# Algebraic and Geometric Surgery

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# Contents

CHAPTER 1

# Differentiable Manifolds

## 1. Differentiable maps on Euclidean spaces

Consider a differentiable map $f : \mathbb{R}^n \to \mathbb{R}^m$. The derivative of $f$ at $x \in \mathbb{R}^n$ is the linear map defined by the Jacobian $m \times n$ matrix of first partial derivatives

$$df(x) = \left(\frac{\partial f_i}{\partial x_j}\right) : \mathbb{R}^n \to \mathbb{R}^m \; ; \; h = (h_1, h_2, \ldots, h_n) \to$$
$$df(x)(h) = \left( \sum_{j=1}^{n} \frac{\partial f_1}{\partial x_j} h_j, \sum_{j=1}^{n} \frac{\partial f_2}{\partial x_j} h_j, \ldots, \sum_{j=1}^{n} \frac{\partial f_m}{\partial x_j} h_j \right) .$$

If $g \colon \mathbb{R}^m \to \mathbb{R}^s$ is a second differentiable map and $gf \colon \mathbb{R}^n \to \mathbb{R}^s$ is the composition then

$$d(gf) = dg\, df.$$

DEFINITION 1. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable map.*

(1) *A* **regular point** *of $f$ is a point $x \in \mathbb{R}^n$ where the linear map $df(x)$ is of maximal rank, i.e.*

$$\operatorname{rank}(df(x)) = \min(m, n) .$$

(2) *A* **critical point** *of $f$ is a point $x \in \mathbb{R}^n$ which is not regular.*
(3) *A* **regular value** *of $f$ is a point in the image $y \in \mathbb{R}^m$ such that every $x \in f^{-1}(y) \subseteq N$ is regular or $f^{-1}(y)$ is empty.*
(4) *A* **critical value** *of $f$ is a point $y \in \mathbb{R}^n$ which is not regular.*

THEOREM 1. (Implicit Function) *For any differentiable map $f : \mathbb{R}^n \to \mathbb{R}^m$ and any regular point $x \in \mathbb{R}^n$ of $f$ there exist a neighborhood $N_x \subset \mathbb{R}^n$ and local coordinates $(z_1, \ldots, z_n)$ in $N_x$ with $x$ at $(0, \ldots, 0)$, and a neighborhood $N_{f(x)} \subset \mathbb{R}^m$ of $f(x)$ with local coordinates $(w_1, \ldots, w_m)$ with $f(x) \sim (0, \ldots, 0)$, such that :*

- *if $n \leq m$, then*

$$f(z_1, \ldots, z_n) = (z_1, \ldots, z_n, \underbrace{0, \ldots, 0}_{m-n \text{ times}})$$

- *if $n \geq m$ then*

$$f(z_1, \ldots, z_n) = (z_1, \ldots, z_m) .$$

Thus, in a neighborhood of a regular point the map either looks like an embedding or a projection. In particular, the set of regular points is open in $\mathbb{R}^n$.

As regards the regular and singular values of $f$ we have the fundamental theorem of Sard :

THEOREM 2. (Sard) *The set of singular values of $f$ has measure 0 in $\mathbb{R}^m$ for any $C^\infty$ map $f\colon \mathbb{R}^n \to \mathbb{R}^m$.*

In the case where $n < m$ this says that the image of $f$ cannot be something like a space filling curve. But in the case where $n \geq m$ it is even more restrictive. For example the implicit function theorem immediately implies :

COROLLARY 1. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be any $C^\infty$ map for which the measure of $\mathrm{im}(f) > 0$. Then there is a regular value, $y \in \mathbb{R}^n$ of $f$, and $f^{-1}(y)$ is a discrete set in $\mathbb{R}^n$.*

In particular, if $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is such that $\mathrm{im}(f)$ contains an open set in $\mathbb{R}^n$ then $\mathrm{im}(f)$ has measure $> 0$.

More generally, the implicit function theorem 1 gives :

COROLLARY 2. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be any $C^\infty$ map for which the measure of $\mathrm{im}(f) > 0$, with $n \geq m$. Then there is a regular value $y \in \mathbb{R}^m$ for $f$ and $f^{-1}(y)$ has the property that for each $x \in f^{-1}(y)$ there is an open neighborhood $N_x \subset \mathbb{R}^n$ of $x$ and local coordinates $(z_1, \ldots, z_n)$ there so that $f^{-1}(y) \cap N_x = \{(z_1, \ldots, z_m, 0, \ldots, 0)\}$ with $\sum_1^m z_i^2 < \epsilon$.*

## 2. Singular points and Morse functions

In a neighborhood of a singular point things are much more complex. Indeed, in general the situation is far from being understood. However, in the extreme case where $f$ is a function $f\colon \mathbb{R}^n \to \mathbb{R}$ we have a fairly good understanding of what happens – at least when the singular point is isolated!

The Taylor expansion through degree $k$ of a differentiable function $f$ : $\mathbb{R}^n \longrightarrow \mathbb{R}$ at $x \in \mathbb{R}^n$ is given by

$$\begin{aligned} f(x+h) \;=\; & f(x) + \sum_{j=1}^{k} \frac{1}{j!} \sum_{1 \leq i_1, i_2, \ldots, i_j \leq n} \frac{\partial^j f}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_j}} h_{i_1} h_{i_2} \ldots h_{i_j} \\ & + O(|h|^{k+1}) \in \mathbb{R} \\ & (x = (x_1, x_2, \ldots, x_n), h = (h_1, h_2, \ldots, h_n) \in \mathbb{R}^n)\ , \end{aligned}$$

so that

$$f(x+h) \;=\; f(x) + df(x)(h) + \ldots\ .$$

The linear term is determined by the gradient vector

$$df(x) \;=\; \left( \frac{\partial f}{\partial x_1}\ \frac{\partial f}{\partial x_2}\ \ldots\ \frac{\partial f}{\partial x_n} \right) \in \mathbb{R}^n\ ,$$

corresponding to the linear map

$$df(x) : \mathbb{R}^n \to \mathbb{R} \; ; \; h \; = \; (h_1, h_2, \ldots, h_n) \to \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} h_i \; ,$$

which is either 0 or has the maximal rank 1. Thus $x \in \mathbb{R}^n$ is a regular point of $f$ if and only if $df(x) \neq 0$, and $x \in \mathbb{R}^n$ is a critical point if $df(x) = 0$. The quadratic term in the Taylor expansion is the quadratic function of the symmetric bilinear form $(\mathbb{R}^n, \lambda)$ defined by the Hessian matrix of second partial derivatives

$$\begin{aligned} \lambda \; = \; & \left(\tfrac{\partial^2 f}{\partial x_1 \partial x_2}\right) \; : \; \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \; ; \\ (u, v) \; = \; & ((u_1, u_2, \ldots, u_n), (v_1, v_2, \ldots, v_n)) \to \sum_{1 \leq i,j \leq n} \tfrac{\partial^2 f}{\partial x_i \partial x_j} u_i v_j \; , \end{aligned}$$

namely

$$\begin{aligned} & \mu \; : \; \mathbb{R}^n \to \mathbb{R} \; ; \\ h \; = \; & (h_1, h_2, \ldots, h_n) \to \lambda(h, h)/2 \; = \; \tfrac{1}{2} \sum_{1 \leq i,j \leq n} \tfrac{\partial^2 f}{\partial x_i \partial x_j} h_i h_j \; . \end{aligned}$$

We say that the critical point $x$ of the function $f \colon \mathbb{R}^n \to \mathbb{R}$ is **non-degenerate** if the determinant of the Hessian matrix

$$H_f(x) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_x$$

is non-zero, $\mathrm{Det}(H_f(x)) \neq 0$. Note that if we change coordinates near $x$ so

$$\begin{aligned} (x_1, \ldots, x_n) \; & = \; K(y_1, \ldots, y_n) \\ & = \; (K_1(y_1, \ldots, y_n), \ldots, K_n(y_1, \ldots, y_n)) \end{aligned}$$

with $dK$ invertible at $x$, then

$$\frac{\partial^2 (fh)}{\partial y_i \partial y_j} = \sum_{r,s} \frac{\partial^2 f}{\partial x_r \partial x_s} \frac{\partial K_r}{\partial y_i} \frac{\partial K_s}{\partial y_j}$$

since $\dfrac{\partial f}{\partial x_i} = 0$ at $x$ $(1 \leq i \leq n)$. Consequently,

$$H_{fk} = (dK)^t H_f dK$$

and $H_f$ actually transforms like a symmetric bilinear form. In particular :

(1) non-degeneracy for the critical point $x$ is invariant under local coordinate changes,
(2) since we can diagonalize a symmetric matrix by an orthogonal matrix, $T$, there is an orthogonal transformation centered at $x$ which

changes $H_f$ to the diagonal matrix

$$H_{fT} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix},$$

and non-singularity assures us that all the $\lambda_i$ are non-zero.

The **index** of $H_f$ at $x$, written $I(f)_x$, is the number of negative $\lambda_i$ in 2, and $I(f)_x$ is independent of the way in which we diagonalize $H$, where $H \mapsto A^t H A$ with $A$ non-singular.

The following lemma, due to M. Morse, shows that the index of a non-degenerate critical point completely determines the structure of $f$ in a neighborhood of $x$.

Lemma 1. (Morse Lemma)
*Let $x$ be a non-degenerate critical point of $f\colon \mathbb{R}^n \to \mathbb{R}$ with index $I(f)_x = k$. Then there is a coordinate system, $(z_1, \dots, z_n)$, in a neighborhood of $x$ so that $(0, \dots, 0) \sim x$ and with respect to this coordinate system $f$ takes the form*

$$f(z) = f(x) + z_1^2 + \cdots + z_{n-k}^2 - z_{n-k+1}^2 - \cdots - z_n^2 .$$

(For a proof see e.g., Milnor [43].)

Remark . In particular 1 has the following implications.

(1) Non-degenerate critical points are isolated.
(2) A non-degenerate critical point of index $n$ is a local maximum.
(3) A non-degenerate critical point of index 0 is a local minimum.

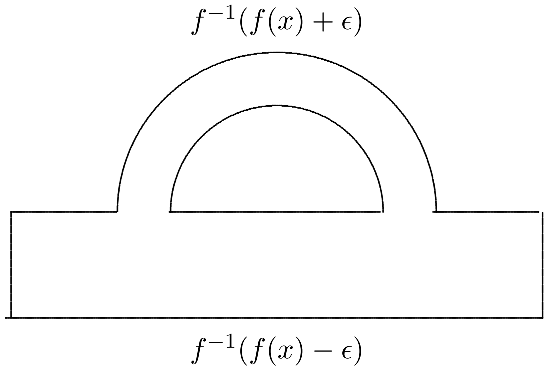
Near a non-degenerate critical point where the index $I(f)_x = k$ lies properly between 0 and $n$ the graph looks like a saddle point. It follows that near $x$ the level surfaces,

$$f^{-1}(f(x) + \epsilon) \cap N(x)$$

have the form

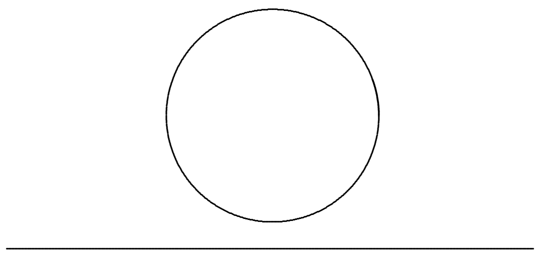
$$\begin{cases} S^{n-k-1} \times D^k & \text{for } \epsilon > 0 \\ D^{n-k} \times S^{k-1} & \text{for } \epsilon < 0. \end{cases}$$

Near an isolated critical point the homotopy type of the level space $f^{-1}(t)$ undergoes a change in homotopy type. Near $x$, $f^{-1}(f(x) + \delta)$ has the property that it looks locally like $\mathbb{R}^{n-1}$ from the implicit function theorem for $\delta \neq 0$. But it need not be a copy of $\mathbb{R}^{n-1}$. In any case, the type of $f^{-1}(t)$ changes near $x$ by removing a copy of $D^k \times S^{n-k-1}$ (which has boundary $S^{k-1} \times S^{n-k-1}$) from $f^{-1}(f(x) - \epsilon)$ and replacing it by $S^{k-1} \times D^{n-k}$ (which has the same boundary).




**The effect on the level surface of passing through a critical point**

To clarify the meaning of 2 assuming that $f^{-1}(f(x)-\epsilon)$ is simply a line segment near $x$ then the new level surface is given as



**The level surface at** $f^{-1}(f(x)+\epsilon)$

Definition 2. *A $C^\infty$ function $f\colon \mathbb{R}^n \to \mathbb{R}$ is* **Morse** *if it's only singularities are non-degenerate critical points.*

Morse proved the following result which shows that the **Morse functions** are generic among all functions.

Lemma 2. (Morse density theorem)
*The set of Morse functions is open and dense in the set of all $C^\infty$ functions $f\colon \mathbb{R}^n \to \mathbb{R}$.*

In particular, given any function $f\colon \mathbb{R}^n \to \mathbb{R}$, it can be uniformly approximated by a Morse function.

**The structure of $f^{-1}((-\infty, t])$ for $f$ Morse.** Suppose that the number of singular points of the Morse function $f$ is finite, or more generally restrict attention to a compact region $N_L \subset \mathbb{R}^n$, where

$$N_L = \{x \in \mathbb{R}^n \mid |x| \le L\} \ .$$

The fact that the singular points are isolated assures that there are only a finite number of critical points. Hence, there are only a finite number if $t \in \mathbb{R}$, say $t_1 < t_2 < \cdots < t_k$, which are critical values for $f$.

We consider the spaces $H_f(t) = f^{-1}((-\infty, t])$ and ask how they change as $t$ passes through the critical values of $f$. In fact, this is the only time the $H_f(t)$ can change since we have

LEMMA 3. *Suppose that there is no critical value of $f$ in the interval $[t_1, t_2]$ then $H_f(t_1)$ is differentiably homeomorphic to $H_f(t_2)$.*

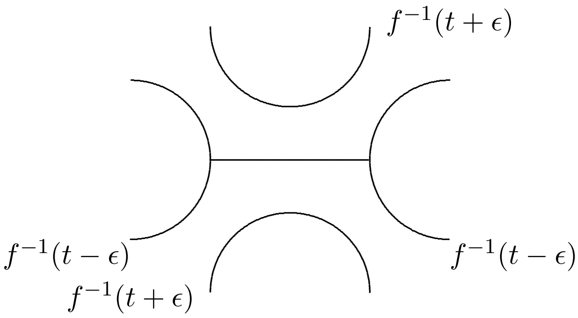
PROOF. The idea here is that the gradient vector field $df$ defines a flow on $\mathbb{R}^n$, $h(x,t)$ for which $f(h(x,t))$ is a monotone increasing function of $t$ and is stationary only at the critical points of $f$. Since there are $\epsilon_1, \epsilon_2 > 0$ so that there are no critical values of $f$ in the interval $(t_1 - \epsilon_1, t_2 + \epsilon_2)$ we squeeze the gradient field to zero for $f(x) \leq t_1 - \epsilon_1$ and then use the resulting flow to expand $H_f(t_1)$ to identify it with $H_f(t_2)$. □

LEMMA 4. *Suppose that $t$ is a critical value of $f$ and $x_1, \ldots, x_r$ are the critical points with $f(x_i) = t$. Suppose that the index of $f$ at $x_i$ is $I_i$. Then $H_f(t+\epsilon)$ has the homotopy type of $H_f(t-\epsilon)$ with one cell $e^{I_j}$ attached for each critical point $x_j$ of index $I_j$ in $f^{-1}(t)$.*

PROOF. We have that $H_f(t+\epsilon)$ is identified with $H_f(t-\epsilon)$ outside of sufficiently small neighborhoods of the critical points from the proof of 3. But in a small neighborhood of $x_i$ we have that $H_f(t+\epsilon)$ is identified with $H_f(t-\epsilon)$ with the disk

$$D^k = (\underbrace{0, \ldots, 0}_{(n-k)\ times}, x_{n-k+1}, \ldots, x_n), \quad \sum_{i=n-k+1}^{n} (x_i)^2 \leq \epsilon$$

attached.




as it is fairly direct from the diagram above to see that the left and right arcs above correspond the the intersection of the level surface $H_f(t-\epsilon)$ with the neighborhood of the critical point $x_i$, and the entire region deformation retracts to the disk above union with the original level set. □

## 3. Differentiable manifolds and differentiable maps

We now globalize the discussion above. An $m$-dimensional topological manifold is a space in which every point has a neighborhood which is homeomorphic to $\mathbb{R}^m$. The study of topological manifolds requires an immense background, not the least of which is a thorough understanding of the more restrictive differentiable manifolds, so in the remainder of this work we concentrate on differentiable manifolds.

DEFINITION 3. *An $m$-dimensional* **differentiable manifold** $M^m$ *is a paracompact Hausdorff topological space together with a covering by open sets $U \subseteq M$ with homeomorphisms $\phi : \mathbb{R}^m \to U$, such that the transition functions*

$$\phi'^{-1}\phi| \ : \ \phi^{-1}(U \cap U') \to U \cap U' \to \phi'^{-1}(U \cap U')$$

*are diffeomorphisms of open subsets of $\mathbb{R}^m$.*

A pair such as

$$(U \subseteq M, \phi : \mathbb{R}^m \to U)$$

is called a coordinate chart of the manifold $M$.

EXAMPLE 1. The simplest examples of compact $m$-dimensional manifolds are the spheres

$$S^m \ = \ \left\{ (x_0, x_1, \ldots, x_m) \in \mathbb{R}^{m+1} \mid \sum_{k=0}^{m} x_k^2 = 1 \right\} \ .$$

We can cover $S^m$ by two open subsets

$$U_+ = S^m \backslash (1, 0, \ldots, 0) \ , \ U_- = S^m \backslash (-1, 0, \ldots, 0) \ .$$

Define a homeomorphism $\phi_+ : \mathbb{R}^m \to U_+$ by

$$\phi_+(x_1, \ldots, x_m) = (f(|x|), \sqrt{1 - f(|x|)^2}\frac{x}{|x|})$$

where $f(t)$ is a monotone increasing $C^\infty$ function of $t$ with $f(0) = -1$ and $\lim_{t \mapsto \infty}(f(t)) = 1$. There is a similar homeomorphism $\phi_- : \mathbb{R}^m \to U_-$, but using the function $g(t) = -f(t)$. In this case the intersection of the two neighborhoods is

$$U_+ \cap U_- \ = \ \phi_+(\mathbb{R}^m \backslash \{0\})$$

and the transition function

$$\begin{aligned} \phi_{+,-} \ &= \ (\phi_-)^{-1}\phi_+| \ : \\ (\phi_+)^{-1}(U_+ \cap U_-) \ = \ \mathbb{R}^m \backslash \{0\} &\to (\phi_-)^{-1}(U_+ \cap U_-) \ = \ \mathbb{R}^m \backslash \{0\} \end{aligned}$$

is given by

$$\phi_{+,-}(x) = \sqrt{t/\bar{t}}x$$

where $t = |x|$ and $f(\bar{t}) = -f(t)$.

The coordinate charts $(U \subseteq M, \phi : \mathbb{R}^m \to U)$ allow points $x \in U$ to be expressed in local coordinates as

$$x = \phi(x_1, x_2, \ldots, x_m) \in \phi(\mathbb{R}^m) = U \subseteq M .$$

We shall usually suppress $(U, \phi)$, writing

$$x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m \subseteq M .$$

The charts also allow us to transport many structures for Euclidean spaces to manifolds. For example we have the definition of a **differentiable map**: $f\colon M^m \to N^n$: $f$ is differentiable if and only if the composition

$$\psi_j^{-1} f \psi_i : U_i \cap \psi_i^{-1} f^{-1} \mathrm{im}(\psi^j) \to \mathbb{R}^n$$

is differentiable for each $i$, $j$. Likewise we have the notion of $C^\infty$ functions on $M^n$. In particular, a $C^\infty$ function $M^m \to \mathbb{R}$ is **Morse** if the compositions $f\psi_i\colon U_i \to \mathbb{R}$ are Morse functions for each $U_i$ in the covering of $M^m$.

The set of Morse functions is open and dense in the function space of all $C^\infty$ differentiable functions $f : M \to \mathbb{R}$ as was the case for the Morse functions from $\mathbb{R}^m$. In particular, we have a very useful pasting theorem for Morse functions.

THEOREM 3. (1) *If $M^m$ is an $m$-dimensional differentiable manifold and $f\colon M^m \to \mathbb{R}$ is any $C^\infty$ differentiable function, then there is a Morse function $g$ arbitrarily close to $f$.*

(2) *Suppose $f\colon M^m \to \mathbb{R}$ is a $C^\infty$ function which is Morse on an open set $U \subset M^m$. Let $W \subset U$ be an open subset such that the closure of $W$ is contained in $U$. Then there is a Morse function $g\colon M^m \to \mathbb{R}$ with $g = f$ on $W$.*

EXAMPLE 2. The height function on the $m$-sphere $S^m$ is a Morse function

$$f : S^m \to \mathbb{R} ; (x_0, x_1, \ldots, x_m) \to x_m$$

with exactly two critical points, the minimum, $(0, \ldots, 0, -1)$, of index 0 and the maximum, $(0, \ldots, 0, 1)$, of index $m$.

EXAMPLE 3. The real $m$-dimensional projective space $\mathbb{RP}^m$ is the $m$-dimensional manifold with one point for each 1-dimensional subspace of the $(m+1)$-dimensional real vector space $\mathbb{R}^{m+1}$, as determined by homogeneous coordinates $[x_0, x_1, \ldots, x_m]$. For any real numbers $\lambda_0 < \lambda_1 < \cdots < \lambda_m$ there is defined a Morse function

$$f : \mathbb{RP}^m \to \mathbb{R} ; [x_0, x_1, \ldots, x_m] \to \frac{\sum\limits_{k=0}^{m} \lambda_k (x_k)^2}{\sum\limits_{k=0}^{m} (x_k)^2}$$

with $(m+1)$ critical points $[0, \ldots, 0, 1, 0, \ldots, 0]$ of index $0, 1, \ldots, m$. ∎

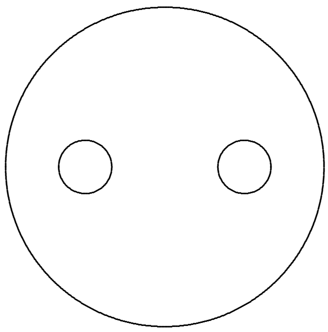
EXAMPLE 4. The complex $m$-dimensional projective space $\mathbb{CP}^m$ is the $2m$-dimensional manifold with one point for each 1-dimensional subspace of the $(m+1)$-dimensional complex vector space $\mathbb{C}^{m+1}$, as determined by homogeneous complex coordinates $[z_0, z_1, \ldots, z_m]$. For any real numbers $\lambda_0 < \lambda_1 < \cdots < \lambda_m$ there is defined a Morse function

$$f \ : \ \mathbb{CP}^m \to \mathbb{R} \ ; \ [z_0, z_1, \ldots, z_m] \to \frac{\sum\limits_{k=0}^{m} \lambda_k |z_k|^2}{\sum\limits_{k=0}^{m} |z_k|^2}$$

with $(m+1)$ critical points $[0, \ldots, 0, 1, 0, \ldots, 0]$ of index $0, 2, \ldots, 2m$. ∎

## 4. Differentiable manifolds with boundary

It is essential in studying cobordism, surgery and other subjects related to the geometry and topology of manifolds, to consider manifolds with boundary.



**A manifold with boundary**

DEFINITION 4. *An* $m$**-dimensional manifold with boundary**,

$$(M^m, \partial M)$$

*is a pair of paracompact Hausdorff topological spaces together with a covering of* $M$ *by open sets*

$$U_i \overset{\psi_i}{\hookrightarrow} M^m$$

*where each* $U_i$ *is either homeomorphic to* $\mathbb{R}^m$ *or*

$$\mathbb{R}^m_+ \ = \ \{(x_1, \ldots, x_m) \in \mathbb{R}^m \,|\, x_m \geq 0\} \ .$$

*The* **boundary** $\partial M$ *is the* $(m-1)$*-dimensional manifold without boundary defined by the union of the images in* $M$ *of the points with* $x_m = 0$ *for those* $U_i$ *homeomorphic to* $\mathbb{R}^m_+$.

One checks directly that $\partial M$ is an $(m-1)$-dimensional manifold and $\psi_j^{-1}\psi_i$ restricted to $\psi_i^{-1}(\partial M \cap \psi_i \mathbb{R}^m_+ \cap \psi_j \mathbb{R}^m_+)$ is contained in the points of $\mathbb{R}^m_+$ with $x_m = 0$. The pair $(M^m, \partial M)$ is a differentiable manifold with boundary if the transition functions $\psi_j^{-1}\psi_i$ are $C^\infty$ diffeomorphisms on the interiors of their domains of definition, and separately on the intersections with $x_m = 0$ in the regions of the form $\mathbb{R}^m_+$.

(1) If $(M^m, \partial M)$ is a differentiable manifold with boundary then there is a **collar neighborhood** of $\partial M$ in $M^m$. That is to say there is a neighborhood of $\partial M \subset M$ diffeomorphic to $\partial M \times [0, 1)$ with $\partial M \subset M$ identified with $\partial M \times 0$.

(2) If $(M^m, \partial M)$ and $(N^n, \partial N)$ are manifolds with boundary then their product

$$(M^m, \partial M) \times (N^n, \partial N) \;=\; (M^n \times N^n, \partial M \times N \cup M \times \partial N)$$

is an $(m+n)$-dimensional manifold with boundary. However, even if the two are differentiable, the boundary of the product has *corners.* But it is quite direct to smooth the corners and obtain a canonical differentiable manifold pair $(M^m \times N^n, \partial(M^m \times N^n))$ with the boundary homeomorphic to $\partial M \times N \cup MN$.

(3) After this smoothing process we have that

$$\begin{aligned}((M^m, \partial M) &\times (N^n, \partial N)) \times (P^p, \partial P) \\ &= (M^m, \partial M) \times ((N^n, \partial N) \times (P^p, \partial P))\ ,\end{aligned}$$

so the process of taking products of differentiable manifolds with boundary is associative.

**Morse functions on manifolds with boundary.** Manifolds with boundary admit special kinds of Morse functions.

Theorem 4. *Let $(M^m, \partial M)$ be a differentiable manifold with boundary. Then there is a function $f \colon M^m \to \mathbb{R}_+$ which satisfies*

(1) *$f$ is Morse on $M^m \backslash \partial M$,*
(2) *$f^{-1}(0) = \partial M$.*
(3) *In a sufficiently small neighborhood of $\partial M$, $\partial M \times [0, \epsilon)$ with $\epsilon > 0$ $f(x, t) = t$.*

Proof. We define a continuous function $g$ on $M^m$ as follows:

$$\begin{cases} \text{if } x = (v, t) \in \partial M \times [0, 1/2] & \text{then } g(x) = 2t, \\ \text{if } x \in M \backslash \{\partial M \times [0, 1/2)\} & \text{then } g(x) = 1. \end{cases}$$

Then $g$ can be approximated arbitrarily closely by a $C^\infty$ function $h$ which is equal to $g$ in $\partial M \times [0, 1/4)$. Finally, using 3 we can replace $h$ by a function $f$ which is Morse and in $\partial M \times (0, 1/8)$ is just $t$. ☐

For $(M, \partial M)$ and $f$ a Morse function as in 4 then for $D > 0$ sufficiently small

$$f^{-1}(-\infty, D] = \partial M \times [0, D].$$

**The Tangent Bundle to a Differentiable Manifold.** The differentiable structure allows us to define the tangent bundle to $M^m$, an invariant of the differentiable structure which is tremendously useful in understanding the global structure of $M^m$. In the next chapter we will generalize this construction considerably.

DEFINITION 5. (1) *The* **tangent bundle** *of an $m$-dimensional differentiable manifold $M^m$ is the $2m$-dimensional manifold*

$$\tau_M \;=\; \big(\coprod_{(U,\phi)} U\times\mathbb{R}^m\big)/\sim$$

*with $\sim$ the equivalence relation defined by*

$$(x\in U, h\in\mathbb{R}^m)\sim(x'\in U', h'\in\mathbb{R}^m)$$

*if $x=x'\in U\cap U'\subseteq M$ and*

$$d(\phi'^{-1}\phi|)(h) \;=\; h'\in\mathbb{R}^m\ .$$

(2) *The* **tangent space** *at $x\in M$ is*

$$\tau_M(x) \;=\; \{x\}\times\mathbb{R}^m\subseteq\tau_M\ .$$

Given any $C^2$ differentiable map $f\colon N^n{\longrightarrow}M^m$ we can extend $f$ to a map $df\colon\tau_N{\longrightarrow}\tau_M$ called the **derivative** of $f$ as follows: for any point $x\in N$ there exist coordinate charts

$$(V\subseteq N,\theta:\mathbb{R}^n{\longrightarrow}V)\ ,\ (U\subseteq M,\phi:\mathbb{R}^m{\longrightarrow}U)$$

with

$$x\in V\ ,\ f(x)\in U\subseteq M\ ,\ f(V)\subseteq U\ ,$$

and $f$ extends to $\tau_M$ as the map

$$\begin{array}{c} df\ :\ \tau_N\longrightarrow\tau_M\ ; \\ (x\in V,h\in\mathbb{R}^n)\longrightarrow(f(x)\in U, d(\phi^{-1}f\theta)(\theta^{-1}(x))(h)\in\mathbb{R}^m)\ . \end{array}$$

This is well defined in view of 1. From the definition we have

PROPOSITION 1. *The derivative of $f: N^n{\longrightarrow}M^m$ restricts to a linear map of tangent spaces*

$$df(x)\ :\ \tau_N(x)\longrightarrow\tau_M(f(x))$$

*for each $x\in N$. If $f$ is given in local coordinates by*

$$f\ :\ \mathbb{R}^n\longrightarrow\mathbb{R}^m\ ;\ x\;=\;(x_1,x_2,\dots,x_n)\longrightarrow(f_1(x),f_2(x),\dots,f_m(x))$$

*the derivative is given in local coordinates by*

$$df(x)\;=\;\left(\frac{\partial f_i}{\partial x_j}\right)\ :\ \tau_N(x)\;=\;\mathbb{R}^n\longrightarrow\tau_M(f(x))\;=\;\mathbb{R}^m\ .$$

The implicit function theorem now extends immediately to manifolds in the following form.

THEOREM 5. **Implicit Function Theorem for Manifolds**
*The inverse image of a regular value* $y \in M$ *of a differentiable map* $f : N^n \to M^m$ *is a submanifold* $P = f^{-1}(y) \subseteq N$ *with*

$$\dim(P) \;=\; n - \min(m,n) \;=\; \begin{cases} n-m & \text{if } m < n \\ 0 & \text{if } m \geq n \end{cases} .$$

We can also extend the definitions of regular value and critical value of maps $\mathbb{R}^n \to \mathbb{R}^m$ in 1 to manifolds word for word as in 1.

Thus, suppose $f\colon M^m \to N^s$ is differentiable with $s \leq m$ and suppose that $n \in N$ is a regular point of $f$. Then $f^{-1}(n) \subset M^m$ is a **submanifold of** $M$ of dimension $m - s$ directly from 2.

CHAPTER 2

# Bundles

Fibre bundles are maps $p\colon E \to X$ which are locally products. As particular examples we have vector bundles and coverings. Overall, they are one of the basic tools for studying manifolds and related spaces as well as the fundamental source of examples in the theory.

## 1. Fibre bundles

DEFINITION 6. *A* **fibre bundle** $\mathcal{E} = \mathcal{E}(X, E, F, p)$ *over a space* $X$ *is a space* $E$*, a projection*

$$p : E \longrightarrow X$$

*and a space* $F$ *such that* $X$ *has a distinguished covering by open neighbourhoods* $U \subset X$ *with homeomorphisms*

$$\phi_U : U \times F \longrightarrow p^{-1}(U)$$

*satisfying*

$$p\phi_U = \text{projection} : U \times F \to U .$$

$F$ *is called the* **fibre***,* $X$ *is called the* **base***, and* $E$ *is called the* **total space** *of the fibre bundle. Also, the maps* $\phi_U$ *are called* **local trivializations** *of the fibre bundle.*

We usually write $F \to E \xrightarrow{p} X$ to denote the fibre bundle $p\colon E \to X$ with fibre $F$.

EXAMPLE 5. A fibre bundle $\mathcal{E}$ over a space $X$ is called a covering if $F$ is discrete.

EXAMPLE 6. The standard example of a covering is the map

$$p\colon S^1 \to S^1 \ ; \ z \mapsto z^n$$

where $S^1$ is regarded as the complex numbers of norm one. The fibre is the discrete space with $n$ elements.

EXAMPLE 7. Given $X$ and $F$ we always have the **product** or **trivial** fibre bundle $X \times F \xrightarrow{p_1} X$.

REMARK . A **section** of a fibre bundle $\mathcal{E}$ is a map $s\colon X \to E$ so that the composition $ps\colon X \to X$ is the identity. Not all fibre bundles have sections, for example, the covering in 6 has no sections.

Note that the set of sections of the trivial bundle is identified with the set (space) of continuous maps $f\colon X \to F$ under the correspondence

$$f \mapsto (s_f\colon X \longrightarrow X \times F), \qquad s_f(x) = (x, f(x)) \ .$$

PROPOSITION 2. *Given fibre bundles $\mathcal{E}_1$ over $X_1$ with fibre $F_1$ and $\mathcal{E}_2$ over $X_2$ with fibre $F_2$ the product map $p_1 \times p_2\colon E_1 \times E_2 \to X_1 \times X_2$ defines a fibre bundle,* (written $\mathcal{E}_1 \times \mathcal{E}_2$)*, with fibre $F_1 \times F_2$, total space $E_1 \times E_2$ and base $X_1 \times X_2$.*

PROOF. All we need to do is to define the relevant open cover of $X_1 \times X_2$ as the $U_i(X_1) \times U_j(X_2)$ and the maps $\phi_{U_i(X_1)\times U_j(X_2)}$ as $\phi_{U_i(X_1)} \times \phi_{U_j(X_2)}$. □

PROPOSITION 3. *Let $\mathcal{E}$ be a bundle over $X$ and suppose that $f\colon Y \to X$ is any continuous map. The fibre bundle over $Y$* **induced** *from $\mathcal{E}$ by $f$ is given by*

$$f^{\,!}(E) \;=\; \{(y,e) \in Y \times E \mid f(y) = p(e)\}$$

*with the projection*

$$p^f\colon f^{\,!}(E) \to Y \ ; \ (y,e) \mapsto y \ .$$

*The construction defines a fibre bundle over $Y$ with fibre $F$, where the $U_i(Y)$ are the open sets $f^{-1}(U_i)$.*

PROOF. Define $\phi_{U_i(Y)}\colon U_i(Y) \times F \to (p^f)^{-1}(U_i(Y))$ as

$$\phi_{U_i(Y)}(y,g) = (y, \phi_{U_i}(g(y), g)) \ .$$

The verification that this is a homeomorphism is direct. □

REMARK . There is also a projection

$$p_2\colon f^{\,!}(E) \to E \ ; \ (y,e) \to e$$

and we have the commutative diagram

$$\begin{array}{ccc} f^{\,!}(E) & \xrightarrow{p_2} & E \\ \big\downarrow{\scriptstyle p^f} & & \big\downarrow{\scriptstyle p} \\ Y & \xrightarrow{f} & X \ . \end{array}$$

Conversely, if we are given fibre bundles $F \to E \xrightarrow{p} X$ and $F' \to E' \xrightarrow{p'} X$ together with a map of bundles

$$\begin{array}{ccc} E' & \xrightarrow{g} & E \\ \big\downarrow{\scriptstyle p'} & & \big\downarrow{\scriptstyle p} \\ Y & \xrightarrow{f} & X \end{array}$$

such that the restrictions $g| : F'_x \to F_{f(x)}$ $(x \in X)$ are homeomorphisms then $g : E' \to E$ is a homeomorphism, and we can identify $\mathcal{E}' = f^!(\mathcal{E})$.

REMARK . Define an equivalence relation on fibre bundles by setting $E \xrightarrow{p} X$ equivalent to $E' \xrightarrow{p'} X$ if and only if there exists a homeomorphism $g : E' \to E$ such that $pg = p'$. When we don't need the explicit construction of the fibre bundle from a set of local trivializations any mention of a fibre bundle over $X$ really refers to the equivalence class. However, in actually working with fibre bundles a system of local trivializations is usually very helpful. The following constructions will illustrate this.

**The homotopy invariance of induced bundles.** Suppose that $F \to E \xrightarrow{p} X$ is a given fibre bundle and $f\colon Y \to X$, $g\colon Y \to X$ are two maps. Then we have

THEOREM 6. *Suppose that $Y$ is a simplicial complex and that the maps $f, g : Y \to X$ are homotopic. Then the induced bundle $f^!(\mathcal{E})$ is equivalent to the induced bundle $g^!(\mathcal{E})$.*

This result, 6, is equivalent to the following more geometric result which is proved by induction over the skeleta of $Y$.

PROPOSITION 4. *Suppose that $F \to E \xrightarrow{\pi} Y \times I$ is a fibre bundle. Let*

$$F \to \pi^{-1}(Y \times 0) \xrightarrow{\pi|} Y \times 0$$

*be the restriction of this fibre bundle to $Y \times 0$. Then the original fibre bundle is equivalent to the product fibre bundle*

$$F \longrightarrow \pi^{-1}(Y \times 0) \times I \xrightarrow{\pi| \times id} Y \times I \ .$$

These results have the basic role of reducing the classification of the equivalence classes of induced $F$-bundles over $Y$ basically to questions in homotopy theory, and they form the basic geometric input into the construction of *classifying spaces* for fibre bundles. For example we have :

COROLLARY 3. *Let $F \to E \xrightarrow{p} X$ be a fibre bundle with $X$ a contractible simplicial complex. Then $E \xrightarrow{p} X$ is equivalent to the trivial fibre bundle $F \to X \times F \xrightarrow{p_1} X$.*

PROOF. The identity map $id\colon X \to X$ induces the original fibre bundle $E \xrightarrow{p} X$ from $E \xrightarrow{p} X$, but, since $X$ is contractible, it follows that $id\colon X \to X$ is homotopic to the trivial map $pt\colon X \to * \hookrightarrow X$, and clearly, $pt^!(E \xrightarrow{p} X)$ is the trivial fibre bundle over $X$. □

COROLLARY 4. *Let $X$ be a simplicial complex and suppose that $\{U\}$ is the open cover of $X$ by the open stars of the simplices. Then* **any** *fibre bundle $F \to E \xrightarrow{p} X$ can be trivialized over each $U$, so that $\{U\}$ can be taken to be the distinguished cover of $X$ by open sets.*

**Homotopy lifting and the homotopy exact sequence.** The results above have some strong consequences which enable us to relate the homotopy groups of the base, fibre and total space. The key result is the following :

LEMMA 5. *(Homotopy Lifting Lemma). Let $Y$ be a simplicial complex, $F \to E \overset{p}{\to} X$ a fibre bundle and $f\colon Y \to E$ any continuous map. Suppose that $H\colon Y \times I \to X$ is any homotopy of the projection, $pf$, i.e., $H(y, 0) = pf(y)$ for all $y \in Y$. Then there is a homotopy $L(H)\colon Y \times I \to E$ with $pL(H) = H$ and $L(H)(y, 0) = f(y)$ for all $y \in Y$.*

PROOF. Using the map $H$ we induce a bundle

$$F \to \overline{E} = H^!(E) \to Y \times I$$

over $Y \times I$. From the definition of the induced bundle 3 we have a section of $H^!(E)$ over $Y \times 0$ defined as the set of pairs $(y, f(y)) \in H^!(E)$. From 4 we have $H^!(E) = (pf)^!E \times I$ with product projection, and we can extend the section over $Y \times 0$ over $H^!(E)$ as $(y, t) \mapsto ((y, f(y)), t)$ in the product structure above. Then use $p_1$ to give the extension. □

REMARK . The homotopy extension property described in 5 is central in studying the deeper properties of fibre bundles. Consequently, it has been isolated out for special treatment.

DEFINITION 7. *A* **Serre fibration** *(or* **fibration** *for short) with total space $E$ and base space $X$ is a map*

$$p : E \longrightarrow X$$

*which satisfies the homotopy extension property of 5.*

It is not hard to see that if $X$ is path-wise connected then the (weak) homotopy type of $p^{-1}(x)$ is the same as that of $p^{-1}(x')$ for any $x, x' \in X$. Consequently, we define the **fibre of a Serre fibration** to be any space having the weak homotopy type of $p^{-1}(x)$ and we denote it $F(p)$. (Note that any two spaces of the same weak homotopy type have isomorphic homotopy groups.)

The Homotopy Lifting Lemma has as a direct corollary the following result

COROLLARY 5. *Let $p\colon E \to X$ be a Serre fibration, with fibre $F(p) = F$. Then there are homomorphisms $\partial\colon \pi_i(X) \to \pi_{i-1}(F)$ $(i \geq 1)$ and a long exact sequence*

$$\cdots \longrightarrow \pi_i(F) \xrightarrow{i_*} \pi_i(E) \xrightarrow{\pi_*} \pi_i(X) \xrightarrow{\partial} \pi_{i-1}(F) \longrightarrow \cdots$$

*where $i$ is the inclusion $p^{-1}(*) \subset E$ and $* \in X$ is the base point.*

The proof of 5 depends on using the homotopy extension theorem to prove that the projection $p_*\colon \pi_i(E, F) \to \pi_i(X, *) = \pi_i(X)$ is an isomorphism for each $i$. Then this isomorphism is used to replace $\pi_*(E, F)$ in the long homotopy exact sequence of the pair $(E, F)$.

**Structure maps for fibre bundles.** Now we turn to the description of the local structure of a fibre bundle and how to rebuild the fibre bundle just using appropriate local data.

PROPOSITION 5. *Let $F \to E \xrightarrow{p} X$ be a fibre bundle and suppose that $U_i$, $U_j$, $U_k$ are three elements of the distinguished covering, then the maps*

$$\phi_{U_i \cap U_j} : (U_i \cap U_j) \times F \longrightarrow (U_i \cap U_j) \times F$$

*defined as $\phi_{U_i}\phi_{U_j}^{-1}$ on the indicated domain of definition satisfy*

(1) *$\phi_{U_i \cap U_j}$ is a homeomorphism of the form $(x, f) \mapsto (x, \psi_{U_i \cap U_j}(x)f)$ where $\psi_{U_i \cap U_j} : (U_i \cap U_j) \to \mathrm{Homeo}(F)$ is a continuous map into the space of homeomorphisms of $F$ to itself.*
(2) *Over $U_j \cap U_i$ we have $\psi_{U_j \cap U_i} = \psi_{U_i \cap U_j}^{-1}$.*
(3) *In the triple intersection $U_i \cap U_j \cap U_k$ the compatibility conditions $\psi_{U_i \cap U_j}\psi_{U_j \cap U_k} = \psi_{U_i \cap U_k}$ are satisfied.*

PROOF. This is a formal consequence of the fact that $\phi_{U_i}$ is a homeomorphism, hence a homeomorphism from $F$ to $p^{-1}(x)$ for each $x \in U_i$. The continuity of the associated map into $\mathrm{Homeo}(F)$ is also clear. □

DEFINITION 8. *In the situation of 5 the $\psi_{U_i \cap U_j}$ are called the structure maps of the fibre bundle $F \to E \xrightarrow{p} X$ with distinguished cover $\{U_i\}$ and local trivializations $\phi_{U_j} : U_j \times F \to E$.*

EXAMPLE 8. In the case of a covering with finite fibre $F$, the group $\mathrm{Homeo}(F)$ is the symmetric group $\mathcal{S}_{|F|}$ and the covering can be described by specifying continuous maps $\psi_{U_i \cap U_j} : X \to \mathcal{S}_{|F|}$ which satisfy the two compatibility conditions of 5. In the case of the $n$-fold covering $S^1 \to S^1; z \mapsto z^n$ of 6 we can cover $S^1$ by the two neighborhoods $U_1 = S^1 \backslash \{1\}$ and $U_2 = S^1 \backslash \{-1\}$ with intersection $U_1 \cap U_2 = S^1 \backslash \{\pm 1\}$ consisting of two components, the open upper and lower semicircles. Moreover, the map $U_1 \cap U_2 \to \mathcal{S}_n$ is continuous, hence constant on the two path components, and can be taken to be the identity on the upper hemisphere while it is the cyclic permutation $(1, 2, 3, \ldots, n)$ on the lower semicircle.

Actually, the covering $\{U_i\}$ for $X$ and the structure maps $\psi_{U_i \cap U_j}$ are sufficient data to rebuild the total space $E$ of the fibre bundle $F \to E \xrightarrow{p} X$. Indeed we have

THEOREM 7. *Let $X$ together with an open cover $\{U_j\}$ be given, and suppose that we are also given homeomorphisms*

$$\psi_{U_i \cap U_j} : (U_i \cap U_j) \times F \longrightarrow (U_i \cap U_j) \times F$$

*satisfying 5(i), 5(ii), and 5(iii). Then there is a fibre bundle $F \to E \xrightarrow{p} X$ together with local trivializations $\phi_{U_j} : U_j \times F \to E$ so that the $\psi_{U_i \cap U_j}$ are the structure maps for $F \to E \xrightarrow{p} X$.*

Proof. Define an equivalence relation on the space $H$ given as the disjoint union of the $U_i \times F$ by decreeing that $(u_i, f) \in U_i \times F$ shall be equivalent to $(u_j, f') \in U_j \times F$ if and only if $u_i = u_j \in U_i \cap U_j$ and $\psi_{U_i \cap U_j}(f) = f'$. Let $E$ be the resulting quotient space of $H$. Note 5(ii) and 5(iii) show the the resulting relation is symmetric and transitive without introducing any further relations. Also, 5(i) shows that there are no further identifications on individual fibres, so that the map sending the points of $U_i \times F$ to their equivalence classes in $E$ gives a homeomorphism onto the inverse image of $U_i$ under the obvious projection $E \to X$. □

## 2. Vector bundles

A **$k$-plane bundle** $\eta$ over a space $X$ is a fibre bundle

$$p \colon E(\eta) \to X$$

with fibre the $k$-dimensional euclidean space $\mathbb{R}^k$ and so that the structure maps

$$\psi_{U_i \cap U_j} \colon U_i \cap U_j \to \text{Homeo}(\mathbb{R}^k)$$

have image contained in $GL_k(\mathbb{R})$. A fibre bundle $\mathcal{E}$ over $X$ is called a vector bundle if it is a $k$-plane bundle over $X$ for some $k$.

Example 9. Recall that the real projective space $\mathbb{RP}^n$ is the $n$-sphere $S^n$ factored out by identifying antipodal points,

$$\mathbb{RP}^n = S^n/(x \sim -x) .$$

It can also be identified with the space of lines through the origin in $\mathbb{R}^{n+1}$. From this point of view there is a natural 'tautological' 1-plane bundle $\gamma_{n,1} \to \mathbb{RP}^n$, defined as the set of pairs $(\lambda, w) \in \mathbb{RP}^n \times \mathbb{R}^{n+1}$ with $w$ contained in the line $\lambda$.

Given two vector bundles $E(\eta)$ and $E(\tau)$ over $X$ we define their **Whitney sum**, $E(\eta) \oplus E(\tau)$, as the induced bundle $\Delta^!(E(\eta) \times E(\tau))$ where $\Delta \colon X \to X \times X$ is the diagonal map.

Remark . We denote the product vector bundle $X \times \mathbb{R} \to X$ as $\epsilon$. Note that the Whitney sum $\epsilon \oplus \epsilon$ is the product bundle $X \times \mathbb{R}^2 \to X$, and more generally, $n\epsilon = \epsilon \oplus (n-1)\epsilon$ is the product bundle $X \times \mathbb{R}^n$.

Example 10. For the tautological 1-plane bundle $\gamma_{n,1} \to \mathbb{RP}^n$ of 9, from the definition, there is a second bundle $\xi_n \to \mathbb{RP}^n$, namely the $n$-plane bundle $\subset \mathbb{RP}^n \times \mathbb{R}^{n+1}$ consisting of all pairs $(v, w)$ with $w \perp v$. Clearly, the Whitney sum $\gamma_{n,1} \oplus \xi_n = (n+1)\epsilon$.

Remark . Note that every vector bundle has a well defined 0-section, as homeomorphisms in $GL_n(\mathbb{R})$ always preserve the 0-element in $\mathbb{R}^n$. Moreover, if $s_1$ and $s_2$ are sections of a vector bundle $\eta$, then we can multiply them by scalars $r \in \mathbb{R}$ or add them (locally) to get well defined sections written $rs$ or $s_1 + s_2$, so the set of sections is a vector space over $\mathbb{R}$.

DEFINITION 9. (i) *A* **stable isomorphism** *between a $k$-plane bundle $\eta$ and a $k'$-plane bundle $\eta'$ over the same space $X$ is a bundle isomorphism*

$$\eta \oplus \epsilon^j \;\cong\; \eta' \oplus \epsilon^{j'}$$

*for some $j,\ j' \geq 0$ with $j + k = j' + k'$.*
(ii) *A* **stable bundle** *over $X$ is an equivalence class of bundles $\eta$ over $X$, subject to the equivalence relation*

$$\eta \sim \eta' \text{ if there exists a stable isomorphism } \eta \oplus \epsilon^j \;\cong\; \eta' \oplus \epsilon^{j'} \text{ for some } j, j' \geq 0\ .$$

(iii) *A $k$-plane bundle $\eta$ is* **stably trivial** *if $\eta \oplus \epsilon^j$ is trivial for some $j \geq 0$.*

**The tangent bundle of a differentiable manifold.** The tangent bundle of an $m$-dimensional differentiable manifold $M^m$ is an $m$-plane bundle $\tau_M$, an invariant of the differentiable structure which is tremendously useful in understanding the global structure of $M^m$.

DEFINITION 10. (1) *The* **tangent bundle** *of an $m$-dimensional differentiable manifold $M^m$ is the $m$-plane bundle $\tau_M$ over $M$ with total space the $2m$-dimensional manifold*

$$E(\tau_M) \;=\; \big(\coprod_{(U,\phi)} U \times \mathbb{R}^m\big)/\sim$$

*with $\sim$ the equivalence relation defined by*

$$(x \in U, h \in \mathbb{R}^m) \sim (x' \in U', h' \in \mathbb{R}^m)$$

*if $x = x' \in U \cap U' \subseteq M$ and*

$$d(\phi'^{-1}\phi|)(h) \;=\; h' \in \mathbb{R}^m\ .$$

*The projection of the tangent $m$-plane bundle is*

$$p : E(\tau_M) \to M; (x,h) \mapsto x\ .$$

(2) *The* **tangent space** *at $x \in M$ is*

$$\tau_M(x) \;=\; p^{-1}(x) \;=\; \{x\} \times \mathbb{R}^m \subseteq \tau_M\ .$$

EXAMPLE 11. The tangent bundle of $S^n$ is trivial (i.e. $\tau_{S^n} = n\epsilon$) for $n = 1, 3, 7$ and non-trivial for $n \neq 1, 3, 7$, with $\tau_{S^n} \oplus \epsilon = (n+1)\epsilon$ in general.

EXAMPLE 12. We have an alternative description of the tautological 1-plane bundle $\gamma_{n,1}$ over $\mathbb{RP}^n$ (9) as

$$E(\gamma_{n,1}) \;=\; S^n \times \mathbb{R}/\{(v,t) \sim (-v,-t)\}\ ,$$

since we can think of $(v,w)$ as $(\{v\}, tv)$. This implies that we have two subbundles of

$$E((n+1)\gamma_{n,1}) \;=\; S^n \times \mathbb{R}^{n+1}/(\pm 1),$$

the first given as the set of elements $(\pm 1)(v, tv)$, and the second as the pairs $(\pm 1)(v,w)$ with $w \perp v$. Clearly, the first subbundle is trivial and the second is just the tangent bundle of $\mathbb{RP}^n$. Thus we have the basic bundle equivalence

$$\tau_{\mathbb{RP}^n} \oplus \epsilon \ = \ (n+1)\gamma_{n,1}$$

and $\tau_{\mathbb{RP}^n}$ is stably equivalent to $(n+1)\gamma_{n,1}$.

The tangent bundle can be used to define the derivative $df : \tau_N \to \tau_M$ of a differentiable map $f : N^n \to M^m$ of manifolds, and to extend the implicit function theorem 1 to such maps.

Given any $C^2$ differentiable map $f\colon N^n \to M^m$ we can extend $f$ to a bundle map $df\colon \tau_N \to \tau_M$ called the **derivative** of $f$ as follows: for any point $x \in N$ there exist coordinate charts

$$(V \subseteq N, \theta : \mathbb{R}^n \to V) \ , \ (U \subseteq M, \phi : \mathbb{R}^m \to U)$$

with

$$x \in V \ , \ f(x) \in U \subseteq M \ , \ f(V) \subseteq U \ ,$$

and $f$ extends to $\tau_M$ as the map

$$\begin{array}{c} df \ : \ \tau_N \longrightarrow \tau_M \ ; \\ (x \in V, h \in \mathbb{R}^n) \longrightarrow (f(x) \in U, d(\phi^{-1} f \theta)(\theta^{-1}(x))(h) \in \mathbb{R}^m) \ . \end{array}$$

This is well defined in view of 1. From the definition we have :

Proposition 6. *The derivative of $f : N^n \to M^m$ is a map of vector bundles $df : \tau_N \to \tau_M$ which restricts to a linear map of tangent spaces*

$$df(x) \ : \ \tau_N(x) \longrightarrow \tau_M(f(x))$$

*for each $x \in N$. If $f$ is given in local coordinates by*

$$f \ : \ \mathbb{R}^n \longrightarrow \mathbb{R}^m \ ; \ x \ = \ (x_1, x_2, \ldots, x_n) \longrightarrow (f_1(x), f_2(x), \ldots, f_m(x))$$

*the derivative is given in local coordinates by*

$$df(x) \ = \ \left(\frac{\partial f_i}{\partial x_j}\right) \ : \ \tau_N(x) \ = \ \mathbb{R}^n \to \tau_M(f(x)) \ = \ \mathbb{R}^m \ .$$

We can also extend the definitions of regular value and critical value of maps $\mathbb{R}^n \to \mathbb{R}^m$ in 1 to manifolds word for word as in 1.

The implicit function theorem now extends immediately to manifolds in the following form.

Theorem 8. **Implicit Function Theorem for Manifolds**
*The inverse image of a regular value $x \in M$ of a differentiable map $f : N^n \to M^m$ is a submanifold $P = f^{-1}(x) \subseteq N$ with*

$$\dim(P) \ = \ n - \min(m,n) \ = \ \begin{cases} n-m & \text{if } m \le n \\ 0 & \text{if } m > n \end{cases} .$$

In particular, if $m \le n$ and $x \in M$ is a regular value of $f$ then $f^{-1}(x) \subset N^n$ is an $(n-m)$-dimensional submanifold, as is also clear directly from 2.

**Riemannian metrics and the tubular neighbourhood theorem.** It is frequently of importance to be able to put a metric on a vector bundle $\eta$. A metric on $\eta$ is a continuous map

$$d\colon E(\eta) \to \mathbb{R}\ ;\ v\ \mapsto\ d(v)$$

such that for all $v, w \in E(\eta)$ with $p(v) = p(w)$ and all $a \in \mathbb{R}$

- $d(v) \geq 0$, with $d(v) = 0$ if and only if $v = 0$,
- $d(av) = |a|d(v)$,
- $d(v + w) \leq d(v) + d(w)$.

When $\eta = \epsilon^k$ is the trivial $k$-plane bundle we obtain a metric on $\eta$ by simply taking a metric on $\mathbb{R}^k$. Whenever $X$ is a paracompact Hausdorff space we can put a metric on $\eta$ by taking a partition of unity, $\lambda$, subordinate to the open cover $U_i$ associated to $\eta$, taking trivial metrics over each $U_i$ and adding them up by the formula

$$d(v_x) = \sum_{U_i, x \in U_i} \lambda_{U_i}(x) d_{U_i}(\psi_{U_i}^{-1}(v_x)) \in \mathbb{R}^+ \ .$$

In particular, if $X = M^m$ is a $C^\infty$ manifold of dimension $m$, we can assume that the vector bundle has a compatible differentiable structure so that the projection to $M$ and the metric are both $C^\infty$ maps. For example, consider the tangent bundle, $\tau_M$. If $N^n \subset M$ is an $n$-dimensional submanifold, the induced map on the tangent bundle injects $\tau_N$ as an $n$-plane **subbundle** of $\tau_M$. Then, the set of vectors orthogonal to $\tau_N$ in $\tau_M$ give a second bundle over $N$, an $(m - n)$-plane bundle, $\nu_{N \subset M}$, called the **normal bundle of** $N$ **in** $M$. We clearly have

$$i^!(\tau_M) = \tau_N \oplus \nu_{N \subset M}$$

where $i\colon N \hookrightarrow M$ is the inclusion.

Actually more is true than this.

THEOREM 9. [**Tubular Neighbourhood Theorem**] *The inclusion* $f : N^n \subset M^m$ *of a submanifold extends to a codimension 0 embedding* $E(\nu_f) \subset M^m$ *of the total space of the normal* $(m-n)$*-plane bundle* $\nu_f$*, such that*

$$\tau_N \oplus \nu_f \ = \ f^* \tau_M$$

*with* $\tau_M$*,* $\tau_N$ *the respective tangent bundles.*

**The stable normal bundle of a manifold.** By the Whitney Embedding theorem any (paracompact) differentiable manifold $M^m$ has a differentiable embedding $f\colon M^m \hookrightarrow \mathbb{R}^{2m+1}$, and any two such embeddings are connected by an embedding $H\colon M^m \times I \hookrightarrow \mathbb{R}^{2m+1} \times I$, where $H|(M^m \times \{0\})$ is the embedding $f_1$ and $H|(M^m \times \{1\})$ is the embedding $f_2$. Consequently, using the tubular neighbourhood theorem, 9, and 4 we see that the normal bundle of the embedding $f_1$ is equivalent to the normal bundle of the embedding

$f_2$. Moreover, and more generally, let $g\colon M^m \hookrightarrow \mathbb{R}^{m+j}$, $h\colon M^m \hookrightarrow \mathbb{R}^{m+k}$ be embeddings with $k + s \geq m + 1$. The composite embeddings

$$M \stackrel{g}{\to} \mathbb{R}^{m+j} \to \mathbb{R}^{m+j+k}\ ,\ M \stackrel{h}{\to} \mathbb{R}^{m+k} \to \mathbb{R}^{m+j+k}$$

are related by an embedding $M^m \times I \hookrightarrow \mathbb{R}^{m+j+k} \times I$, so that

$$\nu_g \oplus k\epsilon \cong \nu_h \oplus j\epsilon$$

and $M^m$ has a unique stable normal bundle which we call $\nu_M$. For example, $\nu_{S^m}$ is the trivial bundle since the normal bundle of the usual embedding $S^m \hookrightarrow \mathbb{R}^{m+1}$ is already trivial.

## 3. Associated bundles

The result of 7 shows that a fibre bundle is entirely determined by its structure maps. The structure maps $\psi_{U_i \cap U_j}\colon (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$ have the special form of 5(i), and are consequently equivalent to continuous maps

$$h_{U_i \cap U_j}\ : U_i \cap U_j \longrightarrow \mathrm{Homeo}(F)$$

which have the adjoint properties:

(1) $h_{U_i \cap U_j}(u)^{-1} = h_{U_j \cap U_i}(u)$.

(2) $h_{U_i \cap U_j}(u) h_{U_j \cap U_k}(u) = h_{U_i \cap U_k}(u)$ for $u \in U_i \cap U_j \cap U_k$.

Conversely, given a collection of maps $h_{U_i \cap U_j}$ satisfying 3(i) and 3(ii) there are defined structure maps

$$\overline{\phi}_{U_i \cap U_j}\ : (U_i \cap U_j) \times \mathrm{Homeo}(F) \longrightarrow (U_i \cap U_j) \times \mathrm{Homeo}(F)$$

satisfying 5(i), 5(ii), 5(iii), by the rule

$$\overline{\psi}_{U_i \cap U_j}(u, f) = (u, h_{U_i \cap U_j}(u) f),$$

The fibre bundle given by 7

$$\mathrm{Homeo}(F) \longrightarrow \overline{E} \stackrel{p}{\longrightarrow} X$$

is **the principal** Homeo($F$)**-fibre bundle** associated to the fibre bundle $F \to E \stackrel{p}{\to} X$.

Actually, this fibre bundle 3 has an additional property:

PROPOSITION 7. *There is a fibre preserving action*

$$\mu\ : \overline{E} \times \mathrm{Homeo}(F) \longrightarrow \overline{E}$$

*so that, with respect to the local trivializations above we have the following commutative diagram*

$$\begin{array}{ccc} (U_i \times \mathrm{Homeo}(F)) \times \mathrm{Homeo}(F) & \stackrel{id \times *}{\longrightarrow} & U_i \times \mathrm{Homeo}(F) \\ \Big\downarrow{\scriptstyle \overline{\psi}_{U_i} \times id} & & \Big\downarrow{\scriptstyle \overline{\psi}_{U_i}} \\ \overline{E} \times \mathrm{Homeo}(F) & \stackrel{\mu}{\longrightarrow} & \overline{E} \end{array}$$

*where* $*\colon \mathrm{Homeo}(F) \times \mathrm{Homeo}(F) \to \mathrm{Homeo}(F)$ *is just multiplication (composition of maps).*

Proof. This is obvious. □

Example 13. From the definition of a $k$-plane bundle there is an associated principal $GL_k(\mathbb{R})$ bundle $GL_k(\mathbb{R}) \to \overline{E} \overset{p}{\to} X$, so it is not always necessary to take the entire group $\mathrm{Homeo}(F)$.

Example 14. In the covering $S^1 \to S^1; z \mapsto z^n$ of 5 the group $\mathrm{Homeo}(F)$ is the symmetric group $\mathcal{S}_n$ but the subgroup $\mathbb{Z}/n$ generated by the cycle $(1, 2, 3, \ldots, n)$ contains all the images of the $h_{U_i}(u)$ and so there is an associated principal bundle for the group $\mathbb{Z}/n$. In fact, this associated principal fibre bundle is equivalent to the original covering.

On the other hand, if we are given a principal fibre bundle with group $G$ and a homomorphism $h\colon G \to \mathrm{Homeo}(F)$ then we can construct an associated bundle with fibre $F$ from a principal $G$ bundle over $X$ by defining

$$\psi_{U_i \cap U_j} : (U_i \cap U_j) \times F \longrightarrow (U_i \cap U_j) \times F$$

as $\psi_{U_i \cap U_j}(u, f) = (u, h\overline{\psi}_{U_i \cap U_j}(u) f)$.

## 4. Reduction of the group of a fibre bundle

We say that given a fibre bundle $F \to E \overset{p}{\to} X$ with structure data $\{U_i\}$, $\psi_{U_i \cap U_j}$, the group of the fibre bundle and structure data is the subgroup of $\mathrm{Homeo}(F)$ generated by the elements $h_{U_i \cap U_j}(u)$, $u \in U_i \cap U_j$.

Remark . The group of a bundle with given structure data is, at best, only defined up to conjugacy, since, if we replace $F$ by $f(F)$ where $f\colon F \to F$ is an element of $\mathrm{Homeo}(F)$, then the $h_{U_i \cap U_j}$ change to

$$h'_{U_i \cap U_j} = f^{-1} h_{U_i \cap U_j} f \ .$$

More generally, given any family of continuous maps $H_{U_j}\colon U_j \to \mathrm{Homeo}(F)$, we can change the structure maps by

$$h'_{U_i \cap U_j}(u) = H_{U_j}^{-1}(u) h_{U_i \cap U_j} H_{U_i}(u) \ .$$

In any case, we say the the group of the bundle $F \to E \overset{p}{\to} X$ **reduces** to $G \subset \mathrm{Homeo}(F)$ if and only if there is an equivalent bundle where the structure maps $h_{U_i \cap U_j}(u) \in G$ for all $u \in U_i \cap U_j$, and all intersections $U_i \cap U_j$. It is evident that if $F \to g^!(E) \overset{p_1}{\to} Y$ is induced from $F \to E \overset{p}{\to} X$ by the map $g\colon Y \to X$, then the group of $g^!\mathcal{E}$ is contained in the group of $E$.

Example 15. Let $\mathcal{E}_1$ be an $F_1$-fibre bundle over $X_1$ with group $G_1$ and $\mathcal{E}_2$ be an $F_2$-fibre bundle over $X_2$ with group $G_2$. Then the group of the product bundle is a subgroup of $G_1 \times G_2$. In particular, the Whitney bundle sum $\eta_k \oplus \nu_s$ of a $k$-plane bundle and an $s$-plane bundle over $X$ has group $GL_k \times GL_s$ thought of as the subgroup of matrices

$$\begin{pmatrix} A_k & 0 \\ 0 & B_s \end{pmatrix} \in GL_{k+s}(\mathbb{R}) \ .$$

EXAMPLE 16. Let a Riemannian metric $d\colon E(\eta) \to \mathbb{R}$ be given on the $k$-plane bundle $\eta$. Then $d$ defines a reduction of the group from the general linear group $GL_k(\mathbb{R})$ to to the orthogonal group $O_k(\mathbb{R})$, by noting first that the structure maps must preserve $d$. Consequently, at each point the structure map $h_{U_i \cap U_j}$ lies in the isotropy group of $d$ at that point. Second, we can assume given reductions to $O_k(\mathbb{R})$ over each $U_i$. Then, the homeomorphisms $U_i \times \mathbb{R}^k \to p^{-1}U_i$ define a map

$$\kappa_i\colon U_i \to GL_k(\mathbb{R})/O_k(\mathbb{R}),$$

the space of distinct metrics on $\mathbb{R}^k$, by comparing metrics. But

(1) There is a fibre bundle

$$O_k(\mathbb{R}) \longrightarrow GL_k(\mathbb{R}) \ \stackrel{\pi}{\longrightarrow}\ GL_k(\mathbb{R})/O_k(\mathbb{R})$$

(2) The quotient space $GL_k(\mathbb{R})/O_k(\mathbb{R})$ is contractible, in fact convex since, given two Riemannian metrics $d_1$, $d_2$ on $\mathbb{R}^k$ then the family $td_1 + (1-t)d_2$, $0 \le t \le 1$, lies in this quotient.

Consequently, the $\kappa_i$'s all lift to maps

$$\tilde{\kappa}_i\colon U_i \longrightarrow GL_k(\mathbb{R})$$

and $\tilde{\kappa}_i(x)(\tilde{\kappa}_i(x))^t$ gives the metric $d(x)$ for each $x \in U_i$. Then replacing the structure maps $\phi_i$ by $\phi_i\tilde{\kappa}_i$ gives an explicit reduction of the group to $O_k(\mathbb{R})$.

DEFINITION 11. *Let $GL_n^+(\mathbb{R})$ be the (index 2) subgroup of $GL_n(\mathbb{R})$ consisting of all matrices $A \in GL_n(\mathbb{R})$ with $Det(A) > 0$. Then we say that the k-plane bundle $\eta$ over $X$ is* **orientable** *if the group of the bundle reduces to $GL_n^+(\mathbb{R})$. A particular reduction of the group will be called an* **orientation** *of $\eta$.*

EXAMPLE 17. The tangent bundle of a manifold $M^m$ is orientable if and only if the manifold is orientable in the usual sense.

EXAMPLE 18. A 1-plane bundle $\mathbb{R} \to E \stackrel{p}{\to} X$ is orientable if and only if it is trivial. This can be seen as follows. The group $GL_1(\mathbb{R})$ is just the non-zero reals and $GL_1^+(\mathbb{R})$ is convex. Hence, by the argument in 16, if the bundle, $E \stackrel{p}{\to} X$ reduces to $GL_1^+(\mathbb{R})$ then it reduces to the trivial group $\{1\}$ and is, consequently, trivial.

## 5. Classifying spaces for fibre bundles

**The universal bundles over Grassmannians.** The Grassmannian manifold $G_{n,k}$ is the space of all $n$-planes through the origin in $\mathbb{R}^{n+k}$. It is given as the quotient $GL_{n+k}(\mathbb{R})/GL_{n,k}(\mathbb{R})$ where $GL_{n,k}(\mathbb{R})$ is the subgroup of matrices of the form

$$\begin{pmatrix} A & N \\ 0 & B \end{pmatrix}$$

with $A \in GL_n(\mathbb{R})$ and $B \in GL_k(\mathbb{R})$. $GL_{n,k}(\mathbb{R})$ is exactly the subgroup of $GL_{n+k}(\mathbb{R})$ which takes the $n$-plane consisting of all vectors $\begin{pmatrix} a \\ 0 \end{pmatrix}$ to itself, where $a$ is a $1 \times n$-tuple. It is well known that $G_{n,k}$ is a closed, compact manifold of dimension $nk$.

Example 19. *A metric on* $\mathbb{R}^n$ *defines a correspondence between* $k$*-planes and* $(n-k)$*-planes,* $v \leftrightarrow v^{\perp}$*, which induces diffeomorphisms,* $G_{n,k} \leftrightarrow G_{k,n}$*. As a special case* $G_{n,1} = G_{1,n} = \mathbb{RP}^n$*.*

As was the case with $\mathbb{RP}^n$ and the vector bundle $\nu_n$, there is a natural $n$-plane bundle over $G_{n,k}$:

Definition 12. Universal $n$-Plane Bundle *The* $n$*-plane bundle* $\gamma_{n,k}$ *over* $G_{n,k}$ *is given by*

$$E(\gamma_{n,k}) \;=\; \{(v,w) \in G_{n,k} \times \mathbb{R}^{n+k} \,|\, w \in v\}\ .$$

There is a natural inclusion $i_{n,k}\colon G_{n,k} \hookrightarrow G_{n,k+1}$ defined by $v \mapsto v$, and $i^!_{n,k}(\gamma_{n,k+1}) = \gamma_{n,k}$. There is also an inclusion $j_{n,k}\colon G_{n,k} \to G_{n+1,k}$ defined by $v \mapsto \langle v, e_{n+k+1}\rangle$, the $n+1$-dimensional subspace of $\mathbb{R}^{n+k+1}$ spanned by $v \subset \mathbb{R}^{n+k}$ and the last vector in a basis for $\mathbb{R}^{n+k+1}$ where the first $n+k$ are contained in $\mathbb{R}^{n+k}$. In this case

$$j^!_{n,k}(\gamma_{n+1,k}) = \gamma_{n,k} \oplus \epsilon\ .$$

The total space of the associated principal $GL_n(\mathbb{R})$-fibre bundle over $G_{n,k}(\mathbb{R})$ is just $GL_{n+k}(\mathbb{R})/H$ where $H \subset GL_{n,k}(\mathbb{R}) \subset GL_{n+k}(\mathbb{R})$ is the subgroup of matrices

$$\begin{pmatrix} I & A \\ 0 & B \end{pmatrix}$$

with $B \in GL_k(\mathbb{R})$, and the $GL_n(\mathbb{R})$ action is just given by right multiplication by elements $\begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix}$ with $C \in GL_n(\mathbb{R})$.

Lemma 6. *The total space of the principal fibre bundle over* $G_{n,k}$*,*

$$GL_n(\mathbb{R}) \longrightarrow GL_{n+k}(\mathbb{R})/H \xrightarrow{\ p\ } G_{n,k}$$

*associated to* $\gamma_{n,k}$ *is* $(k-2)$*-connected.*

Proof. First, $H$ is homotopy equivalent to $GL_k(\mathbb{R})$ via the projection

$$\begin{pmatrix} I & A \\ 0 & B \end{pmatrix} \to B$$

since this is a fibre bundle with fibre $\mathbb{R}^{nk}$ which is contractible. Second, the fibre bundles

$$GL_j(\mathbb{R}) \hookrightarrow GL_{j+1}(\mathbb{R}) \xrightarrow{p} GL_{j+1}(\mathbb{R})/GL_j(\mathbb{R}) = \mathbb{R}^{j+1}\backslash\{0\} \simeq S^j$$

and the homotopy exact sequence of the fibre bundle show that the inclusion $GL_j(\mathbb{R}) \subset GL_{j+1}(\mathbb{R})$ induces isomorphisms in homotopy,

$$\pi_l(GL_j(\mathbb{R})) \to \pi_l(GL_{j+1}(\mathbb{R}))$$

for $l < j-1$. Thus, the iterate inclusion $GL_k(\mathbb{R}) \hookrightarrow GL_{n+k}(\mathbb{R})$ induces isomorphisms in homotopy, $\pi_l(GL_k(\mathbb{R})) \to \pi_l(GL_{n+k}(\mathbb{R}))$, for $l < k-1$. Third, using the homotopy exact sequence of the fibre bundle

$$GL_k(\mathbb{R}) \to GL_{n+k}(\mathbb{R}) \to E(\gamma_{n,k})$$

we see that $\pi_j(E(\gamma_{n,k})) = 0$ for $j \leq k-2$ as asserted. □

**Classifying spaces.** Let $\eta$ be a (differentiable) $k$-plane bundle over a manifold $M^m$. Using the Whitney embedding theorem, $E(\eta)$ – thought of as a differentiable manifold – embeds in $\mathbb{R}^{2m+2k+1}$, and using the Tubular Neighbourhood Theorem 9 we see that we have $\nu_M = \eta \oplus \chi$ where $\chi$ is the normal bundle of $E(\eta)$ restricted to the 0-section $M$. Consequently, since $\nu_M \oplus \tau_M$ is trivial, there is an embedding $E(\eta) \hookrightarrow M \times \mathbb{R}^{2m+2k+1}$, which has the property that the fibre $\mathbb{R}^k_x$ over the point $x \in M$ embeds linearly as a $k$-dimensional vector subspace

$$e(x) \subset \{x\} \times \mathbb{R}^{2m+2k+1} \subset M \times \mathbb{R}^{2m+2k+1} \ .$$

It follows that there is a well defined map

$$u_\eta \colon M^m \to G_{k,2m+k+1} \ ; \ x \ \mapsto \ e(x)$$

such that

$$u_\eta^!(\gamma_{k,2m+k+1}) = \eta \ .$$

More generally, suppose that $E(\eta) \xrightarrow{p} X$ is any $k$-plane bundle $\eta$ over the space $X$ so that there is an $s$-plane bundle $\mu$ over $X$ with $\eta \oplus \mu = (k+s)\epsilon$. Then, the identification $E(\eta \oplus \mu) = X \times \mathbb{R}^{n+s}$ embeds $\eta$ as points of the form $(x, w)$ where the vectors $w$ lie in a $k$-plane $w_x$, depending only on $x$, and once more we have a map

$$h \colon X \to G_{k,s} \ ; \ x \mapsto w_x$$

such that

$$h^!(\gamma_{k,s}) = \eta \ .$$

More generally yet, we have the following theorem of Milnor:

THEOREM 10. *Given a topological group $G$ there exists a space $BG$ and a principal $G$ fibre bundle with total space $EG$,*

$$G \longrightarrow EG \xrightarrow{\pi} BG,$$

*so that the following properties hold.*

(1) *If $G \to E \to X$ is any principal $G$-fibre bundle over a finite dimensional $CW$ complex $X$ then there is a unique homotopy class of maps*
$$f\colon X \to BG$$
*so that $f^!$ induces the given bundle over $X$.*

(2) *Given a (continuous) homomorphism $h\colon G_1 \to G_2$ there is a map of fibre bundles*

$$\begin{array}{ccccc} G_1 & \longrightarrow & EG_1 & \xrightarrow{\pi} & BG_1 \\ \big\downarrow h & & \big\downarrow Eh & & \big\downarrow Bh \\ G_2 & \longrightarrow & EG_2 & \xrightarrow{\pi} & BG_2 \end{array}$$

*extending $h$.*

(3) *If $h$ above embeds $G_1$ as a closed subgroup of $G_2$ so that*
$$G_1 \to G_2 \to G_2/G_1$$
*is a principal fibre bundle then $Bh$ in (2) is a fibre bundle with fibre $G_2/G_1$.*

Any principal fibre bundle $G \to E \to X$ which has the property in 10 is called $G$-classifying, and the base space $X$ is written $BG$ and called a $G$-classifying space, or classifying space for short when $G$ is understood. $BG$ has the property that, for $X$ a finite dimensional $CW$ complex or simplicial complex, then the set of homotopy classes of maps $[X, BG]$ is identified with the set of equivalence classes of principal $G$-bundles over $X$.

There is a direct method for determining whether a given principal $G$-fibre bundle is a $G$-classifying space :

Theorem 11. Steenrod Recognition Principle *The principal bundle*
$$G \longrightarrow E \xrightarrow{p} X$$
*is a $G$-classifying space if and only if $E$ is contractible.*

Thus, applying 6 to the limit space, the Grassmannian of $n$-planes in $\mathbb{R}^\infty$,
$$G_n = \varinjlim k\, G_{n,k}$$
we have

Corollary 6. *The space $BGL_n(\mathbb{R})$ above can be identified with $G_n$. Consequently, the set of isomorphism classes of $n$-plane bundles over any finite dimensional $CW$ complex $X$ is identified with the set of homotopy classes $[X, G_n]$.*

Every finite $CW$ complex is paracompact so that $k$-plane bundles on $X$ automatically admit metrics. Consequently, they all reduce to $O_k(\mathbb{R})$-bundles and it follows that $[X, BGL_k(\mathbb{R})] = [X, BO_k(\mathbb{R})]$, so we can restrict attention to the classifying space for this smaller group. Note that a map $f\colon V \to W$ is a weak equivalence if and only if $f$ induces isomorphisms of homotopy sets

$$f_* : [X,V] \xrightarrow{\cong} [X,W]$$

for each finite $CW$ complex $X$, and, when $V$, $W$ also have the homotopy type of $CW$ complexes then weak equivalence implies equivalence. In particular $BGL_k(\mathbb{R}) \simeq BO_k(\mathbb{R})$.

**Classification of stable vector bundles.** Let $BGL(\mathbb{R}) = \lim_{\rightarrow} n\, BGL_n(\mathbb{R})$ and $BO = \lim_{\rightarrow} n\, BO_n(\mathbb{R})$. As above $BO \simeq BGL(\mathbb{R})$.

The stable bundles over a finite $CW$ complex $X$ are in one-one correspondence with the homotopy classes of maps $X \longrightarrow BO$ to the classifying space $BO$.

Remark . The homotopy groups $\pi_*(BO)$ are 8-periodic by the Bott periodicity theorem, and are given by:

| $n(\mathrm{mod}\, 8)$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $\pi_n(BO)$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}$ | 0 | 0 | 0 |

∎

Proposition 8. *If $k > m$ then two $k$-plane vector bundles $\eta$, $\eta'$ over an $m$-dimensional finite $CW$ complex $X$ are isomorphic if and only if they are stably isomorphic.*

Proof. The inclusion $BO_k \longrightarrow BO_{k+1}$ fits into a fibre bundle sequence

$$S^k \longrightarrow BO_k \longrightarrow BO_{k+1} \ ,$$

the sphere bundle of the tangent bundle of the $(k+1)$-sphere

$$\tau_{S^{k+1}} \ : \ S^{k+1} \longrightarrow BO_{k+1} \ .$$

(See 3 for some discussion of sphere bundles and spherical fibre bundles). Thus the pair $(BO_{k+1}, BO_k)$ is $k$-connected, with

$$\pi_j(BO_{k+1}, BO_k) \ = \ \pi_{j-1}(S^k) \ = \ 0 \text{ for } j \leq k \ .$$

It follows that for $k > m$ each map

$$[X, BO_k] \longrightarrow [X, BO_{k+1}] \longrightarrow \ \ldots \ \longrightarrow [X, BO]$$

is a bijection. □

**The relation between stable and unstable bundles.** An $n$-plane bundle, $\Omega$, on an $n$-dimensional complex, $X_n$, which is stably trivial can, by 8, be assumed to be trivial and trivialized (this means a particular homotopy of the classifying map is given) on the $(n-1)$-skeleton $X_{n-1}$. Hence the classifying map $B_\Omega \colon X \to BO_n$ can be assumed to factor in the form

$$X \longrightarrow X/X_{n-1} = \bigvee_{i \in \mathcal{I}} S^n_i \xrightarrow{\bigvee \mu_i} BO_n,$$

where the $\mu_i \colon S^n \to BO_n$ represent bundles over $S^n$ which, together, induce stably trivial bundles over $X$.

In particular, the stably trivial $n$-plane bundles over $S^n$ are given by the kernel of the map of homotopy groups

$$\pi_n(BO_n) \xrightarrow{(i_n)_*} \pi_n(BO)$$

induced by the inclusion $i_n \colon BO_n \hookrightarrow BO$. The structure of this kernel is well known.

Theorem 12. (1) *Suppose that $n$ is odd, $n \neq 1, 3, 7$. Then the kernel of the map $(i_n)_*$ in (5) is a copy of $\mathbb{Z}/2$ with the non-zero element given as the tangent bundle of $S^n$.*

(2) *Suppose that $n$ is even. Then the kernel of the map $(i_n)_*$ in (5) is a copy of the integers with the element $n$ generated by the following composition*

$$S^n \xrightarrow{n} S^n \xrightarrow{B_\tau} BO_n$$

*where $n : S^n \to S^n$ is the map of degree $n$ from $S^n$ to itself, and $B_\tau$ is the classifying map for the tangent bundle of $S^n$.*

(3) *Suppose that $n = 1, 3,$ or $7$. Then the kernel of $(i_n)_*$ in (5) is $0$.*

## 6. Thom spaces and transversality

Definition 13. *The* **Thom space** *of a $k$-plane bundle $\eta$ is the pointed space*

$$T(\eta) \;=\; E(\eta)/E_1(\eta)$$

*defined by the quotient of the total space $E(\eta)$ by the subspace $E_1(\eta) \subset E(\eta)$ of vectors of length $\geq 1$ with respect to some metric. The zero section gives a well defined inclusion $X \hookrightarrow T(\eta)$. Moreover, by rescaling we can easily show that the pair $X \hookrightarrow T(\eta)$ is independent of the choice of metric.*

This construction is natural for induced bundles in the sense that given $f \colon Y \to X$, then the pair of maps

$$(p_2, f) \,:\, (f^!(\eta), Y) \longrightarrow (\eta, X)$$

gives a map of Thom complexes $T(f) \colon T(f^!(\eta)) \to T(\eta)$

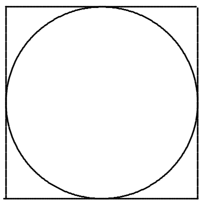
REMARK . Note that if we have a product bundle

$$\mathbb{R}^k \times \mathbb{R}^s \to E(\eta \times \zeta) \xrightarrow{p\times p} X \times Y$$

then

$$E_1(\eta) \times E(\zeta) \bigcup E(\eta) \times E_1(\zeta) \subset E_1(\eta \times \zeta),$$

with the inclusion on a single fibre having the form



where the exterior of the circle represents $E_1(\eta \times \zeta)$ and the exterior of the square represents $E_1(\eta) \times E(\zeta) \cup E(\eta) \times E_1(\zeta)$. In particular, there is an evident radial deformation retraction from $E_1(\eta\times\zeta)$ to $E_1(\eta)\times E(\zeta)\cup E(\eta)\times E_1(\zeta)$ and

$$T(\eta \times \zeta) \simeq E(\eta \times \zeta)/(E_1(\eta) \times E(\zeta) \cup E(\eta) \times E_1(\zeta) = T(\eta) \wedge T(\zeta).$$

where the base point of $T(\eta)$ is $* = \{E_1(\eta)\}$, so that the $\wedge$ makes sense.

