Studies in Economic Theory 27

Svetlana Boyarchenko
Sergei Levendorskiĭ

Irreversible Decisions under Uncertainty

Optimal Stopping Made Easy

Springer

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Editors

Charalambos D. Aliprantis
Purdue University
Department of Economics
West Lafayette, IN 479072076
USA

Nicholas C. Yannelis
University of Illinois
Department of Economics
Champaign, IL 61820
USA

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Prof. Dr. Svetlana Boyarchenko
Prof. Dr. Sergei Levendorskiĭ
University of Texas at Austin
Department of Economics
1 University Station C3100
Austin, TX 78712-0301
USA
sboyarch@eco.utexas.edu
leven@eco.utexas.edu

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To our children, Nina and Mitya

Preface

[Cleopatra is listening to a slave girl who is playing the harp]

Cleopatra. [to the old musician] I want to learn to play the harp with my own hands. Caesar loves music. Can you teach me?

Musician. Assuredly I and no one else can teach the queen. Have I not discovered the lost method of the ancient Egyptians, who could make a pyramid tremble by touching a bass string? . . .

Cleopatra. Good: you shall teach me. How long will it take?

Musician. Not very long: only four years. Your majesty must first become proficient in the philosophy of Pythagoras.

Cleopatra. Has she [indicating the slave] become proficient in the philosophy of Pythagoras?

Musician. Oh, she is but a slave. She learns as a dog learns.

Cleopatra. Well, then, I will learn as a dog learns; for she plays better than you . . .

From *Caesar and Cleopatra* by Bernard Shaw[1]

In real life, as well as in economic models, individuals often make decisions in an uncertain environment. In many cases, a problem which an optimizing agent faces can be formulated or reformulated as a problem of optimal timing of a certain irreversible or partially reversible action. Some of the standard examples are: optimal exercise of American put and call options, timing investment, timing exit or default, incremental capital expansion program, etc.; the list of examples can be easily extended. For an individual, we can also add the timing of acceptance of a job offer or buying a house; for a statesman - the timing of important political decisions such as ratifying the Kyoto protocol, overhauling Social Security, or exiting from a dangerous foreign engagement. In other words, an agent chooses an optimal time to stop waiting and perform a certain action; she solves an *optimal stopping problem*. The

[1] The beginning of Act IV; we omitted a couple of sentences to make quotation shorter. The full version illustrates the idea of the monograph even better.

incremental capital expansion program is an example of a stochastic control problem but can be reduced to a family of optimal stopping problems. The general theory of optimal stopping problems is rich and non-trivial, and it is based on many difficult and deep results of the theory of stochastic processes. A proper understanding of this theory requires significant effort and time on the part of the student, nothing to say about a fairly advanced mathematical background. Thus, the entry cost is very high (although potential rewards are generous).

In the present monograph, we present an alternative approach to optimal stopping problems, whose basic ideas and techniques can be demonstrated on a much simpler level than in the standard literature on optimal stopping problems. Although the systematic study of the basics of the theory of stochastic processes and optimal stopping is useful, the reading of the long list of mathematical definitions and results can be rather tiresome. To alleviate this difficulty, we introduce necessary notions and results of the theory of stochastic processes on the piece-meal basis, when they are needed the first time, and on a rather informal level, as a part of the study of examples from economics and finance. We believe that this “learning-by-doing" approach will teach the reader to apply the technique to many problems in economics and finance, new ones including. Certainly, the reading of the book is easier for the reader familiar with the basics of the theory of stochastic processes.

We hope that not only novices to the theory of optimal stopping will benefit from reading the monograph but established specialists in economics, finance, mathematical finance, and operations research as well.

Parts I-II of the monograph and, possibly, Part III, can be used as a basis for an introductory course (the first author taught such a course at The University of Texas at Austin in 2002 and 2005), and the monograph as the whole - as a basis for an advanced course, for students who are familiar with the basic notions of financial mathematics and theory of stochastic processes and have a good mathematical background.

Austin, Texas,
February 2007

Svetlana Boyarchenko
Sergei Levendorskiĭ

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Introduction

1.1 Uncertainty and (partial) irreversibility

The main goal of this monograph is two-fold: to convey the importance of uncertainty, which individuals, firms or policy-makers face when making irreversible and partially reversible decisions, and explain useful general tools, which help to make rational decisions in such an environment. One can come up with dozens of real life situations, which involve decision-making of this sort, and they multiply in the rapidly changing world. Here are some examples: buying a house, accepting a job offer, timing retirement, adopting higher environment protection standards, switching from oil to environmentally cleaner sources of energy, choosing between outsourcing and using local labor, entry into growing markets, exiting from declining industries, natural resource extraction, new technology adoption, default on corporate or sovereign debt, mergers and acquisitions, human capital acquisition and protection, overhaul of the Social Security...

All seemingly different situations listed above have several essential common features:

(i) an uncertain environment;
(ii) a decision maker has a right but not the obligation to perform a certain action;
(iii) it is costly, sometimes, prohibitively costly, to reverse the decision after it has been made;
(iv) however, typically, an agent is under no obligations to make a decision instantly: she can wait.

Each item in the list calls for comments. In finance, a right but not obligation to buy or sell an asset is called an option. By analogy, a right but not obligation to perform a given action in an uncertain environment is also called an option. To distinguish the “real life options" from their financial counterparts, the former are called *real options*. Decision-making under uncertainty admits an interpretation as pricing of financial options or real options. The natural question is: what is so specific about options, and why should anybody care about option pricing at all? The answer is: because it is valuable to be under no obligation to perform a certain action, and, at the same time, to have the right to exercise it if the action is profitable. In the case of financial options, the irreversibility of the decision to exercise the option is evident, but in the case of real options, the (partial) irreversibility needs some justification. Suppose, to decrease the dependence on the foreign oil, the law will be passed which allows the drilling near the shore; then it will be difficult to reverse it. Similarly, if it is allowed to construct new nuclear plants, for the same purpose, then the closure of these plants in the near future will be impossible. For a firm, the standard explanation for a partial irreversibility is that if the economic environment is good for a plant of a given type, there is no need to reverse the decision to invest. However, if the economic environment deteriorates, then nobody wants to buy a plant of the given profile; therefore, the best the firm can hope for is to sell some pieces of the plant at a scrap value, which is much lower than the price the firm paid for these pieces; as the limiting case, the firm can recover nothing (the investment is completely irreversible). New technology adoption requires, typically, the scrapping of the old one. It is impossible to reverse default under Chapter 7. The list of explanations for the (partial) irreversibility assumption can be easily extended.

1.2 Option value of waiting

The interplay of irreversibility and uncertainty in decision making becomes especially significant if we take into account the possibility of waiting. In many situations, the value of a now or never opportunity is smaller than the value of the same opportunity if an individual can wait and make a decision later: there is an *option value of waiting.* In Chap. 2, we demonstrate this effect with several examples. A basic example is an investment in a plant, which will yield an uncertain stream of profits starting the year after the investment is made. For simplicity, it is assumed that the uncertainty will be realized in a year from now. In one variant of the future, the profit flow will be 100 mln ($) per year, in the other one – zero. The fixed investment cost is 490 mln, and the *riskless rate* is $r = 0.1$ per year. The manager of the firm believes that the odds of the two scenarios of the future are equal. Let $q = 1/(1+r)$ be the *discount factor* per year. If the manager regards the investment as now or never opportunity, he calculates the expected profit flow $E[\Pi_t] = 0.5 \cdot 100 + 0.5 \cdot 0 = 50$, next, the expected present value (EPV) of profits

$$V = qE[\Pi_1] + q^2 E[\Pi_2] + \cdots = \frac{q}{1-q} E[\Pi_1] = \frac{50}{0.1} = 500,$$

and then the net present value of the project at time 0 assuming that the investment is made at time 0: $NPV_{0;0} = 500 - 490 = 10$ mln. Since the NPV is positive, the manager decides to invest. The manager follows the

NPV-rule, which Dixit and Pindyck [39] call naive. Indeed, a sophisticated manager would wait a year, so that the uncertainty of the future will be resolved (in more realistic examples, the uncertainty will be never completely resolved, of course, but additional pieces of information do arrive, and the uncertainty becomes partially resolved). If $\Pi_1 = 0$, it will be non-optimal to invest, otherwise, $\Pi_1 = \Pi_2 = \cdots = 100$, and the time-1 present value of investment at time $t \ge 1$ is $q^{t-1}(100/0.1 - 490) = q^{t-1}510$. Since $q < 1$, it is non-optimal to wait further. Thus, the manager invests at $t = 1$, and the time-0 expected present value of this investment is $NPV_{0;1} = q(0.5 \cdot 510 + 0.5 \cdot 0) = 231.8$. Since $231.8 > 10$, it is optimal to wait a year and invest iff the good scenario of the future is realized. The difference between the rational expected present value of the project and naive net present value, $231.8 - 10 = 221.8$, is called the option value of waiting.

A useful interpretation is as follows. The option value of waiting is the value of the option to postpone the decision. If the manager invests today, this option will be killed, hence, 221.8 mln is the opportunity cost of investing today. Therefore, the total cost of investing today is $490 + 221.8 = 711.8$, which exceeds the expected present value of the future profits, 500. Hence, it is unreasonable to invest today.

1.3 Bad news and good news principles

One of the main results of the monograph is the general form for the optimal investment rule and the corresponding general rule for the optimal disinvestment (optimal entry and exit; both are assumed irreversible), which are very similar to the naive NPV rule. In the discrete time model, the naive NPV rule prescribes to invest when the EPV of the profits

$$EPV_\Pi = \sum_{t=1}^{\infty} q^t E[\Pi_t]$$

exceeds the investment cost, I. Similarly, if the plant operates already but the economic conditions deteriorate, and the plant can be closed and a certain scrap value, Sc, can be recovered, then the naive NPV rule recommends to close the plant when the EPV of profits falls below the scrap value.

Define the *supremum and infimum processes* as $\bar{\Pi}_t = \sup_{0 \le s \le t} \Pi_s$ and $\underline{\Pi}_t = \inf_{0 \le s \le t} \Pi_s$, respectively; in discrete time models, sup and inf can be replaced with max and min, respectively. Note that $\bar{\Pi}_t(\omega) = \sup_{0 \le s \le t} \Pi_s(\omega)$ and $\underline{\Pi}_t(\omega) = \inf_{0 \le s \le t} \Pi_s(\omega)$ are calculated along each sample path of the process. We will prove that if the profit stream Π_t is a monotone increasing function of a process with i.i.d. increments, then the investment must be made when the EPV of the infimum stream $\underline{\Pi}_t$ equals or exceeds the investment cost for the first time:

$$EPV_{\underline{\Pi}} := \sum_{t=1}^{\infty} q^t[\underline{\Pi}_t] \geq I.$$

We call this rule the *record setting bad news principle.* In 1983, Bernanke stated that option value of investment is determined by the expected value of "bad news". In other words, the critical price that triggers new investment depends on downward moves in prices because the ability to avoid the consequences of bad news lead us to wait. The (record setting) bad news principle proved in the monograph can be formulated as follows: in making irreversible investment decisions, one has to disregard all temporary increases in profits. In other words, the manager contemplating an irreversible investment must be very pessimistic.

However, the same manager becomes very optimistic when she contemplates an irreversible disinvestment. The optimal exit rule is: disinvest when the EPV of the supremum stream $\bar{\Pi}_t$ reaches the scrap value or falls below it for the first time. This is the *(record setting) good news principle.* Thus, this time, the investor disregards all temporary decreases in profits.

The bad and good news principles imply that when an optimizing individual, firm or politician has to make a decision in an uncertain environment, they wait more than the standard cost-benefit analysis (naive NPV-rule) recommends, and the discrepancy can be very large indeed if the uncertainty is significant.

1.4 Optimal stopping and stochastic control: capital expansion program

Consider a risk neutral, competitive firm that maximizes its present value net of installation cost of capital. The firm's manager contemplates an increase of the capital stock. Assume that G, the production function of the firm, is differentiable, increasing, concave, and satisfies the Inada conditions, and that the investment is irreversible. For simplicity, assume that all the uncertainty is on the demand side, i.e., the price of firm's output, P, is stochastic, and the marginal cost of capital, C, is constant. The firm's objective is to maximize the expected present value of profits

$$\Pi(K, P) = \max_{\{K_t\}} E\left[\int_0^{+\infty} \mathrm{e}^{-rt}(P_t G(K_t) - rCK_t)dt \mid P_0 = P\right] \qquad (1.1)$$

Here $r > 0$ is the riskless interest rate, (K, P) are state variables, and $\{K_t\}_{t>0}$ is the control variable. Due to the irreversibility of investment, $K_t \geq K$ for any realization of the process.

Following the tradition in the literature, we divide the state space into two disjoint regions: inaction and action ones. For all pairs (K, P) belonging to the inaction region, it is optimal to keep the capital stock unchanged. In the action region, investment becomes optimal. Assume that the inaction region is closed and denote by $\Gamma = \{(H(K), K) \mid K > 0\}$ its boundary. The stochastic control problem under consideration is more complicated than the investment problem considered above because now it is necessary to decide not only when to increase the stock of capital but also by how much. Fortunately, it is possible to reduce the solution of the incremental capital expansion program to the one-shot investment problem of a chunk of capital, and show that once the boundary between the action and inaction regions is found, the investment rule is: whenever, after a price jump, the firm finds itself in the action region, it invests up to the boundary Γ.

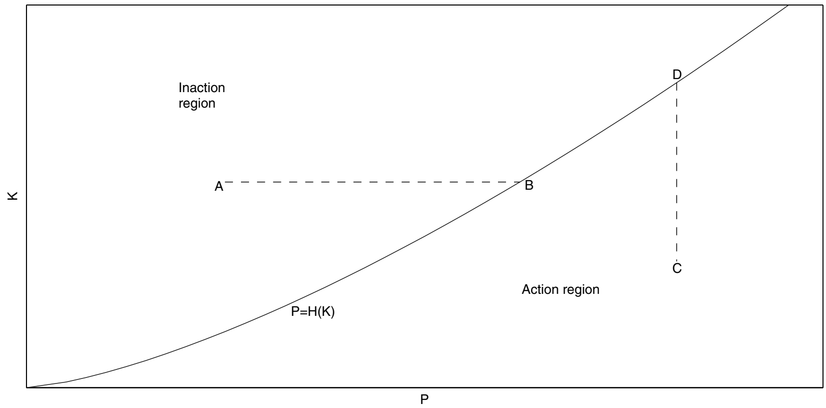

Fig. 1.1. Action and inaction regions. If the price process jumps and (P, K) is at C, capital is adjusted so that (P, K) is at the boundary point D. If the state variable (P, K) was at A, and the price changes so that (P, K) is at B or to the left of B, it is non-optimal to increase capital.

The investment strategy can be formalized as follows

(i) do not invest if $P \leq H(K)$;
(ii) invest when $P > H(K)$ and increase the capital stock till the level K' such that (P, K') is at the boundary Γ.

The question is: how to choose the *investment threshold* $H(K)$. If there is no uncertainty and the stock of capital can be costlessly adjusted, then the firm chooses K to maximize the instantaneous profit

$$\Pi(K) = PG(K) - rCK,$$

hence the necessary first order condition defining the investment triggering price is

$$PG'(K) = rC. \tag{1.2}$$

The solution $H_J = rC/G'(K)$ gives the *Jorgensonian rule*: invest when the marginal revenue equals the marginal cost of investment. A simple naive way to take uncertainty into consideration is to compute the EPV of the both parts in (1.2). The result is the *Marshallian law* or the *NPV-rule*: invest when the EPV of the marginal revenue equals the marginal cost of investment. The equation for the *Marshallian threshold* $H(K) = H_M$ is

$$E\left[\int_0^{+\infty} \mathrm{e}^{-rt} P_t G'(K)dt \mid P_0 = H_M(K)\right] = C \tag{1.3}$$

As in the case of the one-shot investment problem, the Marshallian rule does not take into account the option value of waiting. We show that if P_t is a non-decreasing function of a process with i.i.d. increments (Brownian motion or more general Lévy process), then the correct threshold is defined from an analog of (1.3) with the infimum process substituted for the original price process:

$$E\left[\int_0^{+\infty} \mathrm{e}^{-rt} \underline{P}_t G'(K)dt \mid P_0 = P = H(K)\right] = C \tag{1.4}$$

– the bad news principle once again. In the discrete time model, the prescription is similar. It suffices to replace the integration in (1.4) with summation.

1.5 Discounted utility anomalies

The situations considered so far are quite standard, and the importance of the real option approach to these investment problems – and many other problems – was understood and demonstrated a long time ago. However, we have a feeling that the ideas of the real option theory are still not as popular as they should be. This remains true even 13 years after publication of the first edition of a monograph *Investment Under Uncertainty* by A. Dixit and R. Pindyck [39], whose main purpose was to demonstrate the importance of optimal stopping ideas (*real options approach*) in many fields of economics. We hope that the next example will demonstrate that real option ideas can lead to new results in quite unexpected situations.

One of the current hot topics in economics are the so-called exotic preferences and discounted utility anomalies . The Samuelson discounted utility (DU) model [78] calculates the value of consumption over time interval $[0, T]$ as

$$U = \int_0^T \mathrm{e}^{-rt} u(c_t)dt, \quad r > 0,$$

in continuous time models, and

$$U = \sum_{t=0}^{T} \beta^T u(c_t), \quad 0 < \beta < 1,$$

in discrete time models. In experiments, the following DU anomalies were recorded:

1. hyperbolic discounting (discount rate for gains decreases with time)
2. the sign effect (gains are discounted more than losses)
3. the delay-speedup asymmetry (an individual is willing to pay less to expedite the delivery than she asks as a compensation for the delay)
4. the negative discounting for losses: an individual may prefer to expedite a payment
5. the magnitude effect (small payoffs are discounted more than large ones)

In order to explain these anomalies, various alternative models have been suggested [37, 42, 43, 71]. All these models either postulate a non-standard dependence of the discount factor on time, or deduce this dependence from some axioms of time preference. There are also models that explain DU anomalies as a Nash equilibrium in games of the current short-run self against multiple future selves (so-called dual self models).

In fact, all the aforementioned DU anomalies can be explained as rational optimizing behavior of an individual who perceives the future as uncertain and believes that she can wait till it becomes optimal to exchange the future consumption gain/loss for the current one. For details, see [24].

Hopefully, the above examples show that the real option approach to various real life situations is promising.

1.6 Models of uncertainty

When modeling uncertainty, several important issues must be addressed. The first one is the trade-off between the tractability of the model vs. empirical realism. In many fields of economics, uncertainty is modeled as a random draw from a given distribution. Thus, the uncertainty, which an economic agent will face tomorrow, is independent of the realization of uncertainty today unless the parameters and/or type of the probability distribution are not known for sure and the agent updates her prior beliefs about the distribution. Therefore, there is no option value of waiting, and the naive NPV rule can be applied with impunity.

In the real options theory, the uncertainty is modeled as a *stochastic process*, that is, as a collection of random variables $\{X_t\}_{0\le t\le T}$, where T is the *time horizon* of the model. In discrete time models, $\{t : 0 \le t \le T\}$ is the set of integers, and in continuous time models, it is the segment $\{t : 0 \le t \le T\} = [0, T]$. On the formal level, a model with random draws from the same distribution can be interpreted as a stochastic process with i.i.d. random variables X_t. In more complex models, random variables X_t are not independent. Of course, the present and the past are independent of the future, therefore, for $t > 0$, X_s, $s \le 0$, are independent of X_t. If the future depends only on the present but not on the past, we have a *Markov process*.

In the monograph, we will use only Markov processes to model the evolution of the underlying stochastic factors, and, in the main body of the monograph, we will introduce and explain, with examples, the necessary definitions and results of the theory of stochastic processes for the Markov processes only. In fact, the majority of the results are obtained for processes with i.i.d. increments, that is, in discrete time, for random walks on an integer lattice $\mathbb{Z}$ or on $\mathbb{R}$, and, in continuous time, for the Brownian motion or more general Lévy processes. An additional flexibility is added by modeling the underlying price process, stochastic demand, etc. with an arbitrary monotone function of a process with i.i.d. increments. The resulting processes may have diffusion and jump components, and they may exhibit mean-reverting and/or regime-switching features. The method of the monograph admits an extension to processes with i.i.d. increments modulated by a Markov chain. That is, in each state of the Markov chain, the type or characteristics of the process may change, as well as the riskless rate and the payoff function. This makes our approach very flexible. In some cases, we study models with two sources of uncertainty, and a regime-switching version of our method can be used to approximate 2-3 factor models of various kind, e.g., stochastic volatility models and models with stochastic interest rates.

One of the main points which we want to convey is that different choices of the model for uncertainty may lead not only to quantitatively different results but to new qualitative effects as well. We illustrate this point with three examples. First, in the capital expansion program for a monopolist who faces the demand uncertainty, the standard geometric Brownian motion (GBM) model for the demand shocks (or, more generally, geometric Lévy model) leads to the monotone dependence of the capital expansion threshold $K = H^{-1}(P)$ on the price that the monopolist charges for its product. A typical shape is shown in Fig. 1.1. In other words, each time that the demand shock makes it optimal to increase the production, the monopolist increases both the production capacity and price of the output. We show that if the demand shock is not an exponential function of the underlying stochastic factor but, instead, the rapid growth of demand at low and moderate levels slows down at higher levels, then it is possible that the monopolist finds it optimal to increase the capital stock but, simultaneously, decrease the price. The second example is the technology adoption problem with two sources of uncertainty, one being the price of the output, the other – the stochastic technology frontier. We show that if both factors contain diffusion and jump components, and these components of the two factors correlate, then the technology adoption frontier decreases if the correlation between jump components increases but increases if the correlation between diffusion components increases. The third example is the behavior of the early exercise boundary for the American put on a non-dividend paying stock at expiry. It is well-known that, in diffusion models, the limit of the early exercise boundary is the strike, and, it seems, this property is taken for granted for other continuous time models. We show that for processes with jumps, in many cases, there is a gap between the early exercise boundary and the strike, and, for typical parameter values documented in empirical studies of financial markets, the gap is of order 5-15% of the strike. These examples clearly demonstrate that the word "uncertainty" does not mean much in itself: qualitatively different results can be obtained for different specifications of uncertainty.

1.7 Choice of the probability measure

Suppose that we have chosen a reasonable model for the underlying stochastic factor. The natural question is: to which extent is the choice arbitrary? The situation is clear only in the case of the so-called *complete markets*. In a complete market, there exists a unique probability measure (*equivalent martingale measure*, *EMM*) which must be used to price the assets in the market; otherwise, arbitrage opportunities arise. For a financial market with one riskless asset (bond) and a risky asset (stock), the binomial model in discrete time and the GBM model in continuous time are examples of complete markets. The trinomial model and the Brownian motion with embedded jumps are incomplete. If there is no arbitrage, an EMM exists but it is, typically, not unique. As far as the pricing of assets that are traded in the market already is concerned, this is not a serious problem: an appropriate EMM can be inferred from the data. However, it is important to be able to price new securities, such as the European and American options, and other contingent claims, which are not traded in the market as yet. Due to the theoretical and practical importance of this problem, there exist numerous approaches. All of them assume, however, that the price processes of the underlying securities already traded in the market will not change after a new security is introduced. However, from the viewpoint of general equilibrium theory, there is no reason to believe that the price process of the underlying will not change. Thus, a correct choice of an EMM remains a very difficult problem, which has no satisfactory solution from the point of view of general equilibrium theory so far. Optimal stopping problems become especially interesting and difficult in this context but these situations are outside the scope of the book. We simply presume that the agents use the same EMM to price all contingent claims: "contingent claims are priced under an EMM chosen by the market".

1.8 Techniques used in the monograph

Dixit and Pindyck [39] did a wonderful job explaining the standard techniques of the optimal stopping theory on a rather elementary level, in the framework of the geometric Brownian motion model and some more involved models. Nevertheless, the economic profession did not embrace the ideas of the real options theory as fully as it could have done[1]. One can suggest several explanations. One of these is the use of the GBM model as the workhorse. The GBM model is powerful and tractable but its application to optimal stopping problems relies heavily on two deep and highly non-trivial things: Ito's lemma and smooth pasting principle; in addition, the technique of ordinary differential equations, and, sometimes, partial differential equations, is used. Although Dixit and Pindyck [39] explain both Ito's lemma and smooth pasting principle on an intuitive level, the explanation itself requires understanding of the Taylor formula, with an additional twist necessary for the formal application of Ito's lemma. It seems that pieces of mathematics used in [39] were too many to make the real options ideas widely popular.

In the present monograph, we present an alternative approach to optimal stopping problems, whose basic ideas and techniques can be demonstrated on a much simpler level. Contrary to the standard approaches to optimal stopping problems, which consider the American options with instantaneous payoffs as the model examples of options, we regard as primitives of the model streams of payoffs, and the model options are the options to abandon or acquire a stream. This economically meaningful approach allows us to obtain simpler and shorter proofs than the standard approaches allow for and to study options of a more complicated structure. In simple discrete time – discrete space models, which, in many cases, suffice to explain many important results in economics, our approach is almost as simple as in models where the uncertainty is modelled as a draw from a probability distribution, and a decision rule is of the simplest form: if the expected value of a payoff tomorrow is higher than the current (spot) price then buy the asset now. The general option exercise rules, which we obtain, can be formulated in a similar form. In the simplest binomial and trinomial models, we derive optimal exercise rules using the elementary algebra only; the most advanced tools are summation of a geometric series and the Bellman equation. The same simple ideas are the cornerstone of our approach to more advanced models, including models in continuous time. To treat these more advanced models, we derive general formulas for the expected present values of a stream g_t of payoffs that will be acquired (or abandoned) as a certain exercise boundary is reached or crossed, in terms of operators $\mathcal{E}^{\pm}$, which calculate the (normalized) expected present value of the supremum stream $\bar{g}_t = \sup_{0 \le s \le t} g_s$ and infimum stream $\underline{g}_t = \inf_{0 \le s \le t} g_s$. The formulas are proved if $g_t = g(X_t)$ is a function of a process with i.i.d. increments, the *Wiener–Hopf factorization* technique being used. The supremum and infimum processes and the Wiener–Hopf factorization are explained first in the simple set-up of discrete time – discrete space models. In continuous time models, we do not use Ito's lemma. Instead, we use Dynkin's formula, which is, essentially, a refined version of the Bellman equation. Finally, we do

[1] Of course, ideas of the optimal stoping theory are routinely applied in asset pricing and several related areas in finance, and, in economics, to study investment under uncertainty.

not resort to the smooth pasting principle; and the method provides a simple and straightforward explanation why and when the smooth pasting principle fails.

In simple situations which do not have the features of embedded options, we apply the following scheme.

(1) fix a candidate for the optimal exercise boundary, and write down the Bellman equation for the value function (price of an option, investment opportunity or value of a firm, etc.);
(2) employ the Wiener–Hopf factorization method in the form used in analysis rather than in probability to solve the Bellman equation;
(3) using the explicit form of the solution in terms of operators $\mathcal{E}^{\pm}$, find the optimal exercise boundary.

Typically, the solution of a problem under consideration simplifies after reduction to an option to abandon a stream. This simplification becomes especially important if a sequence of embedded options is considered. A special important example is the solution of a sequence of embedded perpetual American options that arise in the framework of the Carr's randomization approach to the American options in finite time horizon continuous time models. In the study of regime-switching models, the reduction to an option to abandon a stream proves to be crucial.

The method presented in the monograph leads to a general economically meaningful interpretation of the optimal stopping rules - the bad and good news principles, which are new even in the geometric Brownian motion case. The proofs are simpler than standard proofs in the GBM-case, and they remain essentially at the same level of technical difficulty for wide classes of processes with jumps. The method is applicable to numerous types of optimal stopping problems, embedded options of arbitrary length including. In the latter case, contrary to the standard approach to optimal stopping problems, we do not have to reduce the optimal stopping problem to a system of highly non-linear equations with several unknowns; each time, we need to solve only one equation, typically, with a monotone function, which is an easy computational task. For several basic types of optimal stopping problems, explicit solutions are obtained not only for the standard exponential or linear payoffs (as functions of the underlying stochastic factor) but for any payoff representable as the expected present value of a monotone stream.

We do not promise, however, that the reader will find all important variants of optimal stopping problems in the book. Although the technique developed can be adjusted to more complex situations, we work in the partial equilibrium framework assuming that the underlying uncertainty is modelled as a monotone function of a 1D process with i.i.d. increments, that is, a random walk on $\mathbb{R}$ or $\mathbb{Z}$ or a Lévy process on $\mathbb{R}$. In all cases, we assume that all agents agree on the probability measure on the set of all possible states of the future, and that their actions do not change this measure.

1.9 Overview of the monograph

Chap. 2 contains several examples which illustrate the basic ideas and types of real options and their relations to the American call and put options, in the simplest set-up of two-period and three-period models. We consider models, which are, formally, set in the infinite time horizon but admit a simple reduction to two- and three-period models, which can be easily solved by backward induction. In Chap. 3, we give a short overview of the main definitions and results of financial economics in the set-up of finite state-finite time horizon models. The reader with the knowledge of basic financial economics and real options can safely skip Chaps. 2 and 3 but for the reader, who was not exposed to these ideas, Chaps. 2 and 3 can be the most important part of the book. Although the book is designed to be self-contained, the reader, who is not familiar with the basic ideas of financial economics, is strongly recommended to read additional basic texts at the level of discrete time models, and there are many good ones, e.g., [57, 74, 81]. However, the additional reading is unnecessary for the understanding of the following chapters.

From the technical point of view, all problems considered in Chaps. 2 and 3 can be solved using backward induction. In Part II, the underlying stochastic factor is a random walk on an integer lattice, and the time horizon is infinite. Therefore, backward induction is no longer applicable. Naturally, in this case, we use the reduction to the Bellman equation, and we solve the latter using the Wiener-Hopf factorization. Discrete time - continuous state space models are considered in Part III, and continuous time - continuous state space models - in Part IV. In Part IV, Dynkin's formula plays the part of the Bellman equation. The reader can develop an analog of the method for continuous time - discrete space models, $\mathbb{Z}$ being the state space, which is not considered in the book.

The basic types of real options and perpetual American options, namely, options to abandon or acquire a payoff stream, and options to acquire an instantaneous payoff, are studied for all three types of models of uncertainty considered in the book. In each of Parts II-IV, we demonstrate how the method works using a simple process such as the binomial model or Brownian Motion. After that, we formulate and prove three sequences of general theorems about:

1. calculation of the EPV of a stream or instantaneous payoff which will be acquired or abandoned when a certain (fixed) boundary is reached or crossed (from above or below, depending on a situation);
2. optimal stopping rules in the class of stopping rules of the threshold type;
3. optimality in the class of all stopping times.

To prove the theorems of the first two sequences, only the most basic properties of the EPV operators are needed, the conditions on the payoff and process are very weak and the proofs are very short. The third sequence is more involved in each respect, especially in the case of Lévy models; however, even in this case, the proofs are 1-2 pages long, at most, and the conditions which we impose on the payoff and process are fairly weak. The reader will see that the theorems and proofs are very similar in all three Parts but there are some differences. We were tempted to make the monograph shorter by giving the complete proofs in Part II only and indicating the necessary changes in the corresponding theorems in Parts III-IV but we decided against it feeling that the text might become too unwieldy. In addition, the reader may be interested in only one class of models; therefore, she may be unwilling to read the other parts of the monograph.

Somewhat contrary to the last objective, additional types of optimal stopping problems and related questions are considered only at one, sometimes, two places in one of Parts II-IV. We hope that this makes the exposition less boring (albeit less systematic). The reader can reformulate and solve each additional type of problems in the framework of the other models of uncertainty as an exercise; some of these exercises are formulated explicitly.

In Part III, additional problems are investment lags, the expected waiting time for the investment, and capital dynamics in the model of the incremental capital expansion. With these models, we demonstrate certain advantages of the discrete time modeling against continuous time modeling. For instance, we calculate the expected waiting time and stochastic dynamics of the capital installed using simple analytical tools. In the continuous time models, the calculation of these quantities requires the Fourier inversion and calculation of complicated integrals in the complex domain.

In Part IV, we calculate the investment threshold for the capital expansion program under non-standard specifications of the demand shocks and show that a monopolist may find it optimal to increase the production and, simultaneously, decrease the price. We also consider the entry problem with the embedded option to exit or default, two-stage investment problem, and solve certain sequences of embedded options of arbitrary length (*Russian dolls*). Finally, we consider a model of the technology adoption, which is a two-factor model. One factor is the price uncertainty, and the other one describes the evolution of the frontier technology.

In Part V, we derive an explicit backward procedure for pricing American options in continuous time models with finite time horizon (the procedure is approximate, of course). Then, we show that for processes with jumps, typically, the early exercise boundary is separated from the strike by a non-zero margin. Next, we show that the general option exercise rules (bad and good news principles), which we prove for cases when the underlying stochastic factor is a monotone function of a process with i.i.d. increments, can be used as a rule of thumb for more involved models of uncertainty. We solve the standard problems in the model with mean-reversion (the geometric Ornstein-Uhlenbeck model) and compare the exact result with the formal results which the bad and good news principles provide. We show that for the standard perpetual American put and call options, the general formulas coincide with the exact results, and in exit and entry problems, the option values obtained from the general formulas differ from the exact ones by no more than 3-5% unless the mean reversion coefficient is large.

Each of the problems in Parts II-IV can be reformulated as a problem for any other Part. Problems that can be formulated as qualitative ones are collected, for the most part, in Part II, whereas the ones that require more calculations are given in Parts III-V. The reason is that calculations with several infinite sums are more messy than calculations involving integrations - although on the theoretical level, the former are simpler than the latter.

The majority of the results presented in the monograph are published in [15, 20, 16, 23, 24, 25, 29, 30, 58, 59, 60, 61]; some of the proofs are simplified. For other approaches to optimal stopping problems in general and to the Wiener-Hopf factorization as a tool in the theory of optimal stopping, see [75, 72, 73] and [52], respectively, and the bibliography therein.

1.9.1 Extensions

The technique of the monograph is applicable to the Bermudan options [22], two-point free boundary problems such as the perpetual straddles and strangles [17] and related problem of incremental capital expansion/contraction with two boundaries. Also, the technique has been successfully applied to a model with jumps, in the case when the inaction region consists of a semi-infinite interval and a finite interval [18]. The technique can be used to study real options with strategic interactions not only in the Brownian motion setting as in [44, 82] but in jump-diffusion environments as well. See [18]. Other wide fields of possible applications are credit risk models and models with learning. Finally, the technique is applicable to regime-switching models and models with stochastic interest rates and volatility; the latter should be approximated by finite state Markov chains. See [26, 27, 28].

1.9.2 Notation

The labels $\mathcal{E}$, $\mathcal{E}^{\pm}$ for the EPV–operators under the original process and the supremum and infimum processes, and $\kappa_q^{\pm}(z)$ for the factors in the Wiener-Hopf factorization formula are used in all part of the monograph because they denote the objects of, essentially, the same nature. However, the analytical representations are different in Parts II, III and IV-V. To help the reader to find an appropriate formula, in the index, we indicate the part of the monograph. We adopt the same policy on constants $\lambda^{\pm}, c^{\pm}, \beta^{\pm}, \lambda_j^{\pm}, \beta_j^{\pm}$, which denote the parameters of the basic models in parts III and IV-V.

2

Real options and American options

In this Chapter, we give a comprehensive albeit far from exhaustive list of the simplest options in finance and option-like problems in economics, and introduce some basic terminology and technical tools.

2.1 Basic examples

2.1.1 Investment problem, two scenarios of the future

The manager of an energy firm contemplates investment in a plant which will produce 5 mln barrels of ethanol from corn starting a year after the investment is made. The revenue is proportional to the price of ethanol. At the current prices, say, $S_0 = 50$ (\$), the plant will yield revenue $R_0 = 50 \cdot 5 = 250$ (mln \$) per year; the variable cost is $C = 200$ (mln \$) per year, hence, the operational profit is $\Pi = 50$ (mln \$) per year. To make the investment decision, the firm's manager calculates the present value of the operational profits, and compares with the fixed investment cost, $I = 490$ (mln \$). (Taxes are assumed away for simplicity). Assume that the firm discounts the future at rate $r = 0.1$ per year, equivalently, the discount factor is $q = 1/(1+r) = 1/1.1$ per year. Using

$$\sum_{t=0}^{\infty} q^t = \frac{1}{1-q}, \tag{2.1}$$

the manager calculates the present value of the operational profit stream as

$$V = q\sum_{t=0}^{\infty} q^t \Pi = \frac{q}{1-q}\Pi = \frac{\Pi}{r} = 50/0.1 = 500 \text{ (mln \$)}.$$

Since the *net present value* (*NPV*) $V - I = 500 - 490 = 10$ is positive, the manager decides to invest. Assuming that the current prices will remain fixed, the investment is optimal; however, the prices of ethanol and corn may change

$$-I = -490 \longrightarrow \Pi = 50 \longrightarrow \Pi = 50 \longrightarrow \cdots$$

t=0 t=1 t=2

Fig. 2.1. Deterministic future. Profits are the same each period $t = 0, 1, \dots$

(and they do change in the real life). Should the prices of oil and natural gas rise, one may expect that the price of ethanol will rise. On the other hand, it is also possible that the price will fall because either cleaner sources of energy will appear or new large natural gas and oil fields or tar sands will be developed. For the time being, we disregard all other sources of uncertainty in the future but the price of ethanol. To account for the price uncertainty of the output of the plant, we will use a simplifying device, which is common in qualitative models in economics and finance. Namely, we assume that the price will change in a year from now (at $t = 1$) but will remain constant thereafter. One says: *the uncertainty is resolved at* $t = 1$. Assume that two levels of the price are possible: $S_1 = S_2 = \cdots = 60$ and $S_1 = S_2 = \cdots = 40$, and the variable cost is the same in all cases: $C = 200$ (mln \$). Then two levels of the operational profit are possible:

$$\Pi_1(\omega_1) = \Pi_2(\omega_1) = \cdots = 5 \cdot 60 - 200 = 100,$$

and

$$\Pi_1(\omega_2) = \Pi_2(\omega_2) = \cdots = 5 \cdot 40 - 200 = 0,$$

where ω_j label the scenarios of the future, equivalently, the possible *sample paths* of the price process for ethanol.

The manager calculates the present value of the operational profit for each possible scenario of the future:

$$V(\omega) = \sum_{t=1}^{\infty} q^t \Pi_t(\omega) = \frac{\Pi_1(\omega)}{r}, \tag{2.2}$$

which gives $V(\omega_1) = 1000$ and $V(\omega_2) = 0$. In this subsection, we assume that the investment is completely irreversible, so that if the price of ethanol drops and the plant will yield zero profit flow, the firm will be unable to sell the plant and recover a part of the fixed investment cost. (This assumption will

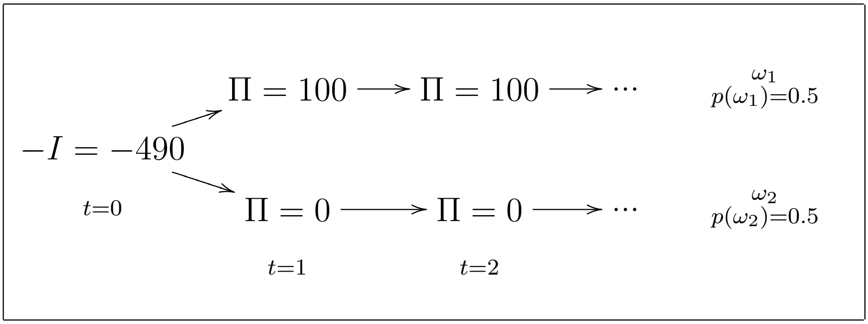

Fig. 2.2. Uncertain future: two possible scenarios. The probability space, $\Omega = \{\omega_1, \omega_2\}$, consists of two sample paths of the price process (equivalently, of the profit process). NPV$= -I + \Sigma_{\omega\in\Omega} p(\omega)[q\Pi_1(\omega) + q^2\Pi_2(\omega) + \cdots]$.

be relaxed). Suppose, the manager believes that the probability of the price increase is $p(\omega_1) = P(S_1 = 60) = 0.5$, hence, the probability of the fall of the price $p(\omega_2) = P(S_1 = 40) = 0.5$ as well[1]. The manager calculates the (expected) present value of the future profits of the investment project as

$$V_0 = \sum_{\omega\in\Omega} p(\omega)V(\omega), \tag{2.3}$$

which in our case gives

$$V_0 = 0.5 \cdot 1000 + 0.5 \cdot 0 = 500. \tag{2.4}$$

Since the NPV of the project, $NPV_{0;0} = V_0 - I = 500 - 490 = 10$, is positive, the manager decides to invest. The manager applies the *NPV-rule*, which Dixit and Pindyck (1994) call the naive investment rule. The manager is naive because he does not take into account an opportunity to wait.

Indeed, assume that the manager waits one period. Then, if the first variant of the future is realized, the manager invests at time $t = 1$ (it makes no sense to wait longer because the uncertainty is resolved already, and the future profits are discounted), and the time-1 NPV of the project is

$$NPV_{1;\geq 1}(\omega_1) = \sum_{s=1}^{\infty} q^s 100 - 490 = \frac{100}{r} - 490 = \frac{100}{0.1} - 490 = 510.$$

The first subscript indicates the time when the NPV is calculated, and the second subscript indicates that the investment is not made before $t = 1$ – if it

[1] In Chap. 3, we explain how the manager can infer these probabilities from the data; they are called *risk-neutral probabilities.* The corresponding probability measure over the set of possible sample paths of the price process is called a *risk-neutral measure* or *equivalent martingale measure* (*EMM*). It will also be shown that in this model, an EMM is unique.

is ever made. If the second scenario of the future is realized, the manager does not invest, and $NPV_{1;\geq 1}(\omega_2) = 0$. Discounting and calculating expectations, we obtain the date-0 (expected) net present value of the project *assuming the decision to invest is postponed for one period*:

$$NPV_{0;\geq 1} = q \sum_{\omega \in \Omega} p(\omega) NPV_{1;\geq 1}(\omega) \tag{2.5}$$
$$= \frac{1}{1.1}(0.5 \cdot 510 + 0.5 \cdot 0) = 231.8$$

(mln \$, approximately). Since $231.8 > 10$, it is optimal to wait and not to invest at time 0. Another interpretation is: the option value of waiting, that is, the difference between the rational value of the project, and the naive NPV, is positive:

$$V_{\text{opt}} = NPV_{0;\geq 1} - NPV_{0;0} = 231.8 - 10 = 221.8.$$

Note that (2.5) can be written as

$$NPV_{0;\geq 1} = E^{\mathbb{Q}}[q \cdot NPV_{1;\geq 1}], \tag{2.6}$$

where $\mathbb{Q}$ is the risk-neutral measure over the set set of possible sample paths of the price of ethanol.

2.1.2 Investment problem as an American call option

The calculations above admit a different interpretation. The manager knows that, after time $t = 1$, she will make no decisions. In this situation, at time $t = 1$, there is no need to think about the details of the price evolution: only the present value of the stream of profits at time 1, conditioned on information available at time 1, matters (and the fixed investment cost, I). Hence, the investment problem is equivalent to pricing of an option to acquire a risky asset, in the two-period model. The asset price at time t is the expected present value (EPV) of the stream of profits which will start to accrue at time $t+1$, $t = 0, 1, \dots$ We have calculated $EPV_0 = 500$, and $EPV_t(\omega_1) = 1000$, $EPV_t(\omega_2) = 0$, $t = 0, 1, 2, \dots$. The firm's manager has the right but not obligation to get the instantaneous payoff EPV_t paying I at any time moment $t \geq 0$ (effectively, either at $t = 0$ or at $t = 1$). Hence, the firm has the *American call option* with the random price of the underlying, EPV_t, and *strike price*, I. Recall that the American call option on an asset S_t, with the strike price K and *expiry date* (or *maturity date*) T is a contract, which gives its owner the right but not obligation to buy the asset at any time $t \leq T$ for K (\$). Formally, the option in question is perpetual: $T = +\infty$, which means that it can be exercised at any moment $t \geq 0$, but, since it is non-optimal to exercise the option after date 1, we may regard it as an option with the expiry date $T = 1$. The optimal exercise rule is found by backward induction. First, the firm's manager calculates the time-1 EPV of the future profits, $EPV_1 = \Pi_1/r$, and compares it to the fixed investment cost. If the former exceeds the latter, the investment at time $t = 1$ is optimal (if the investment had not been made at time 0). If the investment is made at time $t = 1$, its time-1 NPV is $(EPV_1 - I)_+ := \max\{EPV_1 - I, 0\}$ (the result depends on the realization of uncertainty). At time $t = 0$, the manager compares the instantaneous payoff $EPV_0 - I$, that is, the NPV of the project assuming that the investment is made today, and the NPV of the project if the decision to invest is postponed for one period, which is given by (2.5). If the former exceeds the latter, then the option is exercised (investment is made) today, and the date-0 option price is $V_{\text{opt}}(0) = EPV_0 - I$, otherwise the option is not exercised today, and its price

$$V_{\text{opt}}(0) = E^{\mathbb{Q}}[q(EPV_1 - I)_+] \tag{2.7}$$

coincides with the price of the *European call option* with the expiry date $T = 1$ and strike price I, on the asset with the price EPV_1 at time $t = 1$. In all cases,

$$V_{\text{opt}}(0) = \max\{EPV_0 - I,\ E^{\mathbb{Q}}[q(EPV_1 - I)_+]\}.$$

2.1.3 Exit or option to abandon a stream

Consider the same ethanol plant. Assume that the firm has invested at time 0, and, at time $t = 1$, the price of ethanol decreased, so that the realized price turns out to be $S_1 = 40$, and the profit – zero. This time, we assume that the manager believes that the price of ethanol may recover to the time-0 level, $S_0 = 50$, or it may decrease further, so that at time $t = 2$ and thereafter, the price will be either $S_t = 50$ or $S_t = 30$, with equal probability (under the risk-neutral measure). We also assume that the investment is partially reversible in the sense that the firm can terminate production at time t collecting the profit Π_t and sell the equipment for the scrap value $Sc = 150$ (mln \$). We assume that the scrap value can be obtained in the period following the period the decision to exit is made. Suppose that the manager waits a year. Then, if the price for ethanol rises, it will be non-optimal to terminate the production and sell the equipment because the time-2 present value of the future profits

$$V_2 = \sum_{s=1}^{\infty} q^s(R_2 - C) = \frac{R_2 - C}{r} = \frac{250 - 200}{0.1} = 500 \text{ (mln, \$)}$$

will be greater than the scrap value (even if we assume that the latter will increase 3-fold due to the increase in the price of ethanol). If the price falls, the profit will become negative: $\Pi_2 = -50$ (mln \$), the factory will be closed, and the scrap value $Sc' = 100$ (mln \$) will be recovered at time $t = 2$ (it is prudent for the manager to expect that the price of the second hand equipment for the ethanol production falls together with the price of ethanol); clearly, $V_2 = -50 + 100/1.1 = 40.91$ (mln, \$) in this case. The time-1 EPV of the plant, which is not closed at time $t = 1$, is

$$V_1 = E^{\mathbb{Q}}[qV_2] = \frac{1}{1.1}(0.5 \cdot 500 + 0.5 \cdot 40.91) = 245.87 \text{ (mln, \$)}.$$

Should the manager decide to close the plant at time $t = 1$, at the price level $S_1 = 40$, after the profit $\Pi_1 = 0$ is collected, the firm recovers the scrap value $Sc = 150$ at $t = 2$. Hence, $NPV_1 = q \cdot 150 = 150/1.1 = 136.36$ (mln \$). Since $245.87 > 136.36$, it is optimal to wait till $t = 2$, and disinvest at time $t = 2$ if and only if the price falls.

Recall that the subscript t in $E_t^{\mathbb{Q}}$ means the expectation conditioned on the information available at time t. If the future realizations of uncertainty depend on the current value, s, of the underlying stochastic factor only (here, the price S_t), that is, the underlying stochastic process is Markovian, we may (and will) write

$$E_t^{\mathbb{Q}}[f(S_T)] = E^{\mathbb{Q}}[f(S_T) \mid S_t = s]$$

or simply $E^{\mathbb{Q}}[f(S_T) \mid S_t]$.

2.1.4 Exit as an American put option

If we make a simplifying assumption that the scrap value Sc remains the same as at time $t = 1$, for all realizations of uncertainty, then we can interpret the exit problem above as the *American put option*, which gives the right but not the obligation to sell the asset (the EPV of the future profits) for the strike price Sc at any moment $t \le T$, where T is the expiry date. Formally, the exit problem is equivalent to the American put option with the infinite time horizon (*perpetual American put option*). However, since the uncertainty is resolved in the next time period, the perpetual American put option becomes, in effect, the American put option with the finite time horizon $T = 2$, $t = 1$ being the current date. This allows one to find the optimal exercise rule and calculate the option price quite easily using backward induction. Pricing of the perpetual American options becomes more involved in a more realistic set-up, when the uncertainty is never completely resolved. We consider "genuine" perpetual American options in Part II.

2.2 Expected present value of a stream

Equation (2.5) can be written as

$$NPV_0 = E^{\mathbb{Q}}\left[\sum_{t=0}^{\infty} q^t \Pi_t\right] \tag{2.8}$$

$$= \sum_{\omega \in \Omega} p(\omega) \sum_{t=0}^{\infty} q^t \Pi_t(\omega), \tag{2.9}$$

where $\Pi_0 = -I$, and $\mathbb{Q}$ is the risk-neutral measure over the probability space of possible sample paths of the price of ethanol. We will call the expression on the

RHS of (2.8) the *expected present value* (*EPV*) of the stream $\{\Pi_t\}$, and denote it EPV_Π. Calculation of the expected present values of different streams will be the first step in many situations in the book. The representation (2.9) is natural from the viewpoint of probability, and it can be applied whenever the set of sample paths is countable and time is discrete:

- calculate the present value of the discounted gains Π_t along each sample path of the price process, and
- calculate the weighted sum of the present values, the weights being the probabilities of the sample paths (under the risk-neutral measure).

Changing the order of summation in (2.9), we obtain an equivalent representation of EPV_Π:

$$EPV_\Pi = \sum_{t=0}^{\infty} q^t E^{\mathbb{Q}}[\Pi_t], \tag{2.10}$$

which is natural from the analytical viewpoint, and convenient for computation, if the probability distribution function of each of the random variables Π_t is known[2].

In Part II, the state space (the space where the price process or profit process assumes values) is continuous, and the set of sample paths is uncountable. The representation (2.9) needs the evident adjustment: on the formal level, the summation over $\omega \in \Omega$ must be replaced by the integration over Ω

$$EPV_\Pi = \int_\Omega d\mathbb{Q}(\omega) \sum_{t=0}^{\infty} q^t \Pi_t(\omega), \tag{2.11}$$

but the construction of the probability measure $\mathbb{Q}$ and the rigorous definition of the stochastic expression (2.8) are subtler than in the case of finite or countable number of sample paths. However, representation (2.10) remains applicable. Finally, when both time and space become continuous, the definition of the analog of the stochastic expression (2.8)

$$EPV_\Pi = E^{\mathbb{Q}}\left[\int_0^{+\infty} e^{-rt}\Pi_t dt\right] \tag{2.12}$$

becomes highly non-trivial, and the evident analog of the representation (2.11) with the iterated integration

$$EPV_\Pi = \int_\Omega d\mathbb{Q}(\omega) \int_0^{+\infty} dt\, e^{-rt}\Pi_t(\omega) \tag{2.13}$$

makes sense for a rather narrow class of stochastic processes. Nevertheless, as a source of intuition, this representation remains very useful. The analog of the representation (2.10), with the integration over t,

[2] The change of order of summation is justified if the stream is non-negative or the iterated sum converges absolutely.

$$EPV_{\Pi} = \int_0^{+\infty} q^t E^{\mathbb{Q}}[\Pi_t]\, dt, \tag{2.14}$$

is valid and will be used as the basis for calculations.

Note that in Chap. 3 and the following Parts of the monograph, we will use the normalized expected present values

- $\mathcal{E}\Pi = (1-q)^{-1}EPV_{\Pi}$, in the discrete time models, and
- $\mathcal{E}\Pi = r^{-1}EPV_{\Pi}$, in the continuous time model.

This is convenient because $\mathcal{E}\Pi(x)$ can be interpreted as expectation $E[\Pi(x+Y)]$, where Y is a random variable, and, therefore, the standard intuition can be used to work with the *EPV-operators* $\mathcal{E}$.

2.3 Further examples and extensions

2.3.1 New job offer or option to swap streams of payoffs

A US national considers a job offer from a UK-based firm. Her current contract requires a notice one year in advance should she decide to leave her job. The term of the contract with the foreign firm, two years, and wage, $w = 30,000$ GBP per year, are fixed but the exchange rate fluctuates. As above, assume for simplicity that there are no taxes, the riskless rate (for the money account in USD) per year is constant: $r = 0.1$, and, in the future, there can be two levels of the exchange rate, $R^h = \$2.0$ and $R^l = \$1.8$ per 1 GBP. The uncertainty is revealed at time $t = 1$, and, under an EMM chosen by the market, the probability of the high exchange rate is $p = 0.6$.

Assume that the contract is signed at $t = 0$. At time $t = 1$, the present value of the contract for the individual is

$$V_{F,1} = R_1 w + q \cdot R_1 w = R_1 w \cdot (1+q),$$

therefore, the expected present value of the contract at time $t = 0$ is

$$\begin{aligned}
V_{F,0} &= E^{\mathbb{Q}}[qV_{F,1}] \\
&= qpR^h w \cdot (1+q) + q(1-p)R^l w \cdot (1+q) \\
&= q(1+q)w(pR^h + (1-p)R^l) \\
&= \frac{1}{1.1}\left(1 + \frac{1}{1.1}\right) \cdot 30000 \cdot (0.6 \cdot 2.0 + 0.4 \cdot 1.8) \\
&= 99,967.
\end{aligned}$$

The current wage of the individual is \$ 55, 000 per year, and she believes that she will be able to get the same wage after the end of the new contract with the foreign firm should she decide to accept the offer or after the end of the current contract should she decline the offer. We assume that the contract with the foreign firm covers the cost of the relocation so that no additional cost for the individual is involved, and that she is indifferent between living in the USA or the UK.

If the individual considers the offer as a now or never opportunity, she compares the expected present value of the contract with the UK firm, $V_{F,0} = 99,967$, to the discounted present value of the 2-year contract with the local firm starting at time $t = 1$,

$$V_{L,0} = q(1+q)w = \frac{1}{1.1}\left(1 + \frac{1}{1.1}\right) \cdot 55,000 = 95,455(\$).$$

Since the former is larger than the latter: $V_{F,0} - V_{L,0} = 99,967 - 95,455 = 4,512(\$)$, she accepts the offer of the foreign firm.

Assume now that the individual is highly qualified in her field of expertise and believes that the foreign firm will repeat the offer the next year should the individual decline the offer this year. In a year from now, the individual will know the exchange rate for sure. If the exchange rate is 2.0, the difference between the wages (in \$) is $2.0 \cdot 30,000 - 55,000 = 5,000$, and if the former is 1.8, then the latter is $1.8 \cdot 30,000 - 55,000 = -1,000$. Clearly, the individual will reject the offer in the second case but she will accept the offer in the first case. The time-0 EPV of the gains from switching from the local job to the job abroad is

$$\begin{aligned} G &= q^2(1+q)[p \cdot 5,000 + (1-p) \cdot 0] \\ &= \left(\frac{1}{1.1}\right)^2 \left(1 + \frac{1}{1.1}\right) \cdot 0.6 \cdot 5,000 = 4,733. \end{aligned}$$

Since $4,733 > 4,512$, a rational individual must wait a year.

2.3.2 Embedded options. Partially reversible investment

In this subsection, we study how the investment decision may change if investment is partially reversible : should the price of ethanol fall too low, the firm can close the plant and sell the equipment for a *scrap value*, which is smaller than the fixed investment cost. A natural guess is that, if the scrap value is sufficiently high, the option value of waiting to invest becomes too small to offset the forgone opportunity to start collecting profits earlier, and the firm will invest at time 0.

Another important aspect of the problem is a natural interpretation of the post-investment value V_t^{ex} of the plant as the value of the option to get the scrap value in exchange for the stream of future profits (which can be negative). Since the production starts one period after the fixed investment is made, it is natural not to include time-t profit into V_t^{ex}. To calculate V_t^{ex}, the manager must calculate the price of this *embedded option* using backward induction, for each time period. Then she considers the American call option with the strike price I on the asset with value V_t^{ex} and solves the pricing problem for the latter option, also using backward induction.

Example 1

Consider the possibility of investment into an ethanol plant that will produce 5 mln barrels of ethanol per year. The fixed and operational costs are $I = 500$ and $C = 200$ (mln \$), the discount rate per period is $r = 0.1$, and the uncertainty will be resolved in two periods:

- at time $t = 0$, the price of a barrel of ethanol is $S_0 = 50$ (\$);
- at time $t = 1$, the price rises to the level $S_1 = 60$ or drops to $S_1 = 40$, with equal probabilities;
- in the former case, the price remains at the level $S = 60$ forever, and in the latter case, at time $t = 2$, it either decreases further, to the level $S_2 = 30$, with probability $p' = 0.4$, or recovers to the level $S_2 = 50$;
- in all cases, the revenue will not change after $t = 2$: $S_2 = S_3 = \cdots$.

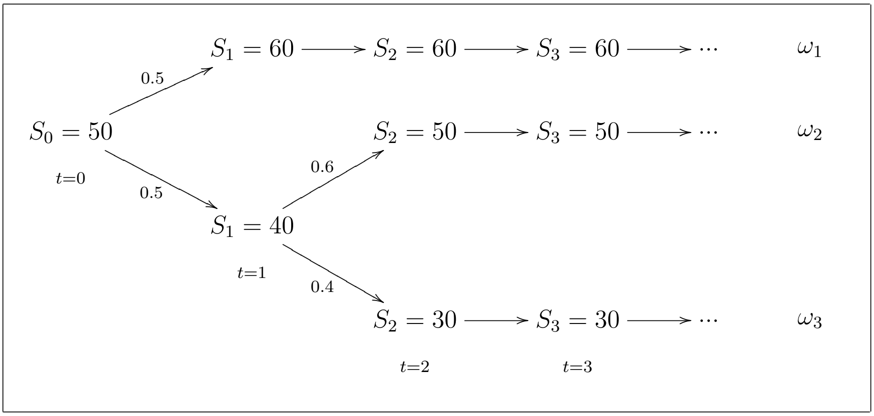

Fig. 2.3. Investment with an embedded option to exit. Example 1.

The event tree is shown in Fig. 2.3; the probability space consists of three sample paths: $\Omega = \{\omega_1, \omega_2, \omega_3\}$, where

- $\omega_1 = \{50, 60, 60, 60, \ldots\}$;
- $\omega_2 = \{50, 40, 50, 50, \ldots\}$;
- $\omega_3 = \{50, 40, 30, 30, \ldots\}$.

Using the conditional probabilities

$$P(S_{t+1} = S_t \mid S_t) = 1, t = 2, 3, \ldots; \quad P(S_2 = 60 \mid S_1 = 60) = 1;$$

$$P(S_2 = 50 \mid S_1 = 40) = 0.6; \quad P(S_2 = 30 \mid S_1 = 40) = 0.4;$$

$$P(S_1 = 60 \mid S_0 = 50) = 0.5; \quad P(S_1 = 40 \mid S_0 = 50) = 0.5,$$

we can calculate conditional expectations and expectations of events of interest, and the probabilities of sample paths ω_j, $j = 1, 2, 3$, under the risk-neutral measure:

$$p(\omega_1) = P(S_1 = 60 \mid S_0 = 50) \cdot P(S_2 = 60 \mid S_1 = 60) = 0.5 \cdot 1 = 0.5,$$
$$p(\omega_2) = P(S_1 = 40 \mid S_0 = 50) \cdot P(S_2 = 50 \mid S_1 = 40) = 0.5 \cdot 0.6 = 0.3,$$
$$p(\omega_3) = P(S_1 = 40 \mid S_0 = 50) \cdot P(S_2 = 30 \mid S_1 = 40) = 0.5 \cdot 0.4 = 0.2.$$

A more difficult question, which we ask, is: assuming that the scrap value, Sc, is the same at any price level of ethanol but smaller than 440 mln (which is 88% of the fixed investment cost, I), what is the smallest scrap value, which makes the investment at time $t = 0$ optimal?

The investment problem in Subsect. 2.1.1 was simple in the sense that once the decision to invest was made, no further decisions were involved, and the effective time horizon was only two periods. This time, the firm's manager must take into account an embedded option to exit. Assuming that the investment was made at time $t-1$ or earlier and the time-t profits have been collected, the manager calculates the EPV of the future gains (profits and the discounted scrap value which the firm can collect should the decision to exit is made later), and compares to the discounted scrap value which the firm can collect the next period should the decision to exit be made at time t. We solve the problem using backward induction. Since the uncertainty will be completely resolved at time $t = 2$, and the future is discounted at a positive rate, it is optimal to exit at $t = 2$ or earlier, if ever. At time $t = 2$, if $S_2 = 60$ or 50, then the EPV of future profits equals 100/0.1=1000 or 50/0.1=500, which exceeds the discounted scrap value, $q \cdot Sc \leq 440/1.1$. Thus, in these two cases, exit is not optimal, and $V_2^{\mathrm{ex}}(60) = 1000$, $V_2^{\mathrm{ex}}(50) = 500$. If $S_2 = 30$, the profit is negative: -50, and it will remain negative forever. Hence, it is optimal to exit at $t = 2$ and collect the scrap value at $t = 3$. We conclude that $V_2^{\mathrm{ex}}(30) = Sc/1.1 - 50$.

The time-2 value V_2^{ex} having being calculated, the manager moves one period back and calculates V_1^{ex}. If $S_1 = 60$, then it is not optimal to exit, and $V_1^{\mathrm{ex}}(60) = 1000$. If $S_1 = 40$, then the EPV of the future profits, the scrap value including, is

$$\begin{aligned} E_1^{\mathbb{Q}}[qV_2^{\mathrm{ex}}] &= E^{\mathbb{Q}}[qV_2^{\mathrm{ex}} \mid S_1 = 40] \\ &= \frac{1}{1.1}\,[0.6 \cdot 500 + 0.4(-50 + Sc/1.1)] \\ &= 254.55 + 0.3306Sc. \end{aligned}$$

Should the firm decide to exit, it will gain $Sc/1.1$ instead. It is easy to see that $254.55 + 0.3306Sc > Sc/1.1$ iff $Sc < 440$, therefore, the firm does not exit at time $t = 1$, and $V_1^{\mathrm{ex}}(40) = 254.55 + 0.3306Sc$.

Now we can calculate the EPV of future gains, at time 0:

$$\begin{aligned} V_0^{\text{ex}} &= E^{\mathbb{Q}}[qV_1^{\text{ex}}] \\ &= \frac{1}{1.1}[0.5 \cdot 1000 + 0.5 \cdot (254.55 + 0.3306Sc)] \\ &= 570.25 + 0.1503 * Sc. \end{aligned}$$

At this stage, the firm's manager has solved the first part of the problem: the value of the embedded option has been calculated for each time period and all realizations of uncertainty.

At the second step, the manager finds the optimal exercise rule for the (perpetual) American call option with the strike price I, on the asset with the price V_t^{ex}. We solve this problem using backward induction. At time $t = 2$, $V_2^{\text{ex}} \geq I = 500$ iff $S_2 = 60, 50$. Hence, the firm invests only in this case, and the payoff of the option to entry at time $t = 2$ is $G_2(60) = 1000 - 500 = 500$, $G_2(50) = 500 - 500 = 0$, $G_2(30) = 0$.

At time $t = 1$, the firm calculates the EPV of the investment at time $t = 2$, and compares with the instantaneous gain $V_1^{\text{ex}} - I$. If $S_1 = 60$, then the former, $q \cdot 500$, is smaller than the latter, 500, therefore the firm invests. If $S_1 = 40$, then the instantaneous gain from investment at time $t = 1$ equals $254.55 + 0.3306Sc - 500 = 0.3306Sc - 245.45$, which is negative for $Sc \leq 440$. Hence, the firm does not invest in this case. The time-1 value of the investment opportunity is given by $G_1(60) = 500$ and $G_1(40) = 0$.

Finally, the manager calculates the EPV of the investment at time $t = 1$ or later

$$qE^{\mathbb{Q}}[G_1] = \frac{1}{1.1}[0.5 \cdot 500 + 0.5 \cdot 0] = 227.27,$$

and compares with the instantaneous gain

$$V_0^{\text{ex}} - I = 570.25 + 0.1503Sc - 500 = 0.1503Sc + 70.25.$$

The latter is larger than the former iff $Sc \geq (227.27 - 70.25)/0.1503 = 1,044.7$. Since Sc does not satisfy this condition, the firm does not invest at time 0. The analysis above shows that the firm will invest at time $t = 1$ iff $S_1 = 60$, and at time $t = 2$ iff $S_1 = 40$ and $S_2 = 50$.

Example 2

Consider the possibility of investment into an ethanol plant that will produce 5 mln barrels of ethanol per year. The discount rate per period is $r = 0.1$, the fixed and variable costs are $I = 300$ and $C = 200$, and scrap value is $Sc = 200$ (mln \$). The uncertainty will be resolved in two periods: at time $t = 1, 2$, the price increases by \$ 10, with probability $p = 0.6$ (under the risk-neutral measure), and decreases by \$ 10, with probability $1 - p = 0.4$. The manager observes $S_0 = 50$ and finds the optimal investment timing by backward induction. This time, we use an argument which differs slightly from the one in Example 1. The probability space consists of 4 sample paths:

- $\omega_1 = \{50, 60, 70, 70, \ldots\}$; $\omega_2 = \{50, 60, 50, 50, \ldots\}$;

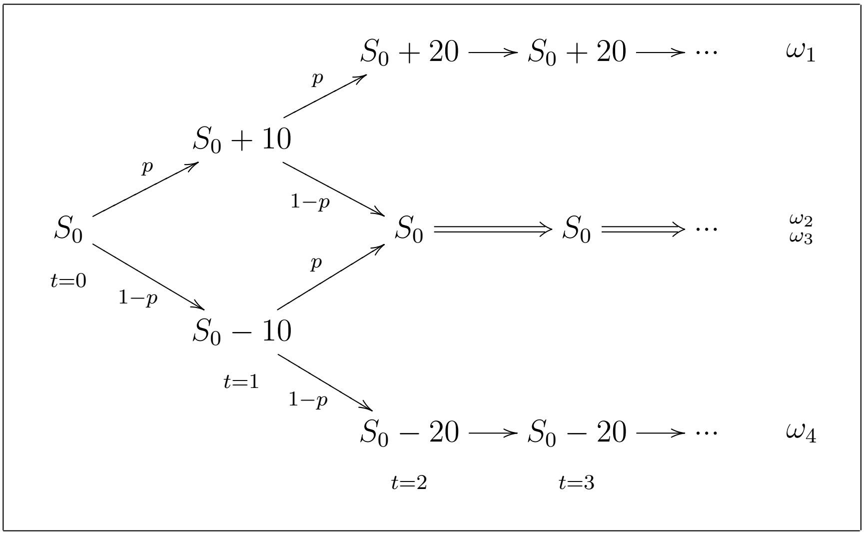

Fig. 2.4. Investment with an embedded option to exit. Example 2.

- $\omega_3 = \{50, 40, 50, 50, \ldots\}$; $\omega_4 = \{50, 40, 30, 30, \ldots\}$,

and, under the risk neutral measure,

$$p(\omega_1) = 0.6 \cdot 0.6 = 0.36; \quad p(\omega_2) = 0.6 \cdot 0.4 = 0.24;$$
$$p(\omega_3) = 0.4 \cdot 0.6 = 0.24; \quad p(\omega_4) = 0.4 \cdot 0.4 = 0.16.$$

Since the uncertainty will be resolved at time $t = 2$, and the future profits are discounted, it is not optimal to invest at time $t > 2$.

At time $t = 2$, if $S_2 = S_0 - 2 \cdot 10 = 30$, the operational profit is negative; therefore, it is not optimal to invest, and $NPV_{2;\geq 2}(30) = 0$. If $S_2 = 50$ or 70, then the investment is optimal, and

$$NPV_{2;\geq 2}(50) = -300 + (5 \cdot 50 - 200)/0.1 = 200,$$
$$NPV_{2;\geq 2}(70) = -300 + (5 \cdot 70 - 200)/0.1 = 1200.$$

Now consider $t = 1$. If $S_1 = 60$, then time-1 NPV of investment at time $t \geq 2$ equals

$$NPV_{1;\geq 2}(60) = \frac{1}{1.1}[0.6 \cdot 1200 + 0.4 \cdot 200] = 727.27,$$

and time-1 NPV of investment at time $t = 1$ is larger:

$$NPV_{1;1}(60) = -300 + 0.6 \cdot 150/0.1 + 0.4 \cdot 50/0.1 = 800.$$

Hence, if $S_1 = 60$, and the investment had not been made at time 0, the firm invests at time $t = 1$.

If $S_1 = 40$, then time-1 NPV of investment at time $t = 2$ equals

$$NPV_{1;2}(40) = \frac{1}{1.1}[0.6 \cdot 200 + 0.4 \cdot 0] = 109.09,$$

and time-1 NPV of investment at time $t = 1$ is smaller:

$$NPV_{1;1}(40) = -300 + 0.6 \cdot 50/0.1 + 0.4 \cdot \frac{1}{1.1} \cdot (-50 + 200/1.1) = 47.93.$$

Hence, if $S_1 = 40$, and the investment had not been made at time 0, the firm does not invest at $t = 1$ and waits one period. It invests at time $t = 2$ if $S_2 = 50$ and never if $S_2 = 30$.

We conclude that if the firm does not invest at time 0, then the time-0 NPV of the investment project is

$$NPV_{0,\geq 1}(50) = \frac{1}{1.1} \cdot [0.6 \cdot 800 + 0.4 \cdot 109.09] = 476.03.$$

Finally, if the investment is made at time 0, then the NPV is

$$\begin{aligned} NPV_{0;0} &= -300 + \frac{1}{1.1} \cdot 0.6 \cdot [5 \cdot 60 - 200 + 0.6 \cdot 150/0.1 + 0.4 \cdot 50/0.1] \\ &\quad + \frac{1}{1.1} \cdot 0.4 \cdot [5 \cdot 0 + 0.6 \cdot 50/0.1 + 0.4 \cdot (-50 + 200/1.1)] \\ &= 482.81. \end{aligned}$$

Since $482.81 > 476.03$, the investment must be made at time 0. We leave to the reader the verification that if the investment is made at time 0, the time-1 expected value of profits exceeds the scrap value, hence, the firm never exits at time $t = 1$, and the calculations made above are consistent.

2.4 General analysis of the basic types of options

2.4.1 Options to acquire or abandon a perpetual stream vs. options with instantaneous payoffs

The options studied in Sect. 2.1 are model examples of 4 basic types of options. In all cases, an optimizing agent chooses an optimal time to

I. abandon a stream of payoffs g_t;
II. acquire a stream of payoffs g_t;
III. abandon an asset of a random value G_t;
IV. acquire an asset of a random value G_t.

Note that, in order to facilitate the theoretical treatment of these model types of options and other, more involved, types, we mention neither the price we pay for the stream of payoffs or instantaneous payoff that we may acquire nor the price we get for the stream of payoffs. Similarly, we do not mention the strike price for options with instantaneous payoffs. The reason is that we can incorporate this price, call it K, both into the stream, as an additional term rK payable from time $t = 1$ until infinity, and into the instantaneous payoff. For instance, if a stream g_t^0 can be acquired for K, then the equivalent model problem is the *call-like Option* II with the stream of payoffs $g_t = g_t^0 - rK$, and if an asset with the random price G_t^0 can be bought for K, then it is the call-like Option IV with the instantaneous payoff $G_t = G_t^0 - K$.

Options of types I-IV are reducible to one another, with the notable exception of American options on a non-dividend paying stock, which will be discussed in Subsect. 2.4.3. Indeed,

- the option to abandon an asset with the spot price G_t is equivalent to the option to acquire an asset with the spot price $-G_t$;
- if the EPV of the stream g_t is finite and can be calculated

$$G_t = E_t^{\mathbb{Q}} \left[\sum_{s=1}^{\infty} q^s g_{t+s} \right], \tag{2.15}$$

  then the option to abandon (acquire) the stream g_t is reducible to the option to abandon (acquire) the instantaneous payoff G_t;
- similarly, if the instantaneous payoff can be represented as the EPV of a stream, then options with instantaneous payoffs can be reduced to options with payoff streams; what can be done in cases when the reduction is impossible is discussed in Subsect. 2.4.3;
- options to abandon and acquire a stream of payoffs are also reducible to one another.

2.4.2 Stopping times and equivalence of an option to acquire a stream g_t and the option to abandon the stream $-g_t$

To prove the last equivalence rigorously, we need the definition of a *stopping time*. A stopping time τ is a random variable on the probability space Ω of sample paths of the underlying process, such that, for each sample path $\omega \in \Omega$ and each time $t \le T$, where $T \le +\infty$ is the time horizon of the model, one knows if $\tau(\omega) \le t$ or not. A typical example of a stopping time is the time when the price of the underlying factor – price of ethanol, revenue, profit, etc. – crosses a certain level from above (or from below, depending on the situation). The standard form of calculations which involve a stopping time τ is the use of the indicator function $\mathbf{1}_{\tau \le t}$, which is a random variable on Ω. For a sample path ω, $\mathbf{1}_{\tau \le t} = 1$, if $\tau(\omega) \le t$ (the event has happened), and 0 otherwise. In particular, we can write the discounted stream of payoffs payable up to a stopping time τ as a random variable on Ω

$$\sum_{t=0}^{\infty} \mathbf{1}_{t \le \tau} q^t g_t = \sum_{t=0}^{\tau} q^t g_t.$$

Using the notion of a stopping time, we can formulate problems of the optimal exercise of Options I-IV as follows.

I. Find a stopping time τ, which maximizes

$$V_I = \sup_\tau E^{\mathbb{Q}} \left[\sum_{t=0}^{\tau-1} q^t g_t \right] \tag{2.16}$$

$$= \sup_\tau \sum_{\omega \in \Omega} p(\omega) \sum_{t=0}^{\tau(\omega)-1} q^t g_t(\omega). \tag{2.17}$$

II. Find a stopping time τ, which maximizes

$$V_{II} = \sup_\tau E^{\mathbb{Q}} \left[\sum_{t=\tau}^{\infty} q^t g_t \right] \tag{2.18}$$

$$= \sup_\tau \sum_{\omega \in \Omega} p(\omega) \sum_{t=\tau(\omega)}^{\infty} q^t g_t(\omega). \tag{2.19}$$

III–IV. Find a stopping time τ, which maximizes

$$V_{\text{instant}} = \sup_\tau E^{\mathbb{Q}} \left[q^\tau G_\tau \right] \tag{2.20}$$

$$= \sup_\tau \sum_{\omega \in \Omega} p(\omega) q^{\tau(\omega)} G_{\tau(\omega)}. \tag{2.21}$$

Now we can prove the equivalence of optimal stopping problems (2.16) and (2.18). For any stopping time τ and any $\omega \in \Omega$,

$$\sum_{t=0}^{\tau(\omega)-1} p(\omega) q^t g_t(\omega) + \sum_{t=\tau(\omega)}^{\infty} p(\omega) q^t g_t(\omega) = \sum_{t=0}^{\infty} p(\omega) q^t g_t(\omega),$$

and, hence,

$$E^{\mathbb{Q}} \left[\sum_{t=0}^{\tau-1} q^t g_t \right] = E^{\mathbb{Q}} \left[\sum_{t=\tau}^{\infty} q^t (-g_t) \right] + E^{\mathbb{Q}} \left[\sum_{t=0}^{\infty} q^t g_t \right].$$

The second term on the RHS is independent of τ, therefore, the same τ maximizes $V_I = V_I(g)$ and $V_{II} = V_{II}(-g)$.

2.4.3 American options on a non-dividend paying stock

From the point of view of the standard financial economics, the price of a share of a stock must equal the EPV (under an EMM) of the discounted stream of dividends, that is, it must be given by (2.15). In the real life, the price of a share can differ significantly from (2.15). There are many reasons for this discrepancy, including numerous imperfections in real financial markets (frictions) and interactions between traders in the market. The interactions may increase the stock value, and the extreme case is the stock which pays no dividends but is, nevertheless, traded in the market. Certainly, it is highly unlikely that such a stock will be traded forever unless it will start paying dividends at some moment in the future. Nevertheless, in finance, options on a non-dividend paying asset (typically, on an index) are among the main objects. The price of such an asset, S_t, cannot be represented as the EPV of a stream of payoffs (dividend stream). Since the difference between S_t and the payoff $(S_t - K)_+$ of the call option is bounded, this difference can be represented as the EPV of a certain stream. Hence, the payoff of the call option cannot be represented as the EPV of a stream, and the equivalence of Option IV to Option II fails. In a certain sense, this is not important because it is non-optimal to exercise the American call option on a non-dividend paying stock before the expiry, and the optimal stopping problem becomes trivial. The optimal stopping rule is: wait till the expiry date, T, and exercise the option iff $S_T \geq K$. Thus, an American call option on a non-dividend paying stock is equivalent to the corresponding European call option. See Subsect. 3.6.

Problems

2.1. In the set-up of Subsect. 2.1.1, assume that the fixed investment cost, I, is not given. For which levels of I, is it optimal to

a) never invest;
b) invest at time $t = 1$;
c) invest at time $t = 0$?

2.2. In the set-up of Subsect. 2.1.3, assume that the scrap value Sc_h, at price level $S_h = 50$, and Sc_l, at price level $S_l = 30$, are not given. For which values of Sc_h and Sc_l will it be optimal to disinvest at time

a) $t = 1$;
b) $t = 2$?

2.3. Solve the model in Subsect. 2.3.2 with $p' = 0.8$.

2.4. Consider the problem of investing into the plant that will produce 50 mln barrels of ethanol per year. The uncertainty is as in Example 1 of Subsect. 2.3.2, only $p' = 0.8$ is different. The variable cost is $C = 200$, and the scrap value is $Sc_h = 150$, if $S_t \geq 50$, and $Sc_l = 100$, if $S_t = 40$ (mln ($).

For which levels of the investment cost, I, is it optimal to

a) never invest;
b) invest at time $t = 2$;
c) invest at time $t = 1$;
d) invest at time $t = 0$?

2.5. In the model in Subsect. 2.3.2, find S_0 such that it is optimal to

a) never invest;
b) invest at time $t = 2$;
c) invest at time $t = 1$;
d) invest at time $t = 0$.

2.6. In the model in Subsect. 2.3.2, let $S_0 = 50$. Find p such that it is optimal to

a) never invest;
b) invest at time $t = 2$;
c) invest at time $t = 1$;
d) invest at time $t = 0$.

3

Risk-neutral pricing. Finite time horizon case

In this Chapter, we recall definitions of a risk-neutral probability measure, or equivalent martingale measure (EMM), on the set of sample paths of the underlying process, notions of complete and incomplete markets, and the fundamental theorem about the equivalence of no-arbitrage and existence of an EMM. We introduce the binomial and trinomial models and calculate prices of European and American options. We also briefly discuss shortcomings of the risk-neutral pricing in incomplete markets, and serious difficulties, which prevent a researcher from constructing a consistent pricing theory in incomplete markets.

3.1 No-arbitrage and EMM

We consider the same manager as in Subsect. 2.1.1. Fitting a model of uncertainty to a real financial market, the manager may infer probabilities of the rise and fall from the historic data; hence the name: *historic probabilities* or the *historic measure*, $\mathbb{P}$, over Ω, the set of all sample paths of the price process. Denote these probabilities $p_H(\omega), \omega \in \Omega$. It is tempting to calculate the EPV of the project using (2.3) with the historic probabilities. However, the manager is aware that the historic probabilities may not be used; instead of the historic probabilities one should use *risk-neutral probabilities* . Equivalently, instead of the historic measure on Ω, a *risk-neutral measure*, $\mathbb{Q}$, must be used. Another name is an *equivalent martingale measure, EMM.* The appearance of an EMM can be explained as follows. Assume that the forward contracts and ethanol on the spot can be bought and sold in arbitrary quantities without transaction costs, and the market does not admit arbitrage. Forward contracts are examples of *contingent claims* of the European type. Recall that a *contingent claim of the European type* pays a random payoff g_T at the deterministic[1]

[1] For other types of contingent claims, the expiry date, τ, can be a random variable (stopping time). For contingent claims of the American type, an optimal stopping time (*early exercise*) is chosen by an optimizing agent.

expiry date T; for the receiver *forward contract*, $g_T = S_T - K$ (at time T, the holder of the forward contract pays K for the delivery of the asset which she can sell on the spot for S_T. If the difference is negative, the holder may wish to but cannot refuse to buy the asset because the counterparty will not allow her to walk away). Clearly, at the expiry date, T, the price of a contingent claim equals the payoff: $V_T = g_T$. To explain how to price the contingent claim at time 0, we need to introduce several definitions.

Consider a two-period model: $T = 1$. Let $S^1, S^2, \ldots, S^n$ be contingent claims traded in the market, and let θ_j be the number of shares of contingent claim S^j in the investor's portfolio $\theta = (\theta_1, \ldots, \theta_n)$. If $\theta_j > 0$ (resp., $\theta_j < 0$), the investor is said to have a *long position* (resp., *short position*) in security j; both types of positions, non-integer positions including, are allowed. The portfolio θ has the market value $\theta \cdot S_0 = \sum_{j=1}^n \theta_j S_0^j$ and the random payoff $\theta \cdot S_1$. We can interpret the random variable on the set $\Omega = \{\omega_1, \ldots, \omega_m\}$ of the possible states tomorrow as a row vector. Then $\theta \cdot S_1 = \theta D$, where D is the payoff matrix (row D_j is the vector of payoffs of S^j). A portfolio θ is called an *arbitrage portfolio* if $\theta \cdot S_0 \le 0$ and $\theta \cdot S_1 > 0$ (that is, $\theta \cdot S_1$ is a nonnegative random variable, which assumes positive values with non-zero probability), or $\theta \cdot S_0 < 0$ and $\theta \cdot S_1 \ge 0$. It can be shown that an economic agent whose utility function is increasing in consumption today and non-decreasing in consumption tomorrow will increase her utility buying arbitrage portfolios. If there is no bound on borrowing and lending and no other frictions such as transaction costs, as it is assumed for simplicity at the basic level of the theory of financial markets, such an agent will be willing to buy all arbitrage portfolios (realize arbitrage opportunities). Therefore, all arbitrage opportunities will disappear. In real financial markets, arbitrage opportunities may appear but they are promptly eliminated due to the activity of *arbitrageurs*, who make money by looking for those opportunities. Thus, the no-arbitrage assumption is sufficiently realistic, and it can be used as a cornerstone of the pricing theory. The following theorem is the fundamental theorem of the financial economics for the two-period model with m states of the nature tomorrow; thus, the set of sample paths of the price process can be identified with the list of the states of the nature tomorrow. We formulate it for the market, which contains a riskless asset (bond), denoted B, with the deterministic payoff $B_1 = (1 + r)B_0$, where $r \ge 0$. Set $q = 1/(1 + r)$.

Theorem 3.1.1 *Two statements are equivalent:*

(i) an arbitrage portfolio does not exist;
(ii) there exists a probability measure $\mathbb{Q}$ over Ω such that time-0 price of each asset traded in the market, V, equals the expectation of the discounted price at time $t = 1$:

$$V_0 = E^{\mathbb{Q}}[qV_1] = \sum_{k=1}^{m} p(\omega_k) q V_1(\omega_k), \tag{3.1}$$

where $p(\omega_k) > 0$ is the probability of ω_k under $\mathbb{Q}$.

We explain Theorem 3.1.1 for the market with $m = 2$ and two securities: a riskless bond yielding the rate of return $r \geq 0$, and the forward contract for delivery of a barrel of ethanol at time $t = 1$, with the delivery price K. We demonstrate how to find the no-arbitrage bounds for S_0, given possible values of S_1, and calculate an EMM if S_0 satisfies these bounds. In this example, an EMM is unique but, in general, an EMM is non-unique. We will prove the following implications:

1) if no arbitrage portfolio exists, then $40 < K < 60$;
2) if $40 < K < 60$, then there exists a unique EMM;
3) if an EMM exists, then no arbitrage portfolio exists.

Assume that the price tomorrow can be either $S_1 = 60$ or $S_1 = 40$ (sample paths ω_1 and ω_2, respectively). Since it costs nothing to enter a forward contract, its time-0 price is $F_0 = 0$, and time-1 price is either $F_1(\omega_1) = 60 - K$ or $F_1(\omega_2) = 40 - K$. Assuming that the market for forward contracts and spot markets for ethanol are liquid and do not admit arbitrage, we conclude that it must be that $40 < K < 60$. Indeed, if $K \leq 40$, then a rational agent (arbitrageur) enters the forward contract that allows her to buy a barrel for K dollars at time $t = 1$. At time 1, she pays K dollars for a barrel and sells it on the spot. The difference $S_1 - K$ is non-negative in all states of the future, and positive in state ω_1. The agent realizes an *arbitrage opportunity.* Similarly, if $K \geq 60$, the agent enters the forward contract that allows her to sell a barrel for K dollars at time $t = 1$. At time 1, she buys a barrel for S_1 dollars, and sells for K; the difference $K - S_1$ is non-negative in all states of the future and positive in state ω_2.

If $40 < K < 60$, then there exist positive $p(\omega_1)$ and $p(\omega_2)$ such that $p(\omega_1) + p(\omega_2) = 1$, equivalently, a probability measure on $\Omega = \{\omega_1, \omega_2\}$, such that

$$0 = F_0 = E^{\mathbb{Q}}[qF_1].$$

Indeed, solving

$$0 = p(\omega_1)(60 - K) + p(\omega_2)(40 - K)$$

for $p = p(\omega_1)$, we find $20p = K - 40$, and $p(\omega_1) = K/20 - 2$, $p(\omega_2) = 3 - K/20$. For instance, if $K = 50$, then $p(\omega_1) = p(\omega_2) = 0.5$, and if $K = 48$, then $p(\omega_1) = 0.4$, $p(\omega_2) = 0.6$: the market believes that the probability of the price decline is higher than the probability of the rise. We see that the price of the forward contract satisfies (3.1). Since the riskless bond yields the riskless rate r of return, its price $B_1 = B_0(1 + r)$ also satisfies (3.1).

Now, if there exist positive $p(\omega_1)$ and $p(\omega_2)$ such that $p(\omega_1) + p(\omega_2) = 1$ and (3.1) holds, then any portfolio with a non-negative value at time 1 has a non-negative value at time 0, and if, in addition, the former is positive for some state of the nature tomorrow, then the latter is positive. Thus, there is no arbitrage portfolio.

3.2 Replication and complete markets

Consider the same example, with the bond price at time 0 normalized to 1: $B_0 = 1$. Let V be a contingent claim which promises a payoff $d(\omega)$ in state ω, $\omega \in \Omega$. Since there are two states of the world at time $t = 1$, and the vectors of payoffs $[60 - K\ 40 - K]$ and $[1 + r\ 1 + r]$ on the forward contract and riskless bond are linearly independent, we can represent the vector $d = [d(\omega_1)\ d_2(\omega_2)]$ as a unique linear combination

$$d = \theta_1(S_1 - K) + \theta_2(1 + r).$$

This means that the payoff of this contingent claim is the same as the payoff of the portfolio of θ_1 forward contracts and θ_2 bonds; using (3.1), we conclude that the time-0 price of the contingent claim, V_0, equals θ_1 times the time-0 price of the forward contract plus θ_2 times the bond price; thus, $V_0 = \theta_2$. One says: the contingent claim can be *replicated*, or spanned, by securities traded in the market; the claim is *redundant*. If, for any stopping time τ, any contingent payoff g_τ can be replicated using assets traded in the market, then the market is called *complete*. In a complete market, an EMM is unique, as in the example above.

3.3 European call and put options in a two-period model

European options are the simplest options traded in the market. A *European call option* on an asset S with the expiry date T and strike price K is a contract which gives the right but not an obligation to buy the asset at time T for K units of account ($). The option will not be exercised unless $S_T \geq K$, therefore, the payoff is $(S_T - K)_+ = \max\{S_T - K, 0\}$. In a two-period model, the time-0 price is

$$V_{\text{call}}(0) = E^{\mathbb{Q}}[q(S_1 - K)_+] \tag{3.2}$$

$$= \sum_{\omega \in \Omega} p(\omega) q(S_1(\omega) - K)_+. \tag{3.3}$$

A *European put option* on S with the expiry date T and strike price K is the contract which gives the right but not an obligation to sell the asset at time T for K units of account. Thus, the payoff is $(K - S_T)_+$, and, in a two-period model, the time-0 price is

$$V_{\text{put}}(0) = E^{\mathbb{Q}}[q(K - S_1)_+] \tag{3.4}$$

$$= \sum_{\omega \in \Omega} p(\omega) q(K - S_1(\omega))_+. \tag{3.5}$$

Note that in complete markets, European options can be replicated using the risky asset and riskless bond, and, therefore, pricing formulas (3.2) and (3.4) are valid.

In the example below, two securities are traded in the market. The first one is risky, and the second one is a riskless bond, with the price today normalized to 1. Hence, the second row of the matrix qD consists of ones. The time-t price of the risky security is denoted S_t.

Example 3.1. Let $n = m = 2$. The first security is risky, equivalently, $S_1(\omega_1) \neq S_1(\omega_2)$. We assume that $S_1(\omega_1) > S_1(\omega_2)$. Hence D_1 and D_2 are linearly independent, and the system (3.8) has a unique solution p, which is calculated as follows:

$$\begin{bmatrix} p(\omega_1) \\ p(\omega_2) \end{bmatrix} = \begin{bmatrix} qS_1(\omega_1) & qS_1(\omega_2) \\ 1 & 1 \end{bmatrix}^{-1} \times \begin{bmatrix} S_0 \\ 1 \end{bmatrix}$$
$$= \frac{1}{q(S_1(\omega_1) - S_1(\omega_2))} \begin{bmatrix} 1 & -qS_1(\omega_2) \\ -1 & qS_1(\omega_1) \end{bmatrix} \times \begin{bmatrix} S_0 \\ 1 \end{bmatrix}.$$

Finally,

$$\begin{bmatrix} p(\omega_1) \\ p(\omega_2) \end{bmatrix} = \frac{1}{q(S_1(\omega_1) - S_1(\omega_2))} \begin{bmatrix} S_0 - qS_1(\omega_2) \\ -S_0 + qS_1(\omega_1) \end{bmatrix}. \tag{3.6}$$

We see that $p(\omega_1)$ and $p(\omega_2)$ are positive, that is, an EMM exists (and it is unique) iff time-0 price of the risky security satisfies

$$qS_1(\omega_2) < S_0 < qS_1(\omega_1). \tag{3.7}$$

Remark 3.2. Inequalities (3.7) are the *no-arbitrage bounds* on the price of the risky asset. If, at time 0, the market prices S incorrectly so that (3.7) fails, an arbitrage opportunity exists. We leave the construction of an arbitrage portfolio to the reader.

Remark 3.3. Note that p is homogeneous of degree 0 in (S_1, S_0): if we scale the payoffs for the risky security and its time-0 price but leave the riskless rate as it is, the vector of state prices will not change. This observation is important for the simplest multi-period generalization of this example: the binomial model.

If (3.7) is satisfied, then the risk-neutral probabilities are defined by (3.6), and the prices of the European call and put options are calculated from (3.2) and (3.4). Note that if $S_1(\omega_1) > K$, so that the payoff of the European call assumes different values in the two states, then the knowledge of the time-0 call price instead of S_0 can be used to infer the risk-neutral probabilities and calculate S_0. Similarly, if $S_1(\omega_2) \leq K$, then the time-0 price of the European put option can be used to infer the risk-neutral probabilities and calculate S_0. We leave the study of these possibilities as an exercise for the reader.

Option pricing becomes rather ambiguous if the market is incomplete.

3.4 Complete and incomplete markets

For future calculations, we reformulate Theorem 3.1.1 as a statement about solutions of a certain linear system, add a statement about complete and *incomplete markets*, and give a proof. Let S^j, $j = 1, \dots, n$, be securities traded in the market. For each security, we write the list of its possible values tomorrow (payoffs) as the row $D_j = [S_1^j(\omega_1) \ \dots S_1^j(\omega_m)]$ and construct the $n \times m$ matrix D of these rows. We write the prices S_0^j today and probabilities $p(\omega_j)$ as columns S_0 and p, and (3.1) as

$$S_0 = qDp. \tag{3.8}$$

Theorem 3.4.1 *a) An arbitrage portfolio does not exist iff system (3.8) has a solution $p \in \mathbb{R}^m_{++}$.*
b) The market is complete iff a solution $p \in \mathbb{R}^m_{++}$ is unique.

Proof. a) Introduce the augmented payoff matrix

$$\mathcal{R} = [-S_0 \ qD].$$

An arbitrage portfolio exists iff

(1) *there exists a row vector $\theta \in \mathbb{R}^n$ such that $\theta\mathcal{R} > 0$.*

Equality (3.8) can be written as $\mathcal{R}[1 \ p]^T = 0$, therefore an EMM exists iff

(2) *there exists $\tilde{p} \in \mathbb{R}^{m+1}_{++}$ such that $\mathcal{R}\tilde{p} = 0$.*

According to the Farkas lemma, for any matrix $\mathcal{R}$, one and only one of the statements (1) and (2) is true. This proves a).

b) Denote by n the row rank of D. It cannot exceed m, the number of columns. Hence, two cases are possible: $n = m$ (the number of non-reduntant securities equals the number of states tomorrow), and $n < m$ (the former is smaller than the latter). If $n = m$, then the rows of D span $\mathbb{R}^m$, and any contingent claim can be replicated; the market is complete. Further, if $n = m = \operatorname{rank} D$, a solution to (3.8) is unique.

If $n < m$, then the columns of D do not span $\mathbb{R}^m$ (hence, there exist securities that cannot be replicated), and a solution to (3.8) is not unique (if it exists). If one solution $p >> 0$ exists, there are infinitely many solutions $p >> 0$.

