

# Quantifiers in Language and Logic

STANLEY PETERS  
DAG WESTERSTÅHL

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# QUANTIFIERS IN LANGUAGE AND LOGIC



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In memory of our colleague and friend Jon Barwise  
1942–2000



## *Preface*

What is so special about the area of quantification that motivates a book of this length? Quantifiers are one of very few expressive devices of language for which it is known how to break out of the circle of language and explain what a word means other than essentially in terms of other words' meanings. It is possible to explain the meaning of quantifiers in mathematical and other non-linguistic terms. This foundation not only provides a satisfyingly clear account of the meaning of quantifiers themselves, but also lies behind the widespread use of quantifiers in analyzing the meaning of an extensive range of non-logical expressions, including tenses and temporal adverbs, modal verbs, conditionals, attitude verbs, and some noun phrases that may not be explicitly quantified.

This book is intended for everyone with a scholarly interest in the exact treatment of meaning. It presents a broad view of the semantics and logic of quantifier expressions in natural languages and, to a slightly lesser extent, in logical languages. The exposition starts from a fairly elementary level and progresses to considerable depth over the course of sixteen chapters. It is meant to be accessible and valuable to readers ranging from those with a rudimentary knowledge of linguistic semantics and of 'predicate calculus' (first-order logic), to ones with advanced knowledge of semantics, logic, philosophy of language, and knowledge representation in artificial intelligence.

We introduce the subject with brief surveys of both the range of means by which human languages express quantification and the ancient Greek discovery of the logic of quantifiers. From this elementary beginning, the book recounts the development of modern concepts of quantification, including so-called generalized quantifiers—mainly a product of the past century and a quarter.

Precise ways of analyzing quantifier meanings by the use of model theory are then introduced. Important distinctions are described and analyzed—concerning what is quantified over, how to restrict quantification to specified domains, and what may be thought of as the number of variables a quantifier binds. The presentation combines intuitive discussion and numerous examples with precise formulation and demonstration of concepts, properties thereof, and proofs of precisely stated results. We aim throughout to give the whole range of readers access to the rigor of logical analysis and demonstration while revealing the rich variety of quantification in natural languages, and to disclose the power of logical techniques to expose significant and sometimes surprising features of quantification in both natural and logical languages.

Attention is paid to properties possessed by substantial subclasses of quantifiers, as well as to properties all quantifiers share. Both kinds of property are related to recognized facts of language. For example, we give an account in depth of various monotonicity properties of quantifiers, and of linguistic facts related to monotonicity, such as the distribution of so-called polarity items. Likewise, we discuss at length the

quantifiers acceptable in existential-there sentences, and we study in detail possessive quantifiers and quantifiers occurring in exception phrases. We also present different means for combining quantifiers, exemplified both in natural and in logical languages.

The book furthermore provides a detailed—and as far as we know unprecedented—abstract account of concepts related to the expressive power of languages, natural as well as logical: synonymy, translation, and definition. This account is used to transfer technical results for logical languages to natural languages. The closing chapters present a rich panoply of logical tools and techniques for demonstrating non-expressibility and undefinability in logical languages—a result that is much harder to show than expressibility, for which it suffices to exhibit a defining expression.

The aim of the book is to give a comprehensive picture of the whole area of quantification, without the reader having to retrace all the steps in the literature. But we also present new material, analyses, and results. In the detailed table of contents, we have marked with an asterisk \* the chapters or sections that are particularly heavy with new material. We have attempted to provide equal value to a broad range of readers: linguists, philosophers, logicians, and others, both in presentation and in choice of material. Our method is always to combine linguistic and logical aspects, rather than focusing primarily on one or the other. A subsidiary aim has been to provide non-logicians with access to some of the main tools that logicians have developed for proving facts of semantic as well as logical properties of quantifiers. So, particularly in the last three chapters, the reader will find a more pedagogic and detailed account of various expressivity results than in corresponding mathematical-logical texts.

A reader primarily interested in a particular aspect of quantification—say, possessive quantifiers—doesn't have to be familiar with all the material in the chapters preceding the one on possessives, but notation, definitions, and results from some of these will be used. Figure 1 indicates how each chapter depends on preceding ones. (A dotted line indicates that the dependence is less strict.)

Numerous people provided invaluable help during our writing of this book, and we are grateful to them all. We especially want to thank Wilfrid Hodges, as well as two anonymous referees, who gave detailed and constructive comments on an early version of the text. Others have read and commented on individual chapters; we are particularly grateful to Johan van Benthem, William Ladusaw, and Barbara Partee. A number of people provided inspiration, gave us references, or pointed out mistakes large and small—among them Robin Cooper, Sol Feferman, Dagfinn Føllesdal, Ivan Garcia-Álvarez, Fritz Hamm, Lauri Hella, Martin Karlsson, Phokion Kolaitis, Hannes Leitgeb, Iddo Lev, Roussanka Loukanova, Lisa Matthewson, Peter Pagin, Jouko Väänänen, and Tim Williamson. We also thank the audiences of numerous talks and seminars where we have presented parts of the material in the book: the joint Stanford–Göteborg–Stockholm seminar (connected by video link) in the fall of 1998, where some of this material was first presented; the Logic and Language seminar in Göteborg; the seminar on Logic and Philosophy of Language in Stockholm; at Stanford: the Logical Methods in the Humanities seminar, the Cognitive Science Lunch seminar at the Center for the Study of Language and Information, our course at NASSLLI 2002, and the Semantics Fest 2000 and 2005; furthermore, the



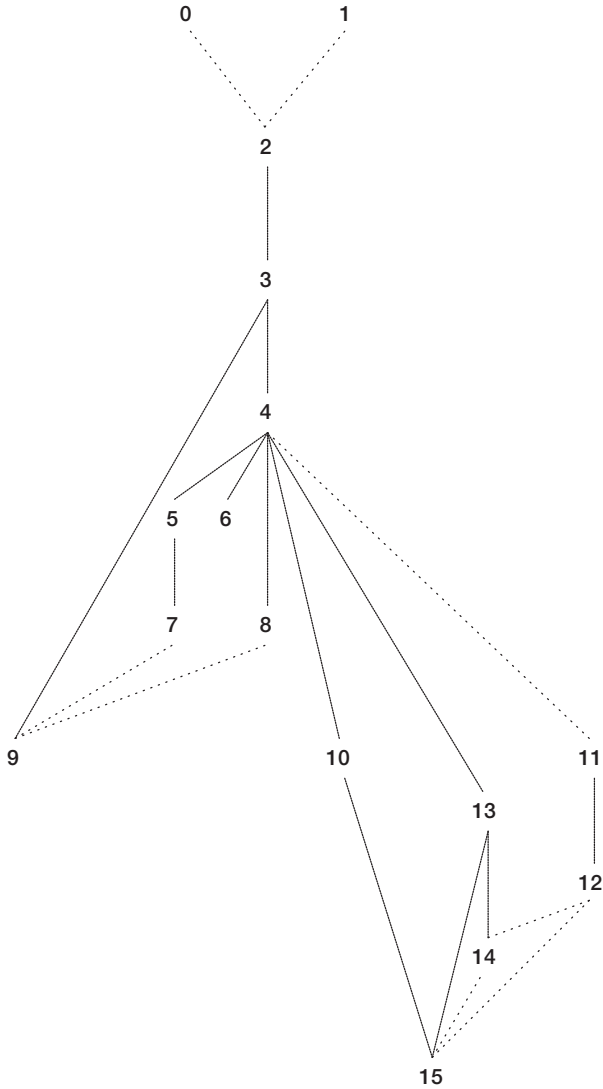


Figure 1 Dependency diagram for the chapters of the book

Content and Context Conference 2005 in Stockholm, the Philosophy Department at the University of Bologna, the Dublin Computational Linguistics Research Seminar, the Institut für Sprachwissenschaft at Tübingen University, the Linguistics Department at UCLA, and probably others that we have forgotten. Finally, Stanley Peters gratefully acknowledges the support of the Andrew W. Mellon Foundation, and Dag

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# 0

## Quantification

This chapter gives a sketch of quantification, as the phenomenon is viewed by linguists, to provide background for the book's study of the meaning and the properties of natural language expressions for quantification.

The logical study of quantification is as old as logic itself, beginning in the work of Aristotle. The logical study focuses on the meaning and inferential characteristics of quantifiers. An outline of how logicians and philosophers have viewed quantification from Aristotle onwards is given in the next chapter. Linguistic study of quantifiers began much later, and until recently focused mainly on the grammatical expression of quantification rather than its meaning.

Linguists have borrowed heavily from logicians in semantically analyzing what quantifiers mean. Logicians found it most natural to analyze the meaning of quantification over discrete individuals rather than over pluralities of them or parcels of non-discrete stuff. In linguistic terms, the logical focus has been on quantification over domains denoted by individual count nouns such as *person* and *table*, rather than domains denoted by collective count nouns such as *crowd* and *suite*<sup>1</sup> or by mass nouns such as *protoplasm* and *furniture*. For this reason, the semantics of plural quantification and of mass quantification have only recently begun to be developed formally (Pelletier 1979; Link 1983; Lønning 1997). This historical order of development seems in fact to be the natural order of conceptual development, and our exposition will follow it. This book treats quantifiers over discrete individuals extensively. It does not, however, undertake to cover what is known about collections and quantities of stuff, although we will briefly touch on them in this chapter.

The first section of this chapter (0.1) provides examples illustrating the wide range of quantifiers that are expressed in natural languages, and to some extent also the variety of means which natural languages employ to express them. Section 0.2 discusses ways in which quantifiers vary in their syntactic expression and their semantic interpretation, and then turns to differences in the semantic quantifier type. In particular, some useful terminology for referring to the most important natural language expressions that signify quantifiers is introduced. Section 0.3 distinguishes between explicitly quantifying expressions and expressions whose meaning implicitly involves quantification in various degrees. Section 0.4, finally, deals briefly with quantifier scope.

<sup>1</sup> As in dining room suite.

## 0.1 SOME QUANTIFIER EXPRESSIONS OF NATURAL LANGUAGES

Natural languages use quantifier expressions for talking about quantity of things or amount of stuff, such as dozens of eggs or liters of milk. These quantifier expressions include some of the ones Aristotle was concerned with, which are discussed further in Chapter 1.

English		Swedish	
<i>count</i>	<i>mass</i>	<i>count</i>	<i>mass</i>
no	no	ingen/inget/inga	ingen/inget
some	some	någon/något/några	någon/något
both		båda/bägge	
few	little	få	lite
(1) a few	a little	ett fåtal/några få	lite
several		flera	
enough	enough	tillräckligt många	tillräckligt mycket
many	much	många	mycket
most	most	de flesta	den/det mesta
each		varje	
every		varje	
all	all	alla	all/allt

(2) a. Every dog barks.  
b. Varje hund skäller.

(3) a. Susan likes most French films.  
b. Susan tycker om de flesta franska filmer.

(4) a. Little water contains deuterium.  
b. Lite vatten innehåller deuterium.

(5) a. Harry drank a little water.  
b. Harry drack lite vatten.

For some of the English quantifier expressions (*viz.* *each* and *every*), the count noun that follows must be singular in number; for others, it must be plural (e.g. *both*, *all*). And for a few the count noun that follows may be either singular or plural: *some*, *no*. In the Swedish examples above, on the other hand, the quantifier expression always determines the grammatical number of the count noun. But the semantic significance of grammatical number appears to be the same in both languages. For example, a difference in meaning is generally felt to exist between

- (6) a. Some woman objected to John's presentation.  
b. Någon kvinna invände mot Johns presentation.

and

- (7) a. Some women objected to John's presentation.  
 b. Några kvinnor invände mot Johns presentation.

Sentences (7) signify that more than one woman objected. One might take these correlated differences in meaning and grammatical number to show that in (7) the expression *some* (*några*) quantifies over pluralities rather than individual women. However, no such difference is evident in

- (8) a. No woman objected to Bill's presentation.  
 b. Ingen kvinna invände mot Bills presentation.
- (9) a. No women objected to Bill's presentation.  
 b. Inga kvinnor invände mot Bills presentation.

Moreover,

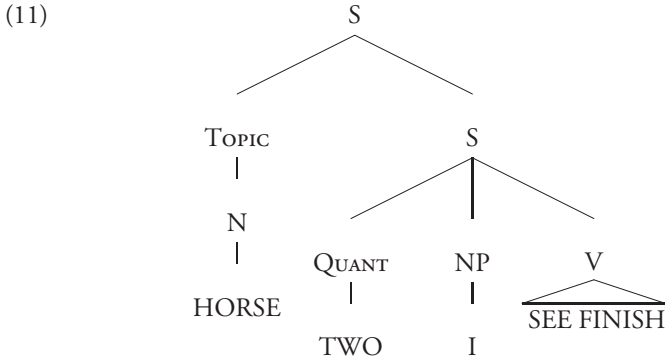
- (10) a. Every/each woman objected to Peter's presentation.  
 b. Varje kvinna invände mot Peters presentation.

ordinarily indicates that more than one woman is under discussion, as well as that they all objected. So grammatical number of the count noun does not in fact correlate in any simple way with the number of instances that the quantifier applies to. This point is borne out by many more examples that could be given with the quantifier expressions discussed below. Therefore, we do not take grammatical number to be semantically significant in and of itself. Such meaning differences as appear between (6) and (7), or between *the woman* (*kvinnan*) and *the women* (*kvinnorna*), need more complicated explanations than just the difference between quantifying over individuals and quantifying over pluralities. We place no explanatory burden on the count noun's grammatical number, and generally treat both singular and plural forms of each count noun as denoting the same set of individuals (see Chapter 4.2.2 for further discussion).

The English and Swedish expressions tabulated above and modified forms like the English *(in) finitely many, too many, too few, or surprisingly few* appear in noun phrases as determiners of nominal expressions, or as part of such determiners.<sup>2</sup> In some languages the morphemes or words expressing these quantifiers do not occur as determiners of the nouns they quantify but rather as agreement markers on verbs or with various other grammatical functions. American Sign Language (ASL) has at least three other ways of expressing quantification besides using determiners in noun phrases (see Petronio 1995). An explicit quantifier, e.g. *TWO* in (11), occurs in the scope clause when the domain is denoted by a non-agreeing

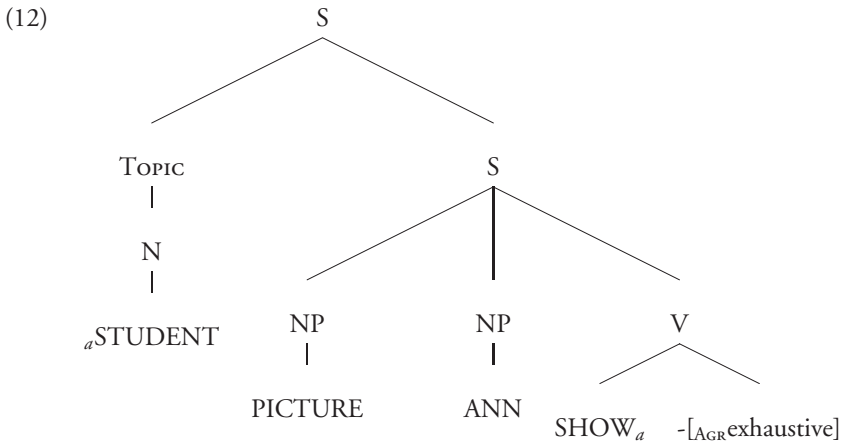
<sup>2</sup> We refer to the syntactic category of phrases like English *every Greek* as noun phrase (sometimes NP) throughout this book for convenience. We are not aware of anything significant to the purposes of this book which hinges on whether these phrases are headed by nouns or by determiners as hypothesized by linguists who call them determiner phrases.

noun for which the two special cases in (12) and (13) don't apply, in (11) the topic noun HORSE:<sup>3</sup>



I saw two horses.

The general form just stated doesn't apply to the agreeing argument. Quantification whose domain is denoted by the agreeing argument of an agreement verb is indicated by the verb's agreement affix, as in (12), or the theme argument of spatial verbs.



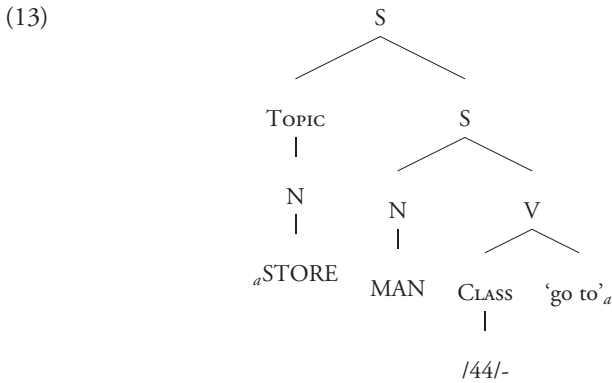
Ann showed each student the picture(s).

Here the exhaustive suffix Agr expresses universal quantification, and the agreeing argument is the topic noun STUDENT, and restricts the quantification to students.

<sup>3</sup> The following examples from ASL and Straits Salish illustrate various syntactic means that these languages employ to express quantification. It is not necessary to understand the examples in detail to appreciate the point they illustrate; so do not be discouraged if linguistic terminology or phonetic symbols are unfamiliar. We use trees to display the structure, in order to make the syntactic categories more evident.

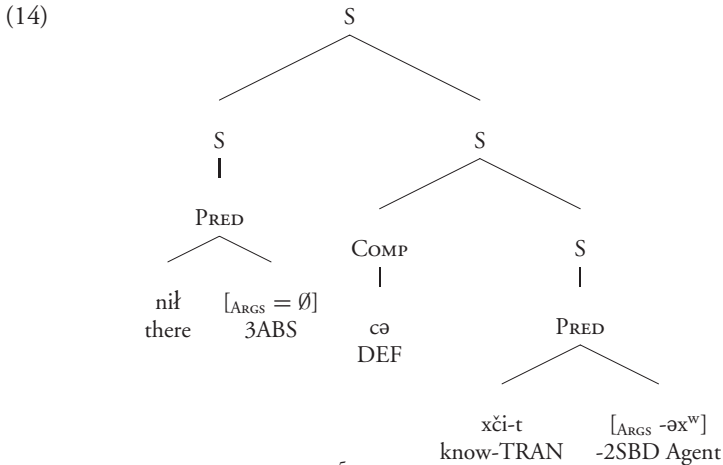


In (13), quantification is indicated by a classifier, /44/-, meaning *many*, for the theme argument of a spatial verb of motion when that noun, MAN, denotes the domain.



The/many men went to the store.<sup>4</sup>

Another language, Straits Salish (a Salishan language spoken on the northwest coast of North America) expresses claims like “You know that man”, “We ate every fish”, “We all ate the fish”, “I saw many of them”, and “There are fish” without benefit of nouns, determiners, or noun phrases (see Jelinek 1995), instead using complementizers, predicate modifiers, and ordinary predicates in the following ways.



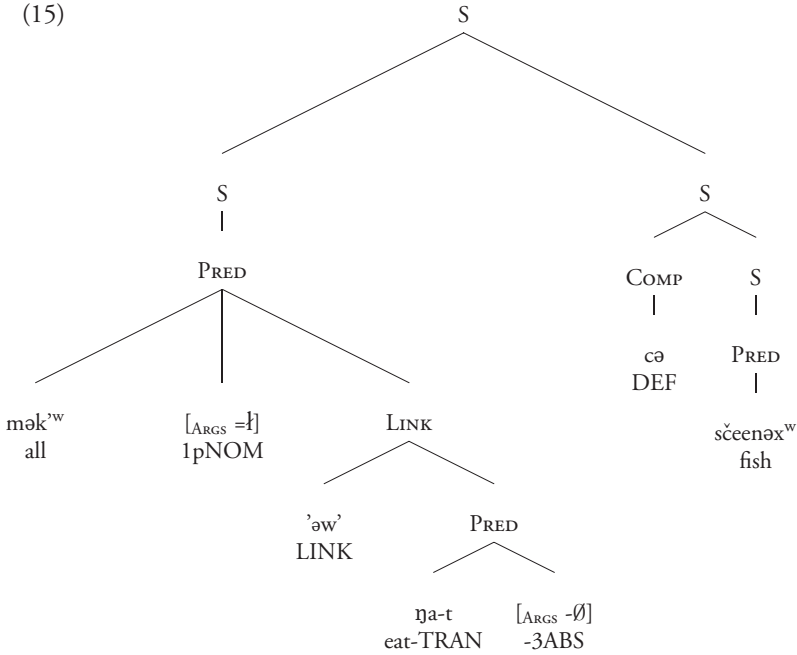
That's him, the one you know.<sup>5</sup>

In (14), the demonstrative-like complementizer *cə* expresses quantification over people you know.

<sup>4</sup> The classifier /44/- indicates a large number of animate objects standing or moving upright on two legs.

<sup>5</sup> The symbols = and - denote different kinds of boundaries between morphemes within a word.

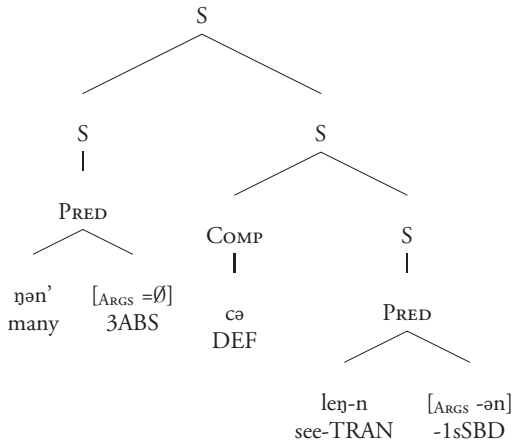
(15)



Ambiguously: We all ate the fish. We ate all the fish.

The adverb-like predicate modifier  $mæk^w$  expresses quantification over either fish or us in (15). The syntax does not mark which is quantified over.

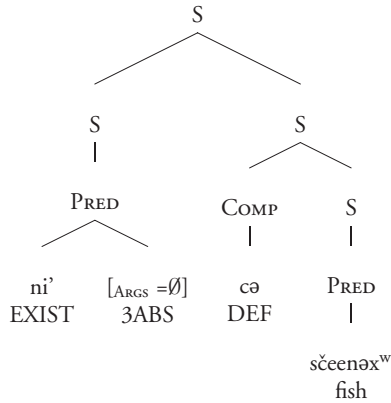
(16)



They are many, the ones I saw.

In (16) and (17), quantification is expressed by ordinary predicates  $\eta\text{ən-}$  and  $\text{ni-}$  in the main clause.

(17)



There's the fish.

In the next section, we address the importance of the syntactic differences that these languages manifest. For the remainder of this section, our illustrative examples are drawn from English and Swedish.

Discontinuous parts of determiners like the following also express quantifiers.

	English		Swedish	
	<i>count</i>	<i>mass</i>	<i>count</i>	<i>mass</i>
(18)	more ... than	more ... than	fler(a) ... än	mer(a) ... än
	fewer ... than	less ... than	färre ... än	mindre ... än
	as many ... as	as much ... as	lika många ... som	lika mycket ... som

(19) a. More doctors than dentists are millionaires.

b. Fler läkare än tandläkare är miljonärer.

(20) a. As much sand as glass is silicon.

b. Lika mycket sand som glas är silikon.

Possessive determiners express quantifiers too.

	English		Swedish	
	<i>count</i>	<i>mass</i>	<i>count</i>	<i>mass</i>
(21)	my	my	min/mitt/mina	min/mitt
	your	your	din/ditt/dina	din/ditt
	Christer's	Christer's	Christers	Christers
	the chemistry professor's	the chemistry professor's	kemiprofessorns	kemiprofessorns
	every student's	every student's	varje students	varje students
	my sister's five		min systers fem	

Numerals, which have syntactic properties different from each group of expressions above, can also be used to express quantifiers.

	English	Swedish
	<i>count</i> <i>mass</i>	<i>count</i> <i>mass</i>
(22)	zero	noll
	one	en/ett
	two	två
	three	tre

So can modified forms of numerals.

	English		Swedish	
	<i>count</i>	<i>mass</i>	<i>count</i>	<i>mass</i>
	about five		omkring fem	
	at least one		minst en/ett	
(23)	at most six		högst sex	
	exactly three		exakt tre	
	more than two		fler/mer än två	
	fewer than four		färre/mindre än fyra	
	no more than five		inte fler/mer än fem	
	no fewer than five		inte färre/mindre än fem	

Likewise complex combinations like *between six and twelve*. In addition, proportional quantifiers exist for count and mass terms alike.<sup>6</sup>

	English		Swedish	
	<i>count</i>	<i>mass</i>	<i>count</i>	<i>mass</i>
	half of the	half of the	hälften av	hälften av
	at least a	at least a	minst en	minst en
(24)	third of the	third of the	tredjedel av	tredjedel av
	at most two-	at most two-	högst två	högst två
	thirds of the	thirds of the	tredjedelar av	tredjedelar av
	more than	more than	mer än	mer än hälften av
	half of the	half of the	hälften av	
	less than three-	less than three-	mindre än tre	mindre än tre
	fifths of the	fifths of the	femtedelar av	femtedelar av

Quantifiers are related to the indefinite and definite articles.

<sup>6</sup> In Swedish as well as English, the nouns need to be definite; e.g.

- (i) a. Hälften av mjölken återstår.  
 b. Half of *the* milk remains.

	English		Swedish	
(25)	<i>count</i>	<i>mass</i>	<i>count</i>	<i>mass</i>
	a(n)		en/ett	
	the	the	-n/-t/-na	-n/-t

In addition, some adjectives express quantifiers.

	English		Swedish	
(26)	<i>count</i>	<i>mass</i>	<i>count</i>	<i>mass</i>
	numerous		talrika	
	innumerable		oräkneliga	

Interestingly, some quantificational determiners also function as adjectives.

- (27) a. (The reasons for doing it are) many/few/three/at most five.  
 b. (Skälen att göra det är) många/få/tre/högst fem.

Some adverbs can be used to express quantifiers.

	English		Swedish	
	<i>count</i>	<i>mass</i>	<i>count</i>	<i>mass</i>
(28)	always	always	alltid	alltid
	mostly	mostly	oftast	oftast
	usually	usually	vanligen	vanligen
	mainly	mainly	huvudsakligen	huvudsakligen
	frequently		(ofta)	
	often	often	ofta	ofta
	seldom	seldom	sällan	sällan
	rarely	rarely	sällan	sällan
	sometimes	sometimes	ibland	ibland
	never	never	aldrig	aldrig

(29) Quadratic equations usually have two distinct solutions.

(30) Men seldom make passes at girls who wear glasses.

(31) Oil always floats on water.

Phrases like the following also express quantifiers.

		English		Swedish	
		<i>count</i>	<i>mass</i>	<i>count</i>	<i>mass</i>
		a couple of		några stycken	
		a lot of	a lot of	en hel del	en hel del
(32)		a small number of	a small amount of	ett litet antal	en liten mängd
		a large number of	a large amount of	ett stort antal	en stor mängd
		a finite number of	a finite amount of	ett ändligt antal	en ändlig mängd
		an infinite number of	an infinite amount of	ett oändligt antal	en oändlig mängd
		a odd number of		ett udda antal	
		an even number of		ett jämnt antal	

Moreover, quantification may be expressed by non-phrasal constructions.

		English		Swedish	
		<i>count</i>	<i>mass</i>	<i>count</i>	<i>mass</i>
(33)		there is/are	there is	det finns/är	det finns/är

- (34) a. There are children living under the railway viaduct.  
b. Det finns barn som bor under järnvägsviadukten.

- (35) a. There is sand in the gears.  
b. Det är sand i växlarerna.

The foregoing examples demonstrate how much wider a range of quantifiers natural languages provide than the universal and existential quantifiers from first-order logic.

## 0.2 VARIETIES OF QUANTIFICATION

Quantifiers vary in several ways. A useful way to group their differences initially is into syntactic variation versus semantic variation.

### 0.2.1 Syntactic variation in quantifier expressions

As we saw in the preceding section, some quantifier expressions are determiners. In fact, most of the examples we presented were. However, other quantifier expressions were adverbs. These are quite common among the world's languages. And other quantifier expressions are noun phrases; for example:

- (36) a. Something (is behind that vase)  
b. Everyone (likes Mary)  
c. Nobody (is there now)  
d. Most tigers (are Siberian)

We will see in the next subsection that the quantifiers expressed by noun phrases differ semantically as well as syntactically from quantifiers expressed by determiners and adverbs.

While all readers of this book have direct knowledge of the English quantifier expressions we discuss, it is important to remember that some languages express quantification by means very different from determiners or noun phrases, which constitute our most frequent examples. We illustrated a few of these other expressions in (11)–(17). In the past decade or thereabouts, these linguistic differences have received considerable attention. Some languages—for instance, Straits Salish (Jelinek 1995)—are known not to express quantifiers with determiners. The next subsection examines the significance of this fact in some detail. Let us here simply emphasize that in the relatively brief time during which scholars have been investigating linguistic variation in the expression of quantification, quite a number of languages have been identified which do not use determiners and/or do not use noun phrases to anything approaching the extent to which English and Swedish do for expressing quantification. Straits Salish uses predicates, auxiliaries, and words adjoined to determiners to express quantification, whereas it is controversial whether it does or doesn't use noun phrases (see Matthewson 1998).

Languages that do not use determiners for quantification are sometimes said to employ *A-quantification* (see Bach *et al.* 1995: Introduction), as opposed to *D-quantification*: the expression of quantification by determiners such as *every/varje*, *most/(de) flesta*, and *no/ingen(/inget/inga)*. The “A” in A-quantification is mnemonic for adverbs, auxiliaries, affixes, and argument-structure adjusters. As we have seen, English and Swedish employ such devices for quantification, having adverbs of quantification like those in (28) in addition to quantificational determiners. The rubric of A-quantification covers quite a broad range of quantificational devices, as the list of ‘A’s and the examples in (11)–(17) show.

### 0.2.2 Semantic types of NL quantification

In itself, the fact that certain languages scarcely use D-quantification does not make them either more capable or less capable of expressing quantified propositions. The examples in (11)–(17) show ASL and Straits Salish doing fine at expressing with A-quantification the same sorts of proposition that English expresses with D-quantification in (36). To appreciate what the significance of A- versus D-quantification is, we need to focus more closely on the meanings of natural language quantifiers.

These meanings can be classified according to semantic types, of which a number are manifested in natural languages. The quantifiers exemplified in (1), (21)–(26), (28), and (32), as well as those used in (11)–(17), are semantically of type  $\langle 1, 1 \rangle$ ; that is, they are binary relations between sets of things or stuff.<sup>7</sup> Thus a quantified sentence like

<sup>7</sup> The type notation will be explained in Ch. 2.

(37) Every dog is barking.

states that *every* ( $D, B$ ) holds, where  $D$  is the set of dogs,  $B$  is the set of things that are barking, and *every* is a relation between sets. The type  $\langle 1, 1 \rangle$  quantifier *every* is a particularly simple relation to describe: it is just the subset relation  $\subseteq$ . By the same token, (15) states, ambiguously, that the set of us is included in the set of eaters of the fish, or that the set of fish is included in the set of things we ate. The next chapters explain more precisely what this amounts to.

By contrast, the English noun phrases *everything*, *something*, *nothing*, *everyone*, *someone*, *no one*, and others in (36) express type  $\langle 1 \rangle$  quantifiers, as do the Swedish noun phrases *allting*, *någonting*, *ingenting*, *var/t och en/ett*, *någon/t*, *ingen/t*. These quantifiers are properties of sets of things (or stuff), and many, though perhaps not all, languages have phrases that express such quantifiers.

It is important to recognize that in natural languages type  $\langle 1, 1 \rangle$  quantifiers are more basic than type  $\langle 1 \rangle$  quantifiers. So far as is known, all languages have expressions for type  $\langle 1, 1 \rangle$  quantifiers, though some languages have been claimed not to have any expressions for type  $\langle 1 \rangle$  quantifiers. A-quantification expressions and D-quantification expressions alike denote type  $\langle 1, 1 \rangle$  quantifiers. Many of these expressions are lexical; that is, they are semantically basic within their language. The meanings of phrases and sentences in which these lexical items occur are composed by combining the type  $\langle 1, 1 \rangle$  quantifier meaning with the meanings of other expressions, regardless of whether the quantifier is expressed by an A-word or a D-word.

This is no less true for English, Swedish, and ASL, which have D-quantification and noun phrases, than for Straits Salish and other languages that lack D-quantification and might lack noun phrases. The meaning of a quantificational noun phrase of English is a type  $\langle 1 \rangle$  quantifier composed of the type  $\langle 1, 1 \rangle$  quantifier that the determiner expresses and the set that the nominal expression denotes. The intermediate step of constructing a type  $\langle 1 \rangle$  quantifier on the way to constructing the meaning of the whole sentence is not really necessary to getting to the full sentence's meaning, and may well be 'skipped' when the basic type  $\langle 1, 1 \rangle$  quantifier is expressed by an A-word instead of a D-word. Skipping it is no less an option for A-quantifiers in languages that possess D-quantification, e.g. English, Swedish, and ASL, than it is in languages that lack D-quantification altogether.

While type  $\langle 1 \rangle$  quantifiers are theoretically important for a number of reasons, which the following chapters discuss in detail, it appears to us that too exclusive a focus has been placed on them as expressions of quantification in natural languages, making the scarcity of quantificational noun phrases in some languages seem more mysterious than it is. Even in languages that have D-quantification, the expression of quantification by means of A-words—without the intervention of quantificational noun phrases—might seem unnecessarily mysterious. In actuality, both D-quantification and A-quantification start with an expression for a type  $\langle 1, 1 \rangle$  quantifier and compose the quantifier with a *restriction* and a *scope* set. The issue of how the restriction set is determined may be simpler in the case of D-quantification, where the nominal expression following the D-word denotes it, than in the case of A-quantification; but



the principal difference between the two means for expressing quantification is the added complexity of determining the restrictor for A-quantification when semantically interpreting sentences. However, for many varieties of A-quantification (e.g. agreement affixes on verbs, classifiers of nouns, and topic-oriented quantifier morphemes), it is no more difficult to determine what the restrictor is than for D-quantification. So even the complexity of interpreting sentences containing A-quantification is no greater when syntax clearly marks what the restrictor is.<sup>8</sup>

In summary, whether there is a phrase whose meaning is the type  $\langle 1 \rangle$  quantifier composed of the type  $\langle 1, 1 \rangle$  A- or D-quantifier plus its restrictor is not really central to understanding how natural languages express quantified propositions.<sup>9</sup> This issue may have assumed an exaggerated importance, partly because of the common phenomenon of familiar things seeming obvious and unfamiliar ones seeming strange. Quantifiers have mostly been studied by people who speak languages with rich systems of quantificational determiners and noun phrases (e.g. Greek, German, English), and the discovery of other grammatical means for expressing quantification may have been surprising mainly because some easy and familiar assumptions were thus found to be dispensable.

### 0.2.3 Quantirelations

The semantic distinction between type  $\langle 1, 1 \rangle$  quantifiers and type  $\langle 1 \rangle$  quantifiers is very helpful, as we have seen, in illuminating the similarities and differences between syntactically distinct D-quantification and A-quantification. It is a fact that D-quantification forms and A-quantification forms both express type  $\langle 1, 1 \rangle$  quantifiers (and sometimes quantifiers of type  $\langle 1, 1, 1 \rangle$  etc.), but not type  $\langle 1 \rangle$ . However, we do not want to create a mistaken impression that *all* type  $\langle 1, 1 \rangle$  quantifiers are potential meanings for D-forms or A-forms of natural languages. Defining all possible quantifiers of type  $\langle 1, 1 \rangle$  is a logical exercise, which we carry out in Chapter 2.4. The following are examples of type  $\langle 1, 1 \rangle$  quantifiers—i.e. binary relations (on a given domain) between sets of things—that are perfectly respectable from a logical point of view, but are not potential meanings of D-forms or A-forms.

- (38) a.  $MO(A, B)$  iff  $A$  has more elements than  $B$   
 b.  $Div(A, B)$  iff the size of  $A \cap B$  divides the size of  $A$   
 c.  $Sqt(A, B)$  iff the size of  $A \cap B$  is greater than the square root of the size of  $A$

<sup>8</sup> In some cases, an A-quantifier binds several variables simultaneously, for example, in

- (i) Politicians are usually willing to help constituents.

In (i) the quantifier expression has one variable ranging over politicians, and another ranging over constituents. We discuss in Ch. 10.2 the interpretation of such cases, which complicate somewhat the determination of the A-quantifier's domain restrictor.

It is sometimes asserted that A-quantifiers can, moreover, unselectively bind variables with little or no constraint from the syntactic structure. If this is true, it adds further complexity to determining the domain restriction for such quantifiers.

<sup>9</sup> We certainly do not deprecate Montague's accomplishment in showing that English noun phrases are interpreted as type  $\langle 1 \rangle$  quantifiers. This innovation was a major advance, as Cooper (1977), Barwise and Cooper (1981), and Keenan and Faltz (1985) pointed out.

The relation *MO* fails to satisfy a condition that holds for all denotations of D-forms and A-forms in natural languages (Chapter 4.5). As to *Div* and *Sqt*, it is simply empirically implausible that natural language determiners or A-forms would denote them.

What meanings natural languages allow D-forms and A-forms to express is of course an empirical question. The answer must be determined by careful observation and analysis of facts. Much of the remainder of this book is devoted to various sorts of efforts directed toward answering this empirical question. As we marshal facts and arguments in our quest for an answer, it will be convenient to have a succinct statement of the question. How can we conveniently refer to the relevant expressions and their denotations? “D-expressions and A-expressions for quantification” is too long-winded to be repeated as often as we’ll need to. So we introduce the term of art *quantirelation* for expressions of natural languages that signify a type  $\langle 1, 1 \rangle$  quantifier. Here is the terminology we will use:

#### quantirelations etc.

- A *quantirelation* is an expression in some natural language that signifies a type  $\langle 1, 1 \rangle$  quantifier.
- A *quantifier expression* is any expression in a natural language that can reasonably be taken to signify a quantifier (of any type).
- A *natural language quantifier* is a quantifier that is signified by some quantifier expression.

Quantirelations are *expressions*, and we’ll use “quantirelation expression” and “quantirelation term” synonymously when it is important to emphasize their syntactical character. Expressions of D-form and A-form are thus prime examples of quantirelations. By contrast, for the binary relation between sets that is the semantic entity signified by these expressions, the quantifier, we often use the term *quantirelation denotation*. Analogously, “natural language quantifier” is synonymous with the more cumbersome “quantifier expression denotation”.

Thus, quantirelation denotations are natural language quantifiers, but the latter class is wider. For example, the denotations of noun phrases, which are type  $\langle 1 \rangle$  quantifiers, belong there. And, as we will see below, there are many quantifiers of more complex types involved in natural language quantification, and they all belong to the somewhat vaguely specified class of natural language quantifiers.<sup>10</sup>

So every, most, at least a third of the, usually,  $m\acute{a}k^w$ , etc. are quantirelations (or quantirelation terms), and the quantifiers they signify are quantirelation denotations. But the relations in (38) are not quantirelation denotations: actually *MO* is still a natural language quantifier, because it is involved in

<sup>10</sup> There is no established terminology here. Some authors, e.g. Westerståhl (1989), use “natural language quantifier” for what we would here call “determiner denotation”. We feel the present terminology, although artificial, is more helpful.

more ... than constructions like *There are more women than men*, whereas *Div* and *Sqt* are presumably not even natural language quantifiers. To repeat, the quantirelation denotations are a restricted subclass of the class of all type  $(1, 1)$  quantifiers. One of our main tasks will be to investigate the extent of this class and the properties of its members.

### 0.3 EXPLICIT AND IMPLICIT QUANTIFICATION

It is useful to distinguish natural language phrases like the quantirelations and quantified noun phrases discussed so far, which express quantification over explicitly mentioned things or stuff, from other expressions whose meaning may involve quantification implicitly, but does not involve it explicitly. The latter expressions include a wide range of items that are nowadays commonly analyzed in terms of quantification. For instance, the past tense may be analyzed as existential quantification over times preceding the present. The deontic modal *must* may be analyzed as universal quantification over possible worlds in which all obligations are complied with. The conditional adverb *if* may be analyzed as universal quantification restricted to a set of possible worlds in which the clause it subordinates (the antecedent) is true. The attitude verb *believe* may be analyzed in terms of universal quantification over possible worlds in which everything a subject believes is true. And so on.

These implicitly quantificational expressions have in common that sentences containing them typically do not also contain an expression that explicitly denotes the entities over which these expressions implicitly quantify. There need not be any time-denoting expression in a past tense sentence.

(39) Mary was at home.

Modal, conditional, and belief statements do not contain expressions denoting possible worlds.

(40) John must be on time.

(41) If life is discovered on Mars, it will revolutionize biology.

(42) I believe you are skeptical.

In terming such quantification “implicit”, we are not advocating an analysis in terms of times or possible worlds.<sup>11</sup> Rather, we mean simply to point out that one who chooses to employ quantification in the semantic analysis of these expressions will quantify over entities that are not mentioned explicitly. These cases contrast sharply with the explicitly quantificational statements exemplified in section 0.1.

Between the above instances of clearly implicit quantification and the fully explicit examples from section 0.1, there are cases which seem to have a certain degree of implicitness.

(43) The men slept.

<sup>11</sup> Nor in terms of such alternatives as events, situations, or sets thereof.

is sometimes treated as explicitly referential, like

(44) These men slept.

and sometime as explicitly quantificational like

(45) Each man slept.

The truth conditions of sentence (43) are close if not identical to those of (45). However, this does not necessarily show that the definite article *the* in (43) is a quantifier. The sentence might acquire its truth conditions by a very different route than (45) does. The sentence

(46) No man didn't sleep.

clearly acquires the same truth conditions as (45) by a different route. While (45) and (46) are logically equivalent, they are not alike in all respects; (46) contains more 'logical operators' than (45) does, for example. By the same token, (43) might be analyzed as containing one plurality-denoting term (the plural noun phrase *the men*) and distributing a collective predication over the members of that set. Thus (43) would come to have the same truth conditions as (45) and (46) even if it did not contain any item that explicitly expresses quantification. The distribution of predication could be analyzed in terms of universal quantification, analogous to what one does with *must*, *if*, and *believe*; but the quantification would be implicit to some degree, rather than explicit, in the meaning of sentence (43). We do not try to settle the puzzle of whether *the* with plural count nouns explicitly expresses quantification, as this would take us to the topic of plural quantification, which we have no space for here. And we take a liberal attitude in this book about interpreting expressions as quantifiers if it makes sense to. However, we do wish to remind ourselves and the reader that one should resist the temptation to call something a quantifier expression simply on the grounds that its meaning *can* be analyzed in terms of quantification, and should inquire further whether the things it would quantify over are mentioned explicitly.

#### 0.4 MONADIC AND POLYADIC QUANTIFIERS

Natural language expressions for type ⟨1⟩ quantifiers such as *everyone* and *most French films* need one additional ingredient to yield a statement. This ingredient is commonly called the scope (some linguists call it the nuclear scope) of the quantifier expression. In contrast with logical languages, natural language syntax does not always indicate the scope of quantifiers unambiguously.

(47) John said that he returned each book.

- (a) For each book, he said (perhaps at different times) that he returned it.
- (b) He said (perhaps only once) that he returned each book.

Ambiguities resulting from lack of explicitly demarcated scope may be one indicator of explicit quantifiers.<sup>12</sup>

The following sentence also exhibits a typical scope ambiguity:

(48) Every student solved at least one problem.

This means either that there is at least one problem such that every student solved it ('narrow-scope reading'), or that each student solved at least one problem, but different students may have solved different problems (wide-scope reading). In this case the first reading logically entails the second (but not vice versa): we have a *scope dominance*. But there are many cases when the readings are logically independent: for example,

(49) John didn't eat one cookie.

On one reading, there is one cookie that John didn't eat (but he may have eaten all the others); on the other reading he ate no cookies at all. Neither of these readings implies the other. We look further at scope dominance in Chapter 10.1.

The quantifier expressions discussed so far all 'bind one variable' in their scope; that is, the quantifiers they express are properties of or relations between sets. The logical notion of a generalized quantifier, however, allows for the possibility of quantifiers that 'bind two or more variables' in their scope—that is, are relations between binary or *n*-ary relations. Do such quantifiers occur in natural languages?

They do indeed seem to exist. One possible example was given in note 8. Another is the reciprocal expression *each other* in English, as in this sentence:

(50) Congressmen must refer to each other indirectly.

Clearly quantification is called for in semantically analyzing (50), whose meaning is something like: 'Each congressman must refer to every other congressman indirectly'. But does *each other* involve quantification only implicitly, like the past tense, the

<sup>12</sup> Also, lack of ambiguity might indicate that something is not really a quantifier. For example, the sentence (i),

(i) The novices chose a mentor.

which is similar in structure to (43), unambiguously entails that all novices have the same mentor. Likewise, sentence (ii) is unambiguous and entails nothing about whether or not the mentors are all different.

(ii) The novices chose mentors.

In striking contrast, sentence (iii), which contains both an explicit quantifier and the indefinite article, exhibits quantifier scope ambiguity.

(iii) Every novice chose a mentor.

Just such facts constitute reasons for questioning the hypothesis that the definite plural noun phrases in (43), (i), and (ii) express universal quantifiers, and suspecting that universal quantification instead figures in the analysis of these sentences' meaning only in explaining how the predication expressed by the verb is distributed over members of the collection that the noun phrase denotes.

modal *must*, and the verb *believe*? It seems not, as sentences with *each other* explicitly mention the things being quantified over—congressmen in (50)—and furthermore the scope of the polyadic quantifier *each other* is not uniquely determined by its position in syntactic structure. For example, the scope of the reciprocal in sentence (51) can be the subordinate clause (i.e. the complement of *think*), giving the silly interpretation (52a), and alternatively, the scope can be the entire sentence, giving the interpretation which is the obviously sensible one in this case.

(51) John and Bill think they are taller than each other,

(52) a. Each thinks: We're taller than each other.

(52) b. Each thinks: I'm taller than him.

We analyze the very interesting type  $\langle 1, 2 \rangle$  quantifier expressed by reciprocal phrases like *each other* in more detail in Chapter 10. For now, we note simply that the following examples show that its meaning does not always decompose into two universal quantifications.

(53) a. Five Boston pitchers sat alongside each other.

b. The pirates stared at each other in surprise.

c. They stacked tables on top of each other to reach a high window.

This is just one kind of polyadic quantifier that occurs in natural languages; other kinds will be discussed in Chapter 10.

We hope that the examples of different kinds of natural language quantifiers indicated in this chapter have at least made it clear that quantification is a subject vastly bigger than the familiar  $\forall$  and  $\exists$  from elementary logic. To approach this subject, we will avail ourselves of a wider logical notion: that of a *generalized quantifier*. But, though the range of natural language quantifiers is very broad, it is easy to find examples of generalized quantifiers that have nothing to do with natural languages. The exact range of natural language quantifiers, within the class of generalized quantifiers, is therefore a serious object of study, to which semantic and logical tools can be applied. At the same time it calls for substantial empirical and theoretical investigations. This book focuses on the general semantic and logical properties of natural language quantifiers, but we pay close attention to several empirical issues as well. In Chapters 3 and 4 we present detailed analyses of many of the quantifiers listed in the present chapter, after introducing in Chapters 1 and 2 the analytic concepts needed for such analyses.

PART I  
THE LOGICAL CONCEPTION  
OF QUANTIFIERS AND  
QUANTIFICATION





# 1

## A Brief History of Quantification

We start approaching the concept of quantification through some glimpses of its historical development, right from the Aristotelian beginnings. This is not just a soft way to introduce a technical concept. To understand how people have tried to think about quantification earlier, both their insights and their difficulties, is useful for us who try to think about it today. In fact, the evolution of notions of quantification is quite interesting, both from a historical and a systematic perspective. We hope this will be evident in what follows. But our own perspective is mainly systematic. We shall use these glimpses from the history of ideas as occasions to introduce a number of semantic and methodological issues that will be recurring themes of this book.

Section 1.1 looks at the early history of quantification, beginning with Aristotle, who initiated the logical study of the four quantifiers *all*, *some*, *not*, and *not all*, that medieval philosophers arranged in the ‘square of opposition’. A closer look at that square reveals interesting facts about how negation combines with quantification, and also brings up the still debated issue of whether the truth of *all As are B* requires that there are some *As*. In the rich medieval discussion of quantification, we focus only on the distinction between *categorematic* and *syncategorematic* terms, a distinction closely tied to the issue of the meaning or significance of the quantifier expressions themselves (not just the sentences containing them), and interestingly related to several modern themes. From the Middle Ages we jump to the emergence of modern logic at the end of the nineteenth century (section 1.2). The tentative accounts of quantification in Peirce, Peano, and Russell, and the clear and explicit account in Frege, gradually turn, via Tarski’s formal treatment of truth, into the modern model-theoretic perspective, where the truth conditions of quantified sentences in formal languages are relative to interpretations or *models* (section 1.3). In particular, we discuss the role of the universe of quantification, and the subtle mechanisms by which natural languages are able to restrict that universe.

Much of the discussion, and the confusion, around quantification since medieval times concerns what quantifier expressions signify or denote, or make larger expressions (like *all men*, *some sailors*, *no cats*) signify or denote. The most obvious candidates are individuals or sets of individuals, but all attempts along these lines seem to meet with insuperable difficulties. In a postscript to the chapter (section 1.4), we give a formal proof that, indeed, no systematic account along these lines *can* succeed. One needs to go one level up in abstraction (to sets of sets of individuals), and that is where the modern theory of quantification begins.

## 1.1 EARLY HISTORY OF QUANTIFIERS

## 1.1.1 Aristotelian beginnings

When Aristotle invented the very idea of logic more than 2,300 years ago, he focused precisely on the analysis of quantification. Operators like *and* and *or* were added later (by Stoic philosophers). Aristotle's syllogisms can be seen as a formal rendering of certain inferential properties, hence of aspects of the meaning, of the expressions *all*, *some*, *no*, *not all*. These provide four prime examples of the kind of quantifiers that this book is about.

A *syllogism* has the form:<sup>1</sup>

$$Q_1 A B$$

$$\underline{Q_2 B C}$$

$$Q_3 A C$$

where each of  $Q_1, Q_2, Q_3$  is one of the four expressions above. Later on, these expressions were often presented diagrammatically in the *square of opposition*; see Fig. 1.1.<sup>2</sup> Positions in the square indicate certain logical relations between the quantifiers involved. Thus, there are really two kinds of logical 'laws' at work here: the (valid) syllogistic inference schemes with their characteristic form, and the 'oppositions' depicted in the square, which do not have that form when written as inference schemes, but are nevertheless considered to be valid. As to the former, here are two typical examples of syllogistic schemes:

$$(1.1) \text{ all } A B$$

$$\underline{\text{no } B C}$$

$$\text{no } A C$$

This scheme is clearly *valid*: no matter what the properties  $A, B, C$  are, it always holds that *if* the two premises are true, then so is the conclusion.

$$(1.2) \text{ all } A B$$

$$\underline{\text{some } B C}$$

$$\text{some } A C$$

This too has the stipulated syllogistic form, but it is *invalid*: one may easily choose  $A, B, C$  so as to make the premises true but the conclusion false.

<sup>1</sup> This is the so-called *first figure*—three more figures are obtained by permuting  $AB$  or  $BC$  in the premises. We are simplifying Aristotle's mode of presenting the syllogisms, but not distorting it. Observe in particular that Aristotle was the first to use *variables* in logic, and thus to introduce the idea of an *inference scheme*.

<sup>2</sup> Apparently it was Apuleios of Madaura (2nd century AD) who first introduced this diagrammatic representation.

A syllogism is a particular instantiation of a syllogistic scheme:

All Greeks are sailors  
No sailors are scared  
 No Greeks are scared

is a valid syllogism, instantiating the scheme in (1.1), and

All whales are mammals  
Some mammals have fins  
 Some whales have fins

is an invalid one instantiating (1.2).

It was perfectly clear to Aristotle (though not always to his followers) that the *actual* truth or falsity of its premises or conclusion is irrelevant to the (in)validity of a syllogism—except that no valid inference can have true premises and a false conclusion. What matters is whether it is *possible* that the premises are true and the conclusion false: if it is possible, the syllogism is invalid; if not, it is valid. In particular, a valid syllogism can have false premises. For example, in the first (valid) syllogism above, all three statements involved are false, whereas in the second (invalid) one they are all true.

As to the logical relations depicted in the square of opposition, there is a small but crucial difference between the classical and the modern version of this square. Since this point has often been misunderstood, and since it serves to illustrate an important issue in the semantics of quantification, it is worthwhile to get clear about it.

The classical square of opposition, i.e. the square as it appears in the work of Aristotle (though he did not use the diagram) and in most subsequent work up to the advent of modern logic in the late nineteenth century, is as in Fig. 1.1. The A and E

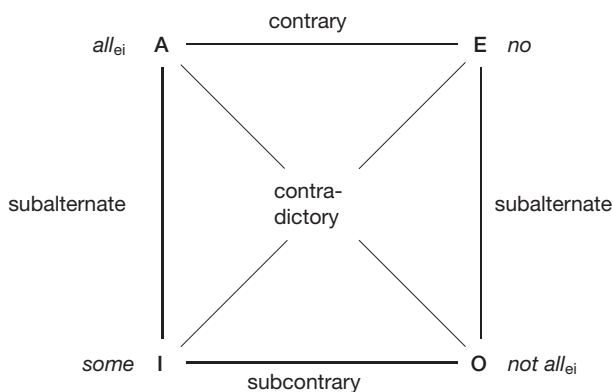


Figure 1.1 The classical square

quantifiers are called *universal*, whereas the I and O quantifiers are *particular*. Also, the A and I quantifiers are called *affirmative*, and the E and O quantifiers *negative*. Now, the important point is that the quantifier in the A position is what we have here called  $all_{ei}$ , i.e. the quantifier *all* with *existential import*. So  $all_{ei}(A, B)$  in effect means that all *As* are *B* and *there are some As*. This is explicit with many medieval authors, but also clearly implicit in Aristotle's work: for example, in the fact that he (and almost everyone else doing syllogistics before the age of modern logic) considered the following scheme as valid:

$$(1.3) \begin{array}{l} all\ AB \\ \underline{all\ BC} \\ some\ AC \end{array}$$

The logical relations in the classical square are as follows: Diagonals connect *contradictory* propositions, i.e. propositions that cannot have the same truth value. The A and E propositions are *contrary*: they cannot both be true (note that this too presupposes that the A quantifier has existential import). The I and O propositions are *subcontrary*: they cannot both be false. Finally, the I proposition is *subalternate* to the A proposition: it cannot be false if the A proposition is true, in other words, it is *implied* by the A proposition (again presupposing the latter quantifier has existential import). Similarly for the O and E propositions.

In addition, the convertibility—or as we shall say, the *symmetry*—of the I and E positions, i.e. the fact that *no A's are B* implies that *no B's are A*, and similarly for *some*, was also taken by Aristotle and his followers to belong to the basic logical facts about the square of opposition. We discuss symmetry in Chapter 6.1.

Notice that it follows that the A and O propositions are negations of each other (similarly for the I and E propositions). Thus, the quantifier at the O position means that either something in *A* is not in *B*, or *there is nothing in A*. So the O proposition is *true* when *A* is empty; i.e. contrary to the modern usage, the quantifier *not all* does not have existential import. (*Q* has existential import if  $Q(A, B)$  implies that *A* is non-empty.)

It seems, however, that during the late nineteenth and twentieth centuries this fact was often forgotten, and consequently it was thought that the logical laws described by the classical square of opposition were actually inconsistent. For example, people wondered how  $no(A, B)$  could imply  $not\ all(A, B)$ . And of course it doesn't—with the modern understanding of  $not\ all(A, B)$ . But it does imply  $not\ all_{ei}(A, B)$ —either there is nothing in *A*, in which case  $not\ all_{ei}(A, B)$  holds, or there is something in *A*, which then cannot be in *B* by assumption, so again  $not\ all_{ei}(A, B)$  holds—and that is all the classical square claims. Furthermore, it is often supposed that the problem arose from insufficient clarity about *empty terms*, i.e. expressions denoting the empty set.<sup>3</sup>

<sup>3</sup> For a recent statement of this view, see Spade 2002: 17. Another example is Kneale and Kneale 1962: 55–60. Consequently, it is also often assumed that the problems go away if one restricts attention to non-empty terms. But actually one has to disallow their complements, i.e. universal terms, as well, which seems less palatable. In any case, neither restriction is motivated; see n. 4.

For a detailed argument that most of this later discussion simply rests on a mistaken interpretation of the classical square, we refer to Parsons 2004. The upshot is that, apparently, neither Aristotle nor (with a few exceptions) medieval philosophers disallowed empty terms, and some medieval philosophers explicitly endorsed them.<sup>4</sup> And as long as one remembers that the **O** quantifier is the negation of the quantifier  $all_{ei}$ , nothing is wrong with the logic of the classical square of opposition.

A totally different issue, however, is which interpretation of words like *all* and *every* is ‘correct’ or, rather, most adequate for linguistic and logical purposes. Nowadays, *all* is used without existential import, and the modern square of opposition is as in Fig. 1.2.

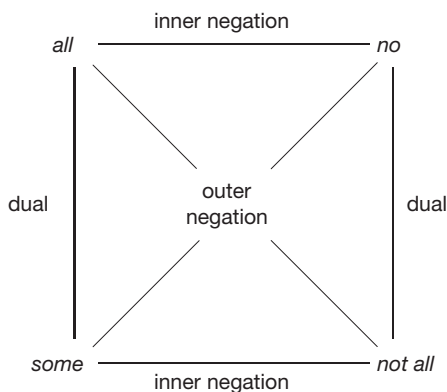


Figure 1.2 The modern square

Parsons (2004) appears to think this square is impoverished and less interesting, but we disagree on this point. The main virtue of the modern square is that it depicts three important forms of *negation* that appear in natural (and logical) languages.

### three forms of negation

As in the classical square, the diagonals indicate ‘contradictory’ negation, or as we shall say, *outer negation*. When  $Q_i$  and  $Q_j$  are at the ends of a diagonal, the proposition  $Q_i$  As are B is simply the negation of  $Q_j$  As are B; i.e. it is equivalent to It is not the case that  $Q_j$  As are B. This propositional negation ‘lifts’ to the (outer) negation of a quantifier, and we can write  $Q_i = \neg Q_j$  (and hence  $Q_j = \neg\neg Q_i = \neg Q_i$ ).

<sup>4</sup> In Paul of Venice’s *Logica Magna* (c. 1400), he gives

(i) Some man who is a donkey is not a donkey.

as an example of a sentence which is *true* since the subject term is empty; see Parsons 2004: sect. 5. So he allows empty terms, and confirms the interpretation of the **O** quantifier given above.

A horizontal line between  $Q_i$  and  $Q_j$  now stands for what we shall call *inner negation*: here  $Q_i$  As are B is equivalent to  $Q_j$  As are not B, which can be thought of as applying the inner negation  $Q_j \neg$  to the denotations of A and B.

Finally, a vertical line in the square indicates that the respective quantifiers are each other's *duals*, where the dual of  $Q_i$  is the outer negation of its inner negation (or vice versa):  $Q_i^d = \neg(Q_i \neg) = (\neg Q_i) \neg = \neg Q_i \neg$

The modern square is *closed* under these forms of negation: applying any number of these operations to a quantifier in the square will not lead outside it. For example,  $(no^d) \neg = \neg no \neg \neg = \neg no = some$ . (It is not closed under other Boolean operations; e.g. the quantifier *some but not all*, which is the conjunction of *some* and its inner negation, is not in the square.)

As we will see later on (Chapter 3.2.3), all three forms of negation have natural manifestations in real languages. Moreover, this notion of a square of opposition applies to quantifiers other than those in Aristotle's square; indeed, any quantifier generates a square. That is not true of the classical square: only outer negation is present there, but not the other two forms.<sup>5</sup>

The issue at stake here is not whether empty terms should be allowed or not, but whether *all* and *every* have existential import or not. Do they have existential import? Everyone is familiar with the fact that it is usually odd to affirm that every *A* is *B* when one knows there are no *As*. The dispute (and the confusion) between the two squares of opposition nicely puts the finger on the meaning of *every*. In modern terms, the main question is whether the existential import that is often felt with uses of *every* belongs to the meaning—the truth conditions—or rather is a presupposition or a Gricean implicature. We return to this issue in Chapter 4.2.1. However, we have already given here one kind of argument for choosing the modern interpretation without existential import: in this way the meaning of *every* fits nicely with the

<sup>5</sup> Though the difference between the classical Aristotelian square and the modern version might at first seem small—*all* instead of *all<sub>ei</sub>*, and similarly for *not all*—the principled differences are huge. First, whereas outer negation is presented in both squares, neither inner negation nor dual is contained in the classical square. For example, the dual of the quantifier *all<sub>ei</sub>* is the quantifier which holds of *A* and *B* iff *either* some *A* is *B* *or* *A* is empty. The latter is rather unnatural, and may not even be a quantirelation in the sense of Ch. 0.2.3.

Second, the logical relations depicted in the two diagrams are quite different. For example, the relation holding between two contrary propositions, that they cannot both be true (but can both be false), amounts to one implying the outer negation of the other (but not being implied by that negation). That relation neither entails nor is entailed by one of the propositions being the inner negation of the other.

Third, the classical square is not generated by any of its members. To make this statement precise, let us define a *classical square* as consisting of four quantifiers arranged as in Fig. 1.1 and with the same logical relations—contradictories, contraries, subalternates—holding between the respective positions. Then each position will determine the quantifier at the diagonally opposed position, i.e. its outer negation, but *not* the quantifiers at the other two positions. For example, the reader may easily verify that

(ii) For every  $k$  and every  $n \geq k$ , [*A*: at least  $n$ ; *E*: fewer than  $k$ ; *I*: at least  $k$ ; *O*: fewer than  $n$ ] is a classical square.

other three quantifiers in the modern square of opposition, and thus with the three kinds of negation that any semantics of natural languages has to account for anyway. In short, *logical coherence* speaks in favor of the modern interpretation (see n. 5).

We have dwelt at some length on the logic of the square of opposition, but we also noted that the inferences which were the main focus of Aristotle's attention were the syllogisms. There are 256 syllogistic schemes. Aristotle characterized which ones of these are valid, not by enumeration but by an axiomatic method whereby all valid ones—and no others—were deducible from just two syllogisms. Apart from being the first example of a deductive system, and of a metalogical investigation of such a system, it was an impressive contribution to the logico-semantic analysis of quantification.

However, it must be emphasized that Aristotle's analysis, even when facts about the square of opposition are added, does not exhaust the meaning of these four quantifier expressions,<sup>6</sup> since there are many valid inference schemes involving them which do not have these forms: for example,

John knows every professor

Mary is a professor

John knows Mary

No student knows every professor

Some student knows every assistant

Some professor is not an assistant

These inferences go beyond syllogistic form in at least the following ways: (i) names of individuals play an essential role; (ii) there are not only names of properties (such as adjectives, nouns, intransitive verbs) as in the syllogisms, but also names of binary relations (such as transitive verbs); (iii) quantification can be iterated (occur in both the subject and the object of a transitive verb, for example). While none of these features may seem, at least in hindsight, like a formidable challenge, it is certainly much harder than in the syllogistic case to describe the logical structures needed to account for the validity of inferences of these kinds. At the same time, it is rather clear that a logic which cannot handle such inferences will not be able to render, say, the structure of proofs in elementary geometry (like Euclid's), or, for that matter, much of everyday reasoning.

The failure to realize the limitations of the syllogistic form, together with the undisputed authority of Aristotle among medieval philosophers and onward, is part of the explanation why logic led a rather stagnant and unproductive existence after Aristotle, all the way up to the late nineteenth century. Only when the syllogistics is extended to modern predicate logic do we in fact get a full set of inference schemes which, in a

<sup>6</sup> Aristotle didn't claim it did; in fact, he was well aware that there are other forms of logically valid reasoning. It is not really known why he attached such importance to the syllogisms. Perhaps he was (justly) amazed by the fact that he could apply the axiomatic method from geometry—deriving valid syllogisms from others—to objects that were not mathematical but linguistic.

precise sense, capture *all* valid inferences pertaining to the four quantifiers that Aristotle studied.<sup>7</sup>

### 1.1.1.1 Proof-theoretic and model-theoretic semantics

Approaching meaning via inference patterns is characteristic of a *proof-theoretic* perspective on semantics. The idea is that an expression's meaning is contained in a specific set of inference rules involving that expression.<sup>8</sup> In the case of our four quantifiers, however, it seems that whenever a certain system of such rules is proposed—such as the syllogisms—we can always ask if these schemes are *correct*, and if they are exhaustive or *complete*. Since we understand these questions, and likewise what reasonable answers would be, one may wonder if there isn't some other more primary sense in which we know the meaning of these expressions. But, however this (thorny) issue in the philosophy of language is resolved, it is clear that in the case of Aristotelian quantifiers there is indeed a different and more direct way in which their meaning can be given. Interestingly, that way is also clearly present, at least in retrospect, in the syllogistics.

For, on reflection, it is clear that each of these four quantifier expressions stands for a particular binary relation between properties, or, looking at the matter more extensionally, a *binary relation between sets of individuals*. When  $A, B, C$  are arbitrary sets, these relations can be given in standard set-theoretic notation as follows:<sup>9</sup>

$$\begin{aligned} \text{all}(A, B) &\iff A \subseteq B \\ \text{some}(A, B) &\iff A \cap B \neq \emptyset \\ \text{no}(A, B) &\iff A \cap B = \emptyset \\ \text{not all}(A, B) &\iff A - B \neq \emptyset^{10} \end{aligned}$$

<sup>7</sup> That is, all valid inferences using only the machinery of predicate logic—this is Gödel's *completeness theorem* for that logic.

<sup>8</sup> In particular, in intuitionistic logic, the explanations of what is a *proof* of a proposition of a certain form are taken to embody the meaning of those propositions. Thus the main semantic concept is an epistemic one. Truth is seen as a derived concept, amounting to the existence of a proof. See Ranta 1994 for an approach to semantics along these lines, within the framework of constructive type theory, and Sundholm 1989 for a similar attempt to deal with generalized quantifiers. Applying these ideas to natural languages, one obtains the *verificationist* semantics which represents one strand in modern philosophy of language; see Dummett 1975; Prawitz 1987.

These approaches place proof prior to truth in the order of semantic explanation. By contrast, many uses of proof theory are perfectly compatible with the reverse order. Proof systems for first-order logic enrich our understanding of that logic, and can be seen to be adequate by Gödel's completeness theorem. For generalized quantifiers, a classic example is Keisler 1970, which deals with an axiomatization of the quantifier *there are uncountably many*; again, the main objective is to prove a completeness theorem.

<sup>9</sup> We are here, as in most of this book, using a standard relational notation. Instead of  $\text{all}(A, B)$ , one could equally well write  $(A, B) \in \text{all}$ , or, using characteristic functions,  $\text{all}(A, B) = 1$  (= True).

<sup>10</sup> As we saw, the relations in effect considered by Aristotle were *some*, *no*, and

$$\begin{aligned} \text{all}_{\text{ci}}(A, B) &\iff \emptyset \neq A \subseteq B \\ \text{not-all}_{\text{ci}}(A, B) &\iff A - B \neq \emptyset \text{ or } A = \emptyset \end{aligned}$$



So, for example,

(1.4) All Greeks are sailors.

simply *means* that the set of Greeks stands in the inclusion relation to the set of sailors.

Such a formulation is consonant with a *model-theoretic* perspective on meaning, where truth and reference are basic concepts, rather than epistemic ones such as the concept of proof. In this book we consistently apply a model-theoretic approach to quantification. This is not to say that we think proof-theoretic considerations are irrelevant. For example, one case where such considerations are important concerns the notion of a *logical constant*, as we will argue in Chapter 9.4. However, our approach to the meaning of quantifier expressions is always in terms of the model-theoretic objects that these expressions denote.

### 1.1.1.2 *Quantifier expressions and their denotations*

In this connection, let us note something that has been implicit in the above. Quantifier *expressions* are syntactic objects, different in different languages. On the present account, some such expressions ‘stand for’, or ‘denote’, or have as their ‘extensions’, particular relations between sets. So, for example, the English `no` and the Swedish `ingen` both denote the relation that holds between two sets if and only if they are disjoint. There is nothing language-dependent about these relations. But, of course, to talk about them, we need to use language: a *meta-language* containing some set-theoretic terminology and, in the present book, English. In this meta-language, we sometimes use the handy convention that an English quantifier expression, in italics, names the corresponding relation between sets. Thus, *no* as defined above is the disjointness relation, and hence the relation denoted by both the Swedish expression `ingen` and the English expression `no`.

We note that on the present (Aristotle-inspired) analysis, each of the main words in a sentence like (1.4) has an extension; it denotes a set-theoretic object. Just as `sailor` denotes the set of sailors and `Greek` the set of Greeks, so `all` denotes the inclusion relation.

In this book we mainly use the term *denote* for the relation between a word and a corresponding model-theoretic object (its extension or denotation), with the idea that this is a modern and fairly neutral term. So we can maintain, for example, that practically everyone agrees that predicates like `sailor` and `Greek` have sets as denotations, *even though* (a) sets are a modern invention; (b) medieval philosophers, who thought a lot about the relations between words and the world, rarely talked about denotation; (c) nominalists of various ilks—medieval or later—deny that reference to anything other than individuals occurs, and sometimes also the existence of abstract objects like sets. The point is that even if you claim, as a nominalist like Ockham did, that `sailor` only refers to individual sailors, it does refer to all and only sailors, so implicitly at least, the denotation, i.e. the set of sailors, plays a role in the semantics.<sup>11</sup>

<sup>11</sup> Likewise, the criticism of so-called name theories of meaning—ridiculed by Ryle (the ‘Fido’–Fido theory) for resting on a category mistake (Ryle 1949)—does not have much force

For quantifier expressions, on the other hand, one cannot say that anyone before Frege (section 1.2.4 below) thought of them as denoting. Their status in this respect is at best unclear, and some logicians explicitly held that it was incoherent to think of them as denoting anything; for example, Russell (who did use the term ‘denote’). One plausible reason for this is that there seemed to be no good candidates for the denotation of these expressions, so they had to be dealt with in another way. However, we already saw that Aristotle’s account of the four quantifiers mentioned so far does point to one such candidate:

*Quantifier expressions denote relations between sets of individuals.*

While it may be anachronistic to attribute that idea to Aristotle himself, it certainly is consistent with his approach and his focus on the syllogistic schemes. And, as it turns out, this simple idea resolves the problems encountered by earlier logicians, and provides the foundation of a coherent and fruitful account of quantification.

### 1.1.2 The Middle Ages

Medieval logicians and philosophers devoted much effort to the semantics of quantified statements, essentially restricted to syllogistic form. For a modern light introduction to (late) medieval logic and semantics we refer the reader to Spade 2002. Here we shall recall just one important distinction that was standard in those days—between words that have an independent meaning and words that don’t. The former were called *categorematic*, the latter *syncategorematic*.

#### *1.1.2.1 Categorematic and syncategorematic terms*

The following quote is illustrative:<sup>12</sup>

Categorematic terms have a definite and certain signification, e.g. this name ‘man’ signifies all men, and this name ‘animal’ all animals, and this name ‘whiteness’ all whitenesses. But syncategorematic terms, such as are ‘all’, ‘no’, ‘some’, ‘whole’, ‘besides’, ‘only’, ‘in so far as’ and such-like, do not have a definite and certain signification, nor do they signify anything distinct from what is signified by the categoremata. . . . Hence this syncategoremata ‘all’ has no definite significance, but when attached to ‘man’ makes it stand or suppose for all men . . . . And the same is to be held proportionately for the others, . . . though distinct functions are exercised by distinct syncategoremata . . . . (William of Ockham, *Summa Logicae*, i, c. 1320; quoted from Bocheński 1970: 157–8)

against our use of “denote”. Model-theoretic semantics takes no serious metaphysical stand on *naming*. And if one can associate extensions, such as sets, functions, or other abstract objects, with linguistic expressions in a systematic way, the value of such an enterprise is to be measured by its fruitfulness, its power of prediction and explanation, etc., rather than on a priori grounds.

<sup>12</sup> We are grateful to Wilfrid Hodges for substantial advice on medieval semantics, and in particular on what Ockham is really saying in this passage.

The word “signify” is a standard medieval term for the relation between signs and the things they are signs of. That  $x$  *signifies*  $y$  means roughly that  $x$  establishes an understanding of  $y$  in the mind of the speaker or hearer (Spade 2002: ch. 3, sect. E). Signification is thus a kind of psychologico-causal relation. The other main term was *supposit*, or “suppose” (for). The quote illustrates the interaction between the two: a word supposits for something only in a given linguistic context (and may supposit for another thing in another context), whereas signification is more absolute and context-independent (Spade 2002: ch. 8, Sect. A).<sup>13</sup> Now the first sentence of the quote should not, given Ockham’s nominalist persuasion, be taken to mean that nouns signify any universal or abstract objects, but rather that each man is signified by man, etc. But the status of universals is not the issue here—the point is that quantifier expressions and other syncategoremata do not establish the understanding of anything “definite and certain” in our minds.

Ockham is not saying that a word like *a11* doesn’t stand for anything, but that it doesn’t have the signification relation to anything. However, when combined with a noun, it *makes the noun stand* (supposit) *for* something. Syncategorematic words never signify or supposit for anything; instead, they have a systematic effect on what other words stand for.

This general contrast is fairly clear. Less clear is what it is that expressions like *a11*, *some*, *no* make their adjacent nouns—or, alternatively, the respective *noun phrases*<sup>14</sup>—stand for. Taking “stand for” now in the weak sense of our “denote”,<sup>15</sup> it would seem that *a11 men* again denotes the set of men (in Ockham’s case, it stands for each man), and we might similarly assume that *some man* denotes a particular man. But what, then, would *no man* denote? It could hardly be the empty set, for then there would be no difference between *no man* and *no dog*. As we show in a postscript to this chapter (section 1.4), trying to make quantified noun phrases stand for individuals or sets of individuals is in fact an impossible task.

<sup>13</sup> Basic idea: In

(a) Every man is an animal.

man supposits for each man if you are a nominalist (and for something more abstract, like a set, if you are a realist); this is usually called *personal* supposition. In

(b) Man is a species.

it supposits for a universal concept (which in Ockham’s case is a thing in the mind); *simple* supposition. And in

(c) Man has three letters

the word man supposits for itself; *material* supposition. The context may be a preceding quantifier expression, or something else about the form of the sentence, or a restricted universe of quantification (a *context set* in the sense to be explained in sect. 1.3.4 below).

<sup>14</sup> It is not clear that Ockham took noun phrases to be meaningful linguistic units, and in general medieval logicians did not have a notion of complex constituents of sentences (although they did think of sentences as complex objects). But presumably it is no great distortion of the facts to assume that what Ockham thinks the word *man*, when preceded by *a11*, stands for is what he would have thought the noun phrase *a11 men*—had he had that notion—stands for.

<sup>15</sup> The closest medieval correspondent would seem to be the way realists took the relation of personal supposition (see n. 13).

The next quote too begins with an attempt to explain the semantic function of the Aristotelian quantifier expressions.

The universal sign is that by which it is signified that the universal term to which it is adjoined stands copulatively for its suppositum (*per modum copulationis*) . . . The particular sign is that by which it is signified that a universal term stands disjunctively for all its supposita. . . . Hence it is requisite and necessary for the truth of this: ‘some man runs’, that it be true of some (definite) man to say that he runs, i.e. that one of the singular (propositions) is true which is a part of the disjunctive (proposition): ‘Socrates (runs) or Plato runs, and so of each’, since it is sufficient for the truth of a disjunctive that one of its parts be true. (Albert of Saxony, *Logica Albertucii Perutilis Logica*, iii (Venice, 1522; quoted from Bocheński 1970: 234))

The latter part of the quote, on the other hand, is a way of stating the truth conditions for quantified sentences, in terms of (long) disjunctions and conjunctions. Whatever one may think of the proposed definition, it brings out the important point that it is perfectly possible to state adequate truth conditions for quantified *sentences* without assuming that the quantifier expressions themselves denote anything. Indeed, this is how the Tarskian truth definition is usually formulated in current textbooks in first-order logic. But although medieval logicians may have taken the position—and, for example, Russell (see below) certainly did—that it is *necessary* to proceed in this way, generalized quantifier theory shows that it is in fact possible to treat quantifier expressions as denoting. Such an approach, which, as we have seen, is quite in line with Aristotle’s syllogistics, has definite advantages: it identifies important syntactic and semantic categories, and it conforms better to the Principle of Compositionality, according to which the meaning of a complex expression is determined by the meanings of its parts and the mode of composition. More about this later.

### 1.1.2.2 Modern variants

What is the modern counterpart of the medieval distinction between categorematic and syncategorematic terms? It might be tempting to define *logic* as the study of the syncategorematic terms. But such an identification should be resisted, we think. What makes a term merit the attribute logical (or, for that matter, constant) is one thing, having to do with particular features of its semantics; we return to the issue of logicality in Chapter 9. However, the fact that a word or morpheme does not have independent semantic status is quite another thing, which applies to many expressions other than those traditionally seen as logical.<sup>16</sup>

A possibly related linguistic notion is that of a *grammatical morpheme*. Such a word may not belong in a dictionary at all, or if it does, there could be a description of its phonological and syntactic features (e.g. valence features: which other words it combines with), but not directly of its meaning. The meaning arises, as it were, via a

<sup>16</sup> When Spade suggests (2002: 116) that, from a modern point of view, categorematic terms are those that get interpreted in models, whereas the syncategorematic ones get their semantic role from the corresponding clauses in the truth definition, he comes close to identifying syncategorematicity with logicality: the standard modern idea is that non-logical constants are interpreted in models, but not logical ones.

grammatical rule; hence the name. Typical English examples might be the progressive verb ending *-ing*, the word *it* in sentences such as

(1.5) It is hard to know what Ockham meant.

and the infinitive particle *to* (the first occurrence below):

(1.6) To lie to your teacher is bad.

But, as these examples also indicate, the words that the medievals thought of as syncategorematic, such as *every*, *no*, *besides*, *only*, *whole*, and the copula *is*, are in general *not* grammatical morphemes: you do find them in dictionaries, along with attempts at explaining their meaning.<sup>17</sup>

It seems that there are really two basic aspects of syncategorematic terms in the medieval sense, or two senses in which the meaning of such terms is “not independent”. One is the lack of ability to produce clear ideas in our minds. The other is that they are functions or *operators*: they need an *argument* to act on. Given the argument, the combined phrase gets a definite meaning (or, as the medievals preferred to put it, the argument itself gets a possibly new definite meaning).

These two aspects are partly orthogonal, it appears to us, and from a modern viewpoint it is the latter one which is most interesting. What you think can be presented clearly to the mind obviously depends crucially on your theory of the mind, and it is not at all evident that only predicate expressions have this property. That something is an operator requiring an argument, on the other hand, seems like a much more robust notion, at least in most of the cases that the medievals looked at. Today we can easily agree that their syncategoremata are indeed operators. But this does not entail that we do not have clear ideas of *how* these operators work.

A further modern issue is how you *present* the workings of these operators. In particular, can you see the quantifier expressions as *denoting*, or even “signifying”, given operators? The conceptually simplest way to do this is to take the Aristotelian hint that they denote relations between sets. These are (second-order) relations; as operators, they map sets of individuals (predicate denotations) to sets of sets of individuals (noun phrase denotations). So the denotations are fairly abstract, and have only become standard with the modern application of generalized quantifier theory to natural language semantics. (It is worth recalling that irrational numbers and infinite sets were once anything but clear ideas in people’s minds, however unproblematic the concepts are to modern minds.)

The alternative is to describe the operators contextually: instead of saying what a word like *every* denotes, you give uniform truth conditions for *sentences* beginning with *every*. This is the standard procedure in logic. The net result is the same, but now you need one clause for each quantifier expression, whereas with the other

<sup>17</sup> An extensionally better match is the linguistic distinction between *open* and *closed classes*: words belonging to closed classes often qualify as syncategorematic. But the idea behind closed classes—that they contain a small number of words to which new ones are not easily added—seems very different from the idea of not having signification.

approach you have only one general clause for quantifiers. We will see later how all of this works in detail.

Summing up, we would say that the medieval idea of syncategorematic terms as operators was quite viable, but that with (generalized) quantifiers, truth functions, etc., one also has the option of seeing such expressions as standing for “definite and certain” ideas, contrary to what the medievals (understandably) thought.

## 1.2 QUANTIFIERS IN EARLY PREDICATE LOGIC

Predicate logic was invented at the end of the nineteenth century. A number of philosophers and mathematicians had similar ideas, partly independently of each other, at about the same time, but pride of place goes without a doubt to Gottlob Frege. Nevertheless, it is interesting to see the shape which these new ideas about quantification took with some other logicians too, notably Peirce, Peano, and Russell.<sup>18</sup>

One crucial addition in the new logic was variable-binding: the idea of variables that could be bound by certain operators, in this case the universal and existential quantifiers. The idea came from mathematics—it is not something that can be directly ‘copied’ from natural languages—and it took some time to crystallize.<sup>19</sup>

<sup>18</sup> As to the history of the English word “quantifier”, we can do no better than quote the following from Hodges’ comments (pers. comm.):

William Hamilton of Edinburgh claimed to have ‘minted’ the words ‘quantify’ and ‘quantification’, presumably in lectures in Edinburgh around 1840. De Morgan (1847), Appendix, 312, confirms Hamilton’s ‘minting’. However, Hamilton’s usage is that to quantify is to *interpret* a phrase as saying something about quantity, whether or not it does on the surface. Thus he talks of ‘quantification of the predicate’, meaning a particular theory that the predicate contains its own quantification. The modern usage seems to begin with De Morgan himself, who in De Morgan 1862 says a good deal about ‘quantifying words’, but then quietly shortens this to ‘quantifiers’. Thus ‘We are to take in both all and some-not-all as quantifiers.’

<sup>19</sup> In the notation for integrals, like

$$(a) \int_a^b f(x) dx$$

which was introduced by Leibniz in the late seventeenth century, the variable  $x$  cannot be assigned values; it is not free but bound. Likewise, algebraic equalities like

$$(b) x + (y + z) = (x + y) + z$$

have implicit universal quantifiers at the front. But here you seemingly can replace the variables by constants—by universal instantiation. Seeing, as Frege did, that (a) and (b) use the same mechanism, and describing that mechanism in a precise way, was no small feat.

As to natural languages, pronouns sometimes resemble bound variables, for example, in

(c) Every girl who meets him realizes that she will soon forget him.

Here *she* can be seen as a variable bound by *every girl*. *Him*, on the other hand, is deictic and somewhat similar to a *free* variable. But this analogy between pronouns and individual variables in logic was noted only well into the twentieth century, perhaps for the first time by Quine. (Quine

In a way, variable-binding belongs to the *syntax* of predicate logic, though it of course engenders the *semantic* task of explaining the truth conditions of sentences with bound variables (an explanation which took even longer to become standard among logicians). In this respect, too, Frege was unique. His explanation, though perfectly correct and precise, did not take the form that was given fifty years later by Tarski's truth definition. Instead, he treated—no doubt because of his strong views about compositionality—the quantifier symbols as categorematic, standing for certain second-order objects. This is closely related to the idea of quantifiers as relations between sets that we have traced back to Aristotle, though in Frege's case combined with a much more expressive formal language than the syllogistics and containing the mechanism of variable-binding. Nothing even remotely similar can be found with the other early predicate logicians, so let us begin with them.

### 1.2.1 Peirce

Peirce in fact designed two systems of predicate logical notation. One was two-dimensional and diagrammatic, employing so-called existential graphs. What the other one looked like is indicated in the following quote:

[T]he whole expression of the proposition consist[s] of two parts, a pure Boolean expression referring to an individual and a Quantifying part saying what individual this is. Thus, if  $k$  means 'he is king' and  $h$ , 'he is happy', the Boolean

$$(\bar{k} + h)$$

means that the individual spoken of is either not a king or is happy. Now, applying the quantification, we may write

$$\text{Any}(\bar{k} + h)$$

to mean that this is true of any individual in the (limited) universe. . . .

In order to render this notation as iconical as possible we may use  $\Sigma$  for *some*, suggesting a sum, and  $\Pi$  for *all*, suggesting a product. Thus  $\Sigma xi$  means that  $x$  is true of some one of the individuals denoted by  $i$  or

$$\Sigma xi = xi + xj + xk + \text{etc.}$$

In the same way,  $\Pi xi$  means that  $x$  is true of all these individuals, or

$$\Pi xi = xixjxk, \text{ etc.}$$

. . . It is to be remarked that  $\Sigma xi$  and  $\Pi xi$  are only *similar* to a sum and a product; they are not strictly of that nature, because the individuals of the universe may be innumerable. (Peirce 1885; quoted from Bocheński 1970: 349)

Thus  $k, h, x$  are formulas here. Note Peirce's use of English pronouns as (bindable) variables in the informal explanation. Quantified sentences are divided into a (Boolean) formula and a quantifier symbol, similarly to the modern notation. The

1950 employed pronouns to explain to logic students the use of variables in logic. Quine 1960 used variables to explain how pronouns work in natural language.)

quantifier symbols are chosen in a way to indicate their meaning (in terms of conjunction and disjunction), but Peirce does not yet have quite the notation for bound variables. In  $\Sigma xi$  it would seem that  $i$  is a variable whose occurrences in the formula  $x$  get bound by the quantification, but in  $x_i + x_j + x_k + \dots$  the  $i, j, k$  look more like names of individuals. One sees that a formally correct expression of these ideas is still a non-trivial matter.

Note also that Peirce (in contrast with Frege; see below) seems to allow that the (discourse?) *universe* may vary from occasion to occasion.

### 1.2.2 Peano

One feature of standard predicate logic is that the same variables that occur free in formulas can get bound by quantification. It appears that the first to introduce this idea was Peano:

If the propositions  $a, b$ , contain undetermined beings, such as  $x, y, \dots$ , i.e. if there are relationships among the beings themselves, then  $a \supset_{x,y,\dots} b$  signifies: whatever  $x, y, \dots$ , may be,  $b$  is deduced from the proposition  $a$ . (Peano 1889; quoted from Bocheński 1970: 350)

Here we have a consistent use of variable-binding, albeit only for universally quantified conditionals.

### 1.2.3 Russell

Bertrand Russell devoted much thought to the notion of *denotation*. In his early philosophy he subscribed to an almost Meinongian view:<sup>20</sup> all expressions of a certain form had to denote *something*, and it was the logician's task to say what these denotations were. Here is a quote from Russell (1903: 59):

In the case of a class  $a$  which has a finite number of terms [members]—say,  $a_1, a_2, a_3, \dots a_n$ , we can illustrate these various notions as follows:

- (1) *All a's* denotes  $a_1$  and  $a_2$  and  $\dots$  and  $a_n$ .
- (2) *Every a* denotes  $a_1$  and denotes  $a_2$  and  $\dots$  and denotes  $a_n$ .
- (3) *Any a* denotes  $a_1$  or  $a_2$  or  $\dots$  or  $a_n$ , where *or* has the meaning that it is irrelevant which we take.
- (4) *An a* denotes  $a_1$  or  $a_2$  or  $\dots$  or  $a_n$ , where *or* has the meaning that no one in particular must be taken, just as in *all a's* we must not take any one in particular.
- (5) *Some a* denotes  $a_1$  or denotes  $a_2$  or  $\dots$  or denotes  $a_n$ , where it is not irrelevant which is taken, but on the contrary some one particular  $a$  must be taken.

<sup>20</sup> Alexius Meinong was an Austrian philosopher who took the theory of intentionality (directedness) of his teacher Franz Brentano to extreme consequences in his theory of objects (*Gegenstandstheorie*). Every thought or judgment has an object, no matter if it is a thought about Mont Blanc or a golden mountain or a round square; the only difference in the latter two cases is that these objects don't *exist* (the last one is even impossible), but they are objects no less, according to Meinong. Russell later became one of Meinong's most severe critics. In recent years Meinong's philosophy has had a certain revival; see e.g. Parsons 1980.



This is a bold attempt to explain the denotation of (what we would now call) certain quantified noun phrases; nevertheless, it is clear that the account is beset by problems similar to those in the medieval tradition.<sup>21</sup> It is a useful reminder of how hard this problem really was.

Later on, Russell explained the quantifiers in terms of a propositional function's being 'always true', 'sometimes true', etc., with a syntax using the notion of 'real' (free) versus 'apparent' (bound) variables.

Russell's modern view of denotation begins with his famous paper 'On denoting' (Russell 1905). In it, he still talks of "denoting phrases", but now emphatically denies that they have any meaning "in isolation". Likewise, they are not assigned any denotation.<sup>22</sup> Instead, Russell uses the tools of modern logic to rewrite sentences that *appear* to have a denoting phrase as, say, the subject—in the *real* logical form, according to Russell, no such phrases are left.

The reasons for Russell's change of view about denotation were chiefly logical: he found that the earlier position was incoherent and could not be consistently upheld. His main occupation was with definite descriptions and the problems arising when the purported described object did not exist. But the translation into logical form disposed of other quantified noun phrases as well (containing *every*, *some*, etc.), and the problem of their denotation disappeared. The divergence between surface form and logical form was a crucial discovery for Russell, with far-reaching consequences for logic, epistemology, and philosophy of language.<sup>23</sup> Though his arguments were quite forceful, it would seem that later developments in formal semantics, and in particular the theory of (generalized) quantifiers, have seriously undermined them. Roughly, this theory provides a logical form which does treat the offending expressions as denoting, and which thus brings out a closer structural similarity between surface and logical structure. One may debate which logical form is the correct one (to the extent that this question makes sense), but one can no longer claim that no

<sup>21</sup> For example, what is the difference between (1) and (2)? It seems that *all a's* denotes the set  $\{a_1, \dots, a_n\}$ , whereas *every a* denotes—ambiguously—each one of the  $a_i$ . But if the latter were the case, one ought to be able to have an utterance of "Every a is blue" mean that, say,  $a_2$  is blue, which clearly is impossible. A charitable interpretation may note that Russell is on to a linguistic insight that "every" is *distributive* in some way that "all" is not, but clearly he has not managed to express it adequately.

Similar comments can be made about the difference that Russell tries to establish between *some* and the indefinite article *a*, but here his intuitions seem to run counter to the prevailing view: i.e. that the indefinite article can be used to talk about a particular individual, whereas *some* usually involves a mere existence claim.

<sup>22</sup> Except that successful definite descriptions, i.e. descriptions that manage to single out a unique object, are sometimes said to denote that object. But there is nothing similar for other quantified noun phrases in Russell's paper.

<sup>23</sup> For example, Russell's epistemology is built on it. The main problem for Russell was how we can know anything, and talk, about objects that we are not *acquainted* with, given that he allowed very few objects that we are acquainted with (sense data, ourselves, perhaps some universals, but not physical objects or other people). On Russell's new theory, it is not necessary to know these objects directly. For the descriptions we use to be successful, it suffices to know *that* certain sentences about objects are true; and these sentences ultimately need not contain reference to anything with which we are not acquainted.

precise logical form which treats quantifier expressions or noun phrases as denoting is available. Indeed, the basic concepts needed for such a treatment were provided already by Frege.

### 1.2.4 Frege

Already in *Begriffsschrift* (1879), Frege was clear about the syntax as well as the semantics of quantifiers. But his two-dimensional logical notation did not survive, so below we use a modernized variant.<sup>24</sup>

*First-level* ( $n$ -ary) functions take ( $n$ ) objects as arguments and yield an object as value. *Second-level* functions take first-level functions as arguments, and so on; values are always objects.<sup>25</sup> Frege was the first to use the trick—now standard in type theory<sup>26</sup>—of reducing predicates (*concepts*) to functions: an  $n$ -ary first-level predicate is a function from  $n$  objects to the *truth values* the True and the False (or 1 and 0), and similarly for higher-level predicates.

For example, from the sentence

(1.7) John is the father of Mary.

we can obtain the two unary first-level predicates designated by the expressions

$\xi$  is the father of Mary

and

John is the father of  $\eta$

as well as the binary first-level predicate designated by

(1.8)  $\xi$  is the father of  $\eta$

(In modern predicate logic we would write *father-of* ( $\xi, \text{Mary}$ ), *father-of* ( $\text{John}, \eta$ ), and *father-of* ( $\xi, \eta$ ), instead.) Here we have abstracted away one or two of the proper nouns in (1.7). We can also abstract away *is the father of*, obtaining the unary second-level predicate denoted by

(1.9)  $\Psi(\text{John}, \text{Mary})$

where  $\Psi$  stands for any binary first-level predicate. ('Mixed' predicates, like  $\Psi(\xi, \text{Mary})$ , were not allowed by Frege.) For example, (1.8) denotes the function

<sup>24</sup> Frege's full-blown theory of quantification and higher-level functions and concepts was presented in *Grundgesetze* (Frege 1893); see esp. §§22–3.

<sup>25</sup> The dichotomy function/object is fundamental with Frege. Functions, as seen above, are 'unsaturated'; they have holes or places, marked with variables in the notation, that can be filled. The filler can be object or (in the case of e.g. second-level functions) again a function, but the result (value) is always an object. Objects have by definition no places to saturate. Ordinary physical objects, numbers, extensions (sets), linguistic expressions, and the two truth values are examples of Fregean objects.

<sup>26</sup> A type theorist might not call this a trick at all, claiming instead that functions are more fundamental mathematical objects than sets or relations.

which sends a pair of objects to the True if the first object is the father of the second, and all other pairs of objects to the False. Equivalently, we can say that (1.8) denotes the relation *father of*.<sup>27</sup> (1.9) can be interpreted as the set of all binary relations that hold between the individuals John and Mary.<sup>28</sup>

Now suppose that

$$(1.10) \ A(\xi)$$

is a syntactic name of a unary first-level predicate. According to Frege, the (object) variable  $\xi$  does not belong to the name; it just marks a place, and we could as well write

$$A(\cdot)$$

The sentence

$$(1.11) \ \forall x A(x)$$

is obtained, according to Frege, by ‘inserting’ the name (1.10) into the second-level predicate name

$$(1.12) \ \forall x \Psi(x)$$

(1.12) is a primitive name denoting the *universal quantifier*, i.e. the unary second-level predicate which is true (gives the value the True) for precisely those first-level (unary) predicates which are true of every object. (Again the (first-level) variable  $\Psi$  in (1.12) is just a place-holder.) So (1.11) denotes the *value* of the universal quantifier (1.12) applied to the predicate (denoted by) (1.10), i.e. a truth value. The result is that

$$\forall x A(x) \text{ is true iff } A(\xi) \text{ is true for any object } \xi$$

Other quantifiers can be given as primitive, or defined in terms of the universal quantifier and propositional operators. For example,

$$\neg \forall x \neg \Psi(x)$$

is the existential quantifier, and

$$\forall x (\Psi(x) \rightarrow \Phi(x))$$

<sup>27</sup> Expressions like *father of* are not phrases or constituents in any linguistic sense. Rather, *father* is usually thought to require a prepositional phrase (PP) complement (so *father* is a lexical item, and *father of Mary* but not *father of* is a phrase). But Frege should not be taken to make any syntactic claims here. His point is the semantic one that you can obtain a function by abstracting any element out of any phrase.

<sup>28</sup> These formulations slur over an important (Fregean) distinction. All the parametric expressions above are unsaturated, and denote concepts (functions), not objects. Sets and relations, on the other hand, are the *extensions* of concepts, and they are objects, according to Frege, who has a special notation for the extension (he calls it *Wertverlauf*, “course-of-values”) of a concept. One may compare this to modern  $\lambda$  notation, where the parametric expressions are just formulas with free variables, and  $\lambda$  abstracts like  $\lambda \xi, \eta$  *father-of*( $\xi, \eta$ ) denote the corresponding relations.

is the binary quantifier (second-level predicate) *all*—one of the four Aristotelian quantifiers. Summarizing, we note in particular

- Frege’s clear distinction between names (formulas, terms) and their denotations;
- the distinction between free and bound variables (Frege used different letters, whereas nowadays we usually use the same), and that quantifier symbols are variable-binding operators;
- the fact that quantifier symbols are not syncategorematic, but denote well-defined entities, *quantifiers*, i.e. second-order (second-level) relations.<sup>29</sup>

### 1.3 TRUTH AND MODELS

Tarski (1935) was the first to formulate a formal *theory of truth*, thus making the notion acceptable to mathematicians, and, more importantly, enabling a clear distinction between simple truth and notions such as validity and provability. In this definition, logical symbols such as quantifiers and propositional connectives are explained contextually (see Chapter 1.1.2.2); they are not assigned independent interpretations; instead, truth conditions are given for each corresponding *sentence* form. Since the number of logical symbols needed for the mathematical purposes for which the logic was originally intended (such as formalizing set theory or arithmetic) is quite small, and there is one clause in the truth definition for each symbol, the truth definition is compact and easy to manage. This contextual or sentential format of the truth definition is still standard practice; see Chapter 2.2.

#### 1.3.1 Absolute and relative truth

In one important respect Tarski’s original truth definition is half-way between Frege’s conception and a modern one. Frege’s notion of truth is *absolute*: all symbols (except variables and various punctuation symbols) have a given meaning, and the universe of quantification is always the same: the class of *all* objects. The modern notion, on the other hand, is that of *truth in a structure* or *model*. Tarski too, in 1935, considers only symbols with a fixed interpretation, and although he mentions the possibility of relativizing truth to an arbitrary domain,<sup>30</sup> he does not really formulate the notion of truth in a structure until his model-theoretic work in the 1950s. The switch from an

<sup>29</sup> We should observe, however, that although Frege had the resources to deal with arbitrary quantifiers, the only primitive one in his *Begriffsschrift* was the universal quantifier  $\forall$ . Indeed, with Russell he strongly criticized traditional logic for its adherence to subject-predicate form, rightly pointing out that such adherence had hampered the development of an adequate logical formalism. He did not note that a formalization of natural language sentences which *preserves* this form is possible via systematic employment of quantifiers. That insight came about 100 years later, with the work of Richard Montague (1974).

<sup>30</sup> In connection with “present day . . . work in the methodology of the deductive sciences (in particular . . . the Göttingen school grouped around Hilbert)” Tarski 1935: 199 in Woodger’s translation).

absolute to a relative notion of truth is in itself quite interesting; see Hodges 1986 for an illuminating discussion.

The model-theoretic notion of truth is relative to two things: an interpretation of the non-logical symbols and a universe of quantification. Let us consider these in turn.

### 1.3.2 Uninterpreted symbols

We are by now accustomed to thinking of logical languages as containing, in addition to *logical* symbols, whose meaning is explained contextually, such as  $\neg$ ,  $\wedge$ ,  $\vee$ ,  $\rightarrow$ ,  $\exists$ ,  $\forall$ , etc., also uninterpreted, or *non-logical*, symbols, which receive a meaning, or rather an extension, by an *interpretation*.

The point to note in the present context is that the notion of an interpretation applies nicely to quantification in natural languages, keeping in mind that an interpretation (in this technical sense) *assigns extensions* to certain words. For it is characteristic of most quantifier expressions that only the extensions of their ‘arguments’ are relevant. That is, in contrast with many other kinds of phrases in natural languages, *most quantifier phrases are extensional*. Furthermore, phrases that provide arguments to quantifiers, like nouns or adjectives or verb phrases, do have sets as denotations, and it is often natural to see these denotations as depending on what the facts are. The extensions of `dog` and `blue` are certain sets; under different circumstances they would have been other sets. It makes sense, then, to have a model or interpretation assign the appropriate extensions to them. Quantifier phrases, on the other hand, do not appear to depend on the facts in this way, so their interpretation is *constant*.

The above description is preliminary. It is a main theme of this book to emphasize and make precise these characteristic features of quantifiers in natural languages: constancy, extensionality (with a few notable exceptions), and the topic neutrality which contrasts them with, say, verbs, adjectives, and nouns. And it is largely these features that make model theory a suitable instrument for a general account of quantification.

*Methodological digression.* We are claiming that a first-order framework allows us to get at the meaning of quantifier expressions, but we are *not* making the same claim for, say, nouns. Here the claim is only that it provides all the needed denotations—but it may provide many more (e.g. various uncountable sets), and it says nothing about how these denotations are constrained. For example, nothing in predicate logic prevents us from interpreting `bachelor` in such a way that it is disjoint from the interpretation of `man`.

But this is not a problem; rather, it illustrates a general feature of all modeling.<sup>31</sup> We are modeling a linguistic phenomenon—quantification. The model should

<sup>31</sup> Where “model” is here used in the sense in which one can build a model of an aircraft to test its aerodynamical properties, not in the logical sense which is otherwise used in this book: viz. that of a mathematical structure or interpretation.

reflect the relevant features of that phenomenon, but may disregard other features of language. Abstracting away from certain features of the world, it allows us, if successful, to get a clearer view of others, by means of systematizations, explanations, even predictions that would otherwise not be available. By way of an analogy, to build a model that allows you to find out if a certain ship will float—before putting it to sea—you need to model certain of its aspects, like density, but not others, like the material of its hull—provided, of course, that the material doesn't have effects on floatability which you didn't foresee. That it doesn't is part of the claim that the model is successful. *End of digression.*

Incidentally, the treatment of individual constants in predicate logic fits well with the use of proper names in natural languages. Here the reason is not that the denotation of a name depends on what the world is like, but, first, that the main purpose of a name is to denote an individual, and second, that although languages in general contain a large number of names, most names denote many different individuals. Thus, in these respects they resemble uninterpreted symbols in a formal language. Selecting a denotation for a name is a way of fixing linguistic content for a certain occasion, and it accords with important aspects of how names are actually used, though, again, it does not constitute—and is not intended to constitute—anything like a full-scale analysis of this use.

The application of model theory allows for three degrees of fixity or variation in analyzing meanings. Certain meanings are completely fixed, given by the semantic rules for truth in arbitrary models. Others are relatively fixed, given by a model's choice of universe and interpretation function. The remaining ones are hardly fixed at all, given only by assignments to free variables. This machinery provides, we claim, just the analytic tool for an account of quantification in general.

### 1.3.3 Universes

The second relativization of truth is to a *universe*. In contrast with Frege's approach, where the universe is always the totality of all objects, such relativization is now a standard feature of formal mathematical languages. Also, as we shall see, it is a fundamental characteristic of natural (as opposed to logical) languages. In most common circumstances, one does not talk about (quantify over) *everything* (but see section 1.3.5 below.) For example, one restricts attention to a *universe of discourse*. This universe need not be referred to or marked in any way by syntactic features of the sentences used. It can be wholly implicit, provided by the relevant context of the discourse.

Suppose, for example, we want to talk about the structure of natural numbers with addition, multiplication, etc. To say that adding 0 to a number does not increase it, we write, in mathematics,

$$\forall x(x + 0 = x)$$

Only the context can insure that this sentence is about natural numbers and not, say, real numbers.

Of course, if one wants, the universe can be made explicit. Suppose we are in some bigger set-theoretic universe and want to express the same fact about natural numbers. We introduce a one-place predicate  $N$  for the natural numbers, and write instead:

$$\forall x(N(x) \rightarrow (x + 0 = x))$$

Now quantification is over all sets (including numbers), but the form of the sentence in fact restricts it to natural numbers.

The technical term for such restriction in logic is *relativization*. Any sentence  $\varphi$  written in first-order logic has a relativized version  $\varphi^{(P)}$ , for any one-place predicate  $P$  (not occurring in  $\varphi$ ), obtained by restricting the quantifiers  $\forall$  and  $\exists$  to  $P$ .<sup>32</sup> Then  $\varphi^{(P)}$  is true in a structure  $\mathcal{M}$  if and only if  $\varphi$  is true in the structure  $\mathcal{M} \upharpoonright P$  obtained by ‘cutting down’  $\mathcal{M}$  to the subuniverse determined by  $P$ .

Now, it is a remarkable fact about most natural languages that relativization is, as it were, built into them, by means of determiners and other quantirelations. An indication of this is that the relativized versions are more easily expressed than the unrelativized ones. Compare the rather awkward

- (1.13) a. For every thing (in the universe), if it is a number then it . . .  
 b. There is some thing (in the universe) which is a number and it . . .

which use English renderings of the unrelativized  $\forall$  and  $\exists$ , with the much simpler but equivalent

- (1.14) a. Every number . . .  
 b. Some number . . .

where the relativized versions are used, by means of the determiners *every* and *some*. Indeed, determiners have the semantic effect of restricting quantification to the subset denoted by the noun in the corresponding nominal phrase. (In addition, they have the syntactic effect of accomplishing *binding* in a way which is often more elegant than in formal languages, without the overt use—at least in simpler cases—of pronouns or variables.)

Quantirelations’ function of restricting quantification is a fundamental fact about natural language. It will be amply discussed in Chapter 4.5. For the moment, however, our point is another. One might think, at first blush, that the mechanism of relativization would render unnecessary the use of universes in natural discourse. In principle, at least for quantification by means of determiners, if these restrict quantification to their noun arguments anyway, couldn’t one assume once and for all a constant discourse universe consisting of absolutely everything? However, that is very far from the way in which natural languages actually work. Limited discourse universes that depend on context are the rule rather than the exception.

Here is an obvious example. If one claims on a certain occasion that

- (1.15) All professors came to the party.

<sup>32</sup> Essentially, this means replacing occurrences of  $\forall x \dots$  and  $\exists y \dots$  by  $\forall x(P(x) \rightarrow \dots)$  and  $\exists y(P(y) \wedge \dots)$ , respectively.

it is understood that one is not talking about all professors in the world, but perhaps only those in, say, the department where the party took place. So the universe of discourse might be the set of people either working or studying in that department. Getting this universe right matters crucially to the truth or falsity of the sentence, and thus to what is being claimed by that use of it. But none of the words in it, nor its grammatical form, carries this information. The universe is provided from somewhere else.

In principle, it would be possible to obtain the correct interpretation of (1.15) by making the denotation of *professor*, rather than the discourse universe, depend on context, so that on this occasion it denoted the set of professors in that particular department. We note in the next subsection that mechanisms of this kind are indeed operative in natural discourse. But although such mechanisms could take care of (1.15), we shall also see that in general they do not eliminate the need for context-dependent discourse universes.

We conclude that the notion of truth in a structure or model with a corresponding universe is well suited to natural language semantics. Should we go further and simply *identify* the discourse universe with the universe of the model? This is partly a methodological question, but we feel there is no need for such a stipulation. Often enough it makes sense to identify them, and a handy convention might be that if nothing else is said, the two are identical. But sometimes the discourse universe might be taken to be a subset of the universe of the model, or of some other set built from that universe.

Think of a model—a universe plus an interpretation—as representing what we want to keep fixed at a certain point. Other semantically relevant things may be allowed to vary. For example, in logic, assignments (of values to variables) do not belong to the model. In language, various sorts of context may provide crucial information, such as the universe of discourse or other means of restricting domains of quantification (see the next subsection), or the reference of certain pronouns, or, to take a different kind of example, the ‘thresholds’ of certain quantifier expressions like *most*.<sup>33</sup> A discourse universe is presumably something fairly constant —perhaps it should stay the same during a whole piece of discourse—but nothing in principle prevents one from accounting for several pieces of discourse within one model.

#### 1.3.4 Context sets

At this point it is appropriate to mention that the mechanisms for restricting quantification in natural languages are in fact more subtle than has been hinted at so far. On an occasion of uttering (1.15), the restriction on all *professors* *could* be the whole discourse universe. But in (1.16), *most children* is presumably restricted to English children, whereas *several pen pals* is *not* so restricted. The discourse universe must contain people from all over the world. It must also contain countries, since they are quantified over as well here.

<sup>33</sup> How many *As* must be *B* in order for *Most As are B* to be true? Sometimes any number more than half seems enough, but other times a larger percentage is required.



- (1.16) The English love to write letters. Most children have several pen pals in many countries.

Thus, we have contextually given restrictions on quantified phrases that cannot be accounted for by means of discourse universes. Such ‘local universes’ were called *context sets* in Westerståhl 1985*a*.

This phenomenon is by no means unusual. Consider

- (1.17) The philosophy students this year are quite sociable. Almost all freshmen already know several graduate students in other departments.

In (1.17) it would in principle be possible to cook up a discourse universe—consisting of philosophy freshmen, graduate students in other departments, the departments themselves, but nothing else—so that restricting quantification to it yields the right truth conditions. But in other cases such a strategy will not work, and anyway it would be completely artificial. It is in fact quite natural to use contextually given subuniverses limited to a particular occurrence of a noun phrase.<sup>34</sup>

While some version of context sets is undoubtedly used, there has been a recent discussion of whether the phenomenon is semantic or pragmatic. A semantic account presumably has to represent context sets by some sort of parameters or variables.<sup>35</sup> This was proposed in Westerståhl 1985*a* and argued for at length by Stanley and Szabo (2000), who gave additional support for the idea from *binding* phenomena, as in

- (1.18) Whenever John shows up, most people tend to leave.

Here most people has to be restricted in a way that depends on the occasions of John showing up—a fixed context set will not do. This looks like an even stronger case for context set variables than examples like (1.16). Still, while presumably everyone agrees that binding occurs *somewhere* within an utterance of (1.18), pragmatic accounts of this effect have also been proposed.

One such account, propounded by Kent Bach (1994, 2000) among others, denies any contextual restriction at all in the semantics of (1.15) and (1.18), for example, which are taken to express the implausible propositions that all professors in the universe came to the party, and that whenever John shows up, most people in the universe tend to leave. However, pragmatic factors are said to insure that these propositions are not the ones *communicated* by utterances of (1.15) and (1.18); so the contextual restrictions show up only in the communicative situation. We cannot enter into this discussion here or do justice to the various arguments that have been put forward, but we do feel that an account like Bach’s fails to capture the intuitively clear

<sup>34</sup> In the examples here, we have let a linguistic context provide the relevant context sets. This is for expository purposes; one can easily imagine the linguistic context replaced by a non-linguistic one.

<sup>35</sup> There is also the alternative of thinking of sentences requiring contextual restriction as elliptical or (surface) incomplete, so that the sentence ‘really’ used in an utterance of (1.15) would be, say, All professors in John’s department came to the party. This idea seems empirically unmotivated to us; see Stanley and Szabo 2000 for discussion.

difference between standard contextual restriction, on the one hand, as in the above sentences, and sentences like

(1.19) Everyone was fighting.

on the other hand, uttered, say, during a description of a barroom brawl. Here the speaker may be well aware that perhaps not *everyone* in the bar participated in the fight—some bystanders might have been huddling under tables—and so she is well aware that the sentence may be literally false. Nevertheless, she succeeds in communicating another content, perhaps that *a lot of* people were fighting. But this case of using a literally false sentence to communicate something by hyperbole is quite different, it seems, from (1.15) and (1.18), where the supposed literal content is not even remotely entertained by the speaker.

The issue, however, is complex, and concerns syntactic and semantic form, as well as where the line between semantics and pragmatics should be drawn. For a recent overview of this debate, and a particular instance (very different from Bach's) of the pragmatic position in it, see Recanati 2004. As indicated, we ourselves tend to favor semantic accounts whenever they are feasible.

Another question is whether context sets belong to the determiner or to the noun in a quantified phrase. In Westerståhl 1985*a* the former view was taken, but Stanley and Szabo (2000) and Stanley (2002) claim that context sets rather restrict the head noun. Thus, on a given occasion, an utterance of a certain noun which normally denotes a set  $A$  will actually denote  $A \cap X$  instead, where  $X$  is a context set. It may also be that both mechanisms occur. Linguists often argue that the definite article *the* has a given slot for contextual restriction, for example, in

(1.20) Every toothpaste tube was missing the cap.

The idea is that the meaning of *the* leads one to look for a context set (whose intersection with the set denoted by the noun has just one element in the singular case), a sort of 'familiarity' requirement. Other determiners may have a similar feature, whereas yet others lack it.

We note, however, that even in the cases where the determiner might have a context set parameter, that parameter has a different role from the other two arguments, which we will call the *restriction* and the *scope*, i.e. the first and the second arguments of the Aristotelian quantifiers in section 1.1.1. As we have already said and will make even clearer later (Chapter 4 onwards), it is most natural to regard determiners as denoting *binary* relations between sets. The possibility that some determiners may have context set parameters does not alter this fact. We recognize that to account for binding phenomena as in (1.18) and (1.20), some context mechanism must be made explicit. But for describing the meanings and stating the properties of determiners in the most natural and efficient way, these mechanisms can for most purposes be left implicit. For this reason, we will usually ignore context sets in this book.<sup>36</sup>

<sup>36</sup> A further issue is if a *set* parameter is always sufficient. One can argue that an intensional property is sometimes required. Stanley and Szabo (2000) give examples; moreover, they argue

### 1.3.5 Quantifying over everything?

A common idea nowadays is that Russell's paradox has shown that there cannot be a totality of everything—contrary to what Frege thought—and that quantification over everything is for this reason an incoherent notion. But this conclusion is by no means straightforward. Set theorists standardly grant that the paradox shows there can be no *set* of all sets, but call that totality a (proper) *class* instead, and happily quantify over it in the object language of set theory (at least those who think that language has an intended interpretation). Model theorists, who usually require models to have sets as universes, happily talk about “all models”, again quantifying over the same totality. Is such quantifying over all sets, or over everything, just loose and imprecise talk that must be abandoned?

Williamson (2003) makes a forceful case that quantification over everything is not only possible but a necessary and fundamental feature of language. He argues that the opposite position, ‘generality-relativism’, is self-defeating: *it* cannot be coherently stated, he claims, and moreover cannot provide adequate accounts of natural laws, universal kind statements, and, notably, truth and meaning. To avoid a threatening Russell-like paradox in semantics, which he formulates in terms of interpretations rather than sets, his solution is basically to give up the idea that interpretations are *things*, and hence can be quantified over like other things, i.e. with first-order quantification. Instead, the semantics of an object language must be done in an irreducibly *second-order* way.

Others are not convinced and continue to claim that quantification over everything is incoherent. For example, Glanzberg (2004) uses the Russellian reasoning to argue that for every large domain, in particular every domain purporting to contain everything, there are in fact things falling outside the domain. This is a version of what Dummett (1991, 1993) has called *indefinite extensibility*, again with the conclusion that it is impossible to quantify over everything.

We shall not go into the philosophical dispute here. But the debate touches on linguistic issues too, and we make some brief comments about these.

First, as we said above, it is a fact about natural languages—in contrast with standard logical languages—that they employ *restricted* quantification, in the sense that they have a built-in slot for a noun that restricts the universe of quantification. Even *everything* decomposes into the determiner *every* and the noun *thing* going into the restriction slot. Williamson stresses that this does *not* entail that one cannot quantify over everything: simply treat *thing* as a logical noun that denotes the totality of everything. But it does indicate, we think, that restricted quantification is primary in natural language: in other words, that quantification over a limited domain is the rule rather than the exception.

that even in an extensional treatment, domain restriction parameters should instead have the form  $f(i)$ , where  $i$  is an individual and  $f$  a function from individuals to context sets, in order to handle quantified contexts such as (1.18).

Second, even if it were the case that one cannot quantify over everything, there is nothing that cannot be quantified over with restricted quantification, since, trivially, every thing is included in some (small) domain.

These points are small. The next point is more important, and one on which we are in sympathy with Williamson: although restricted in the above sense, natural language quantification is over everything in that the meaning of determiners like *every*, *at least five*, *most*, etc. does not involve any particular domains. We do not have to learn separately the meaning of *at least five cows*, *at least five colors*, *at least five integers*, *at least five electrons*, etc. The meaning of *at least five* and the respective nouns suffice. *There are no restrictions on the eligible domains* (except that they contain things that can be counted). Similarly, there are no restrictions on the implicit discourse universes or context sets that can be attached to quantified sentences.

This is why we interpret determiners as *operators* that *with each domain* associate a binary relation between arbitrary subsets of that domain.<sup>37</sup> It cannot be emphasized enough that determiners have *global* meanings in this sense. As we will see, this has important consequences for the semantics of determiners and other quantifier-denoting expressions, consequences that are sometimes obscured in the literature, because linguists often argue within a *local* perspective, where a universe is fixed by context. But certain features of quantifiers are simply not visible from such a local perspective. This point will return again and again in later chapters; see Chapter 3.1.1 for further discussion.

In this book, we follow the standard model-theoretic tradition of treating universes as *sets*. In one sense, this is less general than required; (proper) classes can be universes too, and perhaps even the totality of everything. However, our purpose is an account of the meaning of quantifier expressions. An essential part of that meaning is that these expressions have built-in slots for domains and for subsets of those domains. But, equally essentially, nothing is presupposed about the nature of the domains. Therefore, the philosophical and logical problems about very large domains are in a sense irrelevant to the meaning of quantifier expressions. You have to explain how these expressions behave on *any* domain. What domains really exist is a question for metaphysics, but not for the meaning of determiners and similar expressions.

To take an example, suppose you state, as we did above, that every thing is an element of some small domain. As Williamson forcefully points out, if there is no totality of everything, then the intended meaning of that claim simply doesn't get through! So, in that case, you are either not making any claim at all, or making a different

<sup>37</sup> So instead of saying that *every* denotes the inclusion relation as in sect. 1.1.1.2 above, we will say that *on each universe M* it denotes the inclusion relation *over M*. If there really were a totality of everything, we could say instead that it denotes the inclusion relation over that totality; but we don't want to prejudge that issue, and in any case we have seen that the formulation in terms of universes is congenial to both natural and mathematical language.

If one thinks of quantifiers instead as operators on noun arguments, as medieval philosophers like Ockham can be interpreted as recommending (sect. 1.1.2), the present treatment means that one more argument is added to those operators: viz. an argument for the universe.

claim from the one you wished to make. But—and this is our main point—the eventual resolution of the issue of whether you do or not tells us nothing new about the meaning of *every*. That meaning is already described in a fully adequate way by saying that it denotes a quantifier that with any domain associates the inclusion relation over that domain. What domains there are is a separate matter, and this is why we are content to use standard model-theoretic semantics for the purposes of this book.

#### 1.4 POSTSCRIPT: WHY QUANTIFIERS CANNOT DENOTE INDIVIDUALS OR SETS OF INDIVIDUALS

There is a simple argument, familiar from elementary courses in first-order logic, that quantified noun phrases such as *every woman* and *some man* cannot denote individuals. It hinges on the observation that noun phrases that may denote individuals, such as proper names, exhibit certain semantic behavior in sentences. Specifically,

(1.21) John is brave or John is not brave.

can't be false, and

(1.22) John is brave and John is not brave.

can't be true. This behavior is necessitated by the fact that the set of brave individuals is the complement of the set of individuals who are not brave. In contrast to (1.21) and (1.22), the sentence

(1.23) Every woman is brave or every woman is not brave.

clearly can be false, and

(1.24) Some man is brave and some man is not brave.

clearly can be true. So *every woman* and *some man* obviously do not behave semantically or logically as they would have to if they denoted individuals.

To demonstrate that quantified noun phrases like *every woman* and *some man* can't denote *sets* of individuals requires a more sophisticated argument, which we present below. As we will see, the argument shows *a fortiori* that in a compositional semantics, quantified noun phrases can't denote individuals. The demonstration is based on ideas in Suppes 1976.

Let us consider how many distinct things are denoted by quantified noun phrases. To make the argument, it suffices to consider only two type  $\langle 1, 1 \rangle$  quantifiers. We'll take *every* and *some*, the relations denoted by *every* and *some*. The argument starts from the straightforward observations that

(1.25) Every A is B is true iff  $A \subseteq B$

(1.26) Some A is B is true iff  $A \cap B \neq \emptyset$

where  $A$  is the set denoted by  $\mathbb{A}$ , and  $B$  the set denoted by  $\mathbb{B}$ . These truth conditions on the sentences impose a fundamental requirement on the semantic adequacy of whatever *every*  $\mathbb{A}$  and *some*  $\mathbb{A}$  are taken to denote.

We introduce the notation  $[[\text{every } \mathbb{A}]]$  for whatever *every*  $\mathbb{A}$  does denote. Similarly for  $[[\text{some } \mathbb{A}]]$  and *some*  $\mathbb{A}$ . The truth value of a sentence of the form *every*  $\mathbb{A}$  *is*  $\mathbb{B}$  is produced by a rule that composes the denotation  $[[\text{every } \mathbb{A}]]$  with  $B$ , the denotation of *is*  $\mathbb{B}$ , and similarly for *some*  $\mathbb{A}$  *is*  $\mathbb{B}$ ,  $[[\text{some } \mathbb{A}]]$  and  $B$ . So if the semantics is compositional, there must be a relation  $R$  that holds in both cases between the denotations of the quantified noun phrase and the predicate for the sentence to be true. Then the truth conditions (1.25) and (1.26) translate into the following fundamental requirements on the semantic adequacy of the denotations  $[[\text{every } \mathbb{A}]]$  and  $[[\text{some } \mathbb{A}]]$ .

$$(1.27) \quad R([[ \text{every } \mathbb{A} ]], B) \text{ iff } A \subseteq B$$

$$(1.28) \quad R([[ \text{some } \mathbb{A} ]], B) \text{ iff } A \cap B \neq \emptyset$$

We will use these semantic adequacy requirements to determine how many distinct denotations are required for all noun phrases of the forms *every*  $\mathbb{A}$  and *some*  $\mathbb{A}$ . As will be seen when we have done this, more denotations are needed than there are sets of individuals. Therefore, whatever the denotations of these quantified noun phrases are, they cannot be sets of individuals. *A fortiori*, they cannot be individuals, of which even fewer exist than sets of individuals.

To facilitate counting, let us define operators  $q_{\text{every}}$  and  $q_{\text{some}}$  such that

$$q_{\text{every}}(A) = [[ \text{every } \mathbb{A} ]]$$

and

$$q_{\text{some}}(A) = [[ \text{some } \mathbb{A} ]]$$

where  $A$  is the denotation of  $\mathbb{A}$ . No assumption is made about the *values* of these operators, i.e. about what sort of things *every*  $\mathbb{A}$  and *some*  $\mathbb{A}$  might denote. We first show that the operators must be one-to-one.

### Lemma 1

(1.27) implies that  $q_{\text{every}}$  is 1-1, and (1.28) implies that  $q_{\text{some}}$  is 1-1.

*Proof.* Suppose (1.27) holds, and  $q_{\text{every}}(A_1) = q_{\text{every}}(A_2)$ . Then we have:

$$\begin{aligned} A_1 \subseteq A_1 &\implies R(q_{\text{every}}(A_1), A_1) \\ &\implies R(q_{\text{every}}(A_2), A_1) \\ &\implies A_2 \subseteq A_1 \end{aligned}$$

Thus,  $A_2 \subseteq A_1$ . By an exactly symmetric argument,  $A_2 \subseteq A_2$  implies  $A_1 \subseteq A_2$ . Thus  $A_1 = A_2$ , and  $q_{\text{every}}$  is one-to-one.

Suppose (1.28) holds and  $q_{\text{some}}(A_1) = q_{\text{some}}(A_2)$ . Then:

$$\begin{aligned}
 A_1 \subseteq A_1 &\implies A_1 \cap \overline{A_1} = \emptyset \\
 &\implies \text{not } R(q_{\text{some}}(A_1), \overline{A_1}) \\
 &\implies \text{not } R(q_{\text{some}}(A_2), \overline{A_1}) \\
 &\implies A_2 \cap \overline{A_1} = \emptyset \\
 &\implies A_2 \subseteq A_1
 \end{aligned}$$

So again,  $A_2 \subseteq A_1$ , and by a symmetric argument,  $A_1 \subseteq A_2$ , i.e.  $A_1 = A_2$ , and  $q_{\text{some}}$  is one-to-one.  $\square$

Next we show that the ranges of  $q_{\text{every}}$  and  $q_{\text{some}}$  intersect only in the images of singleton sets; that is, if  $[[\text{every } A_1]] = [[\text{some } A_2]]$ , then  $A_1$  and  $A_2$  denote the same set of one individual.

**Lemma 2**

(1.27) and (1.28) jointly imply that if  $q_{\text{every}}(A_1) = q_{\text{some}}(A_2)$ , then  $A_1 = A_2 = \{a\}$  for some individual  $a$ .

*Proof.* Suppose (1.27) and (1.28) hold, and  $q_{\text{every}}(A_1) = q_{\text{some}}(A_2)$ . Then, for every set  $B$ ,

$$\begin{aligned}
 A_1 \subseteq B &\iff R(q_{\text{every}}(A_1), B) \\
 &\iff R(q_{\text{some}}(A_2), B) \\
 &\iff A_2 \cap B \neq \emptyset
 \end{aligned}$$

With  $B = A_1$ , we obtain  $A_2 \cap A_1 \neq \emptyset$ , since  $A_1 \subseteq A_1$ . Let  $a$  be a member of  $A_2 \cap A_1$ . With  $B = \{a\}$ , we obtain  $A_1 \subseteq \{a\}$ , since  $A_2 \cap \{a\} \neq \emptyset$ . Thus  $A_1 = \{a\}$ , since  $a \in A_1$ . And since  $A_1 \not\subseteq \overline{\{a\}}$ , with  $B = \overline{\{a\}}$ , we obtain  $A_2 \cap \overline{\{a\}} = \emptyset$ . Thus  $A_2 \subseteq \{a\}$  and so  $A_2 = \{a\}$ .  $\square$

What we set out to establish now follows from these two lemmas. Consider any finite universe of discourse  $M$  with  $m$  individuals as its elements. Then exactly  $2^m$  sets of individuals exist.  $q_{\text{every}}$  maps each of these sets to something, mapping different sets to different things. Thus  $2^m$  different values are needed for  $q_{\text{every}}(A)$  as  $A$  ranges over all the subsets of  $M$ . Likewise,  $q_{\text{some}}$  maps each of the  $2^m$  subsets of  $M$  to different things, requiring  $2^m$  distinct values for  $q_{\text{some}}(A)$  as  $A$  ranges over all subsets of  $M$ . The key point of the argument is this: At most  $m$  of the values they require can be shared by  $q_{\text{every}}$  and  $q_{\text{some}}$ . Only when  $A_1 = A_2 = \{a\}$  for some  $a \in M$ , can  $q_{\text{every}}(A_1)$  be identical to  $q_{\text{some}}(A_2)$ . Thus  $q_{\text{every}}$  and  $q_{\text{some}}$  between them require at least  $2^m + 2^m - m = 2^{m+1} - m$  distinct values. However, there are only  $2^m$  sets of individuals in the universe, and  $2^m < 2^{m+1} - m$ . Therefore the denotations of every  $A$  and some  $A$  cannot be sets of individuals, as there are not enough such sets

to meet the semantic adequacy requirements (1.27) and (1.28). *A fortiori*, the denotations of every  $A$  and some  $A$  cannot be individuals, as even fewer individuals exist than sets of individuals.<sup>38</sup>

Note that this argument does not depend on the language containing lots of quantifier-denoting expressions. We chose two of the most simple and familiar determiners, every and some, and, by calculating the number of denotations, of whatever kind, needed jointly for every  $A$  and some  $A$  as  $A$  varies over arbitrary subsets of the universe, showed that this number exceeds the number of such subsets, let alone the number of individuals. The only assumptions required to reach this conclusion are (a) that every subset of a finite universe can in principle be denoted by some expression, and (b) that the semantics is compositional, so that the semantic values of every  $A$  and some  $A$ , and of sentences containing them, are given by the *same* rule.

Recall that in section 1.1.2.1 above we said, in connection with Ockham's account of syncategorematic terms, that one might *try* to let every  $A$  denote the set  $A$  (the denotation of  $A$ ), and let some  $A$  denote some individual  $a$  in  $A$  (or perhaps  $\{a\}$ ), but that problems were bound to appear with the denotation of no  $A$ . We now see that not even the seemingly simple suggestion for every and some works out, if the semantics is to be compositional.

Indeed, treating every  $A$  and some  $A$  as denoting *sets of sets* of individuals—the proposal discussed in the next chapter—is the simplest semantically adequate choice among those available.

<sup>38</sup> Even if some denotations were individuals, and other denotations sets of individuals, only  $2^m + m$  denotations would be available; and when  $m > 2$ , this number is less than the  $2^{m+1} - m$  distinct denotations that are needed.



## 2

# The Emergence of Generalized Quantifiers in Modern Logic

In this chapter we present our principal semantic tool, the concept of a (generalized) quantifier introduced by logicians in the mid-twentieth century. As we indicated in the previous chapter, it is in most respects a *reintroduction*—in a modern model-theoretic setting—of a concept already discovered by Frege.

Logicians call these objects “generalized” quantifiers, since they were originally generalizations of the universal and the existential quantifiers from first-order logic. But once the naturalness and the ubiquity of the concept is appreciated, it becomes natural to drop the qualification, and just call them *quantifiers*. This is what we officially do. However, the original terminology, as well as abbreviations such as “GQ theory”, have become quite entrenched, so we occasionally insert “generalized” within parentheses to remind the reader that we are talking about the general concept, not just  $\forall$  and  $\exists$ .

Thus, after a brief presentation of the syntax and semantics of first-order predicate logic (*FO*), we present, first, Mostowski’s concept of a quantifier, as a very natural generalization of  $\forall$  and  $\exists$ , and then (following the historical development) Lindström’s general concept, which is the official concept of a (generalized) quantifier in this book. We also sketch another way to generalize the standard logical quantifiers, originally due to Henkin, in terms of branching or partially ordered prefixes of these quantifiers, and relate this generalization to our official concept.

We observe that all of these generalized quantifiers are still first-order, not in the sense of being definable in *FO*, but simply by quantifying over individuals, not over higher-type objects. This looks like a trivial point, but we shall insist on it several times in this book, since we think it is actually an important characteristic of quantification. Thus, we begin with an elaboration of the crucial difference between these two senses of “first-order”.

### 2.1 FIRST-ORDER LOGIC VERSUS FIRST-ORDER LANGUAGES

Predicate logic for Frege or Russell was not first-order; it was, rather, higher-order, since one could (universally or existentially) quantify not just over individuals but also over properties of individuals, properties of properties of individuals, etc. But the first-order version, i.e. the restriction to quantification over individuals, is particularly

well-behaved and well understood, and today it forms the basic logical system, which we call *first-order logic*, or simply *FO*. This logic has

- atomic formulas, including identities;
- propositional operators;
- the usual universal and existential quantifiers over individuals.

But the attribute “first-order” need not be restricted to *FO*. New quantifiers can be added to *FO*, and while expressivity may thereby increase greatly, the result is still first-order, as long as these quantifiers too are taken to range over individuals. In this sense, *first-order languages* extend far beyond *FO*.<sup>1</sup>

### first-order languages

In this book, a *first-order language* is one whose constants name individuals, whose predicate or function expressions denote relations between individuals or functions from individuals to individuals, and whose quantifiers range over individuals.

By contrast, a second-order language has in addition quantifiers ranging over relations (or functions) on individuals, predicate symbols that may denote a set of sets of individuals, etc.

Moreover, for quantification, we maintain, the first-order case is not just a special instance of  $n$ 'th order languages (or  $\omega$ 'th order languages, or type theory), that we begin with for simplicity. On the contrary, there is a sense in which higher-order languages are special cases of first-order languages. In this sense, the first-order case is the paradigmatic case. *Any* non-empty collection of any objects—chairs, tables, numbers, thoughts, electrons, sets, sets of sets, etc.—might be chosen as universe.<sup>2</sup> And to explain the meaning of a quantifier expression, we (usually) need to make no assumptions at all about that universe.

Linguistic and other circumstances sometimes impose various constraints on the range of the quantifiers. In a mathematical case, we might restrict attention to universes with a built-in total order, or with some arithmetic structure. In the case of natural languages, we find on occasion that quantification is over  $n$ -tuples of objects rather than single ones (adverbial quantification). Or it might be over sets of objects, or a Boolean algebra, or some sort of lattice structure (plurals and collective quantification). Furthermore, the forms of implicit quantification we have mentioned

<sup>1</sup> This terminology is not quite settled. Often “first-order language” is taken to mean a language using only the resources of *FO*. The wider sense adopted here is, we believe, both natural and illuminating.

<sup>2</sup> These remarks presuppose a standard view of second- or higher-order logic. It is not consonant with the view taken in Williamson 2003, where second-order quantification is different in kind from first-order quantification, because one does not quantify over *things* but properties, and it is crucial for him to maintain that properties are *not* things. That view of second-order logic is largely unexplored, however. And regardless of the metaphysical issue, the linguistic point (below) that all means the same even in a sentence like *All properties are allowed in the definition* still stands.

involve adding additional structure to the things (like times or possible worlds) quantified over.

Such maneuvers can change the expressivity of the language: new things may be expressible, others may get lost. But here is the bottom line: understanding what quantifier expressions *mean* normally doesn't appeal to any features of the domain of quantification. Consider

- (2.1) a. All cats like milk.  
 b. All electrons have negative charge.  
 c. All natural numbers have a successor.  
 d. All twins like each other.  
 e. All compact subsets of Hausdorff spaces are closed.

Obviously, the meaning of  $\forall$  has *nothing* to do with cats or electrons or numbers or twins or Hausdorff spaces, nor with the discourse universes that may be associated with the above examples. *It simply stands for the inclusion relation, regardless of what we happen to be talking about.*

The distinction between languages of various orders, and the idea of a universe of functions of various types built over a universe of individuals, make perfect sense. But these concepts do not help explain the meaning of  $\forall$ . Suppose we are quantifying over such a type-theoretic universe, or over objects in it of a fixed type. That universe of quantification may again be taken as a new universe of 'individuals', with first-order quantification. *Whichever perspective we take,  $\forall$  still means the same.* This is the sense in which the first-order case is paradigmatic for quantification.

At the end of the previous chapter we noted that quantification in natural languages is restricted in various ways: (a) determiners have a restriction argument; (b) there is always an implicit universe in the background (which can often be identified with a discourse universe); (c) implicit context sets may be used. This means that a standard model-theoretic apparatus can be applied directly. The universe is that of a model, not signaled in the quantified sentence itself but crucial for its truth conditions. The restriction argument is usually visible in the sentence, and interpreted in the model as a subset of the universe. (Context sets could be handled by an additional parameter; see Chapter 1.3.4.) The quantifiers that are taken to be the denotations of determiners and similar expressions are model-theoretic objects, but not local to models and not interpreted in them. Their interpretations are global, associating with *each* universe a suitable (second-order) relation over that universe. For example,  $\forall$  is interpreted as the function that with each universe  $M$  associates the inclusion relation between subsets of  $M$ . No structural or other constraints on  $M$  are assumed, since no such constraints belong to the meaning of  $\forall$ .

Thus, from a meaning-theoretic as well as from a methodological point of view, we claim that *for the study of quantification*, first-order languages are the right place to start. But it cannot be stressed enough that this in no way means confining attention to *FO*. Indeed, natural languages easily express many simple quantifiers that go far beyond the expressive powers of *FO*. What claims like this mean exactly, and how one proves them, are the subject of Chapters 11–15. For the moment, we need to be

a bit more precise about the syntax and semantics of *FO*, as well as the logical notion of a (generalized) quantifier that can be added to it.

## 2.2 FIRST-ORDER LOGIC (*FO*)

To specify a logic, you basically need to fix three things: a language (syntax), a class of models or interpretations, and a truth relation between sentences of the language and models. In fact, the class of models will be the same for all the logics we consider.

As explained in the previous section, a *model* consists of two things: a universe and an interpretation which assigns the right sort of objects to (non-logical) symbols of various kinds. Let us be precise:

### vocabularies and models

A (first-order) *vocabulary* is a set  $V$  of non-logical symbols: individual constants, predicate symbols (of various arities), and function symbols (also of various arities).  $V$  is allowed to be empty. It is *relational* if it has only predicate symbols.

A *model* (for the vocabulary  $V$ ) has the form

$$\mathcal{M} = (M, I)$$

where  $M$  is a (usually non-empty) set—the universe—and  $I$  is an interpretation function, which assigns a suitable interpretation  $I(u)$  to each item  $u$  in  $V$ :  $I(c) \in M$  if  $c$  is an individual constant;  $I(P) \subseteq M^n$  if  $P$  is an  $n$ -ary predicate symbol, etc.

One sometimes writes  $u^{\mathcal{M}}$  for  $I(u)$ , and

$$\mathcal{M} = (M, c_1^{\mathcal{M}}, c_2^{\mathcal{M}}, \dots, P_1^{\mathcal{M}}, P_2^{\mathcal{M}}, \dots)$$

for a model for  $V = \{c_1, c_2, \dots, P_1, P_2, \dots\}$ . Or, if it is clear that, say,  $V = \{P_1, P_2, c\}$ , where  $P_1, P_2$  are 1-place, we may write simply

$$\mathcal{M} = (M, A_1, A_2, a)$$

where  $A_1, A_2 \subseteq M$ , and  $a \in M$ . Other variants are also used. For example, we often let

$$\mathcal{M} = (M, A, B)$$

$A, B \subseteq M$ , be a model for an unspecified vocabulary with two unary predicate symbols, sometimes even using “ $A$ ”, “ $B$ ” for these symbols as well.

We assume that the syntax and semantics of *FO* are familiar. For later reference though, here is a quick account: *logical symbols* are  $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, =, \forall, \exists$  (or a subset of these from which the others can be defined); *non-logical symbols* are those that can appear in vocabularies as above; in addition there are *individual variables*  $v_0, v_1, v_2, \dots$ , plus some punctuation marks such as commas and parentheses. The notion of a *formula* in a given vocabulary is as follows:

**formulas and sentences**

If  $V$  is a relational vocabulary, the  $V$ -formulas are defined inductively by the following rules:

1. If  $P$  is an  $n$ -ary predicate symbol in  $V$  and  $x_1, \dots, x_n$  are variables, then  $P(x_1, \dots, x_n)$  is a  $V$ -formula.
2. If  $x$  and  $y$  are variables, then  $(x = y)$  is a  $V$ -formula.
3. If  $\varphi$  and  $\psi$  are  $V$ -formulas, then so are  $\neg\varphi$ ,  $(\varphi \wedge \psi)$ ,  $(\varphi \vee \psi)$ ,  $(\varphi \rightarrow \psi)$ , and  $(\varphi \leftrightarrow \psi)$ .
4. If  $\varphi$  is a  $V$ -formula and  $x$  a variable, then  $\forall x\varphi$  and  $\exists x\varphi$  are  $V$ -formulas.

It is understood that nothing else is a  $V$ -formula. Parentheses can be deleted according to standard conventions. Equally standard is the notion of *free* and *bound* variable occurrences. A  $V$ -sentence is a  $V$ -formula without free variables.

These definitions are readily amended to cover arbitrary vocabularies. Often the vocabulary is not mentioned explicitly. Note that if  $\varphi$  is a  $V$ -formula, and  $V_\varphi$  is the set of non-logical symbols occurring in  $\varphi$ , then  $V_\varphi \subseteq V$ , and  $\varphi$  is a  $V_\varphi$ -formula.

The crucial semantic relation is the *satisfaction* relation

$$\mathcal{M} \models \varphi(a_1, \dots, a_n)$$

where  $\mathcal{M}$  is a model for a vocabulary  $V$ ,  $\varphi = \varphi(x_1, \dots, x_n)$  is a  $V$ -formula containing at most the variables  $x_1, \dots, x_n$  free, and  $a_1, \dots, a_n \in M$  are assigned to  $x_1, \dots, x_n$ , respectively.<sup>3</sup> If  $\varphi$  has no free variables, i.e. if  $\varphi$  is a *sentence*, then

$$\mathcal{M} \models \varphi$$

says that  $\varphi$  is *true in*  $\mathcal{M}$ .

Writing  $\vec{a}$  for a sequence  $a_1, \dots, a_n$ , and continuing to assume for simplicity that the vocabulary is relational, the *truth definition* for  $FO$ —i.e. the definition of the satisfaction relation—can then be expressed as follows, by an induction following the inductive definition of formulas:

**standard truth definition for  $FO$** 

$$(2.2) \quad \mathcal{M} \models P(a_1, \dots, a_n) \iff (a_1, \dots, a_n) \in I(P)$$

when  $P(x_1, \dots, x_n)$  is an atomic formula.

<sup>3</sup> Alternatively, the satisfaction relation is written  $\mathcal{M} \models \varphi[f]$  and holds between a model, a formula, and an *assignment*  $f$  of values in  $M$  to the variables, such that in particular  $f(x_i) = a_i$ ,  $1 \leq i \leq n$ . Yet another notation, familiar from natural language semantics, is to assign to each expression  $u$  a value  $[[u]]_{\mathcal{M},f}$  relative to a model and an assignment. In particular, formulas are assigned truth values (say 0 and 1), and  $\mathcal{M} \models \varphi[f]$  becomes  $[[\varphi]]_{\mathcal{M},f} = 1$ .

Generalizing, we may (as Hodges reminded us) think of this as assigning to each expression  $u$  a value  $[[u]]$ , which is a *mapping* taking a model argument and an assignment argument.

(2.3)  $\mathcal{M} \models a_i = a_j \iff a_i$  is the same member of  $M$  as  $a_j$   
for an atomic formula  $x_i = x_j$ .

(2.4)  $\mathcal{M} \models \neg\varphi(\bar{a}) \iff \mathcal{M} \not\models \varphi(\bar{a})$

(2.5)  $\mathcal{M} \models (\varphi \wedge \psi)(\bar{a}) \iff \mathcal{M} \models \varphi(\bar{a})$  and  $\mathcal{M} \models \psi(\bar{a})$   
and similarly for the other connectives.

(2.6)  $\mathcal{M} \models \forall x\varphi(x, \bar{a}) \iff$  for all  $b \in M$ ,  $\mathcal{M} \models \varphi(b, \bar{a})$

(2.7)  $\mathcal{M} \models \exists x\varphi(x, \bar{a}) \iff$  for some  $b \in M$ ,  $\mathcal{M} \models \varphi(b, \bar{a})$

This fixes the relation  $\models$  for all arguments. In terms of it, the usual logical notions are then defined in a uniform way. For example, a sentence  $\varphi$  is *logically true* if it is true in all ( $V_\varphi$ -)models;  $\varphi$  is a *logical consequence* of sentences  $\psi_1, \dots, \psi_n$  in  $FO$  if there is no model (for the union of the vocabularies involved) making  $\psi_1, \dots, \psi_n$  true and  $\varphi$  false, etc.

The last two clauses in the truth definition explain the meanings of the universal and existential quantifiers, without assuming that they denote anything.<sup>4</sup> Similarly for (2.4) and (2.5), which only indirectly utilize the fact that  $\neg$  and  $\wedge$  denote truth functions. For the propositional connectives, it is easy to reformulate the relevant clauses of the truth definition in a more immediately compositional way, where the connectives denote truth functions. For example, using the second format mentioned in note 3, one would write (2.5) as

$$[[\varphi \wedge \psi]]_{\mathcal{M},f} = [[\wedge]]([[\varphi]]_{\mathcal{M},f}, [[\psi]]_{\mathcal{M},f})$$

where  $[[\wedge]]$  is the familiar truth function of conjunction:

$$[[\wedge]](u, v) = \begin{cases} 1 & \text{if } u = v = 1 \\ 0 & \text{otherwise} \end{cases}$$

<sup>4</sup> To be sure, this explanation is only intelligible to someone who already understands what *all* and *some* mean in English. The point is to explain the meaning of certain formal symbols. Is there a non-circular explanation of these two quantifiers, i.e. one that does not rely on understanding some corresponding natural language expressions?

If this asks for an explanation in terms of some even more fundamental concepts, we doubt that such an explanation exists. (It might be possible to explain  $\forall$  in terms of other equally basic notions. For example, in the  $\lambda$  calculus  $\forall x\varphi$  is equivalent to  $\lambda x\varphi = \lambda x(x = x)$ . The latter presupposes understanding of the notion of the extension of a  $\lambda$  term.) This being the case, it presumably won't be straightforward to explain how we ever come to *learn* such concepts.

What then is the value of a formal account of the quantifiers? One answer might go roughly as follows. Given that people somehow do learn to use *every* and *some*, how can they know that they attach the *same* meaning to these words? Answer: by agreeing on a 'definition' such as the one above. One might dispute what such agreement amounts to, but notice how much easier it seems to be to establish agreement for *every* and *some* than for, say, *believe* and *suggest*. In the latter case one might have a hard time establishing that a speaker of a foreign language meant 'believe' by a certain word rather than 'suggest'. And it might be even harder for words like *dog* or *lake*. One reason for this could be precisely that there are no 'definitions', at least none as obvious and straightforward as those for *every* and *some*.

Or, with the last alternative mentioned in note 3, where  $[[\varphi]]$  is a function from models and assignments, we obtain the even more general compositionality claim

$$[[\varphi \wedge \psi]] = [[\wedge]]([[ \varphi ]], [[ \psi ]])$$

provided the truth function  $[[\wedge]]$  is ‘lifted’ to the appropriate functions too:

$$([[ \wedge ]](F, G))(\mathcal{M}, f) = \begin{cases} 1 & \text{if } F(\mathcal{M}, f) = G(\mathcal{M}, f) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Clearly, these differences in formulation are largely cosmetic. What about a corresponding reformulation of (2.6) and (2.7)? We take that up next.

### 2.3 MOSTOWSKI QUANTIFIERS

Even disregarding issues of compositionality or elegance of formulation, it is very natural to ask from a purely logical point of view: What do the expressions  $\forall$  and  $\exists$  have in common? What category or type do they belong to? What kinds of things are their meanings? What would other specimens of the same category be? The first logician to look at these questions within a model-theoretic perspective was Mostowski in 1957.

We begin by rephrasing (2.6) and (2.7) in a way which prepares for the desired generalizations.

A simple example of a quantifier distinct from  $\forall$  and  $\exists$  is, say,  $\exists_{\geq 5}$ , where

$$\mathcal{M} \models \exists_{\geq 5} x \varphi(x, \bar{a}) \iff \text{there are at least five } b \text{ in } M \text{ s.t. } \mathcal{M} \models \varphi(b, \bar{a})$$

Once this example is appreciated, any number of others come to mind.<sup>5</sup> To express the general idea, we introduce the following notation.<sup>6</sup>

#### extension of a formula in a model

For a formula  $\psi = \psi(x, y_1, \dots, y_n) = \psi(x, \bar{y})$ , and a sequence  $\bar{a}$  of  $n$  objects in  $M$ , let

$$\psi(x, \bar{a})^{\mathcal{M}, x} = \{b \in M : \mathcal{M} \models \psi(b, \bar{a})\}$$

More generally, if  $\psi = \psi(x_1, \dots, x_k, y_1, \dots, y_n) = \psi(\bar{x}, \bar{y})$ ,

$$(2.8) \quad \psi(\bar{x}, \bar{a})^{\mathcal{M}, \bar{x}} = \{\bar{b} \in M^k : \mathcal{M} \models \psi(\bar{b}, \bar{a})\}$$

Here  $k \geq 1$  and  $n \geq 0$ ; for  $k = 0$  we may stipulate

$$(2.9) \quad \psi(\bar{a})^{\mathcal{M}} = \begin{cases} 1 = \text{True}, & \text{if } \mathcal{M} \models \psi(\bar{a}) \\ 0 = \text{False}, & \text{otherwise} \end{cases}$$

<sup>5</sup> Some such examples were in fact discussed early on in predicate logic. For example, Russell and Whitehead (1910–13) use  $\exists!$  for the quantifier ‘there is exactly one thing such that’; this is still standard.

<sup>6</sup> The notation is a variant of the one in n. 3. We could write something like  $[[\psi]]_{\mathcal{M}, f, \bar{x}}$ .

This notation will be used extensively in the rest of this book.<sup>7</sup>

Now (2.6) and (2.7) become

$$\begin{aligned}\mathcal{M} \models \forall x\varphi(x, \bar{a}) &\iff \varphi(x, \bar{a})^{\mathcal{M}, x} = M \\ \mathcal{M} \models \exists x\varphi(x, \bar{a}) &\iff \varphi(x, \bar{a})^{\mathcal{M}, x} \neq \emptyset\end{aligned}$$

Likewise, writing  $|X|$  for the cardinality of the set  $X$ ,

$$\mathcal{M} \models \exists_{\geq 5}x\varphi(x, \bar{a}) \iff |\varphi(x, \bar{a})^{\mathcal{M}, x}| \geq 5$$

We can go on *ad lib*. For example, the following two quantifiers have been studied in model theory:

$$\begin{aligned}\mathcal{M} \models Q_{0x}\varphi(x, \bar{a}) &\iff \varphi(x, \bar{a})^{\mathcal{M}, x} \text{ is infinite} \\ \mathcal{M} \models Q^C_x\varphi(x, \bar{a}) &\iff |\varphi(x, \bar{a})^{\mathcal{M}, x}| = |M|\end{aligned}$$

(On finite universes,  $Q^C$  is identical to  $\forall$ , but not on infinite ones.)

We have arrived at Mostowski's notion of a (generalized) quantifier. Here is a general formulation.

### type (1) quantifiers

For each universe  $M$ , let  $Q_M$  be any set of subsets of  $M$ , and use at the same time (to simplify notation) ' $Q$ ' as a new symbol for a corresponding variable-binding operator. Then  $Q$  is a (generalized) *quantifier of type (1)*, whose meaning is given by

$$(2.10) \quad \mathcal{M} \models Qx\varphi(x, \bar{a}) \iff \varphi(x, \bar{a})^{\mathcal{M}, x} \in Q_M$$

Thus, a quantified sentence  $Qx\varphi(x)$  is true in a model  $\mathcal{M}$  iff the extension of  $\varphi(x)$  in  $\mathcal{M}$  belongs to  $Q_M$ .

For the examples considered above we have:

$$\begin{aligned}\forall_M &= \{M\} \\ \exists_M &= \{A \subseteq M : A \neq \emptyset\} \\ (\exists_{\geq 5})_M &= \{A \subseteq M : |A| \geq 5\} \\ (Q_0)_M &= \{A \subseteq M : A \text{ is infinite}\}^8 \\ (Q^C)_M &= \{A \subseteq M : |A| = |M|\} \text{ (the Chang quantifier)}\end{aligned}$$

<sup>7</sup> For linguists the notation  $\lambda x\psi(x, \bar{y})$  may be more familiar. This denotes a characteristic function, and  $\psi(x, \bar{a})^{\mathcal{M}, x}$  is the set for which  $\lambda x\psi(x, \bar{a})^{\mathcal{M}}$  is the characteristic function.

A minor detail is that (2.9) can be seen if one wants as a special case of (2.8): First, stipulate that  $0 = \emptyset$ ,  $1 = \{\emptyset\}$ , and that  $M^0 = M^\emptyset =$  the set of functions from  $\emptyset$  to  $M$  (or the set of sequences in  $M$  of length 0)  $= \{\emptyset\} = 1$ . Then, if  $\mathcal{M} \not\models \psi(\bar{a})$ , no sequence in  $M^0$  satisfies  $\psi(\bar{a})$  in  $\mathcal{M}$ , so  $\psi(\bar{a})^{\mathcal{M}} = \emptyset = 0$ , but if  $\mathcal{M} \models \psi(\bar{a})$ , all sequences in  $M^0$  do, so  $\psi(\bar{a})^{\mathcal{M}} = M^0 = 1$ .

<sup>8</sup> This is an example of a *cardinality quantifier*; in general,

$$(Q_\alpha)_M = \{A \subseteq M : |A| \geq \aleph_\alpha\}$$

( $\aleph_0, \aleph_1, \dots, \aleph_\alpha, \dots$  are the infinite cardinals in order of magnitude). This notation is somewhat unfortunate, however, and will not be used in this book—we want to be able to use  $Q_1, Q_2$ , etc. for arbitrary quantifiers. But for historical reasons we keep  $Q_0$  for the (type (1)) infinity quantifier.



Two other examples, which will play a role in what follows, are

$$(Q_{\text{even}})_M = \{A \subseteq M : |A| \text{ is an even natural number}\}$$

$$(Q^R)_M = \{A \subseteq M : |A| > |M - A|\} \text{ (the Rescher quantifier)}^9$$

On finite universes,  $Q^R$  is a typical example of a *proportional* quantifier: we have

$$(Q^R)_M(A) \Leftrightarrow |A| > 1/2|M|.$$

Let us introduce the following notation.

(2.11) For natural numbers  $p, q$  such that  $0 < p < q$ , the type  $\langle 1 \rangle$  quantifier  $(p/q)$  is defined by:<sup>10</sup>

$$(p/q)_M(A) \Leftrightarrow q \cdot |A| > p \cdot |M|$$

Similarly,

$$[p/q]_M(A) \Leftrightarrow q \cdot |A| \geq p \cdot |M|$$

So, on finite universes,  $Q^R = (1/2)$ . These examples are from logic, but  $\forall$  and  $\exists$  are naturally taken as the denotations of the English *everything* and *something*, respectively (and  $\neg\exists$  is the denotation of *nothing*). The idea is that a noun phrase (NP) denotes, on a given universe  $M$ , a set of subsets of  $M$ , and that a sentence of the form [NP VP] (for verb phrase) is true iff the extension of the VP is in the denotation of the NP. Thus,

(2.12) Something is blue.

says that the set of blue things (in  $M$ ) belongs to  $\exists_M$ , i.e. that this set is non-empty. Likewise,

(2.13) Everything is colored.

says that the set of colored things belongs to  $\forall_M$ , i.e. that this set equals the whole universe.

One could tell a similar story for at least five things, infinitely many things, at least half of the things, etc., but here it becomes clear that thing is used as a generic expression for ‘object in the universe’, and that what we really want is a uniform interpretation of noun phrases like *every girl*, *at least five cats*, *infinitely many stars*, *more than half of the students*, etc. To this end, we need to extend the notion of a quantifier.

<sup>9</sup> There is some terminological confusion in the literature about which quantifier is to be called “the Rescher quantifier”. In this book, it is  $Q^R$  as defined above. That is also how Rescher (1962) defines it (he calls it “ $M$ ”), so the name is quite apt. See also n. 11.

<sup>10</sup> Formally, these definitions work for all sets, since multiplication is well-defined for finite as well as infinite cardinals. But we only get the intended proportional readings with finite sets. If  $k$  is a finite number and  $X$  an infinite set,  $k \cdot |X| = |X|$ . It follows that if at least one of  $A$  and  $M$  is infinite,  $(p/q)_M(A)$  is always false (since  $|A| \leq |M|$ ). Likewise, if  $A$  is infinite,  $[p/q]_M(A)$  is true iff  $|A| = |M|$ . These consequences are purely stipulative; the reader might prefer other stipulations, e.g. that  $(p/q)_M(A)$  and  $[p/q]_M(A)$  never hold for infinite  $M$ . In any case, when we in the sequel discuss or use proportional quantifiers, we always make the tacit presupposition that only finite universes are considered.

We also note that Mostowski imposed, as part of the definition, a constraint on the sets  $Q_M$ , which is satisfied by the previous examples, but not by arbitrary sets of subsets of  $M$ . The condition is roughly that the quantifier should be insensitive to which particular individuals happen to make up the universe. For example, the constraint does *not* hold for the quantifier  $Q$  defined by

$$Q_M = \{A \subseteq M : \text{Pope Benedict XVI} \in A \text{ and Queen Elizabeth II} \in A\}$$

This constraint is called *isomorphism closure* or *ISOM*. While we do not make it part of the general concept of quantifier, *ISOM* turns out to play an important role also for natural language quantification. We give definitions and examples in Chapter 3.3, and discuss the role of *ISOM* in Chapter 9.

## 2.4 LINDSTRÖM QUANTIFIERS

Mostowskian quantifiers are, on each  $M$ , unary relations among subsets of  $M$ . We can generalize immediately to binary or  $n$ -ary relations among subsets. This turns out to increase expressivity considerably. For example, it was noted early by logicians that whereas the Rescher quantifier  $Q^R$  says of a set  $A$  that most elements of the universe belong to  $A$ , it cannot be used to say of two sets  $A$  and  $B$  that most elements of  $A$  belong to  $B$ . In other words, it cannot be used to express the relation

$$\text{most}_M(A, B) \iff |A \cap B| > |A - B|^{11}$$

Notice by the way that, on finite universes, we get

$$\begin{aligned} (Q^R)_M(A) &\iff |A| > 1/2 \cdot |M| \\ \text{most}_M(A, B) &\iff |A \cap B| > 1/2 \cdot |A| \end{aligned}$$

so there *most* means ‘more than half of’, and  $Q^R$  means ‘more than half of the elements of the universe’.<sup>12</sup> Here are some more examples of quantifiers as relations between subsets of  $M$ :

$$\begin{aligned} \text{all}_M(A, B) &\iff A \subseteq B \\ \text{no}_M(A, B) &\iff A \cap B = \emptyset \\ \text{some}_M(A, B) &\iff A \cap B \neq \emptyset \\ \text{not all}_M(A, B) &\iff A - B \neq \emptyset \end{aligned}$$

<sup>11</sup> Rescher distinguished  $Q^R$  from *most* and claimed (without proof) that “It is readily shown that . . . “Most A’s are B’s” cannot be defined by means of the usual resources of  $[FO]$ ; not even when these are supplemented by  $[Q^R]$ , or any other type of quantification, for that matter” (Rescher 1962: 374). We shall see precisely what this claim amounts to later on. The claim is true, but the proof is not trivial. The only published proof, to our knowledge, of the full claim (if taken to mean that *most* is not definable in terms of  $FO$  with any finite number of type (1) quantifiers added) appeared in Kolaitis and Väänänen 1995; see Theorem 14 in Ch. 14.

<sup>12</sup> As noted earlier, this relation *most* is one meaning of the English word *most*, although in some contexts it could mean the same as, say, at least 75 percent of.

These are of course the four Aristotelian quantifiers from Chapter 1.1.1.

$$\begin{aligned} MO_M(A, B) &\iff |A| > |B| \\ I_M(A, B) &\iff |A| = |B| \text{ (the H\"artig, or equicardinality quantifier)} \\ more\_than_M(A, B, C) &\iff |A \cap C| > |B \cap C| \end{aligned}$$

All of these are of type  $\langle 1, 1 \rangle$ , except the last, which is of type  $\langle 1, 1, 1 \rangle$ .<sup>13</sup>

We also see that for sentences of the form  $[\text{Det } N] \text{ VP}$ , since the extensions of both the noun and the verb phrase are sets of individuals, what we need for the determiner is precisely a type  $\langle 1, 1 \rangle$  quantifier. For example,

(2.14) No sailors drink.

says that the set of sailors and the set of drinkers, in a universe of discourse  $M$ , stand in the  $no_M$  relation to each other; i.e. they are disjoint. And

(2.15) Most students speak Spanish.

says that the set of students and the set of Spanish speakers stand in the  $most_M$  relation to each other. But not all type  $\langle 1, 1 \rangle$  quantifiers are eligible here. For example, there are no determiners, in any language, corresponding to the quantifiers  $I$  and  $MO$  above, although from a logical point of view they are perfectly natural. The reason is that determiners denote *relativized* quantifiers (see Chapter 1.3.3); we come back to this in Chapter 4.5.

Going back to artificial logical languages again, how should the corresponding variable-binding operators work? A type  $\langle 1, 1 \rangle$  quantifier symbol  $Q$  will now apply to a *pair* of formulas  $(\varphi, \psi)$ , and we may write

$$Qx, y(\varphi, \psi)$$

where all free occurrences of  $x$  in  $\varphi$  are bound in  $Qx, y(\varphi, \psi)$ , and similarly for all free occurrences of  $y$  in  $\psi$ . Actually, it is possible, and sometimes simpler, to use the same variable, writing instead

$$Qx(\varphi, \psi)$$

where all free occurrences of  $x$  in  $\varphi$  or  $\psi$  are bound in  $Qx(\varphi, \psi)$ . Thus we have, for example,

$$\mathcal{M} \models most\ x(\varphi(x, \bar{a}), \psi(x, \bar{a})) \iff most_M(\varphi(x, \bar{a})^{M,x}, \psi(x, \bar{a})^{M,x})$$

and similarly for the other examples.<sup>14</sup> Let us summarize:

<sup>13</sup>  $MO$  was called *more* in Westerståhl 1989, but we now find that name misleading. Unlike the H\"artig quantifier, it has no established name in the literature. The word *more* occurs in various comparative constructions, but not properly as an English determiner expression; *more\_than*, on the other hand, can be seen as a two-place determiner, as was noted in Ch. 0.2.2 and will be further explained in Ch. 4.7.

<sup>14</sup> When  $Q$  is the denotation of a determiner, the first formula,  $\varphi$ , in  $Qx(\varphi, \psi)$  is often called the *restriction*, and the second,  $\psi$ , the *scope*, reflecting the syntactic form of a corresponding natural language sentence. One sometimes chooses to mirror this in the logical formalism, by writing, for example,

$$[Qx : \varphi]\psi$$

**type (1, 1) quantifiers**

A (generalized) *quantifier*  $Q$  of type  $\langle 1, 1 \rangle$  associates with each universe  $M$  a binary relation  $Q_M$  between subsets of  $M$ . Using the same symbol as a variable-binding operator, the meaning of a quantified formula  $Qx(\varphi, \psi)$ , where  $\varphi = \varphi(x, x_1, \dots, x_n)$  and  $\psi = \psi(x, x_1, \dots, x_n)$  have at most the free variables shown, is given by

$$(2.16) \quad \mathcal{M} \models Qx(\varphi(x, \bar{a}), \psi(x, \bar{a})) \iff Q_M(\varphi(x, \bar{a})^{\mathcal{M}, x}, \psi(x, \bar{a})^{\mathcal{M}, x})$$

Similarly for type  $\langle 1, 1, 1 \rangle$  quantifiers, etc. Observe that on the left-hand side of (2.16) a syntactic expression—a formula in a formal language—is said to be true in a model  $\mathcal{M}$  (more precisely, satisfied in  $\mathcal{M}$  by a sequence  $\bar{a}$  of elements of  $M$ ). On the right-hand side, by contrast, we are saying that two sets stand in a certain relation, namely the relation  $Q_M$ , to each other.

This generalization of Mostowski's notion of quantifier is due to Lindström (1966). Lindström further observed that there is no principled reason to restrict attention to relations between *sets*; the general concept is that of a relation between relations, or a second-order relation.<sup>15</sup> Here are three examples. For  $A \subseteq M$  and  $R, S \subseteq M^2$ , define

$$\begin{aligned} W_M^{\text{rel}}(A, R) &\iff R \text{ is a well-ordering of } A \\ \text{Ram}_M(A, R) &\iff \exists X \subseteq A [X \text{ is infinite and any two distinct elements of } X \\ &\text{are related by } R] \text{ (a Ramsey quantifier)} \\ (\text{Res}^2(MO))_M(R, S) &\iff |R| > |S| \text{ (the resumption of } MO \text{ to pairs)} \end{aligned}$$

The first two have type  $\langle 1, 2 \rangle$ , and the last  $\langle 2, 2 \rangle$ ; the principle should be clear. The well-ordering quantifier  $W^{\text{rel}}$  is a typical mathematical quantifier; it is useful, and not expressible in first-order logic, to be able to say of a relation that it is a well-ordering. (The significance of the superscript  $\text{rel}$  will be explained in Chapter 4.4.) One would not expect it to turn up in a natural language context. *Ram* and  $\text{Res}^2(MO)$ , on the other hand, are useful mathematically and linguistically as well. *Ram* is of a kind that shows up in the context of *reciprocals* (Chapter 10.4), and resumption is a typical feature of *adverbial quantification* (Chapter 10.2).

As to the syntax of the formal language corresponding to such quantifiers, the variable-binding operator symbol now applies to a  $k$ -tuple of formulas if the second-order relation is  $k$ -place, and in each formula it binds the number of variables

instead of  $Qx(\varphi, \psi)$ . But here we want a logical notation suitable for all quantifiers, whether denoted by determiners or not.

<sup>15</sup> This is of course *very* similar to Frege's treatment of quantifiers as second-level relations which was sketched in Ch. 1.2.4. Indeed, Frege (1893) gives several examples of quantifiers other than  $\forall$  and  $\exists$ , such as *some*, of type  $\langle 1, 1 \rangle$ , and the type  $\langle 2 \rangle$  quantifier which says of a binary relation that it is non-empty. In fact, the only real difference is the model-theoretic treatment, with the universe entering as a parameter, whereas Frege had a fixed total universe and no uninterpreted symbols in the language—see Ch. 1.3.1. However, neither Mostowski nor Lindström seems to have been aware of Frege's definition.

corresponding to the arity of that relation argument. Here, then, is the official concept of a (generalized) quantifier in logic:

### arbitrary (generalized) quantifiers

A *type* is a finite sequence  $\tau = \langle n_1, \dots, n_k \rangle$  of natural numbers  $\geq 1$ .

A (generalized) *quantifier*  $Q$  of *type*  $\tau$  associates with each universe  $M$  a  $k$ -ary second-order relation  $Q_M$  over  $M$ , where the  $i$ 'th argument of  $Q_M$  is an  $n_i$ -ary relation between individuals in  $M$ . Using the same symbol as a variable-binding operator, the meaning of a quantified formula  $Q \bar{y}_1, \dots, \bar{y}_k (\varphi_1(\bar{y}_1, \bar{x}), \dots, \varphi_k(\bar{y}_k, \bar{x}))$ , where each  $\varphi_i(\bar{y}_i, \bar{x}) = \varphi_i(y_{i1}, \dots, y_{in_i}, x_1, \dots, x_n)$  has at most the free variables indicated and  $a_1, \dots, a_n \in M$ , is given by

$$(2.17) \quad \mathcal{M} \models Q \bar{y}_1, \dots, \bar{y}_k (\varphi_1(\bar{y}_1, \bar{a}), \dots, \varphi_k(\bar{y}_k, \bar{a})) \iff \\ Q_M(\varphi_1(\bar{y}_1, \bar{a})^{M, \bar{y}_1}, \dots, \varphi_k(\bar{y}_k, \bar{a})^{M, \bar{y}_k})$$

Here  $Q$  binds in each  $\varphi_i$  all free occurrences of  $y_{i1}, \dots, y_{in_i}$ .

We note that the variables in  $\bar{y}_i$  must be distinct. Normally, one chooses  $\bar{y}_i$  and  $\bar{y}_j$  to be disjoint when  $i \neq j$ , but that is not strictly necessary. For illustration, here is the case of a type  $\langle 2, 3, 1 \rangle$  quantifier, which thus associates with each universe  $M$  a ternary relation  $Q_M$  between a binary relation, a ternary relation, and a unary relation (subset) over  $M$ . We have

$$(2.18) \quad \mathcal{M} \models Qxy, zuv, w(\varphi(x, y, \bar{a}), \psi(z, u, v, \bar{a}), \theta(w, \bar{a})) \iff \\ Q_M(\varphi(x, y, \bar{a})^{M, x, y}, \psi(z, u, v, \bar{a})^{M, z, u, v}, \theta(w, \bar{a})^{M, w})$$

A simpler example, relevant to the English sentence

(2.19) The boys like each other.

would be

$$(2.20) \quad \text{each other } x, y_1 y_2 (\text{boy}(x), \text{like}(y_1, y_2)) \iff \\ (\text{each other})_M (\text{boy}(x)^{M, x}, \text{like}(y_1, y_2)^{M, y_1, y_2}).$$

We discuss what the type  $\langle 1, 2 \rangle$  quantifier *each other* could be in Chapter 10.4.

Already in Chapter 0 we distinguished between monadic and polyadic quantifiers. Here is the formal definition:

### monadic and polyadic quantifiers

A *monadic* quantifier is one of type  $\langle 1, 1, \dots, 1 \rangle$ . Semantically, and it is (on each  $M$ ) a relation between subsets of  $M$ . Syntactically, it binds one variable in each of the formulas to which it applies. One may choose one and the same variable for each formula (as in (2.16)), or distinct variables (as in (2.17)); the difference is just one of bookkeeping.

A (properly) *polyadic* quantifier is of type  $\langle n_1, \dots, n_k \rangle$ , where  $n_i > 1$  for at least one  $i$ . Semantically it is (on each  $M$ ) a relation between  $k$  relations over  $M$ , where the  $i$ th relation is  $n_i$ -ary. Syntactically it binds  $n_i$  variables in the  $i$ th formula it applies to.

Adding generalized quantifiers to  $FO$  is a minimal and well-behaved means to increase its expressive power. There are other ways to strengthen  $FO$ , e.g. by going to second-order logic. But with the generalized quantifier approach we can add an expressive tool for a particular purpose that cannot be served in  $FO$ , while remaining in a first-order language.

Here is an example. Suppose you want to be able to say, of an ordering relation  $<$ , that each element has at most finitely many predecessors.<sup>16</sup> In second-order logic, you can say that a set  $A$  is infinite by a sentence of the following sort:

(2.21) There is an element  $a$  of  $A$  and a one–one function  $f$  from  $A$  to  $A - \{a\}$ .

But to use the machinery of second-order logic only to express infinity looks like a case of overkill. Instead, add to  $FO$  the quantifier  $Q_0$  from the previous section. Then you can say directly that a set is infinite, and now the sentence

$$\forall x \neg Q_0 y (y < x)$$

expresses the desired proposition.  $FO$  with  $Q_0$  added is stronger than  $FO$ , and therefore lacks some of its pleasant properties (such as a completeness theorem; see section 2.6 below, and also Chapter 14.2), but it is much weaker than second-order logic.

## 2.5 BRANCHING QUANTIFIERS

The foregoing explains what we mean by “generalized quantifier” in this book. Historically, there is an even earlier idea about how to generalize  $\forall$  and  $\exists$ , in terms of *branching* or *partially ordered* quantifiers. We describe this idea in the present section, and also explain how it relates to generalized quantifiers in our sense.

Consider formulas in *prenex form*, that is, formulas beginning with a quantifier *prefix* consisting of a sequence of expressions of the form  $\forall x$  or  $\exists y$ , followed by a quantifier-free formula. It is a well-known fact about  $FO$  that each formula is logically equivalent to a formula in prenex form.

When a universal quantifier precedes an existential one, a *dependency* is created. In

$$(2.22) \quad \forall x \exists y \forall z \exists w \varphi(x, y, z, w)$$

$y$  depends on  $x$ , and  $w$  depends on both  $x$  and  $z$ . Another order in the quantifier prefix may change the dependencies: in

$$(2.23) \quad \forall x \forall z \exists y \exists w \varphi(x, y, z, w)$$

<sup>16</sup> For example, given a Big Bang theory and a discrete time ordering, this would be true of that ordering, even if each time point has infinitely many successors.

both  $y$  and  $w$  depend on  $x$  and  $z$ . A convenient way of making the dependencies explicit is by means of *Skolem functions*:<sup>17</sup> (2.22) and (2.23) are equivalent, respectively, to the following formulas in *second-order* logic:

$$(2.24) \quad \exists f \exists g \forall x \forall z \varphi(x, f(x), z, g(x, z))$$

$$(2.25) \quad \exists f \exists g \forall x \forall z \varphi(x, f(x, z), z, g(x, z))$$

Now another way to generalize ordinary quantification is to lift the restrictions that are created by linear dependencies and allow, for example, a sentence with truth conditions

$$(2.26) \quad \exists f \exists g \forall x \forall z \varphi(x, f(x), z, g(z))$$

This was proposed by Henkin (1961), who also introduced the following graphical way of writing (2.26) as a first-order sentence:

$$(2.27) \quad \begin{array}{l} \forall x \text{ --- } \exists y \\ \forall z \text{ --- } \exists w \end{array} \left. \vphantom{\begin{array}{l} \forall x \text{ --- } \exists y \\ \forall z \text{ --- } \exists w \end{array}} \right\} \varphi(x, y, z, w)$$

The partial ordering of the quantifier prefix clearly shows that  $y$  depends only on  $x$ , and  $w$  only on  $z$ . This syntax can be extended to all partially ordered prefixes of existential and universal quantifiers; the particular prefix in (2.27) is called the *Henkin prefix*.

It is not obvious that the dependencies exhibited in (2.26) (or (2.27)) cannot be expressed by an ordinary *FO* sentence. But this can be seen by the following simple but ingenious argument, due to Ehrenfeucht. No *FO*-sentence in two one-place predicate symbols  $A$  and  $B$  expresses that  $|A| \leq |B|$ .<sup>18</sup> That is, no *FO* sentence can say that there is a 1-1 function  $f$  from  $A$  to  $B$ . But this can be expressed with Skolem functions as follows:

$$(2.28) \quad \exists f \forall x \forall z [(x = z \leftrightarrow f(x) = f(z)) \ \& \ (A(x) \rightarrow B(f(x)))]$$

Now observe that (2.28) is equivalent to

$$(2.29) \quad \exists f \exists g \forall x \forall z [(x = z \leftrightarrow f(x) = g(z)) \ \& \ (A(x) \rightarrow B(f(x)))]$$

(the universally quantified first conjunct implies that  $f = g!$ ), which in turn can be written by means of the Henkin prefix:

$$\begin{array}{l} \forall x \text{ --- } \exists y \\ \forall z \text{ --- } \exists w \end{array} \left. \vphantom{\begin{array}{l} \forall x \text{ --- } \exists y \\ \forall z \text{ --- } \exists w \end{array}} \right\} [(x = z \leftrightarrow y = w) \ \& \ (A(x) \rightarrow B(y))]$$

<sup>17</sup> Some form of the *Axiom of Choice* (AC) is needed to see that the two descriptions are equivalent. If one knows that for each  $x$  there exists  $y$  such that  $\varphi(x, y)$ , AC entails that it is possible to select exactly one such  $y$  for each  $x$ ; i.e. that there is a function  $f$  such that, for all  $x$ ,  $\varphi(x, f(x))$ .

<sup>18</sup> This is equivalent to saying that the quantifier *MO* from sec. 2.4 is not *FO*-definable. In Ch. 14 we will see how to prove such facts.

So with the Henkin prefix we *can* say that  $|A| \leq |B|$ . Similarly, we can also express the sentence (2.21) from the previous section.

*Second-order logic*, or *SO*, is like *FO* except that in addition it has variables and (universal and existential) quantification over sets, relations, or functions. Note that the second-order sentences involving Skolem functions above, (2.24)–(2.26), (2.28)–(2.29), all have the same simple form: a sequence of existential second-order quantifiers followed by an *FO*-formula. The formulas of this form constitute the  $\Sigma_1^1$  fragment of *SO*.<sup>19</sup>

Above we indicated another set of formulas: those consisting of a quantifier-free *FO*-formula preceded by a partially ordered quantifier prefix, i.e. a prefix consisting of a finite number of expressions of the form  $\forall x$  and  $\exists y$ , all with distinct variables, arranged in some partial order. Let *PO* be this set of formulas. We didn't attempt to give a truth definition for *PO*-formulas, except in the case of the Henkin prefix. However, it was proved by Enderton (1970) and Walkoe (1970) that the set of *PO*-formulas is expressively equivalent to the set of  $\Sigma_1^1$ -formulas. In fact, each partially ordered *PO*-prefix corresponds to a linear prefix consisting of a sequence of existential function quantifiers followed by a sequence of universal quantifiers over individuals, as in the examples above. Now the meaning of  $\Sigma_1^1$ -formulas, or second-order formulas in general, is in principle clear; these formulas have standard linear syntax and universal and existential quantification over individuals as well as over relations and functions. So we have *de facto* an indirect account of the meaning of all *PO*-formulas.

