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Third Edition

B D Gupta

Mathematical Physics

PREFACE TO THE THIRD EDITION

I have once again availed the opportunity of revising this work designed for the requirements of the students of physics and technical courses at undergraduate and postgraduate levels. Many new topics such as conformal mapping in complex analysis, inversion of complex matrices, error functions, factorial functions, etc. and specially the solution of. Schrödinger's wave equations in Quantum Mechanics as required by the UGC syllabus as well as syllabi of several other universities of India have been added.

In this revision, I have freely consulted the contribution of Indian and foreign authors in the form of titles as Mathematical Physics, Quantum Mechanics or Complex Analysis. I am deeply grateful to the authors and publishers of all these books and pay my heartiest thanks to all of them.

I do hope that the work will be more useful to readers for whom this has been intended. Although I have tried to make it free from errors and omissions, if readers still find any, I will be grateful to be informed.

Any suggestions for the further improvement of the work will be thankfully accepted and executed.

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MATHEMATICAL PHYSICS

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THIRD REVISED EDITION

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PREFACE TO THE FIRST EDITION

It was in the year 1965, that myself and my colleague B.S. Rajput, wrote a book entitled *Mathematical Physics,* to meet the requirements of Honours, Postgraduate and Engineering students of various Indian universities. Since then the same work has appeared in six editions with the previous publishers. But for many reasons, we could not continue the co-authorship.

Here I would like to offer my sincere thanks to my former co-author and previous publisher for giving their consents to get it revised and published independently.

Now, keeping in mind the needs of the changing syllabi of various universities and the requirements of changing knowledge which is almost doubled in every decade, I prepared this work myself alone about five years back, in view of new and advanced standpoint. But owing to some unavoidable circumstances this could not be published earlier, and so some of its portions have been further revised in order to render it up-to-date, hence more useful.

The value and scope of the present work have obviously been considerably increased, because, in the first place, several portions contributed by myself, in all my books written with different coauthors, have been reproduced in a more expository style and in the second place many more portions have been rewritten. The entire matter has been rearranged and many new topics have been added. Every section has been supplemented with a large number of worked-out problems and a set of additional miscellaneous problems selected from the question papers set in various university examinations.

I do not claim any originality of matter and this is at best a compiled work, with a novel presentation. The subject matter has been so arranged that even a layman can understand how to apply the mathematical operations to the problems of Physics. During the preparation of this work, I have freely consulted a number of books on Mathematics and Physics written by foreign and Indian authors. It goes without saying that I am deeply indebted to all of them, although, I am sorry not to acknowledge my gratitude to them individually—the number being too big to be accommodated in the little space that can be spared for that purpose in a work of this nature.

While revising this Volume, I have developed chapter one on 'Vectors' from all my contributions to our 'Vector Analysis', 'Vector Calculus' and 'Elements of Mechanics', chapter two on 'Matrices' from my contributions to our 'Mathematical Physics', chapter three on 'Tensors' from my contributions to 'Relativistic Mechanics' after the publication of which a number of sections on tensors were taken to our 'Mathematical Physics', chapter four on 'Group Theory' (written a new); chapters five to ten on 'Complex Variables', 'Beta, Gamma and Error Functions', 'Differential Equations', 'Harmonics', 'Fourier Series', 'Integrals and Transforms' and 'Laplace's Transforms' from my contributions to our 'Mathematical

Physics'. Chapters eleven to fourteen on 'Hankel Transforms', 'Diffusion, Wave and Laplace's Equation', 'Maxwell's Electromagnetic Field Equations' and 'Special Theory of Relativity' have been written quite new. Chapter fifteen on 'Statistical Probability' has been developed from my contributions to our 'Mathematical Statistics'. In the end the three Appendices A, B, C on 'Some formulated results in Basic Mathematics', 'Asymptotic Expansion of Error Function', and 'Character Tables in Group Theory' have been added in order to enhance the utility of the work.

In the preparation of Appendix C, I have been much guided by 'Elements of Group Theory for Physicists' by A.W. Joshi, Published by Wiley Eastern Ltd., New Delhi, I acknowledge my indebtedness to both author and the publisher of this book. Moreover, I acknowledge my indebtedness to all my past and present coauthors, chiefly my colleagues Messrs O.P. Gupta, H.C. Sharma , J.P. Agarwal, Satya Prakash (formerly a student of mine), and the publisher M/s Kedar Nath Ram Nath, Meerut.

My thanks are also due to A.W. Joshi of the Institute of Advanced Studies, Meerut Univerity and to Naresh Kumar, my colleague and a former pupil of mine, for their concrete suggestions during the preparation of this book. I would be failing in my duty if I do not acknowledge my deep gratitude to my colleagues P.C. Jain, M.P. Tyagi, B. Singh and V.P. Arora for not only rendering me their best help and cooperation but also encouraging and inspiring me all along.

Any suggestions for further improvement of the book from any corner will be thankfully accepted and executed.

B.D. Gupta

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CHAPTER 1

VECTORS

1.1. INTRODUCTION

It is generally observed that there exist two types of physical measurements in applied mathematics, physics and mechanics : one involving *only magnitude and no direction* in the space of three dimensions, such as volume, mass, length, speed, temperature, potential, electric charge etc., and the other involving *a definite direction in space associated with their magnitudes* such as velocity, acceleration, momentum, force, electric or magnetic field intensities etc., the former being called *scalar quantities* or simply *scalars* and the latter, *vector quantities* or simply *Vectors.*

A little consideration will exhibit that the complete characterization of a scalar quantity requires length and support, *i.e.*, a specified unit and a number stating how many times that unit is contained in that quantity, while the complete characterization of a vector quantity requires length, support and sense, *i.e.*, a specified unit, a number stating how many times that unit is contained in that quantity and the statement of the direction.

Stating in a precise manner a *vector* means '*a directed line segment*'. In other words we can state that a vector is a quantity having direction as well as magnitude. In Astronomy a vector means an imaginary straight line that joins a planet moving round a centre (generally the focus of an elliptic orbit) to that centre.

1.2. REPRESENTATION OF VECTORS

Since a vector is the result of abstraction, its magnitude and direction may be represented by a line *OP* directed from the *initial point O* to the *terminal point P* and denoted by $\overrightarrow{OP}$.

Here the length of vector $\overrightarrow{OP}$ denoted by $|\overrightarrow{OP}| = OP$ is called *magnitude* or *module* or *modulus* of the vector and the direction in space is indicated by an arrow head on the line.

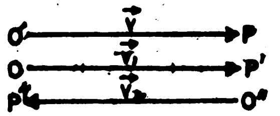


Fig. 1.1

In Fig. 1.1, the vector $\overrightarrow{OP}$ has been shown by $\overrightarrow{V}$ (or in Clarendon print by **V**) while its scalar magnitude is stated by *V*. Thus *OP* is the length of the vector **V**, while the line of indefinite length of which the directed line segment $\overrightarrow{OP}$ is only a part is the support of **V** and the sense is from *O* to *P*.

