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Ethelbert Nwakuche Chukwu

Economic Dynamics of All Members of the United Nations

Second Edition

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Economic Dynamics of All Members of the United Nations

Second Edition



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Preface

After the meetings of Asian and European Leaders in Beijing, China on April 12, 2008, and their joint stand on the solution of the world economic crises at the Washington, November 15, 2008, G20 meeting and the subsequent economic summit of World Leaders in London, Chief Financial Leaders from the world's top seven economic powers pledged, as reported by Jeannine Aversa of the Associated Press, Saturday, April 25, 2009, in the News and Observer:

“The group of seven participants Japan, Germany, France, Britain, Italy and Canada promised to provide the necessary fiscal tonic to turn around their troubled economies as follows—tax cuts or increased government spending; commitment to act together to restore jobs and growth.”

The USA Treasury Secretary, Timothy Geithner and his counterparts made this promise in a joint statement. The news report touched upon new financial commitments to raise \$500 billion for IMF lending and the difficulty of getting China, Brazil, Russia, India, and Saudi Arabia on board. A bigger say in the operation of IMF was needed to be put in place.

The creation of jobs and the restoration of economic growth are the main arguments embodied in Chukwu's current book. The path of cooperation is emphasized. Implicitly they embraced and accepted the conclusions of the book, “The Omega Problem of All Members of the United Nations”.

By affirming the joint statement “We will take whatever actions are necessary” to bring that about—to prevent “a crisis of this magnitude from occurring again”, “fixing financial institutions in the U.S. and World wide and jump-starting lending”. What is accepted implicitly is the analysis of Chukwu on Differential Models and Neutral Systems for controlling the Wealth of Nations, World Scientific 2001, Singapore, pp. 293–296 as highlighted in the forward.

Acknowledgments

I thank Emeka Chukwu, my MATLAB Programming Assistant.

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Dr. C. K. Chui of Stanford University, Editor in Chief of Atlantis is incomparable and great. To the publisher of Atlantis Press, Dr. Keith Jones, and the Founder Zeger Karssen, my gratitude is beyond belief. Thank you all.

For her Secretarial Assistance and excellent typing of the manuscript, I owe gratitude beyond words to Mrs. Joyce Sorensen.

The Product Manager of Atlantis Press, Willie van Berkum, whose careful review of this book was indeed professional. It made Atlantis to avoid numerous problems before the typesetting of the book. I owe him huge gratitude.

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Introduction

In a recent paper the author postulated that the most important applied mathematics problem since the world began is connected with the worldwide conquest of scarcity subject to the values of love and goodness. (See Genesis 3:17–18; Isaiah 58:6–8; 9–12; Malachi 3:10–12; and Matthew 25:31–40). The production of goods and services and their distribution are considered to be the core of applied mathematics problem in the sense that mathematical economic state of all nations—the gross domestic product, interest rate, employment, value of capital stock, prices (and therefore inflation), and cumulative balance of payment—is mathematically derived and identified as a dynamical system with interacting solidarity matrix. Included in the model are government strategies—generalized taxes, autonomous government outlay, exchange rate, tariff, trade policy or distance between trading nations, money supply and its flows, autonomous net capital outflow to foreigners by government. Included also is the representative private firms strategies—autonomous consumption, autonomous investment, autonomous net export, autonomous money demand, labor productivity, wage rate, autonomous income consumption intercept, and autonomous price intercept. By “autonomous” we mean that these do not depend on variables of the economic state. It is possible to include nongovernment organization (NGO) contributions. The production of abundant goods and services driven by government and private strategies is the fundamental aim. With the mathematical economic model which joins all economies of nations, one asks the question whether it is possible to steer each economic state of a nation from low growth of GDP, high interest rate, low employment, low value of capital stock, high inflation, and little cumulative balance of payment to the state of abundance of goods and services. It is then possible, if humans are willing, to feed the hungry, clothe the naked, shelter the homeless, take care of the sick and heal them, visit and care for prisoners, welcome strangers and foreigners to our home and to our countries. Thus, we uphold the uplifting value of love and goodness. How can this be done? First, we take the underlying Applied Mathematical economic problem of improving the economic state of nations using government and representative firm’s strategies. A measure of economic performance which is to be improved is the gross domestic product GDP. It is a measure of the total income generated through the production of goods and services, the so-called output y —the supply. Assuming the differential principle of supply and demand and the principle of

rational expectation, we derive the dynamics of the six components of each country's economic state. Using MATLAB computer program and IMF, UN data, all the coefficients of the dynamics are identified. The system is a differential game of pursuit of the form

$$\dot{x}(t) - A_{-1}\dot{x}(t - h) = A_0x(t) + A_1x(t - h) + A_2(x)x(t - h) + B_1p + B_2q,$$

with government strategy q as the quarry and the representative firms' control function as the pursuer, p , which can be supplemented by nongovernment organization contribution, NGO. By the method of Hájek [1] and Chukwu [2] the system is converted to a control system with 191 state variables and a control variable u a member of the Pontryagin difference of the control set of government Q and the control set of the private firms representative P .

Once this is set up, one studies the controllability and the optimal growth of the economic state of

$$\dot{x}(t) - A_{-1}\dot{x}(t - h) = A_0x(t) + A_1x(t - h) + A_2(x(t))x(t - h) + Bu,$$

where x has 191 components corresponding to member states of the UN. Thus,

$$x = [y_1, y_2, y_3, \dots, y_{191}],$$

and y_i is the GDP of country i . We aim to have material abundance. For this we need huge resources. These are currently available, e.g., from the Sun. It will take at least 3 billion years to exhaust the energy of the Sun. What is available from the waters of the oceans is tremendous. Included are the tons of sand of the deserts, and of the sea—shores—the sand for the silicon and computer chips, and for building, shelter. Ndu Chukwu in a project at St. Augustines College, Raleigh, North Carolina, a follow-up of NASA Project: "Colonization of the Moon," indicated that even the deserts of Peru, Arizona, California can be watered, colonized, and made to bloom. The down pour of the rainstorm caused by El Niño proved him right. It was documented by CBS which witnessed the fulfillment of the prophecy of Isaiah 35:1–2: "The desert will bloom." There is more arable land for humankind to grow its economy than is recognized. We now present the economic model which joins all economies of all nations and present how it is possible to steer each nation's economic state of low growth of GDP, high interest rate, low employment, low value of capital stock, high inflation, and negative cumulative balance of payment to the state of paradise—high growth rate of GDP, low interest, full employment, low prices (or small inflation), and great cumulative balance of payment. It will be shown that it is possible to steer to this target with minimum investment, and do it in minimum time. All nations of the world can then use this model to implement an optimal economic strategy and usher in universal abundance and prosperity. The needed huge resources are already available in our world. Closely linked with this first section of our study is the issue of longevity. It seems that life can be prolonged beyond our imagination: 500 years and going strong. By caloric restriction and manipulating the anti-aging

genes, and by exercise, humans can live longer. It has been observed that there was a 30 years gain in the age of humans in the last century and that longevity and health are related to the generation of wealth. Countries that have a 5-year advance in longevity compared to other countries have a greater Gross Domestic Product. Increasing health and longevity will enable countries to create greater wealth and prosperity. Citizens will live longer as healthy productive individuals. Thus, GDP increases. The value which was earlier articulated, “Take care of the sick,” now assumes a prominent position as a component of the omega problem of all members of the United Nations.

We briefly outline the contents of the book. In Chap. 6, we use the principle of rational expectations and the principle of differential supply and demand to derive differential mathematical models of the economic state of all members of the United Nations. Later in Chap. 14, these equations are validated with historical data from the International Statistical Yearbook and United Nations National Accounts Statistics: Main Aggregates and Detailed Tables, UN, New York. The generic type of these equations is reproduced for each nation from Chukwu, Optimal Control of the Growth of Wealth of Nations, Taylor and Francis 2005 and with Matlab identified in Chap. 14 below, or Chukwu, Stability and Time-Optimal Control of Hereditary Systems with Application to Economic Dynamics of the US, 2nd Edition, World Scientific, 2001, Singapore Chapter 1:10. The Economic State consists of the popular ones, six components-Gross Domestic Product, (GDP) y , Interest Rate R , Employment (or Unemployment) L , Value of Capital Stock K , Prices, (or inflation \dot{p}), and cumulative Balance of Payment, E . The Control Strategies of Government and of the representative private firms are also identified. From these, several real strategies for growth in a nation are highlighted for implementation. The usual analysis for stability, controllability, and permanence are made.

When all the nations are linked up together with an interconnecting function $f(t) = A_2(x(t))x(t - h)$, the emerging dynamics is

$$\dot{x}(t) - A(x(t - h)) = A_0x(t) + A_1x(t - h) + A_2x(t)x(t - h) + Bu,$$

in which the Matlab programs identify A , A_0 , A , A_2 and B . Here u is a member of the Pontrygin difference of sets P and Q . Here x has 191 components corresponding to member states of the UN. Thus,

$$x = [y_1, y_2, \dots, y_{191}].$$

y_i is the GDP of country i .

Cooperation studies can be continued in the direction of the insight of T. G. Hallam, “Community Dynamics in a Homogeneous Environment in Mathematical Ecology,” Biomathematics, Vol. 17, p242.

Biomathematics ed. T. G. Hallam and S. A. Levin, Springer-Verlag, Berlin-Heidelberg, 1986, and the emergence, if conditions permit, of the so-called “orgy of mutuality.”

In [Chaps. 9](#) and [1](#), we study together the two components of the national economic state, Employment and Gross Domestic when the two nations are interacting and the interacting functions are explicitly defined. The control strategies of the representative firms and the government are explicit and can be used to deduce economic policies. China and USA are two examples when their parameters have been identified. An Appendix of numerical work by E. N. Chukwu and E. Chukwu is displayed.

Detailed studies of the 184 nations are continued in Chap. 12. The book concludes with an overview of studies on longevity and the possibility of eternal life and the resource and life style implications of such an eventuality.

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1. Hájek O (1975) Pursuit games. Academic Press, New York
2. Chukwu EN (1996) Universal laws for the control of global economic growth with nonlinear hereditary dynamics. Appl Math Comput 78:19–82

Chapter 1

Full Employment: Private and Government Policies

Employment is currently decaying in the United States of America. Policies of private firms and government to arrest the situation are now hotly debated. This paper is a readable reflection and extraction from Chukwu's books, *Stability and Time-Optimal Control of Hereditary Systems with Application to the Economic Dynamics of the USA*, 2nd Edition, World Scientific 2001 [1], and "The Omega Problem of all Members of the United Nations", *Atlantis Studies in Mathematics for Engineering and Science*, Volume 6. Series editor, C. K. Chui, Stanford University, U.S.A. and World Scientific [2]. Reasonable and verifiable economic policies are deduced from Chukwu.

The dynamics of employment is given by Eq. (1.10.67) of [1]

$$\dot{L} + \ell_5 p(t) + \sigma_3(t),$$

with the economic state

$$x = [y, R, L, k, p, E].$$

Here

- y is the Gross Domestic Product,
- R is the interest rate,
- L is employment,
- k is the value of capital stock,
- p denotes prices,
- E is the cumulative balance of payment,
- τ is tariff,
- e exchange rate,
- T taxes,
- n is labor productivity of the firms.

In this section we highlight the analysis of employment dynamics of the USA.

The profit function is

$$F = y - wL - rk,$$

where w is the wage of L , labor per unit time, r the rent of the use of Capital k . For the maximization of profit F we define

$$m(w) = \left[(1 - \alpha) \frac{1}{w} \right]^{1/\alpha}.$$

Using IMF data and MATLAB program US2.m the following output emerges.

$$\begin{aligned} L_0 &= a_0 = 1.554 \\ \ell_1 &= a_1 = -0.1870, \\ \ell_2 &= m(w)a_2 = 7.7325e^{-(004)} \cdot (01870), \\ \ell_4 &= m(w)a_4 = 7.8295 \cdot (209668), \\ \ell_5 &= m(w)a_5 = 7.8295 \cdot (-161365), \\ \sigma_3 &= m(w)\sigma_4 = 7.7325 \cdot e^{-004} \cdot \sigma_4, \\ \sigma_4 &= x_0 + y_0 + I_0, \end{aligned}$$

x_0 firms private net export, y_0 firms autonomous consumption and I_0 autonomous investment

$$q_3 = m(w)q_4 = 7.7325e^{-004} q_4,$$

government control strategy,

$$\begin{aligned} q_4 &= g_0 + z_{s13}M - z_{s14}T(t) + z_{s15}e(t) + z_{s16}\tau(t) + z_{s17}d(t) \\ &= 675.2828 + 0.3447M - 5.490e^{-009}T(t) + 0.0956e(t) + 1.0456e^{-005}\tau(t) \\ z_{s13}M &= 4.787e^{-007}M \\ g_0 &= 675.2828 \end{aligned}$$

Economic Policy for the Growth of Employment, $L(t)$.

We observe that

$$\begin{aligned} \sigma_3 &= 7.7325 \cdot e^{-004} (x_0 + y_0 + I_0) \\ &= 7.7325 \cdot e^{-004} (-891936) \end{aligned}$$

is the representative private firms strategy. Increase of autonomous net export; autonomous consumption and autonomous investment will make employment bigger. If the private sector was unable to do this, “government has to step in” by increasing g_0 . For example with $m(w)$ g_0 positive, innovative ways for domestic

food production can be explored decreasing import, promoting export. Local consumption y_0 and investment I_0 can also be pursued, by granting loans to small businesses. President Obama is currently touring Durham, NC to encourage a plant of Cree, Inc., a maker of energy efficient LED Lighting for domestic use and export. The company hopefully with Government can initiate policies that will help the company to produce more cheaply for domestic consumption and export, making net export bigger. An economic stimulus package may be a way to go. In general government strategy can be applied.

$$\begin{aligned} q_3 &= m(w)q_4 \\ &= 7.7325e^{-004} [675.2828 + 0.3447M - 5.490e^{-009}T(t) \\ &\quad + 0.0956e(t) + 1.045^{-005}\tau(t)] \end{aligned}$$

First

$$m(w) = \left[(1 - \alpha) \frac{1}{w} \right]^{1/\alpha}$$

and be increased by decreasing wages domestically or by outsourcing plants to countries or regions where wages are small and repatriating the huge profit made by smaller taxes on their products by foreign countries. For example, Cree received 39 million dollars in federal stimulus money, opened a plant in China in 2010. Since $-5.490e^{-009}\bar{T}(t)$ is negative, government can decrease taxes and accelerate the growth of employment $L(t)$. This is consistent with increasing the taxes of the very upper class and decreasing those of the middle class and small firms with average decrease of $T(t)$.

It may be possible to better manage exchange rate or tariff to incite the bigger growth rate of employment.

A positive trading policy d may help. In our initial analysis we set $d = 0$. All these recommendations are effective when subject to the condition that the private firms control set P dominates government's Q : $Q \subseteq p$. Details of more complicated situation with other nations' impact on the USA can be found in the latest book by Chukwu. "The Omega Problem of all Members of the United Nations," Atlantis Press World Scientific, 2010. The symbol g_0 can be part of government economic stimulus strategy. We remark that the debate that government should not be too big has been verified. Chukwu's research on the Chinese economy showed how effective the reduction in China of Government with 95 % G.D.P. to 49 %, see E. N. Chukwu, "Modeling and Optimal Control of the Growth of Wealth of Nations with Austria, Australia, and Chinese Examples", 1999. IFAC 14th Triennial World Congress, Beijing, P. R. China, M-5-e-0 2-0657, Italy 5-9, 1999. Government control set of the USA is currently not too big only 22 % of the GDP. Hope is promised for the acceleration of employment growth.

The IMF's Switch in Time

Joseph E. Stiglitz

Project Syndicate, 5 May 2011

This report is due to Professor Stiglitz.

NEW YORK—The annual spring meeting of the International Monetary Fund was notable in marking the Fund's effort to distance itself from its own long-standing tenets on capital controls and labor-market flexibility. It appears that a new IMF has gradually, and cautiously, emerged under the leadership of Dominique Strauss-Kahn.

Slightly more than 13 years earlier, at the IMF's Hong Kong meeting in 1997, the Fund had attempted to amend its charter in order to gain more leeway to push countries towards capital-market liberalization. The timing could not have been worse: the East Asia crisis was just brewing—a crisis that was largely the result of capital-market liberalization in a region that, given its high savings rate, had no need for it.

That push had been advocated by Western financial markets—and the Western finance ministries that serve them so loyally. Financial deregulation in the United States was a prime cause of the global crisis that erupted in 2008, and financial and capital-market liberalization elsewhere helped spread that “made in the USA” trauma around the world.

The crisis showed that free and unfettered markets are neither efficient nor stable. They also did not necessarily do a good job at setting prices (witness and real-estate bubble), including exchange rates (which are merely the price of one currency in terms of another).

Iceland showed that responding to the crisis by imposing capital controls could help small countries manage its impact. And the US Federal reserve's “quantitative easing” (QEII) made the demise of the ideology of unfettered markets inevitable: money goes to where markets *think* returns are highest. With emerging markets booming, and America and Europe in the doldrums, it was clear that much of the new liquidity being created would find its way to emerging markets. This was especially rue given that America's credit pipeline remained clogged, with many community and regional banks still in a precarious position.

The resulting surge of money into emerging markets has meant that even finance ministers and central-bank governors who are ideologically opposed to intervening believe that they have no choice but to do so. Indeed, country after country has now chosen to intervene in one way or another to prevent their currencies from skyrocketing in value.

Now the IMF has blessed such interventions—but, as a sop to those who are still not convinced, it suggests that they should be used only as a last resort. On the contrary, we should have learned from the crisis that financial markets *need* regulation, and that cross-border capital flows are particularly dangerous. Such regulations should be a key part of any system to ensure financial stability; resorting to them *only* as a last resort is a recipe for continued instability.

There is a wide range of available capital-account management tools, and it is best if countries use a portfolio of them. Even if they are not fully effective, they are typically far better than nothing.

But an even more important change is the link that the IMF has finally drawn between inequality and instability. This crisis was largely a result of America's effort to bolster an economy weakened by vastly increased inequality, through low interest rates and lax regulation (both of which resulted in many people borrowing far beyond their means). The consequences of this excessive indebtedness will take years to undo. But, as another IMF study reminds us, this is not a new pattern.

The crisis has also put to the test long-standing dogmas that blame labor-market rigidity for unemployment, because countries with more flexible wages, like the US, have fared worse. Consumption will remain restrained, while strong and sustainable recovery cannot be based on another debt-fueled bubble.

As unequal as America was before the great recession, the crisis, and the way it has been managed, has led to even greater income inequality, making a recovery all the more difficult. America is setting itself up for its own version of a Japanese-style malaise.

But there are ways out of this dilemma: strengthening collective bargaining, restructuring mortgages, using carrots and sticks to get banks to resume lending, restructuring tax and spending policies to stimulate the economy now through long-term investments, and implementing social policies that ensure opportunity for all. As it is, with almost one-quarter of all income and 40 % of US wealth going to the top 1 % of income earners, America is now less a "land of opportunity" than even "old" Europe.

For progressives, these abysmal facts are part of the standard litany of frustration and justified outrage. What is new is that the IMF has joined the chorus. As Strauss-Kahn concluded in his speech to the Brookings Institution shortly before the Fund's recent meeting: "Ultimately, employment and equity are building blocks of economic stability and prosperity, of political stability and peace. This goes to the heart of the IMF's mandate. It must be placed at the heart of the policy agenda."

Strauss-Kahn is proving himself a sagacious leader of the IMF. We can only hope that governments and financial markets heed his words. Professor Stiglitz rests his case in his usual wisdom.

Chapter 2

A Mathematical Solution to Boost the Positive Value of the Cumulative Balance of Payment of the USA

Very recently the President of the USA, set up a bipartisan commission to generate strategies to boost the positive value of the Balance of Payment of the USA. A mathematical solution can be deduced from the following research monographs of the author, E. N. Chukwu.

1. E. N. Chukwu, *Stability and Time Optimal Control of Hereditary Systems with Application to the Economic Dynamics of the US*, 2nd edition, World Scientific, 2001, ISBN 981-02-4674-9. Singapore, New Jersey, London.
2. E. N. Chukwu, *Differential Models and Neutral Systems for Controlling the Wealth of Nations*, World Scientific, 2001, ISBN 9810243812, Singapore, New Jersey, London.
3. *Optimal Control of the Growth of Wealth of Nations*, Taylor and Francis, 2003, ISBN 0-415-26966-0, London, New York.
4. E. N. Chukwu, *A Mathematical Treatment of Economic Cooperation and Competition Among Nations with Nigeria, USA, UK, China and Middle East Examples*, *Mathematics in Science and Engineering*, v 203, Elsevier, 2005, ISBN-13978-0-444-51859-0, Amsterdam, The Netherlands.
5. E. N. Chukwu, *The Omega Problem of all members of the United Nations*, *Atlantis Studies in Mathematics for Engineering and Science*, v 7, World Scientific, pub. Date scheduled Summer 2010.
6. E. N. Chukwu, *Stability and Time Optimal Control of Hereditary Systems*, Academic Press, Boston, 1992.

In these publications, the author invokes the economic “Differential demand and supply principle”, and the “Rational expectations hypothesis” which assumes that the expected values of economic variables are functions of the current and past values. With these assumptions the author derives this hereditary model of cumulative balance of payment,

$$\begin{aligned}
& \dot{E}(t) - b_{17}\dot{E}(t-h) - b_8\dot{R}(t-h) - b_{12}\dot{L}(t-h) - b_4\dot{y}(t-h) \\
& = b_0 + b_7e(t) + b_{15}d(t) + b_{13}\tau(t) + b_1y(t) + b_5R(t) + b_9L(t) + b_3p(t) \quad (2.1) \\
& + b_2y(t-h) + b_6R(t-h) + b_{10}L(t-h) + q_6(t) - r_6(t).
\end{aligned}$$

or

$$\begin{aligned}
& \dot{E}(t) - b_{17}\dot{E}(t-h) - b_8\dot{R}(t-h) - b_{12}\dot{L}(t-h) - b_4\dot{y}(t-h) \\
& = b_1y(t) + b_5R(t) + b_9L(t) + b_3p(t) + b_2y(t-h) + b_6R(t-h) \quad (2.2) \\
& + b_{10}L(t-h) - r_6(t) + q_6(t),
\end{aligned}$$

where

$$-r_6(t) = X_0 \text{ (autonomous net export)} \quad (2.3)$$

$$q_6(t) = b_7e(t) + b_8\tau(t) + b_{15}d(t) - f_0 \quad (2.4)$$

where

$$X_0 = x_0 - M_0 \text{ (autonomous net export), } b_0 = X_0 - f_0, \quad (2.5)$$

$$\begin{aligned}
& -r_6(t) = x_0 - M_0 = X_0, \text{ (autonomous net export)} \\
& q_6(t) = b_7e(t) + b_8\tau(t) + b_{15}d(t) - f_0. \quad (2.6)
\end{aligned}$$

Here f_0 denotes preferential arrangement (which reduce trade barriers and enhance trade flows between nations)

X_0 = autonomous net export,

$q_6(t)$ = government control instruments: exchange rate, e , tariffs, τ , foreign credit interest equalization tax, f_0 , preferential arrangement (which reduce trade barriers and hence trade flows between nations, d , transportation and distance between partners.).

Indeed if

x = $[y, R, L, k, p, E]^T$, (Economic state),

A_{-1} is 6×6 matrix,

A_0 an 6×6 matrix, $A_1 = 6 \times 6$ matrix,

B_1 an 6×8 matrix,

B_2 an 6×9 matrix,

as recorded in [1, [Sect. 1.10](#)], and if

$$q = [q_1 \ q_2 \ q_3 \ q_4 \ q_5 \ q_6 \ q_7 \ q_8]^T$$

$$= \begin{bmatrix} q_1 \\ g_0 \\ e \\ \tau \\ d \\ M_1 \\ \dot{M}_1 \\ f_0 \end{bmatrix}$$

where

$$T_1 = -z_{14}T(t) + z_{19}T(t-h) \\ - z_{20}\dot{T}(t) - z_{21}\dot{T}(t-h),$$

and if $\sigma = [C_0, I_0, M_0, n, w, x_0, y_{10}, p_0]$,
then

$$\dot{x}(t) - A_{-1}\dot{x}(t-h) = A_0x(t) + A_1x(t-h) + B_q + B_\sigma \quad (2.7)$$

More lucidly, government control vector,

$$q = \begin{bmatrix} T_1 \\ g_0 \\ e \\ \tau \\ d \\ M_1 \\ \dot{M}_1 \\ f_0 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{bmatrix},$$

where

$$T_1 = -z_{14}T(t) + z_{19}T(t-h) - z_{20}\dot{T}(t) - z_{21}\dot{T}(t-h).$$

Also the representative firms control vector

$$\sigma = \begin{bmatrix} C_0 \\ I_0 \\ X_0 \\ M_0 \\ n \\ w \\ x_0 \\ y_{10} \\ p_0 \end{bmatrix} = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \\ \sigma_7 \\ \sigma_8 \\ \sigma_9 \end{bmatrix},$$

and from the analysis

$$\begin{aligned}
q_4(t) &= g_0 + z_{s13}M - z_{s14}T(t) + z_{s15}e(t) + z_{s16}\tau(t) + z_{s17}d(t), \\
\sigma_4(t) &= x_0 + y_{10} + I_0, \\
q_3(t) &= m(w)q_4(t) \\
&= m(w)[g_0 + z_{s13}M - z_{s14}T(t) + z_{s15}e(t) + z_{s16}\tau(t) + z_{s17}d(t)], \\
\sigma_3(t) &= m(w)\sigma_4(t) = m(w)(x_0 + y_0 + I_0), \\
q_6 &= b_7e(t) + b_8\tau(t) + b_{15}d(t) - f_0, \\
-\sigma_6(t) &= X_0.
\end{aligned} \tag{2.8a}$$

If we identify the derivative of the state economic component E and its control component q_6 we obtain

$$b_7(m(w)q_4) + b_8(x_0 + y_0 + I_0) + b_{15}(p_1p^f(t)e(t) + p_5\dot{M}1 + p_6M1 - f_0,$$

or

$$\begin{aligned}
q_6 &= b_7(m(w))(g_0 + z_{s13}M - z_{s14}T(t) + z_{s15}e(t) + z_{s16}\tau(t) + z_{s17}d(t)) \\
&\quad + b_8(x_0 + y_0 + z_0) + b_{15}(p_1p^f(t)e(t) + p_5\dot{M}1 + p_6M1) - f_0,
\end{aligned}$$

where

$$b_{15} = 0,$$

With MATLAB identification program we generate the following coefficients,

$$\begin{aligned}
b_7 &= -154.5053, \\
b_8 &= 5.9898, \\
b_{15} &= 0, \\
m(w) &= 7.7325e^{-004}, \\
g_0 &= -9.6126, \\
f_0 &= -347.68, \\
mwz_{s13} &= 4.7587e^{-007}, \\
mwz_{s14} &= 5.490e^{-009}, \\
mwz_{s15} &= 0.0956, \\
mwz_{s16} &= 1.0145e^{-005}, \\
mwz_{s17} &= 0, \\
x_0 &= -291.5135, \\
y_{10} &= 0.1225, \\
I_0 &= -53.5756, \\
b_6 &= 32.1284, \\
b_{10} &= -20.5989, \\
b_{17} &= -0.0267,
\end{aligned}$$

Using MATLAB identified coefficients and equations, the following were the output controls derived from US2.m in [1]

$$\begin{aligned} X_0 &= 89.1936 = -\sigma_6(t); \\ q_6(t) &= (-154.5053)[(7.7325e^{-004}) + (-9.6126 + 4.7587e^{-009}M - 5.490e^{-009}T(t)) \\ &\quad + (0.0956e(t) + 1.0145e^{-005}\tau(t)) + 5.9898(-291.5135 + 0.225 - 53.5756) - 347.68. \end{aligned}$$

Clearly increase of net current taxes $T(t)$ will increase $(-154.503)(7.7325e^{-004})$ $(-5.490e^{-009}T(t))$ and increase $\dot{E}(t)$. Increase of net taxes is consistent with judiciously increasing the very upper income taxes and lowering the taxes of the poor and lower income citizens. A helpful policy is to decrease $0.0956e(t)$, i.e., the exchange rate e and $1.0145e^{-005}\tau(t)$, i.e., tariff τ .

Consider the first component

$$(-154.5053)(7.7325e^{-004})(-9.6126) \equiv b_7 \cdot m(w) \cdot g_0$$

This is positive. Thus, if we increase the numerical autonomous government outlay g_0 or $m(w) = [(1 - \alpha)\frac{1}{w}]^{1/\alpha}$ by lowering wages, in Eq. (6.1.61) of [4] where w is the wage of labor per unit time and this is needed to maximize profit for the representative firm, then $b_7m(w)g_0$ will increase. If this happens the growth rate of cumulative balance of payment $\dot{E}(t)$ will increase. An acceptable, one way of lowering wages, w , and increasing $m(w)$ is by outsourcing industry to countries of acceptable lower wages and allowing more of the maximized profit of firms to be repatriated to their homeland. This requires the cooperation of interacting nations. The research on cooperation is treated in the author's books [4, 5]. It is then clear that GM's outsourcing to China of their enterprise is both advantageous to the USA and to China if properly managed, and if the two countries are in a cooperative mood—China reduces taxes on GM and allows repatriation of profits to the USA in order to benefit from creation of Chinese labor and the upliftment of China's wealth by GM's presence.

Since $4.7587e^{-009}M$ is positive and $b_7 = -154.503$ is negative, and $M > 0$, $(-154.503)(4.7587e^{-009}M)$ is negative, lowering money supply M will increase the value of the l.f.h. of Eq. (1.3) and therefore of the growth rate of cumulative balance of payment $\dot{E}(t)$. Obviously an increase of net export by private firms will increase the rate of increase of cumulative balance of payment. Note that reducing M will increase the growth rate of R , interest rate.

We note that

$$\begin{aligned}
 b_{15} &= 0, \\
 b_8 &= 5.9898, \\
 b_6 &= 32.1284, \\
 b_{10} &= -20.5989, \\
 b_{17} &= -0.0267, \\
 x_0 &= -291.5135, \\
 M_0 &= -1.5231, \\
 f_0 &= -347.68, \\
 r_6 = X_0 &= 89.1936.
 \end{aligned}$$

$$b_7(m(w))g_0 = (-154.5053)(7.7325e^{-004})(-9.6126)$$

If we decrease wages w , $m(w)$ increases, if we transportation and distance of g_0 and thus increase g_0 , then $b_7m(w)g_0$ increases. Thus the growth rate of cumulative balance of payment increases. Now

$$b_7(m(w))z_{s13}M = -154.5053(7.7325e^{-004})(4.7581e^{-007})M$$

If money supply is decreased, what is subtracted is smaller and the growth rate of cumulative balance of payment is increased. Of course the growth rate of interest rate is bigger. See [6.1.19]. Consider

$$b_7(m(w))(-z_{s14})T(t) = (-154.5053)(7.7325e^{-004}) - (5.490e^{-009})T(t)$$

Since this is positive, increasing taxes will increase this value and therefore increase the growth rate of cumulative balance of payment.

$$b_7(m(w))z_{s15}e(t) = (-154.5053)(7.7325e^{-004})(0.0956)e(t)$$

Decreasing exchange rate will make this negative value smaller and therefore the growth rate of cumulative balance of payment bigger. Let

$$b_7m(w)z_{s16}\tau(t) = (-154.503)(7.7325e^{-004})(1.0145e^{-005})\tau(t)$$

Then decreasing tariff will shrink this negative value and therefore enlarge the growth rate of cumulative balance of payment. Since

$$b_8(x_0 + y_0 + I_0) = 5.9898(-291.5135 - 0.1225 - 53.5756)$$

Increasing autonomous net export, autonomous goods and services, and autonomous investment making them all positive by the action of private firms promoted by government will make the cumulative balance of payment grow bigger.

Increase of autonomous net export X_0 by private firms will also strengthen the growth rate of Cumulative Balance of Payment.

The effect of some Economic State Variables on the Cumulative Balance of Payment in Eq. (1.5) or (6.1.44) of [3]

$$\begin{array}{ll}
 b_{17} = -0.0269 & b_1 = -639.1947 \\
 b_4 = -0.1990 & b_5 = 0 \\
 b_8 = 5.9898 & b_9 = 0 \\
 b_{12} = 4.9703 & b_6 = 32.1284 \\
 b_{10} = -20.5989 & b_2 = -0.3164 \\
 b_3 = 0 & b_{15} = 0 \\
 b_7 = -154.33 & b_2 = 0.3164 \\
 b_{12} = 4.9703 & b_2 = 0.3164
 \end{array}$$

Hence

$$\begin{aligned}
 & \dot{E}(t) - 0.0269\dot{E}(t-h) - 0.1990\dot{y}(t-h) - 5.9898\dot{R}(t-h) - 4.9703\dot{L}(t-h) \\
 & = -639.1947y(t) + 32.1284R(t-h) - 20.5989L(t-h) + 0.3164y(t-h) \\
 & \quad - r_6 + q_6(t)
 \end{aligned}$$

This can be written in the following way:

$$\begin{aligned}
 \dot{E}(t) &= 0.0269\dot{E}(t-h) + 0.1990\dot{y}(t-h) + 5.9898\dot{R}(t-h) + 4.9703\dot{L}(t-h) \\
 &\quad - 639.1947y(t) + 32.1284R(t-h) - 20.5989L(t-h) + 0.3164y(t-h) - r_6 + q_6(t).
 \end{aligned} \tag{2.8b}$$

Abundance of y , goods and services, reduces the value of $\dot{E}(t)$. In matrix form this (2.8b) was expressed in the following way.

$$\dot{x}(t) - A_{-1}\dot{x}(t-h) = A_0x(t) + A_1x(t-h) + B_{1q} + B_2\sigma. \tag{2.9}$$

It is demonstrated in the earlier analysis [1–5] that with A_{-1} , A_0 , A_1 , B_1 , B_2 identified the hereditary system of neutral type is function space controllable. See p. 485 of [1].

The control set of private firms is P and those of government is Q . Let

$$B_{1q} \equiv g \quad \text{and} \quad B_2\sigma = p$$

Then $g \in Q$, $p \in P$. The system (2.9) is

$$\dot{x}(t) - A_{-1}\dot{x}(t-h) = A_0x(t) + A_1x(t-h) + g - p(t) \tag{2.10}$$

and this is equivalent to

$$\dot{x}(t) - A_{-1}\dot{x}(t-h) = A_0x(t) + A_1x(t-h) - u(t), \quad (2.11)$$

where

$u(t) \in W \subset E^n$, and W is the Pontryagin difference of sets

$$W = \{P + \ker U(t_1 - s) \mid Q\},$$

with U the matrix solution of

$$\dot{x}(t) - A_{-1}\dot{x}(t-h) = A_0x(t) + A_1x(t-h).$$

We say that (1.8) is Euclidean controllable on the interval $[0, t_1]$ for any initial function $\phi(-\infty, 0] \rightarrow E^n$ which is continuous and bounded with $\phi(0) \in E^n$ and for every target $y_1 \in E^n$ there exists a measurable function $u: [0, t_1] \rightarrow W$ such that the solution $y(t, \phi, u)$ of (2.11) satisfies $y_1 = y(t_1, \phi, u)$. There is Euclidean space capture everywhere at time t_1 for game (2.10) if for each $\phi \in C([-\infty, 0], E^n)$, ϕ bounded and for each $y_1 \in E^n$ and for any government intervention $g \in L_\infty([0, t_1], Q)$ there exists a private initiative (i.e., the firms can choose a control strategy $p \in L_\infty([0, t_1], P)$ subject to the two conditions:

1. For each $t \in [0, t_1]$ the value of $p(t)$ depends on $q(t)$ only (and of course on $x_0 = \phi(0)$, t_1 , A_i $i = -1, 0, 1$);
2. The pair of controls p, q so obtained is such that the solution of (2.10) satisfies

$$x(t_1, \phi, p, q) = x_1, x(0, x_0, p, q) = \phi(0) = x_0$$

In our earlier results we showed that if Q is the totality of government power, and if the algebraic sum of government action maybe zero ($0 \in Q$) and if the firms economic capacity P is limited and small (compact) the (optimal) control strategy of the firms lies in W which is defined by $W(t) = (P + \ker U(t_1, t)) \mid Q$. The following universal principle on the limitation of government economic power is stated as

$$Q \subset \text{Int}(P + \ker U(t_1 - t)).$$

These universal laws are carefully argued in p. 54, principle 1, principle 2 and principle 3 of [2]. The computation of Q and P are easily done and are available in Chukwu's books and papers. All the deduced results and the stated principles offer guidelines for the growth of Cumulative Balance of Payment. Note that the USA economy is controllable. See [1–4] and all the economic variables can be steered from any “bad” initial state to a state of “paradise”, high growth of GDP, low interest rate, full employment, big value of capital stock, low prices or low inflation, and positive good value of cumulative balance of payment.

Chapter 3

Full Hereditary Interacting Model of All Member States of the United Nations

Aggregate demand z_i is given by

$$z_i = I_i + C_i + X_i + G_i, (i = 1, \dots, 191) \quad (3.1)$$

where

$$\begin{aligned} C_i(t) = & C_{i0} + C_{i1}(y_i(t) - T_i(t)) + C_{i2}(y_i(t-h) - T_i(t-h)) \\ & + C_{i3}(\dot{y}_i(t) - \dot{T}_i(t)) + C_{i4}(\dot{y}_i(t-h) - \dot{T}_i(t-h)). \end{aligned} \quad (3.2)$$

Here

C_i denotes private consumption of nation i ,
 T_i denotes taxes,
 $y_i - T_i$ represents after tax income: $y_i - T_i$,
 y_i represents the GDP of country i ,
 I_i represents private investment of nation i ,
 I_i $I_{i0} + I_{i1}y_i(t) + I_{i2}y_i(t-h) - I_{i3}\dot{y}_i(t) + I_{i4}\dot{y}_i(t-h)$,
 X_i is the net export of nation i ($X_i = \text{Export}_i - \text{Import}_i$),
 e_i exchange rate,

$$\begin{aligned} X_i(t) = & X_{i0} + X_{i1}y_i(t) + X_{i2}y_i(t-h) + X_{i3}\dot{y}_i(t) \\ & + X_{i4}\dot{y}_i(t-h) + X_{i5}e_i(t) \\ & \left(\sum_{\substack{j=i \\ j \neq i}}^{191} a_{ij}y_j(t-h) \right) \end{aligned} \quad (3.3)$$

The function

$$f_i(t) = y_i(t) \left(\sum_{\substack{j=1 \\ j \neq i}}^{191} a_{ij} y_j(t-h) \right) \text{ in } X_i \quad (3.4)$$

is the contribution to the country of y_i interacting with countries of $y_j (j \neq i)$ due to cooperation and competition. It mirrors an inward flow of investment from outside into nation of y_i including

(External) lending to and borrowing from other nations,

investing from other nations j into other nations i : an inward flow of investment from outside into nation i .

Here τ_i = tariff, d_i = trade policy and distance between trading nations.

$$G_i = g_{i0} + g_{i1}y_i(t) + g_{i2}y_i(t-h) + g_{i3}\dot{y}_i(t) + g_{i4}\dot{y}_i(t-h) \quad (3.5)$$

(g_{i0} = federal budget net expenditure for country i)

We apply the differential market principle of supply and demand—the rate of growth of gross domestic product of a nation i is directly proportional to the difference of aggregate demand and aggregate supply:

$$\frac{dy_i}{dt} = \lambda_i(z_i(t) - y_i(t)), \quad i = 1, \dots, 191, \quad (3.6)$$

where λ_i is the speed of response of supply to demand the speed of adjustment. The reciprocal of the speed of adjustment ($1/\lambda_i$) is the mean time lag, i.e., the time necessary for about 63 % of discrepancy between y_i and z_i or between the actual and desired value of y_i to be eliminated. Summing

$$\begin{aligned} z_i(t) &= C_i(t) + I_i(t) + X_i(t) + G_i(t) = +(C_{i0} + I_{i0} + X_{i0} + g_{i0}) \\ &\quad + (C_{i1} + I_{i1} + X_{i1} + g_{i1})y_i(t) + (C_{i2} + I_{i2}X_{i2} + g_{i2})y_i(t-h) \\ &\quad + (C_{i3} - I_{i3} + X_{i3} + g_{i3})\dot{y}_i(t) + (C_{i4} + I_{i4} + X_{i4} + g_{i4})\dot{y}_i(t-h) \\ z_i(t) &= C_i(t) + I_i(t) + X_i(t) + G_i(t) = +(C_{i0} + I_{i0} + X_{i0} + g_{i0}) \\ &\quad + (C_{i1} + I_{i1} + X_{i1} + g_{i1})y_i(t) + (C_{i2} + I_{i2}X_{i2} + g_{i2})y_i(t-h) \\ &\quad + (C_{i3} - I_{i3} + X_{i3} + g_{i3})\dot{y}_i(t) + (C_{i4} + I_{i4} + X_{i4} + g_{i4})\dot{y}_i(t-h) \\ &\quad + y_i \left(\sum_{\substack{j=2 \\ j \neq i}}^{191} a_{ij} y_j(t-h) \right) + X_{i5}e_i(t) + X_{i6}\tau_i(t) + X_{i7}d_i(t) \\ &\quad - C_{i1}T_i(t) - C_{i2}T_i(t-h) - C_{i3}\dot{T}_i(t) - C_{i4}\dot{T}_i(t-h) \end{aligned} \quad (3.7)$$

Let

$$\begin{aligned}
 z_{i0} + C_{i0} - I_{i0} + X_{i0} + g_{i0}, \\
 z_{i1} = C_{i1} + I_{i1} + X_{i1} + g_{i1} + a_{i1}, \\
 z_{i2} = C_{i2} + I_{i2} + X_{i2} + g_{i2}, \\
 z_{i3} + C_{i3} - I_{i3} + X_{i3} + g_{i3}, \\
 z_{i4} = C_{i4} + I_{i4} + X_{i4} + g_{i4},
 \end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
 \underline{p}_i &= \lambda_i(C_{i0} + I_{i0} + X_{i0}), \\
 \underline{g}_i &= \lambda_i(g_{i0} + X_{i5}e_i(t) + X_{i6}\tau_i(t) + X_{i7}d_i(t) - (C_{i1}T_i(t) + C_{i2}T_i(t-h) \\
 &\quad + C_{i3}\dot{T}_i(t-h) + C_{i4}\dot{T}_i(t-h))),
 \end{aligned} \tag{3.9}$$

Then

$$\begin{aligned}
 \frac{dy_i(t)}{dt} &= \lambda_i(z_i(t) - y_i(t)) \\
 &= \lambda_i(z_{i1} - 1)y_i(t) + \lambda_i z_{i2} y_i(t-h) + \lambda_i z_{i3} \dot{y}_i(t) \\
 &\quad + \lambda_i z_{i4} \dot{y}_i(t-h) + \lambda_i p_i(t) + \lambda_i g_i(t) + \lambda_i y_i^* \left(\sum_{\substack{j=i \\ j \neq i}}^{191} a_{ij} y_{i(t-h)} \right)
 \end{aligned}$$

Thus

$$\begin{aligned}
 (1 - \lambda_i z_{i3}) \frac{dy_i(t)}{dt} - \lambda_i z_{i4} \frac{dy_i(t-h)}{dt} &= \lambda_i(z_{i1} - 1)y_i(t) + \lambda_i z_{i2} y_i(t-h) \\
 &\quad + \lambda_i y_i^* \sum_{\substack{j=1 \\ j \neq i}}^{191} a_{ij} y_j(t-h) + \lambda_i \underline{p}_i(t) + \lambda_i \underline{g}_i(t).
 \end{aligned} \tag{3.10}$$

Simplifying, and letting

$$\begin{aligned}
 p_i(t) &= \lambda_i \underline{p}_i(t) / (1 - \lambda_i z_{i3}), \\
 g_i(t) &= \lambda_i \underline{g}_i(t) / (1 - \lambda_i z_{i3}), \\
 a_{-1i} &= \lambda_i z_{i4} / (1 - \lambda_i z_{i3}), \\
 a_{0i} &= \lambda_i(z_{i1} - 1) / (1 - \lambda_i z_{i3}), \\
 a_{1i} &= \lambda_i z_{i2} / (1 - \lambda_i z_{i3}), \\
 b_{ij} &= \lambda_i a_{ij} / (1 - \lambda_i z_{i3}), \\
 &= a_{ij}(\lambda_i / (1 - \lambda_i z_{i3})),
 \end{aligned} \tag{3.11}$$

we obtain

$$\begin{aligned} \frac{dy_i(t)}{dt} - a_{-1i} \frac{dy_i(t-h)}{dt} = & a_{0i}y_i(t) + a_{1i}y_i(t-h) + y_i \sum_{\substack{j=1 \\ j \neq i}}^{191} b_{ij}y_j(t-h) \\ & + p_i(t) + g_i(t), \quad i = 1, \dots, 191. \end{aligned} \quad (3.12)$$

We can indicate some of the terms fully with an eye to putting them in matrix form.

Thus, this system of Eq. (3.12), $i = 1, \dots, 191$ for the GDP of the 191 member states of the United Nations can be written in the matrix form as follows:

$$\dot{x}(t) - A_{-1}\dot{x}(t-h) = A_0x(t) + A_1x(t-h) + A_2(x(t))x(t-h) + B_1p + B_2g, \quad (3.13)$$

where

$$\begin{aligned} x &= [y_1 \ y_2 \ \dots \ y_{191}]', \\ A_{-1} &= \begin{bmatrix} -a_{11} & 0 & & & 0 \\ & & & & \\ & & & & \\ & & & & \\ 0 & & & -a_{-1190} & 0 \\ & & & & a_{-1191} \end{bmatrix}, \end{aligned} \quad (3.14)$$

$A_{-1} = a_{191} \times 191$ matrix,

$$\begin{aligned} \dot{x}(t-h) &= [\dot{y}_1(t-h) \ \dots \ \dot{y}_{191}(t-h)]' \\ A_0 &= \begin{bmatrix} a_{01} & 0 & 0 & & 0 & 0 \\ & & & & & \\ & & & & & \\ & & & & & \\ 0 & 0 & 0 & & 0 & a_{0191} \end{bmatrix} \\ A_0 &= a_{191} \times 191 \text{ matrix;} \end{aligned} \quad (3.15)$$

$$A_1 = \begin{bmatrix} a_{11} & 0 & 0 & & 0 & 0 \\ 0 & a_{12} & 0 & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ 0 & 0 & 0 & & 0 & a_{1191} \end{bmatrix} \quad (3.16)$$

$A_1 = a191 \times 191$ matrix;

$$A_2(x) = \begin{bmatrix} 0 & b_{12}y_1(t) & b_{13}y_1(t) & & b_{1191}y_1(t) \\ b_{21}y_2(t) & 0 & b_{23}y_2(t) & & b_{2191}y_2 \\ b_{31}y_3(t) & b_{32}y_3 & 0 & b_{34}y_3 & b_{3191}y_3 \\ & & & & \\ & & & & \\ & & & & \\ b_{1911}y_{191} & b_{1912}y_{191} & 0 & & b_{191190}y_{191} & 0 \end{bmatrix}, \quad (3.17)$$

$A_2(x) = a191 \times 191$ matrix function;

$A_2(x) \equiv A_2(y_{11}, y_2, \dots, y_{191})$.

For the control strategies of government q and of the representative firms p . Let

$$\xi_i = \lambda_i / (1 - \lambda_i z_i z).$$

Define the matrix B_1 , size $(191) \times (191 + 5)$ as follows

$$B_1 = \begin{bmatrix} \zeta_1 & \zeta_1 X_{15} & \zeta_{16} & \zeta_1 X_{17} & -\zeta_1 & . & . & . & . & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta_{191} & \zeta_{191} X_{190} & \zeta_{191} X_6 & \zeta_{191} X_{17} & -\zeta_{191} \end{bmatrix}, \quad (3.18)$$

$$\begin{aligned} g &= [g_{10}, e_1 \tau_1, d_1 T_{a1} \dots g_{1910} e_{191} T_{191} d_{191} T_{a191}], \\ q &= B_1 g. \end{aligned} \quad (3.19)$$

The $(191) \times (191 + 3)$ matrix B_2 is defined as follows

$$B_2 = \begin{bmatrix} \zeta_1 & \zeta_1 & \zeta_1 & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 0 & . & . & 0 & 0 & \zeta_{191} & \zeta_{191} & \zeta_{191} \end{bmatrix}, \quad (3.20)$$

$$p = [C_{10} \quad I_{10} \quad X_{10} \quad C_{20} \quad I_{20} \quad . \quad . \quad -C_{1910} \quad I_{1910} \quad X_{1910}]' \quad (3.21)$$

$$\sigma = B_2 p. \quad (3.22)$$

$$B_1 = \text{matrix } (191) \times (5 + 191)$$

$$\dot{x}(t) - A_{-1}\dot{x}(t-h) = A_0x(t) + A_1x(t-h) + A_2(x(t)x(t-h)) + B_1g + B_2p \quad (3.24)$$

where $A_{-1}, A_0, A_1, A_2, B_1, B_2, x, g, p$ are identified above. We have derived Eq. (3.24) as the dynamics of the 191 GDPs of all member states of the UN. With

some big effort, the economic state can be enlarged to include interest rate, employment value of capital stock, prices, and cumulative balance of payment.

Government strategies' five components for each of the 191 countries are denoted by

$$q = [g_{0i}, e_i, \tau_{1i}, d_{1i}, Ta_{1i}]. \quad (3.25)$$

Chapter 4

Consequences

Suppose at time t the coefficients a_{-1i}, a_{oi}, a_{1i} are fixed, and b_{ij} , the coefficient of competition or cooperation for nation i and nation j is positive. If b_{ij} is positive, the two nations are cooperating. If b_{ij} is increased the net hereditary growth rate of GDP, $\frac{dy_i(t)}{dt} - a_{-1i} \frac{dy_i(t-h)}{dt}$ is increased, i.e. the production of goods and services grows bigger. Cooperation enhances economic growth. Suppose b_{ij} , the coefficient of competition is negative, decreasing competition will increase the net growth rate. Increasing competition will decrease the net growth rate.

Example General Motors goes to China and establishes industries, makes cars cheaply because of cheap labor and sells them, and by volume makes huge profit. China sees the standard of living and other good things GM has raised for her people and the wealth China created. China cooperates with the USA and in a trade agreement taxes GM only little. The huge profit generated for GM is brought to the USA to benefit the USA workers—(health benefits etc.) more factories, more employment. China's wealth increases and then she invests in the USA, buying American goods, uplifting Americans. The effects of positive cooperation is now clear.

Consider representative firms private strategy $p_i(t)$,

$$\begin{aligned} p_i(t) &= \lambda_i \underline{p}_i(t) / (1 - \lambda_i z_{i3}), \\ \underline{p}_i(t) &= \lambda_i (C_{i0} + I_{i0} + X_{i0}). \end{aligned}$$

If λ_i , the speed of response of supply to demand, is increased and autonomous consumption, autonomous investment, and autonomous net export are increased and assuming $(1 - \lambda_i z_{i3})$ is positive and small, then the net growth rate of GDP increases because of the increase of p_i . Consider the government strategy $g_i(t)$,

$$\begin{aligned} g_i(t) &= \lambda_i \underline{g}_i(t) / (1 - \lambda_i z_{i3}) \\ \underline{g}_i(t) &= \lambda_i (g_{i0} X_{i5} e_i(t) + x_{i6} \tau_i(t) + X_{i7} d_i(t) \\ &\quad - (C_{i1} T_i(t) + C_{i2} T_i(t-h) + C_{i3} \dot{T}_i(t) + C_{i4} \dot{T}_i(t-h))). \end{aligned}$$

Suppose X_{i5} is positive, increase in exchange rate will increase $g_i(t)$ provided $(1 - \lambda_i z_{i3})$ is positive. This will increase the net hereditary growth rate of GDP. This is true for tariff τ_i and trade policy d_i . If we decrease the generalized taxes,

$$T_{ai} = C_{i1}T_i(t) + C_{i2}T_i(t-h) + C_{i3}\dot{T}_i(t) + C_{i4}\dot{T}_i(t-h)$$

and $1 - \lambda_i z_{i3}$ is positive, then the net hereditary growth rate of GDP increases.

In the above description the term autonomous means that constants so designated are independent of economic variables such as gross-domestic product, etc. It is suggested that this can be wealth invested in infrastructure: schools, roads, hospitals, etc.

Consider Eq. (3.12)

$$\begin{aligned} \frac{dy_i(t)}{dt} - a_{-1i} \frac{dy_i(t-h)}{dt} &= a_{0i}y_i(t) + a_{1i}y_i(t-h) + y_i(t) \sum_{j=1}^{191} b_{ij}y_j(t-h) \\ &+ p_i(t) + g_i(t). \end{aligned} \quad (4.1)$$

This equation can be rewritten as follows

$$\begin{aligned} \frac{dy_i(t)}{dt} - a_{-1i} \frac{dy_i(t-h)}{dt} &= a_{1i}y_i(t-h) + y_i(t) \left[a_{0i} + \sum_{j=1}^{191} b_{ij}y_j(t-h) \right] \\ &+ p_i(t) + g_i(t). \end{aligned} \quad (4.2)$$

If in the model $a_{1i} = 0$, $a_{-1i} = 0$, $u(t) \equiv p_i(t) + g_i(t) \equiv 0$, then

$$\frac{dy_i(t)}{dt} = y_i(t) \left[a_{0i} + \sum_{j=1}^{191} b_{ij}y_j(t-h) \right]. \quad (4.3a)$$

With the following declared initial conditions,

$$y_i(s) = \psi_i(s) \geq 0, \quad s \in [-h, 0], \quad \psi_i(0) > 0;$$

$$\psi_i \text{ is continuous on } [-h, 0], \quad i = 1, 2, \dots, 191, \quad (4.4)$$

we now state conditions for a positive (component wise) steady state (of (3.14), (4.3a))

$$y^* = (y_1^*, \dots, y_{191}^*)$$

to exist and have y^* to be locally asymptotically stable.

Theorem 4.1 Suppose a_{0i} b_{ij} ($i, j = 1, 2, \dots, 191$) are nonnegative constant such that

$$\begin{aligned} b_{ii} &> 0, & i &= 1, 2, \dots, 191 \\ b_{ij} &\geq 0, & i, j &= 1, 2, \dots, 191; \quad i \neq j \\ a_{0i} &> \sum_{\substack{j=1 \\ j \neq i}}^n b_{ij} (a_{0j}/b_{jj}) & i &= 1, 2, \dots, 191 \end{aligned}$$

Then the system

$$\frac{dy_i(t)}{dt} = y_i(t) \left(a_{0i} - \sum_{j=1}^{191} b_{ij} y_j(t) \right), \quad i = 1, 2, \dots, 191 \quad (4.3b)$$

has a component wise steady state $y^* = (y_1^*, y_{191}^*)$; satisfying the equations $\sum_{j=1}^{191} b_{ij} y_j^* = a_{0i}$ and $y_i^* > 0$; $i = 1, 2, \dots, 191$.

Theorem 4.2 Assume the following conditions

1. $h \geq 0$
2. b_{ij} ($i, j = 1, 2, \dots, 191$)

are nonnegative constants such that

$$\begin{aligned} b_{ii} &> 0; \quad b_{ii} y_i^* h < 1; \quad i = 1, 2, \dots, 191, \\ \left(h y_i^* \sum_{\substack{j=1 \\ j \neq i}}^{191} b_{ji} \right) / (1 - b_{ii} h y_i^*)^{-1} &> \sum_{\substack{j=1 \\ j \neq i}}^{191} b_{ji}, \quad i = 1, 2, \dots, 191 \end{aligned}$$

Then all the roots of $\det \left[\lambda \delta_{ij} + \sum_{j=1}^{191} b_{ij} y_j^* e^{-\lambda h} \right] = 0$, where $\delta_{ij} = 1, i = j, i = 1, \dots, 191$, $\delta_{ij} = 0, i \neq j, i, j = 1, 2, \dots, 191$, have negative real parts. Also y^* is locally asymptotically stable. As observed by Gopalsamy [3, p. 311] if h is positive but sufficiently small then y^* can in fact be globally asymptotically stable (or at least is globally attractive). It is possible to rule out chaotic behavior and to have

$$\lim_{t \rightarrow 0} y_i(t) = y_i^*, \quad i = 1, 2, \dots, 191,$$

where

$$\sum_{j=1}^{191} b_{ij} y_j^* = a_{0i} \quad i = 1, 2, \dots, 191.$$

If there is very little delay in supplying y and meeting the demand of z observe the inequalities of Theorem 4.2 and if competition is limited and held in check then the conditions of the above theorems may be met.

Remark Delays in Competition and Cooperation.

Our main aim in this section is to derive sufficient conditions for all positive solutions of certain classes of autonomous systems of delay differential equations which describe the evolution of gross domestic product of member states of the United Nations to converge to equilibrium states. What is considered is the initial value problem (4.3a).

For a further clarification of the concept of steady state, or equilibrium and convergence to an equilibrium or steady state see [3, 5].

As observed in Chukwu [2, p. 186] A Mathematical Treatment of Economic Cooperation and Competition, neither convergence to a steady state nor a steady state y^* being globally or locally asymptotically stable is the crucial idea for the GDP of nations. It is desirable to have the GDP to grow when controls are introduced to the system and then the full dynamics (3.13) is studied for convergence, controllability, constrained controllability, and optimal control.

