# CHAPTER 1

## INTRODUCTION

One of the most important advances in applied mathematics in the 20th century has been the development of the *Finite Element Method* as a general mathematical tool for obtaining approximate solutions to boundary-value problems. The theory of finite elements draws on almost every branch of mathematics and can be considered as one of the richest and most diverse bodies of the current mathematical knowledge.

### 1. 1 Mathematical Modeling of Physical Systems

Due to the complexity of physical systems, some approximation must be made in the process of turning physical reality into a mathematical model. It is important to decide at what points in the modeling process these approximations are made. This, in turn, determines what type of analytical or computational scheme is required in the solution process. Let us consider a diagram of the two common branches of the general modeling-solution process given in Figure 1:

For many real world problems the second approach is in fact the only possibility. For instance suppose that the aim is to find the thermo-mechanical stresses in an air-cooled turbine blade depicted in Figure 2.

The complex three-dimensional geometry of the blade along with the combined thermal and mechanical loadings makes the analysis of the blade a formidable task. Nevertheless, many powerful commercial finite element packages are available that can be implemented to perform this task with relative ease.

Numerical Methods

1. Finite Difference Method (FDM)
    - Pointwise approximation to differential equation (DE)
    - Array of grid points

2. Finite Element Method (FEM)
    - Global approximation or integral approximation to DE
    - Assembly of finite elements (subdomains, subregions)

3. Boundary Element Method (BEM)
    - Deal with integral equation rather than differential equation
    - Discretization over boundary only

## 1.2 FEM Analysis Process

A model-based simulation process using **FEM** consists of a sequence of steps. This sequence takes two basic configurations depending on the environment in which **FEM** is used. These are referred to as the ***Mathematical FEM*** and the ***Physical FEM***.

***Mathematical FEM***

The centerpiece in the process steps of the Mathematical FEM is the *mathematical model* which is often an ordinary or partial differential equation in space and time. Using the methods provided by the Variational Calculus, a discrete finite element model is generated from of the mathematical model. The resulting **FEM** equations are processed by an equation solver, which provides a discrete solution. In this process we may also think of an *ideal physical system*, which may be regarded as a *realization* of the mathematical model. For example, if the mathematical model is the Poisson's equation, realizations may be a heat conduction problem. In Mathematical **FEM** this step is unnecessary and indeed **FEM** discretizations may be constructed without any reference to physics.

The concept of *error* arises when the discrete solution is substituted in the mathematical and discrete models. This replacement is generically called *verification*. The *solution error* is the amount by which the discrete solution fails to satisfy the discrete equations. This error is relatively unimportant when using computers. More relevant is the *discretization error*, which is the amount by which the discrete solution fails to satisfy the mathematical model.

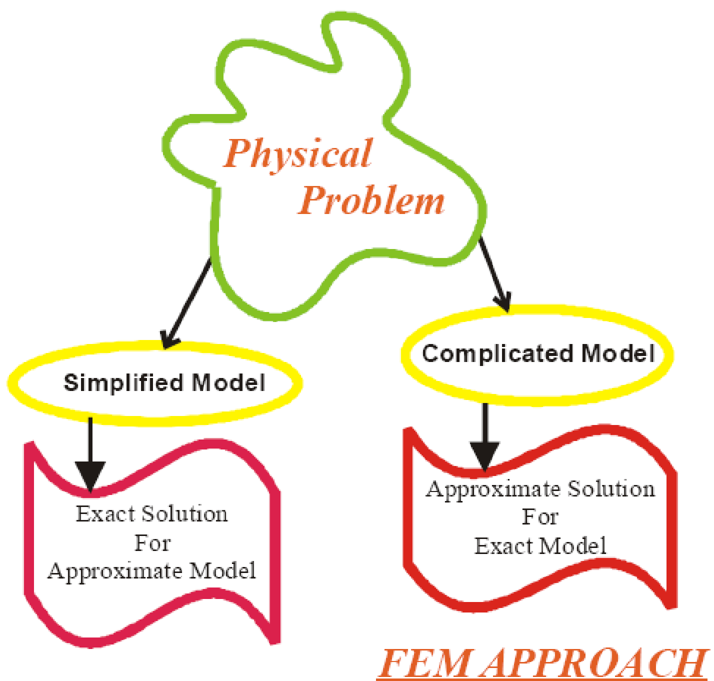



**Figure 1.1** Comparison of Analytical and Computational Model

***Physical FEM***

The processes of idealization and discretization are carried out *concurrently* to produce the discrete model. Indeed **FEM** discretizations may be constructed and adjusted without reference to mathematical models, simply from experimental measurements. The concept of *error* arises in the physical **FEM** in two ways, known as *verification* and *validation*, respectively. Verification is the same as in the Mathematical **FEM**: the discrete solution is replaced into the discrete model to get the solution error. As noted above, this error is not generally important. Validation tries to compare the discrete solution against ***observation*** by computing the *simulation error*, which combines modeling and solution errors. Since the latter is typically insignificant, the simulation error in practice can be identified with the modeling error. Comparing the discrete solution with the ideal physical system would in principle quantify the modeling errors.

