5. Beams and
6. Frames

Beams and frames

- Beams are slender members used for supporting transverse loading.
- Beams with cross sections symmetric with respect to loading are considered.

$$\sigma = -\frac{M}{I} y$$

$$\in = \sigma / E$$

$$d^2 v / dx^2 = M / EI$$

Potential energy approach

Strain energy in an element of length dx is

$$dU = \frac{1}{2}\int_A \sigma\varepsilon dA dx$$

$$= \frac{1}{2}\left(\frac{M^2}{EI^2}\int_A y^2 dA\right)dx$$

$$\int_A y^2 dA \quad \textit{is the moment of inertia } I$$

The total strain energy for the beam is given by-

$$U = \frac{1}{2}\int_0^L EI\left(d^2v/dx^2\right)dx$$

Potential energy of the beam is then given by-

$$\Pi = \frac{1}{2}\int_0^L EI\left(d^2v/dx^2\right)dx - \int_0^L pvdx - \sum_m p_m v_m - \sum_k M_k v_k'$$

Where-

-p is the distributed load per unit length

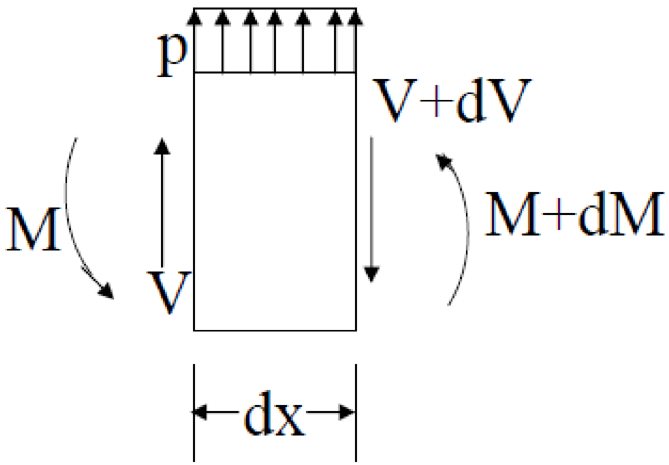
-p_m is the point load at point m.

-M_k is the moment of couple applied at point k

-v_m is the deflection at point m

-v'_k is the slope at point k.

Galerkin's Approach



- Here we start from equilibrium of an elemental length.

dV/dx = p
dM/dx =V

$$d^2v/dx^2 = M/EI$$

$$\frac{d^2}{dx^2}\left(EI\, {d^2v}\big/{dx^2}\right) - p = 0$$

For approximate solution by Galerkin's approach-

$$\int_0^L \left[\frac{d}{dx^2}\left(EI\, {d^2v}\big/{dx^2}\right) - p\right]\Phi\, dx = 0$$

Φ is an arbitrary function using same basic functions as v

Integrating the first term by parts and splitting the interval 0 to L to (0 to x_m), (x_m to x_k) and (x_k to L) we get-

$$\int_0^L EI\frac{d^2v}{dx^2}\frac{d^2\Phi}{dx^2}dx-\int_0^l p\Phi dx+\frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right)\Phi\Bigg|_0^{x_m}$$

$$+\frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right)\Phi\Bigg|_{x_m}^{L}-EI\frac{d^2v}{dx^2}\frac{d\Phi}{dx}\Bigg|_0^{x_k}-EI\frac{d^2v}{dx^2}\frac{d\Phi}{dx}\Bigg|_{x_k}^{L}=0$$

Further simplifying-

$$\int_0^L EI\frac{d^2v}{dx^2}\frac{d^2\Phi}{dx^2}dx-\int_0^L p\Phi dx-\sum_m p_m\Phi_m-\sum_k M_k\Phi_k^{'}=0$$

Φ and M are zero at support..at x_m shear force is p_m and at x_k Bending moment is $-M_k$

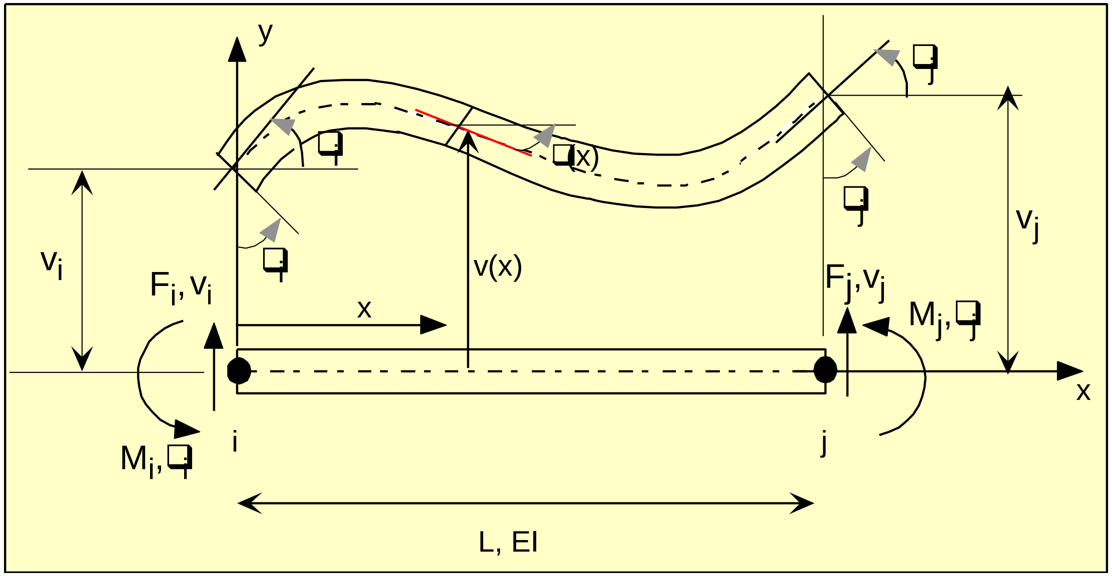
Element Formulation

assume the displacement *w* is a cubic polynomial

$$v(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3$$

a_1, a_2, a_3, a_4 are the undetermined coefficients

L = Length
I = Moment of Inertia of the cross sectional area
E = Modulus of Elsaticity
v = v(x) deflection of the neutral axis
θ= dv/dx slope of the elastic curve (rotation of the section
F = F(x) = shear force
M= M(x) = Bending moment about Z-axis



$$v(x) = \begin{Bmatrix} 1 & x & x^2 & x^3 \end{Bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix}; \quad \theta(x) = \begin{Bmatrix} 0 & 1 & 2x & 3x^2 \end{Bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix}$$

$$x = 0, \quad v(0) = v_1; \quad \left.\frac{dv}{dx}\right|_{x=0} = \theta_1$$

$$x = L, \quad v(L) = v_2; \quad \left.\frac{dv}{dx}\right|_{x=L} = \theta_2$$

$$\begin{Bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{Bmatrix} = \begin{Bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{Bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix}$$

- Applying these boundary conditions, we get

$$\{d\} = [P(x)]\{a\}$$

$$\{a\} = [P(x)]^{-1}\{d\}$$

$$a_1 = v_1; \qquad a_2 = \theta_1$$

$$a_3 = \frac{1}{L^2}(-3v_1 - 2L\theta_1 + 3v_2 - L\theta_2)$$

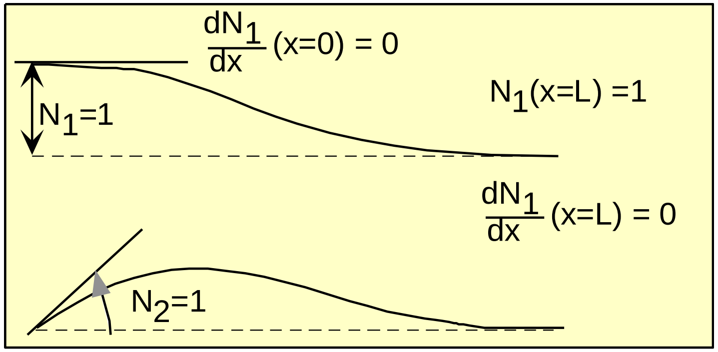
- Substituting coefficients a_i back into the original equation

$$v(x) = \{1 \quad x \quad x^2 \quad x^3\}\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix}$$

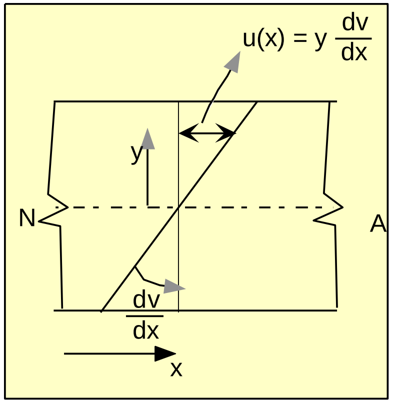
- The interpolation function or shape function is given by

$$v(x) = (1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3})v_1 + (x - \frac{2x^2}{L} + \frac{x^3}{L^2})\theta_1$$
$$+(\frac{3x^2}{L^2} - \frac{2x^3}{L^3})v_2 + (-\frac{x^2}{L} + \frac{x^3}{L^2})\theta_2$$

$$v = \begin{bmatrix} N_1(x) & N_2(x) & N_3(x) & N_4(x) \end{bmatrix} \begin{Bmatrix} v_1 \\ L\theta_1 \\ v_2 \\ L\theta_2 \end{Bmatrix} = [N]\{d\}$$



$$\varepsilon = \frac{du}{dx} = y\frac{d^2v}{dx^2} = \frac{d^2[N]}{dx^2}\{d\} = y[B]\{d\}$$



$$[B] = \left[\frac{12x}{L^3} - \frac{6}{L^2} \quad \frac{6x}{L^3} - \frac{4}{L^2} \quad \frac{6}{L^2} - \frac{12x}{L^3} \quad \frac{6x}{L^3} - \frac{2}{h^2}\right]$$

Internal virtual energy $\delta U^e = \int_{v^e} \delta\{\varepsilon\}^T \{\sigma\} dv$

substitute $\{\sigma\} = [E]\{\varepsilon\}$ in above eqn.

$$\delta U^e = \int_{v^e} \delta\{\varepsilon\}^T [E]\{\varepsilon\} dv$$

$$\delta\{\varepsilon\} = y[B]\delta\{d\}$$

$$\delta U^e = \int_{v^e} \delta\{d\}^T [B]^T [E][B]\{d\} y^2 dv$$

External virtual workdue to body force

$$\delta w_b^e = \int_{v^e} \delta\{d(x)\}^T \{b\} dv = \int_{v^e} \delta\{d\}^T [N]^T \{b_y\} dv$$

External virtual work due to surface force

$$\delta w_s^e = \int_s \delta\{d(x)\}^T \{p\} dv = \int_s \delta\{d\}^T [N]^T \{p_y\} ds$$

External virtual work due to nodal forces

$$\delta w_c^e = \delta\{d\}^T \{P^e\}, \quad \{P^e\}^T = \{P_{yi}, \ M_i, P_{yj}, ...\}$$