We note that controllability property of the system can render competitive system cooperative. For ease of reading the definitions of asymptotic convergence [3, p. 181] locally (or global) asymptotic stability [5] can be found in the cited references of Chukwu or Gopalsamy. The existence of booms and depression, i.e., the existence of oscillation can be studied.

The treatment of Oscillation of linear neutral equation was reported earlier in Chukwu [1, p. 371] Chaotic Economic System with delay was widely explored in Chukwu [1, Sect. 6.4.2].

In Chap. 3 we derived the equation of neutral type as a model of the dynamics of each country i 's gross domestic product,

$$\begin{aligned} \frac{dy_i(t)}{dt} - a_{-1i}\dot{y}_i(t-h) &= a_{0i}y_i(t) + a_{1i}y_i(t-h) + \sum_{j=1}^{191} b_{ij}y_i(t)y_j(t-h) \\ &+ p_i(t) + g_i(t)t > 0 \end{aligned} \quad (4.5a)$$

If $y_i(s) = \varphi_i(s) \geq 0, s \in [-h, 0]$;

$\varphi_i(0) > 0$; φ_i is continuous on $[-h, 0]$; $i = 1, 2, \dots, 191$

where $a_{-1i} = 0, a_{0i}, a_{1j}, -b_{ij}, h, j = 1, \dots, 191$ are nonnegative constants and $h > 0$.

This is an autonomous system of delay differential equation for the gross domestic product.

If $a_{-1i} = 0, a_{1i} = 0, h = 0$ then

$$\begin{aligned}
\frac{dy_i(t)}{dt} - a_{0i}y_i(t) &= \sum_{j=1}^{191} b_{ij}y_i(t)y_j(t) + p_i(t) + g_i(t) \\
&= y_i(t) \left[a_{0i} + \sum_{j=1}^{191} b_{ij}y_j(t) \right] + p_i(t) + g_i(t) \quad b_{ij} > 0
\end{aligned} \tag{4.5b}$$

then (4.5b) is an ordinary differential game of private and government control system for the gross domestic product dynamics of nation i , $i = 1, \dots, 191$ a member of the United Nations. It is a Lotka-Volterra Cooperative system if $b_{ij} > 0$.

We state global stability results for (4.5a, b), a consequence of Hallam [4, p. 274].

Theorem 4.3 *In (7.5b), let $g_i(y) = \left[a_{0i} + \sum_{j=1}^{191} b_{ij}y_j(t) \right]$, and consider*

$$\frac{dy_i(t)}{dt} = y_i(t)g_i(y), \quad i = 1, 2, \dots, 191, \quad y = [y_1, \dots, y_{191}]^T \tag{4.6}$$

where government strategy and the representative firms strategy are absent.

Then (4.6) is a kolmogorov system. Its equilibrium $y^* > 0$ is asymptotically stable and feasible if and only if the interaction matrix $A = \frac{(\partial g_i(y^*))}{\partial y_j}$ is asymptotically stable.

The proof is given by Hallam [4, p. 275].

It is a local stability result.

To prove global stability theorem, a Liapunov function of Volterra type,

$$V(y) = \sum_{i=1}^{191} C_i \left(y_i - y_i^* - y_i^* \ln \frac{y_i}{y_i^*} \right),$$

is utilized.

Thus for any initial value y_0 , the solution $y(t, y_0) \rightarrow y^*$, as $t \rightarrow \infty$.

We do not want to wait forever to attain the equilibrium.

Target. y^* can be attained and indeed attained in finite time by employing the government and private firms strategies $g_i(t)$ and $p_i(t)$. Thus any target can be attained in finite time in a cooperative system by applying government and private strategies which are subject to scarcity; and thus “the freedom of the market is placed in the services of human freedom in its totality” as pontificated by Pope John Paul II and affirmed by President Bush of the USA—a view reported in Chukwu [2, p. 322].

We revisit the gross domestic product dynamics of country i described by (4.5a). If

$$\sum_{j=1}^{191} b_{ij} y_i(t-h) > 0,$$

we have essentially a cooperative game situation, and for fixed i the net growth rate of gross domestic product is increased.

If we consider the effect of government strategy

$$\begin{aligned} g_i(t) = & \frac{\lambda_i}{(1 - \lambda_i z_{13})} [g_{i0} + x_{15} e_i(t) + x_{16} \tau_i(t) \\ & + C_{i3} \dot{T}_i(t) + C_{14} \dot{T}_i(t-h)], \end{aligned} \quad (4.7)$$

we observe that an increase of autonomous government outlay g_{10} (independent of economic variables) for building roads, schools, hospitals, and other infrastructure can increase the growth of gdp. Recently the Chinese Premier Wen Jiabao acknowledged his country was feeling the ripple effects from the global financial meltdown and pledged robust government spending to keep the economy from stalling.

(p. 16A, The News and Observer, Sunday, October 26, 2008). The statement was made at a two-day Asia–Europe meeting in Beijing, where leaders of 48 nations issued a statement for new rules to guide the global economy. Recall the proved assertion that $Q \subseteq \text{Int } P$, Q not empty. Government control set should be dominated by the Private control set P . See [1, p. 127].

The author's student (Song Zhong in a 2005 M.Sc. Thesis used MATLAB 7.0 to prove that for the simple economic Chinese system studied, $Q = 0.6699P$.) She confirms, duplicates and updates in another way the contribution E. N. Chukwu, Modeling and Optimal Control of the Growth of Wealth of Nations, with Austria, Australia and Chinese Examples, 1999 IFAC 14th Triennial World Congress Beijing P. R. China, M-5-e-02-0657-67, July 5–9, 1999.

Nobel Prize winner Dr. Stiglitz had urged the leaders of Botswana to imitate China in using the exchange rate e_i to control the growth of the country's gdp. Trade policies d_i and tariff τ_i can also help as agents of control. The lowering of taxes (if appropriate) can also help. The precise policies will be visible after the identification of parameters using MATLAB. Thus, when and where to apply the economic stimulus will be transparent.

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Chapter 5

Controllability Theory of the GDP of All Member States of the UN

In the earlier chapters we derived the equation,

$$\dot{x}(t) - A_{-1}\dot{x}(t-h) = A_0x(t) + A_1x(t-h) + A_2(x(t))x(t-h) + B_1p + B_2g, \quad (5.1a)$$

where

$$A_{-1}, A_0, A_1, A_2(x(t)) \quad (5.1b)$$

are defined in (3.15) (3.16), (3.17), (3.18), or (3.19), and where $B_1, g, B_2, p, q = B_1g, \sigma = B_2p$ are defined in (3.22), (3.21), (3.22), (3.23), (3.24), and (3.25). Here A_{-1}, A_0, A_1 are 191×191 matrices, $A_2(x(t))$ is 191×191 matrix function, B_1 is an $191 \times (5 + 191)$ matrix, B_2 is an $(191) \times (3 + 191)$ matrix and $p(t) \in P, g(t) \in G$ with

$$E^{(191+3)} \supset P, \quad E^{(191+5)} \supset Q$$

The function $p \in L_\infty([\sigma, t], P), g \in L_\infty([\sigma, t], Q)$ are called the private firms' control strategy and the governments' (solidarity) strategy functions or controls respectively. The strategies p and g are used to steer initial value $\psi \equiv x_0$ of the nations' gross domestic to a target x_1 in time t_1 . Thus the solution of (5.1a, b), $x(\sigma, \psi, p, q)$ with $x_\sigma(\sigma, \psi, p, g) = \psi$ satisfies $x(\sigma, \psi, p, q)(t_1) = x_1$. The dynamics is best described as a differential game of pursuit in the spirit of O. Hájek [1] for autonomous systems and in the spirit of Chukwu (Optimal Control of the Growth of Wealth of Nations, Taylor and Francis]. Equation (5.1a, b) is derived as the dynamics of the 191 GDPs of all member states of the UN. With some big effort the economic state can be enlarged to include interest rate, employment, value of capital stock, prices and cumulative balance of payment. Five components of each of the 191 countries government strategies are designated by

$$q = [g_{0i}, e_i, \tau_{1i}, d_{1i}, T_{ai}].$$

First we consider the simple system,

$$\dot{x} = Ax - p + q \quad p(t) \in P, \quad q(t) \in Q \quad x \in E^n, \quad (5.2)$$

and the associated control system

$$\dot{y} = Ay - u; \quad u(t) \in P_* Q = \{u : u + Q \subset P\}. \quad (5.3)$$

In the game (5.2) where $P_* Q$ is the Pontryagin difference of P and Q . The “private initiative”, $p(t)$ is called a pursuer strategy in the following sense. Consider a mapping $\tau: Q \times E \rightarrow P$, i.e., $\tau(q, t)$ is a point of P whenever q is a point in Q and $t \in E$. If σ is an induced mapping on the collection of “quarry controls” or “solidarity” set then

$$\sigma[q](t) = q(t) + u(t) = p(t) \quad (5.4)$$

is a Stroboscopic Strategy whenever

$$Q + u(t) \subset P \quad \text{for all } t.$$

We call $q \in L_\infty([0, t], E^n)$ $q(t) \in Q$ and $p \in L_\infty([0, t], E^n)$ $p(t) \in P$ “solidarity” and “private initiative” respectively. They are said to steer the initial point $x_0 \in E^n$ to $x_1 \in E^n$ in time t_1 if the solution $x(x_0, p, q)$ with $x(x_0, p, q)(0) = x_0$ satisfies $x(x_0, p, q)(t_1) = x_1$.

The information pattern of our game is described as follows: for any solidarity q .

1. There exists some private initiative p such that for each $s \in [0, t]$, the value of $p(s)$ depends on $q(s)$ (and of course on x_0, A).
2. The pair of controls p, q steer x_0 to x_1 .
3. This is done perhaps in minimum time, with perhaps minimum cost.

The first duality Theorem of Hájek is stated in [1].

We now reproduce Chukwu’s generalization of Hájek’s theorem to cover a hereditary nonlinear differential game.

Consider the nonlinear system of neutral type,

$$\frac{d}{dt} [A_0 x(t) - A_{-1} x(t - h)] = f(t, x_t) - p(t) + q(t), \quad t \geq \sigma \quad (5.5a)$$

To each solution $y(\sigma, \phi)$ of (5.5b),

$$\frac{d}{dt}[A_0x(t) - A_{-1}x(t-h)] = f(t, x_t), \quad (5.5b)$$

we define a linear nonautonomous functional equation of neutral type

$$\frac{d}{dt}[A_0y(t, \sigma, \phi) - A_{-1}y(t-h, \sigma, \phi)]z(t) = D_2f(t, y_t(\sigma, \phi))z_t, \quad (5.6)$$

called the linear variational equation of (5.5a, b) with respect to $y_t(\sigma, \phi)$. Associated with it is a family of continuous linear operator $T(t, \sigma, \phi): C \rightarrow C$, $t \geq \sigma$, defined for all $\psi \in C$, by

$$T(t, \sigma; \phi)\psi = z_t(\sigma, \psi), \quad \phi \in C.$$

$C = C[-h, 0], E^n$, the space of continuous functions from $[-h, 0]$ to E^n .

The nonlinear variation of parameter solution of

$$\frac{d}{dt}[A_0x(t - A_{-1}x(t-h))] = f(t, x_t) - p(t) + q(t), \quad t \geq \sigma \quad (5.7)$$

is given by

$$x_t(\sigma, \phi, p, q) = y(\sigma, \phi) - \int_a^t T(t, s, x_s(\sigma, \phi, p, q))y_0(p(s) - q(s))ds, \quad (5.8)$$

where $y(\sigma, \phi)$ is a solution of

$$\frac{d}{dt}[A_0y(t) - A_{-1}y(t-h)] = f(t, y_t), \quad t \geq \sigma, \quad y_\sigma = \phi \in C. \quad (5.9)$$

(see [1, p. 113]).

With this nonlinear variation of parameter for the system (5.9) the following Theorem was formulated and cited in [1].

Theorem 5.1 *Assume $0 \in Q$ and P compact. There is complete capture everywhere at time t , for game*

$$\frac{d}{dt}[A_0x(t) - A_{-1}x(t-h)] = f(t, x_t) - p(t) + q(t) \quad (5.10)$$

if, and only if, the associated nonlinear control system of neutral type.

$$\begin{aligned} \frac{d}{dt} [A_0 x(t) - A_{-1} x(t-h)] &= f(t, x_t) - v(t), \\ v(t) &\in V(t), \end{aligned} \quad (5.11)$$

$$\begin{aligned} V(t) &\equiv U_{t_1}^+(\cdot, x_t(\sigma, \phi, u, 0), t) U_{t_1}(\cdot, x_t(\sigma, \phi, \xi, q), t) P_-^* \\ &\quad U_{t_1}^+(\cdot, x_t(\sigma, u, 0), t) U_{t_1}(\cdot, x_t(\sigma, \phi, \xi, q), t) Q \end{aligned} \quad (5.12)$$

is controllable at time t_1 . Furthermore, the relation

$$\begin{aligned} U_{t_1}(\cdot, x_t(\sigma, \phi, \xi, q), t) \xi(q, t) &= U_{t_1}(\cdot, x_t(\sigma, \phi, t, \xi, 0) u(t) \\ &\quad + U_{t_1}(\cdot, x_t(\sigma, \phi, \xi, q), t) q, \end{aligned} \quad (5.13)$$

(for all $q \in Q$, $t \in [\sigma, t_1]$) can be used to determine a suitable strategy from $u \in L_\infty([\sigma, t_1], V)$ in (5.13) and vice versa. The set

$$1P = U_{t_1}^+(\cdot, x_t(\sigma, \phi, u, t) U_{t_1}(\cdot, x_{t_1}(\phi, p, q), t) P$$

is called “private initiative” (U_t^+ is the generalized inverse of U_t) and

$$1Q = U_{t_1}^+(\cdot, x_t(\sigma, \phi, u, 0), t) U_{t_1}(\cdot, x_t(\phi, p, q)) Q$$

is “solidarity” set or public initiative. Then

$$Bu(t) \equiv v(t) \in V(t) = 1P(t)^* 1Q(t),$$

is the control set associated with the nonlinear control system (which is equivalent to the game). Consider

$$\frac{d}{dt} [A_0 x(t) - A_{-1} x(t-h)] = f(t, x_t) - B(t)u(t).$$

This is controllable if

$$\text{Rank}(B(t)) = n \quad \text{for all } t.$$

Hale has observed [2, Functional differential equation, p. 279] that if $\det(A_{-1}) = 0$, it is possible that $\ker U(t_1, t, x_t(\sigma, \phi)) = 0$ for the system,

$$\dot{x}(t) - A_{-1} \dot{x}(t-h) = A_0 x(t) + A_1 x(t-h) + A_2(x(t))x(t-h) - B(t)u(t).$$

In this case

$$B(t)u(t) \in V(t) = P_-^* Q.$$

and

$$BV = \{Bv : Bv + B_2G \subset B_1P\}.$$

To see this if

$$BV(t) = P + \ker U(t_1, t, x_t(\sigma, \phi))_-^* Q,$$

for some t_1 where

$$U_t(\cdot, x_s(\sigma, \phi))X_0(\theta) = T(t, x_s(\sigma, \phi)),$$

is the solution operator of

$$\dot{x}(t) - A_{-1}\dot{x}(t-h) = A_0x(t) + A_1x(t-h) + A_2(x(t))x(t-h),$$

i.e.,

$$\begin{aligned} BV &= \{Bv : Bv + B_2Q \subset B_1P + \ker U(t_1, t, x_t(\alpha, \phi)) \\ &= (B_1P + \ker U(t_1, t, x_t(\sigma, \phi))_-^* B_2Q \\ \dot{x}(t) - A_{-1}\dot{x}(t-1) &= A_0x(t) + A_1x(t-h) + A_2(x(t))x(t-h) + Bv \end{aligned}$$

where

$$Bv \in BV.$$

Thus if we use a stroboscopic strategy the game (5.10) is seen to be equivalent to the control system (5.11), and this system is controllable if $\text{rank}(B) = n$.

For the system of all nations' GDP, to be controllable, $\text{rank}(B) = 191$, and there is capture everywhere for the differential game, in a new terminology. Our system identification and numerical work will explore this in detail. More general theoretical situations are treated in [3, Chukwu, p. 146] and [3, Chukwu, Sect. 4.12]. Next we give details of our MATLAB programming used to identify the coefficients in Chukwu [3], A Mathematical Treatment of Economic Cooperation and Competition Among Nations: with Nigeria, USA, UK, China, and Middle East Examples, Elsevier, 2005.

The system (5.1a, b) can be specialized and reduced to form the ordinary Lotka-Volterra system,

$$\frac{dy_i(t)}{dt} = y_i(t) \left[a_{i0} + \sum_{j=1}^{191} b_{ij}y_j(t) \right], \quad a_{i0}, \quad b_{ij} > 0,$$

where the private and government strategies are absent. And if we display the full game, then

$$\frac{dy_i(t)}{dt} = y_i(t) \left[a_{i0} + \sum_{j=1}^{191} b_{ij}y_j(t) \right] + p_i(t) + g_i(t) \quad (5.14)$$

Following the development of the theory of Hallam [4].

A global stability result can now be stated, followed by a global constrained controllability statement.

Theorem 5.2 *The model (5.14) has an asymptotically stable, feasible equilibrium $y^*(y^* > 0)$ if and only if the interaction matrix A is asymptotically stable. Here*

$$A = \frac{(\partial f_i(y^*))}{\partial y_j}$$

and

$$f_i(y) = \left(a_{i0} + \sum_{j=1}^{191} b_{ij}y_j(t) \right)$$

Theorem 5.3 *For the system*

$$\frac{dy_i}{dt} = y_i \left(a_{i0} + \sum_{j=1}^{191} b_{ij}y_j \right), \quad b_{ij} > 0 \quad (5.15)$$

a feasible equilibrium y^ , is globally asymptotically stable if and only if all the principal minors of $-A$ are positive.*

Remark The function

$$V(x) = \sum_{i=1}^{191} C_i \left(y_i - y_i^* - y^* \ln \frac{y_i}{y_i^*} \right)$$

can serve as a Lyapunov function for deriving the stability result. Here $C = \text{diag} [C_1, C_2, \dots, C_n]$, see Hallam [4, p. 275].

The game (5.16) can be proved to be equivalent to the control system

$$\frac{dy_i(t)}{dt} = y_i(t) \left[a_{i0} + \sum_{j=1}^{191} b_{ij}y_j(t) \right] + Bu_i(t) \quad (5.16)$$

where $u_i \in U_i \in P_i^* Q_i$, a potryagin difference of sets P_i, Q_i , P_i is the control set for the private representative firms and Q_i is the control set for government.

If rank $B = 191$, i.e., no component of u_i is zero in a set of positive measure then any initial GDP, y_0 can be steered to $y(t_1 u) = y_1$ in some time T . Since

$$y(y_0, t, 0) \rightarrow y^* \quad \text{as } t \rightarrow \infty,$$

and $y(y_0, t_1, 0) = y_{10}$ at some t_1 and y_{10} is in a good neighborhood of y^* ; then with the control u , steer the solution $y(y_{10}, t_1, u)$ to $y(y_{10}, t_2, u) = y^*$. Thus, $t_1 + t_2 = T$.

Such arguments have been made in other publications by Chukwu [5].

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Chapter 6

Employment and Gross Domestic Product Dynamics and Control of Interacting Nations

6.1 Introduction

In this chapter models of Gross Domestic Product and Employment of two interacting nations are derived. It is a hereditary differential game of pursuit. It utilizes the principles of supply and demand and of rational expectations. A theory of interaction via net export is postulated. The net export function may include trade, debt repayment, etc. International employments can also be included. The UN and IMF data for two nations are used to validate the theory and test for the degree of cooperation and competition. MATLAB and Maple programs are exploited. The consequences of cooperation and competition are studied, and controllability questions answered. Once the system is proved controllable it can be made sufficiently cooperating, and sustained growth of GDP and employment can be guaranteed.

6.2 The Hereditary Model of GDP and Employment

In this section we derive a hereditary model of gross domestic product, y , and employment L by invoking the “demand and supply principal” and the “rational expectations hypothesis” which assumes that the expected values of economic variables are functions of the current and of the past values. Indeed let the aggregate demand function z_i be given by

$$z_i = I_i + C_i + X_i + G_i \quad i = 1, 2 \quad (6.1)$$

where

I_i is investment

C_i is consumption

X_i is net export

G_i is government intervention.

Here,

$$C_i = C_{0i} + C_{1i}(y_i(t) - T_i(t)) + C_2(y_i(t-h) - T_i(t-h)) \\ + C_3(\dot{y}_i(t) - \dot{T}_i(t)) + C_4(\dot{y}_i(t-h) - \dot{T}_i(t-h)) \quad (6.2)$$

Also

C_i = private consumption,
 T_i = Taxes,
 $y_i - T_i$ = after tax income,
 y_i = GDP.

$$I_i = I_{0i} + I_{1i}y_i(t) + I_{2i}y_i(t-h) - I_{3i}\dot{y}_i(t) + I_{4i}\dot{y}_i(t-h), \\ + I_{8i}L_i(t) + I_{9i}L_i(t-h) \quad (6.3)$$

$$X_i = \text{Ex} - \text{Im} = \text{Export} - \text{Import};$$

$$X_i = X_{0i} + X_{1i}y_i(t) + X_{2i}y_i(t-h) + X_{3i}\dot{y}_i(t) + X_{4i}\dot{y}_i(t-h) \\ + X_{8i}L_i(t) + X_{10i}L_i(t-h) + X_{15i}e_i(t) + X_{16i}\tau_i(t) + X_{17i}d_i(t) \\ + y_i(t)[a_{1i}y_{i+1}(t-h)] + L_i(t)[C_{1i}L_{i+1}(t-h)]. \quad i = 1, 2 \quad (6.4)$$

(Assume $y_3 = y_1; L_3 = L_1$)

The last two terms

$$y_1(t)[a_{11} y_2(t-h)] + L_1(t)[C_{11} L_2(t-h)]$$

and

$$y_2(t)[a_{12} y_1(t-h)] + L_2(t)[C_{12} L_1(t-h)] \quad (6.5)$$

are the contribution of interaction between country of y_1 and the country of y_2 due to cooperation and competition measured in terms of trade surplus, trade deficit, debt repayment, grants, debt relief and international employment, etc.

Here τ = tariff, d = distance between trading nations and/or trade policy.

$$G_i = g_{0i} + g_{1i}y_i(t) + g_{2i}y_i(t-h) + g_{3i}\dot{y}_i(t) + g_{4i}L_i(t-h) \quad (6.6) \\ (g_0 = \text{federal budget autonomous net expenditure})$$

Thus, gathering the formulae for $z_i = I_i + C_i + X_i + G_i$, and invoking the demand and supply principle,

$$\frac{dy_i(t)}{dt} = \lambda_i(z_i(t) - y_i(t)), \quad (6.7)$$

and invoking Eqs. (6.1.66 and 6.1.67) of Chukwu [1] with $R \equiv 0$, $k = 0$, $p = 0$ and with q_1 , v , defined in (6.1.16–6.1.17) we deduce the equation

$$\begin{aligned} \begin{bmatrix} \dot{y}_1(t) \\ \dot{L}_1(t) \\ \dot{y}_2(t) \\ \dot{L}_2(t) \end{bmatrix} + \begin{bmatrix} -a_{-11} & -a_{-13} & 0 & 0 \\ -\ell_{-031} & -\ell_{011} & 0 & 0 \\ 0 & 0 & -a_{-22} & -a_{-23} \\ 0 & 0 & -\ell_{022} & -\ell_{-023} \end{bmatrix} \begin{bmatrix} \dot{y}_1(t-h) \\ \dot{L}_1(t-h) \\ \dot{y}_2(t-h) \\ \dot{L}_2(t-h) \end{bmatrix} \\ = \begin{bmatrix} a_1 & a_{14} & 0 & 0 \\ 0 & \ell_0 & 0 & 0 \\ 0 & 0 & b_{01} & b_{14} \\ 0 & 0 & 0 & \ell_{02} \end{bmatrix} \begin{bmatrix} y_1(t) \\ L_1(t) \\ y_2(t) \\ L_2(t) \end{bmatrix} \\ + \begin{bmatrix} a_{11} & a_{15} & 0 & 0 \\ \ell_2 & -\ell_1 & 0 & 0 \\ 0 & 0 & b_{11} & b_{25} \\ 0 & 0 & \ell_{32} & -\ell_{12} \end{bmatrix} \begin{bmatrix} y_1(t-h) \\ L_1(t-h) \\ y_2(t-h) \\ L_2(t-h) \end{bmatrix} \\ + \begin{bmatrix} 0 & 0 & a_{11}y_1(t) & 0 \\ 0 & 0 & 0 & c_{11}L_1(t) \\ a_{12}y_2(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{12}L_1(t) \end{bmatrix} \begin{bmatrix} y_1(t-h) \\ L_1(t-h) \\ y_2(t-h) \\ L_2(t-h) \end{bmatrix} \\ + B_1p + B_2g. \end{aligned} \quad (6.8)$$

The control strategies associated with government are

$$q = [q_1, q_2],$$

and the control strategies associated with the private firms are

$$\sigma = [r_1, r_2].$$

If $[y_1, L_1, y_2, L_2]$ is the prevailing economic state then

$$\begin{aligned} q_1 = \lambda_1 \sigma_1^{-1} [g_{01}, z_{g13}M1, -z_{141}T_1(t) - z_{191}T_1(t-h) - z_{01}\dot{T}_1(t) \\ - z_{211}\dot{T}_1(t-h) - z_{g15}e_1(t) + z_{161}\tau_1(t) + z_{g71}d_1(t)], \end{aligned} \quad (6.9a)$$

and

$$r_1(t) = \lambda_1 \sigma_1^{-1} [(C_{01} + I_{o1} + X_{01}) - M_{01}(I_{151} + C_{71})] \quad (6.9b)$$

are associated with y_1 . The strategy which is related to L_1 is

$$\sigma_{31}(t) = m_1(w)[x_{01} + y_{101} + I_{01}]. \quad (6.9c)$$

Similarly the government strategy associated with y_2 is

$$\begin{aligned} q_2(t) = \lambda_2 \sigma_2^{-1} [& g_{02} + z_{s23} M_2 - z_{s24} T_2(t) - z_{292} T_2(t-h) - z_{02} \dot{T}_2(t) \\ & - z_{212} \dot{T}_2(t-h) - z_{25} e_2(t) - z_{26} T_2(t) + z_{27} d_2(t)] \end{aligned} \quad (6.9d)$$

The representative private firm's strategy associated with L_2 is

$$r_{32}(t) = M_2(w)[x_{02} + y_{102} + I_{02}]. \quad (6.9e)$$

The matrices B_1 and B_2 in (6.8) are identified from the equations $q_1, r_1, q_2, r_2, \sigma_{31}$, and $\sigma_{32}(t)$. In matrix form,

$$\dot{x}(t) - A_{-1} \dot{x}(t-h) = A_0 x(t) + A_1 x(t-h) + A_2(x(t))x(t-h) + B_1 p + B_2 g, \quad (6.9f)$$

where

$$\begin{aligned} A_{-1} &= \begin{bmatrix} -a_{-11} & -a_{13} & 0 & 0 \\ -L_{-031} & -L_{-011} & 0 & 0 \\ 0 & 0 & a_{-22} & -a_{-23} \\ 0 & 0 & -L_{-022} & -L_{-032} \end{bmatrix}, \\ A_0 &= \begin{bmatrix} a_1 & a_{14} & 0 & 0 \\ 0 & \ell_0 & 0 & 0 \\ 0 & 0 & b_0 & b_{14} \\ 0 & 0 & 0 & \ell_{02} \end{bmatrix}, \\ A_1 &= \begin{bmatrix} a_{11} & a_{15} & 0 & 0 \\ \ell_2 & -\ell_1 & 0 & 0 \\ 0 & 0 & b_{11} & a_{25} \\ 0 & 0 & \ell_{32} & -\ell_{12} \end{bmatrix}, \end{aligned} \quad (6.10)$$

$$A_2(x(t)) = \begin{bmatrix} 0 & 0 & a_1 y_1(t) & 0 \\ 0 & 0 & 0 & c_{11} L_1(t) \\ a_{12} y_2(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{12} L_1(t) \end{bmatrix}, \quad (6.11)$$

In [2, p. 27], the derivation of Employment and capital stock equations assumes that national income from the expenditure side in nation 1 is given in (1.44) by

$$y_1 = \bar{C}_1 + \bar{I}_1 + \bar{X}_1 + \bar{G}_1,$$

where

$$\begin{aligned} \bar{X}_1 = & x_0 + x_1 y_1(t) + x_2 y_1(t-h) + x_5 R_1(t) + x_8 L_1(t) + x_{10} \dot{L}_1(t) \\ & + x_{11} e_1(t) + x_{12} \tau_1(t) + x_{13} d_1(t). \end{aligned}$$

If we insert the effects of interaction with nation 2 to \bar{X}_1 , we add

$$y_1(t)[d_{21}y_2(t-h)] + L_1(t)d_{22}L_2(t-1).$$

The interaction matrix term introduced is thus

$$A(x)x(t-1) = \begin{bmatrix} 0 & 0 & a_{15}y_1(t) & c_{16}L_1(t) \\ 0 & 0 & d_{11}y_1(t) & d_{12}L_1(t) \\ a_{25}y_2(t) & a_{26}L_2(t) & 0 & 0 \\ d_{22}y_2(t) & d_{23}L_2(t) & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1(t-h) \\ L_1(t-h) \\ y_2(t-h) \\ L_2(t-h) \end{bmatrix} \quad (6.12)$$

6.3 Controllability

Consider the model

$$\dot{x}(t) - A_{-1}\dot{x}(t-h) = A_0x(t) + A_1x(t-h) + A_2(x(t))x(t-h) + B_1p + B_2q, \quad (6.13)$$

where B_1p , B_2q are as defined in (6.8 and 6.9a-f) and $x = \begin{bmatrix} y_1 \\ L_1 \\ y_2 \\ L_2 \end{bmatrix}$ is the economic state vector of Gross Domestic Product and Employment. Also, A_{-1} is a 4×4 matrix, A_0 a 4×4 matrix, A_1 4×4 matrix $A_2(x(t))$ is a 4×4 matrix function of x .

The strategy $\sigma(t) = B_1p$ is the control function of the private firms: $\sigma(t) \in P \subset E^n$, and $g(t) = B_2q \in Q \subset E^4$, that of government. We are to use p_i, g_i $i = 1 \dots 4$ to steer an initial state value of the nations' gross domestic products x and employment to a desirable target x_1 in time t_1 . Thus the solution of (6.13) $x(\phi, \sigma, g)$ with $x_t(\phi, \sigma, g) = \phi$ satisfies $x(\phi, \sigma; g)(t_1) = x_1$. The terminology of differential game is appropriate. By the method of Hájek [4] and [5] the model can be reduced to a differential control system

$$\dot{x}(t) - A_{-1}\dot{x}(t-h) = A_0x(t) + A_1x(t-h) + A_2(x(t)x(t-h)) + Bv(t), \quad (6.14)$$

where $Bv \subset BV$, by using a Stroboscopic strategy in the game (6.9a-f). Here,

$$BV(t) = (P + \ker U(t_1, t, x_t(t_0, \phi))) \pm Q. \quad (6.15)$$

For some t_1 , where

$$U_t(\cdot, x_s(t_0, \phi))X_0(\theta) = T(t, s, x_s(t_0, \theta)) \quad (6.16)$$

is the solution operator of

$$\dot{x}(t) - A_{-1}\dot{x}(t-h) = A_0x(t) + A_1x(t-h) + A_2(x(t))x(t-h), \quad (6.17)$$

$$\begin{aligned} BV &= \{Bv : Bv + B_2Q \subset B_1P + \ker U(t_1, t, x_t(t_0, \phi))\} \\ &= (B_1P + \ker U(t_1, t, x_t(\sigma, \phi)))^*_-B_2Q. \end{aligned} \quad (6.18)$$

It is possible for Kernel $U(t_1, t, x_t(t_0, \phi))$ to be zero. This happens when $\det(A_{-1}) / = 0$ see Hale [6; p. 279]. In this case

$$BV = B_1P^*_-B_2Q \quad (6.19)$$

In the linear case the proof of conversion of the game into a control system is treated in [3; p. 367]. The nonlinear case uses the nonlinear variation of parameter (Eq. 8.4.3 of 3; p. 111). It is now reasonable to consider the controllability and the Time Optimal Control Theory of the system

$$\dot{x}(t) - A_{-1}\dot{x}(t-h) = A_0x(t) + A_1x(t-h) + A_2(x(t)x(t-h)) + Bu, \quad u \in U. \quad (6.20)$$

The variational control system along the trivial solution is

$$\dot{x}(t) - A_{-1}\dot{x}(t-h) = A_0x(t) + A_1x(t-h) + Bu. \quad (6.21)$$

Theorem [3; p. 450]. *Assume that:*

- (i) $\text{rank}[\Delta(\lambda), B] = 4\forall\lambda$ complex,
where

$$\Delta(\lambda) = I\lambda - A_0 - A_1e^{-\lambda} - A_{-1}\lambda e^{-\lambda h}; \quad (6.22)$$

- (ii) $\text{rank}[\lambda I - A_{-1}, B] = 4\forall\lambda$ complex.

Then (6.20) is locally null controllable and locally null controllable with constraints. That is, there is a neighborhood \mathfrak{V} of zero in $W_\infty^{(1)}$ such that every initial

point of \mathfrak{g} can be driven to zero in some finite time t_1 using some control strategy $u \in U_{ad}$, where

$$U_{ad} = \{u \in L_\infty([t_0, t_1]) : \|u\|_\infty \leq M\}.$$

The bound M on the control strategy can be calculated for the identified system. Let

$$f(x(t), x(t-h)) = A_0x(t) + A_1x(t-h) + A_2(x(t))x(t-h). \quad (6.23)$$

Then

$$f(0, 0) = 0.$$

Let

$$\begin{aligned} H_1 &\triangleq D_1f(x(t), 0), \\ H_2 &\triangleq D_2f(x(t), x(t-h)), \end{aligned}$$

where $D_i f$ is the i -th partial derivative of f . Let $A_0 = D_1f(0, 0)$, $A_1 = D_2f(0, 0)$ and denote by D_a the symmetric 4×4 matrices

$$1/2(H_a + H_a^T)\Delta D_a \quad a = 1, 2$$

Define J_a as follows

$$J_a = AD_a + D_a^T A,$$

where A is a positive definite symmetric 4×4 matrix.

Theorem In (6.20) assume

- (i) $\text{rank}[\Delta(\lambda), B] = 4$ for all $\lambda \in \text{complex}$,
where

$$\Delta(\lambda) = I - \lambda A_{-1}e^{-\lambda h} - A_0 - A_1e^{-\lambda h}.$$

- (ii) $\text{rank}[\lambda I - A_{-1}, B] = 4$, for all of λ complex.

- (iii) All the roots of the equation

$$\det(I - A_{-1}r^{-h}) = 0,$$

have moduli less than 1 and for some positive definite $n \times n$ matrix A we have

$$J_1 \leq -\delta I,$$

where, for some $q > 1$,

$$\delta - q\|J_2 + J_1 A_1\| \equiv \mu > 0.$$

Then (6.20) is null controllable with constraints, i.e., with controls in U_{ad} .

Remark The proofs are contained in Chukwu [3, Chap. 12]. As observed in [3, p. 376] though zero is the target at the final time, the theory incorporates nontrivial targets as well.

Consider the system

$$\frac{d}{dt}x(t) - A_{-1}\dot{x}(t-h) = A_0x(t) + A_1x(t-h) + A_2(x(t)x(t-h) + B_1p + B_2q \quad (6.24)$$

Let

$$f(t, x(t), x(t-h)) \equiv A_0x(t) + A_1x(t-h) + A_2(x(t)x(t-h)); \quad (6.25)$$

and let

$$K(t, x(t), x(t-h), q) = B_2q \quad (6.26)$$

Then we can write

$$\dot{x}(t) - A_{-1}\dot{x}(t-h) = f(t, x(t), x(t-h)) + B_1p + K(t, x(t), x(t-h), q) \quad (6.27)$$

Theorem Assume that

1. $\text{rank}, B_1 = 4$ on $[t_1 - h, t]$;

$$|f(t, x(t), x(t-h))| \leq \alpha F(t, x(t), x(t-h), p),$$

where

$$F: E^4 \times E^4 \rightarrow E^+$$

$$|K(t, x(t), x(t-h), q)| \leq \beta F(t, x(t), x(t-h), p) \quad \text{for all } t, x, p, b < a.$$

Then (6.27) is controllable on $[\sigma, t_1]$ with $t_1 > \sigma + h$. We thus require that private or representative firms control set should dominate the governments. The proof is an adaptation of Theorem 13.1 in p. 253 of [2].

We have presented a model of GDP and Employment for interacting nations. It opens up the possibility of using minimum “investment”, both private and government, to steer growth to the desired target as fast as possible. See [1, pp. 293–299]. We confront our model with UN, data to see how close to the real world our growth model is for the nations considered, i.e., for China and the USA or the USA and UK. See Appendix.

6.4 Simple Interaction

We now assume that the interaction matrix is a simple constant 4×4 matrix, A_1 , and $A_{-1} = 0$ where

$$A_1 = \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{22} & 0 & a_{23} & a_{24} \\ a_{31} & a_{32} & 0 & a_{34} \\ a_{41} & a_{42} & a_{43} & 0 \end{bmatrix}. \quad (6.28)$$

Define A as follows:

$$\begin{aligned} A &= A_0 + A_1 \\ &= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{22} & a_{211} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{311} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{411} \end{bmatrix}. \end{aligned} \quad (6.29)$$

Now consider the ordinary differential control system

$$\dot{x} = Ax + Bu, \quad (6.30)$$

where

$$Bu = B_1P + B_2q = [B_1, B_2] \begin{bmatrix} p \\ q \end{bmatrix}. \quad (6.31)$$

Assume the derived model is a game of pursuit which, by Hájek’s method, can be proved to be equivalent to (6.31), where

$$Bu \in V \equiv B_1 P^* B_2 G.$$

Let

$$\begin{aligned} |C_{oi}| &\leq b_i, \\ |I_{oi}| &\leq c_i, \\ |X_{oi}| &\leq d_i \\ \alpha_i &= \max_{i=1\dots 3} [b_i, c_i, d_i], \\ P &= \prod_{i=1}^4 [-\alpha_i, \alpha_i] \quad i = 1, \dots, 3; \end{aligned}$$

Also

$$\begin{aligned} |g_{oi}| &\leq h_i \\ |T_{ai}| &\leq k_i \\ |\tau_i| &\leq \ell_i \\ |e_i| &\prec n_i \\ Q &= \prod_{i=1}^4 [-\beta_i \beta_i]; \\ \beta_i &= \max_{1 \leq i \leq 4} [h_i, k_i, \ell_i, n_i]. \end{aligned}$$

Then

$$P * Q = \{u : u + Q \subset P\} = V.$$

With V so constructed, a specific optimal control strategy can be deduced from our theory (inspired by) Hermes and LaSalle [7] and Chukwu [3, p. 114]. This investigation can also be viewed from another perspective.

Let

$$Q = \prod_{i=1} Q_i, \tag{6.32}$$

$$P = \prod_{i=1} P_i, \tag{6.33}$$

where

$$\begin{aligned}
Q_1 &= [-\max g_{0i}, +\max g_{0i}], \\
Q_2 &= [-\max T_{ai}, \max T_{ai}], \\
Q_3 &= [-\max \tau_i, \max \tau_i], \\
Q_4 &= [-\max e_i, \max e_i], \\
P_1 &= [-\max C_{oi}, \max C_{oi}], \\
P_2 &= [-\max I_{0i}, \max I_{0i}], \\
P_3 &= [-\max X_{0i}, \max X_{0i}].
\end{aligned}$$

The set

$$U = P_-^* Q,$$

where

$$p = B_1 r, \quad q = B_2 g, \quad (6.34)$$

and

$$|p_i| \leq \alpha_i, \quad |q_i| \leq \beta_i, \quad \text{and we assume } \beta_i < \alpha_i. \quad (6.35)$$

Then

$$U = P_-^* Q = \{u \in U : |u_i| \leq \alpha_i - \beta_i, \quad i = 1, \dots, 4\}. \quad (6.36)$$

With u so constructed a specific optimal control strategy can be constructed from our theory, from the work of Hermes and LaSalle [7] or Chukwu [1, p. 135].

6.4.1 Oscillation

Consider the equivalent control system

$$\dot{x} = Ax - u, \quad (6.37)$$

and its associated autonomous linear system

$$\dot{x}(t) = Ax(t). \quad (6.38)$$

Let

$$\det(A - \lambda I) = 0 \quad (6.39)$$

$$= a_0 \lambda^n + a_1 \lambda^{n-2} + \dots + a_n = 0. \quad (6.40)$$

The solution of (6.38) is oscillatory if the solution λ of (6.40) is complex. If x is non-scalar and nontrivial and $x(t) = (x_1(t), \dots, x_n(t))$ is defined on $[0, \infty)$ then it is oscillatory if and only if at least one component of x has arbitrarily large zeros on $[0, \infty)$. If all components are nonoscillatory, it is nonoscillatory.

Theorem *Let*

$$\dot{x} = Ax, \quad (6.41)$$

and

$$f(\lambda) = |A - \lambda I| = 0,$$

so that

$$= a_0 \lambda^n + a_1 \lambda^{n-1} + \dots + a_n = 0,$$

with $a_n \neq 0$ then the solution of (6.41) is oscillatory whenever

$$\Delta_{n-1} = 0,$$

where

$$\Delta_1 = a_1, \Delta_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix},$$

$$a_k = \begin{bmatrix} a_1 & a_3 & a_5 & \dots & a_{2k-1} \\ a_0 & a_2 & a_4 & \dots & a_{2k} \\ 0 & a_1 & a_3 & \dots & a_{2k-1} \\ 0 & a_0 & a_2 & \dots & a_{2k} \\ 0 & 0 & a_1 & \dots & a_{2k-1} \\ 0 & 0 & 0 & \dots & a_{2k-2} \end{bmatrix} \quad k = 1, 2, \dots, n.$$

Here we substitute a_k for $k > n$; and

$$\Delta_{n-1} = (-1)^{\frac{n(n-1)}{2}} a_0^{n-1} \prod_{i < k}^{1, \dots, n} (\lambda_i + \lambda_k).$$

And

$$\Delta_n = (-1)^{\frac{n(n+1)}{2}} a_0^n \lambda_1 \lambda_2 \dots \lambda_n \prod_{i < k}^{1, \dots, n} (\lambda_i + \lambda_k).$$

Remarks $\Delta_n - 1 = 0$ if, and only if the sum of two roots of $f(\lambda) = 0$ is zero. That is $f(\lambda)$ has at least one pair of conjugate roots or multiple zero roots. If $a_n \neq 0$ we have at least one pair of pure imaginary roots only, and the zero roots are ruled out.

This theorem, translated into economic terms and related to the economic dynamics (6.30) yields policies for movement to depression or/and to boom for GDP and employment. (See p. 191 of Chukwu [1]). Oscillation can be tamed by introducing a feedback control $u(t) = Fx(t)$ so that

$$\dot{x}(t) = (A - F)x(t)$$

is nonoscillatory.

Consider the economic state

$$x = [y_1, L_1, y_2, L_2],$$

of the Gross Domestic Product, and employment for two interacting countries described by the equation

$$\dot{x}(t) = Ax(t) + Bu \tag{6.42}$$

One applies the theorem of Brammer to (6.42) to prove (6.42) null controllable and therefore x_1 —constrained controllable. See p. 131 of Chukwu [1].

With the simple interacting A_1 in (6.28) and the economic state

$$x = [y_1, L_1, y_2, L_2],$$

we obtain the model

$$\dot{x}(t) - A_{-1}\dot{x}(t - h) = Ax(t) + A_1x(t - h) + B_1p + B_2q.$$

Inspired by stroboscopic strategy as in Hájek's Analysis and Chukwu, A Mathematical Treatment of Economic Cooperation and Competition Among Nations with Nigeria, USA, UK, China, and Middle East Examples, Elsevier Academic Press, we deduce an equivalent system

$$\begin{aligned}
\begin{bmatrix} \dot{y}_1(t) \\ \dot{L}_1(t) \\ \dot{y}_2(t) \\ \dot{L}_2(t) \end{bmatrix} &= \begin{bmatrix} b_1 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 \\ 0 & 0 & b_3 & 0 \\ 0 & 0 & 0 & b_4 \end{bmatrix} \begin{bmatrix} y_1(t) \\ L_1(t) \\ y_2(t) \\ L_2(t) \end{bmatrix} + \begin{bmatrix} -a_{11} & -a_{12} & -a_{13} & -a_{14} \\ -a_{21} & -a_{22} & -a_{23} & -a_{24} \\ -a_{31} & -a_{32} & -a_{33} & -a_{34} \end{bmatrix} \begin{bmatrix} y_1(t-h) \\ L_1(t-h) \\ y_2(t-h) \\ L_2(t-h) \end{bmatrix} \\
&+ \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} r_1(t) \\ r_2(t) \\ r_3(t) \\ r_4(t) \end{bmatrix}, \\
r = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} &= B_1 p
\end{aligned}$$

$$B_1 p = \begin{bmatrix} \xi_1 & \xi_1 & \xi_2 & \xi_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_2 & \rho_2 & \rho_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi_3 & \xi_3 & \xi_3 & \xi_{31} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_4 & \rho_4 & \rho_4 & 0 \end{bmatrix} \begin{bmatrix} C_{01} \\ I_{01} \\ X_{01} \\ M_{01} \\ x_{02} \\ y_{02} \\ I_{02} \\ C_{03} \\ I_{03} \\ X_{03} \\ M_{03} \\ x_{04} \\ y_{04} \\ I_{04} \end{bmatrix}.$$

Let

$$T_{ai} = C_{1i}T_i + C_{2i}\dot{T}_1(t) - I_{2i}T_i - I_{2i}\dot{T}_i(t), \quad (6.43)$$

and

$$q = B_1 g = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} =$$

$$B_1 g = \begin{bmatrix} \lambda_1, & \lambda_1, & \lambda_1 X_{161}, & \lambda_1 X_{151}, & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2, & \lambda_2, & \lambda_2 X_{162}, & \lambda_2 X_{152} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_3, & \lambda_3, & \lambda_3 X_{163}, & \lambda_3 X_{153}, & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4, & \lambda_4, & \lambda_4 X_{164}, & \lambda_4 X_{154} \end{bmatrix} \begin{bmatrix} g_{01} \\ T_{a1} \\ \tau_1 \\ e_1 \\ g_{02} \\ T_{a2} \\ \tau_2 \\ e_2 \\ g_{03} \\ T_{a3} \\ \tau_3 \\ e_3 \\ g_{04} \\ T_{a4} \\ \tau_4 \\ e_4 \end{bmatrix} \begin{bmatrix} \lambda \sigma_1^{-1} = \xi_3 \\ \lambda_2 = M_2(\omega) \\ z_{a26} = X_{162} \\ z_{a25} = X_{152} \\ \lambda_3 \sigma_1^{-1} = \xi_3 \\ -z_{35} = X_{153} \\ z_{36} = X_{163} \\ M_4(\omega) = \lambda_4 \\ T_{a4} = z_{a44} T_4 \\ T_{a4} = z_{a44} T_4 \\ M_4(\omega) z_{a45} = \lambda_4 X_{154} \\ m_4 z_{a46} = \lambda_4 X_{164} \end{bmatrix} \quad (6.44)$$

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + B_1 g + B_2 p.$$

$$\dot{x}(t) = A_0(t) + A_1(x(t))x(t-h) + B_2 g + B_1 p$$

where

$$x = [y_1, L_1, y_2, L_2].$$

We now use data on the US and China from the World Bank, and IMF Publications to confront our theoretical model, and examine whether it is realistic.

US: $y_1 = \text{GDP}$
 $L_1 = \text{unemployment/employment}$

China: $y_2 = \text{GDP}$
 $L_2 = \text{unemployment/employment}$

U.S.A.

$I_1 = \text{investment},$
 $C_1 = \text{consumption},$
 $X_1 = \text{net export, Export-Import},$
 $G_1 = \text{government intervention, outlay},$
 $T_1 = \text{Taxes},$
 $e_1 = \text{exchange rate},$
 $\tau_1 = \text{tariff},$
 $d_1 = \text{distance between nations(policy)},$

Aggregate demand

$$\begin{aligned}
 Z_1 &= C_1 + I_1 + X_1 + G_1 \\
 C_1 &= C_{01} + C_{11}(y_1(t) - T_1(t)) + C_{21}(y_1(t-1) - T_1(t-1)) \\
 &\quad + C_{31}(\dot{y}_1(t) - \dot{T}_1(t)) + C_{41}(\dot{y}_1(t-1) - \dot{T}_1(t-1)) \\
 I_1 &= I_{01} + I_{11}y_1(t) + I_{21}y_1(t-1) - I_{31}\dot{y}_1(t) + I_{41}\dot{y}_1(t-1) \\
 &\quad + I_{81}L_1(t) + I_{91}L_1(t-h) + I_{10}\dot{L}_1(t-h) \\
 X_1 &= Ex_1 - Im_1 = \text{Export} - \text{Import}; \\
 X_1 &= X_{01} + X_{11}y_1(t) + X_{21}y_1(t-1) + X_{31}\dot{y}_1(t) + X_{41}\dot{y}_1(t-1) \\
 &\quad + X_{11}L_1(t) + X_{11}L_1(t-h) + X_{151}e_1(t) + X_{161}\tau_1(t) + X_{171}d_1(t) \\
 &\quad + X_{101}\dot{L}_1(t-1) + y_1(t)[a_{11}y_2(t-1)] + L_1(t)[C_{11}L_2(t-1)]. \\
 G_1 &= g_{01} + g_{11}y_1(t) + g_{21}y_1(t-1) + g_{31}\dot{y}_1(t) + g_{41}\dot{y}_1(t-1) + g_{61}L(t).
 \end{aligned}$$

Use the principle of supply and demand:

$$\frac{dy_1}{dt} = \lambda_1(z_1(t) - y_1(t)),$$

where

$$\begin{aligned}
 Z_1(t) &= (C_1 + I_1 + X_1 + G_1) \\
 &= (C_{01} + I_{01} + X_{01} + g_{01}) + (C_{11} + I_{11} + X_{11} + g_{11})y_1(t) \\
 &\quad + (C_{12} + I_{12} + X_{12} + g_{12})y_1(t-h) + (C_{13} - I_{13} + X_{13} + g_{13})\dot{y}_1(t) \\
 &\quad + (C_{14} + I_{14} + X_{14} + g_{14})\dot{y}_1(t-h) \\
 &\quad + (I_{101} + X_{101})\dot{L}_1(t-h) + y_1(t)[a_{11}y_2(t-1)] \\
 &\quad + L_1(t)[C_{11}L_2(t-1)] + [I_{81} + g_{61} + X_{91}]L_1(t) + X_{161}\tau_1(t) \\
 &\quad + X_{151}e_1(t) + [I_{91} + X_{101}]L_1(t-1) + X_{171}d_1(t) + C_{21}y_1(t-1) \\
 &\quad - C_{21}T_1(t-1) + C_{31}y'_1(t) - C_{31}T'_1(t) + C_{41}T'_1(t-h) + C_{41}y'_1(t-1).
 \end{aligned}$$

Let

$$\begin{aligned}
 Z_{01} &= C_{01} + I_{01} + X_{01} + g_{01}, \\
 Z_{11} &= C_{11} + I_{11} + X_{11} + g_{11}, \\
 Z_{12} &= C_{12} + I_{12} + X_{12} + g_{12}, \\
 Z_{13} &= C_{13} - I_{13} + X_{13} + g_{13}, \\
 Z_{14} &= C_{14} + I_{14} + X_{14} + g_{14}, \\
 Z_{15} &= I_{101} + X_{101}, \\
 Z_{16} &= a_{11}, \\
 Z_{17} &= C_{11}, \\
 Z_{18} &= I_{81} + g_{61} + X_{91}, \\
 Z_{19} &= I_{91} + X_{101}.
 \end{aligned}$$

Note that income from the expenditure side is

$$y_s = z_{s0} + z_{s1}y(t) + z_{s2}y(t-h) + z_{s4}\dot{y}(t) + z_{s8}L(t) + z_{s10}\dot{L}(t) \\ - z_{s14}T(t) + z_{s15}e(t) + z_{s16}\tau(t).$$

Invoke the cited differential principle of supply and demand,

$$\frac{dy_1}{dt} = \lambda_1(z_1 - y_1),$$

to obtain

$$(1 - \lambda_1)Z_{13}\frac{dy_1}{dt} - \lambda_1 z_{14}\frac{dy_1}{dt}(t-1) - \lambda_1 z_{15}\dot{L}_1(t-h) = \lambda_1(z_{11}y_1 + z_{01} \\ + z_{21}y_1(t-h) + z_{61}L_1(t) + z_{71}L_1(t-1) + C_{01}y_1(t)y_2(t) \\ + C_{11}L_1(t)L_2(t-1) + C_{21}y_1(t-1) - C_{31}\dot{y}_1(t) \\ + T_{1a} + X_{151}e_1(t) + X_{161}\tau_1(t) + X_{171}d_1(t)).$$

Here

$$T_{1a} = -C_{21}T_1(t-1) - C_{31}T'_1(t) + C_{41}T_1(t-1) + C_{41}\dot{T}_1(t-h) - C_{11}T_1(t).$$

Let

$$(1 - \lambda_1)z_{13} = \sigma_1 \\ \frac{\lambda_1 z_{14}}{(1 - \lambda_1)z_{13}} = a_{-11} \\ \frac{\lambda_1 z_{15}}{(1 - \lambda_1)z_{13}} = a_{-12} \\ a_{11} = \lambda_1 z_{11}, \quad \lambda_1 X_{151}/\sigma_1 = a_{18}, \lambda_1 z_{01}, \\ a_{12} = \lambda_1 z_{21}, \quad \lambda_1 X_{161}/\sigma_1 = a_{19}, \\ a_{13} = \lambda_1 z_{61}, \quad \lambda_1 z_{171}/\sigma_1 = a_{20}, \\ a_{14} = \lambda_1 z_{71}, \\ a_{15} = \lambda_1 C_{01}, \\ a_{16} = \lambda_1 C_{11}, \\ a_{17} = \lambda_1 C_{21},$$

$$\begin{aligned}
\frac{dy_1(t)}{dt} - a_{-11} \frac{dy_1}{dt}(t-1) - a_{-12} \dot{L}_1(t-h) &= a_{11}y_1(t) + a_{12}y_1(t-1) \\
&+ a_{13}L_1(t) + a_{14}L_1(t-1) + a_{15}y_1(t)y_2(t-1) + a_{16}L_1(t)L_2(t-1) \\
&+ T_{1a} + \lambda_1 z_{01} + a_{18}e_1(t) + a_{19}\tau(t) + a_{20}d(t).
\end{aligned} \tag{6.45}$$

China's GDP dynamics is obtained in the same way:

$$\begin{aligned}
\frac{dy_2(t)}{dt} - a_{-21} \frac{dy_2}{dt}(t-1) - a_{-22} \dot{L}_2(t-h) &= a_{21}y_2(t) + a_{22}y_2(t-1) \\
&+ a_{23}L_2(t) + a_{24}L_2(t-1) + a_{25}y_2(t)y_1(t-1) + a_{26}L_2(t)L_1(t-h) \\
&+ T_{2a} + \lambda_2 z_{02} + a_{29}e_2(t) + a_{28}\tau_2(t) + a_{30}d_2(t).
\end{aligned} \tag{6.46}$$

See [3, Eq. 1.10.64] for employment equation, with indicated interaction

$$\begin{aligned}
\dot{L}_1(t) - l_{011} \dot{L}_1(t-h) - l_{031} \dot{y}_1(t-h) &= l_{01}L_1(t) - l_{11}L_1(t-h) \\
&+ l_{12}y_1(t-h) + l_{51}L_1(t-h) + \sigma_{31}(t) + q_{41}(t) + d_{11}y_1(t)y_2(t-h) \\
&+ d_{12}L_1(t)L_2(t-h);
\end{aligned} \tag{6.47}$$

$$\begin{aligned}
\dot{L}_2(t) - l_{022} \dot{L}_2(t-h) - l_{032} \dot{y}_2(t-h) &= l_{02}L_2(t) - l_{12}L_1(t-h) \\
&+ l_{12}y_2(t-h) + l_{52}L_2(t-h) - \sigma_{32}(t) + q_{42}(t) \\
&+ d_{22}y_2(t)y_1(t-h) + d_{23}L_2(t)L_1(t-h).
\end{aligned} \tag{6.48}$$

Use MATLAB to identify the coefficients. This is done in the same as in Chukwu [1–3]. Thus we observe that

$$y = \tilde{C} + \tilde{I} + \tilde{X} + \tilde{G},$$

where

$$\begin{aligned}
\tilde{C} &= y_{10} + y_{11}y(t) + y_{12}\dot{y}(t) + y_{18}(y-T), \\
\tilde{I} &= I_0 + I_1 \\
I_1(t) &= \frac{1}{h}[k(t+h) - k(t)]; \\
\frac{dk}{dt} &= D(t-h)
\end{aligned}$$

is the rate of delivery of capital equipment.

$$\begin{aligned}
 D(t) &= a(1 - e)y(t) - k_0k(t) + L_3L(t) + k\dot{y}(t), \\
 \bar{X} &= x_0 + x_1y(t) + x_2y(t - h) + x_8L(t) + x_{10}\dot{L}(t) + x_{11}e(t) + x_{12}\tau(t) \\
 &\quad + x_{13}d(t) + x_{14}y_1(t)y_2(t - h) + x_{15}L_1(t)L_2(t - h). \\
 G &= g_{s0} + g_{s1}y(t) + g_{s4}\dot{y}(t) + g_{s8}L(t).
 \end{aligned}$$

Assuming profit maximization of the firm where the profit function is

$$P = y - wL - rK$$

and w is the wage of labor per unit time and r = the rent per unit time for the use of capital

$$m(w) = \left[(1 - \alpha) \frac{1}{w} \right]^{1/\alpha}. \quad (6.49)$$

The constant α is obtained from the Cobb-Douglas equation

$$y = f(k, L) = k^\alpha L^{1-\alpha},$$

then

$$\begin{aligned}
 \dot{L}(t) - l_{-01}\dot{L}(t - h) - l_{-03}\dot{y}(t - h) &= l_0L(t) - l_1L(t - h) + l_2y(t - h) \\
 &\quad + l_5L(t - h) + \sigma_3(t) + q_3(t).
 \end{aligned} \quad (6.50)$$

The coefficients are identified in [3 Eqs. 1.10.55–1.10.57 and 1.10.66], Chukwu [1].