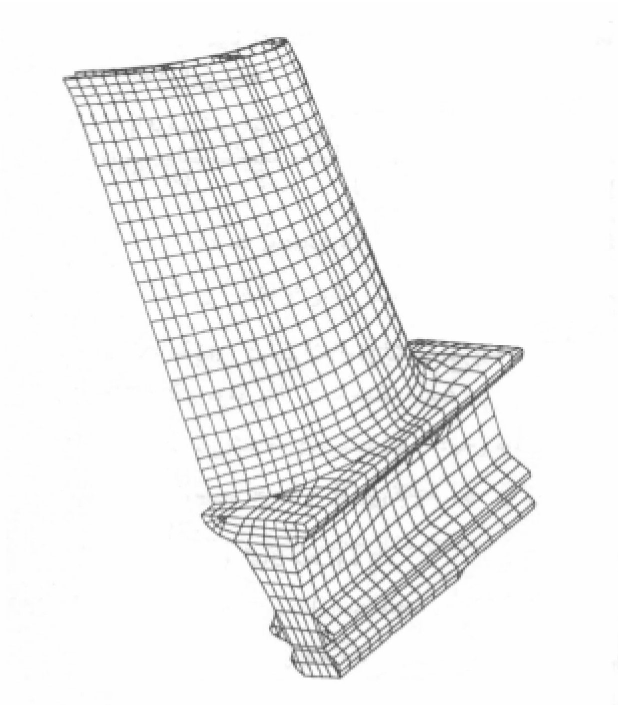


**Figure 1.2** Finite Element Discretization

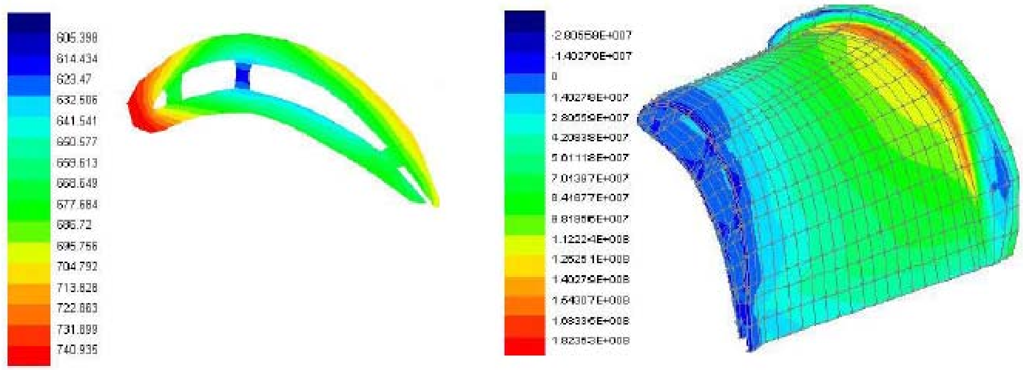



**Figure 1.3** Finite Element Solution

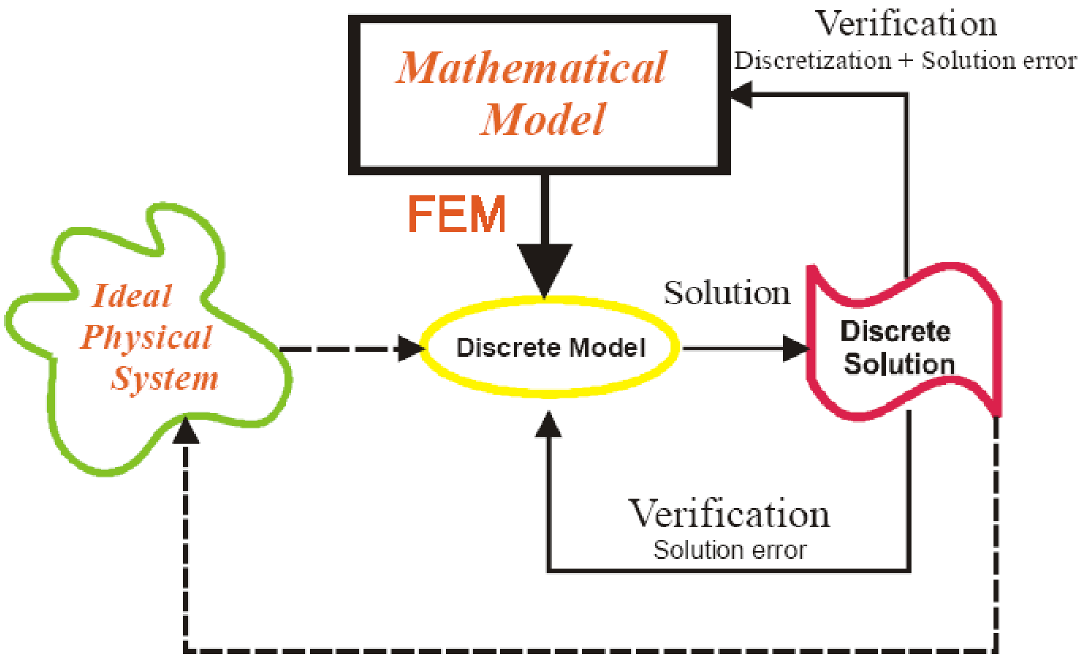



**Figure 1.4** Deviation of the solution from the mathematical model

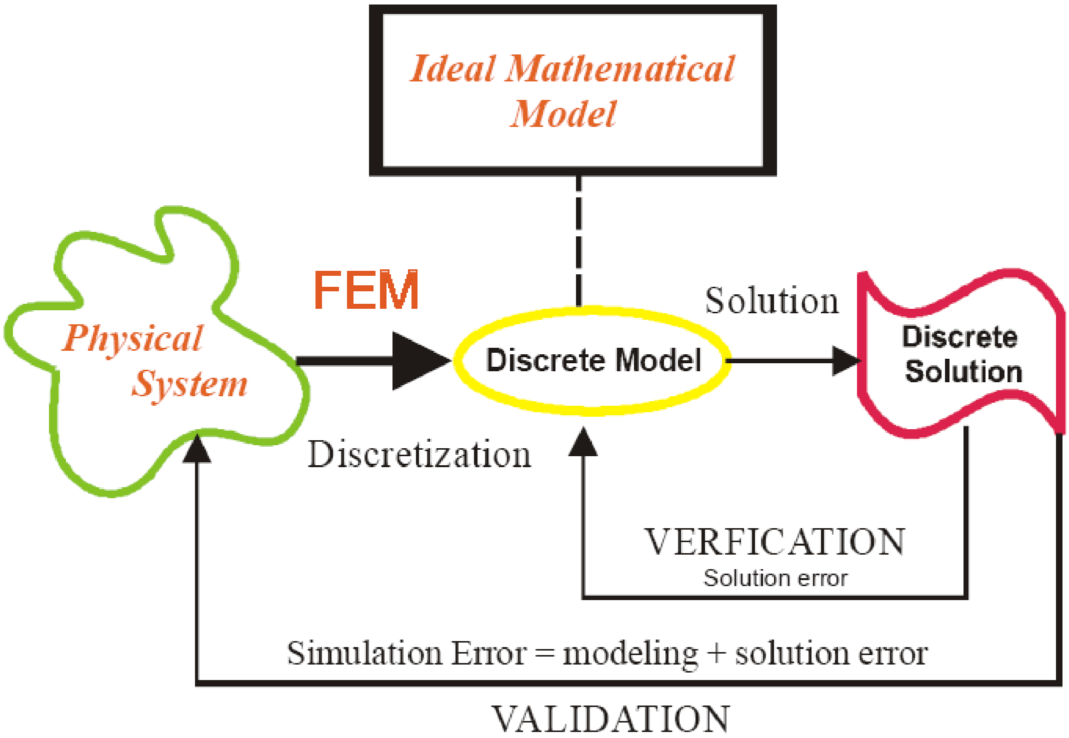



**Figure 1.5** Deviation of the solution from physical system

**Application of of FEM**

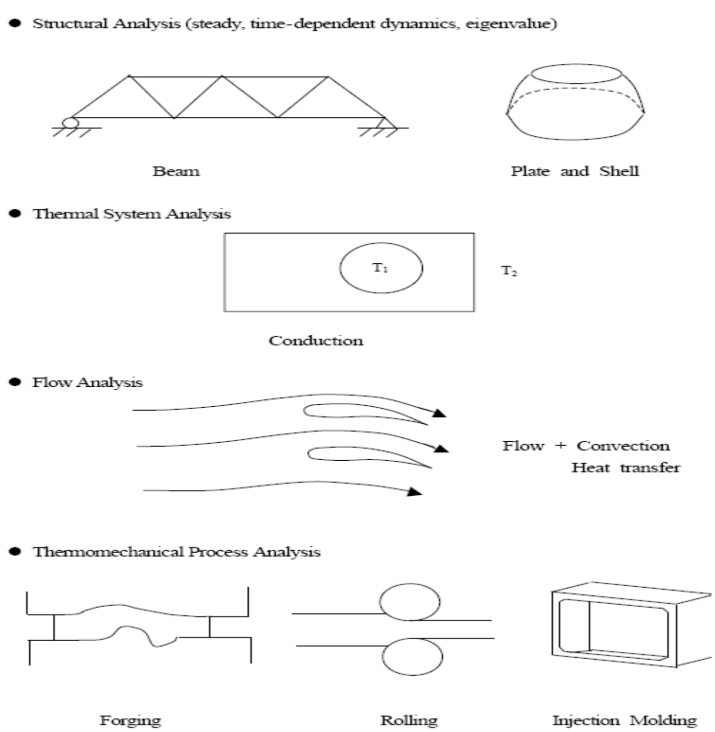



**Figure 1.6** Application Examples of FEM

**FEM PROCEDURE**

1. Identify the system (governing) equation. (usually DE)

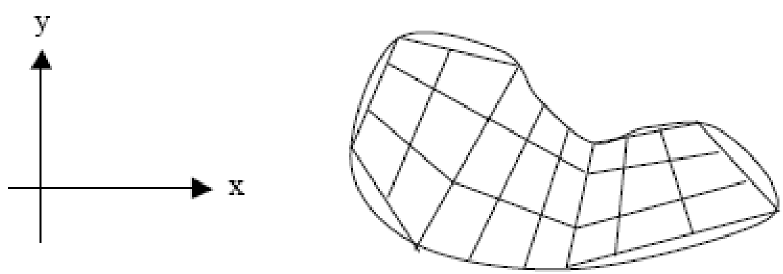
$$L(\phi) = 0.$$

2. Introduce an integral form equation. (Weak form equation) → FEM Formulation

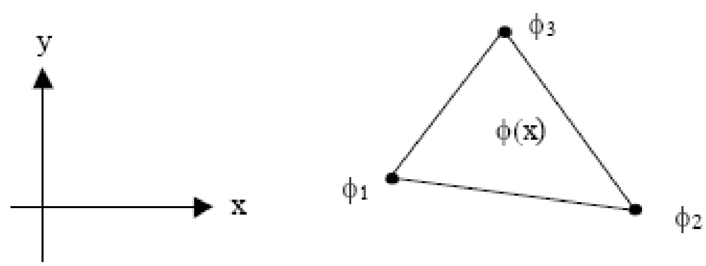
- Direct Approach
- Variational Approach : weak form
- Method of Weighted Residual Approach : weak form

$$\int_{\Omega} \psi L(\phi) d\Omega = 0 \quad \Rightarrow \quad \int_{\Omega} L'(\psi) L''(\phi) d\Omega = 0$$

3. Discretize the domain of interest into elements. → Element Types




4. Introduce an approximation of the field variable over an element. → Interpolation




$$\phi(\mathbf{x}) = N_1(\mathbf{x})\phi_1 + N_2(\mathbf{x})\phi_2 + N_3(\mathbf{x})\phi_3$$

