Chapter -3

INTERPOLATION(SHAPE) FUNCTION

Yilma T(PhD)

Shape functions

- The vertices of finite element (e.g.triangular)are numbered to indicate that these points are nodes. A *node* is a specific point in the finite element at which the value of the field variable is to be explicitly calculated.

- *Exterior* nodes are located on the boundaries of the finite element and may be used to connect an element to adjacent finite elements.

- Nodes that do not lie on element boundaries are *interior* nodes and cannot be connected to any other element.

Example

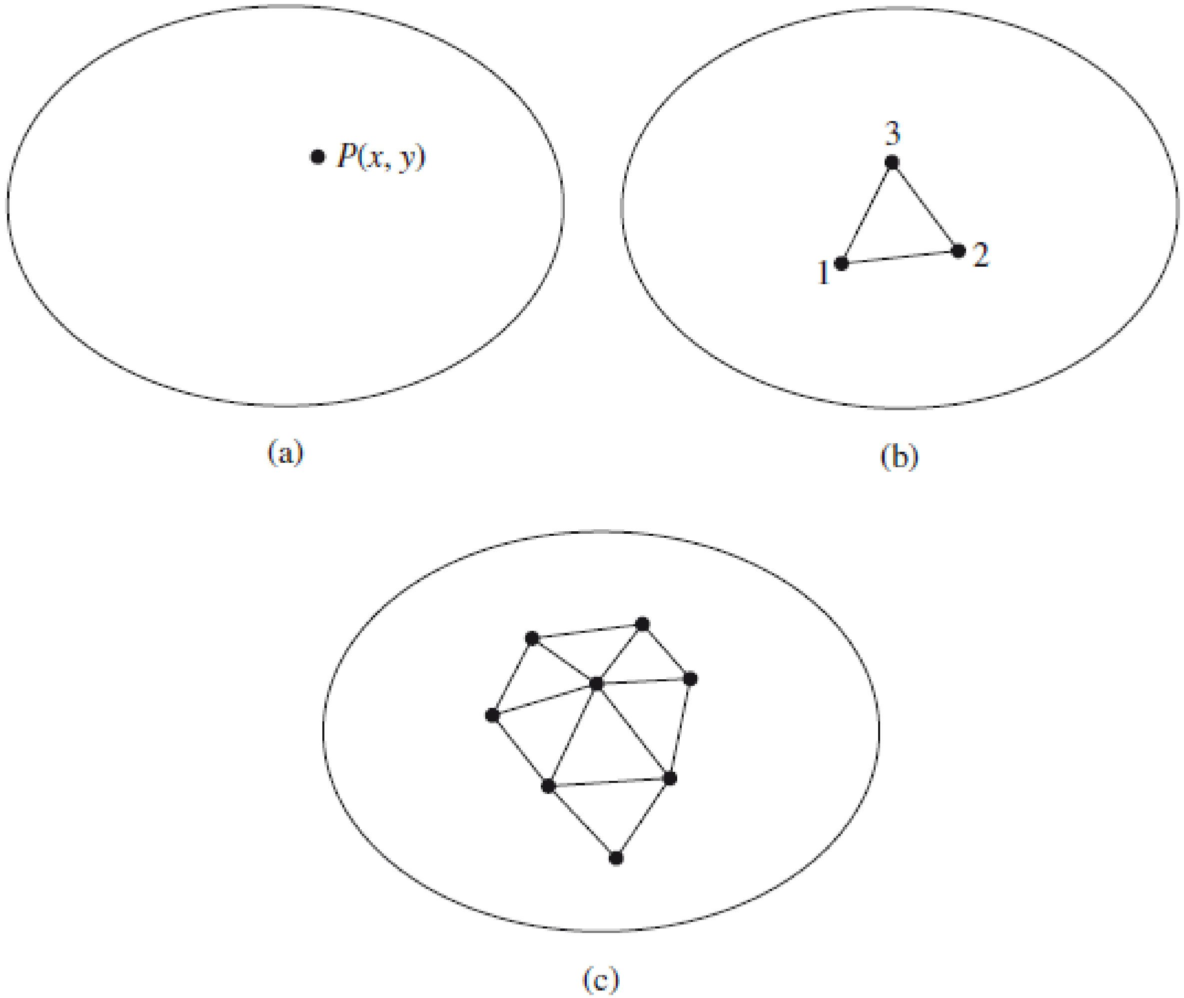


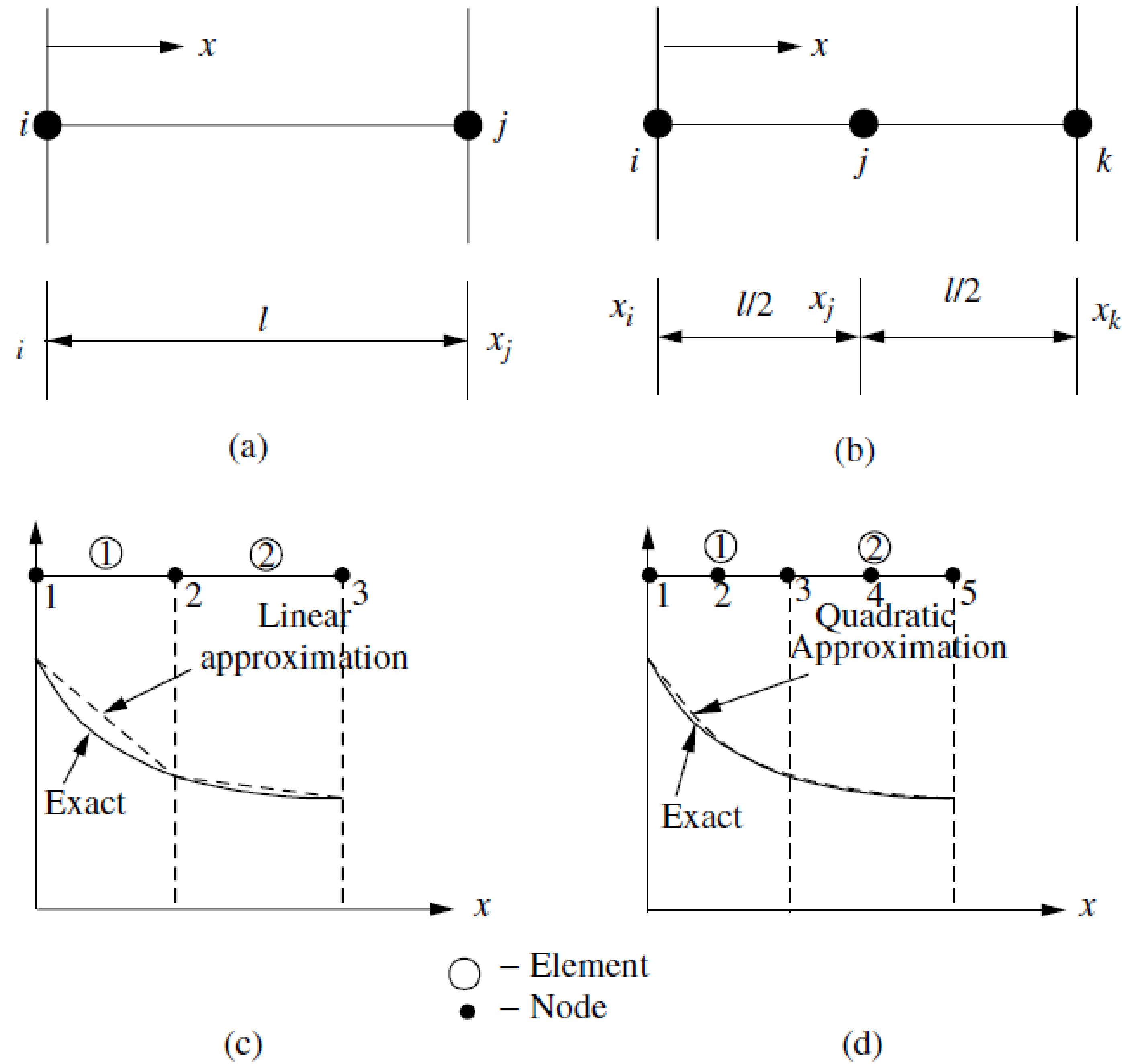
Figure 1.1
(a) A general two-dimensional domain of field variable $\phi(x, y)$.
(b) A three-node finite element defined in the domain. (c) Additional elements showing a partial finite element mesh of the domain.

Shape functions cont..

- Finite element method involves the discretization of both the domain and the governing equations. In this process, the variables are represented in a piece-wise manner over the domain.
- By dividing the solution region into a number of small regions, called elements, and approximating the solution over these regions by a suitable known *function*, a relation between the differential equations and the elements is established.
- The functions employed to represent the nature of the solution within each element are called shape functions, or interpolating functions, or basis functions.

- They are called interpolating functions as they are used to determine the value of the field variable within an element by interpolating the nodal values.
- They are also known as basis functions as they form the basis of the discretization method.
- Polynomial type functions have been most widely used as they can be integrated, or differentiated, easily and the accuracy of the results can be improved by increasing the order of the polynomial as shown in Figure 3.3(c) and (d).

- Figure 3.3 One-dimensional finite elements. (a) A linear element, (b) a quadratic element, (c) linear and (d) quadratic variation of temperature over an element