6.4.2 Parameter Identification

We use the methods of Chukwu, Differential Models and Neutral Systems for Controlling the Wealth of Nations, World Scientific 2001, E. N. Chukwu, Optimal Control of the Growth of Wealth of Nations, Taylor and Francis to identify the coefficients of the dynamic model. It is similar to Chukwu's contribution in "Cooperation and Competition in Modeling the Dynamics of the Gross Domestic Products of Nations," AMC, Elsevier. A careful judicious mix of private and government policies can ensure the growth of GDP and employment to a very desirable state. For example if (6.45) a_{15} is negative, while a_{-11} , a_{-12} are fixed constants then the growth rate of the GDP y_1 of Nation 1 can be magnified by the positive cooperation of nation 1 and nation 2, i.e., add $+a_{15}y_1(t)y_2(t - h)$. One

trade strategy which is used can be illustrated by the case of General Motors. It can transfer its manufacturing business to China where there is cheap well trained abundant labor. It can afford to sell its products cheap, and by volume make enormous profit. China can reduce or eliminate taxes on the profit in return for the jobs created and the standard of living raised. The USA can tax GM moderately to obtain the funds needed to create jobs at home, e.g., public jobs, etc. Thus $-a_{15}$ can become $+a_{15}$ contributing

$$+a_{15}y_1(t)y_2(t-h) \text{ to } \frac{dy_1(t)}{dt}$$

to ensure positive growth rate. According to Chairman Richard Wagoner CEO, General Motors, NCSU Engineering Issues Forum, February 9, 2004, GM is number one company in America and in the World and it uses cooperation strategies similar to what is described. The control strategies B_{2g} , B_{1p} can be exploited to steer USA GDP and employment to a very pleasant level.

Without introducing employment in the equation, the following equations were generated for the GDP y_2 of the USA and the GDP y_4 of China. Both are interacting with Nigeria and UK.

6.5 A Theory

Data generated for United States' Economy relative to the other countries interacting yields the following:

$$\begin{aligned} dy_2(t)/dt - a_{22} \times y_2'(t-h) = & a_{02} \times y_2(t) + a_{12} \times y(t-h) + y_2(t) \\ & \times (a_{21} \times y_1(t-h) + a_{23} \times y_3(t-h) + a_{24} \times y_4(t-h) + p_2(t) + g_2(t)) \end{aligned}$$

$$\begin{aligned} p_2 = & 11 \times (\text{CC}(1) + \text{II}(1) + \text{XX}(1)) / (1 - 11 \times \text{Z4}) = 632.12 \\ g_2(t) = & (1,387,063,249.13 - 397,726.95 \times e(t) + 0 \times \text{ta}(t) \\ & + 0 \times d(t) - 179.16 \times T(t) - 73,506.69 \times T(t-h) \\ & + 120,485.89T'(t) - 1.7624916 \times T'(t-h) / (4,851,492.752) \end{aligned}$$

$$\begin{aligned} dy_2(t)/dt + 1.250 \times y_2'(t-h) = & -9.277 \times 10^{-5} \times y_2(t) + 0.01661 \times y_2(t-h) \\ & + y_2(t) \times (-0.002687 \times y_1(t-h) + 4.9979 \times 10^{-4} \times y_3(t-h) \\ & - 7.3097 \times 10^{-4} \times y_4(t-h)) + p_2(t) + g_2(t). \end{aligned}$$

The following equations were Chukwu's model of the gross domestic product of US interacting with China, Nigeria, UK [8]. The impact of China on the rate of growth of US GDP is negative. From the equation one can remedy this by imposing tariffs (with its consequences and other countries' reaction), or by reducing exchange rate. Senator Elizabeth Dole said in Winston-Salem that she plans to push a bill to slap a 27.5 % tariff on Chinese imports if China persists in what she calls unfair trade practices. It is easy to see how tempting this can be. Mathematically it can increase the value of $dy_2(t)/dt$, the growth rate of GDP. Tax reduction in the past and the present can increase the growth rate of GDP.

The centerpiece of President Bush's Asia-Pacific trip (Friday October 17, 2003, News and Observer: Nation) is a two-day summit of Asia-Pacific Economic Cooperation organization in Bangkok. Bush said that he will urge the leaders of Japan and China to stop manipulating currency markets to keep the value of their currencies low in relation to the dollar. This makes American-made goods expensive abroad. American manufacturers say the strong dollar has badly hurt their foreign sales and brought cuts in jobs at home. The U.S. economy has lost nearly 3 million jobs since President Bush took office. President Bush has chosen the way of exchange rate as a way favorable to the USA—the centerpiece of his Asia-Pacific trip. This can also help. It may be easier to use it to increase the growth rate of GDP. But the most effective way to reduce or make positive the draining of $dy_2(t)/dt$ by $-6y_2(t)y_4(t-h)$ is to add $+6.1y_2(t)y_4(t-h)$ to it. This is the way of cooperation. The US economic state is controllable, so is that of China. The interacting countries studied, Nigeria, US, UK, China are controllable. A high GDP can be attained by a judicious choice of all admissible controls, 3–9 of the representative firm, and 4–8 of the government.

From the additional growth we can invest effective $7y_2(t)y_4(t-h)$ in “cooperative” ventures, making $dy_2(t)/dt$ positive, and $y_2(t)$ increasing. Overall since controllability has been proved and cooperation is possible, this is one good way to go. Selecting only one policy can hurt. It requires all the control strategies.

6.5.1 Remark on Employment

The strategies

$$\begin{aligned}\sigma_{31}(t) &= m(w)_1 \sigma_{41}(t) \\ \sigma_{41}(t) &= x_0 + y_{10} + I_0; \quad m(w)_1 = \left[(1 - \alpha) \frac{1}{w} \right]^{1/\alpha} \\ q_{41}(t) &= q_{s0} + z_{s13} M1 - z_{s14} T(t) + z_{s15} e(t) + z_{s16} \tau(t) + z_{s17} d(t),\end{aligned}$$

can be used to steer to a very low level of unemployment L_1 . This can be done in a similar way for L_2 using $\sigma_{32}(t)$ and $q_{42}(t)$. Because of the controllability of the state $[y_1, L_1, y_2, L_2]$ by means of the controls displayed $d_{11}, d_{12}, d_{22}, d_{23}$ can be made positive and thus render $L_1(t), L_2(t)$ positive and big, and make employment growing in interacting countries. Well paid Chinese workers can consume American goods and services, making jobs to grow in America, and vice versa. The following is a newspaper report after this research was completed.

6.5.1.1 Confirmation

“Software and technology companies that hire workers in Low Cost Countries such as India will add \$3.3 billion to North Carolina’s economy and create 9,699 new jobs by 2008, a new industry—funded study says,” according to News and Observer Business Wednesday, March 31, 2004, p-1 D-60. The cost savings from paying workers in those countries less than those in the United States already have led to 2,555 new jobs in the state since 2000. The study relies on economic models and information from technology companies to measure the effect of off-shore hiring in all 50 states. It predicts that by 2008, companies that send at least some of their technology functions off shore will add \$124 billion to the US economy and create a net gain of 317, 367 jobs.”

This study confirms the main contribution of this chapter which was completed earlier.

The confirmation is continued in June 04.

Tuesday, June 8, 2004

The News and Observer Business, p 1D–8D

GM to double its capacity in china

Chinese Vehicle sales jumped 75 % in 2003

Signaling its confidence in booming Chinese economy, General Motors said Monday it plans to spend \$3 billion in China in the next three years in a challenge to rival Volkswagen for dominance of the Worlds fastest growing auto market. Success in China is crucial to GM’s global success”, Phil Murtaugh, Chairman and Chief Executive of General Motors China Group said GM is the worlds biggest auto maker.

General Motors Ford Volkswagen and Toyota have announced plans in the past eight months to invest about \$10 billion in China.

Stability Remarks [9, Theorem 4.3.2] Suppose

$$\begin{aligned}
 y_1 &= x_1 \\
 L_1 &= x_2 \\
 y_2 &= x_3 \\
 L_2 &= x_4 \\
 \frac{dx_1(t)}{dt} &= x_1(t)[b_1 - a_{11}x_1(t - \tau_{11}) + a_{12}x_2(t - \tau_{12}) \\
 &\quad - a_{13}x_3(t - \tau_{13}) - a_{14}x_4(t - \tau_{14})], \\
 \frac{dx_2(t)}{dt} &= x_2(t)[b_2 - a_{21}x_1(t - \tau_{21}) + a_{22}x_2(t - \tau_{22}) \\
 &\quad - a_{23}x_3(t - \tau_{23}) - a_{24}x_4(t - \tau_{24})], \\
 \frac{dx_3(t)}{dt} &= x_3(t)[b_3 - a_{31}x_1(t - \tau_{31}) + a_{32}x_2(t - \tau_{32}) \\
 &\quad - a_{33}x_3(t - \tau_{33}) - a_{34}x_4(t - \tau_{34})], \\
 \frac{dx_4(t)}{dt} &= x_4(t)[b_4 - a_{41}x_1(t - \tau_{41}) + a_{42}x_2(t - \tau_{42}) \\
 &\quad - a_{43}x_3(t - \tau_{43}) - a_{44}x_4(t - \tau_{44})], \\
 x_o(s) &= \rho_i(s) \geq 0 \quad s \in \{-T, 0\}; \\
 \tau &= \max_{1 \leq i, j \leq n} \tau_{ij} \\
 \rho_i(0) &> 0 \quad \rho_i \text{ continuous on } [-\tau, 0]; \\
 b_i, a_{ij}, \tau_{ij} &(i, j = 1, 2, \dots, n)
 \end{aligned}$$

are nonnegative constants and $\tau_{ii} > 0$ for one or more $i = (1, 2, 3, \dots, n)$.

Suppose that $b > 0$, $a_{ii} > 0$ ($i = 1, 2, \dots, n$) and $a_{ij} \geq 0$, $i, j = 1, 2, \dots, n$ if $i \neq j$.

Furthermore, Let

$$b_i > \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij}(b_j/a_{jj}) \quad i = 1, 2, \dots, n.$$

Then there exists a component wise positive steady state $x^* = (x_1^*, \dots, x_n^*)$, satisfying the equations

$$\sum_{j=1}^n a_{ij}x_j^* = \lambda_i \text{ and } x_i^* > 0; \quad i = 1, 2, \dots, n.$$

H_1 Assume the real constants $\tau_{ij} \geq 0$ ($i, j = 1, 2, \dots, 4$) satisfy

$$\tau_{ii} \leq \min_{\substack{j \leq 4 \\ j \neq i}} \tau_{ji}; i = 1, 2, \dots, 4;$$

$$\text{if } \sum_{j=1}^n \tau_{ji} \neq 0$$

H_2 a_{ij} ($i, j = 1, 2, \dots, 4$) are nonnegative constants such that

$$a_{ii}^{+b} > 0; \quad a_{ii}x_i^* \tau_{ii} < 1; \quad i = 1, 2, \dots, 4;$$

$$\left(\tau_{ii}x_i^* \sum_{\substack{j=1 \\ j \neq i}}^4 a_{ji} \right) (1 - a_{ii}\tau_{ii}x_i^*)^{-1} < \pi/2 \quad i = 1, 2, \dots, n;$$

$$a_{ii} \cos \left(\frac{\{ \tau_{ii}x_i^* \sum_{\substack{j=1 \\ j \neq i}}^4 a_{ji} \}}{\{ 1 - a_{ii}\tau_{ii}x_i^* \}} \right) > \sum_{\substack{j=1 \\ j \neq i}}^4 a_{ji} \quad u = 1, 2, \dots, 4.$$

The positive steady state $x^* = (x_1^* \dots x_4^*)$ of () exists then x^* is locally asymptotically stable.

Remarks 1 The conditions imply that competition should be regulated and the delays small, to ensure that $x^* = [y_1^*, L_1^*, y_2^*, L_2^*]$ is locally asymptotically stable.

Remarks 2 The relative strength of P—the private control constraint strategy set and the government control constraint strength Q was demonstrated to be

$$Q \subset P,$$

where Q is not empty. See [3; p. 372]. The philosophy of privatization, drastic reduction of Q was furiously advocated by IMF and the World Bank. Very recently the following statement was made by the President of the World Bank. “There is a view that we don’t need this official development assistance, because it is the private sector at the end of the day that produces the jobs. But the private sector needs roads, the private sector needs an educated population and private sector needs agricultural extension services.” Paul Wolfowitz: A Conversation with Paul Wolfowitz—v Date 05/30/2007 Charlie Rose.

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Chapter 7

Overview of Each Nations Economic State Dynamics and Control: Equations of GDP, Interest Rate, Employment, Value of Capital Stock, Prices, Cumulative Balance of Payment

7.1 Member Nations' Economic State and Strategies: Existing Results

The economic state is $x = [y, R, L, k, p, E]$. We now recall our earlier equations for interest rate [2, p. 29]—Stability and Time Optimal Control of Hereditary Systems with Applications to Economic Dynamics of the USA. Let M denote money supply (e.g., $M1$)

$$\bar{L} = \text{money demand} \approx Md$$

Quasimoney, set

$$ML = Md - M$$

Assume

$$\bar{L} = M_0 + M_1 y(t-h) + M_3 R(t) + M_4 R(t-h) + M_5 \dot{R}(t-h) + M_6 p(t).$$

The rate of interest is determined by the typically Kenesian dynamics

$$\frac{dR}{dt} = \lambda_2(\bar{L} - M),$$

which yields

$$\begin{aligned} \dot{R}(t) - a_{-22}\dot{R}(t-h) &= a_{21}y(t) + a_{22}y(t-h) + a_{33}\dot{R}(t) \\ &\quad + a_{24}R(t-h) + a_{25}p(t) - \sigma_2(t) + q_2(t) \end{aligned} \quad (7.1)$$

where

$$\begin{aligned}
 q_2(t) &= -\lambda_2 M_1, & \sigma_2(t) &= \lambda_2 M_0 \\
 a_{-22} &= \lambda_2 M_5, & a_{21} &= \lambda_2 M_1 \\
 a_{22} &= \lambda_2 M_2, & a_{23} &= \lambda_2 M_3 \\
 a_{25} &= \lambda_2 M_6, & a_{24} &= \lambda_2 M_4
 \end{aligned} \tag{7.2}$$

Employment dynamics is derived as

$$\begin{aligned}
 \dot{L}(t) &- \ell_{-01}\dot{L}(t-h) - \ell_{-03}\dot{y}(t-h) \\
 &= \ell_0 L(t) - \ell_1 L(t-h) + \ell_2 y(t-h) \\
 &\quad + \ell_4 R(t-h) + \ell_5 L(t-h) + \sigma_3(t) + q_3(t),
 \end{aligned} \tag{7.3}$$

where

$$\begin{aligned}
 \ell_{-01} &= a_{-1} - m(w)a_6, \quad a_3 m(w) = \ell_{-03} \\
 \ell_0 &= a_0 \\
 \ell_1 &= -a_1 \\
 \ell_2 &= m(w)a_2 \\
 \ell_4 &= m(w)a_4 \\
 \ell_5 &= m(w) \\
 \ell_8 &= m(w)a_8 \\
 -\sigma_3(t) &= m(w)\sigma_4(t) \\
 q_3(t) &= m(w)q_4(t) \\
 \sigma_4(t) &= x_0 + y_{10} + I_0 \\
 q_4(t) &= g_0 + z_{s13}M - z_{s14}T(t) + z_{s15}e(t) + z_{s16}\tau(t) + z_{17}d(t).
 \end{aligned} \tag{7.4}$$

Following the treatment of Gandolfo and Padoan in “The Italian Continuous Time Model, Theory and Empirical Model” in Butterworth and Co. Publishers Ltd., pp. 91–132, 1990, the dynamic results of domestic prices is given in Chukwu “Optimal Control of the Growth of wealth of Nations”, Taylor and Francis, 2003.

$$\begin{aligned}
 \dot{p}(t) &= p(t)[(M_6 + p_4)p(t) - p_6 M_1 y(t) - p_6 y(t-h) - p_6 M_3 R(t) \\
 &\quad M_4 p_6 R(t-h) - p_6 M_7 \dot{R}(t-h) + [q_5(t) + \sigma_5(t)]p(t)
 \end{aligned} \tag{7.5}$$

where

$$\begin{aligned}
 q_5(t) &= p_1 p^f(t) \cdot e(t) + p_5 \dot{M}_1(t) + p_6 M_1(t), \\
 \sigma_5(t) &= p_0 - p_3 n(t) + p_2 w(t) - p_6 M_0(t)
 \end{aligned}$$

and M_1, M_3, M_4, M_6 , are identified in (3.1.4) and $p_1, p_2, p_3, p_4, p_5, p_6$ are identified in (6.1.30) of “Optimal Control of the Growth of Wealth of Nations”, [2]. The dynamics of capital stock is given in [1] (1.10.58) and [2] (6.1.57)

$$\begin{aligned} \frac{dk(t)}{dt} + a_{-31}\dot{k}(t-h) - a_{-33}\dot{y}(t-h) - a_{-36}\dot{L}(t-h) \\ = 30k(t) - a_{31}k(t-h) + a_{32}y(t-h) \\ + a_{34}R(t-h) + a_{35}L(t-h) + a_{38}p(t) + \sigma_4(t) + q_4(t) \end{aligned} \quad (7.6)$$

where

$$\begin{aligned} q_4 &= g_0 + z_{s13}M - z_{s14}T(t) + z_{s15}e(t) + z_{s16}\tau(t) + z_{s17}d(t) \\ \sigma_4 &= x_0 + y_{10} + I_0 \end{aligned}$$

Employment dynamics was derived in [2] Eq. (6.1.67) as

$$\begin{aligned} \dot{L}(t) + a_{-41}\dot{L}(t-h) - a_{-42}\dot{y}(t-h) \\ = a_{40}L(t) - a_{41}L(t-h) + a_{42}y(t-h) \\ + a_{43}R(t-h) + a_{45}p(t) - \sigma_3(t) + q_3(t) \end{aligned} \quad (7.7)$$

Finally the dynamics of cumulative balance of payment E is given in (6.1.44) of [2]:

$$\begin{aligned} \dot{E}(t) - a_{-61}\dot{y}(t-h) - a_{16}\dot{R}(t-h) \\ = a_{60}y(t)a_{61}y(t-h) + a_{62}L(t-h) + a_{63}R(t) \\ + a_{66}B(t-h) + a_{64}R(t-h) + a_{65}p(t) - \sigma_6(t) + q_6(t) \end{aligned} \quad (7.8)$$

where

$$\begin{aligned} q_6(t) &= b_7e(t) + b_8\tau(t) + b_{15}d(t) - f_0 \\ \sigma_6(t) &= -X_0 \end{aligned} \quad (7.9)$$

Using the MatLab regression methods, the coefficients of the equations for R, L, k, p, and E are identified. We return to a new dynamics of $y(t)$, the GDP which is now deduced from the formulae for C, I, X, G, and

$$z = C + I + X + G,$$

as follows:

$$\begin{aligned}
 C = & C_0 + C_1(y(t) - T(t)) + C_2(y(t-h) - T(t-h) + C_3(\dot{y}(t) - \dot{T}(t)) \\
 & + C_4(\dot{y}(t-h) - \dot{T}(t-h)) + C_5R(t) + C_6R(T-h) + C_7(M_0 + M_1y(t) \\
 & + M_2y(t-h) + M_3R(t) + M_4R(t-h) + M_5\dot{R}(t-h) + M_6p(t) - M)
 \end{aligned} \tag{7.10}$$

$$\begin{aligned}
 I = & I_0 + I_1y(t) + I_2y(t-h) - I_3\dot{y}(t) + I_4\dot{y}(t-h) + I_5R(t) + I_6R(t-h) \\
 & + I_8L(t) + I_9L(t-h) - I_{11}k(t) - I_{13}(M - M_0 - M_1y(t-h) - M_3R(t) \\
 & - M_4R(t-h) - M_5\dot{R}(t-h) - M_6p(t)
 \end{aligned} \tag{7.11}$$

$$\begin{aligned}
 X = & X_0 + X_1y(t) + X_2y(t-h) + X_3y(t) + X_4\dot{y}(t-h) + X_5R(t) + X_8L(t) \\
 & + X_{10}\dot{L}(t-h) + X_{12}p(t) + X_{16}\tau(t) + X_{15}e(t) + X_{17}d(t)
 \end{aligned} \tag{7.12}$$

$$G = g_0 + g_1y(t) + g_2y(t-h) + g_3\dot{y}(t) + g_4\dot{y}(t-h) + g_5R(t) + g_6L(t) \tag{7.13}$$

$$\begin{aligned}
 Z = & C + I + X + G \\
 = & C_0 + I_0 + X_0 + g_0 + M_0(C_7 + I_{13}) - M(C_7 + I_{13}) + (C_1 + I_1 + X_1 + g_1 + M_1(C_7 + I_{13}))y(t) \\
 & - (C_1T(t) + C_2T(t-h) + C_3\dot{T}(t) + C_4\dot{T}(t-h)) + (C_2 + I_2 + (C_7 + I_{13})M_2 + X_2 + g_2)y(t-h) \\
 & + (C_3 - I_3 + X_3 + g_3)\dot{y}(t) + (C_4 + I_4 + X_4 + g_4)\dot{y}(t-h) + (C_5 + (C_7 + I_{13})M_3 + I_5 + g_5)R(t) \\
 & + (C_6 + (C_7 + I_{13})M_4 + I_6)R(t-h) + (M_5(C_7 + I_{13}))\dot{R}(t-h) + (I_8 + X_8 + g_6)L(t) \\
 & + I_9L(t-h) + X_{10}\dot{L}(t-h) - I_{11}k(t) + (C_7M_6 + I_{13}M_6 + X_{12})p(t)
 \end{aligned} \tag{7.14}$$

$$\begin{aligned}
 & X_{15}e(t) + X_{16}\tau(t) + X_{17}d(t) \\
 = & C_0 + I_0 + M_0(C_7 + I_{13}) + (g_0 - M(C_7 + I_{13})) \\
 & + X_{15}e(t) + X_{16}\tau(t) + X_{17}d(t) + [C_1 + I_1 + X_1 + g_1 + M_1(C_7 + I_{13})]y(t) \\
 & + (C_2 + I_2 + (C_7 + I_{13})M_2 + X_2 + g_2)y(t-h) + (C_3 - I_3 + X_3 + g_3)\dot{y}(t) \\
 & + (C_4 + I_4 + X_4 + g_4)\dot{y}(t-h) + [C_5 + (C_7 + I_{13})M_3 + I_5 + g_5]R(t)
 \end{aligned} \tag{7.15}$$

Let

$$\begin{aligned}
 z_0 &= C_0 + I_0 + X_0 + g_0 + M_0(C_7 + I_{13}) - M(C_7 + I_{13}) \\
 r_0 &= C_0 + I_0 + X_0 + M_0(C_7 + I_{13}) \\
 q_0(t) &= g_0 - M(C_7 + I_{13}) \\
 z_0 &= r_0 + g_0 \\
 z_1 &= C_1 + I_1 + X_1 + g_1 + M_1(C_1 + I_{13}) \\
 T_1 &= C_1T(t) + C_2T(t-h) + C_3\dot{T}(t) + C_4\dot{T}(t-h) \\
 z_2 &= C_2 + I_2 + (C_7 + I_{13})M_2 + X_2 + g_2 \\
 z_3 &= C_3 - I_3 + X_3 + g_3 \\
 z_4 &= C_4 + I_4 + X_4 + g_4 \\
 z_5 &= C_5 + (C_7 + I_{13})M_3 + I_5 + g_5 \\
 z_6 &= (C_6 + (C_7 + I_{13})M_4 + I_6) \\
 z_7 &= (C_7 + I_{13})M_5 \\
 z_8 &= (I_8 + X_8 + g_6) \\
 z_9 &= I_9 \\
 z_{10} &= X_{10} \\
 z_{11} &= -I_{11} \\
 z_{12} &= (C_7 + I_{13})M_6 + X_{12}.
 \end{aligned} \tag{7.16}$$

Hence

$$\dot{y}(t) = \lambda_1(z(t) - y(t)). \tag{7.17}$$

It follows that

$$\begin{aligned}
 \dot{y}(t) &= \lambda_1(z(t) - y(t)) = \lambda_1r_0 + \lambda_1q_0 - \lambda_1T_1 - \lambda_1M(C_7 + I_{13}) + z_1\lambda_1y(t) \\
 &\quad + z_2\lambda_1y(t-h) + \lambda_1z_3\dot{y}(t) + \lambda_1z_4\dot{y}(t-h) + \lambda_1z_5R(t) + \lambda_1z_6R(t-h) \\
 &\quad + \lambda_1z_7(\dot{R}(T-h)) + \lambda_1z_8L(t) + \lambda_1z_9L(t-h) + \lambda_1z_{10}\dot{L}(t-h) \\
 &\quad + \lambda_1z_{11}k(t) + z_{12}\lambda_1p(t) + \lambda_1X_{15}e(t) + X_{16}\lambda_1\tau(t) + X_{17}\lambda_1d(t) - \lambda_1y(t).
 \end{aligned}$$

Thus

$$\begin{aligned}
\dot{y}(t) & - \frac{\lambda_1 z_4}{1 - \lambda_1 z_3} \dot{y}(t-h) - \frac{\lambda_1 z_{10}}{1 - \lambda_1 z_3} \dot{L}(t-h) - \frac{\lambda_1 z_7}{1 - \lambda_1 z_3} \dot{R}(t-h) \\
& = \lambda_1 (z_1 - 1)/(1 - \lambda_1 z_3) y(t) + \lambda_1 z_5/(1 - \lambda_1 z_3) R(t) \\
& \quad + \lambda_1 z_8 L(t)/(1 - \lambda_1 z_3) + \lambda_1 z_{11}/(1 - \lambda_1 z_3) k(t) + \lambda_1 z_{12}/(1 - \lambda_1 z_3) p(t) \\
& \quad + \frac{\lambda_1 z_2 y(t-h)}{(1 - \lambda_1 z_3)} + \lambda_1 z_6 \left/ \frac{R(t-h)}{(1 - \lambda_1 z_3)} \right. + (\lambda_1 z_9) \left/ \frac{L(t-h)}{(1 - \lambda_1 z_3)} \right. \\
& \quad + \frac{\lambda_1 X_{15} e(t)}{(1 - \lambda_1 z_3)} + \frac{\lambda_1 X_{16} \tau(t)}{(1 - \lambda_1 z_3)} + \frac{\lambda_1 X_{17}}{(1 - \lambda_1 z_3)} d(t) + \frac{\lambda_1 r_0}{(1 - \lambda_1 z_3)} \\
& \quad + \frac{\lambda_1 (q_0 - T_1)}{(1 - \lambda_1 z_3)} - \frac{M(C_7 + I_{13})}{(1 - \lambda_1 z_3)}
\end{aligned} \tag{7.18}$$

Writing compactly, we have

$$\begin{aligned}
\dot{y}(t) & - a_{-11} \dot{y}(t-h) - a_{-12} \dot{R}(t-h) - a_{-13} \dot{L}(t-h) \\
& = a_{01} y(t) + a_{02} R(t) - a_{03} L(t) + a_{04} k(t) + a_{05} p(t) + a_{11} y(t-h) \\
& \quad + a_{12} R(t-h) + a_{13} L(t-h) + a_{14} k(t-h) + a_{15} e(t) \\
& \quad + a_{16} \tau(t) + a_{17} d(t) + a_{18} r_0 + a_{19} [(q_0 - T_1)] - a_{20} M
\end{aligned} \tag{7.19}$$

where

$$\begin{aligned}
a_{-11} & = \lambda_1 z_4/(1 - \lambda_1 z_3), & a_{-12} & = \lambda_1 z_7/(1 - \lambda_1 z_3), & a_{-13} & = \lambda_1 z_{10}/(1 - \lambda_1 z_3) \\
a_{01} & = \lambda_1 (z_1 - 1)/(1 - \lambda_1 z_3), & a_{02} & = \lambda_1 z_5/(1 - \lambda_1 z_3), & a_{03} & = \lambda_1 z_8/(1 - \lambda_1 z_3) \\
a_{04} & = \lambda_1 z_{11}/(1 - \lambda_1 z_3), & a_{05} & = \lambda_1 z_{12}/(1 - \lambda_1 z_3), & a_{11} & = \lambda_1 z_2/(1 - \lambda_1 z_3) \\
a_{12} & = \lambda_1 z_6/(1 - \lambda_1 z_3), & a_{13} & = \lambda_1 z_9/(1 - \lambda_1 z_3), & a_{14} & = 0, \\
a_{15} & = \lambda_1 X_{15}/(1 - \lambda_1 z_3), & a_{16} & = \lambda_1 X_{16}/(1 - \lambda_1 z_3), & a_{17} & = \lambda_1 X_{17}/(1 - \lambda_1 z_3) \\
a_{18} & = \lambda_1/(1 - \lambda_1 z_3), & a_{19} & = \lambda_1/(1 - \lambda_1 z_3), & a_{20} & = \lambda_1/(1 - \lambda_1 z_3)(C_7 + I_{13})
\end{aligned} \tag{7.20}$$

To put this in matrix form, let

$$x = \begin{bmatrix} y \\ R \\ L \\ k \\ p \\ E \end{bmatrix}$$

y GDP
R Interest rate

- L Employment/or unemployment
- k Value of capital stock
- p Prices
- E Cumulative balance of payment.

Let

$$q_1(t) = a_{19}(g_0 - (C_7 + I_{13})M) + a_{15}e(t) + a_{16}\tau(t) + a_{17}d(t) - a_{19}T_1, \quad (7.21)$$

$$\begin{aligned} r_1(t) &= a_{18}r_0 \\ &= [a_{18}, \quad a_{18}, \quad a_{18}, \quad a_{18}, \quad a_{18}(C_7 + I_{13})] \begin{bmatrix} C_0 \\ I_0 \\ X_0 \\ M_0 \end{bmatrix} \end{aligned} \quad (7.22)$$

$$\begin{aligned} q_1(t) &= \begin{bmatrix} a_{19}, & -a_{19}(I_{13} + C_7), & a_{15}, & a_{16}, & a_{17}, & -a_{19} \end{bmatrix} \begin{bmatrix} g_0 \\ M \\ e \\ \tau(t) \\ d(t) \\ -T_1(t) \end{bmatrix} \\ r_1(t) &= \begin{bmatrix} a_{18}, & a_{18}, & a_{18}, & a_{18}, & a_{18}(C_7 + I_{13}) \end{bmatrix} \begin{bmatrix} C_0 \\ I_0 \\ X_0 \\ M_0 \end{bmatrix} \end{aligned} \quad (7.23)$$

The following are generated equations from country2.m programs for several countries:

The United States of America USA2.m generates

$$\begin{aligned} \dot{x}(t) - A_{-1}\dot{x}(t-h) &= A_0x(t) + A_1x(t-h) + B_1q + B_2p \\ x &= [y, R, L, k, p, E], \quad p \in P, \quad q \in Q. \end{aligned}$$

This differential game is converted into a control system

$$\dot{x}(t) - A_{-1}\dot{x}(t-h) = A_0x(t) + A_1x(t-h) + Bu \quad (7.24)$$

$u \in U$ —a pontryagin difference of sets P and Q , P representative set firms control set and Q government control. It is controllable in the following sense:

Definition 7.1 The system (7.24) is controllable (respectively, Euclidean controllable) on the interval $[\sigma, t_1]$, if for each $\phi, \psi \in W_2^{(1)}([-h, 0], E^n)$ (respectively $\phi \in W_2^{(1)}([-h, 0], E^n)$, $x_1 \in E^n$) there is a controller $u \in L([\sigma, t_1], E^n)$, such that

$x_\sigma(\cdot, \sigma, \phi, u) = \phi$ and $x_{t_1}(t, \sigma, \phi, u) = \psi$ (respectively, $x(t_1, \sigma, \phi, u) = x_1$). The system (7.24) is null controllable (respectively, Euclidean null controllable on $[\sigma, t_1]$, if in the above definitions $\psi \equiv 0$ (respectively, Euclidean null controllable on $[\sigma, t_1]$) $x_1 = 0$). If the above kinds of controllability hold on every interval $[\sigma, t_1]$, with $t_1 > \sigma + h$, we drop the qualifying phrase “on the interval $[0, t_1]$ ”. In Euclidean controllability we drop “on the interval $[\sigma, t_1]$ ” if it is Euclidean controllable on every interval $[\sigma, t_1]$, $t_1 > \sigma$. The main result proved for the USA economy that any function economic target $x_T = [y_T, R_T, L_T, k_T, P_T, E_T]$, can be reached from any economic position if and only if the number of effective control instruments is equal to the number S of target functions. Thus there exists a set of policy instruments which are capable of steering the initial economic state to a desired position in finite time. Thus levels of gross domestic product, interest rate, employment prices, value of capital stock and cumulative balance of payment can be controlled simultaneously, provided all the control instruments are in force. The control instruments of the economic state are of two kinds: g —the control instrument of government (taxes, money supply, public consumption, exchange rate, subsidy, preferential trade arrangement, tariff) and p —the representative private firms’ initiative (autonomous consumption, investment, net export, money holding, wages, productivity).

Realistically the controls have hard constraints,

$$u \in U = P_-^* Q = \{u: u + Q \subset P\}.$$

where $_-$ denotes the so-called Pontryagin difference of P and Q . For more details of definition in the language of differential games see [1, p. 289].

The data generated in this US2.m prove that the USA economy is controllable. The component I , of $x = [y, R, L, k, p, E]'$, has dynamics

$$\begin{aligned} \dot{y}(t) &- 4.3105\dot{y}(t-h) - 814.855\dot{L}(t-h) \\ &= -31.4655y(t) - 276.787R(t) + 226.4519L(t) + 5.5489k(t) \\ &\quad + 271.1380p(t) + 12.2672y(t-h) + 185.5172R(t-h) - 158.1694L(t-h) \\ &\quad - 17.9606T_1 + 17.9606g_0 140.6221e + 271.1380\tau + 0d - I_{13}M + r_1(t), \end{aligned}$$

r_1 defined in (7.23).

Thus if net aggregate current and hereditary taxes and their flows T_1 are lowered and government autonomous investment (e.g., roads and infrastructure, schools, hospitals etc.) are increased, if exchange rate, tariffs, are positively increased in this period, and trade policies d are effectively positive for the country, the growth rate of GDP may be enhanced. These three government control components e , exchange rate, tariffs τ , and trade policy/or distance and transportation between trading partners, d , are issues of trade which are better negotiated in “authentic solidarity” and cooperation between nations. This is the

only way now; otherwise nationalistic isolationism will tear the world apart. As noted by academic experts in Davos-Klosters, Switzerland, 31 January 2009—the Doha Round of multilateral trade talks is a most valuable step global leaders can take to keep this from happening, because the mathematics proves that competition should be kept in check and cooperation promoted. Negotiations are currently being discussed in The World Economic Forum, Annual Meeting 2009, 28 January–1 February 2009. Trade officials urge G20 to accelerate Doha Round negotiations to combat economic crisis from triggering a destructive protectionist backlash as well as financial protectionism. Contraction in World Trade can hurt regional and international economic growth because net export will decrease. The Japanese PM in the World forum's Annual meeting 2009 announced to the world that \$17bn is being given for development aid. UK's Brown warns financial protectionism and urges cooperation—"cooperation between major powers and global financial institutions is vital to ensure a flow of credit (and therefore capital) to developing and smaller countries, which are likely to be the biggest victims of world secession." 2009 www.weforum.org/en/events/Annualmeeting. This forces us to focus on the dynamics of all member states of the UN.

The other component of control strategy for growth—increase in the private representative firms contribution (autonomous consumption, investment net export C_0, I_0, X_0 , productivity, n and autonomous money holding M_0) can, if in aggregate it dominates government intervention ($G \subseteq P$) incite greater growth rate of the gross domestic product, ushering in the bliss and flow of goods and services. Mankind will then make bold to feed the hungry, to clothe the naked, to house the homeless, to heal the sick, to care for prisoners, etc.

We now explore in detail the cooperation and competition function

$$fi(t) = \sum_{\substack{j=1 \\ i \neq j}}^{191} a_{ij} y_i(t) y_j(t-h)$$

between nation i and nation j , $i, j = 1, \dots, 191$ and its contribution to economic growth of nations i , $i = 1, \dots, 191$.

If $fi(t)$ is measured in the money value of nation i and $y_i(t)$ is in this money value we can make y_j uniform in nation i money value by using monetary exchange rates from a fixed date for the appropriate time. For example, consider Albania which has

$$aa = \text{Leks/SDR},$$

or

$$ae = \text{leks/Usdollar}.$$

Algeria's Dinars per SDR = aa , and

$$ae = \text{Dinars/dollar},$$

so that

$$\text{dollar} = \text{Dinars}/ae.$$

Assuming the developments in [Chaps. 3](#) and [7](#) the dynamics of the gross domestic product equation of y_i for nation i with interaction with other nations j is derived as

$$\begin{aligned} \frac{dy_i(t)}{dt} - a_{-1i} \frac{dy_i(t-h)}{dt} = & a_{0i}y_i(t) + a_{1i}y_i(t-h) \\ & + y_i \sum_{\substack{j=1 \\ j \neq i}}^{191} b_{ij}y_j(t-h) + p_i(t) + g_i(t) \end{aligned} \quad (3.12)$$

where the coefficients are given in [\(3.11\)](#), or

$$\begin{aligned} & y_i(t) - a_{11i}\dot{y}_i(t-h) - a_{-12i}\dot{R}_i(t-h) - a_{13i}\dot{L}_i(t-h) \\ & = a_{01i}y_i(t) + a_{02i}R_i(t) + a_{03i}L_i(t) + a_{04i}k(t) + a_{05i}p_i(t) \\ & + a_{11i}y_i(t-h) + a_{12i}R_i(t-h) + a_{13i}L_i(t-h) + a_{14i}k(t-h) + a_{15i}e_i(t) \\ & + a_{16i}\tau_i(t) + a_{17i}d_i(t) + a_{18i}r_{0i} + a_{19i}[(q_{0i} - T_{1i})] \\ & - a_{20i}M_i + y_i(t) \sum_{\substack{j=1 \\ j \neq i}}^{191} b_{ij}y_j(t-h) \end{aligned} \quad (7.25)$$

where the coefficients are identified in [\(7.25\)](#) and

$$\begin{aligned} r_{0i} &= C_{0i} + I_{0i} + X_{0i} + M_{0i}(C_{7i} + I_{13i}), \\ r_i &= a_{18i}r_{0i}. \end{aligned} \quad (7.26)$$

Let

$$\begin{aligned} q_{1i}(t) &= a_{19i}(g_{0i} - (C_{7i} + I_{13i})M_i + a_{15i}e_i(t) + a_{16i}\tau_i(t) \\ &+ a_{17i}d(t) - a_{19i}T_{1i}, \end{aligned} \quad (7.27)$$

$$T_{1i} = C_{1i}T_i(t) + C_{2i}T_i(t-h) + C_{3i}\dot{T}_i(t) + C_{4i}\dot{T}_i(t-h). \quad (7.28)$$

7.2 Economic Dynamics and Control

In this chapter we extract the following already derived equations of the economic state.

$$x = [y; R, L, k, p, E].$$

where y is the gross domestic product, R is interest rate, L is employment, k is the value of capital stock, p denotes prices ($\dot{p}(t)$, inflation) and E is the cumulative balance of payment. The derivation is made in Chukwu [1]. Chapter 1.10 where we assumed aggregate demand is z , the sum of investment I , consumption C , net export X , and government outlay G which are all explicitly defined:

$$z = I + C + X + G.$$

$$\begin{aligned} \frac{dy(t)}{dt} - a_{-11}\dot{y}(t-h) - a_{-13}\dot{L}(t-h) \\ = a_{01}y(t) + a_{11}y(t-h) + a_{12}R(t) \\ + a_{13}R(t-h) + a_{14}L(t) + a_{15}L(t-h) \\ + a_{16}k(t) - a_{18}p(t) + q_1(t) + r_1(t) \end{aligned} \quad (7.29)$$

where

$$\begin{aligned} q_1(t) = \lambda_1 \sigma_1^{-1} (g_0 - z_{14}T(t) - z_{19}T(t-h) - z_{20}\dot{T}(t) \\ - z_{21}\dot{T}(t-h) - z_{15}e(t) + z_{16}\tau(t) + z_{17}d(t); \end{aligned} \quad (7.30)$$

$$r_1(t) = \lambda_1 \sigma_1^{-1} [(C_0 + I_0 + X_0) - M_0(I_{13} + C_7)]. \quad (7.31)$$

The coefficients are displayed in (3.1.12, 3.1.15) of Optimal Control of the Growth of Wealth of Nations: by Chukwu in [2, pp. 261–263]

$$\begin{aligned} \frac{dR(t)}{dt} - a_{22}\dot{R}(t-h) = a_{21}y(t) + a_{22}y(t-h) + a_{22}y(t-h) + a_{23}R(t) \\ + a_{24}R(t-h) + a_{25}p(t) - \sigma_2(t) + q_2(t), \end{aligned} \quad (7.32)$$

where

$$\begin{aligned} q_2(t) &= \lambda_2 M_1 \\ -\sigma_2(t) &= \lambda_2 M_0 \end{aligned} \quad (7.32)$$

and the coefficients are

$$\begin{aligned} a_{-22} &= \lambda_2 M_5, & a_{21} &= \lambda_2 M_1, & a_{22} &= \lambda_2 M_2, \\ a_{24} &= \lambda_2 M_3, & a_{25} &= \lambda_2 M_6 \end{aligned} \quad (7.34)$$

displayed in Eq. 6.1.21 and equation of Chukwu [2] (6.1.22). Employment dynamics is displayed as

$$\begin{aligned} \dot{L}(t) - \ell_{-01}\dot{L}(t-h) - \ell_{-03}\dot{y}(t-h) \\ = \ell_0 L(t) - \ell_1 L(t-h) + \ell_2 y(t-h) \\ + \ell_4 R(t-h) + l_5 L(t-5) + \sigma_3(t) + q_3(t) \end{aligned} \quad (7.35)$$

where

$$\begin{aligned} \sigma_3(t) &= m(w)\sigma_4(t) \\ q_3(t) &= m(w)q_4(t) \\ q_4(t) &= g_0 + z_{s13}M(t) - z_{s14}T(t) + z_{s15}e(t) + z_{s16}\tau(t) + z_{s17}d(t), \end{aligned} \quad (7.36)$$

where

$$\sigma_4 = x_0 + y_0 + I_0,$$

and with

$$y = Ak^\alpha L^{1-\alpha} \quad 0 < \alpha < 1, \quad m(w) = \left[(1-\alpha) \frac{1}{w} \right]^{1/\alpha}. \quad (7.37)$$

The coefficients are displayed in (6.1.66) of [2], or 1.10.66 of [1]. The equation of the value of Capital Stock k is given by

$$\begin{aligned} \dot{k}(t) - a_{-1}\dot{k}(t-h) - a_3\dot{y}(t-h) - a_6\dot{L}(t-h) \\ = a_0 k(t) - a_1 k(t-h) + a_2 y(t-h) + a_4 R(t-h) a_5 L(t-h) \\ + a_8 p(t) + \sigma_4(t) + q_4(t) \end{aligned} \quad (7.38)$$

where

$$\begin{aligned} \sigma_4(t) &= x_0 + y_{10} + I_0 \\ q_4 &= g_0 + z_{s13}M_1 - z_{s14}T(t) + z_{s15}e(t) + z_{s16}\tau(t) + z_{s17}d(t). \end{aligned} \quad (7.39)$$

Here the coefficients are identified in [1] (1.10.55). Their numerical values are obtained by a simple MATLAB linear regression using the arx command in (1.10.47, 1.10.48) and (1.10.49–1.10.51) and (1.10.52) of Chukwu [1] or [2]. The price equation is given by

$$\begin{aligned}\dot{p}(t) = & p(t)[(m_6 + p_4)p(t) - p_6M_1y(t) - p_6y(t-h) - p_6M_3R(t) \\ & + M_4p_6R(t-h) - p_6M_7\dot{R}(t-h) + [q_5(t) + \sigma_5(t)]p(t),\end{aligned}$$

where

$$\begin{aligned}q_5(t) &= p_1pf(t)e(t) + p_5\dot{M}1 + p_6M1, \\ \sigma_5(t) &= p_0 - p_3n(t) + p_2w(t) - p_6M_0\end{aligned}$$

and the coefficients are contained in Chukwu [2, p. 264]. The cumulative balance of payment equation is extracted as

$$\begin{aligned}\dot{E}(t) = & b_1y(t) + b_2y(t-h) + b_4\dot{y}(t-h) + b_5R(t) + b_6R(t-h) + b_8\dot{R}(t-h) \\ & + b_9L(t) + b_{10}L(t-h) + b_{12}\dot{L}(t-h)b_{17}B(t-h) - r_6(t) + q_6(t),\end{aligned}\tag{7.40}$$

where

$$\begin{aligned}\frac{dE}{dt} &= B(t), \\ -r_6(t) &= X_0 \\ q_6(t) &= b_7e(t) + b_8\tau(t) + b_{15}d(t) - f_0\end{aligned}$$

and the rest of the symbols are defined in Chukwu of [2, p 266].

7.3 BOTSWANA Example

The Botswana Example. In our economic analysis of Botswana we were inspired by an intense longing for peace and prosperity for that land. This longing inspires a unique common goal “Love for all” expressed with a working concept of “goodness”—feed the hungry, give drink to the thirsty, welcome strangers in our homes, clothe the naked, heal and take care of the sick, visit and care for prisoners, all in solidarity with foreigners. Implicit in Botswana’s pursuit of goodness is the possibility and desirability of high growth of Gross Domestic Product, of national “goods and services”, full employment, appropriate low interest rate, low prices (or small inflation) solid high value of capital stock and strong and positive cumulative balance of payment. Even with abundant supply of diamond and excellent education system the conquest of scarcity and the entrenchment of prosperity rest on a good design of a national economic model with realistic mathematical economic state which interacts with other nations and which mirrors government and private strategies. This model is then tested for controllability—the ability to steer existing (perhaps not too bad) state to a better one—to a state of

paradise—full employment, appropriate low interest rate, low prices (or small inflation), high value of capital stock and strong cumulative balance of payment.

In the derivation, the economic state is defined as

$$x = [y, R, L, k, p, E]'$$

where y is the Gross Domestic Product, R is the interest rate, L is employment, k is the value of capital stock, p represents prices, and E denotes the cumulative balance of payment. The Government strategy is a vector of eight things:

$$q = [T, g_0, e, \tau, d, M1, M1^*, f_0]'$$

where T denotes generalized taxes and its flows, g_0 denotes autonomous government outlay (i.e., independent of y , etc.) into education, roads, health care, agricultural extension services, etc. The symbol e denotes exchange rate, τ is tariff, d preferential trade agreement and transportation or trade policy, $M1$ is money supply and $M1^*$ is its flow and f_0 denotes foreign credit equalization tax. The control instrument of the representative firm is a vector of nine things:

$$\sigma = [C_0, I_0, X_0, M_0, n, w, y_{10}, p_0]'$$

where C_0 is autonomous consumption, I_0 is autonomous investment, X_0 is autonomous net export, M_0 is autonomous money demand, n is labor productivity, w is the wage rate, y_{10} is autonomous income consumption intercept and p_0 is autonomous price intercept. These terms are defined in Chap. 6 and also in Chukwu [2] and in Chukwu [1]. Using principle of supply and demand and of “rational expectation” and using some realistic formula for C , I , X , G as well as Cobb Doglass well know formula we derive the economic state equation as

$$\dot{x}(t) - A_{-1}\dot{x}(t-h) = A_0x(t) + A_1x(t-h) + B_1q(t) + B_2s(t). \quad (7.41)$$

where

$$B_1q(t) = g(t), \quad B_2s(t) = \sigma(t)$$

For any country considered and with UN, IMF data we identify the coefficients

$$B_1, \quad B_2, \quad A_{-1}, \quad A_0, \quad \text{and} \quad A_1.$$

Let us extract the employment equation from (7.35) where we disregard the five other components of the economic state and where w is the wage of labor per unit time which maximizes profit for the representative firm and where we let

$$m(w) = \left[(1 - \alpha) \frac{1}{w} \right]^{1/\alpha},$$

and

$$y = Ak^{1-\alpha}L^\alpha, \quad 0 < \alpha < 1.$$

The equation of employment is

$$\begin{aligned} \dot{L}(t) + (a_{-1} - m(w)a_6L(t-h)) &= a_0L(t) - a_1L(t-h) + m(w)[x_0 + y_{10} + I_0] \\ &\quad + m(w)[g_0 + z_{s13}M1_{s14}T(t) + z_{s15}e(t) \\ &\quad + z_{s16}\tau(t) + z_{s17}d(t)]. \end{aligned} \quad (7.42)$$

All the coefficients can be computed. Note that

$$m(w)[x_0 + y_{10} + I_0],$$

is the representative firms' strategy and

$$m(w)[g_0 + z_{s13}M - z_{s14}T(t) + z_{s15}e(t) + z_{s16}\tau(t) + z_{s17}d(t)],$$

is government strategy. From work done [1, p. 89] the constants determined by regression for the USA economy for example are

$$\begin{aligned} m(w)z_{s14} &= 5.4905^{-009}, \\ m(w)z_{s15} &= 0.0956, \\ m(w)z_{s16} &= 1.0145e^{-0.005}, \\ m(w)z_{s17} &= 0, \\ m(w)z_{s13} &= 4.7587e^{-0.007}, \\ w &= 63.1704, \\ m(w) &= 7.7325e^{-0.04}. \end{aligned} \quad (7.43)$$

To make $\dot{L}(t)$ bigger in this example and therefore employment grows bigger increase private firms' strategy $[x_0 + y_{10} + I_0]$. For example increase autonomous net export; increase autonomous private investment and autonomous consumption y_{10} . Professor Joseph Stiglitz a Nobel Prize Laureate in economics at Columbia University visited Botswana, September 11, 2008 as reported in Daily News, No. 17 Thursday, September 2008, recommended that if the private sector was unable to bring food into the country cheaply, "government had to step in". Thus he advised an increase of g_0 . He suggested also a better management of exchange rate, e . He cautioned against targeting interest rate through (the author assumes manipulating M) which may lead to higher inflation.

Professor Stiglitz suggested that without the creation of employment there will always be the possibility of political and social turbulence.

For Botswana, the dynamics of employment is as derived earlier in [1, 2], and [3]

$$\begin{aligned}\dot{L}(t) + L_{-10}\dot{L}(t-h) &= a_0L(t) - a_1L(t-h) + m(w)[x_0 + y_{10} + I_0] \\ &\quad + m(w)[g_0 + z_{s13}M - z_{s14}T(t) + z_{s15}e(t) \\ &\quad + z_{s16}\tau(t) + z_{s17}d(t)],\end{aligned}$$

where

$$m(w)[x_0 + y_{10} + I_0]$$

is the representative private firms strategy and

$$m(w)[g_0 + z_{s13}M - z_{s14}T(t) + a_{s15}e(t) + z_{s16}\tau(t) + z_{s17}d(t)].$$

is the government strategy. From regression this last equation is equal to

$$\begin{aligned}&= 89.9055(-129.4490 + -0.0108M(t) + -0.0070T(t) + 366.6797^*e(t) \\ &\quad + -(1.9634e + 005\tau(t) + z_{s17}^*d(t)))\end{aligned}$$

Note that

$$d(t) = \text{distance} = 0.$$

For Botswana, the constants obtained from E. Chukwu's regression calculations for the employment equation

$$\begin{aligned}\dot{L}(t) - \dot{L}_{-01}(t-h) - l_{-03}\dot{y}(t-h) &= l_0L(t) - l_1L(t-h) + l_2y(t-h) \\ &\quad + l_4R(t-h) + l_5L(t-h) + \sigma_3(t) + q_3(t)\end{aligned}$$

yield

$$\begin{aligned}\sigma_3(t) &= m(w)\sigma_4(t) \\ &= 89.9055^*(x_0 + y_{10} + I_0) \\ &= 89.9055(-560.7791 + -43.9296 + -3.6583^{+006}) \\ q_3(t) &= m(w)q_4(t) = 89.9055[g_0 + z_{s13}M(t) + z_{s14}T(t) \\ &\quad + z_{s15}e(t) + z_{s16}ta(t) + z_{s17}d(t)].\end{aligned}\tag{7.44}$$

We observe that

$$m(w)[x_0 + y_{10} + I_0]$$

the representative private firms strategy is

$$-43.9296 + -3.658e^{+006}$$

a negative value. This observation is perhaps what made Professor Stiglitz to suggest that “the more developing countries kept on buying food, prices would continue to increase a situation which skyrockets inflation... Among things that can be done to decrease food prices is to figure out how food can be transported from the port to their destinations at less cost”. This will decrease x_0 . If private sector is unable to bring food into the country cheaply, government can step in (perhaps with $m(w)g_0$ positive). Of course innovative ways for domestic food production can be explored, decreasing import. The desert parts of Botswana “may be made to bloom”, in accordance with the “prophecy of Isaiah 35”. Local consumption and investment y_{10}, I_0 can also be pursued.

Also government strategy can be applied:

$$q_3(t) = 890.9055(g_0 + (-0.9743M(t) + (-0.6283T(t)] \quad (7.45) \\ + (3.2967e^{+004}e(t) + (-1.7672e^{+007}\tau(t)).$$

Increasing exchange rate in Eq. (7.45) may increase the net employment growth, and decreasing tariffs can increase net employment growth. The same is true for decrease in taxes. Since the equivalent control system for Botswana’s economic state is controllable using U (the Pontryagin difference of sets $1P$ and $1Q$, $1P \pm 11Q = U$ of private control set and government control strategy set) with $u = Bv$, $u \in U$ where B has full rank, creation of employment is possible as well as avoidance of political and social turbulence as observed by Professor Stignitz.

There is another way of promoting employment: decrease wages. Within the countries borders this may be unacceptable. But Botswana “outsourced” firms can be transferred to the rural areas, and outside countries where wages are lower so that in average wage w is lower by increasing $m(w)$. Taxation on huge profits made by the outsourced firms can be negotiated and repatriated to Botswana. This can be used to create more employment and benefits to Botswana workers. The private initiative can be increased by increasing $(x_0 + y_{10} + I_0)$, argued by the wise professor Stignitz, by increasing food supply a portion of x_0 at a cheaper price, for example. This can increase the growth rate of employment. If this does not happen government can decrease money supply, M , Taxes, τ , tariffs and manage exchange rate better or more positively. This will increase $q_3(t)$ and therefore the growth rate of employment. A positive trading policy d may also help. In earlier argument we assumed $d = 0$.

The rate of growth of GDP of Botswana can be increased by government strategy,

$$q_1(t) = \lambda_1 \sigma^{-1}(g_0 - 3.4633e^{-004}T(t) + 0.0013T(t-h) - 0.0054\dot{T}(t) \\ - 0.0018\dot{T}(t-h) - 50.2838e(t) - 0.0038\tau(t)).$$

by decreasing current taxes and the growth rate of current taxes. It can be increased by decreasing current exchange rate and current tariffs. If conditions are such that employment decreases somewhat and the value of capital stock increase somewhat the growth rate of GDP may become somewhat bigger.

The equation of the value of capital stock k shows that the net growth rate of k can be increased by positive government intervention with autonomous g_0 , government spending and reduction of taxes and the use of other monetary policies.

Since the economic system of Botswana is controllable subject to scarcity, with private and government strategy it is possible to steer it to a good desired target.

The possibility of inciting high inflation is seen from the equation for $\dot{p}(t)$ if we target interest rate through money supply. Control with exchange rate seems to be preferable.

Work has begun in University of Botswana on strategies for economic growth and diffusion of wealth for Botswana using a model such as

$$\frac{\partial u}{\partial t} = c[u_{xx} + u_{yy}] + f(x, y, u, t)$$

where $u(x, y, t)$ is wealth at (x, y) in time t and $f(x, y, t) =$ the gross domestic product monitored by our model above.

Hopefully principles and strategies will emerge to facilitate the creation and diffusion of wealth in this country.

