \dfrac{\phi(z)}{(z-a)^n}, \phi(a) \neq 0$, where $\phi(z)$ is analytic everywhere in a region including $z = a$, and if n is a positive integer, then $f(z)$ has an isolated singularity at $z = a$ which is called a *pole of order* n. If $n = 1$, the pole is often called a *single pole*; if $n = 2$, it is called a *double pole*, and so on.

For example, $f(z) = \dfrac{z}{(z-3)^2(z+1)}$ has two singularities; a pole of order 2 at $z = 3$ and a single pole at $z = -1$. Similarly $f(z) = \dfrac{3z-1}{z^2+4} = \dfrac{3z-1}{(z+2i)(z-2i)}$ has two poles at $z = \pm 2i$.

An alternative definition is that

$$\lim_{z \to a} [(z-a)^n \phi(z)] = z_0 \tag{1.39}$$

If z_0 exists and finite, we call such singularity a *removable singularity*. If z_0 is not finite, we have an *essential singularity*.

If $f(a) = 0$, then $z = a$ is called a *zero of the function* $f(z)$.

For instance,

$$f(z) = \frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)$$
$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

therefore $\lim_{z\to 0} f(z) = 1$, and so $f(z)$ has a removable singularity at $z = 0$.

Exercise 1.24 Find the singularities, if any, and describe their type (use power series if needed)

1. $\frac{z^2}{(z+1)^3}$
2. $\frac{\sin mz}{z^2+2z+1}, m \neq 0$
3. $\frac{1-\cos z}{z}$
4. $e^{-\frac{1}{(z-1)^2}}$
5. $\frac{\sin z}{z-\pi}$ ◀

1.8 Complex Integrals

If $f(z)$ is defined, single-valued and continuous in a region $\mathcal{R}$, we define the integral of $f(z)$ along some path C in $\mathcal{R}$ from point $z_1 = x_1 + iy_1$ to point $z_2 = x_2 + iy_2$ as

$$\int_C f(z)dz = \int_{(x_1,y_1)}^{(x_2,y_2)} (u+iv)(dx+idy) = \int_{(x_1,y_1)}^{(x_2,y_2)} (udx-vdy)+i\int_{(x_1,y_1)}^{(x_2,y_2)} (vdx+udy) \tag{1.40}$$

which shows that the evaluation of the line integral of a complex function can be reduced to the evaluation of two line integrals in real functions.

Example 1.13 Evaluate $\int_0^{1+i}(x^2 - iy)dz$ along paths a) $y = x$, b) $y = x^2$.

Exercise 1.25 Evaluate

1. $\displaystyle\int_0^{2+i} (\overline{z})^2 dz$
   (a) along the real axis to 2 and then vertically to $2+i$

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Nov 2013.

(b) along the line $2y = x$

2. $\oint_C |z|^2 dz$, around the square with vertices at $(0,0), (1,0), (1,1), (0,1)$.

3. Show that $\oint_C (z+1)dz = 0$, where C is the boundary of the square whose vertices are at the points $z = 0, z = 1, z = 1+i, z = i$.

4. $\oint_C \ln z \, dz$, where C is the unit circle $|z| = 1$.

5. Prove that $\oint_C \frac{dz}{z-a} = 2\pi i$, $\oint_C (z-a)^n dz = 0$ where n is an integer $\neq -1$ and C is the circle $|z-a| = r$. ◀

1.9 Cauchy's Theorem

Let C be a simple closed curve. If $f(z)$ is analytic within the region bounded by C as well as on C, then we have Cauchy's theorem

$$\oint_C f(z) \, dz = 0 \tag{1.41}$$

Expressed in another way, (1.41) is equivalent to the statement that $\int_{z_1}^{z_2} f(z) \, dz$ has a value *independent of the the path* joining z_1 and z_2.

Exercise 1.26 Prove Cauchy's theorem (1.41).