This has the basic corollary

COROLLARY 7. *The Thom space of the sum $\eta \oplus \epsilon$ of a vector bundle $\eta$ and the trivial bundle $\epsilon$ is the suspension of the Thom space of $\eta$,*

$$T(\eta \oplus \epsilon) = \Sigma T(\eta) \ .$$

Similarly, but more directly from the definition we have

COROLLARY 8. *The Thom space of the trivial bundle $\epsilon$ over $X$ is $T(\epsilon) = \Sigma X_+$, the suspension of $X$ with a disjoint basepoint attached.*

EXAMPLE 20. In the case of $\gamma_{1,n}$ over $G_{1,n} = \mathbb{RP}^n$ we have that the Thom space is exactly $\mathbb{RP}^{n+1}$. This can be seen as follows. First consider the usual inclusion $\mathbb{RP}^n \hookrightarrow \mathbb{RP}^{n+1}$, ($j_{1,n}$ in the discussion after 12). We may write $S^{n+1}$ as the join of $S^n$ and $S^0$ where the $S^0$ lies along the line associated to $e_{n+1}$. Here, recall that the join $X * Y$ is the quotient of the disjoint union

$$X \sqcup X \times I \times Y \sqcup Y$$

by the relations $(x, 0, y) \sim x$, $(x, 1, y) \sim y$, and the lines in $\mathbb{R}^n$ are indexed by the points of $S^n$ while the line through $e_{n+1}$ is indexed by $S^0$. The intermediate lines corresponding to $t \neq 0, 1$ are indexed by the points $\gamma_{1,n}$ of length $< 1$, where $(v, w) \leftrightarrow \{w/|w|, |w|\}$ with the obvious identification with $\mathbb{RP}^n$ in the extreme case $w = 0$, and, for $|w| = 1$ we identify with $\{e_{n+1}\}$. But this is, as asserted the Thom space of $\gamma_{1,n}$.

EXAMPLE 21. Analogous to the bundle $\gamma_{1,n}$ there is the complex line bundle $\zeta_n$ over $\mathbb{C}\mathbb{P}^n$, consisting of pairs $(v, w)$ where $v$ represents a complex line through the origin in $\mathbb{C}^{n+1}$, and $w$ is a vector contained in $v$. An argument totally analogous to the above, using the representation of $S^{2n+1}$ as $S^{2n-1} * S^1$ shows that $T(\zeta_n) = \mathbb{C}\mathbb{P}^{n+1}$.

**The Pontrjagin-Thom construction.** One of the main uses of the Thom complex is in the context of the Tubular Neighborhood Theorem, 9. Suppose that we have an embedded submanifold $M^m \subset N^{m+k}$ with normal $k$-plane bundle $\nu_k$. Then we may identify a sufficiently small open neighborhood of $M \subset N$ with the vectors of length $< 1$ in $\nu_k$. Consequently, collapsing the complement of this open neighborhood to a single point gives a map of $N^{n+k}$ to the Thom space of the normal bundle,

$$p_M : N^{m+k} \longrightarrow T(\nu_k) .$$

This map is called the **Pontrjagin-Thom construction** and is one of the fundamental tools in differential and geometric topology.

A special case involves the embeddings $M^m \hookrightarrow S^{m+k}$ for $k \geq n+1$ given by the Whitney $2n+1$-embedding theorem 10 where $M^m$ is closed and compact. In this special case it states that up to isotopy an embedding $M^n \hookrightarrow \mathbb{R}^{2n+2}$ exists and is unique.

In this case the normal bundle is of the form

$$\nu_k(M) \ = \ (k-m-1)\epsilon \oplus \nu_{m+1}(M)$$

with the Thom space of the form

$$T(\nu_k(M)) \ = \ \Sigma^{k-m-1} T(\nu_{m+1}(M))$$

(by 7). The Pontrjagin-Thom construction now gives a map

$$p_M \colon S^{m+k} \to \Sigma^{k-m-1} T(\nu_{m+1}(M))$$

and since any two embeddings in this range are isotopic (there is an embedding

$$M^m \times I \subset S^{m+k} \times I \ ; \ (x,t) \mapsto (f_t(x), t) \ .$$

it follows that $p_M$ is well defined up to homotopy for each $k$. Moreover, by looking at the particular embedding $M^m \hookrightarrow S^{m+k}$ given by the composition

$$M^m \hookrightarrow S^{2m+1} \hookrightarrow S^{m+k}$$

we see that $p_M$ for $k+1$ is just the suspension of $p_M$ for $k$ as long as $k \geq m+1$. Thus we get a well defined stable homotopy class of maps

$$S^{m+k} \to \Sigma^{k-m-1} T(\nu_{m+1}(M))$$

and hence a well defined element in the $m^{th}$ stable homotopy group

$$\pi_m^s(T(\nu_M)) = \lim_{k\to\infty} \pi_{m+k}(T(\nu_k(M))) \ .$$

**Transversal maps to the Thom complex.** Given a map from a differentiable manifold $N^n$ into the Thom space of a $k$-plane bundle $\eta$ over a complex $X$, $N^n \to T(\eta)$, it is possible to partially reverse the Pontrjagin-Thom construction.

Note that a neighborhood of the zero section in the Thom complex, $T(\eta)$ is homeomorphic to $E(\eta)$. In this context we can consider a map $g\colon N^n \to T(\eta)$ from an $n$-dimensional manifold to the Thom space of the $k$-plane bundle $\eta$.

Definition 14. *A map $g : N^n \longrightarrow T(\eta)$ from an $n$-manifold $N$ to the Thom space of a $k$-plane bundle $\eta$ is* **transverse** *at the zero section $X \subset T(\eta)$ if the inverse image is a closed $(n-k)$-dimensional submanifold*

$$M \ = \ g^{-1}(X) \subset N$$

*with normal $k$-plane bundle*

$$\nu_{M\subset N} \ = \ f^*\eta \ : \ M \ \xrightarrow{f} \ X \ \xrightarrow{\eta} \ BO_k$$

*the pullback of $\eta$ to $M$ along the restriction $f = g| : M \longrightarrow X$, so that there is defined a bundle map*

$$(f, b) \ : \ (M, \nu_{M\subset N}) \longrightarrow (X, \eta) \ .$$

Theorem 13. (Sard, Thom [66])
*Every continuous map $g : N^n \longrightarrow T(\eta)$ from an $n$-dimensional manifold to the Thom space $T(\eta)$ of a $k$-plane bundle $\eta : X \longrightarrow BO_k$ is homotopic to a differentiable map $\overline{g} : N \longrightarrow T(\eta)$ which is transverse at the zero section $X \subset T(\eta)$.*

In particular, for $k < n$ there is an $n-k$ dimensional submanifold $V^{n-j} \subset N^n$ with a map $f\colon V^{n-j} \to X$ pulling the bundle $\eta$ over $X$ back to the normal bundle of the embedding $V^{n-j} \hookrightarrow N^n$, and it follows that the original map $g\colon N^n \to T(\eta)$ is homotopic to the following composition

$$N^n \xrightarrow{P_V} T(f^!(\eta)) \xrightarrow{T(F)} T(\eta)$$

where $P_V$ is the Pontrjagin-Thom construction associated to the embedding $V \hookrightarrow N^n$.

CHAPTER 3

# Immersions and embeddings

An immersion $f\colon N^n \to M^m$ is a $C^\infty$ map which satisfies the condition that the induced map of tangent bundles $df\colon \tau_N \to \tau_M$ has rank $n$ at every point of $N$. From the implicit function theorem it follows that if $f$ is an immersion it is locally an embedding, but there may be a fairly complex set of points in the image with inverse images consisting of multiple points. This set is called the singular set of the immersion.

## 1. Embeddings and Immersions in $\mathbb{R}^m$

Particularly important are immersions and embeddings $f\colon M^n \to \mathbb{R}^{n+s}$. In this case the test for immersion is particularly easy since $\tau_{\mathbb{R}^{n+s}} = \mathbb{R}^{n+s} \times \mathbb{R}^{n+s}$, and there is a projection onto the second coordinate so that $f$ is an immersion if and only if $p_2 df$ takes $\tau_m$ to an $n$-plane in $\mathbb{R}^{n+s}$ for each $m \in M$.

We begin by studying imbeddings and immersions of disks in Euclidian space into $\mathbb{R}^m$. Thus, let $D_r$ be the open disk of radius $r$ centered at the origin in $\mathbb{R}^n$ and consider the space $\mathcal{F}(n, m, r)$ defined as the set of $C^\infty$ maps $f\colon D_r \to \mathbb{R}^m$. A length function on $\mathcal{F}(n, m, r)$ is given by setting

$$||f|| = \max_{x \in \bar{D}_{r-\tau}, 1 \le i,j \le n} \left( |f(x)|, \left| \frac{\partial f_j}{\partial x_i} \right| \right)$$

for some fixed $\tau$ with $0 < \tau < r$. Then we have the following result.

LEMMA 7. *Let $f \in \mathcal{F}(n, m, r)$ and suppose that $\tau$ is given. Suppose that $f$ is an immersion or an embedding when restricted to $D_{r-\tau}$, then there is an $\epsilon > 0$ depending on $f$ so that if $||f' - f|| < \epsilon$ then $f'$ is an immersion or an embedding as well.*

PROOF. We begin with the case of an immersion. Since we are assuming that $f$ is an immersion we must have $m \ge n$. Also the sum

$$\sum_{I \in \mathcal{I}} \mathrm{Det}(df_I)^2(x) \neq 0$$

and is continuous for each $x \in \bar{D}_{r-\tau}$, where $\mathcal{I}$ is the set of $n \times n$ minors of $df$ (so $I$ is a sequence $(1 \le i_1 < i_2 < \cdots < i_n \le m)$ identifying a set of $n$ distinct columns in $df$). Consequently, since $\bar{D}_{r-\tau}$ is compact there is a $d > 0$ so that the sum in 1 is $> d$ for each $x \in \bar{D}_{r-\tau}$. Now, consider the sum of two $m \times n$ matrices, $D + X$. Then there is a polynomial $p_D(X_{1,1}, \ldots, X_{m,n})$ which gives the sum in 1 for $D + X$, and the coefficients are sums of determinants of the

$k \times k$ minors in $D$. In particular, the constant term is 1 for $D$. Consequently, letting $D$ be $df(x)$, there are bounds on all the coefficients with the constant term $\geq d$ on $\bar{D}_{r-\tau}$. It follows that there is an $\epsilon > 0$ so that for $|X_{i,j}| < \epsilon$, then the term 1 for $df(x) + X > 0$ at every $x \in \bar{D}_{r-\tau}$. This proves 7 for the case of immersions.

Now we give the argument for embeddings. So suppose that $f$ is an embedding. First, choose $\epsilon$ so small that $f'$ will be an immersion for $||f' - f|| < \epsilon$. In particular, there will be an $\delta > 0$ so that $f'(x) - f'(x + \vec{v}) \neq \vec{0}$ for $|\vec{v}| < \delta$ and $x \in \bar{D}_{r-\tau}$ with $\delta$ just a function of $f$ and $\epsilon$. Consider the composite map $h = (-)(f' \times f')$,

$$\mathbb{R}^n \times \mathbb{R}^n \xrightarrow{f' \times f'} \mathbb{R}^m \times \mathbb{R}^m \xrightarrow{-} \mathbb{R}^m$$

where $(-1)(\vec{y}_1, \vec{y}_2) = \vec{y}_1 - \vec{y}_2$. We thus have that the distance from the complement of the diagonal in $\{(-)(f' \times f')\}^{-1}(\vec{0})\}$ to the diagonal is $> \delta$.

Let $N_d(\Delta)$ be the set of $(x, y) \in \bar{D}_{r-\tau} \times \bar{D}_{r-\tau}$ with $|x - y| < d$ be a neighborhood of the diagonal. Then, since $f$ is an embedding, the map $(-1)(f \times f)$ restricted to $\bar{D}_{r-\tau} \times \bar{D}_{r-\tau} - N_d(\Delta)$ has distance $> e(d) > 0$ from $\vec{0}$. Assume now that $\epsilon$ is also smaller than $\delta/3$. Then it follows that for $|x - y| < \delta/3$ we have $f'(x) - f'(y) \neq \vec{0}$, so, in fact, $f'$ is an embedding on $\bar{D}_{r-\tau}$. □

LEMMA 8. *Suppose that* $m \geq 2n$, *and that* $f \in \mathcal{F}(n, m, r)$. *Then, for any* $\epsilon > 0$ *there is an* $f' \in \mathcal{F}(n, m, r)$ *with* $||f' - f|| < \epsilon$ *and* $f'$ *is an immersion.*

PROOF. We denote by $M(m, n)$ the space of all $m \times n$ matrices with coefficients in $\mathbb{R}$. Note that as a space $M(m, n) = \mathbb{R}^{mn}$, is a manifold of dimension $mn$.

The $f'$ that we will find during the course of the proof will have the form $x \mapsto f(x) + Mx$, where $M \in M(m, n)$, and such a map has derivative $df' = df + M$. Thus, we are searching for an $M$ arbitrarily close to 0, so that $df + M$ is non-singular on $\bar{D}_{r-\tau}$. To find such an $M$ reverse the equation, $M = Q - df$, and let $Q$ vary over all the $m \times n$ matrices having rank $< n$. Let $M(m, n, k)$ be the subset of $M(m, n)$ consisting of matrices of rank $k$ with $k < n$. We will show that the image of the map,

$$D_r \times M(m, n, k) \xrightarrow{s} M(m, n)$$

defined by $s(x, Q) = Q - df(x)$ has measure zero in $M(m, n)$ for each $k < n$. Thus, choosing $M$ not in the image gives an immersion.

PROPOSITION 9. *The subset* $M(m, n, k) \subset M(m, n)$ *is a manifold of dimension*

$$\mathrm{Dim}(M(m, n, k)) = k(m + n - k) .$$

PROOF. Define an action of the group $GL_m(\mathbb{R}) \times GL_n(\mathbb{R})$ on $M_{m,n}(\mathbb{R})$ by the rule

$$(A, B)(\alpha) = A\alpha B^{-1}.$$

Under this action, every $m \times n$ matrix of rank $k$ is equivalent to the matrix

$$J = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} .$$

Invertible matrices

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \in GL_m(\mathbb{R}) \ , \ B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \in GL_n(\mathbb{R})$$

with $A_1, B_1 \in M_{k,k}(\mathbb{R})$ satisfy the equation $AJB^{-1} = J$ (or equivalently $AJ = JB$) if and only if

$$A_1 = B_1 \ , \ A_3 = 0 \ , \ B_2 = 0 \ .$$

The isotropy group of $J$

$$\{(A,B) \in GL_m(\mathbb{R}) \times GL_n(\mathbb{R}) \mid AJB^{-1} = J\}$$

is thus the group of pairs of the form

$$(A,B) = \left( \begin{pmatrix} L & M \\ 0 & N \end{pmatrix} , \begin{pmatrix} L & 0 \\ P & S \end{pmatrix} \right)$$

where $L$ is $k \times k$, $M$ is $k \times (m-k)$, $N$ is $(m-k) \times (m-k)$, $P$ is $(n-k) \times k$, and $S$ is $(n-k) \times (n-k)$. The dimension of this isotropy group is

$$k^2 + k(m-k) + (m-k)^2 + (n-k)k + (n-k)^2 = m^2 + n^2 - k(m+n-k) ,$$

so that the dimension of the orbit is $k(m+n-k)$ as asserted. □

Consequently, since $k(n+m-k)$ is maximal for $k = (n+m)/2$ which is greater than $n$ since $m \geq 2n$, it follows that $k(n+m-k)$ is monotone increasing for $1 \leq k < n$, and $k(n+m-k) \leq (n-1)(m+1)$. Consequently

$$n + k(n+m-k) \leq n + (n-1)(m+1) = nm + 2n - m - 1$$

is strictly less than $nm$ for $m \geq 2n$, and for each $k$ the measure of the image of $D_r \times M(m,n,k) \subset M(m,n)$ is zero. □

## 2. Constructing immersions $f\colon N^n \to M^m$ with $m \geq 2n$

We can globalize some of the above results directly to apply to compact manifolds.

LEMMA 9. *Suppose that $N^n$ is compact. Let $f\colon N^n \to M^m$ be any map with $m \geq 2n$, then $f$ is arbitrarily close to an immersion.*

PROOF. To begin we replace $f$ by a $C^\infty$ map very close to $f$.

Next we cover $M^m$ by coordinate patches and then cover $N^n$ by a finite set of coordinate patches $U_1, \ldots, U_k$ with $f(D^n_{6/5} \subset U_j)$ contained in one of the coordinate patches $V_{s(j)} \subset M^m$ and the $D^n_1 \subset U_j$ already a covering of $N^n$.

Then we use the modifications above to change $f$ to an immersion on $\bar{D}^n_1 \subset U_1$, modify $f$ on $D^n_{6/5} \subset U_2$ so that on the intersection with $\bar{D}^n_1 \subset U_1$

the modified map remains an immersion, and continue this process with $D^n_{6/5} \subset U_3$, and so on. □

Remark . Note that for $f'$ sufficiently close to $f$ it follows that $f'$ is homotopic to $f$. Indeed, since $M^m$ is paracompact there is a Riemann metric on $M$, and for $f'$ sufficiently close to $f$ there is a unique geodesic joining $f(n)$ and $f'(n)$ for each $n$. These geodesics, if $f(n) \neq f'(n)$, are parametrized by the length of the vector $v \in \tau_{f(n)}(M^m)$ so that $exp(v)$ runs along the geodesic, and renormalizing so the geodesic is traversed in unit time gives the homotopy.

Corollary 9. *Let $f\colon N^n \longrightarrow M^m$ be an arbitary map, then $f$ is homotopic to an immersion for $2n \leq m$.*

## 3. The Singular Set of an Immersion

The **singular set** of an immersion $f\colon N^n \longrightarrow M^m$ is the closure of the set of points $x \in N^n$ so that $f^{-1}f(x)$ is strictly larger than $\{x\}$. The **singular values** are the images of the singular set. Note that an immersion $f$ is an embedding if and only if the singular set is empty. One would expect that generically, the dimension of the singular set would be $2n - m$ if this is greater than or equal to zero, and would be empty otherwise. This is justified by the following result.

Lemma 10. *Let $f\colon N^n \longrightarrow M^m$ be an immersion with, as usual $N^n$ compact with no boundary. Then arbitrarily close to $f$ there is an immersion $f'$ so that the composite map*

$$g\colon N^n \times N^n - \Delta(N^n) \xrightarrow{f' \times f'} M^m \times M^m$$

*is transverse to $\Delta(M^m)$. In particular, the singular set of $f'$ has dimension $2n - m$ if $2n \geq m$ and is empty otherwise.*

Proof. Given $f$ we consider a covering of $N^n$ having the type considered in the proof of 9. Then for each disk $\bar{D}_1 \subset D_2$ there there is $C^\infty$ function on $N^n$, $\lambda(x)$, which is identically 1 on $\bar{D}_1$, decreases to 0 in $D_2$ and is identically 0 in $N^n - D_2$.

The cover has been constructed so that $f(\bar{D}_2) \subset M^m$ is contained in a particular coordinate patch $\mathbb{R}^m$. Then there is an $\epsilon > 0$ so that the map $h\colon D^m_\epsilon \times \bar{D}_2 \longrightarrow \mathbb{R}^m$ given by

$$h((\vec{y}, x)) \;=\; f(x) + \lambda x \vec{y}$$

is entirely contained in the coordinate patch and we can consider the associated map, $K$, given as the composite

$$D^m_\epsilon \times \left(\bar{D}_2 \times \bar{D}_2 - \Delta(\bar{D}_2)\right) \xrightarrow{h \times f} \mathbb{R}^m \times \mathbb{R}^m \xrightarrow{(-)} \mathbb{R}^m.$$

Given $\delta > 0$ there is a $\vec{y} \in \mathbb{R}^m$ with $|\vec{y}| < \delta$ so that $\vec{y}$ is a regular value of $K$ by Sard's theorem. Now, set

$$f'(x) = f(x) + \lambda(x)\vec{y}$$

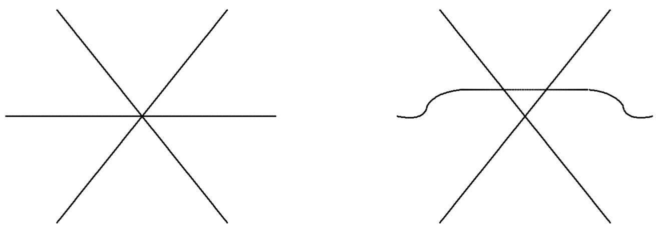
which is well defined as a map into $M^m$. It agrees with $f$ outside of $D_2$ and, since $\vec{y}$ is regular for $K$ it follows that the intersection of the singular set of $f'$ with $\bar{D}_1$ is a submanifold of dimension $2n-m$ or is empty. On $(\bar{D}_2 - D_1)$ intersected with the unit disks in the previous coordinate patches we can assume that $f$ has the same property, and it is direct to see that there is some $\epsilon > 0$ so that $\epsilon$-close immersions have regular singular sets as well. Thus, by choosing $\vec{y}$ sufficiently small we can assure that $f'$ has singular set of the correct type on the union of the first $i$ patches $D_{1,1}, \dots, D_{1,i}$. □

COROLLARY 10. **Whitney Embedding Theorem** *Let* $f\colon N^n \to M^m$ *be any continuous map, with* $N^n$ *a compact manifold with empty boundary, and suppose that* $m \geq 2n+1$. *Then* $f$ *is arbitrarily close to an embedding.*

COROLLARY 11. **The $2n$-Immersion Theorem** *Let* $f\colon N^n \to M^{2n}$ *be any continuous map with* $N^n$ *as above. Then* $f$ *is arbitrarily close to an immersion with singular set consisting only of isolated points. Moreover, we can assume*

(1) *the cardinality of the inverse image of each singular value is* 2,
(2) *the span of* $df(\tau_N(x)), df(\tau_N(x'))$ *is* $\tau_M(f(x))$ *for each pair of* **double points** $(x, x')$ *in* $N^n$ *with* $f(x) = f(x')$.

PROOF. Most of this is evident. Suppose that $f(x)$ is a multiple point with the cardinality of $f^{-1}(f(x)) > 2$. Then we can push away the third and later points creating a series of isolated double points.



A triple point The deformation

There is a similar argument for the structure of the span of the tangent spaces.

□

## 4. Immersions of $S^n$ in $\mathbb{R}^{2n}$

In this section we will construct some basic immersions of $S^n$ into $\mathbb{R}^{2n}$, immersions with a single double point as their only singularities. But before we do this we will find the definition of the **index** of a double point in general position to be useful.

Definition 15. *Let $v \in M^{2n}$ be a double point in general position for an immersion $f\colon N^n \to M^{2n}$. Suppose $f^{-1}(v) = \{n_1, n_2\}$ is given an ordering, $(n_1, n_2)$, so $n_1$ preceeds $n_2$. Then the* **index at** *$v$ for the ordered pair $(n_1, n_2)$ and the given orientations of $N^n$, $M^{2n}$ is either $\pm 1$. Take an oriented basis for $\tau_{n_1}(N)$, $e_1, \ldots, e_n$ and an oriented basis $h_1, \ldots, h_n$ for $\tau_{n_2}(N)$ (with respect to the given orientation of $N$). An orientation on $\tau_v(M)$ is given by the basis*

$$df(e_1), \ldots, df(e_n), df(h_1), 0ts, df(h_n)$$

- *The index is $+1$ if the orientation on $\tau_v(M)$ above agrees with that of $\tau_v(M)$ given by the orientation of $M$.*
- *It is $-1$ otherwise.*

Remark . Note that the index is independent of the choices for the bases $e_1, \ldots, e_n$, $h_1, \ldots, h_n$ as long as the orientations they define agree with that of $N$. Also, note that if $n$ is odd and we reverse the order of $n_1$, $n_2$, then the index changes sign, but for $n$ even it is independent of the ordering of the points in $f^{-1}(v)$. Finally, note that it is independent of the choice of orientation on $N$, but changes sign if we change the orientation of $M$.

Proposition 10. *For each $n \geq 1$ and $\epsilon = \pm 1$ there exists an immersion $f_n^\epsilon\colon S^n \to \mathbb{R}^{2n}$ with a single ordered double point $\vec{v}$ having index*

$$I(\vec{v}) \ = \ \epsilon \ .$$

*The normal bundle to $f_n^\epsilon(S^n)$ is equal to $\epsilon \tau_{S^n}$.*

Proof. For $n = 1$, we can take $f_1^+ = f_1^-\colon S^1 \to S^2$ to be the figure 8 immersion.

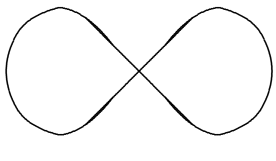


**Figure** 8 **curve**

For $n \geq 2$ we use the following construction. Start with

$$\begin{aligned} D_1^n &= \{(x_1, \ldots, x_n, 0, \ldots, 0) \in \mathbb{R}^{2n} \mid \textstyle\sum x_i^2 \leq 1\} \ , \\ D_2^n &= \{(0, \ldots, 0, y_1, \ldots, y_n) \in \mathbb{R}^{2n} \mid \textstyle\sum y_i^2 \leq 1\} \ . \end{aligned}$$

The intersection of the unit sphere $S^{2n+1} \subset \mathbb{R}^{2n}$ with $D_1^n \sqcup D_2^n$,

$$S^{2n-1} \cap D_1^n \ = \ S_1^{n-1} \ , \ S^{2n-1} \cap D_2^n \ = \ S_2^{n-1}$$

is a pair of linked spheres in $S^{2n-1}$. Next we connect them together by a tube in the complement of the unit ball. For $0 \leq t \leq 1$ let

$$\phi_t \ = \ \begin{pmatrix} \cos(\pi t/2) I_n & \sin(\pi t/2) I_n \\ -\sin(\pi t/2) I_n & \cos(\pi t/2) I_n \end{pmatrix} \ : \ \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n} \ .$$

The embeddings

$$\bar{\phi}_t \colon S_2^{n-1} \longrightarrow \mathbb{R}^{2n} \; ; \; x \longrightarrow \sqrt{1+\sin^2(\pi t)} \cdot \phi_t(x) \; (0 \le t \le 1)$$

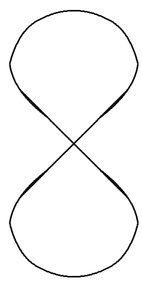
define an isotopy between $\bar{\phi}_0(S^{2n-1}) = S_1^{2n-1}$ and $\bar{\phi}_1(S^{2n-1}) = S_2^{2n-1}$ in the complement of the unit ball in $\mathbb{R}^{2n}$. The map

$$f_n^+ \colon S^n \; = \; D^n \times \{0,1\} \cup S^{n-1} \times I \longrightarrow \mathbb{R}^{2n} \; ; \quad \begin{cases} (x_1, \dots, x_n, 0) \longrightarrow (x_1, \dots, x_n, 0, \dots, 0) \\ (y_1, \dots, y_n, 1) \longrightarrow (0, \dots, 0, y_1, \dots, y_n) \\ (z,t) \longrightarrow \bar{\phi}_t(z) \end{cases} ,$$

after smoothing to make it $C^\infty$ in a small neighborhood of $S_1^{n-1} \sqcup S_2^{n-1}$, is the desired immersion of $S^n$ in $\mathbb{R}^{2n}$. To obtain the immersion $f_n^-$ with a double point of complementary index modify $\phi_t$ to

$$\phi'_t \; = \; \mu_t \phi_t \text{ where } \mu_t \; = \; \begin{pmatrix} I_{n-1} & 0 & 0 & 0 \\ 0 & \cos(\pi t) & \sin(\pi t) & 0 \\ 0 & -\sin(\pi t) & \cos(\pi t) & 0 \\ 0 & 0 & 0 & I_{n-1} \end{pmatrix} .$$

This corresponds to the replacement of the figure 8 immersion in 4 with the immersion



**Index** $-1$ **Figure** $8$ **curve**

It remains to check the normal bundles. We do this in the first case as the second is entirely similar. The tangent bundle to $S^n$ can be described as follows. Take the trivial bundles $\mathbb{R}^n \times D^n$ on the upper and lower hemispheres and glue them together along the common boundary, $S^{n-1}$, by identifying $(\vec{y}, \vec{x}) \sim (\vec{y}, \vec{x})$ for $\vec{y} \perp \vec{x}$, and $(\lambda\vec{x}, \vec{x}) \sim (-\lambda\vec{x}, \vec{x})$ for each $\vec{x} \in S^{n-1}$. This can be seen by using the explicit maps of $\mathbb{R}^n \times D^n$ to the tangent bundles at the respective hemispheres by using the rotation $R_{\theta,\vec{x}}$ which rotates through the angle $\theta$ in the subspace of $\mathbb{R}^{n+1}$ spanned by $(\vec{x}, 0)$ and $(\vec{0}, 1)$ and is the identity in the perpendicular complement.

On the other hand, consider the map $f \colon \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{2n}$ defined by

$$(t, \vec{v}) \mapsto \begin{pmatrix} cos(2\pi t)\vec{v} \\ -sin(2\pi t)\vec{v} \end{pmatrix} .$$

The differential *df* at the point $(t, \vec{x})$ is given as the $2n \times n+1$ matrix

$$\begin{pmatrix} -2\pi sin(2\pi t)\vec{x} & cos(2\pi t)I_n \\ -2\pi cos(2\pi t)\vec{x} & -sin(2\pi t)I_n \end{pmatrix}.$$

Consequently, the image of *df* at $(t, \vec{x})$ is spanned by vectors of the form $e_{1,\vec{x}} = \begin{pmatrix} sin(2\pi t)\vec{x} \\ cos(2\pi t)\vec{x} \end{pmatrix}$, and

$$e_{\vec{y}} = \begin{pmatrix} cos(2\pi t)\vec{y} \\ -sin(2\pi t)\vec{y} \end{pmatrix}.$$

Note that $e_{1,\vec{x}} \perp e_{2,\vec{y}}$ for any $\vec{y} \in \mathbb{R}^n$ so *df* has rank $(n+1)$ as long as $\vec{x} \neq 0$ and $f$ is an immersion on $\mathbb{R} \times (\mathbb{R}^n - \vec{0})$.

Clearly, vectors of the form

$$\begin{pmatrix} sin(2\pi t)\vec{z} \\ cos(2\pi t)\vec{z} \end{pmatrix}$$

are perpendicular to $df(\lambda, \vec{y})$ if $\vec{z} \perp \vec{x}$. This defines an $(n-1)$-dimensional subspace of $\tau(\mathbb{R}^{2n})$ at $f(t, \vec{x})$ and these subspaces together give the normal bundle to $f(\mathbb{R} \times (\mathbb{R}^n - \vec{0}))$. Now, restricting to $\mathbb{R} \times S^{n-1}$ we see that the vectors of type $e_{\vec{y}}$ in 4 are tangent to $df(\mathbb{R} \times S^{n-1}$ if and only if $\vec{y} \perp \vec{x}$. Consequently, the vector

$$e_{2,\vec{x}} = \begin{pmatrix} cos(2\pi t)\vec{x} \\ -sin(2\pi t)\vec{x} \end{pmatrix}$$

is also in the normal bundle to $f(\mathbb{R} \times S^{n-1})$ at $f(t, \vec{x})$, so the normal bundle is the Whitney sum of the trivial bundle above and the normal bundle to $f(\mathbb{R} \times (\mathbb{R}^n - \vec{0}))$.

Thus the normal bundle to the immersion consists of the trivial bundle $\mathbb{R}^n \times D_1^n$, points of the form $((\vec{y}, 0), (0, \vec{z}))$ with $|\vec{z}| \leq 1$ over the first disk, points of the form $((0, \vec{y}), (\vec{z}, 0))$ over the second disk and points of the form above over the tube connecting their boundaries. But gluing these together gives exactly the description above of the tangent bundle to $S^n$. □

REMARK . Work of Smale classified immersions of $M^n$ into $\mathbb{R}^{n+k}$ in terms strictly of the normal bundle. A $k$-plane bundle $\eta$ is a normal bundle to $M^n$ if and only if $\eta + \tau_{M^n} = (n+k)\epsilon$. Then, for $k \geq 1$ there is a one to one correspondence between pairs which consist of

(1) an isomorphism classes of normal $k$-plane bundles, $\eta$, on $M^n$ together with
(2) a specific homotopy class of trivializations of $\eta + \tau_{M^n}$,

and **regular homotopy classes of immersions** $f\colon M^n \to \mathbb{R}^{n+k}$.

Here, two immersions $f, g\colon M^n \to N^{n+k}$ are in the same regular homotopy class of immersions if and only if there is a $C^\infty$ map

$$H\colon I \times M^n \to I \times N^{n+k}$$

so that $H|t \times M^n$ is an immersion for each $t$ and $H(0 \times M^n)$ is $f$, while $H(1 \times M^n)$ is $g$.

From the point of view of classiying spaces Smale's result converts the problem of determining the regular homotopy classes of immersions of $M^n$ into $\mathbb{R}^{n+k}$ with $k \geq 1$ to the problem of determining the set of homotopy classes of maps $I \times M^n \rightarrow BO_{n+2}$ with the image of $0 \times M^n$ contained in $BO_k$ and the image of $1 \times M^n$ a fixed classifying map for the stable normal bundle to $M^n$.

Note that in the special case of immersions of $S^n$ into $\mathbb{R}^{n+k}$ Smale's result shows that the set of regular homotopy classes of immersions is in one to one correspondence with the relative homotopy set (group for $n \geq 1$), $\pi_{n+1}(BO_{n+2}, BO_k)$. Of course, this is just $\pi_n(V_{k,n+2-k})$, where $V_{k,n+2-k}$ is the fiber of the map $B_i \colon BO_k \rightarrow BO_{n+2}$. But $V_{k,m}$ has the homotopy type of the Stiefel manifold of $m$-frames in $\mathbb{R}^{k+m}$ a space which is very well understood.

In particular we have

$$\pi_n(V_{n,2} = \begin{cases} \mathbb{Z}/2 & \text{for } n \text{ odd} \\ \mathbb{Z} & \text{for } n \text{ even.} \end{cases}$$

When $n \neq 1, 3, 7$ we also have that $pi_n(V_{n,2})$ **injects into** $\pi_n(BO_n)$ and the generator of $\pi_n(V_{n,2})$ can be uniquely chosen so that its image represents the tangent bundle to $S^n$. Consequenly, in these cases the isomorphism class of the $n$-dimensional normal bundle completely determines the regular homotopy class of the associated immersion, $S^n \rightarrow \mathbb{R}^{2n}$. However, in the special cases where $n = 1, 3, 7$ the only stably trivial $n$-plane bundle over $S^n$ is the trivial bundle. Here, the non-trivial homotopy class in $\pi_n(V_{n,2})$ represents a *choice of framing* on the trivial bundle.

As a special case the figure eight immersion of $S^1$ in $\mathbb{R}^2$, 4 represents the non-trivial regular immersion class of $S^1$ in $\mathbb{R}^2$ while the usual embedding as the boundary of the unit disk represents $0 \in \pi_1(V_{1,1})$.

REMARK . The addition in $\pi_n(V_{n,2})$ gives a method of "adding stably trivial $n$-plane bundles on $S^n$" together to get further stably trivial $n$-plane bundles on $S^n$. This is just a type of connected sum operation. Specifically, regard $\pm\tau(S^n)$ as given by a specific choice of gluing of the trivial bundle, $\mathbb{R}^n \times D^n$ along $\mathbb{R}^n \times \partial(D^n)$ to a second copy of the trivial bundle on $D^n$. Then if we take the sphere and cut out $k$ disjoint copies of the disk $D^n$, take

$$\mathbb{R}^n \times (S^n - \sqcup_1^k D_j^n)$$

and glue $\mathbb{R}^n \times \sqcup_1^k D_j^n$ in by the gluing above for $\pm\tau$ on *each* $\partial(D_j^n)$ we get a representation of the bundle $\pm k\tau(S^n)$.

Consequently, for $n$ odd and $n \neq 1, 3, 7$, our construction of $k\tau(S^n)$ gives the trivial bundle for $k$ even, but for $n$ even each of these bundles is distinct.

In fact they can be distinguished by a type of self intersection number. Specifically, take the 0-section of the bundle and deform it slightly so that at each point of intersection between the zero section and the deformed 0-section the intersection is in the general position. Choose the orientation of the total space of the tangent bundle to $S^n$ (regarded as a $2n$-dimensional manifold) to be that given by the orientation of $S^n$ together with the orientation of the space of tangents at $x$ corresponding to that orientation. Then the index of each intersection point is well defined and their sum is $2k$ for the bundle $k\tau(S^n)$.

As this last fact will be very important to us in the sequel we will give a proof in REFERENCE after we have introduced the basic homology tools needed.

Remark . It is worth noting that we can construct immersions of $S^{2n}$ in $\mathbb{R}^{4n}$ having index $k$ by simply taking $k$ disjoint immersions of index $sgn(k)$ as constructed in 10 and connecting each of the second through $k^{th}$ to the first with thin disjoint tubes to obtain an immersion of $S^{2n}$ in $\mathbb{R}^{4n}$ in general position having $|k|$ double points. Comparing the construction of $k\tau(S^{2n})$ with the construction of the normal bundle to this immersion we see that the normal bundle is, in fact, $k\tau(S^{2n})$, and, by Smale's theorem we have obtained representatives for all of the regular immersion classes of $S^{2n}$ in $\mathbb{R}^{4n}$.

## 5. The Whitney trick

We now return to the question of embeddings of $N^n$ in $M^{n+k}$. For the special case $k = n$ we will prove the Whitney $2n$-embedding theorem:

Theorem 14. **Whitney, [75]**. *Let $N^n$ be a compact, oriented, differentiable manifold and let $f\colon N^n \to M^{2n}$ be any continuous map. Then $f$ is homotopic to an embedding provided that $M^{2n}$ is simply connected and $n \geq 3$.*

Remark . The constraint that $M^{2n}$ be simply connected is essential here. For $\pi_1(M^{2n}) \neq 0$ it can well happen that there are homotopy classes of maps $f\colon N^n \to M^{2n}$ which contain no embeddings. However, a generalization valid when $n \geq 3$ in the non-simply connected case due to Wall will be the critical ingredient in higher dimensional surgery $m \geq 5$. We will prove this generalization in REFERENCE.

There is also a generalization to $n = 2$ with $M^4$ simply connected due to M. Freedman but this generalization only works for topological embeddings under special circumstances. Of course Freedman's theorem lies at the heart of the classification of 4-dimensional manifolds, but it also lies considerably beyond the focus of our work here so we do not discuss it further.

Proof. **The proof of 14**. We can assume that $f$ has already been deformed to an immersion with isolated double points in general position as its only singularities.

The idea is to try to cancel the double points two at a time. In order to do this we will need to occasionally introduce double points to modify the immersion, in particular to make the total number of double points even.

Choose a pair of double points for $f(N)$, together with orderings on their inverse images so their indexes have opposite signs. If this is not possible for the immersion as it stands, choose a double point and introduce a small $n$-sphere in $M^{2n}$ disjoint from the image of $f$ with a single double point of index minus that of the choosen double point using 10. Then attach a small tube connecting this immersed sphere to $N$ so the chosen double point and the newly introduced one have opposite indices.

As $N$ is path connected and has dimension $\geq 2$, there are differentiable maps

$$\gamma_i \colon I \;=\; [0,1] \longrightarrow N \;(i=1,2)$$

with endpoints

$$\gamma_i(0) \;=\; x_i \;,\; \gamma_i(1) \;=\; y_i \;,$$

so that both curves avoid the double points of $f$ except at their endpoints. Consequently, the map

$$\omega \colon S^1 \;=\; I/\{0=1\} \longrightarrow M^{2n} \;;\; t \longrightarrow \begin{cases} f\gamma_1(2t) & \text{if } 0 \leq t \leq 1/2 \\ f\gamma_2(2(1-t)) & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

is an embedding – though with two corners.

We need to study a small neighborhood of this embedded $S^1 \in M^{2n}$ and we begin by determining the structure of $k$-plane bundles on $S^1$.

It follows directly from 6, 3 that every $k$-dimensional vector bundle over $S^n$ is constructed by glueing together two copies of the trivial $k$-dimensional vector bundle over $D^n$ along $S^{n-1}$, using an isomorphism of the form

$$\phi \colon S^{n-1} \times \mathbb{R}^k \longrightarrow S^{n-1} \times \mathbb{R}^k \;;\; (u,v) \longrightarrow (u, \theta_u(\mathrm{v}))$$

with $\theta_u \colon \mathbb{R}^k \to \mathbb{R}^k$ a linear isomorphism.

For $n=1$ we have $S^0 = \{0\} \sqcup \{1\}$, and we can assume $\theta_1$ is the identity. Call $\theta_0 \colon \mathbb{R}^k \to \mathbb{R}^k$ the characteristic map for the bundle over $S^1$.

LEMMA 11. *Let $k \geq 1$. There are exactly two isomorphism classes of $k$-plane bundles over $S^1$. Moreover, a $k$-plane bundle over $S^1$ is trivial if and only if the characteristic map is orientation preserving.*

PROOF. $\pi_0(GL_1(\mathbb{R})) = \mathbb{Z}_2$, and the inclusion $GL_1(\mathbb{R}) \subset GL_k(\mathbb{R})$ induces an isomorphism of $\pi_0$'s for $k \geq 2$. Moreover, the two path components in $GL_k(\mathbb{R})$ are distinguished by the sign of the determinant. Thus, from 10 we have that $\pi_1(BGL_k(\mathbb{R})) = \pi_0(GL_k(\mathbb{R})) = \mathbb{Z}/2$. Consequently, since $BGL_k(\mathbb{R})$ is path connected, it follows that the set of (unbased) homotopy classes of maps from $S^1$ to $BGL_k(\mathbb{R})$ is the set of conjugacy classes of elements in $\pi_1(BGL_k(\mathbb{R}))$. Thus there are exactly two homotopy classes of maps from $S^1 \to BGL_k(\mathbb{R})$ and, using 5 together with 11, it follows that the determinant of the characteristic map distinguishes them as asserted. □

Thus the isomorphism classes of $k$-plane bundles over $S^1$ are described as follows: the non-trivial one is the Whitney sum of the trivial bundle with the Möbius band, and the other is the trivial bundle itself.

We now study the restriction of the tangent bundle $\tau_M$ to $\omega(S^1) \subset M^{2n}$. Let $g\colon M^{2n} \to BO_{2n}$ classify the tangent bundle to $M$, so $g\omega\colon S^1 \to BO_{2n}$ classifies the restriction of $\tau_M$ to $\omega(S^1)$.

As we have described $\omega$, the image, $\omega(S^1)$, does not have a normal bundle in $M$, since $\omega(S^1)$ is not differentiable at the two double points. However it can be deformed by an arbitrarily small deformation in neighbourhoods of the two double points to make it differentiable. Then we can argue that the normal neighbourhood of the original circle is the same as that for the deformed one, and the homeomorphism defines the bundle over the original circle since the total spaces of the two distinct bundles over $S^1$ are not homeomorphic. Thus it makes sense to talk of the normal bundle to $\omega(S^1)$.

Now, choose a metric on $M^{2n}$ so that at the double point $x$ the two $n$-dimensional subspaces $df(x_1)\tau_N(x_1)$, $df(x_2)\tau_N(x_2)$ of the $2n$-dimensional tangent space $\tau_M(f(x))$ are perpendicular, and similarly at the double point $y$.

Then we can split $\tau(M^{2n})$ into a Whitney bundle sum of two $n$-plane bundles along $\omega(S^1)$ as follows. Choose the first subbundle to be $df(\tau_{\gamma_1(2t)})$ along the image of $f\gamma_1$ and the perpendicular bundle to $df(\tau_{\gamma_2(2t-1)})$ along the image of $f\gamma_2$. Similarly, the second bundle is given by interchanging the roles of $\gamma_1$ and $\gamma_2$ above. Alternately, it is the perpendicular subbundle of $\tau_M$ along $\omega(S^1)$ to the subbundle just constructed. Note that since the indexes of the two double points are opposite, each of these subbundles is **non-orientable** and so has the form of a Whitney bundle sum $\eta \oplus (n-1)\epsilon$.

We now construct an explicit $\eta$ in each of the bundles above. Extend the 1-dimensional tangent section along $\gamma_1$ to a Möbius band section $\beta_1 \subset \alpha_1$ in some way, and the 1-dimensional tangent section on $\gamma_2$ to a 1-dimensional Möbius band section $\beta_2 \subset \alpha_2$, defining non-trivial line bundles $\beta_1$ and $\beta_2$ on $S^1$ so that

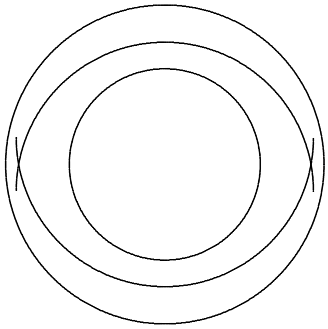
$$\alpha_1 \;=\; \beta_1 \oplus \epsilon^{n-1}\ ,\ \alpha_2 \;=\; \beta_2 \oplus \epsilon^{n-1}\ .$$

Moreover, since $\beta_i \perp \beta_2$ at every point, the span of the two Möbius band sections is their Whitney bundle sum, and

$$\beta_1 \oplus \beta_2 \;=\; 2\epsilon,$$

the trivial 2-plane bundle on $S^1$, because the determinant of the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is positive. This defines, using the tubular neighbourhood theorem 9, an embedding $D^2 \times S^1 \hookrightarrow M^{2n}$ extending $\omega(S^1)$ which we can use to construct a differentiable embedding $D^1 \times S^1 \hookrightarrow M^{2n}$ containing $\omega(S^1)$ in its interior and contained in the image of $D^2 \times S^1$.

We regard this $D^1 \times S^1$ as a closed collar neighborhood of the boundary in the unit ball $D^2 \subset \mathbb{R}^2$, and we now want to extend this embedding to an embedding $D^2 \subset M^{2n}$.



**$D^1 \times S^1$ and the inverse image of $\omega(S^1)$**

LEMMA 12. *Let $V \subset M^m$ be any subcomplex of the manifold $M^m$ having dimension $\leq m-3$ and suppose $m \geq 5$. Then*

$$\pi_1(M \backslash V) \ = \ \pi_1(M) \ .$$

PROOF. The morphism $\pi_1(M\backslash V) \to \pi_1(M)$ induced by inclusion is surjective, since every map $S^1 \to M$ can be moved away from $V$ by general position. In order to prove that the morphism is injective consider an element $x \in \ker(\pi_1(M\backslash V) \to \pi_1(M))$, which may be represented by a commutative square

$$\begin{array}{ccc} S^1 & \xrightarrow{f} & M\backslash V \\ \big\downarrow & & \big\downarrow \\ D^2 & \xrightarrow{g} & M \end{array}$$

with $f$ a differentiable embedding. Since $m \geq 5$ $g$ is homotopic to an embedding leaving $f$ fixed. Now we ensure that $V \cap g(D^2) = \emptyset$. By general position we can move $g(D^2)$ away from $V$ by an arbitrarily small perturbation leaving $g$ an embedding, and leaving $f$ alone on $S^1$. The result is an embedded $g(D^2) \subset M\backslash V$ with $\partial(g(D^2)) = f(S^1)$, so that $x = 1 \in \pi_1(M\backslash V)$. $\square$

Returning to the proof of 14, apply 12 to obtain a differentiably embedded disk $D^2 \subset M\backslash f(N)$ with $\partial(D^2) = \tilde{\gamma}$ and boundary one of the two boundary components of the embedded $D^1 \times S^1$ above.

We have that $\nu_{D^2 \subset M} = \mathbb{R}^{2(n-2)} \times D^2$ and on $\partial(D^2)$ a splitting $\mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1} \times S^1$ is already given. The obstruction to extending this splitting to the entire normal bundle of $D^2$ is an element in $\pi_1(O(2(n-1))) = \mathbb{Z}_2$. Moreover, $\pi_1(O(n-1))$ maps onto this group since $n \geq 3$. Thus we can change the framing on one of the two $\mathbb{R}^{n-1} \times S^1$ bundles to extend the framing across $D^2$.

Finally, leaving the immersion alone near $\gamma_1$ in $N$ change it near $\gamma_2$ to $\gamma_2'$ extending away from $\gamma_2$ to $D^{n-1} \times \gamma_2' \cup D^2 \times \partial(\nu_N(\gamma_2))$. This defines a regular homotopy to an immersion without this pair of unordered double points, completing the proof of 14 $\square$

CHAPTER 4

# Signature Invariants and characteristic classes

Consideration of pairs $(M^n, \partial M)$ consisting of a closed oriented manifold $M^n$ with boundary $\partial M$ is often crucial in surgery. In particular, invariants of closed compact manifolds without boundaries which vanish when the manifold is a boundary are particularly important.

Perhaps the first and most important of these invariants is the signature invariant defined for closed compact manifolds of dimension $4k$.

The definition is as follows. Let $M^{4k}$ be a closed, compact, oriented manifold with empty boundary. Then $H^i(M^{4k}; \mathbb{Z})$ is finitely generated for each $i$ and $H^{2k}(M; \mathbb{Z}) = (\mathbb{Z})^s \oplus Tor$ where $Tor$ is the torsion subgroup. Write the orientation class of $M$ in $H_{4k}(M; \mathbb{Z})$ as $[M]$ and consider the symmetric bilinear form on $H^{2k}(M; \mathbb{Z})$:

$$\langle a, b \rangle \ = \ \langle a \cup b, [M] \rangle \in \mathbb{Z}.$$

First note that this pairing vanishes identically on the torsion subgroup so it factors through $H^{2k}(M; \mathbb{Z})/Tor \cong (\mathbb{Z})^s$. Next, we claim that it is **integrally non-singular**. This means that the adjoint map

$$\phi \colon H^{2k}(M; \mathbb{Z})/Tor \rightarrow Hom(H^{2k}(M; \mathbb{Z})/Tor, \mathbb{Z})$$

defined by $\phi(a)(b) = \langle a, b \rangle$ is an isomorphism. Indeed, this is just a direct consequence of Poincaré duality, since $H_{2k}(M; \mathbb{Z})/Tor = H^{2k}(M; \mathbb{Z})/Tor)^*$ and $a(b \cap [M]) = \langle a \cup b, [M] \rangle$. Then the signature of this bilinear form is the usual signature of the form after tensoring with the rationals $\mathbb{Q}$, namely the number of $+1$-eigenvalues minus the number of $-$-eigenvalues of the symmetric matrix associated to the form after choosing a basis for $\mathbb{Q}^s$.

Lemma 13. **R. Thom** *If $M^{4k}$ is a boundary then the signature of $M$ is* 0.

Proof. Suppose that $M^{4k} = \partial N^{4k+1}$. Then Poincar'e duality applied to the middle dimensional region with $\mathbb{Q}$-coefficients,

$$\cdots \longrightarrow H^{2k}(N; \mathbb{Q}) \xrightarrow{i^*} H^{2k}(M^{4k}; \mathbb{Q}) \xrightarrow{\delta} H^{2k+1}(N, M; \mathbb{Q}) \longrightarrow \cdots$$

shows that $H^{2k}(N; \mathbb{Q})/im(j^*)$ is dual to $im(\delta) \subset H^{2k+1}(N, M; \mathbb{Q})$, so

$$H^{2k}(M^{4k}; \mathbb{Q}) = H^{2k}(N; \mathbb{Q})/im(j^*) \oplus im(\delta)$$

splits $H^{2k}(M;\mathbb{Q})$ into two isomorphic summands. Moreover, for every element in the image of $i^*$ we have

$$\begin{aligned}\langle i^*(a)\cup i^*(b),[M]\rangle &= \langle a\cup b, i_*([M])\rangle \\ &= 0,\end{aligned}$$

since $[M]=\partial[N,M]$, the orientation class of $N$.

Thus, after a change of basis the matrix of the form on $H^{2k}(M;\mathbb{Q})$ becomes $\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}$ or the same becomes a diagonal matrix

$$\begin{pmatrix} T & 0 & 0 & \dots & 0\\ 0 & T & 0 & \dots & 0\\ 0 & 0 & T & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & T\end{pmatrix}$$

where $T=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$. But the eigenvalues of this matrix are $+1$ and $-1$ so it has signature 0, and it follows that the signature of $M$ is zero as well. □

We write the signature of $M^{4k}$ as $I(M)$, and we have the following lemma which is routinely verified.

LEMMA 14. (1) *Let $-M$ be $M$ with the orientation $-[M]$ assigned, then $I(-M)=-I(M)$.*
(2) *Let $M^{4k}$ and $L^{4k}$ be two closed, compact, oriented manifolds without boundary, then, for the disjoint union $M\sqcup L$ we have $I(M\sqcup L)=I(M)+I(L)$.*
(3) *Let $L^{4s}$ be closed, compact, oriented with empty boundary. Then $I(M\times L)=I(M)I(L)$ where the orientation on $M\times L$ is $[M]\otimes[L]$.*

REMARK . The reason one only considers the signature for $4k$-dimensional manifolds is that in the $4k+2$-dimensional case the form is skew symmetric and hence there is always an integral basis so the the matrix of the form becomes $\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}$. Consequently, there isn't much to distinguish the situation where $M$ is not a boundary from that where $M$ is a boundary here. However, in the case where the manifold has additional structure such as a non-trivial finite group action and we wish to consider only manifold pairs $(N, M=\partial N)$ where the group action extends to $N$ even in the $4k+2$-case there may be non-trivial invariants.

The first step is to take account of the group action by defining an enriched bilinear form

$$\bar{b}\colon H^*(M^{2k};\mathbb{Q})\otimes H^*(M^{2k};\mathbb{Q})\longrightarrow\mathbb{Q}(G)$$

by setting $\bar{b}(a,b)=\sum_{g\in G} b(a,Tb)T^{-1}$. In particular, this enriched bilinear form has the following properties

(1) $\bar{b}(a, gc) = g\bar{b}(a, c)$
(2) $\bar{b}(ga, c) = g^{-1}\bar{b}(a, c)$

so that it is $\mathbb{Q}(G)$-linear in the second variable and conjugate $\mathbb{Q}(G)$-linear in the first variable. It is again non-singular in the sense that the adjoint homomorphism

$$\mu(\bar{b})\colon H^*(M^{2k};\mathbb{Q}) \longrightarrow Hom_{\mathbb{Q}(G)}(H^k(M^{2k};\mathbb{Q});\mathbb{Q}(G))$$

is an isomorphism, and it breaks $H^k(M^{2k};\mathbb{Q})$ into a direct sum of orthogonal summands, one for each simple summand of the semi-simple ring $\mathbb{Q}(G)$.

A basic case where this happens is when the group $G$ is abelian. Then $H^k(M^{2k};\mathbb{Q})$ is a module over the group ring $\mathbb{Q}(G)$ which is just a sum of copies of cyclotomic fields $\mathbb{Q}(\zeta_d)$, one for each distinct normal subgroup $N \triangleleft G$ with cyclic quotient. For example, if $G = \mathbb{Z}/2$ then there are two such summands, $\mathbb{Q}(G) = \mathbb{Q}_+ \oplus \mathbb{Q}_-$ and $H^k(M^{2k};\mathbb{Q})$ splits into two orthogonal pieces, the first the subspace of eigenvectors having eigenvalue 1 for the generator $T$ of $G$, and the second the subspace of eigenvectors for the eigenvalue $-1$. In particular, in this case if $k$ is even, the signature of $M^{2k}$ is the sum of two separate signatures $I(M^{2k}) = I^+(M^{2k}) + I^-(M^{2k})$ where the first is the signature of the form restricted to the $+1$-eigenspace and the second is the signature of the form restricted to the $-1$-eigenspace.

Similarly if $G = \mathbb{Z}/3$ then there are again two summands, one consisting of the eigenspace of the generator $T$ with eigenvalue 1 and the other the image of the idempotent projector

$$\frac{1}{3}(2 - T - T^2).$$

In this case the form on the second subspace may be regarded as a $\pm$-symmetric non-singular **Hermitian** form

$$\mathbb{Q}(\zeta_3)^m \times \mathbb{Q}(\zeta_3)^m \longrightarrow \mathbb{Q}(\zeta_3),$$

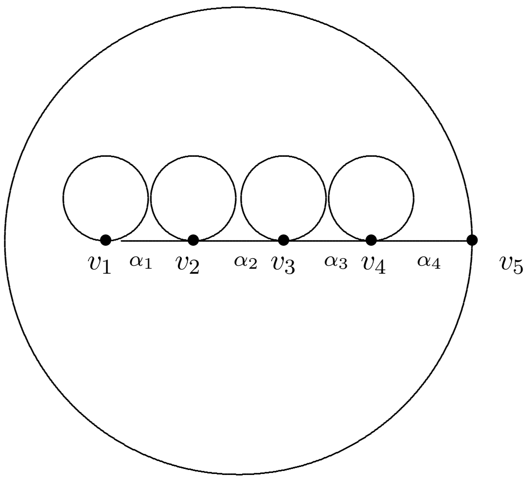
and as such can be diagonalized in both the cases $k = 2r$, $k = 2r + 1$, the only difference being that in the second case the diagonal elements have the property that $\bar{\lambda} = -\lambda$. But such elements exist in $\mathbb{Q}(\zeta_3)$, for example $\sqrt{-3} = \zeta_3 - \zeta_3^2$. Of course, for the off diagonal elements we also have $\lambda_{i,j} = -\bar{\lambda}_{j,i}$, but this is never a constraint.

In general $H^k(M^{2k};\mathbb{C})$ can be written as an orthogonal direct sum of the form $\bigoplus_{\pi_i} \mathbb{C}^{j_i}$ where the $i^{th}$ summand $\mathbb{C}^{j_i}$ is a module over the $i^{th}$ copy of $\mathbb{C}$ in $\mathbb{C}(G)$, and the form restricted to $\mathbb{C}^{j_i}$ is $\pm$-Hermitian and non-singular.

## 1. Some Two Dimensional Examples – the Fermat Surfaces

As a basic example let us consider the two dimensional manifold with boundary which consists of the closed disk $\bar{D}^2$ with $(n-1)$ small disjoint open disks removed. We give it a cellular decomposition by assuming the removed disks are all tangent to a diameter of $\bar{D}^2$ and join one edge of the diametor to the first, then the first to the second, and so on till the $(n-2)^{nd}$

is joined to the $(n-1)^{st}$. Consequently the vertices are the points of tangency and the first intersection of the diameter with the original boundary. The one cells are the arcs around the boundary components starting and ending at the intersection with the diameter as well as the arcs joining each to the next, and there is a single two cell with interior the complement of the one skeleton.

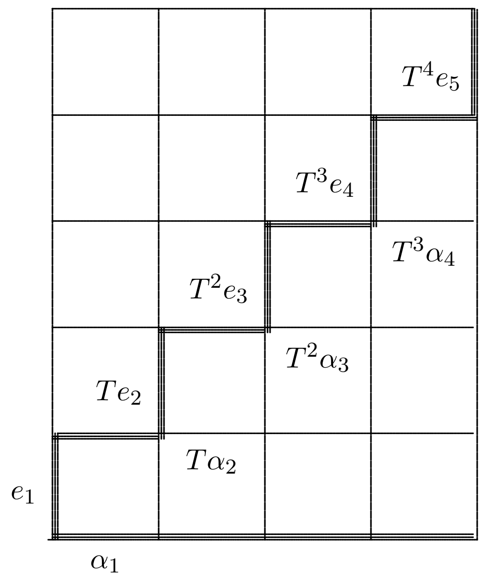



**The example for** $n = 5$

The fundamental group of this manifold is the free group on $(n-1)$-generators corresponding to the arcs in the positive sense around the $(n-1)$ deleted disks. Also, with respect to these generators, the arc in the opposite direction around the boundary of the original disk represents the class

$$(g_1 g_2 \cdots g_{n-1})^{-1}$$

in the fundamental group. Next, a homomorphism of this group into $\mathbb{Z}/n$ is given by sending each $g_i$ to a fixed generator $T$ of $\mathbb{Z}/n$. Consequently, the element on the outside boundary above also maps to $T$. It follows that the associated $\mathbb{Z}/n$-cover has one skeleton of the form




**The 1-Skeleton for $n = 5$**

where the only identification is that the top edge is identified with the bottom edge. The darker line represents the attaching of the generating 2-cell, the other two cells being attached by shifting the attaching map of the first cell by powers of $T$. Here, $T$ acts by shifting the graph up one level.

We now determine the homology of the resulting covering space which we denote $M_n$ and call the **free Fermat surface of level** $n$, a nomenclature explained by . Note that, since $\mathbb{Z}/n$ acts on this space the homology and cohomology groups become modules over the group ring $\mathbb{Z}(bbz/n)$, and we will describe the homology groups as modules.

LEMMA 15. *As a module over $\mathbb{Z}(\mathbb{Z}/n)$ we have that*

$$H_1(M_n; \mathbb{Z}) = \mathbb{Z}(\mathbb{Z}/n)^{n-2} \oplus \mathbb{Z}_+$$

*while $H_0(M_n; \mathbb{Z}) = \mathbb{Z}_+$ and all other homology groups are trivial. Here $\mathbb{Z}_+$ is $\mathbb{Z}$ with the trivial $\mathbb{Z}(\mathbb{Z}/n)$ action, $(\sum n_i T^i)\mathbf{k} = (\sum n_i)\mathbf{k}$ for any $\mathbf{k} \in \mathbb{Z}$).*

PROOF. Let $v_i$, $1 \leq i \leq n$, be the vertex on the $i^{th}$ boundary circle, and $\alpha_j$ $1 \leq j \leq n-1$ be the edje connecting $v_i$ and $v_{i+1}$. Also, let $e_i$, $1 \leq i \leq n$ be the arcs of the bounding circles, oriented so that, for $H$ the generating 2-cell, we have

$$\begin{aligned}\partial(H) &= e_1 + Te_2 + T^2e_3 + \cdots T^{n-1}e_n \\ &\quad + (T-1)\alpha_1 + (T^2-1)\alpha_2 + \cdots + (T^{n-1}-1)\alpha_{n-1}.\end{aligned}$$

We may thus assume that

$$\begin{array}{rcl} \partial(e_i) & = & (T-1)v_i \\ \partial(\alpha_i) & = & v_{i+1} - v_i, \end{array}$$

and this gives the complete structure of the chain complex form $M_n$:

$$\mathbb{Z}(\mathbb{Z}/n) \xrightarrow{\partial_2} \mathbb{Z}(\mathbb{Z}/n)^{2n-1} \xrightarrow{\partial_1} \mathbb{Z}(\mathbb{Z}/n)^n.$$

Clearly, using the $\alpha_i$ we can get rid of all all the $v_k$ but $v_1$. Then, replacing $e_i$ by

$$e_i - e_1 - (T-1)\sum_1^{i-1} \alpha_j = f_i$$

for $i > 1$, we have

$$\begin{array}{rcl} \partial(H) & = & e_1 + Tf_2 + T^2 f_3 + \cdots + T^{n-1} f_n \\ \partial(f_i) & = & 0 \\ \partial(e_1) & = & (T-1)v_1 \end{array}$$

so the role of $H$ is to suppress $f_n$, and the complex takes the form of the direct sum of the trivial complex $\mathbb{Z}(\mathbb{Z}/n)^{n-2}$ in dimension one with the complex

$$\mathbb{Z}(\mathbb{Z}/n) \xrightarrow{T-1} \mathbb{Z}(\mathbb{Z}/n).$$

But this is just the complex of the circle with the free action of $\mathbb{Z}/n$ where the action is rotation by $2\pi/n$-degrees, and the lemma follows. □

REMARK . Consider the Fermat variety $F_n \subset \mathbb{P}^2$ given as the set of zeros of the homogeneous polynomial in three variables $x^n + y^n - z^n = 0$. The points in the intersection of $F_n$ with the at the $\mathbb{P}^1$ at infinity given by $z = 0$,

$$F_n^\infty = F_n \cap \mathbb{P}^1(z=0)$$

are solutions of

$$x^n + y^n = \prod_{j=0}^{n-1} (x + \zeta_n^j y) = 0$$

and hence there are exactly $n$ points at this infinity. In the complement we can normalize by taking $x/z$, $y/z$ as affine coordinates so the equation becomes $X^n + Y^n = 1$, and we can project $F_n - F_n^\infty$ to the complex plane by the projection $(X, Y) \mapsto X$ for each pair $(X, Y) \in F_n - F_n^\infty$. The inverse image of $X$, with $X^n \neq 1$ consists of exactly $n$ points and the group $\mathbb{Z}/n$ acting by $(x, y, z) \mapsto (x, \zeta_n y, z)$ acts freely in the complement of the $n$ points $(0, 1, \zeta_n^i)$ while it fixes each of these $n$ points. If we delete small disk neighborhoods of these $n$ points we obtain the space $M_n$ as well as the action of $\mathbb{Z}/n$ given above.

**The intersection pairing for the Fermat Surfaces.** By duality the relative homology groups

$$H_j(M_n, \partial(M_n); \mathbb{Z}) \;=\; \begin{cases} \mathbb{Z}_+ & \text{in dimension 2} \\ \mathbb{Z}(\mathbb{Z}/n)^{n-2} \oplus \mathbb{Z} & \text{in dimension 1} \\ 0 & \text{in dimension 0} \end{cases}$$

and the map $H_1(M_n; \mathbb{Z}) \longrightarrow H_1(M_n, \partial(M_n); \mathbb{Z})$ must have kernel exactly

$$\mathbb{Z}_+ \oplus (N_n\mathbb{Z}(\mathbb{Z}/n))^{n-2} \cong \mathbb{Z}_+^{n-1}$$

with image $(\mathbb{Z}(\mathbb{Z}/n)/(N\mathbb{Z}(\mathbb{Z}/n)))^{n-2}$.

Let $R_n = \mathbb{Q}(\mathbb{Z}/n)/(N)$. Then, with $R_n$ coefficients, we have an isomorphism

$$j_* \colon H_1(M_n; R_n) \xrightarrow{\cong} H_1(M_n, \partial(M_n); r_n),$$

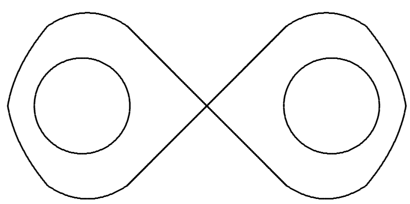
and consequently a non-singular skew-symmetric pairing

$$B \colon R_n^{n-2} \times R_n^{n-2} \longrightarrow R_n.$$

Lemma 16. *There is a basis for* $H_1(M_n; R_n)$, $f_1, \ldots, f_{n-2}$ *so that the pairing* $B$ *of 1 is given by the tri-diagonal Toeplitz matrix*

$$\begin{pmatrix} T - T^{-1} & 1 - T^{-1} & 0 & 0 & \ldots & 0 \\ T - 1 & T - T^{-1} & 1 - T^{-1} & 0 & \ldots & 0 \\ 0 & T - 1 & T - T^{-1} & 1 - T^{-1} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 - T^{-1} \\ 0 & 0 & 0 & 0 & \ldots & T - T^{-1} \end{pmatrix}.$$

Proof. The $f_i$ given in 1 are the generators we require. As it stands they are given as boundary components in $M_n$. However, they can be systematically deformed to lie in the interior as follows:



**The image of the deformed curve in the quotient** $M_n/(\mathbb{Z}/n)$

Then the self-intersection corresponds to the intersection of the deformed curve (which is embedded in $M_n$) with its translates by $T$ and $T^{-1}$. Similarly, shifting this deformed curve to the right we get the deformed $f_{i+1}$ which clearly intersects the deformed $f_i$ in two points. Again, when we lift this corresponds to single point intersections with $f_{i+1}$ and $T^{-1}f_{i+1}$. □

Note that $(T+1)(1-T^{-1}) = T - T^{-1}$, $(T^{-1}+1)(T-1) = T - T^{-1}$. Also, note that $T - T^{-1}$ is invertible in $R_n$ for $n$ odd, since

$$R_n = \coprod_{\substack{d|n \\ d\neq 1}} \mathbb{Q}(\zeta_d)$$

and $\zeta_d - \zeta_d^{-1} \neq 0$ unless $\zeta_d = \pm 1$ and $-1$ is not an $n^{th}$ root of unity for $n$ odd. Consequently, in this case we can divide by $T - T^{-1}$ and the matrix $B$ of 16 takes the form

$$(T - T^{-1}) \begin{pmatrix} 1 & T+1 & 0 & \dots & 0 \\ T^{-1}+1 & 1 & T+1 & \dots & 0 \\ 0 & T^{-1}+1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

where the matrix $C$ in 1 is Hermitian symmetric, *not Hermitian skew-symmetric*, and the multisignature invariants will be determined by the signatures of the various embeddings of $C$ into $M_{n-2}(\mathbb{C})$ obtained from the distinct homomorphisms $R_n \to \mathbb{C}$.

## 2. The Multisignature for Finite Group Actions

In the previous section we saw how to define a non-trivial bilinear form on $H^n(M;\mathbb{C})$ in the case where the manifold structure on $M^{2n}$ is augmented by the action of a finite group, $G$. The form splits as an orthogonal direct sum of $(-1)^n$-symmetric Hermitian forms, one for each irreducible complex representation of $G$. We associate to each such form and irreducible complex representation two integers, first the dimension of the direct summand corresponding to the irreducible representation, and second the generalized signature of the form. This is the ordinary signature of the Hermitian form if $n$ is even; the number of positive eigenvalues of the matrix minus the number of negative eigenvalues, and is the signature of $i$ times the Hermitian form in case $n$ is odd.

We may define a geometric cobordism group of manifolds with $G$-actions by making two oriented manifolds $M^n$ and $\bar{M}^n$ with $G$-actions $G \times M \xrightarrow{\mu_1} M$, $G \times \bar{M} \xrightarrow{\mu} \bar{M}$, equivalent if there is an oriented $n+1$ dimensional manifold $W$ with $G$-action $G \times W \xrightarrow{\eta} W$ which defines an ordinary cobordism between $M$ and $\bar{M}$ and so that the $G$-action on $W$ when restricted to $M$ or $\bar{M}$ is the $G$-action.

Under these conditions an easy extension of the arguments which showed the cobordism invariance of the ordinary signature give the cobordism invariance of these new signature invariants.

Definition 16. *Given $M^{2n}$ together with a $G$-action, $G \times M \xrightarrow{\mu} M$, for $G$ a finite group, then the signature $\sigma_R(M)$ at the irreducible representation*

*$R$ of $G$ over $\mathbb{C}$ is the number of positive eigenvalues minus the number of negative eigenvalues of the form above at this representation. Also the multisignature of $M^{2n}$ is given as the sum*

$$\sum_{R_i} \sigma_{R_i}(M) R_i$$

*taken over all the irreducible representations of $G$.*

**The Multisignatures for the Fermat Surfaces.** Given an $m \times m$ Hermitian symmetric matrix one method of determining the signature is to count the sign changes of the diagonal minors, $D_1$, $D_2$, …, $D_n$, where $D_i$ is the determinant of the $i \times i$-minor consisting of the first $i$ rows intersected with the first $i$ columns.

The various homomorphisms $\mathbb{Z}/n \rightarrow \mathbb{C}^*$ are given by sending $T$ to $\zeta_n^i$ for $i = 0, 1, \ldots, n-1$. Now we evaluate these determinants and the resulting signatures. Note that

$$D_{i+1} = D_i - (T+1)(T^{-1}+1)D_{i-1}$$

gives a recursive evaluation of the $D_i$, and write

$$\begin{aligned} (T+1)(T^{-1}+1) &= T + T^{-1} + 2 \\ &= (T^{n+1/2} + T^{n-1/2})^2 \end{aligned}$$

for $n$ odd. Each $D_i$ is recursively given as a polynomial in $\lambda = (T + T^{-1} + 2)$ which we write $D_i(\lambda)$ and we have

$$\begin{aligned} D_1(\lambda) &= 1 \\ D_2(\lambda) &= 1 - \lambda, \\ D_3(\lambda) &= 1 - 2\lambda, \\ D_4(\lambda) &= 1 - 3\lambda + \lambda^2, \end{aligned}$$

in the first few cases. Note that the root of $D_2(\lambda)$ is $1 = 2cos(2\pi/3)$, the roots of $D_4(\lambda)$ are $(\zeta_5 + \zeta_5^{-1})^2$ and $(\zeta_5^2 + \zeta_5^3)^2$. In fact we have

LEMMA 17. *The roots of $D_{2n}(\lambda)$ are exactly the real numbers $(\zeta_{2n+1}^k + \zeta_{2n+1}^{-k})^{-2}$. Moreover, the roots of $D_{2n+1}(\lambda)$ are the inverses of the* **non-zero** *numbers of the form $(\zeta_{2n+2}^k + \zeta_{2n+1}^{-k})^2$.*

PROOF. Define the generating function

$$F(\lambda, w) = 1 + w + \sum_{i=2}^{\infty} D_i(\lambda) w^i.$$

Consequently, by the recursion rule for the $D_i(\lambda)$, we have

$$1 = F(\lambda, w) - wF(\lambda, w) + \lambda w^2 F(\lambda, w)$$

so

$$F(\lambda, w) = \frac{1}{(\lambda w^2 - w + 1)}$$

and

$$
\begin{aligned}
F(\lambda, \lambda^{-1/2}\bar{w}) &= \tfrac{1}{1-\tau\bar{w}+\bar{w}^2} \\
&= \left(\tfrac{1}{r_1-r_2}\right)\left(\tfrac{1}{\bar{w}-r_1} - \tfrac{1}{\bar{w}-r_2}\right) \\
&= \tfrac{1}{r_1-r_2}\sum_{i=0}^{\infty}\left(r_1 r_2^{-i} - r_2 r_1^{-i}\right)\bar{w}^i \\
&= \tfrac{r_1 r_2}{r_1-r_2}\sum_{i=0}^{\infty}\left(r_2^{-(i+1)} - r_1^{-(i+1)}\right)\bar{w}^i
\end{aligned}
$$

where $\tau = \lambda^{-1/2}$, while $r_1$ and $r_2$ are the roots of $\bar{w}^2 - \tau\bar{w} + 1$. Hence, comparing coefficients we have

$$D_i(\lambda) = \frac{(\lambda)^{i/2}}{r_1 - r_2}\left(r_2^{-(i+1)} - r_2^{-(i+1)}\right).$$

But the roots of $1 - \tau\bar{w} + \bar{w}^2$ are $r$, $r^{-1}$ where $r + r^{-1} = \tau$. Hence, the only time that this is sure to be zero is when $r_1 = \zeta_{i+1}^j$ and $r_2 = \zeta^{-1}$.

There are also two cases where the possibility of extraneous roots occurs, $r = 0$ and $r = 1$ (where $r_1 - r_2 = 0$). So we do a separate analysis of these two cases. For $\lambda = 0$ all of the $D_i(1) = 1$, so no root occurs. For $\lambda = 1$ we see directly from the recursion formula that

$$F(1, w) \;=\; 1 + w - w^3 - w^4 + w_6 + w_7 - w^9 - \cdots$$

so the only zeros occur for $D_{3i-1}$ which agrees with the formula. This completes the proof. □

## 3. The Wall Embedding Theorem

The embedded curve in the $\mathbb{Z}/n$-cover of the Fermat curve in (1) has image curve with a single double point in the Fermat curve itself. Moreover, we will see now that no curve in the $\mathbb{Z}/n$-cover in the homology class of the curve of (1) will have image which imbeds into the Fermat curve. This will be a consequence of C.T.C. Wall's extension of the Whitney $2n$-embedding theorem to the case of non-simply connected manifolds.

Let $N^n$ be any compact, closed, and simply connected $n$-manifold, and suppose $f\colon N^n \to M^{2n}$ is any map, where $M^{2n}$ and $M^{2n}$ is not necessarily compact, closed, simply connected, or oriented. Then $f$ lifts to a map into the universal cover of $M^{2n}$ since $N^n$ is simply connected:

$$
\begin{array}{ccc}
N^n & \xrightarrow{\tilde{f}} & \tilde{M}^{2n} \\
\big\downarrow{=} & & \big\downarrow{p} \\
N^n & \xrightarrow{f} & M^{2n}
\end{array}
$$

where $\tilde{M}^{2n} \xrightarrow{p} M^{2n}$ is the universal covering. Of course, $\tilde{f}$ is not unique, but any two liftings differ by translation, using one of the covering transformations $g\colon \tilde{M}^{2n} \to \tilde{M}^{2n}$ with $g \in \pi_1(M^{2n})$. Hence, $\tilde{f}$ is well defined if we choose a particular point $\tilde{n}_0$ in the fiber over $f(n_0) \in f(N)$ and set $\tilde{f}(n_0) = \tilde{n}_0$.

Since $\tilde{M}$ is $2n$-dimensional and simply connected it follows that we can deform $\tilde{f}$ to a $C^\infty$ embedding $g$. Moreover, since the tangent map

$$d(p)\colon \tau(\tilde{M})\longrightarrow\tau(M)$$

is an isomorphism at each point of $\tilde{M}$ it follows that the composite $pg$ is an immersion. However, as pointed out above, it may well be the case that this immersion is not an embedding into $M^{2n}$.

EXAMPLE 22. *Consider the map $2\colon S^1\longrightarrow S^1$, $2(\zeta) = \zeta^2$. (2) lifts to an embedding of $S^1$ into the two fold cover of $S^1$ (indeed, it is the 2-fold cover), but can never be represented by an embedding.*

LEMMA 18. *For $n \geq 2$ we have that arbitrarily close to $g$ is an embedding so that the only singularities of $pg$ are isolated transverse double points.*

PROOF. Arbitrarily close to $pg$ is an immsersion with only transverse isolated double points as its singularities. Moreover, the lift of such a map, properly based, is arbitrarily close to $g$, and differentiable maps sufficiently close to an embedding are again embeddings. □

COROLLARY 12. *Suppose given an embedding $g\colon N \hookrightarrow \tilde{M}$ so that $pg$ has transverse double points as its only singularities. Then, for $h \in \pi_1(M)$, if $x \in hg(N) \cap g(N)$, $g(N)$ and $hg(N)$ are transverse at $x$. In particular, there are only a finite number of such $x$ for each $h$ and only a finite number of $h$ so that the intersection $g(M) \cap hg(M) \neq \emptyset$.*

PROOF. The double points in $pg(N)$ are the images of the intersections of $g(N)$ with $h(N)$ as $h$ runs over $\pi_1(M)$. Consequently the result follows from the compactness of $N$. □

We now define a twisted intersection form for the embedding $g$, compare (4):

$$\psi\colon [g]\longrightarrow \sum\langle hg(N), g(N)\rangle h^{-1}$$

as $h$ runs over the non-identity elements of $\pi_1(M)$, and $\langle hg(N), g(N)\rangle$ is the sum over the points of intersection of the index $I(h(x), y)$ defined in the proof of the Whitney $2n$-embedding theorem.

By symmetry (when $M$ is oriented) we have

$$\langle h^{-1}g(N), g(N)\rangle = (-1)^n\langle hg(N), g(N)\rangle,$$

while in the case where $M$ is non-oriented we have

$$\langle h^{-1}g(N), g(N)\rangle = (-1)^n(-1)^h\langle hg(N), g(N)\rangle$$

where $(-1)^h$ is $+1$ if $h$ is orientation preserving and is $-1$ otherwise.

Note also that when $T \in \pi_1(M)$ satisfies $T^2 = 1$, then if $g(x) = Tg(y) \in \mathrm{im}(g(N))$ we have that $g(y) = Tg(x)$ as well, and there are an even number of intersections in this case. In particular, when $n$ is even and $T$ is orientation preserving or $n$ is odd and $T$ is orientation reversing, then the coefficient of $T$ in $\psi(g)$ is even, while in the remaining two cases it is zero.

It follows that we can (sort of) halve $\psi(g)$. Specifically, pick, for each pair $\{h, h^{-1}\}$ where $h,\ h^{-1} \in \pi_1(M)$ and $h \neq h^{-1}$, one of $h,\ h^{-1}$. Call the chosen element $\bar{h}$. Also, for $h$ with $h^2 = 1$, $h \neq 1$ choose for each set of ordered pairs of points $(x, y)$ and $(y, x)$ with $hg(y) = g(x)$, $hg(x) = g(y)$, one of the two, say $(y, x)$, and associate to $h$ the sum of the signed intersections, $I(hg(y), g(x))$, as we run over the selected pairs. Then this sum is well defined in the cases where $n$ is even and $T$ is orientation preserving or $n$ is odd and $T$ is orientation reversing, while it is only well defined mod(2) in the remaining cases. Thus, we set

$$\hat{\psi}(g) \ = \ \sum_{h, h^2 \neq 1} \langle \bar{h}^{-1} g(N), g(N) \rangle \bar{h} + \sum_{h, h^2 = 1, h \neq 1} J(h, g) h$$

where $J(h, g)$ is the sum desribed above with values either in $\mathbb{Z}$ or $\mathbb{Z}/2$, depending on $n$, and the orientation properties of $h$.