Example 3.4. Let $n = 2, m = 3$. If the first security is risky, then D_1 and D_2 are linearly independent, therefore solutions to (3.8) exist; since $m > n$, there are infinitely many of them. In this example, there exist infinitely many EMM provided the time-0 price of the risky asset is within certain no-arbitrage bounds. Assuming $S_1(\omega_1) > S_1(\omega_3) > S_1(\omega_2)$, the no-arbitrage bounds for S_0 are (3.7). We leave the proof and calculation of the set of possible EMM to the reader.

Introduce the call option on the risky security, with the strike price K such that $S_1(\omega_2) < K < S_1(\omega_3) < S_1(\omega_1)$. At date $t = 1$, in state ω_2, the strike price is lower than the asset price, the option is not exercised, and the payoff is 0. In states $\omega = \omega_1, \omega_3$, the asset price is higher, the option is exercised, and the payoff is $S_1(\omega) - K$. The payoff matrix is

$$D = \begin{bmatrix} S_1(\omega_1) & S_1(\omega_2) & S_1(\omega_3) \\ S_1(\omega_1) - K & 0 & S_1(\omega_3) - K \\ 1+r & 1+r & 1+r \end{bmatrix}.$$

The reader can easily verify that $\text{rank} D = 3$, therefore, the market becomes complete. Using (3.8), one can derive the no-arbitrage bounds for the time-0 prices of the underlying security S and call option, S_0 and $V_{\text{call}}(0)$. In Finance and Mathematical Finance, the standard practice is to assume that the asset price S_0 does not change after a derivative security is introduced, and use the no-arbitrage bounds on the pair $(S_0, V_{\text{call}}(0))$ as the no-arbitrage bounds on $V_{\text{call}}(0)$ given S_0. However, in equilibrium, one should expect that S_0 will change, generically, after a new security is introduced.

Therefore, in incomplete markets, attempts to derive the price of a new derivative security (which has not been traded in the market earlier) assuming that the price process of the underlying asset does not change, are inconsistent with the very basics of financial economics. We conclude that the no-arbitrage pricing using a risk-neutral measure is really justified only in complete markets. Unfortunately, so far, there is no consistent pricing theory in incomplete markets; before this theory appears, the no-arbitrage pricing seems to be sufficiently reasonable, and we will use it in the monograph.

3.5 Multi-period model

3.5.1 Self-financing dynamic portfolios

Consider the market with trades at dates $0, 1, \dots, T$, where $T > 1$. Now the investor buys a portfolio of assets traded in the market at time 0, and she can rebalance the portfolio at $t = 1, 2, \dots, T-1$. Since the transactions costs are assumed away, we may imagine that each time, she sells the portfolio and buys a new one. The sequence of portfolios the investor buys, $\{\theta_0, \theta_1, \dots, \theta_{T-1}\}$, is called a *dynamic portfolio.* Assume that the assets do not pay dividends, and that the investor neither consumes a portion of proceeds from each transaction nor infuses additional funds. Then $\theta_{t-1} \cdot S_t = \theta_t \cdot S_t$, for $t = 1, 2, \dots, T-1$. This is a special case of a *self-financing* dynamic portfolio. A self-financing dynamic portfolio is called an *arbitrage portfolio*, if either $\theta_0 \cdot S_0 < 0$ and $\theta_{T-1} \cdot S_T \ge 0$ or $\theta_0 \cdot S_0 = 0$ and $\theta_{T-1} \cdot S_T > 0$ in the sense that for each sample path $\theta_{T-1} \cdot S_T(\omega) \ge 0$, and for some sample paths, $\theta_{T-1} \cdot S_T(\omega) > 0$.

3.5.2 No-arbitrage and EMM in a multi-period model

Theorem 3.5.1 *Two statements are equivalent:*

(i) an arbitrage self-financing dynamic portfolio does not exist;
(ii) there exists a probability measure $\mathbb{Q}$ over Ω such that for any asset traded in the market that pays dividends only at expiry, with the expiry date $0 < \tau \le T$, any $t < \tau$, and any stopping time τ', $t \le \tau' \le \tau$,

$$S_t = \sum_\omega p(\omega) q^{\tau' - t} S_{\tau'}(\omega), \tag{3.9}$$

where $p(\omega)$ is the probability of a sample path ω under $\mathbb{Q}$, and the summation is over all paths through S_t at time t.

Note that

(i) measures $\mathbb{Q}$ and $\mathbb{P}$ are equivalent, that is, zero-probability events are the same under each measure; in the simple example with Ω identified with the finite or countable set of sample paths, this means that each sample path of the process under $\mathbb{P}$ is a sample path under $\mathbb{Q}$, and vice versa;
(ii) (3.9) means that the discounted price process $S^*_t = q^t S_t$ for the bundle of the securities traded in the market is a (local) martingale under $\mathbb{Q}$, whence the name *equivalent martingale measure*;
(iii) Adjective *risk-neutral* comes from an observation that a risk-neutral agent prices assets using (3.9) with the historic probabilities.

Proof. Suppose the multi-period market does not admit arbitrage. Then any embedded two-period market, with $(t, S_t(\omega))$ in place of $(0, S_0)$ in the standard two-period model, does not admit arbitrage. Therefore, for each two-period market, there exists an EMM, which can be used to price, at time t, the payoffs due at time $t+1$ conditioned on the event $\{S_t = S_t(\omega)\}$. Using these conditional expectations, we can calculate the probabilities of all sample paths and derive (3.9). Finally, if (3.9) holds, then, using backward induction, the self-financing property, and the same argument as in the proof of Theorem 3.4.1, one shows that an arbitrage portfolio does not exist.

3.5.3 Replication and complete markets

A contingent claim, with the payoff G_τ at expiry date τ, can be *replicated* if there exists a dynamic self-financing portfolio θ such that $\theta \cdot S_\tau = G_\tau$. The market is called *complete* if any contingent claim with the expiry date $\tau \le T$ can be replicated. If the market is complete (and does not admit arbitrage) then an EMM is unique, and the price of any contingent claim at time $t \le \tau$ equals

$$V(G; t) = E^{\mathbb{Q}}_t [q^{\tau - t} G_\tau]. \tag{3.10}$$

If the only source of uncertainty is the evolution of the price vector of the underlying assets S, the payoff G_τ is a function of S_τ: $G_\tau = G(S_\tau)$, and the future evolution of the price S of the assets depends only on the current prices, that is, the process $\{S_t\}$ is a *Markov process*, then we can (and will) write

$$V(G; t, S_t) = E_t^{\mathbb{Q}}[q^{\tau-t} G(S_\tau)] = E^{\mathbb{Q}}[q^{\tau-t} G(S_\tau) \mid S_t]. \tag{3.11}$$

3.5.4 Binomial model

Consider a multi-period market of a riskless bond and a risky security. In the *binomial model*, at each moment $t = 1, 2, \ldots$, the asset price changes by a factor U or D, $U > D$. The no-arbitrage condition for each embedded

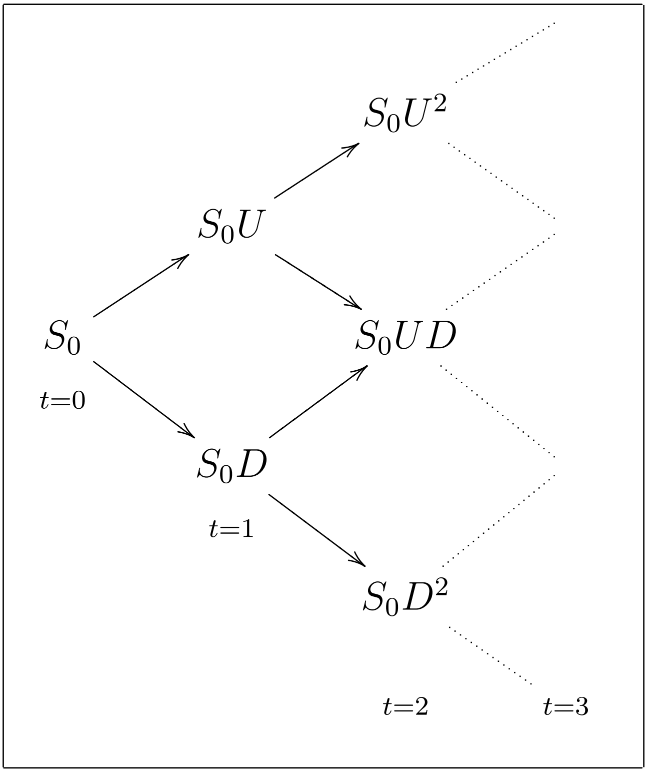

Fig. 3.1. Binomial model.

two-period model, with a spot market (t, S_t) as the new standard spot market $(0, S_0)$, is $D < 1+r < U$. Under this condition, there exists a unique $0 < p < 1$ such that for any S_t (of the form $S_t = S_0 U^k D^{t-k}$, $0 \le k \le t$),

$$S_t = q(pS_tU + (1-p)S_tD), \tag{3.12}$$

where $q = 1/(1+r)$. Namely,

$$p = \frac{q^{-1} - D}{U - D}. \tag{3.13}$$

It is important that in the binomial model, the risk-neutral probabilities are uniquely defined and they are the same for all embedded two-period markets – even if the historic probabilities depend on time, spot price and some other directly unobservable factors. Thus, the market is complete. We will see that in the case of the trinomial model, the situation is opposite.

We can interpret (3.12) as the conditional expectations

$$S_t^* = E_t^{\mathbb{Q}}[S_{t+1}^*] = E^{\mathbb{Q}}[S_{t+1}^* \mid S_t], \tag{3.14}$$

and where $S_t^* = q^t S_t$, and use these conditional expectations to define a unique EMM $\mathbb{Q}$ on the probability space of the sample paths of the process.

Using (3.10)–(3.11), we can write time-t price of the *European call option* with the expiry date T and strike K in two forms:

$$V_{\text{call}}(t, S_t) = E_t^{\mathbb{Q}}[q^{T-t}(S_T - K)_+] = E^{\mathbb{Q}}[q^{T-t}(S_T - K)_+ \mid S_t]. \tag{3.15}$$

Similarly, for the *European put option*,

$$V_{\text{put}}(t, S_t) = E_t^{\mathbb{Q}}[q^{T-t}(K - S_T)_+] = E^{\mathbb{Q}}[q^{T-t}(K - S_T)_+ \mid S_t]. \tag{3.16}$$

If the conditional probability distribution of S_T under an EMM $\mathbb{Q}$ is known, we can easily calculate the expectations in (3.15) and (3.16). For instance, in the binomial model, for $k = 0, 1, \dots, T-t$,

$$P(S_T = S_t U^k D^{T-t-k} \mid S_t) = \binom{T-t}{k} p^t (1-p)^{T-t-k},$$

where

$$\binom{T-t}{k} = \frac{(T-t)!}{k!(T-t-k)!},$$

and, therefore,

$$V_{\text{call}}(t, S_t) = \sum_{k=0}^{T-t} q^{T-t} \binom{T-t}{k} p^t (1-p)^{T-t-k} (S_t U^k D^{T-t-k} - K)_+, \tag{3.17}$$

$$V_{\text{put}}(t, S_t) = \sum_{k=0}^{T-t} q^{T-t} \binom{T-t}{k} p^t (1-p)^{T-t-k} (K - S_t U^k D^{T-t-k})_+. \tag{3.18}$$

3.5.5 Incomplete markets and the trinomial model

In the *trinomial model*, at time $t = 1, 2, \ldots$, the price of the risky asset changes by a constant factor, that assumes one of the three values: $U > M > D$. There is no arbitrage iff $U > 1 + r > D$. Contrary to the binomial model, in the trinomial model, an EMM is not unique, and the conditional risk-neutral probabilities $P(S_{t+1} = S_t U \mid S_t)$, $P(S_{t+1} = S_t M \mid S_t)$, $P(S_{t+1} = S_t D \mid S_t)$ may depend on time t, the spot price S_t, prehistory of the price and on any additional stochastic factor even if the probabilities are the same under the historic measure.

3.6 American options

Contrary to the European option, the corresponding American option can be exercised at any moment at and before the expiration date, T. Consider the American option with the expiry date T and payoff G_t at time $t \le T$. Typically, the payoff is a function of another (underlying) stochastic factor, call it S_t. For instance, the American call option obtains with $G_t = (S_t - K)_+$, and the American put option – with $G_t = (K - S_t)_+$, where S_t is the time-t price of the underlying asset. We will use the notation G_t for $S_t - K$ and $K - S_t$, respectively, because it is not optimal to exercise the option when the payoff is negative, and, in the following chapters, this notation will simplify formulations of theorems and proofs. Denote by $V(t)$ the option price at time t. Assuming that G_t is priced under an EMM $\mathbb{Q}$, the option owner chooses a stopping time that maximizes

$$V(0) = \sup_{\tau} E^{\mathbb{Q}}[q^{\tau} G_{\tau}]. \tag{3.19}$$

At time T, $V(T) = (G_T)_+$, and the option is exercised iff $G_T \ge 0$. The price at time $t = T-1, T-2, \ldots$, and the time-t exercise rule can be easily found by backward induction:

(1) set $V(T) = (G_T)_+$;
(2) the option is exercised at time $t (= T-1, T-2, \ldots)$, iff $G_t \ge E^{\mathbb{Q}}[qV(t+1)]$, and $V(t) = \max\{G_t; E^{\mathbb{Q}}[qV(t+1)]\}$.

This procedure can be easily programmed but the analytical calculations become quite messy (although mathematically trivial) if $T \ge 2$.

Consider the American call option with strike price K and expiration date T. If the option is exercised at time $t \le T$, the option owner receives $S_t - K$. Should the owner decide not to exercise the option till expiry, she has, in effect, the European call option, whose time-t price is

$$\begin{aligned} V_{\text{call}}(t, S_t) &= E_t^{\mathbb{Q}}[q^{T-t}(S_T - K)_+] \\ &\ge E^{\mathbb{Q}}[q^{T-t}(S_T - K)] \\ &= S_t - q^{T-t} K, \end{aligned}$$

because S_t is a martingale under $\mathbb{Q}$. If $q < 1$, then $S_t - q^{T-t}K > S_t - K$, therefore, it is not optimal to exercise the American call option before expiry. If $q = 1$, then, in the binomial model, it may be the case that for certain S_t, $V_{\text{call}}(t, S_t) = S_t - K$, and the option owner is indifferent between early exercise and waiting, but, for more complex models of uncertainty, typically, $V_{\text{call}}(t, S_t) > S_t - K$ for any $t < T$ and any S_t.

Similar argument shows that if $q = 1$, then it is not optimal to exercise the American put before expiry. Consider the American put option assuming $q < 1$, in a two period – two state model. At $t = 0$, the option owner compares the value of exercising the option now, which is $K - S_0 = K - q(p(\omega_1)S_1(\omega_1) + p(\omega_2)S_1(\omega_2))$, and the continuation value, which is the price of the European put option. Assume that $S_1(\omega_2) < K < S_1(\omega_1)$. Then the payoff of the European put is $K - S_1(\omega_2)$, if the state ω_2 is realized, and 0 otherwise. Hence, the date-0 price of the European put is $E^{\mathbb{Q}}[q(K - S_1)_+] = qp(\omega_2)(K - S_1(\omega_2))$, and the American put option is exercised at $t = 0$ iff $K - S_0 \geq qp(\omega_2)(K - S_1(\omega_2))$. Equivalently, using (3.6), the option is exercised at time $t = 0$ iff

$$S_0 \leq H_0 := (1 - q)S_1(\omega_1)\frac{K - S_1(\omega_2)}{S_1(\omega_1) - K} + S_1(\omega_2). \tag{3.20}$$

H_0 is the *early exercise boundary* at $t = 0$: it is optimal to exercise the American put at $t = 0$ iff the asset price S_0 is at $H(0)$ or below it. Since $S_1(\omega_2) < K < S_1(\omega_1)$, the early exercise boundary decreases as the riskless rate $r \to +0$ (equivalently, $q \to 1 - 0$), and in the limit, coincides with the lower bound for payoffs at expiry, $S_1(\omega_2)$. These properties of the early exercise boundary of the American put are typical for other models of uncertainly, too.

Similarly, in a multi-period model with a positive riskless rate and under a typical price process, the early exercise rule can be formulated in terms of the early exercise boundary $H_{\text{put}}(t)$:

exercise the American put option at time t iff $S_t \leq H_{\text{put}}(t)$.

Similarly, if the stock pays dividends at rate $d_t > 0$ so that its time-t (after time-t dividend) price equals the EPV of future dividends

$$S_t = E_t^{\mathbb{Q}}\left[\sum_{s=t+1}^{\infty} q^{t-s}d_t\right], \tag{3.21}$$

then it may be optimal to exercise the option before expiry. For typical specifications of the dividend process, the early exercise rule can be formulated in terms of the early exercise boundary $H_{\text{call}}(t)$:

exercise the American call option at time t iff $S_t \geq H_{\text{call}}(t)$.

See Prob. 3.6.

Problems

3.1. In the model of Subsect. 2.1.1, the riskless rate is $r = 0.1$, $S_0 = 50$, and $S_t = 60$ or $S_t = 40$, $t = 1, 2, \ldots$, and, under the risk-neutral measure, $p(S_t = 60) = 0.5$, $t = 1, 2, \ldots$.

(a) Find the delivery price of a barrel of ethanol, with the date of delivery $t = 1, 2 \ldots$, of the forward contract signed at time 0.
(b) Calculate $E^{\mathbb{Q}}[qS_1]$ and explain why it is different from S_0.

3.2. In the two-period – two-state model of the financial market of a riskless bond and risky asset, assume that (3.7) fails, and construct an arbitrage portfolio.

3.3. In the two-period – three-state model of the financial market of a riskless bond and risky asset, prove that there is no arbitrage iff (3.7) holds, and construct the set of risk-neutral measures.

3.4. Let S be the price of a stock. At time 0, $S_0 = 50$ and at time $t = 1$, the price can assume two values: $S_1 = 60$ or $S_2 = 40$, with nonzero probabilities. Assuming the existence of a riskless bond with the rate of return 0.1 per year,

(a) calculate the risk-neutral probabilities;
(b) explain why they are different from probabilities in Subsect. 2.1.1.

3.5. Assume that at date $t = 1$, the risky asset may assume only two values, U and D, with non-zero probabilities, the discount rate is $r \geq 0$, and a European call option with the expiry date $T = 1$ and strike price K, $D < K < U$, is traded in the market, the call price being $V_{\text{call}}(0)$. Assuming that there is no arbitrage,

(i) calculate the risk-neutral probabilities;
(ii) calculate the price of the risky asset at time 0;
(iii) assuming that there is also a European put option, with the expiry date $T = 1$ and strike price K', with the price today $V_{\text{put}}(0)$, find the relation between the parameters of the model, which exclude arbitrage;
(iv) if $K = K'$ and there is no arbitrage, derive the put-call pairity

$$S_0 - qK = V_{\text{call}}(0) + V_{\text{put}}(0);$$

(v) if the relation derived in (iii) does not hold, construct an arbitrage portfolio.

3.6. The stock S pays dividends at rate d_t. Under an EMM chosen by the market, $E^{\mathbb{Q}}[d_{t+1} \mid d_t] = kd_t$, where $k < 1 + r$.

(i) Using (3.21), calculate the the stock price, S_t, as a function of d_t.
(ii) Consider the American call option on the stock with strike K and expiry date $T = 1$. Show that the exercise at time 0 may be optimal, and find the early exercise boundary.

Part II

Discrete time – discrete space models. Infinite time horizon

4

Random walks on $\mathbb{Z}$

4.1 Definition and main examples

In the framework of the binomial model of Subsect. 3.5.4, consider the log-price $X_t = \log S_t$ instead of the price S_t. Then the increments $Y_t = X_t - X_{t-1}$, $t = 1, 2, \dots$, are independent random variables assuming values $u = \log U$ and $d = \log D$ with probabilities $p = \mathrm{Prob}\,(Y_t = u)$ and $1 - p = \mathrm{Prob}\,(Y_t = d)$; these probabilities are the same for all Y_t. Thus, X_t is a process with independent identically distributed (i.i.d) increments. We can write

$$X_t = X_0 + Y_1 + \cdots + Y_t, \tag{4.1}$$

where the initial location X_0 is either deterministic or, more generally, distributed independently of $Y_1, Y_2, \dots$. A stochastic process (4.1) with i.i.d. $Y_1, Y_2, \dots$ and X_0 independent of $Y_1, Y_2, \dots$ is called a *random walk* (on $\mathbb{R}$, since X_t assume values in $\mathbb{R}$). Technically, the study of options with payoffs depending on X_t simplifies if X_t assumes values in a discrete lattice. In the framework of the binomial model, let $d = -u$; then X_t assumes values in $u\mathbb{Z}$. Normalizing u to 1, we obtain the binomial model on $\mathbb{Z}$. We have

$$p_j := p(Y_t = j) = \begin{cases} p, & j = 1, \\ 1 - p, & j = -1, \\ 0, & \text{otherwise}. \end{cases}$$

where $p > 0, 1 - p > 0$. In the trinomial model,

$$p_j := p(Y_t = j) = \begin{cases} p_+, & j = 1, \\ p_0, & j = 0, \\ p_-, & j = -1, \\ 0, & |j| \ge 2, \end{cases}$$

where $p_\pm > 0$, $p_0 > 0$, and $p_- + p_+ + p_0 = 1$. In the next sections, we will treat the binomial model as a limiting case of the trinomial model with $p_0 = 0$.

For a general random walk on $\mathbb{Z}$, all p_j are non-negative, and $\sum_{-\infty}^{+\infty} p_j = 1$.

4.2 Transition operator and EPV-operator $\mathcal{E}$

Let $q \in (0, 1)$ be the discount factor per period. The *transition operator*, P, is defined by

$$(Pg)(x) = E^x[g(X_1)] = E[g(X_1) | X_0 = x] = E[g(x + Y_1)] = \sum_{j=-\infty}^{+\infty} p_j g(x + j).$$

Using the equality $\sum_{-\infty}^{+\infty} p_j = 1$, it is straightforward to show that P maps bounded functions into bounded ones; moreover, it is a bounded linear operator in $l_\infty(\mathbb{Z})$, with the operator norm 1. Indeed, for any $g \in l_\infty(\mathbb{Z})$,

$$||Pg||_{l_\infty(\mathbb{Z})} = \sup_{x\in\mathbb{Z}} \left| \sum_{-\infty}^{+\infty} p_j g(x + j) \right| \leq \sup_{x\in\mathbb{Z}} \sum_{-\infty}^{+\infty} p_j \sup_{y\in\mathbb{Z}} |g(y)| = 1 \cdot ||g||_{l_\infty(\mathbb{Z})},$$

and for $g(x) \equiv 1$, $||Pg||_{l_\infty(\mathbb{Z})} = ||g||_{l_\infty(\mathbb{Z})}$. If the probability distribution $\{p_j\}$ decays at infinity sufficiently fast, the transition operator P is bounded in spaces of functions, which grow at infinity. We will return to this question later.

Given $p = (p_j)_{j\in\mathbb{Z}}$, one calculates the EPV of a stochastic payoff tomorrow:

$$E^x[qg(X_1)] = q(Pg)(x) = q \sum_{-\infty<j<+\infty} p_j g(x + j).$$

Since $Y_1, Y_2, \ldots$ are i.i.d. random variables, we can use the law of iterated expectations and calculate

$$E^x[g(X_t)] = E^x\left[E^{X_{t-1}}[g(X_t)]\right] = E^x[Pg(X_{t-1})] = \cdots = (P^t g)(x).$$

Here $P^t = P \circ P \circ \cdots \circ P$ is the composition of t copies of P. Therefore, the EPV of a stochastic payoff $g(X_t)$ is

$$E^x[q^t g(X_t)] = q^t P^t g(x)$$

The next step is the calculation of the normalized EPV of a stream of payoffs:

$$(\mathcal{E}g)(x) = (1 - q)E^x\left[\sum_{t=0}^{\infty} q^t g(X_t)\right] = (1 - q)\sum_{t=0}^{\infty} q^t (P^t g)(x). \tag{4.2}$$

The normalization is convenient because

$$(\mathcal{E}\mathbf{1})(x) = (1 - q)E^x\left[\sum_{t=0}^{\infty} q^t \mathbf{1}(X_t)\right] = (1 - q)\sum_{t=0}^{\infty} q^t = 1. \tag{4.3}$$

Since (4.2) holds for any $g \in l_\infty(\mathbb{Z})$, we can write (4.2) as the equality for operators acting in $l_\infty(\mathbb{Z})$:

$$\mathcal{E} = (1-q)\sum_{t=0}^{\infty} q^t P^t. \tag{4.4}$$

We use (4.4) and the properties of the operator norm:

$$||A+B|| \le ||A|| + ||B||, \quad ||AB|| \le ||A|| \cdot ||B||,$$

and obtain

$$||\mathcal{E}|| \le (1-q)\sum_{t=0}^{\infty} q^t \cdot 1^t = 1.$$

In view of (4.3), we conclude that the norm of $\mathcal{E}$ is 1. We will also need another interpretation of (4.2). Let T be the geometric random variable on $\mathbb{Z}_+$, independent of the process $X = \{X_t\}$ (that is, independent of each random variable X_t), with $\text{Prob}(T = t) = (1-q)q^t$. Then

$$\mathcal{E}g(x) = E[g(x + X_T)].$$

In the theory of stochastic processes, the map $g \mapsto (1-q)^{-1}\mathcal{E}g$ is called the resolvent or potential operator.

4.3 Bellman equation and calculation of $\mathcal{E}g$ using factorization

In order to find $u(x) = (\mathcal{E}g)(x)$, we write the *Bellman equation*

$$\begin{aligned} u(x) &= (1-q)g(x) + q(1-q)E^x\left[\sum_{t=1}^{\infty} q^{t-1} g(X_t)\right] \\ &= (1-q)g(x) + q(1-q)\sum_{t=1}^{\infty} q^{t-1} E^x[E^{X_1}[g(X_t)]] \\ &= (1-q)g(x) + qE^x[u(X_1)] \end{aligned}$$

in the form

$$u(x) = (1-q)g(x) + qPu(x),$$

and then as

$$(1-q)^{-1}(I - qP)u(x) = g(x), \ \forall \ x. \tag{4.5}$$

One can view (4.5) as an infinite system of linear algebraic equations with an infinite matrix (operator)

$$A = (1-q)^{-1}(I - qP).$$

The norm of the operator qP (as an operator in $l_\infty(\mathbb{Z})$) equals $q < 1$, hence the operator A is invertible in $l_\infty(\mathbb{Z})$, and we have

$$u(x) = (\mathcal{E}g)(x) = (A^{-1}g)(x), \tag{4.6}$$

for a bounded g. Under additional conditions on the transition probabilities, (4.6) can be extended to unbounded g. The case of unbounded streams will be considered later.

We introduce a simple factorization of the operator A in the binomial and trinomial models, which allows us to invert A and calculate the action of $\mathcal{E}$ quite easily. The factorization will be especially useful in Chap. 5, where we will calculate the EPV of a stochastic stream which is lost when the stochastic factor crosses a certain barrier. Let $p_j = 0$, $|j| \ge 2$, $p_{\pm 1} := p_\pm > 0$, and $p_0 \ge 0$. Then

$$(Pg)(x) = p_0 g(x) + qp_+ g(x+1) + qp_- g(x-1).$$

The case $p_0 = 0$ is the binomial model, and the case $p_0 > 0$ is the trinomial one; we consider both cases simultaneously because the argument and calculations below are essentially the same. Introduce the translation operators S and S^{-1} by

$$(Sg)(x) = g(x+1), \quad (S^{-1}g)(x) = g(x-1),$$

and represent A in the form

$$A = (1-q)^{-1}\left((1-qp_0)I - qp_+S - qp_-S^{-1}\right).$$

With the operator A, we associate a function

$$a(z) = (1-q)^{-1}\left(1 - qp_0 - qp_+z - qp_-z^{-1}\right),$$

which is called the symbol of A. Since $a(z) \to -\infty$ as $z \to +\infty$ and $z \to 0$, and $a(1) = (1-q)^{-1}(1 - qp_0 - qp_+ - qp_-) = 1 > 0$, the symbol a has a zero on $(1, +\infty)$, denote it by $1/q_+$, and a zero on $(0,1)$, denote it by q_-. See Fig. 4.1 for an example. Explicitly,

$$q_+ = \frac{2qp_+}{1 - qp_0 + \sqrt{(1-qp_0)^2 - 4q^2p_+p_-}},$$
$$q_- = \frac{1 - qp_0 - \sqrt{(1-qp_0)^2 - 4q^2p_+p_-}}{2qp_+}.$$

We factorize $a(z)$:

$$\begin{aligned} a(z) &= \frac{-qp_+}{(1-q)z}\left(z - q_+^{-1}\right)(z - q_-) \\ &= \frac{qp_+}{(1-q)q_+}(1 - q_+z)\left(1 - q_-z^{-1}\right). \end{aligned}$$

Since

$$a(1) = 1 = \frac{qp_+}{(1-q)q_+}(1-q_+)(1-q_-),$$

we may write

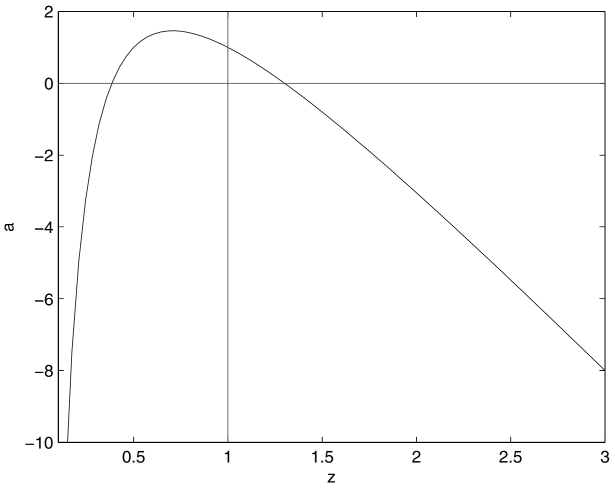

Fig. 4.1. Graph of $a(z)$. Parameters: $q = 0.9, p_0 = 0.1, p_+ = 0.6, p_- = 0.3$.

$$a(z) = a^+(z)a^-(z), \tag{4.7}$$

where

$$a^+(z) = (1 - q_+)^{-1}(1 - q_+z), \tag{4.8}$$
$$a^-(z) = (1 - q_-)^{-1}(1 - q_-z^{-1}). \tag{4.9}$$

Substituting S for z in (4.7), we obtain the factorization of the operator A:

$$A = a^+(S)a^-(S). \tag{4.10}$$

Clearly, both S and S^{-1} have norm 1, and since $q_\pm < 1$, operators $a^+(S)$ and $a^-(S)$ are invertible as operators in $l_\infty(\mathbb{Z})$. The action of the inverses $\mathcal{E}^+ := a^+(S)^{-1}$ and $\mathcal{E}^- := a^-(S)^{-1}$ can be calculated quite easily:

$$\mathcal{E}^+ = (1 - q_+)(I + q_+S + q_+^2S^2 + \cdots) \tag{4.11}$$

and

$$\mathcal{E}^- = (1 - q_-)(I + q_-S^{-1} + q_-^2S^{-2} + \cdots), \tag{4.12}$$

and we have

$$\mathcal{E} = A^{-1} = \mathcal{E}^+\mathcal{E}^- = \mathcal{E}^-\mathcal{E}^+. \tag{4.13}$$

Using (4.13), the solution to the Bellman equation (4.5) can be found in two simple steps:

1. Calculate $w = \mathcal{E}^+g$:

$$w(x) = (1 - q_+)(g(x) + q_+g(x + 1) + q_+^2g(x + 2) + \cdots);$$

2. Calculate $u = \mathcal{E}^- w = \mathcal{E}^- \mathcal{E}^+ g = \mathcal{E} g$:

$$u(x) = (1 - q_-)(w(x) + q_- w(x-1) + q_-^2 w(x-2) + \cdots).$$

Of course, one may calculate $w_2 = \mathcal{E}^- g$ first, and then

$$u = \mathcal{E}^+ w_2 = \mathcal{E}^+ \mathcal{E}^- g = \mathcal{E} g.$$

Thus, we have two variants:

$$u = \mathcal{E}^+ \mathcal{E}^- g, \tag{4.14}$$
$$u = \mathcal{E}^- \mathcal{E}^+ g. \tag{4.15}$$

The action of the operators $\mathcal{E}^\pm$ admits the following interpretation:

$$\mathcal{E}^+ g(x) = E^x[g(x + Y^+)] = \sum_{y \geq 0} p^+(y) g(x+y),$$
$$\mathcal{E}^- g(x) = E^x[g(x + Y^-)] = \sum_{y \leq 0} p^-(y) g(x+y),$$

where Y^+ and Y^- are random variables on $\mathbb{Z}_+ = \{0, 1, \ldots\}$ and $\mathbb{Z}_- = \{\ldots, -1, 0\}$, respectively, with the probability distribution functions

$$p^+(y) = (1 - q_+) q_+^y, \quad \text{and} \quad p^-(y) = (1 - q_-) q_-^{-y}.$$

Assuming that Y^+ and Y^- are independent, we have

$$\mathcal{E} g(x) = E[g(x + Y^+ + Y^-)].$$

Example 4.1. Let $g(x) = b_-, x < 0$, and $g(x) = b_+, x \geq 0$. We calculate $w(x) = \mathcal{E}^+ g(x)$, first, for $x \geq 0$:

$$w(x) = (1 - q_+) \sum_{j=0}^{\infty} q_+^j g(x+j) = (1 - q_+) \sum_{j=0}^{\infty} q_+^j b_+ = b_+,$$

and then for $x < 0$:

$$\begin{aligned} w(x) &= (1 - q_+) \left[\sum_{j=0}^{-x-1} q_+^j g(x+j) + \sum_{j=-x}^{\infty} q_+^j g(x+j) \right] \\ &= (1 - q_+) \left[\sum_{j=0}^{-x-1} q_+^j b_- + \sum_{j=-x}^{\infty} q_+^j b_+ \right] \\ &= b_-(1 - q_+^{-x}) + b_+ q_+^{-x} \\ &= b_- + (b_+ - b_-) q_+^{-x}. \end{aligned}$$

At the second step, we calculate $\mathcal{E} g(x) = \mathcal{E}^- w(x)$. For $x < 0$,

$$\begin{aligned}\mathcal{E}^{-}w(x) &= (1-q_-)\sum_{j=0}^{\infty} q_-^j w(x-j)\\ &= (1-q_-)\sum_{j=0}^{\infty} q_-^j (b_- + (b_+ - b_-)q_+^{-x+j})\\ &= b_- + (b_+ - b_-)\frac{1-q_-}{1-q_-q_+}q_+^{-x},\end{aligned}$$

and for $x \geq 0$,

$$\begin{aligned}\mathcal{E}^{-}w(x) &= (1-q_-)\sum_{j=0}^{\infty} q_-^j w(x-j)\\ &= (1-q_-)\left[\sum_{j=0}^{x} q_-^j b_+ + \sum_{j=x+1}^{\infty} q_-^j (b_- + (b_+ - b_-)q_+^{-x+j})\right]\\ &= b_+(1-q_-^{x+1}) + b_- q_-^{x+1} + (b_+ - b_-)q_+^{-x}(q_-q_+)^{x+1}\frac{1-q_-}{1-q_-q_+}\\ &= b_+ + (b_- - b_+)q_-^{x+1}\frac{1-q_+}{1-q_-q_+}.\end{aligned}$$

4.4 Calculation of $\mathcal{E}g$ for exponentially increasing g

In the standard models of the theory of real options, the payoff function g grows exponentially as $x \to +\infty$: $g(x) = A\gamma^x - B$, where $\gamma > 1$. To ensure that the EPV of the stream $g(X_t)$ were finite, we need the following condition:

$$a(\gamma) = (1-q)^{-1}(1 - qP(\gamma)) > 0, \tag{4.16}$$

where $P(z) = p_0 + p_+ z + p_- z^{-1}$. Indeed, if we apply P to the function γ^x, we obtain

$$P\gamma^x = p_0\gamma^x + p_+\gamma^{x+1} + p_-\gamma^{x-1} = P(\gamma)\gamma^x, \tag{4.17}$$

and, therefore,

$$\sum_{t=0}^{\infty} q^t P^t \gamma^x = \left[\sum_{t=0}^{\infty} q^t P(\gamma)^t\right]\gamma^x.$$

The series on the RHS converges if and only if condition (4.16) is satisfied, and then the normalized EPV is

$$\mathcal{E}\gamma^x = a(\gamma)^{-1}\gamma^x = \frac{1-q}{1-qP(\gamma)}\gamma^x. \tag{4.18}$$

If (4.16) holds, then $0 < q_- < \gamma < 1/q_+$ (see Fig. 4.1), and, therefore, applying (4.11) and (4.12) to $g(x) = \gamma^x$, we obtain the series convergent to

$$\mathcal{E}^+\gamma^x = \kappa_q^+(\gamma)\gamma^x, \tag{4.19}$$
$$\mathcal{E}^-\gamma^x = \kappa_q^-(\gamma)\gamma^x, \tag{4.20}$$

respectively, where we set

$$\kappa_q^+(z) = a^+(z)^{-1} = \frac{1-q_+}{1-q_+z}, \tag{4.21}$$
$$\kappa_q^-(z) = a^-(z)^{-1} = \frac{1-q_-}{1-q_-z^{-1}}. \tag{4.22}$$

A similar argument shows that if (4.16) holds and g is bounded on $\mathbb{Z}_-$ and satisfies an estimate

$$|g(x)| \le C\gamma^x, \quad x \ge 1, \tag{4.23}$$

then functions $\mathcal{E}g$, $\mathcal{E}^+g$ and $\mathcal{E}^-g$ are bounded on $\mathbb{Z}_-$, and admit the bound (4.23). Indeed, there exist $C_1, C_2 > 0$ such that

$$|g(x)| \le C_1 + C_2\gamma^x,$$

therefore, using the notation $\gamma^\cdot$ for the function $x \mapsto \gamma^x$,

$$|\mathcal{E}g(x)| \le (\mathcal{E}|g|)(x) \le \mathcal{E}(C_1 + C_2\gamma^\cdot)(x) \le C_1 + C_2 a(\gamma)^{-1}\gamma^x,$$

and the same argument applies with $\mathcal{E}^\pm$ and $a^\pm(\gamma)$ instead of $\mathcal{E}$ and $a(\gamma)$.

For future calculations, we note the following equivalent form of (4.7):

$$\frac{1-q}{1-qP(z)} = \kappa_q^+(z)\kappa_q^-(z). \tag{4.24}$$

This is an example of the *Wiener-Hopf factorization.*

Problems

In each problem, X_t is either binomial or trinomial.

4.1. Calculate the EPV of a stream

(a) $g(X_t) = \max\{\gamma^{X_t}, 1\}$, where $\gamma > 1$;
(b) $g(X_t) = \max\{\gamma^{X_t}, 1\}$, where $0 < \gamma < 1$;
(c) $g(X_t) = X_t$;
(d) $g(X_t) = \max\{X_t, 1\}$.

Formulate conditions on the random walk which ensure that the EPV is finite and prove their necessity.

4.2. The operational profit flow of the firm is $Ge^{X_t} - C$. Operational profits are taxed at rate $\tau_\Pi > 0$. Calculate the EPV of the profits net taxes. (Caution: profits are taxed when they are positive)

4.3. The profit flow of the firm per share evolves as Ge^{X_t}, where $G > 0$. A firm pays dividends at rate δ_1, when the profit flow is below G, and at rate $\delta_2 > \delta_1$, when the profit flow is above G.

Calculate the rational value of the share.

5

Options in the binomial and trinomial models

Here we explain how to calculate option prices in a simple set-up of the binomial and trinomial models, with several examples. Should the reader prefer to read a systematic exposition of the general theory first and examples afterwards, she may wish to read Chap. 6, and then this Chapter.

5.1 EPV of a stream, which is abandoned when X_t falls to a certain level

Assume that the payoff stream $g(X_t)$ is a non-decreasing function of X_t, a typical example being a firm facing demand uncertainty and non-zero variable cost. Let G be the rate of output, and C the variable cost. The price of a unit of output evolves as e^{X_t}, where X_t is the random walk in the binomial or trinomial model. At high levels of the log-price of the firm's output, X_t, the profit flow $g(X_t) = G\mathrm{e}^{X_t} - C$ is positive, and at low levels, it is negative. Should the (log) price fall sufficiently low, to a certain level h, it may become optimal to exit. Fix h, a candidate for the exit threshold (the optimal choice of h will be analyzed in the next section), and denote by $V(x; h)$ the value of the firm with this choice of the exit threshold. We assume that at the moment when the decision to exit is made, the firm collects no operational profits and suffers no costs. This assumption describes a firm that can make the production decision (continue production or exit) at the beginning of each period. We leave to the reader the reformulation and proof of the results below for a firm which commits to production in the next period. Mathematically, this problem is equivalent to the exit problem of a firm that can collect some scrap value after the exit; we consider a problem of this kind later.

Denote by τ_h^- the first time X_t reaches h from above[1]. Certainly, $\tau_h^- = \tau_h^-(\omega)$ depends on a sample path ω of the process. Thus, τ_h^- is a random variable on the probability space Ω of the sample paths of the process. We have

$$V(x;h) = E^x\left[\sum_{0\le t<\tau_h^-} q^t g(X_t)\right].$$

On the strength of (4.16), the condition

$$1 - qP(\mathrm{e}) > 0. \tag{5.1}$$

is necessary and sufficient for the EPV of the stream $G\mathrm{e}^{X_t} - C$ to be finite.

Lemma 5.1.1 *Let (5.1) hold. Then there exists $c > 0$ such that for any h,*

$$V(x;h) \le c(1+\mathrm{e}^x), \quad \forall\, x, \tag{5.2}$$

and the firm's value admits the same bound (possibly, with a different c).

Proof. We have

$$|g(x)| \le c_g(1+\mathrm{e}^{X_t}), \tag{5.3}$$

where c_g is a constant independent of x, therefore

$$\begin{aligned} V(x;h) &\le c_g \sum_{t=0}^{\infty} q^t E^x[1+\mathrm{e}^{X_t}] = c_g \sum_{t=0}^{\infty}(q^t + q^t P(\mathrm{e})^t \mathrm{e}^x) \\ &= \frac{c_g}{1-q} + \mathrm{e}^x \frac{c_g}{1-qP(\mathrm{e})}. \end{aligned}$$

The proof is valid for any exit time τ, therefore the firm's value is finite.

In the region $x > h$, $V(h;x)$ obeys the Bellman equation

$$V(x;h) = g(x) + qE^x[V(h;X_1)],$$

equivalently,

$$(I - qP)V(x;h) = g(x), \quad x > h. \tag{5.4}$$

After the exit, the value of the firm is zero:

$$V(x;h) = 0, \quad x \le h. \tag{5.5}$$

We introduce the normalized value function $\mathcal{V}(x;h) = (1-q)V(x;h)$, and solve the problem

[1] Note that in the binomial and trinomial models, the process cannot jump over a point on the lattice, this is why we simply say: "reaches h". For a general random walk, we will say: "reaches or crosses h".

$$(1-q)^{-1}(I-qP)\mathcal{V}(x;h)=g(x),\ x>h, \tag{5.6}$$
$$\mathcal{V}(x;h)=0,\quad x\le h, \tag{5.7}$$

which is equivalent to (5.4)–(5.5). The Bellman equation (5.6) is similar to the Bellman equation (4.5) for the value of the firm which never exits but (5.6) holds for $x>h$ only.

Let $\mathbf{1}_{(h,+\infty)}$ denote the indicator function of the subset $\{h+1,h+2,\dots\}\subset\mathbb{Z}$ and the multiplication operator by the same function. The next theorem, which demonstrates the essence of the Wiener–Hopf method in the form used in analysis, states that $\mathcal{V}$ can be calculated using a formula, which is similar to (4.14); the new element is the operator $\mathbf{1}_{(h,+\infty)}$, which must be inserted between $\mathcal{E}^-$ and $\mathcal{E}^+$.

Theorem 5.1.2 *Let (5.1) hold. Then a solution of the problem (5.6)–(5.7) in the class of functions bounded on the negative half-axis and satisfying (5.2) exists. The solution is unique and it is given by*

$$\mathcal{V}(x;h)=(\mathcal{E}^-\mathbf{1}_{(h,+\infty)}\mathcal{E}^+g)(x). \tag{5.8}$$

In other words, from the technical point of view, the calculation of the solution of the problem (5.6)–(5.7) is no more difficult than the calculation of the value of the firm which never exits:

(1) calculate $g_1=\mathcal{E}^+g$: for all x,

$$g_1(x)=(1-q_+)\left(g(x)+q_+g(x+1)+q_+^2g(x+2)+\cdots\right);$$

(2) set $g_2(x)=g_1(x)$ for $x>h$, and $g_2(x)=0$ for $x\le h$;
(3) calculate $\mathcal{V}=\mathcal{E}^-g_2$: for all x,

$$\mathcal{V}(x;h)=(1-q_-)\left(g_2(x)+q_-g_2(x-1)+q_-^2g_2(x-2)+\cdots\right).$$

Notice that now we may not reverse the order of application of $\mathcal{E}^+$ and $\mathcal{E}^-$; the reverse order appears when we solve the problem for a stream which is abandoned as X_t reaches a certain threshold h from below, and then we use the indicator function $\mathbf{1}_{(-\infty,h)}$ instead of $\mathbf{1}_{(h,+\infty)}$ ($\mathbf{1}_{(-\infty,h)}$ denotes the indicator function of the subset $\{\dots,h-2,h-1\}\subset\mathbb{Z}$ and the multiplication operator by the same function).

Proof of Theorem 5.1.2. First, rewrite (5.6) as

$$(a(S)\mathcal{V})(x;h)=g(x)+g^-(x),$$

where $g^-:=a(S)\mathcal{V}-g\in l_\infty$ vanishes above h. Equivalently,

$$a^+(S)a^-(S)\mathcal{V}(x;h)=g(x)+g^-(x),\quad\forall\ x. \tag{5.9}$$

Multiply (5.9) by the inverse $\mathcal{E}^+$ to the (infinite) matrix $a^+(S)$:

$$a^-(S)\mathcal{V}(x;h)=\mathcal{E}^+g(x)+\mathcal{E}^+g^-(x),\quad\forall\ x, \tag{5.10}$$

and note the important property of operators $a^\pm(S)$ and $\mathcal{E}^\pm$:

- If $u(x) = 0 \ \forall \ x \le h$, then, for the same x,

$$(a^-(S)u)(x) = (1 - q_-)^{-1}\left(u(x) - q_- u(x-1)\right) = 0, \tag{5.11}$$

and

$$(\mathcal{E}^- u)(x) = (1 - q_-)\left(u(x) + q_- u(x-1) + q_-^2 u(x-2) + \cdots\right) = 0. \tag{5.12}$$

- If $u(x) = 0 \ \forall \ x > h$, then, for the same x,

$$(a^+(S)u)(x) = (1 - q_+)^{-1}\left(u(x) - q_+ u(x+1)\right) = 0, \tag{5.13}$$

and

$$(\mathcal{E}^+ u)(x) = (1 - q_+)\left(u(x) + q_+ u(x+1) + q_+^2 u(x+2) + \cdots\right) = 0. \tag{5.14}$$

Since $g^-(x) = 0$ for $x > h$, we apply (5.14) and obtain $\mathcal{E}^+ g^-(x) = 0, \quad x > h$. From $\mathcal{V}(x;h) = 0, \ x \le h$, we have $a^-(S)\mathcal{V}(x;h) = 0, \ x \le h$ (see (5.11)). Therefore if we multiply (5.10) by $\mathbf{1}_{(h,+\infty)}$, the LHS does not change and the RHS becomes $\mathbf{1}_{(h,+\infty)}\mathcal{E}^+ g$:

$$a^-(S)\mathcal{V}(x;h) = \mathbf{1}_{(h,+\infty)}(x)(\mathcal{E}^+ g)(x), \ \forall \ x. \tag{5.15}$$

Now it remains to apply the inverse $\mathcal{E}^- = a^-(S)^{-1}$ to obtain (5.8). To prove the uniqueness, it suffices to note that the operators $\mathcal{E}^\pm$ are one-to-one correspondences in the space of functions satisfying the bound (5.3). This follows from the inequalities $0 < q_- < e < q/q_+$ implied by (5.1). For the detailed proof, see the next chapter, where more general random walks and payoff streams are considered. Finally, note that (5.5) holds in view of (5.12). Theorem 5.1.2 has been proved.

5.2 Timing exit

Consider the problem of an optimal choice of the exit boundary h. We have

$$g \text{ is non} - \text{decreasing}; \tag{5.16}$$
$$g(+\infty) := \lim_{x \to +\infty} g(x) > 0; \tag{5.17}$$
$$g(-\infty) := \lim_{x \to -\infty} g(x) < 0 \tag{5.18}$$

From (5.8), we have

$$\mathcal{V}(x;h) = E[(\mathbf{1}_{(h,+\infty)}w)(x + Y^-)], \tag{5.19}$$

where

$$w(x) = \mathcal{E}^+ g(x) = E[g(x + Y^+)], \tag{5.20}$$

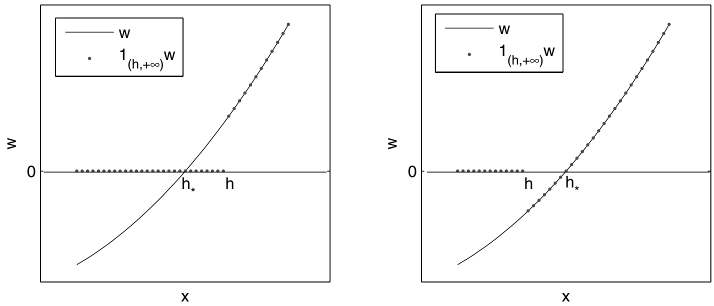

Fig. 5.1. Graphs of w and $\mathbf{1}_{(h,+\infty)}w$. Left panel: exiting too early ($h > h_*$). Right panel: exiting too late ($h < h_*$).

and Y^+, Y^- are the random variables on the positive and negative half-axis, respectively, defined in Sect. 4.3. Clearly, the larger the value of the product $\mathbf{1}_{(h,+\infty)}w$, the larger is the value $\mathcal{V}(x;h)$. Hence, the optimal choice of h should replace all negative values of w by zero, and leave positive ones as they are. Since g is non-decreasing, w is non-decreasing as well. Further, passing to the limit as $x \to \pm\infty$ in (5.20), we obtain that w satisfies (5.17)–(5.18) since g does. Moreover, it is easy to see that if g is increasing in a neighborhood of $+\infty$, then w is increasing on $\mathbb{Z}$, and if g is constant on $\{x_+, x_+ + 1, \ldots\}$ but $g(x_+ - 1) < g(x_+)$, then w is increasing below x_+. We conclude that there exists a unique h_* such that $w(x) > 0$ for all $x > h_*$, and $w(x) \le 0$ for all $x \le h_*$. Generically, w has no zero (recall that we consider a random walk on a lattice) although it changes sign. The optimal exit boundary can be defined as the smallest integer h s.t. for all $x > h$,

$$w(x) = E[g(x + Y^+)] \ge 0. \tag{5.21}$$

It may be the case that $w(h_*) = 0$, and then $h_* - 1$ is also optimal.

Remark 5.1. Notice that we can interpret the exit rule (5.21) as follows: the manager calculates the normalized expected present value of the stream of payoffs assuming that the payoff will grow deterministically and forever:

$$w(x) = (1 - q_+) \sum_{t \ge 0} q_+^t g(x + t),$$

and exits only when, at the current price level, the EPV becomes negative. However, to compensate for such an over-optimistic approach, she discounts the future heavily: it is straightforward to check that $q_+ < q$.

After an optimal exit threshold h_* had been found, the manager calculates the normalized value of the firm $\mathcal{V}(x) = \mathcal{V}(x; h_*)$ for $x > h_*$ as

$$\mathcal{V}(x) = (1 - q_-) \sum_{j=0}^{x-h_*-1} q_-^j w(x - j). \tag{5.22}$$

It follows from (5.22) that the normalized value of the firm is the normalized present value of a deterministic stream of payoffs $w(x+y)$, where y decreases by one every time period starting from zero at the initial date. The stream discontinues at the date $x-h_*$. The discount factor is now q_-. Notice that $q_+ < q_-$ iff $p_+ < p_-$, hence the payoff increases are discounted more heavily than decreases iff the probability of upward jumps is smaller than the probability of downward jumps.

Example 5.2. Let $g(x) = Ge^x - C$, and $1 - qP(e) > 0$. Then the firm's value is finite. We calculate

$$\begin{aligned} w(x) &= \mathcal{E}^+ g(x) \\ &= (1 - q_+) \left(G[e^x + q_+ e^{x+1} + q_+^2 e^{x+2} + \cdots] - C[1 + q_+ + q_+^2 + \cdots]\right) \\ &= (1 - q_+) \left(\frac{Ge^x}{1 - q_+ e} - \frac{C}{1 - q_+}\right). \end{aligned}$$

Hence, h_* is the maximal integer such that $Ge^{h_*}(1 - q_+)/(1 - q_+ e) \le C$ or, using $\kappa_q^+(e) = (1 - q_+)/(1 - q_+ e)$,

$$G\kappa_q^+(e)e^{h_*} \le C. \tag{5.23}$$

At $x > h_*$, the value of the firm is

$$\begin{aligned} V(x) &= (1 - q)^{-1}\mathcal{V}(x) \\ &= (1 - q)^{-1}\mathcal{E}^- \mathbf{1}_{(h_*,+\infty)} w(x) \\ &= \frac{1 - q_-}{1 - q} \sum_{j=0}^{x-h_*-1} q_-^j w(x - j) \\ &= \frac{(1 - q_-)(1 - q_+)}{1 - q} \left\{ \frac{G}{1 - q_+ e} \sum_{j=0}^{x-h_*-1} q_-^j e^{x-j} - \frac{C}{1 - q_+} \sum_{j=0}^{x-h_*-1} q_-^j \right\} \\ &= \frac{(1 - q_-)(1 - q_+)}{1 - q} \left\{ \frac{Ge^x(1 - (q_-/e)^{x-h_*})}{(1 - q_+ e)(1 - q_-/e)} - \frac{C(1 - q_-^{x-h_*})}{(1 - q_+)(1 - q_-)} \right\}. \end{aligned}$$

Using (4.21), (4.22) and (4.24), we obtain

$$\frac{(1 - q_-)(1 - q_+)}{(1 - q)(1 - q_+ e)(1 - q_-/e)} = \frac{\kappa_q^+(e)\kappa_q^-(e)}{1 - q} = \frac{1 - q}{(1 - qP(e))(1 - q)},$$

and simplify

$$V(x) = \frac{Ge^x(1-(q_-/e)^{x-h_*})}{1-qP(e)} - \frac{C(1-q_-^{x-h_*})}{1-q} \tag{5.24}$$

$$= \frac{Ge^x}{1-qP(e)} - \frac{C}{1-q} + V_{\rm opt}(x), \tag{5.25}$$

where the first two terms on the RHS are the value of the firm which produces forever, and

$$V_{\rm opt}(x) = \left(\frac{C}{1-q} - \frac{Ge^{h_*}}{1-qP(e)}\right) q_-^{x-h_*}$$

is the option value to exit.

5.3 Interpretation in terms of EPV-operators under supremum and infimum processes

In Subsect. 6.1.2, we will show that the operators $\mathcal{E}^\pm$ admit another interpretation as normalized EPV-operators under supremum and infimum processes $\bar{X}_t = \sup_{0\le s\le t} X_s$ and $\underline{X}_t = \inf_{0\le s\le t} X_s$:

$$\mathcal{E}^+ g(x) = (1-q)E\left[\sum_{t\ge 0} q^t g(\bar{X}_t) \mid X_0 = x\right], \tag{5.26}$$

$$\mathcal{E}^- g(x) = (1-q)E\left[\sum_{t\ge 0} q^t g(\underline{X}_t) \mid X_0 = x\right]. \tag{5.27}$$

Recall that that $\bar{X}_t$ and $\underline{X}_t$ are defined path-wise: for each sample path ω, $\bar{X}_t(\omega) = \sup_{0\le s\le t} X_s(\omega)$ and $\underline{X}_t(\omega) = \inf_{0\le s\le t} X_s(\omega)$. Now, the optimal exit rule can be formulated as follows: exit the first time $(\mathcal{E}^+ g)(X_t)$ becomes non-positive. If $g(X_t)$ is a non-decreasing function of X_t, as we presumed, we have $g(\bar{X}_t) = \bar{g}_t \equiv \max_{0\le s\le t} g_s$, where $g_t = g(X_t)$, therefore we can reformulate the exit rule in terms of the supremum process: exit at the level g if

$$E\left[\sum_{t=0}^{\infty} q^t \bar{g}_t \mid g_0 = g\right] \le 0.$$

In other words, the rule is: consider all sample paths of the process, and along each sample path, disregard all *temporary drops* of the output price. Then calculate the EPV of profits, and if it is non-positive, abandon the stream. Thus, the hope for the best dies hard: we exit only when the EPV is non-positive even after this rosy adjustment. It looks as if a firm's manager contemplating an exit is too optimistic. However, we will see that the same manager becomes overpessimistic when contemplating an investment.

5.4 Exit under supply uncertainty

Suppose that the price of the firm's output, P, is constant, but the variable cost follows the geometric random walk: $C = \mathrm{e}^{X_t}$. The instantaneous profit $g(X_t) = PG - \mathrm{e}^{X_t}$ is a decreasing function of X_t, and it is positive at low levels of X_t and negative at high levels of X_t. It may be optimal to exit should the cost become too high. To reduce to the case of an increasing payoff stream, which we have considered already, we introduce the dual process $\tilde{X}_t = -X_t$, equivalently, we change the direction on the real line. In terms of $\tilde{X}_t$, the profit flow is $\tilde{g}(\tilde{X}_t) = PG - \mathrm{e}^{-\tilde{X}_t}$, which is an increasing function. Let $\tilde{\mathcal{E}}^{\pm}$ be the EPV-operators under the supremum and infimum processes of $\tilde{X}_t$. The firm exits when $\tilde{X}_t$ reaches the threshold $\tilde{h}_*$ from above; the threshold is the maximal $\tilde{h}$ such that $\tilde{\mathcal{E}}^{+}\tilde{g}(\tilde{h}) < 0$. Set $h^* = -\tilde{h}_*$. Clearly, $\tilde{X}_t$ reaches $\tilde{h}_*$ from above when X_t reaches h^* from below and vice versa, and the supremum process of $\tilde{X}_t$ is the opposite to the infimum process of X_t (provided both start at 0). Hence, the exit rule is formulated in terms of the infimum process of X_t: exit when $\mathcal{E}^{-}g(X_t)$ becomes non-positive. If we formulate the exit rule in terms of the profit flow g_t itself rather than in terms of the underlying process X_t, then the optimal exit rule, the good news principle, is the same as in Sect. 5.3. The normalized value of the firm is

$$\tilde{\mathcal{V}}(x) = (\mathcal{E}^{+}\mathbf{1}_{(-\infty,h^*)}\mathcal{E}^{-}g)(x), \tag{5.28}$$

and the firm's value is

$$V(x) = (1-q)^{-1}(\mathcal{E}^{+}\mathbf{1}_{(-\infty,h^*)}\mathcal{E}^{-}g)(x). \tag{5.29}$$

The argument above used the reduction to the case of the increasing payoff. We could have deduced (5.28) independently, assuming that the exit threshold is given, writing down the Bellman equation, and repeating all the steps in Sect. 5.1 and Sect. 5.2 with $\mathcal{E}^{-}$ in place of $\mathcal{E}^{+}$, and vice versa. We leave this possibility as an exercise for the reader.

5.5 Entry in the binomial and trinomial models

5.5.1 Entry under demand uncertainty

The firm's manager contemplates investment into a plant that will produce G units of output at no variable cost starting the moment the investment is made[2]. The price of a unit of output evolves as e^{X_t}, where X_t is the random walk in either the binomial model or trinomial one. The fixed investment

[2] This assumption simplifies the argument below but it is unnecessary. We consider a more realistic case when the stream of profits are delayed for one period or more in the next Part.

cost is I. Should the price of output rise sufficiently high, it will be optimal to invest. The manager has to find an optimal investment threshold, denote it h^*. To solve this problem, we may interpret the fixed investment cost as the present value of the coupon payments $(1-q)I$ starting the moment the investment is made[3]. Then the optimal timing of investment is equivalent to the problem of an optimal exercise of the (perpetual) option to acquire the stream of payoffs $g(X_t) = Ge^{X_t} - (1-q)I$, with zero strike price. Let h be a candidate for the optimal investment threshold, and denote by τ_h^+ the first time X_t reaches h from below. The EPV of the investment opportunity is

$$V(x;h) = E^x\left[\sum_{t=\tau_h^+}^{\infty} q^t g(X_t)\right].$$

Assuming (5.1) holds, $V(x;h)$ is finite, and visa versa. The proof is essentially the same as the one of Lemma 5.1.1. We have

$$\begin{aligned} V(x;h) &= E^x\left[\sum_{t=0}^{\infty} q^t g(X_t)\right] + W(x;h) \\ &= (1-q)^{-1}\mathcal{E}g(x) + W(x;h), \end{aligned} \tag{5.30}$$

where the first term on the RHS is independent of h, and

$$W(x;h) = E^x\left[\sum_{0\le t<\tau_h^+} q^t(-g(X_t))\right]$$

is the EPV of the stream $-g(X_t)$ which is abandoned the first time X_t reaches h from below. Therefore, an optimal h that maximizes $V(h;x)$ maximizes $W(h;x)$, and vice versa. Since $-g$ is non-increasing, the maximization of $W(h;x)$ is, essentially, the exit problem under supply uncertainty. Using (5.28), we obtain

$$W(x;h) = (1-q)^{-1}\mathcal{E}^+\mathbf{1}_{(-\infty,h)}\mathcal{E}^-(-g)(x). \tag{5.31}$$

But $\mathcal{E} = \mathcal{E}^+\mathcal{E}^-$, therefore, substituting (5.31) into (5.30), we obtain, for the normalized value function $\mathcal{V} = (1-q)V$,

$$\begin{aligned} \mathcal{V}(x;h) &= \mathcal{E}^+\mathcal{E}^-g(x) - \mathcal{E}^+\mathbf{1}_{(-\infty,h)}\mathcal{E}^-g(x) \\ &= \mathcal{E}^+(\mathbf{1}_{(-\infty,h)} + \mathbf{1}_{[h,+\infty)})\mathcal{E}^-g(x) - \mathcal{E}^+\mathbf{1}_{(-\infty,h)}\mathcal{E}^-g(x), \end{aligned}$$

and, finally,

$$V(x;h) = (1-q)^{-1}\mathcal{E}^+\mathbf{1}_{[h,+\infty)}\mathcal{E}^-g(x). \tag{5.32}$$

[3] This interpretation presumes that the firm will never default although it may be optimal to do so.

Since g is an increasing function that changes sign, $\mathcal{E}^- g$ also enjoys these properties. Therefore, there exists a unique integer h^* such that $\mathcal{E}^- g(x) \geq 0$ for all $x \geq h^*$, and $\mathcal{E}^- g(x) < 0$ for all $x < h^*$. We conclude that h^* is an optimal investment threshold (if $\mathcal{E}^- g(h^*) = 0$, then $h^* + 1$ is also optimal). The value of the investment opportunity is

$$V(x) = (1-q)^{-1}\mathcal{E}^+ \mathbf{1}_{[h^*,+\infty)}\mathcal{E}^- g(x). \tag{5.33}$$

Now, the optimal investment rule can be formulated as follows: invest the first time $(\mathcal{E}^- g)(X_t)$ becomes non-negative. If $g(X_t)$ is a non-decreasing function of X_t, as we presumed, we have

$$g(\underline{X}_t) = \underline{g}_t \equiv \min_{0\leq s\leq t} g_s,$$

where $g_t = g(X_t)$, therefore we can reformulate the investment rule in terms of the infimum process: invest at level g if

$$E\left[\sum_{t=0}^{\infty} q^t \underline{g}_t \mid g_0 = g\right] \geq 0.$$

In other words, the rule is: consider all sample paths of the process, and along each sample path, disregard all *temporary increases* of the profit flow. Then calculate the EPV of profits, and if it is non-negative, acquire the stream. Thus, the manager is extremely cautious or too pessimistic: she invests only when the EPV is non-negative even after this worst-case scenario adjustment. We say that she uses the *bad news principle.*

Using (4.20), we derive an explicit condition for the case $g(x) = Ge^x - (1-q)I$: the investment threshold, h^*, is the smallest integer such that

$$\kappa_q^-(\mathrm{e})Ge^{h^*} \geq (1-q)I. \tag{5.34}$$

Applying further (4.11), we calculate the value of the investment opportunity for $x < h^*$:

$$\begin{aligned}
V(x) &= \frac{1-q_+}{1-q}\sum_{j=0}^{\infty} q_+^j \mathbf{1}_{[h^*,+\infty)}(x+j)(\kappa_q^-(\mathrm{e})Ge^{x+j} - (1-q)I) \\
&= \frac{1-q_+}{1-q}\sum_{j=h^*-x}^{\infty} q_+^j(\kappa_q^-(\mathrm{e})Ge^{x+j} - (1-q)I) \\
&= \frac{1-q_+}{1-q}q_+^{h^*-x}\left[\kappa_q^-(\mathrm{e})Ge^{h^*}\sum_{j=0}^{\infty}(q_+\mathrm{e})^j - (1-q)I\sum_{j=0}^{\infty} q_+^j\right] \\
&= q_+^{h^*-x}\left[\frac{1-q_+}{1-q}\frac{\kappa_q^-(\mathrm{e})Ge^{h^*}}{1-q_+e} - I\right] \\
&= q_+^{h^*-x}\left[\frac{Ge^{h^*}}{1-qP(\mathrm{e})} - I\right].
\end{aligned}$$

At the last step, we used

$$\frac{1-q_+}{1-q}\frac{\kappa_q^-(\mathrm{e})G\mathrm{e}^{h^*}}{1-q_+\mathrm{e}} = \frac{\kappa_q^+(\mathrm{e})\kappa_q^-(\mathrm{e})}{1-q}G\mathrm{e}^{h^*} = \frac{G\mathrm{e}^{h^*}}{1-qP(\mathrm{e})}.$$

5.5.2 Entry under supply uncertainty

The firm's manager contemplates the investment into a plant that will yield the constant revenue flow R starting the moment the investment is made. The variable cost is stochastic: $C(X_t) = \min\{\mathrm{e}^{X_t}, C_m\}$, where $0 < C_m < R$. The fixed investment cost is I. Should the variable cost fall sufficiently low, it will be optimal to invest. The manager has to find an optimal investment threshold, denote it h_*. To make the investment irreversible, we assume that the scrap value, Sc, that can be recovered should the firm decide to exit, does not exceed the present value of the lowest profit flow $R-C_m$. We also assume that if the highest level of variable cost, C_m, is presumed to persist forever, then the present value of profits is smaller that the fixed investment cost. On the other hand, to make the investment problem non-trivial, we assume that if the variable cost is zero, then the present value of profits is higher than the fixed investment cost, and it is optimal to invest. Equivalently,

$$R - C_m < (1-q)I < R. \tag{5.35}$$

To solve this investment problem, we may interpret the fixed investment cost as the present value of the coupon payments $(1-q)I$ starting at the moment the investment is made. Then the optimal timing of investment is equivalent to the problem of an optimal exercise of the perpetual option on the stream of payoffs $g(X_t) = R - \min\{\mathrm{e}^{X_t}, C_m\} - (1-q)I$, with zero strike. Under condition (5.35), g is non-increasing and changes sign. Let h be a candidate for the optimal investment threshold. Then the EPV of the investment opportunity is

$$V(x;h) = E^x\left[\sum_{t=\tau_h^-}^{\infty} q^t g(X_t)\right].$$

We have

$$\begin{aligned} V(x;h) &= E^x\left[\sum_{t=0}^{\infty} q^t g(X_t)\right] + W(x;h) \\ &= (1-q)^{-1}\mathcal{E}g(x) + W(x;h), \end{aligned} \tag{5.36}$$

where the first term on the RHS is independent of τ_h^-, and

$$W(x;h) = E^x\left[\sum_{0\le t<\tau_h^-} q^t(-g(X_t))\right]$$

is the EPV of the stream $-g(X_t)$ which is abandoned the first time X_t reaches or crosses h from above. Therefore, an optimal h that maximizes $V(h;x)$ maximizes $W(h;x)$, and vice versa. Since $-g(X_t)$ is non-decreasing, the maximization of $W(h;x)$ is a problem similar to the exit problem considered in Sect. 5.2. We obtain

$$W(x;h) = (1-q)^{-1}\mathcal{E}^{-}\mathbf{1}_{(h,+\infty)}\mathcal{E}^{+}(-g)(x). \tag{5.37}$$

But $\mathcal{E} = \mathcal{E}^{+}\mathcal{E}^{-}$, therefore, substituting (5.37) into (5.36), we obtain, for the normalized value function $\mathcal{V} = (1-q)V$,

$$\begin{aligned}\mathcal{V}(x;h) &= \mathcal{E}^{-}\mathcal{E}^{+}g(x) - \mathcal{E}^{-}\mathbf{1}_{(h,+\infty)}\mathcal{E}^{+}g(x)\\ &= \mathcal{E}^{-}(\mathbf{1}_{(h,+\infty)} + \mathbf{1}_{(-\infty,h]})\mathcal{E}^{+}g(x) - \mathcal{E}^{-}\mathbf{1}_{(h,+\infty)}\mathcal{E}^{+}g(x),\end{aligned}$$

and finally,

$$V(x;h) = (1-q)^{-1}\mathcal{E}^{-}\mathbf{1}_{(-\infty,h]}\mathcal{E}^{+}g(x). \tag{5.38}$$

Since g is a non-decreasing function that changes sign, $\mathcal{E}^{+}g$ enjoys these properties as well. Moreover, $\mathcal{E}^{+}g$ decreases on $(-\infty,h']$, where h' is the smallest integer such that on $[h',+\infty)$, functions g and $\mathcal{E}^{+}g$ equal to the negative constant $R-(1-q)I-C_m$. Therefore, there exists $h_* < h'$ such that $\mathcal{E}^{+}g(x) \ge 0$ for all $x \le h_*$, and $\mathcal{E}^{+}g(x) < 0$ for all $x > h_*$. We conclude that h_* is an optimal investment threshold (if $\mathcal{E}^{+}g(h_*) = 0$, then $h_* - 1$ is also optimal). Note that now the investment rule is formulated in terms of the supremum process of X_t; however, if we reformulate it in terms of the profit flow, we will obtain the same bad news principle for investment decisions. In the case under consideration, we can explicitly calculate $\mathcal{E}^{+}g(x)$, for $x < h'$,

$$\begin{aligned}\mathcal{E}^{+}g(x) &= (1-q_+)\Bigg[\sum_{j=0}^{h'-x-1} q_+^j(R-(1-q)I-\mathrm{e}^{x+j})\\ &\qquad + \sum_{j=h'-x}^{\infty} q_+^j(R-(1-q)I-C_m)\Bigg] \\ &= R-(1-q)I - \frac{\mathrm{e}^x(1-q_+)(1-(q_+\mathrm{e})^{h'-x})}{1-q_+\mathrm{e}} - C_m q_+^{h'-x}.\end{aligned} \tag{5.39}$$

Since we know that the function $\mathcal{E}^{+}g$ decreases on $(-\infty,h']$ and changes sign, we can find h_* numerically quite easily. The investment threshold having being found, we use (5.38) to calculate the value of the investment opportunity, $V(x)$, for $x > h_*$ (for $x \le h_*$, $V(x)$ is the EPV of the stream $g(X_t)$. We leave the explicit calculation to the reader because this kind of problems has been studied in the previous chapter). On $(-\infty,h_*] \subset (-\infty,h']$, the function $\mathcal{E}^{+}g$ is given by (5.39), therefore, for $x > h_*$,

$$\begin{aligned} V(x) &= \frac{1-q_-}{1-q} \sum_{-\infty<j\le h_*-x} q_-^{-j}\mathcal{E}^+ g(x+j) \\ &= q_-^{x-h_*}\frac{1-q_-}{1-q} \sum_{-\infty<j\le 0} q_-^{-j}\mathcal{E}^+ g(h_*+j) \\ &= A\frac{1-q_-}{1-q} q_-^{x-h_*}, \end{aligned}$$

where the constant A equals

$$\sum_{-\infty<j\le 0} q_-^{-j}\left[R-(1-q)I-C_m q_+^{h'-h_*-j}+\frac{\mathrm{e}^{h_*+j}(1-q_+)(1-(q_+\mathrm{e})^{h'-h_*-j})}{1-q_+\mathrm{e}}\right].$$

We leave the simplification of the expression for A to the reader.

5.6 Perpetual American options

5.6.1 Perpetual American call options

Assume that $G(X_t)$, the instantaneous payoff, is an increasing function of X_t. For example, $G(X_t) = S(X_t) - K$ for the call option, where $S(X_t) = \mathrm{e}^{X_t}$ is the price of the underlying asset. Should X_t rise sufficiently high, it may be optimal to exercise the option with the instantaneous payoff $G(X_t)$. Assume that we can express $G(X_t)$ in terms of the EPV of a stream g_t: $G = (1-q)^{-1}\mathcal{E}g$. Since $(I - qP)(1 - q)^{-1}\mathcal{E} = I$, we can find g:

$$g(X_t) = (I - qP)G(X_t). \tag{5.40}$$

Note that the representation of the instantaneous payoff G as the EPV of a stream is impossible in the case of the call option on a stock that pays no dividends because the discounted price process of the stock must be a martingale, and, therefore, $\mathrm{e}^x - E^x[q\mathrm{e}^{X_1}] = (I-qP)\mathrm{e}^x$ must be 0. If $1-qP(\mathrm{e}) > 0$, then the stock pays dividends. If, at time t, the dividends are paid after time-t trades are made, then, to exclude arbitrage opportunities, the dividends must be equal to the difference between the stock price today and expected discounted price tomorrow:

$$\delta(x) = \mathrm{e}^x - E^x[q\mathrm{e}^{X_1}] = \mathrm{e}^x - qP(\mathrm{e})\mathrm{e}^x = (1 - qP(\mathrm{e}))\mathrm{e}^x.$$

If the fraction δ of the asset price is distributed as dividends before the trades are made, then, to avoid arbitrage, it must be that

$$\mathrm{e}^x - E^x[q(1-\delta)\mathrm{e}^{X_1}](= (1-(1-\delta)qP(\mathrm{e}))\mathrm{e}^x) = 0,$$

and $\delta = 1/qP(\mathrm{e}) - 1$. Assume that $1 - qP(\mathrm{e}) > 0$. Then, from (5.40) and (4.17), we obtain that $G(X_t)$ is the EPV of the stream

$$g(X_t) = (1 - qP(\mathrm{e}))\mathrm{e}^{X_t} - (1 - q)K.$$

Therefore, the results of Sect. 5.5.1 are applicable. Let h be a candidate for the exercise boundary. Applying (5.32), we obtain the American call price

$$V_{\mathrm{am.call}}(x; h) = (1 - q)^{-1}\mathcal{E}^+ \mathbf{1}_{[h,+\infty)}\mathcal{E}^- g(x). \tag{5.41}$$

Using the Wiener–Hopf factorization formula and (5.40), we derive

$$\mathcal{E}^- g(x) = (1 - q)\mathcal{E}^-(1 - q)^{-1}(I - qP)G(x) = (1 - q)(\mathcal{E}^+)^{-1}G(x),$$

and rewrite (5.41) as

$$V_{\mathrm{am.call}}(x; h) = \mathcal{E}^+ \mathbf{1}_{[h,+\infty)}(\mathcal{E}^+)^{-1}G(x). \tag{5.42}$$

Function $(\mathcal{E}^+)^{-1}G(x) = \kappa_q^+(\mathrm{e})^{-1}\mathrm{e}^x - K$ is an increasing function that changes sign only once. Hence, the smallest integer h^* such that

$$\mathrm{e}^{h^*} \geq K\kappa_q^+(1) \tag{5.43}$$

is an optimal exercise boundary (if (5.43) holds with the equality, then $h^* + 1$ is also optimal), and the rational call option price is given by

$$V_{\mathrm{am.call}}(x) = \mathcal{E}^+ \mathbf{1}_{[h^*,+\infty)}(\mathcal{E}^+)^{-1}G(x). \tag{5.44}$$

Explicitly, for $x < h^*$,

$$\begin{aligned}
V_{\mathrm{am.call}}(x) &= (1 - q_+)\sum_{j=0}^{\infty} q_+^j \mathbf{1}_{[h^*,+\infty)}(x + j)(\kappa_q^+(\mathrm{e})^{-1}\mathrm{e}^{x+j} - K) \\
&= (1 - q_+) \sum_{j=h^*-x}^{\infty} q_+^j(\kappa_q^+(\mathrm{e})^{-1}\mathrm{e}^{x+j} - K) \\
&= q_+^{h^*-x}(1 - q_+)\sum_{j=0}^{\infty}(\kappa_q^+(\mathrm{e})^{-1}\mathrm{e}^{h^*}q_+^j\mathrm{e}^j - Kq_+^j) \\
&= q_+^{h^*-x}\left[\frac{\mathrm{e}^{h^*}(1 - q_+)}{\kappa_q^+(\mathrm{e})(1 - q_+\mathrm{e})} - K\right].
\end{aligned}$$

Since $\kappa_q^+(z) = (1 - q_+)/(1 - q_+z)$, we simplify

$$V_{\mathrm{am.call}}(x) = (\mathrm{e}^{h^*} - K)q_+^{h^*-x}, \quad x < h^*. \tag{5.45}$$

5.6.2 Perpetual American put options

Let $G(X_t)$ be the instantaneous payoff which is a decreasing function of X_t. For example, $G(X_t) = K - S(X_t)$ for the put option, where $S(X_t) = \mathrm{e}^{X_t}$ is the price of the underlying security. Should X_t fall sufficiently low, it may be optimal to exercise the option with the instantaneous payoff $G(X_t)$. Assume that we can express $G(X_t)$ in terms of the EPV of a stream g_t: $G = (1-q)^{-1}\mathcal{E}g$. Then $g = (I - qP)G$. If the stock does not pay dividends, we cannot apply this procedure with $G(X_t) = K - \mathrm{e}^{X_t}$, but, since the option is not exercised if the payoff is negative, we may replace $K - \mathrm{e}^{X_t}$ with $G_1(X_t) := (G(X_t))_+ = (K - \mathrm{e}^{X_t})_+$. Being bounded, $G_1(X_t)$ is representable as the EPV of the stream $g(X_t) = (I-qP)G_1(X_t)$. Let h be a candidate for the exercise boundary. Then, applying (5.38), we obtain the following formula for the American put price

$$V_{\mathrm{am.put}}(x; h) = (1-q)^{-1}\mathcal{E}^{-}\mathbf{1}_{(-\infty,h]}\mathcal{E}^{+}g(x). \tag{5.46}$$

Using the equality $g = (I-qP)G_1$ and the Wiener–Hopf factorization formula, we derive

$$\mathcal{E}^{+}g(x) = (1-q)\mathcal{E}^{+}(1-q)^{-1}(I-qP)G_1(x) = (1-q)(\mathcal{E}^{-})^{-1}G_1(x),$$

and rewrite (5.46) as

$$V_{\mathrm{am.put}}(x; h) = \mathcal{E}^{-}\mathbf{1}_{(-\infty,h]}(\mathcal{E}^{-})^{-1}G_1(x). \tag{5.47}$$

Since it is not optimal to exercise the option unless $G(h) \ge 0$ and G is decreasing, we have $G_1(x) = G(x)$ for all $x \le h$. For these x,

$$\begin{aligned}(\mathcal{E}^{-})^{-1}G_1(x) &= (1-q_-)^{-1}(G_1(x) - q_-G_1(x-1)) \\ &= (1-q_-)^{-1}(G(x) - q_-G(x-1)) \\ &= (\mathcal{E}^{-})^{-1}G(x),\end{aligned}$$

and, therefore, we can replace G_1 in (5.47) with G. Assume that the function $(\mathcal{E}^{-})^{-1}G$ is a decreasing function that changes sign only once. Then the largest integer h_* such that

$$(\mathcal{E}^{-})^{-1}G(h_*) \ge 0 \tag{5.48}$$

is an optimal exercise boundary (if (5.48) holds with the equality, then $h_* - 1$ is also optimal), and the rational put option price is

$$V_{\mathrm{am.put}}(x) = \mathcal{E}^{-}\mathbf{1}_{(-\infty,h_*]}(\mathcal{E}^{-})^{-1}G(x). \tag{5.49}$$

For the standard American put option, $(\mathcal{E}^{-})^{-1}G(x) = K - \kappa_q^{-}(\mathrm{e})^{-1}\mathrm{e}^{x}$ is decreasing, and, therefore, the exercise boundary is the maximal h_* such that

$$\mathrm{e}^{h_*} \le K\kappa_q^{-}(\mathrm{e}), \tag{5.50}$$

and, for $x > h_*$,

$$V_{\mathrm{am.put}}(x) = (K - \mathrm{e}^{h_*})q_-^{x-h_*}. \tag{5.51}$$

We leave the details of calculations to the reader (cf. (5.45)).

5.6.3 General exercise rules for perpetual American options in the binomial and trinomial models

Consider first call-like options, which are optimal to exercise when the price of the underlying crosses a certain threshold from below. Assume that the function

$$w(x) := (\mathcal{E}^+)^{-1}G(x) = a_+(S)G(x) = (1-q_+)^{-1}(G(x) - q_+G(x+1))$$

increases and changes sign. Then it follows from (5.41) that it is optimal to exercise the option the first time

$$G(X_t) \geq q_+G(X_t + 1). \tag{5.52}$$

Interpretation: imagine that the stochastic factor increases deterministically with the speed 1 per period, and the discount factor per period is q_+. Then the rule (5.52) means that it is optimal to exercise the option the first time the present value of the payoff tomorrow becomes equal or less than the current value of the payoff.

The treatment of a put-like option, which is optimal to exercise when the price of the underlying crosses a certain threshold from above, is similar. Assume that the function

$$w(x) := (\mathcal{E}^-)^{-1}G(x) = a_-(S)G(x) = (1-q_-)^{-1}(G(x) - q_-G(x-1))$$

decreases and changes sign. Then it follows from (5.46) that it is optimal to exercise the option the first time

$$G(X_t) \geq q_-G(X_t - 1). \tag{5.53}$$

Interpretation: imagine that the stochastic factor decreases deterministically with the speed 1 per period, and the discount factor per period is q_-. Then the rule (5.53) means that it is optimal to exercise the option the first time the present value of the payoff tomorrow becomes equal or less than the current value of the payoff.

5.7 Partially reversible investment

Consider the investment in a plant that will yield the revenue stream Ge^{X_t}. The fixed and variable costs, I and C, are positive. In order that the EPV of the profit flow were finite, we need to require that $1 - qP(\mathrm{e}) > 0$. Then the value of the firm is finite as well because it does not exceed the EPV of the stream Ge^{X_t}, which is $(1 - qP(\mathrm{e}))^{-1}Ge^{X_0}$. At high levels of the log-price of the firm's output, X_t, the profit flow $g(X_t) = Ge^{X_t} - C$ is positive, and at low levels, it is negative. Should the (log) price fall sufficiently low, to a certain level h, it may become optimal to exit. To make the situation more realistic, assume that in the event of exit, the firm can collect the scrap value which is proportional to the revenue flow: $Sc(X_t) = \alpha G e^{X_t}$, where $\alpha < 1/(1 - qP(\mathrm{e}))$. The last restriction means that the scrap value does not exceed the EPV of the revenue stream.

The firm's manager solves the investment problem with the embedded option to exit, and she solves the investment problem backward. First, assuming that the plant operates already, she calculates the value of the plant taking into account the option to exit. The manager must find h_* that maximizes

$$V^{\mathrm{ex}}(x;h) = E^x\left[\sum_{0 \le t < \tau_h^-} q^t (G\mathrm{e}^{X_t} - C)\right] + E^x\left[q^{\tau_h^-}\alpha G\mathrm{e}^{X_{\tau_h^-}}\right]. \qquad (5.54)$$

Using $(1-q)^{-1}(I - qP)\mathcal{E} = I$ and (4.18), we obtain

$$\mathrm{e}^x = (1-q)^{-1}(1 - qP(\mathrm{e}))\mathcal{E}\mathrm{e}^x = (1 - qP(\mathrm{e}))E^x\left[\sum_{t=0}^{\infty} q^t \mathrm{e}^{X_t}\right].$$

Inserting the last equality with $X_{\tau_h^-}$ instead of x into the second term on the RHS of (5.54) and using the law of iterated expectations, we obtain

$$\begin{aligned} V^{\mathrm{ex}}(x;h) &= E^x\left[\sum_{0\le t<\tau_h^-} q^t(G\mathrm{e}^{X_t} - C) + \alpha(1 - qP(\mathrm{e}))\sum_{t=\tau_h^-}^{\infty} q^t G\mathrm{e}^{X_t}\right] \\ &= E^x\left[\sum_{0\le t<\tau_h^-} q^t(G_\alpha \mathrm{e}^{X_t} - C)\right] + R(x), \end{aligned}$$

where $G_\alpha = G - G\alpha(1 - qP(\mathrm{e}))$, and

$$R(x) := G\alpha(1 - qP(\mathrm{e}))E^x\left[\sum_{t=0}^{\infty} q^t\mathrm{e}^{X_t}\right] = G\alpha\mathrm{e}^x.$$

The last term is independent of h and positive, therefore it is optimal to exit when it is optimal to abandon the stream $g(X_t) = G_\alpha \mathrm{e}^{X_t} - C$. This problem has been solved in Sect. 5.2 (see (5.23) and (5.24) in Example 5.2). An optimal h_* is the maximal integer such that

$$\kappa_q^+(\mathrm{e})G_\alpha \mathrm{e}^{h_*} \le C. \qquad (5.55)$$

At $x > h_*$, the value of the plant is

$$V^{\mathrm{ex}}(x) = \frac{G_\alpha \mathrm{e}^x(1 - (q_-/\mathrm{e})^{x-h_*})}{1 - qP(\mathrm{e})} - \frac{C(1 - q_-^{x-h_*})}{1-q} + G\alpha\mathrm{e}^x$$

(an additional term $G\alpha\mathrm{e}^x$ was absent in (5.24)), and, finally,

$$V^{\mathrm{ex}}(x) = \frac{C}{1-q}[A\mathrm{e}^{x-h_*} + q_-^{x-h_*}(1-\kappa_q^-(1)) - 1], \tag{5.56}$$

where

$$A = \frac{\mathrm{e}^{h_*}G}{C} \times \frac{1-q}{1-qP(\mathrm{e})}.$$

The value of the plant with the embedded option to exit having being found, the firm's manager needs to find the optimal exercise time for the perpetual American call option with the payoff $G(X_t) = V^{\mathrm{ex}}(X_t) - I$. The natural assumption is that, at the exit boundary, the scrap value plus one-period-profit are smaller than the investment cost: $(1+\alpha)G\mathrm{e}^{h_*} - C < I$. The payoff is defined for $x > h_*$ only, but this complication is not essential because the investment boundary must be larger than the exit threshold. As we showed in Subsect. 5.6.1, among $h > h_*$, the smallest integer that satisfies

$$(\mathcal{E}^+)^{-1}G(h) = a_+(S)G(h) \geq 0$$

is an optimal investment threshold. Since

$$a_+(S)G(h) = (1-q_+)^{-1}(1-q_+S)G(h) = (1-q_+)^{-1}(G(h) - q_+G(h+1))$$

is independent of values of G below $h(> h_*)$, we may assume that the payoff is defined by the analytical expression

$$G(x) = \frac{C}{1-q}[A\mathrm{e}^{x-h_*} + q_-^{x-h_*}(1-\kappa_q^-(1)) - 1 - (1-q)I/C]$$

on the whole axis. Applying $(\mathcal{E}^+)^{-1}$ and using (4.19), we conclude that an optimal investment threshold can be defined as the smallest integer greater than h_*, denote it h^*, such that

$$\frac{A}{\kappa_q^+(\mathrm{e})}\mathrm{e}^{h^*-h_*} + \frac{1-\kappa_q^-(1)}{\kappa_q^+(q_-)}q_-^{h^*-h_*} - 1 - \frac{(1-q)I}{C} \geq 0. \tag{5.57}$$

Since $\kappa_q^+(\mathrm{e})\kappa_q^-(\mathrm{e}) = (1-q)/(1-qP(\mathrm{e}))$, we can simplify the first coefficient:

$$A_1 := A/\kappa_q^+(\mathrm{e}) = \frac{\mathrm{e}^{h_*}G}{C}\kappa_q^-(\mathrm{e}).$$

Set $A_2 = (1-\kappa_q^-(1))/\kappa_q^+(q_-)$ and introduce the function

$$F(y) = A_1\mathrm{e}^y + A_2q_-^y - 1 - (1-q)I/C.$$

The investment threshold exists iff F has a positive zero. Since $A_1 > 0$ and $A_2 > 0$, we have $F''(y) > 0$ for all y; hence, F is convex. As $y \to +\infty$, $F(y) \to +\infty$, therefore, three cases are possible:

(1) $F(y) \geq 0$ for all $y \geq 0$;

(2) there exists $y^* > 0$ such that $F(y) > 0$ for all $y > y^*$, and $F(y) < 0$ for all $0 \le y < y^*$;
(3) there exists $y^* > y' \ge 0$ such that $F(y) > 0$ for all $y > y^*$ and all $0 \le y < y'$, and $F(y) < 0$ for all $y' < y < y^*$.

In Case (1), it is optimal to enter at any level $h \ge h_*$. In Cases (2) and (3), set $h^* = h_* + y^*$. It is optimal to enter if the current log-price is at h^* or above h^*. However, in Case (2), it is not optimal to enter at any other level, whereas in Case (3), we cannot exclude the possibility that the firm may wish to enter at some level below h^*.

To exclude rather unnatural Cases (1) and (3), it suffices to presume that $F(0) < 0$; then F changes sign on $\mathbb{R}_+$ from "-" to "+", and only once. Using (5.55), we obtain

$$A_1 \le \frac{\kappa_q^-(\mathrm{e})G}{\kappa_q^+(\mathrm{e})G_\alpha} = \frac{\kappa_q^-(\mathrm{e})}{\kappa_q^+(\mathrm{e})(1-\alpha(1-qP(\mathrm{e})))},$$

therefore,

$$F(0) \le \frac{\kappa_q^-(\mathrm{e})}{\kappa_q^+(\mathrm{e})(1-\alpha(1-qP(\mathrm{e})))} + \frac{1-\kappa_q^-(\mathrm{e})}{\kappa_q^+(q_-)} - 1 - \frac{(1-q)I}{C}.$$

We conclude that if I/C is sufficiently large, then $F(0) < 0$, and only the natural case (2) is possible.

Problems

5.1. Solve the exit problem in Subsect. 5.4 repeating the argument in Subsections 5.1 and 5.2. Calculate the firm's value (5.29) explicitly.

5.2. Calculate explicitly the value of the investment opportunity assuming that the uncertainty is on the demand side and the price of the output is capped: $\min\{\mathrm{e}^{X_t}, M\}$. Study the dependence of the investment threshold and value of the firm on M.

5.3. Calculate explicitly the value of the investment opportunity assuming that the uncertainty is on the demand side and there is a floor for the price of the output: $\max\{\mathrm{e}^{X_t}, m\}$. Study the dependence of the investment threshold and value of the firm on the floor m. Compare with the results of Prob. 5.2 and comment on the difference.

5.4. The firm is financed by debt. The uncertainty is on the demand side, and the variable cost, C, is constant. The coupon payments are C_d. Default is determined by the debt covenant: the firm is declared insolvent the first time its operational profits $G\mathrm{e}^{X_t} - C$ fall below 0. Calculate

(a) the value of the equity (that is, the EPV of the profit flow which shareholders will collect before default happens);
(b) the value of the debt assuming that in the event of default, the debt holders will get nothing;
(c) the value of the debt assuming that in the event of default, the debt holders recover the fraction α of the EPV of the revenue stream of the firm;
(d) the value of the firm, which is the sum of the value of equity and value of debt, in Cases (b) and (c).

5.5. The firm is financed by debt. The coupon payments are C, and there is no variable cost. The uncertainty is on the demand side. There are no debt covenants. If the profit flow becomes negative, the shareholders may infuse additional funds; equivalently, the profit flow to shareholders may be negative. However, should the losses become too large, the shareholders may find it optimal to declare bankruptcy. For simplicity, we consider the bankruptcy under Chapter 7, so no renegotiation is possible. Calculate

(a) an optimal default threshold;
(b) the value of the equity (that is, the EPV of the profit flow which shareholders will collect before default);
(c) the value of the debt assuming that in the event of default, the debt holders will get nothing;
(d) the value of the debt assuming that in the event of default, the debt holders recover the fraction α of the EPV of the revenue stream of the firm;
(e) the value of the firm, which is the sum of the value of equity and value of debt, in Cases (c) and (d).

5.6. Calculate the value of the investment opportunity into the firm described in Prob. 5.4. Assume that the lenders are competitive so that their expected profit is 0.

The following three problems have two versions: uncertainty is on the demand side, and uncertainty is on the supply side.

5.7. Assume that the operational profits are taxed at rate $\tau_\Pi > 0$. Solve the exit problem and study how the exit threshold and firm's value depend on the the tax rate.

5.8. Assume that the operational profits are taxed at rate $\tau_\Pi > 0$. Solve the entry problem and study how the entry threshold and the value of the investment opportunity depend on the the tax rate.

5.9. Assume that the operational profits are taxed at rate $\tau_\Pi > 0$ below $\bar{\Pi} > 0$, and a rate $\tau'_\Pi > \tau_\Pi$ above $\bar{\Pi}$. Show that if $\bar{\Pi}$ is sufficiently large, then the entry threshold is independent of the higher tax rate, but the value of investment opportunity depends on it.

Show that in the exit problem, both the exit threshold and firm's value depend on the higher tax rate.