We now extract from earlier works the economic state equations of several countries. They are contained in E. N. Chukwu, *Stability and Time-Optimal Control of Hereditary Systems with Application to the Economic Dynamics of the US*, 2nd Edition, World Scientific, 2001; *Optimal Control of the Growth of Wealth of Nations*, E. N. Chukwu, Taylor and Francis, 2003. *A Mathematical Treatment of Economic Cooperation and Competition Among Nations: With Nigeria, USA, UK, China and Middle East Examples*, Elsevier, 2005. To smoothen the flow of ideas and link up with the past we identify the following symbols:

C	Consumption
I	Net Export
G	Government Outlay
Y	Gross Domestic Product
R	Interest Rate
L	Employment
K	Value of Capital Stock
p	Prices
E	Cumulative Balance of Payment
M1	Money supply
M	Money demand
Z	$C + I + X + G$
	Aggregate demand

We postulate the following formula for C, I, X, G.

$$\begin{aligned} C = & C_0 + C_1(y(t) - T(t)) + C_2(y(t-h) - T(t-h)) + C_3(\dot{y}(t) - \dot{T}(t)) \\ & + C_4(\dot{y}(t-h) - \dot{T}(t-h)) + C_5R(t) + C_6R(t-h) + C_7(M_0 + M_1y(t) \\ & + M_2y(t-h) + M_3R(t) + M_4R(t-h) + M_5\dot{R}(t-h) + M_6p(t) - M) \end{aligned}$$

$$\begin{aligned} I = & I_0 + I_1y(t) + I_2y(t-h) - I_3\dot{y}(t) + I_4\dot{y}(t-h) + I_5R(t) + I_6R(t-h) + I_8L(t) \\ & + I_9L(t-h) - I_{11}k(t) - I_{13}(M - M_0 - M_1y(t) - M_2y(t-h) - M_3R(t) \\ & - M_4R(t-h) - M_5\dot{R}(t-h) - M_6p(t)) \end{aligned}$$

$$\begin{aligned} X = & X_0 + X_1y(t) + X_2y(t-h) + X_3y(t) + X_4\dot{y}(t-h) + X_5R(t) + X_8L(t) \\ & + X_{10}\dot{L}(t-h) + X_{12}p(t) + X_{16}\tau(t) + X_{15}e(t) + X_{17}d(t) \end{aligned}$$

$$G = g_0 + g_1y(t) + g_2y(t-h) + g_3\dot{y}(t) + g_4\dot{y}(t-h) + g_5R(t) + g_6L(t)$$

$$\begin{aligned} Z = & C + I + X + G \\ = & C_0 + I_0 + X_0 + g_0 + M_0(C_7 + I_{13}) - M(C_7 + I_{13}) \\ & + (C_1 + I_1 + X_1 + g_1 + M_1(C_7 + I_{13}))y(t) \\ & - (C_1T(t) + C_2T(t-h) + C_3\dot{T}(t) + C_4\dot{T}(t-h)) \\ & + (C_2 + I_2 + (C_7 + I_{13})M_2 + X_2 + g_2)y(t-h) + (C_3 - I_3 + X_3 + g_3)\dot{y}(t) \\ & + (C_4 + I_4 + X_4 + g_4)\dot{y}(t-h) + (C_5 + (C_7 + I_{13})M_3 + I_5 + g_5)R(t) \\ & + (C_6 + (C_7 + I_{13})M_4 + I_6)R(t-h) + (M_5(C_7 + I_{13}))\dot{R}(t-h) \\ & + (I_8 + X_8 + g_6)L(t). \end{aligned}$$

Hence by the principle of supply and demand, we have

$$\begin{aligned} \dot{y}(t) = & \lambda_1(Z(t) - y(t) + I_9L(t-h) + X_{10}\dot{L}(t-h) - I_{11}k(t) \\ & + (C_7M_6 + I_{13}M_6 + X_{12})p(t) \\ & + x_{15}e(t) + X_{16}\tau(t) + X_{17}d(t) = C_0 + I_0 + M_0(C_7 + I_{13}) \\ & + (g_0 - M(C_7 + I_{13}) + X_{15}e(t) + X_{16}\tau(t) + X_{17}d(t) \\ & + [C_1 + I_1 + X_1 + g_1 + M_1(C_7 + I_{13}))y(t) \\ & + (C_2 + I_2 + (C_7 + I_{13})M_2 + X_2 + g_2)y(t-h) \\ & + (C_3 - I_3 + X_3 + g_3)\dot{y}(t) + (C_4 + I_4 + X_4 + g_4)\dot{y}(t-h) \\ & + [C_5 + (C_7 + I_{13})M_3 + I_5 + g_5]R(t) \\ & + (C_6 + (I_{13} + C_7)M_4 + I_6 + g_5)R(t-h) + M_5(C_7 + I_{13})\dot{R}(t-h). \end{aligned}$$

Let

$$\begin{aligned}
Z_0 &= C_0 + I_0 + X_0 + g_0 + M_0(C_7 + I_{13}) - M(C_7 + I_{13}), \\
r_0 &= C_0 + I_0 + X_0 + M_0(C_7 + I_{13}), \\
z_0 &= r_0 + g_0 \\
z_1 &= C_1 + I_1 + X_1 + g_1 + M_1(C_1 + I_{13}) \\
T_1 &= C_1 T(t) + C_2 T(t - h) + C_3 \dot{T}(t) + C_4 \dot{T}(t - h) \\
z_2 &= C_2 + I_2 + (C_7 + I_{13})M_2 + X + g_2 \\
z_3 &= C_3 - I_3 + X_3 + g_3 \\
z_4 &= C_4 + I_4 + X_4 + g_4 \\
z_5 &= C_5 + (C_7 + I_{13})M_3 + I_5 + g_5 \\
z_6 &= (C_6 + (C_7 + I_{13})M_4 + I_6) \\
z_7 &= (C_7 + I_{13})M_5 \\
z_8 &= (I_8 + X_8 + g_6) \\
z_9 &= I_9 \\
z_{10} &= X_{10} \\
z_{11} &= -I_{11} \\
z_{12} &= (C_7 + I_{13})M_6 + X_{12}
\end{aligned}$$

Hence

$$\begin{aligned}
\dot{y}(t) &= \lambda_1(z(t) - y(t)) \\
&= \lambda_1 r_0 + \lambda_1 g_0 - \lambda_1 T_1 z_1 \lambda_1 y(t) + z_2 \lambda_1 y(t - h) + \lambda_1 z_3 \dot{y}(t) \\
&\quad + \lambda_1 z_4 \dot{y}(t - h) + \lambda_1 z_5 R(t) + \lambda_1 z_7 (\dot{R}(t - h) + \lambda_1 z_6 R(t - h) + \lambda_1 z_8 L(t) \\
&\quad + \lambda_1 z_9 L(t - h) + \lambda_1 z_{10} \dot{L}(t - h) + \lambda_1 z_{11} k(t) + z_{12} \lambda_1 p(t) \\
&\quad + \lambda_1 X_{15} e(t) + X_{16} \lambda_1 \tau(t) + X_{17} \lambda_1 d(t) - \lambda_1 y(t).
\end{aligned}$$

Therefore

$$\begin{aligned}
\dot{y}(t) &- \frac{\lambda_1 z_4}{1 - \lambda_1 z_3} \dot{y}(t - h) - \frac{\lambda_1 z_{10}}{1 - \lambda_1 z_3} \dot{L}(t - h) - \frac{\lambda_1 z_7}{1 - \lambda_1 z_3} \dot{R}(t - h) \\
&= \lambda_1 (z_1 - 1)/(1 - \lambda_1 z_3) y(t) + \lambda_1 z_5/(1 - \lambda_1 z_3) R(t) + \lambda_1 z_8 L(t)/(1 - \lambda_1 z_3) \\
&\quad + \lambda_1 z_{11}/(1 - \lambda_1 z_3) k(t) + \lambda_1 z_{12}/(1 - \lambda_1 z_3) p(t) \\
&\quad + \frac{\lambda_1 z_2 y(t - h)}{(1 - \lambda_1 z_3)} + \lambda_1 z_6 \left/ \frac{R(t - h)}{(1 - \lambda_1 z_3)} \right/ + (\lambda_1 z_9) \left/ \frac{L(t - h)}{(1 - \lambda_1 z_3)} \right/ + \lambda_1 \left/ \frac{X_{15} e(t)}{(1 - \lambda_1 z_3)} \right/ \\
&\quad + \lambda_1 r_0/(1 - \lambda_1 z_3) + \lambda_1 (q_0 - T_1)/(1 - \lambda_1 z_3).
\end{aligned}$$

Writing Compactly,

$$\begin{aligned} \dot{y}(t) - a_{-11}\dot{y}(t-h) - a_{-12}\dot{R}(t-h) - a_{-13}\dot{L}(t-h) \\ = a_{01}y(t) + a_{02}R(t) + a_{03}L(t) + a_{04}k(t) + a_{05}p(t) \\ + a_{11}y(t-h) + a_{12}R(t-h) + a_{13}L(t-h) + a_{14}k(t-h) + a_{15}e(t) \\ + a_{16}\tau(t) + a_{17}d(t) + a_{18}r_0 + a_{19}(q_0 - T_1), \end{aligned}$$

where

$$\begin{aligned} a_{-11} &= \lambda_1 z_4 / (1 - \lambda_1 z_3), \quad a_{-12} = \lambda_1 z_7 / (1 - \lambda_1 z_3), \quad a_{-13} = \lambda_1 z_{10} / (1 - \lambda_1 z_3), \\ a_{01} &= \frac{\lambda_1(z_1 - 1)}{(1 - \lambda_1 z_3)}, \quad a_{02} = \lambda_1 z_5 / (1 - \lambda_1 z_3), \quad a_{03} = \lambda_1 z_8 / (1 - \lambda_1 z_3) \\ a_{04} &= \lambda_1 z_{11} / (1 - \lambda_1 z_3), \quad a_{05} = \lambda_1 z_{12} / (1 - \lambda_1 z_3), \quad a_{11} = \lambda_1 z_2 / (1 - \lambda_1 z_3) \\ a_{12} &= \lambda_1 z_6 / (1 - \lambda_1 z_3), \quad a_{13} = \lambda_1 z_9 / (1 - \lambda_1 z_3), \quad a_{14} = \lambda_1 \cdot 0 = 0, \\ a_{15} &= \lambda_1 X_{15} / (1 - \lambda_1 z_3), \quad a_{16} = \lambda_1 X_{16} / (1 - \lambda_1 z_3), \quad a_{17} = \lambda_1 X_{17} / (1 - \lambda_1 z_3) \\ a_{18} &= \lambda_1 / (1 - \lambda_1 z_3), \quad a_{19} = \lambda_1 / (1 - \lambda_1 z_3). \end{aligned}$$

Here

$$x = \begin{bmatrix} y \\ R \\ L \\ k \\ p \\ E \end{bmatrix}$$

$$\begin{aligned} r_0 &= C_0 + I_0 + X_0 + M_0(C_7 + I_{13}) \\ &= \begin{bmatrix} 1, & 1, & 1, & C_7 + I \end{bmatrix} \begin{bmatrix} C_0 \\ I_0 \\ X_0 \\ M_0 \end{bmatrix} \quad q_0 = g_0 - M(C_7 + I_{13}). \end{aligned}$$

The equation of gross domestic product is now derived as

$$\begin{aligned} \dot{y}(t) - a_{-11}\dot{y}(t-h) - a_{-12}\dot{R}(t-h) - a_{-13}\dot{L}(t-h) \\ = a_{01}y(t) + a_{02}R(t) + a_{03}L(t) + a_{04}k(t) + a_{05}p(t) \\ + a_{11}y(t-h) + a_{12}R(t-h) + a_{13}L(t-h) + a_{14}k(t-h) + a_{15}e(t) \\ + a_{16}\tau(t) + a_{17}d(t) + a_{18}r_0 + a_{19}(q_0 - T_1). \end{aligned}$$

Here

- e Exchange rate
- τ Tariff
- d Trade policy or distance between trading nations
- r_0 Autonomous private strategy (consumption, investment and net export)_
- q_0 Autonomous government strategy or spending
- T_1 Generalized taxes.

The equations from earlier works are identified in us2.m for the USA, aus-1r.22.m for Austria, southaf.22.m for South Africa, Italy2.m for Italy, India2.m for India, Egypt2.m for Egypt, Jordan2.m for Jordan, Israel2.m for Israel, China2.m for China and Nigeria2.m for Nigeria. The data are mined from International Financial Statistics and United Nations Statistics as follows

Statistics name	Full name	Statistics number
DIA	Direct investment abroad	78 bdd
DII	Direct investment into reporting economy	
YEAR	Year	78bed
EX	Export of goods and services	90c
IMP	Import of goods and services	98c
X	Net export	EX-IMP
TX	Revenue	81
K	Gross fixed capital formation	93e
KP	Derivative gross fixed capital formation	$K(t + 1) - K(t)$
D	Changes in inventories	93i
I	Investments	$K + D$
Kk	Derivative gross fixed capital formation	$K(t + 1) - K(t)$
i	Incremental	by one
d	trade policy/distance between trading nations	
L	Employment	67e
X	[y, R, L, K, p, E] national economic state	
E	Exchange rate market rate	aa
P	Consumer prices	64
Ta	Tariff	71
E	Cumulative balance of payment $B = dE/dt$	
B	Balance of payment	
F	Other items (Net)	37r
B	Balance of payment	X-F-TX
BD	Shift left in time balance of payment	$B(t - 1)$
T	Net transfer of capital to foreign firms	X-B-F
L	Employment	67e
LD	Delay employment by 1	$L(t - 1)$
LP	Derivative of employment	$L(t + 1) - y(t)$

(continued)

(continued)

Statistics name	Full name	Statistics number
LPD	Delay in time derivative of employment	$LP(t - 1)$
w	Wages: average monthly earnings	65
y	Gross domestic product (GDP)	99b
yD	Shift left in time gross domestic product	$y(t - 1)$
GDP	Gross domestic product (GDP)	99b
yP	Derivative of gross domestic product	$y(t + 1) - y(t)$
yPD	Delay in time derivative of (GDP)	$yP(t - 1)$
G	Government consumption expenditure	91 f.c
C	Private consumption expenditure Househ Cons. expend. incl. NPSIsh	96 f.c
M	Money	34
MD	Shift left in time money	$M(t - 1)$
MP	Derivative of banking supply money	$M(t + 1) - M(t)$
MPD	Delay in time derivative of banking money	$MP(t - 1)$
M1	Money	34
R	Interest rates bank rate (end of period)	60
RP	Derivative of interest rates bank (end)	$R(t + 1) - R(t)$
RPD	Derivative of interest rates (end) delayed	$RP(t - 1)$
RD	Interest rates bank rate delayed	$R(t - 1)$
YT	GDP—revenue	$y - TX$
YTD	YT delayed	$YT(t - 1)$
YTP	Derivative YT	$YT(t + 1) - YT(t)$
YTPD	YTP delayed	$YTP(t - 1)$
RD	Interest rates bank rate delayed	$R(t - 1)$
RD	Interest rates bank rate delayed	$R(T - 1)$
RD	Interest rates bank rate delayed	$R(T - 1)$
Price	Consumer prices	64
for i = 1:12		
pp (i) = price (i + 1) - price (i);		
end		
pie = [pp]';		
% DP = pie.\P; represents pie*DP = P solving for DP		
DP = pie.\P;		

BOTWANA Appendix: Date Code, MATLAB Program Results and Graphs

C:\DOCUME~1\Owner\LOCALS~1\Temp\Botswana2.m

December 8, 2008

%Botswana2.m

% Date: 05-Dec-2008

%Botswana2.m data is millions of pula.

%Adopted from the International Financial Statistics Yearbook 2006, except where noted

%The 2006 Statistics Yearbook publication range between year 1994-2005 (a total of 12

%years) and as a results lead to extrapolation of more data for 1992, 1993, 2006 and 2007

%to give additional statistics for running regression for y(t) and Balance of payment,

%B(t). Please see the two model equation described below, their lengths were out of

%ranged beyond what the 12 years could handle.

% Use NDU.m to obtain the delay and derivative matrix entries.

% Use Alphacomputation.m to estimate "a" and "1-a"

% The rest of the results obtained from the regression analysis can

% be found in botswana2_output_report_latest_release.m.

% The final results of each differential equation based on the regression analysis (arx)

% have been reproduced with the missing constants included in the section after the plot

% section.

Year= [1992 1993 1994 1995 1996 1997 1998 1999 2000 2001 2002 2003 2004
2005 2006 2007]';

Y =[8600.5 9820.9 11041.3 12261.7 14203.9 17740.3 20162.7 21523.8 27649
34787.1 35693.4 38688.3 42580.4 48753.1 54925.8 61098.5]';

GDP =[8600.5 9820.9 11041.3 12261.7 14203.9 17740.3 20162.7 21523.8 27649
34787.1 35693.4 38668.3 42580.4 48753.1 54925.8 61098.5]';

yD =[7380.1 8600.5 9820.9 8600.5 9820.9 11041.3 12261.7 14203.9 17740.3
20162.7 21523.8 27649 34787.1 35693.4 38688.3 42580.4]';

```

yP =[1220.4 1220.4 1220.4 -1220.4 1220.4 1220.4 1220.4 1942.2 3536.4 2422
41361.1 6125.2 7138.1 906.3 2994.9 3892.1]';
yPD =[1220.4 1220.4 1220.4 1220.4 -1220.4 1220.4 1220.4 1942.2 3536.4
2422.4 1361.1 6125.2 7138.1 906.3 2994.9]';
DummyTX=[4396.8 5427.3 7311.8 8169.2 7539.93 11837.04 13920.76 12527.87
13999.19 NaN NaN NaN]';
TX =[2335.8 3366.3 4396.8 5427.3 7311.8 8169.2 7539.93 11837.04 13920.76
12527.87 13999.19 15471.0 16942.0 18413.0 19884 21355]';
% YT= y-DummyTX;
YT = y-TX;
YTD = [6074.8 6264.7 6454.6 6644.5 6834.4 6892.1 9571.1 12622.77 9686.76
13728.24 22259.23 21694.21 23217.3 25638.4 30340.1 35041.8]';
YTP = [189.9 189.9 189.9 189.9 57.7 2679.0 3051.67 -2936.01 4041.48 8530.99
-565.02 1523.09 2421.1 4701.7 4701.7 4701.7]';
YTPD= [189.9 189.9 189.9 189.9 189.9 57.7 2679.0 3051.67 -2936.01 4041.48
8530.99 -565.02 1523.09 2421.1 4701.7 4701.7]';
XX =[4093.2 4752.6 5412.0 6071.4 7411.6 9881.6 11392.8 10051.6 12163.2
17826.4 16399.2 19181.6 17875.4 24266.4 30657.4 37048.4]';
IMP =[3235.9 3748.1 4260.3 4772.5 5300.1 6771.1 8875.3 9960.6 7862.9
10805.9 12044.2 14908.9 14624.1 17104.5 19584.9 22065.3]';
X =XX-IMP;
d =[ ]';
ta =[2611.7 3509.5 4407.3 5305.1 5742.9 8250.0 9803.8 10164.4 12646.8
10613.1 10556.9 12840.1 16237.4 20441.4 24645.4 28849.4]';
E =[ ]';
F =[-419 -478 -1382 -537 596 -1004 -1260 -1759 -678 -222 -90 -45 -121 73
267 461]';
D =[585.5 394.9 204.3 13.7 -261.4 328.2 886.0 1653.9 1126.1 6811.9 7054.7
6721.5 9382.9 7021.1 11744.7 14106.5]';
K =[2171 2492.4 2813.8 3135.2 3632.4 4275.9 5170.1 6263.3 6751.1 6898.2
7743.2 8735.7 9017.8 9937.2 10856.6 11776]';
I =K+D
B =X-F-TX'
BD =[-2522.7 -1954.6 -818.4 -1386.5 -3591.4 -5796.3 -4054.7 -3762.43 -9987.04
-8942.46 -5287.37 -9554.19 -11153.3 -13569.7 -11324.1 -9078.5]';
C =[2053.6 2551.3 3049.0 3546.7 4006.7 4711.0 5452.9 6578.8 7524.5 6517.9
7348.0 8967.4 9161.7 11103.9 13046.1 4988.3]';
QM =[1145 1382 1619 1856 2239 3066 4209 5454 5432 7595 7318 8763 9037
10676 12315 13954]';
QMD =[908 1145 1382 1619 1856 2239 3066 4209 5454 5432 7595 7318 8763
9037 10676 12315]';
QMP =[237 237 237 237 383 827 1143 1245 -22 2163 -277 1445 274 1639
1639 1639]';
QMPD =[237 237 237 237 237 383 827 1143 1245 -22 2163 -277 1445 274
1639 1639]';

```

```

M    =[664 719 774 829 951 1038 1513 1775 1897 1354 2581 2882 4225 3998
3771 3544]';
MD   =[609 664 719 774 829 951 1038 1513 1775 1897 2354 2581 2882 4225
3998 3771]';
MP   =[55 55 55 55 122 87 475 262 122 457 227 301 1343 -227 -227 -227]';
MPD  =[55 55 55 55 55 122 87 475 262 122 457 227 301 1343 -227 -227]';
ML   = QM-M;
L     =[227 229 231 233 238 230 242 257 265 271 279 287 295 303 311
319]';
LD   =[225 227 229 231 233 238 230 242 257 265 271 279 287 295 303
311]';
LP   =[2 2 2 2 5 -8 12 15 8 6 8 8 8 8 8]';
LPD  =[2 2 2 2 2 5 -8 12 15 8 6 8 8 8 8]';
% w = National accounts statistics (United Nations), page 116 2003 edition, defined
% as compensation of employees from the rest of the world. See statistics entry number
"65".
w     =[165 164 163 162 134.8 133.0 150.4 136.4 107.0 99.0 113.0 127.0 141.0
155.0 169 183]';
e     =[3.5122 3.7396 3.967 4.1944 5.2404 5.1400 6.2774 6.3572 6.9861 8.7760
7.4331 6.6014 6.6482 7.879 9.1098 10.3406]';
P     =[48.0 54.3 60.6 66.9 73.7 80.1 85.5 92.1 100.0 106.6 115.1 125.7 134.4
146.1 157.8 169.5]';
Pf    =P.*e;
Pfe   =Pf.*e;
R     =[14.5 14 13.5 13 13 12 12.75 13.75 14.25 14.25 15.25 14.25 14.25 14.5
14.75 15]';
RD    =[15 14.5 14 15.5 15 14.5 13.5 13 13 12 12.75 13.75 14.25 14.25 15.25
14.25]';
RP    =[-0.50 -0.50 -0.50 -0.50 0 -1.00 0.75 1.00 0.50 0 1.0 -1.0 0 0.25 0.25
0.256]';
RPD   =[-0.50 -0.50 -0.50 -0.50 -0.50 0 -1.00 0.75 1.00 0.50 0 1.0 -1.0 0 0.25
0.25]';
G     =[2804.1 3596.6 4389.10 5181.6 9283.5 7188.1 9199.78 10608.94 11102.91
13404.19 15693.45 17191.18 18688.91 20186.64 21684.37 23182.1]';
N     =y.\L;
% Or N = y.\epl;
price=[41.7 48.0 54.3 60.6 66.9 73.7 80.1 85.5 92.1 100.0 106.6 115.1 125.7
134.4 146.1 157.8 169.5 181.2]';
kk     =[1528.2 1849.6 2171 2492.4 2813.8 3135.2 3632.4 4275.9 5170.1 6263.3
6751.1 6898.2 7743.2 8735.7 9017.8 9937.2 10856.6 11776]';
for il: 16
pp(i) =price (I+1) -price (i)
kprim (i) =kk (i+1) -kk (i);
end
pie= [pp]';
KP = kprim';
DP =pie.\P;

```


$n=y \cdot \backslash L;$

% Statistics Number, Statistics Name, Full Name

% 81, TX, Tax revenue

% TX =total current receipts of general government (millions of Botswana pula currency)

% 65, w, Wages: Average Monthly earnings (termed compensation according to National accounts statistics, United Nations publication,

% page 116, 2003 edition) defined as compensation of employees from the rest of the world.

% see statistics entry number "65".

% 90c, EX, Exports of good and services

% 98c, IMP, Imports

% X, EX-IMP (Net export)

% 67e, n Labor productivity ($A/1$) = GDP. \emp; FOR BOTSWANA WE USE THE employment available for

1994-2005. Other years were gotten by considering the lab or activity (percentage)

factor to population for that year.

% 93e, K, Capital stock (Gross fixed capital formation)

% $K(t+1) - K(t)$, KP, Derivative Gross Fixed Capital Formation

% $K+D$, I, Investment (Gross capital formation= $K+D$)

% 93i, D, Increase in stock (changes in inventories)

The UN(National accounts statistics) defines Gross capital formation

% as the sum of the increase in stocks and gross fixed capital

formation

%

% 99b, y or GDP, Gross domestic product

% $y(t-1)$, yD, Shift "right" in time Gross Domestic Product

% $y(t+1) - y(t)$, $y'(t)$, yP, Derivative of Gross Domestic Product

% $yP(t+1)$, yPD, $y'(t+1)$, Delay in time Derivative of (GDP)

% ignore, d, (Dummy) Differential trade agreement

transportation

% Pf Import price level in foreign currency

% 64, p, Consumer prices

% 71, ta, Tariffs (Import duties = custom duties plus other import charges)

% ignore, E, Cumulative balance of payment

% X-F-TX, B, Balance of payment (Deposit money banks= $E'(t)$)

% X-B-F, T, Net government transfer of capital to foreigners and firms

% 37r, F, Net private outflow of capital

% 34, M Money supply

% 35, QM, Money demand (Quasi Money)

% 67e, L, Employment, Labour (Industrial Production)

% emp =Employment (Labor force at equilibrium)

unemp =Unemployment

% 34, ML, QM-M Money

% 60,	R,	Interest rate Bank Rate (Discount rate or end of period)
% R(t+1),	RD,	Interest Rates Bank Rate delayed
% R(t+1) -R(t),	RP,	Derivative of interest Rates (End)
% R(t+1),	RPD,	Derivative of interest Rates (End) delayed
% 82,	G,	Expenditure
% 91f,	C,	Government consumption Expenditure
% aa	e,	Exchange rate (Market rate=aa)
% J3		Direct investment in Republic Economy
% GNI		Gross National income
% PL		Population
% YT	GDP-Revenue,	y-TX
% YTD,	YT delayed	YT(t+1)
% YTD,	Derivative YT,	YT(t+1) -YT (t)
% YTPD,	YTP delayed,	YTP (t+1)

format short;

```
%Bertl= [X, G, I, C, GDP, PGDP, R, M, QM, YEAR]
```

```
[years, nn]=size (year) ;
```

```
X1 = [857.3; X (1:years-1)];
```

```
G1 = [2804.1; G (1:years-1)];
```

```
I1 = [2756.5; I (1:years-1)];
```

```
C1 = [2053.6; C (1:years-1)];
```

```
QM1 = [1145; M (1:years-1)];
```

```
R1 = [14.5; R (1:years-1)]
```

```
GDP1 = [7380.1; GDP (1:years-1)];
```

```
DGDP=GDP-GDP1;
```

```
Z=X+G+I+C;
```

```
Z1=X1+G1+I1+C1;
```

```
%temp=[YEAR X1 G1 I1 C1 Z1 y1 yD M1 RD1 yPD1 RP RPD T1 P P-PE P1  
P-PF1 G QM QMD QMP QMPD QM1 QMD1 QMP1 QMPD1 ta tal e el d  
d1];
```

```
%save Bertl.asc Temp -ascii -double -tabs
```

```
%clear Bertl temp;
```

```
%ML = QM-M = m1-m0 -m1*y(t) -m2*y(t-h) -m3*R(t) -m3*R(t) -m4*R(t-h) -m6*P(t) -  
m7*R'(t-h)
```

```
%QM(t) = m0 +m1*y(t) +m2*y(t-h) +m3*R(t) +m4*R(t-h) +m5*R'(t-h) +m6*P(t);
```

```
temp=[QM ones (size (y)), y, yD, R, RD, RPD, P];
```

```
thqm=arx ( temp, [0, 1 1 1 1 1 1 1 0 0 1 0 1 1 0]);
```

```
QMp=predict ( [QM, ones ( size (y)), y, yD, R, RD, RPD, P], thqm, 4);
```

```
QMM=thqm ( 6, 1:7);
```

```
%C(t) = c0 + c1**YT(t) +c2*YT(t-h) +c3*YT'(t) +c4*YT'(t-h) +c5*R(t) +c6*R(t-h)  
+c7*(a1+a2.*y(t) +a3.*6D(t-h) +a4.*R(t) +a5.*RD(t) +a6.*P(t) +a.*RPD(t)) -M(t);
```

```
temp=[C ones (size(R)), YT, YTD, YTP, YTPD, R, RD, ML];
```

```
thc=arx ( temp, [0 1 1 1 1 1 1 1 1 0 0 1 0 1 0 1 0]);
Cp=predict ( C, ones (size (R)), YT, YTD, YTP, YTPD, R, RD, ML], thc, 4);
CC=thc (7, 1:8);
```

```
%I(t) = i0 +i1*y(t) +i2*y(t-h) -i3*y(t) +i4*y'(t-h) +i5*R(t) +i6*R(t-h) +i8*L(t) +i9*L(t-h)
-i11*K(t) -i13*ML(t)
temp= [I ones (size (y)), y, yD, yP, yPD, R, RD, QM, QMD, K, ML];
thi=arx(temp, [0, 1 1 1 1 1 1 1 1 1 1 1 0 0 1 0 1 0 1 0 0], 11);
Ip=predict ([I, ones(size(y)), y, yD, yP, yPD, R, RD, QM, QMD, K, ML], thi, 4);
II=thi (10, 1:11);
```

```
%X(t) =x0 +x1*y(t) +x2*y(t-h) +x3*y'(t) +x*y'(t-h) + x5*R(t) +x8*L(t) +x10*L'(t-h)
+x12*P(t) +x16*ta(t) +x15*e(t) +x17*d(t)
temp= [X ones(size (y)), y, yD, yP, yPD, R, QM, QMPD, P, ta, e];
thx=arx(temp, [0, 1 1 1 1 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1 0 0 0]);
XP=predict ([X, ones (size (y)), y, yD, yP, yPD, R, QM, QMPD, P, ta, e], thx, 4);
XX=thx(11, 1:12);
```

```
%G(t) = g0 +g1*y(t) +g2*y(t-h) +g3*y'(t) +g4*y'(t-h) +g5*R(t) +g8*L(t)
temp= [G ones (size(y)), y, yD, yP, yPD, R, QM];
thg=arx (temp, [0, 1 1 1 1 1 1 1 0 0 1 0 1 0 0]);
Gp=predict ([G, ones (size (y)), y, yD, yP, yPD, R, QM], thg, 4);
GG=thg (6, 1:7);
```

```
%y(t) zs0 +zs1*y(t) +zs2*y(t-h) +zs4*y'(t) zs5*R(t) +zs8*L(t) +zs10*L'(t) +zs13*M1(t)
-zs14YT(t) +zs15*e(t) +zs16*ta(t) +zs17*d(t) * (0) +1/h*K'(t); h=1
temp= [y ones (size (y)), y, yD, yP, R, QM, QMP, M, TX, e, ta, KP];
thy=arx (temp, [0, 1 1 1 1 1 1 1 1 1 1 1 0 0 1 0 0 0 0 0 0 0 0 0]),
yp=predict ([y, ones (size (y)), y, yD, yP, R, QM, QMP, M, TX, e, ta, KP], thy, 4);
yy=thy (11, 1:12);
```

```
%cy(t) = y10 +y11*y(t) +y12*y'(t) +y13*R(t) +y15*ML(t) +y18*YT(t)
temp=[C ones (size (y)), y, yP, R, ML, YT];
they=arx (temp, [0, 1 1 1 1 1 1 0 0 0 0 0 0 0]);
cyp=predict ([C, ones (size (y)), y, yP, R, ML, YT], they, 4);
ccy=they (5, 1:6);
```

```
%Iy(t) = I0 + 1/h*KP(t)
temp= [I ones (size (KP)), KP];
thIy=arx (temp, [0, 1 1 0 0]);
Iyp=predict ([I, ones (size (y)), KP], thIy, 4);
IIy=thIy (1, 1:2);
```

```
%Gy (t) = gs0 +gs1*y(t) +gs4*y'(t) +ys5*R(t) +gs8*QM(t)
temp= [G ones (size (y)), y, yP, R, QM];
thgy=arx (temp, [0, 1 1 1 1 1 1 0 0 0 0 0]);
gyp=predict ([G, ones (size (y)), y, yP, R, QM], thgy, 4);
```

```
ggy=thgy (4, 1:5) ;
```

```
%Xy(t) x0 +x1*y(t) +x2*y(t) +x5*R(t) +x8*QM(t) +x10*QM'(t) +x11*e(t) +x12*ta(t)
+x13*d(t) * (0)
```

```
temp= [X ones (size (y)), y, yD, R, QM, QMP, e, ta] ;
```

```
thxy=arx (temp, [0, 1 1 1 1 1 1 1 1 0 0 1 0 0 0 0 0] ) ;
```

```
xyp=predict ([X, ones (size (y)), y, yD, R, QM, QMP, e, ta], thxy, 4);
```

```
xxxy=thxy (7, 1:8) ;
```

```
%B(t) = b0 +b1*y(t) +b2*y(t-h) +b3*P(t) +b4*y'(t-h) +b5*R(t) +b6*R(t-h) +b7*e(t)
+b8*R'(t-h) +b9*QM(t) +b10*QM(t-h) +b12*L'(t-h) +b13*ta(t) +b15*d(t) +b17*B(t-h)
```

```
temp= [B ones (size (y)), y, yD, P, yPD, R, RD, e, RPD, QM, QMD, QMPD, ta, BD] ;
```

```
thb=arx (temp, [0, 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 1 0 1 0 1 0 1 0 1 1 0 1] );
```

```
Bp=predict ([B, ones (size (y)), y, yD, P, yPD, R, RD, e, RPD, QM, QMD, QMPD, ta,
BD], thb, 4) ;
```

```
BB=thb (13, 1:14) ;
```

```
%y(t) = A*K^a*L^(1-a) ==> ln (t)) = ln(A) + a*ln(K) + (1-a)*ln(L)
```

```
temp= [log(y) ones (size (y)), log(K), log(L) ] ;
```

```
thly=arx (temp, [0, 1 1 1 0 0 0] ) ;
```

```
lyp=predict ([log(y) ones (siz (y)), log (K), log (L)], thly, 4) ;
```

```
lyy=thly (2, 1:3) ;
```

```
%w = real wage rate per unit time
```

```
%m(w) = a(1-lyy(2)/(w))^(1/1yy(1)) % Commenting out for now. Please see
Alphacomputation.m
```

```
% program for estimation
```

```
procedure used.
```

```
%D(t) = a(1-c)*y(t) – k0*K(t) + k13*K'(t) + L4*R(t) + L5*QM(t) + L6*P(t) + v*y'(t)
```

```
temp= [D ones (size (y)), y, K, KP, R, QM, P, yP] ;
```

```
thd=arx (temp, [0, 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0] ) ;
```

```
Dp=predict ([D, ones (size (w)), y, K, KP, R, QM, P, yP], thd, 4) ;
```

```
DD=thd (7, 1:8) ;
```

```
%DP(t) = p0 + p1*Pfe(t) p2*w(t) – p3*n(t) – p4*P(t) + p5*M'(t) + p6*ML(t)
```

```
%M'(t) = direvative of money supply
```

```
temp= [DP ones (size (w)), Pfe, w, n, P, MP, ML] ;
```

```
thp=arx (temp, [0, 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0] ) ;
```

```
dlp=predict ([DP, ones (size (w)), Pfe, w, n, P, MP, ML], thp, 4) ;
```

```
dp=thp (6 ,1:7) ;
```

```
%-----
```

```
subplot (2, 1, 1), plot (year, C, year, Cp, '- -')
```

```
title ('BOTSWANA - - GOVERNMENT CONSUMPTION EXPENDITURE - -') ;
```

```
ylabel ('C') ;
```

```
xlabel ('year') ;
```

```

subplot (2, 1, 2), plot (year, X, year, Xp, '- -')
title ('BOTSWANA - - EXPORTS - -')
ylabel ('X') ;
xlabel ('year') ;
pause
clf
subplot (2, 1, 1), plot (year, I, year, Ip, '- -')
title ('BOTSWANA - - INVESTMENT - -')
ylabel ('I') ;
xlabel ('year') ;
subplot (2, 1, 2), plot (year, G, year, Gp, '- -')
title ('BOTSWANA - - GOVERNMENT CONSUMPTION - -') ;
ylabel ('G') ;
xlabel ('year') ;
pause
clf
subplot (2, 1, 1), plot (year, I, year, Ip, '- -')
title ('BOTSWANA - - INVESTMENT - -') ;
ylabel ('I') ;
xlabel ('year') ;
subplot (2, 1, 2), plot (year, G, year, Gp, '- -')
title ('BOTSWANA - - GOVERNMENT CONSUMPTION - -') ;
ylabel ('G') ;
xlabel ('year') ;
pause
clf
subplot (2, 1, 1), plot (year, QM, year, QMp, '- -')
title ('BOTSWANA - - MONEY DEMAND - -') ;
ylabel ('QM')
xlabel ('year') ;
subplot (2, 1, 2), plot (year, Z, year, [Xp+Ip+Gp+Cp], '- -')
title ('BOTSWANA - - AGGREGATE DEMAND - -') ;
ylabel ('Z')
xlabel ('year') ;
pause
clf
%-----

subplot (2, 1, 1), plot (year, P, year, dlp, '- -')
title ('BOTSWANA - - CONSUMER PRICES - -') ;
ylabel ('P') ;
xlabel ('year') ;

subplot (2, 1, 2), plot (year, y, year, yp, '- -')
title ('BOTSWANA - - INCOME - -') ;
ylabel ('GDP')
xlabel ('year') ;

```

```

pause
clf
subplot(2, 1, 1), plot(year, C, year, cyp, '--')
title('BOTSWANA- -INCOME/CONSUMPTION--');
ylabel('cy');
xlabel('year');
subplot(2, 1, 2), plot(year, I, year, Iyp, '--')
title('BOTSWANA- -INCOME/INVESTMENT--');
ylabel('Iyp');
xlabel('year');
pause
clf
subplot(2, 1, 1), plot(year, G, year, gyp, '--')
title('BOTSWANA- -INCOME/GOVERNMENT--');
ylabel('gyp');
xlabel('year');
subplot(2, 1, 2), plot(year, X, year, xyp, '--')
title('BOTSWANA- -INCOME/EXPORT--');

ylabel('xyp');
xlabel('year');
pause
clf
%-----
subplot(2, 1, 1), plot(year, B, year, Bp, '--')
title('BOTSWANA- -BALANCE OF PAYMENT--');
ylabel('B');
xlabel('year');
subplot(2, 1, 2), plot(year, y, year, lyp, '--')
title('BOTSWANA- -LOG(income)--');
ylabel('lny');
xlabel('year');
pause
clf
subplot(2, 1, 1), plot(year, D, year, Dp, '--')
title('BOTSWANS- -INCREASE IN STOCK-');
ylabel('D');
xlabel('year');
%-----

% The economic dynamics can be put in marix form as
% follows:  $x'(t) - A-1 * x'(t-h) = A0 * x(t) + A1 * x(t-h) + B * u(t)$ 
%
%  $x = [y \ R \ L \ K \ P \ E]'$ ;
%
%  $A-1 = [a-11 \ 0 \ a-13 \ 0 \ 0 \ 0; \ 0 \ a-22 \ 0 \ 0 \ 0 \ 0; \ L-03 \ 0 \ -L-01 \ 0 \ 0 \ 0; \ a3 \ 0 \ a6 \ a-1 \ 0 \ 0; \ 0 \ -M7p6P(t) \ 0 \ 0 \ 0 \ 0; \ -b4 \ -b8 \ -b12 \ 0 \ 0 \ -b14];$ 

```

```

%
% A0=[a01 a12 a14 a16 a18 0; a21 a23 0 0 a25 0; 0 0 a33 0 a35 0; 0 0 0
a44 a45 0; a51 a52 0 0 a55 0; a61 a62 a63 0 0 0];
%
% A1=[a111 a112 a113 0 0 a116; a121 a122 0 0 0 0; a131 a132 a133 0 0 0;
a141 a142 a143 0 0 0; a151 a152 a153 a154 a155 a156; a161 a162 a163 a164
a165 a166];
%
% q=[T g0 e ta d M M' f0]';
% S=[C0 I0 X0 (p0-p6M0) n w x0 y10]
%
% w = real wage rate per unit time
% m(w)=[1-1yy(2)/(w)]^(1/1yy(1))~89.9055 % See Alphacomputation.m
program for estimation procedure used.
% where w = average of w(t) = 164.4714

%
% B1=[-blz14 b1 -blz15 blz16 blz17 blz13 0 0; 0 0 0 0 0 -11 0 0; -m(w)
zs14 m(w) m(w) zs15 m(w) zs16 m(w) zs17 m(w) zs13 0 0; -zs14 1 zs15 zs16
zs17 zs13 0 0; 0 0 plpf(t) 0 0 p6 p5 0; 0 0 b7 b8 b15 0 0 1];
%
% B2=[ss1 ss1 ss1 -ss1 (I13+c7) 0 0 0 0 0; 0 0 0 -12 0 0 0 0 0; 0 m(2) 0 0
0 0 m m 0; 0 1 0 0 0 0 0 0 0; 0 0 0 -p6 -p3 p2 0 0 1; 0 0 0 -1 0 0 1 0 0]
%
% B=[B1 B2];
% u=[q S];
%
%-----
%
% The program BOTSWANA2.m was executed to achieve the following matrices
%
%-----
%
% Am1 =
1.0e+006 *
%
% Columns 1 through 4
%
% [-0.00001180884210 -0.00000000293417 -0.00000000293417 0
% 0 0.00000381795089 0 0
% 0.23540173115125 0 2.31874740331576 0
% 0.00261832436260 0 0.02579122398335 0.00002519068300
% 0 0.000000000000075 0 0
% 0.00000006018755 0.00022254979060 0.00000091481251 0
%
% Columns 5 through 6
%
% 0 0

```

%	0		0	
%	0		0	
%	0		0	
%	0		0	
%	0		0]	
%				
% A0 =				
% 1.0e+003 *				
%				
% Columns 1 through 4				
%				
% [0.04318742184766	-2.20528464608898	-0.00293026640772	0.00153456829427	
% -0.00000133319754	-0.00234524115650		0	0
%	0			0
%			0	
%			0	-3.46302065571323
% 0.00000000003027	0.00000005325629		0	0
% 0.00015811190951	0.11874229442062	0.00074729124196		0
%				
% Columns 5 through 6				
%				
% 0.04840971433568		0		
% -0.00002845889202		0		
% -0.01136934409781		0		
% -0.00012645884332		0		
% 0.00243971491714		0		
%	0	0]		
%				
%				
%				
% A1 =				
% 1.0e+010 *				
%				
% Columns 1 through 4				
%				
% [0.00000000013823	-0.00000037032083	-0.00000000022404		0
% 0.00000000000022	-0.00000000024441		0	0
% -0.000000001502881	-1.08670041780358	0.00000084419331		0
% -0.00000049031329	-0.01208714198853	0.00000035388001		0
% -0.000000000000000	0.000000000000001		0	0
% 0.00000000001706	0.00000002225498	-0.00000000020105		0
%				
% Columns 5 through 6				
%				
%	0		0	
%	0		0	
%	0		0	
%	0		0	
%	0		0	
%	0	0.00000000001624]		
%				
% B1 =				
%				


```

% 1.0e+007 *
%
% Columns 1 through 4
%
% [0.00000000225907    -0.00000000225907    -0.00000000023770    0.00000000043785
%          0          0          0          0
% 0.00000006282879    0.00000899054886    -1.76522868899960    0.00079399730420
% 0.0000000069883    0.00000010000000    -0.01963427056955    0.00000883146643
%          0          0    -0.00000000003033          0
%          0          0    0.00001187422944    0.00002225497906
%
% Columns 5 through 8
%
% -0.00000000029342          0          0          0
%          0    -0.00000009000000          0          0
%          0    -0.00000009745494          0          0
%          0    -0.00000000108397          0          0
%          0    0.000000000000003    0.000000000000003          0
% -0.00000002307468          0          0    -0.00000010000000]
%
%
% B2 =
%
% Columns 1 through 4
%
% [-0.02259074658698    -0.02259074658698    -0.02259074658698    -0.13847621107715
%          0          0          0    -0.900000000000000
%          0    89.90548860711905          0          0
%          0    1.000000000000000          0          0
%          0          0          0    0.00000026551946
%          0          0          0    -1.000000000000000
%
% Columns 5 through 8
%
%          0          0          0          0
%          0          0          0          0
%          0          0    89.90548860711905    89.90548860711905
%          0          0    1.000000000000000    1.000000000000000
% -0.38212257629773    0.00005482396119          0          0
%          0          0    1.000000000000000          0
%
% Column 9
%
%          0
%          0
%          0
%          0
% 1.000000000000000
%          0]
%
%
```

```
% q(t) = [T - 129.4490 e ta d M M' - 5.9641e+003]';
% S(t) = [-105.3554 -3.6586e+006 -560.7791 0.0038 n w -560.7791 -43.9296]
%
%-----
```

% It follows then that the differential equation from our regression analysis (arx) treated above can
% be rewritten with their missing constants included as:

```
%QM(t) = 77.0778 + (-0.0015)*y(t) + (0.0024)*y(t-h) + (-2.6058)*R(t) + (-2.7157)*R(t-h) +
(4.2422)*R'(t-h) + (-0.009316)*P(t);
%C(t) = -105.3554 + (-0.00220)*YT(t) + (0.00130)*YT(t-h) + (0.00540)*YT'(t) + (0.0018)*YT'(-
h) + (6.2165)*R(t) + (1.8798)*R(t-h) + (-3.4633e-004)*(a1+a2.*y(t) +a3.*yD(t-h) +a4.*R(t)
+a5.*RD(t) +a6.*P(t) + a.*RPD(t)) -M(t);
```

```
%I(t) = -3.6586e_006 + (9.8409)*y(t) + (-60.7588)*y(t-h) - (-45.2408)*y'(t) + (3.7142)*y'(t-h) +
(9.7400e+004)*R(t) + (1.6392e+005)*R(t-h) + (129.5614)*L(t) + (99.1740)*L(t-h) -
(67.9291)*K(t) - (-193.8511)*ML(t)
%X(t) = -560.7791 + (-0.0071)*y(t) + (0.0220)*y(t-h) + (0.0142)*y'(t) + (0.0048)*y'(t-h) +
(6.8764)*R(t) + (-0.1052)*L(t) + (0.1299)*L'(t-h) + (5.0129)*P(t) + (-0.0038)*ta(t) +
(50.2838)*e(t) + (0)*d(t)
%G(t) = -129.4490 + (-0.0174)*y(t) + (0.0195)*y(t-h) + (0.0054)*y'(t) + (-0.0087)*y'(t-h) +
(12.3622)*R(t) + (0.0196)*L(t)
%y(t) -4.1260e-024 +zsl*y(t) +zs2*y(t-h) +zs4*y'(t) +zs5*R(t) +zs8*L(t) +zs10*L'(t)
+zs13*M1(t) -zs14YT(t) +zs15*e(t) +zs16*ta(t) +zs17*d(t) * (0) +1/h*K'(t); h=1
%cy(t) = -43.9296 + (0.0066)*y(t) + (-0.0029)*y'(t) + (1.7161)*R(t) + (-0.0108)*ML(t) + (-
0.0070)*YT(t)
%ly(t) = 4.0962e+007 + (1)*KP(t)
%Gy(t) = -44.6035 + (0.0316)*y(t) + (9.1402e-004)*y'(t) + (-6.4610)*R(t) + (0.1291)*QM(t)
```

```
%Xy(t) = -5.5401e+006 + (94.8314)*y(t) + (-69.7089)*y(t) + (4.5674e+005)*R(t) + (-
390.7215)*QM(t) + (366.6796)*QM'(t) + (-1.9634e+005)*e(t) + (88.3147)*ta(t) + (0)*d(t)
%B(t) = (-6.5249e+003) + (0.1581)*y(t) + (0.1706)*y(t-h) + (45.4795)*P(t) + (0.0602)*y'(t-h) +
(118.7423)*R(t) + (222.5498)*R(t-h) + (-457.8390)*e(t) + (648.2353)*R'(t-h) + (0.7473)*QM(t) +
(-2.0105)*QM(t-h) + (0.9148)*L'(t-h) + (0.2307)*ta(t) + (0.1624)*d(t) + (0.1624)*B(t-h)
```

```
%D(t) = (-3.4630e+003) + (0.0280)*y(t) - (-0.3581)*K(t) = (1.2608)*K'(t) + (183.7397)*R(t) + (-
0.1265)*QM(t) + (26.0837)*P(t) + (-0.1046)*y'(t)
%DP(t) = 0.0038 + (-7.4178e-008)*Pfe(t) + (5.4824e-005)*w(t) - (-0.3821)*n(t) - (-7.5379e-
005)*P(t) + (3.0645e-007)*M'(t) + (2.6552e-007)*ML(t)
```

% Euclidean controllability condition for:

```
% x'(t) - A-1*x'(t-h) = A0*x(t) + A1*x(t-h) + B*u(t)
```

```
%
```

```
% B = [ B1 B2]
```

```
%
```

```
% Define for s = [sigma, t1]
```

```
%
```

```
% Q0 = {B s=h
```

```
% {0 otherwise
```

```
%
```

```
% Qk(s) = A0Qk-1(s-h) + Am1Qk(s-h) + A1Q(s-h), k=1, 2, 3, 4, 5, 6 .....n-1
```

```

%
%   k = 1, 2, 3, 4, 5, ..... n-1
%
% 1) while k=1, s=h,      Q1(0)      = A0*Q0(0) + Am1*Q1(0) + A1*Q(0) =
A0*B + Am1*B + A1*B
% 2) while k=1, s=not=h, Q1(s=not=h) = A0*Q0(s=not=h) + Am1*Q1(s=not=h) +
A1*Q(s=not=h) = 0
% 3) while k=2, s=h,      Q2(0)      = A0*Q1(0) + Am1*Q2(0) + A1*Q(0) =
A0*B + Am1*B + A1*B
% 4) while k=2, s=not=h, Q2(s=not=h) = A0*Q0(s=not=h) + Am1*Q1(s=not=h) +
A1*Q(s=not=h) = 0
% 5) while k=1, s=h,      Q3(0)      = A0*Q0(0) + Am1*Q1(0) + A1*Q(0) =
A0*B + Am1*B + A1*B
% 6) while k=1, s=not=h, Q3(s=not=h) = A0*Q0(s=not=h) + Am1*Q1(s=not=h) +
A1*Q(s=not=h) = 0
% 7) while k=2, s=h,      Q4(0)      = A0*Q1(0) + Am1*Q2(0) + A1*Q(0) =
A0*B + Am1*B + A1*B
% 8) while k=2, s=not=h, Q4(s=not=h) = A0*Q0(s=not=h) + Am1*Q1(s=not=h) +
A1*Q(s=not=h) = 0
%
%   :           :           :           :
%   :           :           :           :
%
% 13) While k=1, s=h      Q13(0)      = A0*Q12(0) + Am1*Q13(0) + A1*Q(0) =
A0*B + Am1*B + A1*B
% 14) while k=1, s=not=h, Q13(s=not=h) = A0*Q12(s=not=h) + Am1*Q13(s=not=h)
+ A1*Q(s=not=h) = 0
% 15) while k=2, s=h,      Q14(0)      = A0*Q13(0) + Am1*Q14(0) + A1*Q(0) =
A0*B + Am1*B + A1*B
% 16) while k=2, s=not=h, Q14(s=not=h) = A0*Q13(s=not=h) + Am1*Q14(s=not=h)
+ A1*Q(s=not=h) = 0
%
% Euclidean controllable if full rank:
% rank[ Qn(t1) =      n           % Note: Qn(t1) has a hipchan
% n = [Q(s) ..... Qn(s)]         s=[0 t], where n also has a hipchan
%
% c = [A0*B Am1*B A1*B , . . . , . . . ]
%
% rank(c) = 6
%
% Since Rank(B) is 6, and rank(c) is 6, the system is function space controllable.
% Euclidean controllable.
%
% Function space controllability condition for:
%
% x'(t) - A-1*x'(t-h) = A0*x(t) + A1*x(t-h) + B*u(t)
%

```

% We use the rank condition of Salamon[10, p. 480] and the full rank of B to deduce the

% controllability of the Botswana linear neutral model:

%

% Function space $W^{1,p}$ controllability is equivalent to rank

%

% $\text{rank}[\Delta(j), B] = n?$

% $\Delta(j) = \text{eye}(\text{size}(A_1)) \cdot j - A_{m1} \cdot (j \cdot 2.7814^{(-j)}) - A_0 - A_1 \cdot (2.7814^{(-j)})$

%

% $cc = [\Delta(j) \ B]$

%

% For all complex number, j.

% $\text{rank}(cc) = 6$

%

% For nonzero eigenvalues of A-1 to be controllable via matrix B,

% $GG = \text{eye}(\text{size}(A_{m1})) \cdot j - A_{m1}$

% $G = [GG \ B]$

%

% For all j =

% $\text{rank}(c) = 6$

% For $y(t) = A \cdot K^a \cdot L^{(1-a)}$

a = -0.84378439 % Initial estimates of a = alpha, see Alphacomputation.m program

aa = 0.21252776 % Initial estimates of aa = (1-a) = beta, see Alphacomputation.m program

W = sum(w)/14

mw = (aa/W.^(1/a))

% Cobb Douglas states: $aa+a = 1 \Rightarrow$ constant return to scale

% $aa+a > 1 \Rightarrow$ increasing return to scale

% $aa+a < 1 \Rightarrow$ decreasing return to scale

%

% Increasing return to scale if doubling L & K more than doubles y.

% constant return to scale if doubling L & K exactly doubles y

% decreasing return to scale if doubling L & K less y.

%

% In Alphacomputation.m program log(A) is found to be 14.85432768 to give the

% best approximation for y(t).

%

% Hence the dynamics which we seek follows that:

%

% $L'(t) - \text{lm01} \cdot L'(t-h) - \text{lm03} \cdot y'(t-h) = \text{l0} \cdot L(t) - \text{l1} \cdot L(t-h) - \text{l2} \cdot y(t-h) + \text{l4} \cdot R(t-h) + \text{l5} \cdot L(t-h) + \text{sigma } 3(t) + q3(t)$

% where as given by the "mw" computation and regression analysis above:

```

%
% lm01 = am1-m(w)*a6 = 1.2959e+006
% l03 = a3*m(w) = -6.6218
% 10 = a0 = -3.4630e+003
% 11 = -a1 = ? TBD below
% 12 = a2*m(w) = ? TBD below
% 14 = a4*m(w) = -2.5031e+009
% 15 = a5*m(w) = 1.5149e+009
%
% sigma3(t) = m(w)*sigma4(t) = 89.9055*(x0 + y10 + I0) =
89.9055*(-560.7791 + -43.9296 + -3.6586e+006)
% q3(t) = m(w)*q4(t) = 89.9055*(g0 + zs13*M(t) + zs14*T(t) + zs15*e(t) +
zs16*ta(t) + zs17*d(t))
%
% OR
%
% sigma3(t) = m(w)*sigma4(t) = 89.9055*(-560.7791 + -43.9296 + -3.6586e+006) = -
3.2898e+008
% q3(t) = m(w)*q4(t) = 89.9055*(-129.4490 + -0.0108*M(t) + -0.0070*T(t) +
366.6796*e(t) + -1.9634e+005*ta(t) + zs17*d(t))
%
% Please note that:
%
% d(t) = distance = 0
%
%
% We extract the following equation of the economic state:  $x = [y, R, L, K, P, E]'$ .
%
%  $dy/dt + (-11.8088)*y'(t-h) + (-0.0029)*L'(t-h) = (43.1874)*y(t) + (1.3823)*y(t-h) +$ 
 $(-2.2053e+003)*R(t) + (-3.7032e+003)*R(t-h) + (-29303)*L(t) + (-2.2404)*L(t-h) +$ 
 $(1.5346)*K(t) - (48.4097)*p(t) + q1(t) + R1(t)$ 
%
% Where,
%  $q1(t) = (-44.2659)*(-129.4490) - (3.4633e-004)*T(t) + (0.0013)*T(t-h) -$ 
 $(0.0054)*T'(t) - (0.0018)*T(t-h) - (50.2838)*e(t) + (-0.0038)*tar(t) + 0*d(t)$ 
%
% The coefficients are displayed in equations (6.1.12), (6.1.15) of “Optimal control of
the growth of Nations” by E. N. Chukwu, pp261-263.
%
%  $R'(t) - (0.0022)*R'(t-h) = (-0.0013)*y(t) + (0.0022)*y(t-h) + (-2.3452)*R(t) + (-$ 
 $2.4441)*R(t-h) + (-0.0285)*p(t) - sig2(t) + q2(t)$ 
%
% Note:  $q2(t) = -12*M1(1)$ ,  $sig2(t) = 12*M0 = 2.9270e+008$ ,
%
% The coefficients were derived using:
%
%  $a-22=12*M5$ ,  $a21=12*M1$ ,  $a22=12*M2$ ,  $a24=12*M3$ ,  $a25=12*M6$ , etc.