From virtual work principle $\delta U^e = \delta W^e$

$$\delta\{d\}^T (\int_{v^e} [B]^T [E][B] y^2 dv \{d\} = \delta\{d\}^T \left(\int_{v^e} [N]^T \{b_y\} dv + \int_s [N]^T \{p_y\} dv + \{P^e\} \right)$$

$$\Rightarrow [K_e]\{U^e\} = \{F_e\}$$

where

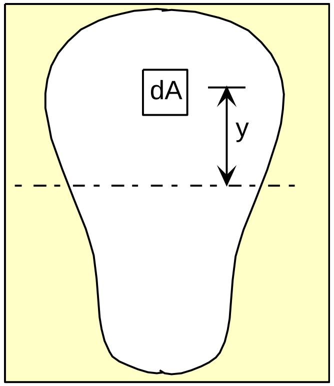
$$[K_e] = \int_{v^e} [B]^T [D][B] y^2 dv = \text{Element stiffness matrix}$$

$$\{F_e\} = \int_{v^e} [N]^T \{b_y\} dv + \int_s [N]^T \{p_y\} ds + \{P^e\} = \text{Total nodal force vector}$$

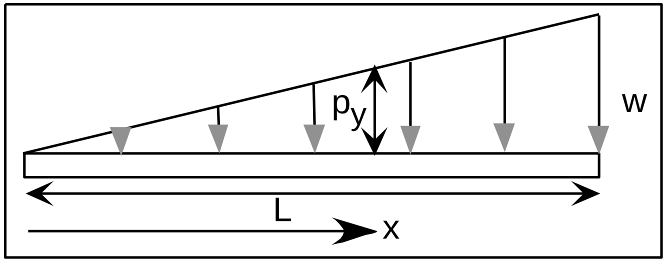
◆the stiffness matrix [k] is defined

$$[k] = \int_V [B]^T E [B] dV = \int_A \left(dAy^2\right) E \int_0^L [B]^T [B] dx$$

$$= \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$



To compute equivalent nodal force vector for the loading shown



$$\{F_e\} = \int_s [N]^T \{p_y\} ds$$

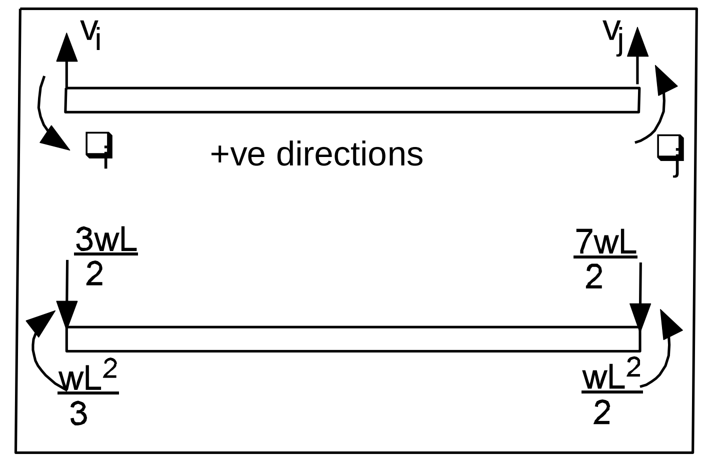
From similar triangles

$$\frac{p_y}{x} = \frac{w}{L}; \quad p_y = \frac{w}{L}x; \quad ds = 1 \cdot dx$$

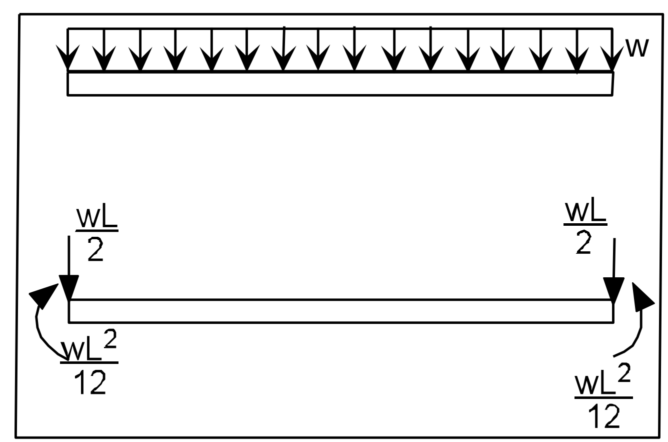
$$[N] = \left\{ \left(1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}\right) \quad \left(x - \frac{2x^2}{L} + \frac{x^3}{L^2}\right) \quad \left(\frac{3x^2}{L^2} - \frac{2x^3}{L^3}\right) \quad \left(-\frac{x^2}{L} + \frac{x^3}{L^2}\right) \right\}$$

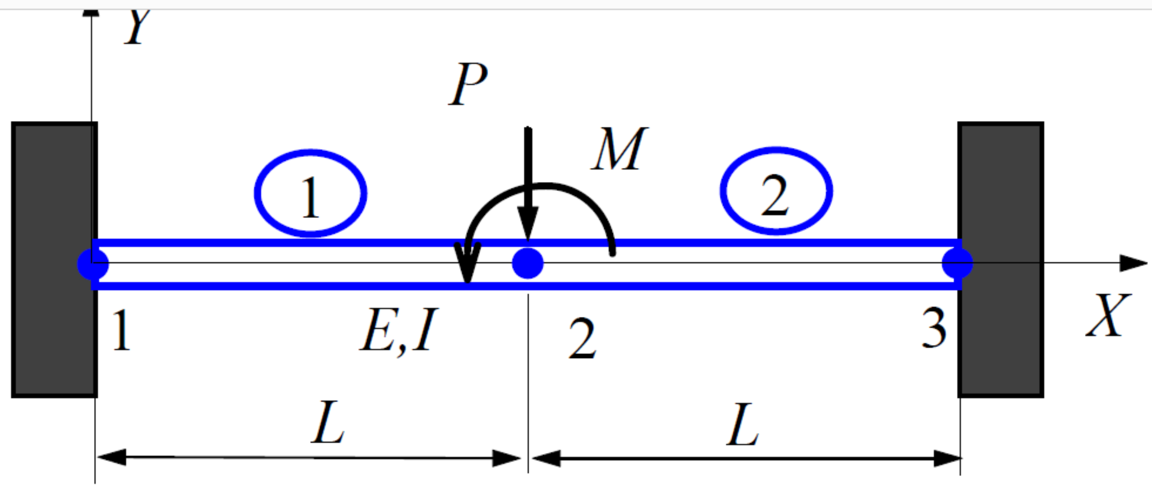
$$\{F_e\} = \int_s [N]^T \{p_y\} ds$$

$$\{F_e\} = \int_L \begin{Bmatrix} (1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}) \\ (x - \frac{2x^2}{L} + \frac{x^3}{L^2}) \\ (\frac{3x^2}{L^2} - \frac{2x^3}{L^3}) \\ (-\frac{x^2}{L} + \frac{x^3}{L^2}) \end{Bmatrix} \left\{ -\frac{wx}{L} \right\} dx = \begin{Bmatrix} -\frac{3wL}{20} \\ -\frac{wL^2}{30} \\ -\frac{7wL}{20} \\ \frac{wL^2}{20} \end{Bmatrix}$$



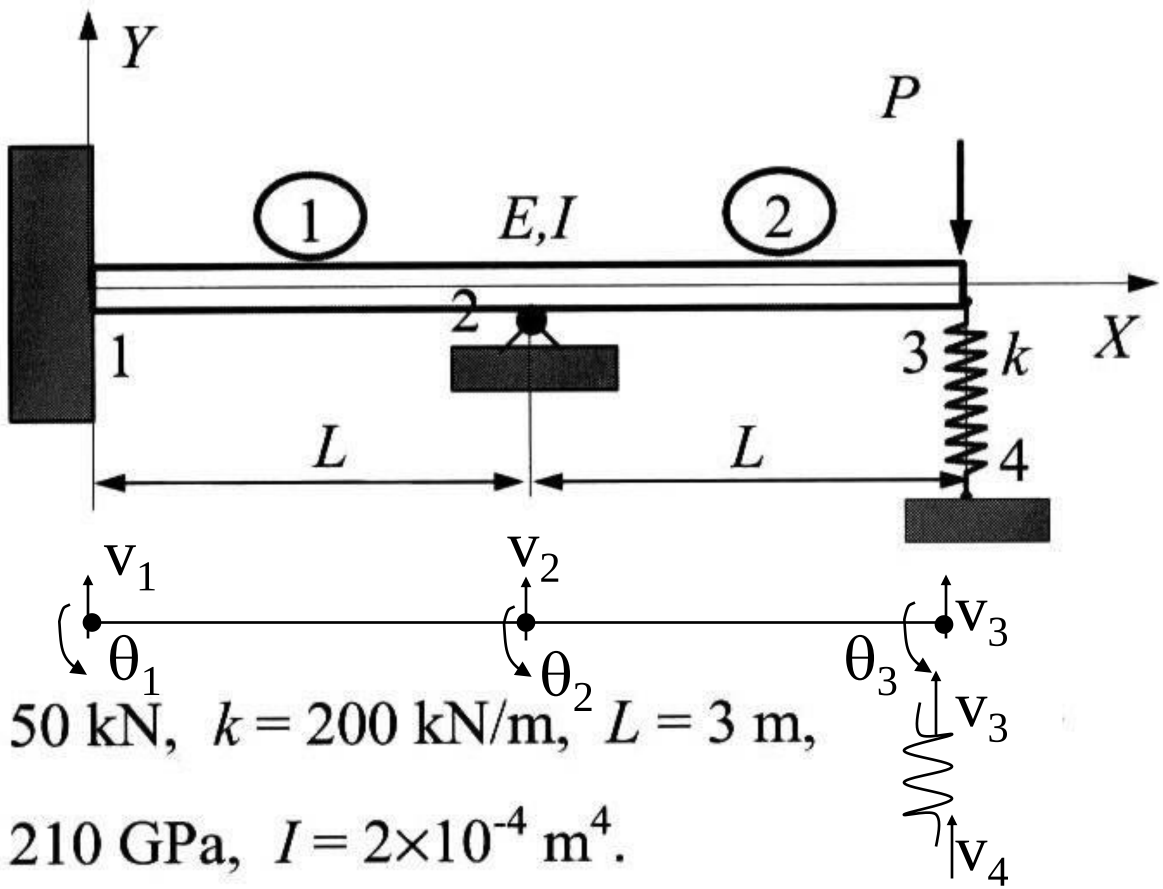
Equivalent nodal force due to
Uniformly distributed load w





Given: The beam shown above is clamped at the two ends and acted upon by the force P and moment M in the mid-span.

Find: The deflection and rotation at the center node and the reaction forces and moments at the two ends.



Given: $P = 50$ kN, $k = 200$ kN/m, $L = 3$ m,

$E = 210$ GPa, $I = 2\times10^{-4}$ m^4.

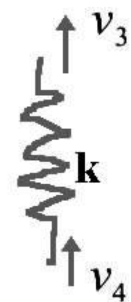
Find: **Slope, deflection, reactions and member end forces**

Solution:

The beam has a roller (or hinge) support at node 2 and a spring support at node 3. We use two beam elements and one spring element to solve this problem.