CHAPTER 5

# The Homology of Fibre Bundles

## 1. Four homology theories

There are 4 distinct ways of constructing homology and cohomology:

(1) The singular homology groups $H_*(X)$ of a topological space $X$ are the homology of the singular chain complex $C = C(X)$

$$H_i(X) \;=\; H_i(C) \;=\; \ker(d : C_i \to C_{i-1})/\mathrm{im}(d : C_{i+1} \to C_i) \;,$$

with $C_i$ the free abelian group generated by all the continuous maps $\sigma : \Delta^i \to X$ from the standard $i$-simplex $\Delta^i$. The differentials are given by

$$d \;:\; C_i \longrightarrow C_{i-1} \;;\; \sigma \longrightarrow \sum_{j=0}^{i} (-1)^j \sigma \partial_j$$

with $\partial_j : \Delta^{i-1} \to \Delta^i$ the $j$th face inclusion. The singular cohomology groups are defined by

$$H^i(X) \;=\; H^i(C) \;=\; \ker(d^* : C^i \to C^{i+1})/\mathrm{im}(d^* : C^{i-1} \to C^i) \;,$$

with

$$C^i \;=\; C_i^* \;=\; \mathrm{Hom}_{\mathbb{Z}}(C_i, \mathbb{Z}) \;,\; d^*(f) \;=\; fd \;.$$

(2) The simplicial homology groups of an ordered simplicial complex

$$X \;=\; \bigcup_{i=0}^{\infty} \bigcup \Delta^i$$

are the homology groups $H_*(X) = H_*(C)$ of the simplicial chain complex $C$, with $C_i$ the free abelian group generated by the $i$-simplexes $\Delta^i$ and the differentials given by

$$d \;:\; C_i \longrightarrow C_{i-1} \;;\; (v_0 v_1 \dots v_i) \longrightarrow \sum_{j=0}^{i} (-1)^j (v_0 \dots v_{j-1} v_j \dots v_i)$$

with $(v_0 v_1 \dots v_i)$ the $i$-simplex spanned by the vertices $v_0, v_1, \dots, v_i$. The cohomology groups are then defined by $H^*(X) = H^*(C)$ as in (i).

(3) The cellular homology of a $CW$ complex

$$X = \bigcup_{i=0}^{\infty} \bigcup D^i$$

is defined to be the homology of the cellular chain complex $C$, with

$$C_i = H_i(X^{(i)}, X^{(i-1)})$$

the free abelian group generated by all the $i$-cells $D^i$ and

$$d : C_i = H_i(X^{(i)}, X^{(i-1)}) \to C_{i-1} = H_{i-1}(X^{(i-1)}, X^{(i-2)})$$

the boundary in the homology of the triple $(X^{(i)}, X^{(i-1)}, X^{(i-2)})$, with $X^{(i)}$ the union of all the cells of dimension $\leq i$. As in (i) and (ii) the cohomology groups are defined by $H^*(X) = H^*(C)$.

(4) The deRham cohomology groups $H^*(M;\mathbb{R})$ of a differentiable manifold $M$ are defined using the cochain complex $\Omega^*(M)$

$$\begin{aligned} H^i(M;\mathbb{R}) &= H^i(\Omega^*(M)) \\ &= \ker(d : \Omega^i(M) \to \Omega^{i+1}(M))/\mathrm{im}(d : \Omega^{i-1}(M) \to \Omega^i(M)) , \end{aligned}$$

with $\Omega^i(M)$ the real vector space with one basis element for each $i$-form. An $i$-form is a section $\omega : M \to \Lambda^i(M)$ of the bundle $\Lambda^i(M)$ over $M$ with

$$\begin{aligned} \Lambda^i(M)(x) &= \Lambda^i(\tau_M(x)^*) \\ &= \{\text{alternating linear maps} : \bigotimes_i \tau_M(x) \to \mathbb{R}\} \ (x \in M) . \end{aligned}$$

The differentials

$$d : \Omega^i(M) \to \Omega^{i+1}(M) ; \ \omega \to d\omega$$

are defined by the exterior derivative, given locally by

$$\begin{aligned} d\omega_x(h_1 \otimes h_2 \otimes & \cdots \otimes h_{i+1}) \\ &= \sum_{j=0}^{i} (-1)^j \omega_x(h_1 \otimes \cdots \otimes h_{j-1} \otimes h_{j+1} \otimes \cdots \otimes h_{i+1}) . \end{aligned}$$

The homology groups are obtained from the deRham cohomology groups by applying the universal coefficient theorem, with

$$H_i(M;\mathbb{R}) = H^i(M;\mathbb{R})^* .$$

Remark . The various homology theories have their own advantages and disadvantages: (i) is topologically invariant, but is rather removed from geometry; (ii) requires a triangulation, but is combinatorially invariant; (iii) requires a $CW$ structure; (iv) only applies to differentiable manifolds, with $\mathbb{R}$-coefficients only. In any case, (i), (ii), (iii) and (iv) agree whenever they are defined on the same space and with the same coefficients. The homology and cohomology groups $H_*(X)$, $H^*(X)$ are finitely generated for a finite simplicial (or $CW$) complex $X$, such as a triangulation of a compact manifold.

However, in the main body of this work we shall have to consider extensively the relations between the homology and cohomology of a space $X$ and of various covers of $X$. So, to begin, we must discuss this topic in some detail.

## 2. The homology and cohomology of a covering

Given a group $\pi$ let $\mathbb{Z}[\pi]$ be the group ring, with elements finite linear combinations

$$\sum_{g\in\pi} n_g g \ (n_g \in \mathbb{Z})$$

such that $\{g \in \pi \mid n_g \neq 0\}$ is finite.

Let $X$ be a $CW$ complex, and let $\widetilde{X} \xrightarrow{p} X$ be a regular covering of $X$ with group of covering translations $\pi$. The cover restricted to the interior of any cell is just a product cover $p^{-1}\dot{e}^k = \dot{e}^k \times \pi$, that the cellular chain complex of $\widetilde{X}$ satisfies

$$C_k(\widetilde{X}) \ = \ H_k(\widetilde{X}^{(k)}, \widetilde{X}^{(k-1)}) \ = \ \mathbb{Z}[\pi] \otimes_{\mathbb{Z}} C_k(X)$$

for each $k$, being the free $\mathbb{Z}[\pi]$-module generated by the $k$-cells of $X$. The cellular boundary map in the cover,

$$C_k(\widetilde{X}) \xrightarrow{\tilde{\partial}} C_{k-1}(\widetilde{X})$$

is

1. $\mathbb{Z}[\pi]$ equivariant, $(\tilde{\partial}((\sum_i n_{i,j} g_{i,j} e^k_j) = \sum_{i,j} n_{i,j} g_{i,j} \tilde{\partial}(e^k_j))$,
2. on tensoring, over $\mathbb{Z}[\pi]$ with $\mathbb{Z}$, so $(\sum_i n_i g_i) e^k_j$ in $C_k(\widetilde{X})$ is identified with $\sum n_i e^k_j$ in $C_k(X)$, that $\tilde{\partial}$ goes to $\partial$.
3. In particular, multiplication by $g \in \pi$ induces a chain equivalence of $C_{\#}(\widetilde{X})$.

Given any right $\mathbb{Z}[\pi]$-module $\mathcal{M}$ the usual homology groups with twisted coefficients in $\mathcal{M}$ $H^{\pi}_*(X;\mathcal{M})$ are defined as the homology groups of the complex $\mathcal{M} \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X})$. Given any left $\mathbb{Z}[\pi]$-module $\mathcal{N}$, the **cohomology groups with twisted coefficients in** $\mathcal{N}$ $H^*_\pi(X;\mathcal{N})$ are the homology groups of the complex

$$Hom_{\mathbb{Z}[\pi]}(C_*(X);\mathcal{N}) \ .$$

EXAMPLE 23. Let $\mathcal{A}$ be any abelian group. Give it the structure of a $\mathbb{Z}[\pi]$-module via the rule $(\sum n_g g)(k) = (\sum n_g)k$. This is the **trivial** $\mathbb{Z}[\pi]$ action on $\mathcal{A}$. Then $H^*_\pi(X;\mathcal{A}) = H^*(X;\mathcal{A})$, the ordinary cohomology of $X$ with $\mathcal{A}$-coefficients.

EXAMPLE 24. Let $F \to E \to X$ be any Serre fibration. Every element $\alpha \in \pi_1(X)$ is represented by a based map $f\colon (S^1, 1) \to (X, *)$, with an induced fibration over $S^1$, $F \to f^!(E) \xrightarrow{p} S^1$. Lift the map

$$F \times I \xrightarrow{p_2} I \xrightarrow{j} S^1$$

to $f^!(E)$ by extending the identity lifting over $F \times 0$, where $j(t) = e^{2\pi i t}$. This gives a homotopy equivalence $l_1(f)\colon F \to F$ which up to homotopy depends only on $\alpha$, with $f^!E$ homotopy equivalent to the mapping torus

$$T(l_1(f)) \;=\; F \times I/\{(x,0) = (l_1(f)(x),1) \,|\, x \in F\}\ .$$

As $\alpha$ runs over the elements in $\pi_1(X)$ the automorphisms $l_1(f)_* : H_*(F;\mathcal{A}) \to H_*(F;\mathcal{A})$ define an action

$$\mathbb{Z}[\pi_1(X)] \otimes_{\mathbb{Z}} H_*(F;\mathcal{A}) \longrightarrow H_*(F;\mathcal{A})\ .$$

Then the $i^{th}$ twisted homology group of $X$ with coefficients in $H_j(F;\mathcal{A})$ is $H_i^{\pi_1(X)}(X;H_j(F;\mathcal{A}))$.

Example 25. Let $\pi = \mathbb{Z}/2$ and define an action of $\pi$ on the abelian group $\mathcal{A}$ by the rule $(m1 + nT)a = (m-n)a$. The group $\mathcal{A}$ with this action is denoted $\mathcal{A}_-$, and $H_*^{\mathbb{Z}/2}(X;\mathcal{A}_-)$ occurs very frequently in applications.

More generally, let $f\colon \pi \to \mathbb{Z}/2$ be a non-trivial homomorphism. (The set of all such distinct homomorphisms is in one to one correspondence with the non-zero elements of $Hom(\pi/\pi', \mathbb{Z}/2)$.) Then $\mathcal{A}_{(f,-)}$ is the $\mathbb{Z}[\pi]$-module induced by $f$ from $\mathcal{A}_-$. $((\sum n_g g)a = \sum n_g f(g)a.)$

The action of $\pi$ on $\widetilde{X}$ also induces actions on the ordinary homology and cohomology of $\widetilde{X}$.

The action of $\pi$ on $\widetilde{X}$ induces an action in homology by the evident rule

$$\mathbb{Z}[\pi] \times H_*(\widetilde{X}) \longrightarrow H_*(\widetilde{X})\ ;\ (g,x) \longrightarrow g_*x\ ,$$

so that the homology groups $H_*(\widetilde{X})$ are (left) $\mathbb{Z}[\pi]$-modules. The ordinary cohomology groups $H^*(\widetilde{X})$ are $\mathbb{Z}[\pi]$-modules via

$$\mathbb{Z}[\pi] \times H^*(\widetilde{X}) \longrightarrow H^*(\widetilde{X})\ ;\ (g,x) \longrightarrow (g^{-1})^*x\ .$$

Remark . Actually, this action is just a special case of the action in 24 since, when we convert the classifying map $X \to B(\pi_1(X))$ of the universal covering space of $X$ into a Serre fibration the fibre is $\widetilde{X}$ and the action of $\pi_1(X)$ on $H_*(\widetilde{X})$ is exactly the action of $\pi_1(B(\pi_1(X))) = \pi_1(X)$ on the fiber.

**Anti-automorphisms, involutions, and actions on cohomology.** The situation for cohomology requires more discussion. In general, if $\widetilde{X}$ is a covering of $X$ with the group of covering translations $\pi$ acting on the left, and $\mathcal{A}$ is an abelian group, then the cochain complex with

$$C^i(\widetilde{X},\mathcal{A}) = Hom_{\mathbb{Z}}(C_i(\widetilde{X}),\mathcal{A})$$

becomes a right $\mathbb{Z}[\pi]$-module under the action $fg(x) = f(gx)$.

Definition 17. *An* **anti-automorphism** *of* $\mathbb{Z}[\pi]$ *is a map*

$$\chi\colon \mathbb{Z}[\pi] \to \mathbb{Z}[\pi]$$

*that satisfies the following three conditions:*

- $\chi(\alpha\beta) = \chi(\beta)\chi(\alpha)$,

- $\chi(\alpha+\beta) = \chi(\alpha) + \chi(\beta)$,
- $\chi(n1) = n1$.

Given an anti-automorphism $\chi$, we can convert $C^i(\widetilde{X}, K)$ into a left $\mathbb{Z}[\pi]$-module by setting $g(f) = f\chi(g)$. Thus, for each anti-automorphism, there is a distinct left action of $\mathbb{Z}[\pi]$ on $H^*(\widetilde{X}, K)$.

Perhaps the simplest anti-automorphism of the group ring $\mathbb{Z}[\pi]$ is the one given by $\sum n_g g \leftrightarrow \sum n_g g^{-1}$. This is an example of a special type of anti-automorphism – an involution.

DEFINITION 18. *An* **involution** *on a ring $A$ is a function*

$$A \longrightarrow A \; ; \; a \longrightarrow \overline{a}$$

*satisfying*

$$\overline{(a+b)} \;=\; \overline{a} + \overline{b} \; , \; \overline{(ab)} \;=\; \overline{b}\,.\,\overline{a} \; , \; \overline{\overline{a}} \;=\; a \; , \; \overline{1} \;=\; 1 \in A \; (a, b \in A) \; .$$

In practice, the types of anti-automorphisms which actually occur are involutions.

EXAMPLE 26. (1) A commutative ring $A$ admits the identity involution $\overline{a} = a$.
(2) Complex conjugation defines an involution on the ring of complex numbers $\mathbb{C}$

$$\mathbb{C} \longrightarrow \mathbb{C} \; ; \; z \;=\; a + ib \longrightarrow \overline{z} \;=\; a - ib \; .$$

In the topological applications $A = \mathbb{Z}[\pi]$ with the involution given via a slight twisting of the involution $g \leftrightarrow g^{-1}$ as follows.

DEFINITION 19. *An* **orientation character** *on a group $\pi$ is a group morphism*

$$w \; : \; \pi \longrightarrow \mathbb{Z}_2 \;=\; \{\pm 1\} \; .$$

*The $w$-***twisted involution** *on the group ring $\mathbb{Z}[\pi]$ is given by*

$$\mathbb{Z}[\pi] \longrightarrow \mathbb{Z}[\pi] \; ; \; a \;=\; \sum_{g\in\pi} n_g g \longrightarrow \overline{a} \;=\; \sum_{g \in \pi} n_g w(g) g^{-1} \; (n_g \in \mathbb{Z}) \; .$$

*In the untwisted case $w(g) = +1$ ($g \in \pi$) this is the* **oriented involution** *on $\mathbb{Z}[\pi]$.*

DEFINITION 20. *Let $A$ be a ring with involution.*

(1) *The* **dual** *of a left $A$-module $K$ is the left $A$-module*

$$K^* \;=\; \mathrm{Hom}_A(K, A) \; ,$$

*with $A$ acting by*

$$A \times K^* \longrightarrow K^* \; ; \; (a, f) \longrightarrow (x \longrightarrow f(x).\overline{a}) \; .$$

(2) *The* **dual** *of an $A$-module morphism $f : K \to L$ is the $A$-module morphism, $f^*$*

$$f^* \; : \; L^* \longrightarrow K^* \; ; \; g \longrightarrow (x \longrightarrow g(f(x))) \; .$$

Thus duality is a contravariant functor

$$* \; : \; \{\text{left } A\text{-modules}\} \longrightarrow \{\text{left } A\text{-modules}\} \; ; \; K \longrightarrow K^* \; .$$

From now on we shall be mainly concerned with left $A$-modules, so "$A$-module" will mean "left $A$-module", unless it is specified as a "right $A$-module".

Remark . In case $A = \mathbb{Z}[\pi]$ and $K$ is a finitely generated free $A$-module, then, with the definition of $K^*$ given in 20, $K^*$ is also a finitely generated, free $A$-module. In particular, it is distinct from the usual dual $Hom_{\mathbb{Z}}(K, \mathbb{Z})$.

Suppose that $X$ is a compact manifold with boundary. Then, $\widetilde{X}$ is also a manifold, but for $\pi$ of infinite order it will not be compact. Consequently, for finite $\pi$ the ordinary cohomology $H^*(\widetilde{X})$ is adequate for Poincaré duality, since then $\widetilde{X}$ is compact. However, for infinite $\pi$ it is the compactly supported cohomology $H^*_{\text{cpt}}(\widetilde{X})$ which must be used to restore Poincaré duality.

Using an involution on $\mathbb{Z}[\pi]$ and cellular chain complexes it is in fact possible to give a uniform treatment of the ordinary cohomology for finite $\pi$ and the compactly supported cohomology for infinite $\pi$, as follows.

Definition 21. *Given a ring with involution $A$ and an $A$-module chain complex*

$$C \; : \; \dots \longrightarrow C_{r+1} \overset{d}{\longrightarrow} C_r \overset{d}{\longrightarrow} C_{r-1} \longrightarrow \dots$$

*write the dual $A$-modules as*

$$C^r \; = \; (C_r)^* \; (r \in \mathbb{Z}) \; .$$

*The* **cohomology** *$A$-modules of $C$ are defined by*

$$H^r(C) \; = \; \ker(d^* : C^r \to C^{r+1})/\text{im}(d^* : C^{r-1} \to C^r) \; (r \in \mathbb{Z}) \; .$$

The functions

$$H^r(C) \longrightarrow \text{Hom}_A(H_r(C), A) \; ; \; f \longrightarrow (x \longrightarrow f(x))$$

are $A$-module morphisms.

Definition 22. *For any $m \in \mathbb{Z}$ define the* $m$**-dual** *of an $A$-module chain complex $C$ to be the $A$-module chain complex $C^{m-*}$ with*

$$d_{C^{m-*}} \; = \; (-)^r (d_C)^* \; : \; (C^{m-*})_r \; = \; C^{m-r} \longrightarrow (C^{m-*})_{r-1} \; = \; C^{m-r+1} \; .$$

The $m$-dual chain complex is such that

$$H_r(C^{m-*}) \; = \; H^{m-r}(C) \; (r \in \mathbb{Z}) \; .$$

In §3 it will be shown that the cellular chain complex $C(\widetilde{M})$ of an oriented cover $\widetilde{M}$ of a closed $m$-dimensional manifold $M$ is chain equivalent to the $m$-dual $C(\widetilde{M})^{m-*}$, with respect to an appropriate involution on the group ring $\mathbb{Z}[\pi]$ of the group of covering translations $\pi$.

For any connected $CW$ complex $X$ there is a natural isomorphism

$$\begin{aligned} H^1(X;\mathbb{Z}_2) \ = \ & [X,\mathbb{RP}^\infty] \xrightarrow{\cong} \mathrm{Hom}(\pi_1(X),\mathbb{Z}_2) \ ; \\ & (w : X \to \mathbb{RP}^\infty) \to (w_* : \pi_1(X) \to \pi_1(\mathbb{RP}^\infty) = \mathbb{Z}_2) \ , \end{aligned}$$

so that the orientation characters $w : \pi_1(X) \to \mathbb{Z}_2$ are in one-one correspondence with the elements $w \in H^1(X;\mathbb{Z}_2)$ classifying double covers $X^w$ of $X$.

DEFINITION 23. *An* **oriented cover** $(\widetilde{X},\pi,w)$ *of a space* $X$ *is a regular covering of* $X$ *with group of covering translations* $\pi$*, together with an orientation character* $w : \pi \to \mathbb{Z}_2$.

Note that for an oriented cover $(\widetilde{X},\pi,w)$ of a space $X$ the group $\pi$ is an intrinsic property of the covering projection $\widetilde{X} \to X$, but the orientation character $w : \pi \to \{\pm 1\}$ can be arbitrary.

Given an oriented cover $(\widetilde{X},\pi,w)$ of $X$ it is clear that the homology groups $H_*(\widetilde{X})$ are $\mathbb{Z}[\pi]$-modules (irrespective of $w$). The action of $\mathbb{Z}[\pi]$ is by

$$\mathbb{Z}[\pi] \times H_*(\widetilde{X}) \to H_*(\widetilde{X}) \ ; \ (\sum_{g\in\pi} n_g g, x) \to \sum_{g\in\pi} n_g g_*(x) \ ,$$

with $g \in \pi$ acting by the $\mathbb{Z}$-module automorphism $g_* : H_*(\widetilde{X}) \to H_*(\widetilde{X})$ induced by the covering translation $g : \widetilde{X} \to \widetilde{X}$. We shall now define cohomology $\mathbb{Z}[\pi]$-modules $H^*_{(\pi,w)}(\widetilde{X})$ which will depend on $w$. Let $C(\widetilde{X})$ be the cellular $\mathbb{Z}[\pi]$-module chain complex of $\widetilde{X}$, with

$$\begin{aligned} C(\widetilde{X})_n \ & = \ H_n(\widetilde{X}^{(n)},\widetilde{X}^{(n-1)}) \\ & = \ \text{free left } \mathbb{Z}[\pi]\text{-module generated by the } n\text{-cells of } X \end{aligned}$$

and $d : C(\widetilde{X})_n \to C(\widetilde{X})_{n-1}$ the boundary map of the triple $(\widetilde{X}^{(n)},\widetilde{X}^{(n-1)}, \widetilde{X}^{(n-2)})$. The homology $\mathbb{Z}[\pi]$-modules of $C(\widetilde{X})$

$$H_n(C(\widetilde{X})) \ = \ \ker(d : C(\widetilde{X})_n \to C(\widetilde{X})_{n-1})/\mathrm{im}(d : C(\widetilde{X})_{n+1} \to C(\widetilde{X})_n)$$

are just the ordinary integral homology groups of $\widetilde{X}$

$$H_*(C(\widetilde{X})) \ = \ H_*(\widetilde{X}) \ .$$

Use the $w$-twisted involution on $\mathbb{Z}[\pi]$ to define the dual $\mathbb{Z}[\pi]$-modules

$$C(\widetilde{X})^n \ = \ (C(\widetilde{X})_n)^* \ (n \geq 0) \ .$$

If $X$ is a finite $CW$ complex then $C(\widetilde{X})$ is a finite f.g. free $\mathbb{Z}[\pi]$-module chain complex.

DEFINITION 24. *The* $(\pi,w)$**-cohomology** $\mathbb{Z}[\pi]$*-modules of a* $CW$ *complex* $X$ *with respect to an oriented cover* $(\widetilde{X},\pi,w)$ *are defined by*

$$\begin{aligned} H^n_{(\pi,w)}(\widetilde{X}) \ & = \ H^n(C(\widetilde{X})) \\ & = \ \ker(d^* : C(\widetilde{X})^n \to C(\widetilde{X})^{n+1})/\mathrm{im}(d^* : C(\widetilde{X})^{n-1} \to C(\widetilde{X})^n) \ (n \geq 0) \end{aligned}$$

*using the* $w$*-twisted involution on* $\mathbb{Z}[\pi]$.

Evaluation defines $\mathbb{Z}[\pi]$-module morphisms

$$H^n_{(\pi,w)}(\widetilde{X}) \longrightarrow H_n(\widetilde{X})^* \; ; \; f \longrightarrow (x \longrightarrow f(x)) \; .$$

EXAMPLE 27. For a finite group $\pi$ and a finite $CW$ complex $X$ the $(\pi,+1)$-cohomology $\mathbb{Z}[\pi]$-modules $H^*_{(\pi,w)}(\widetilde{X})$ are just the ordinary cohomology groups $H^*(\widetilde{X})$ with the $\mathbb{Z}[\pi]$ action

$$\mathbb{Z}[\pi] \times H^*(\widetilde{X}) \longrightarrow H^*(\widetilde{X}) \; ; \; (\sum_g n_g g, x) \longrightarrow \sum_g n_g (g^{-1})^*(x) \; .$$

EXAMPLE 28. Let $X = \mathbb{RP}^2$ be the projective plane, and for $\epsilon = \pm 1$ define the oriented cover

$$(\widetilde{X}, \pi, w) \;=\; (S^2, \mathbb{Z}_2, \epsilon) \; .$$

Let

$$A \;=\; \mathbb{Z}[\mathbb{Z}_2] \;=\; \mathbb{Z}[T]/(T^2-1)$$

with the $w$-twisted involution $\overline{T} = \epsilon T$. The cellular $A$-module chain complex of $S^2$ is

$$C(S^2) \;:\; \dots \longrightarrow 0 \longrightarrow A \xrightarrow{1+T} A \xrightarrow{1-T} A \; ,$$

and the dual $A$-module chain complex is

$$C(S^2)^* \;:\; A \xrightarrow{1-\overline{T}} A \xrightarrow{1+\overline{T}} A \longrightarrow \dots \; ,$$

so that $H^0_{(\mathbb{Z}_2,\epsilon)}(S^2) = \mathbb{Z}^\epsilon$ is the $\mathbb{Z}[\mathbb{Z}_2]$-module defined by $\mathbb{Z}$ with the generator $T \in \mathbb{Z}_2$ acting by $\epsilon$. ∎

The terminology of 24 is somewhat elaborate, so we shall write

$$H^*_{(\pi,w)}(\widetilde{X}) \;=\; H^*(\widetilde{X})$$

on the understanding that the $(\pi,w)$-cohomology modules may not be the usual integral cohomology groups of $\widetilde{X}$ with the induced $\mathbb{Z}[\pi]$-action, and may depend on the choice of orientation character $w$.

REMARK . For a finite $CW$ complex $X$ the $(\pi,+1)$-cohomology $\mathbb{Z}[\pi]$-modules $H^*(\widetilde{X})$ are the cohomology groups $H^*_{\mathrm{cpt}}(\widetilde{X})$ defined by integral cochains with compact support (i.e. taking non-zero values on only a finite number of cells) with the induced $\mathbb{Z}[\pi]$-module structure

$$H^*(\widetilde{X}) \;=\; H^*_{\mathrm{cpt}}(\widetilde{X}) \; .$$

∎

EXAMPLE 29. *The homology and $(\mathbb{Z},+1)$-cohomology $\mathbb{Z}[\mathbb{Z}]$-modules of the oriented cover $(\mathbb{R},\mathbb{Z},+1)$ of $\mathbb{R}/\mathbb{Z} = S^1$ are given by*

$$H_n(\mathbb{R}) \;=\; \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0 \end{cases} \; , \; H^n(\mathbb{R}) \;=\; \begin{cases} \mathbb{Z} & n = 1 \\ 0 & n \neq 1 \end{cases} \; .$$

∎

PROPOSITION 11. *Given a CW complex $X$ and an oriented cover $(\widetilde{X},\pi,w)$ there are defined* **cap products**

$$\cap \ : \ H_m(X;\mathbb{Z}^w)\times H^i(\widetilde{X}) \longrightarrow H_{m-i}(\widetilde{X}) \ ; \ (x,y) \longrightarrow x\cap y$$

*such that for any $a\in\mathbb{Z}[\pi]$*

$$x\cap ay \ = \ a(x\cap y)\in H_{m-i}(\widetilde{X}) \ .$$

PROOF. The usual acyclic model argument gives a $\pi$-equivariant diagonal chain approximation

$$\widetilde{\Delta} \ : \ C(\widetilde{X}) \to C(\widetilde{X})\otimes_{\mathbb{Z}} C(\widetilde{X}) \ ,$$

with $\mathbb{Z}[\pi]$ acting by

$$\begin{array}{ll}\mathbb{Z}[\pi]\times C(\widetilde{X})\otimes_{\mathbb{Z}} C(\widetilde{X}) & \to C(\widetilde{X})\otimes_{\mathbb{Z}} C(\widetilde{X}) \ ; \\ & (\sum_{g\in\pi} n_g g, y\otimes z) \to \sum_{g\in\pi} n_g(gy\otimes gz) \ .\end{array}$$

Apply $\mathbb{Z}^w\otimes_{\mathbb{Z}[\pi]} -$ to obtain a $\mathbb{Z}$-module chain map

$$\begin{array}{l}\Delta \ = \ 1\otimes\widetilde{\Delta} \ : \ \mathbb{Z}^w\otimes_{\mathbb{Z}[\pi]} C(\widetilde{X}) \ = \ C(X;\mathbb{Z}^w) \to \\ \mathbb{Z}^w\otimes_{\mathbb{Z}[\pi]} (C(\widetilde{X})\otimes_{\mathbb{Z}} C(\widetilde{X})) \ = \ C(\widetilde{X})^t\otimes_{\mathbb{Z}[\pi]} C(\widetilde{X})\end{array}$$

where $C(\widetilde{X})^t$ denotes the right $\mathbb{Z}[\pi]$-module cellular chain complex $C(\widetilde{X})$ with the same additive structure and

$$C(\widetilde{X})^t\times\mathbb{Z}[\pi] \to C(\widetilde{X})^t \ ; \ (x,a) \to \overline{a}x \ .$$

Given an $m$-chain $x\in C(X;\mathbb{Z}^w)_m$ let

$$\begin{array}{ll}\Delta(x) & = \ \sum_i x_i'\otimes x_i'' \\ \in (C(\widetilde{X})^t\otimes_{\mathbb{Z}[\pi]} C(\widetilde{X}))_m & = \ \sum_{p+q=m}(C(\widetilde{X})^t_p\otimes_{\mathbb{Z}[\pi]} C(\widetilde{X})_q) \ .\end{array}$$

The cap product of $x$ and an $i$-cochain $y\in C(\widetilde{X})^i$ is the $(m-i)$-chain

$$x\cap y \ = \ \sum_j \overline{y(x_j')}\, x_j'' \ \in C(\widetilde{X})_{m-i} \ ,$$

with $y(x_i')=0\in\mathbb{Z}[\pi]$ if the degree of $x_i'$ is $\neq i$. The composite

$$\begin{array}{ll}H_m(X;\mathbb{Z}^w) & \xrightarrow{\Delta} H_m(C(\widetilde{X})^t\otimes_{\mathbb{Z}[\pi]} C(\widetilde{X})) \\ & \to H_m(\mathrm{Hom}_{\mathbb{Z}[\pi]}(C(\widetilde{X})^{-*},C(\widetilde{X})))\end{array}$$

sends a homology class $x\in H_m(X;\mathbb{Z}^w)$ to a chain homotopy class of $\mathbb{Z}[\pi]$-module chain maps

$$x\cap - \ : \ C(\widetilde{X})^{m-*} \to C(\widetilde{X})$$

inducing the cap products

$$x\cap - \ : \ H^i(\widetilde{X}) \to H_{m-i}(\widetilde{X}) \ ; \ y \to x\cap y \ .$$

□

REMARK . For any $CW$ complex $Y$ there are defined cap product pairings

$$\cap \ : \ H^{\rm lf}_m(Y) \times H^i_{\rm cpt}(Y) \to H_{m-i}(Y) \ ,$$

with $H^{\rm lf}_*(Y)$ the homology groups defined using infinite but locally finite chains in $Y$. If $(\widetilde{X}, \pi, w)$ is an oriented cover of a finite $CW$ complex $X$ with $w = +1$ there are defined infinite transfer maps

$$p^{\,!} \ : \ H_*(X;\mathbb{Z}^w) \ = \ H_*(X) \to H^{\rm lf}_*(\widetilde{X}) \ ; \ x \to \widetilde{x} \ .$$

The cap products of 11 can be expressed as the composites

$$\cap \ : \ H_m(X) \times H^n(\widetilde{X}) \xrightarrow{p^! \times 1} H^{\rm lf}_m(\widetilde{X}) \times H^i_{\rm cpt}(\widetilde{X}) \xrightarrow{\cap} H_{m-i}(\widetilde{X}) \ ,$$

using to identify $H^*(\widetilde{X}) = H^*_{\rm cpt}(\widetilde{X})$. ∎

## 3. Poincaré duality

Poincaré duality for an oriented $m$-dimensional manifold $M^m$ can be expressed in one of three equivalent ways:

(1) the isomorphism between cohomology and homology defined by the evaluation of the cap product on the fundamental class $[M] \in H_m(M)$

$$[M] \cap - \ : \ H^i(M) \xrightarrow{\cong} H_{m-i}(M) \ ,$$

(2) the cohomology pairing defined by the evaluation of the cup product on $[M] \in H_m(M)$

$$H^{m-i}(M) \times H^i(M) \to \mathbb{Z} \ ; \ (x,y) \to \langle x \cup y, [M] \rangle$$

(3) the homology pairing

$$\lambda \ : \ H_i(M) \times H_{m-i}(M) \to \mathbb{Z}$$

defined using (i) and (ii)

$$\lambda([M] \cap x, [M] \cap y) \ = \ \langle x \cup y, [M] \rangle \ ,$$

or else using geometric intersection numbers of cycles represented by submanifolds, which will be considered in Chapter **??**.

The intersection pairing satisfies the symmetry property

$$\lambda(y,x) \ = \ (-)^{i(m-i)}\lambda(x,y) \in \mathbb{Z} \ (x \in H_i(M) \, , \, y \in H_{m-i}(M)) \ .$$

THEOREM 15. (Poincaré duality) *For any closed $m$-dimensional manifold $M$ and oriented cover $(\widetilde{M}, \pi, w)$ cap product with the fundamental class $[M] \in H_m(M;\mathbb{Z}^w)$ defines $\mathbb{Z}[\pi]$-module isomorphisms*

$$[M] \cap - \ : \ H^*(\widetilde{M}) \xrightarrow{\cong} H_{m-*}(\widetilde{M}) \ .$$

PROOF. Regard $M$ as a cobordism $(M;\emptyset,\emptyset)$. By 15 a handle decomposition of $M$ determines a $CW$ structure with a cellular $\mathbb{Z}[\pi]$-module chain complex is $C(\widetilde{M})$, and the dual handle decomposition of $M$ determines another $CW$ structure, with cellular $\mathbb{Z}[\pi]$-module chain complex the $m$-dual $C(\widetilde{M})^{m-*}$. Applying the cellular approximation theorem to the identity map it is possible to approximate $1 : M \to M$ by a cellular homotopy equivalence from a subdivision of the dual handlebody $CW$ structure to the handlebody $CW$ structure. The corresponding $\mathbb{Z}[\pi]$-module chain equivalence $C(\widetilde{M})^{m-*} \to C(\widetilde{M})$ is given by the cap product $[M] \cap -$, up to chain homotopy. (One way to see this is to triangulate $M$ as a $PL$ manifold, choosing the handles of $M$ to be subcomplexes of a second derived subdivision (Rourke and Sanderson [61]), and to define the chain level cap product using the Alexander-Whitney diagonal chain approximation.) □

COROLLARY 13. (Poincaré-Lefschetz duality) *For any* $(m+1)$*-dimensional cobordism* $(W;M,M')$ *and oriented cover* $((\widetilde{W};\widetilde{M},\widetilde{M}'),\pi,w)$ *cap product with the fundamental class* $[W] \in H_{m+1}(W,\partial W;\mathbb{Z}^w)$ *defines* $\mathbb{Z}[\pi]$*-module isomorphisms*

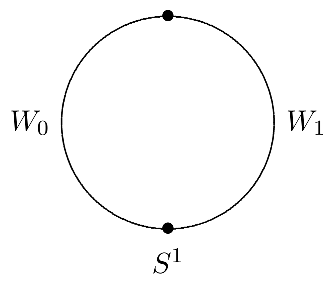
$$[W] \cap - \ : \ H^*(\widetilde{W},\widetilde{M}) \xrightarrow{\cong} H_{m+1-*}(\widetilde{W},\widetilde{M}') \ .$$

PROOF. As for 15, using a handle decomposition and its dual. □

EXAMPLE 30. The circle $S^1$ has a handle decomposition

$$S^1 \ = \ W_0 \cup W_1$$

with one 0-handle $W_0$ and one 1-handle $W_1$.




The fundamental group of $S^1$ is the free abelian group on 1 generator

$$\pi_1(S^1) \ = \ \mathbb{Z} \ = \ \langle z \rangle \ ,$$

with group ring

$$\mathbb{Z}[\pi_1(S^1)] \ = \ \mathbb{Z}[z,z^{-1}] \ .$$

The universal cover is $\widetilde{S}^1 = \mathbb{R}$ with $\pi_1(S^1)$ acting by

$$z \ : \ \mathbb{R} \to \mathbb{R} \ ; \ x \to x+1 \ .$$

The $\mathbb{Z}[z,z^{-1}]$-module Poincaré duality chain equivalence of $\widetilde{S}^1$ is given by

$$\begin{array}{ccccccccc}
C(\tilde{S}^1)^{1-*} & : & \cdots \longrightarrow & 0 & \longrightarrow & \mathbb{Z}[z,z^{-1}] & \xrightarrow{1-z^{-1}} & \mathbb{Z}[z,z^{-1}] \\
\big\downarrow [S^1]\cap - & & & & & \big\downarrow id & & \big\downarrow -z \\
C_*(\tilde{S}^1) & : & \cdots \longrightarrow & 0 & \longrightarrow & \mathbb{Z}[z,z^{-1}] & \xrightarrow{1-z} & \mathbb{Z}[z,z^{-1}]
\end{array}$$

∎

EXAMPLE 31. The torus $T^2 = S^1 \times S^1$ has a handle decomposition

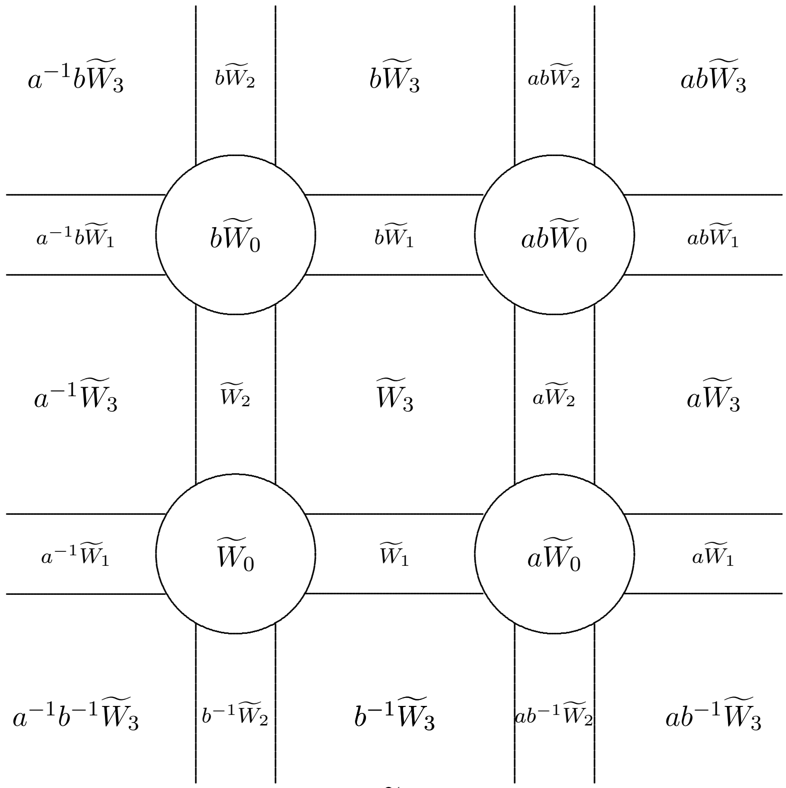
$$T^2 \;=\; W_0 \cup W_1 \cup W_2 \cup W_3$$

with one 0-handle $W_0$, two 1-handles $W_1, W_2$ and a 2-handle $W_3$. The fundamental group of $T^2$ is the free abelian group on 2 generators

$$\pi_1(T^2) \;=\; \mathbb{Z}^2 \;=\; \langle a, b \,|\, aba^{-1}b^{-1}\rangle \;.$$

The universal cover is $\widetilde{T}^2 = \mathbb{R}^2$ with $\pi_1(T^2)$ acting by

$$\begin{array}{l}
a \;:\; \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \;;\; (x,y) \longrightarrow (x+1,y) \;, \\
b \;:\; \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \;;\; (x,y) \longrightarrow (x,y+1) \;.
\end{array}$$




$\widetilde{T}^2$

The cellular $\mathbb{Z}[\mathbb{Z}^2]$-module chain complex of $\widetilde{T}^2$ associated to this handle decomposition of $T^2$ is

$$C(\widetilde{T}^2)\ :\ \mathbb{Z}[\mathbb{Z}^2\,] \xrightarrow{\begin{pmatrix}1-b\\ a-1\end{pmatrix}} \mathbb{Z}[\mathbb{Z}^2\,]\oplus\mathbb{Z}[\mathbb{Z}^2\,] \xrightarrow{\begin{pmatrix}a-1 & b-1\end{pmatrix}} \mathbb{Z}[\mathbb{Z}^2\,]\ .$$

∎

EXAMPLE 32. The real projective plane $\mathbb{RP}^2$ is obtained from $\emptyset$ by first attaching a 0-handle $D^0\times D^2$ to obtain $D^2$, then attaching a 1-handle $D^1\times D^1$ to $D^2$ to obtain a Möbius band, which is closed by attaching a 2-handle $D^2\times D^0$. The fundamental group of $\mathbb{RP}^2$ is the cyclic group of order 2

$$\pi_1(\mathbb{RP}^2)\ =\ \mathbb{Z}_2\ =\ \langle T\,|\,T^2\rangle\ .$$

The cellular $\mathbb{Z}[\mathbb{Z}_2\,]$-module chain complex of the universal cover $\widetilde{\mathbb{RP}^2}=S^2$ associated to this handle decomposition of $\mathbb{RP}^2$ is

$$C(\widetilde{\mathbb{RP}^2})\ :\ \mathbb{Z}[\mathbb{Z}_2\,]\xrightarrow{1+T}\mathbb{Z}[\mathbb{Z}_2\,]\xrightarrow{1-T}\mathbb{Z}[\mathbb{Z}_2\,]\ .$$

∎

Poincaré duality and the associated pairings play an essential role in surgery theory:

EXAMPLE 33. If $M^{4k}$ is a $4k$-dimensional manifold the intersection pairing defines a symmetric form on $H_{2k}(M)$

$$\lambda\ :\ H_{2k}(M)\times H_{2k}(M)\to\mathbb{Z}\ ;\ (x,y)\to\lambda(x,y)\ =\ \lambda(y,x)\ .$$

In order for it to be possible to kill an element $x\in\pi_{2k}(M)$ by surgery on $M^{4k}$ it is necessary (but not in general sufficient) for the Hurewicz image $x\in H_{2k}(M)$ to be such that

$$\lambda(x,x)\ =\ 0\in\mathbb{Z}\ .$$

Moreover, if surgery on $x\in\pi_{2k}(M)$ is possible and the effect $M'^{4k}$ is oriented then

$$H_{2k}(M')\ =\ \{y\in H_{2k}(M)\,|\,\lambda(x,y)=0\in\mathbb{Z}\}/\{nx\,|\,n\in\mathbb{Z}\}\ ,$$

the quotient of the subgroup of the homology classes orthogonal to $x\in H_{2k}(M)$ by the subgroup of the classes parallel to $x$. The symmetric form on $H_{2k}(M')$ is the one inherited from the form on $H_{2k}(M)$

$$\lambda'\ :\ H_{2k}(M')\times H_{2k}(M')\to\mathbb{Z}\ ;\ ([y],[z])\to\lambda(y,z)\ .$$

(See Chapter **??** for a detailed discussion of the homology effect of surgery.)

∎

## 4. The homology of a fibration

The four homology theories above all satisfy Künneth theorems so the homology or cohomology of a product $X \times Y$ is easily determined from the homology or cohomology of $X$ and $Y$ separately.

The homology of a space $X$ with coefficients in an abelian group $\mathcal{A}$ are defined by

$$H_*(X;\mathcal{A}) \ = \ H_*(C(X;\mathcal{A}))$$

with $C(X;\mathcal{A}) = \mathcal{A} \otimes_{\mathbb{Z}} C(X)$ the $\mathcal{A}$-coefficient singular chain complex of $X$.

THEOREM 16. *For any spaces $X, Y$ and abelian group $\mathcal{A}$ there is defined a chain equivalence*

$$C(X \times Y;\mathcal{A}) \simeq C(X) \otimes_{\mathbb{Z}} C(Y;\mathcal{A}) \ .$$

*The natural homomorphisms*

$$\bigoplus_{i+j=*} H_i(X;H_j(Y;\mathcal{A})) \longrightarrow H_*(X \times Y;\mathcal{A})$$

*are isomorphisms in the case where $\mathcal{A}$ is the additive group of a field $\mathcal{F}$, so that*

$$H_*(X \times Y;\mathcal{F}) = \bigoplus_{i+j=*} H_i(X;\mathcal{F}) \otimes_{\mathcal{F}} H_j(Y;\mathcal{F}) \ ,$$

*and similarly for cohomology.*

In the more general situation of a fibre bundle $F \to E \to X$ 16 describes the local situation but some work needs to be done to patch things together so as to relate $H_*(E)$ with $H_*(F)$ and $H_*(X)$.

In the case of a sphere fibration $S^{n-1} \to E \xrightarrow{p} X$ we have the Gysin sequence

$$\cdots \longrightarrow H^i(X) \xrightarrow{\cup\chi} H^{n+i}(X) \xrightarrow{p^*} H^{n+i}(E) \xrightarrow{t} H^{i+1}(X) \xrightarrow{\cup\chi} \ \cdots$$

where $\chi \in H^n(X)$ is the Euler class of the fibration. (So called because in the special case where $X$ is a closed, compact, oriented manifold without boundary and $p = \tau_X : E \to X$ is the tangent sphere bundle

$$\chi \ = \ \chi(X)[X]^* \in H^n(X) \ ,$$

the ordinary Euler number $\chi(X) \in \mathbb{Z}$ multiplied by the dual of the orientation class of $X$.)

But in the case where the fibre is more complex the usual techniques of calculation involve spectral sequences, a subject somewhat distant from the aims of this book. Consequently, we shall suppress the details of the methods of calculation and merely record the cohomology of the various spaces we shall need in the remainder of this work. Suffice it to say that given a **Serre fibration**, $F \to E \xrightarrow{p} X$ then for any abelian group $\mathcal{A}$ there is a spectral sequence with $E_2$-term

$$E_2^{i,j} = H^i_{\pi_1(X)}(X, H^j(F;\mathcal{A})\}$$

which converges to $H^*(E;\mathcal{A})$, where $H^j(F;\mathcal{A})$ is the $\mathbb{Z}[\pi_1(X)]$-module defined in 24. More generally, if $F_1 \to E_1 \xrightarrow{p} X$ is a sub-Serre-fibration, so we have inclusions

$$\begin{array}{ccccc} F_1 & \longrightarrow & E_1 & \xrightarrow{p} & X \\ \Big\downarrow{\scriptstyle j|} & & \Big\downarrow{\scriptstyle j} & & \Big\downarrow{\scriptstyle =} \\ F & \longrightarrow & E & \longrightarrow & X \end{array}$$

the there is a spectral sequence with $E_2$-term

$$E_2^{i,j} = H^i_{\pi_1(X)}(X, \{H^j(F, F_1; \mathcal{A})\})$$

which converges to $H^*(E, E_1; \mathcal{A})$. A **Serre type** spectral sequence is a sequence of terms $E_r^{i,j}$ where, for each $r$ there is a differential

$$d^r_{i,j} \colon E_r^{i,j} \to E_r^{i+r,j-r+1}$$

with $d^r_{i+r,j-r+1} d^r_{i,j} = 0$ for all $(i,j)$, and $E^{i,j}_{r+1} = H_*(E^{i,j}_r; d^r)$. Moreover, in the limit $E_\infty$ is defined and the groups $E_\infty^{i,j-i}$ are the associated graded groups of a filtration of the $j^{th}$ cohomology group of the total space $E$ or of the pair $(E, E_1)$.

As a particular application of the results sketched above we have the Thom Isomorphism Theorem:

COROLLARY 14. Thom Isomorphism Theorem *Let* $\mathbb{R}^n \to E(\zeta) \to X$ *be an n-plane bundle over* $X$ *with* $X$ *path connected and orientation character*

$$\chi \colon \pi_1(X) \to \mathbb{Z}/2 \ .$$

*Then we have*

(1) $H^{n+i}(T(\zeta);\mathbb{Z}) \cong H^i_{\pi_1(X)}(X;\mathbb{Z}_\chi)$ *for each* $i$.
(2) *In the case where the orientation character is trivial then*

$$H^n(T(\zeta);\mathbb{Z}) = \mathbb{Z}$$

*with generator* $U$*, and the isomorphism is given explicitly by*

$$\alpha \in H^i(X;\mathbb{Z}) \mapsto \alpha \cup U \in H^{n+i}(T(\zeta);\mathbb{Z}) \ .$$

(3) *For* $\mathbb{Z}/2$*-coefficients then regardless of the orientation character* $\chi$*,* $H^n(T(\zeta);\mathbb{Z}/2) \cong \mathbb{Z}/2$ *with generator* $U$ *and*

$$\alpha \in H^i(X;\mathbb{Z}/2) \mapsto \alpha \cup U \in H^{n+i}(T(\zeta);\mathbb{Z}/2)$$

*is an isomorphism for all* $i$.

PROOF. For a vector bundle $\mathbb{R}^n \to E(\zeta) \to X$ include $X \subset E(\zeta)$ by the zero section. The Serre spectral sequence of the pair $(E(\zeta), E(\zeta)\backslash X)$ has the form

$$E_2^{i,j} = H^i_{\pi_1(X)}(X; H^j(\mathbb{R}^n, \mathbb{R}^n\backslash\{0\}; \mathcal{A}))$$

which is non-zero only if $j = n$, and there

$$H^n(\mathbb{R}^n, \mathbb{R}^n\backslash\{0\}; \mathcal{A}) = \mathcal{A}_\chi \ .$$

But since only one row is non-zero at $E_2$ it follows that all the differentials are zero and $E_2 = E_\infty$. Moreover, a filtration of $H^*(E(\zeta), E(\zeta)\backslash X; \mathcal{A})$ of length one is just

$$H^*(E(\zeta), E(\zeta)\backslash X; \mathcal{A}) = H^*(T(\zeta); \mathcal{A}) \ (* > 0) \ .$$

$\square$

Remark . The inclusion $X \hookrightarrow E(\zeta)$ as the 0-section of $\zeta$ is a homotopy equivalence $X \simeq E(\zeta)$. Consequently, the usual cup-product pairing

$$H^i(Y; \mathcal{A}) \otimes H^j(Y, W; \mathcal{M}) \overset{\cup}{\to} H^{i+j}(Y, W; \mathcal{A} \otimes_{\mathbb{Z}} \mathcal{M})$$

in the case of $\zeta$ and $p\colon E(\zeta) \to X$ gives the cup product pairing of 14 by $\alpha \cup U = p^*(\alpha) \cup U$.

Definition 25. *Let $\mathbb{R}^n \to E(\zeta) \to X$ be an $n$-plane bundle, and let $h\colon X \to T(\zeta)$ be the inclusion of $X$ as the 0-section of $\zeta$. The* **Euler class** *of $\zeta$ is the cohomology class*

$$\chi(\zeta) \ = \ h^*(U) \in H^n(X; \mathbb{Z}/2) \ .$$

*For oriented $\zeta$ there is an* **integral Euler class**

$$\chi(\zeta) \ = \ h^*(U) \in H^n(X; \mathbb{Z}) \ .$$

The Thom space of the product of two vector bundles $\mathbb{R}^m \to E(\lambda) \to X$, $\mathbb{R}^n \to E(\mu) \to Y$ (2)

$$\mathbb{R}^{m+n} \to E(\lambda) \times E(\mu) \to X \times Y$$

is the smash product of the Thom spaces

$$T(\lambda \times \mu) = T(\lambda) \wedge T(\mu)$$

and the 0-section is the product of the two 0-sections. It follows that the Euler class of $\lambda \times \mu$ is

$$(U(\lambda) \otimes 1) \cup (1 \otimes U(\mu)) = U(\lambda) \otimes U(\mu) \in H^*(X \times Y; \mathcal{A}) \ .$$

Remark . The Euler class of the canonical 1-plane bundle $\gamma_1$ over $\mathbb{RP}^\infty$ is the non-zero class in $H^1(\mathbb{RP}^\infty)$ since $T(\gamma_1) = \mathbb{RP}^\infty$ and the inclusion associated to the 0-section is homotopic to the identity. (The Thom space of $\gamma_{1,k}$ over $\mathbb{RP}^k$ is $\mathbb{RP}^{k+1}$ and the inclusion $\mathbb{RP}^k \hookrightarrow \mathbb{RP}^{k+1}$ is just the usual inclusion.) This shows the non-triviality of the Euler class for all $k$-plane bundles by the observation above.

Remark . Basically the same argument as above shows that the integral Euler class of the canonical 2-plane bundle over $\mathbb{CP}^\infty$ is one of the generators of $H^2(\mathbb{CP}^\infty; \mathbb{Z}) = \mathbb{Z}$, which one depending on the original choice of $U \in H^2(T(\gamma_2); \mathbb{Z})$ over $\mathbb{CP}^\infty$. Again, the observation above shows that for each $k$ there is a $2k$-plane bundle with non-trivial integral Euler class.

Using these remarks, the fibre bundles $S^{n-1} \to BO_{n-1} \to BO_n$ which are just the associated bundles to the $n$-plane bundle

$$\mathbb{R}^n \to E(\gamma_n) \to BO_n ,$$

and induction it can be proved that

$$H^*(BO_n; \mathbb{Z}/2) = \mathbb{Z}/2[w_1, w_2, \ldots, w_n]$$

a polynomial algebra on generators $w_i$ where $w_i$ has dimension $i$ and $1 \leq i \leq n$. Moreover, $w_n$ is $\chi(\gamma_n)$, and the restriction map

$$H^*(BO_n; \mathbb{Z}/2) \to H^*(BO_{n-1}; \mathbb{Z}/2)$$

is surjective with kernel exactly the ideal $(w_n)$.

Definition 26. *The class $w_i$ above is called the $i^{th}$* **(universal) Stiefel-Whitney class**. *Every $n$-plane bundle $E(\zeta) \to X$ induced by a map*

$$f \colon X \to BO_n ,$$

*and $f^*(w_i)$ is called the $i^{th}$* **Stiefel-Whitney class** *of the bundle $\zeta$. The* **total Stiefel-Whitney class** *of $\zeta$ is*

$$W(\zeta) = 1 + w_1(\zeta) + w_2(\zeta) + \cdots + w_n(\zeta) \in \bigoplus_{i=0}^{\infty} H^i(X; \mathbb{Z}/2) .$$

Similarly, we can show

$$H^*(BSO_n; \mathbb{Z}/2) = \mathbb{Z}/2[w_2, \ldots, w_n],$$

and, when we regard $SO_2$ as $U_1$, a similar induction on the fibre bundles

$$S^{2n-1} \longrightarrow BU_{n-1} \longrightarrow BU_n$$

which are just the sphere bundles associated to the complex $n$-plane bundle $\mathbb{C}^n \to \gamma_n \to BU_n$, gives

$$H^*(BU_n; \mathbb{Z}) = \mathbb{Z}[c_1, \ldots, c_n]$$

where $c_i \in H^{2i}(BU_n; \mathbb{Z})$ and $c_n = \chi(\gamma_n)$. Here again the restriction map $H^*(BU_n; \mathbb{Z}) \to H^*(BU_{n-1}; \mathbb{Z})$ is surjective with kernel exactly the ideal $(c_n)$.

Definition 27. *The class $c_i \in H^{2i}(BU_n; \mathbb{Z})$ is called the $i^{th}$* **(universal) Chern class**, *and given a $\mathbb{C}^n$-bundle $E(\zeta) \to X$ induced by $f \colon X \to BU_n$ then $f^*(c_i) \in H^{2i}(X; \mathbb{Z})$ is the $i^{th}$* **Chern class** *of $\zeta$. The* **total Chern class** *of $\zeta$ is*

$$C(\zeta) = 1 + c_1(\zeta) + c_2(\zeta) + \cdots + c_n(\zeta) \in \bigoplus_{i=0}^{\infty} H^{2i}(X; \mathbb{Z}) .$$

We can tensor the universal bundle $\gamma_n$ over $BO_n$ with $\mathbb{C}$ to obtain a complex $n$-plane bundle $\mathbb{C} \otimes_{\mathbb{R}} \gamma_n$ (associated to the usual inclusion $GL_n(\mathbb{R}) \subset GL_n(\mathbb{C})$) and thus a map

$$B_i : BO_n \longrightarrow BU_n .$$

There is also a map in the opposite direction, $B_r\colon BU_n \to BO_{2n}$ which comes from the inclusion $GL_n(\mathbb{C}) \hookrightarrow GL_{2n}(\mathbb{R})$ which simply forgets the complex structure. The composites respectively induce $\gamma_n + \bar{\gamma}_n$ over $BU_n$ and $2\gamma_n$ over $BO_n$.

DEFINITION 28. *The $i^{th}$* **(universal) Pontrjagin class** *is*

$$p_i \;=\; c_{2i}(\mathbb{C} \otimes_{\mathbb{R}} \gamma_n) \in H^{4i}(BO_n; \mathbb{Z}) \; (i \geq 0) \; .$$

*Given an $n$-plane bundle $E(\zeta) \to X$ induced by $f\colon X \to BO_n$ then $f^*(p_i) \in H^{4i}(X; \mathbb{Z})$ is the $i^{th}$* **Pontrjagin class** *of $\zeta$.*

CHAPTER 6

# Cobordism and Handle Decompositions

## 1. Foundations of cobordism

We recall the foundations of the cobordism theory of Thom [66]. The handle and surgery techniques for constructing cobordisms will be described in detail in §2, **??**.

Here is the basic definition:

Definition 29. *A* **cobordism** *of closed $m$-dimensional manifolds $M^m$, $N^m$ is an $(m+1)$-dimensional manifold $W^{m+1}$ with boundary*

$$\partial W \;=\; M \sqcup N \;.$$

Poincaré's definition of homology was motivated by the invariance of integration on cobordant submanifolds. If $\omega$ is a closed differential $m$-form on an $n$-manifold $V^n$ then for any closed $m$-dimensional oriented submanifold $M^m \subset V^n$ it is possible to define the integral

$$\int_M \omega \in \mathbb{R} \;.$$

For disjoint submanifolds $M^m, N^m \subset V^n$ related by an oriented cobordism $(W; M, M')$ which is also a submanifold $W^{m+1} \subset V^n$

$$\int_M \omega \backslash \int_N \omega \;=\; \int_W d\omega \;=\; 0 \in \mathbb{R}$$

by Stokes' theorem. In modern terminology we are dealing with the isomorphism of the universal coefficient theorem

$$H^m(V;\mathbb{R}) \to \mathrm{Hom}_{\mathbb{R}}(H_m(V;\mathbb{R}),\mathbb{R}) \;;\; [\omega] \to ([M] \to \int_M \omega) \;,$$

with $[\omega] \in H^m(M;\mathbb{R})$ the deRham cohomology class of the form $\omega$, and $[M] \in H_m(V;\mathbb{R})$ the homology class of the submanifold $M$.

Proposition 12. *Cobordism is an equivalence relation on manifolds.*

Proof. Every manifold $M$ is cobordant to itself by the product cobordism

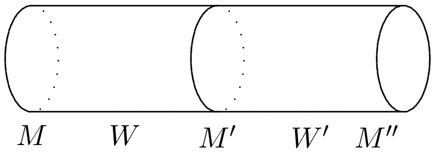
$$M \times (I;\{0\},\{1\}) \;=\; (M \times I; M \times \{0\}, M \times \{1\}) \;,$$

with $I = I$ the unit interval. The union of adjoining cobordisms

$$(W; M, M') \;,\; (W'; M', M'')$$

is a cobordism

$$(W; M, M') \cup (W'; M', M'') = (W \cup_{M'} W'; M, M'') .$$





□

Diffeomorphic manifolds are trivially cobordant.

Definition 30. *The* **unoriented cobordism ring**

$$\mathcal{N}_* = \sum_{m=0}^{\infty} \mathcal{N}_m$$

*is the graded ring with $\mathcal{N}_m$ the abelian group of cobordism classes $[M]$ of closed $m$-dimensional manifolds $M^m$. The addition is by disjoint union*

$$[M^m] + [M'^m] = [(M \sqcup M')^m] \in \mathcal{N}_m ,$$

*and the multiplication is by cartesian product*

$$\mathcal{N}_m \times \mathcal{N}_n \longrightarrow \mathcal{N}_{n+m} ; ([M^m], [N^n]) \longrightarrow [M][N] = [(M \times N)^{m+n}] .$$

INSERT THOM'S BASIC RESULTS HERE.

By reducing the geometry to homotopy theory and algebra Thom computed $\mathcal{N}_*$ to be the polynomial algebra over $\mathbb{Z}_2$

$$\mathcal{N}_* = \mathbb{Z}_2[x_i \mid i \geq 1 , i \neq 2^{j-1}]$$

with one generator $x_i$ in each degree $i \neq 2^j - 1$, with $x_i = [\mathbb{RP}^i]$ if $i$ is even.

Definition 31. *The* **oriented cobordism ring**

$$\Omega_* = \sum_{m=0}^{\infty} \Omega_m$$

*is the graded ring with $\Omega_m$ the abelian group of cobordism classes $[M]$ of closed oriented $m$-dimensional manifolds $M^m$.*

Thom showed that

$$\Omega_* \otimes \mathbb{Q} = \mathbb{Q}[y_{4i} \mid i \geq 1]$$

with one generator $y_{4i} = [\mathbb{C}\,\mathbb{P}^{2i}]$ in degree $4i$ for each $i \geq 1$.

Remark . The low-dimensional cobordism groups are given by:

| $m$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $\mathcal{N}_m$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}_2$ | 0 | $(\mathbb{Z}_2)^2$ | $\mathbb{Z}_2$ | $(\mathbb{Z}_2)^3$ | $\mathbb{Z}_2$ | $(\mathbb{Z}_2)^5$ |
| $\Omega_m$ | $\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}$ | $\mathbb{Z}_2$ | 0 | 0 | $\mathbb{Z}^2$ |

∎

## 2. Handles

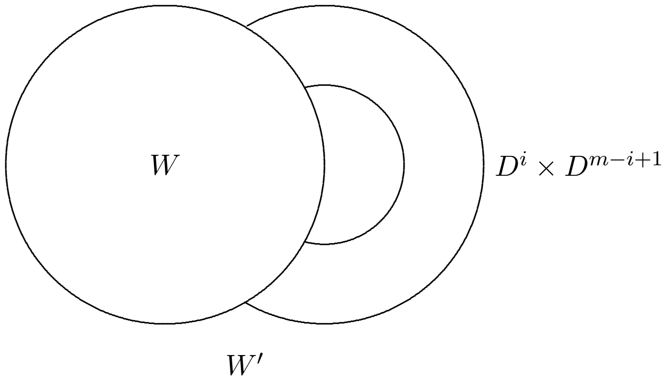
Handles are the building blocks of manifolds and cobordisms. We shall now establish that every $(m+1)$-dimensional cobordism $(W;M,M')$ has a handlebody decomposition, and then use it to obtain Poincaré duality isomorphisms

$$H_*(W,M) \;\cong\; H^{m+1-*}(W,M')$$

using appropriately twisted cohomology groups in the nonorientable case.

DEFINITION 32. *Given an $(m+1)$-dimensional manifold with boundary $(W,\partial W)$ and a framed $(i-1)$-embedding $S^{i-1}\times D^{m-i+1}\subset \partial W$ $(0\le i\le m)$ define the $(m+1)$-dimensional manifold with boundary $(W',\partial W')$ obtained from $W$ by* **attaching an $i$-handle** *to be*

$$W' \;=\; W\cup_{S^{i-1}\times D^{m-i+1}} D^i\times D^{m-i+1}\ .$$




DEFINITION 33. (i) *An* **elementary** $(m+1)$**-dimensional cobordism of index** *$i$ is the cobordism $(W;M,M')$ obtained from $M\times I$ by attaching an $i$-handle at $S^{i-1}\times D^{m+1-i}\subset M\times\{1\}$, with*

$$W \;=\; M\times I\cup D^i\times D^{m+1-i}\ .$$

(ii) *The* **dual** *of an elementary $(m+1)$-dimensional cobordism of index $i$ $(W;M,M')$ is the elementary $(m+1)$-dimensional cobordism $(-W;M',M)$ of index $(m-i+1)$ obtained by reversing the ends, and regarding the $i$-handle attached to $M\times I$ as an $(m-i+1)$-handle attached to $M'\times I$.*

LEMMA 19. *For any $0\le i\le m+1$ the Morse function*

$$f\ :\ \mathbb{R}^{m+1}\to\mathbb{R}\ ;\ (x_1,x_2,\dots,x_{m+1})\to -\sum_{j=1}^{i}x_j^2+\sum_{j=i+1}^{m+1}x_j^2$$

*has a unique critical point $0\in\mathbb{R}^{m+1}$, which is of index $i$. The $(m+1)$-dimensional manifolds with boundary defined for any $\epsilon>0$ by*

$$W_{-\epsilon} \;=\; f^{-1}(-\infty,-\epsilon]\ ,\ W_\epsilon \;=\; f^{-1}(-\infty,\epsilon]$$

*are such that $W_\epsilon$ is obtained from $W_{-\epsilon}$ by attaching an $i$-handle*

$$W_\epsilon \;=\; W_{-\epsilon}\cup D^i\times D^{m-i+1}\ .$$

*(Warning: if $i \geq 1$ then $W_{-\epsilon}$ and $W_\epsilon$ are not compact.)*

PROPOSITION 13. *Let $f : W^{m+1} \to I$ be a Morse function on an $(m+1)$-dimensional manifold cobordism $(W; M, M')$ with $f^{-1}(0) = M$, $f^{-1}(1) = M'$, and such that all the critical points of $f$ are in the interior of $W$.*
(i) *If $f$ has no critical points then $(W; M, M')$ is a trivial h-cobordism, with a diffeomorphism*

$$(W; M, M') \;\cong\; M \times (I; \{0\}, \{1\})$$

*which is the identity on $M$.*
(ii) *If $f$ has a single critical point of index $i$ then $W$ is obtained from $M \times I$ by attaching an $i$-handle at a framed $i$-embedding $S^{i-1} \times D^{m-i+1} \subset M \times \{1\}$, and $(W; M, M')$ is an elementary cobordism of index $i$ with a diffeomorphism*

$$(W; M, M') \;\cong\; (M \times I \cup D^i \times D^{m-i+1}; M \times \{0\}, M') \; .$$

PROOF. (i) See Milnor [43].
(ii) In a neighbourhood of the unique critical point $p \in W$

$$f(x_1, x_2, \ldots, x_{m+1}) \;=\; c - \sum_{j=1}^{i} x_j^2 + \sum_{j=i+1}^{m+1} x_i^2$$

with respect to a coordinate chart $\mathbb{R}^{m+1} \subset W$ such that $0 \in \mathbb{R}^{m+1}$ corresponds to $p \in W$, with $c = f(p) \in \mathbb{R}$ the critical value. For any $\epsilon > 0$ there are defined diffeomorphisms

$$f^{-1}(-\infty, c-\epsilon] \;\cong\; M \times I \; , \; f^{-1}[c+\epsilon, \infty) \;\cong\; M' \times I$$

by (i), and by 19 there is defined a diffeomorphism

$$f^{-1}[c-\epsilon, c+\epsilon] \;\cong\; M \times I \cup D^i \times D^{m-i+1} \; .$$

□

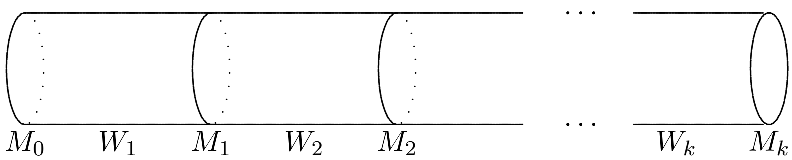
Somewhat by analogy with the result that every finite-dimensional vector space has a finite basis we have:

THEOREM 17. **[Handle Decomposition Theorem]** (Thom, Milnor [41]) *Every cobordism $(W^{m+1}; M^m, M'^m)$ has a handle decomposition as the union of a finite sequence*

$$(W; M, M') \;=\; (W_1; M_0, M_1) \cup (W_2; M_1, M_2) \cup \;\ldots\; \cup (W_k; M_{k-1}, M_k)$$

*of adjoining elementary cobordisms $(W_j; M_{j-1}, M_j)$ with index $i_j$, such that*

$$0 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq m+1 \; , \; M_0 \;=\; M \; , \; M_k = M' \; .$$



PROOF. Any cobordism admits a Morse function

$$f \ : \ (W; M, M') \to I$$

with

$$M \ = \ f^{-1}(0) \ , \ M' \ = \ f^{-1}(1) \ ,$$

and such that all the critical values are in the interior of $I$. Since $W$ is compact there are only a finite number of critical points: label them $p_j \in W$ $(1 \leq j \leq k)$. Write the critical values as $c_j = f(p_j) \in \mathbb{R}$, and let $i_j$ be the index of $p_j$. It is possible to choose $f$ such that

$$0 < c_1 < c_2 \cdots < c_k \ < 1 \ , \ 0 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq m+1 \ .$$

Let $r_j \in I$ $(0 \leq j \leq k)$ be regular values such that

$$0 \ = \ r_0 < c_1 < r_1 < c_2 < \cdots < r_{k-1} < c_k < r_k \ = \ 1 \ .$$

By 13 (i) each

$$(W_j; M_{j-1}, M_j) \ = \ f^{-1}([r_{j-1}, r_j]; \{r_{j-1}\}, \{r_j\}) \ (1 \leq j \leq k)$$

is an elementary cobordism of index $i_j$. □

COROLLARY 15. *Every closed $m$-dimensional manifold $M^m$ can be obtained from $\emptyset$ by attaching handles.*

PROOF. Apply 17 to the cobordism $(M; \emptyset, \emptyset)$. □

EXAMPLE 34.

(1) The $m$-sphere $S^m$ has a handle decomposition consisting of a 0-handle and an $m$-handle

$$S^m \ = \ D^0 \times D^m \cup D^m \times D^0 \ ,$$

given by the upper and lower hemispheres.

(2) The cobordism $(D^{m+1}; \emptyset, S^m)$ has a handle decomposition with one 0-handle. The dual cobordism $(D^{m+1}; S^m, \emptyset)$ has a handle decomposition with one $(m+1)$-handle.

We assume that the reader is already acquainted with $CW$ complexes, which are spaces obtained from $\emptyset$ by successively attaching cells of increasing dimension: the space obtained from a space $X$ by attaching an $i$-cell along a map $f : S^{i-1} \to X$ $(i \geq 0)$ is the identification space

$$Y \ = \ X \cup_f D^i \ .$$

PROPOSITION 14. *The homotopy theoretic effect of attaching an $i$-handle to an $(m+1)$-dimensional manifold with boundary $(W, \partial W)$ is that of attaching an $i$-cell, that is*

$$W' \ = \ W \cup D^i \times D^{m-i+1} \ \simeq \ W \cup D^i \ .$$

Let $X = (A, B \subseteq A)$ be a relative $CW$ pair with

$$A \ = \ B \cup D^{i_1} \cup D^{i_2} \cup \ldots$$

such that $i_1 \leq i_2 \leq \ldots$ . The relative skeleta of $X$

$$B \subset X^{(0)} \subset X^{(1)} \subset \ \ldots \ \subset A$$

are defined by

$$X^{(i)} \;=\; B \cup \bigcup_{i_j \le i} D^{i_j} \;\; (r \ge 0) \;.$$

Given a regular cover $\widetilde{A}$ of $A$ with group of covering translations $\pi$ let $\widetilde{B}$, $\widetilde{X}^{(i)}$ be the corresponding covers of $B$, $X^{(i)}$, such that

$$\widetilde{X}^{(i)} \;=\; \widetilde{B} \cup \bigcup_{i_j \le i} \pi \times D^{i_j} \;.$$

The cellular $\mathbb{Z}[\pi]$-module chain complex of $\widetilde{X} = (\widetilde{A}, \widetilde{B})$

$$C \;=\; C(\widetilde{X}) \;=\; C(\widetilde{A}, \widetilde{B})$$

is defined by

$$\begin{aligned} C_i \;&=\; H_i(\widetilde{X}^{(i)}, \widetilde{X}^{(i-1)}) \\ &=\; \text{the free } \mathbb{Z}[\pi]\text{-module generated by the } i\text{-cells} \;, \end{aligned}$$

with the differentials

$$d \;:\; C_{i+1} \;=\; H_{i+1}(\widetilde{X}^{(i+1)}, \widetilde{X}^{(i)}) \to C_i \;=\; H_i(\widetilde{X}^{(i)}, \widetilde{X}^{(i-1)})$$

the homology boundary maps of the triple $(\widetilde{X}^{(i+1)}, \widetilde{X}^{(i)}, \widetilde{X}^{(i-1)})$. The homology of $C$

$$H_*(C) \;=\; H_*(\widetilde{X}) \;=\; H_*(\widetilde{A}, \widetilde{B})$$

consists of the relative homology $\mathbb{Z}[\pi]$-modules of $(\widetilde{A}, \widetilde{B})$.

PROPOSITION 15. *A handle decomposition of a cobordism*

$$(W; M, M') \;=\; (W_1; M_0, M_1) \cup (W_2; M_1, M_2) \cup \;\dots\; \cup (W_k; M_{k-1}, M_k)$$

*as a union of $k$ adjacent elementary cobordisms determines a relative CW structure on the pair $X = (W, M)$*

$$W \;\simeq\; M \cup D^{i_1} \cup D^{i_2} \cup \dots \cup D^{i_k} \;.$$

*with $i_j$ the index of $(W_j; M_{j-1}, M_j)$. The handle decomposition thus determines the cellular $\mathbb{Z}[\pi]$-module chain complex $C(\widetilde{W}, \widetilde{M})$ and hence the homology $\mathbb{Z}[\pi]$-modules $H_*(\widetilde{W}, \widetilde{M})$, for any regular cover $\widetilde{W}$ of $W$ with group of covering translations $\pi$.*

Let $(W; M, M')$ be an $(m+1)$-dimensional cobordism with a handle decomposition

$$(W; M, M') \;=\; (W_1; M_0, M_1) \cup (W_2; M_1, M_2) \cup \;\dots\; \cup (W_k; M_{k-1}, M_k) \;,$$

and let there be $b_i$ handles $W_j$ with index $i_j = i$. For each $0 \le i \le m+1$ let $(W(i); M(i-1), M(i))$ be the cobordism defined by the union of all the $i$-handles, with

$$\begin{gathered} W(i) \;=\; \bigcup_{i_j = i} W_j \;,\; M(-1) \;=\; M \;,\; M(m+1) \;=\; M' \;, \\ W(i) \;=\; M(i-1) \times I \;,\; M(i-1) \;=\; M(i) \text{ if there are no } i\text{-handles} \;, \\ W \;=\; \bigcup_{i=0}^{m+1} W(i) \;. \end{gathered}$$

It is possible to arrange the handles in such a way that all the handles $W_j$ of index $i$ are attached to $M(i-1)$ simultaneously, with disjoint framed $(i-1)$-embeddings $S^{i-1}\times D^{m-i+1}\subset M(i-1)$ such that

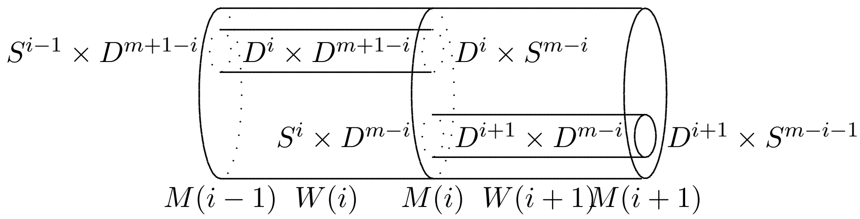
$$W(i) \;=\; M(i-1)\times I\cup\bigcup_{b_i} D^i\times D^{m-i+1}\ .$$

Reverse the $i$-handles, and regard them as dual $(m-i+1)$-handles in $(W(i);M(i),M(i-1))$, so that

$$W(i) \;=\; M(i)\times I\cup\bigcup_{b_i} D^i\times D^{m-i+1}\ .$$

DEFINITION 34. *The* **algebraic intersection** *of handles $W_j, W_{j'}$ of index $i,i+1$ is the algebraic intersection (??) of the cores $S^{m-i}, S^i\subset M(i)$ of the corresponding framed $(m-i)$- and $i$-embeddings $D^i\times S^{m-i}$, $S^i\times D^{m-i}\subset M(i)$*

$$\lambda^{alg}(W_j,W_j') \;=\; \lambda^{alg}(S^{m-i},S^i)\in\mathbb{Z}[\pi]\ .$$




PROPOSITION 16. *The differentials in the cellular $\mathbb{Z}[\pi]$-module chain complex $C=C(\widetilde{W},\widetilde{M})$ of a handle decomposition of a cobordism $(W;M, M')$ are given by the algebraic intersection numbers of the handles of adjoining index*

$$\begin{aligned} d\ :\ C_{i+1} &= H_{i+1}(\widetilde{X}^{(i+1)},\widetilde{X}^{(i)}) \;=\; \mathbb{Z}[\pi]^{b_{i+1}}\\ &\to C_i \;=\; H_i(\widetilde{X}^{(i)},\widetilde{X}^{(i-1)}) \;=\; \mathbb{Z}[\pi]^{b_i}\ ;\\ &[W_{j'}]\to \sum_{i_j=i}\lambda(W_j,W_{j'})[W_j]\ . \end{aligned}$$

PROPOSITION 17. *Let $(W;M,M')$ be an $(m+1)$-dimensional cobordism with an oriented cover $((\widetilde{W};\widetilde{M},\widetilde{M'}),\pi,w)$. Given a Morse function $f : W\to I$ and a corresponding handle decomposition*

$$(W;M,M') \;=\; (W_1;M_0,M_1)\cup(W_2;M_1,M_2)\cup\ \dots\ \cup(W_k;M_{k-1},M_k)$$

*let $f'=1-f : W\to I$ be the opposite Morse function, corresponding to the dual handle decomposition*

$$(W';M',M) \;=\; (W_1';M_0',M_1')\cup(W_2';M_1',M_2')\cup\ \dots\ \cup(W_k';M_{k-1}',M_k')$$

*with*

$$W' \;=\; -W\ ,\ W_i' \;=\; -W_{k-i}\ ,\ M_i' \;=\; M_{k-i}\ (0\leq i\leq k)\ .$$

*The relative cellular $\mathbb{Z}[\pi]$-module chain complexes of $(\widetilde{W}, \widetilde{M})$, $(\widetilde{W}, \widetilde{M}')$ are related by*

$$C(\widetilde{W}, \widetilde{M}') \;=\; C(\widetilde{W}, \widetilde{M})^{m+1-*} \;,$$

*where duality is taken with respect to the $w$-twisted involution on $\mathbb{Z}[\pi]$.*

Proof. The dual of an $n_i$-handle $W_i$ is the $(m+1-n_i)$-handle $W'_{k-i}$. The algebraic intersection of an $n_i$-handle $W_i$ and an $n_j$-handle $W_j$ with $n_j = n_i + 1$ is related to the algebraic intersection of the dual $(m+1-n_i)$-handle $W'_{k-i}$ and the dual $(m+1-n_j)$-handle $W'_{k-j}$ by

$$\lambda(W_i, W_j) \;=\; \overline{\lambda(W'_{k-j}, W'_{k-i})} \in \mathbb{Z}[\pi] \;.$$

□

## 3. Lagrangians and Even Forms

Definition 35. *Let $R$ be a ring with involution $\mu$ and $b\colon R^{2k} \times R^{2k} \to R$ a non-singular $R$-bilinear $\pm$-symmetric form. Then a direct summand $R^k \subset R^{2k}$ is called a* **Lagrangian** *of $b$ if $b|R^k \times R^k \to R$ is identically $0$.*

Definition 36. *Let an $R$-bilinear form $b\colon R^k \times R^k \to R$ be given, then it is* **even** *if there is an $R$-linear map $l(b)\colon R^k \to (R^k)^*$ so that the adjoint of $b$ has the form $l(b) \pm (l(b))^*$.*

Remark . If $(1/2) \in R$ then every $\pm$-symmetic form is even so the distinction is immaterial in this case. However, even in the case of the integers the distinction is important. For example, the smallest even non-singular symmetric form over the integers is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ while the smallest form is $(1)\colon \mathbb{Z} \to \mathbb{Z}^*$.

Remark . Note that if the $\pm$ form $A$ is even, then the matrix, $B$, of the form with respect to any basis is even ($B = C \pm C^*$).

Lemma 20. *Suppose that $R^k = A \subset R^{2k}$ is a Lagrangian for the even $\pm$-symmetric bilinear form $b\colon R^{2k} \times R^{2k} \to R$. Then there is a second Lagrangian $R^k = B$ for $b$ so that $A \oplus B = R^{2k}$ and bases for $A$ and $B$ so that the form becomes $\begin{pmatrix} 0 & I \\ \pm I & 0 \end{pmatrix}$ with respect to these bases.*

Proof. Non-singularity implies that the adjoint of $b$ gives an isomorphism $r^{2k}\colon A \xrightarrow{\cong} A^*$. Consequently, given any basis for $A$ there is a dual basis for $A^*$ and lifting back, a direct summand $B'$ so that $A + B' = R^{2k}$ and the matrix of $b$ with respect to these bases has the form $\begin{pmatrix} 0 & I \\ \pm I & L \end{pmatrix}$. By the evenness of the form we can write $L = M \pm M^*$ where $M^*$ is $M$-transpose with the entries replaced by their images under the involution.

Now, there is a unique $R$-linear map $\tau\colon A^* \to A$ so that $\beta^* M \lambda = \tau(\beta)(\lambda)$ for each $(\beta, \tau) \in B \times B$. Then replacing the vectors of $B$ by the vectors $(-\tau(\beta), \beta)$ gives a new direct summand which is, in fact a Lagrangian. □

Remark . The unoriented cobordism class of an $m$-dimensional manifold $M^m$ is determined by the Stiefel-Whitney numbers

$$w_I(M) \;=\; \langle w_1(M)^{i_1} w_2(M)^{i_2} \dots w_n(M)^{i_n}, [M]\rangle \in \mathbb{Z}_2$$

defined for any sequence $I = (i_1, i_2, \dots, i_n)$ of integers $i_j \geq 0$ such that

$$i_1 + 2i_2 + \cdots + ni_n \;= m \;,$$

with $w_i(M) \in H^i(M;\mathbb{Z}_2)$ the $i$th Stiefel-Whitney class.

Remark . The oriented cobordism class of an oriented $m$-dimensional manifold $M^m$ is determined by the Stiefel-Whitney numbers $w_I(M) \in \mathbb{Z}_2$ and the Pontrjagin numbers

$$p_J(M) \;=\; \langle p_1(M)^{j_1} p_2(M)^{j_2} \dots p_n(M)^{j_n}, [M]\rangle \in \mathbb{Z}$$

defined for any sequence $J = (j_1, j_2, \dots, j_n)$ of integers $j_k \geq 0$ such that

$$4j_1 + 8j_2 + \cdots + 4nj_n \;=\; m \;,$$

with $p_j(M) \in H^{4j}(M)$ the $j$th Pontrjagin class.