```

%

Also from p264, p262 equati0n (6.1.21) and equation (6.1.9):

%

$$p'(t) = p(t) * [(4.2422) + (-7.5379e+005)] * p(t) - (-2.6552e-007) * (77.0778) * y(t) - (-2.6552e-007) * y(t-h) - (-2.6552e-007) * (0.0024) * R(t) - (-2.6058) * (-2.6552e-007) * R(t-h) - (-2.6552e-007) * (-0.0316) * R'(t-h) + [q5(t) + sig3(t)] * p(t)$$

%

Where,

%

$$q5(t) = (-7.4178e-008) * pf(t) * e^t + (3.0645e-007) * M1'(t) + (0.0038) * (M1(t))$$

$$sig5 = (0.0038) - (0.3821) * n(t) + (5.4824e-005) * w(t) - (-2.6552e-007) * M0(t)$$

%

%

From p265, p264 equation (6.1.36):

%

$$E'(t) = b1 * y(t) + b2 * y(t-h) + b4 * y'(t-h) + b5 * R(t) + b6 * R(t-h) - b8 * R'(t-h) + b9 * L(t) + b10 * L(t-h) + b12 * L'(t-h) + b17 * B(t-h) - r6 + q6(t)$$

%

$$E'(t) = (0.1581) * y(t) + (0.1706) * y(t-h) + (45.4795) * P(t) + (0.0602) * y'(t-h) + (118.7423) * R(t) + (222.5498) * R(t-h) + (648.2353) * R'(t-h) + (0.7473) * QM(t-h) + (-2.0105) * QM(t-h) + (0.9148) * QM'(t-h) + (0.1624) * B(t-h) - (-560.7791) + (-457.8390) * e(t) + (-0.2307) * ta(t) + (0) * d(t) - (-5.9641e+003)$$

%

—> see equation (6.1.44)

%

Where:

%

$$-r6(t) = x0 = (-560.7791)$$

%

q6(t) = b7 * e(t) + b8 * ta(t) + b15 * d(t) - f0; b7, b8, b15 are given in the above equation. And f0 = -5.9641e+003 —> equation (6.1.43)

%

$$K'(t) + (25.1907) * K'(t-h) - (2.6183e+003) * y'(t-h) - (2.5791e+004) * L'(t-h) = (-3.4630e+003) * K(t) - (4.9031e+003) * K(t-h) + (-4.9031e+003) * y(t-h) + (-1.2405e+008) * R(t-h) + (3.5388e+003) * L(t-h) + (-0.1265 * p(t) + sig4(t) + q4(t)$$

%

equation (6.1.57)

%

Where,

%

$$sig4(t) = x0 + y0 + I0 = -560.7791 + (-43.9296) + (-3.6586e+066) = -3.6592e+006$$

%

$$q4(t) = (-129.4490) + -0.0108 * M(t) + (-0.0070) * T(t) + (-1.9634e+005) * e(t) + (88.3147) * ta(t) + zs17 * d(t)$$

%

$$L'(t) - (-2.3187e+006) * L'(t-h) - (2.3540e+005) * y'(t-h) = -3.4630e+003 * L(t) - (-4.9031e+003) * L(t-h) + (4.4082e+005) * y(t-h) + (-1.1153e+010) * R(t-h) + (3.1816e+005) * L(t-h) + sig3(t) + q3(t)$$

%

$$sig3(t) = -m(w) * sig4(t) = -89.9055 * (-560.7791 + -43.9296 + -3.6586e+006) = +3.2898e+008$$

```
%      q3(t)=m(w)*q4(t)   = 89.9055*(g0 + zs13*M(t)  zs14*T(t) + zs15*e(t) +
zs16*ta(t) + zs17*d(t)
%      q3(t)=m(w)*q4(t)   = 89.9055*(-129.4490)  +  -0.0108*M(t)  +  (-
0.0070)*T(t) + (-1.9634e+005)*e(t) + (88.3147)*ta(t) + zs17*e(t)
%
%
%      m(w) = [(1-a)^1/w]^1/a = ~ 89.9055
%
%      Note:
%
%      x=[y R L K p E]’, While: A-1, A0, and A1 is as defined above
%
%      Is det(A-1) = not = 0? No, The determinant of Am1 is 0.
```

```
format short;
```

```
% Note: L1(lamda1) is the speed of response of supply to demand, the speed of
adjustment.
```

```
L1 =1
L2 =0.9
L5 =DD(5)
L6 =DD(6)
h =1
z0 = CC(1) + II(1) – QMM(1)*(CC(8) + II(11)) + CC(1) + XX(1)
z1 = GG(2) + II(2) – QMM(2)*(CC(8) + II(11)) + CC(2) + XX(2)
z2 = GG(3) + II(3) – CC(3) + XX(3) + QMM(3)*(CC(8) + II(11))
z3 = GG(4) + II(4) + CC(4) + XX(4)
z4 = GG(5) + II(5) + CC(5) + XX(5) + QMM(5)*(CC(8) + II(11))
z5 = GG(6) + II(6) + CC(6) + XX(6) – CC(8)*QMM(6)
z6 = II(7) + CC(7) – CC(8)*QMM(7)
z8 = GG(7) + II(8) + XX(7)
z9 = II(9)
z10 = XX(8)
z11 = -II(10)
z13 = (II(11) + CC(8))*QMM(7)
z14 = -CC(8)

z15 = XX(11)
z16 = XX(10)
z17 = 0
z18 = (CC(8) + z13)*QMM(7)
z19 = CC(3)
z20 = CC(4)
z21 = CC(5)
zs0 = ccy(1) + Ily(1) + ggy(1) + xxy(1) ; %constant term
zs1 = ccy(2) + ggy(2) + xxy(2) ; %y(t) term
zs2 = xxy(3) ; %y(t-h) term
```

```

zs4 = ccy(3) +          ggy(3)          ; %y'(t) term
zs5 = ccy(4 + xxy(4) +  ggy(4)          ; %R(t) term
zs8 = xxy(5) +          ggy(5)          ; %L(t) term
zs10 = xxy(6)           ; %L'(t) term
zs13 = ccy(5)           ; %ML(t) term
zs14 = ccy(6)           ; % T(t) term
zs15 = xxy(7)           ; %e(t) term
zs16 = xxy(8)           ; % ta(t) term

```

```

s1 = 1-L1*(GG(4) - II(4) + CC(4) + XX(4))

```

```

%assumptions

```

```

yy1  = zs0
yy2  = zs1
yy3  = zs2
yy4  = zs4
yy5  = zs5
yy6  = zs8
yy7  = zs10
yy8  = zs13
yy9  = zs14
yy10 = zs15
yy11 = zs16

```

```

zs0  = GG(1) + XX(1) + CC(1) + II(1)
a0   = DD(1)/(h*(1-yy(2)))
a1   = -DD(1)*yy3/(1-yy2)*h)
a2   = -a1*h
a3   = a0*h*yy4
a4   = DD(1)*yy5/((1-yy2) + DD(4))
a5   = DD(1)*yy6/(1-yy2) + DD(5)
a6   = a0*h*yy7
a8   = DD(6)

```

```

%A-1 starts here; am11 = a-11

```

```

am11 = L1*(GG(5) - II(5) + CC(5) + XX(5) + QMM(5)*(CC(8) + II(11)))/s1
am22 = L2*QMM(6)
am13 = L1*z10/s1
am14 = L1*z10/s1
Lm03          = DD(1)*yy4*mw/(1-yy2)
lm01 = -DD(3)*DD(1)/(1-yy2) - mw*DD(1)*yy7/(1-yy2)
a3   = DD(1)*yy4/(1-yy2)
a6   = DD(1)*yy7/(1-yy2)
am1  = -DD(3)*DD(1)/(1-yy2)
m7p6 = QMM(7)*dp(6)*sum(P)/21
b4   = BB(5)
b8   = BB(9)

```


b12 = BB(12)

b13 = 0

%A0 Starts here

a01 = (L1*(GG(2) + II(2) - QMM(2))*(II(11) + CC(8)) + CC(2) + XX(2) + CC(8) - 1 - (II(11) + CC(8))*QMM(7))/s1

a12 = L1*(GG(6) + II(6) + CC(6) + XX(6) - CC(8)*QMM(7)) - (II(11) + CC(8)) - QMM(3)*QMM(7)/s1

a14 = L1*(GG(7) + II(8) + XX(8))/s1

a16 = -L1*(II(10))/s1

a18 = -L1*(QMM(4) * QMM(6) * (II(11) + CC(8)))/s1

a21 = L2*QMM(2)

a23 = L2*QMM(4)

a25 = L2*QMM(7)

a33 = a0

a35 = mw*a8

a44 = a0

a45 = a8

a51 = -dp(7) * QMM(2) * sum(P)/21 %a51*P(t)

a52 = -dp(7) * QMM(4) * sum(P)/21 %a52*P(t)

a55 = -(QMM(7) + dp(5)) * sum(P)/21 %a55*P(t)

a61 = BB(2)

a62 = BB(6)

a63 = BB(10)

%A1 Starts here!

a11 = L1*(GG(3) + II(3) + CC(3) + XX(3) + QMM(3) * (II(11) + CC(8)))/s1

a13 = L1*(II(7) + CC(7) - CC(8) * QMM(7))/s1

a15 = L1*(II(9))/s1

a22 = L2*QMM(3)

a24 = L2*QMM(5)

a1 = -DD(1) * yy3/((1-yy2) * h)

a2 = -a1*h

l1 = -a1

l2 = -mw*DD(1) * yy3/(1-yy2)

l4 = mw*DD(1) * yy5/(1-yy2 + DD(4))

b2 = BB(3)

b6 = BB(7)

b10 = BB(11)

b17 = BB(13)

a111 = L1*(GG(3) + II(3) + XX(3) + QMM(3) * (II(11) + CC(8)))/s1

a112 = a13

a113 = a15

a114 = 0

a116 = 0

a121 = a22

```

a122 = a24
a131 = mw*DD(1) *XX(3)/(1-(XX(2) + ggy(2) + ccy(2)))
a132 = mw*DD(1) * yy5/(1-yy2) - DD(4)
a133 = a5-11*h
a141 = DD(1) *yy3/(1-yy2)
a142 = DD(1) *yy5/(1-yy2) + DD(4)
a143 = DD(1) *yy6/(1-yy2) = DD(5)
a144 = 0
a145 = 0
a146 = 0
a151 = -dp(7) * sum(P)/21          %a151*P(t)
a152 = -dp(7)*QMM(5) * sum(P)/21   %a152*P(t)
a161 = BB(3)
a162 = BB(7)
a163 = BB(11)
a164 = 0
a165 = 0
a166 = BB(14)

```

```

% B1          Styarts here!
ss1           = L1/s1
ss1z13        = ss1*(II(11) + CC(8)) * QMM(7)
ss1z15        = -ss1*XX(7)
ss1z16        = ss1*XX(11)
ss1z17        = ss1*XX(8)
ss1z18        = ss1*z18
b1z14         = -BB(2) * z14
b1            = BB(1)
b1z15         = BB(2) * z15
b1z16         = BB(2) * z16
b1z17         = BB(2) * z17
b1z13         = BB(2) * z13
mwzsl4        = mw*yy9

```

```

mwzsl5 = mw*yy10
mwzsl6 = mw*yy11
mwzsl7 = 0
mszsl3 = mw*yy8
p1pf    = dp(2) * sum(Pfe)/21
P4       = dp(5)
P5       = dp(6)
b7       = BB(6)
b8       = BB(7)
b15      = BB(13)

```

```

%B2    Starts here!

```

```

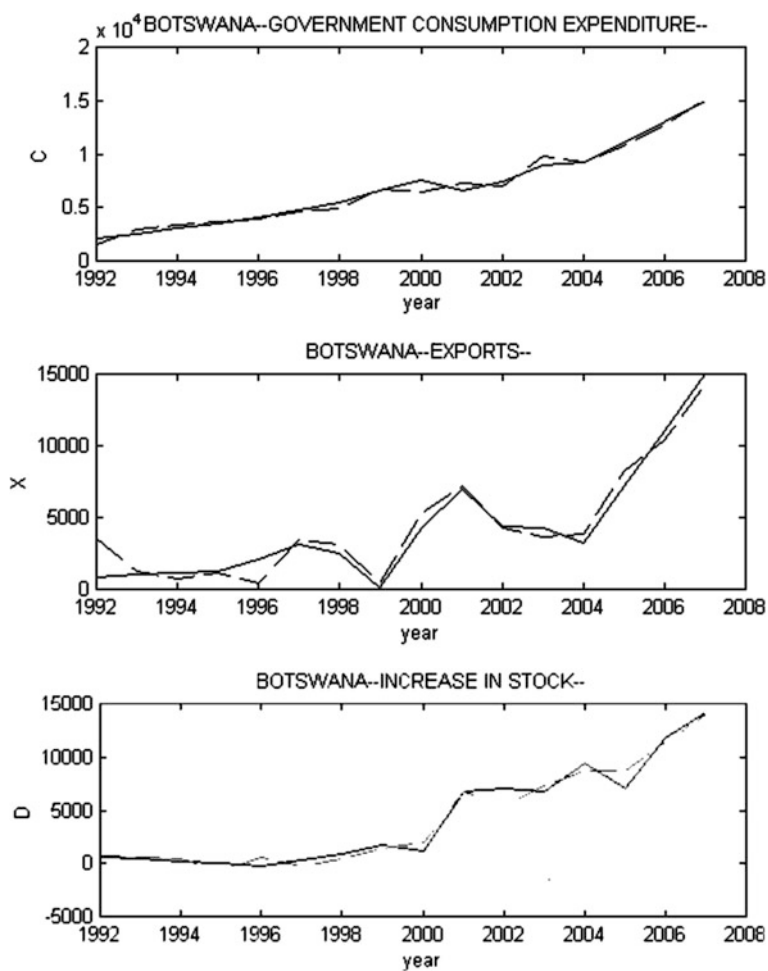
ss2      = -ss1*(II(11) + CC(8))
L2       = 0.9
P6       = -dp(7)
P3       = -dp(4)
P2       = dp(3)

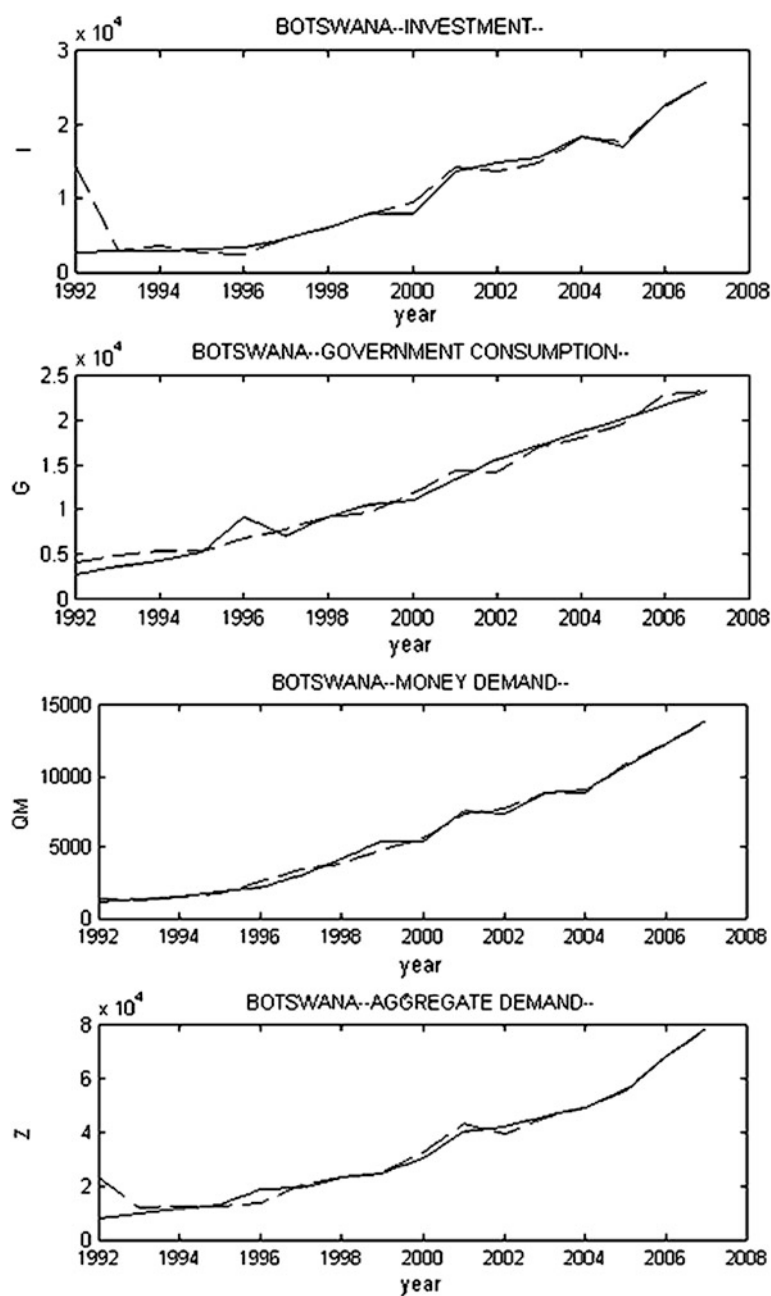
% ln(y) = a*ln(K) + (1-a) * ln(L)
% lyy(1)      = a ;
% lyy(2)      = 1-a
a17       = L1*CC(2)/s1
a45       = a8
c0        = CC(1)
I0        = II(1)
x0        = XX(1)
p0        = dp(1)
qm0       = QMM(1)
y10       = ccy(1)
p0p6m0    = dp(1) - dp(7) * QMM(1)
g0        = GG(1)
f0        = BB(1) - XX(1)
g2        = -L2*QMM(1)
a25       = L2*QMM(7)

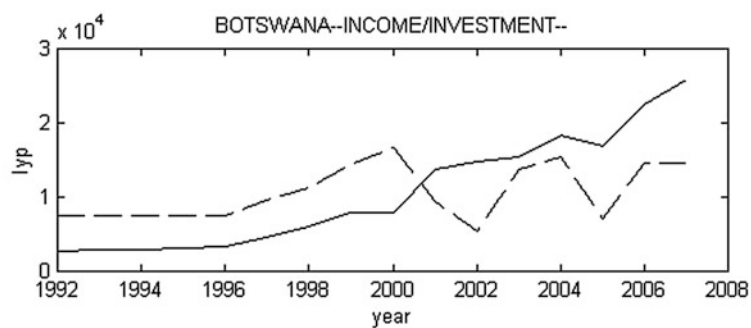
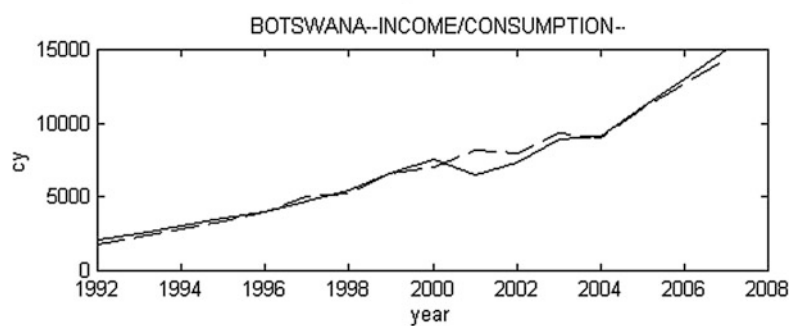
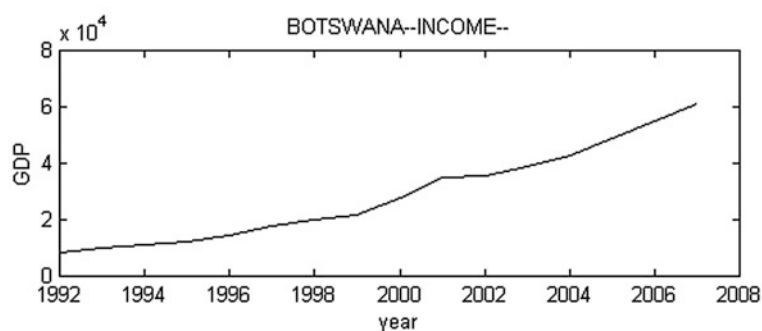
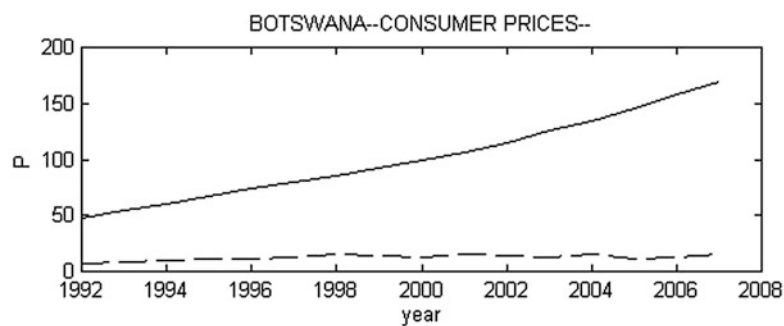
a115      = 0 ;
a124      = 0 ;
a125      = 0 ;
a126      = 0 ;
a134      = 0 ;
%-----

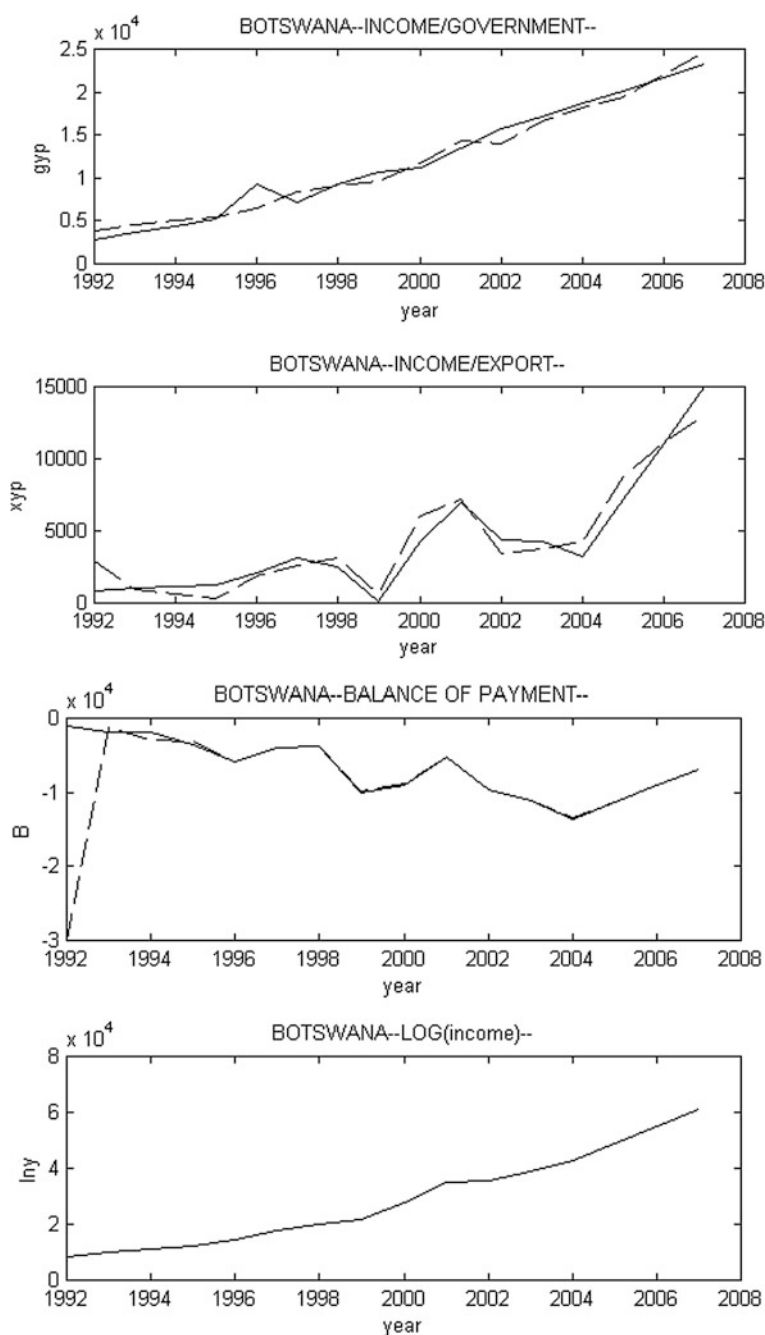
```