Decisive point in FEM

- If the values of the field variable are computed only at nodes, how are values obtained at other points within a finite element? The answer contains the crux of the finite element method:
- The values of the field variable computed at the nodes are used to approximate the values at non nodal points (that is, in the element interior) by *interpolation* of the nodal values.
- For the three-node triangle example, the nodes are all exterior and, at any other point within the element, the field variable is described by the approximate relation

Shape functions cont..

$$\phi(x, y) = N_1(x, y)\phi_1 + N_2(x, y)\phi_2 + N_3(x, y)\phi_3 \qquad (1.1)$$

- Where $\phi_1, \phi_2, \text{and } \phi_3$ are the values of the field variable at the nodes, and $N_1, N_2, \text{and } N_3$ are the interpolation functions, also known as shape functions or blending functions. In the finite element approach, the nodal values of the field variable are treated as unknown *constants* that are to be determined.

- The interpolation functions are most often polynomial forms of the independent variables, derived to satisfy certain required conditions at the nodes.

Shape functions cont..

- The major point to be made here is that the interpolation functions are predetermined, *known* functions of the independent variables; and these functions describe the variation of the field variable within the finite element.
- This would be the case if the field variable represents a scalar field, such as temperature in a heat transfer problem. If the domain of Figure 1.1 represents a thin, solid body subjected to plane stress, the field variable becomes the displacement vector and the values of two components must be computed at each node.
- In the latter case, the three-node triangular element has 6 degrees of freedom.

Shape functions cont..

- In general, the number of degrees of freedom associated with a finite element is equal to the product of the number of nodes and the number of values of the field variable (and possibly its derivatives) that must be computed at each node.