6

General random walks on $\mathbb{Z}$: Option pricing

6.1 Wiener–Hopf factorization

6.1.1 Three forms of the Wiener–Hopf factorization

Let T be the geometric random variable on $\mathbb{Z}_+$ with $\mathrm{Prob}(T = t) = (1-q)q^t$. The Wiener–Hopf factorization formula states that for z on the unit circle in the complex plane $\mathbb{C}$,

$$E[z^{X_T}] = E[z^{\bar{X}_T}]E[z^{\underline{X}_T}]. \tag{6.1}$$

Equation (6.1) follows from:

- $X_T = \bar{X}_T + X_T - \bar{X}_T$;
- $\bar{X}_T$ and $X_T - \bar{X}_T$ are independent;
- the characteristic function of the sum of two independent random variables is the product of the characteristic functions;
- probability distributions of $\underline{X}_T$ and $X_T - \bar{X}_T$ are the same.

See [77], Sect. I.29, and the references therein. Introduce the notation

$$P(z) = \sum_{j=-\infty}^{+\infty} p_j z^j, \tag{6.2}$$

$$\kappa_q^+(z) = (1-q)E\left[\sum_{t=0}^{\infty} q^t z^{\bar{X}_t}\right], \tag{6.3}$$

$$\kappa_q^-(z) = (1-q)E\left[\sum_{t=0}^{\infty} q^t z^{\underline{X}_t}\right]. \tag{6.4}$$

The LHS in (6.1) being $(1-q)/(1-qP(z))$, we can write the Wiener-Hopf factorization formula in an equivalent form

$$\frac{1-q}{1-qP(z)} = \kappa_q^+(z)\kappa_q^-(z). \tag{6.5}$$

To obtain the third form, *define* the EPV operators $\mathcal{E}^{\pm}$ by (5.26)–(5.27), and, assuming that X starts at 0, introduce random variables $Y^+ = \bar{X}_T$ and $Y^- = X_T - \bar{X}_T \sim \underline{X}_T$ on $\mathbb{Z}_+$ and $\mathbb{Z}_-$, respectively. Then

$$\mathcal{E}^+ g(x) = E[g(x + Y^+)], \quad \mathcal{E}^- g(x) = E[g(x + Y^-)]. \tag{6.6}$$

Apply $\mathcal{E}$ and the product of operators $\mathcal{E}^{\pm}$ to a function $g(x)$ of the form $g(x) = g(z; x) = z^x$, where $z \in \mathbb{C}, |z| = 1$. Assuming that X_t starts at 0, we have

$$(\mathcal{E} z^{\cdot})(x) = (1-q)E\left[\sum_{t=0}^{\infty} q^t z^{x+X_t}\right] = z^x (1-q)E\left[\sum_{t=0}^{\infty} q^t z^{X_t}\right]$$
$$(\mathcal{E}^+ z^{\cdot})(x) = (1-q)E\left[\sum_{t=0}^{\infty} q^t z^{x+\bar{X}_t}\right] = z^x (1-q)E\left[\sum_{t=0}^{\infty} q^t z^{\bar{X}_t}\right]$$
$$(\mathcal{E}^- z^{\cdot})(x) = (1-q)E\left[\sum_{t=0}^{\infty} q^t z^{x+\underline{X}_t}\right] = z^x (1-q)E\left[\sum_{t=0}^{\infty} q^t z^{\underline{X}_t}\right],$$

which gives

$$(\mathcal{E} z^{\cdot})(x) = \frac{1-q}{1-qP(z)} z^x, \tag{6.7}$$
$$(\mathcal{E}^+ z^{\cdot})(x) = \kappa_q^+(z) z^x, \tag{6.8}$$
$$(\mathcal{E}^- z^{\cdot})(x) = \kappa_q^-(z) z^x. \tag{6.9}$$

Using (6.5), (6.7), (6.8) and (6.9), we obtain

$$\mathcal{E} g = \mathcal{E}^- \mathcal{E}^+ g = \mathcal{E}^+ \mathcal{E}^- g. \tag{6.10}$$

To show that (6.10) holds for $g \in \mathcal{L}_\infty(\mathbb{Z})$ and for g from wider classes of functions, note that

$$\begin{aligned} E[g(x + X_T)] &= E[g(x + \bar{X}_T + X_T - \bar{X}_T)] \\ &= E[g(x + Y^+ + Y^-)] = (\mathcal{E}^+ g)(x + Y^-) = (\mathcal{E}^+ \mathcal{E}^- g)(x), \end{aligned}$$

which gives $\mathcal{E} g = \mathcal{E}^+ \mathcal{E}^- g$. The second equality in (6.10) is proved similarly. Thus, we have the operator form of the Wiener–Hopf factorization

$$\mathcal{E} = \mathcal{E}^+ \mathcal{E}^- = \mathcal{E}^- \mathcal{E}^+, \tag{6.11}$$

where each operator is understood as an operator in $\mathcal{L}_\infty(\mathbb{Z})$ (or in a wider function space).

6.1.2 Uniqueness of the Wiener–Hopf factorization

There exist general analytical formulas for $\kappa_q^{\pm}(z)$ in terms of the transition probabilities $p_t(j) = \text{Prob}\,(X_t = j \mid X_0 = 0)$:

$$\kappa_q^+(z) = \exp\left[\sum_{t=1}^{\infty} \frac{q^t}{t} \sum_{0<j<\infty} (z^j - 1)p_t(j)\right], \tag{6.12}$$

$$\kappa_q^-(z) = \exp\left[\sum_{t=1}^{\infty} \frac{q^t}{t} \sum_{-\infty<j<0} (z^j - 1)p_t(j)\right]. \tag{6.13}$$

See, e.g., [49] p.72. Formulas (6.12)–(6.13) are rather involved. Fortunately, the following general result allows one to guess explicit formulas for $\kappa_q^{\pm}(z)$ without calculating the double sums in (6.12)–(6.13). Recall that a function is called analytic in an open subset U of $\mathbb{C}$, if it is differentiable at each point of U. Following [49], we will say that a function is analytic in the closure of an open set $U \in \mathbb{C}$ if it is continuous on the closure U and analytic in U.

Lemma 6.1.1 *Let f be a continuous function on the unit circle $\{z \mid |z| = 1\}$ that admits a factorization*

$$f(z) = f_+(z)f_-(z), \ \forall\ |z| = 1, \tag{6.14}$$

where

- *f_+ is analytic in $\{z \mid |z| \le 1\}$, and $1/f_+$ is bounded there;*
- *f_- is analytic in $\{z \mid |z| \ge 1\}$, and f_- and $1/f_-$ are bounded there;*
- *$f_+(1) = f_-(1) = 1$.*

Let

$$f(z) = f_{1,+}(z)f_{1,-}(z), \ \forall\ |z| = 1, \tag{6.15}$$

be another factorization with the same properties.

Then $f_{1,\pm} = f_\pm$.

Proof. Dividing (6.14) by (6.15) and rearranging, we obtain

$$\frac{f_+(z)}{f_{1,+}(z)} = \frac{f_{1,-}(z)}{f_-(z)}, \quad |z| = 1.$$

The LHS (resp., the RHS) is analytic and bounded on $\{z \mid |z| \le 1\}$ (resp., on $\{z \mid |z| \ge 1\}$), therefore, we can define a continuous bounded function on $\mathbb{C}$, call it F, by the LHS on the unit disc, and by the RHS on the exterior of the unit disc. F is analytic in $C \setminus \{z \mid |z| = 1\}$, hence, by Morera's theorem, F is constant. Since $F(1) = 1$, $F(z) = 1$ for all z.

It is evident from (6.12)–(6.13) that $\kappa_q^+(z)$ admits the analytic continuation to the unit disc $\{z \mid |z| \le 1\}$, and $\kappa_q^-(z)$ admits the analytic continuation to $\{z \mid |z| \ge 1\}$. In addition, $\kappa_q^+(z)$ and $1/\kappa_q^+(z)$ are bounded in $\{z \mid |z| \le 1\}$, and $\kappa_q^-(z)$ and $1/\kappa_q^-(z)$ are bounded in $\{z \mid |z| \ge 1\}$. Finally, $\kappa_q^{\pm}(1) = 1$. Thus, the Wiener–Hopf factorization (6.5) satisfies the conditions of Lemma 6.1.1. Therefore, if we guess a factorization of $1/a(z) := (1-q)/(1-qP(z))$ with the same properties, the factors will be $\kappa_q^+(z)$ and $\kappa_q^-(z)$. For the binomial and trinomial models, we derived the factorization (4.24) with $\kappa_q^\pm(z)$ given by (4.21)–(4.22). These $\kappa_q^\pm(z)$ satisfy all conditions of Lemma 6.1.1, therefore, they are identical with $\kappa_q^\pm(z)$ defined by (6.3)–(6.3). Now, the argument about the equivalence of the two forms (6.5) and (6.11) shows that the operators $\mathcal{E}^\pm$ defined in Sect. 4.3 by (4.11)–(4.12) and operators defined by (5.26)–(5.27) are identical. (It suffices to check that their actions on functions of the form $g(x) = z^x$ are identical; but this is the statement about the functions $\kappa_q^\pm(z)$).

6.2 Properties of EPV operators $\mathcal{E}^+$ and $\mathcal{E}^-$

6.2.1 Explicit formulas for $\mathcal{E}^+$ and $\mathcal{E}^-$

For calculations in applications, it is necessary to obtain computationally effective formulas for the action of $\mathcal{E}^\pm$. If explicit formulas for $\kappa_q^\pm(z)$ are available, we can use the representation $\mathcal{E}^\pm = \kappa_q^\pm(S)$ (for the proof, it suffices to apply the Fourier transform). General formulas for $\kappa_q^\pm(z)$ involving integration can be found in, e.g., [14]. If $P(z)$ (hence, $1 - qP(z)$) is a rational function, the representations $\mathcal{E}^+ = \sum_{j\ge 0} \mu_j^+ S^j$ and $\mathcal{E}^- = \sum_{j\le 0} \mu_j^- S^j$ can be calculated quite easily. We demonstrate the calculation for the following random walk, which is an analog of Kou's jump-diffusion model [50] as the binomial model is an analog of the Brownian motion. Take $p_\pm > 0$, $p_0 \ge 0$, $c_\pm > 0$, and $\lambda_\pm > 1$, and consider the transition operator of the form

$$P = p_0 + p_+ S + p_- S^{-1} + c_+ \sum_{j\ge 0} \lambda_+^{-j} S^j + c_- \sum_{j\ge 0} \lambda_-^{-j} S^{-j}. \tag{6.16}$$

The requirement $\sum_j p_j = 1$ is equivalent to

$$p_+ + p_- + p_0 + c_+\lambda^+/(\lambda^+ - 1) + c_-\lambda^-/(\lambda^- - 1) = 1.$$

We have

$$P(z) = p_0 + p_+ z + p_- z^{-1} + \frac{c_+}{1 - z/\lambda_+} + \frac{c_-}{1 - z^{-1}/\lambda_-}, \tag{6.17}$$

and therefore, $1 - qP(z)$ is the ratio of a polynomial of degree 4, and the one of degree 3. To calculate $\kappa_q^\pm(z)$, it suffices to

(1) factorize the numerator and denominator into products of factors of the form $b_+ - z$, where $|b_+| > 1$ (these factors have no zeroes in the unit disc $\{z \mid |z| \le 1\}$), and factors of the form $b_- - z^{-1}$, where $|b_-| > 1$ (these factors have no zeroes in $\{z \mid |z| \ge 1\}$);
(2) collect the factors that have no zeroes in the unit disc; this gives $1/\kappa_q^+(z)$, up to a constant factor;
(3) collect the factors that have no zeroes in $\{z \mid |z| \ge 1\}$; this gives $1/\kappa_q^-(z)$, up to a constant factor;

(4) take the reciprocals and normalize so that $\kappa_q^+(1) = \kappa_q^-(1) = 1$.

The proof that we obtain $\kappa_q^\pm$ is the same as for the binomial and trinomial models at the end of Subsect. 6.1.2.

For the model given by (6.17), the realization of Steps (1)–(4) is quite straightforward. The roots of the denominator of $(1-qP(z))/(1-q)$ are $0, \lambda_+$ and $1/\lambda_-$, and the numerator has 4 real roots, one root on each of the intervals $(0, 1/\lambda_-)$, $(1/\lambda_-, 1)$, $(1, \lambda_+)$, $(\lambda_+, +\infty)$. To see this, it suffices to recall that $1 - qP(1) = 1 - q > 0$, and

$$\begin{aligned}
&1 - qP(z) \to -\infty \text{ as } z \to +0,\ z \to \lambda_+ - 0,\ z \to 1/\lambda_- + 0,\ z \to +\infty,\\
&1 - qP(z) \to +\infty \text{ as }\ z \to \lambda_+ + 0,\quad z \to 1/\lambda_- - 0.
\end{aligned}$$

See Fig. 6.1. Denote these roots $q_{-,2}, q_{-,1}, 1/q_{+,1}$ and $1/q_{+,2}$, respectively, and

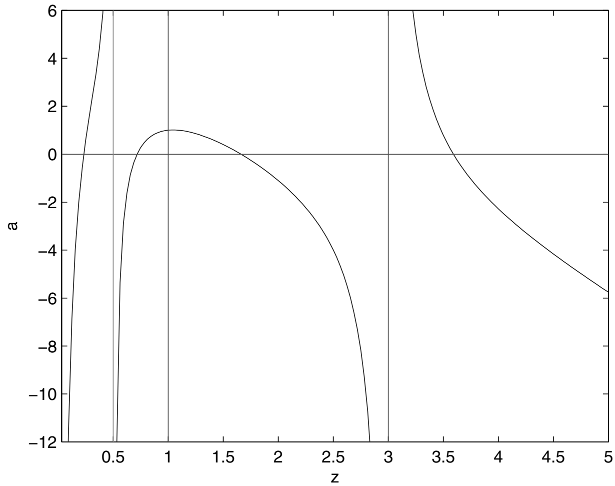

Fig. 6.1. Graph of $a(z)$. Parameters: $q = 0.9, p_0 = 0.2, p_+ = 0.3, p_- = 0.2$, $c_+ = 0.067$, $c_- = 0.1$, $\lambda^+ = 3$, $\lambda^- = 2$.

notice that if condition (4.16) holds for some $\gamma \geq 1$, then

$$q_{-,2} < 1/\lambda_- < q_{-,1} < 1 \leq \gamma < 1/q_{+,1} < \lambda_+ < 1/q_{+,2}. \tag{6.18}$$

It follows that $a(z) = (1-q)^{-1}(1 - qP(z))$ admits the factorization

$$a(z) = a^+(z)a^-(z), \tag{6.19}$$

where

$$a^+(z) = \frac{(\lambda_+ - 1)(1 - q_{+,1}z)(1 - q_{+,2}z)}{(\lambda_+ - z)(1 - q_{+,1})(1 - q_{+,2})},$$
$$a^-(z) = \frac{(\lambda_- - 1)(1 - q_{-,1}z^{-1})(1 - q_{-,2}z^{-1})}{(\lambda_- - z^{-1})(1 - q_{-,1})(1 - q_{-,2})}.$$

For computations, it is convenient to write $\kappa_q^{\pm}(z) := (1/a^{\pm}(z))^{-1}$ as

$$\kappa_q^+(z) = \sum_{j=1,2} a_j^+ \frac{1 - q_{+,j}}{1 - q_{+,j}z}, \quad \kappa_q^-(z) = \sum_{j=1,2} a_j^- \frac{1 - q_{-,j}}{1 - q_{-,j}z^{-1}},$$

where

$$a_1^+ = \frac{(1 - q_{+,2})(\lambda_+ q_{+,1} - 1)}{(\lambda_+ - 1)(q_{+,1} - q_{+,2})}, \quad a_2^+ = \frac{(1 - q_{+,1})(\lambda_+ q_{+,2} - 1)}{(\lambda_+ - 1)(q_{+,2} - q_{+,1})},$$
$$a_1^- = \frac{(1 - q_{-,2})(\lambda_- q_{-,1} - 1)}{(\lambda_- - 1)(q_{-,1} - q_{-,2})}, \quad a_2^- = \frac{(1 - q_{-,1})(\lambda_- q_{-,2} - 1)}{(\lambda_- - 1)(q_{-,2} - q_{-,1})}$$

are positive. Hence,

$$\mathcal{E}^+ = \kappa_q^+(S) = \sum_{k=1,2} a_{+,k}(1 - q_{+,k}) \sum_{j=0}^{\infty} q_{+,k}^j S^j,$$
$$\mathcal{E}^- = \kappa_q^-(S) = \sum_{k=1,2} a_{-,k}(1 - q_{-,k}) \sum_{j=0}^{\infty} q_{-,k}^j S^{-j},$$

and therefore,

$$(\mathcal{E}^+ u)(x) = \sum_{k=1,2} a_{+,k}(1 - q_{+,k}) \sum_{j=0}^{\infty} q_{+,k}^j u(x + j), \tag{6.20}$$
$$(\mathcal{E}^- u)(x) = \sum_{k=1,2} a_{-,k}(1 - q_{-,k}) \sum_{j=0}^{\infty} q_{-,k}^j u(x - j). \tag{6.21}$$

6.2.2 Action in $l_\infty(\mathbb{Z})$

In the following lemma, “monotone" means “increasing", “non-decreasing", “decreasing" or “non-increasing".

Proposition 6.2.1 *The operators $\mathcal{E}^{\pm}$ enjoy the following properties:*

(a) If $g(x) = 0 \ \forall\, x \geq h$, then for the same x, $(\mathcal{E}^+ g)(x) = 0$.
(b) If $g(x) = 0 \ \forall\, x \leq h$, then for the same x, $(\mathcal{E}^- g)(x) = 0$.
(c) If $g(x) \geq 0 \ \forall\, x$, then $(\mathcal{E}^+ g)(x) \geq 0, \ \forall\, x$.
(d) If $g(x) \geq 0 \ \forall\, x$, then $(\mathcal{E}^- g)(x) \geq 0, \ \forall\, x$.
(e) If g is monotone, then $\mathcal{E}^+ g$ and $\mathcal{E}^- g$ are also monotone.

Proof. Since $\mathcal{E}^+g(x) = E[g(x+Y^+)]$ and $\mathcal{E}^-g(x) = E[g(x+Y^-)]$, where $Y^\pm$ is a random variable assuming values in $\mathbb{Z}_\pm$, all properties are immediate.

Proposition 6.2.2 *Operators $\mathcal{E}^+$ and $\mathcal{E}^-$ are invertible operators in $l_\infty(\mathbb{Z})$, with the bounded inverses*

$$(\mathcal{E}^+)^{-1} = (1-q)^{-1}\mathcal{E}^-(I-qP) = (1-q)^{-1}(I-qP)\mathcal{E}^-, \qquad (6.22)$$
$$(\mathcal{E}^-)^{-1} = (1-q)^{-1}\mathcal{E}^+(I-qP) = (1-q)^{-1}(I-qP)\mathcal{E}^+. \qquad (6.23)$$

Proof. In Sect. 4.3, we showed that $\mathcal{E}$ is the bounded inverse to the operator $(1-q)^{-1}(I-qP)$:

$$\mathcal{E} = (1-q)(I-qP)^{-1}. \qquad (6.24)$$

Using (6.24) and the Wiener-Hopf factorization formula (6.11), we obtain

$$\mathcal{E}^+\mathcal{E}^-(1-q)^{-1}(I-qP) = I = (1-q)^{-1}(I-qP)\mathcal{E}^-\mathcal{E}^+,$$

which means that $(1-q)^{-1}(I-qP)\mathcal{E}^-$ is the left inverse to $\mathcal{E}^+$, and $\mathcal{E}^-(1-q)^{-1}(I-qP)$ is the right one. Hence, $\mathcal{E}^+$ is invertible. Since an inverse is unique, we have (6.22). The statement about $\mathcal{E}^-$ is proved similarly.

Proposition 6.2.3 *Operators $(\mathcal{E}^\pm)^{-1}$ enjoy the following properties:*

(a) if $g(x) = 0\ \forall\ x \ge h$, then, for the same x, $((\mathcal{E}^+)^{-1}g)(x) = 0$;
(b) if $g(x) = 0\ \forall\ x \le h$, then, for the same x, $((\mathcal{E}^-)^{-1}g)(x) = 0$;
(c) $\mathcal{E}^\pm$ and $(\mathcal{E}^\pm)^{-1}$ are mutual inverses as operators in $l_\infty(\mathbb{Z}_\pm)$.

Proof. We can identify S with an infinite matrix $\sum_{j\in\mathbb{Z}} e_{j,j+1}$, where $e_{j,k}$ is the matrix with the only non-zero element, 1, in row j and column k. Then $\mathcal{E}^+$ is an infinite matrix of the form $\mathcal{E}^+ = b_+I + K_+$, where $b_+ > 0$ and all elements of K_+ at the diagonal and below it are zeroes. The inverse of such a matrix is of the same form: $(\mathcal{E}^+)^{-1} = (b_+)^{-1} + \sum_{j=1}^{+\infty}\mu_j S^j$, and (a) is immediate. Part (b) is proved similarly.

(c) $\mathcal{E}^\pm$ and $(\mathcal{E}^\pm)^{-1}$ are mutual inverses as operators in $l_\infty(\mathbb{Z})$, and, by (a)–(b), both map $l_\infty(\mathbb{Z}_\pm)$ into itself.

Recall that in Chap. 5, to prove the existence of the optimal exercise boundary, we needed to know that the function $\mathcal{E}^+g$ (or $\mathcal{E}^-g$, depending on a situation) changed sign. The next proposition gives sufficient conditions in the case of general random walks and payoff functions.

Proposition 6.2.4 *a) If $g(-\infty) < 0$, then $\mathcal{E}g(-\infty) < 0$ and $\mathcal{E}^\pm g(-\infty) < 0$;*
b) If $g(-\infty) > 0$, then $\mathcal{E}g(-\infty) > 0$ and $\mathcal{E}^\pm g(-\infty) > 0$;
c) statements a)–b) hold with $+\infty$ instead of $-\infty$.

Proof. (a) Without loss of generality, we may assume that $|g(x)| \le 1$, $\forall\, x \in \mathbb{Z}$. If $g(-\infty) < 0$, then there exist $N > 0$ and $c > 0$ such that $g(x) < -c$ for all $x < -N$. For given N, $c \in (0,1)$ and any s, there exists N_1 such that for any $x < -N_1$, $\text{Prob}(\bar{X}_s > -N \mid X_0 = x) < c/2$. Therefore, for these x,

$$\begin{aligned}
\mathcal{E}^+ g(x) &= E^x \left[(1-q) \sum_{t=0}^{s} q^t g(\bar{X}_t) + (1-q) \sum_{t=s+1}^{\infty} q^t g(\bar{X}_t) \right] \\
&\le \text{Prob}\,(\bar{X}_s \le -N \mid X_0 = x)(1-q) \sum_{t=0}^{s} (-c) q^t \\
&\quad + \text{Prob}\,(\bar{X}_s \le -N \mid X_0 = x)(1-q) \sum_{t=0}^{s} q^t + (1-q) \sum_{t=s+1}^{\infty} q^t \\
&\le -(1-c/2)c(1-q) \sum_{t=0}^{s} q^t + \frac{c(1-q)}{2} \sum_{t=0}^{s} q^t + (1-q) \sum_{t=s+1}^{\infty} q^t \\
&= -\frac{c(1-c)}{2} + \left(1 + \frac{c(1-c)}{2}\right)(1-q) \sum_{t=s+1}^{\infty} q^t,
\end{aligned}$$

and if s is sufficiently large, then the RHS is negative. It follows that $\mathcal{E}^+ g(x) < 0$ in a neighborhood of $-\infty$. The proof for $\mathcal{E}g(x)$ is the same because $X_t \le \bar{X}_t$, and for $\mathcal{E}^- g(x)$, the result is evident, because the sample paths of the infimum process are not increasing.

(b) is (a) for $-g$, and (c)–(d) are the mirror reflections of (a)–(b).

6.2.3 The case of payoffs exponentially growing at infinity

We formulated Propositions 6.2.1–6.2.4 for bounded functions g. These propositions can be extended for g growing at $+\infty$ and/or $-\infty$, if we impose the following related conditions on the random walk and a function g: there exist $0 < \gamma_- \le 1 \le \gamma_+$ and $C, c > 0$ such that

$$1 - qP(\gamma) > 0 \quad \forall\, \gamma \in [\gamma_-, \gamma_+] \tag{6.25}$$

(this presumes that $P(\gamma)$ is finite for all $\gamma \in [\gamma_-, \gamma_+]$), and

$$|g(x)| \le C(\gamma_-^x + \gamma_+^x), \quad \forall\, x \in \mathbb{Z}. \tag{6.26}$$

The spaces $l_\infty(\mathbb{Z})$ and $l_\infty(\mathbb{Z}_\pm)$ must be replaced with the spaces

- $l_\infty(\gamma_-, \gamma_+; \mathbb{Z})$, which consists of functions having finite norm
$$||u||_{\infty;\gamma_-,\gamma_+} = \sup_{x\in\mathbb{Z}} \left(\gamma_-^x + \gamma_+^x\right)^{-1} |u(x)|; \tag{6.27}$$
- $l_\infty(\gamma_+; \mathbb{Z}_+)$, which consists of functions vanishing below 0 and having finite norm
$$||u||_{\infty;\gamma_+} = \sup_{x\in\mathbb{Z}_+} \gamma_+^{-x} |u(x)|; \tag{6.28}$$
- $l_\infty(\gamma_-; \mathbb{Z}_-)$, which consists of functions vanishing above 0 and having finite norm
$$||u||_{\infty;\gamma_-} = \sup_{x\in\mathbb{Z}_-} \gamma_-^{-x} |u(x)|. \tag{6.29}$$

The exact statements follow. The reader who is not interested in the technical regularity issues can safely skip their proofs.

Lemma 6.2.5 *Let (6.25) hold. Then*

(a) $P(z)$ *is well-defined and analytic in the annulus* $\{z \mid \gamma^- \le |z| \le \gamma_+\}$*; moreover, both* $1 - qP(z)$ *and* $1/(1 - qP(z))$ *are uniformly bounded on this annulus;*
(b) $\kappa_q^+(z)$ *admits the analytic continuation to* $\{z \mid |z| \le \gamma_+\}$*, and does not vanish there; hence,* $\kappa_q^+(z)$ *and* $1/\kappa_q^+(z)$ *are bounded on* $\{z \mid |z| \le \gamma_+\}$*;*
(c) $\kappa_q^-(z)$ *admits the analytic continuation to* $\{z \mid |z| \ge \gamma_-\}$*, and does not vanish there; moreover, both* $\kappa_q^-(z)$ *and* $1/\kappa_q^-(z)$ *are bounded on* $\{z \mid |z| \ge \gamma_-\}$*;*
(d) the Wiener–Hopf factorization formula (6.5) holds on the annulus $\{z \mid \gamma_- \le |z| \le \gamma_+\}$.

Proof. (a) follows from the bound $|P(z)| \le P(|z|))$. (b) From (6.5), for $|z| = 1$,

$$\kappa_q^+(z) = \frac{1-q}{1-qP(z)} \times \frac{1}{\kappa_q^-(z)}. \tag{6.30}$$

Under condition (6.25), the first fraction on the RHS is analytic in the annulus $\{z \mid \gamma_- \le |z| \le \gamma_+\}$, whereas the second one is analytic in $\{z \mid |z| \ge 1\}$. Moreover, both fractions and their reciprocals are bounded on the corresponding sets. Hence, we can use (6.30) to define the analytic extension of $\kappa_q^+(z)$ to the annulus $\{z \mid 1 \le |z| \le \gamma_+\}$. This proves (b). Part (c) is proved similarly, and (d) follows from (6.5), (6.25) and (b)–(c).

Lemma 6.2.6 *Let (6.25) hold. Then*

(a) if g *satisfies the bound (6.26), then* $(I - qP)g$ *and* $\mathcal{E}g$ *satisfy the same bound (with different constants* C*);*
(b) operators $A := (1-q)^{-1}(I - qP)$ *and* $\mathcal{E}$ *are mutual inverses as operators in* $l_\infty(\gamma_-, \gamma_+; \mathbb{Z})$*;*
(c) if g *vanishes below 0 and satisfies*

$$|g(x)| \le C\gamma^x, \quad \forall\, x, \tag{6.31}$$

where $\gamma \le \gamma_+$*, then* $\mathcal{E}^+ g$ *and* $(\mathcal{E}^+)^{-1} g$ *satisfy the same two conditions;*
(d) if g *vanishes above 0 and satisfies (6.31) with* $\gamma \ge \gamma_-$*, then* $\mathcal{E}^- g$ *and* $(\mathcal{E}^-)^{-1} g$ *satisfy the same two conditions;*
(e) for any $\gamma \le \gamma_+$*,* $\mathcal{E}^+$ *and* $(\mathcal{E}^+)^{-1}$ *are mutual inverses as operators in* $l_\infty(\gamma; \mathbb{Z}_+)$*;*
(f) for any $\gamma \ge \gamma_-$*,* $\mathcal{E}^-$ *and* $(\mathcal{E}^-)^{-1}$ *are mutual inverses as operators in* $l_\infty(\gamma; \mathbb{Z}_-)$*.*
(g) the Wiener-Hopf factorization formula (6.11) is valid with $\mathcal{E}$ *and* $\mathcal{E}^\pm$ *acting in* $l_\infty(\gamma_-, \gamma_+; \mathbb{Z})$*.*

Proof. (a) Clearly, $u \in l_\infty(\gamma_-, \gamma_+; \mathbb{Z})$ iff u belongs to the the intersection of spaces $l_\infty(\gamma_+; \mathbb{Z})$ and $l_\infty(\gamma_-; \mathbb{Z})$. Moreover, the norm $|| \cdot ||_{\gamma_-,\gamma_+;l_\infty(\mathbb{Z})}$ is equivalent to the norm $||u||'_{\infty;\gamma_-,\gamma_+} = \max\{||u||_{\infty;\gamma_+},\ ||u||_{\infty;\gamma_-}\}$ of the intersection, that is, there exist positive constants C, c such that for any $u \in l_\infty(\gamma_-, \gamma_+; \mathbb{Z})$,

$$c||u||'_{\infty;\gamma_-,\gamma_+} \le ||u||_{\infty;\gamma_-,\gamma_+} \le C||u||'_{\infty;\gamma_-,\gamma_+}.$$

Therefore, it suffices to prove that P and $\mathcal{E}$ are bounded operators in $l_\infty(\gamma; \mathbb{Z})$, for any $\gamma \in [\gamma_-, \gamma_+]$. We have

$$\begin{aligned} ||Pu||_{\infty;\gamma} &= \sup_{x\in\mathbb{Z}} \left| \gamma^{-x} \sum_{j\in\mathbb{Z}} p_j u(x+j) \right| \le \sup_{x\in\mathbb{Z}} \gamma^{-x} \sum_{j\in\mathbb{Z}} p_j |u(x+j)| \\ &\le \sup_{x\in\mathbb{Z}} \gamma^{-x} \sum_{j\in\mathbb{Z}} p_j \gamma^{x+j} ||u||_{\infty;\gamma} = \sum_{j\in\mathbb{Z}} p_j \gamma^j ||u||_{\infty;\gamma} = P(\gamma)||u||_{\infty;\gamma}, \end{aligned}$$

which proves that P is bounded, with the norm less than or equal to $P(\gamma)$. For $u(x) = \gamma^x$, the above inequalities are equalities, hence, the norm equals $P(\gamma)$. Since

$$\mathcal{E}u(x) = (1-q) \sum_{j\in\mathbb{Z}} q^j P^j u(x),$$

we obtain

$$||\mathcal{E}u||_{\infty;\gamma} \le (1-q) \sum_{j\in\mathbb{Z}} q^j P(\gamma)^j ||u||_{\infty;\gamma}.$$

Under condition (6.25), the series above converges:

$$(1-q) \sum_{t\ge 0} q^j P(\gamma)^j = (1-q)/(1-qP(\gamma)),$$

hence, $\mathcal{E}$ is bounded.

(b) is proved as in the case $l_\infty(\mathbb{Z})$ (see Sect. 4.3), the estimate for the norms obtained in part (a) being used.

(c) and (d) are proved exactly as for $g \in l_\infty(\mathbb{Z})$.

(e) Let $\gamma \le \gamma_+$ and $g \in l_\infty(\gamma; \mathbb{Z})$. Define $v(x) = \gamma^{-x} u(x)$. Then $||v||_{l_\infty(\mathbb{Z})} = ||u||_{\infty;\gamma}$, and

$$\begin{aligned} ||\mathcal{E}^+ g||_{\infty;\gamma} &= \sup_{x\in\mathbb{Z}} |\gamma^{-x} \mathcal{E}^+ g(x)| = \sup_{x\in\mathbb{Z}} |\gamma^{-x} \mathcal{E}^+ \gamma^x v(x)| \le ||v||_{l_\infty(\mathbb{Z})} \\ &\le \sup_{x\in\mathbb{Z}} |\gamma^{-x} \mathcal{E}^+ \gamma^x| = ||v||_{l_\infty(\mathbb{Z})} \kappa_q^+(\gamma). \end{aligned}$$

In view of (c), this estimate proves the boundedness of $\mathcal{E}^+$ as an operator in $l_\infty(\gamma; \mathbb{Z})$. Since $1/\kappa_q^+(z)$ is bounded on the disc $\{z \mid |z| \le \gamma_+\}$, we can replace in the above estimate $\mathcal{E}^+$ and $\kappa_q^+(\gamma)$ with $(\mathcal{E}^+)^{-1}$ and $1/\kappa_q^+(\gamma)$, and conclude that $(\mathcal{E}^+)^{-1}$ is bounded as an operator in $l_\infty(\gamma; \mathbb{Z})$. Finally, for any z in the disc $|z| \le \gamma_+$,

$$(\mathcal{E}^+)^{-1}\mathcal{E}^+ z^\cdot(x) = \mathcal{E}^+(\mathcal{E}^+)^{-1}z^\cdot(x) = \kappa_q^+(z)(1/\kappa_q^+(z))z^x = z^x,$$

therefore $(\mathcal{E}^+)^{-1}$ and $\mathcal{E}^+$ are mutual inverses as operators in $l_\infty(\gamma;\mathbb{Z}_+)$.

(f) is proved similarly.

(g) If g satisfies (6.26), then both sides of the equality

$$E[g(x + X_T)] = E[g(x + Y^+ + Y^-)] \tag{6.32}$$

are well-defined, and (6.32) holds since X_T and $Y^+ + Y^-$ are the same in law.

6.3 EPVs of a stream and instantaneous payoff that are acquired or lost at a random time

In Chap. 5, we solved several optimal stopping problems in the binomial and trinomial model. The first step was the calculation of the EPV of a stream or instantaneous payoff that was acquired or lost when a certain boundary fixed in advance had been reached or crossed. At the second step, the boundary was chosen to maximize the option value. Since there exist problems with the exit or entry thresholds given exogenously, an example being the bankruptcy specified by the debt covenants, we start with the list of main theorems for the case of an exogenously given boundary. The standing assumption about the random walk is (6.25), where $0 < \gamma_- \le 1 \le \gamma_+$, and about the stream - (6.26). When we consider an instantaneous payoff $G(X_t)$, the standing assumption is weaker than (6.26). If the payoff $G(X_t)$ is due when a certain boundary is crossed from below, then $G(x)$ must grow not too fast as $x \to +\infty$:

$$|G(x)| \le C\gamma_+^x, \quad \forall\, x \in \mathbb{Z}_+; \tag{6.33}$$

if the payoff is due when a certain boundary is crossed from above, then the bound is imposed in a neighborhood of $-\infty$:

$$|G(x)| \le C\gamma_-^x, \quad \forall\, x \in \mathbb{Z}_-. \tag{6.34}$$

The conditions for an instantaneous payoff are weaker because if a stream is acquired then its EPV may depend on values of $g(x)$ at arbitrary large (in modulus) x, whereas for options with an instantaneous payoff $G(X_t)$ only values $G(x)$ for x in the action region matter.

6.3.1 EPV of a stream that is abandoned when the threshold is reached or crossed from above

Denote by τ_h^- the first time X_t reaches $h \in \mathbb{Z}$ or crosses h from above.

Theorem 6.3.1 *Let X_t and g satisfy (6.25) and (6.26), respectively.*

Then the EPV of the stream that is lost when X_t reaches or crosses $h \in \mathbb{Z}$ from above is given by

$$V^-_{\text{loss}}(x;h) = E^x\left[\sum_{t=0}^{\tau^-_h-1} q^t g(X_t)\right] \tag{6.35}$$

$$= (1-q)^{-1}\mathcal{E}^-\mathbf{1}_{(h,+\infty)}\mathcal{E}^+ g(x). \tag{6.36}$$

Note that (6.35) is the definition, and (6.36) is the statement of the theorem.

Proof. This is, essentially, the situation which we considered in the framework of the binomial model in Theorem 5.1.2. Now we allow for a more general random walk than the binomial or trinomial model, and condition on the behavior of the stream at $\pm\infty$ is more general, but the underlying idea of the proof remains the same. However, now we need to use the general definitions (5.26)–(5.27) of the EPV operators $\mathcal{E}^\pm$ in terms of the supremum and infimum processes from the very beginning. Lemma 6.2.6 allows us to reproduce the proof of Theorem 5.1.2 with evident changes. The Bellman equation for $V^-_{\text{loss}}(x;h)$ is

$$V^-_{\text{loss}}(x;h) = g(x) + E^x[V^-_{\text{loss}}(X_1;h)], \quad x > h,$$

equivalently,

$$(I - qP)V^-_{\text{loss}}(x;h) = g(x), \quad x > h,$$

and $V^-_{\text{loss}}(x;h) = 0$ for $x \le h$. Normalize $V^-_{\text{loss}}(x;h)$, that is, introduce $\mathcal{V} = (1-q)V^-_{\text{loss}}$. The normalized value function satisfies

$$(1-q)^{-1}(I - qP)\mathcal{V}(x;h) = g(x),\ x > h, \tag{6.37}$$

$$\mathcal{V}(x;h) = 0, \quad x \le h. \tag{6.38}$$

Set

$$g^-(x) = (1-q)^{-1}(I - qP)\mathcal{V}(x;h) - g(x),$$

and write (6.37) as an equation on $\mathbb{Z}$:

$$(1-q)^{-1}(I - qP)\mathcal{V} = g + g^-,$$

where g^- vanishes on $(h,+\infty)$. Apply $\mathcal{E}^+$ and use the Wiener–Hopf factorization formula (6.11):

$$(\mathcal{E}^-)^{-1}\mathcal{V} = \mathcal{E}^+ g + \mathcal{E}^+ g^-.$$

Lemma 6.2.6 ensures that for $x > h$, $\mathcal{E}^+ g^-(x) = 0$, and for $x \le h$, $(\mathcal{E}^-)^{-1}\mathcal{V}(x;h) = 0$. Therefore, multiplying by $\mathbf{1}_{(h,+\infty)}$, we obtain

$$(\mathcal{E}^-)^{-1}\mathcal{V} = \mathbf{1}_{(h,+\infty)}\mathcal{E}^+ g.$$

Finally, applying $(1-q)^{-1}\mathcal{E}^-$, we arrive at (6.36).

6.3.2 EPV of a stream that is abandoned when the threshold is reached or crossed from below

Denote by τ_h^+ the first time X_t reaches or crosses $h \in \mathbb{Z}$ from below.

Theorem 6.3.2 *Let X_t and g satisfy (6.25) and (6.26), respectively. Then the EPV of the stream that is lost when X_t reaches or crosses $h \in \mathbb{Z}$ from below is given by*

$$V_{\text{loss}}^+(x;h) = E^x\left[\sum_{t=0}^{\tau_h^+-1} q^t g(X_t)\right] \tag{6.39}$$

$$= (1-q)^{-1}\mathcal{E}^+\mathbf{1}_{(-\infty,h)}\mathcal{E}^- g(x). \tag{6.40}$$

The proof is the mirror reflection of the proof of Theorem 6.3.1. We leave it as an exercise for the reader.

6.3.3 EPV of a stream that is acquired when the threshold is reached or crossed from above

Theorem 6.3.3 *Let X_t and g satisfy (6.25) and (6.26), respectively. Then the EPV of the stream that is acquired when X_t reaches or crosses $h \in \mathbb{Z}$ from above is given by*

$$V_{\text{gain}}^-(x;h) = E^x\left[\sum_{t=\tau_h^-}^{\infty} q^t g(X_t)\right] \tag{6.41}$$

$$= (1-q)^{-1}\mathcal{E}^-\mathbf{1}_{(-\infty,h]}\mathcal{E}^+ g(x). \tag{6.42}$$

Proof. We have

$$(1-q)^{-1}\mathcal{E}g(x) = E^x\left[\sum_{t=0}^{\infty} q^t g(X_t)\right] = E^x\left[\sum_{t=0}^{\tau_h^- -1} q^t g(X_t)\right] + E^x\left[\sum_{t=\tau_h^-}^{\infty} q^t g(X_t)\right]$$

By Theorem 6.3.1, the first term on the RHS equals

$$(1-q)^{-1}\mathcal{E}^-\mathbf{1}_{(h,+\infty)}\mathcal{E}^+ g(x),$$

and by the Wiener–Hopf factorization formula, the LHS can be represented as

$$(1-q)^{-1}\mathcal{E}^-(\mathbf{1}_{(-\infty,h]} + \mathbf{1}_{(h,+\infty)})\mathcal{E}^+ g(x).$$

Now (6.42) is immediate.

6.3.4 EPV of a stream that is acquired when threshold is reached or crossed from below

Theorem 6.3.4 *Let X_t and g satisfy (6.25) and (6.26), respectively. Then the EPV of the stream that is acquired when X_t reaches or crosses $h \in \mathbb{Z}$ from below is given by*

$$V^+_{\text{gain}}(x;h) = E^x\left[\sum_{t=\tau_h^+}^{\infty} q^t g(X_t)\right] \tag{6.43}$$

$$= (1-q)^{-1}\mathcal{E}^+\mathbf{1}_{[h,+\infty)}\mathcal{E}^- g(x). \tag{6.44}$$

The proof is the mirror reflection of the proof of Theorem 6.3.3. We leave it as an exercise for the reader.

6.3.5 EPV of an instantaneous payoff that is acquired when the threshold is reached or crossed from above

Theorem 6.3.5 *Let X_t and G satisfy (6.25) and (6.34), respectively. Then the EPV of the payoff $G(X_t)$ that is acquired when X_t reaches or crosses $h \in \mathbb{Z}$ from above is given by*

$$E^x\left[q^{\tau_h^-} G(X_{\tau_h^-})\right] = \mathcal{E}^-\mathbf{1}_{(-\infty,h]}(\mathcal{E}^-)^{-1}G(x). \tag{6.45}$$

Proof. Since only the values $G(x)$ for $x \le h$ matter, we may replace G with G_1, where G_1 coincides with G on $(-\infty, h]$ and is bounded on $\mathbb{Z}_+$. Then $g := (I - qP)G_1$ satisfies (6.26) with the same γ_- and $\gamma_+ = 1$. We have

$$E^x\left[q^{\tau_h^-} G(X_{\tau_h^-})\right] = E^x\left[q^{\tau_h^-} E^{X_{\tau_h^-}}\left[\sum_{s=0}^{\infty} q^s g(X_s)\right]\right] = E^x\left[\sum_{t=\tau_h^-}^{\infty} q^t g(X_t)\right],$$

where the last equality follows from the law of iterated expectations. Applying Theorem 6.3.3, we continue

$$= (1-q)^{-1}\mathcal{E}^-\mathbf{1}_{(-\infty,h]}\mathcal{E}^+ g(x) = \mathcal{E}^-\mathbf{1}_{(-\infty,h]}\mathcal{E}^+(1-q)^{-1}(I-qP)G_1(x),$$

then, using the Wiener-Hopf factorization formula,

$$= \mathcal{E}^-\mathbf{1}_{(-\infty,h]}(\mathcal{E}^-)^{-1}G_1(x).$$

Finally, we use Proposition 6.2.2, which implies that for $x \le h$, $(\mathcal{E}^-)^{-1}G_1(x) = (\mathcal{E}^-)^{-1}G(x)$. Hence,

$$\mathbf{1}_{(-\infty,h]}(\mathcal{E}^-)^{-1}G_1 = \mathbf{1}_{(-\infty,h]}(\mathcal{E}^-)^{-1}G,$$

and (6.45) follows.

6.3.6 EPV of an instantaneous payoff that is acquired when the threshold is reached or crossed from below

Theorem 6.3.6 *Let X_t and G satisfy (6.25) and (6.33), respectively. Then the EPV of the payoff $G(X_t)$ that is acquired when X_t reaches $h \in \mathbb{Z}$ or crosses h from below is given by*

$$E^x \left[q^{\tau_h^+} G(X_{\tau_h^+}) \right] = \mathcal{E}^+ \mathbf{1}_{[h,+\infty)} (\mathcal{E}^+)^{-1} G(x). \tag{6.46}$$

The proof is the mirror reflection of the proof of Theorem 6.3.6. We leave it as an exercise for the reader.

6.4 Main types of options. Optimality in the class of optimal stopping rules of the threshold type

In this Section, we find optimal exercise rules in the class of optimal stopping rules of the threshold type. A stopping rule of the threshold type is a rule of the form: exercise the option the first time a certain boundary is reached or crossed (from above or below, depending on the situation).

6.4.1 Optimal time to abandon an increasing stream

A model example is the exit problem for a firm with uncertainty on the demand side and the profit flow $g(X_t) = G\mathrm{e}^{X_t} - C$.

Theorem 6.4.1 *Assume that*

(i) X_t and g satisfy (6.25) and (6.26), respectively;
(ii) there exists h_* such that $\mathcal{E}^+ g(x) \le 0 \ \forall\, x \le h_*$, and $\mathcal{E}^+ g(x) \ge 0 \ \forall\, x > h_*$.

Then it is optimal to abandon the stream $g(X_t)$ the first time $X_t \le h_*$, and the EPV of the stream with the option to abandon it is

$$V(x) = (1-q)^{-1} \mathcal{E}^- \mathbf{1}_{(h_*,+\infty)} \mathcal{E}^+ g(x). \tag{6.47}$$

Proof. Let h be a candidate for the exercise threshold. Then by (6.36), the option value is

$$V(x;h) = (1-q)^{-1} \mathcal{E}^- \mathbf{1}_{(h,+\infty)} \mathcal{E}^+ g(x).$$

The choice $h = h_*$ replaces all negative values of $\mathcal{E}^+ g$ by zero, and leaves positive ones intact. By Proposition 6.2.1, $\mathcal{E}^-$ is a monotone operator. Hence, h_* is optimal.

Example 6.1. Let $g(x) = G\mathrm{e}^x - C$. Then $\mathcal{E}^+ g(x) = G\kappa_q^+(\mathrm{e})\mathrm{e}^x - C$. Hence, if $\log(C/(G\kappa_q^+(\mathrm{e}))$ is non-integer, then the optimal threshold is unique, and it is $h_* = \max\{h \mid h < \log(C/(G\kappa_q^+(\mathrm{e}))\}$. Otherwise, $\log(C/(G\kappa_q^+(\mathrm{e}))$ and $\log(C/(G\kappa_q^+(\mathrm{e})) - 1$ are optimal.

6.4.2 Optimal time to abandon a decreasing stream

A model example is the exit problem for a firm with uncertainty on the demand side and the profit flow $g(X_t) = R - e^{X_t}$.

Theorem 6.4.2 *Assume that*

(i) X_t *and* g *satisfy (6.25) and (6.26), respectively;*
(ii) there exists h^* *such that* $\mathcal{E}^- g(x) \le 0 \, \forall \, x \ge h^*$, *and* $\mathcal{E}^- g(x) \ge 0 \, \forall \, x < h^*$.

Then it is optimal to abandon the stream $g(X_t)$ *the first time* $X_t \ge h^*$, *and the EPV of the stream with the option to abandon it is*

$$V(x) = (1-q)^{-1} \mathcal{E}^+ \mathbf{1}_{(-\infty, h^*)} \mathcal{E}^- g(x). \tag{6.48}$$

Proof. Let h be a candidate for the exercise threshold. Then by (6.40), the option value is

$$V(x; h) = (1-q)^{-1} \mathcal{E}^+ \mathbf{1}_{(-\infty, h)} \mathcal{E}^- g(x).$$

The choice $h = h^*$ replaces all negative values of $\mathcal{E}^- g$ by zero, and leaves positive ones intact. By Proposition 6.2.1, $\mathcal{E}^+$ is a monotone operator. Hence, h^* is optimal.

Example 6.2. Let $g(x) = C - Ge^x$. Then $\mathcal{E}^- g(x) = C - G\kappa_q^-(e)e^x$. Hence, if $\log(C/(G\kappa_q^-(e))$ is non-integer, then the optimal threshold is unique, and it is $h^* = \min\{h \mid h > \log(C/(G\kappa_q^-(e))\}$. Otherwise, $\log(C/(G\kappa_q^-(e))$ and $\log(C/(G\kappa_q^-(e)) + 1$ are optimal.

6.4.3 Optimal time to acquire an increasing stream

A model example is the irreversible investment with uncertainty on the demand side, the profit flow $g(X_t) = Ge^{X_t} - C$, and zero fixed investment cost. Non-zero fixed investment cost I can be incorporated by assuming that the project is financed by debt, and the firm precommits not to default on the debt obligations. In this case, the following theorem is applicable with $g(X_t) = Ge^{X_t} - C - (1-q)I$.

Theorem 6.4.3 *Assume that*

(i) X_t *and* g *satisfy (6.25) and (6.26), respectively;*
(ii) there exists h^* *such that* $\mathcal{E}^- g(x) \ge 0 \, \forall \, x \ge h^*$, *and* $\mathcal{E}^- g(x) \le 0 \, \forall \, x < h^*$.

Then it is optimal to acquire the stream $g(X_t)$ *the first time* $X_t \ge h^*$, *and the value of the option to acquire the stream is*

$$V(x) = (1-q)^{-1} \mathcal{E}^+ \mathbf{1}_{[h^*, +\infty)} \mathcal{E}^- g(x). \tag{6.49}$$

Proof. Let h be a candidate for the exercise threshold. Then by (6.44), the option value is

$$V(x;h) = (1-q)^{-1}\mathcal{E}^{+}\mathbf{1}_{[h,+\infty)}\mathcal{E}^{-}g(x).$$

The choice $h = h^*$ replaces all negative values of $\mathcal{E}^{-}g$ by zero, and leaves positive ones intact. By Proposition 6.2.1, $\mathcal{E}^{+}$ is a monotone operator. Hence, h^* is optimal.

Example 6.3. Let $g(x) = Ge^x - C$. Then $\mathcal{E}^{-}g(x) = G\kappa_q^{-}(e)e^x - C$. Hence, if $\log(C/(G\kappa_q^{-}(e))$ is non-integer, then the optimal threshold is unique, and it is $h^* = \min\{h \mid h > \log(C/(G\kappa_q^{-}(e))\}$. Otherwise, $\log(C/(G\kappa_q^{-}(e))$ and $\log(C/(G\kappa_q^{-}(e)) + 1$ are optimal.

6.4.4 Optimal time to acquire a decreasing stream

A model example is the investment project with uncertainty on the supply side, the profit flow $g(X_t) = R - e^{X_t}$ and zero fixed investment cost. Non-zero fixed investment cost I can be incorporated by assuming that it is financed by debt, and the firm precommits not to default on the debt obligations. In this case, the following theorem is applicable with $g(X_t) = R - (1-q)I - e^{X_t}$.

Theorem 6.4.4 *Assume that*

(i) X_t and g satisfy (6.25) and (6.26), respectively;
(ii) there exists h_* such that $\mathcal{E}^{+}g(x) \le 0\ \forall\, x \ge h_*$, and $\mathcal{E}^{+}g(x) \ge 0\ \forall\, x < h_*$.

Then it is optimal to acquire the stream $g(X_t)$ the first time $X_t \le h_*$, and the value of the option to acquire the stream is

$$V(x) = (1-q)^{-1}\mathcal{E}^{-}\mathbf{1}_{(-\infty,h_*]}\mathcal{E}^{+}g(x). \tag{6.50}$$

Proof. Let h be a candidate for the exercise threshold. Then by (6.42), the option value is

$$V(x;h) = (1-q)^{-1}\mathcal{E}^{-}\mathbf{1}_{(-\infty,h]}\mathcal{E}^{+}g(x).$$

The choice $h = h_*$ replaces all negative values of $\mathcal{E}^{+}g$ by zero, and leaves positive ones intact. By Proposition 6.2.1, $\mathcal{E}^{-}$ is a monotone operator. Hence, h_* is optimal.

Example 6.4. Let $g(x) = C - Ge^x$. Then $\mathcal{E}^{+}g(x) = C - G\kappa_q^{+}(e)e^x$. Hence, if $\log(C/(G\kappa_q^{+}(e))$ is non-integer, then the optimal threshold is unique, and it is $h_* = \max\{h \mid h < \log(C/(G\kappa_q^{+}(e))\}$. Otherwise, $\log(C/(G\kappa_q^{+}(e))$ and $\log(C/(G\kappa_q^{+}(e)) - 1$ are optimal.

6.4.5 Perpetual call-like American options

Consider an option with the instantaneous payoff $G(X_t)$ which is an increasing function of the underlying stochastic factor. The standard examples are $G(X_t) = X_t - K$ or $G(X_t) = \mathrm{e}^{X_t} - K$; the following theorem is applicable to much wider classes of payoffs.

Theorem 6.4.5 *Assume that*

(i) X_t *and* G *satisfy (6.25) and (6.33), respectively;*
(ii) there exists h^* *such that*
$(\mathcal{E}^+)^{-1}G(x) \geq 0 \ \forall\ x \geq h^*$, *and* $(\mathcal{E}^+)^{-1}G(x) \leq 0 \ \forall\ x < h^*$.

Then it is optimal to exercise the option the first time $X_t \geq h^*$, *and the option value is*

$$V(x) = \mathcal{E}^+ \mathbf{1}_{[h^*,+\infty)} (\mathcal{E}^+)^{-1} G(x). \tag{6.51}$$

Proof. Let h be a candidate for the exercise threshold. Then by (6.46), the option value is

$$V(x;h) = \mathcal{E}^+ \mathbf{1}_{[h,+\infty)} (\mathcal{E}^+)^{-1} G(x).$$

The choice $h = h^*$ replaces all negative values of $(\mathcal{E}^+)^{-1}G$ by zero, and leaves positive ones intact. By Proposition 6.2.1, $\mathcal{E}^+$ is a monotone operator. Hence, h^* is optimal.

Example 6.5. Let $G(x) = \mathrm{e}^x - K$. Then $(\mathcal{E}^+)^{-1}G(x) = (\kappa_q^+(\mathrm{e}))^{-1}\mathrm{e}^x - K$. Hence, if $\log(K\kappa_q^+(\mathrm{e}))$ is non-integer, then the optimal threshold is unique, and it is $h^* = \min\{h \mid h > \log(K\kappa_q^+(\mathrm{e}))\}$. Otherwise, $\log(K\kappa_q^+(\mathrm{e}))$ and $\log(K\kappa_q^+(\mathrm{e})) + 1$ are optimal.

6.4.6 Perpetual put-like American options

Consider an option with the instantaneous payoff $G(X_t)$ which is a decreasing function of the underlying stochastic factor. The standard examples are $G(X_t) = K - X_t$ or $G(X_t) = K - \mathrm{e}^{X_t}$; the following theorem is applicable to much wider classes of payoffs.

Theorem 6.4.6 *Assume that*

(i) X_t *and* G *satisfy (6.25) and (6.34), respectively;*
(ii) there exists h_* *such that*
$(\mathcal{E}^-)^{-1}G(x) \geq 0 \ \forall\ x \leq h_*$, *and* $(\mathcal{E}^-)^{-1}G(x) \leq 0 \ \forall\ x > h_*$.

Then it is optimal to exercise the option the first time $X_t \leq h_*$, *and the option value is*

$$V(x) = \mathcal{E}^- \mathbf{1}_{(-\infty,h_*]} (\mathcal{E}^-)^{-1} G(x). \tag{6.52}$$

Proof. Let h be a candidate for the exercise threshold. Then by (6.45), the option value is

$$V(x;h) = \mathcal{E}^- \mathbf{1}_{(-\infty,h]}(\mathcal{E}^-)^{-1}G(x).$$

The choice $h = h_*$ replaces all negative values of $(\mathcal{E}^-)^{-1}G$ by zero, and leaves positive ones intact. By Proposition 6.2.1, $\mathcal{E}^-$ is a monotone operator. Hence, h_* is optimal.

Example 6.6. Let $G(x) = K - \mathrm{e}^x$. Then $(\mathcal{E}^-)^{-1}G(x) = K - (\kappa_q^-(\mathrm{e}))^{-1}\mathrm{e}^x$. Hence, if $\log(K\kappa_q^-(\mathrm{e}))$ is non-integer, then the optimal threshold is unique, and it is $h_* = \max\{h \mid h < \log(K\kappa_q^-(\mathrm{e}))\}$. Otherwise, $\log(K\kappa_q^-(\mathrm{e}))$ and $\log(K\kappa_q^-(\mathrm{e})) - 1$ are optimal.

6.5 Optimality in the class of all stopping times

6.5.1 General verification lemmas

Consider the perpetual American option with the instantaneous payoff G. We will prove optimality using the following lemma on p.1364 in [36]. In [36], X_t is a random walk on $\mathbb{R}$ but the proof is valid for random walks on $\mathbb{Z}$ as well.

Lemma 6.5.1 *Let a function V satisfy the following two conditions*

$$V(x) \geq \max\{G(x), 0\}, \tag{6.53}$$

$$V(x) \geq qE^x[V(X_1)], \tag{6.54}$$

for any x. Then V is the option value.

Note the interpretation of conditions (6.53) and (6.54). Equation (6.53) means that the option value is non-negative, and it is greater than or equal to the payoff. Equation (6.54) states that the option price is a supermartingale. We will use (6.54) in the form

$$(1-q)^{-1}(I - qP)V(x) \geq 0, \quad \forall\, x. \tag{6.55}$$

Consider now the option to acquire a stream $g(X_t)$.

Lemma 6.5.2 *Assume that*

(i) the EPV of the stream $g(X_t)$ is finite;
(ii) a function V can be represented as the EPV of a stream $W(X_t)$:

$$V = (1-q)^{-1}\mathcal{E}W;$$

(iii) W is non-negative;
(iv) $V(x) \geq (1-q)^{-1}\mathcal{E}g(x)$, for any x.

Then V is the option value.

Proof. Define $G = (I - qP)^{-1}g$. The option in question is equivalent to the option with the instantaneous payoff $G(X_t)$, therefore it suffices to verify conditions (6.53) and (6.55). Since W is non-negative and the operator $\mathcal{E}$ is monotone, $V = (1-q)^{-1}\mathcal{E}W$ is non-negative as well, and condition (iv) means that $V \geq G$. Thus, (6.53) holds. Finally,

$$(I - qP)V = (I - qP)(1 - q)^{-1}\mathcal{E}W = W \geq 0,$$

therefore (6.55) holds as well.

Consider $V_{1,*}$, the value of a stream g with the option to abandon it.

Lemma 6.5.3 *Assume that*

(i) the EPV of the stream $g(X_t)$ is finite;
(ii) a function V_1 can be represented as the EPV of a stream $W_1(X_t)$:

$$V_1 = (1 - q)^{-1}\mathcal{E}W_1;$$

(iii) $W_1(x) \geq g(x)$, for any x;
(iv) $\mathcal{E}W_1(x) \geq 0$, for any x.

Then V_1 is the option value.

Proof. Let $G = (1-q)^{-1}\mathcal{E}g$ be the EPV of the perpetual stream g, and denote by V_* the value of the option to acquire the stream $-g$. Then $G = V_{1,*} - V_*$, and, therefore, $V_1 = V_{1,*}$ iff $V := V_1 - G$ equals V_*. We have $V = (1-q)^{-1}\mathcal{E}W$, where $W := W_1 - g \geq g - g = 0$. Hence, conditions (ii)–(iii) of Lemma 6.5.2 are satisfied. Condition (i) is equivalent to condition (i) of Lemma 6.5.2. Condition (iv) of the same lemma is satisfied as well:

$$V = (1 - q)^{-1}\mathcal{E}W = (1 - q)^{-1}\mathcal{E}W_1 - (1 - q)^{-1}\mathcal{E}g \geq (1 - q)^{-1}\mathcal{E}(-g).$$

From Lemma 6.5.2, $V = V_*$, hence, $V_1 = V_{1,*}$.

We will check the sufficient conditions of optimality for the main types of options considered in Sect. 6.4, under additional conditions. The standing assumption for the random walk is (6.25), and of the payoff stream $g(X_t)$, we require that it is a monotone function of X_t that changes sign. In the case of the perpetual American options on a dividend-paying stock, with the instantaneous payoff $G(X_t)$, we require that the stream $g(X_t) = (I - qP)G(X_t)$ enjoys the same property. The perpetual American put on a non-dividend paying stock is treated separately.

6.5.2 Option to acquire an increasing stream

Consider the option to acquire a stream of payoffs $g(X_t)$ that is a non-decreasing function of the underlying stochastic factor.

Theorem 6.5.4 *Assume that*

(i) X_t and g satisfy (6.25) and (6.26), respectively;
(ii) g does not decrease and changes sign.

Then

(a) there exists h^* such that $\mathcal{E}^- g(x) \geq 0 \ \forall \ x \geq h^*$, and $\mathcal{E}^- g(x) \leq 0 \ \forall \ x < h^*$;
(b) it is optimal to exercise the option the first time $X_t \geq h^*$;
(c) the option value is given by

$$V = (1-q)^{-1}\mathcal{E}^+ \mathbf{1}_{[h^*,+\infty)}\mathcal{E}^- g; \tag{6.56}$$

(d) the option value can be represented as the EPV of the stream $W(X_t)$, where W is a non-decreasing function vanishing below h^*.

Proof. Since the operator $\mathcal{E}^-$ is monotone, $w = \mathcal{E}^- g$ is a non-decreasing function. From Proposition 6.2.4, w changes sign. This proves (a). By Theorem 6.4.3, $\tau^+_{h^*}$ is an optimal stopping time in the class of stopping times of the threshold type. To prove optimality in the class of all stopping times, we need to show that the function $W := (I - qP)V$, where V is defined by (6.56), satisfies conditions (iii)–(iv) of Lemma 6.5.2. Then by this lemma, V is the option value, that is, (b) and (c) hold. Part (d) will be proved in the process of the verification of (iii)–(iv) of Lemma 6.5.2.

First, we verify (iii). In the inaction region $x < h^*$, the Bellman equation

$$V(x) = qPV(x)$$

holds, which proves that $W(x) = 0$ for $x < h^*$. Further,

$$\begin{aligned} W &= (I-qP)(1-q)^{-1}\mathcal{E}^+\mathbf{1}_{[h^*,+\infty)}\mathcal{E}^- g \\ &= (I-qP)(1-q)^{-1}\mathcal{E}^+\mathcal{E}^- g - (I-qP)(1-q)^{-1}\mathcal{E}^+\mathbf{1}_{(-\infty,h^*)}\mathcal{E}^- g. \end{aligned}$$

By the Wiener–Hopf factorization formula, the first term on the RHS equals g, and, by Proposition 6.2.1, $\mathcal{E}^+\mathbf{1}_{(-\infty,h^*)}\mathcal{E}^- g$ vanishes on $[h^*, +\infty)$. Hence, for $x \geq h^*$,

$$W(x) = g(x) + (1-q)^{-1} qP\mathcal{E}^+\mathbf{1}_{(-\infty,h^*)}\mathcal{E}^- g(x).$$

The multiplication-by-$\mathbf{1}_{(-\infty,h^*)}$-operator replaces positive values of $\mathcal{E}^- g$ by zero and leaves the other values as they are. Since $\mathcal{E}^- g$ is non-decreasing, $\mathbf{1}_{(-\infty,h^*)}\mathcal{E}^- g$ is non-decreasing. Since $\mathcal{E}^+$ is monotone and g is non-decreasing, W is non-decreasing on $[h^*, +\infty)$. To prove that W is non-decreasing on $\mathbb{Z}$ (hence, non-negative on $\mathbb{Z}$), it remains to show that $W(h^*) \geq 0$. Suppose, on the contrary, that $W(h^*) < 0$. Applying $\mathcal{E}^-$ to the equality

$$W = (I-qP)V = (I-qP)(1-q)^{-1}\mathcal{E}^+\mathbf{1}_{[h^*,+\infty)} w$$

and using the Wiener–Hopf factorization formula (6.11), we obtain

$$\mathcal{E}^{-}W = \mathbf{1}_{[h^*,+\infty)}w.$$

Since W vanishes below h^*, and, by our assumption, $W(h^*) < 0$, we have $\mathcal{E}^{-}W(h^*) = E[W(h^* + Y^{-})] < 0$. But $\mathbf{1}_{[h^*,+\infty)}w(h^*) \geq 0$ by the definition of h^*, contradiction. Now we verify (iv). Applying the Wiener–Hopf factorization formula (6.11), we have

$$\begin{aligned} V &= (1-q)^{-1}\mathcal{E}^{+}\mathbf{1}_{[h^*,+\infty)}\mathcal{E}^{-}g \\ &= (1-q)^{-1}\mathcal{E}^{+}\mathcal{E}^{-}g + (1-q)^{-1}\mathcal{E}^{+}\mathbf{1}_{(-\infty,h^*)}(-\mathcal{E}^{-}g) \\ &= (1-q)^{-1}\mathcal{E}g + (1-q)^{-1}\mathcal{E}^{+}\mathbf{1}_{(-\infty,h^*)}(-\mathcal{E}^{-}g). \end{aligned}$$

By construction, $-\mathcal{E}^{-}g$ is positive on $(-\infty, h^*)$, and since $\mathcal{E}^{+}$ is monotone, $V \geq (1-q)^{-1}\mathcal{E}g$.

6.5.3 Option to acquire a decreasing stream

Consider the option to acquire a stream of payoffs $g(X_t)$ that is a non-increasing function of the underlying stochastic factor.

Theorem 6.5.5 *Assume that*

(i) X_t and g satisfy (6.25) and (6.26), respectively;
(ii) g does not increase and changes sign.

Then

(a) there exists h_* such that $\mathcal{E}^{+}g(x) \geq 0\ \forall\ x \leq h_*$, and $\mathcal{E}^{+}g(x) \leq 0\ \forall\ x > h_*$;
(b) it is optimal to exercise the option the first time $X_t \leq h_*$;
(c) the option value is given by

$$V = (1-q)^{-1}\mathcal{E}^{-}\mathbf{1}_{(-\infty,h_*]}\mathcal{E}^{+}g; \tag{6.57}$$

(d) the option value can be represented as the EPV of the stream $W(X_t)$, where W is a non-increasing function vanishing above h_*.

The proof is the mirror reflection of the proof of Theorem 6.5.4. We leave it as an exercise for the reader.

6.5.4 Option to abandon an increasing stream

Consider the option to abandon a stream of payoffs $g(X_t)$ that is a non-decreasing function of the underlying stochastic factor.

Theorem 6.5.6 *Assume that*

(i) X_t and g satisfy (6.25) and (6.26), respectively;
(ii) g does not decrease and changes sign.

Then

(a) there exists h_* *such that*
$\mathcal{E}^+g(x) \ge 0 \ \forall\ x \ge h_*$, *and* $\mathcal{E}^+g(x) \le 0 \ \forall\ x < h_*$*;*

(b) it is optimal to exercise the option the first time $X_t \le h_*$*;*

(c) the value of the stream with the option to abandon it is given by

$$V_1 = (1-q)^{-1}\mathcal{E}^-\mathbf{1}_{(h_*,+\infty)}\mathcal{E}^+g; \tag{6.58}$$

(d) the value of the stream with the option to abandon it can be represented as the EPV of the stream $W(X_t) + g(X_t)$*, where* W *is a non-increasing function vanishing above* h_**.*

Proof. Denote by V_* the value of the option to acquire the stream $-g(X_t)$, and by $V_{1,*}$ the value of the stream $g(X_t)$ with the option to abandon it. We have $V_{1,*} - V_* = (1-q)^{-1}\mathcal{E}g$, therefore V_1 is the option value $V_{1,*}$ iff $V_1 - (1-q)^{-1}\mathcal{E}g$ equals V_*, and $\tau^-_{h_*}$ is an optimal time to abandon the stream $g(X_t)$ iff it is an optimal time to acquire the stream $-g(X_t)$. Using the Wiener–Hopf factorization formula (6.11), we obtain

$$\begin{aligned} V_1 - (1-q)^{-1}\mathcal{E}g &= (1-q)^{-1}\mathcal{E}^-\mathbf{1}_{(h_*,+\infty)}\mathcal{E}^+g - (1-q)^{-1}\mathcal{E}g \\ &= (1-q)^{-1}\mathcal{E}^-\mathbf{1}_{(-\infty,h_*]}\mathcal{E}^+(-g), \end{aligned}$$

which is V_* by Theorem 6.5.5. Thus, (c) and (b) are proved. By the same theorem,

$$V_1 - (1-q)^{-1}\mathcal{E}g = (1-q)^{-1}\mathcal{E}W,$$

where W is a non-increasing function which vanishes on $(h_*, +\infty)$. This is (d).

6.5.5 Option to abandon a decreasing stream

Consider an option to abandon a stream of payoffs $g(X_t)$ that is a non-increasing function of the underlying stochastic factor.

Theorem 6.5.7 *Assume that*

(i) X_t *and* g *satisfy (6.25) and (6.26), respectively;*

(ii) g *does not increase and changes sign.*

Then

(a) there exists h^* *such that* $\mathcal{E}^-g(x) \ge 0 \ \forall\ x \le h^*$, *and* $\mathcal{E}^-g(x) \le 0 \ \forall\ x > h^*$*;*

(b) it is optimal to exercise the option the first time $X_t \ge h^*$*;*

(c) the value of the stream with the option to abandon it is given by

$$V_1 = (1-q)^{-1}\mathcal{E}^+\mathbf{1}_{(-\infty,h^*)}\mathcal{E}^-g; \tag{6.59}$$

(d) the value of the stream with the option to abandon it can be represented as the EPV of the stream $W(X_t) + g(X_t)$*, where* W *is a non-decreasing function vanishing below* h^**.*

The proof is the mirror reflection of the proof of Theorem 6.5.6. We leave the details as an exercise for the reader.

6.5.6 Perpetual call-like American options

Consider an option with the instantaneous payoff $G(X_t)$ that is an increasing function of the underlying stochastic factor.

Theorem 6.5.8 *Assume that*

(i) X_t *and* G *satisfy (6.25) and (6.33), respectively;*
(ii) function $g = (I - qP)G$ *does not decrease and changes sign.*

Then

(a) there exists h^* *such that*
$(\mathcal{E}^+)^{-1}G(x) \geq 0 \ \forall\ x \geq h^*$, *and* $(\mathcal{E}^+)^{-1}G(x) \leq 0 \ \forall\ x < h^*$*;*
(b) it is optimal to exercise the option the first time $X_t \geq h^*$*;*
(c) the option value is given by

$$V(x) = \mathcal{E}^+ \mathbf{1}_{[h^*,+\infty)}(\mathcal{E}^+)^{-1}G(x); \tag{6.60}$$

(d) the option value can be represented as the EPV of the stream $W(X_t)$*, where* W *is a non-decreasing function vanishing below* h^**.*

Proof. Apply Theorem 6.5.4 and the Wiener–Hopf factorization formula.

6.5.7 Perpetual put-like American options on a dividend-paying stock

Consider an option with the instantaneous payoff $G(X_t)$ that is a decreasing function of the underlying stochastic factor.

Theorem 6.5.9 *Assume that*

(i) X_t *and* G *satisfy (6.25) and (6.33);*
(ii) function $g = (I - qP)G$ *does not increase and changes sign.*

Then

(a) there exists h_* *such that*
$(\mathcal{E}^-)^{-1}G(x) \geq 0 \ \forall\ x \leq h_*$, *and* $(\mathcal{E}^-)^{-1}G(x) \leq 0 \ \forall\ x > h_*$*;*
(b) it is optimal to exercise the option the first time $X_t \leq h_*$*;*
(c) the option value is given by

$$V(x) = \mathcal{E}^- \mathbf{1}_{(-\infty,h_*]}(\mathcal{E}^-)^{-1}G(x); \tag{6.61}$$

(d) the option value can be represented as the EPV of the stream $W(X_t)$*, where* W *is a non-increasing function vanishing above* h_**.*

Proof. Apply Theorem 6.5.5 and the Wiener–Hopf factorization formula.

6.5.8 Perpetual put-like American options on a non-dividend-paying stock

Condition (ii) of Theorem 6.5.9 cannot hold if we consider the perpetual American put on a stock which pays no dividends. Assuming that we model the stock price using the geometric random walk model, $S_t = e^{X_t}$, and the stock pays no dividends, we must have $e^x = qE^x[e^{X_1}]$, equivalently, $e^x = qP(e)e^x$, or, simplifying, $1 - qP(e) = 0$. Thus, for the instantaneous payoff $G(X_t) = K - e^{X_t}$, we have $g = (I - qP)G = (1 - q)K$, and, therefore, G cannot be expressed as the EPV of the stream $g(X_t)$. Nevertheless, a natural modification of Theorem 6.5.9 holds, and the proof of the latter needs only a slight adjustment.

Theorem 6.5.10 *Assume that* $1 \ge qP(e)$. *Then*

(a) it is optimal to exercise the perpetual American put option with strike price K *the first time* X_t *reaches or crosses the level* $h_* := \log(K\kappa_q^-(e))$ *from above;*
(b) the option value is given by

$$V = \mathcal{E}^- \mathbf{1}_{(-\infty,h_*]} w, \tag{6.62}$$

where $w(x) = K - \kappa_q^-(e)^{-1} e^x$*;*
(c) part (d) of Theorem 6.5.9 holds.

Proof. We verify conditions (6.53) and (6.55). The function V is non-negative since w is non-negative on $(-\infty, h_*]$ and the operator $\mathcal{E}^-$ is monotone. Set $G = K - e^x$. Since $-w$ is non-negative on $(h_*, +\infty)$ and $\mathcal{E}^-$ is monotone,

$$V = \mathcal{E}^- w + \mathcal{E}^- \mathbf{1}_{(h_*,+\infty)}(-w) = G + \mathcal{E}^- \mathbf{1}_{(h_*,+\infty)}(-w) \ge G.$$

Thus, (6.53) holds. Introduce $W = (I - qP)V$. From the Bellman equation for V, $W(x) = 0,\ x > h_*$, and for $x \le h_*$,

$$\begin{aligned} W(x) &= (I - qP)G(x) + (I - qP)\mathcal{E}^- \mathbf{1}_{(h_*,+\infty)}(-w)(x) \\ &= (1 - q)K - (I - qP(e))e^x + qP\mathbf{1}_{(h_*,+\infty)} w(x). \end{aligned}$$

Since $1 - qP(e) \ge 0$, the first two terms do not increase, and the third one does not increase on $(-\infty, h_*]$ because the multiplication-by-$\mathbf{1}_{(h_*,+\infty)}$ operator replaces negative values of the non-decreasing function w with zero and leaves the other values as they are, and P is monotone. To prove that W is non-negative, it remains to prove that $W(h_*) \ge 0$. Suppose, on the contrary, that $W(h_*) < 0$. Applying $\mathcal{E}^+$ to the equality

$$W = (I - qP)V = (I - qP)(1 - q)^{-1}\mathcal{E}^- \mathbf{1}_{(-\infty,h_*]} w$$

and using the Wiener-Hopf factorization formula (6.11), we obtain

$$\mathcal{E}^+ W = \mathbf{1}_{(-\infty,h_*]} w.$$

Since W vanishes above h_*, we have $\mathcal{E}^+ W(h_*) = E[W(h_* + Y^+)] < 0$ but by the definition of h_*, $\mathbf{1}_{(-\infty,h_*]}(h^*)w(h_*) \ge 0$, contradiction.

Problems

6.1. Assume that the agents in the market have become more pessimistic so that for each $j > 0$, the transition probability p_j decreased or remained the same, and for each $j < 0$, p_j increased or remained the same. Consider models of irreversible entry and exit. Using the good and bad news principles, explain how the investment and disinvestment thresholds will change.

6.2. Consider a random walk with the transition operator

$$P = p_0 + p_+ S + p_- S^{-1} + c_- \sum_{j \geq 0} \lambda_-^{-j} S^{-j}, \tag{6.63}$$

where $c_- > 0$ and $\lambda_- > 1$. This random walk is an analog of the Brownian motion with embedded downward jumps; the sizes of jumps are exponentially distributed on $\mathbb{R}_-$. From the technical point of view, the model (6.63) is simpler that the model (6.17) but more involved than the binomial and trinomial models. Model (6.63) can be regarded as a discretization of the Brownian motion with embedded negative jumps, jump sizes being exponentially distributed. Then c_- and λ_- characterize the intensity of jumps and the relative intensity of large jumps: the larger λ_-, the smaller is the proportion of large jumps. Solve

(a) the irreversible investment problem for a firm with the operational profit stream $G\mathrm{e}^{X_t}$, the fixed investment cost being I;
(b) exit problem for the firm with the operational profits $GP - \mathrm{e}^{X_t}$;
(c) study the dependence of the entry and exit thresholds, value of the investment opportunity and firm's value on c_- and λ_-.

6.3. Prove Theorem 6.5.5.

6.4. Find optimal exercise thresholds and option values for

(a) options to acquire or abandon a stream $g(X_t) = X_t - C$;
(b) options to acquire or abandon a stream $g(X_t) = C - X_t$;
(c) perpetual American call option on a stock with the price dynamics $S_t = X_t$ and strike K;
(d) perpetual American put option on a stock with the price dynamics $S_t = X_t$ and strike K.

Hint: use the limit

$$x = \lim_{\epsilon \to 0} \frac{\mathrm{e}^{\epsilon x} - 1}{\epsilon}.$$

Part III

Discrete time – continuous space models

7

Random walks on $\mathbb{R}$

7.1 Definitions and examples

We assume that under an EMM chosen by the market, X_t is a process on $\mathbb{R}$ with independent identically distributed (i.i.d) increments:

$$X_t = X_0 + Y_1 + \cdots + Y_t, \tag{7.1}$$

where the initial location X_0 is either deterministic or, more generally, a random variable distributed independently of $Y_1, Y_2, \ldots$. For simplicity, we will assume that the probability distribution $P(dy)$ of (any of) Y_t has a density, denote it $p(y)dy$, which is of the following special form

$$p(y)dy = \mathbf{1}_{(-\infty,0)}(y)p_-(y)dy + \mathbf{1}_{[0,+\infty)}(y)p_+(y)dy, \tag{7.2}$$

where $p_\pm(y)$ are exponential polynomials:

$$p_+(y) = \sum_{j=1}^{n_+} p_{+,j}(x)\mathrm{e}^{-\lambda_j^+ x}, \tag{7.3}$$

$$p_-(y) = \sum_{j=1}^{n_-} p_{-,j}(x)\mathrm{e}^{-\lambda_j^- x}. \tag{7.4}$$

Above, $p_{\pm,j}$ are polynomials, and $\lambda_k^- < 0 < \lambda_j^+$ for all j, k. We will see that the model (7.2)-(7.4) is fairly simple for proofs and calculations. At the same time, our standing assumption is not too restrictive, since an arbitrary probability distribution on $\mathbb{R}$ can be approximated by a sequence of probability distributions of the form (7.2)–(7.4).

As the matter of fact, one can obtain the probability density $p(y)dy$ of a desired shape using (7.2)–(7.4) with few terms. The simplest version is with one term in both (7.3) and (7.4) and constants instead of polynomials:

$$p(y) = c^+\lambda^+\mathrm{e}^{-\lambda^+ y}\mathbf{1}_{(0,+\infty)}(y) + c^-(-\lambda^-)\mathrm{e}^{-\lambda^- y}\mathbf{1}_{(-\infty,0]}(y), \tag{7.5}$$

where $c^+, c^- > 0$, and $\lambda^- < 0 < \lambda^+$. Here parameter c^+ is the probability of an upward jump per time period. If an upward jump has happened, then the probability that it is a jump from the current state x into an infinitesimally small interval $[x+y,\, x+y+dy]$ is $\lambda^+ \mathrm{e}^{-\lambda^+ y} dy$. Parameters c^- and λ^- admit a similar interpretation. If we want to have a continuous p, we must require that $c^+\lambda^+ + c^-\lambda^- = 0$, and then the requirement

$$\int_{-\infty}^{+\infty} p(y)dy = 1$$

leads to $c^+ = \lambda^-/(\lambda^- - \lambda^+)$, and $c^- = \lambda^+/(\lambda^+ - \lambda^-)$. We will use the model (7.5) to explain the essence of our method because from the analytical point of view, this model is as simple as the binomial model and Brownian motion. Note, however, that the density (7.5) has a kink (and maximum) at the origin. If we want to have a smooth p (and allow for the maximum to be not at the origin), we need to use more than two exponential functions. Suppose that we want to model a density which has the maximum on the positive half-axis. Then we use one exponential on the negative half-axis, and two on the positive half-axis:

$$(c_1^+\lambda_1^+ \mathrm{e}^{-\lambda_1^+ x} - c_2^+\lambda_2^+ \mathrm{e}^{-\lambda_2^+ x})\mathbf{1}_{(0,+\infty)}(x) - c^-\lambda^- \mathrm{e}^{-\lambda^- x}\mathbf{1}_{(-\infty,0]}(x), \tag{7.6}$$

where c_1^+, c_2^+ and c^- are positive, and $\lambda^- < 0 < \lambda_1^+ < \lambda_2^+$.

Lemma 7.1.1 *For any choice of $\lambda^- < 0 < \lambda_1^+ < \lambda_2^+$, expression (7.6) with c^-, c_1^+, c_2^+ given by*

$$c^- = \frac{\lambda_1^+\lambda_2^+}{(\lambda_1^+ - \lambda^-)(\lambda_2^+ - \lambda^-)}, \quad c_j^+ = \frac{-\lambda^-\lambda_k^+}{(\lambda_2^+ - \lambda_1^+)(\lambda_j^+ - \lambda^-)}, \tag{7.7}$$

where $j,k = 1,2$ and $k \neq j$, defines a probability density, which has the maximum on the positive half-axis.

Proof. Three conditions: $\int_{-\infty}^{+\infty} p(x)dx = 1$, p is continuous at 0, and p is smooth at 0, give a linear system of three equations

$$\begin{aligned} c_1^+ - c_2^+ + c^- &= 1, \\ c_1^+\lambda_1^+ - c_2^+\lambda_2^+ + c^-\lambda^- &= 0, \\ c_1^+(\lambda_1^+)^2 - c_2^+(\lambda_2^+)^2 + c^-(\lambda^-)^2 &= 0. \end{aligned} \tag{7.8}$$

Using Cramer's rule, we find a unique solution (c_1^+, c_2^+, c^-) to (7.8), which is (7.7). It is easy to see that c^- and $c_j^+, j = 1,2$, are positive.

See Fig. 7.1 for an example. Similarly, one can construct a 3-parameter family of probability densities which have the maximum on the negative half-axis. Should one wish to have a smooth probability density which has the maximum at the origin, one can use two exponential functions on each half-line or exponential polynomials of the form $(ax+b)\mathrm{e}^{\gamma x}$.

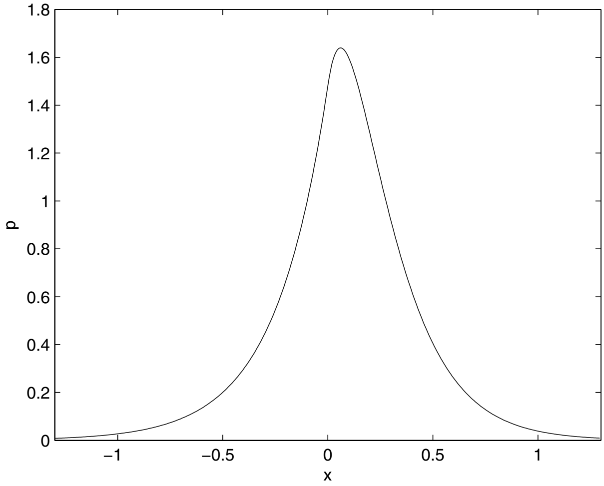

Fig. 7.1. Graph of the probability density (7.6). Parameters: $\lambda^- = -4$, $\lambda_1^+ = 5$, $\lambda_2^+ = 8$, $c^- = 0.3704$, $c_1^+ = 1.1852$, $c_2^+ = 0.5556$.

7.2 Transition operator and EPV-operator $\mathcal{E}$

Let $q \in (0,1)$ be the discount factor per period. The transition operator, P, is defined by

$$\begin{aligned}(Pg)(x) &= E^x[g(X_1)] = E[g(X_1)\,|\,X_0 = x]\\ &= E[g(x+Y_1)] = \int_{-\infty}^{+\infty} p(y)g(x+y)dy.\end{aligned}$$

It is straightforward to show that P maps bounded functions into bounded ones; moreover, it is a bounded linear operator in $\mathcal{L}_\infty(\mathbb{R})$, the space of uniformly bounded measurable functions. Indeed, for any $g \in \mathcal{L}_\infty(\mathbb{R})$,

$$||Pg||_{\mathcal{L}_\infty(\mathbb{R})} = \sup_{x\in\mathbb{R}} \left| \int_{-\infty}^{+\infty} p(y)g(x+y)dy \right| \le \sup_{x\in\mathbb{R}} \int_{-\infty}^{+\infty} p(y) \sup_{z\in\mathbb{R}} |g(z)|dy$$

is bounded by $1 \cdot ||g||_{\mathcal{L}_\infty(\mathbb{R})}$, and for $g(x) \equiv 1$, the equality holds. Hence, P has norm 1. If the probability distribution function $p(y)$ decays at infinity sufficiently fast, the transition operator P is bounded in spaces of measurable functions, which grow at infinity. We will return to this question later.

Given the probability distribution function p, one calculates the EPV of a stochastic payoff tomorrow:

$$E^x[qg(X_1)] = q(Pg)(x) = q \int_{-\infty}^{+\infty} p(y)g(x+y)dy.$$

Since $Y_1, Y_2, \dots$ are i.i.d. random variables, we can use the law of iterated expectations and calculate

$$E^x[g(X_t)] = E^x\left[E^{X_{t-1}}[g(X_t)]\right] = E^x[Pg(X_{t-1})] = \cdots = (P^t g)(x).$$

Here $P^t = P \circ P \circ \cdots \circ P$ is the composition of t copies of P. Therefore, the EPV of a stochastic payoff $g(X_t)$ is

$$E^x[q^t g(X_t)] = q^t P^t g(x).$$

The next step is the calculation of the normalized EPV of a stream of payoffs:

$$(\mathcal{E}g)(x) = (1-q)E^x\left[\sum_{t=0}^{\infty} q^t g(X_t)\right] = (1-q)\sum_{t=0}^{\infty} q^t (P^t g)(x). \tag{7.9}$$

The normalization is convenient because

$$(\mathcal{E}\mathbf{1})(x) = (1-q)E^x\left[\sum_{t=0}^{\infty} q^t \mathbf{1}(X_t)\right] = (1-q)\sum_{t=0}^{\infty} q^t = 1. \tag{7.10}$$

Since (7.9) holds for any $g \in \mathcal{L}_\infty(\mathbb{R})$, we can write (7.9) as the equality for operators acting in $\mathcal{L}_\infty(\mathbb{R})$:

$$\mathcal{E} = (1-q)\sum_{t=0}^{\infty} q^t P^t. \tag{7.11}$$

We use (7.11) and the properties of the operator norm:

$$||A + B|| \le ||A|| + ||B||, \quad ||AB|| \le ||A|| \cdot ||B||,$$

and obtain

$$||\mathcal{E}|| \le (1-q)\sum_{t=0}^{\infty} q^t \cdot 1^t = 1.$$

In view of (7.10), we conclude that the norm of $\mathcal{E}$ is 1. We will also need another interpretation of (7.9). Let T be the geometric random variable on $\mathbb{Z}_+$, independent of the process $X = \{X_t\}$ (that is, independent of each random variable X_t), with $\text{Prob}(T = t) = (1-q)q^t$. Then

$$\mathcal{E}g(x) = E[g(x + X_T)].$$

In the theory of stochastic processes, the map $g \mapsto (1-q)^{-1}\mathcal{E}g$ is called the resolvent or potential operator.

If g is an exponential function: $g(x) = e^{zx}$, then $\mathcal{E}g$ can be expressed in terms of the moment-generating function

$$M(z) := \int_{-\infty}^{+\infty} e^{zy} p(y) dy.$$

Namely,

$$\mathcal{E}e^{zx} = \sum_{t=0}^{\infty} q^t P^t e^{zx} = \sum_{t=0}^{\infty} q^t M(z)^t e^{zx}.$$

Assuming that z is real and $1 - qM(z) > 0$, the series converges, and

$$\mathcal{E}e^{zx} = (1 - qM(z))^{-1} e^{zx}. \tag{7.12}$$

For a complex z, (7.12) holds if $1 - qM(\operatorname{Re} z) > 0$.

7.3 Bellman equation and calculation of $\mathcal{E}g$ using factorization

To find $u(x) = (\mathcal{E}g)(x)$ for a general g, we write the Bellman equation

$$\begin{aligned} u(x) &= (1-q)g(x) + q(1-q)E^x\left[\sum_{t=1}^{\infty} q^{t-1} g(X_t)\right] \\ &= (1-q)g(x) + q(1-q)\sum_{t=1}^{\infty} q^{t-1} E^x[E^{X_1}[g(X_t)]] \\ &= (1-q)g(x) + qE^x[u(X_1)] \end{aligned}$$

in the form

$$u(x) = (1-q)g(x) + qPu(x),$$

and then as an integral equation

$$(1-q)^{-1}(I - qP)u(x) = g(x), \ \forall\, x. \tag{7.13}$$

The norm of the operator qP (as an operator in $\mathcal{L}_\infty(\mathbb{R})$) equals $q < 1$, hence the operator $A = (1-q)^{-1}(I - qP)$ is invertible in $\mathcal{L}_\infty(\mathbb{R})$, and we have

$$u(x) = (\mathcal{E}g)(x) = (A^{-1}g)(x), \tag{7.14}$$

for a bounded g. Under additional conditions on the transition density p, (7.14) can be extended to unbounded g. Consider, for instance, an exponential function $g(x) = e^{zx}$, where $z \in \mathbb{C}$ satisfies $1 - qM(\operatorname{Re} z) > 0$. We have

$$Ag(x) = a(z)g(x), \tag{7.15}$$

where

$$a(z) = (1-a)^{-1}(1 - qM(z)), \tag{7.16}$$

and from (7.12),

$$\mathcal{E}g(x) = a(z)^{-1}g(x). \tag{7.17}$$

Thus, (7.14) holds. In Chap. 9, we will prove (7.14) for more general g.

For applications, we need explicit formulas for $\mathcal{E}g$ for general g. To make our exposition as simple as possible, we start with the model (7.5). We introduce a simple factorization of the operator A, which allows us to invert A and calculate the action of $\mathcal{E}$ quite easily. The factorization will be especially useful in Chap. 8, where we will calculate the EPV of a stochastic stream which is lost when the stochastic factor crosses a certain barrier. First, using $c^+\lambda^+ + c^-\lambda^- = 0$, we calculate the moment generating function

$$\begin{aligned} M(z) &= c^-(-\lambda^-)\int_{-\infty}^{0} \mathrm{e}^{zy}\mathrm{e}^{-\lambda^+ y}dy + c^+\lambda^+\int_0^{+\infty} \mathrm{e}^{zy}\mathrm{e}^{-\lambda^+ y}dy \\ &= \frac{c^+\lambda^+}{\lambda^+ - z} + \frac{c^-\lambda^-}{\lambda^- - z} \\ &= \frac{-\lambda^-\lambda^+}{\lambda^+ - \lambda^-}\left[\frac{1}{\lambda^+ - z} - \frac{1}{\lambda^- - z}\right]. \end{aligned} \tag{7.18}$$

Define $a(z)$ by the analytic expressions (7.16)-(7.18). Then it is well-defined on $\mathbb{C} \setminus \{\lambda^-, \lambda^+\}$, and not only in the strip $\mathrm{Re}\, z \in (\lambda^-, \lambda^+)$, where the moment generating function is defined. It is easy to see that $a(z)$ has zeroes $\beta^+ = \beta^+(q) \in (0, \lambda^+)$ and $\beta^- = \beta^-(q) \in (\lambda^-, 0)$, which are the roots of the quadratic equation

$$z^2 - (\lambda^+ + \lambda^-)z + (1-q)\lambda^+\lambda^- = 0. \tag{7.19}$$

We find

$$\beta^\pm(q) = 0.5 \cdot \left(\lambda^+ + \lambda^- \pm \sqrt{(\lambda^+ + \lambda^-)^2 - 4(1-q)\lambda^-\lambda^+}\right), \tag{7.20}$$

and factorize $a(z)$ as

$$a(z) = (1-q)^{-1}\frac{(\beta^+ - z)(\beta^- - z)}{(\lambda^+ - z)(\lambda^- - z)}.$$

Since

$$a(0) = (1-q)^{-1}\frac{\beta^+\beta^-}{\lambda^+\lambda^-} = 1,$$

we may write

$$a(z) = a^+(z)a^-(z),$$

where

$$a^+(z) = \frac{\lambda^+(\beta^+ - z)}{\beta^+(\lambda^+ - z)}, \quad a^-(z) = \frac{\lambda^-(\beta^- - z)}{\beta^-(\lambda^- - z)}.$$

Introduce

$$\kappa_q^+(z) = a^+(z)^{-1} = \frac{\beta^+(\lambda^+ - z)}{\lambda^+(\beta^+ - z)} \tag{7.21}$$

and

$$\kappa_q^-(z) = a^-(z)^{-1} = \frac{\beta^-(\lambda^- - z)}{\lambda^-(\beta^- - z)}, \tag{7.22}$$

and notice that

$$\frac{1-q}{1-qM(z)} = a(z)^{-1} = a^+(z)^{-1}a^-(z)^{-1} = \kappa_q^+(z)\kappa_q^-(z). \tag{7.23}$$

Next, define operators $A^\pm$ and $\mathcal{E}^\pm$ as follows. For $g(x) = \mathrm{e}^{zx}$,

$$A^\pm g(x) = a^\pm(z)g(x), \quad \mathcal{E}^\pm g(x) = \kappa^\pm(z)g(x). \tag{7.24}$$

Using (7.15), (7.17) and (7.24), we obtain for functions of the form $g(x) = \mathrm{e}^{zx}$, where $z \in \mathbb{C}$ satisfies $1 - qM(\operatorname{Re} z) > 0$,

$$Ag = A^+A^-g = A^-A^+g, \tag{7.25}$$

$$\mathcal{E}g = \mathcal{E}^+\mathcal{E}^-g = \mathcal{E}^-\mathcal{E}^+g. \tag{7.26}$$

The use of the notation $\kappa_q^+(z)$ and $\kappa_q^-(z)$ instead of the explicit analytical expressions in terms of the parameters of the model is an important ingredient of our approach because they are the convenient short-hand notation when it is necessary to calculate the action of operators $\mathcal{E}^+$ and $\mathcal{E}^-$ on exponential functions, and in basic models in the theory of real options, typically, the payoffs are linear combinations of exponential functions. Furthermore, equality (7.23) allows us to see useful simplifications of complicated expressions.