The following diagrams are the graphical results from the Botswana2.m regression analysis program. The plots shown give the original raw data in solid lines while the estimated results from the regression are given in dash lines (—).









```

% Botswana2.m results
% The results given below is the screen capture while
% running Botswana2.m program. NDU.m can be used to data mine the results
to form
% matrices of A-1, A0, B1 and
» Botswana2
MM(1) = -2.2520e+003, MM(2) = -6.7949e-004, MM(3) =13.3576
MM(4) = 42.0128, CC(1) =1.2557e+007, CC(2) = 82.2923, CC(3)= 52.9686
CC(4)= -9.9897e+005, CC(5)=-114.3026
II(1)= 786.4825,II(2)=0.0160, II(3)=0.0262, II(4)=26.4893, II(5)=-5.3238
II(6)= -0.0444, II(7)=-0.0075, XX(1)= 1.0487e+006,XX(2)=-53.4008,
XX(3)= -46.5804, XX(4)=6.1809e+005, XX(5)= -5.1687e+004,
XX(6)=5.3602e+004
XX(7)=60.9531, XX(8)= -6.0095e+004, GG(1)=1.6587e+008,GG(2)=1.7333e+003
GG(3)= 1.4062e+003,GG(4)=3.3090e+006, GG(5) -1.0093e+006, yy(1)=
4.8479e-024
yy(2)= -4.6685e-028, yy(3)= -1.1681e-027,yy(4)=9.6864e-025,yy(5)= -
7.7817e-026
yy(6)=-8.5271e-026, yy(7)=1.5957e-027, yy(8)=1.2827e-027,yy(9)= -5.0421e-
025
yy(10)=4.7506e-028,yy(11)= 2.9657e-028, ccy(1)=40.6354,ccy(2)=0.0213,
ccy(3)=-0.0216, ccy(4)= -5.2669, ccy(5)=0.0257,ccy(6) -0.0330, IIy(1) =
2.7790e+007
IIy(2)=1, ggy(1)=1.7333e+003, ggy(2)=0.0200, ggy(3)= 0.0149, ggy(4)=
54.1316
ggy(5)=-11.7798, ggy(6)= 0, xxy(1)= 1.6163e+006, xxy(2)= 25.2768,
xxy(3)= 2.4977e+005, xxy(4)= -2.1994e+004, xxy(5)= -2.7553e+003, xxy(6)-
5.5496e+004
xxy(7)=36.2100, BB(1)=2.5376e+005, BB(2)=-13.7728, BB(3)= 1.2710e+004
BB(4)= 5.6231e+004, BB(5)=3.1753e+003,BB(6)=-7.1359e+003,BB(7)= -6.4054
lly(1)= 0.0096, lyy(2)= 20.0800, lyy(3)=8, DD(1)=-398.6842, DD(2)=
0.2604
DD(3)=-0.5137, DD(4)=1.4195, DD(5)=-495.8289, DD(6)=55.7444, DD(7)=-
126.6473
DD(8)= 0.1193, dp(1)=-0.0384, dp(2)= 6.6795e-007, dp(3)= 1.1159e-004
dp(4)= 0.1084, dp(5)= 3.8057e-004, dp(6)= -3.7521e-006, dp(7)=-8.1385e-
006
s1=-3.9271e+006, l1 = 0, l2 = -1.3477e-005, l4 = 3.5750, aaal1 =
4.2465e+013
aaal2 =-1.1499e+013, aaal3 = 3.9633e+012, aaal4 = -1.7427e+005,
aaal5 = -2.5192e+011, aaal6 = -2.3600e+011, aaal21 = -2.0268e+003, aaal22
=-12.0218
aaal25 = 37.8115, aaal33 =16.3945, aaal44 =16.3945, aaal45 = 55.7444,
aaal46 = -5.5496e+004, aaal51 = -0.0183, aaal52 =1.0871e-004, aaal55 =
42.0132,
aaal56 = 6.6795e-007, aaal61 = -13.7728, aaal62 = 5.6231e+004, aaal63 = -
7.1359e+003
aaal66 = 3.1753e+003, bbb11 = -114.3026,bbb13 = 6.0095e+004, bbb14 =
60.9531,
bbb26 = 1.3477e-005, bbb33 = 6.6795e-007, bbb36 = -8.1385e-006,
bbb37 = -3.7521e-006, bbb43 = 3.1753e+003, bbb44 = 0, bbb45 =-6.4054,
bbb53 = -5.5496e+004,bbb54 =36.2100, bbb55 = 0, bbb56 = 0.0257,
bbb61 = 2.7136e-008, bbb62 = 8.2207e-007, bbb63 = -0.0456, bbb64 =
2.9767e-005
bbb65 = 0, bbb66 = 0.0257, ccc24 =1.3477e-005,ccc32 = 8.2207e-007, ccc55
=-0.1084,
ccc56 = 1.1159e-004, c0 =1.2557e+007, I0 =786.4825, x0 =1.0487e+006, p0 =-
0.0384,
M0 = -2.2520e+003, y10 = 40.6354, p0p6m0 = -0.0567, g0 =1.6587e+008,
f0 = -7.9495e+005, g2 = 2.0268e+003

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Chapter 8

The Hereditary Model of GDP and Employment

8.1 Derivation

In this section we derive a hereditary model of gross domestic product, y , and employment L by invoking the “demand and supply principle”, and the “rational expectations hypothesis” which assumes that the expected values of economic variables are functions of the current and past values. Indeed let the aggregate demand function z_i be given by

$$z_i = I_i + C_i + X_i + G_i, \quad i = 1, 2, \quad (8.1)$$

where

I_i is investment
 C_i is consumption,
 X_i is net export,
 G_i is government intervention.

Here,

$$\begin{aligned} C_i = & C_{0i} + C_{1i}(y_i(t) - T_i(t)) + C_2(y_i(t-h) - T_i(t-h)) \\ & + C_3(\dot{y}_i(t) - \dot{T}_i(t) + C_4(\dot{y}_i(t-h) - \dot{T}_i(t-h))). \end{aligned} \quad (8.2)$$

Also

C_i = private consumption,
 T_i = Taxes,
 $y_i - T_i$ = after tax income,
 y_i = GDP,

$$I_i = I_{0i} + I_{1i}y_i(t) + I_{2i}y_i(t-h) - I_{3i}\dot{y}_i(t) + I_{4i}\dot{y}_i(t-h) + I_{8i}L_i(t) + I_{9i}L_i(t-h), \quad (8.3)$$

$$X_i = \text{Ex} - \text{Im} = \text{Export} - \text{Import};$$

$$X_i = X_{0i} + X_{1i}y_i(t) + X_{2i}y_i(t-h) + X_{3i}\dot{y}_i(t) + X_{4i}\dot{y}_i(t-h) + X_{8i}L_i(t) + X_{10i}L_i(t-h) + X_{15i}e_i(t) + X_{16i}\tau_i(t) + X_{17i}d_i(t) + y_i(t)[a_{1i}y_{i+1}(t-h)] + L_i(t)[C_{1i}L_{i+1}(t-h)]. \quad i = 1, 2 \quad (8.4)$$

$$(\text{Assume } y_3 = y_1; L_3 = L_1)$$

The last two terms

$$y_1(t)[a_{11} y_2(t-h) + L_1(t)[c_{11} L_2(t-h)],$$

and

$$y_2(t)[a_{12} y_1(t-h)] + L_2(t)[C_{12} L_2(t-h)], \quad (8.5)$$

are the contribution of interaction between country of y_1 and the country of y_2 due to cooperation and competition measured in terms of trade surplus, trade deficit, debt repayment, grants, debt relief and international employment, etc.

Here τ = tariff, d = distance between trading nations and/or trade policy.

$$G_i = g_{0i} + g_{1i}y_i(t) + g_{2i}y_i(t-h) + g_{3i}\dot{y}_i(t) + g_{4i}L_i(t-h). \quad (8.6)$$

$$(g_{0i} = \text{federal budget autonomous net expenditure}).$$

Thus, gathering the formulae for $z_i = I_i + C_i + X_i + G_i$, and invoking the demand and supply principle,

$$\frac{dy_i(t)}{dt} = \lambda_i(z_i(t) - y_i(t)), \quad (8.7)$$

and invoking Eq. (6.1.66, 6.1.67) of Chukwu [1] with $R \equiv 0$, $k = 0$, $p = 0$, and with q_1, v , defined in (6.1.16, 6.1.17) we deduce the equation,

$$\begin{aligned}
& \begin{bmatrix} \dot{y}_1(t) \\ \dot{L}_1(t) \\ \dot{y}_2(t) \\ \dot{L}_2(t) \end{bmatrix} + \begin{bmatrix} -a_{-11} & -a_{-13} & 0 & 0 \\ -\ell_{-031} & -\ell_{011} & 0 & 0 \\ 0 & 0 & -a_{-22} & -a_{-23} \\ 0 & 0 & -\ell_{022} & -\ell_{-022} \end{bmatrix} \begin{bmatrix} \dot{y}_1(t-h) \\ \dot{L}_1(t-h) \\ \dot{y}_2(t-h) \\ \dot{L}_2(t-h) \end{bmatrix} \\
&= \begin{bmatrix} a_1 & a_{14} & 0 & 0 \\ 0 & \ell_0 & 0 & 0 \\ 0 & 0 & b_{01} & b_{14} \\ 0 & 0 & 0 & \ell_{02} \end{bmatrix} \begin{bmatrix} y_1(t) \\ L_1(t) \\ y_2(t) \\ L_2(t) \end{bmatrix} \\
&+ \begin{bmatrix} a_{11} & a_{15} & 0 & 0 \\ \ell_2 & -\ell_1 & 0 & 0 \\ 0 & 0 & b_{11} & b_{25} \\ 0 & 0 & \ell_{32} & -\ell_{12} \end{bmatrix} \begin{bmatrix} y_1(t-h) \\ L_1(t-h) \\ y_2(t-h) \\ L_2(t-h) \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 0 & a_{11}y_1(t) & 0 \\ 0 & 0 & 0 & c_{11}L_1(t) \\ a_{12}y_2(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{12}L_1(t) \end{bmatrix} \begin{bmatrix} y_1(t-h) \\ L_1(t-h) \\ y_2(t-h) \\ L_2(t-h) \end{bmatrix} \\
&+ B_1p + B_2g.
\end{aligned} \tag{8.8}$$

The control strategies associated with government are

$$q = [q_1 \quad q_2],$$

and the control strategies associated with the private firms are

$$\sigma = [r_1 \quad r_2].$$

If $[y_1, \quad L_1, \quad y_2, \quad L_2]$ is the prevailing economic state, then

$$\begin{aligned}
q_1 = & \lambda_1 \sigma_1^{-1} [g_{01}, z_{g13} M_1, -z_{141} T_1(t) - z_{191} T_1(t-h) - z_{01} \dot{T}_1(t) \\
& - z_{211} \dot{T}_1(t-h) - z_{g15} e_{11}(t) + z_{161} \tau_{11}(t) + z_{g71} d_1(t)],
\end{aligned}$$

and

$$r_1(t) = \lambda_1 \sigma_1^{-1} [(C_{01} + I_{o1} + X_{01}) - M_{01}(I_{151} + C_{71})]$$

are associated with y_1 . The strategy which is related to L_1 is

$$\sigma_{31}(t) = m_1(w)[x_{01} + y_{101} + I_{01}].$$

Similarly the government strategy associated with y_2 is

$$q_2(t) = \lambda_2 \sigma_2^{-1} [g_{02} + z_{s23} M_2 - z_{s24} T_2(t) - z_{292} T_2(t-h) - z_{02} \dot{T}_2(t) \\ - z_{212} \dot{T}_2(t-h) - z_{25} e_{11}(t) - z_{26} T_2(t) + z_{27} d_2(t)].$$

The representative private firm's strategy associated with L_2 is

$$r_{32}(t) = -M_2(w)[x_{02} + y_{102} + I_{02}].$$

The matrices B_1 and B_2 in (5.8) are identified from the equations $q_1, r_1, q_2, r_2, \sigma_{31}$, and $\sigma_{32}(t)$.

$$\dot{x}(t) - A_{-1}\dot{x}(t-h) = A_0x(t) + A_1x(t-h) + A_2(x(t))x(t-h) + B_1p + B_2q, \quad (8.9)$$

where

$$\begin{aligned} A_{-1} &= \begin{bmatrix} -a_{-11} & -a_{13} & 0 & 0 \\ -L_{-031} & -L_{-011} & 0 & 0 \\ 0 & 0 & a_{-22} & -a_{-23} \\ 0 & 0 & -L_{-022} & -L_{032} \end{bmatrix}, \\ A_0 &= \begin{bmatrix} a_1 & a_{14} & 0 & 0 \\ 0 & \ell_0 & 0 & 0 \\ 0 & 0 & b_0 & b_{14} \\ 0 & 0 & 0 & \ell_{02} \end{bmatrix}, \\ A_1 &= \begin{bmatrix} a_{11} & a_{15} & 0 & 0 \\ \ell_2 & -\ell_1 & 0 & 0 \\ 0 & 0 & b_{11} & a_{25} \\ 0 & 0 & \ell_{32} & -\ell_{12} \end{bmatrix}, \\ A_2(x(t)) &= \begin{bmatrix} 0 & 0 & a_1 y_1(t) & 0 \\ 0 & 0 & 0 & c_{11} L_1(t) \\ a_{12} y_2(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{12} L_1(t) \end{bmatrix}, \end{aligned} \quad (8.10)$$

In [2, p. 27], the derivation of Employment and capital stock equations assumes that national income from the expenditure side in a nation 1 is given in Eq. (1.44) by

$$y_1 = \bar{C}_1 + \bar{I}_1 + \bar{X}_1 + \bar{G}_1,$$

where

$$\begin{aligned}\bar{X}_1 = & x_0 + x_1 y_1(t) + x_2 y_1(t-h) + x_5 R_1(t) + x_8 L_1(t) + x_{10} \dot{L}_1(t) \\ & + x_{11} e_1(t) + x_{12} \tau_1(t) + x_{13} d_1(t).\end{aligned}$$

If we insert the effects of interaction with nation 2 to \bar{X}_1 , we add

$$y_1(t)[d_{21}y_2(t-h)] + L_1(t)d_{22}L_2(t-1).$$

The interaction matrix term introduced is thus

$$A(x)x(t-1) = \begin{bmatrix} 0 & 0 & a_{15}y_1(t) & c_{16}L_1(t) \\ 0 & 0 & d_{11}y_1(t) & d_{12}L_1(t) \\ a_{25}y_2(t) & a_{26}L_2(t) & 0 & 0 \\ d_{22}y_2(t) & d_{23}L_2(t) & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1(t-h) \\ L_1(t-h) \\ y_2(t-h) \\ L_2(t-h) \end{bmatrix}.$$

8.2 Controllability

Consider the model

$$\dot{x}(t) - A_{-1}\dot{x}(t-h) = A_0x(t) + A_1x(t-h) + A_2(x(t))x(t-h) + B_1p + B_2q, \quad (8.11)$$

where B_1p, B_2q are as defined in (8.7) and (8.8), and $x = \begin{bmatrix} y_1 \\ L_1 \\ y_2 \\ L_2 \end{bmatrix}$ is the economic

state vector of Gross Domestic Product and Employment. Also, A_{-1} is a 4×4 matrix A_0 4×4 matrix, A_1 4×4 matrix $A_2(x(t))$ is a 4×4 matrix function of x .

The strategy $\sigma(t) = B_1p$ is the control function of the private firms: $\sigma(t) \in P \subset E^n$, and $g(t) = B_2q \in Q \subset E^4$, that of government. We are to use $p_i, g_i; i = 1 \dots 4$ to steer an initial state value of the nations' gross domestic products x and employment to a desirable target x_1 in time t_1 . Thus the solution of (11.2.1) $x(\phi, \sigma, g)$ with $x_\tau(\phi, \sigma, g) = \phi$ satisfies $x(\phi, \sigma; g)(t_1) = x_1$. The terminology of differential game is appropriate. By the method of Hájek [3] and [4] the model can be reduced to a differential control system

$$\dot{x}(t) - A_{-1}\dot{x}(t-h) = A_0x(t) + A_1x(t-h) + A_2(x(t))x(t-h) + Bv(t), \quad (8.12)$$

where $Bv \in BV$, by using a Stroboscopic strategy in the game (8.9). Here,

$$BV(t) = (P + \ker U(t_1, t, x_t(t_0, \phi)) \pm Q. \quad (8.13)$$

For some t_1 , where

$$U_t(\cdot, x_s(t_0, \phi))X_0(\theta) = T(t, s, x_s(t_0, \theta)) \quad (8.14)$$

is the solution operation of

$$\dot{x}(t) - A_{-1}\dot{x}(t-h) = A_0x(t) + A_1x(t-h) + A_2(x(t)x(t-h)). \quad (8.15)$$

$$\begin{aligned} BV &= \{Bv : Bv + B_2Q \subset B_1P + \ker U(t_1, t, x_t(t_0, \phi))\} \\ &= (B_1P + \ker U(t_1, t, x_t(\sigma, \phi)))^*B_2Q. \end{aligned} \quad (8.16)$$

It is possible for Kernel $U(t_1, t, x_t(t_0, \phi))$ to be zero. This happens when $\det(A_{-1}) \neq 0$ see Hale [5; p. 279]. In this case

$$BV = B_1P^*B_2Q. \quad (8.17)$$

In the linear case the proof of conversion of the game into a control system is treated in [6; p. 367]. The nonlinear case uses the nonlinear variation of parameter Eq. (4.4.3) of [6; p. 111]. It is now reasonable to consider the controllability and the Time Optimal Control Theory of the system

$$\dot{x}(t) - A_{-1}\dot{x}(t-h) = A_0x(t) + A_1x(t-h) + A_2(x(t)x(t-h) + Bu, u \in U. \quad (8.18)$$

The variational control system along the trivial solution is

$$\dot{x}(t) - A_{-1}\dot{x}(t-h) = A_0x(t) + A_1x(t-h) + Bu. \quad (8.19)$$

Theorem ([6; p. 450]) *Assume that:*

1. $\text{rank}[\Delta(\lambda), B] = 4 \forall \lambda \text{ complex}$

where

$$\Delta(\lambda) = I\lambda - A_0 - A_1e^{-\lambda} - A_{-1}\lambda e^{-\lambda h};$$

2. $\text{rank}[\lambda I - A_{-1}, B] = 4 \forall \lambda \text{ complex}.$

Then (8.18) is locally null controllable and locally null controllable with constraints. That is, there is a neighborhood \mathfrak{V} of zero in $W_\infty^{(1)}$ such that every initial point of \mathfrak{V} can be driven to zero in some finite time t_1 using some control strategy $u \in U_{ad}$, where

$$U_{ad} = \{u \in L_\infty([t_0, t_1]) : \|u\|_\infty \leq M\}.$$

The bound M on the control strategy can be calculated for the identified system. Let

$$f(x(t), x(t-h)) = A_0x(t) + A_1x(t-h) + A_2(x(t))x(t-h). \quad (8.20)$$

Then

$$f(0, 0) = 0.$$

Let

$$\begin{aligned} H_1 &\triangleq D_1f(x(t), 0), \\ H_2 &\triangleq D_2f(x(t), x(t-h)), \end{aligned}$$

where $D_i f$ is the i -th partial derivative of f . Let $A_0 = D_1f(0, 0)$, $A_1 = D_2f(0, 0)$, and denote by D_a the symmetric $n \times n$ matrices

$$1/2(H_a + H_a^T) \triangleq D_a \quad a = 1, 2$$

Define J_a as follows

$$J_a = AD_a + D_a^T A,$$

where A is a positive definite symmetric 4×4 matrix.

Theorem In (7.19) assume

1. $\text{rank}[\Delta(\lambda), B] = 4$ for all $\lambda \in \text{complex}$,

where

$$\Delta(\lambda) = I - \lambda A_{-1} e^{-\lambda h} - A_0 - A_1 e^{-\lambda h}.$$

2. $\text{rank}[\lambda I - A_{-1} B] = 4$, for all of λ complex.

3. All the roots of the equation

$$\det(I - A_{-1} r^{-h}) = 0,$$

have moduli less than 1 and for some positive definite $n \times n$ matrix A we have

$$J_1 \leq -\delta I,$$

where, for some $q > 1$,

$$\delta - q\|J_2 + J_1A_1\| \equiv \mu > 0.$$

Then (8.20) is null controllable with constraints, i.e., with controls in U_{ad} .

Remark The proofs are contained in Chukwu [6], Chapter 12. As observed in [6; p. 376] though zero is the target at the final time, the theory incorporates nontrivial targets as well.

Consider the system

$$\frac{d}{dt} x(t) - A_{-1}\dot{x}(t-h) = A_0x(t) + A_1x(t-h) + A_2(x(t)x(t-h) + B_1p + B_2q). \quad (8.21)$$

Let

$$f(t, x(t), x(t-h)) \equiv A_0x(t) + A_1x(t-h) + A_2(x(t))x(t-h); \quad (8.22)$$

and let

$$K(t, x(t), x(t-h), q) = B_2q \quad (8.23)$$

Then we can write

$$\dot{x}(t) - A_{-1}\dot{x}(t-h) = f(t, x(t), x(t-h)) + B_1p + K(t, x(t), x(t-h), q) \quad (8.24)$$

Theorem Assume that

1. $\text{rank}, B_1 = 4$ on $[t_1 - h, t]$; $|f(t, x(t), x(t-h))| \leq \alpha F(t, x(t), x(t-h), p)$,

where

$$F : E^4 \times E^4 \rightarrow E^+ \\ |K(t, x(t), x(t-h), q)| \leq \beta F(t, x(t), x(t-h), p) \text{ for all } t, x, p, \beta, < \alpha.$$

Then (8.24) is controllable on $[\sigma, t_1]$ with $t_1 > \sigma + h$. We thus require that private or firms control set should dominate the governments. The proof is an adaptation of Theorem 13.1 in p. 253 of [2].

We have presented a model of GDP and Employment for interacting nations. It opens up the possibility of using minimum “investment”, both private and government, to steer growth to the desired target as fast as possible. See [1; pp. 293–299]. We confront our model with UN, data to see how close to the real world our growth model is for the nations considered, i.e., for China and the U.S.A. or the USA and UK. See Appendix 1.

8.3 Simple Interaction

We now assume that the interaction matrix is a simple constant 4×4 matrix, A_1 , and $A_{-1} = 0$ where

$$A_1 = \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{22} & 0 & a_{23} & a_{24} \\ a_{31} & a_{32} & 0 & a_{34} \\ a_{41} & a_{42} & a_{43} & 0 \end{bmatrix}. \quad (8.25)$$

Define A as follows:

$$\begin{aligned} A &= A_0 + A_1 \\ &= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{22} & a_{211} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{311} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{411} \end{bmatrix}. \end{aligned} \quad (8.26)$$

Now consider the ordinary differential control system

$$\dot{x} = Ax + Bu, \quad (8.27)$$

where

$$Bu = B_1P + B_2q = [B_1, B_2] \begin{bmatrix} p \\ q \end{bmatrix}. \quad (8.28)$$

Assume the derived model is a game of pursuit which, by Hájek's method, is equivalent to (8.27), where

$$Bu \in V \equiv B_1P^*B_2G,$$

Let

$$\begin{aligned} |C_{oi}| &\leq b_i \\ |I_{oi}| &\leq c_i \\ |X_{oi}| &\leq d_i \\ \alpha_i &= \max_{i=1, \dots, 3} [b_i, c_i, d_i] \\ P &= \prod_{i=1}^4 [-\alpha_i, \alpha_i] \quad i = 1, \dots, 3; \end{aligned}$$

Also

$$\begin{aligned}
|g_{oi}| &\leq h_i, \\
|T_{ai}| &\leq k_i, \\
|\tau_i| &\leq \ell_i, \\
|e_i| &\leq n_i, \\
Q &= \prod_{i=1}^4 [-\beta_i, \beta_i], \\
\beta_i &= \max_{1 \leq i \leq 4} [h_i, k_i, \ell_i, n_i].
\end{aligned}$$

Then

$$P * Q = \{u : u + Q \subset P\} = V.$$

With V so constructed a specific optimal control strategy can be deduced from our theory applied to Hermes and LaSalle [7] and Chukwu [6, p. 114]. This investigation can also be viewed from another perspective.

Let

$$Q = \prod_{i=1}^4 Q_i, \quad (8.29)$$

$$P = \prod_{i=1}^4 P_i, \quad (8.30)$$

where

$$\begin{aligned}
Q_1 &= [-\max g_{0i}, +\max g_{0i}], \\
Q_2 &= [-\max T_{ai}, \max T_{ai}], \\
Q_3 &= [-\max \tau_i, \max \tau_i], \\
Q_4 &= [-\max e_i, \max e_i], \\
P_1 &= [-\max C_{0i}, \max C_{0i}], \\
P_2 &= [-\max X_{0i}, \max X_{0i}], \\
P_3 &= [-\max I_{0i}, \max I_{0i}].
\end{aligned}$$

The set

$$U = P_-^* Q,$$

where

$$p = B_1 r, \quad q = B_2 g \quad (8.31)$$

and

$$|p_i| \leq \alpha_i, \quad |q_i| \leq \beta_i, \quad \text{and we assume } \beta_i < \alpha_i. \quad (8.32)$$

Then

$$U = P_-^* Q = \{u \in U; |u_i| \leq \alpha_i - \beta_i, \quad i = 1, \dots, 4\}. \quad (8.33)$$

With u so constructed a specific optimal control strategy can be constructed from our theory, from the work of Hermes and LaSalle [7] or Chukwu [1; p. 135].

8.3.1 Oscillation

Consider the equivalent control system

$$\dot{x} = Ax - u, \quad (8.34)$$

and its associated autonomous linear system

$$\dot{x}(t) = Ax(t). \quad (8.35)$$

Let

$$\det(A - \lambda I) = 0 \quad (8.36)$$

$$= a_0 \lambda^n + a_1 \lambda^{n-2} + \dots + a_n = 0. \quad (8.37)$$

The solution of (8.35) is oscillatory if the solution λ of (8.37) is complex. If x is non-scalar and nontrivial and $x(t) = (x_1(t), \dots, x_n(t))$ is defined on $[0, \infty)$ then it is oscillatory if and only if at least one component of x has arbitrarily large zeros on $[0, \infty)$. If all components are nonoscillatory, it is nonoscillatory.

Theorem *Let*

$$\dot{x} = Ax, \quad (8.38)$$

and

$$f(\lambda) = |A - \lambda I| = 0,$$

so that

$$= a_0 \lambda^n + a_1 \lambda^{n-1} + \dots + a_n = 0,$$

with $a_n \neq 0$ then the solution of (8.38) is oscillatory whenever

$$\Delta_{n-1} = 0,$$

where

$$\Delta_1 = a_1, \Delta_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix},$$

$$a_k = \begin{bmatrix} a_1 & a_3 & a_5 & & \\ a_0 & a_2 & a_4 & & \\ 0 & a_1 & a_3 & & \\ 0 & a_0 & a_2 & a_4 & \\ 0 & 0 & a_1 & a_3 & \\ 0 & 0 & 0 & a_0 & a_2 \\ & & & & a_k \end{bmatrix}, \quad k = 1, 2, \dots, n.$$

Here we substitute a_k for $k > n$; and

$$\Delta_{n-1} = (-1) \frac{n(n-1)}{2} a_0^{n-1} \prod_{i < k}^{1, \dots, n} (\lambda_i + \lambda_k).$$

and

$$\Delta_n = (-1) \frac{n(n+1)}{2} a_0^n \lambda_1 \lambda_2 \dots \lambda_n \prod_{i < k}^{1, \dots, n} (\lambda_i + \lambda_k).$$

Remarks $\Delta_n - 1 = 0$ if, and only if the sum of two roots of $f(\lambda) = 0$ is zero. That is $f(\lambda)$ has at least one pair of conjugate roots or multiple zero roots. If $a_n \neq 0$ we have at least one pair of pure imaginary roots only, and the zero roots are ruled out.

This theorem, translated into economic terms and related to the economic dynamics (8.27) yields policies for movement to depression or/and to boom for GDP and employment (See p. 191 of Chukwu [1]). Oscillation can be tamed by introducing a feedback control $u(t) = Fx(t)$ so that

$$\dot{x}(t) = (A - F)x(t)$$

is nonoscillatory.

Consider the economic state

$$x = [y_1, L_1, y_2, L_2],$$

of the Gross Domestic Product, and employment for two interacting countries described by the equation

$$\dot{x}(t) = Ax(t) + Bu \quad (8.39)$$

One applies the theorem of Brammer to (8.40) to prove (8.40) null controllable and therefore x_1 —constrained controllable. See p. 131 of Chukwu [1].

With the simple interacting A_1 in (8.25) and the economic state

$$x = [y_1, L_1, y_2, L_2],$$

we obtain the model

$$\dot{x}(t) - A_{-1}\dot{x}(t-h) = Ax(t) + A_1x(t-h) + B_1p + B_2q.$$

With stroboscopic strategy as in Hájek's Analysis we deduce an equivalent system

$$\dot{x}(t) - A_{-1}\dot{x}(t-h) = A_0x(t) + A_1x(t-h) + Bu.$$

See [1, 6.3.5, pp. 317–320] for test for stability and constrained controllability. We write out the equation in some detail:

$$\dot{x}(t) - A_{-1}\dot{x}(t) = A_0x(t) + A_1x(t-h) + B_1p + B_2q.$$

We note that

$$\begin{aligned} q &= [q_1, q_2, q_3, q_4], \\ x &= [y_1, L_1, y_2, L_2], \\ r &= \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} = B_2p. \end{aligned}$$

Each component of q is of the form

$$\begin{aligned} q &= [g_{01}, Ta_1, e_1, \tau_1, M(\omega) g_{02}, M(\omega) Ta_2, M(\omega) e_2, M(\omega) \tau_2, \\ &\quad g_{03}, Ta_3, e_3, \tau_3, M_4 g_{04}, M_4 Ta_4 \tau_4, M_4 e_4], \end{aligned}$$

and if $A_{-1} = 0$, then

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} \lambda_1 \sigma_1^{-1} & \lambda_1 \sigma_1^{-1} & -z_{15} & z_{16} & z_{17} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} g_{01} \\ Ta_1 \\ \tau_1 \\ e_1 \\ M_2(\omega)g_{02} \\ M_2(\omega)Ta_2 \\ M_2(\omega)e_2 \\ M_2(\omega)\tau_2 \\ g_{03} \\ Ta_3 \\ e_3 \\ \tau_3 \\ M_4g_{04} \\ M_4Ta_4 \\ M_4e_4 \\ M_4\tau_4 \end{bmatrix}$$

and if we assume $d_i \equiv 0$, $i = 1, \dots, 4$.

$$\begin{bmatrix} \dot{y}_1(t) \\ \dot{L}_1(t) \\ \dot{y}_2(t) \\ \dot{L}_2(t) \end{bmatrix} = \begin{bmatrix} b_1 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 \\ 0 & 0 & b_3 & 0 \\ 0 & 0 & 0 & b_4 \end{bmatrix} \begin{bmatrix} y_1(t) \\ L_1(t) \\ y_2(t) \\ L_2(t) \end{bmatrix} + \begin{bmatrix} -a_{11} & -a_{12} & -a_{13} & -a_{14} \\ -a_{21} & -a_{22} & -a_{23} & -a_{24} \\ -a_{31} & -a_{32} & -a_{33} & -a_{34} \end{bmatrix} \begin{bmatrix} y_1(t-h) \\ L_1(t-h) \\ y_2(t-h) \\ L_2(t-h) \end{bmatrix} + \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} r_1(t) \\ r_2(t) \\ r_3(t) \\ r_4(t) \end{bmatrix},$$

$$r = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} = B_2 p$$

$$= \begin{bmatrix} \xi_1 & \xi_1 & \xi_2 & \xi_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_2 & \rho_2 & \rho_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi_3 & \xi_3 & \xi_3 & \xi_{31} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_4 & \rho_4 & \rho_4 \end{bmatrix} \begin{bmatrix} C_{01} \\ I_{01} \\ X_{01} \\ M_{01} \\ x_{02} \\ y_{02} \\ I_{02} \\ C_{03} \\ I_{03} \\ X_{03} \\ M_{03} \\ x_{04} \\ y_{04} \\ I_{04} \end{bmatrix},$$

Let

$$T_{ai} = -C_{1i}T_i - C_{2i}\dot{T}_i(t) - I_{2i}T_i - I_{2i}\dot{T}_i(t), \quad (8.40)$$

and

$$q = B_1 g = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \quad (8.41)$$

$$B_1 g = MNP$$

$$B_1 g = \begin{bmatrix} \lambda_1, & \lambda_1, & \lambda_1 X_{161}, & \lambda_1 X_{151}, & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2, & \lambda_2, & \lambda_2 X_{162}, & \lambda_2 X_{152} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_3, & \lambda_3, & \lambda_3 X_{163}, & \lambda_3 X_{153}, & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4, & \lambda_4, & \lambda_4 X_{164}, & \lambda_4 X_{154} \end{bmatrix} \begin{bmatrix} g_{01} \\ T_{a1} \\ \tau_1 \\ e_1 \\ g_{02} \\ T_{a2} \\ \tau_2 \\ e_2 \\ g_{03} \\ T_{a3} \\ \tau_3 \\ e_3 \\ g_{04} \\ T_{a4} \\ \tau_4 \\ e_4 \end{bmatrix} \begin{bmatrix} \lambda \sigma_1^{-1} = \zeta_3 \\ \lambda_2 = M_2(\omega) \\ z_{s26} = X_{162} \\ z_{s25} = X_{152} \\ \lambda_3 \sigma_3^{-1} = \zeta_3 \\ -z_{35} = X_{153} \\ z_{36} = X_{163} \\ M_4(\omega) = \lambda_4 \\ T_{a4} = z_{s44} T_4 \\ T_{a4} = z_{s44} T_4 \\ M_4(\omega) z_{s45} = \lambda_4 X_{154} \\ m_4 z_{s46} = \lambda_4 X_{164} \end{bmatrix}, \quad (8.42)$$

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + A_1 x(t-h) + B_1 g + B_2 p. \\ \dot{x}(t) &= A_0(t) + A_1(x(t))x(t-h) + B_1 g + B_2 p, \end{aligned}$$

where

$$x = [y_1, L_1, y_2, L_2].$$

We now use data on the US and China from the World Bank, and IMF Publications to confront our theoretical model, and examine whether it is realistic.

$$\begin{aligned} \text{US} \quad & y_1 = \text{GDP}, \\ & L_1 = \text{unemployment/employment}, \\ \text{China} \quad & y_2 = \text{GDP}, \\ & L_2 = \text{unemployment/employment}. \end{aligned}$$

U.S.A.

I_1 = investment,

C_1 = consumption,

X_1 = net export, Export – Import,

G_1 = government intervention, outlay,

T_1 = Taxes,

e_1 = exchange rate,

τ_1 = tariff,

d_1 = distance between nations (policy).

Aggregate demand

$$Z_1 = C_1 + I_1 + X_1 + G_1,$$

$$C_1 = C_{01} + C_{11}(y_1(t) - T_1(t)) + C_{21}(y_1(t-1) - T_1(t-1)) \\ + C_{31}(\dot{y}_1(t) - \dot{T}_1(t)) + C_{41}(\dot{y}_1(t-1) - \dot{T}_1(t-1)),$$

$$I_1 = I_{01} + I_{11}y_1(t) + I_{21}y_1(t-1) - I_{31}\dot{y}_1(t) + I_{41}\dot{y}_1(t-1) \\ + I_{81}L_1(t) + I_{91}L_1(t-h) + I_{10}\dot{L}_1(t-h),$$

$$X_1 = Ex_1 - Im_1 = \text{Export} - \text{Import};$$

$$X_1 = X_{01} + X_{11}y_1(t) + X_{21}y_1(t-1) + X_{31}\dot{y}_1(t) + X_{41}\dot{y}_1(t-1) + X_1L_1(t) \\ + X_1L_1(t-h) + X_{151}e_1(t) + X_{161}\tau_1(t) + X_{171}d_1(t) + X_{101}\dot{L}_1(t-1) \\ + y_1(t)[a_{11}y_2(t-1)] + L_1(t)[C_{11}L_2(t-1)].$$

$$G_1 = g_{01} + g_{11}y_1(t) + g_{21}y_1(t-1) + g_{31}\dot{y}_1(t) + g_{41}\dot{y}_1(t-1) + g_{61}L(t).$$

Use the principle of supply and demand:

$$\frac{dy_1}{dt} = \lambda_1(z_1(t) - y_1(t)),$$

where

$$Z_1(t) = (C_1 + I_1 + X_1 + G_1) \\ = (C_{01} + I_{01} + X_{01} + g_{01}) + (C_{11} + I_{11} + X_{11} + g_{11})y_1(t) \\ + (C_{12} + I_{12} + X_{12} + g_{12})y_1(t-h) + (C_{13} - I_{13} + X_{13} + g_{13})\dot{y}_1(t) \\ + (C_{14} + I_{14} + X_{14} + g_{14})\dot{y}_1(t-h) + (I_{101} + X_{101})\dot{L}_1(t-h) \\ + y_1(t)[a_{11}y_2(t-1)] + L_1(t)[C_{11}L_2(t-1)] + [I_{81} + g_{61} + X_{91}]L_1(t) \\ + X_{161}\tau_1(t) + X_{151}e_1(t) + [I_{91} + X_{101}]L_1(t-1) + X_{171}d_1(t) \\ + C_{21}y_1(t-1) - C_{11}T_1(t) - C_{21}T_1(t-1) + C_{31}\dot{y}_1(t) - C_{31}T_1'(t) \\ - C_{41}T_1'(t-h) + C_{41}\dot{y}_1(t-1).$$

Let

$$\begin{aligned}
 Z_{01} &= C_{01} + I_{01} + X_{01} + g_{01}, \\
 Z_{11} &= C_{11} + I_{11} + X_{11} + g_{11}, \\
 Z_{12} &= C_{12} + I_{12} + X_{12} + g_{12}, \\
 Z_{13} &= C_{13} - I_{13} + X_{13} + g_{13}, \\
 Z_{14} &= C_{14} + I_{14} + X_{14} + g_{14}, \\
 Z_{15} &= I_{101} + X_{101}, \\
 Z_{16} &= a_{11}, \\
 Z_{17} &= C_{11}, \\
 Z_{18} &= I_{81} + g_{86} + X_{91}, \\
 Z_{19} &= I_{91} + X_{101}
 \end{aligned}$$

Note that income from the expenditure side is

$$\begin{aligned}
 y_s &= z_{s0} + z_{s1}y(t) + z_{s2}y(t-h) + z_{s4}\dot{y}(t) + z_{s8}L(t) + z_{s10}\dot{L}(t) \\
 &\quad - z_{s14}T(t) + z_{s15}e(t) + z_{s16}\tau(t).
 \end{aligned}$$

Invoke the cited differential principle of supply and demand,

$$\frac{dy_1}{dt} = \lambda_1(z_1 - y_1),$$

to obtain

$$\begin{aligned}
 (1 - \lambda_1)Z_{13} \frac{dy_1}{dt} - \lambda_1 z_{14} \frac{dy_1}{dt} (t-1) - \lambda_1 z_{15} \dot{L}_1(t-h) \\
 = \lambda_1(z_{11}y_1 + z_{01} + z_{21}y_1(t-h) \\
 + z_{61}L_1(t) + z_{71}L_1(t-1) + C_{01}y_1(t)y_2(t) + C_{11}L_1(t)L_2(t-1) \\
 + C_{21}y_1(t-1) - C_{31}\dot{y}_1(t) + T_{1a} + X_{151}e_1(t) + X_{161}\tau_1(t) + X_{171}d_1(t)).
 \end{aligned}$$

Here

$$\begin{aligned}
 T_{1a} &= -C_{21}T_1(t-1) - C_{31}T'_1(t) - C_{41}T_1(t-1) - C_{41}\dot{T}_1(t-h) \\
 &\quad - C_{11}T_1(t).
 \end{aligned}$$

Let

$$\begin{aligned}
 (1 - \lambda_1) z_{13} &= \sigma_1 \\
 \frac{\lambda_1 z_{14}}{(1 - \lambda_1) z_{13}} &= a_{-11} \\
 \frac{\lambda_1 z_{15}}{(1 - \lambda_1) z_{13}} &= a_{-12} \\
 a_{11} &= \lambda_1 z_{11}, \quad \lambda_1 X_{151} / \sigma_1 = a_{18}, \\
 &\quad \lambda_1 z_{01}, \\
 a_{12} &= \lambda_1 z_{21}, \quad \lambda_1 X_{161} / \sigma_1 = a_{19}, \\
 a_{13} &= \lambda_1 z_{61}, \quad \lambda_1 z_{171} / \sigma_1 = a_{20}, \\
 &\quad a_{14} = \lambda_1 z_{71}, \\
 &\quad a_{15} = \lambda_1 C_{01}, \\
 &\quad a_{16} = \lambda_1 C_{11}, \\
 &\quad a_{17} = \lambda_1 C_{21},
 \end{aligned}$$

$$\begin{aligned}
 \frac{dy_1(t)}{dt} - a_{-11} \frac{dy_1}{dt}(t - 1) - a_{-12} \dot{L}_1(t - h) \\
 = a_{11}y_1(t) + a_{12}y_1(t - 1) \\
 + a_{13}L_1(t) + a_{14}L_1(t - 1) + a_{15}y_1(t)y_2(t - 1) + a_{16}L_1(t)L_2(t - 1) \\
 + T_{1a} + \lambda_1 z_{01} + a_{18}e_1(t) + a_{19}\tau(t) + a_{20}d(t).
 \end{aligned} \tag{8.43}$$

China's GDP dynamics is obtained in the same way:

$$\begin{aligned}
 \frac{dy_2(t)}{dt} - a_{-21} \frac{dy_2}{dt}(t - 1) - a_{-22} \dot{L}_2(t - 1) \\
 = a_{21}y_2(t) + a_{22}y_2(t - 1) + a_{23}L_2(t) + a_{24}L_2(t - 1) + a_{25}y_2(t)y_1(t - 1) \\
 + a_{26}L_2(t)L_1(t - h) + T_{2a} + \lambda_2 z_{02} + a_{28}e_2(t) + a_{29}\tau(t) + a_{30}d(t).
 \end{aligned} \tag{8.44}$$

See [6], Eq. (1.10.64) for employment equation, with indicated interaction

$$\begin{aligned}
 \dot{L}_1(t) - l_{011} \dot{L}_1(t - h) - l_{031} \dot{y}_1(t - h) \\
 = l_{01}L_1(t) - l_{11}L_1(t - h) + l_{12}y_1(t - h) + l_{51}L_1(t - h) \\
 + \sigma_{31}(t) + q_{41}(t) + d_{11}y_1(t)y_2(t - h) + d_{12}L_1(t)L_2(t - h);
 \end{aligned} \tag{8.45}$$

$$\begin{aligned}
 \dot{L}_2(t) - l_{022} \dot{L}_2(t - h) - l_{032} \dot{y}_2(t - h) = l_{02}L_2(t) - l_{12}L_1(t - h) + l_{12}y_2(t - h) \\
 + l_{52}L_2(t - h) - \sigma_{32}(t) + q_{42}(t) + d_{22}y_2(t)y_1(t - h) + d_{23}L_2(t)L_1(t - h)
 \end{aligned} \tag{8.46}$$

Use MATLAB to identify the coefficients. This is done in the same way as in Chukwu [1, 2] and [6]. Thus we observe that

$$y = \tilde{C} + \tilde{I} + \tilde{X} + \tilde{G},$$

where

$$\begin{aligned}
\tilde{C} &= y_{10} + y_{11}y(t) + y_{12}\dot{y}(t) + y_{18}(y - T), \\
\tilde{I} &= I_0 + I_1, \\
I_1(t) &= \frac{1}{h} [k(t + h) - k(t)]; \\
\frac{dk}{dt} &= D(t - h),
\end{aligned}$$

the rate of delivery of capital equipment.

$$\begin{aligned}
D(t) &= a(1 - e)y(t) - k_0k(t) + L_3L(t) + k\dot{y}(t), \\
\bar{X} &= x_0 + x_1y(t) + x_2y(t - h) + x_8L(t) + x_{10}\dot{L}(t) + x_{11}e(t) + x_{12}\tau(t) \\
&\quad + x_{13}d(t) + x_{14}y_1(t)y_2(t - h) + x_{15}y_1(t)L_2(t - h). \\
G &= g_{s0} + g_{s1}y(t) + g_{s4}\dot{y}(t) + g_{s8}L(t).
\end{aligned}$$

Assuming profit maximization of the firm where the profit function is

$$P = y - wL - rK,$$

and w is the wage of labor per unit time and r = the rent per unit time for the use of capital

$$m(w) = \left[(1 - \alpha) \frac{1}{w} \right]^{1/\alpha}. \quad (8.47)$$

The constant α is obtained from the Cobb-Douglas equation

$$y = f(k, L) = k^\alpha L^{1-\alpha}.$$

$$\begin{aligned}
\dot{L}(t) - l_{-01}\dot{L}(t - h) - l_{-03}\dot{y}(t - h) &= l_0L(t) - l_1L(t - h) + l_2y(t - h) \\
&\quad + l_5L(t - h) + \sigma_3(t) + q_3(t)
\end{aligned} \quad (8.48)$$

We can insert interaction effects on this equation. The coefficients are identified in (1.10.55)–(1.10.57), and (1.10.66) of [6].

$$\% y2(t)^*a21*y1(t-h) + a23^* \cdot y3(t-h) + a24*y4(t-h) + p2(t) + g2(t)$$

$$\begin{aligned}
p2 &= 11 * (CC(1) + \Pi(1) + XX(1)) / (1 - 11 * Z4) = 632.12 \\
g2(t) &= (1387063249.13 - 397726.95 * e(t) + 0 * ta(t) + 0 * d(t) \\
&\quad - 179.16 * T(t) - 73506.69 * T(t-h) + 120485.89 * T'(t) \\
&\quad - 1.7624916 * T'(t-h)) / (4851492.752) \\
dy2(t)/dt + 1.250 * y2'(t-h) &= -9.277 \times 10^{-5} * y2(t) + 0.01661 * y2(t-h) \\
&\quad + y2(t) * (-0.002687 * y1(t-h) + 4.9979 \times 10^{-4} * y3(t-h) \\
&\quad - 7.3097 \times 10^{-4} * y4(t-h) + p2(t) + g2(t).
\end{aligned}$$

The following equations were Chukwu's model of the gross domestic product of US interacting with China, Nigeria, UK [8]. The impact of China on the rate of growth of US GDP is negative. From the equation one can remedy this by imposing tariffs (with its consequences and other countries' reaction), or by reducing exchange rate. Senator Elizabeth Dole said in Winston-Salem that she plans to push a bill to slap a 27.5 % tariff on Chinese imports if China persists in what she calls unfair trade practices. It is easy to see how tempting this can be. Mathematically it can increase the value of $\frac{dy_2(t)}{dt}$, the growth rate of GDP. Tax reduction in the past and the present can increase the growth rate of GDP.

The centerpiece of President Bush's Asia-Pacific trip (Friday October 17, 2003, News and Observer: Nation) is a two-day summit of Asia-Pacific Economic Cooperation organization in Bangkok. Bush said that he will urge the leaders of Japan and China to stop manipulating currency markets to keep the value of their currencies low in relation to the dollar. This makes American-made goods expensive abroad. American manufacturers say the strong dollar has badly hurt their foreign sales and brought cuts in jobs at home. The U.S. economy has lost nearly 3 million jobs since President Bush took office. President Bush has chosen the way of exchange rate as a way favorable to the USA—the centerpiece of his Asia-Pacific trip. This can also help. It may be easier to use it to increase the growth rate of GDP. But the most effective way to reduce or make positive the draining of $\frac{dy_2}{dt}$ by $-6y_2(t)y_4(t-h)$ is to add $+6.1y_2(t)y_4(t-h)$ to it. This is the way of cooperation. The US economic state is controllable, so is that of China. The interacting countries studied, Nigeria, US, UK, China are controllable. A high GDP can be attained by a judicious choice of all admissible controls, 3–9 of the representative firm, and 4–8 of the government.

From the additional growth we can invest effective $7y_2(t)y_4(t-h)$ in “cooperative” ventures, making $\frac{dy_2}{dt}$ positive, and $y_2(t)$ increasing. Overall since controllability has been proved and cooperation is possible, this is one good way to go. Selecting only one policy can hurt. It requires all the control strategies.

Remark on Employment

The strategies

$$\sigma_{31}(t) = m(w)_1 \sigma_{41}(t)$$

$$\sigma_{41}(t) = x_0 + y_{10} + I_0; m(w)_1 = \left[(1 - \alpha) \frac{1}{w} \right]^{1/\alpha}.$$

$$q_{41}(t) = q_{s0} + z_{s13}M1 - z_{s14}T(t) + z_{s15}e(t) + z_{s16}\tau(t) + z_{s17}d(t),$$

can be used to steer to a very low level of unemployment L_1 . This can be done in a similar way for L_2 using $\sigma_{32}(t)$ and $q_{42}(t)$. Because of the controllability of the state $[y_1, L_1, y_2, L_2]$ by means of the controls displayed $d_{11}, d_{12}, d_{22}, d_{23}$ can be made positive and thus render $\dot{L}_1(t), \dot{L}_2(t)$ positive and big, and make employment growing in interacting countries. Well paid Chinese workers can consume American goods and services, making jobs to grow in America, and vice versa. The following is a newspaper report after this research was completed.

Confirmation

“Software and technology companies that hire workers in Low Cost Countries such as India will add \$3.3 billion to North Carolina’s economy and create 9,699 new jobs by 2008, a new industry—funded study says,” according to News and Observer Business Wednesday, March 31, 2004, p-1 D-60. The cost savings from paying workers in those countries less than those in the United States already have led to 2,555 new jobs in the state since 2000. The study relies on economic models and information from technology companies to measure the effect of off-shore hiring in all 50 states. It predicts that by 2008, companies that send at least some of their technology functions off shore will add \$124 billion to the US economy and create a net gain of 317, 367 jobs.”

This study confirms the main contribution of this paper which was completed earlier.

The confirmation is continued in June 04.

Tuesday, June 8, 2004

The News and Observer Business, p 1D-8D

GM to double its capacity in China

Chinese Vehicle sales jumped 75 % in 2003

“Signaling its confidence in booming Chinese economy, General Motors said Monday it plans to spend \$3 billion in China in the next three years in a challenge to rival Volkswagen for dominance of the Worlds fastest growing auto market. Success in China is crucial to GM’s global success”, Phil Murtaugh, Chairman and Chief Executive of General Motors china Group said BM is the worlds biggest auto maker.

General Motors Ford Volkswagen and Toyota have announced plans in the past eight months to invest about \$10 billion in China.

Stability Remarks [9; Theorem 4.3.2]

$$\begin{aligned}
 y_1 &= x_1 \\
 L_1 &= x_2 \\
 y_2 &= x_3 \\
 L_2 &= x_4 \\
 \frac{dx_1(t)}{dt} &= x_1(t)[b_1 - a_{11}x_1(t - \tau_{11}) + a_{12}x_2(t - \tau_{12}) \\
 &\quad - a_{13}x_3(t - \tau_{13}) - a_{14}x_4(t - \tau_{14})], \\
 \frac{dx_2(t)}{dt} &= x_2(t)[b_2 - a_{21}x_1(t - \tau_{21}) + a_{22}x_2(t - \tau_{22}) \\
 &\quad - a_{23}x_3(t - \tau_{23}) - a_{24}x_4(t - \tau_{24})], \\
 \frac{dx_3(t)}{dt} &= x_3(t)[b_3 - a_{31}x_1(t - \tau_{31}) + a_{32}x_2(t - \tau_{32}) \\
 &\quad - a_{33}x_3(t - \tau_{33}) - a_{34}x_4(t - \tau_{34})], \\
 \frac{dx_4(t)}{dt} &= x_4(t)[b_4 - a_{41}x_1(t - \tau_{41}) + a_{42}x_2(t - \tau_{42}) \\
 &\quad - a_{43}x_3(t - \tau_{43}) - a_{44}x_4(t - \tau_{44})], \\
 x_o(s) &= \rho_i(s) \geq 0 \quad s \in \{-T, 0\}; \\
 \tau &= \max_{1 \leq i, j \leq n} \tau_{ij} \\
 \rho_i(0) &> 0, \quad \rho_i \text{ continuous on } [-\tau, 0]; \\
 &\quad b_i, a_{ij}, \tau_{ij} (i, j = 1, 2, \dots, n)
 \end{aligned} \tag{8.49}$$

are nonnegative constants and $\tau_{ii} > 0$ for one or more $i = (1, 2, 3, \dots, n)$.

Suppose that $b_i > 0, a_{ii} > 0$ ($i = 1, 2, \dots, n$) and $a_{ij} \geq 0, i, j = 1, 2, \dots, n$, if $i \neq j$.

Furthermore, Let

$$b_i > \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij}(b_j/a_{jj}) \quad i = 1, 2, \dots, n.$$

Then there exists a component wise positive steady state $x^* = (x_1^*, \dots, x_n^*)$, satisfying the equations

$$\sum_{j=1}^n a_{ij}x_j^* = \lambda_i \quad \text{and} \quad x_i^* > 0; \quad i = 1, 2, \dots, n.$$

H_1 Assume the real constants $\tau_{ij} \geq 0$ ($i, j = 1, 2, \dots, 4$) satisfy

$$\tau_{ii} \leq \min_{\substack{j \leq j \leq 4 \\ j \neq i}} \tau_{ji}; \quad i=1,2,\dots,4;$$

$$\text{if } \sum_{j=1}^n \tau_{ji} \neq 0.$$

H_2 $a_{ij}(i, j = 1, 2, \dots, 4)$ are nonnegative constants such that

$$a_{ii}^{+b} > 0; a_{ii}x_i^* \tau_{ii} < 1; \quad i = 1, 2, \dots, 4;$$

$$\left(\tau_{ii}x_i^* \sum_{\substack{j=1 \\ j \neq i}}^4 a_{ji} \right) (1 - a_{ii}\tau_{ii}x_i^*)^{-1} < \pi/2 \quad i = 1, 2, \dots, n.$$

$$a_{ii} \cos \left(\left\{ \tau_{ii}x_i^* \sum_{\substack{j=1 \\ j \neq i}}^4 a_{ji} \right\} / \{1 - a_{ii}\tau_{ii}x_i^*\} \right) > \sum_{\substack{j=1 \\ j \neq i}}^4 a_{ji} \quad ij = 1, 2, \dots, 4.$$

The positive steady state $x^* = (x_1^* \dots x_4^*)$ of (11.3.23) exists then x^* is locally asymptotically stable.

Remarks 1 The conditions imply that competition should be regulated and the delays small, to ensure that $x^* = [y_1^*, L_1^*, y_2^*, L_2^*]$ is locally asymptotically stable.

Remarks 2 The relative strength of P—the private control constraint strategy set and the government control constraint strength Q was demonstrated to be

$$Q \subset P,$$

where Q is not empty. See [6; p. 372]. The philosophy of privatization, drastic reduction of Q was furiously advocated earlier by IMF and the World Bank. Very recently the following statement was made by the President of the World Bank. There is a view that we don't need this official development assistance, because it is the private sector at the end of the day that produces the jobs. But the private sector needs roads, the private sector needs an educated population and private sector needs agricultural extension services. Paul Wolfowitz: A Conversation with Paul Wolfowitz—v Date 05/30/2007 Charlie Rose.

The free-market ideology was also prevalent in financial market as reported in www.satoday.com Money Section B Friday October 24, 2008.

There, the former Federal Reserve Chairman Alan Greenspan told angry lawmakers Thursday October 24th 2008, he was “shocked” to discover—as a once in

a century financial crisis spread- that his bedrock belief that financial firms could police themselves turned out to be “flawed.” I made a mistake in presuming that the **self-interest of organizations**, specifically banks and others were such as that they were best capable of protecting their own shareholders and their equity. He said the nation faces a “once-in-a century credit tsunami.” Rep. Henry Waxman, D. Calif. said: “you had the authority to prevent irresponsible lending ... and now our whole economy is paying the price.” Waxman asked Greenspan of his **free-market ideology** “pushed you to make decisions that you wish you had not made.” Greenspan, for example opposed efforts to **regulate** derivatives; financial products whose value depends on the movements of other assets like commodities or stock indexes.

Chukwu [2, pp. 293–296].

Fundamental Principle 1, 2, 3. These observations seem to have a bearing as an explanation of what is happening in the East and what may happen in the West as a consequence of the 1980–1989 USA dismantling of “regulations” on the economy. In the East economic target cannot be reached because of too much centralization: to make firms controllable market forces are being advocated. In the third world, the popular policy prescription of dismantling of “solidarity function, “which rides the wave of deregulations of the 1980’s in the West is at work with fury. Aside from its hardships, it does not seem to have a solid theoretical foundation. A certain amount of “solidarity function” is effective for economic growth. This can be provided by central governments, which cherish individual initiatives. As this book is being written, the economies of Thailand and South Korea are threatened to collapse. It is now being recognized that the string of large corporate bankruptcies and bank defaults are due to weak solidarity functions-bad government, under-regulated banking systems too much “private initiative”, insider dealings and book keeping practices. It is being recognized that the problem can be solved perhaps internally by reforms enforced by “solidarity function”—more effective government intervention with perhaps some cushion from other countries. This is the insight the principles provided here...” failure to enforce regulation bringing to bear the right q when some isolated system behaves and is locally uncontrollable in “individualistic” systems may make the composite system locally uncontrollable and may trigger an economic depression.

Countries have been trying to recover from the global economic slow down. When this book was in the production stage the News and Observer of Monday, August 24, 2009 reported that “Asia ushers in recovery.” “The economy of China is surging after Chinese banks doled out more than \$1 trillion in loans in the first half of the year in addition to a nearly \$600 billion government stimulus program.” The benefits are manifest, as government spending is increased, and “China’s powerful economy churns ahead as the West slowly digs out.” We realize that the author’s lecture at the 14th World Congress of IFAC International Federation of Automatic Control of the Beijing International Convention Center, Beijing, P. R. China, July 5–9, 1999 required $Q \subset \text{Int } P$ which was made Chinas national policy by increasing P to 51 % and reducing Q to 49 %. For the years 1991–2002 Song Zhong discovered that $Q = 0.6699P$.

8.4 A Proposed Economic Plan of the USA

Chukwu studied Economic plan for USA future Government as extracted from this book and researches [1, 5] undertaken at the Wealth of Nations Institute. Cutting taxes for middle and low income families can increase the aggregate growth rate of the gross domestic product—goods and services of the USA to impact the meaningful growth rate of the GDP. It decreases the net income from taxes. The redistribution of wealth implicit in this proposal can force up consumption by the huge middle and lower middle class population and consequently force up the increase of the growth of GDP. This is most likely to occur if this policy is combined carefully with increase of taxes on very high-income families and if this produces a net decrease in total taxes. Thus a positive increase of the growth rate of GDP is not only possible, but likely in Chukwu's model. Chukwu's studies reported in page 1014, *Applied Mathematics and Computation*, Vol. 162, Issue 215, April 2005, shows that the net income generated and which is allowed to increase autonomous consumption, autonomous investment, autonomous net export will trigger an increase of GDP. It can also generate good government intervention to augment social security outlay for Medicare funds, construction of roads and schools and employment. Careful calculation begun in the Wealth of Nations Institute can predict how long its good effects will last. See [1, 2] and [5]. After President Hoover, President Rosevelt applied this insight to good effect. Proposed National Investment in Renewable Energy, National infrastructure and access to early education follows this tradition. The conservative argument against this is: Government should not be too big. But the theories so far developed for national economic growth prove beyond doubt that private economic initiative P should dominate government strategy G, [1]. Available analysis shows that in the case of USA $G = 0.22P$ and USA has not reached the threshold of Big Government: $G \geq 51\%P$. It is interesting to recall the impact of Tennessee River Authority undertaking, the stimulating effect of NASA Moon undertaking to science and technology and all levels of education in the nation. It is interesting to recall the following news from CHINA: It was reported in *The News and Observer*, Sunday, October 26, 2008, p. 16A that the Chinese Premier Wen Jiabao on Saturday acknowledged his country was feeling ripple effects from the global financial meltdown and pledged robust government spending to keep the economy from stalling. The remarks were delivered at the close of the two-day Asia-Europe Meeting in Beijing where leaders of 43 nations issued a statement calling for new rules to guide the global economy and urging the International Monetary Fund to take a leading role to aid crisis-stricken countries. It seems to have been aimed to galvanize the policy of these nations and to tame the IMF usual policy of privatizations. Participants said the statement would provide the basis of a joint Asian-European approach at November 15 summit on the crisis in Washington involving the world's 20 largest economies. China plans to ward off the looming global recession by spending 586 billion dollars on affordable housing, rural roads, airports and railways. The aim: to keep the Chinese economy from slowing so much

that social stability is endangered. Sharing the same kind of thought various states in the USA plan to cushion slumping economy by “adding to spending and jobs” see Tuesday, November 11, 2008, www.usatoday.com.

The problem of government and institutional intervention for better regulations of financial and economic practices was discussed at the G-20 leaders meeting. The author highlighted this problem of lack of government intervention and its consequences in Chukwu [2; pp. 293–295]. The leaders agreed to establish by March 31, 2009 supervisory colleges that would include all the world’s major financial system regulators who would meet regularly to discuss the status of the world’s biggest banks. The group also pledged to work for better regulations of derivatives, including credit default swaps complex financial instruments that were blamed for contributing to the market meltdown.

The News and Observer Sunday November 16, 2008 p. 3A.

8.5 Economic Stimulus

It has been proposed that at the down turn of the economy and when it is in a recession some economic stimulus can be introduced as a strategy. There are general principles that govern how big and when to apply economic stimulus. The stimulus, the dirac function is mathematically applied to controls in simple or complex examples. See Chukwu [6; p. 183]. The following is a universal idea. If y is the gross domestic product which is flowing under impulse control, i.e., economic stimulus, i.e., dirac function, the system is allowed a period of free oscillation or free fall during which the natural damping present in the system removes energy from the system. When at the last instant the system displacement is identically zero or near zero, impulse—a stimulus of magnitude equal to the GDP’s momentum is applied. The instantaneous change of velocity abruptly transfers the system to the origin in finite times. Zero can be replaced by an arbitrary target. The timing of the stimulus for the most effective impact is extremely important. It is applied when the potential energy is minimum, i.e., near zero, and the kinetic energy, i.e., the mass multiplied by the speed of flow—of the GDP is a local maximum, i.e., is the fastest. Economic stimulus is applied when the GDP is moving fastest towards zero or recession and depression. An initial positive stimulus is applied to impact velocity to the system. It drifts until a final impulse is applied to terminate the flow at the required better target. Economic stimulus, big and of very short duration—can effectively kick in immediately by reversing the slow down. A reserve of 45 billion dollars which are suggested can be ejected periodically into the economy as a stimulus at appropriate times by the government. Of its own alone it will not prevent future further recession or depression. Controlling the economy does require a judicious combination of government G and private strategies, P , the so called Pontryagin difference of sets: $U = P * Q$. When Chukwu designed the model economy of the USA, there are eight components of government control strategy and nine of the representative

private firms. The so called Pontryagin difference of sets, U has control matrix B full rank and applied effectively. Partnership of government and industry is useful. The control strategies required by the firm are donated by

$$p = [C_0, I_0, X_0, M_0, n, w, y_{10}, p_0].$$

Here C_0 is autonomous consumption, X_0 is autonomous net export, I_0 is autonomous investment, M_0 is autonomous money demand, w is wage rate, y_{10} is autonomous income consumption intercept and p_0 is autonomous price intercept. By “autonomous” we mean the strategy does not depend on the economic variables of the State, namely,

$$x = [y, R, L, k, p, E],$$

where the gross domestic product is y , interest rate is R , employment is L , value of capital stock is k , price is p and cumulative balance of payment is E , government control strategy is

$$g = [T, g_0, e, \tau, d, M_1, M'_1, f_0].$$

Here T is taxation, g_0 is autonomous government outlay, e exchange rate τ = tariff, d is trade agreement/transportation, M_1, M'_1 denote money supply and its flow and f_0 denotes interest equalization tax. The suggested partnership of government and industry, for example, with GM, Chrysler and Ford is elucidated in Chukwu's papers in Paris, Ethelbert N. Chukwu, Optimal Economic Growth of All Member States of the United Nations. A Mathematical Control Theory Treatment, 13th IFAC Workshop on Control Applications of Optimization CAO of Paris-Cachan, France, April 26–28, and, Berlin. Ethelbert N. Chukwu, Employment and Gross Domestic Product of Two States of the United Nations, The 61st International Atlantic Economic Conference, Berlin, Germany 15–19 March 2006. The analysis is as follows: Lowering wages can increase profit and employment. But American workers may not accept lower wages. The opposition can be overcome by outsourcing to, e.g., China or Africa where wages are lower and the market is potentially huge and can create jobs for bigger profit. Here also the USA government as partner of GM and other firms can often negotiate with China or Africa and India to lower taxes for USA firms as a good inducement for the USA firms to operate in China, Africa or India and with a promise of raising their standard of living. USA firms' profit becomes bigger and is allowed to bring a large portion of it home to the USA. The government can tax the firms a little more on this imported profit and then use the acquired revenue to cover improved health care for retirees in the auto sector. Also in return the auto industry will be encouraged to now commit to more fuel efficient, hybrid vehicles—vehicles at low cost. In partnership with private firms government can invest in green energy initiatives including wind and solar using inventions. With good citizens' ingenuity, and global outlook, with generous awards to useful inventors world-wide

economic growth can be ensured provided government investment in aggregate is less than private. See [1, 6].

Here are some details. The Secretary of Commerce can negotiate a trade agreement with China. As a consequence, General Motors can set up factories in China to design and produce low cost, more fuel efficient hybrid vehicles. Because of the big market, GM will make huge profit by volume, which China will not tax too much because of the trade agreement, and the economic improvement GM will bring to China. In the USA the huge profit can be taxed a little more to create jobs, for government in partnership with private firms to invest in innovation, in green energy including wind and solar capacity—Economic growth can be ensured provided government part of the investment is less than the private firms. Thus by cooperation and partnership the two can increase the GDP. Consult www4.ncsu.edu/~chukwu for more detailed studies of this applied economics proposed.

Investment of Revenues from cut of earmarks to pre-2001 level, and taxes on oil companies huge profits.

If the earmarks are not wealth generating the revenue can be used to generate jobs. Revenue from taxes can be used by government to partner with private firms for innovation in wealth generating devices, in road construction, urban renewal, etc. The history of Roosevelt's presidency after President Hoover can be a guide. As we observed earlier government intervention is currently far from being "too much" [10].

Remark After this manuscript was submitted and accepted American Workers have been forced to accept lower wages. Consult Shaw University, Raleigh, North Carolina, November 25, 2008. Presidents Bulletin.

Sometime after this wage lowering action by Shaw University, Governor Beverly Perdue decided to shave a half-percent from state employees' salaries. There are some problems and some opposition. For example, can the State stake claim to salaries paid by federal agencies or private foundations? Nevertheless, Governor Beverly Perdue has signed an executive order implementing a flexible furlough program for all State employees. All State employees compensation will be reduced by an annualized amount equivalent to 0.5 % for May and June. This will equal about 5 h per month that each employee will need to take off without pay.

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Chapter 9

Economic Plans, Programs Policies and Recommendation of 184 Nations

9.1 Universal Laws for the Growth of Gross Domestic Product of Nations

For each nation i

Let

$$p_i(t) = \lambda_i(C_{i0} + I_{i0} + X_{i0}),$$

denote the representative private firms' strategy where C_{i0} is autonomous private consumption, I_{i0} autonomous private investment and X_{i0} autonomous net export.

Let $p_i \in P_i$, P_i the control strategy constraint set. We assume that the government strategy control set is G_i , where

$$g_i(t) \in G_i.$$

We derived the equation

$$\begin{aligned} \frac{dy_i(t)}{dt} - a_{-1i} \frac{dy_i(t-h)}{dt} &= a_{0i}y_i(t) + a_{1i}y_i(t-h) \\ &+ \sum_{\substack{j=1 \\ j \neq i}}^{191} b_{ij}y_i(t)y_j(t-h) + p_i(t) + g_i(t) \end{aligned}$$

as the dynamics of the Gross Domestic Product in (3.12). A more refined equation is given later.

Let

$$\begin{aligned} g_i &= \lambda_i(g_{i0} + X_{15}e_i(t) + X_{16}\tau_i(t) + X_{17}d_i(t) - (C_{i1}T_i(t) + C_{12}T_i(t-h) \\ &+ C_{13}\dot{T}_i(t) + C_{14}\dot{T}(t-h)). \end{aligned}$$

This is the government control strategy of nation i ,

g_{i0} is autonomous government outlay, government spending, i.e., it does not depend on economic variables of the economic state, and
 $e_i(t)$ is the exchange rate,
 $\tau_i(t)$ denotes tariffs,
 $d_i(t)$ represents trade valued policy or distance between trading nations
 T_i is the net aggregate taxes and

$$T_{ai} = (C_{i1}T_i(t) + C_{i2}T_i(t - h) + C_{i3}\dot{T}_i(t) + C_{i4}\dot{T}_i(t - h))$$

is the generalized net aggregate taxes.

λ_i is the speed of response of supply to demand and b_{ij} is given by

$$b_{ij} = \lambda_i a_{ij} / (1 - \lambda_i (C_{i3} - I_{i3} + X_{i3} + g_{i3})),$$

and this is the coefficient of cooperation or if negative, of competition.

The following principles are now evident. If the net aggregate gross domestic product growth rate is

$$\frac{dy_i(t)}{dt} - a_{i1} \frac{dy_i(t - h)}{dt} \equiv g_{ri},$$

then g_{ri} increases if there is cooperation between nation i and nation j and the coefficient of cooperation b_{ij} is positive. But g_{ri} decreases with b_{ij} negative and the nations are in sharp competition.

In general if g_i is increased, and this is done by decreasing T_{ai} , the generalized aggregate GDP increases.

If autonomous government outlay g_{i0} is increased (road construction, schools, hospitals, and health facilities, etc.) so does the growth rate of GDP.

Suppose by regression X_{i5} , X_{i6} and X_{i7} are determined as real numbers. If X_{i5} is negative a decrease in exchange rate enhances the growth rate of the net GDP. The same is true for tariffs.

If there is "authentic solidarity" and X_{i7} is determined to be positive from a negotiated trade policy there will be an increase in the net growth rate of GDP. If autonomous consumption C_{oi} , autonomous investment, and autonomous net export are increased, so is the growth rate. If there is excess demand over supply and λ_i made bigger then so is p_i where

$$p_i \in P_i.$$

Finally in our analysis we shall demonstrate that for controllability we require generally that

$$G \subset \text{Int} (P + \ker X(t_1, t)).$$

This means that private strategy should dominate government strategy. If losses, or private waste due to inefficiency or gross theft ($\ker X(t_1, t)$) by the private strategy are taken into account the second inclusion is true. It is now important to do the regression and determine the coefficients for some countries, others are similarly done. In Chukwu, “A Mathematical Treatment of Economic Cooperation and Competition Among Nations: with Nigeria, USA, UK, China, and Middle East examples,” Elsevier 2005, p. 92 it was discovered that

$$G \subset \text{Int}(p + \ker X(t_1, t)).$$

In 1992 China made this a policy by increasing P to 51 % and reducing G to 49 %, see New York Times, July 17, 1992. Chukwu introduced this theory at the 14 Triennial World Congress, Beijing, P.R. China, July 5–9.

To the surprise of many the economy of China has been booming. Similar treatment for the economic state variables $R(t)$, $L(t)$, $k(t)$, $p(t)$, and $E(t)$ are available.

9.2 The Study of Longevity and Economic Growth

In the previous chapter we studied economic growth by modeling through MATLAB regression the growth of gross domestic product, interest rate, employment, value of capital stock, prices, and cumulative balance of payment. It is now a common observation that there is a 30-year gain in the last century of longevity which is related to good health and generation of wealth. Dr. Robert Butler of the International Longevity Center who has studied this for over fifty years affirms this 30-year gain. Furthermore those nations that have a five-year advance in longevity compared to other countries actually have a greater GDP. Thus, increasing health and longevity will create greater wealth and greater prosperity for a society. There are known ways to extend human life-regular physical activity, a diet high in fiber and low in saturated fats, and appropriate consumption of vitamins, such as B-12, and Resveratrol (found in the skin of grapes), caloric restriction (eating approximately 30 % fewer calories than normal) all have very big promise in increasing longevity. From the insight obtained from the study of lower forms of life, genetic manipulation seems to also have big promise. Professors Paul Nurse, Leonard Guarente, Cynthia Kenyon, Richard Weindruch, Jay Olshanky, Robert Butler in “A discussion concerning the study of longevity” with Charlie Rose, MSNBC 03.28/2007 highlights close community and family ties as playing a central role in promoting longevity. Indeed in an earlier book, “A Mathematical Treatment of Economic Cooperation and Competition Among Nations: With Nigeria, USA, UK, China and Middle East example,” Elsevier, 2005 it was mathematically demonstrated that if cooperation between two groups of people is complete eternal life is possible for both. The dynamics between the two are

$$\begin{aligned}\frac{dy}{dt} &= y(-1 + x), \\ \frac{dx}{dt} &= x(-1 + y).\end{aligned}\tag{9.1}$$

Without the presence of the other.

$$\begin{aligned}\frac{dy}{dt} &= -y, \quad \frac{dx}{dt} = -x \\ y(0) &= y_0, \quad x(0) = x_0,\end{aligned}\tag{9.2}$$

and

$$\begin{aligned}x(t) &= x_0 e^{-t} \rightarrow 0 \quad \text{as } t \rightarrow \infty, \\ y(t) &= y_0 e^{-t} \rightarrow 0 \quad \text{as } t \rightarrow \infty.\end{aligned}$$

There is extinction after a long time. But if x helps y , and y helps x in community it can be proved that there is a finite, T , $T < \infty$, such that each pair of solutions $(x(t), y(t))$ of (9.1) satisfies

$$\lim_{t \rightarrow T} x(t) = \infty, \quad \lim_{t \rightarrow T} y(t) = \infty.\tag{9.3}$$

The community interaction is so beneficial that each grows unbounded, i.e., there is a stupendous orgy of mutuality.” To see this let

$$\begin{aligned}w &= x - y \\ \dot{x} - \dot{y} &= -(x - y) = -w.\end{aligned}$$

The solution is given by $w(t) = w(0)e^{-t}$.

Suppose $r = 1/x$ then $\dot{r} = -1/x^2 \dot{x} = -1/x^2(x(-1 + y))$.

So that

$$\begin{aligned}\dot{r} - 1/x + y/x &= 0, \\ \dot{r} - r + ry &= 0,\end{aligned}$$

Solving this equation in r whose integrating factor is

$$h = e^{\int (-1 + w(0)e^{-t}) dt},$$

we have

$$\begin{aligned}
d/dt(re^{-(t+w(0)e^{-t})}) &= -e^{-(t+w(0)e^{-t})} \\
w(0) &= w_0 \\
re^{-(t+w(0)e^{-t})} &= - \int e^{-(s+w(0)e^{-s})} ds = c \\
re^{-(t+w(0)e^{-t})} &= c - e^{-w_0 e^{-t}}
\end{aligned}$$

Therefore

$$r(t) = c e^{(t+w(0)e^{-t})} - 1/w_0 e^t$$

But $r(t) = 0$ if $ce^{w_0 e^{-t}} = 1/w_0$.

This is possible if there's a time T such that

$$w_0 e^{-T} = \ln\left(\frac{1}{e w_0}\right) \text{ or } e^{-T} = \frac{\ln\left(\frac{1}{e w_0}\right)}{w_0},$$

i.e.,

$$-T = \ln\left(\frac{\ln\left(\frac{1}{e w_0}\right)}{w_0}\right)$$

But then if $r = 0$ we have $x = \infty$.

Therefore there exists a finite time T such that

$$\lim_{t \rightarrow T} x(t) = \infty. \quad (9.4)$$

The proof that $\lim_{t \rightarrow T} y(t) = \infty$.

The Maple plots of the equations are displayed in the earlier book by Chukwu, E. N. [2, pp. 260–262]. “A Mathematical Treatment of Economic Cooperation and Competition Among Nations with Nigeria, USA, UK, China and Middle East Examples.” The simple example which is touched upon here has a long history dating back from the garden of Eden in Genesis and the last words of Jesus of Nazareth in John Chap. 17:0 = 9–10, 11, 20–21. The program and graphs are contained in [pp. 327–329 of 2]. In the author's Wealth of Nations Institute, we exposed the role of regular physical activity, family ties, Caloric restrictions and diet, genetic manipulation and the effect of Sirtuins and Reservatrol as worthy of study in the issue of Longevity. These studies are linked up which the economic implications of aging. The treatment in [2, p. 327] and of E. N. Chukwu. “Goodness Through Optimal Dynamics of the Wealth of Nations”, Third World Con—of Nonlinear Analysts [3] Catania, Italy, July 20, 2000 opens up the

possibility of very long life through love and cooperation and the creation of huge resources to sustain life in the planet. It is known through elementary calculus course that if we trapped the energy of the sun for our need it will take about three billion years to exhaust the sun. The sands of the rivers and oceans' shore can be taped to fabricate silicon and computer chips for fast and efficient industrialization. The deserts of the world can be colonized and made to bloom using water from the ice covered parts of the earth.

All Nations of the World Unite. You have nothing to lose but your chains: your hate, your revenge, your greed, unrestrained pursuit of self interest, short longevity, and miserable scarcity.

You have everything to gain: A gift of Light, Love, and Goodness; and through thought and technology, and authentic solidarity, abundant goods and services and Everlasting Life.

9.3 Some Simple Results of Some Interacting Nations

In Chap. 6 we derived the general equation of cooperative and competitive systems for the gross domestic product of member states of the United Nations:

$$\begin{aligned} \frac{dy_i(t)}{dt} - a_{-1i} \frac{dy_i(t-h)}{dt} = & a_{0i}y_i(t) + a_{1i}y_i(t-h) \\ & + y_i \sum_{\substack{j=1 \\ j \neq i}}^{191} b_{ij}y_j(t-h) + p_i(t) + g_i(t). \end{aligned} \quad (9.1)$$

In the earlier book, [2] “A Mathematical Treatment of Economic Cooperation and Competition among Nations: With Nigeria, USA, UK, China, and Middle East Examples we studied the USA GDP in relation with and interacting with China, Nigeria and UK. The MATLAB program, data and output is displayed earlier in Appendix 1.

We extracted the following from [2, p. 129]

$$\begin{aligned} \frac{dy_2(t)}{dt} - a_{22} * y_2^1(t-h) = & a_{02} * y_2(t) + a_{12} * y_2(t-h) \\ & + y_2(t) * (a_{21} * y_1(t-h) + a_{23} * y_3(t-h) \\ & + a_{24} * y_4(t-h)) + p_2(t) + g_2(t). \end{aligned}$$

Here y_2 is the USA GDP. p_2 is USA private representative strategy and g_2 is the government strategy. Thus,

$$\begin{aligned}
p_2 &= 11 * (CC(1) + II(1)XX(1))/1 - 11 * Z_4 = 632.12, \\
g_2(t) &= 1387063249.13 - 397726.95 \cdot e(t) + 0 \cdot ta(t) + 0d(t) \\
&\quad - 279.16 * T(t) - 73506.69 * T(t-h) + 120485.89T'(t) \\
&\quad - 1.7624916 * T'(t-h)/(4851492.752),
\end{aligned}$$

and

$$\begin{aligned}
\frac{dy_2(t)}{dt} + 1.250 * y_2^1(t-h) &= -9.277 \times 10^{-5} * y_2(t) \\
y_2(t) * (-0.002687 * y_1(t-h) + 4.9979 \times 10^{-4} * y_3(t-h) \\
&\quad - 7.309 \times 10^{-4} * y_4(t-h)) + g_2(t) + p_2(t).
\end{aligned}$$

This model of the gross domestic product of USA interacting with China, Nigeria, and UK provides the following policy. To increase the growth rate of gdp—goods and services—increase the current autonomous government spending (1387063249.13) decrease exchange rate and possibly taxes and the flowing taxes of the earlier year. The growth rate will be enhanced by a positively negotiated trade policy or distance between trading partners and distance between trading nations.

Increase of cooperative ventures between the USA and China and reduction of competition will increase the growth rate of USA GDP. Analogous policies can be deduced from the model equations for dy_1/dt and dy_2/dt .

For example within the period 1980–1997 with data from the International financial Statistics Year Book 2002, the net growth rate of gdp of China can be increased by decreasing the current taxes in Ta_1 and for the USA by decreasing the current tax rate in T_2a .

Appropriate policies can be deduced also from the programs in the Appendix.

Indeed in [Chap. 14](#), the data and the MATLAB programs can yield usable economic policies for all members of the United Nations. The following steps can be taken. The execution of regression on

$$r_i(t) = y_i(t) \sum_{j=1}^{141} b_{ij} y_j(t-h),$$

where r_i is the inflow of wealth into nation i from nations $j = 1; 141$, y_i is the GDP of nation i and y_j is that of nation j can help generate the dynamics and the realistic coefficient of interaction b_{ij} .

[Chapter 5](#) in “The System Identification Problem” Eqs. (5.15), (5.16) and the Arx models in [Sect. 7.1](#) give a good insight of what to do. See “System Identification Toolbox for Use with MATLAB, by Lennart Ljung. A little touching over of the data for each nation maybe helpful. It will be a good project for the reader to execute the programs. A built-in MODSTRUCT analysis and predict command

can help better forecasting. See page I-35 of Lennart Ljung and, Chukwu, “Optimal Control for the Growth of Wealth of Nations”, [Sect. 2.6](#), Taylor and Francis, ISBN 0-415-26966-0. In the first step we let

$$y_{ir} = \frac{dy_i(t)}{dt} - a_{-1i} \frac{dy_i(t-h)}{dt},$$

$$\ln_i = y_i(t) \sum_{\substack{j=1 \\ j \neq i}} b_{ij} y_i(t-h),$$

then

$$s_{ir} = a_{0i} y_i(t) + a_{1i} y_i(t-h) + \ln_i + p_i(t) + g_i(t),$$

is equivalent to (12.2.1). Let

$$z_i = [y_{ir}, s_{ir}].$$

We have now formed an array z_i that consists of y_{ir} and s_{ir} . We then estimate the parameter

$$a_{-1i}, a_{0i}, a_{1i}, b_{ij}, \lambda_i / (1 - \lambda_i z_{i3})$$

With the data in [Chap. 11](#), or Appendix, or Atlantis Press Website we can identify completely the Eq. (3.12) for each nation i , 1, ..., 191. Economic policies are then apparent.

For the time intervals cited in the earlier book [5] the executed programs of Austria, Austri22.m, pp. 151–162, South Africa, Southa22.m, pp. 163–180, USA, US2.m, pp. 305–316, Italy, Italy2.m, pp. 325–346, India, India2.m can be mined for economic and financial policies. In the same way the other nations of the UN can be studied with data and program in [Chap. 11](#), Appendix, or Atlantis Press Website.

References

1. Charlie Rose, MSNBC 03/28/2007, The study of longevity
2. Chukwu EN (2005) A mathematical treatment of economic cooperation and competition among nations, with Nigeria, USA, UK, China, and Middle East examples. Elsevier, Amsterdam
3. Chukwu EN (2000) Goodness through optimal dynamics of the wealth of nations. Third world congress of nonlinear analysis (WCNA 2000) Catania, Italy, 20 July 2000

Chapter 10

Program Results of Some Nations: Austria and France

Just as in Botwana2.m policies can be deduced from the equation for y . It was executed by E. Chukwu. The raw data was taken from International Yearbook publications for 2003 and 2007.

The equation $q_1(t)$ yield the government strategies and the equation $r_1(t)$ the representative firms strategy.

Just as in (7.21) and (7.23)

$$q_1(t) = a_{19}(g_0 - (C_7 + I_{13}))M + a_{15}e(t) + a_{16}\tau(t) + a_{17}d(t) - a_{19}T_1$$

$$r_1(t) = a_{18}r_0 = \begin{bmatrix} a_{18} & a_{18} & a_{18} & a_{18} & a_{18} & (C_7 + I_{13}) \end{bmatrix} \begin{bmatrix} C_0 \\ I_0 \\ X_0 \\ M_0 \end{bmatrix}$$

The coefficients for q_1 and r_1 can be identified from the regression and program France2.m. Economic policies for growth can be deduced.

In this Section, Program Result for Some Nations, Austria and France, we display samples of what can be generated from the data, the programs and their execution from the contents of CD-Rom of the 141 nations which is on the Atlantis Web page. The identified government strategies and the equation of the representative firms' strategies yield testable policies for the economic growth of each nation. For example, the identified dynamics of the gross domestic product, y , of Austria is

$$\begin{aligned} \dot{y}(t) &- (-0.0226) \times y'(t-h) - (0.3520) \times L'(t-h) \\ &= (-1.0389) \times y(t) + (0.0465) \times y(t-h) + (-0.674) \times R(t) \\ &\quad + (4.3995) \times R(t-h) + (0.2759) \times L(t) + (0.0305) \times L(t-h) \\ &\quad + (-0.1535) \times k(t) - (-7.5416e^{-011}) \times p(t) + q_1(t) + r_1(t), \end{aligned}$$

where the quarry, government strategy

$$\begin{aligned} q_1(t) = & (0.0011) \times [(-0.0032) - (-3.1322e^{-005}) \times T(t) \\ & + (-2.9347e^{-007}) \times T(t-h) - (5.4070e^{-007}) \times T'(t) \\ & - (9.1153e^{-009}) \times T(t-h) - (0.8547) \times e(t) \\ & + (0.0029) \times tar(t) + 0.d(t), \end{aligned}$$

and the representative private firm's strategy

$$\begin{aligned} r_1(t) = & (0.0011)[((-0.0017) + (-8.0393) + (-15.7276)) \\ & - (-0.0038)(-0.3996 + 3.1322e^{-005})]. \end{aligned}$$

Thus if the hereditary growth rate of employment is steady the hereditary growth rate of GDP,

$$\frac{dy(t)}{dt} + (0.0226) \times y'(t-h),$$

is increased when autonomous government outlay or spending g_0 is increased. Decrease in current growth rate of taxes, decrease in exchange rate and previous year tax rate will increase the hereditary growth rate of GDP. Current increase in taxes, current increase in tariffs and a positive trade policy and/or distance between trading nations which is carefully negotiated will enhance the growth rate of GDP.

A strong increase of autonomous private spending or investment, autonomous consumption, autonomous net export, will reduce the negative values identified for this country or make them positive just as it was said about Botswana, and thus increase the growth of GDP.

Detailed analysis can be made from the identified equations for interest rate, employment, value of capital stock, growth of prices (inflation) and cumulative balance of payment, and good policies deduced.

The time duration is important in the simulation and identification of the economic dynamics.

If the IMF data is incomplete, as presented, one can extract more data from outside the interval considered or from interpolation or from other sources such as the UN. The program provides a guide. Examination of the execution and display of USA.2 m, UK.2 m, etc. in the author's earlier book, "A Mathematical Treatment of Economic Cooperation and Competition Among Nations with Nigeria, USA, UK, China, and Middle East Examples," Elsevier, 2005, is a very good resource.

Running the programs and deducing policies is very good training for emerging policy makers. It may help to download "An Electronic Appendix containing algorithms and data available from <http://www.elsevier.nl/locale/ame>"

E. N. Chukwu, "On the Controllability of Non-linear Economic Systems with Delay: The Italian Example," Applied Mathematics and Computation, Vol. 95 (1998) pp. 245-274.

```
% FRANCE2.m
% Note wherever possible the following exchange currency:
% 1 Euro = 6.55957 French Franc
% 1 French Franc (FRF) = 0.15245 Euro (EUR)
% 0.15245 Euro = 0.19718 US Dollar
% 0.15245 US Dollar (USD) = 0.11787 Euro (EUR)

% Date: 25-May-2009
% France2.m data is Billions of Francs.
% Adopted from the International Financial Statistics Yearbook 2006, except where noted.
% The 2006 Statistics Yearbook publication range between year 1994-2005(a total of 12
% years) and as a results lead to extrapolation of more data for 1992, 1993, 2006 and
% 2007 to give additional statistics for running regression for y(t) and Balance of
% payment, B(t). Please see the two model equation described below, their lengths were
% out of ranged beyond what the 12 years could handle.

% Use NDU.m to obtain the delay and derivative matrix entries.
% Use Alphacomputation.m to estimate "a" and "1-a"
% The rest of the results obtained from the regression analysis can
% be found in France2_output_report_latest_release.m.