Proposition 18. *The signature of closed oriented $4k$-dimensional manifolds is an oriented cobordism invariant, with*

$$\sigma(M) \;=\; \sigma(N) \in \mathbb{Z}$$

*for any oriented $(4k+1)$-dimensional cobordism $(W; M, N)$.*

Proof. It suffices to prove that $\sigma(\partial W) = 0$ for an oriented $(4k+1)$-dimensional manifold with boundary $(W, \partial W)$. Let

$$(V, \lambda) \;=\; (H^{2k}(\partial W;\mathbb{R}), [\partial W] \cap -)$$

be the intersection form of $\partial W$, with $[\partial W] \in H_{2k}(\partial W;\mathbb{R})$. The Poincaré duality isomorphisms define an isomorphism of exact sequences

$$\begin{array}{ccccccccc}
\cdots \to & H^{2k}(W;\mathbb{R}) & \xrightarrow{i^*} & H^{2k}(\partial W;\mathbb{R}) & \to & H^{2k+1}(W,\partial W;\mathbb{R}) & \to \cdots \\
& \Big\downarrow{\scriptstyle [W]\cap -} & & \Big\downarrow{\scriptstyle [\partial W]\cap -} & & \Big\downarrow{\scriptstyle [W]\cap -} & \\
\cdots \to & H_{2k+1}(W,\partial W;\mathbb{R}) & \to & H_{2k}(\partial W;\mathbb{R}) & \xrightarrow{i_*} & H_{2k}(W;\mathbb{R}) & \to \cdots
\end{array}$$

with $i : \partial W \to W$ the inclusion. It follows that the inclusion $j : L \to V$ of the subspace

$$L \;=\; \mathrm{im}(i^* : H^{2k}(W;\mathbb{R}) \to H^{2k}(\partial W;\mathbb{R})) \subset V \;=\; H^{2k}(\partial W;\mathbb{R})$$

is such that there is defined an exact sequence

$$0 \to L \xrightarrow{j} V \xrightarrow{j^*\lambda} L^* \to 0 \;.$$

Thus $L$ is a 'lagrangian' (= maximal isotropic subspace) in $(V, \lambda)$, $j$ extends to an isomorphism of symmetric bilinear forms

$$H(L) \;=\; (L \oplus L^*, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) \to (V, \lambda) \;,$$

and the signature of $\partial W$ is

$$\sigma(\partial W) = \sigma(V,\lambda) = \sigma(H(L)) = 0 \in \mathbb{Z} .$$

(See 55 for the general definition of a lagrangian in a form over an arbitrary ring with involution). □

In particular, the signature defines morphisms

$$\sigma : \Omega_{4k} \to \mathbb{Z} ; [M] \to \sigma(M) .$$

The expression of $\Omega_* \otimes \mathbb{Q}$ in terms of the Pontrjagin numbers led to the original proof of the celebrated:

THEOREM 18. **[Signature Theorem]** (Hirzebruch, 1952) *The signature of a closed oriented $4k$-dimensional manifold $M^{4k}$ is given by*

$$\sigma(M) = \langle \mathcal{L}_k(M), [M] \rangle \in \mathbb{Z}$$

*with $\mathcal{L}_k(M)$ a polynomial in the Pontrjagin classes $p_j(M) = p_j(\tau_M) \in H^{4j}(M)$ of the tangent bundle.*

See Hirzebruch [26] for the classic account of the signature theorem.

EXAMPLE 35. The Pontrjagin classes of the tangent bundle $\tau_{\mathbb{CP}^n} : \mathbb{CP}^n \to BSO_{2n}$ of the $n$-dimensional complex projective plane $\mathbb{CP}^n$ are given by

$$p_j(\mathbb{CP}^n) = \binom{n+1}{j} \in H^{4j}(\mathbb{CP}^n) = \mathbb{Z} \ (0 \leq j \leq n/2)$$

(Milnor and Stasheff [47,p.177]). Th signature of $\mathbb{CP}^{2k}$ is

$$\begin{aligned}\sigma(\mathbb{CP}^{2k}) &= \text{signature}(H^{2k}(\mathbb{CP}^{2k}), \phi_0) \\ &= \text{signature}(\mathbb{Z}, 1) = 1 \in \mathbb{Z} .\end{aligned}$$

The signature theorem gives the following calculations for the signature of $\mathbb{CP}^{2k}$ for low values of $k$. For $k = 1$

$$\begin{aligned}p_1(\mathbb{CP}^2) &= 3 , \ \mathcal{L}_1 = p_1/3 , \\ \sigma(\mathbb{CP}^2) &= \langle \mathcal{L}_1(\mathbb{CP}^2), [\mathbb{CP}^2] \rangle = 1 .\end{aligned}$$

For $k = 2$

$$\begin{aligned}p_1(\mathbb{CP}^4) = 5 , \ p_2(\mathbb{CP}^4) = 10 , \ \mathcal{L}_2 &= \frac{1}{45}(7p_2 - p_1^2) , \\ \sigma(\mathbb{CP}^4) &= \langle \mathcal{L}_2(\mathbb{CP}^4), [\mathbb{CP}^4] \rangle = 1 .\end{aligned}$$

For $k = 3$

$$\begin{aligned}p_1(\mathbb{CP}^6) = 7 , \ p_2(\mathbb{CP}^6) &= 21 , \ p_3(\mathbb{CP}^6) = 35 , \\ \mathcal{L}_3 &= \frac{1}{945}(62p_3 - 13p_2p_1 + 2p_1^3) , \\ \sigma(\mathbb{CP}^6) &= \langle \mathcal{L}_3(\mathbb{CP}^6), [\mathbb{CP}^6] \rangle = 1 .\end{aligned}$$

∎

CHAPTER 7

# The Homotopy and PL Classification of Generalized Lens Spaces

The classical lens spaces are the quotients of the sphere $S^3$ by the various free linear actions of the cyclic groups $\mathbb{Z}/m$ given by regarding $S^3$ as the set of pairs of complex numbers $(\zeta_1, \zeta_2)$ with $|\zeta_1|^2 + |\zeta_2|^2 = 1$ and defining the actions by

$$T(z_1, z_2) \;=\; (\zeta_m^{i_1} z_1, \zeta_m^{i_2} z_2)$$

with $\zeta_m = e^{2\pi i/m}$ and $(i_1, m) = (i_2, m) = 1$. We denote the quotient by $L_m(i_1, i_2)$. These three dimensional lens spaces can be immediately generalized to the higher dimensional lens spaces, $L_m(i_1, i_2, \ldots, i_n)$, given as the quotients of $S^{2n-1} = \{(z_1, \ldots, z_n) \mid \sum_1^n |z_i|^2 = 1\}$ by the free actions of $\mathbb{Z}/m$ given by

$$T(z_1, \ldots, z_n) \;=\; (\zeta_m^{i_1} z_1, \ldots, \zeta_m^{i_n} z_n)$$

with $(i_j, m) = 1$, $1 \leq j \leq n$.

Lemma 21. *Let $v \in \mathbb{Z}/m$ satisfy $vi_1 \equiv 1 \bmod (m)$ with $(i_1, m) = 1$. Then we have*

1. $L_m(i_1, \ldots, i_n) = L_m(1, vi_2, vi_3, \ldots, vi_n)$.
2. $L_m(i_1, \ldots, i_n) = L_m(i_{\sigma(1)}, i_{\sigma(2)}, \ldots, i_{\sigma(n)})$ *for any permutation $\sigma \in \mathcal{S}_n$, the symmetric group on $n$ letters.*

(This is evident.)

Of course one can generalize still further and consider *all* manifolds $M^{2n-1}$ given as the quotients of the odd dimensional spheres by the free action of the finite cyclic groups, $\mathbb{Z}/m$, to be the set of generalized lens spaces. In fact, this will be what we mean by generalized lens spaces in the sequel. In this context we will call the lens spaces constructed above the **linear models**.

One of the main motivations for the development of the non-simply connected surgery theory was the question of classifying the generalized lens spaces. This turns out to be a very difficult problem, not so much conceptually but technically. However, at this time the solution is basically complete. We will discuss aspects of the classification throughout this book as we develop the relevant techniques.

In the present chapter we review the more classical aspects of the classification, first the homotopy classification of these spaces, and then Whitehead's piecewise linear classification.

## 1. The homotopy classification of Generalized Lens Spaces

To do the homotopy classification we replace the sphere $S^{2n-1}$ by a space $X$ homotopy equivalent to $S^{2n-1}$ and consider the homotopy classification of quotients of $X^{2n-1}$ via free actions of the groups $\mathbb{Z}/m$. To avoid truely pathological situations we also assume throughout that our generalized lens spaces are given as $CW$-complexes so that there are cell decompositions of $X$ for which the action of $\mathbb{Z}/m$ becomes cellular and free. Specifically, we assume that the action of $\mathbb{Z}/m$ takes each cell homeomorphically to a different cell in the decompostion. Consequently, the cell complex of $X$ becomes a complex of *free* $\mathbb{Z}(\mathbb{Z}/m)$-modules with the boundary $\partial$ a $\mathbb{Z}(\mathbb{Z}/m)$-module map.

EXAMPLE 36. Cell decompositions for the linear models are relatively easy to construct. Thus, suppose $L_m(i_1, \ldots, i_n)$ is given. We define the 0-cells of the associated cellular decomposition of $S^{2n-1}$ as the elements $(\zeta_m^i, 0, \ldots, 0)$, the one cells as the element

$$e_1 = \{(z_1, 0, \ldots, 0) \mid 0 \leq arg(z_1) \leq 2\pi/m\}$$

so $T^i(e_1) = \zeta_m^i e_1$ and

$$S^1 = \{(z_1, 0, 0, \ldots, 0) \mid |z_1| = 1\} = \sum_{i=0}^{m-1} T^i(e_1).$$

Next, the two cells are given as

$$e_2 = \{(z_1, t, 0, \ldots, 0) \mid 0 \leq t \leq 1\}$$

and $T^i e_2 = (\zeta_m^{i_1} z_1, \zeta_m^{i_2} t, 0, \ldots, 0)$. Note that the boundary of $e_2$ is equal to the boundary of $T^i(e_2)$ and is the circle $S^1 = \sum_{i=0}^{m-1} T(e_1)$ in each case. This procedure generalizes directly so that the cellular chain complex of $S^{2n-1}$ is given as exactly one copy of the group ring $\mathbb{Z}(\mathbb{Z}/m)$ in each dimension $i$, $0 \leq i \leq 2n-1$. Moreover, the boundary map from $C_i$ to $C_{i-1}$ is described as follows:

$$\begin{cases} \partial(e_{2j-1}) = (T^{i_j} - 1)e_{2j-2} & \text{going from odd to even,} \\ \partial(e_{2i} = N e_{2i-1}, \ i > 0 & \text{going from even to odd,} \\ \partial(e_0) = 0 & \text{in dimension 0.} \end{cases}$$

Here, $N \in \mathbb{Z}(\mathbb{Z}/m)$ is the sum of all the elements in the group,

$$N = 1 + T + T^2 + T^3 + \cdots + T^{m-2} + T^{m-1}.$$

Now we turn to the homotopy classification of the generalized lens spaces.

LEMMA 22. (1) *The homology of a generalized lens space is isomorphic to that of $B\mathbb{Z}/m$ through dimensions $\leq 2n-2$, and is a single copy of $\mathbb{Z}$ in dimension $2n-1$.*

(2) *The (ordinary) homology of the classifying space $B\mathbb{Z}/m$ is given as*

$$\begin{cases} \mathbb{Z} & \textit{in dimension } 0, \\ \mathbb{Z}/m & \textit{in odd dimensions}, \\ 0 & \textit{in even dimensions} > 0. \end{cases}$$

(3) *The (ordinary) cohomology ring $H^*(B\mathbb{Z}/m;\mathbb{Z}) = \mathbb{Z}[b]/(mb = 0)$, the quotient of the integral polynomial ring on the two dimensional generator $b$, modulo the relation $mb = 0$.*

PROOF. Let $M^{2n-1}$ be a generalized lens space with universal cover $X$ having the homotopy type of the sphere $S^{2n-1}$ and having fundamental group $\mathbb{Z}/m$. Let $f\colon M \longrightarrow B\mathbb{Z}/m$ be a map classifying the universal cover. The homotopy fiber of $f$ is $X$ which is $2n-2$ connected. It follows, since by assumption $M$ is a $CW$ complex, that $f$ is a homotopy equivalence through dimension $2n-2$. This gives the first statement of $22(i)$. Before we can prove the second it is most convenient to prove $22(ii)$ and $22(iii)$.

Note that since $2n-1$ is odd, the action of $\mathbb{Z}/m$ on $H_*(X)$ is trivial. Also, the homotopy fiber of $f$ is $X$, and the Serre spectral sequence for the fibering takes the form

$$E_2^{i,j} = \begin{cases} H^i(B\mathbb{Z}/m;\mathbb{Z}), & \text{if } j = 0 \text{ or } 2n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, only one differential is possible, namely

$$d_i^{2n}\colon E_2^{i,2n-1} \longrightarrow E_2^{i+2n,0}.$$

But $M^{2n-1}$ is *finite dimensional*. Hence $E_{2n+1}^{i,j} = 0$ for $i+j \geq 2n$. It follows that $d_i^{2n}$ must be an isomorphism for $i \geq 1$. Thus,

$$H^{2n+i}(B\mathbb{Z}/m;\mathbb{Z}) \cong H^i(B\mathbb{Z}/m;\mathbb{Z}) \text{ for } i > 0.$$

We now calculate these groups explicitly for $i = 1, 2$ using $L_m(1,1,1,1)$ which has chain complex starting as

$$\mathbb{Z}(\mathbb{Z}/m) \xrightarrow{T-1} \mathbb{Z}(\mathbb{Z}/m) \xrightarrow{N} \mathbb{Z}(\mathbb{Z}/m) \xrightarrow{T-1} \mathbb{Z}(\mathbb{Z}/m) \longrightarrow 0.$$

Tensoring over the action of $\mathbb{Z}/m$ with $\mathbb{Z}$ regarded as the trivial $\mathbb{Z}(\mathbb{Z}/m)$-module $(\sum n_i T^i)1 = \sum n_i)$, gives the complex

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0.$$

and the homology is

$$\begin{cases} 0 & \text{in dimension } 2, \\ \mathbb{Z}/m & \text{in dimension } 1, \\ 0 & \text{in dimension } 0. \end{cases}$$

Hence, by the universal coefficient theorem

$$\begin{cases} H^0(B\mathbb{Z}/m;\mathbb{Z}) = \mathbb{Z}, \\ H^1(B\mathbb{Z}/m;\mathbb{Z}) = 0, \\ H^2(B\mathbb{Z}/m;\mathbb{Z}) = \mathbb{Z}/m. \end{cases}$$

Let $b$ be a generator for $H^2(B\mathbb{Z}/m;\mathbb{Z})$. Then the periodicity result, (refperiod 2) gives the result for $H^i(B\mathbb{Z}/m;\mathbb{Z})$ for all $i$.

In the Serre spectral sequence, when the coefficients are acted on trivially by the fundamental group of the base, $d^{2n}$ is a derivation, $d(w \cup j_{2n-1}) = w \cup d(j_{2n-1})$ where $j \in E_2^{0,2n-1} = \mathbb{Z}$ is a generator. Consequently, $b^n$ is a generator for $H^{2n}(B\mathbb{Z}/m;\mathbb{Z})$, and 22($ii$), 22($iii$) follow.

Moreover, the differential

$$d^{2n} \colon H^{2n-1}(X;\mathbb{Z}) \to H^{2n}(B\mathbb{Z}/m;\mathbb{Z})$$

has been identified as a surjective map $\mathbb{Z} \to \mathbb{Z}/m$ where $d^{2n}(j) = kb^n$ for some $k$ with $(k,m) = 1$. It follows that $H^{2n-1}(M;\mathbb{Z}) = Ker(d^{2n}) = \mathbb{Z}$ with generator $mj$. □

We can look at the argument above more closely to give the homotopy classification of the generalized lens spaces.

THEOREM 19. *Let $V \subset \mathbb{Z}/m$ be the multiplicative units in $\mathbb{Z}/m$. Then there is a one to one correspondence between the elements in the quotient $V/\langle \pm 1, V^n \rangle$ and the set of homotopy classes of generalized lens spaces. Here, $V^n$ denotes the subgroup of $V$ generated by all $n^{th}$ powers of elements in $V$.*

PROOF. There are three steps in the proof. The first is to construct a homotopy invariant of $X$ in the quotient above. The second is to show that if $M$ and $M'$ are generalized lens spaces having the same invariant then they are homotopy equivalent. Finally, the third is to show that for each element in the quotient of $V$ there is a generalized lens space which has that element as an invariant.

For the first step we refine the cohomology calculation of 22 to obtain

LEMMA 23. *Let $M$ be a generalized lens space with fundamental group $\mathbb{Z}/m$ and universal cover $X$ having the homotopy type of the sphere $S^{2n-1}$. The the following three statements are true.*

(1)
$$H^i(M;\mathbb{Z}/m) = \begin{cases} \mathbb{Z}/m & 0 \le i \le 2n-1, \\ 0 & \textit{otherwise.} \end{cases}$$

(2) *The $\mathbb{Z}/m \to \mathbb{Z}/m$ Bochstein operation $\beta$ gives an isomorphism*
$$\beta \colon H^{2j-1}(M;\mathbb{Z}/m) \to H^{2j}(M;\mathbb{Z}/m)$$
*for $j \le n-1.$,*

(3) *If $b$ is a generator of $H^2(M;\mathbb{Z}/m)$ then $b^i$ is a generator of the group $H^{2i}(M;\mathbb{Z}/m)$ for $i < n$.*

PROOF. The first statement is just a direct consequence of the universal coefficient theorems. The second statement follows similarly from the definition of the Bochstein map and the determination in 22 of the integral cohomology of $M$. Finally, the third statement follows directly from 22($iii$). □

Using this result we note that given any generator $e \in H^1(M;\mathbb{Z}/m)$ we can look at $e \cup (\beta(e))^{n-1} \in H^{2n-1}(M;\mathbb{Z}/m)$. Note that if $f$ is any other generator then $f = ue$ with $u \in V$, and $f \cup (\beta(f))^{n-1} = u^n(e \cup (\beta(e))^{n-1})$. Consequently, the element $e \cup (\beta(e))^{n-1}$ is independent of the choice of generator $e$ in $V/\langle V^n\rangle$.

On the other hand, from 22 we have that $H_{2n-1}(M;\mathbb{Z}) = \mathbb{Z}$ with generator either $\pm[M]$. Then, evaluating $e \cup (\beta(e))^{n-1}$ on the image of a generator for $H_{2n-1}(M;\mathbb{Z})$ gives a well determined element in the quotient group $V/\langle \pm 1, V^n\rangle$ which is clearly a homotopy type invariant for $M$. This is the invariant we will show completely determines the homotopy type of $M$.

REMARK . *To make this invariant well defined we need to identify the quotient above with a canonical quotient. Do this by choosing* $f^*(e\beta(e)^{n-1})$ *as the generator for* $V$*. Now,* $f$ *is well defined only up to a choice of generator for* $\mathbb{Z}/m$*, so the set of distinct* $f$ *which pull the universal covering on* $B\mathbb{Z}/m$ *back to the universal covering of* $M$ *are given as the compostions*

$$M \xrightarrow{f} B\mathbb{Z}/m \xrightarrow{Bu} B\mathbb{Z}/m$$

*where* $u \in V$ *andu*: $\mathbb{Z}/m \to \mathbb{Z}/m$ *is the automorphism* $T^j \mapsto T^{ju}$*. Consequently, this gives a well defined identification of* $H^{2n-1}(M;\mathbb{Z}/m)/\langle V^m\rangle$ *with* $H^{2n-1}(B\mathbb{Z}/m;\mathbb{Z}/m)/\langle V^m\rangle$*.*

We now consider again the classifying space $B\mathbb{Z}/m$. From 22($i$) and the description of the cell decomposition for $L_m(i_1,\dots,i_n)$ given in 36 we see that $B\mathbb{Z}/m = L_m(1,1,\dots,1)$ together with one $2n$-cell, one $2n+1$-cell, etc. In particular, $L_m(1,1,\dots,1)$ can be regarded as the $2n-1$-dimensional skeleton of $B\mathbb{Z}/m$. It follows that the map $f\colon M^{2n-1} \to B\mathbb{Z}/m$ factors as a composition

$$M^{2n-1} \xrightarrow{g} L_m(\underbrace{1,1,\dots,1}_{n-times}) \xrightarrow{i} B\mathbb{Z}/m$$

where $i$ embeds $L_m(1,\dots,1)$ as the $2n-1$ skeleton of $B\mathbb{Z}/m$. In particular, the map $g$ is a homotopy equivalence through dimension $2n-2$ from 22 and consequently induces a map of universal covers:

$$\begin{array}{ccc} X & \xrightarrow{\tilde{g}} & S^{2n-1} \\ \downarrow \pi & & \downarrow \pi \\ M^{2n-1} & \xrightarrow{g} & L_m(1,\dots,1) \end{array}$$

LEMMA 24. *The map $g$ in (1) is a homotopy equivalence if and only if*

$$\tilde{g}\colon X \to S^{2n-1}$$

*is a homotopy equivalence.*

PROOF. Both $M$ and $L_m(1,\ldots,1)$ are $CW$-complexes. Consequently a map between them will be a homotopy equivalence if and only if it induces isomorphisms of homotopy groups $g_*\colon \pi_i(M) \to \pi_i(L_m(1,\ldots,1))$ for all $i$. We already know this is the case for $i \le 2n-2$. But the diagram (1) extends to the following diagram

$$\begin{array}{ccccc} X & \xrightarrow{\pi} & M & \xrightarrow{f} & B\mathbb{Z}/m \\ \Big\downarrow{\scriptstyle \tilde{g}} & & \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle id} \\ S^{2n-1} & \xrightarrow{\pi} & L_m(1,\ldots,1) & \xrightarrow{i} & B\mathbb{Z}/m \end{array}$$

where the horizontal lines are fibrations. Consequently, by the five lemma, $g$ is a homotopy equivalence if and only if $\tilde{g}$ is a homotopy equivalence. □

Note that $g$ can be modified by pinching off a bubble on $M$ and mapping it to $S^{2n-1}$ with degree $k$. This will change $g$ to $g'$ and the map in homology from $H_{2n-1}(X;\mathbb{Z})$ to $H_{2n-1}(S^{2n-1};\mathbb{Z})$ will change from degree $s$ to degree $s+km$. But $g'$ will still be a perfectly good lifting of the classifying map $f$. Moreover, it is evident that the invariant for $M$ is exactly the image in the quotient group $V/\langle V^n, \pm 1\rangle$ of the degree of $\tilde{g}$ mod $(m)$. Consequently, $M$ is homotopy equivalent to $L_m(1,\ldots,1)$ if and only if the invariant in this quotient group is one.

Now suppose that $M$ and $M'$ have the same invariant. Exactly as we constructed the map $g\colon M \to L_m(1,\ldots,1)$ we can construct a map $\kappa\colon M \to M'$ which is a homotopy equivalence through dimension $2n-2$. And up to choice of $g$, $g$ factors as the composite

$$M \xrightarrow{\kappa} M' \xrightarrow{g'} L_m(1,\ldots,1).$$

Thus, the map $\kappa_*$ must take the invariant for $M$ to the invariant for $M'$. Moreover, it follows that lifting the map to fibers $\tilde{\kappa}\colon X \to X'$ will have degree $\pm 1 \in V/\langle V^n\rangle$, so, after changing $\kappa$ by a self map $M \to M$ of degree $u^n$ and further modifying by the pinching construction above, we can assume that the degree of $\tilde{\kappa}$ is actually $\pm 1$ and $M \simeq M'$.

To complete the proof it remains to show that all possible invariants are realized.

LEMMA 25. *The invariant for $L_m(1,1,\ldots,1,k)$ is $k \in V/\langle \pm 1, V^n\rangle$.*

PROOF. We determine the map explicitly by looking at the chain complexes. We have that the chain complexes of $L_m(1,1,\ldots,k)$ and $L_m(1,\ldots,1)$

are equal in dimensions less than $2n-2$. Then in this degree we have the following diagram

$$\begin{array}{ccccccc} C_{2n-1} & \xrightarrow{T^k-1} & C_{2n-2} & \xrightarrow{N} & C_{2n-3} & \longrightarrow \cdots \\ \Big\downarrow h_{2n-1} & & \Big\downarrow id & & \Big\downarrow id & \\ C_{2n-1} & \xrightarrow{T-1} & C_{2n-2} & \xrightarrow{N} & C_{2n-3} & \longrightarrow \cdots \end{array}$$

where each $C_i$ is a copy of the group ring $\mathbb{Z}(\mathbb{Z}/m)$. Then, in order to make the chain map commute $h_{2n-1}$ can be defined as

$$h_{2n-1} = 1 + T + T^2 + \cdots + T^{k-1},$$

and $h_{2n-1}(N) = kN$ in the chain complex of $L_m(1, \ldots, 1)$. □

This complete the proof of 19. □

## 2. The Torsion of Chain Complexes with Trivial Homology

Now we move away from homotopy classification to give a more delicate invariant that distinguishes spaces within a homotopy type. Thus, we suppose given a homotopy equivalence $f\colon X \to Y$ of $CW$ complexes. Whitehead explored the question of trying to refine the cell decompositions of $X$, $Y$ and deforming $f$ so as to construct an isomorphism of cell complexes. Based on earlier work of Reidemeister, Whitehead constructed an invariant in a group which depends only on the fundamental group of $X$, that was later called the Whitehead group $Wh_1(\pi_1(X))$.

**The torsion of a long exact sequence of free $R$-modules.** Throughout the remainder of this chapter we assume that $R$ is a unitary ring and suppose that

(1) $\mathcal{C} = \{\sqcup C_i \mid 0 \le i \le n < \infty, \partial\}$ is a chain complex of **finitely generated, free, and based** $R$-modules, with $\partial\colon C_i \to C_{i-1}$ $R$-linear.
(2) Suppose, moreover, that the resulting sequence
$$0 \longrightarrow C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \longrightarrow 0$$
is **long exact**, i.e., the homology of $\mathcal{C}$ is identically 0.

Let $S$ be a collapsing homotopy for $\mathcal{C}$, i.e., $S$ is a collection of $R$-linear maps $S_i\colon C_i \to C_{i+1}$ so that we have

$$S\partial + \partial S = id.$$

The existence of such an $S$ is established inductively and we may even assume that $S^2 = 0$. Note that $\mathcal{C}$ splits as the direct sum $Ker(\partial) \oplus \mathcal{W}$, and by a direct inductive argument we see that both summands are stably free. Moreover, in the case $S^2 = 0$ we can assume that $\mathcal{W} = Im(S)$. Hence, if we add a certain number of trivial based complexes $R \xrightarrow{id} R$ in dimensions $i+1$

and $i$ we may assume that this splitting decomposes $\mathcal{C}$ into the direct sum of two finitely generated free modules.

LEMMA 26. *Suppose that a triple $\{\mathcal{C}, \partial, S\}$ satisfying the conditions above is given. Let*

$$\begin{array}{lcl} \mathcal{C}_{ev} & = & \bigoplus_{i=0}^{\infty} C_{2i} \\ \mathcal{C}_{od} & = & \bigoplus_{i=0}^{\infty} C_{2i+1}. \end{array}$$

*Then*

(1) *$\mathcal{C}_{ev}$ and $\mathcal{C}_{od}$ are both finitely generated, free, and based $R$-modules.*
(2) *The map $\partial + S \colon \mathcal{C}_{od} \to \mathcal{C}_{ev}$ is an $R$-linear isomorphism.*

PROOF. The first statement is immediate. To see the second statement note that $\partial + S$ also gives a map $\mathcal{C}_{ev} \to \mathcal{C}_{od}$, and composing them in either order gives $\partial S + S\partial + S^2$ which has the form $1 + S^2$. This is the identity if $S^2 = 1$. If $S^2 \neq 1$ it is still an isomorphism since $1 - S^2 + S^4 - S^6 + \cdots$ is a well defined map from either piece to itself, and is clearly an inverse for $1 + S^2$. □

DEFINITION 37. *The* **torsion matrix** *of the triple $(\mathcal{C}, \partial, S)$ and the given basing of $\mathcal{C}$ is the matrix of the isomorphism $(\partial + S) \colon \mathcal{C}_{od} \to \mathcal{C}_{ev}$ with respect to the given bases.*

In the definition of the torsion matrix the basing of $\mathcal{C}$ and $\partial$ are given, but $S$ is arbitrary. We now consider the effect of varying our choice of $S$ on the isomorphism $\partial + S$.

Assume, as above, that a finite number of trivial based complexes $R \xrightarrow{id} R$ in dimensions $i+1$, $i$ have been added to $\mathcal{C}$ to give a complex which we again call $\mathcal{C}$ that has the property $Ker(\partial)$ and $\mathcal{W}$ are both free. Moreover, we assume that $S^2 = 0$ and $W = Im(S)$. Now, write

$$\mathcal{C}_{ev} = C_0 \oplus W_1 \oplus Ker(\partial_1) \oplus W_2 \oplus Ker(\partial_2) \oplus \cdots,$$

with a similar decomposition for $\mathcal{C}_{od}$. Finally, suppose a basis chosen which respects this splitting in each case.

LEMMA 27. *Let $S'$ be a second collapsing homotopy ($S'\partial + \partial S' = id$). Let $\Theta \colon \mathcal{C}_{ev} \to \mathcal{C}_{ev}$ and $\Theta' \colon \mathcal{C}_{od} \to \mathcal{C}_{od}$ satisfy*

$$\begin{array}{lcl} (\partial + S)\Theta & = & \partial + S' \\ \Theta'(\partial + S) & = & \partial + S'. \end{array}$$

*Then both $\Theta$ and $\Theta'$ are upper triangular matrices with ones along the diagonal in the bases above.*

PROOF. Note that

$$\begin{array}{rcl} \Theta & = & (\partial + S)^{-1}(\partial + S') \\ & = & (\partial + S)(\partial + S') \\ & = & \partial S' + S\partial + SS' \\ & = & 1 + \partial(S' - S) + SS', \\ & = & 1 + (S - S')\partial + SS'. \end{array}$$

Thus, for $v \in Ker(\partial)$ we have $\Theta(v) = v + SS'(v)$. In the case where $v \in Im(S)$ we have

$$(S' - S)\partial(v) = -\partial S'(v)$$

and $v \mapsto v + \partial\lambda + SS'(v)$. The argument for $\Theta'$ is similar. □

**The definition of the $K$-group $K_1(R)$ and Torsion.** Let $R$ be a ring. Let $GL_n(R)$ be the group of invertible $R$-linear maps of the free module of rank $n$, $R^n$ to itself. Let $E_n(R)$ be the subgroup generated by the "elementary transformations" of the form $E_{p,w}$, $v \mapsto v + p(v)w$ for some $R$-linear map $p\colon R^n \to R$ and a fixed vector $w \in Ker(p_i)$. Then the inverse of $E_{p,w}$ is $E_{p,-w}$.

LEMMA 28. *$E_n(R)$ is a normal subgroup of $GL_n(R)$.*

PROOF. Let $G \in GL_n(R)$. Then $G^{-1}E_{p,w}G = E_{pG^{-1},Gw}$. □

Consider an inclusion as direct summand with free complement, $R^n \subset R^{n+1}$. That is to say $R^{n+1} = Im(R^n) \oplus R$. Then the pair, (inclusion, splitting) induces an inclusion $I\colon GL_n(R) \subset GL_{n+1}(R)$, by $G \mapsto (G, 1)$. Moreover, clearly $I(E_n(R)) \subset E_{n+1}(R)$, so $I$ induces

$$I'\colon GL_n(R)/\langle E_n(R), GL_n(R)'\rangle \to GL_{n+1}(R)/\langle E_{n+1}(R), GL_n(R)'\rangle,$$

where $GL_n(R)'$ is the commutator subgroup of $GL_n(R)$.

LEMMA 29. *The inclusion $I'$ of (2)is independent of the pair (inclusion, splitting), $R^n \oplus R = R^{n+1}$.*

PROOF. Let $R^n \oplus R = R^{n+1}$ be any other pair (inclusion, splitting). Then there is an element $G \in GL_{n+1}(R)$ so that $G$ takes the first splitting to the second. It follows that $I(GL_n(R))$ for the first is conjugate to $I(GL_n(R))$ for the second in $GL_{n+1}(R)$. It follows that $G$ induces a conjugation action on the quotient $GL_{n+1}(R)/\langle E_{n+1}(R), GL_{n+1}(R)'\rangle$, taking the image of the first $I'$ to the image of the second. But since the commutator subgroup is contained in the normal subgroup the conjugation action of $G$ on the quotient is the identity. □

COROLLARY 16. *Passing to limits over $n$ is well defined for the quotient in (2)*

DEFINITION 38. *The limit group in 16 is the first algebraic $K$-group of the ring $R$ and is written $K_1(R)$.*

Consequently we have the following corollary/definition.

COROLLARY 17. *The class of the torsion matrix in 37 in $K_1(R)$ is independent of the choice of collapsing homotopy $S$ and thus gives a well defined invariant,* **the torsion of** $(\mathcal{C}, \partial)$, *which depends on the based complex $\mathcal{C}$ up to the addition of a finite number of trivial complexes of the form $R \xrightarrow{id} R$ in dimensions $i+1$ and $i$.*

**Some Calculational Results for $K_1(R)$.** Now we give some basic results which enable us to calculate the groups $K_1(R)$ for many types of unitary rings $R$.

LEMMA 30. (1) *Let $f\colon R \to S$ be a ring homomorphism. Then $f$ induces a well defined homomorphism $f_1\colon K_1(R) \to K_1(S)$.*
(2) *Let $g\colon S \to A$ be a second ring homomorphism then $(gf)_1 = g_1 f_1$.*
(3) *Let $M_n(R)$ be the ring of $n \times n$ matrices over $R$. Then the inclusions*

$$GL_{nr}(R) \xrightarrow{=} GL_r(M_n(R)),$$

*pass to limits to give isomorphisms $K_1(R) = K_1(M_n(R))$ for each $n \geq 1$.*
(4) *Let $R$ be a commutative ring. Then there is a well defined "determinant map", $K_1(R) \to R^*$, where $R^*$ is the group of units in $R$.*

PROOF. Only the last assertion needs any kind of explanation. Note that if we represent $GL_n(R)$ as the group of invertible matrices in $M_n(R)$ by choosing a basis for $R^n$ then the matrices $E_{p,w}$ are conjugates of elementary matrices $E_{i,j,r} = I + rE_{i,j}$, $i \neq j$ with $E_{i,j}$ given as the matrix which is all zeros except for a single one in the $(i,j)$-position. Consequently, the determinant of any element of $E_n(R)$ is 1. More directly, since the determinant map $GL_n(R) \to R^*$ is a group homomorphism to an abelian group, the image of any commutator is also one. Thus, the determinant map factors through the quotient in (2) Similarly, it is evident that we have the following commutative diagram

$$\begin{pmatrix} GL_n(R) & & \xrightarrow{I} & & GL_{n+1}(R) \\ & \searrow & & \swarrow & \\ & & R^* & & \end{pmatrix}$$

so the determinant map does, indeed, pass to limits. □

The following result is comparatively obvious but is also quite useful.

LEMMA 31. *Let $R$ be the direct sum of the rings $B$, $C$, $R = B \oplus C$. Then $K_1(R) = K_1(B) \oplus K_1(C)$.*

The following result is very useful in calculations.

LEMMA 32. *Let $R$ be any field or a Dedekind ring with quotient a finite extension of the rationals $\mathbb{Q}$. Then the determinant map $Det\colon K_1(R) \to R^*$ is an isomorphism.*

PROOF. For the case of fields it is standard that $E_n(R)i = SL_n(R)$ and is exactly the kernel of the determinant map. For the case of Dedekind rings the argument is much more complex. But see for example [Milnor- Algebraic $K$-Theory, §16, in particular 16.3 and the following discussion]. □

**The structure of $K_1(\mathbb{Z}(\zeta_d))$.** The case of most interest here is the ring of integers in the cyclotomic number field $\mathbb{Q}(\zeta_d)$, where $\zeta_d$ is a primitive $d^{th}$ root of unity, for example $e^{2\pi i/d} \in \mathbb{C}$. Here, for definiteness, since $\pm 1 \in \mathbb{Z}$, we assume that either $d \equiv 0 \bmod (4)$ or $d$ is odd.

(1) The degree of $\mathbb{Q}(\zeta_d)$ as an extension of $\mathbb{Q}$, is $\mu(d)$, the number of units in $\mathbb{Z}/d$.
(2) The maximal real subfield of $\mathbb{Q}(\zeta_d)$ is $\mathbb{Q}(\zeta_d + \zeta_d^{-1})$ and has degree $\mu(d)/2$ over $\mathbb{Q}$ provided $d \neq 2$.
(3) The ring of algebraic integers in $\mathbb{Q}(\zeta_d)$ is

$$\mathbb{Z}(\zeta_d) = \mathbb{Z} + \mathbb{Z}\zeta_d + \mathbb{Z}\zeta_d^2 + \cdots + \mathbb{Z}\zeta_d^{d-1}.$$

See e.g., [L. Washington, Introduction to Cyclotomic Fields, GTM, Springer-Verlag, # **83**, 1982, p. 11].

Thus, $K_1(\mathbb{Z}(\zeta_d)) = \mathbb{Z}(\zeta_d)^*$, the group of units in $\mathbb{Z}(\zeta_d)$.

THEOREM 20. **(Dirichlet)** *The group of units in the ring of algebraic integers $\mathcal{O}_K$, for $K$ any finite extension of $\mathbb{Q}$, has the form $\mathbb{Z}/l \times \mathbb{Z}^{s+r-1}$ where $2s + r$ is the degree of $K$ over $\mathbb{Q}$, and, for $\kappa$ any primitive generator of $K$, so $K = \mathbb{Q}(\kappa)$, then $r$ is the number of real roots of the minimal polynomial for $\kappa$. Of course, $\mathbb{Z}/l$ is the subgroup of roots of unity contained in $K$.*

(See e.g., [Lang, Algebraic Number Theory, Addison-Wesley, (1970)] for a proof.)

COROLLARY 18.

$$K_1(\mathbb{Z}(\zeta_d)) = \begin{cases} \mathbb{Z}/d \oplus \mathbb{Z}^{\mu(d)/2-1} & \textit{if } d \textit{ is even,} \\ \mathbb{Z}/2d \oplus \mathbb{Z}^{\mu(d)/2-1} & \textit{if } d \textit{ is odd.} \end{cases}$$

Certain units here arise very naturally. For example, if $a$ is prime to $d$, then

$$\frac{\zeta_d^a - 1}{\zeta_d - 1} = 1 + \zeta_d + \zeta_d^2 + \cdots + \zeta_d^{a-1}$$

is always a unit, [L. Washington, loc. cit., p.2], and for $d$ *not prime*, $\zeta_d - 1$, itself is a unit. These are examples of cyclotomic units.

DEFINITION 39. *Let $W_d \subset \mathbb{Q}(\zeta_d)^*$ be the multiplicative subgroup generated by the elements $\pm 1, \zeta_d, \zeta_d - 1, \zeta_d^2 - 1, \ldots, \zeta_d^{d-1} - 1$. Then the* **cyclotomic units***, $C_d \subset \mathbb{Z}(\zeta_d)$, are the elements in $W_d$ intersected with the units of $\mathbb{Z}(\zeta_d)$.*

Certainly, the cyclotomic units contain all the roots of unity in $\mathbb{Z}(\zeta_d)$, but they very nearly give all the units of $\mathbb{Z}(\zeta_d)$. Explicitly, we have the following results:

THEOREM 21. **(Ramachandra)** *The cyclotomic units have finite index in the set of all units of $\mathbb{Z}(\zeta_d)$.*

It is of some interest to give this index explicitly. To this end, let $h_2(d)$ be the class number of the real subfield $\mathbb{Q}(\zeta_d + \zeta_d^{-1})$ over $\mathbb{Q}$. (This is the size of the torsion subgroup of the Grothendieck group of finitely generated projective modules over the ring of integers in $\mathbb{Q}(\zeta_d + \zeta_d^{-1})$.) Fairly little is known about $h_2(d)$, but we have

- If $d$ is a prime power, $p^t$ with $(p-1)p^{t-1} < 66$ then $h_2(d) = 1$.
- If $d$ is composite with $d \leq 200$ and $\mu(d) \leq 72$, then $h_2(136) = 2$, $h_2(148)$, $h_2(152)$ are not known but for the rest $h_2(d) = 1$.
- $h_2(163) = 4$.
- For the prime $p = 11,290,018,777$ $h_2(p) > p$.
- Assume that $n$ is divisible by at least four distinct primes. Then, $h_2(n) > n^{3/2-\varepsilon}$, $\varepsilon > 0$ for infinitely many such $n$.

Then we have the following result which gives the index of the cyclotomic units in the entire unit group of $\mathbb{Z}(\zeta_d)$.

THEOREM 22. **(W. Sinnot)** *The index* $[K_1(\mathbb{Z}(\zeta_d)), C_d] = 2^b h_2(d)$*, where* $b = 0$ *if* $d$ *is a prime power, and* $b = 2^{g-2} + 1 - g$ *where* $g$ *is the number of distinct primes which divide* $d$ *otherwise.*

(See, e.g., [L. Washington, loc. cit., p. 147, 352] for details and references.)

**The Structure of $K_1(\mathbb{Z}(\mathbb{Z}/m))$.** Suppose $G = \mathbb{Z}/m$ is a cyclic group. Then

$$\mathbb{Q}(G) = \coprod_{d \text{ divides } m} \mathbb{Q}(\zeta_d)$$

contains the maximal order $\coprod_{d \text{ divides } m} \mathbb{Z}(\zeta_d)$ which in turn contains $\mathbb{Z}(G)$ since the natural inclusion

$$\mathbb{Z}(\mathbb{Z}/m) \hookrightarrow \mathbb{Q}(\mathbb{Z}/m) = \coprod_{d \text{ divides } m} \mathbb{Q}(\zeta_d)$$

has image contained in $\mathbb{Z}(\zeta_d)$ because projection to each factor is simply given as the map

$$\mathbb{Z}(\mathbb{Z}/m) \longrightarrow \mathbb{Q}(\zeta_d), \ \sum n_i T^i \mapsto \sum n_i \zeta_d^i.$$

Quite a bit deeper is the result of [Bass, Milnor, Serre, *Solution of the congruence subgroup problem for* $SL_n$ $(n \geq 3)$ *and* $Sp_{2n}$*,* $(n \geq 2)$, I.H.E.S. Publications Math., **33**(1967), 59-137, Proposition 4.14]

THEOREM 23. *The natural inclusion of rings induces an injection*

$$K_1(\mathbb{Z}(\mathbb{Z}/m)) \overset{i}{\hookrightarrow} \coprod_{d \text{ divides } m} \mathbb{Z}(\zeta_d)^*.$$

*for each cyclic group* $\mathbb{Z}/m$.

This amounts to the assertion that $K_1(\mathbb{Z}(\mathbb{Z}/m))$ is exactly the group of units in $\mathbb{Z}(\mathbb{Z}/m)$. Indeed, we have

COROLLARY 19. *An element $\theta \in \mathbb{Z}(\mathbb{Z}/m)$ is a unit if and only its image in $\mathbb{Z}(\zeta_d)$ is a unit for each $d$ dividing $m$.*

PROOF. This is immediate from the naturality of the determinant map, 30, and the fact that the determinant detects $K_1(\mathbb{Z}(\zeta_d))$, 32. □

This gives an effective way of constructing and recognizing all the units in $\mathbb{Z}(\mathbb{Z}/m)$.

EXAMPLE 37. Suppose that $p$ is a prime. Then there is a pull-back diagram

$$\begin{array}{ccc} \mathbb{Z}(\mathbb{Z}/p) & \xrightarrow{\pi_1} & \mathbb{Z}(\zeta_p) \\ \Big\downarrow \epsilon & & \Big\downarrow p \\ \mathbb{Z} & \xrightarrow{p} & \mathbb{Z}/p \end{array}$$

so $\alpha = \sum_0^{p-1} n_i T^i$ is a unit in $\mathbb{Z}(\mathbb{Z}/p)$ if and only if $\epsilon(\alpha) = \sum n_i = \pm 1$ and $\pi_1(\alpha) = \sum n_i \zeta_p^i$ is a unit in $\mathbb{Z}(\mathbb{Z}/p)$. For example, when $p = 5$, $\mathbb{Z}(\zeta_5)^* = \mathbb{Z}/10 \times \mathbb{Z}$ with generators $-\zeta_5$ and $\zeta_5 + \zeta_5^{-1} = \frac{1+\sqrt{5}}{2}$. Thus, $\mathbb{Z}(\mathbb{Z}/5)^* = \mathbb{Z}/(10) \times \mathbb{Z}$, and the pair $(-1, (\zeta_5 + \zeta_5^{-1})^2)$ is the image of a generator for the torsion free summand of $K_1(\mathbb{Z}(\mathbb{Z}/5))$. Indeed, the explicit class in $\mathbb{Z}(\mathbb{Z}/5)$ with this image is given as

$$(T^2 + 2 + T^{-2}) - (1 + T + T^2 + T^3 + T^4) = 1 - T - T^{-1}.$$

## 3. The Invariance of Torsion Under Cellular Subdivision

We continue to consider a long exact sequence of finite length of finitely generated based $R$-modules, $(\mathcal{C}, \partial)$. The torsion of this complex is defined in 17 as an element in $K_1(R)$, and now we want to consider the way in which the torsion changes under subdivision.

DEFINITION 40. *Assume that $X$ is a finite CW-complex with a cellular and free action of a finite group $G$. Then the cellular chain complex of $X$, $\mathcal{C}_i = H_i(X, X_{i-1}; \mathbb{Z})$ is a free $\mathbb{Z}(G)$-complex with a geometrically natural choice of basis, namely the image of one of the generating cells.*

Of course, this is only well defined up to multiplication by an element of $G$, and up to sign. But $\mathcal{C}$ as above always will have homology so is not suitable for defining torsion. However, it may be possible to make the complex exact by tensoring with a quotient ring. Another way in which we get exact complexes is in case there is a $G$-equivariant $CW$-subcomplex $Y \subset X$ and the inclusion is a homotopy equivalence. In this case the quotient complex $\mathcal{C}(X)/\mathcal{C}(Y)$ is naturally based up to multiplication by elements in $G$ and $\pm 1$, and is exact, so will have a well defined torsion class in $K_1(\mathbb{Z}(G))/\langle \pm 1, G \rangle$.

DEFINITION 41. *Assume that $X$ is a finite CW-complex with a cellular and free action of a finite group $G$, and $\mathcal{C}$ is is the cellular chain complex* *of $X$.* **A cellular subdivision** *of $X$ is a second CW-decomposition of $X$, $X'$, so that the interior of each $i$-cell of $X'$ is contained in a unique cell of $X$ of dimension at least $i$.*

THEOREM 24. **J.H.C. Whitehead** *Assumptions as above for the CW-pair $(X, Y)$. Let*

$$((\mathcal{C}(X'), \partial)$$

*be a subdivison of $\mathcal{C}(X)$. Then the torsion of $\mathcal{C}(X')/\mathcal{C}(Y')$ is equal to the torsion of $\mathcal{C}(X)/\mathcal{C}(Y)$ in $K_1(\mathbb{Z}(G))/\langle \pm 1, G\rangle$.*

PROOF. Order the generators of $\mathcal{C}(X)/(Y)$ so that if $dim(e_i) > dim(e_j)$ then $e_i < e_j$ in the ordering. Now, filter $X$ so that $F_s(X) = F_{s-1}(X) \cup_{g\in G} g e_j$ where $e_j$ is the $s^{th}$-element in the ordering of the basis and $F_0(X) = Y$. Associated to this is a filtration of $X'$, where $F_s(X')$ is the $G$-subcomplex of $X'$ generated by all the elements $h_{j,k}$ with $h_{j,k}$ in the support of $e_{i,l}$ for some $e_{i,l}$ in $F_s(X)$. Then consider the cell complex

$$X \cup_{0\times F_s(X)} I \times F_s(X) \cup_{1\times F_s(X)} F_s(X').$$

Clearly,

$$H_*(I \times F_s(X) \cup_{0\times F_s(X)} X, X; \mathbb{Z}) \equiv 0$$

for each $s$, so the torsion of each of the quotients

$$\mathcal{C}(F_s(X')) \bigcup I \times \mathcal{C}(F_s(X))$$

with boundary

$$\partial I \times e = \sum_{h\subset \bar{e}} \pm g_h \bar{h} - I \times \partial e$$

is defined, where $\bar{h} \in \{h, g_1 h, \ldots, h_n h\}$ is the chosen generator.

Now, factor out by $\mathcal{C}(Y)$, $\mathcal{C}(Y')$ so that the resulting quotient complexes have the form, separately, of exact complexes. Then it is clear that the torsion of the quotient is, on the one hand the torsion of $\mathcal{C}(X')/\mathcal{C}(Y')$ divided by the torsion of $\mathcal{C}(X)/\mathcal{C}(Y)$, and, on the other hand is the product of the torsions of the relative complexes

$$\big[\mathcal{C}I \times F_s(X) \cup \mathcal{C}F_s(X')\big] \,/\, \big[\mathcal{C}I \times F_{s-1}(X) \cup \mathcal{C}F_{s-1}(X')\big]$$

for $s \neq 0$. But, by excision, each of these complexes is – relatively – a disjoint union of the form $G \times (I \times D^i)$ where $D^i$ is a single cell. Consequently, it has torsion of the form $\pm g$ multiplied by the torsion of $I \times D^i \in K_1(\mathbb{Z}) = \pm 1$. ☐

EXAMPLE 38. The usual situation for the above construction is when we have a cellular map $f\colon X \to Y$ which is a homotopy equivalence. Then the mapping cylinder, $MC(\tilde{f})$ of the induced homotopy equivalence of universal covers

$$\tilde{f}\colon \tilde{X} \to \tilde{Y}$$

has the two subcomplexes $\mathcal{C}(\tilde{X})$ and $\mathcal{C}(\tilde{Y})$ both of which are homotopy equivalent to the total space $MC(\tilde{f})$ and there are two torsions defined: $\tau_1$, the torsion of $\mathcal{C}(MC(\tilde{f})/\mathcal{C}(\tilde{X})$ and $\tau_2 = \tau_1^{-1}$, the torsion of $\mathcal{C}(MC(\tilde{f})/\mathcal{C}(\tilde{Y})$.

This torsion is well defined as an invariant of the pair of $CW$-complexes up to cellular subdivision, $(X, Y)$ when we factor out the subgroup of all torsions which can occur in a self homotopy equivalence of $X$.

## 4. The Reidemeister-Whitehead Classification of Lens Spaces

The other situation where a torsion invariant could be defined for $X$ is if we tensor $\mathcal{C}(X)$ over $\mathbb{Z}(G)$ with a quotient ring $R$ so that the homology of $\mathcal{C}(X) \otimes_{\mathbb{Z}(G)} R \equiv 0$. Then the cellular basing of $\mathcal{C}(X)$ leads to a torsion class in $K_1(R)/\langle \pm 1, G \rangle$. The argument above in the proof of 24 extends in an evident way to show that the resulting torsion for $X$ is unchanged under cellular subdivision and hence provides an invariant of the equivalence class of the cellular decomposition of $X$.

THEOREM 25. *Let $X$ be a generalized lens space homotopic to*

$$L_m(1, \ldots, 1, k).$$

*Let $R = \mathbb{Z}(1/m)(G)/N$ where, as in (36), $N = 1 + T + \cdots + T^{m-1}$. Then, in the $CW$-complex of the universal cover $\tilde{X}$, $\mathcal{C}(\tilde{X})$ we have that*

$$\mathcal{C}(\tilde{X}) \otimes_{\mathbb{Z}(\mathbb{Z}/m)} R$$

*is an exact complex and consequently has a well defined torsion invariant in $K_1(R)/\langle \pm 1, G \rangle$.*

CHAPTER 8

# Surgery in Low Dimensions

## 1. Types of Modifications of Manifolds

**Blowing up submanifolds.** Let $V$ a complex and $\eta$ a $\mathbb{C}^k$-bundle over $V$. There is an associated $\mathbb{CP}^{k-1}$-bundle over $V$, given as the set of pairs $\bar{\eta} = \{(h, v)\}$ where $h \subset \eta_v$ is a complex line through the origin in the fiber of $\eta$ over $v \in V$. There is also a natural complex line bundle $L(\eta)$ over $\bar{\eta}$, given as the set of triples $L(\eta) = \{(w, h, v) \mid w \in h\}$.

LEMMA 33. *Suppose that $\eta$ has a hermitian metric, and let $W \subset \eta$ be the subspace of non-zero vectors of length $\leq 1$. Similarly, let $WL(\eta)$ be the subspace of the line bundle $L(\eta)$ consisting of the triples $(w, h, v)$ with $\vec{0} \neq w$ and $||w|| \leq 1$, then the map $p\colon WL(\eta) \to W$ sending $(w, h, v)$ to $w$ is a homeomorphism.*

PROOF. A non-zero vector $w \in \eta$ determines a unique complex line through the origin. ☐

Now, given any manifold $M^n$, a compact submanifold $V^{n-2k} \subset M^n$ and a complex structure on the normal bundle $\eta_M(V)$ to $V$ in $M^n$, then we can regard an open neighborhood of $V$ in $M$ as the subspace $W(\eta_M(V))$ and using the identification above we can replace this neighborhood by $WL(\eta_M(V))$. This construction is called **blowing up along** $V \subset M$ with respect to the given complex structure on $\eta_M(V)$. This is one of the oldest methods of obtaining new manifolds from old ones.

EXAMPLE 39. If $V$ is a point so $\eta = \mathbb{C}^k$ then $L(\eta)$ is the cannonical line bundle over $\mathbb{CP}^{k-1}$. Consequently, if $M^{2n}$ is a complex manifold and $p \in M^{2n}$ is a point, then $\eta_M(p) = \mathbb{C}^n$ and the blow-up of $M^{2n}$ at $p$ replaces $p$ by a copy of $\mathbb{CP}^{k-1}$.

If $V \subset M$ is a closed manifold with a complex structure given on $\eta_M(V)$ then $\partial(WL(\eta_M(V))) = \partial W(\eta_M(V))$ and we can regard the blow-up of $M$ over $V$ as obtained from $M - Int(W(\eta_M(V)))$ by identifying $\partial(WL(\eta_M(V)))$ with $\partial\{M - Int[W(\eta_M(V))]\}$.

There is a degree one map from the blow-up of $M$ over $V$ to $M$ which collapses the $\mathbb{CP}^{k-1}$-bundle over $V$ to $V$ and is otherwise $1 \leftrightarrow 1$. But, in general, the blow-up over $V$ is not even cobordant with $M$. For example if $n = 4k$ and we blow up a point, then the signature of the blow-up is $\sigma(M) - 1$, and since $\sigma(M)$ is a bordism invariant the two manifolds cannot be bordant.

**Framed surgery and spherical replacement in dimension 3.** Let $K \subset M^3$ be a knot, i.e., a differentiable embedding of $S^1 \subset M^3$. If the normal bundle to $K$ is trivial and a framing is given, then this gives an embedding $D^2 \times S^1 \subset M^3$ extending the original embedding $K \subset M^3$. Now, $\partial(D^2 \times S^1) = S^1 \times S^1$ and this is also the boundary of $S^1 \times D^2$. Consequently, we can replace the neighborhood $Int(D^2 \times S^1) \subset M^3$ of $K$ by $S^1 \times D^2$ by simply identifying the two $S^1 \times S^1$ boundary components. This is called **framed surgery on the knot** $K$.

LEMMA 34. *Let* $\mathbb{C} \times S^1 \xrightarrow{p} S^1$ *be the trivial* $\mathbb{R}^2$*-bundle over* $S^1$*, (with a choice of complex structure), then the set of framings is identified with the integers, where the framing associated to* $m$ *is given by* $\psi_m(v, \zeta) = (\zeta^m v, \zeta)$*. In particular, the map in homology* $(\psi_m)_*\colon H_1(S^1 \times S^1; \mathbb{Z}) \to H_1(S^1, \times S^1; \mathbb{Z})$ *is given by*

$$\begin{aligned} (\psi_m)_*(e_1) &= e_1 \\ (\psi_m)_*(e_2) &= me_1 + e_2 \end{aligned}$$

EXAMPLE 40. We can write $S^3 = \partial(D^2 \times D^2) = S^1 \times D^2 \cup_{S^1 \times S^1} D^2 \times S^1$. Consequently, replacing the second $D^2 \times S^1$ by $S^1 \times D^2$ after using $\psi_m$ to reframe it, we get

$$W^3 = S^1 \times D^2 \cup_{S^1 \times S^1} S^1 \times D^2$$

where the $D^2$ in the second $S^1 \times D^2$ is attached to $e_1 + me_2$, while the first $D^2$ is attached to $e_1$. Consequently, $H_1(W^3; \mathbb{Z}) = \mathbb{Z}/m$, and it is not hard to see that $W^3$ is the ordinary lens space $L^3_m(1)$.

REMARK . The most general type of replacement for $K$ is obtained by gluing $D^2 \times S^1$ to $M^3 - Int(D^2 \times S^1)$ via a homeomorphism

$$\psi\colon S^1 \times S^1 \to S^1 \times S^1.$$

Clearly, if $\psi$ is concordant to $\lambda$ then the resulting manifolds

$$(M^3 - Int(D^2 \times S^1)) \cup_\psi (D^2 \times S^1)$$

and

$$(M^3 - Int(D^2 \times S^1)) \cup_\lambda (D^2 \times S^1)$$

are diffeomorphic, so the distinct constructions are indexed by a quotient of the concordance classes of diffeomorphisms $\psi\colon S^1 \times S^1 \to S^1 \times S^1$.

It is relatively direct to verify that the set of these concordances is identified with $SL_2(\mathbb{Z})$: if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ then the associated diffeomorphism of $S^1 \times S^1$ is given by

$$(\zeta, \nu) \mapsto (\zeta^a \nu^b, \zeta^c \nu^d),$$

and in homology the map is given by $e_1 \mapsto ae_1 + ce_2$, $e_2 \mapsto be_1 + de_2$. In particular, if $a = m$, $b = j$, then the result of attaching $D^2 \times S^1$ to itself by the above diffeomorphism is the lens space $L^3_m(j)$.

**Spherical modification or surgery in general.** Suppose that an embedding

$$\psi\colon S^k \times D^{n-k} \hookrightarrow M^n$$

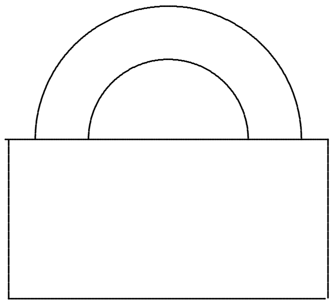
is given. Then $\partial(S^k \times D^{n-k}) = S^k \times S^{n-k-1}$ and this is also the boundary of $D^{k+1} \times S^{n-k-1}$. Consequently, we can replace $im(\psi)$ by $D^{k+1} \times S^{n-k-1}$ as above:

$$\bar{M}^n = M - Int(S^k \times D^{nk}) \cup_{S^k \times S^{n-k-1}} D^{k+1} \times S^{n-k-1}.$$

Such a modification is called **a spherical modification or surgery on** $\psi$.

Consider the $n+1$ dimensional manifold with boundary, $W(\psi)$, constructed from $I \times M^n$ and $D^{k+1} \times D^{n-k}$ by gluing the component $S^k \times D^{n-k}$ of $\partial(D^{k+1} \times D^{n-k})$ to $1 \times M^n$ by $\psi$:

$D^{k+1} \times D^{n-k}$



$I \times M$

Clearly, $W(\psi)$ is obtained from $I \times M^n$ by attaching a handle, 2, on

$$\psi(S^k \times D^{n-k} \subset 1 \times M^n$$

contained in $\partial(I \times M^n)$. We call $W(\psi)$ the **elementary bordism from** $M$ **to** $M'$ associated to the embedding $\psi$. $W(\psi)$ is alos called the **trace of** $\psi$. Also, $\partial(W(\psi)) = M \sqcup \bar{M}$ where $\bar{M}$ is the manifold obtained from $M$ by doing surgery on $\psi(S^k \times D^{n-k})$. Consequently, from the handle decomposition theorem, 17, we have

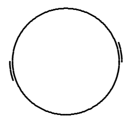
Theorem 26. *Let $M^n$ and $(M^n)'$ be closed compact differentiable manifolds without boundaries, then $(M^n)'$ is obtained from $M^n$ by a sequence of surgeries if and only if $M^n$ is cobordant to $(M^n)'$.*

In fact, given a bordism $W^{n+1}$ from $M^n$ to $(M^n)'$, 17 shows that $W$ is obtained from $I \times M$ by a sequence of elementary bordisms.

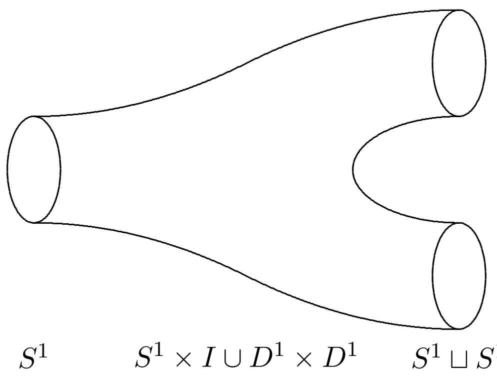
Remark . The following *duality property* of surgery and elementary bordisms is often useful. Clearly, if $M'$ is obtained from $M$ by a single surgery on $S^k \times D^{n-k} \hookrightarrow M$, then $M$ is obtained from $M'$ by a single surgery on $\psi'\colon D^{k+1} \times S^{n-k-1} \hookrightarrow M'$ and $W(\psi) = W(\psi')$.

## 2. Further Examples of the effect of surgery

The special case of surgery on the circle is both easy and very illuminating. First consider the case where we embed $S^0 \times D^1$ with the $D^1$ embedded in such a way that the induced orientation from the orientation of $S^1$ agrees with the that on $S^0 \times D^1$.



In this case the result of the surgery is two disjoint circles, and $W(M)$ is the pair of pants



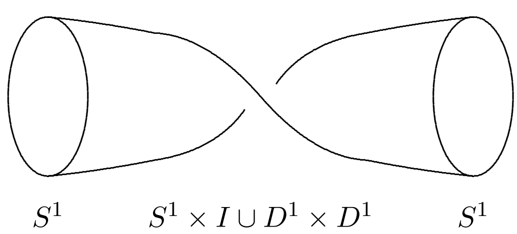

Modify the framed 0-embedding $e : S^0 \times D^1 \to S^1$ in (ii) by twisting one of the two embeddings of $D^1$ by the orientation-reversing diffeomorphism

$$\omega \ : \ D^1 \to D^1 \ ; \ t \to -t \ ,$$

defining a different 0-embedding

$$e_\omega \ : \ S^0 \times D^1 \xrightarrow{1 \sqcup \omega} S^0 \times D^1 \xrightarrow{e} S^1$$

with the same core as $e$. The 0-surgery on $S^1$ removing $e_\omega(S^0 \times D^1)$ has effect a single circle $S^1$. The trace is the non-orientable cobordism $(N^2; S^1, S^1)$ obtained from the Möbius band $M^2$ by punching out the interior of an embedding $D^2 \subset M \backslash \partial M$.

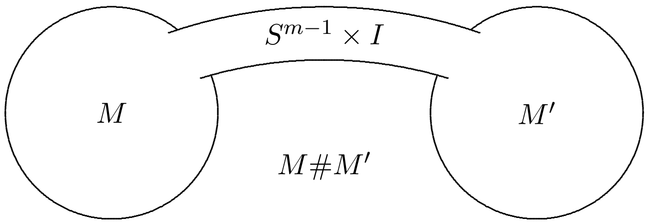


We have already seen in FRAMED SURGERY ON A KNOT the importance of the choice of the framing, i.e., the way in which the map on the core $S^m$ is extended to $S^m \times D^{n-m}$. The general situation for the variation of the framing is that the map is changed by a map $\tau\colon S^m \to O(n-m)$, so $\psi_\tau(s,v) = (s, \tau(s)v)$.

Definition 42. *The* **connected sum** *of connected m-dimensional manifolds $M^m$, $M'^m$ is the connected m-dimensional manifold*

$$(M \# M')^m \;=\; \mathrm{cl.}(M\backslash D^m) \cup S^{m-1} \times I \cup \mathrm{cl.}(M'\backslash D^m)$$

*obtained by excising the interiors of embedded discs $D^m \subset M$, $D^m \subset M'$ and joining the boundary components $S^{m-1} \subset \mathrm{cl.}(M\backslash D^m)$, $S^{m-1} \subset \mathrm{cl.}(M'\backslash D^m)$ by $S^{m-1} \times I$.*




Example 41. The connected sum $M\#M'$ is the effect of the 0-surgery on the disjoint union $M \sqcup M'$ which removes the framed 0-embedding $S^0 \times D^m \subset M \sqcup M'$ defined by the disjoint union of embeddings $D^m \to M$, $D^m \to M'$.

We can push this last example a bit further. Suppose we embed our $S^m \times D^{n-m}$ inside a ball $D^n$ in $M^n$:

$$S^m \times D^{n-m} \subset D^n \subset M.$$

Then, when we do surgery on $S^m \times D^{n-m}$ the result is the connected sum $M^n\#S^{m+1} \times S^{n-m-1}$ since we can regard $M^n$ as the connected sum $M^n\#S^n$ with the $S^n - *$ being identified with $D^n$, and we can write

$$S^n = S^m \times D^{n-m} \cup_{S^m \times S^{n-m-1}} D^{m+1} \times S^{n-m-1}$$

and replacing the first $S^m \times D^{n-m}$ by a second copy of $D^{m+1} \times S^{n-m-1}$ and gluing by the identity map on $S^m \times S^{n-m-1}$ gives $S^{m+1} \times S^{n-m-1}$ as asserted.

For example, doing surgery on $S^{m-1} \times D^{m+1} \subset D^{2m} \subset M^{2m}$ results in the connected sum

$$M^{2m}\#S^m \times S^m$$

and, in $M^{2m}\#S^m \times S^m$, if we do surgery on one of the cores,

$$S^m \times D^m \subset S^m \times S^m$$

this undoes the previous surgery, giving $M^{2m}$ as the result.

Of course, this last cancelling surgery assumes that we took the correct framing on $S^m$, a change of framing by a map $\tau\colon S^m \to O(m)$ could, perhaps, change the result. We will see in the next section that, in this dimension, it cannot change the effect on homology of the surgery. But it could, conceivably, change the diffeomorphism type of the resulting manifold. We can see this more clearly in the next example.

The map $t\colon S^{2m-1}{\to}O(2m)$ which gives the clutching map for the tangent bundle $\tau(S^{2m})$ has the property that if we take the associated map

$$S^{2m-1}\times S^{2m-1}\xrightarrow{Adj(t)}S^{2m-1}\times S^{2m-1},$$

$Adj(t)(\vec{x},\vec{y}) = (\vec{x}, t(\vec{x}(\vec{y}))$, then in homology

$$Adj(t)_*(e_{\vec{x}}) = e_{\vec{x}} + 2e_{\vec{y}},\ \ Adj(t)_*(e_{\vec{y}}) = e_{\vec{y}}).$$

One way of seeing this is to note that the Euler class of $\tau(S^{2m}) = \chi(S^{2m}) = 2$, and it follows that in the usual inclusion of $S^{2m} \hookrightarrow T(\tau(S^{2m}))$ the Thom class $U_{\tau(S^{2m})}$ pulls back to the Euler class, $2[S^{2m}]^*$. On the other hand, if $S^{2m-1}{\to}E{\to}S^{2m}$ is the associated sphere bundle to $\tau(S^{2m})$ then $T(\tau(S^{2m}))$ is the quotient $S^{2m}/E$ and, from the long exact sequence in cohomology

$$\cdots\longrightarrow H^*(T(\tau(S^{2m}))\longrightarrow H^*(S^{2m})\longrightarrow H^*(E)\xrightarrow{\delta}H^{*+1}(T(\tau(S^{2m}))\longrightarrow\cdots$$

we have

$$H^*(E) \;=\; \begin{cases}\mathbb{Z} & \text{if } * = 0 \text{ or } 4m-1,\\ \mathbb{Z}/2 & \text{if } * = 2m,\\ 0 & \text{otherwise.}\end{cases}$$

On the other hand $E$ can be given as

$$D^{2m}\times S^{2m-1}\cup_{S^{2m-1}\times S^{2m-1}} D^{2m}\times S^{2m-1}$$

where the identifiying map on the $S^{2m-1}\times S^{2m-1}$ is $t$ above. Comparing the result of the Mayer-Vietoris sequence for this description of $E$ with the cohomology calculation above gives the assertion.

It follows that the composition $kt$:

$$S^{2m-1}\xrightarrow{k}S^{2m-1}\xrightarrow{t}O(2m)$$

where $k\colon S^{2m-1}{\to}S^{2m-1}$ is the degree $k$ map, gives $Adj(kt)(e_{\vec{x}}) = e_{\vec{x}}+2ke_{\vec{y}}$, $Adj(kt)(e_{\vec{y}}) = e_{\vec{y}}$.

Now, suppose we take $S^{4m-1}$ and inside $S^{4m-1}$ we do surgery on $S^{2m-1}\times D^{2m}$, but, instead of simply replacing, we first reframe via the map $kt$. Then, the manifold resulting from the surgery on the reframed $S^{2m-1}\times D^{2m}$ is $D^{2m}\times S^{2m-1}\cup_{S^{2m-1}\times S^{2m-1}} D^{2m}\times S^{2m-1}$, with associated Mayer-Vietoris sequence

$$\begin{aligned}\cdots\qquad &\xrightarrow{\delta}H_{2m-1}(S^{2m-1}\times S^{2m-1})\\ &\xrightarrow{i}H_{2m-1}(D^{2m}\times S^{2m-1})\oplus H_{2m-1}(D^{2m}\times S^{2m-1})\\ &\longrightarrow H_{2m-1}(M)\xrightarrow{\delta}\cdots\end{aligned}$$

and the map $i_*$ has the property $e_{\vec{x}} \mapsto 2ke_1$, $e_{\vec{y}} \mapsto e_1 + e_2$, which has matrix

$$\begin{pmatrix} 2k & 0 \\ 1 & 1 \end{pmatrix}$$

with determinant $2k$. It follows that $H_{2m-1}(M) = \mathbb{Z}/2k$, which shows that in this case, the reframing does change the homology of the resulting manifold.

## 3. The effect of surgery on homology

Consider the effect of surgery on $\psi(S^k \times D^{n-k}) \subset M^n$. We have that the homotopy type of the elementary cobordism $W(\psi)$ is that of $M^n$ with a single $(k+1)$-cell attached from 14,

$$W(\psi) \simeq M^n \cup_\alpha e^{k+1}.$$

But, since $W(\psi)$ is the same as the elementary cobordism from $\bar{M}^n$ to $M^n$ obtained by attaching $D^{k+1} \times D^{n-k}$ on $D^{k+1} \times S^{n-k-1} \subset \bar{M}^n$ it follows that

$$W(\psi) \simeq \bar{M}^n \cup_\beta e^{n-k}.$$

Consequently, comparing these two expressions for $W(\psi)$ we have

LEMMA 35. *Suppose that $k+1 \leq \left[\frac{n}{2}\right]$. Then, given $\psi\colon S^k \times D^{n-k} \hookrightarrow M^n$, the manifold $\bar{M}^n$ obtained from $M^n$ by surgery on $\psi$ has the homotopy type of $M^n \cup_{\psi(S^k)} e^{k+1}$ through dimensions less than $n-k$.*

Surgery in dimension $m$ for $M$ of dimension $n = 2m-1$ or $n = 2m$ is more complex. In both cases, in order to understand the effect of surgery we need to consider the manifold with boundary $W(M) = M - (S^m \times D^{n-m})$. We have, by excision,

$$H_*(W(M), \partial(W(M))) = H_*(M, S^m)$$

and there is an exact sequence

$$\cdots \longrightarrow H_*(S^m) \xrightarrow{\psi_*} H_*(M) \longrightarrow H_*(M, S^m) \xrightarrow{\partial} H_{*-1}(S^m) \longrightarrow \cdots$$

which in only non-trivial in the range around $* = m$:

$$0 \longrightarrow H_{m+1}(M) \longrightarrow H_{m+1}(M, S^m) \xrightarrow{\partial}$$
$$H_m(S^m) = \mathbb{Z} \xrightarrow{\psi_*} H_m(M) \longrightarrow H_m(M, S^m) \longrightarrow 0.$$

More generally, in the non-simply connected case we can consider the universal cover and the associated embedding

$$\pi_1(M) \times (S^m \times D^{n-m}) \overset{\tilde{\psi}}{\hookrightarrow} \tilde{M}$$

which leads to the generalization of 3

$$0 \longrightarrow H_{m+1}(\tilde{M}) \longrightarrow H_{m+1}(\tilde{M}, \pi_1(M) \times S^m) \xrightarrow{\partial}$$
$$\mathbb{Z}(\pi_1(M)) \xrightarrow{\tilde{\psi}_*} H_m(\tilde{M}) \longrightarrow H_m(\tilde{M}, \pi_1(M) \times S^m) \longrightarrow 0.$$

Then the boundary map from $H_*(W(\tilde{M}),\partial)$ to $\partial(\pi_1(M)\times S^m\times D^{n-m})$ is related to the $\partial$-map in 3 through the commutative diagram

$$\begin{array}{ccc}
H_{*+1}(W(\tilde{M}),\pi_1\times S^m\times S^{n-m-1}) & \xrightarrow{\partial} & H_*(\pi_1\times S^m\times S^{n-m-1}) \\
\Big\downarrow \cong & & \Big\downarrow p \\
H_{*+1}(\tilde{M},\pi_1(M)\times S^m) & \xrightarrow{\partial} & H_*(\pi_1(M)\times S^m)
\end{array}$$

where $p$ is projection on the middle factor (homotopy equivalent to the inclusion $S^m\times S^{n-m-1}\hookrightarrow S^m\times D^{n-m}\simeq S^m$). On the other hand, there are two Mayer-Vietoris sequences which can be used to connect the homology of $M'$ and $\tilde{M}'$ with the groups discussed above, the usual one,

$$\cdots\longrightarrow H_{*+1}(\tilde{M}')\xrightarrow{\partial}H_*(\pi_1(M)\times S^m\times S^{n-m-1})\longrightarrow$$
$$H_*(W(\tilde{M}))\oplus H_*(D^{m+1}\times S^{n-m-1})\longrightarrow H_*(\tilde{M}')\xrightarrow{\partial}\cdots$$

and the sequence of relative groups (which is Poincaré dual to the usual cohomology Mayer-Vietoris sequence for 3),

$$\cdots\longrightarrow H_{*+1}(\tilde{M}')\longrightarrow$$
$$H_{*+1}(W(\tilde{M}),\pi_1(M)\times S^m\times S^{n-m-1})\oplus$$
$$H_{*+1}(\pi_1(M)\times(D^{m+1}\times S^{n-m-1},S^m\times S^{n-m-1}))$$
$$\xrightarrow{\delta}H_*(\pi_1(M)\times S^m\times S^{n-m-1})\longrightarrow H_*(\tilde{M}')\longrightarrow\cdots$$

Away from the middle dimensions we do not need these tools as the result of 35 will be sufficient for our needs here, but for surgery on $S^m\times D^{n-m}$ where $n=2m$ or $n=2m+1$ we do since

(1) In 3, for $n=2m+1$ we have

$$\tilde{H}_*(\pi_1(M)\times S^m\times S^m)=\begin{cases}0 & *\neq m,\\ \mathbb{Z}(\pi_1)\oplus\mathbb{Z}(\pi_1) & *=m.\end{cases}$$

Consequently, the effect of surgery is determined by the image of $\partial$ in 3 on the second factor in the upper right hand corner, a map not determined directly by the image of $\psi(S^m)$.

(2) For $n=2m$ we can replace $H_*(W(\tilde{M}),\pi_1(M)\times S^m\times S^{n-m-1})$ by $H_*(\tilde{M},\pi_1(M)\times S^m)$ in 3, and, in dimension $m$ we obtain the exact sequence

$$0\longrightarrow H_m(\tilde{M}')\longrightarrow H_m(\tilde{M},\pi_1(M)\times S^m)\xrightarrow{\delta}\mathbb{Z}(\pi_1(M))$$

where $\delta$ is again determined by more than just the image of $\psi_*$.

In the case $n=2m$ we have the following result which shows that the most difficult part of understanding the homology effect of surgery in this dimension is to be able to embed a sphere with trivial normal bundle where we can control the intersection numbers.

LEMMA 36. *The image of $\delta$ in 2 is the ideal of all elements $\mu \in \mathbb{Z}(\pi_1(M))$ which occur as $\psi(S^m) \cap e$ for $e \in H_m(\tilde{M};\mathbb{Z})$.*

PROOF. Let $v \in H_m(\tilde{M})$ be represented by a map of an oriented $m$-dimensional manifold $V^m \to \tilde{M}$, and consider the image of $v$ under the composite map

$$H_m(\tilde{M}) \xrightarrow{j_*} H_m(\tilde{M}, \pi_1 \times S^m \times D^m) \xrightarrow{\cong} H_m(W(\tilde{M}), \pi_1 \times S^m \times S^{m-1}).$$

Here $\cong$ is the inverse of the excision isomorphism, and composing with $\partial\colon H_m(W(\tilde{M}), \pi_1 \times S^m \times S^{m-1}) \to H_{m-1}(\pi_1 \times S^m \times s^{m-1}) = \mathbb{Z}(\pi_1)$ is just the intersection pairing $V^m \cap \pi_1 \times S^m$. Thus the result works for classes represented by manifolds. However, by Thom's representation theorem, REFERENCE, a finite multiple of every element in $H_m(\tilde{M})$ has this property, so the lemma follows by the linearity of the composite map. □

REMARK . Since $\psi(S^m) \subset S^m \times D^m \subset M$ it follows that the self-intersection of $\psi_*([S^m])$ with itself in $H_*(\tilde{M};\mathbb{Z})$ is zero.

REMARK . In the case where the ideal in $\mathbb{Z}(\pi_1(M))$ given by capping with $\psi_*([S^m])$ is $\mathbb{Z}(\pi_1(M))$ we see that $H_m(\tilde{M}')$ is given as

$$Ker[\cap(\psi_*([S^m]))] \subset H_m(\tilde{M})/\mathbb{Z}(\pi_1)\psi_*([S^m]).$$

The situation when $n = 2m+1$ turns out to be harder. What matters is the image of

$$\begin{aligned}\partial\colon H_{m+1}(W(\tilde{M}), \pi_1(M) \times S^m \times S^m) \quad &\longrightarrow H_m(\pi_1 \times S^m \times S^m)\\ &= \mathbb{Z}(\pi_1(M)) \oplus \mathbb{Z}(\pi_1(M)).\end{aligned}$$

Let us define an orientation class for $\pi_1(M) \times S^m \times S^m$ as $\partial([W(\tilde{M})])$. Then this defines a non-singular cap-product pairing on $\mathbb{Z}(\pi_1(M)) \oplus \mathbb{Z}(\pi_1(M))$ which is $(-1)^m$-Hermitian symmetric with respect to the involution $\sum n_i g_i \leftrightarrow \sum n_i g_i^{-1}$ on $\mathbb{Z}(\pi_1(M))$:

$$\begin{aligned}\langle \alpha, \beta \rangle &= (-1)^m \overline{\langle \beta, \alpha \rangle},\\ \langle v\alpha, \beta \rangle &= \bar{v}\langle \alpha, \beta \rangle,\end{aligned}$$

and we have

LEMMA 37. *Let $e = \partial(w) \in H_m(\pi_1(M) \times S^m \times S^m)$, then $\langle e, e \rangle = 0$ under the cap-product pairing above.*

PROOF. Let $v \in Hom_{\mathbb{Z}(\pi_1)}(H_m(\pi_1 \times S^m \times S^m), \mathbb{Z}(\pi_1))$ be dual to $e$ so $e = v \cap \partial[W(\tilde{M})]$. Then

$$\begin{aligned}\langle e, e \rangle &= v(e)\\ &= v(v \cap \partial[W(\tilde{M})]\\ &= \delta(v)(i_*(e))\\ &= 0.\end{aligned}$$

□

This turns out to be about all that we can say about the image, but it will turn out that it's almost as much as we will need.

## 4. Normal maps and surgery below the middle dimension

DEFINITION 43. *Suppose that $X$ is a locally finite CW-complex and $M^n$ is an $n$-dimensional compact, closed manifold with empty boundary. Then a continuous map $f\colon M^n \to X$ is said to be a* **normal map** *if there is a $k$-dimensional vector bundle $\chi$ over $X$ so that $f^!(\chi)$ is a stable normal bundle for $M^n$.*

Precisely, this means of course, that $f^!(\chi) \oplus \tau_M = (n+k)\epsilon$, the trivial bundle with fiber $\mathbb{R}^{n+k}$ over $M^n$. But the data does not include a preferred trivialization.

The most important property of normal maps is given by the following lemma.