It should be noted that formulation of a law of physics in terms of vectors is however independent of the choice of axes of reference, *i.e.*, the vector representation has a physical content without reference to any coordinate system.

1.3. KINDS OF VECTORS

Equal vectors. Two given vectors may be equal only when they have the same magnitude and the same direction, *i.e.*, the given two vectors are equal if and only if they have the same or parallel support with equal length and the same sense. For example in Fig. 1.1, we have

$$\mathbf{V}\left(=\overrightarrow{OP}\right)=\mathbf{V}_1\left(=\overrightarrow{O'P'}\right)=-\mathbf{V}_2\left(=\overrightarrow{O''P''}\right)$$

where $\mathbf{V}_1$ and $\mathbf{V}_2$ have the same scalar magnitude as $\mathbf{V}$ but $\mathbf{V}_1$ has the same and $\mathbf{V}_2$ the opposite sense to that of $\mathbf{V}$.

Null vector. A vector having the initial and the terminal points coincident is termed as a *zero vector* or a *null vector*. Thus a null vector has its module zero.

Unit vector. A vector having its modulus as unity is called a *unit vector*.

If $\mathbf{a}$ is a vector and 'a' its modulus, then unit vector $\mathbf{a}$ is denoted by $\hat{\mathbf{a}}$ (read as 'a hat' or 'a caret') defined as

$$\hat{\mathbf{a}}=\frac{\mathbf{a}}{|\mathbf{a}|}=\frac{\mathbf{a}}{a}$$

Polar vectors. The line vectors representing the quantities like force, velocity etc., in which merely a linear action in a particular direction is involved, are termed as *polar* vectors.

Axial vectors. The line vectors representing the quantities like angular velocity, angular acceleration etc., in which some rotational action is involved about an axis and which are drawn parallel to the axis of rotation in order that the magnitude of the quantity is determined by the length of the vector and the direction by the rule of right handed screw (*i.e*,. rotation being considered in clockwise direction), are termed as *axial* vectors.

Free vector. Evidently a vector can be represented by an infinite number of equal vectors by drawing parallel supports. Such a vector which can be transported from place to place such that it remains of the same magnitude and keeps up the same direction is termed as a *free* vector. In fact a free vector is assumed to remain the same through transportation, irrespecitve of its position in space.

Localised or Line vector. We have defined that the value of a free vector depends only on its length and direction, but if it depends also on its position in space, *i.e.*, if a vector is restricted to pass through a given origin, then it is termed as a *localized* vector.

Collinear vectors. The vectors parallel to the same line, regardless of their magnitudes and sense of directions are termed as *collinear* vectors. In other words the vectors having the same or parallel supports are known as collinear vectors. Such vectors are parallel to each other and they may coincide in a special case. As such there exists a scalar ratio say λ between any two collinear vectors $\mathbf{a}$ and $\mathbf{b}$ of the form

$$\mathbf{b}=\lambda\mathbf{a}$$

which follows that one of the two collinear vectors can be expressed as the scalar multiple of the other.

Non-collinear vectors. The vectors whose directions are neither parallel nor coincident are said to be *non-collinear*.

Like vectors or co-directional vectors. The vectors which are collinear and have the same sense of directions *i.e.*, the vectors directed in the same sense irrespective of their magnitudes are termed as *like* vectors.

Unlike vectors. The vectors which are collinear but have opposite sense of directions from each other are termed as *unlike* vectors.

Coplanar vectors. A system of vectors lying in the parallel planes or which can be made to lie in the same plane are said to be *coplanar* vectors. Evidently any two vectors are always coplanar.

Non-coplanar vectors. A system of vectors consisting of three or more vectors which cannot be made to lie in the same plane are called *non-coplanar* vectors.

Reciprocal vector. Any vector having its direction the same as that of a given vector **a**, but its magnitude as the reciprocal of the magnitude of **a** is termed as the *reciprocal* vector of a and written as $\mathbf{a}^{-1}$ or $\frac{1}{\mathbf{a}}$. As such

$$\mathbf{a}^{-1} = \frac{1}{a}\hat{\mathbf{a}} = \frac{a}{a^2}\hat{\mathbf{a}} = \frac{\mathbf{a}}{a^2} \text{ (by definition of a unit vector).}$$

In this connection it is notable that the magnitude and so the reciprocal of the magnitude of a unit vector being unity, the unit vector is reciprocal to itself and it is said to be *self-reciprocal.*

Negative vector. The vector having the same magnitude as the vector **a** but opposite direction, is known as the *negative* of **a** and written as – **a**.

Position vector. If a vector $\overrightarrow{OP}$ specifies the position of a point P relative to an arbitrarily chosen point O, then $\overrightarrow{OP}$ is called the *Position* vector of P with respect to O, the origin of vectors.

Problem 1. *If* {**a**, **b**, **c**} *is a right handed set, which of the following sets are right handed?*

(*i*) **a**, **c**, **b** ; (*ii*) **b**, **c**, **a** ; (*iii*) **b**, **a**, **c** ; (*iv*) **c**, **a**, **b** ; (*v*) **c**, **b**, **a**

It is clear that the sets (*ii*) and (*iv*), *i.e.*, **b**, **c**, **a** ; and **c**, **a**, **b** are right handed.

Problem 2. *Discuss the geometrical significance of* $a\mathbf{A} + b\mathbf{B} = 0$.

We have $a\mathbf{A} + b\mathbf{B} = 0$, a, b being scalars.

This can be written as $\mathbf{A} = -\frac{b}{a}\mathbf{B}$

$$= \lambda\mathbf{B} \quad \text{if} \quad \lambda = -\frac{b}{a}$$

i.e., **A** is expressible as a scalar multiple of $\overrightarrow{\mathbf{B}}$ so that the vectors **A** and **B** are parallel or collinear.

1.4. ADDITION OF VECTORS

The characterisation of process of summation is inherited in the composition of two or more displacements of a point. Suppose that we have two vectors **a** and **b** acting at a point O as shown in Fig 1.2. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

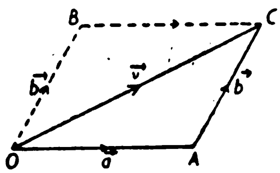


Fig. 1.2

Clearly the resultant effect of the vectors **a** and **b** is the same as that of their vector sum v obtained by setting off the vector **b** at the end of **a** and then joining the beginning of **a** to the end of **b**. This geometrical construction utilised to find the vector sum of two vectors **a** and **b** is known as *the parallelogram law of addition of vectors.*

$$\text{Thus } \mathbf{v} = \vec{OC} = \vec{OA} + \vec{AC} = \mathbf{a} + \mathbf{b} \qquad ...(1)$$

A similar result follows by starting with **b** and setting off the vector **a** on **b**, *i.e.*,

$$\mathbf{v} = \vec{OC} = \vec{OB} + \vec{BC} = \mathbf{b} + \mathbf{a}. \qquad ...(2)$$

Conclusively the result of adding two co-initial vectors is the vector represented by the diagonal of the parallelogram having the two given vectors as its adjacent sides.

