CHAPTER 9

Heat Conduction Solution by FEM

Steady Heat Conduction Equations in apalne

Poisson's equation

$$k\frac{\partial^2 T}{\partial x^2}+k\frac{\partial^2 T}{\partial y^2}+q'''=0 \quad \text{on } \Omega$$

Boundary condition

a) Dirchlet BC (Natural BC)

$$T=T_s \qquad \text{on } \Gamma_1$$

b) Neuman BC (Essential BC)

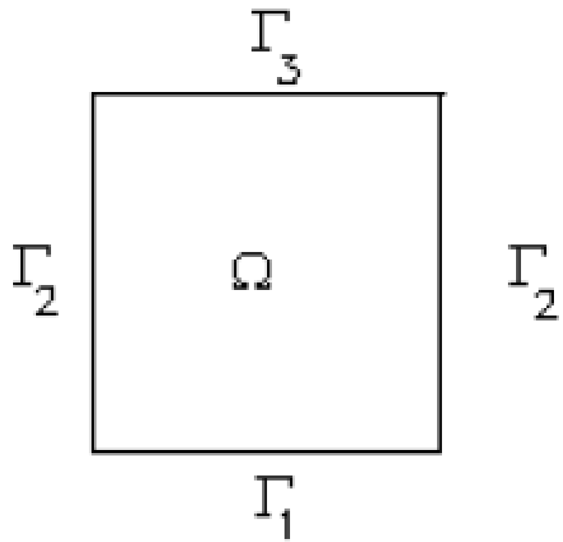
Convection heat transfer at the surface

$$-k\frac{\partial T}{\partial n}=h(T-T_\infty) \text{ on } \Gamma_2$$

c) Specfied heat flux at the surface

$$-k\frac{\partial T}{\partial n}=\dot{q} \qquad \text{on } \Gamma_3$$

Steady Heat Conduction Equations in apalne



Ω - domain Γ - boundary

Steady Heat Conduction Equations in a palne

In Glarkin's method, the weighting function is the same as the shape or interpolation function. Hence, the weighted residual of poisson's equation is

$$\int_{\Omega} N_i \left(k \frac{\partial^2 T}{\partial x^2} + k \frac{\partial^2 T}{\partial y^2} + q_v \right) d\Omega = 0$$

T – must be twice differentiable and it should satisfy all the boundary conditions on $\Gamma_e and\ \Gamma n$

Steady Heat Conduction Equations in apalne

Integration by Parts

$$\int_{\Omega} N \frac{\partial^2 T}{\partial x^2} dxdy = -\int_{A} \frac{\partial N}{\partial x} \frac{\partial T}{\partial x} dxdy + \oint N \frac{\partial T}{\partial x} l ds$$

$$\int_{\Omega} N \frac{\partial^2 T}{\partial y^2} dxdy = -\int_{A} \frac{\partial N}{\partial y} \frac{\partial T}{\partial y} dxdy + \oint N \frac{\partial T}{\partial y} m ds$$

$$\int_{\Omega} \left(Nk \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) dxdy \right) = -\oint_{A} \left(k \left(\frac{\partial N}{\partial x} \frac{\partial T}{\partial x} + \frac{\partial N}{\partial y} \frac{\partial T}{\partial y} \right) dxdy \right) + \oint_{\Gamma} \left(kN \left(\frac{\partial T}{\partial n} \right) \right) ds$$

Steady Heat Conduction Equations in a plane

Convective boundary at the surface

$$-k\left(\frac{\partial T}{\partial n}\right) = h(T - T_\infty)$$

Inseting the baove equation

$$-\oint_A\left(k\left(\frac{\partial N}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial N}{\partial y}\frac{\partial T}{\partial y} \right) - N_i q_v \right) dxdy + \oint_\Gamma \left(N\left(h(T_\infty - T\right)\right) ds = 0$$

Introducing interpolation of temperature and its spatial derivatives

Steady Heat Conduction Equations in apalne

Introducing interpolation of temperature and its spatial derivatives

$$-\int_{\Omega} k\left(\frac{\partial N_i}{\partial x}\sum_{j=1}^{nno} T_j \frac{\partial N_j}{\partial x} + k\frac{\partial N_i}{\partial y}\sum_{j=1}^{nno} T_j \frac{\partial N_j}{\partial y} - N_i q_v\right)d\Omega + \int_{\Gamma} N_i\left(h(T_\infty - \sum_{j=1}^{nno} N_j T_j\right)ds = 0 \qquad i = 1$$