LEMMA 38. *Let $f\colon M^n \to X$ be a normal map as defined in 43. Let $g\colon S^m \hookrightarrow M^n$ be a differential embedding so that the composition $fg$ is homotopic to the trivial map. Then the normal bundle to $g(S^m)$ in $M^n$, $\eta_{M^n}(g(S^m))$, is stably trivial. In particular, if $m < \left[\frac{n}{2}\right]$, then $\eta_{M^n}(g(S^m))$ is the trivial bundle $\mathbb{R}^{n-m} \times S^m$.*

PROOF. The normal bundle to $M^n$ restricted to $g(S^m)$ is $f^!(\chi)$ restricted to $g(S^m)$ and this is $(fg)^!(\chi)$ which is trivial, since $fg$ is homotopic to the trivial map. But the Whitney bundle sum of $\eta_M(g(S^m))$ with the normal bundle to $M$ restricted to $g(S^m)$ is the normal bundle to the composite embedding

$$S^m \overset{g}{\hookrightarrow} M^n \hookrightarrow \mathbb{R}^{n+l}$$

for $l \geq n+1$. Consequently, this bundle sum is trivial:

$$\eta_M(S^m) \oplus (fg)^!(\chi) = (l+n-m)\epsilon$$

and $\eta_M(S^m)$ is stably trivial as required. Moreover, if $m < \left[\frac{n}{2}\right]$ then $n-m > m$ and $\eta_M(S^m)$ being stably trivial guarantees that it is actually trivial by 8. □

This, given a normal map we have a reasonably efficient method for checking when we can thicken an embedding $S^m \hookrightarrow M^n$ to an embedding $D^{n-m} \times S^m \hookrightarrow M^n$ so as to be able to do surgery. Of course this method does not give much information for $m > \left[\frac{n}{2}\right]$ since then there can be a large number of stably trivial but non-trivial $(n-m)$-plane bundles over $S^m$. However, when $n = 2m$ the classes of stably trivial bundles are quite restrictive as indicated in 12, forming a group

$$\begin{cases} \mathbb{Z} & \text{for } m \text{ even,} \\ 0 & \text{if } m = 1, 3, 7, \\ \mathbb{Z}/2 & \text{for} m \text{ odd, } m \neq 1, 3, 7. \end{cases}$$

The next lemma gives us conditions when we can extend the normal map $f\colon M^n \to X$ to a normal map of the elementary cobordism $W(f)$, and in particular, obtain a normal map of the surgered manifold $\bar{M}^n$ to $X$ with respect to the same bundle over $X$.

LEMMA 39. *Let $f\colon M^n \to X$ be a normal map with respect to the bundle $\alpha$ on $X$. Suppose that $m \leq n-m$ and we have an embedding $g\colon S^m \hookrightarrow M^n$ with $fg$ homotopically trivial. Then there is an embedding $\phi\colon D^{n-m} \times S^m \hookrightarrow M^n$ so that $f$ extends to a map $\hat{f}\colon W(\phi) \to X$ with $\hat{f}^!(\alpha)$ equal to the stable normal bundle to $W(\phi)$.*

PROOF. Note first that a $k$-plane bundle on the sphere $S^t$ is trivial on the upper hemisphere as well as the lower hemisphere. Hence it is completely determined by a map $S^{t-1} \to GL_k(\mathbb{R})$ which describes the gluing of these two trivial bundles over the equatorial $S^{t-1} \subset S^t$. Moreover, if two such maps, $S^{t-1} \to GL_k(\mathbb{R})$ are homotopic then the resulting bundles are isomorphic.

Now consider the elementary bordism $W(\phi)$. By 35 $W(\phi) \simeq M \cup_g e^{m+1}$, and, since $fg \simeq 0$ it follows that $f$ extends to

$$\hat{f}\colon M \cup_g e^{m+1} \longrightarrow X$$

and hence to $W(\phi)$, and this is independent of the choice of framing of $\eta_M(g(S^m))$ and consequent embedding $D^{n-m} \times S^m \hookrightarrow M$ extending $g$. Now, use this extension to pull back $\alpha$. Clearly, $\hat{f}^!(\alpha)$ and the stable normal bundle to $W(\phi)$ differ only on $D^{n-m} \times e^{m+1}$, and hence since both are trivial (and identified) on $S^m$ it follows that their difference corresponds to a given framing $\lambda\colon S^m \to GL_{n-m}(\mathbb{R})$. If we use $\lambda$ to get a new embedding $D^{n-m} \times S^m \hookrightarrow M^n$, $(\vec{v}, s) \mapsto \phi(\lambda(s)\vec{v}, s)$, then the resulting normal bundle to $W(\phi')$ will be exactly $\hat{f}^!(\alpha)$. $\square$

REMARK . The process of modification described in the proof of 39 is very similar to the process used in proving 10, and, indeed, it is a good exercise to adjust the argument above to prove 10.

DEFINITION 44. *Given a normal map $f\colon M^n \to X$ as in 43 with respect to the bundle $\alpha$ on $X$, and a bordism $W^{n+1}$ between $M^n$ and $\bar{M}^n$. An extension of $f$ to $\hat{f}\colon W^{n+1} \to X$ is said to be a normal bordism of $f$ if $\hat{f}^!(\alpha)$ is the stable normal bundle to $W^{n+1}$.*

In particular, a normal bordism of $f$ restricts to give a normal map, $\hat{f}|\colon \bar{M}^n \to X$ with respect to $\alpha$.

COROLLARY 20. *Let $f\colon M^n \to X$ be a normal map as in 43, then there is a normal bordism $\hat{f}\colon W \to X$ extending $f$, with $\hat{f}$ restricted to $\bar{M}$ a homotopy equivalence to $X$ through dimension $\left[\frac{n}{2}\right] - 1$.*

## 5. Degree one normal maps.

Suppose that $M^n$ is a compact manifold and $f\colon M^n \to X$, is a normal map. We saw in the last section how surgery can be done in this situation to make $f$ into a homotopy equivalence through one less than half the dimension of $M$. But, to proceed to make $f$ an equivalence through any greater range of dimensions is much more difficult. The first reason is that $H_*(M^n)$ satisfies Poincaré duality, so $H_*(X)$ would have to at least look like the homology of a space which satisfies Poincaré duality. The second reason is that it is much harder to embed spheres, let alone thickened spheres $S^m \times D^{n-m}$ for $m \geq \left[\frac{n}{2}\right]$.

On the other hand, suppose that there is a homology class $[X] \in H_n(X;\mathbb{Z})$ so that $H_*(X)$ satisfies Poincaré duality with respect to $[X]$. Then we have

$$\begin{aligned} f_*(f^*(\alpha) \cap [M]) &= \alpha \cap f_*([M]) \\ &= \alpha \cap k[X] \\ &= k\alpha \cap [X], \end{aligned}$$

where $f_*([M]) = k[X]$. One way of thinking of this is that Poincaré duality connects cohomology and homology, so the map $f^*$, by duality, gives a map from $H_*(X)$ to $H_*(M)$, and $f_*([M])$ determines the effect of the composite map,

$$H_*(X) \longrightarrow H_*(M) \xrightarrow{f_*} H_*(X).$$

It follows that, if $f_*([M]) = \pm[X]$, then $f_*$ is automatically surgective, and, if $f_*$ is an isomorphism through the middle dimension, then it is an isomorphism through all dimensions.

Of course, this is not enough to guarantee that $f$ is a homotopy equivalence unless we can obtain the same conclusions for the universal covers of $M$ and $X$. Fortunately, the discussion of Poincaré duality for non-simply connected manifolds 3 extends the discussion above directly to the universal covers, and it follows that if we have a degree $\pm 1$ map of $n$-dimensional Poincaré duality spaces which induces isomorphisms of fundamental groups, and isomorphisms in homology of the universal covers through the middle dimension, then it automatically induces a homotopy equivalence of the entire spaces.

Definition 45. *Let $(X, Y)$ be a pair of finite complexes which satisfy Poincaré duality 3 in dimension $n$, and $(M^n, \partial M)$ an $n$-dimensional manifold with boundary. A degree one normal map*

$$f\colon (M^n, \partial M) \longrightarrow (X, Y)$$

*is a continuous map $f$ so that $f_*([M, \partial M]) = [X, Y]$ together with a stable vector bundle $\psi$ on $X$ and a bundle isomorphism*

$$g\colon \eta_M \longrightarrow f^!(\psi).$$

**The homotopy interpretation of degree one normal maps.** Let $X$ be an oriented Poincaré duality complex of dimension $n$ with orientiation class $[X]$. Suppose that $f\colon M^n \to X^n$, $g\colon \eta_M \to f^!(\psi)$ is a degree one normal map then, since $\eta_M$ is only stably well defined, we may as well assume that we are free to modify $\psi$ by adding arbitrary copies of the trivial bundle $\epsilon$, and that the dimension of the fibers of $\psi$, $d$, is greater than $n$.

Since the Thom space of $\epsilon \oplus \psi = \Sigma T(\psi)$ 7 it follows from 6 that the degree one normal data gives a map

$$S^{n+d} \xrightarrow{PT} T(\eta_M) \xrightarrow{T(g)} T(\psi)$$

which is well defined up to suspension, and consequently gives a well defined element in the stable homotopy group

$$\lim_{s \mapsto \infty} \pi_{n+s+d}(\Sigma^s T(\psi)) = \pi_n^s(T(\psi)).$$

Lemma 40. *The Hurewicz image of the homotopy class above is the image of* $[X]$ *under the Thom isomorphism* $H_n(X) \to H_{n+d}(T(\psi))$.

Proof. We have that the Hurewicz image of the Pontrajagin-Thom map $PT$ is the image of $[M]$ under the Thom isomorphism $H_n(M) \to H_{n+d}(T(\eta_M))$. Hence, since $f$ is degree 1 the result follows. □

On the other hand, given any map $\alpha\colon S^{n+d} \to T(\psi)$ with $\alpha_*([S^{n+d}])$ equal to the image of $[X]$ under the Thom isomorphism, it follows that when we make $\alpha$ transverse regular over $X \subset T(\psi)$, that restricting to the imverse image of $\alpha$ gives a degree one normal map to $X$, and if $\alpha$ is homotopic to $\beta$, then, again applying transversality this time to the homotopy, we have that the degree one normal problem associated to $\alpha$ is normally bordant to the degree one normal problem associated to $\beta$.

Here the definition of a degree one normal bordism is obvious: a degree one normal map of pairs $H\colon (W, \partial W) \to (V, X \sqcup -X)$, $G\colon \eta_W \to H^!(\nu)$, where $H$, $G$, restrict to $M$, $M'$ as the previously given normal maps.

Thus we have

Theorem 27. *Suppose that*

$$\left\{ f\colon M^n \to X,\ g\colon \eta_M \to f^!(\psi) \right\}$$

*is a degree one normal map with $M$ closed and oriented. Then the set of normal bordism classes of degree one normal maps over $(X, \psi)$ is identified with the Kernel of the stable Hurewicz map,*

$$Ker(h\colon \pi_n^s(T(\psi)) \to H_n^s(T(\psi))).$$

*In the case of degree one normal maps of pairs*

$$f\colon (M, \partial M) \to (X, Y),\ g\colon \eta_M \to f^!(\psi),$$

*the set of degree one normal maps is identified with the kernel of the Hurewicz map of relative groups*

$$Ker(h\colon \pi_n^s(T(\psi), T(\psi|Y)) \longrightarrow H_n^s(T(\psi), T(\psi|Y)).$$

EXAMPLE 42. The stable bundles over the sphere $S^n$ are identified with $\pi_n(B_O)$ which are given by Bott periodicity as $\mathbb{Z}$ if $n \equiv 0 \bmod (4)$, $\mathbb{Z}/2$ if $n \equiv 1, 2 \bmod (4)$, and 0 otherwise. Let $\phi\colon S^{n-1} \longrightarrow SO(d)$ give the attaching map for one of these bundles, $\sigma(\phi)$. Then the Thom complex is the 2-cell complex

$$S^d \cup J(\phi) e^{d+n}$$

where $e^{d+n} = D^d \times D^n$, and $J(\phi)$ is given as

$$\begin{cases} J(\phi)(\vec{v}, \vec{w}) = \phi(\vec{w})(\vec{v}) & \text{when } (\vec{v}, \vec{w}) \in (D^d \times S^{n-1}, \\ ( & \text{if } (\vec{v}, \vec{w}) \in S^{d-1} \times D^n. \end{cases}$$

In particular, here $S^d$ is $D^d/(\partial D \sim *)$.

Clearly, if $[\lambda] = k[\phi] \in \pi_{n-1}(O(d))$ we have that the attaching map $J(\lambda)$ is homotopic to $k$ times the attaching map $J(\phi)$. It follows that we have constructed a homomorphism $\pi_n(B_O) \longrightarrow \pi^s_{n-1}(S^0)$, the $(n-1)^{st}$ stable homotopy group of spheres, and we can construct a degree one normal map,

$$f\colon M^n \longrightarrow S^n, \ g\colon \eta_M \longrightarrow f^!(\sigma(\phi))$$

if and only if $J(\phi) = 0$. But deep results during the 1960's of J. F. Adams, [THE $J(X)$ PAPERS], M. Mahowald, D. Sullivan [ANNALS PAPER], and D. Quillen, [ADAMS CONJECTURE] completely determined the kernel of the homomorphism $J$. The kernel is zero in dimensions not congruent to 0 mod (4) and the $p$-primary part in dimension $4n$ is given as the $p$-primary part of the greatest common division of the numbers $(q^{2n} - 1)$ as $q$ runs over all primes distinct from $p$.

In fact, these powers of $p$ are given as follows. First, if $p$ is odd, then $|Im(J)_{4n-1}|$ is divisible by $p$ if and only if $(p-1)$ divides $2n$. Moreover, $p^t$ divides $|Im(J)_{4n-1}|$ if and only if $(p-1)p^{t-1}$ divides $2n$. When $p = 2$ we have that, $2n = 2^{1+s}c$ with $c$ odd if and only if $2^{3+s}$ is the power of 2 in $|Im(J)_{4n-1}|$. Thus, in particular we have

$$Im(J)_7 \ = \ \mathbb{Z}/(240)$$

where $240 = 3 \cdot 5 \cdot 16$.

For each $\phi \in Ker(J)$ we see that the Thom complex has the form of a wedge $S^d \vee S^{n+d}$ with $d > n$, so

$$\pi_{n+d}(T(\sigma\phi)) = \pi^s_n(S^0) \oplus \mathbb{Z}$$

and the set of degree one normal maps associated to $\phi$ is identified with $\pi^s_n(S^0)$.

EXAMPLE 43. We specialize the above discussion to the case where $\phi$ is the trivial bundle.

PROPOSITION 19. *If $f\colon M^n \rightarrow S^n$ is a normal map as above, associated to the trivial bundle over $S^n$, then $M^n$ is an oriented boundary. In particular the signature of $M^n$ is 0 if $n = 4k$.*

PROOF. Thom's correspondence between elements in the bordism group and elements in $\pi_n^s(T(BO))$ is obtained as in 5, by taking the homotopy class of the composite

$$S^{2n+k} \xrightarrow{PT} T(\eta_M) \xrightarrow{T(c(\eta_M))} T(\zeta_{n+k}(BO(n+k))$$

where $PT$ is now the Pontrajagin-Thom map associated to an embedding $M^n \hookrightarrow S^{2n+k}$ with normal bundle $\eta_M$ classified by the map

$$c(\eta_M)\colon M \to BO(n+k).$$

Here, $\zeta_{n+k}(BO(n+k))$ is the universal $n+k$ bundle over the classifying space $BO(n+k)$, and $T(c(\eta_M))$ is the associated map of Thom spaces.

In the case here $c(\eta_M) \simeq 0$ since the normal bundle to $M$ is trivial, and consequently the same is true of $T(c(\eta_M))$, so the composite above is homotopy trivial and $M$ represents 0 in the oriented bordism group. □

Note that in this case a degree one normal map is simply a choice of framing of $\eta_M$, well defined up to a reframing of the trivial bundle over $S^n$, which is the same as an element in $\pi_n^s(S^0)$, the $m^{th}$ stable homotopy group of the stable sphere taken modulo the elements in $\pi_n^s(S^0)$ which come from stable framings of the $n$-sphere. This can be seen by simply assuming that the framing is changed only over a small region in $S^n$, a region where the map $f$ is a homeomorphism.

But this group can be identified with the image of the $J$-homomorphism of the previous example. Thus, independent of the choice of framing of the trivial bundle over $S^n$ there is a map of normal bordism classes to $\pi_n^s(S^0)/Im(J)$. Alternatively, we can understand this by noting that a re-framing, $\tau$, of the trivial bundle over $S^n$ induces a self map of the Thom complex which is the identity on the $S^d$, while on the $S^{n+d}$ the map is $(id, J(\tau))$.

Now, assume that we have a degree one normal map as above,

$$f\colon M^n \longrightarrow S^n,\ g\colon \eta_M \longrightarrow f^!(d\epsilon)$$

which is normally bordant to 0. By this I mean that it extends to a normal bordism

$$F\colon (W^{n+1}, \partial(W) = M) \longrightarrow (D^{n+1}, S^n),\ G\colon \eta_W \longrightarrow F^!(d\epsilon).$$

This is an element in $\pi_{n+1}^s(S^d \vee D^{n+d+1}, S^{n+d})$.

Note that if we have any two such extensions, $(\hat{f}, W, \hat{\psi})$ and $\bar{f}, \bar{W}, \bar{\psi})$ then we can glue them together over $(f, M, \psi)$ to give a degree one normal map $(g, W \cup_M \bar{W}, \hat{\psi} \cup \bar{\psi})$ to $S^{n+1}$. But, the normal bundle for this normal map is the pull-back of a bundle over $S^{n+1}$ which need not be trivial. This remark will become important later when we connect the discussion here with the determination of the set of distinct diffeomorphism classes of manifolds $\Sigma^n$ which are homeomorphic to $S^n$.

**Fiber homotopy trivial bundles and** $Ker(J)$**.** In the first example a crucial role was played by those bundles over $S^n$ which, while non-trivial as bundles had Thom complexes homotopic to the Thom complex of the trivial bundle, $\Sigma^d(S^n_+) \simeq S^2 \vee S^{n+d}$. One way in which we can guarantee that this will happen for the stable version of a bundle is that when we consider the sphere bundle, $S(\mu)$, with fiber $S^{d-1}$ associated to the $\mathbb{R}^d$-vector bundle $\mu + k\epsilon$, there is a fiber preserving homotopy equivalence, $H\colon S(\mu) \to S^{d-1} \times X$. Since the Thom space of $\mu + k\epsilon$ is the mapping cone of the fibration

$$S(\mu) \longrightarrow X$$

it follows that $H$ gives a cannonical identification of $T(\mu + k\epsilon)$ with $\sigma^d(X_+)$. We call $H$ a **stable fiber homotopy trivialization of** $\mu$, and consider the subset of the set of stable vector bundles over $X$ which have a stable fiber homotopy trivialization.

LEMMA 41. *Suppose that $X$ is a finite complex and $\mu$ is a stably fiber homotopy trivial vector bundle over $X$, then for any other bundle $\gamma$ over $X$ we have*

$$T(\gamma + \mu) = \Sigma^d T(\gamma).$$

*Also, the subset of $K^*_O(X)$ consisting of stably fiber homotopy trivial vector bundles over $X$ is a subgroup.*

PROOF. The sphere bundle associated to $\gamma + \mu$ is the fiberwise join $S(\gamma) * S(\mu) = S(\gamma) * (S^{d-1} \times X)$ and each fiber is simply the join of the fiber over $X$ in $S(\gamma)$ with $S^{d-1}$, so, when we take the mapping cone, the coordinates in the $S^{d-1}$ identify the Thom space with $\Sigma^d(T(\gamma))$.

To show that the Whitney sum of two fiber homotopy trivial bundles is again fiber homotopy trivial we use the same argument, noting that the fiberwise join of $S^{d-1} \times X$ with $S^{l-1} \times X$ is just

$$S^{d-1} * S^{l-1} \times X = S^{d+l-1} \times X.$$

Now, suppose that $\mu$ is fiber homotopy trivial and $\gamma + \mu = (d + m)\epsilon$. We assume that $m > dim(X)$. The trivialization of $\gamma + \mu$ gives a projection

$$p\colon S(\gamma + \mu) \longrightarrow S^{d+m-1},$$

which together with the fiber projection $\pi\colon \gamma + \mu \to X$, gives a homotopy equivalence $S(\gamma + \mu) \stackrel{p \times \pi}{\longrightarrow} S^{d+m-1} \times X$. Embedding $S(\mu) \subset S(\gamma + \mu)$ and composing with $p$ gives a map

$$S^{d-1} \times X \longrightarrow S^{d+m-1}$$

which, for dimensional reasons is homotopic to the constant map

$$S^{d-1} \times X \to S^{d-1} \hookrightarrow S^{d+m-1}.$$

Using this homotopy and the original map $p$ we obtain a map

$$S(\gamma) \longrightarrow S^{m-1}$$

which is degree on on the fiber $S^{m-1}$ and gives a homotopy trivialization of $\gamma$. □

It will turn out that the quotient of $K_O(X)$ by the group of fiber homotopy trivial bundles is always finite for $X$ a finite complex, and from the lemma if $\gamma$ is associated with a degree one normal map to $X$, then $\gamma + \mu$ is associated to degree one normal maps for any $\mu$ in this subgroup. In analogy with the situation for spheres we call this subgroup $Ker(J)$ for $K_O(X)$.

**Homology properties of degree one normal maps.** For now let us consider the structure of degree one normal maps $M \to X$ where $M$ and $X$ have empty boundaries.

From 5 we have that $f_*\colon H_*(M^n) \to H_*(X^n)$ is surjective. But more is true: since the case of degree 1-normal maps gives

$$f_*(f^*(\alpha) \cap [M]) = \alpha \cap [X]$$

it follows that $Ker(f)$ is a split summand of $H_*(M)$, where $f^*(H^{n-1}(X)) \cap [M]$ is the complementary summand. Moreover, we define the kernel of $f$ as

$$K_*(f) = Ker(f_*), \quad \text{and } K^*(f) \ = \ \{\theta \in H^*(M) \mid \theta \cap [M] \in K_*(f)\}.$$

In words $K^*(f)$ is those $\theta$ in $H^*(M)$ so that $\theta$ capped with the orientation class lies in the kernel of $f_*$.

It follows directly that $H^i(M) = f^*(H^i(X)) \oplus K^i(f)$ for each $i$, and capping with $[M]$ preserves this splitting: $K^i(f) \cap [M] = K_{n-i}(f)$, so that associated to a degree one normal map $f\colon M \to X$, $f^!(\psi) = \eta_M$, we have a splitting $H_*(M) = H_*(X) \oplus K_*(f)$, $H^*(M) = H^*(X) \oplus K^*(f)$ and Poincaré duality preserves this splitting so $K^{n-*}(f)$ and $K_*(f)$ are dually paired.

In the case of pairs $K_*(f)$ is defined as before, while

$$K_*(M, \partial M, f) \text{ is } Ker(f_*\colon H_*(M, \partial M) \longrightarrow H_*(X, Y).$$

Once more the formula

$$f_*(f^*(\alpha) \cap [M, \partial M]) = \alpha \cap [X, Y]$$

shows that $f_*$ is surjective and split for both $H_*(M)$ and $H_*(M, \partial M)$. Finally, we define $K^*(f)$ and $K^*(M, \partial M, f)$ as above, and Poincaré duality gives dual parings $K^*(f) \cong K_{n-*}(M, \partial M, f)$, $K^*(M, \partial M, f) \cong K_{n-*}(f)$.

It is also true that the universal coefficient formula relates $K^*(f)$ with $K_*(f)$, and

$$K^*(M, \partial M, f) \text{ with } K_*(M, \partial M, f).$$

Precisely, we have

$$\begin{aligned} K^*(f) &= Hom(K_*(f), \mathbb{Z}) \oplus Ext(K_{*-1}(f), \mathbb{Z}), \\ K^*(M, \partial M, f) &= Hom(K_*(M, \partial M, f), \mathbb{Z}) \oplus Ext(K_{*-1}(M, \partial M, f), \mathbb{Z}). \end{aligned}$$

This discusssion works equally well for covers, though here

$$K_*(f),\ K^*(f),\ K_*(M, \partial M, f) \text{ and } K^*(M, \partial M, f)$$

have the additional structure of modules over $\mathbb{Z}(\pi)$ where $\pi$ is the group of the cover.

From the definition and the results of the last chapter we have

Corollary 21. *Let $f\colon M \to X$ be a degree one normal map as above. Then there is a cobordism $W$ from $(M, \partial M)$ to $(M', \partial(M'))$ together with an extension of $f$ to a degree one normal map of the triple*

$$\hat{f}\colon (W, (M \sqcup M'), W') \longrightarrow (I \times X, (0 \times X \sqcup 1 \times X), I \times \partial X)$$

*and restricting $\hat{f}$ to*

$$(M', \partial M') \to (X, \partial X)$$

*gives a homotopy equivalence through one less than the middle dimension. Moreover, the induced map in homology is surgective in all dimensions.*

Similarly, in the relative case we first construct bordisms on the boundary to a degree one normal map which is a homotopy equivalence through one less than $\left[\frac{n}{2}\right]$, and then extend this bordism to give a bordism to a normal map $F\colon (M^{n+1}, \partial M) \to (X, Y)$ where all the kernels fit together in an exact sequence:

$$0 \longrightarrow K_{\left[\frac{n}{2}\right]+1} \qquad (F|\partial M) \longrightarrow K_{\left[\frac{n}{2}\right]+1}(F)$$
$$\longrightarrow K_{\left[\frac{n}{2}\right]+1}(M, \partial M, F) \xrightarrow{\partial} K_{\left[\frac{n}{2}\right]}(F|\partial M) \longrightarrow 0$$

for $n$ odd and

$$0 \longrightarrow K_{\frac{n}{2}+1}(F) \qquad \longrightarrow K_{\frac{n}{2}+1}(M, \partial M, F) \xrightarrow{\partial}$$
$$K_{\frac{n}{2}}(F|\partial M) \longrightarrow K_{\frac{n}{2}}(F) \longrightarrow K_{\frac{n}{2}}(M, \partial M, F) \longrightarrow 0$$

for $n$ even. Later we will see that we can make the extreme terms in this last sequence 0 as well.

In particular, in the case where $M$ is closed, the data of a degree one normal map shows that the obstruction to obtaining a cobordism of pairs from the original map to a degree one normal map which is a homotopy equivalence occurs only in the middle dimension if $n$ is even, and in the two dimensions $\left[\frac{n}{2}\right]$, $\left[\frac{n}{2}\right] + 1$ if $n$ is odd.

CHAPTER 9

# Surgery for simply connected manifolds

We break the discussion here into cases depending on the dimension of $M$ mod (4). In the first three sections we determine the form of the obstruction to completing surgery – for $n$ odd there is none, while for $n \equiv 0 \bmod (4)$ there obstruction is the index of a certain non-singular symmetric bilinear form on $\mathbb{Z}^r$, and for $\equiv 2 \bmod (4)$ the obstruction turns out to be the class of a certain quadratic refinement of a non-singular skew-symmetric form on $(\mathbb{Z})^{2r}$.

## 1. The case $n \equiv 0 \bmod (4)$

The types of forms which appear in $K_{2n}$ for a $4n$-dimensional surgery problem after surgery has been done to make $K_j = 0$ for $j \neq 2n$ can be described as follows:

LEMMA 42. *Under the conditions above the intersection pairing on $K_{2n}$, (alternately, the $\cup$-product pairing on $K^{2n}$), satisfies the conditions*
*1. The associated adjoint map $\varphi\colon K_{2n} \to K^{2n}$ is an isomorphism,*
*2. $\alpha \cap \alpha \in 2\mathbb{Z}$ for all $\alpha \in K_{2n}$.*
*3. $\varphi^* = \varphi$, i.e., the intersection pairing is symmetric.*

PROOF. : The first statement is a direct consequence of Poincaré duality. The third statement follows since the dimension of $M$ is $4n$. The second statement occurs as follows. In cohomology we have $H^{2n}(M;\mathbb{Z}) = K^{2n} \oplus H^{2n}(X;\mathbb{Z})$, and this splitting preserves cup products. Reducing mod (2) we have $Sq^{2n}(x) = x^2$ for $x \in H^{2n}(M;\mathbb{Z}/2)$, but $Sq^{2n}(x) = x \cup v_{2n}$ where $v_{2n}$ is the Wu class of the normal bundle to $M$. On the other hand, since $\pi$ is a normal map, there is a bundle $\phi$ on $X$ so $\pi^!(\phi) = \eta_M$, and $v_{2n} = f^*(v_{2n}(\phi)$. Finally, note that $K^{2n}$ is orthogonal to $f^*(H^*(X;\mathbb{Z}/2))$ in $H^*(M;\mathbb{Z}/2)$ so, for any element $\alpha \in K^{2n}$, we have $\alpha \cup v_{2n} = 0$, which completes the proof. □

We now list a number of basic properties of forms satisfying the conditions (1)-(3) above.

LEMMA 43. *Let $\alpha \in K_{2n}$ be any element of $K_{2n}$ which is indecomposable and also satisfies the condition $\alpha \cap \alpha = 0$. Then there is a $\beta \in K_{2n}$ so that $\alpha \cap \beta = 1$ and $\beta \cap \beta = 0$. In particular, $K_{2n}$ can be written as the orthogonal direct sum $K'_{2n} \perp \langle \alpha, \beta \rangle$, and the intersection form on $\langle \alpha, \beta \rangle$ is given by the matrix* $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

PROOF. Since $\alpha$ is indecomposable it follows that there is a basis for $K_{2n}$ containing $\alpha$ as one of the basis elements. Since the adjoint map $\varphi\colon K_{2n} \to K^{2n}$ is an isomorphism there is a $\beta \in K_{2n}$ so that $\varphi(\beta)$ evaluates as 0 on all the elements of the basis except for $\alpha$, where it evaluates to 1. We must show that $\beta$ can be now modified to satisfy the condition $\beta \cap \beta = 0$. But consider $\beta' = \beta - \frac{(\beta \cap \beta)}{2}\alpha$. $\beta' \in K_{2n}$ since $\beta \cap \beta \in 2\mathbb{Z}$, and $\beta' \cap \beta' = \beta \cap \beta - 2\left(\frac{\beta \cap \beta}{2}\right)\beta \cap \alpha = 0$, since $\alpha \cap \alpha = 0$.

It remains to show that $K_{2n}$ splits as an orthogonal direct sum. But this is a consequence of the more general

PROPOSITION 20. *Let $\varphi\colon A \to A^*$ be any symmetric isomorphism where $A$ is a finitely generated free abelian group. Let $B \subset A$ satisfy the condition that the composite $B \hookrightarrow A \xrightarrow{\varphi} A^* \xrightarrow{i^*} B^*$ is an isomorphism, then $A$ splits as an orthogonal direct sum $B \perp B^{\perp}$, where $B^{\perp}$ is the set of all elements $a \in A$ which satisfy $\langle a, b\rangle = \varphi(a)(b) = 0$ for all $b \in B$.*

PROOF OF PROPOSITION. Let $a \in A$ be arbitrary, then $a$ defines an element $b(a) \in B^*$ by $b(a)(b) = \langle a, b\rangle$. Since the composite $i^*\varphi i$ is an isomorphism there is an element $b'(a) \in B$ so that $\langle b'(a), b\rangle = b(a)(b)$ for all $b \in B$, and $a - b'(a) \in B^{\perp}$. Consequently, writing $a = b'(a) + (a - b'(a))$ shows that $A = B + B^{\perp}$. It remains to note that $B \cap B^{\perp} = 0$, which is true since, given any element in $B$, there is an element $e(b) \in Hom(B, \mathbb{Z})$ so $e(b)(b) \neq 0$. But since $i^*\varphi i$ is an isomorphism there is an element $c \in B$ so that $i^*\varphi i(c) = e(b)$, and $b$ is not contained in $B^{\perp}$. □

□

Returning to $K_{2n}$ we see that after applying the remarks above we can write $K_{2n} = V \perp W$ where the form on $V$ has the special form $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. A key result of class field theory shows that if there are elements $w_1 \in W$, $w_2 \in W$ with $w_1 \cap w_1 > 0$ and $w_2 \cap w_2 < 0$, then there must be an element $w \in W$ with $w \cap w = 0$. Assuming this, we can iterate the procedure above until $W$ is *definite*, i.e. either $\langle w, w\rangle > 0$ for each $w \in W$ or $\langle w, w\rangle < 0$.

Thus the form on $K_{2n}(f)$ splits as an orthogonal direct sum $W \perp sH$ where $W$ is definite and $H$ is an orthogonal direct sum of $(\mathbb{Z}^2)$'s with form given by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

From the arguments of the last chapter if we embed an $S^n$ representing one of the generating $\mathbb{Z}$'s for such an $H$, then the embedding extends to an embedding $S^{2n} \times D^{2n} \subset M$, and doing surgery on this embedding simply removes the copy of $H$. It follows that after doing surgery we can assume that the intersection form on $K_{2n}(f)$ is definite.

We now show that the dimension of $K_{2n}(f)$ when the form is definite is independent of the choices made.

First, note that when we tensor $K_{2n}(f)$ with the rationals, $\mathbb{Q}$, the form remains definite. Indeed, given any element $\alpha \in K_{2n}(f) \otimes \mathbb{Q}$, then there is some non-zero $v \in \mathbb{Q}$ so that $v\alpha \in K_{2n}(f)$. Hence, since $\langle v\alpha, v\alpha\rangle = v^2\langle \alpha, \alpha\rangle$, the claim follows.

LEMMA 44. *Let $\mathbb{F}$ be any field which contains $\frac{1}{2}$, and $\varphi\colon \mathbb{F}^n \longrightarrow (\mathbb{F}^n)^*$ any non-singular symmetric form, then there is a change of basis so that $\varphi$ becomes diagonal.*

PROOF. If $\alpha \in \mathbb{F}^n$ satisfies $\langle \alpha, \alpha\rangle \neq 0$, then $\langle \alpha\rangle$ is a non-singular direct summand of $\mathbb{F}^n$ and we can write $\mathbb{F}^n = \langle \alpha\rangle \oplus \langle \alpha\rangle^{\perp}$ as above. On the other hand, if $\langle \alpha, \alpha\rangle = 0$ there is $\beta$ so that $\langle \alpha, \beta\rangle = 1$ by non-singularity, and if $\langle \beta, \beta\rangle = 0$ as well, $\langle \alpha+\beta, \alpha+\beta\rangle = 2 \neq 0$, so we have a sufficient number of vectors with $\langle \alpha, \alpha\rangle \neq 0$ in order to iterate the splitting procedure above and complete the proof of the lemma. □

Thus, when we diagonalize the form on $W \otimes \mathbb{Q}$ we see that the diagonal elements are either all positive or all negative and the index is $\pm(Dim(W \otimes \mathbb{Q}))$ On the other hand, the index of the form $H \otimes \mathbb{Q}$ is zero, so the index of $K_{2n}(f) \otimes \mathbb{Q}$ is, in absolute value, the dimension of $W \otimes \mathbb{Q}$.

Finally, note that since

$$H^{2n}(M^{4n}, \mathbb{Q}) = K_{2n}(f) \otimes \mathbb{Q} \perp H^{2n}(X; \mathbb{Q})$$

and the index is additive for orthogonal sums it follows that

$$Dim(W \otimes \mathbb{Q}) \;=\; |I(M) - I(X)|,$$

which is, indeed, independent of the choices made.

We have proved

LEMMA 45. *If $f\colon M^{4n} \longrightarrow X$, $g\colon \eta_M \longrightarrow f^!(\phi)$, is a degree one normal map with $M$ a closed manifold and $\pi_1(X) = 0$, then $f$ is normally bordant to a homotopy equivalence if and only if $I(X) = I(M)$.*

## 2. The case $n = 4k + 2$ and $\pi_1(M) = 0$

Most of the arguments of the previous section for $n = 4k$ carry over to this case. First we have that $K_{2k+1} = \mathbb{Z}^{2m}$ with a non-singular $(-1)$-symmetric form, which must be even since $\langle x, x\rangle = 0$ for any $x \in K_{2k+1}$. Consequently, we can write $K_{2k+1} = B \perp B^*$ with $\langle x, y\rangle = 0$ if $x, y \in B$ or $x, y \in B^*$. Now, assuming that $k \geq 1$ every element in $K_{2k+1}$ is represented by an embedded sphere which is unique up to isotopy when we assume that $f_*(\sigma) = 0$ in $\pi_{2k+1}(X)$. However, all we know about the normal bundle to this embedded sphere is that it is *stably* trivial, so $\epsilon \oplus \eta_\sigma = (2k+2)\epsilon$. As a consequence, $\eta_\sigma$ is either trivial or is a copy of $\tau_{S^{2k+1}}$ (which is trivial if and only if $k = 0, 1, 3$). So for now let us also assume that $k \neq 0, 1, 3$.

LEMMA 46. *Suppose that $\alpha, \beta \in K_{2k+1}$. Define a function $\chi\colon K_{2k+1} \longrightarrow \mathbb{Z}/2$ by assigning $1 \in \mathbb{Z}/2$ to $\alpha$ if $\eta_M(\alpha) = \tau_{S^{2k+1}}$. Otherwise, assign $0 \in \mathbb{Z}/2$ to*

*$\alpha$. Then we have*

$$\chi(\alpha+\beta)=\chi(\alpha)+\chi(\beta)+\langle\alpha,\beta\rangle$$

*where $\langle\alpha,\beta\rangle$ is the intersection number of $\alpha$ and $\beta$ taken* mod(2).

PROOF. Suppose that $\alpha$ is represented by the embedding $f\colon S^{2k+1}\to M$ and $\beta$ is represented by $g\colon S^{2k+1}\to M$. Suppose, moreover that $f$ and $g$ are in general position so that their images intersect transversally in $M$. Then the parity of the number of intersection points in exactly $\langle\alpha,\beta\rangle\in\mathbb{Z}/2$. Now, $\alpha+\beta$ is represented by $f\cup g$, when we connect the two images by a tube disjoint from their images, and this gives an immersion $f\cup g$ representing $\alpha+\beta$ with normal bundle the sum of the normal bundle to $\alpha$ and the normal bundle to $\beta$. On the other hand, in order to use the Whitney trick to change $f\cup g$ into an embedding we have to add $\langle\alpha,\beta\rangle$ single double point immersions. It follows that the normal bundle to the embedded sphere representing $\alpha+\beta$ is

$$\eta_M(\alpha)+\eta_M(\beta)+\langle\alpha,\beta\rangle\tau_{S^{2k+1}}$$

and the lemma follows. □

Thus the normal bundle data to the embedded spheres gives a $\mathbb{Z}/2$ quadratic extension of the intersection form on $K_{2k+1}$, and we have

COROLLARY 22. *We can do surgery to kill the kernel $K_{2k+1}$ if and only if we can find a splitting $K_{2k+1}=B\perp B^*$ with the quadratic extension restricted to $B$ identically zero.*

PROOF. Indeed, we've already seen that the effect of surgery in this situation is to remove a copy of $\mathbb{Z}^2$ generated by $\alpha$ and some dual to $\alpha$. So the only real difficulty is to obtain enough embeddings of $S^{2k+1}\times D^{2k+1}$ to represent half the generators of $K_{2k+1}$. □

REMARK . Of course, at this stage we don't know if two different ways of doing surgery will arrive at different $K_{2k+1}$'s, for one of which we can do surgery and for the other we can't. But later we will, in fact, show that if the obstruction above does not vanish for one way of surgering $f\colon M\to X$ to be an equivalence in dimensions $<2k+1$, then it does not vanish for any choice.

We now define an invariant of a non-singular skew symmetric form on $\mathbb{Z}^{2m}$ together with a quadradic function $\psi\colon\mathbb{Z}^{2m}\to\mathbb{Z}/2$ satisfying the formula of the previous lemma.

DEFINITION 46. *Let $\langle x,y\rangle$ be a non-singular skew-symmetric form on $\mathbb{Z}^{2m}$ with $\psi\colon\mathbb{Z}^{2m}\to\mathbb{Z}/2$ a quadratic refinement. Then if $\mathbb{Z}^{2m}=B\oplus B^*$ with $\langle\alpha,\beta\rangle=0$ for $\alpha,\beta\in B$ or $B^*$ we can choose a basis $e_1,\dots,e_m$ for $B$, and the dual basis $e_1^*,\dots,e_m^*$ for $B^*$, and the Arf invariant of the quadratic refinement is the sum*

$$Arf(\psi)=\sum_1^m\psi(e_i)\psi(e_i^*).$$

Of course, a priori, it is far from obvious that $Arf(\psi)$ does not depend on the choices of $B$, $B^*$ and the basis $e_1, \ldots, e_m$. But what is clear is that $\psi(2\lambda) = 0$ for any $\lambda \in K_{2k+1}$. Consequently, $\psi$ factors through the mod(2) reduction $K_{2k+1} \mapsto K_{2k+1} \otimes \mathbb{Z}/2$ where it becomes a quadratic reduction of the non-singular bilinear form $\langle \bar{\alpha}, \bar{\beta} \rangle = \langle \alpha, \beta \rangle$ mod (2). We also have the relations:

LEMMA 47.
*(a.) The Arf invariant of the orthogonal direct sum $(K_1, \psi_1) \perp (K_1, \psi_2)$ is the sum of the Arf invariants $Arf(\psi_1) + Arf(\psi_2)$.*
*(b.) Let $H = \mathbb{Z}^2$ with skew-symmetric non-singular bilinear form given by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $\psi(e) = \psi(e^*) = 0$, while $A$ is $\mathbb{Z}^2$ with the same non-singular skew-symmetric form, but $\psi(e) = \psi(e^*) = 1$. Then $H \perp H \cong A \perp A$, the isomorphism being with respect to both the bilinear form and $\psi$.*

PROOF. Consider $A \perp A$ with basis $e_1$, $e_1^*$, $e_2$, $e_2^*$ preserving the summands. Then a new basis is given as $e + 1 + e_2$, $e_1 + e_2 + e_1^*$, and $e_1^* + e_2^*$, $e_1^* + e_2^* + e_2$. One easily checks that this gives a splitting into orthogonal summands and the space spanned by the first two is a copy of $H$, as is the space spanned by the third and fourth. □

COROLLARY 23. *Let $\mathbb{Z}^{2m}$ be given with a non-singular skew-symmetric bilinear form together with a quadratic reduction $\psi$. Then there is a basis for $\mathbb{Z}^{2m}$ which makes this triple isomorphic to either*

$$\underbrace{H \perp H \perp \cdots \perp H}_{m-times}$$

*or*

$$A \perp \underbrace{H \perp H \perp \cdots \perp H}_{(m-1)-times}.$$

Finally, we have

LEMMA 48. *The non-singular skew-symmetric bilinear form together with quadratic reduction*

$$A \perp \underbrace{H \perp H \perp \cdots \perp H}_{(m-1)-times}$$

*is not isomorphic to $\underbrace{H \perp \cdots \perp H}_{m-times}$ for any $m$.*

PROOF. Here we use the reduction to $(\mathbb{Z}^{2m}) \otimes \mathbb{Z}/2 = (\mathbb{Z}/2)^{2m}$. We have that the total number of elements of $(\mathbb{Z}/2)^{2m}$ for which $\psi(a) = 1$ in the first case is $2^{2m-1} + 2^{m-1}$, while it is $2^{2m-1} - 2^{m-1}$ in the second case. Indeed, in the first case the total number of elements for which $\psi(a) = 1$ is given as the sum

$$3(N_1) + N_2$$

where $N_1$ is the total number of elements in $(m-1)H \otimes \mathbb{Z}/2$ with $\psi(a) = 0$ while $N_2 = 2^{2(m-1)} - N_1$ is the total number of elements with $\psi(a) = 1$. Similarly, the total number of elements in $mH \otimes \mathbb{Z}/2$ with $\psi(a) = 1$ is given as

$$3N_2 + N_1.$$

Now, the count is an easy induction. □

Returning to the surgery problem in dimension $4k+2$ we see that we can complete the surgery on the degree one normal map to make it a homotopy equivalence if and only if the quadratic refinement of $K_{2k+1}$ has Arf invariant 0. Thus, there is (at most) a $\mathbb{Z}/2$ obstruction to completing the surgery. In 7 we will complete the argument here by showing that the Arf invariant is, in fact, independent of the way in which the surgery problem is made $2k+1$ independent.

### 3. Surgery on degree one maps for $n$ odd and $\pi_1(X) = 0$

We now prove a remarkable result:

THEOREM 28. *Let $f\colon M^{2k+1} \to X$, $g\colon \eta_M \to f^!(\zeta)$ be a degree one normal map with $\pi_1(X) = 0$. Then $f$ is normally bordant to a homotopy equivalence.*

PROOF. The proof builds on the comments in the last chapter relating to the effect of a single surgery in dimension $k$. Let $e_1, \ldots, e_r$ be a generating set for $K_k(f)$, and embed representing spheres with trivial normal bundle for each $e_i$. We can assume all these spheres are disjoint and we consider the resulting manifold, $\hat{M}$, obtained by deleting disjoint $S^k \times D^{k+1}$'s obtained by thickening these embedded $S^k$'s.

For convenience we can assume that $f$ restricted to each $S^k \times D^{k+1}$ has image $* \in X$, a given basepoint. We may also assume that $X = Y \cup e^{2k+1}$, where $Y$ is a $2k-1$-complex and $e^{2k+1}$ represents the class $[X] \in H_{2k+1}(X)$. Then a further homotopy of $f$ modifies it to a map $f'$ which takes $(S^k \times D^{k+1}, S^k \times S^k)$ to $(D^{2k+1}, S^{2k}) \subset e^{2k+1}$, and on $S^k \times S^k$ is just the pinching map, collapsing the skeleton $S^k \vee S^k$ to point. In particular, restricting $f'$ to $\hat{M}$ gives a degree one normal map of pairs,

$$(\hat{M}, \partial(\hat{M})) \longrightarrow (X - Int(D^{2k+1}), S^{2k}).$$

LEMMA 49. *There is a short exact sequence of kernels*

$$0 \longrightarrow K_{k+1}(\hat{M}, \partial(\hat{M})) \xrightarrow{\partial} H_k(\partial(\hat{M})) \xrightarrow{i_*} K_k(\hat{M}) \longrightarrow 0$$

*Here $H_k(\partial(\hat{M})) = (\mathbb{Z})^{2r}$ with non-singular $(-1)^k$-symmetric intersection form given by the matrix*

$$\begin{pmatrix} K & 0 & 0 & \dots & 0 \\ 0 & K & 0 & \dots & 0 \\ 0 & 0 & K & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & K \end{pmatrix}$$

*with $K = \begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix}$, and, with respect to this form and the identification using Poincaré duality in $\hat{M}$ of $K_{k+1}(\hat{M}, \partial(\hat{M}))$ with $K_k(\hat{M})^*$, we have that $im(\partial) = \mathbb{Z}^k$ is a Lagrangian Kernel and $i_* = (\partial)^*$.*

Proof. We note first the following facts about $K_k(f)$ and $K_{k+1}(f)$. Since $K_{k-1}(f) = 0$ it follows that $K^{k+2}(f) = 0$ so $K_{k+1}(f)$ must be torsion free. (By the universal coefficient theorem, if $T \subset K_{k+1}(f)$ is the torsion part, it follows that $K^{k+2}(f) = T^*$ direct summed with the torsion free part of $K_{k+2}(f)$, where $T^* = Hom(T, \mathbb{Q}/\mathbb{Z}) \cong T$.) Consequently $K^k(f) = K_{k+1}(f)^*$ is also torsion free. It follows that since $i_* \colon H_k(\coprod_1^r S^k) \to H_k(M)$ is surjective onto $K_k(f)$, dually, $i^* \colon H^k(M) \to H^k(\coprod_1^r S^k)$ factors through $K^k(f)$ and gives an injection $K^k(f) \hookrightarrow H^k(\coprod_1^r S^k) = \mathbb{Z}^r$. Now consider the diagram

$$\begin{array}{ccccccc}
\mathbb{Z}^r & \xleftarrow{i^*} & K^k(f) & & & & \\
\uparrow \cong & & \uparrow & & & & \\
H^k(\coprod_1^r S^k) & \longleftarrow & H^k(M) & \longleftarrow & H^k(M, \coprod_1^r S^k) & \longleftarrow & 0 \\
 & & \uparrow f^* & & \uparrow f^* & & \\
 & & H^k(X) & \xleftarrow{\cong} & H^k(X, D^{2k+1}) & \longleftarrow & 0
\end{array}$$

Since the middle vertical sequence is exact it follows that

$$f^* \colon H^k(X, D^{2k+1}) \to H^k(M, \coprod_1^r S^k)$$

is an isomorphism, and by duality $H_{k+1}(\hat{M}) \to H_{k+1}(X)$ is an isomorphism as well. Likewise, we can dualize the diagram above to the homology diagram and conclude that $H_k(M, \coprod_1^r S^k) \cong H_k(X, D^{2k+1}) \cong H_k(X)$. Hence by duality

$$f^* \colon H^{k+1}(X - int(D^{2k+1})) \to H^{k+1}(\hat{M})$$

is an isomorphism as well.

Now we need to determine $H^{k+1}(M, \coprod_1^r S^k)$. Note that

$$H_{k+1}(M, \coprod_1^r S^k) = H_{k+1}(X) \oplus L$$

where $L$ is finitely generated from the exact sequence

$$0 \longleftarrow K_k(f) \longleftarrow \mathbb{Z}^r \xleftarrow{\partial} H_{k+1}(M, \coprod_1^r S^k) \longleftarrow H_{k+1}(M) \longleftarrow 0.$$

Indeed, rationally, this shows that $L \otimes \mathbb{Q} = \mathbb{Q}^r$. Consequently, from the universal coefficient theorem we have

$$\begin{aligned} H^{k+1}(M, \textstyle\coprod_1^r S^k) &= Hom(H_{k+1}(M, \textstyle\coprod_1^r S^k), \mathbb{Z}) \oplus Ext(H_k(M, \textstyle\coprod_1^r S^k), \mathbb{Z}) \\ &= Hom(H_{k+1}(X), \mathbb{Z}) \oplus Hom(L, \mathbb{Z}) \oplus Ext(H_k(X), \mathbb{Z}) \\ &= H^{k+1}(X, \mathbb{Z}) \oplus Hom(L, \mathbb{Z}) \end{aligned}$$

and this has the form $H^{k+1}(X, \mathbb{Z}) \oplus \mathbb{Z}^r$ since $L$ is finitely generated.

By excision we have isomorphisms

$$H_*(M, \coprod_1^r S^k) \cong H_*(M, \coprod_1^r S^k \times D^{k+1}) \cong H_*(\hat{M}, \coprod_1^r S^k \times S^k),$$

and similarly in cohomology.

From the calculation above and the fact that $H^{k+1}(\hat{M}) = H^{k+1}(X)$ we now see that

$$\delta \colon H^k(\coprod_1^r S^k \times S^k) \longrightarrow Hom(L, \mathbb{Z})$$

must be onto, hence split since $Hom(L, \mathbb{Z}) = \mathbb{Z}^r$. The remainder of the proof of this lemma is now direct. □

The next step is to point out, as was observed in the previous chapter, that the effect of surgery can be thought of as exchanging a certain number of the $e_i$ with their duals, $e_i^*$, and then taking the quotient by projection of $im(\partial)$ on the resulting first summand. Thus, surgery can be completed if and only if we can modify the embeddings and framings of the $S^k \times D^{k+1}$ while controlling the image $im(\partial)$ so that after switching a subset of the $e_i$, $e_i^*$, the resulting projection is onto. In particular, in such a case we must have a basis for $K_{k+1}(\hat{M}, \partial(\hat{M}))$, $f_1, \ldots, f_r$ so that the image of $f_i$ in the new $H_k(\partial(\hat{M}))$ has the form

$$e_i + \sum a_{i,j} e_j^*$$

and, since the image is a Lagrangian Kernel the $a_{i,j}$ are not arbitrary but must satisfy the symmetry constraint $a_{i,j} = (-)^{k+1} a_{j,i}$.

In the next subsection we show that it is always possible to modify the embeddings and framings to achieve this. □

## 4. The geometric moves

We now consider the effect of certain fairly simple changes in the embedded $S^k$'s and their framing on the exact sequence of 49. First, we can modify the embedding of the $t^{th}$ sphere $S^k$ by taking a path disjoint from the other $S^k$ going from $S^k_t$ to $S^k_s$ and then thickening the path to a tube,

$I \times S^{k-1}$ and using this tube to take the connected sum of $S^k_t$ with a sphere in one of the fibers $D^{k+1}$ of $S^k \times D^{k+1}$ which links $S^k_s$. The effect of this modification is to change the form so that $e_t$ is replaced by $e_t + e^*_s$ and $e_s$ by $e_s \pm e^*_t$ as long as $t$ is distinct from $s$. The matrix associated to such a change is

$$T_{N(t,s)} = \begin{pmatrix} I & J_{ts} \\ 0 & I \end{pmatrix}$$

where $J_{ts}$ is the matrix which is zero except for a 1 in the $(t,s)$-position and a $\pm 1$ in the $(s,t)$-position. The sign being $+$ if $k$ is odd, and $(-)$ if $k$ is even. By this we mean that the resulting image of $K_{k+1}(\hat{M}, \partial(\hat{M}))$ is given by the composition $T_{N(t,s)}\partial$.

Next, we can reorder the $e_j$, which induces, simultaneously a reordering of the $e^*_j$, with matrix of the form

$$T_\sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-t} \end{pmatrix}.$$

As before $T_\sigma \partial$ gives the new image of $\partial$ with respect to this variation in the embeddings. This will remain true for the following variations as well.

Also we can change the framing of over $S^k_t$. The constraint that we must be able to extend the degree one normal map over the trace of a surgery constrains us to take framing modifications which induce stably trivial bundles over $S^{k+1}$, provided that the original framing satisfies the condition that when we do surgery with respect to this framing we are able to extend the normal map to a normal map on the trace of the surgery.

In particular, if we assume this, then for $k$ odd the possible reframings correspond to multiples of the tangent bundle and take $e_t$ to $e_t + 2\lambda e^*_t$, leaving the remaining $e_j$ and $e^*_j$ fixed. Here the matrix is

$$T_{F(\lambda,t)} = \begin{pmatrix} I & D_t \\ 0 & I \end{pmatrix}$$

where $D_t$ is the matrix which is identically zero except for a $2\lambda$ in the $(t,t)$-position.

On the other hand, if $k$ is even, then the only possible reframing corresponds to the tangent bundle to $S^{k+1}$, and, since the Euler class is zero in this case, reframing, in this case has no effect in homology.

Finally, we can proceed as in the first case, taking a path from $S^k_t$ to $S^k_s$ disjoint from the other $S^k$'s, translate $S^k_s$ away from itself to $(S^k, v)$ where $v \in D^{k+1}$ is a given fixed vector, and taking the connected sum of $S^k_t$ and $(S^k_s, v)$ along $I \times S^{k-1}$ thickening the path. This move will have the matrix

$$\begin{pmatrix} I + E_{ts} & 0 \\ 0 & I - E_{st} \end{pmatrix}$$

where, as usual, $E_{rm}$ is the matrix which is 0 except for a single 1 in position $(r, m)$.

These matrices generate a subgroup $H_k \subset SP_r^{(-)^k}(\mathbb{Z})$, where $SP_r^{(-)^k}(\mathbb{Z})$ is the group of isometries of the form over $\mathbb{Z}^{2r}$ given by the matrix

$$J_k = \begin{pmatrix} 0 & I \\ (-)^k I & 0 \end{pmatrix}.$$

In particular, $T \in SP_r^{(-1)^k}(\mathbb{Z})$ if and only if

$$T J_k J^t = J_k,$$

and if we add in the matrix $S_i$, which exchanges $e_i$ with $(-)^k e_i^*$ while leaving everything else fixed (the move associated with surgery on the $i^{th}$ $S^k \times D^{k+1}$), then we have

THEOREM 29. *The group $SP_r^{(-)^k}(\mathbb{Z})$ is generated by the subgroup $H_k$ together with $S_1$. Consequently, a sequence of moves of the type above together with at most $r$ surgeries will kill $K_k(f)$. It follows that for $n$ odd and $\pi_1(X) = 0$, then any degree one normal map $f\colon M^n \to X$, is normally cobordant to a homotopy equivalence.*

(The theorem will follow directly once it is shown that $\langle S_i, H_k \rangle = SP_r^{(-)^k}(\mathbb{Z})$. The proof SHOULD APPEAR in [Coxeter]. However, this result is sufficiently important that we give a proof here in APPENDIX to this chapter.)

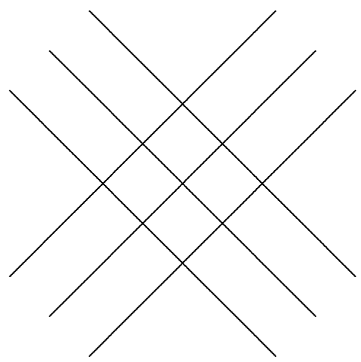
## 5. Plumbing and the Browder-Novikov Theorem

We now construct some examples of degree one normal maps over the pair $(D^{k+1}, S^k)$ which are very useful, and in fact, represent the heart of simply connected surgery theory. Using these examples and our discussion of surgery above we will then prove the most basic theorem in the subject – the Browder-Novikov theorem.

**Plumbing disk bundles.** Given two disk bundles over $S^k$, $D^k \to E \to S^k$ and $D^k \to F \to S^k$, we *plumb them together* by taking a trivialization of $E$ over a small $D^k \subset S^k$, so $\pi^{-1}(D^k) = D^k_f \times D^k_b$, where $D_f$ lies on the fiber and $D_b$ lies parallel to the base, and a similar trivialization of $F$ over a second small $D^k \subset S^k$. Then we make a single manifold with boundary by identifying

$D^k_f \times D^k_b \subset E$ with $D^k_f \times D^k_b \subset F$, by exchanging base and fiber.

**The Plumbing of two copies of $I \times I$**



Clearly, the total space of the resulting plumbing has the homotopy type of the wedge of two spheres $S^k \vee S^k$, with intersection numbers given by the matrix

$$A(E,F) \;=\; \begin{pmatrix} \lambda_1 & 1 \\ (-1)^k & \lambda_2 \end{pmatrix}$$

where $\lambda_i$ is the self intersection number of $S^k$ in $E$ or $F$. Now denote the plumbing by $P(E,F)$, then there is a long exact sequence

$$\begin{aligned}\cdots \longrightarrow H_i( \quad P(E,F)) &\longrightarrow H_i(P(E,F), \partial(P(E,F))) \\ &\xrightarrow{\partial} H_{i-1}(\partial(P(E,F))) \longrightarrow \cdots\end{aligned}$$

which is non-trivial only for $i = k$. Here the sequence becomes

$$0 \longrightarrow H_k(\partial(P)) \longrightarrow \mathbb{Z}^2 \xrightarrow{A(E,F)} (\mathbb{Z}^2)^* \xrightarrow{\partial} H_{k-1}(\partial(P)) \longrightarrow 0,$$

and this allows us to determine the homology of $\partial(P)$.

Standard general position arguments show that $\pi_1(\partial P(E,F)) = 0$ provided $k \geq 3$, but for $k = 2$ $\pi_1(\partial P(E,F))$ is generally non-trivial. Consequently, in the examples below we will assume that $k \geq 3$.

Example 44. The first example occurs for $k$ odd, and $E = F$, the tangent disk bundle over $S^k$. Then the matrix $A(E,F) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is non-singular and $H_i(\partial(P(E,F))) = 0$ for $i \neq 0, 2k-1$. Consequently, from the Poincaré conjecture in dimensions $\geq 5$ it follows that $\partial(P(E,F))$ is homeomorphic to the sphere $S^{2k-1}$. However, Browder, [8], has shown that it cannot be diffeomorphic to the ordinary $S^{2k-1}$ unless $k$ is one less than a power of 2. In these cases, it is known that $\partial(P(E,F))$ is diffeomorphic to $S^{2k-1}$ for $k = 1, 3, 7, 15, 31$, but it is unknown at present whether this is true for $k = 63$.

Now consider the situation where $k$ is even and $E$ is the disk bundle of $\lambda_1$ copies of the tangent disk bundle (so the self intersection number of $S^k$ in $E$ is $2\lambda_1$, and $F$ is $\lambda_2$ copies of the tangent disk bundle. Here

$$A = \begin{pmatrix} 2\lambda_1 & 1 \\ 1 & 2\lambda_2 \end{pmatrix}$$

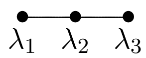
which has determinant $4\lambda_1\lambda_2 - 1$, and the exact sequence above becomes

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{A} \mathbb{Z}^2 \xrightarrow{\partial} \mathbb{Z}/(4\lambda_1\lambda_2 - 1) \longrightarrow 0.$$

In particular, if $E$ and $F$ are both simply the tangent bundle to $S^k$ then the matrix is $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ with determinant 3.

One can also iterate the plumbing construction, using three or more disk bundles over $S^k$ and plumbing them together over disjoint $D^k_f \times D^k_b$'s according to a graph (or better, a tree). For example, in the situation where $k$ is even, $E_i$ is $\lambda_i$ copies of the tangent disk bundle, and the graph is

**Plumbing 3 bundles along a linear tree**



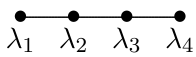

we get the matrix

$$A = \begin{pmatrix} 2\lambda_1 & 1 & 0 \\ 1 & 2\lambda_2 & 1 \\ 0 & 1 & 2\lambda_3 \end{pmatrix}$$

with determinant $8\lambda_1\lambda_2\lambda_3 - 2(\lambda_1 + \lambda_3)$. Consequently, when all the $\lambda_i$ are one, then the determinant is 4.

Likewise, for the linear graph with 4 vertices

**Plumbing 4 bundles along a linear tree**




we get the matrix

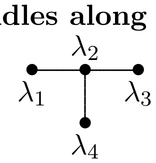
$$A = \begin{pmatrix} 2\lambda_1 & 1 & 0 & 0 \\ 1 & 2\lambda_2 & 1 & 0 \\ 0 & 1 & 2\lambda_3 & 1 \\ 0 & 0 & 1 & 2\lambda_4 \end{pmatrix}$$

which has determinant

$$(1 - 4\lambda_1\lambda_2)(1 - 4\lambda_3\lambda_4) - 4\lambda_1\lambda_4.$$

In particular, if all the $\lambda_i$ are 1 then the determinant is 5. Similarly, for the non-linear graph

**Plumbing 4 bundles along a non-linear tree**



the matrix is

$$\begin{pmatrix} 2\lambda_1 & 1 & 0 & 0 \\ 1 & 2\lambda_2 & 1 & 1 \\ 0 & 1 & 2\lambda_3 & 0 \\ 0 & 1 & 0 & 2\lambda_4 \end{pmatrix}$$

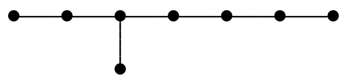
with determinant

$$16\lambda_1\lambda_2\lambda_3\lambda_4 - 4(\lambda_1\lambda_4 + \lambda_1\lambda_3 + \lambda_3\lambda_4),$$

which again gives 4 in case all the $\lambda_i = 1$.

Perhaps the most important example of this kind is associated with the graph of the $E_8$-lattice

**The graph of the $E_8$-lattice**



In the case where all the $\lambda_i = 1$ this has matrix

$$A_{E_8} = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

with determinant equal to 1. Consequently, the plumbing associated to the $E_8$-graph with all the $\lambda_i = 1$ has boundary $\Sigma_8^{4k-1}$ which has the homotopy type of the sphere $S^{4k-1}$, and consequently, is homeomorphic to $S^{4k-1}$ for $k \geq 2$.

LEMMA 50. *The signature of $A_{E_8}$ is* 8.

PROOF. The signature of $V$ is the signature of the matrix $A$ above for the $E_8$-lattice. But the signature is the number of positive eigenvalues minus the number of negative eigenvalues. Moreover, the number of negative eigenvalues corresponds to the number of changes of sign of the diagonal minors given as the intersection of the first $t$ rows with the first $t$ columns. But it is easily checked that these determinants are

$$2,\ 3,\ 4,\ 5,\ 4,\ 3,\ 2,\ 1$$

respectively in this case. □

Actually, it turns out that any non-singular even form on $\mathbb{Z}^r$ must have signature divisible by 8:

THEOREM 30. *Let $W$ be an even, integral, unimodular, symmetric form, then $I(W)$ is divisible by* 8.

PROOF. We introduce the characteristic element of $W$ in $W \otimes \mathbb{Z}/2$, which has a non-singular, even, form induced from the form on $W$. $v$ is defined by $\langle v, w\rangle = \langle w, w\rangle$ for all $w \in W \otimes \mathbb{Z}/2$. This makes sense since $\langle w + s, w + s\rangle = \langle w, w\rangle + \langle s, s\rangle \bmod (2)$, so $\langle w, w\rangle$ defines a homomorphism from $W \otimes \mathbb{Z}/2 \rightarrow \mathbb{Z}/2$, and the non-singularity of $W \otimes \mathbb{Z}/2$ guarantees that $v$ exists and is unique.

Note that if $V$ is any lift of $v$ we have $\langle V, V\rangle \in \mathbb{Z}$ is well defined $\bmod (8)$, since $\langle V + 2s, V + 2s\rangle = \langle V, V\rangle + 4\langle V, s\rangle + 4\langle s, s\rangle$, and $\langle V, s\rangle \cong \langle s, s\rangle \bmod (2)$. Hence $\langle V, V\rangle \in \mathbb{Z}/8$ is a well defined invariant of $W$.

It remains to identify this invariant with the index of $W \bmod (8)$. In order to do this, following an argument of Serre we introduce a certain equivalence relation on non-singlar but not necessarily even forms on free $\mathbb{Z}$-modules.

We now consider unimodular, symmetric forms on $(\mathbb{Z})^n$. Suppose there is an isotropy vector $\alpha$ which we can assume is irreducible. Consider the subspace $\langle\alpha\rangle \subset \langle\alpha\rangle^{\perp}$. I claim that the original form induces a form of the same type on the quotient $\langle\alpha\rangle^{\perp}/\langle\alpha\rangle \cong \mathbb{Z}^{n-2}$. Indeed, for $\beta_i \in \langle\alpha\rangle^{\perp}/\langle\alpha\rangle$, $i = 1, 2$, define $\langle\beta_1, \beta_2\rangle = \langle b_1, b_2\rangle$ for a choice of $b_i \in \langle\alpha\rangle^{\perp}$ which projects onto $\beta_i$. Since $\langle b_1 + k\alpha, b_2\rangle = \langle b_1, b_2\rangle$ for all $b_i \in \langle\alpha\rangle^{\perp}$ it follows that this pairing is well defined. On the other hand, suppose that $\lambda \in Hom(\langle\alpha\rangle^{\perp}/\langle\alpha\rangle, \mathbb{Z})$. By projection this defines an element $\lambda \in Hom(\langle\alpha\rangle, \mathbb{Z})$ which vanishes on $\langle\alpha\rangle$. Thus, when we look at $\varphi^{-1}(\lambda)$, it belongs to $\langle\alpha\rangle^{\perp}$ and it follows that the resulting form on the quotient is non-singular. Moreover, it is even if the original form was.

We define an equivalence relation on integral unimodular forms on $\mathbb{Z}^n$, $n = 1, 2, \ldots$ by setting $\mathbb{Z}^n \equiv \langle\alpha\rangle^{\perp}/\langle\alpha\rangle$ for any indecomposable isotropy vector in $\mathbb{Z}^n$. Note that $I(W) = I(\langle\alpha\rangle^{\perp}/\langle\alpha\rangle)$, so $I(W)$ is an invariant of the equivalence class.

LEMMA 51. *The set of isomorphism classes of unimodular, symmetric forms on $\mathbb{Z}^n$, $n = 1, 2, \ldots$ under the equivalence relation induced by the construction above is a copy of $\mathbb{Z}$, where the generator is $\langle 1\rangle$, and the inverse of this generator is $\langle -1\rangle$. Moreover, $I(W)$ completely determines the class of $W$ in $\mathbb{Z}$.*

PROOF. Note first that the orthogonal direct sum $\langle 1\rangle \perp \langle -1\rangle$ contains the irreducible isotropy vector $e_1 + e_2 = \alpha$ and it is directly seen that $\langle\alpha\rangle^{\perp} = \langle\alpha\rangle$ in this case. Thus the sum above is trivial. Suppose now that $W$ is given, then, suppose that $w \in W$ is a generator. If it is an isotropy vector we can reduce. If not, consider $W \perp \langle -1\rangle \perp \langle 1\rangle$, which is equivalent to $W$. Suppose that $\langle w, w\rangle = k > 0$. Then $w + kf$ is an isotropy vector, where $f$ is a basis element of the summand $\langle -1\rangle$. Consequently, the quotient can be written $W' \perp \langle 1\rangle$, where $Dim(W') = Dim(W) - 1$. (A similar argument holds if $k < 0$, in which case just replace $\langle -1\rangle$ by $\langle 1\rangle$ and conversely. After iterating this construction we see that $W \equiv \underbrace{\langle 1\rangle \perp \cdots \perp \langle 1\rangle}_{k\ times}$ if $I(W) = k$, or $\underbrace{\langle -1\rangle \perp \cdots \perp \langle -1\rangle}_{k\ times}$ if $I(W) = -k$. □

Returning to the proof of the theorem we must show that this invariant is preserved under the relation of equivalence defined above. But note that if $\langle \alpha, \alpha\rangle = 0$, then, by definition a lift of $v$ for $W$ can be chosen in $\langle\alpha\rangle^\perp$. Moreover, again from the definition, the lift of the characteristic element for $\langle\alpha\rangle^\perp/\langle\alpha\rangle$ can be chosen as the lifted element for $W$, and it follows that the invariants are the same for $W$ and $\langle\alpha\rangle^\perp/\langle\alpha\rangle$. Additionally, this invariant is additive in the sense that $\langle V(W \perp W', V(W \perp W'\rangle = \langle V(W), V(W)\rangle + \langle V(W'), V(W')\rangle$, so it defines and is defined by a homomorphism from the set of equivalence classes to $\mathbb{Z}/8$.

Finally, we claim $\langle V, V\rangle = I(W) \bmod (8)$. Indeed, it suffices to evaluate on generators. But, ie $e$ is a generator for $\langle 1\rangle$ it can be chosen as the lift of the characteristic class $v$ in this case, and the invariant is 1.

The theorem follows since, for an even form $\langle w, w\rangle \equiv 0$ for all $w \in W \otimes \mathbb{Z}/2$, so, in this case, $v = 0$ and $V$ can be taken to be zero. □

**The sphere $\Sigma^7_8$ is exotic.**

THEOREM 31 (Milnor). *The manifold $\Sigma^7_8$ is not diffeomorphic to the ordinary sphere $S^7$.*

PROOF. Note first that if $E$ is the disk bundle of $\lambda$ copies of the tangent bundle, $\tau(S^{2k})$, then $E$ is parallelizable. Indeed,

$$\tau(E) = \pi^!(\tau(S^{2k})) \oplus \pi^!(E)$$

where $\pi\colon E \to S^{2k}$ is the bundle map. But this bundle is stably trivial. On the other hand $E \simeq S^{2k}$ and any stably trivial vector bundle of fiber dimension $\geq 2k+1$ is already trivial on $E$. Similarly, since we are plumbing along a tree it is easy to see that the total space of the plumbing is also parallelizable.

Consequently, the plumbing construction realizes $\Sigma^{4k-1}_8$ as the boundary of a parallelizable manifold, $V$.

Using this preliminary remark we prove the theorem by assuming, to the contrary, that $\Sigma^7_8$ is diffeomorphic to the ordinary sphere $S^7$. In this case we can attach a disk to $V$ over $\partial V$, resulting in a closed manifold, $M^8$, with tangent bundle trivial on the 7-skeleton, so the normal map factors through the pinching map $M^8 \to S^8$,

$$M^8 \longrightarrow S^8 \xrightarrow{\psi} B_{SO}.$$

Now, Bott has proved that any such map $\psi\colon S^8 \to B_{SO}$ must have Pontrajagin class $P_2$ divisible by 3!.

Remark . More generally Bott showed that the Pontrajagin class $P_k$ of the bundle induced by the map $f\colon S^{4k}\to B_O$ which represents the generating sphere in $\pi_{4k}(B_O) = \mathbb{Z}$ is $a_k(2k-1)!$, where $a_k = 2$ if $k$ is odd and is 1 otherwise.

Additionally, since the pinching map $M^8\to S^8$ becomes a degree one normal map we must have that the Thom space of the bundle over $S^8$ has the form $S^{d+8}\vee S^d$, and consequently, the associated generator $\sigma^{-1}[f]\in\pi_7(O)$ lies in the kernel of the $J$-homomorphism, and hence is divisible by $15\times 16 = 240$ in $\pi_7(O)$, so $[f]\in\pi_8(B_O)$ is divisible by 240 as well as by 3!. As a result, $P_k[f]$ is divisible by

$$3!\times 240 = 2^5\times 9\times 5.$$

On the other hand the $L$-genus evaluates as

$$\langle\frac{7}{45}P_2, [M]\rangle$$

so the signature of $M$ must be divisible by $7\times 2^5$, and this is a contradiction. □

Remark . Actually, the above argument shows that there are at least 28 distinct diffeomorphism classes of manifolds homeomorphic to $S^7$. Indeed, we know that the signature of $M^8$ is always divisible by 8, and from the Browder-Novikov theorem which we prove in the next section, if $\Sigma^7$ bounds a parallelizable manifold, then the signature of $M$ completely determines the diffeomorphism class of $\Sigma^7$. Thus, the various connected sums of the manifold pair $(V^8, \partial V)$ must give distinct differentiable structures on the boundary until the signature is divisible by $7\times 2^5$, which happens exactly when we've taken 28 summands.

## 6. The Browder-Novikov Theorem

We have shown that in the simply connected case, if we are given a degree one normal map $f\colon M^n\to X^n$, then, for $n$ odd there is no obstruction to completing surgery to construct a normal bordism to a homotopy equivalence. On the other hand, for $n = 4k$ there is a single obstruction to completing the surgery, the difference of the signatures $I(M^{4k}) - I(X)$ which takes values in the integers $\mathbb{Z}$, while for $n = 4k+2$ we've shown that the obstruction is *at most* an element of the group $\mathbb{Z}/2$ – the Arf invariant of a quadratic refinement of the intersection form on $K_{2k+1}(f)$ after $f$ is made $2k$-connected.

On the other hand, in the last section we constructed a parallelizable $4k+2$-dimensional manifold with boundary homeomorphic to $S^{4k+1}$, so that the quadratic refinement of the intersection form on the interior has Arf invariant one. We also constructed a parallelizable $4k$-manifold with boundary homeomorphic to $S^{4k-1}$ and index 8, for $k\geq 2$.

These allow us to change the resulting normal map by gluing in the above manifolds via a (piecewise linear) homeomorphism of their boundaries to the boundary of a small disk in $M^n$ at the cost of possibly losing the differentiability of $M^n$. However, this loss of differentiability occurs only in a small neighborhood of a single point of $M$. Consequently, we can continue to do surgery using our previous techniques of embedding spheres with trivial normal bundles – simply taking mild care to make sure the embeddings avoid this point. This gives the following result:

THEOREM 32. *[Browder-Novikov] Let $X$ be a simply connected Poincaré duality complex of dimension $n \geq 5$, and let $f\colon M^n \to X$ be a degree one normal map. Then $X$ is homotopy equivalent to a differentiable manifold for $n$ odd, and is, otherwise either homeomorphic to a differentiable manifold or a piecewise linear manifold which is differentiable in the complement of a single point.*

There is also a relative version of this theorem.

THEOREM 33 (Browder-Novikov). *Let $X$ be a simply connected Poincaré duality complex of dimension $n$ with $n \geq 5$ and odd. Then, given a degree one normal map, $f\colon M^n \to X$ with $f$ a homotopy equivalence, $M^n$ is unique up to connected sum with the boundary of the $K(n+1)$-plumbing for $n = 4k+1$, or a finite number of copies of $\pm\Sigma_8^{4k-1}$ if $n = 4k$.*

PROOF. If $f_1\colon M_1^n \to X$ and $f_2\colon M_1^n \to X$ are both degree one normal maps which are homotopy equivalences and are normally bordant, then let

$$\begin{array}{ccc} M_1^n \sqcup M_2^n & \xrightarrow{f_1 \sqcup f_2} & 0 \times X \sqcup 1 \times X \\ \Big\downarrow \partial & & \Big\downarrow i_0 \sqcup i_1 \\ W^{n+1} & \xrightarrow{H} & I \times X \end{array}$$

be a normal bordism. We can do surgery on $H$ away from the boundary to make it a homotopy equivalence in dimensions less than $\frac{n+1}{2}$, so we obtain the short exact sequence of kernels in this dimension:

$$0 \longrightarrow K_{\frac{n+1}{2}}(H) \longrightarrow K_{\frac{n+1}{2}}(W, \partial W) \xrightarrow{\partial} 0$$

and it follows that the intersection pairing on $K_{\frac{n+1}{2}}(W)$ is non-singular.

On the other hand, if we can complete surgery to a homotopy equivalence of pairs $(W, \partial W) \to (I \times X, (\partial I) \times X)$, it follows that $W$ is an $h$-cobordism from $M_1^n$ to $M_2^n$ and, consequently, $M_1^n$ and $M_2^n$ will be diffeomorphic since they are simply connected and of dimension $\geq 5$. But if we are willing to take the connected sum of $M_2^n$ with one of the manifolds described in the theorem we can extend the $H$ to a copy of the Arf invariant one plumbing or the appropriate number of copies of $\pm K_{E_8}^{n+1}$, now thought of as given by a degree one normal map to the pair $(D^{n+1}, S^n)$, so as to modify the obstruction to completing surgery on $W$, and the theorem follows. □

Remark . *:* In the case where $n \geq 6$ is even, if the degree one normal map is normally bordant to a homotopy equivalence,

$$f\colon M^n \longrightarrow X$$

then the same argument as above, doing surgery on the normal bordism, shows that $M^n$ is unique up to diffeomorphism. Similarly, in the case where it is necessary to modify $M$ to a piecewise linear manifold in order to get the homotopy equivalence we again conclude that any two which are normally bordant are piecewise linearly homeomorphic by using a piecewise linear variant of the $h$-cobordism theorem. [REFERENCE].

## 7. The Arf-invariant for surgery in dimension $4k+2$ is well defined

As was indicated in the proof of the second Browder-Novikov theorem in the previous section we can also consider surgery problems on manifolds with boundaries, i.e., degree one normal maps to Poincaré pairs $(X, Y)$, (obvious definitions). Thus, suppose that $f_1\colon M_1^{4k+2} \to X^{4k+2}$ and $f_2\colon M_2^{4k+2} \to X^{4k+2}$ are $2k$-connected degree one normal maps which are normally bordant, and suppose, as above that

$$(W^{4k+3}, \partial(W) = M_1 \sqcup -M_2) \xrightarrow{H} (I \times X, 0 \times X \sqcup 1 \times X)$$

is a normal bordism. We can assume that surgery has been done on the interior of $W$ to make the map $H$ $2k$-connected. Consequently, applying the techniques of 49 we obtain the following short exact sequence of kernels,

$$0 \longrightarrow K_{2k+2}(W, \partial W) \xrightarrow{\partial} K_{2k+1}(f_1) \perp -K_{2k+1}(f_2) \xrightarrow{i_*} K_{2k+1}(W) \longrightarrow 0$$

with $K_{2k+1}(W)$, $K_{2k+2}(W, \partial W)$ torsion free and dual to each other. We modify the situation above by taking a path from $M_1$ to $M_2$ in $W$ and deleting the interior of a small regular neighborhood of the path to obtain a new degree one normal map from $(W_1, \partial W_1)$ to $(I \times (X - Int(D^{4k+2})), \partial)$, where $\partial W_1$ is the connected sum $M_1 \# - M_2$. Note that the map of kernels is unchanged in homology. But, using the relative Whitehead theorem, it follows that the elements of $K_{2k+2}(W_1, \partial W_1)$ are now represented by maps of pairs $(D^{2k+2}, S^{2k+1})$. Choosing these maps of pairs to be immersions which are embeddings on the boundaries, we see that the spheres in the image of $\partial$ are represented by embeddings $S^{2k+1} \hookrightarrow M_1 \# M_2$ with trivial normal bundles.

Consequently, these give a $\frac{1}{2}$-rank direct summand of

$$K_{2k+1}(M_1) \perp -K_{2k+1}(M_2)$$

for which the quadratic refinement vanishes, which implies that the Arf invariant of $K_{2k+1}(M_1) \perp -K_{2k+1}(M_2)$ is zero. On the other hand the Arf invariant is additive for orthogonal sums so the Arf invariant for $K_{2k+1}(M_1)$ is equal to the Arf invariant for $K_{2k+1}(M_2)$ and the result follows. ∎

## 8. The classification of homotopy spheres

As an application of the Browder-Novikov theorem we give the classification of the diffeomorphism classes of manifolds homeomorphic to the sphere $S^n$ for $n \geq 5$. This is work of Milnor, and Kervaire-Milnor [REFERENCES] and represents the origins of the ideas of surgery on degree one normal maps.

LEMMA 52. *Let $\Gamma_n$ be the set of diffeomorphism classes of manifolds homeomorphic to $S^n$. Then the operation of connected sum*

$$(\Sigma_1, \Sigma_2) \mapsto \Sigma_1 \# \Sigma_2$$

*makes $\Gamma_n$ into an abelian group for $n \geq 5$.*

PROOF. First, it is clear that $\Sigma_1 \# \Sigma_2$ is diffeomorphic to $\Sigma_2 \# \Sigma_1$ and both are homeomorphic to $S^n$, while $\Sigma_1 \# (\Sigma_2 \# \Sigma_3)$ is equal to $(\Sigma_1 \# \Sigma_2) \# \Sigma_3$. Consequently connected sum defines a commutative, associative pairing on $\Gamma_n$. Thus, it remains to construct inverses.

Delete a little disk $D^n$ from $\Sigma$. Then $\Sigma - Int(D^n)$ is diffeomorphic to $D^n$ and $\Sigma$ is given as $D^n \cup_{S^{n-1}} D^n$ for a given diffeomorphism $\lambda \colon S^{n-1} \to S^{n-1}$ which identifies the two copies of $S^{n-1}$. If we now define $\Sigma'$ as $D^n \cup_{\lambda^{-1}} D^n$, then taking the connected sum of these two spheres along disks centered on the equator (and where $\lambda$, $\lambda^{-1}$ are just the identity on the intersection), it is quite direct to see that the resulting diffeomorphism for the connected sum is $\lambda \# \lambda^{-1}$, but by a standard argument this diffeomorphism is isotopic to the identity.

Finally, an isotopy between to diffeomorphisms of $S^n$ defines an $h$-cobordism between the resulting homotopy spheres. Consequently, using the $h$-cobordism theorem we see that the two elements are diffeomorphic. □

The next result relates $\Gamma_n$ with degree one normal maps to $S^n$.