From (1) and (2) it follows that

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$

i.e., the two vectors obey the **commutative law of addition**, according to which the vector sum of two vectors is independent of their order.

We now propose to find the sum of three vectors say **a**, **b**, **c**. Let $\vec{OA} = \mathbf{a}$, $\vec{AB} = \mathbf{b}$, $\vec{BC} = \mathbf{c}$ as shown in Fig. 1.3. Then

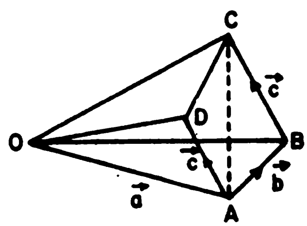


Fig. 1.3

$$\mathbf{v} = \vec{OC} = \vec{OA} + \vec{AB} + \vec{BC} = \mathbf{a} + \mathbf{b} + \mathbf{c}. \qquad ...(3)$$

$$\text{Also} \quad \mathbf{v} = \vec{OC} = \vec{OB} + \vec{BC}$$

$$= (\vec{OA} + \vec{AB}) + (\vec{BC})$$

$$= (\mathbf{a} + \mathbf{b}) + \mathbf{c}. \qquad ...(4)$$

$$\text{Similarly} \quad \mathbf{v} = \mathbf{a} + (\mathbf{b} + \mathbf{c}) \qquad ...(5)$$

$$\text{and} \quad \mathbf{v} = (\mathbf{a} + \mathbf{c}) + \mathbf{b} \qquad ...(6)$$

It follows from (3), (4), (5) and (6) that

$$\mathbf{v} = \mathbf{a} + \mathbf{b} + \mathbf{c} = (\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{c}) + \mathbf{b}.$$

i.e., the three vectors obey the **associative law of addition**, *according to which the vector sum of three vectors is independent of the mode in which component vectors are associated in different groups.*

In general, if there are *n* vectors **a**, **b**, **c**...**n**, then their vector sum v is given by

$$\mathbf{v} = \mathbf{a} + \mathbf{b} + \mathbf{c} + ... + \mathbf{n}.$$

1.5. SUBTRACTION OF VECTORS

If there are two vectors **a** and **b**, then

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}),$$

i.e., the subtraction of **b** from **a** may be regarded as the addition of − **b** to **a**. Thus to subtract **b** from **a**, reverse the direction of **b** and add to **a**, (Fig. 1.4).

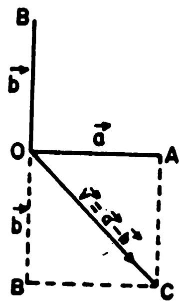


Fig. 1.4

1.6. MULTIPLICATION OF A VECTOR BY A SCALAR

If **a** be any given vector and s a given scalar, then $s\mathbf{a}$ or $\mathbf{a}s$ is defined as a vector whose magnitude is $|s|$ times the magnitude of the vector **a**, *i.e.* $|s|$ times the length of **a**, the support being the same or parallel to that of **a** and direction being the same or opposite to that of **a**, according as s is positive or negative. We thus have

(*i*) $s(-\mathbf{a}) = (-s)\mathbf{a} = -s\mathbf{a}$.

(*ii*) $(-s)(-\mathbf{a}) = s\mathbf{a}$.

(*iii*) $(s+t)\,\mathbf{a} = s\mathbf{a} + t\mathbf{a}$, t being another scalar.

(*iv*) $(st)\,\mathbf{a} = s\,(t\mathbf{a}) = t\,(s\mathbf{a})$.

(*v*) $0\mathbf{a} = \mathbf{0}$, $\mathbf{0}$ being the null vector.

(*vi*) If two non-zero vectors **a** and **b** are collinear, then there exists a non-zero scalar m, such that

$$\mathbf{a} = m\mathbf{b}.$$

Conversely the relation of this type implies that **b** is parallel or collinear to **a**.

(*vii*) If $\hat{\mathbf{a}}$ is the unit vector co-directional with **a** while $a = |\mathbf{a}|$, then

$$\mathbf{a} = a\hat{\mathbf{a}} \text{ or } s\mathbf{a} = s\,(a\hat{\mathbf{a}}) = sa\hat{\mathbf{a}}.$$

Also as $\hat{\mathbf{a}} = \dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{\mathbf{a}}{a}$ and if **b** is parallel to **a**, then $\mathbf{b} = \pm\, b\dfrac{\mathbf{a}}{a}$ according as **b** and **a** have the same or opposite directions.

Note. Division of a vector **a** by a non-zero scalar s is regarded as the multiplication of the vector **a** by a scalar $1/s$.

Problem 3. *If there are two vectors* **a** *and* **b** *represented by the sides AB and BC of a triangle, then show that their resultant is represented by the third side AC. Why is this method equivalent to the parallelogram law of addition?*

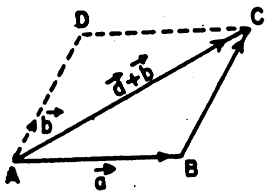


Fig. 1.5

As shown in Fig. 1.5, the vectors **a** and **b** are represented by the sides AB and BC of the triangle. Here $\overrightarrow{AC}$ is a vector drawn between the initial point of **a** and terminal point of **b** and thus may be obtained by parallelogram law of addition, for if we complete the parallelogram $ABCD$, then $\overrightarrow{AC}$ represents a vector along the diagonal of the parallelogram and passing through the common point of the adjacent sides AB and AD representing the vectors **a** and **b**. As such the vector addition obeys the parallelogram law of forces.

Problem 4. *What vector must be added to the two vectors* $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ *and* $2\mathbf{i} + \mathbf{j} - \mathbf{k}$, *so that the resultant may be a unit vector along the x-axis ?*

Suppose that

$$\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k} \text{ and } \mathbf{b} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}.$$

Then $$\mathbf{a} + \mathbf{b} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}.$$

Hence, in order that the resultant of **a** and **b**, *i.e.* $\mathbf{a} + \mathbf{b}$ be **i**, we have to add a vector.

$$\mathbf{i} - (3\mathbf{i} - \mathbf{j} + \mathbf{k}). \; i.e., \; -2\mathbf{i} + \mathbf{j} - \mathbf{k}.$$

1.7. VECTOR SPACE OR LINEAR SPACE

A vector space (or linear space) over a field F is a set V of elements called vectors which may be combined by two primary operations—addition and scalar multiplication ; such that

(*A*) (*i*) If the vectors **a** and **b** belong to V, then **a** + **b** also belongs to V. (This is known as *closure property*).