How do *PO*-formulas relate to generalized quantifiers? One can easily see, via the correspondence established by Enderton and Walkoe, that each partially ordered prefix, say with  $n$  variables, corresponds to a generalized quantifier of type  $\langle n \rangle$ . We illustrate with the Henkin prefix. Define a quantifier  $Q^H$  of type  $\langle 4 \rangle$  by

$$(Q^H)_M(R) \iff \text{there are functions } f, g \text{ on } M \text{ s.t. for all } a, b \in M, \\ R(a, f(a), b, g(b))$$

Note that the right-hand side is just the  $\Sigma_1^1$ -prefix employed in (2.26). Clearly (2.26) and (2.27) are both equivalent to the following sentence:

$$Q^H xyzw \varphi(x, y, z, w)$$

### 2.5.1 First-order sentences making second-order claims

In the sense introduced in section 2.1, the sentences in *PO*, like the sentences obtained by adding generalized quantifiers to *FO*, are clearly first-order: they quantify only over individuals. The sentences in  $\Sigma_1^1$  are clearly second-order. So we have a case of first-order sentences making claims that are not expressible in *FO*. This is what

<sup>19</sup> Classification of quantifier prefixes by means of the symbols  $\Sigma$  (standing for  $\exists$ ) and  $\Pi$  (standing for  $\forall$ ) is standard in logic. Usually,  $\Sigma_1$  indicates a sequence of existential quantifiers,  $\Sigma_2$  a sequence of existential quantifiers followed by a sequence of universal quantifiers (and thus two alternations), etc., and similarly for  $\Pi_1$ ,  $\Pi_2$ , etc. Superscripts stand for the type of these quantifiers; thus  $\Sigma_1^0$ ,  $\Sigma_2^0$ , etc. indicate the corresponding prefixes of first-order quantifiers, whereas  $\Sigma_1^1$ ,  $\Sigma_2^1$ , etc. are used for second-order quantifiers.



one would expect from generalized quantifiers; the language stays first-order, but the expressive power may increase. As we have seen, some claims made with generalized quantifiers can also be naturally expressed in *SO*.

The interesting fact about the branching *PO*-sentences is that they do not employ new quantifier symbols, but instead enrich the syntactic means of arranging  $\forall$  and  $\exists$ , which turns out to result in a significant increase of expressive power. At least three questions become relevant in this context.

First, is the non-linear arrangement of quantifiers really necessary? Most logical languages, like those of *FO* and *SO*, employ a linear syntax, and so do natural languages. Especially if you think of *spoken* language, a linear syntax seems much preferable.

It turns out that a linear syntax is also possible. Jaakko Hintikka has introduced a variant of the linear language of *FO* (see Hintikka 1996 and Hintikka and Sandu 1997), where *independencies* are marked explicitly, so that, for example, (2.27) is written

$$(2.30) \quad \forall x \exists y \forall z (\exists w / \forall x) \varphi(x, y, z, w)$$

This is called *IF* (independence-friendly) logic, and is expressively equivalent to the logic with *PO*-formulas. Without going into details, we see that the (new) existential quantifier ( $\exists w / \forall x$ ), although it comes after  $\forall x$  in the sentence, is marked to *not* depend on that universal quantifier, creating the same independence as the non-linear Henkin prefix did.

But a second question now is whether the *IF* language is *compositional*, in the sense that the occurrences of a universal or (modified) existential quantifier can be dealt with one by one, as is the case for *FO*, or for *FO* with added generalized quantifiers.<sup>20</sup> This is by no means obvious. Indeed, Hintikka gave the semantics for *IF* logic in *game-theoretic* terms, where truth in a model becomes the existence of a *winning strategy* in a certain game. On the face of it, that semantics is not compositional. However, Hodges (1997) shows that a Tarski-style truth definition is in fact possible for *IF* logic; see also Väänänen 2002. So you *can* ‘peel off’ *IF* quantifiers one at a time, but the price is that whereas the clauses in the truth definition for *FO* quantify over assignments (of individuals to variables), the *IF* clauses need to quantify over *sets* of assignments. This truth definition is accordingly more complex than the one for *FO*, and we shall not try to reproduce it here.<sup>21</sup>

The third question concerns the closure properties of the language with branching quantifiers. Normally one thinks of a language as *closed* under its logical operations,

<sup>20</sup> For *FO* we saw how this is done in (2.6) and (2.7), sect. 2.2. When a generalized quantifier *Q* is added, we need an additional clause of the form (2.17) in sect. 2.4. So if we treat each quantifier prefix of an *IF*-formula, or the corresponding *PO*-prefix, as a unit, we get a compositional truth definition, since each prefix corresponds to a generalized quantifier. But there are infinitely many prefixes, so we would get a truth definition with infinitely many clauses, which is not very satisfactory.

<sup>21</sup> Indeed, Hintikka asserts that the intended game-theoretic semantics is, by contrast, simple and intuitive. Moreover, he thinks the restrictions on permissible independencies that the *FO* syntax enforces are just *ad hoc*, and that *IF* is the ‘real’ first-order logic. This claim is very controversial, however, partly because of the limitations on the *IF*-language that the next question discusses.

so that if  $\varphi$  and  $\psi$  are formulas, so are, for example,  $\neg\varphi$ ,  $\varphi \wedge \psi$ ,  $\forall x\varphi$ , and  $Qx(\varphi, \psi)$  (if  $Q$  is a type  $\langle 1, 1 \rangle$  quantifier symbol in the language). However, the set of *PO*- or *IF*-formulas are, characteristically, *not closed under negation*. This is most clear if we recall that each such formula is equivalent to a  $\Sigma_1^1$ -formula of second-order logic, but the class of  $\Sigma_1^1$ -formulas is certainly not closed under negation: if you negate such a formula, you turn the existential second-order quantifiers into universal ones—the formula becomes  $\Pi_1^1$ —and there are many things expressible with  $\Pi_1^1$ -formulas that are not expressible with  $\Sigma_1^1$ -formulas (see note 19).

Of course you can ‘close’ the *PO*- or *IF*-formulas under negation, but the above remarks show that you then get a much stronger language. This language fails to have some of the nice properties that have been claimed for *IF* logic.<sup>22</sup>

Summing up, there is no mystery in the fact that you can make second-order claims with a first-order language, as long as one remembers that the first-orderness of the *language* just means that only quantification over individuals is allowed. Generalized quantifiers, as we have defined them in this chapter, yield first-order formulas in this sense, and the same holds for *PO*-formulas. A totally different issue is the *expressive power* of such a language. What we have hinted at here, and will amply demonstrate in Chapters 13 and following, is that these syntactic additions to *FO* may increase expressive power, in fact to what is also expressible by means of the second-order  $\Sigma_1^1$ -sentences. There are various notions of first and second order in use, and it is important to be clear about what the facts are.<sup>23</sup>

We also pointed out that branching quantification is in one sense already subsumed under the notion of generalized quantifiers, since every branching prefix corresponds to a quantifier. But the syntax of branching quantification has a particular elegance and simplicity that gets lost when translated in to the language of generalized quantifiers.

### 2.5.2 Branching generalized quantifiers

One question about branching quantifiers that we have not yet taken up is their connection to natural languages. A main tenet of this book is that (generalized) quantifiers are omnipresent in *natural* languages as well as logic. Is branching quantification also something that natural languages somehow employ?

<sup>22</sup> See e.g. van Benthem and Doets 1983 for examples of things expressible with  $\Pi_1^1$ - but not with  $\Sigma_1^1$ -sentences.

<sup>23</sup> To begin, there is the use of *first-order logic* to mean simply *FO*, and correspondingly for second-order logic (*SO*). There is also the notion of a *first-order theory*, which is a theory formulated with only the logical resources of *FO*, and similarly for second-order theory. Then, there is our use of *first-order language* (and second-order language) in this book. Finally, there are more elusive uses related to the *claims* expressible by a certain language. As we said, “second-order” can indicate that these claims amount to what is expressible in (some significant fragment of) *SO*. But one can instead attach importance to the fact that the *metalanguage* is second-order (in our sense), e.g. that the clauses in an inductive Tarski-style truth definition essentially involve second-order quantification. This is indeed true for the truth definition that Hodges gave for *IF* logic. Likewise, the game-theoretic truth definition for both *FO* and *IF* involve quantification over winning strategies, which again is second-order.

Indeed, Hintikka (1973) famously claimed that the Henkin quantifier is needed for the semantics of English. He gave several examples, the best-known of which is

(2.31) Some relative of each villager and some relative of each townsman hate each other.

This claim engendered much controversy; see, for example, Gabbay and Moravcsik 1974; Guenther and Hoepelman 1974; Fauconnier 1975; Stenius 1976. Barwise (1978) gave a thorough analysis of this issue and concluded (convincingly, in our opinion) that (2.31) does involve branching, though not the Henkin quantifier, but another partially ordered prefix which is in fact definable in *FO*. He also gave other English examples to illustrate essential branching of  $\forall$  and  $\exists$ . We shall not go into that discussion here. However, Barwise also argued that the branching of *most* as in (2.32) below is a much clearer example of essential branching in English.

(2.32) Most boys in your class and most girls in my class have all dated each other.

This can be rendered with a branching structure of the form

$$\begin{array}{l} \text{most } x A(x) \\ \text{most } y B(y) \end{array} \rangle R(x, y)$$

where the intended reading is that there is a subset  $X$  of  $A$  containing most of the elements of  $A$ , and a similarly large subset  $Y$  of  $B$ , such that each pair  $(a, b)$  where  $a \in X$  and  $b \in Y$  belongs to the relation  $R$ . Quite plausibly, this is a reading of (2.32). But it is not expressible by *most* alone (used linearly). In fact, as we will see in Chapter 15.3, it is not expressible by any form of monadic quantification at all.

Again we see that the two quantified noun phrases here are in a sense independent of each other, and this independence is nicely illustrated by the branching prefix. In principle, since we know how to branch  $\forall x$  and  $\exists y$ , one may ask if arbitrary (generalized) quantifiers can also be branched; and if so, whether this takes us outside the domain of the usual languages with linearly ordered generalized quantifiers.

Most boys and most girls are both *increasing* (if most boys are  $B$  and  $B \subseteq B'$  then most boys are  $B'$ ), and simple branching of two increasing quantifiers makes good sense. One can also make clear sense of the decreasing case, and a few others. But Barwise argues that *mixed* cases (branching an increasing and a decreasing quantifier) make no sense; see Barwise 1978 and Westerståhl 1987 for a discussion. Sher (1997) proposes a definition that would work in general, but it is not obvious in what sense such a definition would be correct or not; nor is it clear, in the few cases where we do have some intuitions about the meaning, that her definition always agrees with those intuitions. Thus, the questions of the preceding paragraph are not really well defined in their full generality.

But if we focus on the increasing case, a clear and undisputed truth definition can be given. In Chapter 10.3 we come back to this, and to the issue of whether natural languages in fact employ branching of this kind. It will be seen that what is defined is really a *polyadic* quantifier, in the above case of type  $\langle 1, 1, 2 \rangle$ . Indeed, branching (of

increasing quantifiers) is naturally seen as a *lift* operation, from monadic to polyadic quantifiers. We study various such polyadic lifts in Chapter 10. So these lifts do not take us outside the domain of generalized quantifiers, although in many cases, such as branching, they take us outside what is expressible by monadic quantifiers (see Chapter 15.3).

In sum, in this section, which has been something of an excursion, we have seen what branching (or independence-friendliness) of quantifiers amounts to, and also that, although the syntax is different, it doesn't produce any quantifiers falling outside the concept of a (generalized) quantifier that is the basic theoretical tool of this book.

## 2.6 DIGRESSION: WHY LOGICIANS LIKE *FO*

Given the immense variety of logical quantifiers that strengthen the expressive power of *FO*, and the fact that many of these are employed by natural languages as well (all of these facts will become clear later on), one might well wonder why logicians have spent so much effort on *FO*, which has only  $\forall$  and  $\exists$ . There are at least two kinds of answers to this question.

First, although the logical resources of *FO* may seem poor, they are nevertheless strong enough to permit formulations of in principle all of mathematics in the language of *FO*. In particular, set theories (such as *ZF*; Zermelo–Fraenkel set theory) are usually formulated as *first-order theories*, by which is meant theories using only the logical resources of *FO*. Therefore, in *one* sense, *FO* is already strong enough to do 'everything'. For example, all our generalized quantifiers are definable in set theory, a first-order theory.

This is a very significant property of *FO*. But it should not obscure the fact that the definitions of the notion of infinity, or of various cardinality quantifiers, that can be carried out in *ZF* are not *logical*—they depend on a mathematical theory. For the purposes of this book, it is not very significant that the semantics of quantification in natural languages can be done in set theory. The significant fact is, rather, that natural languages use quantifier expressions 'logically'; see Chapters 9 and 12 for discussion. But, to repeat, from the viewpoint of the mathematical logician, the fact that *FO* can in principle express all of classical mathematics is one of its most important traits.

The second answer is not in terms of expressivity, but in terms of *properties* of *FO*. *FO* has a host of nice properties that many other logics lack. Here are just a few examples:

- The *Löwenheim property*: If a sentence is true in some infinite model, it is also true in some countable model.
- The *Tarski property*: If a sentence is true in some countably infinite model, it is also true in some uncountable model.
- The *compactness property*: If no model makes every element of the set of sentences  $\Phi$  true, then there is a finite subset  $\Psi$  of  $\Phi$  such that no model makes every sentence in  $\Psi$  true.
- The *completeness property*: The set of valid sentences is recursively enumerable.

It is largely because of properties like these that *FO*, and first-order theories, have been amenable to the very rich study undertaken by model theorists in the last century (see any modern textbook in model theory, e.g. Hodges 1993). For logics lacking these properties, there just isn't a comparable number of *results*, and what logicians like most of all is to prove mathematically non-trivial results *about* logics or languages or theories.

Moreover, *FO* is *characterized* by these properties. This is the content of Lindström's Theorem,<sup>24</sup> of which we give some instances below. Most of the following notions will be explained formally in Chapters 11.2.4 and 4.4, but an informal account suffices for now. A *logic*  $L$  consists of a language and a truth relation, relating  $L$ -sentences to (first-order) models (and satisfying certain minimal conditions).  $L'$  *extends*  $L$  if it is at least as expressive as  $L$ , i.e. if every  $L$ -sentence is equivalent to (true in the same models as) an  $L'$ -sentence.  $L \equiv L'$  if the converse holds too.  $L$  *relativizes* if for every  $L$ -sentence  $\varphi$  and every unary predicate symbol  $P$  (not in  $\varphi$ ), you can find another  $L$ -sentence saying about the universe restricted to  $P$  what  $\varphi$  says about the whole universe.<sup>25</sup> Below,  $L$  is any logic extending *FO*.

**Theorem** (Lindström)

*If  $L$  is compact and has the Löwenheim property, then  $L \equiv FO$ . Also, provided  $L$  relativizes, if  $L$  is complete and has the Löwenheim property, or if  $L$  has both the Löwenheim and the Tarski properties, then  $L \equiv FO$ .*

Thus, if logicians want to study logics with these properties, they have no choice but to study *FO*, or something equivalent in expressive power.

## 2.7 SUMMARY

In this and the previous chapter we have sketched the history of the *logical* notion of quantifiers, as second-order relations. The application of this notion to linguistic issues begins in the next chapter, though several hints about how it might go have already been given. Although the notion is discernible already in Aristotle's syllogistics, and although quantification has been a topic of discussion among philosophers since the Middle Ages, the full-fledged concept appeared with Frege, in the context of the invention of modern predicate logic. But since the general concept was not really needed for predicate logic, it appears to have been forgotten, only to be rediscovered by model theorists (Mostowski and Lindström) almost 100 years later. Since then, however, it has been a standard logical tool.

We gave fairly precise definitions and many examples of the model-theoretic concept of a (generalized) quantifier, and of how such quantifiers can be added, syntactically and semantically, to first-order logic (*FO*). But we also took the opportunity

<sup>24</sup> First proved in Lindström 1969. Accessible proofs can now be found in some logic textbooks: e.g. Ebbinghaus *et al.* 1994.

<sup>25</sup> In Ch. 1.3.3 we indicated that *FO* relativizes, and that most natural language quantifiers are relativized (see Ch. 4.4), which entails that the logics you obtain by adding them to *FO* also relativize.

to introduce and discuss some important theoretical and methodological issues. Here is a recapitulation of these.

- We contrasted a *proof-theoretic* with a *model-theoretic* perspective on semantics in general, and quantification in particular. Although one can study proof-theoretic aspects of various quantifiers, the general notion of a quantifier is model-theoretic in nature, or so it seems to us. This model-theoretic notion is point of departure for the present book.
- Another point of departure is that certain expressions, in particular determiners, can be taken to *denote* (generalized) quantifiers. Historically, it has been a vexed question whether these expressions have denotation at all—a question related to, but not identical with, the question of whether they are categorematic or syncategorematic. We emphasized that, given a small number of fixed quantifiers, this is a methodological issue that can be resolved either way. It is perfectly possible to give precise truth conditions for sentences with the given quantifiers without assuming that they denote anything. Indeed, this is standard practice in predicate logic, with its two quantifiers  $\forall$  and  $\exists$ . However, the concept of a (generalized) quantifier provides all the required denotations—and many more—for the expressions in question. Moreover, in (most) natural languages there is an unlimited supply of such expressions. Once these things are realized, a case by case treatment of quantification in natural languages becomes methodologically unsound. This may rightly be called one of Frege's great insights about quantification. As to modern logical languages, a natural (in retrospect) extension of the variable-binding mechanisms of predicate logic allows one to add symbols denoting arbitrary quantifiers to first-order logic.
- We also mentioned the concept of *logicality* as a possible characteristic of quantifier expressions, pointing out, however, that logicality should not be confused with syncategorematicity. We return to the issue of logicality and logical constants in Chapter 9.
- The basic notion of truth in model theory is *truth in a structure*. This means regimenting semantic interpretation in a certain way, which, as model theorists noted, is often congenial for mathematics. But note that without a universe and an interpretation of non-logical symbols, no proposition at all is expressed by a (quantified) sentence. So the model-theoretic notion is not relativistic but really a way to formulate absolute truth. Moreover, this notion is congenial for natural language as well. First, at least for a theory of quantification, it makes sense to treat verbs, nouns, adjectives, etc. as symbols that receive an interpretation—an extension—in a model. Second, and more importantly, restriction to a discourse universe is ubiquitous in natural language communication, and such universes are similar to (though not necessarily identical with) the universes of models. And in addition to restriction to a context-given discourse universe, languages employ other significant means of quantifier domain restriction. In particular, determiners characteristically restrict quantification by means of relativization to the noun argument (Chapter 4.4).

- Even if truth is relativized to models, the quantifiers themselves are absolute objects in that they have, in each universe, a given extension. However, it seems that determiner denotations—in contrast with arbitrary quantifiers—in some sense to be determined have the ‘same’ extension in all universes; i.e. they are ‘independent’ of the universe. This somewhat elusive property of *constancy* will come into focus several times later on (e.g. Chapters 3.4 and 9.3).
- We distinguished *first-order* from *second-order languages* in terms of what sort of variables their sentences quantify over. We also claimed that the first-order approach is the right one to take for a theory of quantification. This is because the meanings of most quantifier expressions are completely independent of what the universe is made up of, and *any* set can be the universe of a first-order model. Therefore, all quantifiers dealt with in this book are first-order in the sense that they quantify over individuals. But this does not prevent them from generating logics that are much stronger than first-order *logic* (*FO*) in terms of expressive power.





PART II  
QUANTIFIERS OF NATURAL  
LANGUAGE



# 3

## Type $\langle 1 \rangle$ Quantifiers of Natural and Logical Languages

This chapter begins our exposé of quantifiers related to natural languages with the logically simplest case: type  $\langle 1 \rangle$ . These quantifiers are, on each universe, sets of subsets of that universe. Here we find, along with a few of the classical logical quantifiers, the denotations of *noun phrases*, a fundamental syntactic category in many—if not in all—languages. Among noun phrases are the *proper names*, which, although they denote individuals, are sometimes conveniently treated as type  $\langle 1 \rangle$  quantifiers as well.

Of course, languages that do not have any other noun phrases than pronouns might still have constituents that can be taken to denote type  $\langle 1 \rangle$  quantifiers. But however a natural language may use type  $\langle 1 \rangle$  quantification, if it does at all, our claim is that the linguistically important type is  $\langle 1, 1 \rangle$ . This is in stark contrast with standard logical languages where the type  $\langle 1 \rangle$  quantifiers  $\forall$  and  $\exists$  are ubiquitous. We start with type  $\langle 1 \rangle$  because it is logically simpler, and because most type  $\langle 1 \rangle$  quantification that you find in natural languages is systematically related to the more fundamental type  $\langle 1, 1 \rangle$  quantification, and vice versa, in ways that are worthwhile to spell out.

We begin (section 3.2) with various examples of phrases denoting type  $\langle 1 \rangle$  quantifiers. Next, we discuss in detail two fundamental properties of quantifiers: isomorphism closure or ISOM (section 3.3), and extension or EXT (section 3.4). It appears that all quantifiers that one encounters in natural languages either are ISOM themselves or are systematically related to ISOM quantifiers, and similarly for EXT. This will become clear as we go on; a precise statement concerning ISOM is given in Chapter 9.2. So these properties are highly relevant from a linguistic perspective, but they are also familiar from mathematical logic and computer science. They apply to quantifiers of any type, and though we begin by stating them for the type  $\langle 1 \rangle$  case, we also give the general definitions here.

Before we start, however, some preliminary terminological and methodological points, also pertaining to quantifiers of any type, will be raised in section 3.1. One of these concerns the fact that whereas a claim made by a logical quantifier only requires a *universe* to be fixed, the interpretation of natural language expressions like the noun phrase *three cats* or the determiner *John's* also needs input from a (background) *model*. It is nevertheless both convenient and useful to interpret the latter quantifier expressions too as denoting unique quantifiers, i.e. as mappings from

universes to second-order relations over those universes. This issue is usually not raised in the literature, but at the end of the chapter (section 3.5) we show that such a uniform treatment is in fact possible (relative to a background model), provided EXT is assumed.

### 3.1 PRELIMINARY CONCEPTS AND DISTINCTIONS

#### 3.1.1 Global and local quantifiers

A quantifier  $Q$  is on each universe  $M$  a second-order relation  $Q_M$  (a relation between relations) over  $M$ . When it is necessary to make the distinction, we call such second-order relations *local* quantifiers, since they are local to a particular universe. The quantifier  $Q$  itself, on the other hand, i.e. the operator which associates with each  $M$  a local quantifier  $Q_M$  on  $M$ , can then be called a *global* quantifier.

A property  $P$  of local quantifiers extends automatically to the corresponding global quantifier according to the following scheme.

(G)  $Q$  is  $P \iff_{\text{def}} \text{for each } M, Q_M \text{ is } P$

Many properties of quantifiers that we will discuss are like this—it is more natural to define them locally, but the definition lifts immediately via (G) to a global one. For example, we call a local type  $\langle 1 \rangle$  quantifier  $Q_M$  *increasing* (Chapter 5), if  $(Q_M(A) \ \& \ A \subseteq A' \subseteq M)$  implies  $Q_M(A')$ . Then,  $Q$  is increasing iff each  $Q_M$  is increasing.

Some properties, however, are genuinely local, in that the corresponding global property is uninteresting or even unintelligible. Here are a few examples.

##### 3.1.1.1 Trivial quantifiers

Call any relation (over some universe) *trivial* if it is either the universal relation or the empty relation. This gives us the following, for local quantifiers of any type  $\tau = \langle n_1, \dots, n_k \rangle$ :

##### the two trivial quantifiers of type $\tau$ on $M$

$\mathbf{1}_{\tau, M}$  is the universal relation of type  $\tau$  over  $M$ . That is, whenever  $R_i$  is an  $n_i$ -ary relation on  $M$ , for  $1 \leq i \leq k$ ,

$$\mathbf{1}_{\tau, M}(R_1, \dots, R_k)$$

holds. Likewise,  $\mathbf{0}_{\tau, M}$  is the empty relation of type  $\tau$  over  $M$ , so  $\mathbf{0}_{\tau, M}(R_1, \dots, R_k)$  never holds. (When there is no risk of confusion, one or both of the subscripts  $\tau$  and  $M$  will be dropped.)

Now, one *could* lift this notion of triviality to  $Q$  itself via (G). Then there would be just two trivial global quantifiers (of any given type), but that would be a rather uninteresting notion of triviality. More interestingly,  $Q$  might associate trivial quantifiers with some universes (say, with all universes of size less than a fixed number), and

non-trivial ones with other universes. More fine-grained notions of (non-)triviality are called for in the global case, and would typically be adapted to a particular class of quantifiers or a particular investigative purpose.<sup>1</sup>

### 3.1.1.2 *The number of quantifiers*

Suppose you ask *how many* quantifiers (of a certain type) with a certain property there are. If you mean global quantifiers, this question risks being senseless. Such a quantifier associates with each set  $M$  a second-order relation over  $M$ , so there would normally be at least as many such quantifiers as there are sets, but there is no such thing as ‘the number of all sets’. The class of all sets is not itself a set—it is what logicians call a *proper class* (see also Chapter 1.3.5)—and hence cannot be counted.<sup>2</sup>

If, on the other hand, you mean local quantifiers on a particular universe, the question has an answer that one might be able to figure out using set theory in the infinite case, or some piece of finite combinatorics in case the universe was finite. For example, suppose your question was: How many distinct type  $\langle 1, 1 \rangle$  quantifiers are there in all on a given universe with just two elements? (The answer is:  $2^{16}$  (= 65,536)!) In general, there are  $2^{(4^n)}$  type  $\langle 1, 1 \rangle$  quantifiers on a universe with  $n$  elements.) How many with property P? Now there are definite answers, though they may be easy or hard to find.

### 3.1.1.3 *Two perspectives on language and logic*

The global/local contrast concerns not just different kinds of properties but different perspectives on quantifiers, and more generally on language and on the use of logic for linguistic purposes. Very roughly, one may say that the more linguistically oriented literature on quantification has tended towards a local perspective, where discussion takes places within a particular universe of discourse which is taken for granted or presupposed, whereas the more logically oriented literature has used a global perspective. But there is really no need to make a choice here. What is needed, rather, is always to be clear about whether a particular claim or generalization or definition or result is to be taken globally or locally.

We prefer global versions whenever possible, not because our orientation is more logical than linguistic, but because natural language quantifiers are themselves global objects, defined for every universe. An exclusively local study of them risks losing sight of important properties.

<sup>1</sup> Examples are the condition VAR from van Benthem 1984 for type  $\langle 1, 1 \rangle$  quantifiers, and what is called  $(m_1, \dots, m_k)$ -non-triviality in Westerståhl 1994 for certain quantifiers of type  $\langle 1, 1, \dots, 1, k \rangle$ . The point of such global notions of non-triviality is to be able to formulate results or generalizations in the most perspicuous and illuminating way, without having to make exceptions or add extra clauses for various less interesting cases.

<sup>2</sup> What if you restrict attention to, say, *finite* universes? This is a step in the right direction, but the class of finite sets is again not a set. To get a meaningful question, you also need to impose the condition ISOM (sect. 3.3). Then, if you fix a cardinal number  $\kappa$  and ask how many *isomorphism types* there are of quantifiers (of the given type) on a universe of cardinality at most  $\kappa$ , you have an (in principle) answerable question.

Another reason is that global versions are more general: local versions follow by instantiation. But in many cases, the relationship between global and local is more complex. It may require work or non-trivial reformulation to extract a global statement from a local fact. And sometimes there may be more than one global version, or, as we have seen, none at all.

One area where this contrast is particularly acute is expressive power. To say that a certain quantifier is expressible or definable by such and such means may be true in a local sense but false globally. For example, existential quantification is definable locally on a fixed finite universe as a long disjunction, as logicians through the ages have realized (see Chapter 1), but it is certainly not definable globally from sentential operators. So for a logician of today, *FO* is (much) more expressive than propositional logic, though for a linguist working in a local finite universe the difference may not matter much.

Thus, broad claims about expressive power can be seriously ambiguous if not carefully stated. And more than just resolving the ambiguity, one needs to reflect carefully on which concept of expressivity is the correct one for the occasion at hand. We will return to these questions in Chapter 11.3, but clarity about the global/local distinction is a theme that runs throughout this book.

#### 3.1.1.4 *A finer distinction*

For some quantifier expressions, a finer-grained version of the global/local distinction can be made. These are expressions like *John's*, *the three cats*, *no \_ except John*, *more male than female*, which contain individual or predicate expressions that normally get interpreted in models, but which can still usefully be regarded as denoting global quantifiers, *relative* to some background model which fixes the interpretation of the individual and predicate expressions. This statement requires some qualification and a small argument; the details will be given in section 3.5.

The point is that even though a model consists of a universe *and* an interpretation function, it is often useful to fix the interpretation function but let the universe vary. The (discourse) universe is more flexible, as it were, than the rest of the interpretation. For example, fixing the interpretation of *cat*, we can think of *the three cats* as denoting a global quantifier of type  $\langle 1 \rangle$ , which enables us to observe its behavior when the universe, but not the interpretation of *cat*, varies. We will see that it has certain interesting properties—properties that could not even be stated if we were to apply a purely local perspective.

Similarly, thinking of *John's* as a determiner, just like *every* or *most*, is quite natural, and makes various semantic properties of determiners applicable. This is standard practice, and we do it all the time in this book. Usually one doesn't reflect on the principled difference between *John's* and, say, *all but three*: namely, that the former but not the latter requires a background model fixing the interpretation of an individual-denoting expression. They are both seen as determiners, denoting type  $\langle 1, 1 \rangle$  quantifiers (Chapter 4). But to see that such talk is indeed permissible, one should check that there is a *unique* global quantifier that *John's* denotes, given

the background model. In section 3.5 we verify that, under a reasonable condition, this is indeed the case.

### 3.1.2 Quantifier expressions versus predicate expressions

In Chapter 0.2.3 we introduced the term *quantifier expression* for any natural language expression that can reasonably be taken to denote a quantifier. Typical examples can be determiners, quantificational adverbs, noun phrases, reciprocals, etc.

We shall also sometimes use the term of art *predicate expression* for expressions that can be taken to denote sets of or relations between individuals.

This terminology is natural for a logician, but somewhat artificial from a linguistic point of view, since the term bundles together expressions of very different kinds. For example, many lexical adjectives, nouns, and transitive as well as intransitive verbs are predicate expressions, as well as many complex phrases headed by words of these categories. We may also include various expressions in natural languages that aren't normally treated as (grammatical) phrases at all, such as *brother of*, *is above*, or *gave a rose to*. (In logical languages, predicate symbols are lexical predicate expressions, and formulas are complex predicate expressions.)

The important thing about what we call predicate expressions is that their extensions contain *individuals*, or tuples of individuals. Thus, although expressions like *every* and *few* denote (on each universe) binary relations, these relations are between *sets* of individuals—they are second-order—so they are not predicate expressions in our sense.

Similarly, though numerals can be treated as adjectives in predicate position, these predicates apply, again, to sets of individuals, not to the individuals themselves:

- (3.1) a. The boys were five.  
b. The boys were happy.

Not each boy was five, but the set of boys. The set of boys wasn't happy, but each boy was.

Recall that by “individual” we do *not* mean a particular kind of object. *Anything* could be an individual, i.e. an element of the universe of a model: people, middle-size objects, electrons, thoughts, propositions, numbers, sets, . . . The point is instead that the universe of a first-order model is *not structured*; it does not besides its individuals contain other objects, such as sets of or relations between individuals.

It is intuitively clear that there are basic semantic differences between quantifier expressions and predicate expressions. These differences will be explicitly formulated and demonstrated in the chapters to come, but the following is a brief summary:

- Quantifier expressions and predicate expressions have different categories or types, and
  - (i) there are very few *logical* predicate expressions—where by “logical” we mean subject-independent or topic-neutral, a requirement expressed formally by the condition of *isomorphism closure*, or ISOM—whereas
  - (ii) almost all natural language quantifiers do satisfy ISOM.

ISOM is introduced in section 3.3 below. Many examples illustrating claims (i) and (ii) will appear in this chapter and the next, whereas the final and precise statements of these claims is given in Chapter 9.2.

- Quantifier expressions, but not predicate expressions, are *constants*, in the sense of belonging to the *form* of intuitively sound inferences. This will be argued in detail in Chapter 9.3, in connection with a discussion of the notion of a *logical constant*.
- Quantifier expressions are largely *rigid*, in contrast with predicate expressions. That is, coextensive predicate expressions are not always substitutable in intensional or modal contexts, whereas co-extensive quantifier expressions usually are. The latter claim requires some qualification with respect to which equivalence relation is claimed to be preserved by such substitution; various such relations are discussed in Chapter 11.3.

These differences motivate a very different model-theoretic treatment of quantifier expressions from that of predicate expressions. Briefly, (lexical) predicate expressions are interpreted in models, but quantifier expressions are not. This is a fundamental methodological principle, usually adhered to in practice but not always understood in theory, or so it seems to us; see Chapter 9.3.

### 3.1.3 Relational and functional views of quantifiers

We have defined quantifiers (locally) as second-order relations, but from a type-theoretic perspective they would be viewed as functions instead. Recall that Frege defined all relations, including quantifiers, in terms of their characteristic functions. So for example, a (local) type  $\langle 1 \rangle$  quantifier  $Q_M$  would be a function from  $P(M)$ , the powerset (set of subsets) of  $M$ , to the set of truth values  $\mathbf{2} = \{0, 1\}$ , and instead of  $Q_M(A)$  one would write

$$Q_M(A) = 1$$

(The global  $Q$  then associates with each universe  $M$  such a function.) The set of all such functions is often written

$$[P(M) \longrightarrow \mathbf{2}]$$

(or  $\mathbf{2}^{P(M)}$ ). Similarly, a type  $\langle 1, 1 \rangle$  quantifier on  $M$  could be seen as a function from  $P(M)$  to  $[P(M) \longrightarrow \mathbf{2}]$ , i.e. as an object in  $[P(M) \longrightarrow [P(M) \longrightarrow \mathbf{2}]]$ .<sup>3</sup>

Mostly, this choice is a practical matter, not a principled one. For quantifiers, there is even a third choice: Lindström (1966) took them to be *classes of models*, so that, for example, one would then write

$$(M, A) \in Q$$

rather than  $Q_M(A)$ . But there is no claim that quantifiers really *are* objects of one of these three kinds; rather, these are three ways of modeling quantifiers. As long as

<sup>3</sup> Equivalently, it could be seen as a function in  $[(P(M) \times P(M)) \longrightarrow \mathbf{2}]$ .



one can go back and forth between these representations, the choice is practical, or sometimes merely a matter of taste.

There is a certain austere elegance to a type-theoretic framework, where everything is a function. Also, linguists have often preferred a functional view, since it allows an intuitively pleasing formulation of compositionality in terms of function application. This too goes back to Frege, and in modern times is especially prominent in the tradition from Montague. Moreover, the use of Boolean functions of various kinds, not just quantifiers, makes possible an elegant and uniform treatment of large tracts of semantics; see Keenan and Faltz 1985. (But compositionality itself in no way depends on function application; see Chapter 12.2.)

One advantage of the functional approach to quantifiers would be if there were cases where not just characteristic functions were involved. This has been suggested for *interrogatives*, when, for example, *which*( $A$ ) is seen as a function whose value is (a representation of) the answer to the question Which  $A$ s are  $B$ ?. The simplest solution would be to put

$$\textit{which}(A)(B) = A \cap B$$

but this runs into certain problems, and there are other alternatives.<sup>4</sup> However, we shall not discuss interrogatives in this book.

Another difference is that the functional view gives the opportunity to treat quantifiers as *partially defined*. This adds to the distinctions available, since, for example,  $Q_M(A, B)$ , seen as a function of  $A$ , can be either true, false, or *undefined*. More generally, this fits with using *three-valued logic*, rather than the classical two-valued one, for natural language semantics. Among the reasons that have been suggested is that one better captures *presupposition* in this way: when a presupposition of a statement is not satisfied, the statement does not receive a truth value.

Barwise and Cooper (1981) make a limited use of partially defined quantifiers (and three-valued logic), allowing type ⟨1, 1⟩ quantifiers, seen, as above, as functions from sets to type ⟨1⟩ quantifiers, to be undefined for certain arguments. For example, a *definite* determiner like *the three* denotes a function that is undefined when there aren't exactly three things in the restriction argument. Our treatment of definites, on the other hand (which otherwise is the same as Barwise and Cooper's; see Chapter 4.6), instead gives the value *false* in this case. It seems to us that not much is gained by allowing partiality here. In fact, the only instance where Barwise and Cooper make a substantial use of partiality, we think, is in their notion of a *positive strong* quantifier, in connection with the issue of which quantifiers are acceptable in so-called existential-there sentences. We discuss this question in detail in Chapter 6.3. The outcome, however, is that this particular use of partiality is dubious.

For these reasons, as well as for reasons of simplicity, we shall stick to the (total) relational perspective on quantifiers in this book.

<sup>4</sup> For example, the account above cannot distinguish between the questions Which men are unmarried? and Which bachelors are unmarried?. For more about questions see e.g. Hamblin 1973; Ginzburg and Sag 2000; Groenendijk and Stokhof 1997.

### 3.2 PHRASES DENOTING TYPE $\langle 1 \rangle$ QUANTIFIERS

Type  $\langle 1 \rangle$  quantifiers are, on each universe, sets of subsets of that universe. The terminology here varies somewhat in the literature; at times they are called “unary” quantifiers (then type  $\langle 1, 1 \rangle$  quantifiers are “binary”). But “unary” can refer to the arity of the arguments instead of the second-order relation, and the word has been used<sup>5</sup> for the case when all of the arguments are one-place, i.e. for what we here call *monadic* quantifiers. In that case type  $\langle 1 \rangle$  has been called “simple unary”. There is also some diversity as regards the type notation.<sup>6</sup> However, we shall stick to the original notation and terminology from Lindström 1966 introduced in Chapter 2.4.

Among the phrases in natural languages that denote type  $\langle 1 \rangle$  quantifiers, the category of *noun phrases* (among which we include proper names) is by far the most studied and best understood. Therefore our examples in this chapter are taken from this category, even if in some languages there are expressions of other categories which can be taken to denote type  $\langle 1 \rangle$  quantifiers as well.

We are not claiming that *all* noun phrases denote type  $\langle 1 \rangle$  quantifiers. Pronouns usually do not denote quantifiers. Even for certain noun phrases formed with a determiner (for example, *this man*, *those dogs*), an analysis by means of type  $\langle 1 \rangle$  quantifiers may not be the best choice. Some words usually thought of as noun phrases may not denote anything at all—for example, *it* in *It is obvious that she is hurt*, or *there* in *There are cats in the garden*. And some noun phrases denote quantifiers of other types, such as *each other*, which denotes a quantifier of type  $\langle 1, 2 \rangle$  (Chapter 10.4).

#### 3.2.1 Examples of noun phrase denotations

Of the type  $\langle 1 \rangle$  quantifiers that logicians care about, and that were defined in Chapter 2.3, such as

$$(3.2) \quad \forall, \exists, \exists_{\geq 5}, \exists_{\leq 7}, \exists_{=3}, Q_{\text{even}}, Q_0, Q^C, Q^R$$

only the following three appear to be denotations of simple noun phrases in natural languages:

$$\forall, \exists, \text{ and } \neg\exists$$

(where  $\neg\exists_M(A) \Leftrightarrow A = \emptyset$ ). These can be taken to be denoted by

everything, something, nothing

respectively (or congeners such as *everybody*, *someone*, or *nobody*). To say *Everybody has been warned* is to say that every individual in the (discourse)

<sup>5</sup> E.g. in Väänänen 1997.

<sup>6</sup> E.g. the notation  $\langle\langle 1, 1 \rangle, 1\rangle$  was used for certain type  $\langle 1, 1, 1 \rangle$  quantifiers in Keenan and Westerstähl 1997 (to emphasize the special role of the first two noun arguments for quantifiers like *more-than*). Väänänen (1997) instead uses  $\langle 1; 3 \rangle$  here.

universe has been warned. That is, it is a claim of the form  $\forall_M(A)$ . Similarly, to say Nothing has been forgotten is to make a claim of the form  $\neg\exists_{M'}(B)$ . Notice that thing here can be taken as a logical noun denoting the universe. And note further that the phrases everybody, nothing, etc. are not tied to a particular universe: they associate a (local) quantifier with each universe in a uniform way, i.e. they denote global type ⟨1⟩ quantifiers.<sup>7</sup>

Alongside the small number of fixed noun phrases just listed, many languages, including English and Swedish, have productive ways of forming a wide range of other NPs by using any of the language's nominal expressions rather than thing and any of its determiners in place of every. This yields a rich class of (quantified) NPs, such as

every cat, some birds, no dog, at least three students, exactly two senators, more than five congressmen, most sailors, less than two-thirds of the voters, an odd number of guests, . . .

A proper analysis of these involves the interpretation of the corresponding determiner expressions, for which type ⟨1, 1⟩ quantifiers are used. We come to this in Chapter 4, but in fact it turns out that in a precise sense type ⟨1⟩ quantifiers are the right place to start. We therefore make the following definition here, which begins to explore the fact that in such noun phrases, quantification is *restricted* to the set provided by the noun.

#### restricted type ⟨1⟩ quantifiers

Let  $Q$  be a type ⟨1⟩ quantifier and  $A$  a fixed set. The quantifier  $Q^{[A]}$  is defined as follows, for any  $M$  and any  $B \subseteq M$ :

$$(3.3) \quad (Q^{[A]})_M(B) \iff Q_A(A \cap B)$$

For example, let some *background model* fix  $[[cat]]$  to  $A = cat$  (the relevant set of cats). Then, relative to any (discourse) universe  $M$ ,  $(\exists_{=3}^{[cat]})_M$  is the set of subsets  $B$  of  $M$  such that  $(\exists_{=3})_{cat}(cat \cap B)$ , i.e. such that  $|cat \cap B| = 3$ . This makes  $\exists_{=3}^{[cat]}$  a suitable interpretation of the noun phrase *exactly three cats* (given the background model). If *hungry* is the set of hungry things in  $M$  (including, say, hungry cats, hungry dogs, hungry people, etc.),  $(\exists_{=3}^{[cat]})_M(hungry)$  holds if and only if sentence (3.4) is true.

(3.4) Exactly three cats are hungry.

In this way, restrictions of the quantifiers exemplified in (3.2) above provide a plethora of type ⟨1⟩ quantifiers that turn up in linguistic contexts.

<sup>7</sup> This is an oversimplification (though not an uncommon one). Everybody usually picks out only the *persons* in the universe. And everything normally quantifies only over *non-animate* things in the universe. It is, however, quite possible to use the phrase *everything* in the universe, which truly seems to denote  $\forall$ . Treating *everything*, *nothing*, etc. in the same way is a regimentation of language, as Quine would have said, but not quite true to how these words are ordinarily used.

If one were only interested in the truth conditions of (3.4), the background model and the model with the (discourse) universe  $M$  would presumably be the same. But for a study of the semantics of quantifier expressions like the three cats *per se*, it is useful to make the distinction. Then one needs to think of the three cats as denoting a global quantifier (relative to the background model). This quantifier, we claim, is  $\exists_{=3}^{[cat]}$ , defined as in (3.3).

$Q^{[A]}$  has the effect of *switching* (discourse) *universes*, from  $M$  to  $A$ . Only when  $A \subseteq M$  is the universe literally *restricted* to  $A$ . But we did not make that part of the definition. In principle, there are several different ways of defining restriction to  $A$ , which are all equivalent when  $A \subseteq M$ —this is presumably why their differences are rarely if ever discussed in the literature. We discuss the options further in Chapter 4.5.5. But the main reason for choosing (3.3) is that this is the *only* definition that, strictly speaking, allows one to treat the three cats as a global quantifier. This will be shown in section 3.5.

Restricted type  $\langle 1 \rangle$  quantifiers interpret an important class of noun phrases, but not all NP interpretations have this form. In section 3.2.4 we will see that *proper names* can be taken to denote type  $\langle 1 \rangle$  quantifiers too. Another case is so-called *bare plurals*, as in

- (3.5) a. Freshmen need to apply for a library card.  
 b. John doesn't like cats.  
 c. Firemen are public servants.

Here a noun in plural form serves as a noun phrase. The exact semantics of bare plurals is complicated,<sup>8</sup> but an obvious first stab at an interpretation, which is literally correct in many cases, is by means of the following quantifier:

#### the quantifier $C^{pl}$

For a given set  $C$ , let  $C^{pl}$  be the type  $\langle 1 \rangle$  quantifier defined, for any  $M$  and any  $B \subseteq M$ , by

$$(3.6) \quad C_M^{pl}(B) \iff \emptyset \neq C \subseteq B$$

In mathematical terms,  $C_M^{pl}$  is the *principal filter over  $M$  generated by  $C$* . For example, (3.5a) is interpreted as saying that the set of freshmen is included in the set of individuals that need to apply for a library card—a good enough interpretation for most purposes.

<sup>8</sup> For one thing, they often serve to make *generic* statements, as in

- (i) Swedes are melancholic.

This is not truly universal, but roughly says that most Swedes are melancholic, or that the 'typical' Swede is. For another thing, bare plurals sometimes have an existential rather than a universal interpretation; compare (3.5c) with

- (ii) Firemen are available.

Note that, on our definition, if  $C_M^{\text{pl}}(B)$  holds for some  $B$ , then  $C \subseteq M$  (since  $B \subseteq M$ ). It follows that  $C_M^{\text{pl}}$  is trivial (in fact equal to  $\mathbf{0}_M$ ) if and only if  $C = \emptyset$  or  $C \not\subseteq M$ . We are defining a global quantifier, so we must stipulate something about the case  $C \not\subseteq M$ , and one natural thing to say is that then  $C_M^{\text{pl}}(B)$  is always false.

An alternative definition could be:  $C_M^{\text{pl}}(B) \Leftrightarrow C \cap M \subseteq B$ . But then  $C_M^{\text{pl}}(B)$  is *true* when  $C$  is disjoint from  $M$ , which seems extremely unintuitive. In fact, we show in section 3.5 that not only is definition (3.6) intuitively more natural; it is the only possible definition, given that we want to treat bare plural NPs like *firemen* as denoting type ⟨1⟩ quantifiers.

Presumably, many linguists would add the condition  $|C| > 1$  to the truth condition in (3.6). It is indeed true that when a noun is used as a bare plural, it needs to be in plural form. But this need not automatically entail that the *denotation* has more than one element. For example, you may say

(3.7) Swedish students in his department like meat balls.

without having any idea of how many Swedish students there actually are in his department, and this statement would hardly be falsified if there turned out to be only one. In any case, if one so wishes, it is easy to add a plural condition to (3.6).

### 3.2.2 Quantifiers living on sets

Barwise and Cooper (1981) used the following suggestive terminology (taken from mathematics) for a characteristic trait of restricted quantifiers:

#### type ⟨1⟩ quantifiers living on a set

Suppose  $Q$  is a type ⟨1⟩ quantifier,  $M$  a universe, and  $A$  any set. Then  $Q_M$  *lives on*  $A$  iff, for all  $B \subseteq M$ ,

$$(3.8) \quad Q_M(B) \iff Q_M(A \cap B)$$

Often one assumes  $A \subseteq M$ , but that is not needed for the definition. Observe that living on a set is a property of a *local* quantifier. It is related to an important global property, called *conservativity*, that will be introduced in the next chapter. If  $Q_M$  lives on  $A$ , then to know for any subset  $B$  of  $M$  whether or not the quantifier holds of it, you need only look at the part of  $B$  which is also a part of  $A$ . The next lemma contains useful information about type ⟨1⟩ quantifiers and the sets they live on.

#### Lemma 1

- (a)  $Q_M$  *always lives on*  $M$ , but need not live on any proper subset of  $M$ .
- (b)  $Q_M$  *lives on*  $\emptyset$  if and only if it is trivial ( $= \mathbf{0}_M$  or  $\mathbf{1}_M$ ).
- (c) If  $Q_M$  *lives on*  $C_1$  and  $C_2$ , it *lives on*  $C_1 \cap C_2$ . Hence, if  $M$  is finite, there is always a smallest set on which  $Q_M$  *lives*. This fails, however, when  $M$  is infinite.

- (d) If  $C_M^{\text{pl}}$  is non-trivial, it lives on  $X$  if and only if  $X$  includes  $C$ . So  $C$  is the smallest set that  $C_M^{\text{pl}}$  lives on.
- (e)  $(Q^{[A]})_M$  lives on  $A$  and its supersets. When  $(Q^{[A]})_M$  is non-trivial,  $A$  is often, but not always, the smallest set on which it lives.

*Proof.*

(a) The first claim is obvious from the definition. To see that the second claim is true, consider any  $M \neq \emptyset$  and any  $A \subset M$ . Let  $B = M - A$ ; so  $B \neq \emptyset$ . Then  $\exists_M(B)$  but  $A \cap B = \emptyset$ , so  $\neg \exists_M(A \cap B)$ . Hence,  $\exists_M$  does not live on any proper subset of  $M$ .

(b)  $Q_M$  lives on  $\emptyset$  iff, for all  $B \subseteq M$ ,  $Q_M(B) \Leftrightarrow Q_M(\emptyset)$ . This means that  $Q_M$  is trivial.

(c) Suppose  $Q_M$  lives on  $C_1$  and  $C_2$ . Then, for any  $B \subseteq M$ ,  $Q_M(B)$  iff  $Q_M(C_1 \cap B)$  iff  $Q_M((C_1 \cap B) \cap C_2)$  iff  $Q_M((C_1 \cap C_2) \cap B)$ . Thus  $Q_M$  lives on  $C_1 \cap C_2$ . If  $M$  is finite, it follows that  $Q_M$  lives on

$$\bigcap_{C \subseteq M} \{C : Q_M \text{ lives on } C\}$$

But when  $M$  is infinite, it can happen that this intersection is empty, even if  $Q_M$  is not trivial; this will follow from the proof of (e) below.

(d) Assume  $C_M^{\text{pl}}$  is non-trivial, i.e.  $\emptyset \neq C \subseteq M$ .  $C_M^{\text{pl}}$  lives on  $X$  iff, for all  $B \subseteq M$ ,  $C \subseteq B \Leftrightarrow C \subseteq X \cap B$ . It is straightforward to see that, under the assumption, the latter condition is equivalent to  $C \subseteq X$ .

(e) Suppose  $A \subseteq C$ , and  $B \subseteq M$ . Then, by definitions (3.3) and (3.8),  $(Q^{[A]})_M(B)$  iff  $Q_A(A \cap B)$  iff  $Q_A(A \cap (B \cap C))$  iff  $(Q^{[A]})_M(B \cap C)$ , so  $(Q^{[A]})_M$  lives on  $C$ .

Next, consider the quantifier  $Q = \exists_{=3}$ . We have, for  $B \subseteq M$ ,  $(Q^{[A]})_M(B)$  iff  $|A \cap B| = 3$ . Suppose  $(Q^{[A]})_M$  is non-trivial. Then  $|A \cap M| \geq 3$ . (Otherwise, if  $B \subseteq M$ ,  $|A \cap B| < 3$ , so  $(Q^{[A]})_M(B)$  is false.) Now suppose that  $C$  is a set that  $(Q^{[A]})_M$  lives on, and suppose further that there is an element  $a$  in  $A - C$ . Take any set  $B \subseteq M$  such that  $a \in B$  and  $|A \cap B| = 3$ ; this is possible by assumption. But then  $|(A \cap B) \cap C| < 3$ , which contradicts the assumption that  $(Q^{[A]})_M$  lives on  $C$ . Thus, we have shown that if  $(Q^{[A]})_M$  is non-trivial and lives on  $C$ , then  $A \subseteq C$ . It follows that  $A$  is the smallest set on which  $(Q^{[A]})_M$  lives.

By similar arguments, the same conclusion holds for  $Q = \exists_{=n}$ ,  $Q = \exists_{\geq n}$ , and  $Q = \exists_{\leq n}$ , for example. However, consider  $Q = Q_0$ , so that  $(Q^{[A]})_M(B)$  iff  $A \cap B$  is infinite. Suppose  $A$  is infinite. Then it is clear that if  $(Q^{[A]})_M$  lives on  $C$  and  $a \in C$ ,  $(Q^{[A]})_M$  also lives on  $C - \{a\}$ . Thus, there is no smallest set on which  $(Q^{[A]})_M$  lives.  $\square$

The fact that  $(Q^{[A]})_M$  lives on  $A$  is precisely why quantifiers of the form  $Q^{[A]}$  are the correct interpretations for noun phrases of the form [Det N], as we will see in the next chapter. It is possible to be more specific about when  $A$  is the smallest set on which

$(Q^{[A]})_M$  lives, but for that we need the properties ISOM and EXT, defined later in this chapter. Using them, we shall improve on Lemma 1(e) (in section 3.4.1.2, Proposition 13).

In general, the fact that  $Q_M$  lives on a certain set tells us nothing about which sets  $Q_{M'}$  lives on for  $M \neq M'$ . But most quantifiers that turn up in logical or linguistic contexts are constant over universes in ways that make the live-on behavior more regular. Lemma 1 (d) and (e) provide examples. We return to this issue on several occasions later on, e.g. in Chapters 3.4.1.2, 4.6, and 8.13.

### 3.2.3 Boolean operations on quantifiers

One way of forming complex noun phrases is by means of Boolean expressions (or expressions that have a Boolean ‘effect’):

not all residents, some but not all primes, not more than four departments, more than three but fewer than eight books, between one hundred and two hundred students, between one-third and two-thirds of the applicants . . .

On the semantic side, we can define Boolean combinations of quantifiers in the obvious way. In fact, this works for quantifiers of any fixed type  $\tau = \langle n_1, \dots, n_k \rangle$ :

$$\begin{aligned} (Q \wedge Q')_M(R_1, \dots, R_k) &\iff Q_M(R_1, \dots, R_k) \text{ and } Q'_M(R_1, \dots, R_k) \\ (Q \vee Q')_M(R_1, \dots, R_k) &\iff Q_M(R_1, \dots, R_k) \text{ or } Q'_M(R_1, \dots, R_k) \\ (\neg Q)_M(R_1, \dots, R_k) &\iff \text{not } Q_M(R_1, \dots, R_k) \end{aligned}$$

Note that these definitions work globally as well as locally.

So, on the one hand, there are the syntactic Boolean operations on noun phrases, and on the other, the semantic Boolean operations on their denotations, i.e. type ⟨1⟩ quantifiers. In both cases we can ask if the relevant class of objects is *closed* under these operations. In the first case we ask if the noun phrases of a language allow arbitrary combinations with *and*, *or*, and *not*. In the second case we ask if the class of noun phrase denotations—of a given language, or of all languages—is such that if  $Q_1$  and  $Q_2$  belong to it, so do  $Q_1 \wedge Q_2$  and  $\neg Q_1$ . (Note that this question has a global and a local version.)

These questions are different, although a positive answer to the first would imply a positive answer to the second. But it is generally thought that the first question has a negative answer. Some combinations, like *not most students*, are simply not noun phrases. But this in no way prevents the second question from being answered in the affirmative. For example, if *most students* denotes  $(Q^R)^{[student]}$ , which on finite universes means ‘more than half of the students’, then although  $\neg(Q^R)^{[student]}$  is not denoted by *not most students*, it *is* denoted by the noun phrase *at most half of the students*. Still more obviously, *not not every professor* is not a phrase, even though the class of noun phrase denotations is trivially closed under double negation. But even if these were not counterexamples, it is not clear that the answer to the second question is really affirmative. It is an interesting issue, and we will come back to it in Chapter 4.3.

The negation operation above is called *outer negation* or *complement*. For type  $\langle 1 \rangle$  quantifiers there is another important negation operation, *inner negation* or *post-complement*. The inner negation of  $Q$  is written  $Q\neg$  and defined below. In terms of these operations, we also get the *dual* of  $Q$ , written  $Q^d$ .

### negative operations on quantifiers

For  $Q$  of type  $\langle 1 \rangle$ ,  $M$  a universe, and  $A \subseteq M$ , define

$$\begin{aligned} (\neg Q)_M(A) &\iff \text{not } Q_M(A) \text{ (as before)} \\ (Q\neg)_M(A) &\iff Q_M(M - A) \\ (Q^d)_M(A) &\iff (\neg(Q\neg))_M(A) \iff ((\neg Q)\neg)_M(A) \end{aligned}$$

It does not matter where we put the parentheses in the definition of the dual, so we can write simply  $Q^d = \neg Q\neg$ .

This definition works for  $Q$  of type  $\langle n \rangle$  as well. For types with more than one argument, one must select the argument(s) to which the complement operation should apply. A main example comes from the (modern version of the) Aristotelian square of opposition, discussed in Chapter 1.1.1. We return to this in the next chapter.

For inner negation (and dual) there are again the issues of closure. These operations are indeed present in natural languages. For example, the inner negation of *every professor* is *no professor*, and the dual is *some professor*. The inner negation of *all but at most two deans* is *at most two deans*, and the dual is *more than two deans*. And the inner negation of *most students* is *fewer than half of the students*, and the dual is *at least half of the students*.

The following terminology will prove useful on several occasions later on:

### co-properties

Let  $\mathbf{P}$  be a property of quantifiers of a type  $\tau$  for which the notion of inner negation is well-defined. Then we say that

$$(3.9) \quad Q \text{ is co-}\mathbf{P} \text{ iff } Q\neg \text{ is } \mathbf{P}$$

For example, if  $Q$  of type  $\langle 1 \rangle$  is *increasing* whenever  $Q_M(A)$  and  $A \subseteq A' \subseteq M$  implies  $Q_M(A')$ , then it is clear that  $Q$  is *decreasing* (in the obvious sense) if and only if  $Q\neg$  is increasing, so decreasing = co-increasing. In this case the terms “increasing” and “decreasing” are well established. Examples where there is no established name for the co-property will follow later (a first example is the property co-EXT in section 3.4 below).

Inner negation is a kind of Boolean operation too. Taking the complement of a set (or a set of  $n$ -tuples) is the same kind of operation as negating a sentence—they are both the complement operation in the corresponding Boolean algebra. So it is to be expected that Boolean-type laws hold for inner negations and duals as well: for example, the following ones.



**Fact 2**

- (a)  $\neg(Q_1 \wedge Q_2) = \neg Q_1 \vee \neg Q_2$ ;  $(Q_1 \wedge Q_2)\neg = Q_1\neg \wedge Q_2\neg$ ;  
 $(Q_1 \wedge Q_2)^d = Q_1^d \vee Q_2^d$
- (b)  $\neg(Q_1 \vee Q_2) = \neg Q_1 \wedge \neg Q_2$ ;  $(Q_1 \vee Q_2)\neg = Q_1\neg \vee Q_2\neg$ ;  
 $(Q_1 \vee Q_2)^d = Q_1^d \wedge Q_2^d$
- (c)  $Q = \neg\neg Q = Q\neg\neg = (Q^d)^d$

*Proof.* The first claim of (a) is just propositional logic. The second can be directly verified using only the definitions of inner negation and conjunction of quantifiers. The third follows from the definition of duals, using the first two claims. (b) and (c) are similar.  $\square$

Also, one checks that the restriction operation commutes with the outer Boolean operations:

**Fact 3**

$$(Q_1 \wedge Q_2)^{[A]} = Q_1^{[A]} \wedge Q_2^{[A]}, (Q_1 \vee Q_2)^{[A]} = Q_1^{[A]} \vee Q_2^{[A]},$$

and  $(\neg Q)^{[A]} = \neg(Q^{[A]})$

But not with inner negation:  $(Q\neg)^{[A]}(B)$  is equivalent to  $Q_A(A - B)$ , but  $(Q^{[A]}\neg)_M(B)$  amounts to  $Q_A(A \cap (M - B))$ , which is not the same thing, when  $A$  is not a subset of  $M$ .

In general, there are *inner* and *outer* versions of all Boolean operations for quantifiers. If we talk about Boolean operations without specification, we mean the outer ones.

These operations of course work for any quantifiers of the relevant types, regardless of the eventual connections to natural languages. Here are some more examples:

$$\begin{aligned} \forall^d &= \exists \\ (\exists_{=3}\neg)_M(A) &\Leftrightarrow |M - A| = 3 \\ (\exists_{\geq 2} \wedge \exists_{\leq 7})_M(A) &\Leftrightarrow 2 \leq |A| \leq 7 \\ (Q^R)_M^d(A) &\Leftrightarrow |M - A| \not\asymp |A| \Leftrightarrow |A| \geq 1/2 \cdot |M| \text{ (when } M \text{ is finite);} \end{aligned}$$

In general,

$$(3.10) \quad (p/q)^d = [(q - p)/q] \text{ on finite universes (see Chapter 2.3 for definitions)}$$

### 3.2.4 Montagovian individuals

Proper names constitute a particular kind of noun phrase. In the treatment of proper names introduced in Montague 1970b, a noun phrase like  $\text{John}$  referring to an individual  $j$  is interpreted as the set of subsets containing  $j$ . Thus, in a sense we have two interpretations of proper names. To separate these, one may think of the *lexical item*  $\text{John}$  as denoting the individual  $j$  (in a model  $\mathcal{M}_0$ ; i.e.  $[[\text{John}]]_{\mathcal{M}_0} = \text{John}^{\mathcal{M}_0} = j$ ), whereas the *noun phrase*  $\text{John}$  denotes in each universe  $M$  a set of subsets of  $M$ .

**Montagovian individuals**

Define, for each individual  $j$ ,

$$(3.11) (I_j)_M = \{A \subseteq M : j \in A\}$$

The motive is to obtain a uniform treatment of all noun phrases as (locally) sets of subsets of the universe. Since proper names can be conjoined with quantified noun phrases into more complex noun phrases, as in JOHN and MOST STUDENTS, the uniformity resulting from a Montagovian treatment of names is substantial.

$I_j$  is a global quantifier. With every universe  $M$  it associates a local quantifier, given by (3.11). The definition does not presuppose that  $j \in M$ : if  $j \notin M$ , it follows from the definition that  $(I_j)_M$  is the trivial quantifier  $\mathbf{0}_M$  on  $M$ . What about the alternative of defining instead  $(I_j)_M(A) \Leftrightarrow \{j\} \cap M \subseteq A$ ? Then  $(I_j)_M(A)$  would be *true* when  $j \notin M$ , which seems much more unintuitive. And in fact, just as for bare plurals  $C^{\text{pl}}$  and restricted quantifiers of the form  $Q^{[A]}$ , the chosen definition is the *only one* that properly lets us treat the noun phrase JOHN as denoting a global quantifier, given a background model  $\mathcal{M}_0$  that fixes  $j$ ; see section 3.5.<sup>9</sup>

Quantifiers of the form  $I_j$  are independent of the universe in a strong sense. If JOHN is a professor is true (false), it remains so if the universe of discourse is extended, provided the set of professors is the same. The same holds for SOMEbody is a professor, but not, for example, for EVERYbody is a professor. Universe independence in this sense is a property of (some) global quantifiers. It will be made precise and discussed in section 3.4.

In a context where it is clear that JOHN denotes  $j$ , we may write  $I_j$  simply as *John*. And we may go on:

$$(3.12) \text{John and Mary} = I_j \wedge I_m$$

$$(3.13) \text{John or Mary} = I_j \vee I_m$$

Then a set  $A \subseteq M$  belongs to  $(\text{John and Mary})_M$  (to  $(\text{John or Mary})_M$ ) iff  $j \in A$  and  $m \in A$  ( $j \in A$  or  $m \in A$ ), so using these quantifiers will give correct truth conditions for both

$$(3.14) \text{John and Mary used to like pizza.}$$

and

$$(3.15) \text{John or Mary used to like pizza.}^{10}$$

<sup>9</sup> Our truth conditions, however, should not be confused with those for sentences containing *fictional* names, about which we say nothing in this book.

<sup>10</sup> In contrast with OR, there are other uses of *and* than the one illustrated here, which does not work for

(i) John and Mary met at the party.

for example. Met at the party is a collective predicate, and can be used to form expressions denoting ‘collections’. These matters are not treated in this book.

Finally, we note that it follows directly from our definitions that

$$(3.16) \quad I_j \wedge I_m \wedge \dots \wedge I_b = \{j, m, \dots, b\}^{\text{pl}}$$

This seems all right. While it doesn't make much sense to think of a proper name as a bare plural (it is rather a technical fact about the *denotations* of names that  $I_j = \{j\}^{\text{pl}}$ ), it could make some sense to treat the expression John and Mary and ... and Henry as a bare plural generated by those names: although grammatically different, they do similar semantic work.

### 3.3 ISOMORPHISM CLOSURE

Mostowski imposed a condition on type ⟨1⟩ quantifiers: namely, that they “should not allow us to distinguish between elements [of  $M$ ]” (Mostowski 1957: 13). Lindström used the same condition for the general case, and it is usually taken for granted by logicians. The condition, which we call *isomorphism closure*, or simply ISOM, expresses the idea that in logic only structure counts, not individual objects, sets, or relations. This means that if a sentence in a logical language is true in one model, it is true in all isomorphic models. Put differently, logic is *topic-neutral*. We discuss topic neutrality further in Chapter 9.1.

What is interesting in the present context is that although natural languages are anything but topic-neutral, their use of quantification is essentially based on ISOM quantifiers. This will become clear as we go on, but it motivates a proper presentation of the notion of isomorphism closure at this point. We first give a definition for type ⟨1⟩ quantifiers and then present the general case. Notably, for type ⟨1⟩ quantifiers (and more generally, for monadic quantifiers) there is no need to mention isomorphic models; it is enough to talk about the *sizes* of certain sets.

#### 3.3.1 ISOM for type ⟨1⟩ quantifiers

If  $Q$  is of type ⟨1⟩, whether  $Q_M(A)$  holds or not depends on only two sets:  $A$  and  $M - A$ . ISOM says that it depends only on the *sizes* of these sets:

##### ISOM for type ⟨1⟩ quantifiers

A type ⟨1⟩ quantifier  $Q$  satisfies ISOM iff for any universes  $M, M'$ , and any  $A \subseteq M$ ,  $A' \subseteq M'$ :

$$(3.17) \quad \text{If } |A| = |A'| \text{ and } |M - A| = |M' - A'|, \text{ then } Q_M(A) \Leftrightarrow Q_{M'}(A').$$

For example,  $\exists_M(A)$  holds iff  $|A| > 0$ ,  $\forall_M(A)$  holds iff  $|M - A| = 0$ ,  $(Q^R)_M(A)$  holds iff  $|A| > |M - A|$ ; these are all ISOM.

Clearly, an ISOM quantifier  $Q$  can be identified with a *binary relation between cardinal numbers*: namely, those pairs  $(|M - A|, |A|)$  for which  $Q_M(A)$  holds. For simplicity, this relation is also called  $Q$ :

**ISOM type ⟨1⟩ quantifiers as binary relations**

If  $Q$  is an ISOM type ⟨1⟩ quantifier, define, for any cardinal numbers  $k, m$ :

$$(3.18) \quad Q(k, m) \iff \text{there are } M \text{ and } A \subseteq M \text{ such that } |M - A| = k, \\ |A| = m, \text{ and } Q_M(A)$$

Note that  $k, m$  here can be natural numbers as well as infinite cardinal numbers.<sup>11</sup> Conversely, given any binary relation between cardinal numbers, there is a unique corresponding type ⟨1⟩ quantifier:

**Fact 4**

If  $R$  is any binary relation between cardinal numbers, and we define the type ⟨1⟩ quantifier  $Q$  by

$$Q_M(A) \iff R(|M - A|, |A|)$$

then  $Q$  is ISOM, and the binary relation corresponding to  $Q$  according to (3.18) is  $R$ . Thus ISOM type ⟨1⟩ quantifiers can simply be identified with binary relations between cardinal numbers, and we have, for each such quantifier  $Q$ :

$$(3.19) \quad Q_M(A) \iff Q(|M - A|, |A|)$$

*Proof.* If  $Q$  is defined from  $R$  in this way, it clearly satisfies (3.17), and so is ISOM. Now suppose  $Q$  is also the corresponding binary relation given by (3.18). Let  $k, m$  be cardinal numbers. If  $Q(k, m)$ , there are  $M$  and  $A \subseteq M$  such that  $|M - A| = k$ ,  $|A| = m$ , and  $Q_M(A)$ , and hence  $R(k, m)$  holds. Conversely, if  $R(k, m)$ , choose  $M'$  and  $A' \subseteq M'$  such that  $|M' - A'| = k$  and  $|A'| = m$ . Then  $Q_{M'}(A')$  holds, so, by (3.18),  $Q(k, m)$ . Thus,  $R = Q$ .  $\square$

Thus, ISOM type ⟨1⟩ quantifiers only care about *quantities*, or sizes of sets, not the sets themselves. The condition can be extended to all monadic quantifiers; it is sometimes called Q<sub>UANT</sub> in the literature.

All the quantifiers mentioned in (3.2) satisfy ISOM, as we can see by identifying the corresponding relations between numbers:

$$\begin{aligned} \forall(k, m) &\iff k = 0 \\ \exists(k, m) &\iff m \neq 0 \\ \exists_{\geq 5}(k, m) &\iff m \geq 5 \\ Q_{\text{even}}(k, m) &\iff m \text{ is an even natural number} \\ Q_0(k, m) &\iff m \text{ is infinite} \\ Q^C(k, m) &\iff m = k + m \\ Q^R(k, m) &\iff k < m \\ [p/q](k, m) &\iff qm \geq p(m + k) \end{aligned}$$

<sup>11</sup> Why is  $|M - A|$  the first argument and  $|A|$  the second, and not the other way around? No reason; we are just following a usage in the literature.

The following fact is easily established.

**Fact 5**

The class of ISOM type ⟨1⟩ quantifiers is closed under Boolean operations, including inner negations and duals. In particular, using the terminology introduced in section 3.2.3,  $\text{co-ISOM} = \text{ISOM}$ . Moreover, viewed as relations between numbers, the inner negation of a quantifier is simply its converse, i.e.

$$Q\neg(k, m) \iff Q(m, k)$$

However, most type ⟨1⟩ quantifiers denoted by expressions in natural languages are *not* ISOM. As we will see, they are in a sense reducible to ISOM quantifiers, but they are not ISOM themselves. For example, if a quantifier involves a particular individual or a particular set, ISOM usually fails. In particular, we have the following.

**Fact 6**

Quantifiers of the form  $I_a$  are never ISOM. Similarly for quantifiers of the form  $C^{\text{Pl}}$  when  $C \neq \emptyset$ .

*Proof.* The first statement is obvious, but let us check with the definition. Take two distinct objects  $a, b$ , and a universe  $M_0$  which contains them both. Then  $|\{a\}| = |\{b\}| = 1$ , and  $|M_0 - \{a\}| = |M_0 - \{b\}| = |M_0| - 1$ . On the other hand,  $(I_a)_{M_0}(\{a\})$  holds, since  $a \in \{a\}$ , but  $(I_a)_{M_0}(\{b\})$  fails, since  $a \notin \{b\}$ . Thus, we have a counter-instance to (3.17) (with  $M = M' = M_0$ ,  $A = \{a\}$ , and  $A' = \{b\}$ ), so ISOM fails. The second claim follows similarly; see (3.16).  $\square$

Likewise, restricted quantifiers of the form  $Q^{[A]}$  are usually not ISOM, even if  $Q$  is ISOM. Intuitively, this is again clear. Even if it is a fact that

At least four students smoke.

and also that the number of smokers (in a given finite universe of discourse) is the same as the number of drinkers, it certainly does not follow that

At least four students drink.

But it *would* follow, if the quantifier  $\exists_{\geq 4}^{\text{[student]}}$  were ISOM. The next proposition extracts the idea of this argument.

**Proposition 7**

If  $A \neq \emptyset$  and  $Q_A$  is not trivial, then  $Q^{[A]}$  is not ISOM.

*Proof.* Since  $Q_A$  is not trivial, there are subsets  $C_1, C_2$  of  $A$  such that  $Q_A(C_1)$  but not  $Q_A(C_2)$ . We now claim that by possibly adding some new elements to  $A$ , we can form a universe  $M'$  and subsets  $B_1, B_2$  of  $M'$  such that

$$\begin{aligned} |B_1| &= |B_2| \text{ and } |M' - B_1| = |M' - B_2| \\ A \cap B_1 &= C_1 \text{ and } A \cap B_2 = C_2 \end{aligned}$$

Thus,  $Q_A(A \cap B_1)$  and  $\neg Q_A(A \cap B_2)$ , i.e.  $(Q^{[A]})_{M'}(B_1)$  and  $\neg(Q^{[A]})_{M'}(B_2)$ , which contradicts ISOM for  $Q^{[A]}$ .

To verify the claim, suppose  $|C_1 - C_2| = k_1$ ,  $|C_2 - C_1| = k_2$ ,  $|C_1 \cap C_2| = n$ , and  $|A - (C_1 \cup C_2)| = m$ . Furthermore, suppose that  $k_1 \leq k_2$  (the case when  $k_2 \leq k_1$  is similar). We then add a set  $D$  of sufficiently many new elements to  $A$ , so that  $B_1 = C_1 \cup D$  has exactly  $k_2 + n$  elements. (If  $k_2$  is a finite number, then  $|D| = k_2 - k_1$ ; otherwise  $|D| = k_2$ .) Also, we let  $B_2 = C_2$  and  $M' = A \cup D$ . Now  $|B_1| = k_1 + n + (k_2 - k_1) = k_2 + n = |B_2|$ , and  $|M' - B_2| = k_1 + m + (k_2 - k_1) = k_2 + m = |M' - B_1|$ . (These calculations hold also when  $k_2$  is infinite and  $k_2 - k_1$  is replaced by  $k_2$ .) The claim is proved.  $\square$

We may conclude that, with the exception of phrases like *everything* and *nothing*, noun phrase denotations of the form  $Q^{[A]}$  are usually not ISOM. However, the quantifiers  $Q$ , from which these noun phrase denotations are formed by restriction, usually *are* ISOM. In Chapter 4 we will see that a type  $\langle 1, 1 \rangle$  quantifier  $Q^{\text{rel}}$ , the relativization of  $Q$ , is the proper interpretation of the corresponding determiner phrase, and  $Q^{\text{rel}}$  is ISOM if  $Q$  is.

### 3.3.2 ISOM for arbitrary quantifiers

To formulate ISOM for quantifiers of arbitrary type, one needs the notion of isomorphism between structures or models.

A *structure* in general consists of a universe  $M$ , some relations over  $M$ , some distinguished elements of  $M$ , and some operations on  $M$ . Thus it is essentially what we in Chapter 2.2 called a *model* of a first-order language (with corresponding non-logical predicate symbols, individual constants, and function symbols). Now consider *relational* structures, i.e. structures which have only relations. They can be classified (typed) in the same way as quantifiers.

#### isomorphic structures

A (relational) structure of type  $\tau = \langle n_1, \dots, n_k \rangle$  has the form  $\mathcal{M} = (M, R_1, \dots, R_k)$ , where each  $R_i$  is a  $n_i$ -ary relation over  $M$ . Two such structures  $\mathcal{M}$  and  $\mathcal{M}' = (M', R'_1, \dots, R'_k)$  are *isomorphic*, in symbols,

$$\mathcal{M} \cong \mathcal{M}'$$

iff there is a 1-1 function  $f$  from  $M$  onto  $M'$  (a *bijection* from  $M$  to  $M'$ ) such that, for each  $i$  between 1 and  $k$ , and for all  $a_1, \dots, a_{n_i} \in M$ ,

$$(3.20) \quad R_i(a_1, \dots, a_{n_i}) \iff R'_i(f(a_1), \dots, f(a_{n_i}))$$

$f$  is then called an *isomorphism* from  $\mathcal{M}$  to  $\mathcal{M}'$ .

The connection with our first version of ISOM is the following.

#### Fact 8

If  $\mathcal{M} = (M, A)$  and  $\mathcal{M}' = (M', A')$  are type  $\langle 1 \rangle$  structures, then

$$(3.21) \quad \mathcal{M} \cong \mathcal{M}' \iff |A| = |A'| \text{ and } |M - A| = |M' - A'|$$

Thus,  $\text{Isom}$  for a type  $\langle 1 \rangle$  quantifier  $Q$  amounts to the condition that whenever  $\mathcal{M} \cong \mathcal{M}'$ ,  $Q_{\mathcal{M}}(A) \Leftrightarrow Q_{\mathcal{M}'}(A')$ .

*Proof.* By definition, two sets have the same cardinality iff there is a bijection between them. Now if  $\mathcal{M} \cong \mathcal{M}'$  via a bijection  $f$ , then  $f$  restricted to  $A$  is a bijection from  $A$  to  $A'$ , and  $f$  restricted to  $M - A$  is a bijection from  $M - A$  to  $M' - A'$ . To see this, note that we already know that  $f$  is 1-1, and that  $a \in A$  iff  $f(a) \in A'$  by (3.20). Also, if  $b \in A'$  ( $b \in M' - A'$ ),  $b = f(a)$  for some unique  $a$ , and, again by (3.20),  $a \in A$  ( $a \in M - A$ ). So it follows that  $|A| = |A'|$  and  $|M - A| = |M' - A'|$ .

Conversely, if there is a bijection  $g$  from  $A$  to  $A'$ , and a bijection  $h$  from  $M - A$  to  $M' - A'$ , then the union  $f = g \cup h$  of the two (viewed as sets of ordered pairs) is a bijection from  $M$  to  $M'$  such that for all  $a \in M$ ,  $a \in A$  iff  $f(a) \in A'$ . Thus,  $\mathcal{M} \cong \mathcal{M}'$ .  $\square$

It is now clear what  $\text{Isom}$  should amount to for an arbitrary quantifier.

### Isom for arbitrary quantifiers

A quantifier  $Q$  of type  $\tau = \langle n_1, \dots, n_k \rangle$  satisfies  $\text{Isom}$  iff whenever  $(M, R_1, \dots, R_k)$  and  $(M', R'_1, \dots, R'_k)$  are isomorphic structures of type  $\tau$ ,

$$Q_{\mathcal{M}}(R_1, \dots, R_k) \iff Q_{\mathcal{M}'}(R'_1, \dots, R'_k)$$

Or, to put it slightly differently,  $Q$  is  $\text{Isom}$  iff for every structure  $(M, R_1, \dots, R_k)$  and every 1-1 function  $f$  with domain  $M$ ,

$$(3.22) \quad Q_{\mathcal{M}}(R_1, \dots, R_k) \iff Q_{f(\mathcal{M})}(f(R_1), \dots, f(R_k))$$

where  $f(R_i) = \{(f(a_1), \dots, f(a_{n_i})) : (a_1, \dots, a_{n_i}) \in R_i\}$ .

In other words,  $\text{Isom}$  quantifiers cannot distinguish between isomorphic structures. It should be clear that this is a precise way of expressing Mostowski's idea that they do not allow us to distinguish between elements of the universe, i.e. the idea of topic neutrality. Furthermore, we may also observe that if one defines quantifiers as Lindström (1966) did, i.e. as classes of structures of the same type, so that instead of writing

$$Q_{\mathcal{M}}(R_1, \dots, R_k)$$

one writes

$$(M, R_1, \dots, R_k) \in Q$$

then  $\text{Isom}$  is literally closure under isomorphism: if  $\mathcal{M} \in Q$  and  $\mathcal{M} \cong \mathcal{M}'$ , then  $\mathcal{M}' \in Q$ .

There is a slightly different way of making Mostowski's idea precise. With  $\text{Isom}$  one cannot even distinguish between elements of different universes, if the shared structure is the same. Another version restricts attention to elements of one universe at a time.

- An *automorphism on  $\mathcal{M}$* , where  $\mathcal{M}$  is of type  $\tau$ , is an isomorphism from  $\mathcal{M}$  to itself. The corresponding condition on a type  $\tau$  quantifier  $Q$  of being closed under automorphisms is called PERM, or *permutation closure*,<sup>12</sup> since a bijection from a set  $M$  to itself is often called a permutation on  $M$ . ISOM implies PERM, but not vice versa.
- Note that on a given universe  $M$ , permutation closure can be stated as a condition on  $Q_M$ . If we call this local version PERM<sub>M</sub>, then PERM is the corresponding global version (see (G) in section 3.1.1). ISOM has no such local version.

To see that PERM is weaker than ISOM, consider a type  $\langle 1 \rangle$  quantifier  $Q$  such that for all  $M$  and all  $A \subseteq M$ ,

$$Q_M(A) \iff 3 \in M$$

Clearly,  $Q$  satisfies PERM: if  $3 \in M$ ,  $Q_M = 1_M$ , and if  $3 \notin M$ ,  $Q_M = 0_M$ . But  $Q$  is not ISOM: a bijection from a universe containing the number 3 to one which doesn't show this.

Finally, we note that ISOM (and PERM) applies not only to quantifiers but also to arbitrary operations on universes, i.e. operations  $O$  that associate an object  $O_M$  of some type (say, a higher-order object; quantifiers are just a special case) with each universe  $M$ . As often happens, the general formulation is simpler than its special cases: it just says that if  $f$  is an arbitrary bijection with domain  $M$ , then

$$f(O_M) = O_{f(M)}$$

(where  $f$  has been 'lifted' to higher-order objects over  $M$ ). This will be explained in Chapter 9.1.

### 3.4 EXTENSION

We now come to a second property characteristic of natural language quantifiers, called *extension* (EXT) or, sometimes, *domain independence*. This is a strictly global property, with no local counterpart. Roughly, it is the property that the behavior of the quantifier doesn't change when you extend the universe. We have already noted (Chapter 2.1) that words like all, at least five, no, mean the same thing regardless of which universe you are quantifying over, and in fact EXT goes a long way towards capturing this sort of *constancy*.

We will see in this chapter that most English noun phrase denotations satisfy EXT (on our interpretation), and in the next that the same holds for all determiner denotations. But a few familiar quantifiers fail to satisfy EXT. The logical quantifier  $\forall$  is not EXT, since "everything" means precisely 'everything in the universe'. The quantifier  $\exists$ , on the other hand, is EXT. It turns out (Chapter 4.5) that all type  $\langle 1 \rangle$  quantifiers correspond to EXT type  $\langle 1, 1 \rangle$  quantifiers, where we find the denotations of determiners, for example, and furthermore (Chapter 6.1) that the type  $\langle 1 \rangle$  quantifiers

<sup>12</sup> Or PI, or *permutation invariance*.



that are themselves EXT correspond to an especially interesting subclass of type  $\langle 1, 1 \rangle$  quantifiers.

Interestingly, the property EXT (under the label “domain independence”) has also been studied by computer scientists in connection with database languages, without any connection to natural language. In that context it is a property of sentences or formulas. In the next subsection we define EXT, first for the type  $\langle 1 \rangle$  case, and then for quantifiers of arbitrary type, and begin looking at its significance for natural language. The following subsection deals with the connection to database languages.

### 3.4.1 EXT as a property of quantifiers

#### 3.4.1.1 EXT for type $\langle 1 \rangle$ quantifiers

The name “extension” derives from the first version of the following definition.

##### EXT for type $\langle 1 \rangle$ quantifiers

A quantifier  $Q$  of type  $\langle 1 \rangle$  satisfies EXT if and only if

$$(3.23) \quad A \subseteq M \subseteq M' \text{ implies } Q_M(A) \Leftrightarrow Q_{M'}(A)$$

That is, extending the universe has no effect. Equivalently,

$$(3.24) \quad \text{If } A \text{ is a subset of both } M \text{ and } M', \text{ then } Q_M(A) \Leftrightarrow Q_{M'}(A)$$

Clearly (3.24) implies (3.23). Conversely, if  $A \subseteq M, M'$ , let  $M'' = M \cap M'$ . Since  $A \subseteq M'' \subseteq M$  and  $A \subseteq M'' \subseteq M'$ , we get, using (3.23) twice,  $Q_M(A) \Leftrightarrow Q_{M''}(A) \Leftrightarrow Q_{M'}(A)$ .

For ISOM type  $\langle 1 \rangle$  quantifiers, which, as we saw, correspond to binary relations between numbers, EXT says that only one of these numbers matters:

##### Fact 9

If  $Q$  is ISOM, EXT is equivalent to the following condition:

$$(3.25) \quad \text{If } Q(k, m), \text{ then for all cardinal numbers } k', Q(k', m).$$

This means that  $Q$  can be seen as a class  $S$  of cardinal numbers,<sup>13</sup> i.e. the cardinal numbers of those sets which satisfy the condition of the quantifier:

$$(3.26) \quad m \in S \iff \text{for some } k, Q(k, m) \\ \iff \text{for some } M, \text{ and some } A \subseteq M \text{ s.t. } |A| = m, Q_M(A)$$

In other words,  $Q$  is EXT iff there is a class  $S$  such that (3.26) holds, or (equivalently), for all  $M$  and all  $A \subseteq M$ ,

$$Q_M(A) \iff |A| \in S$$

<sup>13</sup>  $S$  can be a proper class; e.g. for  $Q = \exists$ ,  $S$  is the class of all cardinal numbers except 0. If one restricts attention to universes smaller than a certain size, e.g. to finite universes, then  $S$  is a set.

*Proof.* Suppose EXT holds and that  $Q(k, m)$ . So for some  $M$  and some  $A \subseteq M$  such that  $|A| = m$  and  $|M - A| = k$ ,  $Q_M(A)$ . Now if  $k'$  is any number, take a universe  $M'$  such that  $A \subseteq M'$  and  $|M' - A| = k'$ . This is always possible, and it follows by (3.24) that  $Q_{M'}(A)$ , i.e. that  $Q(k', m)$ . Thus, (3.25) holds.

Conversely, given (3.25), suppose that  $A \subseteq M \subseteq M'$ . Then  $Q_M(A) \Leftrightarrow Q(|M - A|, |A|)$ , and  $Q_{M'}(A) \Leftrightarrow Q(|M' - A|, |A|)$ . But then (3.25) implies that  $Q_M(A) \Leftrightarrow Q_{M'}(A)$ .

We have shown that when  $Q$  is seen as a relation between cardinal numbers, EXT means that the first argument of this relation is immaterial. Thus,  $Q$  can be identified with the class of numbers that occur as the second argument (i.e. with the range of the relation).  $\square$

The following are examples of ISOM and EXT quantifiers:

$$\exists, \exists_{\geq 5}, \exists_{\leq 7}, \exists_{=3}, Q_{\text{even}}, Q_0$$

Thus, by Fact 9,  $\exists$  can be identified with the class of cardinal numbers  $> 0$ ,  $\exists_{\geq 5}$  with those  $\geq 5$ ,  $\exists_{\leq 7}$  with  $\{0, 1, \dots, 7\}$ ,  $\exists_{=3}$  with  $\{3\}$ ,  $Q_{\text{even}}$  with  $\{0, 2, 4, \dots\}$ ,  $Q_0$  with  $\{\aleph_0, \aleph_1, \aleph_2, \dots\}$ , etc.

For type  $\langle 1 \rangle$  natural language quantifiers, which, as we saw, are usually not ISOM, we note the following fact.

#### Fact 10

- (a) All quantifiers of the form  $I_a$  or of the form  $C^{\text{Pl}}$  are EXT.
- (b) Restricted quantifiers of the form  $Q^{[A]}$  are EXT.
- (c) The class of EXT type  $\langle 1 \rangle$  quantifiers is closed under conjunction, disjunction, and outer negation.

*Proof.* (a) follows, since  $a \in B$  ( $C \subseteq B$ ) or not regardless of the surrounding universe. (b) follows from the definition (3.3) of  $Q^{[A]}$  in section 3.2.1: If  $B \subseteq M \subseteq M'$ , then  $(Q^{[A]})_M(B)$  iff  $Q_A(A \cap B)$  iff  $(Q^{[A]})_{M'}(B)$ . (c) is completely straightforward.  $\square$

This gives us a host of EXT quantifiers, including some quite odd ones, such as  $\exists_{=3} \vee I_a$ , which holds of a set  $A$  iff  $A$  either has exactly three elements or contains  $a$ . In particular, it goes to show that most English NP denotations are EXT, since proper names (or conjunctions of proper names), bare plurals, and quantified NPs of the form [Det N] cover a large part of English NPs (see section 3.5 below for a precise version of this claim).

In connection with Fact 10 (b) you might ask: Can every thing really be EXT when everything is not EXT? Well, it depends. If you think of thing as denoting a fixed set  $A$  (say, the set of non-animate objects in the universe), then every thing means  $\forall^{[A]}$  and is indeed EXT: for  $B \subseteq M$ ,

$$\begin{aligned}
(\forall^{[A]})_M(B) &\iff \forall_A(A \cap B) \\
&\iff A \cap B = A \\
&\iff A \subseteq B \\
&\iff \text{every}_M(A, B)
\end{aligned}$$

that is,  $\forall^{[A]}$  is simply the quantifier *every* with the first argument frozen to  $A$ . But if you think of `thing` as a logical constant that always denotes the universe, then *every thing* and *everything* both denote  $\forall$ . In general,

$$(Q^{[M]})_M(B) \iff Q_M(B)$$

Next, observe that if  $Q_M(B)$  depends only on  $B$ , then  $Q_{\neg M}(B)$  and  $Q_M^d(B)$  depend only on  $M - B$ . In other words, they are co-EXT ((3.9) in section 3.2.3). So, taking inner negations and duals of the previous examples, we get co-EXT type ⟨1⟩ quantifiers, where the earlier condition on  $B$  is replaced by the corresponding condition, or its negation, on  $M - B$ . The foremost example is

$\forall$

As in Fact 9, co-EXT and ISOM type ⟨1⟩ quantifiers can also be identified with classes of cardinal numbers: this time the sizes of the sets  $M - B$  satisfying the condition of the quantifier.

Proportional type ⟨1⟩ quantifiers are typically neither EXT nor co-EXT, since both  $|A|$  and  $|M - A|$  are essential. As one might expect, we will see that such quantifiers are more complex in various senses than the EXT or co-EXT ones. We can also get examples from Boolean combinations of EXT and co-EXT quantifiers, such as

$\exists \wedge \neg \forall$

which we can read as *something but not everything*. Then  $(\exists \wedge \neg \forall)_M(M)$  is false, but if  $M$  is a non-empty proper subset of  $M'$ ,  $(\exists \wedge \neg \forall)_{M'}(M)$  is true. This shows that EXT fails, and by a similar argument co-EXT fails too.

### 3.4.1.2 EXT and sets that quantifiers live on

As pointed out at the end of section 3.2.2, the live-on behavior of  $Q$  on  $M$  can be totally unrelated to its live-on behavior on  $M'$ . With EXT, this behavior becomes less chaotic. These issues are hardly ever discussed in the literature, presumably because the perspective is usually local, with a fixed universe. But as soon as one takes a global perspective on quantification, they require attention. Of particular interest is the *smallest* set that a (local) quantifier lives on, if there is one. We shall need access to this set later on, and therefore introduce the following notation:<sup>14</sup>

<sup>14</sup> “W” is for “witness”, since  $W_{Q_M}$  is, roughly, what Barwise and Cooper (1981) would call the smallest witness set for  $Q_M$ .

**the set  $W_{Q_M}$**

For  $Q$  of type (1) and any universe  $M$ , let

$$(3.27) \quad W_{Q_M} = \begin{cases} \text{the smallest set } Q_M \text{ lives on,} & \text{if there is such a set} \\ \text{undefined,} & \text{otherwise} \end{cases}$$

From Lemma 1 in section 3.2.2 we see that: (a) when  $M$  is finite,  $W_{Q_M}$  is always defined; (b)  $W_{Q_M} = \emptyset$  if and only if  $Q_M$  is trivial; and (c)  $W_{C_M^{\text{pl}}} = C$ , when  $C_M^{\text{pl}}$  is non-trivial. Likewise, in many cases we have  $W_{(Q^{[A]})_M} = A$  (but not always).

The effect of EXT here is the following:

**Lemma 11**

If  $Q$  is EXT,  $M \subseteq M'$ , and  $Q_{M'}$  lives on  $X$ , then  $Q_M$  lives on  $X$ .

*Proof.* For  $B \subseteq M$  we have:

$$\begin{aligned} B \in Q_M &\iff B \in Q_{M'} \quad [\text{EXT}] \\ &\iff B \cap X \in Q_{M'} \quad [Q_{M'} \text{ lives on } X] \\ &\iff B \cap X \in Q_M \quad [\text{EXT}] \end{aligned} \quad \square$$

**Corollary 12**

If  $Q$  is EXT,  $M \subseteq M'$ , and  $W_{Q_M}$  and  $W_{Q_{M'}}$  are both defined,  $W_{Q_M} \subseteq W_{Q_{M'}}$ .

*Proof.* By the lemma,

$$W_{Q_M} = \cap \{X : Q_M \text{ lives on } X\} \subseteq \cap \{X : Q_{M'} \text{ lives on } X\} = W_{Q_{M'}} \quad \square$$

To see that  $W_{Q_M}$  may sometimes be a proper subset of  $W_{Q_{M'}}$ , consider the quantifier

$$Q = I_j \vee I_m$$

which is EXT by Fact 10. Here  $W_{Q_M}$  is always defined; in fact, one easily verifies that

$$\begin{aligned} \text{if } j, m \in M, \quad W_{Q_M} &= \{j, m\} \\ \text{if } j \in M \text{ and } m \notin M, \quad W_{Q_M} &= \{j\} \end{aligned}$$

So if  $M \subseteq M'$ ,  $j \in M$ , and  $m \in M' - M$ ,  $W_{Q_M}$  is a proper subset of  $W_{Q_{M'}}$ . By contrast,  $W_{(I_j \wedge I_m)_M}$  is always equal to  $\{j, m\}$  whenever  $(I_j \wedge I_m)_M$  is non-trivial—this follows from Lemma 1 and the fact that  $I_j \wedge I_m = \{j, m\}^{\text{pl}}$ .

We can now improve Lemma 1(e) concerning the circumstances when  $A$  is the smallest set that a quantifier  $Q_M^{[A]}$  lives on. Suppose  $Q$  of type (1) is ISOM and EXT. By Fact 9, there is a class  $S$  of cardinal numbers such that for all  $M$  and all  $A \subseteq M$ ,

$$Q_M(A) \iff |A| \in S$$

Let us say that  $Q$  has *finite action* if some finite number is in  $S$  and some finite number is not in  $S$ . In other words,  $Q$  is not trivial (always true or always false) on finite

arguments. For example,  $\exists_{=n}$ ,  $\exists_{\geq n}$ ,  $\exists_{\leq n}$ , and  $Q_{\text{even}}$  all have finite action when  $n$  is a natural number, but  $Q_0$  does not.

**Proposition 13**

Suppose  $Q$  is ISOM, EXT, and has finite action. Then, if  $Q_M^{[A]}$  is not trivial,  $W_{(Q_M^{[A]})_M} = A$ .

*Proof.* With  $S$  as above, there must exist a smallest natural number  $k$  such that  $k \notin S$  and  $k + 1 \in S$ , or else a smallest natural number  $k$  such that  $k \in S$  and  $k + 1 \notin S$ . Suppose the first case holds (in the second case the argument is similar). This means that  $0, 1, \dots, k \notin S$ . So if  $Q_M^{[A]}$  is non-trivial,  $|A \cap M| \geq k + 1$ . But then we can apply exactly the argument in the proof of Lemma 1(e) to show that if  $Q_M^{[A]}$  lives on  $C$  then  $A \subseteq C$ . Since  $Q_M^{[A]}$  always lives on  $A$ ,  $A$  is the smallest set that the quantifier lives on.  $\square$

Lemma 1(e) can be improved a lot more, but we content ourselves with the following illustrative example. Note that the Rescher quantifier  $Q^R$  is not EXT, but that in the obvious sense it has finite action.

(3.28) If  $A$  is finite and  $(Q_M^R)^{[A]}$  is not trivial,  $A$  is the smallest set on which the quantifier lives.

*Proof.* Let  $|A| = k$ . Since  $(Q_M^R)^{[A]}$  is not trivial,  $A \cap M \neq \emptyset$ . We have, for any  $B \subseteq M$ ,

$$(Q_M^R)^{[A]}(B) \iff |A \cap B| > 1/2 \cdot k$$

Suppose  $(Q_M^R)^{[A]}$  lives on  $C$ ; it is enough to show that  $A \subseteq C$ . Suppose  $a \in A - C$ . Choose  $B \subseteq M$  such that  $a \in B$  and  $|A \cap B| - 1 \leq 1/2 \cdot k < |A \cap B|$ . This is possible by our assumptions. But then  $(Q_M^R)^{[A]}(B)$  and not  $(Q_M^R)^{[A]}(C \cap B)$ , a contradiction.  $\square$

### 3.4.1.3 EXT for arbitrary quantifiers

EXT expresses a notion of universe independence. This notion is not in any way tied to type ⟨1⟩ quantifiers, and it is immediate how to formulate EXT for arbitrary types. Here is the version corresponding to (3.23).

**EXT for arbitrary quantifiers**

A quantifier  $Q$  of type  $\langle n_1, \dots, n_k \rangle$  satisfies EXT iff the following holds:

$$(3.29) \text{ If } R_i \subseteq M^{n_i} \text{ for } 1 \leq i \leq k, \text{ and } M \subseteq M', \text{ then } Q_M(R_1, \dots, R_k) \Leftrightarrow Q_{M'}(R_1, \dots, R_k).$$

EXT is a strong constraint: naively one could say that, by far, most quantifiers do not satisfy it. A quantifier associates a second-order relation (of a certain type) with

each universe. In principle, it can associate wildly different relations with different universes. With universes containing John and at least three dogs it could associate the universal quantifier, with those containing John but fewer than three dogs it could associate  $I_j$ , and with other universes it could associate  $\exists_{\geq 7}^{[cat]}$ . Nothing in the general definition of a quantifier prevents this. Even if ISOM is imposed,  $Q_M$  could still be completely different for different *sizes* of  $M$ . But in practice such examples do not seem to occur ‘naturally’, and in particular they do not occur in natural languages.

As we have said, EXT entails an idea of constancy; of a quantifier being ‘the same on each universe’. This is of course an imprecise way of speaking; if  $M \neq M'$ ,  $Q_M$  and  $Q_{M'}$  are usually *different* collections of sets.<sup>15</sup>

What, then, does *constancy* of a quantifier  $Q$  mean? Intuitively, it means that  $Q$  ‘means the same’, or is *given by the same rule*, on every universe. But the identity conditions for meanings, or for rules, are notoriously difficult to lay down. Indeed, it is not clear that the concept of constancy can be explained fully in an extensional framework like ours. But some things can be said.

A special case is when the rule does not mention the universe  $M$  at all. This seems to be what EXT amounts to. It would follow that EXT is a *sufficient* condition for constancy. But we have seen that it is not necessary. Quantifiers like  $\forall$  or  $Q^R$  are also given by the same rule on each universe, but these rules do mention  $M$ , and EXT fails. We shall argue in Chapter 9.3 that constancy is reflected by a more palpable notion of *constancy in inferences*.  $\forall$  and  $Q^R$ —or rather, the type  $\langle 1, 1 \rangle$  determiner denotations *every* and *most* that they give rise to—are constant in this sense.

We note further that being given by the same rule on each universe entails *not being candidates for interpretations in models*. The point of model-theoretic semantics is to interpret just symbols whose denotation is not fixed in models; the other expressions use a fixed interpretation rule.

Summing up, EXT is a powerful notion of constancy. Even if it is too strong to capture exactly the somewhat elusive idea of being given by the same rule in every universe, it is a condition that appears to be satisfied by most quantifiers related to natural languages. We discuss its status further in section 3.5 below, and in Chapters 4.5.3 and 9.3.

Before we end this section, three final remarks about EXT are in order. The first is a notational one. Since EXT says precisely that the universe is irrelevant, one may drop the subscript indicating the universe for such quantifiers, and write simply

$$Q(R_1, \dots, R_k)$$

rather than

$$Q_M(R_1, \dots, R_k)$$

More formally, we can *define*

$$Q(R_1, \dots, R_k) \text{ iff for some } M \text{ s.t. } R_i \subseteq M^{n_i} \text{ for } 1 \leq i \leq k, Q_M(R_1, \dots, R_k)$$

<sup>15</sup> E.g. if  $M$  and  $M'$  are disjoint, then  $Q_M$  and  $Q_{M'}$  are always different except in the one case when both are the empty set.

It then follows that this is well-defined (independent of the choice of  $M$ ) if  $Q$  satisfies EXT.

Second, at the end of the last section we saw that PERM is a weaker constraint than ISOM. In the presence of EXT, however, they are equivalent. The verification of the following fact is left as an exercise.<sup>16</sup>

**Fact 14**

*Let  $Q$  be a quantifier (of any type) satisfying EXT. If PERM holds for  $Q$ , then so does ISOM.*

Third, just like ISOM, EXT is not restricted to quantifiers, but has an obvious generalization to arbitrary operators on universes (see Chapter 9.1.1). The above fact extends to the general case.

### 3.4.2 EXT and database languages

This subsection is a digression from our present theme of quantification in natural languages, and will not be used in what follows. Still, it provides interesting information about the condition EXT.

As we said, this condition is familiar from a completely different context: viz. that of languages for describing and querying databases.<sup>17</sup> A relational structure  $\mathcal{M} = (M, R_1, \dots, R_k)$ , where  $R_i$  is a finite  $n_i$ -ary relation (i.e. one containing finitely many  $n_i$ -tuples), can be seen as a database where the complete (extensional) information about these relations is stored.<sup>18</sup> Consequently, the language of first-order logic, FO (Chapter 2.2), or even of FO with added generalized quantifiers (we began to discuss this in Chapter 2.3 and will describe it in more detail in Chapter 13.1), is a suitable language for talking about databases. An FO sentence  $\varphi$ , say,

$$(3.30) \quad \forall x \forall y \forall z (R(x, y) \wedge R(y, z) \rightarrow R(x, z))$$

makes a claim about a database  $\mathcal{M} = (M, R)$ , in this case that the relation  $R$  is transitive. Or we can think of  $\varphi$  as a *query* to  $\mathcal{M}$ , which has answer YES if  $\mathcal{M} \models \varphi$ , i.e., if  $R$  is transitive, and NO otherwise. More generally, a formula  $\varphi(x_1, \dots, x_n) = \varphi(\bar{x})$  with  $n$  free variables can be seen as a query—Which  $n$ -tuples of elements in the database

<sup>16</sup> Hint: first show that under EXT, PERM implies that whenever  $(M, R_1, \dots, R_k)$  and  $(M', R'_1, \dots, R'_k)$  are isomorphic structures such that  $M$  and  $M'$  are *disjoint*,

$$Q_{\mathcal{M}}(R_1, \dots, R_k) \iff Q_{\mathcal{M}'}(R'_1, \dots, R'_k)$$

Then show that the general case follows from this.

<sup>17</sup> Thanks to Phokion Kolaitis for drawing our attention to this.

<sup>18</sup> Often the database also has a set of *attributes* or one-place predicates, so that to the  $n_i$  argument places of  $R_i$  correspond attributes  $A_{i1}, \dots, A_{in_i}$ , respectively, and if  $R_i(a_1, \dots, a_{n_i})$  holds, then  $a_j \in A_{ij}$ ,  $1 \leq j \leq n_i$ . Also, the language may contain individual constants, as well as additional predicates with fixed interpretations. We can ignore these extra features here, since they are irrelevant for the points we wish to make. For surveys of the theory of relational databases, see Ullman 1988 or Kanellakis 1990.

satisfy  $\varphi(\bar{x})$ ?—whose answer in  $\mathcal{M}$  is the  $n$ -ary relation (set of  $n$ -tuples)

$$\varphi(\bar{x})^{\mathcal{M}, \bar{x}} = \{\bar{b} \in M^n : \mathcal{M} \models \varphi(\bar{b})\}$$

in the notation introduced in Chapter 2.3 (definition (2.8)).

When  $\mathcal{M}$  is seen as a database and emphasis is put on how a user may query it, there is a marked informational asymmetry between the tuples which are *in* a relation  $R$  and those which are *outside* it. Tuples in  $R$  can in principle be inspected, since  $R$  is finite, whereas a search through tuples outside  $R$  may be infinite. And even if we require that the universe  $M$  be finite, there is something amiss with a query that requires us to search through  $M^n - R$ , since the answer may *change* if new elements are added to  $M$  even if  $R$  itself is not touched. Such a query is ‘unsafe’ in an obvious sense. Adding new elements to a database  $(M, R_1, \dots, R_k)$  without changing the given relations  $R_1, \dots, R_k$  is a very natural operation, for example, if we want to store information about *other* relations in our database. The ‘safe’ queries about  $R_1, \dots, R_k$  are those which are not disturbed by such extensions.

It should be clear by now that the property EXT is precisely about such extensions, and indeed it says that certain facts about the given relations do not change. To bring this out, we generalize EXT to arbitrary formulas, as follows:

#### EXT for formulas

Let  $\varphi = \varphi(x_1, \dots, x_n)$  be a formula with exactly the free variables displayed, in a language with non-logical relation symbols  $P_1, \dots, P_k$ , where  $P_i$  is  $n_i$ -ary.  $\varphi$  can be a formula in  $FO$ , or in a language  $FO(Q_1, \dots, Q_m)$  with added generalized quantifiers (see Chapter 13.1 for precise definitions). We say that  $\varphi$  satisfies EXT iff for every model  $\mathcal{M} = (M, R_1, \dots, R_k)$ , and every  $\mathcal{M}'$  such that  $M \subseteq M'$ , letting  $\mathcal{M}' = (M', R_1, \dots, R_k)$  it holds that

$$(3.31) \quad \varphi^{\mathcal{M}, \bar{x}} = \varphi^{\mathcal{M}', \bar{x}}$$

That is, when the universe  $M$  is extended, but the relations  $R_1, \dots, R_k$  remain the same, the answer to the query  $\varphi(x_1, \dots, x_n)$  does not change. We note in particular that if  $\varphi$  is EXT and  $\mathcal{M}, \mathcal{M}'$  are as above, then

$$(3.32) \quad \text{if } \bar{b} \in (M')^n \text{ and } \mathcal{M}' \models \varphi(\bar{b}), \text{ then } \bar{b} \in M^n$$

For example, an atomic formula

$$P(x_1, \dots, x_m)$$

is EXT, but not its negation. However, formulas involving negation can be EXT, such as

$$(3.33) \quad \neg P_1(x) \wedge P_2(x, y)$$



Here only the elements of the complement of  $P_1$  which belong (as first elements) to some pair in  $P_2$  matter, so EXT holds.

The sentence (3.30) expressing transitivity is EXT: to determine whether a binary relation  $R$  is transitive, you have only to look at the pairs in  $R$ , not at the ones outside  $R$ . The sentence

$$\exists x P(x)$$

is EXT, but not the sentence

$$\exists x \neg P(x)$$

If  $\varphi$  and  $\psi$  are both EXT formulas, then it is easy to see that so is their conjunction  $\varphi \wedge \psi$ . But this fails for disjunction in general. For example,

$$P_1(x) \vee P_2(y)$$

is not EXT, even though both disjuncts are. For take a model  $\mathcal{M} = (M, R_1, R_2)$  and  $b \in R_2$ . Now add any  $a \notin M$ , i.e. let  $M' = M \cup \{a\}$ , and consider  $\mathcal{M}' = (M', R_1, R_2)$ . Then  $(a, b)$  satisfies  $P_1(x) \vee P_2(y)$  in  $\mathcal{M}'$  (since  $R_2(b)$  holds) but not in  $\mathcal{M}$  (since  $(a, b)$  is not a tuple of elements of  $M$ ). That is,

$$(P_1(x) \vee P_2(y))^{\mathcal{M}, x, y} \neq (P_1(x) \vee P_2(y))^{\mathcal{M}', x, y}$$

On the other hand, it is easy to see that

(3.34) if  $\varphi$  and  $\psi$  are EXT and have the same free variables, then  $\varphi \vee \psi$  is EXT.

We may also note that logical truth does not guarantee EXT, since, for example,

$$(P(x) \vee \neg P(x))^{\mathcal{M}, x} = M, \text{ but } (P(x) \vee \neg P(x))^{\mathcal{M}', x} = M'$$

But logical falsity trivially implies EXT: if  $\varphi$  is unsatisfiable, then

$$\varphi^{\mathcal{M}, \vec{x}} = \varphi^{\mathcal{M}', \vec{x}} = \emptyset$$

To relate EXT for formulas to our previous notion of EXT for quantifiers, note that to a (Lindström) quantifier  $Q$  (Chapter 2.4) corresponds a defining sentence  $\varphi_Q$ ; for example, if  $Q$  is of type ⟨1⟩,  $\varphi_Q$  is of the form

$$Qx P(x)$$

If it is of type ⟨2, 3, 1⟩,  $\varphi_Q$  is

$$Qxy, zuv, w(P_1(x, y), P_2(z, u, v), P_3(w))$$

The idea should be clear, and it is straightforward to verify the following directly from the definitions:

**Fact 15**

*$Q$  satisfies EXT if and only if the sentence  $\varphi_Q$  is EXT.*

We can also show things like the next proposition, which generalizes the observation that if  $\varphi$  is an EXT formula, so is  $\exists x\varphi$ . Here it is stated for type  $\langle 1 \rangle$  quantifiers, but similar facts hold for other types. Call a type  $\langle 1 \rangle$  quantifier  $Q$  *positive* if for all  $M$ ,  $\emptyset \notin Q_M$ .

**Proposition 16**

If  $Q$  is a positive and EXT type  $\langle 1 \rangle$  quantifier and  $\varphi = \varphi(x, \bar{y})$  is an EXT formula, then  $Qx\varphi(x, \bar{y})$  is also an EXT formula.

*Proof:* Let  $\mathcal{M}$  and  $\mathcal{M}'$  be as in definition (3.31) above, and let  $\bar{b} \in M^n$ . Then

$$\bar{b} \in Qx\varphi(x, \bar{y})^{\mathcal{M}, \bar{y}} \Leftrightarrow \mathcal{M} \models Qx\varphi(x, \bar{b})$$

$$(3.35) \quad \Leftrightarrow \varphi(x, \bar{b})^{\mathcal{M}, x} \in Q_M$$

$$(3.36) \quad \Leftrightarrow \varphi(x, \bar{b})^{\mathcal{M}', x} \in Q_M \quad [\text{since } \varphi \text{ is EXT}]$$

$$\Leftrightarrow \varphi(x, \bar{b})^{\mathcal{M}', x} \in Q_{M'} \quad [\text{since } Q \text{ is EXT}]$$

$$\Leftrightarrow \bar{b} \in Qx\varphi(x, \bar{y})^{\mathcal{M}', \bar{y}}$$

The equivalence of (3.35) with (3.36) uses the fact that  $\varphi(x, \bar{b})^{\mathcal{M}, x} = \varphi(x, \bar{b})^{\mathcal{M}', x}$ . This fact is a consequence of our assumption that  $\varphi$  is EXT, which by definition means that

$$\varphi(x, \bar{y})^{\mathcal{M}, x, \bar{y}} = \varphi(x, \bar{y})^{\mathcal{M}', x, \bar{y}}$$

Fixing the assignment  $\bar{b}$  to  $\bar{y}$ , we get, for all  $a \in M'$ ,

$$\mathcal{M} \models \varphi(a, \bar{b}) \Leftrightarrow \mathcal{M}' \models \varphi(a, \bar{b})$$

which is what is needed.

We have thus shown that  $Qx\varphi(x, \bar{y})^{\mathcal{M}, \bar{y}}$  and  $Qx\varphi(x, \bar{y})^{\mathcal{M}', \bar{y}}$  coincide on tuples in  $M^n$ . To see that they in fact coincide on all tuples, it is enough to demonstrate that

$$(3.37) \quad \text{if } \bar{b} \in (M')^n \text{ and } \bar{b} \in Qx\varphi(x, \bar{y})^{\mathcal{M}', \bar{y}}, \text{ then } \bar{b} \in M^n$$

This is where the positivity of  $Q$  is needed. For if  $\mathcal{M}' \models Qx\varphi(x, \bar{b})$ , then  $\varphi(x, \bar{b})^{\mathcal{M}', x} \in Q_{M'}$ , and hence  $\varphi(x, \bar{b})^{\mathcal{M}', x} \neq \emptyset$ . So there is some  $a \in M'$  such that  $\mathcal{M}' \models \varphi(a, \bar{b})$ , and then it follows from (3.32) that  $(a, \bar{b})$  is a tuple in  $M^{n+1}$ .  $\square$

The assumption of positivity is necessary here, as witnessed by the example

$$\psi(y) = \neg\exists x(P(x, y) \wedge \neg P(x, y))$$

For  $\neg\exists$  is a quantifier which is EXT but not positive, and  $P(x, y) \wedge \neg P(x, y)$  is an EXT formula (since it is a contradiction). But  $\psi(y)$  is satisfied by any  $b$  in  $M' - M$ , so it is not an EXT formula.

Thus, the concept of EXT for formulas generalizes the one we already had for quantifiers. In the literature on relational databases, EXT formulas have been called “definite”, or, more commonly (and more aptly), *domain-independent*. It is clear that

domain independence expresses the notion of safety mentioned above. However, there is a problem: it can be very hard to see if a formula is domain-independent by mere inspection of its form. The examples given above were easy, but not all cases are like that. In fact, there is the following result:

**Theorem** (Di Paola 1969)

*The problem of determining whether a formula is EXT (domain-independent) or not is undecidable.*

The reason for this is precisely that to decide whether a formula is EXT involves deciding whether it is logically false (unsatisfiable), and that is a known undecidable problem.<sup>19</sup>

This makes the notion of domain independence less interesting from a practical computational point of view. Therefore, database theorists have tried to identify easily recognizable (hence decidable) subclasses of the domain-independent formulas that still cover the cases one runs into in practice. The above examples give a pretty good idea of which constructions are allowed and which are not, though there is much room for fine-tuning.<sup>20</sup> The class of *safe FO*-formulas from Ullman 1988 is a case in point. (Here “safe” is a precise technical term.) They are defined inductively by allowing, essentially, atomic formulas, arbitrary conjunctions, disjunctions of formulas with the same free variables, existential but not universal quantification, and only a ‘stratified’ form of negation (exemplified by (3.33) above). These are all domain-independent (EXT), and one can show that such syntactically defined classes

<sup>19</sup> This is seen by the following proof. (Di Paola proves something slightly stronger.) We use the undecidability of validity in *FO*, which entails that satisfiability in *FO* is also undecidable (since  $\varphi$  is satisfiable iff  $\neg\varphi$  is not valid). If  $\varphi$  is any *FO*-sentence, let  $P$  be a one-place predicate symbol not in  $\varphi$ , and let  $\psi(x)$  be the formula

$$\varphi \wedge (P(x) \vee \neg P(x))$$

We claim that

$$\varphi \text{ is satisfiable iff } \psi(x) \text{ is not EXT}$$

Here, if the class of EXT formulas in *FO* were decidable, the same would hold for *FO*-satisfiability (since  $\psi(x)$  is found effectively from  $\varphi$ ), a contradiction. To prove the claim, suppose first that  $\varphi$  is not satisfiable. Then for all models  $\mathcal{M}$  with the vocabulary of  $\psi(x)$ ,  $\psi(x)^{\mathcal{M},x} = \emptyset$ , so  $\psi(x)$  is EXT. In the other direction, suppose  $\varphi$  is satisfiable and let  $\mathcal{M}$  be a model with the vocabulary of  $\varphi$  such that  $\mathcal{M} \models \varphi$ . Expand  $\mathcal{M}$  to  $\mathcal{M}_R = (\mathcal{M}, R)$ , where  $R \subseteq M$  interprets  $P$ . Let  $a$  be any object not in  $M$ , and let  $\mathcal{M}'$  and  $\mathcal{M}'_R$  be the result of adding  $a$  to  $M$ , without touching the relations of  $\mathcal{M}$  and  $\mathcal{M}_R$ , respectively. Now  $\mathcal{M}_R \models \varphi$ , since  $\mathcal{M} \models \varphi$  and  $P$  does not occur in  $\varphi$ , so  $\psi(x)^{\mathcal{M}_R,x} = M$ . If  $\mathcal{M}'_R \models \varphi$ , then  $\psi(x)^{\mathcal{M}'_R,x} = M \cup \{a\} \neq M$ . And if  $\mathcal{M}'_R \not\models \varphi$ , then  $\psi(x)^{\mathcal{M}'_R,x} = \emptyset \neq M$ . In both cases, the extension of  $\psi(x)$  changes between  $\mathcal{M}_R$  and  $\mathcal{M}'_R$ , so  $\psi(x)$  is not EXT.  $\square$

If attention is restricted to finite models (as is usually done in database theory), one uses instead a classical result by Trakhtenbrot, according to which *FO*-validity restricted to finite models is undecidable (in fact, not recursively enumerable).

<sup>20</sup> An investigation of how far one can go while ‘staying safe’ can be found in van Gelder and Topor 1987.

of formulas capture classes of domain-independent ones up to logical equivalence. We quote a recent such result by Wilfrid Hodges.<sup>21</sup>

Call an *FO*-formula *protected* if all occurrences of  $\forall$  and  $\exists$  are restricted by atomic formulas (i.e. of the form  $\forall \bar{x}(P(\bar{x}) \rightarrow \varphi)$  or  $\exists \bar{x}(P(\bar{x}) \wedge \varphi)$ ). Next, a *protective hat* for a variable  $y$  has the form

$$\exists x_1 \dots x_{j-1} x_{j+1} \dots x_k P(x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_k)$$

for some predicate symbol  $P$ . Finally, let a *covered* formula consist of a protected formula  $\psi$  conjoined with a disjunction of protective hats, one for each of the free variables in  $\psi$ .

**Theorem** (Hodges)

*A relational FO-formula is EXT iff it is logically equivalent to a covered formula.*

That each covered formula is EXT is clear from our previous discussion. Hodges proves the converse by an argument using tools characteristic of first-order model theory (compactness, elementary chains, Löwenheim–Skolem theorem). Note that this result does not contradict Di Paola’s theorem, since the relation of logical equivalence in *FO* is not decidable. Thus, the fact that for each EXT formula there exists an equivalent covered formula doesn’t mean that we have an effective method of finding one (indeed, Di Paola’s theorem shows that there is no such method).

From Proposition 16 we see that one can extend notions such as that of being safe or covered to certain formulas in languages with generalized quantifiers, and still be sure that all such formulas are EXT. We do not know, however, if (or under what circumstances) it is still true that every EXT formula in this language is logically equivalent to a formula of that form.

### 3.5 HOW NATURAL LANGUAGE QUANTIFIER EXPRESSIONS ALWAYS DENOTE GLOBAL QUANTIFIERS

In this book we have been speaking somewhat loosely of quantifier expressions (quantirelations, NPs, etc.) *denoting* quantifiers; we also used “signifying”, “expressing”, and “meaning” similarly. This is unproblematic for expressions like *every*, *most*, *between three and five*, but it can seem confusing for expressions like *John*, *three cats*, *every student’s* which involve names or predicate expressions that get interpreted in models. In this section we introduce a more precise terminology, and explain how our usage of “denote” relates to it.

#### 3.5.1 Sense and local denotation

The notation below was introduced in Chapter 2.2 for the special case of *FO*-expressions.

<sup>21</sup> Hodges, personal communication.

### senses and local denotations of expressions

Let  $u$  be a simple or complex expression in a language  $L$ . The *sense* of  $u$  is a mapping  $[[u]]$  from models  $\mathcal{M} = (M, I)$  to suitable semantic values over  $M$ . The value of  $[[u]]$  at  $\mathcal{M}$  is written  $[[u]]_{\mathcal{M}}$  and is called the *denotation of  $u$  in  $\mathcal{M}$* .

For example,

$$\begin{aligned} [[\text{cat}]]_{\mathcal{M}} &= I(\text{cat}) \subseteq M \\ [[\text{three}]]_{\mathcal{M}} &= \{(A, B) : A, B \subseteq M \text{ and } |A \cap B| = 3\} \\ [[\text{three cats}]]_{\mathcal{M}} &= \{B \subseteq M : |I(\text{cat}) \cap B| = 3\} \end{aligned}$$

This is perfectly clear except for one thing: the class of relevant models for  $u$  should be specified. More precisely, one must decide (a) which primitive expressions of  $L$  get interpreted in models, and (b) which models are relevant for  $u$ .<sup>22</sup>

As to (a), we have been arguing for models in which only predicate expressions and individual-denoting expressions get interpreted, *viz. first-order models*.<sup>23</sup> The implication for quantifier expressions like *three cats*, for example, is that only the primitive expression *cats*, and not *three*, gets interpreted in models. The further question of what *three cats* denotes in first-order models is answered in section 3.5.2.

So we continue to look at first-order models, and we use the standard term “non-logical” for the expressions that get interpreted in such models.<sup>24</sup>

As to (b), there are two options. Either you take the models for the whole (non-logical) vocabulary of  $L$ , or you take only those for the (non-logical) primitive expressions that occur in  $u$ . The former is simpler in the sense that the class of models is always the same irrespective of  $u$ , but it is cumbersome in bringing in a lot of unnecessary interpretation. The choice is insignificant, but if one wants to be able to speak about *the* sense of  $u$ , one has to make it. And for that purpose, the latter option seems preferable, so we choose it here.<sup>25</sup>

<sup>22</sup> Actually,  $[[u]]$  should probably be a mapping from models *and assignments* (see Ch. 2.2), but we disregard this complication here.

<sup>23</sup> In principle, you *could* also think of the interpretation function  $I$  as assigning values to words like *every*, *three*, *most*. But we have pointed out that these interpretations are not free: there is a *rule* saying that  $I(\text{every})$  has to be the inclusion relation over  $M$ , etc. So there is no point in letting  $I$  apply to them. This is uncontroversial for *every* and *three*, but it has been disputed for, say, *most*. We will argue at length in Chapter 9 that *most* and similar quantifier expressions should all be treated as constants, and hence not be interpreted in models. But that is not the main issue at the moment.

<sup>24</sup> Even though that terminology is a bit misleading. We shall also argue in Ch. 9 that it is the constancy rather than the logicity of expressions that decides whether they should be interpreted in models or not.

Also, we emphasize here once more that the restriction to first-order models in no way means restricting attention to things expressible in *FO*. In fact, as we will elaborate in Chs. 11 and 12, English can be treated as a first-order language: this means that one is disregarding or trivializing things like non-quantificational adverbs (since these are higher-order functions), which can be acceptable if one is focusing on quantification, but it does *not* mean restricting attention to English sentences that are formalizable in *FO*.

<sup>25</sup> The function  $[[u]]$  still has to be defined for models for more inclusive vocabularies, since one wants to compositionally interpret larger expressions or sentences in which  $u$  occurs. But this

The following terminology is convenient.

### pure quantifier expressions

A *pure quantifier expression* is one not containing any non-logical expressions.

So pure expressions have *empty* (non-logical) vocabulary, which means that for them,  $\mathcal{M} = M$ . Now we can make two observations:

- (3.38) The sense of a pure quantifier expression  $u$  is exactly what we have called its denotation, i.e. a (global) quantifier. In other words, there is a quantifier  $Q$  of the appropriate type such that  $[[u]] = Q$ .
- (3.39) Every quantifier expression  $u$ , whether pure or not, denotes, *in* a model  $\mathcal{M}$ , a (local) quantifier *on*  $M$ . That is, for each suitable model  $\mathcal{M}$ , there is a (global) quantifier  $Q$  of the appropriate type such that  $[[u]]_{\mathcal{M}} = Q_M$ .

Thus, our terminology is already adequate for pure quantifier expressions, and for the others, *local* quantifiers are precisely what they denote in models. The question now is if there is a sense in which they can also be said to denote global quantifiers.

### 3.5.2 Using global quantifiers

Given (3.39), one might think that local quantifiers are enough for the semantics of natural language quantification. But this far from true. First, it would be completely misleading to think of pure quantifier expressions as local, since, as we have been at pains to underscore, they mean *the same* on each universe, and that meaning is given by a general rule. This is an empirical fact about quantifier expressions in natural languages; there is no problem artificially construing quantifiers that mean different things on different universes.

Second, non-pure quantifier expressions are of the same (syntactic and semantic) categories as the pure ones. It would be misleading to treat them as members of a totally different species. Similarly, and third, several important properties of quantifier expressions, notably EXT and ISOM, make sense only in a global setting. If one were to treat non-pure quantifier expressions as only local, those properties could not be applied. But there are in fact several situations where one wants to consider how the same quantifier expression, whether pure or not, behaves on different universes. Of course, this presupposes that the non-logical parts of the quantifier expression remain fixed. The most natural way of fixing these is to assume that a *background model* is given, whose interpretation of these non-logical expressions fixes them while the universe is varied.

The question, then, is if one can select in a principled way a suitable global quantifier for a non-pure quantifier expression  $u$ , given that its sense is known, and that its non-logical parts are fixed by means of some background model  $\mathcal{M}_0$ . We show that this can indeed be done, under one assumption.

is trivial given that, if  $V_u$  is the vocabulary of  $u$ ,  $[[u]]_{(\mathcal{M}, I)} = [[u]]_{(\mathcal{M}, I|V_u)}$  whenever the domain of  $I$  includes  $V_u$ .

To begin, observe that if we ‘abstract’  $u$  so as to regard also its non-logical parts as arguments of the second-order relation that  $u$  expresses on each universe, we obtain a pure quantifier expression whose sense is again completely determined. Let  $u_G$  be a symbol for that relation.<sup>26</sup> For example, suppose  $u$  is of type ⟨1, 1⟩, and suppose a one-place predicate expression  $P$  and an individual-denoting expression  $c$  are the non-logical parts of  $u$ . Then  $u_G$  is, on each  $M$ , a four-place relation, with three set arguments and one individual argument, such that, for all  $A, B, C \subseteq M$  and  $a \in M$ ,

$$[[u_G]]_M(C, a, A, B) \iff [[u]]_{(M, C, a)}(A, B)^{27}$$

Clearly, since  $[[u]]_M$  is supposed to be determined for all suitable  $M$ ,  $[[u_G]]_M$  is also determined for all universes  $M$ .

An unproblematic extra condition would be to require that  $u_G$  is *constant* in the sense of being given by the same rule on each universe as we discussed in section 3.4.1.3 above. This holds for all quantifier expressions in natural languages that we are aware of. It is convenient to use something slightly stronger. Recall that EXT is a sufficient condition for constancy, even though there are a few type ⟨1⟩ quantifiers (e.g.  $\forall$  and  $Q^R$ ) that seem constant but fail to satisfy EXT. But in natural languages, with the exception of quantifier expressions involving a logical expression (like *thing*) that can be taken to stand for the universe, it seems that EXT always holds. In Chapter 4.5.3 we formulate this as a linguistic universal. For now, it motivates taking EXT as the extra assumption.

Let  $\mathcal{M}_0 = (M_0, I_0)$  be a fixed model for the non-logical vocabulary,  $V_u$ , of  $u$ . The *field of  $u$  in  $\mathcal{M}_0$*  is the set of all individuals in  $M_0$  ‘affected’ by  $I_0$ , i.e. all those of the form  $I_0(c)$  or belonging to a tuple in some relation  $I_0(P)$ , for  $c, P \in V_u$ .

### using global quantifiers

Let  $Q$  be a (global) quantifier of the same type as  $u$ . We say that  $u$  *uses  $Q$  relative to  $\mathcal{M}_0$*  if

$$(3.40) \text{ for every } M \text{ containing the field of } u \text{ in } \mathcal{M}_0, [[u]]_{(M, I_0)} = Q_M$$

### Proposition 17

*Let  $u$  be a quantifier expression whose sense is given, and such that  $u_G$  is EXT. Let  $\mathcal{M}_0$  be a fixed model for  $V_u$ , as above. Then there is a unique quantifier  $Q$  satisfying EXT, such that  $u$  uses  $Q$  relative to  $\mathcal{M}_0$ .*

<sup>26</sup>  $u_G$  is a symbol introduced here; it need not be an expression in the language to which  $u$  belongs. In type theory one uses  $\lambda$ -abstraction to form  $u_G$ ; e.g. in the example below, if  $u = u(j, C)$  was an expression containing the primitive non-logical symbols  $j$  and  $C$ , one would have  $u_G = \lambda x \lambda Y u(x, Y)$ .

<sup>27</sup> We are using the notation  $(M, A, c)$  for the model  $\mathcal{M} = (M, I)$ , where  $I(P) = C$  and  $I(c) = a$  (see Ch. 2.2).

Note that  $u_G$  is not strictly a quantifier expression in our sense, since we have not been using quantifiers that take individuals as arguments. But this is a straightforward extension of the notion of a (generalized) quantifier, which we employ in this section only.

*Proof.* Suppose  $V_u = \{P, c\}$  where  $P$  is one-place, and suppose  $u$  is of type  $\langle 1, 1 \rangle$ ; other cases are similar. Let  $I_0(P) = C$  and  $I_0(c) = j$ , so the field of  $u$  in  $\mathcal{M}_0$  is  $C \cup \{j\}$ . Define  $Q$  of type  $\langle 1, 1 \rangle$ , for any  $M$ , as follows:

$$(3.41) \quad Q_M(A, B) \iff [[u]]_{(M \cup C \cup \{j\}, C, j)}(A, B)$$

Clearly,  $u$  uses  $Q$  relative to  $\mathcal{M}_0$ . To see that  $Q$  is EXT, suppose  $A, B \subseteq M \subseteq M'$ . We have  $Q_M(A, B) \Leftrightarrow [[u]]_{(M \cup C \cup \{j\}, C, j)}(A, B) \Leftrightarrow [[u_G]]_{M \cup C \cup \{j\}}(C, j, A, B) \Leftrightarrow [[u_G]]_{M' \cup C \cup \{j\}}(C, j, A, B)$  (since  $u_G$  is EXT)  $\Leftrightarrow [[u]]_{(M' \cup C \cup \{j\}, C, j)}(A, B) \Leftrightarrow Q_{M'}(A, B)$ .

Now suppose  $Q'$  is EXT and used by  $u$  relative to  $\mathcal{M}_0$ . Take any  $M$ , and any  $A, B \subseteq M$ .  $Q_M(A, B) \Leftrightarrow [[u]]_{(M \cup C \cup \{j\}, C, j)}(A, B) \Leftrightarrow Q'_{M \cup C \cup \{j\}}(A, B)$  (since  $u$  uses  $Q'$ )  $\Leftrightarrow Q'_M(A, B)$  (since  $Q'$  is EXT). Thus,  $Q' = Q$ .  $\square$

Note that if  $u$  is pure,  $\mathcal{M}_0$  and the assumption about EXT are not needed: (3.41) just gives us the global quantifier that we already know that  $u$  expresses, i.e. its sense. But when  $u$  is not pure, we see that there is, relative to a fixed interpretation of the non-logical primitive expressions in  $u$ , a unique global quantifier  $Q$  used by  $u$ , provided EXT is assumed. So when in this book we say, somewhat sloppily, that  $u$  denotes  $Q$ , what we really mean is that  $u$  uses  $Q$ , relative to an assumed background model.

Furthermore, the practice in this book (and elsewhere) of writing *cat*, *student*, etc. for the interpretation of *cat*, *student*, etc. can be seen as another aspect of the same convention: relative to some background model  $\mathcal{M}_0$ ,  $\text{cat}^{\mathcal{M}_0} = \text{cat}$ ,  $\text{student}^{\mathcal{M}_0} = \text{student}$ , etc.

### 3.5.3 Examples

Consider *John*, and suppose that  $\text{John}^{\mathcal{M}_0} = I_0(\text{John}) = j \in M_0$ . Recall that the interpretation function of a model takes proper names to individuals, but that we interpret the *noun phrase* *John*, using  $[[\cdot]]$ , as a type  $\langle 1 \rangle$  quantifier. The sense of this noun phrase is well known: for all  $M$  and all  $a \in M$ ,

$$[[\text{John}]]_{(M, a)} = \{B \subseteq M : a \in B\}$$

Moreover, the pure relation  $\text{John}_G$  (given by  $[[\text{John}_G]]_M(a, B) \Leftrightarrow a \in B$ ) is clearly EXT. So Proposition 17 tells us that given  $\mathcal{M}_0$ ,  $\text{John}$  uses a unique EXT type  $\langle 1 \rangle$  quantifier  $Q$ . Indeed, this quantifier is precisely the Montagovian individual  $I_j$ :

$$Q_M(B) \Leftrightarrow [[\text{John}]]_{(M \cup \{j\}, j)}(B) \Leftrightarrow j \in B \Leftrightarrow (I_j)_M(B)$$

So this is what justifies our earlier claim that the NP *John* denotes  $I_j$ . In section 3.2.4 we considered another candidate, with the truth condition  $M \cap \{j\} \subseteq B$  instead, but ruled it out because (a) this becomes true when  $j \notin M$ , and (b) it is not EXT. Now we see that the EXT requirement in fact uniquely selects  $I_j$  as the only possible candidate.

Does one ever have occasion to consider the denotation of *John* on different universes but where the same individual  $j$  bears the name? Of course one does! We saw a technical circumstance in section 3.4.2 where this is so. But also in ordinary situations of speech or linguistic communication, the discourse universe can be much



more variable than the reference of names, say. We want to say, for example, that nothing depending on  $\text{JOHN}$  changes when the discourse universe is extended. That is, we want to say that  $\text{JOHN}$  is EXT. Likewise, we want to say that, in contrast with *three*,  $\text{JOHN}$  is not ISOM. But we *cannot* say these things unless a global quantifier is associated with  $\text{JOHN}$ . And that quantifier is  $I_j$ , given  $\mathcal{M}_0$ .

Similarly, consider a bare plural like *firemen*. Let a background model  $\mathcal{M}_0$  be given such that  $\text{fireman}^{\mathcal{M}_0} = C$ . By reasoning parallel to that for  $\text{JOHN}$ , one sees that the unique EXT quantifier used by *firemen* relative to  $\mathcal{M}_0$  is  $C^{\text{pl}}$  (section 3.2.1), given that the sense of *firemen* is

$$[[\text{firemen}]]_{(\mathcal{M}, \mathcal{A})} = \{B \subseteq M : \emptyset \neq A \subseteq B\}$$

Someone might want to question whether this is really the sense of *firemen*. For example, one would remove the requirement  $A \neq \emptyset$  if one regarded *unicorns are fierce* as true simply because there are no unicorns. This would be a dispute about the sense of *firemen*. But *given* that sense, there is no dispute about which type ⟨1⟩ quantifier *firemen* denotes (uses), relative to a background model.

Next, consider the example we started with, *three cats*. With quantified NPs, it is perhaps harder to see why one would want to say that they denote (i.e. use) global quantifiers. The obvious analysis of *three cats* employs the pure quantifier expression (quantirelation) *three*, which takes two arguments, a restriction and a scope. We devote the next chapter to quantirelation denotations. But in languages like English, a determiner and a restriction argument form a constituent (an NP) most naturally interpreted as a type ⟨1⟩ quantifier. So the question arises which global quantifier is the most reasonable choice, given that in a background model  $\mathcal{M}_0$ , *cat* denotes a set  $C$ . This quantifier can be described either as the *freezing* of the type ⟨1, 1⟩ quantifier *three* to  $C$ , or, as in section 3.2.1 above, as the *restriction* of the type ⟨1⟩ quantifier  $\exists_{=3}$  to  $C$ . Clearly one wants these descriptions to amount to the same thing. Now there are different definitions of these notions that achieve this; we discuss the options in Chapter 4.5.5. But we can already see that our definition of global restricted type ⟨1⟩ quantifiers of the form  $Q^{[A]}$  in section 3.2.1 is forced by the requirement of EXT.

First, it is perfectly clear what the *sense* of *three cats* is: viz.

$$(3.42) \quad [[\text{three cats}]]_{(\mathcal{M}, \mathcal{A})} = \{B \subseteq M : |A \cap B| = 3\}$$

Second, the pure relation  $(\text{three cats})_G$  is EXT; in fact, this relation is nothing but the type ⟨1, 1⟩ quantifier *three* (notice that *cats* here just functions as a placeholder for a one-place predicate expression<sup>28</sup>). So, third, it follows by Proposition 17, given that  $\text{cat}^{\mathcal{M}_0} = C$ , that there is a unique EXT global type ⟨1⟩ quantifier  $Q$  which *three cats* uses. More exactly, by (3.41),

<sup>28</sup> This becomes clearer if we write (3.42) instead as

$$[[\text{three cats}]]_{\mathcal{M}} = \{B \subseteq M : |\text{cat}^{\mathcal{M}} \cap B| = 3\}$$

$$\begin{aligned}
Q_M(B) &\iff [[\text{three cats}]_{(M \cup C, C)}(B)] \\
&\iff |C \cap B| = 3 \\
&\iff (\exists_{=3}^{[C]})_M(B)
\end{aligned}$$

That is,  $Q$  is precisely  $\exists_{=3}^{[C]}$ , or, with the notational convention we have been using,  $\exists_{=3}^{[cat]}$ . In particular, for  $Q_M(B)$  to hold, it is *not* required that  $C \subseteq M$ . To always require this might seem natural, but it is not compatible with EXT.

The clearest examples where one wants a non-pure quantifier expression to denote (use) a global quantifier are quantirelations, in particular determiners (to be discussed in the next chapter). Treating  $\text{John}'s$  or *some student's*, or, for that matter, *every* – except *John*, or *more male than female*, as denoting, on every universe, binary relations between sets simply is an obvious thing to do.

Consider  $\text{John}'s$ . Again the sense is fairly clear (provided certain parameters are set; see Chapter 7), and the corresponding pure relation is

$$(3.43) \quad [[\text{John}'s_G]]_M(a, A, B) \iff \emptyset \neq A \cap R_a \subseteq B$$

where  $R$  is some ‘possessor relation’ and  $R_a$  is the set of things  $R$ 'd by  $a$ . Interestingly, it is natural to think of the relation defined in (3.43) as the denotation of the possessive morpheme  $'s$  (note again that in (3.43),  $\text{John}$  just functions as an arbitrary individual-denoting expression). Indeed, the analysis we give of possessives in Chapter 7 is even more general than this. But the fact that such an analysis is possible is fully compatible with the fact that  $\text{John}'s$  is a determiner, and that its behavior can be fruitfully compared with that of other determiners. For example, it is EXT, but not ISOM. Also it has the property of *definiteness*. But these properties only apply if you fix  $a$  in (3.43). Doing that, we know (since  $\text{John}'s_G$  is EXT) that there is a unique EXT type  $\langle 1, 1 \rangle$  quantifier that  $\text{John}'s$  uses.

Definiteness is interesting from the present perspective. In Chapter 4.6 we present Barwise and Cooper's definition of that notion, according to which definite determiners are such that as soon as you freeze the restriction argument, the resulting NP denotes a *principal filter* (unless it is trivial); that is, there is a generating set  $A$  such that, over a universe  $M$ , the NP denotes exactly the set of subsets of  $M$  that include  $A$ . For example, the *ten boys* is generated by the set of boys (provided there are ten of them), and  $\text{John}'s$  *books* is generated by the set of books that  $\text{John}$   $R$ 's (provided he  $R$ 's at least one of them). Now an obvious question is whether the generating set depends on the universe. The definition allows this, but there is a theorem saying that under reasonable circumstances, the generating set does *not* vary with the universe (Proposition 10 in Chapter 4.6). These facts about definites seem at least mildly interesting, but note that they couldn't even be stated unless we allowed  $\text{John}'s$  as well as the *ten boys* to denote global type  $\langle 1, 1 \rangle$  quantifiers, and  $\text{John}'s$  *books* as well as the *ten boys* to denote global type  $\langle 1 \rangle$  quantifiers. That is, they presuppose that it makes sense to treat non-pure quantifier expressions as denoting (using) quantifiers. And the way this makes sense, we suggest, is by assuming that some model in the background fixes the denotations of the non-logical parts of these quantifier expressions, but not the universe.

## Type $\langle 1, 1 \rangle$ Quantifiers of Natural Language

We now come to the most important quantifiers for natural language semantics: the class of type  $\langle 1, 1 \rangle$  quantifiers. The term “quantirelation” was introduced for expressions in natural languages that denote in this class. Prime examples of quantirelations are determiners, quantificational adverbs, and in some languages classifiers of nouns, quantificational agreement affixes on verbs, etc. (see Chapter 0.1). Although syntactically varied, their common mode of operation is the following: they identify in the sentence a *restriction*, providing a subset of the (discourse) universe as the domain of quantification, and a *scope*, supplying another subset. These two sets are the arguments of the (local) quantifier that the quantirelation signifies.

By far the richest and most studied class of quantirelations are the determiners, and our examples are mainly taken from these. Determiners have the special feature of forming a syntactic constituent together with the restriction: a noun phrase, which, as we have seen, can be taken to denote a type  $\langle 1 \rangle$  quantifier. However, the particular semantic behavior of quantirelation denotations depends not on this syntactic feature, but rather on the fact that the restriction gives, in a sense that we shall make precise, the domain of quantification. This seems to be common to all quantirelations, so the fact that our examples are determiners is actually not a limitation. Neither is it a limitation of expressivity, since it seems plausible that all type  $\langle 1, 1 \rangle$  quantifiers denoted by some quantirelation are in fact denoted by some determiner.

That the restriction *restricts* the domain of quantification to the set it denotes—and hence deserves its name—has as a consequence that all quantirelation denotations are of a particular form: they are so-called *relativizations* of type  $\langle 1 \rangle$  quantifiers. An equivalent characterization, as it turns out, is that they satisfy two basic properties: EXT, introduced in the previous chapter, and conservativity, CONSERV. This is a fundamental observation. The notion of conservativity and the mechanism of relativization are presented in detail in sections 4.4 and 4.5. We establish the fundamental fact just alluded to and its ramifications, as well as the precise connection between relativization and the restricted type  $\langle 1 \rangle$  quantifiers defined in Chapter 3.2.1.

Section 4.6 presents a linguistically important class of determiners (and corresponding quantifiers)—the *definite* ones. We use a semantic definition of definiteness (due to Barwise and Cooper). Later on, in connection with partitive constructions (Chapter 7.11), we come back to the issue of the adequacy of that definition. In section 4.7 we briefly look at determiners which take more than one restriction argument, and the quantifiers they denote.

The final section, 4.8, discusses the effect of ISOM. The relativization of an ISOM type  $\langle 1 \rangle$  quantifier  $Q$  may be seen as a binary relation between numbers—indeed, the very same relation as the one corresponding to  $Q$ . Over finite universes, these quantifiers can be perspicuously represented in the so-called *number triangle*. The representation significantly facilitates stating, as well as proving, various facts about quantifiers. We present and illustrate the number-theoretic representation here; several applications will follow later.

The chapter begins, however, with numerous examples of quantirelations—indeed, determiners—and some discussion of the quantifiers they can be taken to denote, in section 4.1. We also reexamine in greater detail (section 4.2) the issue about the existential import of *all* and *every* which was raised in Chapter 1.1.1 in connection with the Aristotelian square of opposition, as well as the relation, or lack of relation, between syntactic and semantic number indicated in Chapter 0.1. Again with reference to the Aristotelian square (or rather its modern version), we look in section 4.3 at negations and other Boolean operations on type  $\langle 1, 1 \rangle$  quantifiers.

#### 4.1 EXAMPLES OF DETERMINERS

Here are some examples of English determiner expressions, of increasing complexity.

- (4.1) a. some, a, all, every, no, several, most, neither, the, both, this, these, my, John's, many, few, enough, a few, a dozen, ten
- b. at least/more than/fewer than/at most/exactly ten, all but ten, all but at most five, infinitely many, uncountably many, at most finitely many, all but finitely many, about two hundred, nearly a hundred, almost all, an even number of, an infinite number of, the ten, John's ten, each professor's, most students', no child's, at least two-thirds of the, less than 10 percent of the, exactly three-quarters of the, half of the, half of John's, most of Henry's, two-thirds of Mary's several, all... except John, no... but Mary, all conservative, John's liberal, exactly two green, more male than female
- c. not all, not every, not many, no fewer than ten, between five and ten, not more than half of the, some/most but not all, at least two and no more than ten, either fewer than five or else more than a hundred, neither John's nor Mary's, John's but not Mary's, not one of John's, more of John's than of Mary's, fewer of the male than of the female, fewer than five of the ten or more, more than half of the twenty, several of each professor's, at least one of most students', none of several teachers', most male and all female, neither the red nor the green

The list is intended to show the richness of English determiner expressions that *can* be interpreted as type  $\langle 1, 1 \rangle$  quantifiers.<sup>1</sup> In various cases one might argue for another

<sup>1</sup> Some but not all of these examples were given in Ch. 0.1. For a more exhaustive list, illustrating numerous constructions of complex determiners, see Keenan and Stavi 1986: sect. 1.1. Some

treatment. For example, it has been claimed that the indefinite *a* or the definite *the* (see Chapter 0.3), or numerals like *ten*, should not be treated as quantifiers. But our point here is just that they *can* be so treated, and that it is then often obvious which quantifiers they denote:

$$the_M(A, B) \iff A \subseteq B \text{ and } |A| = 1$$

(this is the singular *the*, also called *the<sub>sg</sub>*; for the plural case, *the<sub>pl</sub>*, the second conjunct is  $|A| > 1$ , but see section 4.2.2 below for further discussion);

$$ten_M(A, B) \iff |A \cap B| = 10$$

Another reading that might be possible here is *at least ten*:

$$at\ least\ ten_M(A, B) \iff |A \cap B| \geq 10$$

$$all\ but\ ten_M(A, B) \iff |A - B| = 10$$

$$all\ but\ at\ most\ ten_M(A, B) \iff |A - B| \leq 10$$

$$more\ than\ ten\ but\ no\ more\ than\ twenty_M(A, B) \iff 10 < |A \cap B| \leq 20$$

$$the\ ten_M(A, B) \iff |A| = 10 \text{ and } A \subseteq B$$

$$John's\ ten_M(A, B) \iff |A \cap \{a : R(j, a)\}| = 10 \text{ and } A \cap \{a : R(j, a)\} \subseteq B$$

(where *R* is a contextually given relation, cf. *John's ten bikes*, *John's ten exams*, *John's ten friends*, *John's ten toes*).

Note that for quantifiers with an explicit number condition we have included that condition in the truth conditions. The alternative is to allow *partial* quantifiers, so that e.g. *John's ten<sub>M</sub>(A, B)* would be *undefined* when  $|A \cap \{a : R(j, a)\}| \neq 10$ . The advantage would be that one might want a sentence of that form to lack truth value under those circumstances, rather than being false. The disadvantage would be that partial quantifiers are somewhat more unwieldy to deal with.<sup>2</sup>

The determiners listed in (4.1c) all have *one* restriction argument. There are also determiners with two or more restrictions, like those italicized here:

- (4.2) a. *More* students *than* teachers attended the meeting.  
 b. *Twice as many* men *as* women came to the party.  
 c. *Proportionately more* students *than* professors smoke.

Discussion of these is deferred to section 4.7 below.

As we said, in most cases it is clear what the denotations of the determiners in (4.1c) are. But sometimes the interpretation needs further discussion and analysis. We mention a few cases here.

cases, such as quantifier and adjective combinations, are not uncontroversially agreed to be determiners by all syntacticians.

<sup>2</sup> Barwise and Cooper (1981) use partial quantifiers, whereas Keenan and Stavi (1986), van Benthem (1986), and Keenan and Westerståhl (1997) follow the route taken here. For some discussion, see Keenan and Stavi 1986: 298–9 and Westerståhl 1989: sect. 3.7.

Going back first to possessive determiners, we take *John's* to be *John's one or more*:<sup>3</sup>

$$(4.3) \text{ John's}_M(A, B) \iff \emptyset \neq A \cap \{a : R(j, a)\} \subseteq B$$

Also, here is our interpretation of a typical complex possessive determiner. We use the notation  $R_c = \{a : R(c, a)\}$  for any binary relation  $R$ .

$$(4.4) \text{ most graduates}'_M(A, B) \iff \\ \text{most}(\{c \in \text{graduate} : A \cap R_c \neq \emptyset\}, \{c : A \cap R_c \subseteq B\})$$

That is, the truth conditions are such that only graduates 'possessing' something in  $A$  matter. This means, for example, that the truth of *Most graduates' job offers were acceptable* is unaffected by those graduates that didn't get any offers. Otherwise, any such graduates would trivially satisfy the condition that all their job offers were acceptable, so the sentence would be true if most graduates didn't get any offers, which is clearly wrong. This and other aspects of possessives are discussed in detail in Chapter 7.

The interpretation of basic *exceptive determiners* is fairly clear.

$$\text{no\_but John}_M(A, B) \iff A \cap B = \{j\} \\ \text{every\_except John}_M(A, B) \iff A - B = \{j\}$$

More debatable, perhaps, is the meaning of, say, *no\\_except C*, as in *No students except freshmen were invited to the party*. The interpretation of *exceptives* has been much discussed in the literature; we give an overview, and a partly new account, in Chapter 8.

In some of the above examples, like *about two hundred*, *vagueness* may be an obstacle to giving an interpretation. But this is the problem of finding an adequate semantics for vague expressions in general, and is not specific to determiners.

Some of the determiners above require input from the *context* for their interpretation. As already noted, *most* sometimes means *more than half of the*—this is the default interpretation we use here—but sometimes the 'threshold' is higher. There is no problem in thinking of context as providing this. In other cases, context dependence is so strong that one may doubt if extensional (generalized) quantifiers are adequate as interpretations. Consider

- (4.5) a. Too many doctors attended the reception.  
b. Too many lawyers attended the reception.

Even in a situation where the set of lawyers at the reception was identical to the set of doctors at the reception, these two sentences could have different truth values. One may thus decide to treat *too many* as *intensional* and exclude it from the present treatment. Similarly for expressions like *surprisingly many* or *not enough*. But

<sup>3</sup> We give here the *universal* reading of possessives; there are also existential and other readings. Even *John's* sometimes has an existential reading: *John's fingers are dirty*. In Ch. 7 we present an analysis of possessives which covers all of these readings.

One might also distinguish the singular *John's = John's one* from the plural *John's = John's two or more*; see sect. 4.2.2.

one may also maintain that the nouns of the respective sentences, and perhaps other relevant material, form part of the context, so that different standards for being too many are generated. In the literature, this question has been discussed especially for *many* and *few*; some authors exclude them on the ground of intensionality whereas others attempt a context-dependent extensional interpretation.<sup>4</sup> Again, the issue is part of the more general one of how to deal with context dependence and intensionality. We do not need to take a stand on it here, but will return to the interpretation of *many* and *few* in Chapter 6.1.

Although our examples so far were all determiners, there are other quantifier expressions with similar interpretations. For example,

- (4.6) a. Dogs never laugh.  
b. Graduate students usually do some teaching.

mean roughly that no dogs laugh, and that most graduate students do some teaching; so *never* and *usually* can here be taken to denote the quantifiers *no* and *most*, respectively. However, these adverbs of quantification have a wider use, as exemplified in

- (4.7) Cats usually dislike dogs.

where, in one reading, *most* is still the quantifier, but it now quantifies over *pairs* of cats and dogs, saying that most such pairs are such that the first element dislikes the second. This quantifier is no longer of type  $\langle 1, 1 \rangle$ , but rather of the polyadic type  $\langle 2, 2 \rangle$ , and we save the discussion of these until Chapter 10.2.

## 4.2 ON EXISTENTIAL IMPORT AND RELATED MATTERS

Before pursuing our presentation of the properties of type  $\langle 1, 1 \rangle$  quantifiers, we will make a few comments concerning requirements that certain quantifier relations appear to impose on the number of elements in the restriction argument. The first issue was raised already by Aristotle's version of the square of opposition (Chapter 1.1.1) and concerns the meaning of *every*, *all*, *no*, and *not all*: namely, whether these determiners, and in particular *all* and *every*, have *existential import*, i.e. if they satisfy this condition:

$$Q_M(A, B) \Rightarrow A \neq \emptyset$$

One should carefully distinguish existential import from a notion we call *positivity*, which says that

$$Q_M(A, B) \Rightarrow A \cap B \neq \emptyset$$

<sup>4</sup> Attempts at such interpretations can be found in Barwise and Cooper 1981; Westerståhl 1985*b*; Cohen 2001; among others, whereas Keenan and Stavi (1986) exclude *many* and *few* from the class of (extensional) determiners. Fernando and Kamp (1996) give an intensional analysis in terms of possible worlds.

For example, *some*, *at least five*, *exactly three*, *most* denote positive quantifiers, whereas *no*, *at most four*, *few* do not. Also, *not all* has existential import (since  $A - B \neq \emptyset$  implies  $A \neq \emptyset$ ) but is not positive, whereas *not all<sub>ei</sub>* has neither property. But the point here is that positivity is not a problematic or controversial property in the way that existential import has been.

#### 4.2.1 Existential import

‘Naive’ speakers of English, who are not trained in logic or linguistics, typically feel that:

1. Statements with *every* or *all* imply that they have instances; i.e. the restricted domain is non-empty, and the generalization holds non-vacuously.
2. In *not every* and *not all*, *not* is ordinary negation (termed outer negation in this book and elsewhere).
3. Statements with *not every* or *not all* imply that there are witnesses to the failure of the negated universal generalization.
4. When pressed, they concede that it is impossible to support with a counterexample the notion that a vacuous generalization with *every* or *all* is false. Usually, they then reluctantly agree that it is less problematic to regard such a statement as true than to regard it as false.

Many readers will remember such intuitions from the time when they began to study logic or linguistic semantics.

A tension exists between these intuitions, as the first is in conflict with the last three. Holding onto the first and second requires giving up the third and fourth—and, moreover, accepting something that seems highly counterintuitive to naive speakers today:

5. Statements with *not every* or *not all* are true if the restricted domain is empty.

As we saw, Aristotle and many medievals seemed ready to accept this price for holding onto the first and second intuitions. Nowadays it is clear that alternative means exist for reconciling all these intuitive judgments, by providing an explanation for the first one that does not depend on the sentence uttered being false.

One available alternative involves distinguishing clearly between what a statement communicates and the truth conditions of the sentence used in making it. Most statements communicate everything required for truth of the sentence that is uttered; many statements communicate more. Grice (1967) famously proposed a theory of how this is possible—indeed inevitable—considering the fact that statements are meant to communicate pertinent information, not simply to recite facts. The difference between mere record keeping (whether in a natural language or an artificial database language) and communication has as a consequence that statements made in communicating with another person are selected not simply to be true but also to be informative, relevant, and even cooperatively phrased. The



point of stating something in communication is to influence one's interlocutor in accord with a purpose that is (usually) mutually accepted. And achieving this end requires a more stringent selection of statements than merely choosing an arbitrary true one.

Because a statement's normality involves not only its truth (or falsehood, in the case of denial), but, moreover, that the statement is informative, relevant, and cooperatively worded, it will be odd in a wide range of circumstances to assert a universally quantified sentence for which it is known that the restricted domain is empty. The oddness is adequately explained by the statement's lack of informativeness, its truth being guaranteed by the already known emptiness of the quantifier's domain. Thus the naive intuition (1) that universal statements imply non-emptiness of the domain is explained by the fact that such statements are meant to communicate new information, not by their being false when the domain is empty.

Indeed, statements with `all` and `every` seem fully acceptable when it is unknown whether or not the domain is empty—provided they are true, relevant, etc.

(4.8) All solutions to this system of equations are integers.

Clearly this sentence could be acceptable (for example, provable in a certain theory) regardless of whether there were any solutions at all to the system of equations in question, or at least regardless of whether one has any knowledge about the existence of such solutions. Moreover, this sentence quite clearly would be true—rather than false—should the domain turn out to be empty.

Similarly for statements of laws or rules:

(4.9) All trespassers will be prosecuted.

There don't have to be any actual trespassers for this to hold.

Another strategy for assessing intuitions like the first one above is to consider whether the implication of non-emptiness can be explicitly *canceled*. This strategy gives some definite results. First, a test case:

(4.10) # It is true that at least two graduate students at the party were drunk, because there were in fact no graduate students at the party.

Speakers would not accept this: they would claim that the second part of the sentence *contradicts* the first. This is in agreement with our strong intuition that at least two does have existential import, which is obvious from the fact that it entails at least one.

Next, a fairly clear negative case:

(4.11) It is true that no graduate students at the party were drunk, because there were in fact no graduate students at the party.

The first part of the sentence, if uttered without the second part, would normally lead one to assume that there were graduate students at the party. In this respect, the negative universal quantifier `no` is just like the positive ones `every` and `all`. The second part of the sentence contradicts this implication, but in doing so does not render the

first part *false*. Instead, the implication is merely canceled. The first part of the sentence remains true (if requiring special circumstances to assert). These observations lead us to conclude that apparent existential import with *no* is an implicature (or something like that) arising from communication with the word, and not part of the quantifier expression's meaning.

Finally consider

- (4.12) It is true that all graduate students at the party were drunk, because there were in fact no graduate students at the party.

Again, it seems that the implication that there were graduate students at the party is cancelable, though perhaps with slightly more difficulty than in the previous case.<sup>5</sup>

At least, that is a common verdict, although the issue is somewhat subtle. To appreciate the force of this verdict, compare the following:

- (4.13) a. # It is true that the graduate students at the party were drunk, because there were in fact no graduate students at the party.  
 b. # It is true that Henry's graduate students were drunk, because Henry doesn't have any graduate students.

Most English speakers find these incoherent enough to warrant concluding that the and Henry's do have existential import, in contrast with *every* and *all*.

Perhaps one can sum up the situation as follows. *Assertions* of universal statements have varying degrees of existential import. Every assertion needs some sort of *warrant*. Sometimes the warrant has no information about the emptiness or not of the first argument of the determiner (the restriction term), as in (4.9), or may even be explicitly neutral about its emptiness, as in (4.8). But these cases are a bit special. Usually, the warrant is some observation or inference, and then the assertion can imply rather strongly that the restriction argument is non-empty. But note that all of these remarks apply to assertions. If one thinks of the linguistic meaning of an expression as, roughly, what is common to all assertions involving that expression, it makes sense *not* to endow *all* with existential import. The existential import of assertions of universal statements is a matter for pragmatics, not semantics.

A related point concerns the fact that statements with *every*, *all*, and *no* ordinarily carry stronger implications about the restricted domain of quantification than

<sup>5</sup> Imagine the following dialogue:

—All graduate students at the party were drunk.

—I'm glad I didn't go!

—But nobody was drunk at that party.

—But you just *said* there were drunk people there!

—No I *didn't say* that; I only *said* that all graduate students at the party were drunk, which happens to be true because there were no graduate students at the party!

Clearly the first speaker is seriously misleading the second. But the reason he can do this, and thus be judged uncooperative or even devious while still not being incoherent, is precisely that implying something is not the same as saying it and, in particular, conversationally implicating something does not commit the speaker to it even though the hearer is invited to impute it as a belief of the speaker's.

that it is non-empty. An ordinarily informed person would think it odd to assert (4.14a) and (4.14b) instead of (4.14c), and to assert (4.14d).

- (4.14) a. Every natural satellite of the Earth has been visited by humans.  
 b. Some natural satellite of the Earth has been visited by humans.  
 c. The natural satellite of the Earth has been visited by humans.  
 d. No natural satellite of the Earth has been colonized by humans.

Every and no are normally used to generalize over a substantial number of instances, and some is normally used when referencing a specific instance would be awkward. Although (4.14a), (4.14b), and (4.14d) are all true, they are odd things to say because one instance is not ordinarily enough to generalize over, and referencing this instance is not at all difficult. So these three sentences are not cooperatively worded, in the sense used here; they falsely suggest the existence of more than one natural satellite of the Earth.

In fact, two instances are not ordinarily enough to justify use of every, all, or no—since both and neither cover this situation handily. Although many readers will not find an assertion of (4.15a), (4.15b), or (4.15c) odd,

- (4.15) a. Every natural satellite of Mars has been photographed.  
 b. Some natural satellite of Mars has been visited by a robotic spacecraft.  
 c. No natural satellite of Mars has been visited by humans.

astronomers would, knowing as they do that Mars has but two moons: Phobos and Deimos.

The first naive intuition about universal claims—that their domain is non-empty—is thus really just part of a broader intuition: that the domain contains a significant number of entities, certainly more than two. The broader intuition can be explained by, for instance, Grice's theory of conversational implicature—in a closely related way to its explanation of the narrower case (non-emptiness of the domain). We need not treat either intuition as a reason for taking statements with every, all, or no to be false if their domain is empty, or has just one member, or just two.<sup>6</sup>

These observations demonstrate how cautious one should be in drawing conclusions about what quantifiers require by way of the cardinality of their domain. In particular, care must be taken to distinguish the fact that if one *knows* that the noun A has empty denotation, it would often be *odd to utter* a sentence of the form  $\forall x A x$  are B, from facts about the perceived falsity (or truth) of the sentence in this case.

<sup>6</sup> The intuition with no has apparently never been regarded as justifying such treatment, even by Aristotle or his followers. The broader intuitions with every and all likewise have apparently never been regarded as justifying such treatment, although the implication that the domain contains more than one (or even two) members seems just as hard to cancel (or easy to cancel) as the implication that it contains at least one member. In this light, it may seem odd that the latter intuition about every and all is the only one ever codified into truth conditions.

## 4.2.2 Syntactic and semantic number

A related question concerns the relation, if any, between the syntactic or grammatical number of a noun and the quantificational claim made by a determiner preceding that noun. Already in Chapter 0.1 we said that we take the singular and the plural form of count nouns to denote the same set of individuals, and that grammatical number is not considered to be significant in and of itself; in particular, it does not in any simple way indicate whether one is quantifying over individuals or over pluralities. However, even when one restricts attention to quantification over individuals—as we do in this book—there is sometimes an issue of whether the claim made by a quantified statement about the number of individuals in the set denoted by the restriction noun is related to the grammatical number of that noun. It is instructive to look at some examples.

Let us say that the *number condition* of  $Q$  is the strongest claim about  $|A|$  that logically follows from  $Q(A, B)$ .<sup>7</sup> For example, the number condition of *at least seven* is  $|A| \geq 7$ , the number condition of *exactly two* is  $|A| \geq 2$  (unless  $A$  has at least two elements, the statement *exactly two* ( $A, B$ ), i.e.  $|A \cap B| = 2$ , cannot be true), and the number condition of *at most five* is the trivial condition  $|A| \geq 0$ . In the latter case we simply say that there is *no* number condition.

Note that the number condition of the denotation of a quantirelation expression is part of the meaning of that expression, and does not concern implicatures, etc. (even if, as we saw above, the border is not always crystal clear). Consider first the following examples, where we also indicate the syntactic number of the noun:

	<i>quantified expression</i>	<i>number condition</i>
	zero As are B	—
	at most one A is B	—
	more than one A is B	$ A  > 1$
(4.16)	exactly one A is B	$ A  \geq 1$
	no A is B	—
	no As are B	—
	all As are B	—
	every A is B	—

Zero and all require plural form but have no number condition. The determiners ending with one above require singular form, as does every, but nothing follows from this about the number condition. No can take both forms, but in neither case is there a number condition. In all of these examples, the grammatical number is semantically irrelevant.

<sup>7</sup> We can omit reference to the universe  $M$  here, as will be explained in sect. 4.5 below. It is generally obvious what  $Q(A, B)$  implies about  $|A - B|$  and  $|A \cap B|$ , so we discuss only the sometimes disputed question of what  $Q(A, B)$  implies about  $|A - B| + |A \cap B|$ .

A more interesting case is the definite article. In many languages, the choice of article is correlated with the grammatical number of the noun, and one often assumes that the singular and the plural article have different meanings, and correspondingly that the English *the* is ambiguous, or at least that its meaning depends on the grammatical number of the noun:

	<i>quantified expression</i>	<i>number condition</i>
	$the_{sg}$ A is B	$ A  = 1$
(4.17)	$the_{pl}$ As are B	$ A  > 1$
	$all_{ei}$ As are B	$ A  \geq 1$
	the ten As are B	$ A  = 10$

It is common to claim both that *the* means either  $the_{sg}$  or  $the_{pl}$ , and that the syntactic number of the noun determines which one is meant. Note that this requires that context sets (Chapter 1.3.4) or some notion of salience are used: one can clearly say *The table has been cleaned* even when there are many tables in the discourse universe. But combined with context sets or salience, the two claims may seem quite tenable.

However, we shall have occasion to question both claims in connection with one interesting class of determiners: the possessives, discussed in Chapter 7. We argue there (Chapter 7.8) that if the popular account of possessives, which takes these to essentially involve the definite article, is to be correct, it needs to interpret *the* as our old Aristotelian friend  $all_{ei}$ , and not as  $the_{sg}$  or  $the_{pl}$ .<sup>8</sup>

The possessive determiners themselves have a characteristic behavior with respect to syntactic number, in that they allow both forms:

	<i>quantified expression</i>	<i>number condition</i>
	John's A is B	$ A  \geq 1$
(4.18)	John's As are B	$ A  \geq 1$
	each girl's A is B	—
	each girl's As are B	—

There is no number condition on *each girl's*, since the empty set stands in the relation *each* (the subset relation) to any set. The condition on *John's* comes from the requirement that John 'possess' at least one thing in  $A$ , i.e. that  $A \cap R_j \neq \emptyset$  ((4.3) in section 4.1). But the singular case does not in general require  $|A \cap R_j| = 1$ , and the plural case not always  $|A \cap R_j| > 1$  (see Chapter 7.2). Consider:

(4.19) Each girl's brother feels protective of her.

This sentence does not make a claim only about girls with a single brother; nor does it entail that each girl has only one brother. Likewise, there is no way in which a

<sup>8</sup> All this presupposes that *the* should be treated as denoting a quantifier. See Ch. 0.3 for some hints about a different approach.

particular brother for each girl, when there is more than one, would be salient. Rather, the truth conditions appear to be the same as for

(4.20) Each girl's brothers feel protective of her.

By the same token, the claim that sentence (4.20) makes is not limited to girls with more than one brother. So in general we shall take possessive determiners not to have conditions tied to the syntactic number of the noun (although all of them will involve conditions of the form  $|A \cap R_a| \geq 1$ ). This is not to deny that additional requirements on  $A \cap R_a$  may sometimes be conversationally implicated, nor that grammatical number plays a role. But such extra requirements are not part of the truth conditions for possessive determiners (Chapter 7).<sup>9</sup>

Let us look at one more example:

<i>quantified expression</i>	<i>number condition</i>
(4.21) some A is B	$ A  \geq 1$
some As are B	$ A  > 1$ (?)

Some takes both forms, and the number condition in the singular case is clear. But what about the plural case? Logicians standardly take *some* to denote *at least one*, thus making no distinction between the singular and the plural case. In this book too, *some* = *at least one*. But we should at least ask if the use of, for example, *some women* entails or merely implicates that more than one woman is involved. In the previous subsection we saw, mostly for the case of *all* and *every*, some issues that such a query will require one to investigate. We shall not repeat these for *some*, only note that in this case we are unsure of the result. In fact, it seems to us that it might very well be that 'plural' *some* has the number condition  $|A| > 1$ .<sup>10</sup> In that case one should, when interpreting the English *some*, distinguish between *some<sub>sg</sub>* = *some* = *at least one* and *some<sub>pl</sub>* = *more than one*.

### 4.3 BOOLEAN OPERATIONS

The list of examples in section 4.1 above shows that in many cases complex determiners are Boolean combinations of simpler determiners. This raises natural questions

<sup>9</sup> Likewise, the possessed noun's number may interact with semantic constraints on which objects are 'possessed'. For example, *John and Mary's sister* can only stand for a sister of both John and Mary, whereas *John and Mary's sisters* can be either sisters of both or, when John and Mary are not siblings, sisters of either of them.

<sup>10</sup> One has to make a judgment about issues like these: (1) Is the following a contradiction or does the second sentence merely cancel an implicature?

- (i) Some women are at the door. Only one woman is there.
- (2) If I promise to give you colored pencils and give you just one, have I then broken my promise?
- (3) It is, of course pedantic to assert
- (ii) Bush didn't invade some countries. He invaded only one.

But is *Bush invaded some countries* false, or merely misleading?

about *closure* under Boolean operations. As we pointed out in Chapter 3.2.3, where the same questions were raised for NPs, there is, on the one hand, the syntactic issue of whether a certain kind of expression allows arbitrary combinations with *and*, *or*, *not*, and the like, and on the other hand, the distinct semantic issue whether the class of denotations of these expressions is such that if  $Q$  and  $Q'$  belong to it, then so do  $Q \wedge Q'$ ,  $\neg Q$ , and possibly  $Q \neg$ .

Here we are mostly interested in the semantic question. For determiners, it is often practical to make the following

*idealizing assumption:*

( $\text{BCI}_{\text{Det}}$ ) The class of determiner denotations in a given language, as a subclass of the class of type  $\langle 1, 1 \rangle$  quantifiers, is closed under Boolean operations.<sup>11</sup>

In reality, this may not be strictly true. Note, however, that the issue is not which expressions of the form *Det* and *Det* or *not Det* (in the case of English) are clumsy or long-winded and therefore hardly ever used, but which ones are meaningful. And then the question is whether, if such an expression is judged not meaningful, there is *another* determiner expression that denotes the quantifier. After all, this is all that is required. For example, *not some* and *not most* may be judged ill-formed and hence meaningless, but the negation of *some* is expressed by *no*, and the negation of *most* is expressed by *at most half of the*, so both are determiner denotations.

Barwise and Cooper (1981: 194–200) discuss Boolean operations on NPs and determiners, but mostly seem to focus on the syntactic issue. The corresponding semantic assumption for NPs is

( $\text{BCI}_{\text{NP}}$ ) The class of NP denotations in a given language, as a subclass of the class of type  $\langle 1 \rangle$  quantifiers, is closed under Boolean operations.

So we have two semantic claims, and two parallel syntactic claims. As we indicated in Chapter 3.2.3, the syntactic claims are almost certainly not literally correct, although they too might sometimes serve as useful idealizations. The semantic claims are empirically much harder to assess, since there are so many phrases that in principle could serve as determiners. For example, Barwise and Cooper cite some data concerning conjoining an increasing and a decreasing NP, indicating that this might not always be possible. But as they themselves admit, the data are not clear. For example, when the NPs involve complex determiners, conjunction seems fine, as in

(4.22) At most three women and a least five men came.

The line between clumsiness or long-windedness and ungrammaticality is not always sharp. Note also that the putative difficulties concern NPs; for determiner denotations, there seems to be no corresponding problem: *more than five and at most twelve, between three and six*.

<sup>11</sup> If our assumption that every quantifier denotation is also a determiner denotation is holds within every language, then ( $\text{BCI}_{\text{Det}}$ ) can equally well be named ( $\text{BCI}_{\text{Quantifier}}$ ).

It may be thought that inner and outer *negation* provide clearer cases of operations that transcend the class of determiner or NP denotations. These operations were defined for type  $\langle 1, 1 \rangle$  quantifiers in Chapter 3.2.3, and are very natural for determiner denotations (or quantirelation denotations) too:

#### negations and duals of type $\langle 1, 1 \rangle$ quantifiers

For  $Q$  of type  $\langle 1, 1 \rangle$ , define:

$$(\neg Q)_M(A, B) \iff \text{not } Q_M(A, B) \quad (\text{as usual})$$

$$(Q\neg)_M(A, B) \iff Q_M(A, M - B)$$

$$Q^d = \neg(Q\neg) = (\neg Q)\neg \quad (\text{as usual})$$

A sentence of the form  $[[\text{Det } N]_{\text{NP}} \text{ VP}]_S$  can be syntactically negated by putting *it is not the case that* in front, *or* by negating the VP. Sometimes these negations can be effected by choosing another Det. For example, applying the first kind of negation to *Most students passed*, we get *It is not the case that most students passed*, but this can also be expressed by *At most half of the students passed* (outer negation). In the second case we obtain *Most students did not pass*, and this could instead be put *Fewer than half of the students passed* (inner negation, or, in Keenan's term, *post-complement*).

Thus, the inner and outer negation, as well as the dual of a determiner denotation, is sometimes also a determiner denotation.

$$\neg \text{some} = \text{no}; \text{some}\neg = \text{not all}; \text{some}^d = \text{all}$$

$$\neg \text{more than half of the} = \text{at most half of the}; \text{more than half of the}\neg = \text{fewer than half of the}; \text{more than half of the}^d = \text{at least half of the}$$

(on finite universes, as usual). Other cases have seemed more doubtful. Consider *at least three*.

$$\neg \text{at least three} = \text{fewer than three}; \text{at least three}^d = \text{all but at most two}; \text{at least three}\neg = ?$$

Westerståhl (1989) attempted a principled argument that *at least three* $\neg$ , i.e. the quantifier  $Q$  defined by

$$(4.23) \quad Q_M(A, B) \iff |A - B| \geq 3$$

along with similar quantifiers, was not a determiner denotation—the reason being that phrases of the form *all but Q As are B*, where  $Q$  is *three*, *at most five*, *at least two*, etc., should imply *some As are B*. But that may only be a pragmatic implication; alternatively, it might instead concern the syntactic question of which determiner expressions are well-formed. In any case, observe that if one allows the (single) phrase *not all*, *not all but one*, and *not all but two* as a determiner, it denotes precisely the  $Q$  in (4.23).



Likewise, consider exception determiners, like *no—but John*. We have

$$(no\_but\ John)\neg = every\_except\ John$$

but what about the outer negation? Here one needs to express the condition

$$A \cap B \neq \{j\}$$

Does any determiner express that? The following suggestion is at least logically possible: if John then some other. The idea is that If John then some other A is B says that either John is not an A that is B, or someone else too is an A that is B, which seems to be the desired truth condition.

On the other hand, for a possessive determiner like John's, the outer negation, given our definition in section 4.1, expresses the condition

$$A \cap R_j = \emptyset \text{ or } (A \cap R_j) - B \neq \emptyset$$

This time it may be harder to cook up a corresponding determiner.

Finally, consider proper names and the claim ( $\mathbf{BCI}_{NP}$ ). It might seem that no NP denotes  $\neg I_j$ , since not John is certainly not an NP. But take instead *only things other than John*. This is an English NP, and it does have the desired denotation.

These examples illustrate the difficulty of empirically deciding ( $\mathbf{BCI}_{Det}$ ) and ( $\mathbf{BCI}_{NP}$ ), and also that their chances of being true are greater than it might seem at first sight. But regardless of whether they are strictly true or not, they are sometimes useful and harmless idealizations.<sup>12</sup>

As to the logical relations between ( $\mathbf{BCI}_{NP}$ ) and ( $\mathbf{BCI}_{Det}$ ), it seems plausible that the former implies the latter. In section 4.5.5 below we show that this is indeed true, given some reasonable assumptions.

A quantifier, together with its inner and outer negation and its dual, forms a natural unit, a *square of opposition*. Let us make this official:

#### squares of opposition

For  $Q$  of type  $\langle 1, 1 \rangle$  (or type  $\langle 1 \rangle$ ), the *square of opposition for  $Q$* ,  $\text{square}(Q)$ , is

$$\text{square}(Q) = \{Q, \neg Q, Q\neg, Q^d\}$$

As we saw in Chapter 1.1.1, the Aristotelian square of opposition, which is 'almost' identical to  $\text{square}(\text{some}) = \{\text{some}, \text{no}, \text{not all}, \text{all}\}$ , was the start of the theory of quantifiers. But every quantifier spans a square of opposition. Here are some simple facts about squares:

<sup>12</sup> In Keenan's work, e.g. Keenan and Stavi 1986, it is a crucial feature that NP denotations are closed under Boolean operations. But this is claimed only from a *local* perspective. In a given (finite) universe whose individuals are *John, Mary, Bill, . . . , Harriet*, it is true that  $\neg \text{John}$  can be expressed as  $Mary \vee Bill \vee \dots \vee Harriet \vee \text{nobody}$ . But this definition doesn't work in a larger universe. You have to use a different definition in each universe, which is precisely why Keenan's local claim does not concern the global claim ( $\mathbf{BCI}_{NP}$ ).