LEMMA 53. *Let $\Sigma \in \Gamma_n$ and $f \colon \Sigma \to S^n$ be a degree one homotopy equivalence. Then $f$ is a degree one normal map for $\phi = (n+1)\epsilon$ over $S^n$.*

PROOF. Since $f$ is a homotopy equivalence there is some stable bundle, $\psi$, over $S^n$ so that $f^!(\psi) = \eta_\Sigma$. On the other hand, we know that the bundle must be in $Ker(J)$ since the Thom space has to be a wedge of two spheres, $S^d \vee S^{n+d}$. Unless this bundle is zero, it must have a non-trivial Pontrajagin class (and $n = 4k$). But then the $L$-genus for $\Sigma$ would not be zero and the Signature of $\Sigma$ would be non-trivial, which is impossible. □

Now, by the Browder-Novikov theorem and the remarks of 42, 43, and especially 19, the diffeomorphism type of $\Sigma$ is determined by its normal cobordism class up to connected sums with elements obtained as the boundary of the Kervaire invariant plumbing in dimension $4k+1$, and plumbing of tangent disk bundles over $S^{2k}$ via the $E_8$-lattice in dimension $4k-1$. Also, the discussion in 7 relating to the $\mathbb{Z}/2$-Arf invariant surgery obstruction in dimension $4n-2$ gives

LEMMA 54. *The surgery obstruction gives a well defined homomorphism of* $\pi_n^s(S^0)$ *thought of as the set of degree one normal maps over* $(S^n, (n+1)\epsilon)$ *to* $\mathbb{Z}/2$ *which is trivial in dimensions* $\neq 2k+2$*, and in* $2k+2$ *is the Arf invariant of the resulting surgery problem. This homomorphism is trivial on* $im(J)$*.*

Let $B_n$, the $n^{th}$ Bernoulli number, be the coefficient of $P_n$ in the $L$-genus $L_{4n}$, so

$$L_{4n} \;=\; B_n P_n + \text{ decomposables.}$$

$B_n$ is rational with known denominator, see, e.g. [ADAMS, J(X)]. Then we set

$$W(4n-1) \;=\; \frac{1}{8} B_n a_n (2n-1)! |im(J)_{4n-1}|.$$

See the discussion around  for $a_n(2h-1)!$, the discussion around 42 for $|im(J)_{4n-1}|$, and we have

LEMMA 55. *The rational number* $W(4n-1)$ *is actually an integer.*

PROOF. Indeed, imitating the arguments in the proof that $S^7_8$ was not the ordinary sphere, and that, in fact, the operation of connected sum gives a $\mathbb{Z}/28 \subset \Gamma_7$ generated by connected sums of $S^7_8$ with itself, we see that $8W(4n-1)$ is the signature of the manifold with normal bundle induced from a the generating bundle for $Ker(J)$ over $S^{4n}$ by the (degree one) pinching map, $M^{4n} \rightarrow S^{4n}$. Consequently, $8W(4n-1)$ is an integer. But by 30 this signature is divisible by 8, so $W(4n-1)$ is an integer as well. □

Finally, putting all this together we are able to determine the structure of the groups $\Gamma_n$ for $n \geq 5$ as follows.

THEOREM 34. *Suppose that* $n \geq 5$*. Then there are exact sequences of groups*

$$0 \longrightarrow \mathbb{Z}/W(4n-1) \longrightarrow \Gamma_{4n-1} \longrightarrow \pi^s_{4n-1}(S^0)/Im(J) \longrightarrow 0$$

$$0 \longrightarrow \Gamma_{4n-2} \xrightarrow{\cong} Ker(Arf) \longrightarrow 0$$

*where* $Arf\colon \pi^s_{4n-2}(S^0)/(im(J)) \rightarrow \mathbb{Z}/2$ *is the surgery obstruction. Also, we have the further exact sequences*

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \Gamma_{4n-3} \longrightarrow \pi^s_{4n-3}(S^0)/(Im(J)) \longrightarrow 0$$

*provided that* $Arf\colon \pi^s_{4n-2}(S^0) \rightarrow \mathbb{Z}/2$ *is identically zero, and*

$$0 \longrightarrow \Gamma_{4n-3} \xrightarrow{\cong} \pi^s_{4n-3}(S^0)/(Im(J)) \longrightarrow 0$$

*when* $Arf(\colon \pi^s_{4n-2}(S^0) \rightarrow \mathbb{Z}/2$ *is non-trivial. Finally, we have*

$$0 \longrightarrow \Gamma_{4n} \longrightarrow \pi^s_{4n-3}(S^0)/(Im(J)) \longrightarrow 0$$

*in the remaining case.*

The homotopy spheres in the group $\mathbb{Z}/W(4n-1)$ are called the **Milnor spheres**. They are connected sums of the homotopy sphere in dimension $4n-1$ obtained as the boundary of the plumbing on the $E_8$-lattice, 5, for $n \geq 2$. Similarly, the homotopy spheres in dimension $4n+1$ obtained as the boundaries of the plumbing of two copies of the tangent bundle to $S^{2n+1}$ are called the **Kervaire spheres**.

The Milnor spheres are always exotic, while the Kervaire sphere is not exotic if and only if the dimension $4k+2$ has the form $2^s-2$ and there is a non-trivial element in the 2-primary part of $\pi^s_{4k+2}(S^0)$ which is detected in filtration two in the Adams spectral sequence. Such an element, if it exists, is called an Arf-invariant class, and whether or not these classes exist in the stable homotopy of spheres is one of the chief open questions in homotopy theory. They are known to exist for $s \leq 6$, [MAHOWALD], [MINAMI], and it is conjectured that they exist in each dimension $2^s-2$.

CHAPTER 10

# The Algebraic Analysis of Surgery Groups when $\pi_1(X) = 0$

In this chapter we do two things. First we give a complete proof that the odd surgery groups, $L_{2n+1}(1)$, for surgery on simply connected manifolds vanish. Then we give a deeper analysis of the proof, developing an exact sequence which connects both the even and odd surgery obstruction groups in this case with Witt rings of quadratic forms over the rationals and modules of torsion quadratic forms.

The second topic provides the foundation for techniques which allow us to determine the structure of the surgery obstruction groups in the case of surgery on manifolds with finite fundamental group later in this work.

The first step in the proof of vanishing is to show that we can do surgery to make the surgery kernel into a finite torsion module which has a non-singular linking form together with a quadratic refinement. This is accomplished by some fairly crude moves. However, the next step – showing that we can do surgery to kill the torsion form – is somewhat more delicate. It becomes a case by case analysis where we show that in each case we can reduce the order of the kernel if it is not zero until we finally arrive at a kernel which has order a power of 2. Here we use an explicit surgery to reduce the order further and thus we complete the proof.

Then, as indicated we study the set of torsion forms together with a non-singular $\pm$ - symmetric form and a quadratic refinement. Under orthogonal direct sum and a relation 62 implied by surgery considerations, the sets of equivalence classes of forms with a given $\pm$ symmetry become abelian groups, $L^{Tor}_{\pm}(\mathbb{Z})$, and we determine them completely.

When we study surgery with general fundamental group we will construct exact sequences

$$L_{2n}(\mathbb{Z}(\pi)) \longrightarrow L_{2n}(\mathbb{Q}(\pi)) \longrightarrow L^{Tor}_{2n}(\mathbb{Z}(\pi)) \longrightarrow L_{2n-1}(\mathbb{Z}(\pi)) \longrightarrow \cdots$$

which give us effective tools for studying surgery problems at least for finite fundamental groups. The analysis in the current chapter provides the key motivation for these constructions as well as basic calculations.

## 1. The Vanishing of the Odd Surgery Groups when $\pi_1(X) = 0$

We are given a $\pm$-symmetric non-singular form over $\mathbb{Z}$ together with a quadratic refinement. Precisely, a $q$-refinement is a linear map $T\colon \mathbb{Z}^k \longrightarrow (\mathbb{Z}^k)^*$ so that the original bilinear form is $T \pm T^*$. Note that one more or less automatically has a certain degree of freedom in defining $T$ since $T+(B\mp B^*)$ will satisfy the same criterion as $T$ for any $B$. In the case of $+$-symmetric forms, the constraint that the form is $T+T^*$ simply says that the form is **even** as we've seen is true for the middle dimension kernel form for a $4s$-dimensional degree one normal map. Moreover the variation of $T$ by $B-B^*$ does not have much effect since the diagonal terms of $B-B^*$ are all zero, so the value of $q(X) = X^*TX$ does not change.

However, for $(-)$-symmetric forms the situation is different. Here $B+B^*$ has even terms on the diagonal so that $X^*TX$ changes by even integers and the quadratic refinement is only well defined as an integer mod(2). In particular, this agrees with the structure needed for surgery on $4n+2$-dimensional manifolds where the quadratic refinement is given as

$$q(e_i) = \begin{cases} 0 \\ 1 \end{cases}$$

depending on whether the normal bundle to the embedded sphere corresponding to $e_i$ has trivial or non-trivial normal bundle in the degree one normal map situation.

In the situation we have in mind here we will assume that the form has a Lagrangian so $k = 2s$: indeed, we will assume that $T$ is, in both cases, the matrix

$$T = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$$

at least after modifying by a matrix of the form $B \mp B^*$. Then, the group that we wish to analyze is the group which preserves $T$ up to the same indeterminacy.

Thus, assume the form above with quadratic refinement $T$ is given and fixed. Let $V \cong \mathbb{Z}^s$ be any **quadratic Lagrangian** contained in $\mathbb{Z}^{2s}$ with respect to this form. This means that

- first, $V$ is a direct summand of $\mathbb{Z}^{2s}$,
- second, that the bilinear form is identically zero on $V$,
- third, the value of the quadratic refinement on each $v \in V$ is an even integers if the form is $(-)$-symmetric, and is identically zero if the form is $+$-symmetric.

Write $\mathbb{Z}^{2s}$ as $(\mathbb{Z}^s) \oplus (\mathbb{Z}^s)^*$, and choose a basis $e_1, \dots e_s$ for the first summand and the dual basis $e_1^*, \dots e_s^*$ for the second summand, so that $T$ takes the form above, 1, for this splitting. Then we may write a basis for $V$ in the form $(\vec{l}_1, \vec{t}_1^*), \dots, (\vec{l}_s, \vec{t}_s^*)$ in terms of the splitting above.

We want to modify our choices of basis elements for $\mathbb{Z}^s$ and for $V$ to give the vectors $\vec{l}_1, \dots, \vec{l}_s$ the simplest possible form to make it easier to do surgery.

Note that $GL_s(\mathbb{Z})$ is generated by $\mathcal{S}_s$, the symmetric group acting as permutations of the coordinates, and $E^1_{1,2}$, the transformation which takes

$e_1$ to $e_1 + e_2$, while $e_i \mapsto e_i$ for $i \neq 1$. Consequently the geometric moves of 4 show that the subgroup consisting of all matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & A^{-t} \end{pmatrix}$$

for $A \in GL_s(\mathbb{Z})$ is contained in the group generated by the geometric moves. It follows that we can change basis at will in the first $\mathbb{Z}^s$.

Consider the projection $\pi_1 \colon \mathbb{Z}^{2s} \to \mathbb{Z}^s$ which has kernel the second summand $(\mathbb{Z}^s)^* \subset \mathbb{Z}^{2s}$, and let $I_1(V)$ be the image of the composite map

$$V \hookrightarrow \mathbb{Z}^{2s} \xrightarrow{\pi_1} \mathbb{Z}^s,$$

and $K_1(V)$ be the kernel.

Lemma 56. *A series of geometric moves enables us to assume that $K_1(V)$ is 0, so the composite above is injective and has finite index in $\mathbb{Z}^s$.*

Proof. We identify the image $\mathbb{Z}^s$ of 1 with the first summand $\mathbb{Z}^s$ of $\mathbb{Z}^{2s}$ in what follows. Tensoring with the rationals, $\mathbb{Q}$, $I_1(V)$ defines a sub-vector space of $\mathbb{Q}^s = \mathbb{Z}^s \otimes \mathbb{Q}$. If this is the entire $\mathbb{Q}^s$ we are done. If not, it is a proper subspace, and we have $\mathbb{Z}^{s-k} \subset \mathbb{Z}^s$ defined as

$$\mathbb{Z}^{s-k} \;=\; \mathbb{Z}^s \cap I_1(V) \otimes \mathbb{Q}.$$

Now choose a new basis for $\mathbb{Z}^s$ so that the first $s-k$ basis elements span $\mathbb{Z}^{k-s}$ and the corresponding dual basis for $(\mathbb{Z}^s)^*$. This basis change is realized by geometric moves due to the remarks above. Now, consider the composite projection

$$V \hookrightarrow \mathbb{Z}^{2s} \xrightarrow{\pi_1^*} (\mathbb{Z}^s)^* \longrightarrow (\mathbb{Z}^{s-k})^{\perp}.$$

Restricting to $K_1(V)$, I claim that this map must be injective. Indeed, if not, since $V$ injects into $\mathbb{Z}^{2s}$, we must have that for any $T \in K_1(V)$ which is also in the kernel of the composite map 1, it has image $(0, T_2)$ with $T_2 \in (\mathbb{Z}^{s-k})^*$. But this directly contradicts the assumption that $V$ is a kernel in $\mathbb{Z}^{2s}$. Consequently, 1 injects on $K_1(V)$ as asserted.

Now we can use the surgery moves to exchange $(\mathbb{Z}^{s-k})^{\perp}$ and its dual space in $\mathbb{Z}^s$. □

Now that we are able to assume that the projection in 1 is injective with finite index we modify it still further to make the index as small as possible; in this case 1. (For more general rings, such as the integral group rings of finite groups, this may not always be possible.)

Consider the quotient $\mathbb{Z}^s / I_1(V) = W$, and let $w \in W$ be given. Then, choosing $\bar{w} \in \mathbb{Z}^s$ so that it has image $w$, there is a finite $k \neq 0$ so that $k\bar{w} \in I_1(V)$. Hence, there is a *unique* element $h(w, k) \in V$ with $k\bar{w}$ given as the image of $h(w, k)$, so

$$h(w, k) = (k\bar{w}, \lambda(w, k)) \in \mathbb{Z}^s \oplus (\mathbb{Z}^s)^*.$$

DEFINITION 47. *Let $w_1$, $w_2$ be contained in $W$, then $\langle w_1, w_2\rangle \in \mathbb{Q}/\mathbb{Z}$ is the rational number*

$$\frac{1}{k_1 k_2}\lambda(w_1, k_1)(k_2\bar{w}_2)$$

*taken in the (additive) quotient $\mathbb{Q}/\mathbb{Z}$.*

Note that this is, in fact, well defined. If $\bar{w}'$ is a second lifting of $w$, then $\bar{w}' = \bar{w} + s$ where $s \in I_1(V)$. Consequently,

$$(k\bar{w}, \lambda) + k(s_1, s_2)$$

is the resulting element in $V$, and the value in 47 is changed by adding an element in $\mathbb{Z}$.

LEMMA 57. *The bilinear form on the quotient $\mathbb{Z}^s/I_1(V)$ defined above is $\mp$-symmetric depending on whether the original form on $\mathbb{Z}^{2s}$ is $\pm$-symmetric, and is non-singular, in the sense that adjoint map to the form*

$$W \rightarrow Hom(W, \mathbb{Q}/\mathbb{Z}) = W^*$$

*is an isomorphism.*

PROOF. First we verify that the form is $\mp$-symmetric. Since $V$ is a Lagrangian kernel it follows that

$$\langle(k_1\bar{w}_1, \lambda(\bar{w}_1, k_1)), (k_2\bar{w}_2, \lambda(\bar{w}_2, k_2))\rangle = 0$$

but expanding this out we have

$$\lambda(\bar{w}_1, k_1)(k_2\bar{w}_2) \pm \lambda(\bar{w}_2, k_2)(k_1\bar{w}_1) = 0$$

so the $\mp$-symmetry follows.

It remains to show that the form is non-singular. For this it suffices to show that if $\langle w, w'\rangle = 0$ for all $w' \in W$, then $w = 0$. So suppose $k\bar{w} \in I_1(V)$. Then

$$\frac{1}{k}\lambda(\bar{w}, k)(s) \in \mathbb{Z}$$

for every $s \in \mathbb{Z}^s$. Consequently, $\lambda(\bar{w}, k) = k\tau$ for some $\tau \in (\mathbb{Z}^s)^*$, and

$$(k\bar{w}, \lambda(\bar{w}, k) = k(\bar{w}, s) \in V.$$

But $V$ is a direct summand so $(\bar{w}, s) \in V$, and $w = 0$ as asserted. □

EXAMPLE 45. Here we assume that we have the $+$-symmetric form on $\mathbb{Z}^2 \oplus (\mathbb{Z}^2)^*$, and let $V$ be the Lagrangian,

$$V = \begin{pmatrix} 5 & 0 & | & 0 & 3 \\ 0 & 5 & | & -3 & 0 \end{pmatrix}.$$

Then the quotient $W = (\mathbb{Z}/5)^2$ with $(-)$-symmetric form

$$\frac{1}{5}\begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}.$$

EXAMPLE 46. Now we assume that we have the $(-)$-symmetric form on $\mathbb{Z} \oplus \mathbb{Z}^*$ and

$$V = \langle (5,2) \rangle$$

so $W = \mathbb{Z}/5$ with generator $T$ and $\langle T, T \rangle = \frac{2}{5}$.

REMARK . The construction above of the non-singular form is modeled on the geometric linking form on the torsion in $H_{n-1}(M^{2n-1}, \mathbb{Z})$. However, there is one important difference here: there should be a quadratic refinement reflecting the fact that not all Lagrangian kernels for the linking form are acceptable. They must also have "trivial" quadratic refinement where the quadratic refinement reflects the structure of the normal bundle to the corresponding embedded sphere. As a special case, in the last example,

$$V' = \langle (5,1) \rangle$$

is also a Lagrangian kernel for the linking form, where now $\langle T, T \rangle = \frac{1}{5}$, but on $S^{2n-1} \times S^{2n-1}$ if we take the embedded sphere which corresponds to $(5,1)$ it has normal bundle equal to the tangent bundle to $S^{2n-1}$. (Just take $1(5) \equiv 1 \bmod (2)$.) So $V'$ is not an acceptable kernel for surgery.

This motivates us to introduce the quadratic refinement of the bilinear form above on $W$ as

$$q(w) = \frac{1}{k^2}(\lambda(\bar{w}, k)(k\bar{w}),$$

which is well defined mod$(2\mathbb{Z})$ for the original form $(-)$-symmetric, and must be identically zero for the original form $+$-symmetric.

**The Kervaire-Milnor proof.** What follows is a slight modification of the original proof that surgery can always be completed in odd dimensions. Using the pair, $(\mathbb{Z}^s \oplus (\mathbb{Z}^s)^*, V)$ and the geometric moves we are able to suppress all further geometric arguments, and the proof becomes purely an algebraic manipulation with (1) algebraic surgery (exchanging a generator of $\mathbb{Z}^s$ with its dual in $(\mathbb{Z}^s)^*$), and (2) using the geometric moves to systematically reduce the order of the quotient $\mathbb{Z}^s/I_1(V)$.

Note first that we can change the basis of $V$ at will, and that, using the matrices of the form 1 we can change the basis of $\mathbb{Z}^s$ at will. Thus, the inclusion $J\colon V \to I_1(V) \subset \mathbb{Z}^s$ may be modified to the form $AJB$ with $A, B \in GL_s(\mathbb{Z})$ and $J$ an integral matrix which is rationally invertible. But under such variation $J$ is uniquely equivalent to a matrix of the form

$$N(J) = \begin{pmatrix} m_1 & 0 & 0 & \dots & 0 \\ 0 & m_2 & 0 & \dots & 0 \\ 0 & 0 & m_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & m_r \end{pmatrix}$$

where each $m_i$ is a positive integer and $m_i$ divides $m_{i+1}$ for $i = 1, \ldots, r-1$. Consequently, the image of $V$ in $\mathbb{Z}^s \oplus (\mathbb{Z}^s)^*$ has the form

$$(N(J) \mid A)$$

with $A$ also an $r \times r$ integral matrix. Moreover, since $A$ defines the quadratic form on the quotient, and $V$ is a quadratic kernel, this puts strong conditions on the coefficients in $A$. All we will need is this: if $m_i = m_j$ then $a_{i,j} = \mp a_{j,i}$, and if the form on $\mathbb{Z}^s \oplus (\mathbb{Z}^s)^*$ is $+$-symmetric, then $a_{i,i} = 0$ for each $i$, while, if the form is $(-)$-symmetric, then $a_{i,i}$ is divisible by 2.

**The case $\mathbb{Z}^s \oplus (\mathbb{Z}^s)^*)$ $(-)$-symmetric.** The critical invariant that we will use is $|Det(J)| = m_1 m_2 \cdots m_r$. If this determinant is 1, then surgery has been completed. Otherwise we show that we can always reduce the size of $|Det(J)|$. The argument breaks up into three steps depending on whether

$$a_{r,r} \text{ is } \begin{cases} \neq 0, m_r \\ = 0 \\ = m_r \end{cases} .$$

Suppose that $a_{r,r} \neq 0, m_r$. Then we can write

$$a_{r,r} = 2km_r + v$$

with $s \neq 0$ and $|v| < m_r$. Hence change basis by replacing $e_r$ by $e_r + 2ke_r^*$ and leaving all other basis elements alone, which is one of the geometric moves. This replaces $a_{r,r}$ by $v$, and leaves everything else alone. Now, do surgery – exchanging the $r^{th}$ columns of $N(J)$ and $A$, so the matrix replacing $N(J)$ becomes

$$\begin{pmatrix} m_1 & 0 & 0 & \ldots & 0 & a_{1,r} \\ 0 & m_2 & 0 & \ldots & 0 & a_{2,r} \\ 0 & 0 & m_3 & \ldots & 0 & a_{3,r} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & m_{r-1} & a_{r-1,r} \\ 0 & 0 & 0 & \ldots & 0 & v \end{pmatrix}$$

which has determinant $m_1 m_2 \ldots m_{r-1} v$ which is non-zero and less than $|Det(N(J))|$.

Next consider the case where $a_{r,r} = 0$. The non-singularity of the form on the quotient shows that there must be a second $m_r$ in $N(J)$, say associated to $e_{r-1}$ for definiteness, and

$$\langle \bar{e}_{r-1}, \bar{e}_r \rangle = \frac{a_{r-1,r}}{m_r}$$

with $a_{r-1,r}$ equal to a unit $\mathrm{mod}(m_r)$. Again we can write $a_{r-1,r} = km_r + u$ where $0 < u < m_r$, and the basis change $e_r \mapsto e_r + ke_{r-1}$, $e_{r-1} \mapsto e_{r-1} + ke_r$ with all other basis elements fixed is a geometric move which replaces $a_{r-1,r}$

by $u$, and leaves all other coefficients in $(N(J)|A)$ fixed. Now do surgery on *both* the last two columns. This replaces $N(J)$ by

$$\begin{pmatrix} m_1 & 0 & 0 & \dots & a_{1,r-1} & a_{1,r} \\ 0 & m_2 & 0 & \dots & a_{2,r-1} & a_{2,r} \\ 0 & 0 & m_3 & \dots & a_{3,r-1} & a_{3,r} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{r-1,r-1} & u \\ 0 & 0 & 0 & \dots & u & 0 \end{pmatrix}$$

which has determinant $-m_1m_2\dots m_{r-2}u^2$, and the absolute value has again decreased.

It remains only to consider the case where $a_{r,r} = m_r$. This implies that for the bilinear form we have $\langle \bar{e}_r, \bar{e}_r\rangle = 0$, so there must be a second $m_r$ occuring in $N(J)$ say for $e_{r-1}$ with $\langle \bar{e}_{r-1}, \bar{e}_r\rangle = \frac{u}{m_r}$ with $u$ a unit mod$(m_r)$. If $a_{r-1,r-1} \neq m_r$ we can exchange $r, r-1$, and we are in the previous case. Hence, assume $a_{r-1,r-1} = m_r$ as well. But then

$$q(\bar{e}_{r-1} + \bar{e}_r) = \frac{2u}{m_r}$$

and this is $1 \in \mathbb{Q}/(2\mathbb{Z})$ if and only if $m_r = 2$. Otherwise, we replace $e_r$ by $e_r + e_{r-1}$, leaving the remaining basis elements (except for $e_r^*$ and $e_{r-1}^*$) alone, and we reduce to the previous case.

Finally, we need to consider the case where $m_r = m_{r-1} = 2$, while $a_{r-1,r}$ is odd and $a_{r,r}, a_{r-1,r-1}$ are both congruent to 2 mod (4). Then after a basis change of the form

$$\begin{aligned} e_{r-1} &\mapsto e_{r-1} + l_1e_{r-1}^* + ke_r^* \\ e_r &\mapsto e_r + l_2e_r + ke_{r-1}^*, \end{aligned}$$

with all other basis elements fixed, the relevant parts of $N(J)$ and $A$ become

$$\left(\begin{matrix} 2 & 0 \\ 0 & 2 \end{matrix} \middle| \begin{matrix} 2 & 1 \\ 1 & 2 \end{matrix}\right).$$

We again exchange these two columns, and the determinant on the left side becomes $3m_1m_2\dots m_{r-2}$ which is again less than $Det(N(J))$. This completes the proof that surgery can be completed for $(-)$-symmetric forms..

**The case where $\mathbb{Z}^s \oplus (\mathbb{Z}^s)^*$ is $+$-symmetric.** Here the arguments are simpler since all the $a_{i,i} = 0$ for $m_i > 1$. Indeed, since this is the case, we must always have $m_{r-1} = m_r$, and we can assume that $\langle \bar{e}_{r-1}, \bar{e}_r\rangle = \frac{u}{m_r}$ for some unit mod$(m_r)$.

Then making a basis change $e_r \mapsto e_r + ke_{r-1}^*$, $e_{r-1} \mapsto e_{r-1} - ke_r^*$, we can assume that $0 < a_{r-1,r} < m_r$, and $a_{r-1,r} = a_{r,r-1}$. Hence, exchanging the last two columns as above the determinant of $N(J)$ is replaced by $m_1\dots m_{r-2}(a_{r-1,r})^2$ which is non-zero and less than the determinant of $N(J)$. This completes the proof that surgery can be completed in this case as well.

## 2. Torsion forms and signature

We begin by showing that every non-singular $\pm$-symmetric non-singular torsion form with quadratic extension is geometrically realizable as a kernel form for a degree one normal map. Then, using the result above, we will consider the trace of the sequence of surgeries which make the degree one normal map normally bordant to a homotopy equivalence. This will construct an important relationship between these torsion forms and rational quadratic forms.

LEMMA 58. *Let $(W, b\colon W \xrightarrow{\cong} W^*, q)$ be a non-singular $\pm$-symmetric torsion form with (even) quadratic extension. Then there is a degree one normal map*

$$f\colon M^{2n+1} \to S^{2n+1}$$

*with surgery kernel equal to $(W, b, q)$, where $n \geq 3$ and $n$ is even if $b$ is $(-)$-symmetric, while $n$ is odd if $b$ is $+$-symmetric.*

PROOF. Suppose that $\mathbb{Z}^m \to W$ is a surjective map with kernel $K \cong \mathbb{Z}^m$. Start with the connected sum

$$\#_1^m S^n \times D^{n+1},$$

and attach $n+1$-handles $D^{n+1} \times D^n$ to the boundary according to the choice of basis for the kernel when projecting down onto the $S^n$'s, and at the same time wind them around the normals to the $S_i^n$'s according to the linking pairing and quadratic refinement. This gives a new manifold with boundary, and the non-singularity of $b$ implies that the boundary is a homology sphere. On the other hand, since $n \geq 2$ so $2n+1 \geq 7$ we see that the boundary is simply connected, so it has the homotopy type of a sphere $S^{2n}$, $2n \geq 6$. In the usual way we can modify the framing of the attached handles to assure that the resulting manifold is parallelizable, so the resulting sphere is the ordinary sphere by 32, and we can attach a $D^{2n+1}$ over it. The resulting manifold obviously has a degree one normal map to $S^{2n+1}$ and surgery kernel isomorphic to $(W, b, q)$ as desired. $\square$

Now consider the trace, $T$, of a sequence of surgeries on the degree one normal map above, which surger it to a homotopy equivalence. We may assume the surgeries are all concentrated in middle dimension. Then we have the short exact sequence of kernels

$$0 \longrightarrow K_{n+1}(T) \xrightarrow{J} K_{n+1}(T, \partial T) \xrightarrow{\partial} W \longrightarrow 0$$

where $K_{n+1}(T)$ is torsion free, and we can identify

$$K_{n+1}(T, \partial T) \text{ with } K_{n+1}(T)^*$$

by Poincaré duality. Moreover, if we choose the dual basis for $K_{n+1}(T, \partial T)$ with respect to a chosen basis for $K_{n+1}(T)$ then the map $J$ becomes a $(-1)^{n+1}$-symmetric, even matrix, $A(J)$.

DEFINITION 48. *Let $(\mathbb{Q}^s, A)$ be a rational non-singular quadratic form, where $A\colon \mathbb{Q}^s \to (\mathbb{Q}^s)^*$ is a $\pm$-symmetric isomorphism. Then an even integral sublattice for $(\mathbb{Q}^s, A)$ is an integral lattice in $\mathbb{Q}^s$ (a copy of $\mathbb{Z}^s \subset \mathbb{Q}^s$ so that $\mathbb{Z}^s \otimes \mathbb{Q} = \mathbb{Q}^s$), so that there is an integral matrix $B$ with $A(m)m' = (B + B^t)(m)m'$ for all $m$, $m'$ in $L$.*

LEMMA 59.
*(1) Even integral lattices exist for $(\mathbb{Q}^s, A)$.*
*(2) If $L_1$ and $L_2$ are even integral lattices for $(\mathbb{Q}^s, A)$, then $L_1 \cap L_2$ is an even integral lattice for $(\mathbb{Q}^s, A)$.*

PROOF. Note that for any vector $\vec{v} \in \mathbb{Q}^s$ and any basis $e_1, \ldots, e_s$ for $\mathbb{Q}^s$ there is an integer $n(e_1, \ldots, e_s, \vec{v}) = n$ so that

$$n\vec{v} = n_1e_1 + n_2e_2 + \cdots + n_se_s$$

with each $n_i \in \mathbb{Z}$. Using this we can find a positive integer $n$ so that, with respect to the new basis $ne_1, \ldots, ne_s$ for $\mathbb{Q}^s$ and $\frac{1}{n}e_1^*, \ldots, \frac{1}{n}e_s^*$ for $(\mathbb{Q}^s)^*$, $A$ becomes integral. But then, if $A$ is not already even, if we chose $2ne_1, \ldots, 2ne_s$ and $\frac{1}{2n}e_1^*, \ldots, \frac{1}{2n}e_s^*$ the resulting matrix will be $4A$ which is certainly even.
The second statement is clear. □

DEFINITION 49. *Let $L \subset \mathbb{Q}^s$ be any lattice in $\mathbb{Q}^s$, and suppose the non-singular $\pm$-symmetric form $(\mathbb{Q}^s, A)$ is given. Then the dual lattice to $L$, $L^\# \subset (\mathbb{Q}^s)^*$ is the set of elements $w \in (\mathbb{Q}^s)^*$ so that $w(m) \in \mathbb{Z}$ for each $m \in L$.*

Again, it is direct to see that the dual lattice exists.

LEMMA 60. *Suppose that $L \subset \mathbb{Q}^s$ is an even integral lattice for the non-singular $\pm$-symmetric form $(\mathbb{Q}^s, A)$.*
*(1) Then $A(L) \subset L^\#$, and choosing any basis for $L$ and the corresponding dual basis for $L^\#$, $A$ becomes an integral matrix of the form $B \pm B^t$.*
*(2) The map $A^{-1}\colon L^\#/L \to Hom(L^\#/L, \mathbb{Q}/\mathbb{Z})$ is an isomorphism, defining a non-singular linking form on the quotient $L^\#/L$ with quadratic reduction:*

$$q\colon L^\#/L \to \mathbb{Q}/2\mathbb{Z}$$

*defined by $B$.*

(This is a direct exercise using the stated basis elements. Recall that the form is defined on the quotient by $\langle m, m'\rangle = \frac{1}{k}m(A^{-1}k\bar{m}')$ for some $k \in \mathbb{Z}$ so that $k\bar{m}' \in A(L)$. And similarly for the quadratic reduction.)

REMARK . In the situation above, given $A$, realized as an even $\pm$-symmetric matrix with respect to a basis for $L$ and the corresponding dual basis for $L^\#$, our previous construction realizes $(L^\#/L, A^{-1}, q)$ as the result of taking the Lagrangian kernel

$$(A \mid I) \subset \mathbb{Z}^s \oplus (\mathbb{Z}^s)^*$$

with the associated $(\mp)$-symmetric form.

Summarizing the arguments above we have shown the following result.

THEOREM 35. *Let $(T, C, q)$ be a triple consisting of a fintely generated torsion module $T$ over $\mathbb{Z}$, and $C\colon T \to Hom(T, \mathbb{Q}/\mathbb{Z})$ a $\pm$-symmetric isomorphism together with a quadratic reduction, $q$. Then there is a non-singular rational $\pm$-symmetric form $(\mathbb{Q}^s, A)$ together with an even lattice $L \subset \mathbb{Q}^s$ with respect to $A$, so that $(T, C, q)$ is identified with*

$$(L^{\#}/L, A^{-1}, q)$$

*as described above.*

We now turn to the question of the dependence of the form $(\mathbb{Q}^s, A)$ on $(T, C, q)$, and conversely, the dependence of $(L^{\#}/L, A^{-1}, q)$ on the choice of $L$ when $(\mathbb{Q}^s, A)$ is fixed.

Note first the obvious fact that if $(T_1, C_1, q_1)$ is associated to $(\mathbb{Q}^{s_1}, A_1, L_1)$ while $(T_2, C_2, q_2)$ is associated to $(\mathbb{Q}^{s_2}, A_2, L_2)$, then the orthogonal direct sum $(T_1 \oplus T_2, C_1 \oplus C_2, q_1 \oplus q_2)$ is associated to $\mathbb{Q}^{s_1+s_2}, A_1 \oplus A_2, L_1 \oplus L_2)$. So we have an addition in this set.

Now let us look at the possible variation in the $(T, C, q)$ associated to a fixed $(\mathbb{Q}^s, A)$.

Consider $L_1, L_1^{\#}$, and $L_2$, $L_2^{\#}$, both even lattices with respect to $A$ in $(\mathbb{Q}^s, A)$. We also have $L_1 \cap L_2$ and consequently the diagram of inclusions

$$\begin{array}{ccccccc} & & L_1 & \longrightarrow & L_1^{\#} & & \\ & \nearrow & & & & \searrow & \\ L_1 \cap L_2 & & & & & & (L_1 \cap L_2)^{\#}. \\ & \searrow & & & & \nearrow & \\ & & L_2 & \longrightarrow & L_2^{\#} & & \end{array}$$

LEMMA 61. *In $(L_2 \cap L_2)^{\#}/L_2 \cap L_2$ let $V_1$ be the image of $L_1$ so $V_1 = L_1/(L_1 \cap L_2)$ and similarly let $V_2 = L_2/(L_1 \cap L_2)$. Then $V_1$ and $V_2$ are Lagrangian kernels in the sense that both the bilinear forms and their quadratic reductions vanish identically on them.*

(Since $A$ and $B$ restricted to $L_1$ and $L_2$ are integral this is immediate.)

Note also that $V_1^{\perp}$ is the image of $L_1^{\#}$ in the quotient while $V_1^{\perp}$ is the image of $L_2^{\#}$. Here, as usual, the perpendicular to a sub-module, $M$, is the set of elements $v$ so that $\langle v, m\rangle = 0 \in \mathbb{Q}/\mathbb{Z}$ for each $m \in M$.

LEMMA 62. *Let $(T, A, q)$ be a non-singular $\pm$-symmetric torsion form with quadratic reduction. Let $V \subset T$ be a Lagrangian kernel in the sense above. Then $(V^{\perp}/V, A, q)$ is a non-singular $\pm$-symmetric torsion form.*

PROOF. Note that $(V^{\perp})^{\perp} = V$ since the original form is non-singular. In particular, there is an exact sequence

$$0 \longrightarrow V^{\perp} \overset{A}{\hookrightarrow} Hom(T, \mathbb{Q}/\mathbb{Z}) \overset{r}{\longrightarrow} Hom(V, \mathbb{Q}/\mathbb{Z}) \longrightarrow 0$$

where $r$ is restriction. Moreover, the elements in the image of $V^\perp$ may be characterized as the elements in $Hom(T, \mathbb{Q}/\mathbb{Z})$ which anhilate $V$. Thus this group maps to $Hom(V^\perp/V, \mathbb{Q}/\mathbb{Z})$, and non-singularity amounts to saying that this map is onto.

Consider the extension sequence

$$0 \longrightarrow V^\perp \hookrightarrow T \longrightarrow Hom(V, \mathbb{Q}/\mathbb{Z}) \longrightarrow 0$$

and suppose that an element in $Hom(V^\perp/V, \mathbb{Q}/\mathbb{Z})$ is given. This element extends in the obvious way to $V^\perp$, giving a homomorphism $V^\perp \rightarrow \mathbb{Q}/\mathbb{Z})$. But note that $\mathbb{Q}/\mathbb{Z}$ is **injective**! This implies that the homomorphism extends to all of $T$, and this proves that the restriction map above is onto $Hom(V^\perp/V, \mathbb{Q}/\mathbb{Z})$. The rest of the argument is direct. □

Let us now define an equivalence relation on non-singular $+$ or $(-)$ symmetric forms via the relation above: if $V \subset (T, A, q)$ is a quadratic kernel, then

$$(T, A, q) \sim (V^\perp/V, A|, q|).$$

Clearly, this equivalence relation preserves the direct sum structure provided that $V$ is, itself a direct sum. Consequently, the set of equivalence classes forms a monoid under orthogonal direct sum. Write $L_{4n}^{Tor}(\mathbb{Z})$ for the monoid of equivalence classes of $+$-symmetric forms, and $L_{4n+2}^{Tor}(\mathbb{Z})$ for the monoid of equivalence classes of $(-)$-symmetric forms. Then we have

Lemma 63. *Both $L_{4n}^{Tor}(\mathbb{Z})$ and $L_{4n+2}^{Tor}(\mathbb{Z})$ are abelian groups under orthogonal direct sum.*

Proof. $(T, -A, -q) \perp (T, A, q)$ has a Kernel $\Delta(T)$ with $\Delta(T)^\perp = \Delta(T)$. So additive inverses exist. □

We now define the Witt ring of $\pm$-symmetric $q$-forms over $\mathbb{Q}$ as the monoid generated by the isomorphism classes of non-singular $\pm$-symmetric forms on finite dimensioal $\mathbb{Q}$-vector spaces under the same equivalence relation: Let $V \subset \mathbb{Q}^s$ be a vector subspace and also a quadratic kernel, then $(\mathbb{Q}^s, A, q)$ is equivalent to $(V^\perp/V, A)$, and we define $L_4(\mathbb{Q})$ to be the set of equivalence classes of $+$-symmetric forms while $L_2(\mathbb{Q})$ is the set of equivalence classes of $(-)$-symmetric forms. As above these are abelian groups.

Theorem 36. *The correspondence above $(\mathbb{Q}^s, A)$ maps to*

$$\{(L^\#/L, A, q)\} \in L_{2*}^{Tor}(\mathbb{Z})$$

*for an even integral lattice $L \subset \mathbb{Q}^s$, gives a well defined, surjective mapping $p\colon L_{2*}(\mathbb{Q}) \rightarrow L_{2*}^{Tor}(\mathbb{Z})$ with kernel the subgroup generated by the even $\pm$-symmetric forms which come from non-singular, even forms over $\mathbb{Z}$.*

Proof. If $V \subset \mathbb{Q}^s$ is a quadratic kernel for $(\mathbb{Q}^s, A)$ then we can find a subset $V^* \subset \mathbb{Q}^s$ which is also a kernel and is dual to $V$ under $A$. DOES THIS

NEED TO BE EXPANDED, OR HAS IT ALREADY BEEN DISCUSSED? Consequently, $(\mathbb{Q}^s, A)$ can be written as an orthogonal direct sum

$$(V \oplus V^*) \perp (\mathbb{Q}^{s-2t}, A')$$

and we can construct a sub-lattice of $\mathbb{Q}^s$ of the form $(L(V) \oplus L(V)^*) \perp \bar{L}$ where $\bar{L}$ is an integral, even lattice in $\mathbb{Q}^{s-2t}$. It follows that $p(\mathbb{Q}^s, A) = p(\mathbb{Q}^{s-2t}, A')$ since $L(V) \oplus L(V)^*$ is self dual. Consequently $p$ descends to give a map of equivalence classes.

Clearly, if $(\mathbb{Q}^s, A)$ admits a self-dual, integral, even lattice, then $p(\mathbb{Q}^s, A) = 0$. On the other hand, suppose that $p(\mathbb{Q}^s, A) = 0$. Then, for an arbitrary integral, even lattice $L$ we have that $(L^{\#}/L, A^{-1}, q)$ must have a quadratic kernel $V$ with $V = V^{\perp}$. Let $\bar{L}$ be the inverse image of $V$ in $L^{\#}$. Then $\bar{L}$ is a non-singular, even, self-dual lattice and we are done. $\square$

Remark . We have already seen that the surgery obstruction groups $L_{2k}(\mathbb{Z})$ are $\mathbb{Z}$ when $k$ is even, and $\mathbb{Z}/2$ when $k$ is odd. They are given as the sets of equivalence classes of self-dual even integral lattices modulo the equivalence relation $(L, A) \sim (L, A) \oplus H$ where $H$ is given by the matrix $\begin{pmatrix} 0 & I \\ \pm I & 0 \end{pmatrix}$. But it is easily seen that this gives exactly the same set as requiring that $(L, A) \sim (V^{\perp}/V, A)$ for any quadratic kernel $V \subset L$ which is a $\mathbb{Z}$-direct summand.

Consequently, the kernels of $p$ are directly seen to be the images in $L_{2*}(\mathbb{Q})$ of $L_{2s}(\mathbb{Z})$. Moreover, in the case of $L_4(\mathbb{Q})$, since $L_4(\mathbb{Z})$ is detected by the signature, which factors through $L_4(\mathbb{R})$ it follows that the map

$$L_4(\mathbb{Z}) \xrightarrow{\otimes\mathbb{Q}} L_4(\mathbb{Q})$$

is injective and we have the exact sequence

$$0 \longrightarrow L_4(\mathbb{Z}) \xrightarrow{\otimes\mathbb{Q}} L_4(\mathbb{Q}) \xrightarrow{p} L_4^{Tor}(\mathbb{Z}) \longrightarrow 0$$

connecting all these groups.

Remark . Note that since $\frac{1}{2} \in \mathbb{Q}$ the Arf invariant does not have an analogue for $(-)$-symmetric forms over $\mathbb{Q}$. Consequently it is fairly clear that the image of $L_2(\mathbb{Z})$ in $L_2(\mathbb{Q})$ is zero. Indeed, here is an explicit proof. The $q$-form on $(\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, q)$ which gives Arf-invariant one is $q(e) = q(f) = q(e+f) = 1 \in \mathbb{Z}/2$. Here $\mathbb{Z}/2 = \mathbb{Z}/(a+a)$, but for $\mathbb{Q}$, $\mathbb{Q}/(a+a) = 0$. Consequently, when we tensor with $\mathbb{Q}$ the Arf invariant one form becomes identified with the trivial form, and the image is zero, as asserted. It follows that $L_2(\mathbb{Q}) \cong L_2^{Tor}(\mathbb{Z})$.

**The determination of the groups $L_{2*}^{Tor}(\mathbb{Z})$.** Let $(V, A, q)$ be a torsion form of the type considered in the definition of $L_{2*}^{Tor}(\mathbb{Z})$, then the decomposition of $V$ into its $p$-primary components is an orthogonal decomposition with respect to $A$ and $q$. Consequently, if we define $L_{2*}^{Tor,p}(\mathbb{Z})$ to be the set of equivalence classes of triples $(V, A, q)$ as above with $V$ a finite $p$-torsion module, then we have

$$L_{2*}^{Tor}(\mathbb{Z}) \;=\; \bigoplus_{p\ prime} L_{2*}^{Tor,p}(\mathbb{Z}).$$

We now have

LEMMA 64.
*(1)* $L_2^{Tor,p}(\mathbb{Z}) = 0$ *for each* $p$.
*(2)* $L_4^{Tor,p}(\mathbb{Z}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ *for* $p \equiv 1 \bmod (4)$.
*(3)* $L_4^{Tor,p}(\mathbb{Z}) = \mathbb{Z}/4$ *for* $p \equiv 3 \bmod (4)$
*(4)* $L_4^{Tor,2}(\mathbb{Z}) = \mathbb{Z}/8 \oplus \mathbb{Z}/2$

PROOF. For part one note that the only non-singular $(-)$-symmetric $\mathbb{Q}$ forms are given as

$$(\mathbb{Q}^{2s}, \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

up to isomorphism, and these represent 0 in $L_2(\mathbb{Q})$. Consequently, $L_2(\mathbb{Q}) = 0$ and, since $p\colon L_2(\mathbb{Q}) \to L_2^{Tor}(\mathbb{Z})$ is onto, it follows that $L_2^{Tor}(\mathbb{Z}) = 0$ as well.

For parts (2), (3), note that we can diagonalize every bilinear form on a $p$-torsion module for $p$ odd. So it suffices to consider the basic forms $(\mathbb{Z}/p^r, \frac{2\lambda}{p^r})$. Note that if $r > 1$ and $T$ is a generator of $\mathbb{Z}/p^r$ then $\langle p^{r-1}T\rangle = V$ is a quadratic kernel. Thus

$$(\mathbb{Z}/p^r, \frac{2\lambda}{p^r}) \sim (\mathbb{Z}/p^{r-2}, \frac{2\lambda}{p^{r-2}}).$$

Consequently, the generators for $L_4^{Tor,p}(\mathbb{Z})$, for $p$ odd are the forms $(\mathbb{Z}/p, \frac{2\lambda}{p})$. By a change of basis, $T \mapsto \tau T$, such a form is equivalent to the form $(\mathbb{Z}/p, \frac{2\tau^2\lambda}{p})$, and so, since $\mathbb{F}_p^*$ splits into exactly two cosets under multiplication by squares, the non-squares and the squares, it follows that there are at most two generators for $L_4^{Tor,p}(\mathbb{Z})$,

$$(\mathbb{Z}/p, \frac{2}{p}), \text{ and } (\mathbb{Z}/p, \frac{2\lambda}{p})$$

where $\lambda$ is a non-square in $\mathbb{F}_p^*$. If $-1$ is a non-square, then there is only the one generator, and this happens if and only if $p \equiv 3 \bmod (4)$. In this case $L_4^{Tor,p}(\mathbb{Z})$ is a cyclic group $\mathbb{Z}/l(p)$. In the case $p \equiv 1 \bmod (4)$ it follows that $L_4^{Tor,p}(\mathbb{Z}) = \mathbb{Z}/l_1(p) \oplus \mathbb{Z}/l_2(p)$, and it remains to determine these $l_i(p)$.

In case (2), where $p \equiv 1 \bmod (4)$ it follows that $-1$ is a square so $(\mathbb{Z}/p, \frac{\theta}{p}) \sim (\mathbb{Z}/p, \frac{-\theta}{p})$ so $L_4^{Tor,p}(\mathbb{Z})$ is a quotient of $(\mathbb{Z}/2)^2$.

In case (3) we have that $-1 = \tau^2 + \lambda^2$ for a pair of non-zero elements in $\mathbb{F}_p^*$. Consequently, changing the basis in

$$\left((\mathbb{Z}/p)^2, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

to $\tau e_1+\lambda e_2 = f_1$, $\lambda e_1-\tau e_2 = f_2$ we have that $\langle f_1, f_2\rangle = 0$ while $\langle f_i, f_i\rangle = -1$. It follows that

$$\left((\mathbb{Z}/p)^2,\ \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right) \cong \left((\mathbb{Z}/p)^2,\ \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\right)$$

so $L_4^{Tor,p}(\mathbb{Z})$ is a quotient of $\mathbb{Z}/4$.

It remains to show that these are exactly the groups in the two cases. For this we will need a sensitive invariant. Thus, given $(T, A, q)$ where $q\colon T{\to}\mathbb{Q}/2\mathbb{Z}$ is a quadratic refinement of the non-singular $+$-symmetric form $(T, A)$, set

$$\chi(T, A, q) = \sum_{\tau\in T} e^{\pi i q(\tau)}.$$

This invariant does not factor through $L_4^{Tor}(\mathbb{Z})$ but we do have the following results. First note that

Proposition 21.
*(1) Let $(T, A, q)$ be the orthogonal direct sum $(T_1, A_1, q_1) \perp (T_2, A_2, q_2)$. Then*

$$\chi(T, A, q) \ = \ \chi(T_1, A_1, q_1)\chi(T_2, A_2, q_2).$$

*(2)* $\chi(T, -A, -q) = \overline{\chi(T, A, q)}$.

Proof. For $w_1 \in T_1$, $w_2 \in T_2$ we have $q(w_1 + w_2) = q_1(w_1) + q_2(w_2)$ so

$$\sum_{w_1\in T_1}\sum_{w_2\in T_2} e^{\pi i q(w_1+w_2)} \ = \ \sum_{w_1\in T_1} e^{\pi i q(w_1)} \sum_{w_2\in T_2} e^{\pi i q(w_2)},$$

but this is the result. The second statement is clear. □

Proposition 22. *Let $V \subset T$ be a quadratic kernel for $(T, A, q)$, then*

$$\chi(T, A, q) = |V|\chi(V^{\perp}/V, A, q).$$

Proof. Break $T$ up into cosets of $V^{\perp}$ as follows. The elements of a general coset have the form

$$v^* + (\lambda + v)$$

where $\lambda$ represents a coset of $V$ in $V^{\perp}$. Also, $q(v^*+(\lambda+v)) = q(v^*)+q(\lambda)+2\langle v^*, \lambda\rangle + 2\langle v^*, v\rangle$ so

$$e^{\imath\pi i q(v^*+\lambda+v)} = e^{\pi i q(v^*+\lambda)}e^{2\pi i\langle v^*,v\rangle}.$$

Now, fixing $v^*$, $\lambda$ an varying $v$ over $V$, if the image of $v^*$ in $Hom(V, \mathbb{Q}/\mathbb{Z})$ is non-trivial, then the sum of these terms is zero since the sum of all the $(p^i)^{th}$ roots of 1 is always zero for $p > 1$. Hence the only non-zero term correspond to the coset $V^{\perp}$ itself. Here $v^* = 0$ and the sum is

$$\sum_{v\in V}\sum_{\lambda\in V^{\perp}/V} e^{\pi i q(\lambda)} = |V|\chi(V^{\perp}/V, A, q)$$

as asserted. □

Thus we have

Corollary 24.
*(1)* $\chi(T, A, q) = |T|^{\frac{1}{2}} e^{2\pi i\theta}$ *for some real number* $\theta$.
*(2) If* $|T|$ *is odd, then* $\theta$ *is rational and* $4\theta$ *is an integer.*

Proof. First, note that $(T, A, q) \perp (T, -A, q)$ has a kernel $V$ given as the diagonal image of $T$, with $V^{\perp} = V$. Consequently, $\chi(T, A, q)\overline{\chi(T, A, q)} = |T|$, and $\chi(T, A, q) = |T|^{\frac{1}{2}} w$ where $w$ is a complex number of modulus 1.

Now suppose that $|T|$ is odd. Then $T$ breaks up into an orthogonal direct sum of odd prime components, and for each of these it is true by the preliminary calculations above that $4T_p$ has a kernel $V$ with $V^{\perp} = V$. Thus the same must be true for $T$ and the corollary follows. □

In fact we can evaluate $\chi(T, A, q)$ more or less explicitly for $|T|$ odd. It suffices to evaluate it for the basic classes $(\mathbb{Z}/p, 1/p)$ and $(\mathbb{Z}/p, \lambda/p)$ where $\lambda$ is a non-square. We have

Proposition 23.
*(1) If* $p \equiv 1 \bmod (4)$ *then* $\chi(\mathbb{Z}/p, 2/p) = -\chi(\mathbb{Z}/p, 2\lambda/p)$, *so one is* $p^{\frac{1}{2}}$ *and the other is* $-p^{\frac{1}{2}}$.
*(2) If* $p \equiv 3 \bmod (4)$ *then* $\chi(\mathbb{Z}/p, 2/p) = \pm i p^{\frac{1}{2}}$.

Proof. In the sum $\sum_1^{p-1} e^{2\pi i \tau k^2/p}$ each term in the sum appears twice. Thus

$$\begin{aligned}\chi(\mathbb{Z}/p, 2/p) + \chi(\mathbb{Z}/p, 2\lambda/p) &= 2\textstyle\sum_0^{p-1} e^{2\pi i j/p} \\ &= 0.\end{aligned}$$

Thus, case (1) follows directly. In case (2), writing $S_1 = \chi(\mathbb{Z}/p, 2/p)$, $S_2 = \chi(\mathbb{Z}/p, -2/p)$ we have $S_1 S_2 = -S_1^2 = p$ and $S_1 = \pm i p^{\frac{1}{2}}$ as claimed. □

But this completes the proof of parts (2) and (3) in the theorem, since the argument of $\chi(T, A, q)$ give a non-trivial surjective homomorphism from $L_4^{Tor,p}(\mathbb{Z}) \to \mathbb{Z}/4$ when $p \equiv 3 \bmod (4)$, and $\chi$ distinguishes the two generators when $\equiv 1 \bmod (4)$.

It remains to demonstrate (4). So for the remainder of the argument we assume that $T$ is a 2-torsion module. By downward induction, starting with the elments of largest order in $T$ we see that we can always reduce order unless we have $\mathbb{Z}/4$'s or $\mathbb{Z}/2$'s as the elements of largest order. Morover, again using the existence of kernels for direct sums we can assume that at most one element of order 4 appears in a generating set, and we can assume that the form is diagonalized. Consequently, the generators for $L_4^{Tor,2}(\mathbb{Z})$ are

$$\begin{cases} (\mathbb{Z}/2, \frac{1}{2}) & \text{written } \mathcal{A}_1 \\ (\mathbb{Z}/4, \frac{1}{4}) & \text{written } \mathcal{A}_2 \\ (\mathbb{Z}/4, \frac{3}{4}) & \text{written } \mathcal{A}_3. \end{cases}$$

Proposition 24. *We have the following relations among the generators* $\mathcal{A}_2 \perp \mathcal{A}_3 \sim 4\mathcal{A}_1$, $2\mathcal{A}_2 \sim 2\mathcal{A}_1$, $8\mathcal{A}_1 \sim 0$. *Thus,* $L_4^{Tor,2}(\mathbb{Z})$ *is at most* $\mathbb{Z}/8 \oplus \mathbb{Z}/2$.

PROOF. To begin let us consider $4\mathcal{A}_1$. In $4(\mathbb{Z}/2, 1/2)$ there is a $\mathbb{Z}/2$ kernel generated by the diagonal $\mathbb{Z}/2$, $V = \langle(1,1,1,1)\rangle$. Here $V^\perp/V = (\mathbb{Z}/2)^2$ with generators $(1,1,0,0)$ and $(1,0,0,1)$. Consequently, we have

$$q(1,1,0,0) = q(1,0,0,1) = 1$$

while

$$\langle(1,1,0,0),(1,0,0,1)\rangle = \frac{1}{2}.$$

Thus the resulting form on the quotient is the Arf invariant one form, $\mathcal{B}$. Now, note that the diagonal copy of $\mathcal{B}$ in $\mathcal{B} \perp \mathcal{B}$ is a kernel with $\Delta(\mathcal{B})^\perp = \Delta(\mathcal{B})$, so $2\mathcal{B} \sim 8\mathcal{A}_1$ is zero in $L_4^{Tor,2}(\mathbb{Z})$ as asserted.

Next, note that $\mathcal{A}_2 \perp \mathcal{A}_3$ has the diagonal $V \cong \mathbb{Z}/2 = \langle(2,2)\rangle$ as a kernel and $V^\perp/V = \mathbb{Z}/2 \perp \mathbb{Z}/2$ with generators $(1,1)$ and $(2,0)$. The associated bilinear form has matrix $\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ but $q(1,1) = q(2,0) = 1$ which is again the Arf invariant one form. Thus we have verified the relation $\mathcal{A}_2 \perp \mathcal{A}_3 \sim 4\mathcal{A}_1$ in $L_4^{Tor,2}(\mathbb{Z})$.

Finally, we consider the form $\mathcal{A}_2 \perp \mathcal{A}_2$ which again has the diagonal $V = \langle(2,2)\rangle$ as a quadratic kernel. Here a suitable basis for $V^\perp$ is $(1,1)$ and $(1,3)$ and $V^\perp/V = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ $(1,1) \perp (1,3)$ and $q(1,1) = q(1,3) = \frac{1}{2}$. This shows that $2\mathcal{A}_2 \sim 2\mathcal{A}_1$ and completes the proof. $\square$

To complete the proof of (4) we must show that $L_4^{Tor,2}(\mathbb{Z})$ is at least $\mathbb{Z}/8 \oplus \mathbb{Z}/2$. To do this note that

$$\chi(\mathbb{Z}/2, 1/2) = 1 + i = 2^{\frac{1}{2}} e^{2\pi/8}$$

while

$$\chi(\mathbb{Z}/4, 1/4) = (1 + i - 1 + i) = 2i.$$

On the other hand, $\chi$ really has two invariants in it. The first is the modulus modulo integral powers of 2, so it is either 1 or $2^{\frac{1}{2}}$. The second is the argument which takes its values in $\mathbb{Z}/8$. Thus it defines a homomorphism from $L_4^{Tor,2}(\mathbb{Z})$ to $\mathbb{Z}/2 \oplus \mathbb{Z}/8$ and

$$\begin{array}{lcl} \mathcal{A}_1 & \mapsto & (2^{\frac{1}{2}}, \frac{1}{8}) \\ \mathcal{A}_2 & \mapsto & (1, \frac{1}{8}) \end{array}$$

and these two images together generate the entire $\mathbb{Z}/2 \oplus \mathbb{Z}/8$. $\square$

**A reciprocity law.** We have seen that we have a short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow L_4(\mathbb{Q}) \longrightarrow L_4^{Tor}(\mathbb{Z}) \longrightarrow 0$$

but we also have a homomorphism

$$L_4(\mathbb{Q}) \xrightarrow{\otimes\mathbb{R}} L_4(\mathbb{R})$$

where $\mathbb{R}$ is the reals. On the other hand every real quadratic form is diagonalizable to a direct sum $k\langle 1\rangle \perp s\langle -1\rangle$, so the only invariant, when we factor out by kernels, is $\sigma = k - s$, the signature. It thus follows that $L_4(\mathbb{R}) = \mathbb{Z}$, and, since the signature detects the kernel of the map $L_4(\mathbb{Q}) \to L_4^{Tor}(\mathbb{Z})$, it follows that the sum map

$$L_4(\mathbb{Q}) \longrightarrow L_4^{Tor}(\mathbb{Z}) \oplus L_4(\mathbb{R}) = \mathbb{Z}$$

is injective.

THEOREM 37. *Define a map*

$$s_1 \colon L_4^{Tor}(\mathbb{Z}) \longrightarrow \mathbb{Z}/8$$

*by sending* $(T, A, q)$ *to the argument of* $\chi(T, A, q)$*. Also define* $s_2 \colon L_4(\mathbb{R}) \to \mathbb{Z}/8$ *by taking the signature* mod(8)*. Then* $(v, w) \in L_4^{Tor}(\mathbb{Z}) \oplus L_4(\mathbb{R})$ *is in the image from* $L_4(\mathbb{Q})$ *if and only if* $s_1(v) = s_2(w)$*.*

(We don't prove this here. An analytic proof is given in an appendix to [46], but, being careful with the proof of 36 above leads to a constructive proof.)

## 3. The construction of an exact sequence for the surgery groups

LEMMA 65. *Suppose a non-singular* $\pm$*-torsion form* $W$ *with a quadratic refinement is given as above, and suppose that* $L \subset W$ *is a quadratic kernel (this means that* $\langle l_1, l_2 \rangle = 0$ *for each pair* $l_1, l_2 \in L$ *and* $q(l) = 0$ *as well for each* $l \in L$*). Then there is a well defined non-singular* $\pm$*-form with quadratic refinement on* $L^\perp/L$*.*

PROOF. Indeed, for $m_1, m_2 \in L^\perp$ we define $\langle \bar{m}_1, \bar{m}_2 \rangle = \langle m_1, m_2 \rangle$, and $q(\bar{m}) = q(m)$. These are both well defined since, for example, $q(m + l) = q(m) + q(l) + \langle m, l \rangle = q(m)$, and similarly for the bilinear form.

Now, consider the composite map $W \xrightarrow{B} W^* \xrightarrow{p} L^*$. By definition, $L^\perp$ is precisely the kernel. Using this remark we can see that the induced bilinear form on $L^\perp/L$ is non-singular.

For $m \in L^\perp$, representing a non-trivial element in the quotient, suppose that

$$\langle m, m' \rangle = 0$$

for all $m' \in L^\perp$. It follows that $m \in (L^\perp)^\perp$. Now consider $w \in W$. We have $\langle m, w \rangle = \langle m, w + m' \rangle$ for any $m' \in L^\perp$, so $m \in Hom(L^*, \mathbb{Q}/\mathbb{Z}) \cong L$, and there is a unique element $l(m) \in L$ so that $\langle l(m), w \rangle = \langle m, w \rangle$ for all $w \in W$. But, since the form is non-singular, this implies that $m = l(m) \in L$ which represents 0 in the quotient. $\square$

EXAMPLE 47. In the 46 above, if we set $L = \mathbb{Z}/5$, then $L = L^\perp$, so the quotient is zero. In the first example, 45, there is no such $L$.

The reason this sub-quotient construction is useful is the following lemma:

LEMMA 66. *Suppose the element* $w \in W$ *satisfies* $q(w) = 0$*. Then a series of geometric moves will replace the original surgery problem by a problem with* $W$ *replaced by* $L^\perp/L$*.*

CHAPTER 11

# The Global Structure of Surgery when $\pi_1(X) = 0$

In this chapter we survey the global theory of surgery in the case where $\pi_1(X) = 0$. To begin we construct a topological invariant, the Spivak normal fibration, associated to any closed Poincaré duality complex. This invariant is the homotopy type of the normal sphere bundle over $X$ if $X$ is a manifold, and is classified by a map $f\colon X \to B_G$ where $B_G$ is a space which classifies homotopy sphere bundles – Serre fibrations where the fiber has the homotopy type of a sphere.

Given a lifting of this Spivak normal bundle to an actual vector bundle over $X$ we get a normal bordism class of degree one normal maps over $X$. Moreover, in this context, by taking difference constructions we get a new description of the set of normal bordism classes of degree one normal maps as the set of homotopy classes of maps $[X, G/O]$ where $G/O$ is the fiber of the Serre fibration associated to the natural map $B_O \to B_G$.

From this we obtain the surgery exact sequence

$$[\Sigma X, G/O] \xrightarrow{\sigma} L_{n+1}(1) \longrightarrow \mathcal{H}D(X) \longrightarrow [X, G/O] \xrightarrow{\sigma} L_n(1)$$

where $\mathcal{H}D(X)$ is the set of $h$-cobordism classes of pairs $(M^n, f)$ with $M^n$ an $n$ dimensional differentiable manifold and $f\colon M^n \to X$ a homotopy equivalence. Also, $\sigma$ is the surgery obstruction map. A key property of $\sigma$ is that it only depends on the bordism class of $(M^n, g\colon M^n \to G/O)$ is case $M^n$ is a closed, compact differentiable manifold with empty boundary.

Next we review work of Sullivan which identifies the value of $\sigma(M^n, g)$ in terms of reasonably accessible homology data for the map $g\colon M^n \to G/O$. In particular, these formulae depend crucially on the existence of product formulae – also due to Sullivan – which identify the surgery obstruction of a normal map of the form

$$N^k \times \bar{M}^n \xrightarrow{id \times f} N^k \times M^n$$

in terms of characteristic classes of $N$ and the surgery obstruction of $f$. These product formulae and their generalizations to the non-simply connected case play a crucial role in applications of surgery.

## 1. The Spivak normal fibration in the simply connected case

We assume that $X$ is a closed $n$-dimensional, simply connected and finite Poincaré duality complex throughout this section, and $f\colon X^n \to \mathbb{R}^{2n+k}$ is an embedding of $X$ as a subcomplex of Euclidian space where $k \geq 2$[1].

Then, in the second barycentric subdivision of the triangulation of $\mathbb{R}^{2n+k}$, by taking the stars of the simplices of $X$ and smoothing slightly we obtain a regular neighborhood, $N(X)$ of $X$, which (of course) collapses to $X$ and which has a manifold boundary $\partial N(X)$.

EXAMPLE 48. Let $M^n \hookrightarrow \mathbb{R}^{2n+k}$ be an embedding of a closed, compact, differentiable manifold. Then, by the tubular neighborhood theorem, 9, we can identify such a neighborhood of $M$ with the normal disk bundle to the embedding,

$$D^{n+k} \hookrightarrow N(M^n) \longrightarrow M^n,$$

and the boundary with the sphere bundle of this normal bundle,

$$S^{n+k-1} \hookrightarrow \partial N(M^n) \longrightarrow M^n.$$

Up to homotopy a similar result is true in much greater generality.

THEOREM 38. *The homotopy fiber of the inclusion $\partial N(X) \hookrightarrow N(X) \simeq X$ has the homotopy type of the sphere $S^{n+k-1}$ for $X$ as above and $k$ sufficiently large.*

PROOF. The homology groups $H_*(\partial N(X))$ fit into the exact sequence

$$\cdots \to H_*(X) \xrightarrow{j} H_*(X, \partial N(X)) \xrightarrow{\partial} H_{*-1}(\partial N(X)) \to H_{*-1}(X) \longrightarrow \cdots .$$

Also we have the Poincaré duality isomorphisms

$$H_{2n+k-*}(X, \partial N(X)) \cong H^*(X) \cong H_{n-*}(X).$$

It is convenient to assume that $j$ is the zero map in 1 so that 1 shows that in dimensions less than $n+k$ we have isomorphisms

$$H_*(\partial N(X)) \cong H_*(X)$$

while in dimensions greater than $n+k-1$ we have

$$\partial\colon H_*(X, \partial(N(X)) \longrightarrow H_*(\partial N(X))$$

is an isomorphism.

In fact we can always achieve this by increasing $k$ by one so we have

$$X \subset \mathbb{R}^{2n+k} \subset I \times \mathbb{R}^{2n+k} \subset \mathbb{R}^{2n+k+1}.$$

Indeed,

$$\partial(I \times N(X)) \ = \ I \times \partial(N(X)) \bigcup \partial I \times N(X)$$

so that the map $X \to I \times N(X)$ factors up to homotopy through $\partial(I \times N(X))$.

[1] Sucha an embedding is always possible since $X$ is a subcomplex of the simplex $\Delta^{m-1}$ where $m$ is the number of vertices of $X$

Summarizing, $\partial\colon H_*(X, \partial N(X)) \to H_{*-1}(\partial N(X))$ is an isomorphism and

$$H_i(\partial N(X)) = \begin{cases} H_i(X) & \text{if } i \le n \\ H_{i-n-k+1}(X) & \text{for } i \ge n+k-1. \end{cases}$$

LEMMA 67. *The boundary $\partial I \times N(X)$ is simply connected.*

PROOF. Indeed, the above description of $\partial N(X)$ gives it as a double:

$$\partial(I \times N(X)) \simeq N(X) \cup_{\partial N(X)} N^{(}X).$$

By the van Kampen theorem $\pi_1(\partial N(X))$ is the amalgamated product of $\pi_1(N'(X))$ with itself over $\pi_1(\partial N'(X))$. But $\pi_1(N'(X)) = 0$ so the lemma follows. ☐

The relative Hurewicz theorem now shows that $\pi_i(X, \partial N(X)) = 0$ for $i \le n+k-1$ and

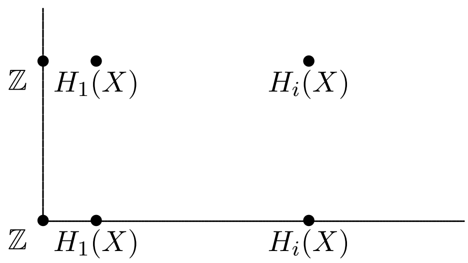
$$\pi_{n+k}(X, \partial N(X)) \cong H_{n+k}(X, \partial N(X)) = \mathbb{Z}.$$

Consequently, if $V \in \pi_{n+k}(X, \partial N(X))$ is a generator, then

$$\partial V \in \pi_{n+k-1}(\partial N(X))$$

is non-trivial, lies in the kernel of $i_*\colon \pi_{n+k-1}(\partial N(X)) \to \pi_{n+k-1}(X)$, and thus represents a non-trivial $\mathbb{Z}$-generator in $\pi_{n+k-1}(F)$ where $F$ is the homotopy fiber of $i$.

On the other hand, in the Serre spectral sequence of the fibration we see that the fiber has $H_*(F) = H_*(S^{n+k-1})$ since $\pi_i(F) = \pi_{i+1}(X, \partial N(X))$ and is consequently 0 for $i < n+k-1$. Then in $n+k-1$ we have constructed the $\mathbb{Z}$ generator, and the resulting two lines of the Serre spectral sequence



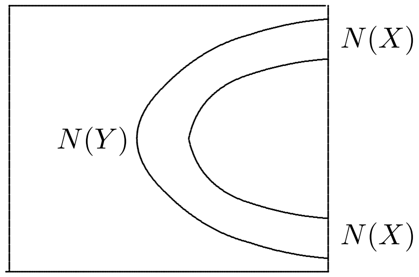

**The Serre spectral sequence of the fibration**

completely describe $H_*(\partial N(X))$. Consequently, there can be no further non-zero lines in the Spectral sequence and the homology of the fiber is that of the sphere. Thus, by applying the Whitehead theorem the fiber has the homotopy type of $S^{n+k-1}$ as asserted. ☐

There is also a relative version of this result. Suppose that $(Y^{n+1}, X^n)$ is a Poincaré pair with both $X$, $Y$ simply connected and

$$h\colon (Y, X) \to (\mathbb{R}_+^{2n+k+1}, \mathbb{R}^{2n+k})$$

is a simplicial embedding with $N(Y) \subset \mathbb{R}_+^{2n+k+1}$, $N(X) \subset \mathbb{R}^{2n+k}$ as regular neighborhoods with manifold boundaries:




THEOREM 39. *Define $W$ as $\partial N(Y) - N(X)$, so $\partial N(Y) = N(X) \cup_{\partial N(X)} W$. Then the inclusion $W \hookrightarrow N(Y)$ has homotopy fiber $S^{n+k}$ if $X$, $Y$ as above are simply connected and $k$ is sufficiently large.*

PROOF. We have

$$H_*(Y, X) \cong H_*(N(Y), N(X)) \cong H^{n-*}(N(Y)) \cong H_{n+k+*+1}(N(Y), W)$$

and the remainder of the proof goes as before. □

DEFINITION 50. *Let $(X, \partial X) \hookrightarrow (\mathbb{R}_+^{2n+k+1}, \mathbb{R}^{2n+k})$ be an embedding of compact Poincaré pairs, or $X \hookrightarrow \mathbb{R}^{2n+k}$ an embedding of a closed compact Poincaré duality complex, and $N(X)$ a regular neighborhood of $X$ with manifold boundary. Then the inclusion $\partial N(X) - N(\partial X) \hookrightarrow N(X)$ is called a* **Spivak normal bundle to** *$X$.*

We have shown above that in the simply connected case the Spivak normal bundle has the homotopy type of a spherical fibering with fiber $S^{n+k-1}$. In the next section we will give Browder's extension of this result to the non-simply connected case.

Then in the following sections we will show that the Spivak normal bundle is unique up to equivalence of Serre fibrations for $k \geq 3$. (Two Serre fibrations $E_1 \xrightarrow{\pi_1} X$ and $E_1 \xrightarrow{\pi_2} X$ are equivalent if and only if there is a homotopy equivalence $h\colon E_1 \rightarrow E_2$ so that $\pi_1 = \pi_2 h$.)

There is also a Thom space construction for fiber homotopy $S^m$-bundles.

DEFINITION 51. *Let $f\colon E \rightarrow X$ be any Serre fibration with fiber the sphere $S^m$, then the Thom space $T(f)$ is the mapping cone of $f$.*

LEMMA 68. *The Thom isomorphism theorem holds with $\mathbb{Z}/2$ coefficients for all Serre fibrations $f\colon E \rightarrow X$ with fiber the sphere $S^m$. If the fibration is oriented – in particular if $\pi_1(X) = 0$ – then the Thom isomorphism theorem holds with integer coefficients.*

PROOF. The Serre spectral sequence for the pair $(X, E)$ has $E_2$-term

$$H^*(X, H^*(D^{m+1}, S^m)))$$

and consequently only has one non-zero row, $E_2^{*,m+1}$ with $E_2^{*,m+1} = H^*(X)$. Here, the coefficients are twisted by an action of $\pi_1(X)$ on

$$H_{m+1}(D^{m+1}, S^m) = \mathbb{Z}$$

which, consequently factors through

$$Aut(\mathbb{Z}) = \mathbb{Z}/2.$$

If we take $\mathbb{Z}/2$-coefficients the twisting is trivial, of course. In any case it follows that $E_2 = E_\infty$. Moreover, by applying cup products, cupping with the class corresponding to a generator of $H^{n+1}(D^{m+1}, S^m)$ gives the isomorphism in the case of $\mathbb{Z}/2$ coefficients, or in the case where the action of $\pi_1(X)$ is trivial. □

On the other hand, by taking the inclusion $\mathbb{R}^{2n+k} \hookrightarrow S^{2n+k}$ and collapsing the complement of $Int(N(X))$ to a point we obtain Pontrajagin-Thom maps for the Spivak normal bundles:

$$S^{2n+k} \longrightarrow S^{2n+k}/(S^{2n+k} - \dot{N}(X)) \simeq N(X)/\partial N(X)$$

when $X$ is a closed, finite, $n$-dimensional Poincaré complex, and

$$\begin{array}{rcl}(D^{2n+k+1}, S^{2n+k}) & \longrightarrow & \mathbb{R}_+^{2n+k+1}/(\mathbb{R}_+^{2n+k+1} - \dot{N}(X)) \\ & & \simeq N(X)/(\partial N(X) - N(\partial X))\end{array}$$

which are direct generalizations of the Pontrajagin-Thom maps for differential manifolds or differential manifolds with boundary. Thus, the top class of the Thom complex for the Spivak normal bundle is always spherical.

## 2. Browder's extension of the Spivak normal bundle to $\pi_1(X) \neq 0$

Suppose that $X$ is a closed Poincaré duality complex of dimension $n$ with $\pi_1(X) \neq 0$. We can still embed $X \hookrightarrow \mathbb{R}^{2n+s}$ for some sufficiently large $s$ provided that $X$ is a finite complex – which we assume in what follows. We would again like to conclude that the inclusion $\partial N(X) \hookrightarrow N(X)$ of the manifold boundary of a regular neighborhood of $X \hookrightarrow \mathbb{R}^{2n+s}$ is again a spherical fibering. However, even though the homology groups are correct for this to happen, the possibility of *non-trivial* twisted coefficients on the fiber prevents us from making this conclusion when we just try to argue as in the proof of 38.

Browder's idea was to embed $X \subset Y$ for $Y$ an $n+1$-dimensional simply connected Poincaré complex so that a neighborhood of $X$ in $Y$ has the form $I \times X$. Then restricting the Spivak normal bundle to $Y$ to this neighborhood and using a Whitney sum construction he showed that the inclusion of the $\partial$ in the thickening of this neighborhood is a spherical fibering.

We modify is idea somewhat in what follows constructing the Spivak normal bundle for an $(n+1)$-dimensional simply connected Poincaré duality space which contains $X$ as a boundary component, and noting that the restriction of this bundle to $X$ together with projection to $X$ must be the homotopy type of the pair $(N(X), \partial N(X))$, and consequently, the homotopy fiber of the inclusion is the sphere $S^{n+k-1}$ for sufficiently large $k$ as desired.

Here are the details. We assume $n \geq 5$ in what follows.

Let $e_1, \ldots, e_r$ generate $\pi_1(X)$ and suppose that maps $f_i\colon S^1 \to X$, $i = 1, \ldots, r$ are given to represent the $e_i$. Then for each $i$

$$(f_i)_*[S^1] = [e_i] = \alpha_i \cap [X]$$

with $\alpha_i \in H^{n-1}(X)$. Let $V$ be the union over $X$ of the mapping cones of the $f_i$ (that is, we attach $r$ 2-cells to $X$ with attaching maps the $f_i$). The fundamental group of $V$ is thus 0. Moreover, $H^{n-1}(V) \cong H^{n-1}(X)$ since $n \geq 5$, so the $\alpha_i$ are still present in $H^{n-1}(V)$.

For each $\alpha_i$ we have a map $\alpha_i\colon V \to K(\mathbb{Z}, n-1)$ with $\alpha_i^*(\iota) = \alpha_i$, and, since

$$K(\mathbb{Z}, n-1) \simeq S^{n-1} \cup e^{n+1} \cup \cdots$$

as a cell complex, it follows that we have maps

$$\bar{\alpha}_i\colon V \longrightarrow S^{n-1}$$

with $\bar{\alpha}_i^*([S^{n-1}]^*) = \alpha_i$, $1 \leq i \leq r$, and thus a map

$$\bigvee_1^r \bar{\alpha}_i\colon V \longrightarrow \bigvee_1^r S^{n-1}.$$

We now study the fiber of $\bigvee_1^r \bar{\alpha}_i$ which we write $F$. There is a fibration

$$\Omega \bigvee_1^r S^{n-1} \longrightarrow F \longrightarrow V$$

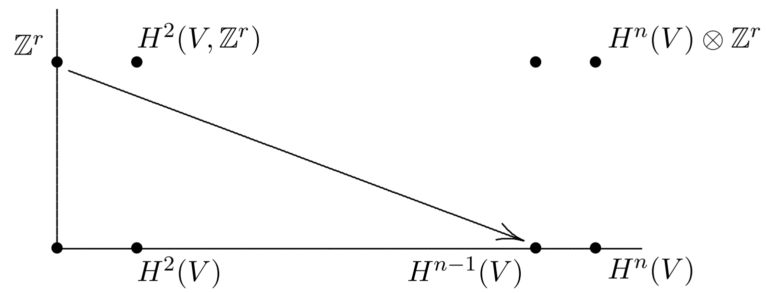
and

$$\Omega \bigvee_1^r S^{n-1} = \left(\bigvee_1^r S^{n-2}\right) \cup_1^{r^2} e^{2n-4} \cup \cdots .$$

Thus in the range of dimensions $< 2n-4$ the Serre spectral sequence only has two rows, $E^*_{*,0}$ and $E^*_{*,n-2}$ and only one differential

$$d^{n-1}\colon H_{n-1}(X) = E_{n-1,0} \to E_{0,n-2} = \mathbb{Z}^r.$$

**The spectral sequence for $H^*(F)$**



As a consequence $H^n(F) = \mathbb{Z} \oplus H^2(V, \mathbb{Z}^r)$ and $H^{n-1}(F) = 0$ since $H^1(V) = 0$.

It remains to choose a single class in $H^n(F)$ which will give Poincaré duality and kill the others.

In homology $H_{n-2}(F)$ is torsion free. We can write

$$H_{n-2}(V) = \mathbb{Z}/n_1 \oplus \mathbb{Z}/n_2 \oplus \cdots \oplus \mathbb{Z}/n_s \oplus \mathbb{Z}^l$$

but the cohomology coboundary $d^{n-1}$ above is surjective onto

$$H^{n-1}(V) = \mathbb{Z}/n_1 \oplus \cdots \mathbb{Z}/n_s \oplus \mathbb{Z}^w.$$

Thus, dually, the classes $e_i$, $1 \leq i \leq s$ responsible for the torsion truncations in the homology of $V$ lift in the fiber to classes with boundaries $n_i f_i + g_i$ where $g_i$ is the dual of the appropriate cohomology class on the fiber with coboundary the Bochstein of $f_i^*$.

Also, $H^2(F) = H^2(V)$ is torsion free since $\pi_1(V) = 0$. Now, the elements in

$$H^2(V, H^{n-2}(F)) = H^2(V) \otimes_{\mathbb{Z}} H^{n-2}(F) = E_\infty^{2,n-2}$$

correspond to cup products if the terms in $H^{n-2}(F)$ lie in $E_\infty^{0,n-2}$ which has finite index in the entire term (being generated by the elements $n_1 e_1^*, \ldots, n_s e_s^*$, and a direct summand). But we have already seen the extension structure in homology which indicates that there is, for each generator $\alpha_i \in H^2(V)$ (which corresponds to the summand $\mathbb{Z}/n_i$), an element in $E^\infty_{2,n-2}$ which, when capped with $\alpha_i$ gives the homology generator in $H_{n-2}(V)$ which corresponds. The sum of these elements for each $i$, together with the class corresponding to $[X]$ gives Poincaré duality. Consequently we have constructed a simply connected Poincaré duality complex by adding this class to the $n-1$-skeleton of the fiber $F$.

Denote the resulting space $X'$.

The inclusions $X \subset V$ and $X' \subset V$, the latter obtained as the composition of inclusion $X' \subset F \rightarrow V$, define an $(n+1)$-dimensional Poincaré pair,

$$(V, X \sqcup -X'),$$

as is easily verified, and consequently a Spivak normal bundle exists for $V$ which restricts to the Spivak normal bundle for $X'$ and, for $X$ we have the following situation. The restriction of this Spivak normal bundle factors through the boundary of a regular neighborhood of $X$ in a sufficiently high dimensional euclidian space, and there gives a homology isomorphism. Again, for sufficiently large dimension, we can assume that the map also induces isomorphisms in homotopy for $i = 1, 2$. But from this, by the Whitehead theorem and the 5-lemma it induces homotopy equivalences on the respective fibers as well. It follows that the construction of 38 works in the non-simply connected case as well provided that $X$ is an oriented, closed Poincaré duality space.

REMARK . The argument above is modeled on the process of doing surgery on embedded $S^1 \times D^{n-1}$'s in $M^n$. However, attempting to directly generalize this to $S^r \times D^{n-r} \hookrightarrow X$ requires that $X$ be connected through dimension $r-1$ if we are to use the fiber of an associated dual map $V \to S^{n-r}$ as a model for the result of surgery. This is not satisfactory. Better theories of surgery on Poincaré duality complexes have been constructed by L. Jones, N. Levitt, Housemann and Vogel and others, [REFERENCE], [REFERENCE], [REFERENCE], by proving various kinds of "patching" theorems which enable one to give models for Poincaré duality spaces which are reasonably close to being $n$-dimensional manifolds.

