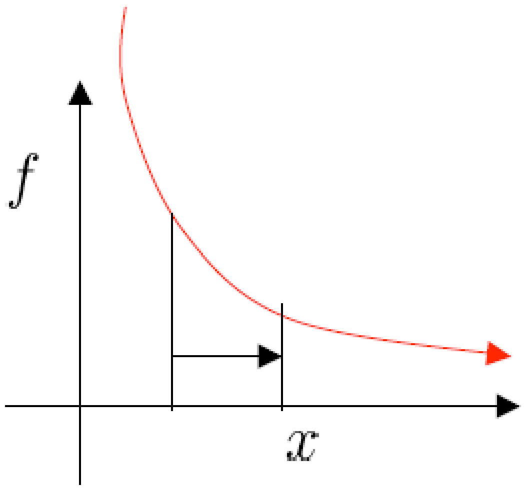
Chapter 4 : Numerical Integration

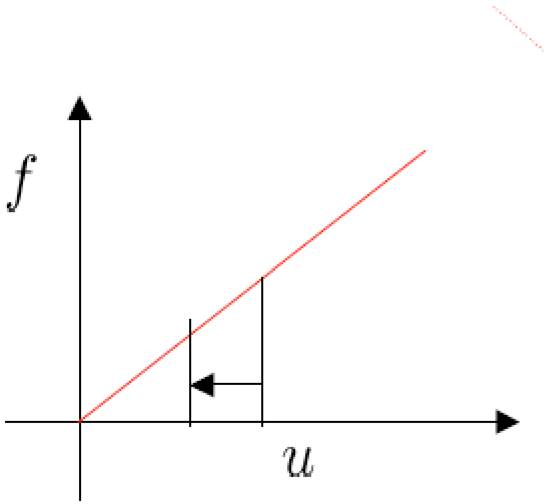
Jacobians- Mathematical Review

- In 1D problems we are used to a simple change of variables, e.g. from *x* to *u*

$$\int_a^b f(x)\mathrm{d}x = \int_\alpha^\beta f(x(u))\frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u$$

1D Jacobian





Example: $\int_1^2 \frac{1}{x}\mathrm{d}x = \ln(2)$ Substitute $x = u^{-1} \longrightarrow \frac{\mathrm{d}x}{\mathrm{d}u} = -u^{-2}$

$$= -\int_1^{\frac{1}{2}} \frac{u}{u^2}\mathrm{d}u = [\ln u]_{\frac{1}{2}}^{1} = \ln(2)$$

2D Jacobian

- For a continuous 1-to-1 transformation from (*x,y*) to (*u,v*), then

$$x = x(u,v) \text{ and } y = y(u,v)$$

$$\int\int_R f(x,y)\mathrm{d}x\mathrm{d}y = \int\int_{R'} f(x(u,v),y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \mathrm{d}u\mathrm{d}v$$

2D Jacobian

Where Region (in the *xy* plane) maps onto region R in the *uv* plane R'
maps areas d*x*d*y* to areas d*u*d*v*

- For a continuous 1-to-1 transformation from (x,y) to (u,v)
- Then $x = x(u,v)$ and $y = y(u,v)$

$$\int\int_R f(x,y)\mathrm{d}x\mathrm{d}y = \int\int_{R'} f(x(u,v),y(u,v)) \boxed{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|} \mathrm{d}u\mathrm{d}v$$

- Where Region (in the *xy* plane) maps onto region R in the *uv* plane R'

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

2D Jacobian

maps areas d*x*d*y* to areas d*u*d*v*

$$= \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = x_u y_v - x_v y_u$$

- Hereafter call such terms x_u

Relation between Jacobians

- The Jacobian matrix $\frac{\partial(x,y)}{\partial(u,v)}$ is the ***inverse matrix*** of $\frac{\partial(u,v)}{\partial(x,y)}$ i.e.,

$$\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- Because (and similarly for d*y*)

$$\mathrm{d}x = x_u \mathrm{d}u + x_v \mathrm{d}v = x_u \mathrm{d}u + x_v (v_x \mathrm{d}x + v_y \mathrm{d}y)$$

$$\begin{array}{ll} x \text{ constant} & \Rightarrow \mathrm{d}x = 0 \Rightarrow 0 = x_u u_y + x_v v_y \\ y \text{ constant} & \Rightarrow \mathrm{d}y = 0 \Rightarrow 1 = x_u u_x + x_v v_x \end{array}$$