Expressing in matrix form

$$(\int_{\Omega} k\left(\left\{\frac{\partial N}{\partial x}\right\}\left\{\frac{\partial N}{\partial x}\right\}^T + \left\{\frac{\partial N}{\partial y}\right\}\left\{\frac{\partial N}{\partial y}\right\}^T\right)d\Omega + \int_{\Gamma}(h\{N\}\{N\})^T ds)\{T\} = \int_{\Gamma}(\{N\}q_v)d\Omega + \int_{\Gamma}(\{N\}hT_\infty)ds$$

$$[K^e]\{T\} = \{F^e\}$$

Steady Heat Conduction Equations in apalne

$$K^e{}_{i,j} = \int_{\Omega} k\left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right)d\Omega + \int_{\Gamma} hN_iN_j ds = 0 \qquad i = 1 \ldots nno \;\; j = 1 \ldots nno$$

$$F_i = \int_{\Omega} N_i q_v d\Omega + \int_{\Gamma} hT_{\infty} ds = 0 \qquad i = 1 \ldots nno$$

Where:

$[K]$ is thermal stiffness Matrix

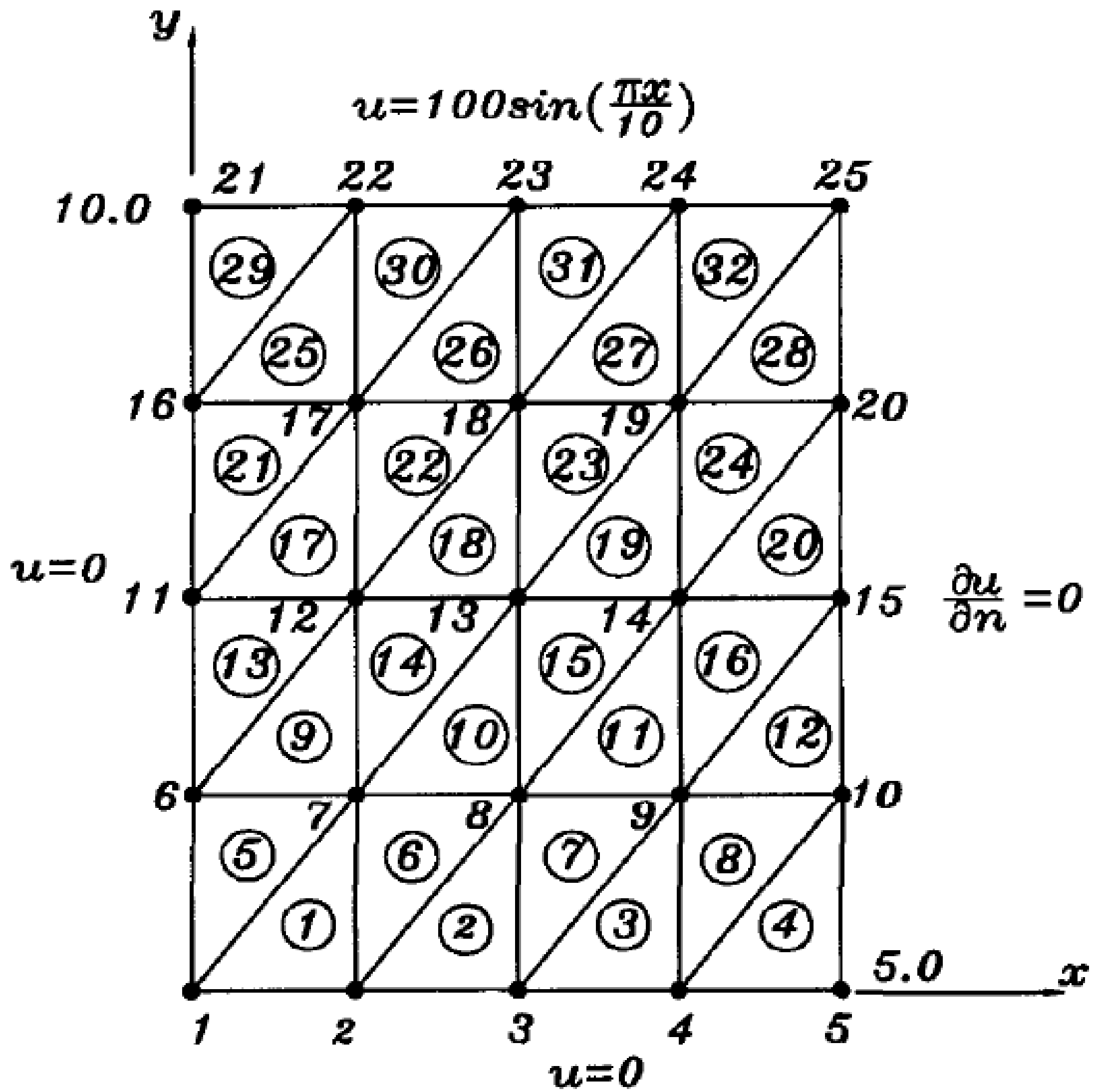
$\{F\}$ is thermal load vector

$$[K] = [K]_{cond} + [K]_{conv}$$

$[K]_{cond}$ - thermal stiffness matrix only due to conduction

$[K]_{conv}$ -Contribution of thermal stiffness matrix due to convection. Only for boundary with convective boundary condition.