FEM objective

- The objective of the finite element method is to discretize the domain into a number of finite elements for which the governing equations are algebraic equations.
- Solution of the resulting system of algebraic equations then gives an *approximate solution to the problem*
- As with any approximate technique, the question, How accurate is the solution? must be addressed.

FEM Solution accuracy

- In finite element analysis, solution accuracy is judged in terms of convergence as the element "mesh" is refined.
- There are two major methods of mesh refinement.

i) *h-refinement*, *mesh refinement refers to the* process of increasing the number of elements used to model a given domain, consequently, reducing individual element size.

FEM Solution accuracy

ii) *p-refinement* : element size is unchanged but the order of the polynomials used as interpolation functions is increased.

- The objective of mesh refinement in either method is to obtain sequential solutions that exhibit asymptotic convergence to values representing the exact solution.

Unstructured meshes

- Mathematical proofs of convergence of finite element solutions to correct solutions are based on a specific, regular mesh refinement procedure.
- Although the proofs are based on regular meshes of elements, irregular or unstructured meshes (such as in Figure 1.2) can give very good results.

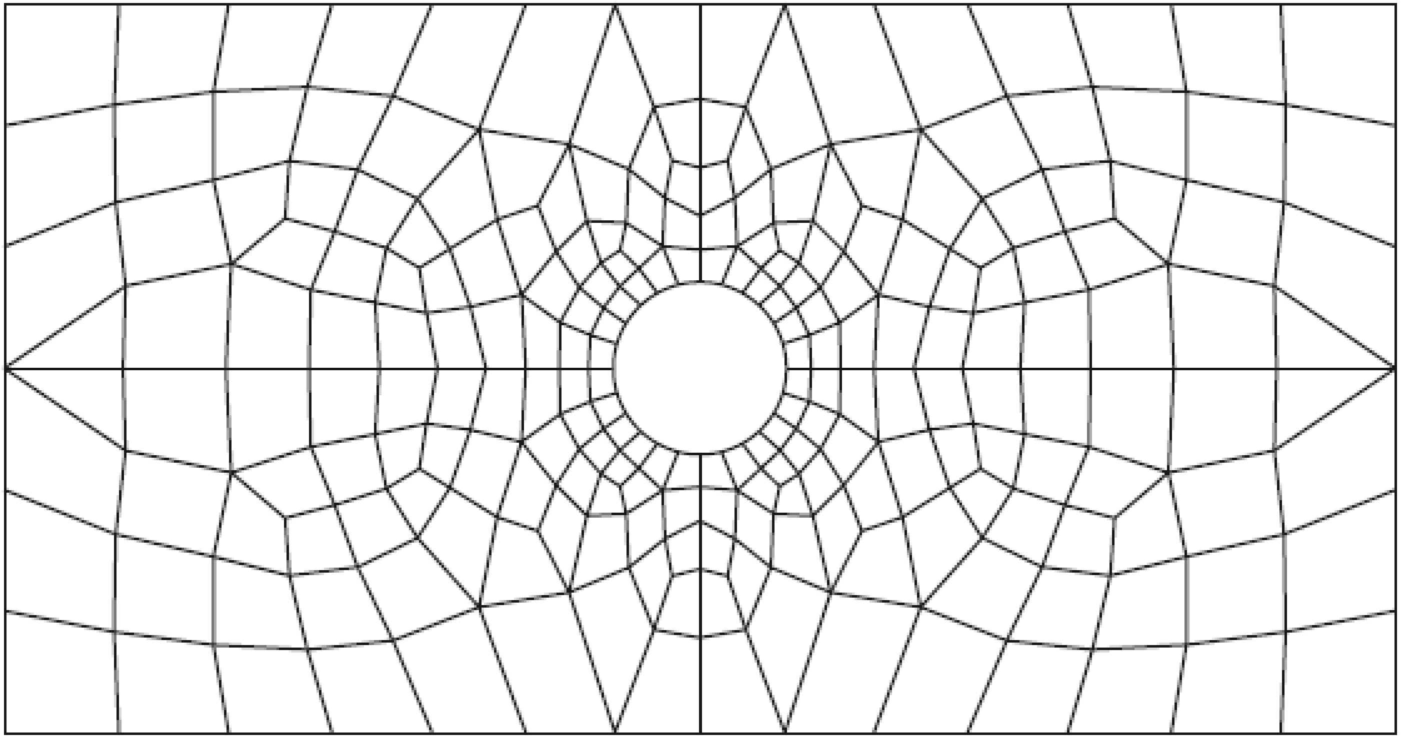


Fig.1.2 A mesh of finite elements over a rectangular region having a
central hole.

Unstructured meshes cont..

- In fact, use of unstructured meshes is more often the case, since

  (1) The geometries being modeled are most often irregular and

  (2) The auto meshing features of most finite element software packages produce irregular meshes.

- An example illustrating regular *h-refinement as well as solution convergence* is shown in Figure 1.3a, which depicts a rectangular elastic plate of uniform thickness fixed on one edge and subjected to a concentrated load on one corner.