To extend the definitions (7.24) and equalities (7.25)–(7.26) to wider classes of functions, we note that

$$a^+(z) = \frac{\lambda^+}{\beta^+} + \frac{\beta^+ - \lambda^+}{\beta^+} \cdot \frac{\lambda^+}{\lambda^+ - z},$$

and

$$\int_0^{+\infty} \lambda^+ \mathrm{e}^{-\lambda^+ y}\mathrm{e}^{z(x+y)}dy = \frac{\lambda^+}{\lambda^+ - z}\mathrm{e}^{zx}.$$

Hence, A^+ acts on exponential functions g as follows

$$(A^+g)(x) = \frac{\lambda^+}{\beta^+}g(x) + \frac{\beta^+ - \lambda^+}{\beta^+}\int_0^{+\infty} \lambda^+\mathrm{e}^{-\lambda^+ y}g(x+y)dy. \tag{7.27}$$

Similarly,

$$(A^-g)(x) = \frac{\lambda^-}{\beta^-}g(x) + \frac{\beta^- - \lambda^-}{\beta^-}\int_{-\infty}^0 (-\lambda^-)\mathrm{e}^{-\lambda^- y}g(x+y)dy, \tag{7.28}$$

$$(\mathcal{E}^+g)(x) = \frac{\beta^+}{\lambda^+}g(x) + \frac{\lambda^+ - \beta^+}{\lambda^+}\int_0^{+\infty} \beta^+\mathrm{e}^{-\beta^+ y}g(x+y)dy, \tag{7.29}$$

$$(\mathcal{E}^-g)(x) = \frac{\beta^-}{\lambda^-}g(x) + \frac{\lambda^- - \beta^-}{\lambda^-}\int_{-\infty}^0 (-\beta^-)\mathrm{e}^{-\beta^- y}g(x+y)dy. \tag{7.30}$$

We can use (7.27)–(7.30) to define $A^{\pm}g(x)$ and $\mathcal{E}^{\pm}g(x)$ for any measurable g that satisfies

$$|g(x)| \leq C\exp(\sigma^{+}x), \quad x \geq 0, \tag{7.31}$$

$$|g(x)| \leq C\exp(\sigma^{-}x), \quad x \leq 0, \tag{7.32}$$

where constant $C > 0$ is independent of x, and $\sigma^{-} < 0 < \sigma^{+}$ satisfy

$$1 - qM(\sigma) > 0, \quad \forall\, \sigma \in [\sigma^{-}, \sigma^{+}]. \tag{7.33}$$

(These conditions are necessary if g is monotone on each half-axis.) Indeed, by linearity, formulas (7.27)–(7.30) extend to functions that can be represented as integrals of exponentials $g(x) = \int \mathrm{e}^{zx}\hat{g}(z)dz$ (with the integration over an appropriate line in the complex plane) or more generally, as sums of such integrals. If g is continuous, then, under conditions (7.31)–(7.32), the representations of its positive and negative parts $g_{+}(x) = \max\{g(x), 0\}$ and $g_{-} = g - g_{+}$ as integrals exist, and (7.27)–(7.30) follow. If g is only measurable, than the representations hold a.e., which implies that (7.27)–(7.30) hold in this case as well. Using the integral representation once again, we easily find that $\mathcal{E}^{\pm}A^{\pm}g = g$ and $A^{\pm}\mathcal{E}^{\pm}g = g$. Therefore, $\mathcal{E}^{\pm}$ and $A^{\pm}$ are mutual inverses in appropriate function spaces:

$$\mathcal{E}^{+} = (A^{+})^{-1}, \quad \mathcal{E}^{-} = (A^{-})^{-1}. \tag{7.34}$$

(For details, see Chap. 9). Similarly, (7.25)–(7.26) can be written as

$$A = A^{+}A^{-} = A^{-}A^{+}, \tag{7.35}$$

$$\mathcal{E} = \mathcal{E}^{+}\mathcal{E}^{-} = \mathcal{E}^{-}\mathcal{E}^{+}. \tag{7.36}$$

Note the following useful interpretation of (7.29)–(7.30). Define Y^{+} and Y^{-} as the random variables on $\mathbb{R}_{+}$ and $\mathbb{R}_{-}$, respectively, with the probability distributions

$$P_{+}(dy) = \frac{\beta^{+}}{\lambda^{+}}\delta_0 + \frac{\lambda^{+} - \beta^{+}}{\lambda^{+}}\beta^{+}\mathrm{e}^{-\beta^{+}y}dy,$$

$$P_{-}(dy) = \frac{\beta^{-}}{\lambda^{-}}\delta_0 + \frac{\lambda^{-} - \beta^{-}}{\lambda^{-}}(-\beta^{-})\mathrm{e}^{-\beta^{-}y}dy,$$

where δ_0 is the unit mass at zero. Then

$$\mathcal{E}^{+}g(x) = E[g(x + Y^{+})], \quad \mathcal{E}^{-}g(x) = E[g(x + Y^{-})]. \tag{7.37}$$

Using (7.36), we can find the solution to the Bellman equation (7.13) in two steps: first, calculate $w(x) = (\mathcal{E}^{+}g)(x)$, and then

$$u(x) = \mathcal{E}^{-}\mathcal{E}^{+}g(x) = (\mathcal{E}^{-}w)(x). \tag{7.38}$$

Alternatively, we may calculate $w_2(x) = (\mathcal{E}^{-}g)(x)$ first, and then

$$u(x) = \mathcal{E}^{+}\mathcal{E}^{-}g(x) = (\mathcal{E}^{+}w_2)(x). \tag{7.39}$$

Example 7.1. Let $g(x) = \max\{e^x - 1, 0\}$. The EPV $\mathcal{E}g$ is finite iff $1 - qM(1) > 0$. Under this condition, $\lambda^- < \beta^- < 0 < 1 < \beta^+ < \lambda^+$. We calculate $w = \mathcal{E}^+ g$ using (7.29) and (7.37). For $x \geq 0$,

$$w(x) = \mathcal{E}^+ g(x) = E[g(x + Y^+)] = E[e^{x+Y^+} - 1] = \kappa_q^+(1)e^x - 1,$$

and for $x < 0$, changing the variables $y \mapsto -x + y$, we obtain

$$\begin{aligned}
(\mathcal{E}^+ g)(x) &= \frac{\beta^+}{\lambda^+} g(x) + \frac{\lambda^+ - \beta^+}{\lambda^+} \int_0^{+\infty} \beta^+ e^{-\beta^+ y} g(x+y)dy \\
&= 0 + \frac{\lambda^+ - \beta^+}{\lambda^+} \int_{-x}^{+\infty} \beta^+ e^{-\beta^+ y}(e^{x+y} - 1)dy \\
&= e^{\beta^+ x} \frac{\lambda^+ - \beta^+}{\lambda^+} \int_0^{+\infty} \beta^+ e^{-\beta^+ y}(e^y - 1)dy \\
&= e^{\beta^+ x} A,
\end{aligned}$$

where

$$A = \frac{\lambda^+ - \beta^+}{\lambda^+} \left[\frac{\beta^+}{\beta^+ - 1} - 1\right] = \frac{\lambda^+ - \beta^+}{\lambda^+(\beta^+ - 1)}.$$

Now we calculate $\mathcal{E}g = \mathcal{E}^- w$. For $x \leq 0$,

$$\mathcal{E}^- w(x) = E[w(x + Y^-)] = E[Ae^{\beta^+(x+Y^-)}] = Ae^{\beta^+ x}\kappa_q^-(\beta^+),$$

and for $x > 0$, we use (7.30):

$$\begin{aligned}
\mathcal{E}^- w(x) &= \frac{\beta^-}{\lambda^-} w(x) + \frac{\lambda^- - \beta^-}{\lambda^-} \int_{-\infty}^{0} (-\beta^-)e^{-\beta^- y} w(x+y)dy \\
&= \frac{\beta^-}{\lambda^-}(\kappa_q^+(1)e^x - 1) + \frac{\lambda^- - \beta^-}{\lambda^-} \int_{-\infty}^{-x} (-\beta^-)e^{-\beta^- y} Ae^{\beta^+(x+y)}dy \\
&\quad + \frac{\lambda^- - \beta^-}{\lambda^-} \int_{-x}^{0} (-\beta^-)e^{-\beta^- y}(\kappa_q^+(1)e^{x+y} - 1)dy \\
&= \frac{\beta^-}{\lambda^-}(\kappa_q^+(1)e^x - 1) + e^{\beta^- x} \frac{\lambda^- - \beta^-}{\lambda^-} A \frac{-\beta^-}{\beta^+ - \beta^-} \\
&\quad + \frac{\lambda^- - \beta^-}{\lambda^-} \kappa_q^+(1) \frac{-\beta^-}{1 - \beta^-}(e^x - e^{\beta^- x}) - \frac{\lambda^- - \beta^-}{\lambda^-}(1 - e^{\beta^- x}).
\end{aligned}$$

Using (7.21), (7.22) and (7.23), we simplify the coefficient at e^x:

$$\begin{aligned}
&\left[\frac{\beta^-}{\lambda^-} + \frac{\lambda^- - \beta^-}{\lambda^-} \cdot \frac{-\beta^-}{1 - \beta^-}\right] \kappa_q^+(1) = \frac{\beta^-}{\lambda^-} \left[1 - \frac{\lambda^- - \beta^-}{1 - \beta^-}\right] \kappa_q^+(1) \\
&= \frac{\beta^-}{\lambda^-} \frac{1 - \lambda^-}{1 - \beta^-} \kappa_q^+(1) = \kappa_q^-(1)\kappa_q^+(1) = \frac{1 - q}{1 - qM(1)}.
\end{aligned}$$

The constant term is -1, and the coefficient at $e^{\beta^- x}$ does not admit a significant simplification. The result is: for $x > 0$,

$$\mathcal{E}g(x) = \frac{1-q}{1-qM(1)}e^x - 1 + \frac{\lambda^- - \beta^-}{\lambda^-} \cdot \left[\frac{-\beta^-}{\beta^+ - \beta^-}A + 1 - \kappa_q^+(1)\frac{\beta^-}{\beta^- - 1}\right] e^{\beta^- x}.$$

Problems

In each problem, the transition probability density is given by (7.5).

7.1. Calculate the EPV of a stream

(a) $g(X_t) = \max\{e^{\alpha X_t}, 1\}$, where $\alpha > 0$;
(b) $g(X_t) = \max\{e^{\alpha X_t}, 1\}$, where $\alpha < 0$;
(c) $g(X_t) = X_t$;
(d) $g(X_t) = \max\{X_t, 1\}$.

Formulate conditions on the random walk which ensure that the EPV is finite, and prove their necessity.

7.2. The operational profit flow of the firm is $Ge^{X_t} - C$. Operational profits are taxed at rate $\tau_\Pi > 0$. Calculate the EPV of the profits net taxes. (Caution: profits are taxed when they are positive)

7.3. The profit flow of the firm per share evolves as Ge^{X_t}, where $G > 0$. A firm pays dividends at rate δ_1, when the profit flow is below G, and at rate $\delta_2 > \delta_1$, when the profit flow is above G.

Calculate the rational value of the share.

8

Basic options in the model (7.5)

Here we explain how to calculate option prices in the simple set-up of the model (7.5), with several examples. Should the reader prefer to read a systematic exposition of the general theory first and examples afterwards, she may wish to read Chap. 9, and then this Chapter.

8.1 EPV of a stream, which is abandoned when X_t falls to a certain level

Assume that the payoff stream $g(X_t)$ is a continuous non-decreasing function of X_t, the typical example being a firm facing demand uncertainty and a constant variable cost. Let G be the rate of output, and C the variable cost. For high levels of the log-price of the firm's output, X_t, the profit flow $g(X_t) = Ge^{X_t} - C$ is positive, and for low levels, it is negative. Should the (log) price fall sufficiently low, to a certain level h, it may become optimal to cease production. Fix h, a candidate for the exit threshold (the optimal choice of h will be analyzed in the next section), and denote by $V(x;h)$ the value of the firm with this choice of the exit threshold. Denote by τ_h^- the first time X_t reaches or crosses h from above. Certainly, $\tau_h^- = \tau_h^-(\omega)$ depends on a sample path ω of the process. Thus, τ_h^- is a random variable on the probability space Ω of the sample paths of the process. We have

$$V(x;h) = E^x \left[\sum_{0 \le t < \tau_h^-} q^t g(X_t) \right].$$

In the region $x > h$, the value of the firm, $V(x;h)$, obeys the Bellman equation

$$V(x;h) = g(x) + qE^x[V(X_1;h)],$$

which we can write as

$$(1 - qP)V(x; h) = g(x), \quad x > h. \tag{8.1}$$

After exit, the firm's value is zero:

$$V(x; h) = 0, \quad x \leq h. \tag{8.2}$$

We introduce the normalized value function $\mathcal{V}(x; h) = (1 - q)V(x; h)$, and solve the problem

$$(1 - q)^{-1}(I - qP)\mathcal{V}(x; h) = g(x), \ x > h, \tag{8.3}$$

$$\mathcal{V}(x; h) = 0, \qquad x \leq h. \tag{8.4}$$

The Bellman equation (8.3) is similar to the Bellman equation (7.13) for the value of the firm which never exits but (8.3) holds for $x > h$ only.

For a set U, denote by $\mathbf{1}_U$ the indicator function of U and the multiplication operator by the same function. The next theorem, which demonstrates the essence of the Wiener-Hopf method in the form used in analysis, states that $\mathcal{V}$ can be calculated using a formula, which is similar to (7.38); the new element is the operator $\mathbf{1}_{(h,+\infty)}$, which must be inserted between $\mathcal{E}^-$ and $\mathcal{E}^+$.

Theorem 8.1.1 *Assume that (7.31), (7.32) and (7.33) hold. Then*

$$\mathcal{V}(x; h) = (\mathcal{E}^- \mathbf{1}_{(h,+\infty)} \mathcal{E}^+ g)(x). \tag{8.5}$$

Remark 8.1. a) From the technical point of view, the calculation of the solution (8.5) is no more difficult than the calculation of the value of the firm which never stops producing:

(1) calculate $w = \mathcal{E}^+ g$:

$$w(x) = \frac{\beta^+}{\lambda^+} g(x) + \frac{\lambda^+ - \beta^+}{\lambda^+} \int_0^{+\infty} \beta^+ \mathrm{e}^{-\beta^+ y} g(x + y) dy; \tag{8.6}$$

(2) set $g_2(x) = w(x)$ for $x > h$, and $g_2(x) = 0$ for $x \leq h$;

(3) calculate $\mathcal{V} = \mathcal{E}^- g_2$:

$$\mathcal{V}(x) = \frac{\beta^-}{\lambda^-} g_2(x) + \frac{\lambda^- - \beta^-}{\lambda^-} \int_{-\infty}^0 (-\beta^-) \mathrm{e}^{-\beta^- y} g_2(x + y) dy;$$

(4) set $V = (1 - q)^{-1}\mathcal{V}$.

Notice that now we may not inverse the order of application of $\mathcal{E}^+$ and $\mathcal{E}^-$; the inverse order appears when we solve the problem for a stream which is abandoned as X_t reaches a certain threshold h from below; and then we use the indicator function $\mathbf{1}_{(-\infty,h)}$ instead of $\mathbf{1}_{(h,+\infty)}$.

b) Using the (independent) random variables Y^+ and Y^- on the positive and negative half-axis, respectively, defined in Subsect. 7.3, we can write (8.5) in another form

$$V(x; h) = (1 - q)^{-1} E \left[\mathbf{1}_{(h,+\infty)}(x + Y^-) g(x + Y^- + Y^+) \right]. \tag{8.7}$$

Proof of Theorem 8.1.1. On the strength of (7.31)–(7.32), the firm's value is bounded by the EPV of a stream of the form $C(e^{\sigma^+ X_t}+e^{\sigma^- X_t})$. Due to (7.33), this EPV admits the bound of the form (7.31)–(7.32):

$$\mathcal{E}(C(e^{\sigma^+ \cdot}+e^{\sigma^- \cdot}))(x) = C\left(\frac{1-q}{1-qM(\sigma^+)}e^{\sigma^+ x} + \frac{1-q}{1-qM(\sigma^-)}e^{\sigma^- x}\right).$$

Therefore, it suffices to prove that:

1) a solution to the problem (8.3)–(8.4) in the class of functions satisfying (7.31)–(7.32) exists;
2) it is unique and given by (8.5).

We rewrite (8.3) as $(A\mathcal{V})(x;h) = g(x)+g^-(x)$, where $g^- := A\mathcal{V} - g \in \mathcal{L}_\infty(\mathbb{R})$ vanishes above h. Equivalently, using (7.35),

$$A^+A^-\mathcal{V}(x;h) = g(x) + g^-(x), \quad \forall\ x. \tag{8.8}$$

Apply the inverse $\mathcal{E}^+$ to the operator A^+:

$$A^-\mathcal{V}(x;h) = \mathcal{E}^+g(x) + \mathcal{E}^+g^-(x), \quad \forall\ x, \tag{8.9}$$

and note the important property of operators $A^\pm$ and $\mathcal{E}^\pm$, which are immediate from the representations (7.27)–(7.30):

- If $u(x) = 0\ \forall\ x \le h$, then, for the same x,

$$(A^-u)(x) = 0, \quad \text{and} \quad (\mathcal{E}^-u)(x) = 0. \tag{8.10}$$

- If $u(x) = 0\ \forall\ x > h$, then, for the same x,

$$(A^+u)(x) = 0, \quad \text{and} \quad (\mathcal{E}^+u)(x) = 0. \tag{8.11}$$

Since $g^-(x) = 0$ for $x > h$, we apply (8.11) and obtain $\mathcal{E}^+g^-(x) = 0, \quad x > h$. From $\mathcal{V}(x;h) = 0,\ x \le h$, we have $A^-\mathcal{V}(x;h) = 0,\ x \le h$ (see (8.10)). Therefore if we multiply (8.9) by $\mathbf{1}_{(h,+\infty)}$, the LHS does not change and the RHS becomes $\mathbf{1}_{(h,+\infty)}\mathcal{E}^+g$:

$$A^-\mathcal{V}(x;h) = \mathbf{1}_{(h,+\infty)}(x)(\mathcal{E}^+g)(x),\ \forall\ x. \tag{8.12}$$

Now it remains to apply the inverse $\mathcal{E}^-$ to obtain (8.5). Note that (8.2) holds in view of (8.10). Finally, it is easy to show that the RHS in (8.5) admits the bounds (7.31)–(7.32). Indeed,

$$\begin{aligned}(\mathcal{E}^-\mathbf{1}_{(h,+\infty)}\mathcal{E}^+g)(x) &\le \mathcal{E}^-\mathcal{E}^+C(e^{\sigma^+ \cdot}+e^{\sigma^- \cdot})(x) = \mathcal{E}(C(e^{\sigma^+ \cdot}+e^{\sigma^- \cdot}))(x)\\ &= C\left(\frac{1-q}{1-qM(\sigma^+)}e^{\sigma^+ x} + \frac{1-q}{1-qM(\sigma^-)}e^{\sigma^- x}\right).\end{aligned}$$

Theorem 8.1.1 has been proved.

8.2 Timing exit

Consider the problem of an optimal choice of the exit boundary h. We assume that

$$g \text{ is continuous and non} - \text{decreasing}; \tag{8.13}$$

$$g(+\infty) := \lim_{x\to+\infty} g(x) > 0; \tag{8.14}$$

$$g(-\infty) := \lim_{x\to-\infty} g(x) < 0 \tag{8.15}$$

(one limit or both may be infinite; in the exit problem above, only $g(+\infty)$ is infinite). From (8.5), we have

$$\mathcal{V}(x;h) = E[(\mathbf{1}_{(h,+\infty)}w)(x+Y^-)], \tag{8.16}$$

where $w(x) = \mathcal{E}^+g(x) = E[g(x+Y^+)]$. Clearly, the larger the value of the product $\mathbf{1}_{(h,+\infty)}w$, the larger is the value $\mathcal{V}(x;h)$. Hence, the optimal choice of h should replace all negative values of w by zero, and leave positive ones as they are. Clearly, $w(x) = E[g(x+Y^+)]$ is continuous. Since g is non-decreasing, w is non-decreasing as well. Further, passing to the limit as $x \to \pm\infty$ in the equation $w(x) = E[g(x+Y^+)]$, we obtain that w satisfies (8.14)–(8.15) since g does. Moreover, it is easy to see that if g is increasing in a neighborhood of $+\infty$, then w is increasing on $\mathbb{R}$, and if g is constant on $[x_+, +\infty)$ but $g(x) < g(x_+)$, $\forall\, x < x_+$, then w is increasing below x_+. We conclude that w has a unique zero, call it h_*, $w(x) > 0$ for all $x > h_*$, and $w(x) < 0$ for all $x < h_*$. Hence, h_* is the optimal exit threshold. The optimal exit threshold h_* having been found, the manager calculates the normalized value of the firm $\mathcal{V}(x) = \mathcal{V}(x;h_*)$ for $x > h_*$ using (8.5) with $h = h_*$:

$$\mathcal{V}(x;h_*) = (\mathcal{E}^-\mathbf{1}_{(h_*,+\infty)}\mathcal{E}^+g)(x), \tag{8.17}$$

and then (7.30):

$$\begin{aligned}\mathcal{V}(x) &= \frac{\beta^-}{\lambda^-}w(x) + \frac{\lambda^- - \beta^-}{\lambda^-}\int_{-\infty}^0 (-\beta^-)\mathrm{e}^{-\beta^- y}\mathbf{1}_{[h_*,+\infty)}(x+y)w(x+y)dy \\ &= \frac{\beta^-}{\lambda^-}w(x) + \frac{\lambda^- - \beta^-}{\lambda^-}\int_{h_*-x}^0 (-\beta^-)\mathrm{e}^{-\beta^- y}w(x+y)dy. \end{aligned}\tag{8.18}$$

Example 8.2. Let $g(x) = G\mathrm{e}^x - C$. Then the EPV of the stream $g(X_t)$, hence, the firm's value is finite iff $1 - qM(1) > 0$. Under this condition, $\lambda^- < \beta^- < 0 < 1 < \beta^+ < \lambda^+$. Since

$$w(x) = \mathcal{E}^+(G\mathrm{e}^{\cdot} - C)(x) = G\kappa_q^+(1)\mathrm{e}^x - C, \tag{8.19}$$

the optimal exit threshold is defined from

$$G\kappa_q^+(1)\mathrm{e}^{h_*} = C. \tag{8.20}$$

Substituting (8.19) into (8.18), we calculate for $x > h_*$:

$$\begin{aligned}\mathcal{V}(x) &= \frac{\beta^-}{\lambda^-}(G\kappa_q^+(1)\mathrm{e}^x - C)\\ &+\frac{\lambda^- - \beta^-}{\lambda^-}\int_{h_*-x}^0 (-\beta^-)\mathrm{e}^{-\beta^- y}(G\kappa_q^+(1)\mathrm{e}^{x+y} - C)dy \qquad (8.21)\\ &= \frac{\beta^-}{\lambda^-}(G\kappa_q^+(1)\mathrm{e}^x - C)\\ &+\frac{\lambda^- - \beta^-}{\lambda^-}\left[\frac{-\beta^-}{1-\beta^-}\kappa_q^+(1)G(\mathrm{e}^x - \mathrm{e}^{h_*+\beta^-(x-h_*)}) - C(1-\mathrm{e}^{\beta^-(x-h_*)})\right]\end{aligned}$$

Using (8.20) and then (7.22), we simplify

$$\begin{aligned}\mathcal{V}(x) &= C\left[\left(\frac{\beta^-}{\lambda^-} + \frac{\lambda^- - \beta^-}{\lambda^-}\cdot\frac{-\beta^-}{1-\beta^-}\right)\mathrm{e}^{x-h_*} - \left(\frac{\lambda^- - \beta^-}{\lambda^-} + \frac{\beta^-}{\lambda^-}\right)\right.\\ &\left.+\frac{\lambda^- - \beta^-}{\lambda^-}\left(1 - \frac{-\beta^-}{1-\beta^-}\right)\mathrm{e}^{\beta^-(x-h_*)}\right]\\ &= C\left[\kappa_q^-(1)\mathrm{e}^{x-h_*} - 1 + \frac{\lambda^- - \beta^-}{\lambda^-(1-\beta^-)}\mathrm{e}^{\beta^-(x-h_*)}\right]\\ &= C\left[\kappa_q^-(1)\mathrm{e}^{x-h_*} - 1 + (1-\kappa_q^-(1))\mathrm{e}^{\beta^-(x-h_*)}\right]. \qquad (8.22)\end{aligned}$$

Using (8.20) once again and the Wiener-Hopf factorization formula (7.23), we can decompose the firm's value into the sum of the value of the firm which produces forever and the option value to exit, call it $V_{\mathrm{opt}}(x)$:

$$\begin{aligned}V(x) &= (1-q)^{-1}(\kappa_q^+(1)\kappa_q^-(1)G\mathrm{e}^x - C) + V_{\mathrm{opt}}(x)\\ &= (1-qM(1))^{-1}G\mathrm{e}^x - (1-q)^{-1}C + V_{\mathrm{opt}}(x),\end{aligned}$$

where

$$V_{\mathrm{opt}}(x) = \frac{C(1-\kappa_q^-(1))}{1-q}\mathrm{e}^{\beta^-(x-h_*)}.$$

8.3 Continuous pasting principle and smooth pasting principle

As we will see in Sect. 10.5, in continuous time models with diffusion component, the value function is continuous for any choice of the exercise boundary, and the optimal one can be chosen using the *smooth pasting principle* or the *principle of smooth fit*, which states that the value function must be smooth (that is, of the class C^1). If the payoff (or payoff stream) is sufficiently regular, then the value function is differentiable at any point but the exercise boundary, for any choice of the latter, therefore, the smooth pasting principle can be used to guess the optimal exercise boundary; and it is widely used for this purpose. Notice, however, that the smooth fit principle fails for certain classes of processes without the diffusion component (for further discussion and references, see [20, 21, 4]). For such processes, the exercise boundary can be guessed using the *continuous pasting principle*, which states that the value function must be continuous (an incorrect choice of the exercise boundary leads to a value function, which is discontinuous at the boundary). The same continuous pasting principle holds in discrete time – continuous state space models, and the smooth pasting principle fails. Consider the example above. Below h_*, the firm's value is 0, and at $x = h_*$, the RHS of (8.22), $\mathcal{V}(h_*) = C\left[\kappa_q^-(1) - 1 + 1 - \kappa_q^-(1)\right]$, is zero as well. Thus, the value function is continuous at h_*. Due to the same reasons, the left derivative of $\mathcal{V}$ at h_* is zero. Evaluating the derivative of the RHS of (8.22) at h_*, we find the right derivative

$$
\begin{aligned}
\mathcal{V}'(h_* + 0) &= C\left[\kappa_q^-(1) + \beta^-(1 - \kappa_q^-(1))\right] \\
&= C\left[\frac{(\lambda^- - 1)(-\beta^-)}{\lambda^-(1 - \beta^-)}(1 - \beta^-) + \beta^-\right] \\
&= C\beta^-\left[1 - \frac{\lambda^- - 1}{\lambda^-}\right] \\
&= C\frac{\beta^-}{\lambda^-} > 0,
\end{aligned}
$$

therefore, the smooth pasting principle fails. We could have guessed the optimal exit boundary using the continuous pasting principle. Indeed, using (8.21) with an arbitrary h instead of h_*, we find that

$$
\mathcal{V}(h) = \frac{\beta^-}{\lambda^-}(G\kappa_q^+(1)\mathrm{e}^h - C)
$$

is zero iff $h = h_*$ solves $G\kappa_q^+(1)\mathrm{e}^h - C = 0$.

8.4 Continuous and discontinuous payoff functions

In the preceding sections, the following subtle point has been brushed under the carpet. From the very beginning, we assumed that the payoff function was continuous. This implied that the function $w = \mathcal{E}^+ g$ was continuous, too, hence, 0 at the threshold. Therefore, in Sect. 8.2, it did not matter whether the firm exits the first time X_t reaches or crosses h_* or only when X_t crosses h_*. However, if g is monotone but discontinuous it may be the case that $w = \mathcal{E}^+ g$ has no zero. In this case, the natural substitute for h_*, the zero of w, is a unique h_* such that $\mathcal{E}^+ g(x) \geq 0 \ \forall\ x \geq h_*$, and $\mathcal{E}^+ g(x) \leq 0 \ \forall\ x \leq h_*$. It is possible that $w(h_*) = \mathcal{E}^+ g(h_*) > 0$, and then (8.17) is not optimal. Indeed,

$$
\mathcal{V}(x; h_*) = (\mathcal{E}^- \mathbf{1}_{[h_*, +\infty)} \mathcal{E}^+ g)(x) \tag{8.23}
$$

exceeds the value given by (8.17) at $x = h_*$: the RHS in (8.23) equals $\frac{\beta^-}{\lambda^-}w(h_*) > 0$, and the RHS in (8.17) equals 0. Modifying the proof of (8.17), we obtain that (8.23) is the value of the firm that exits the first time X_t *crosses* h_* from above (but *does not exit at* h_*). Note that in this case, the continuous pasting condition fails: $\mathcal{V}(x; h_*) = 0$ for $x < h_*$ but $\mathcal{V}(x; h_*) = \frac{\beta^-}{\lambda^-}w(h_*) > 0$. Similar reservations must be made for the other optimal stopping problems, which we will consider in this chapter and the following one.

8.5 Interpretation in terms of the EPV-operators under the supremum and infimum processes

In Subsect. 9.1.2, we will show that the operators $\mathcal{E}^\pm$ admit another interpretation as normalized EPV-operators under supremum and infimum processes $\bar{X}_t = \sup_{0\le s\le t} X_s$ and $\underline{X}_t = \inf_{0\le s\le t} X_s$:

$$\mathcal{E}^+ g(x) = (1-q)E\left[\sum_{t\ge 0} q^t g(\bar{X}_t) \mid X_0 = x\right], \tag{8.24}$$

$$\mathcal{E}^- g(x) = (1-q)E\left[\sum_{t\ge 0} q^t g(\underline{X}_t) \mid X_0 = x\right]. \tag{8.25}$$

Recall that that $\bar{X}_t$ and $\underline{X}_t$ are defined path-wise: for each sample path ω, $\bar{X}_t(\omega) = \sup_{0\le s\le t} X_s(\omega)$ and $\underline{X}_t(\omega) = \inf_{0\le s\le t} X_s(\omega)$. Now, the optimal exit rule can be formulated as follows: exit the first time $(\mathcal{E}^+ g)(X_t)$ becomes non-positive. If $g(X_t)$ is a non-decreasing function of X_t, as we presumed, we have $g(\bar{X}_t) = \bar{g}_t \equiv \max_{0\le s\le t} g_s$, where $g_t = g(X_t)$, therefore we can reformulate the exit rule in terms of the supremum process: exit at level g if

$$E\left[\sum_{t=0}^{\infty} q^t \bar{g}_t \mid g_0 = g\right] \le 0.$$

In other words, the rule is: consider all sample paths of the process, and along each sample path, disregard all *temporary drops* of the output price. Then calculate the EPV of profits, and if it is non-positive, abandon the stream. Thus, the hope for the best dies hard: we exit only when the EPV is non-positive even after this rosy adjustment. It looks as if a firm's manager contemplating an exit is too optimistic. However, we will see that the same manager becomes overpessimistic when contemplating an investment.

8.6 Exit under supply uncertainty

Suppose that the price of the firm's output, P, is constant, but the variable cost follows the geometric random walk: $C = e^{X_t}$. The instantaneous profit

$g(X_t) = PG - e^{X_t}$ is a decreasing function of X_t, and it is positive at low levels of X_t and negative at high levels of X_t. It may be optimal to exit should the cost become too high. Assuming the exit threshold, h, is chosen, one can calculate the firm's value, $V(x; h)$, equivalently, the EPV of the stream $g(X_t)$ with the option to abandon it, using the following theorem. Its formulation and proof are the mirror reflections of Theorem 8.1.1 and its proof.

Theorem 8.6.1 *Assume that (7.31), (7.32) and (7.33) hold. Then*

$$V(x; h) = (1 - q)^{-1}(\mathcal{E}^+ \mathbf{1}_{(-\infty,h)} \mathcal{E}^- g)(x). \tag{8.26}$$

Consider an optimal choice of the exit boundary h. We assume that

$$g \text{ is non} - \text{increasing and continuous}, \tag{8.27}$$

$$g(+\infty) < 0 < g(-\infty) \tag{8.28}$$

(one limit or both may be infinite; in the exit problem above, only $g(+\infty)$ is infinite). From (8.26), we have

$$V(x; h) = (1 - q)^{-1}E[(\mathbf{1}_{(-\infty,h)} w)(x + Y^+)], \tag{8.29}$$

where $w(x) = \mathcal{E}^- g(x) = E[g(x + Y^-)]$. The larger the value of the product $\mathbf{1}_{(-\infty,h)} w$, the larger is the value $V(x; h)$. Hence, the optimal choice of h should replace all negative values of w by zero, and leave positive ones as they are. Clearly, function $w(x) = E[g(x + Y^-)]$ is continuous. Since g is non-increasing, w is non-increasing as well. Further, passing to the limit as $x \to \pm\infty$ in the equality $w(x) = E[g(x+Y^-)]$, we obtain that w satisfies (8.28) since g does. Moreover, it is easy to see from (7.30) that if g is decreasing in a neighborhood of $-\infty$, then w is decreasing on $\mathbb{R}$, and if g is constant on $(-\infty, x_-]$ but $g(x) < g(x_-)$, $\forall\ x > x_-$, then w is decreasing above x_-. We conclude that w has a unique zero, call it h^*, and $w(x) > 0$ for all $x < h^*$, and $w(x) < 0$ for all $x > h^*$. Hence, h^* is the optimal exit threshold.

After the optimal exit threshold h^* had been found, the manager calculates the normalized firm's value for $x < h^*$ using (8.26) and (7.29):

$$\begin{aligned}\mathcal{V}(x) &= \frac{\beta^+}{\lambda^+} w(x) + \frac{\lambda^+ - \beta^+}{\lambda^+} \int_0^{+\infty} \beta^+ e^{-\beta^+ y} \mathbf{1}_{(-\infty,h^*)}(x + y) w(x + y) dy \\ &= \frac{\beta^+}{\lambda^+} w(x) + \frac{\lambda^+ - \beta^+}{\lambda^+} \int_0^{h^* - x} \beta^+ e^{-\beta^+ y} w(x + y) dy. \end{aligned} \tag{8.30}$$

Example 8.3. Let $g(x) = PG - e^x$, and assume for simplicity that $1 - qM(1) > 0$. Under this condition, $\lambda^- < \beta^- < 0 < 1 < \beta^+ < \lambda^+$. Since $w(x) = \mathcal{E}^-(PG - e^{\cdot})(x) = PG - \kappa_q^-(1)e^x$, the optimal exit threshold is defined from

$$PG = \kappa_q^-(1)e^{h^*}. \tag{8.31}$$

Using (8.30), we calculate the normalized value of the firm for $x < h^*$:

$$\begin{aligned}\mathcal{V}(x) &= \frac{\beta^+}{\lambda^+}(PG - \kappa_q^-(1)e^x) \\ &\quad + \frac{\lambda^+ - \beta^+}{\lambda^+}\int_0^{h^*-x} \beta^+ e^{-\beta^+ y}(PG - \kappa_q^-(1)e^{x+y})dy \\ &= \frac{\beta^+}{\lambda^+}(PG - \kappa_q^-(1)e^x) + \frac{\lambda^+ - \beta^+}{\lambda^+}\Big[-PG(e^{\beta^+(x-h^*)} - 1) \\ &\qquad + \frac{\beta^+}{\beta^+ - 1}\kappa_q^-(1)(e^{h^*+\beta^+(x-h^*)} - e^x)\Big].\end{aligned} \tag{8.32}$$

Using (8.31) and then (7.22), we simplify

$$\begin{aligned}\mathcal{V}(x) &= PG\Big[-\left(\frac{\beta^+}{\lambda^+} + \frac{\lambda^+ - \beta^+}{\lambda^+}\cdot\frac{\beta^+}{\beta^+ - 1}\right)e^{x-h^*} + \left(\frac{\lambda^+ - \beta^+}{\lambda^+} + \frac{\beta^+}{\lambda^+}\right) \\ &\qquad + \frac{\lambda^+ - \beta^+}{\lambda^+}\left(\frac{\beta^+}{\beta^+ - 1} - 1\right)e^{\beta^+(x-h^*)}\Big] \\ &= PG\left[1 - \kappa_q^+(1)e^{x-h^*} + \frac{\lambda^+ - \beta^+}{\lambda^+(\beta^+ - 1)}e^{\beta^+(x-h^*)}\right] \\ &= PG\left[1 - \kappa_q^+(1)e^{x-h^*} + (\kappa_q^+(1) - 1)e^{\beta^+(x-h^*)}\right].\end{aligned} \tag{8.33}$$

Using (8.31) once again and the Wiener–Hopf factorization formula (7.23), we can decompose the firm's value into the sum of the value of the firm, which produces forever, and option value to exit, call it $V_{\mathrm{opt}}(x)$:

$$\begin{aligned}V(x) &= (1-q)^{-1}(PG - \kappa_q^+(1)\kappa_q^-(1)e^x) + V_{\mathrm{opt}}(x) \\ &= (1-q)^{-1}PG - (1 - qM(1))^{-1}e^x + V_{\mathrm{opt}}(x),\end{aligned}$$

where

$$V_{\mathrm{opt}}(x) = \frac{PG}{1-q}(\kappa_q^+(1) - 1)e^{\beta^+(x-h^*)}.$$

8.7 Entry

8.7.1 Entry under demand uncertainty

The firm's manager contemplates the investment into a plant that will produce G units of output at no variable cost starting the moment the investment is made[1]. The price of a unit of output evolves as e^{X_t}. The fixed investment cost is I. Should the price of output rise sufficiently high, it will be optimal to invest. The manager has to find an optimal investment threshold, denote it h^*. To solve this problem, we may interpret the fixed investment cost as the present value of the coupon payments $(1-q)I$ starting the moment the

[1] This assumption simplifies the argument below but it is unnecessary.

investment is made[2]. Then the optimal timing of investment is equivalent to the problem of an optimal exercise of the (perpetual) option to acquire the stream of payoffs $g(X_t) = Ge^{X_t} - (1-q)I$, with zero strike. Let h be a candidate for the optimal investment threshold, and denote by τ_h^+ the first time X_t reaches or crosses h from below. Then the EPV of the investment opportunity is

$$V(x;h) = E^x\left[\sum_{t=\tau_h^+}^{\infty} q^t g(X_t)\right] = E^x\left[\sum_{t=0}^{\infty} q^t g(X_t)\right] + W(x;h),$$

where

$$W(x;h) = E^x\left[\sum_{0\le t<\tau_h^+} q^t(-g(X_t))\right]$$

is the EPV of the stream $-g(X_t)$ which is abandoned the first time X_t reaches or crosses h from below. Thus,

$$V(x;h) = (1-q)^{-1}\mathcal{E}g(x) + W(x;h). \tag{8.34}$$

The first term on the RHS is independent of h, therefore, an optimal h that maximizes $V(h;x)$ maximizes $W(h;x)$, and vice versa. Since $-g$ is non-increasing, the maximization of $W(h;x)$ is, essentially, the exit problem under supply uncertainty. Using (8.26), we obtain

$$W(x;h) = (1-q)^{-1}\mathcal{E}^+\mathbf{1}_{(-\infty,h)}\mathcal{E}^-(-g)(x). \tag{8.35}$$

But $\mathcal{E} = \mathcal{E}^+\mathcal{E}^-$, therefore, substituting (8.35) into (8.34), we obtain, for the normalized value function $\mathcal{V} = (1-q)V$,

$$\begin{aligned}\mathcal{V}(x;h) &= \mathcal{E}^+\mathcal{E}^-g(x) - \mathcal{E}^+\mathbf{1}_{(-\infty,h)}\mathcal{E}^-g(x)\\ &= \mathcal{E}^+(\mathbf{1}_{(-\infty,h)} + \mathbf{1}_{[h,+\infty)})\mathcal{E}^-g(x) - \mathcal{E}^+\mathbf{1}_{(-\infty,h)}\mathcal{E}^-g(x),\end{aligned}$$

and, finally,

$$V(x;h) = (1-q)^{-1}\mathcal{E}^+\mathbf{1}_{[h,+\infty)}\mathcal{E}^-g(x). \tag{8.36}$$

Since g is an increasing function which changes sign, $\mathcal{E}^-g$ also enjoys these properties. Therefore, there exists a unique h^* such that $\mathcal{E}^-g(x) > 0$ for all $x > h^*$, and $\mathcal{E}^-g(x) < 0$ for all $x < h^*$. We conclude that h^* is the optimal investment threshold. The value of the investment opportunity is

$$V(x) = (1-q)^{-1}\mathcal{E}^+\mathbf{1}_{[h^*,+\infty)}\mathcal{E}^-g(x). \tag{8.37}$$

Now, the optimal investment rule can be formulated as follows: invest the first time $(\mathcal{E}^-g)(X_t)$ becomes non-negative. If $g(X_t)$ is a non-decreasing function of X_t, as we presumed, we have

[2] This interpretation presumes that the firm will never default although it may be optimal to do so.

$$g(\underline{X}_t) = \underline{g}_t \equiv \min_{0 \le s \le t} g_s,$$

where $g_t = g(X_t)$, therefore we can reformulate the investment rule in terms of the infimum process: invest at level g if

$$E\left[\sum_{t=0}^{\infty} q^t \underline{g}_t \mid g_0 = g\right] \ge 0.$$

In other words, the rule is: consider all sample paths of the process, and along each sample path, disregard all *temporary increases* of the profit flow. Then calculate the EPV of profits, and if it is non-positive, give up the right for the stream. Thus, the manager is extremely cautious or too pessimistic: she invests only when the EPV is non-negative even after this worst-case scenario adjustment. We say that she uses the *bad news principle*.

Example 8.4. In the case $g(x) = Ge^x - (1-q)I$, $w(x) = \kappa_q^-(1)Ge^x - (1-q)I$, therefore the investment threshold, h^*, is a unique solution of the equation

$$\kappa_q^-(1)Ge^{h^*} = (1-q)I. \tag{8.38}$$

Applying further (8.36) and (7.29), and then (8.38), we calculate the value of the investment opportunity for $x < h^*$:

$$\begin{aligned} V(x) &= \frac{\lambda^+ - \beta^+}{(1-q)\lambda^+} \int_0^{+\infty} \beta^+ e^{-\beta^+ y} \mathbf{1}_{[h^*,+\infty)} w(x+y)dy \\ &= \frac{\lambda^+ - \beta^+}{(1-q)\lambda^+} \int_{h^*-x}^{+\infty} \beta^+ e^{-\beta^+ y} (\kappa_q^-(1)Ge^{x+y} - (1-q)I)dy \\ &= \frac{\lambda^+ - \beta^+}{(1-q)\lambda^+} e^{\beta^+(x-h^*)} \left[\frac{G\kappa_q^-(1)\beta^+}{\beta^+ - 1} e^{h_*} - (1-q)I\right] \\ &= I \frac{\lambda^+ - \beta^+}{\lambda^+(\beta^+ - 1)} e^{\beta^+(x-h^*)}. \end{aligned} \tag{8.39}$$

8.7.2 Entry under supply uncertainty

The firm's manager contemplates the investment into a plant that will yield the constant revenue flow R starting the moment the investment is made. The variable cost is stochastic: $C(X_t) = \min\{e^{X_t}, C_m\}$, where $0 < C_m < R$. The fixed investment cost is I. Should the variable cost fall sufficiently low, it will be optimal to invest. The manager has to find an optimal investment threshold, denote it h_*. To make the investment irreversible, we assume that the scrap value, Sc, that can be recovered should the firm decide to exit, does not exceed the present value of the lowest profit flow $R - C_m$. We also assume that if the highest level of variable cost, C_m, is presumed to persist forever, then the present value of profits is smaller that the fixed investment cost. On the other hand, to make the investment problem non-trivial, we assume that if the variable cost is zero, then the present value of profits is higher than the fixed investment cost, and it is optimal to invest. Equivalently,

$$R - C_m < (1-q)I < R. \tag{8.40}$$

To solve this investment problem, we may interpret the fixed investment cost as the present value of the coupon payments $(1-q)I$ starting at the moment the investment is made. Then the optimal timing of investment is equivalent to the problem of an optimal exercise of the perpetual option on the stream of payoffs $g(X_t) = R - \min\{e^{X_t}, C_m\} - (1-q)I$, with zero strike. Under condition (8.40), g is non-increasing and changes sign. Let h be a candidate for the optimal investment threshold. Then the EPV of the investment opportunity is

$$V(x;h) = E^x\left[\sum_{t=\tau_h^-}^{\infty} q^t g(X_t)\right] = E^x\left[\sum_{t=0}^{\infty} q^t g(X_t)\right] + W(x;h),$$

where

$$W(x;h) = E^x\left[\sum_{0\le t<\tau_h^-} q^t(-g(X_t))\right]$$

is the EPV of the stream $-g(X_t)$ which is abandoned the first time X_t reaches or crosses h from above. Thus,

$$V(x;h) = (1-q)^{-1}\mathcal{E}g(x) + W(x;h). \tag{8.41}$$

The first term on the RHS is independent of τ_h^-, therefore, an optimal h that maximizes $V(h;x)$ maximizes $W(h;x)$, and vice versa. Since $-g(X_t)$ is non-decreasing, the maximization of $W(h;x)$ is a problem similar to the exit problem considered in Sect. 8.2. We obtain

$$W(x;h) = (1-q)^{-1}\mathcal{E}^-\mathbf{1}_{(h,+\infty)}\mathcal{E}^+(-g)(x). \tag{8.42}$$

But $\mathcal{E} = \mathcal{E}^+\mathcal{E}^-$, therefore, substituting (8.42) into (8.41), we obtain, for the normalized value function $\mathcal{V} = (1-q)V$,

$$\begin{aligned}\mathcal{V}(x;h) &= \mathcal{E}^-\mathcal{E}^+g(x) - \mathcal{E}^-\mathbf{1}_{(h,+\infty)}\mathcal{E}^+g(x)\\ &= \mathcal{E}^-(\mathbf{1}_{(h,+\infty)} + \mathbf{1}_{(-\infty,h]})\mathcal{E}^+g(x) - \mathcal{E}^-\mathbf{1}_{(h,+\infty)}\mathcal{E}^+g(x),\end{aligned}$$

and finally,

$$V(x;h) = (1-q)^{-1}\mathcal{E}^-\mathbf{1}_{(-\infty,h]}\mathcal{E}^+g(x). \tag{8.43}$$

Since g is a non-decreasing function that changes sign, $\mathcal{E}^+g$ enjoys these properties as well. Moreover, $\mathcal{E}^+g$ decreases on $(-\infty,h']$, where h' is the smallest number such that on $[h',+\infty)$, functions g and $\mathcal{E}^+g$ equal to the negative constant $R-(1-q)I-C_m$. Therefore, there exists $h_* < h'$ such that $\mathcal{E}^+g(x) > 0$

for all $x < h_*$, and $\mathcal{E}^+g(x) < 0$ for all $x > h_*$. We conclude that h_* is an optimal investment threshold. Note that now the investment rule is formulated in terms of the supremum process of X_t; however, if we reformulate it in terms of the profit flow, we will obtain the same bad news principle for investment decisions. In the case under consideration, we can explicitly calculate $\mathcal{E}^+g(x)$, for $x < h'$,

$$\begin{aligned}\mathcal{E}^+g(x) &= \frac{\beta^+}{\lambda^+}(R-(1-q)I-\mathrm{e}^x)\\ &\quad+\frac{\lambda^+-\beta^+}{\lambda^+}\int_0^{h'-x}\beta^+\mathrm{e}^{-\beta^+y}(R-(1-q)I-\mathrm{e}^{x+y})dy\\ &\quad+\frac{\lambda^+-\beta^+}{\lambda^+}\int_{h'-x}^{+\infty}\beta^+\mathrm{e}^{-\beta^+y}(R-(1-q)I-C_m)dy\\ &= \frac{\beta^+}{\lambda^+}(R-(1-q)I-\mathrm{e}^x)+\frac{\lambda^+-\beta^+}{\lambda^+}\\ &\quad\times\left[R-(1-q)I-C_m\mathrm{e}^{\beta^+(x-h')}+\frac{\beta^+}{\beta^+-1}(\mathrm{e}^{x+(\beta^+-1)(x-h')}-\mathrm{e}^x)\right]\end{aligned}$$

Since we know that the function $\mathcal{E}^+g$ decreases on $(-\infty, h']$ and changes sign, we can find h_* numerically quite easily. The investment threshold having being found, we use (8.43) and (7.30) to calculate the value of the investment opportunity, $V(x)$, for $x > h_*$ (for $x \le h_*$, $V(x)$ is the EPV of the stream $g(X_t)$). We leave the simplification of the expression for $\mathcal{E}^+g(x)$ and the explicit calculation of $V(x)$ to the reader.

8.8 Perpetual American options

8.8.1 Perpetual American call options

Let $G(X_t)$ be the instantaneous payoff which is an increasing function of X_t. For example, $G(X_t) = S(X_t) - K$ for the call option, where $S(X_t) = \mathrm{e}^{X_t}$ is the price of the underlying asset. Should X_t rise sufficiently high, it may be optimal to exercise the option with the instantaneous payoff $G(X_t)$. Assume that we can express $G(X_t)$ in terms of the EPV of a stream g_t: $G = (1-q)^{-1}\mathcal{E}g$. Since $(1-qP)(1-q)^{-1}\mathcal{E} = I$, we can find g:

$$g(X_t) = (I - qP)G(X_t). \tag{8.44}$$

Note that the representation of the instantaneous payoff G as the EPV of a stream is impossible in the case of the call option on a stock that pays no dividends because the discounted price process of the stock must be a martingale, and, therefore, $\mathrm{e}^x - E^x[q\mathrm{e}^{X_1}] = (I-qP)\mathrm{e}^x$ must be 0. If $1 - qM(1) > 0$, then the stock pays dividends. If, at time t, the dividends are paid after time-t trades are made, then, to exclude arbitrage opportunity, the dividends must equal the difference between the stock price today and expected discounted price tomorrow:

$$\delta(x) = \mathrm{e}^x - E^x[q\mathrm{e}^{X_1}] = \mathrm{e}^x - qM(1)\mathrm{e}^x = (1 - qM(1))\mathrm{e}^x.$$

If the fraction δ of the asset price is distributed as dividends before the trades are made, then, to avoid arbitrage, it must be that

$$\mathrm{e}^x - E^x[q(1-\delta)\mathrm{e}^{X_1}](= (1 - q(1-\delta)M(1))\mathrm{e}^x) = 0,$$

and $\delta = 1/qM(1) - 1$. Assume that $1 - qM(1) > 0$. Then, from (8.44), we obtain that $G(X_t)$ is the EPV of the stream

$$g(X_t) = (1 - qM(1))\mathrm{e}^{X_t} - (1-q)K.$$

Therefore, the results of Sect. 8.7.1 are applicable. Let h be a candidate for the exercise boundary. Then, applying (8.36), we obtain the following formula for the American call price

$$V_{\mathrm{am.call}}(x; h) = (1-q)^{-1}\mathcal{E}^+\mathbf{1}_{[h,+\infty)}\mathcal{E}^- g(x). \tag{8.45}$$

Using the Wiener-Hopf factorization formula and (8.44), we derive

$$\mathcal{E}^- g(x) = (1-q)\mathcal{E}^-(1-q)^{-1}(1-qP)G(x) = (1-q)(\mathcal{E}^+)^{-1}G(x),$$

and rewrite (8.45) as

$$V_{\mathrm{am.call}}(x; h) = \mathcal{E}^+\mathbf{1}_{[h,+\infty)}(\mathcal{E}^+)^{-1}G(x). \tag{8.46}$$

Function $(\mathcal{E}^+)^{-1}G(x) = \kappa_q^+(1)^{-1}\mathrm{e}^x - K$ is an increasing function that changes sign only once. Hence, the unique solution of the equation

$$\mathrm{e}^{h^*} = K\kappa_q^+(1) \tag{8.47}$$

is an optimal exercise boundary, and the rational call option price is given by

$$V_{\mathrm{am.call}}(x) = \mathcal{E}^+\mathbf{1}_{[h^*,+\infty)}(\mathcal{E}^+)^{-1}G(x). \tag{8.48}$$

Explicitly, using (8.48) and (8.47), we obtain for $x < h^*$, similarly to (8.39):

$$V_{\mathrm{am.call}}(x) = \mathrm{e}^{\beta^+(x-h^*)}(\mathrm{e}^{h^*} - K). \tag{8.49}$$

We leave the details to the reader.

8.8.2 Perpetual American put options

Let $G(X_t)$ be the instantaneous payoff which is a decreasing function of X_t. For example, $G(X_t) = K - S(X_t)$ for the put option, where $S(X_t) = \mathrm{e}^{X_t}$ is the price of the underlying security. Should X_t fall sufficiently low, it may be optimal to exercise the option with the instantaneous payoff $G(X_t)$. Assume that we can express $G(X_t)$ in terms of the EPV of a stream g_t: $G = (1-q)^{-1}\mathcal{E}g$. Then $g = (1 - qP)G$. If the stock does not pay dividends, we cannot apply this procedure with $G(X_t) = K - e^{X_t}$, but, since the option is not exercised if the payoff is negative, we may replace $K - e^{X_t}$ with $G_1(X_t) := (G(X_t))_+ = (K - e^{X_t})_+$. Being bounded, $G_1(X_t)$ is representable as the EPV of the stream $g(X_t) = (I - qP)G_1(X_t)$. Let h be a candidate for the exercise boundary. Then, applying (8.43), we obtain the following formula for the American put price

$$V_{\mathrm{am.put}}(x; h) = (1 - q)^{-1}\mathcal{E}^-\mathbf{1}_{(-\infty,h]}\mathcal{E}^+g(x). \tag{8.50}$$

Using the equality $g = (I - qP)G_1$ and the Wiener-Hopf factorization formula, we derive

$$\mathcal{E}^+g(x) = (1 - q)\mathcal{E}^+(1 - q)^{-1}(1 - qP)G_1(x) = (1 - q)(\mathcal{E}^-)^{-1}G_1(x),$$

and rewrite (8.50) as

$$V_{\mathrm{am.put}}(x; h) = \mathcal{E}^-\mathbf{1}_{(-\infty,h]}(\mathcal{E}^-)^{-1}G_1(x). \tag{8.51}$$

It is non-optimal to exercise the option unless $G(h) \geq 0$ but for $x \leq h$, we find from (7.34) and (7.28) $(\mathcal{E}^-)^{-1}G_1(x) = A^-G_1(x) = A^-G(x) = (\mathcal{E}^-)^{-1}G(x)$. (Clearly, $A^-G(x)$ is independent of values $G(y)$ for $y > x$.) Therefore, we may replace G_1 in (8.51) with G. Assume that the function $(\mathcal{E}^-)^{-1}G$ is a decreasing function that changes sign only once. Then the optimal exercise boundary is the solution of the equation

$$(\mathcal{E}^-)^{-1}G(h_*) = 0, \tag{8.52}$$

and the rational put option price is

$$V_{\mathrm{am.put}}(x) = \mathcal{E}^-\mathbf{1}_{(-\infty,h_*]}(\mathcal{E}^-)^{-1}G(x). \tag{8.53}$$

For the standard American put option, $(\mathcal{E}^-)^{-1}G(x) = K - \kappa_q^-(1)^{-1}e^x$ is decreasing, and, therefore, the exercise boundary is the solution of the equation $e^{h_*} = K\kappa_q^-(e)$. For $x > h_*$, similarly to (8.49),

$$V_{\mathrm{am.put}}(x) = (K - e^{h_*})e^{x-h_*}. \tag{8.54}$$

We leave details of the calculation to the reader.

8.9 Expected waiting time

Consider an option to acquire a stream of payoffs $g(X_t)$, where g is an increasing function. Assume that the current value $X_0 = x < h^*$, and consider the waiting time R_x till the option will be exercised. This is the random variable defined by $R_x = \min\{t > 0 \mid X_t \geq h^*\}$. The expected waiting time (under the historic measure $\mathbb{P}$) can be calculated as follows:

$$E_{\mathbb{P}}[R_x] = E^x_{\mathbb{P}}\left[\sum_{t=0}^{\infty} \mathbf{1}_{(-\infty,h^*)}(\bar{X}_t)\right] = \lim_{q\to 1-0} E^x_{\mathbb{P}}\left[\sum_{t=0}^{\infty} q^t \mathbf{1}_{(-\infty,h^*)}(\bar{X}_t)\right]$$
$$= \lim_{q\to 1-0} \frac{1}{1-q}\mathcal{E}^+\mathbf{1}_{(-\infty,h^*)}(x) = \lim_{q\to 1-0} \frac{1}{1-q}\int_0^{h^*-x} p^+_{\mathbb{P},\,q}(y)dy,$$

where $p^+_{\mathbb{P},\,q}$ is the probability density of the random variable $Y^+_{\mathbb{P}} = Y^+_{\mathbb{P}}(q)$ defined as Y^+ but using the historic measure instead of the risk-neutral measure. The q above is just an auxiliary parameter needed for computational purposes, which will tend to 1 below, and not the discount factor used in the preceding subsections. If the transition density $p_{\mathbb{P}}$ is given by exponential polynomials on each of half-axis, then the limit can be easily calculated. In particular, if the transition density $p_{\mathbb{P}}$ is given by (7.5) with parameters $\lambda^{\pm} = \lambda^{\pm}_{\mathbb{P}}, c^{\pm} = c^{\pm}_{\mathbb{P}}$ (generally, they are different from the ones for $p = p_{\mathbb{Q}}$), we obtain for $x < h^*$:

$$\frac{1}{1-q}\int_0^{h^*-x} p^+_{\mathbb{P},\,q}(y)dy = \frac{(1-\mathrm{e}^{-\beta^+(q)(h^*-x)}) + \mathrm{e}^{-\beta^+(q)(h^*-x)}\beta^+(q)/\lambda^+}{1-q},$$

where $\beta^+(q) = \beta^+(\mathbb{P}, q)$ is given by (7.20). Both terms in the numerator are positive, therefore, if $\beta^+(q)/(1-q)$ is unbounded as $q \to 1$, the limit is clearly infinite, and hence, the expected waiting time is infinite. From (7.20), we find that $\beta^+(q)/(1-q)$ is bounded as $q \to 1$ iff $\lambda^- + \lambda^+ < 0$. If $\lambda^- + \lambda^+ < 0$, we obtain

$$\beta^+(q) = \frac{\lambda^+\lambda^-}{\lambda^+ + \lambda^-}(1-q) + O((1-q)^2),$$

and therefore

$$E_{\mathbb{P}}[R_x] = \frac{1}{m}(h^* - x + 1/\lambda^+), \tag{8.55}$$

where $m = 1/\lambda^+ + 1/\lambda^- = E_{\mathbb{P}}[X_1 - X_0]$. Condition $\lambda^- + \lambda^+ < 0$ admits a clear interpretation: the expected waiting time is finite iff under the historic measure, the drift of the underlying factor, m, is positive, and if it is positive, then (8.55) says that the expected waiting time is inversely proportional to the drift. It is also proportional to the distance to the barrier *plus a constant term* $1/(m\lambda^+) = 1/(1 - c^-/c^+) > 0$, which increases with the frequency of downward jumps.

Problems

8.1. Derive (8.54).

8.2. Solve Problems 5.2-5.9 assuming that X_t is the random walk with the transition density (7.5).

9

Optimal stopping for general random walks

9.1 Wiener-Hopf factorization

9.1.1 Three forms of the Wiener-Hopf factorization

Let T be the geometric random variable on $\mathbb{Z}_+$ with $\mathrm{Prob}(T = t) = (1-q)q^t$. Then The Wiener–Hopf factorization formula states that for purely imaginary z, $\mathrm{Re}\, z = 0$,

$$E[\mathrm{e}^{zX_T}] = E[\mathrm{e}^{z\bar{X}_T}]E[\mathrm{e}^{z\underline{X}_T}]. \tag{9.1}$$

Equation (9.1) follows from:

- $X_T = \bar{X}_T + X_T - \bar{X}_T$;
- $\bar{X}_T$ and $X_T - \bar{X}_T$ are independent;
- the characteristic function of the sum of two independent random variables is the product of the characteristic functions;
- probability distributions of $\underline{X}_T$ and $X_T - \bar{X}_T$ are the same.

See Sect. I.29 in [77] and the references therein. Introduce the notation

$$\kappa_q^+(z) = (1-q)E\left[\sum_{t=0}^{\infty} q^t \mathrm{e}^{z\bar{X}_t}\right], \tag{9.2}$$

$$\kappa_q^-(z) = (1-q)E\left[\sum_{t=0}^{\infty} q^t \mathrm{e}^{z\underline{X}_t}\right]. \tag{9.3}$$

The LHS in (9.1) being $(1-q)/(1-qM(z))$, we can write the Wiener-Hopf factorization formula in an equivalent form

$$\frac{1-q}{1-qM(z)} = \kappa_q^+(z)\kappa_q^-(z). \tag{9.4}$$

To obtain the third form, *define* the EPV-operators $\mathcal{E}^\pm$ by (8.24)-(8.25), and, assuming that X starts at 0, introduce random variables $Y^+ = \bar{X}_T$ and $Y^- = X_T - \bar{X}_T \sim \underline{X}_T$ on $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively. Then

$$\mathcal{E}^+ g(x) = E[g(x + Y^+)], \quad \mathcal{E}^- g(x) = E[g(x + Y^-)]. \tag{9.5}$$

Apply $\mathcal{E}$ and the product of operators $\mathcal{E}^\pm$ to a function $g(x)$ of the form $g(x) = g(z; x) = \mathrm{e}^{zx}$, where $z \in \mathbb{C}, \operatorname{Re} z = 0$. Assuming that X_t starts at 0, we have

$$(\mathcal{E}\mathrm{e}^{z\cdot})(x) = (1-q)E\left[\sum_{t=0}^{\infty} q^t \mathrm{e}^{z(x+X_t)}\right] = \mathrm{e}^{zx}(1-q)E\left[\sum_{t=0}^{\infty} q^t \mathrm{e}^{zX_t}\right]$$
$$(\mathcal{E}^+\mathrm{e}^{z\cdot})(x) = (1-q)E\left[\sum_{t=0}^{\infty} q^t \mathrm{e}^{z(x+\bar{X}_t)}\right] = \mathrm{e}^{zx}(1-q)E\left[\sum_{t=0}^{\infty} q^t \mathrm{e}^{z\bar{X}_t}\right]$$
$$(\mathcal{E}^-\mathrm{e}^{z\cdot})(x) = (1-q)E\left[\sum_{t=0}^{\infty} q^t \mathrm{e}^{z(x+\underline{X}_t)}\right] = \mathrm{e}^{zx}(1-q)E\left[\sum_{t=0}^{\infty} q^t \mathrm{e}^{z\underline{X}_t}\right],$$

which gives

$$(\mathcal{E}\mathrm{e}^{z\cdot})(x) = \frac{1-q}{1-qM(z)}\mathrm{e}^{zx}, \tag{9.6}$$
$$(\mathcal{E}^+\mathrm{e}^{z\cdot})(x) = \kappa_q^+(z)\mathrm{e}^{zx}, \tag{9.7}$$
$$(\mathcal{E}^-\mathrm{e}^{z\cdot})(x) = \kappa_q^-(z)\mathrm{e}^{zx}. \tag{9.8}$$

Therefore, using (9.4), we obtain

$$\mathcal{E}g = \mathcal{E}^-\mathcal{E}^+ g = \mathcal{E}^+\mathcal{E}^- g. \tag{9.9}$$

To show that (9.9) holds for $g \in \mathcal{L}_\infty(\mathbb{R})$ and for g from wider classes of functions, note that

$$\begin{aligned} E[g(x + X_T)] &= E[g(x + \bar{X}_T + X_T - \bar{X}_T)] \\ &= E[g(x + Y^+ + Y^-)] = (\mathcal{E}^+ g)(x + Y^-) = (\mathcal{E}^+\mathcal{E}^- g)(x), \end{aligned}$$

which gives $\mathcal{E}g = \mathcal{E}^+\mathcal{E}^- g$. The second equality in (9.9) is proved similarly. Thus, we have the operator form of the Wiener–Hopf factorization

$$\mathcal{E} = \mathcal{E}^+\mathcal{E}^- = \mathcal{E}^-\mathcal{E}^+, \tag{9.10}$$

where each operator is understood as an operator in $\mathcal{L}_\infty(\mathbb{R})$ (or in a wider function space).

9.1.2 Uniqueness of the Wiener–Hopf factorization

There exist general analytical formulas for $\kappa_q^\pm(z)$ in terms of the probability density $P_t(dy) = p_t(y)dy$ of X_t started at 0:

$$\kappa_q^+(z) = \exp\left[\sum_{t=1}^{\infty} \frac{q^t}{t} \int_0^{+\infty} (\mathrm{e}^{zy} - 1)P_t(dy)\right], \tag{9.11}$$
$$\kappa_q^-(z) = \exp\left[\sum_{t=1}^{\infty} \frac{q^t}{t} \int_{-\infty}^{0} (\mathrm{e}^{zy} - 1)P_t(dy)\right]. \tag{9.12}$$

See, e.g., [49], p.72. Formulas (9.11)–(9.12) are rather involved. Fortunately, the following general result allows one to guess explicit formulas for $\kappa_q^{\pm}(z)$ without calculating the integrals and infinite sums in (9.11)–(9.12). Recall that a function is called analytic in an open subset U of $\mathbb{C}$, if it is differentiable at each point of U. Following [49], we will say that a function is analytic in the closure of an open set $U \in \mathbb{C}$ if it is continuous on the closure U and analytic in U.

Lemma 9.1.1 *Let f be a continuous bounded function on the imaginary axis $\{z \mid \operatorname{Re} z = 0\}$ that admits a factorization*

$$f(z) = f_+(z) f_-(z), \ \forall \operatorname{Re} z = 0, \tag{9.13}$$

where

- *f_- is analytic in the closed right half-plane $\{z \mid \operatorname{Re} z \geq 0\}$, and f_- and $1/f_-$ are bounded there;*
- *f_+ is analytic in the closed left half-plane $\{z \mid \operatorname{Re} z \leq 0\}$, and f_+ and $1/f_+$ are bounded there;*
- *$f_+(0) = f_-(0) = 1$.*

Let

$$f(z) = f_{1,+}(z) f_{1,-}(z), \ \forall \operatorname{Re} z = 0, \tag{9.14}$$

be another factorization with the same properties.

Then $f_{1,\pm} = f_\pm$.

Proof. Dividing (9.13) by (9.14) and rearranging, we obtain

$$\frac{f_+(z)}{f_{1,+}(z)} = \frac{f_{1,-}(z)}{f_-(z)}, \quad \forall \operatorname{Re} z = 0.$$

The LHS (resp., the RHS) is analytic and bounded in left (resp., right) half-plane, therefore, we can define a continuous bounded function on $\mathbb{C}$, call it F, by the LHS on the left half-plane, and by the RHS on the right half-plane. F is analytic in $C \setminus \{z \mid \operatorname{Re} z = 0\}$, hence, by Morera's theorem, F is constant. Since $F(0) = 1$, $F(z) = 1$ for all z.

It is evident from (9.11)–(9.12) that $\kappa_q^+(z)$ admits the analytic continuation to the left half-plane, and $\kappa_q^-(z)$ admits the analytic continuation to the right half-plane. In addition, $\kappa_q^+(z)$ and $1/\kappa_q^+(z)$ (resp., $\kappa_q^-(z)$ and $1/\kappa_q^-(z)$) are bounded on the left (resp., right) half-plane. Finally, $\kappa_q^{\pm}(0) = 1$. Since $1 - qM(z)$ is a non-vanishing continuous function on the imaginary axis, the Wiener–Hopf factorization (9.4) satisfies the conditions of Lemma 9.1.1. Therefore, if we guess a factorization of $1/a(z) := (1-q)/(1-qM(z))$ with the same properties, the factors will be $\kappa_q^+(z)$ and $\kappa_q^-(z)$. For the model (7.5), we derived the factorization (7.23) with $\kappa_q^{\pm}(z)$ given by (7.21)–(7.22). These $\kappa_q^{\pm}(z)$ satisfy all the conditions of Lemma 9.1.1, therefore, they are identical with $\kappa_q^{\pm}(z)$ defined by (9.2)–(9.2). Now, the argument about the equivalence of the two forms (9.4) and (7.36) shows that the operators $\mathcal{E}^{\pm}$ defined in Sect. 7.3 by (7.29)–(7.30) and operators defined by (8.24)–(8.25) are identical. (It suffices to check that their actions on functions of the form $g(x) = e^{zx}$ are identical; but this is the statement about the functions $\kappa_q^{\pm}(z)$).

9.2 Properties of EPV operators $\mathcal{E}^{+}$ and $\mathcal{E}^{-}$

9.2.1 Explicit formulas for $\mathcal{E}^{+}$ and $\mathcal{E}^{-}$

Thus, for a general random walk, we use (8.24)–(8.25) as the *definition* of the EPV-operators $\mathcal{E}^{\pm}$. For calculations in applications, it is necessary to obtain computationally effective formulas for the action of $\mathcal{E}^{\pm}$. The first step is the calculation of $\kappa_q^{\pm}(z)$. For general formulas for $\kappa_q^{\pm}(z)$ involving integration, see, e.g., [14]. If the transition density p is given by exponential polynomials on the positive and negative half-axis, then $M(z)$ is a rational function, and factors $\kappa_q^{\pm}(z)$ and operators $\mathcal{E}^{\pm}$ can be calculated as follows:

(1) represent $(1-q)^{-1}(1-qM(z))$ as the ratio of polynomials $P(z)$ and $Q(z)$;
(2) find the roots of the denominator, which are $\lambda_j^{\pm}$ from (7.3)-(7.4), with the multiplicities depending on the degree of the polynomials $p_{\pm,j}$;
(3) find the roots of the numerator; since $1-qM(z) > 0$ for z on the imaginary axis, one group of the roots is in the left half-plane, denote these roots $\beta_k^{-}, k = 1, 2, \ldots$, and the other group of roots is in the right half-plane, denote these roots $\beta_k^{+}, k = 1, 2, \ldots$;
(4) set

$$\kappa_q^{+}(z) = \prod_j \frac{\lambda_j^{+} - z}{\lambda_j^{+}} \prod_k \frac{\beta_k^{+}}{\beta_k^{+} - z}, \tag{9.15}$$

$$\kappa_q^{-}(z) = \prod_j \frac{\lambda_j^{-} - z}{\lambda_j^{-}} \prod_k \frac{\beta_k^{-}}{\beta_k^{-} - z}, \tag{9.16}$$

where each factor is taken with the multiplicity of the corresponding root of the denominator or numerator, respectively.

The proof is the same as in the special case of the density (7.5) above: by construction, the RHS of (9.15)–(9.16) define functions which are bounded and analytic in the left and right half-planes, respectively, with their reciprocals, and (9.4) holds. By Lemma 9.1.1, these functions are $\kappa_q^{\pm}(z)$.

For simplicity, consider the case of the densities of positive and negative jumps given by linear combinations of exponentials, so that no polynomial factors in (7.3)–(7.4) are involved; two simple examples are (7.5) and (7.6). We order the lambda's in (7.3)–(7.4): $\lambda_{n_-}^{-} < \lambda_{n_- -1}^{-} < \cdots < \lambda_1^{-} < 0 < \lambda_1^{+} < \cdots < \lambda_{n_+}^{+}$. The moment generating function is defined for z in the strip

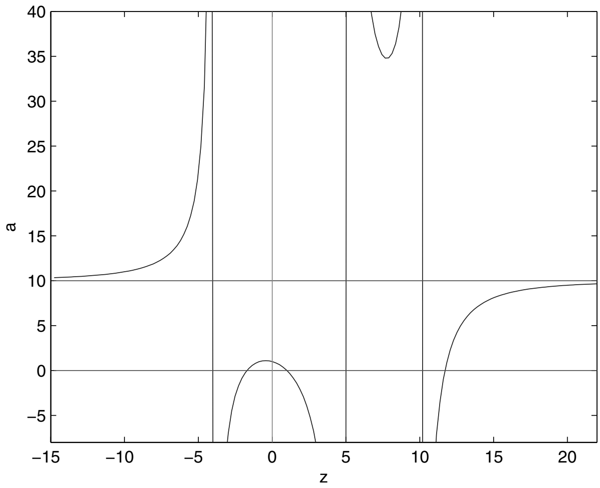

Fig. 9.1. Graph of $a(z) = (1-q)^{-1}(1-qM(z))$ for model (7.6). Parameters: $q = 0.9, \lambda^- = -4,\ \lambda_1^+ = 5,\ \lambda_2^+ = 10,\ c^- = 0.3968,\ c_1^+ = 0.8889,\ c_2^+ = 0.2857$.

$\mathrm{Re}\, z \in (\lambda_1^-, \lambda_1^+)$. It is a linear combination of simple fractions $\lambda/(\lambda - z)$, where $\lambda \in \Lambda := \{\lambda_1^-, \dots, \lambda_1^+, \dots\}$. For the model (7.5), see (7.18), and for the model (7.6), similar calculations give

$$M(z) = c^-\lambda^-/(\lambda^- - z) + c_1^+\lambda_1^+/(\lambda_1^+ - z) - c_2^+\lambda_2^+/(\lambda_2^+ - z). \tag{9.17}$$

We conclude that $M(z)$, hence, $a(z) = (1-q)^{-1}(1-qM(z))$, admit the analytic continuation to $\mathbb{C} \setminus \Lambda$ defined by the same analytic expression.

Since $1 - qM(0) = 1 - q > 0$ and $1 - qM(z) \to -\infty$ as $z \to \lambda_1^- + 0$ or $z \to \lambda_1^+ - 0$, function $a(z)$ has roots both on $(\lambda_1^-, 0)$ and $(0, \lambda_1^+)$. Since $M(z)$ is convex on $(\lambda_1^-, \lambda_1^+)$, these roots are unique; call them β_1^- and β_1^+, respectively. In the case of model (7.5), there are no other roots, and in the case of model (7.6), there is the third root, on $(\lambda_2^+, +\infty)$. Indeed, $a(z) \to 1/(1-q) > 0$ as $z \to \pm\infty$, and $1 - qM(z) \to -\infty$ as $z \to \lambda_2^+ + 0$ (note the sign in front of c_2^+ in (7.6)). There are no more roots because $a(z)$ is the rational function, and the degree of the numerator is 3. See Fig. 9.1.

Below, we assume that the roots of the numerator and denominator of $a(z)$ are simple. Then one can represent $\kappa_q^\pm(z)$ in the form

$$\kappa_q^+(z) = B^+ + \sum_j a_j^+ \frac{\beta_j^+}{\beta_j^+ - z}, \tag{9.18}$$

$$\kappa_q^-(z) = B^- + \sum_j a_j^- \frac{\beta_j^-}{\beta_j^- - z}, \tag{9.19}$$

where β_j^+ (resp., β_j^-) are the roots of the numerator of $a(z)$ in the right (resp., left) half-plane, and

$$a_j^+ = \prod_k \frac{\lambda_k^+ - \beta_k^+}{\lambda_k^+} \prod_{k\neq j} \frac{\beta_k^+}{\beta_k^+ - \beta_j^+}, \quad a_j^- = \prod_k \frac{\lambda_k^- - \beta_k^-}{\lambda_k^-} \prod_{k\neq j} \frac{\beta_k^-}{\beta_k^- - \beta_j^-}$$

$$B^+ = \prod_j \frac{\beta_j^+}{\lambda_j^+}, \quad B^- = \prod_j \frac{\beta_j^-}{\lambda_j^-}.$$

The argument used above for model (7.6) shows that there exist unique roots $\beta_1^+ \in (0, \lambda_1^+)$ and $\beta_1^- \in (\lambda_1^-, 0)$; it is easy to see that the other roots satisfy $\mathrm{Re}\,\beta_j^- < \beta_1^-$, $\mathrm{Re}\,\beta_j^+ > \beta_1^+$. It follows that the no-bubble condition

$$1 - qM(z) > 0, \quad \sigma^- \le z \le \sigma^-, \tag{9.20}$$

is equivalent to

$$\beta_1^- < \sigma^- < \sigma^+ < \beta_1^+. \tag{9.21}$$

For $z < \beta_1^+$, and any $j = 1, 2, \ldots, n_+$,

$$\int_0^{+\infty} \beta_j^+ \mathrm{e}^{-\beta_j^+ y} \mathrm{e}^{z(x+y)} dy = \frac{\beta_j^+}{\beta_j^+ - z} \mathrm{e}^{zx},$$

therefore, for a measurable locally bounded function g satisfying the bound (7.31) with $\sigma^+ < \beta_1^+$ in a neighborhood of $+\infty$,

$$\mathcal{E}^+ g(x) = B^+ g(x) + \sum_j a_j^+ \int_0^{+\infty} \beta_j^+ \mathrm{e}^{-\beta_j^+ y} g(x+y) dy. \tag{9.22}$$

Similarly, for a measurable locally bounded function g satisfying the bound (7.32) with $\sigma^- > \beta_1^-$ in a neighborhood of $-\infty$,

$$\mathcal{E}^- g(x) = B^- g(x) + \sum_j a_j^- \int_{-\infty}^0 (-\beta_j^-) \mathrm{e}^{-\beta_j^- y} g(x+y) dy. \tag{9.23}$$

If in (7.3)–(7.4) functions $p_{\pm,j}$ are (non-constant) polynomials, then one can derive the representations of $\mathcal{E}^\pm$ similar to (9.22) and (9.23) with additional polynomial factors $f_{\pm,j}(y)$ under the integral signs. If both functions $p_\pm$ given by (7.3)–(7.4) are non-zero, we have the representations of the form

$$\mathcal{E}^+ g(x) = B^+ g(x) + \int_0^{+\infty} k^+(y) g(x+y) dy, \tag{9.24}$$

$$\mathcal{E}^- g(x) = B^- g(x) + \int_{-\infty}^0 k^-(y) g(x+y) dy, \tag{9.25}$$

where $B^{\pm}$ are positive constants, and k^+ and k^- are positive functions on $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively, which vanish at infinity. Indeed, comparing (9.24)–(9.25) with (9.5), we see that

$$P^-(dy) = B^-\delta_0 + k^-(y)dy \quad \text{and} \quad P^+(dy) = B^+\delta_0 + k^+(y)dy$$

are the probability distributions of the random variables Y^- and Y^+ on $\mathbb{R}_-$ and $\mathbb{R}_+$, respectively. The reciprocals of $\kappa_q^{\pm}(z)$ admit the representation

$$1/\kappa_q^{\pm}(z) = 1/B^{\pm} + \sum_j b_j^{\pm}\lambda_j^{\pm}/(\lambda_j^{\pm} - z), \tag{9.26}$$

therefore, $\mathcal{E}^{\pm}$ are invertible and their inverses act as follows:

$$(\mathcal{E}^+)^{-1}g(x) = \frac{1}{B^+}g(x) + \int_0^{+\infty} K^+(y)g(x+y)dy, \tag{9.27}$$

$$(\mathcal{E}^-)^{-1}g(x) = \frac{1}{B^-}g(x) + \int_{-\infty}^{0} K^-(y)g(x+y)dy, \tag{9.28}$$

where $K^{\pm}$ are continuous functions which decay exponentially at infinity. Unlike the functions $k\pm$ in (9.24)–(9.25), $K^{\pm}$ may change sign.

To prove that $\mathcal{E}^{\pm}(\mathcal{E}^{\pm})^{-1} = (\mathcal{E}^{\pm})^{-1}\mathcal{E}^{\pm} = I$, we note that both products are of the form I plus a convolution operator with the kernel of the class $L_1(\mathbb{R})$, therefore it suffices to show that both products act as the identity on any function of the form $g(x) = e^{zx}$, where $\operatorname{Re} z < \beta_1^+$ for the sign "+", and $\operatorname{Re} z > \beta_1^-$ for the sign "-". We have

$$\kappa_q^{\pm}(z)(\kappa_q^{\pm}(z))^{-1}e^{zx} = (\kappa_q^{\pm}(z))^{-1}\kappa_q^{\pm}(z)e^{zx} = e^{zx}.$$

Finally, note that

$$(1-q)^{-1}(I - qP) = (\mathcal{E}^+)^{-1}(\mathcal{E}^-)^{-1} = (\mathcal{E}^-)^{-1}(\mathcal{E}^+)^{-1}. \tag{9.29}$$

9.2.2 Action in $\mathcal{L}_\infty(\mathbb{R})$

In the following lemma, "monotone" means "increasing", "non-decreasing", "decreasing" or "non-increasing".

Proposition 9.2.1 *The operators $\mathcal{E}^{\pm}$ enjoy the following properties:*

(a) If $g(x) = 0\ \forall\ x \ge h$, then for the same x, $(\mathcal{E}^+g)(x) = 0$.
(b) If $g(x) = 0\ \forall\ x \le h$, then for the same x, $(\mathcal{E}^-g)(x) = 0$.
(c) If $g(x) \ge 0\ \forall x$, then $(\mathcal{E}^+g)(x) \ge 0,\ \forall x$.
(d) If $g(x) \ge 0\ \forall x$, then $(\mathcal{E}^-g)(x) \ge 0,\ \forall x$.
(e) If g is monotone, then $\mathcal{E}^+g$ and $\mathcal{E}^-g$ are also monotone.
(f) If g is continuous, then then $\mathcal{E}^+g$ and $\mathcal{E}^-g$ are also continuous.

(g) More generally, if g is continuous at a point a, then $\mathcal{E}^+g$ and $\mathcal{E}^-g$ are also continuous at a.

Proof. Since $\mathcal{E}^+g(x) = E[g(x+Y^+)]$ and $\mathcal{E}^-g(x) = E[g(x+Y^-)]$, where $Y^\pm$ is a random variable assuming values in $\mathbb{R}_\pm$, properties (a)–(f) are immediate. To prove (g), note that $\mathcal{E}^\pm = B^\pm I + K^\pm$, where $B^\pm$ are constants and $K^\pm$ are convolution operators with the continuous kernels of the class $L_1(\mathbb{R})$. But these operators map measurable bounded functions to continuous bounded functions.

Proposition 9.2.2 *Operators $\mathcal{E}^+$ and $\mathcal{E}^-$ are invertible operators in $\mathcal{L}_\infty(\mathbb{R})$, with the bounded inverses*

$$(\mathcal{E}^+)^{-1} = (1-q)^{-1}\mathcal{E}^-(I-qP) = (1-q)^{-1}(I-qP)\mathcal{E}^-, \tag{9.30}$$
$$(\mathcal{E}^-)^{-1} = (1-q)^{-1}\mathcal{E}^+(I-qP) = (1-q)^{-1}(I-qP)\mathcal{E}^+. \tag{9.31}$$

Proof. In Sect. 7.3, we showed that $\mathcal{E}$ is the bounded inverse to the operator $(1-q)^{-1}(I-qP)$:

$$\mathcal{E} = (1-q)(1-qP)^{-1}. \tag{9.32}$$

Using (9.32) and the Wiener-Hopf factorization formula (9.10), we obtain

$$\mathcal{E}^+\mathcal{E}^-(1-q)^{-1}(1-qP) = I = (1-q)^{-1}(I-qP)\mathcal{E}^-\mathcal{E}^+,$$

which means that $(1-q)^{-1}(I-qP)\mathcal{E}^-$ is the left inverse to $\mathcal{E}^+$, and $\mathcal{E}^-(1-q)^{-1}(I-qP)$ is the right one. Hence, $\mathcal{E}^+$ is invertible. Since an inverse is unique, we have (9.30). The statement about $\mathcal{E}^-$ is proved similarly.

Proposition 9.2.3 *Operators $(\mathcal{E}^\pm)^{-1}$ enjoy the following properties:*

(a) if $g(x) = 0\ \forall\ x \ge h$, then for the same x, $((\mathcal{E}^+)^{-1}g)(x) = 0$;
(b) if $g(x) = 0\ \forall\ x \le h$, then for the same x, $((\mathcal{E}^-)^{-1}g)(x) = 0$;
(c) $\mathcal{E}^\pm$ and $(\mathcal{E}^\pm)^{-1}$ are mutual inverses as operators in $\mathcal{L}_\infty(\mathbb{R}_\pm)$.

Proof. (a)–(b) are immediate from (9.27)-(9.28).

(c) $\mathcal{E}^\pm$ and $(\mathcal{E}^\pm)^{-1}$ are mutual inverses as operators in $\mathcal{L}_\infty(\mathbb{R})$, and, by (a)–(b), both map $\mathcal{L}_\infty(\mathbb{R}_\pm)$ into itself.

Recall that in Chap. 8, to prove the existence of the optimal exercise boundary, we needed to know that the function $\mathcal{E}^+g$ (or $\mathcal{E}^-g$, depending on a situation) changed sign. The next proposition gives sufficient conditions in the case of general random walks and payoff functions.

Proposition 9.2.4 *a) If $g(-\infty) < 0$, then $\mathcal{E}g(-\infty) < 0$ and $\mathcal{E}^\pm g(-\infty) < 0$;*
b) If $g(-\infty) > 0$, then $\mathcal{E}g(-\infty) > 0$ and $\mathcal{E}^\pm g(-\infty) > 0$;
c) statements a)–b) hold with $+\infty$ instead of $-\infty$.

The prof is the same as the proof of Proposition 6.2.4.

Proposition 9.2.5 *Assume that X_t allows for both upward and downward movements, and g is monotone and changes sign. Then*

(a) there exists a unique h^+ (resp., h^-) such that $\mathcal{E}^+g(x)$ (resp., $\mathcal{E}^-g(x)$) changes sign as x passes h^+ (resp., h^-);
(b) if g is continuous at h^+ (resp., h^-) then h^+ (resp., h^-) is the unique zero of $\mathcal{E}^+g$ (resp., $\mathcal{E}^-g$).

Proof. (a) From Propositions 9.2.1 and 9.2.4, we know that $\mathcal{E}^{\pm}g$ is monotone and it changes sign. Now we need to prove that $\mathcal{E}^{\pm}g$ is strictly monotone on an interval where it changes sign. Since X_t allows for both upward and downward movements, $\text{Prob}\,(\bar{X}_1 > 0 \mid X_0 = 0) > 0$, $\text{Prob}\,(\underline{X}_1 < 0 \mid X_0 = 0) > 0$, and operators $\mathcal{E}^{\pm}$ are non-trivial, that is, differ from I. Moreover, since we consider random walks with transitional densities given by exponential polynomials, representations (9.24)–(9.25) hold with $k^{\pm}$ that are positive on $\mathbb{R}_{\pm}$, a.e. Define $x^+ = \inf\{x \mid \mathcal{E}^+g(x) = \mathcal{E}^+g(+\infty)\}$ and $x^- = \sup\{x \mid \mathcal{E}^-g(x) = \mathcal{E}^-g(-\infty)\}$. Since $k^{\pm}$ are positive on $\mathbb{R}_{\pm}$, a.e., $\mathcal{E}^+g$ is strictly monotone on $(-\infty, x^+)$, and $\mathcal{E}^-g$ is strictly monotone on $(x^-, +\infty)$.

(b) Apply (a) and part (g) of Proposition 9.2.1.

9.2.3 The case of payoffs exponentially growing at infinity

We formulated Propositions 9.2.1–9.2.5 for bounded functions g. These propositions can be extended for g growing at $+\infty$ and/or $-\infty$, if we impose the following related conditions on the random walk and a function g: there exist $\sigma^- \le 0 \le \sigma^+$ and $C, c > 0$ such that

$$1 - qM(\sigma) > 0 \quad \forall\, \sigma \in [\sigma^-, \sigma^+] \tag{9.33}$$

(this presumes that $M(\sigma)$ is finite for all $\sigma \in [\sigma^-, \sigma^+]$), and

$$|g(x)| \le C(\mathrm{e}^{\sigma^- x} + \mathrm{e}^{\sigma^+ x}), \quad \forall\, x \in \mathbb{R}. \tag{9.34}$$

The spaces $\mathcal{L}_\infty(\mathbb{R})$ and $\mathcal{L}_\infty(\mathbb{R}_\pm)$ must be replaced with the spaces

- $\mathcal{L}_\infty(\sigma^-, \sigma^+; \mathbb{R})$, which consists of functions having finite norm

$$||u||_{\infty;\sigma^-,\sigma^+} = \sup_{x\in\mathbb{R}} \left(\mathrm{e}^{\sigma^- x} + \mathrm{e}^{\sigma^+ x}\right)^{-1} |u(x)|; \tag{9.35}$$

- $\mathcal{L}_\infty(\sigma^+; \mathbb{R}_+)$, which consists of functions vanishing at and below 0 and having finite norm

$$||u||_{\infty;\sigma^+} = \sup_{x\in\mathbb{R}_+} \mathrm{e}^{-\sigma^+ x}|u(x)|; \tag{9.36}$$

- $\mathcal{L}_\infty(\sigma^-; \mathbb{R}_-)$, which consists of functions vanishing at and above 0 and having finite norm

$$||u||_{\infty;\sigma^-} = \sup_{x\in\mathbb{R}_-} \mathrm{e}^{-\sigma^- x}|u(x)|. \tag{9.37}$$

The exact statements follow. The reader who is not interested in the technical regularity issues can safely skip their proofs.

Lemma 9.2.6 *Let (9.33) hold. Then*

(a) $M(z)$ *is well-defined and analytic in the strip* $\mathrm{Re}\, z \in [\sigma^-, \sigma^+]$*; moreover, both* $1 - qM(z)$ *and* $1/(1 - qM(z))$ *are uniformly bounded on this strip;*
(b) $\kappa_q^+(z)$ *admits the analytic continuation to the half-plane* $\mathrm{Re}\, z \le \sigma^+$*, and does not vanish there; moreover, both* $\kappa_q^+(z)$ *and* $1/\kappa_q^+(z)$ *are bounded on this half-plane;*
(c) $\kappa_q^-(z)$ *admits the analytic continuation to the half-plane* $\mathrm{Re}\, z \ge \sigma^-$*, and does not vanish there; moreover, both* $\kappa_q^-(z)$ *and* $1/\kappa_q^-(z)$ *are bounded on this half-plane;*
(d) the Wiener–Hopf factorization formula (6.5) holds for z *on the strip* $\sigma^- \le \mathrm{Re}\, z \le \sigma^+$.

Proof. (a) We have $\left|E[\mathrm{e}^{zX_t}]\right| \le E[\left|\mathrm{e}^{zX_t}\right|] \le E\left[\mathrm{e}^{\mathrm{Re}\, zX_t}\right]$, therefore $M(z)$ is well-defined if $M(\mathrm{Re}\, z)$ is, and it is uniformly bounded on the strip. Similarly, one shows that the derivative of M is well-defined. Hence, $M(z)$ is analytic in the strip $\mathrm{Re}\, z \in [\sigma^-, \sigma^+]$. Similarly, $\mathrm{Re}\, M(z) \le M(\mathrm{Re}\, z)$, therefore $\mathrm{Re}\,(1 - qM(z)) \ge 1 - qM(\mathrm{Re}\, z)$ is bounded away from 0.

(b) From (9.4), for $\mathrm{Re}\, z = 0$,

$$\kappa_q^+(z) = \frac{1-q}{1-qM(z)} \times \frac{1}{\kappa_q^-(z)}. \tag{9.38}$$

Under condition (9.33), the first fraction on the RHS is analytic in the strip $\sigma^- \le \mathrm{Re}\, z \le \sigma^+$, whereas the second one is analytic in the half-plane $\mathrm{Re}\, z \ge 0$. Moreover, both fractions and their reciprocals are bounded on the corresponding set. Hence, we can use (9.38) to define the analytic extension of $\kappa_q^+(z)$ to the strip $0 \le \mathrm{Re}\, z \le \sigma^+$. Part (c) is proved similarly, and (d) follows from (6.5), (9.33) and (b)–(c).