(*ii*) The vector sum of two vectors **a** and **b** belonging to V, is *commutative, i.e.*

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}.$$

(*iii*) The vector sum of three vectors **a**, **b**, **c** belonging to V, is *associative, i.e.*

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = (\mathbf{a} + \mathbf{b}) + \mathbf{c} = (\mathbf{a} + \mathbf{c}) + \mathbf{b} = \mathbf{a} + (\mathbf{b} + \mathbf{c}).$$

(*iv*) In vector addition there exists an *additive identity vector* known as null vector, such that

$$\mathbf{a} + \mathbf{0} = \mathbf{a}.$$

(*v*) To every vector **a** in V, there corresponds a vector − **a** known *as additive inverse vector*, such that

$$\mathbf{a} + (-\mathbf{a}) = \mathbf{0}.$$

(*B*) (*i*) If m, n are any two scalars and **a** is a vector in V, then *distributive law* holds, *i.e.*,

$$(m + n)\,\mathbf{a} = m\mathbf{a} + n\mathbf{a}.$$

(*ii*) If m is any scalar and **a**, **b** are two vectors belonging to V, then *distributive law of scalar multiplication* holds, *i.e.*

$$(\mathbf{a} + \mathbf{b})\,m = \mathbf{a}m + \mathbf{b}m.$$

(*iii*) If m, n are any two scalars and **a**, is a vector belonging to V, then *associative law* holds, *i.e.*

$$m\,(n\mathbf{a}) = (mn)\,\mathbf{a} = n\,(m\mathbf{a}).$$

(*iv*) To every vector **a** in V, there corresponds *a multiplicative identity scalar*, such that

$$1\mathbf{a} = \mathbf{a}.$$

Note. In case of *scalar quantities* m, n, p, we have the following laws of combination:

(*i*) The *addition is commutative, i.e.*

$$m + n = n + m.$$

(*ii*) The *addition is associative, i.e.*

$$m + n + p = (m + n) + p = (m + p) + n = m + (n + p).$$

(*iii*) There exists *an additive identity scalar* 0, which when added to another scalar, leaves it unchanged, such as

$$m + 0 = m.$$

(*iv*) To every scalar m, there corresponds an *additive inverse scalar* $-m$, such that

$$m + (-m) = 0.$$

In fact m and $-m$ are inverse of each other as their sum is zero (identity scalar).

(*v*) The *multiplication is distributive, i.e.*

$$m.(n + p) = m.n + m.p.$$

(*vi*) The *multiplication is commutative, i.e.*,

$$m.n = n.m.$$

(*vii*) The *mulplication is associative, i.e.*,

$$m.(n.p) = (m.n).p = n.(m.p).$$

(*viii*) There exists *a multiplicative identity scalar* 1, such that

$$m.1 = m.$$

(*ix*) To every non-zero scalar m, there corresponds a *multiplicative inverse scalar* $\frac{1}{m}$, such that

$$m.\frac{1}{m} = 1 \text{ (the identity scalar).}$$

Interpretation: Due to directional properties of vector these laws cannot be applied to vectors and laws for vectos are consistent with the physical problems in which vector quantities occur.

1.8. CONDITIONS FOR A PHYSICAL QUANTITY TO BE REPRESENTABLE BY A VECTOR

We have already mentioned that the vector sum of two or more vectors is inherited in the combination of two or more displacements of a point in flat space. Besides displacements there are other many more physical quantities which enter into combination in accordance with the same invariance properties as displacements. Such quantities are also representable as vectors. Precisely a physical quantity representable by a vector must satisfy the two conditions : (*i*) It must obey the parallelogram law of addition, and (*ii*) It must have a magnitude as well as direction independent of any choice of co-ordinate axes.

Examples of physical quantities representable by a vector are : velocity, acceleration, electric field intensity and magnetic fields, etc.

Note. It should be noted carefully that all quantities having magnitude and direction are not necessarily vectors. For example, consider the rotation of a rigid body about an axis fixed in space. It has got the magnitude as the angle of rotation and the direction as the direction of the axis. But two or more such rotations do not obey the parallelogram law of addition as they cannot be combined according to the vector law of addition, unless the angles of rotation are vanishingly small. Hence the finite rotations cannot be represented as vectors as may be seen by experimental verification.

Problem 5. *Classify which of the following physical measurements are vectors and scalars*:

Volume, velocity, mass, acceleration, length, speed, temperature, momentum, force, power, pressure of a gas, temperature gradient, displacement, work, potential, kinetic energy, electric charge, electric or magnetic intensities, magnetic moment.

In the light of the discussion made in §1.8 and the definitions of vectors and scalars so far introduced the above measurements may be classified as follows :

Scalars	Vectors
Volume	Velocity
Mass	Accerleration
Length	Momentum
Speed	Force
Temperature	Power
Pressure of a gas	Temperature gradient
Work	Displacement
Potential	Electric and magnetic field intensities
Kinetic energy	
Electric charge	Magnetic moment

Problem 6. *Which of the following have representations as vectors?*

(*a*) *Weight*, (*b*) *Specific heat*, (*c*) *Momentum*, (*d*) *Energy*, (*e*) *Speed*, (*f*) *Velocity*, (*g*) *Magnetic field intensity*, (*h*) *Gravitational force*, (*i*) *Kinetic energy*, (*j*) *Age*, (*k*) *Flux*.

Applying the conditions for a physical quantity to be representable as vector as mentioned in §1.8, we observe that the following quantities have representations as vectors :

(*i*) Momentum, (*ii*) Velocity, (*iii*) Magnetic field intensity, (*iv*) Gravitational force.

1.9. RESOLUTION OF VECTORS

(*i*) **Coplanar vectors.** If there are two non-collinear vectors **a** and **b**, then a third vector **r** which is non-collinear with neither of **a** and **b** but can be made to lie in the same plane in which **a** and **b** lie, can be uniquely expressed in terms of **a** and **b** in the manner

$$\mathbf{r} = m\mathbf{a} + n\mathbf{b}$$

where m, n are scalars.

As shown in Fig. 1.6, consider two coplanar vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ such that $\overrightarrow{OP} = \mathbf{a}$ and $\overrightarrow{OQ} = \mathbf{b}$. Now take another vector $\overrightarrow{OC}$ coplanar with $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ such that $\overrightarrow{OC} = \mathbf{r}$.