**Fact 1**

- (a)  $\text{square}(0) = \text{square}(1) = \{0, 1\}$ .
- (b) *If  $Q$  is non-trivial, so are the other quantifiers in its square.*
- (c) *Each quantifier in a square spans that same square. That is, if  $Q' \in \text{square}(Q)$ , then  $\text{square}(Q) = \text{square}(Q')$ .*
- (d)  *$\text{square}(Q)$  has either two or four members.*

*Proof.* (a) is obvious. For (b), non-triviality of  $Q$  means that there are  $M$  and  $A, B \subseteq M$  such that  $Q_M(A, B)$ , and there are  $M'$  and  $A', B' \subseteq M'$  such that not  $Q_{M'}(A', B')$ . It is easy to see that this extends to the other quantifiers in the square. For example, let  $B_1 = M - B$  and  $B_2 = M' - B'$ . Then  $Q_M(A, M - B_1)$  and not  $Q_{M'}(A', M' - B_2)$ . That is,  $(Q\neg)_M(A, B_1)$  and not  $(Q\neg)_{M'}(A', B_2)$ , so  $Q\neg$  is non-trivial too.

(c) is a simple calculation, using the usual laws for negations and duals (Chapter 3.2.3), by which, for example,  $(Q\neg)^d = \neg Q$  and  $(Q\neg)\neg = Q$ .

As to (d), note that any quantifier is distinct from its (outer) negation, so there are at least two elements in a square. And if (say)  $Q = Q^d$ , then (by the above laws)  $\neg Q = Q\neg$ , so there are either exactly two or exactly four elements.<sup>13</sup>  $\square$

## 4.4 RELATIVIZATION

Every statement involving a generalized quantifier  $Q$  takes place within some universe  $M$ . Sometimes it is useful to be able to mirror this relativization to a universe *inside*  $M$ . This means defining a new quantifier with one extra set argument which says that  $Q$  behaves on the universe restricted to that argument exactly as it behaves on  $M$ . The relativized quantifier,  $Q^{\text{rel}}$ , explicitly restricts the domain of quantification of  $Q$  to that argument. Here is the definition.

**definition of  $Q^{\text{rel}}$** 

If  $Q$  is of type  $\langle n_1, \dots, n_k \rangle$ ,  $Q^{\text{rel}}$  has the type  $\langle 1, n_1, \dots, n_k \rangle$  and is defined, for  $A \subseteq M$  and  $R_i \subseteq M^{n_i}$ ,  $1 \leq i \leq k$ , as follows:

$$(4.24) \quad (Q^{\text{rel}})_M(A, R_1, \dots, R_k) \iff Q_A(R_1 \cap A^{n_1}, \dots, R_k \cap A^{n_k})$$

<sup>13</sup> Normally a square is not degenerate, i.e. there are four members. But consider, as an example, a type  $\langle 1 \rangle$  quantifier  $Q$  that on each  $M$  consists of the subsets of  $M$  containing a certain element  $a$  of  $M$  (the principal filter over  $M$  generated by  $a$ ). Then, for each  $M$  and each  $A \subseteq M$ ,  $Q_M(A) \iff a \in A \iff a \notin M - A \iff Q_M^d(A)$ . So  $Q_M = Q_M^d$ , and hence  $Q = Q^d$ . Our Montagovian individuals  $I_a$  are not quite like this, since we defined them globally, i.e. for a fixed  $a$ , regardless of whether  $a \in M$  or not. Indeed, if  $a \notin M$  and  $A \subseteq M$ , then  $a \notin A$  and  $a \notin M - A$ , i.e. not  $(I_a)_M(A)$ , but  $(I_a)_M^d(A)$ , so  $I_a \neq (I_a)^d$ .

In particular, for  $Q$  of type  $\langle 1 \rangle$

$$(4.25) \quad (Q^{\text{rel}})_M(A, B) \iff Q_A(A \cap B)$$

Here we shall mainly be interested in the type  $\langle 1 \rangle$  case, so if we write  $Q^{\text{rel}}$ , it is assumed unless otherwise stated that  $Q$  is of type  $\langle 1 \rangle$ .

#### 4.4.1 Examples

The following examples show that the relativization operation is in fact already familiar.

$$\begin{aligned} \forall^{\text{rel}} &= \textit{every} \\ \exists^{\text{rel}} &= \textit{some} \\ \exists_{\geq 4}^{\text{rel}} &= \textit{at least four} \\ \exists_{=5}^{\text{rel}} &= \textit{exactly five} \\ (\exists_{=5} \neg)^{\text{rel}} &= \textit{all but five} \\ Q_0^{\text{rel}} &= \textit{infinitely many} \\ (Q^R)^{\text{rel}} &= \textit{most} \\ (p/q)^{\text{rel}} &= \textit{more than } p \textit{ q'ths of the} \\ [p/q]^{\text{rel}} &= \textit{at least } p \textit{ q'ths of the} \\ (Q_{\text{even}})^{\text{rel}} &= \textit{an even number of} \end{aligned}$$

These are all ISOM quantifiers, but relativization applies also to the non-ISOM case: If  $NB_j$  and  $EB_j$  are the type  $\langle 1 \rangle$  quantifiers *nobody but John* and *everybody but John*, taken as, for  $B \subseteq M$ ,

$$\begin{aligned} (NB_j)_M(B) &\iff B = \{j\} \\ (EB_j)_M(B) &\iff B = M - \{j\} \end{aligned}$$

then

$$\begin{aligned} (NB_j)^{\text{rel}} &= \textit{no\_but John} \\ (EB_j)^{\text{rel}} &= \textit{every\_except John} \end{aligned}$$

If  $JT$  is the type  $\langle 1 \rangle$  quantifier *John's things*, defined by

$$JT_M(B) \iff \emptyset \neq \{a \in M : R(j, a)\} \subseteq B$$

(where  $R$  is a contextually given 'possession' relation), then

$$(JT)^{\text{rel}} = \textit{John's}$$

Similarly, if  $JT_{=10}$  is *John's ten things*, then

$$(JT_{=10})^{\text{rel}} = \text{John's ten}$$

So it seems that all the interpretations of familiar determiners are in fact relativizations of type  $\langle 1 \rangle$  quantifiers; we elaborate on this in section 4.5 below. But although all type  $\langle 1 \rangle$  quantifiers can be relativized, not all such relativizations are possible interpretations of natural language determiners. Besides relativizations of purely mathematical type  $\langle 1 \rangle$  quantifiers, such as

$$D_M(A) \iff |A| \text{ divides } |M|$$

whose relativization is the quantifier *Div* from Chapter 0.2.3, we have the case of Montagovian individuals  $I_j$  (Chapter 3.2.4):

$$(I_j^{\text{rel}})_M(A, B) \iff j \in A \cap B$$

But there appears to be no English determiner, say, *John-d*, such that

John-d students smoke.

means that John is a student who smokes, i.e. such that *John-d students* means 'the students who are identical with John'.

We may also relativize restricted type  $\langle 1 \rangle$  quantifiers of the form  $Q^{[A]}$  ((3.3) in Chapter 3.2.1), obtaining

$$\begin{aligned} (Q^{[A]})_M^{\text{rel}}(B, C) &\iff (Q^{[A]})_B(B \cap C) \\ &\iff ((Q^{[A]})^{[B]})_M(C) \end{aligned}$$

So this is roughly the same as restricting the quantifier once more. That restriction is indeed a form of relativization will be clear in section 4.5.5 below.

Next, we calculate

$$\begin{aligned} (Q^{\text{rel}})_M^{\text{rel}}(A, B, C) &\iff Q_A^{\text{rel}}(A \cap B, A \cap C) \\ &\iff Q_{A \cap B}(A \cap B \cap C) \\ &\iff Q_M^{\text{rel}}(A \cap B, C) \end{aligned}$$

Relativizing first to  $A$  and then to  $B$  is the same as relativizing to  $A \cap B$ .

Another observation is that the relativization of  $Q$  is always stronger than  $Q$ , in the sense that  $Q$  is definable from it:

$$(4.26) \quad Q_M(B) \iff Q_M^{\text{rel}}(M, B), \text{ and similarly for } Q \text{ of any other type.}$$

In Chapter 14 we will see that the converse of this may fail:  $Q^{\text{rel}}$  may be strictly stronger than  $Q$ . (Note that in (4.24), the universes are different.)

Let us finally note that the right-hand side of definition (4.25) (or the definition for arbitrary types) does not mention the universe at all, so it is clear that the following holds.

### Fact 2

*Relativized quantifiers, of any type, satisfy EXT.*

#### 4.4.2 Empty universes

For definition (4.25) to work properly, one needs to observe the following stipulation:

- Quantifiers are also defined on the empty universe.

For most familiar quantifiers, the truth conditions in that case are obvious: they are obtained by simply applying the usual defining condition to  $\emptyset$ . For example,

- $\exists_{\emptyset}(\emptyset)$  is false, since  $\emptyset \neq \emptyset$  is false.
- $\forall_{\emptyset}(\emptyset)$  is true, since  $\emptyset = \emptyset$  is true.
- $(Q^R)_{\emptyset}(\emptyset)$  is false, since it is false that  $0 = |\emptyset| > |\emptyset - \emptyset| = 0$ . But  $(Q^R)_{\emptyset}^d(\emptyset)$  is true.
- etc.

These stipulations may seem gratuitous in the type  $\langle 1 \rangle$  case, but they are indispensable—or rather, they avoid totally unnecessary complications—for relativized quantifiers, since even if one often assumes that the universe of discourse is non-empty, the *restriction argument* (as well as the scope) of a determiner must be allowed to be empty.

Still, one might feel that these stipulations, even if practical, are rather consequential, and one may also wonder if they are the ‘right’ ones. But it should be noted that in the case of relativized quantifiers it is just a matter of choosing between one of two possibilities (unless one admits the third possibility of a truth value gap). In general, the truth of

$$Q_M(\emptyset, B)$$

may vary with  $B$ . But when the quantifier is of the form  $Q^{\text{rel}}$ , there are only two cases; it follows immediately from (4.25) that:

- Either  $Q_M^{\text{rel}}(\emptyset, B)$  holds for all  $B \subseteq M$  (viz. when  $Q_{\emptyset}(\emptyset)$  is true), or it holds for no such  $B$  (when  $Q_{\emptyset}(\emptyset)$  is false).

Thus, the stipulations are in fact not so momentous. Are they ‘correct’? This is essentially just the question of whether certain quantifiers have *existential import* or not, a question that has to be answered empirically from case to case. In section 4.2.1 above we discussed what sort of empirical factors are relevant here, with particular emphasis on the distinction between the truth conditions of sentences and the implicatures that they often carry when used in communication.

#### 4.5 CONSERVATIVITY, EXTENSION, AND RELATIVIZATION

We now come to the most basic characteristic of quantifier denotations, which ties together relativization with the properties EXT and CONSERV.

### 4.5.1 Conservativity

That determiner denotations satisfy EXT was (we believe) first noted by van Benthem (1986: ch. 1). Earlier on it had been observed<sup>14</sup> that they also satisfy another condition:

#### conservativity for type $\langle 1, 1 \rangle$ quantifiers

A type  $\langle 1, 1 \rangle$  quantifier  $Q$  is called *conservative* (CONSERV) iff, for all  $M$  and all  $A, B \subseteq M$ ,

$$(4.27) \quad Q_M(A, B) \iff Q_M(A, A \cap B)$$

Barwise and Cooper expressed this property in terms of the live-on property (Chapter 3.2.1). Seeing a determiner  $D$  as denoting (on each  $M$ ) a function  $[[D]]_M$  associating with noun denotations (sets)  $A \subseteq M$  a type  $\langle 1 \rangle$  quantifier, their claim was that  $[[D]]_M(A)$  always lives on  $A$ . If we define the type  $\langle 1, 1 \rangle$   $Q$  by

$$Q_M(A, B) \iff B \in [[D]]_M(A)$$

this is precisely the claim that  $Q$  is conservative. The term “conservativity” is from Keenan 1981.

### 4.5.2 A quantirelation universal

Putting together the above observations, we obtain the following important *quantirelation universal*:

(QU) Type  $\langle 1, 1 \rangle$  quantifiers that interpret quantirelations in natural languages satisfy CONSERV and EXT.

A few remarks about this claim are in order.

First, observe that (QU) places real constraints on the eligible type  $\langle 1, 1 \rangle$  quantifiers. A priori, nothing would seem to prevent some quantirelation in some language from denoting a quantifier meaning *some* on universes with fewer than ten elements, and *at least ten* on larger universes. That quantifier is CONSERV, but not EXT. It seems, in fact, that no language has such quantirelations. Likewise, many mathematically natural type  $\langle 1, 1 \rangle$  quantifiers are (EXT but) not CONSERV, for example,  $MO_M(A, B) \iff |A| > |B|$ , or the Härtig quantifier  $I_M(A, B) \iff |A| = |B|$ . At first sight at least, nothing in principle would seem to exclude these as quantirelation denotations. But in fact they are excluded.

Second, a host of empirical data supports CONSERV. Speakers generally find the following sentence pairs (logically) equivalent—indeed, the second sentence is just a clumsily redundant way of expressing the first.

<sup>14</sup> Barwise and Cooper 1981; Keenan 1981; Higginbotham and May 1981.

- (4.28) a. At least four students smoke.  
b. At least four students are students who smoke.
- (4.29) a. More than half of the balls are red.  
b. More than half of the balls are red balls.
- (4.30) a. All but five teams made it to the finals.  
b. All but five teams are teams that made it to the finals.
- (4.31) a. John's two bikes were stolen.  
b. John's two bikes are bikes that were stolen.
- (4.32) a. Cats never bark.  
b. Cats never are cats and bark.

Not just English determiners, but quantirelations of other languages all appear to be limited to expressing quantifiers that satisfy CONSERV and EXT. For instance, native speakers of the Salish language St'át'imcets (also known as Lillooet) report that the sentence

- (4.33) [ta:kəm ÷i °qayq'iqyçxΣ-a] ·ixΣ ma i~tçq  
[all<sub>pl</sub>.det man(redup)-exis] go walk(redup)  
'All the men went walking'

is true just in case all the men in the domain went walking, regardless of who else did or did not go walking (Matthewson 1998). This fact shows that ta:kəm really does mean *all*, and *a fortiori* satisfies CONSERV, even though St'át'imcets is so different syntactically from English as to preclude constructing a sentence with an NP predicate containing a relative clause (like the English *All the men are men who went walking*).

Third, systematic sampling supports (QU);<sup>15</sup> moreover, various operations for forming complex determiners from simpler ones can be seen to preserve CONSERV and EXT (see Keenan and Westerståhl 1997: 854–5).

<sup>15</sup> Two possible exceptions to (QU) that have been discussed in the literature are (a) certain uses of many and few, and (b) treatment of only, just, and mostly as determiners, in e.g.

- (i) Only liberals voted for Smith.

where the interpretation would be  $only_M(A, B) \Leftrightarrow B \subseteq A$ , which is not conservative.

We return to many and few in Ch. 6. As to only, there seems to be a general consensus that it is not really a determiner, but an expression with a much wider and more varied use. A general account would have to cover cases like:

- (ii) a. I only read two papers today.  
b. Only John came to the opening.

which seem to indicate that the argument of only is not a noun but an NP, and this works for (i) too if liberals is treated as a bare plural (as in *Liberals do not vote for Jones*). But only also modifies other types of phrases, such as verbs, as in *I only TOUCHED him* (with stress on touched), and, as noted in Keenan and Stavi 1986, determiners: only appears in perfectly good complex determiners which are CONSERV and EXT, like *only John's*, *only two*, *only SOME*, etc. The overwhelming weight of evidence is that only does not express a type  $\langle 1, 1 \rangle$  quantifier, but rather expresses a polymorphic operation on type  $\langle 1 \rangle$  quantifiers, type  $\langle 1, 1 \rangle$  quantifiers, properties of and relations between individuals, etc. Similar remarks apply to just and mostly.

But, fourth, it would be somewhat unsatisfactory to see (QU) merely as an empirical generalization based on examples such as the above. Rather, (QU) should itself be *explained*. The most obvious idea is that one function of quantirelations is to restrict the domain of quantification to the denotation of the corresponding noun, i.e. to the restriction argument. Indeed, this was the idea behind CONSERV and EXT all along. We will see below how that idea can be made precise with the help of the concept of relativization.

### 4.5.3 An extension universal

EXT in itself has nothing to do with quantirelations, but applies to quantifier expressions of any type. It is a strong form of *constancy* (Chapter 3.4.1.3), where constancy in general means being given by the same rule on each universe. A universal stating that all quantifier expressions in natural languages are constant in this sense seems undoubtedly correct. Even though constancy has not been precisely defined, such a universal would still make a substantial claim, since it is easy to artificially construe quantifiers that behave very differently on different universes: for example, the one exemplified in the previous section, meaning *some* on universes with fewer than ten elements and *at least ten* on other universes. This quantifier presumably seems so far-fetched that no one would even raise the issue of whether some quantifier expression denotes it. But there is nothing logically wrong with it, and no notion of constancy is built into the concept of a (generalized) quantifier. So the far-fetchedness of the question is precisely an indication that a constancy universal is indeed true.

Can we replace constancy by EXT in such a universal? The answer seems to be Yes, with one caveat.

To begin, we observed that all relativized quantifiers are EXT (Fact 2). Not only quantirelation denotations, but many other natural language quantifiers taking more than one argument are relativized: their ‘action’ is limited to an explicitly given subset of the universe. We will see this in Chapter 10; for example, the various polyadic quantifiers involved in reciprocal constructions are like that. More generally, all the polyadic quantifiers presented in that chapter are EXT. Likewise, the monadic quantifiers with more than two arguments that interpret certain determiners (section 4.7 below) turn out to be EXT (even though they are not relativized).

Likewise, a vast majority of the type ⟨1⟩ noun phrase interpretations are EXT: proper names, bare plurals, quantified NPs (Chapter 3.2). At least, that is how we defined them; and we saw in Chapter 3.5 that this definition is the only one that guarantees that one may think of these expressions as denoting (using) global quantifiers, even though they contain expressions that get interpreted in models. Without the constancy insured by EXT, these global quantifiers would not be well-defined.

In fact, the only non-EXT quantifier expressions in natural languages that we have found are some involving a logical predicate expression, such as *thing*, that can (sometimes) be taken to stand for the universe. English examples are *everything*, *all but two things*, *most things*. In logic, the main example is  $\forall$ , which certainly is a fundamental quantifier (it was Frege’s only primitive quantifier). In one



sense, however, these are all avoidable, if one so wishes.  $\forall$  can be defined in terms of the EXT quantifier  $\exists$  and negation, which is also an EXT operation.<sup>16</sup> Similarly, everything and all but two things can be defined in terms of Boolean operations and something, which, although it contains thing, is EXT. Most things (which denotes  $Q^R$ ), on the other hand, does not seem to be definable from EXT type  $\langle 1 \rangle$  quantifiers. However, all type  $\langle 1 \rangle$  quantifiers are definable from their relativizations ((4.26) in section 4.4.1 above), which are EXT type  $\langle 1, 1 \rangle$  quantifiers. To the extent that the supply of quantifiers is a matter of methodological choice, such a choice can always be made from EXT quantifiers.

These observations motivate the following tentative universal:

- (EU) Quantifiers interpreting natural language expressions are EXT if the expressions contain no part that always denotes the universe of discourse (such as English thing).

(EU) fits the idea of constancy as being given by the same rule in every universe. Such a rule may or may not mention the universe. When it doesn't, EXT holds. When it does, as with the rules for *all but two things* and *most things*, EXT may fail. And it may be that in this case the corresponding quantifier expressions in natural languages must use a predicate that denotes the universe.

#### 4.5.4 The relativization characterization

We now return to quantirelations, and to the claim that the precise role of CONSERV and EXT here is to restrict the domain of quantification to the restriction argument. The relevant observation is contained in the next proposition. The first part of the proposition contains the essential fact. The second part gives a technically more informative statement that will be used in later chapters. In general, although this chapter is mostly expository, it contains a few technical results, in particular in sections 4.5.5 and 4.6, that will be used later on.

##### Proposition 3

- (a) *Quantifiers of the form  $Q^{\text{rel}}$  satisfy CONSERV and EXT. Conversely, each CONSERV and EXT type  $\langle 1, 1 \rangle$  quantifier is equal to  $Q^{\text{rel}}$  for some type  $\langle 1 \rangle$  quantifier  $Q$ .*
- (b) *In more detail: The operation  $\cdot^{\text{rel}}$  is a bijection between the class of type  $\langle 1 \rangle$  quantifiers and the class of CONSERV and EXT type  $\langle 1, 1 \rangle$  quantifiers, a bijection which in fact is an isomorphism in that it preserves a number of features of these quantifiers, among them ISOM, monotonicity, and Boolean combinations (including inner negations and duals).*

<sup>16</sup> EXT is a property applicable to all kinds of operations, not just quantifiers; see Ch. 9.2. In particular, sentential operations like the usual Boolean connectives are trivially EXT, since they do not mention the universe at all.

*Proof.* The verifications of these claims are all straightforward, but in view of the importance of the result, we go through them in some detail. We already observed (Fact 2) that  $Q^{\text{rel}}$  satisfies EXT. It also satisfies CONSERV, since

$$\begin{aligned} (Q^{\text{rel}})_M(A, B) &\iff Q_A(A \cap B) \quad [\text{by definition (4.25)}] \\ &\iff (Q^{\text{rel}})_M(A, A \cap B) \quad [\text{again by (4.25),} \\ &\quad \text{since } A \cap (A \cap B) = A \cap B] \end{aligned}$$

Conversely, suppose  $Q_1$  is a type  $(1, 1)$  quantifier which is CONSERV and EXT. Define a type  $(1)$  quantifier  $Q$  by

$$Q_M(B) \iff (Q_1)_M(M, B)$$

for all  $M$  and all  $B \subseteq M$ . It follows that

$$\begin{aligned} (Q^{\text{rel}})_M(A, B) &\iff Q_A(A \cap B) \\ &\iff (Q_1)_A(A, A \cap B) \quad [\text{by definition of } Q] \\ &\iff (Q_1)_M(A, A \cap B) \quad [\text{since } Q_1 \text{ is EXT}] \\ &\iff (Q_1)_M(A, B) \quad [\text{since } Q_1 \text{ is CONSERV}] \end{aligned}$$

This means that  $Q_1 = Q^{\text{rel}}$ , and we have proved part (a) of the proposition.

For part (b), we have just seen that the operation  $\cdot^{\text{rel}}$  is onto (surjective), so to prove that it is a bijection, we need only establish:

$$Q_1^{\text{rel}} = Q_2^{\text{rel}} \implies Q_1 = Q_2$$

But if  $Q_1^{\text{rel}} = Q_2^{\text{rel}}$  and  $M$  and  $B \subseteq M$  are arbitrary, then

$$\begin{aligned} (Q_1)_M(B) &\iff (Q_1^{\text{rel}})_M(M, B) \quad [\text{by definition (4.25)}] \\ &\iff (Q_2^{\text{rel}})_M(M, B) \quad [\text{by assumption}] \\ &\iff (Q_2)_M(B) \end{aligned}$$

Thus,  $Q_1 = Q_2$ .

Next, that  $\cdot^{\text{rel}}$  preserves ISOM means that

$$Q \text{ is ISOM} \iff Q^{\text{rel}} \text{ is ISOM}$$

This too is straightforward, but we come back to a fuller discussion of the role of ISOM in section 4.8 below. Likewise, the preservation of monotonicity, or more precisely that

$$Q \text{ is increasing} \iff Q^{\text{rel}} \text{ is increasing in the second argument}$$

(and similarly for decreasing) is straightforwardly verified once the relevant notions are in place (Chapter 5).

Finally, preservation of Boolean operations means that the following hold:

- (a)  $(Q_1 \wedge Q_2)^{\text{rel}} = Q_1^{\text{rel}} \wedge Q_2^{\text{rel}}$
- (b)  $(Q_1 \vee Q_2)^{\text{rel}} = Q_1^{\text{rel}} \vee Q_2^{\text{rel}}$
- (c)  $(\neg Q)^{\text{rel}} = \neg Q^{\text{rel}}$
- (d)  $(Q\neg)^{\text{rel}} = Q^{\text{rel}}\neg$ , and hence,  $(Q^{\text{d}})^{\text{rel}} = (Q^{\text{rel}})^{\text{d}}$

We check (d); the others are similar. For any  $M$  and any  $A, B \subseteq M$  we get, using (4.25) and the definition of inner negation (Chapter 3.2.3),

$$\begin{aligned} (Q\neg)_M^{\text{rel}}(A, B) &\iff (Q\neg)_A(A \cap B) \\ &\iff Q_A(A - B) \quad [\text{since } A - (A \cap B) = A - B] \\ &\iff Q_M^{\text{rel}}(A, M - B) \quad [\text{since } A \cap (M - B) = A - B] \\ &\iff (Q^{\text{rel}}\neg)_M(A, B) \end{aligned}$$

The second claim in (d) now follows from the first claim and (c), via the definition of duals.  $\square$

This proposition, then, clarifies the import of the properties of conservativity and extension as aspects of domain restriction, since the CONSERV and EXT type  $\langle 1, 1 \rangle$  quantifiers are precisely the relativizations of the type  $\langle 1 \rangle$  quantifiers. It gives substance to the universal (QU). The syntactic asymmetry between the restriction and the scope which is characteristic of most natural languages—but not most logical languages!—is shown to have a precise semantic counterpart in the domain-restricting (relativizing) function of the restriction.<sup>17</sup>

Thus, although quantirelation denotations are irrevocably binary relations between sets, the class of type  $\langle 1 \rangle$  quantifiers is still fundamental for their interpretations. Proposition 3 shows how the whole class of type  $\langle 1 \rangle$  quantifiers is mirrored in a subclass of the type  $\langle 1, 1 \rangle$  quantifiers, and we shall explore this correspondence on several occasions in the sequel.

One property which is *not* preserved by  $\cdot^{\text{rel}}$  is EXT. That  $Q^{\text{rel}}$  is EXT does not imply that  $Q$  is EXT; indeed,  $Q^{\text{rel}}$  is *always* EXT. So it is natural to ask which property among CONSERV and EXT type  $\langle 1, 1 \rangle$  quantifiers corresponds to EXT for type  $\langle 1 \rangle$  quantifiers. That question too has a simple but interesting answer, which we present in Chapter 6.1.

## 4.5.5 Restricted quantifiers once again

### 4.5.5.1 Restriction and freezing

We have seen that for quantirelations there is a significant difference between the first argument, or *restriction*, which restricts the domain of quantification, and the second

<sup>17</sup> This account of domain restriction is by now standard. Fernando (2001) offers a different and more elaborate (but as far as we can see not incompatible) explanation of CONSERV, set in a constructive type theory framework. A discussion of his idea falls outside the scope of the present book.

argument, i.e. the *scope*, which comes from the rest of the sentence. Furthermore, if the quantirelation is a determiner and we *freeze* the restriction to the set denoted by the associated noun, we obtain a type  $\langle 1 \rangle$  quantifier which serves as the interpretation of the noun phrase. Such freezing is a mechanism for reducing type  $\langle 1, 1 \rangle$  quantifiers to type  $\langle 1 \rangle$  quantifiers. Formally we define it as follows.

#### freezing the restriction argument

If  $Q$  is any type  $\langle 1, 1 \rangle$  quantifier, and  $A$  is any set, the type  $\langle 1 \rangle$  quantifier  $Q^A$  is defined, for all  $M$  and all  $B \subseteq M$ , by

$$(4.34) \quad (Q^A)_M(B) \iff Q_{A \cup M}(A, B)$$

Thus, we do not assume that  $A$  is a subset of  $M$ , but *expand* the universe (if necessary) so that  $Q(A, B)$  becomes meaningful.

Now, as the reader will recall, we already defined, in Chapter 3.2.1, definition (3.3), a notion of the restriction  $Q^{[A]}$  of a *type*  $\langle 1 \rangle$  quantifier  $Q$ , and argued that such restricted quantifiers were appropriate interpretations of noun phrases. So now we have proposed *two* ways of interpreting noun phrases. In fact, they amount to exactly the same thing:

#### Fact 4

Let  $Q$  be any type  $\langle 1 \rangle$  quantifier, and  $A$  any set. Then

$$(Q^{\text{rel}})^A = Q^{[A]}$$

*Proof.* Take any  $M$  and any  $B \subseteq M$ . Then

$$\begin{aligned} ((Q^{\text{rel}})^A)_M(B) &\iff (Q^{\text{rel}})_{A \cup M}(A, B) \quad [\text{by definition (4.34)}] \\ &\iff Q_A(A \cap B) \quad [\text{by definition (4.25)}] \\ &\iff (Q^{[A]})_M(B) \quad [\text{by definition (3.3)}] \end{aligned}$$

□

Recall the (slightly idealizing) assumptions  $(\mathbf{BCI}_{\text{NP}})$  and  $(\mathbf{BCI}_{\text{Det}})$  of closure under Boolean operations of the classes of NP denotations and determiner denotations, respectively, discussed in section 4.3. Is there any relation between these two? In fact there is, provided the following (presumably also slightly idealizing) assumption is made:

(NPD) For any language, if  $Q^{[A]}$  is an NP denotation (in a given language), then  $Q^{\text{rel}}$  is a determiner denotation.

#### Proposition 5

Under (QU) and (NPD),  $(\mathbf{BCI}_{\text{NP}})$  implies  $(\mathbf{BCI}_{\text{Det}})$ , but not vice versa.

*Proof.* Suppose  $(\mathbf{BCI}_{\text{NP}})$  holds, i.e. that the class of NP denotations (in a given language  $L$ ) is closed under Boolean operations. Take any two type  $\langle 1, 1 \rangle$  determiner denotations; by (QU) and Proposition 3, they are of the form  $Q_1^{\text{rel}}$  and  $Q_2^{\text{rel}}$  for some type  $\langle 1 \rangle$  quantifiers  $Q_1$  and  $Q_2$ . Let  $A$  be any set (denoted by some predicate expression in  $L$ ). Then  $(Q_i^{\text{rel}})^A = Q_i^{[A]}$  are NP denotations,  $i = 1, 2$ ; so  $Q_1^{[A]} \wedge Q_2^{[A]}$  is an NP denotation by  $(\mathbf{BCI}_{\text{NP}})$ . But  $Q_1^{[A]} \wedge Q_2^{[A]} = (Q_1 \wedge Q_2)^{[A]}$  (Fact 3 in Chapter 3.2.3), so, by (NPD),  $(Q_1 \wedge Q_2)^{\text{rel}} = Q_1^{\text{rel}} \wedge Q_2^{\text{rel}}$  is a determiner denotation. Similarly for  $Q_1^{\text{rel}} \vee Q_2^{\text{rel}}$ . But note that we cannot go in the other direction: If  $Q_1^{[A]}$  and  $Q_2^{[B]}$  are NP denotations and  $A \neq B$ , there is no guarantee that we can obtain  $Q_1^{[A]} \wedge Q_2^{[B]}$  as the freezing of a determiner denotation.

This shows that the class of determiner denotations in  $L$  is closed under conjunction and disjunction. The case of outer negation is similar, using the fact that  $\neg Q^{[A]} = (\neg Q)^{[A]}$  and that  $(\neg Q)^{\text{rel}} = \neg Q^{\text{rel}}$ .  $\square$

#### 4.5.5.2 Ways of freezing

Fact 4 shows that the operations of restriction and freezing correspond in a natural way. Looking closely, in fact, there are actually at least three possible ways of defining these operations. Each way preserves Fact 4, but they diverge on other properties, in particular EXT. These distinctions are not heeded in the literature, as far as we know. There is a plausible reason for this, as we will see shortly, and it might seem that the differences between the various notions are technical and rather inconsequential. We shall see in the next section, however, and in Chapter 8, that for a proper account of *definite* determiners and *exceptive* determiners, it appears to be essential to get the notions of freezing and restriction right. Furthermore, we already showed in Chapter 3.5 that if one wants to treat quantified NPs as denoting global quantifiers, the only possible definition of restriction and freezing is the one chosen here, i.e. the first of the following three.

##### three notions of restriction

If  $Q$  is of type  $\langle 1 \rangle$ ,  $A$  is any set, and  $B \subseteq M$ , define

- (a1)  $Q_M^{[A]}(B) \iff Q_A(A \cap B)$  (as before)
- (a2)  $Q_M^{[A]}(B) \iff Q_{A \cap M}(A \cap B)$
- (a3)  $Q_M^{[A]}(B) \iff Q_A(A \cap B)$  and  $A \subseteq M$

(a1) is our official notion of restriction. We have not found clear uses of the other two notions, although they have been suggested in the literature.

**three notions of freezing**

If  $Q$  is of type  $\langle 1, 1 \rangle$ ,  $A$  is any set, and  $B \subseteq M$ , define

$$(b1) \quad Q_M^A(B) \iff Q_{A \cup M}(A, B) \text{ (as before)}$$

$$(b2) \quad Q_M^A(B) \iff Q_M(A \cap M, B)$$

$$(b3) \quad Q_M^A(B) \iff Q_M(A, B) \text{ and } A \subseteq M$$

Again, when we just talk about freezing, we mean the notion in (b1). Now, one easily verifies:

$$(4.35) \quad \text{For each pair } ((ai), (bi)), \text{ Fact 4 holds. That is, } (Q^{\text{rel}})^A = Q^{[A]}, (Q^{\text{rel}})^{\underline{A}} = Q^{\underline{[A]}}, \text{ and } (Q^{\text{rel}})^{\underline{\underline{A}}} = Q^{\underline{\underline{[A]}}}.$$

So the respective notions correspond in the right way, but they implement different ideas. Ordinary freezing *expands* the universe to include  $A$ .<sup>18</sup> The (b2) version instead *cuts down* the universe, i.e. (given CONSERV and EXT) the restriction argument, to  $A \cap M$ . The (b3) version does nothing to the universe but makes the corresponding statement *false* whenever  $A \not\subseteq M$ .

Next, observe:

$$(4.36) \quad \text{When } A \subseteq M, \text{ all three notions of restriction are equivalent, and likewise all three notions of freezing.}$$

This presumably explains why the difference between them is not noticed. Often, freezing or restriction is defined a bit sloppily by considering only the case when the universe  $M$  is given and  $A$  is a subset of  $M$ . That is, one defines, say, freezing as a *local* quantifier  $Q^A$  on  $M$  by

$$Q^A(B) \iff Q_M(A, B) \text{ for all } B \subseteq M$$

and ignores how this could be turned into a global definition.<sup>19</sup> For many applications, the sloppy version is sufficient, since only the case when  $A \subseteq M$  is used. But not always, as indicated above. Besides, global definitions are preferable whenever possible.

The choice between the different definitions may not matter much when one is simply interpreting noun phrases in some language. But the differences show up in the *properties* of the quantifiers used. One example is *repeated* freezing: What happens if you freeze, or restrict, twice? Here the various notions give rather different answers, as follows (the verifications are straightforward):

<sup>18</sup> This is clear for freezing, but perhaps less so for restriction. In restriction we *switch* to  $A$  and don't need to care about  $M$ , since even though  $B$  is a subset of  $M$ , the argument  $A \cap B$  is a subset of  $A$ . In any case, Fact 4 shows that (a1) is the correct definition in this case.

<sup>19</sup> This is the overwhelmingly standard procedure. An exception is Westerståhl 1994, where the (b3) version was chosen. (But for the application to iterated quantifiers there, this had no effect; see Ch. 10.1.)

- (4.37) a.  $(Q^{[A]})^{[C]} = Q^{[A \cap C]}$   
 b. For  $B \subseteq M$ ,  $((Q^{[A]})^{[C]})_M(B) \Leftrightarrow Q_A(A \cap C \cap B)$

Roughly, restricting first to  $A$  and then to  $C$  amounts to restricting both the universe and the argument to  $A \cap C$  in the (a2) case, whereas only the argument is thus restricted in the (a1) case. Exactly analogous facts hold for freezing.

The most conspicuous way in which these notions differ is with regard to EXT.

### Fact 6

Let  $Q$  be any type  $\langle 1 \rangle$  quantifier and  $A$  any set.

- (i)  $Q^{[A]}$  is always EXT.
- (ii)  $Q^{[A]}$  is EXT whenever  $Q$  is, but usually not otherwise.
- (iii)  $Q^{\underline{A}}$  is usually not EXT.

Let  $Q$  be a type  $\langle 1, 1 \rangle$  quantifier and  $A$  any set.

- (iv)  $Q^A$  is EXT whenever  $Q$  is EXT.
- (v)  $Q^{\underline{A}}$  and  $Q^{\underline{\underline{A}}}$  need not be EXT.

*Proof.* (i) is Fact 10 in Chapter 3.4.1.1. For (ii), if  $Q$  is EXT, and  $B \subseteq M \subseteq M'$ , then  $(Q^{[A]})_M(B) \Leftrightarrow Q_{A \cap M}(A \cap B) \Leftrightarrow Q_{A \cap M'}(A \cap B)$  (by EXT)  $\Leftrightarrow (Q^{[A]})_{M'}(B)$ . But, for example,  $(\forall^{[A]})_M(B) \Leftrightarrow A \cap M \subseteq B$ , so  $\forall^{[A]}$  is not EXT (in contrast with  $\forall^A$ :  $(\forall^A)_M(B) \Leftrightarrow A \subseteq B$ ). As to (iii), the requirement that  $A \subseteq M$  will usually prevent EXT, since one can find  $M \subseteq M'$  such that  $A \not\subseteq M$  but  $A \subseteq M'$ .

Now suppose  $Q$  is of type  $\langle 1, 1 \rangle$  and EXT, and  $B \subseteq M \subseteq M'$ . Then  $(Q^A)_M(B) \Leftrightarrow Q_{A \cup M}(A, B) \Leftrightarrow Q_{A \cup M'}(A, B) \Leftrightarrow (Q^A)_{M'}(B)$ . This proves (iv). For (v), see the examples below.  $\square$

### Examples

1.  $Q^{\text{rel}}$  is always EXT, so the same holds by (iv) for  $(Q^{\text{rel}})^A = Q^{[A]}$  (Fact 4). That is, (i) follows from (iv).
2.  $three^{\underline{A}}$  is EXT; in fact,  $three^{\underline{A}} = three^A$ , since both express the condition  $|A \cap B| = 3$ . On the other hand,  $(three^{\underline{\underline{A}}})_M \Leftrightarrow |A \cap B| = 3 \ \& \ A \subseteq M$ , so  $three^{\underline{\underline{A}}}$  is not EXT.
3.  $(all \ but \ three)^A$ ,  $(all \ but \ three)^{\underline{A}}$ , and  $(all \ but \ three)^{\underline{\underline{A}}}$  express the conditions  $|A - B| = 3$ ,  $|(A \cap M) - B| = 3$ , and  $|A - B| = 3 \ \& \ A \subseteq M$ , respectively. These are all different, and only the first is EXT.
4.  $(the \ three)^A$  is EXT by (iv); moreover  $(the \ three)^A = (the \ three)^{\underline{A}}$ , since both express the condition  $|A| = 3 \ \& \ A \subseteq B$  (adding  $A \subseteq M$  here has no effect since, by assumption,  $B \subseteq M$ ). But  $(the \ three)^{\underline{\underline{A}}}$  expresses that  $|A \cap M| = 3 \ \& \ A \cap M \subseteq B$ , which is not EXT.

#### 4.5.5.3 Type $\langle 1 \rangle$ quantifiers as frozen type $\langle 1, 1 \rangle$ quantifiers

Another advantage of our chosen way of defining restriction and freezing is that many common type  $\langle 1 \rangle$  quantifiers that interpret non-quantified noun phrases can

nevertheless be construed as frozen type  $\langle 1, 1 \rangle$  quantifiers. This is just a technical trick, but we shall have use for it in Chapters 7 and 8. The cases we will be interested in are the following (see Chapter 3.2.1 and 3.2.4):

**Fact 7**

- (a)  $I_j = all_{ei}^{(j)}$
- (b)  $I_j \wedge I_m = all_{ei}^{(j,m)}$
- (c)  $C^{Pl} = all_{ei}^C$
- (d)  $I_j \vee I_m = some^{(j,m)}$

*Proof.* (a) and (b) are special cases of (c). For (c), take any  $M$  and any  $B \subseteq M$ :

$$\begin{aligned} (C^{Pl})_{M(B)} &\iff \emptyset \neq C \subseteq B \\ &\iff (all_{ei})_{M \cup C}(C, B) \\ &\iff (all_{ei}^C)_M(B) \end{aligned}$$

(d) is similar. □

So proper names, conjunctions and disjunctions of proper names, and bare plurals, can be seen as frozen type  $\langle 1, 1 \rangle$  quantifiers.<sup>20</sup> But we observe that this in no way extends to all type  $\langle 1 \rangle$  quantifiers, or even all NP denotations. Here are some simple counterexamples:

**Proposition 8**

*There is no CONSERV, EXT, and ISOM type  $\langle 1, 1 \rangle$  quantifier  $Q_1$  such that, for some set  $C$ ,  $I_j \wedge (I_m \vee I_s) = Q_1^C$ . Likewise for the quantifier  $O_j$  defined by  $(O_j)_M(B) \iff B = \{j\}$ , or more generally the quantifier  $O_D$  defined by  $(O_D)_M(B) \iff B = D$ .*

*Proof.* We prove the first claim; the other two are similar. Suppose, for contradiction, that  $I_j \wedge (I_m \vee I_s) = Q_1^C$ . A set  $X$  is in  $I_j \wedge (I_m \vee I_s)$  if and only if  $j \in X$  and either  $m \in X$  or  $s \in X$  (or both). So none of the following sets are in  $I_j \wedge (I_m \vee I_s)$ :  $\emptyset, \{j\}, \{m\}, \{s\}$ . It follows that when  $X$  is any of these four sets,  $\neg Q_1(C, X)$ . However,  $\{j, m\} \in I_j \wedge (I_m \vee I_s)$ , so  $Q_1(C, \{j, m\})$ , and thus by CONSERV,  $Q_1(C, C \cap \{j, m\})$ . Since  $C \cap \{j, m\}$  is distinct from the above four sets, it has to be equal to  $\{j, m\}$ , and so  $j \in C$  and  $m \in C$ . By an analogous argument, since  $\{j, s\} \in I_j \wedge (I_m \vee I_s)$ , it follows that also  $s \in C$ .

However,  $Q_1$  is ISOM. Let  $f$  be a permutation of the underlying universe  $M$  (by EXT, we may assume  $C \subseteq M$ ) which leaves everything as it is except that it permutes  $j$  and  $s$ . By the above results,  $f(C) = C$  and  $f(\{j, m\}) = \{s, m\}$ . Since  $Q_1(C, \{j, m\})$

<sup>20</sup>  $all_{ei}^C$  expresses the condition  $\emptyset \neq C \cap M \subseteq B$ , so it is distinct from  $C^{Pl}$ . But  $all_{ei}^C = all_{ei}^C$ . On the other hand,  $some^{(j,m)} = some_{\{j,m\}}^{(j,m)}$ , whereas  $some_{\{j,m\}}^{(j,m)}$  expresses that  $(j \in B \vee m \in B) \& (j, m \in M)$ , which is distinct from  $I_j \vee I_m$ . Only our chosen notion of freezing handles both examples.



holds,  $Q_1(C, \{s, m\})$  holds too, by ISOM. But this contradicts the fact that  $\{s, m\} \notin I_j \wedge (I_m \vee I_s)$ .  $\square$

Thus, the noun phrase *John*, and *Mary or Sue* cannot be construed as an ISOM determiner denotation with a frozen restriction argument. Similarly for the noun phrases *only John* and *only cats*.

#### 4.6 DEFINITENESS

We are now ready to address a property that has received much attention in the linguistic literature: definiteness. The basic intuitions behind definiteness are fairly clear, at least in languages with both a definite and an indefinite article.<sup>21</sup> NPs with the definite article behave differently in characteristic ways from NPs with the indefinite article, syntactically as well as semantically. In these cases we are dealing with NPs that can be taken to *refer*—to objects or sets of objects. But it has also seemed that other NPs have similar properties, even quantified NPs. For example, linguists agree that *John* is definite, and similarly *John's boats*, whereas *three boats* is indefinite. What exactly does this mean?

For a recent overview of how linguists have approached this question, we refer to Abbott 2004. Essentially, two characteristics have been used to identify definiteness. One is in terms of *familiarity*: the object or objects referred to by a definite must already have been ‘introduced’ in the discourse. Indefinites, on the other hand, don't refer back to already introduced objects, but themselves serve as such introductions. The other idea focuses on *uniqueness* instead. Note, however, that these ideas make sense only for NPs that can be taken to refer to individual objects in the first place. A more ambitious enterprise is to try to find a semantic property that singles out, among *all* quantified NPs, or among *all* determiners, the definite ones.

One such attempt, from Barwise and Cooper 1981, has won some acceptance. We will use (essentially) that definition here, partly because we think it is rather successful, and partly because it is the *only* attempt at a precise definition of (in)definiteness for arbitrary determiners that we are aware of. It is not undisputed, but those who dispute it are not, as far as we know, opposing it to another precise notion, but rather to certain intuitions, or paradigmatic cases, or tests that it conflicts with. We shall have occasion to come back to the issue of what the right concept of definiteness is in Chapter 7, in connection with possessive NPs in partitive constructions. But

<sup>21</sup> The articles can be determiners, as in English (*the boat*, *a boat*), or one of them can be a suffix, as in Swedish (*båten*, *en båt*). Swedish, by contrast with English, also marks adjectives for definiteness, but then a determiner is needed as well as the suffix: *den stora båten*, *en stor båt* (*the big boat*, *a big boat*). The facts are similar for plural definites: *båtarna*, *de stora båtarna* (*the boats*, *the big boats*), although in the Swedish case the adjectival marking coincides with the plural marking: *tre stora båtar* (*three big boats*).

It is well known that native speakers of languages that do not mark for definiteness usually find it quite hard to choose the correct form (definite or indefinite) when they learn to speak a language that does.

in this section, we simply present Barwise and Cooper's notion, and some of its properties:<sup>22</sup>

### definite quantifiers

- (a) A type  $\langle 1, 1 \rangle$  quantifier  $Q$  is *definite* iff for every set  $A$ , and every universe  $M$ ,  $(Q^A)_M$  is either empty or a non-trivial principal filter on  $M$ , i.e. such that for some non-empty subset  $X_M$  of  $M$ ,  $(Q^A)_M = \{B \subseteq M : X_M \subseteq B\}$ .
- (b) A quantirelation is *definite* iff it denotes a definite quantifier. An NP is *definite* if it is formed with a definite determiner.

Typical definite determiners that we have already encountered are the *ten* and *John's*; using the interpretations given earlier and our definition of freezing we calculate:

$$(the\ ten^A)_M = \begin{cases} \{B \subseteq M : A \subseteq B\} & \text{if } |A| = 10 \\ \emptyset & \text{otherwise} \end{cases}$$

$$(John's^A)_M = \begin{cases} \{B \subseteq M : A \cap R_j \subseteq B\} & \text{if } A \cap R_j \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases}$$

It is important to note that a definite quantifier can be the trivial  $0_M$  on some (or even all) universes  $M$ , but never  $1_M$ , since the generating set is required to be non-empty. It follows that *every* is not definite on this definition, since although it is a principal filter on every universe  $M$ , that filter may sometimes be the trivial  $P(M)$ . This reflects a difference in existential commitment between, say, *The unicorns were restless* and *Every unicorn was restless*.

On the other hand, the three alternative interpretations of the definite article *the* discussed in section 4.2.2 above, i.e. *the<sub>sg</sub>*, *the<sub>pl</sub>*, and *all<sub>ci</sub>*, are all definite.<sup>23</sup>

This raises a small issue about the definition above. Usually, definiteness applies to both determiners and NPs, but Barwise and Cooper define definiteness only for determiners and *quantified* NPs. Thus, the definiteness of *John*, or *John and Mary*, about which presumably everyone agrees, is not captured by their definition. It does not help to use the fact, noted in the preceding section, that  $I_j = all_{ci}^{(j)}$ , for even though *all<sub>ci</sub>* is definite, the definition as stated doesn't extend the notion of definiteness to NPs that are not formed with a determiner. However, if one agrees that *John* and *John and Mary* are definite, they are presumably definite in the same sense as

<sup>22</sup> Barwise and Cooper (1981) allow partial quantifiers, and they say that  $(Q^A)_M$  is *undefined* when we say it is empty. That difference is not important here.

Also, Barwise and Cooper do not raise the issue of the uniqueness of the generating set over different universes that we address here (Proposition 10 below), presumably because their account is not clearly a global one.

<sup>23</sup> According to Abbott (2004), the universal determiners *every*, *each*, *all* are often classified as definite. Presumably, this is when they are interpreted with existential import, i.e. as *all<sub>ci</sub>*.

above, i.e. by the fact that (on a given universe) they denote a non-trivial principal filter.

We therefore extend Barwise and Cooper's definition slightly (but surely in their spirit) so that the definiteness of an NP means that it denotes a type  $\langle 1 \rangle$  quantifier which on each universe is either empty or a non-trivial principal filter. This coincides with the given definition for quantified NPs, but adds some unquantified definite NPs. It also makes it more reasonable to say that a quantifier, or quantirelation, or NP, is *indefinite* iff it is not definite.

Bare plurals are usually regarded as definite in their universal or generic reading (Firemen are brave), and indefinite in their existential reading (Firemen are available today). The universal interpretation that we represented with the quantifiers  $C^{\text{pl}}$  (Chapter 3.2.1) is indeed definite in the above sense.<sup>24</sup>

The idea behind Barwise and Cooper's definition is simple and clear. In particular, one should note that just as definiteness and indefiniteness were originally applied to referring NPs, so Barwise and Cooper's definition in a sense singles out among the general class of NP denotations those that can be seen as referring, although it is actually plural reference to a set of individuals. Definite NPs can be taken to refer to the generator set  $X_M$  of the principal filter (or, equivalently, to the intersection of all sets belonging to  $Q_M$ ). In this sense, *John* refers to John (technically, to  $\{j\}$ ), *John and Mary* refers to the set of John and Mary, *John's books* refers to the set of books that John owns (say), *the ten boys* refers to the set of boys in the discourse universe, provided there are exactly ten of them, etc.

Observe next that the definition in principle allows the generating sets  $X_M$  to be different for different  $M$ . That is just a consequence of adopting a global view of semantics. But this might appear to make the above idea about reference problematic. If the reference of *the ten boys* could change when a set of girls was added to the universe, for example, it would be a very shaky notion of reference. But of course this does not happen. And in fact we can now show that in the linguistically interesting cases, it *cannot* happen.

### Lemma 9

*If  $Q$  is definite and  $\text{EXT}$ ,  $(Q^A)_M$  is generated by  $X_M$ , and  $(Q^A)_{M'}$  is generated by  $X_{M'}$ , then  $X_M = X_{M'}$ .*

*Proof.* We first show that under these circumstances,

$$(4.38) \text{ if } M \subseteq M', \text{ then } X_M = X_{M'}$$

To see this, note that  $X_M \in (Q^A)_M$ , and so  $X_M \in (Q^A)_{M'}$ , since  $Q^A$  is  $\text{EXT}$ , by Fact 6 (iv) in the preceding section. This means that  $X_{M'} \subseteq X_M$ . Therefore,  $X_{M'} \subseteq M$ . But  $X_{M'} \in (Q^A)_{M'}$ , so it follows by  $\text{EXT}$  that  $X_{M'} \in (Q^A)_M$ , i.e.  $X_M \subseteq X_{M'}$ . This proves (4.38).

<sup>24</sup> Bare plurals are sometimes analyzed as having a null determiner. That determiner would denote *all<sub>ei</sub>* in the universal case and *some* in the existential case.

For the general case, note that since  $X_M$  is defined, so is  $X_{M \cup M'}$ . This also follows from EXT: since  $M \in (Q^A)_M$ ,  $M \in (Q^A)_{M \cup M'}$  by EXT, so  $(Q^A)_{M \cup M'}$  is non-empty. Hence, using (4.38) twice, we obtain  $X_M = X_{M \cup M'} = X_{M'}$ .  $\square$

Clearly, the fact that  $Q$  is EXT is crucial in this proof.

**Proposition 10**

*If  $Q$  is CONSERV, EXT, and definite, then  $Q^A$  is generated by the same set on each universe where it is non-trivial. On such a universe  $M$ , this is the smallest set on which  $(Q^A)_M$  lives. Extending the notation from Chapter 3.4.1.2, we may write the set  $W_{Q^A}$  (rather than  $W_{(Q^A)_M}$ , since  $M$  is immaterial). Also,  $W_{Q^A}$  is a subset of  $A$ .*

*Proof:* From the above lemma it follows that  $Q^A$  is generated by the same set—call it  $X$ —on universes where it is non-trivial.

Next, note that

(a) if  $(Q^A)_M$  is non-trivial, then  $X \subseteq M$

Now suppose  $(Q^A)_M$  lives on  $Z$ . This means that for all  $B \subseteq M$ ,  $X \subseteq B$  if and only if  $X \subseteq B \cap Z$ . Taking  $B = X$  it follows that  $X \subseteq Z$ . Since clearly  $(Q^A)_M$  lives on  $X$ ,  $X$  is the smallest set that the quantifier lives on, i.e.  $X = W_{Q^A}$ .

For the final claim, if  $Q^A$  is trivial on every universe,  $W_{Q^A} = \emptyset \subseteq A$  (Lemma 1(b) in Chapter 3.2.2). So we may assume that  $Q^A$  is non-trivial on at least one universe. By (a), it suffices to show that  $(Q^A)_A$  is non-trivial. But  $(Q^A)_M$  is non-trivial for some  $M$ , so there is  $B \subseteq M$  such that  $(Q^A)_M(B)$ , i.e.  $Q_{M \cup A}(A, B)$ . By CONSERV and EXT,  $Q_A(A, A \cap B)$ , so  $(Q^A)_A$  is non-empty, and hence non-trivial.  $\square$

Since quantirelation denotations are CONSERV and EXT, we always get a unique generating subset of  $A$  for definite quantirelations (on universes where they are non-trivial). That set can be  $A$  itself, as in the ten boys, but it can also be a proper subset, as in John's books. It is always the smallest set that  $Q^A$  lives on.

Roughly, Proposition 10 shows that definite determiners behave as they are supposed to. It is not a completely straightforward observation, however, and it requires that frozen relativized quantifiers are EXT, which in turn depends on choosing the right notion of freezing. With the (b2) version of freezing in the previous section, for example,  $Q^A$  is normally not EXT in the relevant cases. For example, we saw in Example 4 after Fact 6 that *(the three)<sup>A</sup>* is not EXT.

The uniqueness of the generating set for definites is not an issue that has been much discussed in the literature. The reason for this is not that for quantifiers of the form  $(Q^A)_M$  one always assumes that  $A \subseteq M$ . Rather, it is that one assumes a *local* perspective, and thus does not notice the role of the universe (of discourse). But as soon as one takes seriously the fact that it is a standard feature of natural languages to use the same expressions with different universes, it becomes clear that adequate accounts of many linguistic phenomena have to factor in the universe as a parameter. That is, they have to be *global* accounts. In the particular case of definite determiners, the most obvious question from a global perspective is how the generating set depends

on the universe. The appeal of Proposition 10 is that it shows that this set does *not* depend on the universe. So the local analysis turns not to distort the facts in this case, after all. But this is an insight that needs *proof*, not just assumption.

One feature of definite NPs is that they are good in *exception phrases*: *except the two girls* or *except Deirdre's friends*. Given that definites refer to sets, this fits easily into a common form of analysis of exception phrases. However, many non-definite NPs are also good in such phrases, so in fact a more general analysis is required. We discuss these matters further in Chapter 8.

Two properties often thought to characterize definite NPs uniquely are (1) that they can occur in *partitive* constructions, *most of the boys*, *some of these cats*, *several of Mary's cars*, and (2) that they *cannot* occur in existential-there sentences: \**There are John's/the three bikes in the garden*. In Chapter 7.11 we shall take issue with these claims, however, or more precisely with the claim that only definites have these properties. Chapter 6.3 discusses existential-there sentences in some detail.

#### 4.7 TYPE $\langle 1, 1, 1 \rangle$ QUANTIFIERS AND BEYOND

We have now seen a number of examples of quantirelations—mostly determiners—denoting type  $\langle 1, 1 \rangle$  quantifiers, and some fundamental properties they all share. In this section we briefly mention a few English expressions that also appear to behave like determiners but take two or more restriction arguments, which means that their type is  $\langle 1, 1, 1 \rangle$  or beyond. The main syntactic reason for thinking of them as determiners is that when you fix both restriction arguments, you get what seems to be an ordinary noun phrase, denoting a type  $\langle 1 \rangle$  quantifier. And the main semantic reason is that these determiners also have the basic properties—CONSERV, EXT, ISOM—that quantirelations have. An extended study of such expressions is Keenan and Moss 1984, to which we refer the reader for many more examples and details.<sup>25</sup>

Roughly, these expressions either involve comparatives or Boolean operations (or both). Here are some examples, with the putative determiner phrase italicized.

- (4.39) a. *More* students *than* teachers went to the ball game.  
 b. *As many* doctors *as* lawyers were at the cocktail party.  
 c. *Twice as many* women *as* men voted for Smith.  
 d. *Proportionately more* Danes *than* Swedes smoke.
- (4.40) a. *Every* student *and* teacher was there.  
 b. *The eighty-six* men *and* women started singing.  
 c. *Some student's* cats *and* dogs were locked in the apartment.

<sup>25</sup> But that paper takes a *local* perspective on quantifiers, as does most of Keenan's work. This means that the results about expressive power, which constitute a substantial part of Keenan and Moss 1984, and similarly for Keenan and Stavi 1986, are largely unrelated to the global expressivity results in the final part of this book, and more generally to the global view of quantification that we take. See Ch. 11.3.1 for further comments on this issue.

- (4.41) a. *Between three and six teachers and more than twice as many students saw the accident.*  
 b. *More lawyers than either doctors or businessmen came.*

If treated as determiners, these expressions are syntactically discontinuous, which might seem problematic. However, especially in the comparative cases, arguments for a determiner analysis are quite strong. As we saw: (a) syntactically, when combined with their restriction arguments, the results are typical NPs; (b) semantically, they have the usual properties of determiners, CONSERV and EXT (see below); (c) other analyses are *ad hoc* or less uniform. But we shall not dwell too much on these arguments here, noting only that such an analysis is indeed possible. A few interpretations and some comments are given below.<sup>26</sup>

- (4.42) a.  $more\_than_M(A, B, C) \iff |A \cap C| > |B \cap C|$   
 b.  $as\_many\_as_M(A, B, C) \iff |A \cap C| = |B \cap C|$   
 c.  $propmore\_than_M(A, B, C) \iff |B| \cdot |A \cap C| > |A| \cdot |B \cap C|$   
 d.  $every\_and_M(A, B, C) \iff A \cup B \subseteq C$   
 e.  $some\ student's\_and_M(A, B, C) \iff \exists a \in student[A \cap R_a \neq \emptyset \ \& \ B \cap R_a \neq \emptyset \ \& \ (A \cup B) \cap R_a \subseteq C]$   
 f.  $between\ three\ and\ six\_and\ more\ than\ twice\ as\ many_M(A, B, C) \iff 3 \leq |A \cap C| \leq 6 \ \& \ |B \cap C| > 2 \cdot |A \cap C|$

Beginning with comparatives, a first question is if other analyses are possible, and how in particular *more\_than* relates to the comparative quantifiers we have encountered earlier. Recall that the type  $\langle 1, 1 \rangle$  quantifier *MO* from Chapter 2.4 expresses the basic comparison relation:  $MO_M(A, B) \iff |A| > |B|$ . Clearly *more\_than* is definable in terms of *MO*:

$$more\_than_M(A, B, C) \iff MO_M(A \cap C, B \cap C)$$

But also, conversely,

$$MO_M(A, B) \iff more\_than_M(A, B, M)$$

So these two are interdefinable and thus have the same strength. Incidentally, although *MO* is not a determiner denotation, since it is not CONSERV, it is easily expressible in English with an existential-there construction: the following are equivalent:

- (4.43) a.  $|A| > |B|$   
 b. More *As* than *Bs* exist.  
 c. There are more *As* than *Bs*.

<sup>26</sup> These examples are all of type  $\langle 1, 1, 1 \rangle$ , but other monadic types can also be exemplified. Sometimes one writes these types so as to indicate that the first (two) arguments are the restrictions and the last is the scope (see Keenan and Westerståhl 1997: 866–7), but we use the standard type notation here.

So sentences of the form *There are more\_than\_* are naturally expressed with the quantifier *more\_than*.<sup>27</sup> Could we also define *more\_than* in terms of *most*? We will see in Chapter 14 that if infinite sets are allowed, *most* is strictly weaker than *more\_than*, and that on finite universes, although they have the same logical expressive power, there is no *simple* way in English of rendering *more\_than* in terms of *most*.<sup>28</sup>

Continuing our exploration of alternative analyses, is it not possible to analyze (4.39a) in another way, using type  $\langle 1, 1 \rangle$  quantifiers, along one of the following lines?

- (4.44) a. *More students than* teachers went to the ball game.  
 b. *More* students *than* teachers went to the ball game.

But this seems *ad hoc*. Roughly, there are three slots to fill in these sentences, two of which clearly have the character of restrictions, whereas the remaining one is the scope. Freezing just one of the restrictions produces rather awkward results. But if you freeze both simultaneously, you get a well-behaved noun phrase. Moreover, it is not hard to see that the type  $\langle 1, 1 \rangle$  quantifiers that would result if the first option were chosen, namely:

$$\begin{aligned} \text{more students than}(A, B) &\iff |student \cap B| > |A \cap B| \\ \text{more\_than teachers}(A, B) &\iff |A \cap B| > |teacher \cap B| \end{aligned}$$

are not CONSERV, in contrast with *more\_than* (see below).

Keenan and Moss (1984) give further arguments for the viability of this analysis, as well as counter-arguments against other proposals. Further, expressions other than nouns can be arguments of the expression *more\_than*: for example, adjectives, as in

- (4.45) More male than female students went to the ball game.

(where *more male than female* looks like an ordinary determiner taking one restriction argument), and verb phrases, as in

- (4.46) a. More students came early than left late.  
 b. More students came early than teachers left late.

<sup>27</sup> As a determiner word, *more* seems to be related to the comparative form of adjectives, roughly via expressions like *more numerous than*:  $MO(A, B)$  means exactly that  $A$  is more numerous than  $B$ . (However, the determiner *most* does *not* mean *most numerous*.)

<sup>28</sup> Keenan and Moss (1984) prove that *more\_than* is what they call *inherently two-place* by means of the following (local) result. Think of a type  $\langle 1, 1 \rangle$  quantifier on  $M$  as a function from subsets of  $M$  to type  $\langle 1 \rangle$  quantifiers on  $M$  (see Ch. 3.1.3), and a type  $\langle 1, 1, 1 \rangle$  quantifier on  $M$  as a function from pairs of subsets of  $M$  to type  $\langle 1 \rangle$  quantifiers on  $M$ . Then, on a given finite universe  $M$  with at least three elements, the *range* of the two-place function  $\text{more\_than}_M$  cannot be identical to the range of any one-place function, simply because, as Keenan and Moss show (their Theorem 5), the former range contains more type  $\langle 1 \rangle$  quantifiers on  $M$  than one-place functions can supply.

This, however, does not show that *more\_than* is not *definable* (locally or globally) *from* (as opposed to identical with) one-place functions using Boolean or other first-order notions. Indeed, we will see in Ch. 14 that *more\_than* is logically (and globally) definable in terms of the two determiner denotations *most* and *infinitely many*. But it is doubtful what, if anything, such a fact says about expressivity in natural languages (see Ch. 12.5).

Beghelli (1994) makes a general study of such comparatives, extending the approach of Keenan and Moss. Here we merely conclude that we do need the lexical item *more\_than* and its cognates as type  $\langle 1, 1, 1 \rangle$  quantifiers in the analysis of the sentences (4.39).

A final comment about comparatives is that the quantifier *propmore\_than* is sometimes used even when no word like *proportionally* appears. For example, (4.39d) could equally well be expressed as

(4.47) More Danes than Swedes smoke.

since someone familiar with the fact that the Swedish population is about twice the size of the Danish would be unlikely to take it in any other than the proportional reading.

As to the Boolean combinations in (4.40), we shall only note here that even though an analysis in terms of type  $\langle 1, 1, 1 \rangle$  quantifiers is certainly possible for some readings, these sentences are often ambiguous in systematic ways which an adequate account should somehow make clear. Take the simple (4.40a), but replace *every* by *three*, which allows the different readings to become clearer:

(4.48) Three students and teachers were there.

First, there is the reading that the three who were there are *both* students and teachers (the intersection reading). This is less plausible for (4.48), but quite plausible for, say, *Three mothers and doctors were there*. Second, there is the union reading similar to the one in (4.42d), except that it may further be required in this case that each of the two sets in the union is non-empty. These two readings apply to (4.40a) as well, as Keenan and Moss note. But (4.48) also has the third reading that six people, three students and three teachers, were there. We come back to the distribution of readings for these sentences in Chapter 5.8.

The mixed cases (4.41), finally, are even more complex. However, a good case can be made that while an analysis in terms of type  $\langle 1, 1, 1 \rangle$  quantifiers goes relatively smoothly, alternative analyses risk running into various problems.

Having now surveyed some candidates for determiners with more than one restriction argument, let us now look at the properties of the quantifiers they would denote. Clearly, each of the above examples satisfies EXT and ISOM. Further, observe that CONSERV for a type  $\langle 1, 1 \rangle$  quantifier  $Q$  means that if two scope arguments  $B$  and  $B'$  coincide on the restriction  $A$ , i.e. if  $A \cap B = A \cap B'$ , then  $Q_M(A, B) \Leftrightarrow Q_M(A, B')$ . This generalizes directly to quantifiers with more than one restriction argument, for example, in the type  $\langle 1, 1, 1 \rangle$  case.

CONSERV For  $A, B, C, C' \subseteq M$ , if  $A \cap C = A \cap C'$  and  $B \cap C = B \cap C'$ , then  $Q_M(A, B, C) \Leftrightarrow Q_M(A, B, C')$ .

It is easily verified that this amounts to restricting to the union of the restriction arguments, i.e. to the condition



For  $A, B, C \subseteq M$ ,  $Q_M(A, B, C) \Leftrightarrow Q_M(A, B, (A \cup B) \cap C)$ .<sup>29</sup>

Also, it is clear that all of our examples so far are CONSERV.<sup>30</sup>

Certain other properties of type  $\langle 1, 1 \rangle$  quantifiers generalize too. Anticipating Chapter 6, we say that a type  $\langle 1, 1 \rangle$   $Q$  is *intersective* if only the intersection of the restriction and the scope matters, i.e. if  $A \cap B = A' \cap B'$  implies  $Q_M(A, B) \Leftrightarrow Q_M(A', B')$ . Similarly for a type  $\langle 1, 1, 1 \rangle$   $Q$ .

INT For  $A, B, C, A', B', D \subseteq M$ , if  $A \cap C = A' \cap D$  and  $B \cap C = B' \cap D$ , then  $Q_M(A, B, C) \Leftrightarrow Q_M(A', B', D)$ .

For example, *more\_than* and *as\_many\_as* are intersective, but not *propmore\_than*.

Another group of properties that immediately apply to type  $\langle 1, 1, 1 \rangle$  quantifiers are *monotonicity* properties (Chapter 5). These are related to the distribution of *polarity items* in sentences using comparative or other type  $\langle 1, 1, 1 \rangle$  quantifiers (see Chapter 5.9).

This ends the present chapter's survey of quantirelations and closely related expressions that denote CONSERV, EXT, and (sometimes) ISOM quantifiers. In the following two chapters we shall go more deeply into certain significant properties that some but not all of these quantifiers have. After that, we shall make a detailed study of two special kinds of determiner expressions that are richly productive in many languages: possessive and exceptive determiners. But we end this chapter with the presentation of an extremely useful technical device for studying CONSERV, EXT, and ISOM type  $\langle 1, 1 \rangle$  quantifiers, or equivalently, as will follow from Proposition 3 in Section 4.5, of ISOM type  $\langle 1 \rangle$  quantifiers.

#### 4.8 ISOM AND THE NUMBER TRIANGLE

The definition of ISOM for type  $\langle 1 \rangle$  quantifiers ((3.17) in Chapter 3.3) can be directly generalized to type  $\langle 1, 1 \rangle$  quantifiers. This time, given a universe  $M$  and  $A, B \subseteq M$ , there are four sets to consider (Fig. 4.1). As before, ISOM says that only the *size* of these sets matter for whether  $Q_M(A, B)$  holds or not.

<sup>29</sup> Unlike the type  $\langle 1, 1 \rangle$  case, CONSERV plus EXT for type  $\langle 1, 1, 1 \rangle$  quantifiers does not mean that they are relativizations in any straightforward sense. We could define a new relativization operation that took two set arguments and relativized to their union:  $Q_M^{2\text{rel}}(A, B, C) \Leftrightarrow Q_{A \cup B}((A \cup B) \cap C)$ . Then  $Q^{2\text{rel}}$  is always CONSERV and EXT, but it would not follow that any CONSERV and EXT type  $\langle 1, 1, 1 \rangle$  quantifier is equal to such a relativized quantifier.

<sup>30</sup> Keenan and Moss (1984) prove general results showing that only CONSERV quantifiers are generated in the ways they consider.

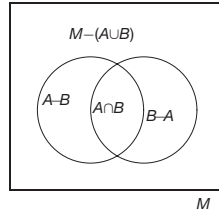


Figure 4.1 The four sets relevant to a type  $\langle 1, 1 \rangle$  quantifier on  $M$

### ISOM for type $\langle 1, 1 \rangle$ quantifiers

A type  $\langle 1, 1 \rangle$  quantifier  $Q$  satisfies ISOM iff for any universes  $M, M'$  and any  $A, B \subseteq M, A', B' \subseteq M'$ :

$$(4.49) \text{ if } |A - B| = |A' - B'|, |A \cap B| = |A' \cap B'|, |B - A| = |B' - A'|, \text{ and } |M - (A \cup B)| = |M' - (A' \cup B')|, \text{ then } Q_M(A, B) \Leftrightarrow Q_{M'}(A', B').$$

Now, CONSERV says precisely that the set  $B - A$  does not matter, and EXT that  $M - (A \cup B)$  does not matter. In other words, together they say that only  $A - B$  and  $A \cap B$  matter:

### Fact 11

- (a) A type  $\langle 1, 1 \rangle$  quantifier  $Q$  is CONSERV and EXT iff the following condition holds: Whenever  $A, B \subseteq M$  and  $A', B' \subseteq M'$  are such that  $A - B = A' - B'$  and  $A \cap B = A' \cap B'$ ,  $Q_M(A, B) \Leftrightarrow Q_{M'}(A', B')$ .
- (b) Thus,  $Q$  is CONSERV, EXT, and ISOM iff the following condition holds: Whenever  $A, B \subseteq M$  and  $A', B' \subseteq M'$  are such that  $|A - B| = |A' - B'|$  and  $|A \cap B| = |A' \cap B'|$ ,  $Q_M(A, B) \Leftrightarrow Q_{M'}(A', B')$ .

*Proof.* (b) follows from (a) and (4.49). For (a), suppose first that  $Q$  satisfies the stated condition. To show that  $Q$  is CONSERV, take  $A, B \subseteq M$ , and let  $A' = A, B' = A \cap B$ , and  $M' = M$ . Then  $A - B = A' - B'$  and  $A \cap B = B' = A' \cap B'$ , so  $Q_M(A, B) \Leftrightarrow Q_{M'}(A', B') \Leftrightarrow Q_M(A, A \cap B)$ . For EXT, take  $A, B \subseteq M \subseteq M'$ , and let  $A' = A$  and  $B' = B$ . By the condition,  $Q_M(A, B) \Leftrightarrow Q_{M'}(A', B') \Leftrightarrow Q_{M'}(A, B)$ .

Conversely, suppose  $Q$  is both CONSERV and EXT, and let  $A, B \subseteq M$  and  $A', B' \subseteq M'$  be as in the antecedent of the condition. Note that this implies that  $A = A'$ . Then  $Q_M(A, B) \Leftrightarrow Q_M(A, A \cap B)$  [by CONSERV]  $\Leftrightarrow Q_M(A', A' \cap B')$  [by assumption]  $\Leftrightarrow Q_{M'}(A', A' \cap B')$  [by EXT]  $\Leftrightarrow Q_{M'}(A', B')$  [by CONSERV].  $\square$

As an application, let us verify that relativization (the operation  $\cdot^{\text{rel}}$ ) does indeed preserve ISOM, as was stated in Proposition 3. Let  $Q$  be any type  $\langle 1 \rangle$  quantifier:

$$(4.50) \text{ } Q \text{ is ISOM iff } Q^{\text{rel}} \text{ is ISOM.}$$

*Proof.* Suppose first  $Q$  is ISOM. To verify that the condition in Fact 11 (b) holds, take  $A, B \subseteq M$  and  $A', B' \subseteq M'$  such that  $|A - B| = |A' - B'|$  and  $|A \cap B| = |A' \cap B'|$ . Then  $Q_M^{\text{rel}}(A, B) \Leftrightarrow Q_A(A \cap B) \Leftrightarrow Q(|A - B|, |A \cap B|)$  [by Fact 4 in Ch. 3.3, since  $Q$  is ISOM and  $A - (A \cap B) = A - B \Leftrightarrow Q(|A' - B'|, |A' \cap B'|)$  [by assumption]  $\Leftrightarrow Q_{A'}(A' \cap B') \Leftrightarrow Q_{M'}^{\text{rel}}(A', B')$ .

Now suppose  $Q^{\text{rel}}$  is ISOM. Take  $A \subseteq M$  and  $A' \subseteq M'$  such that  $|M - A| = |M' - A'|$  and  $|A| = |A'|$ . Then  $Q_M(A) \Leftrightarrow Q_M^{\text{rel}}(M, A)$  [by the definition of relativization]  $\Leftrightarrow Q_{M'}^{\text{rel}}(M', A')$  [by assumption, since  $Q^{\text{rel}}$  is ISOM]  $\Leftrightarrow Q_{M'}(A')$ . Thus,  $Q$  is ISOM.  $\square$

The proof used the fact, pointed out in Chapter 3.3, that an ISOM type  $\langle 1, 1 \rangle$  quantifier  $Q$  can be identified with a binary relation between (cardinal) numbers, also named  $Q$ , such that

$$(4.51) \quad Q_M(A) \iff Q(|M - A|, |A|)$$

From Fact 11 (b) we see that ISOM type  $\langle 1, 1 \rangle$  quantifiers that are CONSERV and EXT can also be identified with binary relations between numbers. But these quantifiers are (Proposition 3) exactly those of the form  $Q^{\text{rel}}$ , for some ISOM type  $\langle 1 \rangle$   $Q$ . Indeed,  $Q$  and  $Q^{\text{rel}}$  correspond to the *same* binary relation between numbers. Let us state this fact explicitly.

#### CONSERV, EXT, and ISOM type $\langle 1, 1 \rangle$ quantifiers as binary relations

If  $Q$  is a CONSERV, EXT, and ISOM type  $\langle 1, 1 \rangle$  quantifier, define a binary relation between cardinal numbers, also called  $Q$ , by

$$(4.52) \quad Q(k, m) \iff \text{there are } M \text{ and } A, B \subseteq M \text{ such that } |A - B| = k, \\ |A \cap B| = m, \text{ and } Q_M(A, B)$$

#### Fact 12

If  $R$  is any binary relation between cardinal numbers and the type  $\langle 1, 1 \rangle$  quantifier  $Q'$  is defined by

$$Q'_M(A, B) \iff R(|A - B|, |A \cap B|)$$

then  $Q'$  is CONSERV, EXT, and ISOM, and the numerical relation corresponding to  $Q'$  by (4.52) is  $R$ . Thus, if  $Q$  is any ISOM type  $\langle 1 \rangle$  quantifier, we have

$$(4.53) \quad Q_M^{\text{rel}}(A, B) \iff Q^{\text{rel}}(|A - B|, |A \cap B|)$$

for all  $M$  and all  $A, B \subseteq M$ . Furthermore, for any (cardinal) numbers  $k, m$ ,

$$(4.54) \quad Q(k, m) \iff Q^{\text{rel}}(k, m)$$

*Proof.* That  $Q'$  is CONSERV, EXT, and ISOM follows immediately from Fact 11 (b). Then  $Q'(k, m) \Leftrightarrow$  there are  $M$  and  $A, B \subseteq M$  such that  $|A - B| = k$ ,  $|A \cap B| = m$ ,

and  $Q'_M(A, B)$  [by (4.52)]  $\Leftrightarrow R(k, m)$  [by definition of  $Q'$ ]. So if  $Q$  is ISOM and of type  $\langle 1 \rangle$ ,  $Q^{\text{rel}}$  has these properties (using (4.50)), and we get (4.53).

Furthermore, (4.54) follows from (4.53): Take  $k, m$ , and  $M$  and  $A, B \subseteq M$  such that  $|A - B| = k$ ,  $|A \cap B| = m$ . Then  $Q^{\text{rel}}(|A - B|, |A \cap B|) \Leftrightarrow Q^{\text{rel}}_M(A, B)$  [by (4.53)]  $\Leftrightarrow Q_A(A \cap B) \Leftrightarrow Q(|A - B|, |A \cap B|)$  [by (4.51)].  $\square$

Thus, the correspondence between ISOM type  $\langle 1 \rangle$  quantifiers and CONSERV, EXT, and ISOM type  $\langle 1, 1 \rangle$  quantifiers is indeed close; when seen as numerical relations, there is no difference at all.

Now, let us restrict attention to the case of *finite universes*—a restriction one can often make in natural language contexts. We let FIN be the assumption that all universes are finite. Then these quantifiers can be seen as binary relations between natural numbers, i.e. as subsets of  $\mathbb{N} \times \mathbb{N}$ . Using this, we can obtain a graphical representation which is very useful for stating, illustrating, and proving properties of ISOM type  $\langle 1 \rangle$  quantifiers (or CONSERV, EXT, and ISOM type  $\langle 1, 1 \rangle$  quantifiers). Think of  $\mathbb{N} \times \mathbb{N}$  as a *number triangle* with  $(0, 0)$  on top, as in Fig. 4.2. Then represent  $Q$  by putting a “+” at those  $(k, m)$  which belong to  $Q$ , and a “−” elsewhere. Familiar quantifiers can now be seen as patterns in the number triangle, as in Fig. 4.3.

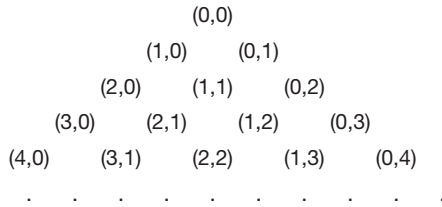


Figure 4.2 The number triangle

We call the lines  $(0, 0), (1, 0), (2, 0), (3, 0), \dots$  and  $(0, 0), (0, 1), (0, 2), (0, 3), \dots$  the *edges* of the number triangle, and the horizontal line  $(n, 0), (n - 1, 1), \dots, (n - m, m), \dots, (1, n - 1), (0, n)$  the *n*'th *level*; see Fig. 4.4. In the case of ISOM type  $\langle 1 \rangle$  quantifiers  $Q$ , the *n*th level corresponds to universes of size  $n$ . For quantifiers of the form  $Q^{\text{rel}}$ ,  $n$  is instead the size of the first argument.

The usefulness of representing quantifiers in the number triangle was first pointed out by van Benthem (1984). It sometimes enables one to perspicuously visualize various properties of quantifiers, to discover patterns that would otherwise be harder to discern, and to give simple proofs of quantificational facts. We will see several examples of this later on. For now, we just note the following:

- Boolean operations on quantifiers are easily visualizable in the number triangle: conjunction is intersection (of +s), disjunction is union, and outer negation is complement, i.e. sign switch. Further, the inner negation of  $Q$  is obtained by rotating the  $Q$  triangle 180 degrees around the vertical axis; see Fact 5 in Chapter 3.3. To

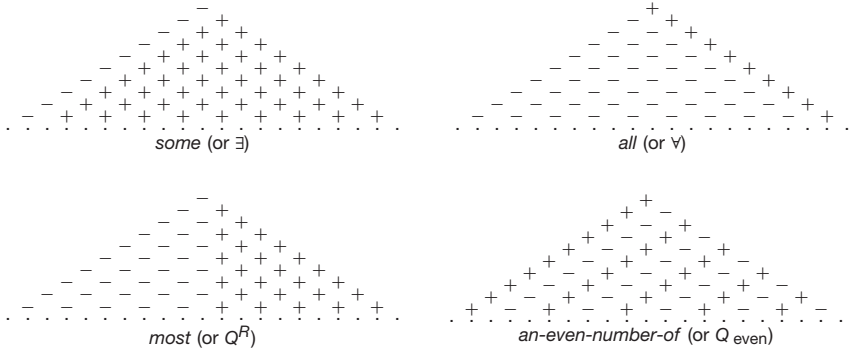


Figure 4.3 Some quantifiers in the number triangle

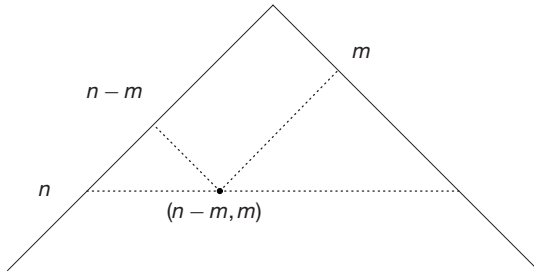


Figure 4.4 A point at level  $n$

get the dual of  $Q$ , one switches sign after the rotation; an example is afforded by the first two triangles in Fig. 4.3.

- The quantifier  $0$  has only  $-$ s, and  $1$  has only  $+$ s. Thus, a quantifier (of the kind representable in the number triangle) is *non-trivial* iff its representation has at least one  $+$  and at least one  $-$ . Stronger non-triviality requirements are easily formulated. Call  $Q$  *trivial at level  $n$*  if that level consists of only  $+$ s or only  $-$ s. The condition  $\text{VAR}$  (from van Benthem 1984) says that each level (except level  $0$ ) is non-trivial. This rules out many common quantifiers, such as *at least five*, which is trivial on levels up to  $4$ . But *at least five* is *eventually VAR*, in the sense that from a certain level on, it is non-trivial. In general, if  $\mathbf{P}$  is a property of quantifiers which makes sense when restricted to levels, the weaker property of being *eventually P* is sometimes useful.
- Similarly, we can say that two quantifiers are *eventually equal* if their patterns in the triangle coincide from a certain level on. This notion is significant in connection with logical definability, which we come to in Chapter 13. If a quantifier is definable in some logical language, then so is each quantifier eventually equal to it.

This is because the logical languages we consider all contain  $FO$ , and any pattern of +s and -s up to some finite level is describable in  $FO$ .

- To see that the last claim is true, note that the type  $\langle 1 \rangle$  quantifier  $\exists!_{k,m}$  defined by

$$(\exists!_{k,m})_M(A) \iff |A| = m \text{ and } |M - A| = k$$

(or its relativization *exactly  $m$  and all but  $k$* ) has a + at the point  $(k, m)$  and -s at all other points. It is clearly definable in  $FO$ , and any pattern below a given level is describable as the finite disjunction of the  $\exists!_{k,m}$ , for all the + points  $(k, m)$  below that level, and so is definable in  $FO$  too.

# 5

## Monotone Quantifiers

Monotonicity is a phenomenon that turns up in many different contexts, linguistic as well as mathematical. Abstractly, it says of some function  $F$  that it is *increasing*: relative to an ordering  $\leq_1$  of the arguments and an ordering  $\leq_2$  (which may be the same as  $\leq_1$ ) of the values:

$$\text{if } x \leq_1 y, \text{ then } F(x) \leq_2 F(y)$$

Such functions are well-behaved in ways that make them have interesting properties.

For quantifiers, being second-order relations, the relevant order for arguments is *inclusion*, and for values *implication*, and the general concept of monotonicity applies straightforwardly to quantifiers. However, we shall see that for the quantifiers that show up as denotations of NPs or determiners in natural languages, there are more fine-grained notions of monotonicity which play significant roles.

In a way, Aristotle discovered logic by studying monotonicity. Each of the four Aristotelian quantifiers has strong monotonicity properties, and, moreover, many valid syllogisms can be seen as *statements* of these properties. For example, the syllogism (called *Barbara* by medieval philosophers<sup>1</sup>)

$$(5.1) \quad \begin{array}{l} \text{all } BC \\ \hline \text{all } AB \\ \hline \text{all } AC \end{array}$$

states that the quantifier *all* is (what we will call) monotone increasing in the right argument (and also that it is decreasing in the left argument), and

$$(5.2) \quad \begin{array}{l} \text{no } BC \\ \hline \text{all } AB \\ \hline \text{no } AC \end{array}$$

can be read as stating that the quantifier *no* is monotone decreasing in the left argument.

After defining and exemplifying standard monotonicity for quantifiers of various types (section 5.1), we focus in section 5.2 on the type  $\langle 1, 1 \rangle$  case, emphasizing that

<sup>1</sup> This was a mnemonic for **AAA**, meaning that the two premises as well as the conclusion use the **A** form in the square of opposition (Ch. 1.1.1): the quantifier *all*. Similarly, the syllogism (5.2) below was called *Celarent*: **EAE** (see Spade 2002: 21–4 for further explanation of these mnemonics).

for CONSERV and EXT quantifiers, and hence for denotations of determiners and other quantirelations, there is a significant difference between monotonicity in the *left* argument—often called *persistence* in the increasing case—and monotonicity in the *right* one, a difference which manifests itself in various ways. Section 5.3 briefly summarizes some facts about the distribution of monotonicity in natural languages in the form of monotonicity universals, three from Barwise and Cooper 1981 and one which is new here.

If ISOM is assumed, monotonicity properties become even more perspicuous, especially when depicted in the number triangle. This is presented in section 5.4. Inspired by that representation, but in fact not dependent on it (i.e. not dependent on the assumption of ISOM and FIN), we deepen the analysis of monotonicity in section 5.5, isolating six basic forms of monotonicity: apart from upward and downward monotonicity in the right argument, also four left monotonicities. Standard left monotonicity combines these, but there are further interesting combinations. In particular, section 5.6 deals with *smooth* quantifiers, which combine two of the basic left properties. Smooth quantifiers are significant in several ways. In particular, smoothness implies right upward monotonicity, and it turns out that many right upward monotone determiner denotations are in fact smooth, e.g. all the proportional quantifiers. These are not left monotone, but a complete description of their monotonicity behavior involves smoothness rather than right monotonicity, a fact usually missed in the literature, perhaps because ordinary monotonicity is easier to *test empirically*. Nevertheless, smoothness can also be tested, and we show what is involved.

With the six basic properties a richer picture of monotonicity in natural languages emerges, compared to the one usually finds in the literature. We also see that almost every determiner denotation has at least one of these properties, and that other seemingly unrelated properties, notably *symmetry*, also are combinations of two of the basic monotonicities.

The final three sections apply facts about monotonicity directly to linguistic issues. Section 5.7 answers a question by Keenan concerning a particular inference scheme involving proportional natural language quantifiers. Section 5.8 proves a result that sheds light on the semantics of quantified NPs with conjoined nouns, as in *Three men and women are sitting over there*, discussed in Chapter 4.7. Section 5.9, finally, overviews ongoing research into what monotonicity and other related properties have to do with the occurrence of (positive and negative) *polarity items* in natural languages. Using techniques and results from this chapter, we are able to give a more precise assessment of how certain hypotheses about the distribution of polarity items could be tested.

## 5.1 STANDARD MONOTONICITY

Let us begin with the type ⟨1⟩ case.

### monotonicity for type ⟨1⟩ quantifiers

$Q_M$  of type ⟨1⟩ is (monotone) *increasing* iff the following holds:



(5.3) if  $A \subseteq A' \subseteq M$ , then  $Q_M(A)$  implies  $Q_M(A')$

$Q_M$  is (monotone) *decreasing* iff

(5.4) if  $A' \subseteq A \subseteq M$ , then  $Q_M(A)$  implies  $Q_M(A')$

These local notions extend immediately to the global case:  $Q$  is monotone increasing (decreasing) if *each*  $Q_M$  is.

Monotone increasing quantifiers are often called just *monotone* in the literature.<sup>2</sup> Most of the type (1) quantifiers we have seen so far are increasing or decreasing; here are some relevant observations. Of the quantifiers exemplified in Chapter 2.3,

- $\forall, \exists, \exists_{\geq n}, Q_0, Q^C, Q^R, (p/q), [p/q], (p/q)^d$  are monotone increasing,
- $\exists_{\leq n}$  is monotone decreasing,
- $\exists_{=n}, \exists_{=n}\neg, \exists_{\geq n} \wedge \exists_{\leq m} (n \leq m), Q_{\text{even}}$  are neither increasing nor decreasing.

There might seem to be fewer decreasing than increasing ones, but that is just an impression created by the vocabulary of natural languages, which influences which quantifiers people find natural to take as primitive, as the following observations illustrate:

(5.5)  $Q$  is increasing iff  $\neg Q$  is decreasing.

[Since if  $Q_M(A)$  implies  $Q_M(A')$ , then  $\neg Q_M(A')$  implies  $\neg Q_M(A)$ .]

(5.6)  $Q$  is increasing iff  $Q\neg$  is decreasing.

[Since if  $A \subseteq A'$ , then  $M - A' \subseteq M - A$ . Hence:]

(5.7)  $Q$  is increasing (decreasing) iff  $Q^d$  is increasing (decreasing).

The trivial quantifiers  $\mathbf{1}_M$  and  $\mathbf{0}_M$  are both increasing *and* decreasing. But on a given universe  $M$  they are the *only* quantifiers with this property. [For, if  $Q_M \neq \mathbf{0}_M$ , there is some  $A \subseteq M$  such that  $Q_M(A)$ . Take any  $B \subseteq M$ . Then  $Q_M(A \cap B)$  since  $Q_M$  is decreasing, hence  $Q_M(B)$  since  $Q_M$  is increasing. So  $Q_M(B)$  holds for any  $B$ , i.e.  $Q_M = \mathbf{1}_M$ .]

Montagovian individuals  $I_a$  and bare plurals  $C^{\text{pl}}$  are monotone increasing. Finally,

(5.8)  $Q$  is monotone increasing (decreasing) if and only if every quantifier of the form  $Q^{[A]}$  is monotone increasing (decreasing).

<sup>2</sup> Thus, these quantifiers can be called “monotone increasing”, or just “increasing”, or just “monotone”. It is also common to let “monotone” stand for “either increasing or decreasing”, so that monotonicity is what increasing and decreasing quantifiers have in common, whereas non-monotone quantifiers are neither increasing nor decreasing. Further, the terms *upward* and *downward* monotone are also used. These slight differences in terminology should not cause confusion.

Note also that with the functional view of (local) type (1) quantifiers (Ch. 3.1.3) as functions in  $[P(M) \rightarrow 2]$ , monotonicity is precisely that  $A \leq_1 A'$  implies  $Q_M(A) \leq_2 Q_M(A')$ , where the (partial) ordering on subsets of  $M$  is the inclusion order, and  $0 < 1$ .

[If  $Q$  is increasing,  $A$  is any set,  $B \subseteq B' \subseteq M$ , and  $Q_{M(B)}^{[A]}$ , then  $Q_A(A \cap B)$ , so  $Q_A(A \cap B')$  since  $A \cap B \subseteq A \cap B'$  and  $Q_A$  is increasing, and thus  $Q_{M(B')}^{[A]}$ . Hence,  $Q^{[A]}$  is increasing. On the other hand, if  $Q$  is not increasing, some  $Q_M$  is not increasing, and we can take  $A = M$ :  $Q_M^{[M]} = Q_M$  is not increasing, so  $Q^{[M]}$  is not increasing. Similarly in the decreasing case.]

Notice that (5.8)—in contrast with the other claims above—is an essentially *global* claim: that  $Q$  is increasing means that *every* local  $Q_M$  is increasing, and this is crucial for the argument to work. For example, to conclude that  $Q_M^{[A]}$  is increasing, it is not enough to know that  $Q_M$  is increasing; we need that  $Q_A$  is increasing.

(5.8) explains why many noun phrase denotations are increasing or decreasing. For example, at most four students is decreasing, which is illustrated by facts such that if

(5.9) At most four students came to the opening.

is true, then

(5.10) At most four students came to the opening wearing a tie.

must also be true. Likewise, if

(5.11) Most professors came to the funeral.

is true, then

(5.12) Most professors came to the funeral or attended the service

is also true, witnessing the fact that most professors is monotone increasing.

By means of such examples one can devise *tests* for monotonicity: for example, that sentences of the form At least five A are B should imply At least five A are B or C. Of course, this is not a test that the quantifier  $\exists_{\geq 5}^{[A]}$  is increasing—we already know that it is. Rather, it is a test that we have interpreted noun phrases of the form at least five A correctly, or at least in a way that does not violate linguistic intuitions. Naturally, such tests are not in themselves conclusive. The hypothesis tested concerns meaning, which is about all possible cases, whereas we can test only a finite number of them. And even if we should find a counterexample, the blame may be put elsewhere than on the meaning of the noun phrase. For example, if we should find that English speakers easily envisage situations where they would judge John is at school to be true, but John is at school or at home to be false, then we ought to conclude that something is seriously amiss with our semantics; but it need not be Montague's treatment of proper names—it could be something to do with or instead. Happily, we do not find such intuitions prevalent among English speakers,<sup>3</sup>

<sup>3</sup> Again, we must take care to distinguish pragmatic facts, such that if one *knows* that John is at school, it may feel *odd to say* that he is at school or at home, from judgments about truth and falsity.

and this then is a test of (among other things) that treatment of names, which does interpret them by means of an increasing quantifier.

As regards non-monotone quantifiers (in the sense of being neither increasing nor decreasing) a great many of the ‘natural’ examples are Boolean combinations of monotone ones. In Chapter 4.3 we discussed the claim that NP denotations (and determiner denotations) are *closed* under Boolean operations, and the related—but distinct—claim that (say) English NPs can always be conjoined with *and*, *or*, etc. Barwise and Cooper (1981) use monotonicity to refute the latter claim, arguing that at least sometimes one cannot conjoin an increasing and a decreasing NP. But (as Barwise and Cooper admit) the data are not so clear, and at least as an approximation, the claim about denotations (called  $\mathbf{BCI}_{\text{NP}}$  in Chapter 4.3) seems plausible, as does the corresponding claim ( $\mathbf{BCI}_{\text{Det}}$ ) about determiner denotations.

In any case, there are many obvious cases of Boolean combinations of NP denotations in natural languages. Thus, *between three and six dogs* is the conjunction of *at least three dogs* and *at most six dogs*, and *all but five* is the inner negation of the conjunction of *at least five* and *at most five*. But a notable exception to this pattern is the quantifier  $Q_{\text{even}}$ , and its cognates like *an odd number of cards*, or *all but an even number of balls*. These are not increasing or decreasing, nor are they Boolean combinations of such quantifiers. There is no reason for the parity of a set to be preserved when one adds to or deletes from it an arbitrary number of elements. Of course this claim about  $Q_{\text{even}}$ —that it is not expressible as a Boolean combination of monotone quantifiers—is so far very preliminary: though seemingly plausible when one thinks about it, it needs to be (a) made precise, and (b) verified (or disproved). Indeed, a clear statement (in fact, a much stronger one) can be proved for a suitable formal language, as we will see in Chapter 14.4. And in Chapter 12 we show how such a result transfers to natural languages.

To extend the idea of monotonicity to arbitrary quantifiers is an easy matter. Naturally, if a quantifier as a second-order relation has more than one argument, we must specify which argument we are talking about. This leads to the following definition.

#### monotonicity for arbitrary quantifiers

A quantifier  $Q_M$  of type  $\langle n_1, \dots, n_k \rangle$  is (monotone) *increasing in the  $i$ 'th argument* iff the following holds:

$$(5.13) \text{ If } Q_M(R_1, \dots, R_k), \text{ and if it holds that } R_i \subseteq R'_i \subseteq M^{n_i}, \text{ then} \\ Q_M(R_1, \dots, R_{i-1}, R'_i, R_{i+1}, \dots, R_k)$$

Similarly for the (monotone) *decreasing* case.

Of course, a quantifier can be increasing in one argument and decreasing in another. For example, the type  $\langle 1, 1 \rangle$  quantifier  $MO_M(A, B) \Leftrightarrow |A| > |B|$  is increasing in the first argument and decreasing in the second;  $most_M(A, B) \Leftrightarrow |A \cap B| > |A - B|$  is increasing in the second argument, but neither increasing nor decreasing in the first; and the Härtig quantifier  $I_M(A, B) \Leftrightarrow |A| = |B|$ , and the quantifier *exactly two*, are not monotone in either argument.

The behavior of a conjunction of an increasing and a decreasing quantifier, such as the type  $\langle 1, 1 \rangle$  *between three and six*, is sometimes given a name, as a weaker form of monotonicity:

### continuous quantifiers

A type  $\langle 1 \rangle$  quantifier  $Q_M$  is *continuous* (CONT) iff

$$Q_M(A'), Q_M(A''), \text{ and } A' \subseteq A \subseteq A'' \text{ implies } Q_M(A)$$

Similarly for arbitrary quantifiers (*continuity in the  $i$ 'th argument*).

An increasing or decreasing quantifier is continuous, but not necessarily vice versa. In fact, CONT turns out to characterize exactly conjunctions of an increasing and a decreasing quantifier (a fact first noted in Thijsse 1983):

### Proposition 1

*A quantifier is CONT (in the  $i$ 'th argument) if and only if it is the conjunction of an increasing and a decreasing quantifier (in the  $i$ 'th argument).*

*Proof.* Consider the local type  $\langle 1 \rangle$  case. Obviously, such a conjunction is CONT. In the other direction, suppose  $Q_M$  is CONT. Define  $(Q_1)_M$  and  $(Q_2)_M$  as follows.

$$(Q_1)_M(A) \Leftrightarrow \exists A' [A' \subseteq A \text{ and } Q_M(A')]$$

$$(Q_2)_M(A) \Leftrightarrow \exists A'' [A \subseteq A'' \subseteq M \text{ and } Q_M(A'')]$$

It is straightforward to see that  $(Q_1)_M$  is increasing,  $(Q_2)_M$  is decreasing, and  $Q_M = (Q_1)_M \wedge (Q_2)_M$ .  $\square$

For monotonicity of quantifiers, we adopt “monotone” as the cover term, specifying when needed the direction and argument concerned. A few terms for special cases will also be used. One is *persistence* (from Barwise and Cooper 1981), which stands for (upward) monotonicity in the left argument of a type  $\langle 1, 1 \rangle$  quantifier; we discuss it in the next subsection. Another one that we also like is *smoothness*, introduced in section 5.6.

## 5.2 MONOTONICITY IN TYPE $\langle 1, 1 \rangle$

A type  $\langle 1, 1 \rangle$  quantifier can be increasing or decreasing in no argument, in just one of them, or in both. For the monotone cases, the notation  $\text{MON}\uparrow$ ,  $\downarrow\text{MON}$ ,  $\uparrow\text{MON}\downarrow$ , etc., is often used. Also, we speak of *right* and *left* monotonicity of a type  $\langle 1, 1 \rangle$  quantifier.

Proposition 3 in Chapter 4.5 claimed that the monotonicity (up or down) of a type  $\langle 1 \rangle$  quantifier  $Q$  carries over to the *right* argument of its relativization  $Q^{\text{rel}}$ :

$$(5.14) \quad Q \text{ is increasing (decreasing) iff } Q^{\text{rel}} \text{ is } \text{MON}\uparrow \text{ (MON}\downarrow\text{)}.$$

The proof is immediate; for example, if  $Q$  is increasing,  $Q_{M(A,B)}^{\text{rel}}$  holds, and  $B \subseteq B' \subseteq M$ , then  $Q_A(A \cap B)$ , so  $Q_A(A \cap B')$ , i.e.  $Q_{M(A,B')}^{\text{rel}}$ .

A consequence is that linguistic observations about monotonicity behavior can often be stated for either NP denotations or determiner denotations as one wishes. For NPs formed with a determiner, it is indeed more natural to state the monotonicity behavior for the determiner, and we have seen that the behavior of the right argument is reflected exactly in that of the NP. Let us be precise about this, for the record. For any type (1) quantifier  $Q$ :

$$\begin{aligned} Q^{\text{rel}} \text{ is } \text{MON}\uparrow &\iff Q \text{ is increasing [by (5.14)]} \\ &\iff \text{for every } AQ^{[A]} = (Q^{\text{rel}})^A \text{ is increasing [by (5.8)],} \end{aligned}$$

and similarly for the  $\text{MON}\downarrow$  case.

It is mostly unproblematic to determine the monotonicity behavior of determiners and other quantirelations in natural languages. Here are some English examples, with respect to right monotonicity:

- $\text{MON}\uparrow$ : all, every, a, some, several, the, both, at least four, more than ten, all but at most five, infinitely many, all but finitely many, most (and indeed all the proportional determiners), many, John's, Mary's ten, most children's, some teachers',<sup>4</sup> the eight, at least four of the seven<sup>5</sup>
- $\text{MON}\downarrow$ : no, not all, at most three, fewer than eleven, finitely many, few, at most half of the, neither John's nor Mary's, no students', at most three of the eight
- Neither  $\text{MON}\uparrow$  nor  $\text{MON}\downarrow$ : exactly five, all but nine, some but not all, at least two and no more than ten, either fewer than five or else more than a hundred, an odd number of, between 20 and 40 percent of the, exactly two students', exactly five of the ten, no—but John, every—but Mary

*Left* monotonicity of determiner and other quantirelation denotations is often called *persistence* in the upward case, and *anti-persistence* in the downward case, and we will also use these terms. Right monotonicity concerns the scope: it concerns what happens if we add or remove some elements from the extension of the scope. But left monotonicity concerns the restriction, which, as we know, acts as the universe of quantification. It is natural, for quantifiers denoted by determiners, that these are quite different properties, and we do find that left monotonicity is less frequent in

<sup>4</sup> Possessive determiners are sometimes ambiguous between various readings, as we will see in Ch. 7. The monotonicity claims for the possessives in the present list apply to their usually preferred readings.

<sup>5</sup> Barwise and Cooper (1981) list a *few* as right increasing, but this seems to be a mistake. We think a *few* means 'some but only a few', or 'some but not many', which is why *not a few* means 'many'. If a *few* meant 'at least several', as Barwise and Cooper seem to think, *not a few* should mean 'not even several' or 'at most a couple', but it doesn't. The misconception may be connected with the fact that American *quite a few* and British a *good few* mean 'more than a few', unlike *quite few*, which means 'very few' (just as *quite many* means 'very many'). We therefore believe that a *few* is not right monotone, although it is almost decreasing, in the sense that if  $\emptyset \neq B' \subseteq B$  and a *few* $_M(A, B)$ , then a *few* $_M(A, B')$ .

natural languages—in fact, is limited to a few typical cases. For example, proportional quantifiers are always right but never left monotone: if

$$q \cdot |A \cap B| \geq p \cdot |A|$$

holds, it continues to do so if  $|B|$  is increased, but no conclusion can be drawn about what happens if  $|A|$  is increased or decreased.

Here are some English examples:<sup>6</sup>

- Persistent ( $\uparrow$ MON): a, some, several, not all, at least four, more than ten, infinitely many, some of John's
- Anti-persistent ( $\downarrow$ MON): all, every, no, at most three, fewer than eleven, finitely many, all but at most five, all but finitely many, no students'
- Neither persistent nor anti-persistent: the, both, exactly five, all but nine, some but not all, at least two and no more than ten, either fewer than five or else more than a hundred, an odd number of, between 20 and 40 percent of the, most, more than two-thirds of the, at most half of the, many(?), few(?), the eight, John's, exactly two students', Mary's ten, at least four of the seven, no—but John, every—except Mary

To further describe the (anti-)persistent quantirelation denotations, we use the notion of a *square of opposition* from Chapter 4.3, and the fact that the monotonicity behavior (right and left) of a type  $\langle 1, 1 \rangle$  quantifier completely determines the monotonicity behavior of the other quantifiers in its square of opposition, namely, as follows.

#### Fact 2

Let  $Q$  be any type  $\langle 1, 1 \rangle$  quantifier:

$$\begin{aligned} Q \text{ is MON}\uparrow &\text{ iff } \neg Q \text{ is MON}\downarrow \\ &\text{ iff } Q\neg \text{ is MON}\downarrow \\ &\text{ iff } Q^d \text{ is MON}\uparrow \\ Q \text{ is }\uparrow\text{MON} &\text{ iff } \neg Q \text{ is }\downarrow\text{MON} \\ &\text{ iff } Q\neg \text{ is }\uparrow\text{MON} \\ &\text{ iff } Q^d \text{ is }\downarrow\text{MON} \end{aligned}$$

Similarly (with reversed arrows) for the downward monotone case. In particular, if  $Q$  is doubly monotone, its square of opposition exhibits all of the four possible doubly monotone patterns.

*Proof.* Clearly, outer negation reverses monotonicity, in any argument. Inner negation reverses monotonicity in the second argument, since there we are looking at complements, but not in the first argument. Dual is the outer negation of inner negation,

<sup>6</sup> About the possessives in this list, see n. 4. Ch. 7.13 deals in detail with the monotonicity behavior of possessives. About the question marks for many and few, see Ch. 6.2.

and so reverses its monotonicity. For example, if  $Q$  is  $\uparrow\text{MON}\uparrow$ ,  $\neg Q$ ,  $Q\neg$ , and  $Q^d$  will be  $\downarrow\text{MON}\downarrow$ ,  $\uparrow\text{MON}\downarrow$ , and  $\downarrow\text{MON}\uparrow$ , respectively.  $\square$

Thus, since *at least three* is  $\uparrow\text{MON}\uparrow$ , we find the other three doubly monotone patterns in its square. Likewise, since *most* is  $\text{MON}\uparrow$  but not left monotone, it follows from Fact 2 that  $\neg\text{most}$  and  $\text{most}\neg$  (on finite universes: *at most half of the* and *fewer than half of the*) are  $\text{MON}\downarrow$ , that  $\text{most}^d$  (*at least half of the*) is  $\text{MON}\uparrow$ , and also that no quantifier in this square is left monotone.

In particular, in describing which quantirelation denotations are left monotone, we can restrict attention to, say, the persistent ( $\uparrow\text{MON}$ ) case, at least if we make the idealizing assumption ( $\text{BCI}_{\text{Det}}$ ) that determiner denotations are closed under Boolean operations. However, in reality there seem to be just a few  $\text{ISOM}$  and persistent ones, though it is easy to construct artificially lots of other persistent type  $\langle 1, 1 \rangle$  quantifiers that are also  $\text{CONSERV}$ ,  $\text{EXT}$ , and  $\text{ISOM}$ . So there are severe restrictions on which left monotone quantifiers quantirelations can denote.

We come back to this in the next section, but first, there is one question we have left hanging: Whereas right monotonicity of  $Q^{\text{rel}}$  corresponds to monotonicity of  $Q$ , what property of  $Q$  does left monotonicity of  $Q^{\text{rel}}$  correspond to?

Consider the following property of a type  $\langle 1 \rangle$  quantifier  $Q$ :

(p) If  $Q_M(A)$  and  $A \subseteq M \subseteq M'$ , then  $Q_{M'}(A)$ .

Here are a few observations.

- (p) is half of the  $\text{EXT}$  condition for  $Q$ ; indeed,  $Q$  is  $\text{EXT}$  iff both  $Q$  and  $\neg Q$  satisfy (p).
- If  $Q^{\text{rel}}$  is persistent, then  $Q$  satisfies (p).

[Suppose  $A \subseteq M \subseteq M'$ .  $Q_M(A)$  implies  $Q_{M(M,A)}^{\text{rel}}$ , and hence  $Q_{M'}^{\text{rel}}(M, A)$  (since all relativized quantifiers are  $\text{EXT}$ ), which implies  $Q_{M'}^{\text{rel}}(M', A)$  (by persistence), that is,  $Q_{M'}(A)$ .]

- The converse of this fails, but we do have the following: If  $Q$  satisfies (p) *and* is increasing, then  $Q^{\text{rel}}$  is persistent.

[Suppose  $A \subseteq A' \subseteq M$  and  $B \subseteq M$ .  $Q_M^{\text{rel}}(A, B)$  implies  $Q_A(A \cap B)$ , and so  $Q_{A'}(A \cap B)$  by (p), whence  $Q_{A'}(A' \cap B)$  (since  $Q$  is increasing), that is,  $Q_M^{\text{rel}}(A', B)$ .]

But these are scattered remarks. There ought to be a systematic connection between the monotonicity behavior of  $Q$  and the right *and* left monotonicity behavior of  $Q^{\text{rel}}$ . At least, that should be so if our claim that the operation of relativization exactly mirrors type  $\langle 1 \rangle$  quantifiers into  $\text{CONSERV}$  and  $\text{EXT}$  type  $\langle 1, 1 \rangle$  quantifiers is correct. And indeed there is, though to state the connection, we need a more fine-grained analysis of monotonicity. This analysis is best presented with the aid of the number triangle (even though it actually presupposes neither  $\text{FIN}$  nor  $\text{ISOM}$ ), so we postpone it until section 5.5 below (where we also give the property (p) a more appropriate name).

To give final emphasis to the strength of left—by contrast with right—monotonicity, we state the following two results. Recall (Chapter 4.8) that FIN stands for the assumption that only finite universes are considered.

**Proposition 3** (van Benthem)

*Under FIN, the only left monotone quantifiers satisfying CONSERV, EXT, ISOM, and the non-triviality condition VAR, are the four quantifiers in square (all), i.e. all, some, no, and not all.*

By contrast, there are infinitely many right monotone quantifiers with these properties. The proof of this rather striking proposition is by simple examination of the number triangle, as we will see below in section 5.4.

**Proposition 4**

*Under FIN, left monotone quantifiers satisfying CONSERV, EXT, and ISOM, are first-order definable, i.e. definable in the logic FO.*

A proof, again using the number triangle, can be found in Westerståhl 1989. Right monotone quantifiers (for example, the proportional quantifiers) are usually not at all FO-definable over finite models, as we will see in Chapter 14.

The stark contrast we have observed between left and right monotonicity for natural language quantifiers has no counterpart for quantifiers in logic. This again illustrates the characteristic feature of quantirelations, that their left (restriction) argument serves to restrict the universe of quantification.

### 5.3 MONOTONICITY UNIVERSALS

It is a significant fact that almost every denotation of a determiner or other quantirelation in a natural language, as well as every NP denotation, has some sort of monotonicity property.<sup>7</sup> One way to appreciate this is to realize how many non-monotone quantifiers (of these types) there are. As we will see below, the number triangle is excellent for boosting one's intuitions here. But we can also try to formulate empirical generalizations about monotonicity, based on the actual distribution of monotone quantifiers in natural languages. Such generalizations are a kind of hypothesized *linguistic universal*. Barwise and Cooper (1981) give a number of possible monotonicity universals, for example:

(MU1) *Syntactically simple determiners denote right monotone quantifiers, or conjunctions of such quantifiers.*<sup>8</sup> *That is (Proposition 1 in section 5.1), they denote continuous quantifiers.*

<sup>7</sup> At the moment, this claim might seem dubious, given that we have already encountered many determiner and NP denotations that are neither increasing nor decreasing. But we will introduce more specialized notions of monotonicity below (sect. 5.5), and our intended (albeit somewhat imprecise) claim is that almost all determiner and NP denotations have at least one of these properties.

<sup>8</sup> Actually, Barwise and Cooper (1981) formulate this for NP denotations, but the formulation in terms of determiner denotations is equivalent. Indeed, we think the universals here apply to quantirelations in general and not only to determiners, as is clear from the claim (Ch. 0.2.3) that every quantirelation denotation is also a determiner denotation.



This rules out that a natural language could have a *simple* determiner denoting, say, *exactly three or more than seven* or *an even number of*. A numerical quantifier like *ten* is simple, but it denotes (in one reading) the conjunction of *at least ten* and *at most ten*.

(MU1) has the character of a straightforward generalization from observed facts. Another example from Barwise and Cooper (1981) is

(MU2) *Positive (negative) strong determiner denotations are increasing (decreasing) in the right argument.*<sup>9</sup>

As to left monotonicity, Barwise and Cooper have (essentially) the claim:

(MU3) *Left monotone determiner denotations—whether the determiner is simple or complex—are also right monotone.*

This seems correct also for possessives (not considered by Barwise and Cooper); we will see in Chapter 7 that the left monotone possessive determiners all appear to be right monotone as well. (MU3) is a significant and strong constraint, since it is easy to construct counterexamples for invented languages.

But which are the left monotone determiner denotations? Restricting attention to the ISOM and persistent ones, we venture the following hypothesis:

(MU4) *Every non-trivial, persistent, and ISOM determiner denotation is one of the quantifiers at least  $m$  of the  $k$  or more and their inner negations, for  $0 \leq m \leq k$ .*

In connection with (MU4), note the following: (a) With  $m = k$  we obtain the quantifiers *at least  $m$* . (b)  $m = k = 0$  gives the trivial quantifier **I**, which could have been set aside explicitly. But we want to allow  $0 = m < k$ , since these are (perhaps) denotations of any number of the  $k$  or more. (c) As to the inner negations, with  $m = 1$  we get the quantifier *not all* ( $k = 1$ ), and more generally *not all of the  $k$  or more*. (d) We are mostly thinking of finite  $m$  and  $k$  here. But  $m = k = \aleph_0$  gives the quantifier *infinitely many*, and similarly for *uncountably many*.

Universals like these *describe* the distribution of monotonicity in natural languages. Surely, one would think, there must be some reason for this state of affairs. What is it that *explains* the ubiquity of monotonicity? Intuitively, monotone quantifiers are simple, easily described, and generally well-behaved. But simplicity is a notoriously hard concept to capture. One might try to do it in terms of *learnability*: that it is quicker or in some other way more feasible to learn determiners denoting monotone quantifiers than others. A related attempt originates with Barwise and Cooper (1981), who started exploring the idea that monotone quantifiers are *easier to process* than non-monotone ones. This was developed (a lot) further by van Benthem (1986: chs. 8 and 10).

<sup>9</sup> These notions are defined for *partial* quantifiers:  $Q$  is *positive (negative) strong* iff, for all  $A$  such that  $Q(A, \cdot)$  is defined,  $Q(A, A) (\neg Q(A, A))$  holds. This is one (the only useful?) instance where partiality offers a distinction that cannot be captured within the total (relational) view on quantifiers taken in this book. However, there is some doubt about (a) what Barwise and Cooper's definition really is, and (b) whether the extra freedom it allows is really warranted. We discuss all of this in detail in Ch. 6.3, in the context of which quantifiers are allowed in existential-there sentences, which is what the notion of a strong determiner was intended to explain; see esp. Ch. 6.3.4.

We will not go into this interesting area in this book, instead referring the reader to the texts mentioned. However, we will exemplify the use of monotonicity, and of facts about monotone quantifiers, to explain various linguistic phenomena. This happens in the final three sections of this chapter. Before that, however, we need to dig deeper into the analysis of monotonicity, as it relates to quantification in natural languages.

## 5.4 MONOTONICITY UNDER ISOM

When ISOM is assumed, and even more so over finite universes (FIN), monotonicity properties become especially simple and perspicuous. We can then represent them in the number triangle, and this very representation leads to insights and results that would have been very hard to discover without it. In fact, it sometimes leads to insights that turn out not to depend on ISOM or FIN at all, but hold without extra assumptions (except that CONSERV is usually needed). This is the case with the finer grained monotonicity properties that we introduce in section 5.5. But let us begin with the number-theoretic versions of monotonicity.

### 5.4.1 A format for monotone quantifiers over finite universes

Consider first the type ⟨1⟩ case. A monotone increasing quantifier  $Q$  associates with a universe  $M$  an idea of relative size: ‘big enough’ subsets of  $M$  are in  $Q_M$  in the sense that if  $A$  is in  $Q_M$ , so are all bigger sets. Of course, quite small sets may be ‘big enough’ in this sense; the minimum size requested by  $Q$  can be chosen freely, and in the extreme case when  $\emptyset \in Q_M$ , all subsets of  $M$  are ‘big enough’ (so  $Q_M = \mathbf{1}_M$ ).

In general there need be no smallest set among the ‘big enough’ ones—smallest in the sense that no proper subset of it is ‘big enough’. For example, consider  $Q_0$ , i.e. *infinitely many things*. Removing one element from an infinite set always leaves an infinite set, so there can be no smallest set among the ones in  $(Q_0)_M$  (if  $M$  is infinite). Likewise,  $(Q_0^d)_M$ , i.e. *all but finitely many things*, contains no smallest set, but both of these quantifiers are monotone increasing.

But there is always a smallest *size* of set that is ‘big enough’. This is because any class of cardinal numbers has a smallest element, so indeed for any quantifier  $Q_M$  there is a smallest size of the sets it contains, regardless of whether it is monotone or not. When  $Q$  is monotone increasing and ISOM, this intuitive idea takes a definite and concrete shape. Then the criterion of being ‘big enough’ for  $Q$  depends only on  $|M|$ , and is a cardinal number  $\kappa \leq |M|$ , such that  $Q_M(A)$  holds iff  $|A| \geq \kappa$ . There is a very useful notation for this in the special case of *finite* universes.

So assume  $Q$  is monotone increasing and satisfies ISOM. There is then a corresponding function  $f$  which with each finite universe of size  $n$  associates a natural number  $f(n)$ , the smallest size of sets in  $Q_M$ , when  $|M| = n$ . Thus,  $0 \leq f(n) \leq n$ . This covers all cases except one: namely, when no sets at all are in  $Q_M$  (so  $Q_M = \mathbf{0}_M$ ). But if we stipulate that in this case  $f(n) = n + 1$ , it holds in general that if  $A \subseteq M$ , then

$$Q_M(A) \iff |A| \geq f(|M|)$$

(In particular, if  $f(|M|) = |M| + 1$ ,  $|A| \geq f(|M|)$  is always false.) We summarize this in the following definition and observation.

**the quantifiers  $Q_f$**

If  $f$  is any function from natural numbers to natural numbers such that for all  $n$ ,  $0 \leq f(n) \leq n + 1$ , the type  $\langle 1 \rangle$  quantifier  $Q_f$  is defined as follows (on finite universes): For any finite  $M$  and any  $A \subseteq M$ ,

$$(5.15) \quad (Q_f)_M(A) \iff |A| \geq f(|M|)$$

Then  $Q_f$  is ISOM and monotone increasing. Moreover, any ISOM and monotone increasing quantifier  $Q$  is equal to  $Q_f$  (on finite universes) for some  $f$ , which we may call the *monotonicity function* for  $Q$ .

Here are some of our previous examples (on finite universes) written in this format:

- If  $f_1(n) = 1$  for all  $n$ , then  $Q_{f_1} = \exists$ . More generally, if  $g(n) =$  a constant  $k$  for all  $n$ , then  $Q_g = \exists_{\geq k}$ .
- If  $f_2(n) = n$  for all  $n$ , then  $Q_{f_2} = \forall$ . If  $h(n) = n - k$  for a fixed  $k$ , then  $Q_h$  is *all but at most  $k$  things*.
- If  $f_3(n) =$  the smallest natural number  $> n/2$ , then  $Q_{f_3} = Q^R$ .

We noted earlier that  $Q$  is increasing iff its dual  $Q^d$  is increasing. Now observe that, with  $|M| = n$ ,

$$\begin{aligned} (Q_f^d)_M(A) &\iff |M - A| < f(|M|) \\ &\iff n - |A| < f(n) \\ &\iff |A| > n - f(n) \\ &\iff |A| \geq n - f(n) + 1 \end{aligned}$$

Thus, we have the following

**Fact 5**

Define, for any monotonicity function  $f$ ,

$$f^d(n) = n - f(n) + 1$$

for all  $n$ . Then  $f^d$  is the monotonicity function for the dual of  $Q_f$ , that is,  $Q_{f^d} = (Q_f)^d$ .

Now consider relativizations of type  $\langle 1 \rangle$  quantifiers. We have

$$\begin{aligned} (Q_f^{\text{rel}})_M(A, B) &\iff (Q_f)_A(A \cap B) \\ &\iff |A \cap B| \geq f(|A|) \end{aligned}$$

As we see (and as we already know),  $Q_f^{\text{rel}}$  is  $\text{MON}\uparrow$ . Furthermore, we know (Proposition 3 in Chapter 4.5) that any type  $\langle 1, 1 \rangle$  quantifier which is  $\text{CONSERV}$ ,  $\text{EXT}$ ,  $\text{ISOM}$ ,

and  $\text{MON}\uparrow$ , is of the form  $Q^{\text{rel}}$  for some ISOM and monotone increasing type  $\langle 1 \rangle$  quantifier  $Q$ . That is, over finite universes it is the relativization of some quantifier  $Q_f$ . Let us summarize this:

### the quantifiers $Q_f^{\text{rel}}$

For any finite  $M$  and any  $A, B \subseteq M$ ,

$$(5.16) \quad (Q_f^{\text{rel}})_M(A, B) \iff |A \cap B| \geq f(|A|)$$

$Q_f^{\text{rel}}$  is  $\text{MON}\uparrow$ . Conversely, any CONSERV, EXT, and ISOM type  $\langle 1, 1 \rangle$  quantifier which is  $\text{MON}\uparrow$  is of the form  $Q_f^{\text{rel}}$  (over finite universes) for some monotonicity function  $f$ .

For example,

$$\text{some}(A, B) \iff |A \cap B| \geq f_1(|A|) = 1$$

$$\text{all}(A, B) \iff A \subseteq B \iff |A \cap B| \geq f_2(|A|) = |A|$$

In the last case, note that since the sets are *finite*,  $|A \cap B| = |A|$  means that  $A \cap B = A$ ; i.e. that  $A \subseteq B$ .

All (right) monotone increasing quantifiers we are dealing with here are of the form  $Q_f$  or  $Q_f^{\text{rel}}$  (under ISOM and FIN). But then, we can further isolate various *kinds* of monotonicity by laying down additional *requirements* on  $f$ . For example, we can demand that  $f$  itself is increasing:  $n \leq m \Rightarrow f(n) \leq f(m)$ . This requirement is related to the property of *smoothness* (see Proposition 10 in section 5.6), a property that turns out to be very significant for natural languages.

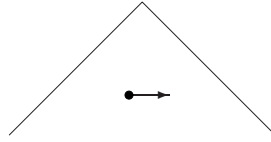
This is one aspect of the fine-grained analysis of monotonicity that we have been promising. To introduce another basic idea, we need to see how monotonicity manifests itself in the number triangle.

### 5.4.2 Monotonicity and the number triangle

If  $Q_M(A)$  and you enlarge  $A$  but stay in the same universe,  $|A|$  increases and  $|M - A|$  decreases, but their sum ( $|M|$ ) is the same. So in the number triangle, you are *moving to the right at level*  $|M|$ . Similarly if  $Q$  is of type  $\langle 1, 1 \rangle$ ,  $Q_M(A, B)$ , and  $B \subseteq B' \subseteq M$ :  $|A \cap B| \leq |A \cap B'|$ ,  $|A - B| \geq |A - B'|$ , but  $|A - B'| + |A \cap B'| = |A|$ . Thus, the *number-theoretic* property of being (right) monotone increasing can be expressed as follows:

$$Q(k, m) \ \& \ k \neq 0 \ \Rightarrow \ Q(k - 1, m + 1)$$

This says that the *next* point to the right of  $(k, m)$  at the same level belongs to  $Q$  if  $(k, m)$  does; by induction, all points to the right of  $(k, m)$  at the same level do. We indicate this by the following simple diagram:



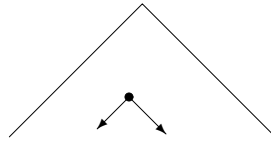
Now consider persistence ( $\uparrow$ MON). We claim that its number-theoretic version is

$$(5.17) \quad Q(k, m) \Rightarrow \bar{Q}(k + 1, m) \text{ and } Q(k, m + 1)$$

or, by induction,

$$(5.18) \quad Q(k, m) \ \& \ k \leq k' \ \& \ m \leq m' \Rightarrow Q(k', m')$$

which we picture as follows:

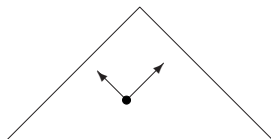


This should be taken as the *conjunction* of the conditions associated with each arrow: if  $(k, m)$  is in  $Q$ , so is every point encountered by moving downward, parallel to the right edge (keeping  $k$  fixed), as well as every point reached by moving downward, parallel to the left edge (keeping  $m$  fixed). But since this applies to all points in  $Q$ , the condition simply means that every point in the downward triangle spanned by  $(k, m)$ —that is, every point  $(k', m')$  with  $k \leq k'$  and  $m \leq m'$ —is in  $Q$ .

The number triangle presupposes ISOM and FIN,<sup>10</sup> so our claim is that, *under these conditions*, a CONSERV and EXT type  $\langle 1, 1 \rangle$  quantifier (the relativization of a type  $\langle 1 \rangle$  quantifier) is persistent if and only if (5.17) or, equivalently, (5.18) holds. This is practically immediate, but here is the verification. Suppose, first, that (5.18) holds, that  $Q_M(A, B)$ , and that  $A \subseteq A' \subseteq M$ . Then  $Q(|A - B|, |A \cap B|)$ ,  $|A - B| \leq |A' - B|$ , and  $|A \cap B| \leq |A' \cap B|$ , so  $Q(|A' - B|, |A' \cap B|)$ ; i.e.  $Q_M(A', B)$ . In the other direction, suppose  $Q$  is persistent, that  $Q(k, m)$  holds, and that  $k \leq k'$  and  $m \leq m'$ . Choose  $M$  and  $A, B \subseteq M$  such that  $|A - B| = k$  and  $|A \cap B| = m$ . Thus,  $Q_M(A, B)$  holds. Also, choose  $A'$  with  $A \subseteq A' \subseteq M$  such that  $|A' - B| = k'$  and  $|A' \cap B| = m'$ . Clearly, this can always be done (provided  $M$  is big enough; if it isn't, EXT insures that  $M$  can be enlarged to a big enough  $M'$ ). By persistence,  $Q_M(A', B)$ , so  $Q(k', m')$ .

It is now also clear how to represent anti-persistence ( $\downarrow$ MON) in the number triangle:

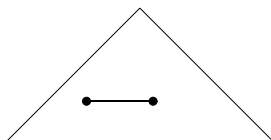
<sup>10</sup> Why FIN? Can't we extend the number triangle by adding infinite cardinals? The problem is that then it is no longer a 'triangle' in any useful sense. Consider the point  $(\aleph_0, 3)$ . Its 'level' is  $\aleph_0 + 3 = \aleph_0$ , but so is its coordinate on the left 'edge'. So the geometric visualization in terms of rows, columns, and levels disappears, and thereby most of the practical use of the number triangle, in particular for representing monotonicity.



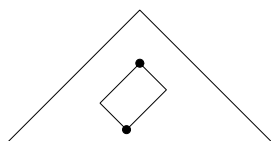
As a first application of number triangle techniques, we are now ready for the following simple proof.

*Proof of Proposition 3, section 5.2:* Since  $Q$  is ISOM, CONSERV, and EXT, and we only consider finite universes, it is representable in the number triangle. Consider level 1. By VAR, this must be either  $+ -$  or  $- +$ . In the first case, if  $Q$  is  $\downarrow$ MON, the point  $(0, 0)$  must have a  $+$  too. Also by  $\downarrow$ MON, no point outside the left edge can have a  $+$  (since that would give  $(0, 1)$  a  $+$ ). But some point at each level must have a  $+$ , by VAR. This means that  $Q$  consists of exactly the left edge, i.e. it is *no*. If, on the other hand,  $Q$  is  $\uparrow$ MON, it follows that the whole downward triangle with  $(1, 0)$  at the top consists of  $+$ s. And, using  $\uparrow$ MON and VAR again, we see that  $Q$  must be precisely this triangle, i.e. *not all*. The second case, when level 1 looks like  $- +$ , is treated analogously, and allows only *all* in the  $\downarrow$ MON case and *some* in the  $\uparrow$ MON case.  $\square$

Let us note too that *continuity* (end of section 5.1) is readily representable in the number triangle. We saw that being (right or left) continuous is the same as being the conjunction of a (right or left) increasing and a decreasing quantifier. In the number triangle, right continuity means that if two points at the same level are in  $Q$ , so are all points in between:



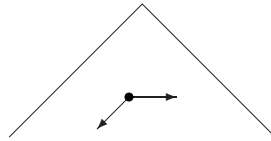
Left continuity, on the other hand, means that for any two points in  $Q$ , all the points in the rectangle spanned by those two points are also in  $Q$ :



## 5.5 SIX BASIC FORMS OF MONOTONICITY

Consider again the diagrammatic picture of (5.18). We see that this is really the conjunction of two more specific monotonicity conditions, one for each arrow. In fact, there are *six directions* in the number triangle relevant to monotonicity: *two for right monotonicity* and *four for left monotonicity*.

The right monotone directions are along the levels. Thinking of the number triangle as a map with  $(0, 0)$  as its northernmost point, these directions are *east* (increasing) and *West* (decreasing). Of the left monotone directions, two are parallel to the right edge (the rows)—*southeast* (SE) and *northwest* (NW); and two parallel to the left edge (the columns)—*southwest* (SW) and *northeast* (NE). Each of these directions corresponds to a specific monotonicity property. And we have just seen that the combination of SW and SE amounts to persistence, and that anti-persistence ( $\downarrow\text{MON}$ ) combines NE and NW. But other combinations are possible as well. For example, we can now see directly in the number triangle that the doubly monotone property  $\uparrow\text{MON}\uparrow$  combines *three* directions. But this clearly is more simply representable as follows (the SE arrow is superfluous):



Similarly for the other three double monotonicities.<sup>11</sup>

Of the six basic monotonicities, the two right ones are familiar, but not the four left ones. Let us formulate these four properties explicitly. Below, we first give the name of the property, then its diagrammatic representation, and finally its formulation for arbitrary quantifiers, first in the type  $\langle 1, 1 \rangle$  case and then in the type  $\langle 1 \rangle$  case; the latter formulations do not assume ISOM or FIN. We shall assume, however, that all quantifiers mentioned here satisfy CONSERV.<sup>12</sup>

$\uparrow_{\text{SE}}\text{MON}$

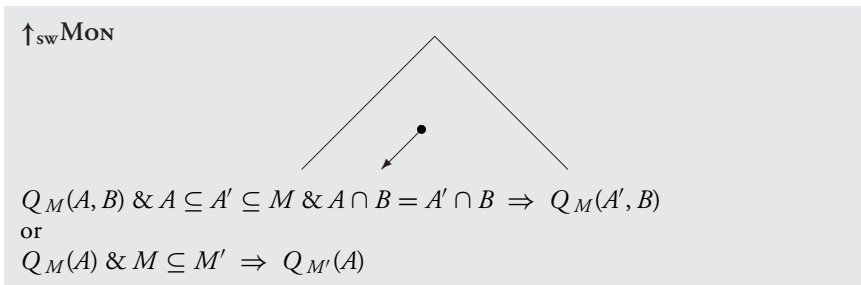
$Q_M(A, B) \ \& \ A \subseteq A' \subseteq M \ \& \ A - B = A' - B \Rightarrow Q_M(A', B)$   
 or  
 $Q_M(A) \ \& \ M \subseteq M' \ \& \ A' \subseteq M' \ \& \ M - A = M' - A' \Rightarrow Q_{M'}(A')$

The chosen name has an up-arrow to the left, since it is a form of upward monotonicity in the left argument; the subscript indicates the direction in the number

<sup>11</sup> Why haven't we defined north and south monotonicity properties? The north property would say that, e.g., if  $(3,2)$  is in  $Q$ , then so are  $(2,1)$  and  $(1,0)$ . This concerns what happens if we decrease each of  $|A - B|$  and  $|A \cap B|$  by one, and hence take away two elements from  $A$ . The property is entailed by  $\downarrow\text{MON}$  (this is clear from the number triangle), but it doesn't seem to have any intrinsic interest.

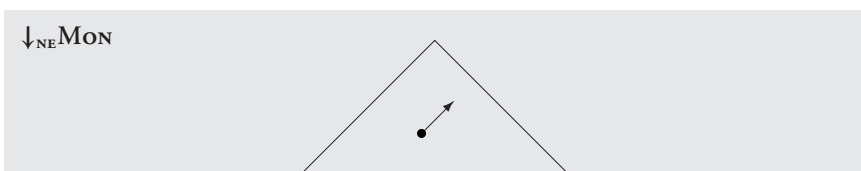
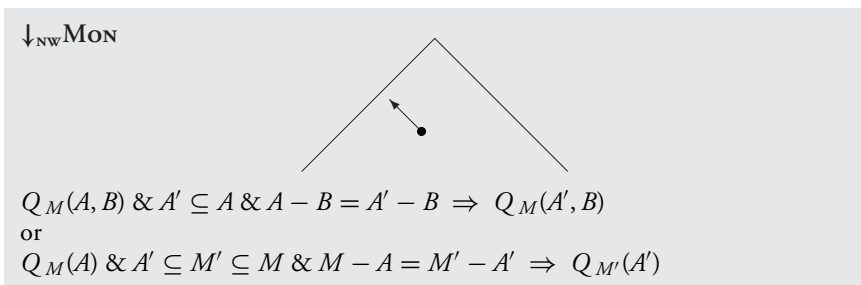
<sup>12</sup> The four left properties were noted in van Benthem 1986 and formulated, under slightly different names than here, in Westerståhl 1989. This book is the first place, to our knowledge, where they are put forward as fundamental components of all kinds of monotonicity related to natural language quantification. About the CONSERV assumption, see n. 13.

triangle. So this property concerns—in the type  $\langle 1, 1 \rangle$  case—what happens when one increases the left argument while keeping the *difference* between the left and the right argument intact. Similarly for the next property, except that there the *intersection* between the two arguments is constant:



It may require some thought to see that the type  $\langle 1 \rangle$  versions do correspond to the type  $\langle 1, 1 \rangle$  versions, in the sense that  $Q$  satisfies the type  $\langle 1 \rangle$  version iff  $Q^{rel}$  satisfies the type  $\langle 1, 1 \rangle$  version.<sup>13</sup> But, when ISOM and FIN hold, it should be clear that both properties are the same as the property depicted in the diagram.

Here are the corresponding two downward properties:



<sup>13</sup> Why isn't the general formulation of, say,  $\uparrow_{se}MON$ , in the type  $\langle 1, 1 \rangle$  case rather

$Q_M(A, B) \ \& \ A \subseteq A' \subseteq M \ \& \ B \subseteq B' \subseteq M \ \& \ A - B = A' - B' \Rightarrow Q_M(A', B')$ ?

That formulation is more general, and independent of CONSERV. Here we do presuppose that CONSERV holds, and then, if  $B$  is as above, it follows that  $A' \cap B = A' \cap B'$ , whence the conclusion  $Q_M(A', B')$  is implied by our simpler condition. The issue of the formulation of basic monotonicity properties in the absence of CONSERV is left open here.



$$Q_M(A, B) \ \& \ A' \subseteq A \ \& \ A \cap B = A' \cap B \ \Rightarrow \ Q_M(A', B)$$

or

$$Q_M(A) \ \& \ A \subseteq M' \subseteq M \ \Rightarrow \ Q_{M'}(A)$$

Note that  $\uparrow_{\text{sw}}\text{MON}$  for the type  $\langle 1 \rangle$  case is the property we called (p) in section 5.2 above; as we noted, it is one half of the EXT property. We can now answer the question left open there, and also see exactly what was behind the observations we made in connection with (p). We showed that if  $Q^{\text{rel}}$  is persistent, then  $Q$  satisfies (p)—and indeed, it is immediate from the number triangle that  $\uparrow\text{MON}$  implies  $\uparrow_{\text{sw}}\text{MON}$ . Further, we proved that if  $Q$  is (p) *and* monotone increasing, then  $Q^{\text{rel}}$  is persistent—this too follows directly from examination of the number triangle; indeed, we see that  $\uparrow_{\text{sw}}\text{MON}$  combined with  $\text{MON}\uparrow$  amounts to  $\uparrow\text{MON}\uparrow$ , which is strictly stronger than persistence.

Thus, in the number triangle, it is *immediate* that upward left monotonicity of  $Q^{\text{rel}}$  corresponds exactly to the combination of  $\uparrow_{\text{sw}}\text{MON}$  and  $\uparrow_{\text{se}}\text{MON}$ , as these are formulated for the type  $\langle 1 \rangle$   $Q$ . This answers our question, *given* that ISOM or FIN hold. However, the answer is valid even without these assumptions, as we now show. This actually requires a small argument:

### Proposition 6

*Let  $Q$  be any type  $\langle 1 \rangle$  quantifier. The following are equivalent:*

- (a)  $Q^{\text{rel}}$  is persistent:  $Q^{\text{rel}}(A, B) \ \& \ A \subseteq A' \ \Rightarrow \ Q^{\text{rel}}(A', B)$
- (b)  $Q$  satisfies  $\uparrow_{\text{sw}}\text{MON}$  and  $\uparrow_{\text{se}}\text{MON}$ , i.e. it has the following two properties:
  - (i)  $Q_M(A) \ \& \ M \subseteq M' \ \Rightarrow \ Q_{M'}(A)$
  - (ii)  $Q_M(A) \ \& \ M \subseteq M' \ \& \ A' \subseteq M' \ \& \ M - A = M' - A' \ \Rightarrow \ Q_{M'}(A')$

*Analogously for the anti-persistent case.*

*Proof.* (a)  $\Rightarrow$  (b): We already showed in section 5.2 that if  $Q^{\text{rel}}$  is persistent, then (p), i.e.  $\uparrow_{\text{sw}}\text{MON}$ , holds of  $Q$ . For  $\uparrow_{\text{se}}\text{MON}$ , suppose  $M \subseteq M'$ ,  $A' \subseteq M'$ , and  $M - A = M' - A'$ . Then

$$\begin{aligned} Q_M(A) &\Rightarrow Q_M^{\text{rel}}(M, A) \\ &\Rightarrow (Q^{\text{rel}})_{\neg M}(M, M - A) \\ &\Rightarrow (Q^{\text{rel}})_{\neg M}(M, M' - A') \quad [\text{assumption}] \\ &\Rightarrow (Q^{\text{rel}})_{\neg M'}(M, M' - A') \quad [\text{since } (Q^{\text{rel}})_{\neg} = (Q_{\neg})^{\text{rel}} \text{ is EXT}] \\ &\Rightarrow Q_{M'}^{\text{rel}}(M, A') \\ &\Rightarrow Q_{M'}^{\text{rel}}(M', A') \quad [\text{by persistence}] \\ &\Rightarrow Q_{M'}(A') \end{aligned}$$

Thus,  $Q$  is  $\uparrow_{\text{se}}\text{MON}$ .

(b)  $\Rightarrow$  (a): Suppose  $Q$  is  $\uparrow_{\text{sw}}\text{MON}$  and  $\uparrow_{\text{se}}\text{MON}$ , and that  $A, B \subseteq M$  and  $A \subseteq A' \subseteq M$ . We extend  $A$  to  $A'$  in two steps. Let  $A'' = A \cup (A' - B)$ . Clearly, we have

- (i)  $A \subseteq A'' \subseteq A'$
- (ii)  $A \cap B = A'' \cap B$
- (iii)  $A' - B = A'' - B$

Then

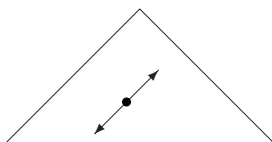
$$\begin{aligned} Q_M^{\text{rel}}(A, B) &\implies Q_A(A \cap B) \\ &\implies Q_{A''}(A \cap B) \quad [\text{by } \uparrow_{\text{sw}}\text{MON and (i)}] \\ &\implies Q_{A''}(A'' \cap B) \quad [\text{by (ii)}] \\ &\implies Q_{A'}(A' \cap B) \quad [\text{by } \uparrow_{\text{se}}\text{MON, (iii), and (i)}] \\ &\implies Q_M^{\text{rel}}(A', B) \end{aligned}$$

This shows that  $Q^{\text{rel}}$  is persistent.  $\square$

We have here a nice illustration of how a result about monotonicity is inspired by the representation in the number triangle—indeed, it would have been hard to even come up with the properties  $\uparrow_{\text{sw}}\text{MON}$  and  $\uparrow_{\text{se}}\text{MON}$  without using that representation—although it actually holds without assuming ISOM or FIN (in fact, without any assumptions at all).

There are many similar cases. Let us first look at what happens if we combine a basic monotonicity property with its opposite along the same axis:

$Q$  is  $\uparrow_{\text{sw}}\text{MON}$  and  $\downarrow_{\text{ne}}\text{MON}$  iff it is symmetric.



$Q$  is  $\uparrow_{\text{se}}\text{MON}$  and  $\downarrow_{\text{nw}}\text{MON}$  iff it is co-symmetric (i.e.,  $Q \neg$  is symmetric).

It is easy to see (Chapter 6.4) that symmetry in the number triangle amounts exactly to the property that if a point is in  $Q$ , so are all points on the axis (column) through that point parallel to the left edge. So the above observations are again immediate from the number triangle. But, again, they hold without FIN or ISOM; only CONSERV is needed.

### Proposition 7

*A CONSERV type  $\langle 1, 1 \rangle$  quantifier is symmetric if and only if it satisfies  $\uparrow_{\text{sw}}\text{MON}$  and  $\downarrow_{\text{ne}}\text{MON}$ . Similarly, it is co-symmetric iff it satisfies  $\uparrow_{\text{se}}\text{MON}$  and  $\downarrow_{\text{nw}}\text{MON}$ .*

*Proof.* One easily verifies (Chapter 6.1) that under CONSERV, symmetry is equivalent to the property that for all  $M$  and all  $A, B \subseteq M$ ,  $Q_M(A, B) \Leftrightarrow Q_M(A \cap B, A \cap B)$ . It

is clear that this property implies both  $\uparrow_{sw}MON$  and  $\downarrow_{ne}MON$ . Conversely, assume that these two left monotonicity conditions hold, and that  $Q_M(A, B)$ . Then  $Q_M(A \cap B, B)$  by  $\downarrow_{ne}MON$ , so  $Q_M(A \cap B, A \cap B)$  by  $CONSERV$ . Similarly,  $Q_M(A \cap B, A \cap B)$  implies  $Q_M(A, B)$  by  $\uparrow_{sw}MON$  and  $CONSERV$ . The case of co-symmetry is similar.  $\square$

It is slightly surprising that symmetry can be seen as a combination of basic monotonicity properties (under  $CONSERV$ ; note, for example, that the Härtig quantifier is symmetric, but fails to satisfy  $\uparrow_{sw}MON$  or  $\downarrow_{ne}MON$ ).

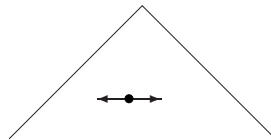
Now suppose we combine the two directions of right monotonicity. Consider quantifiers such that each level consists of either only  $+s$  or only  $-s$ . In other words, they are trivial *on each universe*. Such quantifiers can still say something about the universe, so let us call them *universe quantifiers*. Under  $ISOM$ , they depend only on the *size* of the universe; thus:

- (5.19) An  $ISOM$  type  $\langle 1 \rangle$  quantifier  $Q$  is a universe quantifier iff  $Q_M(A) \Leftrightarrow |M| \in S$  for some class  $S$  of numbers. Similarly for its relativization:  $Q^{rel}(A, B) \Leftrightarrow |A| \in S$ .

Many of these do not seem to be quantirelation denotations. However, consider the English expression *any number of the five*. In the sense of ‘zero or more of the five’ this denotes  $Q_M(A, B) \Leftrightarrow |A| = 5$ . Likewise, *any number of the five or more* denotes  $Q_M(A, B) \Leftrightarrow |A| \geq 5$ . So at least universe quantifiers of this form may be seen as determiner denotations.

In the number triangle, it is immediate that

$Q$  is  $MON\uparrow$  and  $MON\downarrow$  iff it is a universe quantifier.



Next, we note that for each of the six basic monotonicities, the behavior of  $Q$  determines that of the other quantifiers in  $square(Q)$ . We already know this for right monotonicity, so let us state the relevant facts for the four basic left monotonicities.

**Proposition 8**

- (a) *Outer negation reverses direction. That is, it rotates the arrow around the axis perpendicular to it. So if  $Q$  is  $\uparrow_{se}MON$ , then  $\neg Q$  is  $\downarrow_{nw}MON$ , and similarly in the other cases.*
- (b) *Inner negation amounts to rotation around the vertical axis in the number triangle. Thus, it switches between  $\downarrow_{ne}MON$  and  $\downarrow_{nw}MON$ , and between  $\uparrow_{sw}MON$  and  $\uparrow_{se}MON$ .*

- (c) Hence, dual rotates around the horizontal axis (combining the two rotations above), so it switches between  $\downarrow_{\text{NE}}\text{MON}$  and  $\uparrow_{\text{SE}}\text{MON}$ , and between  $\downarrow_{\text{NW}}\text{MON}$  and  $\uparrow_{\text{SW}}\text{MON}$ .

These descriptions use the number triangle, but in fact no extra assumptions are needed, except CONSERV in the type  $\langle 1, 1 \rangle$  case.

*Proof.* (a) is immediate. For (b), suppose  $Q$  is  $\downarrow_{\text{NE}}\text{MON}$ , that  $Q\neg_M(A, B)$ ,  $A' \subseteq A$ , and  $A - B = A' - B$ . Then (by CONSERV)  $Q_M(A, A - B)$ . Since  $A \cap (A - B) = A' \cap (A' - B) = A' \cap (A - B)$ , it follows by  $\downarrow_{\text{NE}}\text{MON}$  that  $Q_M(A', A - B)$ , i.e.  $Q_M(A', A' - B)$ . Thus  $Q\neg_M(A', B)$ , which shows that  $Q\neg$  is  $\downarrow_{\text{NW}}\text{MON}$ . The other cases are similar, and (c) follows from (a) and (b).  $\square$

The second part of Fact 2 (section 2) is a corollary of this. For example, if  $Q$  is persistent ( $\uparrow_{\text{SE}}\text{MON} + \uparrow_{\text{SW}}\text{MON}$ ), both  $\neg Q$  and  $Q^d$  are  $\downarrow_{\text{NW}}\text{MON} + \downarrow_{\text{NE}}\text{MON}$  (anti-persistent), and  $Q\neg$  is  $\uparrow_{\text{SW}}\text{MON} + \uparrow_{\text{SE}}\text{MON}$  (persistent).

Finally, let us consider the issue of testing empirically for these more specialized monotonicities, say,  $\downarrow_{\text{NE}}\text{MON}$ . In principle, one could test when inferences of the form (roughly)

- (5.20) a.  $Q A$  are  $B$   
 b. Every  $A'$  is an  $A$   
 c. Every  $A$  which is  $B$ , is an  $A'$   
 d. Hence:  $Q A'$  are  $B$

are considered valid, where the premises would guarantee that  $A' \subseteq A$  and  $A \cap B = A' \cap B$ . But these inferences are somewhat artificial, and results may be clearer when the required facts are taken for granted by evident empirical circumstances. For example, presumably speakers would normally *not* regard the inference

- (5.21) a. At least one-third of the students smoke.  
 b. Hence: At least one-third of the male students smoke.

as valid, for obvious reasons, but they *would* grant the validity of

- (5.22) a. At least one-third of the students have joined the men's choir.  
 b. Hence: At least one-third of the male students have joined the men's choir.

at least if they take for granted that only men can join the men's choir. If this is granted, then one need only realize that the percentage of chorists among the male students must be at least as high as the percentage of chorists among all the students. That is, one needs to realize this in order to judge the inference valid, and presumably most speakers do realize this.

We would probably get clearer results with a determiner like *at most five* instead. But that would not help, since then we would rather be testing for the stronger property of anti-persistence; note that in this case *both* (5.21) and (5.22)—with *at least one-third of the* replaced by *at most five*—would be judged valid. Thus, the test should use a determiner which (we think) is  $\downarrow_{\text{NE}}\text{MON}$  but *not*  $\downarrow\text{MON}$ .

Now consider also

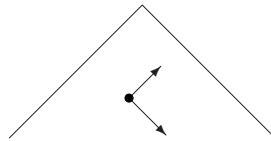
- (5.23) a. Most female students didn't join the heptathlon team.  
 b. *Hence*: Most students didn't join the heptathlon team.

With  $A$  and  $A'$  as before (students and female students, respectively), but with  $C$  as the extension of the property of *not* having joined the heptathlon team—given that this is a women's sport, so only women can join this team—we have  $A' \subseteq A$  and  $A - C = A' - C$ , so this inference should be a good test of  $\uparrow_{SE}MON$ . Prima facie, it might seem to take a bit more calculation to see that, under the circumstances, the conclusion does indeed follow.

Again, we would get clearer results with, say, at least eight, but then we would be testing the stronger property  $\uparrow MON$  (persistence), not  $\uparrow_{SE}MON$ . In conclusion, testing for the more specialized monotonicities, by contrast with testing for ordinary right or left monotonicity, requires some care.<sup>14</sup>

### 5.6 SMOOTH QUANTIFIERS

The choice of monotonicity properties in the last examples,  $\downarrow_{NE}MON$  and  $\uparrow_{SE}MON$ , was no accident.



In fact, *at least one-third of the* and *most* have both of these properties, though neither is left monotone. Moreover, this combination of properties turns out to be ubiquitous.

#### smoothness

A type  $\langle 1, 1 \rangle$  quantifier  $Q$  is *smooth* iff it is  $\downarrow_{NE}MON$  and  $\uparrow_{SE}MON$ ; i.e. if the following two conditions hold:

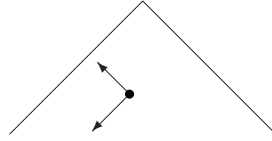
$$(5.24) \quad Q_M(A, B) \ \& \ A' \subseteq A \ \& \ A \cap B = A' \cap B \ \Rightarrow \ Q_M(A', B)$$

$$(5.25) \quad Q_M(A, B) \ \& \ A \subseteq A' \subseteq M \ \& \ A - B = A' - B \ \Rightarrow \ Q_M(A', B)$$

Similarly for the type  $\langle 1 \rangle$  case.

Accordingly, a quantifier is *co-smooth* if its inner negation—or equivalently, in this case, its outer negation—is smooth:

<sup>14</sup> In a preliminary test conducted with 26 first-year philosophy students in Göteborg, around 80 percent claimed that the conclusion does follow (for the corresponding Swedish sentences), with no clear difference between (5.22) and (5.23).



From the number triangle we see directly that all smooth quantifiers are (right) monotone increasing. But again this holds without assuming FIN or ISOM:

**Proposition 9**

*Any CONSERV quantifier satisfying (5.24) and (5.25) is MON $\uparrow$ .*

*Proof.* Suppose  $Q_M(A, B)$  and  $B \subseteq B' \subseteq M$ . Let  $A' = A - (B' - B)$ . Then we have  $A' \subseteq A$  and  $A' \cap B = A \cap B = A' \cap B'$  (draw a picture). By (5.24),  $Q_M(A', B)$ ; so, by CONSERV,  $Q_M(A', B')$ . But also  $A' \subseteq A$  and  $A' - B' = A - B'$ . Thus, by (5.25),  $Q_M(A, B')$ .  $\square$

When FIN and ISOM are satisfied, however, it follows that smooth quantifiers are of the form  $Q_f$  (or  $Q_f^{\text{rel}}$ ). And we can then see that smoothness amounts to a condition that the monotonicity function  $f$  is extra well-behaved. A function  $f$  from natural numbers to natural numbers is called *increasing* if  $n \leq m \Rightarrow f(n) \leq f(m)$ .

**Proposition 10**

(FIN) *If  $Q$  is a CONSERV, EXT, and ISOM type (1, 1) quantifier, the following are equivalent:*

- (a)  $Q$  is smooth.
- (b)  $Q$  is of the form  $Q_f^{\text{rel}}$ , for some monotonicity function  $f$  with the property that for all  $n$ ,

$$f(n) \leq f(n+1) \leq f(n) + 1$$

*In this case,  $f$  too is called smooth.*

- (c)  $Q$  is of the form  $Q_f^{\text{rel}}$ , for some monotonicity function  $f$  such that both  $f$  and  $f^{\text{d}}$  are increasing.

(b) says that when the universe size is increased by 1, the standard for being ‘big enough’ given by  $f$  doesn’t change much: it is either the same or increases by 1. That is, if at level  $n$  the + signs start at  $(n - f(n), f(n))$ , at the next level they start at one of the two immediate successors:  $(n + 1 - f(n), f(n))$  or  $(n - f(n), f(n) + 1)$ . In the number triangle, it is easy to see that this requirement is equivalent to  $\downarrow_{\text{NE}}\text{MON} + \uparrow_{\text{SE}}\text{MON}$ . (c) describes the same requirement in a different way. The details of the proof are left to the reader.

From the above description we also see that the smooth quantifiers are exactly those determined by a ‘branch’ in the number triangle, i.e. a path of + signs starting at some point on one of the edges and going down by always choosing one of the immediate successors of the previous point; the quantifier consists of the ‘branch’ and everything

to the right of it. (We use scare quotes here since the number triangle is not a tree in the mathematical sense: paths like this can separate at some point and join at later points.) This has consequences for the *number* of smooth quantifiers.

First, it is clear that

(5.26) Even under FIN, there are *uncountably* many (in fact  $2^{\aleph_0}$ ) CONSERV, EXT, and ISOM type  $\langle 1, 1 \rangle$  quantifiers.

This is because all such quantifiers are representable as subsets of the number triangle, and there are  $2^{\aleph_0}$  such subsets. Clearly, at most countably many of these can be quantirelation denotations (since there are at most countably many quantirelations in the world's languages).

Further, it is clear that there are as many smooth quantifiers as there are 'branches' of the kind described above: namely, again uncountably many.<sup>15</sup>

(5.27) Under FIN, there are  $2^{\aleph_0}$  CONSERV, EXT, ISOM, and smooth type  $\langle 1, 1 \rangle$  quantifiers.

In contrast, there are only  $\aleph_0$  left monotone quantifiers, since (Proposition 4 in section 5.2) these are all definable by sentences in FO (with two unary predicate symbols), and there are only countably many such FO-sentences.

The last observation emphasizes that there is a huge difference between smoothness and persistence, even though both are characterized in the number triangle by closure under triangle-like areas. For smoothness these triangles extend towards the right edge, whereas for persistence they extend downwards. We remarked earlier that persistence, or left monotonicity, is relatively rare in natural languages, whereas right monotonicity is ubiquitous. In fact, it seems that smoothness is almost as ubiquitous.

The property of smoothness was first isolated—under a different name—by van Benthem, who proved that the quantifiers which in a certain sense are minimally complex to process are exactly the smooth ones and their negations.<sup>16</sup> This is quite interesting, also since it seems that most right monotone increasing determiner denotations are in fact smooth. For example:

(5.28) The proportional quantifiers are smooth.

<sup>15</sup> Let  $T$  be the full *binary tree*, i.e. the (upside-down) tree structure such that every node has exactly two immediate successors, and every node except the top node has exactly one immediate predecessor. So there is one node at level 0, two at level 1, four at level 2, etc., i.e.  $2^n$  at level  $n$ . The infinite branches of  $T$  correspond exactly to infinite sequences of 0s and 1s (at each node write 0 if we take the left successor and 1 if we take the right one), which in turn correspond exactly to the sets of (positive) natural numbers (where  $n \in X$  iff there is a 1 at level  $n + 1$ ), and there are uncountably many of these. But  $T$  can be embedded in the number triangle, letting level  $n$  in  $T$  correspond to what we called level  $2^n$  in the number triangle. It follows that the infinite branches in  $T$  can be mapped in a one-to-one fashion to 'branches' in the number triangle, whence there are at least as many of the latter as there are of the former.

<sup>16</sup> van Benthem 1986 called the property *continuity*. Westerståhl 1989 surveys various uses of this term, and uses the name SUPER CONT for the property in question here. The term "smooth", which we prefer, was introduced in Väänänen and Westerståhl 2002.

*Proof.* Suppose  $Q(A, B) \Leftrightarrow |A \cap B| \geq p/q \cdot |A|$ , where  $0 < p/q < 1$  (and the universe is finite). As we remarked above (for *at least one-third of the*),  $\downarrow_{NE}MON$  is obvious, but let us check anyway. Suppose  $Q(A, B)$ ,  $A' \subseteq A$ , and  $A \cap B = A' \cap B$ . It follows that  $|A' - B| \leq |A - B|$ . Then,

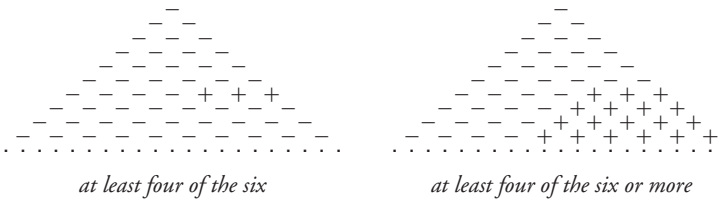
$$\begin{aligned} |A' \cap B| &= |A \cap B| \geq p/q \cdot |A| \\ &= p/q \cdot (|A \cap B| + |A - B|) \\ &\geq p/q \cdot (|A' \cap B| + |A' - B|) \\ &= p/q \cdot |A'| \end{aligned}$$

so  $Q(A', B)$  holds.

$\uparrow_{SE}MON$  is only slightly less obvious. Suppose  $Q(A, B)$ ,  $A \subseteq A'$ , and  $A - B = A' - B$ . So the elements in  $A' - A$ —say there are  $n$  of them—must all be in  $B$ . Then

$$\begin{aligned} |A' \cap B| &= |A \cap B| + n \geq p/q \cdot |A| + n \\ &\geq p/q \cdot (|A| + n) \quad [\text{since } p/q < 1] \\ &= p/q \cdot |A'| \end{aligned} \quad \square$$

One can verify that many other common right monotone increasing determiner denotations are in fact smooth. Väänänen and Westerståhl 2002 conjectured that *all* of them are. That guess was a little too hasty. Some  $MON\uparrow$  determiners make a specific claim about the cardinality of the restriction, and these are not smooth. Typical examples are *at least m of the k* (in particular, for  $m = k$ : *the k*) and *at least m of the k or more* when  $m \leq k$ .



From the number triangle we see that *at least m of the k* is  $MON\uparrow$ , but neither  $\downarrow_{NE}MON$  nor  $\uparrow_{SE}MON$ , and that *at least m of the k or more* is  $\uparrow MON\uparrow$ , but not  $\downarrow_{NE}MON$ .<sup>17</sup>

Another relevant case is possessive determiners, which have interesting monotonicity behavior. We will find in Chapter 7.13 that the most common right monotone

<sup>17</sup> Characteristic of these quantifiers is that they impose a constraint on the size of the restriction argument. One might consider a modified form of the universal suggested in Väänänen and Westerståhl 2002, saying that all right increasing determiner denotations that do not impose such a constraint are smooth. However, in Ch. 7 we will find another kind of counterexample: viz. possessive quantifiers like *at least two of most students*; see Proposition 8 in Ch. 7.13. But note that these quantifiers are not ISOM, so the revised conjecture might still be true in the number triangle. That is, one could propose the following universal:



readings of possessive determiners, such as *John's*, at least three students', and *most of Mary's* are usually smooth, or 'almost' smooth.

An important lesson from this is that mere right monotonicity does not adequately characterize the inference behavior of lots of quantifiers denoted by natural language determiners that are monotone increasing. Each variety of monotonicity amounts to the soundness of certain inferences, but the right monotone quantirelation denotations often support additional stronger, left monotone schemes.

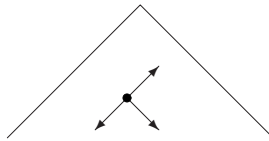
As we saw with *most*, for example, the true description of its monotonicity behavior is that it supports schemes corresponding to smoothness, such as (5.22) and (5.23), discussed in the previous section. These two imply right monotonicity, which may be easier to test, but is strictly weaker. And similarly for many other  $\text{MON}\uparrow$  determiners.<sup>18</sup>

Note next that smoothness *combined* with left monotonicity is a very strong requirement:

**Fact 11**

(FIN) Under CONSERV, EXT, and ISOM, the only smooth and left monotone quantifiers are at least  $k$  and its dual,  $k \geq 0$ .

*Proof.* This is obvious from the number triangle. For example, smoothness plus  $\uparrow\text{MON}$  amounts to



Thus, if  $k$  is the smallest second coordinate of a point in  $Q$ ,  $Q$  has to be *at least*  $k$ . Similarly in the smooth and  $\downarrow\text{MON}$  case.  $\square$

Observe that ISOM is crucial for this result. For example, possessive determiners can be CONSERV, EXT, left monotone, and smooth (e.g., *every student's*; see Chapter 7.13), but not ISOM, which is why they do not constitute counterexamples to the above fact.

(\*) ISOM and  $\text{MON}\uparrow$  determiner denotations that do not put a cardinality constraint on the restriction argument are smooth.

This might be empirically true, and constitute a formulation of the observation, which is still rather striking, that most of the  $\text{MON}\uparrow$  natural language quantifiers that one comes across indeed have the stronger property of smoothness.

<sup>18</sup> We may note that smoothness among  $\text{MON}\uparrow$  quantifiers is not limited to natural language quantifiers. For example, the quantifier

$$\text{Sqrt}_M(A, B) \iff |A \cap B| > \sqrt{|A|}$$

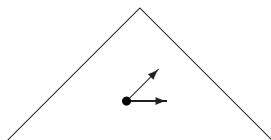
is presumably not a quantirelation denotation, but it is CONSERV, EXT, ISOM, and, by an argument similar to the one for (5.28), smooth.

In conclusion, smoothness is quite a significant property of natural language quantifiers. It was discovered while thinking about quantifiers in the number triangle, but it applies across the board, not just when ISOM and FIN are assumed. Once again, the number triangle functions as an intuition pump, leading to insights also about quantifiers not representable there.

We have now looked at almost all possible combinations of the six basic monotonicity properties, and we have seen how they build up various more familiar monotonicity properties and explain certain connections between them. For example, it is immediate from the representations in the number triangle that

$$\begin{aligned} \text{smoothness} &\implies \text{MON}\uparrow \\ \text{MON}\uparrow + \uparrow_{\text{sw}}\text{MON} &= \uparrow\text{MON}\uparrow \end{aligned}$$

Furthermore, these connections hold in general, without assuming FIN, ISOM, etc. The only combinations we did not mention so far are those where right monotonicity combines with one of its ‘closest’ weak left properties, such as:



We have not found these particular combinations to have a special interest. Quantifiers that have them usually have stronger monotonicity properties as well. It is of course possible to define quantifiers that have exactly the monotonicity property depicted above and no stronger one; consider  $Q(A, B) \Leftrightarrow |A| \leq n \ \& \ |A \cap B| \leq k$  for  $k < n$  (? at least  $k$  of the  $n$  or fewer), but these seem somewhat unnatural.

The picture of how natural language relates to monotonicity that has emerged in this chapter is somewhat different from the one that first comes to mind, and that has been predominant in the literature. The basic monotonicity properties we have isolated for quantifiers are richer than ordinary monotonicity for functions in the mathematical sense (see the introduction to this chapter). More precisely, they coincide for right monotonicity, but the right monotonicities, although primitive, are not strong enough to describe the behavior of  $\text{MON}\uparrow$  or  $\text{MON}\downarrow$  determiner denotations. At least smoothness is also needed, and this is a combination of specialized *left* monotonicity properties.

In a sense, therefore, it is the six basic monotonicity properties that constitute the heart of the monotonicity behavior of quantifiers in natural languages. Ordinary increasing and decreasing monotonicity in the left argument, persistence and anti-persistence, are combinations of the basic left monotonicities, but so is smoothness. Moreover, apparently unrelated but important properties such as symmetry turn out also to be combinations of these four basic monotonicities.

Recall that Aristotle discovered logic partly by studying monotonicity. He dealt with the ordinary left and right properties (for his four quantifiers). The four basic left

monotonicity properties correspond to slightly more complicated inference schemes, not exactly syllogisms (e.g. you need three premises, not two), but almost. So, the picture presented here is readily seen as an elaboration of Aristotle's classical account.

In the remaining sections of this chapter, we will see examples of how monotonicity and related concepts, and the tools and results developed around them, can be applied to the study of some concrete linguistic phenomena.

### 5.7 LINGUISTIC APPLICATION 1: A PECULIAR INFERENCE SCHEME

Keenan 2005 studies four valid inference patterns involving natural language quantifiers, among them the following:

- (5.29) a. More than two-thirds of the students passed the exam.  
 b. At least one-third of the students are athletes.  
 c. *Hence*: At least one student who is an athlete passed the exam.

The pattern is an instance of

- (5.30) a. More than  $p/q$  of the As are Bs.  
 b. At least  $1 - p/q$  of the As are Cs.  
 c. *Hence*: Some A is both a B and a C.

This pattern is quite interesting, since it features the less studied proportional quantifiers in the premises but a familiar Aristotelian quantifier in the conclusion. It therefore exemplifies a form of 'natural logic' that Keenan suggests merits further study (though he does not undertake a further study of this particular scheme in the paper).

Let us see if we can apply our logical tools. First, observe that the pattern amounts to an instance of an even more general scheme:

$$(5.31) \quad Q(A, B) \ \& \ Q^d(A, C) \ \Rightarrow \ \text{some}(A, B \cap C)$$

For it is readily seen that *more than  $p/q$  of the* is the dual of *at least  $1 - p/q$  of the* (on finite universes). So for which quantifiers is (5.31) valid? Interestingly, we can show that it holds *exactly* for the  $\text{MON}\uparrow$  quantifiers. In fact, the left-to-right direction of the next proposition was proved already in Barwise and Cooper 1981 (Proposition C10 in Appendix C). The argument in the other direction is quite easy as well.

**Proposition 12**

*A CONSERV type  $\langle 1, 1 \rangle$  quantifier is  $\text{MON}\uparrow$  if and only if it satisfies (5.31).*

*Proof.* Suppose first that  $Q$  satisfies (5.31), and that  $Q_M(A, B)$  and  $B \subseteq B' \subseteq M$ . We must show  $Q_M(A, B')$ . Suppose instead  $\neg Q_M(A, B')$ . Then,  $Q_M^d(A, M - B')$ . By (5.31),  $A \cap B \cap (M - B') \neq \emptyset$ . In particular,  $B - B' \neq \emptyset$ , but this contradicts the assumption that  $B \subseteq B'$ . Thus,  $Q_M(A, B')$  holds.

In the other direction, suppose  $Q$  is  $\text{MON}\uparrow$ , that  $Q_M(A, B)$  and  $Q_M^d(A, C)$  hold, but that  $B \cap C = \emptyset$ . Then  $B \subseteq M - C$ , so by  $\text{MON}\uparrow$ ,  $Q_M(A, M - C)$ . Thus,

$Q \neg_M(A, C)$ , but this contradicts  $Q_M^d(A, C)$ . Hence,  $B \cap C \neq \emptyset$ . And by CONSERV, we may assume that  $B, C \subseteq A$ , so in fact  $A \cap B \cap C \neq \emptyset$ .  $\square$

Thus, it turns out that the scheme really has nothing to do with proportionality, but instead *characterizes* right upward monotonicity. Note that no properties except conservativity are assumed to hold. In other words, under CONSERV the scheme is equivalent to

$$(5.32) \quad Q(A, B) \ \& \ \text{every}(B, C) \Rightarrow Q(A, C)$$

This is perhaps surprising, and surely not evident, even though the proof turned out to be short. It is a nice illustration of how logical tools and results sometimes provide answers to linguistically motivated questions.

## 5.8 LINGUISTIC APPLICATION 2: LAA QUANTIFIERS

Recall from Chapter 4.7 Keenan and Moss's discussion of quantifiers like those in (5.33).

- (5.33) a. Every student and teacher was there.  
 b. The eighty-six men and women started singing.  
 c. Some student's cats and dogs were locked in the apartment.

We address here a question connected with a couple of the ways they provided for interpreting noun phrases of the form [Det N and N] (or  $Q^{A \text{ and } B}$ ).

First note that *and* can be interpreted as intersection in such cases unless the nouns are incompatible (i.e.  $A \cap B$  is necessarily empty). For instance,

- (5.34) This was the time for *each wife and mother* to bring her finest and greatest delicacies. ([www.rusty1.com/rsbc.history1.htm](http://www.rusty1.com/rsbc.history1.htm))

speaks of some collection of women, each of whom is a wife and mother. The intersective interpretation seems to be available also for sentence (5.33a), if slightly forced in this case. The corresponding interpretations of (5.33b, c) are necessarily false, and thus unavailable.

Keenan and Moss (1984) assign a distributive interpretation (5.36) to sentence (5.33a) by 'lifting' the type  $\langle 1, 1 \rangle$  quantifier  $Q$  expressed by the initial determiner to a type  $\langle 1, 1, 1 \rangle$  quantifier  $Q_{\wedge, \text{and}, 2}$  which they define via the following schema.

$$(5.35) \quad (Q_{\wedge, \text{and}, k})_M(A_1, \dots, A_k, B) \Leftrightarrow Q_M(A_1, B) \ \& \ \dots \ \& \ Q_M(A_k, B)$$

$$(5.36) \quad \text{student} \cup \text{teacher} \subseteq \text{was there}$$

The direct semantic interpretation of sentence (5.33a) via (5.35) is (5.37), which is equivalent to (5.36).

$$(5.37) \quad \text{every}(\text{student}, \text{was there}) \wedge \text{every}(\text{teacher}, \text{was there})$$

The clearly intended interpretation of (5.38) is given by this lift.

- (5.38) Employee Teresa Irwin informed the officer that while she was stocking the cooler, she heard someone enter the store. As she came out of the cooler, she spotted *two men and women* standing at the counter. The two males attempted to make a purchase and while the clerk was distracted, the two female suspects stole fifteen packs of cigarettes. ([www.bcstandard.com/News/2000/0830/Sheriff.html](http://www.bcstandard.com/News/2000/0830/Sheriff.html))

Keenan and Moss (1984) offer a different interpretation for (5.33b): approximately, (5.40), via the following lift.

- (5.39)

$$(Q_{\vee, \text{and}, k})_M(A_1, \dots, A_k, B) \Leftrightarrow Q_M(A_1 \cup \dots \cup A_k, B) \\ \& A_1 \neq \emptyset \dots \& A_k \neq \emptyset$$

- (5.40)  $|man \cup woman| = 86 \ \& \ man \cup woman \subseteq start \ singing$

As we have explained several times, we believe the requirement of non-emptiness on the sets *man* and *woman* (i.e.  $A_1$  and  $A_2$ ) is pragmatic and does not really belong in the truth conditions of this union interpretation. Removing the requirement yields a slightly simpler lift:

- (5.41)  $(Q_{\vee, \text{and}, k})_M(A_1, \dots, A_k, B) \Leftrightarrow Q_M(A_1 \cup \dots \cup A_k, B)$

The clearly intended interpretation of (5.42) is given by this lift.

- (5.42) *Five men and women* from four states have been elected to serve on the University of Iowa Foundation Board of Directors. At its October meeting, the Foundation's Board of Directors elected ... [list of five names]. ([www.ui-foundation.org/news/1999/dec05.shtml](http://www.ui-foundation.org/news/1999/dec05.shtml))

Although the lifts (5.35) and (5.41) assign the same meaning to (5.33a) as well as to (5.34), Dalrymple and Peters (personal communication) noted that both the union and the distributive interpretations are needed, in addition to the intersective interpretation, because sentences can actually be three-ways ambiguous. For instance,

- (5.43) Three linguists and logicians were invited to speak  
can mean any of the following non-equivalent propositions:

- (5.44) a. *three*( $linguist \cap logician$ , *invited to speak*)  
b. *three*(*linguist*, *invited to speak*)  $\&$  *three*(*logician*, *invited to speak*)  
c. *three*( $linguist \cup logician$ , *invited to speak*)

Of course, only two distinct meanings are available for

- (5.45) No linguists and logicians were invited to speak.

because the quantifier *no* makes the distributive and union interpretations equivalent, as the quantifier *every* does for (5.33a) and (5.34). Note that *no* prefers *or* to *and*, underlining that we are talking about the union:

- (5.46) a. No linguists or logicians were invited to speak.  
 b. No linguists were invited to speak and no logicians were invited to speak.

An obvious question is under what circumstances does the equivalence

$$(LAA) \quad Q_M(A, C) \ \& \ Q_M(B, C) \iff Q_M(A \cup B, C)$$

hold. This property is sometimes called *left anti-additivity*, and was introduced in the linguistic literature in another context, which we will discuss in the next section. For the moment, we simply ask: Which quantifiers are LAA?

One readily checks that *every* and *no* qualify, as was illustrated above. Why do we find these examples but, it appears, no others? *Are* there any others? With the help of number triangle techniques, we can show that the answer is (essentially): No.

Indeed, LAA is a very strong property, and closely related to left monotonicity. First, it is practically immediate that

$$(5.47) \quad \text{The right-to-left direction of LAA is equivalent to } \downarrow\text{MON.}$$

Furthermore, the left-to-right direction almost amounts to  $\uparrow\text{MON}$ , but not quite. We have the following facts:

$$(5.48) \quad (\text{FIN}) \text{ If } Q \text{ satisfies CONSERV, EXT, ISOM, and the left-to-right direction of LAA, then, for all } M \text{ and all } A, A', B \subseteq M:$$

$$(5.49) \quad Q_M(A, B) \ \& \ A \subseteq A' \ \& \ A \cap B = A' \cap B \ \& \ A - B \neq \emptyset \Rightarrow Q_M(A', B)$$

or, with the numerical representation,

$$Q(k, m) \ \& \ k \neq 0 \Rightarrow Q(k + 1, m)$$

(Thus, this is  $\uparrow_{\text{sw}}\text{MON}$ , with the extra condition that  $k \neq 0$ .)

$$(5.50) \quad Q_M(A, B) \ \& \ A \subseteq A' \ \& \ A - B = A' - B \ \& \ A \cap B \neq \emptyset \Rightarrow Q_M(A', B)$$

or, with the numerical representation,

$$Q(k, m) \ \& \ m \neq 0 \Rightarrow Q(k, m + 1)$$

( $\uparrow_{\text{se}}\text{MON}$ , with the condition  $m \neq 0$ .)

$$(5.51) \quad Q_M(A, B) \ \& \ A \subseteq A' \ \& \ A - B \neq \emptyset \ \& \ A \cap B \neq \emptyset \Rightarrow Q_M(A', B)$$

or, with the numerical representation,

$$Q(k, m) \ \& \ k \neq 0 \ \& \ m \neq 0 \Rightarrow Q(k, m + 1) \ \& \ Q(k + 1, m)$$

( $\uparrow\text{MON}$ , with the condition  $k, m \neq 0$ .)

*Proof.* Let us focus on the numerical versions. Suppose  $Q(k, m)$  and  $k \neq 0$ . Let  $a_1, \dots, a_k, b_1, \dots, b_m, c$  be distinct objects. By our assumptions, it follows that

$$Q(\{a_1, \dots, a_k, b_1, \dots, b_m\}, \{b_1, \dots, b_m\})$$

and (since  $k > 0$ )

$$Q(\{a_1, \dots, a_{k-1}, c, b_1, \dots, b_m\}, \{b_1, \dots, b_m\})$$

By the left-to-right direction of LAA,

$$Q(\{a_1, \dots, a_k, c, b_1, \dots, b_m\}, \{b_1, \dots, b_m\})$$

which means that  $Q(k + 1, m)$ . This proves (5.49). The proof of (5.51) is similar, and (5.51) follows from (5.49) and (5.51).  $\square$

Using these facts, we can now prove the desired result.