Lemma 9.2.7 *Let (9.33) hold. Then*

(a) if g *satisfies the bound (9.34), then* $(I - qP)g$ *and* $\mathcal{E}g$ *satisfy the same bound (with different constants* C*);*
(b) operators $A := (1-q)^{-1}(I - qP)$ *and* $\mathcal{E}$ *are mutual inverses as operators in* $\mathcal{L}_\infty(\sigma^-, \sigma^+; \mathbb{R})$*;*
(c) if g *vanishes below 0 and satisfies*

$$|g(x)| \le C\mathrm{e}^{\sigma x}, \quad \forall\, x, \tag{9.39}$$

where $\sigma \le \sigma^+$*, then* $\mathcal{E}^+ g$ *and* $(\mathcal{E}^+)^{-1} g$ *satisfy the same two conditions;*

(d) *if g vanishes above 0 and satisfies (9.39) with $\sigma \geq \sigma^-$, then $\mathcal{E}^- g$ and $(\mathcal{E}^-)^{-1} g$ satisfy the same two conditions;*
(e) *for any $\sigma \leq \sigma^+$, $\mathcal{E}^+$ and $(\mathcal{E}^+)^{-1}$ are mutual inverses as operators in $\mathcal{L}_\infty(\sigma; \mathbb{R}_+)$;*
(f) *for any $\sigma \geq \sigma^-$, $\mathcal{E}^-$ and $(\mathcal{E}^-)^{-1}$ are mutual inverses as operators in $\mathcal{L}_\infty(\sigma; \mathbb{R}_-)$.*
(g) *the Wiener–Hopf factorization formula (9.10) is valid with $\mathcal{E}$ and $\mathcal{E}^\pm$ acting in $\mathcal{L}_\infty(\sigma^-, \sigma^+; \mathbb{R})$.*

Proof. (a) Clearly, $u \in \mathcal{L}_\infty(\sigma^-, \sigma^+; \mathbb{R})$ iff u belongs to the the intersection of spaces $\mathcal{L}_\infty(\sigma^+; \mathbb{R})$ and $\mathcal{L}_\infty(\sigma^-; \mathbb{R})$. Moreover, the norm $|| \cdot ||_{\infty;\sigma^-,\sigma^+}$ is equivalent to the norm $||u||'_{\sigma^-,\sigma^+;l_\infty(\mathbb{R})} = \max\{||u||_{\infty;\sigma^+}, \ ||u||_{\infty;\sigma^-}\}$ of the intersection, that is, there exist positive constants C, c such that for any $u \in \mathcal{L}_\infty(\sigma^-, \sigma^+; \mathbb{R})$,

$$c||u||'_{\infty;\sigma^-,\sigma^+} \leq ||u||_{\infty;\sigma^-,\sigma^+} \leq C||u||'_{\infty;\sigma^-,\sigma^+}.$$

Therefore, it suffices to prove that P and $\mathcal{E}$ are bounded operators in $\mathcal{L}_\infty(\sigma; \mathbb{R})$, for any $\sigma \in [\sigma^-, \sigma^+]$. We have

$$\begin{aligned} ||Pu||_{\infty;\sigma} &= \sup_{x \in \mathbb{R}} \left| \mathrm{e}^{-\sigma x} \int_{-\infty}^{+\infty} p(y) u(x+y) dy \right| \\ &\leq \sup_{x \in \mathbb{R}} \mathrm{e}^{-\sigma x} \int_{-\infty}^{+\infty} p(y) |u(x+y)| dy \\ &\leq \sup_{x \in \mathbb{R}} \mathrm{e}^{-\sigma x} \int_{-\infty}^{+\infty} p(y) \mathrm{e}^{\sigma(x+y)} dy \, ||u||_{\sigma;\mathcal{L}_\infty(\mathbb{R})} \\ &= \int_{-\infty}^{+\infty} p(y) \mathrm{e}^{\sigma y} dy \, ||u||_{\infty;\sigma} \\ &= M(\sigma) \, ||u||_{\infty;\sigma}, \end{aligned}$$

which proves that P is bounded, with the norm less than or equal to $M(\sigma)$. For $u(x) = \mathrm{e}^{\sigma x}$, the above inequalities are equalities, hence, the norm equals $M(\sigma)$. Since $\mathcal{E}u(x) = (1-q) \sum_{t \geq 0} q^j P^j u(x)$, we obtain

$$||\mathcal{E}u||_{\infty;\sigma} \leq (1-q) \sum_{t \geq 0} q^j M(\sigma)^j ||u||_{\infty;\sigma}.$$

Under condition (9.33), the series above converges:

$$(1-q) \sum_{t \geq 0} q^j M(\sigma)^j = (1-q)/(1-qP(\gamma)),$$

hence, $\mathcal{E}$ is bounded.

(b) is proved as in the case $\mathcal{L}_\infty(\mathbb{R})$ (see Sect. 7.3), the estimate for the norms obtained in part (a) being used.

(c) and (d) are proved exactly as for $g \in \mathcal{L}_\infty(\mathbb{R})$.

(e) Let $\sigma \le \sigma^+$ and $g \in \mathcal{L}_\infty(\sigma;\mathbb{R})$. Define $v(x) = \mathrm{e}^{-\sigma x}u(x)$. Then

$$\begin{aligned}||\mathcal{E}^+ g||_{\infty;\sigma} &= \sup_{x\in\mathbb{R}} |\mathrm{e}^{-\sigma x}\mathcal{E}^+ g(x)| = \sup_{x\in\mathbb{R}} |\mathrm{e}^{-\sigma x}\mathcal{E}^+ \mathrm{e}^{\sigma x} v(x)| \\ &\le ||v||_{\mathcal{L}_\infty(\mathbb{R})} \sup_{x\in\mathbb{R}} |\mathrm{e}^{-\sigma x}\mathcal{E}^+ \mathrm{e}^{\sigma x}| = \kappa_q^+(\sigma)||v||_{\mathcal{L}_\infty(\mathbb{R})} = \kappa_q^+(\sigma)||u||_{\infty;\sigma}.\end{aligned}$$

In view of (c), this proves that $\mathcal{E}^+$ is bounded in $\mathcal{L}_\infty(\sigma;\mathbb{R})$. Since $1/\kappa_q^+(\sigma)$ is bounded on the half-space $\{z \mid \mathrm{Re}\, z \ge \sigma^-\}$, we can replace in the above estimate $\mathcal{E}^+$ and $\kappa_q^+(\gamma)$ with $(\mathcal{E}^+)^{-1}$ and $1/\kappa_q^+(\gamma)$, and conclude that $(\mathcal{E}^+)^{-1}$ is bounded as an operator in $\mathcal{L}_\infty(\sigma;\mathbb{R})$. Finally, if $\mathrm{Re}\, z \le \sigma$,

$$(\mathcal{E}^+)^{-1}\mathcal{E}^+\mathrm{e}^{z\cdot}(x) = \mathcal{E}^+(\mathcal{E}^+)^{-1}\mathrm{e}^{z\cdot}(x) = \kappa_q^+(z)(1/\kappa_q^+(z))\mathrm{e}^{zx} = \mathrm{e}^{zx},$$

therefore $(\mathcal{E}^+)^{-1}$ and $\mathcal{E}^+$ are mutual inverses as operators in $\mathcal{L}_\infty(\sigma;\mathbb{R}_+)$.

(f) is proved similarly.

(g) Under condition (9.34), both sides of the equality

$$E[g(x + X_T)] = E[g(x + Y^+ + Y^-)] \tag{9.40}$$

are well-defined, and it remains to note X_T and $Y^+ + Y^-$ are the same in law.

9.3 EPVs of a stream and instantaneous payoff that are acquired or lost at a random time

9.3.1 Standing assumptions

In Chap. 8, we solved several optimal stopping problems in the model (7.5). The first step was the calculation of the EPV of a stream or instantaneous payoff that was acquired or lost when a certain boundary fixed in advance had been reached or crossed. At the second step, the boundary was chosen to maximize the option value. Since there exist problems with the exit or entry thresholds given exogenously, an example being the bankruptcy specified by debt covenants, we start with the list of main theorems for the case of an exogenously given boundary. The standing assumption about the random walk is (9.33), where $\sigma^- \le 0 \le \sigma^+$, and about the stream – (9.34). When we consider an instantaneous payoff $G(X_t)$, the standing assumption is weaker than (9.34). If the payoff $G(X_t)$ is due when a certain boundary is crossed from below, then the payoff function may not grow too fast as $x \to +\infty$: for any N, there exists C such that

$$|G(x)| \le C\mathrm{e}^{\sigma^+ x}, \quad \forall\, x > -N; \tag{9.41}$$

if the payoff is due when a certain boundary is crossed from above, then the bound is imposed in a neighborhood of $-\infty$: for any N, there exists C such that

$$|G(x)| \le Ce^{\sigma^- x}, \quad \forall\, x < N. \tag{9.42}$$

For an instantaneous payoff, the conditions on growth are weaker than for payoff streams, because if a stream is acquired then its EPV may depend on values of $g(x)$ at arbitrary large (in modulus) x, whereas for options with an instantaneous payoff $G(X_t)$ only values $G(x)$ for x in the action region matter.

In Sect. 8.4, we showed that if g is discontinuous then two types of optimal exercise rules are possible. For the sake of brevity, in the next sections, we calculate the values of the streams of payoffs and instantaneous payoffs, which correspond to one type of exercise rules. The reader can easily to formulate the counterparts for the other type of exercise rules (cf. (8.17) and (8.23)).

9.3.2 EPV of a stream that is abandoned when the threshold is reached or crossed from above

Denote by τ_h^- the first time X_t reaches $h \in \mathbb{R}$ or crosses h from above.

Theorem 9.3.1 *Assume that g satisfies (9.34). Then the EPV of the stream that is lost when X_t reaches or crosses $h \in \mathbb{R}$ from above is given by*

$$V^-_{\text{loss}}(x;h) = E^x\left[\sum_{t=0}^{\tau_h^- -1} q^t g(X_t)\right] \tag{9.43}$$

$$= (1-q)^{-1}\mathcal{E}^-\mathbf{1}_{(h,+\infty)}\mathcal{E}^+ g(x). \tag{9.44}$$

Note that (9.43) is the definition, and (9.44) is the statement of the theorem.

Proof. This is, essentially, the situation which we considered in the framework of the model (7.5) in Theorem 8.1.1. Now we allow for a more general random walk than the model (7.5), and condition on the behavior of the stream at $\pm\infty$ is more general, but the underlying idea of the proof remains the same. However, now we need to use the general definitions (8.24)–(8.25) of the EPV operators $\mathcal{E}^\pm$ in terms of the supremum and infimum processes from the very beginning. Lemma 9.2.7 allows us to reproduce the proof of Theorem 8.1.1 with evident changes.

The Bellman equation for $V^-_{\text{loss}}(x;h)$ is

$$V^-_{\text{loss}}(x;h) = g(x) + E^x[V^-_{\text{loss}}(X_1;h)], \quad x > h,$$

equivalently,

$$(I - qP)V^-_{\text{loss}}(x;h) = g(x), \quad x > h,$$

and $V^-_{\text{loss}}(x;h) = 0$ for $x \le h$. Normalize $V^-_{\text{loss}}(x;h)$, that is, introduce $\mathcal{V} = (1-q)V^-_{\text{loss}}$. The normalized value function satisfies

$$(1-q)^{-1}(I - qP)\mathcal{V}(x;h) = g(x),\ x > h, \tag{9.45}$$

$$\mathcal{V}(x;h) = 0, \quad x \le h. \tag{9.46}$$

Set $g^-(x) = (1-q)^{-1}(I-qP)\mathcal{V}(x;h) - g(x)$, and write (9.45) as an equation on $\mathbb{R}$:

$$(1-q)^{-1}(I-qP)\mathcal{V} = g + g^-,$$

where g^- vanishes on $(h,+\infty)$. Apply $\mathcal{E}^+$ and use the Wiener-Hopf factorization (7.36):

$$(\mathcal{E}^-)^{-1}\mathcal{V} = \mathcal{E}^+ g + \mathcal{E}^+ g^-.$$

Lemma 9.2.7 ensures that for $x > h$, $\mathcal{E}^+ g^-(x) = 0$, and for $x \le h$, $(\mathcal{E}^-)^{-1}\mathcal{V}(x;h) = 0$. Therefore, multiplying by $\mathbf{1}_{(h,+\infty)}$, we obtain

$$(\mathcal{E}^-)^{-1}\mathcal{V} = \mathbf{1}_{(h,+\infty)}\mathcal{E}^+ g.$$

Finally, applying $(1-q)^{-1}\mathcal{E}^-$, we arrive at (9.44).

9.3.3 EPV of a stream that is abandoned when the threshold is reached or crossed from below

Denote by τ_h^+ the first time X_t reaches or crosses $h \in \mathbb{R}$ from below.

Theorem 9.3.2 *Assume that g satisfies (9.34). Then the EPV of the stream that is lost when X_t reaches or crosses $h \in \mathbb{R}$ from below is given by*

$$V^+_{\text{loss}}(x;h) = E^x\left[\sum_{t=0}^{\tau_h^+-1} q^t g(X_t)\right] \tag{9.47}$$

$$= (1-q)^{-1}\mathcal{E}^+\mathbf{1}_{(-\infty,h)}\mathcal{E}^- g(x). \tag{9.48}$$

The proof is the mirror reflection of the proof of Theorem 9.3.1. We leave it as an exercise for the reader.

9.3.4 EPV of a stream that is acquired when the threshold is reached or crossed from above

Theorem 9.3.3 *Assume that g satisfies (9.34). Then the EPV of the stream that is acquired when X_t reaches or crosses $h \in \mathbb{R}$ from above is given by*

$$V^-_{\text{gain}}(x;h) = E^x\left[\sum_{t=\tau_h^-}^{\infty} q^t g(X_t)\right] \tag{9.49}$$

$$= (1-q)^{-1}\mathcal{E}^-\mathbf{1}_{(-\infty,h]}\mathcal{E}^+ g(x). \tag{9.50}$$

Proof. We have

$$(1-q)^{-1}\mathcal{E}g(x) = E^x\left[\sum_{t=0}^{\infty} q^t g(X_t)\right] = E^x\left[\sum_{t=0}^{\tau_h^--1} q^t g(X_t)\right] + E^x\left[\sum_{t=\tau_h^-}^{\infty} q^t g(X_t)\right]$$

By Theorem 9.3.1, the first term on the RHS equals

$$(1-q)^{-1}\mathcal{E}^{-}\mathbf{1}_{(h,+\infty)}\mathcal{E}^{+}g(x),$$

and by the Wiener–Hopf factorization formula, the LHS can be represented as

$$(1-q)^{-1}\mathcal{E}^{-}(\mathbf{1}_{(-\infty,h]}+\mathbf{1}_{(h,+\infty)})\mathcal{E}^{+}g(x).$$

Now (9.50) is immediate.

9.3.5 EPV of a stream that is acquired when the threshold is reached or crossed from below

Theorem 9.3.4 *Assume that g satisfies (9.34). Then the EPV of the stream that is acquired when X_t reaches or crosses $h \in \mathbb{R}$ from below is given by*

$$V^{+}_{\text{gain}}(x;h) = E^x\left[\sum_{t=\tau_h^+}^{\infty} q^t g(X_t)\right] \tag{9.51}$$

$$= (1-q)^{-1}\mathcal{E}^{+}\mathbf{1}_{[h,+\infty)}\mathcal{E}^{-}g(x). \tag{9.52}$$

The proof is the mirror reflection of the proof of Theorem 9.3.3. We leave it as an exercise for the reader.

9.3.6 EPV of an instantaneous payoff that is acquired when the threshold is reached or crossed from above

Theorem 9.3.5 *Assume that G satisfies (9.42). Then the EPV of the payoff $G(X_t)$ that is acquired when X_t reaches or crosses $h \in \mathbb{R}$ from above is given by*

$$E^x\left[q^{\tau_h^-}G(X_{\tau_h^-})\right] = \mathcal{E}^{-}\mathbf{1}_{(-\infty,h]}(\mathcal{E}^{-})^{-1}G(x). \tag{9.53}$$

Proof. Since only the values $G(x)$ for $x \le h$ matter, we may replace G with G_1, where G_1 coincides with G on $(-\infty,h]$ and is bounded on $\mathbb{R}_+$. Then $g := (I-qP)G_1$ satisfies (9.34) with the same σ^- and $\sigma^+ = 0$. We have

$$E^x\left[q^{\tau_h^-}G(X_{\tau_h^-})\right] = E^x\left[q^{\tau_h^-}E^{X_{\tau_h^-}}\left[\sum_{s=0}^{\infty} q^s g(X_s)\right]\right] = E^x\left[\sum_{t=\tau_h^-}^{\infty} q^t g(X_t)\right],$$

where the last equality follows from the law of iterated expectations. Applying Theorem 9.3.3, we continue

$$= (1-q)^{-1}\mathcal{E}^{-}\mathbf{1}_{(-\infty,h]}\mathcal{E}^{+}g(x) = \mathcal{E}^{-}\mathbf{1}_{(-\infty,h]}\mathcal{E}^{+}(1-q)^{-1}(I-qP)G_1(x),$$

then, using the Wiener–Hopf factorization formula,

$$= \mathcal{E}^{-}\mathbf{1}_{(-\infty,h]}(\mathcal{E}^{-})^{-1}G_1(x).$$

Finally, we use Proposition 9.2.2, which implies that for $x \le h$, $(\mathcal{E}^{-})^{-1}G_1(x) = (\mathcal{E}^{-})^{-1}G(x)$. Hence, $\mathbf{1}_{(-\infty,h]}(\mathcal{E}^{-})^{-1}G_1 = \mathbf{1}_{(-\infty,h]}(\mathcal{E}^{-})^{-1}G$, and (9.53) follows.

9.3.7 EPV of an instantaneous payoff that is acquired when the threshold is reached or crossed from below

Theorem 9.3.6 *Assume that G satisfies (9.41). Then the EPV of the payoff $G(X_t)$ that is acquired when X_t reaches $h \in \mathbb{R}$ or crosses h from below is given by*

$$E^x\left[q^{\tau_h^+} G(X_{\tau_h^+})\right] = \mathcal{E}^+ \mathbf{1}_{[h,+\infty)} (\mathcal{E}^+)^{-1} G(x). \tag{9.54}$$

The proof is the mirror reflection of the proof of Theorem 9.3.6. We leave it as an exercise for the reader.

9.4 Main types of options. Optimality in the class of optimal stopping rules of the threshold type

In this Section, we find optimal exercise rules in the class of optimal stopping rules of the threshold type.

9.4.1 Standing assumptions and notation

The payoff streams and instantaneous payoffs are measurable functions of the underlying stochastic factor X_t, which satisfies (9.33) and allows for both upward and downward movements. Then Proposition 9.2.5 ensures the uniqueness of the optimal exercise threshold in the theorems below. Additional conditions on the payoffs and payoff streams are specified in each theorem. If the payoff (stream) is continuous, then the exercise threshold is a unique zero, h, of a certain continuous monotone function, which is labeled w below. The point h is the optimal exercise boundary, and it does not matter whether the option is exercised the first time h is reached or crossed or only the first time h is crossed. If the payoff stream is not continuous, then it is possible that w does not have a zero although it changes sign as a certain threshold h is crossed. Then, depending on the sign of $w(h)$, it may be optimal to exercise the option the first time h is reached or crossed (exercise rule of type (1)) or only when h is crossed (type (2)). We consider both possibilities in Theorem 9.4.1, and, to simplify the exposition, assume in the theorems following Theorem 9.4.1 that $w(h) = 0$ so that the rules of both types are optimal. Note that in any parametric model for X_t, w has a unique zero, generically.

9.4.2 Optimal time to abandon an increasing stream

A model example is the exit problem for a firm with uncertainty on the demand side and the profit flow $g(X_t) = Ge^{X_t} - C$.

Theorem 9.4.1 *Assume that g satisfies (9.34), and there exists h_* such that either*

(1) $\mathcal{E}^+g(x) \le 0 \ \forall\ x \le h_*$, *and* $\mathcal{E}^+g(x) > 0 \ \forall\ x > h_*$, *or*
(2) $\mathcal{E}^+g(x) < 0 \ \forall\ x \le h_*$, *and* $\mathcal{E}^+g(x) \ge 0 \ \forall\ x > h_*$

Then, in Case (1), it is optimal to abandon the stream $g(X_t)$ *the first time* $X_t \le h_*$, *and the EPV of the stream with the option to abandon it is*

$$V(x) = (1-q)^{-1}\mathcal{E}^-\mathbf{1}_{(h_*,+\infty)}\mathcal{E}^+g(x). \tag{9.55}$$

In Case (2), it is optimal to abandon the stream $g(X_t)$ *the first time* $X_t < h_*$, *and the EPV of the stream with the option to abandon it is*

$$V(x) = (1-q)^{-1}\mathcal{E}^-\mathbf{1}_{[h_*,+\infty)}\mathcal{E}^+g(x). \tag{9.56}$$

If h_* *is a zero of* $\mathcal{E}^+g$ *then the functions defined by (9.55) and (9.56) coincide, and both exercise rules are optimal.*

Proof. Consider Case (1) and assume that the optimal exercise rule is of type (1). Let h be a candidate for the exercise threshold. Then by (9.44), the option value is $V(x;h) = (1-q)^{-1}\mathcal{E}^-\mathbf{1}_{(h,+\infty)}\mathcal{E}^+g(x)$. The choice $h = h_*$ replaces all negative values of $w = \mathcal{E}^+g$ by zero, and leaves positive ones intact. By Proposition 9.2.1, $\mathcal{E}^-$ is a monotone operator. Hence, h_* is optimal. Note that if $w(h_*) < 0$, then it is not optimal not to abandon the stream as $X_t = h_*$, and if $w(h_*) = 0$, then the option value does not change if the stream is not abandoned as $X_t = h_*$.

In Case (2), we assume that the optimal exercise rule is of type (2). Let h be a candidate for the exercise threshold. Then by (8.23), $\mathcal{V}(x;h) = (\mathcal{E}^-\mathbf{1}_{[h,+\infty)}\mathcal{E}^+g)(x)$. The choice $h = h_*$ replaces all negative values of $w = \mathcal{E}^+g$ by zero, and leaves positive ones intact. By Proposition 9.2.1, $\mathcal{E}^-$ is a monotone operator. Hence, h_* is optimal.

Example 9.1. Let $g(x) = G\mathrm{e}^x - C$. Then $\mathcal{E}^+g(x) = G\kappa_q^+(1)\mathrm{e}^x - C$. Hence, $h_* = \log(C/(G\kappa_q^+(1))$ is the optimal exercise threshold.

9.4.3 Optimal time to abandon a decreasing stream

A model example is the exit problem for a firm with uncertainty on the demand side and the profit flow $g(X_t) = R - \mathrm{e}^{X_t}$.

Theorem 9.4.2 *Assume that* g *satisfies (9.34), and there exists* h^* *such that* $\mathcal{E}^-g(x) \le 0 \ \forall\ x \ge h^*$, $\mathcal{E}^-g(x) \ge 0 \ \forall\ x < h^*$, *and* $\mathcal{E}^-g(h^*) = 0$.

Then it is optimal to abandon the stream $g(X_t)$ *the first time* $X_t \ge h^*$, *and the EPV of the stream with the option to abandon it is*

$$V(x) = (1-q)^{-1}\mathcal{E}^+\mathbf{1}_{(-\infty,h^*)}\mathcal{E}^-g(x). \tag{9.57}$$

Proof. Let h be a candidate for the exercise threshold. Then by (9.48), the option value is $V(x;h) = (1-q)^{-1}\mathcal{E}^+\mathbf{1}_{(-\infty,h)}\mathcal{E}^-g(x)$. The choice $h = h^*$ replaces all negative values of $\mathcal{E}^-g$ by zero, and leaves positive ones intact. By Proposition 9.2.1, $\mathcal{E}^+$ is a monotone operator. Hence, h^* is optimal.

Example 9.2. Let $g(x) = C - Ge^x$. Then $\mathcal{E}^- g(x) = C - G\kappa_q^-(1)e^x$. Hence, $h^* = \log(C/(G\kappa_q^-(1))$ is the optimal exercise threshold.

9.4.4 Optimal time to acquire an increasing stream

A model example is the irreversible investment with uncertainty on the demand side, the profit flow $g(X_t) = Ge^{X_t} - C$, and zero fixed investment cost. Non-zero fixed investment cost I can be incorporated by assuming that the project is financed by debt, and the firm precommits not to default on the debt obligations. In this case, the following theorem is applicable with $g(X_t) = Ge^{X_t} - C - (1-q)I$.

Theorem 9.4.3 *Assume that g satisfies (9.34), and there exists h^* such that $\mathcal{E}^- g(x) \geq 0 \ \forall \ x \geq h^*$, $\mathcal{E}^- g(x) \leq 0 \ \forall \ x < h^*$, and $\mathcal{E}^- g(h^*) = 0$.*

Then it is optimal to acquire the stream $g(X_t)$ the first time $X_t \geq h^*$, and the value of the option to acquire the stream is

$$V(x) = (1-q)^{-1}\mathcal{E}^+ \mathbf{1}_{[h^*,+\infty)} \mathcal{E}^- g(x). \tag{9.58}$$

Proof. Let h be a candidate for the exercise threshold. Then by (9.52), the option value is $V(x;h) = (1-q)^{-1}\mathcal{E}^+ \mathbf{1}_{[h,+\infty)} \mathcal{E}^- g(x)$. The choice $h = h^*$ replaces all negative values of $\mathcal{E}^- g$ by zero, and leaves positive ones intact. By Proposition 9.2.1, $\mathcal{E}^+$ is a monotone operator. Hence, h^* is optimal.

Example 9.3. Let $g(x) = Ge^x - C$. Then $\mathcal{E}^- g(x) = G\kappa_q^-(1)e^x - C$. Hence, $h^* = \log(C/(G\kappa_q^-(1))$ is the optimal exercise threshold.

9.4.5 Optimal time to acquire a decreasing stream

A model example is the investment project with uncertainty on the supply side, the profit flow $g(X_t) = R - e^{X_t}$ and zero fixed investment cost. Non-zero fixed investment cost I can be incorporated by assuming that the investment is financed by debt, and the firm precommits not to default on the debt obligations. In this case, the following theorem is applicable with $g(X_t) = R - (1-q)I - e^{X_t}$.

Theorem 9.4.4 *Assume that g satisfies (9.34), and there exists h_* such that $\mathcal{E}^+ g(x) \leq 0 \ \forall \ x \geq h_*$, $\mathcal{E}^+ g(x) \geq 0 \ \forall \ x < h_*$, and $\mathcal{E}^+ g(h_*) = 0$.*

Then it is optimal to acquire the stream $g(X_t)$ the first time $X_t \leq h_*$, and the value of the option to acquire the stream is

$$V(x) = (1-q)^{-1}\mathcal{E}^- \mathbf{1}_{(-\infty,h_*]} \mathcal{E}^+ g(x). \tag{9.59}$$

Proof. Let h be a candidate for the exercise threshold. Then by (9.50), the option value is $V(x;h) = (1-q)^{-1}\mathcal{E}^- \mathbf{1}_{(-\infty,h]} \mathcal{E}^+ g(x)$. The choice $h = h_*$ replaces all negative values of $\mathcal{E}^+ g$ by zero, and leaves positive ones intact. By Proposition 9.2.1, $\mathcal{E}^-$ is a monotone operator. Hence, h_* is optimal.

Example 9.4. Let $g(x) = C - Ge^x$. Then $\mathcal{E}^+ g(x) = C - G\kappa_q^+(1)e^x$. Hence, $h_* = \log(C/(G\kappa_q^+(1))$ is the optimal exercise threshold.

9.4.6 Perpetual call-like American options

Consider an option with the instantaneous payoff $G(X_t)$ which is an increasing function of the underlying stochastic factor. The standard examples are $G(X_t) = X_t - K$ or $G(X_t) = e^{X_t} - K$; the following theorem is applicable to much wider classes of payoffs.

Theorem 9.4.5 *Assume that G satisfies (9.41), and there exists h^* such that $(\mathcal{E}^+)^{-1}G(x) \ge 0 \ \forall\ x \ge h^*$, $(\mathcal{E}^+)^{-1}G(x) \le 0 \ \forall\ x < h^*$, $(\mathcal{E}^+)^{-1}G(x)(h^*) = 0$. Then it is optimal to exercise the option the first time $X_t \ge h^*$, and the option value is*

$$V(x) = \mathcal{E}^+ \mathbf{1}_{[h^*,+\infty)}(\mathcal{E}^+)^{-1}G(x). \tag{9.60}$$

Proof. Let h be a candidate for the exercise threshold. Then by (9.54), the option value is

$$V(x;h) = \mathcal{E}^+ \mathbf{1}_{[h,+\infty)}(\mathcal{E}^+)^{-1}G(x).$$

The choice $h = h^*$ replaces all negative values of $(\mathcal{E}^+)^{-1}G$ by zero, and leaves positive ones intact. By Proposition 9.2.1, $\mathcal{E}^+$ is a monotone operator. Hence, h^* is optimal.

Example 9.5. Let $G(x) = e^x - K$. Then $(\mathcal{E}^+)^{-1}G(x) = (\kappa_q^+(1))^{-1}e^x - K$. Hence, $h^* = \log(K\kappa_q^+(1))$ is the optimal exercise threshold.

9.4.7 Perpetual put-like American options

Consider an option with the instantaneous payoff $G(X_t)$ which is a decreasing function of the underlying stochastic factor. The standard examples are $G(X_t) = K - X_t$ or $G(X_t) = K - e^{X_t}$; the following theorem is applicable to much wider classes of payoffs.

Theorem 9.4.6 *Assume that G satisfies (9.42), and there exists h_* such that $(\mathcal{E}^-)^{-1}G(x) \ge 0 \ \forall\ x \le h_*$, $(\mathcal{E}^-)^{-1}G(x) \le 0 \ \forall\ x > h_*$, and $(\mathcal{E}^-)^{-1}G(h_*) = 0$. Then it is optimal to exercise the option the first time $X_t \le h_*$, and the option value is*

$$V(x) = \mathcal{E}^- \mathbf{1}_{(-\infty,h_*]}(\mathcal{E}^-)^{-1}G(x). \tag{9.61}$$

Proof. Let h be a candidate for the exercise threshold. Then by (9.53), the option value is

$$V(x;h) = \mathcal{E}^- \mathbf{1}_{(-\infty,h]}(\mathcal{E}^-)^{-1}G(x).$$

The choice $h = h_*$ replaces all negative values of $(\mathcal{E}^-)^{-1}G$ by zero, and leaves positive ones intact. By Proposition 9.2.1, $\mathcal{E}^-$ is a monotone operator. Hence, h_* is optimal.

Example 9.6. Let $G(x) = K - e^x$. Then $(\mathcal{E}^-)^{-1}G(x) = K - (\kappa_q^-(1))^{-1}e^x$. Hence, $h_* = \log(K\kappa_q^-(1))$ is the optimal exercise threshold.

9.5 Optimality in the class of all stopping times

9.5.1 General discussion and standing assumptions

The general lemmas formulated and proved in Subsect. 6.5.1 are valid for random walks on $\mathbb{Z}$ and on $\mathbb{R}$. The proofs below are based on these lemmas; the differences from the proofs in Sect. 6.5 are insignificant.

The standing assumptions are the same as in Subsect. 9.4.1. As in Sect. 9.4, in the first theorem below, namely, Theorem 9.5.1, we consider two possible cases: the exercise is optimal the first time the exercise threshold is reached or crossed (exercise rule of type (1)), and the exercise is optimal the first time the threshold is crossed (exercise rule of type (2)). The theorems following Theorem 9.5.1 are formulated for a generic case, when both types of the exercise rules are optimal (therefore, an arbitrary randomization between the two types of rules is also optimal). In order to avoid repetition of this argument in each theorem, we will simply indicate the unique exercise threshold, which is the same for both types of rules. If the option must be exercised as the stochastic factor falls sufficiently low (rises sufficiently high), then we denote the exercise threshold by h_* (by h^*). Under conditions formulated in each theorem below, the threshold is unique.

9.5.2 Option to acquire an increasing stream

Consider the option to acquire a stream of payoffs $g(X_t)$ that is a non-decreasing function of the underlying stochastic factor.

Theorem 9.5.1 *Assume that g does not decrease, changes sign and admits the bound (9.34). Then there exists a unique h^* such that either*

(1) $\mathcal{E}^- g(x) \geq 0 \ \forall\ x \geq h^*$, and $\mathcal{E}^- g(x) < 0 \ \forall\ x < h^*$, or
(2) $\mathcal{E}^- g(x) > 0 \ \forall\ x \geq h^*$, and $\mathcal{E}^- g(x) \leq 0 \ \forall\ x < h^*$.

In Case (1),

(a) it is optimal to exercise the option the first time X_t reaches or crosses h^* from below;
(b) the option value is given by

$$V = (1-q)^{-1}\mathcal{E}^+ \mathbf{1}_{[h^*,+\infty)}\mathcal{E}^- g. \tag{9.62}$$

(c) the option value can be represented as the EPV of the stream $W(X_t)$, where W is a non-decreasing function vanishing on $(-\infty, h^*)$.

In Case (2),

(a) it is optimal to exercise the option the first time X_t crosses h^* from below;
(b) the option value is given by

$$V = (1-q)^{-1}\mathcal{E}^+ \mathbf{1}_{(h^*,+\infty)}\mathcal{E}^- g. \tag{9.63}$$

(c) the option value can be represented as the EPV of the stream $W(X_t)$, where W is a non-decreasing function vanishing on $(-\infty, h^*]$.

If $\mathcal{E}^- g(h^*) = 0$, then the values given by (9.62) and (9.63) coincide, and both exercise rules are optimal.

Proof. The first statement follows from Proposition 9.2.5. Now, consider Case (1). By Theorem 9.4.3, $\tau^+_{h^*}$ is a unique optimal stopping time in the class of stopping times of the threshold type. To prove optimality in the class of all stopping times, we need to show that the function $W := (I - qP)V$, where V is defined by (9.62), satisfies conditions (iii)-(iv) of Lemma 6.5.2. Then by this lemma, V is the option value, that is, (b) holds. Applying Theorem 9.3.6 to (9.62), we obtain (a). Part (c) will be proved in the process of verification of (iii)–(iv) of Lemma 6.5.2.

First, we verify (iii). In the inaction region $x < h^*$, the Bellman equation

$$V(x) = qPV(x)$$

holds, which proves that $W(x) = 0$ for $x < h^*$. Further,

$$\begin{aligned} W &= (I - qP)(1 - q)^{-1}\mathcal{E}^+\mathbf{1}_{[h^*,+\infty)}\mathcal{E}^- g \\ &= (I - qP)(1 - q)^{-1}\mathcal{E}^+\mathcal{E}^- g - (I - qP)(1 - q)^{-1}\mathcal{E}^+\mathbf{1}_{(-\infty,h^*)}\mathcal{E}^- g. \end{aligned}$$

By the Wiener-Hopf factorization formula, the first term on the RHS equals g, and, by Proposition 9.2.1, $\mathcal{E}^+\mathbf{1}_{(-\infty,h^*)}\mathcal{E}^- g$ vanishes on $[h^*, +\infty)$. Hence, for $x \geq h^*$,

$$W(x) = g(x) + (1 - q)^{-1}qP\mathcal{E}^+\mathbf{1}_{(-\infty,h^*)}\mathcal{E}^- g(x).$$

The multiplication-by-$\mathbf{1}_{(-\infty,h^*)}$-operator replaces positive values of $\mathcal{E}^- g$ by zero and leaves the other values as they are. Since $\mathcal{E}^- g$ is non-decreasing, $\mathbf{1}_{(-\infty,h^*)}\mathcal{E}^- g$ is non-decreasing. Since $\mathcal{E}^+$ is monotone and g is non-decreasing, W is non-decreasing on $[h^*, +\infty)$. To prove that W is non-decreasing on $\mathbb{R}$ (hence, non-negative on $\mathbb{R}$), it remains to show that $W(h^*) \geq 0$. Suppose, on the contrary, that $W(h^*) < 0$. Applying $\mathcal{E}^-$ to the equality

$$W = (I - qP)V = (I - qP)(1 - q)^{-1}\mathcal{E}^+\mathbf{1}_{[h^*,+\infty)}w$$

and using the Wiener–Hopf factorization formula (7.36), we obtain

$$\mathcal{E}^- W = \mathbf{1}_{[h^*,+\infty)}w.$$

Since W vanishes below h^*, and $W(h^*) < 0$, we have $\mathcal{E}^- W(h^*) = B^- W(h^*) < 0$ (see (9.25)), but by the definition of h^*, $\mathbf{1}_{[h^*,+\infty)}(x)w(x) \geq 0 \ \forall\ x \geq h^*$, contradiction.

Now we verify (iv). Applying the Wiener–Hopf factorization formula (7.36), we have

$$\begin{aligned} V &= (1-q)^{-1}\mathcal{E}^+\mathbf{1}_{[h^*,+\infty)}\mathcal{E}^-g \\ &= (1-q)^{-1}\mathcal{E}^+\mathcal{E}^-g + (1-q)^{-1}\mathcal{E}^+\mathbf{1}_{(-\infty,h^*)}(-\mathcal{E}^-g) \\ &= (1-q)^{-1}\mathcal{E}g + (1-q)^{-1}\mathcal{E}^+\mathbf{1}_{(-\infty,h^*)}(-\mathcal{E}^-g). \end{aligned}$$

By construction, $-\mathcal{E}^-g$ is positive on $(-\infty, h^*)$, and since $\mathcal{E}^+$ is monotone, $V \ge (1-q)^{-1}\mathcal{E}g$.

In Case (2), the proof is essentially the same. Intervals $[h^*, +\infty)$ and $(-\infty, h^*)$ need to be replaced with $(h^*, +\infty)$ and $(-\infty, h^*]$, respectively. The only not quite evident change needed is in the proof of the inequality $W(x) \ge 0$ for $x > h^*$. This time, it remains to show that $W(h^* + 0) \ge 0$. Suppose, on the contrary, that $W(h^* + 0) < 0$. Then there exists $h' > h^*$ such that $W(x) < 0\ \forall\, x \in (h^*, h')$. Applying $\mathcal{E}^-$ to the equality

$$W = (I - qP)V = (I - qP)(1-q)^{-1}\mathcal{E}^+\mathbf{1}_{(h^*,+\infty)}w$$

and using the Wiener-Hopf factorization formula (7.36), we obtain

$$\mathcal{E}^-W = \mathbf{1}_{(h^*,+\infty)}w.$$

Since W vanishes below h^*, and $W(x) < 0\ \forall\, x \in (h^*, h')$, we have $\mathcal{E}^-W(x) = E[W(x+Y^-)] < 0$ for these x. But by the definition of h^*, $\mathbf{1}_{(h^*,+\infty)}(x)w(x) \ge 0\ \forall\, x \ge h^*$, contradiction.

9.5.3 Option to acquire a decreasing stream

Consider the option to acquire a stream of payoffs $g(X_t)$ that is a non-increasing function of the underlying stochastic factor.

Theorem 9.5.2 *Assume that g does not increase, changes sign, and satisfies (9.34). Then there exists a unique h_* such that $\mathcal{E}^+g(x) > 0\ \forall\, x < h_*$, and $\mathcal{E}^+g(x) < 0\ \forall\, x > h_*$. Assuming that $\mathcal{E}^+g(h_*) = 0$,*

(a) exercise rules defined by the threshold h_* are optimal;
(b) the option value is given by

$$V = (1-q)^{-1}\mathcal{E}^-\mathbf{1}_{(-\infty,h_*]}\mathcal{E}^+g; \tag{9.64}$$

(c) the option value can be represented as the EPV of the stream $W(X_t)$, where W is a non-increasing function vanishing above h_*.

The proof is the mirror reflection of the proof of the last part of Theorem 9.5.1. We leave the proof as an exercise for the reader.

9.5.4 Option to abandon an increasing stream

Consider the option to abandon a stream of payoffs $g(X_t)$ that is a non-decreasing function of the underlying stochastic factor.

Theorem 9.5.3 *Assume that g does not decrease, changes sign and satisfies (9.34). Then there exists a unique h_* such that $\mathcal{E}^+g(x) > 0 \;\forall\; x > h_*$, and $\mathcal{E}^+g(x) < 0 \;\forall\; x < h_*$. Assuming that $\mathcal{E}^+g(h_*) = 0$,*

(a) exercise rules defined by the threshold h_* are optimal;
(b) the value of the stream with the option to abandon it is given by

$$V_1 = (1-q)^{-1}\mathcal{E}^-\mathbf{1}_{(h_*,+\infty)}\mathcal{E}^+g; \tag{9.65}$$

(c) the value of the stream with the option to abandon it can be represented as the EPV of the stream $W(X_t) + g(X_t)$, where W is a non-increasing function vanishing above h_*.

Proof. Denote by V_* the value of the option to acquire the stream $-g(X_t)$, and by $V_{1,*}$ the value of the stream $g(X_t)$ with the option to abandon it. We have $V_{1,*} - V_* = (1-q)^{-1}\mathcal{E}g$, therefore V_1 is the option value $V_{1,*}$ iff $V_1 - (1-q)^{-1}\mathcal{E}g$ equals V_*, and it is optimal to abandon the stream $g(X_t)$ iff it is the optimal to acquire the stream $-g(X_t)$. Using the Wiener–Hopf factorization formula (7.36), we obtain

$$\begin{aligned} V_1 - (1-q)^{-1}\mathcal{E}g &= (1-q)^{-1}\mathcal{E}^-\mathbf{1}_{(h_*,+\infty)}\mathcal{E}^+g - (1-q)^{-1}\mathcal{E}g \\ &= (1-q)^{-1}\mathcal{E}^-\mathbf{1}_{(-\infty,h_*]}\mathcal{E}^+(-g), \end{aligned}$$

which is V_* by Theorem 9.5.2. Thus, (c) and (b) are proved. By the same theorem, $V_1 - (1-q)^{-1}\mathcal{E}g = (1-q)^{-1}\mathcal{E}W$, where W is a non-increasing function which vanishes on $(h_*, +\infty)$. This proves (d).

9.5.5 Option to abandon a decreasing stream

Consider an option to abandon a stream of payoffs $g(X_t)$ that is a non-increasing function of the underlying stochastic factor.

Theorem 9.5.4 *Assume that g does not increase, changes sign and satisfies (9.34). Then there exists a unique h^* such that $\mathcal{E}^-g(x) > 0 \;\forall\; x < h^*$, and $\mathcal{E}^-g(x) < 0 \;\forall\; x > h^*$. Assuming that $\mathcal{E}^-g(h^*) = 0$,*

(a) exercise rules defined by the threshold h^* are optimal;
(b) the value of the stream with the option to abandon it is given by

$$V_1 = (1-q)^{-1}\mathcal{E}^+\mathbf{1}_{(-\infty,h^*)}\mathcal{E}^-g; \tag{9.66}$$

(c) the value of the stream with the option to abandon it can be represented as the EPV of the stream $W(X_t) + g(X_t)$, where W is a non-decreasing function vanishing below h^*.

The proof is the mirror reflection of the proof of Theorem 9.5.3. We leave the details as an exercise for the reader.

9.5.6 Perpetual call-like American options

Consider an option with the instantaneous payoff $G(X_t)$ that is an increasing function of the underlying stochastic factor.

Theorem 9.5.5 *Assume that the function G satisfies (9.41), and function $g = (I - qP)G$ does not decrease and changes sign. Then there exists a unique h^* such that $(\mathcal{E}^+)^{-1}G(x) > 0\ \forall\ x > h^*$, and $(\mathcal{E}^+)^{-1}G(x) < 0\ \forall\ x < h^*$. Assuming that $(\mathcal{E}^+)^{-1}G(h^*) = 0$,*

(a) exercise rules defined by the threshold h^* are optimal;
(b) the option value is given by

$$V(x) = \mathcal{E}^+ \mathbf{1}_{[h^*,+\infty)}(\mathcal{E}^+)^{-1}G(x); \tag{9.67}$$

(c) the option value can be represented as the EPV of the stream $W(X_t)$, where W is a non-decreasing function vanishing below h^*.

This theorem follows from Theorem 9.5.1 and the Wiener–Hopf factorization formula (7.36).

9.5.7 Perpetual put-like options on a dividend-paying stock

Consider an option with the instantaneous payoff $G(X_t)$ that is a decreasing function of the underlying stochastic factor.

Theorem 9.5.6 *Assume that function G satisfies (9.41), and function $g = (I - qP)G$ does not increase and changes sign. Then there exists a unique h_* such that $(\mathcal{E}^-)^{-1}G(x) > 0\ \forall\ x < h_*$, and $(\mathcal{E}^-)^{-1}G(x) < 0\ \forall\ x > h_*$. Assuming that $(\mathcal{E}^-)^{-1}G(h_*) = 0$*

(a) exercise rules defined by the threshold h_* are optimal;
(b) the option value is given by

$$V(x) = \mathcal{E}^- \mathbf{1}_{(-\infty,h_*]}(\mathcal{E}^-)^{-1}G(x); \tag{9.68}$$

(c) the option value can be represented as the EPV of the stream $W(X_t)$, where W is a non-increasing function vanishing above h_*.

The proof is the mirror reflection of the proof of Theorem 9.5.5. We leave it as an exercise for the reader.

9.5.8 Perpetual put-like options on a non-dividend-paying stock

The mirror reflection of the proof of Theorem 9.5.5 breaks down if we consider the perpetual American put on a stock which pays no dividends. Assuming that we model the stock price using the geometric random walk model, $S_t = \mathrm{e}^{X_t}$, and the stock pays no dividends, we must have $\mathrm{e}^x = qE^x[\mathrm{e}^{X_1}]$, equivalently, $\mathrm{e}^x = qP(e)\mathrm{e}^x$, or, simplifying, $1 - qM(1) = 0$. Thus, for the instantaneous payoff $G(X_t) = K - e^{X_t}$, we have $g = (I - qP)G = (1 - q)K$, and, therefore, G cannot be expressed as the EPV of the stream $g(X_t)$. Nevertheless, a natural modification of Theorem 9.5.6 holds, and the proof of the latter needs only a slight adjustment.

Theorem 9.5.7 *Assume that* $1 \geq qM(1)$. *Then*

(a) exercise rules defined by the threshold $h_* := \log(K\kappa_q^-(1))$ *are optimal;*
(b) the option value is given by

$$V = \mathcal{E}^- \mathbf{1}_{(-\infty, h_*]} w, \tag{9.69}$$

where $w(x) = K - \kappa_q^-(e)^{-1} e^x$;
(c) part (d) of Theorem 9.5.6 holds.

Proof. We verify conditions (6.53) and (6.55). The function V is non-negative since w is non-negative on $(-\infty, h_*]$ and the operator $\mathcal{E}^-$ is monotone. Set $G = K - e^x$. Since $-w$ is non-negative on $(h_*, +\infty)$ and $\mathcal{E}^-$ is monotone,

$$V = \mathcal{E}^- w + \mathcal{E}^- \mathbf{1}_{(h_*, +\infty)}(-w) = G + \mathcal{E}^- \mathbf{1}_{(h_*, +\infty)}(-w) \geq G.$$

Thus, (6.53) holds. Introduce $W = (I - qP)V$. From the Bellman equation for V, $W(x) = 0,\ x > h_*$, and for $x \leq h_*$,

$$\begin{aligned} W(x) &= (I - qP)G(x) + (I - qP)\mathcal{E}^- \mathbf{1}_{(h_*, +\infty)}(-w)(x) \\ &= (1 - q)K - (1 - qM(1))e^x + qP\mathbf{1}_{(h_*, +\infty)} w(x). \end{aligned}$$

Since $1 - qM(1) \geq 0$, the first two terms do not increase, and the third one does not increase on $(-\infty, h_*]$ because the multiplication-by-$\mathbf{1}_{(h_*, +\infty)}$ operator replaces negative values of the non-increasing function w by zero and leaves the other values as they are, and P is monotone. To prove that W is non-negative, it remains to demonstrate that $W(h_*) \geq 0$. Suppose, on the contrary, that $W(h_*) < 0$. Applying $\mathcal{E}^+$ to the equality

$$W = (I - qP)V = (I - qP)(1 - q)^{-1} \mathcal{E}^- \mathbf{1}_{(-\infty, h_*]} w$$

and using the Wiener-Hopf factorization formula (6.11), we obtain

$$\mathcal{E}^+ W = \mathbf{1}_{(-\infty, h_*]} w.$$

Since W vanishes above h_* and $W(h_*) < 0$, we have $\mathcal{E}^+ W(h_*) = B^+ W(h_*) < 0$ but by the definition of h_*, $\mathbf{1}_{(-\infty, h_*]}(x) w(x) \geq 0$, contradiction.

9.6 Investment lags

Typically, models of irreversible investment assume that a project is brought on line immediately after the decision to invest is made. In fact, in many instances investments take time, which is referred to as "time-to-build", "construction lag", and "gestation period". In [8], it is shown *numerically* that conventional results on the effect of price uncertainty on investment are weakened or reversed if there are lags in investment and investment is partially reversible. That model is set in continuous time and the underlying stochastic factor follows a geometric Brownian motion. We are going to demonstrate similar effects *analytically* in discrete time, and show that certain general claims made in [8] may fail for some specifications of uncertainty.

Let the project completion take n periods after the decision to invest has been made. When the project is completed, the firm will produce 1 unit of output every period and sell the output at the spot price $P = \mathrm{e}^x$. The marginal cost of production, w, is constant. The fixed cost of production, I, has to be paid in equal installments during the construction period. The present value of the deterministic stream of payoffs I/n that accumulates for n periods is $(1-q)^{-1}(1-q^n)I/n$. Clearly, such a value is generated by a perpetual stream $(1-q^n)I/n$. The future value (at date $t = n$) of this stream is $q^{-n}(1-q^n)I/n$. If the investment is made at the spot price $P = \mathrm{e}^x$, then the expected firm's profit at date $t = n$ will be

$$E^x[\mathrm{e}^{X_n}] - w - \frac{1-q^n}{q^n} \cdot \frac{I}{n} = M(1)^n \mathrm{e}^x - w - \frac{1-q^n}{q^n} \cdot \frac{I}{n}.$$

Discounting n periods back, we may write the payoff flow as

$$g(x) = q^n \left(M(1)^n \mathrm{e}^x - w - \frac{1-q^n}{q^n} \cdot \frac{I}{n} \right).$$

The investment threshold, h^*, is defined by:

$$(\mathcal{E}^- g)(h^*) = q^n \left(M(1)^n \kappa_q^-(1) \mathrm{e}^{h^*} - w - \frac{1-q^n}{q^n} \cdot \frac{I}{n} \right) = 0,$$

equivalently,

$$\kappa_q^-(1)\mathrm{e}^{h^*} = M(1)^{-n}(w + I(q^{-n} - 1)/n). \tag{9.70}$$

It is natural to assume that $M(1) > 1$, which means that the expected revenue increases with time. Thus, the first factor on the RHS decreases with the investment lag. Since $(q^{-n} - 1)/n = \sum_{j \geq 1} (-\log q)^j n^{j-1}/j!$, the second factor on the RHS increases with n, and so one may expect that the overall effect of the investment lag is ambiguous. Since $qM(1) < 1$ (otherwise, the value of the project is infinite), we conclude that in the region of very large investment lags, the investment threshold increases with n, and the intuition is clear: part of the investment cost is suffered in the first period, and although it is the n-th part of the total cost, this part will outweigh the potential benefits which will be exponentially discounted over a long time interval. For moderate investment lags, the situation is more interesting. Assume that the time period in the model is not very large (say, a day, week or month). Then the discount factor per period, q, is close to one. Assume further that n is not very large so that the product $-n \log q$ is small (less than 1/3, say; for reasonable values of the discount factor, this means that the lag is 3–4 years or smaller). Then the product $n \log M(1)$ is also small because $qM(1) < 1$, and we can use the Taylor formula and derive an approximation to the RHS in (9.70) of the form

$$w + I(-\log q) + n[-(w + I(-\log q)) \log M(1) + (\log q)^2 I/2].$$

We see that if the investment lag is moderate, then the investment threshold is an increasing or decreasing function in n depending on the sign of the difference $(\log q)^2 I/2 - (w + I(-\log q)) \log M(1)$. For instance, if the prospects are not very good: $\log M(1)$ is much smaller than $-\log q$, and the fixed cost I is relatively large with respect to the variable cost w, then the investment threshold may increase when the lag increases. However, if the prospects are bright: $\log M(1) > -\log q/2$, then the investment threshold decreases for all w and I. We conclude that depending on the characteristics of the project, "time-to-build" increases or decreases the investment threshold.

The effect of uncertainty (measured by the variance of X) can be described in a simpler fashion. Let the transition density for X be given by (7.5). For simplicity assume that $\lambda^+ = -\lambda^-$ (positive and negative jumps, on average, have the same size), and set $l = (\lambda^+)^{-1} = -(\lambda^-)^{-1}$. (This is the average size of jumps. Since $\lambda^+ > 1$, we have $l \in (0, 1)$.) We have $E[Y_1] = (\lambda^+)^{-1} + (\lambda^-)^{-1} = 0$, and $\mathrm{var}\,(Y_1) = (\lambda^+)^{-2} + (\lambda^-)^{-2} = 2l^2$. Clearly, the bigger the size of an average jump, the larger is the variance. In other words, uncertainty increases if jumps become bigger on average. We rewrite (9.70) as

$$M(1)^n \kappa_q^-(1) \mathrm{e}^{h^*} = w + (q^{-n} - 1)I/n. \tag{9.71}$$

In (9.71), the RHS is independent of l. We only need to study the product $M(1)^n \kappa_q^-(1)$ as a function of l. From (7.18),

$$M(1) = \frac{\lambda^+ \lambda^-}{(\lambda^+ - 1)(\lambda^- - 1)} = \frac{1}{1 - l^2}.$$

Next, using (7.20), we find $\beta^- = -\sqrt{1-q}/l$, so that from (7.22)

$$\kappa_q^-(1) = \frac{(\lambda^- - 1)\beta^-}{\lambda^-(\beta^- - 1)} = \frac{1 + l}{1 - 1/\beta^-} = \frac{1 + l}{1 + l/\sqrt{1-q}}.$$

Straightforward calculations show that $M(1)^n \kappa_q^-(1)$ is decreasing in l (hence h^* is increasing in l) on the interval where

$$2nl^2 + ((2n - 1)\sqrt{1-q} + 1)l - 1 + \sqrt{1-q} < 0. \tag{9.72}$$

Given the "construction lag", n, (9.72) specifies the interval for the level of uncertainty, where the conventional intuition concerning the behavior of the investment threshold applies. When the critical level l_{up} (the positive root of the quadratic polynomial on the LHS) is crossed, the investment threshold starts to decrease with uncertainty. We must observe the condition $qM(1) < 1$, which is equivalent to $\beta^+ = \sqrt{1-q}/l > 1$, therefore we need $l_{\rm up} < \sqrt{1-q}$. Using the explicit formula for $l_{\rm up}$, it is straightforward to derive an equivalent condition $1 < (8n-2)(1-q)+3\sqrt{1-q}$. If the time periods are small, and hence, q is close to 1, then $n(1-q) = Tr$ (with a small error), where r is the discount rate in the corresponding continuous time model, and T is the investment lag in years. We obtain an approximate condition $1 < 8Tr$. This means that for realistic values of r, the effect of a decrease of the investment threshold with the further increase of uncertainty can be observed for investment lags of 2-3 years and more. The intuition for this result is as follows. The naive net present value (NPV) rule gives (9.70) without the correction factor $\kappa_q^-(1)$. When the volatility is low, the factor $M(1)^n$, which is responsible for the growth of the expected value of the future profits (prior to discounting) increases with volatility slower than the correction factor $\kappa_q^-(1)$ decreases. So, for low levels of volatility, the negative effect of uncertainty on investment (waiting is optimal) outweighs the positive one (expected profit becomes larger, and so it is optimal to invest in order to receive high expected profits as soon as possible), but for larger level of uncertainty, the positive effect dominates, and the investment threshold starts to decrease.

We conclude that for realistic lengths of the construction period, and certain specifications of uncertainty, there exists a critical value of the variance of the underlying stochastic factor such that for all the variances below the critical value, the investment threshold increases if the uncertainty measured by the variance increases. For all the variances higher than the critical value, the investment threshold decreases in uncertainty, so that it may even drop below the certainty investment barrier. Thus, a general claim in [8], p.617: "Unless abandonment is possible, an increase in uncertainty always delays investments" is not that universal. The reader may object that in [8], a different specification was used: the cost is incurred in the end of the investment lag, and not in n installments as in the present paper. However, if we had used this specification, only the RHS in (9.71) would have changed: instead of $(q^{-n} - 1)I/n$, we would have had $(1 - q)I$. Now, the RHS in (9.71) is independent of uncertainty both in [8] specification, and the specification in our paper. Therefore, the thresholds e^{h^*} in both models differ by a constant factor which is independent of the uncertainty, and the conclusions which we made in the paper are applicable to the discrete time model similar to the model in [8]. In fact, for different values of the drift in [8], the dependence of the threshold on uncertainty may be non-monotone; and for different specifications of the random walk in the present paper, the threshold may decrease when the uncertainty increases. We conclude that depending on the specification of the process (both in the continuous time and discrete time cases), the threshold may be a monotone or non-monotone function of uncertainty. Probably, Bar-Ilan and Strange [8] did not notice this fact because they just numerically computed the thresholds in some cases. We would like to stress that in the discrete time model, it is possible to describe the region in the parameter space where the non-monotonicity of the threshold as a function of uncertainty is observed, whereas in the continuous time case, one has to produce numerical examples, and one never knows if all the possibilities have been studied.

9.7 Incremental capital expansion

9.7.1 Investment threshold

Consider an operating firm whose production function depends only on capital: $G(K) = dK^\theta$, where $d > 0$ and $\theta \in (0,1)$. At each time period t, the firm receives $\mathrm{e}^{X_t} G(K_t)$ from the sales of its product, and, should it decide to increase the capital stock, suffers the installation cost $C \cdot (K_{t+1} - K_t)$. The firm's objective is to chose the optimal investment strategy $\mathcal{K} = \{K_{t+1}(K_t, X_t)\}_{t\geq 1}, K_0 = K, X_0 = x$, which maximizes the NPV of the firm:

$$V(K,x) = \sup_{\mathcal{K}} E^x \left[\sum_{t\geq 0} q^t (\mathrm{e}^{X_t} G(K_t) - C(K_{t+1} - K_t)) \right]. \tag{9.73}$$

Here we treat the current log price x and capital stock K as state variables, and $\mathcal{K}$ as a sequence of control variables. Due to irreversibility of investment, $K_{t+1} \geq K_t$, $\forall t$. To ensure that firm's value (9.73) is bounded, we assume that $qM(1) < 1$. However, this condition is sufficient only if the firm does not increase the capital stock above some level. We will see, that if unbounded capital expansion is allowed, then a more strigent condition is needed:

$$1 - qM(1/(1-\theta)) > 0. \tag{9.74}$$

Formally, the manager has to choose both the timing and the size of the capital expansion. However, it is well-known (see, for example, [39]) that for each level of the capital stock, it is only necessary to decide when to invest. The manager's problem is equivalent to finding the boundary (the investment threshold), $h(K;C)$, between two regions in the state variable space (K,x): inaction and action ones. For all pairs (K,x) belonging to the inaction region, it is optimal to keep the capital stock unchanged. In the action region, investment becomes optimal. To derive the equation for the investment boundary, suppose first that every new investment can be made in chunks of capital, ΔK, only. In this case, the firm has to suffer the cost $C\Delta K$, and the EPV of the revenue gain due to the investment of a chunk of capital can be represented in the form of the EPV of the stream $g(X_t) = qM(1)(G(K+\Delta K) - G(K))\mathrm{e}^{X_t} - (1-q)C\Delta K$. Thus, the multi-shot investment problem reduces to the one-shot problem studied above. On the strength of Theorem 9.5.1, the optimal exercise boundary is determined from the equation $(\mathcal{E}^- g)(x) = 0$, which can be written as

$$qM(1)(G(K+\Delta K)-G(K))\kappa_q^-(1)e^x = (1-q)C\Delta K. \qquad (9.75)$$

Dividing by ΔK in (9.75) and passing to the limit, we obtain the equation for the optimal threshold, $h^* = h^*(K)$:

$$qM(1)\kappa_q^-(1)G'(K)e^{h^*} = C(1-q),$$

which for the given form of production function reduces to

$$qM(1)\kappa_q^-(1)\theta d K^{\theta-1}e^{h^*} = C(1-q). \qquad (9.76)$$

Set $B = qM(1)\kappa_q^-(1)\theta d/(1-q)$, then the optimal exercise price is

$$e^{h^*} = e^{h^*(K)} = \frac{CK^{1-\theta}}{B}. \qquad (9.77)$$

The rigorous justification of this limiting argument can be made exactly as in the continuous time model in [16].

9.7.2 Value of investment opportunity and firm's value

Let $h = h(K;\Delta)$ be the solution to (9.75). Then, on the strength of Theorem 9.5.1, the normalized option value associated with the increase of the capital by ΔK, at the price level $e^x < e^{h^*}$, is

$$\mathcal{E}^+\mathbf{1}_{[h^*,+\infty)}(\cdot)\left(qM(1)(G(K+\Delta K)-G(K))\kappa_q^-(1)e^{\cdot} - (1-q)C\Delta K\right)(x).$$

As $\Delta K \to 0$, we have $h = h(K;\Delta) \to h^*(K)$; therefore, dividing by ΔK and passing to the limit, we obtain the formula for the derivative of the option value of future investment opportunities w.r.t. K:

$$\begin{aligned} V_K^{\text{opt}}(K,x) &= (1-q)^{-1}\mathcal{E}^+\mathbf{1}_{[h^*,+\infty)}(\cdot)\left(qM(1)G'(K)\kappa_q^-(1)e^{\cdot} - (1-q)C\right)(x) \\ &= \mathcal{E}^+\mathbf{1}_{[h^*,+\infty)}(\cdot)\left(BK^{\theta-1}e^{\cdot} - C\right)(x). \end{aligned} \qquad (9.78)$$

Equations (9.77) and (9.78) imply together that

$$V_K^{\text{opt}}(K,x) = Ce^{-h^*}\mathcal{E}^+\mathbf{1}_{[h^*,+\infty)}(\cdot)\left(e^{\cdot} - e^{h^*}\right)(x). \qquad (9.79)$$

Let $u(x) = \mathbf{1}_{[h^*,+\infty)}(x)\left(e^x - e^{h^*}\right)$. For the sake of brevity, the following calculations are made for the simplest model (7.5). For $x < h^*$,

$$\begin{aligned} (\mathcal{E}^+u)(x) &= \frac{\beta^+(\lambda^+-\beta^+)}{\lambda^+}\int_0^{+\infty} e^{-\beta^+ y}\mathbf{1}_{[h^*,+\infty)}(x+y)\left(e^{x+y} - e^{h^*}\right)dy \\ &= \frac{\beta^+(\lambda^+-\beta^+)}{\lambda^+}\left[e^x\int_{h^*-x}^{+\infty} e^{(1-\beta^+)y}dy - e^{h^*}\int_{h^*-x}^{+\infty} e^{-\beta^+ y}dy\right] \\ &= \frac{(\lambda^+-\beta^+)}{\lambda^+}\cdot\frac{e^{\beta^+(x-h^*)+h^*}}{\beta^+-1}. \end{aligned}$$

Now we may substitute $(\mathcal{E}^+u)(x)$ into (9.79) and using (9.77) derive

$$V_K^{\text{opt}}(K,x) = \frac{(\lambda^+ - \beta^+)C}{\lambda^+(\beta^+ - 1)} \mathrm{e}^{\beta^+(x-h^*)} = \frac{(\lambda^+ - \beta^+)C}{\lambda^+(\beta^+ - 1)} \left(\frac{B}{C}\right)^{\beta^+} K^{\beta^+(\theta-1)} \mathrm{e}^{\beta^+ x}$$

Integrating w.r.t. K, we derive the option value of investment opportunities:

$$\begin{aligned} V^{\text{opt}}(K,x) &= \frac{(\lambda^+ - \beta^+)C}{\lambda^+(\beta^+ - 1)} \left(\frac{B}{C}\right)^{\beta^+} \mathrm{e}^{\beta^+ x} \int_K^{+\infty} k^{\beta^+(\theta-1)} dk \\ &= \frac{(\lambda^+ - \beta^+)CK^{1-\beta^+(1-\theta)}}{\lambda^+(\beta^+ - 1)(\beta^+(1-\theta) - 1)} \left(\frac{B}{C}\right)^{\beta^+} \mathrm{e}^{\beta^+ x} \end{aligned} \tag{9.80}$$

Given the spot price $P = \mathrm{e}^x$, the value of the firm with the capital stock K is the EPV of the stream of revenues, calculated under the assumption that the capital stock remains constant in the future, plus the option value of investment opportunities:

$$V(K,x) = \frac{dK^\theta \mathrm{e}^x}{1 - qM(1)} + V^{\text{opt}}(K,x). \tag{9.81}$$

9.7.3 Capital stock dynamics

Our next goal is to determine the optimal amount of investment and the dynamics of the capital stock. As it was stressed in [48], the benchmark models of uncertainty introduced in [39] do not suggest specific predictions about the level of investment. Since the investment rule itself is not observable, one has to use the data on investment and capital stock to evaluate investment models. In [2] and [15], the behavior of the capital stock of a new born firm in the Gaussian model and Lévy model, respectively, is examined. In both cases, fairly sophisticated mathematical techniques are used. Below, we obtain recurrent formulas for the expected value of capital at any time period in the future by using the elementary calculus.

Direct calculations show that at the moment of entry, the firm will install the stock of capital given by:

$$BK^{\theta-1}\mathrm{e}^x = C. \tag{9.82}$$

Explicitly,

$$K = \left(\frac{B}{C}\right)^{1/(1-\theta)} e^{x/(1-\theta)}.$$

Here is a non-technical proof of (9.82). The firm always chooses a capital stock that is above the boundary $K = (h^*)^{-1}(x)$. If $K_0 > (h^*)^{-1}(X_0)$, and if the firm were given a one-time opportunity to reduce its capital stock, it would choose $K_0' = (h^*)^{-1}(X_0)$. That is, (K_0, X_0) solves (9.82). From (9.82), we conclude that, at the moment of entry, the firm's value is a function of the spot price only:

$$V(x) = C\delta \left(\frac{B}{C}\right)^{\frac{1}{1-\theta}} \mathrm{e}^{\frac{x}{1-\theta}}, \tag{9.83}$$

where

$$\delta = \frac{\kappa_q^+(1)}{qM(1)\theta} + \frac{\lambda^+ - \beta^+}{\lambda^+(\beta^+ - 1)(\beta^+(1-\theta) - 1)}.$$

Let I be the fixed cost of entry. From Theorem 9.5.5, it is optimal to enter the first time

$$(\mathcal{E}^+)^{-1}(V(\cdot) - I))(X_t) \geq 0,$$

equivalently

$$C\delta \left(\frac{B}{C}\right)^{\frac{1}{1-\theta}} \frac{\mathrm{e}^{\frac{x}{1-\theta}}}{\kappa_q^+(1/(1-\theta))} = I,$$

therefore the price that triggers new entry is

$$\mathrm{e}^{h_e} = \left[\frac{\kappa_q^+(1/(1-\theta))I}{C\delta}\right]^{1-\theta} \frac{C}{B}. \tag{9.84}$$

Denote the moment of entry $t = 0$. Since investment is irreversible, the capital stock cannot be decreased, and it is increased when (and only when) the supremum process $\bar{X}_t$ jumps. Therefore, after the entry, the capital stock dynamics is given by

$$K_t = \left(\frac{B}{C}\right)^{\frac{1}{1-\theta}} \mathrm{e}^{\frac{\bar{X}_t}{1-\theta}} = K_0 \mathrm{e}^{\frac{\bar{X}_t - X_0}{1-\theta}},$$

where K_0 is given by (9.82). The expected capital stock at time $t > 0$ is

$$E^{X_0}[K_t] = K_0 E^{X_0}\left[\mathrm{e}^{\frac{\bar{X}_t - X_0}{1-\theta}}\right] = K_0 E\left[\mathrm{e}^{\frac{\bar{X}_t}{1-\theta}} \mid X_0 = 0\right].$$

Using (7.24) and (5.26), we may write

$$\sum_{t=0}^{\infty} q^t E\left[\mathrm{e}^{\frac{\bar{X}_t}{1-\theta}} \mid X_0 = 0\right] = \frac{\kappa_q^+(1/(1-\theta))}{1-q}. \tag{9.85}$$

Equation (9.85) tells us that in order to find the expected stock of capital at any time t, one needs to know the coefficients c_t of the Taylor series of the function $\kappa_q^+(1/(1-\theta))/(1-q)$: if $\kappa_q^+\left(1/(1-\theta)\right)/(1-q) = 1 + \sum_{t=1}^{\infty} c_t q^t$, then

$$E[K_t] = K_0 E\left[\mathrm{e}^{\frac{\bar{X}_t}{1-\theta}} \mid X_0 = 0\right] = c_t K_0.$$

To find c_t, recall that

$$\kappa_q^+\left(\frac{1}{1-\theta}\right) = \frac{\left(\lambda^+ - \frac{1}{1-\theta}\right)\beta^+(q)}{\lambda^+\left(\beta^+(q) - \frac{1}{1-\theta}\right)} = \frac{\lambda^+(1-\theta)-1}{(1-\theta)\lambda^+}\left[1 + \frac{1}{\beta^+(q)(1-\theta)-1}\right] \tag{9.86}$$

where $\beta^+(q)$ is given by (7.20). At the end of this subsection, we prove that

$$(1-\theta)\beta^+(q) - 1 = \left(\lambda^+(1-\theta)-1\right)\left(1 + \gamma\sum_{t=1}^{\infty} a_t q^t\right), \tag{9.87}$$

where $\gamma = (\lambda^+ - \lambda^-)(1-\theta)/\left(\lambda^+(1-\theta)-1\right)/2$, $a_1 = A := 2\lambda^+\lambda^-/(\lambda^+ - \lambda^-)^2$, and $a_t = A^t(-1)^{t-1}(2t-3)(2t-5)\cdots 3\cdot 1/t!$, $t > 1$. Next, we define $b_1, b_2, \ldots$, by

$$\left(1 + \gamma\sum_{t=1}^{\infty} a_t q^t\right)^{-1} = 1 + \sum_{t=1}^{\infty} b_t q^t. \tag{9.88}$$

Straightforward computations show that b_t can be calculated recurrently

$$b_t = -\gamma\sum_{k=1}^{t} a_k b_{t-k}, \tag{9.89}$$

where $b_0 \equiv 1$. Substituting (9.88) into (9.87), and (9.87) into (9.86), we obtain

$$\kappa_q^+\left(1/(1-\theta)\right) = 1 + (\lambda^+(1-\theta))^{-1}\sum_{t=1}^{\infty} b_t q^t.$$

Finally, we write

$$\frac{\kappa_q^+\left(1/(1-\theta)\right)}{1-q} = \kappa_q^+\left(1/(1-\theta)\right)\left(1 + \sum_{t=1}^{\infty} q^t\right) = 1 + \sum_{t=1}^{\infty} c_t q^t,$$

where

$$c_t = 1 + \frac{1}{\lambda^+(1-\theta)}\sum_{n=1}^{t} b_n = E\left[e^{\frac{\bar{X}_t}{1-\theta}} \mid X_0 = 0\right] \tag{9.90}$$

are the coefficients in the formula for the expected value of capital at time t.

Proof of (9.87)

We can write

$$\beta^+(q) = \frac{\lambda^+ + \lambda^-}{2} + \frac{\lambda^+ - \lambda^-}{2}\left(1 + \frac{4\lambda^+\lambda^-}{(\lambda^+ - \lambda^-)^2}q\right)^{1/2}. \tag{9.91}$$

Set $A = 2\lambda^+\lambda^-/(\lambda^+ - \lambda^-)^2$, then the square root on the RHS equals

$$(1 + 2Aq)^{1/2} = 1 + \sum_{t=1}^{\infty} a_t q^t, \tag{9.92}$$

where $a_1 = A$, and for $t > 1$, $a_t = A^t(-1)^{t-1}(2t-3)(2t-5)\cdots 3\cdot 1/t!$. Using (9.91) and (9.92), we derive $\beta^+(q) = \lambda^+ + 0.5(\lambda^+ - \lambda^-)\sum_{t=1}^{\infty} a_t q^t$, and arrive at (9.87).

Problems

9.1. Assume that the agents in the market have become more pessimistic so that for each $x > 0$, the transition density $p^+(x)$ decreased or remained the same, and for each $x < 0$, $p^-(x)$ increased or remained the same. Consider models of the irreversible entry and exit. Using the good and bad news principles, explain how the investment and disinvestment thresholds will change.

9.2. Using the model (7.6), solve

(a) the irreversible investment problem for a firm with the operational profit stream Ge^{X_t}, the fixed investment cost being I;
(b) exit problem for the firm with the operational profits $GP - e^{X_t}$;
(c) study the dependence of the entry and exit thresholds, value of the investment opportunity and firm's value on c_- and λ_-.

9.3. Prove Theorem 9.5.2.

9.4. Find optimal exercise thresholds and option values for

(a) options to acquire or abandon a stream $g(X_t) = X_t - C$;
(b) options to acquire or abandon a stream $g(X_t) = C - X_t$;
(c) perpetual American call option on a stock with the price dynamics $S_t = X_t$ and strike K;
(d) perpetual American put option on a stock with the price dynamics $S_t = X_t$ and strike K.

Part IV

Continuous time - continuous space models

10

Brownian motion case

10.1 Main definitions

A stochastic process on $\mathbb{R}^d$ is a collection of random variables $\{X_t\} := \{X_t \mid t \geq 0\}$ assuming values in $\mathbb{R}^d$. Under additional conditions, one can define sample paths of $\{X_t\}$ and identify the probability space Ω with the set of sample paths of the process. In particular, this is possible for the Brownian motion and, more generally, Lévy processes. Lévy processes are processes in continuous time with i.i.d. increments. The Brownian motion is the only (subclass of) Lévy process(es) with continuous sample paths; sample paths of any other Lévy process exhibit jumps.

In this Part, we will consider processes on $\mathbb{R}$. The Brownian motion with drift b and volatility σ admits several equivalent definitions. In applications to hedging, it is useful to define the Brownian motion as the solution of the stochastic differential equation

$$dX_t = bdt + \sigma dW_t, \tag{10.1}$$

s.t. $X_0 = 0$, where dW_t is the increment of the standard Wiener process with zero drift and unit variance. If it necessary to calculate prices of European options, it suffices to know the probability distribution function p_t of X_t, $t > 0$, under the EMM [1]. Let $q > 0$ be the riskless rate. Then the price $V(t, x)$ of the European option with the expiry date T and payoff $g(X_T)$ is

$$V(x, t) = e^{-q\tau} \int_{\mathbb{R}} p_\tau(y) g(x + y) dy, \tag{10.2}$$

where $\tau = T - t$ is time to expiry. Since the probability distribution function of the Brownian motion is given by a simple analytical expression

$$p_t(y) = \frac{1}{\sqrt{2\pi t\sigma^2}} \exp[-(y - bt)^2/(2t\sigma^2)]. \tag{10.3}$$

[1] Recall that in the financial market with one riskless bond and a stock, whose log-price evolves as the Brownian motion, an EMM is unique (see e.g. [80]).

and p_t decays fast at infinity, the integral (10.2) can be easily calculated.

For our purposes, the third definition, in terms of the moment- generating function, is more important and useful. Using (10.3), it is easy to derive that the moment generating function of the Brownian motion is of the form

$$E\left[e^{zX_t}\right] = e^{t\Psi(z)}, \tag{10.4}$$

where

$$\Psi(z) = \frac{\sigma^2}{2}z^2 + bz \tag{10.5}$$

is called the *Lévy exponent* of the process X_t. Conversely, if the moment generating function of a process admits the representation (10.4)-(10.5), then the process is the Brownian motion with drift b and volatility σ. This representation generalizes for other Lévy processes, and leads to important formulas.

We may (and will) allow the Brownian motion to start at any $x \in \mathbb{R}$; equivalently, we may consider a process $x + X_t$ instead of X_t. We use the standard notation $E^x[g(X_t)] = E[g(X_t) \mid X_0 = x]$. The Brownian motion and Lévy processes in general are Markov processes. The *infinitesimal generator* of a Markov process $\{X_t\}$, denote it L_X or simply L, is defined by

$$Lu(x) = \lim_{t\to+0} \frac{E^x[u(X_t)] - u(x)}{t}. \tag{10.6}$$

The operator L acts from $C_0^2(\mathbb{R})$, the space of twice continuously differentiable functions vanishing at infinity together with their derivatives up to order 2, to $C_0(\mathbb{R})$, the space of continuous functions vanishing at infinity. The infinitesimal generator of the Brownian motion is a differential operator:

$$Lu(x) = \frac{\sigma^2}{2}u''(x) + bu'(x), \tag{10.7}$$

or

$$L = \frac{\sigma^2}{2}\partial^2 + b\partial, \tag{10.8}$$

where $\partial := d/dx$ denotes the differential operator $u \mapsto u'$. Since a differential operator is uniquely defined by its action on exponential functions, it suffices to verify (10.7) for an arbitrary exponential function $u(x) := u(z,x) = e^{zx}$. Applying the definitions of the Lévy exponent and infinitesimal generator, (10.5) and (10.6), we calculate

$$\begin{aligned} Lu(x) &= \lim_{t\to+0} \frac{E^x[e^{zX_t}] - u(x)}{t} \\ &= \lim_{t\to+0} \frac{e^{t\Psi(z)}e^{zx} - e^{zx}}{t} \\ &= \Psi(z)e^{zx}, \end{aligned} \tag{10.9}$$

which equals $\frac{\sigma^2}{2}u''(x) + bu'(x)$.

Let $q > 0$ be the riskless rate. The (normalized) *expected present value operator (EPV-operator)* calculates the normalized EPV of a stream $g(X_t)$:

$$\mathcal{E}g(x) = qE^x \left[\int_0^{+\infty} e^{-qt} g(X_t) dt \right]. \tag{10.10}$$

In the theory of stochastic processes, $\mathcal{E}$ is called the resolvent or q-potential operator. One of the basic facts of the theory of Markov processes is the equality

$$\mathcal{E} = q(q - L)^{-1}, \tag{10.11}$$

where $q - L$ is regarded as an operator acting from $C_0^2(\mathbb{R})$ to $C_0(\mathbb{R})$. Sometimes, it is necessary to allow $\mathcal{E}$ and L to act between wider classes of functions. Clearly, $\mathcal{E}g$ is well-defined by (10.10) for a bounded measurable g. It is a continuous function, which is independent of values of g on a set of zero measure. However, $\mathcal{E}g$ is not twice differentiable at points of discontinuity of g, and, therefore, at these points, $(q - L)\mathcal{E}g$ is not defined. At points of continuity of g, the equality $q^{-1}(q - L)\mathcal{E}g(x) = g(x)$ holds. On the contrary, if u is twice differentiable a.e. and the derivatives up to order two are bounded, then $\mathcal{E}q^{-1}(q - L)u(x) = u(x)$ everywhere. Thus, we have a weak form of (10.11); we will say $\mathcal{E}$ and $q^{-1}(q - L)$ are mutual inverses in the weak sense. We will use the same convention in Subsect. 10.2.3, where the action in spaces of functions exponentially growing at infinity is considered.

In the case of the Brownian motion, it is easy to see why (10.11) should hold by considering the action of $\mathcal{E}$ and $q - L$ on exponential functions. For an exponential function $g(x) = e^{zx}$, the calculation of $\mathcal{E}g$ is easy. If z is real, then the integrand in (10.10) is positive, therefore, the Fubini theorem applies, and we can interchange the order of taking expectation and integration. Using further the definition of the Lévy exponent (10.5), we obtain

$$\begin{aligned}
\mathcal{E}g(x) &= q \int_0^{+\infty} e^{-qt} E^x[g(X_t)] dt \\
&= e^{zx} q \int_0^{+\infty} e^{-(q - \Psi(z))t} dt \qquad (10.12) \\
&= \frac{q}{q - \Psi(z)} e^z x. \qquad (10.13)
\end{aligned}$$

Clearly, the integral in (10.12) converges iff $q - \Psi(z) > 0$, and then (10.13) holds. An equivalent condition, in terms of the positive and negative roots $\beta^\pm := \beta_q^\pm$ of the characteristic equation ("fundamental quadratic", in the terminology of [39]) $q - \Psi(z) = 0$, is $\beta^- < z < \beta^+$. The same argument applies if z is complex, and then the condition is $\beta^- < \operatorname{Re} z < \beta^+$. Comparing (10.13) and (10.9), we see that $\mathcal{E}g = q(q - L)^{-1}g$.

10.2 EPV-operators $\mathcal{E}^{\pm}$

10.2.1 Factorization of $\mathcal{E}$ and EPV-operators $\mathcal{E}^{\pm}$

Let $q > 0$. Define functions $\kappa_q^{\pm}$ and operators $\mathcal{E}^{\pm}$ by

$$\kappa_q^+(z) = \frac{\beta^+}{\beta^+ - z}, \quad \kappa_q^-(z) = \frac{\beta^-}{\beta^- - z}, \tag{10.14}$$

and

$$\mathcal{E}^+ g(x) = \beta^+ \int_0^{+\infty} \mathrm{e}^{-\beta^+ y} g(x+y) dy, \tag{10.15}$$

$$= \beta^+ \mathrm{e}^{\beta^+ x} \int_x^{+\infty} \mathrm{e}^{-\beta^+ y} g(y) dy, \tag{10.16}$$

$$\mathcal{E}^+ g(x) = -\beta^- \int_{-\infty}^0 \mathrm{e}^{-\beta^- y} g(x+y) dy \tag{10.17}$$

$$= -\beta^- \mathrm{e}^{\beta^- x} \int_{-\infty}^x \mathrm{e}^{-\beta^- y} g(y) dy. \tag{10.18}$$

Clearly, for $z \neq \beta^{\pm}$,

$$\frac{q}{q - \Psi(z)} = \kappa_q^+(z)\kappa_q^-(z), \tag{10.19}$$

and the direct calculations show that for an exponential $g(x) = \mathrm{e}^{zx}$,

$$\mathcal{E}^{\pm} g(x) = \kappa_q^{\pm}(z)\mathrm{e}^{zx}. \tag{10.20}$$

It follows from (10.20) and (10.19), that, for the same g,

$$\mathcal{E}g = \mathcal{E}^- \mathcal{E}^+ g = \mathcal{E}^+ \mathcal{E}^- g. \tag{10.21}$$

Using (10.21) and (10.15)-(10.17), we can calculate $\mathcal{E}g$ quite easily:

1) calculate $w = \mathcal{E}^+ g$ using (10.15);
2) calculate $\mathcal{E}g = \mathcal{E}^- w$:

$$\mathcal{E}g(x) = -\beta^- \int_{-\infty}^0 \mathrm{e}^{-\beta^- y} w(x+y) dy.$$

Certainly, we may calculate $w_2 = \mathcal{E}^- g$ first, and then $\mathcal{E}g = \mathcal{E}^+ w_2$.

Operators $\mathcal{E}^{\pm}$ are convolution operators with the kernels of the class $L_1(\mathbb{R})$, therefore their compositions $\mathcal{E}^-\mathcal{E}^+$ and $\mathcal{E}^+\mathcal{E}^-$ are also convolution operators with the kernels of the same class. Since the action of a convolution operator is uniquely defined by its action on exponential functions, we see that $\mathcal{E}$ is a convolution operator with the kernel of the class $L_1(\mathbb{R})$, and

$$\mathcal{E} = \mathcal{E}^- \mathcal{E}^+ = \mathcal{E}^+ \mathcal{E}^-. \tag{10.22}$$

Note that (10.19) and (10.22) are a special case of the Wiener–Hopf factorization. We will also need a similar factorization of the operator $A := q^{-1}(q-L)$:

$$A = A^+A^- = A^-A^+, \tag{10.23}$$

where the operators $A^{\pm}$ act as follows

$$A^+u(x) = (\beta^+)^{-1}(\beta^+ - \partial)u(x) = u(x) - (\beta^+)^{-1}u'(x), \tag{10.24}$$
$$A^-u(x) = (\beta^-)^{-1}(\beta^- - \partial)u(x) = u(x) - (\beta^-)^{-1}u'(x). \tag{10.25}$$

To verify (10.23), it suffices to apply all the operators to an exponential function $u(x) = \mathrm{e}^{zx}$; the result is the identity

$$\frac{q - \Psi(z)}{q}\mathrm{e}^{zx} = \frac{\beta^+ - z}{\beta^+} \cdot \frac{\beta^- - z}{\beta^-}\mathrm{e}^{zx} = \frac{\beta^- - z}{\beta^-} \cdot \frac{\beta^+ - z}{\beta^+}\mathrm{e}^{zx}.$$

Applying A^+ to (10.15), we obtain $A^+\mathcal{E}^+g = g$, provided g is sufficiently regular. Similarly, applying $\mathcal{E}^+$ to (10.24) and integrating by parts, we derive $\mathcal{E}^+A^+u = u$. Thus, $\mathcal{E}^+$ is the inverse to A^+. The same holds for the pair $\mathcal{E}^-$ and A^-. Thus,

$$\mathcal{E}^+ = (A^+)^{-1}, \quad \mathcal{E}^- = (A^-)^{-1}, \tag{10.26}$$

where the operators act between appropriate function spaces. For details, see Subsect. 10.2.3.

The following interpretation of the EPV-operators $\mathcal{E}$ and $\mathcal{E}^{\pm}$ is useful. Let $T \in \mathbb{R}_+$ be an exponentially distributed random variable of mean q^{-1}, independent of the process $X = \{X_t\}_{t \geq 0}$. The probability density of T is $q\mathrm{e}^{-qt}dt$, therefore $\mathcal{E}g(x) = E[g(x + X_T)]$. Introduce the supremum process $\bar{X}_t = \sup_{0 \leq s \leq t} X_s$ and the infimum process $\underline{X}_t = \inf_{0 \leq s \leq t} X_s$. It is well-known (see [13]) that $\bar{X}_T$ is an exponentially distributed random variable on $\mathbb{R}_+$ of mean $1/\beta^+$, and $\underline{X}_T$ is an exponentially distributed random variable on $\mathbb{R}_-$ of mean $-1/\beta^-$. Therefore, (10.15)–(10.17) can be rewritten as

$$\mathcal{E}^+g(x) = E[g(x + \bar{X}_T)] \tag{10.27}$$
$$= qE^x\left[\int_0^{+\infty} \mathrm{e}^{-qt}g(x + \bar{X}_t)\,dt\right], \tag{10.28}$$
$$\mathcal{E}^-g(x) = E[g(x + \underline{X}_T)] \tag{10.29}$$
$$= qE^x\left[\int_0^{+\infty} \mathrm{e}^{-qt}g(x + \underline{X}_t)\,dt\right]. \tag{10.30}$$

In Chap. 11, we deduce (10.27)–(10.30) using the Wiener–Hopf factorization.

10.2.2 Main properties of operators $\mathcal{E}$ and $\mathcal{E}^{\pm}$

In the following propositions, X_t satisfies (10.33), and a function g belongs to an appropriate function space specified in Subsect. 10.2.3 and Lemma 10.2.5

below. The reader may use these proposition adding: *under natural regularity conditions explained in Subsect. 10.2.3.* In Proposition 10.2.1, "monotone" means "increasing", "non-decreasing", "decreasing" or "non-increasing".

Proposition 10.2.1 *The operators $\mathcal{E}^{\pm}$ enjoy the following properties:*

(a) If $g(x) = 0 \; \forall \; x \geq h$, then for the same x, $(\mathcal{E}^{+}g)(x) = 0$.
(b) If $g(x) = 0 \; \forall \; x \leq h$, then for the same x, $(\mathcal{E}^{-}g)(x) = 0$.
(c) If $g(x) \geq 0 \; \forall x$, then $(\mathcal{E}^{+}g)(x) \geq 0, \; \forall x$.
(d) If $g(x) \geq 0 \; \forall x$, then $(\mathcal{E}^{-}g)(x) \geq 0, \; \forall x$.
(e) If g is monotone, then $\mathcal{E}^{+}g$ and $\mathcal{E}^{-}g$ are also monotone.
(f) If g is measurable and bounded, then functions $\mathcal{E}^{\pm}g$ are differentiable, and their derivatives are bounded.

Proof. Since $\mathcal{E}^{+}g(x) = E[g(x+Y^{+})]$ and $\mathcal{E}^{-}g(x) = E[g(x+Y^{-})]$, where $Y^{\pm}$ is a random variable assuming values in $\mathbb{R}_{\pm}$, properties (a)–(e) are immediate. Part (f) follows from (10.16) and (10.18).

Proposition 10.2.2 *Operators $A^{\pm} = (\mathcal{E}^{\pm})^{-1}$ enjoy the following properties:*

(a) if $g(x) = 0 \; \forall \; x > h$, then for $x > h$, $((\mathcal{E}^{+})^{-1}g)(x) = 0$;
(b) if $g(x) = 0 \; \forall \; x < h$, then for $x < h$, $((\mathcal{E}^{-})^{-1}g)(x) = 0$.

Proof. $A^{\pm}$ are differential operators.

Note that above, we can use non-strict inequalities if the derivative at the boundary is understood as the appropriate one-sided derivative.

Proposition 10.2.3 *a) If $g(-\infty) < 0$, then $\mathcal{E}g(-\infty) < 0$, $\mathcal{E}^{\pm}g(-\infty) < 0$;*
b) If $g(-\infty) > 0$, then $\mathcal{E}g(-\infty) > 0$ and $\mathcal{E}^{\pm}g(-\infty) > 0$;
c) statements a)–b) hold with $+\infty$ instead of $-\infty$.

Proof. (a) Without loss of generality, we may assume that $|g(x)| \leq 1, \; \forall \; x \in \mathbb{R}$. If $g(-\infty) < 0$, then there exist $N > 0$ and $c > 0$ such that $g(x) < -c$ for all $x < -N$. For given N, $c \in (0,1)$ and any s, there exists N_1 such that for any $x < -N_1$, $\text{Prob}\,(\bar{X}_s > -N \mid X_0 = x) < c/2$. Therefore, for these x,

$$\begin{aligned}
\mathcal{E}^{+}g(x) &= E^{x}\left[q\int_0^s \mathrm{e}^{-qt}g(\bar{X}_t)dt + q\int_s^{+\infty}\mathrm{e}^{-qt}g(\bar{X}_t)dt\right] \\
&\leq \text{Prob}\,(\bar{X}_s \leq -N \mid X_0 = x)\, q\int_0^s(-c)\mathrm{e}^{-qt}dt \\
&\quad +\text{Prob}\,(\bar{X}_s > -N \mid X_0 = x)\, q\int_0^s \mathrm{e}^{-qt}dt + q\int_s^{+\infty}\mathrm{e}^{-qt}dt \\
&\leq -(1-c/2)cq\int_0^s \mathrm{e}^{-qt}dt + \frac{cq}{2}\int_0^s \mathrm{e}^{-qt}dt + q\int_s^{+\infty}\mathrm{e}^{-qt}dt \\
&= -\frac{c(1-c)}{2} + \left(1 + \frac{c(1-c)}{2}\right) q\int_s^{+\infty}\mathrm{e}^{-qt}dt.
\end{aligned}$$

The right-most expression is negative if s is sufficiently large. The proof for $\mathcal{E}g(x)$ is the same because $X_t \le \bar{X}_t$, and for $\mathcal{E}^- g(x)$, the result is evident, because the sample paths of the infimum process are not increasing.

(b) is (a) for $-g$, and (c)–(d) are mirror reflections of (a)–(b).

Proposition 10.2.4 *Assume that g is monotone and changes sign. Then there exists a unique h^+ (resp., h^-) such that $\mathcal{E}^+ g(h^+) = 0$ (resp., $\mathcal{E}^- g(h_-) = 0$), and $\mathcal{E}^+ g(x)$ (resp., $\mathcal{E}^- g(x)$) changes sign as x passes h^+ (resp., h^-).*

Proof. From Propositions 10.2.1 and 10.2.3, we know that $\mathcal{E}^{\pm} g$ is monotone and it changes sign. Define $x^+ = \inf\{x \mid \mathcal{E}^+ g(x) = \mathcal{E}^+ g(+\infty)\}$ and $x^- = \sup\{x \mid \mathcal{E}^- g(x) = \mathcal{E}^- g(-\infty)\}$. Since $\mathcal{E}^{\pm}$ is a convolution operator whose kernel is positive on $\mathbb{R}_{\pm}$, $\mathcal{E}^+ g$ is strictly monotone on $(-\infty, x^+)$, and $\mathcal{E}^- g$ is strictly monotone on $(x^-, +\infty)$.

10.2.3 The case of payoffs exponentially growing at infinity

The definitions and results of this subsection are purely technical. They are needed to give the exact meaning to the properties of the EPV-operators listed in Subsect. 10.2.2. The reader may skip this subsection and read the following sections having in mind that “all the arguments are valid under certain regularity conditions explained in Subsect. 10.2.3".

In some applications, it is necessary to consider $(q - L)G$ for functions G which are not twice differentiable everywhere, and grow exponentially at infinity. For instance, the value of the perpetual American call option in the Geometric Brownian motion model grows exponentially as $x \to +\infty$ and its second derivative is not defined at the exercise boundary. This motivates the use of function spaces that are more general than C_0 and C_0^2.

Definition 10.1. *Let $\sigma \in \mathbb{R}$ and $m = 0, 1, 2$. We say that a function $u \in \mathcal{L}^m_{\infty;\sigma}(\mathbb{R})$ if for $s = 0, \dots, m$, function $e^{-\sigma x} u^{(s)}(x)$ is defined a.e., measurable and uniformly bounded. The norm in $\mathcal{L}^m_{\infty;\sigma}(\mathbb{R})$ is defined by*

$$||u||_{\infty;\sigma;m} = \max_{0 \le s \le m} \sup_{x \in \mathbb{R}} e^{-\sigma x} |u^{(s)}(x)|. \tag{10.31}$$

The subspace of $\mathcal{L}^m_{\infty;\sigma}(\mathbb{R})$ consisting of functions vanishing below 0 (resp., above 0) is denoted $\mathcal{L}^m_{\infty;\sigma}(\mathbb{R}_+)$ (resp., $\mathcal{L}^m_{\infty;\sigma}(\mathbb{R}_-)$).

Definition 10.2. *Let $\sigma^- \le 0 \le \sigma^+$ and $m = 0, 1, 2$. We say that a measurable function $u \in \mathcal{L}^m_{\infty;\sigma^-,\sigma^+}(\mathbb{R})$ if for $s = 0, \dots, m$, function $(e^{\sigma^- x} + e^{\sigma x})^{-1} u^{(s)}(x)$ is defined a.e., bounded and vanishes as $x \to \pm\infty$. The norm in $\mathcal{L}^m_{\infty;\sigma^-,\sigma^+}(\mathbb{R})$ is defined by*

$$||u||_{\infty;\sigma^-,\sigma^+;m} = \max_{0 \le s \le m} \sup_{x \in \mathbb{R}} (e^{\sigma^- x} + e^{\sigma^+ x})^{-1} |u^{(s)}(x)|. \tag{10.32}$$

Assume that there exist $\sigma^- \le 0 \le \sigma^+$ such that

$$q - \Psi(\sigma) > 0, \quad \forall\, \sigma \in [\sigma^-, \sigma^-]. \tag{10.33}$$

For the Brownian motion, the condition (7.33) is equivalent to

$$\beta^- < \sigma^- \le 0 \le \sigma^+ < \beta^+. \tag{10.34}$$

In the theorems below, we use (10.33) because this condition extends to the case of general Lévy processes which we consider in the next chapter.