The spring stiffness matrix is given by,

$$\mathbf{k}_s = \begin{array}{c} \begin{matrix} v_3 & \quad v_4 \end{matrix} \\ \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \end{array} \begin{matrix} v_3 \\ v_4 \end{matrix}$$



Stiffness matrix for element 1

$$
\mathbf{k}_1 = \frac{EI}{L^3}
\begin{array}{c}
\begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\
\begin{bmatrix}
12 & 6L & -12 & 6L \\
6L & 4L^2 & -6L & 2L^2 \\
-12 & -6L & 12 & -6L \\
6L & 2L^2 & -6L & 4L^2
\end{bmatrix}
\end{array}
\begin{array}{c} \\ v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array}
$$

Stiffness matrix for element 2

$$
\mathbf{k}_2 = \frac{EI}{L^3}
\begin{array}{c}
\begin{array}{cccc} v_2 & \theta_2 & v_3 & \theta_3 \end{array} \\
\begin{bmatrix}
12 & 6L & -12 & 6L \\
6L & 4L^2 & -6L & 2L^2 \\
-12 & -6L & 12 & -6L \\
6L & 2L^2 & -6L & 4L^2
\end{bmatrix}
\end{array}
\begin{array}{c} \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{array}
$$

$$
\frac{EI}{L^3}
\begin{array}{c}
\begin{array}{ccccccc} v_1 & \theta_1 & v_2 & \theta_2 & v_3 & \theta_3 & v_4 \end{array} \\
\begin{bmatrix}
12 & 6L & -12 & 6L & 0 & 0 & 0 \\
 & 4L^2 & -6L & 2L^2 & 0 & 0 & 0 \\
 & & 24 & 0 & -12 & 6L & 0 \\
 & & & 8L^2 & -6L & 2L^2 & 0 \\
 & & & & 12+k' & -6L & -k' \\
 & & & & & 4L^2 & 0 \\
 & Symmetry & & & & & k'
\end{bmatrix}
\end{array}
\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \\ v_4 \end{Bmatrix}
=
\begin{Bmatrix} F_{1Y} \\ M_1 \\ F_{2Y} \\ M_2 \\ F_{3Y} \\ M_3 \\ F_{4Y} \end{Bmatrix}
$$

in which

$$
k' = \frac{L^3}{EI} k
$$

is used to simply the notation.

We now apply the boundary conditions,

$$v_1 = \theta_1 = v_2 = v_4 = 0,$$
$$M_2 = M_3 = 0, \qquad F_{3Y} = -P$$

Deleting'the first three and seventh equations (rows and columns), we have the following reduced equation,

$$\frac{EI}{L^3}\begin{bmatrix} 8L^2 & -6L & 2L^2 \\ -6L & 12+k' & -6L \\ 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P \\ 0 \end{Bmatrix}$$

Solving this equation, we obtain the deflection and rotations at node 2 and node 3,

$$\begin{Bmatrix} \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = -\frac{PL^2}{EI(12+7k')}\begin{Bmatrix} 3 \\ 7L \\ 9 \end{Bmatrix}$$

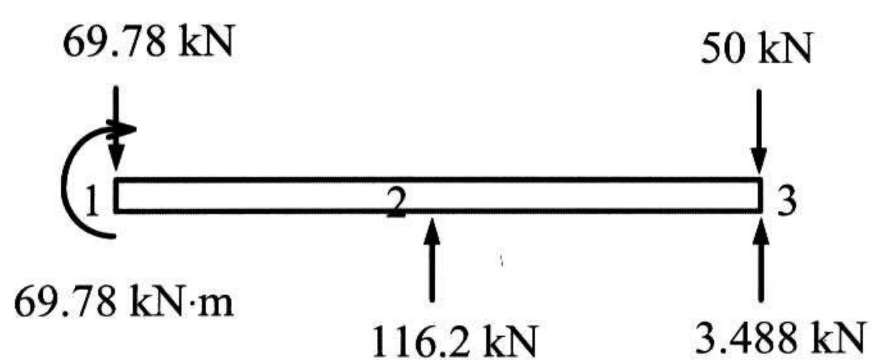
The influence of the spring k is easily seen from this result. Plugging in the given numbers, we can calculate

$$\begin{Bmatrix} \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} -0.002492 \text{ rad} \\ -0.01744 \text{ m} \\ -0.007475 \text{ rad} \end{Bmatrix}$$

From the global FE equation, we obtain the nodal reaction forces as,

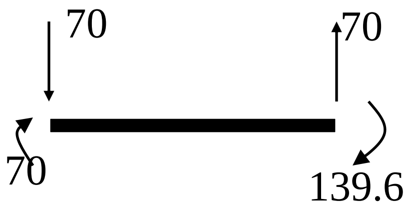
$$\begin{Bmatrix} F_{1Y} \\ M_1 \\ F_{2Y} \\ F_{4Y} \end{Bmatrix} = \begin{Bmatrix} -69.78 \text{ kN} \\ -69.78 \text{ kN}\cdot\text{m} \\ 116.2 \text{ kN} \\ 3.488 \text{ kN} \end{Bmatrix}$$

Checking the results: Draw *free body diagram* of the beam

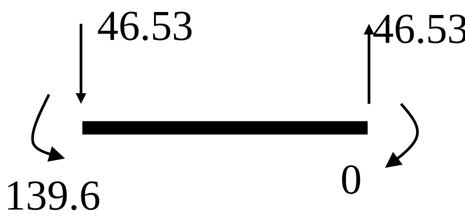


For element 1

$$\begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix} = 1555.6 \begin{bmatrix} 12 & 18 & -12 & 18 \\ 18 & 36 & -18 & 18 \\ -12 & -18 & 12 & -18 \\ 18 & 18 & -18 & 36 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.00249 \end{Bmatrix} = \begin{Bmatrix} -70 \\ -70 \\ 70 \\ -139.6 \end{Bmatrix}$$



$$\begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix} = 1555.6 \begin{bmatrix} 12 & 18 & -12 & 18 \\ 18 & 36 & -18 & 18 \\ -12 & -18 & 12 & -18 \\ 18 & 18 & -18 & 36 \end{bmatrix} \begin{Bmatrix} 0 \\ -0.00249 \\ -0.01744 \\ -0.007475 \end{Bmatrix} = \begin{Bmatrix} -46.53 \\ 139.6 \\ 46.53 \\ 0 \end{Bmatrix}$$



For the beam and loading shown in Fig. E8.1, determine (1) the slopes at 2

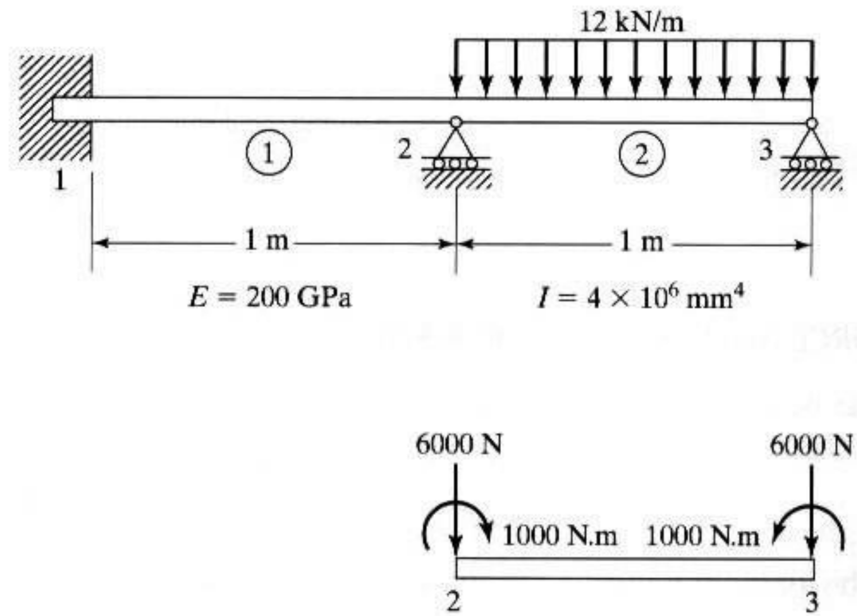


FIGURE E8.1

$$\frac{EI}{\ell^3} = \frac{(200 \times 10^9)(4 \times 10^{-6})}{1^3} = 8 \times 10^5\ \text{N/m}$$

$$\mathbf{k}^1 = \mathbf{k}^2 = 8 \times 10^5 \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}$$

$$e = 1 \qquad v_1 \quad \theta_1 \quad v_2 \quad \theta_2$$

$$e = 2 \qquad v_2 \quad \theta_2 \quad v_3 \quad \theta_3$$

$$\begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \\ V_3 \\ M_3 \end{Bmatrix} = 8x10^5 \begin{bmatrix} 12 & 6 & -12 & 6 & & \\ 6 & 4 & -6 & 2 & & \\ -12 & -6 & 12+12 & -6+6 & -12 & 6 \\ 6 & 2 & -6+6 & 4+4 & -6 & 2 \\ & & -12 & -6 & 12 & -6 \\ & & 6 & 2 & -6 & 4 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix}$$

Boundary condition

$$v_1, \theta_1, v_2, v_3 = 0$$

Loading Condition

$$M_2 = -1000; \quad M_3 = 1000$$

$$8x10^5 \begin{bmatrix} 8 & 2 \\ 2 & 4 \end{bmatrix} \begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} -1000 \\ 1000.0 \end{Bmatrix}$$

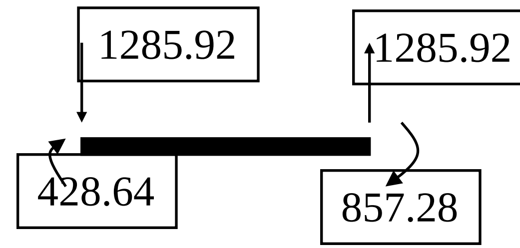
$$\begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix} = \frac{1}{28 * 8x10^5} \begin{bmatrix} 4 & -2 \\ -2 & 8 \end{bmatrix} \begin{Bmatrix} -1000 \\ 1000.0 \end{Bmatrix} = \begin{Bmatrix} -2.679x10^{-4} \\ 4.464x10^{-4} \end{Bmatrix}$$

Final member end forces

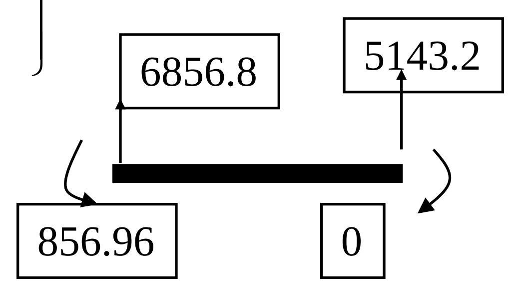
$$\{f\} = [k]\{d\} + \{FEMS\}$$

For element 1

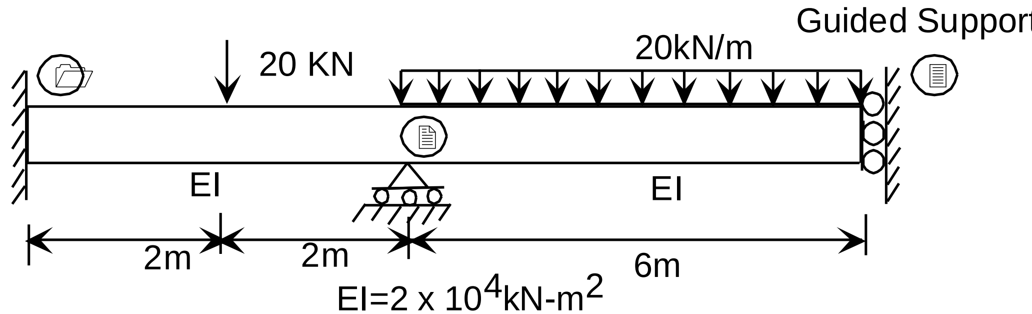
$$\begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + 8X10^5 \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -2.679x10^{-4} \end{Bmatrix} = \begin{Bmatrix} -1285.92 \\ -428.64 \\ 1285.92 \\ -857.28 \end{Bmatrix}$$



