Absolute Stability of Nonlinear Control Systems

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Absolute Stability of Nonlinear Control Systems

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PREFACE

As is well-known, a control system always works under a variety of accidental or continued disturbances. Therefore, in designing and analysing the control system, stability is the first thing to be considered. Classic control theory was basically limited to a discussion of linear systems with constant coefficients. The fundamental tools for such studies were the Routh-Hurwitz algebraic criterion and the Nyquist geometric criterion. However, modern control theory mainly deals with nonlinear problems. The stability analysis of nonlinear control systems based on Liapunov stability theory can be traced back to the Russian school of stability.

In 1944, the Russian mathematician Lurie, a specialist in control theory, discussed the stability of an autopilot. The well-known Lurie problem and the concept of absolute stability are presented, which is of universal significance both in theory and practice. Up until the end of the 1950's, the field of absolute stability was monopolized mainly by Russian scholars such as A. I. Lurie, M. A. Aizeman, A. M. Letov and others. At the beginning of the 1960's, some famous American mathematicians such as J. P. LaSalle, S. Lefschetz and R. E. Kalman engaged themself in this field. Meanwhile, the Romanian scholar Popov presented a well-known frequency criterion and consequently made a decisive breakthrough in the study of absolute stability. Since then, V. A. Yacubovich, R. E. Kalman, K. R. Meyer and others have devoted themselves to the study of equivalent relations between Lurie's method (integral term and quadratic Liapunov function method) and Popov's frequency method.

Although absolute stability has a history of forty years, and hundreds of related articles and quite a number of monographs have been published, these fail to match the requirement of the rapid progress of science and technology. Hitherto, there are more than enough sufficient conditions for absolute stability, but at the same time the number of known necessary and sufficient conditions is rather small and these conditions are mainly limited to the Lurie-type direct control system (i. e. , the elementary condition) and indirect control system (i. e. , the first critical case). The more complicated critical cases are rarely discussed. Among the sufficient conditions obtained, the descriptive results on existence are far more frequent than those of constructive algebraic criteria. No matter whether they are the V-function of the Lurie-type or of the Popov-type, they all contain undetermined matrices or parameters. It is quite difficult to determine these matrices or parameters. Even though the Popov frequency criterion is simple in form, it is rather difficult to put into practice — it is very complicated to set up, to calculate the inverse matrix and to verify the definite signs of the rational fraction of the undetermined parameters on the infinite interval. One of the main causes of the above mentioned difficulties is that outdated old methods are being employed instead of trying to find some newer methods. For example, some modern tools such as M-matrices, K-class functions and Dini-derivatives, the theories and methods of stability of part of the variables, and the stability of sets, have not yet been applied to the study of absolute stability.

The main purpose of this book is to introduce the latest results of the author and some others on developments in the study of absolute stability of nonlinear control systems in recent years. The characteristics of these results are: theoretically, to give as many as possible necessary and sufficient conditions of absolute stability of various nonlinear control systems; in applications, to derive simple enough and even constructive algebraic sufficient conditions from these theoretical necessary and sufficient conditions for use in practical work and in methodology. While promoting the extensive use of modern methods and tools such as M-matrices, K-class functions, Dini-derivatives, partial stability, and set stability, traditional methods and results will not be neglected.

The content of this monograph is the following. Chapter 1 introduces the main tools and the principal results used in this book, such as Liapunov functions, K-class functions, Dini-derivatives, M-matrices, and the principal theorems on global stability. Chapter 2 presents the absolute stability theory of autonomous control systems and the well-known Lurie problem. Chapter 3 gives some simple algebraic necessary and sufficient conditions for absolute stability of several special control systems. Chapter 4 discusses non-autonomous and discrete control systems. Chapter 5 deals with the absolute stability of control systems with m nonlinear control terms. Chapter 6 is devoted to the absolute stability of control systems described by functional differential equations.

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CONTENTS

CHAPTER 1

PRINCIPAL THEOREMS ON GLOBAL STABILITY

In this chapter, we will concentrate on introducing the main tools and the principal results used in this book such as Liapunov functions, wedge functions (or K-class functions), Dini-derivatives, M-matrices, and the principal theorems on global stability, the partial global stability, and the global stability of sets. Those tools and results play a fundamental role in the whole book.

1.1. Liapunov Functions and K-Class Functions

Suppose that $W(x) \in C[\mathbf{R}^n, \mathbf{R}]$, i.e., $W: \mathbf{R}^n \to \mathbf{R}$ is continuous, $W(0) = 0$; $V(t,x) \in C[I \times \mathbf{R}^n, \mathbf{R}]$, $V(t,0) \equiv 0$, where $I = [t_0, +\infty)$.

Definition 1.1.1. The function $W(x)$ is said to be *positive* [*negative*] *definite* if $W(x) \geqslant 0$ [$-W(x) \geqslant 0$] and $W(x) = 0$ if and only if $x = 0$. The function $W(x)$ is said to be *positive* [*negative*] *semi-definite* if $W(x) \geqslant 0$ [$-W(x) \geqslant 0$].

Definition 1.1.2. The function $W(x)$ is said to be *radially unbounded positive definite* if $W(x)$ is positive definite and $\|x\| \to +\infty$ implies $W(x) \to +\infty$.

Definition 1.1.3. The function $V(t,x)$ is said to be *positive definite* if there is a positive definite function $W(x)$ such that $V(t,x) \geqslant W(x)$. The function $V(t,x)$ is said to be *negative definite* if $-V(t,x)$ is positive definite.

Definition 1.1.4. The function $V(t,x)$ is said to have *infinitesimal upper bound* if there exists a positive definite function $W_1(x)$ such that $|V(t,x)| \leqslant W_1(x)$. The function $V(t,x)$ is said to be *radially unbounded positive definite* if there exists a radially unbounded positive definite function $W_2(x)$ such that $V(t,x) \geqslant W_2(x)$.

The positive or negative definite functions are usually called Liapunov functions.

In the following, we will introduce the K-class function.

If a function $\varphi \in [\mathbf{R}^+, \mathbf{R}^+]$ (where $\mathbf{R}^+ \triangleq [0, +\infty)$), φ being continuous, strictly monotone increasing, and $\varphi(0) = 0$, then we call φ a *K-class function*, denoted by $\varphi \in K$.

If $\varphi \in K$ and $\lim\limits_{r \to +\infty} \varphi(r) = +\infty$, then $\varphi(r)$ is called a *radially unbounded K-class function*, denoted by $\varphi \in KR$.

Among the positive definite function and the K-class function some essential equivalence relations exist:

Lemma 1.1.5. *Given a positive definite function $W(x)$, there exist two functions $\varphi_1, \varphi_2 \in K$ such that*

$$\varphi_1(\|x\|) \leqslant W(x) \leqslant \varphi_2(\|x\|). \tag{1.1.1}$$

Proof. For any $R > 0$, we prove that (1.1.1) holds on $\|x\| \leqslant R$. Let $\varphi(r) = \inf\limits_{r \leqslant \|x\| \leqslant R} W(x)$. Evidently we have $\varphi(0) = 0$, $\varphi(r) > 0$ for $r > 0$ and $\varphi(r)$ is a monotone non-decreasing function on $[0, R]$. (It may not be strictly monotone.) Now, we proceed to prove that $\varphi(r)$ is continuous. Since $W(x)$ is continuous, for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\begin{aligned}
\varphi(r_2) - \varphi(r_1) &= \inf_{r_2 \leqslant \|x\| \leqslant R} W(x) - \inf_{r_1 \leqslant \|x\| \leqslant R} W(x) \\
&\triangleq \inf_{r_2 \leqslant \|x\| \leqslant R} W(x) - W(x_0) \\
&\leqslant W(x_1) - W(x_0) \\
&< \varepsilon \qquad (\|x_1 - x_0\| \leqslant r_2 - r_1)
\end{aligned}$$

if $0 \leqslant r_2 - r_1 < \delta(\varepsilon)$, and we take $x_1 = x_0$ for $x_0 \in D_2 = \{x : r_2 \leqslant \|x\| \leqslant R\}$ and x_1 to be an intersection point of the ray Ox_0 and $\|x\| = r_2$ for $x_0 \in D_1 = \{x : r_1 \leqslant \|x\| \leqslant R\}$.

Let $\varphi_1(r) \triangleq \dfrac{r\varphi(r)}{R} \leqslant \varphi(r)$. Evidently, we have $\varphi_1(0) = 0$ and if $0 \leqslant r_1 < r_2 \leqslant R$ we get

$$\varphi_1(r_1) = \frac{r_1\varphi(r_1)}{R} \leqslant \frac{r_1\varphi(r_2)}{R} < \frac{r_2\varphi(r_2)}{R} = \varphi_1(r_2).$$

Thus $\varphi_1(r)$ is strictly monotone increasing and hence $\varphi_1 \in K$. If $\psi(r) \triangleq \max\limits_{\|x\| \leqslant r} W(x)$, then it follows that $\psi(0) = 0$. By the same method, we can prove that ψ is a monotone non-decreasing and continuous function. Choosing $\varphi_2(r) \triangleq \psi(r) + kr$ $(k > 0)$ we have

$$\varphi_2(r_1) = \psi(r_1) + kr_1 \leqslant \psi(r_2) + kr_1 < \psi(r_2) + kr_2 = \varphi_2(r_2).$$

Hence, $\varphi_2(r)$ is strictly monotone increasing and $\varphi_2(r) \in K$. From above, it is inferred

$$\varphi_1(\|x\|) \leqslant \varphi(\|x\|) \triangleq \inf_{|x| \leqslant |\xi| \leqslant R} W(\xi) \leqslant W(x) \leqslant \max_{|\xi| \leqslant |x|} W(\xi)$$
$$\triangleq \psi(\|x\|) < \varphi_2(r_2).$$

Thus,

$$\varphi_1(\|x\|) \leqslant W(x) \leqslant \varphi_2(\|x\|).$$

That proof of Lemma 1.1.5 is completed. ▮

It is not difficult to prove the following lemma by the same method.

Lemma 1.1.6. *For a given radially unbounded positive definite function $W(x)$, there must exist two functions $\varphi_1(r), \varphi_2(r) \in KR$ such that*

$$\varphi_1(\|x\|) \leqslant W(x) \leqslant \varphi_2(\|x\|).$$

Consequently, without loss of generality, we usually replace the positive definite functions and the radially unbounded positive definite functions by K-class functions and radially unbounded K-class functions respectively.

1.2. Dini-Derivatives

Suppose $f(t) \in C[I, \mathbf{R}]$, $I = [t_0, +\infty)$. For any $t \in I$, the following four derivatives:

$$D^+ f(t) \triangleq \overline{\lim_{h \to 0^+}} \frac{1}{h}(f(t+h) - f(t)),$$

$$D_+ f(t) \triangleq \underline{\lim_{h \to 0^+}} \frac{1}{h}(f(t+h) - f(t)),$$

$$D^- f(t) \triangleq \overline{\lim_{h \to 0^-}} \frac{1}{h}(f(t+h) - f(t)),$$

$$D_- f(t) \triangleq \underline{\lim_{h \to 0^-}} \frac{1}{h}(f(t+h) - f(t))$$

are called *right upper derivative*, *right lower derivative*, *left upper derivative and left lower derivative* of $f(t)$ at t respectively. They are all called the *Dini-derivative*.

Sometimes the Dini-derivative may be $\pm\infty$, otherwise there always exists the Dini-derivative. In particular, when $f(t)$ satisfies the local Lipschitz condition, the four Dini-derivatives are finite. Moreover, the derivative of $f(t)$ exists if and only if the four derivatives are equal.

For a continuous function the relation between the monotoneity and the definite sign of the Dini-derivative is as follows.

Theorem 1.2.1. *If $f(t) \in C[I, \mathbf{R}]$, then $f(t)$ is monotone non-decreasing on I if and only if $D^+ f(t) \geqslant 0$ for $t \in I$.*

Proof. The necessity is obvious.

Sufficiency. First, we suppose $D^+ f(t) > 0$ on I. If there are two points α, $\beta \in I$ and $\alpha < \beta$, $f(\alpha) > f(\beta)$, then there exist μ satisfying $f(\alpha) > \mu > f(\beta)$ and some point $t \in [\alpha, \beta]$ such that $f(t) > \mu$.

Let ξ be the supremum of those points. Then $\xi \in [\alpha, \beta)$, and the continuity of $f(t)$ leads to $f(\xi) = \mu$. Therefore for $t \in [\xi, \beta]$, it follows

$$\frac{f(t) - f(\xi)}{t - \xi} < 0.$$

Hence, we have $D^+ f(\xi) \leqslant 0$ which contradicts the hypothesis. Thus, $f(t)$ is monotone non-decreasing. Next, assume that $D^+ f(t) \geqslant 0$. For any $\varepsilon > 0$ we get

$$D^+ (f(t) + \varepsilon t) = D^+ f(t) + \varepsilon \geqslant \varepsilon > 0.$$

As a consequence $f(t) + \varepsilon t$ is monotone non-decreasing on I. Since ε is arbitrary, $f(t)$ is monotone non-decreasing on I. ∎

Remark. If we replace $D^+ f(t) \geqslant 0$ by $D_+ f(t) \geqslant 0$, the sufficiency condition of Theorem 1.1.7 still holds because the latter implies the former.

Similarly, if we replace $D^+ f(t) \geqslant 0$ by $D^- f(t) \geqslant 0$, it suffices to change the supremum of the points satisfying $f(t) > \mu$ to the infimum of the points satisfying $f(t) < \mu$. We may farther intensify $D^- f(t) \geqslant 0$ to be $D_- f(t) \geqslant 0$, and thus any of the four derivatives is not less then zero, each of which implies that $f(t)$ is monotone non-decreasing.

In the following, we will consider the Dini-derivative of a function along the solution of a differential equation.

Let a system of differential equations be given by

$$\frac{dx}{dt} = f(t, x), \tag{1.2.1}$$

where $f(t, x) \in C[I \times \mathbf{R}^n, \mathbf{R}^n]$.

Theorem 1.2.2. *Suppose that $V(t, x) \in C[I \times \Omega, \mathbf{R}]$, where $\Omega \subset \mathbf{R}^n$, Ω is a neighbourhood containing the origin, and $V(t, x)$ satisfies the local Lipschitz condition in x for t (expressed by $V(t, x) \in C_0(x)$), i.e.,*

$$|V(t, x) - V(t, y)| \leqslant L \| x - y \|.$$

Then the right upper derivative and the right lower derivative of $V(t, x)$ along the *solution* $x(t)$ *of* (1. 2. 1) *have the forms*:

$$\mathrm{D}^+ V(t,x(t)) = \overline{\lim_{h\to 0^+}} \frac{1}{h}\{V(t+h,x+hf(t,x)) - V(t,x)\}, \quad (1.2.2)$$

$$\mathrm{D}_+ V(t,x(t)) = \underline{\lim_{h\to 0^+}} \frac{1}{h}\{V(t+h,x+hf(t,x)) - V(t,x)\}. \quad (1.2.3)$$

Proof. Assume that the solution $x(t)$ stays in the region $I \times \Omega$. For $(t,x) \in I \times \Omega$ and $0 < h \ll 1$, there exists a neighbourhood U of (t,x) such that $\overline{U} \subset I \times \Omega$, $(t+h,x+hf(t,x)) \in U$, $(t+h,x(t+h)) \in U$. Let L be the Lipschitz constant of $V(t,x)$ in x on $\overline{U}$. Making use of the Taylor expansion and the Lipschitz condition we arrive at

$$\begin{aligned} &V(t+h,x(t+h)) - V(t,x(t)) \\ &= V(t+h,x+hf(t,x)+h\varepsilon) - V(t,x) \\ &< V(t+h,x+hf(t,x)) + Lh|\varepsilon| - V(t,x), \end{aligned}$$

where $\varepsilon \to 0$ as $h \to 0$. Hence,

$$\begin{aligned} \mathrm{D}^+ V(t,x(t)) &\triangleq \overline{\lim_{h\to 0^+}} \frac{1}{h}[V(t+h,x(t+h)) - V(t,x(t))] \\ &\leqslant \overline{\lim_{h\to 0^+}} \frac{1}{h}[V(t+h,x+hf(t,x)) + Lh|\varepsilon| - V(t,x)] \\ &= \overline{\lim_{h\to 0^+}} \frac{1}{h}[V(t+h,x+hf(t,x)) - V(t,x)]. \qquad (1.2.4) \end{aligned}$$

On the other hand

$$\begin{aligned} &V(t+h,x(t+h)) - V(t,x(t)) \\ &= V(t+h,x+hf(t,x)+h\varepsilon) - V(t,x) \\ &\geqslant V(t+h,x+hf(t,x)) - Lh|\varepsilon| - V(t,x). \end{aligned}$$

Thus

$$\begin{aligned} \mathrm{D}^+ V(t,x(t)) &\triangleq \overline{\lim_{h\to 0^+}} \frac{1}{h}[V(t+h,x(t+h)) - V(t,x(t))] \\ &\geqslant \overline{\lim_{h\to 0^+}} \frac{1}{h}[V(t+h,x+hf(t,x)) - V(t,x)]. \qquad (1.2.5) \end{aligned}$$

Combining (1. 2. 4) with (1. 2. 5), we find that

$$\mathrm{D}^+ V(t,x(t)) = \overline{\lim_{h\to 0^+}} \frac{1}{h}[V(t+h,x+hf(t,x)) - V(t,x)].$$

Thus, (1. 2. 2) is true. The proof of (1. 2. 3) goes along the same line. We have

$$\mathrm{D}_+ V(t,x(t)) = \underline{\lim_{h\to 0^+}} \frac{1}{h}[V(t+h,x+hf(t,x)) - V(t,x)]. \qquad \blacksquare$$

If $V(t,x)$ has a continuous partial derivative of the first order, then along the solution $x(t)$ of (1. 2. 1)

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(1.2.1)} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot f(t,x) = \frac{\partial V}{\partial t} + \operatorname{grad} V \cdot f(t,x)$$

and

$$\mathrm{D}^{+} V(t,x(t)) = \mathrm{D}_{+} V(t,x(t)) = \mathrm{D}^{-} V(t,x(t)) = \mathrm{D}_{-} V(t,x(t)) = \frac{\mathrm{d}V(t,x(t))}{\mathrm{d}t}.$$

By Theorem 1.2.1, $V(t,x(t))$ is non-decreasing [non-increasing] along the solution of (1.2.1) if and only if

$$\mathrm{D}^{+} V(t,x(t)) \geqslant 0 \qquad [\,\mathrm{D}^{+} V(t,x(t)) \leqslant 0\,].$$

The significance of Theorem 1.2.2 lies in the fact that one does not need to know the solution while calculating the Dini-derivative of $V(t,x)$ along the solution of (1.2.1).

1.3. M-Matrices

In the study of differential equations, numerical mathematics, mathematical statistics, and M-matrices are considered to be a fruitful technique for modern development. As far as M-matrices are concerned, there are many conditions equivalent to each other. For the proof of the equivalence, the reader is referred to Ortega and Rheinboldt [1]. Now, we will limit our introduction to some properties of M-matrices used in this book.

Definition 1.3.1. A matrix $A(a_{ij})_{n\times n}$ is an *M-matrix* if it satisfies

1) $a_{ii} > 0$ $(i = 1,\cdots,n)$, $a_{ij} \leqslant 0$ $(i \neq j,\ i,j = 1,\cdots,n)$,

2) $\det \begin{vmatrix} a_{11} & \cdots & a_{1i} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{ii} \end{vmatrix} > 0,\ i = 1,\cdots,n.$

In terms of the M-matrix, we have the following equivalent conditions.

Theorem 1.3.2. *If $A(a_{ij})_{n\times n}$ is an M-matrix, then all the following conditions are equivalent:*

1) $a_{ii}>0$ $(i=1,\cdots,n)$, $a_{ij}\leqslant 0$ $(i\neq j,\ i,j=1,\cdots,n)$, $A^{-1}\geqslant 0$, *i.e., A^{-1} is a nonnegative matrix;*

2) $a_{ii}>0$ $(i=1,\cdots,n)$, $a_{ij}\leqslant 0$ $(i\neq j,\ i,j=1,\cdots,n)$, *$-A$ is a stable matrix;*

3) $a_{ii}>0$ $(i=1,\cdots,n)$, $a_{ij}\leqslant 0$ $(i\neq j,\ i,j=1,\cdots,n)$ *and there exist n positive constants c_j $(j=1,\cdots,n)$ such that* $\sum_{j=1}^{n} c_j a_{ij} > 0$;

4) $a_{ii} > 0$ $(i = 1,\cdots,n)$, $a_{ij} \leqslant 0$ $(i \neq j,\ i,j = 1,\cdots,n)$ *and there are n positive constants d_i $(i = 1,\cdots,n)$ such that* $\sum_{i=1}^{n} d_i a_{ij} > 0$;

5) $a_{ii}>0$ $(i=1,\cdots,n)$, $a_{ij}\leqslant 0$ $(i\neq j,\ i,j=1,\cdots,n)$ *and the spectral radius of the matrix*

$$G \triangleq \begin{bmatrix} 0 & \frac{a_{12}}{a_{11}} & \cdots & \frac{a_{1n}}{a_{11}} \\ \frac{a_{21}}{a_{22}} & 0 & \cdots & \frac{a_{2n}}{a_{22}} \\ \vdots & \vdots & & \vdots \\ \frac{a_{n1}}{a_{nn}} & \frac{a_{n2}}{a_{nn}} & \cdots & 0 \end{bmatrix}$$

is smaller than 1, *that is* $\rho(G)<1$. (*i.e.*, *the norm of eigenvalues of* G *are all smaller than* 1.)

For the proof of Theorem 1.3.2, see Ortega and Rheinboldt [1], where the conditions 3) and 4) are frequently used.

1.4. Principal Theorems on Global Stability

Consider an n-dimensional autonomous system

$$\frac{dx}{dt} = f(x), \qquad f(0) = 0, \tag{1.4.1}$$

where $x \in \mathbf{R}^n$, $f \in C[\mathbf{R}^n,\mathbf{R}^n]$. Suppose that the solution of the initial value problem (1.4.1) is unique.

Definition 1.4.1. The zero solution of (1.4.1) is *globally asymptotically stable* (globally stable for short) if for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\| x(t,t_0,x_0) \| < \varepsilon \quad \text{for all } t \geqslant t_0 \text{ if } \|x_0\| < \delta(\varepsilon)$$

and for any $x_0 \in \mathbf{R}^n$

$$\lim_{t\to+\infty} x(t,t_0,x_0) = 0.$$

Definition 1.4.2. The set $E = \{x: x(t,t_0,x_0), t \geqslant t_0\}$ is called a *positive semi-trajectory* of (1.4.1) through x_0 at $t = t_0$; If $x_0 \neq 0$, then E is a *non-trivial positive semi-trajectory*, x^* is called an *ω-limiting point* of $x(t,t_0,x_0)$ if there is a sequence $\{t_k\}$ with $t_k \to +\infty$ as $k \to +\infty$ such that $x^* = \lim\limits_{t_k\to+\infty} x(t_k,t_0,x_0)$.

Note that $\Omega(x_0)$ is the set of the ω-limiting points of the trajectory through x_0.

Lemma 1.4.3. *Suppose that* x^* *is an ω-limiting point of* $x(t,t_0,x_0)$. *Then the points on the positive semi-trajectory of* $x(t,t_0,x^*)$ *are all the ω-limiting points of* $x(t,t_0,x_0)$.

Proof. From hypothesis, there exists a sequence $\{t_n\}$ with $t_n \to +\infty$ as

$n \to +\infty$ such that

$$x^* = \lim_{n \to +\infty} x(t_n, t_0, x_0).$$

For an arbitrary point $x(t, t_0, x^*)$ on the trajectory through x^*, the property of the group for the solutions of autonomous systems and the relationship of continuous dependence for the solution to initial value lead to

$$\lim_{t_n \to +\infty} x(t_n + t, t_0, x_0) = \lim_{n \to +\infty} x(t, t_0, x(t_n, t_0, x_0)) = x(t, t_0, x^*).$$

In other words $x(t, t_0, x^*)$ is an ω-limiting point of $x(t, t_0, x_0)$. ∎

Theorem 1. 4. 4. *If there exists a radially unbounded positive definite differentiable function $V(x) \in C[\mathbf{R}^n, \mathbf{R}]$ such that*

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(1.4.1)} \leqslant 0 \tag{1.4.2}$$

and the set $M \triangleq \left\{x : \frac{\mathrm{d}V}{\mathrm{d}t} = 0\right\}$ does not contain any entire positive semi-trajectory of non-zero solutions of (1. 4. 1) except $x = 0$, then the zero solution of (1. 4. 1) is globally stable.

Proof. Since $V(x)$ is a radially unbounded positive definite function, there exists $\varphi \in KR$ such that

$$V(x) \geqslant \varphi(\|x\|).$$

From the continuity of $V(x)$ and $V(0) = 0$, $V(x) \geqslant 0$, for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$V(x_0) < \varphi(\varepsilon) \qquad \text{if} \quad \|x_0\| < \delta(\varepsilon).$$

It follows from (1. 4. 2) that

$$\varphi(\|x(t, t_0, x_0)\|) \leqslant V(x(t, t_0, x_0)) \leqslant V(x_0) < \varphi(\varepsilon) \tag{1.4.3}$$

for all $t \geqslant t_0$. Since $\varphi \in KR$, (1. 4. 3) implies that

$$\|x(t, t_0, x_0)\| < \varepsilon.$$

Therefore, the zero solution of (1. 4. 1) is stable.

Similarly to (1. 4. 3), for any $x_0 \in \mathbf{R}^n$, we get

$$\varphi(\|x(t, t_0, x_0)\|) \leqslant V(x(t, t_0, x_0)) \leqslant V(x_0),$$

thus

$$\|x(t, t_0, x_0)\| \leqslant \varphi^{-1}(V(x_0)) \triangleq M.$$

Hence, according to the Weierstrass's accumulation principle, we see that the set $\Omega(x_0)$ is nonempty and bounded.

Now we proceed to prove that $\Omega(x_0) = \{0\}$. If it is not true, then there is a sequence $\{t_n\}$ satisfying $t_n \to +\infty$ as $n \to +\infty$ such that

$$\lim_{n \to +\infty} x(t_n, t_0, x_0) = x^* \neq 0.$$

In virtue of the positive definiteness of $V(x)$ and $\frac{\mathrm{d}V(x(t))}{\mathrm{d}t} \leqslant 0$, we know that $V(x(t,t_0,x_0))$ is monotone non-increasing, continuous, and nonnegative. In our case, this gives rise to

$$\lim_{t \to +\infty} V(x(t,t_0,x_0)) = V(x^*) > 0. \tag{1.4.4}$$

Consider the trajectory $x(t,t_0,x^*)$ through x^*. Since

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(1.4.1)} \leqslant 0,$$

it follows that

$$V(x(t,t_0,x^*)) \leqslant V(x^*).$$

If for every $t \geqslant t_0$, $V(x(t,t_0,x^*)) = V(x^*)$, then there exists

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(1.4.1)} \equiv 0.$$

Thus the set M contains the entire positive semi-trajectory of the non-zero solution $x(t,t_0,x^*)$, which is inconsistency with the hypothesis. Then there exists $t_1 \geqslant t_0$ such that

$$V(x(t_1,t_0,x^*)) < V(x^*).$$

By Lemma 1.4.3, we find that $x(t_1,t_0,x^*)$ is an ω-limiting point of $x(t,t_0,x^*)$. Thus there exists a sequence $\{t_n^*\}$ with $\{t_n^*\} \to +\infty$ as $n \to +\infty$ such that

$$\lim_{n \to +\infty} x(t_n^*,t_0,x_0) = x(t_1,t_0,x^*).$$

Thus we obtain

$$\lim_{n \to +\infty} V(x(t_n^*,t_0,x_0)) = V(x(t_1,t_0,x^*)) < V(x^*)$$

which leads to a contradiction with (1.4.4), and therefore, $\Omega = \{0\}$, that is

$$\overline{\lim_{t \to +\infty}}\, x(t,t_0,x_0) = \underline{\lim_{t \to +\infty}}\, x(t,t_0,x_0) = 0 = \lim_{t \to +\infty} x(t,t_0,x_0). \qquad \blacksquare$$

Corollary 1.4.5. *If there exists a radially unbounded positive definite function* $V(x) \in [\mathbf{R}^n, \mathbf{R}]$ *suth that* $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(1.4.1)}$ *is negative definite, then the zero solution of* (1.4.1) *is globally stable.*

1.5. Partial Global Stability

In the following, a notion of partial global stalility of zero solution for (1.4.1) will be introduced.

Let $x = (y,z)^{\mathrm{T}}$, $y = \mathrm{col}(x_1,\cdots,x_m)$, $z = \mathrm{col}(x_{m+1},\cdots,x_n)$.

Definition 1.5.1. The zero solution of (1.4.1) is said to be *globally stable with respect to y* if for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\|y(t,t_0,x_0)\| < \varepsilon \quad \text{for all } t \geqslant t_0 \text{ if } \|x_0\| \leqslant \delta(\varepsilon)$$

and for any $x_0 \in \mathbf{R}^n$, there exists

$$\lim_{t \to +\infty} \|y(t,t_0,x_0)\| = 0.$$

Definition 1.5.2. A function $V(x) \in C[\mathbf{R}^n, \mathbf{R}]$ is said to be *radially unbounded positive definite with respect to y* if there exists a function $\varphi \in KR$ such that $V(x) \geqslant \varphi(\|y\|)$. A function $W(x) \in C[\mathbf{R}^n, \mathbf{R}]$ is *negative definite with respect to y* if there exists a function $\varphi \in K$ such that $W(x) \leqslant -\varphi(\|y\|)$.

Theorem 1.5.3. *If there is a function $V(x) \in C[\mathbf{R}^n, \mathbf{R}]$ satisfying*

$$a(\|y\|) \leqslant V(x) \leqslant b\left(\left(\sum_{i=1}^{k} x_i^2\right)^{\frac{1}{2}}\right), \quad m \leqslant k \leqslant n \tag{1.5.1}$$

with $a, b \in KR$; and

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(1.4.1)} \leqslant -c\left(\left(\sum_{i=1}^{k} x_i^2\right)^{\frac{1}{2}}\right), \quad c \in K, \tag{1.5.2}$$

then the zero solution of (1.4.1) *is globally stable with respect to y.*

Proof. Since $V(x)$ is continuous and $V(0) = 0$, for given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$V(x_0) < a(\varepsilon) \quad \text{if} \quad \|x_0\| < \delta(\varepsilon).$$

(1.5.1) and (1.5.2) yield

$$a(\|y(t,t_0,x_0)\|) \leqslant V(x(t,t_0,x_0)) \leqslant V(x_0) < a(\varepsilon).$$

Thus we have

$$\|y(t,t_0,x_0)\| < \varepsilon.$$

Therefore the zero solution of (1.4.1) is stable with respect to y.

Next, we prove that

$$\lim_{t \to +\infty} V(x(t,t_0,x_0)) = 0 \quad \text{for any} \quad x_0 \in \mathbf{R}^n,$$

thus the expression

$$a(\|y(t,t_0,x_0)\|) \leqslant V(x(t,t_0,x_0)) \to 0 \quad \text{as} \quad t \to +\infty$$

implies

$$\|y(t,t_0,x_0)\| \to 0 \quad \text{as} \quad t \to +\infty.$$

If this is not true, then there exists $x_0 \in \mathbf{R}^n$ such that

$$V(x(t,t_0,x_0)) \not\to 0 \quad \text{as} \quad t \to +\infty.$$

Using (1.5.2) we derive

$$\lim_{t \to +\infty} V(x(t,t_0,x_0)) = V_\infty,$$

then

$$V(x(t,t_0,x_0)) \geqslant V_\infty > 0.$$

It follows from (1. 5. 1) that

$$\left(\sum_{i=1}^{k} x_i^2(t,t_0,x_0)\right)^{\frac{1}{2}} \geqslant b^{-1}(V_\infty). \tag{1.5.3}$$

Applying the expressions (1. 5. 2) and (1. 5. 3), we write

$$0 \leqslant V(x(t,t_0,x_0)) \leqslant V(x(t_0)) - c(b^{-1}(V_\infty))(t-t_0). \tag{1.5.4}$$

However, when $t \gg t_0$, the expression (1. 5. 4) does not hold; thus

$$\lim_{t\to+\infty} V(x(t,t_0,x_0)) = 0$$

and

$$\lim_{t\to+\infty} \|y(t,t_0,x_0)\| = 0.$$

The proof is completed. ∎

1. 6. Global Stability of Sets

Let $M\subset\mathbf{R}^n$ be a manifold or an arbitrary set of points.

For convenience, we define

$$d(x,M) \triangleq \inf_{y\in M} \|x-y\|,$$

that is, $d(x,M)$ is the distance from x to M.

Definition 1. 6. 1. The solution of (1. 4. 1) is *globally stable with respect to the set M* if for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $d(x_0,M) < \delta(\varepsilon)$ implies

$$d(x(t,t_0,x_0),M) < \varepsilon \qquad \text{for all} \quad t \geqslant t_0,$$

and for any $x_0 \in \mathbf{R}^n$ there exists

$$\lim_{t\to+\infty} d(x(t,t_0,x_0),M) = 0.$$

Theorem 1. 6. 2. *Suppose that* $V(x)\in C[\mathbf{R}^n,\mathbf{R}]$ *and that* $V(x)$ *satisfies*

$$\varphi_1(d(x,M)) \leqslant V(x) \leqslant \varphi_2(d(x,M)), \qquad \varphi_1,\varphi_2 \in KR,$$

$$\left.\frac{\mathrm{d}V(x)}{\mathrm{d}t}\right|_{(1.4.1)} \leqslant -\psi(d(x,M)), \qquad \psi \in K.$$

Then the solution of (1. 4. 1) *is globally stable with respect to the set M.*

Proof. For any $\varepsilon > 0$, choosing $\delta(\varepsilon) \triangleq \varphi_2^{-1}(\varphi_1(\varepsilon))$, we write

$$\varphi_1(d(x(t,t_0,x_0),M)) \leqslant V(x(t,t_0,x_0)) \leqslant V(x_0)$$
$$\leqslant \varphi_2(d(x_0,M)) < \varphi_2(\delta(\varepsilon))$$

if $d(x_0,M) < \delta(\varepsilon)$. Thus

$$d(x(t,t_0,x_0),M) < \varphi_1^{-1}(\varphi_2(\delta(\varepsilon))) = \varepsilon \qquad \text{for all} \quad t \geqslant t_0.$$

In the following, we will prove the validity of

$$d(x(t,t_0,x_0),M) \triangleq d(x(t),M) \to 0 \qquad \text{as} \quad t\to+\infty.$$

For any $\varepsilon > 0$, any $x_0 \in \mathbf{R}^n$ and any $\eta > 0$, since

$$\frac{\mathrm{d}V(x(t))}{\mathrm{d}t} \leqslant -\psi(d(x(t),M)),$$

and

$$\varphi_1(d(x(t),M)) \leqslant V(x(t)) \leqslant \varphi_2(d(x(t),M)),$$

we derive

$$\frac{\mathrm{d}V(x(t))}{\mathrm{d}t} \leqslant -\psi(\varphi_2^{-1}(V(x(t)))) \leqslant 0$$

i. e. ,

$$\frac{\mathrm{d}V(x(t))}{\psi(\varphi_2^{-1}(V(x(t))))} \leqslant -\mathrm{d}t. \tag{1.6.1}$$

Therefore, (1. 6. 1) yields

$$\int_{V(x_0)}^{V(x(t))} \frac{\mathrm{d}V}{\psi(\varphi_2^{-1}(V))} \leqslant -(t-t_0),$$

which is

$$\int_{V(x(t))}^{V(x_0)} \frac{\mathrm{d}V}{\psi(\varphi_2^{-1}(V))} \geqslant t - t_0 .$$

Suppose $V(x_0) \leqslant \varphi_2(d(x_0,M)) \leqslant \varphi_2(\eta)$, then

$$\begin{aligned} t - t_0 &\leqslant \int_{V(x(t))}^{V(x_0)} \frac{\mathrm{d}V}{\psi(\varphi_2^{-1}(V))} \leqslant \int_{\varphi_1(d(x(t),M))}^{\varphi_2(\eta)} \frac{\mathrm{d}V}{\psi(\varphi_2^{-1}(V))} \\ &= \int_{\varphi_1(d(x(t),M))}^{\varphi_1(\varepsilon)} \frac{\mathrm{d}V}{\psi(\varphi_2^{-1}(V))} + \int_{\varphi_1(\varepsilon)}^{\varphi_2(\eta)} \frac{\mathrm{d}V}{\psi(\varphi_2^{-1}(V))}. \end{aligned}$$

Defining

$$T = T(\varepsilon,\eta) > \int_{\varphi_1(\varepsilon)}^{\varphi_2(\eta)} \frac{\mathrm{d}V}{\psi(\varphi_2^{-1}(V))},$$

it is easy to see that

$$\begin{aligned} \int_{\varphi_1(d(x(t),M))}^{\varphi_1(\varepsilon)} \frac{\mathrm{d}V}{\psi(\varphi_2^{-1}(V))} &\geqslant t - t_0 - \int_{\varphi_1(\varepsilon)}^{\varphi_2(\eta)} \frac{\mathrm{d}V}{\psi(\varphi_2^{-1}(V))} \\ &> t - t_0 - T \geqslant 0 \end{aligned}$$

if $t \geqslant t_0 + T$. Hence

$$\varphi_1(\varepsilon) > \varphi_1(d(x(t),M)),$$

i. e. ,

$$d(x(t),M) < \varepsilon \qquad \text{if} \quad t \geqslant t_0 + T(\varepsilon,\eta).$$

The proof is completed. ∎

1. 7. Nonautonomous Systems

Consider an n-dimensional nonautonomous system

$$\frac{dx}{dt} = f(t,x), \qquad f(t,0) \equiv 0, \tag{1. 7. 1}$$

where $x = \text{col}(x_1, \cdots, x_n)$, $f \in C[I \times \mathbf{R}^n, \mathbf{R}^n]$, $I = [t_0, +\infty)$. Suppose that the solution of the initial value problem (1. 7. 1) is unique and let $y = \text{col}(x_1, \cdots, x_m)$, $z = \text{col}(x_{m+1}, \cdots, x_n)$.

In analogy with Definitions 1. 4. 1 and 1. 5. 1, we can formulate the definition of global stability of the zero solution of (1. 7. 1).

Theorem 1. 7. 1. *If there exists a function* $V(t,x) \in C[I \times \mathbf{R}^n, \mathbf{R}]$ *satisfying*

$$\varphi_1(\|x\|) \leqslant V(t,x) \leqslant \varphi_2(\|x\|), \qquad \varphi_1, \varphi_2 \in KR$$

and

$$\left.\frac{dV}{dt}\right|_{(1.7.1)} \leqslant -\psi(\|x\|), \qquad \psi \in K,$$

then the zero solution of (1. 7. 1) *is globally stable.*

We can prove Theorem 1. 7. 1 by the same way as Theorem 1. 4. 4.

Theorem 1. 7. 2. *If there exists a function* $V(t,x) \in C[I \times \mathbf{R}^n, \mathbf{R}]$ *satisfying*

$$\varphi_1(\|y\|) \leqslant V(t,x) \leqslant \varphi_2\left(\left(\sum_{i=1}^{k} x_i^2\right)^{\frac{1}{2}}\right), \qquad \overline{m} \leqslant k \leqslant n, \quad \varphi_1, \varphi_2 \in KR,$$

and

$$\left.\frac{dV}{dt}\right|_{(1.7.1)} \leqslant -\psi\left(\left(\sum_{i=1}^{k} x_i^2\right)^{\frac{1}{2}}\right), \qquad \psi \in K,$$

then the zero solution of (1. 7. 1) *is globally stable with respect to* y.

The proof of this conclusion goes along the same line as in Theorem 1. 5. 3.

1. 8. The Systems with Separable Variables

It will be shown later that by topological transformations a number of automatic control systems of different form can be reduced to systems with separable variables or to systems with generalized separable variables. In this section, we will discuss the systems with separable variables in detail.

Consider a non-linear system with separable variables:

$$\frac{dx}{dt} = \text{col}\left(\sum_{j=1}^{n} f_{1j}(x_j), \cdots, \sum_{j=1}^{n} f_{nj}(x_j)\right), \tag{1. 8. 1}$$

where $f_{ij}(x_j) \in C[\mathbf{R}, \mathbf{R}]$, $f_{ij}(0) = 0$, $i,j = 1, \cdots, n$. Suppose that the solution of the initial value problem (1. 8. 1) is unique .

Let $y = \mathrm{col}(x_1, \cdots, x_m)$, $z = \mathrm{col}(x_{m+1}, \cdots, x_n)$. Then (1. 8. 1) is reduced to

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}t} = \mathrm{col}\left(\sum_{j=1}^{n} f_{1j}(x_j), \cdots, \sum_{j=1}^{n} f_{mj}(x_j) \right), \\ \dfrac{\mathrm{d}z}{\mathrm{d}t} = \mathrm{col}\left(\sum_{j=1}^{n} f_{m+1,j}(x_j), \cdots, \sum_{j=1}^{n} f_{nj}(x_j) \right). \end{cases} \tag{1. 8. 2}$$

Similarly to the Selvester's condition, we first establish a criterion of positive definiteness and negative definiteness of quadratic forms with respect to part of variables.

Let us assume that $A(a_{ij})_{n\times n} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ is symmetric, where A_{11}, A_{12}, A_{21} and A_{22} are $m \times m$, $m \times p$, $p \times m$ and $p \times p$ matrices respectively and $m + p = n$.

Definition 1. 8. 1. The quadratic form $\begin{pmatrix} y \\ z \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$ is said to be *positive* [*negative*] *definite with respect to* y if there are constants $\varepsilon_i > 0$ $(i = 1, \cdots, m)$ such that

$$x^{\mathrm{T}} A x \geqslant \sum_{i=1}^{m} \varepsilon_i x_i^2 \qquad \left[x^{\mathrm{T}} A x \leqslant - \sum_{i=1}^{m} \varepsilon_i x_i^2 \right].$$

Lemma 1. 8. 2. *The quadratic form* $\begin{pmatrix} y \\ z \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$ *is positive* [*negative*] *definite with respect to* y *if and only if there exists a constant* $\varepsilon > 0$ *such that*

$$\begin{bmatrix} A_{11} - \varepsilon E & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ is positive semi-definite}$$

$$\left[\begin{bmatrix} A_{11} + \varepsilon E & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ is negative semi-definite} \right],$$

where E *is an* $m \times m$ *unit matrix.*

Proof. For illustration we prove the positive case. The proof of the other case is similar and is omitted.

Necessity. Since $\begin{pmatrix} y \\ z \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$ is positive definite with respect to y, there exist some constants $\varepsilon_i > 0$ $(i = 1, \cdots, n)$ such that

$$\begin{pmatrix} y \\ z \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \geqslant \sum_{i=1}^{m} \varepsilon_i x_i^2.$$

Let $\varepsilon = \min\limits_{1 \leqslant i \leqslant n} \varepsilon_i$. Then we can find

$$\begin{pmatrix} y \\ z \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \geqslant \sum_{i=1}^{m} \varepsilon_i x_i^2 \geqslant \varepsilon \sum_{i=1}^{m} x_i^2 = \begin{pmatrix} y \\ z \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \varepsilon E & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}.$$

Thus

$$\begin{pmatrix} y \\ z \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} A_{11} - \varepsilon E & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \geqslant 0$$

and $\begin{bmatrix} A_{11} - \varepsilon E & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ is positive semi-definite. In particular, A_{11} is positive definite.

Sufficiency. The assumptions can be reduced to

$$\begin{pmatrix} y \\ z \end{pmatrix}^{T} \begin{bmatrix} A_{11} - \varepsilon E & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \geqslant 0.$$

Thus we have

$$\begin{pmatrix} y \\ z \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \geqslant \sum_{i=1}^{m} \varepsilon x_i^2.$$

This implies our claim. ∎

Lemma 1.8.3. *If there exist functions $\phi_i(x_i)$ on $(-\infty, +\infty)$ $(i=1,\cdots,n)$, which are continuous or have only finite discontinuous points of the first or third kind (i.e., at the discontinuous points, the left and right limits of $\phi_i(x_i)$ exist.) such that*

1) $\phi_i(x_i)x_i > 0$ *for* $x_i \neq 0$, $i=1,\cdots,m$, $\phi_i(x_i)x_i \geqslant 0$, $i=m+1,\cdots,n$;

2) $\int_0^{\pm\infty} \phi_i(x_i)\mathrm{d}x_i = +\infty$, $i = 1,\cdots,m$;

3) *there is a positive definite function $\psi(y)$ satisfying*

$$G(x) \triangleq \sum_{i=1}^{n} \phi_i(x_i \pm 0) \sum_{j=1}^{n} f_{ij}(x_j) \leqslant -\psi(y),$$

then the zero solution of (1.8.2) *is globally stable with respect to* y.

Proof. First of all, we construct the Liapunov function

$$V(x) = \sum_{i=1}^{n} \int_0^{x_i} \phi_i(x_i)\mathrm{d}x_i.$$

Obviously, 1) and 2) imply that

$$V(x) \geqslant \sum_{i=1}^{m} \int_0^{x_i} \phi_i(x_i)\mathrm{d}x_i \triangleq \varphi(y) \to +\infty \quad \text{if} \quad \|y\| \to +\infty.$$

Hence, $V(x)$ is radially unbounded positive definite with respect to y and along the solution of (1.8.2), the Dini-derivative of $V(x)$

$$\mathrm{D}^+ V(x)\big|_{(1.8.2)} = \begin{cases} \sum_{i=1}^{n} \phi_i(x_i) \sum_{j=1}^{n} f_{ij}(x_j) \\ \quad \text{at the continuous points of } \phi_i(x_i),\ i = 1,\cdots,n; \\ \max\left\{ \sum_{i=1}^{n} \phi_i(x_i + 0) \sum_{j=1}^{n} f_{ij}(x_j),\ \sum_{i=1}^{n} \phi_i(x_i - 0) \sum_{j=1}^{n} f_{ij}(x_j) \right\} \\ \quad \text{at the discontinuous points of } \phi_i(x_i),\ i = 1,\cdots,n. \end{cases}$$

Therefore, the condition 3) implies that

$$D^+ V(x)|_{(1.8.2)} \leqslant -\psi(y).$$

As a result, the zero solution of (1.8.2) is globally stable with respect to y. ▮

Remark. If we let $m=n$, then the conditions of Lemma 1.8.3 imply that the zero solution of (1.8.1) is globally stable. In Theorem 1.8.4 below, for $m=n$, the statement follows from global stability of all variables.

Theorem 1.8.4. *If the system* (1.8.2) *satisfies*

1) $f_{ii}(x_i)x_i < 0$ *for* $x_i \neq 0$, $i = 1,\cdots,m$, $f_{ii}(x_i)x_i \leqslant 0$, $i = m+1,\cdots,n$;

2) $\int_0^{\pm\infty} f_{ii}(x_i)\mathrm{d}x_i = -\infty$, $i = 1,\cdots,m$;

3) *there are constants* $c_i > 0$ $(i=1,\cdots,m)$, $c_j \geqslant 0$ $(j=m+1,\cdots,n)$, $\varepsilon > 0$ *such that*

$$A(a_{ij}(x))_{n\times n} + \begin{pmatrix} \varepsilon E_{m\times m} & 0 \\ 0 & 0 \end{pmatrix}_{n\times n} \quad \textit{is negative semi-definite},$$

where

$$a_{ij}(x) = \begin{cases} -\dfrac{1}{2}\left(\dfrac{c_i f_{ij}(x_j)}{f_{jj}(x_j)} + \dfrac{c_j f_{ji}(x_i)}{f_{ii}(x_i)}\right), & x_i x_j \neq 0, \\ 0, & x_i x_j = 0, \end{cases} \quad i,j = 1,\cdots,n,$$

then the zero solution of (1.8.2) *is globally stable with respect to* y.

Proof. We construct the Liapunov function

$$V(x) = -\sum_{i=1}^{n}\int_0^{x_i} c_i f_{ii}(x_i)\mathrm{d}x_i.$$

Clearly, $V(x)$ is radially unbounded positive definite with respect to y. This is because

$$V(x) \geqslant -\sum_{i=1}^{m}\int_0^{x_i} c_i f_{ii}(x_i)\mathrm{d}x_i \triangleq \varphi(y) \to +\infty \qquad \text{as } \|y\| \to +\infty.$$

Now we prove that

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(1.8.2)} \triangleq G(x) = -\sum_{i=1}^{n} c_i f_{ii}(x_i)\sum_{j=1}^{n} f_{ij}(x_j)$$

is negative definite with respect to y.

For any $x = \xi \in \mathbf{R}^n$, without loss of generality we can assume that

$$\prod_{i=1}^{k}\xi_i \neq 0, \quad \sum_{i=k+1}^{n}\xi_i^2 = 0, \quad 1 \leqslant k \leqslant n.$$

Then, we obtain

$$\begin{aligned} G(\xi) &= -\sum_{i=1}^{k} c_i f_{ii}(\xi_i)\sum_{j=1}^{k} f_{ij}(\xi_j) \\ &= -\frac{1}{2}\sum_{i,j=1}^{k}[c_i f_{ii}(\xi_i) f_{ij}(\xi_j) + c_j f_{jj}(\xi_j) f_{ji}(\xi_i)] \end{aligned}$$

$$= -\sum_{i=1}^{k} c_i f_{ii}^2(\xi_i) - \sum_{\substack{i,j=1\\ i\neq j}}^{k} \frac{1}{2}\left(\frac{c_i f_{ij}(\xi_j)}{f_{jj}(\xi_j)} + \frac{c_j f_{ji}(\xi_i)}{f_{ii}(\xi_i)}\right) f_{ii}(\xi_i) f_{jj}(\xi_j)$$

$$= \sum_{i=1}^{k} a_{ii}(\xi) f_{ii}^2(\xi_i) + \sum_{i=1}^{m} \varepsilon f_{ii}^2(\xi_i) + \sum_{\substack{i,j=1\\ i\neq j}}^{k} a_{ij}(\xi) f_{ii}(\xi_i) f_{jj}(\xi_j)$$

$$- \sum_{i=1}^{m} \varepsilon f_{ii}^2(\xi_i)$$

$$\leqslant -\sum_{i=1}^{m} \varepsilon f_{ii}^2(\xi_i) < 0.$$

Since ξ is arbitrary, we obtain that $G(x)$ is negative definite with respect to y. Then the zero solution of (1.8.2) is globally stable with respect to y. ▮

Theorem 1.8.5. *Suppose that* (1.8.2) *satisfies the following conditions:*

1) *the condition* 1) *of Theorem* 1.8.4;

2) *there exist n functions $c_i(x_i)$ $(i=1,\cdots,n)$, which are continuous or have only finite discontinuous points of the first or third kind, and satisfy*

$$c_i(x_i)x_i > 0 \textit{ for } x_i \neq 0 \textit{ and } \int_0^{\pm\infty} c_i(x_i)\mathrm{d}x_i = +\infty, \quad i = 1,\cdots,m,$$

$$c_i(x_i)x_i \geqslant 0,\ i = m+1,\cdots,n;$$

3) *there exist functions $\varepsilon_i(x_i) > 0$ $(i=1,\cdots,n)$ such that*

$$\tilde{A}(\tilde{a}_{ij}(x))_{n\times n} + \begin{pmatrix} \operatorname{diag}(\varepsilon_1(x_1),\cdots,\varepsilon_m(x_m)) & 0 \\ 0 & 0 \end{pmatrix}_{n\times n}$$

is negative semi-definite, where

$$\tilde{a}_{ij}(x) = \begin{cases} \dfrac{1}{2}\left[\dfrac{c_i(x_i)f_{ij}(x_j)}{\sqrt{|f_{ii}(x_i)f_{jj}(x_j)|}} + \dfrac{c_j(x_j)f_{ji}(x_i)}{\sqrt{|f_{jj}(x_j)f_{ii}(x_i)|}}\right], & x_ix_j \neq 0, \\ 0, & x_ix_j = 0, \end{cases}$$
$$i,j = 1,\cdots,n.$$

Then the zero solution of (1.8.2) *is globally stable with respect to y.*

Proof. Let us set

$$V(x) = \sum_{i=1}^{n} \int_0^{x_i} c_i(x_i)\mathrm{d}x_i.$$

We proceed along the lines of Theorem 1.8.4 to complete the proof. ▮

Theorem 1.8.6. *If* (1.8.2) *satisfies*

1) *the condition* 1) *of Theorem* 1.8.4;

2) *there exist constants $c_i > 0$ $(i=1,\cdots,m)$, $c_j \geqslant 0$ $(j=m+1,\cdots,n)$ such that*

$$\begin{cases} -c_j|f_{jj}(x_j)| + \sum\limits_{\substack{i=1\\ i\neq j}}^{n} c_i|f_{ij}(x_j)| < 0 \textit{ for } x_j \neq 0, & j = 1,\cdots,m, \\ -c_j|f_{jj}(x_j)| + \sum\limits_{\substack{i=1\\ i\neq j}}^{n} c_i|f_{ij}(x_j)| \leqslant 0, & j = m+1,\cdots,n, \end{cases}$$

then the zero solution of (1.8.2) *is globally stable with respect to y.*

Proof. We construct the Liapunov function

$$V(x) = \sum_{i=1}^{n} c_i |x_i|.$$

Clearly,

$$V(x) \geqslant \sum_{i=1}^{m} c_i |x_i| \triangleq \varphi(y) \to +\infty \qquad \text{for} \quad \|y\| \to +\infty,$$

and $\varphi(y)$ is positive definite. On the other hand, we have

$$\begin{aligned} D^+ V(x)|_{(1.8.2)} &\leqslant \sum_{j=1}^{n} \Big[-c_j |f_{jj}(x_j)| + \sum_{\substack{i=1 \\ i\neq j}}^{n} c_i |f_{ij}(x_j)| \Big] \\ &\leqslant \sum_{j=1}^{m} \Big[-c_j |f_{jj}(x_j)| + \sum_{\substack{i=1 \\ i\neq j}}^{n} c_i |f_{ij}(x_j)| \Big] \\ &< 0 \qquad \text{if} \quad y \neq 0. \end{aligned}$$

Therefore, the zero solution of (1.8.2) is globally stable with respect to y. ∎

Theorem 1.8.7. *Suppose that* (1.8.2) *satisfies the following conditions:*

1) *the condition* 1) *of Theorem* 1.8.4;

2) $\left| \dfrac{f_{ij}(x_j)}{f_{jj}(x_j)} \right| \leqslant b_{ij} = \mathit{const.}$, $i \neq j$, $i, j = 1, \cdots, n$;

3)

$$\tilde{A} \triangleq \begin{bmatrix} 1 & -b_{21} & \cdots & -b_{n1} \\ -b_{12} & 1 & \cdots & -b_{n2} \\ \vdots & \vdots & & \vdots \\ -b_{1n} & -b_{2n} & \cdots & 1 \end{bmatrix} \triangleq \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix},$$

$\tilde{A}_{11}$, $\tilde{A}_{22}$ *and* $E - \tilde{A}_{11}^{-1}\tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21}$ *are M-matrices, where* $\tilde{A}_{11}$, $\tilde{A}_{12}$, $\tilde{A}_{21}$ *and* $\tilde{A}_{22}$ *are* $m \times m$, $m \times p$, $p \times m$ *and* $p \times p$ *matrices respectively.*

Then the zero solution of (1.8.2) *is globally stable with respect to* y.

Proof. For any $\xi = \mathrm{col}(\xi_1, \cdots, \xi_m) > 0$, $\eta = \mathrm{col}(\eta_1, \cdots, \eta_p) \geqslant 0$ and $m + p = n$, we consider the linear algebraic equations with respect to $c = \mathrm{col}(c_1, \cdots, c_m)$ and $\tilde{c} = \mathrm{col}(\tilde{c}_1, \cdots, \tilde{c}_p)$:

$$\begin{cases} \tilde{A}_{11} c + \tilde{A}_{12} \tilde{c} = \xi, \\ \tilde{A}_{21} c + \tilde{A}_{22} \tilde{c} = \eta, \end{cases}$$

or the equivalent ones:

$$\begin{cases} \tilde{c} = -\tilde{A}_{22}^{-1}\tilde{A}_{21} c + \tilde{A}_{22}^{-1}\eta, \\ c = \tilde{A}_{11}^{-1}\tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21} c - \tilde{A}_{11}^{-1}\tilde{A}_{12}\tilde{A}_{22}^{-1}\eta + \tilde{A}_{11}^{-1}\xi. \end{cases} \tag{1.8.3}$$

Since $\tilde{A}_{11}$, $\tilde{A}_{22}$ are M-matrices, we have

$$\tilde{A}_{11}^{-1} \geqslant 0, \qquad \tilde{A}_{22}^{-1} \geqslant 0.$$

But $\tilde{A}_{12} \leqslant 0$ and $\xi > 0$, $\eta \geqslant 0$, therefore there exist

$$-\tilde{A}_{11}^{-1}\tilde{A}_{12}\tilde{A}_{22}^{-1}\eta \geqslant 0, \qquad \tilde{A}_{11}^{-1}\xi > 0.$$

Since $(E-\tilde{A}_{11}^{-1}\tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21})$ is an M-matrix, the second equation in (1.8.3) has a positive solution with respect to c and the first one in (1.8.3) has a nonnegative solution with respect to $\tilde{c}$. Thus the conditions in Theorem 1.8.6 are satisfied. Therefore we conclude that the zero solution of (1.8.2) is globally stable with respect to y. ▮

Below, we will consider a more specific system:

$$\begin{cases}\dfrac{\mathrm{d}y}{\mathrm{d}t}=\mathrm{col}\left(\sum\limits_{j=1}^{n}a_{1j}f_j(x_j),\cdots,\sum\limits_{j=1}^{n}a_{mj}f_j(x_j)\right),\\ \dfrac{\mathrm{d}z}{\mathrm{d}t}=\mathrm{col}\left(\sum\limits_{j=1}^{n}a_{m+1,j}f_j(x_j),\cdots,\sum\limits_{j=1}^{n}a_{nj}f_j(x_j)\right),\end{cases}\tag{1.8.4}$$

where $f_j(x_j)\in C[\mathbf{R},\mathbf{R}]$, $f_j(0)=0$, $j=1,\cdots,n$. Suppose the solution of the initial value problem (1.8.4) is unique.

Theorem 1.8.8. *Suppose* (1.8.4) *satisfies the following conditions:*

1) $f_i(x_i)x_i>0$ *for* $x_i\neq 0$, $\int_0^{\pm\infty}f_i(x_i)\mathrm{d}x_i=+\infty$, $a_{ii}<0$, $i=1,\cdots,m$, $a_{ii}\leqslant 0$, $f_i(x_i)x_i\geqslant 0$, $i=m+1,\cdots,n$;

2) *there exist constants* $c_i>0$ $(i=1,\cdots,m)$, $c_j\geqslant 0$ $(j=m+1,\cdots,n)$, $\varepsilon>0$ *such that*

$$B(b_{ij})_{n\times n}+\begin{pmatrix}\varepsilon E_{m\times m} & 0\\ 0 & 0\end{pmatrix}_{n\times n}\ \textit{is negative semi-definite},$$

where

$$b_{ij}=\begin{cases}-c_i|a_{ii}|, & i=j=1,\cdots,m;\\ -\dfrac{1}{2}(c_ia_{ij}+c_ja_{ji}), & i\neq j,\ i,j=1,\cdots,n.\end{cases}$$

Then the zero solution of the system (1.8.4) *is globally stable with respect to* y.

Proof. First we construct the Liapunov function

$$V(x)=-\sum_{i=1}^{n}c_i\int_0^{x_i}f(x_i)\mathrm{d}x_i.$$

The proof is analogous to that of Theorem 1.8.4, and is omitted. ▮

Theorem 1.8.9. *Let* (1.8.4) *satisfy the following conditions:*

1) $f_i(x_i)x_i<0$ *for* $x_i\neq 0$, $a_{ii}>0$, $i=1,\cdots,m$, $f_i(x_i)x_i\leqslant 0$, $a_{ii}\geqslant 0$, $i=m+1,\cdots,n$;

2) *there exist functions* $c_i(x_i)$ $(i=1,\cdots,n)$ *which are continuous or have only fintie discontinuous points of the first or third kind such that*

$$c_i(x_i)x_i>0\ \textit{for}\ x_i\neq 0,\ \int_0^{\pm\infty}c_i(x_i)\mathrm{d}x_i=+\infty,\quad i=1,\cdots,m,$$

$$c_i(x_i)x_i\geqslant 0,\quad i=m+1,\cdots,n;$$

3) *there exist functions* $\varepsilon_i(x_i)>0$ $(i=1,\cdots,m)$ *such that*

$$\tilde{B}(\tilde{b}_{ij}(x))_{n\times n} + \begin{pmatrix} \operatorname{diag}(\varepsilon_1(x_1),\cdots,\varepsilon_m(x_m)) & 0 \\ 0 & 0 \end{pmatrix}$$

is negative semi-definite, where

$$\tilde{b}_{ij}(x) = \begin{cases} \dfrac{1}{2}\left[\dfrac{c_i(x_i)a_{ij}f_j(x_j)}{\sqrt{|f_i(x_i)f_j(x_j)|}} + \dfrac{c_j(x_j)a_{ji}f_i(x_i)}{\sqrt{|f_j(x_j)f_i(x_i)|}}\right], & x_ix_j \neq 0, \\ 0, & x_ix_j = 0, \end{cases}$$
$$i,j = 1,\cdots,n.$$

Then the zero solution of the system (1. 8. 4) *is globally stable with respect to* y.

Proof. Construct the Liapunov function

$$V(x) = \sum_{i=1}^{n}\int_0^{x_i} c_i(x_i)\mathrm{d}x_i.$$

The proof is similar to that of Theorem 1. 8. 5 and so is omitted. ∎

Theorem 1. 8. 10. *If* (1. 8. 4) *satisfies*

1) $f_i(x_i)x_i > 0$ *for* $x_i \neq 0$, $a_{ii} < 0$, $i=1,\cdots,m$, $f_i(x_i)x_i \geqslant 0$, $a_{ii} \leqslant 0$, $i=m+1,\cdots,n$;

2) *there exist constants* $c_i > 0$ $(i=1,\cdots,m)$, $c_j \geqslant 0$ $(j=m+1,\cdots,n)$ *such that*

$$\begin{cases} -c_j|a_{jj}| + \sum\limits_{\substack{i=1\\ i\neq j}}^{n} c_i|a_{ij}| < 0, & j = 1,\cdots,m, \\ -c_j|a_{jj}| + \sum\limits_{\substack{i=1\\ i\neq j}}^{n} c_i|a_{ij}| \leqslant 0, & j = m+1,\cdots,n, \end{cases}$$

then the zero solution of (1. 8. 4) *is globally stable with respect to* y.

Proof. We construct the Liapunov function

$$V(x) = \sum_{i=1}^{n} c_i|x_i|.$$

The proof can be completed as in the case of Theorem 1. 8. 6. ∎

Theorem 1. 8. 11. *Suppose* (1. 8. 4) *satisfies the following conditons:*

1) $f_i(x_i)x_i > 0$ *for* $x_i \neq 0$, $i=1,\cdots,m$, $f_i(x_i)x_i \geqslant 0$, $i=m+1,\cdots,n$, $a_{ii} < 0$, $i=1,\cdots,n$;

2)

$$\tilde{A} \triangleq \begin{pmatrix} 1 & -\left|\dfrac{a_{21}}{a_{11}}\right| & \cdots & -\left|\dfrac{a_{n1}}{a_{11}}\right| \\ -\left|\dfrac{a_{12}}{a_{22}}\right| & 1 & \cdots & -\left|\dfrac{a_{n2}}{a_{22}}\right| \\ \vdots & \vdots & & \vdots \\ -\left|\dfrac{a_{1n}}{a_{nn}}\right| & -\left|\dfrac{a_{2n}}{a_{nn}}\right| & \cdots & 1 \end{pmatrix} \triangleq \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix},$$

$\tilde{A}_{11}$, $\tilde{A}_{22}$ *and* $E-\tilde{A}_{11}^{-1}\tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21}$ *are M-matrices, where* $\tilde{A}_{11}$, $\tilde{A}_{12}$, $\tilde{A}_{21}$ *and* $\tilde{A}_{22}$ *represent* $m\times m$, $m\times p$, $p\times m$ *and* $p\times p$ *matrices respectively.*

Then the zero solution of the system (1. 8. 4) *is globally stable with respect to* y.