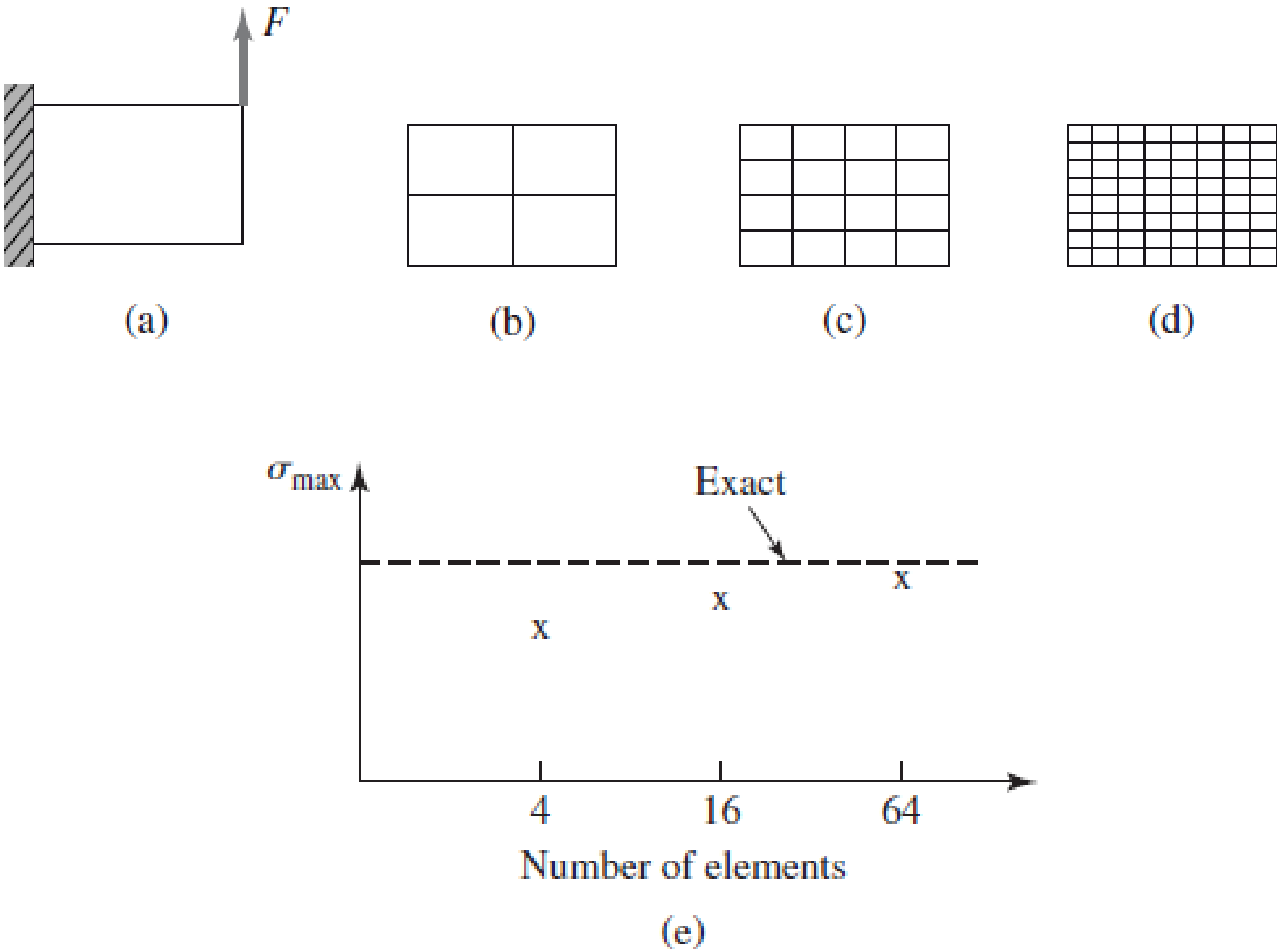


Figure 1.3 Example showing convergence as element mesh is refined.

Shape functions cont..

- The need for convergence during regular mesh refinement is rather clear. If convergence is not obtained, the engineer using the finite element method has absolutely no indication whether the results are indicative of a meaningful approximation to the correct solution. For a general field problem in which the field variable of interest is expressed on an element basis in the discretized form

$$\phi^{(e)}(x, y, z) = \sum_{i=1}^{M} N_i(x, y, z)\phi_i \qquad (1.1)$$

- Where *M is the number of element degrees of freedom*.

The interpolation functions must satisfy two primary conditions to ensure convergence during mesh refinement:

i) The compatibility and

ii) Completeness requirements

Compatibility

- Along element boundaries, the field variable and its partial derivatives up to one order less than the highest-order derivative appearing in the integral formulation of the element equations must be continuous.
- Given the discretized representation of Equation 1.1, it follows that the interpolation functions must meet this condition, since these functions determine the spatial variation of the field variable.

Compatibility

- In addition to satisfying the criteria for convergence, the compatibility condition can be given a physical meaning as well. In structural problems, the requirement of displacement continuity along element boundaries ensures that no gaps or voids develop in the structure as a result of modeling procedure.
- Similarly, the requirement of slope continuity for the beam element ensures that no "kinks" are developed in the deformed structure. In heat transfer problems, the compatibility requirement prevents the physically unacceptable possibility of jump discontinuities in temperature distribution.

Completeness

- *In the limit as element size shrinks to zero in mesh refinement, the field variable and its partial derivatives up to, and including, the highest-order derivative appearing in the integral formulation must be capable of assuming constant values. Again, because of the discretization, the completeness requirement is* directly applicable to the interpolation functions.