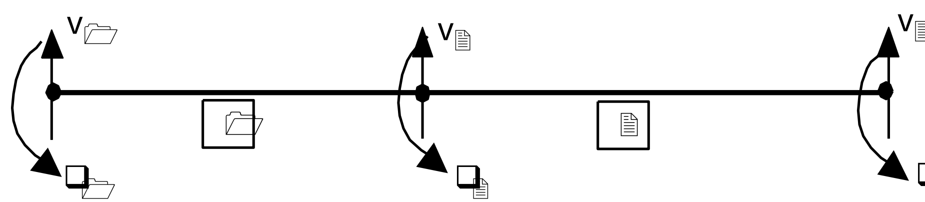
$$\begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix} = \begin{Bmatrix} 6000 \\ 1000 \\ 6000 \\ -1000 \end{Bmatrix} + 8X10^5 \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \begin{Bmatrix} 0 \\ -2.679x10^{-4} \\ 0 \\ 4.464x10^{-4} \end{Bmatrix} = \begin{Bmatrix} 6856.8 \\ 856.96 \\ 5143.2 \\ 0 \end{Bmatrix}$$



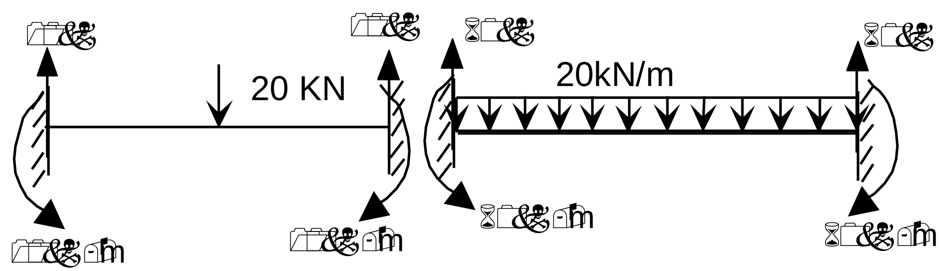
Find slope at joint 2 and deflection at joint 3. Also find member end forces



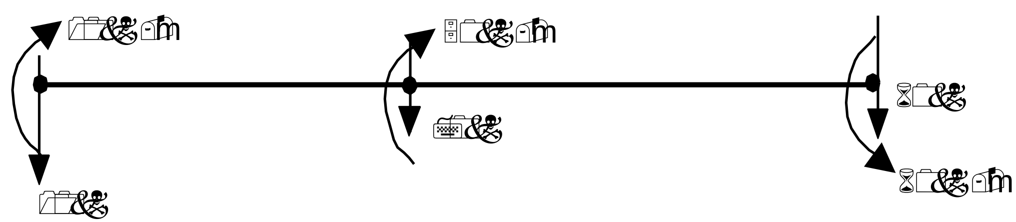
Global coordinates



Fixed end reactions (FERs)



Action/loads at global coordinates



For element 1

$$\begin{Bmatrix} f_1 \\ m_1 \\ f_2 \\ m_2 \end{Bmatrix} = \frac{1X10^4}{4^3} \begin{array}{c} \begin{bmatrix} 12 & 24 & -12 & 24 \\ 24 & 64 & -24 & 32 \\ -12 & -24 & 12 & -24 \\ 24 & 32 & -24 & 64 \end{bmatrix} \\ \begin{matrix} v_1 & \theta_1 & v_2 & \theta_2 \end{matrix} \end{array} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

For element 2

$$\begin{Bmatrix} f_1 \\ m_1 \\ f_2 \\ m_2 \end{Bmatrix} = \frac{1X10^4}{6^3} \begin{array}{c} \begin{bmatrix} 12 & 36 & -12 & 36 \\ 36 & 144 & -36 & 72 \\ -12 & -36 & 12 & -36 \\ 36 & 72 & -36 & 144 \end{bmatrix} \\ \begin{matrix} v_2 & \theta_2 & v_3 & \theta_3 \end{matrix} \end{array} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \\ F_3 \\ M_3 \end{Bmatrix} = \begin{bmatrix} 1875 & 3750 & -1875 & 3750 & & \\ 3750 & 10000 & -3750 & 5000 & & \\ -1875 & -3750 & 1875+555.56 & -3750+1666.67 & -555.56 & 1666.67 \\ 3750 & 5000 & -3750+1666.67 & 10000+6666.67 & -1666.67 & 3333.33 \\ & & -555.56 & -1666.67 & 555.56 & -1666.67 \\ & & 1666.67 & 3333.332 & -1666.67 & 6666.67 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix}$$

Boundary condition

$$v_1, \theta_1, v_2, \theta_3 = 0$$

Loading Condition

$$M_2 = -50; \quad F_3 = -60$$

$$\begin{bmatrix} 16666.67 & -1666.67 \\ -1666.67 & 555.56 \end{bmatrix} \begin{Bmatrix} \theta_2 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} -50 \\ -60 \end{Bmatrix}$$

$$\begin{Bmatrix} \theta_2 \\ v_3 \end{Bmatrix} = \frac{1}{6481481.5} \begin{bmatrix} 555.56 & 1666.67 \\ 1666.67 & 16666.67 \end{bmatrix} \begin{Bmatrix} -50 \\ -60 \end{Bmatrix} = \begin{Bmatrix} -0.019714 \\ -0.16714 \end{Bmatrix}$$

Final member end forces

$$\{f\} = [k]\{d\} + \{FEMS\}$$

For element 1

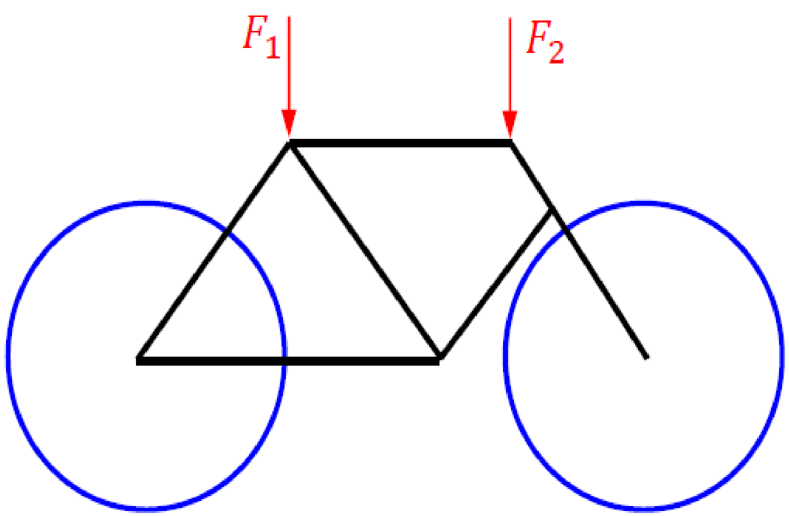
$$\begin{Bmatrix} f_1 \\ m_1 \\ f_2 \\ m_2 \end{Bmatrix} = \begin{Bmatrix} 10 \\ 10 \\ 10 \\ -10 \end{Bmatrix} + \frac{1X10^4}{4^3} \begin{bmatrix} 12 & 24 & -12 & 24 \\ 24 & 64 & -24 & 32 \\ -12 & -24 & 12 & -24 \\ 24 & 32 & -24 & 64 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.019714 \end{Bmatrix} = \begin{Bmatrix} -63.93 \\ -88.57 \\ 83.93 \\ -207.14 \end{Bmatrix}$$

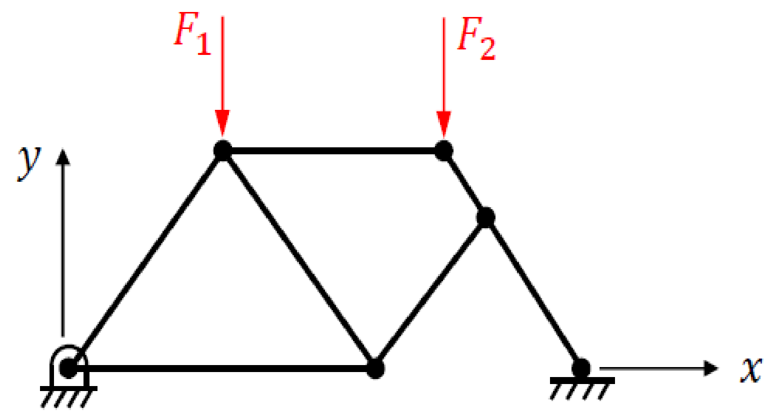
For element 2

$$\begin{Bmatrix} f_1 \\ m_1 \\ f_2 \\ m_2 \end{Bmatrix} = \begin{Bmatrix} 60 \\ 60 \\ 60 \\ -60 \end{Bmatrix} + \frac{1X10^4}{6^3} \begin{bmatrix} 12 & 36 & -12 & 36 \\ 36 & 144 & -36 & 72 \\ -12 & -36 & 12 & -36 \\ 36 & 72 & -36 & 144 \end{bmatrix} \begin{Bmatrix} 0 \\ -0.019714 \\ -0.16714 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 120 \\ 207.14 \\ 0 \\ 152.85 \end{Bmatrix}$$

Planar Frames

- Frames look like trusses, but the connections are rigid, i.e. welded or riveted.
- Each member can carry axial force, shear force and bending moment.



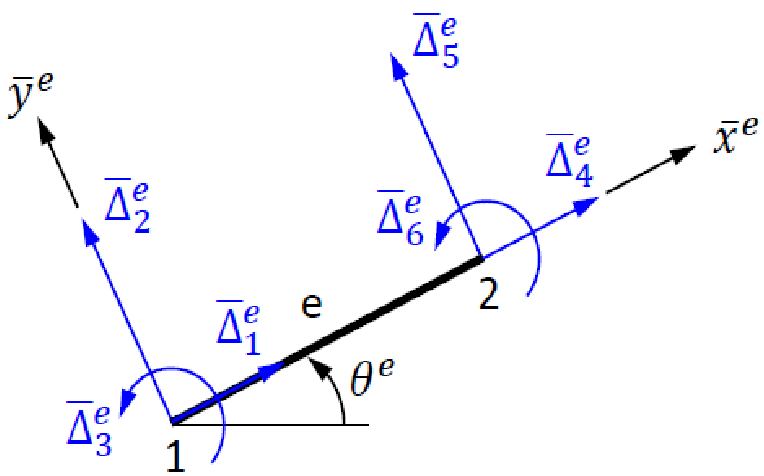


- The above bicycle frame has 7 members.
- Each member can be modeled as a single element or multiple elements.
- It is possible to think of a frame element as the superposition of truss and beam elements.