Example 1.14 Evaluate

1. $\oint_C (x^2 - y^2 + 2ixy)dz$, where C is the contour $|z| = 1$.

2. $\int_C (3z^2 + 4z + 1)dz$, where C is the arc of the cycloid $x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$ between $(0,0)$ and $(2\pi a, 0)$. ◀

1.10 Cauchy's Integral Formulas

If $f(z)$ is analytic within and on a simple curve C and a is any point interior to C, then

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz \tag{1.42}$$

where C is traversed in the positive (counterclockwise) sense. Also the nth derivative of $f(z)$ at $z = a$ is given by

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz \tag{1.43}$$

The Cauchy's integral formulas show that if the function $f(z)$ is known *on* the closed curve C then it is also known *within* C, and the various derivatives at points within C can be calculated. Thus if a function of complex variable has a first derivative, it has all higher derivatives as well. This, of course, is not necessarily true for functions of real variables.

Exercise 1.27 Prove Cauchy's integral formulas (1.42-1.43).

Example 1.15 Evaluate

1. $\oint_C \frac{e^{-z}}{z+1} dz$, where C is the circle (a) $|z| = 2$ (b) $|z| = 1/2$.
2. $\oint_C \frac{3z^2 + z}{z^2 - 1} dz$, C is the circle $|z - 1| = 1$
3. $\oint_C \frac{e^{2z}}{(z+1)^4} dz$, C is the circle $|z| = 2$.

Exercise 1.28

1. Evaluate $\oint_C \frac{3z^2 + 7z + 1}{z+1} dz$, where C is (a) $|z| = 1.5$ (b) $|z + i| = 1$.
2. Evaluate $\oint_C \frac{\cos \pi z}{z^2 - 1} dz$, around the rectangle with vertices (a) $2 \pm i, -2 \pm i$ (b) $-i, 2 - i, 2 + i, i$.
3. Evaluate $\oint_C \frac{\sin^2 z}{(z - \pi/6)^3} dz$, where C is $|z| = 1$.
4. Let C be the unit circle $z = e^{i\theta} \quad (-\pi \le \theta \le \pi)$. First show that for any real constant a,
$$\int_C \frac{e^{az}}{z} dz = 2\pi i$$
Then write the integral in terms of θ to derive the integral formula
$$\int_0^{\pi} e^{a \cos \theta} \cos(a \sin \theta) d\theta = \pi$$
5. If $f(\xi) = \oint \frac{4z^2 + z + 5}{z - \xi} dz$, where C is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$, find $f(1), f(i), f'(-1), f''(-i)$. ◀

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Nov 2013.

1.11 Taylor and Laurent Series

1.11.1 Taylor Series

If $f(z)$ is analytic inside a circle C with center z_0, then for all z we have the Taylor series, the complex analogue of the real Taylor series, as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad \text{where} \quad a_n = \frac{1}{n!} f^{(n)}(z_0) \tag{1.44}$$

or using the Cauchy's integral formula (1.43),

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi \tag{1.45}$$

Exercise 1.29 Verify Taylors series (1.44). ◀

A *Maclaurin* series is a Taylor series with $z_0 = 0$, i.e.,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \text{where} \quad a_n = \frac{1}{n!} f^{(n)}(0) = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi^{n+1}} d\xi \tag{1.46}$$

Putting $z = z_0 + h$ in (1.44), we get

$$\begin{aligned} f(z_0 + h) &= \sum_{n=0}^{\infty} a_n h^n = \sum_{n=0}^{\infty} \frac{h^n}{n!} f^{(n)}(z_0) \\ &= f(z_0) + h f'(z_0) + \frac{h^2}{2!} f''(z_0) + \dots \end{aligned}$$

Example 1.16 Find the Maclaurin series of

1. $\dfrac{1}{1-z}$
2. $\dfrac{1}{1+z^2}$
3. $\tan^{-1} z$.