## 3. The basic properties of homotopy sphere bundles

Given a mapping $f\colon Y \to X$ of finite $CW$-complexes, let $M(f)$ be the mapping cylinder of $f$. We convert $f$ into a Serre fibration by replacing $Y$ by the mapping space

$$E^{M(f)}_{Y,M(f)}$$

consisting of paths $g\colon I \to M(f)$ in the mapping cylinder of $f$ which originate in $Y$, so $f$ is replaced by $h(g) = \pi g(1)$, where $\pi\colon M(f) \to X$ is the projection.

For technical convenience we assume that $f$ is base point preserving and that $M(f)$ is the *reduced* mapping cylinder, where we identify $I \times *_Y$ with $*_X \in X$ and assume that this is the base point $* \in M(f)$.

When we do this the fiber has the homotopy type of the space of paths $E^{M(f)}_{Y,*}$, and if we have $h\colon Z \to X$ we can take the induced fibration $h^!(E) \to Z$ where $h^!(E)$ is the set of pairs $(z, g)$ with $\pi(g(1)) = h(z)$. This is again a Serre fibration with fiber $E^{M(f)}_{Y,*}$.

Similarly, if we have Serre fibrations $f\colon E \to X$ and $f'\colon E' \to X'$ with fibers $F$ and $F'$ respectively, then

$$E \times E' \xrightarrow{f \times f'} X \times X'$$

is a Serre fibration with fiber $F \times F'$. In particular, if $X = X'$, then restricting to the diagonal $\Delta(X) \subset X \times X$ gives us a Serre fibration over $X$ with fiber $F \times F'$.

EXAMPLE 49. Let $I \times X \xrightarrow{p_2} X$ be projection onto the second factor. Then if $f\colon E \to X$ is any Serre fibration the result of the construction above is the composition $I \times E \xrightarrow{p_2} E \xrightarrow{f} X$.

In the case of Serre fibrations where the fibers have the homotopy types of the spheres $S^m$ and $S^t$ respectively, we get a fibration with fiber $S^m \times S^t$ using the construction above. This is not quite what we want. We would like, in analogy with the Whitney bundle sum, to be able to replace the two fibrations above by a Serre fibration with fiber $S^{m+t+1}$. That we can do this is a consequence of the next lemma.

LEMMA 69. *Let $E \xrightarrow{\pi} X$ be a sub-Serre fibration of the Serre fibration $E' \xrightarrow{\pi'} X$ with fibers $F$ and $F'$ respectively. Suppose that $E \xrightarrow{\pi} X$ is also a sub-Serre fibration of the Serre fibration $E'' \xrightarrow{\pi''} X$ with fiber $F''$. Then*

$$E' \cup_E E'' \quad \xrightarrow{\pi'' \cup_\pi \pi'} \quad X$$

*is again a Serre fibration with fiber $F'' \cup_F F'$ provided $E$ is open in both $E'$, $E''$.*

PROOF. We are given the following situation

$$\begin{array}{ccc} V & \xrightarrow{h} & E' \cup_E E'' \\ \Big\downarrow{\scriptstyle i_0} & & \Big\downarrow{\scriptstyle \pi' \cup_\pi \pi''} \\ V \times I & \xrightarrow{H} & X \end{array}$$

and we want to extend $h$ to $V \times I$ so as to cover $H$ where $V$ is a finite simplicial complex. We take a refinement of the triangulation of $V$ so that for each simplex in the refinement is taken by $h$ into $E$, $E'$, or $E''$. Then we construct the extension first over the *subcomplex* generated by the simplexes which map into $E$ so that the extensions lie in $E$. At this point we can extend the extensions to the remaining simplices without difficulty. Thus the map is a Serre fibration and the fiber is manifestly as claimed. □

COROLLARY 25. *Let $f_1\colon E_1 \to X$, $f_2\colon E_2 \to X$ be two Serre fibrations with fibers $S^m$ and $S^t$ respectively. Then the fiberwise join $E_1 *_{f_1,f_2} E_2$ defined as the subset of the join $E_1 * E_2$ consisting of the equivalence classes of triples $(e_1, t, e_2)$ with $f_1(e_1) = f_2(e_2)$ maps to $X$ and, regarded as a Serre fibration has fiber the sphere $S^{m+t+1}$.*

PROOF. Let $E \xrightarrow{f} X$ be a Serre fibration. We regard

$$E^{M(f)}_{E,M(f)} \simeq E^{M(f)}_{E\times[0,1),M(f)}$$

as a subspace of

$$E^{M(f)}_{M(f),M(f)} \simeq X.$$

This gives an open inclusion of the type required in 69. Now apply 69 in the obvious way. □

Thus we have Whitney bundle sums for spherical fibrations. In particular the construction corresponding to adding a trivial line bundle is simply the fiberwise join of the trivial bundle $S^0 \times X \xrightarrow{p_2} X$ with $E \to X$, and this allows us to talk about stabilization.

The major theorem here is the Stasheff classification theorem, which, in our case is the following:

[Stasheff, [65]]

THEOREM 40. *Let $G_m$ be the associative, unitary monoid of homotopy equivalences of the sphere $S^m$, and $SG_m$ the monoid of orientation preserving homotopy equivalences. Then $G_m$ and $SG_m$ have classifying spaces $B_{G_m}$, $B_{SG_m}$ respectively together with universal fibrations*

$$S^m \to E_m \to B_{G_m}, \qquad S^m \to SE_m \to B_{SG_m},$$

*so that, given a finite complex $X$ and a spherical Serre fibration $f\colon E \to X$ with fiber $S^m$ then there is a unique homotopy class of maps $B_f\colon X \to B_{G_m}$ (or $X \to B_{SG_m}$ if the fibration is oriented) so that the original fibrations is the induced bundle.*

(See [65] for details.)

**The structure of the spaces $G_m$, $SG_m$, $G$, and $SG$.** In order to understand the spaces $G_m$ and $SG_m$ note that the map

$$Eval\colon G_m \to S^m$$

defined by $Eval(g) = g(*)$ is a Serre fibration with fiber $\Omega^m(S^m)_\pm$ where $\Omega^m(S^m)$ is the space of *based* maps $g\colon S^m \to S^m$. Here, the decoration $\pm$ means those components which consist of homotopy equivalences, the $+1$ and $-1$ components, where the components are distinguished by the degree of $g$. From the definition of $\Omega^m(S^m)$ we see that the $i^{th}$ homotopy group of $\Omega^m(S^m)$ is $\pi_{i+m}(S^m)$. Moreover, the construction of the $S^{m+1}$-spherical fibration $S^{m+1} \to S^0 \times X *_{p_2,\pi} E \to X$ shows that the corresponding maps

$$G_m \xrightarrow{\sigma} G_{m+1}, \qquad SG_m \xrightarrow{\sigma}, \qquad \Omega^m S^m \xrightarrow{\sigma} \Omega^{m+1} S^{m+1}$$

are given by simply suspending the maps. In particular, for

$$\Omega^m S^m \to \Omega^{m+1} S^{m+1}$$

the map in homotopy is just the usual suspension homomorphism

$$\sigma\colon \pi_i(S^m) \to \pi_{i+1} S^{m+1},$$

which, by a theorem of Freudenthal, is an isomorphism for $i < 2m-1$. On the other hand, from the exact sequence

$$\cdots \xrightarrow{\partial} \pi_{m+i}(S^m) \longrightarrow \pi_i(SG_m) \longrightarrow \pi_i(S^m) \xrightarrow{\partial} \pi_{m+i-1}(S^m) \longrightarrow \cdots$$

associated to the fibration $\Omega^m S^m \to SG_m \to S^m$, we see that the inclusion $\Omega^m S^m \hookrightarrow SG_m$ is a homotopy equivalence through dimension $m-2$. Thus we have

THEOREM 41.
*(1) The associated suspension maps for the classifying spaces*

$$B_\sigma\colon B_{G_m} \to B_{G_{m+1}}, \qquad B_\sigma\colon B_{SG_m} \to B_{SG_{m+1}}$$

*are homotopy equivalences through dimension $m-1$.*

*(2) The limit spaces $B_G$ and $B_{SG}$ which classify stable homotopy spherical fibrations satisfy $\pi_1(B_G) = \mathbb{Z}/2$, $\pi_1(B_{SG}) = 0$ while for $i \geq 2$ we have*

$$\pi_i(B_G) \cong \pi_i(B_{SG}) \cong \lim_{m \mapsto \infty} \pi_{i+m-1}(S^m).$$

DEFINITION 52. *The forgetful maps*

$$J_n\colon O_n(\mathbb{R}) \longrightarrow G_{n-1}, \qquad J_n\colon SO_n(\mathbb{R}) \longrightarrow SG_{n-1}$$

*pass to limits as well giving the maps of classifying spaces*

$$B_J\colon B_O \longrightarrow B_G, \qquad B_J\colon B_{SO} \longrightarrow B_{SG}$$

*and the fiber in both cases is written $G/O$.*

Thus we have the following exact sequence for determining the homotopy groups of $G/O$:

$$\cdots \longrightarrow \pi_n(O) \xrightarrow{j_*} \pi_n(G) \longrightarrow \pi_n(G/O) \overset{\partial}{\pi}_{n-1}(O) \xrightarrow{j_{n-1}} \cdots .$$

Thus, replacing $\pi_*(G)$ by $\pi_*^s(S^0)$, the stable homotopy of spheres, and writing $Coker(j)$ for the cokernel of the map $j\colon \pi_*(O) \to \pi_*^s(S^0)$ this gives

$$0 \longrightarrow Coker(j) \longrightarrow \pi_n(G/O) \longrightarrow \begin{cases} \mathbb{Z} & \text{if } n \text{ is } 4s, \\ 0 & \text{otherwise} \end{cases} \longrightarrow 0.$$

Also, note that $Coker(j)_{8n+4} = \pi_{8n+4}^s(S^0)$ and $Coker(j)_{8n} = \pi_{8n}^s(S^0)/(\mathbb{Z}/2)$.

## 4. The Spivak bundle and degree one normal maps

Any two Spivak normal structures on an $m$-dimensional geometric Poincaré complex $X$ are related by a stable fibre homotopy equivalence $c : \nu \simeq \nu'$ which preserves the Pontrajagin-Thom map $S^{2n+k} \to T(\nu)$. This is a direct consequence of the fact that for complexes any two embeddings $X \hookrightarrow \mathbb{R}^{2n+k}$ with $k \geq 2$ are isotopic in the sense that there is an embedding $H\colon I \times X \hookrightarrow I \times \mathbb{R}^{2n+k}$ so that $H$ restricted to the two ends gives the original embeddings. Indeed, we use the relative version of the existence of Spivak normal bundles and classification to show that the associated spherical fibration on $I \times X$ is a product, and the assertion is direct from this.

Although a homotopy equivalence of manifolds need not preserve the normal bundles it does preserve the Spivak normal fibrations. Indeed, we have

COROLLARY 26. *A homotopy equivalence of geometric Poincaré complexes preserves the Spivak normal fibrations.*

PROOF. Embed the mapping cone of the homotopy equivalence in $\mathbb{R}^{2n+k+3}$ for $k \geq 1$. Then the mapping cone has the homotopy type of $X \times I$, and classification gives the result. □

A normal invariant is a realization of the Spivak normal structure by a vector bundle. Precisely:

DEFINITION 53.
(ii) *A* **normal invariant** $(\eta, \rho)$ *on an $m$-dimensional geometric Poincaré complex $X$ is a vector bundle $\eta$ over $X$ which lifts the Spivak normal bundle to $X$ together with a map $\rho : S^{m+k} \to T(\eta)$ to the Thom space of $\eta$ such that*

$$U_\eta \cap h_*(\rho) \ = \ [X] \in H_m(X; \mathbb{Z}^{w(X)}) \ .$$

(ii) *An* **equivalence** *of normal invariants $(\eta, \rho)$, $(\eta', \rho')$ on an $m$-dimensional geometric Poincaré complex $X$ is a stable bundle isomorphism $c : \eta \simeq \eta'$ such that*

$$T(c)_*(\rho) \ = \ \rho' \in \pi^S_{m+k'}(T(\eta')) \ .$$

(iii) *The* **normal structure set** $\mathcal{T}(X)$ *of an $m$-dimensional geometric Poincaré complex $X$ is the set of equivalence classes of normal invariants.*

EXAMPLE 50. Any embedding $M \subset S^{m+k}$ of an $m$-dimensional manifold $M^m$ with tubular neighbourhood $(Y, \partial Y)$ determines a normal invariant $(\eta, \rho)$ with

$$\begin{aligned} \eta \ &= \ \nu_{M \subset S^{m+k}} \ : \ M \longrightarrow BO(k) \ , \\ (D^k, S^{k-1}) \longrightarrow (E(\eta), S(\eta)) \ &= \ (Y, \partial Y) \longrightarrow M \ , \\ \rho \ &= \ \text{projection} \ : \ S^{m+k} \longrightarrow Y/\partial Y \ = \ T(\eta) \ . \end{aligned}$$

Any two such embeddings determine equivalent normal invariants. The Spivak normal fibration of $M$ is the stable sphere bundle $J\nu_M : M \to BG$ of the stable normal bundle $\nu_M : M \to BO$.

When does a geometric Poincaré complex admit a normal invariant?

THEOREM 42. (Browder [7], [9], Novikov [49])
*The following conditions on an $m$-dimensional geometric Poincaré complex $X$ are equivalent:*
(i) *The normal structure set $\mathcal{T}(X)$ is non-empty, i.e. $X$ admits a normal invariant.*
(ii) *There exists a degree 1 normal map $(f, b) : M^m \to X$.*
(iii) *The Spivak normal fibration $\nu_X : X \to BG$ admits a bundle reduction $\eta : X \to BO$.*

PROOF. (i) $\Longrightarrow$ (ii) Given a normal invariant $(\eta, \rho)$ on $X$ make $\rho : S^{m+k} \to T(\eta)$ transverse at the zero section $X \subset T(\eta)$, obtaining a degree 1 normal map

$$(f, b) \ : \ M^m \ = \ \rho^{-1}(X) \to X \ .$$

(ii) $\Longrightarrow$ (i) Given a degree 1 normal map $(f, b) : M \to X$ with $b : \nu_M \to \eta$ use an embedding $M^m \subset S^{m+k}$ of an $m$-dimensional manifold $M$ to define a normal invariant $(\nu_M, \rho_M)$, with

$$\begin{aligned} \nu_M \ &= \ \nu_{M \subset S^{m+k}} \ : \ M \to BO(k) \ , \\ \rho_M \ &= \ \text{proj.} \ : \ S^{m+k} \longrightarrow S^{m+k}/(S^{m+k} \backslash E(\nu_M)) \ = \ T(\nu_M) \ . \end{aligned}$$

Define a normal invariant $(\eta, \rho)$ on $X$ by

$$\rho \ = \ T(b)\rho_M \ : \ S^{m+k} \xrightarrow{\rho_M} T(\nu_M) \xrightarrow{T(b)} T(\eta) \ .$$

(ii) $\Longrightarrow$ (iii) For any normal invariant $(\eta, \rho)$ of $X$ the sphere bundle $J\eta : X \to BG$ is the Spivak normal fibration $\nu_X$, by 42. □

In view of our previous discussion of degree one normal maps, 45, 5, we can summarize this discussion as follows.

PROPOSITION 25. *Let $X$ be an $m$-dimensional geometric Poincaré complex which admits a normal invariant.*
(i) *The normal structure set $\mathcal{T}(X)$ is in natural bijective correspondence with the set of normal bordism classes of degree 1 normal maps $(f, b) : M \to X$ (for varying reductions $\eta : X \to BO$ of the Spivak normal fibration $\nu_X : X \to BG$).*
(ii) *The normal structure set $\mathcal{T}(X)$ is in unnatural bijective correspondence with the set $[X, G/O]$ of fibre homotopy trivialized stable vector bundles over $X$.*

PROOF. Only (ii) needs to be shown. An element $(\alpha, \beta) \in [X, G/O]$ is a vector bundle $\alpha : X \to BO(j)$ ($j$ large) together with a fibre homotopy trivialization $\beta : J\alpha \simeq \{*\} : X \to BG(j)$. Given a normal invariant $(\eta : X \to BO(k), \rho : S^{m+k} \to T(\eta))$ define a normal invariant $(\eta', \rho')$ by

$$\begin{aligned} \eta' \ &= \ \eta \oplus \alpha \ : \ X \longrightarrow BO(j+k) \ , \\ \rho' \ &: \ S^{m+j+k} \xrightarrow{\Sigma^j \rho} \Sigma^j T(\nu_X) \xrightarrow{1 \oplus T(\beta)} T(\eta') \ . \end{aligned}$$

The construction defines a bijection

$$\iota_{\eta,\rho} \ : \ [X, G/O] \longrightarrow \mathcal{T}(X) \ ; \ (\alpha, \beta) \longrightarrow (\eta', \rho') \ .$$

□

EXAMPLE 51. The Spivak normal fibration $\nu_M : M \to BG$ of a manifold $M$ has a canonical vector bundle reduction, namely the stable normal bundle $\widetilde{\nu}_M : M \to BO$, and the normal structure set $\mathcal{T}(M)$ is in *natural* bijective correspondence with $[M, G/O]$. The bijective correspondence

$$\mathcal{T}(M) \xrightarrow{\simeq} [M, G/O] \ ; \ ((f, b) : N \to M) \to (\alpha, \beta)$$

is defined by sending a normal map $(f, b) : N \to M$ to the fibre homotopy trivialized stable vector bundle $(\alpha, \beta)$ over $M$ with $b : \widetilde{\nu}_N \to \widetilde{\nu}_M \oplus \alpha$.

EXAMPLE 52. The following construction exhibits a geometric Poincaré complex $X$ without a normal invariant, i.e. such that the Spivak normal fibration $\nu_X : X \to BG$ is not reducible to a vector bundle $\widetilde{\nu}_X : X \to BO$. It uses the fact that the total space of a fibration

$$F \longrightarrow E \to B$$

with the basis $B$ an $m$-dimensional geometric Poincaré complex and the fibre $F$ an $n$-dimensional geometric Poincaré complex is an $(m+n)$-dimensional geometric Poincaré complex $E$ (Gottlieb [22]). In particular, the total space of an $n$-spherical fibration over $S^m$ classified by

$$\omega \ : \ S^m \to BG(n+1)$$

is an $(m+n)$-dimensional geometric Poincaré complex $S(\omega)$

$$S^n \to S(\omega) \to S^m \ .$$

If $\omega$ admits a section, say

$$\omega \ = \ \omega_1 \oplus \epsilon \ : \ S^m \to BF(n+1)$$

for some $\omega_1 : S^m \to BG(n)$, the total space $S(\omega)$ has a cell structure

$$S(\omega) \ = \ (S^m \vee S^n) \cup_{[\iota_m,\iota_n]+\theta(\omega)} D^{m+n} \ ,$$

with $[\iota_m, \iota_n] \in \pi_{m+n-1}(S^m \vee S^n)$ the Whitehead product of $\iota_m \in \pi_m(S^m)$ and $\iota_n \in \pi_n(S^n)$ (the attaching map of the top cell in $S(\epsilon^{n+1}) = S^m \times S^n$) and

$$\theta(\omega) \ = \ \text{adjoint of } \omega \ : \ S^{m+n-1} \longrightarrow S^n \ .$$

The Thom space of $\omega$ has a cell structure

$$T(\omega) \ = \ S^{n+1} \cup_{\Sigma\theta(\omega)} D^{m+n+1} \ .$$

The Spivak normal fibration of $S(\omega)$ is classified by

$$\nu_{S(\omega)} \ : \ S(\omega) \longrightarrow S^m \xrightarrow{-\omega} BF(k) \ (k \text{ large}) \ ,$$

and the Thom space of $\nu_{S(\omega)}$ has the cell structure

$$T(\nu_{S(\omega)}) \ = \ (S^k \cup_{\Sigma\theta(-\omega)} D^{m+k}) \vee S^{n+k} \vee S^{m+n+k} \ .$$

In the special case

$$m \ = \ 3 \ , \ n \ = \ 2 \ , \ \omega = 1 \in \pi_3(BF(3)) \ = \ \pi_4(S^2) \ = \ \mathbb{Z}_2$$

there is obtained a 5-dimensional geometric Poincaré complex $X = S(\omega)$ such that the Spivak normal fibration $\nu_X : X \to BG$ does not have a bundle reduction – see Madsen and Milgram [35, p.33]. In this case the composite

$$t(\nu_X) \ : \ X \overset{\nu_X}{\to} BG \longrightarrow B(G/O)$$

does not admit a null-homotopy. *fbox*

**The surgery exact sequence.** With these preliminaries out of the way we can now describe the structure of the set of all simply connected manifolds homotopy equivalent to a given Poincaré complex as follows.

Definition 54. *Let $X$ be a simply connected, closed, compact Poincaré duality complex of dimension $n$. Assume that there is at least one degree one normal problem over $X$, so the structure set $\mathcal{S}(X) \neq \emptyset$. The set of* **homotopy differential structures** *on $X$, written $\mathcal{HD}(X)$ is the set of homotopy equivalences $f \colon M^n \to X$ for $M^n$ a closed compact differentiable manifold, where the equivalence relation is given by $(M, f) \sim (M', f')$ if and only if there is an $h$-cobordism $W$ from $M$ to $M'$ together with a map $H \colon W \to I \times X$ so that $H|M$ is $f$ while $H|M'$ is $f'$.*

Of course, from the Browder-Novikov theorem, $\mathcal{HD}(X) \neq \emptyset$.

THEOREM 43. *Let $X$ be a simply connected closed, finite $n$-dimensional Poincaré duality complex with a vector bundle reduction of its Spivak normal bundle and $n \geq 5$. Then there is a long exact sequence of sets*

$$\cdots \longrightarrow [\Sigma X, G/O] \xrightarrow{\sigma} L_{n+1}(1) \longrightarrow \mathcal{H}D(X) \longrightarrow [X, G/O] \xrightarrow{\sigma} L_n(1)$$

*where $L_n(1)$ is the surgery obstruction group for simply connected surgery problems:*

$$L_n(X) = \begin{cases} \mathbb{Z} & \text{if } n \equiv 0 \bmod (4), \\ \mathbb{Z}/2 & \text{if } n \equiv 2 \bmod (4), \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

PROOF. The only point where we need to make some comments is in analyzing the kernel of the map from $\mathcal{H}D(X)$ to $[X, G/O]$. Here an element in the kernel means, actually, two elements $(M_1, f_1)$ and $(M_2, f_2)$ with the same image in $[X, G/O]$. Thus, there is is an $(n+1)$-dimensional manifold $W$ with boundary $M_2 - M_1$ together with a degree one normal map $H\colon W^{n+1} \to I \times X$ so that $H|M_i$ is $f_i\colon M_i \to X$. We may assume, of course that both $f_1$ and $f_2$ are homotopy equivalences. Thus, we can do surgery on the interior of $W$ to make $H$ a homotopy equivalence if $n$ is even, and it follows that $(M_1, f_1)$ and $(M_2, f_2)$ are $h$-cobordant, hence equal in $\mathcal{H}D(X)$. But if $n$ is odd there will be an index or Kervaire invariant obstruction to completing surgery.

Now, to what degree is this obstruction well defined? Clearly, once the homotopy $I \times X \to G/O$ is specified (so we have a specific choice of surgery problem fixed on the boundary) then the obstruction is well defined. Thus, the variation occurs through the variation of the homotopy. If there are two distinct homotopies from $(M_1, f_1)$ to $(M_2, f_2)$ with different surgury obstructions, then taking the difference gives a homotopy from $(M_1, f_1)$ to itself with surgery obstruction the difference of the two obstructions. And conversely, given a homotopy from $(M_1, f_1)$ to itself with given surgery obstruction, $\sigma$, by adding it to the given homotopy we change the original surgury obstruction by adding $\sigma$ to it.

On the other hand, given the surgery obstruction $\sigma$ it is represented, as we've seen by a surgery problem of the form

$$W^{n+1} \to I \times S^n$$

which, at one end is the identity and at the other end gives a degree one map of a Milnor sphere, 8, onto $S^n$ if $n \equiv 3 \bmod (4)$ while it gives a Kervaire sphere, 8 if $n \equiv 1 \bmod (4)$. By gluing this bordism to the product $I \times M^n \xrightarrow{id \times f_1} I \times X$ we obtain a surgery problem with any desired surgery obstruction on $W$.

This completes the proof. □

EXAMPLE 53. We consider again the case where $X = S^n$, the ordinary $n$-sphere with $n$ even. From 3 we have determined $\pi_n(G/O) = [S^n, G/O]$. In the case $n = 4k$ the surgery obstruction map is injective on the $\mathbb{Z}$-summand and, since $L_{4k+1}(1) = 0$, it follows that $\mathcal{HD}(S^{4k}) = Coker(j)_{4k}$ which is $\pi^s_{4k}(S^0)$ for $k$ odd and $\pi^s_{4k}(S^0)/(\mathbb{Z}/2)$ for $k$ even. In the case of $S^{4n+2}$ the surgery map is onto if and only if the boundary of the Kervaire plumbing is an ordinary sphere (in which case we can close up the surgery problem over the disk $D^{4k+2}$ to a problem over $S^{4k+2}$).

We have that the image of $j$ is $\mathbb{Z}/2$ in dimensions congruent to 2 mod (8) and is 0 in dimensions congruent ot 6 mod (8). It follows that in dimension congruent ot 6 mod (8) there must be a special kind of $\mathbb{Z}/2$-direct summand in $\pi^s_{8n+6}(S^0)$ in order that the surgery obstruction map be non-trivial. (In the remaining case we would need either a $\mathbb{Z}/2$ summand or a $\mathbb{Z}/4$ summand.) In fact Browder proved a very precise result.

THEOREM 44. [Browder] *The only dimensions $n = 4k + 2$ where the surgery obstruction map $\sigma\colon [S^n, G/O] \to L_n(1) = \mathbb{Z}/2$ can be non-trivial are of the form $n = 2^v - 2$. Moreover, the map will be non-trivial if and only if there is an infinite cycle in the Adams spectral sequence for $\pi^s_*(S^0)$ in this dimension occuring in filtration degree two.*

In any case, modulo this indetermininacy, the above discussion determines the set $\mathcal{HD}(S^{4k+2})$.

EXAMPLE 54. The case for the odd spheres $S^{2n+1}$, always has the form

$$\pi_{2n+2}(G/O) \xrightarrow{\Sigma} L_{2n+2}(1) \longrightarrow \mathcal{HD}(S^{2n+1}) \longrightarrow Coker(j)_{2n+1} \longrightarrow 0.$$

Here, the map $\Sigma$ is the surgery map for the even sphere discussed above.

In particular, when $2n + 1 = 4k + 3$ we have $L_{2n+2}(1) = \mathbb{Z}$ and the quotient $\mathbb{Z}/im(\Sigma)$ corresponds to the Milnor spheres of 8.

On the other hand, for $2n+1 = 4k+1$ with $4k+2$ not of the form $2^s - 2$ it follows from Browder's result that $\Sigma$ is the zero map, so the boundary of the Kervaire problem in this dimension is not diffeomorphic to the regular sphere, and taking the connected sum with this class describes the kernel of the map from $\mathcal{HD}(S^{4k+1})$ to $Coker(j)_{4k+1}$.

**The bordism invariance of the surgery obstruction.** The next thing that we need to note is that if we have two maps

$$f_1\colon M_1^n \longrightarrow G/O, \text{ and } f_2\colon M_2^n \longrightarrow G/O$$

where the $M_i$ are closed, compact, differential manifolds without boundary which are **bordant** with bordism

$$H\colon W \to G/O,\ \partial(W) = M_2 - M_1,\ H|M_i = f_i$$

then the associated surgery problem over $W$ can be made highly connected so that we get the following exact sequence of surgery kernels in case $n$ is even:

$$0 \longrightarrow K_{\frac{n}{2}+1}(H) \xrightarrow{\partial} K_{\frac{n}{2}}(f_2) \oplus (-)K_{\frac{n}{2}}(f_1) \longrightarrow K_{\frac{n}{2}}(H) \longrightarrow 0$$

which shows that the surgery obstruction associated to $(M_1, f_1)$ is equal to the surgery obstruction for $(M_2, f_2)$.

Consequently, in the cases where our Poincaré duality spaces are actually manifolds the surgery obstruction map $[M^n, G/O] \to L_n(1)$ actually factors through the bordism of $G/O$, giving a map

$$\sigma\colon \Omega_n(G/O) \to L_n(1)$$

so that the surgery obstruction factors through $\sigma$.

Note that the product construction

$$(M^n, (N^k, f)) \mapsto (M^n \times N^k, fp_2)$$

where $p_2\colon M^n \times N^k \to N^k$ is just projection onto the second factor passes to bordism and defines an action

$$\Omega_n(pt) \times \Omega_k(W) \longrightarrow \Omega_{n+k}(W)$$

for any space $W$. In particular, it is natural to ask how this action affects the surgery invariant in the case where $W = G/O$.

In the case of the signature we know that the signature of $M^{4n} \times N^{4k}$ is just the product $S(M)S(N)$. Consequently, in these dimensions, the surgery obstruction for the product of the surgery problem $\bar{N}^{4k} \xrightarrow{g} N^{4k}$ associated to the map $f\colon N^{4k} ra1 G/O$ is just the surgery obstruction for $f$ multiplied by the signature of $N$.

THEOREM 45. [Sullivan] *There are cohomology classes*

$$K_{4i} \in H^{4i}(G/O, \mathbb{Q}), \quad i = 1, 2, \ldots,$$

*so that given $f\colon N^{4k} \to G/O$, then the surgery obstruction for $(N, f)$ is given by the formula*

$$\sigma(N, f) = \langle L(M) \cup \sum_{i=1}^{k} f^*(K_{4i}).$$

PROOF. We recall that we have the fibration

$$G/O \xrightarrow{j} B_O \longrightarrow B_G,$$

and if $g\colon \bar{M}^{4k} \to M^{4k}$ is the surgery problem associated to the map

$$f\colon M \to G/O,$$

then the normal bundle to $\bar{M}$ is

$$g^!(\nu(M) \oplus j^!(\xi))$$

where $\xi$ is the universal bundle over $B_O$. On the other hand, the $L$-class of $\nu(\bar{M})$ evaluated on $[\bar{M}]$ gives the signature of $\bar{M}$, while the $L$-class of $\nu(M)$ evaluated on $[M]$ gives the signature of $\bar{M}$.

But the $L$-class is multiplicative. Consequently, the $L$-class of $\nu(\bar{M})$,

$$\begin{aligned} L(\nu(\bar{M})) &= g^*(L(M)) \cup g^*(f^* j^*(L)) \\ &= g^*(L(M) \cup f^*(j^*(L)). \end{aligned}$$

Evaluating this on $\bar{M}$ and subtracting $\langle L^*(M), [M]\rangle$ gives the formula in the theorem when we define $K_{4i} = j^*(L_{4i})$. □

Of course, the answer above is just one of four cases in which the product could have a non-trivial surgery obstruction. We can see that one other case is also fairly direct: when the dimension of $N$ is $4k+2$ and the dimension of $M$ is $4n+2$, then the signature of the product is zero so, in this case, whatever the original surgery obstruction, the surgery obstruction for the product is zero.

**The structure of piecewise linear surgery theory.** It remains to understand the cases where the dimension of $N$ is $4k$ but the dimension of $M$ is $4n+2$ and where the dimension of $N$ is $4k+2$ while the dimension of $M$ is $4n$.

Here the arguments are much more subtle as is shown by 44. Sullivan took a more general approach in order to understand this situation. He first considered surgery over piecewise linear manifolds – topological manifolds together with a triangulation so that the open stars of simplices are triangulated $n$-disks. In this case the Poincaré conjecture in dimension $\geq 5$ shows that the boundary of the Kervaire plumbing is the usual piecewise linear sphere, so it can be coned off and one obtains a degree one surgery problem over the sphere $S^{4n+2}$ in the piecewise linear category with surgery obstruction, the non-trivial element in $L_2(1)$.

Work of Milnor and others shows that there is a classifying space for piecewise linear sphere bundles and consequently a fibration

$$G/PL \longrightarrow B_{PL} \longrightarrow B_G$$

which plays the same role for piecewise linear surgery as $G/O$ plays for ordinary surgery theory. Indeed, we can define, as before the set of **homotopy triangulations** $\mathcal{H}D(X^n)$ of a closed, compact, Poincaré complex without boundary as the set of equivalence classes of homotopy equivalences $f\colon M^n \to X$ with $M^n$ a piecewise linear manifold. As before the equivalence here is taken with respect to $h$-cobordisms over $I \times X$. Then the Poincaré conjecture in dimensions $\geq 5$ shows that $\mathcal{H}D(S^n) = \{S^n\}$ consists of a single element in these dimensions. Again, the surgery exact sequence of 43 extends via an almost identical proof to give the exact sequence

$$[\Sigma X, G/PL] \xrightarrow{\sigma} L_{n+1}(1) \longrightarrow \mathcal{H}T(X) \longrightarrow [X, G/PL] \longrightarrow L_n(1)$$

but here as a consequence of the fact that $\mathcal{H}T(S^n) = \{S^n\}$ and the fact that the boundary of the plumbing constructions associated to the Kervaire sphere, 8 in dimensions $m = 4k+2$ and to the $E_8$-lattice in dimensions $m = 4k$ are both $S^{m-1}$ we obtain degree one normal maps of closed piecewise linear manifolds over the spheres $S^m$ and $S^m$ which have surgery obstructions the generators of $L_m(1)$ as long as $m \geq 5$. Thus we have

THEOREM 46. *For $n \geq 5$, we have*

$$\pi_n(G/PL) \cong L_n(1) = \begin{cases} \mathbb{Z} & n \equiv 0 \bmod (4) \\ 0 & n \equiv 1,\ 3 \bmod (4) \\ \mathbb{Z}/2 & n \equiv 2 \bmod (4). \end{cases}$$

Also, this result extends to dimensions $< 5$ as well though the interpretations of the low dimensional classes are different.

THEOREM 47. [Sullivan] *The first 4 homotopy groups of $G/PL$ are*

$$\begin{array}{lcl} \pi_1(G/PL) & = & 0 \\ \pi_2(G/PL) & = & \mathbb{Z}/2 \\ \pi_3(G/PL) & = & 0 \\ \pi_4(G/PL) & = & \mathbb{Z} \end{array}$$

*the $\mathbb{Z}/2$ being represented by the surgery problem given by the Kervaire plumbing over $S^2$ while the generator in dimension 4 is given by an index 16 problem. In particular the image of the generator in dimension 4 under the surgery map is twice the generator of $L_4(1)$.*

**Sullivan's product formula for the Kervaire invariant.** In the case of piecewise linear surgery the discussion above shows that the Kervaire invariant is much closer to the surface! Sullivan exploited this to obtain a complete understanding of the homotopy type of the space $G/PL$. The critical first step was to obtain a closed formula for determining the Kervaire invariant of a surgery problem given in the form

$$(M^{4n+2}, f\colon M^{4n+2} \to G/PL)$$

with $M^{4n+2}$ a differentiable manifold.

First Sullivan proves a second product formula.

THEOREM 48. [Sullivan] *Let $f\colon \bar{M}^{4k+2} \to M^{4k+2}$ be a piecewise linear degree one normal map with $M^{4k+2}$ differentiable, and $N^{4k}$ any differentiable manifold. Then the surgery obstruction for the product normal map*

$$N \times \bar{M} \xrightarrow{id \times f} N \times M$$

*is $I(N)\sigma(f)$, the signature of $N$ multiplied by the surgery obstruction of $f$.*

For the proof see e.g. [8], [REFERENCE-ROURKE, SANDERSON].

This can be rewritten as follows: let $v_{2k}$ be the middle dimensional Wu class of $N^{4k}$, so $v_{2k} \in H^{2k}(N; \mathbb{Z}/2)$ is characterized by the formula

$$\alpha^2 = v_{2k} \cup \alpha = Sq^{2k}(\alpha)$$

for each $\alpha \in H^{2k}(N; \mathbb{Z}/2)$. Then

$$\langle v_{2k}^2, [N] \rangle$$

is directly seen to be the signature of $N$ mod (2). (Compare the arguments in 30 where we show that actually $v_{2k}^2$ is well defined mod (8) and determines the signature of $N$ mod (8).) Thus we have

COROLLARY 27. *Let $V(N)$ be the total Wu class of $N$. Then, for the product problem in 48 the surgery obstruction is given as $\langle\langle V^2(N), [N]\rangle\sigma(f)$.*

The most important thing here is that we have the determination of bordism modulo odd torsion as follows. Let

$$\mathbb{A} \;=\; \mathbb{Z}(\frac{1}{3}, \frac{1}{5}, \dots, \frac{1}{2n+1}, \dots)$$

so we have

$$\Omega_*(X) \otimes \mathbb{A} \;=\; H_*(X; \Omega_*(pt) \otimes \mathbb{A}).$$

This is very close to being the tensor product $H_*(X; \mathbb{A}) \otimes \Omega_*(pt)$. As a consequence, by induction Sullivan proves the product formula

THEOREM 49. [Sullivan] *There are cohomology classes*

$$k_{4k+2} \in H^{4n+2}(G/PL, \mathbb{Z}/2), \qquad k = 0, 1, 2, \dots,$$

*so that, given any differentiable manifold $M^{4n+2}$ and map $g\colon M^{4n+2} \to G/PL$ the Kervaire invariant for the associated degree one normal map is given as*

$$\langle V^2(M) \cup \sum_{k=0}^{\infty} g^*(k_{4k+2}), [M^{4n+2}]\rangle.$$

*Here $V$ is the total Wu class of $M^{4n+2}$, $\sum_0^n v_i{}^2$.*

This has a profound implication for the homotopy type of $G/PL$. Realizing these cohomology classes via maps into Eilenberg-Maclane spaces gives the following result.

COROLLARY 28. [Sullivan] *There is a fibration*

$$W \longrightarrow G/PL \xrightarrow{\prod k_{4i+2}} \prod_{i=0}^{\infty} K(\mathbb{Z}/2, 4i+2)$$

*where the fiber $W$ has homotopy groups*

$$\pi_i(W) = \begin{cases} \mathbb{Z} & \text{if } i \equiv 0 \bmod (4) \\ 0 & \text{otherwise.} \end{cases}$$

*In particular, $W$, when looked at modulo odd torsion (tensoring with $\mathbb{A}$) is a product of Eilenberg-Maclane spaces $K(\mathbb{Z}, 4i)$, while modulo 2-torsion (tensoring with $\mathbb{Z}(\frac{1}{2})$) $W$ is a copy of $B_O$.*

PROOF. The idea of the proof is to use the description of the surgery obstruction as a difference of indexes in dimensions $4n$. This ties it in to the $L$-class and bordism. It is direct to check that the cohomology classes in 45 are actually integral when looked at modulo odd torsion, so the first statement follows by simply mapping out to a product of Eilenberg-Maclane spaces of the form $K(\mathbb{Z}, 4n)$.

[2]The $i^{th}$ Wu class of $M$ is the unique element in $H^i(M; \mathbb{Z}/2)$ so that $v_i \cup \alpha = Sq^i(\alpha)$ for every $\alpha \in H^{n-i}(M)$.

To see the second statment involves checking the denominators of these classes. One finds that they are equal to the denominators of the corresponding classes for $B_O$ up to powers of 2. On the other hand, from homotopy theory these denominators are maximal! That is to say, there is no space with these homotopy groups which has these denominators except for $B_O$ [REFERENCE – ADAMS, MARGOLIS]. Details of the calculations of denominators are given in [35]. □

Finally, putting all this together we have Sullivan's determination of the homotopy type of $G/PL$.

THEOREM 50. [Sullivan] *Ignoring odd torsion $G/PL$ is a product*

$$E \times \prod_{k=1}^{\infty} K(\mathbb{Z}/2, 4k+2) \times K(\mathbb{Z}, 4(k+1))$$

*where $E$ is the unique stable Postnikov system*

$$K(\mathbb{Z}, 4) \longrightarrow E \longrightarrow K(\mathbb{Z}/2, 2)$$

*with $K$-invariant of order 2.*

From these results it is not very difficult to determine the mapping sets $[X, G/PL]$, which makes simply connected surgery in the piecewise linear category quite manageable.

EXAMPLE 55. A case in point is the structure of $[\mathbb{CP}^n, G/PL]$. When we look at the situation ignoring odd torsion this set is just the product

$$[\mathbb{CP}^n, E] \times \prod_{k=3}^{n} \begin{cases} \mathbb{Z}/2 & k \text{ odd} \\ \mathbb{Z} & \text{for } k \text{ even,} \end{cases}$$

while the $K$-theory of $\mathbb{CP}^n$ is just $\sum_1^{[n/2]} \mathbb{Z}$.

Thus, it suffices to understand $[\mathbb{CP}^n, E]$. The effect of the non-trivial $K$-invariant in dimension 5 is to force $Sq^2(\iota_2)$, to be the restriction of a torsion free integral class, where $\iota \in H^2(E; \mathbb{Z}/2)$ is the non-trivial class. In fact $H^4(E; \mathbb{Z}) = \mathbb{Z}$ while $H^4(E; \mathbb{Z}/2) = H^4(E; \mathbb{Z}) \otimes \mathbb{Z}/2$. Consequently, any map $f: \mathbb{CP}^n \to E$ with $f^*(\iota) \neq 0$ must take the torsion free generator in dimension 4 to an odd multiple of $b^2$ in $H^4(\mathbb{CP}^n; \mathbb{Z})$. It follows, using the multiplication in $E$, that $[\mathbb{CP}^n, E] = \mathbb{Z}$ for any $n \geq 2$.

Finally, we should note the commutative diagram of surgery sequences which shows that the control of the surgery obstruction given by Sullivan's results for piecewise linear manifolds gives equally sharp control for differentiable manifolds provided we know something about the map from $G/O$

to $G/PL$.

$$
\begin{array}{ccccccccc}
\xrightarrow{\sigma} & L_{n+1}(1) & \longrightarrow & \mathcal{H}D(X) & \longrightarrow & [X, G/O] & \xrightarrow{\sigma} & L_n(1) \\
 & \downarrow{=} & & \downarrow & & \downarrow & & \downarrow{=} \\
\xrightarrow{\sigma} & L_{n+1}(1) & \longrightarrow & \mathcal{H}T(X) & \longrightarrow & [X, G/PL] & \xrightarrow{\sigma} & L_n(1)
\end{array}
$$

Fortunately, the structure of this map has been pretty well completely analyzed. The details are given in the book [35].

CHAPTER 12

# Surgery When $\pi_1(X) \neq \{1\}$

This chapter completes our discussion of the foundational results in surgery theory by considering the case where the fundamental group is non-trivial. Most of the results here are due to C.T.C. Wall, though the proofs and some of the definitions benefit from later work.

We begin with a short discusssion of duality over non-commutative rings with involutions.

In the next section, in view of the experience we've gained in the the previous four chapters we will start out by directly giving the definitions of the even dimensional surgery obstruction groups. After this we construct a well defined map from even dimensional degree one normal problems to these groups which assigns zero to the problem if and only if it is possible to do surgery to make the map normally bordant to a homtopy equivalence.

Then we repeat this process for the odd groups, first giving their definitions and proving that there is a well defined map from odd dimensional degree one normal problems to these groups which assigns zero if and only if the map is normally bordant to a homotopy equivalence.

Finally, we write down the various versions of the surgery exact sequence which are applicable in this case. There are more than one since, as we've seen, $h$-cobordism in the non-simply connected case does not necessarily give diffeomorphic boundary components here, so it is also necessary to consider $s$-cobordisms, which lead to different obstruction groups. We denote these groups $L^h_n(\mathbb{Z}[\pi], w)$ for degree one normal bordisms in dimension $n$ up to $h$-cobordism, where $\pi$ is the fundamental group and $w$ is the orientation character, and by $L^s_n(\mathbb{Z}[\pi], w)$ for equivalence up to $s$-cobordism. In applications the $L^h$-groups turn out to be easier to handle.

Again, for degree one normal maps over closed manifolds the surgery obstruction will factor through bordism giving maps

$$\Omega_*(G/O \times B_{\pi_1(M)}, w) \xrightarrow{\sigma} L^{h,s}_*(\mathbb{Z}[\pi_1(M)], w).$$

Some applications are indicated in 6 based on the construction of some non-trivial elements in $L^h_n(\mathbb{Z}[\mathbb{Z}/2], 1)$. There it is shown, among other things that there exist smooth free actions of $\mathbb{Z}/2$ on $S^{4n-1}$, $n \geq 2$, which are not even topologically conjugate to the usual antipodal action $\vec{x} \leftrightarrow -\vec{x}$.

The proofs here are fairly close to Wall's original proofs and, as in the simply connected case are based on first making the normal map connected up to the middle dimensions. However, in actual calculations this method is not very useful. The problem is that, unlike the case where $\pi_1 = \{1\}$, the relevant structures in $K_n(f)$ for $2n$-dimensional problems, or $K_n(f)$ and $K_{n+1}(f)$ for $(2n+1)$-dimensional problems are not visible until one has actually done surgery to make the map highly connected. Thus, to do things like prove product formulae in the non-simply connected case one has to have a method of determining the surgery obstruction without first doing surgery. This will require a different point of view on the obstruction, one based on iterated loop space theory and obstruction to desuspension. It will be presented in the next chapter where we begin the discussion of the more recent developments in the theory.

## 1. Preliminaries on Modules over Rings with Involutions

Before we can give the general definitions we have to discuss the structure of non-singular (even) ($\pm$)-symmetric forms over general, in particular *non-commutative* rings. In order to do this the ring $\mathbb{A}$ must be equipped with an involution, 18, $a \leftrightarrow \bar{a}$, $\overline{ab} = \bar{b}\bar{a}$ which we consider fixed in the remainder of our discussion. We also assume throughout that $\mathbb{A}$ satisfies the condition that every finitely generated free module has a well defined rank, that is to say $\mathbb{A}^n \neq \mathbb{A}^m$ if $n \neq m$ and both are finite. Also, all our modules over $\mathbb{A}$ will be assumed to be finitely generated.

Example 56. The most important examples for us are the group rings $\mathbb{Z}[\pi]$ with the $w$-twisted involution of 19,

$$\sum n_i g_i \mapsto \sum w(g_i) n_i g_i^{-1},$$

where $w\colon \pi \to \mathbb{Z}/2 = \{\pm 1\}$ is a fixed homomorphism. These rings all satisfy the dimension condition above since they have homomorphisms $J\colon \mathbb{A} \to \mathbb{F}_p$ $J(\sum n_i g_i) = \sum n_i$ for any finite field of prime order, $\mathbb{F}_p$, and tensoring over $J$ with $\mathbb{F}_p$ takes modules to vector spaces.

We begin our discussion by considering non-singular quadratic forms on finitely generated projective modules over $\mathbb{A}$. Recall that a finitely generated projective over a ring $\mathbb{A}$ is any $\mathbb{A}$-direct summand of a finitely generated free $\mathbb{A}$-module, $\mathbb{A}^m$ for some $m$. The dual of a finitely generated $\mathbb{A}$-module $\mathcal{M}$ is the module

$$\mathcal{M}^* = \mathrm{Hom}_{\mathbb{A}}(\mathcal{M}, \mathbb{A}).$$

Using the involution on $\mathbb{A}$ we define an $\mathbb{A}$-module structure on $\mathcal{M}^*$ by $af(m) = f(m)\bar{a}$ for all $m \in \mathcal{M}$.

Remark . Note here that the (perhaps) expected definition $af(m) = f(am)$ is *not* $\mathbb{A}$-linear unless $\mathbb{A}$ happens to be commutative since $af(bm)$ would be $b(af(m))$ if $(af)$ with this definition were linear, but this is $b(f(am)) = ba(f(m))$. On the other hand $af(bm) = f(abm) = ab(f(m))$. Thus the multiplication on the right is necessary for linearity in general. On the other hand, since $ab(f)$ should equal $a(b(f))$ we see that the involution is needed in order that this give an action.

It is, of course, standard that $\mathcal{M} \mapsto \mathcal{M}^*$ is a contravariant functor on $\mathbb{A}$-modules. Here, given $f\colon \mathcal{M} \to \mathcal{N}$ then $f^*\colon \mathcal{N}^* \to \mathcal{M}^*$ is defined by $f^*(g)(m) = gf(m)$. Additionally,

$$(\mathcal{M} \oplus \mathcal{N})^* \;=\; \mathcal{M}^* \oplus \mathcal{N}^*.$$

Remark . Another thing to note here is that for general modules $\mathcal{M}^*$ is pretty much uncontrolled. For example, in the case that $\mathbb{A} = \mathbb{Z}$ and $\mathcal{M}$ is a torsion module, then $\mathcal{M}^* = 0$. Thus, we have to make stringent assumptions about $\mathcal{M}$ to conclude, for example, that $(\mathcal{M}^*)^* = \mathcal{M}$.

A key result here is this.

Lemma 70. *Let $\mathcal{M}$ be a finitely generated projective, then $\mathcal{M}^*$ is again a finitely generated projective $\mathbb{A}$-module and $(\mathcal{M}^*)^* = \mathcal{M}$.*

Proof. Note first that for $\mathbb{A}$, we have

$$\mathbb{A}^* = \mathrm{Hom}_{\mathbb{A}}(\mathbb{A}, \mathbb{A}) = \mathbb{A}.$$

Consequently, $(\mathbb{A}^n)^* \cong \mathbb{A}^n$, and the result is true in this case. Likewise, since $\mathcal{M}$ is projective, we have $\mathcal{M} \oplus \mathcal{N} \cong \mathbb{A}^n$ for some $n$, $\mathcal{N}$, and $\mathcal{M}^* \oplus \mathcal{N}^* = \mathbb{A}^n$ as well. So $\mathcal{M}^*$ is projective. Also, $\mathcal{M}^*$ is non-trivial if $\mathcal{M}$ is, since, writing $\mathcal{M} \oplus \mathcal{N} = \mathbb{A}^n$, it follows that at least one of the coordinate projections $\mathbb{A}^n \to \mathbb{A}$ defines a non-trivial element of $\mathcal{M}^*$.

It remains to show that $\mathcal{M} = (\mathcal{M}^*)^*$. Note first that the map

$$\mathbb{A} \to (\mathbb{A}^*)^*, \; a \mapsto (f \mapsto \overline{f(a)})$$

for $f \in \mathbb{A}^*$ gives an isomorphism, and is a special case of a general map $\mathcal{M} \to (\mathcal{M}^*)^*$ which has the same definition. Thus, the map

$$\mathcal{M} \oplus \mathcal{N} \cong \mathbb{A}^n \longrightarrow ((\mathbb{A}^n)^*)^*$$

takes $\mathcal{M}$ into $(\mathcal{M}^*)^*$, and since is is an isomorphism overall, it must be an isomorphism between $\mathcal{M}$ and $(\mathcal{M}^*)^*$. □

Next we show that given $\mathcal{M}$ projective with $\mathcal{M}^* \cong \mathcal{M}$ then there is a projective $\mathcal{N}$ so that $\mathcal{N}^* \cong \mathcal{N}$ and $\mathcal{M} \oplus \mathcal{N} \cong \mathbb{A}^k$ for some $k$. But first we need to recall Shanuel's lemma.

[Shanuel]
Lemma 71. *Let $\mathcal{M}$ be any $\mathbb{A}$-module, and $\pi_i\colon \mathbb{A}^{n_i} \to \mathcal{M}$, $i = 1, 2$, be two surjective $\mathbb{A}$-maps. Let $K_1$ be the kernel of $\pi_1$ and $K_2$ be the kernel of $\pi_2$. Then $K_1 \oplus \mathbb{A}^{n_2} \cong K_2 \oplus \mathbb{A}^{n_1}$.*

Proof. The map

$$\pi_1 + \pi_2\colon \mathbb{A}^{n_1+n_2} \longrightarrow \mathcal{M} \qquad (a, b) \mapsto \pi(a) + \pi(b)$$

is also surjective, and depending on our point of view we can identify the kernel with $K_1 \oplus \mathbb{A}^{n_2}$ or $K_2 \oplus \mathbb{A}^{n_1}$. □

COROLLARY 29. *Suppose $\mathcal{M}$ is finitely generated, projective, and isomorphic to $\mathcal{M}^*$. Then there is a finitely generated projective $\mathcal{N}$ so that $\mathcal{N} \cong \mathcal{N}^*$ and $\mathcal{M} \oplus \mathcal{N} \cong \mathbb{A}^m$ with $m < \infty$.*

PROOF. Take any surjective map $\pi\colon \mathbb{A}^n \to \mathcal{M}$ with kernel $\mathcal{N}_1$. Dually, we have

$$0 \longrightarrow \mathcal{M}^* \longrightarrow \mathbb{A}^n \longrightarrow \mathcal{N}_1^* \longrightarrow 0,$$

and since $\mathcal{M}$, $\mathcal{N}_1$ are projective we can reverse the last arrows and obtain a surjection

$$\mathbb{A}^n = \mathcal{M}^* \oplus \mathcal{N}^* \longrightarrow \mathcal{M}^*$$

with kernel $\mathcal{N}^*$. Now, using the isomorphism $\mathcal{M} \cong \mathcal{M}^*$, and applying Shanuel's lemma we have $\mathcal{N}_1 \oplus \mathbb{A}^n \cong \mathcal{N}_1^* \oplus \mathbb{A}^n$. Thus, if we set

$$\mathcal{N} \ = \ \mathcal{N}_1 \oplus \mathbb{A}^n$$

we see that $\mathcal{N}$ satisfies the conditions of the corollary. $\square$

REMARK . As a consequence of 29 the operation $\mathcal{M} \leftrightarrow \mathcal{M}^*$ induces an involution on $K_0(\mathbb{A})$, with the fixed point set identified with those stable isomorphism classes of finitely generated projectives with $\mathcal{M} \oplus \mathbb{A}^n \cong \mathcal{M}^* \oplus \mathbb{A}^n$ for some $n$.

## 2. The Even Surgery Obstruction Groups

We now define the surgery obstruction groups $L_{2n}(\mathbb{A})$ for any ring with involution. There are three special cases which need to be considered. First, the obstruction groups most useful in applications are the groups which are associated to quadratic forms over free $\mathbb{A}$-modules. These groups are denoted $L^h_{2n}(\mathbb{A})$. They are fairly subtle to compute however, and a useful intermediate group which is easier to study, the group associated to quadratic forms over projective $\mathbb{A}$-modules, $L^p_{2n}(\mathbb{A})$, is also introduced. Finally, the orginal group of Wall, the group associated to quadratic forms over free modules together with a choice of preferred bases, written $L^s_{2n}(\mathbb{A})$ is not considered here. This last group is associated to surgery where up to normal $s$-cobordisms, hence the $s$. Likewise, the $L^h$ groups are associated to surgery up to normal $h$-cobordisms. Once we have given the definitions of $L^{h,p}_{2n}(\mathbb{A})$, the modifications for describing $L^s_{2n}(\mathbb{A})$ should be quite clear.

**Non-singular ($\pm$)-symmetric forms and quadratic reductions.** An $\mathbb{A}$-linear map $s\colon \mathcal{M} \to \mathcal{M}^*$ is **non-singular** if it is an isomorphism. It is ($\pm$)-symmetric if and only if $s^* = \pm s$. In what follows we only consider ($\pm$)-symmetric non-singular forms $s\colon \mathcal{M} \to \mathcal{M}^*$. Such a form has a **quadratic reduction** if there is an $\mathbb{A}$-linear map $T\colon \mathcal{M} \to \mathcal{M}^*$ so that $s = T \pm T^*$. Note that if we vary $T$ by any ($\mp$)-symmetric map $V$ – for example a map of the form $B \mp B^*$ – then then $T \pm T^*$ is unchanged. In fact two quadratic reductions, $T$ and $T'$ for $s$ will be called **equivalent** if and only if $T' = T + (B \mp B^*)$ for some $B\colon \mathcal{M} \to \mathcal{M}^*$.

The map $T$ gives a rise to a map

$$q_T\colon \mathcal{M}\longrightarrow\mathbb{A}$$

defined by $q_T(m) = T(m)(m)$. This map has the following properties:

$$\begin{aligned} q_T(am) &= aq_T(m)\bar{a} \\ q_T(m_1+m_2) &= q_T(m_1)+q_T(m_2)+T(m_1)(m_2)+T(m_2)(m_1) \\ &= q_T(m_1)+q_T(m_2)+T(m_1)(m_2)+\overline{T^*(m_1)(m_2)} \\ &= q_T(m_1)+q_T(m_2)+s(m_1)(m_2)+(\nu\mp\bar{\nu}) \end{aligned}$$

where $\nu = \overline{T^*(m_1)(m_2)}$.

In accord with standard usage we introduce the notation

$$\langle m_1, m_2\rangle_s \;=\; s(m_1)(m_2),$$

which defines a mapping $\mathcal{M}\times\mathcal{M}\to\mathbb{A}$ with the properties

$$\begin{aligned} \langle am_1, m_2\rangle_s &= \langle m_1, m_2\rangle_s\bar{a} \\ \langle m_1, am_2\rangle_s &= a\langle m_1, m_2\rangle_s \\ \langle m_1+m_2, m_3\rangle_s &= \langle m_1, m_3\rangle_s + \langle m_2, m_3\rangle_s \\ \langle m_1, m_2+m_3\rangle_s &= \langle m_1, m_2\rangle_s + \langle m_1, m_3\rangle_s. \end{aligned}$$

Moreover, if $s$ is $(+)$-symmetric or $(-)$-symmetric then we have

$$\langle m_1, m_2\rangle_s \;=\; \pm\overline{\langle m_2, m_1\rangle}_s.$$

When $s$ is $T\pm T^*$ with $T$ as above we call $s$ the associated Hermitian form with quadratic reduction $T$.

We now return to consideration of the map $q_T$ in (2). The initial difficulty with $q_T$ is the term $\nu\mp\bar{\nu}$ in the last line of (2). Thus, let $Q_\pm(\mathbb{A})$ be the *abelian group* defined as the quotient

$$\begin{aligned} Q_+(\mathbb{A}) &= \mathbb{A}/(\{a-\bar{a}\}),\ \text{for } a\in\mathbb{A} \\ Q_-(\mathbb{A}) &= \mathbb{A}/(\{a+\bar{a}\}),\ \text{for } a\in\mathbb{A}. \end{aligned}$$

The groups $Q_\pm(\mathbb{A})$ have multiplicative $\mathbb{A}$-actions defined by $a(b) = ab\bar{a}$ and are the natural value groups for the map $q_T$ above. In particular, note that if we redefine **the quadratic reduction of** $s$ as $q_T\colon\mathcal{M}\to Q_\pm(\mathbb{A})$, then

$$\begin{aligned} q_T(am) &= aq_T(m)\in Q_\pm(\mathbb{A}) \\ q_T(m_1+m_2) &= q_T(m_1)+q_T(m_2)+\langle m_1, m_2\rangle_s\in Q_\pm(\mathbb{A}). \end{aligned}$$

Another thing that is important is to note that $q_{T+B\mp B^*} = q_T$ in $Q_\pm(\mathbb{A})$.

EXAMPLE 57. (1) When $\mathbb{A}=\mathbb{Z}$ so the involution is the identity, then $Q_+(\mathbb{Z})=\mathbb{Z}$ while $Q_-(\mathbb{Z})=\mathbb{Z}/2$.

(2) In the case of the group ring $\mathbb{Z}(\mathbb{Z}/2)$ there are two involutions. Letting $\tau$ be the generator of $\mathbb{Z}/2$ they are defined by $\tau\leftrightarrow\pm\tau$ respectively.

(a) In the case where the involution is the identity we have $Q_+(\mathbb{Z}(\mathbb{Z}/2)) = \mathbb{Z}(\mathbb{Z}/2)$ while $Q_-(\mathbb{Z}(\mathbb{Z}/2)) = \mathbb{Z}/2(\mathbb{Z}/2)$.
(b) In the remaining case, $\tau \leftrightarrow -\tau$, we have

$$\begin{array}{rcl} Q_+(\mathbb{Z}(\mathbb{Z}/2)) & = & \mathbb{Z}[1] \oplus \mathbb{Z}/2[\tau] \\ Q_-(\mathbb{Z}(\mathbb{Z}/2)) & = & \mathbb{Z}/2[1] \oplus \mathbb{Z}[\tau]. \end{array}$$

A special case of $(\pm)$-non-singular symmetric form with quadratic reduction is the **hyperbolic form** associated to any finitely generated projective $\mathbb{A}$-module, $\mathcal{M}$:

$$\mathbb{H}_\epsilon(\mathcal{M}) \;=\; \left\{ \mathcal{M} \oplus \mathcal{M}^* \;\middle|\; s = \begin{pmatrix} 0 & [I] \\ \epsilon[I] & 0 \end{pmatrix},\; T = \begin{pmatrix} 0 & [I] \\ 0 & 0 \end{pmatrix} \right\}$$

where $[I]\colon \mathcal{M} \to (\mathcal{M}^*)^* = \mathcal{M}$ is the indentification of $\mathcal{M}$ with $\mathcal{M}^{**}$ discussed above, and $\epsilon$ is $(+)$ or $(-)$ 1 depending on whether we are considering $(+)$-symmetric or $(-)$-symmetric forms.

Definition 55. (1) *Given an $\epsilon$-symmetric form $(\mathcal{M}, s)$ and a submodule $L \subseteq \mathcal{M}$ define the* **orthogonal** *submodule*

$$\begin{array}{rcl} L^\perp & = & \{\, y \in \mathcal{M} \mid \langle x, y\rangle_s = 0 \in A \text{ for all } x \in L \,\} \\ & = & \ker(i^* s\colon \mathcal{M} \to L^*) \end{array}$$

*with $i : L \to \mathcal{M}$ the inclusion.*

(2) *A* **sublagrangian** *of a nonsingular $\epsilon$-quadratic form $(\mathcal{M}, T, s)$ over $A$ is a direct summand $L \subseteq \mathcal{M}$ such that*

$$s(L \times L) \;=\; \{0\} \subset \mathbb{A}\ ,\ T(L) \;=\; \{0\} \subseteq Q_\epsilon(\mathbb{A})\ ,$$

*so that*

$$L \subseteq L^\perp\ .$$

(3) *A* **lagrangian** *of $(\mathcal{M}, s, T)$ is a sublagrangian $L$ such that*

$$L^\perp \;=\; L\ .$$

If $(\mathcal{M}, s)$ is nonsingular and $L$ is a direct summand of $\mathcal{M}$ then so is $L^\perp$.

In the classical theory of quadratic forms over fields a lagrangian is called a “maximal isotropic subspace”. Wall [70] called hyperbolic forms “kernels” and the lagrangians “subkernels”. Novikov [51] called hyperbolic forms “hamiltonian”, and introduced the name “lagrangian”, because of the analogy with the hamiltonian formulation of physics in which the lagrangian expresses a minimum condition.

Example 58. The cases considered in the previous chapter for $\pi_1 = \{1\}$ are typical in many ways. Thus, for example in the hyperbolic form $(H_{(-)}(\mathbb{Z})$ with basis $e_1$, $e_1^*$ the lagrangians are the $\mathbb{Z}$-direct summands of the form $ae_1 + be_1^*$ with $ab$ even. So $\langle 3e_1 + 2e_1^*\rangle$, $\langle 5e_1 - 4e_1^*\rangle$, and so on are all quadratic lagrangians.

Theorem 51. *A nonsingular $\epsilon$-quadratic form $(\mathcal{M}; s, T)$ admits a lagrangian $L$ if and only if it is isomorphic to the hyperbolic form $H_\epsilon(L)$.*

PROOF. An isomorphism of forms sends lagrangians to lagrangians, so any form isomorphic to a hyperbolic form has at least one lagrangian. In proving the converse, it is convenient to use the language of split $\epsilon$-quadratic forms replacing $(s,T)$ by $\psi \in Q_\epsilon(K)$ and choosing a representative $\psi \in S(K)$. Suppose that

$$(i,\theta)\ :\ (L,0) \longrightarrow (K,\psi)$$

is the inclusion of a lagrangian of $(K,s,T)$. An extension of $(i,\theta)$ to an isomorphism

$$(f,\chi)\ :\ H_\epsilon(L) \xrightarrow{\cong} (K,\psi)$$

determines a lagrangian $f(L^*) \subset K$ complementary to $L$. We shall construct an isomorphism $(f,\chi)$ by first choosing a complementary lagrangian to $L$ in $(K,\psi)$. Let $i \in \mathrm{Hom}_A(L,K)$ be the inclusion, and choose a splitting $j' \in \mathrm{Hom}_A(L^*,K)$ of $i^*(\psi+\epsilon\psi^*) \in \mathrm{Hom}_A(K,L^*)$, so that

$$i^*(\psi+\epsilon\psi^*)j' \ =\ 1 \in \mathrm{Hom}_A(L^*,L^*)\ .$$

In general, $j' : L^* \to K$ is not the inclusion of a lagrangian, with $j'^*\psi j' \neq 0 \in Q_\epsilon(L^*)$. Given any $k \in \mathrm{Hom}_A(L^*,L)$ there is defined another splitting

$$j \ =\ j' + ik\ :\ L^* \longrightarrow K$$

such that

$$\begin{aligned} j^*\psi j \quad &=\ j'^*\psi j' + k^*i^*\psi ik + k^*i^*\psi j' + j'^*\psi ik \\ &=\ j'^*\psi j' + k \in Q_\epsilon(L^*)\ . \end{aligned}$$

Choose a representative $\psi \in \mathrm{Hom}_A(K,K^*)$ of $\psi \in Q_\epsilon(K)$ and set

$$k \ =\ -j'^*\psi j'\ :\ L^* \longrightarrow L^*\ .$$

The corresponding splitting $j : L^* \to K$ is the inclusion of a lagrangian

$$(j,\nu)\ :\ (L^*,0) \longrightarrow (K,\psi)$$

which extends to an isomorphism of split $\epsilon$-quadratic forms

$$((i\ j)\,,\begin{pmatrix}\theta & 0\\ j^*\psi i & \psi\end{pmatrix})\ :\ H_\epsilon(L) \xrightarrow{\cong} (K,\psi)\ .$$

□

Theorem 51 is a generalization of Witt's theorem on the extension to isomorphism of an isometry of quadratic forms over fields. The procedure for modifying the choice of complement to be a lagrangian is a generalization of the Gram-Schmidt method of constructing orthonormal bases in an inner product space.

COROLLARY 30. *The inclusion of a lagrangian in a nonsingular $\epsilon$-quadratic form*

$$i\ :\ (L,0,0) \longrightarrow (M,s,T)$$

*extends to an isomorphism of forms*

$$f\ :\ H_\epsilon(L) \xrightarrow{\cong} (M,s,T)\ .$$

PROOF. The proof of 51 gives an explicit extension. □

EXAMPLE 59. In the case of $H_{(-)}(\mathbb{Z})$ with basis $e_1$, $e_1^*$ as in 58 we can consider the quadratic lagrangian $\langle 3e_1 + 2e_1^* \rangle$. Then $4e_1 + 3e_1^*$ generates a second quadratic lagrangian and

$$\langle 4e_1 + 3e_1^*, 3e_1 + 2e_1^* \rangle = 1$$

so the map

$$\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$$

with determinant +1 gives a suitable isomorphism of forms. This reflects that fact that the group $Sp_2(\mathbb{Z}) = SL_2(\mathbb{Z})$.

COROLLARY 31. *For any nonsingular $\epsilon$-quadratic form $(K, s, T)$ the diagonal inclusion*

$$\Delta \ : \ K \longrightarrow K \oplus K \ ; \ x \longrightarrow (x, x)$$

*extends to an isomorphism of $\epsilon$-quadratic forms*

$$f \ : \ H_\epsilon(K) \xrightarrow{\cong} (K, s, T) \oplus (K, -s, -T) \ .$$

PROOF. Apply 30 to the lagrangian $\Delta(K)$ of $(K \oplus K, s \oplus -s, T \oplus -T)$ . For any split $\epsilon$-quadratic structure $\psi \in S(K)$ representing $(s, T)$ define an extension of $\Delta$ to an isomorphism

$$f \ = \ \begin{pmatrix} 1 & -\epsilon\psi^* \\ 1 & \psi \end{pmatrix} \ : \ H_\epsilon(K) \xrightarrow{\cong} (K, s, T) \oplus (K, -s, -T) \ .$$

□

**The definitions of the even $L$-groups.**

DEFINITION 56. *Two $\epsilon$-symmetric non-singular forms on finitely generated projective $\mathbb{A}$-modules with quadratic reductions are Witt equivalent if and only if there is an isomorphism of the first, $\{\mathcal{M} \mid s, T\}$ orthogonal direct sum with $\mathbb{H}_\epsilon(\mathcal{W})$ to the second $\{\mathcal{N} \mid s', T'\}$ orthogonal direct sum with $\mathbb{H}_\epsilon(\mathcal{V})$ so that $s \perp s_{\mathbb{H}(\mathcal{W})}$ is carried to $s' \perp s_{\mathbb{H}(\mathcal{V})}$ and $T$ is carried to $T' + (B - \epsilon B^*)$ for some $B$.*

- The set of Witt equivalence classes of $\epsilon$-symmetric, non-singular forms with quadratic reduction is defined as the **projective $L$-group $L^p_\epsilon(\mathbb{A})$**.
- The set of equivalence classes of $\epsilon$-symmetric, non-singular forms with quadratic reduction under Witt equivalence where we restrict $\mathcal{M}$ to be a finitely generated free module and the hyperbolic forms $H_\epsilon(\mathcal{W})$ to those where $\mathcal{W}$ is free and finitely generated, is defined as the **$L$-group $L^h_\epsilon(\mathbb{A})$**.

The definitions above imply that these sets of equivalence classes have a group structure. Indeed, we have

LEMMA 72. *The sets of equivalence classes above $L^p_\epsilon(\mathbb{A})$ and $L^h_\epsilon(\mathbb{A})$ are both abelian groups under the operation of orthogonal direct sum.*

PROOF. It is clear that orthogonal sum preserves Witt equivalence classes since the orthogonal direct sums of hyperbolic forms are again hyperbolic forms. Thus it suffices to show that given $\mathcal{M}$, together with $T\colon \mathcal{M}\to\mathcal{M}^*$ so that $T \pm T^* = s$ is an isomorphism, there is $\mathcal{N}$ together with $\tau\colon \mathcal{N}\to\mathcal{N}^*$ so that $(\mathcal{M},T) \perp (\mathcal{N},\tau)$ is hyperbolic.

Indeed, suppose that $\{\mathcal{M},T\}$ is given. Then $\{\mathcal{M},T\} \oplus \{\mathcal{M},-T\}$ has the diagonal as a lagrangian, 31, and hence represents the trivial class in $L^p_\epsilon(\mathbb{A})$ or $L^h_\epsilon(\mathbb{A})$. $\square$

DEFINITION 57. *The surgery groups $L^{p,h}_{2n}(\mathbb{A})$ are defined as follows:*

$$L^{p,h}_{2n}(\mathbb{A}) \;=\; L^{p,h}_{(-)^n}(\mathbb{A}).$$

*Thus $L^{p,h}_{4n}(\mathbb{A}) = L^{p,h}_{+}(\mathbb{A})$, and $L^{p,h}_{4n+2}(\mathbb{A}) = L^{p,h}_{-}(\mathbb{A})$. Here the involution is understood. In the special case most useful in applications where $\mathbb{A} = \mathbb{Z}[\pi]$ and the involution is the $w$-twisted involution of 19 we write these groups in the form $L^{h,s}_{2n}(\mathbb{Z}[\pi], w)$.*

EXAMPLE 60. (1) In the case $\mathbb{A} = \mathbb{Z}$ the only projective modules are free so $L^h_{2n}(\mathbb{Z}) = L^p_{2n}(\mathbb{Z})$, and these groups are the surgery obstruction groups already discussed in 43 and the previous two chapters. Consequently

$$L^h_{2n}(\mathbb{Z}) = L^p_{2n}(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n \text{ even,} \\ \mathbb{Z}/2 & \text{for } n \text{ odd.} \end{cases}$$

(2) In case $\frac{1}{2} \in \mathbb{A}$ then the quadratic refinement is unique since, given $s$, setting $T = \frac{1}{2}s$ gives a quadratic reduction and given any second quadratic reduction $T'$ then, if $T'$ is also $\epsilon$-symmetric so $2T = 2T'$ it follows that $T = \epsilon T'$. Thus, setting

$$B = \frac{1}{4}(T' - (T')^*),$$

it follows that $T' - (B \mp \epsilon B^*) = \frac{1}{2}(T' + \epsilon(T')^*)$.

(a) In case $\mathbb{A} = \mathbb{R}$, the real numbers then again projectives are free and $L^h_{4n}(\mathbb{R}) = \mathbb{Z}$, while $L^h_{4n+2}(\mathbb{R}) = 0$.

(b) In case $\mathbb{A} = \mathbb{Q}$, the rational numbers, then $L^h_{4n}(\mathbb{Q})$ is given by the calculation of 36, , , while $L^h_{4n+2}(\mathbb{Q}) = 0$.

EXAMPLE 61. One of the first calculations for a ring which is not an integral domain or a field, is for the group ring $\mathbb{Z}[\mathbb{Z}/2]$ with the oriented involution $\bar{\tau} = \tau$ where $\tau$ is the generator of $\mathbb{Z}/2$ is due to Wall. Here again projectives are free so $L^p = L^h$.

We can construct an explicit element in $L^h_{4n}(\mathbb{Z}[\mathbb{Z}/2], 1)$ as follows. Consider the matrix

$$E_{8,0} \;=\; \begin{pmatrix} 2\tau & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2\tau & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2\tau & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2\tau & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2\tau & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2\tau & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2\tau & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2\tau \end{pmatrix}$$

If we embed $\mathbb{Z}[\mathbb{Z}/2]$ into the direct sum $\mathbb{Z}_+ \oplus \mathbb{Z}_-$,

$$\begin{array}{ccc} \tau & \mapsto & (1, -1) \\ 1 & \mapsto & (1, \;\; 1) \end{array}$$

then $E_{8,0}$ goes to the direct sum of the $E_8$-matrix of 5 and the matrix

$$F \;=\; \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

For $F$ we calculate the determinants of the diagonal minors,

$$-2,\; \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix},\; \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix},\; \text{etc.}$$

as $-2, 3, -4, 5, -4, 3, -2, 1$, respectively, so that the signature of $F$ is zero, (see the discussion in the proof of 50), and the image of the determinant of $E_{8,0}$ is $(1, 1)$. From this we conclude that the determinant of $E_{8,0}$ in $\mathbb{Z}[\mathbb{Z}/2]$ is $1 \in \mathbb{Z}[\mathbb{Z}/2]$, and the multi-signature of $E_{8,0}$ is $(8, 0)$. Moreover, a quadratic reduction of $E_{8,0}$ is given as

$$T \;=\; \begin{pmatrix} \tau & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ & \tau & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tau & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tau & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tau & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tau & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \tau & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tau \end{pmatrix}$$

so $(\mathbb{Z}[\mathbb{Z}/2]^8, E_{8,0}, T)$ represents an element in $L^h_{4n}(\mathbb{Z}[\mathbb{Z}/2], 1)$. Thus, the surgery $L$-group $L^h_{4n}(\mathbb{Z}[\mathbb{Z}/2], 1)$ contains at least a $\mathbb{Z} \oplus \mathbb{Z}$, given by taking orthogonal direct sums of $\pm E_{8,0}$ with sums of the ordinary $\pm E_8$ which has multisignature $\pm(8, 8)$. Consequently, we obtain any pair $(8k, 8w)$ as the multisignature of an element in $L^h_{4n}(\mathbb{Z}[\mathbb{Z}/2], 1)$.

In fact, Wall shows that the elements above generate all of $L^h_{4n}(\mathbb{Z}[\mathbb{Z}/2], 1)$ and

$$\begin{cases} L^h_{4n}(\mathbb{Z}(\mathbb{Z}/2)) = \mathbb{Z} \oplus \mathbb{Z} \\ L^h_{4n+2}(\mathbb{Z}(\mathbb{Z}/2)) = \mathbb{Z}/2. \end{cases}$$

Here the $\mathbb{Z}/2$ is the ordinary Kervaire invariant for $f \colon M \to X$ as will be explained in 63 after we've discussed the induces maps of $L_{2n}(\mathbb{A})$ under ring maps with involutions.

Example 62. In the case of $L_{4n+2}(\mathbb{A})$, consider the group ring $\mathbb{A} = \mathbb{Z}[\mathbb{Z}/4]$ where $\mathbb{Z}/4 = \{\tau \mid \tau^4 = 1\}$, and the involution is given by

$$\sum_0^3 \alpha_i \tau^i \leftrightarrow \sum_0^3 \alpha_i \tau^{-i}.$$

Let

$$T' = \tau \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

so, embedding $\mathbb{A} \subset \mathbb{Z}_+ \oplus \mathbb{Z}_- \oplus \mathbb{Z}(i)$, by

$$1 \mapsto (1,1,1), \quad \tau \mapsto (1,\, -1,\, i), \quad \tau^2 \mapsto (1,1,-1), \quad \tau^3 \mapsto (1,-1,-i),$$

takes $T' - (\bar{T}')^t$ to the triple

$$(H_8, H_8, iE_8)$$

where $H_8$ is the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

which has determinant 1. Consequently, the determinant of $T' - (\bar{T}')^t$ is $1 \in \mathbb{Z}[\mathbb{Z}/4]$ and

$$(\mathbb{Z}[\mathbb{Z}/4]^8; T' - (\bar{T}')^t, T')$$

represents an element of $L^h_{4n+2}(\mathbb{Z}[\mathbb{Z}/4], 1)$, which is non-trivial because the multisignature of the third component, $iE_8$ is 8. In fact it turns out that

$$L^h_{4n+2}(\mathbb{Z}[\mathbb{Z}/4], 1) = \mathbb{Z} \oplus \mathbb{Z}/2,$$

the $\mathbb{Z}$ generated by the element (62), and the $\mathbb{Z}/2$ generated, as in the case of $\mathbb{Z}[\mathbb{Z}/2]$ and $\mathbb{Z}$, by

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

These examples tend to indicate the importance of the multi-signature invariants at least for the $L_{2k}$-groups of the group rings of finite groups. In fact it turns out that for these rings the kernels of the multi-signature maps are always finite 2-torsion groups when $\mathbb{A} = \mathbb{Z}[\pi]$ with a $w$-twisted involution.

We should also note briefly the naturality properties of this definition. Suppose that $\mathbb{A}$ and $\mathbb{B}$ are rings with involution and $f\colon \mathbb{A} \to \mathbb{B}$ is an involution preserving map. Then tensor product

$$\mathcal{M} \mapsto \mathbb{B} \otimes_{\mathbb{A}} \mathcal{M}$$

takes projective $\mathbb{A}$-modules to projective $\mathbb{B}$-modules and non-singular forms with quadratic reduction to non-singular forms with quadratic reduction. It follows that there are induced maps

$$L^{h,p}_{2n}(f)\colon L^{h,p}_{2n}(\mathbb{A}) \to L^{h,p}_{2n}(\mathbb{B})$$

which satisfy the usual naturality conditions with respect to compositions of involution preserving ring maps.

Example 63. (1) In the case of the inclusion $I\colon \mathbb{Z} \hookrightarrow \mathbb{R}$ the induced map is an isomorphism $L^h_{4n}(\mathbb{Z}) \to L^h_{4n}(\mathbb{R})$ while in in dimension $4n+2$ the map is obviously zero.

(2) In the case of where $\mathbb{A} = \mathbb{Z}(\mathbb{Z}/2)$ with the oriented involution, then the inclusion $i\colon \mathbb{Z} \hookrightarrow \mathbb{A}$, $n \mapsto n[I]$, and the projection $\pi\colon \mathbb{Z}(\mathbb{Z}/2) \to \mathbb{Z}$ given by $n[I] + m[T] \mapsto n + m$ are both involution preserving ring maps, and the composition $\pi \circ i$ is the identity. It follows that one of the two $\mathbb{Z}$-summands in $L^h_{4n}(\mathbb{Z}(\mathbb{Z}/2))$ can be identified with the ordinary signature invariant. Similarly,

$$L^h_{4n+2}(\mathbb{Z}(\mathbb{Z}/2)) \xrightarrow{\pi} L^h_{4n+2}(\mathbb{Z})$$

is an isomorphism and identifies the non-trivial element in this group with the ordinary Arf invariant.

## 3. The Even Dimensional Surgery Obstruction

Now we show that for any degree one normal map $f\colon M^{2n}\to X^{2n}$ there is a well defined invariant

$$\sigma(f,g)\in L^h_{2n}(\mathbb{Z}(\pi_1(X),\omega)$$

which vanishes if and only if $f$ is normally bordant to a homotopy equivalence.

To begin we need two lemmas.

LEMMA 73. *Let $f\colon M^{2n}\to X^{2n}$, $b\colon \eta_M\to\xi_X$, be a degree one normal map with $M$ closed and compact. Suppose, moreover that $f$ is $(n-1)$-connected. Then $K_n(f)$ is a finitely generated and stably free $\mathbb{Z}(\pi_1(X))$-module.*

PROOF. We can write $\mathcal{C}_*(M)=\mathcal{C}_*(X)\oplus W$ where both $\mathcal{C}_*(M)$ and $\mathcal{C}_*(X)$ are finitely generated with each group free over $\mathbb{Z}(\pi_1(X))$. It follows that $W$ is also finitely generated and stably free. On the other hand $H_*(W)=K_n(f)$ is concentrated in dimension $n$. From this an easy Euler characteristic argument allows us to replace the complex $W$ by a complex

$$0\longrightarrow\mathcal{C}_{n+1}\xrightarrow{\partial}\mathcal{C}_n\xrightarrow{\partial}\mathcal{C}_{n-1}\longrightarrow 0$$

with $\mathcal{C}_n$ free, $\mathcal{C}_{n\pm1}$ stably free and homology $K_n(f)$. We write $\mathcal{C}_n=\mathrm{Ker}(\partial)\oplus\mathcal{C}_{n-1}$ with $\partial$ the identity on the summand $\mathcal{C}_{n-1}$. Consequently, $\mathrm{Ker}(\partial)$ is stably free and so is $\mathrm{Ker}(\partial)^*$. Dually, we thus have the sequence

$$0\longrightarrow\mathcal{C}^{n-1}\xrightarrow{id}\mathcal{C}^{n-1}\oplus\mathrm{Ker}(\partial)^*\xrightarrow{\delta}\mathcal{C}^{n+1}\longrightarrow 0$$

with $\delta$ the zero map on $\mathcal{C}^{n-1}$. Since the cohomology of this complex is $K_n(f)$ we thus get the short exact sequence

$$0\longrightarrow K_n(f)\longrightarrow\mathrm{Ker}(\partial)^*\longrightarrow\mathcal{C}^{n+1}\longrightarrow 0$$

so $\mathrm{Ker}(\partial)^*=\mathcal{C}^{n+1}\oplus K_n(f)$. But now the result is clear. □

Consequently, by doing surgery on a finite number of small disjointly embedded $S^n\times D^n\subset M^{2n}$ which represent $0\in\pi_n(M^{2n})$ we replace $K_n(f)$ by $K_n(f)\perp H_{(-1)^n}(\mathbb{Z}(\pi_1(X))^k)$, and we can assume that $K_n(f)$ is actually free, at least up to Witt equivalence.