$\phi_i$ : Nodal values of the field variable

$N_i$ : Interpolation functions, Shape functions

5. Evaluate the integral form over each element. ⟶ Numerical Integral

$$[K]^e\{\phi\}^e = \{f\}^e$$

6. Assemble the global matrix equation. ⟶ Assembly Procedure

$$[K]\{\phi\} = \{F\}$$

7. Solve the matrix equation to get the unknowns ⟶ Solution Techniques

$$\{\phi\} = [K]^{-1}\{F\}$$

8. Calculate the values of interest from the approximate solution.

e.g. $\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial x},$ etc.

# CHAPTER 2

# FEM FORMULATION

## 2.1 Direct Formulation

Direct approach has the following features:

- It *applies physical concept* (e.g. force equilibrium, energy conservation, mass conservation, etc.) *directly* to *discretized elements*.
  It is *easy* in its physical interpretation.
- It *does not need elaborate*, sophisticated mathematical manipulation or concept.
- Its applicability is ***limited*** to certain problems for which equilibrium or conservation law can be easily stated in terms of physical quantities one wants to obtain. In most cases, discretized elements are self obvious in the physical sense.

## 2.2 Weighted Residual Methods

Example with a single governing equation with only one independent variable

$$f[T(x)] = 0 \ in \ \Omega$$

T is the function sought , function of x only

$\Omega$ is the domain of the region governed by f

Boundary conditions

$$g_1[T(x)] = 0 \ in \ \Gamma_1$$

$$g_2[T(x)] = 0 \ in \ \Gamma_2$$

$\Gamma_1$ and $\Gamma_2$ are parts of the boundary of $\Omega$

Approximation of the solution with a $T'$ function:

$$T' = T'(x; a_1, a_2, \cdots, a_n) = \sum_{i=1}^{n} a_i N_i(x)$$

which has one or more unknown( but constant) parameters $a_1, a_2 \cdots, a_n$ satisfies exactly the boundary conditions. No surprise if the approximation does not satisfy the equation exactly! We will get a residual error:

$$f[T'(x; a_1, a_2, \cdots, a_n)] = R(x; a_1, a_2, \cdots, a_n)$$

The method of weighted residuals requires that the parameters $a_1, a_2 \cdots, a_n$ be determined satisfying:

$$\int_{\Omega} W_i(x) R(x; a_1, a_2, \cdots, a_n) d\Omega = 0$$

where the functions $W_i(x)$ are the n arbitrary weighting functions

The conditions of the weighting functions is generally left to a personal judgment
The most popular weighted-residual methods are:

1) Point collection
2) Subdomain collection
3) Least squares
4) Galerkin

***Point Collection***

The weighted functions $W_i(x)$ are $\delta(x - x_i)$ and defined such that

$$\int_a^b \delta(x - x_i)dx = 1 \quad for\ x = x_i$$

$$\int_a^b \delta(x - x_i)dx = 0 \quad for\ x \neq x_i$$

Substitution of this choice of $w_i(x)$ gives:

$$\int_\Omega \delta(x - x_i)R(x; a_1, a_2, \cdots a_n)d\Omega = 0 \quad for\ i = 1,2,\cdots,n$$

which is evaluated at n collection points $x_1, x_2, \cdots, x_n$ results n algebraic equations in n unknowns

$$R(x_1; a_1, a_2, \cdots, a_n) = 0$$
$$R(x_2; a_1, a_2, \cdots, a_n) = 0$$
$$\vdots$$
$$R(x_n; a_1, a_2, \cdots, a_n) = 0$$

***Subdomain Collection***

The weighted functions $W_i(x)$ are:

$$W_1(x) = \begin{cases} 1 & for\ x\ in\ \Omega_1 \\ 0 & for\ x\ not\ in\ \Omega_1 \end{cases}$$

$$W_2(x) = \begin{cases} 1 & for\ x\ in\ \Omega_2 \\ 0 & for\ x\ not\ in\ \Omega_2 \end{cases}$$

Substitution of this choice of $w_i(x)$ gives the following n integral equations

$$\int_{\Omega_1} R(x; a_1, a_2, \cdots a_n)d\Omega_1 = 0$$

$$\int_{\Omega_2} R(x; a_1, a_2, \cdots a_n)d\Omega_2 = 0$$

$$\vdots$$

$$\int_{\Omega_n} R(x; a_1, a_2, \cdots a_n)d\Omega_n = 0$$

***Least Squares***

The method of least squares requires that the integral $I$ of the square of the residual R be minimized. That is:

$$I = \int_{\Omega} [R(x; a_1, a_2, \cdots, a_n)]^2 \, d\Omega$$ be a minimum, or equivalently

$$\frac{\partial I}{\partial a_i} = \frac{\partial}{\partial a_i} \int_{\Omega} [R(x; a_1, a_2, \cdots, a_n)]^2 \, d\Omega = \int_{\Omega} \frac{\partial}{\partial a_i} [R(x; a_1, a_2, \cdots, a_n)]^2 \, d\Omega$$

Carrying out the differentiations and simplifying, we have:

$$\int_{\Omega} R \frac{\partial R}{\partial a_i} d\Omega = \int_{\Omega} R \frac{\partial R}{\partial a_2} d\Omega = \cdots = \int_{\Omega} R \frac{\partial R}{\partial a_n} d\Omega = 0$$

which means that the weighting functions are:

$$\int_{\Omega} R \frac{\partial R}{\partial a_i} d\Omega = 0 \quad i = 1, 2, \cdots, n$$

***Galerkin***

The weighting functions $w_i(x)$ are $N_i(x)$

Therefore for the Galerkin method of the weighted residuals we have:

$$\int_{\Omega} N_i(x) R(x; a_1, a_2, \cdots, a_n) d\Omega = 0 \quad i = 1, 2, \cdots, n$$

Remember that our approximation of the $T(x)$ function is:

$$T = T'(x; a_1, a_2, \cdots, a_n) = \sum_{i=1}^{n} a_i N_i(x)$$

**2.2 Galerkin's Method**

Let us consider a steady state continuous physical system described the following system of PDEs:

$$F(u) + f_{\Omega} = 0 \qquad \text{on domain } \Omega$$
$$G(u) + f_{\Gamma} = 0 \qquad \text{on bounder } \Gamma$$

Example: Poisson's equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{q'''}{k} = 0 \quad \text{on } \Omega$$

Boundary condition

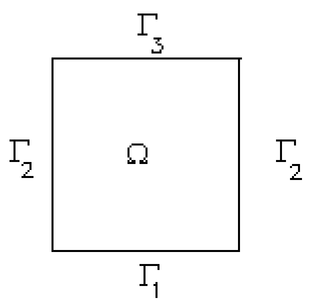
a) Dirchlet BC (Natural BC)

$T = T_s$ on $\Gamma_1$

b) Neuman BC (Essential BC)

$$\frac{\partial T}{\partial n} = h(T - T_{\infty}) \quad \text{on } \Gamma_2$$

c) $$\frac{\partial T}{\partial n} = \dot{q} \qquad \text{on } \Gamma_3$$




$\Omega$ - domain $\Gamma$ - boundary

***Figure 2.1*** The domain of the boundary

The residual is defined as follows:

$R(u) = F(u) + f_{\Omega}$

The residual vanishes when the solution is substituted.

