Application of Finite Difference Method for Heat Conduction

Analytical solution methods are based on solving governing differential equations together with the boundary conditions. They result in the solution functions for the temperature at every point in the medium. Numerical methods on the other hand are based on replacing the differential equations by a set of *n* algebraic equations for the unknown temperatures at *n* selected points in the medium and simultaneous solutions of these equations result in the temperature values at those discrete points in the medium.

Mathematical models of heat conduction problems must often be solved numerically. Three main approximation techniques are available: finite differences (FD), finite elements (FE) and finite volume (FV) methods. These methods are based on the idea of first discretizing the heat equation and then solving the resulting (algebraic) problem. Discretization is accomplished by regarding the medium as constituted by a collection of cells or volumes of finite size. Nodes are usually associated with each cell thus producing a mesh of points.

The separation between any two nodes is the mesh spacing. Temperature at each cell is then represented by the temperature at the corresponding nodal location. A computer and a computer program are then used to solve the resulting algebraic problem.

The Finite Difference Method

The basic idea behind the finite difference method is to replace the various derivatives appearing in the mathematical formulation of the problem by suitable approximations on a finite difference mesh. The simplest derivation of finite difference formulae makes use of Taylor series. The
Taylor series expansions of a function f(x) about a point x are:

$$f(x+\delta x) = f(x) + \delta x f'(x) + \frac{\delta x^2}{2!} f''(x) + \frac{\delta x^3}{3!} f'''(x) + \dots$$

And

$$f(x-\delta x) = f(x) - \delta x f'(x) + \frac{\delta x^2}{2!} f''(x) - \frac{\delta x^3}{3!} f'''(x) + \dots$$

Where δx is the mesh spacing.

Solving the first equation above for $f'(x)$ gives

$$f'(x) = \frac{f(x+\delta x) - f(x)}{\delta x} - \frac{\delta x}{2} f''(x) - \frac{\delta x^2}{6} f'''(x) + \dots$$

and solving the second one

$$f'(x) = \frac{f(x) - f(x-\delta x)}{\delta x} + \frac{\delta x}{2} f''(x) - \frac{\delta x^2}{6} f'''(x) + \dots$$

Finally, from the first two equations

$$f'(x) = \frac{f(x+\delta x) - f(x-\delta x)}{2\delta x} - \frac{\delta x^2}{6} f'''(x) + \dots$$

These are called respectively the forward, backward and central approximations to the derivative of $f(x)$. Note that the second term on the right hand side in the first two equations above is proportional to δx while the same second term in the third equation is proportional to δx^2. Therefore, the first two equations are regarded as leading to **first-order accurate** approximations to the derivative while the last formula leads to a **second-order accurate** approximation.

Note that the neglect of higher order terms in the above formulae for $f'(x)$ produces various approximation schemes for the derivative.

Second order derivative approximations can be similarly obtained. For example, expanding $f(x \pm \delta x)$ about x

$$f(x+2\delta x) = f(x) + 2\delta x f'(x) + 2\delta x^2 f''(x) + \frac{4}{3}\delta x^3 f'''(x) + \dots$$

and

$$f(x-2\delta x) = f(x) - 2\delta x f'(x) + 2\delta x^2 f''(x) - \frac{4}{3}\delta x^3 f'''(x) + \dots$$

Eliminating $f'(x)$ gives

$$f''(x) = \frac{f(x+\delta x) - 2f(x) + f(x-\delta x)}{\delta x^2} - \frac{1}{12}\delta x^2 f''''(x) + \dots$$

Neglecting the higher order terms produces the central difference approximation to $f''(x)$.Note that this leads to a second-order accurate approximation of the second derivative.

Errors:

Errors are always involved in performing any numerical computation. **Round-off errors** appear whenever computing takes place using a finite number of digits. This is the case when using modern computing machines. **Truncation error** is the error that exists even in the absence of round-off error and is the result of neglecting higher order terms in the finite difference approximations obtained from Taylor series expansions. Successful numerical work in conduction heat transfer requires attention to issues of accuracy and error control.