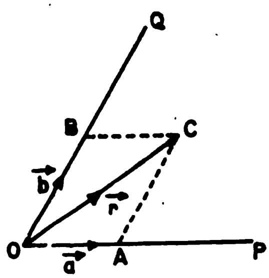


Fig. 1.6

Now take points A and B on OP and OQ respectively, such that $\overrightarrow{OA} = m \, . \, \overrightarrow{OP}$ and $\overrightarrow{OB} = n \, . \, \overrightarrow{OQ}$; m, n being scalars.

It then follows from the parallelogram law of vectors, that

$$\mathbf{r} = \overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$$

$$= \overrightarrow{OA} + \overrightarrow{OB}$$

$$= m\overrightarrow{OP} + \overrightarrow{OQ},$$

$$= m\mathbf{a} + n\mathbf{b}$$

(*ii*) **Non-coplanar vectors.** If **a**, **b**, **c** be three non-coplanar vectors, then any vector **r** can be uniquely expressed as

$$\mathbf{r} = m\mathbf{a} + n\mathbf{b} + p\mathbf{c},$$

where m, n, p are scalars.

Choosing a point O as origin of vectors, let OA, OB, OC be three non-coplanar lines, such that

$$\vec{OA} = \mathbf{a},\ \vec{OB} = \mathbf{b},\ \vec{OC} = \mathbf{c}.$$

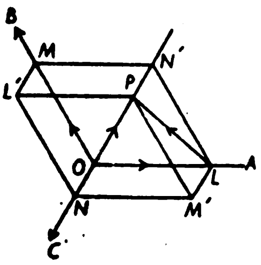


Fig. 1.7

Take any point P, such that $\vec{OP} = \mathbf{r}$, in the space of three dimensions. Through P draw planes parallel to three planes BOC, COA and AOB meeting OA, OB, OC in L, M and N respectively. We thus get a parallelopiped with OP as one of its diagonals. Then

$$\mathbf{r} = \vec{OP} = \vec{OL} + \vec{LP}$$

$$= \vec{OL} + \vec{LN'} + \vec{NP'}$$

$$= \vec{OL} + \vec{OM} + \vec{ON}$$

$$= m\mathbf{a} + n\mathbf{b} + p\mathbf{c},$$

where m, n, p are scalars and $\vec{OL}$, $\vec{OM}$, $\vec{ON}$ being collinear with $\vec{OA}$, $\vec{OB}$, $\vec{OC}$ respectively, we have

$$\vec{OL} = m\mathbf{a},\ \vec{OM} = n\mathbf{b},\ \vec{ON} = p\mathbf{c}.$$

(*iii*) **Components of a vector in three mutually perpendicular directions.** Let OX, OY, OZ be three non-coplanar lines such that each line is perpendicular to the plane containing the other two. The system of axes so chosen form a right handed co-ordinate system such that if OX is turned towards OY about OZ through a small angle, a right handed screw would advance along the positive direction of OZ.

Let $\vec{OP}$ represent the given vector **r** and let the length of orthogonal projections of OP along the axes be x, y, z respectively. It is conventional to take **i**, **j**, **k** as unit vectors along OX, OY, OZ respectively, so that

$$\vec{OA} = x\mathbf{i},\ \vec{OB} = y\mathbf{j},\ \vec{OC} = z\mathbf{k},$$

where OA, OB, OC are collinear with OX, OY, OZ respectively.

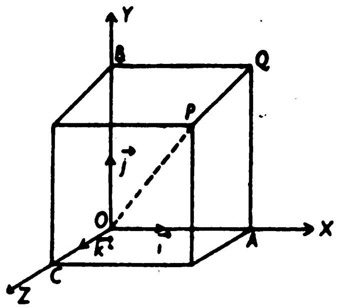


Fig. 1.8

Now $\mathbf{r} = \vec{OP} = \vec{OA} + \vec{AP} = \vec{OA} + \vec{AQ} + \vec{QP}$

$$= \vec{OA} + \vec{OB} + \vec{OC}$$

$$= x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Here $x\mathbf{i}$, $y\mathbf{j}$, $z\mathbf{k}$ are called component vectors of **r** along the directions of **i**, **j**, **k** and are the orthogonal vector projections of **r** along these directions. The scalar projections x, y, z are the rectangular cartesian co-ordinates of the point P referred to O as origin and OX, OY, OZ as axes of reference.

Note 1. **Modulus (magnitude) of a vector.** Modulus of **r**, *i.e.*, $|\vec{OP}|$ is given by

$$\sqrt{(OA^2 + OB^2 + OC^2)},\ i.e.,\ \sqrt{(x^2 + y^2 + z^2)}.$$

Thus if $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then

$$|\mathbf{r}| = \sqrt{(x^2 + y^2 + z)^2} = |\vec{OP}|.$$

Note 2. **Direction Cosines.** If OP makes angles α, β, γ with OX, OY, OZ, respectively, then $x = OP \cos\alpha$, $y = OP \cos\beta$, $z = OP \cos\gamma$, but from note 1, $OP = \sqrt{(x^2 + y^2 + z^2)}$; therefore we have

$$\cos\alpha = \frac{x}{\sqrt{(x^2+y^2+z^2)}}, \cos\beta = \frac{y}{\sqrt{(x^2+y^2+z^2)}}, \cos\gamma = \frac{z}{\sqrt{(x^2+y^2+z^2)}}.$$

The quantities $\cos\alpha, \cos\beta, \cos\gamma$ introduced and defined in this manner are called the *direction cosines* of the line OP with the axes OX, OY, OZ respectively and are usually denoted by l, m, n *respectively i.e.,* $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$

It is obvious that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1,$$

i.e. $$l^2 + m^2 + n^2 = 1.$$

Note 3. **Distance between two points.** Let **a**, **b** be the position vectors of two points A and B whose cartesian coordinates are (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively, then

$$\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k} \; ; \; \mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}.$$

$$\therefore \quad \vec{AB} = \mathbf{b} - \mathbf{a} = (x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j} + (z_2 - z_1)\,\mathbf{k},$$

so that $AB = \left|\vec{AB}\right| = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

Problem 7. *If* $\mathbf{a} = 5\mathbf{i} + 6\mathbf{j} - 4\mathbf{k}$ *and* $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j}$, *find*

(*i*) *Magnitudes of* **a** *and* **b**

(*ii*) *the direction cosines of* **a** *and* **b**.

(*i*) Here $|\mathbf{a}| = \sqrt{\{(5)^2 + (6)^2 + (-4)^2\}} = \sqrt{77}$,

$$|\mathbf{b}| = \sqrt{\{(2)^2 + (3)^2 + (0)^2\}} = \sqrt{13}.$$

(*ii*) Direction cosines of **a** and **b** are respectively

$$\frac{5}{\sqrt{77}}, \frac{6}{\sqrt{77}}, -\frac{4}{\sqrt{77}} \text{ and } \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}, 0.$$ **Ans.**

Problem 8. *If* $\mathbf{A} = 4\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$ *and* $\mathbf{B} = \mathbf{i} + 6\mathbf{j} + \mathbf{k}$, *find the magnitudes and direction cosines of* $(\mathbf{A} + \mathbf{B})$ *and* $(\mathbf{A} - \mathbf{B})$.