**Proposition 13**

(FIN) *The only non-trivial CONSERV, EXT, and ISOM quantifiers satisfying the condition LAA are every, no, and the quantifier  $Q_M(A, B) \Leftrightarrow A = \emptyset$  ('every and no').*

*Proof.* Consider the number triangle. By non-triviality, there is a + somewhere, so by  $\downarrow$ MON, (0,0) has a +. If this is the only +, we have the quantifier mentioned above. If not, the following holds.

- One of (1,0) and (0,1) must have a +.  
[Follows by  $\downarrow$ MON and the existence of a + besides the one on (0,0).]
- No point off the edges can have a +.  
[If it did, the whole downward triangle spanned by that point would have +s, by (5.51), and then the whole number triangle would have +s, by  $\downarrow$ MON.]
- (1,0) and (0,1) cannot both have a +.  
[Otherwise, we would have  $Q(\{a\}, \{b\})$  and  $Q(\{b\}, \{b\})$  with  $a \neq b$ , and so by the left-to-right direction of LAA,  $Q(\{a, b\}, \{b\})$ . But this means that (1,1) has a +, which is forbidden by the previous claim.]

Now if (1,0) has a +, so does the whole left edge, by our assumption and (5.49). No point other than (0,0) on the right edge can have a +, since otherwise (0,1) would have a + by  $\downarrow$ MON. Since no point off the edges can have a +, it follows that  $Q$  is *no*. Similarly, if (0,1) has a +,  $Q$  is *every*.  $\square$

It seems safe to predict that the only ISOM determiner denotations that we will ever encounter satisfying LAA are *every* and *no*. Thus we expect to see three-way ambiguities (or two-way, if the intersective interpretation is excluded by incompatibility of the conjoined nouns) for most other determiners.<sup>19</sup> That is indeed pretty much what we find, though the ambiguity is subtler and less noticeable for determiners that do not involve precise counts, such as in (5.52).

(5.52) *Many banks and lagoons are at very similar depths.*

<sup>19</sup> It seems unlikely that the third quantifier mentioned in the proposition is a determiner denotation. Note, however, that this is another indication that the principle (BCI<sub>Det</sub>) (Ch. 4.1) that determiner denotations are closed under Boolean operations is a bit too strong, since that quantifier is simply the conjunction of *every* and *no*. Another example is the disjunction of *every* and *no*, which consists of the two edges of the number triangle. This quantifier is  $\downarrow$ MON and satisfies (5.49)–(5.51) but not the left-to-right direction of LAA, as can be seen with sets  $A, B, C$  such that  $B \cap C = \emptyset$  and  $\emptyset \neq A \subseteq C$ .

### 5.9 LINGUISTIC APPLICATION 3: POLARITY-SENSITIVE ITEMS IN NATURAL LANGUAGES

Now consider another natural language phenomenon, *polarity sensitivity*, which some linguists argue involves monotonicity in a central way.<sup>20</sup> Natural languages contain expressions that only occur in a negative context, or only have a particular sense in negative contexts, for example, *ever*, *any*, *yet*, and numerous idioms such as *lift a finger*, *budge an inch*, *give a damn*.

- (5.53) a. Mary hasn't ever been to Disneyland.  
b. \*Mary has ever been to Disneyland.
- (5.54) a. John doesn't own any hats.  
b. \*John owns any hats.
- (5.55) a. John will drink any wine.  
b. John won't drink any wine. (ambiguous: John is discriminating (the negation of (5.55a)), or John is a teetotaler)
- (5.56) a. John hasn't left yet.  
b. \*John has left yet.
- (5.57) a. Bill didn't lift a finger to help.  
b. \*Bill lifted a finger to help. (doesn't have the idiomatic sense)
- (5.58) a. Sue won't budge an inch on that question.  
b. \*Sue will budge an inch on that question.
- (5.59) a. Tom didn't give a damn.  
b. \*Tom gave a damn.

A word like *ever*, *any*, *yet*, and an idiom like *lift a finger*, *budge an inch*, *in years*, or *give a damn*, is called a *negative polarity item* (NPI). Each of these NPIs has a meaning that makes the negative sentence containing it maximally strong—compare *ever* with *frequently*, *any* with *many*, *yet* with *often*, *lift a finger* with *do a lot*, *budge an inch* with *yield significantly*, *give a damn* with *care much* in the sentences above.

Languages also have *positive polarity items* (PPIs), which occur only in positive contexts, or have a particular interpretation only in positive contexts. For example, the following facts show that the phrases *would rather*, *already*, *someone*, and *something* are PPIs.

- (5.60) a. W. C. Fields would rather be in Philadelphia.  
b. \*W. C. Fields wouldn't rather be in Philadelphia.<sup>21</sup>

<sup>20</sup> This section has benefited greatly from the survey in Ladusaw 1996.

<sup>21</sup> Of course,

(i) W. C. Fields would rather not be in Philadelphia.

is a perfectly acceptable sentence. It is not, however, the negation of (5.60a); and in (i) *would rather* is outside the scope of *not*.



- (5.61) a. John has already taken out the trash.  
 b. \*John hasn't already taken out the trash.<sup>22</sup>
- (5.62) a. Mary noticed someone/something out of the ordinary.  
 b. Mary didn't notice someone/something out of the ordinary. (cannot have the sense of (5.63))

Although (5.62b) is a well-formed English sentence, it can't have the indicated meaning. The difference between (5.62b) and (5.63)

(5.63) Mary didn't notice anyone/anything out of the ordinary.

can be apprehended by imagining that a hearer asks

(5.64) Who/What didn't Mary notice?

immediately afterward. This question would be perfectly in order as a follow-up to assertion (5.62b), but the question would be distinctly strange after (5.63).

For linguists, a primary concern is to characterize exactly the range of contexts in which NPIs and PPIs may occur with the particular sense at issue, and to explain if possible why just these contexts support the polarity-sensitive items.

The standard view about polarity-sensitive items among linguists goes roughly as follows. There are such things as NPIs and PPIs. There are also such things as *negative positions* (positions that are in a negative context). NPIs occur only in negative positions; if an NPI is in a negative position, no other aspect of its context can make its occurrence there unacceptable. A PPI does *not occur* in negative positions *except* if certain other aspects of its context 'rescue' it.<sup>23</sup> It is clear from this sketch that an analysis of polarity-sensitive items needs to spell out four things clearly:

- what items are NPIs,
- what items are PPIs,
- what positions are negative, and
- what can 'rescue' a PPI in a negative position, allowing the PPI to occur there as it otherwise could not.

### 5.9.1 What are NPIs and PPIs sensitive to?

The reason why negative and positive polarity items are so called is that negation is considered the prototypical conditioning factor controlling the distribution of NPIs

<sup>22</sup> Actually this clause, as well as (5.60b), can occur as part of well-formed sentences such as the following:

- (i) At most three people thought that W. C. Fields wouldn't rather be in Philadelphia.  
 (ii) It is surprising that John hasn't already taken out the trash.

Sentences (5.60b) and (5.61b) are also acceptable as direct denials of the corresponding positive assertions, as is (5.62b).

<sup>23</sup> We saw examples of such 'rescue' in n. 22.

and PPIs in sentences. It has long been recognized, however, that other factors can also condition the distribution of polarity-sensitive items, including interrogatives (questions), comparatives, and 'expressive' contexts such as the complement clause of adjectives like *surprising*.

- (5.65) a. Has Mary ever been to Disneyland?  
 b. Does John own any hats?  
 c. Has John left yet?  
 d. Did Bill lift a finger to help?  
 e. Will Sue budge an inch on that question?  
 f. Does Tom give a damn?
- (5.66) a. John is fatter than Bill ever was.  
 b. Yao Ming is taller than any player on the other basketball team.

These contexts do not, however, all have the full power of negation to 'license' NPIs.

- (5.67) a. It is surprising that Mary has ever been to Disneyland.  
 b. ?It is surprising that John owns any hats.  
 c. ?It is surprising that John has left yet.  
 d. It is surprising that Bill lifted a finger to help.  
 e. ?It is surprising that Sue has visited in years.  
 f. It is surprising that Tom gives a damn.

We return below to distinctions within the class of NPIs that are pertinent to the difference observed between *ever*, *any*, *yet*, and the other NPIs in these examples. The distinctions are also pertinent to which PPIs are allowed in the following cases and why.

- (5.68) a. Would W. C. Fields rather be in Philadelphia?  
 b. Has John already taken out the trash?  
 c. Did Mary notice someone/something out of the ordinary? (cannot have the sense: Did Mary notice anyone/anything out of the ordinary?)
- (5.69) a. It is easier to believe that the Cubs will win the World Series than that W. C. Fields would rather be in Philadelphia.  
 b. \*John is slower than to have already taken out the trash.  
 c. Yao Ming is taller than some player on the other basketball team. (cannot have the sense of (5.66b))
- (5.70) a. It is surprising that W. C. Fields would rather be in Philadelphia.  
 b. It is surprising that John has already taken out the trash.  
 c. It is surprising that Mary noticed someone/something out of the ordinary. (cannot have the sense: It is surprising that Mary noticed anyone/anything out of the ordinary)

We do not here discuss what these other conditioners of polarity items might have in common with negation. For some discussion, see e.g. Hoeksema 1983.

### 5.9.2 What is negative about negation?

Besides the negation morpheme *not*, other expressions with negative-like meanings serve to condition the occurrence of polarity items. For example, NPIs are found in the following monotone decreasing contexts as well as in the scope of the paradigmatic conditioner, negation.

- (5.71) a. No one has ever returned from Disneyland satisfied.  
 b. No child who has ever been to Disneyland dislikes Mickey Mouse.  
 c. Every child who has ever been to Disneyland likes Mickey Mouse.  
 d. At most one hundred people have ever climbed Mt Everest.  
 e. Few people who have ever seen Mt Fuji at sunrise are unimpressed.  
 f. Seldom has anyone ever merited so little sympathy.

It is perhaps not surprising that downward monotone contexts would condition polarity items like negation does. One must nevertheless ask in what specific way or ways the operators in (5.71) are like negation.

Several different properties of negation are candidates for relevance to conditioning the occurrence of polarity-sensitive items. For one, negation is monotone decreasing. Recall that any function  $F$  is *decreasing*, relative to an ordering  $\leq_1$  of the arguments and an ordering  $\leq_2$  (which can be the same as  $\leq_1$ ) of the values, just in case

$$\text{if } x \leq_1 y, \text{ then } F(y) \leq_2 F(x)$$

The relevant ordering of propositions is implication (that is, both  $\leq_1$  and  $\leq_2$  are  $\Rightarrow$ ). For quantifiers, the ordering of arguments is set inclusion ( $\leq_1$  is  $\subseteq$ ), but the ordering of values is still implication.

Another property of negation is anti-additivity. We met this property in the previous section in connection with the restriction argument of type  $\langle 1, 1 \rangle$  quantifiers. It can be present when least upper bounds exist for the ordering  $\leq_1$  and greatest lower bounds exist for  $\leq_2$ . An element  $z$  is the *least upper bound* of elements  $x$  and  $y$  with respect to an ordering  $\leq$  iff

$$x \leq z, y \leq z, \text{ and for all elements } z', \text{ if } x \leq z' \text{ and } y \leq z' \text{ then } z \leq z'$$

Similarly,  $z$  is the *greatest lower bound* of  $x$  and  $y$  iff

$$z \leq x, z \leq y, \text{ and for all } z', \text{ if } z' \leq x \text{ and } z' \leq y, \text{ then } z' \leq z$$

Denoting the least upper bound of  $x$  and  $y$  by  $x \vee y$  and the greatest lower bound by  $x \wedge y$ , we have that a function  $F$  is *anti-additive* just in case

$$F(x \vee_1 y) = F(x) \wedge_2 F(y)$$

For negation,  $\vee_1$  is disjunction,  $\wedge_2$  is conjunction, and anti-additivity is one of De Morgan's laws. For quantifiers,  $\vee_1$  is set union, while  $\wedge_2$  is still conjunction.

Repeating here the definition of left anti-additivity for type  $\langle 1, 1 \rangle$  quantifiers, from section 5.8, we also give the corresponding definition for the scope argument of a type  $\langle 1, 1 \rangle$  and a type  $\langle 1 \rangle$  quantifier.

(LAA)  $Q_M(A, C) \& Q_M(B, C) \iff Q_M(A \cup B, C)$

(RAA)  $Q_M(A, B) \& Q_M(A, C) \iff Q_M(A, B \cup C)$

(AA)  $Q_M(A) \& Q_M(B) \iff Q_M(A \cup B)$

We note that

(5.72) A type  $\langle 1, 1 \rangle$  quantifier  $Q$  satisfies AA if and only if  $Q^{\text{rel}}$  satisfies RAA.

*Examples:* The type  $\langle 1, 1 \rangle$  quantifier *no* satisfies RAA. The type  $\langle 1, 1 \rangle$  quantifier *at most n* does not satisfy RAA (when  $n > 0$ ). This can be seen in the equivalence of the a and b sentences in (5.73) and their non-equivalence in (5.74).

- (5.73) a. No students danced or sang.  
b. No students danced and no students sang.

- (5.74) a. At most three students sang or danced.  
b. At most three students sang and at most three students danced.

Just as we saw in the case of quantifiers satisfying LAA, those satisfying RAA or AA are downward (right) monotone. This follows immediately from the definitions' right-to-left direction.

Anti-additivity is thus a stronger form of negativity than downward monotonicity, one that is shared with negation by some downward monotone quantifiers but not by others.

Now (Zwarts 1981, 1998) and van der Wouden (1997) observed that polarity-sensitive expressions, negative and positive, seem to differ according to 'how negative' the context must be, for NPIs, or is allowed to be, for PPIs. Zwarts argued that for some polarity items downward monotonicity is the operative characteristic. Other polarity items are not sensitive to mere downward monotonicity, he proposed, but instead are sensitive only to the stronger property of anti-additivity.

The NPI *yet* can occur in anti-additive contexts but not in all downward monotone contexts.

- (5.75) a. No one has left the party yet.  
b. \*At most three people have left the party yet.  
c. No medicine yet discovered will cure diabetes.  
d. \*At most three medicines yet discovered will cure diabetes.

By contrast, the NPIs *ever* and *any* occur more freely in monotone decreasing contexts.

- (5.76) a. No one has ever returned from Disneyland satisfied.  
b. At most three children have ever returned from Disneyland satisfied.  
c. No one played with anything in here.  
d. At most three children played with anything in here.
- (5.77) a. No child who has ever been to Disneyland dislikes Mickey Mouse.  
b. At most three children who have ever been to Disneyland dislike Mickey Mouse.

- c. No child who rode any rollercoaster likes dolls.
- d. At most three children who rode any rollercoaster like dolls.

A third property of negation is anti-multiplicativity. A function  $F$  is *anti-multiplicative* just in case

$$F(x \wedge_1 y) = F(x) \vee_2 F(y)$$

For negation,  $\wedge_1$  is conjunction,  $\vee_2$  is disjunction, and anti-multiplicativity is the other one of De Morgan's laws. For quantifiers,  $\wedge_1$  is set intersection, while  $\vee_2$  is still disjunction. Thus the definitions of anti-multiplicativity for quantifiers are:

$$\text{(LAM)} \quad Q_M(A, C) \vee Q_M(B, C) \iff Q_M(A \cap B, C)$$

$$\text{(RAM)} \quad Q_M(A, B) \vee Q_M(A, C) \iff Q_M(A, B \cap C)$$

$$\text{(AM)} \quad Q_M(A) \vee Q_M(B) \iff Q_M(A \cap B)$$

Again we note that quantifiers satisfying LAM are downward left monotone, and quantifiers satisfying RAM or AM are downward (right) monotone. This follows immediately from the definitions' left-to-right direction.

Note that no, though anti-additive, is not anti-multiplicative; (5.78a) does not entail (5.78b), as the former is consistent with some students dancing and other students singing.

- (5.78) a. No students danced and sang.  
 b. No students danced or no students sang.

The combination of anti-additivity and anti-multiplicativity is called anti-morphicity. Negation is anti-morphic. Certain polarity items of Dutch are said by Zwarts (1981) and van der Wouden (1997) to be sensitive to this very strongly negative property.<sup>24</sup>

### 5.9.3 A hypothesis about licensing of NPIs and PPIs

Returning to linguists' concern with characterizing exactly the range of contexts in which polarity items are acceptable and have the applicable sense, we note that some have proposed that *yet* is licensed only in anti-additive contexts, whereas *ever* and

<sup>24</sup> We note that Linebarger (1987) has pointed out that the logical properties of licensors of NPIs may derive partly from pragmatic factors. For example, although neither *exactly four* nor *after* has the negative properties discussed here, both

- (i) a. Exactly four people in the whole room budged an inch when I asked for help.  
 b. He kept writing novels long after he had any reason to believe that they would sell.

are acceptable. This seems to be related to the fact that (ia) implicates (iia) and (ib) implicates (iib) in some way.

- (ii) a. Few people budged an inch when I asked for help.  
 b. He kept writing novels when he no longer had any reason to believe that they would sell.

any are licensed in any downward monotone context. This hypothesis is consistent with the data presented above.<sup>25</sup>

Additionally, the PPI *would rather* and *already* occur pretty freely in contexts that are merely downward monotone, and *someone/something* can be interpreted as existential quantifiers with scope inside these contexts; but none of these is allowed in the anti-additive contexts considered so far.

- (5.79) a. At most three comedians would rather be in Philadelphia.  
 b. At most three comedians have already performed.  
 c. At most three comedians made someone laugh. (can have the sense: At most three comedians made anyone laugh)
- (5.80) a. \*No comedians would rather be in Philadelphia.  
 b. \*No comedians have already performed.  
 c. No comedians made someone laugh. (cannot have the sense: No comedians made anyone laugh)

Besides negation and *no*, the preposition *without* is anti-additive. The latter fact is illustrated in the equivalence of the following sentences.<sup>26</sup>

<sup>25</sup> Note, incidentally, that these operators condition polarity items in only so much of their scope. For example, compare (5.75c) with the following sentence:

- (i) \*No medicine that has been discovered yet will cure diabetes.

Here *yet* is not acceptable in the restriction on *no* because of intervening operators in the relative clause. The same sort of limitation is illustrated by the unambiguity of

- (ii) He didn't budge an inch because he was scared.

as contrasted with

- (iii) He didn't move because he was scared.

The latter can mean that the reason he moved was not that he was scared, but the former does not have this interpretation, on which the *because* operator intervenes between negation and the NPI. Similarly one interpretation of

- (iv) Sam didn't read every child a story.

is that not every child was read a story by Sam, but

- (v) Sam didn't read every child any story.

can only have the other interpretation of sentence (iv): that every child was not read any story by Sam. Formulating this restriction correctly is far from simple, however. Both *any* and *ever* are licensed by the single monotone decreasing adverb *seldom* in (5.71e), as well as in the restriction argument of the LAA determiner *every*.

- (vi) Every customer who had ever purchased anything in the store was contacted.

<sup>26</sup> In fact, *without* is also anti-multiplicative and therefore anti-morphic. The following are equivalent.

- (i) a. She has given parties without food and beverages.  
 b. She has given parties without food or she has given parties without beverages.

- (5.81) a. Bill smokes without inhaling or enjoying it.  
 b. Bill smokes without inhaling and Bill smokes without enjoying it.

As might be expected, the PPI *someone/something* cannot scope inside of *without*, nor is *already* acceptable there.<sup>27</sup>

- (5.82) a. \*Fred left without paying already.<sup>28</sup>  
 b. John won without help from someone (cannot have the sense: John won without help from anyone)

We thus see that the PPIs *would rather* and *someone/something* appear to be prohibited in contexts where the NPI *yet* is allowed;<sup>29</sup> *yet* is fine in the scope of *without* as well as of *not* and *no*.

- (5.83) Mary has been to New York three times without visiting the Statue of Liberty *yet*.

Moreover, the PPIs *would rather* and *someone/something* are permitted in contexts where the NPI *yet* cannot occur: i.e. merely monotone decreasing contexts and positive contexts that are not even monotone decreasing. All of these polarity items, both negative and positive, are sensitive only to pretty strongly negative conditioners; i.e. they occur only in contexts that are at least anti-additive.

In contrast, the NPIs *ever* and *any* are sensitive to mere downward monotonicity, occurring even in the scope of *at most three* and other monotone decreasing quantifiers that are not anti-additive. Thus the conditioning factor for this group of NPIs is different from that of the class containing the NPI *yet* (and the corresponding class of PPIs, which contains *would rather* and *someone/something*). The two groups of NPIs differ in their sensitivity to negativity. Similarly, for groups of PPIs.<sup>30</sup>

These generalizations amount to an analysis, classifying NPIs into two or more classes and specifying what sort of element suffices to license the occurrence of that class of NPIs in the context of that element. For example, *ever* and other NPIs in its class are licensed in any monotone decreasing context (as well as in questions, comparatives, and ‘expressive’ contexts); *yet* and other NPIs in its class are licensed in any anti-additive context (as well as in questions). No NPI can occur except where it is licensed by these statements. Where some element allows NPIs to occur in its context, no wider context surrounding both the element and the NPI can revoke that license.

Similarly, the implicit analysis of PPIs places them in classes and specifies what sort of element excludes each class of PPI from its context. For example, *someone* and

<sup>27</sup> English syntax precludes a test of whether *would rather* is acceptable in the scope of *without*, because this preposition cannot be followed by a finite clause.

<sup>28</sup> Compare *Fred had already paid when he left*.

<sup>29</sup> The only difference we have seen in licensing conditions of the PPIs *would rather* and *someone/something* is in sentences (5.69a, c).

<sup>30</sup> Recall that Zwarts (1998) and van der Wouden (1997) claim that Dutch has a class of polarity-sensitive items that are not sensitive even to (mere) anti-additivity, but only to anti-morphicity.

other PPIs of its class are prohibited from occurring in the scope of an anti-additive operator unless (a) a clause boundary or another operator's scope intervenes or (b) the anti-additive operator in whose immediate scope the PPI occurs is itself in a downward monotone or 'expressive' context, or in a question. Exception (b) is motivated by facts that Jespersen (1943) noticed and Baker (1970) and Szabolcsi (2004) among other have since followed up: namely, that (5.62) and (5.80) are 'rescued' in sentences like the following.

- (5.84) a. At most three people complained that Mary didn't notice someone/something out of the ordinary. (with the sense: At most three people complained that Mary didn't notice anyone/anything out of the ordinary)  
 b. It is surprising that no comedians made someone laugh. (with the sense: It is surprising that no comedians made anyone laugh)<sup>31</sup>

A PPI can occur where its syntactic properties allow except if it is prohibited by these statements.

#### 5.9.4 Testing the hypothesis

Proposed analyses, such as this one, are of course intended to predict facts beyond the ones used to justify the analysis. This makes them empirically testable against new examples. An obvious question about this hypothesis or any other is what sort of examples should be examined to test it. For example, what are the anti-additive quantifiers of English? And what downward monotone quantifiers are not anti-additive? The latter question can be answered with the tools developed in this chapter, allowing one to put the hypothesis to a sharper test than would otherwise be possible.

We saw in Proposition 13 in the preceding section that *no* and *every* are the only non-trivial LAA quantifiers. Something similar is true about RAA quantifiers, though this requirement is slightly less restrictive. Let us make these remarks precise, then return to the issue of testing the hypotheses embodied in the analysis sketched here.

##### Proposition 14

(FIN) *Let  $Q$  be any type  $\langle 1 \rangle$  quantifier satisfying ISOM. Then  $Q$  satisfies AA if and only if for each  $M$ , one of the following three conditions holds:*

- (i)  $Q_M(A)$  for no  $A \subseteq M$
- (ii)  $Q_M(A)$  holds iff  $A = \emptyset$
- (iii)  $Q_M(A)$  holds for all  $A \subseteq M$

*Consequently, by (5.72), if  $Q$  is a CONSERV, EXT, and ISOM type  $\langle 1, 1 \rangle$  quantifier,  $Q$  satisfies RAA if and only if for each  $M$  and each  $A \subseteq M$ , one of the following conditions holds:*

- (i)  $Q_M(A, B)$  holds for no  $B \subseteq M$

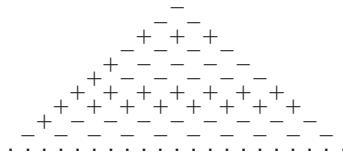
<sup>31</sup> As we saw in n. 22, the strings (5.60b) and (5.61b) are well-formed clauses of English. They simply are unacceptable as complete sentences or embedded anywhere except in the scope of a suitable negative conditioner.



- (ii)  $Q_M(A, B)$  holds for  $B \subseteq M$  iff  $A \cap B = \emptyset$
- (iii)  $Q_M(A, B)$  holds for all  $B \subseteq M$

Remark: We find some familiar determiner denotations here:

- none of the  $k$  or more; in particular ( $k = 0$ ), no
- none of the  $k$
- any number of the  $k$ <sup>32</sup>
- disjunctions of the above (indeed, every RAA quantifier is (under the assumed conditions) a countable disjunction of quantifiers of the last two kinds)



an RAA quantifier

*Proof of Proposition 14.* The proof is a slight variation on the proof of Proposition 13. Let  $Q$  be any type  $\langle 1 \rangle$  quantifier satisfying ISOM.

( $\Rightarrow$ ) Assume  $Q$  satisfies AA, and recall that  $Q$  must therefore be  $\text{MON}\downarrow$ . Consider any finite domain  $M$  and set  $A \subseteq M$ . It suffices to show that if  $Q_M(A)$  and  $A \neq \emptyset$ , then  $Q_M(M)$ ; so suppose  $A \neq \emptyset$  and  $Q_M(A)$ . If  $A = M$  we are done with this direction; so assume  $M - A \neq \emptyset$  and choose  $a' \in M - A$  and  $a \in A$ . Put  $A' = (A - \{a\}) \cup \{a'\}$ , and note that  $|A'| = |A|$  and  $|A \cup A'| = |A| + 1$ . Then  $Q_M(A')$ , because  $Q$  is ISOM, and  $Q_M(A \cup A')$ , because  $Q$  is AA. Since  $M$  is finite,  $Q_M(M)$  follows by induction, completing the proof of this direction.

( $\Leftarrow$ ) Assume that for each finite domain  $M$ , one of (i), (ii), and (iii) holds. We need to show that  $Q_M(A) \ \& \ Q_M(B) \iff Q_M(A \cup B)$  for all  $A, B \subseteq M$ . The right-to-left direction, which is equivalent to  $\text{MON}\downarrow$ , follows immediately from the disjunction of (i), (ii), and (iii). So assume  $Q_M(A)$  and  $Q_M(B)$  hold for  $A, B \subseteq M$ . If  $A = B = \emptyset$ , then  $A \cup B = \emptyset$ , so  $Q_M(A \cup B)$ . And if either  $A$  or  $B$  is not  $\emptyset$ , then  $Q_M(A \cup B)$  holds by (iii); so we are done. □

We need one more linguistic observation about where the hypothesis predicts that NPIs and PPIs can occur. Recall (note 25) that the acceptability of an NPI or PPI is conditioned by an operator in whose immediate scope that polarity item must or cannot occur. Thus for the disjunctions of the basic RAA quantifiers mentioned above, acceptability is determined disjunct by disjunct. Therefore, we can in fact set aside disjunctions and concentrate on the basic RAA quantifiers when testing for polarity sensitivity.

<sup>32</sup> Any number of the  $k$  should be read as ‘zero, one, or more of the  $k$ ’, not as ‘a substantial number of the  $k$ ’.

Now we know where to look for quantifiers to test the hypothesis that anti-additive contexts license NPIs like *yet* and the same contexts preclude PPIs like narrow scope *someone/something* (unless ‘rescued’), while monotone decreasing contexts that are not anti-additive do not accomplish either of these. We already saw some instances of anti-additive contexts and downward monotone contexts that are not anti-additive: the ones that were used above to motivate the hypothesis. We now examine further examples, to determine whether or not they behave as the hypothesis predicts.

Since *every* is LAA, *yet* is predicted to be acceptable in this quantifier’s restriction argument. The facts, however, seem contrary to the prediction.

(5.85) \*Every drug yet tried has failed to cure diabetes.

On the other hand, the PPIs *some{one/thing/N}* are barred from scoping (immediately) inside the domain of *every*.

(5.86) Every linguist trained in some Turkic language received a fellowship. (cannot have the sense: Every linguist trained in any Turkic language received a fellowship)

So the hypothesis about this class of PPI is supported. Of course, we should also compare

(5.87) Some friend of every movie star should introduce me to all of them.

to determine that (5.86) is not excluded simply because the syntactic construction itself cannot be interpreted with the scoping in question. We do indeed see that (5.87) allows *some friend* to scope over *every movie star*; so the impossibility is instead connected with the attempt to scope *some Turkic language* in the domain of *every*. Thus these PPIs may be sensitive to anti-additivity, even if the distribution of the NPI *yet* is not exactly described in terms of anti-additivity.

We also note that the non-ISOM possessive quantifier *every student’s*, defined by

$$\text{every student's}_M(A, B) \iff \forall a \in \text{student } A \cap R_a \subseteq B$$

(where  $R$  is any ‘possessor’ relation and  $R_a$  is the set of things  $R$ ’d by  $a$ ), is LAA, since

$$\forall a \in C A_1 \cap R_a \subseteq B \ \& \ \forall a \in C A_2 \cap R_a \subseteq B \iff \forall a \in C (A_1 \cup A_2) \cap R_a \subseteq B$$

This generates the same kind of counterexamples to the prediction as (5.85):

(5.88) \*Every doctor’s drug yet tried has failed to cure diabetes.

Turning to the RAA cases, we verify the following predictions:

(5.89) None of the five (or more) guests has arrived yet.

(5.90) None of the five (or more) guests brought someone/something unwelcome. (cannot have the sense: None of the five (or more) guests brought anyone/anything unwelcome)

On the other hand, the following fact is contrary to prediction:

(5.91) \*Any number of the six guests have arrived yet.

However, the hypothesis about PPIs like *someone/something* fares better.

- (5.92) Any number of the six guests brought *someone/something* unwelcome. (cannot have the sense: Any number of the six guests brought *anyone/anything* unwelcome)

Finally, we check whether *yet* is acceptable in the scope of a non-ISOM (right) anti-additive quantifier.<sup>33</sup>

- (5.93) Neither of each student's parents has visited *yet*.

In this case, the prediction is successful, although there seem to be very few non-ISOM (right) anti-additive quantifiers available to test.

Overall, the hypothesis that *yet* occurs precisely in anti-additive contexts has a mixed record of success. By contrast, the hypothesis that the PPIs we discussed are excluded from such contexts is more successful. If the hypotheses had no empirical implications beyond a few anti-additive quantifiers, it is unclear how seriously they would deserve to be taken. There are, however, other anti-additive operators besides quantifiers; so more systematic evaluation is possible—although we do not pursue it here as our principal interest is in quantifiers. Our aim in this discussion has been to show how hypotheses like the ones discussed in this section can be tested empirically with respect to quantifiers. Whether or not one rejects any of these particular hypotheses, alternative hypotheses can benefit from similar analysis of their predictions, and systematic testing of predictions against empirical facts.

Many linguists believe that there is some number  $k$  of degrees of negativity, and there are classes  $\text{NPI}_j$  and  $\text{PPI}_j$  of polarity items corresponding to the degrees  $j \leq k$  of negativity, such that items in  $\text{NPI}_j$  occur only in contexts that are negative to degree  $j$  or greater, and items in  $\text{PPI}_j$  occur in contexts that are negative to degree  $j$  or less—more precisely, items in  $\text{PPI}_j$  are excluded from contexts that are negative to degree greater than  $j$ —unless 'rescued'. Such hypotheses differ in their choice of degrees of negativity and their assignment of polarity items to the classes  $\text{NPI}_j$  and  $\text{PPI}_j$ . Each such hypothesis deserves systematic testing, as we showed how to do for the specific hypothesis discussed in this section. Tools like the ones we used here can be developed for testing other hypotheses.

<sup>33</sup> We have the truth conditions

$$\text{neither of each student's}_M(A, B) \iff$$

$$\forall a \in \text{student}(\exists b \in A \text{ s.t. } R(a, b) \implies |A \cap R_a| = 2 \ \& \ A \cap R_a \cap B = \emptyset)$$

from which it is readily seen that this quantifier is (right) anti-additive.

# 6

## Symmetry and Other Relational Properties of Type $\langle 1, 1 \rangle$ Quantifiers

Type  $\langle 1, 1 \rangle$  quantifiers are (on each universe) binary relations between sets, and it is natural to inquire how quantifier denotations behave in terms of ordinary relational properties. This chapter focuses primarily on one such property, *symmetry*. It turns out that symmetric determiners are very common in natural languages, and such determiners are particularly well-behaved. For example, a symmetric determiner can be seen to denote a *property* of the intersection of the restriction and the scope, which in turn explains why such expressions often go well in predicate position too.

Moreover, symmetry has been advanced as the major factor in explaining which determiners are acceptable in existential-there sentences. The bulk of this chapter attempts a thorough analysis of the semantics of such sentences, including a comparative study of various attempts at explaining acceptability that have been proposed in the literature. In the end, we find that the explanation in terms of symmetry is the most successful one so far, although many linguistic facts still remain to be accounted for.

Section 6.1 introduces the concept of symmetry, and gives a number of different characterizations of symmetric quantifiers. In particular, we show that EXT for a type  $\langle 1 \rangle$  quantifier amounts to the symmetry of its relativization. In section 6.2 we touch briefly on the issue of the symmetry of many and few; this is just one aspect of the (exasperating) question of how, if at all, to account for these determiners in the framework of generalized quantifiers. Section 6.3 about existential-there sentences is the main linguistic contribution of the chapter and by far its longest section. The final section (6.4) surveys a few other relational properties of quantifiers, using the number triangle to state and prove various results.

### 6.1 SYMMETRY

Already Aristotle noted that some determiners in natural languages (and their corresponding denotations) are symmetric, and others are not. The (a) and (b) sentences in the following examples imply each other, but the (c) and (d) sentences do not.

- (6.1) a. Some lawyers are crooks.  
b. Some crooks are lawyers.

- c. All lawyers are crooks.
  - d. All crooks are lawyers.
- (6.2)
- a. No lawyers are crooks.
  - b. No crooks are lawyers.
  - c. Most lawyers are crooks.
  - d. Most crooks are lawyers.
- (6.3)
- a. Three lawyers are crooks.
  - b. Three crooks are lawyers.
  - c. All but two lawyers are crooks.
  - d. All but two crooks are lawyers.
- (6.4)
- a. A woman was a witness.
  - b. A witness was a woman.
  - c. The lawyers were crooks.
  - d. The crooks were lawyers.

Thus, *no*, *some*, *three*, as well as modified numerals like *more than six*, at *most four*, and the indefinite article are symmetric, but *all*, *every*, *all but two*, and *most*, and other proportionals, as well as the definite article, are not.

There are some initially puzzling challenges to symmetry, like the intuitive contrast between (6.5a) and (6.5b) below:

- (6.5)
- a. Some men are bachelors.
  - b. Some bachelors are men.

(6.5a) seems informative, whereas (6.5b) is trivial. But this puzzle can be satisfactorily resolved with a pragmatic theory about the point of utterances, such as Grice's (1967) theory of conversational implicature, (6.5a) normally carries the implicature (6.6a), and (6.5a) would carry (6.6b).

- (6.6)
- a. Not all men are bachelors.
  - b. Not all bachelors are men.

The self-contradictory nature of (6.6b) explains the oddness of (6.5b). Conversational implicature is also pertinent to the feeling that symmetry pairs like those above are not fully synonymous. For example, (6.1a) has the implicature (6.7a), and (6.1b) has the non-equivalent implicature (6.7b).

- (6.7)
- a. Not all lawyers are crooks.
  - b. Not all crooks are lawyers.

A similar observation is the following. Determiners have, as we have noted, the property of restricting the domain of quantification to what is accordingly called the restriction argument. This is a semantic property, codified in CONSERV and EXT. But it is natural to assume that there are also pragmatic aspects of domain restriction, such

as making the extension of the corresponding noun the topic at that point of an utterance. If so, the respective sentences in a symmetry pair have different topics, and are already for this reason not fully synonymous.

In any case, aside from these pragmatic aspects, there is an obvious semantic property of generalized quantifiers denoted by symmetric determiners: namely, the symmetry, in the usual sense, of the corresponding second-order relations.

#### symmetry

A type  $\langle 1, 1 \rangle$  quantifier  $Q$  is *symmetric* (SYMM) iff, for all  $M$  and all  $A, B \subseteq M$ ,

$$(6.8) \quad Q_M(A, B) \implies Q_M(B, A)$$

Clearly, the quantifiers corresponding to the symmetry pairs above are all symmetric in this sense, whereas those in the other pairs are not symmetric. But looking at these examples, we can see that there is another way of expressing the characteristic semantic properties of the quantifiers in the symmetry pairs. This is the property that only the intersection of the restriction and the scope matters for whether the sentence is true or not. For example, the only thing that matters for whether THREE LAWYERS ARE CROOKS is true or not is the number of lawyers who are crooks. The number of honest lawyers, say, has no influence at all. This property (in this case of the quantifier *three*) has been called *intersectivity*.

#### intersectivity

A type  $\langle 1, 1 \rangle$  quantifier  $Q$  is *intersective* (INT) iff the truth value of  $Q_M(A, B)$  depends only on  $A \cap B$ . More precisely, for all  $M$  and all  $A, B, A', B' \subseteq M$ ,

$$(6.9) \quad \text{if } A \cap B = A' \cap B', \text{ then } Q_M(A, B) \Leftrightarrow Q_M(A', B')$$

A different way of expressing the same property is the following:

$$(6.10) \quad Q_M(A, B) \iff Q_M(A \cap B, A \cap B)$$

(6.9) implies (6.10), since if we take  $A' = B' = A \cap B$  in (6.9),  $A \cap B = A' \cap B'$ , and so (6.10) follows. Conversely, if  $A \cap B = A' \cap B'$  and (6.10) holds, two applications of this property give the conclusion of (6.9). One advantage of expressing intersectivity as in (6.9) is that it generalizes to determiner denotations of types other than  $\langle 1, 1 \rangle$  (we saw this in Chapter 4.7).

It is also clear that intersectivity implies symmetry: in order to conclude  $Q_M(B, A)$  from  $Q_M(A, B)$ , we let  $A' = B$  and  $B' = A$  in (6.9). The converse implication does not hold in general, but it does hold if  $Q$  is conservative:

#### Fact 1

*Under CONSERV, SYMM and INT are equivalent.*

*Proof.* It remains to verify that symmetry implies intersectivity under CONSERV. Suppose  $A, B \subseteq M$ . Then

$$\begin{aligned} Q_M(A, B) &\iff Q_M(A, A \cap B) \quad [\text{by CONSERV}] \\ &\iff Q_M(A \cap B, A) \quad [\text{by SYMM}] \\ &\iff Q_M(A \cap B, A \cap B) \quad [\text{by CONSERV}] \end{aligned}$$

Thus, (6.10) holds.  $\square$

Since determiner denotations are both CONSERV and EXT, the intuition that the quantifiers in symmetry pairs are both symmetric and intersective is justified. Note that under EXT, (6.10) can be written

$$(6.11) \quad Q_M(A, B) \iff Q_{A \cap B}(A \cap B, A \cap B)$$

Thus, symmetric quantirelations actually express a property of the intersection of their restriction and scope, independent of the universe of discourse.

The fact that determiner denotations are CONSERV and EXT is explained by their being relativizations of type  $\langle 1 \rangle$  quantifiers (Proposition 3 in Chapter 4.5). So what is the property of type  $\langle 1 \rangle$  quantifiers that corresponds to symmetry of their relativizations? The simple answer is as follows.

### Proposition 2

$Q^{\text{rel}}$  is symmetric if and only if  $Q$  satisfies EXT ( $Q$  of type  $\langle 1 \rangle$ ).

*Proof.* Suppose first that  $Q^{\text{rel}}$  is symmetric. If  $A$  is a subset of both  $M$  and  $M'$ , then

$$\begin{aligned} Q_M(A) &\iff Q_M^{\text{rel}}(M, A) \quad [\text{by (4.25)}] \\ &\iff Q_A^{\text{rel}}(A, A) \quad [\text{by (6.11)}] \\ &\iff Q_{M'}^{\text{rel}}(M', A) \quad [\text{by (6.11)}] \\ &\iff Q_{M'}(A) \quad [\text{by (4.25)}] \end{aligned}$$

Thus,  $Q$  satisfies EXT. Conversely, suppose that  $Q$  satisfies EXT, and that  $A, B \subseteq M$ . Then

$$\begin{aligned} Q_M^{\text{rel}}(A, B) &\iff Q_A(A \cap B) \quad [\text{by (4.25)}] \\ &\iff Q_{A \cap B}(A \cap B) \quad [\text{by EXT}] \\ &\iff Q_M^{\text{rel}}(A \cap B, A \cap B) \quad [\text{by (4.25)}] \end{aligned}$$

So (6.10) holds for  $Q^{\text{rel}}$ , and hence it is symmetric.  $\square$

So there is a substantial difference between EXT for type  $\langle 1 \rangle$  quantifiers and EXT for type  $\langle 1, 1 \rangle$  quantifiers, in a natural language context—see the discussion in Chapter 3.4.1.3. The latter property holds of all determiner denotations. But the EXT type  $\langle 1 \rangle$  quantifiers single out a particular class of determiner denotations: namely, the symmetric ones.

*All, most, all but five, the ten* are thus relativizations of non-EXT type  $\langle 1 \rangle$  quantifiers. Recall the universal (EU) (Chapter 4.5.3) which entails that most NPs denote EXT quantifiers. But there is no conflict here, since in fact most determiner denotations are relativizations not of NP denotations, but rather of other type  $\langle 1 \rangle$  quantifiers.

If ISOM holds, we can further characterize symmetry. Keenan and Stavi (1986) call a type  $\langle 1, 1 \rangle$  quantifier *Q cardinal* (CARD) if the following holds, for all  $M$  and all  $A, B \subseteq M$ :

$$(6.12) \text{ If } |A \cap B| = |A' \cap B'|, \text{ then } Q_M(A, B) \Leftrightarrow Q_M(A', B').$$

CARD implies INT (since the antecedent of (6.9) implies the antecedent of (6.12)). The precise relation between the two notions is given in the following proposition.

### Proposition 3

*For any type  $\langle 1, 1 \rangle$  quantifier  $Q$ , CARD is equivalent to ISOM + INT, on finite universes. Hence, under CONSERV, it is equivalent to ISOM + SYMM on finite universes. Under CONSERV and EXT, it is equivalent to ISOM + SYMM on arbitrary universes.*

*Proof.* Suppose  $Q$  satisfies CARD. We already saw that INT follows. Also, it is clear that CARD entails ISOM for type  $\langle 1, 1 \rangle$  quantifiers (in the form of (4.49) in Chapter 4.8). This implication is true whether the universe is finite or not.

Now suppose, conversely, that  $Q$  satisfies ISOM and INT, that  $A, B, A', B' \subseteq M$ , where  $M$  is finite, and that  $|A \cap B| = |A' \cap B'|$ . It follows that  $|M - (A \cap B)| = |M - (A' \cap B')|$  (though this is not necessarily true if  $M$  is infinite). Then

$$\begin{aligned} Q_M(A, B) &\iff Q_M(A \cap B, A \cap B) \quad [\text{by (6.10)}] \\ &\iff Q_M(A' \cap B', A' \cap B') \quad [\text{by ISOM}] \\ &\iff Q_M(A', B') \quad [\text{by (6.10)}]. \end{aligned}$$

Thus CARD holds.

The claim about what happens under CONSERV follows immediately from this and Fact 1. And if EXT also holds, the cardinalities of  $M - (A \cap B)$  and  $M - (A' \cap B')$  are just irrelevant, so the above argument to infer CARD goes through using only the assumption that  $|A \cap B| = |A' \cap B'|$ .  $\square$

We observed in Chapter 3.4, Fact 9, that under ISOM, a type  $\langle 1 \rangle$  quantifier satisfying EXT is characterized simply by a class of cardinal numbers. Combining this with the fact that CONSERV and EXT quantifiers are relativizations of type  $\langle 1 \rangle$  quantifiers, and with Proposition 2, we obtain a final characterization of symmetric quantifiers:

### Corollary 4

*Under ISOM, a CONSERV and EXT type  $\langle 1, 1 \rangle$  quantifier  $Q$  is symmetric iff there is a class  $S$  of cardinal numbers such that for all  $M$  and all  $A, B \subseteq M$ ,*

$$Q_M(A, B) \iff |A \cap B| \in S$$



That symmetric quantifiers can be seen as expressing a property of the intersection of the restriction and the scope explains why expressions denoting such quantifiers usually are fine in predicate position too:

- (6.13) a. The reasons are several / \*every.  
 b. The winning students were three / \*two-thirds.

## 6.2 ON THE SYMMETRY OF *MANY* AND *FEW*

The interpretation of the determiners *many* and *few* has been widely discussed in the literature. It is easy to find instances of what appears to be non-symmetric behavior: (6.14a) intuitively seems not to entail (6.14b), and (6.15a) not to entail (6.15b).

- (6.14) a. Many plutocrats are Americans.  
 b. Many Americans are plutocrats.
- (6.15) a. Few scoundrels are Nobel Laureates.  
 b. Few Nobel Laureates are scoundrels.

This time it is not a matter of total synonymy or pragmatic considerations, since the very truth conditions appear to differ. The above examples seem to admit a proportional reading, though in contrast with at least two-thirds of the or even most, it is hard to say how the intended proportions are arrived at. Moreover, whereas in (6.14) and (6.15) it is the proportion of *B* in *A* that matters, in (6.16) and (6.17) it appears to be the other way around.

(6.16) Many Scandinavians have won the Nobel Prize.

(6.17) Few cooks applied.

For example, (6.17) could say on a given occasion that the number of cooks that applied is smaller than one-fifth of the total number of applicants, but hardly that it is smaller than one-fifth of the total number of cooks.

In view of facts such as these, some authors conclude that the extensional machinery of generalized quantifiers cannot be applied to *many* and *few*. For example, Keenan and Stavi (1986) claim that they are *intensional*, and therefore exclude them from discussion, and Fernando and Kamp (1996) present an analysis of these determiners in terms of possible worlds. However, it is not clear that even the intensions of words and phrases in a sentence where *many* or *few* occurs suffice to fix its interpretation. Perhaps other factors in the context of utterance play a role. Thus, one may also surmise that these determiners are just *extremely context-dependent*, which in principle need not preclude that they are extensional.

The following seems to be a universal fact about the meaning of *many* and *few*: a claim *many(A, B)* sets a lower bound on the size of  $A \cap B$ , and *few(A, B)* sets an upper bound. One indication of this is that an *exception* to the former claim clearly is something in  $A - B$ , whereas an exception to the latter clearly is something in  $A \cap B$  (we discuss exceptions and exceptive determiners at length in Chapter 8).

If context determines each *use* of many, a possible view would be that many A *as* B are B simply *means*

$$|A \cap B| \geq n$$

for some context-dependent  $n$ , and similarly for *few*. This is of course extensional, CONSERV, EXT, ISOM,  $\uparrow$ MON $\uparrow$ , SYMM, etc. The fact that many and *few* are fine in predicate position supports this view:

(6.18) The contestants were many, but the lucky ones were few.

In that position, they clearly denote a property of sets, and hence are symmetric.

However, on this view one could not draw any conclusion about symmetry from examples like (6.14a) and (6.14b), since the context of the two uses of many are different. Of course, that might be just how things are. The fact that it appears to be very hard to find any examples where a sentence of the form Q A *as* B are B is logically equivalent to Q B *as* A does not necessarily show that Q is non-symmetric, since it involves two uses of the determiner in quite different (linguistic) contexts.

Should one nevertheless want to try to capture the uniformities of interpretation that at least appear to hold in cases like (6.14)–(6.17), an alternative might be to think of many *as ambiguous* between, say, a proportional reading

$$|A \cap B| \geq k \cdot |A|$$

and an ‘anti-proportional’ reading

$$|A \cap B| \geq k \cdot |B|$$

where, however, the number  $k$  still has to be provided by context. Note that whereas the proportional reading is conservative but not symmetric, the ‘anti-proportional’ one loses conservativity too, thus threatening our quantifier relation universal (QU) (Chapter 4.5). This might be considered a higher methodological price than one is willing to pay.

Finding an adequate treatment of many and *few* is a complex and interesting task, both empirically and methodologically, but we shall have to leave the matter here with the above brief comments.

### 6.3 EXISTENTIAL-THERE SENTENCES

We now turn our attention to examining how symmetry and other properties of quantifiers shed light on a linguistic puzzle that has exercised minds for several generations (for example, Jespersen 1943).

#### 6.3.1 Natural language talk about existence

Many natural languages have a specialized construction more or less dedicated to talking about the existence or non-existence of things. In English, such talk typically employs sentences with the word *there* as subject and an inflected form of *be* as

verb followed by a noun phrase, the *pivot* noun phrase;<sup>1</sup> there may optionally be a further phrase as the sentence's *coda*. Examples include:

- (6.19) a. There is a man here.  
 b. There is a weasel chasing that rabbit.  
 c. There are two kinds of people.  
 d. There are only two ways to live your life. (Albert Einstein)  
 e. There are few natural opiates.  
 f. There are people coming.  
 g. There are no GIF files on these URL pages.  
 h. There are several solutions to every problem.  
 i. There are many animals in the zoo.  
 j. There are more sheep than people in New Zealand.

Sentences (6.19c, d, e, h) contain just a pivot noun phrase after the copula. Sentences (6.19a, f) additionally contain a coda—here and coming, respectively—following their pivot noun phrase, and the coda must constitute a separate phrase, because neither a man here nor people coming is a noun phrase, as the ungrammaticality of \*A man here is about to leave and \*People coming will arrive soon demonstrates. Seeing that a coda may appear in existential sentences, one comes to realize that sentences (6.19b, g, i, j) might either have a coda or just have a pivot noun phrase that contains a reduced relative clause as postnominal modifier. Scholars have searched for reliable tests to determine which cases are codas and which ones postnominal modifiers within the pivot. The reason why this is so hard to determine appears to be that it makes no difference to the sentence's meaning, and is only definitively determinable when the difference matters for grammaticality.

Some dialects of English employ *it* instead of *there* as the subject of existential sentences, and many other, Germanic languages standardly employ cognates of *it*—for example, German *es gibt* ... and Swedish *det finns* ... Although the grammatical construction varies across languages, quite widely in fact, many languages—perhaps most—use specialized syntax for sentences about the existence of things.

The English syntactic construction is not totally dedicated to talk about existence. The following strings sound unacceptable as statements that something exists, or does not exist.

- (6.20) a. #There is the problem of the cockroaches escaping.  
 b. #There is every book under the bed.  
 c. #There is neither parent in the room.  
 d. #There is John.

<sup>1</sup> Some authors call it the “focus noun phrase”, others simply the “post-verbal noun phrase”. We adopt the “pivot” terminology to avoid confusion about the question whether this noun phrase is included in the informational *focus* of the sentence.

- e. #There are both opinions regarding this. (cf. There are both reform and orthodox services over shabbat.)
- f. #There are most angels in heaven.
- g. #There are the three or more voting members present.
- h. #There are all students' coats hanging in the closet.

Some of these strings are nevertheless good English sentences. For instance, (6.20a, d) present instances of something whose existence has already been asserted or implied. Consider the following:

- (6.21) a. (i) Housing cockroaches in captivity poses two main problems.  
 (ii) There is the problem of the cockroaches escaping. ([www.ex.ac.uk/bugclub/roach.html](http://www.ex.ac.uk/bugclub/roach.html))
- b. (i) Who is available to walk the dog?  
 (ii) There is John.

Sentences like (6.19) are commonly called *existential-there* sentences, to contrast them with *presentational* there sentences (6.21a(ii), b(ii)) as well as with superficially similar sentences whose subject is the place adverb *there*: e.g. *There goes a weasel chasing a rabbit*.

Seen in this light, it seems that at least some of the examples in (6.20) actually are grammatical sentences; they just cannot be interpreted as existential-there sentences. Presenting these examples after setting out a long list of existential-there sentences can bias readers toward looking just for an existential interpretation, which is not available for any sentence in (6.20), even well-formed ones. Existential sentences also differ in meaning from sentences beginning with a place adverb. Sentence (6.19b) does not answer the question “Where is a weasel chasing that rabbit?”, while “There goes a weasel chasing that rabbit” does answer the question. The latter statement does not, however, answer the question “Is there a weasel chasing that rabbit?”<sup>2</sup>

As Jespersen (1943) characterized the existential-there sentences (ch. III, sect. 3.1, p. 107):

The empty, or, as I have called it elsewhere, the existential *there* differs from the local adv *there*

- (1) by having weak stress and consequently having the vowel . . . reduced . . . ,
- (2) by losing its local meaning; hence the possibility of combining it with local advs and other tertiaries, . . . [*But there is no one there. There's some magazines here.*]
- (3) by being a quasi-subject, thus e.g. in an infinitival nexus and with an ing, . . . [*let there be light. account for there being something rather odd*]
- (4) by the tendency to have the vb in sg form with a pl subject, . . .
- (5) by the word-order: *there is nothing wrong*, but *there nothing is wrong* . . .

Thus it is not quite correct to say with NED (*there* 4) [Oxford: *A New English Dictionary*, sense 4 of *there*] that “Grammatically there is no difference between *There comes the train* and *There comes a time when*, etc.; but while in the former *there* is demonstrative and stressed, in

<sup>2</sup> Although it does entail a positive answer to the latter question.

the latter it has been reduced to a mere anticipative element occupying the place of the subject which comes later.”

### 6.3.2 Restrictions on the pivot noun phrase of existential-there sentences

Existential-there sentences are distinguished not only by their meaning but also by limitations on the range of their grammatical formation—in ways that have puzzled and challenged grammarians for a considerable time. The contrast between the existentially interpretable examples (6.19a–j) and examples (6.20a–h), which cannot be interpreted existentially illustrates the grammatical puzzle. The main factor determining whether or not a there sentence is existentially interpretable—examples (6.19) versus (6.20)—appears to be located in the pivot noun phrase. Replacing the noun phrases in existential examples (6.19a–j) by ones from non-existential examples (6.20a–h) yields additional non-existential examples, even if one respects singular/plural number agreement while substituting. This substitution is interesting if the sentence contains a coda in addition to the noun phrase after the copula, so that the result is distinct from already available examples. Replacing the pivot noun phrase in (6.19a) by *John* from (6.20d) yields (6.22a), and similar operations on (6.19f) and (6.20f) produce (6.22b).

- (6.22) a. #There is John here.  
b. #There are most angels coming.

Conversely, replacing the noun phrases in non-existential examples by pivot noun phrases from existential sentences yields further existential sentences if one respects number agreement. Compare (6.23a, b) with (6.20c, e).

- (6.23) a. There is a man in the room.  
b. There are a lot of books regarding this.

For this reason, linguists have directed attention to the issue of what noun phrases can follow the copula in an acceptable existential-there sentence. Jespersen (1943) said that these noun phrases were “indefinite”, or at least usually “more indefinite than ordinary subjects”. For brevity, we will call noun phrases that can occur as the pivot of an existential-there sentence *existentially acceptable*, and noun phrases that cannot occur as the pivot of an existential-there sentence *existentially unacceptable*.<sup>3</sup> An important question, then, is which noun phrases are existentially acceptable.

As Jespersen’s remark indicates, scholars have felt that the crucial feature distinguishing existentially acceptable noun phrases from existentially unacceptable ones is somehow related to what the phrases mean. While differences of meaning apparently

<sup>3</sup> Milsark (1977) introduced the even briefer terms *weak* for what we are calling “existentially acceptable” and *strong* for what we call “existentially unacceptable”. The formal definitions that Barwise and Cooper (1981) subsequently gave for these useful terms seem to have superseded Milsark’s original purely descriptive meaning. Thus we employ new descriptive terminology.

provide a basis for separating existentially acceptable from existentially unacceptable noun phrases, a syntactic superstructure seems to be overlaid on this semantic foundation. Keenan (1987) observed that Boolean combinations of existentially acceptable noun phrases are existentially acceptable (e.g. (6.24a)), but if a Boolean combination of noun phrases includes one that is existentially unacceptable, then the combined noun phrase as a whole is existentially unacceptable (e.g. (6.24b)).

- (6.24) a. There are a man and a lot of books here.  
 b. #There are two kinds of people and most angels.

Thus attention has focused on identifying the semantic characteristics of basic noun phrases that are existentially acceptable, setting them apart from existentially unacceptable basic noun phrases. While proper nouns and pronouns belong to the latter group, attention has focused mainly on determiners as crucial in discriminating between existentially acceptable and existentially unacceptable basic noun phrases.

Here are some English determiners that produce existentially acceptable noun phrases, and some others that produce existentially unacceptable noun phrases:

	<i>Exist acc det</i>	<i>Exist unacc det</i>
	no	neither
	a	the
	two	both
	some	every
(6.25)	several	the three
	few	John's
	a few	each
	many	most
	a lot of	all

Much longer lists can be found in Keenan 1987. The determiner *any* and the null determiner  $\emptyset$  that precedes bare plural nouns<sup>4</sup> are interesting in that they appear in both columns according to their semantic interpretation. Negative polarity *any*, discussed in Chapter 5.9, and the existentially interpreted null determiner  $\emptyset$  produce existentially acceptable noun phrases. By contrast, free choice *any*, found in *John will drink any wine*, and the universally interpreted null determiner  $\emptyset$  produce existentially unacceptable noun phrases.

In section 6.3.4 we compare four accounts of what the crucial semantic difference between these two groups of type  $\langle 1, 1 \rangle$  quantifiers may be. Note here, however, that certain noun phrases that lack a determiner should perhaps nevertheless be seen as falling into one or other of the semantic categories to which the determiners just tabulated belong. This seems particularly appropriate for basic noun phrases such as proper nouns and pronouns, which are existentially unacceptable and perhaps ought to be characterized as relevantly similar in meaning to noun phrases formed from the

<sup>4</sup> If such a determiner exists. Some linguists analyze bare plural noun phrases as containing a null determiner; others analyze such noun phrases as lacking any determiner.

definite article *the*.<sup>5</sup> Accomplishing this is not easy, however. The type ⟨1⟩ quantifiers  $I_j$  and *the*<sup>(j)</sup> are indeed identical; but both are identical to *some*<sup>(j)</sup>. Each of these quantifiers holds of a subset  $A$  of a universe  $M$  if and only if  $j$  is a member of  $A$ . Thus, when one encounters a claim to have explained the existential unacceptability of proper nouns and pronouns, it merits especially close scrutiny.<sup>6</sup>

Other noun phrases that might also be appropriate to categorize directly for existential acceptability include those in (6.26), which illustrate the observation of Johnsen (1987) that noun phrases with *only* or *mostly* preceding a bare plural are existentially acceptable.

- (6.26) a. There are only gray boxes in the photo.  
 b. There are only graves here. (Shujaat Bukhari, <http://www.hindu.com/2005/02/25/stories/2005022505200300.htm>)  
 c. In East Harlem, there are mostly Puerto Ricans.  
 d. There are mostly mosses and ferns on the forest floor.

As we pointed out in Chapter 4.5, n. 15, *only* seems not to be a determiner but instead a modifier that operates fairly generally on noun phrases—including proper nouns and pronouns (*only John* and *you*) as well as most other definite noun phrases (*only the Pope*), and existentially quantified noun phrases (*only an apple*), bare plurals, and certain other noun phrases.<sup>7</sup> Therefore, adding *only* to the left-hand column of (6.25) does not seem an adequate way to account for the fact that sentences like (6.26a, b) have an existential interpretation. In fact, whether the operator *only* and its synonym *just* should be treated as producing existentially acceptable noun phrases or, instead, like *not* and other Boolean operators, as simply preserving the existential acceptability of the noun phrases they operate on depends on whether *only John* and *only the Pope* are existentially acceptable, like *only freshmen* and *only a fly* are. We are inclined to the view that *only* preserves existential acceptability, but the question deserves more extensive investigation than it has received, with due attention to the status of examples such as (6.27).

- (6.27) Now she has left and there is only me to take care of you.  
 (<http://www.circlemagazine.com/issuethirtythree/monster.html>)

The word *mostly* is more plausibly a determiner, syntactically speaking, as it cannot precede any noun phrase other than a bare plural. If it is a determiner, though, it is a semantically troublesome one, for it is the exact inverse of *conservative*. That is,

<sup>5</sup> Furthermore, if bare plural noun phrases lack a determiner, they might be characterized as relevantly similar in meaning to noun phrases formed from the determiners *some* or *every*, according as their interpretation is existential or universal.

<sup>6</sup> E.g. the explanation attempted in Barwise and Cooper (1981) is simply that *John*, say, can be interpreted by *the*<sup>(j)</sup> and that *the* is positive strong (see sect. 6.3.4.2 below) and hence unacceptable. This explanation doesn't work, as we saw, since one could equally well use *some*<sup>(j)</sup>, and *some* is weak and hence acceptable.

<sup>7</sup> In fact, *only* operates much more widely than just on noun phrases—also on verb phrases, adjectives, and some determiners (e.g. *only a few*).

the type  $\langle 1, 1 \rangle$  quantifier  $Q$  that *mostly* might be thought to express satisfies the condition

$$(6.28) \quad Q_M(A, B) \iff Q_M(A \cap B, B)$$

which Keenan (2003) calls  $\text{CONS}_2$ , but it does not satisfy  $\text{CONSERV}$ . An alternative analysis, however, deserves serious investigation: that *mostly* is a quantificational adverb (see Chapters 0.1 and 10.2), not a determiner, and though it can attach to a noun phrase in sentences like

(6.29) The workforce now is mostly immigrants.

and (6.26c, d), the semantic interpretation remains as if the adverb were positioned with auxiliary verbs. Compare

- (6.30) a. In East Harlem, there mostly are Puerto Ricans.  
 b. There mostly are mosses and ferns on the forest floor.  
 c. The workforce now mostly is immigrants.<sup>8</sup>

Thus we are not inclined to put *mostly* or *only* or *just* in the left-hand column of (6.25).

### 6.3.3 What do existential-there sentences mean?

Before turning to an examination of proposals about what semantic property distinguishes determiners that produce existentially acceptable basic noun phrases from determiners that produce existentially unacceptable ones, we consider what the existential interpretation of there sentences is.

Existential-there statements intuitively differ in meaning from presentational there statements. The former assert or deny something's existence; the latter point out things whose existence has already been asserted. Instead of asserting or denying existence, existential-there sentences can also ask about it.

(6.31) Are there cockroaches in his house?

Presentational there sentences do not comfortably take the form of questions.

(6.32) Housing cockroaches in captivity poses two main problems.

\*Is there the problem of the cockroaches escaping?

Nor can presentational there sentences ordinarily be negated; (6.33) cannot even follow (6.21a(i)).

(6.33) \*There is not the problem of cockroaches escaping.

<sup>8</sup> Another determiner-like use of *mostly* that requires accounting for appears in

(i) At its peak the railroad yard employed several thousand workers, mostly immigrants.

where the appositive means "(they) mostly (were) immigrants"; i.e. most of the workers were immigrants.



First-order logic was nicely regimented by Frege and his successors to enable the expression of existence claims in the form  $\exists x P(x)$ , or more generally  $\exists x \varphi(x)$ , which states that  $\exists_M(\varphi(x)^{M,x})$  holds. Natural languages are more baroque in syntax, and possibly also in semantics. The English existential-there sentences (6.19), for example, do not simply state the non-emptiness of the set of things having a property. They appear instead to say something about a collection of sets of things: the extension of the type  $\langle 1 \rangle$  quantifier expressed by the pivot noun phrase. Let us approach the existential interpretation of these sentences in two steps, first considering existential-there sentences with just a pivot noun phrase, and then ones with a coda as well.

The existential interpretation of a coda-less there sentence [There be NP] with a simple pivot noun phrase  $\alpha$  is  $Q_M(M)$ , where  $[[\alpha]]_M = Q$ . When  $\alpha = [\text{Det Nom}]$ , this interpretation is equivalent to  $(Q_1)_M(A, M)$ , where  $[[\text{Det}]]_M = Q_1$  and  $[[\text{Nom}]]_M = A$ . The existential interpretation is, in effect, that NP exists. This straightforward rule of semantic interpretation, which Barwise and Cooper (1981) proposed, is motivated by the fact that the universe  $M$  comprises precisely the things that exist so far as the discourse is concerned. Thus

- (6.34) a. There are two kinds of people.  
 b. There are only two ways to live your life.  
 c. There are few natural opiates.

are true in a model  $\mathcal{M}$  of English just in case  $(Q^A)_M(M)$  holds, where  $Q^A$  is respectively

- (6.35) a. *two*<sup>kind of people</sup>  
 b. *only two*<sup>way to live your life</sup> <sup>9</sup>  
 c. *few*<sup>natural opiate</sup>

and similarly for any other sentence like the ones in (6.19) that contain only a pivot and no coda. As these type  $\langle 1 \rangle$  quantifiers each derive from a type  $\langle 1, 1 \rangle$  quantifier by fixing its restriction set, an equivalent semantic analysis of these existential-there sentences is as stating that  $Q_M(A, M)$  holds,  $Q$  and  $A$  being as immediately above. For sentences with only a pivot NP and no coda, this simple idea works well.

What, then, do existential-there sentences containing a coda mean? As Keenan (1987) pointed out, the existential interpretation of a sentence [There be NP Coda] with pivot noun phrase  $\alpha = [\text{Det Nom}]$  and coda  $\beta$  is  $Q_M(A \cap B, M)$ , where  $[[\text{Det}]]_M = Q$ ,  $[[\text{Nom}]]_M = A$  and  $[[\beta]]_M = B$ . Thus

- (6.36) a. There is a man here.  
 b. There are people coming.

mean respectively that a man who is here exists and that people who are coming exist. This idea works quite generally for existential-there sentences containing a coda, regardless of what existentially acceptable noun phrase occurs as the pivot. The coda

<sup>9</sup> For present purposes, it suffices to regard *only two* as *exactly two*.

joins the pivot's nominal phrase in restricting the quantifier's domain. Note that the meaning is not affected by whether  $\beta$  is a post-nominal modifier in the pivot noun phrase or instead is a separate coda.

These existential interpretations of sentences (6.19) are very different from the presentational interpretations that sentences such as (6.20a, d) can be given. For instance, (6.21a(ii)) can be paraphrased by (6.37a), but not by (6.37b).

- (6.37) a. The problem of the cockroaches escaping is one of the two main problems posed by housing cockroaches in captivity.  
b. The problem of the cockroaches escaping exists.

Similarly, (6.21b(ii)) means roughly (6.38a), and not (6.38b).

- (6.38) a. John is available to walk the dog.  
b. John exists.

An immediate question about the existential interpretation of these sentences containing a coda is how the coda comes to restrict the quantifier in the pivot noun phrase, which is obviously difficult to accomplish with a compositional rule of semantic interpretation. Barwise and Cooper did not address this question, but Keenan proposed an ingenious solution. His semantic rule compositionally interprets existential sentences with a coda by taking the coda to be the scope of the quantification expressed by the pivot noun phrase, i.e. as  $(Q^A)_M(B)$  or  $Q_M(A, B)$ .<sup>10</sup> Then Keenan notes that this compositionally derived proposition is equivalent to the one discussed two paragraphs above, i.e. that  $Q_M(A, B) \Leftrightarrow Q_M(A \cap B, M)$ , when the pivot noun phrase's determiner is one that produces existentially acceptable noun phrases. Thus sentences (6.36) compositionally receive the existential interpretation as meaning that

- (6.39) a.  $some_M(man \cap here, M)$   
b.  $some_M(person \cap coming, M)$

precisely because these propositions are in fact equivalent to the following ones:

- (6.40) a.  $some_M(man, here)$   
b.  $some_M(person, coming)$ <sup>11</sup>

Keenan (1987) termed type  $\langle 1, 1 \rangle$  quantifiers *Q existential* just in case they satisfy the equivalence

- (6.41)  $Q_M(A, B) \Leftrightarrow Q_M(A \cap B, M)$

<sup>10</sup> In the case of coda-less there sentences, the existential interpretation is  $(Q^A)_M(M)$  or  $Q_M(A, M)$  as Barwise and Cooper had it.

<sup>11</sup> As Milsark (1977) argued should follow from an adequate semantic analysis of existential-there sentences, an additional consequence of the fact that existential-there sentences select for the existential interpretation of bare plurals, in contrast to bare plurals' universal interpretation, is that predicates that force bare plurals to be interpreted in terms of universal quantification cannot occur as the coda of existential-there sentences: #There are people generous—cf. People are generous versus People are coming.

for all  $A, B \subseteq M$ , and hypothesized that the determiners that express existential quantifiers, in this sense, produce existentially acceptable basic noun phrases. For reference in the next section we remark that, as Keenan (2003) showed, the existential quantifiers are exactly the intersective quantifiers, which are in turn exactly the quantifiers that satisfy both CONSERV and CONS<sub>2</sub>.

**Fact 5**

Let  $Q$  be any type  $\langle 1, 1 \rangle$  quantifier. The following conditions are equivalent:

- (a)  $Q$  is existential (as defined in (6.41)).
- (b)  $Q$  satisfies INT.
- (c)  $Q$  satisfies both CONSERV and CONS<sub>2</sub> (as defined in (6.28)).

*Proof.* Proving the required equivalences is straightforward, and resembles similar calculations in section 6.1 (e.g., in the proof of Fact 1). As an example, let us verify that CONSERV and CONS<sub>2</sub> together imply INT. If  $A, B \subseteq M$  we have:  $Q_M(A, B) \Leftrightarrow Q_M(A, A \cap B)$  (by CONSERV)  $\Leftrightarrow Q_M(A \cap (A \cap B), A \cap B)$  (by CONS<sub>2</sub>)  $\Leftrightarrow Q_M(A \cap B, A \cap B)$ , and this condition is equivalent to INT.  $\square$

In fact, as Keenan (2003) also observed, this fact holds of all monadic quantifiers  $Q$  under the natural generalization of (6.41) as

$$(6.42) \text{ A monadic quantifier } Q \text{ is existential iff} \\ Q_M(A_1, \dots, A_k, B) \Leftrightarrow Q_M(A_1 \cap B, \dots, A_k \cap B, M) \\ \text{holds for all } M \text{ and } A_1, \dots, A_k, B \subseteq M$$

of INT as

$$\text{If } A_1 \cap B = A'_1 \cap B' \ \& \ \dots \ \& \ A_k \cap B = A'_k \cap B', \text{ then} \\ Q_M(A_1, \dots, A_k, B) \Leftrightarrow Q_M(A'_1, \dots, A'_k, B')$$

(repeated from Chapter 4.7) and of CONS<sub>2</sub> as

$$\text{If } A_1 \cap B = A'_1 \cap B \ \& \ \dots \ \& \ A_k \cap B = A'_k \cap B, \text{ then} \\ Q_M(A_1, \dots, A_k, B) \Leftrightarrow Q_M(A'_1, \dots, A'_k, B)$$

In the next section, we assess the empirical success of Keenan's hypothesis. Here, though, we mention that one should not confuse the semantic equivalence that Keenan takes as his definition of existential quantifiers with the subtly different claim that existential-there sentences and corresponding subject–predicate sentences are fully synonymous. Syntactic differences can affect how the meanings of the pivot/subject noun phrase and other parts of the sentences are allowed to compose semantically. For example, the negation in (6.43a) is ambiguous as to scope,<sup>12</sup> whereas negation can only take wide scope (i.e. outer negation of the quantification) in (6.43b).

<sup>12</sup> For many speakers, although some speakers strongly prefer narrow scope, i.e. inner negation of the quantifier.

- (6.43) a. A lot of horses weren't loose.  
 b. There weren't a lot of horses loose.<sup>13</sup>

So interpreting existential-there sentences as predicating the coda of the pivot NP does not amount to the claim that these sentences have exactly the same interpretations as the corresponding subject–predicate sentences.

In closing this section, we observe that it has also been proposed to assign meaning to existential-there sentences by interpreting *there* as the quantifier  $\exists$  and interpreting the copula *be* in a way that Quine suggested, and Montague adopted: as predicating a type  $\langle 1 \rangle$  quantifier of an individual, i.e. as mapping the quantifier  $Q$  to the set  $\{a : Q(\{a\})\}$ . This approach interprets a codaless sentence [*There is/are Det A*] as  $\exists_M(\{a \in M : (Q^A)_M(\{a\})\})$ , where the interpretation of *Det* is the type  $\langle 1, 1 \rangle$  quantifier  $Q$ . One might wonder wherein the appeal of this approach lies; English appears to have settled on *there* and *be* for talking about existence by an accident of history, but the closely related languages German and Swedish use cognates of *it* and the verbs for 'give' and 'exist' instead. Other languages depart even further from the English mold. That worry aside, we note that this semantic interpretation is correct only for pivot noun phrases with the determiners *a*, *one*, and singular *some*. Almost every determiner that is allowed in the pivot noun phrase of an English existential-there sentence expresses a type  $\langle 1, 1 \rangle$  quantifier  $Q$  for which  $\exists_M(\{a \in M : (Q^A)_M(\{a\})\})$  is true under different conditions than the existential sentence is. For the determiner *few*, for instance,  $\exists_M(\{a \in M : few_M(A, \{a\})\})$  is always true, though most English existential sentences *There are few A*s are not tautologous. In the case of the determiner *no*, the sentence *There is no A* is assigned truth conditions that should be assigned to *Not everything is an A*. When the determiner is *two*, *three*, and other numbers, *several*, *many*, and other permitted determiners, the assigned truth conditions never hold, though English existential sentences with these determiners are not generally self-contradictory. For this reason, advocates of this approach usually abandon first-order quantification and resort to quantifying over sets of individuals. It is then possible at least to state the correct interpretation  $Q_M(A \cap B, M)$  for a sentence of the form [*there is/are Det A B*], for instance by interpreting it as:

for some  $X \subseteq M$ , ( $X = A \cap B$  &  $Q_M(X, M)$ )

In the process, the Quine/Montague idea of *be* forming a predicate from a type  $\langle 1 \rangle$  quantifier must, however, be completely abandoned. We will not discuss these approaches in detail.

<sup>13</sup> Other semantic differences also correlate with the difference in syntactic structure. For example, the adverb *usually* can be interpreted as a resumptive quantifier or a temporal adverb in (i)a, but it can be only a temporal adverb in (i)b.

- (i) a. A visitor is usually twittering in a basket.  
 b. There is usually a visitor twittering in a basket.

These and other differences are clearly discussed in Kim 1997.

### 6.3.4 Four approaches to distinguishing between existentially acceptable determiners and existentially unacceptable ones

Having addressed the question of what the existential interpretation of these sentences is, we now assess four proposals for characterizing the existentially acceptable noun phrases, and explaining why just these noun phrases are existentially acceptable.

#### 6.3.4.1 Milsark 1977

Milsark (1977) pointed out that the traditional characterization of existentially acceptable noun phrases as those that are not definite fails to capture the facts adequately. For instance, *most people* is not definite, but is nevertheless existentially unacceptable. In examining which determiners form existentially acceptable noun phrases, Milsark came to the conclusion that these determiners do not express quantifiers; instead, he proposed, they are what he termed *cardinality words*, which “do nothing more than express the size of the set of entities denoted by the nominal with which they are construed” (Milsark 1977: 23). This hypothesis plays a crucial role in his account of why only they are capable of forming existentially acceptable noun phrases. He supposes that *there be* expresses existential quantification, and semantic rules cannot combine this quantifier with the meaning of any pivot noun phrase whose determiner expresses a quantifier; otherwise the result would be “double quantification of the set denoted by the NP”, which “should certainly be expected to be anomalous under any semantics which makes use of the notion quantification at all” (pp. 24–5). Accordingly, violations “are ruled out of the language by reason of their uninterpretability” (p. 24).

Milsark’s approach to existential-*there* sentences is an important advance over previous approaches, which typically treated the strings in (6.20) as syntactically ill-formed. Milsark’s proposal that these strings are instead syntactically well-formed sentences that are semantically incapable of receiving an existential interpretation fits better with the fact that many of them clearly do have other interpretations. His explanation of existential acceptability in terms of filtering at the syntax/semantics interface appears non-standard from our perspective, both in treating *there be* as expressing quantification and in denying that determiners in the left-hand column of (6.25) express quantifiers. In actuality, he did not present a detailed account of the truth conditions of existential-*there* sentences.<sup>14</sup> Nevertheless, Milsark was the first not only to address systematically the inadequacies of the traditional ‘definiteness’ hypothesis about existentially acceptable noun phrases, but also to essay an explanation of why certain predicates cannot occur as the coda of existential-*there* sentences. And his idea that existentially acceptable determiners are cardinality words strongly

<sup>14</sup> His approach to doing so would apparently have employed existential quantification over sets, asserting that some subset of the coda’s denotation has as its intersection with the set denoted by the pivot NP’s nominal a set whose size is expressed by the pivot NP’s determiner. He asserted that *There are some people in the bedroom* “says nothing more than that the bedroom contains an unspecified number of objects meeting the description ‘people’” (pp. 19–20). This is yet another second-order formulation that is equivalent to  $Q_M(A \cap B, M)$ .

influenced Keenan, who observed that these very determiners actually do denote quantifiers that are cardinal (see definition (6.12) in section 6.1).

#### 6.3.4.2 Barwise and Cooper 1981

Barwise and Cooper (1981) offered a different account of existential acceptability, the first attempt we know of to explain the range of determiners occurring in existentially acceptable noun phrases by means of pragmatic considerations in combination with semantic properties of the determiners. They placed no restriction on the semantic rule of existential interpretation for these sentences, in contrast to Milsark's essentially structural approach. Instead, Barwise and Cooper assign all these sentences an existential meaning, but they noted that this meaning is not communicatively useful when the pivot noun phrase is the sort we have called existentially unacceptable.

The pragmatic idea underlying Barwise and Cooper's proposal relies on a very general principle about the use of language to communicate which Grice (1967) encapsulated in the maxim *Be informative*. That is, make your statement one that adds information to what was already agreed by the speaker and hearer.<sup>15</sup>

Barwise and Cooper simply pointed out that an existential-there statement could not be informative if the determiner of the pivot noun phrase meant something that in itself guaranteed that the statement would be true—or if the determiner's meaning guaranteed that the statement would be false. They called the former kind of determiners positive strong; the latter kind they called negative strong. Their hypothesis was that all determiners are existentially acceptable, except the strong ones (positive or negative), and they termed the remaining determiners *weak*.<sup>16</sup> Thus, the existentially acceptable determiners are those for which an existential-there statement could be true or could be false, insofar as the determiner itself contributes to determining the there sentence's meaning; to this extent, the statement has the possibility of being informative.

Barwise and Cooper's precise definition of positive strong depends crucially on a possibility that the relational treatment of quantifiers does not allow for: leaving a quantifier *undefined* when restricted to certain subsets of the universe of discourse.<sup>17</sup> They defined a type  $\langle 1, 1 \rangle$  quantifier  $Q$  as *positive strong* if and only if  $Q_M(A, A)$  is true for all universes  $M$  and all subsets  $A$  of  $M$  such that  $Q_M(A, \cdot)$  is defined. This enabled Barwise and Cooper to discriminate between existentially unacceptable determiners such as the one or more and existentially acceptable some. Both *the one or more* $_M(A, A)$  and *some* $_M(A, A)$  are true for every non-empty subset  $A$  of  $M$ ; but Barwise and Cooper say that *the one or more* $_M(\emptyset, \emptyset)$  is undefined, and *some* $_M(\emptyset, \emptyset)$  is false. The relational view of quantifiers cannot distinguish the two quantifiers in this

<sup>15</sup> Grice's maxim, and its motivation, extend naturally to the asking of questions: Ask a question only if the answer is not entailed by what is already agreed.

<sup>16</sup> On p. 183 they explicitly claim to explain why all strong determiners, positive and negative, are unacceptable in pivot NPs of existential-there sentences. Their brief two-page discussion does not actually state that all weak determiners are acceptable, though no reason to doubt that they thought this appears in their discussion.

<sup>17</sup> For ISOM quantifiers, on finite sets this amounts to allowing some levels of the number triangle to be entirely blank, while other levels have either a + or a – at each position.

manner, as it treats *the one or more*<sub>M</sub>( $\emptyset, \emptyset$ ) as false. They similarly defined a type  $\langle 1, 1 \rangle$  quantifier  $Q$  as *negative strong* if and only if  $Q_M(A, A)$  is false for all universes  $M$  and all subsets  $A$  of  $M$ .<sup>18</sup> Exploiting partial definedness of quantifiers in the way Barwise and Cooper do seems somewhat in tension with the pragmatic underpinnings of their explanatory strategy; a statement such as *The one or more solutions of this equation are complex numbers* might be informative precisely in telling the listener that the equation has one or more solutions.

Although they interpreted all existential-there sentences as if coda-less, their approach could be extended to existential-there sentences with codas at the cost of stipulating that *There is/are* Det  $A B$  has the truth conditions  $Q_M(A \cap B, M)$ , when Det expresses  $Q$ , as discussed in the preceding section. The reason why each there sentence containing a conservative positive strong determiner would be necessarily true if interpreted existentially, and thus could not constitute an informative statement, is that  $Q_M(A, A) \Leftrightarrow Q_M(A, M)$  for all  $A \subseteq M$  when  $Q$  is conservative. There sentences containing a conservative negative strong determiner could not constitute informative statements if interpreted existentially because they would be necessarily false, and thus could not convey information (as contrasted with misinformation).

This proposal indeed predicts that sentences like *There is the problem of the cockroaches escaping*, *There is every book under the bed*, and *There is neither parent in the room* are not existentially informative. Modulo acceptance of the use of partially defined quantifiers to discriminate positive strong both from weak *exactly two*, positive strong *the five* from weak (*exactly*) *five*, etc., it also accounts for the fact that quantifiers like those in the pivot noun phrase of sentences (6.20) are not existentially informative and, accordingly, are existentially unacceptable in there sentences. Finally, it accounts for the fact that weak quantifiers—i.e. quantifiers that are neither positive nor negative strong, such as those in (6.19)—are existentially informative and thus acceptable in there sentences. We return below to the question of whether resorting to partiality of quantifiers as this approach does is really necessary in order to distinguish existentially acceptable from unacceptable determiners.

A problem which Keenan (1987) pointed out is that the determiners at least zero and fewer than zero are existentially acceptable. These degenerate cases violate the hypothesis of Barwise and Cooper's analysis, as determiners that always produce a trivial quantifier  $1_M$  or  $0_M$  can never be informative.

Despite its limitations, Barwise and Cooper's analysis sparked a surge of interest in explaining precisely which noun phrases are existentially acceptable with the aid of model-theoretic semantics and some kind of general principles.

<sup>18</sup> Barwise and Cooper treated some negative strong quantifiers as partial; however, negative strong quantifiers present no difficulties for the relational approach. Such determiners do not actually require partial definedness, because on any view,  $Q_M(A, A)$  is defined whenever it is false. Thus the relational stratagem of replacing undefined by false does not affect the set of quantifiers picked out by the definition of negative strong. Consider e.g. the negative strong determiner *neither*. When *neither*<sub>M</sub>( $A, B$ ) is relationally defined as  $|A| = 2 \ \& \ A \cap B = \emptyset$ , then *neither*<sub>M</sub>( $A, A$ ) is false whether or not  $|A| = 2$ . This contrasts, as it should, with *no*<sub>M</sub>( $A, A$ ), which is true when  $A = \emptyset$  and false for non-empty subsets  $A$  of  $M$ .

## 6.3.4.3 Keenan 1987

Keenan's approach differs from Barwise and Cooper's in interesting ways. Although Keenan (1987) always assigns these sentences a meaning—often, though not always, the same one they assigned—his account of existential acceptability rests not on the pragmatics of communication but rather on the requirement that semantic interpretation be compositional. Producing the existential interpretation compositionally is not trivial when a there sentence contains a coda. In fact, only for certain quantifiers is the interpretation that Keenan compositionally assigns to a there sentence with a coda equivalent to the sentence's existential interpretation. Which quantifiers these are is an immediate consequence of Fact 5 in its general version: A there sentence whose pivot noun phrase's determiner denotes an intersective quantifier quite simply *has* the existential interpretation; no resort is needed to non-compositional shenanigans in order to construct this meaning. If, however, the quantifier is not intersective, the sentence's interpretation differs from the existential interpretation, which cannot be assigned by compositional means. This is the essence of Keenan's idea.

The element of non-compositionality, which presumably lies behind Keenan's analysis of existential-there sentences with codas, as well as Barwise and Cooper's neglect of codas, is in fact a consequence of the following proposition.

**Proposition 6**

*The relation  $\{(Q^A, B, Q^{A \cap B}) : Q \text{ a type } \langle 1, 1 \rangle \text{ quantifier, } A, B \text{ sets}\}$  is not single-valued, i.e.  $Q^{A \cap B}$  is not a function of the first two arguments.*

*Proof.* Let, for example,  $Q$  be *at least two of the three*, and suppose further that  $|A| = 4$ ,  $|A'| = 5$ ,  $|A \cap B| = 3$ , and  $|A' \cap B| = 4$ . Then  $Q^A = Q^{A'}$  = the trivially false quantifier.  $Q^{A \cap B}$  is also trivially false, but  $Q^{A' \cap B}$  is not, so  $Q^{A \cap B} \neq Q^{A' \cap B}$ . This proves the result.  $\square$

Keenan's approach places great weight on there sentences containing a coda; in coda-less there sentences, no pivot noun phrase poses any difficulty for compositionally assigning the existential interpretation. Only in the presence of a coda does the semantic difference between quantifiers satisfying Keenan's definition (6.42) and other quantifiers manifest its importance for the ability to obtain the existential interpretation compositionally. Keenan 'grammaticalizes' this difference, treating quantifiers that do not satisfy (6.42) as existentially unacceptable in coda-less there sentences as well as in ones with a coda. Grammaticalization of a semantically motivated distinction seems both necessary and appropriate in dealing with well-formedness of strings of a given form, or the availability of a particular semantic interpretation for such strings.<sup>19</sup>

<sup>19</sup> Barwise and Cooper, e.g., would presumably say that the pivot noun phrase in

(i) There is no square table that is not square.

is existentially acceptable, although sentence (i) is necessarily true.



Having identified a semantic characteristic that separates basic determiners that yield existentially acceptable noun phrase from those that produce existentially unacceptable noun phrases, Keenan proposed that the existentially acceptable noun phrases are exactly the Boolean combinations of noun phrases built up from Boolean combinations of existential determiners, in the sense of (6.42).<sup>20</sup> Note that all the cardinal quantifiers, the ones expressed by Milsark's "cardinality words", are intersective, and thus are predicted to yield existentially acceptable noun phrases.

All type  $\langle 1, 1 \rangle$  quantifiers that are existential must be intersective, as shown in Fact 5, and hence symmetric (Fact 1). Thus Keenan's analysis predicts straightforwardly that non-symmetric determiners like *each/every/all*, *the* and *the k*, *both* and *neither*, *most*, and *more than/fewer than p q' <sup>ths</sup> of the* are not allowed in existentially acceptable noun phrases. No recourse is taken to partially defined quantifiers in excluding any of these determiners.

On the other hand, the always trivial positive and negative strong determiners at least zero and fewer than zero are intersective, and thus Keenan's analysis allows them in existentially acceptable noun phrases. It is natural to ask whether Keenan's analysis admits any non-trivial quantifiers into existentially acceptable noun phrases which Barwise and Cooper's does not, or vice versa.

The answer to this simple question seems not to be understood in the literature, and part of the confusion rests on Barwise and Cooper's use of partially defined quantifiers. In fact, they themselves state, in Appendix C (Theorem C6), that non-trivial symmetric quantifiers cannot be positive or negative strong. But in that appendix they do *not* use partiality; instead, they define  $Q$  to be positive strong iff for *all*  $A$ ,  $Q(A, A)$ . Let us use *positive strong<sub>t</sub>* for this notion, and *positive strong<sub>p</sub>* for the notion defined above (their official definition in section 4.6 of Barwise and Cooper 1981).<sup>21</sup> As we noted, the official version is their only means of preventing quantifiers like *the ten* from being weak, and hence wrongly predicted to be acceptable in existential-there sentences. Nevertheless, the part of Barwise and Cooper's claim which concerns positive strong quantifiers relies on using the simplified notion in the appendix. Let us state this explicitly.

#### Fact 7

*Assuming CONSERV: If  $Q$  is non-trivial and symmetric, it cannot be positive strong<sub>t</sub> or negative strong.*

*Proof.* By non-triviality, there are  $A, B \subseteq M$  and  $A', B' \subseteq M'$  such that  $Q_M(A, B)$  and not  $Q_{M'}(A', B')$ . By symmetry and CONSERV, we have  $Q_M(A \cap B, A \cap B)$  and not  $Q_{M'}(A' \cap B', A' \cap B')$ . It follows directly that  $Q$  is neither negative strong nor positive strong<sub>t</sub>.  $\square$

<sup>20</sup> Thus, Keenan claims, proper nouns are not existentially acceptable, because they do not have a determiner. In actual fact, he allowed not only Boolean combinations of existential determiners, but also combinations of them with adjective phrases or with exception phrases.

<sup>21</sup> As we saw (n. 18), there is no need for a similar distinction in the negative strong case.

So if “weak” meant ‘neither negative strong nor positive strong’, then every non-trivial quantifier deemed acceptable by Keenan would also be acceptable according to Barwise and Cooper. But the converse would fail dramatically, since non-symmetric quantifiers like *the ten* are also weak in this sense.

On the other hand, if “weak” means ‘neither negative strong nor positive strong<sub>p</sub>’, as in Barwise and Cooper’s official version, the facts change, as we now show.

**Proposition 8** (CONSERV, EXT, ISOM, FIN)

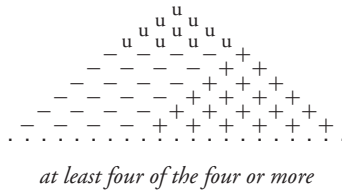
*The non-trivial and symmetric quantifiers which are positive strong<sub>p</sub> are exactly those of the form at least k of the k or more, for some k ≥ 1.*

*Proof.* Because of the assumptions, we can argue in the number triangle. As noted, a partially defined quantifier is represented just as a total one, except that some levels may be (completely) undefined. Now it is easy to show (see section 6.4 below) that symmetry means that  $(0, k) \in Q$  iff  $(n, k) \in Q$ , for all  $k, n$  (this is clear from the fact that symmetry is equivalent to the condition  $Q(A, B) \Leftrightarrow Q(A \cap B, A \cap B)$ ). Thus, if we call lines  $(0, k), (1, k), (2, k), \dots$  *columns*, a symmetric quantifier consists simply of a bunch of columns. Note that for such a quantifier, if  $(0, k)$  is in  $Q$  (and hence so is the corresponding column),  $Q$  must be defined at all levels  $n \geq k$ .

The quantifier mentioned in the proposition is

$$Q(A,B) \Leftrightarrow \begin{cases} |A \cap B| \geq k, & \text{if } |A| \geq k \\ \text{undefined,} & \text{if } |A| < k \end{cases}$$

So with  $k = 4$  it looks as follows (where u stands for undefined).



We see directly that quantifiers of this form are symmetric and positive strong<sub>p</sub>. Conversely, if  $Q$  is non-trivial, symmetric, and positive strong<sub>p</sub>, let  $k$  be the smallest number such that  $(0, k) \in Q$ . By non-triviality,  $k$  exists and is  $> 0$ . Since  $Q$  is positive strong<sub>p</sub>,  $(0, n) \in Q$  for  $n \geq k$ , whereas the levels below  $k$  must be undefined. By symmetry, it follows that  $Q$  has the form indicated in the diagram. □

Thus, whichever version of positive strong one uses, Barwise and Cooper’s account of acceptability in existential-there sentences diverges from Keenan’s (1987) account. With the total version, *the ten*, etc. become weak and hence (wrongly) predicted to be acceptable, but they are not symmetric, so Keenan (correctly) rules them out.

The case of *most* and other strict proportional quantifiers like *more than two-thirds of the* deserves a special comment. Barwise and Cooper in fact treat *most* as total, and declare that  $most(\emptyset, B)$  is always *true*. In this way, *most* becomes positive strong, even

if total. No further motivation is given for this decision, and it is not stated whether the same stipulation applies to, for example, *more than two-thirds of the*. To us, it seems more natural to say that *if* one allows partial quantifiers, then *most*, *more than two-thirds of the*, etc. are undefined when the restriction is empty (and hence still positive strong<sub>p</sub>), but *if not*, they are *false* on the empty set (in which case, however, they become weak). Indeed, that is how we have defined the strict proportional quantifiers in this book.<sup>22</sup> On this view, the only way to prevent *most* from being existentially acceptable by Barwise and Cooper's criterion is to make it partial.

The partial version thus gets these empirical facts right, but weakness still differs from symmetry, and we isolated in Proposition 8 exactly the quantifiers for which this happens. But now we must ask: Do the corresponding determiners exist in English, and what is their actual behavior in existential-there sentences? Actually, although things like *at least two of the two or more* etc. are somewhat clumsy, they can certainly be used, e.g.:

(6.44) ... wherein at least two of the two or more conductivity type layers with low electrical resistance include scattered sources of impurity of a first conductivity type ... (<http://patents.nimblewisdom.com/patent/6551909-Semiconductor-device-with-alternating-conductivity-type-layer-and-method>)

As to their acceptability in existential-there sentences, note first that, plausibly, there ought to be no empirical difference in this respect between determiners of the form *at least k of the k or more* and those of the form *at least m of the k or more* ( $0 < m < k$ ). But whereas both of these are positive strong<sub>p</sub>, only the former (where  $m = k$ ) are symmetric. So if any of these determiners are acceptable in existential-there sentences, Barwise and Cooper's account is going to have major problems. And if any of the latter are existentially acceptable, Keenan's account will have equally major problems. If none were acceptable, Barwise and Cooper would have the upper hand. But it seems in fact that determiners of these and similar forms are sometimes acceptable, giving trouble to *both* accounts.

- (6.45) a. Just from my point of view of the proceedings last Tuesday, I believe that there are at least two of the five supervisors that favor the DRMP proposal. . . . ([www.vmsc.org/forum/post.asp?method=TopicQuote&TOPIC\\_ID=233&FORUM\\_ID=4](http://www.vmsc.org/forum/post.asp?method=TopicQuote&TOPIC_ID=233&FORUM_ID=4))
- b. There are no more than five of the 120 seats in the legislature and perhaps only three of the 53 seats in Congress where the outcome of an election

<sup>22</sup> Robin Cooper (personal communication) indicates that another motivation was the desire to preserve the intuition that *every(A, B)* implies *most(A, B)*. On the other hand, another intuition is that *most(A, B)* and *most(A, C)* jointly imply *some(A, B ∩ C)*; this is lost on Barwise and Cooper's construal of *most*. Besides, it took logicians 2,000 years to eventually realize that the intuition that *all(A, B)* entails *some(A, B)* is in fact best explained in terms of pragmatic rather than semantic considerations. As we saw in Ch. 4.2, intuitions about what determiners mean when the restriction is empty are notoriously unreliable.

Yet another countervailing intuition—at least as strong as that *every* entails *most*—is that *most* and *most of the* have the same truth conditions and differ only in that the latter requires an 'antedecedent' like all definite determiners containing *the* (see sect. 7.11.3, esp. Fact 3).

seems uncertain. ([www.economist.com/surveys/displayStory.cfm?Story\\_id=2609383](http://www.economist.com/surveys/displayStory.cfm?Story_id=2609383))

- c. There were at least two of the remaining four that still had me squirming when I had to make the choice. . . ([www.millarworld.biz/lofiversion/index.php/t37752-50.html](http://www.millarworld.biz/lofiversion/index.php/t37752-50.html))

These examples thus point to a family of *complex* determiners that appear to create serious difficulties for Barwise and Cooper's and Keenan's accounts. In fact, other complex determiners are problematic as well; we come back to this in section 6.3.5 below.

Keenan (1987) points to another possible divergence between the two accounts: namely, for the type  $\langle 1, 1, 1 \rangle$  quantifiers *more than* and *fewer than*, which meet his definition of existential determiner, and do in fact produce existentially acceptable noun phrases.

- (6.46) a. There are more freshmen than sophomores in that class.  
b. There are fewer joggers than runners on the track today.

He suggests that when Barwise and Cooper's definition of negative strong is extended to this type of quantifier, these quantifiers would fall under it, which would be quite bad for their analysis. Similarly, *as many as* meets Keenan's definition of existential, but would, he suggests, count as positive strong under an extension of Barwise and Cooper's definition to type  $\langle 1, 1, 1 \rangle$ . Yet it too yields existentially acceptable noun phrases.

However, Keenan's critique seems to us mistaken, as his suggested extension of positive and negative strong is incorrect. Actually, Barwise and Cooper mean by positive strong 'always true (when defined)', and for conservative type  $\langle 1, 1, 1 \rangle$  quantifiers, like these, this means that  $Q_M(A, B, A \cup B)$  holds for all  $A, B \subseteq M$ . Similarly, negative strong means  $Q_M(A, B, A \cup B)$  fails for all  $A, B \subseteq M$ . It is not sufficient to check merely that  $Q_M(A, A, A)$  always holds or never holds, as Keenan suggested. In fact, all three of these existentially acceptable type  $\langle 1, 1, 1 \rangle$  quantifiers count as weak in Barwise and Cooper's sense.<sup>23</sup>

Returning to the type  $\langle 1, 1 \rangle$  case, we have seen that whereas Keenan makes symmetry the sole requirement for existential acceptability,<sup>24</sup> Barwise and Cooper do not impose this condition. In order to stop the asymmetric proportional quantifiers from being weak, they had to say either that these quantifiers are undefined or that they hold vacuously when restricted to the empty set. This is a serious commitment on their part, as behavior on the empty set is the only thing that makes *no*, *a*, *some*, and *at least one* weak. So nothing fundamental requires weak type  $\langle 1, 1 \rangle$  quantifiers to be symmetric. For instance, if *many* and *few* are ambiguous between intersective and proportional readings, Keenan's analysis predicts that only their intersective reading

<sup>23</sup> For example, since  $\text{more\_than}(A, B, C) \Leftrightarrow |A \cap C| > |B \cap C|$ , we have  $\text{more\_than}(A, B, A \cup B) \Leftrightarrow |A| > |B|$ , which may be true or false, and similarly for the other examples.

<sup>24</sup> Symmetry implies intersectivity for any type  $\langle 1, 1 \rangle$   $Q$  that satisfies either CONSERV or CONS<sub>2</sub>, and symmetry is always implied by intersectivity.

can occur in existential-there sentences, while Barwise and Cooper's analysis makes such a prediction only if proportional *many* and *few* are strong. Proportional *few* is weak just in case  $\text{few}(\emptyset, \emptyset)$  is deemed to hold, because 100 percent is too large a proportion of a non-empty set to constitute few of its members (proportionally speaking). Proportional *many* is weak just in case it enforces an 'absolute' lower bound on  $|A \cap B|$  (i.e. one that is not a percentage of  $|A|$ ), which would render  $\text{many}(A, A)$  false when  $|A|$  is too small to reach this bound—because otherwise 100 percent is a big enough proportion to constitute many. Thus, whether Barwise and Cooper's analysis predicts existential acceptability of the proportional interpretations of *many* and *few* is open to stipulation in accord with one's intuitions about whether  $\text{many}(A, A)$  and  $\text{few}(A, A)$  are true or not when  $A = \emptyset$ .<sup>25</sup>

What are the empirical facts? Many linguists say *many* and *few* must be interpreted intersectively in the pivot noun phrase of existential-there sentences. If this is correct, Keenan's analysis succeeds in predicting correctly that English forgoes every single one of the many possible opportunities for one or another non-symmetric quantifier to appear in existential-there sentences (except the complex determiners in (6.45)). By contrast, Barwise and Cooper's analysis hardly makes such a prediction, though it is fully consistent with the facts about existential unacceptability of non-symmetric quantifiers (apart from the exceptional determiners in (6.45)), because it allows one to stipulate individually for each non-symmetric quantifier that it is undefined on the sets that would otherwise make it weak. There is, in fact, an embarrassingly large number of quantifiers for which such a stipulation would be needed. But are the facts about existential acceptability of *many* and *few* as is commonly said?

In section 6.2, we saw that there are cases where *many* and *few* appear to have a proportional interpretation.

- (6.47) a. Many plutocrats are Americans.  
 b. Few scoundrels are Nobel Laureates.  
 c. Many American Presidents are millionaires.
- (6.48) a. Many Americans are plutocrats.  
 b. Few Nobel Laureates are scoundrels.  
 c. Many millionaires are American Presidents.

We may accordingly ask whether each of these sentences is equivalent to the corresponding sentence in (6.49) and (6.50).

- (6.49) a. There are many plutocrats who are Americans.  
 b. There are few scoundrels who are Nobel Laureates.  
 c. There are many American Presidents who are millionaires.

<sup>25</sup> Barwise and Cooper are somewhat ambivalent concerning *many* and *few*, and do not distinguish proportional from intersective readings, but what they say is consistent with the view that these determiners have both readings, and that the proportional reading too is weak.

- (6.50) a. There are many Americans who are plutocrats.  
 b. There are few Nobel Laureates who are scoundrels.  
 c. There are many millionaires who are American Presidents.

To us this seems in fact to be the case, suggesting that Keenan's characterization of existentially acceptable determiners as intersective is actually too narrow. However, Barwise and Cooper's analysis is consistent with these facts. We return to a comparison of the two analyses in the final part of this section.

#### 6.3.4.4 Keenan (2003)

Keenan was seriously concerned about the fact, which Johnsen (1987) pointed out, that sentences like

- (6.51) a. There are only tradeoffs.  
 b. There are mostly bad options.

have existential interpretations—because his 1987 analysis firmly committed him to existentially acceptable type  $(1, 1)$  quantifiers being symmetric, and he assumed that *only* and *mostly* are determiners in (6.51). In the face of such examples, Keenan changed his mind about what existential-there sentences mean. Inspired by a proposal of Zucchi (1995), Keenan (2003) decided that the existential interpretation is  $Q_M(A \cap B, B)$  instead of  $Q_M(A \cap B, M)$ . That is, he took the coda's denotation not only to restrict the quantifier, but also to be the universe about which the existential claim is made. The rest of his 1987 theory remained the same: what makes a quantifier existentially acceptable is equivalence of the compositional proposition  $Q_M(A, B)$  to the existential proposition, but now the compositional interpretation should be equivalent to  $Q_M(A \cap B, B)$  in existential-there sentences.

Note that this condition on existentially acceptable quantifiers is simply  $\text{CONS}_2!$  Thus Keenan's new analysis predicts that an existentially acceptable quantifier satisfies  $\text{CONSERV}$  if and only if it is intersective. Empirically, he abandoned his earlier analysis in favor of this one because of three apparent determiners—*only*, *just*, and *mostly*—which are not conservative if they actually are determiners; the new analysis's predictions are identical to the earlier one's except with regard to such putative non-conservative determiners.

In section 6.3.2, we explained our skepticism about the view that *only*, *just*, and *mostly* are determiners and, correspondingly, also about the implications of the existential acceptability of noun phrases like *only tradeoffs*, *just diamonds*, and *mostly bad options*. Further investigation will in due course disclose just what their existential acceptability entails, if anything, for the question of which determiners produce existentially acceptable noun phrases. Regardless of how that turns out, note that Keenan's new Zucchi-inspired hypothesis does nothing to mitigate the problems mentioned at the end of the preceding section about existentially acceptable noun phrases deriving from undisputed determiners that are non-symmetric, because these determiners are conservative. In summary, the new analysis adds few successes, if any, to the considerable number which Keenan's old analysis enjoyed.

### 6.3.5 Some additional data and conclusions

We noted that the proportional readings of many and few do seem to be allowed in at least some existential-there sentences. Are explicitly proportional determiners like at least two-thirds of the also existentially acceptable? It appears that they are.

- (6.52) a. The abbot cannot validly open the assembly unless there are at least two-thirds of the members of the assembly present.  
 b. The chair can be removed from office at any time between the annual elections but only if the governing body pass resolutions to remove him or her at two separate meetings and there are at least two-thirds of the governors at both meetings.

Non-symmetric existentially acceptable determiners like this satisfy CONSERV, but not CONS<sub>2</sub>; on their face, they are quite problematic for Keenan's analyses. They are equally problematic for Barwise and Cooper's analysis (note that at least two-thirds of the is positive strong whether or not you think it is undefined on the empty set).

Recall also the earlier examples in (6.45), which, though not symmetric, also caused trouble for Keenan's analyses and Barwise and Cooper's alike.

One other collection of existentially acceptable determiners deserves discussion here: possessives such as no child's, a neighbor's, etc. The first thing to note about possessive determiners is that some are existentially acceptable and others are not.

- (6.53) a. There is no child's pet here.  
 b. There is a neighbor's dogs that bark incessantly.  
 c. There is one man's name that is heard everywhere.  
 d. There are no competitors' applications on file.  
 e. There are two men's voices that are very deep.  
 f. There are several people's ideas involved.  
 g. There are few cities' mayors who support the bill.  
 h. There are some people's faces that I can't stand.  
 i. There are many employees' livelihoods at stake.
- (6.54) a. #There is the teacher's car in the parking lot.  
 b. #There is every competitor's application on file.  
 c. #There is John's voice that is very deep.  
 d. #There are most employees' livelihoods at stake.

In Chapter 7 we discuss in detail the interpretation of possessive determiners. Among other things, we will see that there is a distinction between universal and existential readings (and others), but that there is a general format into which most of these determiners fit. Applied to (6.53a, b), for example, we get the default interpretations below. Here, when  $R$  is a binary relation,  $R_a = \{b : R(a, b)\}$ , and, for a set  $A$ ,  $dom_A(R)$  is the set of things that  $R$  something in  $A$ :  $\{a : \exists b \in A R(a, b)\}$ .

- (6.55) a.  $no(child \cap dom_A(owns), \{a : A \cap owns_a \cap B \neq \emptyset\})$   
 where  $A = pet$  and  $B = here$
- b.  $some(neighbor \cap dom_A(owns), \{a : A \cap owns_a \subseteq B\})$   
 where  $A = dog$  and  $B = bark\ incessantly$

So (6.53a) says that no pet-owning child is such that even one of her pets are here; this is the existential reading. The universal reading would claim that no such child has *all* of her pets here; i.e. it would erroneously make (6.53a) true if there were children who owned several pets only some of which are not here—a much less plausible reading. The universal reading, on the other hand, is more plausible for (6.53b): if the neighbor in question owns several dogs, the sentence may claim that all of them are barking incessantly.

It is clear from the data in (6.53) and (6.54) that the possessor noun phrase is decisive for whether or not the possessive determiner, and the noun phrase of which it is the determiner, are existentially acceptable. However, semantic tests applied to the possessive determiner's meaning (in terms of strong/weak, symmetry, or  $CONS_2$ ) hardly depend at all for their results on the meaning  $Q$  of the possessor noun phrase. This fact throws doubt on whether any attempt like Barwise and Cooper's or Keenan's to explain which determiners are existentially acceptable could succeed for complex determiners such as these.

What all these problematic cases—(6.45) and (6.52) as well as (6.53) and (6.54)—have in common is that they involve complex determiners which seem to incorporate two quantifiers, one taking scope over the other. The data suggest to us that the existential acceptability of many such determiners depends primarily, or even exclusively, on whether the semantically widest scope quantifier is existentially acceptable. Thus we are disposed to assess such a complex determiner for existential acceptability not taken as a whole, but rather just in terms of the outermost operator. This line of investigation seems promising enough to warrant further research.

We note that there have also been attempts to explain which noun phrases are existentially acceptable in terms of presuppositions triggered by constituents of noun phrases; see, for example, De Jong and Verkuyl 1985; Zucchi 1995; and Moltmann 2005.

Finally, and in summary, if one gives some such special treatment to complex determiners and to noun phrases introduced by non-determiners like *only*, *just*, and *mostly*, then both the Barwise and Cooper and the Keenan 1987 analyses do a reasonably good job empirically of accounting for which determiners are existentially acceptable. One empirical difference, which Keenan (1987) pointed out, is that his analysis admits the trivial determiners *at least zero* and *fewer than zero*, but Barwise and Cooper's does not. The other is that Barwise and Cooper's analysis is compatible with the existential acceptability of *proportional many* and *few*, whereas Keenan's is not.

The theoretical differences between the analyses seem greater than the empirical ones. Both have strengths and each has liabilities. Barwise and Cooper's use of partiality to prevent some existentially unacceptable determiners from being weak, making them positive strong instead, diminishes the predictive power of their theory



significantly. Moreover, their lack of a compositional rule to provide semantic interpretations for existential-there sentences containing a coda is a notable lacuna. While Keenan's approach has neither of these faults, it suffers from a different theoretical blemish. In compositionally assigning an interpretation to every there sentence with a coda, his rule assigns many sentences with existentially unacceptable pivot noun phrases an interpretation that these sentences cannot actually have. For instance, the sentences (6.56) do not mean (6.57).

- (6.56) a. There is every student missing.  
b. There are most firemen outside.

- (6.57) a. Every student is missing.  
b. Most firemen are outside.

This problem is built into Keenan's reliance on semantic equivalence for certain quantifiers of the meaning he assigns compositionally to the existential meaning.

#### 6.4 OTHER RELATIONAL PROPERTIES OF CONSERV AND EXT TYPE $\langle 1, 1 \rangle$ QUANTIFIERS

Once one realizes that type  $\langle 1, 1 \rangle$  quantifiers are (locally) binary relations between sets, it is natural to inquire if other familiar relational properties besides symmetry, such as reflexivity, transitivity, linearity, etc., provide interesting classifications of quantifiers. There are a number of results here, sometimes inspired by empirical observations that some property does not seem to be instantiated among determiner denotations, or that it is instantiated by just a few particular determiners, or that all determiner denotations that have a certain relational property also seem to have a certain other relational property. Such observations can often be explained, it turns out, by the fact that determiner denotations are CONSERV, EXT, and (usually) ISOM.

We give below a sample of results and techniques in this area, referring the reader to van Benthem 1984 and Westerståhl 1984, Westerståhl 1989 for a fuller treatment. CONSERV, EXT, and ISOM are standardly assumed, though some of the results require less. Furthermore, representation in the number triangle (Chapter 4.8) is often efficient, as we will illustrate. But recall that this representation deals only with *finite* universes, i.e. it assumes FIN.

Let us start with symmetry properties. Symmetry was characterized and applied in the previous section, but what about, say, asymmetry or anti-symmetry?

(a1) In the proof of Proposition 8 in the preceding section we noted that

$$(6.58) \quad Q \text{ is symmetric iff for all (cardinal) numbers } k, m, Q(k, m) \Leftrightarrow Q(0, m).$$

This follows directly from ISOM and the fact (see (6.10) in section 6.1) that symmetry is the property:  $Q(A, B) \Leftrightarrow Q(A \cap B, A \cap B)$ .<sup>26</sup> In the number triangle, lines

<sup>26</sup> Since EXT is assumed throughout, we can omit the subscript in  $Q_M(A, B)$ .

$(0, m), (1, m), (2, m), \dots$ , parallel to the left edge of the triangle are called *columns* (as they would be if the triangle were tilted 45 degrees to a ‘number square’). Similarly, lines parallel to the right edge are *rows*. Then  $Q$  is symmetric if, whenever a point  $(k, m)$  is in  $Q$ , so is the whole column through  $(k, m)$ . So a symmetric quantifier consists of (the union of a) set of columns in the triangle.

Recall that the number triangle represents both type  $\langle 1 \rangle$  quantifiers and CONSERV and EXT type  $\langle 1, 1 \rangle$  quantifiers. Now consider the property EXT for type  $\langle 1 \rangle$  quantifiers. It says that with fixed  $A, M$  can be varied arbitrarily (as long as it contains  $A$ ) without affecting  $Q_M(A)$ . Clearly, in the number triangle this is the property that if a point belongs to  $Q$ , so does the whole column through that point. So it is *immediate* that EXT for type  $\langle 1 \rangle$  quantifiers in the number triangle is the same as symmetry for their relativizations. But the more general statement in Proposition 2 above, which does not assume ISOM or FIN, needs proof.

(a2) *Asymmetry* is the property  $Q(A, B) \Rightarrow \neg Q(B, A)$ . Here is a perhaps surprising fact:

(6.59) (van Benthem) There are no asymmetric quantifiers (satisfying CONSERV, EXT, and ISOM), except the trivial  $0$ .

To see this, recall that only  $|A - B|$  and  $|A \cap B|$  matter for whether  $Q(A, B)$  holds or not. So as long as  $A - B$  and  $A \cap B$  are not touched, we can change  $B - A$  in any way we want without effect. Now suppose  $Q(A, B)$  holds, and take  $B'$  such that  $A - B = A - B'$ ,  $A \cap B = A \cap B'$ , and  $|B' - A| = |A - B|$ . Then  $Q(A, B')$ , but also  $Q(B', A)$ , since the relevant cardinalities are the same. So  $Q$  cannot be asymmetric, if it is different from  $0$ .

(a3) *Anti-symmetry* is:  $Q(A, B) \ \& \ Q(B, A) \Rightarrow A = B$

(6.60)  $Q$  is anti-symmetric iff for all (cardinal) numbers  $k, m$ ,  $Q(k, m) \Rightarrow k = 0$ . Thus, it is equivalent to the property:  $Q(A, B) \Rightarrow A \subseteq B$ .

For a proof, note first that a direct translation of anti-symmetry to a number-theoretic property gives:

(6.61)  $Q(k, m) \ \& \ Q(k', m) \Rightarrow k = k' = 0$

For, suppose  $Q(k, m)$  and  $Q(k', m)$  hold. Choose  $A, B$  such that  $|A - B| = k$ ,  $|B - A| = k'$ , and  $|A \cap B| = m$ . Then  $Q(A, B)$  and  $Q(B, A)$ , so  $A = B$  by anti-symmetry, and hence  $k = k' = 0$ . Conversely, (6.61) straightforwardly implies anti-symmetry. Now, (6.61) is clearly equivalent to the simpler property  $Q(k, m) \Rightarrow k = 0$ , which in turn is equivalent to  $Q(A, B) \Rightarrow A \subseteq B$ .

Thus, anti-symmetric quantifiers are all subrelations of the quantifier *all*. In the number triangle, their  $+$ s all lie on the right edge. Recall the condition VAR saying that there is a  $+$  and a  $-$  on each level except level 0. Examining the triangle, we see that

(6.62) The only anti-symmetric quantifiers satisfying VAR are *all* and *all<sub>ci</sub>*.

Next, a quick look at *reflexivity* properties:

- (b1)  $Q$  is *reflexive* if, for all sets  $A$ ,  $Q(A, A)$ ; *irreflexive* if  $\neg Q(A, A)$  for all  $A$ . As we saw in section 6.3.4.2, modulo the admittance of partially defined quantifiers, being either reflexive or irreflexive is what Barwise and Cooper (1981) called (positive or negative) *strong*.

Numerically, (ir)reflexivity means that for all  $k$ , (not)  $Q(0, k)$ , i.e. in the number triangle, the whole right edge of the triangle is (not) in  $Q$ .

- (b2) A slightly weaker property than reflexivity is *quasi-reflexivity*:  $Q(A, B) \Rightarrow Q(A, A)$  for all  $A, B$ . Numerically:  $Q(k, m) \Rightarrow Q(0, k + m)$ ; i.e. if there is a + at level  $n (= k + m)$ , then  $(0, n)$  has a + too. For example, *most* is quasi-reflexive but not reflexive, as is *at least  $n$*  ( $n \geq 1$ ).

We can also look at combinations of properties.

- (6.63) No non-trivial quantifier is both symmetric and (ir)reflexive.

- (6.64) (van Benthem) The non-trivial, symmetric and quasi-reflexive quantifiers are exactly the quantifiers *at least  $k$* , where  $k$  is any cardinal number  $> 0$ .

- (6.65) The only reflexive and anti-symmetric quantifier is *all*.

*Proof.* For (6.63), suppose  $Q$  is reflexive, and take any  $A, B$ . Since  $Q(A \cap B, A \cap B)$  holds, so does  $Q(A, B)$  by symmetry, so  $Q$  is the trivial quantifier 1. Similarly, if  $Q$  is irreflexive, it must be 0.

For (6.64), let us argue under FIN, i.e. in the number triangle, even though the result without FIN is also true. Let  $m$  be the smallest number such that some pair  $(k, m)$  is in  $Q$  (there is one, by non-triviality). By symmetry, the whole column through  $(k, m)$  is in  $Q$ . But then, by quasi-reflexivity, each of  $(0, m)$ ,  $(0, m + 1)$ ,  $(0, m + 2)$ , ... is in  $Q$ , and thus, by symmetry, so are the corresponding columns. This means all the points to the right of the column through  $(0, m)$  also have a +, i.e.  $Q$  is *at least  $m$* .

For (6.65), we have  $Q(A, B) \Rightarrow A \subseteq B$  by (6.60). And if  $A \subseteq B$ ,  $Q(A, A)$  implies  $Q(A, B)$  by CONSERV. Thus  $Q = \text{all}$ .

What about *transitivity*, i.e. the property:  $Q(A, B) \ \& \ Q(B, C) \Rightarrow Q(A, C)$ ? It too has a version in the number triangle, though it is slightly complex.<sup>27</sup> But we can look at some examples:

- (c1) Examples of transitive quantifiers are:

- *all*, *all<sub>ei</sub>*, and *all but finitely many*.
- *all<sub>n</sub>*, defined as  $\text{all}_n(A, B) \Leftrightarrow A \subseteq B$  or  $|A| < n$  ( $n > 0$ ). (Note that  $\text{all}_1 = \text{all}$ .)
- Universe quantifiers (see Chapter 5.5) of the form  $Q(A, B) \Leftrightarrow |A| = n$ , or, more generally  $Q(A, B) \Leftrightarrow |A| \in S$ , for some class  $S$  of (cardinal) numbers.
- $Q(A, B) \Leftrightarrow (A \subseteq B \ \& \ |A| \geq 5)$  or  $|A| = 3$

<sup>27</sup> Westerståhl (1984) shows how to translate any universal property of  $Q$  to the number triangle, under FIN.

Let us check the last case: Suppose  $Q(A, B)$  and  $Q(B, C)$ . If  $|A| = 3$ ,  $Q(A, C)$  follows, so assume  $A \subseteq B$  &  $|A| \geq 5$ . But then  $|B| \neq 3$ , so it must be that  $B \subseteq C$  &  $|B| \geq 5$ . Thus,  $A \subseteq C$  &  $|A| \geq 5$ , so  $Q(A, C)$ .

(c2) This last case turns out to be characteristic of transitivity; we quote without proof (the proof uses number triangle techniques) the following result from Westerståhl 1984:

(6.66) (FIN)  $Q$  is transitive iff there are sets  $X, Y$  of natural numbers such that  $X < Y$  (meaning that every number in  $X$  is smaller than every number in  $Y$ ) and  $Q(A, B) \Leftrightarrow (A \subseteq B \text{ \& } |A| \in Y) \text{ or } |A| \in X$ .

The example *all but finitely many* shows that FIN is a necessary requirement here. Next, it is clear that the right-hand-side condition can never hold for both  $Q(A, B)$  and  $Q(B, A)$ . Thus:

(6.67) There are no non-trivial symmetric and transitive quantifiers.

But this can also be seen in a simpler way, and without using FIN. Any symmetric and transitive relation must be quasi-reflexive (if  $Q(A, B)$ , then  $Q(B, A)$  by symmetry, so  $Q(A, A)$  by transitivity), and therefore, by (6.64),  $Q$  must be *at least*  $k$ , for some  $k > 0$ . But it is easy to see that none of these quantifiers are transitive.

Here is another corollary of (6.66), first proved in van Benthem 1984:

(6.68) (FIN) The only reflexive and transitive quantifiers are *all<sub>n</sub>*, for  $n > 0$ .

For it is not hard to see that reflexivity implies that on the right-hand side of the condition in (6.66), we must have  $X = \{0, 1, \dots, n-1\}$  and  $Y = \{n, n+1, \dots\}$ , for some  $n > 0$ .

There are a number of similar results, for example:

(6.69) (van Benthem) There are no (non-trivial) *circular* quantifiers, in the sense of satisfying the condition  $Q(A, B) \text{ \& } Q(B, C) \Rightarrow Q(C, A)$ .

Again, facts like these bear witness to the power of the basic characteristics—CONSERV, EXT, and ISOM—of determiner denotations. For type  $\langle 1, 1 \rangle$  quantifiers in general, there are no constraints on the combinations of relational properties that they might exhibit (except those given by the logic of binary relations).

We end this section by noting that there are several connections between monotonicity properties and the relational properties of CONSERV and EXT type  $\langle 1, 1 \rangle$  quantifiers discussed here. For example, a symmetric quantifier is clearly right monotone if and only if it is left monotone. Here is a final example, from Zwarts 1981.

### Proposition 9

Let  $Q$  be a CONSERV type  $\langle 1, 1 \rangle$  quantifier.

- (a) If  $Q$  is reflexive and transitive, then it is  $\downarrow$ MON $\uparrow$ .
- (b) If  $Q$  is symmetric, then it is quasi-reflexive iff it is MON $\uparrow$ .

*Proof.* Under EXT, FIN, and ISOM, this follows from (6.68) and (6.64), respectively. But the result holds without these assumptions. For (a), suppose  $Q$  is reflexive and

transitive, and that  $Q_M(A, B)$  holds. Let  $A' \subseteq A$  and  $B \subseteq B' \subseteq M$ . By reflexivity and CONSERV, we get  $Q_M(A', A')$ , and hence  $Q_M(A', A)$ . Thus, by transitivity,  $Q_M(A', B)$ . Similarly, it follows that  $Q_M(B, B')$ , and hence  $Q_M(A, B')$ .

For (b), if  $Q$  is  $\text{MON}\uparrow$  and  $Q_M(A, B)$  holds, then  $Q_M(A, A \cap B)$  by CONSERV, so  $Q_M(A, A)$  by  $\text{MON}\uparrow$ . Thus,  $Q$  is quasi-reflexive—this does not use the symmetry of  $Q$ . But in the other direction, suppose  $Q$  is symmetric and quasi-reflexive, that  $Q_M(A, B)$  holds, and that  $B \subseteq B' \subseteq M$ . We must show  $Q_M(A, B')$ . By Fact 1 in section 6.1 (which uses only CONSERV),  $Q_M(A \cap B', A \cap B)$  holds, since  $A \cap B = (A \cap B') \cap (A \cap B)$ . By quasi-reflexivity, it follows that  $Q_M(A \cap B', A \cap B)$ . Again by Fact 1,  $Q_M(A, B')$ .  $\square$

# 7

## Possessive Quantifiers

This chapter and the next one deal with two groups of quantifiers that have received much attention in the linguistics literature: those involved in the interpretation of *possessive* expressions and those related to *exceptive* constructions. Possessive and exceptive constructions occur in great variety in the world's languages. Our ambition is not to survey such constructions, but rather to focus on those possessive and exceptive expressions that can be seen as *determiners*. Possessive determiners constitute a major form of possessive expressions in English and Swedish, whereas they do not occur in, for example, Italian or French, which instead predominantly use a postnominal (prepositional) form. Still, we hope and expect that an account of the semantics of possessive determiners will shed some light on other possessive forms as well.

We look in some detail at the quantifiers denoted by such expressions: at their properties, and at the mechanisms by which they become the interpretations of simple or complex possessive and exceptive determiners. The members of these important classes of natural language quantifiers are CONSERV and EXT, but usually not ISOM, since they involve a fixed noun phrase, and we know that NP denotations are in general not ISOM.<sup>1</sup> Nevertheless, they have logical properties, such as monotonicity, that are worth studying in some detail.

As to possessives, we have not seen a treatment with the scope and generality attempted here, although there are several recent detailed studies of particular instances of possessive constructions.<sup>2</sup>

Section 7.1 presents the forms of possessive determiners that are the focus of this chapter. In section 7.2 we discuss to what extent the grammatical number of the noun following a possessive determiner influences interpretation. This is especially relevant, since possessives are often taken to be *definite*, or at least to contain a definite determiner. When the noun is singular, as in JOHN'S book, the whole NP looks like a definite description. Nevertheless, we argue that the systematic connections between syntactic and semantic number are very weak.

Section 7.3 introduces another important, and in our opinion neglected, theme: even if the universal reading of possessives is often the default, so that JOHN'S chairs are antique means that *all* his chairs are antique, sometimes existential

<sup>1</sup> However, they will be seen to result from applying higher-order operations to quantifiers, and those operations are ISOM—a general phenomenon we come back to in Ch. 9.2.

<sup>2</sup> In terms of coverage, perhaps the treatment of possessives in Keenan and Stavi 1986 is closest to ours.

and other readings are used as well. We use a parameter for the mode of quantification; this also fits well with partitive constructions like *three of John's books*, where a semantic rule allows that same parameter to be set by the first determiner.

After mentioning in section 7.4 certain compound noun-like possessive constructions—only to exclude them from the present discussion—section 7.5 presents another phenomenon that our account takes to be crucial: *narrowing*. This means that a quantified possessive, such as the one in *most people's grandchildren*, in effect quantifies not over all people but only over those with grandchildren. This was observed in Barker 1995 and, in our opinion, needs to be incorporated in any successful account of possessives.

Section 7.6 introduces a much discussed characteristic of possessives: the freedom in how the 'possessor' relation is fixed. Although there are various soft constraints here, these can always be overruled, and we therefore choose to have a parameter for this relation too. We are then ready to present in section 7.7 the semantics proposed here for (a large part of the) possessive determiners, in terms of a higher-order operator *Poss*, that we take to be the denotation of the possessive morpheme 's. In the next section, 7.8, we discuss a popular alternative idea about the meaning of possessives: namely, the definiteness account alluded to above, and compare it to ours. That is, we try to formulate such an account and isolate the role which definiteness plays in it. Differences show up with respect to narrowing, to existential readings of possessives, and to the alleged influence of syntactic number. Still, we show that it is possible to recast our account as a—suitably modified—definiteness account (and vice versa), and this enables a sharp comparison between the two.

Sections 7.9–7.12 discuss extensively how possessives are built syntactically, and what the corresponding semantic rules look like. This has been debated in fine detail for particular cases; instead, we suggest a format aimed at (almost) full generality. The issue of definiteness comes to the fore again at this point, since it is often claimed that possessives are definite, and that this is what makes partitive constructions like *at least one of most student's parents well-formed*. Using the notion of definiteness from Chapter 4.6, we show that in fact very few possessive determiners are definite, and conclude, after a methodological discussion of the criteria for definiteness, that it is more reasonable to say that *both* definites and possessives are allowed after [Det of]. Moreover, we suggest two different semantic rules for the two cases, and show how these rules generate the desired readings of certain sentences with iterated possessive constructions, but (apparently) no other readings.

Section 7.13 describes the monotonicity behavior, including smoothness (Chapter 5.6), of possessives in some detail. In particular, we show how the monotonicity properties of a possessive quantifier depend on those of its two constituent quantifiers (for example, for *two of each student's teachers*, the quantifiers *every* and *two*). This has not been investigated before, as far as we know. The results are rather intricate, but not entirely without interest.

Finally, section 7.14 lists some questions and issues for further study.

Obviously, our own proposals in this chapter build on the work of others, and we certainly don't pretend to provide the last word. We do hope to add some fuel to the

debate. But our main concern is not the linguistic debate, but rather a sufficiently general account of the semantics of possessive determiners.<sup>3</sup>

## 7.1 POSSESSIVE DETERMINERS AND NPs

The morphosyntax of possessive and genitive constructions varies across the world's languages. They usually take a noun to some genitive phrase, but the genitive itself can be prenominal as in the English *John's book*, the Swedish *Evas cyklar* (Eva's bikes), the Russian *Mašin učitel* (Maša's teacher), or postnominal as in the English *car of Harry's*, the Russian *drug Maši* (Maša's friend), or the Italian *cani di Gianni* (Gianni's dogs). Sometimes the genitive can occur as (what at least seems to be) a predicate, as in *These books are John's* or *Questi cani sono di Gianni* (These dogs are Gianni's). But it is not always clear that similar-looking constructions in different languages really are the same, and in any case linguists often disagree about the preferred analysis of possessives even in a particular language. For example, English has all of the following forms:

- (7.1) a. John's portrait  
 b. a portrait of John's  
 c. a portrait of John  
 d. That portrait is John's.

Linguists working in a transformational framework tend to regard (7.1b) or (7.1c) as basic (minimizing the difference between the two) and (7.1a) derived by movement; see Storto 2003: chs. 3.1 and 6.1, for a recent overview. Stockwell, Schachter, and Partee (1973) and Partee (1983/97) also find (7.1b) (and also (7.1d)) more basic. But note that some languages only have the third and fourth form of possessives (e.g. Italian), whereas others only have the first and fourth form (e.g. Swedish<sup>4</sup>). It is certainly possible to hold that the forms above have distinct analyses. Analyses taking the prenominal genitive as basic can be found in Barker 1995 and in a series of recent papers by Jensen and Vikner (e.g. Vikner and Jensen 2002).

Any general account will presumably find much similarity, but also some significant differences, between the various genitive forms. This book is about quantifiers, so we focus on the 'Saxon genitive' of the prenominal kind, since these forms can most easily be seen as quantirelations (determiners), denoting type  $\langle 1, 1 \rangle$  quantifiers. Sometimes they are not called determiners in the literature, but genitive phrases (GP) or possessive phrases (PossP). But Keenan and Stavi (1986) and Partee (1983/97),

<sup>3</sup> We thank Barbara Partee for extensive and very helpful comments on an early draft of this chapter. In particular, she helped us realize that a main contender to our approach to possessives is the above-mentioned definiteness account.

<sup>4</sup> With some nouns, such as *porträtt* (portrait) and *syster* (sister), but not e.g. *lag* (team), Swedish has the postnominal forms *porträtt av John*, *syster till John*, but it is not clear that these should be construed as possessives. The form (7.1b) does not exist for any noun.



for example, observe that in most respects they behave exactly like determiners. We follow this usage here, and call them *possessive determiners*, and NPs formed by combining them with a noun *possessive NPs*. A *possessive quantifier* is one that interprets a possessive determiner (this terminology will be made more precise in section 7.12). We begin with the simplest forms of possessive determiners, exemplified in English by

(7.2) John's, no doctors', at least five teachers', exactly three students', most children's

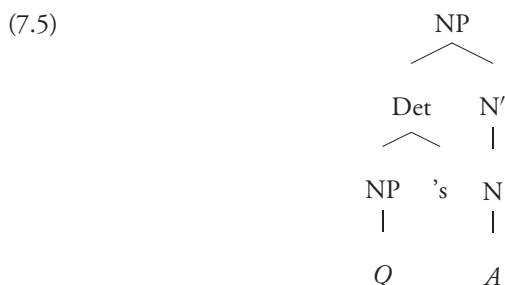
Here, a possessive determiner consists of what we may call a *possessor NP* and the *possessive morpheme* 's. The possessor NP can be unquantified—for example, a proper name or a bare plural—or it can be quantified, consisting of a possessor determiner and a possessor noun. But we will see that it is profitable to think of simple possessive determiners, as in (7.2), as special cases of the more general forms:

(7.3) few of John's, all but five of Mary's, each of most students'

Another variant of possessive NPs has a numerical expression following 's:

(7.4) Henry's three, most professors' few, some of Linda's several

Syntactic fine structure will not matter crucially, but let us lay down the syntax in (7.5) for possessive NPs formed by determiners as in (7.2).<sup>5</sup>



This seems compatible with most of the literature. Given this (rudimentary) syntax, there are still various options for how to derive compositionally the meaning of a possessive NP from the meanings of its parts. It is a remarkable fact, however, that there appear to be *few limitations* on *Q* here. Most English NPs can combine with 's to form a possessive determiner, or so it seems. Quantified NPs, proper names, conjoined names like *John and Mary*, bare plurals like *firemen*, pronouns—all allow the possessive morpheme. This suggests that an adequate account of the truth conditions of possessive sentences should be *uniform in Q*. We discuss the possibility of such an account in sections 7.7 and 7.8. There are certain NPs, however, that do not combine with 's, and we return to the restrictions on the lower NP of (7.5),

<sup>5</sup> As to the forms (7.3), one may dispute whether they are really constituents (determiners) or not. We will treat them as such in sect. 7.9, but in fact very little in our account hinges on them being determiners rather than, say, split into a determiner and a prepositional phrase.

and to the adequacy of the account we actually give (which is not quite uniform), in section 7.12.4. But first we shall note a few important facts about what possessive NPs can mean.

## 7.2 NUMBER AND UNIQUENESS

Possessive determiners characteristically combine with both singular and plural nouns—John's bike, Mary's cars, most students' final exam, some doctors' patients—but there appears to be no straightforward correlation between this and the number of 'possessed' objects. John's bike was stolen might in some context indicate that he had only one bike, but in other contexts not. And Mary's brother came home, or Henry's finger was hurt, normally does *not* presuppose that Mary has only one brother or Henry only one finger.

This is related to an assumption usually made by people writing about possessives (though not Keenan and Stavi (1986)): a *uniqueness assumption*. Since possessives are thought of as essentially involving *definiteness*, it is assumed that every use of a possessive NP presupposes that a unique object, or in the plural case a unique set of objects, is somehow *salient* in the utterance situation.

There are several issues involved. One concerns the size of the domain of 'possessed' objects. Another has to do with the type of quantification over that domain. A third issue is whether syntactic number gives any information about either of the first two issues. And a fourth question, bearing on all the others, is precisely what, if anything, definiteness has to do with possessives.

We come back to the first and second issues in the next section. As to the information we get from syntactic number, the only way to get around apparent examples like the ones cited at the beginning of this section is to maintain that in those situations, some contextual restriction or some notion of salience is operative, selecting a unique brother or a unique finger. But that often seems gratuitous. One can perfectly well say Henry's finger was hurt without thinking of a particular finger, or even knowing which it was, and without implying that only one of his fingers was hurt (although that would sometimes be conversationally implicated). So his hurt finger or fingers need not be salient in any other sense than that they were hurt.

As we observed in Chapter 4.2.2, this is even clearer with examples like

(7.6) Each girl's brother feels protective of her.

This sentence makes a claim about girls with one or more brothers, and it makes no sense to single out one brother for each girl as most salient.

This is not to deny that contextual restriction of the domain of possessed objects often occurs. If I say

(7.7) My friend met your friends.

I am not implying that I have only one friend, or that my friend met absolutely all of your friends—some selection of friends is made in the context of utterance. But from

this it does not follow that the meaning of the possessive phrase *requires* such a restriction. Contextual restriction, by salience or other means, is a general phenomenon in linguistic communication. It can be handled semantically with some form of context sets (see Chapter 1.3.4). In our account of possessives here, contextual restrictions do not play a role different from the one they play in general for quantified sentences. In particular, we do not take the singular form of the noun to indicate that a unique object is salient.<sup>6</sup>

There are, however, accounts of possessives for which it is an essential feature that a uniqueness assumption is always made. Let us call these, somewhat loosely, *definiteness accounts*. For example, Vikner and Jensen (2002) explicitly take the meaning of *each girl's teacher* to be such that, say,

(7.8) Each girl's teacher is nice.

entails that every girl has exactly one teacher. This seems counter-intuitive (as proponents of similar accounts sometimes note, e.g. Partee 1983/97: 467), for the reasons already given. In fact, (7.8) is compatible with some girls having several teachers, and some girls having no teacher at all. From our perspective, a uniqueness assumption is neither desirable nor necessary in general. After we have presented our own account (section 7.7), we come back to the definiteness accounts (section 7.8), and discuss how the seemingly counter-intuitive consequences of the treatment of (7.8) suggested above could be avoided.

For now, we merely note that any analysis of possessives needs to be wary of putting too much weight on simple cases like *his brother* or *John's car*—which are by themselves definite and do seem to involve uniquely singled-out objects—because other examples, like *no student's books* or *a few of most girls' teachers* denote type (1) quantifiers that are neither definite themselves nor obviously built from definite quantifiers.<sup>7</sup>

### 7.3 UNIVERSAL READINGS AND OTHERS

A common interpretation of possessives in the literature is the following: the possessive NP singles out some set (perhaps a unit set), or, when a quantified possessive determiner is involved, a family of such sets, and then claims something about *all* the members of those sets. Let us call this the *universal reading*. Thus,

(7.9) John's bikes were stolen.

<sup>6</sup> Likewise, we do not take plural form to indicate the presence of more than one possessed object; see Ch. 4.2.2. For example, *No students' cars are parked outside* doesn't entail that there is more than one car owned by students—in fact it doesn't entail that there are any such cars.

<sup>7</sup> Definiteness accounts do take them to involve definites. But we don't think this is *obvious*, and other treatments are in fact possible. As to the tendency of focusing on definite possessives, Lyons (1986) e.g. simply claims: "The [possessive] NP is, in fact, understood as definite" (p. 124). And Vikner and Jensen (2002), who are well aware that a possessive determiner need not itself be definite, nevertheless say things like: "A genitive construction, that is a [possessive] NP, is typically used to refer to some item" (p. 194).

(7.10) At least three students' parents were present.

say that each of John's bikes was stolen, and that at least three students were such that all of their parents were present, respectively.

Definiteness accounts employ a seemingly slight (but in fact significant) variant of this. Roughly, the definite quantifier  $all_{ei}$  is used instead, so that all the relevant sets are in addition required to be *non-empty*.<sup>8</sup>

However, other modes of quantification are also used. Consider

(7.11) At most two cars' tires were slashed.

The most reasonable reading of this sentence, we claim, uses *existential* quantification over the 'possessed' objects.

Suppose there are  $n$  cars, each with four tires. (7.11) clearly allows no more than two cars to have any slashed tires, and thus no more than eight slashed tires in all. This is the existential reading: at most two cars had *at least one* of their tires slashed. The universal reading (which in this case coincides with the  $all_{ei}$  or the  $the_{pl}$  reading), on the other hand, would allow all  $n$  cars to have some tires slashed and up to  $2 + 3n$  tires to be slashed. With seven cars, it allows twenty-three slashed tires, but that is clearly *not* (normally) allowed by (7.11).<sup>9</sup>

In other cases, there may be an ambiguity between a universal and an existential reading:

(7.12) Some cars' tires were slashed.

(7.13) No students' books were returned in time.

It seems that both (7.12) and (7.13) have straightforward existential readings. Then (7.13) entails that no books at all (borrowed by students) were returned in time. But one can think of situations where a universal reading would apply.<sup>10</sup> Suppose returning late means having to pay a fine, and the statement was made by a librarian interested in who must pay fines: on the universal reading no student returned *all* borrowed books in time, which implies that every student had to pay a fine.

We don't think these interpretational differences have to do with vagueness. There might be some vagueness as to how universal the universal (or definite) reading really is: i.e. if all, or only 'almost all' elements of the relevant set need to satisfy the condition given by the scope. But the existential reading is not vague at all. Sometimes it is the only reasonable reading, sometimes there is an ambiguity. These facts make it natural to think that there is an implicit parameter for the quantificational force of the possessive.

<sup>8</sup> In fact, the quantifier  $the_{sg}$  is used with nouns in singular form, and  $the_{pl}$  with plural form. In sect. 7.8.3 we shall argue that this doesn't give the desired results, and that  $all_{ei}$  should be used instead.

<sup>9</sup> Couldn't (7.11) be read so as to allow at most two slashed tires? We come back to this reading in the next section.

<sup>10</sup> The following scenario was suggested by Barbara Partee.

This idea gets strong additional support from the fact that slightly more complex possessive NPs specify the quantificational force explicitly:

(7.14) Several of John's CDs were stolen.

(7.15) Each of most people's children expects to inherit.

(7.16) Three of each country's athletes carried a banner.

Here again the possessive NP provides a set or a family of sets (the set of CDs belonging to John, for each country the set of its athletes present at the ceremony, etc.), but now an explicit determiner tells us how to quantify over these sets: universally, existentially, or with other quantifiers. When that determiner is not overtly expressed, the nature of the quantification still has to be specified.

Definiteness accounts of possessives stipulate that the implicit parameter is always *all<sub>ei</sub>* (see note 8). They will then have problems with existential readings like the ones exhibited above. We come back to this in section 7.8.5.

#### 7.4 SCOPE AMBIGUITIES?

Consider again (7.11). The existential reading is compatible with a situation where, for example, five tires in all were slashed. There may also seem to be a reading which excludes the possibility that the total number of slashed tires exceeds two. Such a reading is obtained if *cars' tires* is read like a *compound noun* (i.e. is interpreted as *car tires*), denoting the set of tires belonging to (or used for) cars, and the determiner at most two simply has that set as its restriction.

That may be a less plausible reading for (7.11), but other cases are clearer:

(7.17) I'm looking for a children's book for my niece.

(7.18) Many child's toys have been recalled by their manufacturers because they were dangerous.

Here the syntactic number of *children* and *child* shows that the structure is not that of (7.5).<sup>11</sup>

<sup>11</sup> But with other uses of these same phrases, a compound noun analysis is not intended or even available:

- (i) Those children's book was stolen.
- (ii) This child's toys were neatly put away.
- (iii) Adults buy most children's books for them.

Note that in (7.17) and (7.18), but not in (i) or (ii), both *child's toys* and *children's book* have so-called compound stress (i.e. are accented on the first word rather than the second). The following are some naturally occurring examples:

- (iv) The London library houses a copy of most children's books published within the previous 24 months. ([www.roehampton.ac.uk/subject/childlit.asp](http://www.roehampton.ac.uk/subject/childlit.asp))

Apparently everyone agrees that these are not true possessives (in spite of the possessive morpheme). In particular, they do not involve possessive determiners, and we therefore exclude them from the present discussion.

Thus, the ambiguity of simple possessives, if there is one, derives from the choice of an implicit quantifier; it is not a scope ambiguity—although *iterated* possessives can have genuine scope ambiguities, as will become clear in section 7.10 below.

## 7.5 NARROWING

An important feature of possessive determiners, in our opinion, is what Barker (1995) calls *narrowing*: in a quantified possessor NP, that quantifier's restriction is narrowed to the ones that 'possess' objects of the kind named. We can make this clear with an example:

- (7.19) a. Most people's grandchildren hate them.  
 b. Most people's grandchildren love them.

Presumably, (7.19a) is false, and (7.19b) is true. Now, it is also the case that most people in the world don't have grandchildren (they are too young for that). But it is obvious that the grandchild-less people are simply *irrelevant* to the truth or falsity of (7.19a, b). Only people in the domain of the grandparent relation (i.e. the relation  $R(x, y)$  iff  $y$  is a grandchild of  $x$ ) matter, and the issue is whether more than half of *these* are such that all their grandchildren hate them or love them, as the case may be.

Here we are assuming a universal reading. On that reading, if we don't narrow, (7.19a) becomes *true*, since most people  $x$  are such that the set of  $x$ 's grandchildren is a subset of the set of people that hate  $x$ , simply because in most cases, the set of  $x$ 's grandchildren is *empty*. Without narrowing, we just get the wrong truth conditions.

Recall that definiteness accounts use (something like)  $all_{ei}$  instead of *all*. Then people without grandchildren don't count, since the empty set is not in the domain of that quantifier. However, without narrowing, we get another, equally undesirable effect: (7.19a) and (7.19b) each logically imply that most people have grandchildren! But this is false, whereas (7.19b) should be true, and in any case it definitely is something that should not follow from either (7.19a) or (7.19b). Thus, independently of whether you favor a definiteness account or not, narrowing is needed.

Consider another example.

- (7.20) Every professor's children attend the university where he teaches.

- (v) The library catalogue will be where most students' books will be lent. ([www.zipworld.com.au/rms/virtual/library/lending.html](http://www.zipworld.com.au/rms/virtual/library/lending.html))  
 (vi) Most students' books included highlighting or underlining. ([education.ucsc.edu/faculty/gwells/network/journal/Vol2\(2\).1999oct/article3.html](http://education.ucsc.edu/faculty/gwells/network/journal/Vol2(2).1999oct/article3.html))

Presumably, a compound noun reading was intended for the first two, but not the third, of these.

Whatever the truth value of this sentence, it should *not* entail that every professor has children. But that is precisely what it does on the definite reading, *unless* you narrow *professor* to *professor with children*.

On the universal reading, it doesn't actually matter if you narrow or not in (7.20). That is just because the childless professors are trivially such that all their children go to the university where they teach, so the claim with *every* remains true if you add the professors without children. But consider instead

(7.21) Some professor's children attend the university where he teaches.

In the non-narrowed universal reading, (7.21) is true if at least one professor has no children—again obviously incorrect truth conditions. This time, the definite reading is equivalent to the narrowed (universal) reading; in fact, this holds in general when the main determiner is *some* (see Fact 1, section 7.8).

Examples like these can be multiplied *ad libitum*: i.e. cases where the narrowed truth conditions appear correct but not the non-narrowed ones. Moreover, it can be done for all kinds of possessive determiners, and all kinds of (implicit or explicit) quantificational force over the possessed objects. We believe that narrowing is a characteristic feature of possessive determiners, and we build it into our official truth definition in section 7.7. But we recognize that there are some cases where it seems doubtful whether narrowing is wanted. Barbara Partee (personal communication) has suggested counterexamples, the most convincing of which, to our minds, is the following. Suppose that at a party half of the guests came by bike, leaving their bikes in the driveway, and the other half by car, parking on the street. With narrowing in force, one should be able to say

(7.22) Everyone's bike was parked in our driveway, and everyone's car was parked on the street.

But we agree that (7.22) rather seems to entail that every guest at the party had a bike *and* a car. We do not at present know how to deal with this sort of example, except by saying that narrowing is usually, but not always, in force. In any case, once you have a truth definition for possessive determiners with narrowing, it is easy to see what a non-narrowed version looks like; we discuss this in section 7.8.

Before we get to truth definitions, however, we need to say a few words about another and better-known characteristic of possessives: namely, the freedom of choice that appears to exist for the relation of 'possession'.

## 7.6 THE POSSESSOR RELATION

For a long time people have been impressed by the variety of different relations that can figure in possessive NPs. For example, *JOHN'S books* can be the books that John owns, that he has on his shelf, that he has written, or borrowed, or read, or is now standing on, etc. Let us call it the *possessor relation*. It seems that with enough contextual background, almost any relation can be a possessor relation. Even if in the case of true relational nouns, such as *sister* or *husband*, we can get the possessor

relation immediately from linguistic material in the possessive NP, many times we cannot. Therefore, a common view has been that, apart from *inherent* relations of this kind, the choice of possessor relation is *free*, depending on context, and hence an issue for pragmatics rather than semantics. This, then, is another major source of ambiguity—or context dependence—of possessive NPs.

A lot of recent work, however, questions this view. Starting from observations of Chomsky (1970) that many possessive NPs normally *exclude* certain relations, various attempts to describe the semantics of that relation have been made. For example, the table's leg normally involves a part-whole relation, but the leg's table can hardly ever use *that* relation. Likewise, John's gift can be the gift he bought or the gift he received (of course, many other relations are possible too), but John's purchase is at least by default about something John bought.<sup>12</sup>

Consider also the contrast between the possible interpretations of the forms (7.1) with different nouns:

- (7.23) a. John's picture  
 b. a picture of John's  
 c. a picture of John  
 d. That picture is John's.

- (7.24) a. John's team  
 b. a team of John's  
 c. \*a team of John  
 d. That team is John's.

- (7.25) a. John's sister  
 b. a sister of John's  
 c. a sister of John  
 d. That sister is John's.

While a full analysis of this sort of data is far from easy, certain things are fairly systematic.<sup>13</sup> For example, whereas almost any kind of possessor relation, including inherent ones when they exist, is possible in the (a) forms, inherent relations are *not* possible in the (d) forms, and often not in the (b) forms either.<sup>14</sup> Thus, except in a heavily elliptical context, (7.25d) can never mean that the sister in question is John's sister (but she could be, among the sisters in a certain family, the one that John is dating,

<sup>12</sup> But in a strong enough context, it could be about something I bought for John. Suppose, as Partee (personal communication) suggested, that I made lots of purchases and am trying to figure out which ones to have the store gift-wrap and send. Lexical rules can specify defaults for possessor relations, but it seems that these defaults can always be overridden.

<sup>13</sup> The following observations originate in Stockwell, Schachter, and Partee 1973, and have been developed on many occasions since, e.g. in Partee and Borschev 2003.

<sup>14</sup> E.g. one cannot use the (b) form for a picture of which John is the motive (and not the (d) form either); the (c) form (or the (a) form) is required here. The distribution between (b) and (c) forms is somewhat messy. For the inherent *sister*, both (b) and (c) forms are possible; the (c) form seems to be preferred for long nouns. See also Lyons 1986.



for example). The (c) forms, on the other hand, seem *only* to allow inherent relations, so that when there is none, as in (7.24c), the phrase is even ungrammatical.

Jensen and Vikner (2002) use rich information from the *lexicon* (organized with the lexical semantics of Pustejovsky 1995) to derive various possessor relations in a compositional manner. Besides pragmatic or free interpretations, they isolate four semantic interpretations: the above inherent one, plus control, producer, and part-whole. For *Ann's picture*, for example, we have in addition to free readings a three-way *semantic ambiguity*: (i) control reading: 'the picture that Ann controls' (e.g. owns); (ii) inherent reading: 'the picture of Ann'; (iii) producer reading: 'the picture that Ann painted'; the part-whole reading is ruled out in this case by lexical information, since pictures cannot be parts of persons. A similar view is defended in Storto 2003, where the basic ambiguity is between a(n underspecified) control relation generated independently of context and a free interpretation, whose existence, however, is constrained in significant ways.<sup>15</sup>

Do these insights and proposals show that the classical view was false? First, note that if this view was that

(Free<sub>1</sub>) Any possessive NP can use any binary relation as its possessor relation,

it seems doubtful that anyone has held it (even though some slightly sloppy formulations might suggest that). A more charitable formulation could be that

(Free<sub>2</sub>) Any binary relation can be used as the possessor relation of some possessive NP in some context.

This is not contradicted by the above examples, even should they really show that certain possessive NPs exclude certain relations. However, we think that the most significant formulation of the freedom of the possessor relation is rather the following:

(Free<sub>3</sub>) For any possessive NP, however predictable and semantically describable its *usual* possessor relation is, circumstances can always be found where that same possessive NP is used with *another* possessor relation, not derivable from grammatical or lexical information, but provided only by the context of utterance.

As far as we know, everyone agrees to (Free<sub>3</sub>) for prenominal possessives.<sup>16</sup> Even *John's sister can*, as we just saw, under the right circumstances be used with a possessor relation completely different from that of sisterhood.

To us, this has an important consequence. If we want an account which is general enough to cover all possible cases, there is no choice but to simply use a *relation parameter* in the semantics of possession and possessive NPs. That is what we do in the account below.

<sup>15</sup> Jensen and Vikner (2002) also present an empirical study according to which the pragmatic readings of possessive NPs (in their sense), at least in written corpora, are very rare compared to the others.

<sup>16</sup> One might consider strengthening (Free<sub>3</sub>) by combining it with (Free<sub>2</sub>), so that all relations, or almost all, are possible possessor relations. But (Free<sub>3</sub>) suffices for our purposes here.

The idea is that such an account is *compatible* with the empirical facts about possessor relations alluded to above, although it does not attempt to explain *all* of these facts. In particular, it is compatible with the case of inherent possessor relations. And in that case, the mechanism deriving the inherent reading can in fact be explained, as we will see, by means of an *addition* to the general account. Similar mechanisms could be added for other ‘standardized’ ways of determining (or ruling out) possessor relations. But quite possibly, some empirical facts would remain unexplained. This is not necessarily a shortcoming of the account. In fact, given the strong pragmatic influence on the interpretation of possessives, it is what one would expect.

## 7.7 THE MEANING OF POSSESSIVE DETERMINERS

Not everyone draws the same conclusion as we do from (Free<sub>3</sub>)—even if they agree that there are limits to what one should expect semantics to explain. The question is to a large extent methodological. It concerns one’s views on fundamental concepts such as *ambiguity*, *structure*, and *compositionality*. We come back to a discussion of the issue in section 7.9. For now, we present our general account of the truth conditions of possessive determiners, which uses a free parameter for the possessor relation, and by itself says nothing about how the value of that parameter gets fixed.

We assume that possessive determiners—like other determiners—take two arguments (the restriction and the scope), whose extensions are sets of individuals, and that in addition a binary possessor relation is (somehow) available. The case when the restriction is a relational noun, which actually delivers (the converse of!) the possessor relation, can be obtained with a simple additional mechanism.

It has sometimes been proposed that (at least in the free case) the possessor relation is in fact the denotation of the possessive morpheme ‘s, which then acts like an indexical. But this seems hard to fit in with relational nouns, and also with the fact that the possessor relation often comes somehow from the restriction noun, and not just from the possessive determiner.

Instead, we think of the possessive morpheme ‘s as denoting—in line with the most obvious compositional rule corresponding to the form (7.5)—an operation *Poss* from the relevant arguments to type  $\langle 1, 1 \rangle$  quantifiers.

The following notation (introduced in Chapter 4.1) is handy. If *R* is any binary relation and *a* any individual, then

$$(7.26) \quad R_a = \{b : R(a, b)\}$$

is the set of *R*-successors of *a*. Further, if *A* is any set, let

$$(7.27) \quad \text{dom}_A(R) = \{a : A \cap R_a \neq \emptyset\}$$

be the set of objects having *R*-successors in *A*. Thus, for example,  $\text{dom}_{\text{car}}(\text{own})$  is the set of individuals who own at least one car. This set is used to obtain narrowing.

Narrowing is in fact the only crux in the definition of *Poss*. We will see in the next section that things become simpler without narrowing, but narrowing is usually needed to get the right truth conditions. Now it would be nice if we could say that

narrowing is simply restriction to  $dom_A(R)$ . That is certainly the general idea, but the formal details are slightly more complex.

The possessive morpheme 's is always attached to an NP, denoting a type  $\langle 1 \rangle$  quantifier  $Q$ . The simplest general definition would be one that applied to *any* such  $Q$ . But in many cases we need access to the restriction set and, as we have seen (for example, Proposition 6 in Chapter 6.3.4), this set cannot be extracted from  $Q$ . We shall therefore proceed as follows. We take the case when  $Q$  is of the form  $(Q_1)^C$ , i.e. when we have a quantified NP, as basic. This means that both  $Q_1$  and  $C$  will be arguments of *Poss*. This handles straightforwardly the case where the NP has the form [Det N]. But what if the NP isn't quantified? In Chapter 4.5.5.3 it was pointed out that it is possible to construe proper names, conjunctions and disjunctions of proper names, bare plurals, and many other type  $\langle 1 \rangle$  quantifiers as having the form  $(Q_1)^C$ , and we will see that our definition of *Poss* covers these cases as well, giving the desired truth conditions. In this way the definition handles a large number of actually used possessive determiners. We think it covers far more cases than any comparable definition in the literature, and probably all possessive determiners of the form considered here. This statement will be made more precise in section 7.12, where we also discuss briefly some cases that are not dealt with.

The other two arguments of *Poss* are the possessor relation  $R$  and the type  $\langle 1, 1 \rangle$  quantifier  $Q_2$ , sometimes implicit and sometimes explicit in the syntax, saying what sort of quantification over the possessed objects is operative. The universal reading of the possessive determiner is obtained simply by letting  $Q_2 = \textit{every}$ . The definiteness approach to possessives fixes  $Q_2 = \textit{all}_{ei}$ .

The formal definition of the operation *Poss* is as follows: Let  $Q_1, Q_2$  be type  $\langle 1, 1 \rangle$  quantifiers,  $C$  a set, and  $R$  a binary relation. Put  $C_0 = C \cap dom(R)$ .  $Poss(Q_1, C, Q_2, R)$  is the type  $\langle 1, 1 \rangle$  quantifier defined, for each  $M$  and each  $A, B \subseteq M$ , by

$$(7.28) \quad Poss(Q_1, C, Q_2, R)_M(A, B) \iff (Q_1)_{M \cup C_0}(C \cap dom_A(R), \{a \in M \cup C_0 : (Q_2)_M(A \cap R_a, B)\})$$

$C$  and  $R$  are fixed, but  $M$  is any universe, so we need to consider  $Q_1$  on  $M \cup C_0$  in order that the two arguments are always subsets of the universe of quantification. (The universe for  $Q_2$  doesn't have to be extended, since  $A \cap R_a \subseteq A \subseteq M$ .) However, in practice, we can disregard this complication, due to the following readily verified observation:

$$(7.29) \quad \text{If } Q_1 \text{ and } Q_2 \text{ are CONSERV and EXT, so is } Poss(Q_1, C, Q_2, R).$$

Henceforth we always assume that  $Q_1$  and  $Q_2$  are CONSERV and EXT. Then we don't need to mention universes, and can formulate the definition more simply:

**the operation *Poss***

$$(7.30) \quad Poss(Q_1, C, Q_2, R)(A, B) \iff Q_1(C \cap dom_A(R), \{a : Q_2(A \cap R_a, B)\})$$

We also note that, due to the presence of the fixed set  $C$  and relation  $R$ ,  $Poss(Q_1, C, Q_2, R)$  is almost never ISOM. However,  $Poss$  itself (or more exactly, an operation closely related to  $Poss$ ) is ISOM, as we will see in Chapter 9.2.

The first argument of  $Q_1$  is ‘narrowed’ to  $dom_A(R)$ . To see how this happens, we work through a few examples.

- (7.31) a. Most students’ tennis rackets are cheap. (universal reading)  
 b.  $Poss(most, student, every, own)(tennis\ racket, cheap)$   
 c.  $most(student \cap dom_{tennis\ racket}(own), \{a : tennis\ racket \cap own_a \subseteq cheap\})$

Quantification is in effect restricted to students owning at least one tennis racket. This guarantees, as it should, that the sentence might well be true, even though many students don’t own any tennis rackets at all. It just says that a majority of those students who do own tennis rackets in fact own only cheap ones.

- (7.32) a. Some cars’ tires were slashed. (existential reading; see section 7.3)  
 b.  $Poss(some, car, some, (part\ of^{-1})(tire, slashed))$   
 c.  $car \cap dom_{tire}(part\ of)^{-1} \cap \{a : tire \cap (part\ of)_a^{-1} \cap slashed \neq \emptyset\} \neq \emptyset$

Here narrowing has no effect, because only existential quantifiers are used. Actually, this is an instance of a more general fact that will be stated in the next section (Fact 1).

Now consider a bare plural like `police`men. In the reading of bare plurals that we deal with here,  $police$ men<sup>pl</sup> =  $all_{ei}^{police$ man (Chapter 4.5.5.3). So we use the following interpretation:

- (7.33) a. Policemen’s cars are dented. (universal reading)  
 b.  $Poss(all_{ei}, policeman, every, own)(car, dented)$   
 c.  $all_{ei}(policeman \cap dom_{car}(own), \{a : car \cap own_a \subseteq dented\})$   
 d.  $policeman \cap dom_{car}(own) \neq \emptyset \ \& \ \bigcup\{car \cap own_a : a \in policeman\} \subseteq dented$

The last line is a simple calculation from the line before. Notice that (7.33a) does not say that all policemen own cars, but we are taking it to imply that some policemen do. The sentence further says that all cars belonging to policemen are dented. These seem to be reasonable truth conditions.

Next, take a proper name like `John`. We have  $I_j = all_{ei}^{j}$ , and proceeding as above we obtain, for example,

- (7.34) a. At most two of John’s dogs are fierce.  
 b.  $Poss(all_{ei}, \{j\}, at\ most\ two, own)(dog, fierce)$   
 c.  $\emptyset \neq dog \cap own_j \ \& \ |dog \cap own_j \cap fierce| \leq 2$

That is, John owns at least one dog, and at most two of his dogs are fierce. In Chapter 4.1 we gave the truth conditions for `John’s A are B` as  $\emptyset \neq A \cap R_j \subseteq B$ , so they imply that John ‘possesses’ at least one thing in  $A$ . Reasonably, the same implication should hold for (7.34a): it cannot be true unless John owns at least one dog.

Here is an example with a conjunction of names. Since  $all_{ei}^{\{j\}} \wedge all_{ei}^{\{m\}} = all_{ei}^{\{j,m\}} = I_j \wedge I_m$ , the analysis is clear:

- (7.35) a. John and Mary's reports look fine. (universal reading)  
 b.  $Poss(all_{ei}, \{j, m\}, every, write)(report, look\ fine)$   
 c.  $\emptyset \neq report \cap (write_j \cup write_m) \subseteq look\ fine$

So all reports written by either John or Mary looked fine (to the teacher), and at least one report was written by one of them. This is a reasonable reading; suppose the reports were anonymous, the teacher didn't know who wrote which report, only that it was either John or Mary, and that it turned out that in fact Mary wrote all the reports. That would not make (7.35a) false—in fact, not even odd to utter under the circumstances.

Under other circumstances there would be a tendency to assume that each wrote a report, but this seems like a pragmatic effect. Consider *Peter's children's reports look fine*. Here there is no requirement that all of his children wrote reports. Suppose the children are John, Mary, Ellen, and Sue. Replacing *Peter's children* by *John, Mary, Ellen, and Sue* may not preserve implicatures, but it should preserve truth value. Going to the trouble of enumerating all the children may conversationally implicate that each wrote a report (except in special circumstances).

But we can also get the stronger reading where each wrote a report: either one of the following two sentences works (see also section 7.12.2).

- (7.36) a. John's reports look fine and Mary's reports look fine.  
 b. John's and Mary's reports look fine

As to *John or Mary's*, since  $I_j \vee I_m = some^{\{j,m\}}$ , we obtain

- (7.37) a. One of John or Mary's reports was rejected.  
 b.  $Poss(some, \{j, m\}, one, write)(report, rejected)$   
 c.  $|report \cap write_j \cap rejected| = 1$  or  $|report \cap write_m \cap rejected| = 1$

The sentence is equivalent to

- (7.38) One of John's reports was rejected or one of Mary's reports was rejected.

Here is a final example: again, one where the existential reading seems natural.

- (7.39) a. More male than female students' reports were rejected.  
 b.  $Poss(more\ male\ than\ female, student, some, write)(report, rejected)$   
 c.  $|male \cap student \cap \{a : report \cap write_a \cap rejected \neq \emptyset\}| >$   
 $|female \cap student \cap \{a : report \cap write_a \cap rejected \neq \emptyset\}|$

These examples illustrate the range of cases where *Poss* can be applied. A more precise statement of how wide that range is will be given in section 7.12.3.

Let us see next how relational nouns can be dealt with on our account. We assume that in, say, *John's sisters*, *sister* is a noun denoting the domain of the '(is a)

sister of' relation:

$$sister_N = \{a : \exists x \text{ sister of}(a, x)\} = \text{dom}(\text{sister of})$$

i.e. the set of people who *are* sisters (to someone). This seems entirely reasonable, and has nothing to do with possessives *per se*. If you say Susan is about to become a mother, you mean exactly that she is about to belong to the domain of the 'mother of' relation. Now we obtain the correct truth conditions for

(7.40) Two of John's sisters are graduate students.

using *sister of*<sup>-1</sup> as the possessor relation,<sup>17</sup> since, clearly,

$$\text{dom}_{sister_N}(\text{sister of}^{-1}) = \text{dom}(\text{sister of}^{-1}) = \text{the set of people who have at least one sister}$$

That is, we analyze (7.40) as

$$Poss(\text{all}_{ei}, \{j\}, \text{two}, \text{sister of}^{-1})(sister_N, \text{gradstudent})^{18}$$

The operation *Poss* handles possessives of the forms (7.2) and (7.3) in section 7.1. In the form (7.4) a further condition, coming from numerals such as *several*, *few*, *three*, etc., is added. The definition of *Poss* is easily adapted to this case, as follows. (If  $[[\text{Num}]]$  is thought of as a type  $\langle 1, 1 \rangle$  quantifier, then instead of  $[[\text{Num}]](X)$ , write  $[[\text{Num}]](X, X)$ .)

First, define

$$(7.41) \text{dom}_A^{[[\text{Num}]]}(R) = \{a : [[\text{Num}]](A \cap R_a)\}$$

That is,  $\text{dom}_A^{[[\text{Num}]]}(R)$  is the set of individuals such that the set of objects in  $A$  that they 'possess' satisfies the given numerical condition. For example,  $\text{dom}_{luxcar}^{[[\text{three}]]}(\text{own})$  is the set of individuals owning exactly three luxury cars.

Next, refine *Poss* by simply narrowing to  $\text{dom}_A^{[[\text{Num}]]}(R)$  rather than to  $\text{dom}_A(R)$ . Since for the numerals usually allowed here,  $[[\text{Num}]](X)$  implies  $X \neq \emptyset$  (so *zero*, for example, is not allowed), this strengthens the narrowing effect, in the sense that

$$\text{dom}_A^{[[\text{Num}]]}(R) \subseteq \text{dom}_A(R) = \text{dom}_A^{\exists \geq 1}(R)$$

Thus, we define a generalized operation *Poss'* as follows (in the simplified case when  $Q_1, Q_2$  satisfy CONSERV and EXT):

<sup>17</sup> Note that taking the inverse of the relation provided by the relational noun corresponds to a transformation that moves the object of a prepositional phrase following the possessed noun to a determiner position (e.g. transforming *sister of John* to *John's sister*).

<sup>18</sup> Thus, in the default case, the restriction of a possessive determiner is a noun, and the possessor relation  $R$  is 'free'. When the noun is relational, based on a relation  $R^{-1}$ , the mechanism described here kicks in. Vikner and Jensen (2002) instead take the relational case as the default. They then need to describe how non-relational nouns like *car* are 'coerced' into relational nouns in possessive contexts.

**the operation *Poss'***

With  $Q_1, Q_2, C, R, M, A, B$  as before,

$$(7.42) \text{ Poss}'(Q_1, C, Q_2, R, [[\text{Num}]]) (A, B) \iff \\ Q_1(C \cap \text{dom}_A^{[[\text{Num}]]}(R), \{a : Q_2(A \cap R_a, B)\})$$

The effect is that, say,

(7.43) Most professors' three courses are given in the spring.

is taken (on the universal reading) to mean that more than half of the professors *who give exactly three courses* (per year), give all of those courses in the spring. This seems to be the preferred reading.<sup>19</sup>

*Poss'* is here seen as the most general operation that the possessive morpheme 's can be taken to denote. It has five arguments, to be set in various ways in an utterance of a possessive sentence. The numerical parameter is either set directly to the denotation of a numerical expression following 's, or, when such an expression is lacking, set by default to  $\exists_{\geq 1}$ :  $\text{Poss}(Q_1, C, Q_2, R) = \text{Poss}'(Q_1, C, Q_2, R, \exists_{\geq 1})$ . In what follows, we restrict attention for simplicity to the case without explicit numerical conditions, i.e. to the operation *Poss*.

## 7.8 ALTERNATIVE ACCOUNTS

### 7.8.1 *Poss* without narrowing

Instead of saying that narrowing is always in place, one might, in order to cover *all* the facts (see the end of section 7.5), have to say that it is usually in place but sometimes is turned off. What is *Poss* without narrowing? A first idea might be simply to drop the intersection with  $\text{dom}_A(R)$  on the right-hand side of (7.30). But note that this right-hand side, since we are assuming CONSERV,<sup>20</sup> is equivalent to

$$Q_1(C \cap \text{dom}_A(R), \text{dom}_A(R) \cap \{a : Q_2(A \cap R_a, B)\})$$

That is, in reality *both the restriction and the scope of  $Q_1$  are narrowed*. For reasons that will appear just below, the appropriate interpretation of *Poss* without narrowing seems to be dropping narrowing in the restriction but keeping it in the scope. This

<sup>19</sup> Suppose each professor gives either two or three courses per year, and only teaches one semester. In this situation, one can unproblematically say

(i) Most professors' three courses are given in the spring, and most professors' two courses are given in the fall.

using narrowing as described here. Thus, (i) and (7.43) neither entail nor presuppose that most professors teach three courses.

<sup>20</sup> By CONSERV (twice):  $Q_1(X \cap Y, Z) \iff Q_1(X \cap Y, X \cap Y \cap Z) \iff Q_1(X \cap Y, Y \cap Z)$ .

means that the right-hand side can be written

$$Q_1^C(\text{dom}_A(R) \cap \{a : Q_2(A \cap R_a, B)\})$$

But then we can define an operation  $\text{Poss}^w$  taking an *arbitrary type*  $\langle 1 \rangle$  quantifier  $Q$  as argument, in addition to  $Q_2$  and  $R$ . Again, the formal definition has to extend the universe (this time to  $M \cup \text{dom}_A(R)$ ) in order to be fully general:

$$(7.44) \quad \text{Poss}^w(Q, Q_2, R)_M(A, B) \iff \\ Q_{M \cup \text{dom}_A(R)}(\text{dom}_A(R) \cap \{a : (Q_2)_M(A \cap R_a, B)\})$$

We shall assume here, however, that  $Q$  is EXT and that  $Q_2$  is CONSERV and EXT,<sup>21</sup> from which it follows that  $\text{Poss}^w(Q, Q_2, R)$  is CONSERV and EXT. Then, more simply,

### the operation $\text{Poss}^w$

$$(7.45) \quad \text{Poss}^w(Q, Q_2, R)(A, B) \iff Q(\text{dom}_A(R) \cap \{a : Q_2(A \cap R_a, B)\})$$

This can be applied to all possessive determiners of the forms we are considering, with the (apparent) advantage that one no longer needs to construe proper names, bare plurals, etc. as restricted quantifiers.

A first observation is that when the possessor determiner is symmetric, narrowing has no effect.

#### Fact 1

If  $Q_1$  is symmetric, then  $\text{Poss}(Q_1, C, Q_2, R) = \text{Poss}^w((Q_1)^C, Q_2, R)$ .

*Proof.* Recall that symmetry (under CONSERV) means that  $Q_1(A, B) \Leftrightarrow Q_1(A \cap B, A \cap B)$ . But then it is clear from (7.45) and (7.30) that, for any  $A, B$ , both operations result in the condition

$$Q_1(C \cap \text{dom}_A(R) \cap \{a : Q_2(A \cap R_a, B)\}, C \cap \text{dom}_A(R) \cap \{a : Q_2(A \cap R_a, B)\})$$

To further see the effect of using  $\text{Poss}^w$ , consider, first, proper names. In this particular case, there is no difference. We calculate: □

$$(7.46) \quad \text{Poss}^w(I_j, Q_2, R) = \text{Poss}(\text{all}_{\text{ei}}, \{j\}, Q_2, R)$$

So  $\text{Poss}^w$  gives the correct truth conditions for proper names, i.e. sentences like (7.34a), repeated here,

(7.34a) At most two of John's dogs are fierce.

<sup>21</sup> Assuming that  $Q_2$  is CONSERV and EXT is unproblematic, since it will always be a quantification denotation. But can we assume that the type  $\langle 1 \rangle$   $Q$  is EXT? Note that (Chs. 3.4 and 4.5.5) proper names, bare plurals, and all frozen relativized quantifiers are EXT, as well as Boolean combinations of these. So we can feel rather safe that the examples we will run into are in fact EXT. Note that this hinges on our having identified the notion of freezing correctly.



even though we dropped narrowing in the first argument. But notice that if we had dropped narrowing in the second argument too, the truth conditions for (7.34a) would have become

$$|\text{dog} \cap \text{own}_j \cap \text{fierce}| \leq 2$$

which is not what we want, since it omits the requirement that John owns at least one dog. It seems reasonable that narrowing should not matter for proper names, and the chosen definition of  $Poss^w$  gives the correct result.

However, for NPs other than proper names, it often gives incorrect results. For example, with a conjunction of proper names as in (7.35a), we don't get the same result as with  $Poss$ , but rather the reading, expressed in (7.36a), where both John and Mary wrote at least one report. That might seem tolerable, but consider a bare plural, as in

(7.47) Firemen's wives worry about their husbands.

With  $Poss^w$ , using the type  $\langle 1 \rangle$  quantifier  $\text{firemen}^{pl}$  for  $Q$ , we get an interpretation with the undesirable consequence that all firemen are married (and that all of them are men). That is not what (7.47) means, or even presupposes. Notice that it also seems unproblematic to say

(7.48) Firemen's wives worry about their husbands, and firemen's husbands worry about their wives.

without any implication that all firemen are both male and female, as the non-narrowed account would have it.

Although formally satisfactory, and uniformly applicable to arbitrary type  $\langle 1 \rangle$  quantifiers,  $Poss^w$  is designed not to handle narrowing, and thus gives the wrong truth conditions in many cases. Moreover, there doesn't seem to exist any straightforward way of obtaining narrowing with a definition of the form (7.45), precisely because you cannot always retrieve  $C$  when  $Q = (Q_1)^C$ .

### 7.8.2 A definiteness account

We take what have here been called *definiteness accounts* of possessives to have the following characteristics:<sup>22</sup>

1. Every possessive determiner contains a definite (though it may not appear explicitly in surface form).
2. When there is no explicit indication of how to quantify over the 'possessed' objects, the quantifier is always *the<sub>sg</sub>* or *the<sub>pl</sub>*.
3. Narrowing is not used in the restriction argument of the possessor determiner.

<sup>22</sup> This is our version of a definiteness account, mostly based on comments from Barbara Partee (personal communication). Published versions such as the one in Jensen and Vikner 2002 mention neither narrowing nor accommodation. Partee's comments sketch the general idea, but do not go into details, so we emphasize that we alone are responsible for this reconstruction.

*Re 3*: We have just seen how to define  $Poss^w$  without narrowing. *Re 2*: We will show presently that there is a good reason to use  $all_{ei}$  instead of  $the_{sg}$  or  $the_{pl}$ . *Re 1*: Consider the case when quantification over the ‘possessed’ objects is given by an explicit determiner. The following example illustrates how (we think) the definiteness account applies to this case.

- (7.49) a. At least two of most students’ books are stained.  
 b. For most students  $x$ , at least two of the books of  $x$  are stained.  
 c. for  $Q x$ ,  $Q_2$  of the  $As$   $R$ ’d by  $x$  are  $B$   
 d.  $Q(\{a : Q_2 \text{ of } the (A \cap R_a, B)\})$

In the last line the quantifier  $Q_2 \text{ of } the$  is used, which contains the definite  $the$  (interpreted here by  $all_{ei}$ ). This is a *partitive* construction, here taken to be the interpretation of a determiner of the form [Det of Det], where the second Det is definite. In section 7.11 below we discuss such constructions, and give a general interpretation rule for them by means of an operation  $Def$ .<sup>23</sup>

Thus we set

$$Q_2 \text{ of } the = Def(Q_2, all_{ei})$$

which, by Fact 3 below, gives the following result:

$$(7.50) \quad Q_2 \text{ of } the (X, Y) \iff X \neq \emptyset \ \& \ Q_2(X, Y)$$

In other words,  $Q_2 \text{ of } the$  is  $Q_2$  with existential import.

Now observe that the following equivalences hold:

$$\begin{aligned} Poss^w(Q, Q_2, R)(A, B) &\Leftrightarrow Q(dom_A(R) \cap \{a : Q_2(A \cap R_a, B)\}) \text{ [def. (7.45)]} \\ &\Leftrightarrow Q(\{a : A \cap R_a \neq \emptyset \ \& \ Q_2(A \cap R_a, B)\}) \\ &\Leftrightarrow Q(\{a : Q_2 \text{ of } the (A \cap R_a, B)\}) \text{ [definiteness account]} \end{aligned}$$

Thus, we see that, when no narrowing of the restriction is required,  $Poss$  can indeed be construed as ‘containing’ a definite quantifier. Truth-conditionally, at least, there is no difference between these two accounts.

### 7.8.3 A problem with $the_{sg}$ and $the_{pl}$

Note, however, that the last result depends on our using  $all_{ei}$  instead of  $the_{sg}$  and  $the_{pl}$ . We can now see that this is in fact the right choice. Consider

- (7.51) Most students’ tennis rackets are cheap.