- This makes sense because Jacobians measure the relative areas of d*x*d*y* and d*u*d*v*, i.e

$$\det(AB) = \det(A)\det(B) = 1 \Rightarrow \det(A) = \frac{1}{\det B}$$

- So

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1}$$

3D Jacobian

$$x = x(u, v, w), y = y(u, v, w), z = z(u, v, w)$$

- maps volumes (consisting of small cubes of volume $\mathrm{d}x\mathrm{d}y\mathrm{d}z$
- ....to small cubes of volume $\mathrm{d}u\mathrm{d}v\mathrm{d}w$

$$\int\int\int_V f(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int\int\int_{V'} F(u, v, w)\left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right|\mathrm{d}u\mathrm{d}v\mathrm{d}w$$

- Where

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

Numerical Integration

- Galerkin method requires integration over the element domain once for each interpolation function (trial solution).
- In fact, an integration is required to obtain the value of every component of the stiffness matrix of a finite element. In addition, integrations are required to obtain nodal equivalents of nonnodal loadings.
- In the finite element context, where large numbers of elements, hence huge numbers of integrations, are required, analytical methods are not efficient.
- Finite element software packages do not incorporate explicit integration of the element formulation equations. Instead, they use numerical techniques, the most popular of which is *Gaussian* (or *Gauss-Legendre*) *quadrature.*

Numerical Integration

- Integration is often called quadrature in one dimension and cubature in higher dimensions however we all refer to all numerical approximations as quadrature rules We all consider integrals and quadrature rules of the form

$$I = \iint_{\Omega_0} f(\xi, \eta) d\xi d\eta \approx \sum_{i=1}^{n} W_i f(\xi_i, \eta_i)$$

Numerical Integration

Two types of elements

Global element

- Spatial derivatives of shape functions needed here (in matrices $\mathbf{K}$)
- Integration boundaries given here.

Local element

- Shape functions defined here
- Numerical integration performed on local element (using Gauss-Legendre quadrature)

Transformation of spatial derivatives

Global element

- Spatial derivatives of shape functions

$$\nabla \mathbf{N}(x,y) = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \cdots & \frac{\partial N_9}{\partial x} \\ \frac{\partial N_1}{\partial y} & \cdots & \frac{\partial N_9}{\partial y} \end{bmatrix}$$

Local element

- Spatial derivatives of shape functions

$$\nabla \mathbf{N}(\xi,\eta) = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \cdots & \frac{\partial N_9}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \cdots & \frac{\partial N_9}{\partial \eta} \end{bmatrix}$$

Transformation

$$\nabla \mathbf{N}(x,y) = \mathbf{J}^{-1} \nabla \mathbf{N}(\xi,\eta)$$

$\boxed{\mathbf{J} \text{ is the Jacobian matrix}}$

Transformation of integration boundaries

Global element

- Integration boundaries

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)\,dx\,dy$$

Local element

- Integration boundaries

$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\,d\xi\,d\eta$$

J is the Jacobian matrix

Tranasformation

$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\,d\xi\,d\eta = \int_{-1}^{1}\int_{-1}^{1} f(x,y)\det(\mathbf{J})\,d\xi\,d\eta$$