Example 1.17 Expand $\cos z$ in Taylor series about $z = \pi/4$.

Exercise 1.30 Find the Taylor series of the following functions with the given points as centers and determine the radius of convergence.

1. $e^z, \quad \pi i$
2. $e^z, \quad 1$
3. $e^{-2z}, \quad 0$

4. $\dfrac{1}{z^2}, \quad 1$
5. $\dfrac{1}{z+2}, \quad 1+i$
6. $\ln z, \quad 1$
7. $\sinh(z-4i), \quad 4i$
8. $\dfrac{z}{(z+1)(z+2)}, \quad 2$
9. $\ln(1+z), \quad 0.$

Exercise 1.31 Find the Maclaurin series by integrating that of the integrand term by term

1. $\displaystyle\int_0^z \frac{e^t-1}{t}dt$
2. $\displaystyle\int_0^z \frac{1-\cos t}{t^2}dt$
3. $\displaystyle\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt \quad$ (the error function)
4. $\displaystyle\mathrm{Si}(z) = \int_0^z \frac{\sin t}{t}dt \quad$ (the Sine integral)
5. $\displaystyle\mathrm{S}(z) = \int_0^z \sin t^2 dt \quad$ (Fresnel integrals)
6. $\displaystyle\mathrm{C}(z) = \int_0^z \cos t^2 dt \quad$ (Fresnel integrals.) ◀

1.11.2 Laurent Series

In various applications it is necessary to expand a function $f(z)$ around points where $f(z)$ is singular. Taylor's series can not be applied in such cases. A new type of series, known as *Laurent series* is necessary. This will be a representation that is valid in an annulus bounded by two concentric circles C_1 and C_2 such that $f(z)$ is analytic in the annulus region R and at each points of C_1 and C_2 (Fig. 1.3). The Laurent representation of $f(z)$ is given by

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n} \qquad (1.47)\\ &= a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots \\ &\quad \ldots + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \ldots \end{aligned}$$

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Nov 2013.

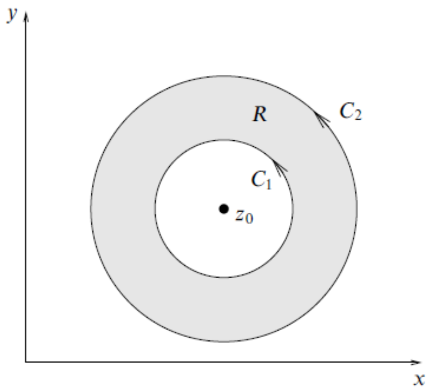

Figure 1.3: The region of convergence R for a Laurent series of $f(z)$ about a point $z = z_0$ where $f(z)$ has a singularity.

where

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(\xi)}{(\xi - z_0)^{n+1}}d\xi, \quad b_n = \frac{1}{2\pi i}\oint_C (\xi - z_0)^{n-1}f(\xi)d\xi$$

Alternatively, the series can be written (denoting b_n by a_{-n}) as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n, \qquad a_n = \frac{1}{2\pi i}\oint_C \frac{f(\xi)}{(\xi - z_0)^{n+1}}d\xi \tag{1.48}$$

Note that the process of finding the coefficients a_n by complex integration is complicated. In practice, we expand the function $f(z)$ by binomial or by some other method to obtain Taylor's or Laurent's series.

Exercise 1.32 Verify Laurent's series.

Example 1.18 Expand $\dfrac{1}{z^2 - 3z + 2}$ in the region

(a) $|z| < 1$ $\quad$ (b) $1 < |z| < 2$ $\quad$ (c) $|z| > 2$ $\quad$ (d) $0 < |z - 1| < 1$.