Given $\mathbf{A} = 4\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$ and $\mathbf{B} = \mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$,

$$(\mathbf{A} + \mathbf{B}) = 4\mathbf{i} + 6\mathbf{j} + 2\mathbf{k} + \mathbf{i} + 6\mathbf{j} + \mathbf{k} = 5\mathbf{i} + 12\mathbf{j} + 3\mathbf{k}.$$

$\therefore$ magnitude of $(\mathbf{A} + \mathbf{B}) = \sqrt{(5^2 + 12^2 + 3^2)}$,

i.e. $$|\mathbf{A} + \mathbf{B}| = \sqrt{(25 + 144 + 9)} = \sqrt{178}$$

$\therefore$ direction cosines are given by

$$\cos\alpha = \frac{|\mathbf{A} + \mathbf{B}|_x}{|\mathbf{A} + \mathbf{B}|} = \frac{5}{\sqrt{(178)}},$$

$$\cos\beta = \frac{|\mathbf{A} + \mathbf{B}|_y}{|\mathbf{A} + \mathbf{B}|} = \frac{12}{\sqrt{(178)}},$$

$$\cos\gamma = \frac{|\mathbf{A}+\mathbf{B}|_z}{|\mathbf{A}+\mathbf{B}|} = \frac{3}{\sqrt{(178)}},$$

where cos α, cos β and cos γ are direction cosines along x, y and z axes respectively.

$$\mathbf{A} - \mathbf{B} = 4\mathbf{i} + 6\mathbf{j} + 2\mathbf{k} - (\mathbf{i} + 6\mathbf{j} + \mathbf{k})$$
$$= 4\mathbf{i} + 6\mathbf{j} + 2\mathbf{k} - \mathbf{i} - 6\mathbf{j} - \mathbf{k}$$
$$= 3\mathbf{i} + \mathbf{k}.$$

∴ magnitude of $(\mathbf{A} - \mathbf{B}) = |\mathbf{A} - \mathbf{B}| = \sqrt{(3^2 + 1^2)} = \sqrt{10}$.

Direction cosines of (**A** − **B**) are given by

$$\cos\alpha = \frac{|\mathbf{A}-\mathbf{B}|_x}{|\mathbf{A}-\mathbf{B}|} = \frac{3}{\sqrt{10}},$$

$$\cos\beta = \frac{|\mathbf{A}-\mathbf{B}|_y}{|\mathbf{A}-\mathbf{B}|} = 0,$$

$$\cos\gamma = \frac{|\mathbf{A}-\mathbf{B}|_z}{|\mathbf{A}-\mathbf{B}|} = \frac{1}{\sqrt{10}}.$$

Problem 9. *A person travelling eastwards at a rate of 3 m.p.h., finds that the wind seems to blow from the north. On doubling the speed it appears to come from north-east. Find the true velocity of the wind.*

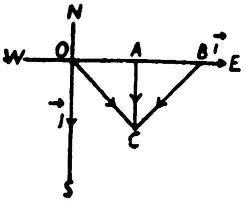


Fig. 1.9

Let **i**, **j** be the unit vectors along east and south, so that by Fig. 1.9, we have

$$\vec{OA} = 3\mathbf{i},\ \vec{OB} = 6\mathbf{i}.$$

The relative velocity of wind is along $\vec{AC} = y\mathbf{j}$ (say)

Actual velocity of the wind

$$= \vec{OA} + \vec{AC} = \vec{OC} = 3\mathbf{i} + y\mathbf{j}$$

Again, the relative velocity of the wind is along BC, such that

$$\angle ABC = 45°,\ \angle BAC = 90° \text{ and so } \angle ACB = 45°,$$
$$\therefore\ AB = AC = 3 \text{ units} = y.$$

∴ Actual velocity of the wind = 3**i** + 3**j** and so

$$\left|\vec{OC}\right| = \sqrt{(9+9)} = 3\sqrt{2} \text{ m.p.h.}$$

Also $\vec{OC}$ makes equal angles with OE and OS, *i.e.*, the wind is blowing from north-west at $3\sqrt{2}$ m.p.h.

Problem 10. *The vectors of magnitudes a, 2a, 3a meet in a point and their directions are along the diagonals of three adjacent faces of a cube. Determine their resultant.*

Consider a cube of unit length and let **i**, **j**, **k** be the unit vectors along the adjacent edges OA, OB, OC as shown in Fig. 1.10.

Also let $\vec{OA}$, $\vec{OQ}$, $\vec{OS}$ be the vectors of magnitudes a, $2a$, $3a$ along the three diagonals, OM, OL, OD of three adjacent faces.

Here $\vec{OL} = \vec{OA} + \vec{AL} = \mathbf{i} + \mathbf{j}$

so that $\hat{OL} = \frac{\vec{OL}}{OL} = \frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}$.

Similarly unit vectors along OM and OD are $\frac{\mathbf{i}+\mathbf{k}}{\sqrt{2}}, \frac{\mathbf{j}+\mathbf{k}}{\sqrt{2}}$.

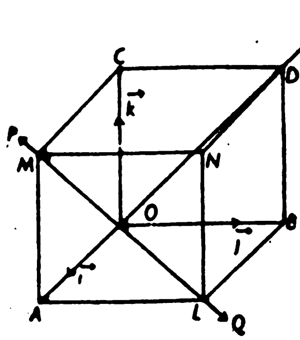


Fig. 1.10

Now

$$\vec{OP} = \left|\vec{OP}\right| \hat{OM} = \frac{a}{\sqrt{2}} (\mathbf{k} + \mathbf{i}).$$

Similarly $\hat{OQ} = \frac{2a}{\sqrt{2}} (\mathbf{i} + \mathbf{j})$ and $\vec{OS} = \frac{3a}{\sqrt{2}}(\mathbf{j} + \mathbf{k})$.

If **R** be the required resultant, then

$$\mathbf{R} = \vec{OP} + \vec{OQ} + \vec{OS} = \frac{3a}{\sqrt{2}}\,\mathbf{i} + \frac{5a}{\sqrt{2}}\,\mathbf{j} + \frac{4a}{\sqrt{2}}\,\mathbf{k}.$$

Its magnitude $= |\mathbf{R}| = \sqrt{\left\{\left(\frac{3a}{\sqrt{2}}\right)^2 + \left(\frac{5a}{\sqrt{2}}\right)^2 + \left(\frac{4a}{\sqrt{2}}\right)^2\right\}} = 5a$ which is inclined at angles $\cos^{-1}\frac{3}{5\sqrt{2}}, \cos^{-1}\frac{1}{\sqrt{2}}, \cos^{-1}\frac{4}{5\sqrt{2}}$ with the edges OA, OB, OC respectively.