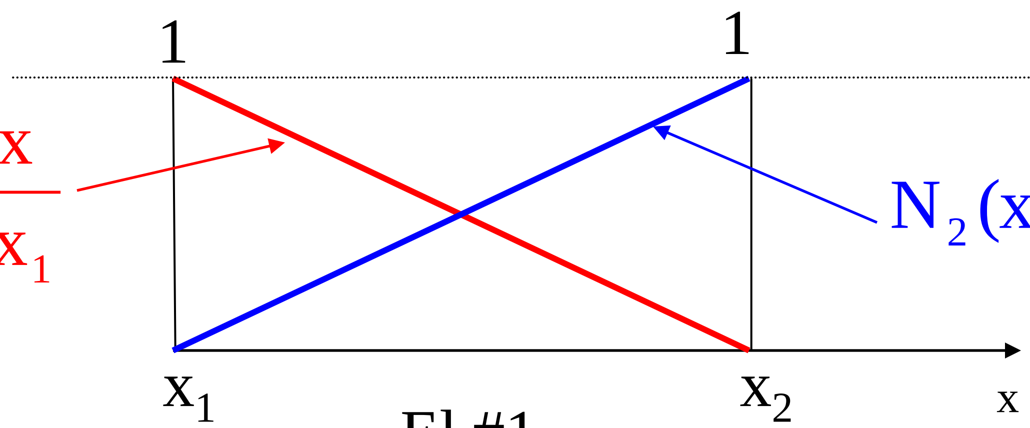
POLYNOMIAL FORMS: ONE-DIMENSIONAL ELEMENTS

- Formulation of finite element characteristics requires differentiation and integration of the interpolation functions in various forms. Owing to the simplicity with which polynomial functions can be differentiated and integrated, polynomials are the most commonly used interpolation functions.

1D LINEAR ELEMENT

1. <u>Kronecker delta property</u>: The shape function at any node has a value of 1 at that node and a value of zero at ALL other nodes.

$$N_1(x) = \frac{x_2 - x}{x_2 - x_1}$$



$$N_2(x) = \frac{x - x_1}{x_2 - x_1}$$

El #1

Check

$$N_1(x) = \frac{x_2 - x}{x_2 - x_1}$$

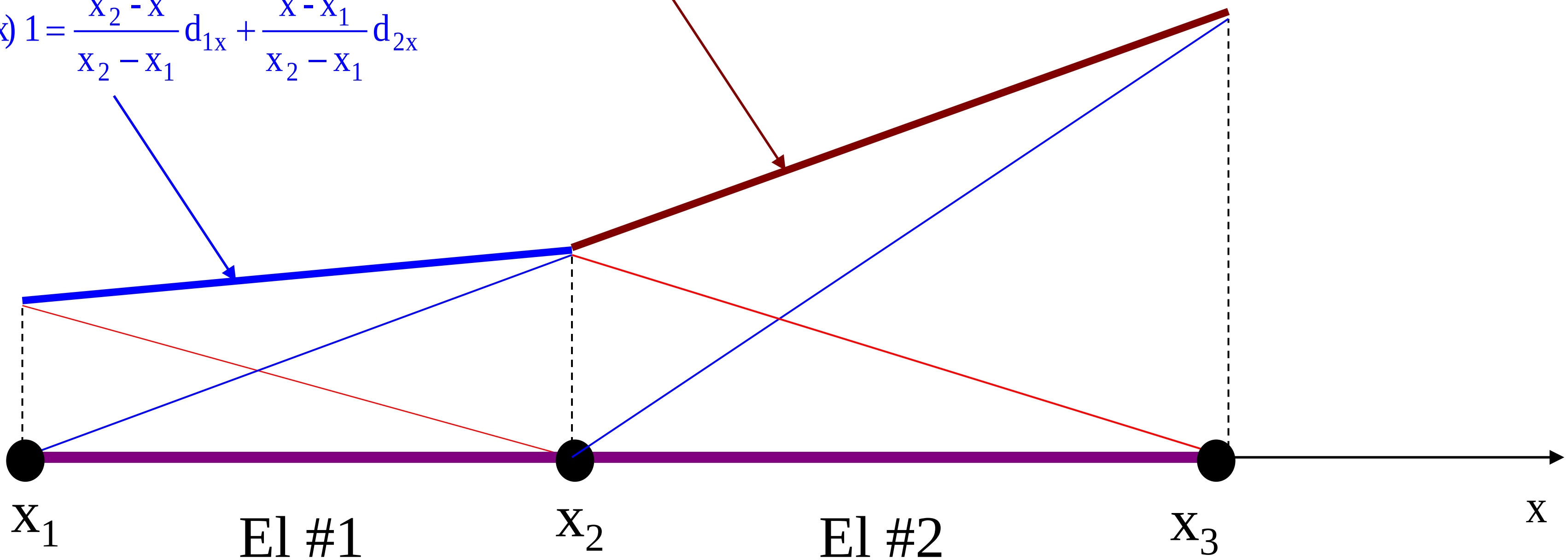
$$\Rightarrow N_1(x = x_1) = \frac{x_2 - x_1}{x_2 - x_1} = 1$$

$$\text{and } N_1(x = x_2) = \frac{x_2 - x_2}{x_2 - x_1} = 0$$

The approximation is **continuous across element boundaries**

$$d^{(2)}(x) = \frac{x_3 - x}{x_3 - x_2} d_{2x} + \frac{x - x_2}{x_3 - x_2} d_{3x}$$

$$d(x)\,1 = \frac{x_2 - x}{x_2 - x_1} d_{1x} + \frac{x - x_1}{x_2 - x_1} d_{2x}$$



Completeness

$$N_1(x) + N_2(x) = 1 \quad \text{for all } x$$
$$N_1(x)x_1 + N_2(x)x_2 = x \quad \text{for all } x$$

