Nonlinear transient heat conduction in a stationary medium

$$\frac{\partial}{\partial x}\left(k(T)\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k(T)\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k(T)\frac{\partial T}{\partial z}\right)+Q=\rho c\frac{\partial T}{\partial t}\text{ , …………..….(1)}$$

Where $k(T)$ is a function of temperature, In terms of Enthalpy, the equation can be modified to give

$$\frac{\partial}{\partial x}\left[\frac{k(T)}{\rho c}\frac{\partial H}{\partial x}\right]+\frac{\partial}{\partial y}\left[\frac{k(T)}{\rho c}\frac{\partial H}{\partial y}\right]+\frac{\partial}{\partial z}\left[\frac{k(T)}{\rho c}\frac{\partial H}{\partial z}\right]+Q=\rho c\frac{\partial H}{\partial t}\text{ , ……………..(2)}$$

Where $H_2-H_1=\int_{T_1}^{T_2}\rho c dT$, the boundary conditions $T=T_b$ on Γ_b

$$k(T)\frac{\partial T}{\partial x}I_x+k(T)\frac{\partial T}{\partial y}I_y+k(T)\frac{\partial T}{\partial z}I_z+q+h(T-T_\infty)=0 \quad \text{On } \Gamma_q$$

Where I_x, I_y and I_z are direction cosines of outward normal

$h=$ Heat transfer coefficient, $T_\infty=$ ambient temperature, the initial condition for the problem is $T=T_0$ at $t=0$

Application of Galerkin's method for nonlinear heat conduction problems

The solution domain is divided in to finite elements in space. The temperature is approximated within each element by

$$T(x,y,z,t)=\sum_{i=1}^{m}N_i(x,y,z)T(t)$$

Where N_i is the usual shape functions defined piecewise or element by element, $T(t)$ is the nodal temperatures considered to be functions of time and m is the number of nodes in the element considered.

The Galerkin representation for the heat conduction problem is

$$\int N_i \left[\frac{\partial}{\partial x}\left(k_x(T)\frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y}\left(k_y(T)\frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z}\left(k_z(T)\frac{\partial T}{\partial z} \right) + Q - \rho c \frac{\partial T}{\partial t} \right] dxdydz = 0$$

…………(3)

Use integration by parts on the first three terms of equation (3) simplifies to

$$-\int \left[k_x(T)\frac{\partial T}{\partial x}\frac{\partial N_i}{\partial x} + k_y(T)\frac{\partial T}{\partial y}\frac{\partial N_i}{\partial y} + k_z(T)\frac{\partial T}{\partial z}\frac{\partial N_i}{\partial z} - N_i Q - N_i \rho c \frac{\partial T}{\partial t} \right] dxdydz$$

$$-\int N_i q d\Gamma_e - \int N_i h (T - T_\infty) d\Gamma_q = 0 \qquad i = 1,2,....m$$

………..(4)

Inserting the temperature approximation equation (4) will simplify to

$$-\int \left[k_x(T)\frac{\partial N_j}{\partial x}\frac{\partial N_i}{\partial x}\{T\} + k_y(T)\frac{\partial N_j}{\partial y}\frac{\partial N_i}{\partial y}\{T\} + k_z(T)\frac{\partial N_j}{\partial z}\frac{\partial N_i}{\partial z}\{T\} + k_z(T)\frac{\partial N_j}{\partial z}\frac{\partial N_i}{\partial z}\{T\} \right] dxdydz$$

$$-\int N_i N_j h \{T\} dT_h + \int N_i Q dxdydz - \int N_i N_j \rho c \, dxdydz \frac{\partial \{T\}}{\partial t}$$

$$-\int N_i q d\Gamma_q + \int N_i h T_\infty d\Gamma_h z = 0$$

……………………………………………………………………………………… (5)

Equation (5) can be put in to more convenient forms as

$M \dfrac{dT}{dt} + kT = f$, where

$$M_{ij} = \int \rho c N_i N_j dxdydz$$

$$K_{ij} = \int \left[k_x(T)\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + k_y(T)\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} + k_z(T)\frac{\partial N_i}{\partial z}\frac{\partial N_j}{\partial z} + k_z(T)\frac{\partial N_i}{\partial z}\frac{\partial N_j}{\partial z} \right] d\Omega + \int N_i N_j h d\Gamma_h$$

$$f_i = \int N_i Q d\Omega - \int N_i q d\Gamma_q + \int N_i h T_\infty d\Gamma_\infty$$

This nonlinear equation set requires an iterative solution. Following the simplest form of iteration method we could start from some initial guess:

$$T = T^0 = \left(T_1^0, T_2^0, T_3^0 ......T_m^0 \right)$$

And obtain an improved solution $T^{'}$ by solving the equation

$$M\frac{dT^{'}}{dt}+k\left(T^{0}\right)T^{'}=f^{0}$$

The general iteration scheme

$$M\frac{dT^{n}}{dt}+k\left(T^{n-1}\right)T^{n}=f^{n-1}$$

is then repeated until convergence. To within a suitable tolerance, is obtained.