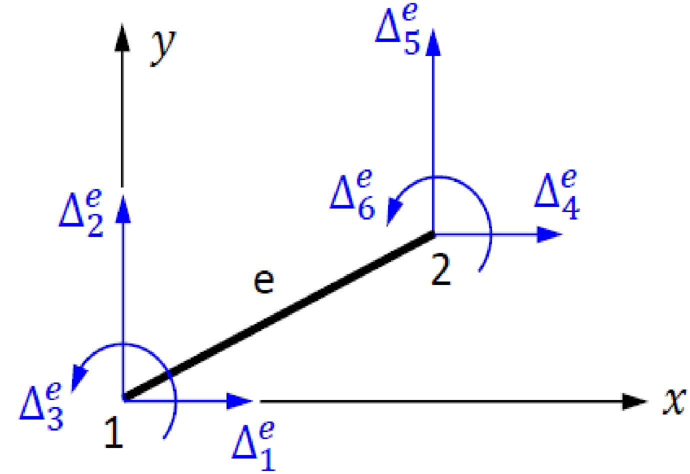
Frame Element

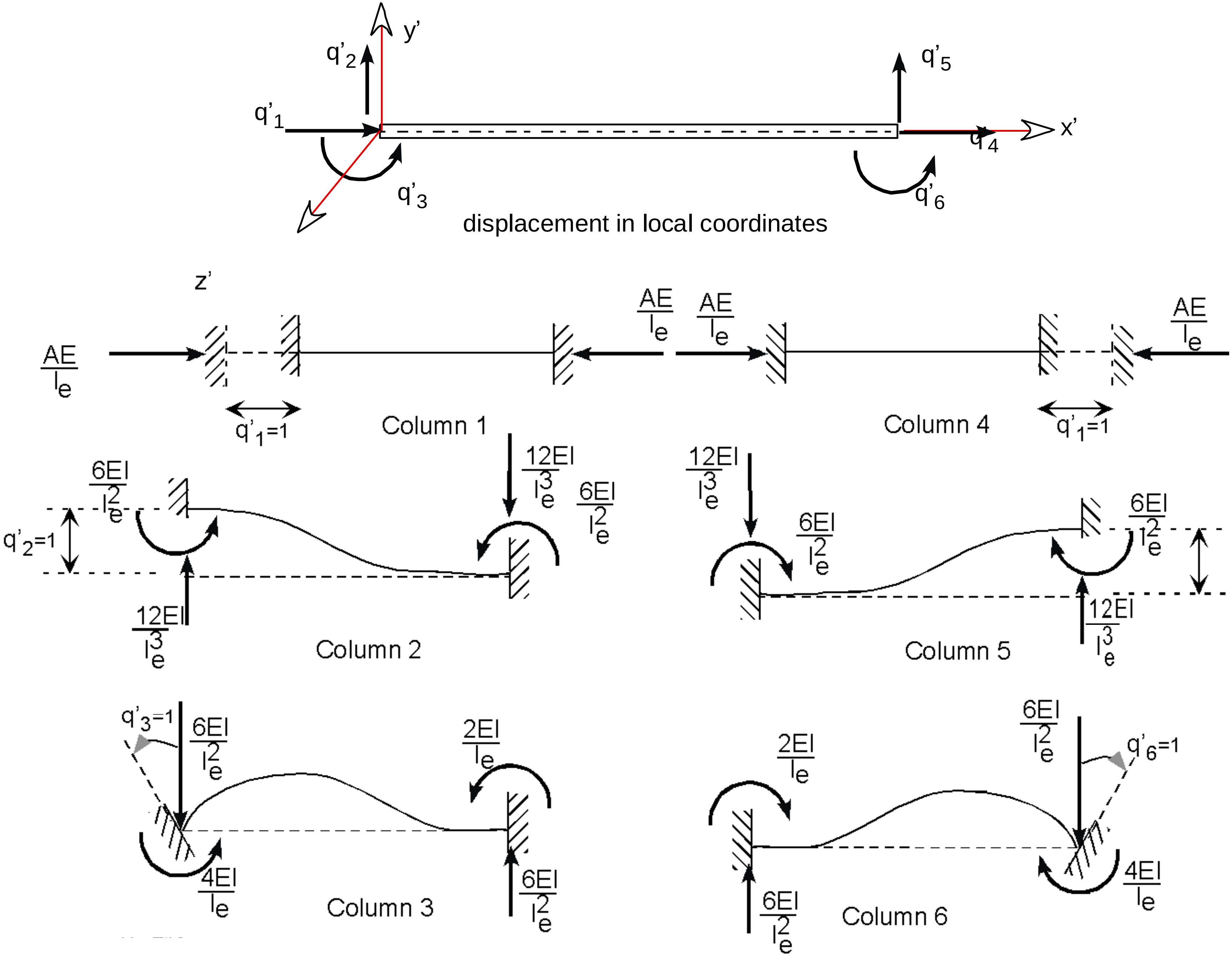
- We now have transformation matrices for arbitrarily oriented beam and truss elements.
- Frame elements carry axial force, shear force and bending moment.
- They can be obtained by the superposition of beam and truss elements.
- Frame element has 3 unknowns at each node.

Frame element in local coordinates



Frame element in global coordinates





y'
q'2
q'5
q'1
x'
q'4
q'3
q'6
displacement in local coordinates
z'
AE / le
q'1=1
Column 1
Column 4
12EI / le3
6EI / le2
q'2=1
Column 2
Column 5
q'3=1
2EI / le
4EI / le
q'6=1
Column 3
Column 6

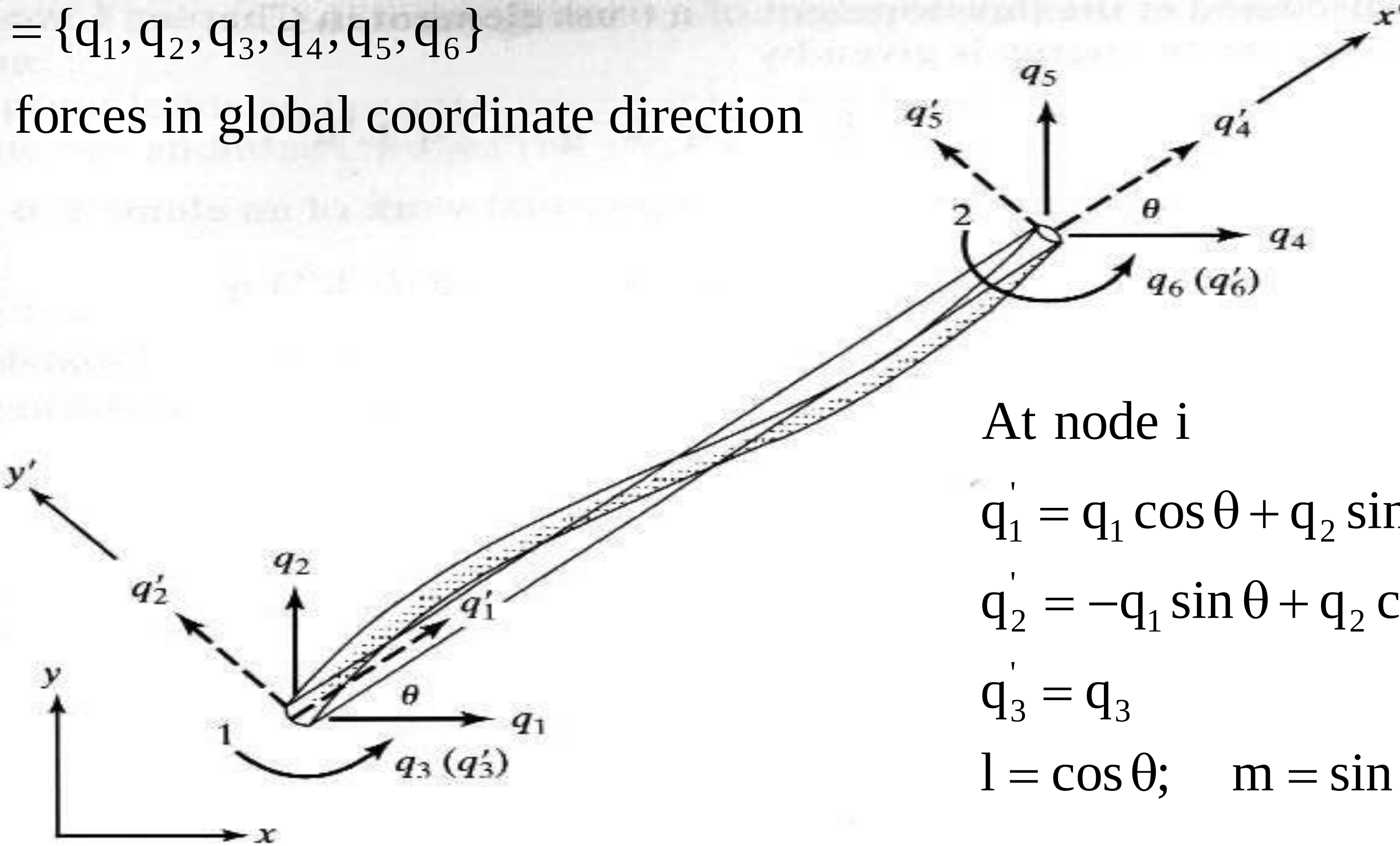
$$[k]=\begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^3} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^3} \\ 0 & \frac{6EI}{L^3} & \frac{4EI}{L^3} & 0 & -\frac{6EI}{L^3} & \frac{2EI}{L^3} \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^3} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^3} \\ 0 & \frac{6EI}{L^3} & \frac{2EI}{L^3} & 0 & -\frac{6EI}{L^3} & \frac{4EI}{L^3} \end{bmatrix}$$

If f' member end forces in local coordinates then

$$\{f'\}=[k']\{q'\}$$

$\{q\} = \{q_1, q_2, q_3, q_4, q_5, q_6\}$

are forces in global coordinate direction



At node i

$$q_1' = q_1 \cos\theta + q_2 \sin\theta$$

$$q_2' = -q_1 \sin\theta + q_2 \cos\theta$$

$$q_3' = q_3$$

$$l = \cos\theta; \quad m = \sin\theta$$

FIGURE 8.9 Frame element.

$$\mathbf{L} = \begin{bmatrix} \ell & m & 0 & 0 & 0 & 0 \\ -m & \ell & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ell & m & 0 \\ 0 & 0 & 0 & -m & \ell & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{8.48}$$

using conditions $\{q'\} = [L]\{q\};$ and $\{f'\} = [L]\{f\}$

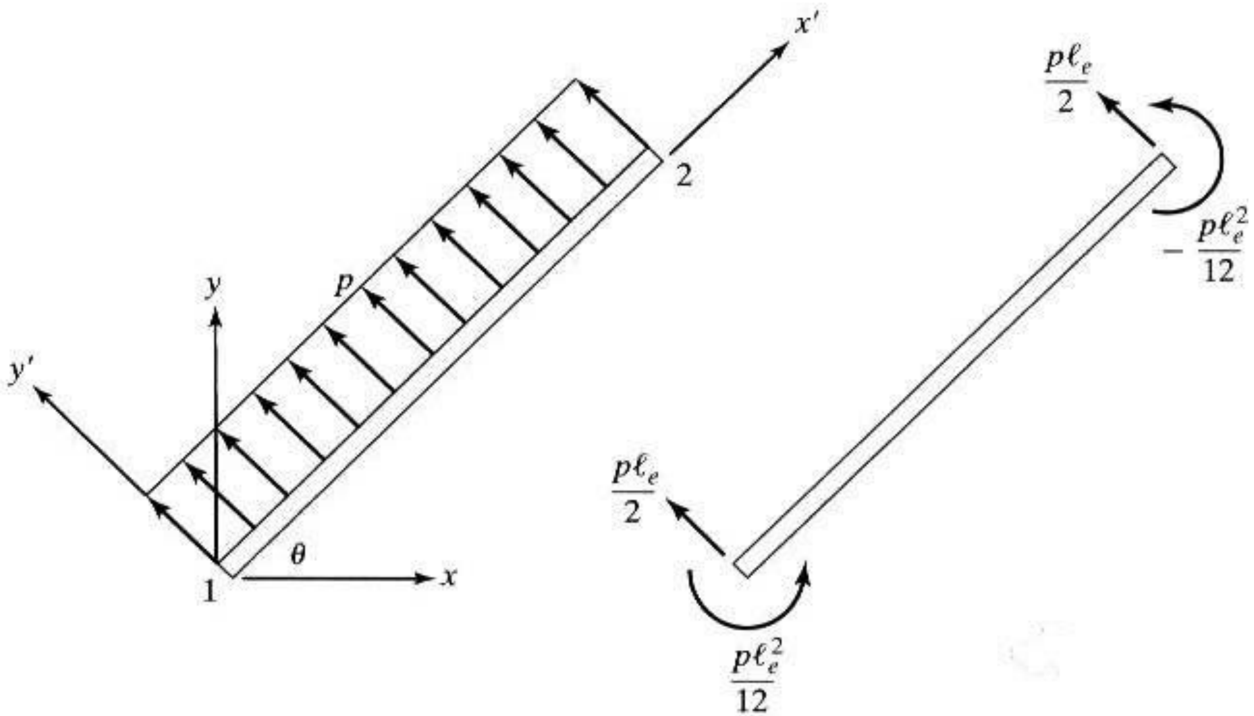
Stiffness matrix for an arbitrarily oriented beam element is given by