Use the expressions $$N_1(x) = \frac{x_2 - x}{x_2 - x_1};$$

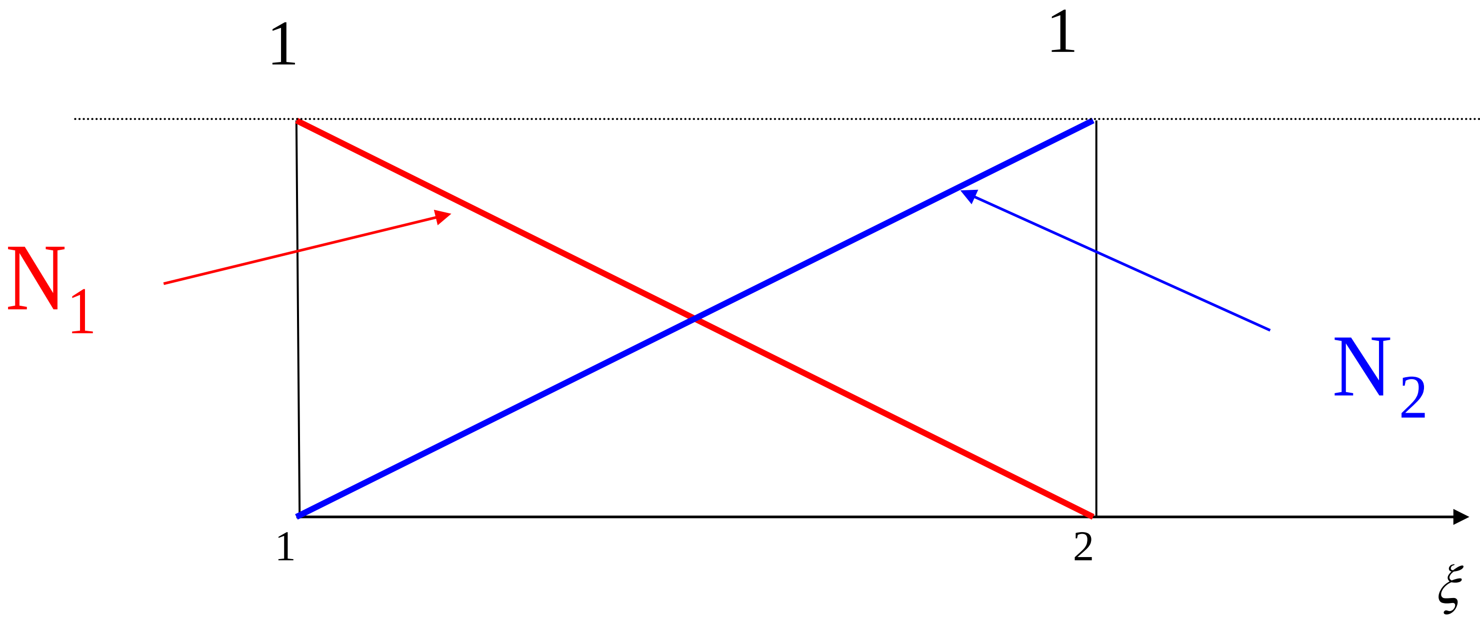
$$N_2(x) = \frac{x - x_1}{x_2 - x_1}$$

And check

$$N_1(x) + N_2(x) = \frac{x_2 - x}{x_2 - x_1} + \frac{x - x_1}{x_2 - x_1} = 1$$

and $$N_1(x)\, x_1 + N_2(x)\, x_2 = \frac{x_2 - x}{x_2 - x_1} x_1 + \frac{x - x_1}{x_2 - x_1} x_2 = x$$

1D LINEAR SHAPE FUNCTIONS ON MASTER ELEMENT



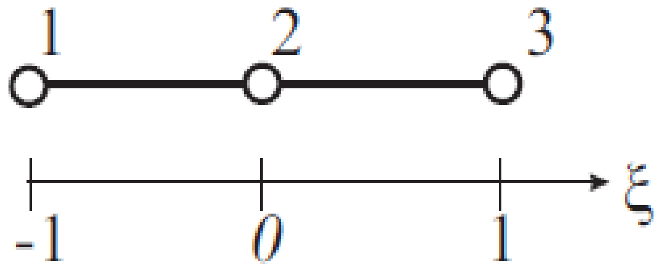
$$N_1(x) = \frac{(1-\xi)}{2}$$

Node at which N_1 is 0

$$N_2(x) = \frac{(\xi - 1\;)}{2}$$

1D QUADRATIC MASTER ELEMENT

Example. Obtain shape functions for the one-dimensional quadratic element three nodes. Use local coordinate system $-1 \le \xi \le 1$.



Solution. With shape functions, any field inside element is presented as:

$$u(\xi) = \sum N_i u_i \ , \quad i = 1,\ 2,\ 3$$

At nodes the approximated function should be equal to its nodal value:

$$u(-1) = u_1$$
$$u(0) = u_2$$
$$u(1) = u_3$$

Since the element has three nodes the shape functions can be quadratic polynomials (with three coefficients). The shape function N_1 can be written as:

$$N_1 = \alpha_1 + \alpha_2\xi + \alpha_3\xi^2$$

Unknown coefficients α_i are defined from the following system of equations:

$$\begin{aligned}
&N_1(-1) = \alpha_1 - \alpha_2 + \alpha_3 = 1 \\
&N_1(0) = \alpha_1 = 0 \\
&N_1(1) = \alpha_1 + \alpha_2 + \alpha_3 = 0
\end{aligned}$$