We now give the quadratic reduction of the self intersection form over $\mathbb{Z}(\pi_1(X))$ on $K_n(f)$, after surgery has been done to make the degree one normal problem $(n-1)$-connected.

Given $\alpha\in K_n(f)$ we define $q_T(\alpha)$ to be the Wall embedding obstruction away from the identity $[1]\in\mathbb{Z}(\pi_1(X))$ while the coefficient of the identity in $Q_n(\mathbb{Z}(\pi_1(X)))$ is determined, as in the simply connected case, by the normal bundle to the embedded sphere representing $\alpha$. In particular, more or less tautologically, with this choice of quadratic reduction, we get an element in $L^h_{2n}(\mathbb{Z}(\pi_1(X)))$, and we have

LEMMA 74. *Under the assumptions of 73, with $n \geq 3$, an element $\alpha \in K_n(f)$ can be represented by an embedded sphere $S^n \hookrightarrow M^{2n}$ with trivial normal bundle if and only if $q_T(\alpha) = 0 \in Q_{(-1)^n}(\mathbb{Z}(\pi_1(X)))$.*

The next step is to show that the element above, $\sigma(f,b) \in L^h_{2n}(\mathbb{Z}(\pi_1(X)))$, is independent of choices, such as the particular method used to make $f$ into an $(n-1)$-connected map.

PROPOSITION 26. *The surgery obstructions of normal bordant $n$-connected $2n$-dimensional degree 1 normal maps*

$$(f,b) : M^{1n} \longrightarrow X, \qquad (f',b') : M'^{2n} \longrightarrow X$$

*are the same so*

$$\sigma_*(f,b) \;=\; \sigma_*(f',b') \in L_{2n}(\mathbb{Z}[\pi_1(X)]) \;.$$

PROOF. Given a $(2n+1)$-dimensional degree 1 normal bordism

$$((g,c);(f,b),(f',b')) \;:\; (W^{2n+1}; M^{2n}, M'^{2n}) \to X \times (I;\{0\},\{1\})$$

use **??** to kill the kernel $\mathbb{Z}[\pi_1(X)]$-modules $K_i(W)$ for $i \leq n-1$ by $i$-surgeries on the interior of $W$ keeping $f$ and $f'$ fixed. Thus it may be assumed that $g$ is $n$-connected, and hence that $W$ has a handle decomposition on $M$ of the type

$$W \;=\; M \times I \cup \cup_m n\text{−handles } D^n \times D^{n+1} \cup \cup_{m'} (n+1)\text{−handles } D^{n+1} \times D^n \;.$$

Let

$$(W; M, M') \;=\; (W'; M, M'') \cup_{M''} (W''; M'', M')$$

with

$$\begin{aligned} W' \;&=\; M \times I \cup \cup_m n\text{−handles } D^n \times D^{n+1} \;, \\ W'' \;&=\; M'' \times I \cup \cup_{m'} (n+1)\text{−handles } D^{n+1} \times D^n \;. \end{aligned}$$

The restriction of $(g,c)$ to $M''$ is an $n$-connected $2n$-dimensional degree 1 normal map

$$(f'',b'') \;:\; M'' \;\cong\; M \#(\mathop{\#}_m S^n \times S^n) \;\cong\; M' \#(\mathop{\#}_{m'} S^n \times S^n) \to X$$

with kernel $(-)^n$-quadratic form

$$\begin{aligned} (K_n(M''), s'', T'') \quad &\cong\; (K_n(M), s, T) \oplus H_{(-)^n}(\mathbb{Z}[\pi_1(X)]^m) \\ &\cong\; (K_n(M'), s', T') \oplus H_{(-)^n}(\mathbb{Z}[\pi_1(X)]^{m'}) \;. \end{aligned}$$

It follows that

$$\begin{aligned} \sigma_*(f,b) \quad &=\; (K_n(M), s, T) \\ =\; (K_n(M'), s', T') \;&=\; \sigma_*(f',b') \in L_{2n}(\mathbb{Z}[\pi_1(X)]) \;. \end{aligned}$$

□

In view of the invariance of the surgery obstruction under normal bordism (26) and the result of Chapter **??** that every $2n$-dimensional degree 1 normal map is normal bordant to an $n$-connected degree 1 normal map the surgery obstruction can be defined in general. Note, also that the definition we give below not only holds for surgery on surgery problems over closed

Poincaré duality complexes with empty boundary, but also over Poincaré complexes with non-trivial boundary where the normal map is assumed to already be a homotopy equivalence on the boundaries and surgery is only done on the interior of $W$.

DEFINITION 58. *The* **surgery obstruction** *of a* $2n$*-dimensional degree 1 normal map* $(f,b): M^{2n} \to X$ *is defined to be the class of the* $(-)^n$*-quadratic form over* $\mathbb{Z}[\pi_1(X)]$

$$\sigma_*(f,b) \;=\; (K_n(M'), s', T') \in L_{2n}(\mathbb{Z}[\pi_1(X)])$$

*of any normal bordant* $n$*-connected* $2n$*-dimensional degree 1 normal map* $(f',b'): M'^{2n} \to X$*.*

PROPOSITION 27. *Let* $n \geq 3$*. Given an* $n$*-connected* $2n$*-dimensional degree 1 normal map* $(f,b): M^{2n} \to X$ *such that*

$$\sigma_*(f,b) = 0 \in L_{2n}(\mathbb{Z}[\pi_1(X)])$$

*there exists an* $n$*-connected* $(2n+1)$*-dimensional degree 1 normal bordism*

$$((g,c);(f,b),(f',b')) \;:\; (W^{2n+1}; M^{2n}, M'^{2n}) \;\to\; X \times (I;\{0\},\{1\})$$

*to a homotopy equivalence* $(f',b'): M' \to X$*.*

PROOF. By hypothesis there exists an isomorphism of forms

$$(K_n(M), s, T) \oplus H_{(-)^n}(\mathbb{Z}[\pi_1(X)]^m) \;\cong\; H_{(-)^n}(\mathbb{Z}[\pi_1(X)]^{m'}) \;.$$

The effect of $k$ $(n-1)$-surgeries on $(f,b)$ is the normal bordant degree 1 map

$$(f_1,b_1) \;=\; (f,b)\,\#\,\text{std.} \;:\; M_1 \;=\; M\,\#\left[\mathop{\#}_{m} S^n \times S^n\right] \;\to X$$

with kernel form

$$(K_n(M_1), s_1, T_1) \;=\; (K_n(M), s, T) \oplus H_{(-)^n}(\mathbb{Z}[\pi_1(X)]^m) \;.$$

The effect of $m'$ $n$-surgeries on $(f_1,b_1)$ killing a basis $x_1, x_2, \ldots, x_{m'} \in K_n(M_1)$ for a lagrangian is a homotopy equivalence $(f',b'): M'^{2n} \to X$ normal bordant to $(f_1,b_1)$, and hence also to $(f,b)$. □

THEOREM 52. *A* $2n$*-dimensional degree 1 normal map of pairs*

$$(f,b) \;:\; (M^{2n}, \partial M) \to (X, \partial X)$$

*with* $\partial f : \partial M \to \partial X$ *a homotopy equivalence has a surgery obstruction*

$$\sigma_*(f,b) \in L_{2n}(\mathbb{Z}[\pi_1(X)])$$

*such that* $\sigma_*(f,b) = 0$ *if (and for* $n \geq 3$ *only if)* $(f,b)$ *is normal bordant rel* $\partial$ *to a homotopy equivalence of pairs.*

PROOF. The surgery obstruction of $(f, b)$ is defined by

$$\sigma_*(f,b) \;=\; (K_n(M'), s', T') \in L_{2n}(\mathbb{Z}[\pi_1(X)])$$

for any $n$-connected degree 1 normal map $(f', b') : (M', \partial M) \rightarrow (X, \partial X)$ bordant to $(f, b)$ relative to the boundary, with $\partial f' = \partial f$, exactly as in the closed case $\partial M = \partial X = \emptyset$ in 58. The rel $\partial$ version of 26 shows that the surgery obstruction is a normal bordism invariant, which is 0 for a homotopy equivalence. Also, if $n \geq 3$ and $\sigma_*(f, b) = 0$ then $(f, b)$ is normal bordant rel $\partial$ to a homotopy equivalence of pairs, proved exactly as in the closed case in 27. □

## 4. The Odd Dimensional Surgery Obstruction Groups

The surgery obstruction of an even-dimensional degree 1 normal map was expressed in section 3 as an equivalence class of nonsingular $\epsilon$-quadratic forms, with the zero class represented by the forms which admit a lagrangian. The different lagrangians admitted by a form correspond to different ways of solving the even-dimensional surgery problem by constructing a normal bordism to a homotopy equivalence. The odd-dimensional surgery obstruction will be expressed in section **??** as an equivalence class of nonsingular $\epsilon$-quadratic 'formations', which are nonsingular $\epsilon$-quadratic forms with ordered pairs of lagrangians, corresponding to two solutions of an even-dimensional surgery problem in codimension 1.

**Formations.** The fundamental algebraic structure determined by a closed $2n$-dimensional manifold $N^{2n}$ is the nonsingular $(-)^n$-quadratic form over $\mathbb{Z}(\pi_1(N)) = \mathbb{A}$ on $H_n(N; \mathbb{A})$. The fundamental algebraic structure determined by a $(2n + 1)$-dimensional manifold with non-empty boundary $(M^{2n+1}, \partial M)$ is the lagrangian of the $(-)^n$-quadratic form on $H_n(\partial M; \mathbb{A})$ defined by

$$L \;=\; \mathrm{im}(H_{n+1}(M, \partial M; \mathbb{A}) \rightarrow H_n(\partial M; \mathbb{A})) \subset H_n(\partial M; \mathbb{A})\ .$$

A decomposition of a closed $(2n+1)$-dimensional manifold $M^{2n+1}$ as a union of two $(2n + 1)$-dimensional manifolds $(M_+^{2n+1}, \partial M_+)$, $(M_-^{2n+1}, \partial M_-)$ with the same boundary $N^{2n} = \partial M_+ = \partial M_-$

$$M^{2n+1} \;=\; M_+ \cup_N M_-$$

determines a $(-)^n$-quadratic formation, namely the $(-)^n$-quadratic form on $K = H_n(N; \mathbb{A})$ with the lagrangians $L_+, L_- \subset K$ defined by

$$L_\pm \;=\; \mathrm{im}(H_{n+1}(M_\pm, N; \mathbb{A}) \rightarrow H_n(N; \mathbb{A}))\ .$$

Such decompositions were first used by Heegard in the study of 3-dimensional manifolds: every closed 3-dimensional manifold can be expressed as $M^3 = M_+ \cup_N M_-$ with $M_+$ and $M_-$ the connected sums of $k$ copies of the solid 3-torus $S^1 \times D^2$, and $N$ the connected sum of $k$ copies of the 2-torus $S^1 \times S^1$. It should be clear that such expressions are not unique, since forming the connected sum of $M$ with $S^3 = S^1 \times D^2 \cup D^2 \times S^1$ increases $k$ by 1 without affecting the diffeomorphism type.

In section 4 we shall only be concerned with the algebraic properties of formations. As before, let $\mathbb{A}$ be a ring with involution, and let $\epsilon = \pm 1$.

DEFINITION 59. *An* **$\epsilon$-quadratic formation over** $\mathbb{A}$ $(K, s, T; F, G)$ *is a nonsingular $\epsilon$-quadratic form $(K, s, T)$ together with an ordered pair of lagrangians $F$,$G$.*

Also, we assume that $\mathbb{A}$ is such that the rank of f.g. free $\mathbb{A}$-modules is well-defined (e.g. $bba = \mathbb{Z}[\pi]$), with

$$\mathrm{rank}_{\mathbb{A}}|(K) \ = \ k \in \mathbb{Z}^+$$

for a f.g. free $\mathbb{A}$-module $K$ if and only if $K$ is isomorphic to $\mathbb{A}^k$. The hypothesis on $\mathbb{A}$ ensures that for every formation $(K, s, TF, G)$ over $\mathbb{A}$ there exists an automorphism $\alpha : (K, s, T) \longrightarrow (K, s, T)$ such that $\alpha(F) = G$ (28 below). In the original work of Wall the odd-dimensional surgery obstruction was defined in terms of such automorphisms, but formations are more algebraically tractable.

DEFINITION 60. *An* **isomorphism** *of $\epsilon$-quadratic formations over $A$*

$$f \ : \ (K, s, T, F, G) \xrightarrow{\cong} (K', T', s'; F', G')$$

*is an isomorphism of forms $f : (K, s, T) \longrightarrow (K', T', s')$ such that*

$$f(F) \ = \ F' \ , \ f(G) \ = \ G' \ .$$

PROPOSITION 28. (1) *Every $\epsilon$-quadratic formation $(K, s, T; F, G)$ is isomorphic to one of the type $(H_\epsilon(F); F, G)$.*

(2) *Every $\epsilon$-quadratic formation $(K, s, T; F, G)$ is isomorphic to one of the type $(H_\epsilon(F); F, \alpha(F))$ for some automorphism*

$$\alpha : H_\epsilon(F) \longrightarrow H_\epsilon(F).$$

PROOF. (i) By 51 the inclusion of the lagrangian $F \hookrightarrow K$ extends to an isomorphism of $\epsilon$-quadratic forms

$$f \ : \ H_\epsilon(F) \xrightarrow{\cong} (K, s, T) \ ,$$

defining an isomorphism of $\epsilon$-quadratic formations

$$f \ : \ (H_\epsilon(F); F, f^{-1}(G)) \xrightarrow{\cong} (K, s, T; F, G) \ .$$

(ii) As in (i) extend the inclusions of the lagrangians $F \hookrightarrow K$, $G \hookrightarrow K$ to isomorphisms of forms

$$f \ : \ H_\epsilon(F) \xrightarrow{\cong} (K, s, T) \ , \ g \ : \ H_\epsilon(G) \xrightarrow{\cong} (K, s, T) \ .$$

Then

$$\mathrm{rank}_A(F) \ = \ \mathrm{rank}_A(K)/2 \ = \ \mathrm{rank}_A(G) \in \mathbb{Z}^+ \ ,$$

so that $F$ is isomorphic to $G$. Choosing an $A$-module isomorphism $\beta : F \to G$ there is defined an automorphism of $H_\epsilon(F)$

$$\alpha \ : \ H_\epsilon(F) \xrightarrow{\begin{pmatrix} \beta & 0 \\ 0 & \beta^{*-1} \end{pmatrix}} H_\epsilon(G) \xrightarrow{g} (K, s, T) \xrightarrow{f^{-1}} H_\epsilon(F)$$

such that there is defined an isomorphism of formations

$$f \ : \ (H_\epsilon(F); F, \alpha(F)) \xrightarrow{\cong} (K, s, T; F, G) \ .$$

□

Later we will associate to an $n$-connected $(2n+1)$-dimensional degree 1 normal map $(f, b) : M^{2n+1} \to X$ a 'stable isomorphism' class of kernel $(-)^n$-quadratic formations $(K, s, T; F, G)$ over $\mathbb{Z}[\pi_1(X)]$. Also, as was the case before for simply connected surgery, the surgery kernels associated to a formation are given as

$$K_n(M) \ = \ K/(F + G) \ , \ K_{n+1}(M) \ = \ F \cap G \ .$$

Stable isomorphism , which is analogous to Witt stabilization in even dimensions, associated to enlarging the surgery kernel by doing a sequence of surgeries over homotopy trivial embedded spheres is defined as follows:

DEFINITION 61. (1) *An $\epsilon$-quadratic formation $T = (K, s, T; F, G)$ is* **trivial** *if it is isomorphic to $(H_\epsilon(F); F, F^*)$.*

(2) *A* **stable isomorphism** *of $\epsilon$-quadratic formations*

$$[f] \ : \ (K, s, T; F, G) \xrightarrow{\cong} (K', T', s'; F', G')$$

*is an isomorphism of $\epsilon$-quadratic formations of the type*

$$f \ : \ (K, s, T; F, G) \oplus T \xrightarrow{\cong} (K', T', s'; F', G') \oplus T'$$

*with $T$, $T'$ trivial.*

DEFINITION 62. *The* **boundary** *of a $(-\epsilon)$-quadratic form $(K, s, T)$ is the* **graph formation**

$$\partial(K, s, T) \ = \ (H_\epsilon(K); K, \Gamma_{(K,T)})$$

*with*

$$\Gamma_{(K,T)} \ = \ \{ \, (x, s(x)) \in K \oplus K^* \, | \, x \in K \, \} \ .$$

REMARK . The $\epsilon$-quadratic formation $\partial(K, s, T)$ depends only on the even $\epsilon$-symmetric form $(K, s)$, and not on the $\epsilon$-quadratic function $T$. ∎

Note that in this situation the form $(K, s, T)$ may be singular, that is the $\mathbb{A}$-module morphism $s : K \to K^*$ need not be an isomorphism.

PROPOSITION 29. *The graphs $\Gamma_{(K,T)}$ of $(-\epsilon)$-quadratic forms $(K, s, T)$ are precisely the lagrangians of $H_\epsilon(K)$ which are the direct complements of $K^*$.*

PROPOSITION 30. (1) *An $\epsilon$-quadratic formation $(K,s,T;F,G)$ is trivial if and only if the lagrangians $F$ and $G$ are direct complements in $K$*

$$F\cap G \;=\; \{0\}\ ,\ F+G \;=\; K\ .$$

(2) *An $\epsilon$-quadratic formation $(K,s,T;F,G)$ is isomorphic to a boundary if and only if $(K,s,T)$ has a lagrangian $H$ which is a direct complement of both the lagrangians $F$, $G$.*
(3) *For any $\epsilon$-quadratic formation $\Phi=(K,s,T;F,G)$ there exists an $\epsilon$-quadratic formation $\Phi'=(K',T',s';F',G')$ such that $\Phi\oplus\Phi'$ is isomorphic to a boundary.*
(4) *A $(-\epsilon)$-quadratic form $(K,s,T)$ is nonsingular if and only if the boundary $\epsilon$-quadratic formation $\partial(K,s,T)$ is trivial.*

PROOF. (i) If $F$ and $G$ are direct complements in $K$ represent $(s,T)$ by a split $\epsilon$-quadratic form $(K,\psi\in S(K))$ with

$$\psi \;=\; \begin{pmatrix}\alpha-\epsilon\alpha^* & \gamma\\ \delta & \beta-\epsilon\beta^*\end{pmatrix}\;:\; K \;=\; F\oplus G\longrightarrow K^* \;=\; F^*\oplus G^*\ .$$

Then $\gamma+\epsilon\delta^*\in \mathrm{Hom}_A(G,F^*)$ is an $\mathbb{A}$-module isomorphism, and there is defined an isomorphism of formations

$$\begin{pmatrix}1 & 0\\ 0 & (\gamma+\epsilon\delta^*)^{-1}\end{pmatrix}\;:\;(H_\epsilon(F);F,F^*)\xrightarrow{\;\cong\;}(K,s,T;F,G)\ .$$

(ii) For the boundary $\partial(F,\phi,\theta)$ of a $(-\epsilon)$-quadratic form $(F,\phi,\theta)$ the lagrangian $F^*$ of $H_\epsilon(F)$ is a direct complement of both the lagrangians $F$, $\Gamma_{(F,\phi)}$. Conversely, suppose that $(K,s,T;F,G)$ is such that there exists a lagrangian $H$ in $(K,s,T)$ which is a direct complement to both $F$ and $G$. By the proof of (i) there exists an isomorphism of formations

$$f\;:\;(H_\epsilon(F);F,F^*)\xrightarrow{\;\cong\;}(K,s,T;F,H)$$

which is the identity on $F$. Now $f^{-1}(G)$ is a lagrangian of $H_\epsilon(F)$ which is a direct complement of $F^*$, so that it is the graph $\Gamma_{(F,\phi)}$ of a $(-\epsilon)$-quadratic form $(F,\phi,\theta)$, and $f$ defines an isomorphism of $\epsilon$-quadratic formations

$$f\;:\;\partial(F,\phi,\theta)\;=\;(H_\epsilon(F);F,\Gamma_{(F,\phi)})\xrightarrow{\;\cong\;}(K,s,T;F,G)\ .$$

(iii) Every lagrangian has a lagrangian direct complement, by 30. Let $(K',T',s')=(K,-T,-s)$, and let $F'$, $G'$ be lagrangian direct complements of $F$, $G$ respectively. The diagonal lagrangian $\Delta$ of $(K,s,T)\oplus(K',T',s')$ (31) is complementary to both $F\oplus F'$ and $G\oplus G'$, so that $\Phi\oplus\Phi'$ is isomorphic to a boundary by (ii).
(iv) The graph lagrangian $\Gamma_{(K,s)}$ of $H_\epsilon(K)$ is a direct complement of $K$ if and only if $s\colon K\to K^*$ is an isomorphism. □

**The odd-dimensional $L$-groups .**

DEFINITION 63. *The* $(2n+1)$**-dimensional $L$-group** $L_{2n+1}(A)$ *of a ring with involution $A$ is the group of equivalence classes of nonsingular $(-)^n$-quadratic formations $(K, s, T; F, G)$ over $A$, subject to the equivalence relation*

$$(K, s, T; F, G) \sim (K', T', s'; F', G')$$
*if there exists an isomorphism of formations*
$$(K, s, T; F, G) \oplus T \oplus B \;\cong\; (K', T', s'; F', G') \oplus T' \oplus B'$$
*for some trivial $T$, $T'$ and boundaries $B$, $B'$.*

*The addition and inverses in $L_{2n+1}(A)$ are given by*

$$\begin{aligned}&(K_1, s_1, T_1; F_1, G_1) + (K_2, s_2, T_2; F_2, G_2)\\ &= \;(K_1 \oplus K_2, s_1 \oplus s_2, T_1 \oplus T_2; F_1 \oplus F_2, G_1 \oplus G_2)\ ,\\ -(K, s, T; F, G) \;&=\; (K, -s, -T; F^*, G^*) \in L_{2n+1}(A)\end{aligned}$$

*with $F^*$, $G^*$ lagrangian direct complements of $F$, $G$.*

It should be clear that $L_{2n+1}(A)$ depends only on the residue $n(\bmod 2)$, so that only two $L$-groups have actually been defined, $L_1(A)$ and $L_3(A)$. Note that 63 uses 30 (iii) to justify

$$(K, T, s; F, G) \oplus (K, -s, -T; F^*, G^*) \sim 0\ .$$

EXAMPLE 64. (1) In the case $\mathbb{A} = \mathbb{Z}$ then these odd $L$-groups are the surgery obstruction groups studied in 10, so

$$L_{2n+1}(\mathbb{Z}) \;=\; 0\ .$$

(2) If $\mathbb{A}$ is a field, $\mathbb{F}$, with arbitrary involution then, after choosing a basis for $F$ and the associated dual basis for $F^*$ a finite number of exchanges $e_i \leftrightarrow e_i^*$ will ensure that the projection of $H_\epsilon(F)$ onto $F$ with kernel $F^*$ will map $G$ isomorphically. But then $F$ and $G$ have $F^*$ as a common complement. It follows that $L_{2n+1}(\mathbb{F}) = 0$

(3) Wall showed that the odd-dimensional $L$-groups of $\mathbb{Z}[\mathbb{Z}_2]$ with the oriented involution $\overline{T} = T$ are given by

$$L_{2n+1}(\mathbb{Z}[\mathbb{Z}_2]) \;=\; \begin{cases} 0 & \text{if } n \equiv 0 (\bmod 2) \\ \mathbb{Z}_2 & \text{if } n \equiv 1 (\bmod 2) \end{cases}$$

(Wall [68]). ∎

PROPOSITION 31. *For any lagrangians $F, G, H$ in a nonsingular $(-)^n$-quadratic form $(K, s, T)$ over $A$*

$$(K, s, T; F, G) \oplus (K, s, T; G, H) \;=\; (K, s, T; F, H) \in L_{2n+1}(A)\ .$$

PROOF. Choose lagrangians $F^*, G^*, H^*$ in $(K, s, T)$ complementary to $F, G, H$ respectively. The $(-)^n$-quadratic formations $(K_i, s_i, T_i; F_i, G_i)$ $(1 \leq$

$i \leq 4$) defined by

$$\begin{aligned}
(K_1, s_1, T_1; F_1, G_1) &= (K, -s, -T; G^*, G^*) \ , \\
(K_2, s_2, T_2; F_2, G_2) &= (K \oplus K, s \oplus -s, T \oplus -T; F \oplus F^*, H \oplus G^*) \\
&\quad \oplus (K \oplus K, -s \oplus s, -T \oplus T; \Delta_K, H^* \oplus G) \ , \\
(K_3, s_3, T_3; F_3, G_3) &= (K \oplus K, s \oplus -s, T \oplus -T; F \oplus F^*, G \oplus G^*) \ , \\
(K_4, s_4, T_4; F_4, G_4) &= (K \oplus K, s \oplus -s, T \oplus -T; G \oplus G^*, H \oplus G^*) \\
&\quad \oplus (K \oplus K, -s \oplus s, -T \oplus T; \Delta_K, H^* \oplus G)
\end{aligned}$$

are such that

$$\begin{aligned}
&(K, s, T; F, G) \oplus (K, s, T; G, H) \oplus (K_1, s_1, T_1; F_1, G_1) \\
&\qquad \oplus (K_2, s_2, T_2; F_2, G_2) \\
&= (K, s, T; F, H) \oplus (K_3, s_3, T_3; F_3, G_3) \oplus (K_4, s_4, T_4; F_4, G_4) \ .
\end{aligned}$$

Each of $(K_i, s_i, T_i; F_i, G_i)$ $(1 \leq i \leq 4)$ is isomorphic to a boundary, since there exists a lagrangian $H_i$ in $(K_i, T_i, s_i)$ complementary to both $F_i$ and $G_i$, so that 30 (ii) applies and $(K_i, s_i, T_i; F_i, G_i)$ represents 0 in $L_{2n+1}(A)$. Explicitly, we can take

$$\begin{aligned}
H_1 &= G \subset K_1 = K \ , \\
H_2 &= \Delta_{K \oplus K} \subset K_2 = (K \oplus K) \oplus (K \oplus K) \ , \\
H_3 &= \Delta_K \subset K_3 = K \oplus K \ , \\
H_4 &= \Delta_{K \oplus K} \subset K_4 = (K \oplus K) \oplus (K \oplus K) \ .
\end{aligned}$$

□

REMARK . It is also possible to express $L_{2n+1}(A)$ as the abelian group of equivalence classes of $(-)^n$-quadratic formations over $A$ subject to the equivalence relation generated by:

(i) $(K, s, T; F, G) \sim (K', s', T'; F', G')$
if $(K, s, T; F, G)$ is stably isomorphic to $(K', s', T'; F', G')$
(ii) $(K, s, T; F, G) \oplus (K, s, T; G, H) \sim (K, s, T; F, H)$

with addition and inverses by

$$\begin{aligned}
&(K, s, T; F, G) + (K', s', T'; F', G') \\
&= (K \oplus K', s \oplus s' T \oplus T'; F \oplus F', G \oplus G') \ , \\
&-(K, s, T; F, G) = (K, s, T; G, F) \in L_{2n+1}(A) \ .
\end{aligned}$$

This is immediate from 30 and the observation that for any $(-)^{n+1}$-quadratic form $(G, \phi, \theta)$ on a f.g. free $A$-module $G$ the lagrangian $G^*$ in $H_{(-)^n}(G)$ is a complement to both $G$ and the graph $\Gamma_{(G,\phi,\theta)}$, so that

$$\begin{aligned}
\partial(G, \phi, \theta) &\sim (H_{(-)^n}(G); G, \Gamma_{(G,\phi,\theta)}) \oplus (H_{(-)^n}(G); \Gamma_{(G,\phi,\theta)}, G^*) \\
&\sim (H_{(-)^n}(G); G, G^*) \sim 0 \ .
\end{aligned}$$

∎

## 5. The Odd Dimensional Surgery Obstruction

Suppose that $(f, b \mid f\colon M^{2n+1} \to X)$ is a degree one normal map which is $(n-1)$-connected. Then, by 4, we've seen how to associate a formation to $(f, b)$.

- First embed a sequence of $S^n \times D^{n+1}$'s disjointly in $\tilde{M}$ to represent the generators of $K_n(f) \hookrightarrow H_n(\tilde{M};\mathbb{Z})$ as a module over $\mathbb{Z}[\pi_1(M)]$ where $\tilde{M}$ is the universal cover of $M$, so that the projections to $M$ are also disjoint embeddings. Then the spaces $\pi^{-1}\pi(S^n \times D^{n+1}$ are all copies of $\pi_1(M) \times S^n \times D^{n+1}$ and are all disjoint. Denote the union of these inverse images by $U \subset \tilde{M}$.
- Regard $\tilde{M}$ as the disjoint union
$$\tilde{M}_0 \cup_{\partial U} U.$$
- Note that $H_*(\partial U) = \mathbb{Z}[\pi_1(X)] \otimes \bigoplus H_*(S^n \times S^n)$, one $S^n \times S^n$ for each $S^n \times D^{n+1}$. In particular $H_n(\partial U)$ is a hyperbolic form $H_{(-1)^n}(\mathbb{Z}[\pi_1(X)]^m)$.
- Finally, the two lagrangians are given as the images under $\partial$,

$$\begin{aligned} F &= \mathrm{im}\partial\colon H_{n+1}(U, \partial U) \longrightarrow H_n(\partial U), \\ G &= \mathrm{im}\partial\colon H_{n+1}(\tilde{M}_0, \partial U) \longrightarrow H_n(\partial U). \end{aligned}$$

Remark . One thing that is evident is that if our original disjoint embeddings $h_j\colon S^n \times D^{n+1}$ are isotoped through disjoint embeddings to new embeddings $\bar{h}_j\colon S^n \times D^{n+1} \hookrightarrow M$, very close to the original ones the resulting formation is identical to the original one.

What we need to do now is to show that this assignment is independent of the choices made,

- first of the particular representatives for $K_n(f)$, and
- second of the particular way the original surgery problem is surgered to one which is $(n-1)$-connected.

In fact, the formations will vary but we now show that their representatives in $L^h_{2n+1}(\mathbb{Z}[\pi_1(X)])$ are all the same.

Theorem 53. *The assignment of a formation to the surgery problem in (5) $(f, b \mid f\colon M^{2n+1} \to X^{2n+1})$ gives a well defined map from surgery problems to $L^h_{2n+1}(\mathbb{Z}[\pi_1(X)])$ by taking the equivalence class of this formation. As before we denote this equivalence class as $\sigma(f, b)$. Moreover, $\sigma(f, b)$ is zero if and only if the orginal problem $(f, b)$ is normally bordant to a homotopy equivalence.*

Proof. We begin by considering the effect of the choice of representing embeddings. So suppose that $h_1, \ldots, h_r$ and $\bar{h}_1, \ldots, \bar{h}_t$ are two sets of disjoint embeddings giving generators for $K_n(f)$. By a sequence of very small moves we may assume that all the embeddings are disjoint on the core spheres, and by shrinking the $D^{n+1}$'s if necessary we can, in fact, assume that the entire images of $S^n \times D^{n+1}$ are disjoint. This gives us three formations:

- the first,
  $$(H_{(-)^n}(\mathbb{Z}[\pi_1(X)]^r), \partial H_{n+1}(U_1, \partial U_1), \partial H_{n+1}(\tilde{M}_1, \partial U_1))$$
  associated to $h_1, \dots, h_r$,
- the second,
  $$(H_{(-)^n}(\mathbb{Z}[\pi_1(X)]^t), \partial H_{n+1}(U_2, \partial U_2), \partial H_{n+1}(\tilde{M}_2, \partial U_2))$$
  associated to $\bar{h}_1, \dots, \bar{h}_t$,
- and the third,
  $$\begin{aligned}(H_{(-)^n}(\mathbb{Z}&[\pi_1(X)]^{r+t}),\\ &\partial H_{n+1}(U_1 \sqcup U_2, \partial(U_1 \sqcup U_2)), \partial H_{n+1}(\tilde{M}_3, \partial(U_1 \sqcup U_2)))\end{aligned}$$
  associated to the union $h_1, \dots, h_r, \bar{h}_1, \dots, \bar{h}_t$.

By excision $H_i(\tilde{M}_i, \partial\tilde{M}_i) \cong H_i(\tilde{M}, U_i)$, so we have commutative diagrams

$$\begin{array}{ccc} H_{n+1}(\mathcal{M}_i, \partial U_i) & \xrightarrow{\partial} & H_n(\partial U_i) \\ \big\downarrow & & \big\downarrow \\ H_{n+1}(\mathcal{M}_3, \partial(U_1 \sqcup U_2)) & \xrightarrow{\partial} & H_n(\partial(U_1 \sqcup U_2)) \end{array}$$

for $i = 1, 2$ with the vertical and horizontal arrows all inclusions as $\mathbb{Z}[\pi_1(X)]$ direct summands. Note that we have exact sequences

$$\begin{aligned} 0 \longrightarrow K_{n+1}(f) \longrightarrow \quad & H_{n+1}(\tilde{M}_i, \partial U_i) \longrightarrow H_n(\partial U_i)/(\mathrm{im}(H_{n+1}(U_i, \partial U_i)) \\ & \longrightarrow K_n(f) \longrightarrow 0 \end{aligned}$$

for $i = 1, 2$. Consequently, we have

LEMMA 75. *There is an exact sequence of* $\mathbb{Z}[\pi_1(X)]$*-modules*

$$\begin{aligned} 0 \longrightarrow K_{n+1}(f) \longrightarrow \quad & H_{n+1}(\tilde{M}_1, \partial U_1) \oplus H_{n+1}(\tilde{M}_2, \partial U_2) \\ & \longrightarrow H_{n+1}(\tilde{M}_3, \partial\tilde{M}_3) \longrightarrow K_n(f) \longrightarrow 0, \end{aligned}$$

*where the inclusions are those in 5.*

(This is direct.)

Consequently, in $G_3 = \mathrm{im}(H_{n+1}(\tilde{M}_3, \partial\tilde{M}_3))$, the intersection of $G_1$ and $G_2$ is $K_{n+1}(f)$. Next, note that the images of both $F_1^*$ and $F_2^*$ under projection of $F_3^* = F_1^* \oplus F_2^*$ are $K_n(f)$. As a result, we can lift $F_1^*$ to $G_3$, by taking, for each generator $e_i^*$, $1 \le i \le r$ , $e_i^* - s_i$ where $s_i \in F_2^*$ has image equal to that of $e_i$ in $K_n(f)$, and lifting this element to $G_3$. Let $V \subset G_3$ be this lifting.

Note that $V$ is perpendicular to $G_2$ since they are both contained in $G_3$. If we write $H_{(-)^n}(F_1 \oplus F_2) = F_1 \oplus F_1^* \oplus G_2 \oplus G_2^*$, then the basis elements $e_i$, $1 \le i \le r$ for $V$ all have the form $(h_i, e_i^*, \theta, 0)$ since otherwise they would not be orthogonal to $G_2$. Since $V_i$ is a kernel, it follows that $(h_i, e_i)$ is a lagrangian kernel for $(F_1 \oplus F_1^*)$, and hence is a graph kernel.

We have that $G_3 = V \oplus G_2$. Hence, replacing $G_3$ by an associated graph formation, we can assume that the generators of $V$ have the form $(0, e_i^*, \theta_i, 0)$,

$1 \le i \le r$. But then a new basis for this kernel is simply $(0, e_i^*, 0, 0)$ together with a basis for $G_2$, and it follows that after modifying by a graph formation we have our formation given as

$$((F_1 \oplus V) \perp (G_2 \oplus G_2^*), F_1 \oplus F_2, V \oplus G_2).$$

This is the sum $(F_1 \oplus V, F_1, V) \oplus (F_2 \oplus F_2^*, F_2, G_2)$ which is equivalent to $(F_2 \oplus F_2^*, F_2, G_2)$.

Finally, exchanging the subscripts 1 and 2 throughout the argument above shows that the big formation is also equivalent to $(F_1 \oplus F_1^*, F_1, G_1)$.

Thus, we have completed first step of showing that for a given $(n-1)$-connected surgery problem $(f, b)$, any two ways of realizing the kernel $K_n(f)$ give rise to the equivalent formations. It remains to consider the relation between the formations associated to two normally bordant $(n-1)$-connected surgery problems, $(f, b)$ and $(f', b')$ over $X$.

At this point we asssume given a normal bordism $(H, B)$, $H \colon W^{2n+2} \to I \times X$ so that $\partial(W) = M \sqcup -M'$ and $(H, B)$ restricted to $M$ is $(f, b)$ while $(H, B)$ restricted to $M'$ is $(f', b')$. To begin we may do surgery on the interior of $W$ to make $H$ into an $n$-connected map so that the exact sequence of kernels becomes

$$0 \longrightarrow K_{n+1}(f) \oplus K_{n+1}( \quad f') \longrightarrow K_{n+1}(W)$$
$$\longrightarrow K_{n+1}(W, \partial W) \xrightarrow{\partial} K_n(f) \oplus K_n(f' \longrightarrow 0.$$

Consequently, $W$ can be given as either $I \times M$ or $I \times M'$ with a number of handles $D^{n+1} \times D^{n+1}$ attached along $1 \times M$ or $0 \times M'$. Suppose, for definiteness that we have $r$ such handles. Then, they are obtained by embedding $r$ disjoint copies of $S^n \times D^{n+1} \hookrightarrow M$ where the images of the core $S^n$'s when lifted to the universal cover, give generators $g_1, \ldots, g_r$ for $K_n(f)$. It follows that the inverse images of this $\bigsqcup_1^r S^n \times D^{n+1} = U \subset \tilde{M}$ is a suitable $U$ for our previous discussion.

It follows that the chain subcomplex of $\mathcal{C}_*(M)$ associated to the surgery kernel is given as $\mathbb{Z}[\pi_1(X)]^{2r} \xrightarrow{\partial} \mathbb{Z}[\pi_1(X)]^{2r}$ or more exactly:

$$K_{n+1}(\tilde{M}_0, \partial U) \oplus H_{n+1}(U, \partial U) \xrightarrow{\partial_1 + \partial_2} H_n(\partial U).$$

Consequently the chain complex needed for determining the homology of $W$ and of $(W, M)$ is the direct sum of the chain complex of the universal cover of $X$ together with the complex associated to the surgery kernels:

$$\left[\bigoplus_1^r \mathbb{Z}[\pi_1(X)] \otimes H_{n+1}((D^{n+1}, S^n)\right] \quad \oplus K_{n+1}(\tilde{M}_0, \partial U) \oplus H_{n+1}(U, \partial U)$$
$$\xrightarrow{\partial_3 + \partial_1 + \partial_2} H_n(\partial U).$$

Note that the sum $\partial_3 + \partial_2$ gives an isomorphism

$$\left[\bigoplus_1^r \mathbb{Z}[\pi_1(X)] \otimes H_{n+1}((D^{n+1}, S^n)\right] \oplus H_{n+1}(U, \partial U) \longrightarrow H_n(\partial U)$$

since the first summand hits the lift of the core spheres of $U$ in $\partial U$, while the second summand hits the transverse spheres. It follows that

$$K_*(H) = K_{n+1}(H) \cong \mathbb{Z}[\pi_1(X)]^r$$

where the generators are represented by elements of the form $e_i + L\partial_2(e_i)$ with $L$ a lift of $\partial_3 + \partial_2$ in 5, while $e_i$ a free generator for $K_{n+1}(\tilde{M}_0, \partial U)$, as $i$ runs from 1 to $r$. Similarly,

$$K_{n+1}(W,M) \cong K^{n+1}(W,M') \cong \mathbb{Z}[\pi_1(X)]^r$$

representing the piece $\bigoplus_1^r \mathbb{Z}[\pi_1(X)] \otimes H_{n+1}((D^{n+1}, S^n)$ in the complex above.

Thus we obtain the formation

$$(K_{n+1}(W,M) \oplus K_{n+1}(W,M'); K_{n+1}(W,M'), K_{n+1}(W))$$

which we can identify with the formation for $(f,b)$ associated to the decomposition $M = M_0 \cup \coprod_1^r S^n \times D^{n+1}$ above. But reversing the roles of $M$ and $M'$ we obtain the formation

$$(K_{n+1}(W,M) \oplus K_{n+1}(W,M'); K_{n+1}(W,M), K_{n+1}(W))$$

for an associated decomposition $M' = M_0' \cup \coprod_1^r S^n \times D^{n+1}$. On the other hand, the formations 5 and 5 are equivalent in $L^h_{2n+1}(\mathbb{Z}[\pi_1(X)], w)$. Thus we have proved the first statement of 53.

It remains to show that the surgery obstruction is zero in $L^h_{2n+1}(\mathbb{Z}[\pi_1(X), w)$ if and only if the original problem is equivalent to a homotopy equivalence.

Thus suppose that in the construction above $(f', b')$ is a homotopy equivalence. In this case, 5 has the property that the projection,

$$p\colon K_{n+1}(W) \to K_{n+1}(W,M')$$

is an isomorphism and $K_{n+1}(W)$ is a graph kernel,

$$(h \mp (-1)^n h^*, id)(K_{n+1}(W,M')).$$

Then the equivalence between 5 and 5 completes the proof of 53. □

## 6. Realizing the Surgery Obstructions

In this section we show how to construct degree one normal problems over manifolds of the form $I \times M$, $(F,b)$ with

$$F\colon (W, \partial W) \to (I \times M, (\partial I) \times M)$$

which are homotopy equivalences on the boundary of $W$ and on the interior realize any arbitrary element in the surgery obstruction group

$$L^h_{n+1}(\mathbb{Z}[\pi_1(M)], w).$$

When $\pi = \pi_1(M) = \{1\}$, we've already seen how to do this, and, at least in the piecewise linear case we were able to actually assume that the boundaries were empty. However, in the case where $\pi \neq \{1\}$ this is not always possible as we will point out with some examples. Throughout this section we assume that $n \geq 5$.

**Realizing the obstructions in odd dimensions.** Suppose that we have $\alpha \in L^h_{2n+1}(\mathbb{Z}[\pi], w)$ and $M^{2n}$ is a compact closed manifold with fundamental group $\pi$, and orientation cover $w$. Then we realize $\alpha$ as an explicit formation

$$\alpha \equiv (H_{(-)^n}(\mathbb{Z}[\pi]^k), F, G)$$

and embed $k$ disjoint copies of $S^{n-1} \times D^{n+1}$, each contained in a disjoint disk $D^{2n} \subset M^{2n}$. Doing surgery on these $S^{n-1} \times D^{n+k}$ gives

$$N^{2n} = M^{2n} \# \underbrace{S^n \times S^n \# \cdots \# S^n \times S^n}_{k-times}$$

together with a manifold

$$W = I \times M^{2n} \cup_{1 \times S^{n-1} \times D_1^{n+1}} D^n \times D^{n+1} \cup \cdots \cup_{S^{n-1} \times D_k^{n+1}} D^n \times D^{n+1}$$

which gives a bordism between the two manifolds. The framings are given so that there is a normal map to $I \times M$ extending the identity map on $0 \times M$. Next, in the $\pi$-cover, embed spheres $S^n \subset N^{2n}$ realizing the generators of the kernel $G$. This makes sense since the surgery kernel of the normal map from $N$ to $M$ is $H_{(-)^n}(\mathbb{Z}[\pi]^k)$ with the first kernel $F$ being the image of the boundaries of the $D^{n+1}$'s in the handles.

Since $G$ is a quadratic lagrangian, we can write $H_{(-)^n}(F) = H_{(-)^n}(G)$ from 28, surgery can be done on the generators of $G$, and the effect of surgery on $G$ cancels the surgery kernel, thus giving a homotopy equivalence of the surgered manifold to $M^{2n}$. The union of the traces of these two surgeries is the manifold $W$, and attempting to do surgery on the interior of $W$ leads to the formation of (6).

**Realizing the obstructions in even dimensions.** Now suppose that we are given $M^{2n-1}$, closed, compact, oriented, with empty boundary so that $\pi_1(M) = \pi$ and $w$ is the orientation covering. Let $\alpha \in L^h_{2n}(\mathbb{Z}[\pi], w)$ be represented by the form

$$\alpha \equiv (\mathbb{Z}[\pi]^k; s, T).$$

Then, as before start with $k$ disjointly embedded $S^{n-1} \times D^n$'s, each embedded in a disjoint disk $D^{2n-1} \subset M$. Take the universal cover, and select lifts of each of these $S^{n-1} \times D^n$. Then, use $T$ to modify these embedding by attaching tubes between the representing $S^{n-1}$ and a linking sphere about $(gS^{n-1})_i$, according to the coefficients in the matrix $T$. All of this can be done so that the modified embeddings remain disjoint of course, and for each sphere let $D^n$ be an embedded disk in $M^{2n}$ for which it is the boundary. This disk is obtained from a disk in the $D^{2n-1}$ containing the original sphere together with obvious disks attached for each linking. Next, modify the framings on the normal bundles to the resulting spheres so that the resulting normal bundles to the core spheres in the handle body given by doing surgery on these $S^{n-1} \times D^n$'s are also given by $T$. (Alternately, we can do this by constructing attaching further self-linkings of the sphere to itself.) In any case, the resulting handle-body, $W$, will have the property that there is a degree one normal map $F\colon W \to I \times M$ with $\partial W = M_1 \sqcup -M$, $F$ restricted to $-M$ is the identity and $F|M_1$ is a homotopy equivalence onto $1 \times M$. Indeed, the exact sequence of kernels becomes

$$0 \longrightarrow K_n(F|M_1) \longrightarrow \mathbb{Z}[\pi]^k = K_n(F) \xrightarrow{s} K_n(W, \partial W) \cong K_n(f)^* \xrightarrow{\partial} K_{n-1}(F|M_1) \longrightarrow 0$$

and since $s$ is an isomorphism, both $K_n(F|M_1)$ and $K_{n-1}(F|M_1)$ are 0.

Thus we have succeeded in realizing $\alpha$ in this case as well, again regarded as the obstruction to doing surgery on the interior of $W$ to obtain a homotopy equivalence of pairs

$$F'\colon (W', \partial W = \partial W') \longrightarrow (I \times M, 1 \times M \sqcup -0 \times M).$$

Example 65. Let us realize the generator of $L_8^h(\mathbb{Z}[\mathbb{Z}/2], 1)$ studied in 61, which is given explicitly as

$$(\mathbb{Z}[\mathbb{Z}/2]^8, E_{8,0}, T)$$

where $T$ is the matrix of (61) using $\mathbb{RP}^7$ for example as the manifold $M^7$. Then the two boundary components of $W$ are homotopy equivalent and we can glue them together using an orientation preserving equivalence.

The resulting complex (there is no reason to assume that this homotopy equivalence is a diffeomorphism) has fundamental group $\mathbb{Z}/2 \times \mathbb{Z}$ and we can embed an $S^1 \times D^7$ in this complex, where the $S^1$ represents the new generator in $\pi_1$, since we can assume that the homotopy equivalence is the identity in a neighborhood of a given point of $\mathbb{RP}^7$. Doing surgery on this $S^1 \times D^7$ gives us a new 8-dimensional oriented Poincaré duality complex $\bar{W}^8$ with $\pi_1(\bar{W}) = \mathbb{Z}/2$. But $\bar{W}^8$ has been so constructed that it has multi-signature $(8, 0)$. (Alternately, this is the same as saying that both $\bar{W}^8$ and its universal cover have index 8.)

However, a theorem of Wall shows that every closed, oriented, $4k$-dimensional differentiable manifold, $M^{4k}$, with $\partial M = \emptyset$, and with fundamental group $\mathbb{Z}/2$ has the property that the index of the universal cover $\tilde{M}$ is twice the index of $M$. (This is quite easy. One simply looks at the torsion free subgroup of $\Omega_*(B_{\mathbb{Z}/2})$ and notes that the quotient of this group by the elements coming from $\Omega_*(pt)$ is, in each dimension, a finite 2-group. But if $M^{4k} \to B_{\mathbb{Z}/2}$ factors through the point map then the associated cover is just the disjoint union of two copies of $M^{4k}$ projecting onto $M$.)

Remark . Later work on the structure of the piecewise linear and topological cobordism rings, [REFERENCE], [REFERENCE], shows that the same property holds here - if $\pi$ is a finite group then the quotient of $\Omega_*^{PL}(B_\pi)$ by $\Omega_*^{PL}(pt)$ is again finite in each dimension, and similarly for $\Omega_*^{TOP}(B_\pi)$. Consequently, if $\bar{W}$ were even a topological manifold it would have to have the property that the signature of the universal cover is twice that of $\bar{W}$.

We conclude that $\bar{W}$ is an example of a closed compact Poincaré duality complex with empty boundary that does not have the homotpy type of any manifold, whether differentiable, piecewise linear, or topological.

As a consequence the two boundary components of $W$, $\mathbb{RP}^7$ and the manifold, $\overline{\mathbb{RP}}^7$, homotopy equivalent to $\mathbb{RP}^7$ cannot be diffeomorphic, piecewise linearly homeomorphic, or even homeomorphic. Of course the universal cover of $\overline{\mathbb{RP}}^7$ is diffeomorphic to the Milnor sphere, but it is piecewise linearly homeomorphic to the ordinary sphere, so this construction shows the existence of two distinct, piecewise linear, free actions of $\mathbb{Z}/2$ on the piecewise linear $S^7$ which are not even topologically conjugate.

Example 66. If we realize the orthogonal sum of 28 copies of the quadratic form $(\mathbb{Z}[\mathbb{Z}/2]^8, E_{8,0}, T)$, then the universal cover of the exotic $\mathbb{RP}^7$ will be the ordinary differentiable $S^7$, but Wall's result still applies, so this exotic $\mathbb{RP}^7$ cannot be diffeomorphic to the ordinary $\mathbb{RP}^7$ and we have constructed a new free action of $\mathbb{Z}/2$ on $S^7$ which is not topologically conjugate to the usual action.

In constructing $\bar{W}$ of 65 we did not consider the question of the triviality or non-triviality of the set of degree one normal maps over $\bar{W}$, which is the same as the question of whether there is a reduction of the Spivak normal bundle to a vector bundle. It turns out that there is only a single obstruction which lies in dimension 3 to this reduction. In fact, the obstruction is represented by a characteristic class $k_3 \in H^3(B_{SG}; \mathbb{Z}/2)$ but we don't know if it is zero or not for this particular bundle.

Thus there are two possibilities. First, if the Spivak normal bundle reduces to a vector bundle, then there is an 8-dimensional Poincaré complex with a degree one normal problem over it that has surgery obstruction the generator $(\mathbb{Z}[\mathbb{Z}/2]^8, E_{8,0}, T)$ above. Second, the exotic characteristic class $k_3$ is non-zero for the Spivak normal bundle to $\bar{W}$.

In either case it appears clear that

$$(\mathbb{Z}[\mathbb{Z}/2]^8, E_{8,0}, T) \perp (\mathbb{Z}[\mathbb{Z}/2]^8, E_{8,0}, T)$$

is the surgery obstruction to doing surgery on a degree normal map to a closed, oriented Poincaré duality complex $\bar{V}$ with fundamental group $\mathbb{Z}/2$.

In general, the question of which elements in $L^h_m(\mathbb{Z}[\pi], w)$ can be the surgery obstruction to degree one normal problems over closed Poincaré duality complexes with empty boundaries and fundamental group $\pi$ appears to be a very difficult problem.

Later we will answer this question completely in the case where $\pi$ is finite but we must assume that the Poincaré duality complex $X$ is actually a closed, compact differentiable or topological manifold.

## 7. Surgery Exact Sequence and Bordism when $\pi_1(X) \neq 0$

In this section we extend the surgery exact sequence of 4, more exactly, 43 and 4, to the non-simply connected case. For the most part the extension is routine. However, two significant differences complicate the results considerably. First is the problem that the two boundary components of an $h$-cobordism need not be homeomorphic or diffeomorphic. This makes it necessary to consider the distinction between $h$ and $s$-cobordisms, and, optionally, to introduce more sensitive surgery obstruction groups, the groups $L_n^s(\mathbb{Z}[\pi_1(X)], w)$. The second difficulty is that, unlike the case $\pi_1(X) = \{1\}$, the elements of $L_n^h(\mathbb{Z}[\pi], w)$ or $L_n^s(\mathbb{Z}[\pi], w)$ need not be realized by surgery problems over closed manifolds, though we will see that they can all be realized by surgery problems of the form

$$[(H, b) \mid H\colon (W^n, \partial W) \longrightarrow (M^n, \partial M)]$$

with $H$ restricted to $\partial W$ a homotopy equivalence to $\partial M$.

As in 4 we have the set $\mathcal{HD}(X^n)$ for any closed Poincaré duality complex with $\partial(X) = \emptyset$, that is homotopy equivalent to a closed compact differentiable manifold with empty boundary, and we have the set $\mathcal{HT}(X^n)$ when $X$ has the homotopy type of a finite piecewise linear manifold with empty boundary. But since, in both cases the equivalence relation is that of normal $h$-cobordisms, and, as observed, the two boundary components in an $h$-cobordism need not be diffeomorphic or even piecewise linearly homeomorphic in the non-simply connected case, the geometric objects represented by these sets are unclear. Thus we also introduce a refinement:

Definition 64. (1) *The set $\mathcal{SD}(X)$ is the set of closed, compact, differentiable manifolds with empty boundary together with degree one normal map $(f, b)$ with $f\colon M^n \to X$ a simple homotopy equivalence $f\colon M^n \to X$, where the equivalence relation is normal $s$-cobordism.*

(2) *The set $\mathcal{ST}(X)$ is defined similarly as the set of closed, compact, piecewise linear manifolds with empty boundary together with a degree one normal map $(f, b)$ as above, with the same notion of equivalence.*

As above, we can define surgery obstruction groups $L_n^s(\mathbb{Z}[\pi_1(X)], w)$, using forms and formations, together with well defined surgery obstruction maps,

$$\begin{aligned} \sigma(f,b)\colon [X, G/O] &\longrightarrow L_n^s(\mathbb{Z}[\pi_1(X)], w) \\ \sigma(f,b)\colon [X, G/PL] &\longrightarrow L_n^s(\mathbb{Z}[\pi_1(X)], w). \end{aligned}$$

Here, the only difference is that we are restricted by the constraint that the normal bordisms between homotopy equivalences must preserve the simple homotopy type of $M$. This puts a restriction on the types of basis changes that can occur. The reader can easily work out the details, or see [WALL].

**Surgery exact sequences.** In as much as the proofs in the simply connected case and the cases where $\pi_1(X) \neq \{1\}$ are identical we simply record the results. We have the following exact sequences:

$$\begin{aligned}
\cdots \longrightarrow [\Sigma X, G/O] &\xrightarrow{\sigma} L^h_{n+1}(\mathbb{Z}[\pi], w) \longrightarrow \\
\mathcal{HD}(X) \longrightarrow [X, G/O] &\xrightarrow{\sigma} L^h_n(\mathbb{Z}[\pi], w) \\
\cdots \longrightarrow [\Sigma X, G/O] &\xrightarrow{\sigma} L^s_{n+1}(\mathbb{Z}[\pi], w) \longrightarrow \\
\mathcal{SD}(X) \longrightarrow [X, G/O] &\xrightarrow{\sigma} L^s_n(\mathbb{Z}[\pi], w) \\
\cdots \longrightarrow [\Sigma X, G/PL] &\xrightarrow{\sigma} L^h_{n+1}(\mathbb{Z}[\pi], w) \longrightarrow \\
\mathcal{HT}(X) \longrightarrow [X, G/PL] &\xrightarrow{\sigma} L^h_n(\mathbb{Z}[\pi], w) \\
\cdots \longrightarrow [\Sigma X, G/PL] &\xrightarrow{\sigma} L^s_{n+1}(\mathbb{Z}[\pi], w) \longrightarrow \\
\mathcal{ST}(X) \longrightarrow [X, G/PL] &\xrightarrow{\sigma} L^s_n(\mathbb{Z}[\pi], w)
\end{aligned}$$

One remark needs to be made here. One can construct an $h$-cobordism over $M$ with any desired Whitehead torsion invariant by simply attaching a number of handles $D^{k+1} \times D^{n-k}$ over trivially embedded $S^k \times D^{n-k}$'s which results in a normal bordism between $M$ and $M$ connected sum with a number of $S^k \times S^{n-k}$'s. Then, in the universal cover, which is the connected sum of $\tilde{M}$ with the same number of copies of $\pi_1(X) \times S^k \times S^{n-k}$, we choose a new basis for the kernel in dimension $k$: $\mathbb{Z}[\pi_1(X)]^r$, which has determinant representing an arbitrary element in $Wh_1(\mathbb{Z}[\pi_1(X)])$, and attach handles over embedded $S^k \times D^{n-k}$'s representing these basis elements. The result gives a normal bordism between $M$ and $M'$ with the requisite torsion (or it's inverse, depending on $k$). Using this, if we can solve the $h$-cobordism problem, then we can solve the $s$-cobordism problem as well. Consequently, in terms of existence, the crucial step is solving the $L^h$-problem.

**Factoring the surgery obstruction through bordism.** As before, 4, when we have a degree one normal map

$$(f, b), \quad f\colon M^{2n+\epsilon} \longrightarrow N^{2n+\epsilon}$$

with $N^{2n+\epsilon}$ a closed, compact manifold with empty boundary and $\pi_1(N) = \pi$, whether differentiable or piecewise linear, then we have two maps associated to $(f, b)$:

- a map $g\colon N \to G/O$ or $g\colon N \to G/PL$ which measures the difference between the given normal problem and the identity problem $(id, id)$ over $N$,
- a map $h\colon N \to B_\pi$ which classifies the universal covering.

Consequently, the product map $g \times h\colon N \to G/CAT \times B_\pi$, defines an element in the bordism group $\Omega^{CAT}_{2n+\epsilon}(G/CAT \times B_\pi, w)$ where $w$ is the orientation character of $N$.

On the other hand, given a bordism of $g \times h$ we can assume, after doing surgery that the fundamental groups of both boundary components and the interior are copies of $\pi$, and, using the map into $G/CAT$ we obtain a degree one normal problem over the bordism which, on the boundary component $M$ is the original surgery problem.

Now, we can do surgery on the interior of the bordism to make it highly connected and the exact sequence of surgery kernels becomes

$$0 \longrightarrow K_{n+1}(W, \partial W) \xrightarrow{\partial} K_n(\partial W) \longrightarrow K_n(W) \longrightarrow 0$$

for $n$ even, and

$$0 \longrightarrow K_{n+1}(\partial W) \longrightarrow K_{n+1}(W) \longrightarrow K_{n+1}(W, \partial W) \xrightarrow{\partial} K_n(\partial W) \longrightarrow 0$$

for $n$ odd. In the first case, the image of $\partial$ is a quadratic lagrangian, so

$$K_n(\partial W) = K_n(M) \perp -K_n(M')$$

is trivial, and in the second case we've seen in the discussion following (5) that this picture results in a trivial surgery obstruction. Thus we have shown that in the case where the surgery problem lies over a manifold the surgery obstruction factors through bordism.

REMARK . In the following chapters we will see how, using product formulae, this fact will enable us to get explicit formulae for the surgery obstructions over closed manifolds with fundamental group $\pi$ and orientation character $w$ for many classes of $\pi$. In particular, explicit formulae exist for all finite groups $\pi$.

CHAPTER 13

# The Instant Surgery Obstruction and Product Formulae

To this point we have discussed the original development of surgery theory, initiated in the late 1950's and carried to the point of understanding the definitions of the surgery obstruction groups and the foundations of the theory. From this point, in order to get further it was necessary to have some control of the $L$-groups, $L_n^h(\mathbb{Z}[\pi], w)$, and to be able to calculate the surgery obstructions to degree one normal problems without having to do sugery to make the maps highly connected. The main objective of the current chapter is to discuss this last point. We will show how ideas and techniques due primarily to the authors allow one to define the surgery obstruction essentially instantly, from the map $f\colon M^n \to X$ and one further bit of homotopy theoretic data which is often readily available. Then we will apply this result to give product formulae for the surgery obstructions to problems of the form $N \times (f, b)$, where $f\colon M^n \to X$ is producted with the manifold $N$ to give

$$id \times f\colon N \times M^n \longrightarrow N \times X,$$

while $b$ is replaced by $id \times b$. In the majority of the applications these product formulae are used starting with $(f, b)$ where $X$ is simply connected and $N$ contains the possibly non-trivial fundamental group.

In any case, in order to develop these results we have to review the basic facts of iterated loop space theory, which will occupy us for most of this chapter.

## 1. Review of Iterated Loop Space Theory

Let $x_1, \ldots, x_r$ be points in the pointed space $X$ which we will assume throughout this chapter is a connected $CW$-complex with base point a 0-cell. Also, assume that $y_1, \ldots, y_r$ are *distinct* points in the interior of the unit disk $D^n \subset \mathbb{R}^n$. Let $w = \min_{i \neq j} ||y_i - y_j||$, and set $N(y_1, \ldots, y_r) = \frac{w}{4}$. As a consequence, the closed disks $D_N(y_1), \ldots, D_N(y_r)$ of radius $N = N(y_1, \ldots, y_r)$ centered at the points $y_i$ are disjoint.

Write local coordinates in the disk, $D_N(y_i)$ in the form $\vec{v} \mapsto \vec{v} + \vec{y}_i$ where $\vec{v}$ is an arbitrary point in the closed disk or radius $N$ about $\vec{0}$.

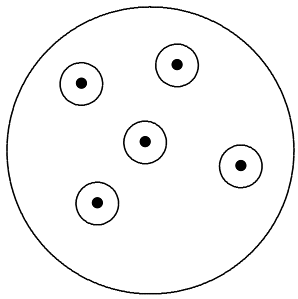
Given an $r$-tuple of points $x_1, \ldots, x_r$ together with the $y_1, \ldots, y_r$ above we can define a map

$$f_{(y_1, \ldots, y_r), (x_1, \ldots, x_r)}\colon S^n \longrightarrow \Sigma^n X$$

by the formula

$$f(\theta) \;=\; \begin{cases} (\frac{1}{N}\vec{v}, x_i) & \text{if } \theta = \vec{y_i} + \vec{v} \in D_N(y_i), \\ * & \text{the base point of } \Sigma^n X \text{ otherwise.} \end{cases}$$

EXAMPLE 67. In the case $n = 2$ and $r = 5$ we have the following situation



where each little disk is linearly expanded to the unit disk $D^2 \times y_i$ for the $y_i$ corresponding to the center of the little disk, and the remaining points in the big disk are all mapped to the base point.

This map has the following two properties as a function of the $x_i$ and the $y_j$.

(1) If $x_i = *$ then

$$f_{(y_1,\ldots,y_r),(x_1,\ldots,x_r)} \sim f_{(y_1,\ldots,\hat{y}_i,\ldots y_r),(x_1,\ldots,\hat{x}_i,\ldots x_r)}.$$

In words, this says that the original map is identical to the map associated to the $(r-1)$-tuples where $x_i$ and $y_i$ are deleted, except possibly for some evident rescaling, since the disks about the $y_j$ may increase in size slightly.

(2) If we permute the $y_j$ by the permutation $\sigma \in \mathcal{S}_r$ and simultaneously permute the $x_j$ by the same permuation, then the associated maps are identical:

$$f_{(y_{\sigma 1},\ldots,y_{\sigma r}),(x_{\sigma 1},\ldots,x_{\sigma r})} = f_{(y_1,\ldots,y_r),(x_1,\ldots,x_r)}.$$

The space of all ordered $r$-tuples of distinct points of $\text{Int}(D^n) = S^n - \infty$ is denoted $F_r(S^n - \infty)$ and the construction above gives a continuous map

$$h_r \colon F_r(S^n - \infty) \times_{\mathcal{S}_r} X^r \longrightarrow \Omega^n \Sigma^n X$$

where $\Omega^n Y$ is the $n$-fold loop space of $Y$, the space of all based continuous maps $f \colon S^n \longrightarrow Y$, with the compact-open topology.

REMARK . There is a generalization of this to give a map,

$$T_r \colon F_r(S^n - \infty) \times_{\mathcal{S}_r} (\Omega^n X)^r \longrightarrow \Omega^n X$$

defined by

$$T_r((y_1,\ldots,y_r),(f_1,\ldots,f_r)) = \begin{cases} f_i(\frac{1}{N}\vec{v}) & \text{if } \theta = \vec{y_i} + \vec{v} \text{ is in the disk } D_N(y_i), \\ * & \text{the base point of } X \text{ otherwise} \end{cases}$$

which just uses the map $f_i$ appropriately scaled in the disk of radius $N$ about $y_i$ and is base point otherwise.

Note that $F_r(S^n - \infty)$ is $(n-2)$-connected since it can be given as an iterate fibration:

$$\begin{array}{ccc}
\mathbb{R}^n - (y_1, \ldots, y_{r-1}) & \longrightarrow & F_r(S^n - \infty) \\
 & & \downarrow \\
\mathbb{R}^n - (y_1, \ldots, y_{r-2}) & \longrightarrow & F_{r-1}(S^n - \infty) \\
 & & \downarrow \\
 & & \vdots \\
 & & \downarrow \\
\mathbb{R}^n - y_1 & \longrightarrow & F_2(S^n - \infty) \\
 & & \downarrow \\
 & & S^n - \infty
\end{array}$$

and each fiber is $(n-2)$-connected while the base is contractible. Also, $F_r(S^n - \infty)$ is free under the action of $\mathcal{S}_r$. Incidently, the notation in the diagram should be clear. Each vertical arrow is a fibration with fiber given by the domain of the horizontal arrow.

For $\epsilon > 0$ sufficiently small we can replace the space $F_r(S^n - \infty)$ by the space $F_r(S^n - \infty, \epsilon)$ consisting of all $r$-tuples $(y_1, \ldots, y_r) \in F_r(S^n - \infty)$ such that $N(y_1, \ldots, y_r) \geq \epsilon$, and, comparing the iterate fibration associated to $F_r(S^n - \infty, \epsilon)$ with that of (1), it is directly shown that the inclusion $F_r(S^n - \infty, \epsilon) \hookrightarrow F_r(S^n - \infty)$ is a homotopy equivalence. Consequently, the inclusion induces a homotopy equivalence

$$F_r(S^n - \infty, \epsilon) \times_{\mathcal{S}_r} X^r \hookrightarrow F_r(S^n - \infty) \times_{\mathcal{S}_r} X^r$$

and for $s \leq r$ we can consider the construction

$$C_r(S^n, \infty, X, *, \epsilon) \;=\; \coprod_1^r F_s(S^n - \infty, \epsilon) \times_{\mathcal{S}_s} X^s / \sim$$

where $\sim$ is the equivalence relation given by

$$\{(y_1, \ldots, y_r), (x_1, \ldots, x_r)\} \sim \quad \{(y_1, \ldots, \hat{y}_i, \ldots, y_r), (x_1, \ldots, \hat{x}_i, \ldots, x_r)\} \quad \text{if } x_i = *.$$

Then, clearly the maps $h_s^\epsilon$ defined on the $F_s(S^n - \infty, \epsilon)$ in the same way as $h_s$ except that we replace $N(y_1, \ldots, y_s)$ by $\epsilon/4$ all fit together to give a map

$$E_r^\epsilon \colon C_r(S^n, \infty, X, *, \epsilon) \longrightarrow \Omega^n \Sigma^n X.$$

Now, define

$$C(S^n, \infty, X, *) \;=\; \coprod_1^\infty F_s(S^n - \infty) \times_{\mathcal{S}_s} X^s / \sim$$

where $\sim$ is the equivalence relation of (1). Passing to limits and checking the existence of compatible homotopies for the varying $\epsilon$'s we obtain a natural map

$$E\colon C_r(S^n, \infty, X, *) \longrightarrow \Omega^n \Sigma^n X$$

and the main result of iterated loop space theory is that this map is a homotopy equivalence when $X$ satisfies our standing assumptions.

Variations of this construction abound. One can replace $S^n$ by any $n$-dimensional connected, closed manifold with empty boundary, $M^n$, and $\infty$ by any point in $M^n$. Then one obtains

$$C(M^n, pt, X, *) = \coprod_1^\infty F_s(M^n - pt) \times_{\mathcal{S}_s} X^s / \sim$$

where $\sim$ is again the equivalence relation of (1). In this case we obtain the following theorem of D. McDuff:

**[D. McDuff]**

THEOREM 54. *Assume that $M^n$ is, as above, a closed, compact, connected manifold with empty boundary which is almost parallelizable. Then there is a natural inclusion $C(M^n, pt, X, *) \hookrightarrow Map_*(M^n, \Sigma^n(X))$ which is a homotopy equivalence of mapping spaces.*

(The idea is that $M^n - pt$ is parallelizable and choosing a trivialization of the tangent bundle $\tau_{M^n}$, the exponential map $exp\colon \tau_{M^n} \longrightarrow M^n$ gives a way of identifying a set of sufficiently small neighborhoods of each point of $M^n - pt$ with the disk $D^n$. Consequently, a collection of $r$ distinct points in $M^n - pt$ together with $r$ corresponding points of $X$ and an $\epsilon$ sufficiently small that the $\epsilon$-disks about the distinct points are all disjoint give a map $M^n, pt) \longrightarrow \Sigma^n X$ as before. Then using a handle body decomposition of $M^n$ reduces this to the previous result.)

Perhaps the best discussion of these results is contained in [CARL-FRIEDRICH BÖDIGHEIMER]

In another direction, we can go to infinity with $n$ in the original construction.

In the limit as $n \mapsto \infty$ we can replace $F_r(S^n - \infty)$ by the total space of the universal cover of a classifying space for $\mathcal{S}_r$, $E_{\mathcal{S}_r}$ since $F_r(S^n - \infty)$ is $(n-1)$-connected and $\mathcal{S}_r$-free.

Define $Q(X, *) = \lim_{n \mapsto \infty} \Omega^n \Sigma^n X$. Then we get a model for $Q(X, *)$ given originally by Dyer-Lashof [REFERENCE],

$$\coprod_1^\infty E_{\mathcal{S}_s} \times_{\mathcal{S}_s} X^s / \sim \simeq Q(X, *).$$

In particular, the first two stages of the construction give the following space

$$X \cup_F S^\infty \times_{\mathbb{Z}/2} X^2$$

where the generator of $\mathbb{Z}/2$ acts as multiplication by $-1$ on $S^\infty$ and to interchange coordinates in $X^2$. Similarly, at the finite levels, the first two stages of $C(S^n, \infty, X, *)$ give

$$X \cup_F S^{n-1} \times_{\mathbb{Z}/2} X^2.$$

The construction has been modified to include the situation where $X$ is not connected but $X$ is the union of a simply conneced space $Y$ with a disjoint base point $*$. In this case $Q(Y_+)$ has the form $\mathbb{Z} \times W(Y)$ where the components are all of the form

$$W(Y) \simeq (E_{\mathcal{S}_\infty} \times_{\mathcal{S}_\infty} X^\infty)^+ .$$

We now explain the $+$ on the space above. To begin we attach one 2-cell to kill the homotopy class of the permutation $t = (1, 2, 3) \in \mathcal{S}_\infty$. The normal closure of this element is the alternating group $\mathcal{A}_\infty$, and so the fundamental group of the resulting space is $\mathbb{Z}/2$. Also, the homology class represented by the 2-cell we just attached becomes spherical, and we can attach a 3-cell to kill this spherical class. Then the plus means the result of attaching this 2-cell and 3-cell.

Remark . The attaching of the two cells above does not change the homology of $E_{\mathcal{S}_\infty} \times_{\mathcal{S}_\infty} (X^\infty)$ so that the suspensions of these two spaces are homotopy equivalent. On the other hand, V. Snaith proved the following splitting theorem.

Theorem 55. *[REFERENCE] [REFERENCE] Let $X$ be a connected CW-complex with base point $*$. Then there is a splitting natural in $X$:*

$$\Sigma^\infty E_{\mathcal{S}_\infty} \times_{\mathcal{S}_\infty} X^\infty \simeq \bigvee_1^\infty \Sigma^\infty E_{\mathcal{S}_n} \times_{\mathcal{S}_n} \underbrace{X \wedge X \wedge \cdots \wedge X}_{n-times}.$$

Thus, given a map $f \colon X \to Q(Y_+)$ we have a sequence of natural projections

$$h_n(f) \colon \Sigma^\infty X \longrightarrow \Sigma^\infty E_{\mathcal{S}_n} \times_{\mathcal{S}_n} \underbrace{X \wedge X \wedge \cdots \wedge X}_{n-times}$$

called the **higher Hopf invariants of $f$.**

We need to make this splitting explicit because we need a variant of it for the case $X = Y_+$. Thus, let

$$I_k = (i_1, \ldots, i_k), \qquad 1 \le i_1 < i_2 < i_3 < \cdots < i_k \le r$$

be any $k$-tuple of distinct points in the set $\{1, 2, \ldots, n\}$, and define the projection $\pi_{I_k} \colon F_r(\mathbb{R}^n) \to F_k(\mathbb{R}^n)$ in the evident way:

$$\pi_{I_k}(y_1, \ldots, y_r) = (y_{i_1}, y_{i_2}, \ldots, y_{i_k}).$$

Note that

$$\pi_{I_k}(y_1, \dots, y_r) \neq \pi_{I'_k}(y_1, \dots, y_r)$$

unless $I_k = I'_k$ since all the $y_j$ are distinct. Consequently, with the lexiographic ordering on the $I_k$, we have that the product map

$$\prod_{I_k} \pi_{I_k} : F_r(\mathbb{R}^n) \longrightarrow \prod_{1}^{\binom{r}{k}} F_k(\mathbb{R}^n)$$

actually has image contained in $F_{\binom{r}{k}}(F_k(\mathbb{R}^n))$.

Further, $F_k(\mathbb{R}^n) \subset \mathbb{R}^{nk}$, and using the diagonal map

$$\Delta \colon F_k(\mathbb{R}^n) \hookrightarrow \mathbb{R}^{nk} \times F_k(\mathbb{R}^n)$$

we extend the map above to a map of the form

$$\Theta_{k,r} \colon F_k(\mathbb{R}^n) \longrightarrow F_{\binom{r}{k}}(\mathbb{R}^{kn}) \times F_k(\mathbb{R}^n)^{\binom{r}{k}}.$$

We now investigate the way the action of $\mathcal{S}_r$ on $F_k(\mathbb{R}^n)$ commutes with $\Theta_{k,r}$. First, let $\alpha \in \mathcal{S}_r$ be an arbitrary permuation. Then it defines a permutation of the elements in the power set of all distinct $k$-tuples $I_k$,

$$\pi(\alpha)(\{i_1, \dots, i_k\}) = \{i_{\alpha^{-1}(1)}, \dots, i_{\alpha^{-1}(k)}\}.$$

Of course, it is not generally true that $i_{\alpha^{-1}(1)} < i_{\alpha^{-1}(2)} < \cdots < i_{\alpha^{-1}(k)}$, so – for each $I_k$, $\alpha$ also defines a $\beta(\alpha, I_k)$ which puts these $i_{\alpha^{-1}(j)}$ into ascending order. Then the map into the wreath product

$$\mathcal{S}_{\binom{r}{k}} \wr \mathcal{S}_k \;=\; \mathcal{S}_{\binom{r}{k}} \times \mathcal{S}_k^{\binom{r}{k}}$$

with multiplication given by

$$(\lambda, (t_1, \dots, t_{\binom{r}{k}}))(\tau, (s_1, \dots, s_{\binom{r}{k}})) = (\lambda\tau, (t_{\lambda^{-1}(1)} s_1, \dots, t_{\lambda^{-1}(\binom{r}{k})} s_{\binom{r}{k}}))$$

defined by

$$\alpha \mapsto (\pi(\alpha), (\beta(\alpha, I_1), \dots, \beta(\alpha, I_{\binom{r}{k}}))$$

is actually a homomorphism, known as the Frobenius homomorphism.

We now extend $\Theta$ to give a map from $F_r(\mathbb{R}^n) \times_{\mathcal{S}_r} X^r$. To do this, let $I_k$ be an element in the power set of all subsets with $k$ elements of the set $\{1, 2, \dots, r\}$, and define

$$I_k \colon X^r \longrightarrow X^k$$

by $I_k(x_1, \dots, x_r) = (x_{i_1}, \dots, x_{i_k})$. Then we have the extension of $\Theta$:

$$\bar{\Theta} \, : F_r(\mathbb{R}^n) \times_{\mathcal{S}_r} X^r \longrightarrow F_{\mathcal{S}_{\binom{r}{k}}}(\mathbb{R}^{nk}) \times_{\mathcal{S}_{\binom{r}{k}}} \left(F_k(\mathbb{R}^n) \times_{\mathcal{S}_k} X^k\right)^{\binom{r}{k}}$$

by

$$((y_1, \dots, y_r), (x_1, \dots, x_r)) \mapsto \Theta(y_1, \dots, y_r), I_1(\vec{x}), \dots, I_{\binom{r}{k}}(\vec{x}).$$

Also, note that the adjoint map

$$\mathrm{Adj}(id) \colon \Sigma^n \Omega^n X \longrightarrow X$$

defined by $\mathrm{Adj}(id)(\vec{t}, f) = f(\vec{t})$ can be applied to the composition $E\bar{\Theta}$ where $E$ is the map of (1) so we obtain a **transfer map**:

$$T_{r,k} \colon \Sigma^{nk}\left(F_r(\mathbb{R}^n) \times_{\mathcal{S}_r} X^r\right) \longrightarrow \Sigma^{nk}\left(F_k(\mathbb{R}^n) \times_{\mathcal{S}_k} X^k\right)$$

which is the core of Snaith splitting. In fact we have

LEMMA 76. *Let $x \in X$ be any base point. Consider the inclusion*

$$J_x \colon F_k(\mathbb{R}^n) \times_{\mathcal{S}_k} X^k \longrightarrow F_r(\mathbb{R}^n) \times_{\mathcal{S}_r} X^r,$$

*defined on points by*

$$\{(y_1, \ldots, y_k), (x_1, \ldots, x_k)\} \mapsto \{(y_1, \ldots, y_k, z_{k+1}, \ldots, z_r), (x_1, \ldots, x_k, x, \ldots, x)\}$$

*where $z_i = (Max(||y_i||) + i, 0, \ldots, 0) \in \mathbb{R}^n$. Next, consider the composition $\pi\Sigma^{nk}J_x$:*

$$\Sigma^{nk}(F_k(\mathbb{R}^n) \times_{\mathcal{S}_k} X^k) \xrightarrow{\Sigma^{nk}J_x} \Sigma^{nk}(F_r(\mathbb{R}^n) \times_{\mathcal{S}_r} X^r) \xrightarrow{Adj(id)} \Sigma^{nk}(F_k(\mathbb{R}^n) \times_{\mathcal{S}_k} X^k) \xrightarrow{\pi} \Sigma^{nk}(F_k(\mathbb{R}^n) \ltimes_{\mathcal{S}_k} X^{(k)})$$

*where $X^{(k)}$ is the smash product of $X$ with itself $k$-times with respect to the base point $x$, and $\ltimes$ denotes that when $X^{(k)}$ is at base point we collapse everything to base point. Then this composition factors through*

$$\Sigma^{nk} F_k(\mathbb{R}^n) \ltimes_{\mathcal{S}_k} X^{(k)},$$

*and on this quotient is a homotopy equivalence.*

PROOF. Note that if we take a point of the form

$$((y_1, \ldots, y_r), (x_1, \ldots, x_k, *, \ldots, *)),$$

then $I_k(x_1, \ldots, x_k, *, \ldots, *)$ contains at least one $*$ unless $I_k = (1, \ldots, k)$. Consequently, when we project to

$$F_{\binom{r}{k}}(\mathbb{R}^{nk}) \times_{\mathcal{S}_{\binom{r}{k}}} \left(F_k(\mathbb{R}^n) \ltimes_{\mathcal{S}_k} X^{(k)}\right),$$

the image has the form

$$\{\Theta(y_1, \ldots, y_r), \{(y_1, \ldots, y_k), (x_1, \ldots, x_k)\}, \underbrace{*, \ldots, *}_{\binom{r}{k}-1 \text{ times}})\}$$

and this is identified with

$$\{(y_1, \ldots, y_k), \{(y_1, \ldots, y_k), (x_1, \ldots, x_k)\}\} \sim \{(y_1, \ldots, y_k), (x_1, \ldots, x_k)\}$$

which completes the proof. □

This result more or less directly gives Snaith splitting. But the prospective application of splitting to surgery required this very precise description. We turn to this application now.

## 2. The quadratic reduction of the surgery kernel

Let the group $G$ act on $X$. Then there is an induced action of $G$ on $F_k(\mathbb{R}^n) \times_{\mathcal{S}_k} X^k$ by

$$g(\{(y_1, \ldots, y_k), (x_1, \ldots, x_k)\}) = \{(y_1, \ldots, y_k), (g(x_1), \ldots, g(x_k))\}$$

and the map $\bar{\Theta}$ of 1 is $G$-equivariant, so it induces a reduced map

$$X/G \longrightarrow F_{\binom{r}{k}}(\mathbb{R}^{nk}) \times_{\mathcal{S}_{\binom{r}{k}}} F_k(\mathbb{R}^n) \times_{\mathcal{S}_k} (X^k/\Delta(G))$$

for each $k$ with $1 \leq k \leq r$.

### 3

In surgery we are interested in the case where $Y$ is the universal cover of a manifold $M^n$ and $\pi_1(M)$ acts freely on $Y$. In this case, letting $\pi_1(M)$ act freely on the disjoint base point, there is a semi-free action of $\pi_1(M)$ on $\Sigma^n(Y_+)$ with the base point as the only fixed point. This action induces an action of $\pi_1(M)$ on $Q(Y_+)$, which restricted to our model is simply given by the identity on $E_{\mathcal{S}_\infty}$ and the diagonal action on $X^\infty$. In particular, the higher Hopf invariant maps above are equivariant with respect to the

One aspect of this naturality will be particularly important to us. Suppose that $X$ has a free