Numerical integration

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)dx_i$$

Approximating integral as summation

$$\int_a^b f(x)dx = \sum_{i=1}^{N} f(x_i)\Delta x_i$$

Can be rearranged and written as by defining Wi

$$\int_a^b f(x)dx = \sum_{i=1}^{M} f(x_i)W_i$$

Gauss-Legendre-Quadrature

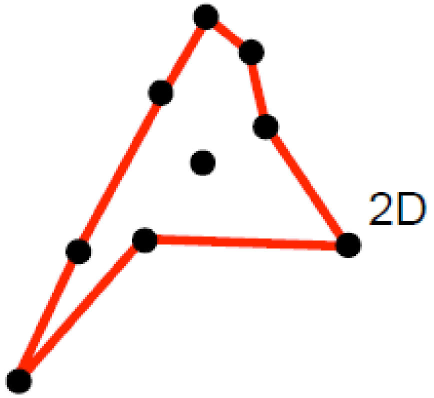
Gauss-Legendre-Quadrature

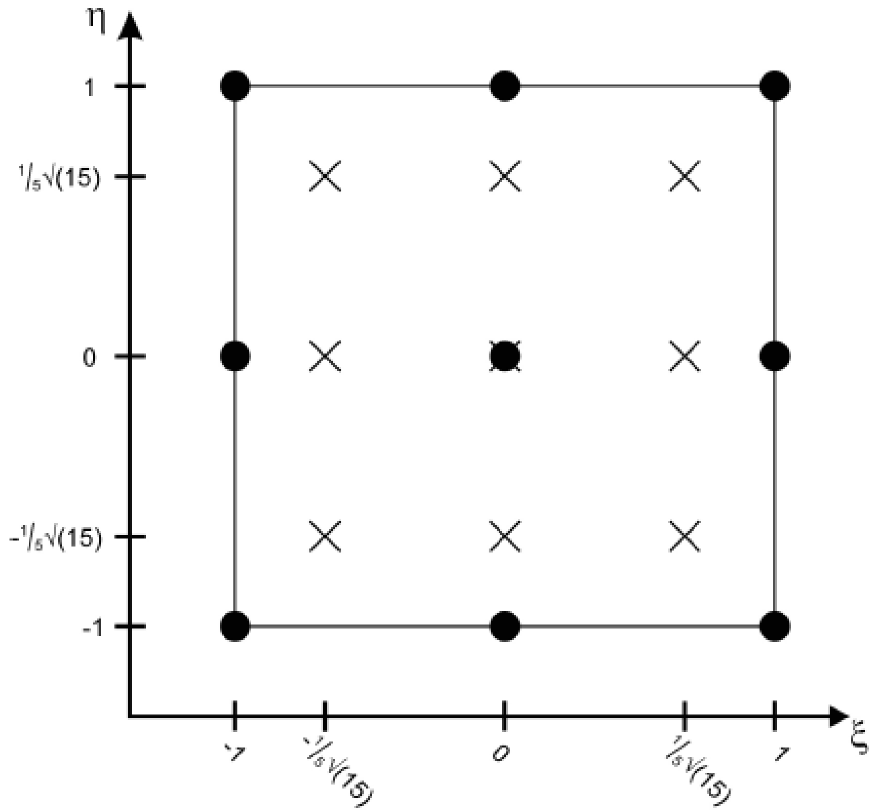
- Numerical integration with Gauss-Legendre-Quadrature only works on an idealized Element
  - For $x = -1$ to 1 in 1D
  - For $x = -1$ to 1 and $y = -1$ to 1 in 2D
- So, it does not solve the problem of the distorted elements, yet.
- A coordinate transformation from the distorted element to the idealized element is needed in addition.

Distorted vs. idealized element

$$\int_0^{dx}\begin{bmatrix}\frac{\partial N_i(x)}{\partial x}\frac{\partial N_i(x)}{\partial x} & \frac{\partial N_i(x)}{\partial x}\frac{\partial N_{i+1}(x)}{\partial x}\\ \frac{\partial N_{i+1}(x)}{\partial x}\frac{\partial N_i(x)}{\partial x} & \frac{\partial N_{i+1}(x)}{\partial x}\frac{\partial N_{i+1}(x)}{\partial x}\end{bmatrix}Adx\begin{Bmatrix}u_i\\ u_{i+1}\end{Bmatrix}-\int_0^{dx}\begin{Bmatrix}N_i(x)\\ N_{i+1}(x)\end{Bmatrix}Bdx=0$$

1D: FEM introduction





- Derivatives of shape functions with respect to global coordinates
- Integral form written in terms of global coordinates (dx)

- Shape functions given in terms of local coordinates ξ
- Numerical integration more convenient in a local coordinate system.