Problem 11. *If the resultant of two forces is equal in magnitude to one of the components and perpendicualr to it in direction, find the other component.*

Let **P**, **Q** be two forces inclined at an angle θ and let their magnitudes be P and Q respectively. Also let **i**, **j** be the unit vectors along the direction of **P** and in a direction perpendicular to it. According to the question the resultant of **P** and **Q** is **P** in a direction perpendicular to that of **P**.

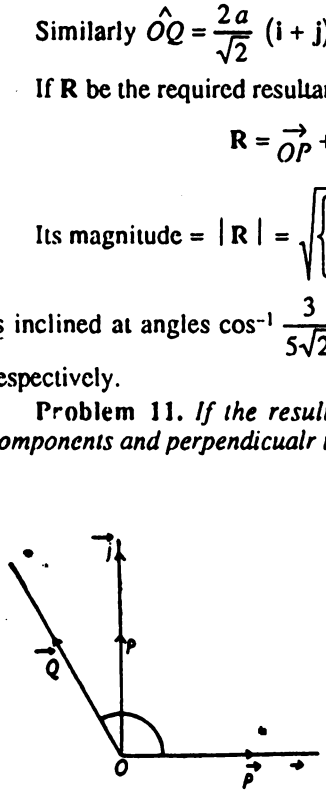


Fig. 1.11

Here

$\mathbf{P} = P\mathbf{i}$ and $\mathbf{Q} = Q\cos\theta\,\mathbf{i} + Q\sin\theta\,\mathbf{j}$, so that $\mathbf{P} + \mathbf{Q} = \mathbf{P}$ gives

$$P\mathbf{i} + Q\cos\theta\,\mathbf{i} + Q\sin\theta\,\mathbf{j} = P\mathbf{j},$$

i.e., $(P + Q\cos\theta)\,\mathbf{i} + (Q\sin\theta - P)\mathbf{j} = 0.$

Equating the coefficients of like vectors on either side, we get

$$P + Q\cos\theta = 0 \text{ and } Q\sin\theta - P = 0$$

i.e. $Q = -\frac{P}{\cos\theta}$ and $Q = \frac{P}{\sin\theta}$.

Dividing, $\tan\theta = -1$, *i.e.*, $\theta = 135°$

Hence $Q = \frac{P}{\sin 135°} = P\sqrt{2}.$

Problem 12. *The base BC of a triangle ABC is divided at G so that mBG = nGC. Show that* $mAB^2 + nAC^2 = mBG^2 + nCG^2 + (m+n)AG^2$.

Taking A as origin of vectors, let position vectors of B and C be **b** and **c** respectively.

Since $mBG = nGC$

i.e. $$\frac{BG}{GC} = \frac{n}{m}$$

or $BG : GC = n : m$,

therefore the position vector of G is given by

$$\vec{AG} = \frac{n\mathbf{c} + m\mathbf{b}}{n+m}.$$

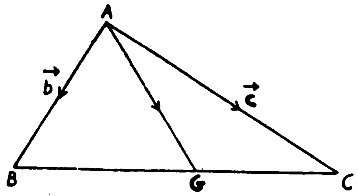


Fig. 1.12

Now

$$mBG^2 + nCG^2 + (m+n)AG^2 = m(\vec{BG})^2 + n(\vec{CG})^2 + (m+n)(\vec{AG})^2$$

$$= m[\vec{BA} + \vec{AG}]^2 + n[\vec{CA} + \vec{AG}]^2 + (m+n)(\vec{AG})^2$$

$$= m\left[-\mathbf{b} + \frac{n\mathbf{c} + m\mathbf{b}}{n+m}\right]^2 + n\left[-\mathbf{c} + \frac{n\mathbf{c} + m\mathbf{b}}{n+m}\right]^2 + (m+n)\left(\frac{n\mathbf{c} + m\mathbf{b}}{n+m}\right)^2$$

$$= \frac{1}{(m+n)^2}[mn^2(\mathbf{c}-\mathbf{b})^2 + nm^2(\mathbf{b}-\mathbf{c})^2 + (m+n)(n\mathbf{c} + m\mathbf{b})^2]$$

$$= \frac{1}{(m+n)^2}[mn(m+n)(\mathbf{b}-\mathbf{c})^2 + (m+n)(n\mathbf{c} + m\mathbf{b})^2]$$

$$= \frac{1}{m+n}[mn\mathbf{b}^2 + mn\mathbf{c}^2 - 2mn\mathbf{b}.\mathbf{c} + n^2\mathbf{c}^2 + m^2\mathbf{b}^2 + 2mn\mathbf{b}.\mathbf{c}],$$

$$= \frac{1}{m+n}[mb^2(n+m) + nc^2(n+m) = mb^2 + nc^2$$

$\therefore$ $\mathbf{b}^2 = b^2$ and $\mathbf{c}^2 = c^2$

$= mAB^2 + nAC^2$ (by properties of dot product)

Problem 13. *The line AB is bisected in* P_1, P_1B *in* P_2, P_2B *in* P_3 *and so on ad infinitum; and the particles of masses* $m, \frac{m}{2}, \frac{m}{2^2}$*...etc., are placed* P_1, P_2, P_3*...etc., respectively. Prove that the distance of their centre of mass from A is equal to one-third of the distance from B to A.*

Taking A as origin let the position vector of B be **b**. The position vectors of the points P_1, P_2, P_3...etc. are $\frac{\mathbf{b}}{2}, \frac{\mathbf{b}}{2^2}, \frac{\mathbf{b}}{2^3}$...... respectively.

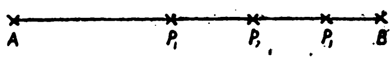


Fig. 1.13

Let G be the required centre of mass; then

$$\vec{AG} = \frac{m\frac{\mathbf{b}}{2} + \frac{m}{2}\cdot\frac{\mathbf{b}}{2^2} + \frac{m}{2^2}\cdot\frac{\mathbf{b}}{2^3} + \ldots}{m + \frac{m}{2} + \frac{m}{2^2} + \ldots}$$

$$= b \left\{ \frac{\frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \ldots}{1 + \frac{1}{2} + \frac{1}{2^2} + \ldots} \right\} = b. \frac{\frac{\frac{1}{2}}{1 - 1/2^2}}{\frac{1}{1 - \frac{1}{2}}} = \frac{b}{3},$$

i.e. $AG = \frac{1}{3}$ of the distance from B to A.