The weighted residual method consists in finding functions u that satisfy the following integral equation

$$\int_{\Omega} W_i R(u) dR = \int_{\Omega} W_i \{F(u) + f_{\Omega}\} d\Omega = 0 \qquad i = 1 \ldots n$$

where $W_i$ is weighting function and u is the solution that satisfy the boundary condition.

***Example***

Integral of Poisson's Equation

$$\int_{\Omega} W_i(x,y)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+f_{\Omega}\right)d\Omega=0$$

u – must be twice differentiable and it should satisfy all the boundary conditions on $\Gamma_u and \Gamma_f$

***Integration by Parts***

*Gradient Theorem*

$$\int_{\Omega} \nabla F\, dx\, dy = \oint_{\Gamma} \vec{n}\, F ds$$

$$\int_{\Omega} (i\frac{\partial F}{\partial x} + j\frac{\partial F}{\partial y}) dx\, dy = \oint_{\Gamma} (n\vec{i} + n_y \vec{j}) F ds$$

*Divergence Theorem*

$$\int_{\Omega} \nabla.\vec{G}\, dx\, dy = \oint_{\Gamma} \vec{n}.G ds$$

$$\int_{\Omega} (\frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y}) dx\, dy = \oint_{\Gamma} (n_x G_x + n_y G_y) ds$$

$$\int_{\Omega} w(\nabla^2 G) dx\, dy = -\int_{\Omega} \nabla w.\nabla G dxdy \; -\oint_{\Gamma} w \frac{\partial G}{\partial n} ds$$

$$\int_{\Omega} w\frac{\partial G}{\partial x} dxdy = -\int_{\Omega} \frac{\partial w}{\partial x} G dxdy + \oint_{\Gamma} n_x wGds$$

$$\int_{\Omega} w\frac{\partial G}{\partial x} dxdy = -\int_{\Omega} \frac{\partial w}{\partial y} G dxdy + \oint_{\Gamma} n_x wGds$$

In one dimension:

$$\int_{x_1}^{x_2} W\frac{du}{dx} dx = -\int_{x_1}^{x_2} \frac{dw}{dx} u dx + Wu \Big|_{x_1}^{x_2}$$

$$\int_{x_1}^{x_2} W\frac{d^2 u}{dx^2} dx = -\int \frac{dW}{dx}\frac{du}{dx} dx + W\frac{du}{dx}\Big|_{x_1}^{x_2}$$

In two dimension:

$$\int_\Omega W\frac{\partial u}{\partial x}dxdy = -\int_\Omega \frac{\partial w}{\partial x}udxdy + \oint_\Gamma Wudy$$

$$= -\int_\Omega \frac{\partial w}{\partial x}udxdy + \int_\Gamma Wulds$$

$$l = \cos\theta$$

$$\int W\frac{\partial u}{\partial y}dxdy = -\int \frac{\partial W}{\partial y}udxdy - \oint_\Gamma Wudx$$

$$-\int \frac{\partial W}{\partial y}udxdy + \oint_\Gamma Wumds$$

$$m = \sin\theta$$

$$\int_\Omega W\frac{\partial^2 u}{\partial x^2}dxdy = -\int_A \frac{\partial W}{\partial x}\frac{\partial u}{\partial x}dxdy + \oint W\frac{\partial u}{\partial x}lds$$

$$\int_\Omega W\frac{\partial^2 u}{\partial y^2}dxdy = -\int_A \frac{\partial W}{\partial y}\frac{\partial u}{\partial y}dxdy + \oint W\frac{\partial u}{\partial x}mds$$

$$\int_\Omega \left( W\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \right)dxdy = -\int_A (\frac{\partial W}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial W}{\partial y}\frac{\partial u}{\partial y})dxdy + \oint W(\frac{\partial u}{\partial x}l + \frac{\partial u}{\partial x}m)ds$$

$$= \int_A (\frac{\partial W}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial W}{\partial y}\frac{\partial u}{\partial y})dxdy + \oint W(\frac{\partial u}{\partial n})ds$$

***Weak Integral Form***

A given integral form may be transformed to obtain a so-called weak form through integration by parts. By this process, the order of the highest derivative can be reduced. Boundary conditions other than u can also be specified. However, the integration by parts introduce derivates of the weighing function W. Thus, the continuity condition of W are more severe

***Example***

$$= -\int_A (\frac{\partial N}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial N}{\partial y}\frac{\partial u}{\partial y})dxdy + \oint N(\frac{\partial u}{\partial n})ds$$