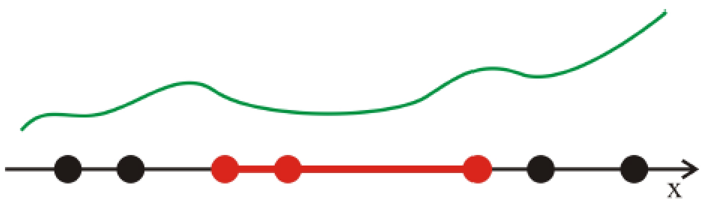
Two transformations are necessary

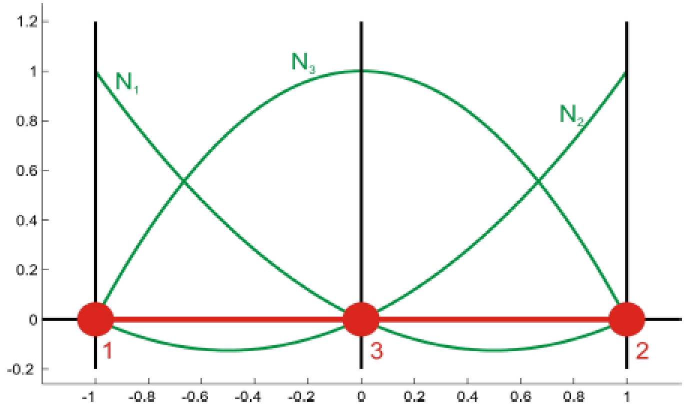
- Transform locally defined derivatives of shape functions to global coordinate system
- Transform locally performed (numerical) integration to global coordinates

First transformation in 1D

- **Derivatives of shape functions from local to global**

  - Global distorted element
    - Coordinate x arbitrary
    - Derivatives of shape functions wanted here
  - Local isoparametric element
    - Coordinate ξ from -1 to 1
    - Shape functions defined here
    - Derivatives of shape functions determinable here





First transformation in 1D

→ Derivatives of shape fcts. from local to global

- Global distorted element
  - Coordinate x arbitrary
  - Derivatives of shape functions wanted here

- Local isoparametric element
  - Coordinate ξ from -1 to 1
  - Shape functions defined here
  - Derivatives of shape functions determinable here

$$\mathbf{N}(\xi)=\begin{Bmatrix} N_1(\xi) \\ N_2(\xi) \\ N_3(\xi) \end{Bmatrix} \qquad \frac{\partial \mathbf{N}(\xi)}{\partial \xi}=\begin{Bmatrix} \partial N_1(\xi)/\partial \xi \\ \partial N_2(\xi)/\partial \xi \\ \partial N_3(\xi)/\partial \xi \end{Bmatrix}$$

Second transformation in 1D

→ Integration form from local to global

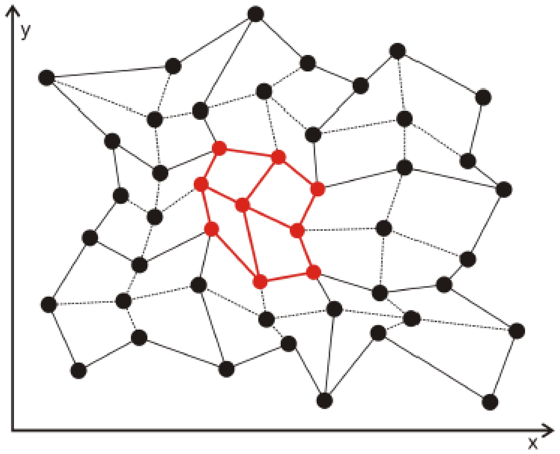
- Global distorted element
  - Coordinate x arbitrary
  - Integral form of system of equations given here

- Local isoparametric element
  - Coordinate ξ from -1 to 1
  - Numerical integration performed here

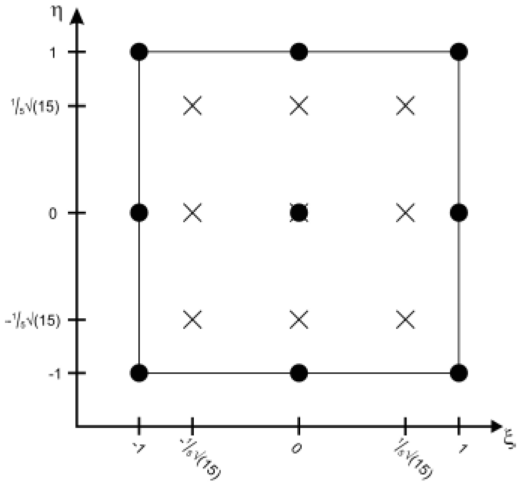
First transformation in 2D

→ Derivatives of shape fcts. from local to global

- Global distorted element
  - Coordinate x and y arbitrary
  - Derivatives of shape functions wanted here



- Local isoparametric element
  - Coordinate ξ and η from -1 to 1
  - Shape functions and their derivatives defined here



First transformation in 2D

→ Derivatives of shape fcts. from local to global

- Global distorted element
  - Coordinate x and y arbitrary
  - Derivatives of shape functions wanted here