Proof. We follow the ideas applied in the proof of Theorem 1. 8. 7. ∎

1. 9. Autonomous Systems with Generalized Separable Variables

Consider the system with generalized separable variables:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t}=\sum_{j=1}^{n}F_{ij}(x)\cdot f_{ij}(x_j),\qquad i=1,\cdots,n, \tag{1.9.1}$$

where $x=\mathrm{col}(x_1,\cdots,x_n)$, $F_{ij}\in C[\mathbf{R}^n,\mathbf{R}]$, $f_{ij}\in C[\mathbf{R},\mathbf{R}]$, $f_{ij}(0)=0$, $i,j=1,\cdots,n$. Suppose the solution of the initial value problem (1. 9. 1) is unique.

Clearly, (1. 8. 2) is a special case of (1. 9. 1) with $F_{ij}(x)\equiv 1$.

In fact, we will see in what follows that both the Lurie direct control system and the Lurie indirect one can be reduced to (1. 9. 1).

We still let $y=\mathrm{col}(x_1,\cdots,x_m)$, $z=\mathrm{col}(x_{m+1},\cdots,x_n)$ and rewrite (1. 9. 1) as

$$\begin{cases}\dfrac{\mathrm{d}y}{\mathrm{d}t}=\mathrm{col}\left(\displaystyle\sum_{j=1}^{n}F_{1j}(x)f_{1j}(x_j),\cdots,\sum_{j=1}^{n}F_{mj}(x)f_{mj}(x_j)\right),\\[2ex] \dfrac{\mathrm{d}z}{\mathrm{d}t}=\mathrm{col}\left(\displaystyle\sum_{j=1}^{n}F_{m+1,j}(x)f_{m+1,j}(x_j),\cdots,\sum_{j=1}^{n}F_{nj}(x)f_{mj}(x_j)\right).\end{cases} \tag{1.9.2}$$

Theorem 1. 9. 1. *If the system* (1. 9. 2) *satisfies the following conditions:*

1) $f_{ii}(x_i)x_i>0$ *for* $x_i\neq 0$ *and* $\int_0^{\pm\infty}f_{ii}(x_i)\mathrm{d}x_i=+\infty$, $i=1,\cdots,m$, $f_{ii}(x_i)x_i\geqslant 0$, $i=m+1,\cdots,n$;

2) *there exist constants* $c_i>0$ $(i=1,\cdots,m)$, $c_j\geqslant 0$ $(j=m+1,\cdots,n)$, $\varepsilon>0$ *such that*

$$B(b_{ij}(x))_{n\times n}+\begin{bmatrix}\varepsilon E_{m\times m} & 0\\ 0 & 0\end{bmatrix}\ \textit{is negative semi-definite},$$

where

$$b_{ij}(x)\triangleq\begin{cases}\dfrac{1}{2}\left(\dfrac{F_{ij}(x)f_{ij}(x_j)}{f_{jj}(x_j)}+\dfrac{F_{ji}(x)f_{ji}(x_i)}{f_{ii}(x_i)}\right), & x_ix_j\neq 0,\\ 0, & x_ix_j=0,\end{cases}\qquad i,j=1,\cdots,n,$$

then the zero solution of (1. 9. 2) *is globally stable with respect to* y.

Proof. Let us set

$$V(x)=\sum_{i=1}^{n}\int_0^{x_i}f_{ii}(x_i)\mathrm{d}x_i.$$

Clearly,

$$V(x) \geqslant \sum_{i=1}^{m}\int_0^{x_i} f_{ii}(x_i)\mathrm{d}x_i \triangleq \varphi(y) \to +\infty \qquad \text{for} \quad \|y\| \to +\infty.$$

Hence $V(x)$ is radially unbounded positive definite with respect to y.

Now we proceed to prove that

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(1.9.2)} \triangleq G(x) = \sum_{i=1}^{n} f_{ii}(x_i)\sum_{j=1}^{n} F_{ij}(x)f_{ij}(x_j)$$

is negative definite with respect to y.

For any $x = \xi \in \mathbf{R}^n$, without loss of generality we can assume that

$$\prod_{j=1}^{k}\xi_j \neq 0, \quad \sum_{i=k+1}^{n}\xi_i^2 = 0, \quad 1 \leqslant k \leqslant n.$$

Then it follows that

$$\begin{aligned}
G(\xi) &= \sum_{i=1}^{k} f_{ii}(\xi_i)\sum_{j=1}^{k} F_{ij}(\xi)f_{ij}(\xi_j) \\
&= \sum_{i=1}^{k} F_{ii}(\xi)f_{ii}^2(\xi_i) \\
&\quad + \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{k}\left[f_{ii}(\xi_i)F_{ij}(\xi)f_{ij}(\xi_j) + f_{jj}(\xi_j)F_{ji}(\xi)f_{ji}(\xi_i)\right] \\
&= \sum_{i=1}^{k} b_{ii}(\xi)f_{ii}^2(\xi_i) + \sum_{i=1}^{m}\varepsilon f_{ii}^2(\xi_i) + \sum_{\substack{i,j=1\\ i\neq j}}^{k} b_{ij}(\xi)f_{ii}(\xi_i)f_{jj}(\xi_j) \\
&\quad - \sum_{i=1}^{m}\varepsilon f_{ii}^2(\xi_i) \\
&\leqslant -\sum_{i=1}^{m}\varepsilon f_{ii}^2(\xi_i) < 0.
\end{aligned}$$

Since ξ is arbitrary, we can find that

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(1.9.2)} \quad \text{is negative definite with respect to } y.$$

Hence, the proof is finished. ▮

Theorem 1.9.2. *If the system* (1.9.2) *satisfies the following conditions*:

1) $F_{ii}(x)f_{ii}(x_i)x_i < 0$ *for* $x_i \neq 0$, $i = 1, \cdots, m$, $F_{ii}(x)f_{ii}(x_i)x_i \leqslant 0$, $i = m+1, \cdots, n$;

2) *there exist constants* $c_i > 0$ $(i = 1, \cdots, m)$, $c_j \geqslant 0$ $(j = m+1, \cdots, n)$ *such that*

$$\begin{cases}
-c_j|F_{jj}(x)f_{jj}(x_j)| + \sum\limits_{\substack{i=1\\ i\neq j}}^{n} c_i|F_{ij}(x)f_{ij}(x_j)| < 0 \ \text{ for } x_j \neq 0, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad j = 1, \cdots, m, \\
-c_j|F_{jj}(x)f_{jj}(x_j)| + \sum\limits_{\substack{i=1\\ i\neq j}}^{n} c_i|F_{ij}(x)f_{ij}(x_j)| \leqslant 0, \quad j = m+1, \cdots, n,
\end{cases}$$

then the zero solution of (1.9.2) *is globally stable with respect to* y.

Proof. We construct the Liapunov function

$$V(x) = \sum_{i=1}^{n} c_i |x_i|$$

and complete the proof as in Theorem 1.8.6. ∎

1.10. Nonautonomous Systems with Separable Variables

Consider the nonautonomous system with separable variables:

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}t} = \mathrm{col}\left(\sum_{j=1}^{n} f_{1j}(t,x_j), \cdots, \sum_{j=1}^{n} f_{mj}(t,x_j)\right), \\ \dfrac{\mathrm{d}z}{\mathrm{d}t} = \mathrm{col}\left(\sum_{j=1}^{n} f_{m+1,j}(t,x_j), \cdots, \sum_{j=1}^{n} f_{nj}(t,x_j)\right), \end{cases} \tag{1.10.1}$$

where $y = \mathrm{col}(x_1, \cdots, x_m)$, $z = \mathrm{col}(x_{m+1}, \cdots, x_n)$, $f_{ij}(t,x_j) \in C[I \times \mathbf{R}, \mathbf{R}]$, $f_{ij}(t,0) \equiv 0$, $i,j = 1,\cdots,n$. Suppose the solution of the initial value problem (1.10.1) is unique.

Lemma 1.10.1. *If there exist functions* $\phi_i(x_i)$ *on* $(-\infty, +\infty)$ $(i=1,\cdots,n)$, *which are continuous or have only finite discontinuous points of the first or third kind such that*

1) $\phi_i(x_i)x_i > 0$ *for* $x_i \neq 0$, $i=1,\cdots,m$, $\phi_i(x_i)x_i \geqslant 0$, $i=m+1,\cdots,n$;

2) $\int_0^{\pm\infty} \phi_i(x_i)\mathrm{d}x_i = +\infty$, $i = 1,\cdots,m$;

3) *there is a positive definite function* ψ *satisfying*

$$G(x) \triangleq \sum_{i=1}^{n} \phi_i(x_i \pm 0) \sum_{j=1}^{n} f_{ij}(t,x_j) \leqslant -\psi(y),$$

then the zero solution of (1.10.1) *is globally stable with respect to* y.

Proof. The proof repeats the one of Lemma 1.8.3 and is omitted. ∎

Theorem 1.10.2. *Let the system* (1.10.1) *satisfy the following conditions:*

1) $f_{ii}(t,x_i)x_i < 0$ *for* $x_i \neq 0$, $i=1,\cdots,m$, $f_{ii}(t,x_i)x_i \leqslant 0$, $i=m+1,\cdots,n$;

2) *there exist functions* $F_{ii}(x_i)$ *on* $(-\infty, +\infty)$ $(i=1,\cdots,n)$ *which are continuous or have only finite discontinuous points of the first or third kind such that*

$$F_{ii}(x_i)x_i > 0 \ \textit{for}\ x_i \neq 0, \quad i = 1,\cdots,m,$$

$$F_{ii}(x_i)x_i \geqslant 0, \quad i = m+1,\cdots,n,$$

$$\int_0^{\pm\infty} F_{ii}(x_i)\mathrm{d}x_i = +\infty, \quad i = 1,\cdots,m,$$

$$|F_{ii}(x_i)| \leqslant |f_{ii}(t,x_i)|, \quad i = 1,\cdots,n;$$

3) *the matrix* $A(a_{ij}(t,x))_{n\times n} + \begin{pmatrix} \varepsilon E_{m\times m} & 0 \\ 0 & 0 \end{pmatrix}_{n\times n}$ *is negative semi-definite, where* $0 < \varepsilon \ll 1$,

$$a_{ij}(t,x) = \begin{cases} -1, & i = j = 1,\cdots,n; \\ \dfrac{1}{2}\left(\dfrac{f_{ij}(t,x_j)}{F_{jj}(x_j)} + \dfrac{f_{ji}(t,x_i)}{F_{ii}(x_i)}\right), & i \neq j,\ x_i x_j \neq 0,\ i,j = 1,\cdots,n; \\ 0, & i \neq j,\ x_i x_j = 0,\ i,j = 1,\cdots,n. \end{cases}$$

Then the zero solution of (1.10.1) *is globally stable with respect to* y.

Proof. Consider the Liapunov function

$$V(x) = \sum_{i=1}^{n}\int_0^{x_i} F_{ii}(x_i)\mathrm{d}x_i.$$

Then

$$V(x) \geqslant \sum_{i=1}^{m}\int_0^{x_i} F_{ii}(x_i)\mathrm{d}x_i \triangleq \varphi(y).$$

Clearly we have

$$\varphi(y) \to +\infty \qquad \text{as} \quad \|y\| \to +\infty.$$

We prove that

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(1.10.1)} \triangleq G(t,x) = \sum_{i=1}^{n} F_{ii}(x_i)\sum_{j=1}^{n} f_{ij}(t,x_j)$$

is negative definite with respect to y.

For any $x = \xi \in \mathbf{R}^n$, without loss of generality we can assume that

$$\prod_{i=1}^{k}\xi_i \neq 0, \quad \sum_{i=k+1}^{n}\xi_i^2 = 0, \quad 1 \leqslant k \leqslant n.$$

Then it follows

$$\begin{aligned} G(t,\xi) &= \sum_{i=1}^{k} F_{ii}(\xi_i)\sum_{j=1}^{k} f_{ij}(t,\xi_j) \\ &\leqslant \sum_{i=1}^{k} a_{ii}(t,\xi)F_{ii}^2(\xi_i) + \sum_{i=1}^{m}\varepsilon F_{ii}^2(\xi_i) \\ &\quad + \sum_{\substack{i,j=1 \\ i\neq j}}^{k} a_{ij}(t,\xi)F_{ii}(\xi_i)F_{jj}(\xi_j) \\ &\quad - \sum_{i=1}^{m}\varepsilon F_{ii}^2(\xi_i) \\ &\leqslant -\sum_{i=1}^{m}\varepsilon F_{ii}^2(\xi_i) < 0. \end{aligned}$$

Since ξ is arbitrary, we derive

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(1.10.1)} \quad \text{is negative definite with respect to } y.$$

Hence the zero solution of (1.10.1) is globally stable with respect to y. ∎

Theorem 1.10.3. *If the system* (1.10.1) *satisfies the following conditions*:

1) $f_{ii}(t,x_i)x_i<0$ *for* $x_i\neq 0$, $i=1,\cdots,m$, $f_{ii}(t,x_i)x_i\leqslant 0$, $i=m+1,\cdots,n$;

2) *there exist constants* $c_i>0$ $(i=1,\cdots,m)$, $c_j\geqslant 0$ $(j=m+1,\cdots,n)$ *such that*

$$-c_j|f_{jj}(t,x_j)|+\sum_{\substack{i=1\\ i\neq j}}^{n}c_i|f_{ij}(t,x_j)|<0 \text{ for } x_j\neq 0,\quad j=1,\cdots,m,$$

$$-c_j|f_{jj}(t,x_j)|+\sum_{\substack{i=1\\ i\neq j}}^{n}c_i|f_{ij}(t,x_j)|\leqslant 0,\qquad j=m+1,\cdots,n,$$

then the zero solution of (1.10.1) *is globally stable with respect to* y.

Proof. Let us set

$$V(x)=\sum_{i=1}^{n}c_i|x_i|.$$

Analogously to the proof of Theorem 1.8.6, we verify the validity of this theorem. ∎

Theorem 1.10.4. *Let the system* (1.10.1) *satisfy the following assertion*:

1) *there exist functions* $\phi_i(x_i)$ *on* $(-\infty,+\infty)$ $(i=1,\cdots,n)$ *which are continuous or have only finite discontinuous points of the first or third kind such that*

$$\phi_i(x_i)x_i>0 \text{ for } x_i\neq 0 \text{ and } \int_0^{\pm\infty}\phi_i(x_i)\,\mathrm{d}x_i=+\infty,\quad i=1,\cdots,m,$$

$$\phi_i(x_i)x_i\geqslant 0,\quad i=m+1,\cdots,n;$$

2) *there are functions* $a_i(x_i)$ *with* $a_i(x_i)>0$ *for* $x_i\neq 0$ $(i=1,\cdots,m)$ *such that*

$$\sum_{i=1}^{n}\phi_i(x_i)f_{ij}(x_j)\leqslant -a_j(x_j),\quad j=1,\cdots,m,$$

$$\sum_{i=1}^{n}\phi_i(x_i)f_{ij}(x_j)\leqslant 0,\qquad j=m+1,\cdots,n.$$

Then the zero solution of (1.10.1) *is globally stable with respect to* y.

Proof. We construct the Liapunov function

$$V(x)=\sum_{i=1}^{n}\int_0^{x_i}\phi_i(x_i)\mathrm{d}x_i.$$

Then we have

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(1.10.1)}\leqslant\sum_{i=1}^{n}\phi_i(x_i)\sum_{j=1}^{n}f_{ij}(t,x_j)\leqslant\sum_{j=1}^{m}\sum_{i=1}^{n}\phi_i(x_i)f_{ij}(t,x_j)$$
$$\leqslant-\sum_{j=1}^{m}a_j(x_j).$$

Thus $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(1.10.1)}$ is negative definite with respect to y. Therefore we conclude that the zero solution of (1.10.1) is globally stable with respect to y. ∎

1.11. Notes

For various definitions of this chapter see Hahn [1], Malkin [1], Rumyantsev and Oziraner [1], Liao Xiaoxin [1]. Theorem 1.2.2 is taken from Yoshizawa [1]. The discussion of M-matrices in Section 1.3 follows Ortega and Rheinboldt [1]. Theorem 1.4.4 is well-known as Barbashin-Krasovskii Theorem. The results of Sections 1.5, 1.6 and 1.7 are due to Rumyantsev and Orizaner [1], Reissig, Sansone and Conti [1] and Malkin [1]. All materials of Section 1.8 are taken from Liao Xiaoxin [2], [11], [14]. The concept and method of Sections 1.9 and 1.10 are due to Liao Xiaoxin [3], [4], [17].

CHAPTER 2

AUTONOMOUS CONTROL SYSTEMS

The stability problem is a basic one in designing autonomous control systems. In fact, the development of autonomous control theory began with the analysis of stability of Watt's centrifugal governor by Maxwell. In designing an autonomous control system, the stability should be considered first. The instability of an autonomous control system means that it can not complete its work.

2. 1. The Expression and Classification of the Problems

Consider the general Lurie control systems

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = Ax + bf(\sigma), \\ \sigma = c^{\mathrm{T}}x = \sum\limits_{i=1}^{n} c_i x_i, \end{cases} \tag{2.1.1}$$

where $x = \mathrm{col}(x_1, \cdots, x_n)$, $b = \mathrm{col}(b_1, \cdots, b_n)$ and $c = \mathrm{col}(c_1, \cdots, c_n)$ are n-dimensional real vectors, or $x, b, c \in \mathbf{R}^n$ for short, and $A = (a_{ij})_{n \times n}$ is a real matrix.

Let F, F_k and F_{k_1, k_2} be the following functional spaces:

$$F = \{f : f(0) = 0, \sigma f(\sigma) > 0 \text{ for } \sigma \neq 0, f \in C[(-\infty, +\infty), \mathbf{R}]\},$$

$$F_k = \{f : f(0) = 0, 0 \leqslant f(\sigma)/\sigma \leqslant k < +\infty \text{ for } \sigma \neq 0,$$
$$f \in C[(-\infty, +\infty), \mathbf{R}]\},$$

$$F_{k_1, k_2} = \{f : f(0) = 0, 0 \leqslant k_1 \leqslant f(\sigma)/\sigma \leqslant k_2 < +\infty \text{ for } \sigma \neq 0,$$
$$f \in C[(-\infty, +\infty), \mathbf{R}]\}.$$

In practical problems, we know only that either $f \in F$, or $f \in F_k$, or $f \in F_{k_1, k_2}$, instead of the expression of f, which can only be obtained with the help of experimental records depending on specific load. So, strictly speaking, the system (2. 1. 1) is indefinite, or is called a *multiple valued system of differential equations* or *differential inclusion*.

Definition 2. 1. 1. The zero solution of (2. 1. 1) is said to be *absolutely stable* if for any f in F, the zero solution of (2. 1. 1) is globally stable. The zero solution of (2. 1. 1) is said to be *absolutely stable in the Hurwitz angle* $[0,k]$ (absolutely stable in $[0,k]$ for short) if for any f in F_k, the zero solution of (2. 1. 1) is globally stable. The zero solution of (2. 1. 1) is said to be *absolutely stable in* $[k_1,k_2]$ if for any f in F_{k_1,k_2}, the zero solution of (2. 1. 1) is globally stable.

The well-known Lurie problem is expressed as follows:

What are the necessary and sufficient conditions for the zero solution of (2. 1. 1) *to be absolutely stable (or absolutely stable in* $[0,k]$*)?*

Lurie didn't find these necessary and sufficient conditions.

When studying the absolute stability of (2. 1. 1), we usually make the following classification.

We say that (2. 1. 1) is a *direct control system*, or the *basic case* (*principal case*) if $\mathrm{Re}\,\lambda(A)<0$, that is, A is stable. In other words, the eigenvalues of A have negative real parts only.

We say that (2. 1. 1) is a *indirect control system*, or *the first critical control system*, if $\mathrm{Re}\,\lambda(A)\leqslant 0$, A has only one zero eigenvalue while the other eigenvalues have negative real parts.

We say that (2. 1. 1) is *other critical case*, if $\mathrm{Re}\,\lambda(A)\leqslant 0$ and (2. 1. 1) does not belong to the second classification.

In the following, we will give some often used necessary conditions for the zero solution of (2. 1. 1) to be absolutely stable.

Theorem 2. 1. 2. *If the zero solution of* (2. 1. 1) *is absolutely stable, then one of the following conditions holds:*

1) $\mathrm{Re}\,\lambda(A)\leqslant 0$, *i. e.* , *$A$ has no eigenvalues with positive real part*;

2) $c^{\mathrm{T}}b\leqslant 0$;

3) *if A is stable then* $c^{\mathrm{T}}A^{-1}b\geqslant 0$.

Proof. 1) Suppose there exists an eigenvalue λ_0 such that $\mathrm{Re}\,\lambda_0(A)>0$. By taking $f(\sigma)=\varepsilon\sigma$ $(0<\varepsilon\ll 1)$, the expression (2. 1. 1) becomes

$$\frac{\mathrm{d}x}{\mathrm{d}t}=(A+\varepsilon bc^{\mathrm{T}})x. \tag{2. 1. 2}$$

Since eigenvalues depend on coefficients continuously, for $0<\varepsilon\ll 1$, the coefficient matrix of (2. 1. 2) must have an eigenvalue $\tilde{\lambda}_0$ such that $\mathrm{Re}\,\tilde{\lambda}_0>0$. This is inconsistent with the fact that the zero solution of (2. 1. 1) is absolutely stable.

2) The trace of coefficient matrix of (2. 1. 2) is expressed as

$$\sum_{i=1}^{n} a_{ii} + \varepsilon \sum_{i=1}^{n} c_i b_i < 0.$$

If $c^{\mathrm{T}} b = \sum_{i=1}^{n} c_i b_i > 0$, then for $\varepsilon \gg 1$, it follows that

$$\sum_{i=1}^{n} a_{ii} + \varepsilon \sum_{i=1}^{n} c_i b_i > 0,$$

which is impossible. Thus we have $c^{\mathrm{T}} b \leqslant 0$.

3) We first verify the identity of linear algebra:

$$\det(I + EF) = \det(I + FE),$$

where E and F are arbitrary matrices for which EF and FE exist. The unit matrices on both sides of the equality can be different order.

Indeed,

$$\begin{aligned}
\det(I + EF) &= \det\begin{pmatrix} I + EF & 0 \\ F & I \end{pmatrix} = \det\begin{pmatrix} I & E \\ 0 & I \end{pmatrix}\begin{pmatrix} I & -E \\ F & I \end{pmatrix} \\
&= \det\begin{pmatrix} I & -E \\ F & I \end{pmatrix}\begin{pmatrix} I & E \\ 0 & I \end{pmatrix} = \det\begin{pmatrix} I & 0 \\ F & I + FE \end{pmatrix} \\
&= \det(I + FE).
\end{aligned}$$

Now, we prove $c^{\mathrm{T}} A^{-1} b \geqslant 0$.

Since A and $A + \varepsilon b c^{\mathrm{T}}$ are stable ($\varepsilon > 0$), for $\lambda = 0$, we have

$$\begin{aligned}
&\det(I\lambda - (A + \varepsilon b c^{\mathrm{T}})) \\
&\quad = \det(I\lambda - A) \cdot \det[I - \varepsilon(I\lambda - A)^{-1} b c^{\mathrm{T}}] \\
&\quad = \det(I\lambda - A) \cdot \det[I - \varepsilon c^{\mathrm{T}}(I\lambda - A)^{-1} b] \neq 0.
\end{aligned} \tag{2.1.3}$$

If $c^{\mathrm{T}} A^{-1} b < 0$, then there must exist $\varepsilon_0 > 0$ such that

$$1 - \varepsilon_0 c^{\mathrm{T}}(-A)^{-1} b = 1 + \varepsilon_0 c^{\mathrm{T}} A^{-1} b = 0.$$

It is in contradiction with (2. 1. 3). As a result, $c^{\mathrm{T}} A^{-1} b \geqslant 0$. ∎

2. 2. Necessary and Sufficient Conditions for Absolute Stability

Consider the general Lurie control systems

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = Ax + bf(\sigma), \\ \sigma = c^{\mathrm{T}} x = \sum_{i=1}^{n} c_i x_i, \end{cases} \tag{2.2.1}$$

where $f(\sigma) \in F$, or $f(\sigma) \in F_k$.

Definition 2. 2. 1. The zero solution of (2. 2. 1) is said to be *absolutely stable for the set $\Omega = \{x : \sigma = 0\}$* [*absolutely stable for Ω in* $[0, k]$] if for every $\varepsilon >$

0, there exists $\delta(\varepsilon) > 0$ such that for any $f(\sigma) \in F$ [for any $f(\sigma) \in F_k$], the solution $x(t) = x(t,t_0,x_0)$ of (2.2.1) satisfies

$$|\sigma(t,t_0,x_0)| \doteq |c^{\mathrm{T}}x(t,t_0,x_0)| < \varepsilon \qquad \text{for all} \quad t \geqslant t_0$$

if $\| x_0 \| < \delta(\varepsilon)$, and for any $x_0 \in \mathbf{R}^n$, there exists

$$\lim_{t\to+\infty} \sigma(t,t_0,x_0) = \lim_{t\to+\infty} c^{\mathrm{T}}x(t,t_0,x_0) = 0.$$

Definition 2.2.2. We say that the function $V(x) \in C[\mathbf{R}^n,\mathbf{R}]$ is *positive definite for the set* $\boldsymbol{\Omega}$ if

$$V(x)\begin{cases} = 0 & \text{when} \quad x \in \Omega, \\ > 0 & \text{when} \quad x \notin \Omega. \end{cases}$$

We say $V(x) \in C[\mathbf{R}^n,\mathbf{R}]$ to be *negative definite for* $\boldsymbol{\Omega}$ if $-V(x)$ is positive definite for $\boldsymbol{\Omega}$. We say $V(x) \in C[\mathbf{R}^n,\mathbf{R}]$ to be *radially unbounded positive definite* if $V(x)$ is positive definite for $\boldsymbol{\Omega}$ and $V(x) \to +\infty$ as $|\sigma| = |c^{\mathrm{T}}x| \to +\infty$.

Theorem 2.2.3. *The necessary and sufficient conditions for the zero solution of* (2.2.1) *to be absolutely stable* [*absolutely stable in* $[0,k]$] *are*

1) $A+bc^{\mathrm{T}}\theta \triangleq B$ *is stable, where* $\theta=0$ *or* $\theta=1$ [$\theta=0$ *or* $\theta=k$];

2) *the zero solution of* (2.2.1) *is absolutely stable for* $\boldsymbol{\Omega}$ [*absolutely stable in* $[0,k]$ *for* $\boldsymbol{\Omega}$].

Proof. *Necessity.* 1) When $\operatorname{Re} \lambda(A)<0$, we take $\theta = 0$; when $\operatorname{Re} \lambda(A) \leqslant 0$, we take $\theta = 1$ [$\theta = k$]. By putting $f(\sigma) = \sigma = c^{\mathrm{T}}x$ [$f(\sigma) = k\sigma = kc^{\mathrm{T}}x$] in (2.2.1) it follows that $B = A + bc^{\mathrm{T}}\theta$ is stable.

2) For any $\varepsilon > 0$, we take $\tilde{\varepsilon} = \varepsilon / \max\limits_{1\leqslant i\leqslant n} |c_i|$. Then there exists $\delta(\varepsilon) > 0$ such that for any $f \in F$[for any $f \in F_k$], $\| x_0 \| < \delta(\varepsilon)$ implies that

$$\| x(t,t_0,x_0) \| \triangleq \| x(t) \| = \sum_{i=1}^{n} |x_i(t)| < \tilde{\varepsilon} \qquad \text{for all} \quad t \geqslant t_0,$$

and further

$$\begin{aligned} |\sigma(t,t_0,x_0)| &\triangleq |\sigma(t)| = \sum_{i=1}^{n} |c_i x_i(t)| \\ &\leqslant \max_{1\leqslant i\leqslant n} |c_i| \sum_{i=1}^{n} |x_i(t)| < \max_{1\leqslant i\leqslant n} |c_i| \cdot \tilde{\varepsilon} = \varepsilon. \end{aligned}$$

Clearly, there exists

$$\lim_{t\to+\infty} |\sigma(t)| \leqslant \lim_{t\to+\infty} \max_{1\leqslant i\leqslant n} |c_i| \sum_{i=1}^{n} |x_i(t)| = 0$$

for any $x_0 \in \mathbf{R}^n$.

Sufficiency. For any $f \in F$ [for any $f \in F_k$], the solution of (2.2.1) can be expressed as

$$x(t) = \mathrm{e}^{B(t-t_0)}x_0 + \int_{t_0}^{t} \mathrm{e}^{B(t-\tau)}[bf(\sigma(\tau)) - \theta b\sigma(\tau)]\mathrm{d}\tau.$$

Since B is stable, there exist constants $M \geqslant 1$ and $\alpha > 0$ such that

$$\| e^{B(t-t_0)} \| \leqslant M e^{-\alpha(t-t_0)} \quad \text{for all} \quad t \geqslant t_0.$$

Since $\sigma(t) \to 0$ as $t \to +\infty$, $\sigma(t)$ continuously depends on x_0, and $f(\sigma(t))$ is a continuous function of x_0 and $f(\sigma(t)) \to 0$ as $t \to +\infty$, for any $\varepsilon > 0$, there exist $\delta_1(\varepsilon) > 0$ and $t_1 > t_0$ such that

$$\int_{t_0}^{t_1} M e^{-\alpha(t-\tau)} [\| bf(\sigma(\tau)) \| + \| b\theta\sigma(\tau) \|] d\tau < \frac{\varepsilon}{3},$$

$$\int_{t_1}^{t} M e^{-\alpha(t-\tau)} [\| bf(\sigma(\tau)) \| + \| b\theta\sigma(\tau) \|] d\tau < \frac{\varepsilon}{3} \quad \text{for all} \quad t \geqslant t_1,$$

$$\| e^{B(t-t_0)} x_0 \| \leqslant M e^{-\alpha(t-t_0)} \| x_0 \| < \frac{\varepsilon}{3} \quad \text{for all} \quad t \geqslant t_0$$

if

$$\| x_0 \| < \delta_1(\varepsilon).$$

Thus, it follows

$$\begin{aligned} \| x(t) \| &\leqslant \| e^{B(t-t_0)} x_0 \| + \int_{t_0}^{t_1} M e^{-\alpha(t-\tau)} [\| bf(\sigma(\tau)) \| + \| \theta b\sigma(\tau) \|] d\tau \\ &\quad + \int_{t_1}^{t} M e^{-\alpha(t-\tau)} [\| bf(\sigma(\tau)) \| + \| \theta b\sigma(\tau) \|] d\tau \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \quad \text{for all} \quad t \geqslant t_0. \end{aligned}$$

For any $x_0 \in \mathbf{R}^n$, by using de L' Hospital rule we write

$$\begin{aligned} 0 &\leqslant \lim_{t \to +\infty} \| x(t) \| \\ &\leqslant \lim_{t \to +\infty} M e^{-\alpha(t-t_0)} \| x_0 \| \\ &\quad + \lim_{t \to +\infty} \int_{t_0}^{t} M e^{-\alpha(t-\tau)} [\| bf(\sigma(\tau)) \| + \| \theta b\sigma(\tau) \|] d\tau = 0. \end{aligned}$$

Therefore, the zero solution of (2.2.1) is absolutely stable [absolutely stable in $[0,k]$]. ▮

Theorem 2.2.4. *The necessary and sufficient conditions for the zero solution of* (2.2.1) *to be absolutely stable* [*absolutely stable in* $[0,k]$] *are that*

1) *the condition* 1) *of Theorem* 2.2.3 *is true;*

2) *there exists a differentiable function* $V_f(x) \in C[\mathbf{R}^n, \mathbf{R}]$ *such that* $V_f(0) = 0$,

$$V_f(x) \geqslant \varphi_f(|\sigma|), \qquad \varphi_f \in KR \tag{2.2.2}$$

and

$$\left. \frac{dV_f}{dt} \right|_{(2.2.1)} \leqslant -\psi_f(|\sigma|), \qquad \psi_f \in K. \tag{2.2.3}$$

Proof. *Sufficiency.* It suffices to prove that condition 2) implies that the zero solution of (2.2.1) is absolutely stable for Ω.

Since $V_f(0)=0$ $(0\in\Omega)$ and $V_f(x)$ is continuous, for any $\varepsilon>0$, there exists $\delta(\varepsilon)>0$ such that

$$V_f(x_0)<\varphi_f(\varepsilon) \qquad \text{for} \quad \|x_0\|<\delta(\varepsilon).$$

From (2.2.2) and (2.2.3), it follows

$$\varphi_f(|\sigma(t)|)\leqslant V_f(x(t))\leqslant V_f(x_0)<\varphi_f(\varepsilon).$$

Hence, we deduce that $|\sigma(t)|<\varepsilon$, that is, the zero solution of (2.2.1) is stable for Ω.

Let us prove that $\lim\limits_{t\to+\infty}\sigma(t)=0$.

For any $x_0\in\mathbf{R}^n$ it follows from (2.2.3) that $V_f(x(t))$ is monotone decreasing and has a lower bound. Thus there exists

$$\inf_{t\geqslant t_0}V_f(x(t))=\lim_{t\to+\infty}V_f(x(t))\triangleq\alpha\geqslant 0$$

and α can only be reached on Ω. If α is reached outside Ω, then there must exist a constant $\beta>0$ such that $|\sigma(t)|\geqslant\beta>0$ for all $t\geqslant t_0$. Otherwise, there exists a sequence $\{t_k\}$ with $t_k\to+\infty$ as $k\to+\infty$ such that $\lim\limits_{t_k\to+\infty}\sigma(t_k)=0$, thus

$$\alpha=\inf_{t\geqslant t_0}V_f(x(t))=\lim_{t\to+\infty}V_f(x(t))=\lim_{t_k\to+\infty}V_f(x(t_k))=\lim_{\sigma(t_k)\to 0}V_f(x(t_k)),$$

namely, α is reached on Ω which is in contradiction with that α is reached outside Ω.

For any $x_0\in\mathbf{R}^n$, from (2.2.3), it follows that

$$|\sigma(t)|\leqslant|c^{\mathrm{T}}x(t_0)|\triangleq h<+\infty.$$

If $\lim\limits_{t\to+\infty}\sigma(t)\neq 0$, then according to the uniform continuity of $\sigma(t)$, there exist constants $\beta>0$, $\eta>0$ and a sequence $\{t_j\}$ such that

$$|\sigma(t)|\geqslant\beta \qquad \text{for} \quad t\in[t_j-\eta,t_j+\eta].$$

We fix $r_f=\inf\limits_{\beta\leqslant|\sigma|\leqslant h}\psi_f(|\sigma|)>0$, then

$$\begin{aligned}V_f(x(t))&=V_f(x(t_0))+\int_{t_0}^{t}\frac{\mathrm{d}V_f}{\mathrm{d}t}\mathrm{d}t\\&\leqslant V_f(x(t_0))-\int_{t_0}^{t}\psi_f(|\sigma(\tau)|)\mathrm{d}\tau\\&\leqslant V_f(x(t_0))-\sum_{j=1}^{n}\int_{t_j-\eta}^{t_j+\eta}\psi_f(|\sigma(\tau)|)\mathrm{d}\tau\\&\leqslant V_f(x(t_0))-2\eta n r_f\to-\infty \qquad \text{as} \quad n\to+\infty.\end{aligned}$$

It is in contradiction with $V_f(x(t))\geqslant 0$, so $\lim\limits_{t\to+\infty}\sigma(t)=0$. Then the zero solution of (2.2.1) is absolutely stable for Ω [absolutely stable in $[0,k]$ for Ω]. The sufficiency follows directly from Theorem 2. 2. 3.

Necessity. Since the zero solution of (2. 2. 1) is absolutely stable, $\mathbf{R}^n$ is a region of attraction. For any $x_0 \in \mathbf{R}^n$, let

$$\varphi_f(x) = \sup_{t \geqslant 0} \| x(t,0,x) \|^2.$$

Obviously, $\varphi(x)$ possesses the following properties:

1) $\varphi(x) \geqslant 0$, where the equality holds if and only if $x = 0$, and $\varphi(x)$ is radially unbounded positive definite;

2) $\varphi_f(x(\eta)) = \sup_{t \geqslant \eta} \| x(t) \|^2$ is monotone decreasing;

3) $\varphi_f(x)$ is continuous on $\mathbf{R}^n$.

Again we define

$$V_f(x) = \int_0^{+\infty} \varphi_f(x(\eta,0,x))e^{-\eta}d\eta,$$

then we get

$$V_f(x(t)) = \int_0^{+\infty} \varphi_f(x(t+\eta))e^{-\eta}d\eta.$$

Denoting

$$\Phi(t+\eta) = \int_0^{t+\eta} \varphi_f(x(\xi))d\xi,$$

we obtain

$$\Phi_\eta' = \Phi_t' = \varphi_f(x(t+\eta)).$$

Using integration by parts,

$$\begin{aligned} V_f(x(t)) &= \int_0^{+\infty} e^{-\eta}d\Phi \\ &= e^{-\eta}\int_0^{t+\eta} \varphi_f(x(\xi))d\xi \Big|_0^{+\infty} + \int_0^{+\infty} \Phi(t+\eta)e^{-\eta}d\eta \\ &= -\int_0^t \varphi_f(x(\xi))d\xi + \int_0^{+\infty} \Phi(t+\eta)e^{-\eta}d\eta. \end{aligned}$$

Since $\varphi(x(\xi))$ is monotone decreasing, it is bounded. Thus

$$\lim_{t\to+\infty} e^{-\eta}\int_0^{t+\eta} \varphi_f(x(\xi))d\xi = 0.$$

Now we take the derivative of V_f along the solution of (2. 2. 1). Clearly,

$$\begin{aligned} \frac{dV_f(x(t))}{dt}\Big|_{(2.2.1)} &= -\varphi_f(x(t)) + \int_0^{+\infty} \Phi_t' e^{-\eta}d\eta \\ &= -\varphi_f(x(t)) + \int_0^{+\infty} \varphi_f(x(t+\eta))e^{-\eta}d\eta \\ &= \int_0^{+\infty} [\varphi_f(x(t+\eta)) - \varphi_f(x(t))]e^{-\eta}d\eta. \end{aligned}$$

Since $\varphi_f(x(t))$ is monotone decreasing, we have

$$\varphi_f(x(t)) \geqslant \varphi_f(x(t+\eta)) \qquad \text{for} \quad \eta \geqslant 0.$$

In particular, if $x(t)$ is not the zero solution, then there exists

$$\varphi_f(x(t)) \not\equiv \varphi(x(t+\eta)).$$

Otherwise, we have

$$\varphi(x(t)) \equiv \varphi(x(t+\eta)) \to 0 \qquad \text{as} \quad \eta \to +\infty,$$

i. e. , $\varphi(x(t)) \equiv 0$, which is a contradiction. Thus if $x(t) \not\equiv 0$, we have

$$\int_0^{+\infty} [\varphi(x(t+\eta)) - \varphi(x(t))] e^{-\eta} d\eta < 0,$$

that is

$$\left.\frac{dV_f}{dt}\right|_{(2.2.1)} < 0 \qquad \text{for} \quad x \neq 0.$$

Therefore, we see that

$$\left.\frac{dV_f}{dt}\right|_{(2.2.1)} \leqslant - W_f(x),$$

where $W_f(x)$ is a positive definite function. From the equivalence relation between the positive definite function and the K-class function, it follows that there exist $\tilde{\varphi}_f(r) \in KR$ and $\tilde{\psi}(r) \in K$ such that

$$\tilde{\varphi}(\|x\|) \leqslant V_f(x), \qquad - W_f(x) \leqslant - \tilde{\psi}_f(\|x\|).$$

Therefore we write

$$\begin{aligned} V_f(x) &\geqslant \tilde{\varphi}_f(\|x\|) \triangleq \tilde{\varphi}_f\left(\sum_{i=1}^n |x_i|\right) \\ &\geqslant \tilde{\varphi}_f\left(\frac{1}{\max\limits_{1\leqslant i\leqslant n} |c_i|} \sum_{i=1}^n |c_i x_i|\right) \\ &= \tilde{\varphi}_f\left(\frac{1}{\max\limits_{1\leqslant i\leqslant n} |c_i|} |\sigma|\right) \triangleq \bar{\varphi}_f(|\sigma|) \in KR, \end{aligned}$$

and confirm $V_f(x)$ is positive definite for Ω. Moreover, there exists

$$\begin{aligned} \left.\frac{dV_f}{dt}\right|_{(2.2.1)} &\leqslant - W_f(x) \leqslant - \tilde{\psi}_f(\|x\|) \\ &\leqslant - \tilde{\psi}\left(\frac{1}{\max\limits_{1\leqslant i\leqslant n} |c_i|} \sum_{i=1}^n |c_i x_i|\right) \\ &= - \tilde{\psi}\left(\frac{1}{\max\limits_{1\leqslant i\leqslant n} |c_i|} |\sigma|\right) \triangleq - \bar{\psi}_f(|\sigma|) \in K. \end{aligned}$$

The condition 2) of Theorem 2. 2. 4 is satisfied.

The proof of condition 1) of Theoren 2. 2. 4, it is trivial. ∎

By imitating Theorems 2. 2. 3 and 2. 2. 4 we formulate

Theorem 2. 2. 5. *The necessary and sufficient conditions for the zero solution of* (2. 2. 1) *to be absolutely stable* [*absolutely stable in* $[0,k]$] *are that*

1) *there exists* $\bar{b} \in \mathbf{R}^n$ *such that* $A + \bar{b}c^{\mathrm{T}}$ *is stable*;

2) *the condition* 2) *of Theorem* 2. 2. 3 *is satisfied.*

Theorem 2. 2. 6. *The necessary and sufficient conditions for the zero solution to be absolutely stable* [*absolutely stable in* $[0,k]$] *are that*

1) *the condition* 1) *of Theorem* 2. 2. 5 *is satisfied*;

2) *the condition* 2) *of Theorem* 2. 2. 4 *is satisfied.*

Theorems 2. 2. 5 and 2. 2. 6 are useful because it is more convenient to verify the stability of $A + \bar{b}c^{\mathrm{T}}$ than that of $A + bc^{\mathrm{T}}\theta$.

Theorem 2. 2. 7. *Suppose the following conditions are satisfied*:

1) $A + \theta bc^{\mathrm{T}}$ *is stable*, $\theta = 0$ *or* $\theta = 1$;

2) *there exist a symmetric matrix* $P_{n\times n}$ *such that*

$$\begin{cases} x^{\mathrm{T}}Px \geqslant \alpha\sigma^2, & \alpha > 0, \\ x^{\mathrm{T}}(PA + A^{\mathrm{T}}P)x + (2Pb + \beta A^{\mathrm{T}}c)^{\mathrm{T}}xf(\sigma) + \beta c^{\mathrm{T}}bf^2(\sigma) \leqslant -\varepsilon\tau, & \beta \geqslant 0, \end{cases}$$

or

$$\begin{cases} x^{\mathrm{T}}Px \geqslant 0, \quad \int_0^{\pm\infty} f(\sigma)\mathrm{d}\sigma = +\infty, \\ x^{\mathrm{T}}(PA + A^{\mathrm{T}}P)x + (2Pb + \beta A^{\mathrm{T}}c)^{\mathrm{T}}xf(\sigma) + \beta c^{\mathrm{T}}bf^2(\sigma) \leqslant -\varepsilon\tau, \quad \beta > 0, \end{cases}$$

where $\tau \in \{\sigma^2, \sigma f(\sigma), f^2(\sigma)\}$, $0 < \varepsilon \ll 1$, $\alpha, \beta, \varepsilon$ *are constants.*

Then the zero solution of (2. 2. 1) *is absolutely stable.*

Proof. It suffices to prove that the condition 2) of Theorem 2. 2. 7 implies the condition 2) of Theorem 2. 2. 4.

In fact, we construct

$$V(x) = x^{\mathrm{T}}Px + \beta\int_0^{\sigma} f(\sigma)\mathrm{d}\sigma,$$

by the condition 2), it bears

$$V(x) \geqslant \alpha\sigma^2 \geqslant \varphi(|\sigma|), \qquad \varphi \in KR$$

or

$$V(x) \geqslant \beta\int_0^{\sigma} f(\sigma)\mathrm{d}\sigma \geqslant \phi(|\sigma|), \qquad \phi \in KR,$$

and

$$\begin{aligned} \left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.2.1)} &= x^{\mathrm{T}}(PA + A^{\mathrm{T}}P)x + (2Pb + \beta A^{\mathrm{T}}c)^{\mathrm{T}}xf(\sigma) + \beta c^{\mathrm{T}}bf^2(\sigma) \\ &\leqslant -\varepsilon\tau \leqslant -\psi(|\sigma|), \qquad \psi \in K, \end{aligned}$$

Thus all the conditions of Theorem 2. 2. 4 are satisfied, and the desired conclusion of the theorem holds. ∎

As a special case, Theorem 2. 2. 7 can contain all the criteria of absolute stability obtained by the Lurie function

$$V(x) = x^{\mathrm{T}}Px + \beta\int_0^{\sigma} f(\sigma)\mathrm{d}\sigma,$$

which makes $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.2.1)}$ negative definite for positive definite P.

Corollary 2. 2. 8. *If A is stable and there exist a constant $\beta \geqslant 0$ and a symmetric positive definite matrix P such that*

$$x^{\mathrm{T}}(PA + A^{\mathrm{T}}P)x + 2\left(Pb + \frac{1}{2}\beta A^{\mathrm{T}}c\right)^{\mathrm{T}} xf(\sigma) + \beta c^{\mathrm{T}}bf^2(\sigma)$$

is negative definite, then the zero solution of (2. 2. 1) *is absolutely stable.*

Proof. It suffices to prove that the conditions of Theorem 2. 2. 7 are satisfied. In fact, since A is stable, the condition 1) of Theorem 2. 2. 7 holds.

Now, we construct the Liapunov function

$$V(x) = x^{\mathrm{T}}Px + \beta\int_0^{\sigma} f(\sigma)\mathrm{d}\sigma.$$

Obviously, $V(0) = 0$ and

$$V(x) \geqslant \lambda_1 x^{\mathrm{T}}x \geqslant \lambda_1 \frac{n\sum_{i=1}^{n}|c_i x_i|^2}{n\max\limits_{1\leqslant i\leqslant n}|c_i|^2} \geqslant \lambda_1 \frac{\left(\sum_{i=1}^{n} c_i x_i\right)^2}{n\max\limits_{1\leqslant i\leqslant n}|c_i|^2}$$

$$= \lambda_1 \frac{\sigma^2}{n\max\limits_{1\leqslant i\leqslant n}|c_i|^2} \triangleq \varphi(|\sigma|) \in KR.$$

Here λ_1 refers to the smallest eigenvalue of P. Thus $V(x)$ is radially unbounded positive definite for Ω. Again

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.2.1)} \leqslant -\psi(\|x\|) \leqslant -\psi\left(\frac{1}{\max\limits_{1\leqslant i\leqslant n}|c_i|}\sum_{i=1}^{n}|c_i x_i|\right)$$

$$= -\psi\left(\frac{1}{\max\limits_{1\leqslant i\leqslant n}|c_i|}|\sigma|\right) \triangleq -\psi_1(|\sigma|),\ .$$

where $\psi_1, \psi \in K$. Therefore, all the conditions of Theorem 2. 2. 7 are satisfied and the corollary follows. ∎

Now we turn to the indirect control systems

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = Ax + bf(\sigma), \\ \dfrac{\mathrm{d}\sigma}{\mathrm{d}t} = c^{\mathrm{T}}x - \rho f(\sigma), \end{cases} \tag{2. 2. 4}$$

where $A \in \mathbf{R}^{n\times n}$, $b \in \mathbf{R}^n$, $c \in \mathbf{R}^n$, $f(\sigma) \in F$.

Corollary 2. 2. 9. *If A and $\begin{pmatrix} A & b \\ c^{\mathrm{T}} & -\rho \end{pmatrix}$ are stable and there exist a symmetric positive semi-definite matrix G and a positive constant p, such that*

$$W(x) = -x^{\mathrm{T}}Gx + f(\sigma)\{2u^{\mathrm{T}}x + pc^{\mathrm{T}}A^{-1}x\} - \rho f^2(\sigma)$$

is negative semi-definite, where $u = Pb + \frac{c}{2}$, P is the solution of the Liapunov equation

$$PA + A^{\mathrm{T}}P = -G,$$

then the zero solution of (2. 2. 4) is absolutely stable.

Proof. We construct the Liapunov function

$$V(x) = x^{\mathrm{T}}Px + \int_0^{\sigma} f(\sigma)\mathrm{d}\sigma + \frac{p}{2(\rho + c^{\mathrm{T}}A^{-1}b)}(c^{\mathrm{T}}A^{-1}x - \sigma)^2,$$

then

$$\begin{aligned} \left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.2.4)} &= -x^{\mathrm{T}}Gx + 2f(\sigma)\left(Pb + \frac{1}{2}c\right)^{\mathrm{T}}x - \rho f^2(\sigma) \\ &\quad + \frac{p}{(\rho + c^{\mathrm{T}}A^{-1}b)}(c^{\mathrm{T}}A^{-1}x - \sigma)(\rho + c^{\mathrm{T}}A^{-1}b)f(\sigma) \\ &= -x^{\mathrm{T}}Gx + f(\sigma)\{2u^{\mathrm{T}}x + pc^{\mathrm{T}}A^{-1}x\} \\ &\quad - \rho f^2(\sigma) - p\sigma f(\sigma) \\ &= W(x) - p\sigma f(\sigma) \\ &\leqslant -p\sigma f(\sigma). \qquad \blacksquare \end{aligned}$$

Example 2. 2. 10. Let's discuss the absolute stability of the zero solution of the following system:

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = x_2 - f(x_1 - x_2), \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = -x_1 + f(x_1 - x_2), \end{cases} \tag{2. 2. 5}$$

where $f \in F$, and the coefficient matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ has a pair of purely imaginary eigenvalues. Thus, Example 2. 2. 10 is neither a Lurie direct control system, nor a Lurie indirect control system, but a more complicated critical case. Therefore, the Liapunov matrix equation

$$AP + A^{\mathrm{T}}P = -G$$

has no solution P of symmetric positive definite matrix for any positive definite matrix G. Thus, the traditional Lurie method, i. e., the method of using the Lurie function

$$V(x) = x^{\mathrm{T}}Px + \beta\int_0^{\sigma} f(\sigma)\mathrm{d}\sigma \qquad (P \text{ being positive definite})$$

which makes $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.2.5)}$ negative definite, cannot be applied. Instead we can use Theorem 2.2.7.

(i) Let $f(x_1 - x_2) = x_1 - x_2$. Then the system (2.2.5) changes into

$$\begin{cases} \frac{\mathrm{d}x_1}{\mathrm{d}t} = -x_1 + 2x_2, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} = -x_2. \end{cases} \tag{2.2.6}$$

Obviously, the coefficient matrix $B = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$ is stable.

(ii) We construct the Liapunov function

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2)$$

then

$$\begin{aligned} \left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.2.5)} &= -x_1 f(x_1 - x_2) + x_2 f(x_1 - x_2) \\ &= -(x_1 - x_2) f(x_1 - x_2) \end{aligned}$$

is negative definite for $\Omega = \{x; \sigma = x_1 - x_2 = 0\}$. Thus, all the conditions of Theorem 2.2.7 are satisfied. The zero solution of (2.2.5) is absolutely stable.

Example 2.2.11. Consider the 3-dimensional system

$$\begin{cases} \frac{\mathrm{d}x_1}{\mathrm{d}t} = -3x_1 + x_2 + x_3 - f(x_1 + 2x_2 + x_3), \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} = x_1 - 2x_2 + x_3 + f(x_1 + 2x_2 + x_3), \\ \frac{\mathrm{d}x_3}{\mathrm{d}t} = x_1 + 3x_2 - 3x_3 - 2f(x_1 + 2x_2 + x_3). \end{cases} \tag{2.2.7}$$

It is easy to verify that

$$A = \begin{bmatrix} -3 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 3 & -3 \end{bmatrix}$$

is stable. We construct

$$V = (x_1 + 2x_2 + x_3)^2,$$

then

$$\begin{aligned} \left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.2.7)} &= 2(x_1 + 2x_2 + x_3)\left(\frac{\mathrm{d}x_1}{\mathrm{d}t} + 2\frac{\mathrm{d}x_2}{\mathrm{d}t} + \frac{\mathrm{d}x_3}{\mathrm{d}t}\right) \\ &= -2(x_1 + 2x_2 + x_3) f(x_1 + 2x_2 + x_3). \end{aligned}$$

Consequently, $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.2.7)}$ is negative definite for $\Omega = \{x; \sigma = x_1 + 2x_2 + x_3 = 0\}$.

Clearly, V is radially unbounded positive definite for $\Omega = \{x : \sigma = x_1 + 2x_2 + x_3 = 0\}$. We conclude that the zero solution of (2. 2. 7) is absolutely stable.

Example 2. 2. 12. Consider the indirect control system

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 - f(\sigma), \\ \dfrac{dx_2}{dt} = - x_1 + f(\sigma), \\ \dfrac{d\sigma}{dt} = x_1 - x_2 - \rho f(\sigma), \end{cases} \tag{2. 2. 8}$$

where $\rho > 0$, $f(\sigma) \in F$ and $\int_0^{\pm\infty} f(\sigma) d\sigma = +\infty$.

Since $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is unstable, the Liapunov matrix equation

$$A^T P + PA = - G$$

has no positive definite matrix solution for any positive definite matrix G . So the traditional Lurie method cannot be applied. Instead, we can use Theorem 2. 2. 7.

(i) Let $f(\sigma) = \sigma$. Then (2. 2. 8) changes into

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 - \sigma, \\ \dfrac{dx_2}{dt} = - x_1 + \sigma, \\ \dfrac{d\sigma}{dt} = x_1 - x_2 - \rho\sigma. \end{cases}$$

Since

$$\det|\lambda E - B| = \begin{vmatrix} \lambda & -1 & 1 \\ 1 & \lambda & -1 \\ -1 & 1 & \lambda + \rho \end{vmatrix} = \lambda^3 + \rho\lambda^2 + 3\lambda + \rho,$$

the necessary and sufficient conditions for the characteristic polynomial to be the Hurwitz one are that

$$\rho > 0, \qquad \Delta_1 = 3 > 0, \qquad \Delta_2 = \begin{vmatrix} 3 & \rho \\ 1 & \rho \end{vmatrix} = 2\rho > 0.$$

Thus

$$B = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & -\rho \end{pmatrix}$$

is stable.

(ii) We construct

$$V(x,\sigma) = \frac{1}{2}(x_1^2 + x_2^2) + \int_0^\sigma f(\sigma)\mathrm{d}\sigma.$$

Then

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.2.8)} = -\rho f^2(\sigma) < 0 \qquad \text{when} \quad \sigma \neq 0.$$

Therefore, the zero solution of (2.2.8) is absolutely stable.

2.3. The S-Method and Modified S-Method

For the direct control system (2.2.1), we construct the Liapunov function consisting of an integral term and a quadratic form:

$$V(x) = x^{\mathrm{T}}Px + \beta\int_0^\sigma f(\sigma)\mathrm{d}\sigma,$$

where P denotes the solution of the Liapunov matrix equation

$$PA + A^{\mathrm{T}}P = -R,$$

R stands for a given $n \times n$ symmetric positive definite matrix and $\beta \geqslant 0$ is a constant.

Provided that

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.2.1)} = x^{\mathrm{T}}(PA + A^{\mathrm{T}}P)x + 2\left(Pb + \frac{1}{2}\beta A^{\mathrm{T}}c\right)^{\mathrm{T}} xf(\sigma) + \beta c^{\mathrm{T}}bf^2(\sigma) \tag{2.3.1}$$

is negative definite, the zero solution of (2.2.1) is absolutely stable. One can regard (2.3.1) as a quadratic form in x and $f(\sigma)$ and estimate its sign by means of the Selvester condition. Thus, (2.3.1) is negative definite in x and $f(\sigma)$ if and only if

$$\det\begin{vmatrix} R & -\left(\frac{1}{2}\beta A^{\mathrm{T}}c + Pb\right) \\ -\left(\frac{1}{2}\beta A^{\mathrm{T}}c + Pb\right)^{\mathrm{T}} & -\beta c^{\mathrm{T}}b \end{vmatrix} > 0. \tag{2.3.2}$$

But the condition (2.3.2) can never be satisfied. It can be shown that

$$\det\begin{vmatrix} R & -\left(\frac{1}{2}\beta A^{\mathrm{T}}c + Pb\right) \\ -\left(\frac{1}{2}\beta A^{\mathrm{T}}c + Pb\right)^{\mathrm{T}} & -\beta c^{\mathrm{T}}b \end{vmatrix} \leqslant 0.$$

Hence, it is impossible to use Selvester condition to find the sign of (2.3.1) regarded as a quadratic form in x and $f(\sigma)$. This was pointed out by Xie

Huimin [2].

To overcome this difficulty a new method called S-process was discovered.

By adding and subtracting $\alpha f(\sigma)\left(\sigma - \frac{1}{k}f(\sigma)\right)$ in (2.3.1) with constant $\alpha \geqslant 0$, we deduce

$$\begin{aligned}
\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.2.1)} &= x^{\mathrm{T}}(PA + A^{\mathrm{T}}P)x + 2\left(Pb + \frac{1}{2}\beta A^{\mathrm{T}}c\right)^{\mathrm{T}} xf(\sigma) \\
&\quad + \beta c^{\mathrm{T}} b f^2(\sigma) + \alpha f(\sigma)\left(\sigma - \frac{1}{k}f(\sigma)\right) \\
&\quad - \alpha f(\sigma)\left(\sigma - \frac{1}{k}f(\sigma)\right) \\
&\triangleq -S(x,\sigma) - \alpha f(\sigma)\left(\sigma - \frac{1}{k}f(\sigma)\right),
\end{aligned}$$

where

$$\begin{aligned}
S(x,\sigma) &= x^{\mathrm{T}}Rx + 2d^{\mathrm{T}}xf(\sigma) + rf^2(\sigma), \\
R &= -PA - A^{\mathrm{T}}P, \\
d &= -\left[Pb + \frac{1}{2}(\alpha c + \beta A^{\mathrm{T}}c)\right], \\
r &= -\beta c^{\mathrm{T}}b + \frac{\alpha}{k}.
\end{aligned}$$

If $S(x,\sigma)$ is positive definite in x and σ, then $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.2.1)}$ is negative definite. The Selvester condition for $S(x,\sigma)$ to be positive definite in x and σ is usually satisfied. Then the following result is valid.

Theorem 2.3.1. (S-process) *Let A be stable and suppose there exist constants $\alpha>0$, $\beta\geqslant 0$ and a real symmetric positive definite matrix P such that*

$$r > 0, \qquad R - \frac{1}{r}dd^{\mathrm{T}} > 0$$

or

$$R > 0, \qquad r - d^{\mathrm{T}}R^{-1}d > 0.$$

Then (2.3.1) *is negative definite, and the zero solution of* (2.2.1) *is absolutely stable.*

Proof. It suffices to prove that

$$S(x,\sigma) = x^{\mathrm{T}}Rx + 2d^{\mathrm{T}}xf(\sigma) + rf^2(\sigma)$$

is positive definite, i.e.,

$$\det\begin{vmatrix} R & d \\ d^{\mathrm{T}} & r \end{vmatrix} > 0.$$

Using linear algebra, we reach to conclusion from the following two relations:

$$\begin{pmatrix} I & -d/r \\ 0 & 1 \end{pmatrix}\begin{pmatrix} R & d \\ d^T & r \end{pmatrix}\begin{pmatrix} I & 0 \\ -d^T/r & 1 \end{pmatrix} = \begin{pmatrix} R - dd^T/r & 0 \\ 0 & r \end{pmatrix},$$

$$\begin{pmatrix} I & 0 \\ -d^TR^{-1} & 1 \end{pmatrix}\begin{pmatrix} R & d \\ d^T & r \end{pmatrix}\begin{pmatrix} I & -R^{-1}d \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R & 0 \\ 0 & r - d^TR^{-1}d \end{pmatrix}. \qquad \blacksquare$$

Naturally, readers can easily see that positive definiteness of $S(x,\sigma)$ is only the sufficient condition for $\left.\frac{dV}{dt}\right|_{(2.2.1)}$ to be negative definite. If this condition is not necessary, we say that the S-method has a defect. Aizeman and Gantmather [1, p. 119] presented two problems. The second one is whether there is an example in which the S-method cannot be applied but one can judge if that $\left.\frac{dV}{dt}\right|_{(2.2.1)}$ is negative definite by other methods.

In Zhao Suxia [3], the author presented the following example.

Example 2.3.2. Consider the system

$$\begin{cases} \dfrac{dx_1}{dt} = -2x_1 + f(x_2), \\ \dfrac{dx_2}{dt} = x_1 - x_2 - \dfrac{1}{2}f(x_2), \end{cases} \qquad k = +\infty, \qquad (2.3.3)$$

where $A = \begin{pmatrix} -2 & 0 \\ 1 & -1 \end{pmatrix}$ is stable; $b = \mathrm{col}(1, -1/2)$; $c = \mathrm{col}(0,1)$, and take

$$V = \frac{1}{2}x_1^2 + x_2^2 + \int_0^{x_2} f(x_2)dx_2.$$

Then we arrive at

$$\left.\frac{dV}{dt}\right|_{(2.3.3)} = -2x_1^2 + 2x_1x_2 - 2x_2^2 + 2x_1f(x_2) - 2x_2f(x_2) - \frac{1}{2}f^2(x_2)$$

$$= \begin{bmatrix} x_1 \\ x_2 \\ f(x_2) \end{bmatrix}^T \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & -1+\alpha \\ 1 & -1+\alpha & -1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ f(x_2) \end{bmatrix} - \alpha x_2 f(x_2).$$

It is not difficult to verify that there is no $\alpha > 0$ such that

$$\det \begin{vmatrix} -2 & 1 & 1 \\ 1 & -2 & -1+\alpha \\ 1 & -1+\alpha & -1/2 \end{vmatrix} > 0.$$

However, in Zhao Suxia [3], the author proved that $\left.\frac{dV}{dt}\right|_{(2.3.3)}$ is negative definite by some another method.

When we judge that the derivative $\dot{V}$ of the function V with the Lurie form is negative definite, the original S-method is the most useful one and is used extensively. This method is also widely used for the nonautonomous systems, multiple adjusted systems etc.. But the original S-method requires that $S(x,\sigma)$ is negative definite in x and σ. If it is only negative semi-definite, one can apply Barbashin-Krasovskii's theorem or LaSalle's invariance principle. But it is rather troublesome. Our goal is to improve the S-method such that it can be widely used.