Lemma 10.2.5 *Let X_t satisfy (10.33). Then*

(a) operators $q^{-1}(q-L) : \mathcal{L}^2_{\infty;\sigma^+,\sigma^-}(\mathbb{R}) \to \mathcal{L}^0_{\infty;\sigma^+,\sigma^-}(\mathbb{R})$, $\mathcal{E} : \mathcal{L}^0_{\infty;\sigma^+,\sigma^-}(\mathbb{R}) \to \mathcal{L}^2_{\infty;\sigma^+,\sigma^-}(\mathbb{R})$ are bounded, and they are mutual inverses in the weak sense;
(b) for any $\sigma \le \sigma^+$, and $s = 0, 1$, operators $A^+ : \mathcal{L}^{s+1}_{\infty;\sigma}(\mathbb{R}_+) \to \mathcal{L}^s_{\infty;\sigma}(\mathbb{R}_+)$ and $\mathcal{E}^+ : \mathcal{L}^s_{\infty;\sigma}(\mathbb{R}_+) \to \mathcal{L}^{s+1}_{\infty;\sigma}(\mathbb{R}_+)$ are bounded, and they are mutual inverses in the weak sense; the same is true with $\mathbb{R}$ instead of $\mathbb{R}_+$;
(c) for any $\sigma \ge \sigma^-$, and $s = 0, 1$, operators $A^- : \mathcal{L}^{s+1}_{\infty;\sigma}(\mathbb{R}_-) \to \mathcal{L}^s_{\infty;\sigma}(\mathbb{R}_-)$ and $\mathcal{E}^+ : \mathcal{L}^s_{\infty;\sigma}(\mathbb{R}_-) \to \mathcal{L}^{s+1}_{\infty;\sigma}(\mathbb{R}_-)$ are bounded, and they are mutual inverses in the weak sense; the same is true with $\mathbb{R}$ instead of $\mathbb{R}_-$;
(d) $\mathcal{E} = \mathcal{E}^-\mathcal{E}^+ = \mathcal{E}^+\mathcal{E}^-$ as operators from $\mathcal{L}^0_{\infty;\sigma^+,\sigma^-}(\mathbb{R})$ to $\mathcal{L}^2_{\infty;\sigma^+,\sigma^-}(\mathbb{R})$;
(e) $A = A^+A^- = A^-A^+$ as operators from $\mathcal{L}^2_{\infty;\sigma^+,\sigma^-}(\mathbb{R})$ to $\mathcal{L}^0_{\infty;\sigma^+,\sigma^-}(\mathbb{R})$.

Proof. The boundedness of the operators $A = q^{-1}(q-L)$, A^+ and A^- is immediate from the definition. Further, if $\sigma \le \sigma^+$, $s = 0, 1, \dots$, and $g \in \mathcal{L}^s_{\infty;\sigma}(\mathbb{R})$, then

$$|\mathcal{E}^+ g(x)| \le ||g||_{\infty;\sigma;s}\mathcal{E}^+ e^{\sigma\cdot}(x) = \frac{\beta^+}{\beta^+ - \sigma}||g||_{\infty;\sigma;s}e^{\sigma x},$$

and, using this estimate and (10.16), we obtain a similar estimate for the derivative of $\mathcal{E}^+ g$. This proves the boundedness of $\mathcal{E}^+$ in (b) with $\mathbb{R}$ instead of $\mathbb{R}_+$. To prove the boundedness in (b) with $\mathbb{R}_+$, it suffices to note that $\mathcal{E}^+ g(x) = E[g(x + Y^+)]$, therefore if g vanishes below 0 then $\mathcal{E}^+ g$ vanishes there as well. Since A^+ is a differential operator and $\mathcal{E}^+$ is a convolution operator, and $A^+\mathcal{E}^+$ and $\mathcal{E}^+A^+$ act as the identity operator in spaces of sufficiently regular functions, A^+ and $\mathcal{E}^+$ are mutual inverses in the weak sense. This proves (b). Part (c) is proved similarly.

To prove that $\mathcal{E}$ in (a) is bounded, take $\sigma \in [\sigma^-, \sigma^+]$, and use the representation of the convolution operator $\mathcal{E}$ as the composition $\mathcal{E} = \mathcal{E}^-\mathcal{E}^+$ and (b) and (c). The proof that A and $\mathcal{E}$ are mutual inverses is the same as for A^+ and $\mathcal{E}^+$. The proofs of parts (d) and (e) are straightforward.

10.3 EPV of a stream, which is abandoned when X_t falls to a certain level

Assume that the payoff stream $g(X_t)$ is a continuous non-decreasing function of X_t, a typical example being a firm facing demand uncertainty and a constant variable cost. Let G be the rate of output, and C the variable cost. For high levels of the log-price of the firm's output, X_t, the profit flow $g(X_t) = Ge^{X_t} - C$ is positive, and for low levels, it is negative. Should the log-price fall sufficiently low, to a certain level h, it may become optimal to cease production. Fix h, a candidate for the exit threshold (the optimal choice of h will be analyzed in the next section), and denote by $V(x; h)$ the value of the firm with this choice of the exit threshold. Denote by $\tau_h^- = \inf\{t \geq 0 \mid X_t \leq h\}$ the hitting time of $(-\infty, h]$; this is the continuous time counterpart of "the first time X_t reaches or crosses h from above" in discrete time case. For the rigorous definition, see [12, 77, 79]. Certainly, $\tau_h^- = \tau_h^-(\omega)$ depends on a sample path ω of the process. Thus, τ_h^- is a random variable on the probability space Ω of the sample paths of the process. We have

$$V(x; h) = E^x \left[\int_0^{\tau_h^-} \mathrm{e}^{-qt} g(X_t) dt \right].$$

In the region $x > h$, the value of the firm, $V(x; h)$, obeys the stationary Kolmogorov equation (another name used in finance and economics is the stationary Black–Scholes equation)

$$(q - L)V(x; h) = g(x), \quad x > h. \tag{10.35}$$

After exit, the firm's value is zero:

$$V(x; h) = 0, \quad x \leq h. \tag{10.36}$$

Introduce the normalized value function $\mathcal{V}(x; h) = qV(x; h)$. In terms of $\mathcal{V}$, the boundary problem (10.35)–(5.7) becomes

$$q^{-1}(q - L)\mathcal{V}(x; h) = g(x), \quad x > h, \tag{10.37}$$
$$\mathcal{V}(x; h) = 0, \quad x \leq h. \tag{10.38}$$

The next theorem, which demonstrates the essence of the Wiener–Hopf method in the form used in analysis, states that $\mathcal{V}(x; h)$ can be calculated using a formula, which is similar to the formula for $\mathcal{V}(x; -\infty)$, the value of the firm that never stops producing: $\mathcal{V}(x; -\infty) = \mathcal{E}g(x) = \mathcal{E}^-\mathcal{E}^+g(x)$. The new element is the operator $\mathbf{1}_{(h,+\infty)}$ between $\mathcal{E}^-$ and $\mathcal{E}^+$.

Theorem 10.3.1 *Assume that g is a measurable function satisfying*

$$|g(x)| \leq C(\mathrm{e}^{\sigma^- x} + \mathrm{e}^{\sigma^+ x}), \tag{10.39}$$

where $\sigma^- \le 0 \le \sigma^+$, *and*

$$q - \Psi(z) > 0, \quad \sigma^- \le z \le \sigma^+. \tag{10.40}$$

Then

$$\mathcal{V}(x; h) = (\mathcal{E}^- \mathbf{1}_{(h,+\infty)} \mathcal{E}^+ g)(x). \tag{10.41}$$

Remark 10.3. a) From the technical point of view, the calculation of the solution (10.41) is no more difficult than the calculation of the value of the firm which never stops producing:

(1) calculate $w = \mathcal{E}^+ g$: $w(x) = \int_0^{+\infty} \beta^+ \mathrm{e}^{-\beta^+ y} g(x+y)dy$;
(2) set $g_2(x) = w(x)$ for $x > h$, and $g_2(x) = 0$ for $x \le h$;
(3) calculate $V = q^{-1}\mathcal{E}^- g_2$: $V(x) = q^{-1} \int_{-\infty}^0 (-\beta^-)\mathrm{e}^{-\beta^- y} g_2(x+y)dy$.

Notice that now we may not inverse the order of application of $\mathcal{E}^+$ and $\mathcal{E}^-$; the inverse order appears when we solve the problem for a stream which is abandoned as X_t reaches a certain threshold h from below; and then we use the indicator function $\mathbf{1}_{(-\infty,h)}$ instead of $\mathbf{1}_{(h,+\infty)}$.

b) Using the (independent) random variables Y^+ and Y^- on the positive and negative half-axis, we can write (10.41) in another form

$$V(x; h) = q^{-1} E\left[\mathbf{1}_{(h,+\infty)}(x + Y^-) g(x + Y^- + Y^+)\right]. \tag{10.42}$$

c) Contrary to discrete time models, for any h, the value function given by (10.41) does not change if we raplace $\mathbf{1}_{(h,+\infty)}$ with $\mathbf{1}_{[h,+\infty)}$. Indeed, $\mathcal{E}^-$ is a convolution operator with the integrable kernel, and, therefore, $\mathcal{V} = \mathcal{E}^- g_2$ is idepenedent of the value of g_2 at any point, in particular, of $g_2(h)$. This remark holds for any h and for wide classes of Lévy processes, in particular, processes which we consider in the next chapter. Note, however, that if a Lévy process does not have the diffusion component then it may be the case that $\mathcal{E}^- g(x) = E[g(x + Y^-)]$, where the probability density of Y^- has a non-zero mass at 0, and then $\mathcal{E}^- \mathbf{1}_{(h,+\infty)} w \ne \mathcal{E}^- \mathbf{1}_{[h,+\infty)} w$. For details, see Boyarchenko and Levendorskiǐ (2002), where the regularity of solutions of similar boundary problems for wide classes of Lévy processes is studied.

Proof of Theorem 10.3.1. On the strength of (10.39), the firm's value is bounded by the EPV of a stream of the form $C(\mathrm{e}^{\sigma^+ X_t} + \mathrm{e}^{\sigma^- X_t})$. Under condition (10.34), this EPV admits the bound of the same form (10.39). Therefore, it suffices to prove that:

1) a solution to the problem (10.37)–(10.38) in the class of functions satisfying (10.39) exists;
2) it is unique and given by (10.41).

We rewrite (10.37) as

$$(A\mathcal{V})(x; h) = g(x) + g^-(x), \quad \forall\, x \ne h, \tag{10.43}$$

where $g^- := A\mathcal{V} - g$ vanishes above h. Using $A = A^+A^-$, $\mathcal{E}^+A^+ = I$ and applying $\mathcal{E}^+$ to (10.43), we obtain

$$A^-\mathcal{V}(x;h) = \mathcal{E}^+g(x) + \mathcal{E}^+g^-(x), \quad \forall\, x \neq h. \tag{10.44}$$

Since $g^-(x) = 0\ \forall x > h$, $\mathcal{E}^+g^-(x) = E[g^-(x + Y^+)] = 0$ for these x as well. Using (10.24), we have the first order ODE on $(h, +\infty)$:

$$\mathcal{V}(x;h) - (1/\beta^+)\mathcal{V}'(x;h) = \mathcal{E}^+g(x), \quad x > h. \tag{10.45}$$

The value function is continuous and vanishes at the boundary h. Therefore, the solution is

$$\mathcal{V}(x;h) = (-\beta^-)\int_{h-x}^{0} \mathrm{e}^{-\beta^- y}(\mathcal{E}^+g)(x+y)dy \tag{10.46}$$

$$= (-\beta^-)\int_{h}^{x} \mathrm{e}^{\beta^-(x-y)}(\mathcal{E}^+g)(y)dy. \tag{10.47}$$

This is a standard fact in the theory of ODE; it can be easily verified substituting (10.46) into (10.45). Finally, using (10.25), we rewrite (10.46) as (10.41). It is straightforward to check that the RHS in (10.47) admits the bound (10.39), and satisfies (10.38).

10.4 Timing exit

Consider the problem of an optimal choice of the exit boundary h. We assume that

$$g \text{ is non} - \text{decreasing}; \tag{10.48}$$

$$g(-\infty) < 0 < g(+\infty) \tag{10.49}$$

(one limit or both may be infinite; in the exit problem above, only $g(+\infty)$ is infinite). From (10.41), we have

$$\mathcal{V}(x;h) = E[(\mathbf{1}_{(h,+\infty)}w)(x + Y^-)], \tag{10.50}$$

where $w(x) = \mathcal{E}^+g(x) = E[g(x + Y^+)]$. Clearly, the larger the value of the product $\mathbf{1}_{(h,+\infty)}w$, the larger is the value $\mathcal{V}(x;h)$. Hence, the optimal choice of h should replace all negative values of w by zero, and leave positive ones as they are. From Proposition 10.2.1, w is continuous (in fact, differentiable). Since g is non-decreasing, w is non-decreasing as well. Further, passing to the limit as $x \to \pm\infty$ in the equation $w(x) = E[g(x + Y^+)]$, we obtain that w satisfies (10.49) since g does. Moreover, it is easy to see that if g is increasing in a neighborhood of $+\infty$, then w is increasing on $\mathbb{R}$, and if g is constant on $[x_+, +\infty)$ but $g(x) < g(x_+)$, $\forall\, x < x_+$, then w is increasing below x_+. We conclude that w has a unique zero, call it h_*, and $w(x) > 0$ for all $x > h_*$, and $w(x) < 0$ for all $x < h_*$. Hence, h_* is the optimal exit threshold.

Note that $w(x) = \mathcal{E}^+ g(x)$ is the EPV of the stream $g(X_t)$ under supremum process, therefore we have the *good news principle*: abandon the stream when the EPV of the stream becomes non-positive. If $g(X_t)$ is a non-decreasing function of X_t, as we presumed, we have $g(\bar{X}_t) = \bar{g}_t \equiv \sup_{0 \le s \le t} g_t$, where $g_t = g(X_t)$, therefore we can reformulate the exit rule in terms of the supremum process: exit at level g if

$$E\left[\int_0^\infty \mathrm{e}^{-qt} \bar{g}_t dt \mid g_0 = g\right] \le 0.$$

In other words, the rule is: consider all sample paths of the process, and along each sample path, disregard all *temporary drops* of the output price. Then calculate the EPV of profits, and if it is non-positive, abandon the stream. Thus, the hope for the best dies hard: we exit only when the EPV is non-positive even after this rosy adjustment. It looks as if a firm's manager contemplating an exit is too optimistic. However, we will see that the same manager becomes overpessimistic when contemplating an investment.

After the optimal exit threshold h_* had been found, the manager calculates the normalized value of the firm $\mathcal{V}(x) = \mathcal{V}(x; h_*)$ for $x > h_*$ using (10.41) with $h = h_*$ and then (10.17):

$$\mathcal{V}(x; h_*) = (\mathcal{E}^- \mathbf{1}_{(h_*, +\infty)} \mathcal{E}^+ g)(x) \tag{10.51}$$

$$= -\beta^- \int_{h_*-x}^0 \mathrm{e}^{-\beta^- y} w(x+y) dy. \tag{10.52}$$

Example 10.4. Let $g(x) = G\mathrm{e}^x - C$. Then the EPV of the stream $g(X_t)$, hence, the firm's value is finite iff $q - \Psi(1) > 0$. Under this condition, $\beta^- < 0 < 1 < \beta^+$. Since

$$w(x) = \mathcal{E}^+(G\mathrm{e}^{\cdot} - C)(x) = G\kappa_q^+(1)\mathrm{e}^x - C, \tag{10.53}$$

the optimal exit threshold is defined from

$$G\kappa_q^+(1)\mathrm{e}^{h_*} = C. \tag{10.54}$$

Substituting (10.53) into (10.52), we calculate for $x > h_*$:

$$\begin{aligned}\mathcal{V}(x) &= -\beta^- \int_{h_*-x}^0 \mathrm{e}^{-\beta^- y} (G\kappa_q^+(1)\mathrm{e}^{x+y} - C) dy \\ &= G\kappa_q^+(1) \frac{-\beta^-}{1-\beta^-} \mathrm{e}^x \left(1 - \mathrm{e}^{(1-\beta^-)(h_*-x)}\right) - C(1 - \mathrm{e}^{-\beta^-(h_*-x)}).\end{aligned}$$

Using $\kappa_q^-(1) = \beta^-/(\beta^- - 1)$, (10.19) and (10.54), we obtain

$$V(x) = \frac{G\mathrm{e}^x}{q - \Psi(1)} - \frac{C}{q} + V_{\mathrm{opt}}(x), \tag{10.55}$$

where $V_{\rm opt}(x)$, the option value to exit, is given by

$$V_{\rm opt}(x) = q^{-1}C(\beta^-/(1-\beta^-)+1)\mathrm{e}^{\beta^-(x-h_*)} = \frac{C}{q(1-\beta^-)}\mathrm{e}^{\beta^-(x-h_*)}.$$

10.5 Smooth pasting principle

It is clear that $\mathbf{1}_{(h,+\infty)}w$ is maximally regular iff h is a zero of w: this choice makes $\mathbf{1}_{(h,+\infty)}w$ continuous, and with any other choice, $\mathbf{1}_{(h,+\infty)}w$ is discontinuous at h. Using (10.18), we conclude that $V(x;h)$ is differentiable at h iff h is a zero of w. We see that the optimal choice of h makes the value function smooth at the exercise threshold, and with a non-optimal choice, (a candidate for) the value function is not smooth at the threshold. The same argument works for wide classes of Lévy processes, in particular, the ones which we consider in the next chapter. However, there are Lévy processes for which the smooth pasting principle fails. Indeed, if the probability distribution of the random variable Y^- associated with $\mathcal{E}^-$ has an atom at 0, then $\mathcal{E}^-w$ will be non-differentiable at h if w is not differentiable at h. For details, see [20, 21, 4].

10.6 Exit under supply uncertainty

Suppose that the price of the firm's output, P, is constant, but the variable cost follows the geometric random walk: $C = \mathrm{e}^{X_t}$. The instantaneous profit $g(X_t) = PG - \mathrm{e}^{X_t}$ is a decreasing function of X_t, and it is positive at low levels of X_t and negative at high levels of X_t. It may be optimal to exit should the cost become too high. Assuming that the exit threshold, h, is chosen, one can calculate the firm's value, $V(x;h)$, equivalently, the EPV of the stream $g(X_t)$ with the option to abandon it, using the following theorem. Its formulation and proof are the mirror reflections of Theorem 10.3.1 and its proof.

Theorem 10.6.1 *Assume that (10.39) and (10.34) hold. Then*

$$V(x;h) = q^{-1}(\mathcal{E}^+\mathbf{1}_{(-\infty,h)}\mathcal{E}^-g)(x). \tag{10.56}$$

Consider the problem of an optimal choice of the exit boundary h. We assume that

$$g \text{ is non} - \text{increasing}, \tag{10.57}$$

and

$$g(+\infty) < 0 < g(-\infty) \tag{10.58}$$

(one limit or both may be infinite; in the exit problem above, only $g(+\infty)$ is infinite). From (10.56), we have

$$V(x;h) = q^{-1}E[(\mathbf{1}_{(-\infty,h)}w)(x+Y^+)], \tag{10.59}$$

where $w(x) = \mathcal{E}^- g(x) = E[g(x + Y^-)]$. Clearly, the larger the value of the product $\mathbf{1}_{(-\infty,h)} w$, the larger is the value $V(x;h)$. Hence, the optimal choice of h should replace all negative values of w by zero, and leave positive ones as they are. From Proposition 10.2.1, $w = \mathcal{E}^- g$ is continuous. Since g is non-increasing, w is non-increasing as well. Further, passing to the limit as $x \to \pm\infty$ in the equality $w(x) = E[g(x + Y^-)]$, we obtain that w satisfies (10.58) since g does. Moreover, it is easy to see from (10.17) that if g is decreasing in a neighborhood of $-\infty$, then w is decreasing on $\mathbb{R}$, and if g is constant on $(-\infty, x_-]$ but $g(x) < g(x_-)$, $\forall\ x > x_-$, then w is decreasing above x_-. We conclude that w has a unique zero, call it h^*, and $w(x) > 0$ for all $x < h^*$, and $w(x) < 0$ for all $x > h^*$. Hence, h^* is the optimal exit threshold.

The optimal exit threshold h^* having been found, the manager calculates the firm's value for $x < h^*$ using (10.56) and (10.15):

$$\begin{aligned} V(x) &= q^{-1}(\mathcal{E}^+ \mathbf{1}_{(-\infty,h^*)} \mathcal{E}^- g)(x) \\ &= q^{-1} \int_0^{h^*-x} \beta^+ e^{-\beta^+ y} w(x+y) dy. \end{aligned} \tag{10.60}$$

Example 10.5. Let $g(x) = PG - e^x$, and assume for simplicity that $q - \Psi(1) > 0$. Under this condition, $\beta^- < 0 < 1 < \beta^+$. Since $w(x) = \mathcal{E}^-(PG - e^{\cdot})(x) = PG - \kappa_q^-(1)e^x$, the optimal exit threshold is defined from

$$\kappa_q^-(1)e^{h^*} = PG. \tag{10.61}$$

Using (10.60), we calculate the normalized value of the firm for $x < h^*$:

$$\begin{aligned} V(x) &= q^{-1} \int_0^{h^*-x} \beta^+ e^{-\beta^+ y}(PG - \kappa_q^-(1)e^{x+y})dy \\ &= q^{-1} \left[PG(1 - e^{\beta^+(x-h^*)}) - \kappa_q^-(1) \frac{\beta^+}{\beta^+ - 1}(e^x - e^{x+(\beta^+-1)(x-h^*)}) \right]. \end{aligned}$$

Applying (10.19) and (10.61), we simplify

$$V(x) = \frac{PG}{q} - \frac{e^x}{q - \Psi(1)} + V_{\text{opt}}(x), \tag{10.62}$$

where the first two terms are the value of the firm which never exits, and

$$V_{\text{opt}}(x) = \frac{PG}{q}(\beta^+/(\beta^+ - 1) - 1)e^{\beta^+(x-h^*)} = \frac{PG}{q(\beta^+ - 1)} e^{\beta^+(x-h^*)}$$

is the option value to exit.

10.7 Model entry problems

10.7.1 Entry under demand uncertainty

The firm's manager contemplates the investment into a plant that will produce G units of output at no variable cost starting the moment the investment is made. The price of a unit of output evolves as e^{X_t}, where X_t is the Brownian motion. The fixed investment cost is I. Should the price of output rise sufficiently high, it will be optimal to invest. The manager has to find an optimal investment threshold, denote it h^*. To solve this problem, we may interpret the fixed investment cost as the present value of the coupon payments qI starting the moment the investment is made[2]. Then the optimal timing of investment is equivalent to the problem of an optimal exercise of the (perpetual) option to acquire the stream of payoffs $g(X_t) = Ge^{X_t} - qI$, with zero strike. Let h be a candidate for the optimal investment threshold, and denote by τ_h^+ the hitting time of $[h, +\infty)$. Then the EPV of the investment opportunity is

$$\begin{aligned} V(x;h) &= E^x\left[\int_{\tau_h^+}^{\infty} e^{-qt} g(X_t)dt\right] \\ &= E^x\left[\int_0^{\infty} e^{-qt} g(X_t)dt\right] + W(x;h) \\ &= q^{-1}\mathcal{E}g(x) + W(x;h), \end{aligned} \tag{10.63}$$

where the first term on the RHS is independent of h, and

$$W(x;h) = E^x\left[\int_0^{\tau_h^+} e^{-qt}(-g(X_t))dt\right]$$

is the EPV of the stream $-g(X_t)$ which is abandoned the first time X_t reaches or crosses h from below. Therefore, an optimal h that maximizes $V(h;x)$ maximizes $W(h;x)$, and vice versa. Since $-g$ is non-increasing, the maximization of $W(h;x)$ is, essentially, the exit problem under supply uncertainty. Using (10.56), we obtain

$$W(x;h) = q^{-1}\mathcal{E}^+\mathbf{1}_{(-\infty,h)}\mathcal{E}^-(-g)(x). \tag{10.64}$$

But $\mathcal{E} = \mathcal{E}^+\mathcal{E}^-$, therefore, substituting (10.64) into (10.63), we obtain, for the normalized value function $\mathcal{V} = q^{-1}V$,

$$\begin{aligned} \mathcal{V}(x;h) &= \mathcal{E}^+\mathcal{E}^- g(x) - \mathcal{E}^+\mathbf{1}_{(-\infty,h)}\mathcal{E}^- g(x) \\ &= \mathcal{E}^+(\mathbf{1}_{(-\infty,h)} + \mathbf{1}_{[h,+\infty)})\mathcal{E}^- g(x) - \mathcal{E}^+\mathbf{1}_{(-\infty,h)}\mathcal{E}^- g(x), \end{aligned}$$

and, finally,

[2] This interpretation presumes that the firm will never default although it may be optimal to do so.

$$V(x;h) = q^{-1}\mathcal{E}^{+}\mathbf{1}_{[h,+\infty)}\mathcal{E}^{-}g(x). \tag{10.65}$$

Since g is an increasing function which changes sign, $\mathcal{E}^{-}g$ is a non-decreasing continuous function that changes sign. Moreover, it is increasing on an interval where it changes sign. Therefore, there exists a unique h^* such that $\mathcal{E}^{-}g(x) > 0$ for all $x > h^*$, and $\mathcal{E}^{-}g(x) < 0$ for all $x < h^*$. We conclude that h^* is the optimal investment threshold. The value of the investment opportunity is

$$V(x) = q^{-1}\mathcal{E}^{+}\mathbf{1}_{[h^*,+\infty)}\mathcal{E}^{-}g(x). \tag{10.66}$$

Now, the optimal investment rule can be formulated as follows: invest the first time $(\mathcal{E}^{-}g)(X_t)$ becomes non-negative. If $g(X_t)$ is a non-decreasing function of X_t, as we presumed, we have

$$g(\underline{X}_t) = \underline{g}_t \equiv \min_{0\le s\le t} g_s,$$

where $g_t = g(X_t)$, therefore we can reformulate the investment rule in terms of the infimum process: invest at level g if

$$E\left[\int_0^{\infty} \mathrm{e}^{-qt}\underline{g}_t dt \mid g_0 = g\right] \ge 0.$$

In other words, the rule is: consider all sample paths of the process, and along each sample path, disregard all *temporary increases* of the profit flow. Then calculate the EPV of profits, and if it is non-positive, give up the right for the stream. Thus, the manager is extremely cautious or too pessimistic: she invests only when the EPV is non-negative even after this worst-case scenario adjustment. We say that she uses the *bad news principle.*

Example 10.6. In the case $g(x) = G\mathrm{e}^{x} - qI$, we have $w(x) = \kappa_q^{-}(1)G\mathrm{e}^{x} - qI$, therefore the investment threshold, h^*, is a unique solution of the equation

$$\kappa_q^{-}(1)G\mathrm{e}^{h^*} = qI. \tag{10.67}$$

Applying further (10.65), and then (10.67), we calculate the value of the investment opportunity for $x < h^*$:

$$\begin{aligned} V(x) &= q^{-1}\int_0^{+\infty} \beta^{+}\mathrm{e}^{-\beta^{+}y}\mathbf{1}_{[h^*,+\infty)}w(x+y)dy \\ &= q^{-1}\int_{h^*-x}^{+\infty} \beta^{+}\mathrm{e}^{-\beta^{+}y}(\kappa_q^{-}(1)G\mathrm{e}^{x+y} - qI)dy \\ &= \mathrm{e}^{\beta^{+}(x-h^*)}\left[\frac{G\kappa_q^{-}(1)\beta^{+}}{q(\beta^{+}-1)}\mathrm{e}^{h_*} - I\right] \\ &= \frac{I}{\beta^{+}-1}\mathrm{e}^{\beta^{+}(x-h^*)}. \end{aligned} \tag{10.68}$$

10.7.2 Investment lags

Consider the same investment project but with an additional twist, which makes the problem more realistic. Now, the firm will yield the profit flow e^{X_t} starting date $T+t$, where t is the time the investment is made. Discounting back, we obtain that the investment problem is equivalent to the perpetual American call option on the stream

$$g(X_t) = E[e^{-qT} G e^{X_{T+t}}] - qI = G e^{(\Psi(1)-q)T} e^{X_t} - qI.$$

Thus, we have the same investment problem but with a different factor $G_1 := G e^{(\Psi(1)-q)T}$. Therefore, using (10.67), we obtain the equation for the investment threshold $\kappa_q^-(1) G e^{(\Psi(1)-q)T} e^{h^*} = qI$. The value of the investment opportunity is given by (10.68) (with the new h_*). Due to the no-bubble condition $q - \Psi(1) > 0$, the investment threshold increases with the investment lag, and the value of the investment opportunity decreases.

Note, however, that if the problem has additional features, for instance, the investment payments are spread during the gestation period or there is an embedded option to exit then the investment threshold can decrease if there is an investment lag. See [8] and Sect. 9.6.

10.7.3 Entry under supply uncertainty

Consider the investment in the plant with the characteristics specified in Sect. 10.6. The investment cost is I. For simplicity, in this subsection, we assume that once the investment is made, the firm never exits the market no matter how large the variable cost will become. In Subsect. 10.9.1, we consider more realistic situations, when the firm may exit should the variable costs rise too high. The profit flow is $\Pi(X_t) = PG - e^{X_t}$, and the optimal timing of investment is equivalent to the problem of an optimal exercise of the (perpetual) option to acquire the stream of payoffs $g(X_t) = PG - qI - e^{X_t}$, with zero strike. Certainly, investment is never optimal if $PG - qI \le 0$, therefore, we assume that $PG - qI > 0$. Should X_t fall sufficiently low, it may be optimal to invest. Let h be a candidate for the optimal investment threshold. Then the EPV of the investment opportunity is

$$\begin{aligned} V(x;h) &= E^x \left[\int_{\tau_h^-}^{\infty} e^{-qt} g(X_t) dt \right] \\ &= E^x \left[\int_0^{\infty} e^{-qt} g(X_t) dt \right] + W(x;h) \\ &= q^{-1} \mathcal{E} g(x) + W(x;h), \end{aligned} \tag{10.69}$$

where the first term on the RHS is independent of h, and

$$W(x;h) = E^x \left[\int_0^{\tau_h^-} e^{-qt} (-g(X_t)) dt \right]$$

is the EPV of the stream $-g(X_t) = e^{X_t} - (PG - qI)$ which is abandoned the first time X_t reaches h from above. Therefore, an optimal h that maximizes $V(h;x)$ maximizes $W(h;x)$, and vice versa. Since $-g$ is increasing, the maximization of $W(h;x)$ is, essentially, the exit problem under demand uncertainty. Using (10.41), we obtain

$$W(x;h) = q^{-1}\mathcal{E}^{-}\mathbf{1}_{(h,+\infty)}\mathcal{E}^{+}(-g)(x). \tag{10.70}$$

But $\mathcal{E} = \mathcal{E}^{+}\mathcal{E}^{-}$, therefore, substituting (10.70) into (10.69), we obtain, for the normalized value function $\mathcal{V} = q^{-1}V$,

$$\begin{aligned}\mathcal{V}(x;h) &= \mathcal{E}^{-}\mathcal{E}^{+}g(x) - \mathcal{E}^{-}\mathbf{1}_{(h,+\infty)}\mathcal{E}^{+}g(x)\\ &= \mathcal{E}^{-}(\mathbf{1}_{(-\infty,h]} + \mathbf{1}_{(h,+\infty)})\mathcal{E}^{+}g(x) - \mathcal{E}^{-}\mathbf{1}_{(h,+\infty)}\mathcal{E}^{+}g(x),\end{aligned}$$

and, finally,

$$V(x;h) = q^{-1}\mathcal{E}^{-}\mathbf{1}_{(-\infty,h]}\mathcal{E}^{+}g(x). \tag{10.71}$$

Since g is a decreasing function which changes sign, $\mathcal{E}^{+}g$ is a decreasing continuous function that changes sign. Therefore, there exists a unique h_* such that $\mathcal{E}^{+}g(x) < 0$ for all $x > h_*$, and $\mathcal{E}^{+}g(x) > 0$ for all $x < h_*$. We conclude that h_* is the optimal investment threshold. The value of the investment opportunity is

$$V(x) = q^{-1}\mathcal{E}^{-}\mathbf{1}_{[h_*,+\infty)}\mathcal{E}^{+}g(x). \tag{10.72}$$

Now, the optimal investment rule can be formulated as follows: invest the first time $(\mathcal{E}^{+}g)(X_t)$ becomes non-negative. If $g(X_t)$ is a decreasing function of X_t, as we presumed, we have

$$g(\bar{X}_t) = \underline{g}_t \equiv \inf_{0\le s\le t} g_s,$$

where $g_t = g(X_t)$, therefore, in terms of the flow g_t, we can reformulate the investment rule as the same bad news principle.

For $g(x) = PG - qI - e^{X_t}$, $w(x) := \mathcal{E}^{+}g(x) = PG - qI - \kappa_q^{+}(1)e^{x}$, hence, the threshold is given by

$$\kappa_q^{+}(1)e^{h_*} = PG - qI. \tag{10.73}$$

Substituting into (10.72), we find the value of the investment opportunity for $x > h_*$:

$$\begin{aligned}V(x) &= q^{-1}(-\beta^{-})\int_{-\infty}^{0} e^{-\beta^{-}y}\mathbf{1}_{(-\infty,h_*]}(x+y)(PG - qI - \kappa_q^{+}(1)e^{x+y})dy\\ &= q^{-1}(-\beta^{-})\int_{-\infty}^{h_*-x} e^{-\beta^{-}y}(PG - qI - \kappa_q^{+}(1)e^{x+y})dy\\ &= q^{-1}(PG - qI)(1 + \beta^{-}/(1-\beta^{-}))e^{\beta^{-}(x-h_*)}\\ &= \frac{PG - qI}{q(1-\beta^{-})}e^{\beta^{-}(x-h_*)}\end{aligned} \tag{10.74}$$

10.8 Perpetual American options

10.8.1 Perpetual American call options

Let $G(X_t)$ be the instantaneous payoff which is an increasing function of X_t. For example, $G(X_t) = S(X_t) - K$ for the call option, where $S(X_t) = \mathrm{e}^{X_t}$ is the price of the underlying asset. Should X_t rise sufficiently high, it may be optimal to exercise the option with the instantaneous payoff $G(X_t)$. Assume that we can express $G(X_t)$ in terms of the EPV of a stream g_t: $G = q^{-1}\mathcal{E}g$. Since $q^{-1}(q - L)\mathcal{E} = I$, we can find g:

$$g(X_t) = (q - L)G(X_t). \tag{10.75}$$

Note that the representation of the instantaneous payoff G as the EPV of a stream is impossible in the case of the call option on a stock that pays no dividends because the discounted price process of the stock must be a martingale, and, therefore, $\mathrm{e}^x - E^x[\mathrm{e}^{-qt}\mathrm{e}^{X_t}] = (1 - \mathrm{e}^{(\Psi(1)-q)t})\mathrm{e}^x$ must be 0. Equivalently, $q - \Psi(1) = 0$. If $\delta = q - \Psi(1) > 0$, then the stock pays dividends, at rate δ. Assume that $q - \Psi(1) > 0$. Then, from (10.75), we obtain that $G(X_t)$ is the EPV of the stream

$$g(X_t) = (q - \Psi(1))\mathrm{e}^{X_t} - qK.$$

Therefore, the results of Sect. 10.7.1 are applicable. Let h be a candidate for the exercise boundary. Then, applying (8.36), we obtain the following formula for the American call price

$$V_{\mathrm{am.call}}(x; h) = q^{-1}\mathcal{E}^+\mathbf{1}_{[h,+\infty)}\mathcal{E}^- g(x). \tag{10.76}$$

Using the Wiener–Hopf factorization formula (10.22) and (10.75), we derive

$$\mathcal{E}^- g(x) = q\mathcal{E}^- q^{-1}(q - L)G(x) = q(\mathcal{E}^+)^{-1}G(x),$$

and rewrite (10.76) as

$$V_{\mathrm{am.call}}(x; h) = \mathcal{E}^+\mathbf{1}_{[h,+\infty)}(\mathcal{E}^+)^{-1}G(x). \tag{10.77}$$

Function $(\mathcal{E}^+)^{-1}G(x) = \kappa_q^+(1)^{-1}\mathrm{e}^x - K$ is an increasing function that changes sign only once. Hence, the unique solution of the equation

$$\mathrm{e}^{h^*} = K\kappa_q^+(1) \tag{10.78}$$

is an optimal exercise boundary, and the rational call option price is given by

$$V_{\mathrm{am.call}}(x) = \mathcal{E}^+\mathbf{1}_{[h^*,+\infty)}(\mathcal{E}^+)^{-1}G(x). \tag{10.79}$$

Explicitly, using (10.79) and (10.78), we obtain for $x < h^*$, similarly to (10.68):

$$V_{\mathrm{am.call}}(x) = \mathrm{e}^{\beta^+(x-h^*)}(\mathrm{e}^{h^*} - K). \tag{10.80}$$

We leave the details as an exercise for the reader. We recover the result due to McKean [65]. For explicit formulas in the case of more general payoff functions, see Example 11.22.

10.8.2 Perpetual American put options

Let $G(X_t)$ be the instantaneous payoff which is a decreasing function of X_t. For example, $G(X_t) = K - S(X_t)$ for the put option, where $S(X_t) = \mathrm{e}^{X_t}$ is the price of the underlying security. Should X_t fall sufficiently low, it may be optimal to exercise the option with the instantaneous payoff $G(X_t)$. Assume that we can express $G(X_t)$ in terms of the EPV of a stream: $G = q^{-1}\mathcal{E}g$. Then $g = (q - L)G$. If the stock does not pay dividends, we cannot apply this procedure with $G(X_t) = K - \mathrm{e}^{X_t}$, but, since the option is not exercised if the payoff is negative, we may replace $K - \mathrm{e}^{X_t}$ with $G_1(X_t) := (G(X_t))_+ = (K - \mathrm{e}^{X_t})_+$. Being bounded, $G_1(X_t)$ is representable as the EPV of the stream $g(X_t) = (q - L)G_1(X_t)$ (the function g is not defined at $x = \log K$ but this causes no problem). Let h be a candidate for the exercise boundary. Then, applying (10.71), we obtain the following formula for the American put price

$$V_{\mathrm{am.put}}(x; h) = q^{-1}\mathcal{E}^-\mathbf{1}_{(-\infty,h]}\mathcal{E}^+g(x). \tag{10.81}$$

Using the equality $g = (q - L)G_1$ and the Wiener–Hopf factorization formula (10.22), we derive $\mathcal{E}^+g(x) = q\mathcal{E}^+q^{-1}(q - L)G_1(x) = q(\mathcal{E}^-)^{-1}G_1(x)$, and rewrite (10.81) as

$$V_{\mathrm{am.put}}(x; h) = \mathcal{E}^-\mathbf{1}_{(-\infty,h]}(\mathcal{E}^-)^{-1}G_1(x). \tag{10.82}$$

It is non-optimal to exercise the option unless $G(h) \geq 0$ but for $x \leq h$, we have $(\mathcal{E}^-)^{-1}G_1(x) = A^-G_1(x) = A^-G(x) = (\mathcal{E}^-)^{-1}G(x)$. Therefore, we may replace G_1 in (10.82) with G. Assume that the function $(\mathcal{E}^-)^{-1}G$ is a decreasing function that changes sign only once. Then the optimal exercise boundary is the solution of the equation

$$(\mathcal{E}^-)^{-1}G(h_*) = 0, \tag{10.83}$$

and the rational put option price is

$$V_{\mathrm{am.put}}(x) = \mathcal{E}^-\mathbf{1}_{(-\infty,h_*]}(\mathcal{E}^-)^{-1}G(x). \tag{10.84}$$

For the standard American put option, $(\mathcal{E}^-)^{-1}G(x) = K - \kappa_q^-(1)^{-1}\mathrm{e}^x$ is decreasing, and, therefore, the exercise boundary is the solution of the equation $\mathrm{e}^{h_*} = K\kappa_q^-(e)$. For $x > h_*$, similarly to (10.74),

$$V_{\mathrm{am.put}}(x) = (K - \mathrm{e}^{h_*})\mathrm{e}^{\beta^-(x-h_*)}. \tag{10.85}$$

We leave details of the calculation to the reader. We recover Merton's [66] result. For explicit formulas in the case of more general payoff functions, see Example 11.20.

10.9 Embedded options

There are many types of options with embedded features. In this section, we consider simple investment problems with embedded options to exit or default assuming that the uncertainty is on the supply side. For similar options with the uncertainty on the demand side, see the problems for this Chapter. Other types of embedded options under Lévy processes will be studied in Chap. 12.

10.9.1 Debt-financed investment. Endogenous default

Consider investment in the same firm as in Subsect. 10.7.1. We modify the set-up of Subsect. 10.7.1. First, we assume that the firm will be financed by debt, the coupon payments being ρI, and, second, the firm may default on its debt obligation. This implies that the interest ρ that the firm pays on its debt must be greater than the riskless rate q. Thus, $q < \rho < PG/I$. Later, we will find ρ assuming that the lenders are competitive and earn zero profit.

Assuming that the investment has been made already, and the firm's manager maximizes the value of equity, the manager finds the optimal default threshold as the exit threshold in Sect. 10.6. The profit flow of the firm net coupon payments is $g(X_t) = PG - \rho I - \mathrm{e}^{X_t}$, therefore, the optimal default threshold is given by

$$\kappa_q^-(1)\mathrm{e}^{h^*} = PG - \rho I \tag{10.86}$$

(cf. (10.61)), and, similarly to (10.62), for $x < h^*$, the value of equity is

$$VE_{\mathrm{post}}(x) = \frac{PG - \rho I}{q} - \frac{\mathrm{e}^x}{q - \Psi(1)} + \frac{PG - \rho I}{q(\beta^+ - 1)}\mathrm{e}^{\beta^+(x-h^*)}; \tag{10.87}$$

for $x \ge h^*$, $VE_{\mathrm{post}}(x) = 0$. The "post-investment" value of equity having been calculated, the manager finds the investment threshold solving

$$w(h) := (\mathcal{E}^-)^{-1} VE_{\mathrm{post}}(h) = 0.$$

Since the investment is non-optimal if $x \ge h^*$, and $(\mathcal{E}^-)^{-1} VE_{\mathrm{post}}(h)$ is independent of values of VE_{post} above h, we can calculate $w(h)$ assuming that VE_{post} is given by (10.87) for all x. Thus, the equation for the investment threshold, denote it h_*, is

$$\frac{PG - \rho I}{q} - \frac{\mathrm{e}^{h_*}}{\kappa_q^-(1)(q - \Psi(1))} + \frac{PG - \rho I}{q\kappa_q^-(\beta^+)(\beta^+ - 1)}\mathrm{e}^{\beta^+(h_* - h^*)} = 0. \tag{10.88}$$

From the Wiener-Hopf factorization formula (10.19) and (10.86),

$$\frac{\mathrm{e}^{h^*}}{\kappa_q^-(1)(q - \Psi(1))} = \frac{(PG - \rho I)\kappa_q^+(1)}{q\kappa_q^-(1)},$$

therefore, we can simplify (10.88):

$$\frac{PG-\rho I}{q} - \frac{e^{h_*-h^*}(PG-\rho I)\kappa_q^+(1)}{q\kappa_q^-(1)} + \frac{PG-\rho I}{q\kappa_q^-(\beta^+)(\beta^+-1)}e^{\beta^+(h_*-h^*)} = 0,$$

and then

$$1 - e^{h_*-h^*}\frac{\kappa_q^+(1)}{\kappa_q^-(1)} + \frac{e^{\beta^+(h_*-h^*)}}{\kappa_q^-(\beta^+)(\beta^+-1)} = 0. \tag{10.89}$$

Set $\delta = h_* - h^*$, and write (10.89) as $e^\delta F(\delta) = 0$, where

$$F(\delta) = e^{-\delta} - \frac{\kappa_q^+(1)}{\kappa_q^-(1)} + \frac{e^{(\beta^+-1)\delta}}{\kappa_q^-(\beta^+)(\beta^+-1)}.$$

We need to show that the equation $F(\delta) = 0$ has a unique negative solution δ_*, and F changes sign as δ passes δ_*. At $\delta = 0$,

$$F(0) = 1 - \frac{\beta^+}{\beta^+-1}\cdot\frac{\beta^--1}{\beta^-} + \frac{\beta^--\beta^+}{\beta^-(\beta^+-1)} = 0,$$

and

$$F'(0) = -1 + \frac{(\beta^+-1)(\beta^--\beta^+)}{\beta^-(\beta^+-1)} = -\beta^+/\beta^- > 0.$$

Hence, $F(y)$ is negative in a left neighborhood of 0. As $\delta \to -\infty$, $F(\delta) \to +\infty$ because $\beta^+ > 1$ due to the condition $q - \Psi(1) > 0$. Hence, F has a zero $\delta_* < 0$. To prove that δ_* is unique and F changes sign as δ passes δ_*, it remains to note that $F''(\delta) > 0$, and, therefore, F is convex.

We conclude that δ_* can be easily calculated numerically, and then we find $h_* = h^* + \delta_*$. After that, we calculate the value of the investment opportunity by integration: for $x > h_*$,

$$\begin{aligned} V(x) &= \mathcal{E}^-\mathbf{1}_{(-\infty,h_*]}(\mathcal{E}^-)^{-1}VE_{\text{post}}(x) \\ &= -\beta^-\int_{-\infty}^{h_*-x} e^{-\beta^-y}VE_{\text{post}}(x+y)dy, \end{aligned} \tag{10.90}$$

where VE_{post} is given by (10.87). The integral in (10.90) can be explicitly calculated because it is easy to see that the RHS in (10.90) is of the form $Ae^{\beta^+(x-h_*)}$, where A is a constant. Since at $x = h_*$, $V(x) = VE_{\text{post}}(x)$, we obtain

$$V(x) = VE_{\text{post}}(h_*)e^{\beta^+(x-h_*)}. \tag{10.91}$$

Note that this simple trick does not work in the case of other Lévy processes, and, therefore, it is necessary to evaluate the integral in (10.90) directly. For Lévy processes with rational Lévy exponents, which we consider in the next chapter, one needs to integrate exponential functions, which is not difficult.

10.9.2 Competitive interest rate for lending

Assume that the lenders are competitive so that the cost of investment, I, equals the value of the debt, $VD(h_*)$, at the moment of its initiation. Clearly, for $x < h^*$, $VD(x) = \rho VD_1(x)$, where

$$\begin{aligned} VD_1(x) &= E^x \left[\int_0^{\tau^+_{h^*}} \mathrm{e}^{-qt} dt \right] = q^{-1}\mathcal{E}^+ \mathbf{1}_{(-\infty,h^*)} \mathcal{E}^- \mathbf{1}(x) \\ &= q^{-1}\mathcal{E}^+ \mathbf{1}_{(-\infty,h^*)}(x) = \frac{\beta^+}{q} \int_0^{h^*-x} \mathrm{e}^{-\beta^+ y} dy = q^{-1}(1 - \mathrm{e}^{\beta^+(x-h^*)}), \end{aligned}$$

therefore the equation for the competitive lending rate is

$$I = \rho q^{-1}(1 - \mathrm{e}^{\beta^+(h_*(\rho)-h^*(\rho))}). \tag{10.92}$$

(Recall that both h_* and h^* are functions of ρ).

10.9.3 Debt-financed investment. Exogenous default

Now we assume that the default is specified by the following debt covenant: the operational profit flow $PG - \mathrm{e}^{X_t}$ may not fall below zero. Thus, the exogenous default boundary is given by

$$\mathrm{e}^{h^{\mathrm{ex}}} = PG. \tag{10.93}$$

Comparing (10.86) and (10.93), we see that the debt covenant is binding iff $PG < (PG - \rho I)/\kappa^-_q(1)$, equivalently,

$$\rho I/(PG) < 1 - \kappa^-_q(1). \tag{10.94}$$

Condition (10.94) admits a natural interpretation. The higher the interest that the firm pays on its debt, the sooner the firm will default. For $x < h^{\mathrm{ex}}$, the post-entry value of equity, $VE_{\mathrm{post}}(x)$, equals

$$\begin{aligned} & q^{-1}\mathcal{E}^+ \mathbf{1}_{(-\infty,h^{\mathrm{ex}})} \mathcal{E}^-(PG - \rho I - \mathrm{e}^{\cdot})(x) \\ &= \frac{\beta^+}{q} \int_0^{h^{\mathrm{ex}}-x} \mathrm{e}^{-\beta^+ y}(PG - \rho I - \kappa^-_q(1)\mathrm{e}^{x+y}) dy \\ &= \frac{PG - \rho I}{q}(1 - \mathrm{e}^{\beta^+(x-h^{\mathrm{ex}})}) - \frac{\beta^+ \kappa^-_q(1)}{q(\beta^+ - 1)} \left(\mathrm{e}^x - \mathrm{e}^{x+(\beta^+-1)(x-h^{\mathrm{ex}})} \right) \\ &= \frac{PG - \rho I}{q} - \frac{PG\mathrm{e}^{x-h^{\mathrm{ex}}}}{q - \Psi(1)} + \mathrm{e}^{\beta^+(x-h^{\mathrm{ex}})} \left[\frac{PG}{q - \Psi(1)} - \frac{PG - \rho I}{q} \right], \end{aligned}$$

and the investment threshold, h_*, is a unique solution of the equation $(\mathcal{E}^-)^{-1} VE_{\mathrm{post}}(h) = 0$. Explicitly, h_* is determined from

$$\frac{PG - \rho I}{q} - \frac{PG e^{h_* - h^{\rm ex}}}{\kappa_q^-(1)(q - \Psi(1))} + \frac{e^{\beta^+(h_* - h^{\rm ex})}}{\kappa_q^-(\beta^+)} \left[\frac{PG}{q - \Psi(1)} - \frac{PG - \rho I}{q} \right] = 0$$

if this equation has a solution $h_* < h^{\rm ex}$. A unique solution exists iff the LHS is negative at $h_* = h^{\rm ex}$ (the proof is similar to the proof of the existence and uniqueness of the investment threshold in Subsect. 10.86). If

$$\frac{PG - \rho I}{q} - \frac{PG}{\kappa_q^-(1)(q - \Psi(1))} + \frac{1}{\kappa_q^-(\beta^+)} \left[\frac{PG}{q - \Psi(1)} - \frac{PG - \rho I}{q} \right] \geq 0,$$

then the investment is never optimal. When h_* is found, the value of the investment opportunity is calculated from (10.91), with $VE_{\rm post}(x)$ calculated in this subsection.

The competitive lending rate can be found from (10.92) with $h^{\rm ex}$ substituted for h^*.

Problems

10.1. Solve Problems 5.2-5.9 assuming that X_t is the Brownian motion.

11

General Lévy processes

11.1 Main definitions

A Lévy process is a continuous time process with i.i.d. increments. The rigorous definition is as follows (see [79], p.3).

Definition 11.1. *A stochastic process* $\{X_t \mid t \ge 0\}$ *on* $\mathbb{R}^d$ *is a* Lévy process *if the following conditions are satisfied:*

(1) for any choice of $n \ge 1$ *and* $0 \le t_0 < t_1 < \cdots < t_n$*, random variables* $X_{t_0}, X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$ *are independent (increments are independent);*
(2) $X_0 = 0$*, a.s;*
(3) the distribution of $X_{s+t} - X_s$ *does not depend on* s *(increments are stationary);*
(4) $\{X_t\}$ *is stochastically continuous or continuous in probability, that is, for every* $t \ge 0$ *and* $\epsilon > 0$*,*

$$\lim_{s \to t} P[|X_t - X_s| > \epsilon] = 0; \tag{11.1}$$

(5) there exists a realization of $\{X_t\}$ *such that, almost surely, a sample path* $X_t(\omega)$ *is right continuous in* $t \ge 0$ *and has left limits.*

The moment-generating function of a Lévy process admits the representation (10.4) for purely imaginary $z \in i\mathbb{R}$, where $i = \sqrt{-1}$; under additional conditions, it is defined for z in a strip around the imaginary axis. The function Ψ appearing in (10.4) is called the Lévy exponent. The Lévy–Khintchine representation or Lévy–Khintchine formula gives a representation of the Lévy exponent of a Lévy process:

$$\Psi(z) = \frac{1}{2}(Az, z) + (b, z) + \int_{\mathbb{R}^d \setminus \{0\}} (e^{(z,x)} - 1 - (z, x)\mathbf{1}_{|\cdot|<1}(x))F(dx), \tag{11.2}$$

where $(\cdot, \cdot)$ is the standard inner product in $\mathbb{R}^d$, A is a symmetric nonnegative-definite $d \times d$ matrix, $b \in \mathbb{R}^d$, and F is the measure on $\mathbb{R}^d$ satisfying

$$\int_{\mathbb{R}^d\setminus\{0\}} \min\{|x|^2, 1\} F(dx) < \infty. \tag{11.3}$$

It is called the *Lévy density* or jump density. Conversely, given the *generating triplet* (A, b, F) with these properties, one can construct the Lévy process with the Lévy exponent given by (11.2). If the measure F satisfies the condition

$$\int_{\mathbb{R}^d\setminus\{0\}} \min\{|x|, 1\} F(dx) < \infty, \tag{11.4}$$

then the representation (11.2) can be simplified

$$\Psi(z) = \frac{1}{2}(Az, z) + (b, z) + \int_{\mathbb{R}^d\setminus\{0\}} (\mathrm{e}^{(z,x)} - 1) F(dx), \tag{11.5}$$

with, possibly, a different b. The Lévy process with the Lévy exponent (11.5) can be interpreted as the Brownian motion (with the variance-covariance matrix A and drift b) with embedded compound Poisson jumps. The Lévy density can be interpreted as follows: during an infinitely small time interval dt, the probability of a jump from 0 to a measurable set $U \subset \mathbb{R}^d \setminus \{0\}$ equals $dt \int_U F(dx)$.

In this Part, we consider processes on $\mathbb{R}$. A fairly flexible and general class of Lévy processes obtains by modelling the Lévy densities by exponential polynomials on each of half-axis. In this case, the characteristic exponent is a rational function, which can be easily seen from (11.5). For instance , consider

$$F(dy) = c^+\lambda^+\mathrm{e}^{-\lambda^+ y}\mathbf{1}_{(0,+\infty)}(y)dy - c^-\lambda^-\mathrm{e}^{-\lambda^- y}\mathbf{1}_{(-\infty,0)}(y)dy, \tag{11.6}$$

where $c^\pm > 0$, and $\lambda^- < 0 < \lambda^+$. The coefficient c^+ (respectively, c^-) characterizes the intensity of upward jumps (respectively, downward jumps). This density was introduced in [40] for processes of the Ornstein-Uhlenbeck type, and used in [50] for Lévy processes. The parameter λ^+ describes the relative intensity of large jumps: the smaller the λ^+, the larger is the probability of large upward jumps as opposed to small ones. Likewise, the smaller the λ^-, the larger is the probability of large downward jumps. If one of the $c^\pm$ is zero, there are no jumps in the corresponding direction. Straightforward calculations give

$$\begin{aligned}
&\int_{-\infty}^{+\infty} (\mathrm{e}^{zx} - 1) F(dx) \\
&= c^+\lambda^+ \int_0^{+\infty} (\mathrm{e}^{zx} - 1)\mathrm{e}^{-\lambda^+ x} dx + c^-(-\lambda^-) \int_{-\infty}^0 (\mathrm{e}^{zx} - 1)\mathrm{e}^{-\lambda^- x} dx \\
&= \frac{c^+ z}{\lambda^+ - z} + \frac{c^- z}{\lambda^- - z},
\end{aligned}$$

and, therefore,

$$\Psi(z) = \frac{\sigma^2}{2}z^2 + bz + \frac{c^+ z}{\lambda^+ - z} + \frac{c^- z}{\lambda^- - z}. \tag{11.7}$$

For simplicity, we will formulate and prove all optimal stopping results for Lévy processes with the non-trivial diffusion component (that is, $\sigma^2 > 0$) and rational characteristic function, which is sufficient for applications to economics and non-sophisticated applications to finance, we believe. All the optimal stopping results admit generalizations for much more general classes of Lévy processes [23], namely, Lévy processes satisfying the (ACP)-condition, or absolute continuity of potential measures (Definition 41.11 in [79]). In Theorem 41.15 in [79], the reader can find several equivalent conditions. One of those is: if f is a bounded Borel-measurable function with compact support, then, for $q > 0$, $\mathcal{E}f(= \mathcal{E}_q f)$ is continuous. This is a fairly weak regularity condition. For example, it is satisfied if, for every $t > 0$, there exists a measurable function p_t such that $E^x[u(X_t)] = \int_{\mathbb{R}} p_t(x+y)u(y)dy$.

For our purposes, it is convenient to allow a Lévy process to start at any point $x \in \mathbb{R}^d$, therefore we modify (2): almost all sample paths starts at x; equivalently, we consider the process $x + X_t$ instead of X_t. We use the notation $E^x[g(X_t)] = E[g(X_t) \mid X_0 = x]$. The infinitesimal generator L and EPV-operator $\mathcal{E}$ are defined by the same formulas (10.6) and (10.10) as in the Brownian motion case, and their action on exponential functions $g(x) = \mathrm{e}^{zx}$ is defined by the same formulas (10.9) and (10.12)–(10.13), respectively. The latter are valid provided the non-bubble condition $q - \Psi(z) > 0$ for real z, and $q - \Psi(\mathrm{Re}\, z) > 0$ for complex z, is satisfied. For a sufficiently regular u, e.g., $u \in C_0^2(\mathbb{R})$, and a Lévy process with the Lévy density satisfying (11.4),

$$Lu(x) = \frac{\sigma^2}{2}u''(x) + bu'(x) + \int_{-\infty}^{+\infty}(u(x+y) - u(x))F(dy). \tag{11.8}$$

If $\sigma > 0$ and the density $F(dy)$ is integrable, as our standing assumptions will be, then (10.11) holds, in the same function spaces, and with the same reservations as in the Brownian motion case. As in the case of the Brownian motion, it is easy to see why (10.11) should hold by considering the action of $\mathcal{E}$ and $q - L$ on exponential functions and using (10.9) and (10.12)–(10.13).

11.2 Wiener–Hopf factorization

11.2.1 Three forms of the Wiener–Hopf factorization

For a Lévy process X_t define the EPV-operators of the supremum and infimum process by (10.28) and (10.30). Let T be the exponential random variable on $\mathbb{R}_+$ of mean $1/q$. Then the representations (10.27) and (10.29) hold.

The Wiener–Hopf factorization formula states that for $z \in \mathrm{i}\mathbb{R}$,

$$E[\mathrm{e}^{zX_T}] = E[\mathrm{e}^{z\bar{X}_T}]E[\mathrm{e}^{z\underline{X}_T}]. \tag{11.9}$$

Equation (11.9) follows from:

- $X_T = \bar{X}_T + X_T - \bar{X}_T$;
- $\bar{X}_T$ and $X_T - \bar{X}_T$ are independent;
- the characteristic function of the sum of two independent random variables is the product of the characteristic functions;
- probability distributions of $\underline{X}_T$ and $X_T - \bar{X}_T$ are the same.

See Sect. I.29 in [77] and the references therein. Introduce the notation

$$\kappa_q^+(z) = qE\left[\int_0^{+\infty} e^{-qt} e^{z\bar{X}_t}\, dt\right], \tag{11.10}$$

$$\kappa_q^-(z) = qE\left[\int_0^{+\infty} e^{-qt} e^{z\underline{X}_t}\, dt\right]. \tag{11.11}$$

The LHS in (11.9) being $q/(q - \Psi(z))$, we can write the Wiener-Hopf factorization formula in an equivalent form

$$q/(q - \Psi(z)) = \kappa_q^+(z)\kappa_q^-(z). \tag{11.12}$$

The third form obtains if we apply $\mathcal{E}$ and the product of operators $\mathcal{E}^\pm$ to a function $g(x)$ of the form $g(x) = g(z; x) = e^{zx}$, where $z \in \mathbb{C}, \mathrm{Re}\, z = 0$. Assuming that X starts at 0, we have

$$(\mathcal{E} e^{z\cdot})(x) = qE\left[\int_0^{+\infty} e^{-qt} e^{z(x+X_t)}\, dt\right] = e^{zx} qE\left[\int_0^{+\infty} e^{-qt} e^{zX_t}\, dt\right]$$

$$(\mathcal{E}^+ e^{z\cdot})(x) = qE\left[\int_0^{+\infty} e^{-qt} e^{z(x+\bar{X}_t)}\, dt\right] = e^{zx} qE\left[\int_0^{+\infty} e^{-qt} e^{z\bar{X}_t}\, dt\right]$$

$$(\mathcal{E}^- e^{z\cdot})(x) = qE\left[\int_0^{+\infty} e^{-qt} e^{z(x+\underline{X}_t)}\, dt\right] = e^{zx} qE\left[\int_0^{+\infty} e^{-qt} e^{z\underline{X}_t}\, dt\right],$$

which gives

$$(\mathcal{E} e^{z\cdot})(x) = \frac{q}{q - \Psi(z)} e^{zx}, \tag{11.13}$$

$$(\mathcal{E}^+ e^{z\cdot})(x) = \kappa_q^+(z) e^{zx}, \tag{11.14}$$

$$(\mathcal{E}^- e^{z\cdot})(x) = \kappa_q^-(z) e^{zx}. \tag{11.15}$$

Therefore, using (11.12), we obtain

$$\mathcal{E} g = \mathcal{E}^- \mathcal{E}^+ g = \mathcal{E}^+ \mathcal{E}^- g. \tag{11.16}$$

To show that (11.12) holds for $g \in \mathcal{L}_\infty(\mathbb{R})$ and for g from wider classes of functions, introduce the independent random variables $Y^+ = \bar{X}_T$ and $Y^- = X_T - \bar{X}_T \sim \underline{X}_T$. Then $\mathcal{E}^\pm g(x) = E[g(x + Y^\pm)]$, and

$$\begin{aligned} E[g(x + X_T)] &= E[g(x + \bar{X}_T + X_T - \bar{X}_T)] \\ &= E[g(x + Y^+ + Y^-)] = (\mathcal{E}^+ g)(x + Y^-) = (\mathcal{E}^+ \mathcal{E}^- g)(x), \end{aligned}$$

which gives $\mathcal{E}g = \mathcal{E}^+\mathcal{E}^-g$. The second equality in (11.16) is proved similarly. Thus, we have

$$\mathcal{E} = \mathcal{E}^+\mathcal{E}^- = \mathcal{E}^-\mathcal{E}^+, \tag{11.17}$$

where each operator is understood as an operator in $\mathcal{L}_\infty(\mathbb{R})$ (or in a wider function space).

11.2.2 Uniqueness of the Wiener–Hopf factorization

There exist general analytical formulas for $\kappa_q^\pm(z)$ in terms of the probability density $P_t(dy)$ of X_t started at 0:

$$\kappa_q^+(z) = \exp\left[\int_0^{+\infty} q^{-1}\mathrm{e}^{-qt}\,dt \int_0^{+\infty} (\mathrm{e}^{zy}-1)P_t(dy)\right], \tag{11.18}$$

$$\kappa_q^-(z) = \exp\left[\int_0^{+\infty} q^{-1}\mathrm{e}^{-qt}\,dt \int_{-\infty}^{0} (\mathrm{e}^{zy}-1)P_t(dy)\right]. \tag{11.19}$$

See, e.g., [79], p.324. Formulas (11.18)-(11.19) are rather involved. Fortunately, the following general result allows one to guess explicit formulas for $\kappa_q^\pm(z)$ without calculating the integrals in (11.18)–(11.19). Recall that a function is called analytic in an open subset U of $\mathbb{C}$, if it is differentiable at each point of U. Following [49], we say that f is analytic in the closure of an open set $U \in \mathbb{C}$ if f is continuous on the closure of U and analytic in U.

Lemma 11.2.1 *Let f be a continuous bounded function on the imaginary axis $\{z \mid \mathrm{Re}\, z = 0\}$ that admits a factorization*

$$f(z) = f_+(z)f_-(z), \;\; \forall\; \mathrm{Re}\, z = 0, \tag{11.20}$$

where

- *f_- is analytic in the closed right half-plane $\{z \mid \mathrm{Re}\, z \ge 0\}$, and does not vanish there;*
- *f_- and $1/f_-$ grow no faster than a polynomial as $z \to \infty$ in the closed right half-plane;*
- *f_+ is analytic in the closed left half-plane $\{z \mid \mathrm{Re}\, z \le 0\}$, and does not vanish there;*
- *f_+ and $1/f_+$ grow no faster than a polynomial as $z \to \infty$ in the closed left half-plane;*
- *$f_+(0) = f_-(0) = 1$.*

Let

$$f(z) = f_{1,+}(z)f_{1,-}(z), \;\; \forall\; \mathrm{Re}\, z = 0, \tag{11.21}$$

be another factorization with the same properties.

Then $f_{1,\pm} = f_\pm$.

Proof. Dividing (11.20) by (11.21) and rearranging, we obtain

$$\frac{f_+(z)}{f_{1,+}(z)} = \frac{f_{1,-}(z)}{f_-(z)}, \quad \forall \operatorname{Re} z = 0.$$

The RHS (resp., the LHS) is analytic in right (resp., left) half-plane, and grows no faster than a polynomial there, therefore, we can define a continuous function on $\mathbb{C}$, call it F, by the RHS on the right half-plane, and by the LHS on the left half-plane. F is analytic in $C \setminus \{z \mid \operatorname{Re} z = 0\}$, and grows at infinity no faster than a polynomial. Hence, by Morera's theorem, F is a polynomial. Since F has no zeroes and $F(0) = 1$, $F(z) = 1$ for all z.

It is evident from (11.18)–(11.19) that $\kappa_q^+(z)$ admits the analytic continuation to the left half-plane, and $\kappa_q^-(z)$ admits the analytic continuation to the right half-plane. From now on, we will restrict ourselves to processes with rational Lévy exponents. Under weak regularity conditions on a Lévy process, which are satisfied for processes with rational Lévy exponents, it is shown in [20, 21] that functions $f_+(z) := \kappa_q^+(z)$ and $f_-(z) := \kappa_q^-(z)$, given by the expressions (11.18) and (11.19), satisfy the conditions of Lemma 11.2.1. Therefore, we can easily derive simple explicit formulas for the Wiener–Hopf factors by factorizing the rational function $q/(q - \Psi(z))$:

(1) represent $q^{-1}(q - \Psi(z))$ as the ratio of polynomials $P(z)$ and $Q(z)$;
(2) find the roots of the denominator; since $\Psi(z)$ is well-defined on the imaginary axis, one group of the roots is in the left half-plane, denote these roots $\lambda_j^-, j = 1, 2, \ldots$, and the other group of roots is in the right half-plane, denote these roots $\lambda_j^+, j = 1, 2, \ldots$;
(3) find the roots of the numerator; since $q - \Psi(z) > 0$ for z on the imaginary axis, one group of the roots is in the left half-plane, denote these roots $\beta_k^-, k = 1, 2, \ldots$, and the other group of roots is in the right half-plane, denote these roots $\beta_k^+, k = 1, 2, \ldots$;
(4) set

$$\kappa_q^+(z) = \prod_j \frac{\lambda_j^+ - z}{\lambda_j^+} \prod_k \frac{\beta_k^+}{\beta_k^+ - z}, \tag{11.22}$$

$$\kappa_q^-(z) = \prod_j \frac{\lambda_j^- - z}{\lambda_j^-} \prod_k \frac{\beta_k^-}{\beta_k^- - z}, \tag{11.23}$$

where each factor is taken with the multiplicity of the corresponding root of the denominator or numerator, respectively.

Example 11.2. Consider the Lévy process with the Lévy exponent (11.7). The function $q - \Psi(z)$ is the ratio of a polynomial of degree 4 and a polynomial of degree 2. The graph is shown in Fig. 11.1. The roots of the latter are λ^+ and λ^-, and the former has 4 real roots, separated by $\lambda^-, 0$, and λ^+. To see

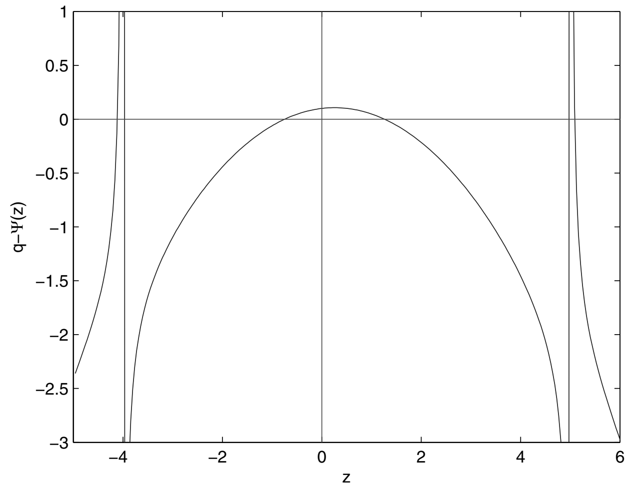

Fig. 11.1. Graph of $q - \Psi(z)$ for model (11.7). Parameters: $q = 0.05, \sigma = 0.45, b = -0.05, \lambda^- = -4$, $\lambda^+ = 5$, $c^- = 0.05$, $c^+ = 0.04$.

this, it suffices to notice that $q - \Psi(0) = q > 0$, $q - \Psi(z) \to -\infty$ as $z \to \pm\infty$, $z \to \lambda^+ - 0$ and $z \to \lambda^- + 0$, and $q - \Psi(z) \to +\infty$ as $z \to \lambda^+ + 0$ and $z \to \lambda^- - 0$. We order these roots as follows: $\beta_2^- < \lambda^- < \beta_1^- < 0 < \beta_1^+ < \lambda^+ < \beta_2^+$.

The factors in the Wiener–Hopf factorization formula are

$$\kappa^{\pm}(z) = \frac{\beta_1^{\pm}}{\beta_1^{\pm} - z} \cdot \frac{\beta_2^{\pm}}{\beta_2^{\pm} - z} \cdot \frac{\lambda^{\pm} - z}{\lambda^{\pm}}. \tag{11.24}$$

11.2.3 Analytic continuation of the factors $\kappa_q^{\pm}(z)$

In order to apply the operator form of the Wiener–Hopf factorization formula in spaces of functions exponentially growing at infinity, we need the following result.

Lemma 11.2.2 *Suppose that, for some $\sigma^- \le 0 \le \sigma^+$, Ψ is well-defined for $z \in [\sigma^-, \sigma^+]$, and*

$$q - \Psi(z) > 0, \quad \forall\, z \in [\sigma^-, \sigma^+]. \tag{11.25}$$

Then

(a) $\Psi(z)$ is well-defined and analytic in the strip $\operatorname{Re} z \in [\sigma^-, \sigma^+]$, and $q - \Psi(z) \neq 0$ in the strip;
(b) $\kappa_q^+(z)$ admits the analytic continuation to the half-plane $\operatorname{Re} z \le \sigma^+$. The analytic continuation has no zeroes in this half-plane, and it is defined on the strip $0 \le \operatorname{Re} z \le \sigma^+$ by

$$\kappa_q^+(z) = \frac{q}{(q - \Psi(z))\kappa_q^-(z)}; \tag{11.26}$$

(c) $\kappa_q^-(z)$ admits the analytic continuation to the half-plane $\mathrm{Re}\, z \geq \sigma^-$. The analytic continuation has no zeroes in this half-plane, and it is defined on the strip $0 \geq \mathrm{Re}\, z \geq \sigma^-$ by

$$\kappa_q^-(z) = \frac{q}{(q - \Psi(z))\kappa_q^+(z)}; \tag{11.27}$$

(d) the Wiener-Hopf factorization formula (11.12) holds for z in the strip $\sigma^- \leq \mathrm{Re}, z \leq \sigma^+$.

Proof. (a) We have $\left|E[e^{zX_t}]\right| \leq E[\left|e^{zX_t}\right|] \leq E\left[e^{\mathrm{Re}\, zX_t}\right]$, therefore $\Psi(z)$ is well-defined if $\Psi(\mathrm{Re}\, z)$ is. Similarly, one shows that the derivative $\Psi'(z)$ is well-defined for z in the open strip. Hence, $\Psi(z)$ is analytic in the strip $\mathrm{Re}\, z \in [\sigma^-, \sigma^+]$. Further, if $\mathrm{Im}\, E[e^{zX_1}] \neq 0$, then $q - \Psi(z)$ is not real, hence, $q - \Psi(z) \neq 0$. Clearly, $q - \Psi(z) \neq 0$ if $\mathrm{Re}\, E[e^{zX_t}] \leq 0$. Finally, if $e^{\Psi(z)} = E[e^{zX_t}]$ is real and positive, then $\Psi(z)$ is real, and $q - \Psi(z) \neq 0$ by assumption.

(b) In the RHS of (11.26), $\kappa_q^-(z)$ is analytic in the half-plane Rez ≥ 0 and does not vanish there, and $q - \Psi(z)$ in analytic in the strip $\mathrm{Re}\, z \in [\sigma^-, \sigma^+]$ and does not vanish there. Hence, the same is true of their product on the intersection, the strip $\mathrm{Re}\, z \in [0, \sigma^+]$. The proof of (c) is similar, and (d) is immediate from (11.26) and (11.27).

11.3 Properties of the EPV-operators $\mathcal{E}$ and $\mathcal{E}^\pm$

11.3.1 Explicit formulas for $\mathcal{E}^+$ and $\mathcal{E}^-$

For applications to economics, and, in many cases, for applications to finance, the class of Lévy processes with the rational Lévy exponents is sufficiently rich. For the sake of brevity, we consider processes for which the roots $\beta_j^\pm$ of $q - \Psi(z)$ are simple. In this case, $\kappa^\pm(z)$ can be decomposed into sums of simple fractions.

Example 11.3. The simplest example is the Brownian motion model. The denominator is 1, and it has no roots, whereas the numerator has two roots $\beta^- < 0 < \beta^+$. Therefore, $\kappa_q^+(z) = \beta^+/(\beta^+ - z)$, $\kappa_q^-(z) = \beta^-/(\beta^- - z)$, the action of the EPV-operators $\mathcal{E}^\pm$ is defined by (10.16) and (10.18), and the action of their inverses – by (10.24) and (10.25).

Example 11.4. In the model (11.7),

$$\kappa^\pm(z) = a_1^\pm \frac{\beta_1^\pm}{\beta_1^\pm - z} + a_2^\pm \frac{\beta_2^\pm}{\beta_2^\pm - z}, \tag{11.28}$$

where $a_{1,2}^\pm > 0$ are given by

$$a_1^+ = \frac{\beta_2^+}{\beta_2^+ - \beta_1^+} \cdot \frac{\lambda^+ - \beta_1^+}{\lambda^+}, \; a_2^+ = \frac{\beta_1^+}{\beta_1^+ - \beta_2^+} \cdot \frac{\lambda^+ - \beta_2^+}{\lambda^+}, \tag{11.29}$$

$$a_1^- = \frac{\beta_2^-}{\beta_2^- - \beta_1^-} \cdot \frac{\lambda^- - \beta_1^-}{\lambda^-}, \; a_2^- = \frac{\beta_1^-}{\beta_1^- - \beta_2^-} \cdot \frac{\lambda^- - \beta_2^-}{\lambda^-}. \tag{11.30}$$

Now it is easy to see that $\mathcal{E}^{\pm}$ act on the exponents $g(x) = \mathrm{e}^{zx}$ as the following integral operators

$$\left(\mathcal{E}^+ g\right)(x) = \sum_{j=1,2} a_j^+ \int_0^{+\infty} \beta_j^+ \mathrm{e}^{-\beta_j^+ y} g(x+y)dy, \tag{11.31}$$

$$\left(\mathcal{E}^- g\right)(x) = \sum_{j=1,2} a_j^- \int_{-\infty}^{0} (-\beta_j^-) \mathrm{e}^{-\beta_j^- y} g(x+y)dy \tag{11.32}$$

Note that the RHS in (11.31)–(11.32) are finite under the no-bubble condition $q - \Psi(z) > 0$ (condition (10.33)). For the model (11.7), we can reformulate it in the form similar to the standard form (10.34) in the BM case, via the roots of the “characteristic equation" $q - \Psi(z) = 0$:

$$\beta_1^- < z < \beta_1^+. \tag{11.33}$$

It can be proved that (11.31)–(11.32) define the action of $\mathcal{E}^+$ and $\mathcal{E}^-$ on bounded measurable functions g.

Decomposing $1/\kappa_q^{\pm}(z)$ into the sum of a polynomial and rational function, we derive the following analytical representation for $A^{\pm} := (\mathcal{E}^{\pm})^{-1}$:

$$A^+ g(x) = c_+^1 g'(x) + c_+^0 g(x) + \int_0^{\infty} b_+ \lambda^+ \mathrm{e}^{-\lambda^+ y} g(x+y)dy \tag{11.34}$$

$$A^- g(x) = c_-^1 g'(x) + c_-^0 g(x) - \int_{-\infty}^{0} b_- \lambda^- \mathrm{e}^{-\lambda^- y} g(x+y)dy \tag{11.35}$$

where $b_{\pm}$, $c_{\pm}^0$, $c_{\pm}^1$ are constants given by $c_{\pm}^1 = \lambda^{\pm}/(\beta_1^{\pm}\beta_2^{\pm})$, $\quad c_{\pm}^0 = (\beta_1^{\pm} + \beta_2^{\pm} - \lambda^{\pm})c_{\pm}^1$, $\quad b_{\pm} = 1 + c_{\pm}^0$. Indeed, if $g(x) = \mathrm{e}^{zx}$, then

$$\begin{aligned} & c_+^1 g'(x) + c_+^0 g(x) + b_+ \int_0^{+\infty} \lambda^+ \mathrm{e}^{-\lambda^+ y} g(x+y)dy \\ &= z c_+^1 \mathrm{e}^{zx} + c_+^0 \mathrm{e}^{zx} + b_+ \frac{\lambda^+}{\lambda^+ - z} \mathrm{e}^{zx} \\ &= \frac{\lambda^+}{\lambda^+ - z} \frac{\beta_1^+ - z}{\beta_1^+} \frac{\beta_2^+ - z}{\beta_2^+} \mathrm{e}^{zx} \\ &= (\kappa_q^+(z))^{-1} \mathrm{e}^{zx} \\ &= (\mathcal{E}^+)^{-1} g(x), \end{aligned}$$

which proves (11.34). Equation (11.35) is verified similarly.

Example 11.5. Let the Lévy density be given by (11.6) with $c_+ = 0$, that is, there are no positive jumps. Then the Lévy exponent is given by (11.7) without the corresponding term, the "characteristic equation" $q - \Psi(z) = 0$ has two negative and one positive root, $\beta_2^- < \lambda^- < \beta_1^- < 0 < \beta^+$, the factor $\kappa_q^-(z)$ and operators $\mathcal{E}^-$ and $(\mathcal{E}^-)^{-1}$ are of the same form as in the model (11.7), and the factor $\kappa_q^+(z) = \beta^+/(\beta^+ - 1)$ and operators $\mathcal{E}^+$ and $(\mathcal{E}^+)^{-1}$ are of the same form as in the Brownian motion model.

11.3.2 Main properties and action in spaces of functions of exponential growth

Note the important properties of the EPV-operators $\mathcal{E}^\pm$ and their inverses, which hold not only for the model (11.7), but for any Lévy process with the rational Lévy exponent provided $\sigma > 0$. These operators are of the form

$$\mathcal{E}^+ g(x) = \int_x^{+\infty} k^+(y-x)g(y)dy \tag{11.36}$$

$$\mathcal{E}^- g(x) = \int_{-\infty}^{x} k^-(y-x)g(y)dy \tag{11.37}$$

$$(\mathcal{E}^+)^{-1}u(x) = c_+^1 u'(x) + c_+^0 u(x) + \int_x^\infty K^+(y-x)u(y)dy \tag{11.38}$$

$$(\mathcal{E}^-)^{-1}u(x) = c_-^1 u'(x) + c_-^0 u(x) + \int_{-\infty}^x K^-(y-x)g(y)dy \tag{11.39}$$

where $k^\pm$, $K^\pm$ and their derivatives are continuous functions that decay exponentially at $\pm\infty$. The first two equalities ensure that for a measurable bounded g, $\mathcal{E}^\pm g$ are differentiable and the derivatives are bounded measurable functions. Equations (11.38)–(11.39) imply that if u is differentiable a.e. and its derivative is bounded, then $(\mathcal{E}^\pm)^{-1}u$ is measurable and bounded, and continuous at each point of continuity of u'. Finally, $\mathcal{E}^\pm$ is the left inverse to $(\mathcal{E}^\pm)^{-1}$ acting from the space of differentiable functions with the derivatives of the class $\mathcal{L}_\infty(\mathbb{R})$ to $\mathcal{L}_\infty(\mathbb{R})$ but $(\mathcal{E}^\pm)^{-1}$ is the left inverse of $\mathcal{E}^\pm$ in the weak sense only: $(\mathcal{E}^\pm)^{-1}\mathcal{E}^\pm g(x) = g(x)$ a.e. At points of continuity of g, the equality holds.

We assume that the Lévy process satisfies (11.25). For Lévy processes with non-trivial diffusion component ($\sigma > 0$), rational Lévy exponents and non-trivial supremum and infimum processes, the proof of Lemma 10.2.5 and Propositions 10.2.1–10.2.4 remain valid. The Wiener–Hopf factorization formula holds in the space $\mathcal{L}_{\infty;\sigma^-,\sigma^+}(\mathbb{R})$ as well. Indeed, under condition (11.25), both sides of the Wiener-Hopf equation $E[g(x + X_T)] = E[g(x + Y^+ + Y^-)]$ are well-defined, and it remains to note that X_T and $Y^+ + Y^-$ are the same in law.

11.4 EPVs of a stream and instantaneous payoff that are acquired or lost at a random time

11.4.1 Standing assumptions

In this Section, we calculate the EPV of a stream or instantaneous payoff that is acquired or lost when a certain threshold fixed in advance is reached or crossed. The threshold is exogenously given, an example being the bankruptcy threshold implied by debt covenants. The standing assumptions about the Lévy process X_t is (11.25), and stream $g(X_t)$ must be measurable and satisfy the bound

$$|g(x)| \le C(\mathrm{e}^{\sigma^- x} + \mathrm{e}^{\sigma^+ x}). \tag{11.40}$$

When we consider an instantaneous payoff $G(X_t)$, the local regularity assumption is stronger but the condition on the rate of growth is weaker than the ones for a stream. For simplicity, we assume that G is twice differentiable a.e., although the form of the results indicates that they must be valid for continuously differentiable functions G. If the payoff $G(X_t)$ is due when a certain boundary is crossed from below, then the payoff function may not grow too fast as $x \to +\infty$: for any N, there exists C such that on $(-N, +\infty)$

$$\sum_{0 \le s \le 2} |G^{(s)}(x)| \le C\mathrm{e}^{\sigma^+ x}, \quad \text{a.e.} \tag{11.41}$$

If the payoff is due when a certain boundary is crossed from above, then the bound is imposed in a neighborhood of $-\infty$: $\forall\ N$, $\exists\ C$ s. t. on $(-\infty, N]$,

$$\sum_{0 \le s \le 2} |G^{(s)}(x)| \le C\mathrm{e}^{\sigma^- x}, \qquad \text{a.e.} \tag{11.42}$$

For an instantaneous payoff, the conditions on growth are weaker than for payoff streams, because if a stream is acquired then its EPV may depend on values of $g(x)$ at arbitrary large (in absolute value) x, whereas for options with an instantaneous payoff $G(X_t)$ only values $G(x)$ for x in the action region matter. In this and following sections, we assume that the Lévy exponent is rational. The proofs of all theorems are valid provided $\kappa_q^{\pm}(z)$ and their reciprocals grow no faster than a polynomial (in the corresponding half-plane). In [23], these theorems are proved for Lévy processes satisfying the (ACP)-condition, a somewhat different method being used.

11.4.2 EPV of a stream that is abandoned when the threshold is reached or crossed from above

Denote by $\tau_h^- = \inf\{t > 0 \mid X_t \le h\}$ the hitting time of $(-\infty, h]$, and consider

$$V(x; h) = E^x \left[\int_0^{\tau_h^-} \mathrm{e}^{-qt} g(X_t) dt \right].$$

The following theorem is a special case of the generalization of the stationary Black–Scholes equation proved in [21].

Theorem 11.4.1 *Assume that X_t satisfies the (ACP)-condition. Then $V(x;h)$ satisfies the equation*

$$(q - L)V(x) = 0, \quad x > h. \tag{11.43}$$

Note that in [21], (11.43) is understood in the sense of generalized functions. One of the standard facts of the theory of boundary problems for integro-differential operators (and, more generally, pseudo-differential operators) is that if L is the infinitesimal generator of a Lévy process with the integrable Lévy density and $\sigma > 0$, then a solution of (11.43) is twice differentiable on $(h, +\infty)$, and (11.43) is satisfied in the usual sense.

Theorem 11.4.2 *Let g be a measurable function that satisfies (11.40). Then*

$$E^x \left[\int_0^{\tau_h^-} \mathrm{e}^{-qt} g(X_t) dt \right] = q^{-1} (\mathcal{E}^- \mathbf{1}_{(h,+\infty)} \mathcal{E}^+ g)(x). \tag{11.44}$$

Proof. The proof is essentially the same as the proof of the identical Theorem 10.3.1 in the Brownian motion case. Equation (11.43) is the same as (10.35), and $V(x;h)$ satisfies the boundary condition (10.36) in both cases. The next steps of the proof are also the same as in the Brownian motion case. Some differences appear after (10.44). Since $g^-(x) = 0 \ \forall x > h$, $\mathcal{E}^+ g^-(x) = E[g(x + Y^+)] = 0$ for these x as well. Using (10.24) and taking the boundary condition (10.38) into account, we conclude that the LHS in (10.44) is 0 below h. Hence, multiplying (10.44) by $\mathbf{1}_{(h,+\infty)}$, we obtain

$$A^- \mathcal{V}(x;h) = \mathbf{1}_{(h,+\infty)} \mathcal{E}^+ g(x), \quad \forall \ x \neq 0.$$

Applying the inverse $\mathcal{E}^-$ to A^-, we arrive at (10.41). It is straightforward to check that the RHS in (10.41) admits the bound (11.40) and satisfies (10.38).

Example 11.6. Let $g(x) = \mathrm{e}^{zx}$, where $z \in [\sigma^-, \sigma^+]$. Then $\mathcal{E}^+ g(x) = \kappa_q^+(z)\mathrm{e}^{zx}$. To make the second step of calculations, we use the integral representation of $\mathcal{E}^-$. In the case of the model with the Lévy density (11.6) and Lévy exponent (11.7), we use (11.32), (11.28) and (11.12), and find for $x > h$

$$\begin{aligned} V(x;h) &= q^{-1} \sum_{j=1,2} a_j^-(-\beta_j^-) \int_{h-x}^0 \mathrm{e}^{-\beta_j^- y} \kappa_q^+(z) \mathrm{e}^{z(x+y)} dy \\ &= q^{-1} \sum_{j=1,2} \frac{a_j^- \beta_j^-}{\beta_j^- - z} \kappa_q^+(z) \left(\mathrm{e}^{zx} - \mathrm{e}^{zx + (z - \beta_j^-)(h-x)} \right) \\ &= \frac{\mathrm{e}^{zx}}{q - \Psi(z)} - \frac{\kappa_q^+(z) \mathrm{e}^{zh}}{q} \sum_{j=1,2} \frac{a_j^- \beta_j^-}{\beta_j^- - z} \mathrm{e}^{\beta_j^-(x-h)}. \end{aligned} \tag{11.45}$$

Note that if the Lévy density is given by (11.6) with $c^- = 0$, then the equation $q - \Psi(z) = 0$ has only one negative root, call it β^-, and (11.45) becomes

$$V(x;h) = \frac{e^{zx} - e^{zh+\beta^-(x-h)}}{q - \Psi(z)}, \quad x > h. \tag{11.46}$$

11.4.3 EPV of a stream that is abandoned when the threshold is reached or crossed from below

Denote by $\tau_h^+ = \inf\{t > 0 \mid X_t \geq h\}$ the hitting time of $[h, +\infty)$.

Theorem 11.4.3 *Let g be a measurable function that satisfies (11.40). Then*

$$E^x \left[\int_0^{\tau_h^+} e^{-qt} g(X_t) dt \right] = q^{-1} (\mathcal{E}^+ \mathbf{1}_{(-\infty,h)} \mathcal{E}^- g)(x). \tag{11.47}$$

The proof is the mirror reflection of the proof of Theorem 11.4.2. We leave it as an exercise for the reader.

Example 11.7. Denote by $V(x;h)$ the LHS in (11.47), and let $g(x) = e^{zx}$, where $z \in [\sigma^-, \sigma^+]$. Then $\mathcal{E}^- g(x) = \kappa_q^-(z) e^{zx}$. To make the second step of calculations, we use the integral representation of $\mathcal{E}^+$. In the case of the model with the Lévy density (11.6) and Lévy exponent (11.7), we use (11.31), (11.28) and (11.12), and find for $x < h$

$$\begin{aligned} V(x;h) &= q^{-1} \sum_{j=1,2} a_j^+ \beta_j^+ \int_0^{h-x} e^{-\beta_j^+ y} \kappa_q^-(z) e^{z(x+y)} dy \\ &= q^{-1} \sum_{j=1,2} \frac{a_j^+ \beta_j^+}{\beta_j^+ - z} \kappa_q^-(z) \left(e^{zx} - e^{zx + (z - \beta_j^+)(h-x)} \right) \\ &= \frac{e^{zx}}{q - \Psi(z)} - \frac{\kappa_q^-(z) e^{zh}}{q} \sum_{j=1,2} \frac{a_j^+ \beta_j^+}{\beta_j^+ - z} e^{\beta_j^+ (x-h)}. \end{aligned} \tag{11.48}$$

Note that if the Lévy density is given by (11.6) with $c^+ = 0$, then the equation $q - \Psi(z) = 0$ has only one positive root, call it β^+, and (11.48) becomes

$$V(x;h) = \frac{e^{zx} - e^{zh+\beta^+(x-h)}}{q - \Psi(z)}, \quad x < h. \tag{11.49}$$

11.4.4 EPV of a stream that is acquired when the threshold is reached or crossed from above

Theorem 11.4.4 *Let g be a measurable function that satisfies (11.40). Then*

$$E^x \left[\int_{\tau_h^-}^{\infty} e^{-qt} g(X_t) dt \right] = q^{-1} (\mathcal{E}^- \mathbf{1}_{(-\infty,h]} \mathcal{E}^+ g)(x). \tag{11.50}$$

Proof. The sum of the LHSs in (11.50) and (11.44) is the EPV of the stream g, which is $q^{-1}\mathcal{E}g = q^{-1}\mathcal{E}^-\mathcal{E}^+g$. Therefore, the RHS in (11.50) is the difference $q^{-1}\mathcal{E}^-\mathcal{E}^+g - q^{-1}\mathcal{E}^-\mathbf{1}_{(h,+\infty)}\mathcal{E}^+g = q^{-1}\mathcal{E}^-\mathbf{1}_{(-\infty,h]}\mathcal{E}^+g$.

Example 11.8. Let $g(x) = e^{zx}$, where $z \in [\sigma^-, \sigma^+]$. Then $\mathcal{E}^+g(x) = \kappa_q^+(z)e^{zx}$. Assuming X_t is the Lévy process with the Lévy density (11.6) and Lévy exponent (11.7), the LHS in (11.50) equals

$$V(x;h) = \frac{\kappa_q^+(z)e^{zh}}{q} \sum_{j=1,2} \frac{a_j^-\beta_j^-}{\beta_j^- - z} e^{\beta_j^-(x-h)}, \quad x > h. \tag{11.51}$$

If, in (11.6), $c^- = 0$, then

$$V(x;h) = \frac{e^{zh+\beta^-(x-h)}}{q - \Psi(z)}, \quad x > h \tag{11.52}$$

(cf. (11.45) and (11.46)).

11.4.5 EPV of a stream that is acquired when the threshold is reached or crossed from below

Theorem 11.4.5 *Let g be a measurable function that satisfies (11.40). Then*

$$E^x \left[\int_{\tau_h^+}^{\infty} e^{-qt} g(X_t) dt \right] = q^{-1}(\mathcal{E}^+\mathbf{1}_{[h,+\infty)}\mathcal{E}^-g)(x). \tag{11.53}$$

The proof is the mirror reflection of the proof of Theorem 11.4.4. We leave the details as an exercise for the reader.

Example 11.9. Let $g(x) = e^{zx}$, where $z \in [\sigma^-, \sigma^+]$. Then $\mathcal{E}^-g(x) = \kappa_q^-(z)e^{zx}$. Assuming X_t is the Lévy process with the Lévy density (11.6) and Lévy exponent (11.7), the LHS in (11.53) equals

$$V(x;h) = \frac{\kappa_q^-(z)e^{zh}}{q} \sum_{j=1,2} \frac{a_j^+\beta_j^+}{\beta_j^+ - z} e^{\beta_j^+(x-h)}, \quad x < h. \tag{11.54}$$

If, in (11.6), $c^+ = 0$, then

$$V(x;h) = \frac{e^{zh+\beta^+(x-h)}}{q - \Psi(z)}, \quad x < h \tag{11.55}$$

(cf. (11.48) and (11.49)).

11.4.6 EPV of an instantaneous payoff that is acquired when the threshold is reached or crossed from above

Theorem 11.4.6 *Assume that G satisfies (11.42). Then*

$$E^x\left[q^{\tau_h^-}G(X_{\tau_h^-})\right] = \mathcal{E}^-\mathbf{1}_{(-\infty,h]}(\mathcal{E}^-)^{-1}G(x). \qquad (11.56)$$

The proof is the same as for the Brownian motion in Subsect. 10.8.2.

Example 11.10. Let $G(x) = e^{zx}$, where $z \geq \sigma^-$. Then $(\mathcal{E}^-)^{-1}G(x) = (\kappa_q^-(z))^{-1}e^{zx}$. Assuming X_t is the Lévy process with the Lévy density (11.6) and Lévy exponent (11.7), the LHS in (11.56) equals

$$V(x;h) = \frac{e^{zh}}{\kappa_q^-(z)}\sum_{j=1,2}\frac{a_j^-\beta_j^-}{\beta_j^- - z}e^{\beta_j^-(x-h)}, \quad x > h. \qquad (11.57)$$

If, in (11.6), $c^- = 0$, then

$$V(x;h) = e^{zh+\beta^-(x-h)}, \quad x > h \qquad (11.58)$$

(cf. (11.51) and (11.52)).