Example 1.19 Expand the following functions at the indicated points

1. $\dfrac{z}{(z+1)(z+2)}, \quad z = -2$
2. $\dfrac{e^z}{(z-1)^2}, \quad z = 1$
3. $z\cos(\frac{1}{z}), \quad z = 0$

4. $\dfrac{1}{z(z+2)^3}$, $\quad z = 0, -2$

Exercise 1.33 Expand the following functions in Laurent's series

1. $\dfrac{1}{z-2}$, $\quad |z| > 2$
2. $\dfrac{1}{z^2-4z+3}$, $\quad 1 < |z| < 3$
3. $\dfrac{1}{z(z-1)(z-2)}$, $\quad |z| > 2$
4. $\dfrac{1-\cos z}{z^3}$, $\quad z = 0$
5. $\dfrac{e^z}{(z-1)^2}$, $\quad z = 1$
6. $\dfrac{4z^2+2z-4}{z^3-4z}$, $\quad 2 < |z-2| < 3$
7. $z^2 \sinh(\frac{1}{z})$, $\quad z = 0$. ◀

1.12 Residue Theorem

We have seen that if $f(z)$ has a singularity at a point $z = z_0$ inside C, but is otherwise analytic on C and inside C, then $f(z)$ has a Laurent series given by

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \dots$$

but

$$b_1 = \frac{1}{2\pi i}\oint_C f(z)dz$$

Now, since we can obtain Laurent series by various methods, without using the integral formulas for the coefficients, we can find b_1 by one of those methods and then use the formula for b_1 for evaluating the integral, i.e.,

$$\oint_C f(z)dz = 2\pi i b_1 \tag{1.49}$$

The coefficient b_1 is called the *residue* of $f(z)$ at $z = z_0$

$$b_1 = \operatorname*{Res}_{z=z_0} f(z) \tag{1.50}$$

Example 1.20 Evaluate $\displaystyle\oint_C \frac{\sin z}{z^4}dz$, around the unit circle C. ◀

The residue theorem is states as:
If $f(z)$ is analytic at all points inside and on a simple closed curve C, except at a finite number of isolated singular points $z_1, z_2, \ldots, z_k$ within C, then

$$\oint_C f(z)dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z) \tag{1.51}$$

Exercise 1.34 Prove the residue theorem (1.51). ◀

If $f(z)$ has a simple pole (i.e., pole of order 1) at $z = z_0$, then

$$\operatorname*{Res}_{z=z_0} f(z) = b_1 = \lim_{z\to z_0} (z - z_0)f(z) \tag{1.52}$$

Alternatively, assuming that $f(z) = p(z)/q(z), p(z_0) \neq 0$, and $q(z)$ has a simple pole at z_0,

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{p(z_0)}{q'(z_0)} \tag{1.53}$$

If $f(z)$ has a pole of order m at $z = z_0$, then

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}[(z - z_0)^m f(z)] \tag{1.54}$$

Exercise 1.35 Verify (1.52), (1.53) and (1.54).

Example 1.21

1. Determine the poles of the function $f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$ and the residue at each pole.
2. Find the sum of the residues of the function $f(z) = \dfrac{\sin z}{z \cos z}$ at its poles inside the circle $|z| = 2$.

Example 1.22 Evaluate $\displaystyle\oint_C \frac{e^z}{(z+1)^2} dz$, where C is the circle $|z - 1| = 3$.

Example 1.23 Evaluate $\displaystyle\oint_C \frac{2z-1}{z(z+1)(z-3)} dz$, where C is the circle $|z| = 2$.

Exercise 1.36 Determine the poles and the residues at each pole.

1. $\dfrac{2z+1}{z^2 - z - 2}$
2. $\dfrac{1 - e^{2z}}{z^4}$
3. $\dfrac{z}{\cos z}$

4. $\tan z$.

Exercise 1.37 Find the residues at $z = 0$

1. $z\cos(\frac{1}{z})$
2. $\csc^2 z$
3. $\dfrac{1+e^z}{\sin z + z\cos z}$.