- Local isoparametric element
  - Coordinate ξ and η from -1 to 1
  - Shape functions and their derivatives defined here

$$\mathbf{N}(\xi,\eta)=\begin{Bmatrix} N_1(\xi,\eta) \\ N_2(\xi,\eta) \\ \cdots \\ N_9(\xi,\eta) \end{Bmatrix}$$

$$\nabla_{\xi,\eta}\mathbf{N}(\xi,\eta)=\begin{bmatrix} \frac{\partial N_1(\xi,\eta)}{\partial \xi} & \frac{\partial N_1(\xi,\eta)}{\partial \eta} \\ \frac{\partial N_2(\xi,\eta)}{\partial \xi} & \frac{\partial N_2(\xi,\eta)}{\partial \eta} \\ \cdots & \cdots \\ \frac{\partial N_9(\xi,\eta)}{\partial \xi} & \frac{\partial N_9(\xi,\eta)}{\partial \eta} \end{bmatrix}$$

First transformation in 2D

→ Derivatives of shape fcts. from local to global

- Derivation of the Jacobian in a FEM manner!
  - Definition $\mathbf{J} = \begin{bmatrix} \partial x/\partial \xi & \partial y/\partial \xi \\ \partial y/\partial \eta & \partial x/\partial \eta \end{bmatrix}$
  - So

$$\mathbf{J} = \begin{bmatrix} \partial x/\partial \xi & \partial y/\partial \xi \\ \partial y/\partial \eta & \partial x/\partial \eta \end{bmatrix} = \nabla_{\xi,\eta}\mathbf{N}^T(\xi,\eta)\mathbf{x} = \begin{bmatrix} \dfrac{\partial N_1(\xi,\eta)}{\partial \xi} & \dfrac{\partial N_2(\xi,\eta)}{\partial \xi} & \cdots & \dfrac{\partial N_9(\xi,\eta)}{\partial \xi} \\ \dfrac{\partial N_1(\xi,\eta)}{\partial \eta} & \dfrac{\partial N_2(\xi,\eta)}{\partial \eta} & \cdots & \dfrac{\partial N_9(\xi,\eta)}{\partial \eta} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \cdots \\ x_9 \end{Bmatrix}$$

Second transformation in 2D

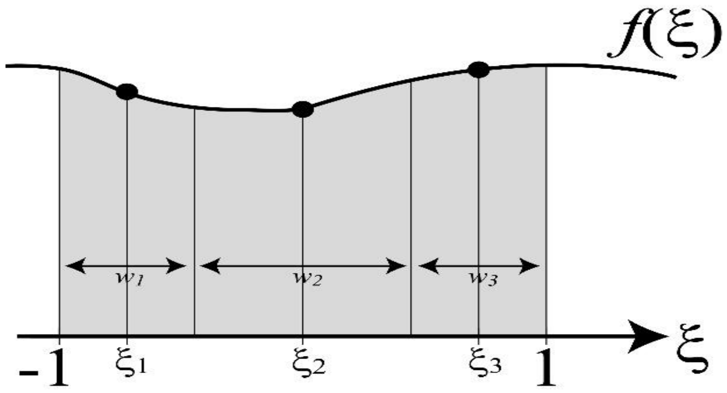
→ Integration form from local to global

- Global distorted element
  - Coordinate x and y arbitrary
  - Integral form of system of equations given here

- Local isoparametric element
  - Coordinate ξ and η from -1 to 1
  - Numerical integration performed here

Gauss-Legendre quadrature

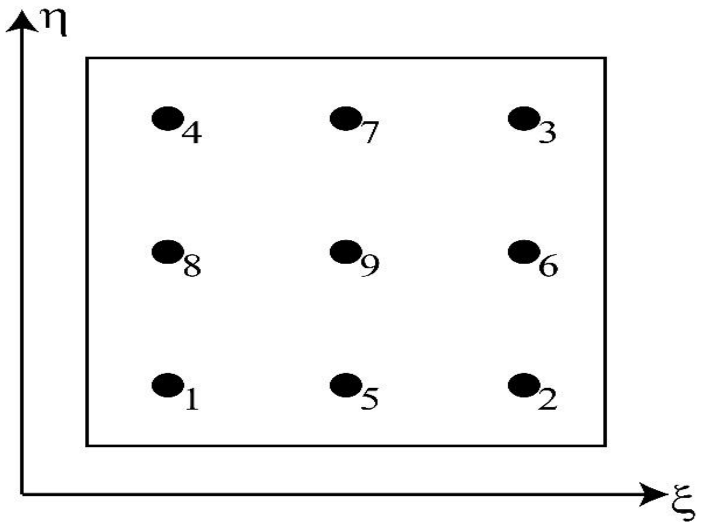
- The integration is carried between 1 and -1
- In 1D (n=3): $$\int_{-1}^{1} f(\xi)\,d\xi \approx \sum_{i=1}^{n} f(\xi_i)\,w_i$$