% The final results of each differential equation based on the regression analysis(arx)have been
% reproduced with the missing constants included in the section after the plot section.
year =[1992 1993 1994 1995 1996 1997 1998 1999 2000 2001
2002 2003 2004 2005 2006 2007]';
EX =[1529 1494.8 1610.2 1784.7 1866.9 2136.8 2278.5 2345.687111 2700.55756
2761.561168 2754.345687 674.319449 801.574287 927.517219 053.460151 79.403083]';
IMP =[1489.2 1385.8 1510.4 1695.4 1745.8 1895.6 2056.7 2158.084618 15.2837
2648.737291 2580.518203 2568.711053 2774.680223 3034.43752
3294.194818 3553.952115]';
TX =[2845.5 2871.3 2983.5 3116.5 3271.2 3438.7 3354.95 3396.83 375.89 3386.36 3381.13
3383.75 3382.44 3383.1 3383.76 3384.42]';
K =[1491 1398.2 1428.4 1419.6 1440.5 1451.4 1555.4 1683.830764 1841.259429
1912.758281 906.198754 1971.138078 2088.553624 2207.937029 2327.320433 2446.703837]';
KD =[1514.6 1491 1398.2 1428.4 1419.6 1440.5 1451.4 1555.4 1683.830764 1841.259429
1912.758281 1906.198754 1971.138078 2088.553624 2207.937029 2327.320433]';
KP =[ -92.8 30.2 -8.8 20.9 10.9 104 128.4307642 157.4286651 71.49885208
-6.559527714 64.93932437 117.4155461 119.3834044 119.3834044 119.3834044
19.3834044]';
D =[ -3.6 -90 -5.2 34.8 -19.7 -5.9 75.5 7.7 14.3 8.9 3.2 -0.2 4.4
6.9 11.1 15.3]';
I =[1487.4 1308.2 1423.2 1454.4 1420.8 1445.5 1630.9 1691.530764 1855.559429
1921.658281 1909.398754 1970.938078 2092.953624 2214.837029 2338.420433
2462.003837]';
kk =[ -92.8 30.2 -8.8 20.9 10.9 104 128.4307642 157.4286651 71.49885208 -6.559527714
64.93932437 117.4155461 119.3834044 119.3834044 119.3834044 119.3834044]';
n =[21609 20705 21875 20233 22311 20413 22479 20864 23262 23759 23942 24691 24784
24919 24745 24571]';
e =[7.5714 8.0978 7.8044 7.2838 7.5306 8.0794 7.9161 8.961626763 9.184650705 9.35388652
8.503771728 7.717284356 7.4791735 7.947523778 8.415874057 8.884224336]';
P =[94.7 96.6 92.5 94.1 96 97.2 97.8 98.3 100 101.7 103.6 105.8 108 110 112
114]';
Pf =[717.01158 782.24748 721.907 685.40558 722.9376 785.31768 774.19458
880.9279108 918.4650705 951.2902591 880.9907511 816.4886848
807.7507379 874.2276156 942.5778944 1012.801574]';
Pfe =[5428.781477 6334.483644 5634.050991 4992.357164 5444.153891 6344.895664
6128.601715 7894.547142 8435.780858 8898.261131 7491.744242 6301.075354 6041.307913
6947.944763 7932.616848
8997.956394]';
ta =[1268.37 1149.13 1298.43 1403.8 1441.62 1585.34 1708.92 1817.185963 2214.955723
2211.872745 2169.891768 2147.523778 2334.601509 2545.031158 2755.460807
2965.890456]';
F =[19417.35505 12376.3665 5335.37795 -1705.6106 -8745.59915
-15785.5877 -22825.57625 -29865.5648 -36905.55335 -48628.1961
-56183.3132 -68436.17705 -75343.839 -71356.3568 -67368.8746
-63381.3924]';
```

```

B =[-838043.1949 -535296.7645 -232487.1342 70360.79606 373150.3264
676015.3567 978992.137 1281828.49 1584659.531 2089071.495 2414214.803 2936118.382
3232288.928 3185437.497 3957228.244 4729018.991]';
BD =[-1140819.925 -838043.1949 -535296.7645 -232487.1342 70360.79606
373150.3264 676015.3567 978992.137 1281828.49 1584659.531 2089071.495 2414214.803
2936118.382 3232288.928 3185437.497 3957228.244]';
T =[2845.5 2871.3 2983.5 3116.5 3271.2 3438.7 3354.95 3396.83 3375.89
3386.36 3381.13 3383.75 3382.44 3383.1 3383.76 3384.42]';
L =[21609 20705 21875 20233 22311 20413 22479 20864 23262 23759 23942 24485 24720
24919 24745 24571]';
LD =[22316 21609 20705 21875 20233 22311 20413 22479 20864
23262 23759 23942 24485 24720 24919 24745]';
LP =[-904 1170 -1642 2078 -1898 2066 -1615 2398 497 183 543 235 199 -174 -174 -
24571]';
LPD =[-707 -904 1170 -1642 2078 -1898 2066 -1615 2398 497 183 543 235 199 -
174 -174]';
w =[93.4 96.3 86.3 87 88.5 91 93.6 95.7 100 104.5 108.4
112.8 116.1 119.2 122.7 126.2]';
Y =[7119.2 7227.2 7488.2 7759.9 7955.2 8206.9 9564.4 8963.594621
9454.903247 9820.924893 10158.08462 10461.1348 10882.25648 11216.79239 11551.3283
11885.86422]';
YD =[6895.3 7119.2 7227.2 7488.2 7759.9 7955.2 8206.9 9564.4 8963.594621
9454.903247 9820.924893 10158.08462 10461.1348 10882.25648 11216.79239 11551.3283]';
GDP =[7119.2 7227.2 7488.2 7759.9 7955.2 8206.9 9564.4 8963.594621
9454.903247 9820.924893 10158.08462 10461.1348 10882.25648 11216.79239 11551.3283
11885.86422]';
YF =[108 261 271.7 195.3 251.7 1357.5 -600.8053788 491.3086258 366.0216464 337.1597245
303.0501804 421.1216792 334.5359134 334.5359134 334.5359134 334.5359134]';
YFD =[223.9 108 261 271.7 195.3 251.7 1357.5 -600.8053788 491.3086258 366.0216464
337.1597245 303.0501804 421.1216792 334.5359134 334.5359134 334.5359134]';
G =[3154.7 3336.9 3458.2 3564.7 3687.2 3789.2 3738.2 13626.77879 14203.346
14680.95019 15584.52567 16281.56961 16935.58615 17474.91499 17969.24384
18484.57269]';
C =[1644.3 1769.4 1797.6 1850.9 1922.1 1984.7 2004.5 2077.402427 2165.300098
2238.110856 2375.860938 2482.125287 2581.830108 2660.544441 2739.258773 2817.973106]';
M =[1603 1626 1671 1800 1815 1933 1993 2042 2111 2163 2253 2297 2360 2417 2474
2523]';
QM =[2807 2854 3003 3246 3363 3624 3781 3948 4065
4256 4419 4568 4729 4890 5047 5194]';
QMD =[2845 2807 2854 3003 3246 3363 3624 3781 3948
4065 4256 4419 4568 4729 4890 5047]';
QMP =[47 149 243 117 261 157 167 117 191 163 149 161 161 157 147 151]';
QMFD =[-38 47 149 243 117 261 157 167 117 191 163 149 161 161 157 147]';
P =[94.7 96.6 92.5 94.1 96 97.2 97.8 98.3 100 101.7 103.6 105.8 108 110 112 114]';
R =[10.35 8.755 6.696 3.533 3.24 3.393 3.593 4.9
3.85 3.91 4.18 4.39 4.38 4.61 4.74]';
RD =[9.49 10.35 8.75 5.69 6.35 3.73 3.24 3.39 3.59 3.49 3.85 3.91 4.18 4.39 4.38
4.61]';
RP =[-1.6 -3.06 0.66 -2.62 -0.49 0.15 0.2 -0.1 0.36 0.060 0.27 0.21 -0.01 0.23 0.13
0.13]';
RPD =[0.87 -1.6 -3.06 0.66 -2.62 -0.49 0.15 0.2 -0.1 0.36 0.06
0.27 0.21 -0.01 0.23 0.13]';
Y =[7119.2 7227.2 7488.2 7759.9 7955.2 8206.9 9564.4 8963.594621 9454.903247
9820.924893 10158.08462 10461.1348 10882.25648 11216.79239 11551.3283 11885.86422]';
TX =[2845.5 2871.3 2983.5 3116.5 3271.2 3438.7 3354.95 3396.83 3375.89 3386.36
3381.13 383.75 3382.44 3383.1 3383.76 3384.42]';
YT =[4273.7 4355.9 4504.7 4643.4 4684 4768.2 6209.45 5566.764621 6079.013247 6434.564893
6776.954618 7077.384798 7499.816478 7833.692391 8167.568304 8501.444218]';
YTD =[4129.1 4273.7 4355.9 4504.7 4643.4 4684 4768.2 6209.45 5566.764621 6079.013247
6434.564893 6776.954618 7077.384798 7499.816478 7833.692391 8167.568304]';
YTF =[82.2 148.8 138.7 40.6 84.2 1441.25 -642.6853788 512.2486258 355.5516464
342.3897245 300.4301804 422.4316792 333.89 333.87 333.9 334]';
YTFD =[144.6 2.2 48.8 138.7 0.6 84.2 1441.25 -642.6853788 512.2486258 355.5516464
342.3897245 300.4301804 422.4316792 333.89 333.87 333.9]';
P =[94.7 96.6 92.5 94.1 96 97.2 97.8 98.3 100 101.7 103.6 105.8 108 110 112 114]';
price =[90.3 92.5 94.7 96.6 92.5 94.1 96 97.2 97.8 98.3 100 101.7 103.6 105.8 108
110 112]';
ML =QM-M;

```

```

X= EX - IMP;

yPD =[223.9  108261  271.7  195.3   251.7  1357.5  -600.8053788  491.3086258  366.0216464
      337.1597245  303.0501804  421.1216792  334.5359134  334.5359134  334.5359134]';

%T= X-B-F; See above for entries
for i=1:16
    pp(i)=price(i+1)-price(i);
    %kprim(i) =kk(i+1)-kk(i);
end
pie=[pp]';
%KP =kprim';
DP =pie.\P;
%n=y.\L;

% Statistics Number, Statistics Name, Full Name
% 81, TX, Tax revenue
% TX = total current receipts of general government(Billions of FRENCE Francs currency)
% 65, w, Wages: Average Monthly earnings(termed compensation according to National
% accounts statistics, United Nations publication,
% page 116, 2003 edition) defined as compesation of employees from the rest
% of the world. see statistics entry number "65".
%
% 90c, EX, Exports of goods and services
% 98c, IMP, Imports
% X, EX-IMP(Net export)
% 67e, n, Labor productivity(A/1)=GDP.\emp; FOR FRANCE WE USE the employment
% available for 1994-2005. Other years were gotten by considering the
% labor activity(percentage)
% factor to population for that year.
% 93e, K, Capital stock(Gross fixed capital formation)
% K(t+1)-K(t), KP, Derivative Gross Fixed Capital Formation
% K+D, I, Investment(Gross capital formation=K+D)
% 93i, D, Increase in stock(changes in inventories)
% The UN(National accounts statistics) defines Gross capital formation
% as the sum of the increase in stocks and gross fixed capital formation
%
% 99b, y or GDP, Gross domestic product
% y(t-1), yD, Shift "right" in time Gross Domestic Product
% y(t+1)-y(t), y'(t), yP, Derivative of Gross Domestic Product
% yP(t+1), yPD, y'(t+1), Delay in time Derivative of (GDP)
% ignore, d, (Dummy)Differential trade agreement transportation
% Pf Import price level in foreign currency
% Pfe Import price level in foreign currency(p*e*e)
% 64, P, Consumer prices
% 71, ta, Tariffs(Import duties = custom duties plus other import charges)
% E, Cumulative balance of payment
% X-F-TX, B, Balance of payment(Deposit money banks=E'(t))
% X-B-F, T, Net government transfer of capital to foreigners and firms
% 37r, F, Net private outflow of capital
% 34, M, Money supply
% 35, QM, Money demand(Quasi Money)
% 59ma, Using M1 for M since Money supply was not available.
% 59mb, Using M2 for QM since Money demand was not available.
% 67e, L, Employment, Labour(Industrial production)
% emp =Employment(Labor force at equilibrium)
% unemp =Unemployment
% 34, ML, QM-M Money
% 60b, R, Interest rate(Money Market rate)
% R(t+1), RD, Interest Rates Bank Rate delayed
% R(t+1)-R(t), RP, Derivative of interest Rates(End)
% R(t+1), RPD, Derivative of interest Rates(End)delayed
% 82, G, Expenditure
% 91f, C, Government consumption Expenditure
% aa, e, Exchange rate(Market rate=aa)

```

```

% J3 Direct investment in Republic Economy
% GNI Gross National income
% PL Population
% YT, GDP-Revenue, y-TX
% YTD, YT delayed, YT(t+1)
% YTP, Derivative YT, YT(t+1)-YT(t)
% YTPD, YTP delayed, YTP(t+1)

format short;
%Bertl=[X,G,I,C,GDP,PGDP,R,M,QM,YEAR]
[years,nn]=size(year);
Xl=[29.2;X(1:years-1)];
G1=[3032.2;G(1:years-1)];
I1=[1305.1;I(1:years-1)];
C1=[1584.4;C(1:years-1)];
QM1=[2031.0;QM(1:years-1)];
R1=[3.05;R(1:years-1)];
GDP1=[6247.4;GDP(1:years-1)];
DGDP=GDP-GDP1;
Z=X+G+I+C;
Z1=X1+G1+I1+C1;

%temp=[YEAR X1 G1 I1 C1 Z1 y1 yD M1 RD1 yPD1 RP RPD T1 P P-PF P1 P-PF1 G QM QMD QMP QMPD QM1
QMD1 QMP1 QMPD1 ta tal e el d dl];
%save Bertl.asc temp -ascii -double -tabs
%clear Bertl temp;
%ML = QM-M = m1-m0 -m1*y(t) - m2*y(t-h) - m3*R(t) - m4*R(t-h) - m6*R(t) - m7*R'(t-h)

%QM(t) = m0 + m1*y(t) + m2*y(t-h) + m3*R(t) +m4*R(t-h) +m5*R'(t-h) +m6*R(t);
temp=[QM ones(size(y)),y,yD,R,RD,RPD,P];
thqm=arx(temp,[0,1 1 1 1 1 1 1 1 0 0 0 0 0 0 1 0]);
QMp=predict([QM,ones(size(y)),y,yD,R,RD,RPD,P],thqm,4);
QM=thqm(6,1:7);

%C(t) = c0 + c1*YT(t) +c2*YT(t-h) + c3*YT'(t) + c4*YT'(t-h) +c5*R(t) +c6*R(t-h)
+c7*(a1+a2.*y(t) +a3.*yD(t-h) + a4.*R(t) +a5.*RD(t) +a6.*P(t) +a.*RPD(t)) -M(t);
temp=[C ones(size(R)),YT,YTD,YTP,YTPD,R,RD,ML];
thc=arx(temp,[0 1 1 1 1 1 1 1 1 1 0 0 0 1 1 0 0 0]);
Cp=predict([C,ones(size(R)),YT,YTD,YTP,YTPD,R,RD,ML],thc,4);
CC=thc(7,1:8);

%I(t) = i0 + i1*y(t) + i2*y(t-h) -i3*y'(t) +i4*y'(t-h) +i5*R(t) +i6*R(t-h) +i8*L(t) +i9*L(t-h) -
i11*K(t)-i13*ML(t)
temp=[I ones(size(y)),y,yD,yP,yPD,R,RD,QM,QMD,K,ML];
thi=arx(temp,[0, 1 1 1 1 1 1 1 1 1 1 0 0 1 0 1 0 1 0 0 0],11);
Ip=predict([I,ones(size(y)),y,yD,yP,yPD,R,RD,QM,QMD,K,ML],thi,4);
II=thi(10,1:11);

%X(t) =x0+ x1*y(t) +x2*y(t-h) +x3*y'(t) +x4*y'(t-h) +x5*R(t) +x8*L(t) +x10*L'(t-h) +x12*P(t)
+x16*ta(t) +x15*e(t) +x17*d(t)
temp=[X ones(size(y)),y,yD,yP,yPD,R,QM,QMPD,P,ta,e];
thx=arx(temp,[0,1 1 1 1 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1 0 0 0]);
Xp=predict([X,ones(size(y)),y,yD,yP,yPD,R,QM,QMPD,P,ta,e],thx,4);
XX=thx(11,1:12);

%G(t) = g0 +g1*y(t) +g2*y(t-h) +g3*y'(t) +g4*y'(t-h) +g5*R(t) +g8*L(t)
temp=[G ones(size(y)),y,yD,yP,yPD,R,QM];
thg=arx(temp,[0,1 1 1 1 1 1 1 0 0 1 0 1 0 0]);
Gp=predict([G,ones(size(y)),y,yD,yP,yPD,R,QM],thg,4);
GG=thg(6,1:7);

%y(t) zs0 +zsl*y(t) +zs2*y(t-h) +zs4*y'(t) +zs5*R(t) +zs8*L(t) +zs10*L'(t) +zs13*M1(t) -
zs14YT(t) +zs15*e(t) +zs16*ta(t) +zs17*d(t)*(0) +1/h*K'(t); h=1
temp=[y ones(size(y)),y,yD,yP,R,QM,QMP,M,TX,e,ta,KP];
thy=arx(temp,[0,1 1 1 1 1 1 1 1 1 1 1 1 0 0 1 0 0 0 0 0 0 0 0]);
yp=predict([y,ones(size(y)),y,yD,yP,R,QM,QMP,M,TX,e,ta,KP],thy,4);
yy=thy(11,1:12);

```

```

%cy(t) = y10 + y11*y(t) + y12*y'(t) + y13*R(t) + y15*ML(t) + y18*YT(t)
temp=[C ones(size(y)),y,yP,R,ML,YT];
thcy=arx(temp,[0,1 1 1 1 1 0 0 0 0 0]);
cyp=predict([C,ones(size(y)),y,yP,R,ML,YT],thcy,4);
ccy=thcy(5,1:6);

%Iy(t) = I0 + 1/h*KP(t)
temp=[I ones(size(KP)),KP];
thIy=arx(temp,[0,1 1 0 0]);
Iyp=predict([I,ones(size(y)),KP],thIy,4);
IIy=thIy(1,1:2);

%Gy(t) = gs0 + gs1*y(t) +gs4*y'(t) +ys5*R(t) + gs8*QM(t)
temp=[G ones(size(y)),y,yP,R,QM];
thgy=arx(temp,[0,1 1 1 1 1 0 0 0 0]);
gyp=predict([G,ones(size(y)),y,yP,R,QM],thgy,4);
ggy=thgy(4,1:5);

%Xy(t) x0 + x1*y(t) + x2*y'(t) + x5*R(t) +x8*QM(t) +x10*QM'(t) +x11*e(t) +x12*ta(t) +x13*d(t)*(0)
temp=[X ones(size(y)),y,yD,R,QM,QMP,e,ta];
thxy=arx(temp,[0,1 1 1 1 1 1 1 1 0 0 1 0 0 0 0]);
xyp=predict([X,ones(size(y)),y,yD,R,QM,QMP,e,ta],thxy,4);
xxy=thxy(7,1:8);

%B(t) = b0 +b1*y(t) +b2*y(t-h) +b3*P(t) +b4*y'(t-h) +b5*R(t) +b6*R(t-h) +b7*e(t) +b8*R'(t-h)
+b9*QM(t) +b10*QM(t-h) +b12*L'(t-h) +b13*ta(t) +b15*d(t) +b17*B(t-h)
temp=[B ones(size(y)),y,yD,P,yPD,R,RD,e,RPD,QM,QMD,QMPD,ta,BD];
thb=arx(temp,[0,1 1 1 1 1 1 1 1 1 1 1 0 0 1 0 1 0 1 0 1 0 1 1]);
Bp=predict([B,ones(size(y)),y,yD,P,yPD,R,RD,e,RPD,QM,QMD,QMPD,ta,BD],thb,4);
BB=thb(13,1:14);

%y(t) = A*K^a*L^(1-a) ==> ln(y(t)) = ln(A)+ a*ln(K) + (1-a)*ln(L)
temp=[log(y) ones(size(y)),log(K),log(L)];
thly=arx(temp,[0,1 1 1 0 0 0]);
lyp=predict([log(y) ones(size(y)),log(K),log(L)],thly,4);
lyy=thly(2,1:3);

%w = real wage rate per unit time
%W = sum(w)/16
%mw = [1-lyy(2)/(W)]^(1/lyy(1))
% Previously commented out. For additional help estimating "aa" and "a" please
% see Alphacomputation.m program for estimation procedure used.

%D(t) = a(1-c)*y(t) -k0*K(t) +k13*K'(t) +L4*R(t) +L5*QM(t) +L6*P(t) +v*y'(t)
temp=[D ones(size(y)),y,K,KP,R,QM,P,yP];
thd=arx(temp,[0,1 1 1 1 1 1 1 1 0 0 0 0 0 0 0]);
Dp=predict([D,ones(size(w)),y,K,KP,R,QM,P,yP],thd,4);
DD=thd(7,1:8);

%DP(t) = p0 +p1*Pfe(t) p2*w(t) -p3*n(t) -p4*P(t) +p5*M'(t) +p6*ML(t)
%M'(t)=differential of money supply
temp=[DP ones(size(w)),Pfe,w,n,P,QMP,ML];
thp=arx(temp,[0,1 1 1 1 1 1 1 0 0 0 0 0 0]);
dip=predict([DP,ones(size(w)),Pfe,w,n,P,QMP,ML],thp,4);
dp=thp(6,1:7);

%-----
subplot (2,1,1),plot(year,C,year,Cp,'--')
title('FRANCE--GOVERNMENT CONSUMPTION EXPENDITURE--');
ylabel('C');
xlabel('year');
subplot(2,1,2),plot(year,X,year,Xp,'--')
title('FRANCE--EXPORTS--');
ylabel('X');
xlabel('year');
pause
clf
subplot (2,1,1),plot(year,I,year,Ip,'--')

```

```

title('FRANCE--INVESTMENT--');
ylabel('I');
xlabel('year');
subplot(2,1,2),plot(year,G,year,Gp,'--')
title('FRANCE--GOVERNMENT CONSUMPTION--');
ylabel('G');
xlabel('year');
pause
clf
subplot(2,1,1),plot(year,QM,year,QMp,'--')
title('FRANCE--MONEY DEMAND--');
ylabel('QM');
xlabel('year');
subplot(2,1,2),plot(year,Z,year,[Xp+Ip+Gp+Cp],'--')
title('FRANCE--AGGREGATE DEMAND--');
ylabel('Z');
xlabel('year');
pause
clf
%-----

subplot(2,1,1),plot(year,P,year,dIp,'--')
title('FRANCE--CONSUMER PRICES--');
ylabel('P');
xlabel('year'); subplot(2,1,2),plot(year,y,year,yp,'--')
title('FRANCE--INCOME--');
ylabel('GDP');
xlabel('year');
pause
clf
subplot(2,1,1),plot(year,C,year,cyp,'--')
title('FRANCE--INCOME/CONSUMPTION--');
ylabel('cy');
xlabel('year');
subplot(2,1,2),plot(year,I,year,Iyp,'--')
title('FRANCE--INCOME/INVESTMENT--');
ylabel('Iyp');
xlabel('year');
pause
clf
subplot(2,1,1),plot(year,G,year,gyp,'--')
title('FRANCE--INCOME/GOVERNMENT--');
ylabel('gyp');
xlabel('year');
subplot(2,1,2),plot(year,X,year,xyp,'--')
title('FRANCE--INCOME/EXPORT--'); ylabel('xyp');
xlabel('year');
pause
clf
%-----
subplot(2,1,1),plot(year,B,year,Bp,'--')
title('FRANCE--BALANCE OF PAYMENT--');
ylabel('B');
xlabel('year');
subplot(2,1,2),plot(year,y,year,lyp,'--')
title('FRANCE--LOG(income)--');
ylabel('lny');
xlabel('year');
pause
clf
subplot(2,1,1),plot(year,D,year,Dp,'--')
title('FRANCE--INCREASE IN STOCK--');
ylabel('D');
xlabel('year');
%-----

% The economic dynamics can be put in matrix form as
% follows:  $x'(t) - A \cdot x(t-h) = A_0 \cdot x(t) + A_1 \cdot x(t-h) + B \cdot u(t)$ 

```

```

%
% x=[y R L K P E]';
%
% A-1=[a-11 0 a-13 0 0 0; 0 a-22 0 0 0 0; L-03 0 -L-01 0 0 0; a3 0 a6 a-1 0 0; 0 -M7p6P(t) 0
% 0 0 0; -b4 -b8 -b12 0 0 -b14];
%
% A0=[a01 a12 a14 a16 a18 0; a21 a23 0 0 a25 0; 0 0 a33 0 a35 0; 0 0 0 a44 a45 0; a51 a52 0 0
% a55 0; a61 a62 a63 0 0 0];
%
% A1=[a111 a112 a113 0 0 a116;a121 a122 0 0 0 0; a131 a132 a133 0 0 0; a141 a142 a143 0 0 0;
% a151 a152 a153 a154 a155 a156; a161 a162 a163 a164 a165 a166];
%
% q = [T g0 e ta d M M' f0]';
% S = [C0 I0 X0 (p0-p6M0) n w x0 y10]
%
% w = real wage rate per unit time
% m(w) = [1-lyy(2)/(w)]^(1/lyy(1)) % See Alphacomputation.m program for estimation procedure
% used.
% where w = average of w(t)
% B1=[-biz14 b1 -biz15 biz16 biz17 biz13 0 0; 0 0 0 0 0 -11 0 0; -m(w)zsl4 m(w) m(w)zsl5
% m(w)zsl6 m(w)zsl7 m(w)zsl3 0 0; -zsl4 1 zsl5 zsl6 zsl7 zsl3 0 0; 0 0 plpf(t) 0 0 p6 p5 0; 0
% 0 b7 b8 b15 0 0 1];
%
% B2=[ss1 ss1 ss1 -ss1(I13+c7) 0 0 0 0 0; 0 0 0 -12 0 0 0 0; 0 m(w) 0 0 0 0 m m 0; 0 1 0 0 0
% 0 0 0 0 -p6 -p3 p2 0 0 1; 0 0 0 -1 0 0 1 0 0]
%
% B = [B1 B2];
% u = [q S];
%
% It follows then that the differential equation from our regression analysis(arx) treated above
% can be
% rewritten with their missing constants included as:
%
% For  $y(t) = A \cdot K^a \cdot L^{1-a}$ 
%
% Cobb Douglas states: aa+a = 1 => constant return to scale
% aa+a > 1 => increasing return to scale
% aa+a < 1 => decreasing return to scale
%
% Increasing return to scale if doubling L & K more than doubles y.
% constant return to scale if doubling L & K exactly doubles y
% decreasing return to scale if doubling L & K less y.
%
% Initial estimates of a =alpha, see Alphacomputation.m program were "solve" is used.
% Initial estimates of aa =(1-a)= beta, see Alphacomputation.m program were "solve" is used.
%
% [logA,a,aa] = solve('log(y(1)) - a*log(K(1)) - (aa)*log(L(1))','(log(y(9)) - logA -
% (aa)*log(L(9)))/log(K(9))','(log(y(16)) - logA - a*log(K(16)))/log(L(16))')
%
% Hence:
logA =(log(L(1))*log(y(9)/y(16))+log(y(1))*log(L(9)))/(log(L(9))*log(K(1))
+ log(K(16))*log(L(1)))
%
% Solve for a & aa:
a = (log(K(16))*log(y(1))+log(y(9)/y(16))*log(K(1)))/(log(L(9))*log(K(1))
+log(K(16))*log(L(1)))
aa =(log(y(9))*log(L(9))*log(K(1))+log(y(9))*log(K(16))*log(L(1)) -
log(L(9))*log(K(16))*log(y(1)) -log(L(9))*log(y(9)/y(16))*log(K(1)))/(log(L(9))*log(K(1)) +
log(K(16))*log(L(1)))
%
After "Solve"
logA = 0.6044
a = 0.4463
aa = 4.6673
%
W = sum(w)/16

```

```

mw = (aa/W.^(1/a))

% Where, W = 102.6063

% We extract the following equation of the economic state: x = [y, R, L, K, P, E]'.

dy/dt - aml1*y'(t-h) - aml3*L'(t-h) = a01*y(t) + a11*y(t-h) + a12*R(t) + a13*R(t-h)
      + a14*L(t) + a15*L(t-h) + a16*k(t) - a18*p(t) + q1(t) + r1(t)

dy/dt - (-4.5207e-006)*y'(t-h) - (-1.2090e-007)*L'(t-h) = (1.000)*y(t)
      + (-1.000)*y(t-h) + (7.0102e-004)*R(t) + (0.0014)*R(t-h)
      + (2.5715e-005)*L(t) + (-5.5078e-007)*L(t-h) + (-1.3845e-007)*K(t)
      - (4.5881e-009)*p(t) + q1(t) + R1(t)

Where,
q1(t) = 11*sigma^-1*[g0 - z14*T(t) + z19*T(t-h) - z20*T'(t) - z21*T'(t-h) - z15*e(t)
      + z16*ta(t) + z17*d(t)]
q1(t) = (-1.1523e-006)*[(-7.1876e+003) - (7.5718)*T(t) + (8.6786e+005)*T(t-h)
      - (8.6786e+005)*T'(t) - (-1.7960e-004)*T(t-h) - (3.3997)*e(t)
      + (-0.0090)*ta(t) + 0*d(t)]

r1(t) = 11*sigma^-1*[(C0+ I0 + X0) - M0(I13 + C7)]
r1(t) = (-1.1523e-006)*[(2.1726e+004) + (-362.6903) + (-20.2539)] - (6.4888)*(-1.0807
      + -1.2580e+003)]

format short;
% Note: L1(lamdal) is the speed of response os supply to deman, the speed of adjustment.
L1 =1
L2 =0.9
L5 =DD(5)
L6 =DD(6)
h = 1
z0=CC(1)+II(1)-QMM(1)*(CC(8)+II(11))+CC(1)+XX(1)
z1=GG(2)+II(2)-QMM(2)*(CC(8)+II(11))+CC(2)+XX(2)
z2=GG(3)+II(3)-CC(3)+XX(3)+QMM(3)*(CC(8)+II(11))
z3=GG(4)+II(4)+CC(4)+XX(4)
z4=GG(5)+II(5)+CC(5)+XX(5)+QMM(5)*(CC(8)+II(11))
z5=GG(6)+II(6)+CC(6)+XX(6)-CC(8)*QMM(6)
z6=II(7)+CC(7)-CC(8)*QMM(7)
z8=GG(7)+II(8)+XX(7)
z9=II(9)
z10=XX(8)
z11=-II(10)
z13=(II(11)+CC(8))*QMM(7)
z14=-CC(8) z15=XX(11)
z16=XX(10)
z17=0
z18=(CC(8)+z13)*QMM(7)
z19=CC(3)
z20=CC(4)
z21=CC(5)
zs0 =ccy(1)+IIy(1)+ggy(1)+xxy(1) ;%constant term
zs1 =ccy(2)+ ggy(2)+xxy(2) ;%y(t) term
zs2 = xxy(3) ;%y(t-h) term
zs4 =ccy(3)+ ggy(3) ;%y'(t) term
zs5 =ccy(4)+xxy(4)+ggy(4) ;%R(t) term
zs8 =xxy(5)+ ggy(5) ;%L(t) term
zs10=xxy(6) ;%L'(t) term
zs13=ccy(5) ;%ML(t) term
zs14=ccy(6) ;%T(t) term
zs15=xxy(7) ;%e(t) term
zs16=xxy(8) ;%ta(t) term

s1 =1-L1*(GG(4)-II(4)+CC(4)+XX(4))

%assumptions
yy1 = zs0

```



```

yy2 = zs1
yy3 = zs2
yy4 = zs4
yy5 = zs5
yy6 = zs8
yy7 = zs10
yy8 = zs13
yy9 = zs14
yy10= zs15
yy11= zs16

zs0= GG(1)+XX(1)+CC(1)+II(1)
a0 = DD(1)/(h*(1-yy(2)))
a1 = -DD(1)*yy3/((1-yy2)*h)
a2 = -a1*h
a3 = a0*h*yy4
a4 = DD(1)*yy5/((1-yy2)+DD(4))
a5 = DD(1)*yy6/(1-yy2)+DD(5)
a6 = a0*h*yy7
a8 = DD(6)

%A-1 starts here; am11 = a-11
am11 =L1*(GG(5)-II(5)+CC(5)+XX(5)+QMM(5)*(CC(8)+II(11)))/s1
am22 =L2*QMM(6)
am13 =L1*z10/s1
am14 =L1*z10/s1
lm03 =DD(1)*yy4*mw/(1-yy2)
lm01 =-DD(3)*DD(1)/(1-yy2)-mw*DD(1)*yy7/(1-yy2)
a3 =DD(1)*yy4/(1-yy2)
a6 =DD(1)*yy7/(1-yy2)
am1 =-DD(3)*DD(1)/(1-yy2)
m7p6 =QMM(7)*dp(6)*sum(P)/21
b4 =BB(5)
b8 =BB(9)
b12 =BB(12)
b13 =0

%A0 Starts here
a01 = (L1*(GG(2)+II(2)-QMM(2))*(II(11)+CC(8))+CC(2)+XX(2)+CC(8)-1-(II(11)+CC(8))*QMM(7))/s1
a12 = L1*((GG(6)+II(6)+CC(6)+XX(6)-CC(8)*QMM(7))-(II(11)+CC(8))-QMM(3)*QMM(7))/s1
a14 = L1*(GG(7)+II(8)+XX(8))/s1
a16 = -L1*(II(10))/s1
a18 = -L1*(QMM(4)*QMM(6)*(II(11)+CC(8)))/s1
a21 = L2*QMM(2)
a23 = L2*QMM(4)
a25 = L2*QMM(7)
a33 = a0
a35 = mw*a8
a44 = a0
a45 = a8
a51 = -dp(7)*QMM(2)*sum(P)/21 %a51*P(t)
a52 = -dp(7)*QMM(4)*sum(P)/21 %a52*P(t)
a55 = -(QMM(7)+dp(5))*sum(P)/21 %a55*P(t)
a61 = BB(2)
a62 = BB(6)
a63 = BB(10)

%A1 Starts here!
a11 =L1*(GG(3)+II(3)+CC(3)+XX(3)+QMM(3)*(II(11)+CC(8)))/s1
a13 =L1*(II(7)+CC(7)-CC(8)*QMM(7))/s1 a15 =L1*(II(9))/s1
a22 =L2*QMM(3)
a24 =L2*QMM(5)
a1 =-DD(1)*yy3/((1-yy2)*h)
a2 =-a1*h
l1 =-a1
l2 =-mw*DD(1)*yy3/(1-yy2)
l4 =mw*DD(1)*yy5/(1-yy2+DD(4))
b2 =BB(3)

```

```

b6      =BB(7)
b10     =BB(11)
b17     =BB(13)
a111    =L1*(GG(3)+II(3)+XX(3)+QMM(3)*(II(11)+CC(8)))/s1
a112    =a13
a113    =a15
a114    =0
a116    =0
a121=a22 a122=a24
a131=mw*DD(1)*XX(3)/(1-(XX(2)+ggy(2)+ccy(2)))
a132=mw*DD(1)*yy5/(1-yy2)-DD(4)
a133=a5-l1*h
a141=DD(1)*yy3/(1-yy2)
a142=DD(1)*yy5/(1-yy2)+DD(4)
a143=DD(1)*yy6/(1-yy2)+DD(5)
a144=0
a145=0
a146=0
a151 = -dp(7)*sum(P)/21 %a151*P(t)
a152 = -dp(7)*QMM(5)*sum(P)/21 %a152*P(t)
a161 = BB(3)
a162 = BB(7)
a163 = BB(11)
a164 = 0
a165 = 0
a166 = BB(14)
% B1 Starts here!
ss1    =L1/s1
ss1z13 =ss1*(II(11)+CC(8))*QMM(7)
ss1z15 =-ss1*XX(7)
ss1z16 =ss1*XX(11)
ss1z17 =ss1*XX(8)
ss1z18 =ss1*z18
blz14  =-BB(2)*z14
b1      =BB(1)
blz15  =BB(2)*z15
blz16  =BB(2)*z16
blz17  =BB(2)*z17
blz13  =BB(2)*z13
mwzs14 =mw*yy9 mwzs15 =mw*yy10
mwzs16 =mw*yy11
mwzs17 =0
mwzs13 =mw*yy8
p1pf   =dp(2)*sum(Pfe)/21
P4      =dp(5)
P5      =dp(6)
b7      =BB(6)
b8      =BB(7)
b15     =BB(13)

%B2 Starts here!
ss2 =-ss1*(II(11)+CC(8))
L2  =0.9
P6  =-dp(7)
P3  =-dp(4)
P2  =dp(3)

% ln(y) =a*ln(K) + (1-a)*ln(L)
% lyy(1) =a;
% lyy(2) =1-a

a17    =L1*CC(2)/s1
a45    =a8
c0     =CC(1)
I0     =II(1)
x0     =XX(1)
p0     =dp(1)
qm0    =QMM(1)

```

```

y10    =ccy(1)
p0p6m0=dp(1)-dp(7)*QMM(1)
g0      =GG(1)
f0      =BB(1)-XX(1)
g2      =-L2*QMM(1)
a25     =L2*QMM(7)

```

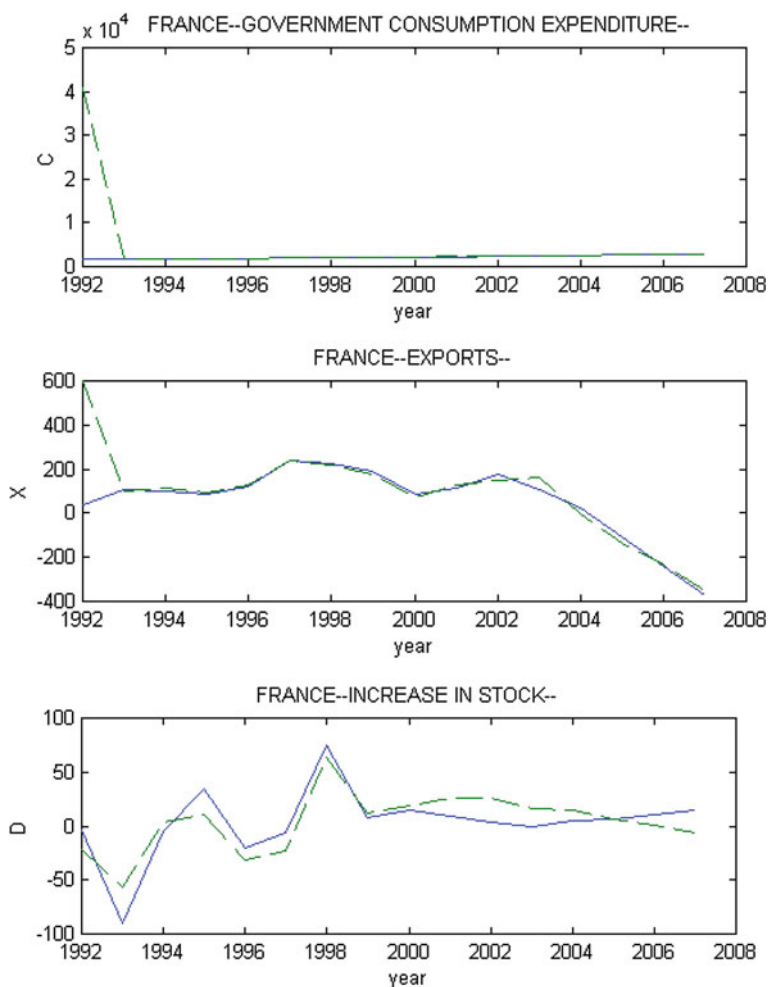
```

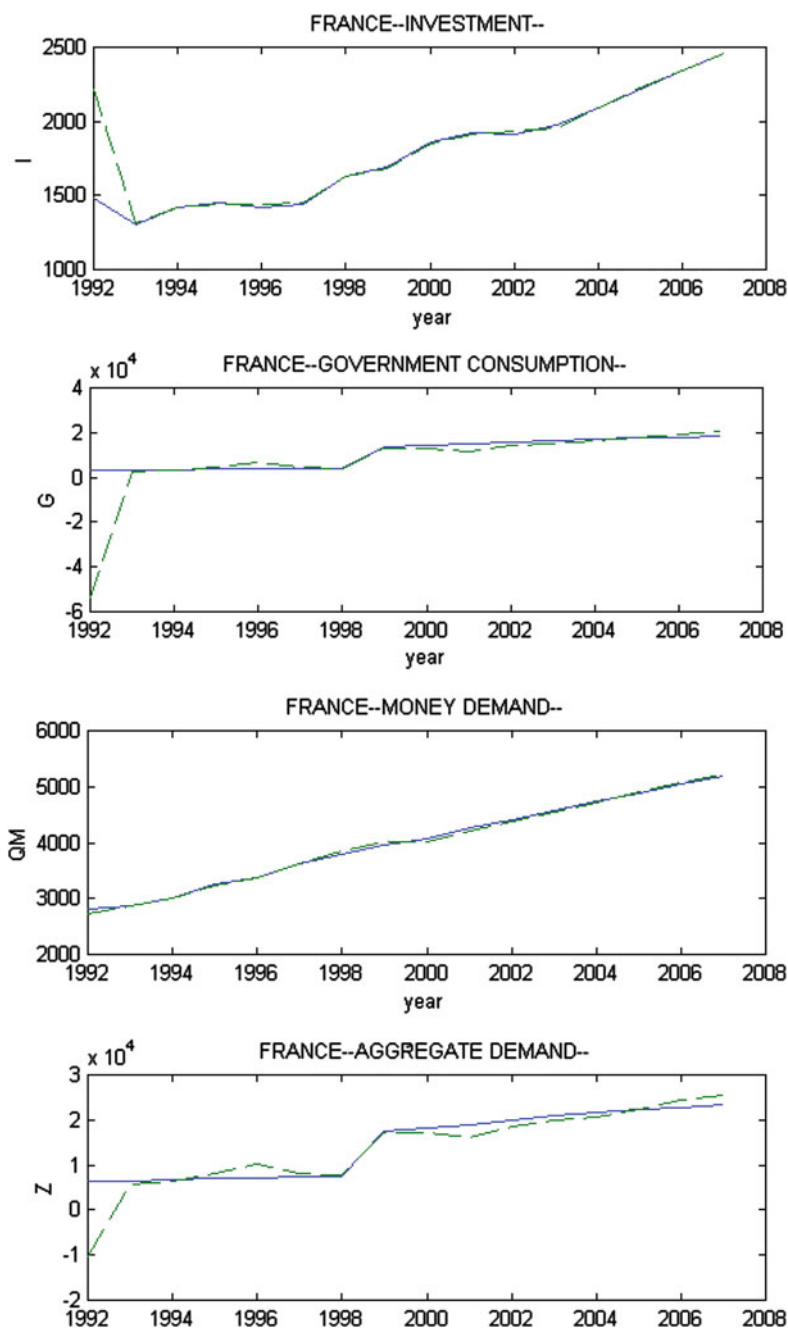
a115    =0;
a124    =0;
a125    =0;
a126    =0;
a134    =0;

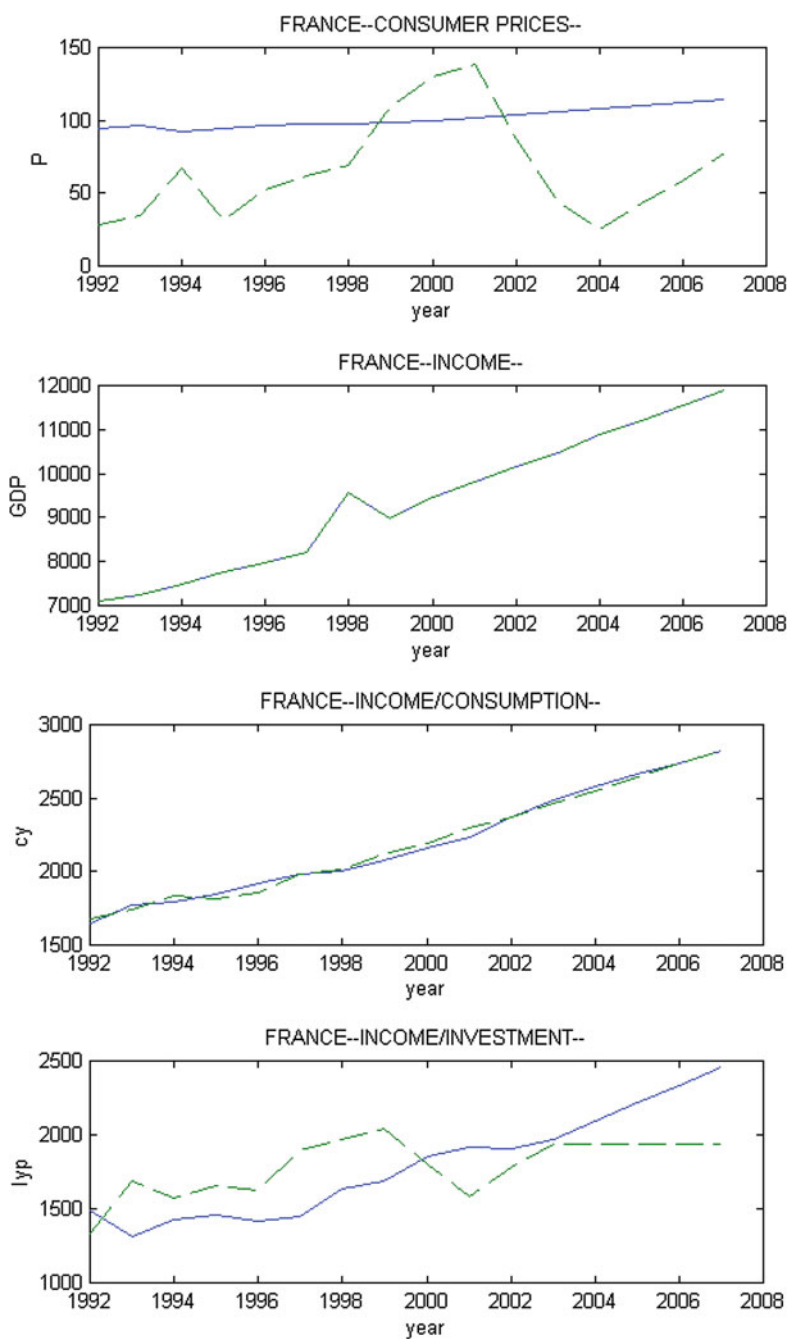
```

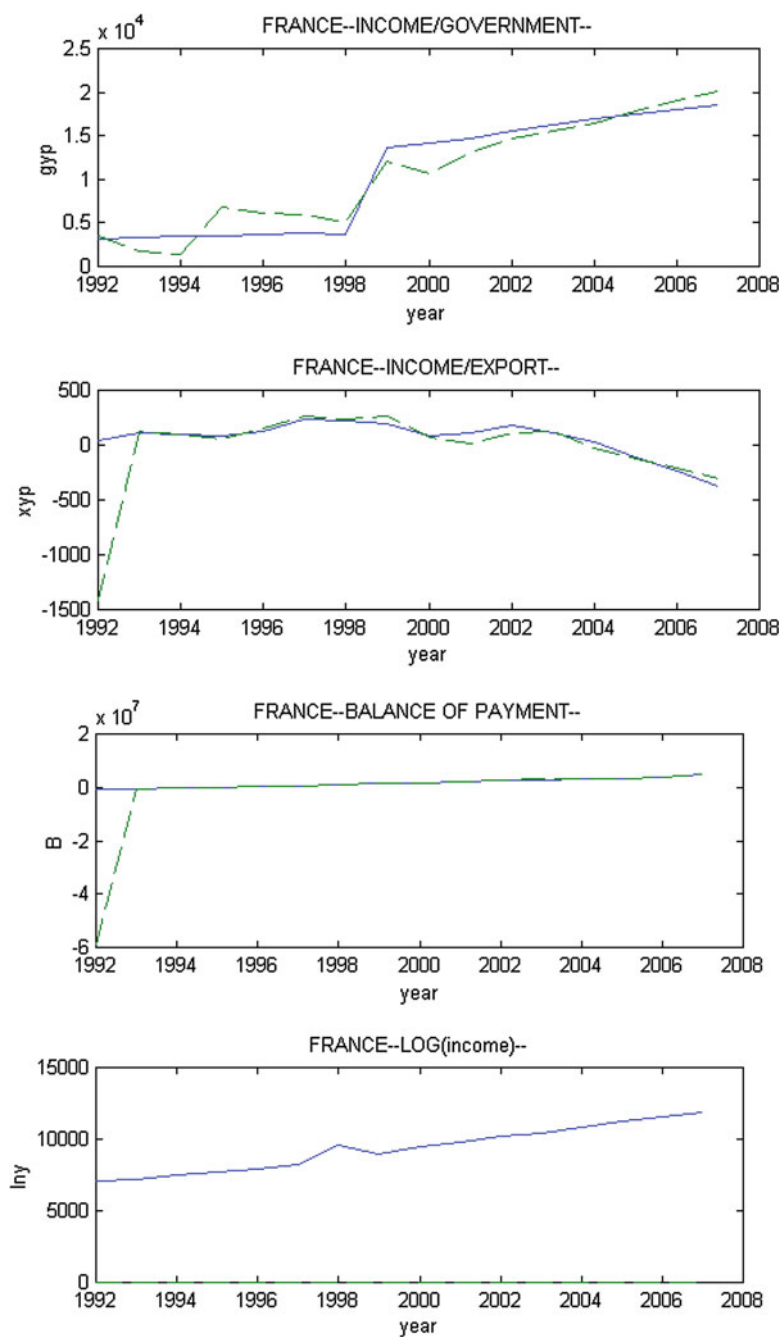
%-----

The following diagrams are the graphical results from the France2.m regression analysis program. The plots shown give the original raw data in solid lines while the estimated results from the regression are given in dash lines (- -).









```

% FRANCE2.m results
% FRANCE2_output_report_latest_release.m
% The results given below is the screen capture while
% running Austria2.m program. NDU.m can be used to data mine the results to form
% matrices of A-1, A0, B1 and B2
%
» FRANCE2
logA = 0.6044,a = 0.4463,aa = 4.6673,W = 102.6063,mw = 1.4534e-004,L1 = 1,L2 = 0.9000,L5 =
0.0303, L6 = -4.6902e-004,h = 1,z0 = 4.3126e+004,z1 = -8.6786e+005,z2 = -8.6786e+005,z3 =
8.6786e+005, z4 = 3.8830,z5 = -616.4716,z6 = -1.2325e+003,z8 = -22.4744,z9 = 0.4780,z10 =
0.1049,z11 = 0.1202, z13 = 1.1531,z14 = 7.5718,z15 = 3.3997,z16 = -0.0090,z17 = 0,
z18 = 0.8554,z19 = 8.6786e+005, z20 = 8.6786e+005,z21 = -0.0100,s1 = -8.6786e+005,yy1 =
4.2345e+007,yy2 = -2.1884e+004,yy3 = -0.5102, yy4 = -1.2338e+004,yy5 = -4.4768e+005,yy6 =
4.2351e+004,yy7 = 4.3679,yy8 = -0.0242,yy9 = -0.0307, yy10 = 157.1991,yy11 = -0.3489,zs0 =
1.4156e+004,a0 = 10.3356,a1 = 2.4093e-004,a2 = -2.4093e-004, a3 = -1.2752e+005,a4 = -
211.42308,a5 = 20.0313,a6 = 45.1451,a8 = -4.6902e-004,am11 = -4.5207e-006
am22 = -0.0523,am13 = -1.2090e-007,am14 = -1.2090e-007,Im03 = -8.4686e-004,Im01 = -2.9385e-006,
a3 = -5.8269,a6 = 0.0021,am1 = -2.6387e-006,m7p6 = -18.4457, b4 = 4.8929e+005,b8 = 4.0242e+008,
b12 = -1.3359e+007,b13 = 0,a01 = 1.0000,a12 = 7.0102e-004,a14 = 2.5715e-005,a16 = -1.3845e-007,
a18 = 4.5881e-009,a21 = -1.7960e-004,a23 = 0.0071,a25 = -0.1199,a33 = 10.3356,a35 = -6.8165e-
008, a44 = 10.3356,a45 = -4.6902e-004,a51 = 0.0096,a52 = -0.3819,a55 = 1.1812e+004,a61 =
8.5896e+005, a62 = -4.4256e+007, a63 = 1.0109e+007,a11 = -1.0000,a13 = 0.0014,a15 = -5.5078e-
007,a22 = 8.2476e-004
a24 = 0.0923,a1 = 2.4093e-004,a2 = -2.4093e-004,l1 = -2.4093e-004,l2 = 3.5016e-008,
l4 = -0.0307, b2 = 1.7374e+006,b6 = 3.5950e+007,b10 = -1.5045e+007,b17 = -1.9051e+006,a111 = -
8.6225e-006, a112 = 0.0014,a113 = -5.5078e-007,a114 = 0,a116 = 0,a121 = 8.2476e-004,a122 =
0.0923,a131 = 6.0021e-010, a132 = -0.0445,a133 = 20.0316,a141 = -2.4093e-004,a142 = -
211.4102,a143 = 20.0313,a144 = 0, a145 = 0,a146 = 0,a151 = -48.2416,a152 = -4.9467,a161 =
1.7374e+006,a162 = 3.5950e+007,a163 = -1.5045e+007
a164 = 0,a165 = 0,a166 = 55.5330,ss1 = -1.1523e-006,ss1z13 = -1.3287e-006,slz15 = -6.0016e-008,
ss1z16 = -3.9173e-006,ss1z17 = -1.2090e-007,ss1z18 = -9.8567e-007,b1z14 = -6.5039e+006,b1 = -
1.2897e+010, b1z15 = 2.9202e+006,b1z16 = -7.7618e+003,b1z17 = 0,b1z13 = 9.9050e+005,mwzs14 = -
4.4623e-006, mwzs15 = 0.0228,mwzs16 = -5.0709e-005,mwzs17 = 0,mwzs13 = -3.5156e-006,plpf = -

252.3180,P4 = -152.7633, P5 = 1.7916,b7 = -4.4256e+007,b8 = 3.5950e+007,b15 = -1.9051e+006,ss2 =
-9.9699e-006,L2 = 0.9000, P6 = -0.6245,P3 = 0.1243,P2 = 73.8894,a17 = 1.0000,a45 = -4.6902e-
004,c0 = 2.1726e+004,I0 = -362.690
x0 = -20.2539,p0 = 9.5917e+003,qm0 = 6.4888,y10 = -117.5546,p0p6m0 = 9.5877e+003,g0 =
-7.1876e+003, f0 = -1.2897e+010,g2 = -5.8399,a25 = -0.1199
»

```

```

% Date: 25-May-2009
% Austria2.m data is Billions of Schillings.
% Adopted from the International Financial Statistics Yearbook 2006, except where noted.
% The 2006 Statistics Yearbook publication range between year 1994-2005 (a total of 12
% years) and
% as a results lead to extrapolation of more data for 1992, 1993, 2006 and 2007 to give
% additional
% statistics for running regression for y(t) and Balance of payment, B(t). Please see the
% two model equation described below, their lengths were out of ranged beyond what the 12 %
% years could
% handle.

% Use NDU.m to obtain the delay and derivative matrix entries.
% Use Alphacomputation.m to estimate "a" and "1-a"
% The rest of the results obtained from the regression analysis can
% be found in Austria2_output_report_latest_release.m.