Exercise 1.38 Evaluate the following integrals

1. $\displaystyle\oint_C \frac{\sin \pi z}{z^4}dz, \quad C: |z-i| = 2$
2. $\displaystyle\oint_C \frac{z}{(z-1)(z-2)^2}dz, \quad C: |z-2| = 1.5$
3. $\displaystyle\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)^2}dz, \quad C: |z| = 3$
4. $\displaystyle\oint_C \frac{e^z}{\cos \pi z}dz, \quad C: |z-i| = 1.5$
5. $\displaystyle\oint_C \frac{dz}{\sinh z}, \quad C: |z| = 4$
6. $\displaystyle\oint_C \frac{\tan \pi z}{z^3}dz, \quad C: |z+1.5i| = 1.$ ◀

1.13 Residue Integration of Real Integrals

The residue theorem provides a simple and elegant method for evaluating many important definite integrals of real variables. Some of these are illustrated below.

$\displaystyle\int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta$

Integrals of the type $\displaystyle\int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta$, where $F(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$:

Let $z = e^{i\theta}$, then

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(z + z^{-1}), \qquad \sin\theta = \frac{1}{2i}(z - z^{-1}), \qquad d\theta = dz/iz$$

As θ varies from 0 to 2π, we move once around the unit circle in the anticlockwise direction. Therefore,

$$\int_0^{2\pi} F(\cos\theta, \sin\theta) d\theta = \oint_C F\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)\frac{dz}{iz} \tag{1.55}$$

where C is the unit circle $|z| = 1$.

Exercise 1.39 Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$. ◀

$\displaystyle\int_{-\infty}^{\infty} \frac{f(x)}{F(x)} dx$

Integrals of the type $\displaystyle\int_{-\infty}^{\infty} \frac{f(x)}{F(x)} dx$, where $f(x)$ and $F(x)$ are polynomials in x such that $\frac{xf(x)}{F(x)} \to 0$ as $x \to \infty$ (degree of F is at least two units higher than f) and $F(x)$ has no zeros on the real axis:

Consider $\displaystyle\oint_C \frac{f(z)}{F(z)} dz$ over the closed contour C consisting of the real axis from $-R$ to R and the semi-circle C_1 of radius R in the upper half plane (Fig. 1.4).

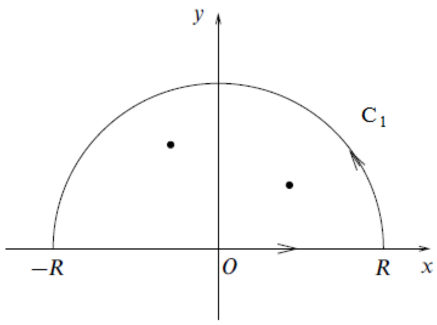

Figure 1.4: A semicircular contour in the upper half-plane.

$$\oint_C \frac{f(z)}{F(z)} dz = 2\pi i (\text{sum of residues of } \frac{f(z)}{F(z)} \text{ in the upper half plane})$$

or

$$\int_{C_1} \frac{f(z)}{F(z)} dz + \int_{-R}^{R} \frac{f(x)}{F(x)} dx = 2\pi i \sum \text{Res } \frac{f(z)}{F(z)}$$

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Let $z = Re^{i\theta}$ in the first integrand,

$$\int_{C_1} \frac{f(z)}{F(z)} dz = \int_0^{\pi} \frac{f(Re^{i\theta})}{F(Re^{i\theta})} Re^{i\theta} i d\theta \to 0 \text{ as } R \to \infty$$

Remember $\lim_{x\to\infty} \frac{xf(x)}{F(x)} \to 0$ in our assumption. Therefore, the required integral becomes

$$\int_{-\infty}^{\infty} \frac{f(x)}{F(x)} dx = 2\pi i (\text{sum of residues of } \frac{f(z)}{F(z)} \text{ in the upper half plane})$$

Exercise 1.40 Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx, \quad (a > 0, b > 0)$ ◀

$\displaystyle\int_{-\infty}^{\infty} \frac{f(x)}{F(x)} dx$, $F(x)$ has zeros on the real axis

In such case we proceed in a manner similar as in the above case except that the singularities on the real axis are encircled in a small semi-circle to avoid their inclusion in C, i.e., the contour C is indented at these singularities (Fig. 1.5).