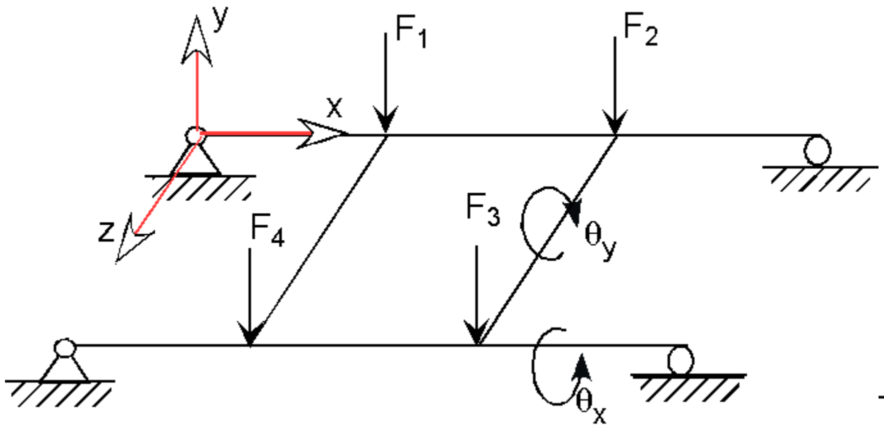
$[k] = [L]^T [k'][L]$

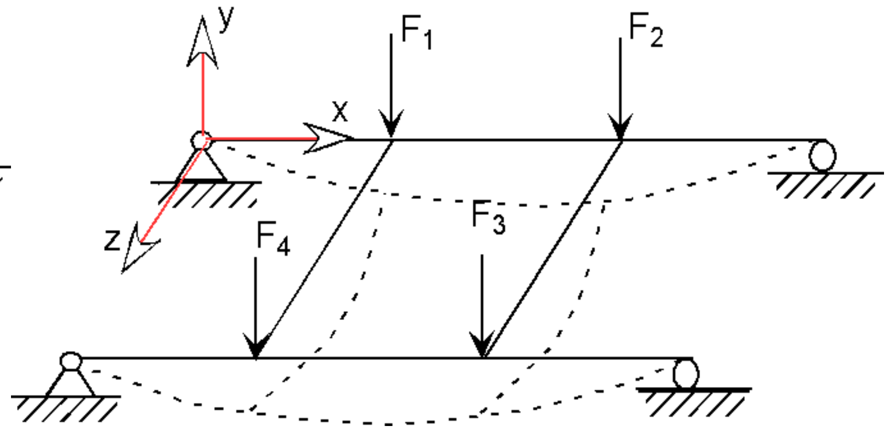
$$\mathbf{f}' = \left[0,\ \frac{p\ell_e}{2},\ \frac{p\ell_e^2}{12},\ 0,\ \frac{p\ell_e}{2},\ -\frac{p\ell_e^2}{12}\right]^{\mathrm{T}}$$

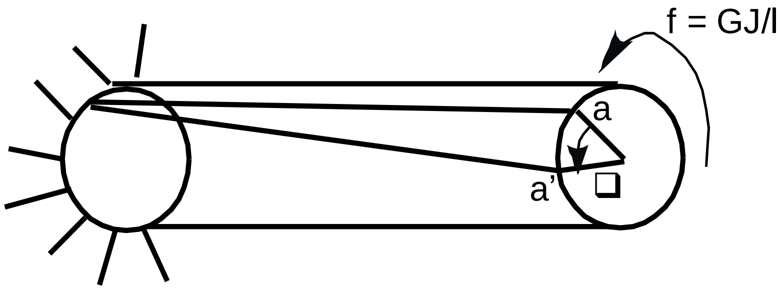
The nodal loads due to the distributed load p are given by

$$\mathbf{f} = \mathbf{L}^{\mathrm{T}}\mathbf{f}'$$

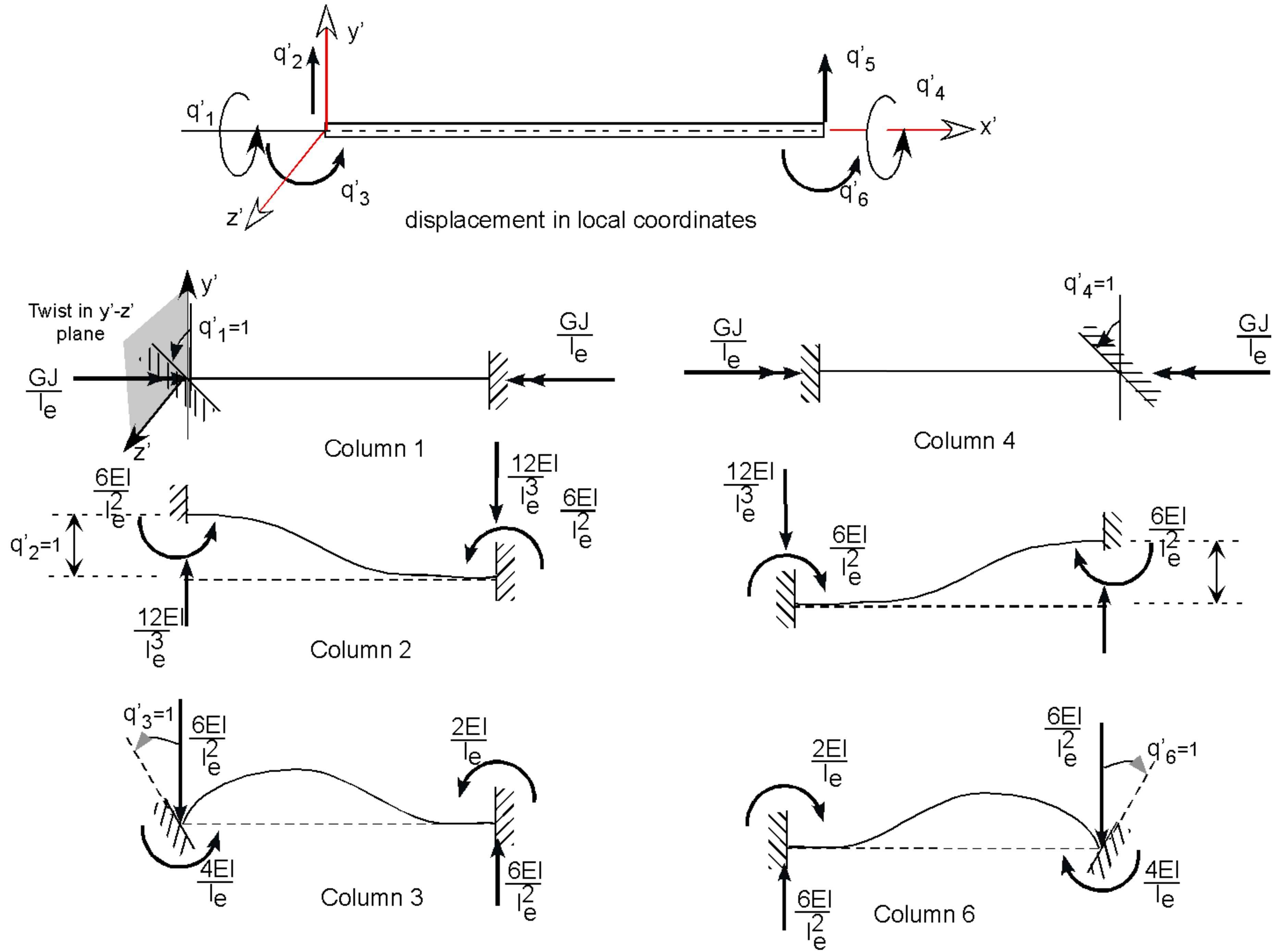








$$\begin{bmatrix} \frac{JG}{L} & -\frac{JG}{L} \\ -\frac{JG}{L} & \frac{JG}{L} \end{bmatrix} \begin{Bmatrix} q_{xi} \\ q_{xj} \end{Bmatrix} = \begin{Bmatrix} f_i \\ f_j \end{Bmatrix}$$



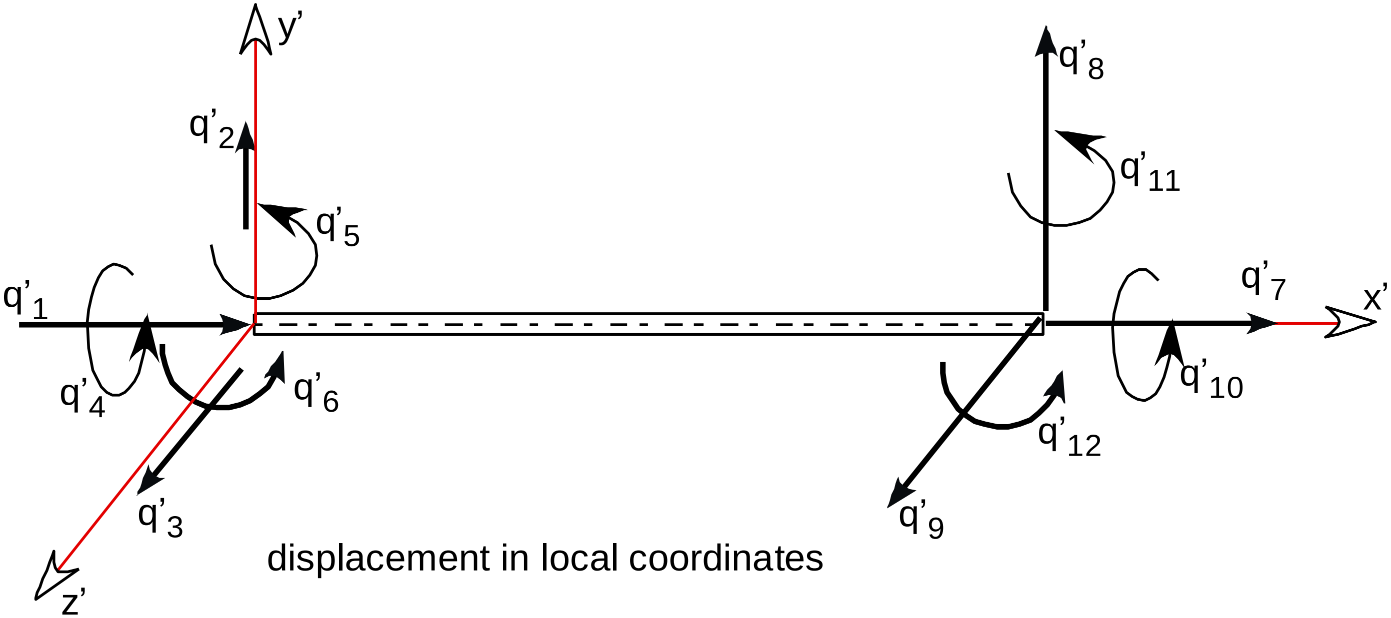
y'
q'2
q'5
q'4
q'1
x'
q'3
q'6
z'
displacement in local coordinates
Twist in y'-z' plane
y'
q'1=1
GJ/le
GJ/le
z'
Column 1
q'4=1
GJ/le
GJ/le
Column 4
6EI/le^2
12EI/le^3
6EI/le^2
q'2=1
12EI/le^3
Column 2
12EI/le^3
6EI/le^2
6EI/le^2
q'3=1
6EI/le^2
2EI/le
4EI/le
Column 3
6EI/le^2
6EI/le^2
q'6=1
2EI/le
4EI/le
6EI/le^2
Column 6

$$
\begin{bmatrix}
\frac{GJ}{L} & 0 & 0 & -\frac{GJ}{L} & 0 & 0 \\
0 & \frac{12EI}{L^3} & \frac{6EI}{L^3} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^3} \\
0 & \frac{6EI}{L^3} & \frac{4EI}{L^3} & 0 & -\frac{6EI}{L^3} & \frac{2EI}{L^3} \\
-\frac{GJ}{L} & 0 & 0 & \frac{GJ}{L} & 0 & 0 \\
0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^3} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^3} \\
0 & \frac{6EI}{L^3} & \frac{2EI}{L^3} & 0 & -\frac{6EI}{L^3} & \frac{4EI}{L^3}
\end{bmatrix}
$$