The definiteness account takes this as: for most students  $x$ , the tennis rackets of  $x$  are cheap. But if this “the” means  $the_{pl}$ , (7.51) entails that most students own at least two

<sup>23</sup> And in sect. 7.9 we point out that the construal of [Det of Det] as a determiner is for most purposes inessential to the analysis; one could equally well have used the more common syntax with an NP of the form [Det of NP].

tennis rackets, and it surely doesn't entail that. That may well be false, while (7.51) is true.

Notice, further, that narrowing will not help. If we narrow the restriction argument but still use *the<sub>pl</sub>*, we get the entailment that most students who own at least one tennis racket in fact own at least two. Again, this may well be false, while (7.51) is true. It is not a claim that (7.51) makes.

We can make the same observation with respect to the earlier example (7.47). With narrowing in force (as we argued that it must be in this case), using *the<sub>pl</sub>* we would get the entailment that firemen are bigamists! Now, here one might argue that wives in (7.47) is a so-called dependent plural,<sup>24</sup> and that in fact one should use *the<sub>sg</sub>* instead. This may work for (7.47), given a society where people have at most one wife (at a time). But it doesn't work for (7.51). Students can own any number of tennis rackets (including zero). With *the<sub>sg</sub>*, (7.51) entails that most students who own a tennis racket own exactly one. That may happen to be true, but surely it isn't something that should follow from (7.51). This is even clearer with the already considered sentence (7.6):

(7.6) Each girl's brother feels protective of her.

It is obviously wrong to analyze this sentence as claiming that every girl has at most one brother.

The conclusion is that we should use *all<sub>ei</sub>*. Only then do we obtain the correct truth conditions in all of these cases—provided narrowing is also in place—for example, in the case of (7.51), that most students who own tennis rackets own only cheap ones.

We see that the quantifier *all<sub>ei</sub>*, although presumably not the denotation of words like *all* and *every* (see Chapters 1.1.1 and 4.2.1), is nevertheless a very useful tool in the analysis of quantification.<sup>25</sup> It is a kind of neutral version of the definite article, and sometimes works better than the singular and plural versions. Not only is the correspondence between syntactic and semantic number in natural languages not fully systematic, but sometimes both choices of semantic number are wrong.

#### 7.8.4 Narrowing versus accommodation

Consider again a case where something like narrowing really seems to be required. Using the definiteness account (and the treatment of relational nouns suggested earlier):

<sup>24</sup> I.e. a plural form that is syntactically motivated by another plural form in the sentence, but has no semantic impact. Thus, the plural form of *wheel* is required in

(i) Unicycles have wheels.

but the sentence does not claim that each unicycle has more than one wheel.

<sup>25</sup> Neale (1990) calls it *wbe*, in order to display its similarity to *the<sub>sg</sub>* and *the<sub>pl</sub>*, and the fact that it can be used in the interpretation of what he calls *numberless* descriptions: *whatever I say, whoever shot John F. Kennedy* (p. 46). He further thinks it should be used when his D-type pronouns are anaphoric to phrases like *every A, all As, each A* (p. 235).

- (7.52) a. Most professors' wives were there.  
 b. For most professors  $x$ , the wife of  $x$  was there.  
 c.  $most^{professor}(\{a : \emptyset \neq \text{husband of } a \subseteq \text{was there}\})$

(7.52c) entails that most professors are married, and we presume that no one really claims that (7.52a) entails that. However, while we have treated the existential import of *the* (or  $all_{ci}$ ) as part of the truth conditions, another view is that it is a presupposition. Say that a presupposition is *accommodated* in virtue of a fact that makes it satisfied. The presupposition in (7.52b) can be accommodated by assuming that all professors are married. That is what (7.52c) does. But one idea is that presuppositions can also be accommodated *locally*,<sup>26</sup> in this case in the restriction of *most*, i.e. by assuming that all professors *in the restriction* are married. This means, in effect, that *most* quantifies only over married (male) professors, which is precisely the effect of narrowing. Thus, without going into further details, we get the rough suggestion that

narrowing  $\approx$  local presupposition accommodation in the possessor's restriction argument

In other words, the suggestion is that a definiteness account equipped with a theory of local accommodation provides a way to handle narrowing, if desired. We note, however, that the problems with  $the_{sg}$  and  $the_{pl}$  referred to in the previous section would remain even if narrowing were accomplished by accommodation.

### 7.8.5 Summing up

We have seen that the definiteness account is less different from the account proposed in section 7.7 than one might have guessed. In particular, when narrowing has no impact, *and* when the mode of quantification (our parameter  $Q_2$ ) over the possessed objects is explicitly given by a determiner, the two accounts seem extensionally equivalent. Also, it is conceivable that local presupposition accommodation could give the same effect as narrowing.

But certain differences remain. First, narrowing is a clearer and simpler idea than local accommodation of presuppositions, we think. We implemented narrowing as the normal case; so our proposal accounts for all the sentences where narrowing is necessary, as well as the sentences where it makes no truth-conditional difference. We think narrowing is the normal case. The local accommodation account needs a *trigger* for cases where narrowing is necessary, and it is not entirely clear what triggers local accommodation. On the other hand, the definiteness account deals with the cases where narrowing is unwanted (assuming local accommodation is not triggered then), whereas our account needs to *block* narrowing, i.e. to use  $Poss^w$  instead.

A second, more important difference is that the definiteness accounts we know of use either  $the_{sg}$  or  $the_{pl}$  to interpret the definite article. That doesn't always work, at least not on our reconstruction of such accounts. You have to use  $all_{ci}$ .

<sup>26</sup> See Beaver 1997 for a useful survey of all matters related to presupposition, in particular accommodation (his sect. 4.4).

Third, and even more crucially, the definiteness account cannot handle the case when  $Q_2$  is implicit but not universal, in particular the existential readings exemplified in section 7.3 and later. In these cases, with or without narrowing/local accommodation, it simply gives the wrong truth conditions.

However, the account in (7.49d) remedies the second and third of these deficiencies, by modifying the definiteness account to (a) use  $all_{ei}$ , and (b) incorporate a parameter for the quantification over the ‘possessed’ objects, which is *always* present, whether that quantification is explicit or not. Even when it is implicit, this parameter can be set to *every* or *some*, as the definite article is already present in the partitive.

Then, not only when  $Q_2$  is explicit, as in at least three of most students’ books, is there a built-in partitive construction, but—according to the modified definiteness account—also when it is implicit, as in at most two cars’ tires. For example, on the existential reading, the implicit partitive in the latter example would be, according to (7.49d), at least one of the tires (belonging to car  $x$ ).<sup>27</sup>

In light of all this, we tentatively conclude that if the definiteness account admits that an implicit quantifier parameter is always present, *and* if it replaces  $the_{sg}$  as well as  $the_{pl}$  by  $all_{ei}$ , it can cover roughly the same data as our account here. There is a (perhaps big) difference regarding how frequently narrowing is judged to occur, and how it is handled, but neither account at present has a mechanism for deciding when it applies and when it doesn’t. We do think that the account presented here is simpler. But what is *right* about the (modified) definiteness account, from our perspective, is that there is indeed a trace of the definite article in possessives: namely, in the condition  $A \cap R_a \neq \emptyset$  found in both  $Poss^w$  and (7.49d). When narrowing is operative, that condition is still present, though it becomes less visible in  $Poss$ , since, because of CONSERV, it is taken care of by the narrowing of the restriction argument (of  $Q_1$ ).

We should add, however, that it is rather *existence* than *definiteness* that is present in possessives. Indeed, the usual idea of definiteness, in terms of either uniqueness or familiarity, is not one that we have found to do much work for possessives. Furthermore, we emphasize again, because it sometimes seems to be forgotten, that the question is not at all whether possessive determiners *themselves* are definite. That is a simpler question. In fact, some of them are, but most of them aren’t; we will see which ones in section 7.11.

In short, definiteness has interesting connections to possessive quantifiers, but it is not a characteristic feature of all of them. The possessives should be seen as a rich and significant class of quantifiers in themselves, with their own characteristic properties. For the rest of this chapter, we use exclusively the analysis of possessives in terms of  $Poss$ , i.e. with built-in narrowing.

<sup>27</sup> Note that *at least one of the* is roughly the same as *some* (perhaps with the extra condition that the restriction has at least two elements). This seems to us to be a better way to look at what are sometimes called *weak* (or even indefinite) definites in the literature; e.g. Barker 2004. Moreover, Barker predicts such readings only for possessives with relational nouns, whereas we have seen that at most two cars’ tires has a straightforward existential reading.

## 7.9 SEMANTIC RULES FOR POSSESSIVES

How do rules of semantic interpretation combine the meaning *Poss* of 's with the meaning of other parts of a sentence containing 's? A determiner Det with 's as an immediate constituent always has an NP preceding 's, as in the structure (7.5) in section 7.1. That structure is essentially built with the two rules

(npdet) NP  $\rightarrow$  Det N  
 (poss) Det  $\rightarrow$  NP 's

The rule (npdet) is completely general, and the semantic rule corresponding to it is the usual one: the set denoted by N is the restriction argument of the type  $\langle 1, 1 \rangle$  quantifier expressed by the Det, so the NP's interpretation, as a type  $\langle 1 \rangle$  quantifier results from freezing that argument.

The rule (poss) is also very general, although there are certain restrictions; we will come back to them in section 7.12 below. As for semantics, in the cases we consider the possessor NP will have been interpreted as a restricted type  $\langle 1 \rangle$  quantifier  $(Q_1)^C$ , and the rule determining the meaning of the possessive Det fills in  $Q_1$  and  $C$  as the first two arguments of *Poss*.<sup>28</sup> The third and fourth arguments that *Poss* wants are filled with a parameter  $Q_2$  for type  $\langle 1, 1 \rangle$  quantifiers and a parameter  $R$  for binary relations on individuals. What does the semantics say about these last two arguments? Here approaches to possessives split into different camps.

The approach we have adopted here has the semantic rules do nothing to them: they remain as parameters of the NP's meaning. Although pragmatic principles of utterance interpretation, and possibly even semantic rules for interpretation of other parts of the sentence may operate on these parameters, they are left untouched by this particular semantic rule.

Other approaches hold that a genuine semantic ambiguity exists in expressions like *John's child*, and postulate two different semantic rules (perhaps triggered by two slightly different syntactic structures), the one leaving the relation parameter free (to be set in utterance context by pragmatic principles), and the other fixing the parameter's value as the inverse of the child-of relation. The 'free' approach readily allows the addition of an extra mechanism to fix this parameter, as we saw in section 7.7.

The issue is largely a methodological one. Everyone agrees that the multiple possibilities for the choice of possessor relation can make a possessive NP highly ambiguous. The main question is how this affects a compositional—or 'quasi-compositional'—semantics for such phrases.

Standardly, compositionality is thought to presuppose disambiguation—otherwise talk of *the* meaning of complex expressions and their parts threatens to become

<sup>28</sup> We recognize that this rule is not strictly (locally) compositional: to apply it, one has to look one step further down the tree; i.e. not only the interpretation of the immediate constituents of the possessive Det but also that of their immediate constituents must be used. If all of these are available in the syntax, there is no problem in using the rule to figure out the meaning of the possessive determiner. The rule for *Poss*<sup>w</sup>, by contrast, is fully compositional, but we have seen that *Poss*<sup>w</sup> doesn't always provide correct interpretations.

empty.<sup>29</sup> However, the almost unlimited range of choices in the case when the possessor relation is free is *not* seen as a threat to compositionality. This freedom does not qualify as ambiguity, but rather as *indexicality* or, more generally, *context dependence*. An indexical pronoun like *you* or *he* can certainly be handled in a compositional semantics, either at the level of denotation or at the level of ‘character’, i.e. before particular individuals have been assigned to them. At that stage, they too are represented by free parameters.

The question, rather, is how to separate the free cases from the non-free ones, sometimes referred to as *inherent* (e.g. Partee 1983/97), sometimes further divided into *inherent* and a few other categories, as in the account of Jensen and Vikner (2002) mentioned earlier. Are there structural and/or lexical ambiguities involved? Which case is most basic: the free or the non-free case? Somewhat (but not completely) analogously: Are relational nouns like *sister* or non-relational ones like *car* the default case in possessive NP’s?<sup>30</sup>

In a series of papers, Partee and Borschev, on the one hand, and Jensen and Vikner, on the other, have debated these issues. Partee and Borschev (2003) classify the approaches as either *split*—with distinct analyses for the two (or more) main cases—or *uniform*—reducing all possessives to one default case. There are two main uniform candidates: one where relational nouns are the default case and non-relational nouns are coerced to denote relations, and one where the opposite holds (as in our suggested treatment of relational nouns in the preceding subsection). In that paper (and recently also Jensen and Vikner 2002), Partee and Borschev tend towards a split approach, at least for English. But they also present (their section 6) a uniform alternative which takes the case of a free relation variable as basic, much as in our account here: theirs is less general, in that quantified possessives are not dealt with at all, but more detailed in describing the mechanisms by means of which the parameter *R* is set. In the end, though, they are doubtful that this alternative can succeed.<sup>31</sup>

<sup>29</sup> In principle, compositionality doesn’t necessarily rule out ambiguity. One possibility is to take the *set* of meanings of an ambiguous expression to be the input to the function that composes (sets of) meanings. So-called Cooper storage (Cooper 1983) is an example of this.

<sup>30</sup> There is also the issue, not discussed here, of which of the possessive forms—prenominal, postnominal, or predicative—is most basic.

<sup>31</sup> One hurdle is how to account for the contrast in

- (i) a. The former mansion was damaged by fire
- b. Mary’s former mansion was damaged by fire.

where the second sentence has an ambiguity lacking in the first, depending on whether *former* modifies *mansion* or the possessor relation (which then becomes ‘formerly own’ in this case). A more serious problem for a uniform ‘free’ analysis, according to Partee and Borschev (2003), is the following contrast:

- (ii) a. Sanderson’s portraits are mostly better than his wife’s.
- b. If Kandinsky’s portraits had all been Gabriele Münter’s, then I suppose they would all be in Munich by now.

These questions are—as these authors admit—highly theoretical, and the empirical evidence is often hard to evaluate. So far, it has not convinced us of the need to posit a semantic ambiguity, or even to have one or more semantic rules that fix the relation parameter's value when a possessive determiner combines with an  $N'$ . So we present only the semantic rule that (Free<sub>3</sub>) shows is needed in any case. In fact, this is nothing more than the normal semantic rule for combining the meaning of a Det with that of an  $N'$ .

What about the strong, though not inviolable, tendency of relational nouns to favor interpreting the relation parameter  $R$  as the inverse of the relation  $R_1$  that gives the noun's meaning (i.e. the noun denotes  $dom(R_1)$  and  $R = (R_1)^{-1}$ )? One possibility seems to be to treat this simply as a pragmatic tendency to assign the parameter  $R$  the value  $(R_1)^{-1}$  when the possessed noun was semantically interpreted as  $dom(R_1)$ . Not only can this pragmatic tendency be overcome in the context of some utterances, but it must also compete with other strong tendencies. Again, *Mary's picture* may involve the relations of ownership, artist, or motive, but sometimes none of these tendencies prevails, and *Mary's picture* denotes a picture to which she bears an *ad hoc* relation, e.g. the picture she was told to discuss. There are undoubtedly many regularities involved in the description of the distribution of the various readings, but it does not necessarily follow from this that these regularities manifest themselves as semantic composition rules.<sup>32</sup>

All this is why we are content for now to accept that the speaker-hearer pragmatic toolkit provides a small array of 'usual and customary' ways to fix the possessor relation's parameter  $R$ , which the speaker-hearer draws upon to place a specific interpretation on an utterance of the possessive NP.

Now what about the parameter  $Q_2$ , which still remains? Here we can only say what we would do, since other approaches do not recognize the need for this parameter. We believe that  $Q_2$  too is frequently fixed by pragmatic principles. For *John's fingers* it seems to be fixed as *some* in the statement *John's fingers are dirty* but as *every* in *John's fingers are clean*. These choices conform to very general tendencies, such as to read *the windows* as 'some windows' in *The windows were open (when it began to rain)* but as 'all windows' in *The windows were closed (when it began to rain)*. There is little doubt that general principles are at work here. What may be doubted is that the principles find their expression as semantic rules.

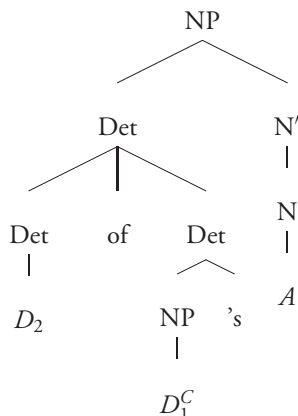
That said, however, there is one way of fixing  $Q_2$  that clearly is governed by semantic rule: namely, when the possessive NP occurs as part of the larger construction given by (7.53).

(ia) allows at least three possessor relations—owner, artist, motive—but requires the *same* relation in the elliptical phrase. (iib), on the other hand, allows *Münter's* to express ownership even when *Kandinsky's* does not, but if *Kandinsky's* uses either artist or subject, then *Münter's* can be the same relation but not the other one. According to Partee and Borschev, a split approach can explain these facts, but a uniform approach will have serious problems.

<sup>32</sup> E.g. the regularities exemplified in n. 31 may have more to do with the meaning of portrait than with the semantics of possessives.



(7.53)



Here we will take this structure to be built by an additional rule

(plex)  $\text{Det} \rightarrow \text{Det of Det}$

that forms complex determiners.<sup>33</sup>

The rule (plex) may seem to wildly (over)generate recursive structures. But we take it to be equipped with severe restrictions, which in fact prevent all but a few kinds of iteration. To state these, we shall use the following terms of art:

<sup>33</sup> We are following Keenan and Stavi 1986 in using a Det-forming rule, rather than the more common NP-forming rule

(part)  $\text{NP} \rightarrow \text{Det of NP}$

Keenan and Stavi argue at length in favor of (plex), one of their arguments being that one then easily generates directly, say,

- (i) Each of John's but not one of Mary's articles was accepted.

which seems harder with (part) and perhaps requires some sort of reduction rule. The rule (plex) goes nicely with the operation *Poss*. The two kinds of possessive determiners mentioned in sect. 7.1 will all denote (perhaps parametric) values of *Poss*. Also, the compositional semantics using (P-rule) below becomes somewhat simpler. A drawback, however, is that we cannot account in this way for unquantified NPs after [Det of], like *two of them* or *each of John, Mary and Sue*.

For most of what we say here, this issue is not really important; e.g. our remarks about the structure and meaning of iterated possessives in the next section are translatable, preserving the points we make, to structures based on (part).

Looking ahead to section 7.11.3, it is however, true that the compositional semantics using (D-rule) becomes somewhat simpler if (part) is adopted. Then the definition appropriate for the operation *Def* changes to the following: When  $Q_2$  is of type  $\langle 1, 1 \rangle$ ,  $Q$  is of type  $\langle 1 \rangle$ ,  $Q$  is definite, and  $B \subseteq M$ , define

$$\text{Def}'(Q_2, Q)_M(B) \iff Q_M \neq \emptyset \ \& \ (Q_2)_M(W_Q, B)$$

It is easily seen that, when  $Q = Q_3^A$ ,  $\text{Def}'(Q_2, Q)_M(B) \iff \text{Def}(Q_2, Q_3)_M(A, B)$ , for all  $M$  and all  $B \subseteq M$ .

**basic and complex possessives; partitives**

- (a) Determiners resulting from applying the rule (poss) are called *basic possessives*.
- (b) Determiners resulting from applying the rule (plex), where the second Det is a basic possessive, are called *complex possessives*.
- (c) Determiners of the form [Det of Det] are called *partitive*.

Thus, John's, most professors', some student's brothers' are basic possessive determiners, whereas most of John's, two of every student's, at least one of Mary's brothers' are complex possessives. No determiner can be both a basic and a complex possessive, but a complex possessive always contains a basic one. Complex possessives are partitive, but so are, for example, two of the and at least five of the ten, and there may be other rules resulting in structures of the form [Det of Det].

We emphasize that these are terms of art, used here to enable us to express certain things more precisely. In section 7.1 "possessive determiner" was used in a more intuitive sense, without giving a definition, and we may continue to use it thus, noting that the examples listed in (7.2) in that section are in fact basic possessives, and the ones in (7.3) complex possessives. (We could easily include the ones in (7.4) too, by a slight modification of the above definitions.) In addition, we noted that there are other possessive determiners, such as more of John's than of Mary's, which don't fall under the above categories, and which we don't treat systematically here (but see section 7.12.3).

Likewise, the term "partitive" is used here only to classify expressions of a certain form, but (so far) without any connotations about the right semantic analysis of these expressions.

Now we can express the restrictions on (plex) as follows:

- (plex-restr) (i) The left Det must *not* be: basic possessive, or definite, or partitive.  
 (ii) The right Det *must* be either basic possessive or definite, but it *cannot* be partitive.

This is corroborated by examples like:

- (7.54) a. few of the boys  
 b. each of the three girls  
 c. two of every student's books  
 d. at most one of Susan's mother's parents  
 e. \*Mary's of the three boys  
 f. \*the of the three boys  
 g. \*the two of the three boys  
 h. \*two of Mary's of the three boys  
 i. ?two of most girls  
 j. \*two of three of Mary's girls  
 k. \*two of each of the ten girls

(7.54e–h) illustrate the restrictions of the left Det. It is not enough to say that it cannot be partitive (in the above sense); clearly one must also exclude basic possessives and non-partitive definites.<sup>34</sup> Since the right Det cannot be partitive either, left and right branching with (plex) is excluded.

We are not saying that (part-restr) gets the constraints on (plex) exactly right. For example, a few more Dets must also be excluded on the left, such as *every* and *a*<sup>35</sup> (though *each*, *all*, and *some* are fine). But we think it gets things roughly right. We also note that it is possible to analyze explicitly proportional determiners, like *more than two-thirds of the* by means of (plex). So far we simply treated these—in accordance with much of the literature—as complex determiners, partly because one can plug in a restriction noun directly after such expressions, but not after *more than two-thirds*. However, if one delegates that problem to syntax, and thinks of *more than two-thirds* as denoting the quantifier

$$Q_{M(A,B)} \iff |A \cap B| > 2/3 \cdot |A|$$

and thus *more than two-thirds of the* as obtained with (plex), the natural corresponding semantic rule will give this partitive (in our sense) determiner the *same* denotation; this will be shown in section 7.11.3. Thus, a more satisfactory account of these proportional determiners is obtained.

The question mark in (7.54i) illustrates that there seem to be certain quantifiers other than basic possessives and definites that are sometimes permitted after [Det of]. We here take the position, however, that these should not be generated by the same rule; see section 7.11.2.

Thus, in the right position, two kinds of Det are permitted: basic possessives and definites (but a complex possessive like *three of John's*, or a partitive definite like *each of the ten* is excluded<sup>36</sup>). This contrasts with the standard accounts in the literature, which permit only definite determiners.<sup>37</sup> Recall that many standard accounts construe possessives as definite, or at least as essentially containing a definite. However, most possessives are in fact not themselves definite (we will see presently which ones are), and to somehow bring the definite that might be ‘contained’ in a possessive to the surface just in order to use a more traditional formulation of (plex) seems awkward at best. We find it more revealing to allow both basic possessives and definites.

Since we allow two kinds of determiners, with rather different semantics, we have *two semantic rules* corresponding to (plex). Here we describe the rule for the possessive case (P-rule); the second rule (D-rule) is introduced in section 7.11. Having two

<sup>34</sup> There is a slight problem with the definite *both*, whose interpretation on our analysis coincides with that of *the two*, but which, in contrast with *the two*, can occur to the left—*both of some student's parents*, *both of the problems*. Inversely, *the two* is fine in the right Det position, but *both* is not. We shall not address that problem here.

<sup>35</sup> This may be related to the fact that they cannot occur with a null noun.

<sup>36</sup> We observe in sect. 7.11 that, in contrast with most partitives, *each of the ten* is itself definite.

<sup>37</sup> Note that the ungrammaticality of *\*each of boys* militates against a putative null determiner in bare plural noun phrases, because that Det would be definite when interpreted as *alle<sub>i</sub>* (see sect. 7.11).

semantic rules may seem odd, and it would be simple to distinguish the two cases syntactically by splitting (plex) into two rules. This is not an important issue from a semantic point of view; the main linguistic point is that basic possessives are perfectly fine after  $[\text{Det} \circ \text{f}]$ , even when they are not definite.

Here is the rule for the possessive case.

(P-rule) A determiner of the form  $[D_2 \circ \text{f} D_1]$ , where  $D_1$  is a basic possessive with  $[[D_1]] = \text{Poss}(Q_1, C, Q_2, R)$ , is interpreted as  $\text{Poss}(Q_1, C, [[D_2]], R)$ .

So (P-rule) just formalizes what we have been arguing all along: namely, that when there is an explicit determiner, it sets the value of the  $Q_2$  parameter in *Poss*. When this determiner comes via the rule (plex), no pragmatic considerations can overturn this assignment of  $Q_2$ .

To see how these rules work, we take a brief look at iterated possessives.

### 7.10 ITERATED POSSESSIVES

The *semantics* of iterated possessive determiners is perfectly clear from our analysis: the type  $\langle 1, 1 \rangle$  quantifier  $Q'_1$  in  $\text{Poss}(Q'_1, C', Q'_2, R')$  can *itself* be the denotation  $\text{Poss}(Q_1, C, Q_2, R)$  of a possessive determiner. Thus, the general truth conditions of such a sentence are

$$(7.55) \text{Poss}(\text{Poss}(Q_1, C, Q_2, R), C', Q'_2, R')(A, B)$$

For the syntax, we suggested the three rules (npdet), (poss), and (plex). We now show that these rules, with their corresponding semantic rules, give an adequate first pass at explaining certain empirical facts concerning iterated possessives. The case when the second Det is definite is taken up in the next section.

Rules (npdet) and (poss) allow iterated possessives as in the following sentences:

(7.56) John's books' pages are stained.

(7.57) Mary's sisters' friends' children were there.

(7.58) Most students' bikes' tires are worn out.

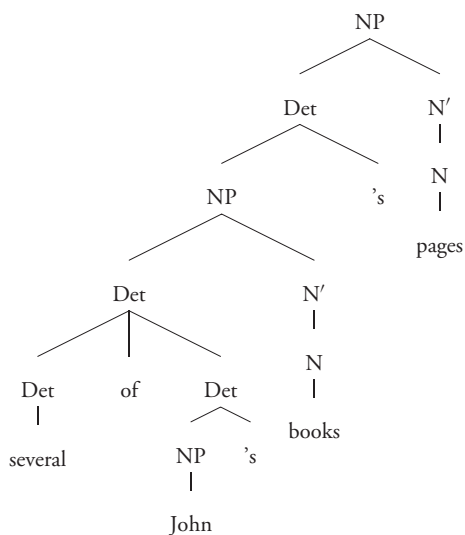
We note first that with iterated possessives we usually have at least two possessor relations. Freedom as described by (Free<sub>3</sub>) seems just as great as before (the two relations need not constrain each other). In (7.56), for example, there is an ownership relation and a part-whole relation in the default case, but as usual, other relations can be made salient by context. In the general case, both relations need to be treated as parameters.

Second, the ambiguity between universal, existential, and other readings already seen in basic possessives multiplies when the construction is iterated. For example, (7.56) could be used to say that each of John's books has every page stained, or, perhaps more naturally, that each book has some pages stained, but other readings may be possible as well. Since readings are fixed by setting the parameters  $Q_2$  and  $Q'_2$  in (7.55), we can unproblematically represent them.

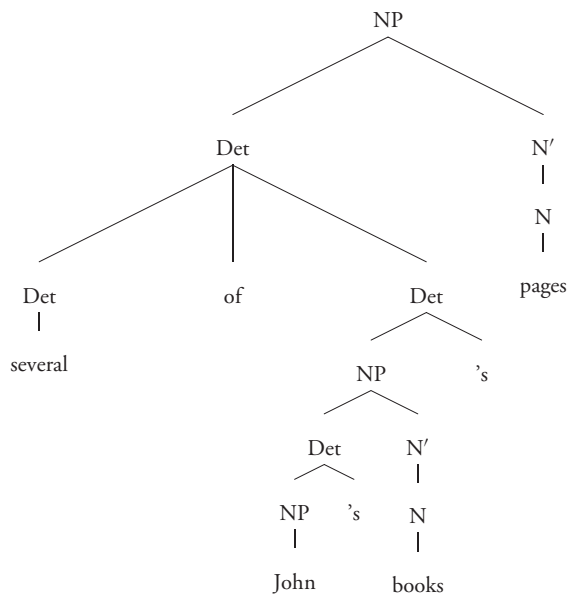
Now consider complex iterations, where rule (plex) is involved. The main NP in the next sentence has two analyses, as follows:

(7.59) a. Several of John's books' pages are stained.

b.



c.



In (7.59b), (P-rule) requires the parameter  $Q_2$  from the interpretation of *John's* to be set to *several*, leaving free the parameter  $Q'_2$  coming from the possessive *'s* applied to the NP *several of John's books*. The universal reading would then say that several of John's books are such that all of their pages are stained. In (7.59c), on the other hand, it is  $Q'_2$  that must be set to *several* by (P-rule), so the universal reading (for  $Q_2$ ) would say that each of John's books has several of its pages stained. This seems to capture the ambiguity of (7.59a) correctly.

In the following variant, morphosyntax suggests that only one book is involved, i.e. that only the structure of the form b is present.

(7.60) Several of John's book's pages are stained.

The next example has the same structure, but here number agreement distinguishes between the two readings.

- (7.61) a. One of John's ex-wives' previous husbands were millionaires.  
 b. One of John's ex-wives' previous husbands was a millionaire.

(7.61a) can only mean that of John's ex-wives, one of them is such that her previous husbands are millionaires. (Presumably all of her husbands were millionaires, and we have a universal reading.) For (7.61b), the only reading is: among the husbands of John's ex-wives, one was a millionaire. That is, *one* quantifies over husbands here, by (P-rule).

Thus, on the present account, the *scope* ambiguities arising from the interaction of partitives with iterated possessive determiners are reflected in a difference of syntactic structure, as between (7.59b) and (7.59c), in contrast with the ambiguities that arise from quantifier parameters that are not set by explicit determiners.

Next, we observe that certain iterations of possessives do not make sense. Let us see how we can account for these facts. Consider first

(7.62) #Many of some of John's books are stained.

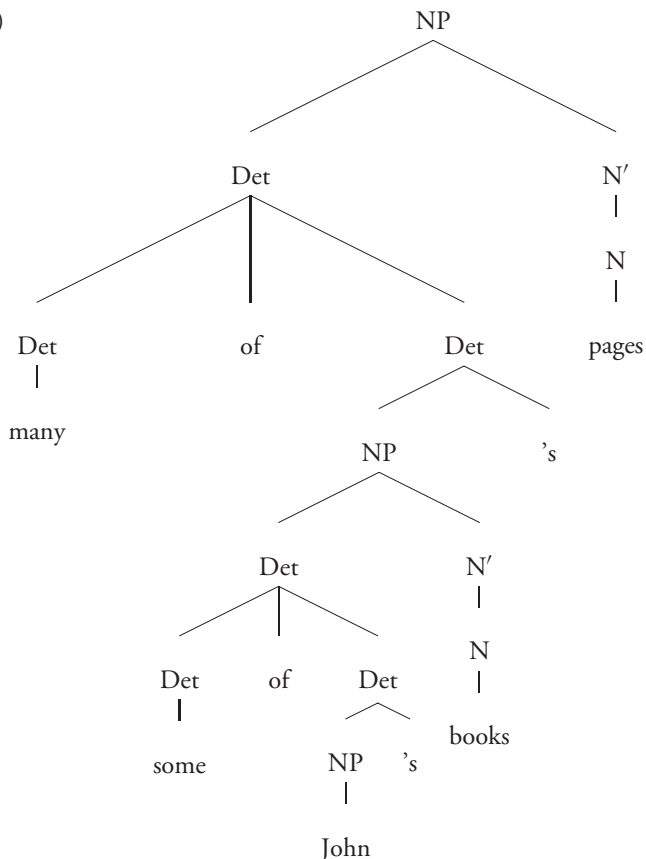
There is no way to derive this sentence by our rules. *Many of some* cannot be derived, since *some* is not a basic possessive or definite Det. *Some of John's books* can be derived, but it can never be the right Det in an application of (plex), since it is partitive. And indeed, even if (7.62) were somehow syntactically derivable, it would make no sense, as the idea behind (P-rule) shows. The  $Q_2$  parameter from *John's books* would have to be set to *some*, and the form also requires that many set some such parameter, but there is none left.

By contrast, take

(7.63) Many of some of John's books' pages are stained.

The string *some of John's* is derivable, but, being partitive, it cannot be appended to *many of* by (plex). But *some of John's books'* is derivable by (poss), as a basic possessive, so there is one (and only one) way of deriving (7.63) that complies with the requirements in (plex-restr):

(7.64)



The Det *some* of *John's* is derivable, and the  $Q_2$  parameter is set to *some*. So *some* of *John's books'* is derivable with (npdet) and (poss); it is a basic possessive, with parameter  $Q'_2$ , say. Therefore (plex) applies again, and  $Q'_2$  is set to *many*. We get the interpretation that some of John's books are such that many of their pages are stained. And this seems indeed to be the only possible reading of (7.64).

Here is a slightly more natural example:

(7.65) Both of many of my friends' parents work.<sup>38</sup>

<sup>38</sup> An editor would want to change this to *Both parents of many of my friends work*, but clumsiness is not meaninglessness. However, it is interesting to note in this connection that while not all possessive NPs may have a postnominal as well as a determiner form (e.g. *John's friends are hard to find* does not seem synonymous with *Friends of John ('s) are hard to find*), many do; cf. *Most of John's books are pristine* and *Most books of John ('s) are pristine*. In the latter form, the ordinary determiner position of the NP containing the postnominal possessive expresses the  $Q_2$  parameter, an effect that (P-rule) achieves in the former case.

## 7.11 DEFINITES AND POSSESSIVES

In section 7.8 we tried to clarify the way in which possessive determiners can be considered to *contain* a definite. Now let us ask which possessive determiners *are themselves* definite. Although it is not uncommon to claim that possessive determiners are definite,<sup>39</sup> this is in fact usually not true. We begin by showing this, and then move on to the question of what to make of the fact that basic possessives nevertheless occur freely after [Det  $\circ$ f], and finally to the semantic rule (D-rule) that interprets (plex) when the right Det is definite.

## 7.11.1 Which possessives are definite?

We gave a semantic definition of definite quantifiers and determiners in Chapter 4.6, and showed there that when  $Q_1$  is definite, the principal filter  $(Q_1)_M^A$  is generated by the *same* subset of  $A$  on each  $M$  where it is non-trivial—indeed, by the set  $W_{(Q_1)^A}$ , which is also the smallest set that  $(Q_1)^A$  lives on. With a precise semantics for possessives we can now check which of the possessive determiners are definite.

**Proposition 2**

(CONSERV, EXT) *If  $Q_1$  is definite, so is  $Poss(Q_1, C, \text{every}, R)$ .*

*Proof.* By definition (7.30) we have

$$Poss(Q_1, C, \text{every}, R)(A, B) \iff (Q_1)^{C \cap \text{dom}_A(R)}(\{a : A \cap R_a \subseteq B\})$$

Now fix  $A$ , a subset of a universe  $M$  that (we may assume) includes also  $C \cap \text{dom}_A(R)$ . If  $(Q_1)_M^{C \cap \text{dom}_A(R)}$  is empty, so is  $Poss(Q_1, C, \text{every}, R)^A$ . Otherwise, there is (Prop. 10 in Ch. 4.6) a (unique) generating set  $X \subseteq C \cap \text{dom}_A(R)$  such that

$$(Q_1)_M^{C \cap \text{dom}_A(R)} = \{B \subseteq M : X \subseteq B\}$$

We calculate

$$\begin{aligned} B \in Poss(Q_1, C, \text{every}, R)^A &\iff X \subseteq \{a : A \cap R_a \subseteq B\} \\ &\iff \forall a \in X (A \cap R_a \subseteq B) \\ &\iff \bigcup \{A \cap R_a : a \in X\} \subseteq B \end{aligned}$$

That is, the generating set for  $Poss(Q_1, C, \text{every}, R)^A$  is  $Y = \bigcup \{A \cap R_a : a \in X\}$ . Also, since  $X$  is a non-empty subset of  $\text{dom}_A(R)$ ,  $A \cap R_a \neq \emptyset$  for  $a \in X$ ; so it follows that  $Y$  is non-empty. This completes the proof.  $\square$

For example, the universal reading of *John's* and the *ten students'* is definite, and each of *John's* and each of the *ten students'* are always definite. For

<sup>39</sup> “Possessive determiners . . . are almost universally considered to be definite” (Abbott 2004: 123).



(each of) the ten students' cars, the filter is generated by the set of student-owned cars (provided there are exactly ten car-owning students).

But note that the above result depends crucially on the fact that (i) the universal reading is used, i.e.  $Q_2 = every$ , and (ii)  $Q_1$  is itself definite. It is clear from the proof that when one of these two conditions is not satisfied, the possessive determiner will in general *not* be definite.

In particular, the result fails for  $Q_2 = the_{pl}$ . For that case we get (with  $X$  as above)

$$B \in Poss(Q_1, C, the_{pl}, R)^A \Leftrightarrow X \subseteq \{a : |A \cap R_a| \geq 2 \ \& \ A \cap R_a \subseteq B\}$$

but this does *not* imply that  $Poss(Q_1, C, the_{pl}, R)^A$  is a principal filter, since there may be individuals  $a$  such that  $|A \cap R_a| = 1$  and  $A \cap R_a \not\subseteq B$ , and then it may happen that  $\bigcup \{A \cap R_a : a \in X \ \& \ |A \cap R_a| \geq 2\}$  is a subset of  $B$ , even though  $X$  is not a subset of  $\{a : |A \cap R_a| \geq 2 \ \& \ A \cap R_a \subseteq B\}$ .

On the other hand, the result holds for  $Q_2 = all_{ei}$ . This is because, as should be clear from the discussion in section 7.8.1,

$$(7.66) \quad Poss(Q_1, C, every, R) = Poss(Q_1, C, all_{ei}, R)$$

(for all  $Q_1, C, R$ ). Again, we have a context where the quantifier  $the_{pl}$  doesn't quite give the expected results (cf. section 7.8.3).

But  $the_{sg}$ ,  $the_{pl}$ , and  $all_{ei}$  are all definite, so the possessive determiners  $the_{sg}$  boy's,  $the_{pl}$  boys', and  $all_{ei}$  boys', in their universal reading, are also definite. For example,

$$(7.67) \quad The_{sg} \text{ boy's tennis rackets were stolen.}$$

on this reading says that there is a unique boy who owns tennis rackets, and all of his rackets were stolen.

Likewise, we noted in Chapter 4.6 that since  $C^{pl} = all_{ei}^C$ , (universal interpretations of) bare plurals are definite. Hence, universal readings of possessives built from bare plurals, like *policemen's*, also become definite.

### 7.11.2 What makes an expression definite?

What conclusion should be drawn from the fact that both definites and (non-definite basic) possessives occur freely after [Det  $\circ$ f]? If one were desperate to uphold the entrenched view that only definites go there, one might try to argue that either this fact *by definition* makes basic possessives definite, or that possessives do not 'really' appear in that position. To us it seems more natural to abandon the standard view.

As noted in Chapter 4.6, the main attempts to characterize definiteness, apart from Barwise and Cooper's more general (since applicable to all determiners) and purely semantic definition, are in terms of either *uniqueness* or *familiarity*. Based on these ideas, various *tests* for definiteness have been proposed. One looks at linguistic contexts where definites are barred and indefinites permitted, or vice versa. We consider just two examples. The first is the already mentioned partitive construction: it is

claimed that only (plural) definites are allowed after [Det  $\circ$   $\&$ ]. Let us call this diagnostic the *partitive test*. The second test concerns acceptability in existential-there sentences, the claim being that indefinites go well there, but not definites. Call this the *existential test*. In Chapter 6.3 we compared various attempts to characterize the existentially acceptable quantifiers. Though these disagree slightly, they all agree that definite determiners are not acceptable.<sup>40</sup>

Suppose for the moment that the results of these two tests agreed (they don't). Would one then be justified in concluding that each determiner or NP fitting after [Det  $\circ$   $\&$ ], and not fitting in existential contexts, was definite?

It seems that this conclusion would follow only under one of two conditions: *either* one claims to know for *independent* reasons that these tests are indeed tests for definiteness and nothing else, *or* one takes the tests themselves to *define* definiteness. But neither of these options seems attractive.

First, whereas all basic possessives are fine after [Det  $\circ$   $\&$ ], as we have seen, the outcome of the existential test is mixed. Unquantified possessives, as well as many quantified possessives, don't fit:

- (7.68) a. \*There are John's books on the table.  
 b. \*There were most/all/the three students' scarves hanging in the closet.

But other quantified possessives are fine:<sup>41</sup>

- (7.69) a. There were two students' scarves hanging in the closet.  
 b. There is a woman's car reported stolen.

Thus the two tests don't give exactly the same results, which already throws doubt on the position that they are simply tests for definiteness. Since [Det  $\circ$   $\&$ ] is our main concern here, we focus on the partitive test.<sup>42</sup>

Second, linguists' stance on the definiteness of basic possessives is less than clear, but those who explicitly address the issue, such as Vikner and Jensen (2002: 201 n. 5),

<sup>40</sup> Non-trivial definite quantifiers are positive strong, and not symmetric. Note that indefiniteness is not given as the criterion for acceptability by Barwise and Cooper (1981), but rather the property of being *weak*. For example, *every* is indefinite but positive strong by their definitions, and it is not existentially acceptable.

<sup>41</sup> Naturally occurring examples of the same kind are:

- (i) In reading the *NY Times* this morning I noted that there were two men's deaths last week that left many of us poorer for their passing. ([http://www.reynardfox.com/content/REF\\_billandal1.htm](http://www.reynardfox.com/content/REF_billandal1.htm))  
 (ii) The view is great, the food was good, but the service was poor and there were two children's birthday parties within a couple tables of ours. ([www.digitalcity.com/stlouis/dining/venue.adp?sbid=111716011&ctype=userreviews](http://www.digitalcity.com/stlouis/dining/venue.adp?sbid=111716011&ctype=userreviews))  
 (iii) I did that, explaining that there was a woman's car broken down on my road, and he should call right away about it. ([www.asstr.org/~jeffzephyr/stories/PickingBerries-Part1-JZ.html](http://www.asstr.org/~jeffzephyr/stories/PickingBerries-Part1-JZ.html))

<sup>42</sup> Woisetschlaeger (1983) gives an early discussion of definiteness in connection with possessives, focusing on the existential test. Since he only looks at possessive determiners headed by the indefinite article, such as

- (i) an old man's book

usually take care to distinguish the claim that such possessives *contain* a definite from the claim that they *are* definite. We know of no one who seriously *defines* definiteness by the partitive test. It is taken as a test for a property that is already known. For example, Keenan and Stavi (1986), who are among the few who discuss possessive determiners in partitive constructions, explicitly choose Barwise and Cooper's notion of definiteness, according to which, as we saw in the previous subsection, many basic possessives occurring after [Det  $\circ$   $\mathcal{E}$ ] are *indefinite*.

But there may be one option left for those who still wish to defend the partitive test: argue that, when a possessive comes after [Det  $\circ$   $\mathcal{E}$ ], the structure isn't really partitive any more, even though it seems to be on the surface.

Keenan and Stavi attempt this route. They note the conflict with Barwise and Cooper's definition in the case of possessives, as in

(7.70) They called the police because seven of some professor's manuscripts were missing.

but suggest (p. 298) that the construction *seven of some A's* could be eliminated in favor of a form where only definites occur after [Det  $\circ$   $\mathcal{E}$ ]. Although the particular rewriting they suggest isn't, to our minds, very convincing,<sup>43</sup> it could perhaps be seen as a version of the definiteness approach to possessives, which (at least on our version of that approach) systematically rewrites sentences of this form. Recall from section 7.8.2 that (7.70) would be recast as

(7.71) For some professors  $x$ , seven of the manuscripts  $R'$ d by  $x$  were missing.

Here, the manuscripts  $R'$ d by  $x$  is definite. And we saw in that section that, *as long as narrowing is ignored*, this rewriting always works.

However, as we have been at pains to argue, one shouldn't ignore narrowing. The simplest conclusion therefore, in our opinion, is that the partitive test does not in fact test for definiteness. It tests for definiteness *or* possessiveness. As a consequence, one cannot use the fact that basic possessives satisfy this test to discredit Barwise and

the partitive test does not apply. His main claims are that the existential test is not reliable for definiteness, and that (i) is definite, in spite of arguments to the contrary by, among others, Jackendoff (1974). Jackendoff had used the fact that two tests gave conflicting results to conclude that (i), although seemingly indefinite, was in fact ambiguous between a definite and an indefinite reading. Woisetschlaeger casts doubt on one of these tests—indeed, the existential test—but also makes clear that the whole discussion is meaningful only if one has access to an independently given notion of definiteness; here he opts for a definition in terms of familiarity.

While we appreciate Woisetschlaeger's presentation of the methodological situation, we disagree with his particular conclusion about (i). It seems to us that phrases of this form sometimes do introduce objects in the discourse: for example,

(ii) A graduate student's scarf was left hanging in the closet.

This scarf doesn't have to be mentioned in previous discourse for (ii) to make sense.

<sup>43</sup> Their proposal is to use

$$\textit{seven of some } A\textit{'s} = \bigvee_{a \in A} \textit{seven of } a\textit{'s}$$

But it is hard to see that this could work in general, say, for *seven of most A's*.

Cooper's definition of definiteness, and we will thus continue to use their definition here.

The issue of the correct analysis of [Det of] constructions is a complex one, theoretically and empirically. We do not claim to have given conclusive arguments for our account here, but we hope to have drawn attention to some pertinent facts, and to have suggested one way of looking at things. To end, we briefly mention some further relevant facts.

It may appear that some indefinite determiners are also admitted after [Det of]. A first set of cases is exemplified by

- (7.72) a. Two of ten interviewed persons have not answered the question. ([www.pts.se/Archive/Documents/EN/112-Misdialling%20of%20Alarm%20Number%20112.pdf](http://www.pts.se/Archive/Documents/EN/112-Misdialling%20of%20Alarm%20Number%20112.pdf))
- b. The newspaper is the moviegoer's PRIMARY source of information—five of ten rely most on the newspaper for show times and six of ten depend on their newspaper to find out where a movie is playing. ([www.beldenassociates.com/articleshowtime.htm](http://www.beldenassociates.com/articleshowtime.htm))

But these are not really partitives, but determiners for frequencies or proportions. (Note that *two out of ten* and *two in ten* can replace *two of ten*.) More interesting examples are as follows.

- (7.73) a. A reputed Haitian druglord charged with coordinating the movement of 33 tons of Colombian drug shipments through Haiti on their way to the United States pleaded guilty Thursday to two of five counts against him. ([http://www.haiti-info.com/article.php3?id\\_article=492](http://www.haiti-info.com/article.php3?id_article=492))
- b. Prior to launch, all but two of five tie-downs were removed, and the mid-geet was given positive buoyancy. (<http://www.usni.org/navalhistory/Articles99/Nhrodgaard.htm>)
- c. At least three of the six for the B.S. degree (two of five for the B.A. degree) must be at the 300-level or higher. (<http://www.roanoke.edu/biology/degree.htm>)

Here the final numeral seems to give the exact cardinality of the restriction. So these determiners actually express definite quantifiers, but the construction is perhaps an *appositive* (numeral) rather than a partitive. Note the alternation in (7.73c) between *three of the six* and *two of five*, indicating that these are interchangeable.

Tentatively, at least, it appears that (plex-restr) can be upheld.

### 7.11.3 A semantic rule for the definite case

In section 7.9 we proposed that (P-rule) should be used with (plex) when the right Det is possessive; this was merely a matter of setting the  $Q_2$  parameter in  $Poss(Q_1, C, Q_2, R)$  to the interpretation of the left Det. To formulate an interpretation rule for the definite case, we define the following operation:

**the operation *Def***

When  $Q_2, Q_3$  are of type  $\langle 1, 1 \rangle$ ,  $Q_3$  is definite, and  $A, B \subseteq M$ , define

$$(7.74) \text{Def}(Q_2, Q_3)_M(A, B) \iff (Q_3)_M^A \neq \emptyset \ \& \ (Q_2)_M(W_{Q_3^A}, B)$$

That  $(Q_3)_M^A \neq \emptyset$  is the non-triviality requirement; if it is satisfied, the generating set  $W_{Q_3^A}$  (the smallest set on which  $(Q_3)_M^A$  lives) exists and is independent of  $M$  (provided  $Q_3$  is also CONSERV and EXT; Proposition 10 in Chapter 4.6). For example, if  $|A| = 10$ ,  $[[\text{the ten}]](A, B) \iff A \subseteq B$  (generating set:  $A$ ). And if John owns at least one thing in  $A$ ,  $[[\text{John's}]](A, B) \iff A \cap R_j \subseteq B$  (generating set:  $A \cap R_j$ ).

It readily follows from Proposition 10 in Chapter 4.6 that

$$(7.75) \text{If } Q_3 \text{ is definite and } Q_2, Q_3 \text{ are CONSERV and EXT, then } \text{Def}(Q_2, Q_3) \text{ is CONSERV and EXT.}$$

Now the required rule is simply

$$(D\text{-rule}) \text{When } D_3 \text{ is definite, } [[D_2 \text{ of } D_3]] = \text{Def}([[D_2]], [[D_3]]).$$

Thus *Two of the ten boys were home* is correctly interpreted as *two (boy, home)*, provided there are exactly ten boys in the discourse universe.

Note that *John's*, in its universal reading, is both definite and possessive. So a determiner like *two of John's* can be interpreted both by (P-rule) and (D-rule). In this case the result is the same; we have

$$(7.76) \text{Poss}(all_{ei}, \{j\}, Q_2, R) = \text{Def}(Q_2, \text{Poss}(all_{ei}, \{j\}, every, R))$$

But this is not typical. In other cases, the choice between the two rules can create an ambiguity. This too is correct, since many such sentences are indeed ambiguous. Consider

$$(7.77) \text{Two of the ten boys' books are missing.}$$

This sentence is *three*-ways ambiguous. First, there is the question of whether *two* quantifies over boys or books. In the preceding section we saw how this can be cashed out as a structural ambiguity, and the semantic rules in our possession now provide the respective readings. (a) Of the ten boys, two are such that all of their books (universal reading) are missing:

$$\text{Poss}(\text{Def}(\text{two}, \text{the ten}), \text{boys}, \text{every}, R)(\text{book}, \text{missing})$$

(b) Each of the ten boys is missing two books:

$$\text{Poss}(\text{the ten}, \text{boys}, \text{two}, R)(\text{book}, \text{missing})$$

But there is yet another reading for the second structure, where *two* again quantifies over books: (c) Of the books owned by any of the ten boys, two are missing. In (b), twenty books are missing in all, whereas only two books are missing in (c), so the

truth conditions are quite different. The (c) reading, which seems entirely plausible, is accounted for by noting that the *ten boys'* is definite, by Proposition 2, *provided* the universal reading is used. So if the  $Q_2$  parameter in  $[[\text{the ten boys}']]$  is set to *every*, (P-rule) can no longer be applied; (D-rule), however, can be applied, giving the truth conditions of (c):

$$\text{Def}(\text{two}, \text{Poss}(\text{the ten, boys, every}, R))(book, \text{missing})$$

Note also that all three readings entail that there are exactly ten boys standing in the relation  $R$  to books.

So on this account, a single partitive structure can be semantically ambiguous in that there is a choice between using (P-rule) and (D-rule). (To repeat, this is one way of accounting for the facts. It would also be possible to enrich the syntax in order to have all the ambiguities come out as structural, and maintain strict compositionality.) It would be interesting to pursue these interpretation strategies in more detail, but we shall leave the matter here.

Finally, let us verify that (D-rule) allows an adequate treatment of determiners of the form  $D$  of *the*, provided *the* is interpreted as  $all_{ei}$ . The following is practically immediate from the definitions.

### Fact 3

$$\text{Def}(Q_2, all_{ei})(A, B) \iff A \neq \emptyset \ \& \ Q_2(A, B)$$

It follows that if  $D$  has existential import,  $D$  of *the* means the same as  $D$ . For example, *most* and *most of the* have the same truth conditions, and so do *some* and *some of the*, *more than two-thirds* and *more than two-thirds of the*, etc. In particular, note that in Chapter 4 and elsewhere we have treated *more than p qths of the*, *at least p q' ths of the*, etc. as unanalyzed determiners, simply because in English these phrases can be followed by a noun. But it seems equally possible to let the determiners be simply *more than p q' ths*, *at least p q' ths*, and add *of the* with the rule (plex). In the strict case, the truth conditions are the same as before. In the non-strict case (*at least p q' ths of the*), they are the same except when  $A = \emptyset$ . So these quantifiers, while not themselves definite, can be seen to contain a definite element, just as the corresponding surface forms indicate.

## 7.12 CLOSURE PROPERTIES OF *Poss*

Recall that a determiner is a basic possessive if it is formed by the rule (poss), and a complex possessive if formed by the rule (plex) where the right Det is a basic possessive. There are also other possessive determiners, such as *more of John's than of Mary's*.

On the semantic side, it is natural to use the following terminology:

- A *quantifier*  $Q$  is *possessive* if it belongs to the range of the operation *Poss*; i.e. if  $Q = \text{Poss}(Q_1, C, Q_2, R)$ , for some  $Q_1, C, Q_2, R$ .

It is then clear from our semantic rules that every basic or complex possessive determiner denotes a possessive quantifier, *provided* every basic possessive can be interpreted using *Poss*. We will attempt a justification of that assumption below.

In this section we take a quick look at closure properties related to possessives. For example, are the possessive quantifiers closed under Boolean operations? It turns out that they are closed under negation, but not, it seems, under conjunction. On the other hand, the class of denotations of basic possessives appears not to be closed under negation. These questions concern the *output* of *Poss*. There are similar questions about its *input*, and there are analogous issues concerning the syntactic range of possessive determiners. We shall only scratch the surface here, but we end up with a possible formulation of the syntactic restrictions on the rule (*poss*), which in turn leads to an assessment as to how adequate *Poss* is for the semantics of possessive determiners.

### 7.12.1 Negations

Pleasantly, the class of possessive quantifiers is closed under both inner and outer negations, and hence duals. This follows from the next fact, which is straightforward from the definition (7.30) of *Poss*.

#### Fact 4

- (a)  $\neg Poss(Q_1, C, Q_2, R) = Poss(\neg Q_1, C, Q_2, R)$   
 (b)  $Poss(Q_1, C, Q_2, R)\neg = Poss(Q_1, C, Q_2\neg, R)$

Thus, the square of opposition of a possessive quantifier consists of possessive quantifiers. But we note, *first*, that applying a syntactic negation does not necessarily amount to moving around inside one square of opposition, and, *second*, that it does *not* follow from Fact 4 that the square of opposition spanned by the denotation of a possessive determiner consists entirely of denotations of possessive determiners. In fact, it seems that the class of possessive determiner denotations is not closed under outer negation.

Consider first

(7.78) Two students' books are not missing.

If the unnegated sentence is taken in its universal reading—*Poss(two, student, every, R)*—then (7.78) can mean that two students are such that none of their books are missing, that is, is inner negation: *Poss(two, student, every  $\neg$ , R)*. Less plausibly, (7.78) could instead mean that it is not the case that two students' books are missing, i.e. that the number of students all of whose books are missing is distinct from two: *Poss( $\neg$ two, student, every, R)*. In addition, however, it seems that it could mean that two students are such that some of their books are not missing. This is *Poss(two, student,  $\neg$ every, R)*, but rather than thinking of this as an outer negation moving inside the possessive, it could be the *existential* reading of (7.78) that has received an inner negation: *Poss(two, student, some  $\neg$ , R)*. The point here is just that this quantifier does not belong to the square of opposition of the original one, but to another square, generated by the existential reading.

For an example that possibly leads outside the class of denotations of complex and basic possessives, consider

(7.79) Mary's friends are nice.

Applying outer negation yields the truth conditions that either Mary has no friends or at least one of her friends is not nice. It is doubtful that even *It is not the case that Mary's friends are nice* expresses this, and it is even more doubtful that some complex or basic possessive determiner would do the job.

The following possessive square of opposition, on the other hand, stays in the class of denotations of complex and basic possessive determiners:

$$\begin{aligned} Q &= \textit{most students}' \text{ (universal reading)} \\ \neg Q &= \textit{at most half of the students}' \text{ (universal reading)} \\ Q\neg &= \textit{none of most students}' \\ Q^d &= \textit{none of at most half of the students}' \end{aligned}$$

### 7.12.2 Conjunctions and disjunctions

Consider the sentence

(7.80) John and Mary's brothers came to the wedding.

In principle, this has four (universal) readings, but we shall immediately set two of these aside. First, the main NP can be the conjunction of two NPs, *John and Mary's brothers*, in which case (7.80) says that John came to the wedding and Mary's brothers too. This reading has nothing to do with conjoined possessives, so we disregard it and similar readings in what follows.

Second, there is the 'joint' reading which talks about brothers of *both* John and Mary. That may not be readily available here, but sometimes such a reading is forced by other factors, as in

(7.81) John and Mary's mother came on Graduation Day.

This can only mean that the mother of both John and Mary came. By contrast,

(7.82) John and Mary's mothers came on Graduation Day.

suggests that they have different mothers. The 'joint' reading appears to be related to the collective use of *John and Mary*, which is not something we deal with in this book; so we set it aside too.

The third reading of (7.80) is the one discussed in section 7.7: the possessive morpheme is attached to the NP *John and Mary*, producing a possessive determiner denoting  $Poss(all_{ci}, \{j, m\}, every, R)$ . As we noted, there is also a fourth reading, only slightly different, corresponding to  $Poss(all_{ci}, \{j\}, every, R) \wedge Poss(all_{ci}, \{m\}, every, R)$ . The only difference is that the latter requires John and Mary to have at least one brother each, whereas the former requires only the existence of a brother to either John or Mary. The fourth reading is expressed by either of the two sentences



- (7.83) a. John's and Mary's brothers came to the wedding.  
 b. John's brothers came to the wedding and Mary's brothers came to the wedding.

A natural suggestion is that the determiner in (7.83a) is obtained with a simple Determining rule,

(detconj) Det  $\rightarrow$  Det and Det

(to which in general various restrictions might apply), with the obvious corresponding semantic rule

$$[[D_1 \text{ and } D_2]] = [[D_1]] \wedge [[D_2]]$$

(7.83a) uses a determiner involving possessives that is not itself a basic or complex possessive (by our definition). So it seems that the class of denotations of basic and complex possessives is not closed under conjunction. The class of possessive quantifiers, however, is closed under this particular conjunction, since, as one easily verifies,

$$(7.84) \text{ Poss}(\text{all}_{\text{ei}}, \{j\}, \text{every}, R) \wedge \text{Poss}(\text{all}_{\text{ei}}, \{m\}, \text{every}, R) = \\ \text{Poss}(\text{the two}, \{j, m\}, \text{every}, R)$$

(From this we can also conclude, by the way, that this quantifier, just like  $\text{Poss}(\text{all}_{\text{ei}}, \{j, m\}, \text{every}, R)$ , is definite: use Proposition 2.) In general, though, we conjecture that the class of possessive quantifiers is not closed under conjunction; perhaps even the denotation of John's, and Mary's or Sue's is outside this class.

Analogous observations can be made for disjunction, *except* that there is no 'joint' reading in this case, and that  $\text{Poss}(\text{some}, \{j, m\}, \text{every}, R)$  is logically equivalent to  $\text{Poss}(\text{all}_{\text{ei}}, \{j\}, \text{every}, R) \vee \text{Poss}(\text{all}_{\text{ei}}, \{m\}, \text{every}, R)$ .

Turning now to the 'input' side, we discussed in section 7.9 the constraints on the rule (plex). As to the rule (poss), we said at the end of section 7.1 that there are remarkably few restrictions on the NP, but that some do exist. One such restriction concerns the case where the NP is conjoined from other NPs. Consider:

- (7.85) a. John and Mary's bikes were stolen.  
 b. John or Mary's bikes were stolen.  
 c. #John and some students' bikes were stolen.  
 d. #Two teachers or some students' bikes were stolen.  
 e. #?John, and Mary or Sue's bikes were stolen.  
 f. #Policemen but not lawyers' cars are dented.

It is clear that the form (7.85a) occurs. (7.85b) is perhaps less frequent, but sampling shows that it occurs as well.<sup>44</sup> For (7.85c–f), we disregard as before the case where the NP is a non-possessive NP conjoined with a possessive one; the # sign concerns other

<sup>44</sup> For example:

- (i) If no gift tax returns are filed, the IRS could question the valuation and assess estate or gift tax when reviewing John or Mary's estates after their deaths. (<http://sanjose.bizjournals.com/sanjose/stories/1999/01/18/focus5.html>)

readings. In all these cases we would get a perfectly good sentence if the possessive morpheme were attached to the first NP(s) too. But this gives a syntactic structure different from the intended one.

Indeed, conjoined NPs in general do not provide suitable inputs to *Poss*: this happens only when  $Q \wedge Q'$  or  $Q \vee Q'$  can be written in the form  $Q_1^C$ , which we have seen not to be possible in general (Proposition 8 in Chapter 4.5.5.3). The situation is different if we forget about narrowing and use  $Poss^w$  instead (section 7.8.1). One easily verifies that

$$\begin{aligned} Poss^w(Q, Q_2, R) \wedge Poss^w(Q', Q_2, R) &= Poss^w(Q \wedge Q', Q_2, R) \\ Poss^w(Q, Q_2, R) \vee Poss^w(Q', Q_2, R) &= Poss^w(Q \vee Q', Q_2, R) \\ \neg Poss^w(Q, Q_2, R) &= Poss^w(\neg Q, Q_2, R) \end{aligned}$$

Mathematically, *Poss* is less well-behaved than  $Poss^w$ . But apparently the restrictions on *Poss* correspond to limitations in natural language on possessor NPs. We just saw that conjoined NPs are not accepted inputs to (poss), except in the case of proper names. That is, their denotations cannot be arguments to *Poss*, the denotation of the possessive morpheme 's. In the next subsection we point to another example of disallowed inputs to *Poss*. Now, in theory there would be no obstacles to inputting these type ⟨1⟩ quantifiers to  $Poss^w$ . The difference between *Poss* and  $Poss^w$  is precisely narrowing. So it appears that what would otherwise be an unexplained gap in the possibilities for NPs to be possessors is explained by reference to the ubiquity of narrowing.

### 7.12.3 Some other determiners involving possessives

As Keenan especially has pointed out, there are more complex forms of determiners involving possessives than the ones we have looked at so far. One kind of example is related to the comparative type ⟨1, 1, 1⟩ quantifiers discussed in Chapter 4.7:

- (7.86) a. More students' than teachers' books were returned late.  
 b. At least as many professors' as lawyers' children go to this school.  
 c. More of John's than of Mary's reports were rejected.

These are not formed by the rule (poss); indeed, attaching the possessive morpheme to only the last of the restriction arguments of a two-place determiner is ungrammatical.<sup>45</sup>

- (ii) Contact John Allan if you have info on John or Mary's first spouse. (<http://www.ogs.on.ca/ogspsi/190/m190k002.htm>)

(The context of (ii) makes it clear that John and Mary are married to each other and that both had other spouses before.)

<sup>45</sup> One could formulate a rule attaching 's to *each* restriction argument and thus generating the first two sentences in (7.86). But, first, that rule would not generate the third sentence, and, second, it is not obvious that the operation *Poss* could be generalized accordingly.

The sentences in (7.86) can be analyzed using the following scheme:

$$(7.87) \text{ Poss}^2(Q_1, C, D, R)_M(A, B) \iff (Q_1)_M(\bigcup_{a \in C} R_a, \bigcup_{a \in D} R_a, A \cap B)$$

where  $Q_1$  is of type  $\langle 1, 1, 1 \rangle$ . For example, in (7.86c),  $Q_1 = \text{more\_than}$ ,  $C = \{j\}$ , and  $D = \{m\}$ . Note that there is no implicit quantifier in this construction. Indeed, attempts to spell out explicitly what that quantifier would be aren't even grammatical:

(7.88) \*Two/\*all/\*none/\*most of more students' than teachers' books were returned late.

So two-place determiners cannot occur after  $[\text{Det } \circ \mathcal{E}]$ , presumably because there would be no quantifier parameter for the determiner to set. Nor can NPs formed with two-place determiners be inputs to (poss), perhaps because, as indicated at the end of the previous subsection, that would conflict with the narrowing requirement.

### 7.12.4 Restrictions on the (poss) rule

If the observations earlier in this section are correct, the following general restriction on the rule (poss) suggests itself:

(poss) Det  $\longrightarrow$  NP 's

(poss-restr) The NP in (poss) cannot be a conjunction or a disjunction of other NPs (except when these are proper names), nor can it be formed with a determiner taking two or more arguments.

No doubt this needs to be refined. But if it is roughly correct, it follows that our interpretation of determiners formed with either (poss) or (plex) using the operator *Poss*—even though not as uniform as one might have wished—actually covers the examples we have encountered. It covers all quantified NPs (in view of the second part of (poss-restr)), and we handled (conjunctions and disjunctions of) proper names and bare plurals by interpreting them with frozen type  $\langle 1, 1 \rangle$  quantifiers. Proposition 8 in Chapter 4.5 showed that such a treatment of type  $\langle 1 \rangle$  quantifiers is not always possible, but, by the first part of (poss-restr), it is possible in the very cases we know to be needed for the interpretation of possessive determiners.<sup>46</sup>

<sup>46</sup> Proposition 8 in Ch. 4.5 also showed that *only John* cannot be treated in this way; but isn't

(i) Only John's books are stained.

a perfectly good sentence? It is, but its determiner isn't formed by attaching 's to the NP *only John*. Rather, *only* modifies the possessive determiner  $\mathcal{J}\text{ohn}'s$  (see n. 15 in Ch. 4.5), so that (i) is interpreted using the scheme

(ii)  $A \cap B \subseteq R_j$

*only John's* is then another determiner involving possessives not covered by *Poss*, but likewise not generated with (poss) or (plex). But since *only John* is a perfectly good NP, we also see that the restrictions in (poss-restr) are *necessary* for the application of the rule (poss), but not quite sufficient.

## 7.13 POSSESSIVES AND MONOTONICITY

Finally, we shall discuss the monotonicity behavior of  $Poss(Q_1, C, Q_2, R)$ , which is quite regular, but in more interesting ways than one might think. *Right* monotonicity is determined completely by the (right) monotonicity of  $Q_1$  and  $Q_2$ . *Left* monotonicity, on the other hand, depends in more complex ways on that of  $Q_1$  and  $Q_2$ . We give an overview, beginning with the right monotone case.

**Proposition 5**

*If  $Q_1$  and  $Q_2$  are right monotone in the same direction (i.e. both are  $MON\uparrow$  or both are  $MON\downarrow$ ), then  $Poss(Q_1, C, Q_2, R)$  is  $MON\uparrow$ . If they are monotone in opposite directions,  $Poss(Q_1, C, Q_2, R)$  is  $MON\downarrow$ .<sup>47</sup>*

*Proof.* Let us look at one of the four cases (the others are similar):  $Q_1$  is  $MON\uparrow$ , and  $Q_2$  is  $MON\downarrow$ . Suppose that  $Poss(Q_1, C, Q_2, R)(A, B)$  holds, i.e. that  $Q_1(C \cap dom_A(R), \{a : Q_2(A \cap R_a, B)\})$ , and  $B' \subseteq B$ . Since  $Q_2$  is  $MON\downarrow$ ,

$$\{a : Q_2(A \cap R_a, B)\} \subseteq \{a : Q_2(A \cap R_a, B')\}$$

Since  $Q_1$  is increasing,  $Q_1(C \cap dom_A(R), \{a : Q_2(A \cap R_a, B')\})$  holds; that is,  $Poss(Q_1, C, Q_2, R)(A, B')$ . □

These facts are illustrated by the soundness of inferences like the following (taking *Mary's* in the universal reading ('all of *Mary's*'), and *no professors'* in the existential reading):

- (7.89) a. *Mary's* bikes are expensive. [ $Q_1: MON\uparrow, Q_2: MON\uparrow$ ]  
 b. *Hence:* *Mary's* bikes are either expensive or new.
- (7.90) a. *No professors'* cars are rusty. [ $MON\downarrow + MON\uparrow$ ]  
 b. *Hence:* *No professors'* cars are gray and rusty.
- (7.91) a. *None of John's* pets is a dog. [ $MON\uparrow + MON\downarrow$ ]  
 b. *Hence:* *None of John's* pets is a black dog.

Turning to left monotonicity, here are some first observations. If (all of) *John's* dogs bark, does it follow that his black dogs bark? Only if he *has* any black dogs! So  $\downarrow JOHN's$  is not  $\downarrow MON$ , but almost. It satisfies the scheme

- (7.92) a. *John's*( $A, B$ )  
 b.  $A' \subseteq A$  and *John* 'possesses' some  $A'$   
 c. *Hence:* *John's*( $A', B$ )

Similarly for some cases when  $Q_2$  is explicit, like all of *John's*, or none of *John's*:

<sup>47</sup> This proposition generalizes Proposition 8 in Keenan and Stavi 1986, which deals only with universal readings, i.e. when  $Q_2 = every$ .

- (7.93) a. None of John's cars is a convertible.  
 b. John owns a black car.  
 c. *Hence*: None of John's black cars is a convertible.

Thus, when  $Q_1$  has existential import (as is the case with  $\text{JOHN}'s$ ), we must avoid  $A$  shrinking so much that all 'possessed' objects are excluded. For the upward case, on the other hand, no weak variants are needed. For example, *some of JOHN's*, or at least *k of JOHN's*, is simply persistent (indeed  $\uparrow\text{MON}\uparrow$ ):

- (7.94) a. At least one of John's black cars is an old convertible.  
 b. *Hence*: At least one of John's cars is a convertible.

Here is a definition of the weak version of left downward monotonicity:

**weak downward left monotonicity for possessives**

$\text{Poss}(Q_1, C, Q_2, R)$  is called *weakly anti-persistent*, or *weakly*  $\downarrow\text{MON}$ , iff  $\text{Poss}(Q_1, C, Q_2, R)(A, B)$ ,  $A' \subseteq A$ , and  $C \cap \text{dom}_A(R) = C \cap \text{dom}_{A'}(R)$  jointly imply  $\text{Poss}(Q_1, C, Q_2, R)(A', B)$ . Similarly for weakly  $\downarrow_{\text{NE}}\text{MON}$ , weakly  $\downarrow_{\text{NW}}\text{MON}$ , and weakly (co-)smooth.

The extra condition is that every  $C$  possessing something in  $A$  also possesses something in  $A'$ . When  $C = \{j\}$ , we have  $A \cap R_j \neq \emptyset \Leftrightarrow A' \cap R_j \neq \emptyset$ .

The above examples all involve possessives with proper nouns. Such possessives have characteristic monotonicity behavior, which the next proposition tries to capture. In Chapter 5.6 we pointed out that right monotone increasing quantifiers very often are smooth, and that an adequate account of the monotonicity behavior of a quantifier should include smoothness whenever applicable. We therefore add some facts about smoothness of possessives with proper nouns.

Recall the notion of symmetry of type  $\langle 1, 1 \rangle$  quantifiers (Chapter 6.1), and the fact that, under  $\text{CONSERV}$  (which we assume here), symmetry is equivalent to intersectivity ( $\text{INT}$ ), i.e. that only the intersection  $A \cap B$  matters for whether  $Q_{M(A,B)}$  holds or not. Also, the following terminology was introduced at the beginning of Chapter 4.2.

- (7.95)  $Q$  is *positive* if  $Q_{M(A,B)}$  implies  $A \cap B \neq \emptyset$ .

For example, *some*, *at least five*, *most* are positive, but *no*, *at most five*, *every* are not.

**Proposition 6**

Let  $Q' = \text{Poss}(\text{all}_{\text{ci}}, \{j\}, Q_2, R)$ .

- (a) If  $Q_2$  is persistent, so is  $Q'$ . If in addition  $Q_2$  is intersective and positive, then  $Q'$  is also smooth. Example: at least  $k$  of John's is persistent and smooth.  
 (b) If  $Q_2$  is smooth,  $Q'$  is weakly smooth. Example: most of John's is weakly smooth. (It is not left monotone.)  
 (c) If  $Q_2$  is anti-persistent,  $Q'$  is weakly anti-persistent. If in addition  $Q_2$  is co-intersective, then  $Q'$  is also weakly smooth. Example: all but at most  $k$  of John's, and hence all of John's, are weakly anti-persistent and weakly smooth.

(d) If  $Q_2$  is co-smooth,  $Q'$  is weakly co-smooth. Example: at most half of John's is weakly co-smooth.

*Proof.* (a) Suppose that  $Q'(A, B)$ , i.e. that  $A \cap R_j \neq \emptyset$  and  $Q_2(A \cap R_j, B)$ . If  $A \subseteq A'$ , it follows that  $A' \cap R_j \neq \emptyset$  and  $Q_2(A' \cap R_j, B)$ , since  $Q_2$  is persistent, i.e.  $Q'(A', B)$ . So  $Q'$  is persistent. Now suppose the extra conditions hold too. Since we have persistence, we need only check  $\downarrow_{\text{NE MON}}$  to verify smoothness. So suppose  $A' \subseteq A$  and  $A \cap B = A' \cap B$ . Then  $A \cap R_j \cap B = A' \cap R_j \cap B$ . Hence, by intersectivity,  $Q_2(A \cap R_j, B) \Leftrightarrow Q_2(A' \cap R_j, B)$ . So  $Q_2(A' \cap R_j, B)$  holds, and hence by positivity,  $A' \cap R_j \neq \emptyset$ . We conclude that  $Q'(A', B)$  holds, so  $Q'$  is  $\downarrow_{\text{NE MON}}$ .

(b) Suppose again  $A \cap R_j \neq \emptyset$  and  $Q_2(A \cap R_j, B)$ . If  $A \subseteq A'$  and  $A - B = A' - B$ , then  $\emptyset \neq A \cap R_j \subseteq A' \cap R_j$ , and  $(A \cap R_j) - B = (A' \cap R_j) - B$ , so by  $\uparrow_{\text{SE MON}}$ ,  $Q_2(A' \cap R_j, B)$ . This shows that  $Q'$  is  $\uparrow_{\text{SE MON}}$ . If instead  $A' \subseteq A$ ,  $A \cap B = A' \cap B$ , and  $\{j\} \cap \text{dom}_A(R) = \{j\} \cap \text{dom}_{A'}(R)$ , then  $\emptyset \neq A' \cap R_j \subseteq A \cap R_j$ , and  $(A \cap R_j) \cap B = (A' \cap R_j) \cap B$ , so by  $\downarrow_{\text{NE MON}}$ ,  $Q_2(A' \cap R_j, B)$ . That is,  $Q'$  is weakly  $\downarrow_{\text{NE MON}}$ .

(c) Similar to the proof of (a).

(d) Use (b) and Fact 4 from the previous section.  $\square$

Clause (a) of the proposition shows the existence of persistent and smooth natural language quantifiers that are not of the form *at least k*. As we remarked after Fact 11 in Chapter 5.6, this is possible because possessives are not ISOM.

Now let us consider arbitrary possessive quantifiers  $\text{Poss}(Q_1, C, Q_2, R)$ . Given the monotonicity behavior of  $Q_1$  and  $Q_2$  (remember that  $Q_2$  may be implicit), what can be concluded? Sometimes, the left *and* right pattern of  $Q_1$  transfers to  $\text{Poss}(Q_1, C, Q_2, R)$ . For example, the following is a sound inference:

- (7.96) a. Some of at least five students' sports cars are black convertibles.  
 b. Hence: Some of at least five students' cars are convertibles.

Here the possessive determiner is  $\uparrow_{\text{MON}}\uparrow$ , just as  $Q_1 = \textit{at least five}$  is. This holds because  $Q_2 = \textit{some}$  is persistent. If we replace *some* by the anti-persistent *none*, we see that *none of at least five students'* is instead weakly anti-persistent. Also, it is  $\text{MON}\downarrow$ , by Proposition 5. In fact, we will see that it has the stronger property of being weakly co-smooth.

Similarly, consider  $Q_2 = \textit{every}$ , and look at the universal reading—which is the preferred one—of the following two sentences:

- (7.97) a. Some students' black dogs are (all) fierce.  
 b. Some students' dogs are (all) fierce.

Neither one implies the other. Even if some students are such that all of their black dogs are fierce, it obviously does not follow that all of their dogs are fierce (these students may have brown dogs too, who are real wimps, and there may be no other students with only fierce dogs).

Conversely, even if some students are such that all of their dogs are fierce, these students need have no black dogs at all. But if each student having a dog also has a

black dog, then it does follow from (b) that all of the relevant black dogs are fierce. So the universal reading of *some students'* is weakly anti-persistent. In addition, we will see that it is weakly smooth.

Let us look at three more examples.

- (7.98) a. Every professor's cars are black convertibles. (universal reading)  
 b. *Hence*: Every professor's sports cars are convertibles.

Here we actually get full anti-persistence, indeed  $\downarrow\text{MON}\uparrow$ . This is because  $Q_1 = \textit{every}$ , and *every* has no existential import. The premise says that all car-owning professors are such that all of their cars are black convertibles. This is trivially true—by our standard reading of *every*—if there are no professors, or no car-owning ones. Thus, while *Mary's*, *some students'*, and *most cats'* do make existential commitments, *every professor's*, and similarly *no professors'* and *at most three professors'*, do not, and this affects monotonicity behavior in the way indicated. Moreover, the point does not depend on the choice of a universal reading.

- (7.99) a. Some of at most three professors' cars are convertibles.  
 b. *Hence*: Some of at most three professors' sports cars are black convertibles.

Again we get full anti-persistence. The premise says that at most three car-owning professors are such that at least one of their cars is a convertible. If there are no car-owning professors at all, or if none own a convertible, this is true. The conclusion follows, since sports car-owning professors are necessarily car-owning professors, so there cannot be more than three of them who own a convertible, let alone a black convertible. *Some of at most three professors'* is  $\downarrow\text{MON}\downarrow$ , just as  $Q_1 = \textit{at most three}$  is.

Thus, it is the anti-persistence of  $Q_1$  that makes the possessive quantifier fully anti-persistent in these cases. To appreciate this, compare with:

- (7.100) a. Some of less than half of the professors' cars are convertibles.  
 b. *Hence*: Some of less than half of the professors' sports cars are black convertibles—provided all car-owning professors own sports cars.

*Less than half of the* is not left monotone. The inference is sound, but now the proviso is necessary. Provided the set of car-owning professors is the same as the set of sports car-owning professors, the proportion of those owning a black convertible sports car cannot be bigger than that of those owning a convertible car of any color, so this possessive quantifier is weakly  $\downarrow\text{MON}\downarrow$ .

$\textit{Poss}(Q_1, C, Q_2, R)$  always inherits some monotonicity behavior from  $Q_1$  and  $Q_2$ . Cataloguing all cases would be a somewhat tedious exercise, but we shall try the reader's patience with the following proposition, which covers several common instances, including the examples mentioned above.

**Proposition 7**

Let  $Q' = \textit{Poss}(Q_1, C, Q_2, R)$ . Suppose first that  $Q_2$  is anti-persistent, i.e.  $\downarrow\text{MON}$ .

- (a) If  $Q_1$  is  $\text{MON}\uparrow$ ,  $Q'$  is weakly  $\downarrow\text{MON}$ .

More in detail, consider ‘universal-like’ readings of  $Q_2$ , i.e. when  $Q_2$  is  $\downarrow\text{MON}\uparrow$ , co-intersective, and non-trivial. (Under ISOM and FIN, it follows that  $Q_2 = \text{at most } k$ , for some  $k \geq 0$ ; see Chapter 6.4.)

- (b1) If  $Q_1$  is smooth and positive,  $Q'$  is weakly  $\downarrow\text{MON}\uparrow$  and weakly smooth. Example: most professors’ (universal reading).
- (b2) If  $Q_1$  is  $\uparrow\text{MON}\uparrow$ ,  $Q'$  is again weakly  $\downarrow\text{MON}\uparrow$  and weakly smooth. Example: at least  $k$  students’ (universal reading).
- (b3) If  $Q_1$  is  $\downarrow\text{MON}\uparrow$ ,  $Q'$  is  $\downarrow\text{MON}\uparrow$ . Example: every teacher’s (universal reading).  
Similarly, consider ‘negative’ readings, i.e.  $Q_2$  is  $\downarrow\text{MON}\downarrow$ , intersective, and non-trivial. (With ISOM and FIN,  $Q_2 = \text{at most } k$ , for some  $k \geq 0$ .)
- (c1) If  $Q_1$  is smooth and positive,  $Q'$  is weakly  $\downarrow\text{MON}\downarrow$  and weakly co-smooth. Example: none of most professors’.
- (c2) If  $Q_1$  is  $\uparrow\text{MON}\uparrow$ ,  $Q'$  is again weakly  $\downarrow\text{MON}\downarrow$  and weakly co-smooth. Example: none of at least  $k$  students’.
- (c3) If  $Q_1$  is  $\downarrow\text{MON}\uparrow$ ,  $Q'$  is  $\downarrow\text{MON}\downarrow$ . Example: none of each teacher’s.

Suppose instead that  $Q_2$  is persistent, i.e.  $\uparrow\text{MON}$ .

- (d) If  $Q_1$  is  $\text{MON}\downarrow$ ,  $Q'$  is weakly  $\downarrow\text{MON}$ . Example: some of less than half of the professors’.  
Consider ‘existential-like’ readings, i.e.  $Q_2$  is  $\uparrow\text{MON}\uparrow$ , intersective, and non-trivial. (Under ISOM and FIN,  $Q_2 = \text{at least } k$ , for some  $k > 0$ .)
- (e1) If  $Q_1$  is  $\uparrow\text{MON}\uparrow$ ,  $Q'$  is  $\uparrow\text{MON}\uparrow$  and weakly smooth. Example: some of at least three students’.
- (e2) If  $Q_1$  is  $\downarrow\text{MON}\downarrow$ ,  $Q'$  is  $\downarrow\text{MON}\downarrow$ . If in addition  $Q_1$  is intersective,  $Q'$  is co-smooth. Example: some of at most three teachers’.

*Proof.* For each case below, we start by assuming that  $Q'(A, B)$  holds, i.e.

$$(7.101) \quad Q_1(C \cap \text{dom}_A(R), \{a : Q_2(A \cap R_a, B)\})$$

The task is to show that if  $A'$  has certain properties,  $Q'(A', B)$  holds as well.

(a) Suppose  $A' \subseteq A$  and  $C \cap \text{dom}_{A'}(R) = C \cap \text{dom}_A(R)$ . Then  $\{a : Q_2(A \cap R_a, B)\} \subseteq \{a : Q_2(A' \cap R_a, B)\}$ , since for each  $a$ ,  $A' \cap R_a \subseteq A \cap R_a$  and  $Q_2$  is  $\downarrow\text{MON}$ . Thus,  $Q'(A', B)$  holds, since  $Q_1$  is  $\text{MON}\uparrow$ .

(b1) Since  $Q_1$  is smooth, it is  $\text{MON}\uparrow$ , so it follows from Proposition 5 and the assumption that  $Q_2$  is  $\text{MON}\uparrow$  that  $Q'$  is also  $\text{MON}\uparrow$ . By (a), all that remains to prove for weak smoothness is the property  $\uparrow\text{SEM}$ . So assume that  $A \subseteq A'$ ,  $A - B = A' - B$ , and (7.101) holds. We make the following claims:

(i)  $Q_2(\emptyset, B)$

This is because, by the positivity of  $Q_1$ , there is some  $a$  such that  $Q_2(A \cap R_a, B)$ , and hence  $Q_2(\emptyset, B)$  since  $Q_2$  is  $\downarrow\text{MON}$ . Next, it follows by definition that  $\text{dom}_{A'}(R) - \text{dom}_A(R) \subseteq \{a : A \cap R_a = \emptyset\}$ , and hence by (i),



$$(ii) \text{ dom}_{A'}(R) - \text{dom}_A(R) \subseteq \{a : Q_2(A \cap R_a, B)\}$$

Now we clearly have

$$(iii) C \cap \text{dom}_A(R) \subseteq C \cap \text{dom}_{A'}(R)$$

and also, by (ii),

$$(iv) (C \cap \text{dom}_A(R)) - \{a : Q_2(A \cap R_a, B)\} = \\ (C \cap \text{dom}_{A'}(R)) - \{a : Q_2(A \cap R_a, B)\}$$

Also, since by assumption  $(A \cap R_a) - B = (A' \cap R_a) - B$  and  $Q_2$  is co-intersective,

$$(v) \{a : Q_2(A \cap R_a, B)\} = \{a : Q_2(A' \cap R_a, B)\}$$

Finally, since  $Q_1$  is smooth, and hence  $\uparrow_{\text{SEMON}}$ , it follows from (iii), (iv), and (v) that

$$Q_1(C \cap \text{dom}_{A'}(R), \{a : Q_2(A' \cap R_a, B)\})$$

i.e.  $Q'(A', B)$ .

(b2) The conclusion is as in (b1), but the argument is simpler. Again, we only need to prove that  $Q'$  is  $\uparrow_{\text{SEMON}}$ . This time, given that  $A \subseteq A'$ ,  $A - B = A' - B$ , and (7.101) holds, we get (iii) and (v) above, as before, and then, since  $Q_1$  is  $\uparrow_{\text{MON}}$ ,  $Q'(A', B)$  follows.

(b3) By (a) and Proposition 5,  $Q'$  is weakly  $\downarrow_{\text{MON}\uparrow}$ . But we actually have full  $\downarrow_{\text{MON}}$ . Suppose (7.101) holds and  $A' \subseteq A$ . We have  $C \cap \text{dom}_{A'}(R) \subseteq C \cap \text{dom}_A(R)$ , and (v) holds as before, so, since  $Q_1$  is  $\downarrow_{\text{MON}}$ ,  $Q'(A', B)$  follows.

(c1)–(c3) If  $Q_2$  is  $\downarrow_{\text{MON}\downarrow}$ , intersective, and non-trivial, then  $(Q_2)\neg$  is  $\downarrow_{\text{MON}\uparrow}$ , co-intersective, and non-trivial. The desired results now follow from (b1)–(b3) using Fact 4.

(d) Similar to the proof of (a).

(e1) That  $Q'$  is  $\uparrow_{\text{MON}\uparrow}$  follows by reasoning of the kind already given. Moreover, it is trivially weakly  $\downarrow_{\text{NEMON}}$  (and hence weakly smooth), since if  $A' \subseteq A$ ,  $A \cap B = A' \cap B$ , and  $C \cap \text{dom}_{A'}(R) = C \cap \text{dom}_A(R)$ , neither of the arguments of  $Q_1$  in (7.101) changes when we replace  $A$  by  $A'$ .

(e2) Again, now familiar reasoning gives  $\downarrow_{\text{MON}\downarrow}$ . We need to prove that  $Q'$  is  $\uparrow_{\text{SWMON}}$  when  $Q_1$  is also intersective. So suppose (7.101) holds,  $A \subseteq A'$ , and  $A \cap B = A' \cap B$ . As before, (v) holds. But also,

$$(vi) C \cap \text{dom}_A(R) \cap \{a : Q_2(A \cap R_a, B)\} = C \cap \text{dom}_{A'}(R) \cap \{a : Q_2(A' \cap R_a, B)\}$$

For suppose  $a \in C \cap \text{dom}_{A'}(R)$  and  $Q_2(A' \cap R_a, B)$ , i.e.  $Q_2(A \cap R_a, B)$ . Since  $Q_2$  is non-trivial and  $\uparrow_{\text{MON}\uparrow}$ , we have  $A \cap R_a \cap B \neq \emptyset$ , and hence  $a \in C \cap \text{dom}_A(R)$ . This proves (vi), and since  $Q_1$  is intersective, it follows that  $Q'(A', B)$  holds.  $\square$

Note that when  $Q_2$  is persistent, our conclusions are slightly weaker than for the anti-persistent case. In particular, there is no analogue of (b1) or (c1). To see this, recall

the discussion in Chapter 5.6 of possible amendments to the (false) empirical conjecture in Väänänen and Westerståhl 2002 that all  $\text{MON}\uparrow$  determiner denotations are smooth. In particular (n. 17 in that chapter), one might guess that the conjecture was true for quantifiers that do not put a cardinality constraint on the restriction argument, i.e. if one excepts quantifiers like *at least  $k$  of the  $n$  (or more)*, which are  $\text{MON}\uparrow$  but not smooth. We now show that, somewhat surprisingly, certain possessives provide a different sort of counterexample. In fact, quantifiers like *at least  $k$  of most students'*, when  $k \geq 2$ , are  $\text{MON}\uparrow$  but not smooth determiner denotations, although the purely existential version is (weakly) smooth. This follows from the next proposition.

**Proposition 8**

*Let  $Q_1$  be any proportional type  $\langle 1, 1 \rangle$  quantifier; hence  $Q_1$  is smooth but not left monotone.*

- (a) (FIN) *some of  $Q_1$  students' is weakly smooth—and thus  $\text{MON}\uparrow$ —but not left monotone.*
- (b) *at least  $k$  of  $Q_1$  students' is  $\text{MON}\uparrow$  for  $k \geq 2$ , but not weakly smooth or left monotone.*

*Proof.* That these quantifiers are  $\text{MON}\uparrow$  follows from Proposition 5, and it is easy to see that they are not left monotone. For example, suppose some of more than half of the students' red bikes were stolen, and suppose further that each student owning a red bike owns a red racing bike. It does not follow that more than half of the students are such that some of their red racing bikes were stolen—the premises are consistent with no racing bikes at all being stolen. So *some of most students'* is not even weakly  $\downarrow\text{MON}$ . But neither does it follow that more than half of the students had some bikes stolen—perhaps only a small proportion of the bike-owning students own red bikes, and perhaps only red bikes were stolen; i.e. *some of most students'* is not  $\uparrow\text{MON}$  either.

Next, the quantifiers in (a) and (b) are trivially weakly  $\downarrow\text{NE}\text{MON}$ , by (one part of) the argument used for (e1) in the proof of the previous proposition.

Thus, for (a) we must show that  $\text{Poss}(Q_1, C, \text{some}, R)$  is  $\uparrow\text{SE}\text{MON}$ , where  $Q_1(X, Y) \Leftrightarrow |X \cap Y| > r \cdot |X|$  for some constant  $r$  such that  $0 < r < 1$ . So suppose

$$|C \cap \text{dom}_A(R) \cap \{a : A \cap R_a \cap B \neq \emptyset\}| > r \cdot |C \cap \text{dom}_A(R)|$$

and that  $A \subseteq A'$  and  $A - B = A' - B$ . Suppose the size of  $C \cap \text{dom}_A(R)$  increases by 1 (if it doesn't increase at all, there is nothing to prove); i.e. that  $C \cap \text{dom}_{A'}(R) = (C \cap \text{dom}_A(R)) \cup \{a_0\}$  for some  $a_0$  which has an  $R$ -successor  $b_0$  in  $A'$  but none in  $A$ . But then  $b_0 \in B$ , since  $A - B = A' - B$  and  $b_0$  is not in  $A$ , so  $A' \cap R_{a_0} \cap B \neq \emptyset$ . Thus,

$$\begin{aligned} & |C \cap \text{dom}_{A'}(R) \cap \{a : A' \cap R_a \cap B \neq \emptyset\}| \\ &= |C \cap \text{dom}_A(R) \cap \{a : A \cap R_a \cap B \neq \emptyset\}| + 1 \\ &> r \cdot |C \cap \text{dom}_A(R)| + 1 \\ &> r \cdot (|C \cap \text{dom}_A(R)| + 1) \\ &= r \cdot |C \cap \text{dom}_{A'}(R)| \end{aligned}$$

Since we have assumed FIN, the result now follows by induction.

Finally, for (b), here is an example showing that the above reasoning cannot be carried through when *some* is replaced by *at least two*. Let the following hold ( $C$  can be set to the universe  $M$ ):

$$\begin{aligned} \text{dom}_A(R) &= \{a_1, \dots, a_5\}, \text{dom}_{A'}(R) = \text{dom}_A(R) \cup \{a_6\} \\ A &= \{b_1, b_2\}, A' = A \cup \{b_3\} = B \\ R(a_i, b_j) &\text{ for } i = 1, 2, 3, j = 1, 2 \\ R(a_i, b_2) &\text{ for } i = 4, 5 \\ R(a_6, b_3) & \end{aligned}$$

(no other individuals are  $R$ -related). Then  $A \subseteq A'$  and  $A - B = A' - B (= \emptyset)$ . Also,  $\{a : |A \cap R_a \cap B| \geq 2\} = \{a_1, a_2, a_3\}$ , so

$$\text{most}(\text{dom}_A(R), \{a : |A \cap R_a \cap B| \geq 2\})$$

But  $\{a : |A' \cap R_a \cap B| \geq 2\} = \{a_1, a_2, a_3\}$ , since  $a_6$  has only one  $R$ -successor, so

$$\neg \text{most}(\text{dom}_{A'}(R), \{a : |A' \cap R_a \cap B| \geq 2\})$$

This shows that  $\text{Poss}(\text{most}, C, \text{at least two}, R)$  is not weakly smooth. The general case in (b) can be dealt with similarly.  $\square$

In conclusion, possessive determiners provide a rich variety of quantifiers with various combinations of monotonicity properties. They can be used to test hypotheses about how monotonicity relates to other linguistic phenomena, in particular to the distribution of polarity-sensitive items in natural languages (Chapter 5.9).

## 7.14 SOME REMAINING ISSUES

One major remaining issue is of course to extend the treatment of possessives suggested here to postnominal and perhaps predicate possessives (see section 7.1). Below we list a few questions about determiners involving possessives which were left open or insufficiently treated.

- Is it possible to define a more uniform version of  $\text{Poss}$  that still handles narrowing? Although we argued in section 7.12.4 that  $\text{Poss}$  actually covers most possessive determiners, whereas the more uniform  $\text{Poss}^w$  fails because of the phenomenon of narrowing, there might be better solutions still. In their absence, one should be open to the possibility that narrowing actually prevents a uniform (and strictly compositional) treatment.
- A technical question where we only hinted at some answers concerns the exact range and the closure properties of  $\text{Poss}$  (section 7.12).
- Narrowing usually applies, or so we have claimed. But there may be cases when one needs to turn it off (see the end of section 7.5). Is there a principled characterization of those cases?

- The issue of the syntax and semantics of partitive phrases is intimately connected to possessives. In sections 7.9–7.11 we sketched one proposal, in terms of the simple context free rule (plex), with its characteristic restrictions, and the semantic rules (P-rule) and (D-rule), but we also noted some possible problems. Although the issue may be partly methodological, we are certainly not claiming to have found the optimal solution. Presumably much more needs to be done here, but we hope that our focus on the (somewhat neglected) role of possessives after [Det of] has at least made some of the relevant questions stand out more clearly.
- We treated the possessor relation  $R$  as a free parameter, since this covers all uses, and some uses do require such a treatment. The idea was that in more constrained cases like relational nouns, special mechanisms can be added. But relegating all such additions to pragmatics could be seen as taking an easy way out of a real problem, and although we hinted at the vast and intricate literature on this issue, we didn't make a serious attempt to integrate such mechanisms with our semantics.

# 8

## Exceptive Quantifiers

Given man's fondness for generalizations—and the circumstance that exception appears to be the rule—it is not surprising that natural languages have a rich supply of handy means of expressing exceptions. *Exception phrases*, like the English *except (for) John, but Mary, except my enemies, other than students, apart from two philosophy professors, besides sailors, but not Deirdre's friends*, often consist of an *exceptive marker* and an NP. Though these markers all have to do with making exceptions, they differ somewhat in their semantic properties. In this chapter we focus on English, and on its presumably most typical exceptive marker: *except*.

*Except* can be inserted in sentences in several different positions. It was noted by Keenan and Stavi (1986) that exception sentences like those in (8.1a–c) can be seen as involving NPs formed by applying an *exceptive determiner* to a noun.

- (8.1) a. Every professor *except* Susan approved.  
b. No teaching assistant *except* John helped grade the papers.  
c. I haven't met anyone *except* Susan and John.  
d. No Bostonians *except* Deirdre's friends were at the party.  
e. No students *except* freshmen were invited.

(8.1d, e) are similar, but here a *set* is excepted rather than just one or two individuals.

We have already mentioned the determiners in (8.1a, b), with the truth conditions from Keenan and Stavi 1986:

$$\begin{aligned} \text{No } A \text{ except } j \text{ is } B &\iff A \cap B = \{j\} \\ \text{Every } A \text{ except } j \text{ is } B &\iff A - B = \{j\} \end{aligned}$$

The other examples are perhaps somewhat less obvious, but all the ones above can be fitted into the scheme

$$(8.2) \quad Q_1 A \text{ except } C B$$

where  $A, B, C$  are sets ( $C$  can be a singleton like  $\{j\}$ ). Thus, it is natural to look for an operator *Exc* that takes a type  $\langle 1, 1 \rangle$  quantifier  $Q_1$  and a set  $C$ , and forms a type  $\langle 1, 1 \rangle$  quantifier  $Exc(Q_1, C)$ . How is this operator defined?

Ultimately, what we want is an operator *Except* that takes a type  $\langle 1, 1 \rangle$  quantifier  $Q_1$  and a type  $\langle 1 \rangle$  quantifier  $Q$ , and produces a type  $\langle 1, 1 \rangle$  quantifier

$$Except(Q_1, Q)$$

This will be needed for more general forms of exception sentences, such as

(8.3) All beach-goers except a few enthusiastic swimmers were fully clothed.

Here *Q* interprets the NP *a few enthusiastic swimmers*. And in the earlier examples too, we see that *except* is most reasonably taken to attach to an NP: e.g. a proper name (*Susan*), a definite NP (*Deirdre's friends* with the universal reading), or a bare plural (*freshmen*).

It makes sense to start with the simpler operator *Exc*, however, and it is these simpler examples that have been discussed most extensively in the literature: e.g. Hoeksema 1996; von Stechow 1993, 1994; Moltmann 1995, 1996; Lappin 1996a, 1996b. This chapter begins by surveying the basic questions involved, and some standard views concerning them. More precisely, section 8.1 introduces the distinction between free and connected exception phrases; here we deal mainly with the latter. Sections 8.2–8.6 review certain claims often taken to be entailed by exception sentences. We cast some doubt on the so-called Inclusion Condition and the Negative Condition (in its various versions), arguing that these are often too strong. But we also suggest a principle of *exception conservativity* that seems to hold generally.

Section 8.7 presents some empirical findings by García-Álvarez (2003) that led him to question another received view, the so-called Quantifier Constraint, which says that only the quantifiers *every* and *no* admit exceptions. In sections 8.9 and 8.10 we present and analyze the influential accounts of exceptives in von Stechow 1993 and Moltmann 1995, respectively. Our conclusion is that neither of these gives a convincing argument for the Quantifier Constraint; moreover, they are not quite right for exceptions with *every* and *no* either, since they entail too strong versions of the Inclusion and Negative Conditions.

All of this leads us to look for an alternative. Our proposal for *Exc* is given in section 8.12; it is based on a simple idea about what an exception is (section 8.8). It handles García-Álvarez's examples, and delivers exception conservativity as well as what we think are suitable versions of the above-mentioned conditions. In fact, *Exc* comes in both a strong and a weak version; the strong version, which is sometimes but not always used, is closer to the previous accounts.

*Exc* is tentatively extended to *Except*—also in a strong and a weak version—in section 8.13. The definition we give develops an idea from Moltmann, and we examine how various sentences with quantified exceptions are interpreted by means of it (such sentences have not been given much attention in the literature), and suggest a principle for choosing between the strong and the weak version. We also note, however, that this operator is not fully general, and in section 8.14 we list various remaining issues for further study.<sup>1</sup>

<sup>1</sup> Many of our remarks in this chapter are inspired by an unpublished draft about exceptives by Iván García-Álvarez (as well as on the published García-Álvarez 2003): e.g. the criticism of the Negative Condition in the form we call (NC<sub>3</sub>) in sect. 8.5 below is essentially his. While our conclusions sometimes differ, they are in the same spirit as his remarks, which we gratefully acknowledge as a source in this chapter.

8.1 CONNECTED AND FREE EXCEPTION PHRASES

In the above examples, the exception phrase is in a clear sense connected to an NP. But such phrases can occur in other positions. Hoeksema introduced the terminology of “free” and “connected” exception phrases, which we follow here. Clear examples of free exception phrases with *except*<sup>2</sup> occur in:

- (8.4) a. Most of Mary’s relatives were there. Except (for) her father, of course.
- b. Except for John, no one has arrived yet.
- c. No one has arrived yet, except (for) John.
- d. No one, except (for) John, has arrived yet.

In writing, commas (or hyphens or parentheses) are often used to ‘disconnect’ the exception phrase.

It is plausible that the syntax of free exception phrases is different from that of connected ones, even when the free exception phrase is adjacent to the relevant NP. Of course it does not automatically follow that the semantics is also different, or that it cannot be reduced to that of connected ones.

Here we focus on connected uses of *except*. We are not interested in arguing that the analysis in terms of exceptive determiners is the correct one. Treating the exception phrase as an NP modifier (as in most of the literature after Keenan and Stavi 1986) is another alternative. Semantically, the difference is between rendering (8.2) as

$$except(Q_1, C)(A, B)$$

and rendering it as

$$except'(Q_1^A, C)(B).^3$$

For much of what we have to say, it will not matter a lot which alternative one uses. The main issue is a uniform account of the truth conditions of sentences of this form.

Now, the sentences in (8.1) obviously make universal claims with exceptions. To begin, one may in each case isolate the following three claims. First, there is what we here dub the *Generality Claim* that, excepting the Cs, the quantifier  $Q_1$  does relate *A* and *B*. Second, there is what Moltmann (1995) calls the *Negative Condition*, which says something about what happens if the exceptions are *not* made. Finally, what she calls the *Inclusion Condition* says, for example, in (8.1a) that Susan is a professor. We start by discussing each of these three conditions.

<sup>2</sup> These free uses appear to have a slight preference for *except for* rather than just *except*. However, it is just a tendency, and *except* is used as well.

<sup>3</sup> But notice that for the latter option to be an alternative, one must verify that this operation is well-defined, i.e. that when  $Q_1^A = Q_1^{A'}$ , one has  $Q_1 A \text{ except } C B$  iff  $Q_1 A' \text{ except } C B$ . In fact, the definition given in Moltmann 1995, to be discussed in sect. 8.10, does have this property, but this is less clear with some other definitions. From this point of view, it is safer to have  $Q_1$  and *A* as separate arguments, as we automatically get with the exceptive determiner analysis.

## 8.2 THE GENERALITY CLAIM

An apparently uncontroversial ingredient in the truth conditions for (8.2)—although there seems to be no accepted name for it<sup>4</sup>—is what we called the Generality Claim: i.e.

$$(GC) \quad Q_1(A - C, B)$$

For example, (8.1a) entails that if we remove Susan from the (relevant) set of professors, all in the remaining set approved, and (8.1d) entails that if we consider the set of Bostonians minus Deirdre's friends, no one in that set was at the party.

## 8.3 THE INCLUSION CONDITION

In the literature, the Inclusion Condition is usually taken to be

$$(IC_1) \quad C \subseteq A$$

(hence the name). This works fine for (8.1a) and (8.1b). But in general it does not seem quite right. Consider again (8.1d). Surely this sentence makes no general claim that *all* of Deirdre's friends are from Boston. There is nothing odd about the following discourse:

(8.5) No Bostonians except Deirdre's friends were at the party. Some of her friends from New York were there too.

The second sentence does not contradict the first; it only adds further information. Or consider the following even more obvious case:

(8.6) No students except foreigners need apply in advance for admission to the course.

There is certainly no assumption here that all foreigners (in the discourse universe) are students, so (IC<sub>1</sub>) is simply too strong.

On the other hand, (IC<sub>1</sub>) could be said to be too weak also, in that it allows  $A \cap C = \emptyset$ . But in fact no account of exceptives in the literature allows this to hold while (8.2) is true. For this reason, we suggest the following simple alternative:

$$(IC_2) \quad A \cap C \neq \emptyset$$

As we will see, this is usually taken to hold when (8.2) is true, although it is not stated explicitly. Note that when  $C$  is a singleton, the two versions are indistinguishable.

Of course we are not saying that (IC<sub>1</sub>) never holds when  $C$  is not a singleton. One case where there is a strong tendency for it to hold is when the elements of  $C$  are *enumerated*. Consider

<sup>4</sup> Garcia-Álvarez calls it the *Circumscribed Quantified Condition*. We have chosen a name that is clearly distinct from what is called the *Quantifier Constraint* in the literature; see sect. 8.8 below.



(8.7) Every student except Fred, George, and Harry was admitted to the tennis course.

Here, it seems clear that all the enumerated exceptions are students. In other cases the nature of the predicates involved can make (IC<sub>1</sub>) true; (8.1e) is a clear example of that. But our point is that, as other examples clearly show, (IC<sub>1</sub>) is not *in general* entailed by sentences of the form (8.2). Therefore it is not really part of the *semantics* of exception sentences, and we propose to replace it by (IC<sub>2</sub>).

#### 8.4 EXCEPTION CONSERVATIVITY

It might be objected that our account of the standard adherence to (IC<sub>1</sub>) is unfair, since in a sentence like (8.6), *foreigner* would be somehow contextually restricted to *foreign students*. That would indeed make (IC<sub>1</sub>) trivially true. There is something to this objection, we think, but it has nothing to do with contextual restrictions. Rather, it points to a general property of exceptives that we will call (for obvious reasons) *exception conservativity*: namely, that of the individuals in *C*, only those that are also in *A* count. That is:

##### exception conservativity

The exceptive determiner (i.e.  $Q_1$  – except *C*) in a sentence of the form (8.2) is *exception conservative* iff

$$Q_1 A \text{ except } C B \iff Q_1 A \text{ except } A \cap C B$$

or, more formally, in terms of the *Exc* operator discussed at the beginning of this chapter,

$$(8.8) \text{Exc}(Q_1, C)_M(A, B) \iff \text{Exc}(Q_1, A \cap C)_M(A, B)$$

It seems trivial that something can only be an *exception* to a claim of this form if it belongs to *A*. Indeed, just like ordinary conservativity for quantrelations, exception conservativity is illustrated by trivial equivalences as in the following pairs. The redundancy felt in the second sentence in these pairs indicates that exception conservativity is really taken for granted.

- (8.9) a. No students except foreigners need visas.  
b. No students except foreign students need visas.
- (8.10) a. Every professor except Susan Jones approved.  
b. Every professor except Professor Susan Jones approved.
- (8.11) a. All logicians except Swedes know this.  
b. All logicians except Swedish logicians know this.

Exception conservativity follows from the semantics for exceptives that we give in section 8.12, but it is *not* a property of the standard accounts, which, as we have said, instead assume (IC<sub>1</sub>), with the resulting need for some *ad hoc* maneuver in cases like (8.6).

## 8.5 THE NEGATIVE CONDITION

In Moltmann 1995, the Negative Condition is informally spelled out as follows: “Applying the predicate [denoted  $B$  in (8.2)] to the exceptions yields the opposite truth value from applying the predicate to non-exceptions” (p. 226). No doubt this has been a basic intuition, but it can be made precise in different ways. Cashing out “opposite truth value” as a negation, that negation can apply either to the whole sentence (outer negation) or to the VP (inner negation). The first alternative could be rendered as follows:

$$(NC_1) \neg Q_1(A \cap C, B)$$

Recall that we know exactly what the truth conditions are of exception sentences like (8.1a, b). These are hard facts that any proposed semantics must accommodate. And in these cases, given the Inclusion Condition,  $\neg \text{every}(A \cap \{s\}, B)$  is equivalent to  $s \notin B$  (Susan didn’t approve), and  $\neg \text{no}(A \cap \{j\}, B)$  is equivalent to  $j \in B$  (John helped grade the papers). Indeed, it is clear that in this simplest case, and as long as  $Q_1$  is *every* or *no*, the conjunction of (GC), (IC<sub>2</sub>) (or (IC<sub>1</sub>)), and (NC<sub>1</sub>) amounts *exactly* to the desired truth conditions.

But there is another rather plausible version of the first alternative mentioned above:

$$(NC_2) \neg Q_1(A, B)$$

This expresses the idea, explicitly endorsed in e.g. von Stechow 1993, that the exception sentence *should be false if modified so that no exceptions are made*. And one can easily check that

(8.12) Under (GC), and provided  $Q_1$  is *every* or *no*, (NC<sub>1</sub>) and (NC<sub>2</sub>) are equivalent.

Finally, the second alternative, with inner negation instead, is presumably the following:

$$(NC_3) Q_1(A \cap C, A - B)$$

This is often stronger than (NC<sub>1</sub>). For example, for (8.1d), (NC<sub>3</sub>) says that all of Deirdre’s Boston friends were at the party, whereas (NC<sub>1</sub>) says only that some of them were.

When  $C$  is a singleton set such that (IC<sub>2</sub>) holds, and  $Q_1$  is *every* or *no*, (NC<sub>1</sub>) and (NC<sub>3</sub>) are easily seen to be equivalent.<sup>5</sup> Then it makes no difference which one we choose. But in general, and in particular for cases like (8.1d), it makes a big difference.

<sup>5</sup> More generally, they are then equivalent for any CONSERV, EXT, and ISOM  $Q_1$  such that, in the number triangle, exactly one of (1,0) and (0,1) is in  $Q_1$ .

The status of the Negative Condition in the literature is somewhat unclear. This is partly because it is not formulated with sufficient clarity. But another reason is that most of the literature maintains, first, that the only allowed values for  $Q_1$  are *every* and *no*, and, second, that the truth conditions for those two cases are essentially the following:

- (8.13) a. *every A except C B*  $\iff \emptyset \neq C = A - B$   
 b. *no A except C B*  $\iff \emptyset \neq C = A \cap B$

We shall see below that this follows from the accounts in von Stechow 1993 and Moltmann 1995. And under *these* assumptions, the truth of *every* (or *no*) *A except C* is *B* logically implies, as one easily verifies, *each* of (IC<sub>1</sub>), (IC<sub>2</sub>), (NC<sub>1</sub>), (NC<sub>2</sub>), and (NC<sub>3</sub>). So under these assumptions, it is not surprising that the distinctions between the various versions of the Inclusion Condition and the Negative Condition have not seemed very important.

But above we pointed to cases where (IC<sub>1</sub>) was too strong. The counterexamples seem indubitable, and already this shows that there is something wrong with the truth conditions (8.13).

Similarly, we find (NC<sub>3</sub>) dubious when *C* is not a singleton. Although there are cases where this stronger version of the Negative Condition seems to fit, as in the case of (IC<sub>1</sub>), there are cases where it doesn't fit. This becomes clear if we look at a greater variety of examples, such as the following:<sup>6</sup>

- (8.14) a. All dishwashers except very low-end models have a water-saving feature. In fact, a few of the very low-end models have this feature as well.  
 b. Harry has a strain of something feminine that no men except creative geniuses possess. Which is not to say, by the way, that all men who are creative geniuses possess it.

The second sentence in each pair does not *contradict* the first; it gives additional information. So the first sentence in (8.14a) should not be taken to *entail* that *no* low-end models have a water-saving feature. But that is precisely what (NC<sub>3</sub>) says it does. (NC<sub>1</sub>), on the other hand, says only that some low-end models lack a water-saving feature. Perhaps it would be pragmatically odd to utter the first sentence unless a large percentage of the low-end models lacked the feature in question. But it seems unlikely, we think, that semantics would tell us how large that percentage had to be. In the case of (8.14b), it is even more obvious that (NC<sub>3</sub>) is too strong.

This discredits (NC<sub>3</sub>) as a plausible general version of the Negative Condition, and *a fortiori* it discredits the truth conditions in (8.13) which, as we saw, entail (NC<sub>3</sub>). (NC<sub>3</sub>) is often not stated explicitly, but it follows from the standard analyses given of exception sentences. As to explicit formulations, (NC<sub>1</sub>) and (NC<sub>2</sub>) are common, and the above examples are indeed consistent with these. For example, in (8.14b), (NC<sub>1</sub>) says that some male creative geniuses possess that particular strain of something feminine that Harry has, and (NC<sub>2</sub>) is just the weaker claim that some men possess it.

<sup>6</sup> The first sentences in these examples are variants of actual examples reported in Garcia-Álvarez 2003, to be presented in sect. 8.7 below.

(NC<sub>3</sub>), on the other hand, states that *all* male creative geniuses possess that strain, and that is just what the second sentence in (8.14b) reminds us need not be true.<sup>7</sup>

However, we shall find reasons to doubt (NC<sub>1</sub>) and (NC<sub>2</sub>) as well (section 8.11). Before we get there, we need to examine critically another cornerstone of the received view of exception sentences. This is the view mentioned above—often referred to as the *Quantifier Constraint*—that the only allowed values for the quantifier  $Q_1$  are *every* and *no*. But first, a brief digression.

## 8.6 ENTAILED OR IMPLICATED?

As is clear from the above, we follow here the common practice of treating the various conditions and claims involved in exceptive sentences as part of the truth conditions of these sentences. It is sometimes suggested, however, that the Negative Condition (and, less often, the Inclusion Condition) is implicated rather than entailed. The following example from Hoeksema 1996 is often cited.

(8.15) Well, except for Dr. Samuels, everybody has an alibi, inspector. Let's go see Dr. Samuels to find out if he's got one too.

But note that although the first sentence may not rule out that Dr. Samuels has an alibi after all, it does rule out that one has been *established* for him at the time of utterance. That indicates that the literal interpretation may not be very plausible in this case. A reasonable alternative is that *has an alibi* is not taken literally but rather as 'is known to have an alibi'.

This point about (8.15) is made by von Stechow (1994: 102–3), who also offers additional evidence that in this case, where there is exactly one exception, no violation or canceling of the Negative Condition occurs. Of course much more can be said about the general issue of whether the Negative Condition is entailed or presupposed or implicated. We shall not discuss this further here, but we make a brief remark on the status of the various claims involved in exception statements at the end of section 8.12 below.

## 8.7 OTHER QUANTIFIERS IN EXCEPTION SENTENCES

García-Álvarez (2003) challenges the Quantifier Constraint: i.e. the claim that the only eligible quantifiers in exception constructions are *every* and *no*. He lists a number of actually occurring sentences where exceptions are made to quantified claims expressed with *most*, *many*, and *few*, for example:

(8.16) a. Johnston noted that most dishwashers except very low-end models have a water-saving feature.

<sup>7</sup> Also, (8.14b) is another clear refutation of (IC<sub>1</sub>), since it certainly does not follow from the first sentence that all creative geniuses are men.

- b. . . . redeemed by a strain of something feminine that most men except creative geniuses lack.
- c. . . . he should eventually be able to walk long distances and do most things except vigorous exercise.
- d. Kate is an actress who played many roles except that of a real woman.
- e. Few people except director Frank Capra expected the 1946 film 'It's a Wonderful Life' to become a classic piece of Americana.
- f. Few except visitors will know that Czechoslovakia produces wine. It has been one of the country's better-guarded secrets.

It seems clear that these sentences are not odd or marginal. But they all at least appear to violate the Quantifier Constraint. A defender of that constraint then has one of the following two options: either insist that these sentences are nevertheless ungrammatical or meaningless, or argue that they are not really exception sentences in the sense discussed here: for example, that the exception phrases in them are free, or that *except* means something else on these occasions. Each of these positions has been taken. We can dismiss the first strategy, but what about the second?

Before trying to say something about that, we stop to note that there is nothing mysterious or strange about the fact that the Quantifier Constraint has seemed so obvious to many theorists. It simply codifies a straightforward idea about the nature of *universal claims*, and about exceptions to such claims.

### 8.8 THE CLASSICAL IDEA OF UNIVERSAL CLAIMS WITH EXCEPTIONS

The following is standard since Aristotle. A *universal proposition* has the logical form

(U) Every  $A$  is  $B$

As we saw in Chapter 1.1.1, Aristotle divided universal propositions into *affirmative* ones, as in (U), and *negative* ones,

No  $A$  is  $B'$

Note further that (if one uses the modern square of opposition; see Chapter 1.1.1) this is just the special case of (U) when  $B = \overline{B'}$ . So one can say that there really is only one (logical) form of universal proposition: (U).

Given this, it is obvious what an *exception* to (U) is: *an  $A$  that is not  $B$* . So in the negative case, it is an  $A$  that is not  $\overline{B'}$ , i.e. an  $A$  that is  $B'$ .

And given this, it seems perfectly obvious what the logical form of exception sentences is, at least when one individual  $j$  is excepted:

(8.17) *every*( $A - \{j\}, B$ ), and  $j$  is an exception

That is, it is the conjunction of the Generality Claim (GC) and an *Exception Claim*. Note that here the latter amounts to the conjunction of either version of the Inclusion Condition we have discussed with any of the three versions of the Negative Condition. And note again that this covers both positive and negative universal claims.

This is crystal clear. Problems appear when  $\{j\}$  is replaced by an arbitrary set  $C$ ; we saw that there is an issue as to whether *all* the elements of  $C$  should be exceptions or only *some* of them. But let us stick to the singleton case for the moment, where intuitions are sharp.

The foregoing, then, is an attractive philosophical theory about universal claims with exceptions. The question, however, is whether this theory is *linguistically* sound; i.e. whether it adequately accounts for how we *speak* about exceptions.

The received view is that it does. The Quantifier Constraint is treated as a given fact that the semantics of exception sentences should somehow explain. The problem, to our minds, is that the standard semantic accounts either simply reiterate that idea—the idea that the only universal claims, and hence the only exception sentences, are those of the logical forms shown above—or else derive it from largely *ad hoc* requirements. These accounts are more like stipulations than explanations. But then one has to ask if the stipulations are adequate, and in particular if the fact that they leave sentences like those in (8.16) unaccounted for isn't a sign that the received view needs to be revised, rather than an indication that these sentences aren't really exception sentences.

To substantiate our concern about the received view, we take a brief look at two of the most prominent standard accounts of exception sentences.

### 8.9 THE ACCOUNT IN VON FINTEL 1993

As to connected exception phrases, von Fintel (1993) considers phrases with *but* rather than *except*. There are some differences, but for many purposes they can be ignored. Applied to sentences of the form (8.2), repeated below,

(8.2)  $Q_1 A \text{ except } C B$

von Fintel's truth conditions are then essentially:

(vFint)  $C \neq \emptyset$  and  $Q_1(A - C, B)$  and  $\forall X(Q_1(A - X, B) \Rightarrow C \subseteq X)$

That is,  $C$  is the (unique) *smallest* set satisfying the Generality Claim, and it is non-empty.

Actually, von Fintel does not explicitly include the requirement that  $C \neq \emptyset$ . But it is required for his explicit claims that the truth conditions entail (NC<sub>2</sub>) (since if  $Q_1(A, B)$  holds,  $\emptyset$  is trivially the smallest set  $Y$  such that if  $Q_1(A - X, B)$  then  $Y \subseteq X$ ), and that consequently a  $\uparrow$ MON  $Q_1$  would make (8.2) automatically false (since then  $Q_1(A - C, B)$  entails  $Q_1(A, B)$ ). Presumably, von Fintel (like many others) simply takes for granted that the set  $C$  talked about is non-empty. In any case, for clarity we add it to the truth conditions here.<sup>8</sup>

<sup>8</sup> (vFint) would in a sense be more elegant without it, since then  $Q_1 A \text{ except } \emptyset B$  is equivalent to  $Q(A, B)$ , i.e. to the quantified claim *without* (with the empty set of) exceptions. But that is not how (8.2) is usually taken, and not how von Fintel intends it, it seems.

(vFint) is an elegant proposal. Let us see how it relates to our previous discussion.

(8.18) (vFint) implies (GC), (IC<sub>1</sub>), (IC<sub>2</sub>), (NC<sub>1</sub>), and (NC<sub>2</sub>).

*Proof.* (GC) is immediate. Hence,  $Q_1(A - (A \cap C), B)$ , which, by minimality, implies  $C \subseteq A \cap C$ , and thus  $C \subseteq A$ , i.e. (IC<sub>1</sub>). Since  $C \neq \emptyset$ , (IC<sub>2</sub>) holds as well. We have already seen that (NC<sub>2</sub>) follows. (NC<sub>1</sub>) follows similarly: if  $Q_1(A \cap C, B)$ , then  $Q_1(A - \overline{C}, B)$ , so  $C \subseteq \overline{C}$ , i.e.  $C = \emptyset$ .  $\square$

Also, as we saw above,

(8.19) (vFint) implies that if  $Q_1$  is persistent, (8.2) is always false.

Next, as von Fintel shows,

(8.20) (vFint) implies that in the special case where  $Q_1$  is *every* or *no*, we get the truth conditions (8.13), i.e.

every A except C B is true iff  $\emptyset \neq C = A - B$

no A except C B is true iff  $\emptyset \neq C = A \cap B$

*Proof.* We check *every*; the case of *no* is similar. Suppose first that *every* A except C B is true, i.e. that (vFint) holds with  $Q_1 = \text{every}$ . First,  $C \neq \emptyset$ . Second, since  $A - C \subseteq B$ , it follows that  $A - B \subseteq C$ . Third, since  $A \cap B = A - (A - B) \subseteq B$ , it follows from the minimality part of (vFint) that  $C \subseteq A - B$ . Hence,  $C = A - B$ .

Now suppose that  $\emptyset \neq C = A - B$ . Clearly,  $A - C \subseteq B$ . Also, if  $A - X \subseteq B$ , then  $(A - B) - X = \emptyset$ , i.e.  $C = A - B \subseteq X$ . This means that (vFint) (with  $Q_1 = \text{every}$ ) holds.  $\square$

Thus, when  $Q_1$  is *every* or *no*, (NC<sub>3</sub>) also follows. As we have seen, von Fintel's truth conditions are correct for the clear cases *every* A except John B and *no* A except John B, but not so when  $C$  is not a singleton, since they imply (IC<sub>1</sub>) and (NC<sub>3</sub>). They also violate exception conservativity (section 8.4), since  $\emptyset \neq C = A - B$  does not follow from  $\emptyset \neq A \cap C = A - B$ , nor  $\emptyset \neq C = A \cap B$  from  $\emptyset \neq A \cap C = A \cap B$ . But let us focus for the moment on the Quantifier Constraint.

Von Fintel claims that, among simple determiners, all but *every*, *no*, and their paraphrases are ruled out, i.e. that his truth definition enforces the Quantifier Constraint. Two arguments for this are given. The first is the observation (8.19) above. The second is that for determiners like *most* there aren't always unique exception sets, which is shown by means of an example. These arguments are presumably not intended to prove the claim but instead to make it plausible. Consider the second one. As Moltmann (1995) observed, it amounts in fact to a general requirement on the quantifier  $Q_1$  occurring in (8.2): namely, that there *always* is a unique exception set. Let us call it the *Uniqueness Condition* (UC) and formulate it explicitly:

(UC) there is a set  $Y$  such that  $Q_1(A - Y, B)$  and  $\forall X(Q_1(A - X, B) \Rightarrow Y \subseteq X)$

Like the other conditions discussed earlier, this is a universal claim: that for *all*  $A, B$  there is a smallest set  $Y$  satisfying the corresponding Generality Claim.<sup>9</sup>

Note that

(8.21) *every* and *no* satisfy (UC).

*Proof.* For *every*, take  $Y = A - B$ , and for *no*, take  $Y = A \cap B$ . □

Using (UC), we can substantiate part of von Fintel's claim:

(8.22) Properly proportional quantifiers do not satisfy (UC).

*Proof.* If  $Q_1$  is a non-trivial proportional quantifier, we can always find finite sets  $A, B$  such that  $|A \cap B| \geq 2$ ,  $\neg Q_1(A, B)$ , but for some  $a \in A \cap B$ ,  $Q_1(A - \{a\}, B)$ . Let  $b$  be a distinct element of  $A \cap B$ . Then  $Q_1(A - \{b\}, B)$  (the proportions are the same). Therefore, a set  $Y$  satisfying (UC) has to be a subset of both  $\{a\}$  and  $\{b\}$ ; i.e.  $Y = \emptyset$ . But that is impossible, since  $\neg Q_1(A - \emptyset, B)$ . So no set  $Y$  satisfies (UC) for this choice of  $A$  and  $B$ . □

By similar methods, one can see that, for example, the quantifiers *at most*  $k$  do not satisfy (UC) for  $k \geq 1$ .

Thus, (UC) is fundamental for von Fintel's claim. But have we formulated it correctly? (UC) does not require that the set  $Y$  is non-empty, but (vFint) requires that  $C$  is, so should we perhaps add that requirement to (UC)? No; then not even *every* or *no* satisfies it (namely, when  $A - B = \emptyset$  or  $A \cap B = \emptyset$ , respectively). But we may note the following: If  $Q_1(A, B)$  holds, then for these  $A, B$  the condition in (UC) is trivially satisfied, with  $Y = \emptyset$ . Disregarding that case, we are left with:

(UC') if  $\neg Q_1(A, B)$ , there is a set  $Y$  such that  $Q_1(A - Y, B)$  holds and  $\forall X (Q_1(A - X, B) \Rightarrow Y \subseteq X)$

Now this set  $Y$ , if it exists, is necessarily non-empty. However, as just noted, it follows that (UC') is only apparently weaker:

(8.23) (UC) and (UC') are equivalent.

So what might have looked like a problem with (UC) wasn't one, after all.

Which quantifiers, then, do satisfy (UC)? Not only *every* and *no*, but von Fintel's claim may still be correct, for the exceptions are rare and do not seem to be denotations of simple determiners. Here are some examples:

(8.24) The quantifiers  $Q_1(A, B) \Leftrightarrow A = \emptyset$  or  $|A| \geq n$  satisfy (UC).

<sup>9</sup> Moltmann (1995) rather takes (UC), for each  $A, B$ , as an assertion condition, and argues that von Fintel's account mixes truth conditions and assertion conditions and therefore has problems with compositionality. Our interpretation of (UC) here as a general constraint on  $Q_1$  is more charitable, in that it clarifies von Fintel's own remarks (but see n.12) and does not cause problems for compositionality.



*Proof.* By (8.23), it is enough to verify (UC'). So suppose  $\neg Q_1(A, B)$ . Then  $1 \leq |A| < n$ . We claim that with  $Y = A$  the minimality condition in (UC') holds. First,  $Q_1(A - A, B)$  by definition. Second, if  $Q_1(A - X, B)$ , then either  $A - X = \emptyset$  or  $|A - X| \geq n$ . But  $|A - X| \leq |A| < n$ . Thus,  $A - X = \emptyset$ , i.e.  $A \subseteq X$ .  $\square$

Other examples include  $Q_1(A, B) \Leftrightarrow A = \emptyset$  and  $Q_1(A, B) \Leftrightarrow A = \emptyset$  or  $(|A| = n \wedge |A \cap B| \leq k)$  or  $|A| \geq n + 1$ . One could give an exact characterization in the number triangle,<sup>10</sup> but we shall not pursue this, since we have already established that (UC) rules out numerous common determiner denotations.

Thus, although one could quibble with von Fintel's claim that his truth conditions for sentences with (connected) exception phrases rule out determiners other than every and no, we may readily concede that (UC) rules out all other simple determiners (and most complex ones as well).

But this brings us to two main questions: (a) Where does (UC) come from? (b) Are the (vFint) truth conditions—which in a sense presuppose (UC)—correct?

Starting with the first question, we have seen that (UC) comes from the desire to have the (vFint) truth conditions rule out e.g. proportional quantifiers. But the question now is whether there is any *independent* motivation for (UC) as a general constraint on quantifiers that can occur in exception sentences. It seems to us that, despite its elegance and impressive consequences, (UC) has an *ad hoc* character. There doesn't seem to be an intuition behind it that clearly relates to the semantics of exception sentences. It is true that von Fintel does suggest one such intuition: roughly that the smaller the exception set, the more information is conveyed by the exception sentence, and that the pragmatic maxim of being maximally informative has thus been lexicalized in *except*.

We do not find this suggestion very convincing, for two reasons. First, even though one might have some success in arguing that a reluctance to convey either tautological or contradictory information may have semantic consequences—this was Barwise and Cooper's idea of how to explain the distribution of determiners permitted in existential-there sentences (Chapter 6.3.2)—in the present case it is not at all a matter of avoiding logical truths or falsehoods. We see no good reason why conversational implicatures in this particular case should have moved from the pragmatic realm where they normally belong into grammar.

But the main reason we do not find the suggestion convincing is that the answer to question (b) above seems to be negative. (vFint) is not even correct for *every* and *no*. We have already seen this in sections 8.3 and 8.5, where we found strong reasons to discredit (IC<sub>1</sub>), and natural examples where (NC<sub>3</sub>) fails. Consider again, say, (8.14a), repeated below:

(8.14a) All dishwashers except very low-end models have a water-saving feature. In fact, a few of the very low-end models have this feature as well.

<sup>10</sup> Incidentally, the above counterexamples to (UC) were found by 'translating' (UC') to the number triangle and looking for possible patterns, and would have been harder to find without that graphical representation.

As usual, there is no controversy about the fact that the first sentence, read as *every A except C B*, entails the Generality Claim:

$$\text{every}(A - C, B)$$

We further know from (8.21) above that (UC) holds for *every*, with  $Y = A - B$ . But the second sentence states precisely that

$$A \cap C \cap B \neq \emptyset$$

Thus,  $C \not\subseteq A - B$ , contrary to what (vFint) implies. Similar remarks hold for the other counterexamples to (IC<sub>1</sub>) and (NC<sub>3</sub>) discussed earlier.<sup>11</sup> But then, if (vFint) is not correct, the motivation for assuming (UC) as a general constraint disappears as well.

Summing up: A crucial reason for von Fintel's particular truth conditions for exception sentences was the desire to enforce the Quantifier Constraint. And, together with (UC), they indeed (almost) enforce that constraint. But the price is too high: they don't give correct results for *every* and *no*. And there is no convincing independent motivation for (UC).

Thus, so far we have found no sound reason for accepting the Quantifier Constraint, except, of course, the classical theory of universal claims with exceptions (section 8.8). But if one wants to use *that* motivation, there are simpler ways to implement it than via (vFint) + (UC), which in any case do not give empirically correct results.

## 8.10 THE ACCOUNT IN MOLTSMANN 1995

Moltmann's idea is that the *except C* operation transforms the NP denotation  $(Q_1)^A$  pointwise: it either subtracts  $C$  from each set in  $(Q_1)^A$ , or adds  $C$  to each set. Thus, *every boy except John* contains the same sets as *every boy* except that John has been subtracted from each of those sets. Similarly, *no boy except John* consists of the sets  $X \cup \{j\}$  for each set  $X$  containing no boys. She gives the following truth conditions (expressed in the terminology used here) for sentences of the form (8.2):

(8.25)

$$(8.2) \Leftrightarrow \begin{cases} C \neq \emptyset \ \& \ \exists X(Q_1(A, X) \ \& \ B = X - C) & \text{if } \forall X(Q_1(A, X) \Rightarrow C \subseteq X) \\ C \neq \emptyset \ \& \ \exists X(Q_1(A, X) \ \& \ B = X \cup C) & \text{if } \forall X(Q_1(A, X) \Rightarrow C \subseteq \bar{X}) \\ \text{undefined} & \text{otherwise} \end{cases}$$

Again we have added the requirement  $C \neq \emptyset$ , not explicit in Moltmann 1995, but needed for her explicit claim that the Negative Condition (in the form of (NC<sub>2</sub>)) follows from her truth definition.

(8.25) too is an interesting proposal. Here are some of its consequences:

(8.26) Suppose (8.2) is true according to definition (8.25). Then

<sup>11</sup> Significantly, von Fintel does not discuss (in the connected case) any examples where  $C$  is not a singleton.

- (a)  $\neg Q_1(A, B)$ , so (NC<sub>2</sub>) holds.
- (b)  $C \subseteq A$ , so (IC<sub>2</sub>) and (IC<sub>1</sub>) hold.

*Proof.* (a) If  $Q_1(A, B)$ , then either  $C \subseteq B$  or  $C \subseteq \bar{B}$ . But in the former case, also  $Q_1(A, X)$  for some  $X$  such that  $B = X - C$ , and thus  $C \subseteq X - C$ , i.e.  $C = \emptyset$ , contradicting the assumption. Similarly, in the latter case,  $Q_1(A, X)$  for some  $X$  such that  $B = X \cup C$ , so again  $C = \emptyset$ .

(b) (Moltmann) Suppose first  $\forall X(Q_1(A, X) \Rightarrow C \subseteq X)$ . Since we also know that  $Q_1(A, X)$  for some  $X$ ,  $Q_1(A, A \cap X)$  holds by CONSERV. It follows that  $C \subseteq A \cap X$  and hence  $C \subseteq A$ . Next, suppose  $\forall X(Q_1(A, X) \Rightarrow C \subseteq \bar{X})$ , and let  $C' = C - A$ . Again take  $X$  such that  $Q_1(A, X)$ . Now

$$A \cap X = A \cap (X \cup C')$$

since  $A \cap C' = \emptyset$ , so, using CONSERV twice, it follows that  $Q_1(A, X \cup C')$ . Hence,  $C \subseteq X \cup C'$ , which implies that  $C \subseteq A$ . □

- (8.27) In the cases  $Q_1 = \textit{every}$  and  $Q_1 = \textit{no}$ , we again obtain the truth conditions in  
 (8.13):  $\textit{every } A \textit{ except } C B \iff \emptyset \neq C = A - B$   
 $\textit{no } A \textit{ except } C B \iff \emptyset \neq C = A \cap B$

*Proof.* We check the case when  $Q_1 = \textit{every}$ ; the other case is similar. Suppose first that (8.2) is true according to definition (8.25). By the definition, and (8.26b), we have  $\emptyset \neq C \subseteq A$ . This means that the first case, but not the second, in (8.25) holds. Thus, there is a set  $X$  such that  $A \subseteq X$  and  $B = X - C$ . But this readily implies that  $A - B = C$ .

Now suppose that  $\emptyset \neq C = A - B$ . Again,  $\emptyset \neq C \subseteq A$ , so the first case (but not the second) in the definition holds. Now let  $X = A \cup B$ . It follows that  $A \subseteq X$  and  $B = X - C$ , so (8.2) holds. □

Notably, (GC) does not follow from Moltmann's truth conditions. But the idea is that there is a requirement implicit in the truth definition which enforces the Quantifier Constraint and hence excludes quantifiers other than *all* and *no*, and then of course (GC) would follow. The same remarks hold for (NC<sub>1</sub>) and (NC<sub>3</sub>).

The implicit requirement is simply that (8.2) be defined. Moltmann calls this the *Homogeneity Condition*, treating it as an assertion condition of exception sentences, rather than an expressly formulated constraint on  $Q_1$ . But there is not a big difference between saying that a sentence of the form (8.2) is assertible (meaningful) only if it gets a definite truth value by definition (8.25), and saying that only quantifiers satisfying the following (universal) condition qualify for use in exception sentences:

$$(HC) \text{ if } C \subseteq A, \text{ either } \forall X(Q_1(A, X) \Rightarrow C \subseteq X) \text{ or } \forall X(Q_1(A, X) \Rightarrow C \subseteq \bar{X})$$

So we shall use (HC) as Moltmann's version of the Quantifier Constraint. Which quantifiers does (HC) allow? This time the answer is easy:

- (8.28)  $Q_1$  satisfies (HC) if and only if for each  $A$ ,  $(Q_1)^A$  (as a set of sets) is a subset of either *all*<sup>A</sup> or *no*<sup>A</sup>.

*Proof.* With  $C = A$  in (HC), we get

$$\forall A[\forall X(Q_1(A, X) \Rightarrow A \subseteq X) \text{ or } \forall X(Q_1(A, X) \Rightarrow A \subseteq \overline{X})]$$

from which the claim readily follows.  $\square$

Again we see that certain quantifiers other than *all* and *no* satisfy the requirement. But they are all fairly ‘unnatural’: e.g. the (CONSERV, EXT, and ISOM) quantifier which is *all<sup>A</sup>* when  $|A|$  is even and *no<sup>A</sup>* (or  $0^A$ ) when  $|A|$  is odd. Simple standard quantifiers such as *at least k*, *at most k*, *exactly k*, *most*, *few*, etc. are all excluded.<sup>12</sup>

Our main points, however, are the same as before. First, it is clear that (HC) essentially reformulates the Quantifier Constraint, but does not independently motivate it. Second, the account leads to the truth conditions (8.13) when  $Q_1 = \textit{every}$  or  $Q_1 = \textit{no}$ , and therefore is not empirically correct.<sup>13</sup>

### 8.11 COUNTER-EVIDENCE TO THE QUANTIFIER CONSTRAINT

To avoid the examples presented in (8.16) from Garcia-Álvarez 2003, one would have to claim that these sentences are not of the same form as those to which the analysis is supposed to apply: say, that the exception phrases in such sentences are free, or that *except* has a different meaning in them. Actually, sentences where *except* is used with *most* or *few* are often discarded as ill-formed, or else given a rather cursory treatment.

Free exception phrases with quantifiers other than *every* and *no* are sometimes discussed. But the analysis offered seems to suffer problems similar to those discussed

<sup>12</sup> Actually, Moltmann argues that in order to exclude quantifiers like *at least two* or *most*, the truth definition needs to be amended. This is because she starts by reading (8.25) locally, and (HC) as an assertion condition. She then observes that in a universe with exactly  $k$  boys, say  $[[\textit{boy}]] = A = \{a_1, \dots, a_k\}$ , each of which except John =  $a_1$  passed the exam, so  $[[\textit{passed the exam}]] = B = \{a_2, \dots, a_k\}$ , the sentence

(i)  $k$  boys except John passed the exam.

will come out true according to (8.25) (since if *at least k*( $\{a_1, \dots, a_k\}$ ,  $X$ ), then  $\{a_1, \dots, a_k\} \subseteq X$ , and so  $C = \{a_1\} \subseteq X$ ). Likewise, if  $k = 2$ ,

(ii) Most boys except John passed the exam.

will come out true (since if *most*( $\{a_1, a_2\}$ ,  $X$ ), then  $\{a_1, a_2\} \subseteq X$ , and so  $C = \{a_1\} \subseteq X$ ). So a local version of (HC) is satisfied, and in order to still be able to claim that these sentences are not assertible, she complicates the truth definition by adding a (somewhat more global) requirement that it must hold not only in the “intended model” but also in all “appropriate extensions”.

Thus, our account with (8.25) and (HC) is a ‘global reconstruction’ of Moltmann’s account, not the one she actually gives. Though perhaps not true to all her intentions, it has the virtue of being simpler, and of directly eliminating (i) and (ii): *at least k* and *most* simply fail to satisfy (HC), when it is taken as a general constraint on the quantifiers allowed. (Somewhat analogous comments apply to the discussion in von Stechow 1993 of essentially the same examples.)

<sup>13</sup> Again, significantly, Moltmann does not discuss examples where the problems with (8.13) show up, i.e. examples where the set  $C$  is not a singleton.

above for connected exception phrases. Consider, for example, von Fintel 1993, which argues that some uses of free exception phrases can be accounted for by a variant of the analysis given there for the connected case.

Von Fintel argues that a sentence like

(8.29) Except for the assistant professors, most faculty members supported the dean. should be analyzed by a weakening of the analysis he provided for connected exception phrases (section 8.9 above): the meaning is essentially given by the conjunction of the Generality Claim ( $Q_1(A - C, B)$ ) and (NC<sub>2</sub>) ( $\neg Q_1(A, B)$ ), but there is no longer a requirement that *C* is the *smallest* such set, and that is why quantifiers like *most* are allowed.

This is again simple and elegant, but is it correct? Consider the following variant of (8.29):

(8.30) Except for the assistant professors, few faculty members supported the dean.

This seems clearly consistent both with few faculty members in total supporting the dean (perhaps there was a very small number of assistant professors), and with this not being the case. And in fact, similar remarks can be made for (8.29).

The main observation here is that *the Negative Condition simply doesn't always follow*. This is even clearer with the examples in (8.16). Consider again (8.16e), repeated here:

(8.16e) Few people except director Frank Capra expected the 1946 film 'It's a Wonderful Life' to become a classic piece of Americana.

Obviously, the truth of this sentence in no way excludes that few people expected the film to be a success. Rather, one would guess the opposite to be true, given that the addition of one more (i.e. Frank Capra) still leaves the total number of optimists few. So (NC<sub>2</sub>) fails. (NC<sub>1</sub>) is outright false, since it claims that *one* optimistic person is not few.

As to (NC<sub>3</sub>), it says in this case that few persons identical to Frank Capra did not expect the film to become a classic piece of Americana. This happens to be true: actually the number of such people is zero, given that (8.16e) is true. But we have already seen that (NC<sub>3</sub>) is not true in general, not even for *every* and *no*.

So even if it could be argued that exceptives like those in (8.16) are free, it would not follow that the standard analysis is correct. But, in fact, no convincing argument that the exceptives in (8.16) are free exists, as far as we know, and neither has it been convincingly shown that the word *except* has a different meaning in these and similar sentences. It seems appropriate to at least look for a revision of the standard account that handles these cases as well.

## 8.12 A MODEST PROPOSAL

The following proposal is, we think, simple, not unreasonable, and consistent with all the examples and observations considered so far in this chapter.

We start from the classical picture of universal statements with exceptions (section 8.8), but we extend it slightly. There seems to be a wider class of *generalization statements*, all of the form  $Q_1 A B$ . Such a statement can be either *positive* or *negative*, and only generalization statements can have *exceptions*, and thus admit sentences of the form

$$(8.2) \quad Q_1 A \text{ except } C B$$

An exception to a positive statement is something in  $A - B$ , and an exception to a negative one is something in  $A \cap B$ . We do not yet know exactly which ones the *generalization quantifiers* are—i.e. we don't know the correct Quantifier Constraint—but we do know that *every*, *most*, and *many* are positive generalization quantifiers, and that *no* and *few* are negative ones. Very roughly, it seems that positive generalization statements say something about how small  $A - B$  is, and negative ones something about how small  $A \cap B$  is.

Now, the semantic content of a sentence of this form simply is, we propose, that the Generality Claim (GC) holds, and in addition an *Exception Claim* saying that *there are exceptions in C*.

Note that, by the definition of the notion of an exception, an exception in  $C$  is always in  $A \cap C$ . We have arrived at a definition of the operator *Exc* that was sought at the beginning of this chapter. Let  $Q_1$  be any type  $\langle 1, 1 \rangle$  generalization quantifier and  $C$  any set:

**the operation  $Exc_w$**

$Exc_w(Q_1, C)$  is the type  $\langle 1, 1 \rangle$  quantifier defined as follows: for each  $M$  and each  $A, B \subseteq M$ ,

$$(8.31) \quad Exc_w(Q_1, C)_M(A, B) \iff (Q_1)_M(A - C, B) \ \& \ \text{something in } A \cap C \text{ is an exception for } Q_1.$$

“w” stands for “weak”, and the reason for the particular wording of (8.31) will become clear when we see the strong version below. But first let us note that provided  $Q_1$  is CONSERV and EXT, so is  $Exc_w(Q_1, C)$  (so we don't need the subscript  $M$ ). It is not ISOM, due to the fixed set  $C$ . Note further that the Inclusion Condition, in the form (IC<sub>2</sub>), follows, but none of the Negative Conditions we have encountered. This is just what we want.

Next, one easily verifies the following:

$$(8.32) \quad \begin{array}{l} \text{a. } Exc_w(\text{every}, C)(A, B) \iff A - C \subseteq B \text{ and } (A \cap C) - B \neq \emptyset \\ \text{b. } Exc_w(\text{no}, C)(A, B) \iff A - C \subseteq \overline{B} \text{ and } (A \cap C) - \overline{B} \neq \emptyset \end{array}$$

In particular, when  $C = \{j\}$ , we obtain the correct conditions  $A - B = \{j\}$  and  $A \cap B = \{j\}$ , respectively. So for *every* and *no*, but not for other quantifiers, (NC<sub>1</sub>) and (NC<sub>2</sub>) do hold, and (NC<sub>3</sub>) does not in general, as we noticed should be the case.

Consider two of the examples in section 8.7:

(8.16a) Johnston noted that most dishwashers except very low-end models have a water-saving feature.

By our proposal, what Johnston noted is that if you consider the dishwashers that are not low-end models, most (but not necessarily all) of these have a water-saving feature, and furthermore that some of the low-end dishwashers lack that feature. As we have argued, this is the semantic content you want for that (part of the) sentence.

(8.16f) Few except visitors will know that Czechoslovakia produces wine.

Thus, excepting the visitors, few people know about Czechoslovakia's wine production, though some visitors know about it. But there is no claim—at least, not in the semantics—about how many visitors know this. In particular, it is not claimed, as (NC<sub>3</sub>) would have it, that few visitors are ignorant of it, and perhaps not even that it is false that few visitors know it, which is what (NC<sub>1</sub>) would say.

There are cases when the Exception Claim seems to say something stronger: not only that *something* in  $A \cap C$  is an exception, but that *everything* is. But even for *every* and *no*, this stronger reading is not quite the one proposed in von Stechow 1993 and Moltmann 1995. This is because (IC<sub>1</sub>), which is implied by those accounts, is in fact too strong, and moreover exception conservativity should always hold. (However, for *every* and *no*, the stronger version will imply all three versions of the Negative Condition.) When we come to the most general form of exceptive determiners in the next section, we will see that the stronger version, defined below via the operator  $Exc_s$ , becomes more prominent.

**the operation  $Exc_s$**

With  $Q_1, C, M, A, B$  as before, define

$$(8.33) \quad Exc_s(Q_1, C)_M(A, B) \iff (Q_1)_M(A - C, B), A \cap C \neq \emptyset, \& \text{every-thing in } A \cap C \text{ is an exception for } Q_1.$$

Clearly,  $Exc_s(Q_1, C)_M(A, B)$  implies  $Exc_w(Q_1, C)_M(A, B)$ . We obtain

$$(8.34) \quad \begin{array}{l} \text{a. } Exc_s(\textit{every}, C)(A, B) \iff \emptyset \neq A \cap C = A - B \\ \text{b. } Exc_s(\textit{no}, C)(A, B) \iff \emptyset \neq A \cap C = A \cap B \end{array}$$

The difference between this and the version we rejected earlier is precisely that  $A \cap C$  replaces  $C$ . Indeed, for both the weak and the strong account here, but not for the accounts discussed earlier, we have:

$$(8.35) \quad Exc_i(Q_1, C) \text{ is } \textit{exception conservative}, \text{ i.e., for } i = w \text{ or } s, \\ Exc_i(Q_1, C)_M(A, B) \iff Exc_i(Q_1, A \cap C)_M(A, B)$$

As to the choice between the weak and the strong version, consider

(8.36) All students except foreigners were invited.

Here it seems plausible that no foreign students were invited, in addition to the fact that the non-invited students were all foreigners. This is the strong reading (but without the claim that all foreigners are students!). On the other hand, consider the following variant:

(8.37) No students except foreigners came to the inauguration.

This says that there were only foreign students at the inauguration, but it is not so clear that it also says that *all* of the foreign students were there. If not, it is the weak reading.

Finally, we note that although we have construed exception statements as conjunctions of the Generality Claim and an Exception Claim, it could be argued that these two claims do not have exactly the same status. Suppose it is obvious that all Swedish logicians know a certain fact (say, that Professor Nilsson has retired). It is then quite odd to ask, Do all logicians except Swedes know this? (Rather, one would ask, Do all logicians apart from Swedes know this?) Likewise, if it is clear that all assistant professors support the dean, it is odd to ask, Do all professors except assistant professors support the dean? But the oddity seems not to be due just to the fact that the answer is known to be No, since the Exception Claim is not fulfilled. Rather, it suggests that the Exception Claim may be presupposed and not simply entailed. We recognize this possibility, but we shall not discuss presupposition further here.

### 8.13 QUANTIFIED EXCEPTION PHRASES

We now discuss briefly how to extend the account to the operator *Except* whose second argument is a type  $\langle 1 \rangle$  quantifier rather than a set. An obvious first idea is to let  $Except(Q_1, Q)(A, B)$  hold if there is  $X \in Q$  such that  $Exc(Q_1, X)(A, B)$ . But this definition does not work: not any set in  $Q$  will do. For example, if  $Q = I_j$ , then  $X \in Q$  amounts to  $j \in X$ , but if  $X$  contains other elements besides  $j$ ,  $Exc(Q_1, X)(A, B)$  will not give the right truth conditions. In that case we obviously need  $X = \{j\}$ .

Is there a way to always find the right sets in  $Q$ ? We don't know, but the most successful proposal we are aware of comes from Moltmann (1995). The idea is to use the smallest set that a quantifier lives on. Even though it provably doesn't cover all cases (see below), it gives the right result for a great majority of them. We use this idea in the following definition of the operation *Except*, in a strong and a weak version:<sup>14</sup>

<sup>14</sup> Moltmann does not discuss the problems stemming from the fact that the smallest set a quantifier lives on may vary between universes, nor the related issue of verifying that exceptive determiners denote EXT quantifiers (see below), nor the fact that sometimes there is no smallest live-on set.



**the operations *Except<sub>s</sub>* and *Except<sub>w</sub>***

When  $Q_1$  is as before, and  $Q$  is a type (1) quantifier such that  $W_{Q_M}$  (Chapter 3.4.1.2) is defined for each  $M$ , let, for each  $M$  and each  $A, B \subseteq M$ ,

$$(8.38) \text{ Except}_i(Q_1, Q)_M(A, B) \iff \exists X \subseteq W_{Q_M}[Q_M(A \cap X) \ \& \ \text{Exc}_i(Q_1, X)_M(A, B)]$$

where  $i$  is  $s$  or  $w$ .

As usual, we first need to check that these operations yield CONSERV and EXT quantifiers. It is clear that if  $Q_1$  is CONSERV, so is  $\text{Except}_i(Q_1, Q)$  (when defined). As to EXT, the live-on behavior of  $Q$  plays a role. Recall (Corollary 12 in Chapter 3.4) that when  $Q$  is EXT,  $M \subseteq M'$ , and  $W_{Q_M}$  and  $W_{Q_{M'}}$  are both defined,  $W_{Q_M} \subseteq W_{Q_{M'}}$ . We introduce a technical term for a slight strengthening of this property:

(8.39)  $Q$  is called *strongly* EXT if and only if

- (a)  $Q$  is EXT
- (b)  $W_{Q_M}$  is defined for every  $M$
- (c) If  $M \subseteq M'$ ,  $W_{Q_M} = W_{Q_{M'}} \cap M$

**Fact 1**

If  $Q_1$  is EXT and  $Q$  is *strongly* EXT, then  $\text{Except}_i(Q_1, Q)$  is EXT.

*Proof.* Suppose  $A, B \subseteq M \subseteq M'$ . We need to show that the following are equivalent:

- (i)  $\exists X \subseteq W_{Q_M}[Q_M(A \cap X) \ \& \ \text{Exc}_i(Q_1, X)_M(A, B)]$
- (ii)  $\exists X \subseteq W_{Q_{M'}}[Q_{M'}(A \cap X) \ \& \ \text{Exc}_i(Q_1, X)_{M'}(A, B)]$

Since  $W_{Q_M} \subseteq W_{Q_{M'}}$  and  $Q_1, Q$  are both EXT (hence also  $\text{Exc}_i(Q_1, X)$ ), it is clear that (i) implies (ii). Conversely, suppose that  $X$  satisfies the condition in (ii). Let  $Y = X \cap M$ . Thus  $A \cap X = A \cap Y$ . Using exception conservativity (8.35) and the assumptions about EXT, it follows that  $Q_M(A \cap Y)$  and  $\text{Exc}_i(Q_1, Y)_M(A, B)$ . Also, by (c) above,  $Y \subseteq W_{Q_M}$ . That is, (i) holds. □

This is helpful, since it appears that in most actual cases (but not all; see below),  $Q$  is in fact *strongly* EXT. ( $Q_1$ , as a determiner denotation, is of course always EXT.) For example, if  $Q$  is a proper name, a conjunction of proper names, a bare plural, or of the form  $(Q_2)^A$  where  $Q_2$  is definite, it follows from previous results that  $Q$  is *strongly* EXT.<sup>15</sup> Likewise, when  $Q_2$  is any quantifier such that, for each  $A$ ,  $A$  is the smallest set that  $(Q_2)_M^A$  lives on (if non-trivial),  $(Q_2)^A$  is *strongly* EXT.<sup>16</sup> For a different example,

<sup>15</sup> Lemma 1 (d) and (3.16) in Ch. 3; Proposition 10 in Ch. 4.6.

<sup>16</sup> See Lemma 1 (e), Proposition 13, in Ch. 3, and the remarks after that proposition.

consider the quantifier  $Q = I_j \wedge (I_m \vee I_s)$ , the denotation of *John*, and *Mary* or *Sue*, discussed in Chapter 4.5.5.3. This is a case when  $W_{Q_M}$  changes with  $M$ ; we leave it to the reader to verify that

$$(8.40) \text{ For } Q = I_j \wedge (I_m \vee I_s), W_{Q_M} = M \cap \{j, m, s\}.$$

It follows that this quantifier too is strongly EXT.

Next, let us verify that the above definition gives the desired result in some familiar cases. The truth conditions are calculated using the definition of *Except<sub>i</sub>*, exception conservativity, and familiar facts about the quantifiers involved; only the final result is shown here.

- $Except_i(Q_1, C^{\text{pl}}) = Exc_i(Q_1, C)$
- $Except_s(Q_1, I_j) = Except_w(Q_1, I_j) = Exc_s(Q_1, \{j\}) = Exc_w(Q_1, \{j\})$

Thus, the truth conditions for proper names are entirely as before. This, of course, is a criterion for any definition of an *Except* operation. For bare plurals too, *Except<sub>i</sub>* gives what we earlier had with *Exc<sub>i</sub>*; here both the strong and the weak versions are used.

Now consider the following:

- (8.41) a. Every student except the three Swedes showed up.  
 b. Every student except three Swedes showed up.

We obtain

- $Except_i(Q_1, \text{the three}^C)(A, B) \iff |C| = 3 \ \& \ Exc_i(Q_1, C)(A, B) \ \& \ C \subseteq A$
- $Except_i(Q_1, \text{three}^C)(A, B) \iff \exists X \subseteq C[|A \cap X| = 3 \ \& \ Exc_i(Q_1, X)(A, B)]$

This seems right, but note that there is a clear preference for the strong version here. (8.41a) says that the set of students who didn't show up consists precisely of the three Swedes; the weak reading which says that only some of the Swedes may have been exceptions is inadequate. Similarly for (8.41b); calculating the particular truth conditions we obtain, in the strong case,

$$(s) \ |A - B| = 3 \ \& \ A - B \subseteq C$$

and in the weak case,

$$(w) \ \exists X \subseteq C[|A \cap X| = 3 \ \& \ A - B \subseteq X \ \& \ (A \cap X) - B \neq \emptyset]$$

(w) is compatible with the number of Swedish students not showing up being exactly two, for example (as long as there are at least three Swedish students): just add one Swedish student (who did show up) to those two and you get a set  $X$  satisfying the condition in (w). This is clearly inadequate. It is also useful to compare (8.41b) with the sentence

- (8.42) Every student except at most three Swedes showed up.

The truth conditions are as in (s) and (w) with “= 3” replaced by “≤ 3”. This is right for the strong reading, but, by the reasoning indicated above, we see that, as long as

there are at least three Swedish students, the weak readings of (8.41b) and (8.42) are in fact equivalent—again a sign of the inadequacy of these weak readings.<sup>17</sup>

Note further that in (8.41a) there have to be exactly three Swedes in the (discourse) universe, and they all have to be students. This is not required in (8.41b): three Swedish students failed to show up (though all other students showed up), but there may be other Swedes around, and not all of them need be students. This accords pretty well with intuitive judgments, we think.<sup>18</sup>

- (8.43) a. No books except John's short stories are missing.  
 b. No books except two of John's short stories are missing.

We calculate, with  $John's = Poss(all_{ei}, \{j\}, every, R)$  and  $two\ of\ John's = Poss(all_{ei}, \{j\}, two, R)$ :

- $Except_i(Q_1, John's^C)(A, B) \iff C \cap R_j \neq \emptyset \ \& \ Exc_i(Q_1, C \cap R_j)(A, B) \ \& \ C \cap R_j \subseteq A$
- $Except_i(Q_1, two\ of\ John's^C)(A, B) \iff \exists X \subseteq C \cap R_j [ |A \cap X| = 2 \ \& \ Exc_i(Q_1, X)(A, B) ]$

Similar remarks can be made in this case; the strong reading is preferred in (8.43b). Note that the exceptions are only short stories that John has written (assuming  $R$  is

<sup>17</sup> There is another possible problem with (8.42), however, even in the strong reading. This is because  $Exc_i$  builds in the condition that there are exceptions, but (8.42) may be compatible with *every* student (including the Swedes if there are any) showing up. That is, a downward monotone  $Q$  may allow that there are no exceptions. If this is true, the definition needs to be amended accordingly. But the situation is not quite clear. Observe e.g. that phrases like *every student except no Swedes* do not make much sense, and an explanation for this might be precisely that the truth conditions should entail, as we have assumed, that there are exceptions.

<sup>18</sup> We note in this connection an additional putative problem with the truth definition. Consider the three sentences

- (i) Most dishwashers except at least three/three/at most three low-end models have a water-saving feature.

We consider only the strong truth conditions, which are, respectively,

- (ii)  $\exists X \subseteq C [ |A \cap X| \geq 3 / = 3 / \leq 3 \ \& \ most(A - X, B) \ \& \ \emptyset \neq A \cap X \subseteq A - B ]$

Each of these conditions allows that there are three low-end dishwashers lacking a water-saving feature, but also *many other* low-end models that possess such a feature. This is completely in order for *at least three*. It seems all right also for *three*; the sentence is a bit odd to utter, but that looks like the effect of an implicature. But for *at most three* (and perhaps also for *exactly three*) it becomes distinctly more odd. We are not sure if this is also a pragmatic effect, but for now we just note the problem.

Garcia-Álvarez (2003) gives no examples of this form, but here are some:

- (iii) a. Yet even that will prove unattainable for most except two or three parties, say analysts. ([www.infid.be/election\\_16feb.htm](http://www.infid.be/election_16feb.htm))  
 b. 16 of us booked on Serenade August 14th. First cruise for most except two of us. ([boards.cruisecritic.com/showthread.php?t=88950&page=3](http://boards.cruisecritic.com/showthread.php?t=88950&page=3))  
 c. Most except three patterns are currently available ... ([link.aip.org/link/?PODIE2/17/202/1](http://link.aip.org/link/?PODIE2/17/202/1))

authorship); other short stories are irrelevant. This effect is obtained precisely because  $C \cap R_j$ , and not  $C$ , is the smallest set that  $John's^C$  lives on, and  $X$  is accordingly required to be a subset of  $C \cap R_j$ . Similarly for *two of John's^C*.<sup>19</sup>

Next, consider the following pair:

- (8.44) a. No professors except John and Mary attended the meeting.  
b. No professors except John or Mary attended the meeting.

For (8.44a), we get the obvious truth conditions, as can also be seen from the remarks above and the fact that  $I_j \wedge I_m = \{j, m\}^{pl}$ . The strong conditions seem obligatory. The sentence would be false if both John and Mary are professors, and John but not Mary attended the meeting, but that scenario is allowed by the weak conditions.

For (8.44b), on the other hand, we have (when  $j, m \in M$ )

- $Except_i(Q_1, I_j \vee I_m)(A, B) \iff \exists X \subseteq \{j, m\}[(j \in X \cap A \vee m \in X \cap A) \& Exc_i(Q_1, X \cap A)(A, B)]$

This allows that John was the only professor attending the meeting, *or* Mary was the only such professor, *or* John and Mary were the only such professors. Contrast this with

- (8.45) No professors except John attended the meeting, or no professors except Mary attended the meeting.

Here the third case, that both attended, is excluded (since it is impossible that both disjuncts are true). Even though intuitions may be less strong here, this aspect of the truth conditions seems fairly reasonable.

A possible problem, however, is that (8.44b) (but not (8.44a)), allows that while John was the only professor attending the meeting, Mary is not a professor at all. But (8.44b) ought to entail that Mary and John are both professors. This problem seems specific to disjunctions; in the next sentence, for example, there is no inclination to conclude that Swedes are professors; only that some Swedish professor attended.

- (8.46) No professors except some Swedes attended the meeting.

This remark is similar to the one made about (IC<sub>1</sub>) in section 8.3. In general, with  $Q_1 A \text{ except } Q_2^C B$ , and even if  $W_{(Q_2^C)_M} = C$ , there is no entailment that  $C \subseteq A$ .

<sup>19</sup> Let us check that  $C \cap R_j$  is the smallest set *two of John's^C\_M* lives on, when non-trivial. Non-triviality in this case means that  $|C \cap R_j| \geq 2$ . Now suppose *two of John's^C\_M* lives on  $D$ ; i.e., for all  $B \subseteq M$ ,

$$|C \cap R_j \cap B| = 2 \iff |C \cap R_j \cap B \cap D| = 2$$

Take any  $a \in C \cap R_j$ . Then choose  $B \subseteq M$  such that  $a \in B$  and  $|C \cap R_j \cap B| = 2$ . Thus,  $|C \cap R_j \cap B \cap D| = 2$ , which means that  $C \cap R_j \cap B \subseteq D$ , and hence  $a \in D$ . Since  $a$  was arbitrary, it follows that  $C \cap R_j \subseteq D$ . And since *two of John's^C\_M* clearly lives on  $C \cap R_j$ , we have  $W_{\text{two of John's}^C_M} = C \cap R_j$ .

However, in the particular case of a disjunction like  $I_j \vee I_m = \text{some}^{\{j,m\}}$ , it seems one does want  $\{j, m\} \subseteq A$ . For now, we merely record this problem.

Finally, we note that the analysis works in the same way for more complex Boolean combinations, as in

(8.47) No professors except John, and Mary or Sue, attended the meeting.

This sentence is well-formed and has a rather clear meaning, which our treatment accounts for—except for the ‘disjunction problem’ just mentioned.

Concerning the choice between weak and strong readings, we have already seen that for bare plurals, the weak reading is the default, but there are also cases where the strong reading is intended. Similarly for simple unquantified possessives like  $\exists \text{John}'s$ . It seems that (8.43a) can be read either weakly, so that only some of John’s short stories are missing, or strongly, so that all of them are missing. However, in (8.41)–(8.44) except for (8.43a), and in (8.47), a strong reading was required. We may tentatively propose the following

*Strong Exceptions Principle*

When the putative exceptions are explicitly enumerated or quantified over, *all* of them are in fact exceptions. That is, in these cases the strong truth conditions for exceptives are used.

In the vast majority of exception sentences, including all of the above examples, the quantifier  $Q$  which is the second argument to  $Except_i$  is strongly EXT, and it seems that definition (8.38) does a pretty good job of accounting for their meaning. But it is also clear that the definition is not completely correct. This is because there are cases with  $Q$  of the form  $Q_1^C$  where there is no smallest set that the quantifier lives on, and yet the exception sentence makes perfectly good sense. For example, a mathematician could unproblematically say:

(8.48) All odd numbers except denumerably many primes have property P.

In the universe of natural numbers, there is no smallest set that *denumerably many*<sup>prime</sup> lives on ( $\text{denumerably many}(A, B) \Leftrightarrow |A \cap B| = \aleph_0$ ). But that is clearly beside the point; it is the set of (all) primes we want. And the truth conditions intended for (8.48) are clear: there is a denumerable set  $X$  of primes such that  $Exc_3(\text{all}, X)(\text{odd number}, P)$ .<sup>20</sup>

This example is mathematical, and we know that cases where  $W_{Q,M}$  is undefined have to involve infinite sets. But the point seems perfectly general: the live-on property is *irrelevant* for this quantifier of the form  $Q_1^C$ ; what is required is merely that  $X$  in definition (8.38) be a subset of  $C$ . And the same could be said of many other quantifiers of that form, even when it happens to be the case that  $C$  is the smallest

<sup>20</sup> (8.48) may be true or false depending on what P is. Since we do not yet know how to extend definition (8.38) to the case when there isn’t always a smallest live-on set, we chose not to define the operator  $Except_i(Q_1, Q)$  for that case, rather than incorrectly stipulating that it always yields, say, false.

live-on set, for example, *ten*<sup>C</sup>, *at least three*<sup>C</sup>, *the five*<sup>C</sup>. However, we also saw above that for possessives like *two of John's*<sup>C</sup> it is essential that  $X$  is a subset of  $C \cap R_j$ , which in those cases is the smallest live-on set.

Thus, this definition of *Except*, like the definition of *Poss* in the previous chapter, still has some drawbacks. The problems are different in the two cases, though. For *Except*, we would like to add some instances when *Except*, as it now stands, is undefined, but where exception phrases still make sense. But there is no obvious way, when just an arbitrary type  $\langle 1 \rangle$  quantifier  $Q$  is given, to locate the set  $C$  we want to use in place of  $W_{QM}$  in the definition.

For *Poss*, the problem was instead that when  $Q$  was not of the form  $Q_1^C$ , there was no obvious way to formulate *narrowing*. We circumvented that by letting both  $Q_1$  and  $C$  be arguments of *Poss*, noting that the type  $\langle 1 \rangle$  quantifiers that apparently occur in that position could in fact be recast in the form  $Q_1^C$ . A similar strategy would not be possible for *Except*, if only because certain quantifiers that are probably *not* of that form, such as  $I_j \wedge (I_m \vee I_s)$ , are still good in exception phrases. In both cases, the operators we defined seem to be on the right track, but they still leave something to be desired.

In conclusion, let us say that although we have been critical of the received views about exceptives, we recognize that those views and ours coincide materially in many cases of exceptives with *every* and *no*. For these determiners, they coincide completely when the exception phrase is a proper name or a conjunction of proper names; they almost coincide when that phrase is a quantified NP (a case dealt with by Moltmann (1995) but not by von Stechow (1993)), the difference being that we insist on  $(IC_2)$ , and exception conservativity, rather than  $(IC_1)$ . They differ, however, when the exception phrase is a bare plural or a non-quantified possessive NP, since then the weak reading is often used. They also differ substantially for exceptions to determiners other than *every* and *no*.

A main novelty here is the *rationale* behind our account, which builds on a simple and classical idea of universal statements with exceptions, but rejects the old Quantifier Constraint and leaves room for the exceptions with *many*, *most*, and *few* that speakers actually use.

## 8.14 FURTHER ISSUES

Clearly, what we have done here only scratches the surface of the semantics of exception sentences. We have looked only at *except*, and only at certain forms of connected exception phrases. Much more is required for a comprehensive account, and a lot more can also be found in the literature. We hope that some of the observations made here have a place in such an account. Below we list some issues for the account given here that would require further study:

- A main issue is the formulation of the *proper* Quantifier Constraint. Clearly, *few* if any quantifiers besides *every*, *no*, *few*, *many*, *most* can occupy the  $Q_1$  position of *Exc<sub>i</sub>* or *Except<sub>i</sub>*, and one would like a principled explanation of this. The standard

accounts, although they attempt to explain this, don't succeed: they leave out *few*, *many*, and *most*, and even for *every* and *no* they get the meaning wrong.<sup>21</sup>

- A related issue is the division of generalization statements into positive and negative ones. The notion is clear for *every* and *no* (see section 8.8), but we had to just stipulate (section 8.12) that *most* and *many* were positive and *few* negative.
- There is also the question of constraints on the  $Q$  argument in  $Except_i$ . Here there seems to be much freedom, but certain quantifiers are forbidden. To take one example, *every A except only John* is clearly unacceptable, so the quantifier denoted by *only John*, i.e.  $(O_j)_M(B) \Leftrightarrow B = \{j\}$ , should somehow be ruled out. Similarly for *every A except no C* (see n. 17).
- Although the Strong Exceptions Principle proposed in the previous section may be part of the truth, one would like a more precise statement, and an explanation, of how the choice between weak and strong readings is made.
- A general truth definition for sentences of the form  $Q_1 A \text{ except } Q B$  is still lacking. The operations  $Except_i$  come close, but are not completely general, since sentences like (8.48) are not handled correctly. They should be seen as part of a definition, not a definition of an operator that is essentially partial with respect to  $Q$ . Also, the possible problems signaled in notes 17 and 18 need to be resolved.
- Similarly, should we think of the operations  $Exc_i$  as partial, i.e. as defined only when  $Q_1$  satisfies the correct Quantifier Constraint, or should we assign the corresponding sentences a truth value in the other cases too?
- Should we think of *except* as *ambiguous* between weak and strong readings, or is one of these, rather, the basic meaning, and the other sometimes enforced by pragmatic factors?

<sup>21</sup> Garcia-Álvarez (2003) proposes as a Revised Quantifier Constraint that  $Q_1$  cannot be persistent and symmetric. The unacceptability of examples like *#Some/Four students except foreigners were invited* corroborates this. He also notes that *many* does have a persistent and symmetric interpretation, but argues that when *many* occurs in exception sentences, it has a proportional interpretation. So this revised constraint may well be correct, but we cannot derive it from the truth conditions without further assumptions. If  $Q_1$  is persistent, and the Generality Claim  $Q_1(A - C, B)$  holds, it follows that  $Q_1(A, B)$  holds too. This contradicts (NC<sub>2</sub>), but we have found reason to doubt (NC<sub>2</sub>) in general: *few A except C B* is fully compatible with *few A B*.

# 9

## Which Quantifiers are Logical?

The question of logicality has already been touched upon several times. There is no unanimity in the literature about what logicality is, nor indeed about whether it is a well-defined notion. But almost everyone would agree, it seems, that *topic neutrality* or *subject independence*, that is, ISOM, is at least a necessary condition. In section 9.1 we present the general version of isomorphism closure for arbitrary operations, and mention some known—though perhaps not widely enough known—facts about it and some of its variations.

As regards natural language quantification, ISOM is quite significant. We express its significance in two claims (section 9.2): one is that simple quantirelations—including determiners and quantificational adverbs—all denote ISOM quantifiers, and the other that those denoting non-ISOM quantifiers all arise from more general ISOM operations by freezing some arguments.

In section 9.3 we consider *constancy*, a property which is here (non-standarily) distinguished from logicality. We already described constancy as being given by the same rule in every universe (Chapter 3.4.1.3), and now we look at what appears to be the empirical manifestation of that property, which concerns the behavior of certain expressions in intuitively valid arguments, and which we call *inference constancy*. Speakers' intuitions about what is constant and what is variable in inferences—i.e. about *form*—are quite robust, and the idea is to use in this way a *given* consequence relation to identify the constants, in a sense reversing Bolzano's (and Tarski's) method of defining consequence, given the constants. One outcome of this is further evidence that quantifier expressions should *not* be interpreted in models, contrary to what is sometimes stated in the literature.

Finally, we come back to the question of logical constants (section 9.4), but rather than attempt a resolution of this vexed issue, we note that at least two precise and informative properties of (most of) these expressions have been identified—ISOM and EXT—and that the two are equivalent to one single condition: invariance under injections.<sup>1</sup>

### 9.1 LOGICALITY AND ISOM

If logic, as most people seem to agree, is independent of subject-matter, then logical constants ought to be invariant under arbitrary bijections between universes (or,

<sup>1</sup> We are grateful to Johan van Benthem for several valuable comments on an earlier draft of this chapter.



more weakly, under arbitrary permutations of the universe). Suppose you map (in a one–one fashion) triangles to file cabinets, lines to ice-cream cones, and “intersects” to “contains”. Then, if

(9.1) Every triangle intersects at least four lines.

is true in a model,

(9.2) Every file cabinet contains at least four ice-cream cones.

must be true in the corresponding isomorphic model. That’s just what it means for every and at least four to be ISOM.

One might feel that, especially in the case of mathematical or geometrical objects, not all such mappings should be allowed, but only those that go between geometrical objects, say, and respect certain requirements. This gives you concepts that are invariant under a restricted class of bijections. For example, Euclidean geometry deals with properties invariant under rigid motions, such as rotation, translation (in the geometric sense), and reflection. And this shows that these properties are *geometrical*, not *logical*: logic treats all objects the same, so logical properties are invariant not only under rigid transformations but under all bijections.

### 9.1.1 ISOM in arbitrary types

ISOM is not restricted to quantifiers. For example, propositional connectives certainly don’t care about subject-matter: a conjunction applies uniformly to two sentences (or sets, or whatever), without regard for what these sentences are about. So it is ISOM too.

In general, over any universe  $M$  of individuals we can form the *objects of finite type over  $M$* : relations over  $M$ , functions on  $M$ , relations over relations over  $M$  (that is, Lindström quantifiers on  $M$ ), functions from such relations to individuals, etc. There are various means of describing the types of these objects: for example, in the following way.

Let the elements (individuals) of  $M$  have type  $e$ , the two truth values (1 and 0) have type  $t$ , and all other types be of the form  $\tau \rightarrow \sigma$  or  $(\tau_1, \dots, \tau_n)$ , the former standing for functions from objects of type  $\tau$  to objects of type  $\sigma$ , and the latter for  $n$ -tuples of objects of type  $\tau_1, \dots, \tau_n$ , respectively. Relations are identified with their characteristic functions from tuples to truth values, so that instead of  $R(a_1, \dots, a_n)$  or  $(a_1, \dots, a_n) \in R$  we can write  $R(a_1, \dots, a_n) = 1$ . For example, sets of individuals have type  $\pi = e \rightarrow t$ , so quantifiers on  $M$  of type  $\langle 1, 1 \rangle$  in our earlier classification now get type  $(\pi, \pi) \rightarrow t$  in this type system, as sets of pairs of sets of individuals.

Any bijection  $f$  between two universes  $M$  and  $M'$  can be *lifted pointwise* to a bijection—still denoted  $f$ —from objects of any finite type over  $M$  to objects of the same type over  $M'$ . For example,

- If  $u$  is a truth value,  $f(u) = u$  (by stipulation).
- If  $R$  is a binary relation between individuals in  $M$ ,  $f(R) = \{(f(a), f(b)) : (a, b) \in R\}$ .

- If  $F$  is a function from binary relations between individuals in  $M$  to sets of individuals in  $M$ ,  $f(F)$  is the corresponding function over  $M'$  defined by  $f(F)(R') = S'$  iff  $F(f^{-1}(R')) = f^{-1}(S')$ . (Thinking of  $F$  as a relation instead, we have  $f(F) = \{(f(R), f(S)) : F(R) = S\}$ .)<sup>2</sup>

Now consider *universal operators*, i.e. functionals  $O$  that—like our quantifiers—associate with each universe  $M$  an object  $O_M$  of a certain type over  $M$ . Generalizing the notion of ISOM for quantifiers, we get the following:

### ISOM and PERM for arbitrary operators on universes

An operator  $O$  on universes in the above sense is ISOM iff for all universes  $M$  and all bijections  $f$  with domain  $M$ :

$$(9.3) \quad f(O_M) = O_{f(M)}$$

In the special case where  $f$  is a permutation on  $M$ , we get the condition PERM (or with fixed  $M$ , the condition PERM <sub>$M$</sub> ):

$$(9.4) \quad f(O_M) = O_M$$

That is,  $O_M$  is a *fixed point* of  $f$ .