Theorem 2. 3. 3. *Assume that A is stable. If there exist constants $\alpha > 0$, $\beta \geqslant 0$ and a real symmetric positive definite matrix P such that*

$$r > 0, \qquad R - \frac{1}{r}dd^{\mathrm{T}} \geqslant 0 \tag{2.3.4}$$

or

$$R > 0, \qquad r - d^{\mathrm{T}}R^{-1}d \geqslant 0, \tag{2.3.5}$$

then the zero solution of (2. 2. 1) is absolutely stable, where

$$R = -PA - A^{\mathrm{T}}P,$$

$$d = -\left[Pb + \frac{1}{2}(\alpha c + \beta A^{\mathrm{T}}c)\right],$$

$$r = -\beta c^{\mathrm{T}}b + \frac{\alpha}{k},$$

$$0 \leqslant \frac{f(\sigma)}{\sigma} \leqslant k \leqslant +\infty.$$

Proof. We construct the Liapunov function

$$V(x) = x^{\mathrm{T}}Px + \beta\int_0^{\sigma} f(\sigma)\mathrm{d}\sigma,$$

then

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.2.1)} = x^{\mathrm{T}}(PA + A^{\mathrm{T}}P)x + 2\left(Pb + \frac{1}{2}\beta A^{\mathrm{T}}c\right)^{\mathrm{T}} xf(\sigma) + \beta c^{\mathrm{T}}bf^2(\sigma).$$

By using S-method, we order $-\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.2.1)}$ as follows

$$-\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.2.1)} = x^{\mathrm{T}}Rx + 2d^{\mathrm{T}}xf(\sigma) + rf^2(\sigma) + \alpha f(\sigma)\left(\sigma - \frac{1}{k}f(\sigma)\right)$$

$$\triangleq S(x,\sigma) + \alpha f(\sigma)\left(\sigma - \frac{1}{k}f(\sigma)\right),$$

where

$$S(x,\sigma) = x^{\mathrm{T}}Rx + 2d^{\mathrm{T}}xf(\sigma) + rf^2(\sigma).$$

When $k = +\infty$, the condition (2. 3. 4) or (2. 3. 5) implies that $S(x,\sigma) \geqslant 0$. Thus

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.2.1)} \leqslant -\alpha f(\sigma)\sigma$$

is negative definite for Ω. Therefore, from Theorem 2. 2. 1 it follows that the conclusion is true. ∎

Corollary 2. 3. 4. *Suppose $k=+\infty$ and one of the following conditions hold:*

1) $\left(\frac{\beta A^{\mathrm{T}}c}{2}+Pb\right)^{\mathrm{T}}R^{-1}\left(\frac{\beta A^{\mathrm{T}}c}{2}+Pb\right)+\beta c^{\mathrm{T}}b<0$;

2) $\left(\frac{\beta A^{\mathrm{T}}c}{2}+Pb\right)^{\mathrm{T}}R^{-1}c<0$ *and*

$$\left[\left(\frac{\beta A^{\mathrm{T}}c}{2}+Pb\right)^{\mathrm{T}}R^{-1}c\right]^2 - (c^{\mathrm{T}}R^{-1}c)\left[\left(\frac{\beta A^{\mathrm{T}}c}{2}+Pb\right)^{\mathrm{T}}R^{-1}\left(\frac{\beta A^{\mathrm{T}}c}{2}+Pb\right)+\beta c^{\mathrm{T}}b\right]\geqslant 0;$$

3) $\left(\frac{\beta A^{\mathrm{T}}c}{2}+Pb\right)^{\mathrm{T}}R^{-1}c\leqslant 0$ *and*

$$\left[\left(\frac{\beta A^{\mathrm{T}}c}{2}+Pb\right)^{\mathrm{T}}R^{-1}c\right]^2 - (c^{\mathrm{T}}R^{-1}c)\left[\left(\frac{\beta A^{\mathrm{T}}c}{2}+Pb\right)^{\mathrm{T}}R^{-1}\left(\frac{\beta A^{\mathrm{T}}c}{2}+Pb\right)+\beta c^{\mathrm{T}}b\right]> 0.$$

Then there exists $\alpha>0$ such that $S(x,\sigma)\geqslant 0$. Therefore, the zero solution of (2. 2. 1) is absolutely stable.

Proof. We have $R>0$. Let $d^{\mathrm{T}}R^{-1}d-r=0$, i. e.,

$$\left(\frac{\beta A^{\mathrm{T}}c}{2}+Pb+\frac{\alpha c}{2}\right)^{\mathrm{T}}R^{-1}\left(\frac{\beta A^{\mathrm{T}}c}{2}+Pb+\frac{\alpha c}{2}\right)+\beta c^{\mathrm{T}}b=0,$$

or

$$\frac{c^{\mathrm{T}}}{2}R^{-1}\frac{c}{2}\alpha^2+\left[\left(\frac{\beta A^{\mathrm{T}}c}{2}+Pb\right)^{\mathrm{T}}R^{-1}\frac{c}{2}+\frac{c^{\mathrm{T}}}{2}R^{-1}\left(\frac{\beta A^{\mathrm{T}}c}{2}+Pb\right)\right]\alpha + \left(\frac{\beta A^{\mathrm{T}}c}{2}+Pb\right)^{\mathrm{T}}R^{-1}\left(\frac{\beta A^{\mathrm{T}}c}{2}+Pb\right)+\beta c^{\mathrm{T}}b=0. \tag{2. 3. 6}$$

(2. 3. 6) has the positive solution α, if and only if one of the conditions 1), 2) and 3) holds. In this case the condition of Theorem 2. 3. 3 holds, so the conclusion is clear. ∎

Corollary 2. 3. 5. *If $k<+\infty$ and the following conditions hold:*

$$\frac{1}{k}-c^{\mathrm{T}}R^{-1}d>0,$$

$$\left(\frac{1}{k}-c^{\mathrm{T}}R^{-1}d\right)^2-c^{\mathrm{T}}R^{-1}c(d^{\mathrm{T}}R^{-1}d+\beta c^{\mathrm{T}}b)>0,$$

then the derivative of $V(x)=x^{\mathrm{T}}Px+\beta\int_0^\sigma f(\sigma)\mathrm{d}\sigma$ along the solution of (2. 2. 1) is negative definite for Ω, thus the zero solution of (2. 2. 1) is absolutely stable in $[0,k]$.

Proof. The conditions imply that there exists ε with $0<\varepsilon\ll 1$ such that

$$\frac{1}{k+\varepsilon} - c^{\mathrm{T}}R^{-1}d > 0, \tag{2.3.7}$$

$$\left(\frac{1}{k+\varepsilon} - c^{\mathrm{T}}R^{-1}d\right)^2 - c^{\mathrm{T}}R^{-1}c(d^{\mathrm{T}}R^{-1}d + \beta c^{\mathrm{T}}b) > 0. \tag{2.3.8}$$

Consequently, we derive

$$-\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.2.1)} = x^{\mathrm{T}}Rx + 2d^{\mathrm{T}}xf(\sigma) + \tilde{r}f^2(\sigma) + \alpha f(\sigma)\left(\sigma - \frac{1}{k+\varepsilon}f(\sigma)\right)$$

$$\triangleq \tilde{S}(x,\sigma) + \alpha f(\sigma)\left(\sigma - \frac{1}{k+\varepsilon}f(\sigma)\right),$$

where

$$d = -\left[Pb + \frac{1}{2}(\alpha c + \beta A^{\mathrm{T}}c)\right],$$

$$\tilde{r} = -\left(\beta c^{\mathrm{T}}b - \frac{\alpha}{k+\varepsilon}\right).$$

Obviously, R is positive definite. Thus, the conditions (2. 3. 7) and (2. 3. 8) guarantee that

$$\det\begin{vmatrix} R & d \\ d^{\mathrm{T}} & \tilde{r} \end{vmatrix} = 0$$

has positive solution for α, so, we find $\tilde{S}(x,\alpha) \geqslant 0$, and therefore the zero solution of (2. 2. 1) is absolutely stable in $[0,k]$. ∎

Example 2. 3. 2 indicates that the original S-method losed efficiency. In the following, we still adopt this example to illustrate that the modified S-method easily judge the absolute stability.

Taking

$$V = \frac{1}{2}x_1^2 + x_2^2 + \int_0^{x_2} f(x_2)\mathrm{d}x_2,$$

we derive

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.3.3)} = -2x_1^2 + 2x_1x_2 - 2x_2^2 + 2x_1f(x_2) - 2x_2f(x_2) - \frac{1}{2}f^2(x_2)$$

$$= \begin{pmatrix} x_1 \\ x_2 \\ f(x_2) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & -1+\alpha \\ 1 & -1+\alpha & -1/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ f(x_2) \end{pmatrix} - \alpha x_2 f(x_2).$$

Taking $\alpha = 1/2$, we get

$$\det\begin{vmatrix} -2 & 1 & 1 \\ 1 & -2 & -1/2 \\ 1 & -1/2 & -1/2 \end{vmatrix} = 0.$$

Thus

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.3.3)} \leqslant -\frac{1}{2}x_2 f(x_2),$$

that is, $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.3.3)}$ is negative definite for $\Omega = \{x; \sigma = x_2 = 0\}$. Therefore the zero solution of (2. 3. 3) is absolutely stable. Evidently, the modified S-method is more general than the standard one.

2. 4. Direct Control Systems

Consider the direct control system

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = Ax + bf(\sigma), \\ \sigma = c^{\mathrm{T}}x, \end{cases} \tag{2. 4. 1}$$

where $A \in \mathbf{R}^{n\times n}$, A is stable; $b,c \in \mathbf{R}^n$ and $f \in F$ [$f \in F_k$].

Theorem 2. 4. 1. *Assume that there exist an* $n\times n$ *symmetric positive definite matrix* P *and a constant* $\beta \geqslant 0$ *such that for*

$$V(x) = x^{\mathrm{T}}Px + \beta\int_0^{\sigma} f(\sigma)\mathrm{d}\sigma, \tag{2. 4. 2}$$

the derivative

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.4.1)} = x^{\mathrm{T}}(PA + A^{\mathrm{T}}P)x + 2\left(Pb + \frac{1}{2}\beta A^{\mathrm{T}}c\right)^{\mathrm{T}} xf(\sigma) + \beta c^{\mathrm{T}}bf^2(\sigma) \tag{2. 4. 3}$$

is negative definite with respect to x, *in particular, there exists* $\alpha > 0$ *such that*

$$S(x,\sigma) \triangleq x^{\mathrm{T}}(PA + A^{\mathrm{T}}P)x + 2\left(Pb + \frac{1}{2}\beta A^{\mathrm{T}}c\right)^{\mathrm{T}} xf(\sigma) + \alpha f(\sigma)\left(\sigma - \frac{1}{k}f(\sigma)\right)$$

is negative definite with respect to x,σ. *Then the zero solution of* (2. 4. 1) *is absolutely stable* [*absolutely stable in* [0, k]].

Proof. For any $f \in F$ [for any $f \in F_k$], the expression (2. 4. 2) is a radially unbounded positive definite function. In addition, the negative definiteness of $S(x,\sigma)$ for x,σ implies that (2. 4. 3) is negative definite, and hence the zero solution of (2. 4. 1) is absolutely stable. ∎

The negative definiteness of $S(x,\sigma)$ for x,σ, is just a sufficient condition for negative definiteness of (2. 4. 3). Below, we describe the necessary and sufficient condition for negative definiteness of (2. 4. 3).

Theorem 2. 4. 2. *Assume that* A *is stable. Given a symmetric negative* *definite matrix* $-B$, *the solution* P *of the Liapunov matrix equation*

$$A^{\mathrm{T}}P + PA = -B$$

provides the Liapunov function

$$V(x) = x^{\mathrm{T}}Px + 2\beta\int_0^{\sigma} f(\sigma)\mathrm{d}\sigma, \qquad f \in F_k \tag{2.4.4}$$

the derivative of which

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.4.1)} = -x^{\mathrm{T}}Bx + 2x^{\mathrm{T}}(Pb + \beta A^{\mathrm{T}}c)f(\sigma) + 2\beta c^{\mathrm{T}}bf^2(\sigma) \tag{2.4.5}$$

is negative definite. Thus, the necessary and sufficient conditions for the zero solution of (2.4.1) *to be absolutely stable in* $[0, k]$ *are*

$$\begin{cases} \dfrac{1}{k} - c^{\mathrm{T}}B^{-1}d > 0, & (2.4.6) \\ \dfrac{1}{c^{\mathrm{T}}B^{-1}c}\left(\dfrac{1}{k} - c^{\mathrm{T}}B^{-1}d\right)^2 - d^{\mathrm{T}}B^{-1}d - 2\beta c^{\mathrm{T}}b > 0, & (2.4.7) \end{cases}$$

where $d = Pb + \beta A^{\mathrm{T}}c$.

Proof. B is symmetric positive definite and so is B^{-1}. In addition, there exists H such that $B = H^{\mathrm{T}}H$ and $B^{-1} = H^{-1}(H^{-1})^{\mathrm{T}}$. Clearly, $V(x)$ is radially unbounded positive definite.

(2.4.5) can be reduced to

$$\begin{aligned} -\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.4.1)} &= x^{\mathrm{T}}Bx - 2x^{\mathrm{T}}(Pb + \beta A^{\mathrm{T}}c)f(\sigma) - 2\beta c^{\mathrm{T}}bf^2(\sigma) \\ &= x^{\mathrm{T}}H^{\mathrm{T}}Hx - 2x^{\mathrm{T}}df(\sigma) - 2\beta c^{\mathrm{T}}bf^2(\sigma) \\ &= (Hx)^{\mathrm{T}}Hx - 2(Hx)^{\mathrm{T}}(H^{-1})^{\mathrm{T}}df(\sigma) - 2\beta c^{\mathrm{T}}bf^2(\sigma) \\ &= [Hx - (H^{-1})^{\mathrm{T}}df(\sigma)]^{\mathrm{T}}[Hx - (H^{-1})^{\mathrm{T}}df(\sigma)] \\ &\quad - (d^{\mathrm{T}}B^{-1}d + 2\beta c^{\mathrm{T}}b)f^2(\sigma) \\ &= \begin{cases} 0 & \text{for} \quad x = 0, \\ (Hx)^{\mathrm{T}}Hx = x^{\mathrm{T}}Bx > 0 & \text{for} \quad f(\sigma) = 0, x \neq 0, \\ vf^2(\sigma) & \text{for} \quad f(\sigma) \neq 0, \end{cases} \end{aligned}$$

where

$$\begin{aligned} v = &\left\{\left[H\frac{x}{f(\sigma)} - (H^{-1})^{\mathrm{T}}d\right]^{\mathrm{T}}\left[H\frac{x}{f(\sigma)} - H^{-1})^{\mathrm{T}}d\right]\right. \\ &\left.- (d^{\mathrm{T}}B^{-1}d + 2\beta c^{\mathrm{T}}b)\right\}. \end{aligned}$$

Hence, it is enough to show that $v > 0$ for any $c^{\mathrm{T}}x \geqslant 1/k$. To complete the proof, we use the topological transformation

$$y = Hx - (H^{-1})^{\mathrm{T}}d$$

to reduce v to

$$v = y^{\mathrm{T}}y - p, \qquad p = d^{\mathrm{T}}B^{-1}d + 2\beta c^{\mathrm{T}}b$$

and the condition $c^{\mathrm{T}}x \geqslant 1/k$ is equivalent to

$$c^{\mathrm{T}}H^{-1}[y + (H^{-1})^{\mathrm{T}}d] \geqslant \frac{1}{k}. \tag{2.4.8}$$

As a result, it only needs to show that in the half space $\tilde{\Omega}$ of y satisfying (2.4.8), the expression $v = y^{\mathrm{T}}y - p$ satisfies $v > 0$.

It can be easily proved that $p \geqslant 0$. Thus, we have $v \leqslant 0$ for $y = 0$. Hence, it needs to guarantee that $y = 0$ is not on the half space $\tilde{\Omega}$, i. e.,

$$c^{\mathrm{T}}H^{-1}[0 + (H^{-1})^{\mathrm{T}}d] = cH^{-1}(H^{-1})^{\mathrm{T}}d = c^{\mathrm{T}}B^{-1}d < \frac{1}{k},$$

this is just the condition (2.4.6).

Clearly, on the half space $\tilde{\Omega}$, v will get its minimun at $y = y^*$, i. e. at the interseition point of the super plane

$$c^{\mathrm{T}}H^{-1}[y + (H^{-1})^{\mathrm{T}}d] = \frac{1}{k}$$

and the normal line passing through $y = 0$.

We put $y^* = \lambda(c^{\mathrm{T}}H^{-1})^{\mathrm{T}}$, where λ is to be determined. The expression

$$c^{\mathrm{T}}H^{-1}[\lambda(c^{\mathrm{T}}H^{-1})^{\mathrm{T}} + (H^{-1})^{\mathrm{T}}d] = \frac{1}{k}$$

leads to

$$\lambda = \frac{\frac{1}{k} - c^{\mathrm{T}}B^{-1}d}{c^{\mathrm{T}}B^{-1}c}.$$

As a result $v(y^*) = (y^*)^{\mathrm{T}}y^* - p > 0 \Leftrightarrow v > 0$ (on the half space $\tilde{\Omega}$) and the expression

$$\begin{aligned} v(y^*) &= [\lambda(c^{\mathrm{T}}H^{-1})^{\mathrm{T}}]^{\mathrm{T}}[\lambda(c^{\mathrm{T}}H^{-1})^{\mathrm{T}}] - p \\ &= \lambda^2 c^{\mathrm{T}}B^{-1}c - (d^{\mathrm{T}}B^{-1}d + 2\beta c^{\mathrm{T}}b) \\ &= \frac{1}{c^{\mathrm{T}}B^{-1}c}\left(\frac{1}{k} - c^{\mathrm{T}}B^{-1}d\right)^2 - d^{\mathrm{T}}B^{-1}d - 2\beta c^{\mathrm{T}}b > 0 \end{aligned}$$

is the conditon (2.4.7). ∎

Imitating the proof of Theorem 2.4.2, we have the following results.

Corollary 2.4.3. *Let the Liapunov function be of the form* (2.4.4) *and* $f(\sigma)$ *satisfies*

$$0 \leqslant f(\sigma)\sigma < k\sigma^2.$$

Then (2.4.5) *is negative definite. Thus, the necessary and sufficient conditions for the zero solution of* (2.4.1) *to be absolutely stable in the Hurwitz angle* $[0, k)$ *are*

$$\begin{cases} \dfrac{1}{k} - c^{\mathrm{T}}B^{-1}d \geqslant 0, \\ \dfrac{1}{c^{\mathrm{T}}B^{-1}c}\left(\dfrac{1}{k} - c^{\mathrm{T}}B^{-1}d\right)^2 - d^{\mathrm{T}}B^{-1}d - 2\beta c^{\mathrm{T}}b \geqslant 0. \end{cases}$$

In case of $k = +\infty$, i. e. , $\dfrac{1}{k} = 0$, we can prove the following corollary along the line of proving Theorem 2. 4. 2 and Corollary 2. 4. 3.

Corollary 2. 4. 4. *Let the Liapunov function be of the form* (2. 4. 4) (*for any* $f \in F$). *Then the expression* (2. 4. 5) *is negative definite. Consequently, the necessary and sufficient conditions for the zero solution of* (2. 4. 1) *to be absolutely stable are*

$$\begin{cases} c^{\mathrm{T}}B^{-1}d \leqslant 0, \\ \dfrac{1}{c^{\mathrm{T}}B^{-1}c}(c^{\mathrm{T}}B^{-1}d)^2 - d^{\mathrm{T}}B^{-1}d - 2\beta c^{\mathrm{T}}b \geqslant 0. \end{cases}$$

Remark. If $\beta > 0$, then without loss of generality, we can assume that $\beta = 1/2$. The function

$$V(x) = x^{\mathrm{T}}Px + 2\beta \int_0^{\sigma} f(\sigma)\mathrm{d}\sigma$$

is a proper Liapunov function if and only if

$$\tilde{V}(x) = x^{\mathrm{T}} \frac{1}{2\beta}Px + \int_0^{\sigma} f(\sigma)\mathrm{d}\sigma$$

is also a proper Liapunov function. In this way, a free parameter can be removed.

Now, let us describe three equivalent necessary and sufficient conditions for constructing

$$V(x) = x^{\mathrm{T}}Px + \int_0^{\sigma} f(\sigma)\mathrm{d}\sigma$$

such that $\left.\dfrac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.4.1)}$ is negative definite. For this purpose, we first prove a lemma.

Lemma 2. 4. 5. *For an arbitrary symmetric positive definite matrix* B, *we have*

$$c^{\mathrm{T}}b + \eta^{\mathrm{T}}B^{-1}\eta \geqslant 0,$$

where $\eta = \dfrac{1}{2}(A^{\mathrm{T}}c + 2Pb)$.

Proof. Choosing

$$V(x) = x^{\mathrm{T}}Px + \int_0^{\sigma} f(\sigma)\mathrm{d}\sigma,$$

where P satisfies

$$A^{\mathrm{T}}P + PA = -B,$$

we deduce

$$-\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.4.1)} = x^{\mathrm{T}}Bx - 2\eta^{\mathrm{T}}xf(\sigma) - c^{\mathrm{T}}bf^{2}(\sigma)$$

$$= (x^{\mathrm{T}}, f(\sigma))\begin{pmatrix} B & -\eta \\ -\eta^{\mathrm{T}} & -c^{\mathrm{T}}b \end{pmatrix}\begin{pmatrix} x \\ f(\sigma) \end{pmatrix}. \tag{2.4.9}$$

Since B is positive definite, there exist

$$\det B > 0,$$

$$\det\begin{pmatrix} B^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \det B^{-1} = (\det B)^{-1} > 0,$$

and

$$\det\begin{pmatrix} B^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} B & -\eta \\ -\eta^{\mathrm{T}} & -c^{\mathrm{T}}b \end{pmatrix} = \det\begin{bmatrix} E & -B^{-1}\eta \\ -\eta^{\mathrm{T}} & -c^{\mathrm{T}}b \end{bmatrix}$$

$$= -c^{\mathrm{T}}b - \eta^{\mathrm{T}}B^{-1}\eta.$$

Thus, we get

$$\det\begin{pmatrix} B & -\eta \\ -\eta^{\mathrm{T}} & -c^{\mathrm{T}}b \end{pmatrix} = (-c^{\mathrm{T}}b - \eta^{\mathrm{T}}B^{-1}\eta)\cdot\det B.$$

If $c^{\mathrm{T}}b + \eta^{\mathrm{T}}B^{-1}\eta < 0$, then (2.4.9) is positive definite for $x, f(\sigma)$, which is impossible. Hence, we conclude that

$$c^{\mathrm{T}}b + \eta^{\mathrm{T}}B^{-1}\eta \geqslant 0. \qquad \blacksquare$$

Theorem 2.4.6. *If B is a symmetric positive definite matrix, then the following conditions are equivalent to each other:*

$$(\mathrm{I})\ \begin{cases} \text{(i)}\ \ u(x) = x^{\mathrm{T}}Bx - 2\eta^{\mathrm{T}}x - c^{\mathrm{T}}b \geqslant 0 \quad \text{when} \quad c^{\mathrm{T}}x = 0, \\ \text{(ii)}\ \ c^{\mathrm{T}}B^{-1}\eta \leqslant 0; \end{cases}$$

$$(\mathrm{II})\ \ c^{\mathrm{T}}B^{-1}\eta + \sqrt{c^{\mathrm{T}}B^{-1}c(\eta^{\mathrm{T}}B^{-1}\eta + c^{\mathrm{T}}b)} \leqslant 0;$$

$$(\mathrm{III})\ \begin{cases} \text{(i)}\ \ c^{\mathrm{T}}b + \eta^{\mathrm{T}}B^{-1}\eta - \dfrac{(c^{\mathrm{T}}B^{-1}\eta)^2}{c^{\mathrm{T}}B^{-1}c} \leqslant 0, \\ \text{(ii)}\ \ c^{\mathrm{T}}B^{-1}\eta \leqslant 0. \end{cases}$$

Proof. We will complete the proof by proving (I)⇒(II)⇒(III)⇒(I).

1) (I)⇒(II)

If (I) is satisfied, it can be shown that $u(x_0) > 0$ for $c^{\mathrm{T}}x_0 > 0$. Choosing

$$\bar{x} = B^{-1}\eta - \frac{c^{\mathrm{T}}B^{-1}\eta - m}{c^{\mathrm{T}}B^{-1}c}B^{-1}c, \qquad m \text{ being a constant},$$

we have

$$c^{\mathrm{T}}\bar{x} = c^{\mathrm{T}}B^{-1}\eta - \frac{c^{\mathrm{T}}B^{-1}\eta - m}{c^{\mathrm{T}}B^{-1}c}c^{\mathrm{T}}B^{-1}c = m.$$

Hence, for any $m \in (0, +\infty)$, $u(\bar{x}) > 0$, and

$$u(\bar{x}) = \bar{x}^{\mathrm{T}}B\bar{x} - 2\eta^{\mathrm{T}}\bar{x} - c^{\mathrm{T}}b$$
$$= \left[\eta^{\mathrm{T}}(B^{-1})^{\mathrm{T}} - \frac{c^{\mathrm{T}}B^{-1}\eta - m}{c^{\mathrm{T}}B^{-1}c}c^{\mathrm{T}}(B^{-1})^{\mathrm{T}}\right]B\left[B^{-1}\eta - \frac{c^{\mathrm{T}}B^{-1}\eta - m}{c^{\mathrm{T}}B^{-1}c}B^{-1}c\right]$$
$$- 2\eta^{\mathrm{T}}\left[B^{-1}\eta - \frac{c^{\mathrm{T}}B^{-1}\eta - m}{c^{\mathrm{T}}B^{-1}c}B^{-1}c\right] - c^{\mathrm{T}}b.$$

Since B is symmetric positive definite and so is B^{-1}, we write

$$u(\bar{x}) = \left[\eta^{\mathrm{T}} - \frac{c^{\mathrm{T}}B^{-1}\eta - m}{c^{\mathrm{T}}B^{-1}c}c^{\mathrm{T}}\right]\left[B^{-1}\eta - \frac{c^{\mathrm{T}}B^{-1}\eta - m}{c^{\mathrm{T}}B^{-1}c}B^{-1}c\right]$$
$$- 2\eta^{\mathrm{T}}\left[B^{-1}\eta - \frac{c^{\mathrm{T}}B^{-1}\eta - m}{c^{\mathrm{T}}B^{-1}c}B^{-1}c\right] - c^{\mathrm{T}}b$$
$$= \frac{(c^{\mathrm{T}}B^{-1}\eta - m)^2}{c^{\mathrm{T}}B^{-1}c} - \eta^{\mathrm{T}}B^{-1}\eta - c^{\mathrm{T}}b$$
$$= \frac{1}{c^{\mathrm{T}}B^{-1}c}m^2 - 2\frac{c^{\mathrm{T}}B^{-1}\eta}{c^{\mathrm{T}}B^{-1}c}m + \frac{(c^{\mathrm{T}}B^{-1}\eta)^2}{c^{\mathrm{T}}B^{-1}c}$$
$$- \eta^{\mathrm{T}}B^{-1}\eta - c^{\mathrm{T}}b,$$

which is a quadratic trinomial in m with the coefficient of quadratic term $\frac{1}{c^{\mathrm{T}}B^{-1}c} > 0$.

The criterion of the equation $u(\bar{x}) = 0$ reads

$$D = 4\frac{(c^{\mathrm{T}}B^{-1}\eta)^2}{(c^{\mathrm{T}}B^{-1}c)^2} - 4\frac{1}{c^{\mathrm{T}}B^{-1}c}\left[\frac{(c^{\mathrm{T}}B^{-1}\eta)^2}{c^{\mathrm{T}}B^{-1}c} - \eta^{\mathrm{T}}B^{-1}\eta - c^{\mathrm{T}}b\right]$$
$$= \frac{4}{c^{\mathrm{T}}B^{-1}c}(\eta^{\mathrm{T}}B^{-1}\eta + c^{\mathrm{T}}b).$$

By Lemma 2.4.5, it follows that $D \geqslant 0$, and so $u(\bar{x})$ has two real roots

$$m_i = c^{\mathrm{T}}B^{-1}\eta \pm \sqrt{c^{\mathrm{T}}B^{-1}c(\eta^{\mathrm{T}}B^{-1}\eta + c^{\mathrm{T}}b)}, \qquad i = 1,2.$$

From (ii) in the condition (I), it follows that

$$m_1 = c^{\mathrm{T}}B^{-1}\eta - \sqrt{c^{\mathrm{T}}B^{-1}c(\eta^{\mathrm{T}}B^{-1}\eta + c^{\mathrm{T}}b)} \leqslant 0,$$

If $m_2 = c^{\mathrm{T}}B^{-1}\eta + \sqrt{c^{\mathrm{T}}B^{-1}c(\eta^{\mathrm{T}}B^{-1}\eta + c^{\mathrm{T}}b)} > 0$, then for any $m \in (0, m_2)$, it follows that $u(\bar{x}) < 0$ in contradiction with the fact that for any $m \in (0, +\infty)$, there exists $u(\bar{x}) > 0$. Thus,

$$m_2 = c^{\mathrm{T}}B^{-1}\eta + \sqrt{c^{\mathrm{T}}B^{-1}c(\eta^{\mathrm{T}}B^{-1}\eta + c^{\mathrm{T}}b)} \leqslant 0$$

which is exactly the condition (II).

2) (II)⇒(I).

If (II) is true, i.e.,

$$c^{\mathrm{T}}B^{-1}\eta + \sqrt{c^{\mathrm{T}}B^{-1}c(\eta^{\mathrm{T}}B^{-1}\eta + c^{\mathrm{T}}b)} \leqslant 0,$$

then we obtain $c^{\mathrm{T}}B^{-1}\eta \leqslant 0$. Consequently,

$$-c^{T}B^{-1}\eta \geqslant \sqrt{c^{T}B^{-1}c(\eta^{T}B^{-1}\eta + c^{T}b)} \geqslant 0,$$

and then

$$(c^{T}B^{-1}\eta)^{2} \geqslant c^{T}B^{-1}c(\eta^{T}B^{-1}\eta + c^{T}b),$$

which implies

$$c^{T}b + \eta^{T}B^{-1}\eta - \frac{(c^{T}B^{-1}\eta)^{2}}{c^{T}B^{-1}c} \leqslant 0.$$

The condition (II) holds.

3) (II)⇒(I).

Since (ii) in the condition (II) is the same as (ii) in the condition (I), it is sufficient to show that (i) of the condition (I) holds.

Since B is positive definite, there exists a non-singular matrix H such that $H^{T}BH = E$, and $B^{-1} = HH^{T}$. Make a full-rank linear transformation

$$x = H\left[y + \frac{1}{2}H^{T}(A^{T}c + 2Pb)\right] = H[y + H^{T}\eta]$$

and let y_0 be the vector corresponding to x_0. The expression $c^{T}x_0 = 0$ gives

$$c^{T}Hy_0 + c^{T}HH^{T}\eta = c^{T}Hy_0 + c^{T}B^{-1}\eta = 0,$$

i. e. ,

$$(c^{T}B^{-1}\eta)^{2} = (c^{T}Hy_0)^{2}.$$

By the Cauchy inequality

$$(c^{T}B^{-1}\eta)^{2} \leqslant \| c^{T}H \|^{2} \cdot \| y_0 \|^{2},$$

we derive

$$y_0^{T}y_0 \geqslant \frac{(c^{T}B^{-1}\eta)^{2}}{(c^{T}H)(H^{T}c)} = \frac{(c^{T}B^{-1}\eta)^{2}}{c^{T}B^{-1}c}$$

Consequently, if $c^{T}x_0 = 0$, then there exists

$$\begin{aligned} u(x_0) &= y_0^{T}y_0 - \eta^{T}B^{-1}\eta - c^{T}b \\ &\geqslant \frac{(c^{T}B^{-1}\eta)^{2}}{c^{T}B^{-1}c} - \eta^{T}B^{-1}\eta - c^{T}b. \end{aligned}$$

(i) in the condition (II) gives $u(x_0) \geqslant 0$.

Hence (I) holds. ∎

The difficulty of the Lurie problem lies not only in the indeterminateness of $f(\sigma)$, but also in the fact that the variables do not separate.

We will simplify (2. 4. 1) by means of two full-rank linear transformations as follows:

Without loss of generality, we can assume that $c_n \neq 0$. (Otherwise by rearrangement of variables and order signs of the equation we can make $c_n \neq 0$.)

We choose a full-rank linear transformation

$$y = G(g_{ij})x, \qquad x,y \in \mathbf{R}^n,\ G \in \mathbf{R}^{n\times n},$$

where

$$g_{ij} = \begin{cases} 1, & i = j = 1,\cdots,n-1, \\ c_j, & j = 1,\cdots,n,\ i = n, \\ 0, & \text{otherwise}. \end{cases}$$

Then (2.4.1) are transformed into

$$\frac{\mathrm{d}y}{\mathrm{d}t} = By + hf(y_n), \tag{2.4.10}$$

where

$$b_{ij} = a_{ij} - \frac{a_{ij}}{c_n}c_j, \qquad i,j = 1,\cdots,n-1,$$

$$b_{in} = \frac{a_{in}}{c_n}, \qquad i = 1,\cdots,n-1,$$

$$b_{nn} = \frac{\sum_{i=1}^{n} c_i a_{in}}{c_n},$$

$$b_{nj} = \sum_{i=1}^{n} c_i a_{ij} - \sum_{i=1}^{n} c_i \frac{a_{in}}{c_n} c_j, \qquad j = 1,\cdots,n-1,$$

$$h_i = b_i, \qquad i = 1,\cdots,n-1,$$

$$h_n = \sum_{i=1}^{n} c_i b_i = c^{\mathrm{T}} b.$$

In the following we will prove that $c^{\mathrm{T}}b$ and $c^{\mathrm{T}}A^{-1}b$ are invariant under arbitrary full-rank linear transformation of (2.4.1).

In case $h_n = c^{\mathrm{T}}b < 0$, we again make a full-rank linear transformation

$$\xi = Ly,$$

where $\xi, y \in \mathbf{R}^n$, $L \in \mathbf{R}^{n\times n}$,

$$l_{ij} = \begin{cases} 1, & i = j = 1,\cdots,n, \\ -\dfrac{h_j}{h_n}, & i = 1,\cdots,n-1,\ j = n, \\ 0, & \text{otherwise}. \end{cases}$$

Then (2.4.10) reduces to a simpler form

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = \tilde{A}\xi + \tilde{b}f(\xi_n), \tag{2.4.11}$$

where $\tilde{A} \in \mathbf{R}^{n\times n}$, $\tilde{b} = \mathrm{col}(\overbrace{0,\cdots,0}^{n-1},h_n)$, $\tilde{A} = LBL^{-1}$.

Since these transformations have no effects on absolute stability, what we need, is to study the absolute stability of (2.4.10) and (2.4.11).

Definition 2.4.7. The zero solution of (2.4.10) [(2.4.11)] is said to be *absolutely stable for partial variables* $y_j, \cdots, y_n[\xi_j, \cdots, \xi_n]$, $i<j\leqslant n$, if for any $f\in F$, the zero solution of (2.4.10) [(2.4.11)] is globally stable for partial variables $y_j,\cdots,y_n[\xi_j,\cdots,\xi_n]$.

The absolute stability for partial variables in the Hurwitz angle $[0,k]$ can be defined analogously.

Proceeding along the line of Theorem 2.2.3, we can prove the following theorems.

Theorem 2.4.8. *The zero solution of* (2.4.10) *is absolutely stable if and only if*

1) *the matrix* $B+O_{n\times(n-1)}h\theta$ *is stable, where* $\theta=0$ *or* 1, *and* $O_{n\times(n-1)}h$ *is an* $n\times n$ *matrix with the last column being* h, *and the other elements being zero*;

2) *the zero solution of* (2.4.10) *is absolutely stable for* y_n.

Theorem 2.4.9. *The zero solution of* (2.4.11) *is absolutely stable if and only if*

1) *the matrix* $\tilde{A}+O_{n\times(n-1)}\tilde{b}\theta$ *is stable, where* $\theta=0$ *or* 1, *and* $O_{n\times(n-1)}\tilde{b}$ *is an* $n\times n$ *matrix with the last column being* $\tilde{b}$, *and the other elements being zero*;

2) *the zero solution of* (2.4.11) *is absolutely stable for* ξ_n.

Theorem 2.4.10. *The necessary and sufficient conditions for the zero solution of* (2.4.10) *to be absolutely stable are that there exists a constant vector* $\eta\in\mathbf{R}^n$ *satisfying*:

1) *the matrix* $B+O_{n\times(n-1)}\eta$ *is stable*;

2) *the condition* 2) *in Theorem* 2.4.8 *holds*.

Proof. *Necessity.* The existence of $\eta\in\mathbf{R}^n$ is evident, for example, one takes $\eta=h$, the condition 1) in Theorem 2.4.8 holds, and the condition 2) is just the condition 2) in Theorem 2.4.8.

Sufficiency. We transform (2.4.10) into

$$\frac{\mathrm{d}y}{\mathrm{d}t}=(B+O_{n\times(n-1)}\eta)y+hf(y_n)-\eta y_n,$$

with the help of the formula of variation of constant

$$\begin{aligned}y(t)=&\exp[(B+O_{n\times(n-1)}\eta)(t-t_0)]\cdot y_0\\&+\int_{t_0}^{t}\exp[(B+O_{n\times(n-1)}\eta)(t-\tau)][hf(y_n(\tau))-\eta y_n(\tau)]\mathrm{d}\tau,\end{aligned}$$

and the proof can be completed in a way similar to Theorem 2.2.3.

Similarly, we note the following theorems.

Theorem 2.4.11. *The zero solution of* (2.4.11) *is absolutely stable if and only if*

1) *there exists* $\eta \in \mathbf{R}^n$ *such that the matrix* $\tilde{A} + O_{n\times(n-1)}\eta$ *is stable*;

2) *the condition* 2) *in Theorem* 2.4.9 *is true.*

Theorem 2.4.12. *The necessary and sufficient conditions for the zero solution of* (2.4.10) *to be absolutely stable are that*

1) *the condition* 1) *in Theorem* 2.4.10 *holds*;

2) *the zero solution of* (2.4.10) *is absolutely stable for* $y_j, \cdots, y_n$ $(1 < j \leqslant n)$.

Proof. When the condition 1) is satisfied, the absolute stability of the zero solution of (2.4.10) implies the absolute stability for y_j, $\cdots$, y_n in particular for y_n. In agreement to Theorem 2.4.10, the absolute stability for y_n implies the absolute stability for all the variables y_1, $\cdots$, y_n, in particular for y_j, $\cdots$, y_n.

In a similar manner, we formulate

Theorem 2.4.13. *The zero solution of* (2.4.11) *is absolutely stable if and only if*

1) *the condition* 1) *of Theorem* 2.4.11 *is satisfied*;

2) *the zero solution of* (2.4.11) *is absolutely stable for* $\xi_j, \cdots, \xi_n$.

In the following, we will provide some simple and practical sufficient conditions derived from Theorems 2.4.8~2.4.13.

First we introduce some notations as follows:

$$B_{(j_0)} = \begin{bmatrix} b_{11} & \cdots & b_{1j_0} \\ \vdots & & \vdots \\ b_{j_01} & \cdots & b_{j_0j_0} \end{bmatrix}, \qquad B^{(n-j_0)} = \begin{bmatrix} b_{1,j_0+1} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{j_0,j_0+1} & \cdots & b_{j_0n} \end{bmatrix},$$

$$h_{(j_0)} = \mathrm{col}(h_1, \cdots, h_{j_0}), \qquad h^{(n-j_0)} = \mathrm{col}(h_{j_0+1}, \cdots, h_n),$$

$$y_{(j_0)} = \mathrm{col}(y_1, \cdots, y_{j_0}), \qquad y^{(n-j_0)} = \mathrm{col}(y_{j_0+1}, \cdots, y_n),$$

$$G_B = ((-1)^{\delta_{ij}}|b_{ij}|)_{n\times n}, \qquad G_{B_{(j_0)}} = ((-1)^{\delta_{ij}}|b_{ij}|)_{j_0\times j_0},$$

$$\tilde{A}_{(j_0)} = \begin{bmatrix} \tilde{a}_{11} & \cdots & \tilde{a}_{1j_0} \\ \vdots & & \vdots \\ \tilde{a}_{j_01} & \cdots & \tilde{a}_{j_0j_0} \end{bmatrix}, \qquad \tilde{A}^{(n-j_0)} = \begin{bmatrix} \tilde{a}_{1,j_0+1} & \cdots & \tilde{a}_{1n} \\ \vdots & & \vdots \\ \tilde{a}_{j_0,j_0+1} & \cdots & \tilde{a}_{j_0n} \end{bmatrix},$$

$$\tilde{b}_{(j_0)} = \mathrm{col}(\overbrace{0, \cdots, 0}^{j_0}), \qquad \tilde{b}^{(n-j_0)} = \mathrm{col}(\overbrace{0, \cdots, 0}^{n-j_0-1}, b_n),$$

$$G_{\tilde{A}} = ((-1)^{\delta_{ij}}|\tilde{a}_{ij}|)_{n\times n}, \qquad G_{\tilde{A}_{(j_0)}} = ((-1)^{\delta_{ij}}|\tilde{a}_{ij}|)_{j_0\times j_0},$$

$$\xi_{(j_0)} = \mathrm{col}(\xi_1, \cdots, \xi_{j_0}), \qquad \xi^{(n-j_0)} = \mathrm{col}(\xi_{j_0+1}, \cdots, \xi_n),$$

where $1 < j_0 \leqslant n$.

Theorem 2.4.14. *If $B_{(j_0)}$ is stable, and the zero solution of (2.4.10) is absolutely stable for $y_{j_0+1}, \cdots, y_n$, then the zero solution of (2.4.10) is absolutely stable. If $\tilde{A}_{(j_0)}$ is stable, and the zero solution of (2.4.11) is absolutely stable for $\xi_{j_0+1}, \cdots, \xi_n$, then the zero solution of (2.4.11) is absolutely stable as well.*

Proof. According to the formula of variation of constant the first j_0 components $y_{(j_0)}(t) = y_{(j_0)}(t, t_0, y_0)$ of the solution of (2.4.10) can be expressed as

$$\begin{aligned} y_{(j_0)}(t) &= \exp B_{(j_0)}(t - t_0) \cdot y_{(j_0)}(t_0) \\ &\quad + \int_{t_0}^{t} \exp B_{(j_0)}(t - \tau) \cdot B^{(n-j_0)} y^{(n-j_0)}(\tau) \mathrm{d}\tau \\ &\quad + \int_{t_0}^{t} \exp B_{(j_0)}(t - \tau) \cdot h^{(n-j_0)} f(y_n(\tau)) \mathrm{d}\tau. \end{aligned}$$

And the first j_0 components $\xi_{(j_0)}(t) = \xi_{(j_0)}(t, t_0, \xi_0)$ of the solution of (2.4.11) can be expressed as

$$\begin{aligned} \xi_{(j_0)}(t) &= \exp \tilde{A}_{(j_0)}(t - t_0) \cdot \xi_{(j_0)}(t_0) \\ &\quad + \int_{t_0}^{t} \exp \tilde{A}_{(j_0)}(t - \tau) \cdot \tilde{A}^{(n-j_0)} \xi^{(n-j_0)}(\tau) \mathrm{d}\tau \\ &\quad + \int_{t_0}^{t} \exp \tilde{A}_{(j_0)}(t - \tau) \cdot \tilde{b}^{(n-j_0)} f(\xi_n(\tau)) \mathrm{d}\tau. \end{aligned}$$

Now we can complete the proof in a manner similar to the 'sufficiency' of Theorem 2.2.3. ∎

It can be seen from the above theorems that as sufficient condition, the condition of absolute stability for the partial variable y_n has been enforced to be the absolute stability for $y_j, \cdots, y_n$, while the other algebraic condition of stability can be reduced to the stability of $B_{(j_0)}$.

Theorem 2.4.15. *Assume that there exist $r_i > 0$ $(i = 1, \cdots, n)$ such that*

$$-r_j b_{jj} > \sum_{\substack{i=1 \\ i \neq j}}^{n} r_i |b_{ij}|, \qquad j = 1, \cdots, n-1,$$

and such that

$$-r_n b_{nn} \geqslant \sum_{i=1}^{n-1} r_i |b_{in}|, \qquad -r_n h_n > \sum_{i=1}^{n-1} r_i |h_i|$$

or

$$-r_n b_{nn} > \sum_{i=1}^{n-1} r_i |b_{in}|, \qquad -r_n h_n \geqslant \sum_{i=1}^{n-1} r_i |h_i|.$$

Then the zero solution of (2.4.10) is absolutely stable.

Proof. Choosing $V(y) = \sum_{i=1}^{n} r_i |y_i|$, we have

$$
\begin{aligned}
D^+ V|_{(2.4.10)} \leqslant & \sum_{j=1}^{n-1}\Big[r_j b_{jj} + \sum_{\substack{i=1\\ i\neq j}}^{n} r_i|b_{ij}|\Big]|y_j| \\
& + \Big[r_n b_{nn} + \sum_{i=1}^{n-1} r_i|b_{in}|\Big]|y_n| \\
& + \Big[r_n h_n + \sum_{i=1}^{n-1} r_i|h_i|\Big]|f(y_n)| \\
< & 0 \quad \text{when} \quad y \neq 0.
\end{aligned}
$$

Clearly, $V(y)$ is radially unbounded positive definite. As a consequence, the zero solution of (2.4.10) is absolutely stable. ∎

Theorem 2.4.16. *Assume that the following conditions are satisfied:*

1) *the matrix $B_{(j_0)}$ is stable;*

2) *there exist constants $r_i \geqslant 0$ $(i=1,\cdots,j_0)$, $r_j > 0$ $(j=j_0+1,\cdots,n)$ such that*

$$
-r_j b_{jj} \geqslant \sum_{\substack{i=1\\ i\neq j}}^{n} r_i|b_{ij}|, \qquad j = 1,\cdots,j_0,
$$

$$
-r_j b_{jj} > \sum_{\substack{i=1\\ i\neq j}}^{n} r_i|b_{ij}|, \qquad j = j_0+1,\cdots,n-1,
$$

and such that

$$
-r_n b_{nn} \geqslant \sum_{i=1}^{n-1} r_i|b_{in}|, \qquad -r_n h_n > \sum_{i=1}^{n-1} r_i|h_i|
$$

or

$$
-r_n b_{nn} > \sum_{i=1}^{n-1} r_i|b_{in}|, \qquad -r_n h_n \geqslant \sum_{i=1}^{n-1} r_i|h_i|.
$$

Then the zero solution of (2.4.10) is absolutely stable.

Proof. 1) We construct the Liapunov function $V(y) = \sum_{i=1}^{n} r_i|y_i|$. Clearly,

$$
V(y) \geqslant \sum_{i=j_0+1}^{n} r_i|y_i| \to +\infty \qquad \text{as} \quad \sum_{i=j_0+1}^{n} |y_i| \to +\infty,
$$

$$
\begin{aligned}
D^+ V|_{(2.4.10)} \leqslant & \sum_{j=1}^{n}\Big[r_j b_{jj} + \sum_{\substack{i=1\\ i\neq j}}^{n} r_i|b_{ij}|\Big]|y_j| \\
& + \Big[r_n h_n + \sum_{i=1}^{n-1} r_i|h_i|\Big]|f(y_n)| \\
\leqslant & \sum_{j=j_0+1}^{n}\Big[r_j b_{jj} + \sum_{\substack{i=1\\ i\neq j}}^{n} r_i|b_{ij}|\Big]|y_j| \\
& + \Big[r_n h_n + \sum_{i=1}^{n-1} r_i|h_i|\Big]|f(y_n)| \\
< & 0 \quad \text{for} \quad \sum_{j=j_0+1}^{n} y_{j+1}^2 \neq 0.
\end{aligned}
$$

Thus, the zero solution of (2.4.10) is absolutely stable for $y_{j_0+1}, \cdots, y_n$.

2) The first j_0 components of the solution of (2.4.10) can be expressed as

$$\begin{aligned} y_{(j_0)}(t) &= \exp B_{(j_0)}(t-t_0) \cdot y_{(j_0)}(t_0) \\ &\quad + \int_{t_0}^{t} \exp B_{(j_0)}(t-\tau) \cdot B^{(n-j_0)} y^{(n-j_0)}(\tau) \mathrm{d}\tau \\ &\quad + \int_{t_0}^{t} \exp B_{(j_0)}(t-\tau) \cdot h^{(n-j_0)} f(y_n(\tau)) \mathrm{d}\tau. \end{aligned}$$

In the remainder of the proof, imitating the proof of the sufficiency part of Theorem 2.2.3, we can verify the absolute stability of the zero solution of (2.4.10) for $y_1, \cdots, y_{j_0}$. ∎

Theorem 2.4.17. *Suppose the following conditions are satisfied:*

1) *the matrix B is stable;*

2) *there exists a constant $\varepsilon > 0$ such that*

$$\begin{bmatrix} -E_{n\times n} & Ph + B_n/2 + \varepsilon e_n \\ (Ph + B_n/2 + \varepsilon e_n)^{\mathrm{T}} & h_n \end{bmatrix}$$

is negative semi-definite, where $B_n = \mathrm{col}(b_{n1}, \cdots, b_{nn})$, $e_n = \mathrm{col}(\overbrace{0, \cdots, 0}^{n-1}, 1)$, P is the solution of the Liapunov matrix equation

$$PB + B^{\mathrm{T}}P = -E.$$

Then the zero solution of (2.4.10) is absolutely stable.

Proof. Choosing the Liapunov function

$$V(y) = y^{\mathrm{T}}Py + \int_0^{y_n} f(y_n)\mathrm{d}y_n,$$

we get

$$\begin{aligned} &\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.4.10)} \\ &= (y^{\mathrm{T}}, f(y_n)) \begin{bmatrix} -E & Ph + B_n/2 + \varepsilon e_n \\ (Ph + B_n/2 + \varepsilon e_n)^{\mathrm{T}} & h_n \end{bmatrix} \begin{pmatrix} y \\ f(y_n) \end{pmatrix} \\ &\quad - (y^{\mathrm{T}}, f(y_n)) \begin{bmatrix} 0_{n\times n} & \varepsilon e_n \\ (\varepsilon e_n)^{\mathrm{T}} & h_n \end{bmatrix} \begin{pmatrix} y \\ f(y_n) \end{pmatrix} \\ &\leqslant -2\varepsilon y_n f(y_n) < 0 \qquad \text{for} \quad y_n \neq 0. \end{aligned}$$

Therefore, the zero solution of (2.4.10) is absolutely stable for y_n. According to Theorem 2.4.14, the conclusion follows. ∎

Theorem 2.4.18. *Assume that*

1) *the matrix $\tilde{A}$ is stable;*

2) *there exist constants $r_i \geqslant 0$ $(i=1, \cdots, n-1)$, $r_n > 0$ such that*

$$-r_j\tilde{a}_{jj} \geqslant \sum_{\substack{i=1\\i\neq j}}^{n} r_i|\tilde{a}_{ij}|, \quad j=1, \cdots, n.$$

Then the zero solution of the system (2.4.11) *is absolutely stable.*

Proof. Using the Liapunov function $V=\sum_{i=1}^{n} r_i|\xi_i|$, we get

$$\begin{aligned} D^+ V|_{(2.4.11)} &\leqslant \sum_{j=1}^{n}\Big[r_j\tilde{a}_{jj}+\sum_{\substack{i=1\\i\neq j}}^{n} r_i|\tilde{a}_{ij}|\Big]|\xi_j| + r_n h_n|f(\xi_n)| \\ &\leqslant -r_n h_n|f(\xi_n)| \\ &< 0 \qquad \text{for} \quad \xi_n \neq 0. \end{aligned}$$

Hence the zero solution of (2.4.11) is absolutely stable. ∎

Theorem 2.4.19. *Assume that*

1) *the matrix* $\tilde{A}+O_{n\times(n-1)}\tilde{b}\theta$ *is stable*, $\theta=0$ *or* 1;

2) *there exist* $r_i \geqslant 0$ $(i=1, \cdots, n-1)$, $r_n>0$ *such that*

$\xi^T(\operatorname{diag}(r_1,\cdots,r_n)\tilde{A}+\tilde{A}^T\operatorname{diag}(r_1,\cdots,r_n))\xi$ *is negative semi-definite.*

Then the zero solution of the system (2.4.11) *is absolutely stable.*

Proof. We construct the Liapunov function which is radially unbounded positive definite for ξ_n:

$$V(\xi)=\xi^T \operatorname{diag}(r_1,\cdots,r_n)\xi,$$

and we obtain

$$\begin{aligned} \left.\frac{dV}{dt}\right|_{(2.4.11)} &\leqslant \xi^T(\tilde{A}^T\operatorname{diag}(r_1,\cdots,r_n)+\operatorname{diag}(r_1,\cdots,r_n)\tilde{A})\xi + 2r_n h_n \xi_n f(\xi_n) \\ &< 0 \qquad \text{for} \quad \xi_n \neq 0. \end{aligned}$$

Hence the zero solution of (2.4.11) is absolutely stable. ∎

Corollary 2.4.20. *Assume that* $\tilde{A}$ *is Volterra-Liapunov stable, i.e., there exists* $R=\operatorname{diag}(r_1, \cdots, r_n)>0$ *such that*

$$\tilde{A}^T R + R\tilde{A} \quad \textit{is negative definite.}$$

Then the zero solution of (2.4.11) *is absolutely stable.*

Corollary 2.4.21. *If* $\tilde{A}+\tilde{A}^T$ *is negative definite, then the zero solution of* (2.4.11) *is absolutely stable.*

Now we rewrite the coefficient matrix B in the expression (2.4.11) in the form:

$$\tilde{A}=\begin{bmatrix}\tilde{A}_{11} & \tilde{A}_{12}\\ \tilde{A}_{21} & \tilde{A}_{22}\end{bmatrix},$$

where

$$\begin{aligned} \tilde{A}_{11} &= (\tilde{a}_{ij})_{(n-1)\times(n-1)}, & \tilde{A}_{12} &= \operatorname{col}(\tilde{a}_{1n},\cdots,\tilde{a}_{n-1,n}), \\ \tilde{A}_{21} &= (\tilde{a}_{n-1},\cdots,\tilde{a}_{n,n-1}), & \tilde{A}_{22} &= \tilde{a}_{nn}. \end{aligned}$$

Theorem 2. 4. 22. *Assume that*

1) *for any given* $(n-1)\times(n-1)$ *symmetric positive definite matrix* Q *(in particular for* E*), the Liapunov matrix equation*

$$P\tilde{A}_{11}+\tilde{A}_{11}^{\mathrm{T}}P=-Q$$

has the symmetric positive definite matrix solution P;

2) $p_{nn}=-\dfrac{1}{\tilde{a}_{nn}}>0$;

3) $(\tilde{A}_{21}P+p_{nn}A_{12}^{\mathrm{T}})Q^{-1}(\tilde{A}_{12}p_{nn}+P\tilde{A}_{21}^{\mathrm{T}})>-1$ *or*

$Q-(\tilde{A}_{12}p_{nn}+P\tilde{A}_{21}^{\mathrm{T}})(\tilde{A}_{21}P+p_{nn}\tilde{A}_{12}^{\mathrm{T}})$ *is positive definite.*

Then the zero solution of (2. 4. 11) *is absolutely stable.*

Proof. We choose the Liapunov function of positive definite quadratic form

$$V=\xi^{\mathrm{T}}\begin{bmatrix} P & 0_{(n-1)\times 1} \\ 0_{1\times(n-1)} & p_{nn} \end{bmatrix}\xi.$$

Since the condition 3) ensures that

$$\det\begin{vmatrix} Q & \tilde{A}_{12}p_{nn}+P\tilde{A}_{21}^{\mathrm{T}} \\ (\tilde{A}_{21}P+p_{nn}\tilde{A}_{12}^{\mathrm{T}} & 1 \end{vmatrix}>0,$$

we have

$$\begin{aligned}\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.4.11)} &=\xi^{\mathrm{T}}\left[\begin{bmatrix} P & 0_{(n-1)\times 1} \\ 0_{1\times(n-1)} & p_{nn} \end{bmatrix}\begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{a}_{nn} \end{bmatrix}\right. \\ &\quad\left.+\begin{bmatrix} \tilde{A}_{11}^{\mathrm{T}} & \tilde{A}_{21}^{\mathrm{T}} \\ \tilde{A}_{12}^{\mathrm{T}} & \tilde{a}_{nn} \end{bmatrix}\begin{bmatrix} P & 0_{(n-1)\times 1} \\ 0_{1\times(n-1)} & p_{nn} \end{bmatrix}\right]\xi \\ &\quad-2\frac{h_n}{\tilde{a}_{nn}}\xi_n f(\xi_n) \\ &<0 \quad \text{for} \quad \xi\neq 0.\end{aligned}$$

Thus, the zero solution of (2. 4. 11) is absolutely stable. ∎

Further, we can formulate the following theorem.

Theorem 2. 4. 23. *Suppose that*

1) *the matrix* $\tilde{A}+O_{n\times(n-1)}\tilde{b}\theta$ *is stable,* $\theta=0$ *or* 1;

2) P *in condition* 1) *of Theorem* 2. 4. 22 *is positive semi-definite*;

3) $p_{nn}=-\dfrac{1}{\tilde{a}_{nn}}>0$;

4) $(\tilde{A}_{21}P+p_{nn}\tilde{A}_{12}^{\mathrm{T}})Q^{-1}(\tilde{A}_{12}p_{nn}+P\tilde{A}_{21}^{\mathrm{T}})\geqslant-1$ *or*

$Q-(\tilde{A}_{12}p_{nn}+P\tilde{A}_{21}^{\mathrm{T}})(\tilde{A}_{21}P+p_{nn}\tilde{A}_{12}^{\mathrm{T}})$ *is negative semi-definite.*

Then the zero solution of (2. 4. 11) *is absolutely stable.*

Example 2. 4. 24. We study the absolute stability of the 3-dimensional system

$$\begin{cases} \dfrac{dx_1}{dt} = -3x_1 + \dfrac{1}{2}x_2 + x_3 - f(x_3), \\ \dfrac{dx_2}{dt} = 2x_1 - 2x_2 - x_3 + f(x_3), \\ \dfrac{dx_3}{dt} = 2x_1 + \dfrac{4}{5}x_2 - 3x_3 - 4f(x_3). \end{cases} \tag{2.4.12}$$

Choosing $V(x) = 2|x_1| + |x_2| + |x_3|$, we get

$$D^+ V(x)|_{(2.4.12)} \leqslant -2|x_1| - \frac{1}{5}|x_2| - |f(x_3)| < 0 \qquad \text{for} \quad x \neq 0.$$

Thus, the zero solution of the system (2. 4. 12) is absolutely stable.

Example 2. 4. 25. Let us discuss the absolute stability of the 4-dimensional system

$$\begin{cases} \dfrac{dx_1}{dt} = -2x_1 - x_2 + x_3 + x_4 + f(x_4), \\ \dfrac{dx_2}{dt} = x_1 - 2x_2 + x_3 + x_4 + \dfrac{6}{5}f(x_4), \\ \dfrac{dx_3}{dt} = \dfrac{1}{2}x_2 - \dfrac{3}{2}x_3 + x_4 - f(x_4), \\ \dfrac{dx_4}{dt} = \dfrac{1}{4}x_1 + \dfrac{1}{5}x_3 - x_4 - \dfrac{6}{5}f(x_4). \end{cases} \tag{2.4.13}$$

Choosing $r_1 = 1$, $r_2 = 1$, $r_3 = 2$, $r_4 = 4$, we find

$$r_1 a_{11} + \sum_{i=2}^{4} r_i |a_{i1}| = -2 + 1 + 1 = 0,$$

$$r_2 a_{22} + \sum_{\substack{i=1 \\ i\neq 2}}^{4} r_i |a_{i2}| = -2 + 1 + 1 = 0,$$

$$r_3 a_{33} + \sum_{\substack{i=1 \\ i\neq 3}}^{4} r_i |a_{i3}| = -3 + 1 + 1 + \frac{4}{5} = -\frac{1}{5} < 0,$$

$$r_4 a_{44} + \sum_{i=1}^{3} r_i |a_{i4}| = -4 + 1 + 1 + 2 = 0,$$

$$r_4 b_4 + \sum_{i=1}^{3} r_i |b_i| = -\frac{24}{5} + 1 + \frac{6}{5} + 2 = -\frac{3}{5} < 0.$$

We construct the Liapunov function $V(x) = \sum_{i=1}^{n} r_i |x_i|$. It follows that

$$D^+ V|_{(2.4.13)} \leqslant -\frac{1}{5}|x_3| - \frac{3}{5}|f(x_4)|.$$

Therefore, we conclude that the zero solution of the system is absolutely stable for x_3, x_4.

Furthermore, since $A_{(2)} = \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix}$ is obviously stable, the zero solution of (2.4.13) is absolutely stable.

2.5. Indirect Control System

Consider the Lurie indirect control system

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}t} = Dy + s\xi, \\ \dfrac{\mathrm{d}\xi}{\mathrm{d}t} = f(\sigma), \\ \sigma = c^{\mathrm{T}}y - \rho\xi, \end{cases} \tag{2.5.1}$$

where $D = (d_{ij})_{n\times n} \in \mathbf{R}^{n\times n}$, $s \in \mathbf{R}^n$, $y \in \mathbf{R}^n$, $\xi \in \mathbf{R}$, $p \in \mathbf{R}$,
$f(\sigma) \in F = \{f{:}\ f(0) = 0,\ \sigma f(\sigma) > 0,\ \sigma \neq 0,\ f \in C[(-\infty, +\infty), \mathbf{R}]\}$.

It can be easily proved that the necessary condition for absolute stability of the zero solution of (2.5.1) is $\rho \geqslant 0$ and $\begin{pmatrix} D & s \\ c^{\mathrm{T}} & -\rho \end{pmatrix}$ stable.

If $\rho \neq 0$, we can use the n-dimensional full-rank linear transformation

$$\begin{cases} x_i = y_i, \qquad i = 1,\cdots,n, \\ x_{n+1} = \sigma = \sum\limits_{i=1}^{n} c_i y_i - \rho\xi. \end{cases} \tag{2.5.2}$$

Then (2.5.2) is transformed into the system with separable variables

$$\begin{cases} \dfrac{\mathrm{d}x_i}{\mathrm{d}t} = \sum\limits_{j=1}^{n+1} a_{ij}x_j, \qquad i = 1,\cdots,n, \\ \dfrac{\mathrm{d}x_{n+1}}{\mathrm{d}t} = \sum\limits_{j=1}^{n+1} a_{n+1,j}x_j - \rho f(x_{n+1}), \end{cases} \tag{2.5.3}$$

where

$$\begin{aligned} a_{ij} &= d_{ij} + \frac{s_i}{\rho}c_j, \qquad i,j = 1,\cdots,n, \\ a_{i,n+1} &= -\frac{s_i}{\rho}, \qquad i = 1,\cdots,n, \\ a_{n+1,j} &= \sum_{i=1}^{n} c_i d_{ij} = \sum_{i=1}^{n} c_i\left(d_{ij} + \frac{s_i c_j}{\rho}\right), \qquad j = 1,\cdots,n, \\ a_{n+1,n+1} &= \sum_{i=1}^{n} c_i d_{i,n+1} = \sum_{i=1}^{n} c_i\left(-\frac{s_i}{\rho}\right). \end{aligned}$$

Since the necessary condition for absolute stability of the zero solution of (2.5.1) is $\begin{pmatrix} D & s \\ c^{\mathrm{T}} & -\rho \end{pmatrix}$ being stable, we take $\begin{pmatrix} D & s \\ c^{\mathrm{T}} & -\rho \end{pmatrix}$ nonsingular.