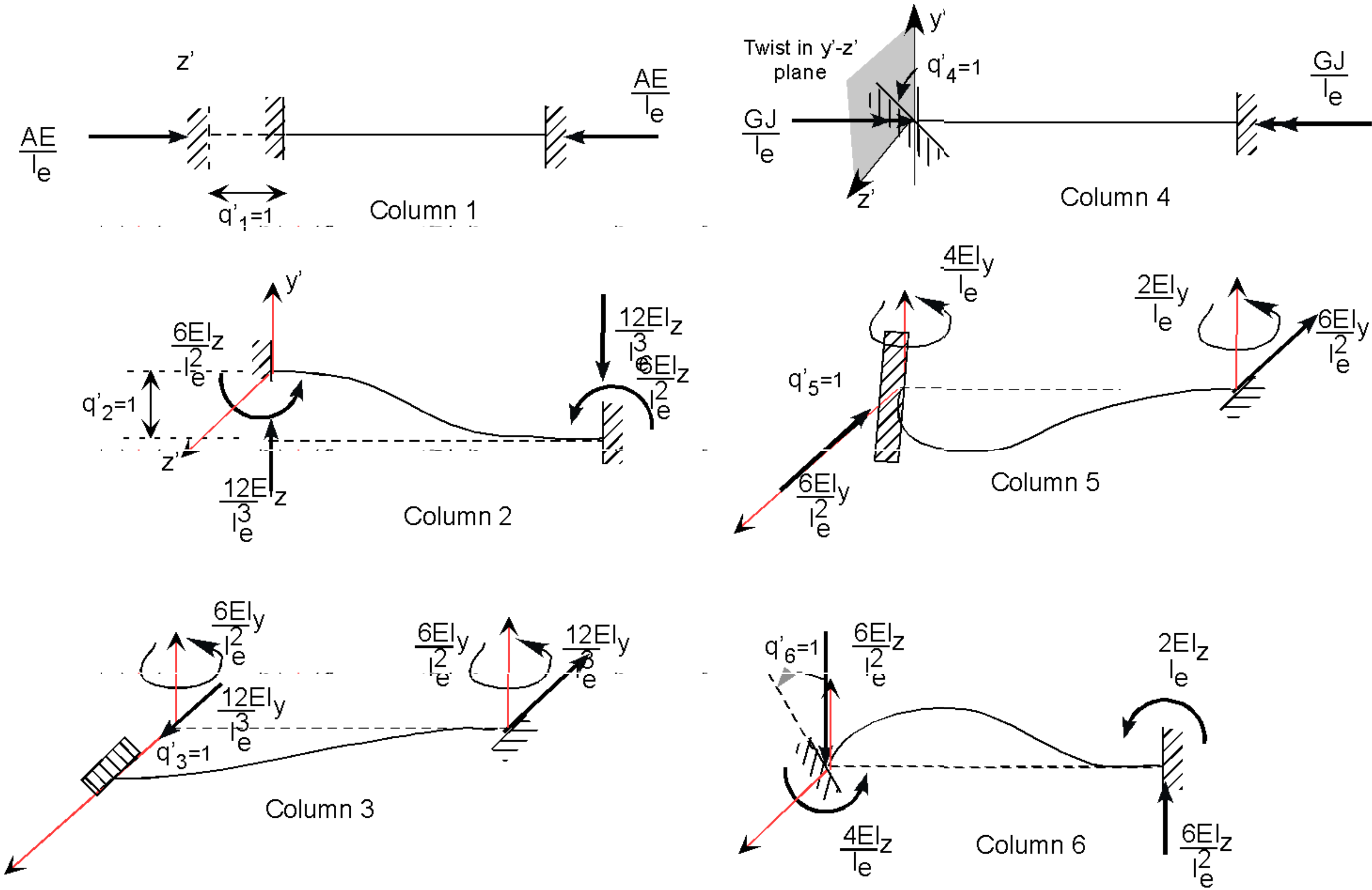
If f' member end forces in local coordinates then

$$\{f'\} = [k']\{q'\}$$

$$[L] = \begin{bmatrix} C & 0 & -s & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -s & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & -s \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & --s & 0 & c \end{bmatrix}$$



displacement in local coordinates



Column 1
z'
$\frac{AE}{l_e}$
$\frac{AE}{l_e}$
$q'_1=1$
Column 2
y'
z'
$q'_2=1$
$\frac{6EI_z}{l_e^2}$
$\frac{12EI_z}{l_e^3}$
$\frac{12EI_z}{l_e^3}$
$\frac{6EI_z}{l_e^2}$
Column 3
$\frac{6EI_y}{l_e^2}$
$\frac{12EI_y}{l_e^3}$
$\frac{6EI_y}{l_e^2}$
$\frac{12EI_y}{l_e^3}$
$q'_3=1$
Column 4
Twist in y'-z' plane
y'
z'
$q'_4=1$
$\frac{GJ}{l_e}$
$\frac{GJ}{l_e}$
Column 5
$\frac{4EI_y}{l_e}$
$\frac{2EI_y}{l_e}$
$\frac{6EI_y}{l_e^2}$
$\frac{6EI_y}{l_e^2}$
$q'_5=1$
Column 6
$q'_6=1$
$\frac{6EI_z}{l_e^2}$
$\frac{2EI_z}{l_e}$
$\frac{4EI_z}{l_e}$
$\frac{6EI_z}{l_e^2}$

$$
\begin{array}{cccccccccccc}
\hat{d}_{1x} & \hat{d}_{1y} & \hat{d}_{1z} & \hat{\phi}_{1x} & \hat{\phi}_{1y} & \hat{\phi}_{1z} & \hat{d}_{2x} & \hat{d}_{2y} & \hat{d}_{2z} & \hat{\phi}_{2x} & \hat{\phi}_{2y} & \hat{\phi}_{2z}
\end{array}
$$

$$
\hat{\underline{k}} = \left[\begin{array}{cccccc|cccccc}
\frac{AE}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{AE}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} & 0 & -\frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} \\
0 & 0 & \frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 & 0 & 0 & -\frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 \\
0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{GJ}{L} & 0 & 0 \\
0 & 0 & -\frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} \\
\hline
-\frac{AE}{L} & 0 & 0 & 0 & 0 & 0 & \frac{AE}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} \\
0 & 0 & -\frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 & 0 & 0 & \frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 \\
0 & 0 & 0 & -\frac{GJ}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 \\
0 & 0 & -\frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L}
\end{array}\right]
$$

The transformation from local to global axis system is accomplished as follows:

$$\underline{k} = \underline{T}^T \hat{\underline{k}} \underline{T}$$

$$\underline{T} = \begin{bmatrix} \underline{\lambda}_{3\times 3} & & & \\ & \underline{\lambda}_{3\times 3} & & \\ & & \underline{\lambda}_{3\times 3} & \\ & & & \underline{\lambda}_{3\times 3} \end{bmatrix}$$

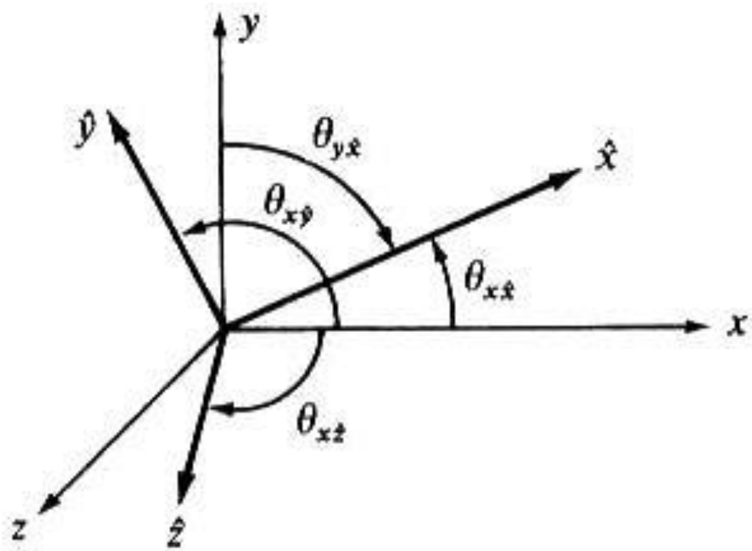


Figure 6–24 Direction cosines associated with the *x* axis

$$\underline{\lambda} = \begin{bmatrix} C_{x\hat{x}} & C_{y\hat{x}} & C_{z\hat{x}} \\ C_{x\hat{y}} & C_{y\hat{y}} & C_{z\hat{y}} \\ C_{x\hat{z}} & C_{y\hat{z}} & C_{z\hat{z}} \end{bmatrix}$$

- If axial load is tensile, results from beam elements are higher than actual $\Rightarrow$ results are conservative

- If axial load is compressive, results are less than actual
  - size of error is small until load is about 25% of Euler buckling load

- for 2-d, can use rotation matrices to get stiffness matrix for beams in any orientation

- To develop 3-d beam elements, must also add capability for torsional loads about the axis of the element, and flexural loading in x-z plane

- to derive the 3-d beam element, set up the beam with the x axis along its length, and y and z axes as lateral directions

- torsion behavior is added by superposition of simple strength of materials solution

= torsional moment about x axis

G = shear modulus

L = length

ϕ_{xi}, ϕ_{xj} are nodal degrees of freedom of angle of twist at each end

T_i, T_j are torques about the x axis at each end

- flexure in x-z plane adds another stiffness matrix like the first one derived
- superposition of all these matrices gives a 12×12 stiffness matrix
- to orient a beam element in 3-d, use 3-d rotation matrices

- for beams long compared to their cross section, displacement is almost all due to flexure of beam
- for short beams there is an additional lateral displacement due to transverse shear
- some FE programs take this into account, but you then need to input a shear deformation constant (value