Weak integral of Poisson's equation

$$\int_\Omega N_i\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f_\Omega\right)dxdy = -\int_\Omega \left(\frac{\partial N_i}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial u}{\partial y} - N_i f_\Omega\right)d\Omega + \int_{\Gamma_f} N_i\frac{\partial u}{\partial n}ds = 0\,.$$

$N_i$ must be twice differentiable

$$-\int_\Omega \left(\frac{\partial N_i}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial u}{\partial y} - N_i f_\Omega\right)d\Omega + \int_\Gamma N_i\left(f_\Gamma - \alpha U\right)ds = 0$$

$N_i$ must be twice differentiable

$$-\int_{\Omega}\left(\frac{\partial N_i}{\partial x}\sum_{j=1}^{nno}u_j\frac{\partial N_j}{\partial x}+\frac{\partial N_i}{\partial y}\sum_{j=1}^{nno}u_j\frac{\partial N_j}{\partial y}-N_i f_{\Omega}\right)d\Omega+\int_{\Gamma}N_i\left(f_{\Gamma}-\alpha\sum_{j=1}^{nno}N_j u_j\right)ds=0 \qquad i=1...nno$$

$$[K]\{U\}=\{F\}$$

$$K_{i,j}=\int_{\Omega}\left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}+\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right)d\Omega+\int_{\Gamma}\alpha N_i N_j ds=0 \qquad i=1...nno \;\; j=1...nno$$

$$F_i=\int_{\Omega}N_i f_{\Omega}d\Omega+\int_{\Gamma}f_{\Gamma}N_i ds=0 \qquad i=1...nno$$

Where:

$[K]$ is stiffness Matrix

$\{F\}$ is load vector

$\{U\}$ is nodal values of function of interest

***Variational Formulation***

A functional is linear

$$\pi(u,\frac{\partial u}{\partial x}) = \int_{\Omega}(a_1 u + a_2\frac{\partial u}{\partial x})d\Omega$$

A functional is quadratic if all terms are of second order, for example

$$\pi = \int_{\Omega}(a_1(\frac{\partial u}{\partial x})^2 + a_2 u^2)d\Omega$$

For a purely quadratic functional

$$\pi = \frac{1}{2}\int_{\Omega}\left\langle u\ \frac{\partial u}{\partial x}\cdots\right\rangle [D]\begin{Bmatrix} u \\ \frac{\partial u}{\partial x} \\ \vdots \end{Bmatrix} d\Omega$$

$[D]$ is a symmetric matrix independent of $u$

$$\delta\pi = \int_{\Omega}\left\langle \delta u\ \frac{\partial(\delta u)}{\partial x}\cdots\right\rangle [D]\begin{Bmatrix} u \\ \frac{\partial u}{\partial x} \\ \vdots \end{Bmatrix} d\Omega$$

$\pi$ is positive definite if D is positive definite matrix that is all the value of D are positive

***Example***

Find the variation of the following one dimensional functional

$$\pi(u,\frac{du}{dx}) = \int_{x_1}^{x_2}(\frac{1}{2}(\frac{du}{dx})^2 - u\ f)dx$$

Its variation

$$\delta\pi = \delta\int_{x_1}^{x_2}\frac{1}{2}(\frac{du}{dx})^2 - u\ f)dx$$

Using properties of $\delta$

$$\delta\pi = \int_{x_1}^{x_2}(\delta(\frac{du}{dx})\frac{du}{dx} - \delta u\ f)dx$$

$$= \int_{x_1}^{x_2}(\frac{d}{dx}(\delta u)\frac{du}{dx} - \delta u\ f)dx$$

The second variation of $\pi$ can be obtained

$$\delta^2\pi = \delta(\delta\pi) = \int_{x_1}^{x_2}(\delta(\frac{du}{dx}))^2 dx$$

$$= \int_{x_1}^{x_2}(\frac{d(\delta u)}{dx})^2 dx = 0$$

$$w = \frac{d}{dx}(\delta u)$$
$$\delta(\delta u) = 0$$

$$0 = \int_{x_2}^{x_1} (\frac{dw}{dx}\frac{du}{dx} - w\, f\,)dx$$
$$\delta\pi \;= \int R = 0$$
$$\int R = \int_\Omega \langle \delta u \rangle \{L(u) + f\,\} d\Omega = 0$$

$L$ is a linear operator

$f_\Omega$ and $f_\Gamma$ are independent of $u$

These conditions are sufficient for a functional to exist

***Example***

Formulate functional for poission's equation

$$F(u) + f_\Omega = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} + f_\Gamma = 0$$

The corresponding integral form is obtained previously

$$\int R = \int_\Omega (\frac{\partial w}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial u}{\partial y} - wf_\Omega)d\Omega + \int_\Gamma w(\alpha\, u - f_\Gamma)d\Gamma = 0$$