Again, by the nonsingular linear transformation:

$$\begin{cases} z = Dy + s\xi, \\ z_{n+1} = c^{\mathrm{T}}y - \rho\xi, \end{cases} \qquad z, y \in \mathbf{R}^n, \tag{2.5.4}$$

the expression (2. 5. 3) is transformed into

$$\begin{cases} \dfrac{\mathrm{d}z}{\mathrm{d}t} = Bz + hf(z_{n+1}), \\ \dfrac{\mathrm{d}z_{n+1}}{\mathrm{d}t} = c^{\mathrm{T}}z - \bar{p}f(z_{n+1}), \end{cases} \tag{2.5.5}$$

where

$$\begin{pmatrix} B & h \\ c^{\mathrm{T}} & -\bar{p} \end{pmatrix} = \begin{pmatrix} D & s \\ c^{\mathrm{T}} & -\rho \end{pmatrix} \begin{pmatrix} D & s \\ 0 & 0 \end{pmatrix} \begin{pmatrix} D & s \\ c^{\mathrm{T}} & -\rho \end{pmatrix}^{-1},$$

$$\begin{pmatrix} h \\ -\bar{p} \end{pmatrix} = \begin{pmatrix} D & s \\ c^{\mathrm{T}} & -\rho \end{pmatrix} \begin{pmatrix} s \\ 1 \end{pmatrix}.$$

Obviously, under condition that $\rho \neq 0$, $\begin{pmatrix} D & s \\ c^{\mathrm{T}} & -\rho \end{pmatrix}$ is nonsingular, the stability of (2. 5. 1), (2. 5. 3) and (2. 5. 4) is equivalent. In analogy with Theorems 2. 2. 3～2. 2. 4 we formulate:

Theorem 2. 5. 1. *The zero solution of* (2. 5. 3) *is absolutely stable if and only if*

1) *the zero solution of* (2. 5. 3) *is absolutely stable for* x_{n+1},

2) *the matrix* $G(g_{ij})_{(n+1)\times(n+1)}$ *is stable, where*

$$g_{ij} = \begin{cases} a_{n+1,n+1} - \rho, & i = j = n+1, \\ a_{ij}, & \textit{otherwise}. \end{cases}$$

Theorem 2. 5. 2. *The zero solution of* (2. 5. 3) *is absolutely stable if and only if*

1) *the zero solution of* (2. 5. 3) *is absolutely stable for the partial variables* $x_j, \cdots, x_{n+1}$ $(i < j \leqslant n+1)$,

2) *the condition* 2) *of Theorem* 2. 5. 1 *holds.*

Theorem 2. 5. 3. *The zero solution of* (2. 5. 5) *is absolutely stable if and only if*

1) *the zero solution of* (2. 5. 5) *is absolutely stable for* z_{n+1},

2) $\begin{pmatrix} B & h \\ c^{\mathrm{T}} & -\bar{p} \end{pmatrix}$ *is stable.*

Theorem 2. 5. 4. *The zero solution of* (2. 5. 5) *is absolutely stable if and only if*

1) *the zero solution of* (2. 5. 5) *is absolutely stable for the partial variables* $z_j, \cdots, z_{n+1}$ $(1 < j \leqslant n+1)$,

2) $\begin{pmatrix} B & h \\ c^{T} & -\bar{p} \end{pmatrix}$ *is stable.*

The proofs of Theorems 2.5.1~2.5.4 can be completed similarly to that of Theorem 2.2.3, and is omited.

In the following, we derive a series of practical sufficient conditions from the above theorems. Henceforth, we always assume $\rho > 0$ $(\bar{p} > 0)$.

Theorem 2.5.5. *If* $a_{ii}<0$, $i=1,\cdots,n+1$, *and*

$$G_a \triangleq -((-1)^{\delta_{ij}}|a_{ij}|)_{(n+1)\times(n+1)} \quad \text{is an } M\text{-matrix},$$

the zero solution of (2.5.3) *is absolutely stable.*

Proof. Since G_a is an M-matrix, and $a_{ii} < 0$ $(i = 1,\cdots,n+1)$, there exist $n+1$ positive constants $r_i > 0$ $(i = 1,\cdots,n+1)$ such that

$$-r_j a_{jj} > \sum_{\substack{i=1\\ i\neq j}}^{n+1} r_i |a_{ij}|, \qquad j = 1,\cdots,n+1.$$

We construct the radially unbounded positive definite Liapunov function

$$V(x) = \sum_{i=1}^{n+1} r_i |x_i|.$$

Then we get

$$\mathrm{D}^{+} V(x)|_{(2.5.3)} \leqslant \sum_{j=1}^{n+1}\Big[r_j a_{jj} + \sum_{\substack{i=1\\ i\neq j}}^{n+1} r_i |a_{ij}|\Big]|x_j| - \rho r_{n+1}|f(x_{n+1})|$$

$$<0 \qquad \text{for} \quad x \neq 0.$$

Consequently the zero solution of (2.5.3) is absolutely stable. ∎

Theorem 2.5.6. *Suppose that*

1) *the matrix A or the matrix* $A + \begin{bmatrix} 0_{n\times n} & 0_{n\times 1} \\ 0_{1\times n} & -\rho \end{bmatrix}$ *is stable,*

2) *there exist constants* $r_i \geqslant 0$ $(i=1,\cdots,n)$, $r_{n+1}>0$ *such that*

$$-r_j a_{jj} \geqslant \sum_{\substack{i=1\\ i\neq j}}^{n+1} r_i |a_{ij}|, \qquad j = 1,\cdots,n+1.$$

Then the zero soluton of (2.5.3) *is absolutely stable.*

Proof. We construct the radially unbounded positive definite Liapunov function

$$V(x) = \sum_{i=1}^{n+1} r_i |x_i|.$$

The argument used in the proof of Theorem 2.5.5 works and

$$\mathrm{D}^{+} V(x)|_{(2.5.3)} \leqslant \sum_{j=1}^{n+1}\Big[r_j a_{jj} + \sum_{\substack{i=1\\ i\neq j}}^{n+1} r_i |a_{ij}|\Big]|x_j| - \rho r_{n+1}|f(x_{n+1})|$$

$$\leqslant -\rho r_{n+1}|f(x_{n+1})| < 0 \qquad \text{for} \quad x_{n+1} \neq 0.$$

Thus the zero solution of (2.5.3) is absolutely stable for x_{n+1}.

According to Theorem 2.5.1 the assertion holds. ∎

Theorem 2.5.7. *If $b_{ii}<0$, $i=1,\cdots,n$, $\bar{p}>0$ and*

$$\Omega \triangleq \begin{pmatrix} |b_{11}| & -|b_{12}| & \cdots & -|b_{1n}| & -|h_1| \\ -|b_{21}| & |b_{22}| & \cdots & -|b_{2n}| & -|h_2| \\ \vdots & \vdots & & \vdots & \vdots \\ -|b_{n1}| & -|b_{n2}| & \cdots & |b_{nn}| & -|h_n| \\ -|c_1| & -|c_2| & \cdots & -|c_n| & \bar{p} \end{pmatrix} \text{ is an M-matrix,}$$

then the zero solution of (2.5.5) is absolutely stable.

Proof. Since Ω is an M-matrix, there exist constants $r_i > 0$ $(i = 1,\cdots, n+1)$ such that

$$r_j|b_{jj}| > \sum_{\substack{i=1\\i\neq j}}^{n} r_i|b_{ij}| + r_{n+1}|c_j|, \qquad j = 1,\cdots,n,$$

$$r_{n+1}\bar{p} > \sum_{i=1}^{n} r_i|h_i|.$$

We construct the Liapunov function

$$V(z) = \sum_{i=1}^{n+1} r_i|z_i|.$$

As in Theorem 2.5.5, we obtain

$$\begin{aligned} D^+ V(z)|_{(2.5.5)} &\leqslant \sum_{j=1}^{n}\Big[r_jb_{jj} + \sum_{\substack{i=1\\i\neq j}}^{n} r_i|b_{ij}|\Big]|z_j| \\ &\quad + \Big[-\bar{p}r_{n+1} + \sum_{i=1}^{n} r_i|h_i|\Big]|f(z_{n+1})| \\ &< 0 \qquad \text{for} \quad z\neq 0. \end{aligned}$$

Thus the zero solution of (2.5.5) is absolutely stable. ∎

Theorem 2.5.8. 1) *Let the matrix* $\begin{pmatrix} B & h \\ c^T & -\bar{p} \end{pmatrix}$ *be stable.*

2) *Suppose that there exist constants $r_i\geqslant 0$ $(i=1,\cdots,n)$, $r_{n+1}>0$ such that*

$$-r_jb_{jj} \geqslant \sum_{\substack{i=1\\i\neq j}}^{n} r_i|b_{ij}| + r_{n+1}|c_j|, \qquad j = 1,\cdots,n,$$

$$r_{n+1}\bar{p} > \sum_{i=1}^{n} r_i|h_i|.$$

Then the zero solution of (2.5.5) is absolutely stable.

Proof. We construct the radially unbounded positive definite Liapunov function

$$V(z) = \sum_{i=1}^{n+1} r_i|z_i|,$$

then

$$\begin{aligned}
D^+ V(z)|_{(2.5.5)} &\leqslant \sum_{j=1}^{n}\Big[r_j b_{jj} + \sum_{\substack{i=1\\ i\neq j}}^{n} r_i|b_{ij}| + r_{n+1}|c_j|\Big]|z_j| \\
&\quad + \Big[-\bar{p}r_{n+1} + \sum_{i=1}^{n} r_i|h_i|\Big]|f(z_{n+1})| \\
&\leqslant \Big[-\bar{p}r_{n+1} + \sum_{i=1}^{n} r_i|h_i|\Big]|f(z_{n+1})| \\
&< 0 \qquad \text{for} \quad z_{n+1}\neq 0.
\end{aligned}$$

Accordingly, $D^+ V(z)|_{(2.5.5)}$ is negative definite for z_{n+1}, and it follows from condition 1) that the conditions of Theorem 2. 5. 3 are satisfied. Hence the conclusion of this theorem holds. ▮

In the following, we take

$$A_{(j_0)} = \begin{bmatrix} a_{11} & \cdots & a_{1j_0} \\ \vdots & & \vdots \\ a_{j_01} & \cdots & a_{j_0j_0} \end{bmatrix}, \qquad A^{(n+1-j_0)} = \begin{bmatrix} a_{1,j_0+1} & \cdots & a_{1,n+1} \\ \vdots & & \vdots \\ a_{j_0,j_0+1} & \cdots & a_{j_0,n+1} \end{bmatrix},$$

$$B_{(j_0)} = \begin{bmatrix} b_{11} & \cdots & b_{1j_0} \\ \vdots & & \vdots \\ b_{j_01} & \cdots & b_{j_0j_0} \end{bmatrix}, \qquad B^{(n+1-j_0)} = \begin{bmatrix} b_{1,j_0+1} & \cdots & b_{1n} & h_1 \\ \vdots & & \vdots & \vdots \\ b_{j_0,j_0+1} & \cdots & b_{j_0n} & h_{j_0} \end{bmatrix},$$

$$x_{(j_0)} = \mathrm{col}(x_1,\cdots,x_{j_0}), \qquad x^{(n+1-j_0)} = \mathrm{col}(x_{j_0+1},\cdots,x_{n+1}),$$

$$z_{(j_0)} = \mathrm{col}(z_1,\cdots,z_{j_0}), \qquad z^{(n+1-j_0)} = \mathrm{col}(z_{j_0+1},\cdots,z_n, f(z_{n+1})).$$

Theorem 2. 5. 9. *Suppose that*

1) *the matrix $A_{(j_0)}$ is stable,*

2) *there exist constants $r_i\geqslant 0$ $(i=1,\cdots,j_0)$, $r_j>0$ $(j=j_0+1,\cdots,n+1)$ such that*

$$-r_ja_{jj} \geqslant \sum_{\substack{i=1\\ i\neq j}}^{n+1} r_i|a_{ij}|, \qquad j=1,\cdots,j_0,$$

$$-r_ja_{jj} > \sum_{\substack{i=1\\ i\neq j}}^{n+1} r_i|a_{ij}|, \qquad j=j_0+1,\cdots,n+1.$$

Then the zero solution of (2. 5. 3) is absolutely stable.

Proof. We construct the radially unbounded positive definite Liapunov function with respect to the partial variables $x_{j_0+1},\cdots,x_{n+1}$:

$$V(x) = \sum_{i=1}^{n+1} r_i|x_i|,$$

then

$$D^+ V|_{(2.5.3)} \leqslant \sum_{j=1}^{n+1}\Big[r_j a_{jj} + \sum_{\substack{i=1\\i\neq j}}^{n+1} r_i |a_{ij}|\Big]|x_j| - \rho r_{n+1}|f(x_{n+1})|$$
$$\leqslant \sum_{j=j_0+1}^{n+1}\Big[r_j a_{jj} + \sum_{\substack{i=1\\i\neq j}}^{n+1} r_i |a_{ij}|\Big]|x_j|$$
$$<0 \quad \text{for} \quad \sum_{j=j_0+1}^{n+1} x_j^2 \neq 0.$$

Thus the zero solution of (2.5.3) is absolutely stable for the partial variables $x_{j_0+1}, \cdots, x_{n+1}$.

The first j_0 components of the solution of (2.5.3) can be expressed as

$$x_{(j_0)}(t,t_0,x_0) = \exp A_{(j_0)}(t-t_0)\cdot x_{(j_0)}(t_0) + \int_{t_0}^{t} \exp A_{(j_0)}(t-\tau)\cdot x^{(n+1-j_0)}(\tau)\mathrm{d}\tau - \int_{t_0}^{t} \rho \exp A_{(j_0)}(t-\tau)\cdot f(x_{n+1}(\tau))\mathrm{d}\tau.$$

Following the proof of the sufficiency in Theorem 2.2.3, we can complete the rest of the proof. ▮

Similarly,

Theorem 2.5.10. *Suppose that*

1) *the matrix $B_{(j_0)}$ is stable;*

2) *there exist constants $r_i \geqslant 0$ $(i=1,\cdots,j_0)$, $r_j > 0$ $(j=j_0+1,\cdots,n+1)$ such that*

$$-r_j b_{jj} \geqslant \sum_{\substack{i=1\\i\neq j}}^{n} r_i |b_{ij}| + r_{n+1}|c_j|, \qquad j=1,\cdots,j_0,$$
$$-r_j b_{jj} > \sum_{\substack{i=1\\i\neq j}}^{n} r_i |b_{ij}| + r_{n+1}|c_j|, \qquad j=j_0+1,\cdots,n,$$
$$r_{n+1}p > \sum_{i=1}^{n} r_i |h_i|.$$

Then the zero solution of (2.5.5) is absolutely stable.

Theorem 2.5.11. *Assume that*

1) *$A(a_{ij})_{(n+1)\times(n+1)}$ or $A + \begin{pmatrix} 0 & 0 \\ 0 & -\rho \end{pmatrix}$ is stable,*

2) *there exist a symmetric positive semi-definite matrix of the form*

$$P = \begin{pmatrix} p_{11} & \cdots & p_{1n} & 0 \\ \vdots & & \vdots & \vdots \\ p_{n1} & \cdots & p_{nn} & 0 \\ 0 & \cdots & 0 & p_{n+1,n+1} \end{pmatrix} \qquad (p_{n+1,n+1} > 0)$$

such that $A^{\mathrm{T}}P+PA$ is negative semi-definite.

Then the zero solution of (2.5.3) *is absolutely stable.*

Proof. Obviously, the condition 1) is the same as the condition 2) in Theorem 2.5.1.

We construct the radially unbounded positive definite Liapunov function with respect to x_{n+1}:

$$V(x) = x^{\mathrm{T}}Px.$$

We fix $l = \mathrm{col}(\overbrace{0,\cdots,0}^{n}, -\rho)$, then

$$\begin{aligned}
\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.5.3)} &= x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x + (l^{\mathrm{T}}Px + x^{\mathrm{T}}Pl)f(x_{n+1}) \\
&= x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x - 2\rho p_{n+1,n+1}x_{n+1}f(x_{n+1}) \\
&\leqslant -2\rho p_{n+1,n+1}x_{n+1}f(x_{n+1}) \\
&< 0 \qquad \text{for} \quad x_{n+1} \neq 0.
\end{aligned}$$

Thus, the condition 1) of Theorem 2.5.1 is satisfied and the conclusion of this theorem is true. ∎

Theorem 2.5.12. *Assume that*

1) *the condition* 1) *of Theorem* 2.5.11 *holds;*

2) *there exist a constant $\varepsilon>0$ and an $(n+1)\times(n+1)$ symmetric, positive semi-definite matrix P such that*

$$\begin{pmatrix} A^{\mathrm{T}}P + PA & Pl + A_{n+1}/2 + \varepsilon e_n \\ (Pl + A_{n+1}/2 + \varepsilon e_n)^{\mathrm{T}} & -\rho \end{pmatrix}$$

is negative semi-definite, where

$$A_{n+1} = \mathrm{col}(a_{n+1,1},\cdots,a_{n+1,n+1}), \qquad l = \mathrm{col}(\overbrace{0,\cdots,0}^{n}, -\rho),$$

$$e_n = \mathrm{col}(\overbrace{0,\cdots,0}^{n-1},1);$$

3) $\displaystyle\int_0^{\pm\infty} f(x_{n+1})\mathrm{d}x_{n+1} = +\infty.$

Then the zero solution of (2.5.3) *is absolutely stable.*

Proof. We construct the Liapunov function

$$V(x) = x^{\mathrm{T}}Px + \int_0^{x_{n+1}} f(x_{n+1})\mathrm{d}x_{n+1}.$$

Obviously, there exists

$$V(x) \geqslant \int_0^{x_{n+1}} f(x_{n+1})\mathrm{d}x_{n+1} > 0 \qquad \text{for} \quad x_{n+1} \neq 0,$$

and $V(x) \to +\infty$ as $x_{n+1} \to +\infty$. Moreover, we have

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.5.3)} = \dot{x}^{\mathrm{T}}Px + x^{\mathrm{T}}P\dot{x} + [A_{n+1}^{\mathrm{T}}x - \rho f(x_{n+1})]f(x_{n+1})$$

$$= x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x + [l^{\mathrm{T}}Px + x^{\mathrm{T}}Pl + A_{n+1}^{\mathrm{T}}x]f(x_{n+1}) - \rho f^2(x_{n+1})$$

$$= (x^{\mathrm{T}}, f(x_{n+1}))\begin{bmatrix} A^{\mathrm{T}}P + PA & Pl + A_{n+1}/2 + \varepsilon e_n \\ (Pl + A_{n+1}/2 + \varepsilon e_n)^{\mathrm{T}} & -\rho \end{bmatrix}\begin{pmatrix} x \\ f(x_{n+1}) \end{pmatrix}$$

$$- (x^{\mathrm{T}}, f(x_{n+1}))\begin{bmatrix} 0_{(n+1)\times(n+1)} & \varepsilon e_n \\ (\varepsilon e_n)^{\mathrm{T}} & 0 \end{bmatrix}\begin{pmatrix} x \\ f(x_{n+1}) \end{pmatrix}$$

$$\leqslant - 2\varepsilon x_{n+1}f(x_{n+1}) < 0 \quad \text{as} \quad x_{n+1} \neq 0.$$

Thus, the zero solution of (2. 5. 3) is absolutely stable. ∎

Remark. Suppose that the condition that the matrix P is positive semi-definite is replaced by the condition that the matrix P is positive definite or that there exists a constant $\varepsilon > 0$ satisfying

$$x^{\mathrm{T}}Px \geqslant \varepsilon x_{n+1}^2.$$

Then the condition 3) of Theorem 2. 5. 12 can be dropped.

Theorem 2. 5. 13. 1) *Let the condition* 1) *of Theorem* 2. 5. 12 *be satisfied.*

2) *Suppose that there exists an* $n\times n$ *symmetric positive semi-definite matrix* P *such that* $B^{\mathrm{T}}P+PB\triangleq -Q$ *is negative semi-definite, and that there exists a constant* $\varepsilon>0$ *such that*

$$\det\begin{vmatrix} Q & -(Ph + c/2) \\ -(Ph + c/2)^{\mathrm{T}} & \bar{p} - \varepsilon \end{vmatrix} \geqslant 0$$

and

$$\int_0^{\pm\infty} f(z_{n+1})\mathrm{d}z_{n+1} = +\infty.$$

Then the zero solution of (2. 5. 5) *is absolutely stable.*

Proof. We construct the Liapunov function

$$V(z) = z^{\mathrm{T}}Pz + \int_0^{z_{n+1}} f(z_{n+1})\mathrm{d}z_{n+1}$$

with $z = \mathrm{col}(z_1, \cdots, z_n)$. Obviously, $V(z)$ is radially unbounded positive definite for z_{n+1}, and

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.5.5)} = z^{\mathrm{T}}(B^{\mathrm{T}}P + PB)z + (h^{\mathrm{T}}Pz + z^{\mathrm{T}}Ph + c^{\mathrm{T}}z)f(z_{n+1}) - \bar{p}f^2(z_{n+1})$$

$$= - z^{\mathrm{T}}Qz + 2f(z_{n+1})(Ph + c/2)^{\mathrm{T}}z - \bar{p}f^2(z_{n+1})$$

$$= (z^{\mathrm{T}}, f(z_{n+1}))\begin{pmatrix} -Q & Ph + c/2 \\ (Ph + c/2)^{\mathrm{T}} & -\bar{p} + \varepsilon \end{pmatrix}\begin{pmatrix} z \\ f(z_{n+1}) \end{pmatrix} - \varepsilon f^2(z_{n+1})$$

$$\leqslant - \varepsilon f^2(z_{n+1}) < 0 \quad \text{for} \quad z_{n+1} \neq 0.$$

In this case, the zero solution of (2.5.5) is absolutely stable for z_{n+1}. Hence all the conditions of Theorem 2.5.3 are satisfied; thus the conclusion of this theorem is valid. ∎

Theorem 2.5.14. *Suppose the following conditions are satisfied*:

1) $A_{(j_0)}$ *is stable*,

2) *there exist an* $n\times n$ *matrix* P *and a constant* $\varepsilon>0$ ($\varepsilon\ll1$) *such that*

$$x^{\mathrm T}Px\geqslant\varepsilon\sum_{j=j_0+1}^{n+1}x_j^2$$

and

$$(x^{\mathrm T},f(x_{n+1}))\begin{pmatrix}A^{\mathrm T}P+PA & Ph+A_{n+1}/2\\ (Ph+A_{n+1}/2)^{\mathrm T} & -\rho\end{pmatrix}\begin{pmatrix}x\\ f(x_{n+1})\end{pmatrix}$$

$$\leqslant\begin{cases}-\varepsilon\sum\limits_{i=j_0+1}^{n}x_i^2-\varepsilon f^2(x_{n+1}), & \text{or}\\ -\varepsilon\sum\limits_{i=j_0+1}^{n+1}x_i^2, & \text{or}\\ -\varepsilon\left[\sum\limits_{i=j_0+1}^{n}x_i^2+x_{n+1}f(x_{n+1})\right],\end{cases}$$

where $A_{n+1}=\mathrm{col}(a_{n+1,1},\cdots,a_{n+1,n+1})$.

Then the zero solution of (2.5.3) *is absolutely stable.*

Proof. We construct the Liapunov function

$$V=x^{\mathrm T}Px+\int_0^{x_{n+1}}f(x_{n+1})\mathrm{d}x_{n+1}.$$

The condition 2) asserts that $V(x)$ is radially unbounded and positive definite with respect to the partial variables $x_{j_0+1},\cdots,x_{n+1}$, and that $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.5.3)}$ is negative definite with respect to $x_{j_0+1},\cdots,x_{n+1}$.

In addition, the first j_0 components of the solution of (2.5.3) can be expressed as

$$x_{(j_0)}(t)=\exp A_{(j_0)}(t-t_0)\cdot x_{(j_0)}(t_0)+\int_{t_0}^{t}\exp A_{(j_0)}(t-\tau)\cdot x^{(n+1-j_0)}(\tau)\mathrm{d}\tau$$

$$+\int_{t_0}^{t}\exp A_{(j_0)}(t-\tau)\cdot f(x_{n+1}(\tau))\mathrm{d}\tau.$$

The rest of the proof can be completed as in Theorem 2.2.3. ∎

Similarly we formulate

Theorem 2.5.15. 1) *Let* $B_{(j_0)}$ *be stable*;

2) *Suppose that there exist an* $n\times n$ *symmetric matrix* P *and a constant* $\varepsilon>0$ *such that*

$$z^{\mathrm{T}}Bz \geqslant \varepsilon \sum_{i=j_0+1}^{n} z_i^2$$

and

$$(z^{\mathrm{T}}, f(z_{n+1}))\begin{bmatrix} B^{\mathrm{T}}P+PB & Ph+c/2 \\ (Ph+c/2)^{\mathrm{T}} & -\bar{p} \end{bmatrix}\begin{pmatrix} z \\ f(z_{n+1}) \end{pmatrix}$$
$$\leqslant -\varepsilon \sum_{i=j_0+1}^{n} z_i^2 - \varepsilon f^2(z_{n+1});$$

3) $\int_0^{\pm\infty} f(z_{n+1})\mathrm{d}z_{n+1} = +\infty$.

Then the zero solution of (2.5.5) *is absolutely stable.*

In the following, we will consider a different method.

Without loss of generality, we can assume that (2.5.5) is transformed into the following form:

$$\begin{cases} \dfrac{\mathrm{d}z_i}{\mathrm{d}t} = \sum\limits_{j=1}^{n} b_{ij}z_j, \quad i=1,\cdots,i_0-1, \\ \dfrac{\mathrm{d}z_i}{\mathrm{d}t} = \sum\limits_{j=1}^{n} b_{ij}z_j + h_i f(z_{n+1}), \quad i=i_0,\cdots,n, \\ \dfrac{\mathrm{d}z_{n+1}}{\mathrm{d}t} = \sum\limits_{j=1}^{n} c_j z_j - \bar{p}f(z_{n+1}). \end{cases} \tag{2.5.6}$$

By the nonsingular linear transformation

$$\tilde{z} = \mathrm{diag}\Big(\overbrace{1,\cdots,1}^{i_0-1}, -\frac{1}{h_{i_0}}, \frac{1}{h_{i_0+1}}, \cdots, \frac{1}{h_n}, -\frac{1}{\bar{p}}\Big) z, \tag{2.5.7}$$

the system (2.5.6) can be transformed into

$$\begin{cases} \dfrac{\mathrm{d}\tilde{z}_i}{\mathrm{d}t} = \sum\limits_{j=1}^{n} \tilde{b}_{ij}\tilde{z}_j, \quad i=1,\cdots,i_0-1, \\ \dfrac{\mathrm{d}\tilde{z}_{i_0}}{\mathrm{d}t} = \sum\limits_{j=1}^{n} \tilde{b}_{i_0 j}\tilde{z}_j - \tilde{f}(\tilde{z}_{n+1}), \\ \dfrac{\mathrm{d}\tilde{z}_i}{\mathrm{d}t} = \sum\limits_{j=1}^{n} \tilde{b}_{ij}\tilde{z}_j + \tilde{f}(\tilde{z}_{n+1}), \quad i=i_0+1,\cdots,n+1, \end{cases} \tag{2.5.8}$$

where

$$\tilde{b}_{ij} = b_{ij}, \quad 1 \leqslant i,j \leqslant i_0-1;$$
$$\tilde{b}_{ii_0} = -b_{ii_0}h_{i_0}, \quad 1 \leqslant i \leqslant i_0-1;$$
$$\tilde{b}_{ij} = b_{ij}h_j, \quad i_0+1 \leqslant j \leqslant n, \quad 1 \leqslant i \leqslant i_0-1;$$
$$\tilde{b}_{i_0 j} = -\frac{b_{i_0 j}}{h_{i_0}}, \quad 1 \leqslant j \leqslant i_0-1;$$
$$\tilde{b}_{i_0 i_0} = -b_{i_0 i_0};$$

$$\bar{b}_{i_0 j} = -\frac{b_{i_0 j}h_j}{h_{i_0}}, \qquad i_0 + 1 \leqslant j \leqslant n;$$

$$\bar{b}_{ij} = \frac{b_{ij}h_j}{h_i}, \qquad i_0 + 1 \leqslant i \leqslant n, \quad i_0 + 1 \leqslant j \leqslant n;$$

$$\bar{b}_{ij} = \frac{b_{ij}}{h_i}, \qquad i_0 + 1 \leqslant i \leqslant n, \quad 1 \leqslant j \leqslant i_0 - 1;$$

$$\bar{b}_{ii_0} = -\frac{b_{ii_0}h_{i_0}}{h_i}, \qquad i_0 + 1 \leqslant i \leqslant n;$$

$$\bar{b}_{n+1,j} = -\frac{c_j}{\bar{p}}, \qquad 1 \leqslant j \leqslant i_0 - 1;$$

$$\bar{b}_{n+1,i_0} = \frac{c_{i_0}h_{i_0}}{\bar{p}};$$

$$\bar{b}_{n+1,j} = -\frac{c_j h_j}{\bar{p}}, \qquad i_0 + 1 \leqslant j \leqslant n;$$

$$\bar{f}(\bar{z}_{n+1}) = f(-\bar{p}\bar{z}_{n+1}) = f(z_{n+1}).$$

We use the n-dimensional full-rank linear transformation

$$z^* = I_{n\times n}\bar{z} + \mathrm{col}(\overbrace{0,\cdots,0}^{i_0-1},\overbrace{\bar{z}_{i_0},\cdots,\bar{z}_{i_0}}^{n+1-i_0}), \tag{2.5.9}$$

where

$$z^* = \mathrm{col}(z_1^*,\cdots,z_{i_0-1}^*,z_{i_0+1}^*,\cdots,z_{n+1}^*),$$

$$\bar{z} = \mathrm{col}(\bar{z}_1,\cdots,\bar{z}_{i_0-1},\bar{z}_{i_0+1},\cdots,\bar{z}_{n+1}).$$

The system (2.5.8) is reduced to the following system

$$\frac{\mathrm{d}z_i^*}{\mathrm{d}t} = \sum_{\substack{j=1\\ j\neq i_0}}^{n} b_{ij}^* z_j^* + b_i^* \bar{z}_{i_0}, \qquad i = 1,\cdots,n+1,\ i \neq i_0, \tag{2.5.10}$$

where

$$b_{ij}^* = \bar{b}_{ij}, \qquad 1 \leqslant i \leqslant i_0 - 1, \quad 1 \leqslant j \leqslant n,\ j \neq i_0;$$

$$b_{ij}^* = \bar{b}_{i_0 j} + \bar{b}_{ij}, \qquad i_0 + 1 \leqslant i \leqslant n + 1, \quad 1 \leqslant j \leqslant n,\ j \neq i_0;$$

$$b_i^* = \bar{b}_{ii_0} - \sum_{j=i_0+1}^{n} \bar{b}_{ij}, \qquad 1 \leqslant i \leqslant i_0 - 1;$$

$$b_i^* = \bar{b}_{i_0}b_{i_0} + \bar{b}_{ii_0} - \sum_{j=i_0+1}^{n} (\bar{b}_{i_0 j} + \bar{b}_{ij}), \qquad i_0 + 1 \leqslant i \leqslant n + 1.$$

We always assume that the initial values of (2.5.6), (2.5.8) and (2.5.10) satisfy (2.5.7) and (2.5.9).

Theorem 2.5.16. *The zero solution of* (2.5.6) *is absolutely stable if and only if*

1) *the zero solution of* (2.5.6) *is partially absolutely stable for* z_{i_0};

2) *the solution* $z^*(t, t_0, z_0^*)$ *of* (2.5.10) *corresponding to an arbitrary solution of* (2.5.6) *is such that: for any* $\varepsilon>0$, *there exists* $\delta(\varepsilon)>0$ *such that*

$$\| z^*(t,t_0,z_0^*) \| < \varepsilon \qquad \textit{for all} \quad t \geqslant t_0$$

if $\| z^*(t_0) \|^2 + \| \tilde{z}_{i_0}(t_0) \|^2 < \delta$, *and*

$$\lim_{t\to+\infty} \| z^*(t,t_0,z_0^*) \| = 0 \qquad \textit{for any} \quad z_0 \in \mathbf{R}^n.$$

Proof. *Necessity.* 1° The absolute stability of the zero solution of (2.5.6) implies partial absolute stability of the zero solution for z_{i_0}.

2° The expressions (2.5.7) and (2.5.9) lead to

$$z^* = \mathrm{diag}\Big(\overbrace{1,\cdots,1}^{i_0-1}, \frac{1}{h_{i_0+1}}, \cdots, \frac{1}{h_n}, \frac{1}{h_{n+1}}\Big)\hat{z}$$

$$- \mathrm{col}\Big(\overbrace{0,\cdots,0}^{i_0-1}, \overbrace{\frac{z_{i_0}}{h_{i_0}}, \cdots, \frac{z_{i_0}}{h_{i_0}}}^{n+1-i_0}\Big), \tag{2.5.11}$$

where $h_{n+1} = -p$, $\hat{z} = \mathrm{col}(z_1, \cdots, z_{i_0-1}, z_{i_0+1}, z_{n+1})$. Let

$$M = \max\left[1, \frac{2}{\min\limits_{i_0+1\leqslant i\leqslant n+1} h_i^2}, \frac{2(n+1-i_0)+1}{h_{i_0}^2}\right].$$

For any $\varepsilon > 0$, $\varepsilon_1 = \varepsilon/M$. Then there exists $\delta_1(\varepsilon)$ such that

$$\sum_{i=1}^{n+1} z_i^2(t,t_0,z_0) < \varepsilon_1 \qquad \text{for all} \quad t \geqslant t_0$$

if

$$\sum_{i=1}^{n+1} z_i^2(t_0) < \delta_1(\varepsilon).$$

We fix $\delta(\varepsilon) = \delta_1(\varepsilon)M$. Defining

$$\sum_{\substack{i=1\\ i\neq i_0}}^{n+1} z_i^{*2}(t_0) + \tilde{z}_{i_0}^2(t_0) = \sum_{i=1}^{i_0-1} z_i^2(t_0) + \sum_{i=i_0+1}^{n+1}\left[\frac{z_i(t_0)}{h_i} - \frac{z_{i_0}(t_0)}{h_{i_0}}\right]^2 + \left[\frac{z_{i_0}(t_0)}{h_{i_0}}\right]^2$$

$$\leqslant \sum_{i=1}^{i_0-1} z_i^2(t_0) + 2\sum_{i=i_0+1}^{n+1}\left(\frac{z_i(t_0)}{h_i}\right)^2$$

$$+ 2(n+1-i_0)\left[\frac{z_{i_0}(t_0)}{h_{i_0}}\right]^2 + \left[\frac{z_{i_0}(t_0)}{h_{i_0}}\right]^2$$

$$\leqslant M\sum_{i=1}^{n+1} z_i^2(t_0) < \delta_1(\varepsilon)M = \delta(\varepsilon),$$

we derive

$$\sum_{\substack{i=1\\ i\neq i_0}}^{n+1} z_i^{*2}(t,t_0,z^*(t_0))$$

$$
\begin{aligned}
&= \sum_{i=1}^{i_0-1} z_i^2(t,t_0,z(t_0)) \\
&\quad + \sum_{i=i_0+1}^{n+1} \left(\frac{1}{h_i} z_i(t,t_0,z(t_0)) - \frac{1}{h_{i_0}} z_{i_0}(t,t_0,z(t_0)) \right)^2 \\
&\leqslant \sum_{i=1}^{i_0-1} z_i^2(t,t_0,z(t_0)) + 2 \sum_{i=i_0+1}^{n+1} \left(\frac{z_i(t,t_0,z(t_0))}{h_i} \right)^2 \\
&\quad + 2(n+1-i_0) \left[\frac{z_{i_0}(t,t_0,z(t_0))}{h_{i_0}} \right]^2 \\
&< M\varepsilon_1 = \varepsilon.
\end{aligned}
$$

Obviously, we have

$$
\begin{aligned}
&\lim_{t\to+\infty} \sum_{\substack{i=1\\ i\neq i_0}}^{n+1} z_i^{*2}(t,t_0,z^*(t_0)) \\
&= \lim_{t\to+\infty} \sum_{i=1}^{i_0-1} z_i^2(t,t_0,z(t_0)) \\
&\quad + \lim_{t\to+\infty} \sum_{i=i_0+1}^{n+1} \left(\frac{1}{h_i} z_i(t,t_0,z(t_0)) - \frac{1}{h_{i_0}} z_{i_0}(t,t_0,z(t_0)) \right)^2 \\
&= 0 \qquad \text{for any} \quad z(t_0) \in \mathbf{R}^n.
\end{aligned}
$$

The necessity of the statement is completed.

Sufficiency. By virtue of (2.5.11), we write

$$
\begin{aligned}
\hat{z} &= \mathrm{diag}(\overbrace{1,\cdots,1}^{i_0-1}, h_{i_0+1},\cdots,h_{n+1}) z^* \\
&\quad + \mathrm{col}\left[\overbrace{0,\cdots,0}^{i_0-1}, \frac{h_{i_0+1} z_{i_0}}{h_{i_0}}, \cdots, \frac{h_{n+1} z_{i_0}}{h_{i_0}} \right].
\end{aligned}
$$

We set

$$
M_1 = \max_{i_0+1\leqslant i\leqslant n+1} [1,\ 2|h_i|^2], \qquad M_2 = 2 \sum_{i=i_0+1}^{n+1} \left(\frac{h_i}{h_{i_0}} \right)^2 + 1.
$$

Since

$$
\begin{aligned}
\sum_{i=1}^{n+1} z_i^2(t) &= \sum_{i=1}^{i_0-1} z_i^{*2}(t) + \sum_{i=i_0+1}^{n+1} \left[h_i z_i^*(t) + \frac{h_i z_{i_0}}{h_{i_0}} \right]^2 + z_{i_0}^2(t) \\
&\leqslant \sum_{i=1}^{i_0-1} z_i^{*2}(t) + \sum_{i=i_0+1}^{n+1} 2\left[h_i^2 z_i^{*2}(t) + \left[\frac{h_i z_{i_0}}{h_{i_0}} \right]^2 \right] + z_{i_0}^2(t) \\
&\leqslant M_1 \sum_{\substack{i=1\\ i\neq i_0}}^{n+1} z_i^{*2}(t) + M_2 \tilde{z}_{i_0}^2(t)
\end{aligned}
$$

and for any $\varepsilon > 0$ there exist $\delta_1(\varepsilon) > 0$, $\delta_2(\varepsilon) > 0$ such that

$$\sum_{\substack{i=1 \\ i \neq i_0}}^{n+1} z_i^{*2}(t) < \frac{\varepsilon}{2M_1}, \qquad z_{i_0}^2(t) < \frac{\varepsilon}{2M_2} \qquad \text{for all} \quad t \geqslant t_0$$

if

$$\sum_{\substack{i=1 \\ i \neq i_0}}^{n+1} z_i^{*2}(t_0) < \delta_1(\varepsilon), \qquad \bar{z}_{i_0}^2(t_0) < \delta_2(\varepsilon),$$

it follows

$$\sum_{i=1}^{n+1} z_i^2(t) \leqslant M_1 \sum_{\substack{i=1 \\ i \neq i_0}}^{n+1} z_i^{*2}(t) + M_2 z_{i_0}^2(t)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \text{for all} \quad t \geqslant t_0$$

if

$$\sum_{i=1}^{n+1} z_i^2(t_0) \leqslant M_1 \sum_{\substack{i=1 \\ i \neq i_0}}^{n+1} z_i^{*2}(t_0) + M_2 \bar{z}_{i_0}^2(t_0)$$

$$< M_1 \delta_1(\varepsilon) + M_2 \delta_2(\varepsilon) \triangleq \delta(\varepsilon).$$

Consequently, the zero solution of (2. 5. 6) is stable.

Obviously, for any $z(t_0) \in \mathbf{R}^{n+1}$ we have

$$0 \leqslant \lim_{t \to +\infty} \sum_{i=1}^{n+1} z_i^2(t) \leqslant \lim_{t \to +\infty} M_1 \sum_{\substack{i=1 \\ i \neq i_0}}^{n+1} z_i^{*2}(t) + M_2 \lim_{t \to +\infty} z_{i_0}^2(t) = 0.$$

The theorem follows. ∎

The following corollary is useful for application.

Corollary 2. 5. 17. 1) *Let condition* 1) *of Theorem* 2. 5. 16 *be true.*

2) *Let the matrix* $B(b_{ij}^*)_{n \times n}$ *be stable, where* b_{ij}^* *is defined in* (2. 5. 10). *Then the zero solution of* (2. 5. 6) *is absolutely stable.*

Proof. The solution of (2. 5. 10) can be written as

$$z^*(t, t_0, z_0^*) = e^{B^*(t-t_0)} z^*(t_0) + \int_{t_0}^{t} e^{B^*(t-\tau)} b^* \bar{z}_{i_0}(\tau) d\tau.$$

On the basis of the condition 2), let

$$\| e^{B^*(t-t_0)} \| \leqslant M e^{-\beta(t-t_0)},$$

where M, β are constants with $M > 1$ and $\beta > 0$. Condition 1) shows that for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\| z_{i_0}(t, t_0, z(t_0)) \| < \frac{\varepsilon |h_{i_0}|}{2\beta M \| b^* \|} \qquad \text{for all} \quad t \geqslant t_0$$

if $\| z(t_0) \| < \delta(\varepsilon)$.

Now, provided that $\| z^*(t_0) \| < \frac{\varepsilon}{2M}$, there exists

$$\begin{aligned}\| z^*(t,t_0,z^*(t_0)) \| &\leqslant M e^{-\beta(t-t_0)} \| z^*(t_0) \| \\ &\quad + M \int_{t_0}^{t} e^{-\beta(t-\tau)} \| b^* \| \cdot \frac{1}{|h_{i_0}|} \cdot \| z_{i_0}(\tau) \| \, d\tau \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

We can further verify that

$$\lim_{t\to+\infty} z^*(t,t_0,z_0^*) = 0 \qquad \text{for any} \quad z_0^* \in \mathbf{R}^n.$$

Thus, the condition of Theorem 2.5.16 are satisfied, which concludes the proof of this theorem. ∎

Corollary 2.5.18. 1) *Suppose that there exist constants* $r_i \geqslant 0$ $(i=1,\cdots,n+1,\ i\neq i_0)$, $r_{i_0}>0$ *such that*

$$-r_j b_{jj} \geqslant \sum_{\substack{i=1\\ i\neq j}}^{n} r_i |b_{ij}| + r_{n+1}|c_j|, \qquad j=1,\cdots,n+1,\ j\neq i_0,$$

$$r_{n+1}\rho \geqslant \sum_{i=1}^{n} r_i |h_i|,$$

$$-r_{i_0} b_{i_0 i_0} > \sum_{\substack{i=1\\ i\neq i_0}}^{n} r_i |b_{ii_0}| + r_{n+1}|c_{i_0}|.$$

2) *Let the condition* 2) *of Corollary* 2.5.6 *be satisfied.*

Then the zero solution of (2.5.6) *is absolutely stable.*

Proof. Since the condition 1) implies partial absolute stability for z_{i_0}, the conditions of Corollary 2.5.17 hold. Thus the conclusion of this corollary is valid. ∎

2.6. Notes

For the concept of absolute stability see Lurie [1], Aizerman and Gantmacher [1]. The expression of Lurie problem see Lefschitz [1, p. 19] and Burton [1]. Theorem 2.1.1 describes the well-known Lurie method, see Xie Huimin [2], or Aizerman and Gantmacher [1]. Theorems 2.2.3~2.2.7, 2.3.3, and Corollary 2.3.4 are taken from Liao Xiaoxin [15]. Theorem 2.4.1 belongs to Lurie, see Xie Huimin [2] or Lefschitz [1]. Theorem 2.4.2 is due to Zhu Siming [1]. Theorem 2.4.6 is due to Peng Lequn [1]. Theorems 2.4.8~2.4.19 are taken from Liao Xiaoxin [6],[7],[9],[16], respectively. For all the results of Section 2.5 see Liao Xiaoxin [8].

CHAPTER 3

SPECIAL CONTROL SYSTEMS

In this chapter, we will give some necessary and sufficient algebraic conditions for absolute stability of several classes of Lurie type control systems. Moreover, the algebraic sufficient conditions for absolute stability of other systems are obtained. All conditions are practical and convenient.

3.1. The Second Order Direct Control Systems

We consider the second order direct control system

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = a_{11}x_1 + a_{12}x_2 + b_1 f(c_1x_1 + c_2x_2), \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = a_{21}x_1 + a_{22}x_2 + b_2 f(c_1x_1 + c_2x_2), \end{cases} \tag{3.1.1}$$

where $f(\sigma) \in F$, $\sigma = c^{\mathrm{T}}x = c_1x_1 + c_2x_2$. Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Ye Baiying[1] gave the necessary and sufficient conditions for absolute stability of the zero solution of the second order direct control system.

Theorem 3.1.1. *If the matrix A is stable, then the necessary and sufficient conditions for absolute stability of the zero solution of* (3.1.1) *are*

$$c^{\mathrm{T}}b \leqslant 0, \qquad c^{\mathrm{T}}A^{-1}b \geqslant 0.$$

To prove the above theorem, a theorem due to Krasovskii must be used and the following lemma is needed.

Lemma 3.1.2. *For the second order nonlinear system*

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f_1(x) + by, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = f_2(x) + dy, \end{cases} \qquad f_1(0) = f_2(0) = 0, \tag{3.1.2}$$

with

1) $[bf_2(x)-df_1(x)]x<0$ *for* $x\neq 0$,

2) $\frac{f_1(x)}{x}+d<0$ *for* $x\neq 0$,

3) $\lim\limits_{|x|\to+\infty}\left\{(f_1(x)+dx)\operatorname{sgn} x-\int_0^x[df_1(x)-bf_2(x)]\mathrm{d}x\right\}=-\infty$,

the zero solution of the system (3.1.2) *is globally stable.*

Now we turn to proof of Theorem 3.1.1.

Proof. *Necessity.* It has been proved for more general situation in Chapter 2.

Sufficiency. Without loss of generality, let $c_1\neq 0$, and we make the transformation

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 1/c_1 & -c_2/c_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

which reduces the system (3.1.1) to

$$\begin{cases} \dfrac{\mathrm{d}y_1}{\mathrm{d}t}=f_1(y_1)+by_2, \\ \dfrac{\mathrm{d}y_2}{\mathrm{d}t}=f_2(y_1)+dy_2, \end{cases}$$

where

$$f_1(y_1)=\left(a_{11}+\frac{c_2}{c_1}a_{21}\right)y_1+(c_1b_1+c_2b_2)f(y_1),$$

$$f_2(y_1)=\frac{a_{21}}{c_1}y_1+b_2f(y_1),$$

$$b=-c_2a_{11}-\frac{c_2^2}{c_1}a_{21}+c_1a_{12}+c_2a_{22},$$

$$d=-\frac{c_2}{c_1}a_{21}+a_{22},$$

$$y_1f(y_1)>0 \quad \text{if } y_1\neq 0, \qquad f(0)=0.$$

Now we prove that the conditions of Theorem 3.1.1 imply the conditions of Lemma 3.1.2. We observe

$$\begin{aligned}&[bf_2(y_1)-df_1(y_1)]y_1\\ &=-\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}y_1^2-(c_1,c_2)\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}y_1f(y_1)\\ &<0 \quad \text{if } y_1\neq 0,\end{aligned} \tag{3.1.3}$$

$$\frac{f_1(y_1)}{y_1} + d = (a_{11} + a_{22}) + (c_1 b_1 + c_2 b_2)\frac{f(y_1)}{y_1}$$
$$< 0 \quad \text{if} \quad y_1 \neq 0. \tag{3.1.4}$$

Obviously, (3.1.3) yields

$$-\int_0^{y_1} [d f_1(y_1) - b f_2(y_1)] \mathrm{d}y_1 \to -\infty \quad \text{as} \quad |y_1| \to +\infty,$$

and (3.1.4) yields

$$\lim_{|y_1| \to +\infty} (f_1(y_1) + d y_1) \operatorname{sgn} y_1 = -\infty.$$

So, the condition 3) of Lemma 3.1.2 is satisfied.

As a result, the zero solution of the system (3.1.1) is absolutely stable.

The conclusion can be proved in the case of $c_2 \neq 0$ as well. ∎

Theorem 3.1.3. *If the eigenvalues of the matrix* $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ *are pure imaginary numbers, then the zero solution of the system* (3.1.1) *is absolutely stable if and only if*

$$c^{\mathrm{T}} b < 0, \qquad c^{\mathrm{T}} A^{-1} b \geqslant 0.$$

Theorem 3.1.4. *If* $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ *has two real eigenvalues, namely, one equals zero and the other negative, then the zero solution of the system* (3.1.1) *is absolutely stable if and only if*

$$c^{\mathrm{T}} b \leqslant 0, \qquad c^{\mathrm{T}} (\operatorname{adj} A) b > 0.$$

where adj$A$ *is the adjaint matrix of* A, *i.e.*,

$$\operatorname{adj} A = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}.$$

We can prove Theorems 3.1.3 and 3.1.4 similarly to Theorem 3.1.1 and therefore, we omit proofs.

Example 3.1.5. Let us discuss absolute stability of the zero solution of the system

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = -2x_1 + x_2 - 2f(x_1 - x_2), \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = -x_1 - x_2 + f(x_1 - x_2), \end{cases} \tag{3.1.5}$$

where

$$A = \begin{pmatrix} -2 & 1 \\ -1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Since

$$c^{\mathrm{T}}b=(1,-1)\begin{pmatrix}-2\\1\end{pmatrix}=-3<0,$$

$$c^{\mathrm{T}}A^{-1}b=(1,-1)\begin{pmatrix}-1&-1\\1&-2\end{pmatrix}\begin{pmatrix}-2\\1\end{pmatrix}=5>0,$$

A has two eigenvalues with negative real parts, and thus, the zero solution of (3.1.5) is absolutely stable on the basis of Theorem 3.3.1.

3.2. A Class of the Third Order Control Systems

Consider the third order control system

$$\begin{cases}\dfrac{\mathrm{d}x}{\mathrm{d}t}=Ax+bf(\sigma),\\ \sigma=c^{\mathrm{T}}x,\end{cases}\tag{3.2.1}$$

where $x\in\mathbf{R}^3$, $A\in\mathbf{R}^{3\times3}$, $b\in\mathbf{R}^3$, $c\in\mathbf{R}^3$, $f(0)=0$,

$$0<f(\sigma)\sigma<k\sigma^2\ \ (k\leqslant+\infty)\qquad\text{for all}\quad\sigma\neq0.$$

Xie Huimin[1] has proved that for a class of the third order control systems including the third order indirect control systems, the conditions of Popov's frequency criterion are not only sufficient for absolute stability but necessary as well. Then the Lurie problem of that class of control system has been solved completely.

Theorem 3.2.1. *In the system* (3.2.1), *let A have at least one eigenvalue with zero real part and have no eigenvalue with positive real part. If there exist two constants $p\geqslant0$ and q which are not both zero such that*

$$\mathrm{Re}\{(p+\mathrm{i}\omega q)c^{\mathrm{T}}(A-\mathrm{i}\omega I)^{-1}b\}+\frac{p}{k}\geqslant0\qquad\textit{for}\quad\omega\in(-\infty,+\infty),\tag{3.2.2}$$

then (3.2.2) *is the necessary and sufficient condition for absolute stability of the zero solution to the system* (3.2.1) *in the Hurwitz angle* $(0,k)$.

Popov's criterion implies sufficiency, while the proof of necessity is too lengthy to be quoted in detail; the reader is referred to Xie Huimin[1].

3.3. Special Direct Control Systems of the nth Order

Consider a special Lurie direct control system

$$\begin{cases}\dfrac{\mathrm{d}x}{\mathrm{d}t}=Ax+bf(\sigma),\\ \sigma=c^{\mathrm{T}}x,\end{cases}\tag{3.3.1}$$

where $A \in \mathbf{R}^{n\times n}$, $c,b \in \mathbf{R}^n$, $f(\sigma) \in F$.

Theorem 3.3.1. *Let the matrix A of the system* (3.3.1) *be of the form*

$$A = \begin{bmatrix} -\lambda & 1 & & 0 \\ & -\lambda & & \\ & & \ddots & \\ 0 & & & -\lambda \end{bmatrix}, \qquad \lambda > 0.$$

Then the necessary and sufficient conditions for absolute stability of the system (3.3.1) *are*

$$c^{\mathrm{T}}b \leqslant 0, \qquad c^{\mathrm{T}}A^{-1}b \geqslant 0.$$

Proof. Necessity. It has been proved in Chapter 2.

Sufficiency. By virtue of Popov's criterion , if there exists a real number $q \geqslant 0$ such that

$$\mathrm{Re}\{(1+\mathrm{i}\omega q)W(\mathrm{i}\omega)\} \geqslant 0 \qquad \text{for all} \quad \omega \geqslant 0, \tag{3.3.2}$$

where $W(z) = -c^{\mathrm{T}}(zI - A)^{-1}b$, then the zero solution of the system (3.3.1) is absolutely stable.

The condition (3.3.2) may be equivalently rewritten as

$$\mathrm{Re}\{(1+\mathrm{i}\omega q)c^{\mathrm{T}}A_{\mathrm{i}\omega}^{-1}b\} \leqslant 0 \qquad \text{for} \quad \omega \geqslant 0,$$

where $A_{\mathrm{i}\omega} = \mathrm{i}\omega I - A$. In this case, we have

$$A_{\mathrm{i}\omega} = \begin{bmatrix} \mathrm{i}\omega+\lambda & -1 & & 0 \\ & \mathrm{i}\omega+\lambda & & \\ & & \ddots & \\ 0 & & & \mathrm{i}\omega+\lambda \end{bmatrix},$$

$$A_{\mathrm{i}\omega}^{-1} = \begin{bmatrix} \dfrac{1}{\mathrm{i}\omega+\lambda} & \dfrac{1}{(\mathrm{i}\omega+\lambda)^2} & & 0 \\ & \dfrac{1}{\mathrm{i}\omega+\lambda} & & \\ & & \ddots & \\ 0 & & & \dfrac{1}{\mathrm{i}\omega+\lambda} \end{bmatrix},$$

thus

$$\begin{aligned} c^{\mathrm{T}}A_{\mathrm{i}\omega}^{-1}b &= (c_1,c_2,\cdots,c_n)A_{\mathrm{i}\omega}^{-1}(b_1,b_2,\cdots,b_n)^{\mathrm{T}} \\ &= \sum_{j=1}^{n}\frac{b_jc_j}{\mathrm{i}\omega+\lambda} + \frac{c_1b_2}{(\mathrm{i}\omega+\lambda)^2} \\ &= \frac{c^{\mathrm{T}}b(\lambda-\mathrm{i}\omega)}{\lambda^2+\omega^2} + \frac{c_1b_2(\lambda^2-\omega^2) - c_1b_2\cdot 2\omega^2\lambda}{(\lambda^2-\omega^2)^2+4\omega^2\lambda^2}. \end{aligned}$$

We conclude

$$\begin{aligned}
&\operatorname{Re}\{(1+i\omega q)c^{\mathrm{T}}A_{i\omega}^{-1}b\} \\
&= \frac{c^{\mathrm{T}}b\lambda}{\lambda^2+\omega^2} + \frac{c_1b_2(\lambda^2-\omega^2)}{(\lambda^2-\omega^2)^2+4\omega^2\lambda^2} + \frac{q\omega^2(c^{\mathrm{T}}b)}{\lambda^2+\omega^2} \\
&\quad + \frac{2q\omega^2c_1b_2\lambda}{(\lambda^2-\omega^2)^2+4\omega^2\lambda^2} \\
&\triangleq \frac{F(\omega^2)}{(\lambda^2+\omega^2)[(\lambda^2-\omega^2)^2+4\lambda^2\omega^2]},
\end{aligned}$$

where

$$\begin{aligned}
F(\omega^2) &= [(\lambda^2-\omega^2)+4\omega^2\lambda^2][(c^{\mathrm{T}}b)\lambda+q\omega^2(c^{\mathrm{T}}b)] \\
&\quad + [\lambda^2+\omega^2][c_1b_2(\lambda^2-\omega^2)+2q\omega^2c_1b_2\lambda] \\
&= (c^{\mathrm{T}}b)[\lambda^4+2\omega^2\lambda^2+\omega^4][\lambda+q\omega^2] \\
&\quad + c_1b_2(\lambda^4-\omega^4)+2q\lambda^3\omega^2c_1b_2+2q\omega^4c_1b_2\lambda \\
&= q(c^{\mathrm{T}}b)\omega^6+[(c^{\mathrm{T}}b)\lambda+2(c^{\mathrm{T}}b)\lambda^2q-c_1b_2+2qc_1b_2\lambda]\omega^4 \\
&\quad + [(c^{\mathrm{T}}b)\lambda^4q+2(c^{\mathrm{T}}b)\lambda^3+2q\lambda^3c_1b_2]\omega^2 \\
&\quad + [(c^{\mathrm{T}}b)\lambda^5+c_1b_2\lambda^4].
\end{aligned}$$

The conditions

$$c^{\mathrm{T}}b \leqslant 0, \qquad -c^{\mathrm{T}}A^{-1}b = \frac{1}{\lambda^2}[(c^{\mathrm{T}}b)\lambda+c_1b_2] \leqslant 0$$

indicate that the coefficients of the first term and the constant term of $F(\omega^2)$ are not positive. Now we discuss the coefficients of ω^4 and ω^2 terms.

1° If $c_1b_2 \leqslant 0$, then, obviously, for any $q \geqslant 0$

$$(c^{\mathrm{T}}b)\lambda^4q+2(c^{\mathrm{T}}b)\lambda^3+2q\lambda^3c_1b_2 \leqslant 0.$$

Choosing $q > \frac{1}{2\lambda}$, it follows that

$$\begin{aligned}
&(c^{\mathrm{T}}b)\lambda+2(c^{\mathrm{T}}b)\lambda^2q-c_1b_2+2qc_1c_2\lambda \\
&\quad = (c^{\mathrm{T}}b)\lambda+2(c^{\mathrm{T}}b)\lambda^2q+c_1b_2(2q\lambda-1) \leqslant 0.
\end{aligned}$$

Therefore, the coefficients of ω^4 and ω^2 terms are not positive.

2° If $c_1b_2 > 0$, we choose $q=0$. Then the coefficient of ω^4 term is $(c^{\mathrm{T}}b)\lambda - c_1b_2 \leqslant 0$ and the coefficient of ω^2 term is $2(c^{\mathrm{T}}b)\lambda^3 \leqslant 0$.

In any case, we can choose $q \geqslant 0$ such that $F(\omega^2) \leqslant 0$; therefore, the zero solution of the system (3. 3. 1) is absolutely stable by Popov's criterion (3. 3. 2). ▮

Corollary 3. 3. 2. *If there exists a real similarity transformation which transforms the matrix A of the system* (3. 3. 1) *into the form presented in Theorem* 3. 3. 1, *then the necessary and sufficient conditions for absolute stability of the zero solution of the system* (3. 3. 1) *are* $c^{\mathrm{T}}b\leqslant 0$, $c^{\mathrm{T}}A^{-1}b\geqslant 0$.

Proof. It is sufficient to prove that $c^T b$ and $c^T A^{-1} b$ are not changed by similarity transformation.

By the nonsingular transformation $x = By$, $B \in \mathbf{R}^{n\times n}$, the system (3. 3. 1) transforms into

$$\frac{dy}{dt} = B^{-1}ABy + B^{-1}bf(c^T By) \triangleq \tilde{A}y + \tilde{b}f(\tilde{c}^T y),$$

where $\tilde{A} = B^{-1}AB$, $\tilde{b} = B^{-1}b$, $\tilde{c} = B^T c$. Thus,

$$\tilde{c}^T\tilde{b} = c^T BB^{-1}b = c^T b,$$

$$\tilde{c}^T\tilde{A}^{-1}\tilde{b} = c^T BB^{-1}A^{-1}BB^{-1}b = c^T A^{-1}b.$$

The corollary follows. ∎

Theorem 3. 3. 3. *In the system* (3. 3. 1), *we assume that*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} = \begin{bmatrix} -\lambda I_1 & 0 \\ 0 & -\rho I_2 \end{bmatrix},$$

where $\lambda > 0$, $\rho > 0$, $I_1 \in \mathbf{R}^{n_1\times n_1}$, $I_2 \in \mathbf{R}^{n_2\times n_2}$ *are unit matrices and* $n_1 + n_2 = n$. *Then the necessary and sufficient conditions for absolute stability of the zero solution of the system* (3. 3. 1) *amount to* $c^T b \leqslant 0$, $c^T A^{-1} b \geqslant 0$.

Proof. The conditions are obviously necessary.

Now we will prove they are also sufficient. From

$$A_{i\omega} = \begin{bmatrix} (i\omega + \lambda)I_1 & 0 \\ 0 & (i\omega + \rho)I_2 \end{bmatrix}, \qquad A_{i\omega}^{-1} = \begin{bmatrix} \dfrac{1}{i\omega + \lambda}I_1 & 0 \\ 0 & \dfrac{1}{i\omega + \rho}I_2 \end{bmatrix},$$

taking

$$\tilde{c}_1 = \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}_{n\times n} c, \qquad \tilde{c}_2 = \begin{pmatrix} 0 & 0 \\ 0 & I_2 \end{pmatrix}_{n\times n} c,$$

$$\tilde{b}_1 = \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix} b, \qquad \tilde{b}_2 = \begin{pmatrix} 0 & 0 \\ 0 & I_2 \end{pmatrix} b,$$

we observe

$$c^T A_{i\omega}^{-1} b = \frac{\tilde{c}_1^T\tilde{b}_1}{i\omega + \lambda} + \frac{\tilde{c}_2^T\tilde{b}_2}{i\omega + \rho}$$

$$= \frac{\tilde{c}_1^T\tilde{b}_1\lambda - i\tilde{c}_1^T\tilde{b}_1\omega}{\omega^2 + \lambda^2} + \frac{\tilde{c}_2^T\tilde{b}_2\rho - i\tilde{c}_2^T\tilde{b}_2\omega}{\omega^2 + \rho^2}.$$

Then

$$\operatorname{Re}\{(1 + i\omega q)c^T A_{i\omega}^{-1} b\} = \frac{\tilde{c}_1^T\tilde{b}_1\lambda + q\omega^2\tilde{c}_1^T\tilde{b}_1}{\omega^2 + \lambda^2} + \frac{\tilde{c}_2^T\tilde{b}_2\rho + q\omega^2\tilde{c}_2^T\tilde{b}_2}{\omega^2 + \rho^2}$$

$$= \frac{F(\omega^2)}{(\omega^2 + \lambda^2)(\omega^2 + \rho^2)}.$$

where

$$F(\omega^2) = q(\bar{c}_1^{\mathrm{T}}\bar{b}_1 + \bar{c}_2^{\mathrm{T}}\bar{b}_2)\omega^4 + [(\bar{c}_1^{\mathrm{T}}\bar{b}_1\lambda + \bar{c}_2^{\mathrm{T}}\bar{b}_2\rho) + q(\bar{c}_1^{\mathrm{T}}\bar{b}_1\rho^2 + \bar{c}_2^{\mathrm{T}}\bar{b}_2\lambda^2)]\omega^2 + \left(\frac{\bar{c}_1^{\mathrm{T}}\bar{b}_1}{\lambda} + \frac{\bar{c}_2^{\mathrm{T}}\bar{b}_2}{\rho}\right)\lambda^2\rho^2.$$

The conditions of Theorem 3. 3. 3 give

$$c^{\mathrm{T}}b = \bar{c}_1^{\mathrm{T}}\bar{b}_1 + \bar{c}_2^{\mathrm{T}}\bar{b}_2 \leqslant 0,$$

$$-c^{\mathrm{T}}A^{-1}b = \frac{\bar{c}_1^{\mathrm{T}}\bar{b}_1}{\lambda} + \frac{\bar{c}_2^{\mathrm{T}}\bar{b}_2}{\rho} \leqslant 0.$$

It is easy to prove that there exists $q \geqslant 0$ such that

$$(\bar{c}_1^{\mathrm{T}}\bar{b}_1\lambda + \bar{c}_2^{\mathrm{T}}\bar{b}_2\rho) + q(\bar{c}_1^{\mathrm{T}}\bar{b}_1\rho^2 + \bar{c}_2^{\mathrm{T}}\bar{b}_2\lambda^2) \leqslant 0,$$

i. e. , there exists $q \geqslant 0$ satisfying $f(\omega^2) \leqslant 0$. Then the zero solution of the system (3. 3. 1) is absolutely stable. ∎

Corollary 3. 3. 4. *Assume that in the system* (3. 3. 1) $A = A^{\mathrm{T}}$ *and that* A *has at most two different real eigenvalues and simple elementary divisor. Then the necessary and sufficient condition for absolute stability of the zero solution of the system* (3. 3. 1) *are* $c^{\mathrm{T}}b \leqslant 0$, $c^{\mathrm{T}}A^{-1}b \geqslant 0$.

Corollary 3. 3. 5. *Suppose that in the system* (3. 3. 1) $A = A^{\mathrm{T}}$, *while* A *has only one real eigenvalue and corresponds to the simple elementary divisor. Then the necessary and sufficient conditions for absolute stability of the zero solution of the system* (3. 3. 1) *are* $c^{\mathrm{T}}b \leqslant 0$.

Theorem 3. 3. 6. *Suppose that in the system* (3. 3. 1) *the vector* b *is an eigenvector of* A, *or the vector* c *is an eigenvector of* A^{T}. *Then the zero solution of the system* (3. 3. 1) *is absolutely stable if and only if* $c^{\mathrm{T}}b \leqslant 0$.