For example, if  $Q$  is a type  $\langle 1, 1 \rangle$  quantifier,

$$\begin{aligned} Q_{f(M)}(f(A), f(B)) &\iff f(Q_M)(f(A), f(B)) \quad [\text{by (9.3)}] \\ &\iff Q_M(A, B) \quad [\text{by the definition of } f(Q_M)] \end{aligned}$$

which is our previous version of ISOM for  $Q$ .

At the lowest types—in particular, for predicate expressions—these conditions have strong effects:

<sup>2</sup> More precisely, define by induction for each type  $\tau$  a bijection  $f$  from  $T_\tau^M$ —the set of objects over  $M$  of type  $\tau$ —to  $T_\tau^{M'}$ , as follows:

- (i) If  $a \in T_e^M = M$ ,  $f(a)$  is already defined.
- (ii) If  $u \in T_t^M = \{0, 1\}$ ,  $f(u) = u$ .
- (iii) If  $\vec{a} = (a_1, \dots, a_n) \in T_{(\tau_1, \dots, \tau_n)}^M$ ,  $f(\vec{a}) = (f(a_1), \dots, f(a_n))$ . If  $f$  is already a bijection from  $T_{\tau_i}^M$  to  $T_{\tau_i}^{M'}$ ,  $1 \leq i \leq n$ , this gives a bijection from  $T_{(\tau_1, \dots, \tau_n)}^M$  to  $T_{(\tau_1, \dots, \tau_n)}^{M'}$ .
- (iv) If  $F \in T_{\tau \rightarrow \sigma}^M$  and  $a' \in T_\tau^{M'}$ , then  $(f(F))(a') = f(F(f^{-1}(a')))$ . In other words,  $(f(F))(a') = b'$  iff  $F(f^{-1}(a')) = f^{-1}(b')$ , which is equivalent to:
  - (a) If  $a \in T_\tau^M$  and  $b \in T_\sigma^M$ , then  $f(F)(f(a)) = f(b)$  iff  $F(a) = b$ .

If  $f$  is already a bijection from  $T_\tau^M$  to  $T_\tau^{M'}$ , and from  $T_\sigma^M$  to  $T_\sigma^{M'}$ , this defines a bijection from  $T_{\tau \rightarrow \sigma}^M$  to  $T_{\tau \rightarrow \sigma}^{M'}$ .

**Proposition 1**

1. No individuals in  $M$  satisfy  $\text{PERM}_M$  (if  $|M| \geq 2$ ). But all truth functions (type  $(t, \dots, t) \rightarrow t$ ) do.
2. The only subsets of  $M$  satisfying  $\text{PERM}_M$  are  $\emptyset$  and  $M$ .
3. The only binary relations on  $M$  satisfying  $\text{PERM}_M$  are  $\emptyset$ ,  $M \times M$ , the identity relation on  $M$ , and its complement.
4. For ternary relations we get a few more cases, such as the relation  $R(x_1, x_2, x_3)$  defined as  $x_1 = x_2 \ \& \ x_1 \neq x_3$ . Etc.

*Proof.* This observation has been made by many people: e.g. Westerståhl (1985*b*), Tarski (1986), Keenan (2000). Let us look here just at the case of a binary relation  $R$  on  $M$ .

Suppose  $R$  is distinct from each of  $\emptyset$ ,  $M \times M$ , the identity relation on  $M$ , and its complement. Then there are  $a, b, c, d, e \in M$  such that  $a \neq b$ ,  $(a, b) \in R$ ,  $(c, c) \in R$ , and  $(d, e) \notin R$ . If  $d = e$ , consider a permutation  $f$  on  $M$  that maps  $c$  and  $d$  onto each other, but leaves everything else as it is. Then  $(c, c) \in R$ , but  $(f(c), f(c)) \notin R$ , so  $\text{PERM}_M$  is violated. If  $d \neq e$ , let  $f$  instead map  $a$  to  $d$  and  $b$  to  $e$  (and vice versa):  $(a, b) \in R$  but  $(f(a), f(b)) \notin R$ , again violating  $\text{PERM}_M$ .  $\square$

*A fortiori*, very few logical constants can be found among predicate expressions in natural languages. (Possible English examples could be the noun `thing`, and one reading of the verb `be` or of adjectives like `same` and `different`.)

Note also that, interestingly, the number of ISOM relations between individuals is not only small but essentially independent of the size of the universe.<sup>3</sup> This is false for objects of higher types: the number of ISOM such objects (of a given type) is relatively large, and increases with the size of the universe. For example, over a universe with  $n$  elements, there are  $2^{(n+1)^3}$  type  $\langle 1, 1 \rangle$  quantifiers satisfying  $\text{PERM}$ , and  $2^{(n+1)(n+2)/2}$  ones satisfying  $\text{CONSERV}$  and  $\text{PERM}$ .

On the other hand, also for higher-type objects there are many more non-ISOM ones than ISOM ones—e.g. there are exactly  $2^{3^n}$   $\text{CONSERV}$  type  $\langle 1, 1 \rangle$  quantifiers over a fixed universe with  $n$  elements. (Over a universe with two elements (!), there are 512  $\text{CONSERV}$  quantifiers, of which 64 are  $\text{PERM}$ .) Thus, in terms of the number of possible denotations, it looks like a significant fact that so many quantification-related expressions in natural languages, in particular determiners, denote ISOM operations or relations.

### 9.1.2 ISOM is necessary for logicity

A number of people have, largely independently of each other, proposed ISOM or  $\text{PERM}$  as a necessary condition for logicity, starting with Mostowski (1957) and

<sup>3</sup> Westerståhl (1985*b*) explicitly calculates the number of  $k$ -ary relations on a universe  $M$  satisfying  $\text{PERM}$ .

Lindström (1966); some also in full type-theoretic generality, e.g. Tarski (1986) and van Benthem (1986). An informative overview of the status of the condition is van Benthem (2002). One noteworthy thing is that practically all formal languages that logicians or computer scientists have proposed indeed have the property that truth is preserved under isomorphism, and that all the operators used, and all those definable from them in the system, satisfy ISOM.

There are also converse results, for example:

**Theorem** (McGee 1996)

*On a fixed universe  $M$ , the PERM operations taking relations over individuals as arguments, i.e. the quantifiers on  $M$ , are essentially the ones definable in the logic  $FO_{\infty,\infty}$ .<sup>4</sup> As to the ISOM ones, you can define in  $FO_{\infty,\infty}$ , for each cardinal  $k$ , such operations restricted to universes of cardinality  $\kappa$ .*

**Theorem** (van Benthem 1991)

*On a finite universe  $M$ , the PERM operations over finite types are exactly the ones definable in finite type theory.<sup>5</sup>*

These results emphasize that ISOM and PERM are not very restrictive requirements, at the level of quantifiers and higher. Do they tell us anything about the logicity of quantifiers? If you consider  $FO_{\infty,\infty}$  or finite type theory as unproblematically belonging to logic, they might. But many are reluctant to do this, because they feel that these systems belong to *mathematics* rather than logic. Mathematics does have a subject-matter, so the argument goes, involving sets, numbers, functions, geometrical objects, etc. And the content of these formal systems essentially hinges on what mathematical objects exist.

Take another, closely related case: second-order logic, *SO*, where one is allowed to quantify also over subsets of (or relations on) the universe. In *SO* one can characterize the natural numbers, and one can formulate many undecided mathematical issues, such as the Continuum Hypothesis, in a way that makes them either logically true or logically false. But the Continuum Hypothesis is concerned with how many sets of natural numbers there are, and so, many people conclude, it is not a logical issue but a mathematical one, and therefore *SO* is not really logic either. (See Väänänen 2001 for a recent statement of this view, and Boolos 1998 for a dissenting opinion.)

### 9.1.3 Strengthening ISOM

If ISOM is necessary, one may consider strengthening it to a necessary and sufficient condition. One obvious way to strengthen ISOM is to weaken the requirement that  $f$  is a bijection. An interesting weakening is proposed by Feferman (1999). Essentially,

<sup>4</sup>  $FO_{\infty,\infty}$  is  $FO$ , but where you allow arbitrarily long conjunctions and disjunctions and arbitrarily long sequences of  $\forall x$  and  $\exists y$ .

<sup>5</sup> This is a logical theory with variables for each type, and the two basic operations  $\lambda$ -abstraction and function application; see e.g. van Benthem 1991 for details.

he considers invariance under *homomorphisms* rather than isomorphisms, or, more precisely, under functions from  $M$  onto  $M'$  that are not necessarily one–one.<sup>6</sup>

The resulting criterion HOM has drastic consequences for *monadic* quantifiers.<sup>7</sup> Onto mappings can collapse any non-empty set to a set with one element; up to HOM there are basically only two (kinds of) sets: empty ones and non-empty ones. One immediate consequence is that the identity relation is not invariant under such mappings. Indeed:

**Theorem** (Feferman 1999)

*The monadic (generalized) quantifiers satisfying HOM are exactly the ones definable in FO without identity.*

The treatment of identity is precisely the difference between ISOM and HOM. ISOM enforces topic neutrality, whereas HOM in addition enforces *number neutrality*: not only is the actual extension of predicate expressions deemed irrelevant to logic, but also the number of objects in those extensions. Only whether the extension is empty or non-empty matters. This is why only the existential quantifier  $\exists$ , and those definable from it in FO without identity, qualify.

The restriction to monadic operators is essential. As Feferman observes, many polyadic operators not usually regarded as logical satisfy HOM: for example, the well-ordering quantifier (see Chapter 2.4).

Interestingly, Feferman's main motivation for HOM is that he wants to resolve an issue that we too have been grappling with in this book: what is it that constitutes a quantifier being the *same* operation on universes of different sizes? ISOM has no effect here, since it only compares universes of the same size. In Chapter 3.4 we explicated constancy roughly as being given by the same rule on each universe, and noted that EXT corresponds to the special case when that rule doesn't mention the universe at all. In Chapter 4.5.3 we argued that although EXT is a sufficient but not necessary condition, it seems to hold for all quantifiers in natural languages that do not involve the logical predicate *thing*. However quantifiers like  $\forall$  or  $Q^R$ , which fail to satisfy EXT,

<sup>6</sup> Such functions are sometimes called *endomorphisms*. Some care is needed to formulate the lifting of such a map  $f$  to objects of higher type. If  $f$  is not one–one, condition (iv) in n. 2 cannot be used: first, because  $f^{-1}$  is not defined, and second, because condition (a), which does not use  $f^{-1}$ , need not define a function unless the further requirement that  $f(a_1) = f(a_2)$  implies  $F(a_1) = F(a_2)$  is added. However, for a set  $A$  (and correspondingly for relations), the proper requirement is:

(b)  $a \in A$  iff  $f(a) \in f(A)$

This is stronger than simply saying that  $f$  maps  $A$  to  $f(A) = \{f(a) : a \in A\}$ , since the latter allows that  $a \notin A$  but still  $f(a) \in f(A)$ ; viz. if  $f(a) = f(b)$  for some  $b \in A$ . Indeed, (b) says that  $f$  maps  $A$  to  $f(A)$  and (when  $A \subseteq M$ )  $M - A$  to  $f(M - A)$ .

<sup>7</sup> HOM says that whenever  $f$  is a mapping from  $M$  onto  $M'$  for which the lift appropriate for a universal operator  $O$  is well-defined,  $f(O_M) = O_{f(M)}$ . In particular, if  $Q$  is a type  $\langle 1, 1 \rangle$  quantifier, and  $f$  is such that (b) in n. 6 holds for  $A, B \subseteq M$ , then

$$Q_M(A, B) \iff Q_{M'}(f(A), f(B))$$

As Feferman's theorem shows, this is a very strong requirement.

seem nevertheless to be given by the same rule on every universe. So a final explication of this idea of sameness was left open.

Is HOM a better alternative? It does resolve—or rather eliminate—the problem. But to us, it seems perfectly reasonable to say that, for example, *at least four* yields the same quantifier on every universe: adding or deleting elements from the universe, as long as we do not alter *A* or *B*, has no effect on the truth value of *At least four A are B*. That is, we do think EXT is a *sufficient* condition. But then HOM cannot be the right criterion. There might be other reasons for denying that *at least four* is *logical*—for example, Feferman's idea that identity is not a logical notion—but even so, it can still be regarded as the *same* over all universes.

In general, we agree with the verdict in van Benthem 2002 that, as criteria of logicality, invariance conditions like the ones we have been discussing here are to some extent *circular*. This is most clear in the case of identity. Suppose we were uncertain whether identity is a logical notion or not. Well, restricting attention to invariance under bijections settles this in one way, and choosing surjections settles it in the opposite way. That is, the question has to be settled beforehand, and neither ISOM nor HOM can tell us anything here that we didn't know already.

However, as van Benthem emphasizes, this in no way precludes the potential for such invariance conditions to be (necessary) properties of logical expressions. To us, it seems that ISOM is the proper explication of the idea of topic neutrality, whereas HOM is too radical in this respect. In any case, the claim that ISOM is a necessary part of logicality appears to be practically uncontroversial.<sup>8</sup>

## 9.2 TWO CLAIMS ABOUT ISOM AND NATURAL LANGUAGE QUANTIFICATION

We have already observed that a great many determiner expressions in natural languages are ISOM. In fact, the following seems true.

(A) *All lexical quantifier expressions in natural languages denote ISOM quantifiers.*

Many complex determiners are ISOM too; for example, all Boolean combinations of ISOM quantifiers are ISOM. But there are some exceptions. About them, we make the following claim:

(B) *All non-ISOM quantifier expressions come from ISOM operations or relations by freezing some arguments.*

Recall that we defined the notion of freezing for type  $\langle 1, 1 \rangle$  quantifiers in Chapter 4.5.5, in fact as an operation from global type  $\langle 1, 1 \rangle$  quantifiers to global type  $\langle 1 \rangle$  quantifiers. We also discussed (three) alternative definitions of freezing, but argued that the version we chose is best suited to an account of natural language quantifiers (e.g. it underpins certain properties of definite quantifiers (Chapter 4.6) and

<sup>8</sup> Tarski (1986) seems to think the issue is a matter of stipulation. Sher (1991) argues that ISOM is both necessary and sufficient for logicality.

exceptive quantifiers (Chapter 8.13)). This notion is easily lifted to arbitrary universal operations.

The *field* of a binary relation over individuals is the union of its domain and range. This can be extended in an obvious way to arbitrary relations, indeed to objects of any type:

$$(9.5) \quad field(u) = \begin{cases} \{u\} & \text{if } u \text{ is an individual} \\ \emptyset & \text{if } u \text{ is a truth value} \\ \{x : x \text{ belongs to some tuple in } u\} & \text{if } u \text{ is a relation} \end{cases}$$

Also, if  $u$  is a function from  $X$  to  $Y$ , the field of  $u$  is the field of the corresponding binary relation ( $u'(x, y) \Leftrightarrow u(x) = y$ ). We now define freezing as follows:

### general freezing

Let  $O$  be a universal operator as above, which for simplicity we think of as relational, so that for each  $M$ ,  $O_M = O_M(X_1, \dots, X_n)$  is a (perhaps higher-order) relation over  $M$  of a given type. If  $S$  is a fixed object of the type of the  $k$ th argument place, define a new universal operator  $O^S$ , with one less argument place, as follows: For all  $M$ , and all  $X_1, \dots, X_{k-1}, X_{k+1}, \dots, X_n$  over  $M$  of suitable types,

$$(9.6) \quad (O^S)_M(X_1, \dots, X_{k-1}, X_{k+1}, \dots, X_n) \iff O_{M \cup field(S)}(X_1, \dots, X_{k-1}, S, X_{k+1}, \dots, X_n)$$

Similarly for simultaneous freezing of several arguments.

Our earlier example of freezing in Chapter 4.5.5,

$$(Q^A)_M(B) \iff Q_{M \cup A}(A, B)$$

is an instance of (9.6). We also noted that  $Q^A$  is EXT when  $Q$  is. This holds in general (recall that EXT is defined for arbitrary universal operators):

$$(9.7) \quad \text{With } O \text{ and } S \text{ as above: if } O \text{ is EXT, so is } O^S.$$

Indeed, this was one of our reasons for choosing the above version of freezing, which extends the universe, if necessary, to the individuals involved in the fixed object  $S$ ; the other two alternatives do not satisfy (9.7).

We illustrate Claim (B) with a number of examples:

### 9.2.1 Proper names

Montagovian individuals of the form  $I_a$  are not ISOM (Chapter 3.2.4). But consider the operation  $E$  that with each  $M$  associates a relation  $E_M$  such that, for all  $a \in M$  and  $B \subseteq M$ :

$$E_M(a, B) \iff a \in B$$

$E$  is ISOM, and, for a fixed individual  $a$ ,  $I_a$  is  $E^a$ , the freezing of  $E$  to  $a$ .  $((E^a)_M(B) \iff E_{M \cup \{a\}}(a, B) \iff a \in B \iff (I_a)_M(B))$  So if, in the Montagovian tradition, one construes proper names as quantifier expressions, i.e. as expressions that

“can be reasonably taken to signify a quantifier” (Chapter 0.2.3), they conform to Claim (B).<sup>9</sup>

### 9.2.2 Restricted noun phrases

We have already seen that noun phrases of the form *every dog*, *most professors*, *all except three bikes* are usually not ISOM (Proposition 7 in Chapter 3), but that they result from ISOM type  $\langle 1, 1 \rangle$  quantifiers by freezing the first (restriction) argument.

### 9.2.3 Possessives

Possessive determiners—discussed at length in Chapter 7—almost never denote ISOM quantifiers. But what about the operation *Poss* that we took to be the denotation of the possessive morpheme ‘*s*’? As defined in Chapter 7.7, *Poss* isn’t really a universal operator in the sense of the present chapter, since it takes (among other things) global type  $\langle 1, 1 \rangle$  quantifiers as arguments. These quantifiers are themselves universal operators, and do not fit in the type hierarchy of section 9.1.1 above. A universal operator can take only typed objects, such as quantifiers *on* a given universe, as arguments. But there is still a sense in which the Claim (B) is satisfied.

For any type  $\langle 1, 1 \rangle$  quantifiers  $Q_1, Q_2$ , let  $UP(Q_1, Q_2)$  be the universal operator defined, for each  $M$  and each  $A, B, C \subseteq M$  and  $R \subseteq M^2$ , by

$$(9.8) \quad UP(Q_1, Q_2)_M(C, R, A, B) \iff (Q_1)_M(C \cap \text{dom}_A(R), \{a \in C \cap \text{dom}_A(R) : (Q_2)_M(A \cap R_a, B)\})$$

(So *UP* is an operator taking universal operators to universal operators.) Now the following facts are easily established:

#### Fact 2

- (a) *If  $Q_1, Q_2$  are ISOM, so is  $UP(Q_1, Q_2)$ .*
- (b) *If  $Q_1, Q_2$  are CONSERV and EXT, then for any  $C, R$ ,*  
 $Poss(Q_1, C, Q_2, R) = UP(Q_1, Q_2)^{C, R}$

Thus, given that possessive determiners denote possessive quantifiers, i.e. quantifiers of the form *Poss* ( $Q_1, C, Q_2, R$ ), where  $Q_1$  and  $C$  come from the determiner

<sup>9</sup> This particular observation is of course trivial. Yet we do not think Claim (B) is completely empty. It is true that in type theory you can always find a ‘logical skeleton’ (van Benthem’s expression, personal communication) behind each expression by taking out all non-logical primitives by  $\lambda$ -abstraction. Inserting the primitives as values then gives you an expression equivalent to the original one. However, natural languages don’t have  $\lambda$ -abstraction in the syntax, and it is interesting to note (as scholars have since Montague) how they manage without it. Also, there is a small issue, as we have observed, about just how the frozen universal operation is to be defined. Finally, in natural languages there are sometimes more complex ‘skeletons’ behind non-ISOM quantifier expressions than those obtained by mere  $\lambda$ -abstraction, as the example of possessives below illustrates.



itself, and  $Q_2$  and  $R$  are set by context (sometimes by other semantic rules), it follows that each possessive determiner denotes a quantifier that results from freezing arguments of an ISOM operation.<sup>10</sup>

Similar observations can be made for the operator  $Poss^w$  in Chapter 7.8.1.

### 9.2.4 Exceptive quantifiers

Exactly analogous things can be said about the operators  $Exc_i$  used in Chapter 8.12 to handle certain exceptive quantifiers (where  $i$  is  $w$  or  $s$ ). Defining, for each type  $\langle 1, 1 \rangle$  quantifier  $Q_1$ , the universal operators  $UE_i(Q_1)$  so that  $UE_i(Q_1)_M(C, A, B) \Leftrightarrow Exc_i(Q_1, C)_M(A, B)$ , we see that the type  $\langle 1, 1 \rangle$  exceptive quantifier  $Exc_i(Q_1, C)$  results from freezing the first argument of  $UE_i(Q_1)$ , and that  $UE_i(Q_1)$  is ISOM if  $Q_1$  is.

As to the more general operators  $Except_i$  introduced in Chapter 8.13, which take a type  $\langle 1, 1 \rangle$  quantifier  $Q_1$  and a type  $\langle 1 \rangle$  quantifier  $Q$  as arguments, we first note that these are *partial* operators, since they are undefined on  $Q$  if  $W_{Q_M}$  (the smallest set that  $Q_M$  lives on) is undefined for any  $M$ . They are, however, total on finite universes.

Now in this case it can again be seen that if  $Q_1$  and  $Q$  are ISOM, so is  $Except_i(Q_1, Q)$  (when defined). But note that when  $Except_i$  is used to interpret exceptive determiners,  $Q$  is *never* ISOM (e.g. when  $Q$  is  $C^{pl}$ , or  $I_j$ , or of the form  $Q_2^C$ ). So what we would like to say here is rather that exceptive quantifiers result by *freezing*  $Q$  in some ISOM higher-order operation. Notice that the situation is different from the one with  $Poss$  or  $Exc_i$ , in that there the quantifier arguments are in general ISOM. However, the present notion of freezing doesn't apply to global quantifiers, so we cannot quite express the fact that exceptive quantifiers conform to Claim (B) in this way.

One option is to resort to the tactics of note 10, defining operators  $UExcept_i$  taking *local* quantifiers as arguments (in addition to the set arguments  $A, B$ ). As noted there, we can then freeze  $Q_M$  rather than  $Q$ , and Claim (B) is almost satisfied.

We conclude that, *roughly*, Claim (B) holds for exceptives too, even though the exact formulation of freezing needs some further technical amendments that we shall not go further into here.

Moreover, similar remarks can be made regarding other higher-order operators that we used for the interpretation of determiners—for example, the operator  $Def$  from Chapter 7.11.

<sup>10</sup> One might let ' $s$ ' denote  $UP$  rather than  $Poss$ , but neither of these is a universal operator. A third option is to have it denote the universal operator  $UPoss$ , defined for each  $M$  by

$$UPoss_M(Q_1, C, Q_2, R, A, B) \Leftrightarrow Q_1(C \cap dom_A(R), \{a \in C \cap dom_A(R) : Q_2(A \cap R_a, B)\})$$

where  $Q_1, Q_2$  are type  $\langle 1, 1 \rangle$  quantifiers on  $M$ , and  $A, B, C, R$  are as before.  $UPoss$  is ISOM, and its relation to  $Poss$  is the following:

- (i) If  $Q_1, Q_2$  are CONSERV and EXT, then for any  $M$  and any  $C, R$ ,

$$Poss(Q_1, C, Q_2, R)_M = (UPoss^{(Q_1)_M, C, (Q_2)_M, R})_M$$

This is almost as in Claim (B), but not quite, since freezing is local in (i).

## 9.3 CONSTANCY

In this book we treat logicality and constancy as distinct properties. This is hardly standard; the literature contains much discussion about the notion of a *logical constant*, but little about constancy. But separating the two makes certain issues clearer, it seems to us.

Consider again the quantifier *most*, and restrict attention to finite universes. Is it a logical constant? It certainly is topic-neutral (ISOM), and constant in the sense of EXT. But this may not settle the question. One might argue that *most* is too mathematical to be logical: for example, it is not definable in *FO* (as will be shown in Chapter 14), and it involves comparison of cardinalities. On finite sets this still seems rather innocent. Perhaps there is no correct answer.

Barwise and Cooper (1981) make a strong claim against the logicality of *most*, however. Part of the argument comes from infinite sets where *most* might be thought of as involving not just size but some notion of measure or topology. We can leave that aside here. More interesting is the conclusion that Barwise and Cooper draw:

The difference between “every man” and “most men” is this. The interpretation of both “most” and “man” depend on the model, whereas the interpretation of “every” is the same for every model (p. 163)

Thus, the issue of logicality is tied directly to *what gets interpreted in models* and what doesn't.<sup>11</sup>

If Barwise and Cooper's remark is meant to apply to finite models and natural language (as their examples seem to indicate), and not just to models where mathematical interpretations of *most* different from the cardinality interpretation might be relevant, then we believe their claim is mistaken. Briefly, whatever the verdict on the logicality of *most*, it is definitely *constant*, and that's why it should *not* be interpreted in models.

How do we know that *most* is constant? Our explication of constancy—as being given by the same rule on every universe—sounds a bit abstract and imprecise. But it seems to us that the constancy of an expression is also reflected in its linguistic behavior, in a way that is in fact easy to test. The idea is that such expressions are *constant in inferences* in a way that clearly sets them apart from other expressions. In the rest of this section we try to explain this idea.

## 9.3.1 Inference constancy

The idea we shall try to bring home is this: Insofar as *consequence* or *entailment* is a matter of *form*, quantifier expressions, but not predicate expressions (with few exceptions), are *fixed* in that form: they are what we shall call inference constant. Such

<sup>11</sup> It is important to note that the issue does *not* concern the vagueness or context dependence of *most*. Barwise and Cooper make clear—as we have done in this book—that a *fixed context assumption* should eliminate this kind of unclarity. In particular, it should fix the appropriate ‘threshold’ (the one we take to be 1/2 here for simplicity). Thus, it is not context dependence that prevents *most* from being logical, according to Barwise and Cooper.

constancy is an empirical fact about languages, not something theorists can stipulate one way or the other. Moreover, it is a phenomenon that can be used to identify the expressions which are constant in the more abstract interpretational sense.

In fact, our point is so obvious that it seems to be forgotten. Consider straightforward entailments like the following. Once the reasoning

- (9.9) a. Most students passed.  
b. *Hence*: Some students passed.

is seen to be correct or valid, it is *self-evident* that

- (9.10) a. Most teachers are taller than Bill.  
b. *Hence*: Some teachers are taller than Bill.

is not only valid as well, but is in fact another instance of the *same* entailment. And it is equally self-evident that

- (9.11) a. At least two students passed.  
b. *Hence*: All students passed.

is not only incorrect, but *completely irrelevant* to the correctness of (9.9). Even if a replacement of quantifiers turns (9.9) into a correct inference (say, if we replace *most* by *at least two*, but keep *some*), that inference still says nothing at all about (9.9).

We have here a very palpable difference between quantifier expressions and predicate expressions. Replacement of predicate expressions preserves the entailment. Replacement of quantifier expressions often destroys it. And the reason is that the latter kind of replacement, but not the former, results in a *different* inference (scheme), without significant relations to the original one. In (9.9), *most* and *some* are fixed, but *students* is not. We now try to describe this idea a little more carefully.

### 9.3.2 Bolzano's method

The idea of identifying a linguistic notion by studying how systematic replacement of expressions preserves certain properties is one that, as far as we know, goes back to Bernard Bolzano in the early nineteenth century (Bolzano 1837). But whereas he used it to *define* consequence, we in a sense reverse his idea, in order to characterize the constants themselves. We first illustrate it with a more recent and more familiar application of the technique: identifying syntactic or semantic categories in a language.

**Example 1:** Let the class of relevant expressions be the set of finite strings of words in some language, and let the property to be preserved be *grammaticality* in the case of syntactic categories, and *meaningfulness* in the case of semantic categories. Roughly, strings  $u$  and  $v$  are said to have the same syntactic (semantic) category if and only if, for every string  $s$  with  $u$  as a substring,  $s$  is grammatical (meaningful) iff  $s[u/v]$  is grammatical. Categories can then be identified with equivalence classes under these equivalence relations.

This idea is simple, powerful, and intuitive. Substitutability of this kind *should* be a mark of sameness of category. But the idea is also problematic in familiar ways. A minor point is that one often cannot simply replace a word, say, by another, without making morphological changes (man cannot replace women in two women came, even though both are nouns). A more substantial point is that if there is no grammatical organization of the strings of words, the technique is very likely to give wrong results: a substitution may succeed, or fail, when it shouldn't. For a trivial example, suppose we try to identify NPs in English, and consider John in

(9.12) Mary met John.

As desired, we may replace John by Lisa, Irene and Henry, some student, the professors, more students than professors, a boy, them, etc., preserving grammaticality, and these are all good NPs. But, we may also replace John by David and went home, for example, which no one wants to call an NP.

It seems likely that for the method to succeed, some sort of grammar must be given, attention must be restricted to strings generated by the grammar, and replacement must be defined in a way that respects the grammar. This shows that the method needs to be refined, not that it is useless or circular. It might be circular if the grammar already mentioned the categories, but that is not necessary. Hodges (2001) gives a precise definition of categories in an abstract framework which is far from trivial, tracing the idea (for semantic categories) back to Husserl (1901).

However, even if it turned out in the end that, for English, no precise definition of syntactic categories in the above format is really adequate, that wouldn't make the substitution technique useless. It would remain a powerful, indeed essential, heuristic for *finding* the categories of various expressions (even if the delimitation in certain cases requires other tools as well).

In what follows, we shall assume that (syntactic or semantic) categories of English words and phrases have been identified in some way or other. By an *appropriate replacement*, we mean a replacement that respects these categories as well as morphological requirements.

**Example 2:** Consider (English, say) *sentences*, under appropriate replacement of *parts*, and see when *truth* is preserved. This is Bolzano's use of the method. Roughly, if a sentence remains true under all appropriate replacements of certain of its parts, its truth is special, or "analytical", dependent not just on facts about the world, but on other factors.<sup>12</sup> Similarly, if such replacements never make the premises of an inference true and the conclusion false, there is some ground for saying that the conclusion *follows* from the premises. But, as Bolzano emphasized, these notions are *relative* to a set of non-replaceable expressions—the constants. In fact, his general notion of consequence can be seen as *three-place*; it holds between a (set of) premises, a conclusion, and a chosen set of constants. Fixing the constant vocabulary in various ways,

<sup>12</sup> Note that this presupposes an analysis of sentences into constituents, which makes Bolzano a pioneer in the field of syntactic structure as well.

we obtain a variety of special notions of consequence, suitable for different purposes, which was precisely Bolzano's idea.<sup>13</sup>

Among these notions, Bolzano only briefly mentions *logical* truth and consequence (e.g. Bolzano 1837: §148(b), (c), where he defines "logically analytical"), saying, however, that the logical concepts—the ones to be held constant in this case—do not form a sharply delimited area.

Bolzano's idea is amazingly rich. A possible problem is that its implementation in natural languages is exposed to 'accidental' effects due to scarce linguistic resources. For example, suitable replacements for the word *and* should be (from a modern standpoint) at least the fifteen other binary truth functions, but it is not clear that English has a reasonable name for each of these. This *might* make us miss some relevant counterexample. Worse, one might want to replace a set-denoting predicate expression with one for any other possible extension (set), which certainly is impossible in any natural language. When Tarski (1936) formulated his definition of logical truth and consequence, 100 years later than Bolzano (and apparently unaware of him), he avoided this problem by considering arbitrary *interpretations* of certain parts, rather than replacing them by other parts. The basic idea, however, is the same.

Tarski too noted that the range of logical expressions—the ones not to replace or interpret—is hard to delimit. (Later he suggested ISOM as a criterion, as we have seen.) Yet it has to be delimited in order for the notion of logical consequence to make sense.

### 9.3.3 Bolzano reversed

We now suggest that a variant of the same method can be used to identify the constants themselves. This time the complex expressions to be considered are *inferences*, construed as finite sequences of sentences in some language, where the last one is the conclusion and the others are premises. The point is that, just as speakers have intuitions about grammaticality or meaningfulness or truth, so they have intuitions about *validity* or *invalidity* of inferences.

Such intuitions can be used, just as in the two examples above, as *data* for the substitution technique. Of course we must not assume that speakers are equipped with sophisticated notions of consequence: for example, that they can distinguish logical from analytical validity. That presumably already presupposes a notion of (logical) constant, and *those* intuitions, if there are any, can clearly not be used in the present context. The empirical fact we can use is simply that speakers have intuitions, in many cases, about whether or not the conclusion of a given inference *follows* from the premises.

Now, as we remarked in section 9.3.1 above, part of the intuition about the validity of an inference is that some of the expressions involved are *schematic*, whereas others

<sup>13</sup> See van Benthem 2003 for a recent and lucid survey of Bolzano's theory of consequence, with many striking comparisons to modern activities in logic. One thing to note is that Bolzano also required the premises to be jointly consistent for the conclusion to follow; this condition is usually dropped nowadays (anything follows from inconsistent premises), but if it is upheld, the logic becomes *non-monotone*, as van Benthem notes.

are not. That an expression is schematic in a given inference means roughly that it can be replaced by a variable without disturbing validity. This results in an *inference scheme*, and the notion of validity extends naturally to schemes. We let “inference scheme” be the most general term here, so that an inference is the special case of an inference scheme without schematic variables. The following notation is used:

- (9.13) When  $u$  is an expression occurring in an inference scheme  $\Pi$ , write  $\Pi[u/x]$  for the result of replacing at least one occurrence of  $u$  in  $\Pi$  by the (new) variable  $x$ .<sup>14</sup>

We can now make the following definition.

#### schematic expressions in valid inferences

An expression  $u$  occurring in a valid inference (scheme)  $\Pi$  is *schematic with respect to*  $\Pi$  iff some scheme of the form  $\Pi[u/x]$  is valid.

It follows from this definition that a variable in a valid scheme is trivially schematic. Now look at plain inferences without variables. Even though the following statements are imprecise, their approximate correctness should be obvious when one considers examples like the ones in section 9.3.1, and innumerable many others like them:

- (I) Predicate expressions and individual-denoting expressions are usually schematic in inferences.  
 (II) Quantifier expressions are often non-schematic in inferences.

Of course, “usually” and “often” need to be qualified. As to (I), one should note two things. First, *identity* is a predicate that usually seems non-schematic; consider inferences like the following:

- (9.14) a. Alex is a horse.  
 b. Henry is not a horse.  
 c. *Hence*: Alex is not identical to Henry.

This is not a problem; on the contrary, it strengthens the intuition that identity is a (logical) constant. But identity is an exception among predicate expressions; we saw this in terms of topic neutrality (ISOM; see Proposition 1), and we see it also in terms of constancy.

Second, for most predicate expressions, *there exist* valid inferences in which they are not schematic. For example,

- (9.15) a. John is a student.  
 b. *Hence*: John is human.

<sup>14</sup> It is not required that all occurrences of  $u$  be replaced by  $x$ ; the motivation for this will appear below. Also, we can think of  $x$  as a semantic rather than a syntactic variable, so it ranges over the *objects* of the kind to which the denotation of  $u$  belongs. For example, if  $u$  is the word *and*,  $x$  might range over binary truth functions, and if  $u$  is a two-place predicate expression,  $x$  might range over binary relations of the appropriate kind.

This is presumably valid—in the sense that the conclusion would be judged to *follow* from the premise.<sup>15</sup>  $\exists$ John is schematic in (9.15), but *student* and *human* are not. But the point is that there are countless *other* valid inferences where these words *are* schematic.

As to (II), one can certainly find valid inferences where quantifier expressions are schematic in the sense used here, but they seem to be of particular kinds. One kind is easy to identify, and hence exclude: when a quantifier expression  $u$  in  $\Pi$  is a proper subexpression of another expression in  $\Pi$  which already is schematic. For example, the following are valid:

- (9.16) a. At least two students laughed.  
 b. *Hence*: At least two students laughed or John loves Mary.
- (9.17) a. Mary is taller than Bill, and Bill is taller than Sue.  
 b. *Hence*: Mary is taller than Sue, or at least two students laughed.

In both of these, *at least two* is schematic: it can be replaced by any other determiner without affecting validity. But these inferences are irrelevant to the issue of the constancy of *at least two*, since the following are also valid:

- (i) a.  $\varphi$   
 b. *Hence*:  $\varphi$  or John loves Mary
- (ii) a. Mary is taller than Bill, and Bill is taller than Sue  
 b. *Hence*: Mary is taller than Sue, or  $\varphi$

We may call such occurrences of expressions in inferences *spurious*. Disregarding inferences where the expression in question has spurious occurrences is one way of focusing on *relevant* inferences. This effect is also seen in the following fact.

- (9.18) If an occurrence of  $u$  is a proper subexpression of a redundant premise in  $\Pi$ , then that occurrence is spurious in  $\Pi$ .

This is immediate, since a redundant premise can simply be replaced by a propositional variable. We give two examples; for simplicity they are taken from logical languages. Consider first the quantifier  $\exists$ . The following is trivially valid.

- (9.19)  $\forall X(QxX(x) \leftrightarrow X \neq \emptyset)$   
 $\frac{P(a)}{QxP(x)}$

<sup>15</sup> It will not do to disqualify the entailment (9.15) for not being a case whose validity is a matter of *form*. First, since  $\exists$ John can be replaced by a variable, one may well say that (9.15) is valid because of its form. Second, to rule out entailments of this particular form would presuppose that one already had a distinction between logical and non-logical expressions. Unless one is careful, this could be a serious case of question begging.

One may avoid question begging by imposing a criterion of logicity independent of the notion of constancy that we are trying to delineate here. Indeed, *ISOM* is such a criterion, and would nicely rule out *human* and *John* (Proposition 1). At the moment, however, we are trying to single out quantifier expressions—if possible—solely by their behavior in intuitively valid inferences, without involving other criteria.

Now replace  $Q$  by  $\exists$ :

$$(9.20) \quad \frac{\forall X(\exists xX(x) \leftrightarrow X \neq \emptyset) \quad P(a)}{\exists xP(x)}$$

In (9.20), we can replace  $\exists$  by a (quantifier) variable while preserving validity:  $\exists$  is schematic. However, the first occurrence of  $\exists$  in (9.20), but not the second, is spurious: the whole premise where it occurs is redundant and can be replaced by a schematic variable:

$$\frac{\varphi \quad P(a)}{\exists xP(x)}$$

And in this inference,  $\exists$  is non-schematic. Here is an analogous argument concerning the identity symbol  $=$ . Since

$$\frac{\text{'R is symmetric'} \quad R(a, b)}{R(b, a)}$$

is trivially valid, so is

$$\frac{\text{'= is symmetric'} \quad a = b}{b = a}$$

where  $=$  is schematic. But again, the first premise is redundant, and its occurrence of  $=$  spurious.

There is, however, one kind of valid inference where (standard) logical constants are schematic without having spurious occurrences. Again we illustrate with a logical example, even though examples from natural languages are easy to construct.

$$(9.21) \quad \frac{\exists xP(x) \quad \forall x(P(x) \leftrightarrow R(x))}{\exists xR(x)}$$

Here  $\exists$  can be replaced by any type (1) quantifier variable  $Q$ .<sup>16</sup> (9.21) is an *extensionality claim*: only the extension of the arguments matters to the operators denoted by

<sup>16</sup> Note also the following variant of (9.21):

$$\frac{\exists xP(x) \quad \neg\exists\neg x(P(x) \leftrightarrow R(x))}{\exists xR(x)}$$

Here too  $\exists$  can be replaced by  $Q$ , *provided* we replace the first and the third occurrence, but *not* the second. This shows why we must allow the possibility, in the definition of schematicity above, that some but not all occurrences of  $u$  are replaced; see n. 14.



these expressions. And in every model where the premises are true,  $P$  and  $R$  have the same extension.

Here is a final and, at first sight, not so obvious example, this time from propositional logic:

$$(9.22) \frac{(p \vee r) \wedge q}{\neg r} \\ p \wedge q$$

Now  $\wedge$  is schematic, since, for any binary truth function  $\#$ , the following is valid:

$$\frac{(p \vee r) \# q}{\neg r} \\ p \# q$$

(9.22) is also an extensionality claim. This is because in all valuations (models) where the premises are true,  $p \vee r$  and  $p$  must have the same truth value, so in fact the conclusion results from replacing, in one of the premises, an expression with one which has the same extension.

We have illustrated in what kind of inferences one may expect quantifier expressions to be schematic, and predicate expressions (apart from identity, etc.) to be non-schematic. Hopefully, it has become clear that these are all *exceptions* in one way or another, and that the “usually” and “often” in (I) and (II) above are well motivated. And that is the reason why we feel confident in saying that quantifier expressions are indeed inference constant.<sup>17</sup>

Whereas Bolzano (and Tarski) started from a set of logical constants and defined logical consequence, we start from an intuitive notion of consequence and try to identify the constants. In this sense it is a reversal. But the underlying method is the same: use facts about preservation of an (in principle) empirically testable property under appropriate replacement of parts of sentences.

### 9.3.4 What to interpret in models

Inference constancy is not the same notion as constancy in the sense of being given by the same rule on every universe, but we take the former to be an indication of the latter. Inference constancy relies on which expressions preserve validity under arbitrary

<sup>17</sup> The general format of the definition of this concept is thus:  $u$  is *inference constant* iff it is non-schematic in *all* relevant inferences. Can this be made into a precise concept? That depends on whether we can pin down the pertinent notion of relevance. We have seen how to identify two kinds of irrelevant inferences. If these are all there are, we could stipulate that (assuming  $u$  occurs in  $\Pi$ )  $\Pi$  is *relevant to  $u$*  iff  $u$  has no spurious occurrences in  $\Pi$ , and  $\Pi$  is not an extensionality claim about  $u$ . (The examples in the text indicate how the concept of an extensionality claim could be precisely defined.)

To test if this works, one should at least verify that the usual logical operators in classical propositional or predicate logic are indeed inference constant in this sense. (It is clear that proposition letters, predicate symbols, and individual symbols are not inference constant.) We leave it as an open question whether this holds or not.

appropriate replacements. If an expression can be thus replaced, there is no fixed rule which tells us how to interpret it. If it cannot, that might be because there is such a rule. And appropriately replacing an expression is of course very similar to interpreting that expression in different models.

We might see the two notions as a proof-theoretic and a model-theoretic version of the same idea. Or we could see inference constancy as an empirical counterpart of the more theoretical idea of being given by the same rule. In any case, since the inference constancy of quantifier expressions seems indubitable, the contention that they are constant *simpliciter* is reinforced. And from this it follows that these expressions are *not* the kind that get interpreted in models, contrary to Barwise and Cooper's recommendation cited at the beginning of this section.

In fact, it turns out that not even Barwise and Cooper themselves followed this recommendation. This becomes clear when one considers issues of expressivity. Barwise and Cooper (1981) proved one of the first significant undefinability results about quantifiers: namely, that *most* is not definable in  $FO(Q^R)$ , i.e.  $FO$  logic with the Rescher quantifier added, not even over finite models (see Proposition 12 in Chapter 14). But in this result and its proof one uses first-order models and lets both *most* and the Rescher quantifier have their constant meanings, given by the familiar rules. The result means that sentences with the determiner *most* cannot be expressed using only the unrelativized version of the quantifier—an important fact about expressive power. All such facts are relative to a notion of synonymy (an insight we elaborate on in Chapter 11). And in this as well as other similar cases, the synonymy in question is standard logical equivalence, defined as sameness of truth value in all *first-order* models.

To be sure, it is possible to define a much stricter notion of logical equivalence, this time with respect to models where certain quantifier expressions as well as predicate expressions are arbitrarily interpreted. This is equivalence in the logic  $FO(q)$  (sometimes called *weak* logic), which has the same formulas as  $FO(Q)$  for a, say, type  $\langle 1 \rangle$  quantifier  $Q$ , but with models now of the form  $(\mathcal{M}, q)$ , where  $\mathcal{M}$  is as before, and  $q$  is any set of subsets of  $M$  (any local type  $\langle 1 \rangle$  quantifier on  $M$ ). So the symbol “ $Q$ ” is interpreted as an arbitrary quantifier, but  $\forall, \exists$ , and the other logical symbols in  $FO$  have their fixed standard meaning. This logic was first studied by Keisler (1970), who proved a completeness theorem for it.<sup>18</sup> Sentences which are equivalent in this sense are also logically equivalent in the usual sense, so  $FO(q)$ -equivalence *refines* standard equivalence. Therefore, the Barwise and Cooper result transfers to it too: no  $FO(q)$ -sentence can define *most*. But that result is weaker, and certainly not as interesting from a linguistic perspective, precisely because a notion of synonymy wherein quantifier expressions are treated as variable or schematic has no intuitive counterpart in

<sup>18</sup> See also Westerståhl 1989: appendix B, for an overview. Keisler's result was just a preliminary step towards his proof that the logic  $FO(Q_1)$ , where  $Q_1$  means 'there exist uncountably many', had a simple, complete axiomatization. But it is interesting to note that the only new axioms required for  $FO(q)$  are precisely *extensionality claims* in the sense of the previous subsection (plus axioms permitting change of bound variables).

natural languages. Significantly, it is not this result that Barwise and Cooper state and prove, but the result with fixed quantifiers and first-order models.

Granted, then, that quantifier expressions should not be interpreted in models, this is further evidence for the adequacy of choosing the class of *first-order models* (models in which only predicate expressions and individual-denoting expressions are interpreted) in an account of quantification in natural languages (see Chapter 2.1). The choice of models obviously affects the notion of *logical equivalence*: having the same truth value in all (those) models. We shall use “logical equivalence” for sameness of truth value in all first-order models. Someone who insists that, say, *most* is not a logical quantifier might deplore this terminology; for example, in her view (9.9) above is not a case of logical consequence. That is, she wants to reserve “logical consequence” or “logical equivalence” for a more restricted relation, one that refines our notion.

We think that this is mostly a terminological question, and we have clearly stated how these terms are used here. (More about logical equivalence will follow in Chapters 11.3.2 and 12.1.) But for results about expressive power there is an important point in using the least refined version of logical equivalence. We already indicated this in connection with Barwise and Cooper’s result above, and we will explain the underlying mechanism in detail in Chapter 12.5.

## 9.4 LOGICAL CONSTANTS

Is it thus the case that logical constants are simply expressions that are logical *and* constant in our sense? Perhaps, but we have not defined logicity, or even claimed that it can be defined; we only said that topic neutrality, i.e. ISOM, is a necessary condition.

### 9.4.1 Logic versus mathematics

As to logicity conditions in addition to ISOM, a common line of argument is that essentially mathematical notions are not logical; in particular, logic should not rely on facts about which sets exist. We already mentioned this in connection with  $FO_{\infty,\infty}$  and second-order logic in section 9.1.2 above, but let us restate the point, since it is an important though difficult one.

Logic, according to this idea, has no subject-matter, but mathematics does: numbers, sets, functions, geometrical objects, etc. The truths of mathematics are *necessary*—since mathematical objects are abstract and independent of the material world—but not *logical*. The border line is not clear, though. For example, if identity is a logical notion, each finite number is characterizable in  $FO$  (in the sense that each quantifier  $\exists_{=n}$  is definable there). So at least things like

if  $A$  has five elements and  $B$  has three elements, some things in  $A$  are not in  $B$

will then come out as logical truths, although they seem to involve numbers. Similarly for

if  $A$  is a subset of  $B$  and  $B$  is a subset of  $A$ , then  $A = B$

However, these borderline cases are perhaps not very serious, since there is no controversy at all about finite numbers and their properties, or about the subset relation. Things get more complicated when mathematical objects whose existence is not clear are involved. A paradigmatic case is size comparisons between *infinite* sets.

Such comparison essentially involves the existence of functions. To say that  $B$  has *at least* the size of  $A$  is to say that there exists a one–one function from  $A$  to  $B$ . To say that  $B$  has *more* elements than  $A$  is to say in addition that there is no one–one function in the other direction. But which functions are there, really? One would think that set theory gives the answer, but in fact all current set theories *leave open* important questions about size comparison (such as the Continuum Hypothesis), and there is no unanimity about if and how these gaps should be filled. Most logicians conclude—though there are exceptions—that size comparison between infinite sets clearly belongs to mathematics, not logic.

For quantifiers, special conditions that in various ways try to separate the mathematical from the logical have been proposed. The idea of using conditions involving size comparison for this purpose was proposed by van Benthem (1986); an overview can be found in Westerståhl 1989. We furthermore mentioned Feferman’s condition HOM (section 9.1.2) with its drastic consequences. It has to be said that choosing between these criteria feels more like stipulation than discovering the truth. And perhaps that is precisely as it should be.

#### 9.4.2 ISOM + EXT

On the bright side, we do have two precise and informative requirements on logical constants: namely, ISOM and EXT. Also, we proposed certain universal generalizations concerning the distribution of these properties: for ISOM the claims (A) and (B) in section 9.2 above, and for EXT the universal (EU) in Chapter 4.5.3.

Interestingly, the two characteristics appear to single out very much the same expressions, although they are based on quite different intuitions. ISOM is necessary, but probably not sufficient, for logicity, and EXT is sufficient, but not necessary, for constancy. But, as (EU) says, the exceptions to EXT appear to be few and of the same kind. If we somehow disregard those, we can think of ISOM + EXT as a necessary property of logical constants. We end by observing that this combined condition can be expressed as a simple strengthening of ISOM.

Recall Feferman’s way of strengthening ISOM by postulating invariance for arbitrary surjections, rather than bijections (section 9.1.3). Now, another obvious way to weaken the requirement on the mapping is to allow it to be an *injection* (i.e. one–one but not necessarily onto). Call the resulting invariance condition INJ.<sup>19</sup> This works for objects of any type; for a generalized quantifier  $Q$  of type  $\langle n_1, \dots, n_k \rangle$ , it can also be expressed as follows:

(INJ) if  $f$  is an injection from  $M$  to  $M'$ , and  $R_i \subseteq M^{n_i}$  for  $1 \leq i \leq k$ , then  
 $Q_M(R_1, \dots, R_k) \Leftrightarrow Q_{M'}(f(R_1), \dots, f(R_k))$

<sup>19</sup> Lifting injections from  $M$  to  $M'$  to injections from objects of type  $\tau$  over  $M$  to objects of type  $\tau$  over  $M'$  requires only small changes to the definition given in n. 2.

Then we have (for arbitrary types):

**Proposition 3**

*INJ is equivalent to ISOM + EXT.*

We leave the (easy) proof as an exercise.

# 10

## Some Polyadic Quantifiers of Natural Language

In this chapter we look at some instances of polyadic natural language quantifiers, beginning in section 10.1 with the most common way of combining monadic quantifiers, in ordinary clauses with more than one noun phrase: *iteration*. Such combination does not really lead outside the class of monadic quantifiers, but the iteration operation has interesting properties both from a logical and a linguistic point of view. Another operation, cumulation, which is also monadically expressible, is also mentioned.

Each of the remaining three sections deals with an operation that *lifts* monadic quantifiers to irreducibly polyadic ones. Section 10.2 discusses *resumption*: i.e. applying monadic quantifiers to pairs, triples, etc. rather than to individuals, as can be done with certain instances of adverbial quantification. Section 10.2.1 reviews the (mainly linguistic) issue of what resumption has to do with ‘donkey’ anaphora, and section 10.2.2 gives some useful (mainly logical) characterizations of resumptive quantifiers in terms of certain properties.

Section 10.3 very briefly discusses the putative examples of *branching* quantification in natural languages that were mentioned already in Chapter 2.5.2. Section 10.4, finally, looks at *reciprocals*. We indicate a uniform approach to various kinds of reciprocal quantification, and end with a suggested semantic interpretation rule that works for reciprocal predicates (*helped each other study for the exam*) and collective predicates (*gathered for the protest*) alike—indeed, the only occasion in this book where we touch upon the issue of collective quantification.

Proofs that these three lifts do transcend monadic quantification will be given in Chapter 15. We are taking these three as typical examples here, but note that several other cases of polyadic natural language quantification have been suggested in the literature (see Keenan and Westerståhl 1997 for some further examples).

### 10.1 ITERATION

Noun phrases and determiners occur not only as subjects in quantified sentences, but as objects, in prepositional phrases, in subordinate clauses, etc. Consider

(10.1) John recognized all but four dogs.

(10.2) Two critics reviewed most films.

(10.3) At most three boys gave more dahlias than roses to Mary.

In each of these cases, the truth conditions of the sentence could be described with a polyadic quantifier. Using our convention that when  $R$  is a binary relation,  $R_a = \{b : R(a, b)\}$ , we would have (suppressing the universe):

$$(10.1)' \quad Q_1(A, R) \iff |A - R_j| = 4. \quad (\text{type } \langle 1, 2 \rangle)$$

$$(10.2)' \quad Q_2(A, B, R) \iff |\{a \in A : |B \cap R_a| > |B - R_a|\}| = 2 \quad (\text{type } \langle 1, 1, 2 \rangle)$$

$$(10.3)' \quad Q_3(A, B, C, R) \iff |\{a \in A : |\{b \in B : R(a, b, m)\}| > |\{b \in C : R(a, b, m)\}|\}| \leq 3 \quad (\text{type } \langle 1, 1, 1, 3 \rangle)$$

However, each of these can also be obtained by one and the same operation from the usual monadic quantifiers:  $I_j, I_m, \textit{all but four}, \textit{two}, \textit{most}, \textit{at most three}, \textit{more\_than}$ . Describing them in this way is preferable, since it allows a compositional semantics, as well as a straightforward account of (some of) the ambiguity in such sentences. The operation is called *iteration*, and can be defined in the following way.

Let  $UN$  ('unary') be the class of quantifiers of type  $\langle n \rangle$  for some  $n$ . It is enough, as we will soon see, to define iteration for arbitrary quantifiers in  $UN$ . The iteration operation has two arguments, but it can be repeated: applying it to two type  $\langle 1 \rangle$  quantifiers yields a type  $\langle 2 \rangle$  quantifier, but this can in turn be iterated with a type  $\langle 1 \rangle$  quantifier, resulting in a type  $\langle 3 \rangle$  quantifier, etc. That is why  $UN$  is the smallest class of quantifiers which is *closed* under iteration.

We generalize the notation  $R_a$  as follows: If  $R$  is  $(n + k)$ -ary, and  $a_1, \dots, a_n$  are individuals,

$$R_{a_1, \dots, a_n} = \{(b_1, \dots, b_k) : R(a_1, \dots, a_n, b_1, \dots, b_k)\}$$

One easily verifies that

$$(10.4) \quad (R_a)_b = R_{ab}$$

**iteration**

Suppose  $Q_1$  is of type  $\langle n \rangle$  and  $Q_2$  of type  $\langle k \rangle$ .  $It(Q_1, Q_2)$ , also written  $Q_1 \cdot Q_2$ , is the type  $\langle n + k \rangle$  quantifier defined, for any  $M$  and any  $R \subseteq M^{n+k}$ , by (suppressing  $M$ ):

$$(10.5) \quad It(Q_1, Q_2)(R) \iff Q_1 \cdot Q_2(R) \iff Q_1(\{(a_1, \dots, a_n) : Q_2(R_{a_1, \dots, a_n})\})$$

The first thing to observe is that iteration is *associative*:

$$(10.6) \quad (Q_1 Q_2) Q_3 = Q_1 (Q_2 Q_3)$$

(The  $\cdot$  is suppressed whenever convenient.) This follows using (10.4). Thus, parentheses are not needed, and we may write without ambiguity

$$Q_1 Q_2 \dots Q_n$$

As an initial example, we see that the truth conditions for John likes Mary are obtainable as the iteration of  $I_j$  and  $I_m$  (in that order):

$$I_j I_m(\textit{like}) \iff I_j(\{a : I_m(\textit{like } a)\}) \iff \textit{like}(j, m)$$

Similarly, we could analyze the sentence Henry respects policemen by means of  $I_b \cdot \text{policemen}^{\text{pl}}(\text{respect})$ , and construe John introduced Mary to Peter as  $I_j I_m I_p(\text{introduced to})$ . Here are a few general facts about the iteration operation.<sup>1</sup>

**Fact 1**

- (a) *If*  $Q_1, Q_2$  *are* ISOM (EXT), *so is*  $Q_1 Q_2$ .  
 (b) *Iteration interacts with inner and outer negation as follows:*  
 (i)  $\neg(Q_1 Q_2) = (\neg Q_1) Q_2$  (ii)  $(Q_1 Q_2) \neg = Q_1 (Q_2 \neg)$  (iii)  $(Q_1 \neg)(\neg Q_2) = Q_1 Q_2$  *and consequently* (iv)  $(Q_1 Q_2)^d = Q_1^d Q_2^d$

The facts in (b) systematize various equivalences between sentences in natural languages. For example, (iii) explains why the following two sentences are equivalent (see also the end of this subsection):

- (10.7) a. Each critic reviewed at most three films.  
 b. No critic reviewed more than three films.

A number of logical questions about iteration arise naturally. One such question is to what extent an iteration determines its components. The sentences in (10.7) show that the components cannot be completely determined, because of facing negations. Another problem is that if one component is trivial (on  $M$ ), the whole iteration becomes trivial. But these are the only obstacles: slightly generalizing a Prefix Theorem in Keenan 1993, Westerståhl (1994) proves that if

$$(Q_1 \dots Q_k)_M = (Q'_1 \dots Q'_k)_M$$

holds, where all  $Q_i, Q'_i$  are of type  $\langle 1 \rangle$ , non-trivial on  $M$ , and *balanced* on  $M$  in the sense that  $(Q_i)_M(\emptyset) \Leftrightarrow (Q'_i)_M(\emptyset)$  (this is what avoids facing negations), then

$$(Q_i)_M = (Q'_i)_M$$

for each  $i$ . (There is also a global version of this result.)

Another issue is when the *order* among the quantifiers is important. More precisely, for which  $Q_1, Q_2$  does the following hold for every binary relation  $R$ ? (We assume that  $Q_1, Q_2$  are of type  $\langle 1 \rangle$ , for simplicity.)

- (10.8) a.  $Q_1 Q_2(R) \implies Q_2 Q_1(R^{-1})$  (*scope dominance*)  
 b.  $Q_1 Q_2(R) \iff Q_2 Q_1(R^{-1})$  (*order independence*)

<sup>1</sup> Westerståhl 1994 is a general study of iteration and its properties. A more general perspective is taken in Keenan and Westerståhl 1997: from the functional perspective of generalized quantifiers (Ch. 3.1.3), a type  $\langle n \rangle$  quantifier  $Q$  can be seen as an *arity reducer* or a *relation reducer*: if  $R$  is  $(n+k)$ -ary,  $Q(R) = \{(a_1, \dots, a_n) : Q(R_{a_1, \dots, a_n})\}$  is an  $n$ -ary relation, so  $Q$  reduces the arity of  $R$  from  $n+k$  to  $n$ . The point of this is that there are arity reducers in natural languages other than quantifiers, one example being the English reflexive *self*( $R$ ) =  $\{a : R(a, a)\}$ .

(i) John criticized himself.

can be obtained by composing  $I_j$  with *self*:  $I_j \text{self}(\text{criticized}) \Leftrightarrow \text{criticized}(j, j)$ . Here we restrict attention to iteration of quantifiers.



Note that we must use  $R^{-1}$  on the right-hand side, since, by definition, the first argument of the relation is linked to  $Q_1$  and the second to  $Q_2$ . This becomes clearer if one writes the iteration in logical notation. Definition (10.5) then becomes

$$Q_1 Q_2 x y R(x, y) \iff_{\text{def}} Q_1 x Q_2 y R(x, y)$$

so the properties above are

$$(10.9) \quad \begin{array}{l} \text{a. } Q_1 x Q_2 y R(x, y) \implies Q_2 y Q_1 x R(x, y) \text{ (} Q_1 \text{ dominates } Q_2 \text{)} \\ \text{b. } Q_1 x Q_2 y R(x, y) \iff Q_2 y Q_1 x R(x, y) \text{ (the pair } (Q_1, Q_2) \text{ is independent)} \end{array}$$

For example,  $\exists$  dominates  $\forall$ , but not vice versa. (Order) independence holds when  $Q_1 = Q_2 = \forall$ , or  $Q_1 = Q_2 = \exists$ , but fails when, for example,  $Q_1 = Q_2 = \exists_{=3}$  (or  $\exists_{\leq 3}$ , or  $\exists_{\geq 3}$ , or  $Q^k$ ). Another case of independence is when one of the quantifiers is a Montagovian individual; one easily shows that

$$(10.10) \quad \text{Every pair } (Q, I_a) \text{ is independent (for arbitrary } Q, \text{ and assuming } a \in M \text{)}.$$

Zimmermann (1993) calls quantifiers satisfying this condition *scopeless*, and proves that the Montagovian individuals are in fact the *only* examples.

The latter result is one of several attempting to *characterize* the scope dominant or independent pairs. For example, Westerståhl (1986) showed that

$$(10.11) \quad \text{On each finite universe } M, \text{ if } Q_1, Q_2 \text{ are non-trivial, PERM, upward monotone, and if } Q_1 \text{ dominates } Q_2, \text{ then either } Q_1 = \exists \text{ or } Q_2 = \forall. \text{ Thus, if in addition } (Q_1, Q_2) \text{ is independent, } Q_1 = Q_2 = \exists \text{ or } Q_1 = Q_2 = \forall.^2$$

This has been generalized in several ways. Altman, Peterzil and Winter (2005) give a characterization of the upward monotone cases of scope dominance that holds on countable universes and does not assume PERM (thus allowing quantifiers like  $I_a$ ). It follows from that characterization that (10.11) holds for countable universes too (under PERM); indeed, it seems likely that it holds for arbitrary universes. Ben-Avi and Winter (2005) instead consider the case when  $Q_1$  is upward monotone and  $Q_2$  downward monotone, as in the following instance of scope dominance:

$$(10.12) \quad Q^R x (\neg \exists) y R(x, y) \implies (\neg \exists) y Q^R x R(x, y)$$

Under these monotonicity assumptions, they give a complete characterization (with or without PERM) of scope dominance over finite universes. The influential paper van Benthem 1989 brought up a number of questions about iteration and proved various results, among them a strengthening of Zimmermann's theorem: Montagovian individuals are in fact the only (non-trivial) quantifiers that *dominate* arbitrary quantifiers.

If the monotonicity requirement is dropped, there are two rather curious further examples of independence assuming PERM, besides the ones mentioned in (10.11): namely,  $Q_1 = Q_2 = Q_{\text{odd}}$  and  $Q_1 = Q_2 = Q_{\text{even}}$ . This is because

$$Q_{\text{odd}} x Q_{\text{odd}} y R(x, y)$$

<sup>2</sup> The restriction to a given universe is necessary. For example, if  $Q_1 = Q_2 = \forall$  on odd-numbered universes and  $= \exists$  on other universes, then (10.9b) clearly holds.

in fact says (on finite universes) that the number of pairs in  $R$  is odd, so  $Q_{\text{odd}}y Q_{\text{odd}}x R(x, y)$  makes the same claim about  $R^{-1}$ , which of course is equivalent.<sup>3</sup> It might well be true that these are the only other non-trivial and PERM cases satisfying (10.1) on a given (finite) universe.

Westerståhl (1996) has a complete solution of the scope dominance problem (without extra assumptions) for the case when  $Q_1 = Q_2$ . Such quantifiers were called *self-commuting* by van Benthem, and it follows from this result that the only non-trivial and PERM self-commuting quantifiers are indeed  $\forall, \exists$ , and, if the universe is finite,  $Q_{\text{odd}}$  and  $Q_{\text{even}}$  (on infinite universes only  $\forall$  and  $\exists$  work).<sup>4</sup> As far as we know, the question of a general characterization of the independent (or, better, scope-dominant) pairs is still open.

Returning now to the interpretation of sentences (10.1)–(10.3), they can be dealt with directly with iteration as defined above, if one uses the type  $\langle 1 \rangle$  quantifiers resulting from *freezing* the relevant restriction arguments. Thus, one sees that the truth conditions (10.1)'–(10.3)' can be expressed as follows:

$$(10.1)'' \quad Q_1(A, R) \iff I_j \cdot (\text{all but four})^A(R)$$

$$(10.2)'' \quad Q_2(A, B, R) \iff \text{two}^A \cdot \text{most}^B(R)$$

$$(10.3)'' \quad Q_3(A, B, C, R) \iff (\text{at most three})^A \cdot (\text{more than})^{B,C} \cdot I_m(R)$$

This gives an elegant compositional analysis of sentences of these forms.<sup>5</sup> Without going into details of meaning assignment and syntactic structure, the point is that for such sentences, each noun phrase can be dealt with separately, bringing its own

<sup>3</sup>  $Q_{\text{even}}x Q_{\text{even}}y R(x, y)$ , on the other hand, says that  $|R|$  is even, if  $|M|$  is also even, but that  $|R|$  is odd, if  $|M|$  is odd. The claim about  $Q_{\text{odd}}$  is verified as follows. Note first that  $R = \bigcup_{a \in M} \{(a, y) : y \in R_a\}$ , and hence, if  $M = \{a_1, \dots, a_m\}$ ,

$$(i) \quad |R| = |R_{a_1}| + \dots + |R_{a_m}|$$

Then one calculates:

$$\begin{aligned} |R| \text{ is odd} &\Leftrightarrow |R_{a_1}| + \dots + |R_{a_m}| \text{ is odd} \\ &\Leftrightarrow |\{a \in M : |R_a| \text{ is odd}\}| \text{ is odd} \\ &\Leftrightarrow Q_{\text{odd}}Q_{\text{odd}}(R) \end{aligned}$$

The case of  $Q_{\text{even}}$  is similar.

<sup>4</sup> The theorem is that on a given universe  $M$ ,  $Q$  is self-commuting iff it is either a union or an intersection of Montaguean individuals (of the form  $I_a$  with  $a \in M$ ), or a finite symmetric difference of such individuals, or the negation of such a symmetric difference. (The symmetric difference of two sets  $X$  and  $Y$ ,  $X \ominus Y$ , is  $(X - Y) \cup (Y - X)$ . Since  $X \ominus (Y \ominus Z) = (X \ominus Y) \ominus Z$ , the notion of a finite symmetric difference  $X_1 \ominus \dots \ominus X_k$  makes sense.) Note that this covers  $Q = \exists_M = \bigcup_{a \in M} I_a$  and  $Q = \forall_M = \bigcap_{a \in M} I_a$ .

<sup>5</sup> Along these lines one may extend the iteration operation to quantifiers of type  $\langle 1, \dots, k \rangle$ . Since the last argument is the scope, and the others are restrictions, conservativity and intersectivity may be defined for such quantifiers (see Ch. 4.7), and one can see that these properties are preserved under iteration: if the arguments are CONSERV (INT), so is the iteration. Also, many of the technical results about scope mentioned earlier have versions specially adapted to the case where  $Q_1, Q_2$  result from freezing the restriction arguments of type  $\langle 1, 1 \rangle$  quantifiers; e.g. such quantifiers are not PERM, but one can often get by assuming that the corresponding type  $\langle 1, 1 \rangle$  quantifiers are PERM or ISOM instead. See Westerståhl 1994.

contribution to the meaning of the sentence, irrespective of what the other noun phrases mean.

Moreover, their scope ambiguities correspond simply to the order of iteration. (10.1) is not ambiguous, and, as we noted above, this is reflected in the fact that  $I_j \cdot (\text{all but four})^A(R)$  is equivalent to  $(\text{all but four})^A \cdot I_j(R^{-1})$ ; i.e. that  $I_j$  is scopeless. The two possible orders in (10.2) correspond to the readings where each of the two critics reviewed most films, and where each of a large number of films was reviewed by two critics, respectively. So scope order is the same as iteration order, and results about scope dominance and independence like those mentioned above are directly relevant to linguistic issues concerning which readings are possible for this kind of sentence.

There is also a third reading of (10.2): the *cumulative* reading, which says that there were two film-reviewing critics and that most films were reviewed by one of those two.

In general, we can define the *cumulation*  $(Q_1, Q_2)^{\text{cl}}$  of two type  $\langle 1 \rangle$  quantifiers  $Q_1$  and  $Q_2$  as follows (using logical notation):

$$(10.13) \quad (Q_1, Q_2)^{\text{cl}} xyR(x, y) \leftrightarrow Q_1 x \exists y R(x, y) \wedge Q_2 y \exists x R(x, y)$$

So  $(Q_1, Q_2)^{\text{cl}}$  is a conjunction of iterations, but is not itself an iteration; in fact, Westerståhl (1994) proves that (disregarding trivial cases) it is an iteration if and only if  $Q_1 = Q_2 = \exists$ . In particular, the cumulative reading of (10.2) is distinct from the two iterative readings.

For (10.3), finally, we see, using the scopelessness of  $I_j$ , that there are two iterative readings, the second one (the one where *(more\_than)*<sup>B,C</sup> comes before *(at most three)*<sup>A</sup>) saying that more dahlias than roses have the property of being given to Mary by at most three boys. This is a very implausible reading of that sentence, of course, since normally a flower can be given only once. But the implausibility comes from facts about the world, not from the structure of the sentence. There may also be a cumulative reading here; we leave it to the reader to formulate such a reading.

The smooth compositional analysis of sentences like (10.1)–(10.3) shows that polyadic quantification can and should be avoided here. As van Benthem (1989) puts it, they are safely inside the “Frege boundary” where monadic quantifiers are all you need. Other cases, like the cumulative reading of (10.2), are perhaps located on that boundary: they use Boolean combinations of iterations. Such sentences will be called *unary complexes*; we restrict attention to type  $\langle 2 \rangle$  quantifiers.<sup>6</sup>

### unary complexes

A quantifier  $Q$  of type  $\langle 2 \rangle$  is a *unary complex* iff there is a Boolean combination  $\Phi$  of iterations of the form  $Q_1 Q_2(R)$  and *inverse iterations* of the form  $Q_3 Q_4(R^{-1})$ , such that for all  $R$ ,  $Q(R) \leftrightarrow \Phi$  (suppressing the universe as before).  $Q$  is a *right complex* (*left complex*) if only iterations (inverse iterations) are used.

<sup>6</sup> That term too is from van Benthem 1989, though he uses it for what we call *right complexes* below.

One reason to include inverse iterations is that sentences denoting iterations can be ambiguous, so one may want access to both readings. Also, we said that cumulations are unary complexes, but note that definition (10.13) uses both ‘straight’ and inverse iterations.

As another example, it is proved in Westerståhl (1994) that  $Res^2(\exists_{\geq 2})$ , the quantifier that says of a binary relation  $R$  that it contains at least two pairs, is not an iteration, but is a right complex. In the next section we discuss such resumptive quantifiers further. Together with some other quantifiers to be discussed later in this chapter, they exemplify natural language quantifiers that are way beyond the Frege boundary.

## 10.2 RESUMPTION

In Chapters 0.1 and 4.1 we noted that, particularly in certain cases of adverbial quantification, ordinary natural language quantifiers are sometimes applied to *tuples* of individuals rather than to individuals. This was first observed by Lewis (1975); witness examples like

- (10.14) a. Men are usually taller than women.  
 b. Men seldom make passes at girls who wear glasses. (Dorothy Parker)  
 c. People usually are grateful to firemen who rescue them.

In effect, this means that we are *lifting* a monadic quantifier  $Q$  of type  $\langle 1, \dots, 1 \rangle$  to a quantifier  $Res^k(Q)$  of type  $\langle k, \dots, k \rangle$ . Put differently, we employ a monadic quantifier to bind multiple variables simultaneously, in order to quantify over tuples. This operation is called *resumption*, and the resulting polyadic quantifiers will be called *resumptive quantifiers*. Another term is *vectorized* quantifiers; this is often used when generalized quantifiers figure in the study of computational complexity in computer science, where resumption is an expressive resource that often comes in handy (see Ebbinghaus and Flum 1995).

$Res^k(Q)$  is just the old  $Q$ , but applied to universes  $M^k$  containing  $k$ -tuples (vectors) of individuals. Like every quantifier,  $Q$  is defined over all universes, including those of the form  $M^k$ . But over the universe  $M$  it is a polyadic quantifier. Let us state the formal definition.<sup>7</sup>

### resumption

Let  $Q$  be any monadic quantifier, of type  $\langle 1, \dots, 1 \rangle$  with  $n$  arguments, and let  $k \geq 1$ . For any universe  $M$ , and any  $R_1, \dots, R_n \subseteq M^k$ ,

$$Res^k(Q)_M(R_1, \dots, R_n) \iff Q_{M^k}(R_1, \dots, R_n)$$

Note that  $Res^1(Q) = Q$ .

<sup>7</sup> Actually, resumption can be applied to quantifiers of any type, roughly by replacing each individual variable by a  $k$ -tuple of new variables. We shall only discuss resumption of monadic quantifiers here.

In some natural language cases,  $Q$  is of type  $\langle 1, 1 \rangle$  and the first argument of  $Res^k(Q)$  is a Cartesian product  $A_1 \times \dots \times A_k$ . In (10.1b), for example, the first argument is the product of the set of men and the set of girls wearing glasses, whereas the second argument is the relation *make passes at*. But not always; (10.1c) expresses a relation between the set of pairs of a person and a fireman such that the latter rescues the former, and the set of pairs where the first element is grateful to the second.

The simplest resumptions of type  $\langle k \rangle$  are  $Res^k(\forall)$  and  $Res^k(\exists)$ . These are also iterations:  $Res^k(\forall)x_1 \dots x_k R(x_1, \dots, x_k)$  is equivalent to  $\forall x_1 \dots \forall x_k R(x_1, \dots, x_k)$ , and simply says that  $R$  is the universal relation (over  $M$ ): every  $k$ -tuple belongs to it. Similarly,  $Res^k(\exists) = \exists \dots \exists$  ( $k$  factors) says of  $R$  that it is non-empty. But usually resumptions are not iterations; in fact, Westerståhl (1994) proves that, for ISOM  $Q$  of type  $\langle 1 \rangle$ , and over finite universes,  $Res^k(Q)$  is an iteration of  $k$  type  $\langle 1 \rangle$  quantifiers if and only if, on each  $M$  where it is non-trivial, it is either  $Res^k(\forall)$  or  $Res^k(\exists)$  or  $Res^k(Q_{\text{odd}})$ , or a negation of one of these.<sup>8</sup> We come back to issues of the expressive power of resumption in Chapter 15.5.

In this section we discuss two aspects of resumption, one linguistic and the other logical. First, we look a little more closely at quantificational adverbs, and at a suggestion that resumption figures in the analysis of so-called ‘donkey’ sentences. Then we discuss some logical properties of resumptions, and in particular show that resumptions can be neatly characterized in terms of certain of those properties.

Before we begin, however, let us go back to typical cases of adverbial quantification like those in (10.1), and see if they really have resumptive readings. In fact, some people find this doubtful, partly for one of the following two reasons: It is argued that (a) these sentences actually quantify over *events* or *situations* and should be analyzed along those lines, or (b) that the resumptive truth conditions simply aren’t the right ones.

As to (a), the issue of the extent to which events or similar entities should be used in natural language semantics has been a complex and controversial one, ever since Davidson (1967) first suggested that many sentences existentially quantify over events, among other things in order to have HARRY runs follow logically from HARRY runs fast.<sup>9</sup> This is not the place to get into that debate, but we can make the observation that if the sentences in (10.14) are taken to quantify over events or situations, these are bound to contain the very elements (men, women, girls with glasses, etc.) of the tuples that the resumptive quantifier would quantify over. So applying e.g. *most* to events is in fact rather similar to applying it to ordered tuples, and in particular would not give the truth conditions of standard iterative quantification. We can therefore concentrate here on the objection (b), which seems to say that some such standard truth conditions do capture the meaning of these sentences.

<sup>8</sup> ISOM is a necessary requirement here; note that e.g.  $I_a \cdot I_b = Res^2(I_{(a,b)})$ .

<sup>9</sup> This follows if the conclusion is taken as  $\exists e(e$  is an event of running and the agent of  $e$  is Harry and  $e$  is fast), and similarly for the premise. If adverbs like *fast* are treated simply as verb modifiers, on the other hand, it is difficult to get this entailment.

Take (10.14a). In this case an event analysis doesn't seem to be called for. Is the resumptive reading an option? Consider some possible iterative alternatives. It seems clear that (10.14a) does *not* imply

(10.15) Most men are taller than all women.

so we can rule out that sentence. Likewise, it seems pretty obvious that (10.14a) implies, but is *not* implied by

(10.16) Most men are taller than some women.

Perhaps the most likely alternative is something like

(10.17) Most men are taller than most women.

Let us for the moment interpret *most* as *more than 75 percent of the*, to make the example more realistic. Now consider a small country where there are 100,000 men and 100,000 women, and 70,000 men are taller than all of the women, but each of the remaining 30,000 men is only taller than 70,000 women. Also, no woman has the same height as any man. It seems to us that (10.14a) is then clearly true. But (10.17) is definitely false in this situation. It is not true that more than 75 percent of the men are taller than more than 75 percent of the women. (Nor is the narrow scope reading of (10.17) true.) However, of the 10 billion man–woman pairs, 9.1 billion are in the *taller than* relation. This seems to us like fair evidence that readings where the number of pairs is what counts, in particular resumptive readings, do occur.<sup>10</sup>

### 10.2.1 Quantificational adverbs and 'donkey' anaphora

Can determiners ever express resumptive quantification like some adverbs are able to? Kamp (1981) and Heim (1982) proposed that determiners can indeed do this. They were responding to a problem pointed out by Geach (1962) about the standard interpretation of anaphoric pronouns having a quantified noun phrase as antecedent. Anaphoric pronouns inside the scope or restriction of a quantified antecedent can be treated as relation reducers in a now familiar way. As Quine (1960) suggested, they function, like bound variables, to insure that several argument positions of a relation are filled identically, thereby reducing the arity of the relation.

In some sentences, however, a pronoun that is anaphoric to a quantified antecedent does not function in this way. Geach argued that

(10.18) Every farmer who owns a donkey beats it.

<sup>10</sup> We are not denying that sentences like these can be vague, and that the resumptive reading may give idealized truth conditions. But vagueness besets iterative quantification too. Our claim is still that the resumptive truth conditions often come much closer to intuitions than any iterative ones. Likewise, we don't deny that adverbial quantification can sometimes have generic or 'average' readings (the typical man is taller than the typical woman). Nor do we deny that quantification over events sometimes seems quite plausible: e.g. *Dogs usually bark at cats* (but as we noted, this does not give iterative truth conditions). We merely claim, with Lewis (1975), that resumptive readings exist.

makes a universal assertion, so the scope of the existential quantification expressed by a donkey is not wider than that of the universal quantification every farmer who owns a donkey expresses,<sup>11</sup> and specifically the assertion is that every farmer who owns a donkey beats every donkey he owns. The standard treatment of anaphoric pronouns with quantified antecedents does not provide this interpretation. Analysis (10.19b) gives wide scope to the existential, not the universal, quantifier, and analysis (10.19a) leaves the variable  $y$  corresponding to the pronoun *it* unbound, thus failing to require that what each farmer beats is even a donkey, let alone one he or she owns.

- (10.19) a. *every*  $x$  (*farmer*( $x$ )  $\wedge$  *some*  $y$  (*donkey*( $y$ ), *owns*( $x$ ,  $y$ )), *beats*( $x$ ,  $y$ ))  
 b. *some*  $y$  (*donkey*( $y$ ), *every*  $x$  (*farmer*( $x$ )  $\wedge$  *owns*( $x$ ,  $y$ ), *beats*( $x$ ,  $y$ ))).

### 10.2.1.1 'Donkey' pronouns and resumptive quantification

Kamp (1981) and Heim (1982) proposed that in (10.18) and other sentences of the form

- (10.20)  $QA$  that  $R$  a  $B$   $R'$   $it$

the quantifier  $Q$  is interpreted resumptively. Specifically, they assign (10.18) the interpretation

- (10.21)  $Res^2(every)xy(farmer(x) \wedge donkey(y) \wedge owns(x, y), beats(x, y))$

and each sentence of the form (10.20) the interpretation:

- (10.22)  $Res^2(Q)xy(A(x) \wedge R(x, y) \wedge B(y), R'(x, y))$

Their individual approaches to accomplishing this are not the same, but share the common feature of not interpreting the indefinite article *a* as a quantifier, in effect leaving the variable  $y$  over donkeys ( $B$ s) free to be bound by the (resumptive) quantifier *every* ( $Q$ ) along with the variable  $x$  over farmers ( $A$ s) that own ( $R$ )  $y$ . The decision not to interpret *a* as a quantifier is supported by the article's apparent lack of quantificational force in sentences with adverbial quantification such as (10.23a, b).

- (10.23) a. A quadratic equation often has two distinct solutions.  
 b. Before 1914 a car seldom had an electric starter.

This approach interprets Geach's sentence (10.18) as he said it should be interpreted, and likewise interprets similar sentences (10.24a, b)

- (10.24) a. Some New Yorker who bought a lottery ticket lost it.  
 b. No actor who insults a critic is favorably reviewed by him.

as most people feel is natural; i.e. as

<sup>11</sup> Analysis (10.19b) says that some donkey has the property that every farmer who owns it beats it, which would, for instance, absurdly be made true vacuously by the existence of a wild donkey, which is not owned by any farmer. Moreover, if two donkeys are owned, say by different farmers, and one of the donkeys is beaten by the farmer who owns it, that fact does not suffice to make (10.18) true, but it does satisfy (10.19b).

- (10.25) a.  $Res^2(at\ least\ one)xy(New\ Yorker(x) \wedge bought(x, y) \wedge lottery\ ticket(y), lost(y, x))$   
 b.  $Res^2(no)xy(actor(x) \wedge insults(x, y) \wedge critic(y), reviews\ favorably(y, x))$

which have the truth conditions (10.26a, b).

- (10.26) a. At least one New Yorker who bought at least one lottery ticket lost at least one lottery ticket that he or she bought.  
 b. No actor who insults one or more critics is favorably reviewed by any critic that he insults.

Does this analysis assign the right truth conditions to all ‘donkey’ sentences, i.e. sentences of the form (10.20)? The answer is generally negative, and for two distinct reasons. For one thing, it gets sentences like (10.27)–(10.29) wrong.

(10.27) At least two farmers who own a donkey are happy.

(10.28) At least two farmers who own a donkey beat it.

(10.29) Most farmers who own a donkey beat it.

As Partee pointed out, statement (10.27) is false if Jones and Smith are the only existing farmers, farmer Jones is unhappy, and farmer Smith is happy. However, the resumptive quantification analysis that Kamp’s and Heim’s rules give (10.27) would make it true if farmer Smith owns at least two donkeys, despite the obvious irrelevance of this fact to the truth value of statement (10.27).<sup>12</sup> Likewise, Kanazawa (1994) pointed out that (10.28) is false if farmer Jones doesn’t beat the one donkey he owns, while farmer Smith beats both of his donkeys (again assuming no other farmers exist). Rooth (1987) drew attention to the corresponding failure of Kamp’s and Heim’s resumptive quantification analysis for (10.29). These examples illustrate what has become known as the *proportion problem*. Far from being exceptional, this problem arises for almost all quantifiers in the ‘farmer’ position of ‘donkey’ sentences. Van der Does (1996), who deals with several of the issues discussed here, gives a characterization of the quantifiers for which the proportion problem for the corresponding ‘donkey’ sentences does *not* arise: essentially these are just the quantifiers *every*, *no*, *some*, and *not every*.

<sup>12</sup> Note that the interpretive rules under discussion may be needed even for sentences that do not actually contain a pronoun anaphoric to and outside the scope of a quantified noun phrase. For instance,

- (i) Some politician who ran for a higher office won.

means that some politician who ran for a higher office won an office for which he ran. Still, as Kanazawa (1994) pointed out in connection with (10.28),

- (ii) At least two politicians who ran for a higher office won.

does not mean that at least two pairs of a politician and a higher office for which he ran are such that he won it. For this misinterpretation could count a politician as many times as he successfully ran for higher office, while sentence (ii)’s truth value is not affected by how many times after the first a given politician ran successfully.



Second, it is widely acknowledged that even for these quantifiers, sentences of the form (10.20) can have an interpretation different from (10.22). For instance, sentences (10.30a–c)

- (10.30) a. Every guest who had a credit card used it to pay his hotel bill.  
 b. At least one boy who had an apple for breakfast didn't give it to his best friend.  
 c. No man who had a credit card failed to use it.

prefer the interpretations (10.31a–c) respectively.

- (10.31) a. Every guest who had a credit card used at least one credit card that he had to pay his hotel bill.  
 b. At least one boy who had an apple for breakfast didn't give every apple that he had to his best friend.  
 c. No man who had a credit card failed to use every credit card that he had.

None of these interpretations results from resumptive quantification by the quantifier *every*, *some* (or *at least one*), or *no*.

In fact, sentences of the form (10.20) are often, though not always, ambiguous between what may be termed *strong* and *weak* readings.

$$(10.32) D_{\text{strong}}(Q)(A, B, R, R') \Leftrightarrow Q(A \cap \{a : B \cap R_a \neq \emptyset\}, \{a : B \cap R_a \subseteq R'_a\})$$

$$(10.33) D_{\text{weak}}(Q)(A, B, R, R') \Leftrightarrow Q(A \cap \{a : B \cap R_a \neq \emptyset\}, \{a : B \cap R_a \cap R'_a \neq \emptyset\})$$

We can see  $D_{\text{strong}}$  as a polyadic lift, taking the type  $\langle 1, 1 \rangle$  quantifier  $Q$  to the type  $\langle 1, 1, 2, 2 \rangle$  quantifier  $D_{\text{strong}}(Q)$ , and similarly for  $D_{\text{weak}}$ .

Kanazawa (1994) and Chierchia (1992) argued convincingly that for every quantifier  $Q$ , sentences of the form (10.20) can mean one or both of the propositions defined in (10.32) and (10.33). Indeed, except when  $Q$  is *most*, in which case (10.20) might possibly also mean

$$(10.34) Q(A \cap \{a : B \cap R_a \neq \emptyset\}, \{a : Q(B \cap R_a, R'_a)\})$$

These are the only possible meanings of sentences having the form (10.20).

Now in the sentences (10.18), (10.24a), (10.24b), and others which served as evidence supporting the resumptive interpretation (10.22) of sentences of the form (10.20),  $Q$  was either *every*, *some*, or *no*. Comparing this to the strong reading above, one easily verifies that

$$(10.35) \text{ When } Q \text{ is } \textit{every} \text{ or } \textit{not every}, \\ \text{Res}^2(Q)((A \times B) \cap R, R') \iff D_{\text{strong}}(Q)(A, B, R, R')$$

$$(10.36) \text{ When } Q \text{ is } \textit{some} \text{ or } \textit{no}, \\ \text{Res}^2(Q)((A \times B) \cap R, R') \iff D_{\text{weak}}(Q)(A, B, R, R')$$

However, both of these fail for most other values of  $Q$ , and each fails for the above values too when “strong” and “weak” are interchanged, or when the reading (10.34) is used. Thus, it becomes clear that the facts which Chierchia, Kanazawa, and others noticed about the possible interpretations of sentences of the form (10.20)

demonstrate that only ‘accidentally’ do ‘donkey’ sentences ever express resumptive quantification: i.e. when their quantifier is one of the few that were considered in the research that led to proposing (10.22) as the general interpretation. So we see that these sentences do not really demonstrate that determiners can express resumptive quantification.

### 10.2.1.2 ‘Donkey’ pronouns as quantifiers

Evans (1977) had a different response to the problem Geach pointed out. He argued that a pronoun outside the scope of a quantified noun phrase to which it is anaphoric expresses a type ⟨1⟩ quantifier. In fact, the strong and weak interpretations are precisely of this sort. In the strong reading, the quantifier corresponding to *it* results from *every* by freezing the restriction argument to the part of the antecedent quantifier’s scope that lies in its restriction:  $B \cap R_a$ . The weak reading simply employs the quantifier *some* in place of *every*. Evans proposed using *the<sub>sg</sub>*, and Neale (1990) suggests *all<sub>ci</sub>* instead. We do not evaluate these alternatives here. However, we should mention that this approach is general in a way that the one using resumptive quantification cannot be, because ‘donkey’ sentences are instances of the more general form (10.37a).

(10.37) a.  $Q_1 A \text{ that } R Q_2 B R' \text{ it}$

$$\text{b. } D(Q_1, Q_2, Q)(A, B, R, R') \iff Q_1(A \cap \{a : Q_2(B, R_a)\}, \{a : Q(B \cap R_a, R'_a)\})$$

(Note that  $D_{\text{strong}}(Q_1) = D(Q_1, \text{some}, \text{every})$  and  $D_{\text{weak}}(Q_1) = D(Q_1, \text{some}, \text{some})$ .) Whereas  $Q_2$  is *some* in ‘donkey’ sentences, in

(10.38) Everyone who owns exactly one Cadillac keeps it very clean.

$Q_2$  is *exactly one*. Although the resumptive interpretation (10.22) is completely wrong for (10.38), this sentence like all others of the form (10.37a), including ‘donkey’ sentences, can be interpreted using the lift in (10.34) according to the strategy Evans proposed, where  $Q$  can be chosen to be *every*, *some*,  $Q_1$ , *the<sub>sg</sub>*, or *all<sub>ci</sub>*, as turns out to be appropriate.<sup>13</sup> (Note that the scope of the quantifier  $Q$  introduced for the ‘donkey’ pronoun is uniquely determined, and thus the quantifier does not participate in scope ambiguities.)

We conclude that determiners cannot in fact express resumptive quantification in English.<sup>14</sup> In particular, sentences of the ‘donkey’ form (10.20) are not interpreted with resumptive quantification. Nor are they support for the notion that the indefinite article *a* does not express a quantifier.

Of course, English still has resumptive quantification; it is just expressed by adverbs.

<sup>13</sup> The lift defined in (10.34) determines exactly what scope is taken by the quantifier  $Q$  that interprets the pronoun *it* outside the scope of its quantified antecedent  $Q_2(B, R_a)$ . The strategy of interpreting ‘donkey’ pronouns as quantifiers need not raise the specter of scope ambiguity for these quantifiers.

<sup>14</sup> Nor can they in any other language, so far as we know.

### 10.2.2 Resumption, orientation, and tuple-isomorphisms

The resumption  $Res^k(Q)$  is not only ISOM if  $Q$  is, it is invariant under isomorphism of  $k$ -tuples. In fact, this characterizes resumption, as we shall see. Closely related are the so-called orientation properties of certain polyadic quantifiers introduced by van Benthem (1989). We start with the case  $k = 2$  and  $Q$  of type  $\langle 1 \rangle$ , and then discuss how to generalize the results.

**pair isomorphisms**

$Q$  of type  $\langle 2 \rangle$  is *closed under pair isomorphisms* (satisfies 2-ISOM) if, for any bijection  $h$  from  $M^2$  to  $(M')^2$  and any  $R \subseteq M^2$ ,

$$Q_M(R) \iff Q_{M'}(h(R))$$

where  $h(R) = \{b(a, b) : R(a, b)\}$ . Similarly for 2-PERM (let  $M' = M$ ). It is immediate how to generalize this to  $k$ -ISOM for any  $k$ , when  $Q$  has type  $\langle k, \dots, k \rangle$ .

Thus ISOM = 1-ISOM, and 2-ISOM implies ISOM, since any bijection  $g$  from  $M$  to  $M'$  induces a bijection  $h$  from  $M^2$  to  $(M')^2$ :  $h(a, b) = (g(a), g(b))$ . Similarly,  $(k + 1)$ -ISOM implies  $k$ -ISOM.

**Proposition 2**

*A type  $\langle 2 \rangle$  quantifier is of the form  $Res^2(Q)$  for some ISOM  $Q$ , if and only if, it satisfies 2-ISOM. More generally, a type  $\langle k, \dots, k \rangle$  quantifier is of the form  $Res^k(Q)$  for some ISOM  $Q$  of type  $\langle 1, \dots, 1 \rangle$  iff it satisfies  $k$ -ISOM. Furthermore, the corresponding claims with "ISOM" replaced everywhere by "PERM" are also valid.*

*Proof.* It is clear from the definitions that  $Res^2(Q)$  satisfies 2-ISOM when  $Q$  satisfies ISOM, since the behavior of  $Res^2(Q)$  depends only on the behavior of  $Q$  on universes of the form  $M^2$ , and for such universes, ISOM talks precisely about bijections mapping pairs to pairs. By the same argument,  $Res^2(Q)$  satisfies 2-PERM if  $Q$  satisfies PERM.

Now suppose  $Q'$  of type  $\langle 2 \rangle$  satisfies 2-ISOM. This means that  $Q'$  is given by a binary relation between cardinal numbers (also denoted  $Q'$ ):

$$Q'(k, m) \iff \exists M \exists R \subseteq M^2 (|M^2 - R| = k \ \& \ |R| = m \ \& \ Q'_M(R))$$

(here  $k, m$  may be infinite cardinals). By 2-ISOM it follows that if  $R \subseteq M^2, S \subseteq (M')^2, |M^2 - R| = |(M')^2 - S|, |R| = |S|$ , and  $Q'_M(R)$ , then  $Q'_{M'}(S)$ . Hence,

$$Q'_M(R) \iff Q'(|M^2 - R|, |R|)$$

Now let  $Q$  be the (ISOM) type  $\langle 1 \rangle$  quantifier defined by the very same relation  $Q'$ : for all  $M$  and all  $A \subseteq M$ ,

$$Q_M(A) \iff Q'(|M - A|, |A|)$$

Then we have  $Res^2(Q)_M(R) \iff Q_{M^2}(R) \iff Q'(|M^2 - R|, |R|) \iff Q'_M(R)$ . In other words,  $Res^2(Q) = Q'$ .

If  $Q'$  instead satisfies 2-PERM, then, on every universe  $M$ ,  $Q'$  is a binary relation between cardinal numbers, which we can denote  $Q'_M$ :

$$Q'_M(k, m) \iff \exists R \subseteq M^2 (|M^2 - R| = k \ \& \ |R| = m \ \& \ Q'_M(R))$$

The argument now proceeds as above, letting  $Q$  be given, on each  $M$ , by the relation  $Q'_M$ .

The cases of  $k$ -ISOM and  $k$ -PERM are similar.  $\square$

This proposition gives us a way of determining if a given quantifier is a resumption or not. An even more convenient way to do the same thing, on finite universes, can be expressed in terms of the notion of *orientation* (introduced by van Benthem).<sup>15</sup>

### orientation

$Q$  of type  $\langle 2 \rangle$  is *right-oriented* if, for any  $M$  and any  $R, S \subseteq M^2$ ,

$$\text{if } |R_a| = |S_a| \text{ for all } a \in M, \text{ and } Q_M(R), \text{ then } Q_M(S)$$

Similarly, it is *left-oriented* if  $|{}_aR| = |{}_aS|$  for all  $a \in M$ , and  $Q_M(R)$  implies  $Q_M(S)$ , where  ${}_aR = \{b : R(b, a)\}$ .

For example,  $Q = \forall\exists$  is right-oriented, since  $Q_M(R)$  says that  $R_a \neq \emptyset$  for all  $a \in M$ , so if  $|R_a| = |S_a|$ , then  $S_a \neq \emptyset$  too. But it is not left-oriented, since we can easily have  $|{}_aR| = |{}_aS|$  for all  $a \in M$  and  $Q_M(R)$ , even though  $S_a = \emptyset$  for some  $a$ . This observation can be generalized. Recall the notions of a unary (right, left) complex introduced at the end of section 10.1.

### Proposition 3

Consider type  $\langle 1 \rangle$  quantifiers satisfying PERM. Over finite universes, iterations of these are right-oriented, and inverse iterations are left-oriented. Moreover, right (left) orientation is preserved under Boolean operations on quantifiers, including inner negation and duals. Hence, right (left) complexes are right- (left-) oriented.

*Proof.* An iteration has the form

$$(Q_1 Q_2)_M(R) \iff (Q_1)_M(\{a : (Q_2)_M(R_a)\})$$

Now, if  $|R_a| = |S_a|$  for  $a \in M$ , and  $M$  is finite, it follows that  $|M - R_a| = |M - S_a|$ . Therefore, by PERM,  $(Q_2)_M(R_a)$  iff  $(Q_2)_M(S_a)$ , and hence  $(Q_1 Q_2)_M(R)$  iff  $(Q_1 Q_2)_M(S)$ . The case of inverse iterations is similar.

It is obvious that right (left) orientation is preserved under conjunction and (outer) negation. (This holds generally, without assuming finite universes or PERM.) For inner negation, we use the fact, mentioned in note 3, that

<sup>15</sup> A model of the form  $(M, R)$  with  $R \subseteq M^2$  can be seen as a (directed) *graph*, and a type  $\langle 2 \rangle$  quantifier  $Q$  as a property of graphs. Right (left) orientation then means that if  $(M, R)$  and  $(M, S)$  are such that each node has the same out-degree (in-degree), and  $(M, R)$  has  $Q$ , then so does  $(M, S)$ .

$$(10.39) \quad |R| = \sum_{a \in M} |R_a|$$

Therefore, if  $|R_a| = |S_a|$  for  $a \in M$ , then  $|R| = |S|$ . Moreover, if  $M$  is finite, we also have  $|M^2 - R| = |M^2 - S|$ . Now

$$(Q\neg)_M(R) \iff Q_M(M^2 - R)$$

and hence, if  $Q$  is PERM, right orientation follows. The argument for left orientation is similar.  $\square$

For examples of non-orientation, consider the branching quantifiers, to be studied in section 10.3 below. It is readily seen that these are usually neither right- nor left-oriented. At the other extreme, however, we have resumptions.

**Lemma 4**

*Over finite universes, resumptions of PERM quantifiers are both right-oriented and left-oriented.*

*Proof.* This follows from (10.39). As above, if  $|R_a| = |S_a|$  for  $a \in M$ , then  $|R| = |S|$  and  $|M^2 - R| = |M^2 - S|$ , since  $M$  is finite. So if  $Res^2(Q)_M(R)$ , then  $Q_{M^2}(R)$ , hence  $Q_{M^2}(S)$ , since  $Q$  satisfies PERM, and therefore  $Res^2(Q)_M(S)$ . The case of left orientation is similar.  $\square$

Note, however, that the assumption of finiteness is essential here.

**Example.** Let  $\mathbb{N} = \{0, 1, 2, \dots\}$ ,  $R = \mathbb{N}^2$ , and  $S = \mathbb{N}^2 - \{(i, 0) : i \in \mathbb{N}\}$ . For each  $i \in \mathbb{N}$ ,  $R_i = \mathbb{N}$  and  $S_i = \mathbb{N} - \{0\}$ . Thus,  $|R_i| = |S_i| = \aleph_0$ . Now let  $Q$  be any binary relation between cardinal numbers such that  $Q(0, \aleph_0)$  holds, but  $Q(\aleph_0, \aleph_0)$  fails.  $Q$  can be seen as a type (1) quantifier satisfying ISOM (hence PERM), and we consider  $Res^2(Q)$ . Since  $|\mathbb{N}^2 - R| = 0$  and  $|\mathbb{N}^2 - S| = \aleph_0$ ,  $Res^2(Q)_{\mathbb{N}}(R)$  is true, but  $Res^2(Q)_{\mathbb{N}}(S)$  is false.

Next we come to a more substantial observation, due to van Benthem (1989).

**Lemma 5 (van Benthem)**

*Over finite universes, if  $Q$  is both right-oriented and left-oriented, then  $Q$  satisfies 2-PERM.*

*Proof.* The idea is that if  $R, S \subseteq M^2$ ,  $|R| = |S|$  (and hence  $|M^2 - R| = |M^2 - S|$ ), and  $Q_M(R)$ , then  $R$  can be transformed into  $S$  by a finite number of elementary operations, each of which preserves membership in  $Q_M$ . In detail, this is done as follows.<sup>16</sup>

$R$  is a set of ordered pairs, or *arrows*. Consider the following two ways of ‘moving an arrow’  $(a, b) \in R$ :

- Replace  $(a, b)$  by  $(a, c)$  for some  $c \in M$  (head move), provided  $(a, c)$  is not in  $R$ . ( $c = a$  is allowed.)

<sup>16</sup> Since van Benthem (1989) only illustrates the proof with an example, we give a full proof here. Impatient readers may skip it without loss.

- Replace  $(a, b)$  by  $(c, b)$  for some  $c \in M$  (tail move), provided  $(c, b)$  is not in  $R$ . ( $c = b$  is allowed.)

Note that a head move does not alter the out-degree (number of successors) of any point, and a tail move leaves the in-degree of each point intact. Therefore, since  $Q$  is both right- and left-oriented, if  $R'$  results from  $R$  by one of these moves,  $Q_M(R')$  holds. Also,  $|R'| = |R|$ . It is useful to allow one further kind of move:

- Replace  $(a, b)$  by  $(b, a)$  ('switch'), provided  $(b, a)$  is not in  $R$ .

If  $R'$  results from  $R$  by a switch, again  $Q_M(R')$ . This is because a switch can be decomposed into a tail move followed by a head move, or vice versa, as follows. Suppose  $R(a, b)$  but not  $R(b, a)$ . If not  $R(b, b)$ , first take  $(a, b)$  to  $(b, b)$  by a tail move (resulting in  $R''$ , say), and then take  $(b, b)$  to  $(b, a)$  by a head move (applied to  $R''$ ). The result is  $R'$ . If, on the other hand,  $R(b, b)$  holds, take  $(b, b)$  to  $(b, a)$  by a head move, and then  $(a, b)$  to  $(b, b)$  by a tail move. Again the result is  $R'$ .

Now assume that  $R, S \subseteq M$ ,  $|R| = |S|$ , and  $Q_M(R)$ . If  $R \neq S$ , there must exist some  $(a, b) \in R - S$ , as well as some  $(c, d) \in S - R$  (since  $R$  and  $S$  have the same finite cardinality). We claim that  $R$  can be transformed into  $R' = (R - \{(a, b)\}) \cup \{(c, d)\}$  by a series of head moves, tail moves, and switches. Given that, and if  $R'$  is not equal to  $S$ , we keep repeating this step until we reach  $S$ .

To prove the claim about  $R'$ , distinguish the case when one of  $a, b$  is identical to one of  $c, d$  from the case when  $a, b \neq c, d$ . As a subcase of the first case, suppose  $a = d$ . If not  $R(a, c)$ , move as follows: (1)  $(a, b) \Rightarrow (a, c)$ ; (2)  $(a, c) \Rightarrow (c, a)$ . If, on the other hand,  $R(a, c)$  holds, do (1)  $(a, c) \Rightarrow (c, a)$ ; (2)  $(a, b) \Rightarrow (a, c)$ . The other subcases of the first case are similar. In the second case, consider

*Subcase A:*  $R(a, c)$  and  $R(c, a)$ . Move (1)  $(c, a) \Rightarrow (c, d)$ ; (2)  $(a, c) \Rightarrow (c, a)$ ; (3)  $(a, b) \Rightarrow (a, c)$ . This sequence of moves is legitimate and results in  $R'$  (as is easily seen if one draws a diagram).

*Subcase B:*  $R(a, c)$  and not  $R(c, a)$ . Move (1)  $(a, c) \Rightarrow (c, a)$ ; (2)  $(c, a) \Rightarrow (c, d)$ ; (3)  $(a, b) \Rightarrow (a, c)$ .

*Subcase C:*  $R(c, a)$  and not  $R(a, c)$ . Move (1)  $(c, a) \Rightarrow (c, d)$ ; (2)  $(a, b) \Rightarrow (a, c)$ ; (3)  $(a, c) \Rightarrow (c, a)$ .

*Subcase D:* Neither  $R(a, c)$  nor  $R(c, a)$ . Move (1)  $(a, b) \Rightarrow (a, c)$ ; (2)  $(a, c) \Rightarrow (c, a)$ ; (3)  $(c, a) \Rightarrow (c, d)$ .

This completes the proof. □

Putting together the two previous lemmas, and Proposition 2, we obtain the following useful characterization of resumption.

### Theorem 6

*Over finite universes, a type  $\langle 2 \rangle$  quantifier  $Q$  is the 2-ary resumption of a PERM type  $\langle 1 \rangle$  quantifier iff  $Q$  is both right-oriented and left-oriented.*

## 10.3 BRANCHING

In Chapter 2.5.2 we mentioned the following sentence,

(10.40) Most boys in your class and most girls in my class have all dated each other,

suggested by Barwise (1978) as a straightforward example of a branching construction in English that cannot be reduced to ordinary iterative quantification. Generalizing to the branching of  $k$   $\text{MON}\uparrow$  type  $\langle 1, 1 \rangle$  quantifiers, which need not be the same, the intended truth conditions are obtained by means of the following polyadic lift to a quantifier of type  $\langle 1, \dots, 1, k \rangle$ . Define:

$$(10.41) \quad Br^k(Q_1, \dots, Q_k)_M(A_1, \dots, A_k, R) \iff \exists X_1 \subseteq A_1, \dots, \exists X_k \subseteq A_k \\ ((Q_1)_M(A_1, X_1) \ \& \ \dots \ \& \ (Q_k)_M(A_k, X_k) \ \& \ X_1 \times \dots \times X_k \subseteq R)$$

It can be shown (see Westerståhl 1994) that  $Br^k(Q_1, \dots, Q_k)(A_1, \dots, A_k, R)$  logically implies the iteration  $Q_1^{A_1} \dots Q_k^{A_k}(R)$ , as well as any iteration of these in any order, but that even the conjunction of these iterations in general does not entail  $Br^k(Q_1, \dots, Q_k)(A_1, \dots, A_k, R)$ . In fact, much stronger undefinability facts hold (see Chapter 15.3).

We also noted in Chapter 2.5.2 that one issue that has been discussed is whether branching of arbitrary generalized quantifiers can be defined. While there is no disagreement about the definition when all arguments of the lift are  $\text{MON}\uparrow$ , Barwise's formulation even in the  $\text{MON}\downarrow$  case has been disputed (by Sher (1997)), as well as his claim that branching of a  $\text{MON}\uparrow$  and a  $\text{MON}\downarrow$  quantifier makes no sense. Westerståhl (1987) suggested a general definition of branching of (right) *continuous* quantifiers (Chapter 5.1), which entails Barwise's (but not Sher's) formulations in the  $\text{MON}\uparrow$  and the  $\text{MON}\downarrow$  case. We do not pursue this debate here, however, but will make a few comments on the  $\text{MON}\uparrow$  case.

What triggers the branching reading in (10.40)? That there are two NPs conjoined with and is part of the explanation, but clearly not all of it. Consider

(10.42) Most students and at least three professors came to the meeting.

There is no temptation to read this in any other way than that most students came to the meeting and at least three professors came to the meeting. To obtain a branched effect, the predicate needs to be (at least implicitly) reciprocal, as in (10.40) and (the branching reading of)

(10.43) At least three students and at least two professors are friends,

or it can be *plural* (collective). Compare

(10.44) Most students and at least three professors gathered for the protest.

This can hardly mean that most students gathered for the protest and at least three professors gathered for the protest, but rather that one set consisting of most students and at least three professors has the property *gathered for the protest*. The similarity

between reciprocal and collective properties seems significant; we come back to it briefly in the next section.

However, mere reciprocity may still not be enough for a branching reading. Wilfrid Hodges has reported (personal communication) that if one changes the relation in (10.40) to one that is not usually taken to be bipartite, as in

(10.45) Most boys in your class and most girls in my class know each other.

the most common interpretation is that there is a set consisting of most boys in your class and most girls in my class, such that any two (distinct) members of this set know each other. In the next section we look further at reciprocals and observe that an expression *R each other* is naturally taken to predicate of a set that each two of its members stand in the relation denoted by *R* to each other; e.g.

(10.46) John, Mary, and Sue know each other.

When the antecedent is quantified, as in

(10.47) Most of my friends know each other.

the set in question is one which contains most of my friends. This uses a so-called *Ramsey quantifier* (see Chapter 15.4). However, the reading of (10.45) noted by Hodges exhibits yet another mode of polyadic quantification: each NP provides a set and *know each other* is predicated of their union.

As Henkin's syntax for branching quantification nicely indicates, there is no significant order between the branched quantifier symbols, or the NPs in the case of natural language. The semantic counterpart of this is precisely the (order) *independence* that we discussed in connection with iteration in section 10.1. In the case of an operation *Lift* that lifts two type  $\langle 1, 1 \rangle$  quantifiers to a type  $\langle 1, 1, 2 \rangle$  quantifier, independence means that for all  $M$ , all  $A, B \subseteq M$ , and all  $R \subseteq M^2$ ,

(10.48)  $Lift(Q_1, Q_2)_M(A, B, R) \iff Lift(Q_2, Q_1)_M(B, A, R^{-1})$

When *Lift* is *It*, this holds very rarely, as we saw, but for *Br*<sup>2</sup> it always holds: branching is independent. But independence is not a sure sign of branching. There are other independent lifts, such as cumulation; in particular, the lift suggested by (10.45) is independent too.

Thus, several factors are needed to obtain a branching reading. We have indicated only some of the issues here that deserve to be further studied. The main ones are (a) whether one can find clear evidence for the correct interpretation (if there is one) for branching of non-MON $\uparrow$  quantifiers; (b) how to obtain the branching readings. The latter question concerns both what it is that triggers branching and, when it is triggered, whether these readings can be obtained by a compositional semantic rule.

#### 10.4 RECIPROCALLS

Virtually all languages have a special purpose means of saying that members of a group are on a more or less equal footing as regards a relationship. In English, for example, one can use the reciprocal expressions *each other* and *one another* to say:



- (10.49) a. John and Mary gave each other books.  
 b. Those guys hung out at one another's homes throughout their childhood.

In this section we shall see that it is illuminating to treat reciprocals as expressing type  $\langle 1, 2 \rangle$  quantifiers. We illustrate with a few typical cases, and refer to Dalrymple *et al.* 1998 for a more complete overview of reciprocal quantifiers.

#### 10.4.1 What do reciprocals mean?

Sentences (10.49a, b) mean roughly (10.50a, b).

- (10.50) a. Each of John and Mary gave the other books.  
 b. Each of those guys hung out at every other (one)'s home throughout their childhood.

Generalizing, a sentence of the form

(10.51) (The) *A R* each other/one another

means (10.52a), that is (10.52b).<sup>17</sup>

- (10.52) a. Each (of the) *A R* every other *A*  
 b. *every x(A(x), every y(A(y) ∧ y ≠ x, R(x, y)))*

Although a number of people (including Peters) have been tempted to try and analyze English reciprocals as if their universal quantificational force somehow derived from the *each of each other*, *one another* is synonymous with *each other* and has equally universal force despite containing *one* in place of *each*. Furthermore, both means for expressing reciprocity are completely fixed expressions, not compositionally interpreted phrases, as is demonstrated for example by the lack of synonymy between (10.49a) and the following.

- (10.53) a. John and Mary gave every other books.  
 b. John and Mary gave all others books.  
 c. \*John and Mary gave each different books.  
 d. \*John and Mary gave each single other books.

In languages other than English, moreover, expressions synonymous with *each other* often do not contain parts resembling the language's universal quantifier expression (nor its adjective meaning 'different'). Therefore we do not think

<sup>17</sup> Some people feel that *each other* and *one another* differ subtly in meaning. Occasionally it is suggested that *each other* is used when only two people or things are involved, *one another* when more than two are. Other conjectures distinguish the two reciprocal expressions by different nuances of meaning. Dalrymple *et al.* (1998) examined 3,319 examples from *The New York Times* and the Hector corpus (a pilot corpus for the British National Corpus), and found no differences of grammaticality or meaning when either reciprocal expression was substituted for the other in any example. If a difference in grammar or meaning does exist between these two English reciprocal expressions, it is very difficult to find.

that anything general is lost by treating reciprocal expressions as units that semantically express type  $\langle 1, 2 \rangle$  quantifiers, which vary little from one language to the next.

Within each language, however, the quantifier expressed by reciprocals can vary significantly. How equal between members of the group in question the footing must be, figuratively speaking, can vary more than one might expect. Of course, (10.52b) is the most common meaning of (10.51), in part because with overwhelming frequency reciprocal sentences make statements about just two individuals. When more individuals are involved, the strong interpretation (10.52b) is still very common, for example with

(10.54) House of Commons etiquette requires *legislators* to address only the speaker of the House and *refer to each other indirectly*.

However, it is quite clear, as many authors including Langendoen (1978) and Dalrymple *et al.* (1998) have noted, that the reciprocal expression can mean very different things, including those exhibited in sentences (10.55)–(10.57).

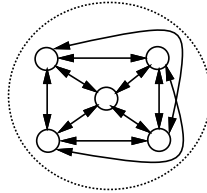
(10.55) As the preposterous horde crowded around, waiting for the likes of Evans and Mike Greenwell, *five Boston pitchers sat alongside each other*: Larry Andersen, Jeff Reardon, Jeff Gray, Dennis Lamp and Tom Bolton.

(10.56) “The captain!” said *the pirates, staring at each other in surprise*.

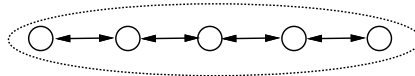
(10.57) He and scores of other inmates slept on *foot-wide wooden planks stacked atop each other*—like sardines in a can—in garage-sized holes in the ground.

Typical models satisfying (10.54)–(10.57) are illustrated in diagrams (10.58)–(10.61), respectively.

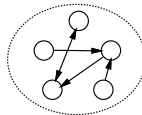
(10.58)



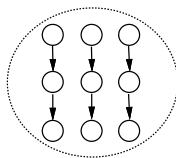
(10.59)



(10.60)



(10.61)



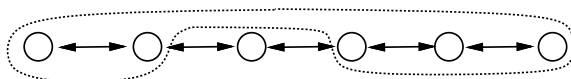
In each diagram, the set indicated by the dotted line comprises the group (denoted by the antecedent of the reciprocal expression) among whom reciprocity is spoken of, and the arrows represent the relation (denoted by the reciprocal expression's scope) with respect to which the members of that group are said to be on a more or less equal footing.

Besides being accounted as true in differing types of circumstances, as illustrated by the typical models above, each sentence except the last is also accounted false in circumstances where others are considered true. For instance, the second clause of

(10.62) #House of Commons legislators refer to each other indirectly; the most senior one addresses the most junior one directly.

contradicts the first one, showing that (10.54) is false in circumstances like any of those depicted in (10.59)–(10.61). Similarly, (10.55) is false in

(10.63)



as well as in (10.60) and (10.61), though it is true in (10.59) and would be true in (10.58) were that model not physically impossible for pitchers sitting alongside pitchers. The circle in (10.63) which is not included in the group indicated by the dotted line represents an individual who is not a Boston pitcher. Finally, the second clause of

(10.64) #The pirates were staring at each other in surprise; one of them wasn't staring at any pirate.

contradicts the first clause, and (10.56) is false in (10.61) even though it is true in (10.60)—and would be true in (10.58) and (10.59) were those physically possible for staring. So, the reciprocal *each other* really does express different quantifiers in examples (10.54)–(10.57). The evident differences are not just manifestations of vagueness, or some such phenomenon.

### 10.4.2 Type ⟨1, 2⟩ reciprocal quantifiers

Each quantifier expressed by *each other* in these examples is a relativization of one of a small number of polyadic quantifiers. Following Dalrymple *et al.* (1998) we define three type ⟨2⟩ quantifiers: *FUL*, *LIN*, and *TOT*.

$$\begin{aligned}
 \text{FUL}_M(R) &\iff M^2 - Id_M \subseteq R \ \& \ |M| \geq 2 \\
 \text{LIN}_M(R) &\iff M^2 - Id_M \subseteq R^+ \ \& \ |M| \geq 2 \\
 \text{TOT}_M(R) &\iff M \subseteq \text{dom}(R - Id_M) \ \& \ |M| \geq 2
 \end{aligned}$$

Here  $Id_M$  is the identity relation on  $M$ , and  $R^+$  is the transitive closure of the binary relation  $R$ : that is, the smallest transitive relation that includes  $R$ .<sup>18</sup>

It is immediately obvious that  $FUL \Rightarrow LIN \Rightarrow TOT$ .<sup>19</sup>

For any type  $\langle 2 \rangle$  quantifier  $Q$  such as these, its relativization  $Q^{\text{rel}}$  is the type  $\langle 1, 2 \rangle$  quantifier defined by

$$(Q^{\text{rel}})_M(A, R) \iff Q_A(A^2 \cap R)$$

The four type  $\langle 1, 2 \rangle$  quantifiers expressed by the reciprocal each other in sentences (10.54)–(10.57) are, respectively, the following:

$$Q_{\text{rcp}_1} = \text{FUL}^{\text{rel}}$$

$$Q_{\text{rcp}_2} = \text{LIN}^{\text{rel}}$$

$$Q_{\text{rcp}_3} = \text{TOT}^{\text{rel}}$$

$$(Q_{\text{rcp}_4})_M(A, R) \iff (\text{TOT}^{\text{rel}})_M(A, R \cup R^{-1})$$

**Fact 7**

$$Q_{\text{rcp}_1} \implies Q_{\text{rcp}_2} \implies Q_{\text{rcp}_3} \implies Q_{\text{rcp}_4}$$

Dalrymple *et al.* (1998) note that reciprocal statements are typically not ambiguous, even if the reciprocal expression they contain can denote any of the type  $\langle 1, 2 \rangle$  reciprocal quantifiers  $Q_{\text{rcp}_i}$  ( $1 \leq i \leq 4$ ). They propose a pragmatic principle to predict which meaning the statement will have: roughly that the reciprocal expression is interpreted as that one of these quantifiers which gives the statement the (logically) strongest truth conditions which are consistent with specified facts—the meanings of other parts of the sentence, and certain extralinguistic facts.

### 10.4.3 Properties of reciprocal quantifiers

All of the type  $\langle 1, 2 \rangle$  reciprocal quantifiers  $Q_{\text{rcp}_i}$  defined above are ISOM, EXT, CONSERV, and  $\text{MON}\uparrow$ , where conservativity for a quantifier of this type is the property

$$Q_M(A, R) \iff Q_M(A, A^2 \cap R)$$

and  $\text{MON}\uparrow$  means that it is increasing in the relation argument. By EXT, we can suppress the subscript  $M$  when it is not needed for clarity.

Each also manifests *indifference to self-relatedness*; that is,

$$Q_M(A, R) \iff Q_M(A, R - Id_M)$$

<sup>18</sup> Note that the condition  $M \subseteq \text{dom}(R - Id_M)$  in  $TOT$  in fact entails that  $|M| \geq 2$  (for non-empty universes).

<sup>19</sup> We write  $Q \Rightarrow Q'$ , meaning ‘for all  $M$  and all  $R \subseteq M^2$ ,  $Q_M(R)$  implies  $Q'_M(R)$ ’, and similarly for other types.

All these quantifiers except  $Q_{\text{rcp}_3}$  and  $Q_{\text{rcp}_4}$ , which are defined from *TOT*, are almost  $\downarrow\text{MON}$ ; i.e. they satisfy  $A' \subseteq A \ \& \ |A'| \geq 2 \ \& \ Q(A, R) \Rightarrow Q(A', R)$ .  $Q_{\text{rcp}_1}$  and  $Q_{\text{rcp}_2}$ , which are defined from *FUL* and *LIN*, are *convertible*; i.e. they satisfy  $Q(A, R) \Leftrightarrow Q(A, R^{-1})$ . So does  $Q_{\text{rcp}_4}$ .

$Q_{\text{rcp}_2}$ , which is defined from *LIN*, is the logically most complex of these interpretations of each other: whereas  $Q_{\text{rcp}_1}$ ,  $Q_{\text{rcp}_3}$ , and  $Q_{\text{rcp}_4}$  are definable in *FO*,  $Q_{\text{rcp}_2}$  is not monadically definable; i.e. it is not definable in terms of any monadic quantifiers whatsoever—let alone just  $\forall$  and  $\exists$ . This will be shown in Chapter 15.2.

#### 10.4.4 Collective predicates formed with reciprocals, and quantification of such predicates

Although we do not treat collective predication generally in this book, the special case of reciprocals is sufficiently pertinent to warrant a brief discussion here.

We have seen that some quantifiers can be viewed as relation reducers (see note 1); for example, a type  $\langle 1 \rangle$  quantifier  $Q$  reduces a binary relation  $R$  to a set,  $\{a : Q(R_a)\}$ , to which another type  $\langle 1 \rangle$  quantifier can apply (iteration). Conservative type  $\langle 1, 2 \rangle$  quantifiers can also be construed as reducers. A notable difference, however, between this case and the ones mentioned earlier is that here two first-order argument roles are collapsed into an argument role that is not first-order but second-order or “collective”, as it is sometimes termed in the linguistic literature. We illustrate the simplest case by defining an operation *EO* that takes conservative type  $\langle 1, 2 \rangle$  quantifiers to a mapping from binary first-order relations to unary second-order properties; specifically,  $EO(Q)$  is defined locally as follows with respect to any set  $M$ , when  $R \subseteq M^2$ ,

$$(10.65) \quad EO(Q)_M(R) = \{A \subseteq M : Q_M(A, R)\}$$

If  $Q$  is *EXT*, as reciprocal quantifiers are, then  $EO(Q)_M$  is independent of  $M$ , so we henceforth suppress  $M$ .

For example, if  $Q$  is expressed by each other and  $R$  is denoted by see, then  $EO(Q)(R)$  is the property of sets expressed by see each other.

Definition (10.65) generalizes straightforwardly to operations  $EO_{ij}$  for  $1 \leq i < j$  whose value on  $Q$  maps each  $n$ -ary relation  $R$  with  $n \geq j$  to a new relation missing  $R$ 's  $i$ th and  $j$ th argument roles and having a new argument role for sets of the things that could fill  $R$ 's  $i$ th and  $j$ th roles.

$$\begin{aligned} EO_{ij}(Q)(R) = \\ \{ (A, a_1, \dots, a_{n-2}) : \\ Q(A, \{(b, c) : R(a_1, \dots, a_{i-1}, b, a_i, \dots, a_{j-2}, c, a_{j-1}, \dots, a_{n-2})\}) \} \end{aligned}$$

For example, recommended each other to relates a set  $A$  and an individual  $a$  such that the members of  $A$  recommended each other to  $a$ .

One reason why this perspective on reciprocals is interesting is that ordinary type  $\langle 1 \rangle$  quantifiers, e.g. the one expressed by many students, can combine with collective predicates, including collective predicates formed from reciprocals.<sup>20</sup>

<sup>20</sup> Some type  $\langle 1 \rangle$  quantifiers resist combining with collective predicates. For example,

- (10.66) a. Many students gathered for the protest.  
 b. Many students helped each other study for the exam.

It is natural to want to interpret both of these by the same compositional rule. That rule must clearly be different from the one used throughout this book for sentences like

- (10.67) Many students took the exam.

Dalrymple *et al.* (1998) build on van der Does 1993 to propose the following rule:

- (10.68)  $CQ(Q_1, A, P) \iff$   
 $\forall Y \subseteq A(P(Y) \Rightarrow \exists X \subseteq A(|X| \geq |Y| \ \& \ |A-X| \leq |A-Y| \ \& \ P(X)$   
 $\ \& \ Q_1(A, X)) \ \& \ (Q_1(A, \emptyset) \text{ or } \exists X \subseteq A P(X))$

This rule is complicated because it has to handle a wide range of type  $\langle 1, 1 \rangle$  quantifiers  $Q_1$ , increasing, decreasing, and non-monotone. But it is not hard to show that, in accord with intuition, if  $Q_1$  is ISOM and  $\text{MON}\uparrow$ , then

- (10.69)  $CQ(Q_1, A, P) \iff$   
 $\exists X \subseteq A(P(X) \ \& \ Q_1(A, X)) \text{ or } (Q_1(A, \emptyset) \ \& \ \neg \exists X \subseteq A P(X))$

(note that in non-trivial cases, the second disjunct will be false).<sup>21</sup>

Similarly, if  $Q_1$  is  $\text{MON}\downarrow$  and ISOM, and if we assume that  $Q_1(A, X)$  is not false for all  $X$ ,

- (10.70)  $CQ(Q_1, A, P) \iff \forall X \subseteq A(P(X) \Rightarrow Q_1(A, X))$

Now we can see that rule (10.68) assigns sentences such as

- (10.71) a. Cats are suspicious of each other.  
 b. The students know each other.

the truth conditions we noted they have in section 10.4.1 above. This follows from the next result.

- (i) a. \*Each student gathered for the protest.  
 b. ?Every student gathered for the protest.

sound highly anomalous. However, the fully acceptable sentences

- (ii) a. All students gathered for the protest.  
 b. The students gathered for the protest.

are close or identical in meaning to (ia) and (ib). It is not completely clear whether the difference in acceptability is related to a difference in meaning that we do not treat, or is instead accountable to fundamentally syntactic facts of English.

<sup>21</sup> The verification uses the following fact about monotonicity:

FACT: (a) If  $Q_1$  is  $\text{MON}\uparrow$  and ISOM,  $Q_1(A, X)$ ,  $X, Y \subseteq A$ , and  $|X| < |Y|$ , then  $Q_1(A, Y)$ .

(b) If  $Q_1$  is  $\text{MON}\downarrow$  and ISOM,  $Q_1(A, X)$ ,  $X, Y \subseteq A$ , and  $|X| > |Y|$ , then  $Q_1(A, Y)$ .

(For infinite sets, this fails if  $|X| < |Y|$  is replaced by  $|X| \leq |Y|$ .)

**Proposition 8**

Suppose  $Q_1$  is CONSERV, EXT, and definite, and that  $Q_1^A$  is non-trivial on the universe at hand, and furthermore that  $W_{Q_1^A} = A$ . Let  $Q$  be  $Q_{rcp_i}$  for some  $i$ . Then

$$Q(A, R) \iff CQ(Q_1, A, EO(Q)(R))$$

*Proof.* Recall from Chapter 4.6 that, under the circumstances, when  $Q_1^A$  is non-trivial, and hence a non-trivial principal filter, the generator  $W_{Q_1^A}$  of that filter is independent of the universe, and always a subset of  $A$ .

( $\Rightarrow$ ) Assume  $Q(A, R)$ , i.e.  $A \in EO(Q)(R)$ , so the second conjunct of the right-hand side follows. For the first conjunct, assume  $Y \in EO(Q)(R)$  for some  $Y \subseteq A$ . Taking  $X = A$ , we have  $|X| \geq |Y|$ ,  $0 = |A - X| \leq |A - Y|$ , and  $X \in EO(Q)(R)$ . Also,  $Q_1(A, X)$ , since  $Q_1$  is definite.

( $\Leftarrow$ ) Assume  $CQ(Q_1, A, EO(Q)(R))$ . By non-triviality,  $\neg Q_1(A, \emptyset)$ , so there is  $Y \subseteq A$  such that  $Y \in EO(Q)(R)$ . Hence there is  $X \subseteq A$  such that  $X \in EO(Q)(R)$  and  $Q_1(A, X)$ , i.e.  $W_{Q_1^A} \subseteq X$ . Since  $W_{Q_1^A} = A$ ,  $X = A$ , and thus we have  $Q(A, R)$ .  $\square$

Under these circumstances the compositional rule gives the truth conditions for reciprocal sentences in terms of  $Q_{rcp_i}$  that we have seen to be correct. For example, when  $Q_1 = \text{the}_{pl}$ , or *the ten*, or when we have the universal reading of a bare plural like *cats*, construed as  $cat^{pl} = all_{ei}^{cat}$ ,  $W_{Q_1^A} = A$ .

There are still problems with the suggested rule  $CQ$ , however. Dalrymple *et al.* (1998) note that the pragmatic principle they propose for finding the correct reciprocal interpretation fails for some cases where  $Q_{rcp_3}$  is used. Related problems with  $Q_{rcp_3}$  appear with  $CQ$ . Consider

(10.72) John's students stared at each other in disbelief.

In a model where the students  $s_1, \dots, s_n$  stare in a 'cycle', i.e.  $s_i$  stares at  $s_{i+1}$  for  $i < n$ ,  $s_n$  stares at  $s_1$ , and no other staring occurs,  $Q_{rcp_3}(X, R)$  holds only if  $X = A = \{s_1, \dots, s_n\}$ . So if John's students are a proper subset of  $A$ , then (10.72) must be false. But (10.68) and (10.69) (note that the universal reading of *John's* is  $MON\uparrow$ ) make the sentence come out true, choosing  $X = A$ , since it is trivially true that *John's*( $A, A$ ). This shows that  $CQ$  does not cover all reciprocal readings, and hence not all collective cases either. The issue of a correct semantic rule for reciprocal and plural predicates as above apparently requires further study.

Likewise, one should further investigate the interesting question of whether branching sentences involving (at least implicit) reciprocals, as discussed in the previous section, can be treated compositionally by a similar semantic rule, and how such a rule then relates to the (correct) rule for simple reciprocals.





PART III  
BEGINNINGS OF A THEORY  
OF EXPRESSIVENESS,  
TRANSLATION,  
AND FORMALIZATION



## The Concept of Expressiveness

The expressive power of languages is an intriguing subject. We are familiar with the fact that logical languages are more or less expressive, and that certain concepts expressible in some of these cannot be expressed in others. For example, each finite cardinality is expressible in *FO* (with identity), but the concept of finitude itself (or of infinity) is not. One question is what such claims *mean*, another is how they are *established*. This will occupy much of the following chapters. Yet our main motivation in this book for studying logical languages is the prospect of applying the results to natural languages. Could logical results about expressivity really be relevant to natural languages? Isn't a plausible view that *everything*—if expressible in words at all—is expressible *somehow* in every natural language? And yet, don't we all know the difficulty, and sometimes impossibility, of *faithful translation* between languages? Indeed, an opposing but also popular view is that of *linguistic relativism*, according to which it frequently happens that what can be said in one language cannot even be *thought* by (monolingual) speakers of another language, let alone faithfully translated to it.

We'd like to convince the reader that there are precise ways of talking about facts of expressivity for natural languages, and moreover that, in the case of facts concerning quantification, they can be established using relatively simple model-theoretic tools. But to make such a claim at all convincing, one must provide not only the concepts needed to formulate the results in a clear way, but also the demonstrations that they hold. This is roughly what we intend to do in the following chapters.

It is easy to see that such demonstrations can be non-trivial. If you want to show that one language is *more expressive* than another, you must not only provide a translation in one direction, but also show that none in the other direction is possible. Failed attempts at translation won't suffice—you must find at least one sentence in the stronger language which is provably not synonymous with *any* sentence in the weaker language—in principle, an infinite task. What is surprising, and fascinating, is that this task can sometimes be carried through.

The first question, then, is: How does one talk about expressive power? It seems to us that although theorists as well as ordinary speakers often have strong intuitions about facts of expressivity, the notion itself is rarely analyzed very far. But it looks like a good idea to have a rather firm grip on the concept of expressivity that one is using before trying to substantiate particular claims for particular languages.

Therefore, we start by focusing on that very concept in the present chapter, on how it can be specified, and on related notions involved. Nothing we have to say is sensational, and much seems straightforward, but we have not seen a comparable

discussion of expressivity anywhere else. This chapter, then, also contains a sketch of a *theory* of expressivity.

## 11.1 PRELIMINARIES

Before presenting a general framework for talking about expressivity and translation in section 11.2, we look at various linguistic items that might be involved in translation and explain why the level of *sentences* is the natural one for matters of expressive power. This underlies the formulation of the basic concepts introduced in 11.2.

We also reemphasize (section 11.1.2) the crucial difference between quantifier expressions on the one hand and predicate expressions on the other. This insight has important consequences for the way in which expressivity issues are framed, as will become clear when we discuss the notion of logical equivalence in section 11.3.2 and in the following chapter.

### 11.1.1 Three levels of expressions

Consider the phenomenon of saying the same thing in different languages, i.e. the notion of *translation* between two languages  $L_1$  and  $L_2$ . A first issue is: Translation of what? Thinking about that leads naturally to distinguishing three levels of linguistic expressions.

#### 11.1.1.1 *The vocabulary level*

There are innumerable examples of obviously correct translations of words in one language into words or phrases in another language. The Swedish *hund* translates as the English *dog*, and the English *woman* as the Swedish *kvinn*.<sup>1</sup> Sometimes no single word is available but a complex phrase is. For example, the Swedish *farfar* can be translated as *paternal grandfather*, and the English *grandfather* as the Swedish *farfar eller morfar* (father's father or mother's father).

Likewise, there are many familiar instances where there is *no* translation in  $L_2$  of some lexical item in  $L_1$ . For a trivial example from logical languages, let  $L_1$  be propositional logic with all the usual connectives, and let  $L_2$  be propositional logic with only  $\wedge$  and  $\neg$ . There is no translation in  $L_2$  of the connective  $\vee$ ; i.e. there is no expression of the same category (binary propositional operator) which means the same. This is not very significant for logical purposes; it may make expressing things in  $L_2$  a little clumsier, but every *sentence* of  $L_1$  has a translation in  $L_2$ . For translation at the

<sup>1</sup> Are *hund* and *dog* really synonymous? In Swedish you can say *Här ligger en hund begrav*, literally *A dog is buried here*, but meaning roughly 'Something is not quite right here', but no direct translation of this idiom works in English. Linguists sometimes take such examples to show that there are no true synonyms in natural languages, or no perfect translations between languages. On the other hand, an interpreter asked to translate the sign *Hundar äga icke tillträde* ('Dogs not allowed') would be foolish to take the example above as an obstacle to translating *hund* by *dog*! What the examples show, we want to insist, is that you need to be careful about *which synonymy relation to choose*. Much of this chapter deals with that issue; see sect. 11.3, and in particular sect. 11.3.6.

vocabulary level, however, a necessary condition appears to be that the category (in some suitable sense) does not change.

For further natural examples, consider *color terms*. The distribution of color terms in the world's languages has been studied extensively. The wide variation across languages was first thought to support linguistic relativism. Careful studies show, however, that the ability not only to *discriminate* among colors but also to *categorize* them is by and large independent of language, and moreover that linguistic variation roughly follows fixed patterns of development, from languages with only two *basic color terms* to languages with about eleven.<sup>2</sup> Nevertheless, the linguistic variation poses issues of translation. There are clear cases when a basic color term in one language corresponds to a basic color term in another, and others when it corresponds to a complex term, such as green or blue, or some other Boolean combination, perhaps involving non-basic terms like pale yellow, deep blue, greenish blue. But it also seems likely that languages with many basic color terms will be hard to translate to languages with few such terms. Such negative claims would be more difficult to substantiate than the positive ones, since one would have to survey somehow all possible simple or complex color terms in a language. However, even if there is no translation at the vocabulary level of a certain basic color term in  $L_2$  to  $L_1$  (because  $L_2$  has more highly developed color terminology than  $L_1$ ), it might still be possible to translate each  $L_2$ -sentence saying that a particular object has that color to an  $L_1$ -sentence: namely, to one saying that the object *has the same color as* . . . , where . . . is some familiar object known to display the color saliently.

Similar facts hold for non-basic color terms, which may stand for highly specific shades of colors (e.g. scarlet, vermilion, and maroon are commonly named shades of red in English). If, for example, a language has a term C for the color of a male mallard duck's neck feathers (example from Kay 2001), *sentential* translation into English poses no problem. Even translation into a language that has no word straightforwardly translating the English color may still work, as Kay points out, since a speaker wishing to characterize a certain fish, say, as C can usually say that the fish "looks like" the neck feathers of a male mallard. But this relies heavily on context to provide the relevant dimension of comparison.

On the other hand, it is not hard to imagine cases where no such translation strategy works. The relevant context may be unavailable, or the comparison object unknown, or the notions of salience too different. Suppose you found an ancient inscription using an unknown color term, but also giving a description of that color. If the description was by means of a closed sentence, you might understand what the

<sup>2</sup> See e.g. the classic Berlin and Kay 1969, or its extended and updated version, Kay *et al.* 2003. Criteria for *basic* color terms are that they together exhaust the color spectrum of the language, that they are disjoint (nothing can be a typical instance of two basic colors), that they are non-composite (not built from meaningful smaller parts, eliminating, say, reddish), that they are predicable of any kind of object (eliminating e.g. blond), etc. These criteria have been the subject of debate, but the idea is sufficiently clear for our purposes here.

When there are only two basic terms, these seem invariably to stand for black and white. English basic color terms could be black, white, red, green, yellow, blue, brown, purple, pink, orange, and gray.

color was, but if it was by means of a demonstration, you might never find out. Of course, these possible epistemic circumstances should not be confused with semantic ones, but they at least make it conceivable that a language  $L_1$  has color-ascribing sentences that cannot be translated by any closed sentences of another language  $L_2$ , perhaps not even if contextual information is allowed. Whether this is the case or not appears to be a wholly empirical question. Suppose it is the case. Then we have a definite fact about relative expressive power between two whole languages, namely, the following: *It is not the case that  $L_2$  is at least as expressive as  $L_1$* . For this just means, it would seem, that there is something that can be said in  $L_1$  but not in  $L_2$ .

In sum, straightforward claims about differences in expressivity between natural languages are easy to find. We started with examples at the level of vocabulary, but have also indicated how these relate to the sentential level, distinguishing cases which essentially use context from those that don't.

#### 11.1.1.2 *The level of closed sentences*

The sentential level is where things are *said*. Sentences are the basic linguistic units for expressing thoughts, propositions, things which can be true or false.<sup>3</sup> This level is rather obviously more fundamental than the vocabulary level. Positive claims about translation at the vocabulary level can be 'lifted' to claims at the sentence level, but the converse does not hold. For example, it would be misguided to conclude from the propositional logic example above that propositional logic with the usual connectives is fundamentally more expressive than propositional logic with only  $\wedge$  and  $\neg$ . The former may be more convenient for certain purposes than the latter, but when it comes to expressible propositions, they have *equal* power. Similar comments apply to the case of color terms. Thus, expressive power should be measured at the sentential level, as long as one is interested in what can, and what cannot, be said in various languages.

A *closed* sentence has no parameters needing to be set by the utterance context in order to determine what is said. As is well known, such sentences are rather rare in natural languages.

#### 11.1.1.3 *The level of open sentences in context*

Context dependence is ubiquitous, and often no obstacle to translation. As long as *the same context applies in the same way* to both source and target, translation may work fine. This is completely trivial; compare the following English to Swedish translations:

- (11.1) a. The room was empty.  
b. Rummet var tomt.

<sup>3</sup> Assertions made with declarative sentences uncontroversially have truth conditions. Other ways of saying, besides making statements, are also associated with truth conditions, via other routes. A directive issued with an imperative sentence requests that the person addressed take action to insure that a certain proposition becomes true (or to omit action so that it remains false). Using a question, one can ask which of a presented collection of propositions are true. So various manners of saying share the feature the text discusses explicitly in connection with declarative sentences.

- (11.2) a. He saw her.  
b. Han såg henne.
- (11.3) a. I am hungry.  
b. Jag är hungrig.

None of these sentences says anything true or false, unless a suitable context is specified; but once that is done, the translation pairs use it in exactly the same way. Therefore, under a *fixed context assumption*, examples like these may be subsumed under the level of closed sentences, at least for the purpose of discussing issues of expressive power. This is a standard and for present purposes unproblematic practice.

But uses of context can be more complex. As a first example, suppose a request such as the English *Close the window!* is to be translated in to a language *L* like Japanese or Korean, which requires explicit indication of the social relations between the speaker and the addressee. Each available alternative is restricted to certain relationships. Thus, context is needed for translation, since without it there is no way to choose the correct version. On the other hand, neither the English sentence nor the relevant *L*-sentences have contextual variables for this: the English one is insensitive to it, and the alternatives in *L* signal it explicitly. A fixed context assumption determines the right translation. But that translation indicates—some might even contend that it *says*—something about social relations that the source sentence does not say or even indicate. If that is included, the source and the target don't really say the same thing, so translation may not be exact after all. A conclusion is that, unsurprisingly, whether translation is possible or not depends a lot on the notion of sameness of meaning that one is working with. When source and target sentences make different use of the same context, characterizing what is meant by “translation” may require appropriate choice of the underlying notion of synonymy.<sup>4</sup>

Consider another familiar example. To say *The elephant ate the peanuts* in Russian, you need to specify whether all the peanuts or just some of them were eaten, whereas in Turkish you must specify whether the eating was witnessed or only hearsay.<sup>5</sup> The relevant facts are there in the world, but one language is indifferent to them, whereas in another it can be compulsory to refer to them in the form used. This sort of case seems slightly different from the previous one. There, one could imagine *L*-speakers agreeing that *the same request* was expressed by both sentences, but that the *L*-sentence had to express something else as well. A suitable notion of synonymy might disregard that something, so meaning would be preserved after all. But in the case of verb aspect, obligatory in Russian but not in English, Russian speakers might

<sup>4</sup> Cases like this example may be described very differently by different persons. Some might agree to talk about translation here, but insist that it does not preserve meaning. Others may reject the label “translation” altogether, claiming that we are only discussing what would be the corresponding thing to say in various situations. Terms like “translation” do not have fixed uses, and we don't want to make unnecessary stipulations here. We will suggest, however, that a translation always has to preserve *something*; i.e. that it is relative to some sameness relation, a relation that may or may not deserve the label “synonymy”.

<sup>5</sup> The example is from Boroditsky 2002.

never call sentences using the two aspectual forms of a verb synonymous, and similarly might find meaning-preserving translation from English impossible.<sup>6</sup>

A final example concerns giving *directions* and has been debated recently in the literature. Most languages can use both absolute compass point-style directions and relative left-right directions, but some languages have only the former. So whereas in English I can tell someone *My brother's house is 20 meters to your left*, in such a language I would say something like *My brother's house is 20 meters north*. Both utterances require the location of the addressee, but the left-right one also requires his *orientation*.<sup>7</sup> This seems like two descriptions of the very same thing, so here the problem is not so much *what* is said, but *how*. Both sentences use the context, but in different ways: the left-right version requires the orientation, although it does not refer to it explicitly.

These differences have been used in arguments for linguistic relativity, arguments that have in turn been disputed.<sup>8</sup> But disregarding what exactly goes on in people's heads when they think about directions, it seems that often the sentences talk about the same features of the described situation; it is just that a correct translation needs to incorporate a (systematic) mechanism by which left-right directions can be turned into north-south directions relative to context, and vice versa. This may be quite feasible, but clearly cannot be handled with a fixed context assumption.

In this book we restrict attention to expressivity at the level of closed sentences, or cases of open sentences that can be treated with a fixed context assumption. At the sentential level, we'll be able to handle most cases of translation of vocabulary, including those in which no direct paraphrase is available but an embedding sentence can be translated. We'll also be able to deal with standard uses of indexicals and other context-referring expressions, as long as the source and the target use the context in the same way. We may even be able to handle the "Close the window!" case above, with a suitably chosen synonymy relation. But we don't pretend to handle all cases of context-sensitive translation, in particular not the directions case. Extending the ideas here to such cases remains a project for further study.

### 11.1.2 Expressibility of quantifier expressions versus predicate expressions

Actually, the examples of expressibility and translatability above are not typical of the ones that most interest us here. The next example illustrates this.

Let  $L_1$  be Swedish and let  $L_2$  be English<sup>-</sup>, which is just like real English except that it has no proportional determiners; in particular, it lacks *most*. Also, let English<sub>0</sub>

<sup>6</sup> They would presumably agree that the two forms have *something* in common, and one might take this as a basis for a technical notion of synonymy, relative to which translation would be possible.

<sup>7</sup> More precisely, it normally requires that he is standing up (not lying down, e.g., or standing on his head) and the direction in which he is *facing*.

<sup>8</sup> E.g. Levinson (1996) argues for linguistic relativity using psychological studies of speakers of Tzeltal (a Mayan language); the arguments are questioned in Li and Gleitman 2002.



lack all determiners except the four Aristotelian ones, and also lack adverbs of quantification, as well as all size-comparing constructions in terms of *more*, *less*, *fewer*, *as many as*, *majority*, etc. Of course, English<sup>-</sup> and English<sub>0</sub> are not natural languages, and have not even been precisely specified here, but the idea should be sufficiently clear to make them *possible* languages.

Now, various precise things can be said about the relative expressive power of these languages. For example, although there is no translation from Swedish to English<sup>-</sup> at the vocabulary level, there is one at our preferred level of closed sentences (provided there is one from Swedish to real English), since sentences of the form

- (11.4) a. De flesta katter är grå.  
b. De flesta lösningar till denna ekvation är heltal.

(meaning that most cats are gray, and that most solutions to this equation are integers, respectively) can be translated<sup>9</sup> as

- (11.5) a. There are more gray cats than cats that are not gray.  
b. There are more integer than non-integer solutions to this equation

which are sentences in English<sup>-</sup>, and similarly (though even more clumsily) for other proportional determiners. Thus, English<sup>-</sup> is at least as expressive as Swedish. But English<sub>0</sub> certainly isn't: no sentence at all in English<sub>0</sub> is an adequate translation of (11.4a). Rather, Swedish has strictly stronger expressive power than English<sub>0</sub> (given that it has at least the expressive power of real English).

Facts of this kind will interest us in what follows, and we will show how to establish them. But we have much less to say about, for example, the color terms in the examples mentioned earlier. What is the crucial difference here, between *all* or *most*, on the one hand, and *gray*, *cat*, *equation*, *integer*, on the other? We have already answered this question, in terms of the distinction between quantifier expressions and predicate expressions, introduced in Chapter 3.1.2. As we have shown, these two kinds of expression have fundamentally different properties, which motivate (Chapter 9.3) that the latter can get arbitrary interpretations in models, whereas the former have fixed interpretations. No wonder, then, that in model-theoretic semantics, there is much less to say about the expressivity of predicate expressions than about that of quantifier expressions.

But what if someone denied that there is such a difference, and maintained that it is our choice of a first-order model-theoretic apparatus that deprives us of the possibility of saying anything significant about the meaning of predicate expressions, and that this poor choice creates an appearance of a difference where there is none?

This objection has the cart before the horse. It is true that first-order models interpret only predicate expressions, but that is not an arbitrary stipulation, or one made in the interest of simplicity or elegance, or for mathematical reasons. Rather, we maintain, it is precisely because there are crucial intrinsic differences between quantifier

<sup>9</sup> On *one* natural reading of *most*; cf. Ch. 2.4. For another reading, one might instead translate by *There are many more* . . . .

expressions and predicate expressions that a first-order account—i.e. one using first-order models, not one using *FO*—is successful. We tried to establish, in Chapter 9, that these differences force upon us a treatment of quantifier expressions as constants, unless we are to violate fundamental intuitions about entailment in natural languages. Likewise, predicate expressions are largely variable in entailments. So there is in fact no reason to expect that expressivity issues for quantifier expressions and for predicate expressions could be dealt with in the same way.

Thus, in the context of natural language quantification, we are in the fortunate position that the technical tools for a study of expressivity that are required by basic facts about natural languages are precisely the simple and familiar ones of first-order (in our extended sense) model theory. Indeed, this is the key to the success of the logical theory of generalized quantifiers in this area.

## 11.2 A FRAMEWORK FOR TRANSLATION

In this section we present a general framework for talking about relative expressive power and translatability. The underlying ideas are completely straightforward and uncontroversial, we believe, but expressing them requires a modicum of precision, which is what we try to provide here.

We begin by noting (11.2.1) that every claim about expressivity is relative to a notion of saying the same thing, and that there are several such notions around. In fact, rather than talking about meanings directly, it will be enough, and simpler, to formulate our framework in terms of such same-saying relations (11.2.2). What these relations all have in common is being partial equivalence relations, or PERs as we shall call them. Section 11.2.3 explores this concept, and in particular the concept of a PER *between* two languages; a concept which is necessary, we argue, when accounting for translation between languages. Another important concept is that of one PER *refining* another.

Armed with these concepts, we proceed to define (11.2.4) the notion of a sentence, or a set of sentences, in one language being *translatable* into another (relative to given PERs), and more generally of a *translation* of the whole source language into the target language. The existence of such a translation amounts precisely to the target language being *at least as expressive as* the source language. We note some properties of translations, and end by introducing *formalization* as a special case of translation.

### 11.2.1 Expressiveness is relative to a notion of saying the same thing

One compares the expressive power of two languages by finding out what can be *said* in them. Thus, a fundamental relation is going to be:

$L_2$  is at least as expressive as  $L_1$ ,

which we shall write  $L_1 \leq L_2$ , and the intended meaning is simply that

everything that can be said in  $L_1$  can also be said in  $L_2$  (but not necessarily vice versa).<sup>10</sup>

Or, put in a different but essentially equivalent way,

there is a translation from  $L_1$  to  $L_2$ .

It will often be the case, however, that neither  $L_1 \leq L_2$  nor  $L_2 \leq L_1$  holds, but that, say, a part of  $L_1$  can be translated into  $L_2$ . Thus, the basic concept is really that of an  $L_1$ -sentence  $\varphi$  being translatable into  $L_2$ , in the sense that there is an  $L_2$ -sentence saying the same thing. Then we can say that a set  $X$  of  $L_1$ -sentences is translatable if each member of  $X$  is translatable.

This idea, we think, is completely natural and straightforward. But note that it depends crucially on a notion of *saying the same thing*. What that means is presumably less clear. Or rather, there are different ways of making it precise, each perhaps reasonable for its own purposes. If that is so, there will be just as many different notions of relative expressivity.

Does that in turn entail that every statement about expressivity becomes equally ambiguous, and thus needs to be indexed with a particular purpose? Not necessarily. The reason is that different notions of saying the same thing are often related, so that one is a *refinement* of the other. Then, to make a positive claim that something in  $L_1$  is expressible in  $L_2$ , relative to *both* notions of saying the same thing, it is enough to establish that claim for the finer-grained notion. On the other hand, a negative claim that something in  $L_2$  is *not* expressible in  $L_1$  only needs to be established for the less fine-grained variant. This simple observation will be crucial in what follows.

Each notion of expressing the same thing uniformly generates a concept of relative expressive power. We need to make that precise. But first, we need to lay down some general properties of the candidate same-saying relations.

### 11.2.2 Sameness of meaning, without meanings

It is familiar wisdom in philosophy and logic that you shouldn't introduce a class of abstract objects unless you are able to say something intelligible about what it is for two such objects to be the same, or different. This is, for example, Quine's main argument against treating meanings or intensions as objects—their identity criteria are obscure—and many others have rejected meanings for similar reasons. But there is no immediate implication in the converse direction: you can very well have a reputable notion of sameness of meaning—synonymy—without having meanings as objects.

In this book too synonymy is the basic relation, and the meanings themselves rarely show up at all. However, our reason is not a conviction that meanings are necessarily obscure. On the contrary, there are well-known theories of formal semantics that we admire and respect, where meanings are perfectly precise model-theoretic objects.

<sup>10</sup> If  $\leq$  is a pre-order, as will be the case here (remark (f) in sect. 11.2.4.1), the relation  $\equiv$  defined by  $L_1 \equiv L_2 \Leftrightarrow L_1 \leq L_2 \ \& \ L_2 \leq L_1$  is an equivalence relation. Thus, we can, if we wish, use the equivalence classes as absolute measures of expressive power. This gives a *partial* order of expressivity, which is presumably all that one can expect.

But there is a different, and stronger, argument why one should nevertheless stay with synonymy as long as this is possible.

First, it *is* possible for much semantic discourse. Facts about relative expressive power and translation can be stated in terms of synonymy. Another example of an area which seemingly involves meanings but can be rephrased in terms of synonymy is *compositionality*: instead of saying that the meaning of a compound expression is determined by the meanings of its parts and the mode of composition, you can say, equivalently, that substitution of synonymous parts preserves meaning (see Chapter 12.2.1).

Second, and no less important, suppose you have developed a theory of meaning for some fragment of a language, associating a complex set-theoretic object, or a collection of uses, or something else, with each well-formed expression. Now you want to extend the theory to a larger fragment, or, within the same fragment, you want to take account of a semantic distinction previously ignored. Often, this is not at all trivial. You may have to begin anew, perhaps with some even more complex objects, of which the ones used earlier can be seen as special instances. The same thing may happen again at the next enlargement, etc.<sup>11</sup>

Thus, totally new concepts of meaning may be needed, but one constraint is usually obeyed: the new notion of meaning *refines* the old one. That is, two expressions that were synonymous under the old notion of meaning may fail to be so under the new one, since a new semantic distinction has been observed, but *not* vice versa: if they had different meanings before, they must have different meanings afterwards too.<sup>12</sup>

This means that when giving an account of a semantic phenomenon in terms of synonymy, you need to be aware that your actual synonymy may be refined later on, and to make your account able to cope with such refinement, but that is all. You do not need to enable it to cope with vastly different technical notions of meaning.

It should be noted, however, that synonymies are less fine-grained than meanings. Many different meaning assignments will result in the same synonymy relation, but few of these will be of interest. For example, there is always the possibility of taking the equivalence classes of the synonymy as meanings. This works, in the sense that the synonymy corresponding in turn to that meaning assignment is the one you started with, but equivalence classes can be rather useless as meanings. In particular, it is unlikely that they will cast much light on the phenomenon of *understanding*, and doing that is sometimes seen as the main goal of a theory of meaning.

Still, the methodological choice of dealing with synonymy rather than meaning works for our purposes here, or so we claim.

<sup>11</sup> E.g. think about how Frege (or for that matter, any present-day semanticist) had to complicate his semantics to account for propositional attitudes, or how truth-conditional 'static' semantics was generalized to dynamic 'input-output' semantics *à la* Heim 1982 or Kamp and Reyle 1993 or Groenendijk and Stokhof 1991.

The advantage of using synonymy relations instead of meanings in this context has been stressed by Wilfrid Hodges (e.g. Hodges 2002).

<sup>12</sup> The ultimate refinement, of course, is the identity relation (within one language). It is sometimes proposed as the only 'real' synonymy. We return to this in sect. 11.3.6 below.

### 11.2.3 Sameness is a partial equivalence relation (PER)

It is important that the same-saying relations we are interested in can hold between expressions in one and the same language, as well as between expressions in two different languages. You could think of the one-language version and the two-language version as two different relations, but in that case some obvious connections between them must hold, which we will get to in a moment. But however you think of that, it is clear that sameness of meaning has to be a *partial equivalence relation*, or a PER, as we abbreviate it. That is, such a relation  $R$  must be:

- (a) reflexive on its domain: if  $uRv$  holds for some  $v$ , then  $uRu$
- (b) symmetric: if  $uRv$ , then  $vRu$
- (c) transitive: if  $uRv$  and  $vRw$ , then  $uRw$

Why partial? Well, even within one language there might be well-formed expressions that are not assigned meanings, according to your favorite semantics, so *a fortiori* they cannot be synonymous with themselves. Just to take one example, your favorite semantics might assign meanings only to sentences, not to their non-sentential parts. A related reason could be that you wish to enforce a principled distinction between grammaticality and meaningfulness, so that only a subset of the grammatically well-formed expressions are meaningful (think of *colorless green ideas* and similar examples). Or, taking a different perspective, you might be working with an unfamiliar language, and so far have only been able to assign meanings to a fragment of it. In this case the synonymy relation reflects your present state of knowledge of the other language. But of course fragments of familiar languages are frequently studied too; the partiality of a corresponding semantics is then due to your decision to restrict attention to a thoroughly analyzed part of the language. We want a necessary condition on sameness of meaning compatible with all of these possibilities; hence the notion of a PER.

The *domain* of any binary relation  $R$  on a set  $X$ ,  $dom(R)$ , is the set of objects in  $X$  related by  $R$  to something in  $X$ . It is immediate that

$$(11.6) \text{ If } \sim \text{ is a PER on } L, \text{ then } dom(\sim) = \{u \in L : u \sim u\}.$$
<sup>13</sup>

We say that  $\sim$  is *total* if  $dom(\sim) = L$ .

#### 11.2.3.1 PERs between languages

Suppose  $L_1$  and  $L_2$  are two languages with corresponding PERs  $\sim_1$  and  $\sim_2$ , respectively. They can be any PERs, but for intuition, think of them as some sort of synonymies. Also, think of languages simply as sets of (well-formed and unambiguous) expressions of some kind. There is an obvious way in which a synonymy, say  $\sim$ , between expressions in  $L_1$  and expressions in  $L_2$  should respect the given synonymies or PERs. For example, if  $u, v$  are both in  $L_1$ , then  $u \sim v$  should imply  $u \sim_1 v$ , and

<sup>13</sup> The *range* of a relation  $R$  is the set of objects in  $X$  to which something in  $X$  is related by  $R$ , and the *field* of  $R$  is the union of the domain and the range. If  $R$  is a PER, the domain, the range, and the field of  $R$  all coincide.

vice versa. Also, if  $u_1 \sim_1 v_1$ ,  $u_1 \sim u_2$ , and  $u_2 \sim_2 v_2$ , then we should have  $v_1 \sim v_2$ . This results in the following definition:

#### PERs between $L_1$ and $L_2$

Consider partial equivalence relations (PERs) on languages (i.e. on sets of expressions). Let  $L_1$  and  $L_2$  be languages with respective PERs  $\sim_1$  and  $\sim_2$ . A PER  $\sim$  on  $L_1 \cup L_2$  is called a PER *between*  $L_1$  and  $L_2$ , or, more precisely, between  $(L_1, \sim_1)$  and  $(L_2, \sim_2)$ , iff

$$(11.7) \quad \sim \cap (L_i \times L_i) = \sim_i, \quad i = 1, 2$$

It easily follows from (11.7) that, in the above case,

$$(11.8) \quad \text{dom}(\sim) = \text{dom}(\sim_1) \cup \text{dom}(\sim_2)$$

Note that we did *not* assume that  $L_1 \cap L_2 = \emptyset$ . That makes sense, since, for example, when  $L_1$  and  $L_2$  are logical languages of the kind used in this book, they have their *FO* part in common. Similarly, if  $L_1$  and  $L_2$  are fragments of the same natural language. This implies of course that no expression belonging to two different languages can mean one thing in the one language and another thing in the other language. That is, it follows that

$$(11.9) \quad \text{if there is a PER between } L_1 \text{ and } L_2, \text{ then for } u, v \in \text{dom}(\sim_1) \cap \text{dom}(\sim_2), \\ u \sim_1 v \Leftrightarrow u \sim_2 v$$

But this restriction seems rather harmless. Some words, like the English *be* and the Swedish *be* (meaning ‘to pray’), may appear to be shared but are really different words (since they have different syntactic properties and meanings), and there is no problem to treating them as different *expressions* in a more technical sense.<sup>14</sup>

On the other hand, precisely because the notion of expression is technical, there is no obstacle to enforcing the disjointness of, say, two fragments of the same language when one so desires; for example, if different kinds of semantics are considered for the respective fragments.

It may be worthwhile to observe that the converse of (11.9) holds too; i.e. we have the following proposition.

#### Proposition 1

*The following are equivalent:*

- (a) *There exists a PER between  $(L_1, \sim_1)$  and  $(L_2, \sim_2)$ .*
- (b) *For  $u, v \in \text{dom}(\sim_1) \cap \text{dom}(\sim_2)$ ,  $u \sim_1 v \Leftrightarrow u \sim_2 v$ .*

<sup>14</sup> Our definition of a PER between  $L_1$  and  $L_2$  allows (as Wilfrid Hodges pointed out) that some  $L_1$ -expression outside the domain of  $\sim_1$  not only belongs to but is meaningful in  $L_2$  (i.e. belongs to  $\text{dom}(\sim_2)$ ). This seems unnatural, and could be avoided with an extra condition, should one so desire.

*Proof.* Exercise.<sup>15</sup>

□

Note that if  $L_1 \cap L_2 = \emptyset$ , condition (b) is trivially satisfied, so there is then always a PER between  $L_1$  and  $L_2$ . Indeed, the smallest (in terms of inclusion) such PER is  $\sim_1 \cup \sim_2$ . This is a rather trivial PER between  $L_1$  and  $L_2$ , since it does not make anything in  $L_1$  synonymous with anything in  $L_2$ . In general, when condition (b) holds, there will be lots of PERs between  $L_1$  and  $L_2$ ; see also the next subsection.

Note finally that it does follow from (11.7) that, as requested above,  $u_1 \sim_1 v_1$ ,  $u_1 \sim u_2$ , and  $u_2 \sim_2 v_2$  together imply that  $v_1 \sim v_2$ .

### 11.2.3.2 PERs and partitions

Equivalence relations (total or partial) *partition* their domains, and it is often useful to think in terms of the partition rather than the relation itself. A PER  $\sim$  over a set  $X$  splits  $dom(\sim)$  into non-empty and pairwise disjoint equivalence classes, where, for  $u \in dom(\sim)$ , the *equivalence class of  $u$*  is

$$[u]_{\sim} = \{v \in X : u \sim v\}$$

In general, a *partition of  $X$*  is a collection  $\{C_i\}_{i \in I}$  of subsets of  $X$  such that

1.  $C_i \neq \emptyset$ , for  $i \in I$
2.  $C_i \cap C_j = \emptyset$ , for  $i, j \in I, i \neq j$
3.  $\bigcup_{i \in I} C_i = X$

We have the following familiar facts:

(11.10) If  $\sim$  is a PER on  $X$ ,  $P_{\sim} = \{[u]_{\sim} : u \in X\}$  is a partition of  $dom(\sim)$ .

(11.11) Conversely, every partition  $\{C_i\}_{i \in I}$  of  $dom(\sim)$  corresponds to a PER on  $X$ , given by

$$u \sim v \iff u, v \in C_i, \text{ for some } i \in I$$

in the sense that  $P_{\sim} = \{C_i\}_{i \in I}$ .

We can also say that a PER  $\sim$  on  $X$  partitions all of  $X$ , into

$$(11.12) \{[u]_{\sim} : u \in X\} \cup \{X - dom(\sim)\}$$

only  $X - dom(\sim)$  is empty if  $\sim$  is total. Put slightly differently, we can always extend a PER  $\sim$  to a total equivalence relation  $\sim^{\text{tot}}$ , where

$$u \sim^{\text{tot}} v \iff u \sim v \text{ or } u, v \in X - dom(\sim)$$

<sup>15</sup> Our purpose here is not to enter into the theory of equivalence relations but to present PERs as a useful and natural tool, so we leave the proof of Proposition 1 as an exercise, with the following hint for the direction (b)  $\Rightarrow$  (a): Let  $R$  be the relation  $\sim_1 \cup \sim_2$ . Show that the transitive closure of  $R$  (the smallest transitive relation containing  $R$ ) is a PER between  $L_1$  and  $L_2$ .

$\sim^{\text{tot}}$  is sometimes called the total *one-point extension* of  $\sim$ , since, thinking of  $\sim$  as sameness of meaning, it gives all expressions in  $X - \text{dom}(\sim)$  the *same* meaning (unspecified which one, except that it has to be different from all the meanings of expressions in  $\text{dom}(\sim)$ ).  $\sim^{\text{tot}} = \sim$  iff  $\sim$  is total, and  $\sim^{\text{tot}}$  partitions  $X$  into (11.12) above, except that, as we said, the empty set is not in the partition when  $\sim$  is total.

Now suppose again that  $\sim$  is a PER between  $L_1$  and  $L_2$ . What are its equivalence classes? A  $\sim$ -class can be identical to  $\sim_1$ -class  $C_1$ , in case nothing in  $C_1$  bears the relation  $\sim$  to anything in  $L_2$ . (Similarly, it can be identical to a  $\sim_2$ -class.) But if there is some  $u \in C_1$  related by  $\sim$  to some  $v \in L_2$ , then it is not hard to see that the  $\sim$ -class of  $u$  has to be the union of the  $\sim_1$ -class of  $u$  and the  $\sim_2$ -class of  $v$ . These observations are contained in the following proposition, whose proof is also left as an exercise.

**Proposition 2**

*A PER on  $L_1 \cup L_2$  is a PER between  $L_1$  and  $L_2$  if and only if the following conditions hold:*

- (a) *Each  $\sim$ -class is either a  $\sim_1$ -class or a  $\sim_2$ -class, or a union of exactly one  $\sim_1$ -class and exactly one  $\sim_2$ -class.*
- (b) *Each  $\sim_i$ -class is a subset of some  $\sim$ -class,  $i = 1, 2$ .*

Note that this holds whether or not  $L_1 \cap L_2 = \emptyset$ . Using this proposition, one can calculate the number of possible PERs between  $L_1$  and  $L_2$ , from the given number of  $\sim_1$ -classes and of  $\sim_2$ -classes, but we abstain from that here.

*11.2.3.3 Examples of PERs*

As we said, many PERs on a language  $L$  are nothing like synonymies. For a trivial example, suppose expressions in  $L$  are made up of letters, and let

$$uRv \iff u \text{ and } v \text{ have the same number of letters}$$

This is a PER, in fact a total one. Less trivial examples concern syntactic structure. Suppose expressions in  $L$  are generated by grammar rules, including phrase structure rules. Let

$$uRv \iff u \text{ and } v \text{ have isomorphic phrase structure trees}$$

This is a perfectly natural PER, if one is interested in form rather than content, but of course it has nothing to do with sameness of meaning.

The following PER, however, comes closer:

$$uRv \iff u \text{ and } v \text{ have the same denotation or extension}$$

This presupposes that at least some expressions of  $L$  have extensions associated with them in some systematic way, but the general idea is clear. A standard view is that sameness of denotation is *not* sufficient for sameness of meaning—examples go back to Aristotle (creature with kidneys versus creature with liver, etc.). Actually this issue is quite complex, and in interesting ways. We come back to it in more



detail in section 11.3.1 below. For now, we note that sameness of denotation is at least a candidate for a weak notion of synonymy.

If this seems overly generous, a less controversial candidate would be

$$uRv \iff u \text{ and } v \text{ are logically equivalent}$$

Again, details need to be filled in, especially since we shall want to talk about logical equivalence between sentences from different languages (Chapter 12.1). We will, however, take logical equivalence as a basic concept of synonymy.

As we will see, many other synonymy concepts are *refinements* of logical equivalence. This is an important notion, so let us be precise about it too.

#### 11.2.3.4 Refinements

Recall from our discussion at the end of section 11.2.2 that the passage from a synonymy  $\sim$  to a new synonymy  $\sim'$  may involve not only making finer semantic distinctions among the given expressions, but also extending the semantics to new expressions. This leads to the following definition:

##### refining a PER

Let  $\sim$  be any PER on  $L$ . We say that another PER  $\sim'$  on  $L$  *refines*  $\sim$  iff the following two conditions hold:

$$(11.13) \quad \text{dom}(\sim) \subseteq \text{dom}(\sim')$$

$$(11.14) \quad \text{for } u, v \in \text{dom}(\sim), u \sim' v \text{ implies } u \sim v$$

Thus, old expressions with the same meaning may get different meanings in the refined semantics, but if the old expressions were already distinguished by  $\sim$ , they are distinguished by  $\sim'$  as well.

One can refine logical equivalence by adding stricter constraints on synonymy: for example, some similarity of syntactic structure, or constraints having to do with cognitive processing. This gives rise to a host of ‘synonymies’, most of which are at least implicit in the literature. We give a survey of this landscape in section 11.3.

The effect of refinement is best appreciated by looking at the equivalence classes. A refinement  $\sim'$  of  $\sim$  partitions the equivalence classes of  $\sim$  into smaller classes, thus giving a finer partition of  $\text{dom}(\sim)$  (hence ‘refinement’). In addition, the  $\sim'$ -classes may stretch into  $\text{dom}(\sim') - \text{dom}(\sim)$ . Formally:

##### Fact 3

Let  $\sim$  and  $\sim'$  be PERs on  $X$ . If  $\text{dom}(\sim) \subseteq \text{dom}(\sim')$ , then  $\sim'$  *refines*  $\sim$  iff, for each  $u \in \text{dom}(\sim)$ ,

$$[u]_{\sim'} \cap \text{dom}(\sim) \subseteq [u]_{\sim}$$

In this case,  $\{[v]_{\sim} \cap \text{dom}(\sim) : v \in [u]_{\sim}\}$  is a partition of  $[u]_{\sim}$ , so

$$\{[v]_{\sim} \cap \text{dom}(\sim) : v \in \text{dom}(\sim)\}$$

is a partition of  $\text{dom}(\sim)$ .

### 11.2.4 Translation and formalization

We now have the means at our disposal to formulate a general account of expressivity in terms of translation. According to our discussion in section 11.2.1, we should start with the concept of a sentence in one language being translatable into another language. The basic notions involved are *language* and *translation*, and, to account for the latter, *sameness of meaning*.

#### 11.2.4.1 Expressive power in terms of translations

As to *languages*, we simply identify these with sets of expressions, as above. No specification of the nature of expressions (for example, that they are finite strings of symbols) is needed at this point. We shall stipulate, however, that each language  $L$  has associated with it a subset of  $L$ , consisting of the *sentences* in  $L$ . Furthermore, we assume that expressions are *unambiguous*, so that synonymy relations can apply to them. So “expression” is best seen here as a technical term, standing roughly for ‘analyzed word or phrase’.

As to *sameness of meaning*, we use the more general notion of a PER from the previous section. Thus, let  $L_1$  and  $L_2$  again be languages with corresponding PERs  $\sim_1$  and  $\sim_2$ , and let  $\sim$  be a PER between  $L_1$  and  $L_2$ .

PERs are partial, and we say that an  $L_i$ -expression is *meaningful* (more exactly,  $\sim_i$ -*meaningful*) if it is in the domain of  $\sim_i$ . We restrict translation to meaningful expressions. Of particular interest will be the set of meaningful  $L_i$ -sentences, which we denote  $LSent_i$ :

$$(11.15) \quad LSent_i = \{\varphi : \varphi \text{ is an } L_i\text{-sentence in } \text{dom}(\sim_i)\}$$

It is natural to assume furthermore—though it will not play a significant role in what follows—that  $L_i$ -sentences stand in the relation  $\sim_i$  only to other sentences; i.e.

$$(11.16) \quad \text{If } u \in LSent_i \text{ and } u \sim_i v, \text{ then } v \in LSent_i.$$

We now lay down the following definitions:

#### translatable sets of sentences

A sentence  $\varphi$  in  $LSent_1$  is  $\sim$ -*translatable into*  $L_2$ , iff there is an  $L_2$ -sentence  $\psi$  such that

$$\varphi \sim \psi$$

(Note that since  $\sim$  is a PER between  $L_1$  and  $L_2$ ,  $\psi \sim \psi$ , hence  $\psi \sim_2 \psi$ , so *a fortiori*  $\psi \in LSent_2$ .)

Likewise, a set  $X \subseteq LSent_1$  of meaningful  $L_1$ -sentences is  $\sim$ -*translatable into*  $L_2$  iff each member of  $X$  is  $\sim$ -translatable into  $L_2$ .

The following observation is practically immediate:

(11.17)  $X \subseteq L_{Sent_1}$  is  $\sim$ -translatable into  $L_2$  iff there is a function  $\pi$  from  $X$  to  $L_{Sent_2}$  such that for all  $\varphi \in X$ ,  $\varphi \sim \pi(\varphi)$ .

The if direction is clear. For the ‘only if’ direction one needs to select, for every member  $\varphi$  of  $X$ , one of its  $\sim$ -equivalents in  $L_2$  as the value of  $\pi$  at  $\varphi$ .<sup>16</sup>

We call the function  $\pi$  a *translation* of  $X$  into  $L_2$ .<sup>17</sup> To extend this notion to that of a translation of  $L_1$  into  $L_2$ , we want to allow for the fact that, even though sentences are fundamental for expressive power (see section 11.1.1), words and phrases other than sentences may also be translatable. For this reason, the general concept of a translation of one language into another is defined here as follows.

**translations**

A  $\sim$ -translation of  $L_1$  into  $L_2$  is a partial mapping  $\pi$  from  $L_1$  to  $L_2$  such that conditions (i)–(iii) hold:

- (i)  $\pi$  maps  $L_1$ -sentences to  $L_2$ -sentences ( $\pi$  is *sentential*)
- (ii)  $L_{Sent_1} \subseteq dom(\pi) \subseteq dom(\sim_1)$
- (iii) for all  $u \in dom(\pi)$ ,  $u \sim \pi(u)$  ( $\pi$  preserves meaning)

Thus, a translation—by stipulation—maps *all* the meaningful  $L_1$ -sentences to synonymous  $L_2$ -sentences. It may also map other meaningful  $L_1$ -expressions to  $L_2$ -expressions. Crucially, we require of a translation  $\pi$  that it *preserves meaning*, as given by the PER *between*  $L_1$  and  $L_2$ :

(11.18)  $u \sim \pi(u)$  (An expression is  $\sim$ -synonymous with its translation.)

A weaker requirement would be that  $\pi$  *preserves synonymy* in the sense that whenever  $u, v$  are  $L_1$ -expressions in its domain:

(11.19)  $u \sim_1 v \iff \pi(u) \sim_2 \pi(v)$  (Two expressions are synonymous in  $L_1$  if and only if their translations are synonymous in  $L_2$ .)

This does not use  $\sim$ , but only the local PERs within each language. Indeed, this requirement is sometimes used as a criterion of translatability. We show below (section 11.2.4.2), however, that in the present context it is too weak, and that preservation of meaning is the correct requirement.

Finally, we can restate the concept of relative expressive power in the framework just introduced.

<sup>16</sup> When  $X$  is infinite, this requires some form of the Axiom of Choice or, as will in practice usually be the case, the existence of a definable well-ordering on  $L_2$ -expressions.

<sup>17</sup> The word “translation” has the familiar ambiguity between, on the one hand, the function or rule mapping  $L_1$ -expressions to  $L_2$ -expressions, and on the other hand, the value of that function for a particular  $L_1$ -expression, as when we say “ $\psi$  is a translation of  $\varphi$ .” This should not cause any confusion.

**relative expressive power**

Define

$$L_1 \leq L_2 \text{ iff there exists a } \sim\text{-translation from } L_1 \text{ into } L_2$$

We read this: “ $L_2$  is at least as expressive as  $L_1$ ” (with respect to  $\sim$ ), or “ $L_2$  is an extension of  $L_1$ ”. We also define

$$L_1 < L_2 \text{ iff } L_1 \leq L_2 \text{ and } L_2 \not\leq L_1$$

$$L_1 \equiv L_2 \text{ iff } L_1 \leq L_2 \text{ and } L_2 \leq L_1$$

Using (11.17), one sees that “translation” and “translatability” have the requisite relation to each other:

$$(11.20) \quad L_1 \leq L_2 \text{ iff } L\text{Sent}_1 \text{ is } \sim\text{-translatable into } L_2.$$

For if  $\pi$  is a  $\sim$ -translation from  $L_1$  into  $L_2$ , then clearly the restriction  $\pi'$  of  $\pi$  to  $L\text{Sent}_1$  is a function into  $L\text{Sent}_2$  such that for all  $\varphi \in X$ ,  $\varphi \sim \pi'(\varphi)$ . Conversely, if  $\pi'$  has the latter property, it is clearly sentential, and  $L\text{Sent}_1 = \text{dom}(\pi') \subseteq \text{dom}(\sim_1)$ . Thus, (i)–(iii) in the definition of a translation are satisfied.

We end with a few further comments on the preceding definitions.

- (a) Every translation preserves synonymy, i.e. (11.19) holds. For take  $u, v \in \text{dom}(\pi)$ . If  $u \sim_1 v$ , then  $u \sim v$ ,  $u \sim \pi(u)$ , and  $v \sim \pi(v)$ , so by transitivity of  $\sim$ ,  $\pi(u) \sim \pi(v)$ , and hence  $\pi(u) \sim_2 \pi(v)$ . The other direction is similar.
- (b) Another observation is that the first part of condition (ii) in the definition of a translation can be strengthened to

$$(11.21) \quad L\text{Sent}_1 = \text{dom}(\pi) \cap \{\varphi : \varphi \text{ is an } L_1\text{-sentence}\}$$

For if  $\varphi$  is an  $L_1$ -sentence in  $\text{dom}(\pi)$ , then  $\varphi \sim \pi(\varphi)$  by (iii), and so  $\varphi \sim \varphi$ ; i.e.  $\varphi \sim_1 \varphi$ , since  $\sim$  is a PER between  $L_1$  and  $L_2$ . Thus,  $\varphi \in L\text{Sent}_1$ , and we have the inclusion from right to left. The inclusion in the other direction is given by (ii).