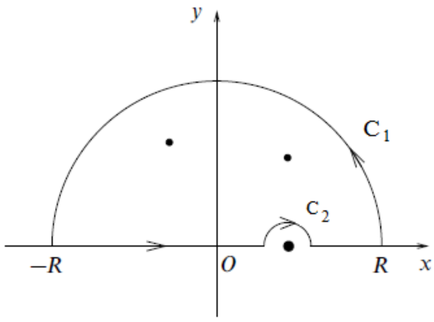

Figure 1.5: An indented contour used when the integrand has a simple pole on the real axis.

Exercise 1.41 Show that if $f(z)$ has a simple pole at $z = a$ on the real axis (Fig. 1.6), then

$$\lim_{r\to 0} \int_{C_2} f(z) dz = \pi i \operatorname*{Res}_{z=a} f(z)$$

Example 1.24 Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2-3x+2)(x^2+1)}$ ◀

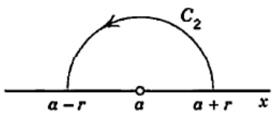

Figure 1.6:

Fourier Integrals

$\int_{-\infty}^{\infty}\frac{f(x)}{F(x)}\cos sx\,dx$ and $\int_{-\infty}^{\infty}\frac{f(x)}{F(x)}\sin sx\,dx$, where $f(x)$ and $F(x)$ are defined as above.

Consider $\oint_C \frac{f(z)}{F(z)}e^{isz}dz$, then

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{f(x)}{F(x)}\cos sx\,dx &= -2\pi\sum \text{Im Res}\left[f(z)e^{isz}\right]\\ \int_{-\infty}^{\infty}\frac{f(x)}{F(x)}\sin sx\,dx &= 2\pi\sum \text{Re Res}\left[f(z)e^{isz}\right]\end{aligned}$$

Exercise 1.42 Evaluate the following integrals

1. $\int_0^{2\pi}\frac{d\theta}{5-3\cos\theta}$
2. $\int_0^{2\pi}\frac{\cos 2\theta}{5+4\cos\theta}d\theta$ Hint: $\cos 2\theta = \frac{1}{2}(z^2+z^{-2})$
3. $\int_0^{2\pi}\frac{d\theta}{1-2p\cos\theta+p^2}, \quad 0<p<1$
4. $\int_0^{2\pi}\frac{\sin^2\theta}{5-4\cos\theta}d\theta$
5. $\int_{-\infty}^{\infty}\frac{dx}{x^4+1}$
6. $\int_0^{\infty}\frac{dx}{x^6+1}$
7. $\int_{-\infty}^{\infty}\frac{\sin x}{x^4+1}dx$
8. $\int_{-\infty}^{\infty}\frac{x^2-x+2}{x^4+10x^2+9}dx$
9. $\int_0^{\infty}\frac{x^2dx}{(x^2+9)(x^2+4)^2}$

10. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^4-1}dx$

11. $\displaystyle\int_{0}^{\infty} \frac{\sin mx}{x(x^2+a^2)}dx, \quad m>0, a>0$

12. $\displaystyle\int_{0}^{\infty} \frac{d\theta}{(5-3\cos\theta)^2}$

13. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x^4+4x+5}dx$

14. $\displaystyle\int_{0}^{\infty} \sin^{2n}\theta \, d\theta \qquad ans. \; \frac{(2n)!}{2^{2n}(n!)^2}\pi \quad (n=1,2,\ldots)$

15. $\displaystyle\int_{0}^{\infty} \frac{\ln x}{(x^2+1)^2}dx \qquad ans. \; -\pi/4$ ◀

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