Proof. The necessity is trivial. We will prove sufficiency only. First, suppose $Ab = -\lambda b$ $(\lambda > 0)$. Then

$$(\mathrm{i}\omega I - A)b = [\lambda + \mathrm{i}\omega]b, \quad \omega \geqslant 0, \qquad (\mathrm{i}\omega I - A)^{-1} = \frac{1}{\lambda + \mathrm{i}\omega}b.$$

Thus

$$c^{\mathrm{T}}(\mathrm{i}\omega I - A)^{-1}b = \frac{c^{\mathrm{T}}b}{\mathrm{i}\omega + \lambda}.$$

It follows from the condition $c^{\mathrm{T}}b \leqslant 0$ that

$$\mathrm{Re}\{c^{\mathrm{T}}(\mathrm{i}\omega I - A)^{-1}b\} = \frac{\lambda c^{\mathrm{T}}b}{\lambda^2 + \omega^2} \leqslant 0.$$

This shows that the zero solution of the system (3. 3. 1) is absolutely stable.

Next, suppose that $A^{\mathrm{T}}c = -\lambda c$ $(\lambda > 0)$. Then we obtain

$$(\mathrm{i}\omega I - A^{\mathrm{T}})^{-1}c = \frac{1}{\mathrm{i}\omega + \lambda}c.$$

Thus

$$c^{\mathrm{T}}(\mathrm{i}\omega I - A)^{-1}b = \frac{c^{\mathrm{T}}b}{\mathrm{i}\omega + \lambda}.$$

We also have

$$\mathrm{Re}\{c^{\mathrm{T}}(\mathrm{i}\omega I - A)^{-1}b\} \leqslant 0.$$

Consequently, the zero solution of the system (3.3.1) is absolutely stable. ∎

Theorem 3.3.7. *Suppose that in the system* (3.3.1) A *is a quasi-diagonal matrix*:

$$A = \mathrm{diag}(A_1, A_2, \cdots, A_m),$$

where A_r $(r=1,\cdots,m)$ *are square submatrices. If* $\bar{b}_r$ *is an eigenvector of* A_r *for any* $r=1,\cdots,m$, *or* $\bar{c}_r$ *is an eigenvector of* A_r^{T}, *then* $\bar{c}_r^{\mathrm{T}}\bar{b}_r \leqslant 0$ $(r=1,\cdots,m)$ *imply absolute stability of the zero solution of the system* (3.3.1), *where* $\bar{c}_r$, $\bar{b}_r$ *refer to column vectors corresponding to* A_r, $b=\mathrm{col}(\bar{b}_1,\cdots,\bar{b}_m)$, $c=\mathrm{col}(\bar{c}_1,\cdots,\bar{c}_m)$ *and* I_r *is the unit matrix corresponding to* A_r.

Proof. If $A_r\bar{b}_r = -\lambda\bar{b}_r$, since

$$\bar{c}_r^{\mathrm{T}}(\mathrm{i}\omega I_r - A_r)^{-1}\bar{b}_r = \frac{\bar{c}_r^{\mathrm{T}}\bar{b}_r}{\lambda + \mathrm{i}\omega},$$

we get

$$\mathrm{Re}\{\bar{c}_r^{\mathrm{T}}(\mathrm{i}\omega I_r - A_r)^{-1}\bar{b}_r\} = \frac{\lambda\bar{c}_r^{\mathrm{T}}\bar{b}_r}{\lambda^2 + \omega^2} \leqslant 0, \qquad r = 1,\cdots,m.$$

When $A_r^{\mathrm{T}}\bar{c}_r = -\lambda\bar{c}_r$, it gives rise to

$$\bar{c}_r^{\mathrm{T}}(\mathrm{i}\omega I_r - A_r)^{-1}\bar{b}_r = \frac{\lambda\bar{c}_r^{\mathrm{T}}\bar{b}_r}{\mathrm{i}\omega + \lambda},$$

$$\mathrm{Re}\{\bar{c}_r^{\mathrm{T}}(\mathrm{i}\omega I_r - A_r)^{-1}\bar{b}_r\} \leqslant 0, \qquad r = 1,\cdots,m.$$

Now we proceed to prove that the expression $\mathrm{Re}\{\bar{c}_r^{\mathrm{T}}(\mathrm{i}\omega I_r - A_r)^{-1}\bar{b}_r\} \leqslant 0$ $(r = 1,\cdots,m)$ implies absolute stability of the zero solution of the system (3.3.1). Since

$$\begin{aligned} A &= \mathrm{diag}(A_1,\cdots,A_m), \\ A_{\mathrm{i}\omega} &= \mathrm{diag}(\mathrm{i}\omega I_1 - A_1,\cdots,\mathrm{i}\omega I_m - A_m), \\ c^{\mathrm{T}}A_{\mathrm{i}\omega}^{-1}b &= (\bar{c}_1^{\mathrm{T}},\cdots,\bar{c}_m^{\mathrm{T}})\cdot\mathrm{diag}((\mathrm{i}\omega I_1 - A_1)^{-1},\cdots,(\mathrm{i}\omega I_m - A_m)^{-1}) \\ &\quad\cdot(\bar{b}_1^{\mathrm{T}},\cdots,\bar{b}_m^{\mathrm{T}})^{\mathrm{T}} \\ &= \sum_{r=1}^{m}\bar{c}_r^{\mathrm{T}}(\mathrm{i}\omega I_r - A_r)^{-1}\bar{b}_r, \end{aligned}$$

it follows that

$$\mathrm{Re}(c^{\mathrm{T}}A_{\omega}^{-1}b) = \sum_{r=1}^{m}\mathrm{Re}\,\bar{c}_r^{\mathrm{T}}(\mathrm{i}\omega I_r - A_r)^{-1}\bar{b}_r \leqslant 0.$$

Therefore, the zero solution of the system (3. 3. 1) is absolutly stable. ∎

Example 3. 3. 8. Consider the 3-dimensional control system

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = -x_1 + x_2 - x_3 + f(x_1 - x_2 - x_3), \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = x_1 - x_2 - x_3 + f(x_1 - x_2 - x_3), \\ \dfrac{\mathrm{d}x_3}{\mathrm{d}t} = x_1 + x_2 - 3x_3 + f(x_1 - x_2 - x_3), \end{cases}$$

where

$$A = \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & 1 & -3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \quad f \in F.$$

It is clear that A is a stable matrix and b an eigenvector which belongs to the eigenvalue $\lambda = -1$. Moreover

$$c^{\mathrm{T}}b = -1 < 0.$$

Thus, the system is absolutely stable by Theorem 3. 3. 6.

3. 4. The First Canonical Form of Control Systems

Letov[1] has considered a class of control system called the first canonical form in which A is diagonal:

$$\begin{cases} \dfrac{\mathrm{d}x_j}{\mathrm{d}t} = -\lambda_j x_j + f(\sigma), \quad \lambda_j > 0, \quad j = 1, \cdots, n, \\ \sigma = c^{\mathrm{T}}x = \sum_{i=1}^{n} c_i x_i. \end{cases} \tag{3. 4. 1}$$

We discuss the absolute stability of the system (3. 4. 1) again.

For a real number $q \geqslant 0$, let

$$a_j = \begin{cases} c_j q & \text{if } c_j(q\lambda_j - 1) > 0, \\ c_j/\lambda_j & \text{if } c_j(q\lambda_j - 1) < 0, \\ c_j/\lambda_j & \text{if } c_j(q\lambda_j - 1) = 0,\ c_j \neq 0, \\ 0 & \text{if } c_j(q\lambda_j - 1) = 0,\ c_j = 0. \end{cases}$$

Theorem 3. 4. 1. *If there exists $q \geqslant 0$ such that $\sum_{j=1}^{n} a_j < \frac{1}{k}$, then the zero solution of the system (3. 4. 1) is absolutely stable in the Hurwitz angle $[0, k]$.*

Proof. Since

$$A=\begin{bmatrix}-\lambda_1 & & & 0\\ & -\lambda_2 & & \\ & & \ddots & \\ 0 & & & -\lambda_n\end{bmatrix},\quad A_{i\omega}^{-1}=\begin{bmatrix}\frac{1}{i\omega+\lambda_1} & & & 0\\ & \frac{1}{i\omega+\lambda_2} & & \\ & & \ddots & \\ 0 & & & \frac{1}{i\omega+\lambda_n}\end{bmatrix},$$

it follows that

$$c^{\mathrm{T}}A_{i\omega}^{-1}b=\sum_{j=1}^{n}\frac{c_j}{i\omega+\lambda_j},$$

$$\mathrm{Re}\{(1+i\omega q)c^{\mathrm{T}}A_{i\omega}^{-1}b\}=\sum_{j=1}^{n}\frac{c_j(\lambda_j+q\omega^2)}{\omega^2+\lambda_j^2}.$$

Let $f_j(\omega)=\dfrac{c_j(\lambda_j+q\omega^2)}{\omega^2+\lambda_j^2}$, and we compute the derivative of $f_j(\omega)$:

$$f_j'(\omega)=\frac{2c_jq\omega(\omega^2+\lambda_j^2)-(c_j\lambda_j+c_jq\omega^2)\cdot 2\omega}{(\omega^2+\lambda_j^2)^2}$$

$$=\frac{2c_j\lambda_j(q\lambda_j-1)\omega}{(\omega^2+\lambda_j^2)^2}.$$

We see that $f_j'(\omega)=0$ only if $\omega=0$. The function $f_j(\omega)$ is monotone increasing on $[0,\infty)$ when $c_j(q\lambda_j-1)>0$, thus $f_j(\omega)\leqslant f_j(\infty)=c_jq$; $f_j(\omega)$ is monotone decreasing on $[0,\infty)$ when $c_j(q\lambda_j-1)<0$, thus $f_j(\omega)\leqslant f_j(0)=c_j/\lambda_j$. Since $f_j'(\omega)=0$, $q=1/\lambda_j$, when $c_j(q\lambda_j-1)=0$, $c_j\neq 0$, we have

$$f_j(\omega)=\frac{c_j(\lambda_j+\omega^2/\lambda_j)}{\omega^2+\lambda_j^2}=\frac{c_j}{\lambda_j}.$$

When $c_j(q\lambda_j-1)=0$, $c_j=0$, it obviously follows that $f_j(\omega)=0$. Choosing appropriate a_j, we always have $f_j(\omega)\leqslant a_j$. From the hypothesis of the theorem, we can deduce that

$$\mathrm{Re}\{(1+i\omega q)c^{\mathrm{T}}A_{i\omega}^{-1}b\}=\sum_{j=1}^{n}\frac{c_j(\lambda_j+q\omega^2)}{\omega^2+\lambda_j^2}$$

$$=\sum_{j=1}^{n}f_j(\omega)\leqslant\sum_{j=1}^{n}a_j<\frac{1}{k},$$

i.e.,

$$\mathrm{Re}\{(1+i\omega q)c^{\mathrm{T}}A_{i\omega}^{-1}b\}-\frac{1}{k}<0.$$

The zero solution of the system (3.4.1) is absolutely stable in the Hurwitz angle $[0,k]$ by Popov's criterion. ∎

We are always able to order equations in (3.4.1) and the unknown functions

x_j such that

$$c_j \begin{cases} <0, & j=1,\cdots,j_1, \\ =0, & j=j_1+1,\cdots,j_2, \\ >0, & j=j_2+1,\cdots,n. \end{cases}$$

Corollary 3. 4. 2. *If* $\sum_{j=j_2+1}^{n} \frac{c_j}{\lambda_j} < \frac{1}{k}$, *then the zero solution of the system* (3. 4. 1) *is absolutely stable in the Hurwitz angle* $[0,k]$.

The conclusion may be easily derived by choosing $q=0$ in Theorem 3. 4. 1.

Corollary 3. 4. 3. *If* $\lambda_j > 2k(n-j_2)c_j$ $(j=j_2+1,\cdots,n)$, *then the zero solution of the system* (3. 4. 1) *is absolutely stable in the Hurwitz angle* $[0,k]$.

Proof. From $\lambda_j > 2k(n-j_2)c_j$ $(j=j_2+1,\cdots,n)$,

$$\sum_{j=j_2+1}^{n} \frac{c_j}{\lambda_j} < \sum_{j=j_2+1}^{n} \frac{1}{2k(n-j_2)} = \frac{1}{2k} < \frac{1}{k}.$$

The condition of Corollary 3. 4. 2 is satisfied. Corollary 3. 4. 3 is proved. ∎

Theorem 3. 4. 4. *If there exists* $q \geqslant 0$ *such that* $\sum_{j=1}^{n} a_j \leqslant 0$, *then the zero solution of the system* (3. 4. 1) *is absolutely stable.*

Proof. Repeating the proof of Theorem 3. 4. 1, we have $f_j(\omega) \leqslant a_j$. From

$$\mathrm{Re}\{(1+\mathrm{i}\omega q)c^{\mathrm{T}}A_{\omega}^{-1}b\} = \sum_{j=1}^{n} f_j(\omega) \leqslant \sum_{j=1}^{n} a_j \leqslant 0,$$

the conclusion is valid by Popov's criterion. ∎

3. 5. Critical Systems

Theorem 3. 5. 1. *In the system* (3. 4. 1), *let* $\lambda_i > 0$ $(i=1,\cdots,n-1)$, $\lambda_n = 0$, $c_n < 0$, $c_i < \frac{\lambda_i}{2k(n-1)}$ $(i=1,\cdots,n-1)$. *Then the zero solution of the system* (3. 4. 1) *is absolutely stable in* $[0,k]$.

Proof. It is clear that the quadratic inequalities for ξ_i:

$$k(n-1)\xi_i^2 + 2[k(n-1)c_i - \lambda_i]\xi_i + k(n-1)c_i^2 < 0, \quad i=1,\cdots,n-1, \tag{3.5.1}$$

have solutions if and only if

$$c_i < \frac{\lambda_i}{2k(n-1)}, \qquad i=1,\cdots,n-1.$$

Suppose that $\xi_i = r_i$ $(i=1,\cdots,n-1)$ is a positive solution of (3. 5. 1). Let

$$\varphi_i(x_i) = \begin{cases} 2r_i x_i, & i=1,\cdots,n-1, \\ -2c_i x_i, & i=n. \end{cases}$$

Then

$$
\begin{aligned}
G(x) &\triangleq \sum_{i=1}^{n} \varphi_i(x_i) \frac{\mathrm{d}x_i}{\mathrm{d}t} \\
&= -2\sum_{i=1}^{n-1} r_i\lambda_i x_i^2 + 2\sum_{i=1}^{n-1} r_i x_i f(\sigma) - 2c_n x_n f(\sigma) \\
&= -2\sum_{i=1}^{n-1} r_i\lambda_i x_i^2 + 2\sum_{i=1}^{n-1} (r_i + c_i) x_i f(\sigma) - 2\sum_{i=1}^{n} c_i x_i f(\sigma) \\
&\leqslant -2\sum_{i=1}^{n-1} r_i\lambda_i x_i^2 + 2\left|\sqrt{k}\sum_{i=1}^{n-1}(r_i + c_i)x_i\right| \cdot \left|\frac{f(\sigma)}{\sqrt{k}}\right| - 2\sigma f(\sigma) \\
&\leqslant -2\sum_{i=1}^{n-1} r_i\lambda_i x_i^2 + k(n-1)\sum_{i=1}^{n-1}(r_i + c_i)^2 x_i^2 + \sigma f(\sigma) - 2\sigma f(\sigma) \\
&\leqslant -\sum_{i=1}^{n-1}[k(n-1)(r_i + c_i)^2 - 2r_i\lambda_i]x_i^2 - \sigma f(\sigma) \\
&= -\sum_{i=1}^{n-1}\{k(n-1)r_i^2 + 2[k(n-1)c_i - \lambda_i]r_i + k(n-1)c_i^2\}x_i^2 - \sigma f(\sigma) \\
&\triangleq W(x).
\end{aligned}
$$

Now we have

$$G(x) \leqslant W(x) \leqslant 0.$$

If $W(\tilde{x}) = 0$, since $\sigma f(\sigma) > 0$ for $\sigma \neq 0$, we have

$$\sum_{i=1}^{n-1} \tilde{x}_i^2 = 0,$$

and therefore $c_n\tilde{x}_n f(c_n\tilde{x}_n) = 0$. This implies $c_n\tilde{x}_n = 0$. From $c_n < 0$, we have $\tilde{x}_n = 0$; thus $\tilde{x} = 0$. It follows that $W(x)$ is negative definite. Therefore the zero solution of the system (3.4.1) is absolutely stable in the Hurwitz angle $[0,k]$. ▮

Corllary 3.5.2. *If* $\lambda_i>0$ $(i=1,\cdots,n-1)$, $\lambda_n=0$, $c_n<0$, $c_i\leqslant 0$ $(i=1,\cdots,n-1)$, *then the zero solution of the system* (3.4.1) *is absolutely stable in the Hurwitz angle* $[0,k]$.

Corollary 3.5.3. *Provided that* $\lambda_i>0$ $(i=1,\cdots,n-1)$, $\lambda_n=0$, $c_i<0$ $(i=1,\cdots,n)$, *the zero solution of the system* (3.4.1) *is absolutely stable.*

Proof. Let $\varphi_i(x_i) = -c_i x_i$ $(i = 1,\cdots,n)$. Then

$$
\begin{aligned}
G(x) &\triangleq \sum_{i=1}^{n} \varphi_i(x_i) \frac{\mathrm{d}x_i}{\mathrm{d}t} = \sum_{i=1}^{n-1} c_i\lambda_i x_i^2 - \sum_{i=1}^{n} c_i x_i f(\sigma) \\
&= \sum_{i=1}^{n-1} c_i\lambda_i x_i^2 - \sigma f(\sigma).
\end{aligned}
$$

Thus, $G(x)$ is negative definite and the zero solution of the system (3.4.1) is absolutely stable. ▮

Theorem 3.5.4. *If the following conditions are satisfied*

1) $\lambda_i>0$ $(i=1,\cdots,r)$, $\lambda_j=0$ $(j=r+1,\cdots,n)$, $c_i\leqslant\frac{\lambda_i}{2kr}$ $(i=1,\cdots,r)$, $c_j<0$ $(j=r+1,\cdots,n)$;

2) *the matrix* $\mathrm{diag}(-\lambda_1,\cdots,-\lambda_n)+bc^{\mathrm{T}}$ *is stable with* $b=\mathrm{col}(1,1,\cdots,1)$,

the zero solution of the system (3.4.1) *is absolutely stable.*

Proof. The condition $c_i\leqslant\frac{\lambda_i}{2kr}$ implies that the equations for η_i:

$$kr\eta_i^2+2(krc_i-\lambda_i)\eta_i+krc_i^2=0,\qquad i=1,\cdots,r$$

have positive number solution $\eta_i=\xi_i>0$, $i=1,\cdots,r$. Let

$$\varphi_i(x_i)=\begin{cases}2\xi_i x_i, & i=1,\cdots,r,\\ -2c_jx_j, & j=r+1,\cdots,n.\end{cases}$$

Then we get

$$\begin{aligned}G(x)&=\sum_{i=1}^{n}\varphi_i(x_i)\frac{\mathrm{d}x_i}{\mathrm{d}t}\\
&=-2\sum_{i=1}^{r}\xi_i\lambda_ix_i^2+2\sum_{i=1}^{r}\xi_ix_if(\sigma)-2\sum_{j=r+1}^{n}c_jx_jf(\sigma)\\
&=-2\sum_{i=1}^{r}\xi_i\lambda_ix_i^2+2\sum_{i=1}^{r}(\xi_i+c_i)x_if(\sigma)-2\sum_{i=1}^{n}c_ix_if(\sigma)\\
&\leqslant-2\sum_{i=1}^{r}\xi_i\lambda_ix_i^2+2\sqrt{k}\sum_{j=1}^{r}(\xi_i+c_j)|x_i|\frac{f(\sigma)}{|k|}-2\sigma f(\sigma)\\
&\leqslant-2\sum_{i=1}^{r}\xi_i\lambda_ix_i^2+kr\sum_{i=1}^{r}(\xi_i+c_i)^2x_i^2+\sigma f(\sigma)-2\sigma f(\sigma)\\
&\leqslant-\sum_{i=1}^{r}[kr(\xi_i+c_i)^2-2\xi_i\lambda_i]x_i^2-\sigma f(\sigma)\\
&\leqslant-\sigma f(\sigma).\end{aligned}$$

Hence, $G(x)$ is negative definite with respect to σ. The conclusion follows. ∎

Corollary 3.5.5. *If the condition* 2) *in Theorem* 3.5.4 *is satisfied, and*

$$\lambda_i>0\ (i=1,\cdots,r),\qquad c_i\leqslant0\ (i=,\cdots,r),$$
$$\lambda_j=0\ (j=r+1,\cdots,n),\qquad c_j<0\ (j=r+1,\cdots,n),$$

then the zero solution of the system (3.4.1) *is absolutely stable in the Hurwitz angle* $[0,k]$.

Corollary 3.5.6. *Let the condition* 1) *in Theorem* 3.5.4 *be satisfied, and*

$$\lambda_i>0\ \ (i=1,\cdots,r),\qquad \lambda_j=0\ \ (j=r+1,\cdots,n),$$
$$c_i<0\ \ (i=1,\cdots,n).$$

Then the zero solution of the system (3.4.1) *is absolutely stable.*

3. 6. The Second Canonical Form of Control Systems

Consider the second canonical form of the control system

$$\begin{cases} \dfrac{\mathrm{d}x_i}{\mathrm{d}t} = -\rho_i x_i + \sigma, & i = 1, \cdots, n, \\ \dfrac{\mathrm{d}\sigma}{\mathrm{d}t} = \sum_{i=1}^{n} \beta_i x_i - p\sigma - rf(\sigma), \end{cases} \tag{3.6.1}$$

where $p > 0$, $r > 0$, $\rho_i > 0$ are constants.

Theorem 3. 6. 1. *Suppose*

$$p \geqslant \sum_{i=1}^{n} \left(\frac{1 + \operatorname{sign} \beta_i}{2} \right) \frac{\beta_i}{\rho_i}.$$

Then the zero solution of the system (3. 6. 1) *is absolutely stable.*

Proof. We construct the Liapunov function

$$V(x, \sigma) = \sum_{i=1}^{n} c_i x_i^2 + \sigma^2.$$

Obviously, $V(x, \sigma)$ is radially unbounded and positive definite for

$$c_i = \begin{cases} -\beta_i & \text{if} \quad \beta_i < 0, \\ \varepsilon_i \ (0 < \varepsilon_i \ll 1) & \text{if} \quad \beta_i = 0, \\ \beta_i & \text{if} \quad \beta_i > 0. \end{cases}$$

Then,

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(3.6.1)} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \sigma \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} -2c_1\rho_1 & 0 & \cdots & 0 & c_1 + \beta_1 \\ 0 & -2c_2\rho_2 & \cdots & 0 & c_2 + \beta_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -2c_n\rho_n & c_n + \beta_n \\ c_1 + \beta_1 & c_2 + \beta_2 & \cdots & c_n + \beta_n & -2p \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \sigma \end{pmatrix}$$
$$- 2r\sigma f(\sigma).$$

Now we prove

$$(-1)^{n+1} D_{n+1} \triangleq \frac{(-1)^{n+1}}{2^{n-1}} \begin{vmatrix} -2c_1\rho_1 & 0 & \cdots & 0 & c_1 + \beta_1 \\ 0 & -2c_2\rho_2 & \cdots & 0 & c_2 + \beta_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -2c_n\rho_n & c_n + \beta_n \\ c_1 + \beta_1 & c_2 + \beta_2 & \cdots & c_n + \beta_n & -2p \end{vmatrix} \geqslant 0.$$

By induction it can be verified that

$$D_{n+1} = (-1)^{n+1} 4 \prod_{i=1}^{n} c_i \rho_i p + (-1)^n \sum_{j=1}^{n} \prod_{\substack{i=1 \\ i \neq j}}^{n} c_i \rho_i (c_j + \beta_j)^2. \tag{3.6.2}$$

For $n + 1 = 2$,

$$D_2 = \begin{vmatrix} -2c_1\rho_1 & c_1+\beta_1 \\ c_1+\beta_1 & -2p \end{vmatrix} = 4c_1\rho_1 p(-1)^2 + (-1)(c_1+\beta_1)^2.$$

Assume that when $n+1=k$

$$D_k = \frac{1}{2^{k-2}} \begin{vmatrix} -2c_2\rho_2 & 0 & \cdots & 0 & c_1+\beta_1 \\ 0 & -2c_3\rho_3 & \cdots & 0 & c_2+\beta_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -2c_k\rho_k & c_k+\beta_k \\ c_2+\beta_2 & c_3+\beta_3 & \cdots & c_k+\beta_k & -2p \end{vmatrix}$$

$$= (-1)^k 4\prod_{i=2}^{k} c_i\rho_i p + (-1)^{k-1}\sum_{j=2}^{k}\prod_{\substack{i=2\\ i\neq j}}^{k} c_i\rho_i(c_j+\beta_j)^2.$$

Then, for $n+1=k+1$, we have

$$D_{k+1} = \frac{1}{2^{k-1}} \begin{vmatrix} -2c_1\rho_1 & 0 & \cdots & 0 & c_1+\beta_1 \\ 0 & -2c_2\rho_2 & \cdots & 0 & c_2+\beta_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -2c_k\rho_k & c_k+\beta_k \\ c_1+\beta_1 & c_2+\beta_2 & \cdots & c_k+\beta_k & -2p \end{vmatrix}$$

$$= \frac{1}{2^{k-1}}\left[-2c_1\rho_1 2^{k-2}D_k + (-1)^{k+2}(c_1+\beta_1)^2(-1)^{k+1}\prod_{i=2}^{k}(-2c_i\rho_i)\right]$$

$$= (-1)^{k+1}4\prod_{i=1}^{k} c_i\rho_i p + (-1)^k\sum_{j=2}^{n}\prod_{\substack{i=1\\ i\neq j}}^{k} c_i\rho_i(c_j+\beta_j)^2$$

$$+ (-1)^{k+2}(-1)^{k+1}(-1)^{k-1}\prod_{i=2}^{k} c_i\rho_i(c_1+\beta_1)^2$$

$$= (-1)^{k+1}4\prod_{i=1}^{k} c_i\rho_i p + (-1)^k\sum_{j=1}^{k}\prod_{\substack{i=1\\ i\neq j}}^{k} c_i\rho_i(c_j+\beta_j)^2.$$

Therefore, for any natural number n, the expression (3. 6. 2) holds. Since,

$$p - \sum_{i=1}^{n}\left(\frac{1+\operatorname{sign}\beta_i}{2}\right)\frac{\beta_i}{\rho_i} \geqslant 0,$$

we write

$$-4p - \sum_{i=1}^{n}\frac{(c_i+\beta_i)^2}{c_i\rho_i} = \sum_{i=1}^{n}\left(\frac{4p}{n} - \frac{(c_i+\beta_i)^2}{c_i\rho_i}\right) \geqslant 0.$$

Consequently,

$$(-1)^{n+1}D_{n+1} \geqslant 0$$

and

$$\left.\frac{dV}{dt}\right|_{(3.6.1)} \leqslant -2r\sigma f(\sigma) \qquad \text{when} \quad \sigma \neq 0.$$

Thus the zero solution of the system (3. 6. 1) is absolutely stable with respect to σ.

On the other hand, let us take $f(\sigma)=\sigma$ in (3. 6. 1). The system (3. 6. 1) is turned to the linear system

$$\begin{cases} \dfrac{\mathrm{d}x_i}{\mathrm{d}t} = -\rho_i x_i + \sigma, \\ \dfrac{\mathrm{d}\sigma}{\mathrm{d}t} = \sum\limits_{i=1}^{n} \beta_i x_i - (p+r)\sigma. \end{cases} \tag{3. 6. 3}$$

For the system (3. 6. 3), using the Liapunov function discussed above, we can prove that

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(3.6.3)} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \sigma \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} -2c_1\rho_1 & 0 & \cdots & 0 & c_1+\beta_1 \\ 0 & -2c_2\rho_2 & \cdots & 0 & c_2+\beta_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -2c_n\rho_n & c_n+\beta_n \\ c_1+\beta_1 & c_2+\beta_2 & \cdots & c_n+\beta_n & -2p-2r \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \sigma \end{pmatrix}$$

is negative definite. Then the zero solution of the system (3. 6. 3) is globally stable and thus the coefficient matrix is stable. This completes the proof . ∎

Example 3. 6. 2. Consider the equations of longitudinal motion of a plane

$$\begin{cases} \dfrac{\mathrm{d}x_i}{\mathrm{d}t} = -\rho_i x_i + \sigma, \qquad i = 1,2,3,4, \\ \dfrac{\mathrm{d}\sigma}{\mathrm{d}t} = \sum\limits_{j=1}^{4} \beta_j x_j - rp_2\sigma - f(\sigma), \end{cases} \tag{3. 6. 4}$$

where $rp_2 > 0$, $\rho_i > 0$ $(i = 1,2,3,4)$, $f(\sigma) \in F$.

The system (3. 6. 4) is clearly a particular case of (3. 6. 1) for $n = 4$. We have the following result.

Corollary 3. 6. 3. *If*

$$rp_2 \geqslant \sum_{i=1}^{4} \frac{1+\operatorname{sign}\beta_i}{2} \cdot \frac{\beta_i}{\rho_i},$$

then the zero solution of the system (3. 6. 4) *is absolutely stable.*

3. 7. Notes

Theorems 3. 1. 1, 3. 1. 2 and 3. 1. 4 are presented in Ye Baiying [1]. Theorem 3. 2. 1 is due to Xie Huimin[1]. Since the Popov's frequency method is well-known, we omit the proof of Popov's theorem, and concentrate on application only. The material of Sections 3. 3, 3. 4 and 3. 5 is taken from Zhang Wei [1] and Liao Xiaoxin [5].

CHAPTER 4

NONAUTONOMOUS AND DISCRETE CONTROL SYSTEMS

4.1. Nonautonomous Systems

Consider the nonlinear nonautonomous control systems

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = A(t)x(t) + bf(\sigma,t), \\ \sigma = c^{\mathrm{T}}x = \sum\limits_{i=1}^{n} c_i x_i, \end{cases} \tag{4.1.1}$$

where $A(t) \in C[[0, +\infty), \mathbf{R}^{n\times n}]$, $b \in \mathbf{R}^n$, $c \in \mathbf{R}^n$, $x \in \mathbf{R}^n$,

$$f \in F_k \triangleq \{f: f(t,0) \equiv 0,\ 0 \leqslant f(\sigma,t)/\sigma \leqslant k < +\infty,$$
$$f \in C[[0, +\infty) \times \mathbf{R}, \mathbf{R}]\}.$$

If for any $f \in F_k$, the zero solution of (4.1.1) is globally stable, we say the zero solution of (4.1.1) to be *absolutely stable in* $[0,k]$.

Definition 4.1.1. We say that the zero solution of (4.1.1) is *absolutely stable with respect to the set* $\Omega = \{x: \sigma = c^{\mathrm{T}}x = 0\}$ *in* $[0,k]$, if for any $f \in F_k$ and any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that the solution $x(t,t_0,x_0)$ of (4.1.1) satisfies:

$$|c^{\mathrm{T}}x(t,t_0,x_0)| = \left|\sum_{i=1}^{n} c_i x_i(t,t_0,x_0)\right| < \varepsilon \qquad \text{for all} \quad t \geqslant t_0$$

if $\| x_0 \| < \delta(\varepsilon)$, and for any $x_0 \in \mathbf{R}^n$,

$$\lim_{t\to+\infty} \sigma(t,t_0,x_0) = \lim_{t\to+\infty} c^{\mathrm{T}}x(t,t_0,x_0) = 0.$$

Definition 4.1.2. We say that the function $V(x) \in C[\mathbf{R}^n, \mathbf{R}]$ is *positive definite* [*negative definite*] *with respect to the set* $\Omega = \{x: \sigma = 0\}$ if

$$V(x)\begin{cases} = 0 & \text{for } x \in \Omega, \\ > 0 & \text{for } x \overline{\in} \Omega \end{cases} \qquad \left[V(x)\begin{cases} = 0 & \text{for } x \in \Omega, \\ < 0 & \text{for } x \overline{\in} \Omega. \end{cases} \right]$$

The function $V(x)$ is said to be *radially unbounded positive definite for* Ω if $V(x)$ is positive definite for Ω, and $V(x) \to +\infty$ as $|\sigma| \to +\infty$.

Theorem 4.1.3. *Suppose the following conditions are satisfied:*

1) *the zero solution of the following linear systems:*

$$\frac{dx}{dt} = A(t)x \tag{4.1.2}$$

is uniformly asymptotically stable;

2) *the zero solution of* (4.1.1) *is absolutely stable for the set Ω in* $[0,k]$.

Then the zero solution of (4.1.1) *is absolutely stable in* $[0,k]$.

Proof. According to the formula of variation of constants, the solution of (4.1.1) can be expressed as

$$x(t) \triangleq x(t,t_0,x_0) = K(t,t_0)x_0 + \int_{t_0}^{t} K(t,\tau)bf(\sigma(\tau),\tau)d\tau, \tag{4.1.3}$$

where $K(t,t_0)$ is the Cauchy matrix solution of (4.1.2), i.e.,

$$\begin{cases} \dfrac{dK(t,t_0)}{dt} = A(t)K(t,t_0), \\ K(t_0,t_0) = E. \end{cases}$$

Since the condition 2) is satisfied, there exist constants $\alpha > 0$ and $M \geqslant 1$ such that

$$\| K(t,\tau) \| \leqslant Me^{-\alpha(t-\tau)}.$$

Since $\sigma(t,t_0,x_0) \to 0$ as $t \to +\infty$, and $\sigma(t,t_0,x_0)$ continuously depends on x_0, $f(\sigma(t,t_0,x_0),t)$ is a continuous function of x_0, and

$$f(\sigma(t,t_0,x_0),t) \to 0 \quad \text{as} \quad t \to +\infty.$$

Thus, for any $\varepsilon > 0$, there exist $\delta_1(\varepsilon) > 0$ and $t_1 > t_0$ such that

$$Me^{-\alpha(t-t_0)} \| x_0 \| < \frac{\varepsilon}{3} \qquad \text{for all} \quad t \geqslant t_1, \tag{4.1.4}$$

$$\int_{t_0}^{t_1} Me^{-\alpha(t-\tau)} \| bf(\sigma(\tau),\tau) \| d\tau < \frac{\varepsilon}{3}, \tag{4.1.5}$$

$$\int_{t_1}^{t} Me^{-\alpha(t-\tau)} \| bf(\sigma(\tau),\tau) \| d\tau < \frac{\varepsilon}{3} \qquad \text{for all} \quad t \geqslant t_1 \tag{4.1.6}$$

if $\| x_0 \| < \delta_1(\varepsilon)$.

Let us take $\delta_2(\varepsilon) = \frac{\varepsilon}{3M}$, $\delta(\varepsilon) = \min(\delta_1(\varepsilon),\delta_2(\varepsilon))$. From (4.1.3), (4.1.4), (4.1.5) and (4.1.6), we have

$$\begin{aligned} \| x(t,t_0,x_0) \| &\leqslant \| K(t,t_0) \| \cdot \| x_0 \| + \int_{t_0}^{t_1} \| K(t,\tau) \| \cdot \| bf(\sigma(\tau),\tau) \| d\tau \\ &\quad + \int_{t_1}^{t} \| K(t,\tau) \| \cdot \| bf(\sigma(\tau),\tau) \| d\tau \\ &\leqslant Me^{-\alpha(t-t_0)} \| x_0 \| + \int_{t_0}^{t_1} Me^{-\alpha(t-\tau)} \| bf(\sigma(\tau),\tau) \| d\tau \\ &\quad + \int_{t_1}^{t} Me^{-\alpha(t-\tau)} \| bf(\sigma(\tau),\tau) \| d\tau \end{aligned}$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$
$$= \varepsilon \quad \text{for all} \quad t \geqslant t_1 > t_0 \quad \text{if} \quad \| x_0 \| < \delta(\varepsilon).$$

Therefore, the zero solution of (4. 1. 1) is stable.

For any $x_0 \in \mathbf{R}^n$, by using de L'Hospital rule, we deduce

$$0 \leqslant \lim_{t \to +\infty} \| x(t) \| \leqslant \lim_{t \to +\infty} M e^{-\alpha(t-t_0)} + \lim_{t \to +\infty} \int_{t_0}^{t} M e^{-\alpha(t-\tau)} \| b f(\sigma(\tau), \tau) \| \mathrm{d}\tau$$
$$= 0 + \lim_{t \to +\infty} \frac{1}{e^{\alpha t}} \int_{t_0}^{t} M e^{\alpha \tau} \| f(\sigma(\tau), \tau) \| \mathrm{d}\tau = 0.$$

Therefore the zero solution of (4. 1. 1) is absolutely stable in $[0, k]$. The proof of the theorem is completed. ∎

Theorem 4. 1. 4. *If there exists a function* $V(x) \in [\mathbf{R}^n, \mathbf{R}]$ *such that*

$$V(0) = 0, \quad V(x) \geqslant \varphi(|\sigma|), \qquad \varphi \in KR,$$
$$\mathrm{D}^+ V(x)|_{(4.1.1)} \leqslant -\psi(|\sigma|), \qquad \psi \in K, \tag{4.1.7}$$

then the zero solution of (4. 1. 1) *is absolutely stable for the set* Ω *in* $[0, k]$.

Proof. Since $V(0) = 0$ ($0 \in \Omega$) and $V(x) \in C[\mathbf{R}^n, \mathbf{R}]$, for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $V(x_0) < \varphi(\varepsilon)$ as $\| x_0 \| < \delta(\varepsilon)$. According to (4. 1. 7), it yields

$$\varphi(|\sigma(t, t_0, x_0)|) \leqslant V(x(t, t_0, x_0)) \leqslant V(x_0) \leqslant \varphi(\varepsilon) \qquad \text{for all} \quad t \geqslant t_0,$$

which implies

$$|\sigma(t, t_0, x_0)| < \varepsilon \qquad \text{for all} \quad t \geqslant t_0.$$

Now, we prove that

$$\lim_{t \to +\infty} \sigma(t, t_0, x_0) = 0 \qquad \text{for any} \quad x_0 \in \mathbf{R}^n.$$

Expression (4. 1. 7) gives

$$\inf_{t \geqslant t_0} V(x(t)) = \lim_{t \to +\infty} V(x(t)) \triangleq \alpha \geqslant 0.$$

We can easily check that the function $V(x(t))$ can reach its inferior limit only in Ω.

For any $x_0 \in \mathbf{R}^n$, we have from (4. 1. 7)

$$|\sigma(t)| \leqslant |c^{\mathrm{T}} x(t_0)| \triangleq h < H < +\infty.$$

Assuming that $\lim_{t \to +\infty} \sigma(t) \neq 0$, from the uniform continuity of $\sigma(t)$, there exist constants $\beta > 0$, $\eta > 0$ and a sequence $\{t_j\}$ such that

$$|\sigma(t)| \geqslant \beta \qquad \text{for} \quad t \in [t_j - \eta, t_j + \eta].$$

Let $r = \inf_{\beta \leqslant |\sigma| \leqslant h} \psi(|\sigma|) > 0$. We have

$$\begin{aligned}0 \leqslant V(t) &\leqslant V(t_0) + \int_{t_0}^{t} \mathrm{D}^+ V(t)\mathrm{d}t \\ &\leqslant V(t_0) - \int_{t_0}^{t} \psi(|\sigma(\tau)|)\mathrm{d}\tau \\ &\leqslant V(t_0) - \sum_{j=1}^{n}\int_{t_j-\eta}^{t_j+\eta} \psi(|\sigma(\tau)|)\mathrm{d}\tau \\ &\leqslant V(t_0) - 2n\eta r \to -\infty \qquad \text{as} \quad n \to +\infty.\end{aligned}$$

This yields a contradiction, thus

$$\lim_{t\to+\infty} \sigma(t,t_0,x_0) = 0.$$

The theorem is proved. ∎

Theorem 4.1.5. 1) *Let the condition* 1) *of Theorem* 4.1.3 *be satisfied, and* $|f(\sigma,t)| \geqslant \psi(|\sigma|) \in K$.

2) *Suppose there exist a symmetric matrix* $B(t)_{n\times n}$ *and constants* $\alpha>0$, $\beta>0$, $\varepsilon>0$ *such that* $x^{\mathrm{T}}B(t)x \geqslant \beta|\sigma|^2$ *and the matrix*

$$\begin{pmatrix} G(t) & g(t) \\ g^{\mathrm{T}}(t) & -\alpha/(k+\varepsilon) \end{pmatrix},$$

where

$$G(t) = A^{\mathrm{T}}(t)B(t) + B(t)A(t) + \dot{B}(t), \tag{4.1.8}$$

$$g(t) = B(t)b + \frac{\alpha}{2}c,$$

is negative semi-definite.

Then the zero solution of (4.1.1) *is absolutely stable in* $[0,k]$.

Proof. We choose the Liapunov function

$$V(t,x) = x^{\mathrm{T}}B(t)x.$$

Then $V(t,x)$ is radially unbounded positive definite for the set Ω. Using the fact that

$$\begin{aligned}\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.1.1)} &= x^{\mathrm{T}}(A^{\mathrm{T}}(t)B(t) + \dot{B}(t) + B(t)A(t))x + 2x^{\mathrm{T}}B(t)bf(\sigma,t) \\ &= x^{\mathrm{T}}(A^{\mathrm{T}}(t)B(t) + \dot{B}(t) + B(t)A(t))x + 2x^{\mathrm{T}}B(t)bf(\sigma,t) \\ &\quad + \alpha c^{\mathrm{T}}xf(\sigma,t) - \frac{\alpha}{k+\varepsilon}f^2(\sigma,t) \\ &\quad - \alpha\left(\sigma - \frac{f(\sigma,t)}{k+\varepsilon}\right)f(\sigma,t) \\ &= (x^{\mathrm{T}}, f(\sigma,t))\begin{pmatrix} G(t) & g(t) \\ g^{\mathrm{T}}(t) & -\alpha/(k+\varepsilon) \end{pmatrix}\begin{pmatrix} x \\ f(\sigma,t) \end{pmatrix} \\ &\leqslant -\alpha\left(\sigma - \frac{f(\sigma,t)}{k+\varepsilon}\right)f(\sigma,t)\end{aligned}$$

$$\leqslant -\frac{\alpha}{k+\varepsilon}\varepsilon\sigma f(\sigma,t)$$

$$\leqslant -\frac{\alpha\varepsilon}{k+\varepsilon}|\sigma|\cdot\psi(|\sigma|).$$

We see that $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{4.1.1)}$ is negative definite for the set Ω. The conditions of Theorem 4.1.4 are satisfied and therefore the zero solution of (4.1.1) is absolutely stable in $[0,k]$. ∎

4.2. The Systems with Separable Variables

In the following, we will study absolute stability for the set Ω by turning this stability into absolute stability for one state variable.

Without loss of generality, we assume that $c_n \neq 0$. Let

$$y_i = x_i, \qquad i = 1,\cdots,n-1,$$

$$y_n = c^{\mathrm{T}}x = \sum_{i=1}^{n} c_i x_i.$$

The system (4.1.1) transforms into

$$\frac{\mathrm{d}y_i}{\mathrm{d}t} = \sum_{j=1}^{n}\tilde{a}_{ij}(t)y_j + \tilde{b}_i f(y_n,t), \qquad i = 1,\cdots,n, \tag{4.2.1}$$

where

$$\tilde{a}_{ij}(t) = a_{ij}(t) - \frac{a_{ij}c_i}{c_n}, \qquad i,j = 1,\cdots,n-1,$$

$$\tilde{a}_{in}(t) = \frac{a_{ij}(t)}{c_n}, \qquad i = 1,\cdots,n-1,$$

$$\tilde{a}_{nn} = \frac{1}{c_n}\sum_{i=1}^{n} c_i a_{in}(t),$$

$$\tilde{a}_{nj}(t) = \sum_{i=1}^{n} c_i a_{ij}(t) - \sum_{i=1}^{n} c_i a_{in}(t)c_i/c_n, \qquad j = 1,\cdots,n-1,$$

$$\tilde{b}_i = b_i, \qquad i = 1,\cdots,n-1,$$

$$\tilde{b}_n = \sum_{j=1}^{n} c_i b_i.$$

Obviously, the stability of the zero solution of (4.1.1) is equivalent to that of (4.2.1).

Definition 4.2.1. We say that the zero solution of (4.2.1) is *absolutely stable in* $[0,k]$ *for part of variables* $y_{j_0},\cdots,y_n$ if for any $f \in F_k$ and any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\| y_{j_0}(t,t_0,y_0)\cdots y_n(t,t_0,y_0) \| < \varepsilon \qquad \text{for all} \quad t \geqslant t_0$$

if $\| y_0 \| < \delta(\varepsilon)$, and for any $y_0 \in \mathbf{R}^n$,

$$\lim_{t\to+\infty} \| y_{j_0}(t,t_0,y_0)\cdots y_n(t,t_0,y_0) \| = 0.$$

Theorem 4.2.2. 1) *Let the zero solution of the following system*

$$\frac{dy}{dt} = \tilde{A}(t)y$$

be uniformly asymptotically stable.

2) *Suppose there exist constants* $r_i \geqslant 0$ $(i=1,\cdots,n-1)$, $r_n > 0$ *and* $\delta > 0$ *such that*

$$\begin{cases} -r_j\tilde{a}_{jj}(t) \geqslant \sum\limits_{\substack{i=1\\i\neq j}}^{n} r_i|\tilde{a}_{ij}(t)|, \qquad j=1,\cdots,n-1, & (4.2.2)\\ -r_n\tilde{a}_{nn}(t) \geqslant \sum\limits_{i=1}^{n} r_i|\tilde{a}_{in}(t)| + \delta, & (4.2.3)\\ -r_n\tilde{b}_n \geqslant \sum\limits_{i=1}^{n-1} r_i|\tilde{b}_i|, & (4.2.4)\end{cases}$$

or

$$\begin{cases} -r_j\tilde{a}_{jj}(t) \geqslant \sum\limits_{\substack{i=1\\i\neq j}}^{n} r_i|\tilde{a}_{ij}(t)|, \qquad j=1,\cdots,n-1, & \\ -r_n\tilde{a}_{nn} \geqslant \sum\limits_{i=1}^{n} r_i|\tilde{a}_{in}(t)|, & (4.2.5)\\ -r_n\tilde{b}_n > \sum\limits_{i=1}^{n-1} r_i|\tilde{b}_i|. & (4.2.6)\end{cases}$$

In other words, either (4.2.2), (4.2.3) *and* (4.2.4), *or* (4.2.2), (4.2.5) *and* (4.2.6) *are simultaneously valid. Then the zero solution of* (4.2.1) *is absolutely stable in* $[0,k]$.

Proof. We construct the Liapunov function

$$V(y) = \sum_{i=1}^{n} r_i|y_i|.$$

$V(y) \geqslant r_n|y_n| \to +\infty$ as $y_n \to +\infty$ and, as a consequence, $V(y)$ is infinitely positive definite for y_n. In addition, there exists

$$\begin{aligned} D^+V(y)|_{(4.2.1)} &\leqslant \sum_{j=1}^{n}\Big[r_j\tilde{a}_{jj}(t) + \sum_{\substack{i=1\\i\neq j}}^{n} r_i|\tilde{a}_{ij}(t)|\Big]|y_j| \\ &\quad + \Big[r_nb_n + \sum_{j=1}^{n} r_jb_j\Big]|f(y_n)| \\ &\leqslant \Big[r_n\tilde{a}_{nn}(t) - \sum_{i=1}^{n} r_i|a_{in}(t)|\Big]|y_n| \\ &\quad + \Big[r_n\tilde{b}_n + \sum_{i=1}^{n-1} r_i|\tilde{b}_i|\Big]|f(y_n,t)|, \end{aligned}$$

thus $D^+V(y)|_{(4.2.1)}$ is negative definite for y_n. According to Theorem 4.1.5, the conclusion of this theorem is valid. ∎

Theorem 4.2.3. 1) *Let the condition* 1) *of Theorem* 4.2.2 *be staisfied.*

2) *Suppose there exist a symmetric matrix* $B(t)_{n\times n}$ *and constants* $\alpha>0$, $\beta>0$, $\varepsilon>0$ *such that*

$$y^{\mathrm{T}}B(t)y \geqslant \beta y_n^2,$$

and either

$$(y^{\mathrm{T}}, f(y_n(t),t))\begin{pmatrix} G(t) & g(t) \\ g^{\mathrm{T}}(t) & -\alpha/(k+\varepsilon)\end{pmatrix}\begin{pmatrix} y \\ f(y_n(t),t)\end{pmatrix} \leqslant -\varepsilon y_n^2,$$

or the matrix $\begin{pmatrix} G(t) & g(t) \\ g^{\mathrm{T}}(t) & -\alpha/(k+\varepsilon)\end{pmatrix}$ *is negative semi-definite, where*

$$G(t) = A^{\mathrm{T}}(t)B(t) + B(t)A(t) + \dot{B}(t),$$

$$g(t) = B(t)\bar{b} + \frac{\alpha}{2}c,$$

and $|f(y_n,t)| \geqslant \psi(|y_n|) \in K$.

Then the zero solution of (4. 2. 1) *is absolutely stable in* $[0,k]$.

Proof. We choose the Liapunov function

$$V(t,y) = y^{\mathrm{T}}B(t)y.$$

According to the hypothesis we know that $V(t,y)$ is radially unbounded positive definite and

$$\begin{aligned}
\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.2.1)} &= y^{\mathrm{T}}(\bar{A}^{\mathrm{T}}(t)B(t) + \dot{B}(t) + B(t)\bar{A}(t))y + 2y^{\mathrm{T}}B(t)\bar{b}f(y_n(t),t) \\
&= y^{\mathrm{T}}(\bar{A}^{\mathrm{T}}(t)B(t) + \dot{B}(t) + B(t)\bar{A}(t))y + 2y^{\mathrm{T}}B(t)\bar{b}f(y_n(t),t) \\
&\quad + \alpha y_n(t)f(y_n(t),t) - \frac{\alpha}{k+\varepsilon}f^2(y_n(t),t) \\
&\quad - \left(\alpha y_n(t) - \frac{\alpha}{k+\varepsilon}f(y_n(t),t)\right)f(y_n(t),t) \\
&\leqslant \begin{cases} -\dfrac{\alpha\varepsilon}{k+\varepsilon}|y_n(t)|\cdot|\psi(y_n(t))|, & \text{or} \\ -\varepsilon y_n^2(t) \end{cases} \\
&< 0 \qquad \text{for} \quad y_n^2 \neq 0.
\end{aligned}$$

As a result, the zero solution of (4. 2. 1) is absolutely stable in $[0,k]$. ∎

Knowing that $\bar{b}_n \leqslant 0$ is the necessary condition for the absolute stability of the system (4. 2. 1), we assume $\bar{b}_n < 0$. By the topological transformation

$$z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-1} \\ z_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 & -b_1/b_n \\ 0 & 1 & \cdots & 0 & -b_2/b_n \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -b_{n-1}/b_n \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix} \triangleq Hy,$$

the system (4. 2. 1) becomes

$$\begin{cases} \dfrac{\mathrm{d}z_i}{\mathrm{d}t} = \sum\limits_{j=1}^{n} p_{ij}(t)z_j, \qquad i = 1,\cdots,n-1, \\ \dfrac{\mathrm{d}z_n}{\mathrm{d}t} = \sum\limits_{j=1}^{n} p_{nj}(t)z_j + \tilde{h}_n f(z_n(t),t), \end{cases} \tag{4.2.7}$$

where $P(t) = (p_{ij}(t))_{n\times n} = H\tilde{A}(t)H^{-1}$.

Theorem 4. 2. 4. *Assume that*

1) *the condition* 1) *of Theorem* 4. 2. 2 *be satisfied*;

2) *there exists a symmetric positive semi-definite matrix*

$$B = \begin{bmatrix} b_{11} & \cdots & b_{1,n-1} & 0 \\ \vdots & & \vdots & \vdots \\ b_{n-1,1} & \cdots & b_{n-1,n-1} & 0 \\ 0 & \cdots & 0 & b_{nn} \end{bmatrix} \qquad (b_{nn} > 0)$$

such that either

$$z^{\mathrm{T}}(P^{\mathrm{T}}(t)B + BP(t))z \leqslant -\varepsilon z_n$$

or

$$z^{\mathrm{T}}(P^{\mathrm{T}}(t)B + BP(t))z \leqslant 0 \quad \textit{and } z_n f(z_n,t) \textit{ is positive definite.}$$

Then the zero solution of (4. 2. 7) *is absolutely stable in* $[0,k]$.

Proof. We choose the Liapunov function

$$V(z) = z^{\mathrm{T}}Bz + b_{nn}z_n^2.$$

Then

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.2.7)} = z^{\mathrm{T}}(P^{\mathrm{T}}(t)B + BP(t))z + 2\tilde{h}_n z_n f_n(z_n,t)$$

$$\leqslant \begin{cases} -\varepsilon z_n^2, & \text{or} \\ 2\tilde{h}_n z_n f(z_n,t). \end{cases}$$

Consequently, the zero solution of (4. 2. 7) is absolutely stable in $[0,k]$. ∎

4. 3. Direct Control Systems

Consider the general nonautonomous direct control system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x + b(t)f(\sigma,t), \qquad \sigma = c^{\mathrm{T}}x, \tag{4.3.1}$$

where $x \in \mathbf{R}^n$, $A(t)$ is an $n \times n$ continuous matrix, $c(t)$ and $b(t)$ are n-dimensional continuous vectors which are bounded and differentiable,

$$f \in F_k \triangleq \{f: f(0,t) \equiv 0,\ 0 \leqslant \sigma f(\sigma,t) \leqslant k\sigma^2,\ 0 < k < +\infty,$$
$$f \in C[[0,+\infty) \times \mathbf{R},\mathbf{R}]\}.$$

Theorem 4. 3. 1. *Suppose that there exists a symmetric differentiable and bounded $n\times n$ matrix $B(t)$ such that $x^{\mathrm{T}}B(t)x$ is radially unbounded positive definite and there exists a constant $\alpha>0$ such that*

$$\begin{pmatrix} x \\ \xi \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} -G(t) & g(t) \\ g^{\mathrm{T}}(t) & -\alpha/k \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix}$$

for

$$-G(t) = A^{\mathrm{T}}(t)B(t) + B(t)A(t) + \dot{B}(t),$$

$$g(t) = B(t)b(t) + \frac{\alpha}{2}c(t)$$

is negative definite.

Then the zero solution of (4. 3. 1) *is absolutely stable in* $[0,k]$.

Proof. We construct the radially unbounded, positive definite Liapunov function $V = x^{\mathrm{T}}B(t)x$. Using the S-method, we deduce

$$\begin{aligned} \left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.3.1)} &= x^{\mathrm{T}}(A^{\mathrm{T}}(t)B(t) + B(t)A(t) - \dot{B}(t))x \\ &\quad + 2b^{\mathrm{T}}(t)B(t)x(t)f(\sigma(t),t) \\ &= -x^{\mathrm{T}}G(t)x + 2b^{\mathrm{T}}(t)B(t)x(t)f(\sigma(t),t) \\ &\quad + \alpha\sigma(t)f(\sigma(t),t) - \alpha\frac{1}{k}f^2(\sigma(t),t) \\ &\quad - \alpha\left(\sigma(t) - \frac{1}{k}f(\sigma(t),t)\right)f(\sigma(t),t) \\ &\leqslant (x^{\mathrm{T}}, f(\sigma(t),t)) \begin{pmatrix} -G(t) & g(t) \\ g^{\mathrm{T}}(t) & -\alpha/k \end{pmatrix} \begin{pmatrix} x \\ f(\sigma(t),t) \end{pmatrix} \\ &< 0 \qquad \text{for} \quad x \neq 0. \end{aligned}$$

Thus, the zero solution of (4. 3. 1) is absolutely stable in $[0,k]$. ∎

4. 4. Indirect Control Systems

Consider the nonautonomous indirect control system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = A(t)x + b(t)\xi + d(t)f(\sigma,t), \\ \dfrac{\mathrm{d}\xi}{\mathrm{d}t} = f(\sigma,t), \quad \sigma = c^{\mathrm{T}}(t)x - r(t)\xi, \end{cases} \tag{4. 4. 1}$$

where $x \in \mathbf{R}^n$, $b(t)$, $d(t)$, $c(t)$ are n-dimensional continuous differentiable vectors, ξ,σ are scalars, and $r(t) \neq 0$ is a scalar continuously differentiable function. Suppose the coefficients of the system (4. 4. 1) are continuously differentiable, and

$$\Delta(t) \triangleq \det \begin{vmatrix} A(t) & b(t) \\ -c^{\mathrm{T}}(t) & r(t) \end{vmatrix} \neq 0 \qquad \text{for all} \quad t \in [0, +\infty),$$

$$f \in F_{k_1,k_2} = \{f: f(0,t) \equiv 0,\ k_1\sigma^2 < \sigma f(\sigma,t) < k_2\sigma^2 \text{ for } \sigma \neq 0,$$
$$f \in C[[0, +\infty) \times \mathbf{R}, \mathbf{R}]\}.$$

We set

$$\sigma = c^{\mathrm{T}}(t)x - r(t)\xi, \qquad \text{i.e.,} \quad \xi(t) = r^{-1}(t)(c^{\mathrm{T}}(t)x - \sigma).$$

Then (4.4.1) can be reduced to

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \tilde{A}(t)x + \tilde{b}(t)\sigma + d(t)f(\sigma,t), \\ \dfrac{\mathrm{d}\sigma}{\mathrm{d}t} = \tilde{c}^{\mathrm{T}}(t)x - \rho(t)\sigma - \gamma(t)f(\sigma,t), \end{cases} \tag{4.4.2}$$

where

$$\begin{aligned} \tilde{A}(t) &= A(t) + b(t)r^{-1}(t)c^{\mathrm{T}}(t), \\ \tilde{b}(t) &= -b(t)r^{-1}(t), \\ \tilde{c}^{\mathrm{T}}(t) &= c^{\mathrm{T}}(t)\tilde{A}(t) + \dot{c}^{\mathrm{T}}(t) - \dot{r}(t)r^{-1}(t)c^{\mathrm{T}}(t), \\ \rho(t) &= -c^{\mathrm{T}}(t)\tilde{b}(t) + \dot{r}(t)r^{-1}(t), \\ \gamma(t) &= -c^{\mathrm{T}}(t)d(t) + r(t). \end{aligned}$$

Let

$$P(t) = \begin{bmatrix} \tilde{A}(t) & \tilde{b}(t) \\ \tilde{c}^{\mathrm{T}}(t) & -\rho(t) \end{bmatrix}, \qquad z = \begin{bmatrix} x \\ \sigma \end{bmatrix},$$

$$l(t) = \begin{bmatrix} d(t) \\ -\gamma(t) \end{bmatrix}.$$

The system (4.4.2) transforms into

$$\frac{\mathrm{d}z}{\mathrm{d}t} = P(t)z + l(t)f(\sigma,t). \tag{4.4.3}$$

Theorem 4.4.1. *Suppose that there exists an $n \times n$ symmetric differetiable matrix $H(t)$ such that $x^{\mathrm{T}}H(t)x$ is radially unbounded, positive definite, and $x^{\mathrm{T}}(G(t)+k_1S(t)S^{\mathrm{T}}(t))x$, where*

$$\begin{aligned} G(t) &= P^{\mathrm{T}}(t)H(t) + H(t)P(t) + \dot{H}(t), \\ \sigma &= S^{\mathrm{T}}(t)z, \\ S(t) &= 2H(t)l(t), \end{aligned}$$

is positive definite.

Then the zero solution of (4.4.3) is absolutely stable in $[k_1, k_2]$.

Proof. We construct the radially unbounded positive definite Liapunov function $V(t,x) = x^{\mathrm{T}}H(t)x$. It follows that

$$-\left.\frac{dV}{dt}\right|_{(4.4.3)} = z^{T}G(t)z + \sigma f(\sigma,t).$$

Since $k_1\sigma^2 < \sigma f(\sigma) < k_2\sigma^2$ for $\sigma \neq 0$, we deduce

$$z^{T}(G(t) + k_1 S(t)S^{T}(t))z \leqslant -\left.\frac{dV}{dt}\right|_{(4.4.3)} \leqslant z^{T}(G(t) + k_2 S(t)S^{T}(t))z.$$

Furthermore, there exists

$$\left.\frac{dV}{dt}\right|_{(4.4.3)} \leqslant -z^{T}(G(t) + k_1 S(t)S^{T}(t))z < 0 \qquad \text{for} \quad z \neq 0.$$

This complete the proof. ∎

In the following, we will use the method of absolute stability for part of variables to determine the absolute stability of (4.4.2).

Suppose that all coefficients of (4.4.3) are bounded, and that

$$f \in F \triangleq \{f:\ f(0,t) \equiv 0,\ 0 < \sigma f(\sigma,t) \leqslant k_2\sigma^2 \ (k_2 \leqslant +\infty) \text{ for } \sigma \neq 0,$$
$$f \in C[[0, +\infty) \times \mathbf{R},\mathbf{R}]\}.$$

For the coefficients of (4.4.3), we adopt the following notations:

$$P(t) = \begin{bmatrix} p_{11}(t) & \cdots & p_{1,n+1}(t) \\ \vdots & & \vdots \\ p_{n+1,1}(t) & \cdots & p_{n+1,n+1}(t) \end{bmatrix},$$

$$P_{(j_0)}(t) = \begin{bmatrix} p_{11}(t) & \cdots & p_{1j_0}(t) \\ \vdots & & \vdots \\ p_{j_01}(t) & \cdots & p_{j_0j_0}(t) \end{bmatrix},$$

$$P^{(n+1-j_0)}(t) = \begin{bmatrix} p_{1,j_0+1}(t) & \cdots & p_{1,n+1}(t) \\ \vdots & & \vdots \\ p_{j_0,j_0+1}(t) & \cdots & p_{j_0,n+1}(t) \end{bmatrix},$$

$$l(t) = \mathrm{col}(l_1(t),\cdots,l_{n+1}(t)),$$
$$l_{(j_0)}(t) = \mathrm{col}(l_1(t),\cdots,l_{j_0}(t)),$$
$$z_{(j_0)}(t) = \mathrm{col}(x_1(t),\cdots,x_{j_0}(t)),$$
$$z^{(n-j_0)}(t) = \mathrm{col}(x_{j_0+1}(t),\cdots,x_n(t),\sigma).$$

Theorem 4.4.2. *Assume that the following conditions are satisfied:*

1) *the zero solution of the system*

$$\frac{dz_{(j_0)}}{dt} = A_{j_0}(t)z_{j_0} \tag{4.4.4}$$

is uniformly asymptotically stable;

2) *there exist constants $r_i \geqslant 0$ $(i=1,\cdots,j_0)$, $r_j > 0$ $(j=j_0+1,\cdots,n+1)$, $\varepsilon >$*

0 *such that*

$$-r_j p_{jj}(t) \geqslant \sum_{\substack{i=1\\ i\neq j}}^{n+1} r_i |p_{ij}(t)|, \qquad j = 1,\cdots,j_0,$$

$$-r_j a_{jj}(t) \geqslant \sum_{\substack{i=1\\ i\neq j}}^{n+1} r_i |p_{ij}(t)| + \varepsilon, \qquad j = j_0+1,\cdots,n+1,$$

$$-l_{n+1}(t) r_{n+1} \geqslant \sum_{i=1}^{n} r_i l_i(t).$$

Then the zero solution of (4. 4. 3) *is absolutely stable in* $[0, k_2]$.