$$\boxed{\begin{array}{lll} \xi_1 = -\sqrt{\frac{3}{5}} \quad , & \xi_2 = 0 \quad , & \xi_3 = \sqrt{\frac{3}{5}} \\ w_1 = \frac{5}{9} \quad , & w_2 = \frac{8}{9} \quad , & w_3 = \frac{5}{9} \end{array}}$$

- In 2D (*Number of Integration Points*=9):

$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\,d\xi\,d\eta \approx \sum_{i=1}^{n\xi}\sum_{j=1}^{n\eta} f\left(\xi_i,\eta_j\right)w_i w_j = \sum_{k=1}^{nip} f\left(\xi_k,\eta_k\right)w_k$$



Gauss-Legendre quadrature

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)\,dxdy =$$

The integral is transformed to integration over master element

$$\int_{-1}^{1}\int_{-1}^{1} g(\xi,\eta)\,d\xi\, d\eta$$

$$= \int_{-1}^{1}\sum_{i=1}^{M_1} W_i\, g(\xi_i,\eta)\,d\eta$$

$$= \sum_{j=1}^{M_2}\bar{W}_j \sum_{i=1}^{M_1} W_i\, g(\xi_i,\eta_j)$$

$$= \sum_{i=1}^{M_1}\sum_{j=1}^{M_2} W_i \bar{W}_j\, g(\xi_i,\eta_j)$$

Integration points and weights

n	*Int.Point*	Weight
1	0. 0.00000 00000 00000	2.00000 00000 00000
2	±0.57735 02691 89626	1.00000 00000 00000
3	±0.77459 66692 41483	0.55555 55555 55556
	0.00000 00000 00000	0.88888 88888 88889

Jacobian Matrix in FEM

It is used to transformer the derivatives of shape function from global to local coordinate system

$$\left\{ \begin{array}{c} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{array} \right\} = \left\{ \begin{array}{cc} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{array} \right\} \left\{ \begin{array}{c} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{array} \right\} = [J] \left\{ \begin{array}{c} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{array} \right\}$$

where $[J]$ is the Jacobian matrix.

Inverse of Jacobian Matrix

It is used to transformer the derivatives of shape function from local to global coordinate system

$$\left\{ \begin{array}{c} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{array} \right\} = [J]^{-1} \left\{ \begin{array}{c} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{array} \right\}$$

,

Elements of Jacobian Matrix

$$\frac{\partial x}{\partial \xi} = \sum \frac{\partial N_i}{\partial \xi} x_i \ , \quad \frac{\partial x}{\partial \eta} = \sum \frac{\partial N_i}{\partial \eta} x_i$$
$$\frac{\partial y}{\partial \xi} = \sum \frac{\partial N_i}{\partial \xi} y_i \ , \quad \frac{\partial y}{\partial \eta} = \sum \frac{\partial N_i}{\partial \eta} y_i$$

The determinant of Jacobean Matrix is used to transform the integral from local to global coordinates.

$$dxdy = |J| d\xi d\eta$$

The components in the *Jacobian* matrix are computed

$$J_{11} = \frac{\partial x}{\partial \xi} = \sum_{i=1}^{4} \frac{\partial H_i(\xi, \eta)}{\partial \xi} x_i$$

$$J_{12} = \frac{\partial y}{\partial \xi} = \sum_{i=1}^{4} \frac{\partial H_i(\xi, \eta)}{\partial \xi} y_i$$

$$J_{21} = \frac{\partial x}{\partial \eta} = \sum_{i=1}^{4} \frac{\partial H_i(\xi, \eta)}{\partial \eta} x_i$$

$$J_{22} = \frac{\partial y}{\partial \eta} = \sum_{i=1}^{4} \frac{\partial H_i(\xi, \eta)}{\partial \eta} y_i$$