Steady Heat Conduction Equations in a plane



Mesh with triangular element

Chapter 10.

Finite Element Discretization of Transient Heat Conduction Equation by Galerkin's Method

The PDE of transient heat conduction in two dimensions is given as

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + q''' \quad \text{on } \Omega$$

$$\int_{\Omega} N_i \rho c \frac{\partial T}{\partial t} dxdy - \int_{\Omega}\left(N_i k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) dxdy \right) = 0$$

Integration by Parts

$$l = \cos\theta$$

$$m = \sin\theta$$

$$\int_{\Omega} N \frac{\partial^2 T}{\partial x^2} dxdy = -\int_{A} \frac{\partial N}{\partial x}\frac{\partial T}{\partial x} dxdy + \oint N \frac{\partial T}{\partial x} l ds$$

$$\int_{\Omega} N \frac{\partial^2 T}{\partial y^2} dxdy = -\int_{A} \frac{\partial N}{\partial y}\frac{\partial T}{\partial y} dxdy + \oint N \frac{\partial T}{\partial y} m ds$$

$$\int_{\Omega} \left(Nk \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) dxdy \right) = -\oint_{A} \left(k \left(\frac{\partial N_i}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial T}{\partial y} \right) dxdy \right) + \oint_{\Gamma} \left(kN \left(\frac{\partial T}{\partial n} \right) \right) ds$$

Convective boundary at the surface

$$-k\left(\frac{\partial T}{\partial n}\right) = h(T - T_{\infty})$$

Inserting the above equation

$$-\oint_A \left(k\left(\frac{\partial N_i}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial T}{\partial y} \right) - N_i q_v \right) dxdy + \oint_\Gamma \left(N_i\left(h(T_\infty - T\right)\right) ds$$

Introducing interpolation of temperature and its spatial derivatives

$$\int_\Omega N_i \rho c \frac{\partial T}{\partial t} dxdy - \oint_A \left(k\left(\frac{\partial N_i}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial T}{\partial y} \right) dxdy \right) + \oint_\Gamma \left(k N_i \left(\frac{\partial T}{\partial n} \right) \right) ds = 0$$

$$\int_\Omega N_i \rho c \frac{\partial T}{\partial t} dxdy = \int_\Omega \rho c \{N\}\{N\}^T \left\{ \frac{\partial T}{\partial t} \right\} \{dxdy) = (\int_\Omega \{N\}\{N\}^T dxdy) \left\{ \frac{\partial T}{\partial t} \right\} = [C] \left\{ \frac{\partial T}{\partial t} \right\}$$

$$[C]\left\{ \frac{\partial T}{\partial t} \right\} + [K(T)]\{T\} = \{F\}$$

$$C_{i,j} = \int_\Omega \rho c N_i N_j d\Omega \qquad i = 1 \ldots nno \; j = 1 \ldots nno$$

$$K_{i,j} = \int_\Omega k \left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} \right) d\Omega + \int_\Gamma h N_i N_j ds = 0, i = 1 \ldots nno \; j = 1 \ldots nno$$

$$F_i = \int_\Omega N_i q_v d\Omega + \int_\Gamma h T_\infty ds = 0 \qquad i = 1 \ldots nno$$

Where:

$[C]$ is thermal Capacitance Matrix

$[K]$ is thermal stiffness Matrix

$\{F\}$ is thermal load vector

$$[K] = [K]_{cond} + [K]_{conv}$$

$[K]_{cond}$ - Thermal stiffness matrix only due to conduction

$[K]_{conv}$ - Contribution of thermal stiffness matrix due to convection,
Only for elements at boundary with convective boundary condition

Where T_s represents the surrounding temperature and

The above equation can finally be written in the following form

$$[C]\left\{\frac{\partial T}{\partial t}\right\}+([K']+[K''])\{T\}=\{F\}----(4)$$

Using generalized θ method

$$c\{\frac{\partial T}{\partial t}\}+[K']\{T\}=\{F\}----(4)$$

$$\{T\}=(1-\theta)\{T(t)\}+\theta\{T(t+\Delta t)\} ------(5)$$

Substituting equation 5 into equation 6 and rearranging,

The following equation will be obtained

$$\underbrace{([C]+\Delta t\theta[K])}_{[A]}\{T\}^{t+\Delta t}=\underbrace{\Delta t\{F\}^{t}+\big([C]-\Delta t[K](1-\theta)\big)}_{(V)}\{T\}^{t}$$

$$\{T\}^{t+\Delta t}=[A]^{-1}(V)$$

$$([C] + \theta\Delta t[K])\{T(t+\Delta t)\} = \Delta t\{F\} - ((1-\theta)\Delta t[K] - [C])\{T(t)\} \text{ ----- } (6)$$