Proof. We construct the radially unbounded positive definite Liapunov function for $z_{j_0+1},\cdots,z_{n+1}$:

$$V(z) = \sum_{i=1}^{n+1} r_i |z_i|.$$

Then

$$\begin{aligned}
D^+ V(z)|_{(4.4.3)} &\leqslant \sum_{j=1}^{n+1}\Big[r_j p_{jj}(t) + \sum_{\substack{i=1\\ i\neq j}}^{n+1} r_i |p_{ij}(t)|\Big]|z_j| \\
&\quad + \Big[r_{n+1} l_{n+1}(t) + \sum_{i=1}^{n} r_i |l_i(t)|\Big]|f(\sigma,t)| \\
&\leqslant \sum_{j=j_0+1}^{n+1}\Big[r_j p_{jj}(t) + \sum_{\substack{i=1\\ i\neq j}}^{n+1} r_i |p_{ij}(t)|\Big]|z_j| \\
&\leqslant -\varepsilon \sum_{j=j_0+1}^{n+1} |z_j| \\
&< 0 \qquad \text{for} \quad \sum_{j=j_0+1}^{n+1} z_j^2 \neq 0.
\end{aligned}$$

As a result, the zero solution of (4. 4. 3) is absolutely stable for variables $z_{j_0+1},\cdots,z_{n+1}$.

Assume that the Cauchy matrix solution of (4. 4. 4) is $K(t,t_0)$. The condition 1) indicates that there exist two constants $M \geqslant 1$ and $\alpha > 0$ such that

$$\| K(t,t_0) \| \leqslant M e^{-\alpha(t-t_0)}. \tag{4.4.5}$$

The first j_0 components of the solution of (4. 4. 3) can be expressed as

$$\begin{aligned}
z_{(j_0)}(t,t_0,z_0) &= K(t,t_0) z_{(j_0)}(t_0) + \int_{t_0}^{t} K(t,\tau) P^{(n+1-j_0)}(\tau) z^{(n+1-j_0)}(\tau)\mathrm{d}\tau \\
&\quad + \int_{t_0}^{t} K(t,\tau) l_{(j_0)}(\tau) f(\sigma(\tau),\tau)\mathrm{d}\tau.
\end{aligned}$$

Using (4. 4. 5) and the method used in Theorem 2. 5. 8, we can easily prove that the zero solution of (4. 4. 3) is absolutely stable for $z_{j_0+1},\cdots,z_n$, as well. This proves the theorem. ∎

We can also prove the following corollary, along the same line.

Corollary 4. 4. 3. 1) *Let the zero solution of the system* $\frac{dz}{dt} = P(t)z$ *be uniformly asymptotically stable.*

2) *Suppose that there exist constants* $r_i \geqslant 0$ $(i=1,\cdots,n)$, $r_{n+1} > 0$, $\varepsilon > 0$ *such that*

$$-r_j p_{jj}(t) \geqslant \sum_{\substack{i=1\\ i\neq j}}^{n+1} r_i |p_{ij}(t)|, \qquad i = 1,\cdots,n+1,$$

$$-l_{n+1}(t)r_{n+1} \geqslant \sum_{i=1}^{n} r_i |l_i(t)| + \varepsilon, \qquad 0 < \varepsilon \ll 1.$$

3) *Let* $|f(\sigma,t)|$ *be positive definite.*

Then the zero solution of (4. 4. 3) *is absolutely stable in* $[0,k_2]$.

Theorem 4. 4. 4. 1) *Let the conditions* 1), 3) *of Corollary* 4. 4. 3 *be satisfied.*

2) *Suppose that there exists a symmetric differentiable bounded* $(n+1)\times(n+1)$ *matrix* $B(t)$ *such that*

$$z^{\mathrm{T}}B(t)z \geqslant \varepsilon z^{\mathrm{T}}z, \qquad 0 < \varepsilon \ll 1.$$

3) *Suppose that there exists a constant* $\alpha > 0$ *such that*

$$(z^{\mathrm{T}}, f(\sigma,t))\begin{pmatrix} -\tilde{G}(t) & \tilde{g}(t) \\ \tilde{g}^{\mathrm{T}}(t) & -\alpha/k \end{pmatrix}\begin{pmatrix} z \\ f(\sigma,t) \end{pmatrix} \leqslant -\delta\tau, \qquad 0 < \delta \ll 1,$$

where

$$-\tilde{G}(t) = P^{\mathrm{T}}(t)B(t) + B(t)P(t) + \dot{B}(t),$$

$$\tilde{g}(t) = B(t)l(t) + \frac{\alpha}{2}\tilde{c}(t),$$

$$\tau \in \{\sigma^2, \sigma f(\sigma,t), f^2(\sigma,t)\}.$$

Then the zero solution of (4. 4. 3) *is absolutely stable in* $[0,k_2]$.

Proof. We construct the radially unbounded positive definite Liapunov function $V(z) = z^{\mathrm{T}}B(t)z$. Differentiating along the solution of (4. 4. 3), making use of the S-method and of the proof of Theorem 4. 3. 1, we get

$$\begin{aligned}\left.\frac{dV}{dt}\right|_{(4.4.3)} &\leqslant (z^{\mathrm{T}}, f(\sigma,t))\begin{pmatrix} -\tilde{G}(t) & \tilde{g}(t) \\ \tilde{g}^{\mathrm{T}}(t) & -\alpha/k \end{pmatrix}\begin{pmatrix} z \\ f(\sigma,t) \end{pmatrix} \\ &\quad - \alpha\left(\sigma(t) - \frac{1}{k}f(\sigma(t),t)\right)f(\sigma(t),t) \\ &\leqslant (z^{\mathrm{T}}, f(\sigma,t))\begin{pmatrix} -\tilde{G}(t) & \tilde{g}(t) \\ \tilde{g}^{\mathrm{T}}(t) & -\alpha/k \end{pmatrix}\begin{pmatrix} z \\ f(\sigma(t),t) \end{pmatrix} \leqslant -\delta\tau.\end{aligned}$$

Therefore, the zero solution of (4. 4. 3) is absolutely stable for σ in $[0,k]$.

Suppose the Cauchy matrix solution of the system $\frac{\mathrm{d}z}{\mathrm{d}t} = P(t)z$ is $K(t,t_0)$. Using the method of variation of constants, we can express the solution of (4. 4. 3) as

$$z(t,t_0,z_0) = K(t,t_0)z_0 + \int_{t_0}^{t} K(t,\tau)l(\tau)f(\sigma(\tau),\tau)\mathrm{d}\tau.$$

A reasoning similar to that used in the proof of Theorem 2. 5. 8, applies to the proof of this theorem. ∎

4. 5. The Systems with Rigid and Revolving Feedback

Consider the control systems with rigid and revolving feedback

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = A(t)x + b(t)\xi + d(t)f(\sigma), \\ \dfrac{\mathrm{d}\xi}{\mathrm{d}t} = f(\sigma), \quad \sigma = c^{\mathrm{T}}(t)x - r(t)\xi - Nf(\sigma). \end{cases} \tag{4. 5. 1}$$

Above, we use the notations of (4. 4. 1). However, we assume that the control function $f(\sigma)$ is differentiable and satisfies

$$-v_1 \leqslant \frac{\partial f(\sigma)}{\partial \sigma} \leqslant v_2,$$

where v_1, v_2 are some positive constants.

Suppose that $r(t) \neq 0$. Using (4. 5. 1), we can write

$$\xi(t) = r^{-1}(t)(c^{-1}(t)x - \sigma - Nf(\sigma)). \tag{4. 5. 2}$$

The substitution of (4. 5. 2) into (4. 5. 1) yields

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \tilde{A}(t)x + \tilde{b}(t)\sigma + \tilde{d}(t)f(\sigma), \\ \omega(\sigma)\dfrac{\mathrm{d}\sigma}{\mathrm{d}t} = \tilde{c}^{\mathrm{T}}x - \rho(t)\sigma - \tilde{N}f(\sigma), \end{cases} \tag{4. 5. 3}$$

where

$$\begin{aligned} \tilde{A}(t) &= A(t) + b(t)r^{-1}(t)c^{\mathrm{T}}(t), \\ \tilde{b}(t) &= -b(t)r^{-1}(t), \\ \tilde{d}(t) &= d(t) - r^{-1}(t)b(t)N, \\ \rho(t) &= -c^{\mathrm{T}}(t)\tilde{b}(t) - \dot{r}(t)r^{-1}(t), \\ \tilde{c}^{\mathrm{T}}(t) &= c^{\mathrm{T}}(t)\tilde{A}(t) + \dot{c}^{\mathrm{T}}(t) - \dot{r}(t)r^{-1}(t)c^{\mathrm{T}}(t), \\ \omega(\sigma) &= 1 + N\frac{\partial f}{\partial \sigma}, \end{aligned}$$

$$\tilde{N} = r(t) - \dot{r}(t)r^{-1}(t)N - c^{T}(t)d(t).$$

Theorem 4.5.1. *Suppose that*

1) *$\omega(\sigma)$ does not change its sign for any $f \in F_{k_1,k_2}$;*

2) *there exists an $(n+1)\times(n+1)$ symmetric matix $H(t)$ such that $x^{T}H(t)x$ is radially unbounded positive definite;*

3)

$$(x^{T},\sigma,\quad \begin{matrix} G(t) & H(t)\tilde{b}(t) & \alpha\tilde{c}(t) + H(t)\tilde{d}(t) \\ \tilde{b}^{T}(t)H(t) & 2\alpha\rho(t)k_1 & \alpha\rho(t) \\ \alpha\tilde{c}^{T}(t) + (H(t)\tilde{d}(t))^{T} & \alpha\rho^{T}(t) & 2\alpha\tilde{N} \end{matrix}\Bigg]\begin{bmatrix} x \\ \sigma \\ f \end{bmatrix}$$

where α is a constant with the same sign as $\omega(\sigma)$, is positive definite.

Then the zero solution of (4.5.1) *is absolutely stable in* $[k_1, k_2]$.

Proof. We take the Liapunov function

$$V(x,\sigma) = x^{T}H(t)x + 2\alpha\int_0^{\sigma} f(\sigma)\omega(\sigma)\mathrm{d}\sigma,$$

where α is a constant having the same sign as $\omega(\sigma)$. Obviously, $V(x,\sigma)$ is radially unbounded positive definite. Differentiating along the solution of (4.5.1), we obtain

$$-\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.5.1)} = x^{T}G(t)x - 2x^{T}H(t)\tilde{b}(t)\sigma - 2x^{T}H(t)\tilde{d}(t)f(\sigma)$$
$$- 2\alpha x^{T}c(t)f(\sigma) + 2\alpha\rho(t)\sigma f(\sigma) + 2\alpha\tilde{N}f^{2}(\sigma), \quad (4.5.4)$$

where $\quad -G(t) = \tilde{A}^{T}(t)H(t) + H(t)\tilde{A}(t) + \dot{H}(t).$

We introduce the following notation:

$y = (x,\sigma,f)^{T}$,

$$G_i = \begin{bmatrix} G(t) & H(t)\tilde{b}(t) & \alpha\tilde{c}(t) + H(t)\tilde{d}(t) \\ \tilde{b}^{T}(t)H(t) & 2\alpha\rho(t)k_i & \alpha\rho(t) \\ \alpha\tilde{c}^{T}(t) + (H(t)\tilde{d}(t))^{T} & \alpha\rho^{T}(t) & 2\alpha\tilde{N} \end{bmatrix}, \quad i = 1,2.$$

Using the facts that $f(0) = 0$ and $k_1\sigma^2 < \sigma f(\sigma) < k_2\sigma^2$, the expression (4.5.4) becomes

$$y^{T}G_1(t)y \leqslant -\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.5.3)} \leqslant y^{T}G_2(t)y.$$

From (4.5.4) we find that $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.5.3)}$ is negative definite. Thus, the zero solution of (4.5.3) is absolutely stable in $[k_1, k_2]$. ∎

Theorem 4.5.2. *Assume that*

1) *the condition* 1) *of Theorem* 4.5.1 *is satisfied;*

2) *the zero solution of the system $\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x$ is uniformly asymptotically*

stable;

3) *there exist constants* $r_i \geqslant 0$ $(i=1,\cdots,n)$, $r_{n+1}>0$ *such that*

$$-r_j\tilde{a}_{jj}(t) \geqslant \sum_{\substack{i=1\\i\neq j}}^{n} r_i|\tilde{a}_{ij}(t)| + r_j|\tilde{c}_j(t)|, \qquad j=1,\cdots,n, \tag{4.5.5}$$

$$r_{n+1}\rho(t) \geqslant \sum_{i=1}^{n} |\tilde{b}_i(t)|r_i + \varepsilon, \tag{4.5.6}$$

$$r_{n+1}\tilde{N} \geqslant \sum_{i=1}^{n} r_i|\tilde{d}_i(t)|, \tag{4.5.7}$$

where ε *is a constant with* $0<\varepsilon\ll 1$;

4) $\int_0^{+\infty} |\omega(\sigma)|\mathrm{d}\sigma = +\infty.$

Then the zero solution of (4.5.3) *is absolutely stable in* $[k_1,k_2]$.

Proof. We construct the Liapunov function

$$V = \sum_{i=1}^{n} r_i|x_i| + r_{n+1}\int_0^{\sigma} \operatorname{sign}\sigma|\omega(\sigma)|\mathrm{d}\sigma.$$

V is radially unbounded positive definite for σ, and there exists

$$\begin{aligned}
\mathrm{D}^+ V|_{(4.5.3)} &\leqslant \sum_{j=1}^{n}\Big[r_j\tilde{a}_{jj}(t) + \sum_{\substack{i=1\\i\neq j}}^{n} r_i|\tilde{a}_{ij}(t)| + r_j|\tilde{c}_j(t)|\Big]|x_j| \\
&\quad + \Big[-r_{n+1}\rho(t) + \sum_{i=1}^{n} |\tilde{b}_i(t)|r_i\Big]|\sigma| \\
&\quad + \Big[-\tilde{N}r_{n+1} + \sum_{i=1}^{n} r_i|\tilde{d}_i(t)|\Big]|f(\sigma)| \\
&\leqslant -\varepsilon|\sigma| \\
&< 0 \qquad \text{for} \quad \sigma \neq 0.
\end{aligned}$$

In this case, $\mathrm{D}^+V|_{(4.5.3)}$ is negative definite in σ. Therefore, the zero solution of (4.5.3) is absolutely stable for σ in $[k_1,k_2]$.

Let the Cauchy matrix solution of the following system be $K(t,t_0)$:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \tilde{A}(t)x.$$

From the condition 2), we know that there exist constants $M \geqslant 1$, $\alpha > 0$ such that

$$\|K(t,t_0)\| \leqslant M\mathrm{e}^{-\alpha(t-t_0)}.$$

The first set of equations in (4.5.3) gives

$$x(t,t_0,x_0) = K(t,t_0)x_0 + \int_{t_0}^{t} K(t,\tau)[\tilde{b}(\tau)\sigma(\tau) + \tilde{d}(\tau)f(\sigma(\tau))]\mathrm{d}\tau.$$

The proof can be completed along the lines of the proof of sufficiency in Theorem 2.5.8. This proves the theorem. ∎

Theorem 4.5.3. *If the conditions* 1), 2) *of Theorem* 4.5.2 *hold, and* (4.5.5) *holds as well, while* (4.5.6), (4.5.7) *are replaced by*

$$r_{n+1}\rho(t) \geqslant \sum_{i=1}^{n} r_i |\tilde{b}_i(t)|,$$

$$r_{n+1}\tilde{N} \geqslant \sum_{i=1}^{n} r_i |\tilde{d}_i(t)| + \varepsilon, \qquad 0 < \varepsilon \ll 1,$$

then the zero solution of (4.5.3) *is absolutely stable in* $[k_1, k_2]$.

Proof. We construct the radially unbounded positive definite Liapunov function

$$V = \sum_{i=1}^{n} r_i |x_i| + r_{n+1} \int_0^{\sigma} \operatorname{sign} \sigma |\omega(\sigma)| \mathrm{d}\sigma,$$

then there exists

$$\begin{aligned} \mathrm{D}^+ V|_{(4.5.3)} \leqslant & \sum_{j=1}^{n} \Big[r_j \tilde{a}_{jj}(t) + \sum_{\substack{i=1 \\ i \neq j}}^{n} r_i |\tilde{a}_{ij}(t)| + r_j |\tilde{c}_j(t)| \Big] |x_j| \\ & + \Big[- r_{n+1}\rho(t) + \sum_{i=1}^{n} |\tilde{b}_i(t)| r_i \Big] |\sigma| \\ & + \Big[- \tilde{N} r_{n+1} + \sum_{i=1}^{n} r_i |\tilde{d}_i(t)| \Big] |f(\sigma)| \\ \leqslant & - \varepsilon |f(\sigma)| \\ < & 0 \qquad \text{for} \quad \sigma \neq 0. \end{aligned}$$

Thus, the zero solution of (4.5.3) is absolutely stable for σ in $[k_1, k_2]$. The rest of the proof is identical to that of Theorem 4.5.2. ∎

4.6. Discrete Control Sysyems

We examine the discrete Lurie control system

$$\begin{cases} x(t_{k+1}) = Ax(t_k) + hf(\sigma(t_k)), \\ \sigma = c^{\mathrm{T}} x = \sum_{i=1}^{n} c_i x_i, \end{cases} \tag{4.6.1}$$

where $x \in \mathbf{R}^n$, $A \in \mathbf{R}^{n \times n}$, $b \in \mathbf{R}^n$, $c \in \mathbf{R}^n$, $f \in F$ or

$$f \in F_{k_1, k_2} \triangleq \{f: f(0) = 0,\ 0 \leqslant k_1 \leqslant f(\sigma)/\sigma \leqslant k_2,\ \sigma \neq 0,$$
$$f \in C(-\infty, +\infty)\}.$$

We choose $J = \{t_k : t_0 < t_1 < \cdots < t_k < \cdots\}$, $N \triangleq \{0, 1, 2, \cdots\}$.

Definition 4.6.1. The zero solution of (4.6.1) is said to be *absolutely stable* [*absolutely stable in* $[k_1, k_2]$] if for any $f \in F$ [$f \in F_{k_1, k_2}$], the zero solution of (4.6.1) is globally asymptotically stable.

Definition 4.6.2. The zero solution of (4.6.1) is said to be *absolutely* *stable for the set* $\Omega=\{x: c^{\mathrm{T}}x=0\}$ [*absolutely stable for* Ω *in* $[k_1, k_2]$] if for any $f \in F$ [$f \in F_{k_1,k_2}$], the zero solution of (4. 6. 1) is globally asymptotically stable for Ω.

Lemma 4. 6. 3. *Let* $x(t_{k+1})$ *be the solution of the following system*

$$\begin{cases} x(t_{k+1}) = Ax(t_k) + F(t_k, x(t_k)), \\ x(t_0) = x_0. \end{cases}$$

Then for any natural number k, *the following formula of variation of constants holds*

$$x(t_{k+1}) = A^{k+1}x_0 + \sum_{l=0}^{k} A^{k-l}F(t_l, x(t_l)).$$

Proof. Lemma can be easily verified by mathematical induction, and the proof is omitted. ∎

Corollary 4. 6. 4. *The solution* $x(t_{k+1}) \triangleq x(t_{k+1}, t_0, x_0)$ *of* (4. 6. 1) *can be written as*

$$x(t_{k+1}) = A^{k+1}x_0 + \sum_{l=1}^{k} A^{k-l}hf(\sigma(t_l)).$$

Corollary 4. 6. 5. *Suppose that* $f(\sigma) \in C(-\infty, +\infty)$. *For arbitrary fixed* m, *the solution* $x(t_m)$ *of* (4. 6. 1) *depends continuously on the initial value* x_0.

Proof. Obviously, when $m = 0$, the following expression

$$x(t_1) = Ax_0 + hf(\sigma(t_0)), \qquad \sigma(t_0) = c^{\mathrm{T}}x_0$$

is a continuous function for x_0. Suppose that

$$x(t_k) = A^k x_0 + \sum_{l=0}^{k-1} A^{k-l}hf(\sigma(t_l))$$

depends continuously on x_0. Since

$$x(t_{k+1}) = Ax(t_k) + hf(\sigma(t_k))$$

is a continuous function for $x(t_k)$, thus $x(t_{k+1})$ depends continuously on x_0. ∎

Theorem 4. 6. 6. *The zero solution of* (4. 6. 1) *is absolutely stable* [*absolutely stable in* $[k_1, k_2]$] *if and only if*

1) $\rho(B)<1$ [$\rho(B)^*<1$], *where* $\rho(B)$ *and* $\rho(B^*)$ *are the respective spectral radius of* B *and* B^*;

2) *the zero solution of* (4. 6. 1) *is absolutely stable for the set* $\Omega=\{x: c^{\mathrm{T}}x=0\}$ [*absolutely stable in* $[k_1, k_2]$]. *Here*

$$B = (b_{ij})_{n\times n} = A + hc^{\mathrm{T}}\theta, \qquad \theta = 0 \quad or \quad \theta = 1,$$

$$B^* = (b_{ij}^*)_{n\times n} = A + hc^{\mathrm{T}}\frac{k_2 - k_1}{2}.$$

Proof. It is enough to prove the necessary and sufficient conditions

(NASC) of the absolute stability because the proof of NASC of absloute stability in $[k_1, k_2]$ is the same.

Necessity. Since the zero solution of (4. 6. 1) is absolutely stable, for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that the solution of (4. 6. 1) satisfies

$$|x(t_k)| < \frac{\varepsilon}{\max\limits_i |c_i|} \qquad \text{for} \quad \|x_0\| < \delta(\varepsilon).$$

Further, we have

$$|\sigma(t_k)| = \left|\sum_{i=1}^n c_i x_i(t_k)\right| \leqslant \max_i |c_i| \sum_{i=1}^n |c_i(t_k)|$$
$$\triangleq \max_i |c_i| \cdot \|x(t_k)\| < \varepsilon.$$

Obviously, $\lim\limits_{k\to+\infty} x(t_k) = 0$ implies $\lim\limits_{k\to+\infty} \sigma(t_k) = 0$, which leads to the fact that the zero solution of (4. 6. 1) is absolutely stable for the set Ω.

If $f(\sigma) = \sigma$, then (4. 6. 1) is transformed into

$$x(t_k) = Bx(t_{k-1}).$$

Since the zero solution of (4. 6. 1) is globally asymptotically stable, there exists $\rho(B) < 1$. Necessity is proved.

Sufficeency. The solution $x(t_{k+1})$ of (4. 6. 1) can be written as

$$x(t_{k+1}) = B^{(k+1)}x_0 + \sum_{l=0}^{k} B^{k-l}(hf(\sigma(t_l)) - h\theta\sigma(t_l)).$$

Since B^m is bounded, we can define

$$\|B^{k+1}\| \leqslant M = \text{const.} \qquad \text{for all} \quad k \in N.$$

For any $\varepsilon > 0$, we take $\delta_1(\varepsilon) = \frac{\varepsilon}{3M}$. Since $\lim\limits_{k\to+\infty} \sigma(t_k) = 0$ and $\lim\limits_{k\to+\infty} f(\sigma(t_k)) = 0$, there exists a constant $k_1 > k_0$ such that the following estimation holds

$$\left\|\sum_{l=k_1+1}^{k} B^{k-1}(hf(\sigma(t_l)) - h\theta\sigma(t_l))\right\| < \frac{\varepsilon}{3}.$$

By virtue of the facts that $\lim\limits_{k\to+\infty} \sigma(t_k, t_0, x_0) = 0$, $\sigma(t_k, t_0, x_0)$ depends continuously on the initial value x_0 and that $f(\sigma(t_k))$ is continuous, there exists a constant $\delta_2(\varepsilon) > 0$ such that

$$\left\|\sum_{l=0}^{k_1} B^{k-1}(hf(\sigma(t_l)) - h\theta\sigma(t_l))\right\| < \frac{\varepsilon}{3} \qquad \text{for} \quad \|x_0\| < \delta_2(\varepsilon).$$

Let $\delta(\varepsilon) = \min(\delta_1(\varepsilon), \delta_2(\varepsilon))$. Then we obtain

$$\|x(t_{k+1})\| \leqslant \|B^{k+1}x(t_0)\| + \left\|\sum_{l=0}^{k_1} B^{k-l}(hf(\sigma(t_l)) - h\theta\sigma(t_l))\right\|$$
$$+ \left\|\sum_{l=k_1+1}^{k} B^{k-l}(hf(\sigma(t_l)) - h\theta\sigma(t_l))\right\|$$

$$<\varepsilon \qquad \text{for} \quad \|x_0\| < \delta(\varepsilon).$$

Therefore the zero solution of (4. 6. 1) is stable.

Since $\lim\limits_{k\to+\infty} f(\sigma(t_k)) = 0$ and $\lim\limits_{k\to+\infty} \sigma(t_k) = 0$ for any $x_0 \in \mathbf{R}^n$, there exists a constant $M_1 > 0$ such that

$$\|hf(\sigma(t_l)) - h\theta\sigma(t_l)\| \leqslant M_1.$$

Taking into account that $\rho(B) > 1$, we have $\sum\limits_{l=1}^{\infty} \|B^l\| < +\infty$. We know that there exists a constant $M_2 > 0$ such that

$$\sum_{l=\left[\frac{k}{2}\right]}^{k} \|B^{k-l}\| \leqslant M_2.$$

Therefore,

$$\begin{aligned}
0 &\leqslant \lim_{k\to+\infty} \|x(t_k)\| \\
&\leqslant \lim_{k\to+\infty} \|B^k x_0\| + \lim_{k\to+\infty} \sum_{l=0}^{\left[\frac{k}{2}\right]} \|B^{k-l}\| \cdot \|\bar{h} f(\sigma(t_l)) - h\theta\sigma(t_l)\| \\
&\quad + \lim_{k\to+\infty} \sum_{l=\left[\frac{k}{2}\right]+1}^{k} \|B^{k-l}\| \cdot \|hf(\sigma(t_l)) - h\theta\sigma(t_l)\| \\
&= M_1 \lim_{k\to+\infty} \sum_{l=0}^{\left[\frac{k}{2}\right]} \|B^{k-l}\| + M_2 \lim_{k\to+\infty} \max_{\left[\frac{k}{2}\right]<l<k} \|hf(\sigma(t_l)) - h\theta\sigma(t_l)\| \\
&= 0. \qquad \blacksquare
\end{aligned}$$

Theorem 4. 6. 7. *The zero solution of* (4. 6. 1) *is absolutely stable* [*absolutely stable in* $[k_1, k_2]$] *if and only if*

1) *the condition* 1) *of Theorem* 4. 6. 6 *is satisfied*;

2) *for any* $f \in F$ [*for any* $f \in F_{k_1,k_2}$], *there exists a function* $V_f(x)$ *which is radially unbounded positive definite with respect to* Ω *such that*

$$\Delta V_f = V_f(x(t_{k+1})) - V_f(x(t_k))$$

is negative definite for Ω.

Proof. It is enough to prove the NASC for absolute stability, because the NASC of absolute stability in $[k_1, k_2]$ can be proved along the same lines.

Sufficiency. On the basis of Theorem 4. 6. 6, what we need is to prove that the condition 2) implies that the zero solution of (4. 6. 1) is absolutely stable with respect to σ.

Since $V_f(x(t_{k+1})) - V_f(x(t_k))$ is negative definite in σ, we find

$$V_f(x(t_{k+1})) - V_f(x(t_k)) \leqslant -\psi(\sigma(t_k)).$$

It can be deduced that

$$
\begin{aligned}
V_f(x(t_{k+1})) &\leqslant V_f(x(t_k)) - \psi(\sigma(t_k)) \\
&\leqslant V_f(x(t_{k-1})) - \psi(\sigma(t_k)) - \psi(\sigma(t_{k-1})) \\
&\leqslant V_f(x(t_0)) - \psi(\sigma(t_k)) - \cdots - \psi(\sigma(t_0)).
\end{aligned}
$$

Consequently,

$$
\varphi(\sigma(t_{k-1})) \leqslant V_f(x(t_{k+1})) \leqslant V_f(x(t_0)),
$$

$$
|\sigma(t_{k+1})| \leqslant \varphi^{-1}(V_f(x(t_0))) \ll 1 \qquad \text{for} \quad \|x_0\| \ll 1.
$$

Now, we show

$$
\lim_{k\to+\infty} \sigma(t_{k+1}) = 0 \qquad \text{for any} \quad x_0 \in \mathbf{R}^n.
$$

If there exist some $x_0 \in \mathbf{R}^n$ satisfying $\lim\limits_{k\to+\infty} \sigma(t_{k+1}) \neq 0$, then there exist $\varepsilon > 0$ and a sequence $\{k_j\}$ such that

$$
\sigma(t_{k_i}) \geqslant \varepsilon, \qquad i = 1,2,\cdots.
$$

Provided $k_1 < k_2 < \cdots < k_j < k+1$, k being a sufficiently large constant, we derive

$$
\begin{aligned}
0 \leqslant \varphi(\sigma(t_{k+1})) &\leqslant V_f(x(t_{k+1})) \\
&\leqslant V_f(x(t_0)) - \psi(\sigma(t_{k_1})) - \cdots - \psi(\sigma(t_0)) \\
&\leqslant V_f(x_0) - \varepsilon - \cdots - \varepsilon \to -\infty \qquad \text{as} \quad k \to +\infty,
\end{aligned}
$$

which yields contradiction. Hence, we have $\lim\limits_{k\to+\infty} \sigma(t_{k+1}) = 0$, i. e., the zero solution of (4. 6. 1) is absolutely stable for Ω.

Necessity. Since the zero solution of (4. 6. 1) is absolutely stable, one can prove that for any $f \in F$, there exists a radially unbounded positive definite function $V_f(x)$ such that

$$
V_f(x(t_{k+1})) - V_f(x(t_0)) \quad \text{is negative definite.}
$$

Accordingly, there exist two functions, namely $\varphi \in KR$, $\psi \in KR$ such that

$$
V_f(x) \geqslant \varphi(\|x\|),
$$

and

$$
V_f(x(t_{k+1})) - V_f(x(t_k)) \leqslant -\psi(\|x(t_k)\|).
$$

By virtue of

$$
\begin{aligned}
|\sigma| = \left|\sum_{i=1}^n c_i x_i\right| &\leqslant \sum_{i=1}^n |c_i| \cdot |x_i| \leqslant \max_i |c_i| \sum_{i=1}^n |x_i| \\
&\triangleq \max_i |c_i| \cdot \|x\|,
\end{aligned}
$$

we obtain

$$
V_f(x) \geqslant \varphi(\|x\|) \geqslant \varphi\left(\frac{|\sigma|}{\max\limits_i |c_i|}\right) \triangleq \varphi_1(|\sigma|),
$$

where $\varphi_1 \in KR$, and

$$
V_f(x(t_{k+1})) - V_f(x(t_k)) \leqslant -\psi(\|x(t_k)\|)
$$

$$\leqslant -\psi\left(\frac{|\sigma|}{\max\limits_i |c_i|}\right) \triangleq -\psi_1(|\sigma|).$$

The condition 2) is satisfied, the condition 1) holds trivially and the conclusion follows. ∎

Theorem 4. 6. 8. *The zero solution of* (4. 6. 1) *is absolutely stable* [*absolutely stable in* $[k_1, k_2]$] *if and only if*

1) *there exists a constant vector* $\eta = \mathrm{col}(\eta_1, \cdots \eta_n)$ *such that* $\rho(B_1(b_{ij})) < 1$, *where* $B_1 = A + \eta c^{\mathrm{T}}$,

2) *the condition* 2) *of Theorem* 4. 6. 4 *holds.*

Proof. *Necessity.* The existence of $\eta = \mathrm{col}(\eta_1, \cdots, \eta_n)$ is obvious, for example, we may take $\eta_i = h_i\left[\eta_i = \frac{k_2 - k_1}{2} h_i\right]$, and the condition 1) is satisfied. The necessity of the condition 2) has been proved in Theorem 4. 6. 4.

Sufficiency. For any $f \in F$ [for any $f \in F_{k_1,k_2}$], we rewrite (4. 6. 1) as

$$x(t_{k+1}) = B_1 x(t_k) + hf(x(t_k)) - \eta f(\sigma(t_k)).$$

In accordance with Lemma 4. 6. 3, we deduce

$$x(t_{k+1}) = B_1^{k+1} x(t_0) + \sum_{l=0}^{k} B_1^{k-l}(hf(\sigma(t_l)) - \eta f(\sigma(t_l))).$$

The proof of the remaining part is similar to that of Theorem 4. 6. 6. ∎

Theorem 4. 6. 9. *The zero solution of* (4. 6. 1) *is absolutely stable* [*absolutely stable in* $[k_1, k_2]$] *if and only if both the condition* 1) *of Theorem* 4. 6. 6 *and the condition* 2) *of Theorem* 4. 6. 5 *hold.*

In the following, we present some useful sufficient conditions for absolute stability.

Without loss of generality, we may assume that $c_n \neq 0$. Let

$$Q = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ c_1 & c_2 & \cdots & c_{n-1} & c_n \end{bmatrix}.$$

A full rank linear transformation gives

$$\xi = Qx.$$

(4. 6. 1) is then transformed into

$$\begin{aligned} \xi(t_{k+1}) &= QAQ^{-1}\xi(t_k) + Qhf(\xi_n(t_k)) \\ &\triangleq \bar{A}\xi(t_k) + \bar{h}f(\xi_n(t_k)), \end{aligned} \tag{4. 6. 2}$$

where $\tilde{A} = QAQ^{-1}$, $\tilde{h} = Qh$.

Obviously, the absolute stability of the zero solution of (4.6.1) and (4.6.2) are equivalent. However, all variables of (4.6.2) are separable. The definition of absolute stability of the zero solution for the variable ξ_n can be stated as in Definition 4.6.2.

Let

$$\frac{f(\xi_n)}{\xi_n} = \begin{cases} g(\xi_n) & \text{when } \xi_n \neq 0, \\ 0 & \text{when } \xi_n = 0. \end{cases} \tag{4.6.3}$$

Accordingly, (4.6.2) can be rewritten as

$$\begin{aligned} \xi_i(t_{k+1}) &= \sum_{j=1}^{n} \tilde{a}_{ij}\xi_j(t_k) + \tilde{h}_i g(\xi_n(t_k))\xi_n(t_k) \\ &= \sum_{j=1}^{n-1} \tilde{a}_{ij}\xi_j(t_k) + (\tilde{a}_{in} + \tilde{b}_i g(\xi_n(t_k)))\xi_n(t_k). \end{aligned} \tag{4.6.4}$$

Theorem 4.6.10. 1) *Let the condition* 1) *of Theorem* 4.6.4 *be satisfied.*

2) *Suppose that for any* $f \in F$ [*for any* $f \in F_{k_1,k_2}$], *there exist positive constants* r_i $(i=1,\cdots,n)$ *which are independent of* f *such that*

$$\max_{1 \leqslant j \leqslant n-1} \left\{ \sum_{i=1}^{n} \frac{r_i}{r_j} |\tilde{a}_{ij}| \right\} \leqslant 1,$$

$$\sum_{i=1}^{n} \frac{r_i}{r_n} (\tilde{a}_{in} + \tilde{h}_i g(\xi_n)) \leqslant \rho < 1.$$

Then the zero solution of (4.6.4) *is absolutely stable* [*absolutely stable in* $[k_1, k_2]$].

Proof. We take the Liapunov function

$$V(\xi) = \sum_{i=1}^{n} r_i |\xi_i|$$

and find that

$$\begin{aligned} V(\xi(t_{k+1})) - V(\xi(t_k)) &= \sum_{i=1}^{n} r_i \left| \sum_{j=1}^{n} \tilde{a}_{ij}\xi_j(t_k) + \tilde{h}_i f(\xi_n(t_k)) \right| - \sum_{i=1}^{n} r_i |\xi_i(t_k)| \\ &\leqslant \sum_{j=1}^{n-1}\sum_{i=1}^{n} \frac{r_i}{r_j} |\tilde{a}_{ij}| \cdot |r_j| \cdot |\xi_j(t_k)| \\ &\quad + \sum_{i=1}^{n} \frac{r_i}{r_n} |\tilde{a}_{in} + \tilde{h}_i g(\xi_n(t_k))| r_n - \sum_{j=1}^{n} r_j |\xi_j(t_k)| \\ &= \sum_{j=1}^{n-1} \left[\sum_{i=1}^{n} \frac{r_i}{r_j} |\tilde{a}_{ij}| - 1 \right] r_j |\xi_j(t_k)| \\ &\quad + \left[\sum_{i=1}^{n} \frac{r_i}{r_n} |\tilde{a}_{in} + \tilde{h}_i g(\xi_n(t_k)) - 1 \right] r_n |\xi_n(t_k)| \\ &\leqslant (\rho - 1) r_n |\xi_n(t_k)| \\ &\triangleq -\delta r_n |\xi_n(t_k)|. \end{aligned} \tag{4.6.5}$$

Thus, it follows

$$r_n|\xi_n(t_{k+1})| \leqslant V(t_{k+1}) < V(t_k) < V(t_{k-1}) < \cdots < V(t_0). \qquad (4.6.6)$$

The expression (4.6.6) tells us that the zero solution of (4.6.4) is stable for the partial variable ξ_n, and (4.6.5) gives

$$\begin{aligned} r_n|\xi_n(t_{k+1})| &\leqslant V(t_{k+1}) < V(t_k) - \delta r_n|\xi_n(t_k)| \\ &\leqslant V(t_0) - \delta r_n|\xi_n(t_k)| - \delta r_n|\xi_n(t_{k-1})| - \cdots - \delta r_n|\xi_n(t_0)|. \end{aligned}$$

Now we will prove that $\lim\limits_{k\to+\infty} \xi_n(t_{k+1}) = 0$. If $\lim\limits_{k\to+\infty} \xi_n(t_{k+1}) \neq 0$, then there exist a constant $\varepsilon > 0$ and a sequence $\{k_i\}$ such that

$$|\xi_n(t_{k_i})| \geqslant \varepsilon, \qquad i = 1, 2, \cdots.$$

Assume that $k_1 < k_2 < \cdots < k_j < L + 1$, where the constant L is large enough. In this case,

$$\begin{aligned} 0 < r_n|\xi_n(t_{k+1})| \leqslant V(t_0) &- \delta r_n|\xi_n(t_k)| \\ &- \delta r_n|\xi_n(t_{k-1})| - \cdots - \delta r_n|\xi_n(t_0)| \\ &\to -\infty \quad \text{as} \quad k \to +\infty, \end{aligned}$$

which leads to contradiction. Thus we have $\lim\limits_{k\to+\infty} \xi_n(t_{k+1}) = 0$. Following the ideas from Theorem 4.6.6, we see that the conclusion of Theorem 4.6.10 holds. ∎

Corollary 4.6.11. 1) *Let the condition* 1) *of Theorem* 4.6.6 *be satisfied.*

2) *Let* $\max\limits_{1\leqslant i\leqslant n}\left\{\sum\limits_{j=1}^{n} |\tilde{a}_{ij}|\right\} \leqslant 1$ *and* $\sum\limits_{j=1}^{n} |\tilde{a}_{in} + \tilde{h}_i g(\xi_n)| \leqslant \rho < 1$ *hold for any* $f(\sigma) \in F$ [*for any* $f \in F_k$].

Then the zero solution of (4.6.4) *is absolutely stable* [*absolutely stable in* $[0, k]$].

Proof. Taking $r_i = 1$ $(i = 1, \cdots, n)$ in Theorem 4.6.10, we see that all the conditions of Theorem 4.6.10 are satisfied. The conclusion is true. ∎

Below, we will use another Liapunov function to study absolute stability.

For any $\varepsilon > 0$, let

$$D = (d_{ij})_{n\times n}, \qquad G_\varepsilon \triangleq \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \varepsilon \end{pmatrix}_{n\times n},$$

where

$$d_{ij} \triangleq \begin{cases} \tilde{a}_{ij}, & 1 \leqslant i, j \leqslant n-1, \\ \tilde{a}_{in} + \tilde{h}_i g(\xi_n), & i = 1, \cdots, n. \end{cases}$$

Theorem 4.6.12. 1) *Let the condition* 1) *of Theorem* 4.6.6 *be satisfied.*

2) *Suppose that there exists a symmetric positive definite matrix* $B(b_{ij})_{n\times n}$ *such that the matrix* $D^{\mathrm{T}}BD-B+G_{\epsilon}$ *is negative semi-definite for any* $f\in F$ [*for any* $f\in F_k$].

Then the zero solution of (4.6.4) *is absolutely stable* [*absolutely stable in* $[0,k]$].

Proof. Choosing the Liapunov function $V=\xi^{\mathrm{T}}B\xi$, we get

$$\begin{aligned} V(t_{k+1})-V(t_k) &= \xi^{\mathrm{T}}(t_{k+1})B\xi(t_k)-\xi^{\mathrm{T}}(t_k)B\xi(t_k)\\ &= (D\xi(t_k))^{\mathrm{T}}BD\xi(t_k)-\xi^{\mathrm{T}}(t_k)B\xi(t_k)\\ &\leqslant -\epsilon\xi_n^2(t_k). \end{aligned}$$

Then there exists a constant $\alpha>0$ such that

$$\begin{aligned} \xi_n^2(t_k)\leqslant \alpha V(t_k) &\leqslant \alpha V(t_{k-1})-\alpha\epsilon\xi_n^2(t_{k-1})\\ &\leqslant \alpha V(t_0)-\alpha\epsilon\xi_n^2(t_{k-1})-\alpha\epsilon\xi_n^2(t_{k-2})-\cdots-\alpha\epsilon\xi_n^2(t_0). \end{aligned}$$

The rest of the proof can be completed similarly to the proof of (4.6.6). ∎

In the case $B=E$, we have the following

Corollary 4.6.13. 1) *Let the condition* 1) *of Theorem* 4.6.6 *be satisfied.*

2) *Suppose the matrix* $D^{\mathrm{T}}D-E+G_{\epsilon}$ *is negative semi-definite for any* $f\in F$ [*for any* $f\in F_k$].

Then the zero solution of (4.6.4) *is absolutely stable* [*absolutely stable in* $[0,k]$].

Example 4.6.14. Consider the 2-dimensional discrete control system

$$\begin{cases} x_1(t_k)=\dfrac{2}{5}x_1(t_{k-1})-\dfrac{1}{5}x_2(t_{k-1})+\dfrac{3}{10}f(x_2(t_{k-1})),\\[2mm] x_2(t_k)=\dfrac{3}{10}x_1(t_{k-1})-\dfrac{3}{10}x_2(t_{k-1})+\dfrac{2}{5}f(x_2(t_{k-1})), \end{cases} \tag{4.6.7}$$

where

$$\begin{aligned} f(x_2)\in F_k=\{f(x_2):\ &f(0)=0,\ 0\leqslant f(x_2)/x_2\leqslant 2,\\ &f(x_2)\in(-\infty,+\infty)\}. \end{aligned}$$

1) Let $f(x_2)=x_2$. Then (4.6.7) transforms into

$$\begin{cases} x_1(t_k)=\dfrac{2}{5}x_1(t_{k-1})+\dfrac{1}{10}x_2(t_{k-1}),\\[2mm] x_2(t_k)=\dfrac{3}{10}x_1(t_{k-1})+\dfrac{1}{10}x_2(t_{k-1}). \end{cases}$$

where $B\triangleq\begin{pmatrix} 2/5 & 1/10\\ 3/10 & 1/10\end{pmatrix}$ and $\rho(B)\leqslant\|B\|<1$. The condition 1) of Theorem 4.6.4 holds true.

2) We choose the Liapunov function $V(x)=|x_1|+|x_2|$. By virtue of

$$
\begin{aligned}
&|\bar{a}_{11}| + |\bar{a}_{21}| = \frac{2}{5} + \frac{3}{10} = \frac{7}{10} < 1,\\
&|\bar{a}_{21} + \bar{h}_1 g(x_2)| + |\bar{a}_{22} + h_2 g(x_2)|\\
&\quad = \left| -\frac{1}{5} + \frac{3}{10}\frac{f(x_2)}{x_2} \right| + \left| -\frac{3}{10} + \frac{2}{5}\frac{f(x_2)}{x_2} \right|\\
&\quad \leqslant \left| -\frac{1}{5} + \frac{3}{5} \right| + \left| -\frac{3}{10} + \frac{4}{5} \right|\\
&\quad = \frac{2}{5} + \frac{1}{2} = \frac{9}{10},
\end{aligned}
$$

the conditions of Corollary 4. 6. 9 are satisfied, and thus the zero solution of (4. 6. 7) is absolutely stable in [0,2].

Example 4. 6. 15. Consider the system

$$
\begin{cases}
x_1(t_{k+1}) = \dfrac{1}{\sqrt{2}} x_1(t_k) - \dfrac{1}{2} x_2(t_k) + \dfrac{1}{3} f(x_2(t_k)),\\[2mm]
x_2(t_{k+1}) = \dfrac{1}{\sqrt{2}} x_1(t_k) + \dfrac{1}{2} x_2(t_k) - \dfrac{1}{3} f(x_2(t_k)),
\end{cases}
\tag{4. 6. 8}
$$

where

$$
f(x_2) \in F_k = \{f(x_2):\ f(0) = 0,\ 0 \leqslant f(x_2)/x_2 \leqslant 7/2,\ f(x_2) \in C(-\infty, +\infty)\}.
$$

Now let us discuss the absolute stability of this sysyem.

1) We fix $f(x_2(t_k)) = x_2(t_k)$. The system (4. 6. 8) changes into

$$
\begin{cases}
x_1(t_{k+1}) = \dfrac{1}{\sqrt{2}} x_1(t_k) - \dfrac{1}{6} x_2(t_k),\\[2mm]
x_2(t_{k+1}) = \dfrac{1}{\sqrt{2}} x_1(t_k) + \dfrac{1}{6} x_2(t_k),
\end{cases}
$$

where

$$
B_1 = \begin{bmatrix} -\dfrac{1}{\sqrt{2}} & -\dfrac{1}{6} \\[2mm] \dfrac{1}{\sqrt{2}} & \dfrac{1}{6} \end{bmatrix}, \qquad \rho(B_1) \leqslant \|B_1\| = \frac{1}{\sqrt{2}} + \frac{1}{6} < 1.
$$

We see that the condition 1) of Theorem 4. 6. 4 is satisfied.

2) We take the Liapunov function

$$
V(x) = x_1^2 + x_2^2, \qquad G_\varepsilon = \begin{pmatrix} 0 & 0 \\ 0 & 1/10 \end{pmatrix}.
$$

By virtue of

$$
-\frac{1}{2} + \left[\frac{2}{9} g^2(x_2) - \frac{2}{3} g(x_2)\right] + \frac{1}{10}
$$

$$\leqslant -\frac{1}{2} + \left[\frac{2}{9} \times \left(\frac{7}{2}\right)^2 - \frac{2}{3} \times \frac{7}{2}\right] + \frac{1}{10}$$
$$\leqslant -\frac{1}{2} + \frac{2}{5} + \frac{1}{10} < 1,$$

it follows that

$$D^T D - E + G_e = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \\ -\dfrac{1}{2} + \dfrac{1}{3} g(x_2) & \dfrac{1}{2} - \dfrac{1}{3} g(x_2) \end{bmatrix} \cdot \begin{bmatrix} \dfrac{1}{\sqrt{2}} & -\dfrac{1}{2} + \dfrac{1}{3} g(x_2) \\ \dfrac{1}{\sqrt{2}} & \dfrac{1}{2} - \dfrac{1}{3} g(x_2) \end{bmatrix}$$
$$- \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1/10 \end{pmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & -\dfrac{1}{2} + \dfrac{1}{10} + \dfrac{2}{9} g^2(x_2) - \dfrac{2}{3} g(x_2) \end{bmatrix}$$

is negative semi-definite. In accordance with Corollary 4. 6. 11, the zero solution of (4. 6. 8) is absolutely stable in [0,7/2].

4. 7. Notes

Theorems 4. 1. 3~4. 1. 5, 4. 2. 2~4. 2. 4 are all new results obtained by the author. Theorems 4. 3. 1, 4. 4. 1 are taken from Maygrin [1]. Theorem 4. 4. 2, Corollary 4. 4. 3, Theorem 4. 4. 1 are new results, too. Theorem 4. 5. 1 is due to Maygrin [1]. Theorems 4. 5. 2 and 4. 5. 3 are new results. The material of Section 4. 6 is taken from Liao Xiaoxin [10].

CHAPTER 5

CONTROL SYSTEMS WITH m NONLINEAR CONTROL TERMS

In this chapter, we will discuss the absolute stability of control system with m nonlinear terms.

5.1. Necessary and Sufficient Conditions for Absolute Stability

Consider the control system with m nonlinear control terms:

$$\begin{cases} \dfrac{dx}{dt} = Ax + \sum_{j=1}^{m} b_j f_j(\sigma_j), \\ \sigma_j = c_j^T x = \sum_{i=1}^{n} c_{ij} x_i, \qquad j = 1, \cdots, m, \end{cases} \tag{5.1.1}$$

where $A \in \mathbf{R}^{n\times n}$, $x = \mathrm{col}(x_1, \cdots, x_n)$, $b_j = \mathrm{col}(b_{1j}, \cdots, b_{nj})$, $c_j = \mathrm{col}(c_{1j}, \cdots, c_{nj})$,

$$f_j \in F \triangleq \{f: f(0) = 0,\ f(\sigma)\sigma > 0,\ \sigma \neq 0,$$
$$f(\sigma) \in [(-\infty, +\infty), \mathbf{R}]\}, \qquad j = 1, \cdots, m,$$

Re $\lambda(A) \leqslant 0$. Let

$$\Omega_i = \{x: \sigma_i = c_i^T x = 0\}, \qquad i = 1, \cdots, m,$$

$$\Omega = \left\{x: \|\sigma\| = \sum_{j=1}^{m} |\sigma_j| = \sum_{j=1}^{m} |c_j^T x| = 0\right\}.$$

Definition 5.1.1. The zero solution of (5.1.1) is said to be *absolutely stable for the set* Ω $[\Omega_j]$ if for any $f_j(\sigma_j) \in F$ $(j = 1, \cdots, m)$ and any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that if $\|x_0\| < \delta(\varepsilon)$, then the distance from the solution $x(t) \triangleq x(t, t_0, x_0)$ to the set Ω $[\Omega_j]$ satisfies

$$\rho(x, \Omega) \triangleq \sum_{j=1}^{m} |c_j^T x(t)| < \varepsilon \qquad [\rho(x, \Omega_j) = |c_j^T x(t)| < \varepsilon]$$

and such that

$$\lim_{t \to +\infty} \sum_{j=1}^{m} |c_j^T x(t)| = 0 \qquad \left[\lim_{t \to +\infty} |c_j^T x(t)| = 0\right]$$

for every $x_0 \in \mathbf{R}^n$.

Definition 5. 1. 2. The function $V(x) \in C[\mathbf{R}^n, \mathbf{R}]$ is said to be *positive definite with respect to the set* Ω $[\Omega_j]$ if

$$V(x)\begin{cases} = 0 & \text{for } x \in \Omega, \\ > 0 & \text{for } x \overline{\in} \Omega \end{cases} \qquad \left[V(x)\begin{cases} = 0 & \text{for } x \in \Omega_j, \\ > 0 & \text{for } x \overline{\in} \Omega_j. \end{cases}\right]$$

The function $V(x) \in C[\mathbf{R}^n, \mathbf{R}]$ is said to be *negative definite with respect to the set* Ω $[\Omega_j]$ if $-V(x)$ is positive definite for Ω $[\Omega_j]$.

Definition 5. 1. 3. The function $V(x) \in C[\mathbf{R}^n, \mathbf{R}]$ is said to be *radially unbounded positive definite for* Ω $[\Omega_j]$ if $V(x)$ is positive definite for Ω $[\Omega_j]$ and $V(x) \to +\infty$ as $\sum_{j=1}^{m} |\sigma_j| \to +\infty$ $[|\sigma_j| \to +\infty]$.

Theorem 5. 1. 4. *The necessary and sufficient conditions of absolute stability for the zero solution of* (5. 1. 1) *are*

1) $B \triangleq A + \sum_{j=1}^{m} \theta_j b_j c_j^T$ *is stable with* $\theta_j = 1$ *or* $\theta_j = 0$, $j = 1, \cdots, m$,

2) *the zero solution of* (5. 1. 1) *is absolutely stable for* Ω.

Proof. *Necessity.* 1) In the case Re $\lambda(A) < 0$, we take $\theta_j = 0$, $j = 1, \cdots, m$ and then $B = A$. B is obviously stable. In the case Re $\lambda(A) \leqslant 0$, we take some $\theta_j = 1$. Let in (5. 1. 1) $f_j(\sigma_j) = \sigma_j = c_j^T x$ $(j = 1, \cdots, m)$; then (5. 1. 1) can be transformed into

$$\frac{dx}{dt} = \left[A + \sum_{j=1}^{m} \theta_j b_j c_j^T\right] x.$$

Therefore,

$$B = A + \sum_{j=1}^{m} \theta_j b_j c_j^T \quad \text{is stable.}$$

2) For any $\varepsilon > 0$, taking

$$\tilde{\varepsilon} = \frac{\varepsilon}{\sum_{j=1}^{m} \| c_j^T \|},$$

there exists $\delta(\tilde{\varepsilon}) > 0$ such that if $\| x_0 \| < \delta(\tilde{\varepsilon})$, then

$$\| x(t) \| \triangleq \| x(t, t_0, x_0) \| \triangleq \sum_{j=1}^{n} |x_j(t)| < \tilde{\varepsilon} \qquad \text{for all} \quad t \geqslant t_0.$$

This implies

$$\sum_{j=1}^{m} \| c_j^T x(t) \| \leqslant \sum_{j=1}^{m} \| c_j^T \| \cdot \| x(t) \| < \sum_{j=1}^{m} \| c_j^T \| \tilde{\varepsilon} = \varepsilon$$

for all $t \geqslant t_0$.

Furthermore, we find $\lim_{t \to +\infty} \| x(t) \| = 0$ for every $x_0 \in \mathbf{R}^n$, and thus

$$0 \leqslant \lim_{t \to +\infty} \sum_{j=1}^{m} \| c_j^T x(t) \| \leqslant \sum_{j=1}^{m} \| c_j^T \| \cdot \lim_{t \to +\infty} \| x(t) \| = 0.$$

Consequently, the zero solution of (5.5.1) is absolutely stable for Ω. The necessity is proved.

Sufficiency. In accordance with the formula of variation of constant, the solution $x(t) \triangleq x(t, t_0, x_0)$ of (5.1.1) satisfies

$$x(t) = e^{B(t-t_0)}x_0 + \int_{t_0}^{t} e^{B(t-\tau)}\left[\sum_{j=1}^{m} b_j f_j(\sigma_j(\tau)) - \sum_{j=1}^{m} \theta_j b_j \sigma_j(\tau)\right] d\tau.$$

Since B is stable, there exist constants $M \geqslant 1$ and $\alpha > 0$ such that

$$\| e^{B(t-t_0)} \| \leqslant M e^{-\alpha(t-t_0)}, \qquad t \geqslant t_0.$$

We define $\sigma_j(t) = \sigma_j(t, t_0, x_0)$. Since $\sigma = \sum_{j=1}^{m} |\sigma_j(t)| \to 0$ as $t \to +\infty$, we have $\lim_{t \to +\infty} \sigma_j(t) = 0$. Because $\sigma_j(t)$ continuously depends on the initial value x_0 and $f_j(\sigma_j(t))$ is a composite continuous function of x_0 and $f_j(\sigma_j(t)) \to +\infty$ as $t \to +\infty$, for any $\varepsilon > 0$ there exist $\delta(\varepsilon) > 0$ and $t_1 > t_0$ such that $\| x_0 \| < \delta(\varepsilon)$ implies

$$\| e^{B(t-t_0)} x_0 \| \leqslant \| e^{B(t-t_0)} \| \, \| x_0 \| < \frac{\varepsilon}{3},$$

$$\int_{t_0}^{t_1} M e^{-\alpha(t-\tau)}\left[\sum_{j=1}^{m} \| b_j f_j(\sigma_j(\tau)) \| + \sum_{j=1}^{m} \theta_j \| b_j \sigma_j(\tau) \|\right] d\tau < \frac{\varepsilon}{3},$$

$$\int_{t_1}^{t} M e^{-\alpha(t-\tau)}\left[\sum_{j=1}^{m} \| b_j f_j(\sigma_j(\tau)) \| + \sum_{j=1}^{m} \theta_j \| b_j \sigma_j(\tau) \|\right] d\tau < \frac{\varepsilon}{3}.$$

Thus, we have

$$\begin{aligned} \| x(t) \| \leqslant{} & \| e^{B(t-t)} x_0 \| \\ & + \int_{t_0}^{t_1} M e^{-\alpha(t-\tau)}\left[\sum_{j=1}^{m} \| b_j f_j(\sigma_j(\tau)) \| + \sum_{j=1}^{m} \theta_j \| b_j \sigma_j(\tau) \|\right] d\tau \\ & + \int_{t_1}^{t} M e^{-\alpha(t-\tau)}\left[\sum_{j=1}^{m} \| b_j f_j(\sigma_j(\tau)) \| + \sum_{j=1}^{m} \theta_j \| b_j \sigma_j(\tau) \|\right] d\tau \\ <{} & \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

For any $x_0 \in \mathbf{R}^n$, by the de L'Hospital rule, we get

$$\begin{aligned} 0 & \leqslant \lim_{t \to +\infty} \| x(t) \| \\ & \leqslant \lim_{t \to +\infty} M e^{-\alpha(t-t_0)} \\ & \quad + \lim_{t \to +\infty} \int_{t_0}^{t} M e^{-\alpha(t-\tau)}\left[\sum_{j=1}^{m} \| b_j f_j(\sigma_j(\tau)) \| + \sum_{j=1}^{m} \theta_j \| b_j \sigma_j(\tau) \|\right] d\tau \\ & = 0. \end{aligned}$$

Thus, the zero solution of (5.1.1) is absolutely stable. The proof of the theorem is completed. ∎

Theorem 5.1.5. *The zero solution of* (5.1.1) *is absolutely stable if and* *only if*

1) $A + \sum_{j=1}^{m} \theta_j b_j c_j^T \triangleq B$ *is stable, where* $\theta_j = 0$ *or* $\theta_j = 1$, $j = 1, \cdots, m$;

2) *there exists a differential function* $V_f \in [\mathbf{R}^n, \mathbf{R}]$, *where* $V_f(x)$ *is radially unbounded positive definite for* Ω, *i. e.*, *there exist* $\varphi_f \in KR$ *and* $\psi_f \in K$ *such that*

$$V_f(x) \geqslant \varphi_f(\sigma),$$

$$\left.\frac{\mathrm{d}^+ V}{\mathrm{d}t}\right|_{(5.1.1)} \leqslant -\psi_f(\sigma). \tag{5.1.2}$$

Proof. *Sufficiency.* It is sufficient to prove that the condition 2) implies that the zero solution of (5. 1. 1) is absolutely stable for Ω.

Since $V_f(0) = 0$, $0 \in \Omega$, and $V_f(x)$ is a continuous function of x, for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$V_f(x_0) < \varphi_f(\varepsilon) \qquad \text{for} \quad \| x_0 \| < \delta(\varepsilon).$$

It follows from (5. 1. 2) that

$$\varphi_f(|\sigma(t)|) \leqslant V_f(x(t)) \leqslant V_f(x_0) \leqslant \varphi_f(\varepsilon),$$

and therefore $|\sigma(t)| < \varepsilon$. Thus, the zero solution of (5. 1. 1) is stable for Ω.

Now, we prove that $\lim\limits_{t \to +\infty} \sigma(t, t_0, x_0) = 0$ for any $x_0 \in \mathbf{R}^n$.

Since $V_f(x(t)) \triangleq V_f(t)$ is a monotone decreasing and bounded function,

$$\inf_{t \geqslant t_0} V_f(x(t)) \triangleq \lim_{t \to +\infty} V_f(x(t)) \triangleq \alpha \geqslant 0.$$

Now, we prove that α can be reached only in Ω. If it can be reached outside Ω, then there must exist a constant $\beta > 0$ such that $|\sigma(t)| \geqslant \beta > 0$, $t_0 \leqslant t < +\infty$, or there must exist a sequence $\{t_k\}$ with $t_k \to +\infty$ for $k \to +\infty$ such that $\lim\limits_{t_k \to +\infty} \sigma(t_k) = 0$. As a result,

$$\alpha = \lim_{t_k \to +\infty} V_f(t_k) = \lim_{\substack{t_k \to +\infty \\ \sigma(t_k) \to 0}} V_f(t_k).$$

In other words, α can be reached in Ω. This conclusion is in contradiction with the preassumption that α can be reached outside Ω.

For any $x_0 \in \mathbf{R}^n$, the expression (5. 1. 2) gives

$$|\sigma(t)| \leqslant |\sigma(t_0)| \triangleq h < H < +\infty.$$

Now, let us prove that $\lim\limits_{t \to +\infty} \sigma(t) = 0$. Assume that $\lim\limits_{t \to +\infty} \sigma(t) \neq 0$. Since $\sigma(t)$ is uniformly continuous, there exist constants $\beta > 0$, $\eta > 0$ and point sequence $\{t_j\}$ such that $|\sigma(t)| \geqslant \beta$ for $t \in [t_j - \eta, t_j + \eta]$, $j = 1, 2, \cdots$.

Setting $r_f = \inf\limits_{\beta \leqslant \sigma \leqslant h} \psi_f(\sigma)$, we deduce

$$
\begin{aligned}
0 \leqslant V_f(t) &\leqslant V_f(t_0) + \int_{t_0}^{t} \frac{\mathrm{d}V_f}{\mathrm{d}t}\mathrm{d}t \\
&\leqslant V_f(t_0) - \int_{t_0}^{t} \psi_f(|\sigma(\tau)|)\mathrm{d}\tau \\
&\leqslant V_f(t_0) - \sum_{j=1}^{n}\int_{t_j-\eta}^{t_j+\eta} \psi_f(|\sigma(\tau)|)\mathrm{d}\tau \\
&\leqslant V_f(t_0) - 2n\eta V_f \to -\infty \qquad \text{as} \quad n \to +\infty.
\end{aligned}
$$

This yields contradiction; thus $\lim\limits_{t\to+\infty} \sigma(t) = 0$. This proves that the zero soluion of (5. 1. 1) is absolutely stable for Ω. The sufficiency is proved.

Necessity. Since the zero solution of (5. 1. 1) is absolutely stable, $\mathbf{R}^n$ is an attractive space. For any $f_j \in F$ $(j = 1,\cdots,m)$ and any $x \in \mathbf{R}^n$, let

$$W_f(x) \triangleq \sup\{\, \| x_f(t,0,x) \|^2, t \geqslant 0\},$$

where $x_f(t)$ denotes a solution of (5. 1. 1). From the Theorem 2. 2 of Bhatia and Szegö [1], we know that $W_f(x)$ has the following properties:

1) $W_f(x) \geqslant 0$, and $W_f(x) = 0$ if and only if $x = 0$, $W_f(x)$ is radially unbounded positive definite;

2) $W_f(x)$ is a monotone decreasing function;

3) $W_f(x)$ is continous in $\mathbf{R}^n$.