- (c) With a similar argument, it follows that

$$(11.22) \quad \text{range}(\pi) \subseteq \text{dom}(\sim_2)$$

- (d) In practice, one often wants to put stronger requirements on the translation  $\pi$ . One reasonable demand is that it be *computable* in some sense, another that it be *compositional*. The latter requirement presupposes some account of how  $L_i$ -expressions are *structured*. We deal with compositionality in detail in the next chapter, but do not include it in our most general notion of a translation.
- (e) It would have been more accurate to write  $L_1 \leq_{\sim} L_2$ , rather than just  $L_1 \leq L_2$ , but for readability we often suppress reference to the PER  $\sim$  (as well as to  $\sim_1$  and  $\sim_2$ ).
- (f) Relative to a fixed  $\sim$  and a class of languages with associated local PERs, such that for any  $L_1, L_2$  in this class,  $\sim$  is a PER between  $L_1$  and  $L_2$ , the relation  $\leq$  is

a *pre-order*, i.e. reflexive and transitive (see also note 10). For  $L \leq L$  via the identity translation, and if  $L_1 \leq L_2$  via  $\pi$  and  $L_2 \leq L_3$  via  $\sigma$ , then  $L_1 \leq L_3$  via the composition  $\sigma\pi$ .

- (g)  $L_1 < L_2$  means that  $L_2$  has *stronger* expressive power than  $L_1$ , in the sense that everything that can be said in  $L_1$  can also be said in  $L_2$  (relative to  $\sim$ ), while there is something that can be said in  $L_2$  but cannot be said (still relative to  $\sim$ ), *by any sentence whatsoever*, in  $L_1$ . Similarly,  $L_1 \equiv L_2$  means that the two languages are intertranslatable and thus have the *same* expressive power (relative to the chosen PER), whereas if both  $L_1 \not\leq L_2$  and  $L_2 \not\leq L_1$  hold, their expressive powers are *incomparable*, in the sense that in each language, some sentence is not synonymous (in the relevant sense) with any sentence in the other.

We have now made completely clear what is meant by one language being at least as expressive as another. As a simple first example, consider a logical language  $L$  which extends  $FO$  by adding the quantifiers ‘there are at least  $n$  things such that’, i.e.  $\exists_{\geq n}$ , for each  $n$ . Since  $FO$  is a part of  $L$ ,  $FO \leq L$  via the identity mapping. But since to each formula of the form  $\exists_{\geq n}x\psi$  there is an (albeit much longer!) logically equivalent formula in  $FO$ , one can also translate from  $L$  to  $FO$ , so  $L \equiv FO$ . In this case, the relevant PER, for  $FO$  and  $L$  as well as between them, is standard logical equivalence, i.e. truth in the same models.

By contrast, if  $FO(Q_0)$  is the logical language obtained by adding the quantifier  $Q_0$  (‘there are infinitely many things such that’) to  $FO$ , then  $FO < FO(Q_0)$  (same PER). None of the infinitely many sentences in  $FO$  is logically equivalent to the sentence  $Q_0xP(x)$  which says that (the denotation of)  $P$  is infinite.

Similarly, recall the languages  $\text{English}^-$  and  $\text{English}_0$  from the beginning of section 11.1.2. We claim that  $\text{English}^- \equiv \text{English}$ , but that  $\text{English}_0 < \text{English}$ , again relative to a suitable notion of logical equivalence.

So far, of course, we are just *saying* that infinity is undefinable in  $FO$ , or that the quantifier *most* is undefinable in  $\text{English}_0$ . For someone versed in  $FO$ , the first claim probably seems highly plausible. But plausibility isn’t always a trustworthy guide to expressive power: some surprises are in store, as we will see. Fortunately, these particular claims can be *proved*, as we will see in Chapter 14.

#### 11.2.4.2 Preservation of meaning and preservation of synonymy

The property (11.19), preservation of synonymy (repeated below),

$$(11.19) \text{ for } u, v \in \text{dom}(\pi), u \sim_1 v \iff \pi(u) \sim_2 \pi(v)$$

is a weaker property than preservation of meaning, (11.18). A mapping  $\pi$  satisfying (11.18) also preserves synonymy, but not vice versa. The property (11.19) involves only the two local PERs on  $L_1$  and  $L_2$ , not the given PER  $\sim$  between them. However, we now observe that every synonymy-preserving mapping  $\pi$  induces a PER  $\sim^{[\pi]}$  between the languages, with respect to which  $\pi$  is a translation in our sense; i.e. it preserves meaning. For simplicity, assume in this subsection and the next that  $L_1 \cap L_2 = \emptyset$ .

**from  $\pi$  to  $\sim^{[\pi]}$** 

Given a partial function  $\pi$  from  $L_1$  to  $L_2$ , define  $\sim^{[\pi]}$  as follows, for  $u, v \in L_1 \cup L_2$  (with  $L_1 \cap L_2 = \emptyset$ ):

$$(11.23) \quad u \sim^{[\pi]} v \quad \text{iff} \quad \begin{array}{l} \text{(a) } u, v \in L_1 \text{ and } u \sim_1 v \\ \text{or (b) } u, v \in L_2 \text{ and } u \sim_2 v \\ \text{or (c) } u \in L_1, v \in L_2, \text{ and } \pi(u) \sim_2 v \\ \text{or (d) } u \in L_2, v \in L_1, \text{ and } \pi(v) \sim_2 u \end{array}$$

Note that condition (c) implies that  $u \in \text{dom}(\pi)$  and that  $\pi(u), v \in \text{dom}(\sim_2)$ , and similarly for (d).

**Proposition 4**

Let  $\pi$  be a sentential map from  $L_1$  to  $L_2$  that preserves synonymy, and furthermore is such that  $L\text{Sent}_1 \subseteq \text{dom}(\pi) \subseteq \text{dom}(\sim_1)$  and  $\text{range}(\pi) \subseteq \text{dom}(\sim_2)$ . Then  $\pi$  is a  $\sim^{[\pi]}$ -translation from  $L_1$  into  $L_2$ .

*Proof.* (Outline) First, show that  $\sim^{[\pi]}$  is a PER between  $L_1$  and  $L_2$ . It is clear from the definition that  $\sim^{[\pi]}$  reduces to  $\sim_i$  on  $L_i$  (condition (11.7)), so it remains to check that  $\sim^{[\pi]}$  is a PER. Symmetry is practically immediate; for reflexivity and transitivity, one considers the four cases (a)–(d) for  $u \sim^{[\pi]} v$  in (11.23) (for transitivity each case has two subcases). We leave this as an exercise.

Second, conditions (i) and (ii) in the definition of a  $\sim^{[\pi]}$ -translation are satisfied by assumption. For condition (iii), suppose  $u \in \text{dom}(\pi)$ . Since  $\text{range}(\pi) \subseteq \text{dom}(\sim_2)$ , also by assumption, we have  $\pi(u) \sim_2 \pi(u)$ . By (11.23), this means that  $u \sim^{[\pi]} \pi(u)$ . So  $\pi$  preserves meaning in the sense of  $\sim^{[\pi]}$ , and is thus a  $\sim^{[\pi]}$ -translation from  $L_1$  into  $L_2$ .  $\square$

This result explains how the two notions of preservation are related. However, a moment's reflection shows that preservation of synonymy only is too weak to correspond to the intuitive idea of translation. The reason is that the induced PER  $\sim^{[\pi]}$  between the two languages is essentially determined by  $\pi$ :  $u$  is treated as meaning the same as  $\pi(u)$  just *because*  $u$  is mapped to  $\pi(u)$ . But  $\pi$  is arbitrary, except for the fact that (11.19) holds. For example, nothing prevents it from 'translating' *Snö är vitt* as *Grass is green* (rather than as *Snow is white*), as long as all Swedish sentences synonymous with *Snö är vitt* are mapped to English sentences synonymous with *Grass is green*.

This becomes most clear in the extreme case when there are no non-trivial synonymies in  $L_1$  or  $L_2$ , so that the local PERs are the identity. Then condition (11.19) just says that  $\pi$  is one–one, and so *any* one–one mapping from  $L_1$  to  $L_2$  preserves synonymy.

A final example clinches the point. Consider the function which maps every sentence of a language  $L$  to its *negation*. As a mapping from  $L$  to  $L$ , it preserves (any reasonable notion of) synonymy in the sense that (11.19) holds, and it satisfies the

other assumptions of Proposition 4, but it certainly does not preserve (any reasonable notion of) meaning.

We conclude that the idea of translation between two languages intuitively requires that there is a *given* synonymy relation between sentences in the one language and sentences in the other, which reduces to the given local synonymy when one restricts to one language, and which the translation mapping must respect. That is precisely what our formal definition of a translation tries to capture.

### 11.2.4.3 Formalization

Formalization can be seen as a special case of translation in our sense. Thus, suppose  $L_1$  is a natural language, or a fragment of such a language, and  $L_2$  is a logical language, equipped with a PER  $\sim_2$  of logical equivalence. Translating  $L_1$  into  $L_2$  can serve the purpose of bringing forth the ‘logical structure’ of  $L_1$ , and also to provide  $L_1$  with a semantics via the translation  $\pi$  and a given model-theoretic semantics for  $L_2$ .

Formalization in this sense need not directly involve a given synonymy on  $L_1$ . Instead, via  $\pi$  we can *pull back* the logical equivalence from  $L_2$  to  $L_1$ . In fact, this is a general operation, not tied to logical equivalence or formalization:

#### **pulling back a PER along $\pi$**

Let  $\sim_2$  be a PER on  $L_2$ , and let  $\pi$  be a partial function from a language  $L_1$  into  $\text{dom}(\sim_2)$ . Define, for  $u, v \in L_1$ ,

$$(11.24) \quad u \sim_1 v \text{ iff } \pi(u), \pi(v) \text{ are both defined and } \pi(u) \sim_2 \pi(v)$$

#### **Lemma 5**

$\sim_1$  is a PER on  $L_1$ , with  $LSent_1 \subseteq \text{dom}(\pi) \subseteq \text{dom}(\sim_1)$ .

*Proof.* If  $u \in \text{dom}(\sim_1)$ , there is  $v$  such that  $u \sim_1 v$ , and hence  $\pi(u) \sim_2 \pi(v)$ . So  $\pi(u) \sim_2 \pi(u)$ , whence  $u \sim_1 u$ . This shows that  $\sim_1$  is reflexive on its domain. That  $\sim_1$  is symmetric and transitive is obvious. Thus,  $\sim_1$  is a PER on  $L_1$ .

Next, if  $u \in LSent_1$ , then  $u$  is an  $L_1$ -sentence in  $\text{dom}(\sim_1)$ , so by (11.24),  $u \in \text{dom}(\pi)$ . Thus,  $LSent \subseteq \text{dom}(\pi)$ .

Finally, if  $u \in \text{dom}(\pi)$ , then  $\pi(u) \in \text{dom}(\sim_2)$  by assumption, so  $\pi(u) \sim_2 \pi(u)$ , and hence  $u \sim_1 u$ . That is,  $\text{dom}(\pi) \subseteq \text{dom}(\sim_1)$ .  $\square$

By definition, the map  $\pi$  in (11.24) preserves synonymy with respect to  $(L_1, \sim_1)$  and  $(L_2, \sim_2)$ . Now we can apply Proposition 4:

#### **Theorem 6**

- (a) Let  $\pi$  be any sentential map from  $L_1$  to  $L_2$  ( $L_1 \cap L_2 = \emptyset$ ), where  $L_2$  has a given PER  $\sim_2$ , with  $\text{range}(\pi) \subseteq \text{dom}(\sim_2)$ .  $\pi$  and  $\sim_2$  induce a PER  $\sim_1$  on  $L_1$  by (11.24), and a PER  $\sim_1^{[\pi]}$  between  $(L_1, \sim_1)$  and  $(L_2, \sim_2)$  by (11.23), in such a way that  $\pi$  is a  $\sim_1^{[\pi]}$ -translation from  $L_1$  into  $L_2$ .

(b) *If on the other hand  $\sim$  is a given PER between  $(L_1, \sim_1)$  and  $(L_2, \sim_2)$  and  $\pi$  is a  $\sim$ -translation from  $L_1$  into  $L_2$ , then the above construction gives back  $\sim_1$  and  $\sim$ . More precisely:*

- (i) *Defining  $u \sim'_1 v$  iff  $u, v \in \text{dom}(\pi)$  and  $\pi(u) \sim_2 \pi(v)$ , we have that  $\sim'_1 = \sim_1 \upharpoonright \text{dom}(\pi)$ .*
- (ii) *If  $\sim^{[\pi]}$  is defined from  $\sim_1, \sim_2$ , and  $\pi$  as in (11.23), then  $\sim^{[\pi]} = \sim$ .*

*Proof.* (a) is immediate from Lemma 5 and Proposition 4. For (b) (i), if  $u, v \in \text{dom}(\pi)$  and  $u \sim'_1 v$ , then  $\pi(u) \sim_2 \pi(v)$ , i.e.  $\pi(u) \sim \pi(v)$ . Also,  $u \sim \pi(u)$  and  $v \sim \pi(v)$ , so  $u \sim v$ , and hence  $u \sim_1 v$ . Similarly,  $u \sim_1 v$  implies  $u \sim'_1 v$ .

For (b) (ii), take  $u, v \in L_1 \cup L_2$ . There are four cases. If  $u, v \in L_1$ , then, by definition,  $u \sim^{[\pi]} v$  iff  $u \sim_1 v$  iff  $u \sim v$ . Similarly if  $u, v \in L_2$ . If  $u \in L_1$  and  $v \in L_2$ , then  $u \sim^{[\pi]} v$  iff  $\pi(u) \sim_2 v$  iff  $\pi(u) \sim v$  iff  $u \sim v$ , since  $u \sim \pi(u)$  by the assumption that  $\pi$  is a  $\sim$ -translation. The case when  $u \in L_2$  and  $v \in L_1$  is similar.  $\square$

Thus, any suitable map  $\pi$  from a language  $L_1$  into a given language  $L_2$  with a given PER can be used to pull back that PER to  $L_1$ , and thereby to define a corresponding translation from  $L_1$  into  $L_2$ . But, as we remarked at the end of the previous section, this translation need not preserve any natural notion of meaning. Whether it does or not depends on further properties of  $\pi$ . If, however, we already know that  $\pi$  is a reasonable translation of  $L_1$  into  $L_2$  relative to some PER between  $(L_1, \sim_1)$  and  $(L_2, \sim_2)$ , then, by (b) in the theorem above, pulling back  $\sim_2$  along  $\pi$  gives nothing new.

This theorem has nothing *in se* to do with formalization. But it makes sense to mention the latter in this context, since a formalization  $\pi$  of a (fragment of a) natural language  $L$  in some logical language with a given semantics has often been thought of as a way to provide a semantics for  $L$ . If there is a given meaning assignment to the logical expressions, then an  $L$ -expression  $u$  can simply be given the same meaning as  $\pi(u)$ . In terms of the corresponding synonymy relations, this is precisely the pulling back operation described here.

For  $\pi$  to merit the label “formalization”, further requirements clearly need to be satisfied, such as:

- The source is a natural language (fragment), the target is a formal or logical language, and the given PER is logical equivalence.
- $\pi$  is compositional.
- Reasonable synonymy relations on the source language need to be appropriately related to the induced PER on that language (in terms of refinement; what relations we have in mind will become clear in the next section, in particular 11.3.8).

We return to these requirements in the next chapter. Formalization in this sense is indeed a main tool of this book, not so much for providing semantics for languages, but rather for transferring results about expressive power from formal to natural languages. How this works in detail will be spelled out in Chapter 12.5.



### 11.3 VARIETIES OF SAMENESS

Expressibility, as defined in the previous section, is always relative to a PER (within a language or between languages). But we are only interested in PERs that capture some notion of *synonymy*. This section looks at some possible candidates, in particular *sameness of denotation* (11.3.1), *logical equivalence* (11.3.2), *analytical equivalence* (11.3.3), and various *cognitive* (11.3.4) and *linguistic* equivalences (11.3.5). One sometimes finds in the literature very definite claims about what “synonymy” means, or ought to mean, including the claim that there aren’t really any non-trivial synonymies, i.e. that the synonymy relation amounts to *identity* (11.3.6). We try to be as undogmatic about this as possible, since we don’t think that there is one right notion but rather several, useful for different purposes. This last point is elaborated in 11.3.7.

In the context of expressivity, we are also interested in how these notions relate to each other, in terms of the *refinement* relation. In 11.3.8 we give a picture of the landscape in which the various synonymies live, and in particular how they are positioned with respect to the synonymy relation that comes from logic: i.e. logical equivalence.

In a clear sense, sameness of denotation is a minimal requirement (in addition to being a PER) on any synonymy, so let us start with that notion.

#### 11.3.1 Sameness of denotation

Suppose  $u$  in  $L_1$  denotes  $m$ . Can we find an expression  $v$  in  $L_2$  which also denotes  $m$ ? This looks like a very weak notion of synonymy, but just how weak it is depends on a number of factors: (a) what kind of object  $m$  is; (b) relative to what class of models one takes the denotation to be; (c) whether one adopts a global or a local perspective.

For example, if  $u$  is a sentence, and sentences are taken to denote truth values, finding  $v$  with the same truth value seems trivial. Trivial, that is, unless the same  $v$  has to work for *all* models, in which case we go from truth *values* to truth *conditions*, and get one standard notion of logical equivalence for sentences. If, on the other hand, sentences denote propositions, finding  $v$  might be non-trivial even if the model (say, of the possible worlds style) is fixed.

A first-order and local perspective with a fixed model  $\mathcal{M}$  trivializes sameness of denotation for sentences, but not necessarily for other kinds of expressions. A predicate expression denotes a relation over  $M$ , a noun phrase a set of subsets of  $M$ . It might not be obvious whether or not we can find something in  $L_2$  denoting the same relation, or the same set of subsets.

Likewise, consider just one language  $L$ . Take *any* set  $X$  of subsets of  $M$ . Can we always find an expression in  $L$  that denotes  $X$ ? This is one of the *effability* questions introduced in Keenan and Stavi 1986. It is a question about the expressive resources of interpreted languages. The answer to this particular question in the case of English, given some assumptions, turns out to be Yes.

It is instructive to see an outline of the proof. First, we need to assume that  $M$  is finite. Second, we also need to assume that for each  $a \in M$ , a name of  $a$  is available, or rather a name of the Montagovian individual (Chapter 3.2.4)  $(I_a)_M = \{Y \subseteq M : a \in Y\}$  is available.<sup>18</sup> Now, for any  $A \subseteq M$ , let  $S_A$  be the unit set  $\{A\}$ . Then

$$S_A = \bigwedge_{a \in A} (I_a)_M \wedge \bigwedge_{a \in M-A} (\neg I_a)_M$$

so each  $S_A$  is denotable, provided we assume, third, that (finite) Boolean combinations of English NPs are English NPs. Now let  $X$  be any set of subsets of  $M$ , i.e. any type  $\langle 1 \rangle$  quantifier on  $M$ . Then, clearly,  $X = \bigvee_{A \in X} S_A$ . Since this disjunction is finite, the proof is complete.  $\square$

### 11.3.1.1 Local and global expressibility

The result just mentioned is local. For issues of expressivity, it is crucial to be clear about the global/local distinction, discussed in Chapter 3.1.1. Let us elaborate the point in greater detail, beginning at the level of sentences.

Suppose one claims that the English sentence

(11.25) Most philosophers smoke.

is expressible in a certain language  $L$ . From a global perspective, this asks for a characterization of the truth conditions of (11.25) which is *uniform* over (1) universes, and (2) the interpretations of philosopher and smoke. A definition which works when restricted to people at my university, but fails for your university, is not what one wants. Likewise, a definition that is correct for (11.25) but can't handle, say,

(11.26) Most students jog.

would be less than satisfactory too.

From a local and first-order perspective, there is no interesting question about the whole sentence (11.25)—it is either true or false. What about the noun philosopher or the verb smoke, which denote subsets of  $M$ , or the noun phrase most philosophers, treated as denoting a local type  $\langle 1 \rangle$  quantifier (set of subsets) on  $M$ ?

$L$  might lack a word for smoke. It might even lack a complex expression with the same denotation as the English verb smoke (in the fixed intended model  $\mathcal{M}$ ). Smoke would then be inexpressible in  $L$ . On the other hand, if  $M$  is finite and the assumptions from Keenan and Stavi's effability result above hold for  $L$ , we could cook up a complex predicate expression  $u$  in  $L$ —a disjunction enumerating the smokers in  $M$ —with the correct denotation. The English smoke and the  $L$ -expression  $u$  would be very different, since smoke is lexical, whereas  $u$  involves various proper names, but they would have the same denotation in  $\mathcal{M}$ .

<sup>18</sup> This might seem unrealistically strong. The assumption can be replaced by a more plausible one about available English NPs; see Keenan and Westerståhl 1997: 844 n. 4.

But clearly, this ‘translation’ would be next to useless in practice. Also, as we have said, it would usually need to be changed every time the discourse universe changes.<sup>19</sup>

For predicate expressions, there appears to be no other reasonable sense in which one can raise expressivity issues in the present framework. The conclusion is that the expressivity issues for predicate expressions that we can deal with in a local and first-order framework are only marginally more interesting than the corresponding (trivial) ones for sentences. Usually, to achieve translation of predicate expressions in any interesting sense, one needs sameness of *intension*.

For quantifier expressions, the situation is different.  $L$  might lack a determiner with the same denotation as *most*. Keenan and Stavi’s result shows that we may nevertheless, if  $L$  has certain basic resources, cook up an  $L$ -expression denoting the set of subsets of  $M$  that *most philosophers* denotes. Again, this is a local, non-uniform fact: the ‘translation’ depends crucially on  $M$ , it works only for finite universes, and it is hard to see how it could have any use in practice.

But for noun phrases and determiners, there *is* a global notion of expressive power. For *most*, this is precisely the issue of the uniform expressibility of sentence (11.25) above; that is, expressibility regardless of what the arguments of the determiner are, and of which universe we are in. As we will see, the global issue concerns the *definability* in  $L$  of the quantifier *most*.

Thus: for quantifier expressions, by contrast with predicate expressions, there are both global and local facts of expressivity. Both kinds can be non-trivial. For example, Keenan and Stavi (1986) show that not all local type  $\langle 1, 1 \rangle$  quantifiers are extensions of English determiner expressions, although all conservative ones are. In this book we focus on global facts. Note that a positive global result about expressivity implies all corresponding local versions.<sup>20</sup> More importantly, if one is interested in principled facts about *language*, local results seem to depend too much on the *world*.

For the unconvinced reader we offer the following final remark. Consider the question: Is existential quantification definable in propositional logic? If you feel inclined to answer Yes, because of the following fact:

$$(11.27) \quad \exists xP(x) \leftrightarrow \bigvee_{a \in M} P(a)$$

then local definability is your preference. Indeed, many philosophers have found the conceptual ties between existential quantification and disjunction very strong—see, for example, the quote from Albert of Saxony in Chapter 1.1.2, or from Russell in 1.2.3, or Peirce in 1.2.1, or the treatment of quantification in Wittgenstein’s *Tractatus*. On the other hand, although *for each*  $M$ , the right hand side of (11.27) is a

<sup>19</sup> Not to mention the fact that *Elisabeth smokes*, if true, would translate to a *necessary* truth in  $L$ .

<sup>20</sup> Conversely, a negative local result in principle implies the corresponding negative global fact. But in the case of quantifier expressions, such negative local facts are unlikely to exist. If the model is fixed, a quantified sentence either holds or not, so it cannot be the case that *no*  $L$ -sentence has the same truth value, but that is seemingly what would be required. In Ch. 13 we will see what is needed to prove a negative global fact of expressibility; it will then be clear that there really are no shortcuts via local facts.

good sentence in propositional logic—provided there are enough names and the universe is finite!—no one of these sentences works for different-size  $M$ . We feel inclined to conclude that existential quantification is *not* expressible with the resources of propositional logic, even though the object (the set of sets) in the world/model denoted by a local existential quantifier can be described by a disjunction.

In conclusion, sameness of denotation in a fixed universe or model is a necessary condition on synonymy, but not a sufficient one, in our opinion. On the other hand, we have explained that the extensions of quantifier expressions are not really tied to universes. Rather, they are objects that with each universe associate second-order relations over that universe. Then, sameness of denotation or extension is a central notion. In fact, it amounts precisely to the *logical equivalence*—truth in the same models—of certain corresponding sentences. And it is precisely logical equivalence that we propose as the weakest reasonable notion of synonymy.

### 11.3.2 Logical equivalence

For sentences, logical equivalence amounts to truth in the same models. Characteristically, *all* models in the relevant class count, not just those constrained by, say, analytical connections between the meanings of certain words, or by mathematical facts. For example, one also counts models where the interpretations of `boy` and `female` have non-empty intersection, or where `prime` and `odd` have empty intersection. This does not mean, of course, that there could be a boy who is female, or that it could be that no prime numbers are odd. Presumably, that is impossible. It just means that for logical equivalence we treat such words as expressions that can receive arbitrary interpretations (of the right kind), unconstrained by certain features of their meaning in English.

But how do we select the relevant class of models? Such a choice can be informed by the expressive means of the language, and the level of analysis desired. And we have already argued at length (see Chapter 9.4) that for a theory of quantification, the right choice is the class first-order models, i.e. those described in Chapter 2.2, consisting of a universe  $M$  and (a function  $I$ , assigning to the relevant atomic expressions) relations over  $M$ , and possibly some selected individuals in  $M$ .

This allows us to apply the tools of standard model theory to the semantics of quantification, for logical as well as natural languages. The former languages are specified exactly and are unproblematic syntactically, whereas the latter are immensely complex and only partly described by current grammars. Nevertheless, a model-theoretic approach works for quantification in natural languages too, given a few reasonable assumptions.

#### 11.3.2.1 *First-order English*

To apply model-theoretic tools to, say, English, it is not necessary to work under the fiction that we have a complete syntactic description of that language, or to restrict attention to a small but exactly specified fragment. What is needed is something like the following.

**fixed languages**

Let us say that a language  $L$  is *fixed* if an appropriate class of models has been selected, i.e. if it has been determined which (lexical) expressions are interpreted in models and which aren't, and, moreover, if a *truth relation* between  $L$ -sentences and models in this class has been somehow specified. (We write  $\mathcal{M} \models \varphi$  for ' $\varphi$  is true in  $\mathcal{M}$ '.)

Note that the same language (set of expressions) can be fixed in different ways, so a fixed language is really a pair consisting of a language and a way to fix the model class.

For formal languages, our notion of a fixed language is essentially the logicians' notion of a (model-theoretic) *logic*: a set of sentences, a class of models, and a truth relation between them (subject to certain conditions).<sup>21</sup> Here we use the term for natural languages too, where these things are less exactly specified. Still, as speakers of such languages, we usually have a pretty good idea.

Let us choose, then, the class of first-order models. It is eminently possible to look at English or any other natural language in this first-order way, as long as there is a workable notion of lexical predicate expression (see Chapter 3.1.2) and individual-denoting expression, and a reasonably fixed truth relation. Since this is an important concept, we state it explicitly.

**first-order English**

*First-order English* is English from the perspective of first-order models (but *not* from the perspective of *FO!*), i.e. where lexical predicate expressions and individual-denoting expressions are interpreted in models, but nothing else is, and the truth relation is the usual one for English speakers.

This truth relation is of course not specified exactly, but this need not be problematic. For atomic sentences it is the obvious one (see below). For a quantified sentence, one may use a truth definition with the corresponding (generalized) quantifier, provided it is fairly clear that the quantifier expression in question denotes that quantifier. For other sentences, first-order model theory may give little or no help, and we must rely on normal speakers' practice.

Observe that first-order English is *English*, not some artificial language. It is English, looked at from a certain perspective.

Given that the lexical predicate expressions are identified, it is rather clear which the *atomic sentences* of first-order English are. For example,

- (11.28) a. John is happy.  
 b. Elsa liked him.<sup>22</sup>  
 c. Bill is taller than me.

<sup>21</sup> See Barwise and Feferman 1985: ch. 1.

<sup>22</sup> A first-order treatment of tense, i.e. not in tense logic but with quantification over temporal variables, is easily carried out in first-order English, but we do not go into that here.

Indexicals or demonstratives can be thought of as individual-denoting expressions, given a fixed context assumption (section 11.1.1.3 above).

In some cases, there are choices to be made. Consider a sentence like

(11.29) John believes that Sara is very bright.

Believes and very are not interpreted in (first-order) models. From the perspective of first-order English there is nothing very illuminating to say about them, but that is to be expected. Now, one choice is to regard (11.29) as a complex sentence of the form

(11.30)  $a$  believes that  $b$  is very  $P_1$

Since belief is intensional, there will presumably be cases where (11.30) is true in  $\mathcal{M}$  and  $P_1^{\mathcal{M}} = P_2^{\mathcal{M}}$ , but

(11.31)  $a$  believes that  $b$  is very  $P_2$

is not true in  $\mathcal{M}$ . If one wants to avoid this, one may instead regard (11.29) simply as an atomic sentence, attributing a certain property to  $a$ .

For another example, consider modal expressions. Here our point can be made even for logical languages. The formal language of *modal predicate logic* is just *FO* extended with modal sentential operators  $\Box$  and  $\Diamond$  and has the same non-logical symbols as *FO*, so it is quite simple to regard it as a first-order language in our sense. This time there is something one can say about the truth conditions of modal sentences from a first-order perspective. We can use an *S5*-style modality, with a clause in the truth definition of the form

$$\mathcal{M} \models \Box\varphi \text{ iff, for all models } \mathcal{M}', \mathcal{M}' \models \varphi$$

This does some justice to intuitions about modality in a first-order framework, though much is left unaccounted for. Similar remarks hold for modal statements in first-order English.

Thus, though details have to be worked out, a first-order view of English, singling out those (lexical) expressions that denote individuals or relations between individuals for interpretation in models, is quite feasible. It handles matters of quantification correctly, and while other semantic phenomena may be less adequately dealt with or not at all, the corresponding sentences are not excluded, just seen in the same first-order way.

### 11.3.2.2 *Logical equivalence across languages*

To sum up: A model-theoretic notion of logical equivalence is relative to a selection of a class of models, i.e. to a choice of which expressions to regard as constant and which not. Such a choice is only partly methodological, since it is constrained by strong intuitions about entailment and form. When it is made, the language, and the notion of logical equivalence, is fixed. First-order models constitute a particular way of fixing languages, formal as well as natural.

But one more thing needs to be said about logical equivalence. Recall that for matters of translation between two languages, we need not one PER but *three*. Thus, we need to be clear about logical equivalence *between* languages.

This issue does not arise in logic, at least as long as the two languages have the same supply of predicate expressions. A model is just a universe and an assignment of interpretations to items in the vocabulary, and then the *same* models can be used for both languages, so the notion of truth in the same models also applies between them.

But this does not apply to real languages, or formal languages with fixed special purpose (non-schematic) vocabularies. Still, it is not hard to see what to do. Logical equivalence needs to be defined relative to a *mapping* between lexical expressions. Such a mapping is presumably provided by considerations about translation not dealt with here, since they concern items interpreted in models. But we can simply regard it as *given*, and adapt the notion of logical equivalence accordingly.

Another way might be to first replace vocabulary expressions by formal symbols, and then apply the usual notion of logical equivalence. But this is just a roundabout way of defining a lexical mapping, since we in any case have to decide when symbols in the respective vocabularies should be replaced by the *same* formal symbol, and when by *different* ones.

Thus, we need to define logical equivalence relative to lexical mappings. In principle, this is straightforward. But lexical mappings turn out to have a wider use, relating directly to translations between languages and to compositionality. We deal with these matters in the next chapter (section 12.1), and therefore defer a precise definition of logical equivalence between languages until then.

### 11.3.3 Analytical/necessary equivalence

Some necessary connections between meanings go beyond the logical ones. It doesn't seem possible that someone has fully understood the English words *brother*, *male*, and *sibling*, for example, without knowing how their meanings are connected, or that someone knew the meaning of *happy* and yet believed that this adjective mostly applies to inanimate things.

A standard way to enforce some such connections is with *meaning postulates*. Then, not just any models count, but only those which are models of certain *theories*, where a theory is simply a set of sentences in the relevant language. This works for mathematics as well. We can account to some extent for the idea that arithmetical truths are necessary—but still not logically true—by restricting attention, in the definition of necessary truth and necessary equivalence, to models that satisfy some arithmetical theory. Call this the *axiomatic approach*.

Sometimes another method is used, and might in the case of mathematics seem preferable in view of the fact that the axiomatic approach usually does not characterize the intended mathematical structures, but allows non-standard ones as well. The idea is roughly to 'attach' the relevant structure to the models one is using. For example, one might add the natural numbers with addition, multiplication, etc., or a structure of sets with the membership relation. We might call this the *structural approach*.

For linguistic analyticity the structural approach is not always an option, since it may not be a matter of characterizing particular objects or structures, but merely of laying down certain connections between the meanings of words. But certain structures may be, as it were, built into natural languages too: for example, the ordered structure of *time*, which underlies the use of tense, characteristic of many natural languages.

Mathematical meaning postulates can safely be assumed to be the same in all languages where the corresponding symbols occur—it is in fact not too much of an idealization to assume that the mathematical expressions (symbols, sentences, etc.) themselves are the same across languages. But for postulates stemming from the meanings of ordinary words, there may in principle arise an issue of how to handle translation from  $L_1$  to  $L_2$  when the two languages have *different* meaning postulates.

If there is a translation  $\pi$  from  $L_1$  to  $L_2$ ,  $L_1$ -postulates are mapped to  $L_2$ -sentences. What if the translated  $L_1$ -postulates differ from the  $L_2$ -postulates? Intuitions do not seem very clear here. Could the postulates contradict each other? Suppose that bachelors are male in  $L_1$  but that bachelors $^\pi$  can be female $^\pi$  in  $L_2$ .<sup>23</sup> Then bachelor and bachelor $^\pi$  would not have the same extension in intended models, and  $\pi$  would simply be a bad translation. So assume that each lexical  $u$  in  $L_1$  is interpreted in the same way as  $u^\pi$ . Still, in principle, nothing prevents the translated  $L_1$ -postulates from selecting a quite different class of models than the  $L_2$ -postulates, or even from contradicting them (in which case there are *no* models satisfying both).

Should we restrict attention to models making *both* sets of postulates true, or to models making *at least one* set true? On the first alternative one would probably have to rule that a translation which preserves analytical equivalence is impossible when the two sets contradict each other. But both alternatives seem *ad hoc*. And there is another problem: A synonymy between languages should reduce to the given local synonymy when restricted to sentences in one of the languages. On either alternative, this is likely to fail if, as seems natural, the local analytical equivalence relation relies only on the meaning postulates of the corresponding language. Thus, literal translation respecting meaning postulates from both sides may be impossible, a fact which in itself is not surprising.

A different approach is the following. Recall that translation goes *from* language  $L_1$  *to* language  $L_2$ . But then it seems reasonable to apply the  $L_1$ -postulates both in  $L_1$  and  $L_2$ , and forget about the  $L_2$ -postulates. That is, *assume* the  $L_1$ -postulates, in  $L_2$  too, and use them to define what is (analytically) equivalent with what. If the  $L_2$ -postulates are somehow at variance with these, the quality of the translation might be in doubt. But at least it is one kind of translation, based on one simple idea. And with only the  $L_1$ -postulates relevant between  $L_1$  and  $L_2$ , as well as inside each of  $L_1$  and  $L_2$ , the general requirements on PERs between languages will be satisfied.

A case where we do have some intuitions is formalization, say of English into a logical language  $L$ . As described in section 11.2.4.3, we can pull back the given relation of logical equivalence in  $L$  to English. There we have various meaning

<sup>23</sup> We use  $u^\pi$  for  $\pi(u)$ , assume that  $L_1$  is English, and express matters a bit sloppily, but the idea should be clear.



postulates, e.g. that bachelors are male, but there are none in *L*. Logic has no meaning postulates, but clearly, if we want to define a notion of analytical consequence or equivalence adequate for this case, we can use the translations of the English postulates. In fact, these postulates are usually formulated—for example, in Montague Grammar—directly in the logical language. So here we do use the one-sided notion of equivalence just described. But if we had used either of the two alternatives sketched two paragraphs up, a different notion would have resulted, one that would not be a PER between English and *L*.

However analytic equivalence is defined, it restricts the relevant class of models. Therefore *logical equivalence refines analytical equivalence*: two sentences that are logically equivalent will also be analytically equivalent, but not vice versa. At first sight, this might lead one to think that we should look for some notion of analytical equivalence as the weakest synonymy, in terms of which (negative) expressibility results should be stated. We shall see, however, that this is not so, and that logical equivalence is the right level for negative facts of expressibility. This will be explained in Chapter 12.5.

### *Philosophical digression*

The notion of analyticity has come under fire in modern philosophy, and many philosophers of language today—though by no means all of them—agree with Quine that it is fruitless to try to separate the contribution of the world to the truth value of a single sentence from the contribution of the meanings of its words. On the other hand, if our starting-point is how dictionaries and grammars (partly) encode ordinary language use, there is no doubt that there do exist various connections between the meanings of words, and that understanding these words entails understanding some of those connections. Indeed, that is not something a Quinean would deny. The disagreement with Quine here is rather whether there are facts of language use that can verify or falsify the statements found in dictionaries. Quine denies this (see Quine 1960: §§ 15–16). We prefer, on the other hand, to regard them as (usually) true descriptions. But even if Quine should be right about the principled issue, our approach is unlikely to lead to vastly incorrect semantic results.

We are not claiming that the meanings of lexical items are commonly *defined* in terms of others. The case of *brother* = *male sibling* is an exception rather than the rule. To say that there are “connections” (encoded by meaning postulates), such as the one that only animate objects can be happy, is a much weaker claim.

A Quinean might even agree to call such connections “postulates”. The question is where they come from, and the Quinean might prefer “meaning/world postulates” instead. The difference is not of great consequence for what we have to say here. But we do call these connections *analytic*, in order to highlight the fact that they seem to be somehow necessary but not logical; see Chapter 9.4, where we discussed the notion of logicity and the border line between logic and mathematics.

### 11.3.4 Varieties of cognitive equivalence

Both logical and analytical equivalence are defined without reference to the cognitive capacities of speakers. They disregard the fact that it may be arbitrarily hard for

humans to *find out* if two sentences are logically (or analytically) equivalent, or if one follows logically from the other. Indeed, unsolved mathematical problems can be given this form. So even if all speakers were mathematicians, this divergence between theory and human performance would not be overcome. But of course speakers need not be mathematicians in order to have full mastery of their language, or translate it in to another language. Thus, in many contexts it is natural to look for a synonymy relation that strengthens logical or analytical equivalence by taking cognitive capacities into account.

Such a relation presumably needs to rely on some way of measuring the cognitive effort or *cost* of processing sentences. It is not necessary here to recount or summarize the vast literature on this subject.<sup>24</sup> It's enough to observe that cognitive considerations yield natural notions of synonymy. Though precise definitions may be difficult, some logical equivalences *clearly* are cognitive ones as well, whereas others are *clearly not*. Here are a few examples of the first kind that involve quantification.

- (11.32) a. Not more than five students came to the party.  
 b. At most five students came to the party.  
 c. Fewer than six students came to the party.
- (11.33) a. Some student didn't sign up for the next course.  
 b. Not every student signed up for the next course.
- (11.34) a. Exactly one of the ten lottery tickets that John bought was a win.  
 b. Exactly nine of the ten lottery tickets that John bought were losing tickets.

Not very much processing seems to be required in order to judge the sentences in these pairs equivalent. On the other hand, individual differences may be huge. Compare, for example, the well-documented<sup>25</sup> differences in attitude towards sentences like

- (11.35) a. There is a 95 percent chance you will win.  
 b. There is a 5 percent chance you will lose.

On the other hand, there are many cases where one quantifier turns out to be expressible in terms of others, but where this fact is not something speakers can easily make use of, because the cognitive effort is too big. One curious kind of example is from Keenan 2005:

- (11.36) a. Between a third and two-thirds of the students laughed at that joke.  
 b. Between a third and two-thirds of the students didn't laugh at that joke.

<sup>24</sup> See e.g. Croft and Cruse 2004; Sperber and Wilson 1995; Lewis 1979; Stenning and Oberlander 1995; Vallduví and Engdahl 1996.

<sup>25</sup> See Tversky and Kahneman 1974 and Kahneman and Tversky 2000.

It takes (for most people) more than a little ‘figuring’ before seeing that these two are equivalent.<sup>26</sup> Other cases are more drastic. Consider

- (11.37) a. There are more linguists than logicians.  
 b. There are more linguists who are not logicians than logicians who are not linguists.

Some thought reveals that these are equivalent, given that the number of linguists who are also logicians is finite.<sup>27</sup> For a case when the intersection is infinite, on the other hand—say, the number of even numbers that are divisible by 3—the correct equivalence is (we have negated the sentences in order to make them true):

- (11.38) a. There aren’t more even numbers than numbers divisible by 3.  
 b. If there are more even numbers non-divisible by 3 than odd numbers divisible by 3, then there aren’t more even numbers non-divisible by 3 than even numbers divisible by 3.

Here more thinking, and some cardinal arithmetic,<sup>28</sup> is required. Obviously, these are not cognitively equivalent.

### 11.3.5 Possible linguistic equivalences

Instead of letting cognitive factors enter into a notion of synonymy, one might strengthen the purely logical or analytical concept by requiring similarity with respect to linguistic features such as

- morphosyntactic structure
- topic/focus relations
- presuppositions
- anaphoric behavior

Accordingly, there are numerous possible linguistic notions of synonymy. Moreover, some of the relevant concepts depend on particular linguistic theories, a fact which tends to multiply the possible synonymy relations.

For example, many linguists would regard

<sup>26</sup> In general, one has to see that the condition

$$1/n \cdot |A| \leq |A \cap B| \leq (n-1)/n \cdot |A|$$

is equivalent to

$$1/n \cdot |A| \leq |A - B| \leq (n-1)/n \cdot |A|$$

<sup>27</sup> For then,

$$\begin{aligned} |A| > |B| &\iff |A - B| + |A \cap B| > |B - A| + |A \cap B| \\ &\iff |A - B| > |B - A| \end{aligned}$$

<sup>28</sup> Viz. that if  $|A \cap B|$  is infinite, then  $|A|$  is the maximum of  $|A - B|$  and  $|A \cap B|$ . See Ch. 13.2.1.

- (11.39) a. Mary picked up the book.  
 b. The book was picked up by Mary.

as synonymous, in virtue of having, at some level of analysis, the same structure. But if further factors are taken into account, such as topic/focus, their meanings are no longer the same. Concerning our previous supposedly cognitively equivalent pairs of quantified sentences, one could argue that at least (11.33a) and (11.33b), as well as (11.34a) and (11.34b), are not linguistically equivalent, since the first but not the second sentence allows later anaphoric reference to the student or the lottery ticket in question.

In the philosophical literature, a well-known early attempt at strengthening logical equivalence is Carnap's (1936) notion of *intensional isomorphism*, which was introduced in order to have a notion strong enough to sustain intersubstitutivity *salva veritate* in intensional contexts. This required synonymous sentences to have the same structure, and simple parts with the same intension.<sup>29</sup>

For formal languages, such notions are also familiar. In *FO*, for example, one could require sameness of structure (i.e. being derived by the same formation rules in the same order), perhaps up to changes of bound variables, and extensionally equivalent atomic parts. In type theory, a less stringent notion is that of reducibility, in terms of a sequence of  $\lambda$ -conversions, bound variable changes, and perhaps other operations as well.<sup>30</sup> One desirable aspect is that such synonymies usually are *decidable*, in contrast with ordinary logical equivalence.

In linguistics, although it is not hard to find discussions of translational difficulties for particular constructions in particular languages, general accounts of the notion of translation, and thereby of a notion of synonymy that translation preserves, seem rarer.<sup>31</sup> Some early work of Keenan, however, though informal, presents a view very much in line with the one in this chapter. Keenan (1974) discusses the concept of sentences in two languages being *exact translations* of each other, defined as being "semantically related to other sentences of their respective languages in exactly the same way" (p. 193). The semantic relations he speaks of include presupposition, and others that differ from identity of truth conditions in that they concern more than 'what is said'. Thus, for example, if  $\psi$  is an exact translation of  $\varphi$ , and  $\varphi$  presupposes  $\theta$ , then  $\psi$  must presuppose an exact translation of  $\theta$ . Thus the synonymy that Keenan requires translation to preserve is an identity of a meaning that he takes to be constituted by a sentence's semantic relations to other sentences in the same language.

<sup>29</sup> See e.g. Lewis 1970; Cresswell 1985; Salmon 1986; Pagin 2003.

<sup>30</sup> Type theories have the primitive operations of function or  $\lambda$ -abstraction—allowing one to form, say, the predicate term  $\lambda x(P_1(x) \wedge \neg P_2(x))$  from  $P_1(x) \wedge \neg P_2(x)$  by abstracting over  $x$ —and function application—allowing the application  $s(t)$  of the function  $s$  to the argument  $t$  (provided they are of the right types). Then an abstraction followed directly by an application is normally redundant; so e.g.  $\lambda x(P_1(x) \wedge \neg P_2(x))(c)$  reduces by  $\lambda$ -conversion to  $P_1(c) \wedge \neg P_2(c)$ .

<sup>31</sup> The collection Guenther and Guentherc-Reuter 1978 has some papers with a principled discussion, in particular the contributions mentioned below, as well as Kamp 1978. In the same volume, Givón 1978 gives a useful survey of a variety of translation problems between various languages, organized along the distribution of certain grammatical morphemes related to definiteness, pronominal reference, and other features.

Keenan goes on to argue that by this criterion, languages can be seen to differ in expressive power, where “two languages are the same in logical expressive power if each sentence of one has an exact translation in the other” (p. 194).

This argument is further elaborated in Keenan 1978, in the form of a polemic against what is called the Exact Translation Hypothesis,

(ETH) Anything that can be said in one natural language can be translated exactly into any other language (p. 157),

a claim defended in Katz 1972 and 1978. For example, Keenan argues that it will often be the case that lexical predicate expressions in one language have no lexical correspondents in other languages, and further that the operations forming complex expressions, such as *passive*, differ so much across languages that it is reasonable to expect exact translation to be hard or impossible. Though not stated explicitly, one could take this to mean that some similarity of syntactic structure is required for exact translation (since, for example, a complex predicate expression apparently does not exactly translate an atomic one; cf. section 11.1.1.1 above).

Keenan lists other necessary conditions on exact translation, in terms of properties required by the relevant synonymy: sameness of speech act, sameness of truth conditions, preservation of presupposition and also of ambiguity.<sup>32</sup> Another condition is called sameness of derived truth conditions, which amounts to substitutability *salva veritate* in various linguistic contexts. For example, though

- (11.40) a. John sold a house to Mary.  
b. Mary bought a house from John.

have the same truth conditions, the sentences in (11.41) can have different truth values,

- (11.41) a. The fact that John sold a house to Mary surprised us.  
b. The fact that Mary bought a house from John surprised us.

and therefore the sentences in (11.40) are taken not to be exact paraphrases of each other. Although failures of exact translation are said to be common between languages, translations that are inexact but still sufficient for actual cases of communication are (implicitly) taken to be usually possible.

Katz (1978) in turn criticizes Keenan (1974), claiming on the one hand that translation with preservation of presupposition is in fact possible where Keenan said it wasn't, and on the other that Keenan does not adequately distinguish different levels of translation. Indeed, Katz explicitly relates translation to notions of synonymy much as we have done, whereas this is at most implicit with Keenan. Our aim here is not to discuss the validity of their respective claims, however, but to point out that the discussion is based on a treatment of synonymy and translation very similar to ours (though less formal), and that linguists tend to impose stronger notions of synonymy (resulting in what Katz calls “non-logical translation”) than philosophers.

<sup>32</sup> In the strong form that if a sentence is ambiguous between several readings, its translation must be ambiguous *in the same way*.

Let us note finally that Keenan (1978) is careful to state that negative claims about expressive power, i.e. claims about *untranslatability*, can be made plausible but not conclusively demonstrated, since “at the moment, it is not possible effectively to enumerate the set of all meanings expressible in any given [natural] language” (p. 173). Thus, it could be just our lack of imagination or ingenuity that prevents us from finding a required translation. For formal languages, as we will see in later chapters, such claims *can* be conclusively demonstrated, even though an infinite number of meanings are expressible. The extent to which this allows us to conclude anything about natural languages will be discussed in Chapter 12.5.

### 11.3.6 The ultimate refinement: identity

If “sameness of meaning” is taken very strictly, and if the meaning of a word or phrase is taken to include *all* linguistically relevant aspects of its role in the language, then it may be that no two distinct expressions have the same meaning.

For example, if substitutability *salva veritate* in *all* contexts is required, including intensional contexts, idioms (see note 1 in section 11.1.1 above), not to mention contexts of quotation, this is a likely consequence. Similarly, if not just extension and intension but also *connotations* count, including perhaps even associations triggered by the sounds or shapes of the words themselves.

The result is that synonymy becomes identity, the ultimate refinement of any PER. From this it is but a short step to the conclusion that meaning-preserving translation between different languages is impossible.

Such a stance is sometimes taken. It is not inconsistent, but it is, to say the least, unhelpful. In general, we would say, a concept of synonymy is helpful only if there are, on the one hand, lots of distinct synonymy classes, but on the other hand, in each (or at least most) of these classes lots of distinct expressions.

Someone could try to argue that the intuition that synonymy is a substantial relation is simply wrong: a closer look at the complexities of meaning reveals that there turn out to be no non-trivial synonymies. But here we disagree, partly because there are familiar arguments in favor of less inclusive notions of meaning or content—such as descriptive content—which do generate interesting synonymy relations; but mostly because we don’t think that sameness of meaning is an absolute notion—rather, it is relative to particular practices of linguistic communication, or, in the case of distinct languages, of translation.

### 11.3.7 Where do relations of synonymy come from?

Notions of *descriptive content* are familiar attempts to abstract away from the subjective associations that usually accompany the production or understanding of utterances, and from certain background facts about the utterance situation, and to focus on what expressions standardly communicate. The motivations, coming from ordinary practices of linguistic communication, are well known and need not be rehearsed here. Connotations are usually private, and thus difficult or impossible to communicate, and even when they could be communicated, they are not meant to be;

the point of communication is (often) another. Background facts affect *how* things are said, but usually not what is said. Abstracting from these, we are left (hopefully) with a communicated *content*. When the purpose of communication is related to information about the world (in a wide sense), that content can be termed “descriptive”.<sup>33</sup>

The concept of descriptive content is not undisputed: Can we be sure that for natural languages there really is such a thing? This question is deceptive. Although language use is an empirical phenomenon, descriptive content is a theoretical concept, and one cannot simply go out and look for it in nature (just as one cannot go out and look for the relation of synonymy). What one can look for are aspects of linguistic communication and understanding for which positing such a level of content is helpful. We have no doubt that such aspects of communication are ubiquitous. Indeed, this book argues that an extensional and model-theoretic treatment of quantification provides a clear and succinct account of significant linguistic practices, so it is in itself a contribution to a notion of descriptive content.

Most of the synonymy relations mentioned previously can be thought of as some sort of sameness of some sort of descriptive content, and are thus related to the ways in which speakers use language to communicate information. Other kinds of sameness of other kinds of content may underwrite other uses of language. Whenever there is a practice of linguistic communication, it is natural to suppose that there is a relevant notion of sameness of meaning.

The point becomes most clear for translation between different languages. We here refer to translation not just as the abstract notion introduced previously, but as an activity through which thousands of translators and interpreters earn their living, helping millions of people to communicate across languages. This may seem trivial, but is very much to the point. We have here a genuinely linguistic activity whose purpose and usefulness are not in any way in doubt. Everyone realizes that without translation, communication and understanding among people would be seriously impeded.

Moreover, everyone knows too that a translation can be *more or less accurate*. Complete accuracy is often felt to be unachievable—see the remarks above on identity as a synonymy—so one selects a level of accuracy suitable for the purpose at hand. However, *within* such a level or practice of translation, it is often absolutely clear that a proposed translation is *correct*, and that another one is simply *wrong*. It is perfectly natural to say that correctness here consists in preservation of something: namely, of that level of meaning or content one desires to communicate. Again, translation is relative to a notion of sameness of meaning, and it succeeds when the source and the target have this relation to each other. This is clearest in factual contexts concerning aspects of descriptive content, but need not be restricted to them. Translation of, say, fiction may follow the same pattern, using other synonymies.

<sup>33</sup> Of course, we are simplifying. For example, there is an ongoing discussion about the very notion of ‘what is said’, about different kinds of content that an utterance may have, and about how background and context determines that content; see e.g. Perry 2001; Stanley 2000; Recanati 2002. However, under the idealization of a fixed context assumption (sect. 11.1.1.3), one can still say that these authors do posit a level of descriptive content(s), or ‘official content’, or ‘what is said’.

To deny this is to hold the view that the activity of translators and interpreters may be practically or pragmatically useful but is linguistically irrelevant. To us, this seems like a refusal to theorize about an important linguistic phenomenon. In this chapter and the next, we are presenting a general way of talking about the activity of translation and thus of expressivity, with a minimum of theoretical tools, the main tool being precisely the notion of a PER or a synonymy relation, within and between languages.

### 11.3.8 A landscape of possible synonymies

To get a feeling for the variety of possible synonymy relations, one can try to picture them under the (pre-)order of refinement (section 11.2.3.4). Fig. 11.1 is designed to help with this. For simplicity, we restrict attention to one fixed language  $L$ , and to total PERs on the set  $LSent$  of its sentences.

Thus, there is a fixed class  $MOD$  of models for  $L$ , and the PERs above the dotted horizontal line are defined in terms of sameness of truth value in certain of these models. When  $X \subseteq MOD$ , we use the notation

$$\varphi \Leftrightarrow_X \psi$$

for truth in the same  $X$ -models, i.e. for all  $\mathcal{M} \in X$ ,  $\mathcal{M} \models \varphi$  iff  $\mathcal{M} \models \psi$ . Thus  $\Leftrightarrow = \Leftrightarrow_{MOD}$  is *logical equivalence*, truth in the same models, and  $\Leftrightarrow_{\emptyset}$  is the universal relation  $LSent \times LSent$ , included in the diagram for completeness.

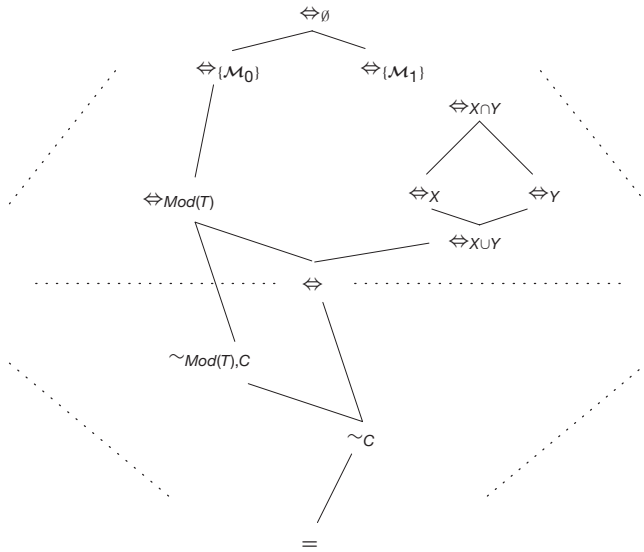


Figure 11.1 A landscape of PERs over  $L$



The next level contains all PERs of the form  $\Leftrightarrow_{\{\mathcal{M}\}}$ . These are the *sameness of denotation* PERs relative to a fixed model.<sup>34</sup>

If  $T \subseteq LSent$  is a theory—a set of meaning postulates—let  $Mod(T) = \{\mathcal{M} : \text{for all } \varphi \in T, \mathcal{M} \models \varphi\}$ . Then  $\Leftrightarrow_{Mod(T)}$  is the *analytical equivalence* corresponding to the postulates in  $T$ .

Refinement goes downward in the diagram, and is sometimes indicated with a line between a PER and another below that refines it. Note that

$$\text{if } X \subseteq Y, \text{ then } \Leftrightarrow_Y \text{ refines } \Leftrightarrow_X$$

(Note also that the diagram assumes that  $\mathcal{M}_0 \in Mod(T)$ .)

Logical equivalence refines all PERs above the horizontal dotted line. In addition, this part of the diagram has a semi-lattice structure, since  $\Leftrightarrow_{X \cup Y}$  is the *greatest lower bound* of  $\Leftrightarrow_X$  and  $\Leftrightarrow_Y$ : it refines both, and if  $\Leftrightarrow_Z$  also refines both  $\Leftrightarrow_X$  and  $\Leftrightarrow_Y$ , then  $\Leftrightarrow_Z$  refines  $\Leftrightarrow_{X \cup Y}$ .<sup>35</sup>

Below the horizontal dotted line are PERs that refine PERs above it by adding extra requirements, not expressible in terms of classes of models. Thus, various *cognitive* and *linguistic equivalences* are found here. At the bottom we have the only PER refining all the others: *identity*.

Note that such extra requirements can be made not only on logical equivalence but also on analytical equivalences. For example, let  $\sim_{Mod(T),C}$  be the PER obtained from  $\Leftrightarrow_{Mod(T)}$  by adding the requirement that  $\varphi, \psi$  also satisfy some cognitive condition  $C$ , and let  $\sim_C$  be obtained by adding the same condition to logical equivalence. Then, as logical equivalence  $\Leftrightarrow$  refines  $\Leftrightarrow_{Mod(T)}$ ,  $\sim_C$  refines  $\sim_{Mod(T),C}$ . In addition, of course,  $\sim_{Mod(T),C}$  refines  $\Leftrightarrow_{Mod(T)}$ , and  $\sim_C$  refines  $\Leftrightarrow$ .

As we have emphasized, if a negative expressibility result holds relative to a particular PER  $\sim$ , it also holds relative to all refinements of  $\sim$ . That is, it persists *going down* in the diagram. Now, all such negative results in the following chapters are primarily established for logical equivalence  $\Leftrightarrow$ . *A fortiori*, they hold for refinements of  $\Leftrightarrow$ , such as  $\sim_C$  in Fig. 11.1. But it might seem from the diagram that they tell us nothing about perfectly reasonable PERs such as  $\Leftrightarrow_{Mod(T)}$  or  $\sim_{Mod(T),C}$ .

We will see in the next chapter, however (section 12.5), that in a certain sense, PERs based on meaning postulates are irrelevant for expressivity issues. Thus, logical equivalence remains the focal point of Fig. 11.1 for negative facts of expressivity.

<sup>34</sup> If  $LSent$  is countably infinite,  $|LSent| = \omega$ , then there are  $2^\omega$  possible PERs on  $LSent$ .  $MOD$ , on the other hand, is a proper class. So there will be innumerable many more classes of models than there are PERs, which means that each PER above the horizontal dotted line will correspond to lots of model classes. For example, if  $L$ -sentences are ISOM-closed—a sentence true in  $\mathcal{M}$  is true in all models isomorphic to  $\mathcal{M}$ —then  $\Leftrightarrow_{\{\mathcal{M}\}} = \Leftrightarrow_{Is(\mathcal{M})}$ , where  $Is(\mathcal{M})$  is the (proper) class of models isomorphic to  $\mathcal{M}$ .

<sup>35</sup> This follows from the fact that  $\varphi \Leftrightarrow_{X \cup Y} \psi$  if and only if  $\varphi \Leftrightarrow_X \psi$  and  $\varphi \Leftrightarrow_Y \psi$ . Also,  $\Leftrightarrow_{X \cap Y}$  is an upper bound of  $\Leftrightarrow_X$  and  $\Leftrightarrow_Y$ , but not necessarily the least one.

# 12

## Formalization: Expressibility, Definability, Compositionality

The motivation for the previous chapter was to present tools and concepts making it possible to infer in a reasonably precise way facts about expressivity for natural languages from facts about expressivity for logical languages. But we also tried to make plausible that some of these concepts are interesting in their own right. To attain the original goal, however, still a few more pieces need to be in place.

To begin, we need a precise account of logical equivalence across languages; this is given in section 12.1, in terms of the concept of a lexical mapping. Next, the issue of compositionality must be addressed. We did not make compositionality a requirement on translation, but it is perhaps the first extra condition that comes to mind. In section 12.2, we discuss compositionality in an abstract setting, with a twist however, in that we distinguish compositionality of a language from compositionality of a translation.

It should normally be possible to extend a lexical mapping to a compositional translation—provided certain conditions on the grammars of the respective languages are fulfilled. We lay down such conditions in section 12.4, and show that the desired extension is then indeed possible. The main condition is that the syntactic rules of the source language should be definable in the target language. Therefore, we first spend section 12.3 talking a bit about definitions in general, motivating a notion of definability closely related to uniform translation.

Finally, in section 12.5 we pull all the threads together to achieve, we hope, a reasonably detailed and coherent picture of the relation between expressivity issues for natural and for formal languages.

### 12.1 LOGICAL EQUIVALENCE REVISITED

Section 11.3.2.2 in the previous chapter noted that although logical equivalence—truth in the same models—is well-defined for one language, we need to extend it to a relation *between* languages, in order to talk about translations that preserve this relation.

The general situation is as follows. We have two languages  $L_1$  and  $L_2$ . They can be formal or natural languages, but they are first-order in our extended sense; i.e. we have distinguished the *predicate expressions* (those denoting relations between individuals) from other expressions. The *atomic* predicate expressions (and atomic

individual-denoting expressions) are the only expressions that get interpreted in *models*.  $L_1$  and  $L_2$  are *fixed*, in the sense that the relation of truth in a model is supposed to be given.

Now, in general, the atomic predicate expressions of  $L_1$  will be different from those in  $L_2$ . A model, however, is defined (Chapter 2.2) as  $\mathcal{M} = (M, I)$ , where  $I$  assigns suitable interpretations to predicate and individual-denoting expressions in a *vocabulary*. Thus, there is no question of a sentence from  $L_1$  and a sentence from  $L_2$  being true in the *same* models. In order to define a relation of logical equivalence between  $L_1$  and  $L_2$ , we need to know when two predicate expressions of the respective languages are to have the same interpretation. We will account for that in terms of a notion of *lexical mapping*.

For the record, let us first state the standard notion of logical equivalence inside one language  $L$ . As before, if  $\varphi$  is an  $L$ -sentence,  $V_\varphi$  is the set of vocabulary expressions occurring in  $\varphi$ .<sup>1</sup>

### logical equivalence inside one language

Let  $L$  be a fixed first-order language (in our extended sense), with vocabulary  $V$ , and let  $\varphi, \psi$  be  $L$ -sentences. Using  $\Leftrightarrow$  for logical equivalence, define

$$(12.1) \quad \varphi \Leftrightarrow \psi \text{ iff for all models } \mathcal{M} \text{ for } V: \mathcal{M} \models \varphi \text{ iff } \mathcal{M} \models \psi$$

Here  $V$  is the total vocabulary of  $L$ . But we also allow models  $(M, I)$  where  $I$  is defined for a subset of  $V$ . Assuming the truth relation is reasonable, one will have

$$(12.2) \quad (M, I) \models \varphi \text{ iff } (M, I \upharpoonright V_\varphi) \models \varphi$$

i.e. only lexical items actually occurring in  $\varphi$  matter for its truth or falsity in a model. Thus, in (12.1), one can restrict attention to models for  $V_\varphi \cup V_\psi$ .

## 12.1.1 Lexical mappings

In the simplest case, a lexical mapping just maps atomic expressions in  $L_1$  to atomic expressions in  $L_2$  in a one–one fashion (such mappings will be called *strict*). This gives an obvious correspondence between the respective models. For example, suppose  $\Gamma$ , from English to Swedish, is such that  $\Gamma(\text{John}) = \text{John}$ ,  $\Gamma(\text{happy}) = \text{lycklig}$ , and  $\Gamma(\text{logician}) = \text{logiker}$ . Then the two sentences

$$(12.3) \quad \begin{array}{l} \text{a. John is happy.} \\ \text{b. John är lycklig.} \end{array}$$

<sup>1</sup> In some suitable sense of “occur”; e.g. we may want to say that the lexical item *buy* occurs in the sentence *Harry bought new shoes*.

make the same claim: if (12.3b) is true in  $\mathcal{M} = (M, I)$ , where  $I(\text{John}) = a$  and  $I(\text{lycklig}) = R$ , then, trivially, (12.3a) is true in the *corresponding* model, the one which interprets  $\text{John}$  as  $a$  and  $\text{happy}$  as  $R$ , i.e. the model

$$\Gamma\mathcal{M} = (M, I \circ \Gamma)$$

But isn't it odd to say that (12.3a) and (12.3b) are *logically* equivalent? No, the equivalence is *relative* to  $\Gamma$ , and relative to  $\Gamma$ , (12.3a) and (12.3b) trivially make the same claim. The fact that these two sentences are logically equivalent relative to  $\Gamma$  is no more remarkable than the fact that (12.3a) is logically equivalent to *itself*.

Now consider

- (12.4) a. Most logicians are happy.  
 b. De flesta logiker är lyckliga.

Again we find that (12.4b) is true in  $(M, I)$  if and only if (12.4a) is true in  $(M, I \circ \Gamma)$ , so these sentences too are logically equivalent, relative to  $\Gamma$ . This time, however, the equivalence does not only depend on  $\Gamma$ .  $\Gamma$  does not map *most* to *de flesta*; lexical mappings, as we defined them, only map predicate expressions and individual-denoting expressions. It just so happens that these two determiners denote the same quantifier, i.e. *most*. But that is a fact about the given truth relation for the respective languages. Of course, since this is the case, we *could* have mapped *most* to *de flesta*—they are, after all, lexical items. From a linguistic point of view, this seems quite natural. But we have already argued at length (Chapter 9.3) that this would generate a very non-standard notion of logical equivalence, and not the one we are usually interested in. Quantifier expressions are *constant*, and so they should *not* be interpreted in models. Hence, lexical mappings used for explicating the notion of logical equivalence between languages do not concern quantifier expressions.<sup>2</sup>

Further, by the same explanation as above, we see (given our default interpretation of *most/de flesta*) that (12.4b) is also logically equivalent (relative to  $\Gamma$ ) to, say,

- (12.5) There are more happy logicians than unhappy ones.

This equivalence could not be reduced to a mapping of lexical quantifier expressions.

However, in the general case, it may not be possible to map lexical predicate expressions to other *lexical* predicate expressions, or to have the mappings be one–one. There might be two distinct words in  $L_1$  for which only one counterpart in  $L_2$  is reasonable. Or, a lexical predicate expression in  $L_1$  may correspond to a complex predicate expression in  $L_2$ .

Such cases cannot be ruled out in advance, it seems. Note that we are not attempting to say anything about where lexical mappings come from, or what makes one such mapping reasonable and another not. Indeed, within a first-order (albeit in our extended sense) and extensional framework one should not expect to be able to say anything about that. Lexical mappings are just *given*. The intuitions that nevertheless make us pretty good judges of such mappings are intuitions about a synonymy

<sup>2</sup> For somewhat similar reasons, we do not take  $\Gamma$  to map the English *be* to the Swedish *vara*. Rather, in (12.4a) and (12.4b) these verbs are seen as part of the atomic predicate expressions.

relation that has nothing to do with logical equivalence, and that has to *precede* an account of logical equivalence between two different languages.

Suppose, for example, that *ophthalmologist* and *eye doctor* both are mapped by  $\Gamma$  to the Swedish *ögonläkare*.<sup>3</sup> We may still say that all of the following *three* sentences are logically equivalent, relative to  $\Gamma$ :

- (12.6) a. Most ophthalmologists are happy.  
 b. Most eye doctors are happy.  
 c. De flesta ögonläkare är lyckliga.

The explanation is as before: (12.6c) is true in  $(M, I)$  if and only if (12.6a), say, is true in  $\Gamma\mathcal{M} = (M, I \circ \Gamma)$ . But now there is an asymmetry: whereas in the one–one case we could as well have started with a model for the English sentence, saying instead that (12.4a) is true in  $(M, I')$  if and only if (12.4b) is true in  $(M, I' \circ \Gamma^{-1})$ , this is not possible when  $\Gamma$  is not one–one. We have to start with models for the target language of  $\Gamma$ , and find the corresponding models for the source language.

Similarly if a language  $L_1$  has a color word *bloog* with no lexical correspondent in English, but with the same extension as *blue* or *green*. Clearly, this correspondence can underwrite a notion of logical equivalence between  $L_1$  and English. Note that now  $\Gamma$  will not map *bloog* directly to an English predicate expression. It is better to think of it as mapping the atomic formula *bloog*( $x$ ) to the English  *$x$  is blue or  $x$  is green*.

There is another consequence. Normally, we would not say that (12.6a) and (12.6b) are *logically* equivalent in English. However, if logical equivalence *between* English and Swedish, relative to  $\Gamma$ , is to reduce to the respective relations of logical equivalence *inside* these languages—and recall that this is the basic requirement of a PER between two languages (Chapter 11.2.3.1)—then it seems that we have to say just that, since both are logically equivalent to (12.6c)!

What happens is that  $\Gamma$  induces a new relation of logical equivalence over  $L_1$ : truth not in the same models but in the same  $\Gamma$ -models (models of the form  $\Gamma\mathcal{M}$ ). (12.6a) and (12.6b) are equivalent in this sense. We see that  $\Gamma$  has introduced a *meaning postulate* in English: namely, one saying that eye doctors are ophthalmologists, and vice versa. In this sense, the induced logical equivalence relation is more like analytical equivalence, although its meaning postulates in this case derive from a translation into another language, i.e. the lexical mapping  $\Gamma$ . As with other relations of analytical equivalence, they reduce the number of allowed models, and hence ordinary logical equivalence *refines* analytical equivalence.

We are now ready for the general concept of a lexical mapping. We do not restrict attention to logical languages. But we still write, whenever  $P$  is an  $n$ -ary predicate expression, the corresponding atomic formula  $P(x_1, \dots, x_n)$ , though this now abbreviates some expression obtained from an actual  $L$ -sentence by replacing names with variables. English examples of such formulas could be  *$x$  is green*,  *$x$  likes  $y$* ,  *$z$  runs fast*, etc. Also, when  $\varphi(\bar{x})$  is an arbitrary  $L$ -formula in this extended sense,

<sup>3</sup> Treating, for simplicity, the compound noun *eye doctor* as an atomic predicate expression.

we continue to use  $\varphi(\bar{x})^{\mathcal{M}, \bar{x}}$  for the set of (sequences of) objects in  $\mathcal{M}$  satisfying it:

$$\varphi(\bar{x})^{\mathcal{M}, \bar{x}} = \{\bar{a} : \mathcal{M} \models \varphi(\bar{a})\}$$

Here is the definition:

### lexical mappings

A *lexical mapping*  $\Gamma$  from  $L_1$  to  $L_2$  associates with each atomic  $L_1$ -formula  $P(x_1, \dots, x_n)$  an  $L_2$ -formula, written  $P_\Gamma(x_1, \dots, x_n)$ , with the same free variables.

$\Gamma$  is *strict* if it can be seen as mapping atomic expressions in  $L_1$  to atomic expressions in  $L_2$  (of the same arity) in a one–one fashion.

If  $\mathcal{M}$  is an  $L_2$ -model, and  $V_1$  a vocabulary of predicate expressions in  $L_1$ , define a model  $\Gamma\mathcal{M}$  for  $V_1$  as follows:

$$\Gamma\mathcal{M} = (M, P^{\Gamma\mathcal{M}})_{P \in V_1} = (M, P_\Gamma(\bar{x})^{\mathcal{M}, \bar{x}})_{P \in V_1}$$

So the extension of  $P$  in  $\Gamma\mathcal{M}$  is the same as the extension of the formula  $P_\Gamma(\bar{x})$  in  $\mathcal{M}$ .

A  $\Gamma$ -*model* is an  $L_1$ -model of the form  $\Gamma\mathcal{M}$ .

Lexical mappings can be seen as special cases of what logicians call “interpretations”: interpretation of one vocabulary in another, and of one structure (model) in another. A comprehensive account is Hodges 1993: ch. 5, whose notation we follow here. Since “interpretation” is already occupied, however, we continue to use “lexical mapping”.<sup>4</sup>

In the general case,  $\Gamma$  thus maps atomic  $L_1$ -formulas to  $L_2$ -formulas. But in the strict case, we can continue to think of  $\Gamma$  as mapping atomic predicate expressions to atomic predicate expressions, and write  $\Gamma(M, I)$  as  $(M, I \circ \Gamma)$ .

### 12.1.2 Logical equivalence relative to a lexical mapping

First, let us spell out how a lexical mapping  $\Gamma$  from  $L_1$  to  $L_2$  restricts the permissible  $L_1$ -models, inducing a  $\text{PER} \Leftrightarrow_1^\Gamma$  which is refined by the given relation  $\Leftrightarrow_1$  of logical equivalence for  $L_1$ .

#### $L_1$ -equivalence relative to $\Gamma$

If  $\Gamma$  is a lexical mapping from  $L_1$  to  $L_2$  and  $\varphi, \psi$  are  $L_1$ -sentences:

$$(12.7) \quad \varphi \Leftrightarrow_1^\Gamma \psi \text{ iff for all } \Gamma\mathcal{M}\text{-models } \Gamma\mathcal{M}: \Gamma\mathcal{M} \models \varphi \text{ iff } \Gamma\mathcal{M} \models \psi.$$

<sup>4</sup> According to the classification in Hodges 1993, a lexical mapping in our sense here is a *one-dimensional, unrelativized, and injective* interpretation of a relational vocabulary  $V_1$  in  $V_2$ . A two-dimensional interpretation, e.g., takes elements of one structure to *pairs* of elements in the other, as when complex numbers are interpreted as pairs of reals (the interpretation then needs to be equipped with a *coordinate map* to this effect). A relativized interpretation takes one structure to a substructure of the other, as when rationals are interpreted as the pairs  $(m, n)$  of integers such that  $n \neq 0$ . Finally, if an interpretation is not injective, identity in one structure can correspond to an equivalence relation in the other.

Clearly  $\Leftrightarrow_1^\Gamma$  is a PER on  $L_1$ . Equally clearly:

(12.8) If  $\Gamma$  is strict, then  $\Leftrightarrow_1^\Gamma = \Leftrightarrow_1$ .

But otherwise,  $\Leftrightarrow_1$  will normally be a proper refinement of  $\Leftrightarrow_1^\Gamma$ .

We can now formulate the appropriate notion of logical equivalence across languages.

### logical equivalence between two languages

Let  $(L_1, \Leftrightarrow_1)$  and  $(L_2, \Leftrightarrow_2)$  be fixed languages as above, with their respective relations of logical equivalence, and let  $\Gamma$  be a lexical mapping from  $L_1$  to  $L_2$ . Furthermore, assume that  $L_1 \cap L_2 = \emptyset$ . Define the relation  $\Leftrightarrow_1^\Gamma$  as follows:

(12.9)  $\varphi \Leftrightarrow_1^\Gamma \psi$  iff (a)  $\varphi, \psi \in L_1$  and  $\varphi \Leftrightarrow_1 \psi$   
 or (b)  $\varphi, \psi \in L_2$  and  $\varphi \Leftrightarrow_2 \psi$   
 or (c)  $\varphi \in L_1, \psi \in L_2$ , and for all  $L_2$ -models  $\mathcal{M}$ ,  
 $\Gamma \mathcal{M} \models \varphi$  iff  $\mathcal{M} \models \psi$   
 or (d)  $\varphi \in L_2, \psi \in L_1$ , and for all  $L_2$ -models  $\mathcal{M}$ ,  
 $\Gamma \mathcal{M} \models \psi$  iff  $\mathcal{M} \models \varphi$

### Proposition 1

When  $L_1$  and  $L_2$  are disjoint,  $\Leftrightarrow_1^\Gamma$  is a PER between  $(L_1, \Leftrightarrow_1^\Gamma)$  and  $(L_2, \Leftrightarrow_2)$ .

*Proof.* We need only check that  $\Leftrightarrow_1^\Gamma$  is a PER. Reflexivity and symmetry are obvious. For transitivity, suppose  $\varphi \Leftrightarrow_1^\Gamma \psi$  and  $\psi \Leftrightarrow_1^\Gamma \theta$ . We must show  $\varphi \Leftrightarrow_1^\Gamma \theta$ . There are four cases, each with two subcases.

(a):  $\varphi, \psi \in L_1$ , so  $\varphi \Leftrightarrow_1 \psi$ .

(a1):  $\theta \in L_1$ . Then  $\psi \Leftrightarrow_1 \theta$ , so  $\varphi \Leftrightarrow_1 \theta$ , i.e.  $\varphi \Leftrightarrow_1^\Gamma \theta$ .

(a2):  $\theta \in L_2$ . Let  $\mathcal{M}$  be an  $L_2$ -model. Then  $\Gamma \mathcal{M} \models \varphi$  iff  $\Gamma \mathcal{M} \models \psi$  (since  $\varphi \Leftrightarrow_1 \psi$ ) iff  $\mathcal{M} \models \theta$  (since  $\psi \Leftrightarrow_1 \theta$ ). That is,  $\varphi \Leftrightarrow_1^\Gamma \theta$ .

(b):  $\varphi, \psi \in L_2$ . Similar to case (a).

(c):  $\varphi \in L_1$  and  $\psi \in L_2$ .

(c1):  $\theta \in L_2$ , so  $\psi \Leftrightarrow_2 \theta$ . If  $\mathcal{M}$  is any  $L_2$ -model,  $\Gamma \mathcal{M} \models \varphi$  iff  $\mathcal{M} \models \psi$  iff  $\mathcal{M} \models \theta$ , as desired.

(c2):  $\theta \in L_1$ . Take any  $\Gamma$ -model, i.e. any model of the form  $\Gamma \mathcal{M}$ . Then,  $\Gamma \mathcal{M} \models \varphi$  iff  $\mathcal{M} \models \psi$  iff  $\Gamma \mathcal{M} \models \theta$ . This shows that  $\varphi \Leftrightarrow_1^\Gamma \theta$ , and so  $\varphi \Leftrightarrow_1^\Gamma \theta$ .

(d):  $\varphi \in L_2$  and  $\psi \in L_1$ . Similar to case (c). □

Thus, we have a *bona fide* relation of logical equivalence between two disjoint languages, relative to a lexical mapping. The restriction to disjoint languages will not be a problem here. For, as we will see later in this chapter: (i) In our applications of this notion, we shall never have to consider lexical mappings between logical languages. Different such languages need not be, and often are not, disjoint, but for them we can use the standard notion of logical equivalence. (ii) We shall in fact be mostly interested in strict lexical mappings for applications (cf. section 12.5). Such mappings just switch labels for predicates, and then the assumption of disjointness is harmless.

It is still possible to question the label “logical” for relations of the form  $\Leftrightarrow^\Gamma$ .<sup>5</sup> One could claim that an equivalence relation based on meaning postulates is analytical, not logical, and consequently one could limit logical equivalence between an  $L_1$ -sentence and an  $L_2$ -sentence to the trivial case where both are logically true (or false). But such a relation would not do the work we need here.

As far as we know, the notion of logical equivalence between languages has not been discussed in the literature, and so it is not surprising if our intuitions about it are less than clear. Our position, roughly, is the following. We propose that the relation  $\Leftrightarrow^\Gamma$  for a *strict*  $\Gamma$ —and this is the case we will use in applications here—is true logical equivalence. In this case, no meaning postulates arise in the source language (see (12.8) above). Further, logical equivalence is truth in the same models: that is, truth however the corresponding (by  $\Gamma$ ) atomic predicate expressions are interpreted, provided they are interpreted *in the same way*. As we have said, in this context those expressions function merely as labels for predicates. Thus, it seems to us that, relative to a strict  $\Gamma$ ,  $\Leftrightarrow^\Gamma$  does indeed preserve the essence of logical equivalence (recall that  $\Gamma$  is supposed to be given, independently justified lexical mapping).

Furthermore, even when  $\Gamma$  is not strict, there is clearly a principled difference between ordinary meaning postulates for a given language  $L_1$ , on the one hand, and meaning postulates resulting from translation into another language  $L_2$ , on the other. The latter are not intrinsic to  $L_1$  but merely due to translation difficulties. Perhaps the label “logical” is not appropriate, but then neither is “analytical”; we should rather invent a new term for this case. However, in this book we continue to talk about logical equivalence relative to  $\Gamma$ .

Leaving now the issue of meaning postulates, let us consider another feature of lexical mappings. Such a mapping has a *direction*; it goes from one language to another. How does this square with the fact that logical equivalence between languages is a PER, hence symmetric, or ‘direction-less’?

We make two remarks about this. First, suppose that a lexical predicate expression  $P$  occurs in an  $L_2$ -sentence  $\psi$ , but not in any formula to which an atomic  $L_1$ -formula is mapped by  $\Gamma$ . Can  $\psi$  be logically equivalent to an  $L_1$ -sentence? Yes, if both  $\psi$  and  $\varphi$  are logically true, or both logically false. Then there is no requirement that their respective vocabularies be the same. The same holds for logical equivalence between

<sup>5</sup> This was pointed out to us by Tim Williamson, and the discussion that follows is inspired by his comments.



two languages; indeed, the meaning postulates induced by  $\Gamma$  may further contribute to making seemingly unrelated sentences logically equivalent (relative to  $\Gamma$ ).<sup>6</sup>

However, let us say that  $P$  occurs *essentially* in  $\psi$  if there are two  $L_2$ -models  $\mathcal{M}_1, \mathcal{M}_2$  which differ only on the interpretation of  $P$ , such that  $\psi$  is true in  $\mathcal{M}_1$  and false in  $\mathcal{M}_2$ . Then we can state:

(12.10) If  $P$  as above occurs essentially in  $\psi$ , then  $\psi$  does not stand in the relation  $\Leftrightarrow^\Gamma$  to any  $L_1$ -sentence.

The reason is that with  $\mathcal{M}_1, \mathcal{M}_2$  as above,  $\Gamma\mathcal{M}_1 = \Gamma\mathcal{M}_2$ , so the condition (12.9) (c) can never obtain. In conclusion: Yes, there is an asymmetry, and in cases like these,  $\Leftrightarrow^\Gamma$  will be only a *partial* equivalence relation on  $L_1 \cup L_2$ .

The second remark is that we could indeed go on to ask if it would be reasonable to strengthen the requirements on lexical mappings so as to make them bidirectional. Or, we can ask what happens when there is a lexical mapping  $\Gamma$  from  $L_1$  to  $L_2$ , and a mapping  $\Delta$  from  $L_2$  to  $L_1$  (which agree in some suitable sense). Some subtleties are known here from the theory of interpretations (Hodges 1993: ch. 5), but we shall have to leave the matter at this point.

Let us finally observe that a lexical mapping  $\Gamma$  generates a *partial*  $\Leftrightarrow^\Gamma$ -translation from  $L_1$  into  $L_2$ . This is because, whenever  $P(c)$ , say, is an atomic  $L_1$ -sentence, we will have

$$P(c) \Leftrightarrow^\Gamma P_\Gamma(c_\Gamma)$$

by definition, provided  $\Gamma$  also maps individual-denoting  $L_1$ -expressions to corresponding  $L_2$ -expressions denoting the same individual. (We glossed over this in the earlier definition.) When can such a mapping be extended to a total translation (preserving logical equivalence), in the sense of Chapter 11.2.4? One would guess that sentences in these languages would have to be generated in a reasonably systematic way, which also reflected their semantics; i.e. that *compositionality* would be required. We deal with this notion next.

## 12.2 COMPOSITIONALITY

We have not made compositionality part of the notion of translation, but it looks like a highly relevant extra requirement. Actually, the issue of compositionality is somewhat controversial, and there are frequent discussions of various aspects of it in the literature. Here we shall not enter into that debate. We do believe that compositionality is an important property, and that it is part of the explanation of how a seemingly infinite (or unbounded) language capacity can be handled by finite means. But in this

<sup>6</sup> Here is an illustration. Suppose the English *bachelor(x)* is mapped to the Swedish *ogift(x) och man(x)* (i.e. to ‘unmarried man’), and *unmarried(x)* is mapped to *ogift(x)*. Then *All bachelors are unmarried* will be true in all  $\Gamma$ -models. Thus, it will be logically equivalent, relative to  $\Gamma$ , to any logically true Swedish sentence.

chapter we merely formulate the notion, and see how it fits with other notions already introduced.

Even so, there is a distinction we think is important, but that is not often made clear. On the one hand, a *language* can be compositional, in the sense of having a compositional semantics. On the other hand, a *translation* of one language into another can be compositional. There are certain historical and theoretical reasons for thinking that conflating these is a good idea. Nevertheless, we shall keep them apart, and begin with the former.

### 12.2.1 Compositional languages

Consider a language as a set  $L$  of linguistic expressions. To even talk about the compositionality, or not, of  $L$ , at least two things need to be in place: first, that the  $L$ -expressions must be *structured*; second, that they must have a *semantics*, usually in the form of an association of some sort of semantic values ('meanings') with them. Only then can the Principle of Compositionality begin to make sense:

(PoC<sub>1</sub>) The meaning of a compound expression is determined by the meanings of its parts and the mode of composition.

Actually, as we have said, you don't really need meanings, but only *synonymy*, if you instead use a principle along the lines of:

(PoC<sub>2</sub>) Substitution of synonymous parts of a compound expression preserves meaning.

The main requirement on syntactic structure here is that there is a(n immediate) *constituent* relation among expressions. Such a relation is standardly induced by syntactic rules *generating* the well-formed expressions.<sup>7</sup> Abstractly, syntactic structure is then *algebraic* structure. The classical (Montague-style) approach uses total many-sorted algebras, where sorts correspond to syntactic categories. A more recent (Hodges-style) approach uses partial algebras instead, without sorts. Here partiality takes care of category constraints, so that a syntactic rule is simply undefined for arguments that don't 'fit'. We rely here on Hodges' approach, though the formal details will not matter much.<sup>8</sup>

Thus, suppose that  $L$  is generated by an algebra of the form  $\mathcal{L} = (L, A, \Sigma)$ , where  $A$  is a subset of  $L$ , the set of *atomic* (lexical) expressions, and  $\Sigma$  is a set of *syntactic rules*,

<sup>7</sup> For simplicity, we presuppose a generative perspective in what follows. But a constituent relation can also be given by grammar formats not based primarily on generation, as in 'model-theoretic syntax' or HPSG-style grammars.

<sup>8</sup> For Montague's approach, see Montague 1970c; Janssen 1986; and Hendriks 2001. For the Hodges-style framework, see Westerståhl 1998; Hodges 2001; and Westerståhl 2004; the latter paper also contains a comparison between the two approaches. Mixed variants are possible; Keenan and Stabler (2004) use partial algebras *and* primitive syntactic categories. Hodges (2003) employs a format where the properties of a constituent relation are described axiomatically, without assuming an underlying generative syntax.

or partial functions from  $L^n$  to  $L$  (for some  $n$ ). That  $L$  is generated means that each of its expressions is either atomic or the result of a finite number of rule applications.

An expression may be generated in different ways, which may correspond to its being *ambiguous*. So it is not really surface expressions that should be assigned meanings, but—for maximum generality—their generation histories or analysis trees. These are represented by terms in the (partial) *term algebra* corresponding to  $\mathcal{L}$ . For example, the term

$$(12.11) \quad \alpha(\beta(a, b, \gamma(c)), \gamma(a))$$

represents an expression obtained by applying rule  $\gamma$  to the atom  $c$  and then applying rule  $\beta$  to atoms  $a, b$  and the result of the first application, and finally applying rule  $\alpha$  to the result of that and the result of applying  $\gamma$  to  $a$ .<sup>9</sup> This term is well-defined—grammatical—if and only if the relevant partial functions are defined for the relevant arguments.

Strictly speaking, we should thus think of  $L$  as a set of analysis trees, possibly with a function *val* mapping these to their surface manifestations. For simplicity, however, we ignore this complication whenever possible.

Next, a *semantics* is a partial function  $\mu$  from  $L$  to some set  $M$ .<sup>10</sup> Elements of  $M$  can be thought of as meanings, but we make no assumption about what they are. The ( $\mu$ -)meaningful  $L$ -expressions are those in the domain of  $\mu$ .  $\mu$  induces a PER  $\equiv_\mu$  on  $L$ :

$$u \equiv_\mu v \text{ iff } \mu(u) \text{ and } \mu(v) \text{ are both defined, and } \mu(u) = \mu(v)$$

( $u, v \in L$ ). Now we can express (PoC<sub>1</sub>) as follows:

Rule( $\mu$ ) For each syntactic rule  $\alpha \in \Sigma$  there is an operation  $r_\alpha$  such that whenever  $\alpha(u_1, \dots, u_n)$  is meaningful,  $\mu(\alpha(u_1, \dots, u_n)) = r_\alpha(\mu(u_1), \dots, \mu(u_n))$ .

Note that this formulation presupposes that whenever  $\alpha(u_1, \dots, u_n)$  is meaningful, so are all of  $u_1, \dots, u_n$ . That is, it presupposes that *dom*( $\mu$ ) is closed under subterms; this is sometimes called the *Domain Principle*.

(PoC<sub>2</sub>) becomes the following principle:

Subst( $\equiv_\mu$ ) For each  $\alpha \in \Sigma$ , if  $\alpha(u_1, \dots, u_n)$  and  $\alpha(v_1, \dots, v_n)$  are both meaningful, and if  $u_i \equiv_\mu v_i$ ,  $1 \leq i \leq n$ , then  $\alpha(u_1, \dots, u_n) \equiv_\mu \alpha(v_1, \dots, v_n)$ .

Subst( $\equiv_\mu$ ) does not mention  $\mu$  but only  $\equiv_\mu$ ; indeed, it can be formulated for any PER on  $L$ .<sup>11</sup> The next fact is practically immediate:

<sup>9</sup> Note that  $\alpha, \beta$ , etc. are used here in two ways: as formal *symbols* (in (12.11)), and as the rules or *functions* these symbols stand for (in the preceding text). We could distinguish them by using  $\alpha$  for the symbol and  $\underline{\alpha}$  for the function, but usually the ambiguous use does not cause confusion.

<sup>10</sup> Sometimes, one may want to count certain ungrammatical expressions as meaningful, e.g., since such expressions are frequently used and still understood. One would then give  $\mu$  a domain stretching outside  $L$ . We ignore such complications in what follows; i.e. we restrict attention to meaningful *grammatical* expressions.

<sup>11</sup> Subst( $\equiv_\mu$ ) says that  $\equiv_\mu$  is a *congruence*. It does not presuppose the Domain Principle. A more general version of Subst( $\equiv_\mu$ ), used in Hodges 2001, allows substitution of arbitrary synonymous subterms, not just immediate ones; this is essentially (12.12) below. If *dom*( $\mu$ ) is closed under subterms, the two versions are equivalent; see Westerståhl 2004.

**Fact 2**

Given that  $\text{dom}(\mu)$  is closed under subterms,  $\text{Rule}(\mu)$  and  $\text{Subst}(\equiv_\mu)$  are equivalent.

$\text{Rule}(\mu)$  (or  $\text{Subst}(\equiv_\mu)$ ) clearly captures a core meaning of compositionality, i.e. (PoC<sub>1</sub>) or (PoC<sub>2</sub>), but it is a rather weak principle. Roughly, it has more bite, the more synonymies there are in  $L$ , but when there are few synonymies, it requires almost nothing of  $L$  and  $\mu$ . This can be seen from the following

**Fact 3**

If  $\mu$  is one-one, i.e. if there are no non-trivial synonymies in  $L$ , then  $\text{Subst}(\equiv_\mu)$  holds.

*Proof.* Since in this case,  $u_i \equiv_\mu v_i$  implies  $u_i = v_i$ ,  $\text{Subst}(\equiv_\mu)$  holds trivially.  $\square$

One may wonder how this can be the case, if  $\text{Rule}(\mu)$  is to capture the idea that the meaning of a complex expression can be ‘figured out’ from the meaning of its parts and the relevant syntactic rule of composition. Even if there are no non-trivial synonymies in  $L$ , a lot of non-trivial ‘figuring out’ may be required to get at the meanings of its complex expressions. So how can (PoC<sub>1</sub>) then be trivially true?

The answer lies in the fact that we required nothing of the semantic operations  $r_\alpha$  except that they *exist*. For a more realistic rendering of compositionality, the  $r_\alpha$  should be *computable* in some relevant sense. However, it is still useful to isolate the core meaning of compositionality as we have done, and then strengthen it by adding further requirements.

It is thus quite clear and, we hope, uncontroversial, what the essence of compositionality of a language amounts to. Our account does not rely on a particular view of grammar. The algebraic approach gives an abstract frame into which almost any current variety of grammar can be fitted. It is a general way of talking about meaningful structured expressions, which abstracts away from particular grammar formats and particular notions of meaning. The intuitive principles (PoC<sub>1</sub>) and (PoC<sub>2</sub>) are in fact formulated at such a general level, and our account here merely gives them some precision.

### 12.2.2 Compositional translation

Now let us return to the situation in Chapter 11.2.4.1, with two languages  $L_i$ ,  $i = 1, 2$ , with associated local PERs  $\sim_i$ , and a PER  $\sim$  between them. A translation  $\pi$  from  $L_1$  into  $L_2$  must respect  $\sim$ , but this alone is not enough to make it in any way systematic. For example, suppose  $\pi$  maps *Alla katter är grå* to *All cats are gray*, but also maps *Alla ankor är vita* (‘All ducks are white’) to *It is not the case that some duck is not white*. Such a translation, although truth-conditionally correct, hardly qualifies as compositional. Or does it?

The motivation for a compositionality requirement on translation is easy to see. Languages are (usually) infinite, but in order to be useful to us, a translation  $\pi$  between two languages needs to be describable in a finite manner, for example in a translation manual. Intuitively, compositionality seems to offer a way to realize this goal.

We just saw what it (minimally) means for  $L_i$  to be compositional—that  $\text{Subst}(\sim_i)$  holds—but what exactly is a compositional *translation*? A first suggestion might be:  $\text{Rule}(\pi)$ . But in this sense the translation suggested above could well be compositional. Indeed, this requirement is so weak that under very general circumstances, if there is a translation at all between  $L_1$  and  $L_2$ , there is also a compositional one. This is the content of the following proposition.

**Proposition 4**

Let  $(L_i, \sim_i)$  be as above, with a PER  $\sim$  between them, and  $\pi$  a  $\sim$ -translation from  $L_1$  to  $L_2$ . For  $u \in L_i$ , let  $[u]_i$  be the  $\sim_i$ -equivalence class of  $u$ , i.e.  $[u]_i = \{v \in L_i : u \sim_i v\}$ . Now assume that for each  $L_1$ -expression  $u$ , there are at least as many  $L_2$ -synonyms of its translation  $\pi(u)$  as there are  $L_1$ -synonyms of  $u$ . That is, assume that  $|[u]_1| \leq |[\pi(u)]_2|$ . Then there is a  $\sim$ -translation  $\pi'$  from  $L_1$  to  $L_2$  satisfying  $\text{Rule}(\pi')$ .

*Proof.* Take  $u \in \text{dom}(\pi)$ , and let  $X_u = [u]_1 \cap \text{dom}(\pi)$ .<sup>12</sup> We know that  $\text{range}(\pi) \subseteq \text{dom}(\sim_2)$  ((11.22) in Chapter 11.2.4.1). It follows that  $\pi$ , restricted to  $X_u$ , maps  $X_u$  onto  $\pi(X_u)$ , which is a subset of  $[\pi(u)]_2$ . Now take any subset of  $[\pi(u)]_2$  of cardinality  $|X_u|$ , and let  $\pi'$ , restricted to  $X_u$ , map  $X_u$  in a one–one fashion onto that subset. By our assumption, this is possible. Doing the same for all sets of the form  $X_u$  clearly results in a one–one mapping  $\pi'$  from  $\text{dom}(\pi)$  to  $\text{dom}(\sim_2)$  meeting the requirements of a translation; in particular,  $u \sim \pi'(u)$ . But since  $\pi'$  is one–one,  $\text{Rule}(\pi')$  holds by Fact 3. □

In practice, the cardinality condition in this proposition might well be satisfied. But the main point is that this condition is totally *irrelevant* to the idea of compositionality. Also, the structure of  $L_i$ -expressions has not been involved at all. We conclude that  $\text{Rule}(\pi)$  (or  $\text{Subst}(\equiv_\pi)$ ) is too weak.

$\text{Rule}(\pi)$  says nothing about the given PERs  $\sim_i$ .<sup>13</sup> An apparently more promising idea is to express the compositionality of  $\pi$  as a substitution condition involving the given PERs  $\sim_1, \sim_2$ , and  $\sim$ . Assume, then, that  $L_i$  is generated by  $\mathcal{L}_i = (L_i, A_i, \Sigma_i)$ ,  $i = 1, 2$ , and that  $A_1 \subseteq \text{dom}(\pi)$ . Consider the following condition:

$\text{Subst}(\sim_{1,2,\pi})$  For each  $\alpha \in \Sigma_1$ , if  $\alpha(u_1, \dots, u_n)$  and  $\alpha(v_1, \dots, v_n)$  are both in  $\text{dom}(\pi)$ , and if  $u_i \sim_1 v_i, 1 \leq i \leq n$ , then  $\pi(\alpha(u_1, \dots, u_n)) \sim_2 \pi(\alpha(v_1, \dots, v_n))$ .

Alas, this condition too is virtually empty, provided  $L_1$  is compositional.

**Fact 5**

If  $L_1$  is compositional, in the sense that  $\text{Subst}(\sim_1)$  holds, and if  $\pi$  is a translation from  $L_1$  to  $L_2$ , then  $\text{Subst}(\sim_{1,2,\pi})$  holds too.

<sup>12</sup> If  $u$  is a sentence,  $X_u = [u]_1$  by condition (ii) in the definition of translation in ch. 11.2.4.1 (assuming condition (11.16)), but if  $u$  is another  $L_1$ -expression,  $X_u$  might be a proper subset of  $[u]_1$ .

<sup>13</sup> It only involves the PER  $\equiv_\pi$  over  $L_1$ , where  $u \equiv_\pi v$  iff  $u$  and  $v$  have the same translation.  $\equiv_\pi$  refines  $\sim_1$  (since  $u \equiv_\pi v \Rightarrow \pi(u) = \pi(v) \Rightarrow \pi(u) \sim_2 \pi(v) \Rightarrow u \sim_1 v$ ), but it is not a very interesting relation.

*Proof.* If  $\alpha(u_1, \dots, u_n)$  and  $\alpha(v_1, \dots, v_n)$  are both in  $\text{dom}(\pi)$ , then they are both meaningful in the sense of  $\sim_1$ , by the requirements on translations. So if  $u_i \sim_1 v_i$ ,  $1 \leq i \leq n$ , it follows by  $\text{Subst}(\sim_1)$  that

$$\alpha(u_1, \dots, u_n) \sim_1 \alpha(v_1, \dots, v_n).$$

Since  $\pi$  is a translation (and thus preserves synonymy),

$$\pi(\alpha(u_1, \dots, u_n)) \sim_2 \pi(\alpha(v_1, \dots, v_n)) \quad \square$$

Let us take stock. It seems that the versions of compositionality for translations that most immediately correspond to what we have called the core version of compositionality for languages are too weak, in the sense that if there exists a translation at all, then (under certain conditions) it can always be taken to be compositional in this sense. This is not really surprising, however. Compositionality for languages was formulated without any regard for the nature of meanings; indeed, a synonymy relation suffices. But in translation from  $L_1$  to  $L_2$ ,  $L_1$ -expressions are mapped not to meanings but to  $L_2$ -expressions. It then seems reasonable *not* to disregard the structure of  $L_2$ -expressions, but that is precisely what we have done so far.

What confuses this issue is that since Montague it is not uncommon to give a semantics for a language by means of a translation into a formal language, i.e. a formalization (see Montague 1974). You map from one syntactic algebra to another, and compositionality is preservation of structure, in the sense that the mapping is a homomorphism. The target algebra comes from a logical language, already assumed to have a transparent model-theoretic semantics, and the formalization is a precise way of giving the semantics for the source language. The logical algebra can be seen as a handy intermediate step, in principle dispensable. But in the general case of translation between two arbitrary languages, there is no such assumption, and the two syntactic algebras should be equally important.

The main difference between the Montague-style approach and the Hodges style approach is not that one uses total many-sorted algebras and the other uses partial one-sorted algebras. Rather, it is that one deals with translation (albeit for the purpose of giving a semantics) between two languages, and the other with the semantics of one language. An achievement of Hodges' approach is the realization that you can say interesting things concerning compositionality independently of the nature of meanings. But if we *are* interested in compositional translation, we need a Montague-style account of compositionality. In fact, we can obtain that without about abandoning the simpler partial algebra framework of Hodges, by suitably strengthening the condition  $\text{Rule}(\pi)$ .

But note that since there is no ambition to define the semantics by means of the translation, we are still assuming that a synonymy (a PER) between the two languages is *given*, and hence that the translation must conform to that relation.

$\text{Rule}(\pi)$  says that for each syntactic rule  $\alpha$  in  $L_1$  there is a corresponding operation  $r_\alpha$  on the expressions in  $L_2$ . What we should add is simply that this operation is *definable* by means of the syntactic operations in  $L_2$ . In this sense, the little example at the beginning of this subsection is an instance of *non*-compositional translation, since

two complex  $L_1$ -expressions obtained with the same syntactic rule were translated as  $L_2$ -expressions formed by different rules. That is, the translation was not *uniform*.

To express the appropriate condition, one needs the notion of a *polynomial* in a partial algebra  $\mathcal{L} = (L, A, \Sigma)$ . This is simply a term in the corresponding term algebra with variables. Instead of giving a precise definition, we explain by means of an example.

Starting with the term (12.11), i.e.

$$\alpha(\beta(a, b, \gamma(c)), \gamma(a))$$

we can obtain, replacing some atoms by variables, the polynomial

$$p(x, y) = \alpha(\beta(x, b, \gamma(y)), \gamma(x))$$

$p(x, y)$  does not stand for a primitive operation of  $\mathcal{L}$  as  $\alpha, \beta, \gamma$  do, but it is definable from them. For every assignment of objects  $u, v$  in  $L$  to  $x, y$  it gives a well-defined object  $\mathbf{p}(u, v)$  in  $L$ , obtained in the obvious way: first apply  $\gamma$  to  $u$  and  $v$ , resulting in  $u_1$  and  $v_1$ , say; then apply  $\beta$  to  $u, b, v_1$ , resulting in  $u_2$ , and finally apply  $\alpha$  to  $u_2, u_1$ . This is assuming that all of the operations are defined for the relevant arguments. If that fails at some point,  $\mathbf{p}(u, v)$  is undefined. So  $\mathbf{p}$  is a partial operation on  $L$ , just as  $\alpha, \beta$ , and  $\gamma$  themselves may be.

Here, then, is what it means for a translation  $\pi$  from  $L_1$  to  $L_2$ , where  $L_i$  is generated by  $\mathcal{L}_i = (L_i, A_i, \Sigma_i)$  and  $A_1 \subseteq \text{dom}(\pi)$ , to be compositional:

**DefRule( $\pi$ )** For each  $n$ -ary syntactic rule  $\alpha \in \Sigma_1$  there is a polynomial  $p_\alpha(x_1, \dots, x_n)$  in the term algebra of  $\mathcal{L}_2$  such that whenever  $\alpha(u_1, \dots, u_n)$  is in the domain of  $\pi$ ,  $\pi(\alpha(u_1, \dots, u_n)) = \mathbf{p}_\alpha(\pi(u_1), \dots, \pi(u_n))$ .<sup>14</sup>

This is a syntactic condition, but in a semantic framework.  $\pi$  is a  $\sim$ -translation, so the polynomial  $p_\alpha$  which acts as a definition of the  $L_1$ -rule  $\alpha$  in  $L_2$  is required to be such that whenever  $\alpha(u_1, \dots, u_n)$  is translatable,  $\alpha(u_1, \dots, u_n) \sim \mathbf{p}_\alpha(\pi(u_1), \dots, \pi(u_n))$ .

We have the following connection between compositionality of the *languages*  $L_1$  and  $L_2$ , and compositionality of a *translation*  $\pi$  between them.

**Proposition 6**

*If there is a compositional translation from  $L_1$  to  $L_2$ , and if  $L_2$  is compositional, then so is  $L_1$ . More precisely, if  $\text{dom}(\sim_1) = \text{dom}(\pi)$ , then DefRule( $\pi$ ) together with Subst( $\sim_2$ ) implies Subst( $\sim_1$ ).*

*Proof.* (Outline) Suppose  $\alpha(u_1, \dots, u_n), \alpha(v_1, \dots, v_n)$  are in  $\text{dom}(\sim_1)$ , and hence in  $\text{dom}(\pi)$ . By DefRule( $\pi$ ), there is a polynomial  $p_\alpha(x_1, \dots, x_n)$  such that  $\pi(\alpha(u_1, \dots, u_n)) = \mathbf{p}_\alpha(\pi(u_1), \dots, \pi(u_n))$  and  $\pi(\alpha(v_1, \dots, v_n)) = \mathbf{p}_\alpha(\pi(v_1), \dots, \pi(v_n))$ . Now Subst( $\sim_2$ ) can be generalized as follows:<sup>15</sup>

<sup>14</sup> In particular,  $\mathbf{p}_\alpha$  is required to be defined for the arguments  $\pi(u_1), \dots, \pi(u_n)$ , under these circumstances. Furthermore, we are assuming a Domain Principle for each  $L_i$ , which means that if  $\alpha(u_1, \dots, u_n)$  is in  $\text{dom}(\sim_i)$ , then so is each  $u_j$ . By the definition of a translation,  $\text{dom}(\sim_1) = \text{dom}(\pi)$ , so we also have a Domain Principle for translation: whenever  $\alpha(u_1, \dots, u_n)$  is in the domain of  $\pi$ , so is each  $u_j$ .

<sup>15</sup> Provided  $\text{dom}(\sim_2)$  is closed under subterms; cf. n. 11. If not, we would replace Subst( $\sim_2$ ) by the stronger (12.12) in the definition of compositionality.

(12.12) If  $p(x_1, \dots, x_n)$  is a polynomial in (the term algebra of)  $\mathcal{L}_2$ ,  $u_i \sim_2 v_i$ ,  $1 \leq i \leq n$ , and if  $\mathbf{p}(u_1, \dots, u_n)$  and  $\mathbf{p}(v_1, \dots, v_n)$  are both defined, then  $\mathbf{p}(u_1, \dots, u_n) \sim_2 \mathbf{p}(v_1, \dots, v_n)$ .

Now, if  $u_i \sim_1 v_i$ , then  $\pi(u_i) \sim_2 \pi(v_i)$ ,  $1 \leq i \leq n$ . Hence, by the above,

$$\begin{aligned} \mathbf{p}_\alpha(\pi(u_1), \dots, \pi(u_n)) &\sim_2 \mathbf{p}_\alpha(\pi(v_1), \dots, \pi(v_n)), \text{ so} \\ \pi(\alpha(u_1, \dots, u_n)) &\sim_2 \pi(\alpha(v_1, \dots, v_n)) \end{aligned}$$

It follows that

$$\alpha(u_1, \dots, u_n) \sim_1 \alpha(v_1, \dots, v_n)$$

so  $\text{Subst}(\sim_1)$  holds. □

One cannot expect that, conversely,  $\text{Subst}(\sim_1)$  implies  $\text{Subst}(\sim_2)$  under  $\text{DefRule}(\pi)$ ; for example, there might be a ‘non-compositional part’ of  $L_2$ , outside the range of the translation  $\pi$ .

Another feature of compositional translation that we might expect is some version of the following: if each operation  $\alpha \in \Sigma_1$  is definable in  $L_2$ , then every mapping from  $A_1$  to  $L_2$  respecting  $\sim$  can be extended to a compositional  $\sim$ -translation of  $L_1$  into  $L_2$ . This is reminiscent of a familiar fact from universal algebra: if  $\mathcal{A}$  is a term algebra (or a free algebra) generated from a set  $X$ , and  $\mathcal{B}$  is any algebra of the same signature (vocabulary), then every mapping from  $X$  to  $B$  can be extended to a homomorphism from  $\mathcal{A}$  to  $\mathcal{B}$ .<sup>16</sup>

In our case however, the signatures of the two algebras may be very different, and somehow one would have to describe how syntactic categories in  $\mathcal{L}_1$  map to categories in  $\mathcal{L}_2$ . Here it seems that one would actually need primitively given categories, as with the total many-sorted framework.

We shall not enter into the details of this, but instead go back to a special case, which is the one we are mostly interested in anyway: namely, where languages have predicate expressions interpreted in models, and the relations  $\sim_1$ ,  $\sim_2$ , and  $\sim$  are logical equivalences. The mapping from atoms to  $L_2$  is then what we have called a lexical mapping. We want to describe conditions under which a lexical mapping can be extended to a compositional translation. First, however, we consider some general requirements of definitions.

### 12.3 REQUIREMENTS ON DEFINITIONS

Definitions are closely connected to uniform translation. Translation maps expressions to expressions, but if you want to do that in any systematic way, expressions of the same form had better translate similarly. The various forms of expressions usually correspond to syntactic operations by means of which complex expressions are built, so you want the operations themselves to be translatable, or, more precisely,

<sup>16</sup> See, in the context of partial algebras, Grätzer 1968: ch. 2.



you want the operations in the source language to correspond to operations in the target language, or at least be definable from such operations. That is exactly what the notion of compositional translation in the preceding subsection required. Let us try to make this notion of definability more concrete.

We lay down a few conditions of definitions, and discuss their effect by means of an example. Since there are many kinds of definitions, these conditions are partly stipulative. But they single out a natural concept which, we claim, is the reasonable one to use in the present context.

Recall the classical terminology, where the *definiendum* in a definition is the thing to be defined, and the *definiens* the thing defining it.

- (i) We consider *explicit* definitions. Such definitions are standardly required to be *non-circular*. Also, defined concepts should be *eliminable*.<sup>17</sup>
- (ii) The *definiens* in a definition is a—usually schematic—*sentence* in the relevant language. Note that it is *one* sentence, not a set of sentences.

By a schematic sentence we mean a real sentence, where at least one subexpression is regarded as a parameter or variable. This is to obtain some degree of uniformity (see section 12.5.2 below). That we chose sentences and not some other kind of expression is consonant with the focus here on sentential translations. It is not a severe restriction.

Sometimes explicit definitions are contrasted with *contextual* ones. If the latter are definitions in which the meaning of some term is given by providing a sense for each sentence containing it, but without giving an explicit sense or reference to the term itself, then that contrast is justified.<sup>18</sup> An explicit definition singles out in a given universe a relation, an operation, a quantifier, etc. (usually uniformly over each model).

<sup>17</sup> By contrast, a *recursive* definition is circular, in the sense that the defined concept appears in both the *definiens* and the *definiendum*. For an example, take the definition of the *transitive closure*  $R^+$  of a binary relation  $R$ :

$$xR^+y \iff \exists z(xRz \ \& \ zR^+y)$$

If correctly formulated (this one is), recursive definitions are perfectly all right and produce well-defined concepts (in fact, with some extra apparatus they can be reformulated as explicit ones). There are logics, stronger than *FO*, which allow some form of recursive definitions, e.g. so-called *fixed-point logics* (see e.g. Ebbinghaus and Flum 1995, and Ch. 15.2 below). They have been extensively studied in logic related to computer science, and several of the definability results presented in the following chapters have versions for these logics, though they are usually more difficult.

In natural languages one can find some instances of recursively definable concepts—e.g. the transitive closure of *parent* is *ancestor*—but, it seems, no simple general mechanisms to form recursive definitions. For this reason, and for reasons of simplicity, we restrict attention to *FO*-based logics in this book, and hence to explicit definitions.

<sup>18</sup> This is related to the famous *context principle* in Frege's *Grundlagen*, (1884), which can be taken to recommend such definitions. Frege is after a definition of the natural numbers, and at one point suggests doing it contextually by explaining the meaning of each sentence with “number words”. He then rejects this and gives an explicit definition, since he wants numbers to be objects over which one can quantify. As Dummett and others have pointed out, this presupposes that the numbers are already there in the domain one is talking about, in fact as classes: the explicit definition singles out certain classes a numbers. The problem about a correct definition is then pushed to classes instead; it was a problem that Frege struggled with, but could never solve satisfactorily.

Our trivial point in (ii) is that this can always be done by means of a sentence-like expression, rather than some other kind of expression.<sup>19</sup>

To see how our requirements on definitions work, consider the following example. Suppose Eric is a staunch defender of first-order logic  $FO$ , and argues as follows: “Infinity is definable in  $FO$  after all, for it is definable in set theory, and e.g. Zermelo-Fraenkel set theory ( $ZF$ ) is a first-order theory!” What is wrong with this argument?

The problem is that it does not distinguish a logic or a language from a *theory* formulated in that language. A definition of infinity in set theory uses the vocabulary and axioms of that theory in an essential way. More precisely, if the definition has the form

$$\text{inf}(P) \iff \psi$$

then  $\psi$  will contain, besides the one-place predicate symbol  $P$  and the usual logical apparatus of  $FO$ , also the two-place predicate symbol  $\in$  (the only non-logical symbol in  $ZF$ ). One may then choose what we called in Chapter 11.3 (when discussing analytic or necessary equivalence) the structural approach, and on top of a model  $\mathcal{M}$  build a set-theoretic structure  $V_{\mathcal{M}}$ , with the elements of  $\mathcal{M}$  as atoms or *urelements*. So  $V_{\mathcal{M}}$  also contains sets of atoms, sets of sets of atoms, functions between sets of atoms, etc., and satisfies some standard axiomatization of set theory with atoms. Then let  $\psi$  be the sentence

$$\exists x \exists f (P(x) \wedge 'f \text{ is a bijection}' \wedge 'dom(f) = P' \wedge 'range(f) = P - \{x\}')$$

where the parts in quotes stand for corresponding  $FO$ -formulas in the predicate symbols  $P$  and  $\in$ . Since a set is infinite if and only if there is a one–one correspondence between that set and the result of removing one element from it, we have, for all  $\mathcal{M} = (M, A)$ ,

$$(12.13) \quad A \text{ is infinite} \iff V_{\mathcal{M}} \models \psi$$

So in a sense this *is* a definition of infinity, but it does not satisfy our requirements: the term *inf* is *not eliminable*, but only expressed in terms of  $\in$ , and  $\in$  is not eliminable at all.

Note that it is necessary to use  $\in$  as a predicate symbol, since both  $\exists x$  and  $\exists f$  are *first-order* quantifiers in  $\psi$ , in the sense (Chapter 2.1) that they range over the individuals—atoms *and* sets—in the universe of the model  $V_{\mathcal{M}}$ .

If, on the other hand, we treat  $\exists f$  as a *second-order* quantifier, then we don't need  $\in$  or the extended universe  $V_{\mathcal{M}}$ . Second-order models are just like first-order ones (at least if there are no second-order predicate constants), but second-order quantifiers by definition range over relations (or functions) over  $M$ . Hence, instead of writing, say,  $\exists P \dots x \in P \dots$  with first-order quantification over a universe which also

Dummett 1991 contains a detailed discussion of Frege's view of the context principle and the role of contextual definitions. We mention this here only to agree with the Dummettian point that an explicit definition presupposes an independently given domain of objects.

<sup>19</sup> E.g., instead of defining *symmetric difference* (an operation on pairs of subsets of the universe) by the expression  $(A - B) \cup (B - A)$ , one can instead use the formula  $(A(x) \wedge \neg B(x)) \vee (B(x) \wedge \neg A(x))$ .

contains sets, we can write  $\exists P \dots P(x) \dots$  with second-order quantification over  $M$ . Indeed, the argument shows that infinity *is* definable in second-order predicate logic, *SO* (Chapter 2.5.1). This is uncontroversial; our claim is only that it is not definable in *FO*.

Sometimes one does consider definitions like (12.13), using extra non-logical symbols with fixed interpretations, and extended models. For example, in computer science logic, one may have reason to restrict attention to models with built-in total order, i.e. models of the form  $(\mathcal{M}, \leq)$ , where  $\mathcal{M}$  is a model in our sense and  $\leq$  is a total order on  $M$ . If one only deals with universes where one can expect to have access to some total order, this is natural. But in the general case, that would be a gratuitous assumption.

A case closer to natural language could involve *time*: suppose that only models with an ordering relation satisfying some reasonable temporal axioms were considered, and some (or all) predicates had an argument place for time. Presumably, infinity would still not be definable, since whether a predicate has an infinite extension or not at a certain time need have nothing to do with its extension at other times. But even if it were, we would merely have reduced infinity to another concept, which in turn is not eliminable.

Now suppose our staunch objector Eric is suitably impressed by these arguments, but instead tries the following line: “Infinity is still definable in *FO* because there are *FO*-sentences which are only true in infinite models!”

This could take two forms. First, let  $\varphi_n$  be an *FO* version of  $\exists_{\geq n} x P(x)$ , saying that (the interpretation of)  $P$  has at least  $n$  elements. We have, for any  $M$  and  $A \subseteq M$ ,

$$(12.14) \quad A \text{ is infinite} \iff \text{for all } n, (M, A) \models \varphi_n$$

But this violates the requirement that the definition should be one sentence, not infinitely many.<sup>20</sup>

The other possibility is to use extra non-logical symbols, though not ones with fixed interpretations, as above. For example, let  $\theta$  be an *FO*-sentence in the one-place  $P_1$  and the two-place  $P_2$  which says that  $P_2$  is a strict dense total ordering of  $P_1$ ; i.e. for all  $M$ , all  $A \subseteq M$ , and all  $R \subseteq M^2$ ,

$$(M, A, R) \models \theta \iff R \text{ is a strict dense total ordering of } A$$

A strict total ordering can only be dense (between two points there is always a third) if its domain is infinite, and conversely, on any infinite set one can always define a dense total ordering.<sup>21</sup> Thus, for any  $M$  and any  $A \subseteq M$ ,

$$(12.15) \quad A \text{ is infinite} \iff \text{there exists } R \subseteq M^2 \text{ s.t. } (M, A, R) \models \theta$$

<sup>20</sup> Unless one takes the whole right-hand side of (12.14) to be the *definiens*. But then one is back to the previous case of relying on a given mathematical structure—again a set-theoretic one including the satisfaction relation and the natural numbers—for the definition to work.

<sup>21</sup> At least if the Axiom of Choice holds.

But this definition is in even worse shape than (12.13): there is no way to eliminate the relation symbol  $P_2$ ; in particular, it cannot be treated as a symbol with a fixed interpretation.<sup>22</sup>

This sort of definability has nevertheless been studied in model theory, where it is called definability with extra predicates, or PC-definability. It is connected to interesting mathematical issues (see, for example, Hodges 1993: chs. 5.2 and 6.6). However, it concerns a particular form of definition in  $SO$ , since it existentially quantifies over relations: (12.15) could be written

$$A \text{ is infinite} \iff (M, A) \models \exists P_2 \theta$$

where  $P_2$  now functions as a second-order variable for binary relations. Again, we have a definition of infinity in  $SO$ , not in  $FO$ .<sup>23</sup>

(12.13)–(12.15) are putative definitions of infinity in  $FO$  that in different ways violate straightforward requirements on definability, the ones we use here. Indeed, infinity is *not* definable in  $FO$ , as we will see in Chapter 14.2. So let us end by giving an example of another logical language, still of the first-order kind, where it *is* definable. Recall the type  $\langle 1, 1 \rangle$  H\"artig quantifier, which says of two sets that they are of equal cardinality:

$$I_M(A, B) \iff |A| = |B|$$

In a language containing this quantifier we can again characterize infinity as preservation of size while removing one element:

$$\begin{aligned} A \text{ is infinite} &\iff \text{there is } a \in A \text{ s.t. } |A| = |A - \{a\}| \\ &\iff (M, A) \models \exists x(P(x) \wedge I_y(P(y), P(y) \wedge y \neq x)) \end{aligned}$$

This definition fulfills all requirements: it uses one sentence in the logic  $FO(I)$  obtained by adding the H\"artig quantifier to  $FO$ , and there are no extra predicate symbols, just the one corresponding to  $A$ . So infinity is indeed definable in  $FO(I)$ .

Note finally that in these definitions in logical languages, the defining sentences are clearly *schematic* as we required: they use formal predicate symbols, which can be arbitrarily interpreted in models. Such symbols are not available in natural languages. There one has to use real sentences with real predicate expressions, but still *treat* them as schematic. We will see shortly how this can be done.

## 12.4 EXTENDING LEXICAL MAPPINGS TO COMPOSITIONAL TRANSLATIONS

Return now to the circumstances described in section 12.1. We have a lexical mapping  $\Gamma$  from  $(L_1, \Leftrightarrow_1)$  to  $(L_2, \Leftrightarrow_2)$ , where  $\Leftrightarrow_i$  is logical equivalence. The  $L_i$  can be

<sup>22</sup> Suppose  $R$  is a fixed binary relation, and we consider only models  $(M, A)$  such that  $M$  contains the field of  $R$ . It is always *false* that for all such models,

$$A \text{ is infinite} \iff (M, A, R) \models \theta$$

<sup>23</sup> More exactly, we have a  $\Sigma_1^1$  definition of infinity; cf. Ch. 2.5.

formal or natural languages, but are treated as first-order languages in our extended sense, so that only atomic predicate expressions (and individual-denoting expressions) get interpreted in models.  $\Gamma$  induces a  $\text{PER} \Leftrightarrow_1^\Gamma$  on  $L_1$ , of which  $\Leftrightarrow_1$  is a refinement, and we defined logical equivalence between  $L_1$  and  $L_2$  (relative to  $\Gamma$ ) as a  $\text{PER} \Leftrightarrow^\Gamma$  between  $(L_1, \Leftrightarrow_1^\Gamma)$  and  $(L_2, \Leftrightarrow_2)$ . Our objective now is to show that if the  $L_i$  are generated by syntactic rules, codified in algebras  $\mathcal{L}_i = (L_i, A_i, \Sigma_i)$ , as in section 12.2 above, and if the  $L_1$ -rules are definable in  $L_2$ , then  $\Gamma$  extends to a  $\Leftrightarrow^\Gamma$ -translation from  $L_1$  to  $L_2$ .

A precise proof of such a fact would need to rely on specific features of the syntactic rules for the two languages. But we can outline the structure of the required argument without going into unnecessary detail.

### 12.4.1 Definable grammar rules

The choice of  $L_1$ -rules depends on whether  $L_1$  is a formal or a natural language (fragment), and on the preferred grammar format. We focus on rules involving predicate expressions and sentences. Let us look at some sample types. Consider, first, rules taking predicate expressions to sentences, or, more generally, to formulas. For instance, suppose  $\alpha_1$  applies to two one-place predicate expressions  $\delta_1$  and  $\delta_2$ , and that (roughly)

$$\alpha_1(\delta_1, \delta_2) = \text{“most } \delta_1 \text{ are } \delta_2\text{”}$$

So in this case  $L_1$  is English (or a fragment of English), and presumably one-place predicate expressions come from different parts of speech (say, intransitive verbs, nouns, and adjectives), and  $\alpha_1$  places various constraints on  $\delta_1$  and  $\delta_2$ . But these details are irrelevant here.

Next, consider a rule  $\alpha_x$ , for each variable  $x$ , such that

$$\alpha_x(\delta) = \text{“}\mathbf{Q}^R_x \delta(x)\text{”}$$

This time,  $L_1$  is a logical language, obtained by adding some quantifiers to  $FO$ , much like the ones we focus on in this book, but here with the additional twist that there is a category of one-place predicate expressions (“set terms”) and that whenever  $\delta$  is such an expression and  $y$  is a variable, “ $\delta(y)$ ” is a formula. Barwise and Cooper (1981) present quantifier languages in this form.

A final example could be a branching construction for English: say,

$$\alpha_2(\delta_1, \delta_2, \rho) = \text{“most } \delta_1 \text{ and } \delta_2 \text{ } \rho \text{ each other”}$$

What about rules involving quantifier expressions as arguments? For example, instead of  $\alpha_1$ , we could have the following more general rule:

$$\alpha_0(Q, \delta_1, \delta_2) = \text{“}Q \delta_1 \text{ are } \delta_2\text{”}$$

One reason to focus instead on one rule of type  $\alpha_1$  for each quantifier is that a rule like  $\alpha_0$  is less likely to be definable. For example, if  $L_2$  is the language English<sup>-</sup> from Chapter 11.1.2, then  $\alpha_1$  is definable (in terms of a comparative `there are more ... than ...`), but it is less clear that  $\alpha_0$  would be. Another reason is that definitions

involving quantifier variables, which would be needed to define  $\alpha_0$ , go beyond our first-order framework.

A grammar with rules such as those above will also need rules for forming predicate expressions. For example, consider a (Montague-style) rule  $\beta_n$ , applying to a one-place predicate expression  $\delta$  and a formula  $\varphi$ , such that

$$\beta_n(\delta, \varphi)$$

is either “ $\delta[x_n]$  such that  $\varphi[x_n]$ ”, where appropriate insertions of the variable  $x_n$  (or the ‘pronoun’  $he_n$ ; see Montague 1970b) have taken place, or, for a Barwise and Cooper (1981)-style formal language, a set term “ $\hat{x}_n[\delta(x_n) \wedge \varphi]$ ”.

In general, whenever  $L_i$  has a category of ( $n$ -ary) *predicate expressions*, we shall assume that it also has the means of forming, for any (simple or complex) such expression  $\delta$ , formulas that we write

$$\delta(\bar{x})$$

saying that  $\bar{x}$  satisfies the predicate in question, and similarly for sentences with individual-denoting expressions in place of the variables.

Next, we must specify an appropriate notion of definability. The rules exemplified above are rules for syntax. But  $L_1$  and  $L_2$  are already equipped with (model-theoretic) semantics. So we take a definition of a rule to be a systematic (uniform) account, in  $L_2$ , of the meaning of expressions (in  $L_1$ ) that the rule generates. By means of such a definition  $L_1$ -expressions of that form are translated into  $L_2$ . It remains to formulate the relevant notions in a precise way, and to verify that if all the rules are definable, all of  $L_1$  can be translated.

### definable operations

Take  $\alpha_1$  and  $\beta_n$  above as examples, and recall that definitions are schematic sentences without circularity or extra predicates.

A *definition* in  $L_2$  of  $\alpha_1$  is an  $L_2$ -sentence  $\theta_{\alpha_1}$  with exactly two one-place atomic predicate expressions  $P'_1, P'_2$ , such that for every  $L_1$ -sentence  $\alpha_1(P_1, P_2)$  with atomic  $P_1, P_2$ , and every model  $(M, A_1, A_2)$ ,

$$(12.16) \quad (M, A_1, A_2) \models \alpha_1(P_1, P_2) \iff (M, A_1, A_2) \models \theta_{\alpha_1}$$

Similarly, a *definition* in  $L_2$  of  $\beta_n$  is an  $L_2$ -formula  $\theta_{\beta_n}$  with one free variable and two one-place atomic predicate expressions  $P'_1, P'_2$ , such that for every predicate expression in  $L_1$  of the form  $\beta_n(P_1, P_2(x_n))$ , where  $P_i$  is atomic, and for every model  $(M, A_1, A_2)$  and every  $a \in M$ ,

$$(12.17) \quad (M, A_1, A_2) \models \beta_n(P_1, P_2(a)) \iff (M, A_1, A_2) \models \theta_{\beta_n}(a)$$

So we use atomic predicate expressions, in both languages, as schematic symbols that can be given arbitrary interpretations in models. (In formal languages, such symbols are already available, and the *same* symbols can be used in both the *definiens*

and the *definiendum*.) Clearly, such definitions must be independent of the choice of atomic predicate expressions. This requires one final piece of equipment to be in place: a notion of *substitution*.

**substitution of formulas for atomic predicate expressions**

Let  $\theta$  be an  $L_i$ -formula containing an  $n$ -ary atomic predicate expression  $P$ , and let  $\varphi$  be an  $L_i$ -formula, not containing  $P$ , with at least  $n$  free variables, of which  $\bar{x} = x_1, \dots, x_n$  correspond to the distinct argument places of  $P$ . A *notion of substitution* for  $L_i$  is a rule that under these circumstances yields a formula  $\theta[P/\varphi]$  s.t. for all models  $\mathcal{M}$  for  $V_{\theta[P/\varphi]}$ ,

$$(12.18) \quad \mathcal{M} \models \theta[P/\varphi] \iff (\mathcal{M}, \varphi(\bar{x})^{\mathcal{M}, \bar{x}}) \models \theta$$

where  $\varphi(\bar{x})^{\mathcal{M}, \bar{x}}$  interprets  $P$  (modulo appropriate assignments to eventual remaining free variables).

In the case when  $\varphi$  is a formula of the form  $\delta(\bar{x})$ , where  $\delta$  is a predicate expression, we write simply  $\theta[P/\delta]$  (in place of  $\theta[P/\delta(\bar{x})]$ ).

Substitution is furthermore required to be independent of the choice of  $P$ , in the sense that if  $P'$  is an  $n$ -ary atomic predicate expression occurring neither in  $\theta$  nor in  $\varphi$ , then

$$(12.19) \quad (\theta[P/P'])[P'/\varphi] = \theta[P/\varphi]$$

That is, if  $\theta'$  is obtained by simply replacing  $P$  by  $P'$  in  $\theta$ , then substituting  $\varphi$  for  $P$  in  $\theta$  is the same as substituting  $\varphi$  for  $P'$  in  $\theta'$ .

Similarly for simultaneous substitution  $\theta[P_1/\varphi_1, \dots, P_k/\varphi_k]$ .

This description is rough, since we have avoided all details concerning appropriate selection of variables (to prohibit clashes, unwanted bindings, etc.), but it suffices for present purposes.

We have given a minimal semantic condition that a notion of substitution must satisfy, without going into the syntax of the sentences  $\theta[P/\varphi]$ , except for the requirement (12.19). For a language with a completely described syntax, substitution could instead be defined syntactically. This is done for formal languages of various kinds; usually it tends to be somewhat intricate. The present semantic characterization has the advantage of being applicable to languages for which we do not have an exactly specified syntax, such as English, or English<sub>0</sub>, as long as we are prepared to believe that, in the cases we encounter, it would be reasonably straightforward to identify the relevant substitution instances.

**12.4.2 An extension theorem**

The result we can now get is not completely general, but uses the sample syntactic rules above. Still, the idea should be clear.

**Theorem 7**

Suppose that  $(L_i, \Leftrightarrow_i)$  are generated by  $(L_i, A_i, \Sigma_i)$  and equipped with notions of substitution,  $i = 1, 2$ , and that each  $L_1$ -rule is definable in  $L_2$ . Then every lexical mapping  $\Gamma$  from  $L_1$  to  $L_2$  can be extended to a compositional  $\Leftrightarrow^\Gamma$ -translation  $\Gamma^*$  of  $L_1$  into  $L_2$  (i.e.  $DRule(\Gamma^*)$  holds).<sup>24</sup>

*Proof.* Note that there is no requirement that  $L_1$  and  $L_2$  have the same categories or the same organization of grammars. For example,  $L_1$  could be a natural language fragment and  $L_2$  a formal language, or vice versa. What is required is that both have a category *sentence*, and it is in terms of this category that facts of definability and expressive power are stated.

$\Gamma^*$  is already defined via  $\Gamma$  for atomic  $L_1$ -formulas, and, through them, for atomic predicate expressions in  $L_1$ . It is extended inductively to complex  $L_1$ -expressions as follows, where we write  $u_{\Gamma^*}$  for the translation of the expression  $u$ .

Suppose  $u = \alpha_1(\delta_1, \delta_2)$ , where  $(\delta_1)_{\Gamma^*}$  and  $(\delta_2)_{\Gamma^*}$  are already defined. Consider the definition  $\theta_{\alpha_1}$  as in (12.16) above. By (12.19) for  $L_2$ , we may assume that  $P'_1, P'_2$  do not occur in  $(\delta_1)_{\Gamma^*}$  or  $(\delta_2)_{\Gamma^*}$ . Let

$$\alpha_1(\delta_1, \delta_2)_{\Gamma^*} = \theta_{\alpha_1}[P'_1/(\delta_1)_{\Gamma^*}, P'_2/(\delta_2)_{\Gamma^*}]$$

Again by (12.19), this is well-defined, i.e. independent of the choice of  $P'_1, P'_2$ . Similarly, if  $u = \beta_n(\delta, \varphi)$ , we let

$$\beta_n(\delta, \varphi)_{\Gamma^*} = \theta_{\beta_n}[P'_1/\delta_{\Gamma^*}, P'_2/\varphi_{\Gamma^*}]$$

With similar definitions for all the rules, this extends  $\Gamma^*$  to the whole of  $L_1$ .

By Proposition 1,  $\Leftrightarrow^\Gamma$  is a PER between  $(L_1, \Leftrightarrow_1^\Gamma)$  and  $(L_2, \Leftrightarrow_2)$ . Therefore, to show that  $\Gamma^*$  is a  $\Leftrightarrow^\Gamma$ -translation, all that remains to verify is that for all  $L_1$ -sentences  $\varphi$ ,  $\varphi \Leftrightarrow^\Gamma \varphi_{\Gamma^*}$ . It suffices to prove by induction that for all  $L_1$ -formulas  $\varphi$ , all models  $\mathcal{M}$  for  $V_{\varphi_{\Gamma^*}}$ , and all appropriate sequences  $\bar{a}$  of elements of  $M$ ,

$$(12.20) \quad \Gamma \mathcal{M} \models \varphi(\bar{a}) \iff \mathcal{M} \models \varphi_{\Gamma^*}(\bar{a})$$

In the following we suppress mention of the sequence  $\bar{a}$  (we are not dealing in detail with variables anyway).

If  $\varphi$  is atomic, (12.20) holds by the definition of a lexical mapping.

Suppose  $\varphi$  is complex, say of the form  $\alpha_1(\delta_1, \delta_2)$ . Note that the induction hypothesis entails that, writing  $\delta_i$  for  $\delta_i(\bar{x})$ ,

$$\delta_i^{\Gamma \mathcal{M}, \bar{x}} = (\delta_i)_{\Gamma^*}^{\mathcal{M}, \bar{x}}$$

<sup>24</sup> This is inspired by the Reduction Theorem (Theorem 5.3.2) of Hodges 1993, adapted to what we call lexical mappings between two different languages with different grammars. Though the context is slightly different, we think Hodges' characterization of this result as "fundamental but trivial" is still to the point.



Choose one-place atomic predicate expressions  $P_1, P_2$  in  $L_1$  not occurring in  $\delta_1$  or  $\delta_2$ , nor in  $\varphi_{\Gamma^*}$ . Then

$$\begin{aligned} \Gamma \mathcal{M} \models \alpha_1(\delta_1, \delta_2) &\Leftrightarrow (\Gamma \mathcal{M}, \delta_1^{\Gamma \mathcal{M}, \bar{x}}, \delta_2^{\Gamma \mathcal{M}, \bar{x}}) \models \alpha(P_1, P_2) \quad [\text{substitution in } L_1] \\ &\Leftrightarrow (\mathcal{M}, \delta_1^{\Gamma \mathcal{M}, \bar{x}}, \delta_2^{\Gamma \mathcal{M}, \bar{x}}) \models \alpha(P_1, P_2) \quad [\text{since } P_i \notin V_{\varphi_{\Gamma}}] \\ &\Leftrightarrow (\mathcal{M}, (\delta_1)_{\Gamma^*}^{\mathcal{M}, \bar{x}}, (\delta_2)_{\Gamma^*}^{\mathcal{M}, \bar{x}}) \models \alpha(P_1, P_2) \quad [\text{induction hyp.}] \\ &\Leftrightarrow (\mathcal{M}, (\delta_1)_{\Gamma^*}^{\mathcal{M}, \bar{x}}, (\delta_2)_{\Gamma^*}^{\mathcal{M}, \bar{x}}) \models \theta_{\alpha} \quad [\text{definability of } \alpha_1] \\ &\Leftrightarrow \mathcal{M} \models \theta_{\alpha}[P'_1/(\delta_1)_{\Gamma^*}, P'_2/(\delta_2)_{\Gamma^*}] \quad [\text{substitution in } L_2] \\ &\Leftrightarrow \mathcal{M} \models \alpha_1(\delta_1, \delta_2)_{\Gamma^*} \end{aligned}$$

Similarly for other forms of complex  $L_1$ -expressions. □

## 12.5 WHY FORMAL DEFINABILITY RESULTS MATTER FOR NATURAL LANGUAGES

This chapter and the previous one have presented a general framework for talking about translation, expressivity, and similar notions, and established a few basic facts about them. We are now ready to use that framework to explain in what way logical results that are formulated and proved for formal languages may tell us something about natural languages as well.

Here is a typical situation. We have two languages  $L_1$  and  $L_2$ , and a sentence  $\varphi$  in  $L_1$  which is claimed to be *inexpressible* in, or *untranslatable* to,  $L_2$ . So the question is:

- (a) What exactly is it that is being claimed?
- (b) How can that claim be established or proved?

One often takes these questions lightly, relying on one's intuitive knowledge of the two languages to surmise what can be said, and what cannot be said, in  $L_2$ . But such questions need not, and sometimes should not, be taken so lightly. A sloppy notion of expressivity threatens to dilute the claim into triviality, or to make it intolerably vague. And even though intuitions about what can (or cannot) be said in a language play an important heuristic role, it is all too easy to forget some complex construction in  $L_2$ . After all, in principle one needs, as Keenan (1978) said, to “effectively . . . enumerate the set of all meanings expressible in any given language” (p. 173), if one is to show that something cannot be expressed in it.

We'll discuss (a) and (b) using a very simple running example as illustration: namely, when  $L_1$  is Swedish,  $\varphi$  is the sentence

(12.21) De flesta hundar skäller.

(most dogs bark), and  $L_2$  is English<sub>0</sub> (beginning of Chapter 11.1.2), i.e. English without proportional determiners or other constructions involving comparison of cardinalities (even adverbs of quantification like `usually`). Of course, this example does not have much interest in itself, involving as it does a language which is not an actual but only a possible natural language. But it can be used to nicely illustrate some basic points.

The claim, then, is that the Swedish (12.21) cannot be expressed in English<sub>0</sub>. What does this mean, and how could we know it? We discuss the first question in the next three subsections, and the second question in section 12.5.4.

### 12.5.1 The role of lexical mappings

As we have stressed, all claims of translatability—and hence of untranslatability—are relative to some notion of synonymy, or, more precisely, to a PER between the two languages. But the formal results that one might use here employ particular logical languages and a particular PER: logical equivalence. Therefore, it makes sense to view  $L_1$  and  $L_2$ , in our case Swedish and English<sub>0</sub>, as fixed first-order languages, in the extended sense of that term used in this book (see Chapter 11.3.2.1).

Note that this does not imply any change in those languages; it just means that we choose to consider lexical predicate expressions (and individual-denoting expressions) as the only expressions that get interpreted in models.

Since  $L_1$  and  $L_2$  are viewed in this way, the respective relations of logical equivalence (truth in the same models), say  $\Leftrightarrow_1$  and  $\Leftrightarrow_2$ , are reasonably well delineated. These may not be the synonymy relations one had in mind when making the original claim about inexpressibility. But we saw in Chapter 11.3.8 that logical equivalence has a special place among the natural PERs of a language, in particular, that many of the other PERs that come up in a linguistic or cognitive context *refine* logical equivalence.

The only exception to this rule appeared to be PERs obtained by imposing meaning postulates. We will argue shortly (section 12.5.3) that meaning postulates are rather irrelevant to questions of expressibility. For the moment, just assume that the PER implicitly underlying the intuitive claim of inexpressibility was one that refines logical equivalence. Then it is often sufficient to restrict attention to logical equivalence, for if inexpressibility of  $\varphi$  in  $L_2$  holds already for logical equivalence, it certainly holds for all of its refinements. If there were an  $L_2$ -sentence  $\psi$  synonymous with  $\varphi$  in the stronger sense, then  $\psi$  would also be logically equivalent to  $\varphi$ .

Now, as we saw in section 12.1, logical equivalence *between* two different languages is relative to a lexical mapping. The next question, then, is this: Is the expressibility claim relative to a particular lexical mapping, or is it enough that *there exists* a lexical mapping allowing one to find a logically equivalent  $L_2$ -sentence?

The latter option is *not* what we want. To see this, consider the following case. Suppose  $\Gamma$  maps *skälla*( $x$ ) to *bark*( $x$ ), but *hund*( $x$ ) to *dog*( $x$ ) and  $x \neq x$ , an unsatisfiable predicate. Of course this is totally unrealistic, but the point is that if we allow lexical mappings to map atomic predicate expressions to complex predicate expressions, we have no control over what those complex expressions might be. In this case, if  $\mathcal{M} = (M, A, B)$  is any model for the vocabulary  $\{\text{dog}, \text{bark}\}$ , then  $\Gamma\mathcal{M} = (M, \emptyset, B)$ . So (12.21) can never be true in  $\Gamma\mathcal{M}$ , which means that if  $\psi$  is *any* logically false sentence in English<sub>0</sub>, then

$$\varphi \Leftrightarrow^{\Gamma} \psi$$

Even if this example is extreme, it shows that the meaning postulates induced by lexical mappings may seriously influence the relation  $\Leftrightarrow^\Gamma$ . Thus, translatability relative to *some* lexical mapping is a trivial notion: all  $L_1$ -sentences will be translatable in this sense. Likewise, translatability relative to *all* lexical mappings would be trivial too; almost no sentences except logical truths would then be translatable.

Instead, the choice of a suitable lexical mapping precedes the expressivity issues we deal with here. As we have said before, such mappings are simply regarded as given. So the expressivity claim we are looking at—that (12.21) is not logically equivalent to any sentence in English<sub>0</sub>—has to concern logical equivalence relative to a *fixed* lexical mapping  $\Gamma$ .

### 12.5.2 Uniform expressivity

The basic notion of translation (Chapter 11.2.4) does not require more than the *existence* in  $L_2$  of sentences that are  $\Leftrightarrow^\Gamma$ -equivalent to the relevant  $L_1$ -sentences. But in practice one wants more. If (12.21) was equivalent to an  $L_2$ -sentence, but

(12.22) De flesta katter jamar.

(most cats meow) was not, we would hesitate to say that (12.21) was expressible in  $L_2$ . In fact, even if (12.22) was equivalent to an  $L_2$ -sentence, but only to one with a completely different form than the translation of (12.21), we would still be hesitant. Expressibility, as we usually think of it, should be *uniform*.

In section 12.4.1 above we introduced a concept that can be used to define the required kind of uniformity: a *notion of substitution*. First, some terminology.

#### schematic sentence pairs

A pair of sentences  $(\varphi, \psi)$  such that  $\varphi \Leftrightarrow^\Gamma \psi$  is called *schematic* (relative to  $\Gamma$ ), iff  $\Gamma$  is strict (section 12.1.1) on the atomic predicate expressions of  $\varphi$  and maps these onto the atomic predicate expressions of  $\psi$ .

The idea is that in schematic sentence pairs, the atomic predicate expressions can be *used* as schematic predicate letters, if one so desires. Now uniformity is defined as follows.

#### uniform translations

Suppose  $L_1$  and  $L_2$  are both equipped with notions of substitution. A  $\Leftrightarrow^\Gamma$ -translation  $\pi$  from  $L_1$  into  $L_2$  is *uniform* iff the following condition holds: If  $(\varphi, \pi(\varphi))$  is a schematic pair, where the atomic predicate expressions of  $\varphi$  are  $P_1, \dots, P_n$ , and the corresponding (by  $\Gamma$ ) atomic predicate expressions of  $\pi(\varphi)$  are  $R_1, \dots, R_n$ , then, for any atomic predicate expressions  $P'_1, \dots, P'_n$  in  $L_1$ ,

$$(12.23) \quad \pi(\varphi[P_1/P'_1(\bar{x}_1), \dots, P_n/P'_n(\bar{x}_n)]) = \pi(\varphi)[R_1/(P'_1)_\Gamma(\bar{x}_1), \dots, R_n/(P'_n)_\Gamma(\bar{x}_n)]$$

Suppose  $\pi$  translates the Swedish (12.21) to the English (not English<sub>0</sub>) sentence Most dogs bark. (De flesta hundar skäller, Most dogs bark) is schematic. That  $\pi$  is uniform means that it should also translate (12.22) to Most cats meow, provided  $\Gamma$  maps katt to cat and jama to meow. Thus, in effect, both hund, skäller and dog, bark are treated as schematic predicate letters.

But note that (12.23) does not require that the  $(P'_i)_\Gamma$  are atomic or that  $(\varphi[P_1/P'_1(\bar{x}_1), \dots, P_n/P'_n(\bar{x}_n)], \pi(\varphi[P_1/P'_1(\bar{x}_1), \dots, P_n/P'_n(\bar{x}_n)]))$  is schematic. For an illustration we consider formal versions of these languages. Suppose the atomic formula in 'formal Swedish' farmor( $x$ ) (paternal grandmother) is mapped by  $\Gamma$  to

$$\theta(x) = \exists y \exists z (\text{mother-of}(x, z) \wedge \text{father-of}(z, y))$$

Then the Swedish De flesta farmödrar röker (most paternal grandmothers smoke), formally,

$$(12.24) \text{ de-flesta } x(\text{farmor}(x), \text{röker}(x))$$

should be translated by  $\pi$  to

$$(12.25) \text{ most } x(\theta(x), \text{smoke}(x))$$

obtained by uniform substitution from the formal translation of (12.21).

Uniform translation starts from a schematic sentence pair, where lexical predicate expressions in the source language correspond to lexical predicate expressions in the target language in a one–one fashion, but allows replacing the former by other lexical predicate expressions in the source language and the latter by the corresponding, *possibly complex*, predicate expressions in the target language.

Notice that we cannot go in the other direction. The fact that (12.24) is translated by  $\pi$  to (12.25) depends essentially on the particular lexical mapping of farmor( $x$ ), and if that predicate expression is replaced by another, it is simply irrelevant what happens when we replace mother-of( $x$ ) and father-of( $x$ ) in (12.25) by other predicate expressions. That is why the condition that  $(\varphi, \pi(\varphi))$  be schematic in the above definition of uniformity is crucial.

The present notion of uniformity expresses a minimal semantic condition that seems eminently reasonable. There is no mention of the syntactic mechanisms behind substitution. As we said, this can be an advantage if it is not clear what exactly the syntactic rules are. But if we know how sentences are constructed, we can be more specific. In section 12.2.2 we discussed an abstract notion of *compositional* translation which used the grammar rules of both target and source language, and in section 12.4 we saw how such translations could be generated from lexical mappings under certain conditions, in precisely the first-order model-theoretic case we are focusing on now.

Compositionality (of the translation) entails uniformity. This is clear for the abstract characterization of compositionality: DefRule( $\pi$ ) from section 12.2.2. The use of variables in the defining polynomial for each  $L_1$ -rule guarantees uniformity. Also, for the present special case of first-order languages (in our extended sense), it is not hard to prove, by induction on the grammar rules for  $L_1$ , the following corollary to Theorem 7.

**Fact 8**

The compositional translation  $\Gamma^*$  obtained in the proof of Theorem 7 by extending a lexical map  $\Gamma$  is uniform, in the sense of satisfying (12.23).

### 12.5.3 Meaning postulates and expressivity

Let us now give the argument promised earlier that meaning postulates are irrelevant to issues of expressivity. Think of analytical or necessary equivalence as obtained by adding meaning postulates to some given PER  $\sim$ . Then analytical equivalence is *coarser* than  $\sim$ .<sup>25</sup> So it could be that an  $L_1$ -sentence that has no logical equivalent in  $L_2$  nevertheless is analytically equivalent to some  $L_2$ -sentence. Doesn't this jeopardize the idea of focusing on logical equivalence as we are doing here?

No. The point is precisely uniformity. Meaning postulates are by definition not uniform; there is no expectation that if one replaces the atomic predicate expressions in a meaning postulate, then another meaning postulate will result.

Given meaning postulates, translation of an individual sentence could succeed 'by accident'. We have seen that, via a certain meaning postulate, our running example (12.21) could become analytically false, and hence translatable to English<sub>0</sub>. Or, more realistically, consider the Swedish

(12.26) Alla ungarlar är ogifta.

(All bachelors are unmarried.) We do not want to say that

(12.27) All unmarried men are unmarried.

is a good translation of this into English (or English<sub>0</sub>): it does not say anything about the translation of other sentences of the same form.

As soon as meaning postulates are introduced, uniformity disappears. But uniformity is an essential ingredient in typical concepts of expressivity, although we did not make it part of the most general concepts of translation and expressive power. Uniformity is what motivates disregarding meaning postulates, and focusing on logical equivalence as the basic (coarsest) synonymy relation.

The upshot is that for issues of translatability, it makes sense to restrict attention to uniform translation, i.e. to schematic sentence pairs. Is this a severe restriction, given that  $\Gamma$  cannot in general be required to be strict? No, because it seems plausible that non-schematic sentence pairs can be obtained from schematic ones by substitution. A given sentence  $\varphi$  in the source language need not have a translation  $\psi$  in the target language such that  $(\varphi, \psi)$  is schematic, because  $\varphi$  may contain primitive predicate expressions that can only be expressed by complex predicate expressions (formulas) in the target language. But it is likely that there is another sentence  $\varphi'$ ,

<sup>25</sup> This can seem confusing. But recall that we are talking about synonymy relations, and that  $\sim$  makes *more distinctions* than analytical equivalence: There will be analytically equivalent sentences which do not stand in the relation  $\sim$  to each other, and are thus *distinguished* by  $\sim$ , but not vice versa.

resulting from  $\varphi$  by replacing primitive expressions with ones that do have primitive correspondents, i.e. on which  $\Gamma$  is strict. For example, instead of *De flesta farmödrar röker* (most paternal grandmothers smoke), one considers *De flesta kvinnor röker* (most women smoke). If  $\varphi'$  has a translation  $\psi'$  such that  $(\varphi', \psi')$  is schematic, we obtain the translation of  $\varphi$  by substitution. If there is no uniform translation, there *might* still be a translation that succeeds ‘by accident’, but that is not the kind of translation we want here. Our focus is not on the translation of predicate expressions (i.e. on  $\Gamma$ ), but on the translation of quantifier expressions, and that is another reason why uniformity with respect to predicate expressions is a reasonable requirement.

#### 12.5.4 The role of formalization

Assume now that  $L_1$  and  $L_2$  are disjoint (possible) natural languages. We have seen that they can be regarded as first-order in the sense that only predicate expressions and individual-denoting expressions are interpreted in models, and thus are equipped with standard relations  $\Leftrightarrow_1$  and  $\Leftrightarrow_2$  of logical equivalence. Furthermore, a lexical mapping  $\Gamma$  from  $L_1$  to  $L_2$  is assumed to be given, which (section 12.1) results in a relation  $\Leftrightarrow^\Gamma$  of logical equivalence *between*  $L_1$  and  $L_2$ .

We also assume that  $L_1$  and  $L_2$  can be *formalized*. In Chapter 11.2.4.3 we defined formalization as translation into logical languages, subject to certain conditions yet to be specified. The logical languages here are of the form  $FO(Q_1, Q_2, \dots)$ , and we may assume they have the same predicate symbols, and therefore the same relation, denoted  $\Leftrightarrow$ , of logical equivalence (truth in the same models).

Our goal is to show that, under reasonable conditions, the claim that an  $L_1$ -sentence is inexpressible in  $L_2$  transfers via the formalization to the corresponding logical languages (where it may become susceptible to a rigorous *proof*). The idea is straightforward, but it may be worthwhile to spell out the details, and we now have the machinery to do so.

We saw in Chapter 11.2.4.3 that  $\Leftrightarrow$  can be pulled back to  $L_1$  and  $L_2$  along the respective mappings, and (Theorem 6) that a reasonable necessary condition for these mappings to be formalizations is that this pulling back of  $\Leftrightarrow$  results precisely in  $\Leftrightarrow_1$  and  $\Leftrightarrow_2$ , respectively. Thus, we assume that

(\*)  $\pi_i$  is a  $\Leftrightarrow^{[\pi_i]}$ -translation of  $(L_i, \Leftrightarrow_i)$  into a logical language  $(L_i^f, \Leftrightarrow)$ ,  $i = 1, 2$ .

Recall that  $\Leftrightarrow^{[\pi_i]}$  is the uniquely determined (by Theorem 6) PER between  $L_i$  and  $L_i^f$  that the formalization  $\pi_i$  respects.

In our running example, we could assume that  $L_1^f$  is  $FO(\textit{most})$ , since we need formalize only a fragment of the source language to which (12.21) belongs. For purposes of illustration, assume that  $L_2^f$  is just  $FO$ .<sup>26</sup>

<sup>26</sup> The target language must be formalized more carefully, since the aim is to prove that no target language sentence expresses the same meaning as the given source language sentence. Because  $L_2 = \text{English}_0$  lacks all comparative constructions, even proportional quantifiers, we may assume for the sake of illustration that  $L_2^f$  is  $FO$ .

Now observe that  $\pi_i$  can also be seen as a lexical mapping (via its mapping of atomic sentences). This leads us to the following two further assumptions:

- (I)  $\pi_1 = \pi_2 \circ \Gamma$
- (II)  $\Gamma$ ,  $\pi_1$ , and  $\pi_2$  are all strict.

The motivation for the strictness of  $\Gamma$  was given at the end of the preceding subsection: we are not really assuming this in general, but rather restricting attention to the fragment of the languages considered on which  $\Gamma$  is strict. Next, as a lexical mapping,  $\pi_i$  maps atomic predicate expressions of  $L_i$  to formal predicate symbols of  $L_i^f$ . Clearly, there can be no reason whatever not to map distinct predicate expressions to distinct formal symbols. Also, since  $\Gamma$  maps atomic  $L_1$ -expressions to atomic  $L_2$ -expressions in a one–one fashion, there is no loss of generality in assuming that if  $\pi_1$  maps an atomic predicate expression  $P$  to a formal predicate symbol, then  $\pi_2$  maps the image of  $P$  under  $\Gamma$  to the *same* predicate symbol. That is,  $\pi_1 = \pi_2 \circ \Gamma$ .

Finally, recall from section 12.1.2 above that, as a lexical mapping,  $\pi_i$  generates a PER, written  $\Leftrightarrow^{\pi_i}$ , of logical equivalence relative to  $\pi_i$ , between  $L_i$  and  $L_i^f$ . The last requirement we need is that the two PERs  $\Leftrightarrow^{\pi_i}$  and  $\Leftrightarrow^{[\pi_i]}$  are suitably related. In fact, under the final assumption we shall make, they become identical.

This assumption is that the formalization is set up so that truth of an  $L_i$ -sentence  $\varphi$  in a model is the same as truth of  $\pi_i(\varphi)$  in the corresponding model. In other words, it is simply the assumption that the formalizations preserve truth conditions:

- (III) For any  $L_i$ -sentence  $\theta$  and any model  $\mathcal{M}$  for the predicate symbols in  $\pi_i(\theta)$  ( $i = 1, 2$ ),  $\mathcal{M} \models \pi_i(\theta)$  iff  $\pi_i\mathcal{M} \models \theta$ .

Here  $\models$  is the truth relation for logical languages, and  $\models_i$  is the truth relation for  $L_i$ . Recall that if  $\mathcal{M} = (M, I)$ , then  $\pi_i\mathcal{M} = (M, I \circ \pi_i)$ . Also, by definition (section 12.1.2 above), (III) amounts to the condition

$$\theta \Leftrightarrow^{\pi_i} \pi_i(\theta)$$

holding for any  $L_i$ -sentence  $\theta$ .

To express the import of (III), it is convenient to define the following PER:

- (12.28) For  $\varphi, \psi$  in  $L_1^f \cup L_2^f$ , define:  $\varphi \Leftrightarrow^{[\pi_1, \pi_2]} \psi$  iff  $\pi_i(\varphi) \Leftrightarrow \pi_j(\psi)$ , where  $i$  ( $j$ ) is 1 if  $\theta$  ( $\psi$ ) is in  $L_1$  and 2 if it is in  $L_2$ .

**Lemma 9**

Conditions (I)–(III) entail that  $\Leftrightarrow^\Gamma = \Leftrightarrow^{[\pi_1, \pi_2]}$  and that  $\Leftrightarrow^{[\pi_i]} = \Leftrightarrow^{\pi_i}$ .

*Proof.* If  $\varphi, \psi \in L_i$ ,  $\varphi \Leftrightarrow^\Gamma \psi$  iff  $\varphi \Leftrightarrow_i \psi$  (by definition) iff  $\pi_i(\varphi) \Leftrightarrow \pi_i(\psi)$  (since  $\pi_i$  is a translation and hence preserves synonymy; chapter 11.2.4.2) iff  $\varphi \Leftrightarrow^{[\pi_1, \pi_2]} \psi$ . Suppose instead that  $\varphi \in L_1$  and  $\psi \in L_2$  (the case  $\varphi \in L_2$  and  $\psi \in L_1$  is symmetric).  $\varphi \Leftrightarrow^\Gamma \psi$  is the condition that for all  $L_2$ -models  $\mathcal{M}$ ,

$$(12.29) \quad \Gamma \mathcal{M} \models_1 \varphi \text{ iff } \mathcal{M} \models_2 \psi$$

If  $\mathcal{M} = (M, I)$  is a model for the logical vocabulary, we have, by strictness (condition (II)), that  $\pi_2 \mathcal{M} = (M, I \circ \pi_2)$ , and hence, by (I),

$$\Gamma(\pi_2 \mathcal{M}) = (M, I \circ \pi_2 \circ \Gamma) = (\pi_2 \circ \Gamma) \mathcal{M} = \pi_1 \mathcal{M}$$

Now  $\varphi \Leftrightarrow^{[\pi_1, \pi_2]} \psi$  means that  $\pi_1(\varphi) \Leftrightarrow \pi_2(\psi)$ ; that is, for all models  $\mathcal{M}$  for the logical vocabulary,  $\mathcal{M} \models \pi_1(\varphi)$  iff  $\mathcal{M} \models \pi_2(\psi)$ . If (12.29) holds, we obtain, using the above and condition (III), for any such model  $\mathcal{M}$ :

$$\mathcal{M} \models \pi_2(\psi) \text{ iff } \pi_2 \mathcal{M} \models_2 \psi \text{ iff } \Gamma(\pi_2 \mathcal{M}) \models_1 \varphi \text{ iff } \pi_1 \mathcal{M} \models_1 \varphi \text{ iff } \mathcal{M} \models \pi_1(\varphi)$$

The converse direction is similar. This proves the first claim of the lemma. The second claim is proved similarly.  $\square$

Now we have what is needed to show that the envisaged strategy for proving inexpressibility works.

### Proposition 10

*Suppose  $\varphi$  is an  $L_1$ -sentence, such that it can be proved that  $\pi_1(\varphi)$  is not logically equivalent to any  $L_2^f$ -sentence in the same predicate symbols. If conditions (I)–(III) hold, it follows that  $\varphi$  does not stand in the relation  $\Leftrightarrow^\Gamma$  to any  $L_2$ -sentence  $\psi$  such that the pair  $(\varphi, \psi)$  is schematic. Thus, there is no uniform  $\Leftrightarrow^\Gamma$ -translation of a part of  $L_1$  containing  $\varphi$  into  $L_2$ .*

*Proof.* Suppose, for contradiction, that  $\varphi$  is  $\Leftrightarrow^\Gamma$ -equivalent to an  $L_2$ -sentence  $\psi$  such that the pair  $(\varphi, \psi)$  is schematic. Thus,  $\varphi \Leftrightarrow^{[\pi_1, \pi_2]} \psi$ , i.e.  $\pi_1(\varphi) \Leftrightarrow \pi_2(\psi)$ . But this contradicts the assumptions, since it follows from them that  $\pi_1(\varphi)$  and  $\pi_2(\psi)$  have the same predicate symbols.  $\square$

In our example, suppose (12.21) is formalized in  $L_1^f = FO(\text{most})$  as

$$(12.30) \quad \text{most } x(P_1(x), P_2(x))$$

and  $L_2^f = FO$ . Then we do have the result—to be proved in Chapter 14.2—that (12.30) is not logically equivalent to any  $FO$ -sentence in the vocabulary  $\{P_1, P_2\}$ . So provided (I)–(III) hold, which seems unproblematic, we can conclude that none of the infinitely many sentences of English<sub>0</sub> in the vocabulary  $\{\text{dog}, \text{bark}\}$  will do as a translation of (12.21).

## 12.5.5 Summing up

Suppose we want to know if a sentence  $\varphi$  in a language  $L_1$  is expressible in another language  $L_2$ , or, more generally, if there is a translation of a set of  $L_1$ -sentences into  $L_2$ . Concerning a *negative* answer, we made the following points:

- Translation only makes sense relative to a synonymy relation (a PER) between  $L_1$  and  $L_2$ . However, we don't have to consider all possible such PERs, but only the coarsest one, logical equivalence.



- We are not attempting (here) to explain how predicate expressions are translated. Rather, a lexical mapping  $\Gamma$  is assumed to be given, mapping some atomic predicate expressions of  $L_1$  to atomic predicate expressions of  $L_2$ , and others perhaps to complex predicate expressions of  $L_2$ .
- We focus on uniform translation, so that a translation of a particular sentence into another serves as a schema, allowing one to replace predicate symbols according to  $\Gamma$ . A consequence is that, once  $\Gamma$  is given, meaning postulates become irrelevant to translation (although they may play a role in finding a suitable  $\Gamma$ ).
- If there are grammars for the two languages (or for relevant fragments of them), one wants translation to be compositional, a requirement which is stronger than uniformity.
- To apply facts from logic, we formalize both  $L_1$  and  $L_2$ , i.e. translate them into logical languages. This induces relations of logical equivalence between  $L_1$ ,  $L_2$ , and the logical languages.
- In order to establish that  $\varphi$  is not (uniformly) translatable into  $L_2$ , it provably suffices to show, given certain natural conditions on the formalizations and the PERs they induce, that the formalization of  $\varphi$  is not expressible in the logical language formalizing  $L_2$ .

Negative facts of expressibility are ‘infinite’: one must show that none of in principle infinitely many sentences works as a translation. But many such facts are known for logical languages, and, under the circumstances indicated, some of them can be transferred to natural languages. Logical expressibility and its linguistic applications are the subject of the remaining chapters.

What about *positive* facts? An excursion into logic *might* be helpful here too, but there are some caveats.

Suppose one can show that the formalization of  $\varphi$  is logically equivalent to some sentence in the language formalizing  $L_2$ . First, that sentence might not be the formalization of anything in  $L_2$ ! The logical language may allow constructions with no natural counterparts in  $L_2$ , in which case nothing about  $L_2$  follows.

But suppose the logical equivalent of  $\varphi$  is in fact the formalization of an  $L_2$ -sentence  $\psi$ . Under the same circumstances as above (conditions (I)–(III)), we may then conclude that  $\varphi$  is logically equivalent to  $\psi$ . But—and this is the second caveat—we only get logical equivalence. Some logical equivalences are linguistically less natural, or cognitively hard to grasp, as we have seen (Chapter 11.3). In other words, there is no guarantee of a translation relative to stronger notions of synonymy. Still, the logical equivalent *may* be adequate, or at least it may serve as an indication that it could be worthwhile to look for a more natural translation.

In the final part of the book, we now turn our attention to the logical languages obtained from *FO* by adding quantifiers, and to techniques and results concerning their expressive power. Via formalization, such results can be directly relevant to issues of expressive power for natural languages.



PART IV  
LOGICAL RESULTS ON  
EXPRESSIBILITY WITH  
LINGUISTIC APPLICATIONS



# 13

## Definability and Undefinability in Logical Languages: Tools for the Monadic Case

Already in Chapter 2 (sections 2.3 and 2.4) we described how new quantifiers could be added syntactically and semantically to  $FO$ . There is a point in being quite precise about this, not least when dealing with matters of definability. To prove a positive definability result, one usually has to exhibit the defining sentence, and it helps if this sentence belongs to a well-defined language, in order to see if it fills the requirements we made on definitions in Chapter 12.3. To prove a negative *undefinability* result, on the other hand, one needs to verify that *no* sentence works as a definition. Then it is crucial that the eligible sentences are generated by a precise grammar and have precise truth conditions.

Thus, section 13.1 recalls the definition of a logic of the form  $FO(Q_1, Q_2, \dots)$ , the already introduced (Chapter 11.2.4) notion of relative expressive power for such logics, and two measures of the syntactic complexity of their formulas, which will be used to prove claims about *all* formulas of a given logic by induction. Section 13.2 presents the appropriate notion of definability of a quantifier in a logic, and observes that relative expressive power can be cashed out in terms of definability; indeed, this is an especially simple instance of Theorem 7 in Chapter 12.4. A number of examples of definable quantifiers are given, and the crucial role of  $ISOM$  in logical languages is pointed out.

The bulk of the chapter, however, concerns *undefinability*. Section 13.3 introduces the principal strategy for proving that a quantifier is not definable in a given logic, and section 13.4 presents the main tool for implementing that strategy: the Ehrenfeucht–Fraïssé technique. A detailed proof that this technique works is given at the end.

This chapter is more technical than the previous ones. As will be explained below, *using* the tools to prove undefinability is often easy. The reader interested primarily in applications can thus focus on the notions introduced, and skip the details of the proofs, at least at first reading.

### 13.1 LOGICS WITH QUANTIFIERS

Below,  $Q$  is a type  $\langle 1, 1 \rangle$  quantifier, but the definitions that follow are easily emended to other types.

**logics**

The logical language, or better, the *logic*  $FO(Q)$  is given by, first, adding ( $Q$ -syn) to the usual formation rules for  $FO$ :

( $Q$ -syn) If  $\varphi, \psi$  are formulas and  $x$  is a variable, then  $Qx(\varphi, \psi)$  is a formula.

Here  $Qx$  binds free occurrences of  $x$  in  $\varphi$  and in  $\psi$ . Second, the truth definition for  $FO$  is extended with the clause:

( $Q$ -sem) If  $\varphi = \varphi(x, y_1, \dots, y_n) = \varphi(x, \bar{y})$  and  $\psi = \psi(x, \bar{y})$  have at most the free variables shown,  $\mathcal{M}$  is a model, and  $a_1, \dots, a_n = \bar{a}$  a sequence of elements of  $M$ , then

$$\mathcal{M} \models Qx(\varphi(x, \bar{a}), \psi(x, \bar{a})) \text{ iff } Q_{\mathcal{M}}(\varphi(x, \bar{a})^{\mathcal{M}, x}, \psi(x, \bar{a})^{\mathcal{M}, x})$$

In  $FO(Q)$ , one quantifier is added, but we can in the same way add several, obtaining logics  $FO(Q_1, \dots, Q_m)$  or even  $FO(\{Q_n : n \in I\})$ , where  $I$  may be infinite, and the  $Q_n$  are quantifiers of any type.<sup>1</sup>

This is the official notion of a logic in the present and the following chapters. The general idea is that a logic has a syntax, i.e. a grammar which generates its well-formed sentences, and a semantics, i.e. a truth relation between models and sentences.<sup>2</sup> In the present case, both the syntax and the semantics are obtained by adding clauses for each additional quantifier to the standard account of  $FO$ .

In Chapter 11.2.4 we defined the notion of relative expressive power between languages, in terms of the existence of a translation that preserves some notion of synonymy. For logics, this notion is standard logical equivalence, i.e. truth in the same models. Thus, it is clear what  $L_1 \leq L_2$ ,  $L_1 < L_2$ , and  $L_1 \equiv L_2$  mean, when  $L_1$  and  $L_2$  are logics in the present sense. To repeat:

- $L_1 \leq L_2$  iff each  $L_1$ -sentence is logically equivalent to  $L_2$ -sentence.

Proving general statements about all formulas in a logic  $L$  often relies on induction over some measure of complexity of such formulas: one first proves the statement for all formulas of complexity 0 (usually the atomic ones), and then that if the statement holds for formulas of complexity less than  $n$ , it also holds for compound formulas of

<sup>1</sup> The syntax of variable-binding can be varied in many ways. In linguistics, for formulas used to express quantified sentences in a natural language, notations like  $[Qx : \varphi]\psi$  are often used, emphasizing that  $\varphi$  is the *restriction* of the quantifier and  $\psi$  its *scope*. But the present notation works for all types. Also, as pointed out in Ch. 2.4, we could have used two variables instead of one, writing  $Qx, y(\varphi, \psi)$  instead (so  $x$  is bound in  $\varphi$  and  $y$  in  $\psi$ ).

<sup>2</sup> There are various notions of a *model-theoretic* or *abstract logic* in the literature; see e.g. Barwise and Feferman 1985. Roughly, a logic consists of a class  $S$  of sentences and a truth relation  $\models$  between elements of  $S$  and models in some specified class, such that  $(S, \models)$  satisfies some basic structural requirements. These requirements (and much more) automatically hold for logics in the sense here, where  $S$  is the set of sentences (or formulas) generated by first-order means using some chosen quantifier symbols as variable-binding operators,  $\models$  is given by the corresponding Tarski-style truth definition, and the models are all first-order.

complexity  $n$ . The simplest complexity measure is what is often called the *degree*: the number of occurrences of logical symbols (except the identity symbol) in a formula. For example, the degree of following formula is 7.

$$(13.1) \quad \exists x P(x) \wedge \forall z (P(z) \rightarrow Qx(x = x, \forall y (R(x, y, z) \rightarrow y = z)))$$

A proof that a property  $\mathbf{P}$  holds of all  $L$ -formulas by induction over the degree then proceeds as follows. First, in the *base step*, one shows that  $\mathbf{P}$  holds for all atomic formulas. Second, the *induction step* consists in showing that if  $\varphi$  and  $\psi$  satisfy  $\mathbf{P}$ , then so do  $\neg\varphi$ ,  $\varphi \wedge \psi$ ,  $\varphi \vee \psi$ ,  $\varphi \rightarrow \psi$ , as well as  $\exists x\varphi$ ,  $\forall x\varphi$ , and  $Qx(\varphi, \psi)$ . Often, matters are further simplified by taking, say,  $\neg$ ,  $\wedge$ , and  $\exists$  as primitive, so that the clauses for  $\vee$ ,  $\rightarrow$ , and  $\forall$  can be omitted in the proof.

We will also have use for another complexity measure on formulas, their so-called *quantifier rank*, which is the number of *nestings* of quantifiers that occur in a formula. For example, although (13.1) has four quantifier symbols, its quantifier rank is 3. Here is a precise inductive definition of the notion of quantifier rank, for the case of a logic  $FO(Q)$  with  $Q$  of type  $\langle 1, 1 \rangle$ :

#### quantifier rank

1.  $qr(\varphi) = 0$ , if  $\varphi$  is atomic
2.  $qr(\neg\varphi) = qr(\varphi)$
3.  $qr(\varphi \wedge \psi) = qr(\varphi \vee \psi) = qr(\varphi \rightarrow \psi) = \max(qr(\varphi), qr(\psi))$
4.  $qr(\exists x\varphi) = qr(\forall x\varphi) = qr(\varphi) + 1$
5.  $qr(Qx(\varphi, \psi)) = \max(qr(\varphi), qr(\psi)) + 1$

## 13.2 DEFINABILITY

The notion of definability is closely tied to expressive power. Below is a precise account of what it means for a quantifier  $Q$  to be definable in a logic  $L$ . We take a quantifier of type  $\langle 1, 1, 2 \rangle$  as an example, but the idea extends to any type.

#### definability of $Q$ in $L$

$Q$  is *definable* in  $L$  iff there is an  $L$ -sentence  $\psi$ , containing two one-place predicate symbols and one two-place predicate symbol, such that, for all  $M$  and all  $A, B \subseteq M$  and  $R \subseteq M^2$ ,

$$Q_M(A, B, R) \iff (M, A, B, R) \models \psi$$

Note how the vocabulary of the defining sentence corresponds exactly to the type of the quantifier (see Chapter 12.3).

In Chapter 12 we said that, under favorable circumstances, questions of relative expressive power boil down to questions of definability of logical constants. Here is a statement of this fact for the present notion of logic.

**Proposition 1**

$FO(Q_1, \dots, Q_n) \leq L$  if and only if each  $Q_i$  is definable in  $L$ .

This is clear from left to right, and in the other direction one can use a routine induction. But note that the right-to-left direction is a special case of Theorem 7 in Chapter 12.4: that  $Q_i$  is definable in  $L$  in the present sense entails that the formation rule for  $Q_i$  is an operation definable in  $L$  in the sense of Chapter 12.4.1. In this case the non-logical vocabulary of the two languages is the same, so the mapping  $\Gamma$  in Theorem 7 is the identity mapping. The PERs involved are just standard logical equivalence, and languages of the form  $FO(Q_1, \dots, Q_m)$  have the standard notion of substitution. So the theorem says that there is a (compositional) translation  $\Gamma^*$  of  $FO(Q_1, \dots, Q_m)$  into  $L$ , i.e. that  $FO(Q_1, \dots, Q_m) \leq L$ .

An even more obvious property of definability is *transitivity*.

**Fact 2**

If  $Q$  is definable in  $FO(Q_1, \dots, Q_m)$ , and each  $Q_i$  is definable in  $L$ , then  $Q$  is definable in  $L$ .

To get some more feeling for the concept of definability, we go through some easy first examples.

### 13.2.1 Examples of definability

*Some* is definable from the existential quantifier, since

$$\text{some}_M(A, B) \iff A \cap B \neq \emptyset \iff \exists_M(A \cap B)$$

Using the above notion of definability, we can also write this:<sup>3</sup>

$$\text{some}_M(A, B) \iff (M, A, B) \models \text{some } x(A(x), B(x))$$

or simply say that the following sentence is valid:

$$\text{some } x(A(x), B(x)) \leftrightarrow \exists x(A(x) \wedge B(x))$$

Likewise, *all* is definable from  $\exists$ ,

$$\text{all } x(A(x), B(x)) \leftrightarrow \neg \exists x(A(x) \wedge \neg B(x))$$

as is, say,  $\exists_{\leq 2}$ :

$$\exists_{\leq 2} x A(x) \leftrightarrow \forall x \forall y \forall z (A(x) \wedge A(y) \wedge A(z) \rightarrow x = y \vee y = z \vee x = z)$$

<sup>3</sup> Here and below we follow the convenient and very slight abuse of language that consists in using the same letter ( $A, B, \dots$ ) for a predicate symbol and the set or relation it denotes.



Thus, we have

$$(13.2) \quad FO \equiv FO(\text{some}) \equiv FO(\text{all}) \equiv FO(\exists_{<2})$$

Recall the Hartig quantifier  $I$  mentioned earlier, and the quantifier  $MO$  with  $MO_M(A, B) \Leftrightarrow |A| > |B|$ . Obviously  $I$  is definable in terms of  $MO$ ,

$$I_M(A, B) \Leftrightarrow |A| \leq |B| \ \& \ |B| \leq |A| \Leftrightarrow \neg MO_M(A, B) \ \& \ \neg MO_M(B, A)$$

(the converse does not hold). The first of the following two facts was explained in Chapter 12.3; the second is obvious.

$$(13.3) \quad Q_0 \text{ is definable from } I.$$

(13.4) *most* is definable from  $MO$  (and Boolean operations):

$$\text{most}_M(A, B) \iff |A \cap B| > |A - B| \iff MO_M(A \cap B, A - B)$$

We will see later that *more* is not definable from *most*. But note that if  $A \cap B$  is *finite*, we have (cf. Chapter 11.3.4)

$$|A| > |B| \Leftrightarrow |A - B| + |A \cap B| > |B - A| + |A \cap B| \Leftrightarrow |A - B| > |B - A|$$

which means that in this case,

$$MO \ x(A(x), B(x)) \leftrightarrow \text{most } x((A(x) \wedge \neg B(x)) \vee (B(x) \wedge \neg A(x)), A(x))$$

Let the right-hand side of this equivalence be  $\psi$ . Now if  $A \cap B$  is instead *infinite*, then  $|A|$  is the maximum of  $|A - B|$  and  $|A \cap B|$ , and  $|B|$  is the maximum of  $|B - A|$  and  $|A \cap B|$ . It follows that in this case,

$$|A| > |B| \iff |A - B| > |B - A| \ \& \ |A - B| > |A \cap B|,$$

which means that

$$MO \ x(A(x), B(x)) \leftrightarrow \psi \ \wedge \ \text{most } x(A(x), \neg B(x))$$

Now this means that if we also have the quantifier  $Q_0$  at our disposal, we can actually define  $MO$ :

$$MO \ x(A(x), B(x)) \leftrightarrow \psi \ \wedge \ (Q_0 x(A(x) \wedge B(x)) \rightarrow \text{most } x(A(x), \neg B(x)))$$

So  $MO$  is definable from *most* and  $Q_0$  together, and we have proved the following not completely obvious proposition.