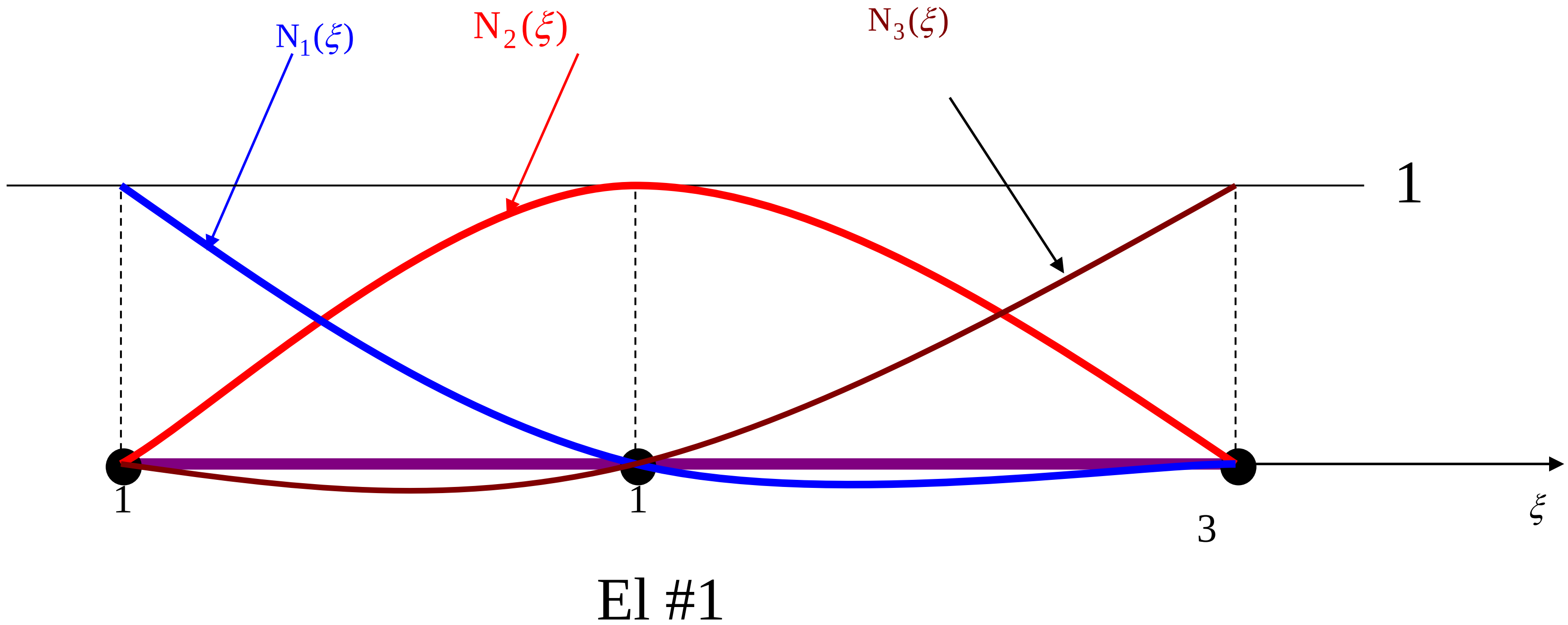
The solution is: $\alpha_1 = 0, \quad \alpha_2 = -1/2, \quad \alpha_3 = 1/2$. Thus the shape function N_1 is equal to:

$$N_1 = -\frac{1}{2}\xi(1-\xi)$$

Similarly it is possible to obtain that the shape functions N_2 and N_3

$$\begin{aligned}
&N_2 = 1 - \xi^2 \\
&N_3 = \frac{1}{2}\xi(1+\xi)
\end{aligned}$$

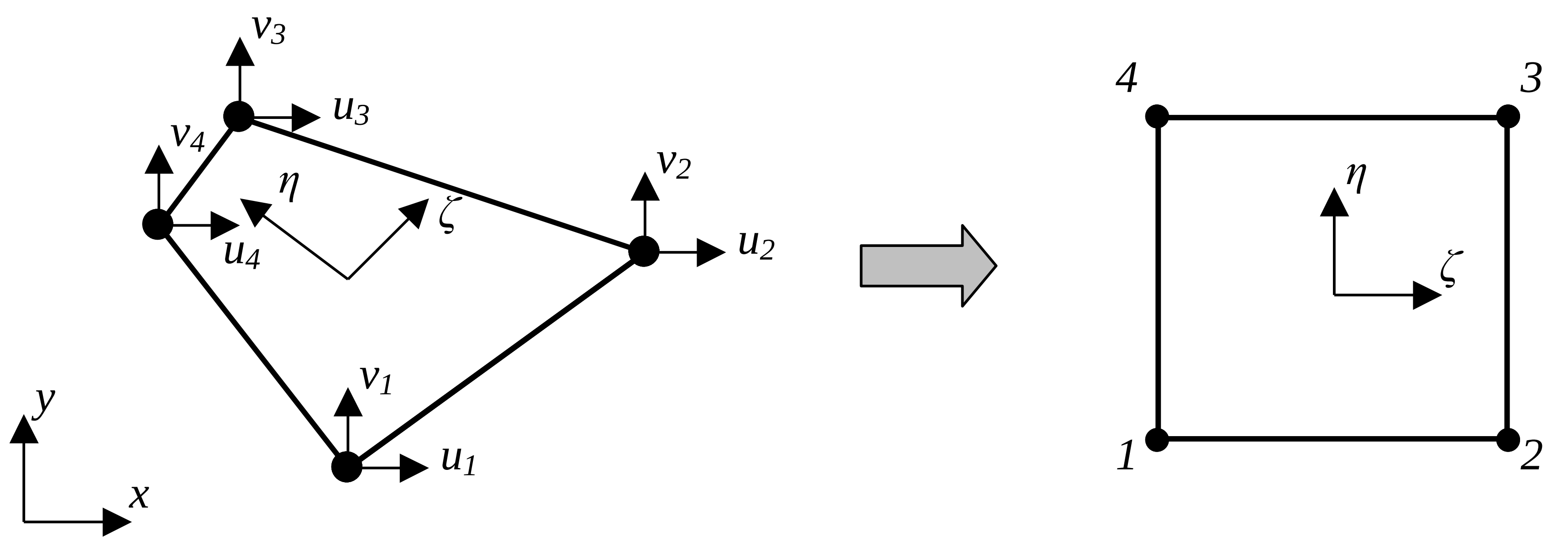
A slightly fancier assumption:
displacement varying **quadratically** inside each bar