Furthermore, we define

$$V_f(x) \triangleq \int_0^{+\infty} W_f(x_f(\eta,0,x))\mathrm{e}^{-\eta}\mathrm{d}\eta.$$

Obviously, $V_f(x)$ is radially unbounded positive definite. Thus, there exists $\bar{\varphi}_f \in KR$ such that

$$V_f(x) \geqslant \bar{\varphi}_f(\| x \|).$$

Let

$$\Phi = \int_0^{t+\eta} W_f(x_f(\xi))\mathrm{d}\xi.$$

It follows that

$$\Phi_\eta' = \Phi_t' = W_f(x_f(t+\eta)).$$

Integrating by parts

$$
\begin{aligned}
V_f(x_f(t)) &= \int_0^{+\infty} \mathrm{e}^{-\eta}\mathrm{d}\Phi = \mathrm{e}^{-\eta}\int_0^{t+\eta} W_f(x_f(\xi))\mathrm{d}\xi \Big|_0^{+\infty} + \int_0^{+\infty} \Phi(t+\eta)\mathrm{e}^{-\eta}\mathrm{d}\eta \\
&= -\int_0^{t} W_f(x_f(\xi))\mathrm{d}\xi + \int_0^{+\infty} \Phi(t+\eta)\mathrm{e}^{-\eta}\mathrm{d}\eta.
\end{aligned}
$$

Since $W_f(x_f(t))$ is a monotone nonincreasing function, $W_f(x_f(t))$ is bounded. Furthermore, we note that

$$\lim_{\eta\to+\infty} e^{-\eta}\int_0^{t+\eta} W_f(x_f(\xi))\mathrm{d}\xi = 0,$$

$$\begin{aligned}\left.\frac{\mathrm{d}V_f}{\mathrm{d}t}\right|_{(5.1.1)} &= -W_f(x_f(t)) + \int_0^{+\infty}\Phi_t' e^{-\eta}\mathrm{d}\eta \\ &= -W_f(x_f(t)) + \int_0^{+\infty} W_f(x_f(t+\eta))e^{-\eta}\mathrm{d}\eta \\ &= \int_0^{+\infty}[W_f(x_f(t+\eta)) - W_f(x_f(t))]\mathrm{d}\eta.\end{aligned}$$

Since $W_f(x_f(t))$ is a monotone nonincreasing function, we abtain

$$W_f(x_f(t)) \geqslant W_f(x_f(t+\eta)), \qquad \eta \geqslant 0.$$

In particular, if $x(t)$ is a non-zero solution of (5. 1. 1),

$$W_f(x_f(t)) \not\equiv W_f(x_f(t+\eta)),$$

or

$$W_f(x_f(t)) \equiv W_f(x_f(t+\eta)) \to 0 \qquad \text{as} \quad \eta \to +\infty.$$

Thus $W_f(x_f(t)) \equiv 0$, which contradicts

$$W_f(x) = \sup\{\|x_f(t,0,x)\|^2, t \geqslant 0\} \not\equiv 0.$$

Therefore, if $x_f(t) \not\equiv 0$, then

$$\int_0^{+\infty}[W_f(x_f(t+\eta)) - W_f(x_f(t))]e^{-\eta}\mathrm{d}\eta < 0,$$

i. e. ,

$$\left.\frac{\mathrm{d}V_f}{\mathrm{d}t}\right|_{(5.1.1)} < 0 \qquad \text{for} \quad x \neq 0.$$

Therefore, we conclude

$$\left.\frac{\mathrm{d}V_f}{\mathrm{d}t}\right|_{(5.1.1)} \leqslant -u_f(x),$$

with $u_f(x)$ being a positive definite function. Thus we have

$$\begin{aligned} u_f(x) &\geqslant \tilde{\varphi}_f(\|x\|) \triangleq \tilde{\varphi}_f\left(\sum_{i=1}^n |x_i|\right) \\ &\geqslant \tilde{\varphi}_f\left(\frac{1}{m}\sum_{j=1}^n \frac{1}{\max\limits_{1\leqslant i,j\leqslant n}|c_{ij}|}\sum_{i,j=1}^n |c_{ij}x_j|\right) \\ &\geqslant \tilde{\varphi}_f\left(\frac{1}{m}\frac{1}{\max\limits_{1\leqslant i,j\leqslant n}|c_{ij}|}\sum_{i,j=1}^n |c_{ij}x_j|\right) \\ &\triangleq \varphi_f(\sigma) \in KR. \end{aligned}$$

Hence, $u_f(x)$ is radially unbounded positive definite for Ω. Further, we have

$$\left.\frac{\mathrm{d}V_f}{\mathrm{d}t}\right|_{(5.1.1)} \leqslant -u_f(x) \leqslant -\tilde{\varphi}_f(\|x\|) \leqslant -\varphi_f(\sigma), \qquad \varphi_f \in K.$$

The condition 2) of Theorem 5.1.5 is satisfied; the condition 1) of this theorem is trivially satisfied as well. The necessity is proved. ∎

Theorem 5.1.6. *The zero solution of* (5.1.1) *is absolutely stable if and only if*

1) *the condition* 1) *of Theorem* 5.1.4 *is satisfied*;

2) *for any* $f_j \in F$ $(j=1,\cdots,m)$, *there exist* m *Liapunov functions* $V_f^{(j)}(x) \in [\mathbf{R}^n,\mathbf{R}]$ $(j=1,\cdots,m)$ *such that*

$$V_f^{(j)}(x) \geqslant \varphi_f^{(j)}(|\sigma_j|), \qquad \varphi_f^{(j)} \in KR, \quad j=1,\cdots,m,$$

$$\left.\frac{\mathrm{d}V_f^{(j)}}{\mathrm{d}t}\right|_{(5.1.1)} \leqslant -\psi_f^{(j)}(|\sigma_j|), \qquad \psi_f^{(j)} \in K, \quad j=1,\cdots,m.$$

Proof. *Necessity.* Theorem 5.1.5 guarantees that the condition 1) is satisfied, and that there exists $V_f(x) \geqslant \varphi_f(\sigma) \in KR$ such that

$$\left.\frac{\mathrm{d}V_f(x)}{\mathrm{d}t}\right|_{(5.1.1)} \leqslant -\psi_f(\sigma), \qquad \varphi_f \in K.$$

Take $V_f^{(j)} = V_f(x)$, $j=1,\cdots,m$. Due to

$$V_f^{(j)} = V_f(x) \geqslant \varphi_f(\sigma) = \varphi_f\left(\sum_{j=1}^{m}|c_j^{\mathrm{T}}x|\right) \geqslant \varphi_f(|c_j^{\mathrm{T}}x|)$$
$$= \varphi_f(|\sigma_j|) \in KR, \qquad j=1,\cdots,m,$$

we have

$$\left.\frac{\mathrm{d}V_f^{(j)}}{\mathrm{d}t}\right|_{(5.1.1)} = \left.\frac{\mathrm{d}V_f}{\mathrm{d}t}\right|_{(5.1.1)} \leqslant -\psi_f(\sigma) = -\psi_f\left(\sum_{j=1}^{m}|c_j^{\mathrm{T}}x|\right)$$
$$\leqslant -\psi_f(|c_j^{\mathrm{T}}x|) = -\psi_j(\sigma_j), \qquad \psi_j \in K, \quad j=1,\cdots,m.$$

This verifies the necessity.

Sufficiency. Similarly to the proof of sufficiency for Theorem 5.1.5, the condition 2) can be proved to imply that the zero solution of (5.1.1) is absolutely stable for Ω_j, $j=1,\cdots,m$. Then, as in Theorem 5.1.4 one can show that the zero solution of (5.1.1) is absolutely stable. This verifies the sufficiency. ∎

Theorem 5.1.7. *Let the following conditions be satisfied*:

1) *the condition* 1) *of Theorem* 5.1.3 *holds*;

2) *there exist an* $n\times n$ *real symmetric matrix* B *and constants* $\beta_i \geqslant 0$ $(i=1,\cdots,m)$, $\alpha>0$ *such that*

$$V(x) = x^{\mathrm{T}}Bx + \sum_{j=1}^{m}\beta_j\int_0^{\sigma_j} f_j(\sigma_j)\mathrm{d}\sigma_j$$

with

$$x^{\mathrm{T}}Bx \geqslant \alpha\sum_{i=1}^{n}x_i^2$$

or

$$V(x) \geqslant \sum_{j=1}^{m}\beta_i\int_0^{\sigma_j} f_j(\sigma_j)\mathrm{d}\sigma_j,\ \ \beta_j > 0,\ \ \int_0^{+\infty} f(\sigma_j)\mathrm{d}\sigma_j = +\infty,\ \ j = 1,\cdots,m;$$

3) $$\frac{\mathrm{d}V}{\mathrm{d}t}\bigg|_{(5.1.1)} \leqslant -\varepsilon\tau,\quad \tau \in \left\{\sigma^2, \sum_{j=1}^{m}\sigma_j f_j(\sigma_j), \sum_{j=1}^{m} f_j^2(\sigma_j)\right\}.$$

Then the zero solution of (5. 1. 1) *is absolutely stable.*

Proof. It is sufficient to prove that the conditions 2) and 3) of this theorem imply the condition of Theorem 5. 1. 5.

In fact, for the Liapunov function

$$V(x) = x^{\mathrm{T}}Bx + \sum_{j=1}^{m}\beta_j\int_0^{\sigma_j} f_j(\sigma_j)\mathrm{d}\sigma_j,$$

the condition 2) implies

$$V(x) \geqslant \alpha\sum_{i=1}^{m}\sigma_j^2 + \sum_{j=1}^{m}\beta_j\int_0^{\sigma_j} f_j(\sigma_j)\mathrm{d}\sigma_j = \varphi(\sigma) \in KR$$

or

$$V(x) \geqslant \sum_{j=1}^{m}\beta_j\int_0^{\sigma_j} f_j(\sigma_j)\mathrm{d}\sigma_j \triangleq \varphi(\sigma) \in KR,$$

$$\frac{\mathrm{d}V}{\mathrm{d}t}\bigg|_{(5.1.1)} = -\varepsilon\tau \triangleq -\psi(\sigma),\qquad \psi \in K.$$

Therefore the conditions of Theorem 5. 1. 5 are satisfied. Theorem 5. 1. 7 is proved. ∎

Corollary 5. 1. 8. *Suppose that there exist constants* $\beta_j \geqslant 0$ $(j = 1,\cdots,m)$ *and a symmetric positive definite matrix P such that the function*

$$V(x) = x^{\mathrm{T}}Px + \sum_{j=1}^{m}\beta_j\int_0^{\sigma_j} f_j(\sigma_j)\mathrm{d}\sigma_j$$

satisfies $\frac{\mathrm{d}V}{\mathrm{d}t}\Big|_{(5.1.1)} < 0$, $x \neq 0$. *Then the zero solution of* (5. 1. 1) *is absolutely stable.*

Proof. It is sufficient to prove that $V(x)$ is radially unbounded positive definite for Ω, while $\frac{\mathrm{d}V}{\mathrm{d}t}\Big|_{(5.1.1)}$ is negative definite for Ω.

Let $\bar{c} = \max\limits_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant m}} |c_{ij}|$, $\underline{\lambda} = \min\limits_{1\leqslant i\leqslant n}\lambda_i(P)$, λ_i being enginvalues of P. Then

$$V(x) \geqslant x^{\mathrm{T}}Px \geqslant \underline{\lambda}x^{\mathrm{T}}x \geqslant \underline{\lambda}\frac{\sum\limits_{j=1}^{m} n\sum\limits_{i=1}^{n}|c_{ij}x_j|}{nm\bar{c}} \geqslant \underline{\lambda}\frac{\sum\limits_{j=1}^{m}\sigma_j^2}{nm\bar{c}} \triangleq \varphi(|\sigma|) \in KR.$$

Therefore, $V(x)$ is radially unbounded positive definite for Ω, and

$$\frac{\mathrm{d}V}{\mathrm{d}t}\bigg|_{(5.1.1)} \leqslant -\psi(\|x\|) \leqslant -\psi\left(\frac{1}{m\bar{c}}\sum_{j=1}^{m}|\sigma_j|\right) \triangleq -\psi_1(\sigma),\qquad \psi_1 \in K.$$

Thus $\left.\frac{dV}{dt}\right|_{(5.1.1)}$ is negative definite for Ω. The conditions of Theorem 5. 1. 7 are satisfied. The proof of Corollary 5. 1. 8 is completed. ∎

5. 2. Some Simple Sufficient Conditions for Absolute Stability

Without loss of generality, we assume that $c_i = \mathrm{col}(c_{i1}, \cdots, c_{in})$ $(i = 1, \cdots, m)$ are linearly independent. By an n-dimensional full-rank linear transformation, (5. 1. 1) can be transformed into the following form:

$$\frac{dx}{dt} = Ax + \sum_{j=n-m+1}^{n} b_j f_j(x_j), \qquad (5.2.1)$$

or into the vector component form:

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij} x_j + \sum_{j=n-m+1}^{n} b_{ij} f_j(x_j), \qquad j = 1, \cdots, m. \qquad (5.2.2)$$

Theorem 5. 2. 1. *Suppose that*

1) $A = (a_{ij})_{n\times n}$ *is stable*,

2) *there exist constants* $r_i \geqslant 0$ $(i = 1, \cdots, n-m)$, $r_i > 0$ $(i = n-m+1, \cdots, n)$ *such that*

$$\begin{cases} -r_j a_{jj} + \sum\limits_{\substack{i=1\\ i\neq j}}^{n} r_i |a_{ij}| \leqslant 0, & j = 1, \cdots, n-m, \\ -r_j a_{jj} + \sum\limits_{\substack{i=1\\ i\neq j}}^{n} r_i |a_{ij}| < 0, & j = n-m+1, \cdots, n, \\ -r_j b_{jj} + \sum\limits_{\substack{i=1\\ i\neq j}}^{n} r_i |b_{ij}| \leqslant 0, & j = n-m+1, \cdots, n, \end{cases}$$

or

$$\begin{cases} -c_j a_{jj} + \sum\limits_{\substack{i=1\\ i\neq j}}^{n} c_i |a_{ij}| \leqslant 0, & j = 1, \cdots, n-m, \\ -c_j a_{jj} + \sum\limits_{\substack{i=1\\ i\neq j}}^{n} c_i |a_{ij}| \leqslant 0, & j = n-m+1, \cdots, n, \\ -c_j b_{jj} + \sum\limits_{\substack{i=1\\ i\neq j}}^{n} c_i |b_{ij}| < 0, & j = n-m+1, \cdots, n. \end{cases}$$

Then the zero solution of the system (5. 2. 1) *is absolutely stable.*

Proof. We construct the Liapunov function

$$V = \sum_{i=1}^{n} c_i |x_i|.$$

Obviously, we have

$$V = \sum_{i=1}^{n} c_i |x_i| \geqslant \sum_{i=n-m+1}^{n} c_i |x_i| \to +\infty \quad \text{as} \quad \sum_{i=n-m+1}^{n} |x_i| \to +\infty.$$

Thus, V is radially unbounded positive definite for $x_{n-m+1}, \cdots, x_n$. Since

$$\begin{aligned} D^+ V(x)|_{(5.2.1)} &\leqslant \sum_{j=1}^{n} \Big[c_j a_{jj} + \sum_{\substack{i=1 \\ i\neq j}}^{n} c_i |a_{ij}| \Big] |x_i| \\ &\quad + \sum_{l=n-m+1}^{n} \Big[c_l b_{ll} + \sum_{\substack{i=1 \\ i\neq l}}^{n} |c_i b_{il}| \Big] |f_l(x_l)| \\ &< 0 \quad \text{for} \quad \sum_{j=n-m+1}^{n} |x_j| \neq 0, \end{aligned}$$

the zero solution of (5. 2. 1) is absolutely stable for $x_{n-m+1}, \cdots, x_n$. Since the matrix A is stable, there exist $M \geqslant 1$ and $\alpha > 0$ such that

$$\| e^{A(t-t_0)} \| \leqslant M e^{-\alpha(t-t_0)}.$$

The solution of (5. 2. 1) can be expressed in the following form:

$$x(t, t_0, x_0) = e^{A(t-t_0)} x_0 + \int_{t_0}^{t} e^{A(t-\tau)} \sum_{j=1}^{m} b_j f_j(x_{n-m+j}(\tau)) d\tau.$$

Following the proof of Theorem 5. 1. 1, we can prove that the zero solution of (5. 2. 1) is absolutely stable. ∎

Let us denote

$$\tilde{a}_{ij} = \begin{cases} a_{ij}, & i = 1, \cdots, n, \quad j = 1, \cdots, n-m, \\ b_{ij}, & i = 1, \cdots, n, \quad j = n-m+1, \cdots, n. \end{cases}$$

Theorem 5. 2. 2. *The zero solution of* (5. 2. 1) *is absolutely stable if either of the following two sets of conditions is satisfied:*

1) $a_{ii} < 0$, $i = 1, \cdots, n$, *the matrix* $((-1)^{\delta_{ij}} |a_{ij}|)_{n\times n}$ *is stable,*

2) *there exists a constant* $k > 0$ *such that*

$$\begin{cases} k b_{ll} \leqslant a_{ll}, & l = n-m+1, \cdots, n, \\ k|b_{il}| \leqslant |a_{il}|, & i = 1, \cdots, n, \ i \neq l, \quad l = n-m+1, \cdots, n; \end{cases}$$

or

1)′ $a_{ii} < 0$, $i = 1, \cdots, n$, *the matrix* $((-1)^{\delta_{ij}} |\tilde{a}_{ij}|)_{n\times n}$ *is stable,*

2)′ *there exists a constant* $r > 0$ *such that*

$$\begin{cases} r a_{ll} \leqslant b_{ll}, & l = n-m+1, \cdots, n, \\ r|a_{il}| \leqslant |b_{il}|, & i = 1, \cdots, n, \ i \neq l, \quad l = n-m+1, \cdots, n. \end{cases}$$

Proof. Since the condition 1) implies that $-((-1)^{\delta_{ij}} |a_{ij}|)_{n\times n}$ is an M-matrix, there exist constants $c_i > 0$ $(i = 1, \cdots, n)$ such that

$$c_i a_{ij} + \sum_{\substack{i=1\\ i\neq j}}^{n} |c_i a_{ij}| < 0, \qquad j = 1, \cdots, n.$$

Condition 2) implies

$$c_l b_{ll} + \sum_{\substack{i=1\\ i\neq l}}^{n} |c_i b_{il}| \leqslant \left(c_l a_{ll} + \sum_{i=1}^{n} c_i |a_{il}| \right) \Big/ k < 0, \qquad l = n - m + 1, \cdots, n.$$

We construct the Liapunov function

$$V = \sum_{i=1}^{n} c_i |x_i|$$

and we deduce that

$$\begin{aligned} D^+ V|_{(5.2.1)} &\leqslant \sum_{j=1}^{n} \Big[c_j a_{jj} + \sum_{\substack{i=1\\ i\neq j}}^{n} |c_i a_{ij}| \Big] |x_j| \\ &\quad + \sum_{l=n-m+1}^{n} \Big[c_l b_{ll} + \sum_{\substack{i=1\\ i\neq l}}^{n} |c_i b_{il}| \Big] |f_l(x_{n-m+l})| \\ &< 0 \qquad \text{for} \quad x \neq 0. \end{aligned}$$

Consequently, the zero solution of (5. 2. 1) is absolutely stable.

The sufficiency of 1)′ and 2)′ can be proved analogously. ∎

Example 5. 2. 3. Consider the problem of absolute stability of the system

$$\begin{cases} \dfrac{dx_1}{dt} = - x_1 - 2f_1(x_1) + 2f_2(x_2), \\[2mm] \dfrac{dx_2}{dt} = - x_2 + 2f_1(x_1) - 2f_2(x_2), \end{cases} \qquad f_1, f_2 \in F. \tag{5. 2. 3}$$

We choose the Liapunov function

$$V = |x_1| + |x_2|.$$

Thus,

$$\begin{aligned} D^+ V|_{(5.2.3)} &\leqslant - |x_1| - |x_2| + [-2 + 2]|f_1(x_1)| \\ &\quad + [-2 + 2]|f_2(x_2)| \\ &\leqslant - |x_1| - |x_2| < 0, \qquad x \neq 0 \end{aligned}$$

and the zero solution of (5. 2. 3) is absolutely stable.

Example 5. 2. 4. Consider the problem of absolute stability of the system

$$\begin{cases} \dfrac{dx_1}{dt} = - 4x_1 + 2x_2 + x_3 + 2f_1(x_2) + 2f_2(x_3), \\[2mm] \dfrac{dx_2}{dt} = 2x_1 - 3x_2 + x_3 - 4f_1(x_2) - 2f_2(x_3), \\[2mm] \dfrac{dx_3}{dt} = - 2x_1 + x_2 - 3x_3 + f_1(x_2) - 6f_2(x_3), \end{cases} \tag{5. 2. 4}$$

where f_1, $f_2 \in F$, and

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} -4 & 2 & 1 \\ 2 & -3 & 1 \\ -2 & 1 & -3 \end{bmatrix},$$

$$\begin{bmatrix} b_{12} & b_{13} \\ b_{22} & b_{23} \\ b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -4 & -2 \\ 1 & -6 \end{bmatrix}$$

It is easy to prove that the matrix $((-1)^{\delta_{ij}}|a_{ij}|)$ is stable. Furthermore, there exist

$$0 > a_{22} > b_{22}, \qquad |a_{i2}| = |b_{i2}|, \qquad i = 1,3,$$

$$0 > a_{33} = \frac{1}{2}b_{33}, \qquad |a_{i3}| = \frac{1}{2}|b_{i3}|, \qquad i = 1,2.$$

Thus, the conditions of Theorem 5. 2. 2 are satisfied. The zero solution of (5. 2. 4) is absolutely stable.

Let

$$\mathscr{F}_{ij}(x_j) = \begin{cases} a_{ij}x_j + b_{ij}f(x_j), & i = 1,\cdots,n, \quad j = n-m+1,\cdots,n, \\ a_{ij}x_j, & i = 1,\cdots,n, \quad j = 1,\cdots,n-m. \end{cases}$$

The system (5. 2. 2) can be rewritten as

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \sum_{j=1}^{n}\mathscr{F}_{ij}(x_j), \qquad i = 1,\cdots,n. \tag{5. 2. 5}$$

Theorem 5. 2. 5. *Let the following conditions be satisfied:*

1) $A=(a_{ij})_{n\times n}$ *is stable;*

2) $\int_0^{\pm\infty}\mathscr{F}_{ii}(x_i)\mathrm{d}x_i = -\infty$, $\mathscr{F}_{ii}(x_i)x_i < 0$ *for* $x_i \neq 0$, $i = 1,\cdots,n$;

3) *there exist constants* $c_i \geqslant 0$ $(i=1,\cdots,n-m)$, $c_i > 0$ $(i=n-m+1,\cdots,n)$, $\varepsilon > 0$, *such that the matrix*

$$G(g_{ij})_{n\times n} + \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon E_{m\times m} \end{pmatrix}$$

is negative semi-definite, where

$$g_{ij}(x) = \begin{cases} -\dfrac{1}{2}\left(\dfrac{c_i\mathscr{F}_{ij}(x_j)}{\mathscr{F}_{jj}(x_j)} + \dfrac{c_j\mathscr{F}_{ji}(x_i)}{\mathscr{F}_{ii}(x_i)}\right), & x_ix_j \neq 0, \\ 0, & x_ix_j = 0, \end{cases} \qquad i,j = 1,\cdots,n.$$

Then the zero solution of (5. 2. 5) *is absolutely stable.*

Proof. We construct the radially unbounded positive definite Liapunov function

$$V(x) = -\sum_{i=1}^{n}\int_0^{x_i} c_i\mathscr{F}_{ii}(x_i)\mathrm{d}x_i.$$

Following the proof of Theorem 5.1.1, we can prove that $\left.\frac{dV}{dt}\right|_{(5.2.5)}$ is negative definite for $x_{n-m+1},\cdots,x_n$. Thus, the zero solution of (5.2.5) is absolutely stable for $x_{n-m+1},\cdots,x_n$, i.e., the zero solution of (5.2.1) is absolutely stable for $x_{n-m+1},\cdots,x_n$.

Using the fact that the matrix A is stable, the solution of (5.2.1) can be expressed:

$$x(t,t_0,x_0) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)} \sum_{j=n-m+1}^{n} b_j f_j(x_j(\tau))d\tau.$$

As in the proof of Theorem 5.5.1, we can prove that the zero solution of (5.2.1) is absolutely stable. Consequently, the zero solution of (5.2.5) is absolutely stable. ∎

Theorem 5.2.6. *Let the following conditions be satisfed:*

1) *the conditions* 1) *and* 2) *of Theorem* 5.2.5 *are satisfied;*

2) $\left|\frac{\tilde{f}_{ij}(x_j)}{\tilde{f}_{jj}(x_j)}\right| \leqslant b_{ij}$;

3)

$$\tilde{B} = \begin{bmatrix} 1 & -b_{21} & \cdots & -b_{n1} \\ -b_{12} & 1 & \cdots & -b_{n2} \\ \vdots & \vdots & & \vdots \\ -b_{1n} & -b_{2n} & \cdots & 1 \end{bmatrix} \triangleq \begin{bmatrix} \tilde{B}_{11} & \tilde{B}_{12} \\ \tilde{B}_{21} & \tilde{B}_{22} \end{bmatrix},$$

with $\tilde{B}_{11}$, $\tilde{B}_{22}$ *and* $E - \tilde{B}_{22}^{-1}\tilde{B}_{21}\tilde{B}_{11}^{-1}\tilde{B}_{12}$ *being M-matrices, where* $\tilde{B}_{11}$, $\tilde{B}_{12}$, $\tilde{B}_{21}$, $\tilde{B}_{22}$ are $(n-m)\times(n-m)$, $(n-m)\times m$, $m\times(n-m)$, $m\times m$ *matrices respectively.*

Then the zero solution of (5.2.5) *is absolutely stable.*

Proof. For any $\xi = \mathrm{col}(\xi_1,\cdots,\xi_{n-m}) \geqslant 0$ and any $\eta = \mathrm{col}(\eta_1,\cdots,\eta_m) > 0$, we consider the linear equations

$$\begin{cases} \tilde{B}_{11}c + \tilde{B}_{12}\tilde{c} = \xi, \\ \tilde{B}_{21}c + \tilde{B}_{22}\tilde{c} = \eta. \end{cases} \tag{5.2.6}$$

Obviously, (5.2.6) is equivalent to

$$c = -\tilde{B}_{11}^{-1}\tilde{B}_{12}\tilde{c} + \tilde{B}_{11}^{-1}\eta, \tag{5.2.7}$$

$$\tilde{c} = \tilde{B}_{22}^{-1}\eta + \tilde{B}_{22}^{-1}\tilde{B}_{21}\tilde{B}_{11}^{-1}\tilde{B}_{12}\tilde{c} - \tilde{B}_{22}^{-1}\tilde{B}_{21}\tilde{B}_{11}^{-1}\xi. \tag{5.2.8}$$

Since $\xi \geqslant 0$, $\tilde{B}_{11}$ and $\tilde{B}_{22}$ are M-matrices, we have $\tilde{B}_{11}^{-1} \geqslant 0$, $\tilde{B}_{22}^{-1} \geqslant 0$. Taking $\tilde{B}_{21} \leqslant 0$, we have

$$-\tilde{B}_{22}^{-1}\tilde{B}_{21}\tilde{B}_{11}^{-1} \geqslant 0, \qquad \tilde{B}_{22}^{-1}\eta > 0.$$

Recalling that $E - \tilde{B}_{22}^{-1}\tilde{B}_{21}\tilde{B}_{11}^{-1}\tilde{B}_{12}$ is an M-matrix, we know that there exist a positive solution of (5.2.8) for $\tilde{c}$ and a non-negative solution of (5.2.7) for c.

We choose the Liapunov function

$$V = \sum_{i=1}^{n} c_i |x_i| \geqslant \sum_{i=n-m+1}^{n} c_i |x_i|.$$

Then

$$\begin{aligned}
D^+ V|_{(5.2.5)} &\leqslant \sum_{j=1}^{n}\Big[-c_j|\tilde{f}_{jj}(x_j)| + \sum_{\substack{i=1\\ i\neq j}}^{n} c_i|\tilde{f}_{ij}(x_j)|\Big] \\
&\leqslant \sum_{j=n-m+1}^{n}\Big[-c_j|\tilde{f}_{jj}(x_j)| + \sum_{\substack{i=1\\ i\neq j}}^{n} c_i|\tilde{f}_{ij}(x_j)|\Big] \\
&\leqslant \sum_{j=n-m+1}^{n}\Big[-c_j + \sum_{\substack{i=1\\ i\neq j}}^{n} c_i b_{ij}\Big]|\tilde{f}_{jj}(x_j)| \\
&< 0 \qquad \text{for} \quad \sum_{j=n-m+1}^{n} |f_{jj}(x_j)| \neq 0.
\end{aligned}$$

Therefore, the zero solution of the system (5.2.5) is absolutely stable for x_{n-m+j} $(j = 1,\cdots,m)$, i.e., the zero solution of the system (5.2.1) is absolutely stable for x_{n-m+j} $(j = 1,\cdots,m)$. ∎

Following the proof of Theorem 5.2.5, we can complete the proof of the remaining part of this theorem.

Example 5.2.7. We examine the absolute stability of the following system:

$$\begin{cases} \dfrac{dx_1}{dt} = -x_1 - \dfrac{1}{2}x_2 - f_1(x_1) - \dfrac{1}{2}f_2(x_2), \\ \dfrac{dx_2}{dt} = \dfrac{1}{2}x_1 - x_2 + \dfrac{1}{2}f_1(x_1) - f_2(x_2), \end{cases} \tag{5.2.9}$$

where f_1, $f_2 \in F$.

Let

$$\tilde{f}_{11}(x_1) = -x_1 - f_1(x_1), \qquad f_{12}(x_2) = -\frac{1}{2}x_2 - \frac{1}{2}f_2(x_2),$$

$$\tilde{f}_{21}(x_1) = \frac{1}{2}(x_1 + f_1(x_1)), \qquad f_{22}(x_2) = -x_2 - f_2(x_2).$$

We choose the Liapunov function

$$V(x) = -\int_0^{x_1}\tilde{f}_{11}(x_1)dx_1 - \int_0^{x_2}\tilde{f}_{22}(x_2)dx_2.$$

It follows that

$$\left.\frac{dV}{dt}\right|_{(5.2.9)} = -\tilde{f}_{11}^2(x_1) - \tilde{f}_{22}^2(x_2) < 0 \qquad \text{for} \quad x_1^2 + x_2^2 \neq 0.$$

Therefore, the zero solution of (5.2.9) is absolutely stable.

5. 3. Discrimination of Definite Sign for Lurie's Functions

Let us denote

$$F_{k_j} \triangleq \{f: f(0) = 0,\ 0 \leqslant \sigma_j f(\sigma_j) \leqslant k_j\sigma_j^2,\ k_j < +\infty,$$
$$f \in [(-\infty, +\infty), \mathbf{R}]\}.$$

If the zero solution of (5. 1. 1) is globally stable for any $f_j \in F_{k_j}$ $(j = 1, \cdots, m)$, we say the zero solution of (5. 1. 1) to be *absolutely stable in the angle* $K = \mathrm{diag}([0, k_1], \cdots, [0, k_m])$.

Suppose the matrix A of the system (5. 1. 1) is stable. Now, we consider the absolute stability of the zero solution of the system (5. 1. 1) by employing the Lurie type V-function

$$V(x) = x^{\mathrm{T}}Px + \sum_{j=1}^{m}\theta_j\int_0^{\sigma_j} f_j(\sigma_j)\mathrm{d}\sigma_j, \qquad \theta_j \geqslant 0\ (j = 1, \cdots, m), \tag{5. 3. 1}$$

where the positive definite matrix P is the solution of the Liapunov matrix equation

$$A^{\mathrm{T}}P + PA = -G,$$

$G = G^{\mathrm{T}}$ is a given positive definite matrix. Now, the difficult problem is how to judge whether $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(5.1.1)}$ is negative definite.

The necessary and sufficient conditions for the negative definiteness of $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(5.1.1)}$ are given in Zhao Suxia [4]. Since the proof is too lengthy to present, we only give the results here. For the proof readers are referred to the original paper.

Theorem 5. 3. 1. *Let* $m \geqslant 2$. $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(5.1.1)}$ *is negative definite if and only if*

1) $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(5.1.1)}$ *is negative definite for* $f(\sigma_1) = \alpha_1\sigma_1$ $(\alpha_1 = 0,\ \alpha_1 = k)$ *and for any* $f_j(\sigma_j) \in F_{k_j}$ $(j = 2, \cdots, m)$;

2) $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(5.1.1)}$ *is negative definite for any* $f_1(\sigma_1) \in F_{k_1}, f_j(\sigma_j) = 0, j = 2, \cdots, m.$

We adopt the following notations:

$$H \triangleq \mathrm{diag}(\alpha_1, \cdots, \alpha_m), \qquad A[H] = P + BHC^{\mathrm{T}},$$

$$P[H] = P + \frac{1}{2}C\Theta HC^{\mathrm{T}}, \quad -G[H] = A^{\mathrm{T}}[H]P[H] + P[H]A[H],$$

$$u_j[H] = P[H]b_i + \frac{1}{2}\theta_j A^{\mathrm{T}}[H]c_j,$$

$$z_j[H] = \{[\theta_j c_j^{\mathrm{T}} b_j + u_j[H]G^{-1}[H]u_j[H]][c_j^{\mathrm{T}}G^{-1}[H]c_j]\}^{\frac{1}{2}}$$
$$+ c_j^{\mathrm{T}}G^{-1}[H]u_j[H],$$

where $B_{n\times m} = (b_1,\cdots,b_m)$, $C_{n\times m} = (c_1,\cdots,c_m)$, $\Theta = \mathrm{diag}(\theta_1,\cdots,\theta_m)$.

Theorem 5. 3. 2. *Let* $m\geqslant 1$. $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(5.1.1)}$ *is negative definite if and only if*

$$\frac{1}{k_j} > \max_{H\in D_j} z_j[H], \qquad j = 1,\cdots,m,$$

where

$$D_j = \{H; \alpha_i = 0 \ for\ i\geqslant j;\ \alpha_i = 0,\ \alpha_i = u_i \ for\ i < j\}.$$

If the conditions of Theorem 5. 3. 1 or Theorem 5. 3. 2 are satisfied, the zero solution of (5. 1. 1) is absolutely stable in $K = \mathrm{diag}([0,k_1],\cdots,[0,k_n])$.

In the following, let

$$F_j \triangleq \{f_j;\ f_j(0) = 0,\ 0\leqslant \sigma_j f(\sigma_j) < +\infty,$$
$$f_j \in (-\infty, +\infty)\}, \qquad j = 1,\cdots,m.$$

Theorem 5. 3. 3. $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(5.1.1)}$ *is negative definite for any* $f_j\in F_j$ *if and only if* $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(5.1.1)}$ *is negative definite for any*

$$f\in \tilde{F} \triangleq \{f;\ f(\sigma) = \mathrm{col}(\lambda\delta_1\sigma_1,\cdots,\lambda\delta_m\sigma_m)\},$$

where $\lambda\in[0,+\infty)$, $\delta_i=0$ *or* $\delta_i=1$, $i=1,\cdots,m$.

We use the following notations:

$$\Delta = \mathrm{diag}(\delta_1,\cdots,\delta_m), \qquad \Theta = \mathrm{diag}(\theta_1,\cdots,\theta_m),$$

$$U = -\left(PB + \frac{1}{2}A^{\mathrm{T}}C\Theta\right)$$
$$= (u_1,\cdots,u_m), \qquad u_i\in \mathbf{R}^n, \quad i = 1,\cdots,m,$$

$$W = -\frac{1}{2}(\Theta C^{\mathrm{T}}B + B^{\mathrm{T}}C\Theta),$$

$M = \{\Delta$: there are $2^m - m - 1$ elements and at least two δ_i such that $\delta_i \neq 0\}$,

$$G[\Delta,\lambda] = G + \frac{1}{2}[U\Delta C^{\mathrm{T}} + C\Delta U^{\mathrm{T}}]\lambda + C\Delta W\Delta C^{\mathrm{T}}\lambda^2,$$

$$f_\Delta(\lambda) = \det G[\Delta,\lambda], \qquad f_0(\lambda) = \prod_{\Delta\in M} f_\Delta(\lambda).$$

Suppose that $f_0(\lambda)$ is an nth order polynomical and the first coefficient $a_0\neq 0$. Let

$$f(\lambda) = \frac{1}{a_0}f_0(\lambda^2).$$

Theorem 5. 3. 4. *The derivative of* (5. 3. 1) *along the solution of* (5. 1. 1) $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(5.1.1)}$ *is negative definite for any* $f_j\in F_j$ *if and only if*

1) $-\max\limits_{1\leqslant i\leqslant n} \rho_i\leqslant 0$, *where*

$$\rho_i = -c_i^T G^{-1} u_i + \{[\theta_i c_i^T b_i + u_i^T G^{-1} u_i][c_i^T G^{-1} c_i]\}^{\frac{1}{2}}, \qquad i = 1, \cdots, m;$$

2) *all the main sub-determinat of the Newton's matrix $S(f)$ determined by $f(\lambda)$ are nonnegative, i.e., $f(\lambda)$ has no real root.*

The Newton's matrix $S(f)$ of the polynomial $f(\lambda) = \lambda^n - a_1\lambda^{n-1} + \cdots + (-1)^n a_n$ reads

$$S(f) = \begin{bmatrix} s_0 & s_1 & \cdots & s_{n-1} \\ s_1 & s_2 & \cdots & s_n \\ \vdots & \vdots & & \vdots \\ s_{n-1} & s_n & \cdots & s_{2n-2} \end{bmatrix},$$

where $s_0, \cdots, s_{2n-2}$ are the Newton's sum of $f(\lambda)$:

$$s_0 = n,$$

$$s_1 = a_1,$$

$$\vdots$$

$$s_k = a_1 s_{k-1} - a_2 s_{k-2} + \cdots + (-1)^k a_{k-1} s_1 + (-1)^{k+1} a_k s_0 \quad \text{for } 1 \leqslant k \leqslant n,$$

$$s_k = a_1 s_{k-1} + \cdots + (-1)^{n+1} a_n s_{k-n} \quad \text{for } k > n.$$

Example 5.3.5. Consider the absolute stability of the zero solution of the system

$$\begin{cases} \dfrac{dx_1}{dt} = -x_1 - 2f_1(x_1) - 2f_2(x_2), \\ \dfrac{dx_2}{dt} = -x_2 - 2f_1(x_1) - 2f_2(x_2), \end{cases} \tag{5.3.2}$$

where

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$G = 2E = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \Theta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows from Theorem 5.3.4 that

$$U = \begin{pmatrix} 2.5 & 2 \\ 2 & 2.5 \end{pmatrix}, \qquad W = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \rho_1 = \rho_2 = 0.$$

Therefore, we find

$$f(\lambda) = \lambda^6 + \frac{25}{4}\lambda^4 + 10\lambda^2 + 4 = 0.$$

Obviously, there is no real root. We conclude that the zero solution of (5.3.2) is absolutely stable.

5.4. Particular Systems

Consider the following particular system

$$\begin{cases}\dfrac{dx}{dt} = Ax + Bf(\sigma), \\ \sigma = C^T x,\end{cases} \tag{5.4.1}$$

where $A \in \mathbf{R}^{n\times n}$, $x \in \mathbf{R}^n$, B and C are $n \times m$ constant matrices, σ and f are m-dimensional vectors, and $f(\sigma)$ satisfies:

$$f(0) = 0, \qquad \sigma^T K_1 \sigma \leqslant \sigma^T K_3 f(\sigma) \leqslant \sigma^T K_2 \sigma, \tag{5.4.2}$$

K_1, K_2 and K_3 being $m \times m$ symmetric constant matrices.

Remark 1) Let $\varphi(\sigma) = K_2 f(\sigma) - K\sigma$, $K = K_2 - K_1$, then (5.4.2) transforms into

$$\varphi(0) = 0, \qquad 0 < \sigma^T \varphi(\sigma) \leqslant \sigma^T K \sigma.$$

2) If K_3 is a nonsingular matrix, then using the transformation, we have

$$f(\sigma) = K_3^{-1} K_1 \sigma + K_3^{-1}\varphi(\sigma).$$

The system (5.4.1) is then transformed into

$$\begin{cases}\dfrac{dx}{dt} = \tilde{A}x + \tilde{B}\varphi(\sigma), \\ \sigma = C^T x,\end{cases} \tag{5.4.3}$$

where $\tilde{A} = A + BK_3^{-1}K_1C^T$ and $\tilde{B} = BK_3^{-1}$.

Theorem 5.4.1. *Assume that*

1) *the matrix $\tilde{A}$ is stable;*

2) *there exists a positive definite matrix $\tilde{G}$ such that the solution of Liapunov matrix equation*

$$\tilde{A}^T\tilde{H} + \tilde{H}\tilde{A} = -2\tilde{G} \tag{5.4.4}$$

satisfies

$$C = -\tilde{H}\tilde{B}. \tag{5.4.5}$$

Then the zero solution of (5.4.3) *is absolutely stable.*

Proof. Since $\tilde{A}$ is stable, the Liapunov matrix equation (5.4.4) has only a symmetric and positive definite solution $\tilde{H}$.

Constructing the Liapunov function

$$V(x) = x^T\tilde{H}x,$$

we deduce

$$\begin{aligned}\left.\frac{dV}{dt}\right|_{(5.4.3)} &= -2x^T\tilde{G}x + 2x^T\tilde{H}\tilde{B}\varphi(\sigma) \\ &= -2x^T\tilde{G}x - 2\sigma^T\varphi(\sigma) \\ &< 0 \qquad \text{for} \quad x \neq 0.\end{aligned}$$

Therefore the zero solution of (5.4.3) is absolutely stable. ∎

The significance of Theorem 5.4.1 lies in the fact that when $\widetilde{A}$ is stable, we can construct a positive definite matrix $\widetilde{H}$ first, by choosing control matrix C such that $\widetilde{H}$ satisfies (5.4.5). This makes a design of a stable system easy.

Theorem 5.4.2. *Assume that*

1) *the matrix $\widetilde{A}$ is stable*;

2) $\varphi^{\mathrm{T}} K^{-1} \varphi \leqslant \varphi\, \sigma$;

3) *there exists a constant $\tau > 0$ such that*

$$\begin{pmatrix} 2\widetilde{G} & -Q \\ -Q^{\mathrm{T}} & \tau K^{-1} \end{pmatrix} \quad \textit{is positive definite},$$

$\widetilde{G}$ being a positive definite matrix defined by (5.4.4), $Q = \widetilde{H}\widetilde{B} + \frac{1}{2}\tau C$.

Then the zero solution of the system (5.4.3) *is absolutely stable.*

Proof. We construct the Liapunov function of positive definite quadratic form

$$V(x) = x^{\mathrm{T}} \widetilde{H} x,$$

where $\widetilde{H}$ denotes a solution of (5.4.4). Then

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(5.4.3)} = -\,2x^{\mathrm{T}}\widetilde{G}x + 2x^{\mathrm{T}}\widetilde{H}\widetilde{B}\varphi(\sigma).$$

Using S-process, we obtain

$$\begin{aligned} \left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(5.4.3)} &= -\,2x^{\mathrm{T}}\widetilde{G}x + 2x^{\mathrm{T}}Q\varphi - \tau\varphi^{\mathrm{T}}K^{-1}\varphi \\ &= -\,(x^{\mathrm{T}}, \varphi^{\mathrm{T}})\begin{pmatrix} 2\widetilde{G} & -Q \\ -Q^{\mathrm{T}} & \tau K^{-1} \end{pmatrix}\begin{pmatrix} x \\ \varphi \end{pmatrix} \\ &< 0 \qquad \text{for} \quad x \neq 0. \end{aligned}$$

This completes the proof of Theorem 5.4.2. ∎

5.5. Nonautonomous Systems

Consider the m-dimensional nonautonomous and nonlinear control system

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}t} = \widetilde{A}(t)y + \sum\limits_{j=1}^{m} \widetilde{b}_j(t) f_j(\sigma_j(t), t), & 1 \leqslant m \leqslant n, \\ \sigma_j = c_j^{\mathrm{T}} y, \end{cases} \tag{5.5.1}$$

where

$$\begin{aligned} f_j \in F_{k_j} \triangleq \{ f:\ & f(0) = 0,\ 0 \leqslant \sigma_j f(\sigma_j, t) \leqslant k_j \sigma_j^2,\ k_j < +\infty, \\ & f \in C[(-\infty, +\infty) \times [0, +\infty), \mathbf{R}] \}, \qquad j = 1, \cdots, m, \end{aligned}$$

$A(t) = (a_{ij}(t))_{n\times n}$ is continuous function matrix on $[0,+\infty)$,

$$b_j(t) = \mathrm{col}(b_{j1}(t),\cdots,b_{jn}(t)) \in C[[0,+\infty),\mathbf{R}^n],$$

$c_j(t) = \mathrm{col}(c_{j1}(t),\cdots,c_{jn}(t))$ is a constant vector. Suppose that c_j $(j = 1,\cdots,m)$ are linearly independence. Without loss of generality, we can assume that

$$\det\begin{bmatrix} c_{1,n-m+1} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{m,n-m+1} & \cdots & c_{mn} \end{bmatrix} \neq 0.$$

By the following nonsingular linear transformation

$$x = Qy,$$

i.e.,

$$x_i = y_i, \qquad i = 1,\cdots,n-m,$$

$$x_i = \sigma_i, \qquad i = n-m+1,\cdots,n,$$

the system (5.5.1) can be transformed into

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x + \sum_{j=n-m+1}^{n} b_j(t)f_j(x_j,t),$$

where

$$A(t) = Q^{-1}\tilde{A}(t)Q, \qquad b_j(t) = Q^{-1}\tilde{b}_j(t),$$

or, in the vector component form

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \sum_{j=1}^{n} a_{ij}(t)x_j + \sum_{j=n-m+1}^{n} b_{ij}(t)f_j(x_j,t), \qquad i = 1,\cdots,n. \qquad (5.5.2)$$

Theorem 5.5.1. 1) *Assume that there exist constants* $r_i \geqslant 0$ $(i = 1,\cdots, n-m)$ *and* $r_i > 0$ $(i = n-m+1,\cdots,n)$, $\delta > 0$ *such that*

$$\begin{cases} -r_ja_{jj}(t) + \sum\limits_{\substack{i=1\\ i\neq j}}^{n} r_i|a_{ij}(t)| \leqslant 0, & j = 1,\cdots,n-m, \\ -r_ja_{jj}(t) + \sum\limits_{\substack{i=1\\ i\neq j}}^{n} r_i|a_{ij}(t)| \leqslant -\delta < 0, & j = n-m+1,\cdots,n, \\ -r_jb_{jj}(t) + \sum\limits_{\substack{i=1\\ i\neq j}}^{n} r_i|b_{ij}(t)| \leqslant 0, & j = n-m+1,\cdots,n. \end{cases}$$

2) *Let the zero solution of the linear system*

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \sum_{j=1}^{n-m} a_{ij}(t)x_j, \qquad i = 1,\cdots,n-m \qquad (5.5.3)$$

be uniformly asymptotically stable.

Then the zero solution of the system (5.5.2) *is absolutely stable in*

$$K = \mathrm{diag}([0,k_1],\cdots,[0,k_m]).$$

Proof. We choose the Liapunov function

$$V(x) = \sum_{i=1}^{n} c_i |x_i|,$$

which is radially unbounded positive definite for $x_{n-m+1}, \cdots, x_n$. Then

$$\begin{aligned} \mathrm{D}^+ V(x)|_{(5.5.2)} &\leqslant \sum_{j=1}^{n} \Big[r_j a_{jj}(t) + \sum_{\substack{i=1 \\ i \neq j}}^{n} r_i |a_{ij}(t)| \Big] |x_i| \\ &\quad + \sum_{j=n-m+1}^{n} \Big[r_j b_{jj}(t) + \sum_{\substack{i=1 \\ i \neq j}}^{n} r_i |b_{ij}(t)| \Big] |f_j(x_j,t)| \\ &\leqslant \sum_{j=n-m+1}^{n} \Big[r_j a_{jj}(t) + \sum_{\substack{i=1 \\ i \neq j}}^{n} r_i |a_{ij}(t)| \Big] |x_i| \\ &\leqslant -\delta \sum_{j=n-m+1}^{n} |x_j|. \end{aligned}$$

Thus $\mathrm{D}^+ V|_{(5.5.2)}$ is negative definite for $x_{n-m+1}, \cdots, x_n$, and the zero solution of (5. 5. 2) is absolutely stable for $x_{n-m+1}, \cdots, x_n$ in $K = \mathrm{diag}([0,k_1], \cdots, [0,k_m])$.

The Cauchy matrix solution $K(t,t_0)$ of (5. 5. 3) satisfies

$$\|K(t,t_0)\| \leqslant M\mathrm{e}^{-\alpha(t-t_0)}.$$

Let

$$\begin{aligned} x^{(n-m)} &= \mathrm{col}(x_1, \cdots, x_{n-m}), \\ A^{(n-m)}(t) &= (a_{ij}(t))_{(n-m)\times(n-m)}, \\ b_j^{(n-m)}(t) &= \mathrm{col}(b_{1j}(t), \cdots, b_{n-m,j}(t)). \end{aligned}$$

Then, the first $n - m$ components of the solution of (5. 5. 2) can be written as

$$x^{(n-m)}(t,t_0,x_0) = K(t,t_0)x_0^{(n-m)} + \int_{t_0}^{t} K(t,\tau) \sum_{j=n-m+1}^{n} b_j^{(n-m)} f_j(x_j(\tau),\tau)\mathrm{d}\tau.$$

Since $0 \leqslant f_j(x_j,t)x_j \leqslant kx_j^2$, we have $f_j(x_j(t),t) \to 0$ as $x_j(t) \to 0$ uniformly in t.

Following the proof of Theorem 5. 1. 1, we can prove that the zero solution of (5. 5. 2) is also absolutely stable for $x^{(n-m)}$ in $K = \mathrm{diag}([0,k_1], \cdots, [0,k_m])$.

Theorem 5. 5. 1 is proved. ▮

Theorem 5. 5. 2. 1) *Suppose that there exist constants* $r_i \geqslant 0$ $(i = 1, \cdots, n-m)$, $r_i > 0$ $(i = n-m+1, \cdots, n)$, $\delta > 0$ *such that*

$$\begin{cases} -r_j a_{jj}(t) + \sum\limits_{\substack{i=1 \\ i \neq j}}^{n} r_i |a_{ij}(t)| \leqslant 0, & j = 1, \cdots, n-m, \\ -r_j b_{jj}(t) + \sum\limits_{\substack{i=1 \\ i \neq j}}^{n} r_i |b_{ij}(t)| \leqslant -\delta < 0, & j = n-m+1, \cdots, n. \end{cases}$$

2) *Let the condition* 2) *of Theorem* 5. 5. 1 *be satisfied.*

3) *Let* $|f_j(x_j,t)|$ *be positive definite*, $j = 1, \cdots, m$.

Then the zero solution of the system (5. 5. 2) *is absolutely stable in*

$$K = \mathrm{diag}([0,k_1],\cdots,[0,k_m]).$$

Proof. We choose the Liapunov function

$$V(x) = \sum_{i=1}^{n} r_i |x_i|,$$

which is radially unbounded positive definite for $x_{n-m+1},\cdots,x_n$. We find

$$\mathrm{D}^+ V(x)|_{(5.5.2)} \leqslant \sum_{j=n-m+1}^{n}\left[r_j b_{jj}(t) + \sum_{\substack{i=1\\ i\neq j}}^{n} r_i |b_{ij}(t)|\right]|f_j(\sigma_j,t)|$$

$$<0 \qquad \text{for} \quad x^{(n-m)} \neq 0.$$

Thus the zero solution of (5. 5. 2) is absolutely stable for $x^{(n-m)}$. The rest of the proof is exactly the same as in Theorem 5. 5. 1 and we do not repeat it. The theorem is verified. ∎

Example 5. 5. 3. We examine the absolute stability of the following system:

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = (-4+\sin t)x_1 + (\sin t)x_2 + \left(\dfrac{1}{2}\cos t\right)x_3 + f_2(x_2) \\ \qquad + (1+\sin t)f_3(x_3), \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = (2\cos t)x_1 - 3x_2 - \dfrac{1}{1+t^2}x_3 - 2f_2(x_2) + 2f_3(x_3), \\ \dfrac{\mathrm{d}x_3}{\mathrm{d}t} = \dfrac{1}{1+t^2}x_1 - (\cos t)x_2 - 2x_3 - f_2(x_2) - 4f_3(x_3), \end{cases} \tag{5.5.4}$$

where $f_2 \in F_{k_2}$, $f_3 \in F_{k_3}$.

Choosing the Liapunov function

$$V = |x_1| + |x_2| + |x_3|,$$

we deduce that

$$\begin{aligned} \mathrm{D}^+ V|_{(5.5.4)} \leqslant & \left[(-4+\sin t) + |2\cos t| + \frac{1}{1+t^2}\right]|x_1| \\ & + [-3 + |\sin t| + |\cos t|]|x_2| \\ & + \left[-2 + \frac{1}{1+t^2} + \frac{1}{2}|\cos t|\right]|x_3| \\ & + [-2+1+1]|f_2(x_2)| \\ & + [-4+2+1+\sin t]|f_3(x_3)| \\ \leqslant & -|x_1| - |x_2| - \frac{1}{2}|x_3|. \end{aligned}$$

The zero solution of the system (5. 5. 4) is absolutely stable in

$$K = \mathrm{diag}([0,k_1],[0,k_2]).$$

Next, let us take

$$\mathcal{F}_{ij}(x_j,t) = \begin{cases} a_{ij}(t)x_j + b_{ij}(t)f_j(t,x_j), & i = 1,\cdots,n,\ j = n-m+1,\cdots,n, \\ a_{ij}(t)x_j, & i = 1,\cdots,n,\ j = 1,\cdots,n-m. \end{cases}$$

Then the system (5.5.2) can be rewritten as

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \sum_{j=1}^{n}\mathcal{F}_{ij}(x_j,t), \tag{5.5.5}$$

where $f_j \in F_{k_j}$, $j = 1,\cdots,n$.

Theorem 5.5.4. *Let the following conditions be satisfied:*

1) $\mathcal{F}_{jj}(x_j,t)x_j<0$, $j=1,\cdots,n$, $x_j\neq 0$;

2) *there exist functions* $F_{jj}(x_j)$ *defined on* $(-\infty,+\infty)$, $j=1,\cdots,m$, *which are continuous or have only finite discontinuous points of the first or third kind such that*

$$F_{jj}(x_j)x_j < 0 \quad \text{for } x_j \neq 0, \qquad |F_{jj}(x_j)| \leqslant |f_{jj}(x_j,t)|,$$

$$\int_0^{\pm\infty} F_{jj}(x_j)\mathrm{d}x_j = -\infty, \qquad j = 1,\cdots,m;$$

3) *the matrix* $G=(g_{ij})_{n\times n}$ *is negative definite, where*

$$g_{ij}\begin{cases} = -1, & i = j = 1,\cdots,n, \\ \geqslant \dfrac{1}{2}\left|\dfrac{\mathcal{F}_{ij}(x_j,t)}{F_{jj}(x_j)} + \dfrac{\mathcal{F}_{ji}(x_i,t)}{F_{ii}(x_i)}\right|, & i\neq j,\ x_ix_j \neq 0,\ i,j = 1,\cdots,n, \\ = 0, & i\neq j,\ x_ix_j = 0,\ i,j = 1,\cdots,n. \end{cases}$$

Then the zero solution of the system (5.5.5) *is absolutely stable in*

$$K = \operatorname{diag}([0,k_1],\cdots,[0,k_m]).$$

Proof. We choose the Liapunov function

$$V(x) = -\sum_{i=1}^{n}\int_0^{x_i}F_{ii}(x_i)\mathrm{d}x_i.$$

Obviously, it is a radially unbounded positive definite function.

Following the proof of Theorem 5.5.1, we can prove that $D^+V|_{(5.5.5)}$ is negative definite. Therefore, the zero solution of (5.5.5) is absolutely stable in $K = \operatorname{diag}([0,k_1],\cdots,[0,k_m])$. ∎

Theorem 5.5.5. *Assume that*

1) *the condition* 1) *of Theorem* 5.5.4 *is satisfied*;

2) $|\mathcal{F}_{jj}(x_j,t)|$ *is positive definite*, $j=1,\cdots,m$;

3) $\left|\dfrac{\mathcal{F}_{ij}(x_j,t)}{\mathcal{F}_{jj}(x_j,t)}\right| \leqslant \tilde{g}_{ij}$, $i\neq j$, $i,j=1,\cdots,n$,

$$G = \begin{pmatrix} 1 & -\tilde{g}_{12} & \cdots & -\tilde{g}_{1n} \\ -\tilde{g}_{21} & 1 & \cdots & -\tilde{g}_{2n} \\ \vdots & \vdots & & \vdots \\ -\tilde{g}_{n1} & -\tilde{g}_{n2} & \cdots & 1 \end{pmatrix}$$

is an M-matrix.

Then the zero solution of the system (5. 5. 5) *is absolutely stable in*

$$K = \mathrm{diag}([0,k_1],\cdots,[0,k_m]).$$

Proof. Since G is an M-matrix, for any $\xi = (\xi_1,\cdots,\xi_n)^T > 0$ the algebra equation $G^T\eta = \xi$ has a positive solution

$$r = \eta = (G^T)^{-1}\xi > 0.$$

Constructing the Liapunov function

$$V(x) = -\sum_{i=1}^{n}[\mathrm{sgn}\, \tilde{f}_{ii}(x_i,t)]r_i|x_i|,$$

we find

$$\begin{aligned} D^+ V(x)|_{(5.5.5)} &\leqslant \sum_{j=1}^{n}\Big[-r_j|f_{jj}(x_j,t)| + \sum_{\substack{i=1\\ i\neq j}}^{n} r_i|f_{ij}(x_j,t)|\Big] \\ &\leqslant \sum_{j=1}^{n}\Big[-r_j + \sum_{\substack{i=1\\ i\neq j}}^{n} r_i\tilde{g}_{ij}\Big]|f_{jj}(x_j,t)| \\ &< 0 \qquad \text{for} \quad x \neq 0. \end{aligned}$$

Therefore the zero solution of the system (5. 5. 5) is absolutely stable in $K = \mathrm{diag}([0,k_1],\cdots,[0,k_m])$. The theorem is proved. ∎

5. 6. Notes

Theorems 5. 1. 4～5. 1. 7 are taken from Liao Xiaoxin [13]. Theorems 5. 2. 1, 5. 2. 2, 5. 2. 5 and 5. 2. 6 are new. The proof of Theorems 5. 3. 1, 5. 3. 2 can be found in Zhao Suxia [4]. Theorems 5. 3. 3, 5. 3. 4 are due to Cheng Ruanqi [1]. Theorems 5. 4. 1, 5. 4. 2 are from Maygarin [1]. All the results of Section 5. 5 are new.

CHAPTER 6

CONTROL SYSTEMS DESCRIBED BY FDE

Since the control systems are very thoroughly studied and the stability theory of differential equations develops rapidly, it is natural to consider Lurie control systems described by functional differential equations. In this chapter, we will present the result concerning such systems.

6.1. The Systems Described by RFDE

Consider the Lurie indirect control system deseribed by retarded functional differential equtions (RFDE):

$$\begin{cases} \dfrac{dx}{dt} = g(t,x_t) + bf(\sigma), \\ \dfrac{d\sigma}{dt} = q(t,x_t) - \rho f(\sigma), \end{cases} \tag{6.1.1}$$

where $g(t,\psi) \in C[[0, +\infty) \times C_n[-r,0],\mathbf{R}^n]$, r is a positive constant, $C_n[-r,0]$ is an n-dimensional vector space of continuous functions defined on $[-r,0]$, $g(t,0) \equiv 0$, $q(t,\psi) \in C[[0,+\infty) \times C_n[-r,0],\mathbf{R}]$, $x \in \mathbf{R}^n$, $b \in \mathbf{R}^n$, $\sigma \in \mathbf{R}$, ρ is a constant and $\rho > 0$. We denote

$$|\psi(t)| \triangleq \left[\sum_{i=1}^{n} \psi_i^2(t)\right]^{\frac{1}{2}} \quad \text{(the norm of an } n\text{-dimensional vector } \psi(t)),$$

$$\| \psi(t) \| \triangleq \sup_{t\in[-r,0]} |\psi(t)| \quad \text{(the norm of } C_n[-r,0], \text{ where } \psi \in C_n[-r,0]),$$

$g(t,\psi)$ is Lipschitzian for ψ, i. e. , for given ψ_1, $\psi_2 \in C_n[-r,0]$, we have

$$|g(t,\psi_1) - g(t,\psi_2)| \leqslant L \| \psi_1 - \psi_2 \| , \qquad L = \text{const.}, \tag{6.1.2}$$

We also let $x_t \in C_n[-r,0]$, $x_t(\theta) \triangleq x(t+\theta)$, $-r \leqslant \theta \leqslant 0$ and $q(t,\psi) \in C[[0,+\infty) \times C_n[-r,0],\mathbf{R}]$,

$$|q(t,\psi)| \leqslant c \| \psi \| , \qquad c = \text{const.}, \tag{6.1.3}$$

$$f \in F \triangleq \{f: f(0) = 0,\ 0 < \sigma f(\sigma) \leqslant k\sigma^2,\ \sigma \neq 0,\ k > 0,$$
$$f \in C(-\infty, +\infty)\}.$$

Somolinos[1] was the first who discussed the absolute stability of the system (6.1.1).

Concerning the absolute stability of the zero solution, one can establish that for any given $f \in F$, the zero solution of (6.1.1) is globally asymptotically stable.

We know that the phase equations of the system (6.1.1) are written as

$$\begin{cases} \dfrac{dx}{dt} = g(t, x_t), \\ x(t) = \psi(t), \qquad t \in [-r, 0]. \end{cases} \tag{6.1.4}$$

Assume that the solution of its Cauchy problem satisfies:

$$|x(t, t_0, \psi)| \leqslant De^{-\alpha(t-t_0)} \|\psi\|, \qquad \alpha > 0, \quad D > 0. \tag{6.1.5}$$

Using Lemma 2.1 in Hale [1], we obtain that for (6.1.1), (6.1.3), given any $q \in (0,1)$, there exists a functional $V(t,\psi) \in C[[0, +\infty) \times C_n[-r, 0], \mathbf{R}^n]$ satisfying

$$\|\psi\| \leqslant V(t,\psi) \leqslant D\|\psi\|, \tag{6.1.6}$$

$$\|V(t,\psi_1) - V(t,\psi_2)\| \leqslant M\|\psi_1 - \psi_2\|, \tag{6.1.7}$$

and

$$\left.\frac{dV}{dt}\right|_{(6.1.4)} \leqslant -\gamma^2 V(t,\psi), \tag{6.1.8}$$

where $\gamma^2 = (1-q)\alpha$, and $M = D^{[L+(1-q)\alpha]/q\alpha}$ are constants.

Theorem 6.1.1. *Suppose that* (6.1.2), (6.1.3) *and* (6.1.5) *hold and let*

$$\int_0^{\pm\infty} f(s)ds = +\infty,$$

$$4\rho\gamma^2 > (M|b| + c)^2. \tag{6.1.9}$$

Then the zero solution of the system (6.1.1) *is absolutely stable.*

Proof. Assume that $V(t,\psi)$ is a functional which satisfies (6.1.5) ~ (6.1.9). It follows from Lemma 26.1 in Yoshizawa [1] that along the solution of (6.1.1) the derivative of V satisfies

$$\left.\frac{dV}{dt}\right|_{(6.1.1)} \leqslant -\gamma^2 V + M|bf(\sigma)|,$$

and thus

$$\left.V\frac{dV}{dt}\right|_{(6.1.1)} \leqslant -\gamma^2 V^2 + MV|bf(\sigma)|.$$

Let

$$W(t,\psi,\sigma) = \frac{1}{2}V^2(t,\psi) + \int_0^\sigma f(\sigma)d\sigma.$$

It is easy to prove that there are two increasing continuous functions h_1 and h_2 such that

$$h_1(\|\psi,\sigma\|) \leqslant W(t,\psi,\sigma) \leqslant h_2(\|\psi,\sigma\|),$$

and for $\|\psi,\sigma\| \to +\infty$, $h_1 \to +\infty$.

If (6. 1. 9) holds, then we deduce that

$$\left.\frac{\mathrm{d}W}{\mathrm{d}t}\right|_{(6.1.1)} \leqslant -\gamma^2V^2 + V(M|b| + c)|f(\sigma)| - \rho|f(\sigma)|^2$$

and there is $\eta > 0$ such that

$$\left.\frac{\mathrm{d}W}{\mathrm{d}t}\right|_{(6.1.1)} \leqslant -\eta(V^2 + |f(\sigma)|^2) \leqslant -\eta(\|x_t\|^2 + |f(\sigma)|^2),$$

$$= -h_3(\|x_t,\sigma\|),$$

where h_3 refers to a positive definite continuous function in the norm $\|x_t,\sigma\|$ of (x_t,σ).