Let, $[A] = ([C] + \theta\Delta t[K])$ and

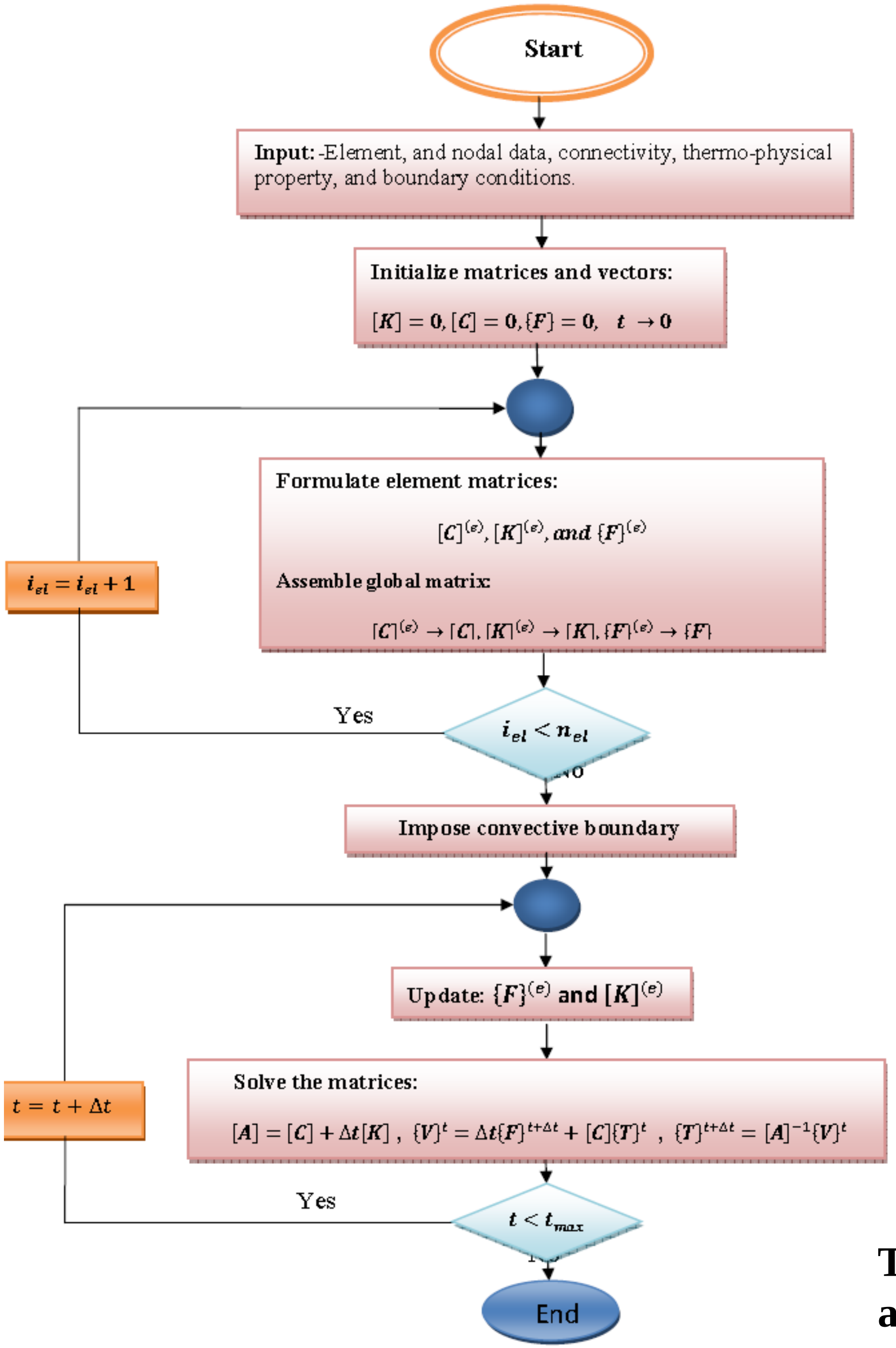
$$\{V\} = \Delta t\{F\} - ((1-\theta)\Delta t[K] - [C])\{T(t)\}$$

Then Substituting the above two equations into equation (6)

$$[A]\{T(t+\Delta t)\} = \{v\}$$

$$\{T(t+\Delta t)\} = [A]^{-1}\{v\}$$

Teta =0 Forward difference – Conditionally stable
Teta =1 Backward difference- Unconditionally Stable
Teta = 0.5 Crank Nicolson – Unconditionally stable with oscillation
Teta= 0.66 Galerkin- Unconditionally stable



Transient heat conduction solution algorithm using FEM