- limitations:
  - same assumptions as in conventional beam and torsion theories
  - no better than beam analysis
  - axial load capability allows frame analysis, but formulation does not couple axial and lateral loading which are coupled nonlinearly analysis does not account for
    - stress concentration at cross section changes
    - where point loads are applied
    - where the beam frame components are connected

Finite Element Model

- Element formulation exact for beam spans with no intermediate loads

  need only 1 element to model any such member that has constant cross section

- for distributed load, subdivide into several elements

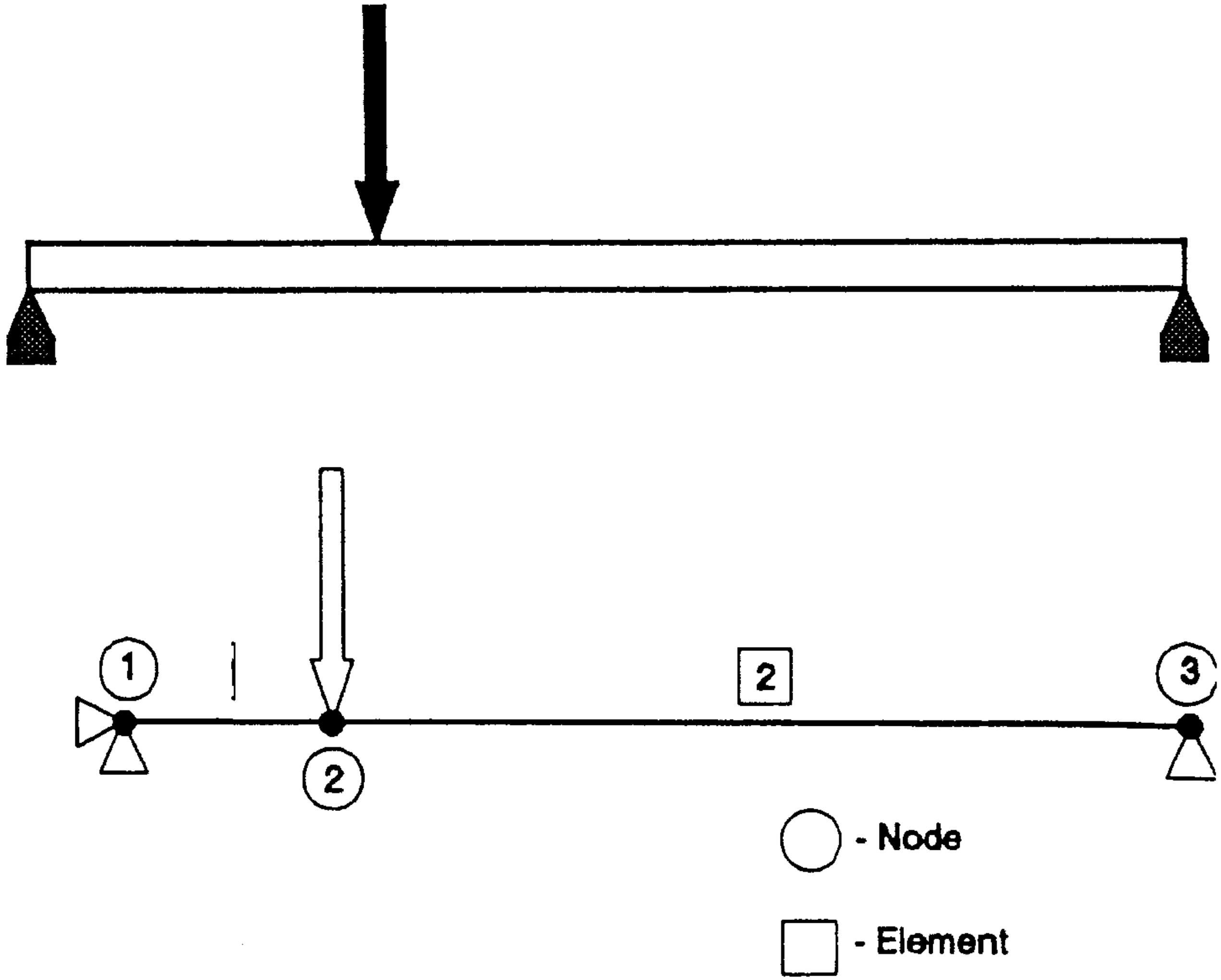
- need a node everywhere a point load is applied

- need nodes where frame members connect, where they change direction, or where the cross section properties change

- for each member at a common node, all have the same linear and rotational displacement

- boundary conditions can be restraints on linear displacements or rotation

- simple supports restrain only linear displacements built in supports restrain rotation also

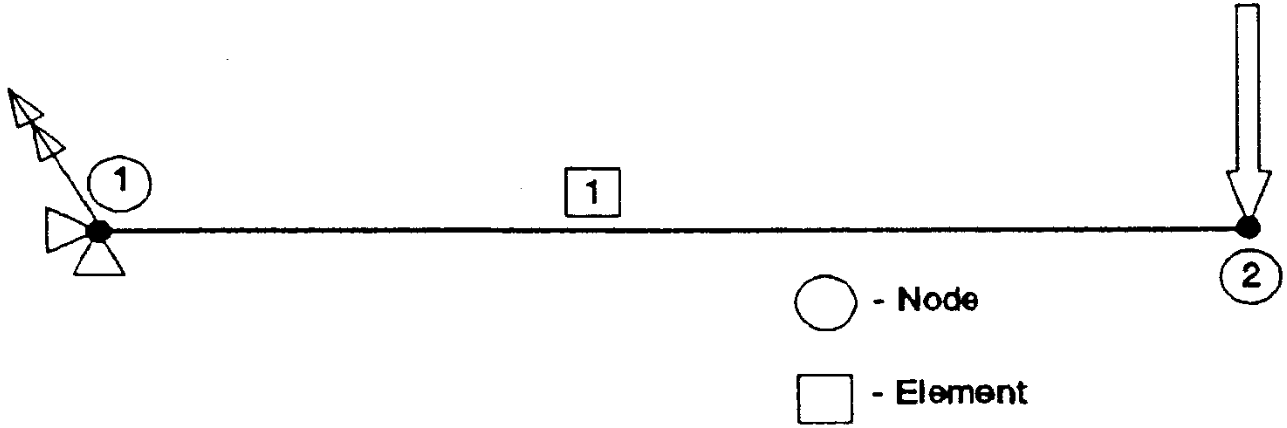

1
2
2
3
- Node
- Element

restrain vertical and horizontal displacements of nodes 1 and 3

no restraint on rotation of nodes 1 and 3

need a restraint in x direction to prevent rigid body motion, even if all forces are in y direction

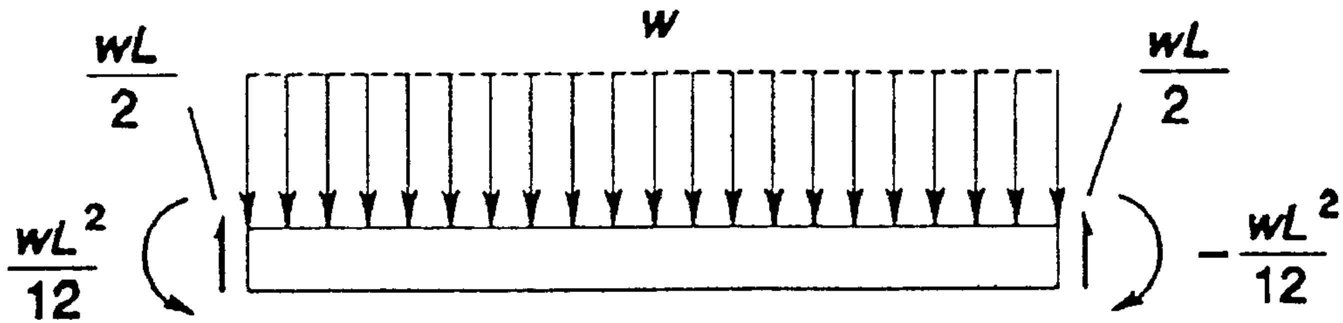
- cantilever beam



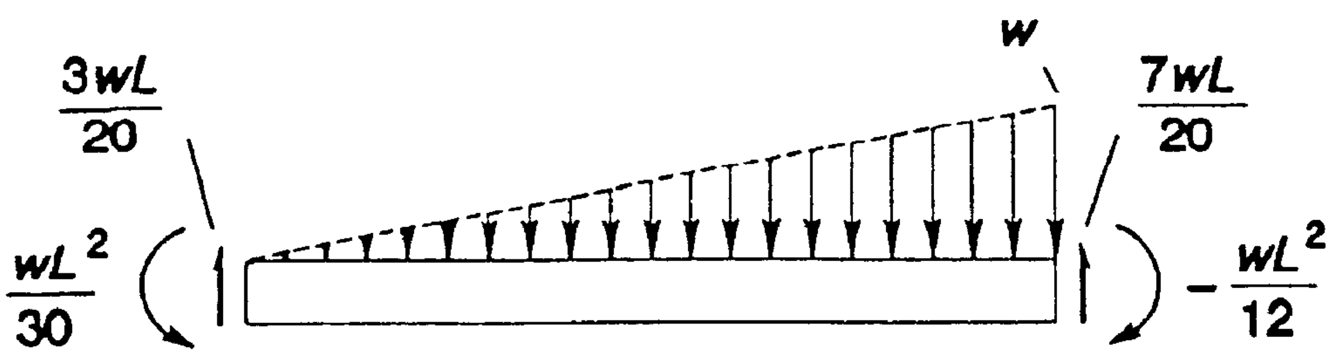
has x and y linear displacements and rotation

- point loads are idealized loads
    - structure away from area of application behaves as though point loads are applied

- only an exact formulation when there are no loads along the span
    - for distributed loads, can get exact solution everywhere else by replacing the distributed load by equivalent loads and moments at the nodes



w
wL/2
wL/2
wL²/12
− wL²/12



w
3wL/20
7wL/20
wL²/30
− wL²/12

Computer Input Assistance

- preprocessor will usually have the same capabilities as for trusses
- a beam element consists of two node numbers and associated material and physical properties

- material properties:
  - modulus of elasticity
  - if dynamic or thermal analysis, mass density and thermal coefficient of expansion
- physical properties:
  - cross sectional area
  - 2 area moments of inertia
  - torsion constant
  - location of stress calculation point

- boundary conditions:

    specify node numbers and displacement components that are restrained

- loads:

    specify by node number and load components

    most commercial FE programs allows application of distributed loads but they use and equivalent load/moment

Analysis Step

- small models and carefully planned element and node numbering will save you from bandwidth or wavefront minimization
- potential for ill conditioned stiffness matrix due to axial stiffness >> flexural stiffness (case of long slender beams)

Output Processing and Evaluation

- graphical output of deformed shape usually uses only straight lines to represent members
- you do not see the effect of rotational constraints on the deformed shape of each member
- to check these, subdivide

- most FE codes do not make graphical presentations of beam stress results
    - user must calculate some of these from the stress values returned
- for 2-d beams, you get a normal stress normal to the cross section and a transverse shear acting on the face of the cross section
    - normal stress has 2 components
        - axial stress

- top or bottom of the cross section
- transverse shear is usually the average transverse load/area
  - does not take into account any variation across the section

- BEAMS
  - normal stress is combination of axial stress, flexural stress from local *y*- and *z*- moments
  - stress due to moment is linear across a section, the combination is usually highest at the extreme corners of the cross section
  - may also have to include the effects of torsion
    - get a 2-d stress state which must be evaluated
  - also need to check for column buckling