From Theorem 11. 1 of Hale [1], we know that Theorem 6. 1. 1 holds. ▮

Somolinos[1] considered also the following direct control system:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = g(t,x_t) + bf(\sigma), \\ \sigma = c^{\mathrm{T}}x, \end{cases} \tag{6.1.10}$$

where $c^{\mathrm{T}}b = -\rho < 0$, and the meaning of $g(t,x_t)$ and $f(\sigma)$ is similar to that of $g(t,x_t)$, $f(\sigma)$ in (6. 1. 1).

Theorem 6. 1. 2. *If the conditions* (6. 1. 2), (6. 1. 3) *hold and*

$$\frac{f(\sigma)}{\sigma} < \frac{\gamma^2}{M|b| \cdot |c|},$$

then the zero solution of the system (6. 1. 10) *is absolutely stable.*

Proof. Suppose that V is a Liapunov functional satisfying (6. 1. 6), (6. 1. 7) and (6. 1. 8). Similarity to the proof of Theorem 6. 1. 1, we find

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(6.1.10)} \leqslant -\gamma^2V + M|b| \cdot |f(\sigma)|$$

$$\leqslant -\gamma^2\|x_t\| + M|b| \cdot |f(\sigma)|. \tag{6.1.11}$$

For $\sigma = 0$, we deduce

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(6.1.10)} \leqslant -\gamma^2\|x_t\|;$$

for $\sigma \neq 0$, the last term on right side in (6. 1. 11) is multiplied by $\dfrac{|c| \cdot \|x_t\|}{c^{\mathrm{T}}x_t}$.

From $\sigma = c^{\mathrm{T}}x$, it follows that

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(6.1.10)} \leqslant -\|x_t\|\left(\gamma^2 - M|b| \cdot |c|\frac{f(\sigma)}{\sigma}\right).$$

Hence, if $\frac{f(\sigma)}{\sigma} < \frac{\gamma^2}{M|b| \cdot |c|}$, then $\left.\frac{dV}{dt}\right|_{(6.1.10)}$ is negative definite. The proof is completed. ▮

Zhu Siming also studied the absolute stabiity of the zero solution of (6.1.10) in $[0,k]$.

Theorem 6.1.3. *Suppose that the conditions* (6.1.2), (6.1.3), (6.1.6), (6.1.7), *and* (6.1.8) *hold and that there is a real number* β *such that*

$$k|c|(M|b| + |\beta|L|c|) < \gamma, \tag{6.1.12}$$

$$2k|c|[M|b| + |c|(|\beta|L + \beta c^T b)] < \gamma. \tag{6.1.13}$$

Moreover, let

$$1 + \beta k|c|^2 > 0 \qquad for \quad \beta < 0.$$

Then the zero solution of the system (6.1.10) *is absolutely stable in* $[0,k]$.

Proof. Let us set

$$u(t,\psi) = \frac{1}{2}V^2(t,\psi) + \beta\int_0^\sigma f(\sigma)d\sigma,$$

where V is a functional satisfying (6.1.6), (6.1.7) and (6.1.8). It follows from $0 < \sigma f(\sigma) \leqslant k\sigma^2$ that

$$0 \leqslant \int_0^\sigma f(\sigma)d\sigma \leqslant \frac{1}{2}k\sigma^2,$$

hence, by (6.1.6) we have

$$\begin{aligned}&\left[\frac{1}{2} + (1 - \operatorname{sgn}\beta)\frac{1}{4}k\beta|c|^2\right]\|\psi\|^2\\ &\leqslant u(t,\psi)\\ &\leqslant \left[\frac{D}{2} - (1 - \operatorname{sgn}\beta)\frac{1}{4}\beta k|c|^2\right]\|\psi\|^2.\end{aligned} \tag{6.1.14}$$

It follows from the conditions that $u(t,\psi)$ is a radially unbounded positive definite functional of ψ.

On the other hand, we have

$$\left.\frac{du}{dt}\right|_{(6.1.10)} \leqslant -\gamma^2V^2 + (M|b| + L|\beta| \cdot |c|)V|f(\sigma)| + \beta c^T b f^2(\sigma).$$

Choosing a constant $\tau > 0$, and taking

$$N = (M|b| + L|\beta| \cdot |c| + \tau|c|)/2\gamma,$$

we deduce

$$\begin{aligned}\left.\frac{du}{dt}\right|_{(6.1.10)} \leqslant &-(\gamma V - N|f(\sigma)|)^2 + \left[N^2 - \left(\frac{\tau}{k} - \beta c^T b\right)\right]|f(\sigma)|^2\\ &+ \tau\left[\frac{1}{k}f(\sigma) - \sigma\right]f(\sigma).\end{aligned} \tag{6.1.15}$$

Clearly, when $f = 0$, the above form is reduced to

$$\left.\frac{du}{dt}\right|_{(6.1.10)} \leqslant -\gamma^2 V^2 \leqslant -\gamma^2 \| x_t \|^2.$$

However, when $f \neq 0$, it follows from $q(t,\psi) \in C[[0,+\infty) \times C_a[-r,0],\mathbf{R}]$ that

$$\left[\frac{1}{k} f(\sigma) - \sigma\right] f(\sigma) = f^2(\sigma)\left[\frac{1}{k} - \frac{\sigma}{f(\sigma)}\right] \leqslant 0.$$

Therefore only if

$$N^2 < \frac{\tau}{k} - \beta c^{\mathrm{T}} b, \tag{6.1.16}$$

we obtain from (6.1.15) that $\left.\frac{du}{dt}\right|_{(6.1.10)}$ is a negative definite functional of ψ. In order to decide the conditions satisfying (6.1.16), we substitute the representation of N to (6.1.16), which can be reduced to

$$|c|^2\Bigg[\tau^2 + 2\left(\frac{M|b|}{|c|} - \frac{\gamma}{k|c|^2} + |\beta| L\right)\tau$$
$$+ \left(\frac{M|b|}{|c|} + |\beta| L\right)^2 + 2\gamma\beta\frac{c^{\mathrm{T}}b}{|c|^2}\Bigg] < 0.$$

Let

$$\lambda = \frac{M|b|}{|c|} - \frac{\gamma}{k|c|^2} + |\beta| L,$$
$$p = \left(\frac{M|b|}{|c|} + |\beta| L\right)^2 - 2\gamma\beta\frac{c^{\mathrm{T}}b}{|c|^2}.$$

The above formula is reduced to

$$|c|^2(\tau^2 + 2\lambda\tau + p) = |c|^2(\tau + \lambda - \sqrt{\lambda^2 - p})(\tau + \lambda + \sqrt{\lambda^2 - p}) < 0.$$

Clearly, when $\lambda < 0$, $\lambda^2 > p$, we can find a range of τ such that the above inequality is satisfied, viz.

$$0 < -\lambda - \sqrt{\lambda^2 - p} < \tau < -\lambda + \sqrt{\lambda^2 - p}.$$

From $\lambda < 0$ it follows that

$$kL|\beta| \cdot |c|^2 < \gamma - kM|b| \cdot |c|.$$

This is just the condition (6.1.12).

From $\lambda^2 > \rho$, we get

$$-2\left(\frac{M|b|}{|c|} + |\beta| L\right)\frac{\gamma}{k|c|^2} + \frac{\gamma^2}{k^2|c|^4} > -2\gamma\beta\frac{c^{\mathrm{T}}b}{|c|^2}.$$

Using the above expression we obtain condition (6.1.13).

Till now we proved that when (6.1.13) is satisfied, $\left.\frac{du}{dt}\right|_{(6.1.10)}$ is a negative definite functional.

Finally, it follows from the inequality (6. 1. 14) that $u(t,\psi)$ is a positive definite functional with an infinitesimal small upper bound and radially unbounded.

Therefore the zero solution of the system (6. 1. 10) is absolutely stable in $[0,k]$. ∎

The consequences of Theorem 6. 1. 2 and Theorem 6. 1. 3 exclude each other. Ruan Jong[2] generalized Theorem 6. 1. 2 and obtained the following theorem.

Theorem 6. 1. 4. *Suppose that the conditions* (6. 1. 2), (6. 1. 3) *hold and let*

$$\alpha^* = \frac{kM|b| \cdot |c|}{\gamma^2}, \qquad \beta^* = L - k\rho, \qquad \rho = -c^{\mathrm{T}}b.$$

If one of the following four conditions holds:

1) $0<\alpha^*<1$;

2) $\alpha^*=1$ *and* $|\rho|>\frac{L}{k}$;

3) $2-\frac{L}{k\rho}<\alpha^*<2$ *and* $|\rho|>\frac{L}{k}$ (>0);

4) $1<\alpha^*<2+\frac{L}{k\rho}$ *and* $\rho<-\frac{L}{k}$ (<0),

then the zero solution of (6. 1. 10) *is absolutely stable in* $[0,k]$.

Proof. First, for the system (6. 1. 1), the criterion of absolute stability is that there exists a real number β such that

$$\gamma^2 > \frac{1}{2}k|c|(M|b| + L|\beta| \cdot |c|), \tag{6.1.17}$$

$$\gamma^2 > k|c|(M|b| + L|\beta| \cdot |c| + k|c|\beta c^{\mathrm{T}}b), \tag{6.1.18}$$

and such that in case of $\beta < 0$, there exists

$$1 + \beta k|c|^2 > 0 \tag{6.1.19}$$

or

$$4\gamma^2\rho\beta > (M|b| + L|\beta| \cdot |c|). \tag{6.1.20}$$

However, the condition (6. 1. 20) makes G a positive definite quadratic form (G will be given below.) This leads to contradiction.

Secondly, we prove that when the conditions of Theorem 6. 1. 4 are satisfied there exists a real number β satisfying (6. 1. 17) or (6. 1. 20). The theorem is proved.

For the system (6. 1. 10), let us set

$$W(t,\psi) = \frac{1}{2}V^2(t,\psi) + \beta\int_0^{c^{\mathrm{T}}\psi(0)} f(s)\mathrm{d}s,$$

where V stands for a functional satisfying (6.1.6), (6.1.7) and (6.1.8). Then we deduce

$$\begin{aligned}&\left[\frac{1}{2}+(1-\operatorname{sgn}\beta)\frac{1}{4}\beta k|c|^2\right]\|\psi\|^2\\&\leqslant W(t,\psi)\\&\leqslant\left[\frac{D^2}{2}+(1+\operatorname{sgn}\beta)\frac{1}{2}\beta k|c|^2\right]\|\psi\|^2.\end{aligned}$$

From the hypothesis of the theorem, we find that W is radially unbounded and have an infinitesimal small upper bound. Below we only find a criterion which W satisfies:

$$\left.\frac{\mathrm{d}W}{\mathrm{d}t}\right|_{(6.1.10)}\leqslant-h_3(|\psi(0)|)$$

with h_3 being a positive definite continuous non-decreasing function.

Following this line of reasoning, we have

$$\begin{aligned}\left.\frac{\mathrm{d}W}{\mathrm{d}t}\right|_{(6.1.10)}&\leqslant-\gamma^2V^2+MV|bf(\sigma)|+\beta f(\sigma)[c^{\mathrm{T}}g(t,x_t)-\rho f(\sigma)]\\&\leqslant-\gamma^2V^2+M|b|V|f(\sigma)|+|\beta|L|c|V|f(\sigma)|-\beta\rho|f(\sigma)|^2\\&=-\gamma^2[V^2-2pV|f(\sigma)|+q|f(\sigma)|^2],\end{aligned}$$

where $\rho=-c^{\mathrm{T}}b>0$, $2p=\dfrac{M|b|+L|\beta|\cdot|c|}{\gamma^2}$ and $q=\dfrac{\beta\rho}{\gamma^2}$. Furthermore, it follows from $q(t,\psi)\in C[[0,+\infty)\times C_a[-r,0],\mathbf{R}]$ that

$$\frac{V(t,x_t)}{|f(c^{\mathrm{T}}x_t(0))|}\geqslant\frac{\|x_t\|}{|f(c^{\mathrm{T}}x(t))|}\geqslant\frac{1}{k|c|}$$

and

$$\left.\frac{\mathrm{d}W}{\mathrm{d}t}\right|_{(6.1.10)}\leqslant-\gamma^2G,$$

where

$$G=[V-(p+\tau)|f(\sigma)|]^2+N|f(\sigma)|^2+2\tau\left[V-\frac{1}{k|c|}|f(\sigma)|\right]|f(\sigma)|$$

and $\tau>0$ is yet undetermined constant such that $N>0$,

$$N=\frac{2\tau}{k|c|}-p^2-2p\tau-\tau^2+q.$$

If $\tau>0$ exists, it follows from the condition $f(0)=0$, $0<\sigma f(\sigma)\leqslant k\sigma^2$, $\sigma\neq0$ that

$$\begin{aligned}G&\geqslant[V-(p+\tau)|f(\sigma)|]^2+N|f(\sigma)|^2\\&\geqslant\eta[V^2(t,x_t)+f^2(c^{\mathrm{T}}x(t))]\\&\geqslant\eta(\|x_t\|^2+|f(c^{\mathrm{T}}x(t))|^2)\end{aligned}$$

$$\geqslant \eta \| x_t \|^2 \geqslant \eta | x(t) |^2,$$

where $[V - (p + \tau) | f(\sigma) |^2]^2 + N | f(\sigma) |^2$ is a positive definite quadratic form in V, $| f(\sigma) |$, viz. $h_3(|\psi(0)|) = \gamma^2 \eta |\psi(0)|^2$ $(\eta > 0)$ can hold. Therefore, from Theorem 2. 1 in Hale [1], we conclude that the zero solution of the system (6. 1. 10) is absolutely stable in $[0, k]$.

Now we are in position to verify that τ with $\tau > 0$ exist. We note that

$$f(\tau) = \tau^2 + 2\lambda^* \tau + \rho^2 = - N,$$

where $\lambda^* = p - \frac{1}{k|c|}$ and $p^* = p^2 - q$. The conditions of existence of τ with $\tau > 0$ and $f(\tau) < 0$ are $p^* < 0$ or $\lambda^{*2} > p^* > 0$ and $\lambda^* < 0$. Expanding them, we obtain (6. 1. 17) or (6. 1. 18) and (6. 1. 19). The first part is completed.

Note the criterion condition for $G > 0$. It can be directly obtained from $\| x_t \| > \frac{1}{k|c|} | f(\sigma) |$, instead of the existence of $\tau > 0$. If $\psi = 0$, then $G = 0$; if $\psi \neq 0$ but $c^T \psi(0) = 0$, then G is a positive definite function $\| \psi \|^2$. Furthermore, if $\psi \neq 0$ and $c^T \psi(0) \neq 0$, then the sufficient conditions for $G > 0$ are

$$\frac{1}{k|c|} > p + \sqrt{p^2 - q^2}, \qquad p^2 - q > 0.$$

Using these conditions, we obtain (6. 1. 17) and (6. 1. 18) which in combination with (6. 1. 19) give rise to the following: $\left.\frac{\mathrm{d}W}{\mathrm{d}t}\right|_{(6.1.10)}$ is a criterion for a negative definite function to be smaller than $|\psi(0)|$.

Let us turn to the second part of the proof.

First, we analyse the necessary and sufficient conditions for existence of a real number β, which satisfies (6. 1. 17), (6. 1. 18) and (6. 1. 19).

1° The case of $\beta > 0$ satisfying (6. 1. 17) and (6. 1. 18).

It follows from (6. 1. 17) that

$$0 < \beta < \frac{2\gamma^2 - kM|c| \cdot |b|}{kL|c|^2}.$$

From (6. 1. 18), we deduce

$$\gamma^2 - kM|c| \cdot |b| > \beta k |c|^2 (L - k\rho).$$

Hence it can be shown that if

$$\gamma^2 > k|c| \cdot M \cdot |b| \tag{6. 1. 21}$$

or if

$$\gamma^2 = k|c| \cdot M \cdot |b|, \qquad L - k\rho < 0 \tag{6. 1. 22}$$

there exists β with $\beta > 0$, which satisfies (6. 1. 17) and (6. 1. 18). However, if

$$2-\frac{L}{k\rho}=1-\frac{\beta^*}{k\beta}<\alpha^*<2, \tag{6.1.23}$$

the existence of β (>0) implies

$$L-k\rho<0 \tag{6.1.24}$$

and

$$\beta>\frac{\gamma^2-k|c|\cdot M\cdot|b|}{k|c|^2(L-k\rho)}>0,$$

$$0<\beta<\frac{2\gamma^2-k|c|\cdot M\cdot|b|}{kL|c|^2}.$$

Since

$$\frac{2\gamma^2-k|c|\cdot M\cdot|b|}{kL|c|^2}>\frac{\gamma^2-k|c|\cdot M\cdot|b|}{k|c|^2(L-k\rho)},$$

we know that there exists a positive number β satisfying (6.1.17) and (6.1.18). If (6.1.17) and (6.1.18) are satisfied, (6.1.21) and (6.1.22) or (6.1.23) and (6.1.24) holds definitely.

2° The case when β with $\beta>0$ satisfies (6.1.17), (6.1.18) and (6.1.19). (6.1.19) is just $0>\beta>-\frac{1}{k|c|^2}$. The form (6.1.17) implies

$$0>\beta>\frac{2\gamma^2-k|c|\cdot M\cdot|b|}{-kL|c|^2},$$

and (6.1.18) implies

$$\gamma^2-k|c|\cdot M\cdot|b|>-\beta k|c|^2(\rho k+L).$$

In the case of $\alpha^*=1$, there exists β with $\beta>0$ satisfying (6.1.17), (6.1.18) and (6.1.19) only if $k\rho+L<0$. Under the condition 4) of the theorem, we can also verify the existence of β with $\beta<0$ by the same way.

3° The case with $\beta=0$ to satisfy (6.1.17) and (6.1.18).

From (6.1.17) and (6.1.18), we have the following independent inequalities

$$\gamma^2>\frac{1}{2}k|c|\cdot M\cdot|b|$$

and

$$\gamma^2>k|c|\cdot M\cdot|b|.$$

Hence if $\gamma^2>k|c|\cdot M\cdot|b|$, there exists $\beta=0$ which satisfies (6.1.17) and (6.1.18), and vice versa.

To summarize the above three cases, we know that if one of the four condition is satisfied, the zero solution of the system (6.1.10) is absolutely stable in $[0,k]$. ∎

6.2. Large-Scale Control Systems Described by RFDE

In this section, we will introduce a notion of the absolute stability of Lurie large-scale systems described by retarded functional differential equations.

Let $x^{\mathrm{T}} = ((x^{(1)})^{\mathrm{T}}, \cdots, (x^{(m)})^{\mathrm{T}}) \in \mathbf{R}^n$, $x^{(i)} \in \mathbf{R}^{n_i}$, $\sum_{i=1}^{m} n_i = n$, $J = [0, +\infty)$.

Assume that $r > 0$, $H_i > 0$ are constants. Let $C_{n_i} = C[[-r, 0], \mathbf{R}^{n_i}]$. For $\varphi^{(i)} \in C_{n_i}$, we define

$$\| \varphi^{(i)} \| = \sup_{-r \leqslant \theta \leqslant 0} | \varphi^{(i)}(\theta) |.$$

Let $C_{n_i}^{H_i} = \{\varphi^{(i)} \in C_{n_i}: \ \| \varphi^{(i)} \| < H_i\}$. Then it follows $C_n^H \subset C_{n_1}^{H_1} \times \cdots \times C_{n_m}^{H_m}$.

Consider a Lurie direct control large-scale system which is described by retatded functional differential equations:

$$\begin{cases} \dfrac{\mathrm{d}x^{(i)}}{\mathrm{d}t} = g_i(t, x_t^{(i)}) - b_i f_i(\sigma_i), \\ y^{(i)} = c_i^{\mathrm{T}} x^{(i)}, \\ \sigma_i = d_i^{\mathrm{T}} y = \sum\limits_{j=1}^{m} d_{ij} y^{(j)}, \qquad i = 1, \cdots, m, \end{cases} \tag{6.2.1}$$

where $y \in \mathbf{R}^m$, $x^{(i)} \in \mathbf{R}^{n_i}$, $b_i \in \mathbf{R}^{n_i}$, $c_i \in \mathbf{R}^{n_i}$, $d_i^{\mathrm{T}} \in \mathbf{R}^m$, $\sigma_i \in \mathbf{R}$, $g_i \in C[J \times C_{n_i}^{H_i}, \mathbf{R}^{n_i}]$, $g_i(t, 0) \equiv 0$. Moreover, $g_i(t, \varphi^{(i)})$ is Lipschitzian in $\varphi^{(i)}$, namely for given $\varphi_1^{(i)}$, $\varphi_2^{(i)} \in C_{n_i}^{H_i}$, we have

$$|g_i(t, \varphi_1^{(i)}) - g_i(t, \varphi_2^{(i)})| \leqslant L_i \| \varphi_1^{(i)} - \varphi_2^{(i)} \|. \tag{6.2.2}$$

$f_i(\sigma_i)$ denotes a scalar continuous function satisfying

$$f_i(0) = 0, \quad 0 < f_i(\sigma_i)\sigma_i \leqslant k_i \sigma_i^2 \ (\sigma_i \neq 0), \qquad k_i > 0, \quad i = 1, \cdots, m.$$

Suppose that the phase equations are

$$\begin{cases} \dfrac{\mathrm{d}x^{(i)}}{\mathrm{d}t} = g_i(t, x_t^{(i)}), \\ x^{(i)}(t) = \varphi^{(i)}, \qquad \tau \leqslant t \leqslant 0. \end{cases} \tag{6.2.3}$$

We assume that the solution of (6.2.3) satisfies

$$|x^{(i)}(t, t_0, \varphi^{(i)})| \leqslant D_i e^{-\lambda_i (t - t_0)} \| \varphi^{(i)} \|, \qquad i = 1, \cdots, m, \tag{6.2.4}$$

where λ_i is a positive constant. For any i there exists a functional $V_i(t, \varphi^{(i)})$ such that

$$\begin{cases} \| \varphi^{(i)} \| \leqslant V_i(t, \varphi^{(i)}) \leqslant D_i \| \varphi^{(i)} \|, \\ |V_1(t, \varphi_1^{(i)}) - V_2(t, \varphi_2^{(i)})| \leqslant M_i \| \varphi_1^{(i)} - \varphi_2^{(i)} \|, \\ \mathrm{D}^+ V_i(t, \varphi^{(i)}) |_{(6.2.3)} \leqslant -\gamma_i^2 \| \varphi^{(i)} \|, \end{cases} \tag{6.2.5}$$

where D_i, M_i are positive constants, and $\gamma_i^2 \triangleq (1 - q_i)\lambda_i$, $M_i \triangleq D_i^{[\lambda_i + (1 - q_i)\lambda_i]/q_i \lambda_i}$, $0 \leqslant q_i < 1$, $i = 1, \cdots, m$.

Theorem 6.2.1. *Suppose that the system* (6.2.1) *satisfies* (6.2.2), *the* *system* (6. 2. 3) *satisfies* (6. 2. 4), *and there are m constants* β_i $(i = 1,\cdots,m)$ *such that the matrix* $V_{m\times m}$ *is positive definite and* $\begin{pmatrix} C & S \\ S^T & R \end{pmatrix}_{2m\times 2m}$ *is negative definite. Here* $V = (v_{jl})_{m\times m}$, $C = (c_{ij})_{m\times m}$, $S = (s_{ij})_{m\times m}$, $R = (r_{ij})_{m\times m}$, *where*

$$v_{jl} = \begin{cases} \frac{1}{2}\alpha_j + \sum\limits_{i=1}^{m} \frac{1}{4}(1 - \operatorname{sgn} \beta_i)\beta_i\alpha_i k_i |c_j d_{ij}^T c_l^T|, & j = l, \\ \sum\limits_{i=1}^{m} \frac{1}{4}(1 - \operatorname{sgn} \beta_i)\beta_i\alpha_i k_i |c_i d_{ij}^T d_{il} c_l^T|, & j \neq l; \end{cases}$$

$$c_{ij} = \begin{cases} -\alpha_i \gamma_i^2, & i = j, \\ 0, & i \neq j; \end{cases}$$

$$s_{ij} = \begin{cases} \frac{1}{2}L_i c_i d_{ii} |\beta_i| \alpha_i + \frac{1}{2}c_i d_{ii}\alpha_i + \frac{1}{2}M_i\alpha_i |b_i| D_i, & i = j, \\ \frac{1}{2}L_i |c_i d_{ji}| \cdot \alpha_i \cdot |\beta_j| + \frac{1}{2}|c_i d_{ji}|\alpha_i, & i \neq j; \end{cases}$$

$$r_{ij} = \begin{cases} \alpha_i |\beta_i| \cdot |d_{ii} c_i^T b_i| - \alpha_i / k_i, & i = j, \\ \frac{1}{2}(\alpha_i |\beta_i| \cdot |d_{ij} c_j^T b_j| + \alpha_j |\beta_j| \cdot |d_{ji} c_i^T b_i|), & i \neq j. \end{cases}$$

Then the zero solution of the Lurie-type system (6. 2. 1) *is absolutely stable.*

Proof. Let us choose the Liapunov functional

$$u(t,\varphi) = \sum_{i=1}^{m} \alpha_i u_i(t,\varphi^{(i)}),$$

where

$$u_i(t,\varphi^{(i)}) = \frac{1}{2}V_i^T(t,\varphi^{(i)}) \cdot V_i(t,\varphi^{(i)}) + \beta_i \int_0^{\sigma_i} f_i(s)\mathrm{d}s, \qquad i = 1,\cdots,m, \tag{6. 2. 6}$$

$V_i(t,\varphi^{(i)})$ being given by (6. 2. 5). It follows from the properties of $f_i(s)$ that

$$0 \leqslant \int_0^{\sigma_i} f_i(s)\mathrm{d}s \leqslant \frac{1}{2}k_i\sigma_i^2.$$

Combining the first form of (6. 2. 5) with (6. 2. 6), we deduce

$$\int_0^{\sigma_i} f_i(s)\mathrm{d}s \leqslant \frac{1}{2}k_i\Big[\sum_{j=1}^{m}(y^{(i)})^T d_{ij}^T\Big]\Big[\sum_{j=1}^{m} d_{ij}y^{(i)}\Big]$$
$$= \frac{1}{2} k_i \sum_{j,l=1}^{m} (y^{(j)}) d_{ij}^T d_{il}^T y^{(l)} = \frac{1}{2} k_i \sum_{j,l=1}^{m} (x^{(j)})^T c_i d_{ij}^T d_{il} c_l^T x^{(l)},$$

hence

$$u(t,\varphi) \leqslant \sum_{i=1}^{m} \alpha_i D_i^2 \,\|\, \varphi^{(i)} \,\|^2 + \sum_{i=1}^{m} \frac{1}{4}(1 + \operatorname{sgn} \beta_i)\alpha_i\beta_i k_i \cdot \sum_{j,l=1}^{m} |c_j d_{ij}^T d_{il} c_l^T| \cdot \|\, \varphi^{(j)} \,\| \cdot \|\, \varphi^{(l)} \,\|,$$

$$
\begin{aligned}
u(t,\varphi) &\geqslant \sum_{i=1}^{m} \frac{1}{2}\alpha_i \| \varphi^{(i)} \|^2 + \sum_{i=1}^{m} \frac{1}{4}(1 - \operatorname{sgn} \beta_i)\alpha_i\beta_i k_i \\
&\quad \cdot \sum_{j,l=1}^{m} |c_j d_{ij}^{\mathrm{T}} d_{il} c_l^{\mathrm{T}}| \cdot \| \varphi^{(j)} \| \cdot \| \varphi^{(l)} \| \\
&= (\| \varphi^{(1)} \|, \cdots, \| \varphi^{(m)} \|) V (\| \varphi^{(1)} \|, \cdots, \| \varphi^{(m)} \|)^{\mathrm{T}} \\
&\geqslant \lambda_{\min}(V) \sum_{i=1}^{m} \| \varphi^{(i)} \|^2,
\end{aligned}
$$

where $\lambda_{\min}$ is the minimum eigenvalue of V.

In addition, along the solution of the system (6. 2. 1) we have

$$
\begin{aligned}
\mathrm{D}^+ u|_{(6.2.1)} &\leqslant \sum_{i=1}^{m} [-\alpha_i \gamma_i^2 \| \varphi^{(i)} \|^2 + \alpha_i M_i |b_i| D_i \| \varphi^{(i)} \| \cdot |f_i(\sigma_i)|] \\
&\quad + \sum_{i=1}^{m} \alpha_i |\beta_i| \cdot |f_i(\sigma_i)| \sum_{j=1}^{m} |d_{ij} c_j^{\mathrm{T}} b_j| \cdot |f_i(\sigma_j)| \\
&\quad + \sum_{i=1}^{m} \alpha_i |\beta_i| \cdot |f_i(\sigma_i)| \sum_{j=1}^{m} |d_{ij} c_j^{\mathrm{T}} L_j| \cdot \| \varphi^{(j)} \| \\
&\quad + \sum_{i=1}^{m} \alpha_i \left(\sigma_i - \frac{f_i(\sigma_i)}{\sigma_i} \right) f_i(\sigma_i) \\
&\quad - \sum_{i=1}^{m} \alpha_i \left(\sigma_i - \frac{f_i(\sigma_i)}{k_i} \right) f_i(\sigma_i) \\
&\leqslant w^{\mathrm{T}} \begin{pmatrix} C & S \\ S^{\mathrm{T}} & R \end{pmatrix} w - \sum_{i=1}^{m} \alpha_i \left(\sigma_i - \frac{f_i(\sigma_i)}{k_i} \right) f_i(\sigma_i),
\end{aligned}
$$

where $w^{\mathrm{T}} = (\| \varphi^{(1)} \|, \cdots, \| \varphi^{(m)} \|)$, $C = (c_{ij})_{m\times m}$, $S = (s_{ij})_{m\times m}$ and $R = (r_{ij})_{m\times m}$ are the matrices given in the theorem. Since the matrix $\begin{pmatrix} C & S \\ S^{\mathrm{T}} & R \end{pmatrix}$ is negative definite, we conclude that the zero solution of (6. 2. 1) is absolutely stable. ∎

6. 3. The Systems Described by NFDE

Consider a Lurie indirect control system described by neutral functional differential equations

$$
\begin{cases}
\dfrac{\mathrm{d}}{\mathrm{d}t}(D(t)x_t) = A(t,x_t) + bf(\sigma), \\
\dfrac{\mathrm{d}\xi}{\mathrm{d}t} = f(\sigma), \\
\sigma(t) = B(t,x_t) - r\xi, \\
x_{t_0} = \varphi, \qquad t_0 \in I, \quad \varphi \in C_n[-h,0],
\end{cases}
\tag{6. 3. 1}
$$

where $I = [t_0, +\infty)$, $C_n[-h,0]$ is a set of continuous functions mapping $[-h,0]$ to $\mathbf{R}^n$, $D(\cdot):[t_0,+\infty) \times C_n[-h,0] \to \mathbf{R}^n$ and

$$
D(t)\varphi = \varphi(0) - g(t,\varphi).
$$

Let $\|\varphi\| \triangleq \sup_{-h\leqslant\theta\leqslant 0} |\varphi(\theta)|$, $x_t \in C_n[-h,0]$ and $x_t(\theta) \triangleq x(t+\theta)$, $-h \leqslant \theta \leqslant 0$.

Suppose that $g(t,\varphi)$ is linear in φ. Using Stieltjes integral, we get

$$g(t,\varphi) = \int_{-h}^{0} [\mathrm{d}_s u(t,s)]\varphi(s),$$

where $u(t,s)$ is an $n \times n$ matrix, $t \in I$, $s \in [-h,0]$, satisfying

$$\left|\int_{-\theta}^{0} [\mathrm{d}_s u(t,s)]\varphi(s)\right| \leqslant l(\theta) \sup_{-\theta\leqslant\xi\leqslant 0} |\varphi(\xi)|,$$

and $l(\theta)$ is a non-decreasing continuous function with $\theta \in [0,h]$ and $l(0) = 0$.

We assume that $A: I \times C_n[-h,0] \to \mathbf{R}^n$ is continuous, b, $c \in \mathbf{R}^n$ and $f(\sigma)$ is a continuous function.

Consider the phase equations

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(D(t)x(t)) = A(t,x_t), \\ x_{t_0} = \varphi, \qquad t_0 \in I, \quad \varphi \in C_n[-h,0]. \end{cases} \tag{6.3.2}$$

Definition 6. 3. 1. An operator D is said to be *uniformly stable* if there exist two constants $\alpha > 0$ and $\beta > 0$ such that the solution of difference equations $D(t)x_t = 0$, $x_{t_0} = \varphi$, $D(t_0)\varphi = 0$ satisfies

$$\|x_t\| \leqslant \beta e^{-\alpha(t-t_0)} \|\varphi\|, \qquad t \geqslant t_0.$$

In the following, we use the existence theorem of neutral functional differential equations, which is uniformly asymptotic stable. This theorem is due to Cruz and Hale [1].

Lemma 6. 3. 2. *Suppose that $D(t)$ and $A(t,\cdot)$ in the system (6. 3. 2) are linear bounded operators mapping $C_n[-h,0]$ to $\mathbf{R}^n$ and there exists $L_1 > 0$ such that for $t \geqslant t_0$, $\varphi \in C_n[-h,0]$,*

$$|D(t)\varphi| \leqslant L_1 \|\varphi\|.$$

If the zero solution of the system (6. 3. 2) is uniformly asymptotically stable, then there exist constants M, γ^2, K and a continuous scalar function $V(t,\varphi)$ on $I \times C_n[-h,0]$ such that for $t \geqslant t_0$, φ, $\psi \in C_n[-h,0]$

$$|D(t)\varphi| \leqslant V(t,\varphi) \leqslant M\|\varphi\|,$$

$$\left.\frac{\mathrm{d}^+ V}{\mathrm{d}t}\right|_{(6.3.2)} \leqslant -\gamma^2 V(t,\varphi),$$

$$|V(t,\varphi) - V(t,\psi)| \leqslant K(\varphi - \psi),$$

where $\left.\dfrac{\mathrm{d}^+ V}{\mathrm{d}t}\right|_{(6.3.2)}$ represents the upper right derivative of V along the solution of (6. 3. 2).

Lemma 6. 3. 3. *Suppose that $A(t,0)=0$, and $A(t,\varphi)$ in the system (6. 3. 3) is locally Lipschiziam in φ (the Lipschitz constant is N, which is uniform in t.) Assume that*

$$|D(t)\varphi| \leqslant L_1 \|\varphi\|, \qquad t \geqslant t_0,$$
$$|g(t,\varphi)| \leqslant l(h)\|\varphi\|, \qquad t \geqslant t_0.$$

If the zero solution of (6. 3. 2) is uniformly asymptotically stable, then there exist $\delta_0>0$, $M=M(\delta_0)>0$ together with two positive function $b(u)$, $c(u)$ $(0\leqslant u\leqslant\delta_0)$ and a continuous scalar function $V(t,\varphi)$ $(t\in I,\ \varphi\in C_n[-h,0])$ such that for $t\geqslant t_0$, φ_1, $\varphi_2\in C_n[-h,0]$ with $\|\varphi_i\|\leqslant\delta_0$ $(i=1,2)$

$$|D(t)\varphi| \leqslant V(t,\varphi) \leqslant b(\|\varphi\|),$$
$$\left.\frac{\mathrm{d}^+V(t,\varphi)}{\mathrm{d}t}\right|_{(6.3.2)} \leqslant -c(|D(t)\varphi|),$$
$$|V(t,\varphi_1)-V(t,\varphi_2)| \leqslant M\|\varphi_1-\varphi_2\|.$$

Lemma 6. 3. 4. *If $D(t)$ in the system (6. 3. 2) satisfies the conditions of Lemma 6. 3. 3, then for arbitrary $r_0>0$, there exists a constant $L=L(r_0)$ such that for φ_1, $\varphi_2\in C_n[-h,0]$ with $\|x_t(t_0,\varphi_1)\|\leqslant r_0$ and $\|x_t(t_0,\varphi_2)\|\leqslant r_0$ and for all $t\geqslant t_0$*

$$\|x_t(t_0,\varphi_1)-x_t(t_0,\varphi_2)\| \leqslant \mathrm{e}^{L(t-t_0)}\|\varphi_1-\varphi_2\|.$$

Chukwu[1] generalized the result of Somolinos[1] to the system of neutral functional differential equations.

Theorem 6. 3. 5. *Suppose that the zero solution of equations (6. 3. 2) is uniformly asymptotically stable. Let γ^2 and K be defined as in Lemma 6. 3. 2. Further, let us assume that*

1) *$D(t)$, $A(t,\cdot)$ and $B(t,\cdot)$ are bounded linear operators mapping $C_n[-h,0]$ to $\mathbf{R}^n$ and*

$$|D(t)\varphi| \leqslant K\|\varphi\|, \qquad t\geqslant t_0,\quad \varphi\in C_n[-h,0],$$
$$|A(t,\varphi)| \leqslant L|D(t)\varphi|, \qquad t\geqslant t_0,\quad \varphi\in C_n[-h,0],$$
$$|B(t,\varphi)| \leqslant c|D(t)\varphi|, \qquad t\in I;$$

2) *$f\in F_k=\{f:\ f(0)=0,\ 0<\sigma f(\sigma)\leqslant k\sigma^2,\ f\in C(-\infty,+\infty)\}$ and*

$$\int_0^{\pm\infty} f(s)\mathrm{d}s = +\infty;$$

3) *for $\theta\in[0,h]$ and a function $l(\theta)$, there exists*

$$4\gamma^2 r > \left(c+\frac{K|b|}{1-l(\theta)}\right)^2;$$

4) *the operator D is uniformly stable.*

Then the zero solution of the system (6. 3. 1) *is absolutely stable in* $[0, k]$.

Proof. 1° From the hypothesis, the conditions of Lemma 6. 3. 1 are satisfide. Hence we use the same Liapunov functional $V(t,\psi)$ as in Lemma 6. 3. 1. Suppose that $x = x(t_0,\varphi)$ is the solution of (6. 3. 2) and that $y = y(t_0,\varphi)$ is the solution of (6. 3. 1). We write $D^+V|_{(6.3.1)}$ as the upper right derivative of $V(f,\psi)$ along the solution of (6. 3. 1). Then we deduce

$$D^+ V|_{(6.3.1)} = \overline{\lim_{\tau\to 0^+}} \frac{1}{\tau}[V(t+\tau, y_{t+k}(t,\psi)) - V(t,\psi)]$$

$$\leqslant \overline{\lim_{\tau\to 0^+}} \frac{1}{\tau}[V(t+\tau, y_{t+k}(t,\psi)) - V(t+\tau, x_{t+\tau}(t,\psi))]$$

$$+ \overline{\lim_{\tau\to 0^+}} \frac{1}{\tau}[V(t+\tau, x_{t+\tau}(t,\psi)) - V(t,\psi)],$$

where $x(t,\psi)$ is a solution of (6. 3. 2) through (t,ψ), i. e. , $x_t = \psi$.

Clearly, the second part of above inequality is equal to $D^+ V|_{(6.3.2)}$.

In the following, we estimate the first part. Noting that

$$\begin{aligned} & D(t+\tau)(y_{t+\tau}(t,\psi) - x_{t+\tau}(t,\psi)) \\ &= D(t+\tau)y_{t+\tau}(t,\psi) - D(t)y_t(t,\psi) \\ &\quad + D(t)x_t(t,\psi) - D(t+\tau)x_{t+\tau}(t,\psi) \\ &= \int_\tau^{t+\tau}[A(\xi,y_\xi) + bf(\sigma(\xi))]d\xi - \int_t^{t+\tau} A(\xi,x_\xi)d\xi \\ &= \int_\tau^{t+\tau}[A(\xi,y_\xi) - A(\xi,x_\xi)]d\xi + \int_t^{t+\tau} bf(\sigma(\xi))d\xi, \end{aligned}$$

and that

$$\begin{aligned} & D(t+\tau)(y_{t+\tau}(t,\psi) - x_{t+\tau}(t,\psi)) \\ &= [y(t+\tau) - x(t+\tau)] - [g(t+\tau, y_{t+\tau}) - g(t+\tau, x_{t+\tau})], \end{aligned}$$

we find

$$\begin{aligned} |y(t+\tau) - x(t+\tau)| &\leqslant |D(t+\tau)(y_{t+\tau} - x_{t+\tau})| + l(h)\, \| y_{t+\tau} - x_{t+\tau} \| \\ &\leqslant KL \int_t^{t+\tau} \| y_\xi - x_\xi \| d\xi + \int_t^{t+\tau} |bf(\sigma(\xi))| d\xi \\ &\quad + l(h)\, \| y_{t+\tau} - x_{t+\tau} \| . \end{aligned}$$

In fact, there exists

$$\begin{aligned} & \sup_{-\tau\leqslant\theta\leqslant 0} |y(t+\tau+\theta) - x(t+\tau+\theta)| = \| y_{t+\tau} - x_{t+\tau} \| \\ &\leqslant \sup_{-\tau\leqslant\theta\leqslant 0}\Big[KL \int_t^{t+\tau+\theta} \| y_\xi - x_\xi \| d\xi + l(h)\, \| y_{t+\tau+\theta} - x_{t+\tau+\theta} \| \\ &\quad + \int_t^{t+\tau+\theta} |bf(\sigma(\xi))| d\xi\Big] \end{aligned}$$

$$\leqslant KL\int_t^{t+\tau+\theta} \| y_\xi - x_\xi \| \mathrm{d}\xi + l(h) \| y_{t+\tau} - x_{t+\tau} \|$$

$$+\int_t^{t+\tau+\theta^*} |bf(\sigma(\xi))|\mathrm{d}\xi, \qquad -\tau \leqslant \theta^* \leqslant 0.$$

Hence,

$$\| y_{t+\tau} - x_{t+\tau} \| \leqslant \frac{1}{1-l(h)}\left[KL\int_t^{t+\tau} \| y_\xi - x_\xi \| \mathrm{d}\xi + \int_t^{t+\tau} |bf(\sigma(\xi))|\mathrm{d}\xi\right]$$

and

$$\overline{\lim_{\tau\to 0^+}} \frac{1}{\tau} \| y_{t+\tau} - x_{t+\tau} \| \leqslant \frac{|bf(\sigma(t))|}{1-l(h)}.$$

Therefore,

$$\mathrm{D}^+ V|_{(6.3.1)} \leqslant -\gamma^2 V + \frac{K}{1-l(h)}|bf(\sigma(t))|.$$

2° Defining

$$W = \frac{1}{2}V^2 + \int_0^\sigma f(s)\mathrm{d}s,$$

we have

$$\mathrm{D}^+ W|_{(6.3.1)} \leqslant -\gamma^2 V^2 - r|f(\sigma)|^2 + V\left(\frac{K|b|}{1-l(h)} + c\right)|f(\sigma)|.$$

The right-hand side of the above expression is a quadratic form with respect to V and $|f(\sigma)|$. From the condition 3), we know that it is negative definite.

Then, there exists a constant m with $m > 0$ such that

$$\mathrm{D}^+ W|_{(6.3.1)} \leqslant -m(V^2 + |f(\sigma)|^2) \leqslant -m|D(t)\psi|^2.$$

Finally, using Theorem 4.1 from Cruz and Hale [1], we conclude that Theorem 6.3.5 is true. ▮

Ruan Jong studied the absolute stability of the Lurie direct control system of neutral type:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}[D(t)x_t] = A(t,x_t) + bf(\sigma(t)), & t \geqslant t_0, \\ \sigma(t) = c^{\mathrm{T}}D(t)x_t, \\ x_{t_0} = \varphi, \quad t_0 \in I, \quad \varphi \in C_n[-h,0]. \end{cases} \tag{6.3.3}$$

Theorem 6.3.6. *Let the following condition hold:*

1) *$D(t)$ and $A(t,\cdot)$ are linear bounded operators mapping $C_n[-h,0]$ to $\mathbf{R}^n$ such that*

$$|D(t)\varphi| \leqslant L_1 \| \varphi \|, \qquad |A(t,\varphi)| \leqslant L|D(t)\varphi|.$$

for all $t \geqslant t_0$.

2) *the zero solution of phase equations* (6.3.2) *is globally uniformly asymptotically stable*;

3) $f \in F_k = \{f: f(0) = 0,\ 0 < \sigma f(\sigma) \leqslant k\sigma^2,\ f \in C(-\infty, +\infty)\}$,

$$\int_0^{\pm\infty} f(s)\mathrm{d}s = +\infty;$$

4) *the operator* D *is uniformly stable and it is atomic at* 0.

In addition, let $l(h)<1$. *Then the zero solution of the system* (6.3.3) *is absolutely stable in* $[0, k]$ *if one of the four following conditions is satisfied*:

1) $0<\alpha^*<1$;

2) $\alpha^*=1$ *and* $|\rho|>\frac{L}{k}>0$;

3) $2-\frac{L}{k\rho}<\alpha^*<2$ *and* $\rho>\frac{L}{k}>0$;

4) $1<\alpha^*<2+\frac{L}{k\rho}$ *and* $\rho<-\frac{L}{k}$ (<0),

where $\alpha^*=kM|b|\cdot|c|/[1-l(h)]\gamma^2$, γ^2 *and* M *are the same as in Lemma* 6.3.2, *and* $\rho=-c^{\mathrm{T}}b$.

Proof. 1° As in Theorem 6.3.5, we obtain

$$\mathrm{D}^+ V|_{(6.3.3)} \leqslant -\gamma^2 V + \frac{M}{1-l(h)}|bf(\sigma(t))|.$$

2° Let

$$W = \frac{1}{2}V^2 + \beta\int_0^{\sigma} f(s)\mathrm{d}s,$$

where β is the undetermined constant. We get

$$\begin{aligned}\mathrm{D}^+ W|_{(6.3.3)} &\leqslant -\gamma^2 V^2 + V\frac{M|b|\cdot|f(\sigma)|}{1-l(h)} + \beta f(\sigma)\dot{\sigma} \\ &\leqslant -\gamma^2 V^2 - \rho\beta|f(\sigma)|^2 + \left[\frac{M|b|}{1-l(h)} + |\beta|\cdot L\cdot|c|\right]V|f(\sigma)| \\ &= -\gamma^2[V^2 - 2pV|f(\sigma)| + q|f(\sigma)|^2],\end{aligned}$$

where

$$2p = \left[\frac{M|b|}{1-l(h)} + |\beta|\cdot|c|L\right]\Big/\gamma^2, \qquad q = \frac{\beta\rho}{\gamma^2}.$$

In the remainder of this proof, we proceed along the line of Theorem 6.1.4 with M replaced by $M/[1-l(h)]$. We obtain that if one of the four conditions is satisfied, then we can choose β such that

$$\mathrm{D}^+ W|_{(6.3.3)} \leqslant -m|D(t)\psi|^2, \qquad m>0.$$

The proof of the theorem is completed. ∎

6.4. Control Systems in Hilbert Spaces

In this section, we introduce a notion of the absolute stability of the Lurie control system in Hilbert space.

Consider the Lurie indirect control system

$$\begin{cases} \dfrac{dx}{dt} = Ax + bu, \\ \dfrac{du}{dt} = \varphi(\sigma), \\ \sigma = (c, x) - \rho u, \end{cases} \tag{6.4.1}$$

where the operator A is either bounded, or is assumed to generate C_0 strongly continuous group $T(t)$, $t \in (-\infty, +\infty) = \mathbf{R}$. On a real Hilbert space X, we denote an inner product by $(\cdot, \cdot)$ and a norm by $|\cdot|$. Above, b, $c \in X$, u, $\rho \in \mathbf{R}$, and $\varphi: \mathbf{R} \to \mathbf{R}$ is a continuously uniformly Lipschitzan nonlinear function which satisfies the following properties:

$$\sigma\varphi(\sigma) > 0 \quad \text{for } \sigma \neq 0, \qquad \varphi(0) = 0,$$
$$|\varphi(\sigma)| \leqslant K(|\sigma|) \qquad \text{for all } \sigma \in \mathbf{R},$$

where $K(s)$ are some monotonically nondecreasing function, $s \in \mathbf{R}_+ = (0, +\infty)$.

Assume that the linear phase equation

$$\frac{dx}{dt} = Ax \tag{6.4.2}$$

is exponentially stable, i.e., that there exist constants $M \geqslant 1$ and $\alpha > 0$ such that

$$|T(t)|_{L(x)} \leqslant Me^{-\alpha}, \qquad t \geqslant 0, \tag{6.4.3}$$

where $L(x)$ is the Banach space of bounded linear operator from X to X. Because of the condition (6.4.3), it follows from Theorem 2.1 and Theorem 2.2 in Pao [1] that there is an unique symmetric positive definite bounded operator P on X such that

$$(PAx, x) + (x, PAx) = -(x, x), \tag{6.4.4}$$

where $T(t)$ is a strongly continuous group satisfying

$$(A^* Px, x) + (PAx, x) \leqslant -\lambda \|x\|^2 \tag{6.4.5}$$

for any λ with $0 < \lambda < 1$. If $T(t)$ is a strongly continuous semigroup and A is bounded, a similar result is given by Walker[1]. When A satisfies

$$(x, (A - wI)x)_X \leqslant 0$$

for real $w \in \mathbf{R}$ and for all x in the domain of A, it is clear that (6.2.1) can be regarded as equations in the Hilbert Space $H = X \times \mathbf{R}$ with the inner product $(\cdot, \cdot)$ defined by

$$((x_1, r_1), (x_2, r_2)) = (x_1, x_2) + r_1 r_2.$$

Theorem 6.4.1. 1) *Let the origin be the only singular point of* (6.4.1);

2) *Let P be a unique symmetric positive definite bounded operator on H given by* (6.4.5);

3) *Let λ in* (6.4.5) *satisfy*

$$\lambda\rho > \left| Pb + \frac{c}{2} \right|^2;$$

4) $\int_0^\sigma \varphi(s)\mathrm{d}s \to +\infty$ *as $\sigma \to +\infty$.*

Then the zero solution of (6.4.1) *is uniformly asymptotically stable in the large.*

Proof. Since (6.4.2) is uniformly exponentially stable, there exsts a unique symmetric positive definite bounded operator on H such that

$$(A^* Px, x) + (PAx, x) \leqslant -\lambda|x|^2$$

for some λ $(0 < \lambda \leqslant 1)$. We use P to define the functional on H:

$$V(x,\mu) = (Ax + b\mu, P(Ax + b\mu)).$$

Let

$$U(x,\mu) = \int_0^\sigma \varphi(s)\mathrm{d}s, \qquad \sigma = (c,x) - \rho\mu,$$

and

$$W = V + U.$$

It is easy to prove that W is positive definite. Since P is symmetric and positive definite if and only if

$$\delta|x|^2 \leqslant |(x,Px)| \leqslant l|x|^2$$

where $|P| \leqslant l$, we have the estimate

$$\delta|Ax + b\mu|^2 + \int_0^\sigma \varphi(s)\mathrm{d}s$$

$$\leqslant W(x,\mu) \leqslant l|Ax + b\mu|^2 + \int_0^\sigma \varphi(s)\mathrm{d}s,$$

so that

$$\begin{aligned}\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(6.4.1)} &= \left(\frac{\mathrm{d}}{\mathrm{d}t}(Ax + b\mu), P(Ax + b\mu)\right) + \left(Ax + b\mu, \frac{\mathrm{d}}{\mathrm{d}t}(Ax + b\mu)\right) \\ &= (A(Ax + b\mu) + b\varphi(\sigma), P(Ax + b\mu)) \\ &\quad + (Ax + b\mu, P(A(Ax + b\mu) + b\varphi(\sigma))) \\ &= (Ax + b\mu, PA(Ax + b\mu)) + (PA(Ax + bu), Ax + bu) \\ &\quad + (Ax + b\mu, Pb\varphi(\sigma)) + (Pb\varphi(\sigma), Ax + b\mu),\end{aligned} \tag{6.4.6}$$

$$\begin{aligned}\left.\frac{\mathrm{d}U}{\mathrm{d}t}\right|_{(6.4.1)} &= \varphi(\sigma)\dot{\sigma} = [(c, Ax + b\mu) - \rho\varphi(\sigma)]\varphi(\sigma) \\ &= (Ax + b\mu, c)\varphi(\sigma) - \rho\varphi^2(\sigma).\end{aligned} \tag{6.4.7}$$

Hence, using (6.4.6), (6.4.7), (6.4.4), we deduce

$$\left.\frac{\mathrm{d}W}{\mathrm{d}t}\right|_{(6.4.1)} = (A^* P(Ax + b\mu), Ax + b\mu) + (PA(Ax + b\mu), Ax + b\mu)$$
$$+ (Ax + b\mu, Pb\varphi(\sigma)) + (Pb\varphi(\sigma), Ax + b\mu)$$
$$+ (Ax + b\mu, c)\varphi(\sigma) - \rho\varphi^2(\sigma)$$
$$\leqslant -\lambda |Ax + b\mu|^2 + 2\left(Ax + b\mu, Pb + \frac{c}{2}\right)\varphi(\sigma) - \rho\varphi^2(\sigma).$$

Therefore, the condition $\lambda\rho > \left|Pb + \frac{c}{2}\right|^2$ leads to

$$\left.\frac{\mathrm{d}W}{\mathrm{d}t}\right|_{(6.4.1)} \leqslant -\lambda\left[\,|Ax + b\mu|^2 + 2\left|Ax + \frac{c}{2}\right|\varphi(\sigma) - \rho\varphi^2(\sigma)\right]$$
$$\leqslant -k[\,|Ax + b\mu|^2 + \rho\varphi^2(\sigma)] \leqslant 0, \qquad 0 < k \ll 1,$$

and $\left.\frac{\mathrm{d}W}{\mathrm{d}t}\right|_{(6.4.1)} = 0$ if and only if $x = 0$ and $\mu = 0$. As a result, we write

$$S \triangleq \left\{(x,\mu) \in H : \left.\frac{\mathrm{d}W}{\mathrm{d}t}\right|_{(6.4.1)} = 0\right\} = \{0,0\},$$
$$W(x(t),\mu(t)) \leqslant W(x(0),\mu(0)), \qquad t \geqslant 0.$$

Therefore, every solution $(x,\mu) \in H$ is bounded. The orbits of (6.4.1) form a precompact subset of H. The invariance principle of Hale [1, p. 50] yields $(x(t), \mu(t)) \to 0$ as $t \to +\infty$. This concludes the proof. ▮

In the following, we will apply the idea of Lakshmikantham to generalize the results to the system with nonlinear phase equation:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(x) + b\mu, \\ \dfrac{\mathrm{d}\mu}{\mathrm{d}t} = \varphi(\sigma), \\ \sigma = (c,x) - \rho\mu, \end{cases} \tag{6.4.8}$$

where b, c, μ are defined as in (6.4.1) and $f: X \to X$ is a continuous Frechet differentiable function whose Frechet derivative at x is $A(x)$. To ensure that solutions of (6.4.8) exist, we shall always assume, for example, that $\varphi: \mathbf{R} \to \mathbf{R}$ is continuous, uniformly Lipschitzian and that $-f$ is a monotone function. In other words, there exists a constant M such that

$$(f(u) - f(v), u - v) \leqslant M|u - v|^2, \qquad u, v \in X.$$

We now state a basic stability comparison theorem for the system

$$\dot{x} = l(t,x), \tag{6.4.9}$$

where l: $\mathbf{R}^+ \times X \to X$ is continuous.

Lemma 6.4.2. *Assume that the following conditions hold:*

1) $V \in (\mathbf{R}^+ \times X, \mathbf{R}^+)$ *and for* (t,x_1) *and* $(t,x_2) \in \mathbf{R}^+ \times X$, *there exists*

$$|V(t,x_1) - V(t,x_2)| \leqslant L(t)|x_1 - x_2|,$$

where $L(t) \geqslant 0$ *and is continuous on* $\mathbf{R}^+$;

2) *there exists a function* $g \in C[\mathbf{R}^+ \times \mathbf{R}^+, \mathbf{R}]$ *such that for each* $(t,x) \in \mathbf{R}^+ \times \mathbf{R}$,

$$\mathrm{D}^+ V(t,x) \triangleq \overline{\lim_{h \to 0^+}} \frac{1}{h}[V(t+h, x+hl(t,x)) - V(t,x)]$$
$$\leqslant g(t, V(t,x));$$

3) *for each* $(t_0, r_0) \in \mathbf{R}^+ \times \mathbf{R}^+$, *the maximal solution* $r(t,t_0,r_0)$ *of the scalar initial value problem*

$$\begin{cases} \dfrac{\mathrm{d}r}{\mathrm{d}t} = g(t,r), \\ r(t_0) = r_0 \end{cases} \tag{6.4.10}$$

exists for $t > t_0$;

4) $f(t,0) \equiv 0$, $g(t,0) \equiv 0$ *and* $V(t,0) \equiv 0$, $t \in \mathbf{R}^+$;

5) *there exist functions* $a(r)$ *and* $b(r) \in K$ *such that*

$$b(\|x\|) \leqslant V(t,x) \leqslant a(\|x\|) \qquad \text{for} \quad (t,x) \in \mathbf{R}^+ \times X.$$

Then if the zero solution of (6.4.10) *is uniformly asymptotically stable in the large then the zero solution of* (6.4.9) *has the same property.*

Theorem 6.4.3. *Let in* (6.4.8) $f(0)=0$ *and* $\varphi(0)=0$. *Again let* $A(x)$ *be the Frechet derivative of* $f(x)$ *at* x. *Suppose that*

1) *there exists a symmetric positive definite operator* P *such that*

$$((PA(x) + A^*(x)P)y, y) \leqslant -\lambda |y|^2$$

for all x *and* y *in* X *and some* $\lambda > 0$, *where* A^* *is the adjoint of* A;

2) $\varphi(s) \cdot \operatorname{sgn} s > 0$, $\varphi(s) \cdot \operatorname{sgn} s \to +\infty$ *as* $|s| \to +\infty$,

$$\varphi'(s) \geqslant \frac{1}{2}\lambda_1 \qquad \text{for some} \quad \lambda_1 \geqslant |P|;$$

3) $|f(x) + b\mu| \to +\infty$ *as* $|x| + |\mu| \to +\infty$;

4) $\lambda\rho > \left| Pb + \dfrac{c}{2} \right|^2$.

Then the origin of (6.4.8) *is uniformly asymptotically stable in the large.*

Proof. Let $H = X \times \mathbf{R}$ be equipped with the inner product $(\cdot,\cdot)$ defined by

$$((x_1, r_1), (x_2, r_2))_H = (x_1, x_2) + r_1 r_2.$$

Let $V: H \to R$ be defined by

$$V = W + U,$$

where

$$W = (f(x) + b\mu, p(f(x) + b\mu)),$$
$$U = \int_0^\sigma \varphi(s)\mathrm{d}s.$$

Since P is positive definite and symmetric, there exist positive constants λ_1 and λ_2 such that

$$\lambda_2 |f(x) + b\mu| \leqslant \lambda_1 |f(x) + b\mu|^2,$$

with $\lambda_1 \geqslant |P|$. Hence

$$\begin{aligned} &\lambda_2 |f(x) + b\mu|^2 + \int_0^\sigma \varphi(s)\mathrm{d}s \\ &\leqslant V(x,\mu) \leqslant \lambda_1 |f(x) + b\mu|^2 + \int_0^\sigma \varphi(s)\mathrm{d}s. \end{aligned} \tag{6.4.11}$$

It follows from the condition 2) that $\int_0^\sigma \varphi(s)\mathrm{d}s \to +\infty$ as $\sigma \to +\infty$. Thus, by 3) and (6.4.11), we find that the condition 5) of Lemma 6.4.2 is satisfied. Noting

$$\begin{aligned} \frac{\mathrm{d}W}{\mathrm{d}t}\bigg|_{(6.4.8)} &= \left(\frac{\mathrm{d}}{\mathrm{d}t}(f(x) + b\mu), P(f(x) + b\mu)\right) \\ &\quad + \left(f(x) + b\mu, P\frac{\mathrm{d}}{\mathrm{d}t}(f(x) + b\mu)\right) \\ &= (A(x)[f(x) + b\mu] + b\varphi(\sigma), P[f(x) + b\mu]) \\ &\quad + (f(x) + b\mu, P\{A(x)[f(x) + b\mu] + b\varphi(\sigma)\}) \\ &= (PA(x)[f(x) + b\mu], f(x) + b\mu) \\ &\quad + (A^*(x)P[f(x) + b\mu], f(x) + b\mu) \\ &\quad + 2(Pb\varphi(\sigma), f(x) + b\mu) \\ &\leqslant -\lambda |f(x) + b\mu|^2 + 2(Pb\varphi(\sigma), f(x) + b\mu), \end{aligned}$$

and

$$\frac{\mathrm{d}U}{\mathrm{d}t}\bigg|_{(6.4.8)} = \varphi(\sigma)\dot{\sigma} = \varphi(\sigma)(c, f(x) + b\mu) - \rho\varphi^2(\sigma),$$

we obtain

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t}\bigg|_{(6.4.8)} &\leqslant \frac{\mathrm{d}W}{\mathrm{d}t}\bigg|_{(6.4.8)} + \frac{\mathrm{d}U}{\mathrm{d}t}\bigg|_{(6.4.8)} \\ &\leqslant -\lambda |f(x) + b\mu|^2 + 2\varphi(\sigma)\left(Pb + \frac{c}{2}, b(x) + b\mu\right) - \rho\varphi^2(\sigma) \\ &\leqslant -\lambda |f(x) + b\mu|^2 + 2|\varphi(\sigma)| \cdot \left|Pb + \frac{c}{2}\right| \cdot |f(x) + b\mu| \\ &\quad - \rho\varphi^2(\sigma). \end{aligned}$$

The condition $\lambda\rho > \left|Pb + \frac{c}{2}\right|^2$ implies that there exists $0 < \lambda_3 \ll 1$ such that

$$\frac{\mathrm{d}V}{\mathrm{d}t}\bigg|_{(6.4.8)} \leqslant -\lambda_3[|f(x) + b\mu|^2 + \varphi^2(\sigma)].$$

By virtue of (6. 4. 11), we have the inequality

$$\left\{V(x,u) - \int_0^{\sigma}\varphi(s)\mathrm{d}s\right\}\frac{1}{\lambda_1} \leqslant |f(x) + b\mu|^2,$$

and thus

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(6.4.8)} \leqslant -\lambda_3\left\{\frac{V}{\lambda_1} - \frac{1}{\lambda_1}\int_0^{\sigma}\varphi(s)\mathrm{d}s + \phi^2(\sigma)\right\}.$$

Since $\phi^2(0) = 0$, it follows that

$$\phi^2(\sigma) = 2\int_0^{\sigma}\phi'(s)\varphi(s)\mathrm{d}s.$$

Consequently,

$$\frac{1}{\lambda_1}\int_0^{\sigma}\varphi(s)\mathrm{d}s - \phi^2(s) = \int_0^{\sigma}\left[\frac{1}{\lambda_1} - 2\phi'(s)\right]\varphi(s)\mathrm{d}s \leqslant 0.$$

Using $\varphi(s)\cdot\operatorname{sgn} s > 0$ and $\frac{1}{2}\lambda_1 - \phi'(s) \leqslant 0$, condition 2), and the above remarks, we obtain the final inequality

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(6.4.8)} \leqslant -\frac{\lambda_3}{\lambda_1}V(x,u).$$

Thus, the comparison equation takes the form

$$\begin{cases} \dot{r}(t) = -\dfrac{\lambda_3}{\lambda_1}r(t), \\ r(t_0) = r_0. \end{cases} \tag{6.4.12}$$

It is easy to prove that the solution

$$r(t) = r_0\exp\left[-\frac{\lambda_3}{\lambda_1}(t - t_0)\right], \qquad t \geqslant t_0$$

of (6. 4. 12) is uniformly asymptotically stable. Theorem 3. 1 of Ladas and Lakshmikantham [1] yields the uniform asymptotic stability of the zero solution of (6. 4. 8). ∎

6. 5. Notes

Theorems 6. 1. 1, 6. 1. 2 are due to Somolinos [1]. Theorem 6. 1. 3 is from Zhu Siming [2]. Theorem 6. 1. 4 is from Ruan Jong [2]. Theorem 6. 2. 1 is based on Ruan Shigui and Wu Jianhong [1]. Lemmas 6. 3. 2 ~ 6. 3. 4 and Theorem 6. 3. 5 belongs to Chukwu [1]. Theorem 6. 3. 6 is due to Ruan Jong [3]. All the results of Section 6. 4 are based on Chukwu [2].

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