**Proposition 3**

$$FO(MO) \equiv FO(Q_0, \text{most})$$

Consider now the type  $\langle 1, 1, 1 \rangle$  quantifier *more-than*, which is denoted by a two-place English determiner. As we pointed out in Chapter 4.7, *more-than* and  $MO$  are interdefinable:

$$\text{more-than}_M(A, B, C) \iff |A \cap C| > |B \cap C| \iff MO_M(A \cap C, B \cap C)$$

$$|A| > |B| \iff \text{more-than}_M(A, B, M)$$

Hence the following fact.

**Fact 4**

$$FO(\text{more\_than}) \equiv FO(MO)$$

We end with some general facts about the expressive power of relativization (Chapter 4.4). It is significant that a relativized quantifier often has greater expressive power than its unrelativized version—more about this later. The relativized quantifier is always at least as strong.

**Fact 5**

$Q$  is definable in  $FO(Q^{\text{rel}})$ , for  $Q$  of any type.

*Proof.* Say  $Q$  is of type  $\langle 1, 1 \rangle$ , so  $Q^{\text{rel}}$  is given by  $(Q^{\text{rel}})_M(A, B, C) \Leftrightarrow Q_A(A \cap B, A \cap C)$ . Thus,  $Q_M(A, B) \Leftrightarrow (Q^{\text{rel}})_M(A, B)$ ; that is,

$$Qx(A(x), B(x)) \Leftrightarrow Q^{\text{rel}}_x(x = x, A(x), B(x))$$

is valid. □

Next, although relativizing once may increase expressive power, doing it once more has no effect.

**Fact 6**

$(Q^{\text{rel}})^{\text{rel}}$  is definable in  $FO(Q^{\text{rel}})$ .

*Proof.* We noted in Chapter 4.4 that relativizing first to one set and then to another is the same as relativizing once to their intersection. For example, when  $Q$  is of type  $\langle 1, 1 \rangle$ , one calculates:

$$((Q^{\text{rel}})^{\text{rel}})_M(A, B, C, D) \iff (Q^{\text{rel}})_M(A \cap B, C, D)$$

That is,

$$(Q^{\text{rel}})^{\text{rel}}_x(A(x), B(x), C(x), D(x)) \Leftrightarrow Q^{\text{rel}}_x(A(x) \wedge B(x), C(x), D(x))$$

is valid. □

From these two facts we directly obtain the following corollary.

**Corollary 7**

$$FO(Q^{\text{rel}}) \equiv FO((Q^{\text{rel}})^{\text{rel}})$$

Finally, the following proposition is slightly less trivial to prove; one uses induction over the degree of the definition of  $Q_1$ , but we omit it here, since we shall only use the result as a heuristic aid (in section 13.4 below).

**Proposition 8**

If  $Q_1$  is definable in  $FO(Q_2)$ , then  $Q_1^{\text{rel}}$  is definable in  $FO(Q_2^{\text{rel}})$ .

Combining this result with the previous corollary, we obtain the following.

**Corollary 9**

$Q_1$  is definable in  $FO(Q_2^{\text{rel}})$  if and only if  $Q_1^{\text{rel}}$  is definable in  $FO(Q_2^{\text{rel}})$ .

### 13.2.2 The role of ISOM

In Chapter 9 we pointed out that ISOM, in addition to being fundamental for natural language quantification, is a necessary condition of logicity. This condition says roughly that it doesn't matter which particular universe we are in and which particular interpretations we give to the relevant non-logical symbols; any isomorphic model gives the same result. In this chapter we assume throughout that all quantifiers talked about satisfy ISOM. The rough idea then takes the following precise form.

**Proposition 10**

*If  $\mathcal{M} \cong \mathcal{M}'$  and  $\mathcal{M} \models \psi$ , where  $\psi$  is any sentence of any logic  $L = FO(\{Q_i : i \in I\})$ , and all the  $Q_i$  satisfy ISOM, then  $\mathcal{M}' \models \psi$ . More generally, if  $f$  is an isomorphism from  $\mathcal{M}$  to  $\mathcal{M}'$ , and  $\varphi = \varphi(x_1, \dots, x_n)$  is an  $L$ -formula, then*

$$\mathcal{M} \models \varphi(a_1, \dots, a_n) \iff \mathcal{M}' \models \varphi(f(a_1), \dots, f(a_n))$$

This is a trivial but fundamental fact about logics. We omit the straightforward proof, which is by induction on the degree of  $\varphi$ .

## 13.3 UNDEFINABILITY

If one is interested in questions of expressive power, it is necessary, as we saw in Chapter 12.5, to be able to *prove* certain facts about definability and undefinability. Although one can get pretty good at guessing what the facts are, guesses can be wrong, and proof is needed in the end. In particular, *undefinability* results need proof. Such proofs range from fairly easy to impossibly difficult. However, logicians have developed a set of tools especially apt to deal with this sort of question (and many other logical issues as well). They were invented in the 1950s by Roland Fraïssé and Andrzej Ehrenfeucht and have been developed and refined ever since. A generic name for these tools is Ehrenfeucht–Fraïssé games (though the game-theoretic formulation is not the one we shall start with). Here we call them simply *EF-tools*.

Especially for undefinability in terms of *monadic* quantifiers, the EF-tools are often very easy to use, and we treat this case in some detail. There is a simple strategy for proving a quantifier, or any notion, to be undefinable in a logic with monadic quantifiers, and the application of this strategy to particular cases does not require further use of or proficiency in mathematical logic. It may require some combinatorics, which in principle ranges from trivial to arbitrarily hard, but several familiar cases from natural language quantification turn out to be located towards the trivial side. It is rather satisfactory, we think, that some results of this sort of complexity—that a quantifier is not definable in a certain logic, so that adding it strictly increases expressive power—can be verified by such simple means, and that is the reason why we take some space to present the method here.

Thus, the practical use of this method does not require that one has understood exactly *why* it works, only *how* it works. Our presentation keeps the ‘what’ and the ‘how’ separate from the ‘why’, and the reader who does not wish to indulge in logical

subtleties may thus be content with learning the strategy below, and then going directly to the applications in Chapter 14 and later.

Let  $Q$  be a quantifier and  $L$  a logic. The idea for proving that  $Q$  is *not* definable in  $L$  is the following. Find two models  $\mathcal{M}$  and  $\mathcal{M}'$  which (a) are *close* in the sense that no  $L$ -sentence can distinguish between them, but (b) are such that  $Q$  does distinguish them; i.e. some sentence  $\psi$  involving  $Q$  is true in one but not in the other. If this holds, we are done, because any putative definition of  $Q$  in  $L$  would be an  $L$ -sentence, so  $\psi$  would also be (equivalent to) some  $L$ -sentence, hence have the same truth value in  $\mathcal{M}$  and  $\mathcal{M}'$  by (a); but this is ruled out by (b).

Note that  $Q$  can be a quantifier of any type. In fact, it can be any concept for which a notion of definability in logics exists. Likewise,  $L$  can be any logic (see note 2), though here we consider only those of the form  $FO(\{Q_n : n \in I\})$ .

To express the strategy succinctly, the following standard terminology is convenient.

### ***L*-equivalence**

Let  $\mathcal{M}$  and  $\mathcal{M}'$  be two models (for the same vocabulary).  $\mathcal{M}$  is *L-equivalent* to  $\mathcal{M}'$ , in symbols,

$$\mathcal{M} \equiv_L \mathcal{M}'$$

if the same  $L$ -sentences are true in both; i.e. if, for every  $L$ -sentence  $\varphi$ ,  $\mathcal{M} \models \varphi \Leftrightarrow \mathcal{M}' \models \varphi$ . If this is required to hold only for  $\varphi$  of quantifier rank at most  $r$  (some  $r > 0$ ), we write

$$\mathcal{M} \equiv_{L^r} \mathcal{M}'$$

The latter notation is a special case of the first one, provided we introduce the *restricted logics*  $L^r$ , where  $L$  is a logic in the earlier sense, and  $L^r$  is simply  $L$  restricted to sentences of quantifier rank at most  $r$ .

The above idea for proving undefinability can be formulated in terms of either  $\equiv_L$  or  $\equiv_{L^r}$ . However, it is quite important for us that the strategy works also in the case of *finite* universes, and then one *has to* use  $\equiv_{L^r}$ . This is a consequence of the next proposition.

### **Proposition 11**

*If  $\mathcal{M}$  is finite and  $\mathcal{M} \equiv_{FO} \mathcal{M}'$ , then  $\mathcal{M} \cong \mathcal{M}'$ .*

So  $L$ -equivalent (hence  $FO$ -equivalent) finite models are too close to be useful in our strategy; they are *isomorphic*, and hence cannot be distinguished by any quantifier  $Q$ , or any concept satisfying ISOM.

Proposition 11 is a standard fact about finite models. To begin, if  $\mathcal{M}$  has  $m$  elements, then the  $FO$ -sentence equivalent to

$$\exists_{=m} x(x = x)$$

is true in  $\mathcal{M}$ , hence also in  $\mathcal{M}'$ , so  $\mathcal{M}'$  has  $m$  elements too. Furthermore, for each relation  $R$  in  $\mathcal{M}$ , one can write a long  $FO$ -sentence of the form  $\exists x_1 \dots \exists x_m[\dots]$ , where  $\dots$  describes exactly which tuples from  $x_1, \dots, x_m$  belong to  $R$  and which don't. This sentence is then true in  $\mathcal{M}'$  as well, and from this it readily follows that  $\mathcal{M} \cong \mathcal{M}'$ .  $\square$

On the other hand, if  $r$  is smaller than the size of  $M$  and we know only that  $\mathcal{M} \equiv_{L^r} \mathcal{M}'$ , it no longer follows that  $\mathcal{M} \cong \mathcal{M}'$ ; note that the first sentence used in the above argument has quantifier rank  $|M| + 1$ . Indeed, in the cases when the strategy works, we must have  $\mathcal{M} \equiv_{L^r} \mathcal{M}'$  but  $\mathcal{M} \not\cong \mathcal{M}'$ .

It thus becomes clear that one main concern in undefinability proofs is to verify that the relation  $\equiv_{L^r}$  holds between two models. A problem here is that the relation concerns only *sentences*, not arbitrary formulas. This means that the method of induction over the degree of formulas cannot easily be applied. So the crucial part of our method will be to formulate a criterion for  $L^r$ -equivalence which is more directly suited to applications. The EF-tools provide precisely such a criterion. It is particularly easy to formulate and apply in the monadic case. We first formulate a version of the criterion, after some motivating discussion, and then prove the theorem that it really does characterize  $L^r$ -equivalence. In the next chapter we consider numerous applications to natural language quantifiers, and in Chapter 15 we look at some applications of EF-tools for polyadic quantifiers.

### 13.4 EF-TOOLS FOR THE MONADIC CASE

We want to show that a particular quantifier, say  $Q$ , is not definable in  $L = FO(\{Q_n : n \in I\})$ , where the  $Q_n$  are monadic. To simplify things, we often deal explicitly only with the following case:

- $Q$  is of type  $\langle 1 \rangle$  and the  $Q_n$  are of type  $\langle 1, 1 \rangle$ .

If one is interested in the interdefinability of natural language quantifiers denoted by quantirelations, this special case is particularly relevant. First, quantirelation denotations are of type  $\langle 1, 1 \rangle$ . Second, they are relativizations of type  $\langle 1 \rangle$  quantifiers (Chapter 4.5), and we saw earlier, in Corollary 9, that a relativized quantifier is definable in a logic with another relativized quantifier if and only if the unrelativized quantifier is also definable in that logic. However, it will be clear how to express the criterion for arbitrary monadic types.

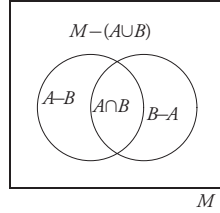
#### 13.4.1 The structure of monadic models

Let  $\mathcal{M} = (M, A_1, \dots, A_m)$  be a monadic model.

When  $m = 1$ , the universe  $M$  of  $\mathcal{M} = (M, A)$  is *partitioned* into  $A$  and  $M - A$ . We call  $A$  and  $M - A$  the *parts* of  $\mathcal{M}$ , and similarly for  $\mathcal{M}'$ . With  $L$ -sentences one can clearly talk about the parts of models, and also about arbitrary unions of these

parts. We call these unions the *blocks* of  $\mathcal{M}$ , and include the empty set among them. Thus, the blocks of  $(M, A)$  are the four sets  $\emptyset, A, M - A, M$ .

This generalizes to any  $m$ . For example, if  $\mathcal{M} = (M, A, B)$ , there are four parts (cf. Chapter 4.8):



We can form  $4^2 = 16$  blocks from these, such as  $B, (A - B) \cup (B - A), M - A$ , etc. Let us use the following notation.

**parts and blocks**

- the *parts* (of the partition induced by  $A_1, \dots, A_m$ ) are  $P_{1,\mathcal{M}}, \dots, P_{2^m,\mathcal{M}}$
- the corresponding *blocks* are  $U_{1,\mathcal{M}}, \dots, U_{2^m,\mathcal{M}}$

both assumed to be enumerated in some fixed order (so some parts and blocks in this numeration may be empty).

Fact 8 in Chapter 3.3 easily generalizes to the general case above; two monadic models (for the same vocabulary) are isomorphic iff their corresponding parts have the same size:

**Fact 12**

$$\mathcal{M} \cong \mathcal{M}' \iff |P_{i,\mathcal{M}}| = |P_{i,\mathcal{M}'}| \text{ for } 1 \leq i \leq 2^m$$

### 13.4.2 A criterion for $L^r$ -equivalence over monadic models

To say in  $FO$  that a set has exactly  $k$  elements in general requires a sentence of quantifier rank  $k + 1$ . Since all logics contain  $FO$ , it is clear that one necessary condition for  $\mathcal{M}$  and  $\mathcal{M}'$  to be  $L^r$ -equivalent to each other is that, for any  $i$ :

- (a) If  $P_{i,\mathcal{M}}$  or  $P_{i,\mathcal{M}'}$  has fewer than  $r$  elements, then  $|P_{i,\mathcal{M}}| = |P_{i,\mathcal{M}'}|$ .

Next, we are assuming  $L = FO(\{Q_n : n \in I\})$ , so the criterion must say something about how  $\mathcal{M}$  and  $\mathcal{M}'$  behave with respect to each  $Q_n$ . The following condition must be satisfied (assuming  $Q_n$  is of type  $\langle 1, 1 \rangle$ ):

- (b) If  $X, Y$  are blocks in  $\mathcal{M}$ , and  $X', Y'$  are corresponding blocks in  $\mathcal{M}'$ , then  $(Q_n)_M(X, Y) \Leftrightarrow (Q_n)_{M'}(X', Y')$ .

Here  $X$  and  $X'$  are said to be *corresponding* if, for some  $i$ ,  $X = U_{i,\mathcal{M}}$  and  $X' = U_{i,\mathcal{M}'}$ . Then  $X = \varphi_i(x)^{\mathcal{M},x}$  and  $X' = \varphi_i(x)^{\mathcal{M}',x}$ , for some  $FO$ -formula  $\varphi_i(x)$  (indeed,  $\varphi_i(x)$  is a Boolean combination of atomic formulas). If also  $Y = \psi_i(x)^{\mathcal{M},x}$  and  $Y' = \psi_i(x)^{\mathcal{M}',x}$ , then (b) says that the sentence

$$Qx(\varphi_i(x), \psi_i(x))$$

which has quantifier rank 1, has the same truth value in  $\mathcal{M}$  and  $\mathcal{M}'$ ; clearly a necessary condition.

Conditions (a) and (b) are *almost* sufficient for  $L^r$ -equivalence. What remains to be accounted for is the fact that one can compare, by means of sentences involving  $FO$ -quantifiers and the  $Q_n$ , not only blocks, but sets resulting from blocks by ‘moving around’ some elements in the universe. Here is an example.

Consider  $\mathcal{M} = (M, A)$ ; let  $a, b$  be distinct elements of  $A$ , and  $c$  an element of  $M - A$ , and suppose the corresponding facts hold for  $a', b', c'$  in  $\mathcal{M}' = (M', A')$ . Suppose further that it holds that

$$(Q_n)_M((A \cup \{c\}) - \{a, b\}, (M - A) \cup \{a\})$$

Then, if  $\mathcal{M} \equiv_{L^r} \mathcal{M}'$  and  $r \geq 4$ , it must be the case that

$$(13.5) \quad (Q_n)_{M'}((A' \cup \{c'\}) - \{a', b'\}, (M' - A') \cup \{a'\})$$

This is because the sentence

$$\begin{aligned} & \exists x \exists y \exists z [A(x) \wedge A(y) \wedge \neg A(z) \wedge x \neq y \wedge \\ & Q_n w ((A(w) \vee w = z) \wedge w \neq x \wedge w \neq y, \neg A(w) \vee w = x)] \end{aligned}$$

of quantifier rank 4, is true in  $\mathcal{M}$ , and hence in  $\mathcal{M}'$ . It follows that there are  $a'', b'', c'' \in M'$  such that  $a'', b'' \in A'$ ,  $c'' \in M' - A'$ , and

$$(13.6) \quad (Q_n)_{M'}((A' \cup \{c''\}) - \{a'', b''\}, (M' - A') \cup \{b''\})$$

And since we have assumed ISOM for  $Q_n$ , (13.5) follows. More in detail: Let  $f$  be a function with domain  $M'$  that maps  $a''$  to  $a'$ ,  $b''$  to  $b'$ , and  $c''$  to  $c'$ , but is the identity on all other arguments. Then  $f$  is an isomorphism from  $\mathcal{M}'$  to itself, and so by ISOM, (13.5) follows from (13.6).

Here is a more precise account of this sort of situation, given two monadic models  $\mathcal{M} = (M, A_1, \dots, A_m)$  and  $\mathcal{M}' = (M', A'_1, \dots, A'_m)$ .

**partial isomorphisms and corresponding block variants**

- $(a_1, \dots, a_k) \in M^k$  and  $(a'_1, \dots, a'_k) \in (M')^k$  are *isomorphic*,  $(a_1, \dots, a_k) \cong (a'_1, \dots, a'_k)$ , if for all  $i, j, l$ :  $a_i \in A_l \Leftrightarrow a'_i \in A'_l$  and  $a_i = a_j \Leftrightarrow a'_i = a'_j$ . Then  $\{(a_i, a'_i) : 1 \leq i \leq k\}$  is a *partial isomorphism* from  $M$  to  $M'$ .
- If  $a_1, \dots, a_k \in M$  and  $Z \subseteq M$ , a subset  $X$  of  $M$  is a  $\{a_1, \dots, a_k\}$ -*variant* of  $Z$  if it is obtained from  $Z$  by adding or deleting some of  $a_1, \dots, a_k$ . That is,  $(X - Z) \cup (Z - X) \subseteq \{a_1, \dots, a_k\}$ . Thus,  $X$  and  $Z$  are the same ‘up to’  $a_1, \dots, a_k$ .

- $X_1, \dots, X_p$  and  $X'_1, \dots, X'_p$  are *corresponding  $r$ -block variants* if there are isomorphic  $a_1, \dots, a_k \in M$  and  $a'_1, \dots, a'_k \in M'$  with  $k < r$ , and blocks  $U_{i_1, \mathcal{M}}, \dots, U_{i_p, \mathcal{M}}$  s.t. for  $1 \leq l \leq p$ ,
  1.  $X_l$  is a  $\{a_1, \dots, a_k\}$ -variant of  $U_{i_l, \mathcal{M}}$ , and  $X'_l$  is a  $\{a'_1, \dots, a'_k\}$ -variant of  $U_{i_l, \mathcal{M}'}$ ;
  2.  $a_j \in X_l \Leftrightarrow a'_j \in X'_l$  for  $1 \leq j \leq k$ .

Armed with these notions, we can formulate the following criterion.

### $L'$ -similarity

We say that  $\mathcal{M}$  and  $\mathcal{M}'$  are  $L'$ -similar, in symbols,

$$\mathcal{M} \approx_{L'} \mathcal{M}'$$

iff the following holds:

(13.7) If  $P_{i, \mathcal{M}}$  or  $P_{i, \mathcal{M}'}$  has fewer than  $r$  elements,  $|P_{i, \mathcal{M}}| = |P_{i, \mathcal{M}'}|$ .

(13.8) For each  $n \in I$ , if  $Q_n$  has  $p$  arguments, and  $X_1, \dots, X_p$  and  $X'_1, \dots, X'_p$  are corresponding  $r$ -block variants, then  $(Q_n)_M(X_1, \dots, X_p) \Leftrightarrow (Q_n)_{M'}(X'_1, \dots, X'_p)$ .

The next theorem is essentially due to Fraïssé and Ehrenfeucht.

### Theorem 13

The following are equivalent, for any monadic models  $\mathcal{M}$  and  $\mathcal{M}'$  with the same vocabulary, and any logic  $L = FO(\{Q_n : n \in I\})$  where the  $Q_n$  are monadic:

- (i)  $\mathcal{M} \equiv_{L'} \mathcal{M}'$
- (ii)  $\mathcal{M} \approx_{L'} \mathcal{M}'$

It should be immediately clear that it can be much easier to verify that  $\mathcal{M} \approx_{L'} \mathcal{M}'$  holds than to directly prove  $\mathcal{M} \equiv_{L'} \mathcal{M}'$ . We give a proof of the theorem in the next section, and proceed to applications to various natural language quantifiers after that.

### 13.4.3 Proof that the criterion is correct

Let  $\mathcal{M}$  be a model. A subset  $X$  of  $M$  is *definable* in  $\mathcal{M}$ , relative to a logic  $L$ , if it is the extension in  $\mathcal{M}$  of some  $L$ -formula. To prove Theorem 13, we need to know which sets are definable in a monadic model.



**L-definable sets**

A set  $X \subseteq M$  is *L-definable* in  $\mathcal{M}$  iff there is some *L*-formula  $\varphi(x)$  with one free variable which defines it, i.e. such that

$$X = \varphi(x)^{\mathcal{M},x} = \{a \in M : \mathcal{M} \models \varphi(a)\}$$

More generally, if  $\bar{a} = a_1, \dots, a_k$  is a sequence of elements of  $M$ ,  $X$  is *L-definable in  $\mathcal{M}$  with parameters  $\bar{a}$*  iff, for some *L*-formula  $\varphi(x, x_1, \dots, x_k)$  with exactly the free variables shown,

$$X = \varphi(x, \bar{a})^{\mathcal{M},x}$$

**Proposition 14**

If  $\mathcal{M}$  is a monadic structure,  $a_1, \dots, a_k \in M$ , and *L* any logic, then the *L*-definable subsets of  $M$  with parameters  $a_1, \dots, a_k$  are precisely the  $\{a_1, \dots, a_k\}$ -variants of the blocks of  $\mathcal{M}$ .

Observe that the (variants of the) blocks depend on  $\mathcal{M}$  but not on the logic *L*. So it follows from the proposition that we might as well take  $L = FO$ ; strengthening the logic does not make more sets definable. This is a special feature of monadic models — it fails completely for polyadic models<sup>4</sup> — that we make use of here.

To prove the proposition, one needs a simple but fundamental lemma. Recall from Chapter 3.3.2 that an *automorphism* on  $\mathcal{M}$  is an isomorphism from a  $\mathcal{M}$  to itself:  $f : \mathcal{M} \cong \mathcal{M}$ .<sup>5</sup> The next lemma, which is a direct consequence of Proposition 10, holds for all structures, monadic or not. Note also that it holds for any logic *L*.

**Lemma 15**

If  $X$  is *L*-definable in  $\mathcal{M}$  with parameters  $a_1, \dots, a_k$ , and  $f$  is an automorphism on  $\mathcal{M}$  such that  $f(a_i) = a_i$  for  $1 \leq i \leq k$ , then  $f$  preserves  $X$  in the sense that, for any  $a \in M$ ,  $a \in X \Leftrightarrow f(a) \in X$ .

*Proof.* Suppose  $\varphi(x, x_1, \dots, x_k)$  defines  $X$  in  $\mathcal{M}$  with parameters  $a_1, \dots, a_k$ . Then  $a \in X \Leftrightarrow \mathcal{M} \models \varphi(a, a_1, \dots, a_k) \Leftrightarrow \mathcal{M} \models \varphi(f(a), f(a_1), \dots, f(a_k))$  (by Proposition 10)  $\Leftrightarrow \mathcal{M} \models \varphi(f(a), a_1, \dots, a_k) \Leftrightarrow f(a) \in X$ . □

*Proof of Proposition 14.* Each block  $U_{i,\mathcal{M}}$  is definable by an *FO*-formula (a Boolean combination of atomic formulas). To define an  $\{a_1, \dots, a_k\}$ -variant of  $U_{i,\mathcal{M}}$ , just add suitable disjuncts of the form  $x = x_j$ , and conjuncts of the form  $x \neq x_l$ .

<sup>4</sup> E.g. a non-standard model of the ordering of the natural numbers contains non-standard ‘numbers’, and the set of standard numbers is not definable in *FO*. It is, however, definable in *FO*( $Q_0$ ), as the set of elements which have finitely many predecessors.

<sup>5</sup> In model theory, the set of automorphisms on a structure is a fundamental object of study. At one extreme are the *rigid* structures which have only one automorphism: the identity function. A typical example is a well-ordered set, i.e. a structure  $(M, <)$ , where  $<$  well-orders  $M$ . If  $\mathcal{M}$  is a monadic structure, on the other hand, then, e.g., any permutation of  $M$  that does not move elements from one part to another will be an automorphism.

Now suppose that  $X$  is  $L$ -definable in  $\mathcal{M}$  with parameters  $a_1, \dots, a_k$ , and let  $X' = X - \{a_1, \dots, a_k\}$ . We prove that  $X'$  is a  $\{a_1, \dots, a_k\}$ -variant of a block in  $\mathcal{M}$ ; it then follows that so is  $X$ .

Let  $i_1, \dots, i_q$  be those  $i$  such that  $P_{i,\mathcal{M}} \cap X' \neq \emptyset$ . Thus,

$$X' \subseteq P_{i_1,\mathcal{M}} \cup \dots \cup P_{i_q,\mathcal{M}} = U_{p,\mathcal{M}}$$

for some  $p$ . Suppose, for contradiction, that  $X'$  is *not* a  $\{a_1, \dots, a_k\}$ -variant of  $U_{p,\mathcal{M}}$ . Then there must be a  $j$  and a  $b \in P_{ij,\mathcal{M}} - X'$  such that  $b \neq a_1, \dots, a_k$ . And by our assumptions, there is some  $a \in P_{ij,\mathcal{M}} \cap X'$ , whence  $a \neq a_1, \dots, a_k$ . Let  $f$  be the permutation of  $M$  which interchanges  $a$  and  $b$  but leaves all other elements as they are.  $f$  is an automorphism on  $\mathcal{M}$  (since  $a$  and  $b$  are in the same part) such that  $f(a_i) = a_i$  for  $1 \leq i \leq k$ . But  $a \in X'$  whereas  $b = f(a) \notin X'$ , contradicting Lemma 15. So  $X'$  is a  $\{a_1, \dots, a_k\}$ -variant of  $U_{p,\mathcal{M}}$ , and the proof is finished.  $\square$

After these preparations, we can finally proceed to the proof of the main result.

*Proof of Theorem 13:* (i)  $\Rightarrow$  (ii): This is the easy direction. We saw an example of the required reasoning in the preceding subsection, but let us now do the general case. Suppose  $\mathcal{M} \equiv_{L^r} \mathcal{M}'$ . Since it takes at most  $k + 1$   $FO$ -quantifiers to say that a set has exactly  $k$  elements, it is clear that (13.7) holds. For (13.8), suppose that  $Q_n$  has type  $(1, 1)$  and that  $X_1, X_2$  and  $X'_1, X'_2$  are as described there, relative to  $a_1, \dots, a_k \in M$  and  $a'_1, \dots, a'_k \in M'$  with  $(a_1, \dots, a_k) \cong (a'_1, \dots, a'_k)$ , for some  $k < r$ . Let  $\varphi(x, x_1, \dots, x_k)$  and  $\psi(x, x_1, \dots, x_k)$  be standard  $FO$ -formulas that define  $X_1$  and  $X_2$ , respectively, in  $\mathcal{M}$ , with parameters  $a_1, \dots, a_k$ . Note that the same formulas define  $X'_1$  and  $X'_2$ , respectively, in  $\mathcal{M}'$ , with parameters  $a'_1, \dots, a'_k$ . Now each  $a_j$  belongs to exactly one  $P_{i,\mathcal{M}}$ ; let that part be defined by  $\theta_j(x)$  in  $\mathcal{M}$ . If  $(Q_n)_M(X_1, X_2)$ , then

$$\exists x_1 \dots \exists x_k (\theta_1(x_1) \wedge \dots \wedge \theta_k(x_k) \wedge Q_n x (\varphi(x, x_1, \dots, x_k), \psi(x, x_1, \dots, x_k)))$$

is true in  $\mathcal{M}$ . Since this sentence has quantifier rank  $k + 1 \leq r$ , it is also true in  $\mathcal{M}'$ , so there are  $b_1, \dots, b_k \in M'$  such that  $b_j \in P_{i_j,\mathcal{M}'}$  and

$$\mathcal{M}' \models Q_{nx} (\varphi(x, b_1, \dots, b_k), \psi(x, b_1, \dots, b_k))$$

Let  $f$  map each  $b_j$  on  $a'_j$  but leave everything else as it is. Then  $f$  is an automorphism on  $\mathcal{M}'$  (since no element is moved from one part to another), so, by Proposition 10,

$$\mathcal{M}' \models Q_{nx} (\varphi(x, a'_1, \dots, a'_k), \psi(x, a'_1, \dots, a'_k))$$

But this means that  $(Q_n)_{M'}(X'_1, X'_2)$ . The other direction is similar, so (13.8) holds.

(ii)  $\Rightarrow$  (i): Suppose that  $\mathcal{M} \approx_{L^r} \mathcal{M}'$ . We shall prove the following statement, for all  $L$ -formulas  $\varphi$ :

(A) $_{\varphi}$  If  $k \leq r$ ,  $a_1, \dots, a_k \in M$ ,  $a'_1, \dots, a'_k \in M'$ , and  $(a_1, \dots, a_k) \cong (a'_1, \dots, a'_k)$ , and if  $\varphi(x_1, \dots, x_k)$  has quantifier rank at most  $r - k$  and at most the free variables shown, then

$$(B) \mathcal{M} \models \varphi(a_1, \dots, a_k) \Leftrightarrow \mathcal{M}' \models \varphi(a'_1, \dots, a'_k)$$

We prove  $(A)_\varphi$  by induction over  $\varphi$ . This is enough, since with  $k = 0$  we get the desired result that  $\mathcal{M} \equiv_{L^r} \mathcal{M}'$ .

In the base step,  $\varphi$  is an atomic formula. But  $(a_1, \dots, a_k) \cong (a'_1, \dots, a'_k)$  means precisely that (B) holds for atomic formulas.

For the induction step, assume  $(a_1, \dots, a_k) \cong (a'_1, \dots, a'_k)$ . If  $k = r$ , then  $\varphi$  has quantifier rank 0, and is thus an atomic formula, or a Boolean combination of atomic formulas. But we already saw that (B) holds for atomic formulas, and it is practically immediate that this extends to Boolean combinations of atomic formulas. (To be meticulous, one should prove this by induction over the degree of the Boolean combinations.)

Thus we may assume  $k < r$ . There are a few cases to consider. If  $\varphi$  is a conjunction  $\psi_1 \wedge \psi_2$ , the conclusion is immediate from the induction hypothesis that  $(A)_{\psi_1}$  and  $(A)_{\psi_2}$  hold. ( $qr(\psi_i) \leq qr(\varphi) \leq r - k$ , and the assumption about variables is satisfied, so (B) holds for each  $\psi_i$ .) Similarly if  $\varphi$  is a negation.

The two remaining cases are when  $\varphi$  is  $\exists x\psi$  and when it is  $Q_n x(\psi_1, \psi_2)$ . The first of these is actually a (simple) special case of the second one, which deals with a quantified formula in general. So we go directly to the second case.

Suppose then  $\varphi(x_1, \dots, x_k) = Q_n x(\psi_1(x, x_1, \dots, x_k), \psi_2(x, x_1, \dots, x_k))$  with at most the free variables shown, and with quantifier rank at most  $r - k$ . Now look at the definable sets  $X_i = \psi_i(x, a_1, \dots, a_k)^{\mathcal{M}, x}$ ,  $i = 1, 2$ . By Proposition 14,  $X_i$  is a  $\{a_1, \dots, a_k\}$ -variant of some block  $U_{p_i, \mathcal{M}}$ . Let  $X'_i = \psi_i(x, a'_1, \dots, a'_k)^{\mathcal{M}', x}$ ,  $i = 1, 2$ . We claim that

(C)  $X_1, X_2$  and  $X'_1, X'_2$  are corresponding  $r$ -block variants.

Since  $\mathcal{M} \approx_{L^r} \mathcal{M}'$ , (C) implies that  $(Q_n)_M(X_1, X_2) \Leftrightarrow (Q_n)_{M'}(X'_1, X'_2)$ . But this means that (B) holds, and we are done.

So it only remains to verify (C). There are two things to check, for  $i = 1, 2$ :

(C1)  $a_j \in X_i \Leftrightarrow a'_j \in X'_i$ , for  $1 \leq j \leq k$ .

(C2)  $X'_i$  is a  $\{a'_1, \dots, a'_k\}$ -variant of  $U_{p_i, \mathcal{M}'}$ .

For (C1), observe that  $(a_j, a_1, \dots, a_k) \cong (a'_j, a'_1, \dots, a'_k)$ , and that  $\psi_i$  has quantifier rank at most  $r - (k + 1)$ . (You can see why we allowed repetitions in partial isomorphism sequences!) This means that the induction hypothesis  $(A)_{\psi_i}$  applies (since  $k + 1 \leq r$ ), and hence

$$\mathcal{M} \models \psi_i(a_j, a_1, \dots, a_k) \Leftrightarrow \mathcal{M}' \models \psi_i(a'_j, a'_1, \dots, a'_k)$$

so (C1) follows.

For (C2), note that in general it need not be the case that if a formula defines one block in a model, it defines the corresponding block in other models. (For example, the formula  $\psi(x) = (x = x) \wedge Q_0 y(y = y)$  defines the universe  $M$  in infinite models, but  $\emptyset$  in finite models.) But here we are assuming  $\mathcal{M} \approx_{L^r} \mathcal{M}'$ , and that is what makes the difference.

Now, 'up to'  $a_1, \dots, a_k$ ,  $X_i$  is the block

$$U_{p_i, \mathcal{M}} = P_{i_1, \mathcal{M}} \cup \dots \cup P_{i_l, \mathcal{M}}$$

We know, by Proposition 14, that, ‘up to’  $a'_1, \dots, a'_k, X'_i$  is *some* block in  $\mathcal{M}'$ ; what we must show is that this is precisely  $U_{p_i, \mathcal{M}'}$ . In view of (C1), it is enough to show that, for each part  $P_{j, \mathcal{M}}$  of  $\mathcal{M}$ ,

$$(D) \quad (X - \{a_1, \dots, a_k\}) \cap P_{j, \mathcal{M}} \neq \emptyset \Leftrightarrow (X' - \{a'_1, \dots, a'_k\}) \cap P_{j, \mathcal{M}'} \neq \emptyset$$

Suppose  $a \neq a_1, \dots, a_k$  and  $a \in P_{j, \mathcal{M}}$ . Since  $\mathcal{M} \approx_{L^r} \mathcal{M}'$  and  $k + 1 \leq r$ , it follows that there is  $a' \neq a'_1, \dots, a'_k$  such that  $a' \in P_{j, \mathcal{M}'}$ . This is because our assumption requires  $P_{j, \mathcal{M}}$  to have at least  $m$  elements, where  $m$  is some number  $\leq k + 1$  (depending on how many of  $a_1, \dots, a_k$  are in  $P_{j, \mathcal{M}}$ ), and then by condition (13.7), so does  $P_{j, \mathcal{M}'}$ . But this means that  $(a, a_1, \dots, a_k) \cong (a', a'_1, \dots, a'_k)$ , so again (A) $_{\psi_i}$  applies, yielding

$$\mathcal{M} \models \psi_i(a, a_1, \dots, a_k) \Leftrightarrow \mathcal{M}' \models \psi_i(a', a'_1, \dots, a'_k)$$

i.e.  $a \in X$  iff  $a' \in X'$ . The other direction is similar, so (D) is proved, whereby the induction step is completed, and the whole proof is finished.  $\square$

This proof is a nice example of a somewhat complex inductive argument. The main induction is over formula complexity. But the (carefully chosen) claim (A) $_{\varphi}$  to be established itself quantifies over all  $k \leq r$ , where  $k$  is the maximal number of free variables and  $r - k$  the maximal quantifier rank. As it were, the proof combines an induction over the complexity of  $\varphi$  with a *downward* induction over  $k$ : the case  $k = r$  is (almost) trivial, but for smaller  $k$  one needs to use the induction hypothesis for  $k + 1$  (as each (monadic) quantifier is ‘peeled off’, the number of free variables increases by 1, and the quantifier rank, as well as the formula complexity, decreases by 1).

# 14

## Applications to Monadic Definability

In this chapter we apply the EF-tools to present a variety of results about the expressive power of certain monadic quantifiers. As promised, to apply the method, it suffices to be familiar with the notions involved (such as equivalence up to quantifier rank  $r$ , etc.), and some of the applications are surprisingly easy, considering that the results themselves are not trivial. Others involve more or less combinatorics. After repeating the general method (section 14.1), we begin with (un)definability relative to  $FO$  (section 14.2), and then relative to stronger logics (section 14.3). Section 14.4 is devoted to issues of expressivity and monotonicity, and section 14.5 gives a number of exercises for the reader who wishes to try his or her own hand at applying the EF-tools.

The direct applications of the method concern logical languages. But we saw in Chapter 12.5 how results about such languages may transfer to natural languages, under certain conditions. In each section with applications below, we also discuss the possible consequences for expressibility issues within natural languages. In particular, section 14.4 explores the prospects of finding a universal statement that would somehow describe or even explain the role of monotonicity in the context of expressive power in natural languages, in view of certain rather striking technical results about definability in terms of monotone quantifiers. So far, however, these prospects seem rather bleak.

### 14.1 THE METHOD

Our general method for undefinability proofs is the following.

#### method for undefinability proofs

To show that  $Q$  of type  $\langle 1 \rangle$  is not definable in  $L$ , find, for each  $r > 0$ , two models  $\mathcal{M} = (M, A)$  and  $\mathcal{M}' = (M', A')$  such that

- (i)  $\mathcal{M} \approx_{L^r} \mathcal{M}'$
- (ii)  $Q_{\mathcal{M}}(A)$  but not  $Q_{\mathcal{M}'}(A')$

More generally, and for  $Q$  of any type, it suffices that there is some sentence  $\psi$  involving  $Q$  and only one-place predicates such that, for each  $r$ , there are models  $\mathcal{M}$  and  $\mathcal{M}'$  such that

- (i)  $\mathcal{M} \approx_{L^r} \mathcal{M}'$   
(ii)  $\mathcal{M} \models \psi$  but  $\mathcal{M}' \not\models \psi$

(In the first case above,  $\psi$  is  $QxP(x)$ .)

This method works because if there were a definition of  $Q$  in  $L$ ,  $\psi$  would be logically equivalent to an  $L$ -sentence  $\theta$ .<sup>1</sup> Choosing  $r$  to be the quantifier rank of  $\theta$ , and using Theorem 13 in Chapter 13.4, it would follow by (i) that  $\theta$ , and hence  $\psi$ , has the same truth value in  $\mathcal{M}$  and  $\mathcal{M}'$ , but this contradicts (ii).

Note that the method works also if  $Q$  is polyadic, as long as the sentence  $\psi$  containing  $Q$  uses only one-place predicate symbols. This is required because we have defined the relation  $\approx_{L^r}$  only for monadic models.

## 14.2 FO-UNDEFINABILITY

For  $L = FO$ , the condition for  $FO^r$ -similarity, and hence  $FO^r$ -equivalence, becomes simply (13.7):

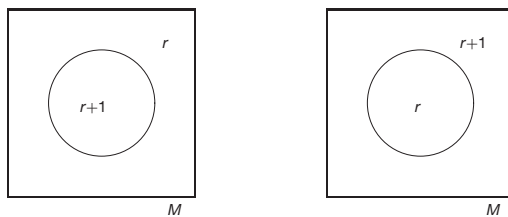
(13.7) If  $P_{i,\mathcal{M}}$  or  $P_{i,\mathcal{M}'}$  has fewer than  $r$  elements, then  $|P_{i,\mathcal{M}}| = |P_{i,\mathcal{M}'}|$ .

It is *very easy* to construct models meeting this condition. Consider the type  $\langle 1 \rangle$  quantifier  $Q_{\text{even}}$  which says of a set that it has an even number of elements. This is a classical example of quantifier that is not definable in  $FO$ .

### Fact 1

$Q_{\text{even}}$  is not first-order definable.

To prove this, we need to exhibit, for each  $r > 0$ , two models  $\mathcal{M} = (M, A)$  and  $\mathcal{M}' = (M', A')$  such that  $|A|$  is even,  $|A'|$  is odd, and  $\mathcal{M} \approx_{FO^r} \mathcal{M}'$ . But this is almost trivial, for example:



That is,  $M = M'$  has  $2r + 1$  elements,  $A$  has  $r + 1$  elements, an  $A'$  has  $r$  elements. So all parts of  $\mathcal{M}$  and  $\mathcal{M}'$  have  $\geq r$  elements, hence  $\mathcal{M} \approx_{FO^r} \mathcal{M}'$ . But only one of  $r, r + 1$  is even.  $\square$

<sup>1</sup>  $\theta$  is obtained from  $\psi$  by systematically eliminating  $Q$  using the definition.

Other choices of  $\mathcal{M}, \mathcal{M}'$  work too. For example, we could have taken sets such that  $|A| = r + 1, |A'| = r$ , but  $|M - A| = |M' - A'|$ . Then, note that regardless of whether the latter number is  $< r$  or  $\geq r$ , condition (13.7) still holds.

An even simpler proof of Fact 1 is this: consider the quantifier ‘ $|M|$  is even’. (Actually, this is not a quantifier in the sense we defined, but we could easily include it: its type would be  $\langle \rangle$ , and it would be expressed by a propositional constant, true on even-numbered universes and false on others.) To show, by our strategy, that this quantifier is not *FO*-definable, we need only, for each  $r$ , choose two universes  $M, M'$  such that  $|M|$  is even,  $|M'|$  is odd, and  $|M|, |M'| \geq r$ . So ‘ $|M|$  is even’ is not *FO*-definable. But then Fact 1 follows, since ‘ $|M|$  is even’ is definable from  $Q_{\text{even}}$ :

$$|M| \text{ is even} \iff (Q_{\text{even}})_M(M)$$

Hence, if  $Q_{\text{even}}$  were *FO*-definable, ‘ $|M|$  is even’ would be too. □

$Q_{\text{even}}$  is perhaps not a natural language quantifier, but its relativization is:  $(Q_{\text{even}})^{\text{rel}} = \text{an even number of}$ . So from Facts 1 and Facts 2 and 5 in the preceding chapter, we can conclude the following.

**Fact 2**

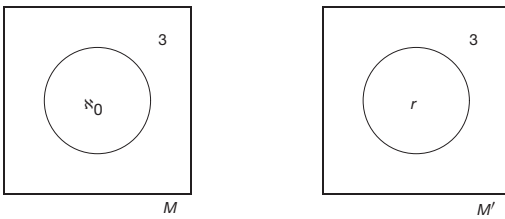
An even number of is not first-order definable.

Next, here is an easy proof of another well-known fact (cf. the discussion in Chapter 12.3).

**Fact 3**

$Q_0$  (and hence an infinite number of) is not *FO*-definable.

*Proof.* Fix  $r$ . Then simply take  $\mathcal{M} = (M, A)$  and  $\mathcal{M}' = (M', A')$  such that  $A$  is infinite,  $|A'| = r$ , and  $|M - A| = |M' - A'|$ . For example, as follows:



Clearly,  $\mathcal{M} \approx_{\text{FOR}} \mathcal{M}'$ ,  $(Q_0)_M(A)$ , but not  $(Q_0)_{M'}(A')$ . □

Notice how extremely simple this proof is (once we have assured ourselves that the method is correct). There are many other ways to obtain the same result. For example, one can observe that the set of sentences

$$\{\exists_{\geq n} x(x = x) : n = 1, 2, \dots\} \cup \{\neg Q_0 x(x = x)\}$$

is such that each finite subset is true in some model, but the whole set cannot be true in any model. This shows that  $FO(Q_0)$  does not have the *compactness property* (see Chapter 2.6). But  $FO$  is compact, so  $Q_0$  cannot be definable in  $FO$ .

Or, one may use the fact that with  $Q_0$  one can characterize the standard ordering  $\mathcal{N} = (\mathbb{N}, <)$  of the natural numbers: one adds to the usual axioms for a strict order with a first but no last element, and where each element has an immediate successor—call their conjunction  $\psi_1$ —the axiom

$$\psi_2 = \forall x \neg Q_0 y (y < x)$$

which guarantees that each element has only a finite number of predecessors. Then it holds that

$$(M, R) \models \psi_1 \wedge \psi_2 \iff (M, R) \cong (\mathbb{N}, <)$$

This implies that the *Tarski property* fails for  $FO(Q_0)$ : it is not true that every sentence which has an infinite model also has an uncountable model:  $\psi_1 \wedge \psi_2$  has only countable models. But the Tarski property holds for  $FO$ , so  $Q_0$  cannot be definable in  $FO$ . Or, one may invoke *Gödel's theorems*: If  $Q_0$  were definable in  $FO$ , one could characterize the standard model of arithmetic by means  $FO$ -sentences, but Gödel's incompleteness theorem, together with his completeness theorem for  $FO$ , has the consequence that this is not possible.

These properties of  $FO$  (which are established in any reasonably advanced logic textbook) are of course highly significant from a logical point of view. But invoking them to prove that the notion of infinity is not expressible in  $FO$  seems like using too heavy artillery. The method using the EF-tools, on the other hand, proves to be exactly what one wants. Moreover, it applies to logics where compactness, completeness, etc. fail, and hence are not available to show facts of undefinability. And furthermore, in contrast to all of the other methods hinted at above, it applies also to *finite* models.

Next, let us state the following fact.

#### Fact 4

$Q^R$  (and hence most) is not first-order definable.

*Proof.* The models chosen in the (first) proof of Fact 1 work here too, since  $|A| > |M - A|$ , but  $|A'| \leq |M - A'|$ .  $\square$

By a slightly more careful choice of models, one shows similarly that none of the properly proportional quantifiers, i.e. the quantifiers  $p/q$  and  $[p/q]$ , where  $0 < p < q$  (see Chapter 2.3)—and hence neither their relativizations such as *more than two-thirds of the*, etc.—are definable in  $FO$ .

Note that this is undefinability even if we should restrict attention to *finite* universes. It has nothing to do with the inability of  $FO$  to express the notion of infinity, but points to another weakness of the expressive power of  $FO$ , one that natural languages do not suffer from.

The facts mentioned above about  $Q_0$ , *infinitely many*, and *an even number of* may not seem too exciting from a natural language perspective. But no one can deny that



most or more than two-thirds of the are natural language expressions, indeed quantirelations (determiners). As we explained in detail in Chapter 12.5, Fact 4 and similar facts have consequences for natural languages, at least if these can be treated as first-order in the sense that one uses first-order models to describe (however roughly) the truth conditions of their sentences. We don't see why any natural language would resist such a treatment—as was stressed in Chapter 11.3.2, the claim is not that this completely captures the basic semantics of any natural language; on the contrary, it will slur over much of that semantics, but the claim is that it provides enough for an adequate treatment of quantification. Then, the language viewed from this perspective can be formalized in logical languages of the kind discussed here, and via the formalization, facts about definability transfer between a language and its formalization.

For example, if one language had no means whatever of making comparisons of size, it (or relevant parts of it) might be formalizable in  $FO$ , and then Fact 4 tells us, by the mechanism described in Chapter 12.5, that Swedish, for example, is not translatable into that language.

More generally, there is a methodological conclusion concerning the adequacy of  $FO$  as a semantic framework. In the early days of formal semantics it was often supposed that the 'logical form' associated with sentences and carrying most of their semantic information would be expressible with  $FO$  means. There have also been contrary claims, using quite involved arguments. For example, Hintikka and Sandu (1997) attempt to discredit  $FO$  by arguing that natural language semantics needs the game-theoretic machinery of  $IF$  logic (see Chapter 2.5.1) instead. The conclusion that  $FO$  is insufficient for natural language semantics is correct, it seems to us, but it can be established much more simply and directly, precisely by the fact that *most* and similar quantirelations are not expressible in  $FO$ .

### 14.3 UNDEFINABILITY IN OTHER LOGICS

That the  $FO$ -undefinability of *most* has nothing to do with infinity is also seen by the following result.

**Fact 5**

$Q^R$  (and hence *most*) is not definable in  $FO(Q_0)$ .

*Proof.* This is intuitively obvious, since we gave a finite counterexample to the  $FO$ -definability of  $Q^R$ , and on finite models  $Q_0$  has no effect (all formulas of the form  $Q_{0x}\psi$  are unsatisfiable). But we immediately get a formal proof as well. In fact, the models in the first proof of Fact 1 work again. This time, we also have condition (13.8) in the definition of  $FO(Q_0)$ -similarity to check. It says that, for any corresponding  $r$ -block variants  $X$  in  $\mathcal{M}$  and  $X'$  in  $\mathcal{M}'$ ,

$$X \text{ is infinite} \iff X' \text{ is infinite}$$

which is trivial, since all subsets of these universes are finite. □

In a similarly trivial way, using finite models as counterexamples, one can establish the following fact, whose proof we leave as an exercise.

**Fact 6**

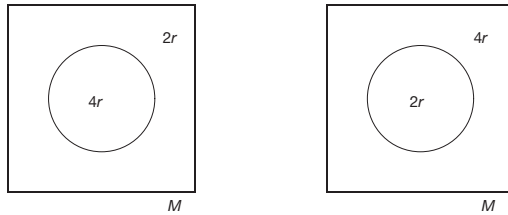
*The H\"artig quantifier  $I$  is not definable in  $FO(Q_0)$ .*

We first proved that  $Q^R$  is not  $FO$ -definable, and then the stronger result that it is not definable in  $FO(Q_0)$ . This fact about  $Q^R$  can be strengthened even further, as follows.

**Fact 7**

*$Q^R$  (and hence most) is not definable in  $FO(I)$ .*

*Proof.* (Note that this implies Fact 5, since  $Q_0$  is definable from  $I$ .) For any  $r > 0$ , use the following models:



To check that  $\mathcal{M} \equiv_{FO(I)^r} \mathcal{M}'$ , we only need to verify that if  $X, Y$  and  $X', Y'$  are corresponding  $r$ -block variants, then

$$|X| = |Y| \iff |X'| = |Y'|$$

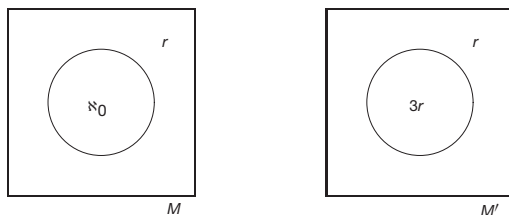
But this is easy to see. Note that, in both models, variants of  $\emptyset$  have size between 0 and  $r - 1$ , i.e. in the interval  $[0, r)$ , and variants of  $M$  have size in  $(5r, 6r]$ . Also, in  $\mathcal{M}$ , variants of  $A$  are in  $(3r, 5r)$ , and variants of  $M - A$  are in  $(r, 3r)$ , whereas in  $\mathcal{M}'$  the situation is reversed. This means that if  $|X| = |Y|$ , then  $X$  and  $Y$  must be variants of the *same* block, and moreover, that block must be either  $\emptyset$  or  $M$ . But then it is clear that  $|X| = |X'|$  and  $|Y| = |Y'|$ . The argument in the other direction is symmetric, and we are done. □

In the converse direction, we have the following.

**Proposition 8**

*$Q_0$  is not definable in  $FO(\text{most})$ .*

*Proof.* This is a little more subtle, since we saw in Chapter 13.2.1 that  $Q_0$  is definable in  $FO(MO)$ . But take  $r > 0$  and choose



So  $(Q_0)_M(A)$  holds, but not  $(Q_0)_{M'}(A')$ . Condition (13.7) is clearly satisfied. To check condition (13.8), we must convince ourselves that if  $X, Y$  and  $X', Y'$  are corresponding  $r$ -block variants, then

$$(14.1) \quad |X \cap Y| > |X - Y| \iff |X' \cap Y'| > |X' - Y'|$$

There are sixteen cases to check here (four cases for each of  $X$  and  $Y$ ). In view of the fact that the result is not immediately obvious, let us be patient and go through them. Certain symmetries will reduce our work. The following lemma, which is a straightforward consequence of our definitions, also helps.

**Lemma 9**

Suppose  $(a_1, \dots, a_{r-1}) \cong (a'_1, \dots, a'_{r-1})$ , and  $X, Y$  and  $X', Y'$  are corresponding  $r$ -block variants, relative to  $\{a_1, \dots, a_{r-1}\}$  and  $\{a'_1, \dots, a'_{r-1}\}$ , and blocks  $U_{i,M}, U_{j,M}$  and  $U_{i,M'}, U_{j,M'}$ . Then:

- (a) If  $|U_{i,M}| = |U_{i,M'}|$ , then  $|X| = |X'|$ .
- (b)  $X \cap Y$  and  $X' \cap Y'$  are also corresponding  $r$ -block variants. Similarly for  $X - Y$  and  $X' - Y'$ .

Now let us verify (14.1).

Case 1.  $X$  is a  $\{a_1, \dots, a_{r-1}\}$ -variant of  $\emptyset$ . By the lemma,  $|X \cap Y| = |X' \cap Y'|$  and  $|X - Y| = |X' - Y'|$ . Thus, (14.1) holds.

Case 2.  $X$  is a  $\{a_1, \dots, a_{r-1}\}$ -variant of  $A$ .

Case 2.1.  $Y$  is a  $\{a_1, \dots, a_{r-1}\}$ -variant of  $\emptyset$ . By the lemma,  $|X \cap Y| = |X' \cap Y'| < r$ . Also,  $|X - Y| = \aleph_0$ , and  $|X' - Y'| > 2r$ . Thus, both sides of (14.1) are *false*.

Case 2.2.  $Y$  is a  $\{a_1, \dots, a_{r-1}\}$ -variant of  $A$ . Now  $|X \cap Y| = \aleph_0$ , and  $|X' \cap Y'| > 2r$ , whereas  $|X - Y| = |X' - Y'| < r$ , so both sides of (14.1) are *true*.

Case 2.3.  $Y$  is a  $\{a_1, \dots, a_{r-1}\}$ -variant of  $M - A$ . This is similar to case 2.1.

Case 2.4.  $Y$  is a  $\{a_1, \dots, a_{r-1}\}$ -variant of  $M$ . Similar to 2.2.

Case 3.  $X$  is a  $\{a_1, \dots, a_{r-1}\}$ -variant of  $M - A$ .

Case 3.1.  $Y$  is a  $\{a_1, \dots, a_{r-1}\}$ -variant of  $\emptyset$ . Again using Lemma 9,  $|X \cap Y| = |X' \cap Y'|$  and  $|X - Y| = |X' - Y'|$ . Note that the latter equality depends on the fact that  $X - Y$  and  $X' - Y'$  are corresponding  $r$ -block variants of  $M - A$  and  $M' - A'$ , respectively, and that in our chosen models,  $|M - A| = |M' - A'|$ .

Case 3.2.  $Y$  is a  $\{a_1, \dots, a_{r-1}\}$ -variant of  $A$ . In this subcase too,  $X \cap Y$  and  $X' \cap Y'$  are corresponding  $r$ -block variants of  $\emptyset$ , and  $X - Y$  and  $X' - Y'$  are corresponding  $r$ -block variants of  $M - A$  and  $M' - A'$ , respectively.

Cases 3.3 and 3.4. These are similar to subcases 3.1 and 3.2, but with the roles of  $X \cap Y$  and  $X - Y$  reversed.

Case 4.  $X$  is a  $\{a_1, \dots, a_{r-1}\}$ -variant of  $M$ .

Case 4.1.  $Y$  is a  $\{a_1, \dots, a_{r-1}\}$ -variant of  $\emptyset$ . As before,  $|X \cap Y| = |X' \cap Y'| < r$ , whereas  $|X - Y| = \aleph_0$ , and  $|X' - Y'| > 3r$ .

Case 4.2.  $Y$  is a  $\{a_1, \dots, a_{r-1}\}$ -variant of  $A$ . We have  $|X \cap Y| = \aleph_0$ , and  $|X' \cap Y'| > 2r$ . Since  $X - Y$  and  $X' - Y'$  are corresponding  $r$ -block variants of  $M - A$  and  $M' - A'$ , respectively, it follows that  $|M - A| = |M' - A'| < 2r$ , so both sides of (14.1) are *true*. This and the next subcase are the only occasions where we need the fact that  $A'$  has  $3r$  elements (for the others  $2r$  would have been enough).

Case 4.3.  $Y$  is a  $\{a_1, \dots, a_{r-1}\}$ -variant of  $M - A$ . This is symmetric to 4.2.

Case 4.4.  $Y$  is a  $\{a_1, \dots, a_{r-1}\}$ -variant of  $M$ . Symmetric to 4.1.

This concludes the proof of (14.1), and thereby of Proposition 8.  $\square$

Observe that (13.8) would not hold if we had  $MO$  instead of *most*. For,

$$|M'| > |A'|, \text{ but } |M| \not\asymp |A|$$

so  $MO$ , but not *most*, can distinguish  $\mathcal{M}$  from  $\mathcal{M}'$ .

Summing up some of the preceding results, we have the following.

### Corollary 10

(a)  $FO(\textit{most}) < FO(MO)$  and  $FO(I) < FO(MO)$ , but  $FO(\textit{most})$  and  $FO(I)$  have incomparable expressive power.

(b)  $FO(Q_0) < FO(I)$

*Proof.* We know (Chapter 13.2.1) that  $FO(\textit{most}) \leq FO(MO)$ , that  $FO(I) \leq FO(MO)$ , and that  $FO(Q_0) \leq FO(I)$ . Fact 7 tells us that *most* is not definable in  $FO(I)$ , and hence neither is  $MO$ , so  $FO(I) < FO(MO)$ . The just proved proposition says that  $Q_0$  is not definable in  $FO(\textit{most})$ . It follows that neither  $I$  nor  $MO$  is definable in  $FO(\textit{most})$ , since  $Q_0$  is definable from each of them. Thus

$FO(most) < FO(MO)$ , and  $I$  and  $most$  are incomparable. Finally, that  $I$  is not definable in  $FO(Q_0)$ , so that  $FO(Q_0) < FO(I)$ , is Fact 6.  $\square$

Furthermore, from this and the fact that  $FO(more\_than) \equiv FO(MO)$  (Fact 4 in Chapter 13), we immediately obtain that the type  $\langle 1, 1, 1 \rangle$  *more\_than*, which is the denotation of a two-place English determiner, is not definable in terms of the one-place determiner denotation *most*.

**Corollary 11**

$FO(most) < FO(more\_than)$ .

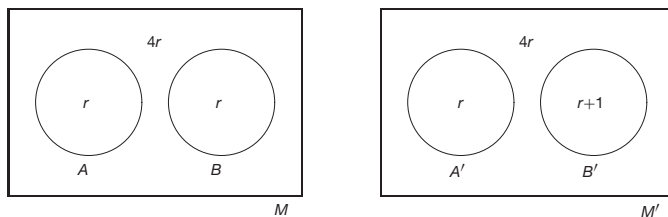
However, as we see using Proposition 3 in Chapter 13, *more\_than* is definable if we are allowed to use *two* English one-place determiner denotations: namely, *most* and *infinitely many*.

Continuing the discussion of *most*, let us now compare it to its type  $\langle 1 \rangle$  counterpart  $Q^R$ .  $Q^R$  is definable from *most*, but not vice versa.

**Proposition 12**

*Most is not definable in  $FO(Q^R)$ .*<sup>2</sup>

*Proof.* To see this we choose, given  $r$ ,



(where  $A$  and  $B$ , as well as  $A'$  and  $B'$ , are disjoint). Observe here that each of  $\emptyset, A, B$ , and  $A \cup B$  has size less than its complement, and that this still holds if at most  $r - 1$  elements are added or deleted. The same holds in  $\mathcal{M}'$ . From this, it is not hard to verify (we leave the details as an exercise) that  $\mathcal{M} \approx_{FO(Q^R)_r} \mathcal{M}'$ . However,

$$\neg most_{\mathcal{M}}(A \cup B, B), \text{ whereas } most_{\mathcal{M}'}(A' \cup B', B'),$$

so  $\mathcal{M}$  and  $\mathcal{M}'$  can be distinguished by means of *most*.  $\square$

Most of the definability results so far in this chapter are summed up in Fig. 14.1.

<sup>2</sup> Apparently (see Barwise and Cooper 1981: 216) this result was first proved by David Kaplan in 1965; the proof, which made essential use of infinite models, was never published. We mentioned in Ch. 2.4 (n. 11) that Rescher (1962) made the even stronger claim—without proof—that *most* isn't definable from any finite number of type  $\langle 1 \rangle$  quantifiers; see Theorem 14 below. The first proof using the EF-tools and finite models was given by Barwise and Cooper (1981).

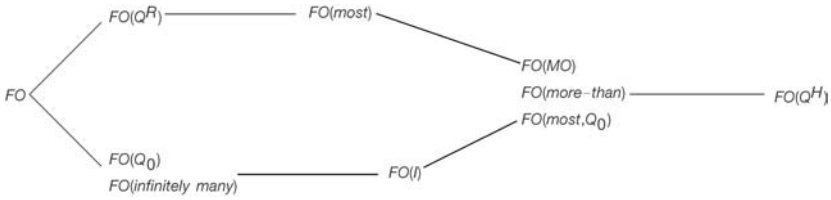


Figure 14.1 Relative strengths of some logics

Here the lines from left to right mean ‘strictly weaker than’, whereas the logics in the upper part of the diagram are incomparable in strength to the logics in the lower part. We also note that all logics in the diagram except  $FO(Q^R)$  relativize: the relativizations of their quantifiers are already definable in them. For example,  $(Q_0)^{\text{rel}} = \textit{infinitely many}$ , which is definable by the sentence  $Q_0x(A(x) \wedge B(x))$ , but  $\textit{most} = (Q^R)^{\text{rel}}$  is not definable in  $FO(Q^R)$ .  $FO(\textit{most})$  relativizes, since  $\textit{most}$  is already a relativized quantifier (Fact 6 in Chapter 13); one easily sees that  $FO(I)$ ,  $FO(MO)$ , and  $FO(Q^H)$  also relativize.

The only fact of Fig. 14.1 that doesn’t follow from results in this chapter is the claim that logic with the Henkin quantifier is stronger than  $FO(MO)$ . We saw in Chapter 2.5 that  $MO$  is definable in  $FO(Q^H)$ . Exercise 4 at the end of this chapter shows that *an even number of* is not definable in  $FO(MO)$ , whereas Exercise 5 shows that it *is* definable in  $FO(Q^H)$ . It follows that  $FO(MO) < FO(Q^H)$ .

The logic  $FO(Q_{\text{even}}) \equiv FO(\textit{an even number of})$  is not itself in Fig. 14.1; this logic is incomparable in strength to all the logics in the diagram except  $FO$  and  $FO(Q^H)$ .

These expressibility facts have consequences for natural language semantics. For example, we see that different forms of size comparison have different strength. It is unlikely that a natural language has a primitive expression (NP) denoting  $Q^R$ ; instead, many languages have quantirelations denoting the stronger *most* (and other proportional quantifiers). But these cannot compare the sizes of arbitrary sets (only of *disjoint* sets: if  $A \cap B = \emptyset$ ,  $\textit{most}(A \cup B, B) \Leftrightarrow |A| > |B|$ ). We also see that comparison only in terms of “as many as” but not “more than” or “at least as many as”, even with the help of Boolean operations and universal and existential quantification, is not sufficient for  $MO$ , or even for *most*.

No natural language is likely to have a quantirelation denoting  $MO$ , because quantirelations are CONSERV. But many languages manage size comparison for arbitrary sets with other constructions, such as the English *More As than Bs exist* (using the two-place *more-than*), or *There are more As than Bs* (same quantifier in an existential-there construction), or simply *The number of As is greater than the number of Bs*. Suppose that a putative language  $L$  lacks all of these. Can we conclude that English is not translatable into  $L$ ? As we saw in Chapter 12.5, we can, if there is an adequate formalization of  $L$  in a logical language  $L^f$  in which  $MO$  is not definable. But note that the conclusion presupposes that  $L^f$  really is adequate:

it must capture a sufficiently large portion of  $L$ , and in particular not leave out any other comparative constructions that  $L$  might employ.

In addition to facts about particular (real or possible) natural languages, there is again a methodological consequence of some importance, signaled by Proposition 12. This proposition is in fact quite significant. We have seen that often  $Q^{\text{rel}}$  is definable from  $Q$ . For example, *all* is definable from  $\forall$ , *at least five* is definable from  $\exists_{\geq 5}$ , *an even number of* is definable from  $Q_{\text{even}}$ . Thus one might be inclined to think that, after all, type  $\langle 1 \rangle$  quantifiers are enough for natural language. That ‘type  $\langle 1 \rangle$  is enough’ has been the received wisdom in logic since Frege, when used to formulate mathematical facts. But that is not how logic is used in natural language semantics. And Proposition 12 tells us that for some very common natural language quantifiers, type  $\langle 1 \rangle$  is *not* enough.

The question “Is type  $\langle 1 \rangle$  enough for natural languages?” is ambiguous, as we can now see. On the one hand, we showed in Chapter 4.5 that all type  $\langle 1, 1 \rangle$  quantification denotations are relativizations of type  $\langle 1 \rangle$  quantifiers (Proposition 3 in that chapter). So in this sense it is enough to have the type  $\langle 1 \rangle$  quantifiers, *and* the operation of relativization, at hand. But on the other hand, we have seen that the relativization operation may increase expressive power. Proposition 12 shows, via a translation argument, that there is no way to express *Most As are B* using only the locution *most things in the universe are A* and the usual *FO* apparatus. In this sense, type  $\langle 1 \rangle$  is definitely not enough.

In fact, this answer to the question can be significantly strengthened, in two directions.

**Theorem 13** (Westerståhl 1991; Kolaitis and Väänänen 1995)

*Suppose  $Q$  is of type  $\langle 1 \rangle$  and monotone (increasing or decreasing). Then, over finite models,  $Q^{\text{rel}}$  is definable in  $FO(Q)$  if and only if  $Q$  is first-order definable.*

The proof, using the EF-tools, relies on the same strategy; one has to go through a number of cases, using some patience and a little combinatorics. Lots of familiar quantifiers are monotone, and for this case we thus know precisely when type  $\langle 1 \rangle$  ‘is enough’. Very often, it is not. But note that the restriction to monotone quantifiers is essential:  $Q_{\text{even}}$  is not *FO*-definable (Fact 1), but *an even number of* =  $(Q_{\text{even}})^{\text{rel}}$  is definable in terms of it.

The second strengthening is the following.

**Theorem 14** (Kolaitis and Väänänen 1995)

*Most is not definable in any logic of the form  $FO(Q_1, \dots, Q_n)$ , where the  $Q_i$  are of type  $\langle 1 \rangle$ . (In fact, the same holds for all the properly proportional quantifiers.)*

To prove this, more combinatorics is required, but in essence it boils down to combining our strategy with a clever use of the pigeon-hole principle.<sup>3</sup> Clearly, the

<sup>3</sup> Suppose you are putting objects in boxes. If there are more objects than boxes, at least one box is going to contain more than one object. This is the simplest form of pigeon-hole principle, but there are many combinatorially more intricate versions.

theorem provides an even stronger sense in which type  $\langle 1 \rangle$  is ‘not enough’ for natural language semantics.

Interdefinability among monadic quantifiers is a vast but still reasonably manageable subject, and several general results are known. A survey containing a large part of the known facts is Väänänen 1997.

#### 14.4 MONOTONICITY AND DEFINABILITY

Monotone quantifiers are ubiquitous in natural languages; see Chapter 5.3. Many simple and complex quantifier expressions in natural languages denote monotone quantifiers or Boolean combinations of monotone quantifiers: e.g. *at most four*, *more than seven*, *at least two-thirds of the*, *exactly one*, *between three and six*, *either exactly three or at least nine*. It is tempting to venture the suggestion that all determiner denotations are such Boolean combinations, but *an even number of* and related quantifiers form conspicuous exceptions. However, could it be that this quantifier is definable from monotone ones in some other way?

Thus, it makes sense to ask for a characterization of which quantifiers are logically definable from monotone ones, and to see if an answer to that question leads to a suitable generalization that pinpoints the role of monotone quantification in natural languages. We discuss these issues, which are a bit more subtle than one might think, in this section. The technical results come from Väänänen 1997 and Väänänen and Westerståhl 2002, to which we refer for proofs and further details.

Unless otherwise indicated, *monotone* here means *monotone increasing*. Also, we restrict attention to *finite* models. Recall the characterization of the (ISOM and) monotone type  $\langle 1 \rangle$  quantifiers in Chapter 5. Each such quantifier has the form  $Q_f$ , where  $f$  is a function from natural numbers to natural numbers such that  $f(n) \leq n + 1$  for all  $n$  (Chapter 5.4.1), and

$$(Q_f)_M(A) \iff |A| \geq f(|M|)$$

Furthermore, the relativizations of these quantifiers are precisely the right monotone CONSERV and EXT type  $\langle 1, 1 \rangle$  quantifiers, by Proposition 3 in Chapter 4. Each one of these thus has the form  $Q_f^{\text{rel}}$ , where

$$Q_f^{\text{rel}}(A, B) \iff |A \cap B| \geq f(|A|)$$

(no need to mention  $M$  here). Recall also the characterization of monotonicity in the number triangle (Chapter 5.4.2), where the quantifiers  $Q_f$  (or  $Q_f^{\text{rel}}$ ) are precisely

Kolaitis and Väänänen prove by these methods that, on finite models, *MO* is not definable from any finite number of type  $\langle 1 \rangle$  quantifiers, not even if  $I$  is added. And on finite models, *most* is expressively equivalent to *MO* (Ch. 13.2.1), so the result follows. The extension to all properly proportional quantifiers is (a special case of a result) proved by Väänänen (1997). Kolaitis and Väänänen also prove that  $I$  is not definable from type  $\langle 1 \rangle$  quantifiers; this requires more combinatorics (van der Waerden’s Theorem, which is a more advanced version of the pigeon-hole principle).



those which, on each level  $n$ , switch from  $-$  to  $+$  at at most one point (namely, the point  $(f(n), n - f(n))$ ).

Let  $Q$  be any type  $\langle 1 \rangle$  quantifier. The characterization question above splits into two:

- When is  $Q$  definable in terms of monotone type  $\langle 1 \rangle$  quantifiers, i.e. quantifiers of the form  $Q_f$ ?
- When is  $Q$  definable in terms of monotone, CONSERV and EXT type  $\langle 1, 1 \rangle$  quantifiers, i.e. quantifiers of the form  $Q_f^{\text{rel}}$ ?

Since (right) monotone quantirelation denotations usually are of the form  $Q_f^{\text{rel}}$ , the second question seems to be of greater linguistic interest.<sup>4</sup> But the first question has a simpler answer, which, moreover, is easily visualizable in the number triangle.

### bounded oscillation

The *oscillation* of a type  $\langle 1 \rangle$  quantifier  $Q$  at level  $n$  in the number triangle is the number of times  $Q$  switches from  $+$  to  $-$ , or vice versa, at that level.  $Q$  has *bounded oscillation* if there is a finite bound  $m$  on the oscillation of  $Q$  (at any level). See Fig. 14.2.

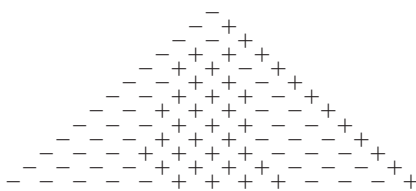


Figure 14.2 A boundedly oscillating quantifier

For example, any monotone quantifier has bounded oscillation (with  $m = 1$ ). The quantifier *either between three and five or more than eight* has bounded oscillation (with  $m = 3$ ). Indeed, any Boolean combination of monotone quantifiers has bounded oscillation. On the other hand, *an even number of* is a typical quantifier with unbounded oscillation.

**Theorem 15** (Bounded Oscillation Theorem, Väänänen 1997)

*$Q$  is definable in terms of quantifiers of the form  $Q_f$  if and only if  $Q$  has bounded oscillation.*

The proof, as usual, uses the EF-tools.

<sup>4</sup> From a linguistic perspective, one should rather ask when  $Q^{\text{rel}}$  is definable in terms of quantifiers of the form  $Q_f^{\text{rel}}$ . But we saw in Corollary 9 in Ch. 13.2 that this question has the same answer as (b).

However the Bounded Oscillation Theorem fails drastically for definability in terms of quantifiers of the form  $Q_f^{\text{rel}}$ . An illustration of this is the fact that  $Q_{\text{even}}$ , and thus *an even number of*, is, rather surprisingly, definable in terms of one such quantifier. The function  $f_0$  needed to express  $Q_{\text{even}}$  is simply

$$(14.2) \quad f_0(n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd} \end{cases}$$

Somewhat less perspicuously,  $Q_{f_0}^{\text{rel}}$  is expressible in English as follows:

$$(14.3) \quad Q_{f_0}^{\text{rel}}(A, B) \text{ iff either some } A \text{ are } B \text{ and there is an even number of } A, \text{ or at least two } A \text{ are } B \text{ and there is an odd number of } A.$$

Now we can define  $Q_{\text{even}}$  with a simple trick:

$$\begin{aligned} (Q_{\text{even}})_M(A) &\iff |A| \text{ is even} \\ &\iff 1 \geq f_0(|A|) \\ &\iff A = \emptyset \text{ or } \exists a \in A (|\{a\}| \geq f_0(|A|)) \\ &\iff A = \emptyset \text{ or } \exists a \in A (Q_{f_0}^{\text{rel}})_M(A, \{a\}) \\ (14.4) \quad &\iff (M, A) \models \neg \exists x A(x) \vee \exists x (A(x) \wedge Q_{f_0}^{\text{rel}} y(A(y), y = x)) \quad \square \end{aligned}$$

In fact, this argument has nothing in particular to do with  $Q_{\text{even}}$ , but works for any EXT type  $\langle 1 \rangle$  quantifier, or equivalently (Proposition 2 in Chapter 6), any symmetric CONSERV and EXT type  $\langle 1, 1 \rangle$  quantifier: recall (Corollary 4 in Chapter 6) that the latter are exactly the quantifiers  $Q$  for which there is a set  $S$  of numbers such that

$$Q_M(A, B) \iff |A \cap B| \in S$$

Thus we have the following result.

### Proposition 16

*All symmetric CONSERV and EXT quantifiers are definable in terms of one quantifier of the form  $Q_f^{\text{rel}}$ .*

Theorem 15 answers question (a) above, but what about (b), which appeared to be linguistically more relevant? Väänänen and Westerståhl (2002) introduce a generalization of the notion of bounded oscillation, called *bounded color oscillation*, and prove, again using the EF-tools, that the quantifiers with that property are exactly the ones definable in terms of quantifiers of the form  $Q_f^{\text{rel}}$ , thus answering (b). Theorem 15 is a special case of this Bounded Color Oscillation Theorem. Bounded color oscillation is also defined with the help of the number triangle, but is less easy to visualize. We shall not give the definition here, only note that it is a very general property. Indeed, it turns out to be quite hard to find examples of quantifiers with unbounded color oscillation. The only known example, at least as far as we know, is a quantifier construed explicitly for that purpose in Väänänen and Westerståhl 2002, using some serious

finite combinatorics (van der Waerden's Theorem). It is most certainly not a natural language quantifier.

Thus, it appears to be quite safe to propose as a linguistic universal that all quantifier-relations are definable, in the first-order logical sense used here, from monotone CONSERV and EXT quantifiers. But such a universal may be almost empty. Can it be improved?

A strengthening that naturally comes to mind is that the quantifiers used in the definition also be quantifier-relations. Is the universal true in that form? A test case is precisely *an even number of*, defined as above from the quantifier  $Q_{f_0}^{\text{rel}}$ . Is the latter a natural language quantifier?

We saw in (14.3) how to express  $Q_{f_0}^{\text{rel}}$  in English. But notice that it is one thing that there exists a reasonable English sentence expressing the defining condition of a particular CONSERV and EXT type  $\langle 1, 1 \rangle$  quantifier, and quite another thing that this quantifier is denoted by some English determiner phrase. Indeed, it is quite clear that (14.3) does not provide such a determiner phrase, and it seems to us implausible that one could be provided.

For example, looking at the first disjunct of (14.3),

(14.5) Some  $A$  are  $B$ , and there is an even number of  $A$ ,

one might be inclined to try a relative construction like

(14.6) Some  $A$  such that there is an even number of them are  $B$ .

But this seems not to work: with a phrase *some  $A$  such that  $\varphi$*  one can restrict  $A$  to those objects satisfying  $\varphi$ , not state a property of  $A$ .

If this is correct, we could say that  $Q_{f_0}^{\text{rel}}$  is perhaps a natural language quantifier, but not a quantifier-relation denotation, which means that the suggested strengthening of the universal about monotone definability is false.<sup>5</sup>

Moreover, even if it were possible to cook up some determiner phrase denoting  $Q_{f_0}^{\text{rel}}$ , there would remain the question of whether the definition of *an even number of* in terms of it, given by (14.4), is 'reproducible' by some reasonable English sentence. This is the issue we mentioned briefly at the end of Chapter 12.5.5, about whether positive definability facts in logic can be transferred to natural languages. Two things can be said. First, we would certainly not get a cognitively synonymous paraphrase, since the trick that (14.4) builds on, though simple, is not one that speakers can be expected to grasp immediately. Second, it appears to be quite hard to render (14.4) in English, due to the fact that the scope of the quantification is a unit set, and it is not obvious how to make a verb phrase denote a unit set.<sup>6</sup>

<sup>5</sup> At least if there is no essentially different definition of *an even number of*, in terms of a monotone CONSERV and EXT type  $\langle 1, 1 \rangle$  quantifier, which seems unlikely.

<sup>6</sup> It may work in special cases: e.g. slightly analogous to the second disjunct of (14.4), one could say

There is some  $A$  such that no  $B$  is identical to it.

Coming back to our search for a non-trivial monotonicity universal, let us, instead of trying to strengthen the requirements on the quantifiers  $Q_f^{\text{rel}}$ , consider if we can strengthen the notion of monotonicity itself. One such strengthening was discussed in Chapter 5.6: *smoothness*.

At one point it was conjectured that all right increasing determiner denotations are in fact smooth, but we have seen that although this seems to hold in a vast majority of the cases, there are some exceptions. We found one kind of exception in Chapter 7.13 (Proposition 8), with  $\text{MON}\uparrow$  possessive determiners like *at least two of most students'*; these are not smooth. But they are not  $\text{ISOM}$  either. If we limit attention to  $\text{ISOM}$  and  $\text{MON}\uparrow$  quantirelation denotations, there are still some that are not smooth: it appears that each of those involves a cardinality condition on the restriction argument (see Chapter 5.6): for example, *at least two of the five (or more)*.

But even if not all right monotone quantirelation denotations are smooth or co-smooth, might it not still be the case that all quantirelation denotations are logical definable from smooth ones? No, *an even number of* still looks like an exception, and indeed it is. This follows from the next result from Väänänen and Westerståhl 2002, which is a corollary to one half of their Bounded Color Oscillation Theorem.

**Theorem 17** (Väänänen and Westerståhl 2002)

*If  $Q$  is definable in terms of quantifiers of the form  $Q_f^{\text{rel}}$ , where  $f$  is smooth, then  $Q$  has bounded oscillation.*

Summing up, our search for a linguistic universal about definability from monotone quantifiers has been somewhat disappointing. We cannot require these quantifiers to be smooth; nor can we require them to be quantirelation denotations. We do have a

(14.7) *General monotonicity universal:* All ( $\text{ISOM}$ ) quantifiers denoted by quantirelations in natural languages are logically definable from monotone  $\text{CONSERV}$  and  $\text{EXT}$  quantifiers.

But this seems to allow far too much. The one remaining option would be to restrict the form of the definition, so that only definitions that are easily rendered in, say, English are allowed. But we already saw that the definition of *an even number of* from  $Q_{f_0}$  is not a likely candidate. So perhaps the proper conclusion is that (14.7) is the best one can do, and that in natural languages, almost all determiner denotations can be obtained rather simply from monotone ones, but that *an even number of* and its variants are exceptions.

We shall leave the discussion of how monotonicity relates to expressivity issues for natural languages here, but presumably it could be taken much further. First, as just indicated, one could restrict the allowed forms of definition to 'linguistically natural' ones. For example, *iteration* is a particularly natural linguistic form for defining quantifiers, or, more generally, Boolean combinations of iterations, i.e. *unary complexes*

i.e.  $\exists x(Ax \wedge \text{no } y(By, y = x))$  (in other words,  $A - B \neq \emptyset$ , or, *not all A are B*). But with other determiners this looks much worse.

(see Chapter 10.1). There are some studies of this kind of definability.<sup>7</sup> But a sufficiently general and viable notion of ‘natural language definability’ is still lacking. Second, we saw in Chapter 5 that right monotonicity is just one of the basic forms of monotonicity occurring in natural languages; the others being the four weak forms of left monotonicity,  $\uparrow_{\text{SEM}}\text{MON}$ , etc. It would be interesting to study how these too relate to issues of expressive power.

#### 14.5 EXERCISES

This chapter has presented and illustrated a method for undefinability proofs. The following exercises are meant to further familiarize the reader with this method, as well as to state some additional facts about definability.

1. Show that  $Q_0$  is not definable in the logic  $FO(\{\exists_{\geq n} : n = 1, 2, \dots\})$ .
2. Show that *more than one-third of* is not  $FO$ -definable. Is it definable in  $FO(\text{most})$ ? [This is harder.]
3. Show that the quantifiers  $p/q$  and  $[p/q]$ , where  $0 < p < q$  (Chapter 2.3) are not  $FO$ -definable.
4. Show that *an even number of* is not definable in  $FO(MO)$ . [Hint: Show that not even ‘ $|M|$  is even’ is so definable.]
5. Show that ‘ $|M|$  is even’ is definable with the Henkin prefix, i.e. in the logic  $FO(Q^H)$ . [Hint: Consider the sentence

$$Q^H u v x y ((u = x \rightarrow v = y) \wedge (u = y \leftrightarrow v = x) \wedge u \neq v)$$

Show (cf. Chapter 2.5) that this says that there is a function  $f$  such that  $f = f^{-1}$  and  $f(a) \neq a$  for all  $a \in M$ . Conclude that  $|M|$  must be even.]

6. Show that  $Q^R$  is not definable in  $FO(Q_{\text{even}})$ .
7. Give two proofs of Fact 6: one with finite models and one with infinite models [this presupposes use of uncountable sets].
8. Fill in the details of the proof of Proposition 12.
9. Show that, over finite models, *more than one-half of* is definable in terms of *at least one-half of*, and vice versa. Is *at least one-third of* definable in terms of *more than one-third of*? [Harder.]
10. *Partial type*  $\langle 1, 1 \rangle$  *quantifiers*: Associate with  $Q$  a *domain* on each  $M$ ,  $\text{dom}(Q_M) \subseteq P(M)$ , for the first argument, and treat  $Q_M(A, B)$  as *undefined* whenever  $A \notin \text{dom}(Q_M)$ . For example, this is how a determiner like *the three* is treated in Barwise and Cooper 1981: *the three* $_M(A, B)$  is defined iff  $|A| = 3$ , and then the truth condition is just  $A \subseteq B$ .

<sup>7</sup> Though some of it concerns local rather than global definability. See e.g. van Benthem 1989; Keenan 1992; Ben-Shalom 1994; Westerståhl 1994.

- (a) Formulate CONSERV, EXT, and ISOM for partial type  $\langle 1, 1 \rangle$  quantifiers. Define a notion of partial type  $\langle 1 \rangle$  quantifier so that the CONSERV and EXT partial type  $\langle 1, 1 \rangle$  quantifiers are precisely the relativizations of these.
- (b) Adapt the notion of a quantifier  $Q$  being definable in a logic  $L$  to the case when  $Q$  is partial.

Any partial type  $\langle 1, 1 \rangle$  quantifier  $Q$  has a total version  $Q^{tot}$  defined by

$$Q_M^{tot}(A, B) \iff A \in \text{dom}(Q_M) \ \& \ Q_M(A, B)$$

- (c) Suppose the condition ' $A \in \text{dom}(Q_M)$ ' is definable in  $L$  (in the usual sense, just like a total type  $\langle 1 \rangle$  quantifier). Show that  $Q$  is definable in  $L$  iff  $Q^{tot}$  is definable in  $L$ .
- (d) Let  $Q$  be such that  $A \in \text{dom}(Q_M) \iff |A|$  is even, and on such  $A$ ,  $Q_M(A, B)$  means  $A \subseteq B$ .
- i. Show that  $Q$  is  $FO$ -definable.
  - ii. Show that  $Q^{tot}$  is not  $FO$ -definable. [Hint: Define  $Q_{\text{even}}$  from  $Q^{tot}$ .]
- (e) Adapt the strategy for proofs of the undefinability of  $Q$  in  $L$  to the case when  $Q$  is partial. Consider the quantifier  $Q$  such that  $A \in \text{dom}(Q_M) \iff |A|$  is odd, and, for such  $A$ ,  $Q_M(A, B) \iff |A \cap B|$  is even. Show that  $Q$  is not  $FO$ -definable.
11. Show that ' $|M|$  is even' is definable in  $FO(Q_{f_0})$ , where  $f_0$  is the function defined by (14.2). Conclude that  $Q_{f_0}$  is not  $FO$ -definable. Is  $Q_{\text{even}}$  definable in  $FO(Q_{f_0})$ ?
  12. Show that  $Q_{\text{even}}$  is definable from a quantifier of the form  $Q_g^{\text{rel}}$ , where  $g$  is non-decreasing. [Try  $g(n) = n$  if  $n$  is even,  $g(n) = n - 1$  if  $n$  is odd.]
  13. A function  $f$  is *eventually constant* if there are  $n_0$  and  $k$  such that for all  $n \geq n_0$ ,  $f(n) = k$ . Call  $f$  *trivial* if either  $f$  or  $f^{\text{d}}$  is eventually constant. Show that  $Q_f$  is  $FO$ -definable iff  $f$  is trivial.
  14. (Väänänen and Westerståhl 2002) Show that if  $f$  is non-trivial and smooth, then it eventually leaves the edges of the number triangle, in the sense that
 
$$\forall r \exists n_0 \forall n > n_0 (r < f(n) < n - r).$$
  15. (Väänänen and Westerståhl 2002) Strengthen Exercise 4 by showing that ' $|M|$  is even' is not definable from smooth quantifiers. [Hint: Suppose it were definable in  $FO(\{Q_{f_1}^{\text{rel}}, \dots, Q_{f_k}^{\text{rel}}\})$ , where the  $f_i$  are smooth. Conclude from Exercise 13 that we can assume that the  $f_i$  are non-trivial. Since ' $|M|$  is even' has empty vocabulary, it suffices to find for each  $r$  sets  $M$  and  $M'$ , where one has even and the other has odd cardinality, such that  $M \approx_{FO(\{Q_{f_1}^{\text{rel}}, \dots, Q_{f_k}^{\text{rel}}\})^r} M'$ . First, given  $r$ , use Exercise 14 to find  $n_1$  such that for  $1 \leq i \leq k$ ,
 
$$\forall n > n_1 (r < f_i(n) < n - r)$$

Then take  $M$  and  $M'$  with  $|M| > n_1 + r$  and  $|M'| = |M| + 1$ , and verify that this works.] Conclude that the converse of Theorem 17 fails.

## EF-tools for Polyadic Quantifiers

In this final chapter we shall see how the EF-tools can be made to work for arbitrary quantifiers and structures. The idea is the same, but the criterion for  $L'$ -equivalence needs to be reformulated. The simplest and most intuitive way to do this is in terms of certain two-person games, and we begin by describing these (sections 15.1 and 15.2). Applying these games to interesting examples turns out to be harder than in the monadic case, but we give some relatively simple illustrations, and then a rather detailed outline of the application to branching quantifiers, in section 15.3. The result is that branching of many familiar quantifier denotations is not definable from any finite number of monadic quantifiers.

There is a similar result for Ramsey quantifiers (section 15.4), which we mention without giving the proof, and which is relevant for the interpretation of certain reciprocal sentences in natural language. For resumptions (section 15.5) there are slightly weaker results; in this case the monadic EF-tools can actually be used. We give a fair amount of detail, and for the rest refer the reader to Hella *et al.* 1997. Also, we discuss the inverse question of under what circumstances a quantifier is definable from its  $k$ -ary resumption.

In each case, we state the conclusions about natural languages that may be drawn from these logical results, via formalization as described in Chapter 12.5.

### 15.1 EF-GAMES

Let two models  $\mathcal{M}, \mathcal{M}'$  for the same vocabulary be given. They do not have to be monadic, but let us suppose (for simplicity) that they are relational; i.e. the vocabulary only has predicate symbols (of various arities).  $EF^r(\mathcal{M}, \mathcal{M}')$  is the simple game played in  $r$  rounds by two players, called *Duplicator* and *Spoiler*, in the following way.

#### the game $EF^r(\mathcal{M}, \mathcal{M}')$

Each round begins by Spoiler making a *move*. He chooses one of  $\mathcal{M}, \mathcal{M}'$ , and then picks an element in the chosen model. Duplicator has to respond by picking an element in the *other* model. After  $r$  rounds, the game is over, and elements  $(a_1, a'_1, \dots, a_r, a'_r)$  have been picked, where  $a_1, \dots, a_r \in \mathcal{M}$  and  $a'_1, \dots, a'_r \in \mathcal{M}'$ . Now Duplicator *wins* if

$$(a_1, \dots, a_r) \cong (a'_1, \dots, a'_r)$$

Otherwise Spoiler wins.

So the idea is that Duplicator is trying to show that  $\mathcal{M}$  and  $\mathcal{M}'$  are as similar as possible, whereas Spoiler is trying to show that they are different. Who can succeed depends on the models  $\mathcal{M}$  and  $\mathcal{M}'$ , of course, but also on how many rounds are played.

For example, let  $\mathcal{M} = (\mathbb{N}, <)$  be the ordered structure of the natural numbers, and  $\mathcal{M}' = (\mathbb{Q}, <)$  the rational numbers with the usual ordering. These models are quite different, so it should be easy for Spoiler to win the corresponding game. In fact, he can win in three rounds, as follows. In the first two rounds he picks natural numbers  $n$  and  $n + 1$ . To avoid losing, Duplicator has to pick two rational numbers  $q_1$  and  $q_2$  with  $q_1 < q_2$ . But then, in the third round, Spoiler picks a rational number *between*  $q_1$  and  $q_2$ . Duplicator should respond with a natural number between  $n$  and  $n + 1$ , but there is none, so she loses.

Of course one may lose a game because of poor playing. What interests us is whether one of the players has a *winning strategy*, which allows him or her to win no matter how the opponent plays. In the example above, we saw that Spoiler has a winning strategy in the game  $EF^3(\mathcal{M}, \mathcal{M}')$ . It is an elementary fact of game theory that in any game  $EF^r(\mathcal{M}, \mathcal{M}')$ , either Duplicator or Spoiler has a winning strategy.

For another example, consider monadic models  $\mathcal{M} = (M, A)$  and  $\mathcal{M}' = (M', A')$ , where  $|A| = k$ ,  $|A'| > k$ , and  $|M - A|, |M' - A'| > k$ . Then Duplicator can always win the game  $EF^k(\mathcal{M}, \mathcal{M}')$ . For the best Spoiler can do (from his point of view) is to pick  $k$  distinct elements in  $A'$ . But Duplicator can always respond with  $k$  distinct elements in  $A$  (and if he picks elements in  $M - A'$ , she responds with elements in  $M - A$ , etc.). So after  $k$  rounds, she wins. But if a  $(k + 1)$ th round is played, and Spoiler picks another element of  $A'$ , Duplicator can respond only with either an element of  $A$  that has already been picked, or an element of  $M - A$ ; in either case she loses. Thus, Duplicator has a winning strategy in  $EF^k(\mathcal{M}, \mathcal{M}')$ , but not in  $EF^{k+1}(\mathcal{M}, \mathcal{M}')$ .

Now consider condition (13.7) (Chapter 13.4), in the definition of  $L^r$ -similarity, for any monadic models. From the example just mentioned it is clear that if the condition holds, then Duplicator has a winning strategy in  $EF^r(\mathcal{M}, \mathcal{M}')$ . On the other hand, if the condition fails, say that  $|P_{i,\mathcal{M}}| < r$  but  $|P_{i,\mathcal{M}'}| \neq |P_{i,\mathcal{M}}|$ , then by similar reasoning, Spoiler can always win.

Let us write

$$\mathcal{M} \sim_{FO^r} \mathcal{M}'$$

to mean that Duplicator has a winning strategy in  $EF^r(\mathcal{M}, \mathcal{M}')$ . From the reasoning just indicated, and Theorem 13 in Chapter 13.4, we get the theorem below, restricted to monadic structures.

### Theorem 1

For all models  $\mathcal{M}, \mathcal{M}'$  for the same vocabulary (monadic or not),

$$\mathcal{M} \equiv_{FO^r} \mathcal{M}' \iff \mathcal{M} \sim_{FO^r} \mathcal{M}'$$

But the game-theoretic formulation works equally well for non-monadic models. Moreover, it too can be generalized to logics with added quantifiers. We now define



the corresponding games, and show that they result in a modified strategy for undefinability proofs.

Consider first  $L = FO(Q)$ , where  $Q$  is of type  $\langle 1 \rangle$ . In a game for this logic, the players have to pick not only elements of the models but also subsets of the universes (belonging to  $Q_M$  or  $Q_{M'}$ ).

**the game  $EF(Q)^r(\mathcal{M}, \mathcal{M}')$**

This time, at each round Spoiler can either make a move as in  $EF^r(\mathcal{M}, \mathcal{M}')$ , called an  $\exists$ -move (and Duplicator must respond with a similar move), or he can make a  $Q$ -move, as follows. Spoiler chooses one of  $\mathcal{M}$  and  $\mathcal{M}'$  (say,  $\mathcal{M}$ ; the other case is symmetric), and a subset  $X$  of  $M$  such that  $Q_M(X)$  holds. Duplicator must respond with  $X' \subseteq M'$  such that  $Q_{M'}(X')$  (if she is unable to do this, she loses). Next, Spoiler picks an element  $d' \in M'$ , and Duplicator responds with  $a \in M$ , subject to the condition

$$a \in X \Leftrightarrow d' \in X'$$

(if this is impossible, she loses). Thus, after  $r$  rounds, elements  $a_1, d'_1, \dots, a_r, d'_r$  have been picked, and Duplicator wins iff

$$(a_1, \dots, a_r) \cong (d'_1, \dots, d'_r)$$

Letting  $\mathcal{M} \sim_{FO(Q)^r} \mathcal{M}'$  mean that Duplicator has a winning strategy in  $EF(Q)^r(\mathcal{M}, \mathcal{M}')$ , we have the following result.<sup>1</sup>

**Theorem 2**

For all models  $\mathcal{M}, \mathcal{M}'$  for the same vocabulary,  $\mathcal{M} \sim_{FO(Q)^r} \mathcal{M}'$  implies that  $\mathcal{M} \equiv_{FO(Q)^r} \mathcal{M}'$ .

*Proof.* Suppose Duplicator has a winning strategy  $\tau$  in  $EF(Q)^r(\mathcal{M}, \mathcal{M}')$ . We show by induction over the complexity of  $\varphi$  that

(A) $_{\varphi}$  If  $k \leq r$ , and  $a_1, \dots, a_k \in M$  and  $d'_1, \dots, d'_k \in M'$  have been played according to  $\tau$ , and if  $\varphi(x_1, \dots, x_k)$  has quantifier rank at most  $r - k$  and at most the free variables shown, then

$$(B) \mathcal{M} \models \varphi(a_1, \dots, a_k) \Leftrightarrow \mathcal{M}' \models \varphi(d'_1, \dots, d'_k)$$

The case  $k = 0$  gives the result. So suppose that  $a_1, \dots, a_k \in M$  and  $d'_1, \dots, d'_k \in M'$  have been played according to  $\tau$ . Since  $\tau$  is winning, Duplicator has not lost at round  $k$ , which implies that  $(a_1, \dots, a_k) \cong (d'_1, \dots, d'_k)$ .

The base step is when  $\varphi(x_1, \dots, x_k)$  is atomic. But then it is immediate that (B) holds. Likewise, for a conjunction or a negation the result follows directly from the

<sup>1</sup> Note to the reader: If you skipped Ch. 13.4.3, you should skip the proof of this theorem too. But if not, the proof will be very easy to follow, since it has the same structure as the second half of the proof of Theorem 13 in that chapter.

induction hypothesis. Furthermore, if  $k = r$ ,  $\varphi(x_1, \dots, x_k)$  has quantifier rank 0 and is thus an atomic formula or a Boolean combination of atomic formulas. Then (B) follows by the reasoning just made. Assume, then, that  $k < r$ .

Of the remaining cases, we look at  $\varphi(x_1, \dots, x_k) = Qx\psi(x, x_1, \dots, x_k)$ , assumed to have quantifier rank at most  $r - k$ . Suppose  $\mathcal{M} \models Qx\psi(x, a_1, \dots, a_k)$ . With  $X = \psi(x, a_1, \dots, a_k)^{\mathcal{M}, x}$ , we have  $Q_M(X)$ . Now let Spoiler play  $X$ .  $\tau$  works whatever set Spoiler plays; assume Duplicator's response is  $X' \subseteq M'$  such that  $Q_{M'}(X')$ . We claim that

$$(C) \quad X' = \psi(x, a'_1, \dots, a'_k)^{\mathcal{M}', x}$$

This means that  $\mathcal{M}' \models Qx\psi(x, a'_1, \dots, a'_k)$ . Suppose (C) is *not* true. There are two cases.

*Case 1:* There is some  $d'$  in  $X' - \psi(x, a'_1, \dots, a'_k)^{\mathcal{M}', x}$ . Let Spoiler play  $d'$ .  $\tau$  tells Duplicator to play some  $a \in M$ . Since  $\tau$  is winning,  $a \in X$ . Also, by the induction hypothesis (A) $_\psi$  (for  $k + 1$  rounds; note that  $\psi$  has quantifier rank at most  $r - (k + 1)$ ),  $a \notin \psi(x, a_1, \dots, a_k)^{\mathcal{M}, x}$ , contradicting the definition of  $X$ .

*Case 2:* There is some  $d'$  in  $\psi(x, a'_1, \dots, a'_k)^{\mathcal{M}', x} - X'$ .  $\tau$  directs Duplicator to pick some  $a \in M$ , and, by the same reasoning as in case 1,  $a \in \psi(x, a_1, \dots, a_k)^{\mathcal{M}, x} - X$ , a contradiction.

The other direction is symmetric, so (B) is proved, and thereby the theorem.  $\square$

This theorem is in principle what we need to reformulate the strategy for undefinability proofs (see below), but we may still ask: What about the converse? The reader who looked at the first half of the proof of Theorem 13 in Chapter 13.4 will recall that it is essential for that argument to work that the sets talked about are *definable* by  $L$ -formulas in the respective models. For monadic models only the (variants of the) blocks mattered, and they were precisely the definable sets (Proposition 14 in Chapter 13.4), so no extra stipulation about definability was needed for the notion of  $L'$ -similarity. In the general case, however, we do need such a stipulation. So consider the following modified game.

#### the game $DEF(Q)^r(\mathcal{M}, \mathcal{M}')$

This game is as  $EF(Q)^r(\mathcal{M}, \mathcal{M}')$ , except that the  $Q$ -moves are as follows: Suppose  $a_1, \dots, a_k \in M$  and  $a'_1, \dots, a'_k \in M'$  have been played ( $k < r$ ). Spoiler chooses one of  $M$  and  $M'$  (say,  $M$ ), and a subset  $X$  of  $M$  such that  $Q_M(X)$ , and *which is  $L^{(r-k)}$ -definable in  $\mathcal{M}$  with parameters  $a_1, \dots, a_k$* . The rest of the move is played as before.

This game is easier for Duplicator to win, since Spoiler's choice of sets is severely restricted. For example, if  $M$  is infinite, there are uncountably many subsets of  $M$ , but only countably many definable ones. But note that the proof above goes through exactly as before, since the set we had Spoiler play at the induction step was in fact  $L^{(r-k)}$ -definable in  $\mathcal{M}$  with parameters  $a_1, \dots, a_k$ . So we may as well use the game

$DEF(Q)^r(\mathcal{M}, \mathcal{M}')$  for undefinability proofs, which is good, since it is in general easier to describe a winning strategy for Duplicator in this game.

Moreover, one can now show that the converse implication holds. The structure is as in the first half of the proof of Theorem 13 in Chapter 13, but the details are somewhat different.<sup>2</sup>

The description of these games is straightforwardly reformulated for quantifiers of other types. For example, if  $Q$  has type  $(1, 1)$ , the  $\exists$ -moves are as before, but a  $Q$ -move requires Spoiler to pick two (possibly suitably definable) sets  $X_1, X_2 \subseteq M$  (say), such that  $Q_M(X_1, X_2)$ , and Duplicator must then respond with  $X'_1, X'_2$  such that  $Q_{M'}(X'_1, X'_2)$ . Then Spoiler picks two elements  $d', b' \in M'$ , and Duplicator responds with  $a, b \in M$ , with the condition that  $a \in X_1 \Leftrightarrow d' \in X'_1$ , and  $b \in X_2 \Leftrightarrow b' \in X'_2$ . If after  $r$  rounds  $k$  elements in each structure have been played (so  $r \leq k \leq 2r$ ), the winning condition for Duplicator is as before, that the respective sequences are partially isomorphic.

Similar  $Q$ -moves apply to Lindström quantifiers of arbitrary type  $\langle n_1, \dots, n_k \rangle$ . Then Spoiler first picks (definable)  $n_i$ -ary relations  $R_i$  in one of the models,  $1 \leq i \leq k$ .

<sup>2</sup> Here is a sketch: Suppose  $\mathcal{M} \equiv_{FO(Q)^r} \mathcal{M}'$ . The required winning strategy for Duplicator in  $DEF(Q)^r(\mathcal{M}, \mathcal{M}')$  will be to maintain the condition, at round  $k$  when  $a_1, \dots, a_k \in M$  and  $d'_1, \dots, d'_k \in M'$  have been played:

- (F)<sub>k</sub> For all  $L$ -formulas  $\varphi(x_1, \dots, x_k)$  with at most the free variables shown and quantifier rank at most  $r - k$ ,  $\mathcal{M} \models \varphi(a_1, \dots, a_k) \Leftrightarrow \mathcal{M}' \models \varphi(d'_1, \dots, d'_k)$ .

For  $k = 0$  this is just the assumption. Suppose  $k < r$  and (F)<sub>k</sub> has been maintained. We describe Duplicator's strategy when Spoiler makes a  $Q$ -move; handling  $\exists$ -moves is similar, but simpler. Suppose he plays  $X = \psi(x, a_1, \dots, a_k)^{M, x}$  such that  $Q_M(X)$ , where  $\psi$  has quantifier rank  $r - (k + 1)$ . Thus,  $\mathcal{M} \models Qx\psi(x, a_1, \dots, a_k)$ . Then Duplicator plays  $X' = \psi(x, a_1, \dots, a_k)^{M', x}$ . By (F)<sub>k</sub>,  $Q_{M'}(X')$  holds. Next, Spoiler plays  $d' \in M'$ .

Case 1:  $d' \in X'$ . Duplicator must pick  $a \in X$  such that, given that  $a, a_1, \dots, a_k \in M$  and  $d', d'_1, \dots, d'_k \in M'$  have been played, (F)<sub>k+1</sub> holds. Suppose this is *not* possible. That means that for all  $a \in X$  there is a formula  $\varphi_a$  of quantifier rank  $r - (k + 1)$  such that  $\mathcal{M} \models \varphi_a(a, a_1, \dots, a_k)$  but  $\mathcal{M}' \not\models \varphi_a(d', d'_1, \dots, d'_k)$ . In other words,

$$\mathcal{M} \models \forall x(\psi(x, a_1, \dots, a_k) \rightarrow \bigvee_{a \in X} \varphi_a(x, a_1, \dots, a_k))$$

This formula has quantifier rank  $r - k$ , so by (F)<sub>k</sub> it is satisfied by  $d'_1, \dots, d'_k$  in  $\mathcal{M}'$ . Hence, in particular,

$$\mathcal{M}' \models \bigvee_{a \in X} \varphi_a(d', d'_1, \dots, d'_k)$$

But this contradicts the assumption about the  $\varphi_a$ . The only problem with this argument is that  $X$  may be infinite, so it looks as if we have an infinite disjunction, and hence not an  $L$ -formula at all. But here we can use the following fact, which is not too hard to establish:

- (i) For a fixed quantifier rank  $n$ , there are only finitely many non-equivalent  $L^n$ -formulas in a given (finite) vocabulary.

Thus we can replace the disjunction above with a finite one, and the argument goes through.

Case 2:  $d' \notin X'$ . Reason as in case 1, but with  $\neg\psi$  instead of  $\psi$  and  $M - X$  instead of  $X$ . This completes the proof.

Duplicator responds with relations  $R'_i$ , after which Spoiler chooses  $n_i$ -tuples  $\bar{a}_i$  (in the other model), and Duplicator correspondingly chooses  $\bar{a}'_i$ , such that  $\bar{a}_i \in R_i \Leftrightarrow \bar{a}'_i \in R'_i$ . The winning condition is the same: the respective sequences of all elements in chosen tuples of the respective models must be partially isomorphic. The theorem generalizes with the same proof to this case. It also generalizes without change to logics of the form  $L = FO(\{Q_j : j \in I\})$ ; then Spoiler can make a  $Q_j$ -move for any  $j$ .

We thus have the following result (of which Theorem 1 is a special case):

**Theorem 3**

*For any logic  $L = FO(\{Q_j : j \in I\})$ , and for all models  $\mathcal{M}, \mathcal{M}'$  with the same vocabulary,  $\mathcal{M} \sim_{L^r} \mathcal{M}' \Leftrightarrow \mathcal{M} \equiv_{L^r} \mathcal{M}'$  (where the left-hand side now refers to a winning strategy in the corresponding DEF game).*

Getting back now to our main concern: as a general strategy for undefinability proofs we have arrived at the following.

**strategy for undefinability proofs: polyadic case**

To show that a certain property  $P$  is undefinable in  $L = FO(\{Q_j : j \in I\})$ , find, for each  $r$ , models  $\mathcal{M}, \mathcal{M}'$  such that  $\mathcal{M}$  has  $P$  but  $\mathcal{M}'$  doesn't, and Duplicator has a winning strategy in  $DEF(\{Q_j : j \in I\})^r(\mathcal{M}, \mathcal{M}')$ .

The strategy is guaranteed to work (by the previous theorem), but in practice it can be quite complicated to describe winning strategies. It seems to require some sort of analysis of the definable sets in the models. But the notion of  $L^r$ -definability is very similar to the notion of  $L^r$ -equivalence, which is the very notion we are trying to characterize! So the characterization is in a sense circular, by contrast with the monadic case.<sup>3</sup> In general, undefinability results for polyadic quantifiers are much harder than in the monadic case, and it is much less easy to find non-trivial illustrative examples. But there are some, and to give more feeling for these games, we now look at two of them.

**Example 4**

Let  $M$  be a denumerable set (e.g. the set of natural numbers), and let, for each  $n$ ,  $\mathcal{M}_n$  be a model  $(M, E)$ , where  $E$  is an equivalence relation on  $M$  with  $n$  equivalence classes, each of which has denumerably many elements. Now consider the games  $EF^r(\mathcal{M}_n, \mathcal{M}_m)$ . Since maintaining a partial isomorphism is just a matter of being able to pick elements in corresponding equivalence classes (there are no quantifier moves), it is clear, using Theorem 2, that

$$(15.1) \text{ If } r \leq m, n, \text{ where } m, n \text{ are natural numbers or } \aleph_0, \text{ then } \mathcal{M}_n \equiv_{FO^r} \mathcal{M}_m.$$

From this it is easy to see, using the strategy, that the type  $\langle 2 \rangle$  quantifiers

<sup>3</sup> Another version of the game, which to some extent eliminates this circularity, can be found in Caicedo 1980.

$(Q_{feq})_M(R) \iff R$  is an equivalence relation on  $M$  with finitely many equivalence classes

$(Q_{eq})_M(R) \iff R$  is an equivalence relation on  $M$  with an even number of equivalence classes

are not definable in  $FO$ . Now the same conclusion could in fact be reached already from our earlier results about monadic quantifiers, since

(15.2)  $Q_{\text{even}}$  is definable from  $Q_{eq}$ , and  $Q_0$  is definable from  $Q_{feq}$ .

This is because for any set  $A \subseteq M$ , the identity relation restricted to  $A$  is an equivalence relation with exactly  $|A|$  equivalence classes (the unit sets  $\{a\}$  for  $a \in A$ ). Therefore,

$|A|$  is even iff  $(M, A) \models Q_{eq}x(A(x) \wedge x = y)$

$A$  is infinite iff  $(M, A) \models \neg Q_{feq}x(A(x) \wedge x = y)$

So one might think that the undefinability of  $Q_{feq}$  and  $Q_{eq}$  in  $FO$  is just a matter of certain corresponding monadic quantifiers being undefinable in  $FO$ . But this is actually far from the case. We shall see (in the next subsection) that *no* monadic quantifiers can help to define  $Q_{feq}$  and  $Q_{eq}$ ; they are essentially polyadic in a strong sense. An indication of this is that it does not help to add  $Q_0$  or  $Q_{\text{even}}$  to  $FO$ . Indeed, (15.1) can be strengthened to

(15.3) If  $r \leq m, n$ , where  $m, n$  are natural numbers or  $\aleph_0$ , then  $\mathcal{M}_n \equiv_{FO(Q_0)^r} \mathcal{M}_m$  and  $\mathcal{M}_n \equiv_{FO(Q_{\text{even}})^r} \mathcal{M}_m$ .

To see this, consider the game  $DEF(Q_{\text{even}})^r(\mathcal{M}_n, \mathcal{M}_m)$ . We must define a winning strategy for Duplicator. The  $\exists$ -moves are handled as indicated above. Suppose  $a_1, \dots, a_k \in \mathcal{M}_n$  and  $b_1, \dots, b_k \in \mathcal{M}_m$  have been played,  $k < r$ , and that Spoiler chooses a set  $X$  in  $\mathcal{M}_n$ , definable from  $a_1, \dots, a_k$ , with an even number of elements. In particular,  $X$  must be a finite set,<sup>4</sup> and it is not hard to verify that the finite sets definable in  $\mathcal{M}_n$  from  $a_1, \dots, a_k$  are precisely the subsets of  $\{a_1, \dots, a_k\}$ . Say  $X$  is  $\{a_{i_1}, \dots, a_{i_s}\}$ ; then Duplicator chooses  $Y = \{b_{i_1}, \dots, b_{i_s}\}$ . Obviously  $|Y|$  is even too. Also, it is clear that however Spoiler chooses an  $a \in \mathcal{M}_n$ , Duplicator can respond with a  $b \in \mathcal{M}_m$  such that  $a \in X \iff b \in Y$ , and the partial isomorphism between chosen elements is maintained. A similar argument works for  $Q_0$ .<sup>5</sup>

**Example 5**

A (non-directed) *graph* is a structure  $\mathcal{M} = (M, R)$  where  $R$  is any symmetric binary relation on  $M$ . The elements of  $M$  are then called *vertices*, and if  $R(a, b)$  holds, there is said to be an *edge* between  $a$  and  $b$ . A *path* from  $a$  to  $b$  is a finite sequence of edges starting from  $a$  and ending in  $b$ .  $\mathcal{M}$  is *connected* if for any  $a, b \in M$ , there is a path from  $a$  to  $b$ . It is a fact that

<sup>4</sup> Here we think of  $Q_{\text{even}}$  as defined for arbitrary universes, and containing the even-numbered subsets, but not the odd-numbered or infinite subsets.

<sup>5</sup> Since  $FO(Q_0) \equiv FO(\neg Q_0)$ , we can consider the game  $DEF(\neg Q_0)^r(\mathcal{M}_n, \mathcal{M}_m)$ , and then exactly the same strategy wins.

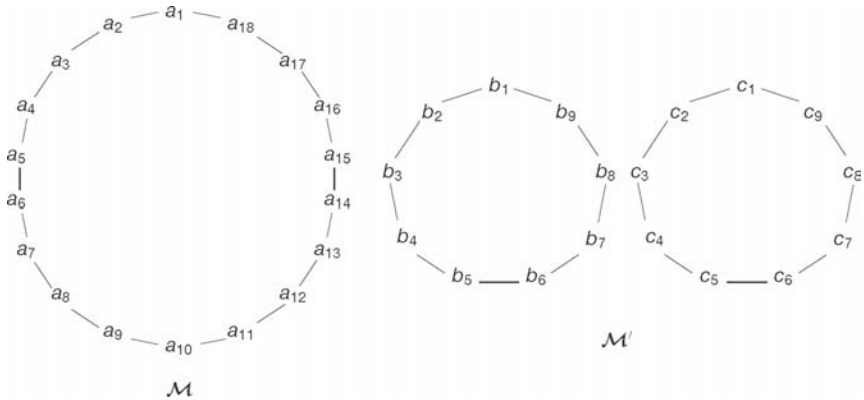


Figure 15.1 A graph with one cycle, and a graph with two cycles

(15.4) Connectedness is not an *FO*-definable property of graphs.

To show this, we use pairs of graphs of the following kind.<sup>6</sup> For  $r > 0$ , the first graph has  $2^r + 2$  vertices, connected in a big cycle. The second graph has the same number of vertices, but divided into two cycles, each of size  $2^{r-1} + 1$ . The case for  $r = 4$  is shown in Fig. 15.1.

Consider the game  $EF^r(\mathcal{M}, \mathcal{M}')$ . We claim that

(15.5) For the graphs above, Spoiler can win in five steps.

Here are hints for a strategy: At round 1, we may suppose  $a_1, b_1$  are played. At round 2, Spoiler picks  $b_5$ . The shortest path from  $b_5$  to  $b_1$  has length 4; we say that the *distance* between these points is 4. Now if Duplicator picks a vertex in  $\mathcal{M}$  with distance  $\leq 3$  from  $a_1$ , then Spoiler wins in at most two more rounds by choosing the vertices between that vertex and  $a_1$ . Duplicator must respond with vertices between  $b_1$  and  $b_5$ , but there are too many of them. So suppose she picks a vertex with distance at least 4 from  $a_1$ , i.e. some  $a_i$  with  $5 \leq i \leq 15$ . Then, at round 3, Spoiler picks  $b_7$ . So Duplicator must pick a vertex  $a_j$  which still has distance at least 4 from  $a_1$  (since there are still two rounds left), but which also has distance 2 from  $a_j$  (since  $b_7$  has distance 2 from  $b_5$ ). But the path  $b_7, b_8, b_9, b_1$  has length 3, whereas the path from  $a_j$  to  $a_1$  that does not involve  $a_i$  has greater length, so Duplicator loses anyway in the remaining two rounds.

However, if only four rounds are allowed, Duplicator has a winning strategy. An analysis of what makes these arguments work will in fact show that

<sup>6</sup> The construction goes back to Fagin 1975.

(15.6) For the general case when  $\mathcal{M}$  and  $\mathcal{M}'$  have  $2^r + 2$  elements, Duplicator has a winning strategy in  $EF^r(\mathcal{M}, \mathcal{M}')$ .

But  $\mathcal{M}$  is connected, whereas  $\mathcal{M}'$  is not, so (15.4) follows.

## 15.2 HELLA'S BIJECTIVE EF-GAME

Recall the various *lifts* of monadic quantifiers that we encountered in Chapter 10. Here the obvious question about expressivity is whether adding the lifts increases expressive power, i.e. if it allows us to say *more* than we could without them. If  $O$  is such a lift, we want to know, for any monadic  $Q_1, \dots, Q_n$  (of the appropriate types):

Is  $O(Q_1, \dots, Q_n)$  definable in  $FO(Q_1, \dots, Q_n)$ ?

When  $O$  is iteration, cumulation, or comes from a unary complex, it is clear that  $O(Q_1, \dots, Q_n)$  is indeed definable in  $FO(Q_1, \dots, Q_n)$ . For example,

$$(Q_1, Q_2)^{\text{cl}} xy(A(x), B(y), R(x, y)) \leftrightarrow Q_1 x(A(x), \exists y(B(y) \wedge R(x, y))) \wedge Q_2 y(B(y), \exists x(A(x) \wedge R(x, y)))$$

So in this sense one cannot say more with the cumulation lift, even though one can say certain things more elegantly. However, for branching, Ramseyfication, and resumption, expressive power does increase. In fact, for these lifts something stronger holds: essentially, no monadic quantifiers can replace them. A variant of the EF-games, due to Hella (1996), allows us in some cases to prove results of this kind.

### the bijective game $BEF^r(\mathcal{M}, \mathcal{M}')$

This game is played in  $r$  rounds as follows. In each round  $k$ , Duplicator chooses a bijection  $g_k$  from  $M$  to  $M'$  (if she cannot do that, i.e. if  $|M| \neq |M'|$ , she loses), and Spoiler responds by an element  $a_i \in M$ . After  $r$  rounds, Duplicator wins if

$$(a_1, \dots, a_r) \cong (g_1(a_1), \dots, g_r(a_r))$$

Otherwise, Spoiler wins.

Think of this as Duplicator proposing a bijection as an isomorphism, daring Spoiler to show that it is not. If the models are in fact isomorphic, she can always choose the isomorphism, and so will always win. But even if they are not isomorphic, she can sometimes win: a chosen bijection may not work, but then in the next round she can choose another one. And she may be able to keep this up for all the rounds of the game.

This will not work if  $\mathcal{M}$  and  $\mathcal{M}'$  are monadic. For then, if the structures are not isomorphic, no bijection is an isomorphism, so for each bijection  $g$  there is some  $a$  and some  $A_i$  such that  $a \in A_i$  but  $g(a) \notin A'_i$ , or vice versa. So if Spoiler chooses  $a$ ,

he wins in one round. But suppose the vocabulary has a binary relation (symbol)  $R$ . There may be  $a_1, a_2$  such that  $(a_1, a_2) \in R$  and  $(g(a_1), g(a_2)) \notin R$ , but Spoiler can pick only one of  $a_1, a_2$  in a given round, and in the next round Duplicator may switch to another bijection.

Let  $FO(\mathbf{Mon})$  be the logic obtained by adding *all* monadic quantifiers to  $FO$ , i.e.  $FO(\{Q : Q \text{ is a monadic quantifier}\})$ .

**Proposition 6**

*If Duplicator has a winning strategy in  $BEF^r(\mathcal{M}, \mathcal{M}')$ , then  $\mathcal{M} \equiv_{FO(\mathbf{Mon})^r} \mathcal{M}'$ .*

*Proof sketch.* Consider a formula  $\varphi(\vec{y}) = Qx(\psi_1(x, \vec{y}), \dots, \psi_n(x, \vec{y}))$  with a monadic quantifier  $Q$ , and sequences  $\vec{a}$  in  $M$  and  $\vec{b}$  in  $M'$ . Now if  $g$  is any bijection from  $M$  to  $M'$  such that for every  $a \in M$  and  $1 \leq j \leq n$ ,

$$\mathcal{M} \models \psi_j(a, \vec{a}) \iff \mathcal{M}' \models \psi_j(g(a), \vec{b})$$

then, simply by Isom,

$$Q_{\mathcal{M}}(\psi_1(x, \vec{a})^{\mathcal{M}, x}, \dots, \psi_n(x, \vec{a})^{\mathcal{M}, x}) \iff Q_{\mathcal{M}'}(\psi_1(x, \vec{b})^{\mathcal{M}', x}, \dots, \psi_n(x, \vec{b})^{\mathcal{M}', x})$$

regardless of which quantifier  $Q$  is. Using this observation, it is not hard to prove the following, by induction on the complexity of  $\varphi$ :

- (A) $_{\varphi}$  If  $g_1, \dots, g_k$  are bijections from  $M$  to  $M'$  which Duplicator has chosen according to her winning strategy ( $k \leq r$ ), and  $a_1, \dots, a_k$  are the elements chosen by Spoiler, and if  $\varphi$  has quantifier rank at most  $r - k$ , then

$$\mathcal{M} \models \varphi(a_1, \dots, a_k) \iff \mathcal{M}' \models \varphi(g_1(a_1), \dots, g_k(a_k))$$

The case  $k = 0$  gives the result.<sup>7</sup> □

We can illustrate the game by returning to two previous examples.

**Example 4 continued**

Consider again the models  $\mathcal{M}_n$  and  $\mathcal{M}_m$ . We can strengthen (15.1) and (15.3) to

(15.7) If  $r \leq m, n$ , where  $m, n \leq \aleph_0$ , then  $\mathcal{M}_n \equiv_{FO(\mathbf{Mon})^r} \mathcal{M}_m$ .

To see this, suppose we have  $a_1, \dots, a_k$  in  $\mathcal{M}_n$  and  $b_1, \dots, b_k$  in  $\mathcal{M}_m$  such that  $(a_1, \dots, a_k) \cong (b_1, \dots, b_k)$ . Let  $[a_i]$  be the equivalence class of  $a_i$ , and similarly for  $b_i$ . Also, let  $A_i = [a_i] - \{a_1, \dots, a_k\}$  and  $B_i = [b_i] - \{b_1, \dots, b_k\}$ . Then each  $A_i$  and each  $B_i$  has cardinality  $\aleph_0$ , and likewise for  $A = M - (A_1 \cup \dots \cup A_k)$  and  $B = M - (B_1 \cup \dots \cup B_k)$ . Thus there is a bijection  $g$  from  $M$  to  $M$  such that  $g(A_i) = B_i$ ,  $1 \leq i \leq k$ , and  $g(A) = B$ . But this means that for each  $a \in M$ ,

$$\{(a_1, b_1), \dots, (a_k, b_k)\} \cup \{(a, g(a))\} \text{ is a partial isomorphism.}$$

<sup>7</sup> If the models are finite, one can show that the converse of Proposition 6 holds too; see Hella 1996.



Hence, at each stage of the game, Duplicator can choose a bijection which preserves the partial isomorphism, whatever Spoiler plays.  $\square$

We thus obtain, by the now familiar method, the result that the type  $\langle 2 \rangle$  quantifiers

(15.8)  $Q_{feq}$  and  $Q_{eeq}$  are not definable in  $FO(\mathbf{Mon})$ .

That is, not only are they not definable from  $Q_0$  or  $Q_{\text{even}}$ ; they are not definable from *any* finite number of quantifiers of *any* monadic types.

**Example 5 continued**

Recall the two models  $\mathcal{M}, \mathcal{M}'$  of size  $2^r + 2$ , one consisting of a big cycle, and the other of two smaller cycles (see Fig. 15.1). We hinted by means of an example that Duplicator can always win  $EF^r(\mathcal{M}, \mathcal{M}')$ . It was observed by Hella and Sandu (1995) that a substantially stronger statement is true:

(15.9) Duplicator has a winning strategy in  $BEF^r(\mathcal{M}, \mathcal{M}')$ .

For a proof, see Hella and Sandu 1995 or Hella *et al.* 1997. Here is the idea. We must show that Duplicator can choose successively  $r$  bijections  $g_1, \dots, g_r$  from  $M$  to  $M'$  such that, no matter how Spoiler successively chooses his  $a_1, \dots, a_r \in M$ ,  $a_i = a_j \Leftrightarrow g_i(a_i) = g_j(a_j)$ , and there is an edge between  $a_i$  and  $a_j$  iff there is an edge between  $g_i(a_i)$  and  $g_j(a_j)$ . The strategy involves, when Spoiler has picked  $a$ , always choosing bijections which cut the big cycle in half, mapping one half onto one small cycle, and the other half onto the other small cycle, and which furthermore are such that the cutting points are as far away from  $a$  (and previously chosen points) as possible. Given the size of the models, it can be shown that this can be kept up for  $r$  rounds, and in such a way that a partial isomorphism is maintained.

It follows from this that connectedness of graphs is not definable in  $FO(\mathbf{Mon})$ . In the next section we apply (15.9) to one of our polyadic lifts: branching. Here is yet a third application. Recall the notion of the transitive closure  $R^+$  of a binary relation  $R$ , i.e. the smallest transitive relation extending  $R$ . It can be defined by recursion, and one way to make it a logical tool is to add it as a closure operator. Then, whenever  $\varphi(u, v)$  is a formula,

$$[TC_{u,v}\varphi(u, v)](x, y)$$

is a new atomic formula, with the truth condition

$$[TC_{u,v}\varphi(u, v)](x, y)^{\mathcal{M},x,y} = (\varphi(x, y))^{\mathcal{M},x,y}+$$

Another way would be to add a type  $\langle 2, 2 \rangle$  quantifier  $Q_{tc}$ , where  $(Q_{tc})_M(R, S) \Leftrightarrow S = R^+$ . It is also common to consider addition of *fixed-point operations* to  $FO$ , which means allowing certain forms of inductive definitions, of which transitive closure is a special case.

**Theorem 7** (Hella)

*Transitive closure, in any of these forms, is not definable in FO(Mon).*

*Proof.* (Outline) This time we consider, for any  $r$ , models  $\mathcal{M} = (M, R)$  and  $\mathcal{M}' = (M', R')$ , where  $\mathcal{M}$  is a *directed cycle* (the edge relation goes from  $a_i$  to  $a_{i+1}$  but not in the other direction) of length  $2^{r+1}$ , whereas  $\mathcal{M}'$  has two directed cycles, each of length  $2^r$ . Again one can show that Duplicator has a winning strategy in the game  $BEF^r(\mathcal{M}, \mathcal{M}')$ . Now  $R^+$  is the universal relation in  $M$ , but  $(R')^+$  is not the universal relation in  $M'$ ; it still consists of two disjoint subgraphs. Thus, the sentence

$$\forall x \forall y [TC_{u,v} R(u, v)](x, y)$$

is true in  $\mathcal{M}$ , but false in  $\mathcal{M}'$ . Or, in terms of the quantifier  $Q_{tc}$ ,

$$Q_{tc} xy (R(x, y), x = x \wedge y = y)$$

is true in  $\mathcal{M}$ , but false in  $\mathcal{M}'$ . □

**Corollary 8**

*The quantifier LIN from Chapter 10.4.2 is not definable in FO(Mon).*

*Proof.*  $LIN_{\mathcal{M}}(R)$  holds iff  $|M| \geq 2$  and  $M^2 - Id_M \subseteq R^+$ , so with  $\mathcal{M}$  and  $\mathcal{M}'$  as above,  $LIN_{xy} R(x, y)$  is true in  $\mathcal{M}$  and false in  $\mathcal{M}'$ . □

Consequently, the quantifier  $LIN^{\text{rel}} = Q_{\text{rcp}_2}$ , which is one interpretation of the reciprocal each other, is not monadically definable either.

## 15.3 APPLICATION TO BRANCHING QUANTIFICATION

Recall the branching lift  $Br^k$  from Chapter 10, which takes  $k$  monotone increasing type  $\langle 1, 1 \rangle$  quantifiers  $Q_1, \dots, Q_k$  and gives a type  $\langle 1, \dots, 1, k \rangle$  quantifier  $Br^k(Q_1, \dots, Q_k)$ . This lift applies most naturally to determiner denotations, but there is a corresponding lift—we may use the same name for it—taking  $k$  monotone increasing type  $\langle 1 \rangle$  quantifiers to a type  $\langle k \rangle$  quantifier:

$$Br^k(Q_1, \dots, Q_k)_M(R) \iff \exists X_1, \dots, X_k \subseteq M [(Q_1)_M(X_1) \ \& \ \dots \ \& \ (Q_k)_M(X_k) \ \& \ X_1 \times \dots \times X_k \subseteq R]$$

Hella *et al.* (1997) use (15.9) to characterize exactly the circumstances under which  $Br^k(Q_1, \dots, Q_k)$  is definable in terms of  $Q_1, \dots, Q_k$ , or indeed any monadic quantifiers. The main outcome is in that in many typical cases, such as branching of the quantifier, *most*, or *at least m n'ths of the*, the branching quantifier is *not* definable from any monadic quantifiers. So branching truly allows one to say more than is possible with ordinary operations like iteration or Boolean combinations. In the rest of this section we sketch how this result is obtained.

### 15.3.1 An instructive example

First, let us see how a special case can be obtained from the facts we already established in Chapter 13.4 about monadic quantifiers. Let  $H = [1/2] = (Q^R)^d$ ; that is,  $H_M(A)$  says, on finite universes, that at least half of the elements of  $M$  are in  $A$ . Consider the sentence

$$Br^2(H, H)_{xy}(x \neq y)$$

This says that there are subsets  $X, Y$  of  $M$ , each of which contains at least half of the elements of  $M$ , and which are *disjoint*. But then  $X$  and  $Y$  have *exactly* half of the elements of  $M$ . And the existence of two such sets is tantamount to saying that  $|M|$  is even. Thus, ‘ $|M|$  is even’ is definable in  $FO(Br^2(H, H))$ .

However, we know that ‘ $|M|$  is even’ is not definable in  $FO(H)$ —by Exercise 4 in Chapter 14.5 (or Exercise 15), it is not definable in  $FO(\text{most})$ , hence not in  $FO(Q^R)$ , which is equivalent to  $FO(H)$ . Thus, we have established the first claim of the following fact:

(15.10)  $Br^2(H, H)$  is not definable in  $FO(H)$ , not even on finite models. Likewise,  $Br^2(H^{\text{rel}}, H^{\text{rel}})$  is not definable in  $FO(H^{\text{rel}})$ .

Note that  $H^{\text{rel}} = \text{at least half of the}$ . The second claim follows from the fact that  $Br^2(H, H)$  is definable in terms of  $Br^2(H^{\text{rel}}, H^{\text{rel}})$ ; cf. (15.13) below.

### 15.3.2 The general result

The monotone type  $\langle 1 \rangle$  quantifiers are those of the form  $Q_f$  (Chapter 5.4), and the CONSERV and EXT monotone type  $\langle 1, 1 \rangle$  quantifiers are their relativizations. We have already seen that by making requirements on  $f$ , one can impose further conditions on  $Q_f$  besides being monotone; an example was smoothness in Chapter 5.6. Now, let us say that  $f$  is *bounded*, if there is a fixed number  $m$  such that for all  $n$ ,

$$f(n) < m, \text{ or } f(n) > n - m$$

Otherwise  $f$  is *unbounded*. That is, a bounded function never leaves the edges of the number triangle by more than a fixed distance.  $\exists$  and  $\exists_{\geq k}$  and  $\forall$  have bounded functions, whereas the function for  $Q^R$  is unbounded. But also very ‘unruly’ quantifiers may be bounded; an example is the quantifier  $Q_{f_0}$  from Chapter 14.4, which has bound 2.

Let us start with a simple case:  $Br^2(Q_f, Q_f)$ . A first observation is:

(15.11) If  $f$  is bounded,  $Br^2(Q_f, Q_f)$  is definable in  $FO(Q_f)$ . Also,  $Br^2(Q_f^{\text{rel}}, Q_f^{\text{rel}})$  is definable in  $FO(Q_f^{\text{rel}})$ .

*Proof.* First note that the claim that  $f(n)$  is a fixed number  $p > 0$  can be expressed using  $Q_f: f(|M|) = p$  iff

$$\exists \text{ distinct } a_1, \dots, a_p \text{ s.t. } (Q_f)_M(\{a_1, \dots, a_p\}) \wedge \neg(Q_f)_M(\{a_1, \dots, a_{p-1}\})$$

This sentence is readily written in  $FO(Q_f)$ ; call it  $\varphi_p$ . Similarly, we can write a sentence  $\psi_q$  saying that  $f(|M|) = |M| - q > 0$ . Also, the case  $f(n) = 0$  is expressed by a sentence  $\xi = Q_f x(x \neq x)$ .

Next, in each of these cases with a fixed value for  $f$ , the branching condition

$$\exists X, Y[|X|, |Y| \geq p \ \& \ X \times Y \subseteq R]$$

is expressible in  $FO$ . For example, if  $f(n) = p > 0$ , this can be written as a sentence  $\eta_p$  in  $FO$ :

$$\exists \text{ distinct } x_1, \dots, x_p, y_1, \dots, y_p \bigwedge_{1 \leq i, j \leq p} R(x_i, y_j)$$

There is a similar sentence  $\zeta_q$  for the case  $f(n) = n - q > 0$ . Note also that when  $f(n) = 0$  ( $f(n) = n + 1$ ), the branching condition is trivially true (false).

Now, normally there are infinitely many values of  $f(n)$  or  $n - f(n)$  to cover, so these conditions cannot be put together in a single sentence. But when  $f$  is bounded, there are only finitely many values. Indeed, when the bound is  $m$ , we can conclude that the sentence  $Br^2(Q_f, Q_f)xyR(x, y)$  is equivalent to

$$\xi \vee \bigvee_{1 \leq p < m} (\varphi_p \wedge \eta_p) \vee \bigvee_{q < m} (\psi_q \wedge \zeta_q)$$

The statement for the branching of  $Q_f^{\text{rel}}$  is proved similarly. This time, to formulate the branching condition  $Br^2(Q_f, Q_f)_M(X_1, X_2, R)$ , we need to express statements of the forms

$$f(|X_i|) = 0, f(|X_i|) = |X_i| + 1, f(|X_i|) = p, f(|X_i|) = |X_i| - q$$

( $p > 0, q \geq 0$ ), and this can be done with the quantifier  $Q_f^{\text{rel}}$ . □

Thus, branching of bounded quantifiers does not increase expressive power. The main result, however, is that for unbounded quantifiers, expressive power does increase, beyond the reach of all monadic quantifiers.

(15.12) If  $f$  is unbounded,  $Br^k(Q_f, \dots, Q_f)$  is not definable in  $FO(\mathbf{Mon})$ , for  $k > 1$ . Likewise,  $Br^k(Q_f^{\text{rel}}, \dots, Q_f^{\text{rel}})$  is not definable in  $FO(\mathbf{Mon})$ .

Note first that the second statement follows from the first: it is an easy exercise to verify that

(15.13)  $Br^k(Q_f, \dots, Q_f)$  is definable in  $FO(Br^k(Q_f^{\text{rel}}, \dots, Q_f^{\text{rel}}))$ .

Second, we may restrict attention to the case  $k = 2$ , because of the following fact:

(15.14)  $Br^k(Q_f, \dots, Q_f)$  is definable in  $FO(Br^{k+1}(Q_f, \dots, Q_f))$ .

This follows, since for all models  $(M, R)$  with  $R \subseteq M^k$ , one has

$$(M, R) \models Br^k(Q_f, \dots, Q_f)x_1 \dots x_k R(x_1, \dots, x_k) \leftrightarrow$$

$$Br^{k+1}(Q_f, \dots, Q_f)x_1 \dots x_{k+1} R(x_1, \dots, x_k)$$

which can be verified by just applying the truth definition for  $Br^k$  and  $Br^{k+1}$ , treating separately the cases (i)  $f(|M|) = 0$ , (ii)  $f(|M|) = |M| + 1$ , and (iii)  $0 < f(|M|) \leq |M|$ .

Thus, returning to the proof of (15.12), our task is to find, given  $r$ , two models  $\mathcal{M}, \mathcal{M}'$  which are equivalent up to quantifier rank  $r$  for sentences in  $FO(\mathbf{Mon})$ , but such that a fixed sentence involving  $Br^2(Q_f, Q_f)$  is true in one and false in the other. To give the idea, suppose  $Q_f = Q^R$ . Then we can use the models of Example 5; see Fig. 15.1. Consider the following sentence:

$$(15.15) \quad \exists x \exists y Br^2(Q^R, Q^R)uv(R(u, v) \rightarrow u = x \vee v = y)$$

First, (15.15) is true in  $\mathcal{M}'$ . Suppose, for illustration, that  $r = 4$ . Let  $x = c_1, y = b_1, X = \{b_1, \dots, b_9, c_1\}$ , and  $Y = \{c_1, \dots, c_9, b_1\}$ . Then  $Q_{\mathcal{M}'}^R(X)$  and  $Q_{\mathcal{M}'}^R(Y)$ , since both sets contain more than half of the elements of the universe. But if  $u \in X$  and  $v \in Y$  and  $R'(u, v)$  holds, then either  $u$  must be  $c_1$  or  $v$  must be  $b_1$ .

Second, (15.15) is not true in  $\mathcal{M}$ . For take any  $x, y \in M$ , and any  $X, Y \subseteq M$  containing at least ten elements each (still for the case  $r = 4$ ). But then there necessarily has to be  $u \in X - \{x\}$  and  $v \in Y - \{y\}$  such that  $R(u, v)$  (since  $|X - \{x\}|, |Y - \{y\}| \geq 9$ ). So the branching condition cannot be satisfied.

Third, we already saw in (15.9) that Duplicator has a winning strategy in  $BEF^r(\mathcal{M}, \mathcal{M}')$ .

This proves (15.12) in the special case of  $Q_f = Q^R$ , or  $Q_f^{\text{rd}} = \text{most}$ . But in fact the argument can be generalized to any unbounded quantifier,  $Q_f$ . Then one chooses, by unboundedness, the size  $n$  of the models such that  $2^{r-1} \leq f(n) \leq n - 2^{r-1}$ . The big cycle still has size  $2^r + 2$ , but now we add two ‘boxes’ (disjoint sets)  $P$  and  $S$  (and corresponding one-place predicates) with  $|P| = f(n) - 2^{r-1}$  and  $|S| = n - f(n) - 2^{r-1}$ , and similarly in  $\mathcal{M}'$ . By suitably extending  $R$  and  $R'$  to these boxes, one can insure that the first two parts of the above argument still go through, and also the third part, since the added boxes will in fact not disturb Duplicator’s winning strategy.

Finally, it remains to generalize all of the above to the branching of  $k$  quantifiers:  $Br^k(Q_{f_1}, \dots, Q_{f_k})$ . It turns out that it is not quite enough to require that each of  $f_1, \dots, f_k$  is unbounded. What one needs is, rather, a condition that guarantees that for each  $m$  there are  $i, j$ , and  $n$  such that

$$m \leq f_i(n), f_j(n) \leq n - m$$

and at the same time all the other  $f_l(n)$  are distinct from 0 and  $n + 1$ . Such a condition, which may be called unboundedness of the tuple  $(f_1, \dots, f_k)$ , can be formulated. Also, the previous arguments can be extended to cover this case; see Hella *et al.* 1997 for details. The result is the following.

**Theorem 9** (Hella *et al.* 1997)

Let  $Q_1, \dots, Q_k$  be either monotone type  $\langle 1 \rangle$  quantifiers, or monotone, CONSERV, and EXT type  $\langle 1, 1 \rangle$  quantifiers ( $k > 1$ ). Then the following are equivalent:

- (a)  $Br^k(Q_1, \dots, Q_k)$  is definable in  $FO(Q_1, \dots, Q_k)$ .
- (b)  $Br^k(Q_1, \dots, Q_k)$  is definable in  $FO(\mathbf{Mon})$ .
- (c)  $(Q_1, \dots, Q_k)$  is bounded.

### 15.3.3 Linguistic conclusions

Natural languages like English have lots of determiners, like *most*, that denote unbounded  $\text{MON}\uparrow$  quantifiers. Granted that certain sentences in such languages have readings where these quantifiers are branched (see Chapter 10.3), it follows, as detailed in Chapter 12.5, that *branching strengthens expressive power*. No English sentence formalizable in  $\text{FO}(\text{Mon})$  can provide the branched reading of, say, Barwise's example

(15.16) Most boys in your class and most girls in my class have all dated each other.

And formalizability in  $\text{FO}(\text{Mon})$  is not a very severe requirement. Any English sentence whose NPs—either non-quantificational or formed with Dets taking one or more restriction arguments—are interpreted in a straightforward compositional way, e.g. allowing Boolean combinations, relative clauses, etc., will be thus formalizable, and hence unable to express branched readings.

The theorem also tells us that if there were a rendering of (15.16) from monadic quantifiers with a construction formalizable in  $\text{FO}(\text{Mon})$ , there would be one already from *most*. There isn't, since *most* is unbounded, but what about the bounded case? Are branched readings with bounded English determiner denotations readily expressible without branching in English? This depends on whether one thinks the definitions in the proof of (15.11) above can be 'naturally' rendered in English. We venture no definite judgment, but note that although the sentences are long and complex, part of the complexity comes from using long conjunctions and disjunctions, and from quantifying explicitly over many variables; which may be a case of practical but not principled complexity.

If one restricts attention to very simple linguistic constructions, the facts change. It is proved by Westerståhl (1994) that  $\text{Br}^k(Q_1, \dots, Q_k)$  ( $Q_i$  of type  $\langle 1 \rangle$ ) is an iteration (i.e. is *reducible* in the sense of Keenan (1992)) if and only if, on each universe where the quantifiers are non-trivial,  $Q_1, \dots, Q_k$  is of the form  $\exists, \dots, \exists, \forall, \dots, \forall$ . For example, the branched reading of

(15.17) At least two boys in your class and at least two girls in my class have all dated each other.

is not reducible. The reading is readily expressed in  $\text{FO}$  by

$$\exists a, b \in A \exists c, d \in B (a \neq b \wedge c \neq d \wedge R(a, c) \wedge R(a, d) \wedge R(b, c) \wedge R(b, d))$$

but saying this in plain English is bound to be somewhat awkward.

## 15.4 APPLICATION TO RAMSEY QUANTIFIERS AND RECIPROCALLS

### 15.4.1 A characterization theorem

The Ramsey quantifiers mentioned in Chapter 2.4, and indirectly in Chapter 10.4 (see below), can be defined as follows. For  $Q$  of type  $\langle 1, 1 \rangle$ ,  $A \subseteq M$ , and  $R \subseteq M^k$ ,

$$\text{Ram}^k(Q)_M(A, R) \iff \exists X \subseteq A(Q_M(A, X) \ \& \ X^k - \text{Id}_X^k \subseteq R)$$

where  $Id_X^k = \{(a, \dots, a) : (a, \dots, a) \in X^k\}$  (we have earlier written  $Id_X$  for  $Id_X^2$ ). There is an analogous definition for  $Q$  of type  $\langle 1 \rangle$ . It is usually assumed that  $Q$  is (right) increasing.

**Theorem 10** (Hella *et al.* 1997)

*If  $Q$  is a monotone type  $\langle 1 \rangle$  quantifier, or the relativization of such a quantifier, then  $Ram^k(Q)$  is definable in  $FO(\mathbf{Mon})$  iff  $Ram^k(Q)$  is definable in  $FO(Q)$  iff  $Q$  is bounded ( $k > 1$ ).*

The right-to-left direction is similar to the proof of (15.11) above, but the other direction requires new constructions (and a new EF-game) that we shall not present here.<sup>8</sup>

### 15.4.2 Linguistic conclusions

We saw in Chapter 10.4 that polyadic quantification enters into *reciprocal* constructions in natural languages in at least two ways. First, the phrases *each other* and *one another* can be taken to express various type  $\langle 1, 2 \rangle$  quantifiers, and one of these,  $Q_{\text{rcp}2} = LIN^{\text{rel}}$ , which occurs in English sentences like

(15.18) Five Boston pitchers sat alongside each other.

is not definable in  $FO(\mathbf{Mon})$ , as Corollary 8 above shows. Granted that this sentence has that reading, there is no way to produce the correct truth conditions by means of only monadic quantifiers and standard operations on these.

This is an instance of the more general observation that natural languages (and  $FO$ ) have no general way of forming the ‘ancestral’, i.e. the transitive closure of a given relation, even though it does have primitive terms for the transitive closure of some relations, such as *ancestor* for *parent*.

The second way in which polyadicity appears with reciprocals is for certain quantified reciprocal sentences, like

(15.19) Most of John’s friends know each other.

Since *most* is  $\mathbf{MON}\uparrow$ , this is naturally interpreted along the lines

$$\exists X \subseteq A(\text{most}(A, X) \ \& \ X^2 - Id_X \subseteq R)$$

(with  $A$  as the set of John’s friends), i.e. as

$$Ram^2(\text{most})(A, R)$$

<sup>8</sup> Actually, the result about  $Ram^k(Q)$  in Hella *et al.* 1997 is stronger than Theorem 10, and concerns definability in the logic  $FO_{\infty, \omega}^{\omega}(\mathbf{Q}_{k-1}) = \bigcup_{n \geq 1} FO_{\infty, \omega}^n(\mathbf{Q}_{k-1})$ , where  $FO_{\infty, \omega}^n(Q_M)$  allows infinite conjunctions and disjunctions but only has a supply of  $n$  variables, and  $\mathbf{Q}_m$  is the collection of all generalized quantifiers whose arguments are at most  $m$ -ary relations (so  $\mathbf{Q}_1 = \mathbf{Mon}$ ). This requires yet another variant of the Ehrenfeucht–Fraïssé technique, so-called *pebble games*. It is shown that if  $Q$  is bounded,  $Ram^k(Q)$  cannot be defined from quantifiers in  $\mathbf{Q}_{k-1}$  even if infinite conjunctions and disjunctions are allowed. The interest of the logic  $FO_{\infty, \omega}^{\omega}$  comes from the fact that various recursive operators like  $[TC_{u,v}\varphi(u, v)]$ , which are not definable in  $FO(\mathbf{Q}_1)$  (Example 5 above), are definable in  $FO_{\infty, \omega}^{\omega}(\mathbf{Q}_1)$ .

Even though in this case each other denotes a first-order definable type  $\langle 1, 2 \rangle$  quantifier (viz.  $Q_{\text{rcp}_1} = \text{FUL}^{\text{rel}}$ ), Theorem 10 shows that the quantification employed in (15.19) is not expressible with any finite number of monadic quantifiers.<sup>9</sup>

Recall that these logical undefinability facts hold (also) for finite models. The reciprocal sentences exemplified above are by no means odd or extreme, and they provide rather striking illustrations of natural language using essentially polyadic quantification.

## 15.5 APPLICATION TO RESUMPTION AND ADVERBIAL QUANTIFICATION

In Chapter 10 we saw that, in addition to the by now standard question of whether adding a polyadic lift allows one to say more than one could without it, there is in the case of resumption also the issue of what we can say using *only* resumptive quantification. Roughly: What is the logical strength of adverbial quantification compared to quantification by means of determiners? In this section, we look at each of the logical questions in turn, and try to assess the significance of the results obtained.

### 15.5.1 Definability in terms of resumptive quantifiers

Recall the definitions of resumption of type  $\langle 1 \rangle$  quantifiers and their relativizations from Chapter 10: For  $R, S \subseteq M^k$  and  $A_1, \dots, A_k \subseteq M$ ,

$$\begin{aligned} \text{Res}^k(Q)_M(R) &\iff Q_{M^k}(R) \\ \text{Res}^k(Q^{\text{rel}})_M(S, R) &\iff (Q^{\text{rel}})_{M^k}(S, R) \iff Q_S(S \cap R) \end{aligned}$$

One may note that the second of these is ‘almost’ the relativization of first, in the following sense:

$$(15.20) \quad (\text{Res}^k(Q))_M^{\text{rel}}(A, R) \iff \text{Res}^k(Q^{\text{rel}})_M(A^k, R)$$

Now, a characteristic feature of the  $k$ -ary resumption of  $Q$  is that it is sensitive only to how  $Q$  behaves on universes whose size is a  $k$ 'th power. Thus, if  $Q'$  is identical to  $Q$  on such universes, but possibly completely different on universes of other sizes, one still has  $\text{Res}^k(Q) = \text{Res}^k(Q')$ . This has consequences for definability issues.

For example, it is trivial to find type  $\langle 1 \rangle$  quantifiers which are *not* definable from their resumptions: Let  $Q$  be equal to the existential quantifier on universes whose size is a square, and equal to, say,  $Q_{\text{even}}$  on other universes. Then  $\text{Res}^2(Q) = \text{Res}^2(\exists)$ , so it is *FO*-definable. But  $Q_{\text{even}}$  is not *FO*-definable, and hence neither is  $Q$ .<sup>10</sup> So  $Q$  is not definable in terms of  $\text{Res}^2(Q)$ .

<sup>9</sup> In (10.69) in Ch. 10.4.4 we gave the truth conditions for sentences like (15.19) by means of an operator  $CQ$ , which (since  $\neg \text{most}(A, \emptyset)$ ) yields exactly  $\text{Ram}^2(\text{most})(A, R)$ .

<sup>10</sup> The counterexamples used in the proof of Fact 1 in Ch. 14.2 can readily be taken to have a size that is not a square.



However, quantifiers that change behavior radically between universes do not seem very natural. We know that EXT is a strong requirement of constancy over universes, so the next result is not surprising.

**Proposition 11**

*If  $Q$  is any EXT monadic quantifier, then  $Q$  is definable from  $Res^k(Q)$ . In particular, all quantifiers of the form  $Q^{rel}$  are definable from their resumptions.*

*Proof.* We look at the type  $\langle 1, 1 \rangle$  case; the others are similar. Suppose  $k = 2$ . We make the following

$$\text{Claim: } Q_M(A, B) \iff \exists a Res^2(Q)_M(A \times \{a\}, B \times \{a\})$$

Thus,  $Q$  is defined by the sentence  $\exists x Res^2(Q)yz(A(y) \wedge z = x, B(y) \wedge z = x)$ , and the result follows. To prove the claim, note first that, under the assumptions (which in this chapter always include ISOM),  $Q$  is a relation between three cardinal numbers:  $|A - B|$ ,  $|A \cap B|$ ,  $|B - A|$ . We have  $Res^2(Q)_M(A \times \{a\}, B \times \{a\}) \iff Q_{M^2}(A \times \{a\}, B \times \{a\})$ . Also, note that, for any set  $Y$ ,

$$|Y \times \{a\}| = |Y|$$

and furthermore,  $(A \times \{a\}) - (B \times \{a\}) = (A - B) \times \{a\}$ ,  $(A \times \{a\}) \cap (B \times \{a\}) = (A \cap B) \times \{a\}$ , etc. So the right- and left-hand sides of the claim in fact express the same relation between cardinal numbers. (Notice that by EXT, we do not have to worry about the cardinality  $|M^2 - ((A \cup B) \times \{a\})|$ —which is in general distinct from  $|M - (A \cup B)|$ —and that is why the argument works.)

For  $k = 3$ , we instead have

$$Q_M(A, B) \iff \exists a Res^3(Q)_M(A \times \{a\} \times \{a\}, B \times \{a\} \times \{a\})$$

etc. □

But many non-EXT quantifiers are also definable from their resumptions. Consider, for example,  $\forall$ , for which we have

$$\forall_M(A) \iff A = M \iff A \times A = M^2 \iff Res^2(\forall)_M(A \times A)$$

Another example is the proportional quantifiers, as we now show.

**Proposition 12**

*The type  $\langle 1 \rangle$  proportional quantifiers are definable from their resumptions.*

*Proof.* Take  $Q = (p/q)$ , so  $Q_M(A) \iff |A| > p/q \cdot |M|$ . Now

$$\begin{aligned} |A| > p/q \cdot |M| &\iff |A| \cdot |M|^{k-1} > p/q \cdot |M| \cdot |M|^{k-1} \\ &\iff |A \times M^{k-1}| > p/q \cdot |M|^k \\ &\iff Res^k(Q)_M(A \times M^{k-1}) \end{aligned}$$

In other words, the definition is

$$QyA(y) \leftrightarrow \text{Res}^k(Q)yx_1 \dots x_{k-1}(A(y) \wedge x_1 = x_1 \wedge \dots \wedge x_{k-1} = x_{k-1}) \quad \square$$

We may also note the following.

**Proposition 13**

*The class of quantifiers definable in terms of their  $k$ -ary resumptions is closed under Boolean operations, as well as inner negations and duals.*

*Proof.* This is because, as is easily verified,

$$\begin{aligned} \text{Res}^k(\neg Q) &= \neg \text{Res}^k(Q) \\ \text{Res}^k(Q_1 \wedge Q_2) &= \text{Res}^k(Q_1) \wedge \text{Res}^k(Q_2) \\ \text{Res}^k(Q\neg) &= \text{Res}^k(Q)\neg \end{aligned} \quad \square$$

*Digression:* Recall our earlier discussion of EXT as a suitable notion of constancy for quantifiers. We found that natural language quantifiers taking more than one argument are normally EXT, but among the type  $\langle 1 \rangle$  ones there were some conspicuous exceptions, like  $\forall$  and  $Q^R$ . In our tentative universal (EU) in Chapter 4.5.3 we suggested that the exceptions are due to the presence of a logical constant *thing* that denotes the universe. The question still remains, however, if there is some slightly weaker property than EXT of type  $\langle 1 \rangle$  quantifiers that better captures the idea of constancy for them. The above results suggest one answer: consider the property of *being definable from its binary resumption*.

The EXT quantifiers have this property, but so do  $\forall$  and proportional quantifiers like  $Q^R$ , which are not EXT but still qualify as intuitively ‘constant’.<sup>11</sup> Proposition 13 shows that it is somewhat well-behaved, and in view of what was said above about trivial examples of quantifiers not being definable from their resumptions due to irregular behavior on universes that are not of the form  $M^k$ , we see that the property indeed captures a form of constancy which is slightly weaker than EXT. But one should not make too much of this. We can in any case see that, *extensionally*, this is a better approximation of constancy for type  $\langle 1 \rangle$  quantifiers than EXT.

### 15.5.2 Linguistic consequences, 1

We saw in Chapter 0 that there are languages such as Straits Salish which do not have D-quantification (using determiners) but only A-quantification (using adverbs, auxiliaries, etc., instead). One might wonder if the expressive power of such languages differs from those having both, or from a language with only D-quantification.

Consider adverbial quantification and assume that these adverbs always denote quantifiers of the form  $\text{Res}^k(Q)$ ,  $k \geq 1$ . As an example easy to think of, one may

<sup>11</sup> Note that instead of the direct argument above that  $\forall$  is definable from  $\text{Res}^2(\forall)$ , we could have used the fact that  $\forall$  is co-EXT, and applied Propositions 11 and 13.

consider the possibility of expressing quantified statements in English by means of adverbs only; no determiners. Proposition 11 tells us that as long as  $Q$  is EXT, it can always be defined from  $Res^2(Q)$ . But the linguistic interest of this is doubtful: *first*, because it is not at all clear that the definition used in the proof of that proposition has any natural rendering in English (see Chapter 12.5.5), and *second*, because  $Q$  is trivially definable from  $Res^1(Q) = Q!$

In general, when comparing the strength of English<sub>A</sub> (English with only adverbial quantification) and English<sub>D</sub> (English with only determiner quantification), and *a fortiori* of, say, Straits Salish and English, one issue is clearly the supply of adverbs and determiners. Sentences like All/some/most dogs bark can be translated roughly as Dogs always/sometimes/usually bark, where we can think of the adverb as denoting  $Res^1(every)$ ,  $Res^1(some)$ , and  $Res^1(most)$ , respectively. Likewise, Few/no cats are green becomes Cats are rarely/never green, but for, say, More than two-thirds of the/All but three dogs bark we presumably have to find corresponding adverbs. In English, this is not necessarily possible. However, it might well be possible for other languages.

Modulo the supply of adverbs, the above renderings of D-quantification by means of A-quantification seem linguistically quite natural. But one should also ask if more complex quantificational constructions can be treated similarly. An obvious case is iteration. It might seem that iteration by means of adverbs is hard or impossible, since two quantificational adverbs cannot occur in the same clause, whereas two or more determiners can easily occur in the same clause. But one can achieve the same effect with relative clauses. Compare

- (15.21) a. Most dogs bark at some cats.  
 b. Dogs usually are things that sometimes bark at cats.

The second sentence is somewhat clumsy, but that may simply be because English has a smooth alternative with determiners. In a language like Straits Salish, the second type of construction may not be clumsy at all.

In conclusion, we have again seen that instances of positive logical definability need not carry over to ‘linguistic definability’, because the synonymy (PER) of logical equivalence is weaker than (refined by) relevant notions of linguistic synonymy. ‘Linguistic definability’ is a less precise, or at least less familiar, notion, but tentatively we may still conjecture that having to use A-quantification rather than D-quantification need not reduce expressive power, as long as the supply of adverbs, or other means of A-quantification, in the one language matches the supply of determiners in the other.

We now turn to *undefinability* results for resumption.

### 15.5.3 Definability of resumption

It is not too hard to show—in fact, using the machinery introduced in chapter 13 for *monadic* structures—that the resumption of some common quantifiers increases logical expressive power, in the sense that, for example,  $Res^2(most)$  is not definable in  $FO(most)$  (a direct proof of this is given at the end of this subsection). For monotone type (1) quantifiers, some more general facts are known. But we do not have, so far,

any general characterization of which resumptions are undefinable in terms of any monadic quantifiers. Such results, although one can make educated guesses as to what they might look like, appear to be very hard to prove. Below, we shall outline what is known at present, including a proof of a main result about resumption in Hella *et al.* 1997. As before, the proofs, although they form excellent illustrations of how to use the EF-tools, can be skipped if one so wishes.<sup>12</sup>

Here is a first complication. As with branching and Ramseyfication, we have, for any type  $\langle 1 \rangle$   $Q$ :

$$(15.22) \text{ Res}^k(Q) \text{ is definable in } FO(\text{Res}^k(Q^{\text{rel}}))$$

since one easily checks from the definitions that

$$\text{Res}^k(Q)_M(R) \Leftrightarrow \text{Res}^k(Q^{\text{rel}})_M(M^k, R)$$

So, to show that  $\text{Res}^k(Q^{\text{rel}})$  is undefinable in  $L$ , it is enough to show that  $\text{Res}^k(Q)$  is undefinable in  $L$ . But this fact is less helpful than before, since this time we do not have  $L = FO(\mathbf{Mon})$  but rather, say,  $L = FO(Q_1, \dots, Q_n)$ , and then it is likely that what we want to show in the relativized case is that  $\text{Res}^k(Q^{\text{rel}})$  is undefinable in  $FO(Q_1^{\text{rel}}, \dots, Q_n^{\text{rel}})$ , and this does not follow in general. In other words, the type  $\langle 1 \rangle$  case and the (relativized) type  $\langle 1, 1 \rangle$  case have to be treated separately. Let us start with the type  $\langle 1 \rangle$  case.

Most of the results concern monotone quantifiers. Consider a typical monotone type  $\langle 1 \rangle$  quantifier, say,

$$Q_f = [1/3]$$

so  $(Q_f)_M(A)$  means that  $A$  contains at least one-third of the elements of  $M$ .<sup>13</sup> Let  $Q_g$  be any monotone type  $\langle 1 \rangle$  quantifier. We will show that

$$(15.23) \text{ Res}^2(Q_f) \text{ is not definable in } FO(Q_g).$$

*Proof.* First, we note that  $f$  clearly has the following property:

$$(a) \forall m \exists n (m \leq f(n^2) \leq \text{int}(n^2/2))$$

where  $\text{int}(x)$  is the integral part of  $x$  (i.e. the biggest integer  $\leq x$ ). We are going to use the EF-tools, as in Chapter 14.1. Fix  $r > 0$ . Given  $n \geq r$ , think of the interval  $[0, n]$  as the corresponding level in the number triangle, but now include also real numbers:  $[0, n]$  consists of all real numbers  $x$  such that  $0 \leq x \leq n$ . For  $x, y \in [0, n]$ , we say that  $x$  is *far away from*  $y$  if  $|x - y| \geq r + 1$ . Our objective is to show:

(b) There are  $n$  and  $x, y \in [0, n]$  such that  $x < y$ ,  $y$  is not an integer, and

(i)  $x$  and  $y$  are far away from each of  $0, n, g(n), n - g(n)$

(ii)  $xy = f(n^2)$

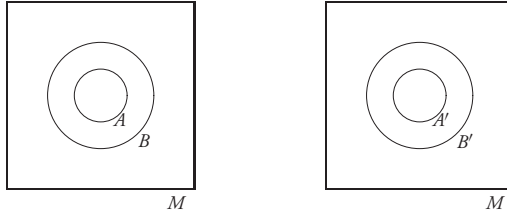
<sup>12</sup> Again, the result about resumption in Hella *et al.* 1997 is stronger than Theorem 14 below, as it concerns definability in  $FO_{\infty, \omega}^0(Q_1, \dots)$ .

<sup>13</sup> Then  $f(n) = \text{int}((n + 2)/3)$ , where  $\text{int}(x)$  is the integral part of  $x$ .

If we let  $a = \text{int}(x)$  and  $b = \text{int}(y)$ , it follows that

$$ab < f(n^2) \leq (a + 1)(b + 1)$$

Therefore, if we take (monadic) models  $\mathcal{M}, \mathcal{M}'$



with  $|M| = n, |A| = a, |B| = b, |A'| = a + 1,$  and  $|B'| = b + 1$  (and  $A \subseteq B, A' \subseteq B'$ ), it is clear that the sentence

$$\text{Res}^2(Q_f)xy(A(x), B(y))$$

which expresses that  $|A \times B| \geq f(|M|^2)$ , is false in  $\mathcal{M}$ , but true in  $\mathcal{M}'$ . Thus, to establish (15.23) it suffices to show that

$$(c) \mathcal{M} \approx_{FO(Q_g)^r} \mathcal{M}'$$

But this follows from (i) above, which guarantees that for corresponding  $r$ -block variants (Chapter 13.4.2)  $X$  in  $\mathcal{M}$  and  $X'$  in  $\mathcal{M}'$ , we always have  $|X| \geq g(n) \Leftrightarrow |X'| \geq g(n)$ . Note that the only blocks of  $\mathcal{M}$  over whose cardinality we have no ‘control’ are  $B - A$  and  $M - (B - A)$ , but for them we have  $|B - A| = |B' - A'|$ , and  $|M - (B - A)| = |M - (B' - A')|$ .

So it only remains to prove (b). This requires calculation; the rest of the proof is mathematics. Let us say that  $x \in [0, n]$  is  $n$ -good if it is far away from each of  $0, n, g(n), n - g(n)$ . Clearly, in any subinterval  $I$  of  $[0, n]$  of length  $c_1 = 3 + 6(r + 1)$ , there has to be some  $n$ -good integer; indeed there has to be an  $n$ -good subinterval of  $I$  of length 1 (where an interval is  $n$ -good if each of its elements are). Similarly, one sees that a subinterval of length  $c_2 = 3c_1 + 6(r + 1)$  has to have an  $n$ -good subinterval of length  $c_1$ . Let  $m = (c_2 + c_2/(\sqrt{2} - 1))^2$ . By (a), there is a number  $n$  such that  $m \leq f(n^2) \leq n^2/2$ . Let  $l = f(n^2), \mu = \sqrt{l}$ , and  $\nu = l/n$ . We now claim that

$$(d) \mu - \nu \geq c_2$$

To prove (d), we distinguish two cases. Let  $d = c_2/(\sqrt{2} - 1)$ .

Case 1:  $l \geq dn$ . Then

$$\begin{aligned} \mu - \nu &= \nu(\sqrt{n^2/l} - 1) \quad [\text{a little calculation}] \\ &\geq l/n(\sqrt{2} - 1) \quad [\text{since } l \leq n^2/2] \\ &\geq d(\sqrt{2} - 1) \quad [\text{by assumption of the case}] \\ &= c_2 \end{aligned}$$

Case 2:  $l < dn$ . Then  $\mu - v = \sqrt{l} - l/n \geq \sqrt{l} - d$  (assumption)  $\geq \sqrt{m} - d$  (since  $m \leq l$ )  $= c_2$ .

This proves (d), so there is an  $n$ -good subinterval  $I$  of  $[v, \mu]$  of length  $\geq c_1$ . Consider the function  $F(x) = l/x$ . One verifies that  $F$  is a bijection from  $[v, \mu]$  to  $[\mu, n]$  which has derivative  $< -1$ , which means that the image  $F(I)$  also is an interval of length  $\geq c_1$ . Thus, there is some  $n$ -good  $y = F(x)$  on  $F(I)$ , and we can choose  $y$  to be a non-integer. Also,  $x$  is  $n$ -good, and  $xy = l = f(n^2)$ . This proves (b), and thereby (15.23).  $\square$

Note that the only property of  $f$  used in this proof was that (a) holds. In fact, the proof can be generalized as follows. *First*, it suffices to assume that  $f$  is *unbounded on squares*, which means that

$$\forall m \exists n (m \leq f(n^2) \leq n^2 - m)$$

For  $f(n^2)$  is either  $\leq \text{int}(n^2/2)$  or  $> \text{int}(n^2/2)$ , and at least one of these alternatives occurs infinitely often, which means that either (a) holds, or

$$(a') \quad \forall m \exists n (\text{int}(n^2/2) < f(n^2) \leq n^2 - m).$$

In case (a') holds, one proceeds as before until the definition of  $n$  and  $l$ . Then one takes, using (a'),  $n$  such that  $\text{int}(n^2/2) < f(n^2) \leq n^2 - m$  and sets  $l = n^2 - f(n^2) + 1 = f^d(n^2)$ . Now  $m \leq l \leq n^2/2$ , so we can continue as above, using the dual sentence

$$\neg \text{Res}^2(Q_f)xy(\neg(A(x), B(y)))$$

instead.

*Second*, it is not very hard to generalize all of this to  $k$ -ary resumption. [One lets  $m = (c_2 + c_2 / (\sqrt[k]{2} - 1))^k$ ,  $l = f(n^k)$  (when the case corresponding to (a) holds),  $\mu = \sqrt[k]{l}$ ,  $v = \sqrt[k-1]{l/n}$ , and  $d = (c_2 / (\sqrt[k]{2} - 1))^{k-1}$ . The calculations work much as before. The function  $F$  to use is  $F(x) = lx^{1-k}$ , and the relevant sentence is  $\text{Res}^k(Q_f)x_1 \dots x_k(A(x_1), \dots, A(x_{k-1}), B(x_k))$ .<sup>14</sup>]

*Third*, one can arrange to have  $x, y$  far away from  $0, n, g_i(n), n - g_i(n)$ , for any given functions  $g_1, \dots, g_i$ ; the proof goes through as before, with a corresponding change in the definition of  $c_1$  and  $c_2$ . And *fourth*, one can show that if  $f$  is instead *bounded* in the relevant sense, i.e. if it is bounded on numbers of the form  $n^k$ ,<sup>15</sup> then  $\text{Res}^k(Q_f)$  is actually definable in  $FO(Q_g, Q_b)$ , where  $g(n) = f(n^k)$  and  $b(n) = n^k - f(n^k)$ . The

<sup>14</sup> One has to generalize the proof in this way, since, in contrast with branching and Ramseyfication, it is not the case in general that  $\text{Res}^k(Q)$  is definable in terms of  $\text{Res}^{k+1}(Q)$ : e.g. the behavior of  $Q$  on universes of the form  $M^2$  need have nothing to do with its behavior on universes of the form  $M^3$ .

<sup>15</sup> We say that  $f$  is *bounded* on a set  $X$  if there is some  $m$  such that for all  $n \in X$ , either  $f(n) < m$  or  $n - f(n) < m$ .

details of all of this are in Hella *et al.* 1997. Summing up, we have the following result, where  $\mathbf{M}_1$  is the class of all monotone type  $\langle 1 \rangle$  quantifiers.

**Theorem 14** (Hella *et al.* 1997)

$Res^k(Q_f)$  is definable in  $FO(\mathbf{M}_1)$  iff  $f$  is bounded on  $\{n^k : n = 1, 2, \dots\}$ .

This theorem is fine as far as it goes, but it is restricted to monotone quantifiers. In principle, resumption has nothing much to do with monotonicity. From a logical point of view, one would like to generalize the theorem to resumption of arbitrary type  $\langle 1 \rangle$  quantifiers, and to definability in terms of arbitrary monadic quantifiers. But we know of no such generalization. The only result in this direction that we know of is the following.

**Theorem 15** (Luosto 2000)

$Res^2(Q^R)$  is not definable in  $FO(\mathbf{Mon})$ . (Hence, by (15.22),  $Res^2(most)$  is not definable in  $FO(\mathbf{Mon})$  either.)

The proof of this theorem combines the EF-tools with the most complex use of finite combinatorics—Ramsey theory—that we are aware of, and we won't even begin to try to explain the idea here. Moreover, Luosto has shown that this complexity seems to be necessary for the proof.<sup>16</sup>

For non-monotone type  $\langle 1, 1 \rangle$  quantifiers we know of no general results, but we quote from Westerståhl 1994 the following fact, alluded to at the beginning of Chapter 10.2 (the proof was sketched in note 3 of Chapter 10).

**Proposition 16**

On finite models,  $Res^k(Q_{\text{even}})$  is definable in  $FO(Q_{\text{even}})$ . In fact,  $Res^k(Q_{\text{odd}}) = Q_{\text{odd}} \cdots Q_{\text{odd}}$  ( $k$  factors) and hence  $Res^k(Q_{\text{even}}) = Q_{\text{even}} Q_{\text{odd}} \cdots Q_{\text{odd}}$  ( $k - 1$  occurrences of  $Q_{\text{odd}}$ ).

But there is no result, as far as we know, corresponding to Theorem 14, let alone anything about non-monotone quantifiers. Väänänen and Westerståhl (2002) had to use van der Waerden's Theorem to find an example of a type  $\langle 1, 1 \rangle$  quantifier not definable in terms of relativizations of monotone type  $\langle 1 \rangle$  quantifiers (Chapter 14.4), and, taking a hint from Luosto's results just mentioned, it may be just as hard to find one whose resumption is not definable in this way.

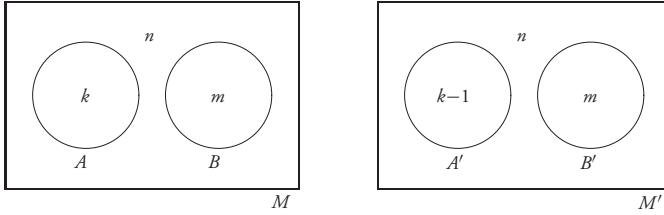
We end this subsection with a much simpler proof (due to Per Lindström) of a (much) weaker result: namely, that  $Res^2(most)$  is not definable in  $FO(most)$ . This provides an ulterior illustration of a straightforward application of the EF-tools:

Fix  $r > 0$ . It is quite easy to see that there are numbers  $k, m, n$  such that

- (i)  $(k - 1)m \leq n < km$
- (ii)  $2r < k < m < n$ , and  $m - k, n - m > 2r$

<sup>16</sup> More precisely, Luosto (1999) shows that (a result similar to) Theorem 15 implies van der Waerden's Theorem.

Now take a pair of models as follows:



Consider the sentence

$$Res^2(most)xy((\neg(A(x) \vee B(x)) \wedge y = x) \vee (A(x) \wedge B(y)), A(x) \wedge B(y))$$

The first disjunct of the first formula in the scope of the quantifier denotes  $\{(a, a) : a \in M - (A \cup B)\}$ , which has the same cardinality as  $M - (A \cup B)$ , i.e.  $n$ . The second disjunct denotes  $A \times B$ . Thus, the sentence in fact claims that  $|A \times B| > |M - (A \cup B)|$ , which is true in  $\mathcal{M}$ , but false in  $\mathcal{M}'$ , by (i). On the other hand, it is quite easy to see from (ii) that  $\mathcal{M} \approx_{FO(most)^r} \mathcal{M}'$ .<sup>17</sup> This proves the result. As with Theorem 14, although the resumptive quantifier is polyadic, the monadic EF-tools work here too, since only one-place predicates are used in the relevant sentence.  $\square$

*Remark:* Surely a similar result can be proved for the relativization of any properly proportional quantifier? What about  $Q_f$  when  $f$  is unbounded on squares? We leave these speculations to the interested reader.

### 15.5.4 Linguistic consequences, 2

Even in the absence of completely general results, the above theorems, and in particular Theorem 15, being undefinability results, have immediate linguistic consequences. Briefly, adverbial quantification increases expressive power.

If we grant that a sentence like (10.14) in Chapter 10.2, repeated here as

(15.24) Men are usually taller than women.

has a resumptive reading

(15.25)  $Res^2(most)(man \times woman, taller\ than)$

it is natural to ask if that reading can be obtained by some linguistic construction using only *most* or other type  $\langle 1, 1 \rangle$  quantifiers. For example, is it *reducible* (an iteration of two type  $\langle 1, 1 \rangle$  quantifiers)? The answer is No: It follows from a result in

<sup>17</sup> Indeed,  $\mathcal{M} \approx_{FO(MO)^r} \mathcal{M}'$ . The difference in size between any two different blocks is at least  $2r$ . So a comparison of size, which is what the quantifier  $MO$  can do, is unaffected by ‘moving around’ fewer than  $r$  elements, and gives parallel results in the two models.



Westerståhl 1994 that (15.24) is not even a unary complex, i.e. not even a Boolean combination of iterations and inverse iterations.<sup>18</sup>

But Luosto's theorem above provides a much more general answer, via the reasoning in Chapter 12.5. *No* construction using *any* monadic quantifiers which is formalizable in  $FO(\mathbf{Mon})$  can give the correct (uniform) truth conditions for (15.24). It seems fairly safe to conclude that a language with only D-quantification could not express (15.24).<sup>19</sup> Thus the fact that determiners do not express resumptive quantification, as we saw in Chapter 10.2.1, really does limit the expressive power of languages that lack A-quantification, which could be why no such language has been found.

<sup>18</sup> This notion was defined in ch. 10.1 for type  $\langle 1 \rangle$  quantifiers, but readily extends to type  $\langle 1, 1 \rangle$  quantifiers. To see how the result in Westerståhl 1994 applies, see Peters and Westerståhl 2002: n. 8.

<sup>19</sup> Theorem 15 provides an acute illustration of the difference between global and local definability. It was proved in van Benthem 1989 that, *on each universe*  $M$ ,  $Res^2(Q^R)$  is a unary complex, and this extends to  $Res^2(\text{most})$ . But the form of the complex depends on  $M$ . This local result gives no information about *uniform* definability.



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