% The final results of each differential equation based on the regression
% analysis(arx) have been reproduced with the missing constants included in the section
% after the plot section.

year=[1992 1993 1994 1995 1996 1997 1998 1999 2000 2001 2002 2003 2004 2005 2006
2007]';
EX=[649.3 715.2 781.1 847 897.5 1012.2 1104.2 85.3 95.6 103.3
107.7 109.6 121.7 133.1 144.5 155.9]';
IMP=[688.8 743.8 798.8 853.8 917.4 1006.5 1066.2 81.9 92.7 97.9
97.2 101.5 111.7 121.4 131.1 140.8]';
TX=[754.5 784.8 815.1 845.4 903.4 874.4 888.9 881.65 885.275
883.463 884.369 883.916 884.143 884.03 883.917 883.804]';
K=[529 530.7 532.4 534.1 553.3 568.2 592.2 44.2 47.9 47.7 45.1
48.1 49.2 50.3 51.4 52.5]';
KD=[534.95 534.1 533.25 532.4 534.1 553.3 568.2 592.2 44.2 47.9 47.7 45.1 48.1 49.2
50.3 51.4]';
KP=[5.95 3.4 0.85 -1.7 -19.2 -14.9 -24 548 -3.7 0.2 2.6 -3 -1.1 -1.1 -1.1 -1.1]';
D=[-66.7 -34.5 -2.329 916 17.320 82.5 1.3 1 0.7 1.1 1
0.7 0.4 0.1]';
I=[462.3 496.2 530.1 564 569.3 585.5 613 46.7 49.2 48.7 45.8 49.2 50.2 51 51.8
52.6]';
kk=[8.5 5.95 3.4 0.85 -1.7 -19.2 -14.9 -24 548 -3.7 0.2 2.6 -3 -1.1 -1.1 -1.1-
1.1]';
n=[3708 3725 3742 3759 3710 3056 3075 3109 3134 3148 3155 3185 3199 3230 3261
3292]';
P=[87.3 89.3 91.3 93.3 95 96.3 97.2 97.7 100 102.7 104.5 105.9 108.1 110.6
113.1 115.6]';
e=[18.047 17.03 16.013 14.996 15.751 17.045 16.54 1.3662 1.4002 1.426 1.2964
1.1765 1.1402 1.2116 1.283 1.3544]';
Pf=[1587.8 1524.9 1462 1399.1 1496.3 1641.4 1607.7 133.5 140 146.5 135.5 124.6
123.3 134 144.7 155.4]';
Pfe=[28271 25841 23411 20981 23569 27978 26591 182 196 209 176 147 141
162 183 204]';
ta=[552.2 590.81 629.42 668.03 712.76 790.25 842.13 65.32 74.94 78.66 77.19
78.187 51 37.97 -11.57 -61.11]';
F=[306.9 330.8 354.7 378.6 405.1 391.85 398.475 15.54 20.35 17.73 20.82
21.21 27.62 36.99 46.36 55.73]';
B=[-1100.9 -1144.2 -1187.5 -1230.8 -1328.4 -1260.6 -1249.4 -893.8 -902.7 -895.8 -
894.7 -897 -901.8 -909.3 -916.8 -924.3]';
BD=[-1252.45 -1230.8 -1209.15 -1187.5 -1230.8 -1328.4 -1260.6
-1249.4 -893.8 -902.7 -895.8 -894.7 -897 -901.8 -906.6 -911.4]';
T=[754.5 784.8 815.1 845.4 903.4 874.45 888.925 881.66 885.25
883.47 884.38 883.89 884.18 884.01 883.84 883.67]';
n=[3708 3725 3742 3759 3710 3056 3075 3109 3134 3148 3155 3185
3199 3230 3261 3292]';
L=[3708 3725 3742 3759 3710 3056 3075 3109 3134 3148 3155 3185 3199 3230 3261
3292]';
LD=[3767.5 3759 3750.5 3742 3759 3710 3056 3075 3109 134 3148
3155 3185 3199 3213 3227]';
LP=[59.5 34 8.5 -17 49 654 -19 -34 -25 -14 -7 -30 -14 -31
-48 -65]';

```



```

LPD =[-29.75 -17 -4.25 8.5 -17 49 54 -19 -34 -25 -14 -7 -30 -14 2 18]';
w =[89.9 90.4 90.9 91.4 91.9 90.9 92.9 95.2 98 100 102.2 104.8
    106.8 108.5 111.4 114.3 117.2]';
y =[2143.5 2234.1 2324.7 2415.3 2502.6 2547.6 2647.3 200 210.4 215.9
    220.8 226.2 235.8 245.1 254.4 263.7]';
yD =[2460.6 2415.3 2370 2324.7 2415.3 2502.62547.6 2647.3 200 210.4
    215.9 220.8 226.2 235.8 245.4 255]';
GDP =[2143.5 2234.1 2324.7 2415.3 2502.6 2547.6 2647.3 200 210.4 215.9 220.8
    226.2 235.8 245.1 254.4 263.7]';

yP =[453 317.1 181.2 45.3 -90.6 -87.3 -45 -99.7 2447.3 -10.4 -5.5 -4.9 -5.4 -9.6
    -9.3 -9 -8.7]';
yPD=[-158.55 -90.6 -22.65 45.3 -90.6 -87.3 -45 -99.7 2447.3 -10.4
    -5.5 -4.9 -5.4 -9.6 -13.8 -18]';
G =[789.02 853.08 917.14 981.2 1009.89 995.545 1002.72 999.133 1000.93 1000.03
    1000.48 1000.26 1000.37 1000.32 1000.27 1000.22]';
C =[407.9 433.7 459.5 485.3 497.2 482.2 498.2 38 38.738.9
    40.141.442.744.546.348.1]';
M =[248.36 301.97 355.58 409.19 431.15 452.3 473.45 494.6 515.75 536.9 558.05
    579.2 600.35 621.5 642.65 663.8]';
MP =[53.61 53.61 53.61 21.96 21.15 21.15 21.15 21.15 21.15 21.15 21.15
    21.15 21.15 21.15 21.15 -663.8]';
QM =[137.7 148 158.3 168.6 176.7 178.8 177.75 13.92 14.54 10.69
    11.01 12.29 14.11 15.917.69 19.48]';
QMD =[173.75 168.6 163.45 158.3 168.6 176.7 178.8 177.75 13.92
    14.54 10.69 11.01 12.29 14.11 15.93 17.75]';
QMP =[36.05 20.65.15-10.3 -8.1-2.11.05163.83 -0.62 3.85
    -0.32 -1.28 -1.82 -1.79 -1.76 -1.73]';
QMPPD=[-18.025 -10.3 -2.575 5.15-10.3 -8.1-2.11.05163.83
    -0.62 3.85-0.32 -1.28 -1.82 -1.76 -1.73]';
ML =[137.7 148 158.3 168.6 176.7 178.8 177.75 13.92 14.54 10.69
    11.01 12.29 14.11 15.917.69 19.48]';
YT =[1389 1449.3 1509.6 1569.9 1599.2 1673.2 1758.4 -681.6 -674.9 -667.6 -663.6
    -657.7 -648.3 -638.9 -629.5 -620.1]';
YTD =[1600.05 1569.9 1539.75 1509.6 1569.9 1599.2 1673.2 1758.4 -681.6
    -674.9 -667.6 -663.6 -657.7 -48.3 561.1 1170.5]';
YTP =[211.05 120.6 30.15 -60.3 -29.3 -74 -85.2 2440-6.7
    -7.3-4 -5.9-9.4-9.4-9.4]';
YTPPD=[-75.525 -40.3 -5.075 30.15 -60.3 -29.3 -74 -85.2 2440 -6.7 -7.3 -4
    -5.9 -9.4 -12.9 -16.4]';
R =[7.5 6 4.5 3 2.5 2.5 2.5 2.5 2.5 2.5 2.5 2.5 2.5 2.5 2.5]';
RD =[2.25 3 3.754.5 3 2.5 2.5 2.5 2.5 2.5 2.5 2.5 2.5 2.5]';
RP =[5.25 -3 -0.75 1.5 0.5 0 0 0 0 0 0 0 0 0 0]';
RPD =[5.25 -3 -0.75 1.5 0.5 0 0 0 0 0 0 0 0 0 0]';
P =[87.3 89.391.393.395 96.397.297.7100 102.7 104.5 105.9 108.1 110.6 113.1
    115.6]';
price =[85.3 87.3 89.3 91.3 93.3 95 96.3 97.2 97.7 100 102.7 104.5 105.9 108.1
    110.6 113.1 115.6]';
X= EX - IMP;
%T= X-B-F; See above entries
for i=1:16
    pp(i)=price(i+1)-price(i);
    kprim(i) =kk(i+1)-kk(i);
end
pie=[pp]';
KP =kprim';
DP =pie.\P;
n=y.\L;

% Statistics Number, Statistics Name, Full Name
% 81, TX, Tax revenue
% TX = total current receipts of general government(Billions of AUSTRIA Schillings
% currency)
% 65, w, Wages: Average Monthly earnings(termed compensation according to National
% accounts statistics, United Nations publication, page 116, 2003
% edition) defined as compesation of employees from the rest of the
% world.
% see statistics entry number "65".

```

```

%
% 90c,EX,      Exports of goods and services
% 98c,IMP,Imports
%      X,      EX-IMP(Net export)
% 67e,  n,      Labor productivity(A/l)=GDP.\emp; FOR AUSTRIA WE USE the employment
%              available for 1994-2005. Other years were gotten by considering the
%              labor activity(percentage) factor to population for that year.
%
% 93e,K,      Capital stock(Gross fixed capital formation)
% K(t+1)-K(t), KP, Derivative Gross Fixed Capital Formation
% K+D,I,      Investment(Gross capital formation=K+D)
% 93i,D,      Increase in stock(changes in inventories)
%              The UN(National accounts statistics) defines Gross capital formation
%              as the sum of the increase in stocks and gross fixed capital formation
%
% 99b,  y or GDP,      Gross domestic product
% y(t-1), yD,      Shift "right" in time Gross Domestic Product
% y(t+1)-y(t), y'(t), yP,      Derivative of Gross Domestic Product
% yP(t+1),      yPD, y'(t+1),      Delay in time Derivative of (GDP)
% ignore,      d,(Dummy)      Differential trade agreement transportation
% Pf          Import price level in foreign currency
%
% 64, P,      Consumer prices
% 71, ta,      Tariffs(Import duties = custom duties plus other import charges)
% E,          Cumulative balance of payment
% X-F-TX, B,      Balance of payment(Deposit money banks=E'(t))
% X-B-F, T,      Net government transfer of capital to foreigners and firms
% 37r, F,      Net private outflow of capital
% 34, M,      Money supply
% 35, QM,Money demand(Quasi Money)
% 67e, L,      Employment, Labour(Industrial production)
%              emp =Employment(Labor force at equilibrium)
%              unemp =Unemployment
%
% 34, ML,QM-M Money
% 60, R,      Interest rate Bank Rate(Discount rate or end of period)
% R(t+1), RD,      Interest Rates Bank Rate delayed
% R(t+1)-R(t),RP,      Derivative of interest Rates(End)
% R(t+1), RPD,      Derivative of interest Rates(End)delayed
% 82, G,      Expenditure
% 91f, C,      Government consumption Expenditure
% aa, e,      Exchange rate(Market rate=aa)
% J3          Direct investment in Republic Economy
% GNI          Gross National income
% PL          Population
% YT, GDP-Revenue, y-TX
% YTD, YT delayed, YT(t+1)
% YTP, Derivative YT, YT(t+1)-YT(t)
% YTFD, YTP delayed, YTP(t+1)

```

```
format short;
```

```
%Bertl=[X,G,I,C,GDP,PGDP,R,M,QM,YEAR]
```

```
[years,nn]=size(year);
```

```
X1=[-39.50;X(1:years-1)];
```

```
G1=[789.02;G(1:years-1)];
```

```
I1=[462.3;I(1:years-1)];
```

```
C1=[407.9;C(1:years-1)];
```

```
QM1=[137.7;QM(1:years-1)];
```

```
R1=[7.5;R(1:years-1)];
```

```
GDP1=[2143.5;GDP(1:years-1)];
```

```
DGDP=GDP-GDP1;
```

```
Z=X+G+I+C;
```

```
Z1=X1+G1+I1+C1;
```

```
%temp=[YEAR X1 G1 I1 C1 Z1 y1 yD M1 RD1 yPD1 RP RPD T1 P P-PF P1 P-PF1 G QM QMD QMP QMPD QM1
```

```
QMD1 QMP1 QMPD1 ta tal e el d dl];
```

```
%save Bertl.asc temp -ascii -double -tabs
```

```
%clear Bertl temp;
```

```
%ML = QM-M = m1-m0 -m1*y(t) - m2*y(t-h) - m3*R(t) - m4*R(t-h) - m6*P(t) - m7*R'(t-h)
```

```

%QM(t) = m0 + m1*y(t) + m2*y(t-h) + m3*R(t) + m4*R(t-h) + m5*R'(t-h) + m6*P(t);
temp=[QM ones(size(y)),y,yD,R,RD,RPD,P];
thqm=arx(temp,[0,1 1 1 1 1 1 0 0 1 0 1 1 0]);
QMp=predict([QM,ones(size(y)),y,yD,R,RD,RPD,P],thqm,4);
QMM=thqm(6,1:7);

%C(t) = c0 + c1*YT(t) + c2*YT(t-h) + c3*YT'(t) + c4*YT'(t-h) + c5*R(t) + c6*R(t-h)
+ c7*(a1+a2.*y(t) + a3.*yD(t-h) + a4.*R(t) + a5.*RD(t) + a6.*P(t) + a.*RPD(t)) -M(t);
temp=[C ones(size(R)),YT,YTD,YTP,YTPD,R,RD,ML];
thc=arx(temp,[0 1 1 1 1 1 1 1 1 0 0 1 0 1 0 1 0]);
Cp=predict([C,ones(size(R)),YT,YTD,YTP,YTPD,R,RD,ML],thc,4);
CC=thc(7,1:8);

%I(t) = i0 + i1*y(t) + i2*y(t-h) -i3*y'(t) +i4*y'(t-h) +i5*R(t) +i6*R(t-h) +i8*L(t) +i9*L(t-h) -
i11*K(t) -i13*ML(t)
temp=[I ones(size(y)),y,yD,yP,yPD,R,RD,QM,QMD,K,ML];
thi=arx(temp,[0, 1 1 1 1 1 1 1 1 1 1 0 0 1 0 1 0 1 0 0 11]);
Ip=predict([I,ones(size(y)),y,yD,yP,yPD,R,RD,QM,QMD,K,ML],thi,4);
II=thi(10,1:11);

%X(t) = x0+ x1*y(t) +x2*y(t-h) +x3*y'(t) +x4*y'(t-h) +x5*R(t) +x8*L(t) +x10*L'(t-h) +x12*P(t)
+x16*ta(t) +x15*e(t) +x17*d(t)
temp=[X ones(size(y)),y,yD,yP,yPD,R,QM,QMPD,P,ta,e];
thx=arx(temp,[0,1 1 1 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1 0 0 0]);
Xp=predict([X,ones(size(y)),y,yD,yP,yPD,R,QM,QMPD,P,ta,e],thx,4);
XX=thx(11,1:12);

%G(t) = g0 +g1*y(t) +g2*y(t-h) +g3*y'(t) +g4*y'(t-h) +g5*R(t) +g8*L(t)
temp=[G ones(size(y)),y,yD,yP,yPD,R,QM];
thg=arx(temp,[0,1 1 1 1 1 1 1 0 0 1 0 1 0 0]);
Gp=predict([G,ones(size(y)),y,yD,yP,yPD,R,QM],thg,4);
GG=thg(6,1:7);

%y(t) zs0 +zs1*y(t) +zs2*y(t-h) +zs4*y'(t) +zs5*R(t) +zs8*L(t) +zs10*L'(t) +zs13*ML(t) -
zs14YT(t) +zs15*e(t) +zs16*ta(t) +zs17*d(t)*(0) +1/h*K'(t); h=1
temp=[y ones(size(y)),y,yD,yP,R,QM,QMP,M,TX,e,ta,KP];
thy=arx(temp,[0,1 1 1 1 1 1 1 1 1 1 1 0 0 1 0 0 0 0 0 0 0]);
yp=predict([y,ones(size(y)),y,yD,yP,R,QM,QMP,M,TX,e,ta,KP],thy,4);
yy=thy(11,1:12);

%cy(t) = y10 + y11*y(t) + y12*y'(t) +y13*R(t) +y15*ML(t) +y18*YT(t)
temp=[C ones(size(y)),y,yP,R,ML,YT];
thcy=arx(temp,[0,1 1 1 1 1 1 0 0 0 0 0 0]);
cyp=predict([C,ones(size(y)),y,yP,R,ML,YT],thcy,4);
ccy=thcy(5,1:6);

%Iy(t) = I0 + 1/h*KP(t)
temp=[I ones(size(KP)),KP];
thIy=arx(temp,[0,1 1 0 0]);
Iyp=predict([I,ones(size(y)),KP],thIy,4);
IIy=thIy(1,1:2);

%Gy(t) = gs0 + gs1*y(t) +gs4*y'(t) +ys5*R(t) + gs8*QM(t)
temp=[G ones(size(y)),y,yP,R,QM];
thgy=arx(temp,[0,1 1 1 1 1 0 0 0 0 0]);
gyp=predict([G,ones(size(y)),y,yP,R,QM],thgy,4);
ggy=thgy(4,1:5);

%Xy(t) x0 + x1*y(t) + x2*y(t) + x5*R(t) +x8*QM(t) +x10*QM'(t) +x11*e(t) +x12*ta(t) +x13*d(t)*(0)
temp=[X ones(size(y)),y,yD,R,QM,QMP,e,ta];
thxy=arx(temp,[0,1 1 1 1 1 1 1 0 0 1 0 0 0 0 0]);
xyp=predict([X,ones(size(y)),y,yD,R,QM,QMP,e,ta],thxy,4);
xxy=thxy(7,1:8);

%B(t) = b0 +b1*y(t) +b2*y(t-h) +b3*P(t) +b4*y'(t-h) +b5*R(t) +b6*R(t-h) +b7*e(t) +b8*R'(t-h)
+b9*QM(t) +b10*QM(t-h) +b12*L'(t-h) +b13*ta(t) +b15*d(t) +b17*B(t-h)
temp=[B ones(size(y)),y,yD,P,yPD,R,RD,e,RPD,QM,QMD,QMPD,ta,BD];
thb=arx(temp,[0,1 1 1 1 1 1 1 1 1 1 1 1 0 0 1 0 1 0 1 0 1 0 1 1 1]);

```

```

Bp=predict([B,ones(size(y)),y,yD,P,yPD,R,RD,e,RPD,QM,QMD,QMPD,ta,BD],thb,4);
BB=thb(13,1:14);

%y(t) = A*K^a*L^(1-a) ==> ln(y(t)) = ln(A)+ a*ln(K) + (1-a)*ln(L)
temp=[log(y) ones(size(y)),log(K),log(L)];
thly=arx(temp,[0,1 1 1 0 0 0]);
lyp=predict([log(y) ones(size(y)),log(K),log(L)],thly,4);
lyy=thly(2,1:3);

%w = real wage rate per unit time
%w = [1-lyy(2)/(w)]^(1/lyy(1)) % Commenting out for now. Please see
                                % Alphacomputation.m program for estimation
                                % procedure used.

%D(t) = a(1-c)*y(t) -k0*K(t) +k13*K'(t) +L4*R(t) +L5*QM(t) +L6*P(t) +v*y'(t)
temp=[D ones(size(y)),y,K,KP,R,QM,P,yP];
thd=arx(temp,[0,1 1 1 1 1 1 1 0 0 0 0 0 0 0]);
Dp=predict([D,ones(size(w)),y,K,KP,R,QM,P,yP],thd,4);
DD=thd(7,1:8);

%DP(t) = p0 +p1*Pfe(t) p2*w(t) -p3*n(t) -p4*P(t) +p5*M'(t) +p6*ML(t)
%M'(t)=direvative of money supply
temp=[DP ones(size(w)),Pfe,w,n,P,MP,ML];
thp=arx(temp,[0,1 1 1 1 1 1 1 0 0 0 0 0 0]);
dlp=predict([DP,ones(size(w)),Pfe,w,n,P,MP,ML],thp,4);
dp=thp(6,1:7);

%-----
subplot(2,1,1),plot(year,C,year,Cp,'--')
title('AUSTRIA--GOVERNMENT CONSUMPTION EXPENDITURE--');
ylabel('C');
xlabel('year');
subplot(2,1,2),plot(year,X,year,Xp,'--')
title('AUSTRIA--EXPORTS--');
ylabel('X');
xlabel('year');
pause
clf
subplot(2,1,1),plot(year,I,year,Ip,'--')
title('AUSTRIA--INVESTMENT--');
ylabel('I');
xlabel('year');
subplot(2,1,2),plot(year,G,year,Gp,'--')
title('AUSTRIA--GOVERNMENT CONSUMPTION--');
ylabel('G');
xlabel('year');
pause
clf
subplot(2,1,1),plot(year,QM,year,QMp,'--')
title('AUSTRIA--MONEY DEMAND--');
ylabel('QM');
xlabel('year');
subplot(2,1,2),plot(year,Z,year,[Xp+Ip+Gp+Cp], '--')
title('AUSTRIA--AGGREGATE DEMAND--');
ylabel('Z');
xlabel('year');
pause
clf
%-----

subplot(2,1,1),plot(year,P,year,dlp,'--')
title('AUSTRIA--CONSUMER PRICES--');
ylabel('P');
xlabel('year'); subplot(2,1,2),plot(year,y,year,yp,'--')
title('AUSTRIA--INCOME--');
ylabel('GDP');
xlabel('year');
pause

```

```

clf
subplot (2,1,1),plot(year,C,year,cyp,'--')
title('AUSTRIA--INCOME/CONSUMPTION--');
ylabel('cy');
xlabel('year');
subplot(2,1,2),plot(year,I,year,Iyp,'--')
title('AUSTRIA--INCOME/INVESTMENT--');
ylabel('Iyp');
xlabel('year');
pause
clf
subplot (2,1,1),plot(year,G,year,gyp,'--')
title('AUSTRIA--INCOME/GOVERNMENT--');
ylabel('gyp');
xlabel('year');
subplot(2,1,2),plot(year,X,year,xyp,'--')
title('AUSTRIA--INCOME/EXPORT--'); ylabel('xyp');
xlabel('year');
pause
clf
%-----

subplot (2,1,1),plot(year,B,year,Bp,'--')
title('AUSTRIA--BALANCE OF PAYMENT--');
ylabel('B');
xlabel('year');
subplot(2,1,2),plot(year,y,year,lyp,'--')
title('AUSTRIA--LOG(income)--');
ylabel('lny');
xlabel('year');
pause
clf
subplot (2,1,1),plot(year,D,year,Dp,'--')
title('AUSTRIA--INCREASE IN STOCK--');
ylabel('D');
xlabel('year');
%-----

% The economic dynamics can be put in matrix form as
% follows:  $x'(t) - A^{-1}x'(t-h) = A0*x(t) + A1*x(t-h) + B*u(t)$ 
%
%  $x=[y \ R \ L \ K \ P \ E]'$ ;
%
%  $A^{-1}=[a-11 \ 0 \ a-13 \ 0 \ 0 \ 0; \ 0 \ a-22 \ 0 \ 0 \ 0 \ 0; \ L-03 \ 0 \ -L-01 \ 0 \ 0 \ 0; \ a3 \ 0 \ a6 \ a-1 \ 0 \ 0; \ 0 \ -M7p6P(t) \ 0 \ 0 \ 0 \ 0; \ -b4 \ -b8 \ -b12 \ 0 \ 0 \ -b14];$ 
%
%  $A0=[a01 \ a12 \ a14 \ a16 \ a18 \ 0; \ a21 \ a23 \ 0 \ 0 \ a25 \ 0; \ 0 \ 0 \ a33 \ 0 \ a35 \ 0; \ 0 \ 0 \ 0 \ a44 \ a45 \ 0; \ a51 \ a52 \ 0 \ 0 \ a55 \ 0; \ a61 \ a62 \ a63 \ 0 \ 0 \ 0];$ 
%
%  $A1=[a111 \ a112 \ a113 \ 0 \ 0 \ a116; \ a121 \ a122 \ 0 \ 0 \ 0 \ 0; \ a131 \ a132 \ a133 \ 0 \ 0 \ 0; \ a141 \ a142 \ a143 \ 0 \ 0 \ 0; \ a151 \ a152 \ a153 \ a154 \ a155 \ a156; \ a161 \ a162 \ a163 \ a164 \ a165 \ a166];$ 
%
%  $q = [T \ g0 \ e \ ta \ d \ M \ M' \ f0]'$ ;
%  $S = [C0 \ I0 \ X0 \ (p0-p6M0) \ n \ w \ x0 \ y10]$ 
%
%  $w =$  real wage rate per unit time
%  $m(w) = [1-lyy(2)/(w)]^{(1/lyy(1))}$  % See Alphacomputation.m program for estimation procedure used.
% where  $w =$  average of  $w(t)$  %
%  $B1=[-b1z14 \ b1 \ -b1z15 \ b1z16 \ b1z17 \ b1z13 \ 0 \ 0; \ 0 \ 0 \ 0 \ 0 \ -11 \ 0 \ 0; \ -m(w)zs14 \ m(w) \ m(w)zs15 \ m(w)zs16 \ m(w)zs17 \ m(w)zs13 \ 0 \ 0; \ -zs14 \ 1 \ zs15 \ zs16 \ zs17 \ zs13 \ 0 \ 0; \ 0 \ 0 \ p1pf(t) \ 0 \ 0 \ p6 \ p5 \ 0; \ 0 \ 0 \ b7 \ b8 \ b15 \ 0 \ 0 \ 1];$ 
%
%  $B2=[ss1 \ ss1 \ ss1 \ -ss1(I13+c7) \ 0 \ 0 \ 0 \ 0; \ 0 \ 0 \ 0 \ -12 \ 0 \ 0 \ 0 \ 0; \ 0 \ m(w) \ 0 \ 0 \ 0 \ 0 \ m \ m \ 0; \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ -p6 \ -p3 \ p2 \ 0 \ 0 \ 1; \ 0 \ 0 \ 0 \ -1 \ 0 \ 0 \ 1 \ 0 \ 0];$ 
%
%  $B = [B1 \ B2];$ 
%  $u = [q \ S];$ 

```

```

%
% Cobb Douglas states:      aa+a = 1 => constant return to scale
%                          aa+a > 1 => increasing return to scale
%                          aa+a < 1 => decreasing return to scale
%
% Increasing return to scale if doubling L & K more than doubles y.
% constant return to scale if doubling L & K exactly doubles y
% decreasing return to scale if doubling L & K less y.
%
% For y(t) = A*K^a*L^(1-a)
% Initial estimates of a =alpha, see Ahphacomputation.m program were "solve" is used.
% Initial estimates of aa =(1-a)= beta, see Ahphacomputation.m program were "solve" is
% used.

% [logA,a,aa] = solve('log(y(1)) - a*log(K(1)) - (aa)*log(L(1))','(log(y(9)) - logA -
% (aa)*log(L(9)))/log(K(9))','(log(y(16)) - logA - a*log(K(16)))/log(L(16))')

% Solving for a & aa:

logA = (-log(L(1))*log(y(9)/y(16)) + log(y(1))*log(L(9)))/(log(L(9))*log(K(1))
+ log(K(16))*log(L(1)))
a = (log(K(16))*log(y(1)) +log(y(9)/y(16))*log(K(1)))/(log(L(9))*log(K(1))
+log(K(16))*log(L(1)))
aa = (log(y(9))*log(L(9))*log(K(1)) +log(y(9))*log(K(16))*log(L(1)) -
log(L(9))*log(K(16))*log(y(1)) -
log(L(9))*log(y(9)/y(16))*log(K(1)))/(log(L(9))*log(K(1)) +log(K(16))*log(L(1)))

After "Solve"
logA = 0.7660
a = 0.3488
aa = 2.5409

W = sum(w)/16
mw = (aa/W.^(1/a))

Where, W = 6.3498e+003

% We extract the following equation of the economic state: x = [y, R, L, K, P, E]'.

dy/dt - am11*y'(t-h) - am13*L'(t-h) = a01*y(t) + a11*y(t-h) + a12*R(t) + a13*R(t-h) +
a14*L(t) + a15*L(t-h) + a16*k(t) - a18*p(t) + q1(t) + r1(t)

dy/dt - (-0.0226)*y'(t-h) - (0.3520)*L'(t-h) = (-1.0389)*y(t) + (0.0465)*y(t-h)
+ (-0.0674)*R(t) + (4.3995)*R(t-h) + (0.2759)*L(t) + (0.0305)*L(t-h)
+ (-0.1535)*K(t) - (-7.5416e-011)*p(t) + q1(t) + R1(t)

% Where,
q1(t) = l1*sigma^-1*[g0 - z14*T(t) + z19*T(t-h) - z20*T'(t) - z21*T'(t-h) - z15*e(t)
+ z16*tar(t) + z17*d(t)]

q1(t) = (0.0011)*[(-0.0032) - (-3.1322e-005)*T(t) + (-2.9347e-007)*T(t-h) - (5.4070e-
007)*T'(t) - (9.1153e-009)*T'(t-h) - (0.8547)*e(t) + (0.0029)*tar(t) + 0*d(t)]

r1(t) = l1*sigma^-1*[(C0+ I0 + X0) - M0(I13 + C7)]
r1(t) = (0.0011)[((-0.0017) + (-8.0393) + (-15.7276)) - (-0.0038)(-0.3996 + 3.1322e-005)]

format short;

% Note: L1(lamdal) is the speed of response of supply to demand, or the speed of adjustment.
L1 =1
L2 =0.9
L5 =DD(5)
L6 =DD(6)
h = 1

```

```

z0=CC(1)+II(1)-QMM(1)*(CC(8)+II(1))+CC(1)+XX(1)
z1=GG(2)+II(2)-QMM(2)*(CC(8)+II(1))+CC(2)+XX(2)
z2=GG(3)+II(3)-CC(3)+XX(3)+QMM(3)*(CC(8)+II(1))
z3=GG(4)+II(4)+CC(4)+XX(4)
z4=GG(5)+II(5)+CC(5)+XX(5)+QMM(5)*(CC(8)+II(1))
z5=GG(6)+II(6)+CC(6)+XX(6)-CC(8)*QMM(6)
z6=II(7)+CC(7)-CC(8)*QMM(7)
z8=GG(7)+II(8)+XX(7)
z9=II(9)
z10=XX(8)
z11=-II(10)
z13=(II(11)+CC(8))*QMM(7)
z14=-CC(8) z15=XX(11)
z16=XX(10)
z17=0
z18=(CC(8)+z13)*QMM(7)
z19=CC(3)
z20=CC(4)
z21=CC(5)
zs0 =ccy(1)+IIy(1)+ggy(1)+xxy(1) ;%constant term
zs1 =ccy(2)+          ggy(2)+xxy(2) ;%y(t) term
zs2 =          xxy(3) ;%y(t-h) term
zs4 =ccy(3)+          ggy(3)          ;%y'(t) term
zs5 =ccy(4)+xxy(4)+ggy(4)          ;%R(t) term
zs8 =xxy(5)+          ggy(5)          ;%L(t) term
zs10=xxy(6)          ;%L'(t) term
zs13=ccy(5)          ;%ML(t) term
zs14=ccy(6)          ;%T(t) term
zs15=xxy(7)          ;%e(t) term
zs16=xxy(8)          ;%ta(t) term

s1 =1-L1*(GG(4)-II(4)+CC(4)+XX(4))

%assumptions
yy1 = zs0
yy2 = zs1
yy3 = zs2
yy4 = zs4
yy5 = zs5
yy6 = zs8
yy7 = zs10
yy8 = zs13
yy9 = zs14
yy10= zs15
yy11= zs16

zs0= GG(1)+XX(1)+CC(1)+II(1)
a0 = DD(1)/(h*(1-yy(2)))
a1 = -DD(1)*yy3/((1-yy2)*h)
a2 = -a1*h
a3 = a0*h*yy4
a4 = DD(1)*yy5/((1-yy2)+DD(4))
a5 = DD(1)*yy6/(1-yy2)+DD(5)
a6 = a0*h*yy7
a8 = DD(6)

%A-1 starts here; am11 = a-11
am11 =L1*(GG(5)-II(5)+CC(5)+XX(5)+QMM(5)*(CC(8)+II(1)))/s1
am22 =L2*QMM(6)
am13 =L1*z10/s1
am14 =L1*z10/s1
Lm03 =DD(1)*yy4*mw/(1-yy2)
lm01 =-DD(3)*DD(1)/(1-yy2) -mw*DD(1)*yy7/(1-yy2)
a3 =DD(1)*yy4/(1-yy2)
a6 =DD(1)*yy7/(1-yy2)
am1 =-DD(3)*DD(1)/(1-yy2)
m7p6 =QMM(7)*dp(6)*sum(P)/21
b4 =BB(5)

```

```

b8      =BB(9)
b12     =BB(12)
b13     =0

%A0 Starts here
a01 = (L1*(GG(2)+ II(2) -QMM(2))* (II(11)+CC(8)) +CC(2)+XX(2)+CC(8) - 1 - (II(11)+CC(8))*QMM(7))/s1
a12 = L1*( (GG(6)+II(6)+CC(6)+XX(6) -CC(8)*QMM(7)) - (II(11)+CC(8)) -QMM(3)*QMM(7))/s1
a14 = L1*(GG(7)+II(8)+XX(8))/s1
a16 = -L1*(II(10))/s1
a18 = -L1*(QMM(4)*QMM(6)*(II(11)+CC(8)))/s1
a21 = L2*QMM(2)
a23 = L2*QMM(4)
a25 = L2*QMM(7)
a33 = a0
a35 = mw*a8
a44 = a0
a45 = a8
a51 = -dp(7)*QMM(2)*sum(P)/21      %a51*P(t)
a52 = -dp(7)*QMM(4)*sum(P)/21      %a52*P(t)
a55 = -(QMM(7)+dp(5))*sum(P)/21    %a55*P(t)
a61 = BB(2)
a62 = BB(6)
a63 = BB(10)

%A1 Starts here!
a11 =L1*(GG(3) +II(3) +CC(3) +XX(3) +QMM(3)*(II(11)+CC(8)))/s1
a13 =L1*(II(7) +CC(7) -CC(8)*QMM(7))/s1 a15 =L1*(II(9))/s1
a22 =L2*QMM(3)
a24 =L2*QMM(5)
a1  =-DD(1)*yy3/((1-yy2)*h)
a2  =-a1*h
l1  =-a1
l2  =-mw*DD(1)*yy3/(1-yy2)
l4  =mw*DD(1)*yy5/(1-yy2 +DD(4))
b2  =BB(3)
b6  =BB(7)
b10 =BB(11)
b17 =BB(13)
a111 =L1*(GG(3)+II(3)+XX(3)+QMM(3)*(II(11)+CC(8)))/s1
a112 =a13
a113 =a15
a114 =0
a116 =0
a121=a22 a122=a24
a131=mw*DD(1)*XX(3)/(1-XX(2)+ggy(2) +ccy(2))
a132=mw*DD(1)*yy5/(1-yy2) -DD(4)
a133=a5-l1*h
a141=DD(1)*yy3/(1-yy2)
a142=DD(1)*yy5/(1-yy2) + DD(4)
a143=DD(1)*yy6/(1-yy2) + DD(5)
a144 =0
a145 =0
a146 =0
a151 = -dp(7)*sum(P)/21      %a151*P(t)
a152 = -dp(7)*QMM(5)*sum(P)/21 %a152*P(t)
a161 = BB(3)
a162 = BB(7)
a163 = BB(11)
a164 = 0
a165 = 0
a166 = BB(14)
% B1 Starts here!
ss1 =-L1/s1
ss1z13 =ss1*(II(11)+CC(8))*QMM(7)
ss1z15 =-ss1*XX(7)
ss1z16 =ss1*XX(11)
ss1z17 =ss1*XX(8)
ss1z18 =ss1*z18

```



```

b1z14  =-BB(2)*z14
b1      =BB(1)
b1z15  =BB(2)*z15
b1z16  =BB(2)*z16
b1z17  =BB(2)*z17
b1z13  =BB(2)*z13
mwzs14 =mw*yy9 mwzs15 =mw*yy10
mwzs16 =mw*yy11
mwzs17 =0
mwzs13 =mw*yy8
p1pf   =dp(2)*sum(Pfe)/21
P4      =dp(5)
P5      =dp(6)
b7      =BB(6)
b8      =BB(7)
b15     =BB(13)

%B2 Starts here!
ss2 =-ss1*(II(11)+CC(8))
L2  =0.9
P6  =-dp(7)
P3  =-dp(4)
P2  =dp(3)

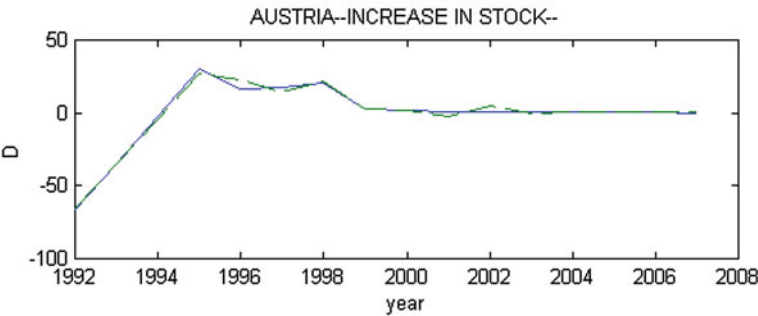
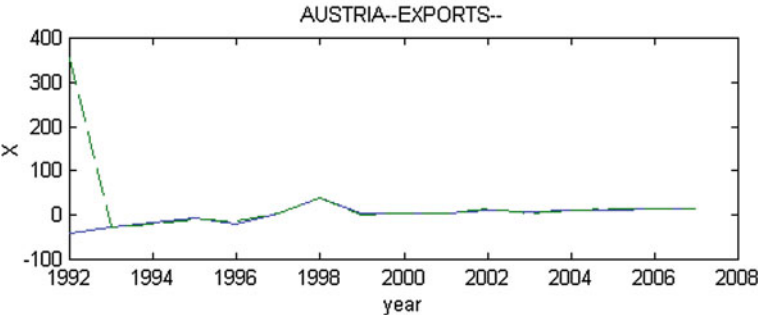
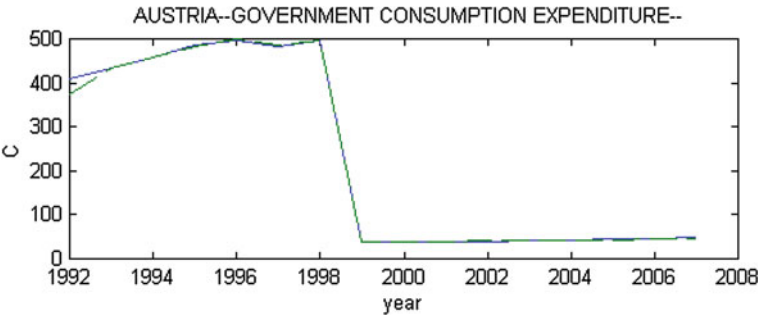
% ln(y)  =a*ln(K) + (1-a)*ln(L)
% lyy(1) =a;
% lyy(2) =1-a

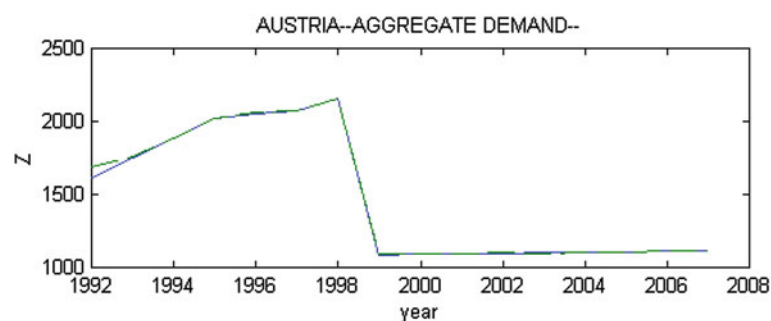
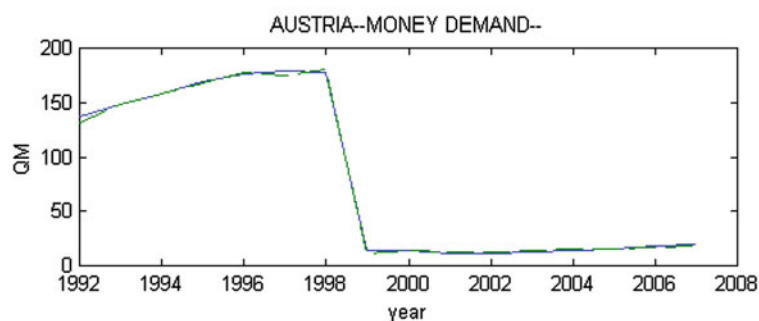
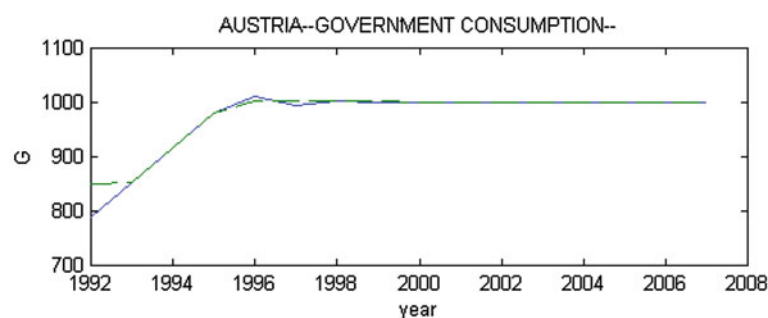
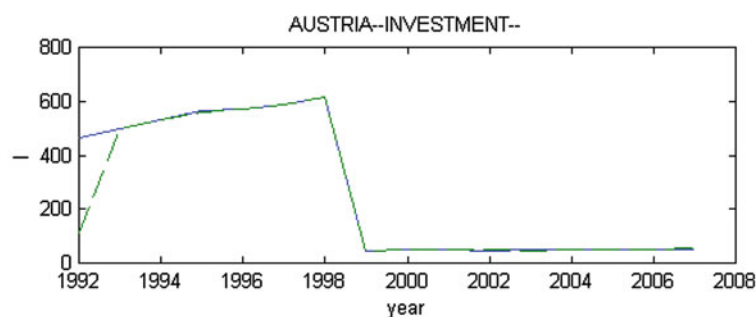
a17  =L1*CC(2)/s1
a45  =a8
c0    =CC(1)
I0    =II(1)
x0    =XX(1)
p0    =dp(1)
qm0   =QMM(1)
y10   =ccy(1)
p0p6m0=dp(1)-dp(7)*QMM(1)
g0    =GG(1)
f0    =BB(1)-XX(1)
g2    =-L2*QMM(1)
a25   =L2*QMM(7)

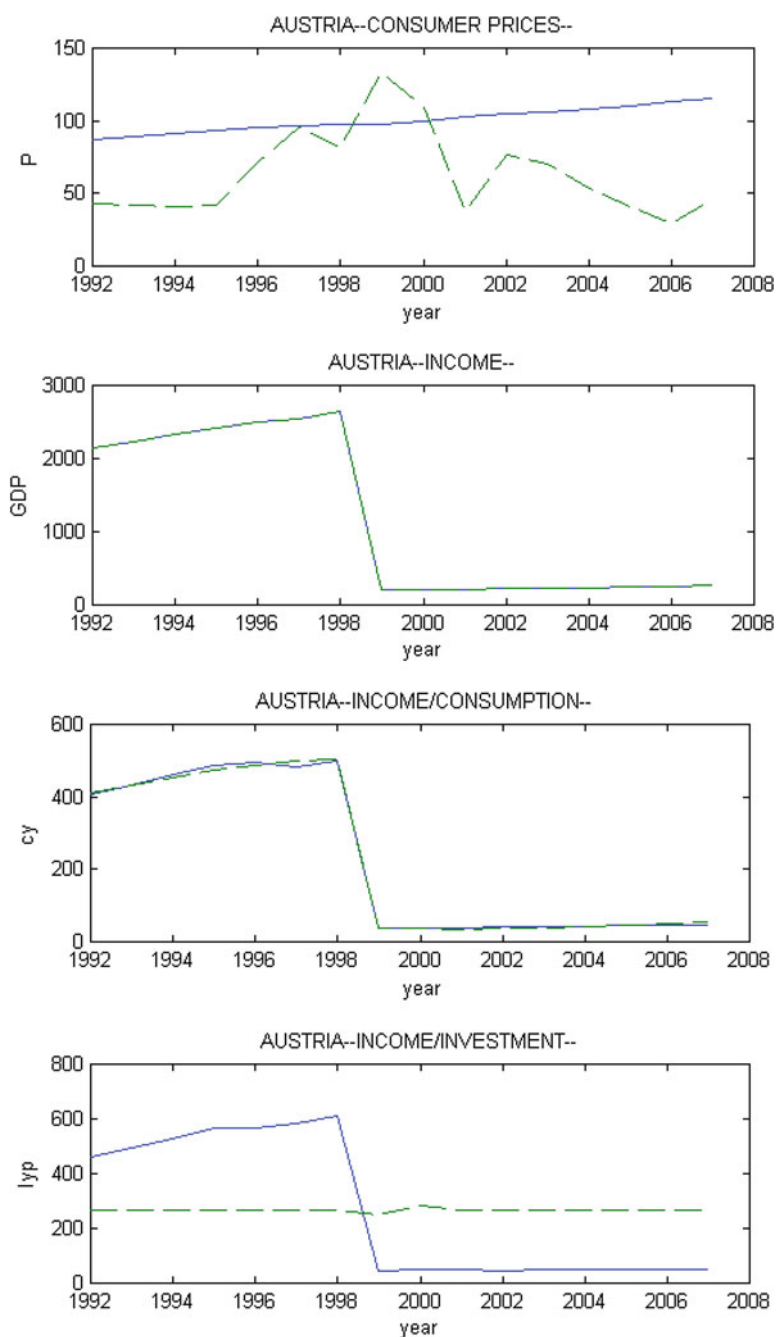
a115  =0;
a124  =0;
a125  =0;
a126  =0;
a134  =0;
%-----

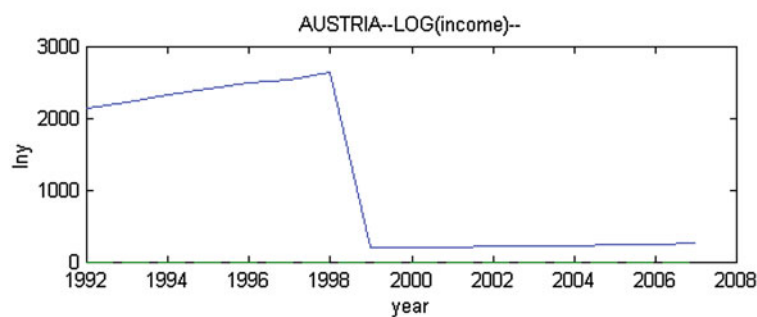
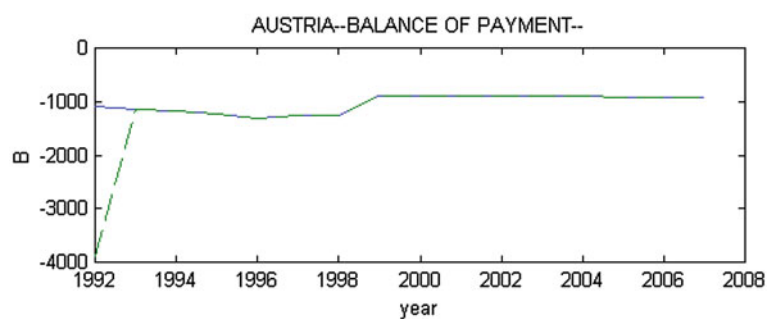
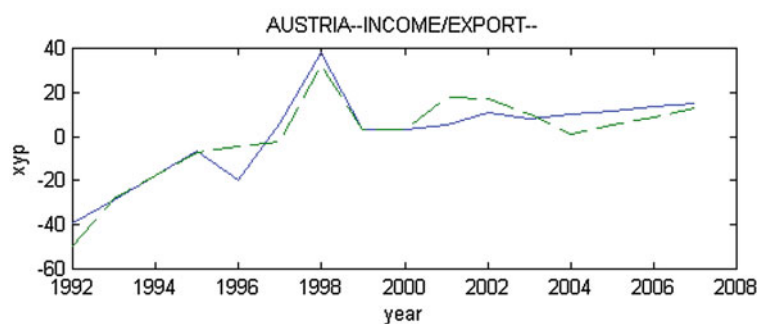
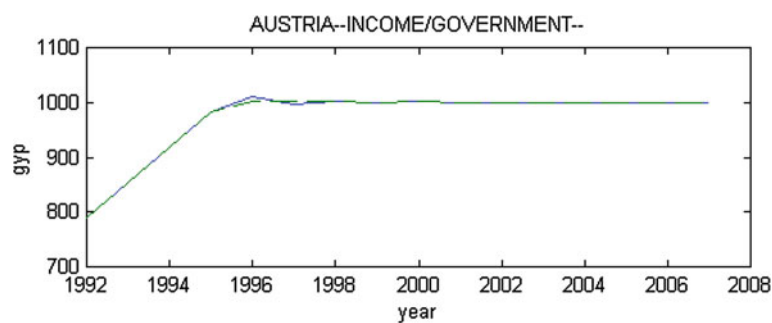
```

The following diagrams are the graphical results from the Austria2.m regression analysis program. The plots shown give the original raw data in solid lines while the estimated results from the regression are given in dash lines (- -).









```
% Austria2.m results
% Austria2_output_report_latest_release.m
% The results given below is the screen capture while
% running Austria2.m program. NDU.m can be used to data mine the results to form
% matrices of A-1, A0, B1 and B2
%
logA = 0.7660, a = 0.3488,aa = 2.5409, W = 6.3498e+003, mw = 3.1877e-011,L1 =1, L2 =0.9000,L5
=0.0080, L6 =-0.0017,h =1,z0 =-23.7717, z1 =-0.0508, z2 =0.0437,z3 =-0.0465,z4 = -0.0227,z5 =-
0.4629,z6 =4.1344,z8 =-0.0692,z9 =0.0287,z10 =0.3308,z11 =-0.1442,z13 =-1.2724e-005,z14 =-
3.1322e-005, z15 =0.8547,z16 =0.0029,z17 =0, z18 =5.9226e-010,z19 =-2.9347e-007,z20 =5.4070e-
007,z21 =9.1153e-009,s1 =0.9397, yy1 = 5.9083e+00,yy2 =-0.3263,yy3 =-0.0091,yy4 = 9.6048e-
005,yy5 =62.4806,yy6 =7.0374,yy7 =-0.1056,yy8 =-0.0105,yy9 =-0.0283,yy10 =-42.4966,yy11
=0.4485,zs0 =-23.7717,a0 =-0.1658, a1 =-0.0011,a2 =0.0011,a3 =-1.5925e-005, a4 =-7.8107,a5 =-
0.8718,a6 = 0.0175,a8 =-0.0017,am11 = -0.0226,am22 = -1.2314e-005,am13 = 0.3520,am14 =
0.3520,Lm03 = -3.8276e-016,lm01 = -1.3389e-004,a3 = -1.2007e-005,a6 = 0.0132,am1 = -1.3389e-
004,m7p6 = -2.0667e-010,b4 =-5.7921e-016, b8 = -2.2086e-013,b12 =8.8069e-015,b13 =0,a01 = -
1.0389,a12 =-0.0674,a14 = 0.2759,a16 =-0.1535,a18 = -7.5416e-011, a21 = -3.8091e-007,a23 =
1.1667e-005,a25 = 2.8661e-005,a33 = -0.1658,a35 = -5.5491e-014, a44 = -0.1658,a45 =-0.0017,a51 =
-1.7894e-010,a52 = 5.4809e-009,a55 =-0.0062,a61 = 3.5436e-015,a62 = -3.9971e-013,a63 = -6.7600e-
014, a11 = 0.0465,a13 = 4.3995,a15 = 0.0305,a22 =5.8485e-007,a24 = 6.4363e-005,a1 =-0.0011,a2
=0.0011,l1 = 0.0011, l2 = -3.6179e-014l4 =-2.4898e-010,b2 = -8.7617e-017,b6 = 2.8909e-013,b10
=-1.5810e-014,b17 = 9.7884e-016, a111 =0.0465,a112 =4.3995,a113 =0.0305,a114 =0,a116 = 0,a121 =
5.8485e-007,a122 = 6.4363e-005,a131 = 4.5948e-014,a132 =-6.3327e-005,a133 = -0.8729,a141
=0.0011,a142 =-7.8110,a143 = -0.8718,a144 = 0,a145 = 0,a146 =0, a151 = 4.2279e-004,a152
=3.0235e-008,a161 =-8.7617e-017,a162 = 2.8909e-013,a163 =1.5810e-014,a164 =0,a165 = 0,a166
=5.6044e-015,ss1 =1.0641,ss1z13 =-1.3540e-005,ss1z15 =-0.0025,ss1z16 = 0.9095,ss1z17
=0.3520,ss1z18 =6.3023e-010,b1z14 =1.1099e-019,b1 = -4.9236e-013,b1z15 = 3.0288e-015,b1z16 =
1.0191e-017,b1z17 =0, b1z13 =-4.5088e-020,mwzs14 = -9.0096e-013,mwzs15 = -1.3547e-009,mwzs16 =
1.4296e-011,mwzs17 =0,mwzs13 = -3.3330e-013,plpf = -2.4117e-004,P4 =4.8477e-005,P5 =-8.4757e-
008,b7 =-3.9971e-013,b8 =2.8909e-013,b15 =9.7884e-016,ss2 =0.4252,L2 =0.9000,P6 =5.5218e-006,P3
=2.4532e-004
P2 =7.5587e-008,a17 = -1.9956e-006,a45 = -0.0017,c0 =-0.0017,i0 =-8.0393,x0 =-15.7276,p0 =-
0.0025,qm0 =-0.0038,y10 = -26.8513,p0p6m0 =-0.0025,g0 =-0.0032,f0 =15.7276,g2 = 0.0034,a25 =
2.8661e-005
»
```

Chapter 11

Program Results: Introduction

In Chap. 10, Program Result for Some Nations: Austria and France, we display samples of what can be generated from the data, the programs and their execution from the contents of CD-Rom of the 141 nations. The identified government strategies and the equation of the representative firms' strategies yield testable policies for the economic growth of each nation. For example, the identified dynamics of the gross domestic product, y , of Austria is

$$\begin{aligned}\dot{y}(t) &= (-0.0226) * y'(t-h) - (0.3520) * L'(t-h) \\ &= (-1.0389) * y(t) + (0.0465) * y(t-h) + (-0.674) * R(t) \\ &\quad + (4.3995) * R(t-h) + (0.2759) * L(t) + (0.0305) * L(t-h) \\ &\quad + (-0.1535) * k(t) - (-7.5416e^{-011}) * p(t) + q1(t) + r_1(t),\end{aligned}$$

where the quarry, government strategy

$$\begin{aligned}q_1(t) &= (0.0011) * [(-0.0032) - (-3.1322e^{-005}) * T(t) \\ &\quad + (-2.9347e^{-007}) * T(t-h) - (5.4070e^{-007}) * T'(t) \\ &\quad - (9.1153e^{-009}) * T(t-h) - (0.8547) * e(t) \\ &\quad + (0.0029) * tar(t) + 0.d(t),\end{aligned}$$

and the representative private firm's strategy

$$\begin{aligned}r_1(t) &= (0.0011)[((-0.0017) + (-8.0393) + (-15.7276)) \\ &\quad - (-0.0038)(-0.3996 + 3.1322e^{-005})].\end{aligned}$$

Thus if the hereditary growth rate of employment is steady the hereditary growth rate of GDP,

$$\frac{dy(t)}{dt} + (0.0226) * y'(t - h),$$

is increased when autonomous government outlay or spending g_0 is increased. Decrease in current growth rate of taxes, decrease in exchange rate and previous year tax rate will increase the hereditary growth rate of GDP. Current increase in taxes, current increase in tariffs and a positive trade policy and/or distance between trading nations which is carefully negotiated will enhance the growth rate of GDP.

A strong increase of autonomous private spending or investment, autonomous consumption, autonomous net export, will reduce the negative values identified for this country or make them positive just as it was said about Botswana, and thus increase the growth of GDP.

Detailed analysis can be made from the identified equations for interest rate, employment, value of capital stock, growth of prices (inflation) and cumulative balance of payment, and good policies deduced.

The time duration is important in the simulation and identification of the economic dynamics.

If the IMF data is incomplete, as presented, one can extract more data from outside the interval considered or from interpolation or from other sources such as the UN. The program provides a guide. Examination of the execution and display of USA.2 m, UK.2 m, etc. in the author's earlier book, "A Mathematical Treatment of Economic Cooperation and Competition Among Nations with Nigeria, USA, UK, China, and Middle East Examples," Elsevier, 2005, is a very good resource.

Running the programs and deducing policies is very good training for emerging policy makers. It may help to download "An Electronic Appendix containing algorithms and data available from <http://www.elsevier.nl/locale/ame>" E. N. Chukwu, "On the Controllability of Non-linear Economic Systems with Delay: The Italian Example," Applied Mathematics and Computation, Vol. 95 (1998) pp. 245–274.

Appendix: Program Results of All the Nations—<http://www.atlantis-press.com/publications/books/chukwu/>

All the Nations

1. Albania
2. Algeria
3. Angola
4. Anguilla
5. Antigua and Barbuda
6. Argentina
7. Republic of Armenia
8. Aruba
9. Australia
10. Austria
11. Republic of Azerbaijan
12. The Bahamas
13. Kingdom of Bahrain
14. Bangladesh
15. Barbados
16. Belarus
17. Belgium
18. Belize
19. Benin
20. Bhutan
21. Bolivia
22. Bosnia and Herzegovina
23. Botswana
24. Brazil
25. Brunei Darussalam
26. Bulgaria
27. Burkina Faso
28. Burundi
29. Cambodia
30. Cameroon
31. Canada

32. Cape Verde
33. CEMAC
34. Central African Republic
35. Chad
36. Chile
37. Hong Kong, P. R. China
38. Macao, P. R. China
39. Mainland, P. R. China
40. Colombia
41. Comoros
42. Democratic Republic of Congo
43. Republic of Congo
44. Costa Rica
45. Côte d'Ivoire
46. Croatia
47. Cyprus
48. Czech Republic
49. Denmark
50. Djibouti
51. Dominica
52. Dominican Republic
53. Eastern Caribbean Currency Union
54. Ecuador
55. Egypt
56. El Salvador
57. Equatorial Guinea
58. Eritrea
59. Estonia
60. Ethiopia
61. Euro Area
62. Fiji
63. Finland
64. France
65. Gabon
66. The Gambia
67. Georgia
68. Germany
69. Ghana
70. Greece
71. Grenada
72. Guatemala
73. Guinea
74. Guinea-Bissau
75. Guyana
76. Haiti

77. Honduras
78. Hungary
79. Iceland
80. India
81. Indonesia
82. Islamic Republic of Iran
83. Iraq
84. Ireland
85. Israel
86. Italy
87. Jamaica
88. Japan
89. Jordan
90. Kazakhstan
91. Kenya
92. Republic of Korea
93. Kuwait
94. Kyrgyz Republic
95. Lao People's Democratic Republic
96. Latvia
97. Lebanon
98. Lesotho
99. Liberia
100. Libya
101. Lithuania
102. Luxembourg
103. Macedonia
104. Madagascar
105. Malawi
106. Malaysia
107. Maldives
108. Mali
109. Malta
110. Mauritania
111. Mauritius
112. Mexico
113. Federated States of Micronesia
114. Moldova
115. Mongolia
116. Montserrat
117. Morocco
118. Mozambique
119. Myanmar
120. Namibia
121. Nepal

122. Netherlands Antilles
123. Netherlands
124. New Zealand
125. Nicaragua
126. Niger
127. Nigeria
128. Norway
129. Oman
130. Pakistan
131. Panama
132. Papua New Guinea
133. Paraguay
134. Peru
135. Philippines
136. Poland
137. Portugal
138. Qatar
139. Romania
140. Russian Federation
141. Rwanda
142. Saint Kitts and Nevis
143. Saint Lucia
144. Samoa
145. San Marino
146. São Tomé and Príncipe
147. Saudi Arabia
148. Senegal
149. Seychelles
150. Sierra Leone
151. Singapore
152. Slovak Republic
153. Slovenia
154. Solomon Islands
155. South Africa
156. Spain
157. Sri Lanka
158. Sudan
159. Suriname
160. Swaziland
161. Sweden
162. Switzerland
163. Syrian Arab Republic
164. Tajikistan
165. Tanzania
166. Thailand

- 167. Togo
- 168. Tonga
- 169. Trinidad and Tobago
- 170. Tunisia
- 171. Turkey
- 172. Uganda
- 173. Ukraine
- 174. United Arab Emirates
- 175. United Kingdom
- 176. United States
- 177. Uruguay
- 178. Vanuatu
- 179. República Bolivariana de Venezuela
- 180. Vietnam
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