$$\boxed{d = N_1(\xi)d_1 + N_2(\xi)d_2 + N_3(\xi)d_3}$$

This is a **<u>quadratic finite element</u>** in 1D and it has three nodes and three associated shape functions per element.

2D elements - bilinear element



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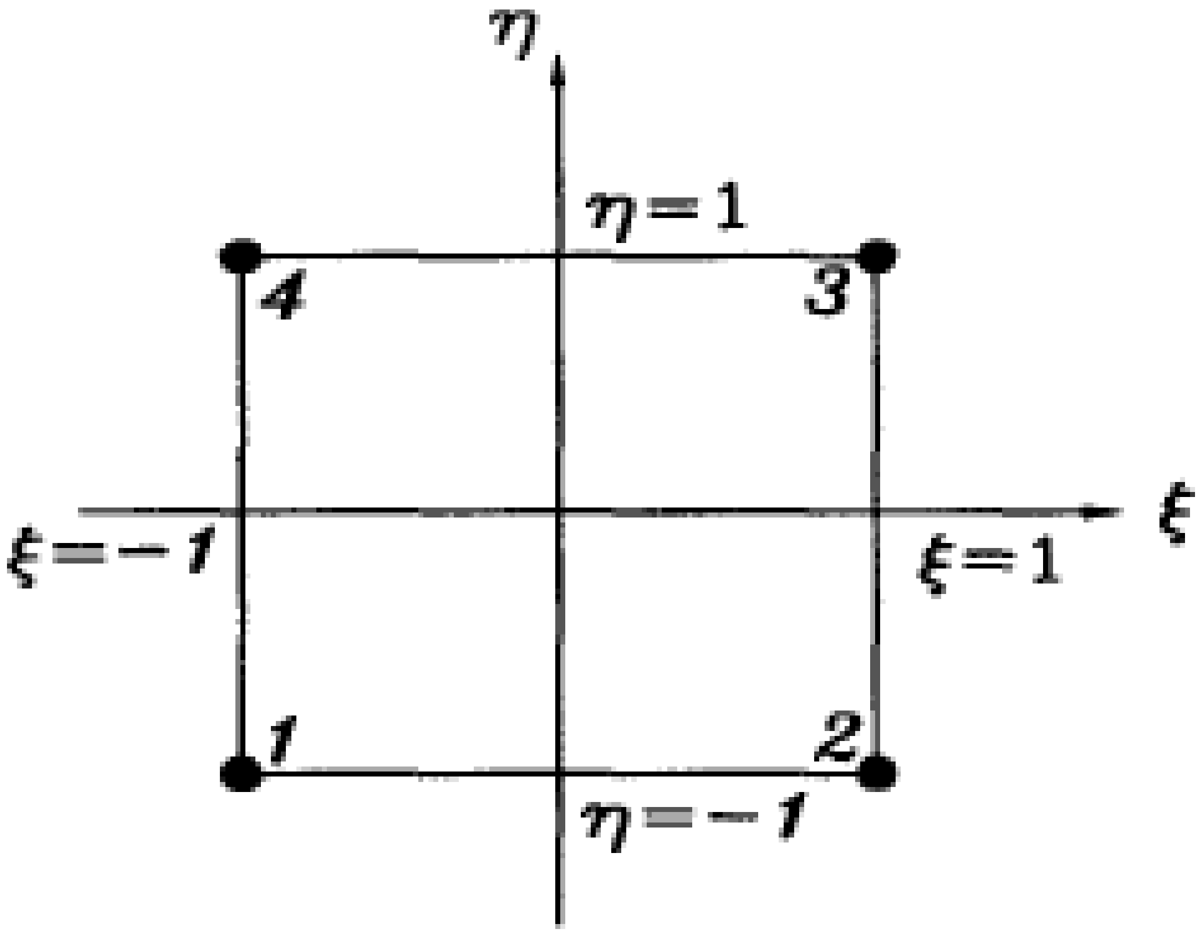
2D linear rectangular element



Shape Functions of rectangular linear element

Bilinear element has shape functions generated by multiplying linear expressions in ζ and η direction. Due to multiplication, the product is not a linear function:

$$N_1 = N_{\zeta 1} \, . \, N_{\eta 1} = (1 - \zeta).(1 - \eta \,) \,/\, 4$$

$$N_2 = N_{\zeta 2} \, . \, N_{\eta 1} = (1 + \zeta).(1 - \eta \,) \,/\, 4$$

$$N_3 = N_{\zeta 2} \, . \, N_{\eta 2} = (1 + \zeta).(1 + \eta \,) \,/\, 4$$

$$N_4 = N_{\zeta 1} \, . \, N_{\eta 2} = (1 - \zeta).(1 + \eta \,) \,/\, 4$$

,