11.4.7 EPV of an instantaneous payoff that is acquired when the threshold is reached or crossed from below

Theorem 11.4.7 *Assume that G satisfies (11.41). Then*

$$E^x\left[q^{\tau_h^+}G(X_{\tau_h^+})\right] = \mathcal{E}^+\mathbf{1}_{[h,+\infty)}(\mathcal{E}^+)^{-1}G(x). \qquad (11.59)$$

The proof is the mirror reflection of the proof of Theorem 11.4.6.

Example 11.11. Let $G(x) = e^{zx}$, where $z \leq \sigma^+$. Then $(\mathcal{E}^+)^{-1}G(x) = (\kappa_q^+(z))^{-1}e^{zx}$. Assuming X_t is the Lévy process with the Lévy density (11.6) and Lévy exponent (11.7), the LHS in (11.59) equals

$$V(x;h) = \frac{e^{zh}}{\kappa_q^+(z)}\sum_{j=1,2}\frac{a_j^+\beta_j^+}{\beta_j^+ - z}e^{\beta_j^+(x-h)}, \quad x < h. \qquad (11.60)$$

If, in (11.6), $c^+ = 0$, then

$$V(x;h) = e^{zh+\beta^+(x-h)}, \quad x < h \qquad (11.61)$$

11.5 Main types of options. Optimality in the class of optimal stopping rules of the threshold type

In this Section, we find optimal stopping times in the class of hitting times of semi-infinite intervals. For the standing assumptions, see Subsect. 11.4.1.

11.5.1 Optimal time to abandon an increasing stream

A model example is the exit problem for a firm with uncertainty on the demand side and the profit flow $g(X_t) = Ge^{X_t} - C$.

Theorem 11.5.1 *Assume that g is a measurable function satisfying (11.40), and there exists h_* such that $\mathcal{E}^+g(x) < 0 \;\forall\; x < h_*$ and $\mathcal{E}^+g(x) > 0 \;\forall\; x > h_*$. Then h_* is a unique optimal threshold, and the rational value of the stream with the option to abandon it is given by*

$$V(x) = q^{-1}(\mathcal{E}^-\mathbf{1}_{(h_*,+\infty)}\mathcal{E}^+g)(x). \tag{11.62}$$

Proof. Let h be a candidate for an optimal exit threshold. Then the rational value equals the RHS of (11.44). Since the operator $\mathcal{E}^-$ is monotone, an optimal choice of h must replace all negative values of $\mathcal{E}^+g$ by zeroes and leave positive ones as they are. Hence, $h = h_*$ is the only optimal threshold, and the option value is given by (11.62).

Example 11.12. Let $g(x) = Ge^{zx} - C$, where $0 < z \le \sigma^+$. Then $\mathcal{E}^+g(x) = G\kappa_q^+(z)e^{zx} - C$. Hence, the exit threshold is defined by

$$G\kappa_q^+(z)e^{zh_*} = C. \tag{11.63}$$

To calculate the rational value, we use the integral representation of $\mathcal{E}^-$. In the case of the model with the Lévy density (11.6) and Lévy exponent (11.7), we apply (11.45) to (11.62), then use (11.63) and find for $x > h_*$

$$\begin{aligned} V(x) &= \frac{Ge^{zx}}{q - \Psi(z)} - \frac{G\kappa_q^+(z)e^{zh_*}}{q} \sum_{j=1,2} \frac{a_j^-\beta_j^- e^{\beta_j^-(x-h_*)}}{\beta_j^- - z} \\ &\quad - \frac{C}{q}(1 - \sum_{j=1,2} a_j^- e^{\beta_j^-(x-h_*)}) \\ &= \frac{Ge^{zx}}{q - \Psi(z)} - \frac{C}{q} + \frac{C}{q} \sum_{j=1,2} \frac{a_j^- z}{z - \beta_j^-} e^{\beta_j^-(x-h_*)} \end{aligned} \tag{11.64}$$

The first two terms on the RHS are the EPV of the perpetual stream, and the third one is the option value to abandon the stream. If the Lévy density is given by (11.6) with $c^- = 0$, then the equation $q - \Psi(z) = 0$ has only one negative root, call it β^-, and (11.64) assumes the form

$$V(x) = \frac{Ge^{zx}}{q - \Psi(z)} - \frac{C}{q} + \frac{Cz}{q(z - \beta^-)} e^{\beta^-(x-h_*)}. \tag{11.65}$$

11.5.2 Optimal time to abandon a decreasing stream

A model example is the exit problem for a firm with uncertainty on the supply side and the profit flow $g(X_t) = R - e^{zX_t}$.

Theorem 11.5.2 *Assume that g is a measurable function satisfying (11.40), and there exists h^* such that $\mathcal{E}^- g(x) > 0 \;\forall\; x < h^*$ and $\mathcal{E}^- g(x) < 0 \;\forall\; x > h^*$. Then h^* is a unique optimal threshold, and the rational value of the stream with the option to abandon it is given by*

$$V(x) = q^{-1}(\mathcal{E}^+ \mathbf{1}_{(-\infty,h^*)} \mathcal{E}^- g)(x). \tag{11.66}$$

The proof is the mirror reflection of the proof of Theorem 11.5.1.

Example 11.13. Let $g(x) = R - e^{zx}$, where $0 < z \le \sigma^+$. Then $\mathcal{E}^- g(x) = R - \kappa_q^-(z)e^{zx}$. Hence, the exit threshold is defined by

$$\kappa_q^-(z)e^{zh^*} = R. \tag{11.67}$$

To calculate the rational value, we use the integral representation of $\mathcal{E}^+$. In the case of the model with the Lévy density (11.6) and Lévy exponent (11.7), we apply (11.48) to (11.66), then use (11.67), and find for $x < h^*$

$$\begin{aligned}
V(x) &= \frac{R}{q}\Big(1 - \sum_{j=1,2} a_j^+ e^{\beta_j^+(x-h^*)}\Big) - \frac{e^{zx}}{q - \Psi(z)} \\
&\quad + \frac{\kappa_q^-(z)e^{zh^*}}{q} \sum_{j=1,2} \frac{a_j^+ \beta_j^+ e^{\beta_j^+(x-h^*)}}{\beta_j^+ - z} \\
&= \frac{R}{q} - \frac{e^{zx}}{q - \Psi(z)} + \frac{R}{q} \sum_{j=1,2} \frac{a_j^+ z}{\beta_j^+ - z} e^{\beta_j^+(x-h^*)}
\end{aligned} \tag{11.68}$$

(cf. (11.64)). The first two terms on the RHS are the EPV of the perpetual stream, and the third one is the option value to abandon the stream. If the Lévy density is given by (11.6) with $c^+ = 0$, then the equation $q - \Psi(z) = 0$ has only one positive root, call it β^+, and (11.68) assumes the form

$$V(x) = \frac{R}{q} - \frac{e^{zx}}{q - \Psi(z)} + \frac{Rz}{q(\beta^+ - z)} e^{\beta^-(x-h^*)}. \tag{11.69}$$

11.5.3 Optimal time to acquire an increasing stream

A model example is the irreversible investment with uncertainty on the demand side, the profit flow $g(X_t) = Ge^{X_t} - C$, and zero fixed investment cost. Non-zero fixed investment cost I can be incorporated by assuming that the project is financed by debt, and the firm precommits not to default on the debt obligations. In this case, the following theorem is applicable with $g(X_t) = Ge^{X_t} - C - qI$.

Theorem 11.5.3 *Assume that g is a measurable function satisfying (11.40), and there exists h^* such that $\mathcal{E}^- g(x) > 0 \;\forall\; x > h^*$ and $\mathcal{E}^- g(x) < 0 \;\forall\; x < h^*$. Then h^* is a unique optimal threshold, and the rational value of the option to acquire the stream is given by*

$$V(x) = q^{-1}(\mathcal{E}^+ \mathbf{1}_{[h^*,+\infty)} \mathcal{E}^- g)(x). \tag{11.70}$$

The proof is a straightforward modification of the proof of Theorem 11.5.1, Theorem 11.4.5 being used.

Example 11.14. Let $g(x) = Ge^{zx} - C$, where $0 < z \le \sigma^+$. Then $\mathcal{E}^- g(x) = G\kappa_q^-(z)e^{zx} - C$. Hence, the entry threshold is defined by

$$G\kappa_q^-(z)e^{zh^*} = C. \tag{11.71}$$

To calculate the rational value, we use the integral representation of $\mathcal{E}^+$. In the case of the model with the Lévy density (11.6) and Lévy exponent (11.7), we apply (11.54) to (11.70), then use (11.71) and find for $x < h^*$

$$\begin{aligned} V(x) &= \frac{G\kappa_q^-(z)e^{zh^*}}{q} \sum_{j=1,2} \frac{a_j^+ \beta_j^+}{\beta_j^+ - z} e^{\beta_j^+(x-h^*)} - \frac{C}{q} \sum_{j=1,2} a_j^+ e^{\beta_j^+(x-h^*)} \\ &= \frac{C}{q} \sum_{j=1,2} \frac{a_j^+ z}{\beta_j^+ - z} e^{\beta_j^+(x-h^*)}. \end{aligned} \tag{11.72}$$

If the Lévy density is given by (11.6) with $c^+ = 0$, then the equation $q - \Psi(z) = 0$ has only one positive root, call it β^+, and (11.72) assumes the form

$$V(x) = \frac{Cz}{q(\beta^+ - z)} e^{\beta^+(x-h^*)}. \tag{11.73}$$

11.5.4 Optimal time to acquire a decreasing stream

A model example is the irreversible investment with uncertainty on the supply side, the profit flow $g(X_t) = R - e^{X_t}$, and zero fixed investment cost. Non-zero fixed investment cost I can be incorporated by assuming that the project is financed by debt, and the firm precommits not to default on the debt obligations. In this case, the following theorem is applicable with $g(X_t) = R - qI - e^{X_t}$.

Theorem 11.5.4 *Assume that g is a measurable function satisfying (11.40), and there exists h_* such that $\mathcal{E}^+ g(x) > 0 \ \forall\, x < h_*$ and $\mathcal{E}^+ g(x) < 0 \ \forall\, x > h_*$. Then h_* is a unique optimal threshold, and the rational value of the option to acquire the stream is given by*

$$V(x) = q^{-1}(\mathcal{E}^- \mathbf{1}_{(-\infty, h_*]} \mathcal{E}^+ g)(x). \tag{11.74}$$

The proof is the mirror reflection of Theorem 11.5.3.

Example 11.15. Let $g(x) = R - e^{zx}$, where $0 < z \le \sigma^+$. Then $\mathcal{E}^+ g(x) = R - \kappa_q^+(z)e^{zx}$. Hence, the entry threshold is defined by

$$\kappa_q^+(z)e^{zh_*} = R. \tag{11.75}$$

To calculate the rational value, we use the integral representation of $\mathcal{E}^-$. In the case of the model with the Lévy density (11.6) and Lévy exponent (11.7), we apply (11.51) to (11.74), then use (11.75) and find for $x > h_*$

$$V(x) = \frac{R}{q} \sum_{j=1,2} \frac{a_j^- z}{z - \beta_j^-} e^{\beta_j^-(x-h_*)}. \tag{11.76}$$

If the Lévy density is given by (11.6) with $c^- = 0$, then the equation $q - \Psi(z) = 0$ has only one negative root, call it β^-, and (11.76) assumes the form

$$V(x) = \frac{Rz}{q(z - \beta^-)} e^{\beta^-(x-h_*)}. \tag{11.77}$$

11.5.5 Perpetual call-like American options

Consider an option with the instantaneous payoff $G(X_t)$ which is an increasing function of the underlying stochastic factor. The standard examples are $G(X_t) = X_t - K$ or $G(X_t) = e^{X_t} - K$; the following theorem is applicable to much wider classes of payoffs.

Theorem 11.5.5 *Assume that G satisfies (11.41), and there exists h^* such that $(\mathcal{E}^+)^{-1}G(x) < 0 \ \forall\ x < h^*$, and $(\mathcal{E}^+)^{-1}G(x) > 0 \ \forall\ x > h^*$. Then h^* is a unique optimal exercise boundary, and the rational option value is given by*

$$V(x) = \mathcal{E}^+ \mathbf{1}_{[h^*,+\infty)} (\mathcal{E}^+)^{-1} G(x). \tag{11.78}$$

Example 11.16. Consider the perpetual American power call option with the payoff $G(X_t) = e^{zX_t} - K$, where $0 < z \le \sigma^+$. Then the optimal exercise boundary is given by

$$e^{zh^*} = K\kappa_q^+(z). \tag{11.79}$$

Assuming X_t is the Lévy process with the Lévy density (11.6) and Lévy exponent (11.7), the option value is given by

$$\begin{aligned} V(x) &= \frac{e^{zh^*}}{\kappa_q^+(z)} \sum_{j=1,2} \frac{a_j^+ \beta_j^+}{\beta_j^+ - z} e^{\beta_j^+(x-h^*)} - K \sum_{j=1,2} a_j^+ e^{\beta_j^+(x-h^*)} \\ &= K \sum_{j=1,2} \frac{a_j^+ z}{\beta_j^+ - z} e^{\beta_j^+(x-h^*)}, \quad x < h^*. \end{aligned} \tag{11.80}$$

If, in (11.6), $c^+ = 0$, then

$$V(x) = \frac{Kz}{\beta^+ - z} e^{\beta^+(x-h^*)} = (e^{h^*} - K) e^{\beta^+(x-h^*)}, \quad x < h^*. \tag{11.81}$$

11.5.6 Perpetual put-like American options

Consider an option with the instantaneous payoff $G(X_t)$ which is a decreasing function of the underlying stochastic factor. The standard examples are $G(X_t) = K - X_t$ or $G(X_t) = K - \mathrm{e}^{X_t}$; the following theorem is applicable to much wider classes of payoffs.

Theorem 11.5.6 *Assume that G satisfies (11.42), and there exists h_* such that $(\mathcal{E}^-)^{-1}G(x) > 0 \ \forall \ x < h_*$, and $(\mathcal{E}^-)^{-1}G(x) < 0 \ \forall \ x > h_*$. Then $\tau^-_{h_*}$ is a unique optimal exercise time, and the rational value of the option with the payoff $G(X_t)$ is given by*

$$V(x) = \mathcal{E}^-\mathbf{1}_{(-\infty,h_*]}(\mathcal{E}^-)^{-1}G(x). \tag{11.82}$$

Example 11.17. Consider the perpetual American power put option with the payoff $G(X_t) = K - \mathrm{e}^{zX_t}$, where $z > 0$. Then the optimal exercise boundary is given by

$$\mathrm{e}^{zh_*} = K\kappa^-_q(z). \tag{11.83}$$

Assuming X_t is the Lévy process with the Lévy density (11.6) and Lévy exponent (11.7), the option value is given by

$$\begin{aligned} V(x) &= K \sum_{j=1,2} a^-_j \mathrm{e}^{\beta^-_j(x-h_*)} - \frac{\mathrm{e}^{zh_*}}{\kappa^-_q(z)} \sum_{j=1,2} \frac{a^-_j \beta^-_j}{\beta^-_j - z} \mathrm{e}^{\beta^-_j(x-h_*)} \\ &= K \sum_{j=1,2} \frac{a^+_j z}{z - \beta^-_j} \mathrm{e}^{\beta^-_j(x-h_*)}, \quad x > h_*. \end{aligned} \tag{11.84}$$

If, in (11.6), $c^- = 0$, then

$$V(x) = \frac{Kz}{z - \beta^-}\mathrm{e}^{\beta^-(x-h_*)} = (K - \mathrm{e}^{h_*})\mathrm{e}^{\beta^-(x-h_*)}, \quad x > h_*. \tag{11.85}$$

11.6 Optimality in the class of all stopping times

In this section, we consider stopping times satisfying $0 \le \tau = \tau(\omega) < \infty$, $\omega \in \Omega$, a.s.; the class of such stopping times will be denoted $\mathcal{M}$. The rational price of the perpetual American option with the payoff G is given by

$$V_*(x) = \sup_{\mathcal{M}} E^x[\mathrm{e}^{-q\tau}G(X_\tau)]. \tag{11.86}$$

For the other types of options considered in the preceding section, the definitions of the value function are similar. The standing assumptions are the same as in Subsect. 11.4.1. The proofs are based on the following general lemmas.

11.6.1 General verification lemmas

Consider the option to swap a measurable stream g_o for another stream g_n. We assume that $\mathcal{E}|g_o|$ and $\mathcal{E}|g_n|$ are finite, and X_t satisfies the (ACP)-condition. Then $\mathcal{E}g_o$ and $\mathcal{E}g_n$ are continuous ([79], pp. 288-289).

Lemma 11.6.1 *Let an open set $U \subset \mathbb{R}$ and a non-negative measurable function W_* satisfy the following conditions:*

$$W_*(x) = g_o(x), \qquad x \in U; \tag{11.87}$$
$$W_*(x) \ge g_o(x), \qquad x \notin U, \text{ a.e.}; \tag{11.88}$$
$$\mathcal{E}W_*(x) = \mathcal{E}g_n(x), \qquad x \notin U; \tag{11.89}$$
$$\mathcal{E}W_*(x) \ge \mathcal{E}g_n(x), \qquad x \in U. \tag{11.90}$$

Then τ_*, the hitting time of U^c, is an optimal stopping time in the class $\mathcal{M}$, and $V_* := q^{-1}\mathcal{E}W_*$ is the rational option price.

Remark 11.18. a) We can reformulate Lemma 11.6.1 as follows: the option price is generated by a measurable stream W_*. Conditions (11.87)–(11.88) state that in the inaction region, this stream coincides with the stream which the option generates prior to exercise, and in the action region, the former equals or exceeds the latter. In the action region, after the action is undertaken, the EPV of the stream matters, and evidently, the option value is generated by stream $g_n(X_t)$ (condition (11.89)). On the other hand, in the inaction region, the option value must be at least as big as the EPV of the stream $g_n(X_t)$ (condition (11.90)).

b) The difference in the formulation between pairs (11.87)–(11.88) and (11.89)–(11.90) is due to the irreversibility of the swap. In the completely reversible case, the option value is generated by the stream $W(x) = \max\{g_o(x), g_n(x)\}$, and conditions (11.89)–(11.90) hold without the EPV operator $\mathcal{E}$. The streams g_o and g_n are on the equal footing.

c) If X_t satisfies the (ACP)-condition, and W_* is measurable and bounded, then $V_* = q^{-1}\mathcal{E}W_*$ is continuous ([79], pp. 288-289). The case of unbounded functions satisfying appropriate conditions on the growth at infinity can be reduced to the case of bounded functions, therefore V_* is continuous.

d) The statement of Lemma 11.6.1 and the remark above are valid under weaker regularity conditions on W_*: universal measurability suffices (for the definition of a universally measurable function, see [79], p. 274). In the setting of the main theorems below, W_* turns out to be monotone hence universally measurable.

Proof of Lemma 11.6.1. Notice that

$$\begin{aligned} V(g_o, g_n; x) &= E^x\left[\int_0^{+\infty} \mathrm{e}^{-qt} g_o(X_t)dt\right] + V(0, g_n - g_o; x) \\ &= q^{-1}\mathcal{E}g_o(x) + V(0, g; x), \end{aligned}$$

where $g = g_n - g_o$, therefore the optimization of $V(g_o, g_n; x)$ is equivalent to the optimization of $V(0, g; x)$. Further, W_* satisfies (11.87)–(11.90) if and only if $\tilde{W} = W_* - g_o$ satisfies the same conditions with $g_o = 0$, $g_n = g$, therefore it suffices to give a proof for this case. Then, on the strength of (11.87)-(11.88), $\tilde{W}$ is non-negative, a.e., and it is measurable, since W_* and g_0 are. Hence, Dynkin's formula (equation (41.3) in [79]) is applicable with $f = \tilde{W}$: for any stopping time τ,

$$\mathcal{E}\tilde{W}(x) = E^x \left[\int_0^\tau q \mathrm{e}^{-qt} \tilde{W}(X_t) dt \right] + E^x \left[\mathrm{e}^{-q\tau} \mathcal{E}\tilde{W}(X_\tau) \right]. \tag{11.91}$$

Since the process satisfies the (ACP)-condition, the q-potential measure of a point is 0. This implies that if we change $\tilde{W}$ on a set of zero measure, all terms in (11.91) will not change. Hence, we may assume that $\tilde{W}$ is non-negative, and drop the integral on the RHS of (11.91). In the result, "=" will be replaced by "≥". Applying (11.89)–(11.90), we arrive at

$$\mathcal{E}\tilde{W}(x) \ge E^x \left[\mathrm{e}^{-q\tau} \mathcal{E}\tilde{W}(X_\tau) \right] \ge E^x \left[\mathrm{e}^{-q\tau} \mathcal{E}g(X_\tau) \right].$$

Hence, $\mathcal{E}\tilde{W}(x) \ge qV(0, g; x)$. If we take $\tau = \tau_*$ and apply (11.87) and (11.89) to (11.91), we obtain

$$\mathcal{E}\tilde{W}(x) = E^x \left[\mathrm{e}^{-q\tau_*} \mathcal{E}\tilde{W}(X_{\tau_*}) \right] = E^x \left[\mathrm{e}^{-q\tau_*} \mathcal{E}g(X_{\tau_*}) \right] \le qV(0, g; x).$$

Lemma has been proved. Notice that Dynkin's formula (11.91), the key element of the proof, has a simple meaning: the EPV of a stream equals the EPV up to a stopping time τ plus the continuation value. Thus, it is a general form of the Bellman equation.

Now, consider the perpetual American option with the continuous payoff $G(X_t)$.

Lemma 11.6.2 *Assume that X_t satisfies the (ACP)-condition, and an open set $U \subset \mathbb{R}$ and a continuous function V_* satisfy the following conditions:*

$$(q - L)V_*(x) = 0, \qquad x \in U; \tag{11.92}$$
$$(q - L)V_*(x) \ge 0, \qquad x \notin U, \text{ a.e.}; \tag{11.93}$$
$$V_*(x) = G(x), \qquad x \notin U; \tag{11.94}$$
$$V_*(x) \ge G(x), \qquad x \in U; \tag{11.95}$$
$$W_* := (q - L)V_* \quad \text{is universally measurable}; \tag{11.96}$$
$$q^{-1}\mathcal{E}W_* = V_*. \tag{11.97}$$

Then τ_*, the hitting time of U^c, is an optimal stopping time in the class $\mathcal{M}$, and V_* is the rational option price.

In all applications in this monograph, W_* turns out to be monotone.

Proof. Due to (11.92)–(11.93), W_* is non-negative, a.e., and by (11.96), it is universally measurable, therefore for any stopping time τ, (11.91) is applicable to W_*. In particular, with $\tau = +\infty$, we obtain $V_* = q^{-1}\mathcal{E}W_*$. Since the process satisfies the (ACP)-condition, the q-potential measure of a point is 0. This implies that if we change W_* on a set of zero measure, all terms in (11.91) will not change. Hence, we may assume that W_* is non-negative, and drop the integral on the RHS of (11.91). In the result, "=" will be replaced by "≥". Applying the equality $\mathcal{E}W_* = qV_*$ and then (11.94)-(11.95), we arrive at

$$\mathcal{E}W_*(x) \geq E^x\left[e^{-q\tau}\mathcal{E}W_*(X_\tau)\right] = qE^x\left[e^{-q\tau}V_*(X_\tau)\right] \geq qE^x\left[e^{-q\tau}G(X_\tau)\right].$$

Therefore, $V_*(x)$ is not smaller than the option value. If we take $\tau = \tau_*$ and apply (11.96) and (11.92) to (11.91), we obtain

$$\mathcal{E}W_*(x) = E^x\left[e^{-q\tau_*}\mathcal{E}W_*(X_{\tau_*})\right] = E^x\left[e^{-q\tau_*}\mathcal{E}(q-L)V_*(X_{\tau_*})\right].$$

According to (11.94), this is $qE^x\left[e^{-q\tau_*}G(X_{\tau_*})\right]$. Lemma has been proved.

11.6.2 Option to abandon an increasing stream

Consider the option to abandon a stream of payoffs $g(X_t)$ that is a non-decreasing function of the underlying stochastic factor.

Theorem 11.6.3 *Assume that g does not decrease, changes sign and satisfies (11.40). Then*

(a) equation $\mathcal{E}^+g(h) = 0$ has a unique solution, call it h_*;
(b) $\tau^-_{h_*}$ is an optimal stopping time;
(c) the value of the stream with the option to abandon it is given by

$$V_* = q^{-1}\mathcal{E}^-\mathbf{1}_{(h_*,+\infty)}\mathcal{E}^+g; \tag{11.98}$$

(d) the value of the stream with the option to abandon it can be represented as the EPV of the stream $W(X_t) + g(X_t)$, where W is a non-increasing function vanishing above h_*.

Proof. (a) Since g satisfies (11.40), $\mathcal{E}g$ is well-defined, and Propositions 10.2.1-10.2.4 apply. Since g is non-decreasing, Proposition 10.2.1 states that $\mathcal{E}^+g$ is a continuous non-decreasing function, and Proposition 10.2.4 gives (a). Moreover, $\mathcal{E}^+g(x) > 0\ \forall\ x > h_*$, and $\mathcal{E}^+g(x) < 0\ \forall\ x < h_*$. According to Theorem 11.4.2, $\tau^-_{h_*}$ is the unique optimal stopping time in the class of hitting times of semi-infinite intervals, and $V(x; h_*)$, the value of the stream g with this choice of the exit threshold, is given by the RHS of (11.98).

(b)–(d) We define $W_* = (q-L)V(x; h_*)$, and verify the conditions of Lemma 11.6.1 with $g_o = g$ and $g_n = 0$. Using the Wiener-Hopf factorization formula, we obtain

$$\begin{aligned} W_* &= q^{-1}(q-L)\mathcal{E}^-\mathbf{1}_{(h_*,+\infty)}\mathcal{E}^+g \\ &= q^{-1}(q-L)\mathcal{E}^-\mathcal{E}^+g - (\mathcal{E}^+)^{-1}\mathbf{1}_{(-\infty,h_*])}\mathcal{E}^+g \\ &= g - (\mathcal{E}^+)^{-1}\mathbf{1}_{(-\infty,h_*]}\mathcal{E}^+g. \end{aligned} \tag{11.99}$$

Since $\mathbf{1}_{(-\infty,h_*]}\mathcal{E}^+g(x) = 0\ \forall\ x > h_*$, the last term on the RHS of (11.99) is zero for $x > h_*$ (see Proposition 10.2.2). Thus, (11.87) holds. Further,

$$\mathcal{E}W_* = \mathcal{E}(q-L)V(x;h_*) = qV(x;h_*) = \mathcal{E}^-\mathbf{1}_{(h_*,+\infty)}\mathcal{E}^+g,$$

which is 0 for $x \le h_*$ by Proposition 10.2.1. Hence, (11.89) holds. On $(h_*,+\infty)$, $\mathcal{E}^+g$ is non-negative, therefore $\mathcal{E}^-\mathbf{1}_{(h_*,+\infty)}\mathcal{E}^+g$ is non-negative as well, and (11.90) holds. Finally, (11.88) follows from part (d). Thus, all conditions of Lemma 11.6.1 are verified, and parts (b)–(c) are proved. Part (d) is immediate from the following lemma.

Lemma 11.6.4 *Function* $W = (\mathcal{E}^+)^{-1}\mathbf{1}_{(-\infty,h_*]}\mathcal{E}^+(-g)$ *vanishes above* h_* *and coincides with a non-increasing function, a.e.*

Proof. By Proposition 10.2.2, W vanishes above h_*. To study W on $(-\infty,h_*)$, we represent W in the form

$$W(x) = -g(x) - q^{-1}(q-L)\mathcal{E}^-\mathbf{1}_{(h_*,+\infty)}\mathcal{E}^+(-g)(x).$$

On $(-\infty,h_*)$, $\mathcal{E}^-\mathbf{1}_{(h_*,+\infty)}\mathcal{E}^+(-g)(x)$ vanishes, therefore $W(x)$ is independent of the local (differential) part of L, and we obtain

$$W(x) = -g(x) + q^{-1}B\mathcal{E}^-\mathbf{1}_{(h_*,+\infty)}\mathcal{E}^+(-g)(x),\ x < h_*,$$

where B acts as follows: $Bu(x) = \int_{-\infty}^{+\infty} u(x+y)F(dy)$. Since the density $F(dy)$ is non-negative, operators $\mathcal{E}^\pm$ are monotone and $-g$ in non-increasing, W is non-increasing as well. To finish the proof, it remains to show that $W(h_*-0) \ge 0$. Suppose that, on the contrary, $W(h_*-0) < 0$. Then there exists $h' < h_*$ such that $W(x) < 0\ \forall\ x \in (h',h_*)$. Applying $\mathcal{E}^+$ to W, we obtain

$$\mathcal{E}^+W(x) = \mathbf{1}_{(-\infty,h_*]}\mathcal{E}^+(-g)(x). \tag{11.100}$$

Since $W(x) = 0$ for $x \in (h_*,\infty)$, and $W(x) < 0$ for $x \in (h',h_*)$, the LHS in (11.100) is negative for these x. But the RHS is non-negative since $\mathcal{E}^+(-g)(x) \ge 0$ for $x \le h_*$, contradiction.

11.6.3 Option to abandon a decreasing stream

Consider the option to abandon a stream of payoffs $g(X_t)$ that is a non-increasing function of the underlying stochastic factor.

Theorem 11.6.5 *Assume that* g *does not increase, changes sign and satisfies (11.40). Then*

(a) equation $\mathcal{E}^- g(h) = 0$ *has a unique solution, call it* h^**;*
(b) $\tau^+_{h^*}$ *is an optimal stopping time;*
(c) the value of the stream with the option to abandon it is given by

$$V^* = q^{-1}\mathcal{E}^+ \mathbf{1}_{(-\infty,h^*)}\mathcal{E}^- g; \tag{11.101}$$

(d) the value of the stream with the option to abandon it can be represented as the EPV of the stream $W(X_t) + g(X_t)$*, where* W *is a non-decreasing function vanishing below* h^**.*

The proof is the mirror reflection of the proof of Theorem 11.6.3

11.6.4 Option to acquire an increasing stream

Consider the option to acquire a stream of payoffs $g(X_t)$ that is a non-decreasing function of the underlying stochastic factor.

Theorem 11.6.6 *Assume that* g *does not decrease, changes sign and satisfies (11.40). Then*

(a) equation $\mathcal{E}^- g(h) = 0$ *has a unique solution, call it* h^**;*
(b) $\tau^+_{h^*}$ *is an optimal stopping time;*
(c) the value of the option to acquire the stream is given by

$$V^* = q^{-1}\mathcal{E}^+ \mathbf{1}_{[h^*,+\infty)}\mathcal{E}^- g; \tag{11.102}$$

(d) the option value can be represented as the EPV of a stream $W(X_t)$*, where* W *is a non-decreasing function vanishing below* h^**.*

Proof. Since the value of the option to acquire the stream $g(X_t)$ equals the EPV of the perpetual stream $g(X_t)$ and the value of the stream $-g(X_t)$ with the option to abandon it, we apply Theorem 11.6.5 and obtain that h^*, the solution of the equation $\mathcal{E}^-(-g)(h) = 0$, is the optimal exercise boundary, and the option value equals

$$q^{-1}\mathcal{E}g(x) + q^{-1}\mathcal{E}^+\mathbf{1}_{(-\infty,h^*)}\mathcal{E}^-(-g) = q^{-1}\mathcal{E}^+\mathbf{1}_{[h_*,+\infty)}\mathcal{E}^- g,$$

which proves (a)–(c). According to part (d) of Theorem 11.6.5, the option value can be represented as the EPV of the sum $g + (-g + W) = W$, where W is a non-decreasing function vanishing below h^*.

The second proof obtains if we note that V^* is the value of the option to acquire the instantaneous payoff $G = q^{-1}\mathcal{E}g$ and apply the theorem about options of this type, which will be formulated and proved in Subsect. 11.6.7.

11.6.5 Option to acquire a decreasing stream

Consider the option to acquire a stream of payoffs $g(X_t)$ that is a non-increasing function of the underlying stochastic factor.

Theorem 11.6.7 *Assume that g does not increase, changes sign and satisfies (11.40). Then*

(a) equation $\mathcal{E}^+g(h) = 0$ has a unique solution, call it h_*;
(b) $\tau^-_{h_*}$ is an optimal stopping time;
(c) the value of the option to acquire the stream is given by

$$V^* = q^{-1}\mathcal{E}^-\mathbf{1}_{(-\infty,h_*]}\mathcal{E}^+g; \tag{11.103}$$

(d) the option value can be represented as the EPV of a stream $W(X_t)$, where W is a non-increasing function vanishing above h_*.

The proof is the mirror reflection of the proof of Theorem 11.6.6.

11.6.6 Put-like options

Consider an option with the non-increasing instantaneous payoff $G(X_t)$.

Theorem 11.6.8 *Assume that*

(i) G is a non-increasing function satisfying (11.41);
(ii) $(\mathcal{E}^-)^{-1}G$ decreases and changes sign;
(iii) $(q - L)G$ is non-increasing.

Then

(a) equation $(\mathcal{E}^-)^{-1}G(h) = 0$ has a unique solution, call it h_*;
(b) $\tau^-_{h_*}$ is an optimal stopping time;
(c) the option value is given by

$$V_* = \mathcal{E}^-\mathbf{1}_{(-\infty,h_*]}(\mathcal{E}^-)^{-1}G; \tag{11.104}$$

(d) the option value can be represented as the EPV of a stream $W(X_t)$, where W is a non-increasing function vanishing above h_*.

Proof. If G can be represented as the EPV of a non-increasing stream g which changes sign, then $(\mathcal{E}^-)^{-1}G = q^{-1}(\mathcal{E}^-)^{-1}\mathcal{E}g = q^{-1}\mathcal{E}^+g$, and Theorem 11.6.8 follows from Theorem 11.6.7. Otherwise, an independent proof is needed. Set $W = (q - L)V_*$ and notice that $W(x)$ is well-defined for $x \neq h_*$.

Lemma 11.6.9 *Function W vanishes above h_* and coincides with a non-increasing function, a.e.*

Proof. On the strength of Theorems 11.4.1 and 11.4.6, $W = (q-L)V_*$ vanishes above h_*. To study W on $(-\infty, h_*)$, we represent W as

$$W = (q - L)(G - \mathcal{E}^- \mathbf{1}_{(h_*,+\infty)}(\mathcal{E}^-)^{-1}G).$$

On $(-\infty, h_*)$, $\mathcal{E}^- \mathbf{1}_{(h_*,+\infty)}(\mathcal{E}^-)^{-1}G(x)$ vanishes, therefore $W(x)$ is independent of the local (differential) part of L, and we obtain

$$W(x) = (q - L)G(x) + B\mathcal{E}^- \mathbf{1}_{(h_*,+\infty)}(\mathcal{E}^-)^{-1}G(x), \; x < h_*,$$

where B acts as follows: $Bu(x) = \int_{-\infty}^{+\infty} u(x+y)F(dy)$. Since the density $F(dy)$ is non-negative, operator $\mathcal{E}^-$ is monotone and $(q-L)G$ and $(\mathcal{E}^-)^{-1}G$ are non-increasing, W is non-increasing as well. To finish the proof, it remains to show that $W(h_* - 0) \geq 0$. Suppose that, on the contrary, $W(h_* - 0) < 0$. Then there exists $h' < h_*$ such that $W(x) < 0 \; \forall \; x \in (h', h_*)$. Applying $q^{-1}\mathcal{E}^+$ to W and using the Wiener-Hopf factorization, we obtain

$$q^{-1}\mathcal{E}^+W(x) = \mathbf{1}_{(-\infty,h_*]}(\mathcal{E}^-)^{-1}G(x). \tag{11.105}$$

Since $W(x) = 0$ on $x \in (h_*, \infty)$, and $W(x) < 0$ for $x \in (h', h_*)$, the LHS in (11.105) is negative for these x. But the RHS is non-negative since $(\mathcal{E}^-)^{-1}G(x) > 0$ for $x < h_*$, contradiction.

We continue the proof of Theorem 11.6.8. It follows from Lemma 11.6.9 that W is universally measurable, hence, $V_* = q^{-1}\mathcal{E}W$, and part (d) holds. Now we check conditions (11.92)–(11.96) of Lemma 11.6.2. Since W coincides with a non-decreasing function, a.e., W is universally measurable. Conditions (11.92)–(11.93) hold since $(q - L)V_*(x) = W(x)$ for $x \neq h_*$. Further, $V_* = G + \mathcal{E}^- \mathbf{1}_{(h_*,+\infty)}(-(\mathcal{E}^-)^{-1}G)$ equals G on $(-\infty, h_*]$ on the strength of Proposition 10.2.1, which proves (11.94). Finally, $(-(\mathcal{E}^-)^{-1}G)$ is positive on $(h_*, +\infty)$, therefore $V_* \geq G$, which is (11.95).

Example 11.19. Consider the American put option with the payoff $G(X_t) = K - \mathrm{e}^{X_t}$. Then $G(x) = K - \mathrm{e}^x$ and $(\mathcal{E}^-)^{-1}G = K - (1/\kappa_q^-(1))\mathrm{e}^x$ are decreasing functions that change sign. If the stock pays dividends, then $q - \Psi(1) > 0$ and if not then $q - \Psi(1) = 0$. In both cases, $q - \Psi(1) \geq 0$, and, therefore, $(q - L)G(x) = qK - (q - \Psi(1))\mathrm{e}^x$ is non-increasing. Thus, all conditions of Theorem 11.6.8 are satisfied, and $\tau_{h_*}^-$, where $h_* = \log(K\kappa_q^-(1))$, is an optimal exercise time.

In Example 11.19, the payoff is standard but the condition on a Lévy process is fairly weak: the (ACP)-condition (in [68, 70, 4], the result is proved for an arbitrary Lévy process and the standard payoff, different methods being used). In the next example, we prove a simple formula for a fairly general payoff but under additional restriction on the process. We assume that there are no negative jumps (*spectrally positive Lévy process*). Then the characteristic equation $q - \Psi(z) = 0$ has a unique negative root (see, e.g., [12]), call it β^-.

Example 11.20. Assume that X_t is a spectrally positive Lévy process satisfying the (ACP)-condition, G is a non-increasing function satisfying (11.41) such that $G-(1/\beta^-)G'$ decreases and changes sign, and $(q-L)G$ is non-increasing. Then the optimal exercise boundary h_* is a unique solution of the equation

$$G(h)-(1/\beta^-)G'(h)=0, \tag{11.106}$$

and, for $x>h_*$, the option value is given by

$$\begin{aligned} V(x) &= -\beta^- \int_{-\infty}^{h_*-x} \mathrm{e}^{-\beta^- y}(G(x+y)-(1/\beta^-)G'(x+y))\,dy \\ &= \mathrm{e}^{\beta^- x}\int_{-\infty}^{h_*} \mathrm{e}^{-\beta^- y}(G'(y)-\beta^- G(y))\,dy. \end{aligned} \tag{11.107}$$

11.6.7 Call-like options

Consider an option with the non-decreasing instantaneous payoff $G(X_t)$.

Theorem 11.6.10 *Assume that*

(i) G is a non-decreasing function satisfying (11.41);
(ii) $(\mathcal{E}^+)^{-1}G$ increases and changes sign;
(iii) $(q-L)G$ is non-decreasing.

Then

(a) equation $\mathcal{E}^+g(h)=0$ has a unique solution, call it h^*;
(b) $\tau^+_{h^*}$ is an optimal stopping time;
(c) the option value is given by

$$V^* = \mathcal{E}^+\mathbf{1}_{[h^*,+\infty)}(\mathcal{E}^+)^{-1}G; \tag{11.108}$$

(d) the option value can be represented as the EPV of a stream $W(X_t)$, where W is a non-decreasing function vanishing below h^*.

Proof is the mirror reflection of the proof of Theorem 11.6.8.

Example 11.21. Consider the American call option with the payoff $G(X_t)=\mathrm{e}^{X_t}-K$. If the stock pays dividends, then $q-\Psi(1)>0$ and if not then $q-\Psi(1)=0$. In the latter case, $\mathcal{E}^+$ is not defined, and Theorem 11.6.10 is not applicable. If the stock pays dividends, then $G(x)=\mathrm{e}^x-K$, $(q-L)G(x)=(q-\Psi(1))\mathrm{e}^x-qK$, and $(\mathcal{E}^+)^{-1}G=(1/\kappa^+_q(1))\mathrm{e}^x-K$ are increasing functions that change sign. Thus, all conditions of Theorem 11.6.10 are satisfied, and $\tau^+_{h^*}$, where $h^*=\log(K\kappa^+_q(1))$, is an optimal exercise time.

In Example 11.21, the payoff is standard but the condition on a Lévy process is fairly weak: the (ACP)-condition (in [68, 70, 4], the result is proved for an arbitrary Lévy process and the standard payoff, different methods being used). In the next example, we prove a simple formula for a fairly general payoff but under additional restriction on the process. We assume that there are no positive jumps (*spectrally negative Lévy process*). Then the characteristic equation $q - \Psi(z) = 0$ has a unique positive root (see, e.g., [12]), call it β^+.

Example 11.22. Assume that X_t is spectrally negative Lévy process satisfying the (ACP)-condition, G is a non-decreasing function satisfying (11.41) such that $G-(1/\beta^+)G'$ increases and changes sign, and $(q-L)G$ is non-decreasing. Then the optimal exercise boundary h^* is a unique solution of the equation

$$G(h) - (1/\beta^+)G'(h) = 0, \tag{11.109}$$

and, for $x < h^*$, the option value is given by

$$\begin{aligned} V(x) &= \beta^+ \int_{h_*-x}^{+\infty} \mathrm{e}^{-\beta^+ y}(G(x+y) - (1/\beta^+)G'(x+y))\,dy \\ &= \mathrm{e}^{\beta^+ x} \int_{h^*}^{+\infty} \mathrm{e}^{-\beta^+ y}(\beta^+ G(y) - G'(y))\,dy. \end{aligned} \tag{11.110}$$

11.6.8 Options to swap a stream for another one

Theorem 11.6.11 *Assume that g_o and g_n satisfy (11.40), and $g := g_n - g_o$ does not decrease and changes sign. Then*

(a) equation $\mathcal{E}^- g(h) = 0$ has a unique solution, call it h^*;
(b) $\tau^+_{h^*}$ is an optimal time for the swap;
(c) the value of an option to swap a stream $g_o(X_t)$ for a stream $g_n(X_t)$ is

$$V(g_o, g_n; x) = q^{-1}\mathcal{E}g_o + q^{-1}\mathcal{E}^+\mathbf{1}_{[h^*,+\infty)}\mathcal{E}^- g \tag{11.111}$$
$$= q^{-1}\mathcal{E}g_n + q^{-1}\mathcal{E}^+\mathbf{1}_{(-\infty,h^*)}\mathcal{E}^-(-g); \tag{11.112}$$

(d) $V(g_o, g_n; x)$ can be represented as the EPV of the stream $W(X_t) + g(X_t)$, where W is a non-decreasing function vanishing below h^*.

Proof. Clearly, $V(g_o, g_n; x) = q^{-1}\mathcal{E}g_o +$ the value of the option to acquire the stream $g = g_n - g_o$, and it suffices to apply Theorem 11.6.6.

Theorem 11.6.12 *Assume that g_o and g_n satisfy (11.40), and $g := g_n - g_o$ does not increase and changes sign. Then*

(a) equation $\mathcal{E}^+ g(h) = 0$ has a unique solution, call it h_*;
(b) $\tau^-_{h_*}$ is an optimal time for the swap;
(c)

$$V(g_o, g_n; x) = q^{-1}\mathcal{E}g_o + q^{-1}\mathcal{E}^-\mathbf{1}_{(-\infty,h_*]}\mathcal{E}^+ g; \tag{11.113}$$

(d) $V(g_o, g_n; x)$ can be represented as the EPV of the stream $W(X_t) + g(X_t)$, where W is a non-increasing function vanishing above h_*.

Proof. Clearly, $V(g_o, g_n; x) = q^{-1}\mathcal{E}g_o +$ the value of the option to acquire the stream $g = g_n - g_o$, and it suffices to apply Theorem 11.6.7.

11.7 Influence of idiosyncratic uncertainty on exit and entry thresholds

Consider a family of firms which face uncertainty represented by processes with the same first two instantaneous moments, $m_1 = \Psi'(0)$ and $m_2 = \Psi''(0)$. Each process has the diffusion component that represents the industry specific uncertainty, and jump component, which models the idiosyncratic risk. If a standard geometric Brownian motion is fitted to each of these price processes, the same Brownian motion is obtained, and entry and exit thresholds will be the same for each firm. However, as we verified in a number of numerical examples, for the firms that face the downward idiosyncratic risk, the entry threshold is lower, and the exit threshold is higher. When the upward jumps prevail, the exit threshold becomes lower. The entry threshold becomes a bit lower as well, which can be explained as follows. We keep the first two moments fixed, therefore, if the positive jumps component increases, the drift of the Gaussian component must decrease, and this negative effect prevails. The difference between the entry and exit thresholds increases when more positive jumps are added. Notice that on average, the effect of positive jumps on the thresholds is smaller than that of negative ones: bad firm-specific news are more important for investment decisions than good ones. The entry threshold is more sensitive to negative jumps, and the exit one – to positive jumps. Both thresholds can change by more than 10 percent even if a moderate jump component is added (for a significant jump component, they can change by dozens percent); and if one averages over many firms, one observes the thresholds which are lower (entry) and higher (exit) than in the standard Brownian motion model. Notice that practitioners are known to be uncomfortable with too high investment thresholds, which the real option approach recommends, and the use of jump-diffusion processes in investment models can alleviate these concerns.

To illustrate these effects, we consider the exit problem for a firm with the profit flow $G\mathrm{e}^{X_t}$ and scrap value C, and the entry problem for a firm with the profit flow $G\mathrm{e}^{X_t}$ and investment cost I. We assume the no-bubble condition $q - \Psi(1) > 0$. The exit problem is equivalent to the pricing of the perpetual American put option on the stock with the price process $G\mathrm{e}^{X_t}/(q-\Psi(1))$ (the EPV of the stream of profits) and strike C. Equivalently, we can regard it as the option to swap the profit stream $G\mathrm{e}^{X_t}$ for the stream qC. In both cases, the exit threshold is

$$H_* = \mathrm{e}^{h_*} = C\kappa_q^-(1)(q - \Psi(1)/G = qC/(G\kappa_q^+(1)). \tag{11.114}$$

The entry problem is equivalent to the problem of pricing of the perpetual American call option on the stock with the price process $G\mathrm{e}^{X_t}/(q-\Psi(1))$ and strike I. Equivalently, we can regard it as the option to swap the stream qI for the profit stream. In both cases, the entry threshold is

$$H^* = \mathrm{e}^{h^*} = I\kappa_q^+(1)(q - \Psi(1)/G = qI/(G\kappa_q^-(1)).$$

In Fig. 11.2, we plot the entry and exit thresholds H^* and H_* in the jump-diffusion model with fixed m_1, m_2, and either positive jumps only: $\lambda^+ = \lambda$ varies from 2 to $+\infty$, or negative jumps only: $\lambda^- = \lambda$ varies from -2 to $-\infty$. In the limit $\lambda \to \pm\infty$ (the point $1/\lambda = 0$), the Gaussian model is obtained. Parameters $c_\pm = c$ are the same for the cases of upward and downward jumps. When m_1, m_2, c and λ are fixed, the parameters σ and b are uniquely defined.

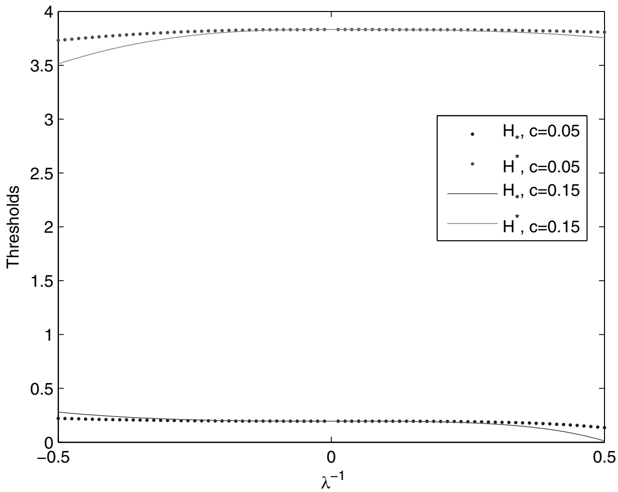

Fig. 11.2. Entry and exit thresholds. Parameters: $C = 7, I = 10.5, G = 0.56, q = 0.08, m_1 = -0.6, m_2 = 0.2$.

Problems

11.1. Give the detailed proof of Theorem 11.5.5.

11.2. Deduce (11.85).

Solve Problems 11.3–11.7 assuming that X_t is
a) the Brownian motion;
b) the Lévy process with the Lévy density (11.6) and $c^- = 0$;
c) the Lévy process with the Lévy density (11.6) and $c^+ = 0$.

11.3. a) Prove Theorem 11.5.5 assuming that G and G' are continuous and satisfy the bound

$$|G(x)| + |G'(x)| \leq C_N e^{\sigma^+ x}, \quad \forall\, x > -N, \tag{11.115}$$

for any N.

b) Consider G of the form $G(x) = 0, x < 0$, $G(x) = x^2, x \geq 0$. Find the optimal exercise boundary and calculate the option price.

11.4. Solve model problems for the payoff stream

a) $g(X_t) = X_t - K$;
b) $g(X_t) = \max\{0, \mathrm{e}^{X_t}\}$;
c) $g(X_t) = \max\{K, \mathrm{e}^{X_t}\}$;
d) $g(X_t) = -1, X_t < 0, g(X_t) = 1, X_t > 0$.

11.5. Solve model problems for the payoff stream a) $g(X_t) = K - X_t$; b) $g(X_t) = \max\{K, \mathrm{e}^{-X_t}\}$.

11.6. Find the optimal exercise threshold and rational price for the perpetual option to exchange the stream e^{X_t} for the stream e^{2X_t}.

11.7. Solve the problems of exogenous and endogenous default in Sect. 10.9.1, and derive the equation for the competitive lending rate. Study the dependence of the endogenous default threshold and riskless rate on parameters of the Lévy density assuming that the first two instantaneous moments $m_1 = \Psi'(0)$ and $m_2 = \Psi''(0)$ remain fixed.

12

Embedded options

In this Chapter, we consider investment problems with an embedded option to exit, multi-stage investment/disinvestment problems, and two types of problems which are reducible to sequences of perpetual options: capital expansion program, and capital expansion with technology adoption. We assume that X_t is a Lévy process satisfying (11.25) and the (ACP)-property.

12.1 Entry with an embedded option to exit

Consider an investor who chooses the time τ to invest capital I into a technology producing output at rate G; the price of a unit of the output is modeled as the geometric Lévy process e^{X_t}, and there is no variable cost. The investor has in mind the option of scrapping the inventory for the value C should the things go badly for the firm. It is reasonable to assume that $I > C$, because second hand inventories are less valuable than new inventories. Assume that the investment has been made. Applying Theorem 11.6.12 with $g_o(x) = Ge^x$ and $g_n = Cq$, the optimal exit threshold is given by (11.114), and the post-investment firm's value equals

$$V(x) = \frac{Ge^x}{q - \Psi(1)} + q^{-1}\mathcal{E}^-\mathbf{1}_{(-\infty,h_*]}\mathcal{E}^+(Cq - Ge^{\cdot})(x). \tag{12.1}$$

In the case of the model with the Lévy density (11.6) and Lévy exponent (11.7), we obtain similarly to (11.76), for $x > h_*$

$$V(x) = \frac{Ge^x}{q - \Psi(1)} + C\sum_{j=1,2}\frac{a_j^-}{1-\beta_j^-}e^{\beta_j^-(x-h_*)}. \tag{12.2}$$

Using (11.114) and the Wiener–Hopf factorization formula, we simplify (12.2):

$$V(x) = C\left[\kappa_q^-(1)e^{x-h_*} + \sum_{j=1,2}\frac{a_j^-}{1-\beta_j^-}e^{\beta_j^-(x-h_*)}\right]. \tag{12.3}$$

It is easy to check that $V(x)$ satisfies the value matching condition $V(h_*-0) = V(h_*+0)$:

$$C = C\left[\kappa_q^-(1) + \sum_{j=1,2} \frac{a_j^-}{1-\beta_j^-}\right],$$

because $\kappa_q^-(z) = \sum_{j=1,2} a_j^- \beta_j^- / (\beta_j^- - z)$ and $\sum_{j=1,2} a_j^- = 1$. The smooth pasting condition $V'(h_* - 0) = V'(h_* + 0)$ is satisfied as well:

$$0 = C\left[\kappa_q^-(1) - \sum_{j=1,2} \frac{a_j^- \beta_j^-}{\beta_j^- - 1}\right]. \tag{12.4}$$

At the time of investment, the investor receives an instantaneous payoff $V(x) - I$. Therefore, the optimal investment threshold, call it h^{**}, is the solution of the equation $w(h) := (\mathcal{E}^+)^{-1}(V(\cdot) - I)(h) = 0$ on $(h_*, +\infty)$, provided a solution is unique and w changes sign from "-" to "+" as h passes h^{**}. On the strength of Proposition 10.2.2, the $w(h)$ is independent of values $V(x) - I$ for $x < h$, therefore we can calculate $w(h)$ assuming that V is given by (12.3) on the whole axis. The result is $w(h) = Cf(h - h_*)$, where

$$f(y) = \frac{\kappa_q^-(1)}{\kappa_q^+(1)} \mathrm{e}^y + \sum_{j=1,2} \frac{a_j^-}{\kappa_q^+(\beta_j^-)(1-\beta_j^-)} \mathrm{e}^{\beta_j^- y} - I/C. \tag{12.5}$$

To show that f has a unique zero on $(0, +\infty)$ and changes sign from "-" to "+" as y increases from 0 to $+\infty$, note that 1) f is positive for large y, 2) f is convex because

$$f''(y) = \frac{\kappa_q^-(1)}{\kappa_q^+(1)} \mathrm{e}^y + \sum_{j=1,2} \frac{a_j^-(\beta_j^-)^2}{\kappa_q^+(\beta_j^-)(1-\beta_j^-)} \mathrm{e}^{\beta_j^- y}$$

is positive, and 3) $f(0) \le 1 - I/C < 0$. To prove 3), we note first that $\sum_{i=1,2} a_i^{\pm} = 1$ and $\beta_2^- < \beta_1^- < 0 < \beta_1^+ < \beta_2^+$. It follows that

$$\kappa_q^+(\beta_j^-) = \sum_{i=1,2} \frac{a_i^+ \beta_i^+}{\beta_i^+ - \beta_j^-} > \sum_{i=1,2} a_i^+ \frac{\beta_1^+}{\beta_1^+ - \beta_j^-} \ge 1 \cdot \frac{\beta_1^+}{\beta_1^+ - \beta_2^-},$$

and, therefore,

$$\sum_{j=1,2} \frac{a_j^-}{\kappa_q^+(\beta_j^-)(1-\beta_j^-)} < \sum_{j=1,2} a_j^- \frac{\beta_1^+ - \beta_2^-}{\beta_1^+(1-\beta_1^-)} = \frac{\beta_1^+ - \beta_2^-}{\beta_1^+(1-\beta_1^-)}.$$

We also need

$$\kappa_q^-(1) < \frac{\beta_2^-}{\beta_2^- - 1} = \frac{-\beta_2^-}{1-\beta_2^-} < \frac{-\beta_2^-}{1-\beta_1^-}, \quad \frac{1}{\kappa_q^+(1)} < \frac{\beta_1^+ - 1}{\beta_1^+}.$$

Gathering these bounds together, we obtain

$$\begin{aligned} f(0) &= \frac{\kappa_q^-(1)}{\kappa_q^+(1)} + \sum_{j=1,2} \frac{a_j^-}{\kappa_q^+(\beta_j^-)(1-\beta_j^-)} - \frac{I}{C} \\ &< \frac{-\beta_2^+(\beta_1^+-1)}{\beta_1^+(1-\beta_1^-)} + \frac{\beta_1^+-\beta_2^-}{\beta_1^+(1-\beta_1^-)} - \frac{I}{C} \\ &= \frac{1-\beta_2^+}{1-\beta_1^+} - \frac{I}{C} \\ &< 1 - I/C. \end{aligned} \tag{12.6}$$

Since f is convex, the zero y^* can be easily calculated, and $h^{**} = h_* + y^*$ found. After that, we can calculate the value of the investment opportunity for $x < h^{**}$:

$$V_{\mathrm{opt}}(x) = q^{-1}(\mathcal{E}^+ \mathbf{1}_{[h^{**},+\infty)} w)(x) = \sum_{j=1,2} D_j e^{\beta_j^+(x-h^{**})},$$

where $D_j, j = 1, 2$, are constants which can be expressed in terms of the parameters of the model and h^{**}. We leave the details of calculations as an exercise for the reader. If there are no positive jumps (that is, $c^+ = 0$ in (11.6)), then the answer simplifies

$$V_{\mathrm{opt}}(x) = (V(h^{**}) - I)e^{\beta^+(x-h^{**})}.$$

Note that on the strength of (12.6), for all $C \le I$, $f(0) \le (1-\beta_2^+)/(1-\beta_1^+) - 1 < 0$, and $f'(0)$ is bounded by a constant independent of C. It follows that the difference $e^{h^{**}} - e^{h_*}$ between the trigger price of investment and the trigger price of scrapping is bounded away from zero even as the investment becomes completely reversible.

12.2 Embedded options: Russian dolls

12.2.1 Expanding dolls

Consider a firm in a growing industry, which contemplates a multi-stage investment project. On each stage, an additional production facility can be added or a new technology adopted, etc. Assume that the number of stages is finite, say, N, the order of stages is fixed, and investment is irreversible. After investment stage k but before stage $k+1$, the profit flow is $g_k(X_t)$, and the sunk cost of investment at stage k (to move to stage $k+1$) is I_k. At stage 0 (no investment has been made yet), $g_0 \equiv 0$. The time τ^k of making investment at stage k is random; it is chosen by a firm to maximize the EPV of the project. Denote by $V_k(X_t)$ the value of the firm at stage k. At stage N no further investment is expected, therefore $V_N(X_t) = q^{-1}\mathcal{E}g_N(X_t)$ is known. The firm needs to solve the following sequence of optimal stopping problems:

$(\mathbf{RD})^+$. *Find the optimal stopping time* τ_k^* *to swap the stream* $g_k(X_t)$ *for the instantaneous payoff* $V_{k+1}(X_t) - I_k$, $k = N-1, N-2, \dots, 0$.

We will call this sequence of embedded options a *Russian doll.* After the completion of all N stages of investment, the firm can be associated with a Russian doll, containing a sequence of smaller dolls inside. We solve the problem of the expanding firm by backward induction, that is by opening the Russian doll: in order to see the smaller doll (option), we must first remove (resolve) the current one. The Russian doll associated with the expansion of investment project will be called an *expanding Russian doll.*

Assume that for each $k = 0, 1, \dots, N-1$,

(i) function $g_{k+1} - g_k$ does not decrease and admits the bound (11.40);
(ii) $(g_{k+1} - g_k)(-\infty) - qI_k < 0$;
(iii) $(g_N - g_k)(+\infty) - q\sum_{j\le k\le N-1} I_j > 0$.

The bound (11.40) is a regularity condition. If it fails, the firm's value may be infinite. The first part of (i) means that as the stochastic factor increases, the next stage becomes more attractive relative to the current one. If the level of the stochastic factor is too low, then the possible investments at the next stages of the project can be ignored (their contribution to the firm's value is too small), and, therefore, only stage-k investment matters. Condition (ii) means that it is not optimal to make this investment at low levels of the stochastic factor. Condition (iii) can be interpreted as follows: starting at any stage k, after a sufficiently vigorous positive jump of the underlying stochastic factor, the firm will find it optimal to make all possible investments and swap stage-k profit flow for the last stage profit flow $g_N(X_t)$.

Theorem 12.2.1 *Under conditions (i)–(iii), for* $k = N-1, N-2, \dots, 0$, *the following statements hold:*

a) the value function V_k *can be represented as the EPV of a stream* $W_k(X_t) + g_k(X_t)$, *where* W_k *satisfies the following conditions:*
1. W_k *does not decrease and vanishes at* $-\infty$;
2. $W_k(+\infty) = (g_N - g_m)(+\infty) - q\sum_{j\le k\le N-1} I_j$;
b) function

$$w_k := \mathcal{E}^-(W_{k+1} + g_{k+1} - g_k - qI_k) = q(\mathcal{E}^+)^{-1}V_{k+1} - qI_k - \mathcal{E}^- g_k$$

has a unique zero, call it h_k;
c) $\tau_k^* = \max\{\tau_{k-1}^*, \tau_{h_k}^+\}$ *is the optimal time for stage k investment;*
d)

$$\begin{aligned} V_k &= q^{-1}\left(\mathcal{E} g_k + \mathcal{E}^+ \mathbf{1}_{[h_k,+\infty)} w_k\right) \\ &= q^{-1}\left(\mathcal{E}(W_{k+1} + g_{k+1} - qI_k) - \mathcal{E}^+ \mathbf{1}_{(-\infty,h_k)} w_k\right) \\ &= V_{k+1} - I_k + q^{-1}\mathcal{E}^+ \mathbf{1}_{(-\infty,h_k)}(-w_k). \end{aligned}$$

Note that c) means that it is optimal to invest at stage k when X_t reaches h_k from below or crosses h_k the first time after stage $k-1$ investment; hence, it is possible that several investment will be made simultaneously.

Proof. We solve the problem by backward induction. At stage $N-1$, the firm chooses an optimal time to swap the stream $g_{N-1}(X_t)$ for the stream $g_N(X_t) - qI_{N-1}$. According to (i)–(iii), the difference $g_N(X_t) - g_{N-1} - qI_{N-1}$ admits the bound (11.40), and it is a non-decreasing function that changes sign. Hence, Theorem 11.6.11 applies, and we obtain, for $k = N-1$, part b) with $W_{k+1} = W_N = 0$, and then parts c), d) and a1). To prove part a2), we apply $q - L$ to the second equality in d) and obtain

$$\begin{aligned} W_{N-1} + g_{N-1} &= (q-L)V_{N-1} \\ &= -qI_{N-1} + W_N + g_N - (\mathcal{E}^-)^{-1}\mathbf{1}_{(-\infty,h_{N-1})}w_{N-1}. \end{aligned}$$

Since $(\mathcal{E}^-)^{-1}$ is the sum of a local differential operator and a convolution operator with the kernel supported on $(-\infty, 0]$, the last term vanishes above h_{N-1}. Since $W_N = 0$, we obtain $W_{N-1}(+\infty) = -qI_{N-1} + (g_N - g_{N-1})(+\infty)$, which is a2).

Assume now that a) is valid for $k = m+1, \dots, N-1$. Stage-m firm's value is the EPV of the stream $g_m(X_t)$ plus the value of the option to swap the stream $g_m(X_t)$ for the stream $-qI_m + W_{m+1}(X_t) + g_{m+1}(X_t)$. Using (i)–(iii) with $k = m$ and the induction hypothesis a) with $k = m+1$, we obtain that $W^1_m := -qI_m + W_{m+1} + g_{m+1} - g_m$ is a non-decreasing function such that $W^1_m(-\infty) = -qI_m + (g_{m+1} - g_m)(-\infty) < 0$ and $W^1_m(+\infty) = -qI_m + (W_{m+1} + g_{m+1} - g_m)(+\infty) = -q\sum_{m \le j \le N-1} I_j + (g_N - g_{m+1} + g_{m+1} - g_m)(+\infty) > 0$. Hence, Theorem 11.6.11 applies, and we obtain b), c), d) and a) with $k = m$.

Algorithm. Theorem 12.2.1 gives a simple algorithm for calculation of the rational price of the expanding Russian doll:

(1) Calculate $V_N = q^{-1}\mathcal{E}g_N$.
(2) For $k = N-1, N-2, \dots, 0$,
  (i) calculate $w_k = q(\mathcal{E}^+)^{-1}V_{k+1} - qI_k - \mathcal{E}^- g_k$;
  (ii) find a unique root of the non-decreasing function w_k;
  (iii) calculate $V_k = V_{k+1} - I_k + q^{-1}\mathcal{E}^+\mathbf{1}_{(-\infty,h_k)}(-w_k)$.

Numerical realization. Steps (1), (2i) and (2iii) are direct calculations, which are simple and straightforward in the Brownian motion model and the Brownian motion model with embedded jumps given by the Lévy density (11.6). If the payoffs are given by exponential polynomials on non-overlaping intervals U_s such that $\mathbb{R} = \cup_s U_s$, then the functions w_k and V_k enjoy the same properties, and it is possible to derive a rule which calculates the coefficients of the exponential polynomials for w_k given the coefficients of V_{k+1}, and the coefficients of V_k given the coefficients of w_k. The rule is the solution of a system of linear algebraic equations, hence, it is very fast and accurate. See [58, 59, 63], where the procedure is suggested, described in detail and applied to pricing of the American put with finite time horizon. One can also discretize the integro-differential procedures needed to calculate w_k and V_k, and the discretizations are fast and accurate. See [26, 27, 28], where the explicit algorithm are written for regime-switching models. Step (2ii) is also very simple to program, and calculations are fast and accurate.

12.2.2 Contracting dolls

Now we consider a firm in a declining industry, which scraps its production facilities in a predetermined order; timing depends on a realization of uncertainty. Such a firm can be viewed as a Russian doll stripped of larger dolls that contained the current one before the contraction had started. We call the multistage contraction option a *contracting Russian doll.* To obtain the solution in this situation, we assume that the characteristics of the smallest doll are known and then use this information to deduce the characteristics of the sequence of larger dolls (in other words, we assemble the Russian doll).

Let C_k be the scrap value that can be recovered should disinvestment at stage k be made, and g_k is the stream of profits when the firm is at stage k. We assume that there exists the last stage N at which no further disinvestment is possible. The simplest interpretation is that after the last disinvestment, the firm disappears, and its stream of profits g_N and value V_N are zero. However, we formulate and solve the multi-stage disinvestment problem for a general g_N. Then V_N is the EPV of the stream $g_N(X_t)$: $V_N = q^{-1}\mathcal{E}g_N$. The firm needs to solve the following sequence of optimal stopping problems:

> $(\mathbf{RD})^-$. *Find the optimal stopping time $\tau_{*,k}$ to exchange stream $g_k(X_t)$ for the instantaneous payoff $C_k + V_{k+1}(X_t)$, $k = N-1, N-2, \dots, 0$.*

Assume that for each $k = N-1, N-2, \dots, 0$, the following conditions hold

(i) function $g_{k+1} - g_k$ does not increase, and satisfies (11.40);
(ii) $qC_k + g_{k+1}(+\infty) - g_k(+\infty) < 0$;
(iii) $q\sum_{k\le j\le N-1} C_j + (g_N - g_k)(-\infty) > 0$.

The statement and proof of the following theorem are similar to Theorem 12.2.1 and its proof.

Theorem 12.2.2 *Let (i)–(iii) hold. Then for $k = N-1, N-2, \dots, 0$,*

a) the value function V_k can be represented as the EPV of a stream $W_k(X_t) + g_k(X_t)$, where W_k satisfies the following conditions:
1. W_k does not increase and vanishes at $+\infty$;
2. $W_k(-\infty) = q(C_{N-1} + C_{N-2} + \cdots + C_k) + (g_N - g_m)(-\infty)$;

b) function

$$w_k := \mathcal{E}^+(qC_k + W_{k+1} + g_{k+1} - g_k) = qC_k + q(\mathcal{E}^-)^{-1}V_{k+1} - \mathcal{E}^+ g_k$$

has a unique zero, call it h_k;

c) $\tau_{*,k} = \max\{\tau_{,*k-1}, \tau^-_{h_k}\}$ *is the optimal time for stage k disinvestment;*
d)

$$\begin{aligned} V_k &= q^{-1}\left(\mathcal{E}g_k + \mathcal{E}^-\mathbf{1}_{(-\infty,h_k]}w_k\right) \\ &= q^{-1}\left(\mathcal{E}(qC_k + W_{k+1} + g_{k+1}) - \mathcal{E}^-\mathbf{1}_{(h_k,+\infty)}w_k\right) \\ &= C_k + V_{k+1} + q^{-1}\mathcal{E}^-\mathbf{1}_{(h_k,+\infty)}(-w_k). \end{aligned}$$

Proof. We solve the problem by backward induction. At stage $N-1$, the firm chooses an optimal time to swap the stream $g_{N-1}(X_t)$ for the stream $g_N(X_t) + qC_{N-1}$. According to (i)-(iii), the difference $g_N(X_t) + qC_{N-1} - g_{N-1}$ admits the bound (11.40), and it is a non-increasing function that changes sign. Hence, Theorem 11.6.12 applies, and we obtain, for $k = N-1$, b) with $W_{k+1} = W_N = 0$, c), d) and a1). To prove part a2), we apply $q - L$ to the second equality in d) and obtain

$$\begin{aligned} W_{N-1} + g_{N-1} &= (q-L)V_{N-1} \\ &= qC_{N-1} + W_N + g_N - (\mathcal{E}^+)^{-1}\mathbf{1}_{(h_{N-1},+\infty)}w_{N-1}. \end{aligned}$$

Since $(\mathcal{E}^+)^{-1}$ is the sum of a local differential operator and a convolution operator with the kernel supported on $[0,+\infty)$, the last term vanishes below h_{N-1}. Since $W_N = 0$, we obtain $W_{N-1}(-\infty) = qC_{N-1} + (g_N - g_{N-1})(-\infty)$, which is a2).

Assume now that a) is valid for $k = m+1, \ldots, N-1$. Stage-m firm's value is the EPV of the stream $g_m(X_t)$ plus the value of the option to swap the stream $g_m(X_t)$ for the stream $qC_m + W_{m+1}(X_t) + g_{m+1}(X_t)$. Using (i)–(iii) with $k = m$ and the induction hypothesis a) with $k = m+1$, we obtain that $W^1_m := qC_m + W_{m+1} + g_{m+1} - g_m$ is a non-increasing function such that $W^1_m(+\infty) = qC_m + (g_{m+1} - g_m)(+\infty) < 0$ and $W^1_m(-\infty) = qC_m + (W_{m+1} + g_{m+1} - g_m)(-\infty) = q\sum_{m\le j\le N-1} C_j + (g_N - g_{m+1} + g_{m+1} - g_m)(-\infty) > 0$. Hence, Theorem 11.6.12 applies, and we obtain b), c), d) and a) with $k = m$.

Algorithm. Theorem 12.2.2 gives a simple algorithm for calculation of the rational price of the contracting Russian doll:

(1) Calculate $V_N = q^{-1}\mathcal{E}g_N$.
(2) For $k = N-1, N-2, \ldots, 0$,
  (i) calculate $w_k = qC_k + q(\mathcal{E}^-)^{-1}V_{k+1} - \mathcal{E}^+ g_k$;
  (ii) find a unique root of the non-decreasing function w_k;
  (iii) calculate $V_k = C_k + V_{k+1} + q^{-1}\mathcal{E}^-\mathbf{1}_{(h_k,+\infty)}(-w_k)$.

As in the case of an expanding doll, the algorithm admits simple efficient realizations.

12.3 Capital expansion program

12.3.1 Timing an investment of a marginal unit of capital

Consider a monopoly whose production function depends only on capital: $Q = Q(K)$. (A generalization to the case of a production function with costlessly adjustable labor as in [2] is straightforward but leads to more involved formulas below). For simplicity, assume that the inverse demand function is factorizable: $D_t = \bar{D}(Q_t)Z_t$, where Z_t is the exogenous demand shock. We assume that

(i) function $G(Q) := Q\bar{D}(Q)$ is differentiable, increasing, concave and satisfies the Inada conditions;
(ii) $Z_t = Z(X_t)$ is a non-decreasing function of a Lévy process X_t with the Lévy exponent Ψ;
(iii) function Z satisfies the estimate

$$Z(x) \le c_1 \mathrm{e}^{\gamma x}, \quad \forall\, x, \tag{12.7}$$

where $c_1 > 0$ and $\gamma \ge 0$ are independent of x, and the no-bubble condition holds:

$$q - \Psi(\gamma) > 0. \tag{12.8}$$

Remark 12.1. a) Under condition (i), when K_t units of capital are in place, the firm finds it optimal to produce the maximal amount $Q_t = Q(K_t)$, and therefore, the revenue flow is

$$R_t = Q(K_t)\bar{D}(Q(K_t))Z(X_t) = G(K_t)Z(X_t).$$

b) For a jump-diffusion with the Lévy exponent (11.6), (12.8) is equivalent to $\gamma < \beta_1^+$.
c) Conditions (12.7)–(12.8) guarantee that if the firm keeps the level of installed capital fixed: $K_t = K_0$, $\forall\, t$, then the EPV of the revenue flow is finite:

$$E\left[\int_0^{+\infty} \mathrm{e}^{-qt} R(X_t)dt\right] \le \frac{c_1 G(K_0)}{q - \Psi(\gamma)} < \infty. \tag{12.9}$$

Should the firm decide to invest a unit of capital, it suffers the installation cost C; the investment is irreversible. The firm's objective is to choose the optimal investment strategy $\mathcal{K} = \{K_{t+1}(K_t, X_t)\}_{t\ge 1}, K_0 = K, X_0 = x$, which maximizes the NPV of the firm:

$$V(K,x) = \sup_{\mathcal{K}} E^x \left[\int_0^{+\infty} \mathrm{e}^{-qt}(Z(X_t)G(K_t) - qCK_t)dt\right]. \tag{12.10}$$

A similar situation was considered in [39, 2] for the geometric Brownian motion model and extended in [16] for geometric Lévy processes. In these papers, $Z(X_t) = \exp X_t$, and therefore, condition (12.7) holds with $\gamma = 1$. In [39], it is shown that the value of the firm is infinite unless an additional restriction on the rate of growth of function $G(K)$ as $K \to +\infty$ is imposed, and this condition is too restrictive. We will show that if $Z(X_t)$ behaves as $\exp X_t$ up to a certain threshold but above the threshold the rate of growth of $Z(X_t)$ decreases then the restriction on the rate of growth of $G(K)$ can be relaxed. As a by-product, we will show that, as the optimal capital increases, the range within which the monopoly price P_t fluctuates widens slower than in the standard geometric Lévy model. Moreover, we will demonstrate that this range may shrink as the demand shock reaches the intermediate region between the intervals of the fast exponential growth and of the slower growth. This means that the firm may find it optimal to simultaneously increase the capital stock and decrease the price of the output.

For the time being, to ensure that firm's value (12.10) is bounded, we impose a resource constraint: there exists $\bar{K} < \infty$ such that $K_t \le \bar{K}$, $\forall t$. Later, we will show that if γ in (iii) is sufficiently small, then the resource constraint is redundant: the expected rate of growth of the optimal capital is not very large, and the value of the firm is finite even if the firm has unlimited access to capital. Notice that if the demand shock Z is bounded ($\gamma = 0$), then there exists $\bar{K}$ such that the firm would never want to choose $K_t > \bar{K}$.

It is well-known (see, for example, [39]) that in order to determine the optimal capital expansion program, it is only necessary to decide when to invest at any given stock of capital. Equivalently, one needs to find the investment threshold $h(K)$, which is the boundary between two regions in the state variable space (K, x): the action and inaction ones. To derive the equation for the investment boundary, suppose first that every new investment can be made in increments of capital, ΔK, only. In this case, the firm has to suffer the cost $C\Delta K$, and the EPV of the profit gain due to this investment can be represented in the form of the EPV of the stream $g(X_t) = (G(K+\Delta K) - G(K))Z(X_t) - qC\Delta K$. We know that it is optimal to invest capital $C\Delta K$ the first time the demand shock crosses the investment barrier $h = h(K; \Delta K)$ that solves the equation $\mathcal{E}^- g(h) = 0$ (the bad news principle). For g defined above, we obtain

$$\mathcal{E}^- \left[(G(K+\Delta K) - G(K))\, Z(\cdot) - qC\Delta K \right](x) = 0,$$

or

$$(G(K+\Delta K) - G(K))\, \mathcal{E}^- Z(x) = qC\Delta K. \tag{12.11}$$

Dividing (12.11) by ΔK and passing to the limit as $\Delta K \to 0$, we obtain the following equation for the optimal investment threshold $h^* = h^*(K)$:

$$G'(K)(\mathcal{E}^- Z)(h^*) = qC, \tag{12.12}$$

or

$$G'(K) E\left[\int_0^{+\infty} \mathrm{e}^{-qt} Z(h^* + \underline{X}_t) dt \big| X_0 = 0 \right] = C. \tag{12.13}$$

The last equation says that it is optimal to invest into a marginal unit of capital the first time the EPV of the marginal profit, calculated under the assumption that the underlying stochastic process $\{X_t\}$ is replaced by the infimum process $\{\underline{X}_t\}$, becomes non-negative[1].

Let $h = h(K; \Delta K)$ be a solution to (12.11). Then at the shock level x, the option value associated with the chunk of capital ΔK is

$$q^{-1}\mathcal{E}^+\mathbf{1}_{[h,+\infty)}\left[(G(K+\Delta K)-G(K))(\mathcal{E}^-Z)(\cdot)-qC\Delta K\right](x).$$

As $\Delta K \to 0$, we have $h(K;\Delta K) \to h^*(K)$. Notice that capital accumulation extinguishes the option value of investment, this means that the option value is decreasing in K (for more detailed discussion, see [1]). Therefore, dividing the above option value by ΔK and passing to the limit as $\Delta K \to 0$, we obtain the following formula for the marginal option value of capital:

$$V_K^{\text{opt}}(K,x) = -q^{-1}\mathcal{E}^+\mathbf{1}_{[h^*,+\infty)}\left(G'(K)(\mathcal{E}^-Z)(\cdot)-qC\right)(x).$$

Substituting C from (12.12) into the above equation, we arrive at

$$qV_K^{\text{opt}}(K,x) = -G'(K)\mathcal{E}^+\mathbf{1}_{[h^*,+\infty)}[(\mathcal{E}^-Z)(\cdot)-(\mathcal{E}^-Z)(h^*)](x). \quad (12.14)$$

Using the independent random variables $Y^+ = \bar{X}_T$ and $Y^- = X_T - \bar{X}_T \sim \underline{X}_T$ supported on the positive and negative half-axes, respectively, we can write (12.12) and (12.14) in the form

$$G'(K)E[Z(h^*+Y^-)] = qC, \quad (12.15)$$

and

$$qV_K^{\text{opt}}(K,x) = -G'(K)E[\mathbf{1}_{[h^*,+\infty)}(x+Y^+)(Z(x+Y^++Y^-)-Z(h^*+Y^-))]. \quad (12.16)$$

We have proved

Theorem 12.3.1 *Let conditions (i)–(iii) hold. Then the optimal capital expansion threshold $h^* = h^*(K)$ is a unique solution of any of equivalent equations (12.12), (12.13) and (12.15), and the marginal option value of capital is given by any of equivalent equations (12.14) and (12.16).*

Consider the case when X is a jump-diffusion process with the Lévy exponent (11.7). We use (11.32), and rewrite the equation for the threshold in the form

$$G'(K)\sum_{j=1,2}a_j^-\int_{-\infty}^{0}(-\beta_j^-)e^{-\beta_j^-y}Z(h^*+y)dy = qC. \quad (12.17)$$

The marginal option value of capital is

$$\begin{aligned} V_K^{\text{opt}}(K,x) &= -G'(K)\sum_{j=1,2}a_j^+\int_{h^*-x}^{+\infty}\beta_j^+e^{-\beta_j^+y}w(x+y)dy \quad (12.18)\\ &= -G'(K)\sum_{j=1,2}a_j^+e^{\beta_j^+(x-h^*)}\int_0^{+\infty}\beta_j^+e^{-\beta_j^+y}w(h^*+y)dy, \end{aligned}$$

[1] For the rigorous justification of the limiting argument see [16].

where

$$w(x) = q^{-1} \sum_{j=1,2} a_j^- \int_{-\infty}^{0} (-\beta_j^-) \mathrm{e}^{-\beta_j^- y} \left(Z(x+y) - Z(h^* + y) \right) dy.$$

12.3.2 Option value

Integrating (12.18) w.r.t. K, we find the option value

$$V^{\mathrm{opt}}(K, x) = -\int_{K}^{\bar{K}} V_K^{\mathrm{opt}}(K', x) dK'. \tag{12.19}$$

If we want to remove the resource constraint $K \leq \bar{K}$, we need to prove that the limit of the integral (12.19) exists as $\bar{K} \to +\infty$, and then the value of the firm is given by (12.19) with $\bar{K} = +\infty$.

Lemma 12.3.2 *Assume that (12.7) holds, and*

$$\int_{1}^{+\infty} G'(K')^{\beta_1^+/\gamma} dK' < +\infty. \tag{12.20}$$

Then the value of the firm is given by (12.19) with $\bar{K} = +\infty$.

Proof. We need to prove that as $\bar{K} \to +\infty$, the limit of the integral (12.19) exists and it is finite. Since $Z(x)$ satisfies (12.7), we have

$$w(x) \leq c_1 E \left[\int_0^{+\infty} \mathrm{e}^{-qt + \gamma(x + \underline{X}_t)} dt \right] \leq c_1 q^{-1} \kappa_q^-(\gamma) \mathrm{e}^{\gamma x} \leq c_1 q^{-1} \mathrm{e}^{\gamma x}.$$

Therefore

$$\int_0^{+\infty} \mathrm{e}^{-\beta_j^+ y} w(h^* + y) dy \leq c_1 q^{-1} \mathrm{e}^{\gamma h^*} \int_0^{+\infty} \mathrm{e}^{-\beta_j^+ y + \gamma y} dy = \frac{c_1 \mathrm{e}^{\gamma h^*}}{q(\beta_j^+ - \gamma)},$$

and

$$V_K^{\mathrm{opt}}(K, x) \leq \frac{c_1 G'(K) \mathrm{e}^{\gamma h^*}}{q} \sum_{j=1,2} \frac{a_j^+ \beta_j^+}{\beta_j^+ - \gamma} \mathrm{e}^{\beta_j^+ (x - h^*)}.$$

Since $\gamma \in (0, 1]$ and $1 < \beta_1^+ < \beta_2^+$, we obtain

$$V_K^{\mathrm{opt}}(K, x) \leq D(x) G'(K) \mathrm{e}^{(\gamma - \beta_1^+) h^*(K)}, \tag{12.21}$$

where $D(x)$ depends on $x \leq h^*(K)$ but not on K. Next, we notice that if $W(X_t)$ is another demand shock such that $Z(x) \leq W(x)$ for any x, then the corresponding thresholds are related as $h^*(K; Z) \geq h^*(K; W)$. This result follows immediately if one compares (12.17), the equation for the threshold under demand shocks $W(X_t)$,

$$G'(K)\sum_{j=1,2} a_j^- \int_{-\infty}^0 (-\beta_j^-)\mathrm{e}^{-\beta_j^- y} W(h^*+y)dy = qC$$

with the corresponding equation for the threshold under Z:

$$G'(K)\sum_{j=1,2} a_j^- \int_{-\infty}^0 (-\beta_j^-)\mathrm{e}^{-\beta_j^- y} Z(h^*+y)dy \le qC.$$

For $W(x) = c_1\mathrm{e}^{\gamma x}$, we derive from (12.17) $G'(K)\kappa_q^-(\gamma)\mathrm{e}^{\gamma h^*(K,W)} = qC$, therefore the RHS in (12.21) admits a bound via $D_1(x)G'(K)^{\beta_1^+/\gamma}$, and we conclude that (12.20) is a sufficient condition for the convergence of the integral (12.19) with $\bar{K} = +\infty$.

In the geometric Lévy case, we obtain $V_K^{\mathrm{opt}}(K,x) = D_1(x)G'(K)^{\beta_1^+/\gamma}$, where $\gamma = 1$, therefore (12.20) is necessary as well. In particular, if $G(K) = dK^\theta$ ($d > 0$, $\theta \in (0,1)$), then, for the convergence of the integral in the case of the jump-diffusion process, we must have $\theta < 1 - \gamma/\beta_1^+$. In other words, θ must be sufficiently less than one, which means that the returns to capital must decrease sufficiently fast. As Dixit and Pindyck (1996) show in the geometric Brownian motion case, for typical parameters of a process, this condition requires for θ to be too small. If the jump component is not very strong, then β_1^+ is close to the one in the geometric Brownian motion case, and the same conclusion holds.

Now, suppose that up to a moderate level of demand, the demand shock is fitted well by a geometric jump-diffusion process with $\gamma = 1$, and $\theta \ge 1 - 1/\beta_1^+$. To ensure that the value of the firm is finite, we may assume that above a certain high level $\bar{Z}$ of the stochastic factor $Z(X_t)$, the rate of growth of $Z(X_t)$ slows down, and (12.7) holds with sufficiently small $\gamma > 0$ so that $\theta < 1 - \gamma/\beta_1^+$. Then the integral (12.20) converges, and the value of the firm is finite, even if the resource constraint is dropped. Finally, assume that Z is uniformly bounded from above: $Z(x) \le c_2$, which implies that the demand shocks are bounded. Then the LHS in (12.13) admits an upper bound via $G'(K)c_2q^{-1}$. Since G satisfies the Inada conditions, $G'(K) \to 0$ as $K \to +\infty$. Hence, for sufficiently large K, the LHS in (12.13) will be smaller than the RHS for any h^*, and it is not optimal to increase the capital stock above a certain level. The resource constraint becomes redundant.

12.3.3 Non-standard shape of the boundary between the action and inaction regions

Consider the Cobb-Douglas production function $Q_t = dK_t^\rho$, where $d, \rho > 0$, and the inverse demand function $P_t = Z_tQ_t^{-1/\epsilon}$, where $Z_t = Z(X_t)$ is the demand shock, and $\epsilon > 1$ is the elasticity of demand. Then $G(K) = d^{1-1/\epsilon}K^{\rho(1-1/\epsilon)}$, and the above results apply provided $\theta := \rho(1-1/\epsilon) \in (0,1)$, and Z satisfies condition (iii). We consider two families of functions Z; the process X_t is a jump-diffusion process with the Lévy exponent (11.6).

Example 12.2. First, consider the geometric Lévy case $Z(X_t) = \mathrm{e}^{\gamma X_t}$, where $\gamma > 0$. Condition (12.8) is equivalent to $\gamma < \beta_1^+$. If there is no exogenous bound on the amount of capital available, then the value of the firm is finite iff $\theta = \rho(1-1/\epsilon) < 1-\gamma/\beta_1^+$. This means that for a given $\gamma > 0$, either ρ or ϵ must be sufficiently small. However, if $\rho \le 1 - \gamma/\beta_1^+$, then the elasticity of demand may assume any value $\epsilon > 1$. The revenue flow is $R_t = (dK^\rho)^{1-1/\epsilon}\mathrm{e}^{\gamma X_t}$, and equation (12.12) for the investment threshold becomes

$$d^{1-1/\epsilon}\rho(1-1/\epsilon)K^{\rho(1-1/\epsilon)-1}\kappa_q^-(\gamma)\mathrm{e}^{\gamma h^*} = Cq. \tag{12.22}$$

The description of the optimal investment policy in terms of the demand shock is standard: when a point (X_t, K) remains to the left of the boundary (12.22) of the inaction region: $X_t \le h^*(K)$, the monopoly keeps the capital level $K_t = K$ fixed and increases or decreases the price of the output as the demand increases or decreases. When the demand shock factor X_t crosses level $h^*(K)$, the firm increases the capital stock to the new level K' so that $X_t = h^*(K')$, and (X_t, K') is on the boundary of the inaction region. At this moment, the firm increases the price, decreases it, or keeps it fixed, if the production technology exhibits decreasing returns to scale, increasing returns to scale, or constant returns to scale, respectively[2]. Indeed, when the demand shock $Z(X_t)$ is at the investment threshold, the monopoly charges price

$$P^* = P^*(K, h^*(K)) = (dK^\rho)^{-1/\epsilon}Z(h^*(K)) = \frac{K^{1-\rho}}{d\rho(1-1/\epsilon)\kappa_q^-(\gamma)},$$

and the RHS increases in K if $\rho < 1$, decreases if $\rho > 1$, and remains constant if $\rho = 1$. The smaller the $\gamma > 0$, the larger is $\kappa_q^-(\gamma) = E[\mathrm{e}^{\gamma Y^-}]$, and the lower is the output price at the moment of investment.

Example 12.3. Consider the following demand shock. As $Z(X_t)$ remains below a certain critical value $\bar{Z}$, the dynamics of the stochastic factor is given by the geometric Lévy process:

$$Z(X_t) = \bar{Z}\mathrm{e}^{X_t}, \quad X_t \le 0. \tag{12.23}$$

However, in the region above the critical level $\bar{Z}$, the rate of growth of $Z(X_t)$ slows down:

$$Z(X_t) = \bar{Z}[\gamma^{-1}(\mathrm{e}^{\gamma X_t} - 1) + 1], \quad X_t > 0, \tag{12.24}$$

where $\gamma \in (0, 1)$. In the limit $\gamma \to 1$, we recover the standard geometric Lévy case; in the limit $\gamma \to 0$, the shock follows the geometric Lévy process below 0, and the Lévy process above 0.

[2] Of course, we understand that the technology may exhibit increasing returns to scale only locally, for small levels of capital. We mention the price behavior for increasing returns to scale production function only because in numerical experiments we observe similar behavior for decreasing returns to scale technology and small rate of growth of the demand shock, when the demand is in a certain range.

Consider the equation (12.13) for the investment threshold. Since function $Z = Z(x)$ is monotone, $(\mathcal{E}^- Z)(x)$ also is. Hence, (12.13) has a unique solution, $h^* = h^*(K)$. If $h^* \le 0$, then the LHS of (12.13) is independent of the values of $Z(x)$ for positive x, hence h^* is determined from the same equation as in the geometric Lévy case:

$$d^{1-1/\epsilon}\rho(1-1/\epsilon)K^{\rho(1-1/\epsilon)-1}E\left[\int_0^{+\infty} e^{-qt}\bar{Z}e^{h^*+\underline{X}_t}dt\right] = C, \tag{12.25}$$

which is

$$d^{1-1/\epsilon}\rho(1-1/\epsilon)K^{\rho(1-1/\epsilon)-1}\kappa_q^-(1)\bar{Z}e^{h^*} = qC. \tag{12.26}$$

From (12.26), it is evident that $h^* \le 0$ iff

$$d^{1-1/\epsilon}\rho(1-1/\epsilon)K^{\rho(1-1/\epsilon)-1}\kappa_q^-(1)\bar{Z} \ge qC.$$

Let $d^{1-1/\epsilon}\rho(1-1/\epsilon)K^{\rho(1-1/\epsilon)-1}\kappa_q^-(1)\bar{Z} < qC$, then (12.26) has no non-positive solutions. Therefore, the investment threshold h^* is positive, and we have to use both (12.23) and (12.24). We calculate $(\mathcal{E}^- Z)(x)$ for $x > 0$:

$$\mathcal{E}^- Z(x) = \bar{Z}\left[\gamma^{-1}\kappa_q^-(\gamma)e^{\gamma x} - \gamma^{-1}(1-\gamma) + \sum_{j=1,2} d_{\gamma,j}e^{\beta_j^- x}\right], \tag{12.27}$$

where $d_{\gamma,j}$ are positive constants (for details and explicit formulas for $d_{\gamma,j}$, see the end of the subsection). The investment threshold is the solution of (12.12). Using (12.27), we write equation (12.12) in the form

$$d^{1-1/\epsilon}\rho(1-1/\epsilon)K^{\rho(1-1/\epsilon)-1}\bar{Z}\left[\frac{\kappa_q^-(\gamma)e^{\gamma h^*}}{\gamma} - \frac{1-\gamma}{\gamma} + \sum_{j=1,2} d_{\gamma,j}e^{\beta_j^- h^*}\right] = qC \tag{12.28}$$

In the upper panel of Fig. 12.1, we plot the graph of $Z(x)$ for $\gamma = 0.999$ (which is close to the geometric Lévy case $\gamma = 1$), $\gamma = 0.6$ and $\gamma = 0.3$. In the middle panel, we plot the boundary of the inaction region in the (Z, K)-plane. Finally, in the lower panel, we plot the boundary of the inaction region in the (P, K)-plane. Here, as a natural technical device, we use the explicit parameterization of the curve (K, P^*) by h^*: $K = K(h^*)$ is found from (12.26) for $h^* \le 0$, and from (12.28) for $h^* > 0$, and after that we calculate $P^* = (dK^*)^{-1/\epsilon}Z(h^*) = (dK(h^*))^{-1/\epsilon}Z(h^*)$. We take $\rho = 0.9$ (decreasing returns to scale case). As $h^*(K) \le 0$ (which implies that K is below a certain level), the threshold is the same for all γ, and the boundary in the (P, K)-plane is upward sloping which means that each increase of capital stock is accompanied by an increase in the price of the output. For larger values of K, the boundary depends on γ, and it may be even locally downward sloping, which means that an *increase* in the capital stock may be accompanied by a *decrease* in the output price. The business returns to normality at sufficiently large levels of capital stock:

once again, an increase in the capital stock is accompanied by an increase in the output price. To see this, we derive an approximate formula for the threshold in the region of large K. As $K \to \infty$, $\mathcal{E}^- Z(h^*) = qC/G'(K) \to \infty$, hence $e^{h^*(K)} \to \infty$, and $\mathcal{E}^- Z(h^*) \sim \bar{Z}\gamma^{-1}\kappa_q^-(\gamma)e^{\gamma h^*}$. Now we can write an approximate equation

$$d^{1-1/\epsilon}\rho(1-1/\epsilon)K^{\rho(1-1/\epsilon)-1}q^{-1}\bar{Z}\gamma^{-1}\kappa_q^-(\gamma)e^{\gamma h^*} = C$$

and obtain

$$P^* = (dK^\rho)^{-1/\epsilon}Z(h^*) \sim (dK^\rho)^{-1/\epsilon}\frac{\bar{Z}}{\gamma}e^{\gamma h^*} \sim \frac{qC}{\kappa_q^-(\gamma)d\rho(1-1/\epsilon)}K^{1-\rho}.$$

The smaller the $\gamma > 0$, the larger is $\kappa_q^-(\gamma)$, and the lower is the output price at the moment of investment. In Fig. 12.1, we see that the effect of the

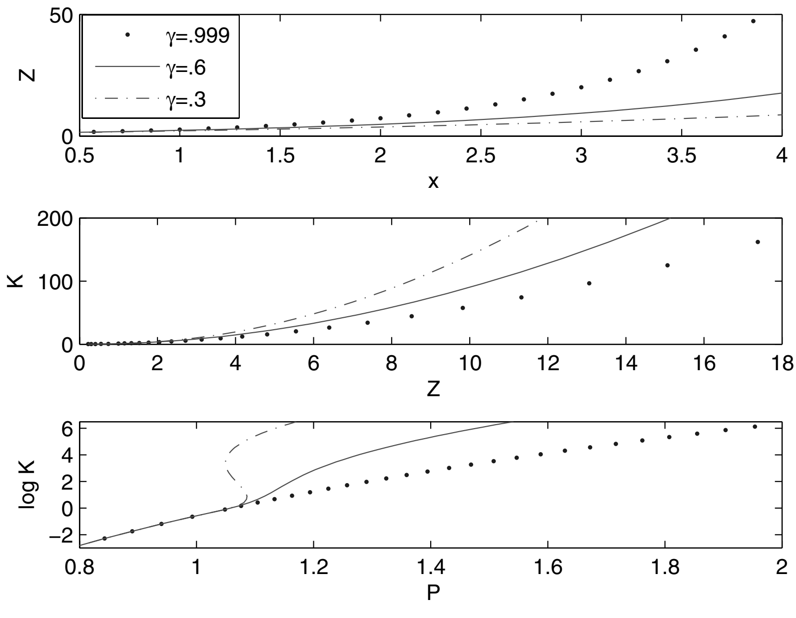

Fig. 12.1. Upper panel: dependence of the demand shock $Z_t = Z(X_t)$ (Example 12.3) on the Lévy process. Middle panel: the boundary of the inaction region in (Z, K)-plane. Lower panel: the boundary of the inaction region in (P, K)-plane. Discount rate: $q = 0.08$. Marginal cost: $C = 2$. Elasticity of demand: $\epsilon = 2$. Parameters of the production function: $d = 1, \rho = 0.9$. Parameters for model (11.7): $\sigma^2 = 0.2, b = -0.6, c^- = 0.10, \lambda^- = -2, c^+ = 0$.

decrease of the monopoly price at the moment of investment is observed when the production function exhibits almost constant returns to scale ($\rho = 0.9$), and the demand shock grows slowly above a certain level ($\gamma = 0.3$). The same effect can be observed for smaller ρ but then the rate of growth of the demand shock must be smaller as well - see Fig. 12.2, where $\rho = 0.85$. The effect is not observed for $\gamma = 0.3$ anymore but it is observed for $\gamma = 0.1$.