Choosing $w = \delta\, u$

$$\int R = \int_\Omega (\frac{\partial(\delta u)}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial(\delta u)}{\partial y}\frac{\partial u}{\partial y} - \delta u\;\, f_\Omega)d\Omega + \int_\Gamma \delta u\,(\alpha\, u - f_\Gamma)d\Gamma = 0$$

Defining the functional

$$\pi(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = \int_\Omega (\frac{1}{2}(\frac{\partial u}{\partial x})^2 + \frac{1}{2}(\frac{\partial u}{\partial y})^2 - uf_\Omega)d\Omega + \int_\Gamma (\frac{1}{2}\alpha u^2 - uf_\Gamma)d\Gamma$$
$$\delta\pi \;= R = 0$$

A solution u for $R = 0$ also renders the functional stationary $\delta\pi \;= 0$. At this condition the functional is either a minimum or a maximum.

To illustrate the process let us consider now a specific example.

Suppose we specify the problem by requiring the stationary of a functional

$$\Pi = \int_\Omega \left[ \frac{1}{2}k\left(\frac{\partial T}{\partial x}\right)^2 + \frac{1}{2}k\left(\frac{\partial T}{\partial y}\right)^2 - \dot{Q}_V\, T \right] d\Omega - \int_{r_q} \bar{q}\;\phi\; d\Gamma \qquad (9.72)$$

In which k and Q depend only on position and $\delta T$ such that $\delta T = 0$ on $\Gamma_\phi$ *where* $\Gamma_\phi$ *and* $\Gamma_q$ are bounding the domain $\Omega$.

We now perform the variation. This can be written following rules of differentiation as

$$\delta\Pi = \int_\Omega \left[ k\frac{\partial T}{\partial x}\delta\left(\frac{\partial T}{\partial x}\right) + k\frac{\partial T}{\partial y}\delta\left(\frac{\partial T}{\partial y}\right) - \dot{Q}_v \delta T \right] d\Omega - \int_{r_q} (\bar{q}\,\delta T) d\Gamma$$

As

$$\delta\left(\frac{\partial T}{\partial x}\right) = \frac{\partial}{\partial x}(\delta T)$$

We can integrate by parts and, noting that $\delta T = 0$ *on* $\Gamma_\phi$, and obtain

$$\delta\Pi = -\int_\Omega \delta T\left[ \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \dot{Q}_v \right] d\Omega$$

$$+ \int_{r_q} \delta T\left(k\frac{\partial T}{\partial n} - \bar{q}\right) d\Gamma = 0$$

We immediately observe that the Euler equations are

$$\mathrm{A}(T) = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \dot{Q}_v \quad \text{in } \Omega$$

$$\mathrm{B}(T) = k\frac{\partial T}{\partial n} - \bar{q} = 0 \qquad \text{on } \Gamma_q$$

If $T$ is so prescribed that $T = \bar{T}$ on $\Gamma_\phi$ and $\delta T = 0$ on that boundary, then the problem is precisely the one we have already discussed and the functional specifies the two-dimensional heat conduction problem in an alternative way.

In this case we have 'guessed' the functional but the reader will observe that the variation operation could have been carried out for any functional specified and corresponding Euler equations could have been established.

Let us continue the problem to obtain an approximate solution of the linear heat conduction problem. Taking, as usual,

$$T \approx \bar{T} = \sum N_i a_i \qquad (9.76)$$

We substitute this approximation into the expression for the functional $\Pi$ and obtain

$$\Pi = \int_\Omega \frac{1}{2} k \left( \sum \frac{\partial N_i}{\partial x} a_i \right)^2 d\Omega + \int_\Omega \frac{1}{2} k \left( \sum \frac{\partial N_i}{\partial y} a_i \right)^2 d\Omega$$

$$- \int_\Omega \dot{Q}_v \sum N_i a_i d\Omega - \int_{r_q} \bar{q} \sum N_i a_i \, d\Gamma$$

On differentiation with respect to atypical parameter $a_j$ we have

$$\frac{\partial \Pi}{\partial a_j} = \int_\Omega k \left( \sum \frac{\partial N_i}{\partial x} a_i \right) \frac{\partial N_j}{\partial x} d\Omega + \int k \left( \sum \frac{\partial N_i}{\partial y} a_i \right) \frac{\partial N_j}{\partial y} d\Omega$$

$$- \int_\Omega \dot{Q}_v N_j d\Omega - \int_{r_q} \bar{q} N_j \, d\Gamma$$

and a system of equations for solution of the problem is

$$[K]\{a\} = \{f\}$$

with

$$K_{ij} = K_{ji} = \int_\Omega k \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} d\Omega + \int_\Omega k \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} d\Omega$$

$$f_i = -\int_\Omega N_j Q \, d\Omega - \int_{r_q} N_j \bar{q} \, d\Gamma$$