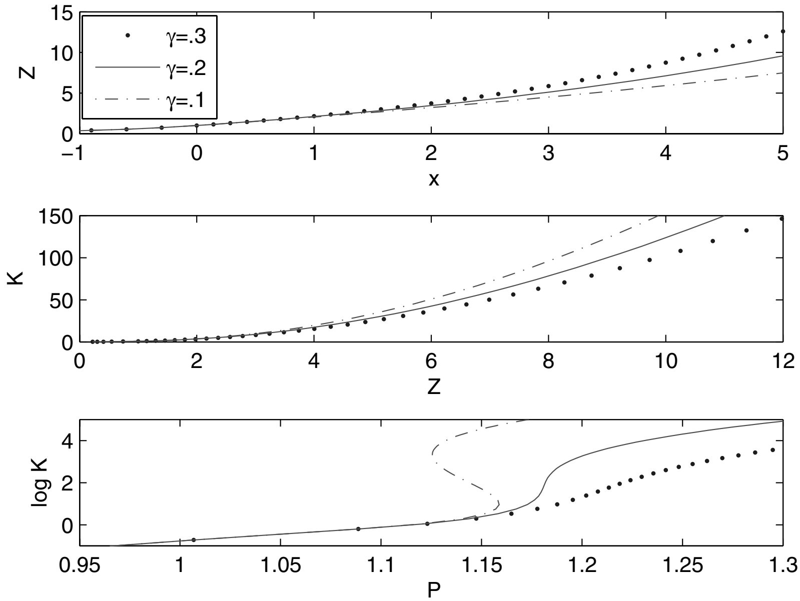

Fig. 12.2. Upper panel: dependence of the demand shock $Z_t = Z(X_t)$ (Example 12.3) on the Lévy process. Middle panel: the boundary of the inaction region in (Z, K)-plane. Lower panel: the boundary of the inaction region in (P, K)-plane. Discount rate: $q = 0.08$. Marginal cost: $C = 2$. Elasticity of demand: $\epsilon = 2$. Parameters of the production function: $d = 1, \rho = 0.85$. Parameters for model (11.7): $\sigma^2 = 0.2, b = -0.6, c^- = 0.10, \lambda^- = -2, c^+ = 0$.

Proof of (12.27). Without loss of generality, $Z_c = 1$. We have

$$\begin{aligned}\mathcal{E}^- Z(x) &= \sum_{j=1,2} a_j^- \int_{-\infty}^{0} (-\beta_j^-) e^{-\beta_j^- y} Z(x+y) dy \\ &= \sum_{j=1,2} a_j^- e^{\beta_j^- x} \int_{-\infty}^{x} (-\beta_j^-) e^{-\beta_j^- y} Z(y) dy = \sum_{j=1,2} a_j^- e^{\beta_j^- x} f_j(x),\end{aligned}$$

where $f_j(x)$ equals

$$\int_{-\infty}^{0}(-\beta_j^-)e^{-\beta_j^- y}e^y dy + \int_0^x(-\beta_j^-)e^{-\beta_j^- y}(\gamma^{-1}e^{\gamma y} + (1-\gamma^{-1}))dy$$
$$= \frac{-\beta_j^-}{1-\beta_j^-} + \frac{-\beta_j^-}{\gamma(\gamma-\beta_j^-)}(e^{(\gamma-\beta_j^-)x} - 1) - \frac{1-\gamma}{\gamma}(e^{-\beta_j^- x} - 1)$$
$$= \frac{(-\beta_j^-)}{1-\beta_j^-} - \frac{(-\beta_j^-)}{\gamma(\gamma-\beta_j^-)} + \frac{1-\gamma}{\gamma} + \frac{(-\beta_j^-)e^{(\gamma-\beta_j^-)x}}{\gamma(\gamma-\beta_j^-)} - \frac{(1-\gamma)e^{(-\beta_j^-)x}}{\gamma}.$$

Using

$$\sum_{j=1,2}\frac{(-\beta_j^-)a_j^-}{\gamma(\gamma-\beta_j^-)} = \gamma^{-1}\kappa_q^-(\gamma), \quad \sum_{j=1,2}a_j^- = \kappa_q^-(0) = 1,$$

we obtain (12.27) with $d_{\gamma,j} = a_j^-(1-\gamma)/(\gamma(1-\beta_j^-))$.

12.4 New technology adoption

In this section, we assume that the manager of a firm chooses not only the optimal capital stock, but also the optimal timing of an upgrade to the frontier technology. This model is more complicated than the ones of the previous sections because it is driven by two factors: one characterizes the dynamics of the technology frontier, and the other incorporates all other shocks in the economy. Powerfully, the method of the paper preserves the tractability even in this two-factor model. Timing new technology adoption is one of the applications where it is essential to model a stochastic technology factor as a process with jumps, because the new technology is not introduced continuously. We believe that the most important component in the evolution of the technology frontier is a compound Poisson process with upward jumps, with possible inclusion of a small diffusion component. One may think about the diffusion component in the technological process as moderate innovations in technology, which may be caused by (or lead to) small fluctuations in non-technological uncertainty; in this case, the interaction between the technological factor and (small) innovations to non-technological factor is modelled as in the standard Gaussian model. However, major technological breakthroughs should be modelled as a jump process, and then it is natural to presume that if there is a correlation between technological and non-technological factors, it should be described by a bivariate jump process.

A natural assumption is that the capital adjustment when the same technology is in place is less costly than the adoption of the new technology; the extreme assumption is that the capital adjustment is costless.

12.4.1 Model specification

We follow fairly closely the setup of [3]. There are no costs of adjustment of the stock of capital, and the stock is chosen optimally, therefore we may concentrate solely on the timing of adoption of the frontier technology. Let A_t be the technology in place, and $\hat{A}_t$ be the frontier technology at date t. Suppose that the updating happens at stopping times $\tau_1 < \tau_2 < \cdots$, so that between the updates the level of technology remains constant: for $t \in [\tau_{i-1}, \tau)$, $A_t = A_{\tau_{i-1}}$. We take the inverse demand function $P_t = Z_t Q_t^{-1/\epsilon}$ as the primitive of the model, assume that the marginal cost of capital is constant (normalized to 1 for simplicity), and the production function is $Q_t = d_t K_t^\rho$, where $\rho > 0$, and d_t is the factor which is determined by the technology in place. Solving for the optimal level of capital between technology updates, we find $C_t = (\alpha\rho)^{-1} d_t^\alpha (\rho(\epsilon-1)/\epsilon Z_t)^\beta$, where $\alpha = (\epsilon-1)/(\epsilon - \rho(\epsilon-1))$ and $\beta = \epsilon/(\epsilon - \rho(1\epsilon - 1))$ are positive constants. Hence, the firm's cash flow is $A_t S_t$, where $A_t = d_t^\alpha$ and $S_t = (\alpha\rho)^{-1}(\rho(\epsilon-1)/\epsilon Z_t)^\beta$.

Updating to the frontier technology is costly, and the cost of updating is proportional to the updated cash stream: $\theta A_{\tau_i} S_{\tau_i}$, $\theta \in (0,1)$. Let $V(A_{\tau_{i-1}}, \hat{A}_t, S_t)$ be the value of the firm net of the value of its capital stock for $t \in [\tau_{i-1}, \tau)$. Following [3], we look for the value function of the form

$$V(A_{\tau_{i-1}}, \hat{A}_t, S_t) = A_{\tau_{i-1}} S_t V^1(\hat{A}_t / A_{\tau_{i-1}}), \tag{12.29}$$

and assume that updating occurs when the ratio $\hat{A}_t / A_{\tau_{i-1}}$ reaches a certain threshold, call it A^*. In [3], the technological factor $\hat{A}_t$ and non-technological factor S_t are modelled as geometric Brownian motions: $\hat{A}_t / A_{\tau_{i-1}} = \mathrm{e}^{a_t}$, $S_t = \mathrm{e}^{X_t}$, where (a_t, X_t) is a two-dimensional Gaussian process with the non-trivial correlation between components. We assume that $\hat{A}_t / A_{\tau_{i-1}} = \mathrm{e}^{X_t^1}$, $S_t = \mathrm{e}^{X_t^2}$, where $X_t = (X_t^1, X_t^2)$ is a two-dimensional Lévy process driven by compound Poisson processes and two independent standard Brownian motions W_t^1 and W_t^2. To be more specific, we model X_t as the solution of the stochastic differential equation

$$d\begin{bmatrix} X_t^1 \\ X_t^2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} dt + \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} dW_t^1 \\ dW_t^2 \end{bmatrix} + \sum_k \begin{bmatrix} 1 \\ \gamma_k \end{bmatrix} dJ_{c_k, \lambda_k; t}, \tag{12.30}$$

where $c_k > 0$, $\lambda_k > 0$, $\gamma_k \in \mathbb{R}$, and $J_{c,\lambda;t}$ denotes the compound Poisson process with the Lévy density $c\mathrm{e}^{-\lambda x} \mathbf{1}_{(0,+\infty)}(x)$. We may identify $\sum_k J_{c_k,\lambda_k;t}$ as the jump component of the innovation process (creation of essentially new technologies), and then γ_k describe the impact of unexpected innovations on the dynamics of the non-technological factor. If $\gamma_k < 0$ (respectively, $\gamma_k > 0$), then a positive jump in the frontier technology is accompanied by a negative (respectively, positive) jump in the non-technological stochastic factor. The diffusion part of the process describes small fluctuations in the non-technological factor and related fluctuations in minor technological improvements. If $\sigma_{12} = 0$, then σ_{21} describes the impact of the process of small technological innovations on small fluctuations in the non-technological uncertainty, and if $\sigma_{21} = 0$, then σ_{12} describes the impact of the latter on the former. The Lévy exponent of X_t, $\Psi(z) = \Psi(z_1, z_2)$, is defined by

$$E\left[e^{\langle z,\, X_t\rangle}\right] = E\left[e^{z_1 X_t^1 + z_2 X_t^2}\right] = e^{t\Psi(z)}.$$

For the process given by (12.30),

$$\Psi(z) = \frac{1}{2}||\Sigma' z||^2 + (b,\, z) + \int_{\mathbb{R}\setminus\{0\}} \left(e^{(z,\, y)} - 1\right) F(dy), \tag{12.31}$$

where $\Sigma = [\sigma_{jk}]$; $b = (b_1, b_2)$ and $\Sigma\Sigma'$ are the drift and variance-covariance matrix of the Gaussian component of the process, and

$$F(dy) = \sum_k c_k \lambda_k e^{-\lambda_k y_1} \mathbf{1}_{[0,\, +\infty)}(y_1) \delta_0(y_2 - \gamma_k y_1) dy_1 \tag{12.32}$$

is the Lévy density. Here δ_0 is the one-dimensional Dirac delta-function.

Without loss of generality, set $\tau_{i-1} = 0$ and denote $\tau = \tau_i =$ $= \inf\{t > 0 |\, X_t^1 \geq h\}$, where $h = \log A^*$. Then the value of the firm satisfies

$$V(A_0, \hat{A}_t, S_t) = E_t\left[\int_t^\tau e^{-q(s-t)} A_0 S_s ds\right]$$
$$+ E_t\left[e^{-q(\tau-t)}\left(V(\hat{A}_\tau, \hat{A}_\tau, S_\tau) - \theta \hat{A}_\tau S_\tau\right)\right].$$

Substitute (12.29) into the last equation and divide the latter by $A_0 S_t$. Let

$$v(X_t) = V^1\left(e^{X_t^1}\right) = V^1(\hat{A}_t / A_0).$$

At the time of updating, $A_\tau = \hat{A}_\tau$, hence $V^1(\hat{A}_\tau / A_\tau) = V^1(1) = v(0)$. Before the next updating, we have

$$v(X_t) = E_t\left[\int_t^\tau e^{-q(s-t) + X_s^2 - X_t^2} ds\right] \tag{12.33}$$
$$+ E_t\left[e^{-q(\tau-t) + X_\tau^2 - X_t^2} e^{X_\tau^1}(v(0) - \theta)\right].$$

12.4.2 One source of uncertainty

First, we consider the case when only innovations to technology occur, i.e., the factor X_t^2 is constant. The underlying stochastic process is a one-dimensional Lévy process. Examining only technological innovations is not only instructive by itself, but as we will show it in the next subsection, the general case reduces to this special case. Of course, the Lévy exponent of a one-dimensional process that appears after the reduction is made depends on the Lévy exponent of the initial two-dimensional process. In Subsect. 12.4.3, we will discuss the impact of interaction between the two components of the process on the new technology adoption threshold.

Let h be the threshold for updating. The objective of the firm is to choose h so as to maximize the value

$$v(x;h) = E\left[\int_t^{\tau} \mathrm{e}^{-q(s-t)} ds |\, X_t = x\right] \qquad (12.34)$$
$$+E\left[\mathrm{e}^{-q(\tau-t)}\mathrm{e}^{X_\tau}(v(0;h)-\theta)|\, X_t = x\right].$$

To ensure that the value of the firm is finite, assume that $q - \Psi(1) > 0$.

Lemma 12.4.1 *Equation (12.34) can be written in the form*

$$v(x;h) = q^{-1} + \left(\mathcal{E}^{+}\mathbf{1}_{[h,+\infty)}\left[\kappa_q^{+}(1)^{-1}(v(0;h)-\theta)\mathrm{e}^{\cdot} - q^{-1}\right]\right)(x), \qquad (12.35)$$

where $\mathrm{e}^{\cdot}$ *denotes the exponential function* $x \mapsto \mathrm{e}^{x}$.

Proof. If $X_t = x$, then

$$E_t\left[\int_t^{\tau} \mathrm{e}^{-q(s-t)} ds\right] = E_t\left[\int_t^{+\infty} \mathrm{e}^{-q(s-t)} ds\right] - E_t\left[\int_\tau^{+\infty} \mathrm{e}^{-q(s-t)} ds\right]$$
$$= q^{-1} - q^{-1}(\mathcal{E}^{+}\mathbf{1}_{[h,+\infty)}\mathcal{E}^{-}1)(x)$$
$$= q^{-1} - q^{-1}(\mathcal{E}^{+}\mathbf{1}_{[h,+\infty)})(x).$$

Next, we use the fundamental relationship between the infinitesimal generator and the EPV-operator to write the payoff $\mathrm{e}^{X_\tau}(v(0;h)-\theta)$ as the normalized EPV of a stream $g(x) = q^{-1}(q-L)\mathrm{e}^{x}(v(0;h)-\theta)$, substitute $\mathcal{E}g(X_\tau)$ into (12.34), and write the second term in (12.34) as

$$\left(\mathcal{E}^{+}\mathbf{1}_{[h,+\infty)}\mathcal{E}^{-}\right)g(x) = \left(\mathcal{E}^{+}\mathbf{1}_{[h,+\infty)}\mathcal{E}^{-}q^{-1}(q-L)(v(0;h)-\theta)\mathrm{e}^{\cdot}\right)(x)$$
$$= \left(\mathcal{E}^{+}\mathbf{1}_{[h,+\infty)}(\mathcal{E}^{+})^{-1}(v(0;h)-\theta)\mathrm{e}^{\cdot}\right)(x)$$
$$= \left(\mathcal{E}^{+}\mathbf{1}_{[h,+\infty)}(v(0;h)-\theta)\kappa_q^{+}(1)^{-1}\mathrm{e}^{\cdot}\right)(x).$$

Now it becomes possible to rewrite (12.34) in the form (12.35).

Introduce

$$v_{\mathrm{opt}}(x;h) = v(x;h) - q^{-1}.$$

Recall that given the new technology is adopted as the threshold h is reached or crossed, the value of the firm is

$$V(A_0, \hat{A}_t, S_t; h) = A_0 S_t v(X_t;h) = \frac{A_0 S_t}{q} + A_0 S_t v_{\mathrm{opt}}(X_t;h).$$

The first term, $A_0 S_t/q$, is the EPV of the stream of profits, which the firm will generate provided the current technology stays in place forever, and the second term is the option value of upgrading to the frontier technology. In order to find the option value, we rewrite (12.35) in terms of $v_{\mathrm{opt}}(x;h)$:

$$v_{\mathrm{opt}}(x;h) = \left(\mathcal{E}^{+}\mathbf{1}_{[h,+\infty)}\left[\frac{(v_{\mathrm{opt}}(0;h)+q^{-1}-\theta)\mathrm{e}^{\cdot}}{\kappa_q^{+}(1)} - \frac{1}{q}\right]\right)(x). \qquad (12.36)$$

Suppose for a moment that we know the value $V_0 := v_{\rm opt}(0; h)$ at the moment of updating. Assuming that $V_0 + q^{-1} - \theta > 0$ (a sufficient condition is $q\theta < 1$, that is, the cost of updating is not too high), and arguing as in the proof of the theorem for the perpetual American call option, we conclude that the optimal updating threshold h satisfies

$$\kappa_q^+(1)^{-1}(v_{\rm opt}(0; h) + q^{-1} - \theta)e^h - q^{-1} = 0. \tag{12.37}$$

Using (12.37), we can simplify (12.36) for $x < h$:

$$\begin{aligned} v_{\rm opt}(x; h) &= e^{-h} \left(q^{-1}\mathcal{E}^+\mathbf{1}_{[h,+\infty)}\left(e^{\cdot} - e^h\right)\right)(x) \\ &= \left(q^{-1}\mathcal{E}^+\mathbf{1}_{[h,+\infty)}\left(e^{\cdot-h} - 1\right)\right)(x). \end{aligned} \tag{12.38}$$

Equation (12.37) has two unknowns: h and $v_{\rm opt}(0; h)$, however we can add the second equation by letting $x = 0$ in (12.38):

$$v_{\rm opt}(0; h) = \left(q^{-1}\mathcal{E}^+\mathbf{1}_{[h,+\infty)}\left(e^{\cdot-h} - 1\right)\right)(0). \tag{12.39}$$

Substituting (12.39) into (12.37), and multiplying by $q\kappa_q^+(1)$, we obtain the equation for h:

$$e^h \left(\mathcal{E}^+\mathbf{1}_{[h,+\infty)}\left(e^{\cdot-h} - 1\right)\right)(0) + (1 - q\theta)e^h - \kappa_q^+(1) = 0. \tag{12.40}$$

We claim that if $q\theta < 1$, then this equation has a unique solution on $(0, +\infty)$. Indeed, as $h \to +\infty$, the LHS tends to $+\infty$, and at $h = 0$, the LHS is negative:

$$(\mathcal{E}^+(e^{\cdot} - 1))(0) + (1 - q\theta) - \kappa_q^+(1) = \kappa_q^+(1) - 1 + (1 - q\theta) - \kappa_q^+(1) = -q\theta < 0.$$

Hence, a solution exists, and to see that it is unique, it suffices to check that the LHS in (12.40) is convex. We will verify this, and obtain explicit formulas for h and $v_{\rm opt}(0; h)$ after we specify a process for the frontier technology.

Suppose that X is a diffusion process with exponentially distributed upward jumps. The Lévy density is

$$F(dy) = c\lambda e^{-\lambda y}\mathbf{1}_{(0,+\infty)}(y)dy, \tag{12.41}$$

where $c > 0$ and $\lambda > 1$ (the last inequality is necessary for the inequality $r - \Psi(1) > 0$ to hold, which ensures the finiteness of the value function). Then the Lévy exponent is $\Psi(z) = \sigma^2 z^2/2 + bz + cz/(\lambda - z)$, and the inequality $r - \Psi(z) > 0$ is satisfied provided $q > \sigma^2/2 + b + c/(\lambda - 1)$. The characteristic equation has three roots: $\beta^- < 0 < 1 < \beta_1^+ < \lambda < \beta_2^+$. The factor $\kappa_q^-(z)$ is defined by $\kappa_q^-(z) = \beta^-/(\beta^- - 1)$, and $\kappa_q^+(z)$ is given by (11.24) or (11.28). The value $v_{\rm opt}(x; h)$ satisfying (12.38) can be computed in exactly the same manner as the value of the perpetual American call option:

$$v_{\rm opt}(x; h) = q^{-1} \sum_{j=1,2} \frac{a_j^+ e^{\beta_j^+(x-h)}}{\beta_j^+ - 1}, \text{ for } x < h,$$

and (12.40) assumes the form

$$\sum_{j=1,2} \frac{a_j^+ e^{(1-\beta_j^+)h}}{\beta_j^+ - 1} + (1 - q\theta)e^h - \kappa_q^+(1) = 0. \tag{12.42}$$

Denote by $f(h)$ the LHS in (12.42). We have shown for the general case above that $f(h)$ changes sign on $(0, +\infty)$, and the root of (12.42) exists. To show the uniqueness of the root, we prove that f is convex:

$$f''(h) = \sum_{j=1,2} a_j^+(\beta_j^+ - 1)e^{(1-\beta_j^+)h} + (1 - q\theta)e^h > 0, \ \forall\ h.$$

12.4.3 Two sources of uncertainty

For simplicity, assume that there is only one term in the jump component. Set $c = c_k, \lambda = \lambda_k, \gamma = \gamma_k$, assume that $\gamma < \lambda - 1$, and denote by a_{jk} the entries of the variance-covariance matrix $\Sigma\Sigma'$. In [25], we show that the new technology adoption threshold in the two-factor model (12.30) is the same as in the one-factor model with the characteristic exponent

$$\Psi^1(z_1) = \frac{a_{11}}{2}z_1^2 + b^1 z_1 + \frac{c^1 z_1}{\lambda^1 - z_1}, \tag{12.43}$$

where $b^1 = a_{12} + b_1$, $c^1 = c\lambda/(\lambda - \gamma)$, and $\lambda^1 = \lambda - \gamma$. To ensure that the value of the firm is finite, we need to impose two conditions, which in the case of one jump component assume the form

$$q^1 := q - \frac{a_{22}}{2} - b_2 - \frac{c\gamma}{\lambda - \gamma} > 0, \tag{12.44}$$

and

$$q - \frac{a_{11}}{2} - a_{12} - \frac{a_{22}}{2} - b_1 - b_2 - \frac{c(1+\gamma)}{\lambda - \gamma - 1} > 0. \tag{12.45}$$

Notice that both (12.44) and (12.45) imply that γ cannot be too close to λ, equivalently, if positive technological jumps are accompanied by vigorous positive jumps in the non-technological factor, then the value of the firm becomes infinite: the prospects are too good to be true. Probably, the advocates of the New Economy had in mind similar models for shocks in technology and non-technological uncertainty. We also need to require $1 - q^1\theta > 0$; if this condition is violated, then new technology adoption is never optimal.

If the Gaussian component in the dynamics of the technology frontier is non-trivial, then the characteristic equation has three roots $\beta^- < 0 < 1 < \beta_1^+ < \lambda < \beta_2^+$, and the equation for the technology adoption frontier is (cf. (12.42))

$$\sum_{j=1,2} \frac{a_j^+ e^{(1-\beta_j^+)h}}{\beta_j^+ - 1} + (1 - q^1\theta)e^h - \kappa_{q^1}^+(1) = 0, \tag{12.46}$$

where a_j^+ and $\kappa_{q^1}^+(1)$ are defined by the same formulas as a_j^+ and $\kappa_q^+(1)$ above but with q^1 in place of q. The existence and uniqueness of the solution h of (12.46) is proved in Subsect. 12.4.2.

12.4.4 Dependence of the new technology adoption threshold, A^*, on diffusion and jump uncertainty

We start with the study of the dependence of A^* on the jump component when the technological process has no Gaussian component: $\sigma_{11} = \sigma_{12} = \sigma_{21} = 0$. For the calculation of A^* in this case, see [25]. First, we fix the Gaussian component of the non-technological factor, σ_{22}, and change c, λ and γ. Then we fix λ, and change c, σ_{22}, and γ. The increase in c means that the total uncertainty of the technological factor increases, the increase in λ^{-1} means that the average jump size becomes larger, and the increase in σ_{22} means the increase in non-technological uncertainty. Finally, the increase in γ means that the correlation between the two factors goes up. The numerical results obtained in [25] show that the new technology adoption threshold is

(a) an increasing function of (c, λ^{-1}), that is, of the uncertainty in the technological factor, and average jump size;
(b) a decreasing function of σ_{22}, that is, of the uncertainty in the non-technological factor;
(c) a decreasing function of the "correlation coefficient", γ, between the jump components in the technological and non-technological factors.

Thus, the uncertainty in the technological factor and uncertainty in the non-technological one affect the threshold in opposite directions. The dependence on the technological uncertainty can be naturally explained in the framework of the record-setting news principles as follows. In a situation similar to the call option with an instantaneous (random) payoff, the record-setting good news principle applies, and the higher the uncertainty of good news, the higher is the threshold. Clearly, this is the situation with new technology adoption: once the new technology is in place, it remains fixed for a lengthy period of time. The feature (b) is not as transparent as (a). According to the record-setting news principles, if the option gives the right to a *stream* of payoffs (a cash flow here), then the record-setting bad news principle applies, and the higher the uncertainty of bad news (the lower the trajectories of the infimum process), the higher is the threshold. It may seem that the increase in σ_{22} means the increase in the overall uncertainty in S_t, the non-technological factor, hence in the uncertainty of bad news, and so the threshold should increase. Notice, however, that the threshold is derived for the technological factor, but not for S_t, and the standard intuition may be non-applicable. If σ_{22} increases, then $b_2 + \sigma_{22}^2/2$, the rate of growth of S_t increases; therefore, the higher the expected rate of growth of S_t (hence, of the revenue), the sooner should the firm take the advantage of adoption of the frontier technology.

The reader may wonder if the difference between the ways the new technology factor and non-technological one influence the threshold is an artifact of the different ways these factors are modelled: pure jump process and diffusion process with embedded jumps, respectively. The next series of numerical examples in [25] demonstrates how the adoption threshold changes if we add the diffusion component to technological process so that the Gaussian uncertainty in the non-technological factor drives the Gaussian uncertainty in the technological factor (similar effects are observed when the latter driver the former). We also show the threshold when there is no Gaussian uncertainty in the technological factor. The conclusions (a)–(c) made above remain valid. The new technology adoption threshold is

(a) an increasing function of the uncertainty in the technological factor;
(b) a decreasing function of the uncertainty in the non-technological factor;
(c) a decreasing function of the “correlation coefficient", γ, between the jump components in the technological and non-technological factors;
(d) an increasing function of the covariance coefficients, σ_{12} and σ_{21}, between the Gaussian components in the technological and non-technological factors.

Notice the important difference between the impact of the “correlation" between the Gaussian and non-Gaussian sources of uncertainty on the threshold: A^* is a *decreasing* function of the “correlation coefficient", γ, between the jump components in the technological and non-technological factors, and an *increasing* function of the correlation coefficients σ_{12} and σ_{21} between the Gaussian components of technological and non-technological innovations. Hence, the interaction between Gaussian sources of uncertainty, and the one between non-Gaussian sources of uncertainty are not just quantitatively different: they are of *opposite signs*.

Problems

12.1. Consider the two-stage investment problem. Assume that on stage $j = 1, 2$, the revenue flow is $G_j e^x$, and variable cost is C_j, where $G_2 > G_1$ and $C_2 > C_1$ (the productivity and variable cost at stage 1 are higher than the ones at stage 0). The fixed investment costs are I_0 and I_1, respectively. Formulate the conditions of Theorem 12.2.1 in terms of G_j, C_j and I_j, solve the investment problem. Derive the necessary and sufficient conditions which ensure that the firm will not invest into the second stage at once (suffering cost $I_0 + I_1$).

12.2. Solve Prob. 12.2 assuming that there is an additional source of uncertainty: at stages 0 and 1, the manager does not know G_2 for sure. She believes that with probability p, G_2 will be high: $G_2 = G_{2,h}$, and with probability $1 - p$, it will be low: $G_2 = G_{2,l}$.

12.3. Solve Prob. 12.3 assuming that the stage-2 profit flow is capped: $\max\{\Pi_2, G_2 e^x - C_2\}$, where $\Pi_2 > 0$ is a constant.

12.4. Formulate and solve the problem of irreversible incremental capital contraction program.

Part V

Extensions

13

American options with finite time horizon

In the finite time horizon case, exact formulas for the early exercise boundary are not available even in the Brownian motion case. There are several approximate methods – see, e.g., the discussion and references in [58, 59, 23]. An approximate method for a finite horizon problem is based on the time discretization. One of the first methods of this kind is the analytical method of lines suggested in [34]; another interpretation of the same pricing procedure was given in [33]. In [34, 33], the Brownian motion case was considered. In Chapt. 6 of [21], the method is generalized for wide classes of Lévy processes, and in [58], an efficient pricing procedure for the put under exponential jump-diffusion processes was suggested. For different generalizations of Carr's randomization method, see [7, 53]. The main results of this Chapter are obtained in [21, 23], however, the proof given in this Chapter is simpler than in op. cit. In the last section, we consider the behavior of the early exercise boundary near maturity. Starting with [9, 54], the behavior of the critical stock price near maturity for American options in diffusion models has been studied in a number of publications – see, e.g., [51, 41, 55, 35]. One of the standard and well-known conclusion is that the early exercise boundary for the American put on a non-dividend paying stock converges to the strike at expiry. However, the dynamics of prices of stocks and bonds in real financial markets has a significant jump component. Therefore, it is important to understand the impact of jumps on the early exercise boundary. We formulate a general result about the gap between the early exercise boundary and the strike and show that in the presence of jumps, for realistic parameter values, the gap may exists even for the American put on a non-dividend paying stock. This is in the stark contrast with diffusion models.

13.1 Call option

Consider the American call-like option with maturity date T and payoff $G(X_t)$, where X_t is a Lévy process satisfying the (ACP)-condition and (11.25). We assume that

(i) $g = (q - L)G$ is defined a.e. and satisfies (11.40);
(ii) g does not decrease and changes sign.

Hence, $G(X_t)$ is the EPV of the stream $g(X_t)$: $G = q^{-1}\mathcal{E}g$.

Example 13.1. For the standard call option, $G(x) = \mathrm{e}^x - K$, therefore we may take $\sigma^- = 0, \sigma^+ = 1$. Condition (11.25) is equivalent to $q - \Psi(1) > 0$, and then $g(x) = (q - L)G(x) = (q - \Psi(1))\mathrm{e}^x - qK$ satisfies (i)–(ii).

Recall that the EPV-operators can be defined for any $q > 0$; since we will use these definition with $q' > q$ which will be defined later, we will explicitly indicate the dependence on q: $\mathcal{E}_q$. Thus, $G = q^{-1}\mathcal{E}_q g = (q - L)^{-1}g$, and $g = g_q := (q - L)G$.

Lemma 13.1.1 *Let g_q satisfy (11.40) and (ii). Then*

a) G does not decrease, changes sign and satisfies (11.40);
b) for any $q' > q$, function $g_{q'} = (q' - L_j)G$ satisfies (11.40) and (ii).

Proof. a) Operator $\mathcal{E}$ is monotone therefore $G = q^{-1}\mathcal{E}_q g_q$ is non-decreasing. From Lemma 10.2.5, G satisfies (11.40) (and its derivatives of order 1 and 2 satisfy this bound as well). Applying Proposition 10.2.3, we find that $G(-\infty) < 0 < G(+\infty)$ because g satisfies these inequalities.

b) Use a) and the equality

$$(q' - L)G = (q - L)G + (q' - q)G = g_q + (q' - q)G.$$

Let T be the maturity date. We divide the period $[0, T]$ into n sub-periods $[t_j, t_{j+1}]$, where $0 = t_0 < t_1 < \cdots < t_n = T$. Set $\Delta_s = t_{s+1} - t_s$, $q^s = \Delta_s^{-1} + q$, $s = 0, 1, \ldots, N - 1$. In the Carr's randomization method, the early exercise boundary is approximated by a "staircase". Over each time interval (t_s, t_{s+1}), the boundary is flat: $x = h^s$. The h^s and approximation $v^s(x)$ to $V(t_s, x)$ are found using backward induction. For $s = N$, $v^s(x) = G(x)_+$, and for $s = N - 1, N - 2, \ldots$, h^s is chosen to maximize

$$v^s(x) = E^x\left[\int_0^{\tau^s} \mathrm{e}^{-q^s t}\Delta_s^{-1} v^{s+1}(X_t)dt\right] + E^x\left[\mathrm{e}^{-q^s\tau^s}G(X_{\tau^s})\right], \qquad (13.1)$$

where τ^s is the hitting time of $[h^s, +\infty)$. (We write $G(X_{\tau^s})$ instead of $G(X_{\tau^s})_+$ because it is not optimal to exercise the option when the payoff is negative). In [21], Chapt. 2, it is shown that v^s given by (13.1) solves the boundary value problem

$$(q^s - L)v^s(x) = \Delta_s^{-1} v^{s+1}(x), \quad x < h^s, \tag{13.2}$$
$$v^s(x) = G(x), \quad x \geq h^s. \tag{13.3}$$

Note that (13.2)–(13.3) is the time-discretization of the free boundary problem for the American put with finite time horizon (the so-called analytical method of lines).

Introduce $\tilde{v}^s = v^s - G$, and substitute $v^s = \tilde{v}^s + G$ and $v^{s+1} = \tilde{v}^{s+1} + G$ into (13.1):

$$\tilde{v}^s(x) + G(x) = E^x \left[\int_0^{\tau^s} e^{-q^s t} \Delta_s^{-1} \tilde{v}^{s+1}(X_t) dt \right]$$
$$+ E^x \left[\int_0^{\tau^s} e^{-q^s t} \Delta_s^{-1} G(X_t) dt \right] + E^x \left[e^{-q^s \tau^s} G(X_{\tau^s}) \right].$$

Using the Dynkin's formula

$$G(x) = E^x \left[\int_0^{\tau^s} e^{-q^s t} (q_s - L) G(X_t) dt \right] + E^x \left[e^{-q^s \tau^s} G(X_{\tau^s}) \right],$$

which is applicable on the strength of part b) of Lemma 13.1.1, we simplify

$$\tilde{v}^s(x) = E^x \left[\int_0^{\tau^s} e^{-q^s t} (\Delta_s^{-1} \tilde{v}^{s+1} + (q - L)(-G))(X_t) dt \right].$$

Clearly, the optimization of v^s given v^{s+1} is equivalent to the optimization of $\tilde{v}^s$ given $\tilde{v}^{s+1}$.

Theorem 13.1.2 *Let (i)–(ii) hold. Then for $s = N - 1, N - 2, \ldots, 0$,*

(a) function $\tilde{w}^s = \mathcal{E}_{q^s}^-(\Delta_s^{-1} \tilde{v}^{s+1} + (q - L)(-G))$ is a non-increasing function that has a unique zero, call it h^s;
(b) $\tau_{h^s}^+$ is an optimal stopping time;
(c) $\tilde{v}^s = (1/q^s)\mathcal{E}_{q^s}^+ \mathbf{1}_{(-\infty,h^s)} \tilde{w}^s$;
(d) $v^s = G + \tilde{v}^s$;
(e) $\tilde{v}^s$ is a non-increasing function that vanishes above h^s.

Proof. For $s = N - 1$, $\tilde{v}^{s+1} = G_+ - G = (-G)+$ is a non-increasing function that is 0 above $h^N := \max\{x \mid G(x) \leq 0\}$. Applying part a) of Lemma 13.1.1, we find that $\Delta_s^{-1} \tilde{v}^{s+1} + (q - L)(-G)$ is a non-increasing function that changes sign. Therefore, the optimization of $\tilde{v}^s (= \tilde{v}^{N-1})$ is the problem of optimal abandonment of a non-increasing stream that changes sign and satisfies (11.40). Hence, for $s = N - 1$, the statements a), b) and c) are the statements of Theorem 11.6.5. Part d) follows from the definition $\tilde{v}^s = v^s - G$. Part e) follows from a) and c). Now, assume that e) has been proved for $s = N - 1, \ldots, m + 1$. Then $\Delta_s^{-1} \tilde{v}^{m+1} + (q - L)(-G)$ is a non-increasing function that changes sign, and the statements a), b) and c) are the statements of Theorem 11.6.5. Part d) follows from the definition $\tilde{v}^s = v^s - G$.

Remark 13.2. Theorem 13.1.2 contains a simple and efficient algorithm for calculation of the early exercise boundary and option price. For a very fast computational realization of the algorithm, see [58, 59, 63].

13.2 Put option

Consider the American put-like option with maturity date T and payoff $G(X_t)$, where X_t is a Lévy process satisfying the (ACP)-condition and (11.25). First, we consider the case which is the mirror reflection of the problem for the call option above. We assume that

(i) $g = (q - L)G$ is defined a.e. and satisfies (11.40);
(ii) g does not increase and changes sign.

Hence, $G(X_t)$ is the EPV of the stream $g(X_t)$: $G = q^{-1}\mathcal{E}g$, which excludes options on a non-dividend paying stock. Set $v^N = G_+$, and denote by v^s the Carr's randomization approximation to the option price for $t \in [t_{s-1}, t_s)$.

Theorem 13.2.1 *Let (i)–(ii) hold. Then for $s = N-1, N-2, \dots, 0$,*

(a) function $\tilde{w}^s = \mathcal{E}^+_{q^s}(\Delta_s^{-1}\tilde{v}^{s+1} + (q-L)(-G))$ is a non-decreasing function that has a unique 0, call it h^s;
(b) $\tau^-_{h^s}$ is an optimal stopping time;
(c) $\tilde{v}^s = (1/q^s)\mathcal{E}^-_{q^s}\mathbf{1}_{(h^s,+\infty)}\tilde{w}^s$;
(d) $v^s = G + \tilde{v}^s$;
(e) $\tilde{v}^s$ is a non-decreasing function that vanishes below h^s.

The proof is the straightforward modification of the proof of Theorem 13.1.2. The details are left as an exercise for the reader.

Now consider the case of the put option on a non-dividend paying stock. In the standard geometric Lévy model, $G(x) = K - e^x$, and, if the stock does not pay dividends, the no-arbitrage condition implies that $g_q := (q-L)G(x) = qK$ and $q - \Psi(1) = 0$. Thus, (11.25) fails and $G \neq q^{-1}\mathcal{E}_q g_q$. However, for any $q' > q$, (11.25) holds with q' in place of q:

$$q' - \Psi(z) > 0 \quad z \in [\sigma^-, \sigma^+], \tag{13.4}$$

and, therefore, $G = (q')^{-1}\mathcal{E}_{q'}g_{q'}$. Since we used this equality with $q^s > q$, the equality $q - \Psi(1) = 0$ is not a problem but we need to reformulate the conditions (i)–(ii). We assume that

(1) G is non-increasing, changes sign and satisfies (11.40);
(2) for $g_q = (q-L)G$ is defined a.e. and satisfies (11.40);
(3) g_q does not increase and $g_q(-\infty) > 0$;
(4) there exists $q' > 0$ such that (13.4) holds.

Example 13.3. Consider the standard put option with the payoff $G(x) = K - e^x$. Then $q - \Psi(1) > 0$ and $(q - L)G(x) = qK - (q - \Psi(1))e^x$ if the stock pays dividends, and $q - \Psi(1) = 0$ and $(q - L)G(x) = qK$ if the stock does not pay dividends. In both cases, (1)–(4) hold.

Theorem 13.2.2 *Let (1)–(4) hold, and Δ_s are sufficiently small. Then the statements (a)–(e) of Theorem 13.2.1 hold.*

Proof. If we take Δ_s small enough, then $q + \Delta_s^{-1} > q'$, and, therefore, (13.4) holds with q^s in place of q'. Further, $\tilde{v}^N := v^N - G = G_+ - G = (-G)_+$ is a non-decreasing function that vanishes in a neighborhood of $-\infty$ and satisfies (11.40). Using (ii), we obtain that $\tilde{v}^N + (q - L)(-G)$ is a non-decreasing function that changes sign and satisfies (11.40). Similarly, if $\tilde{v}^s$ satisfies (11.40), does not decrease and vanishes at $-\infty$, then $\tilde{v}^s + (q - L)(-G)$ is a non-decreasing function that changes sign and satisfies (11.40). This means that we can repeat the proof of Theorem 13.2.1.

13.3 Gap between the early exercise boundary and strike

Consider the American put with the strike price K and maturity T, on a non-dividend paying stock; the riskless rate $q > 0$ is constant. Let $H(t) = H(q, K, T; t)$ be the optimal exercise price of the American put. If the stock log-price $X_t = \log S_t$ follows the Brownian motion and the stock pays no dividends, then the early exercise boundary converges to strike at expiry:

$$\lim_{q \to T} H(q, K, T; t) = K. \tag{13.5}$$

See, e.g., [9, 54]. On the other hand, if $q = 0$, then it is non-optimal to exercise the put before expiry at any price level, and it can be shown that for any $t < T$,

$$H(q, K, T; t) \to 0 \quad \text{as } q \to 0. \tag{13.6}$$

Notice that (13.5) and (13.6) do not agree well when both the riskless interest rate and time to expiry vanish. In [21, 58], as a by-product of the Carr's randomization method, it was shown that for many families of non-Gaussian Lévy processes used in empirical studies of financial markets, the analogs of (13.5) and (13.6) agree much better. Namely,

$$\lim_{t \to T} H(q, K, T; t) = H_T(q, K), \tag{13.7}$$

where $H_T(q, K)$ depends on parameters which characterize the price process, and, in many cases, is *smaller* than the strike price, K. Moreover, it was proved that $H_T(q, K)$ vanishes with q, and therefore,

$$H(q, K, T; t) \to 0, \quad \text{as } q \to 0, \ \ uniformly\ in\ \ t \in [0, T). \tag{13.8}$$

Equation (13.7) states that in the presence of jumps, the gap between the strike and early exercise boundary for the American put on a non-dividend paying stock may increase from 0 to $K - H_T(q, K) > 0$. For the American put and call on a dividend paying stock, non-zero gaps may exist even in the Brownian motion case, and addition of jumps may increase these gaps. We formulate the results for the American call and put options and explain the main idea of the proof used in [60, 59]. In [61], similar results are obtained for wide classes of multi-dimensional Markov processes with jumps.

13.3.1 European options at expiry

Consider the European option with the payoff $g_+(X_T) = \max\{0, g(X_T)\}$; for the European call on a stock with the spot price e^{X_t}, and strike K, $g(x) = e^x - K$, and for the put, $g(x) = K - e^x$. Denote by $\mathcal{C}(g_+; x, \tau)$ the option price at time $\tau > 0$ to expiry and $X_{T-\tau} = x$, and set

$$\mathcal{C}(g_+; x) = \lim_{\tau \to +0} \frac{\mathcal{C}(g_+; x, \tau)}{\tau}. \tag{13.9}$$

In finance, the derivative of the price of an option w.r.t. t is called the option's theta. Thus, $\mathcal{C}(g_+; x) = -\theta$ at expiry. For x in the the out-of-the-money region, using the semi-group representation of the European option price $\mathcal{C}(g_+; x, \tau) = (e^{-\tau(q-L)} g_+)(x)$ and the Taylor formula, we obtain informally:

$$\tau^{-1}\mathcal{C}(g_+; x, \tau) = \tau^{-1}(1 - \tau(q - L) + O(\tau^2))g_+(x),$$

where $O(\tau^2)$ denotes a function that vanishes as τ^2 as $\tau \to +0$. Since $g(x) = 0$ in the out-of-the-money region, we can pass to the limit and obtain $\mathcal{C}(g_+; x) = Lg_+(x)$. The application of the differential part of the infinitesimal generator to g_+ gives 0 in the region $\{x \mid g(x) < 0\}$, and we obtain

$$\mathcal{C}(g_+; x) = \int g_+(x + y)F(dy), \tag{13.10}$$

where $F(dy)$ be the density of jumps of the underlying Lévy process X_t. If X_t is a Lévy process without the diffusion component and drift, and the Lévy density $F(dy)$ is of the class L_1, then L is a bounded operator in $\mathcal{L}_\infty$, and the argument above constitutes a rigorous proof. For more general Lévy processes, the proof is more involved. Using the duality argument, (13.10) is proved in [61] for almost all x in out-of-the money region. Under additional regularity conditions, which are satisfied for many classes of Lévy processes (and, more generally, Markov processes with jumps), in particular, for processes with the Lévy densities given by exponential polynomials, (13.10) holds for all x in the out-of-the money region. For the European call and put options, (13.10) assumes the form

$$\mathcal{C}_{\rm call}(x) = \int_{\mathbb{R}} ({\rm e}^{x+y} - K)_+ F(dy), \quad x < \log K, \tag{13.11}$$

$$\mathcal{C}_{\rm put}(x) = \int_{\mathbb{R}} (K - {\rm e}^{x+y})_+ F(dy), \quad x > \log K. \tag{13.12}$$

For the Lévy process with the Lévy density (11.6), the integrals in (13.11)–(13.12) are easy to calculate. Normalizing K to 1, we have

$$\mathcal{C}_{\rm call}(x) = \frac{c^+ {\rm e}^{\lambda^+ x}}{\lambda^+ - 1}, \quad x < 0 \tag{13.13}$$

$$\mathcal{C}_{\rm put}(x) = \frac{c^- {\rm e}^{\lambda^- x}}{1 - \lambda^-}, \quad x > 0. \tag{13.14}$$

13.3.2 Gap for the American call option

We compare the instantaneous payoff $g(x)$ and the value $\mathcal{C}(g_+; x, \tau)$ of the European option with the effective payoff g_+. Clearly, it is non-optimal to exercise the American option if the value of the European one exceeds $g(x)$. Since $g_+ = g + (-g)_+$, the standard no-arbitrage argument gives the pairity relation $\mathcal{C}(g_+; x, \tau) = \mathcal{C}(g; x, \tau) + \mathcal{C}((-g)_+; x, \tau)$, where $\mathcal{C}(g; x, \tau)$ is the price of the forward contract with the payoff $g(X_T)$, at time $T - \tau$. It is not optimal to exercise the option if the following difference is positive:

$$\begin{aligned}\mathcal{C}(g_+; x, \tau) - g(x) &= \mathcal{C}(g; x, \tau) + \mathcal{C}((-g)_+; x, \tau) - g(x) \\ &= (\mathcal{C}(g; x, \tau) - g(x)) + \mathcal{C}((-g)_+; x, \tau).\end{aligned}$$

From the definition of the infinitesimal generator,

$$\mathcal{C}(g; x, \tau) - g(x) = {\rm e}^{-\tau(q-L)} g(x) - g(x) = \tau(L - q) g(x) + o(\tau),$$

therefore using (13.9) with $(-g)_+$, we obtain

$$\mathcal{C}(g_+; x, \tau) - g(x) = \tau((L - q) g(x) + \mathcal{C}((-g)_+; x)) + O(\tau^2),$$

as $\tau \to +0$. If $(L - q) g(x) + \mathcal{C}((-g)_+; x) > 0$ and $\tau > 0$ is small, then the RHS is positive, and the early exercise is not optimal.

Consider the American call option on a stock that pays dividends. Then $q - \Psi(1) > 0$ (recall that if a stock does not pay dividends, then it is optimal not to exercise the option until expiry). Using (13.12), we obtain that the American call option with strike K should not be exercised in the in-the-money region $x > \log K$ if $\int_{\mathbb{R}} (K - {\rm e}^{x+y})_+ F(dy) + qK - (q - \Psi(1)){\rm e}^x > 0$. Consider the equation

$$\int_{\mathbb{R}} (K - {\rm e}^{x+y})_+ F(dy) + qK - (q - \Psi(1)){\rm e}^x = 0. \tag{13.15}$$

Since $q-\Psi(1)>0$, the LHS in (13.15) decreases and changes sign. Therefore, there exists a unique solution of (13.15), denote it $h^*_{\lim}$. The argument above shows that it is non-optimal to exercise the American call option if $S=e^x<\max\{K, e^{h^*_{\lim}}\}$ up to expiry. Using a similar argument for the discrete time approximation, when the exercise is allowed at times $0, \tau, 2\tau, \dots$ to expiry, we obtain that in the limit $\tau \to +0$, it is optimal to exercise the option if $(K-e^{x+y})_+F(dy)+qK-(q-\Psi(1))e^x<0$ and τ is sufficiently small. Thus, (13.15) defines the limit of the early exercise boundary for the American call at expiry, in the log-price space. In the Brownian motion case, (13.15) gives the well-known result: for the call option, the limit of the early exercise boundary at expiry is $S^*_{\lim}=\max\{K, qK/(q-\Psi(1))\}$.

Consider a family of stocks with the same dividend rate $\delta=q-\Psi(1)$, and the call options on these stocks with the same strike K. It follows from (13.3.2), that an addition of *negative jumps* may lead to a higher limit of the early exercise boundary at expiry; if there is a gap in the Brownian motion case, and sufficiently large negative jumps are possible, then the gap widens. The result may seem counterintuitive: why an addition of negative jumps increases the option value of waiting for American call? The reason is that the dividend rate is fixed. If we increase the negative jumps, then, to keep $q-\Psi(1)$ fixed, we must increase the volatility or drift or positive jump component. Essentially, the result which we obtain means that the contribution of the positive changes is higher than the negative ones (provided $q-\Psi(1)$, the dividend rate, does not change).

Example 13.4. For the model (11.6), the limit of the early exercise boundary is the maximim of K and S^*, the unique solution of the equation

$$\frac{c^-}{1-\lambda^-}(S/K)^{\lambda^-}+q-\delta S/K=0.$$

13.3.3 Gap for the American put option

Using the condition $(L-q)g(x)+\mathcal{C}((-g)_+;x)>0$ for no-exercise in the in-the-money region $g(x)>0$ and (13.12), we obtain that the American put option with the strike K should not be exercised in the in-the-money region $x<\log K$ if

$$\int_{\mathbb{R}}(e^{x+y}-K)_+F(dy)+(q-\Psi(1))e^x-qK>0. \tag{13.16}$$

Consider the equation

$$\int_{\mathbb{R}}(e^{x+y}-K)_+F(dy)+(q-\Psi(1))e^x-qK=0. \tag{13.17}$$

The LHS is negative in a neighborhood of $-\infty$ and does not decrease. If $\int_0^\infty(e^y-1)F(dy)\le q$ and the stock does not pay dividends: $\delta=q-\Psi(1)=0$, then the LHS is negative, and (13.17) has no solution. In this case, the limit of the early exercise boundary is the strike. If either $\delta > 0$ or $\int_0^{+\infty}(e^y - 1)F(dy) > qK$ then (13.17) has a unique solution, call it $h_{*,\lim}$. Then the limit of the early exercise boundary is $S_{*,\lim} = \min\{K, e^{h_{*,\lim}}\}$. In the Brownian motion case, $S_{*,\lim} = \min\{K, qK/(q - \Psi(1))$. In particular, for the put on a non-dividend paying stock, $q - \Psi(1) = 0$, and the limit is the strike.

If $\int_0^{+\infty}(e^y - 1)F(dy) > q$, then the limit is below the strike even if $q - \Psi(1) = 0$. Moreover, if $\int_0^{+\infty}(e^y - 1)F(dy) > 0$ and $q - \Psi(1) = 0$, then $h_{*,\lim} = h_{*,\lim}(q) \to -\infty$ as $q \to +0$. Thus, if the riskless rate vanishes, the density of negative jumps is non-trivial and fixed, and the other parameters of the process change so that the no-arbitrage condition for a stock that does not pay dividends holds: $q - \Psi(1) = 0$, then the early exercise boundary in (t, S) space tends to 0 uniformly in $t \in [0, T)$.

Example 13.5. For the model (11.6), the limit of the early exercise boundary is the minimum of K and S_*, the solution of the equation

$$\frac{c^+}{\lambda^+ - 1}(S/K)^{\lambda^+} - q + \delta S/K = 0.$$

If $\delta = 0$, then $S_* = K(q(\lambda^+ - 1)/c^+)^{1/\lambda^+}$.

14

Perpetual American and real options under Ornstein–Uhlenbeck processes

In this chapter, we calculate optimal exercise boundaries and rational prices for perpetual American call and put options, and solve entry and exit problems when the underlying uncertainty is modelled as an exponential Ornstein–Uhlenbeck process. The solution is almost as simple as in the case of an exponential (geometric) Brownian motion although the equations for the optimal exercise boundary are more involved. Surprisingly, for the standard perpetual American call and put options, the general formulas for the optimal exercise boundary and option price in terms of the EPV-operators under supremum and infimum processes derived in the monograph for processes with i.i.d. increments turns out to be valid for Ornstein–Uhlenbeck processes as well. For the entry and exit problems, the general optimal exercise rules can be used as approximations. We provide numerical examples to demonstrate that the exact and formal approximate results agree reasonably well.

14.1 The model

Let $r > 0$ be the constant riskless rate, and assume that the log-price X_t on the stock follows the Ornstein–Uhlenbeck process given by

$$dX_t = \kappa(\theta - X_t)dt + \sigma dW_t, \tag{14.1}$$

where $\kappa > 0$, $\sigma > 0$ and $\theta \in \mathbb{R}$. We demonstrate below that this model is only marginally more involved than the standard geometric Brownian motion model. Indeed, for a perpetual American option with the (instantaneous) pay-off $G(X_t)$, the solution is obtained in three steps. First, we write the free boundary problem for the value function in the inaction region and exercise boundary. Next, we make a change of the variable and unknown function to obtain a simpler free boundary problem. The new equation is the so-called

Weber equation, and its general solution is a linear combination of the Weber–Hermite parabolic cylinder functions $D_{-\nu}(\pm z)$. Returning to the initial unknown, we express the value function as a linear combination of two functions $e^{z^2/4}D_{-\nu}(\pm z)$. The next two steps are exactly the same as in the Brownian motion case: we notice that $e^{z^2/4}D_{-\nu}(\pm z)$ is unbounded as $z \to \mp\infty$ and vanishes as $z \to \pm\infty$, hence one of the arbitrary constants must be 0, and then we use the value matching and smooth pasting conditions to eliminate the remaining arbitrary constant and obtain the equation for the exercise boundary. The analytical formula for the derivative of $e^{z^2/4}D_{-\nu}(\pm z)$ is available (in terms of $e^{z^2/4}D_{-\nu-1}(\pm z)$), hence both the value matching and smooth pasting conditions are easy to write.

Remark 14.1. The geometric mean-reverting processes typically used in the literature (see e.g. [39, 67, 32]) are given as solutions to the stochastic differential equation of the form

$$dP_t = \kappa(\bar{P} - P_t)dt + \sigma P_t dW_t, \tag{14.2}$$

where W_t is the standard Brownian motion, and $\kappa > 0$ is the coefficient of mean-reversion to the long-run central tendency $\bar{P}$. This type of modelling leads to complicated calculations and serious technical difficulties. For instance, the solution in [67] stops at the final remark 15 on p.1488. This remark states that it remains to use the value matching and smooth pasting condition to find three unknowns, and since unlike the geometric Brownian motion model, there is no obvious way to eliminate one of the unknowns, the authors suggest some unclear numerical procedure to overcome this difficulty.

To solve the exit or entry problems in the real option framework, an additional step is needed: given the profit flow $g(X_t)$ of the project and the current value $x = X_0$, calculate its expected present value $G(x)$. The calculation is straightforward, and the result is expressed in terms on an integral, which can be calculated numerically quite easily.

14.2 Perpetual call option

Consider the perpetual American call option on the stock with the strike price K. Let $H = e^h$ be the optimal exercise price. Since it is not optimal to exercise the option when $P_t - K = e^{X_t} - K$ is negative, we may write the free boundary problem for the rational price $V(X_t)$ of the option as

$$(r - \frac{\sigma^2}{2}\partial^2 - \kappa(\theta - x)\partial)V(x) = 0, \quad x < h, \tag{14.3}$$

$$V(h) = e^h - K, \tag{14.4}$$

$$V'(h) = e^h, \tag{14.5}$$

where $\partial = d/dx$, so that $\partial V(x) = V'(x)$. Naturally, we also impose the condition

$$V(x) \to 0, \quad \text{as } x \to -\infty. \tag{14.6}$$

To solve the free boundary problem (14.3)–(14.6), we fix h, a candidate for the exercise log-price, set $\bar\sigma = \sigma/\sqrt{2\kappa}$, and change the variable $z = (x-\theta)/\bar\sigma$, and unknown function $V(x) = \mathrm{e}^{z^2/4}w(z)$. Equation (14.3) becomes

$$(r - \kappa\partial_z^2 + \kappa z \partial_z)\mathrm{e}^{z^2/4}w(z) = 0, \quad z < \bar h, \tag{14.7}$$

where $\bar h = (h-\theta)/\bar\sigma$. Set $\nu = r/\kappa$, divide (14.7) by $-\kappa \mathrm{e}^{z^2/4}$, and use the commutation relation $\mathrm{e}^{-z^2/4}\partial_z \mathrm{e}^{z^2/4} = \partial_z + z/2$. We obtain

$$(\partial_z^2 + \frac{1}{2} - \nu - \frac{z^2}{4})w(z) = 0, \quad z < \bar h. \tag{14.8}$$

Since $-\nu$ is not a positive integer, the general solution of (14.8) can be represented in the form

$$w(z) = AD_{-\nu}(-z) + BD_{-\nu}(z), \tag{14.9}$$

where $D_{-\nu}(\pm z)$ are the *parabolic cylinder functions* or *Weber-Hermite functions.* For the representations of $D_{-\nu}$ as a series or integral, see, e.g., [31] or [13], A 2.9, p. 639. In numerical examples, we use the built-in procedures in the standard packages. We need the formula for the derivative

$$\left(\mathrm{e}^{z^2/4}D_{-\nu}(z)\right)' = -\nu \mathrm{e}^{z^2/4}D_{-\nu-1}(z) \tag{14.10}$$

(see, e.g., [13], A 2.9, p. 639), and asymptotic formulas, as $z \to +\infty$ (see equations (5a) and (5b) on p. 92 and (25) on p.40 in [31]):

$$D_{-\nu}(z) = z^{-\nu}\mathrm{e}^{-z^2/4}(1 + O(z^{-2})), \tag{14.11}$$

$$D_{-\nu}(-z) = \frac{\sqrt{2\pi}}{\Gamma(\nu)}\mathrm{e}^{z^2/4}|z|^{\nu-1}(1 + O(z^{-2})). \tag{14.12}$$

Notice also that for positive ν, $D_{-\nu}$ has no zeroes on the real line. Hence, from (14.10), we see that $D_{-\nu}$ is decreasing.

From (14.9), $V(x)$ can be represented in the form

$$V(x) = \mathrm{e}^{z^2/4}\left(AD_{-\nu}(-z) + BD_{-\nu}(z)\right), \tag{14.13}$$

and we see from (14.11) and (14.12), that $V(x)$ satisfies (14.6) if and only if $B = 0$. We set $B = 0$, substitute (14.13) into (14.4) and (14.5) and use (14.10):

$$A\mathrm{e}^{\bar h^2/4}D_{-\nu}(-\bar h) = \mathrm{e}^h - K, \tag{14.14}$$

$$A\frac{\nu}{\bar\sigma}\mathrm{e}^{\bar h^2/4}D_{-\nu-1}(-\bar h) = \mathrm{e}^h. \tag{14.15}$$

Now we can exclude A, and obtain the equation for the optimal exercise price:

$$\frac{\nu}{\bar{\sigma}}\frac{D_{-\nu-1}(-\bar{h})}{D_{-\nu}(-\bar{h})} = \frac{e^h}{e^h - K}. \tag{14.16}$$

Recall that $\bar{h} = (h-\theta)/\bar{\sigma}$, hence this is an equation with one unknown; and we look for the solution on $(\log K, +\infty)$. After h (hence, $\bar{h}$) is found, we calculate

$$A = \frac{e^h - K}{e^{\bar{h}^2/4}D_{-\nu}(-\bar{h})}, \tag{14.17}$$

and finally, find the option value in the inaction region $x < h$:

$$V(x) = (e^h - K)\frac{e^{z^2/4}D_{-\nu}(-z)}{e^{\bar{h}^2/4}D_{-\nu}(-\bar{h})},$$

where $z = (x - \theta)/\bar{\sigma}$.

It is interesting and important to compare the scheme above with the geometric Brownian motion case. In the latter case, the counterpart of (14.13) is $V(x) = Ae^{\beta^+ x} + Be^{\beta^- x}$, where $\beta^- < 0 < 1 < \beta^+$ are the roots of the *fundamental quadratic equation*, in the terminology of [39]. The equation for the threshold is

$$\beta^+ = \frac{e^h}{e^h - K},$$

and since $\beta^+ > 1$, the solution exists and it is unique. In the exponential Ornstein–Uhlenbeck model, the LHS of (14.16) is approximately linear for large h (see (14.12)), and therefore, if K is sufficiently large (the other parameters being fixed) then the solution of (14.16) exists, and it is unique. It follows from (14.11) and (14.12) that the LHS is increasing in a neighborhood of $\pm\infty$; we were unable to prove analytically that the LHS increases on the whole axis but we verified this fact for many values of ν – both large and small. Now, the RHS in (14.16) decreases on $(\log K, +\infty)$ from $+\infty$ to 1, hence the solution to (14.16) exists, and it is unique. It can be calculated numerically quite easily.

14.3 Perpetual put option

Consider the perpetual American put option on the stock with the strike price K. Let $H = e^h$ be the optimal exercise price. Since it is not optimal to exercise the option when $K - P_t = K - e^{X_t}$ is negative, we may write the free boundary problem for the rational price $V(X_t)$ of the option as

$$(r - \frac{\sigma^2}{2}\partial^2 - \kappa(\theta - x)\partial)V(x) = 0, \quad x > h, \tag{14.18}$$

$$V(h) = K - e^h, \tag{14.19}$$

$$V'(h) = -e^h. \tag{14.20}$$

Naturally, we also impose the condition

$$V(x) \to 0, \quad \text{as } x \to +\infty. \tag{14.21}$$

To solve the free boundary problem (14.18)–(14.21), we fix h, a candidate for the exercise log-price, and change the variable $z = (x - \theta)/\bar{\sigma}$, and unknown function $V(x) = \mathrm{e}^{z^2/4}w(z)$. Equation (14.3) becomes

$$(r - \kappa\partial_z^2 + \kappa z\partial_z)\mathrm{e}^{z^2/4}w(z) = 0, \quad z > \bar{h}, \tag{14.22}$$

where $\bar{h} = (h - \theta)/\bar{\sigma}$. Set $\nu = r/\kappa$, divide (14.22) by $-\kappa\mathrm{e}^{z^2/4}$, and use the commutation relation $\mathrm{e}^{-z^2/4}\partial_z\mathrm{e}^{z^2/4} = \partial_z + z/2$. We obtain

$$(\partial_z^2 + \frac{1}{2} - \nu - \frac{z^2}{4})w(z) = 0, \quad z > \bar{h}. \tag{14.23}$$

Since $-\nu$ is not a positive integer, the general solution of (14.23) can be represented in the form (14.9), but the condition (14.21) forces $A = 0$ (see (14.11) and (14.12)). We set $A = 0$, substitute (14.13) into (14.19) and (14.20) and use(14.10):

$$B\mathrm{e}^{\bar{h}^2/4}D_{-\nu}(\bar{h}) = K - \mathrm{e}^h, \tag{14.24}$$

and

$$B\frac{\nu}{\bar{\sigma}}\mathrm{e}^{\bar{h}^2/4}D_{-\nu-1}(\bar{h}) = \mathrm{e}^h. \tag{14.25}$$

Now we can exclude B, and obtain the equation for the optimal exercise price:

$$\frac{\nu}{\bar{\sigma}}\frac{D_{-\nu-1}(\bar{h})}{D_{-\nu}(\bar{h})} = \frac{\mathrm{e}^h}{K - \mathrm{e}^h}. \tag{14.26}$$

(Recall that $\bar{h} = (h-\theta)/\bar{\sigma}$). We look for the solution on $(-\infty, \log K)$, and the same argument as for the call option shows that the solution exists, and it is unique. After h (hence, $\bar{h}$) is found, we calculate

$$B = \frac{K - \mathrm{e}^h}{\mathrm{e}^{\bar{h}^2/4}D_{-\nu}(\bar{h})}, \tag{14.27}$$

and finally, find the option value in the inaction region $x > h$:

$$V(x) = (K - \mathrm{e}^h)\frac{\mathrm{e}^{z^2/4}D_{-\nu}(z)}{\mathrm{e}^{\bar{h}^2/4}D_{-\nu}(\bar{h})}, \tag{14.28}$$

where $z = (x - \theta)/\bar{\sigma}$.

14.4 Investment timing

Consider a manager who contemplates investment into a project which will yield the revenue flow e^{X_t} and requires the fixed cost I to implement; for simplicity, there is no variable cost, and hence, the revenue and profit flow are the same. The expected present value of the flow is

$$R(x) = \int_0^\infty \mathrm{e}^{-rt} E^x[\mathrm{e}^{X_t}]dt. \tag{14.29}$$

Applying the Feynman–Kac theorem, we obtain that $v(x,t) = E^x[\mathrm{e}^{X_t}]$ is a solution to the problem

$$(\partial_s + \frac{\sigma^2}{2}\partial_x^2 + \kappa(\theta - x)\partial_x)V(t; x, s) = 0, \quad s < t, \tag{14.30}$$

$$V(x,t) = \mathrm{e}^x, \tag{14.31}$$

evaluated at $s = 0$: $v(x,t) = V(t; x, 0)$. A solution that grows as $x \to +\infty$ not faster than e^x and is bounded as $x \to -\infty$ is unique. We look for the solution of the problem (14.30)–(14.31) in the form

$$V(t; x, s) = \exp[A(\tau)x + B(\tau)],$$

where $\tau = t - s$. Substituting into (14.30)–(14.31), we obtain the system of the Riccati equations

$$A'(\tau) = -\kappa A(\tau), \tag{14.32}$$

$$B'(\tau) = \frac{\sigma^2}{2}A(\tau)^2 + \kappa\theta A(\tau), \tag{14.33}$$

subject to $A(0) = 1$, $B(0) = 0$. The solution is easy to find:

$$A(\tau) = \mathrm{e}^{-\kappa\tau}, \tag{14.34}$$

$$B(\tau) = \frac{\sigma^2}{4\kappa}(1 - \mathrm{e}^{-2\kappa\tau}) + \theta(1 - \mathrm{e}^{-\kappa\tau}), \tag{14.35}$$

and we obtain

$$v(x,t) = \exp\left[\mathrm{e}^{-\kappa t}x + \frac{\sigma^2}{4\kappa}(1 - \mathrm{e}^{-2\kappa t}) + \theta(1 - \mathrm{e}^{-\kappa t})\right]. \tag{14.36}$$

Finally, substituting into (14.29), we find the EPV of the revenue stream

$$R(x) = \int_0^\infty \exp\left[\mathrm{e}^{-\kappa t}x - rt + \frac{\sigma^2}{4\kappa}(1 - \mathrm{e}^{-2\kappa t}) + \theta(1 - \mathrm{e}^{-\kappa t})\right] dt. \tag{14.37}$$

Now we can find the option value of the investment opportunity and investment threshold by solving the problem similar to the problem for the perpetual call option. Indeed, in the argument of Sect. 14.2, we need to replace the payoff function $e^{X_t} - K$ with $R(X_t) - I$, and the derivative e^x of the function $e^x - K$ with the derivative

$$R'(x) = \int_0^{\infty} \exp\left[e^{-\kappa t}x - (r+\kappa)t + \frac{\sigma^2}{4\kappa}(1 - e^{-2\kappa t}) + \theta(1 - e^{-\kappa t})\right] dt. \tag{14.38}$$

The equation for the investment threshold is an analog of (14.16)

$$\frac{\nu}{\bar{\sigma}} \frac{D_{-\nu-1}(-\bar{h})}{D_{-\nu}(-\bar{h})} = \frac{R'(h)}{R(h) - I}, \tag{14.39}$$

where $\bar{h} = (h - \theta)/\bar{\sigma}$, and $R(h)$ and $R'(h)$ are given by (14.37) and (14.38), respectively. We look for the solution on the interval where $R(x) - I$ is positive. Since $R(x)$ is increasing, the interval is well-defined, and it is of the form $(\hat{h}, +\infty)$, where $\hat{h}$ is the solution of the equation $R(h) = I$. As we discussed in Sect. 14.2, the LHS in (14.39) increases from $-\infty$ to $+\infty$, and if we show that the RHS decreases on $(\hat{h}, +\infty)$, then we can conclude that the solution exists on this interval, and it is unique. We represent the RHS as the product of $R(h)/(R(h) - I)$ and $R'(h)/R(h)$. Since R is increasing and positive, $R(h)/(R(h) - K)$ decreases (from $+\infty$) on $(\hat{h}, +\infty)$, therefore it suffices to prove that $R'(h)/R(h)$ decreases. We calculate the derivative $(R'(h)/R(h))' = (R''(h)R(h) - R'(h)^2)/R(h)^2$, and notice that the derivatives of $R(h)$ are of the form

$$\frac{d^j}{dh^j} R(h) = \int_0^{\infty} e^{-jkt} d\mu,$$

where

$$d\mu = \exp\left[e^{-\kappa t}h - rt + \frac{\sigma^2}{4\kappa}(1 - e^{-2\kappa t}) + \theta(1 - e^{-\kappa t})\right] dt.$$

From the Cauchy-Schwartz inequality, $R'(h)^2 \le R''(h)R(h)$, and the proof of the monotonicity of the RHS in (14.39) is completed. The threshold h having been found, we calculate the option value of the investment opportunity

$$V(x) = (R(h) - I)\frac{e^{z^2/4} D_{-\nu}((\theta - x)/\bar{\sigma})}{e^{\bar{h}^2/4} D_{-\nu}(-\bar{h})}. \tag{14.40}$$

14.5 Timing exit

Assume that the revenue flow of a firm is as above but there is the variable cost C. Then the profit flow is $\Pi(X_t) = e^{X_t} - C$. Should the current level of profits fall too low, it would be optimal to abandon the project. The expected present value of the profit flow is $V_0(x) = R(x) - C/r$, where $R(x)$, the EPV of the revenue flow, is given by (14.37). We see that the option to exit can be interpreted as the perpetual American put option on the stock with the price process $P_t = R(X_t)$, and the strike $K = C/r$. Therefore, in the argument of Sect. 14.3, we need to replace the payoff function $K - e^{X_t}$ with $C/r - R(X_t)$, and the derivative $-e^x$ of the function $K - e^x$ with the derivative $-R'(x)$, where $R'(x)$ is given by (14.38). The equation for the investment threshold is an analog of (14.26):

$$\frac{\nu}{\bar{\sigma}} \frac{D_{-\nu-1}(\bar{h})}{D_{-\nu}(\bar{h})} = \frac{R'(h)}{C/r - R(h)}, \tag{14.41}$$

where $\bar{h} = (h - \theta)/\bar{\sigma}$. After h (hence, $\bar{h}$) is found, we find the option value to exit in the inaction region $x > h$:

$$V_{\text{opt}}(x) = (C/r - R(h)) \frac{e^{z^2/4} D_{-\nu}(z)}{e^{\bar{h}^2/4} D_{-\nu}(\bar{h})}, \tag{14.42}$$

where $z = (x - \theta)/\bar{\sigma}$. The value of the firm is

$$V(x) = R(x) - C/r + V_{\text{opt}}(x).$$

14.6 Bad and good news principles as approximations

14.6.1 The case of a general Markov process

Let $T \sim \text{Exp}(r)$ be the exponential random variable with mean $1/r$. Assuming that g is differentiable, we can write the EPV-operators as

$$\begin{aligned}
\mathcal{E}^+ g(x) &= E^x[g(\bar{X}_T)] \\
&= \int_{+\infty}^{x} g(y) d\mathbb{P}_x(\bar{X}_T \geq y) \\
&= \mathbb{P}_x(\bar{X}_T \geq y) g(y) \,\Big|_{+\infty}^{x} - \int_{+\infty}^{x} g'(y) \mathbb{P}_x(\bar{X}_T \geq y) dy \\
&= g(x) + \int_{x}^{+\infty} g'(y) \mathbb{P}_x(\bar{X}_T \geq y) dy,
\end{aligned} \tag{14.43}$$

$$\begin{aligned}
\mathcal{E}^- g(x) &= E^x[g(\underline{X}_T)] \\
&= \int_{-\infty}^{x} g(y) d\mathbb{P}_x(\underline{X}_T \leq y) \\
&= \mathbb{P}_x(\underline{X}_T \leq y) g(y) \,\Big|_{-\infty}^{x} - \int_{-\infty}^{x} g'(y) \mathbb{P}_x(\underline{X}_T \leq y) dy \\
&= g(x) - \int_{-\infty}^{x} g'(y) \mathbb{P}_x(\underline{X}_T \leq y) dy.
\end{aligned} \tag{14.44}$$

Consider a call-like option with the payoff stream $g(X_t)$. The bad news principle, which is proved for processes with i.i.d. increments, states that the optimal exercise boundary, h^*, is the solution of the equation

$$\mathcal{E}^- g(h) = 0. \tag{14.45}$$

Using (14.44), (14.45) can be written as

$$g(h) = \int_{-\infty}^{h} g'(y)\mathbb{P}_h(\underline{X}_T \le y)dy, \tag{14.46}$$

and after its solution, h^*, is found, we calculate the option value in the inaction region $x \le h^*$ from

$$\begin{aligned}
rV^*(x) &= \mathcal{E}^+\mathbf{1}_{[h^*,+\infty)}\mathcal{E}^- g(x)\\
&= \int_{+\infty}^{h^*} \mathcal{E}^- g(y)d\mathbb{P}_x(\bar{X}_T \ge y)\\
&= \mathcal{E}^- g(h^*)\mathbb{P}_x(\bar{X}_T \ge h^*) + \int_{h^*}^{+\infty} \mathbb{P}_x(\bar{X}_T \ge y)\frac{d}{dy}\mathcal{E}^- g(y)dy.
\end{aligned}$$

The first term on the RHS is zero. Using (14.44), we transform the integral into

$$\begin{aligned}
&\int_{h^*}^{+\infty} \left\{g'(y) - g'(y)\mathbb{P}_y(\underline{X}_T \le y) - \int_{-\infty}^{y} g'(\tilde{y})\frac{\partial}{\partial y}\mathbb{P}_y(\underline{X}_T \le \tilde{y})d\tilde{y}\right\}\\
&\times \mathbb{P}_x(\bar{X}_T \ge y)dy,
\end{aligned}$$

and since the first two terms in the curly brackets cancel out, we obtain

$$V^*(x) = -\frac{1}{r}\int_{h^*}^{+\infty}\int_{-\infty}^{y} \mathbb{P}_x(\bar{X}_T \ge y)\frac{\partial}{\partial y}\mathbb{P}_y(\underline{X}_T \le \tilde{y})g'(\tilde{y})d\tilde{y}dy. \tag{14.47}$$

In the case of a put-like option with the payoff stream $g(X_t)$, the calculations are similar. Using (14.43), the equation $\mathcal{E}^+ g(h) = 0$ for the optimal exercise threshold (the good news principle) can be written as

$$g(h) = \int_{+\infty}^{h} g'(y)\mathbb{P}_h(\bar{X}_T \ge y)dy, \tag{14.48}$$

and after its solution, h_*, is found, we calculate the option value in the inaction region $x \ge h^*$ from

$$\begin{aligned}
rV_*(x) &= \mathcal{E}^-\mathbf{1}_{(-\infty,h^*]}\mathcal{E}^+ g(x)\\
&= \int_{-\infty}^{h_*} \mathcal{E}^+ g(y)d\mathbb{P}_x(\underline{X}_T \le y)\\
&= \mathcal{E}^+ g(h_*)\mathbb{P}_x(\underline{X}_T \le h^*) - \int_{-\infty}^{h_*} \mathbb{P}_x(\underline{X}_T \le y)\frac{d}{dy}\mathcal{E}^+ g(y)dy.
\end{aligned}$$

The first term on the RHS is zero. Using (14.43), we transform the integral into

$$\int_{-\infty}^{h_*}\left\{g'(y) - g'(y)\mathbb{P}_y(\bar{X}_T \ge y) - \int_{+\infty}^{y} g'(\tilde{y})\frac{\partial \mathbb{P}_y(\bar{X}_T \ge \tilde{y})}{\partial y}d\tilde{y}\right\}\mathbb{P}_x(\underline{X}_T \le y)dy,$$

and since the first two terms in the curly brackets cancel out, we obtain

$$V_*(x) = -\frac{1}{r}\int_{-\infty}^{h_*}\int_{y}^{+\infty}\mathbb{P}_x(\underline{X}_T \le y)\frac{\partial}{\partial y}\mathbb{P}_y(\bar{X}_T \ge \tilde{y})g'(\tilde{y})d\tilde{y}dy. \quad (14.49)$$

14.6.2 The case of the Ornstein–Uhlenbeck process

In this subsection, U_t stands for the Ornstein–Uhlenbeck process (14.1).

Probability distributions of the supremum and infimum processes

Set $\nu = r/\kappa > 0$, and $z = (x-\theta)/\bar{\sigma}$, $w = (y-\theta)/\bar{\sigma}$. Eqs. 7.1.1.2 and 7.1.2.2 on p. 522 in [13] give the following formulas for the probability distributions of the supremum and infimum processes of $\bar{U}_t = \sup_{0\le s\le t} U_s$, $\underline{U}_t = \inf_{0\le s\le t} U_s$ evaluated at the random time T (and conditioned on $U_0 = x$):

$$\mathbb{P}_x\left(\bar{U}_T \ge y\right) = \frac{e^{z^2/4}D_{-\nu}(-z)}{e^{w^2/4}D_{-\nu}(-w)}, \quad x \le y, \quad (14.50)$$

$$\mathbb{P}_x\left(\underline{U}_T \le y\right) = \frac{e^{z^2/4}D_{-\nu}(z)}{e^{w^2/4}D_{-\nu}(w)}, \quad x \ge y. \quad (14.51)$$

Using (14.50) and (14.51), we rewrite (14.43) and (14.44) as

$$\mathcal{E}^+g(x) = g(x) + \int_x^{\infty}\frac{e^{z^2/4}D_{-\nu}(-z)}{e^{w^2/4}D_{-\nu}(-w)}g'(y)dy, \quad (14.52)$$

and

$$\mathcal{E}^-g(x) = g(x) - \int_{-\infty}^{x}\frac{e^{z^2/4}D_{-\nu}(z)}{e^{w^2/4}D_{-\nu}(w)}g'(y)dy. \quad (14.53)$$

Investment timing: a formal application of the bad news principle

Consider the same problem as in Sect. 14.4. We can represent the NPV of the project, $R(x) - I$, where $R(x)$ is given by (14.37), as the EPV of the stream $g(x) = e^x - rI$. Applying (14.46) formally and using (14.53), we obtain the equation for the investment threshold:

$$e^h - rI - \int_{-\infty}^{h}\frac{e^{\bar{h}^2/4}D_{-\nu}(\bar{h})}{e^{w^2/4}D_{-\nu}(w)}e^y dy = 0, \quad (14.54)$$

where $\bar{h} = (h - \theta)/\bar{\sigma}$. We find the solution, h^*, numerically, and after that, we calculate the (approximate) value of the investment opportunity using (14.47) and (14.10). For x in the inaction region $x < h^*$, the answer is

$$V^*(x) = \mathrm{e}^{z^2/4} D_{-\nu}(-z) M_{\mathrm{approx}}(h^*), \tag{14.55}$$

where

$$M_{\mathrm{approx}}(h^*) = \frac{\nu}{r\bar{\sigma}} \int_{h^*}^{+\infty} \int_{-\infty}^{y} \frac{D_{-\nu-1}(w) g'(\tilde{y})}{\mathrm{e}^{\tilde{w}^2/4} D_{-\nu}(\tilde{w}) D_{-\nu}(-w)} d\tilde{y} dy,$$

and $z = (x - \theta)/\bar{\sigma}$, $w = (y - \theta)/\bar{\sigma}$, $\tilde{w} = (\tilde{y} - \theta)/\bar{\sigma}$. Changing the variables, we obtain

$$M_{\mathrm{approx}}(h^*) = \frac{\nu\bar{\sigma}\mathrm{e}^{\theta}}{r} \int_{z^*}^{+\infty} \int_{-\infty}^{w} \frac{D_{-\nu-1}(w) \mathrm{e}^{\tilde{w}\bar{\sigma} - \tilde{w}^2/4}}{D_{-\nu}(-w) D_{-\nu}(\tilde{w})} d\tilde{w} dw, \tag{14.56}$$

where $z^* = (h^* - \theta)/\bar{\sigma}$. Notice that the exact formula for the value function in the inaction region (14.40) is of the form (14.55):

$$V(x) = \mathrm{e}^{z^2/4} D_{-\nu}(-z) M_{\mathrm{exact}}(h), \tag{14.57}$$

with a different constant multiple

$$M_{\mathrm{exact}}(h) = \frac{R(h) - K}{\mathrm{e}^{\bar{h}^2/4} D_{-\nu}(-(h - \theta)/\bar{\sigma})}, \tag{14.58}$$

where h is the solution of (14.39). Hence, to access the accuracy of the *bad news principle approximation*, it suffices to compare the thresholds h and h^*, and coefficients $M_{\mathrm{exact}}(h)$ and $M_{\mathrm{approx}}(h^*)$. Numerical results obtained in [62] indicate that the relative error is moderate unless κ, the mean reverting coefficient, is sizable w.r.t. σ^2. We would like to stress that we replace the exact formulas by approximate ones not to suggest the use of an approximate and more complicated expression instead of a simpler exact one but to test the validity of the bad news principle in a situation, where this principle can be used as an approximation only.

Timing exit: a formal application of the good news principle

Consider the same problem as in Sect. 14.5. The option to exit can be interpreted as the option to acquire the stream $g(X_t) = C - \mathrm{e}^{X_t}$. Applying the good news principle formally and using (14.52), we obtain the equation for the investment threshold:

$$C - \mathrm{e}^h - \int_h^{+\infty} \frac{\mathrm{e}^{(h-\theta)^2/(4\bar{\sigma}^2)} D_{-\nu}(-(h - \theta)/\bar{\sigma})}{\mathrm{e}^{w^2/4} D_{-\nu}(-w)} \mathrm{e}^y dy = 0. \tag{14.59}$$

We find the solution, h_*, numerically, and after that, we calculate the (approximate) value of the option value to exit. For x in the inaction region $x > h_*$, the answer is

$$V_*(x) = \mathrm{e}^{z^2/4} D_{-\nu}(z) M^-_{\mathrm{approx}}(h_*), \tag{14.60}$$

where

$$M^-_{\mathrm{approx}}(h_*) = -\frac{\nu}{r\bar{\sigma}} \int_{-\infty}^{h_*} \int_y^{+\infty} \frac{D_{-\nu-1}(-w) g'(\tilde{y})}{\mathrm{e}^{\tilde{w}^2/4} D_{-\nu}(-\tilde{w}) D_{-\nu}(w)} d\tilde{y} dy,$$

and $z = (x - \theta)/\bar{\sigma}$, $w = (y - \theta)/\bar{\sigma}$, $\tilde{w} = (\tilde{y} - \theta)/\bar{\sigma}$. Changing the variables, we obtain

$$M^-_{\mathrm{approx}}(h^*) = \frac{\nu\bar{\sigma}\mathrm{e}^{\theta}}{r} \int_{-\infty}^{z_*} \int_w^{+\infty} \frac{D_{-\nu-1}(-w) \mathrm{e}^{\tilde{w}\bar{\sigma} - \tilde{w}^2/4}}{D_{-\nu}(w) D_{-\nu}(-\tilde{w})} d\tilde{w} dw, \tag{14.61}$$

where $z_* = (h_* - \theta)/\bar{\sigma}$. Notice that the exact formula for the option value to exit in the inaction region (14.42) is of the form (14.60):

$$V_{\mathrm{opt}}(x) = \mathrm{e}^{z^2/4} D_{-\nu}(z) M^-_{\mathrm{exact}}(h), \tag{14.62}$$

with a different constant multiple

$$M^-_{\mathrm{exact}}(h) = \frac{C/r - R(h)}{\mathrm{e}^{\bar{h}^2/4} D_{-\nu}(\bar{h})}. \tag{14.63}$$

14.7 Options with instantaneous payoffs

14.7.1 Call option

Consider the perpetual American option with the instantaneous payoff $G(X_t)$. Assume that G is increasing; then we have a call-like option. The standard call option obtains with $G(X_t) = \mathrm{e}^{X_t} - K$. If X_t is a Lévy process, then under weak regularity conditions, the exercise rule can be formulated as follows (see Theorem 11.6.10). Let $(\mathcal{E}^+)^{-1}$ be the inverse operator to the EPV-operator under the supremum process, and assume that $(\mathcal{E}^+)^{-1} G$ is an increasing continuous function, which changes sign. Then the equation

$$(\mathcal{E}^+)^{-1} G(x) = 0 \tag{14.64}$$

has a unique solution, call it h^*. The optimal stopping time in the class of hitting times of semi-finite intervals $[h, +\infty)$ is: exercise the option the first time X_t reaches h^* or crosses it, and the rational option price is

$$V^*(x) = \mathcal{E}^+ \mathbf{1}_{[h^*,+\infty)} (\mathcal{E}^+)^{-1} G(x). \tag{14.65}$$

Surprisingly, (14.64) and (14.65) are valid for X_t the Ornstein–Uhlenbeck process as well although it is not a process with i.i.d. increments. To see this, note that $G_1 := (\mathcal{E}^+)^{-1}G$ is the solution to the (functional) equation $\mathcal{E}^+G_1 = G$, equivalently,

$$\int_{+\infty}^{x} G_1(y)d\mathbb{P}_x(\bar{U}_T \geq y) = G(x), \quad \forall\ x. \tag{14.66}$$

Using (14.50), we write this equation as

$$\int_{+\infty}^{x} \left(\frac{1}{e^{w^2/4}D_{-\nu}(-w)}\right)'_y G_1(y)dy = \frac{G(x)}{e^{z^2/4}D_{-\nu}(-z)}. \tag{14.67}$$

We differentiate (14.67)

$$\left(\frac{1}{e^{z^2/4}D_{-\nu}(-z)}\right)'_x G_1(x) = \left(\frac{G(x)}{e^{z^2/4}D_{-\nu}(-z)}\right)'_x,$$

and find that $G_1(x)$ is proportional to

$$G'(x)e^{z^2/4}D_{-\nu}(-z) - G(x)\frac{\nu}{\bar{\sigma}}e^{z^2/4}D_{-\nu-1}(-z).$$

Therefore, (14.64) is equivalent to

$$G'(x)D_{-\nu}(-z) - G(x)\frac{\nu}{\bar{\sigma}}D_{-\nu-1}(-z) = 0.$$

which gives the same equation as in Sect. 14.2 (written there for the special case $G_1(x) = e^x - K$):

$$\frac{\nu}{\bar{\sigma}}\frac{D_{-\nu-1}(-z)}{D_{-\nu}(-z)} = \frac{G'(x)}{G(x)}.$$

On $x < h^*$, the solution (14.65) is of the same form as the optimal solution in Sect. 14.2:

$$V^*(x) = Me^{z^2/4}D_{-\nu}(-z),$$

where M is a constant. Hence, it remains to check that the value matching condition holds. We represent $V^*(x)$ in the form

$$\begin{aligned} V^*(x) &= \mathcal{E}^+(\mathbf{1} - \mathbf{1}_{(-\infty,h^*]})(\mathcal{E}^+)^{-1}G(x) \\ &= \mathcal{E}^+(\mathcal{E}^+)^{-1}G(x) - \mathcal{E}^+\mathbf{1}_{(-\infty,h^*]}G_1(x) \\ &= G(x) - \mathcal{E}^+f(x), \end{aligned}$$

where $f(x) = \mathbf{1}_{(-\infty,h^*]}G_1(x)$ is continuous and equal zero on $[h^*, +\infty)$. It follows that $\mathcal{E}^+f(x)$ is continuous and equal to zero on $[h^*, +\infty)$. Hence, V^* satisfies the value matching condition.

14.7.2 Put option

Consider the perpetual American option with the instantaneous payoff $G(X_t)$. Assume that G is decreasing; then we have a put-like option. The standard put option obtains with $G(X_t) = K - \mathrm{e}^{X_t}$. If X_t is a Lévy process, then under weak regularity conditions, the exercise rule can be formulated as follows (see Theorem 11.6.8). Let $(\mathcal{E}^-)^{-1}$ be the inverse operator to the EPV-operator under the infimum process, and assume that $(\mathcal{E}^-)^{-1}G$ is an increasing continuous function, which changes sign. Then the equation

$$(\mathcal{E}^-)^{-1}G(x) = 0 \tag{14.68}$$

has a unique solution, call it h_*. The optimal stopping time in the class of hitting times of semi-finite intervals $(-\infty, h]$ is: exercise the option the first time X_t reaches h_* or crosses it, and the rational option price is

$$V_*(x) = \mathcal{E}^-\mathbf{1}_{(-\infty,h_*]}(\mathcal{E}^-)^{-1}G(x). \tag{14.69}$$

As in the case of the perpetual call options, (14.68) and (14.69) are valid for X_t the Ornstein–Uhlenbeck process as well although it is not a process with i.i.d. increments. To see this, note that $G_1 := (\mathcal{E}^-)^{-1}G$ is the solution of the (functional) equation $\mathcal{E}^-[G_1] = G$, equivalently,

$$\int_{-\infty}^{x} G_1(y)d\mathbb{P}_x(\underline{U}_T \le y) = G(x), \quad \forall\ x. \tag{14.70}$$

Using (14.51), we write this equation as

$$\int_{-\infty}^{x} \left(\frac{1}{\mathrm{e}^{w^2/4}D_{-\nu}(w)}\right)'_y G_1(y)dy = \frac{G(x)}{\mathrm{e}^{z^2/4}D_{-\nu}(z)}. \tag{14.71}$$

We differentiate (14.71)

$$\left(\frac{1}{\mathrm{e}^{z^2/4}D_{-\nu}(z)}\right)'_x G_1(x) = \left(\frac{G(x)}{\mathrm{e}^{z^2/4}D_{-\nu}(z)}\right)'_x,$$

and find that $G_1(x)$ is proportional to

$$G'(x)\mathrm{e}^{z^2/4}D_{-\nu}(z) + G(x)\frac{\nu}{\bar{\sigma}}\mathrm{e}^{z^2/4}D_{-\nu-1}(z).$$

Therefore, (14.68) is equivalent to

$$G'(x)D_{-\nu}(z) + G(x)\frac{\nu}{\bar{\sigma}}D_{-\nu-1}(z) = 0,$$

which gives the same equation as in Sect. 14.3 (written there for the special case $G_1(x) = C/r - \mathrm{e}^x$):

$$\frac{\nu}{\bar{\sigma}} \frac{D_{-\nu-1}(z)}{D_{-\nu}(z)} = -\frac{G'(x)}{G(x)}.$$

On $x > h^*$, the solution (14.69) is of the same form as the optimal solution in Sect. 14.3:

$$V_*(x) = M^- e^{z^2/4} D_{-\nu}(-z),$$

where M^- is a constant. Hence, it remains to check that the value matching condition holds. We represent $V_*(x)$ in the form

$$\begin{aligned} V_*(x) &= \mathcal{E}^-(\mathbf{1} - \mathbf{1}_{(h_*,+\infty)})(\mathcal{E}^-)^{-1}G(x) \\ &= \mathcal{E}^-(\mathcal{E}^-)^{-1}G(x) - \mathcal{E}^-\mathbf{1}_{(h_*,+\infty)}G_1(x) \\ &= G(x) - \mathcal{E}^- f(x), \end{aligned}$$

where $f(x) = \mathbf{1}_{(h_*,+\infty)}G_1(x)$ is continuous and equal to zero on $(-\infty, h_*]$. It follows that $\mathcal{E}^- f(x)$ is continuous and equal to zero on $(-\infty, h_*]$. Hence, V_* satisfies the value matching condition.

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