Problem 14. *Prove that*

(*a*) *The internal bisectors of the angles of a triangle are concurrent.*

(*b*) *The medians of a triangle meet in a point of trisection of each other.*

(*Nagpur, 1965*)

(*a*) Consider a triangle ABC, the position vectors of whose vertices A, B, C are $\mathbf{p}, \mathbf{q}, \mathbf{r}$ respectively. Let a, b, c be the lengths of the sides BC, CA, AB respectively.

If AD be the internal bisector of the angle A, then by geometry,

$$BD : DC = AB : AC$$
$$= c : b.$$

$\therefore$ The position vector of

$$D = \frac{b\mathbf{q} + c\mathbf{r}}{b + c}.$$

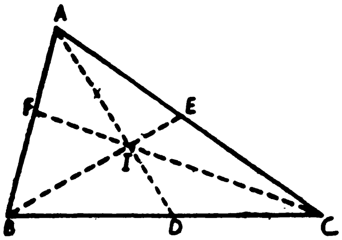


Fig. 1.14

Now the position vector of a point I dividing AD in the ratio $b + c : a$ is

$$= \frac{(b + c). \frac{b\mathbf{q} + c\mathbf{r}}{b + c} + a\mathbf{p}}{b + c + a}$$

$$= \frac{a\mathbf{p} + b\mathbf{q} + c\mathbf{r}}{a + b + c}.$$

The symmetry of this result follows that the point I also lies on the other two internal bisectors, namely BE and CF.

Hence the three internal bisectors of a triangle are concurrent.

(*b*) Consider a triangle ABC, the position vectors of whose vertices are $\mathbf{a}, \mathbf{b}, \mathbf{c}$ respectively. Let D, E, F be the mid. points of the sides BC, CA, AB respectively; then their position vectors are

$$\frac{\mathbf{b} + \mathbf{c}}{2}, \frac{\mathbf{c} + \mathbf{a}}{2}, \frac{\mathbf{a} + \mathbf{b}}{2}$$

respectively.

The position vector of a point G dividing the median AD in the ratio $2 : 1$ is

$$= \frac{2. \frac{\mathbf{b} + \mathbf{c}}{2} + \mathbf{a}}{2 + 1} = \frac{\mathbf{a} + \mathbf{b} + \mathbf{c}}{3}.$$

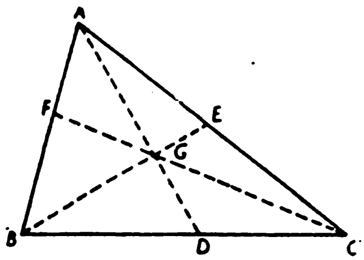


Fig. 1.15

The symmetry of the result follows that the point G also lies on the other two medians, namely BE and CF. Hence the medians of a triangle meet in a point of trisection of each other.

Problem 15. *Forces* **P**, **Q** *act at O and have a resultant R. If any transversal cuts their lines of action at A, B, C respectively, show that*

$$\frac{P}{OA}+\frac{Q}{OB}=\frac{R}{OC}.$$

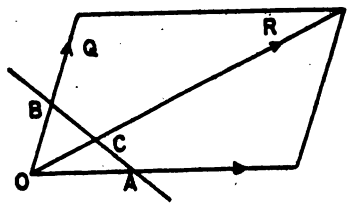


Fig. 1.16

Taking O as origin, let **a**, **b**, **c** be the position vectors of A, B, C respectively; then

$$\hat{\mathbf{a}}=\frac{\mathbf{a}}{OA},\ \hat{\mathbf{b}}=\frac{\mathbf{b}}{OB},\ \hat{\mathbf{c}}=\frac{\mathbf{c}}{OC}.$$

so that

$$\mathbf{P}=P\hat{\mathbf{a}}=\frac{P}{OA}\mathbf{a},\ \mathbf{Q}=\frac{Q}{OB}\mathbf{b},$$

$$\mathbf{R}=\frac{R}{OC}\mathbf{c}.$$

Since $\mathbf{P}+\mathbf{Q}=\mathbf{R},\ \therefore \frac{P}{OA}\mathbf{a}+\frac{Q}{OB}\mathbf{b}=\frac{R}{OC}\mathbf{c}$

or $\frac{P}{OA}\mathbf{a}+\frac{Q}{OB}\mathbf{b}-\frac{R}{OC}\mathbf{c}=0.$

The points A, B, C being collinear, we must have

$$\frac{P}{OA}+\frac{Q}{OB}-\frac{R}{OC}=0,\ i.e.\ \frac{P}{OA}+\frac{Q}{OB}=\frac{R}{OC}.$$

Problem 16. *If the force* $\mathbf{F}_1, \mathbf{F}_2, \ldots \mathbf{F}_n$ *acting in a plane at O are in equilibrium and any transversal cuts their lines of action in points* $L_1, L_2, \ldots, L_n$ *and a length OL is positive when in the same direction as OF, then prove that* $\Sigma\frac{F}{OL}=0$.

Let AB be the given transversal such that the force $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ make angles $\theta_1, \theta_2, \ldots, \theta_n$ with it.

If p be the length of the perpendicular from O to AB, then

$$\sin\theta_1=\frac{p}{OL_1},$$

$$\sin\theta_2=\frac{p}{OL_2}, \ldots, \sin\theta_n=\frac{p}{OL_n}.$$

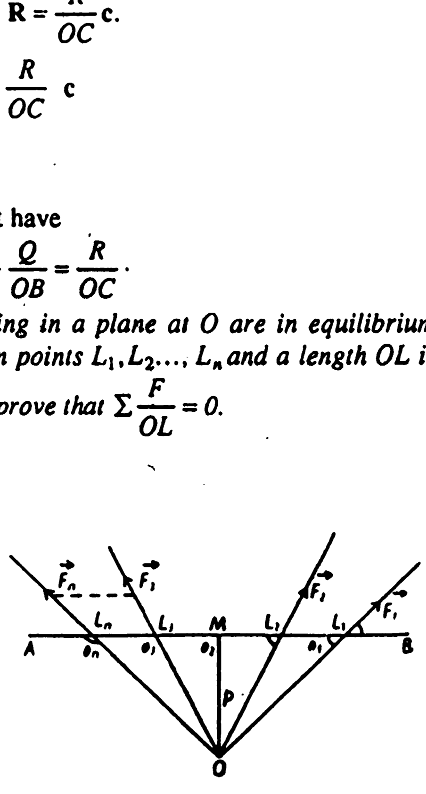


Fig. 1.17

Let **i**, **j** be the unit vectors along and perpendicular to AB and F_1, F_2, F_3, etc., be the magnitudes of the forces, then

$$\mathbf{F}_1=F_1\cos\theta_1\,\mathbf{i}+F_1\sin\theta_1\,\mathbf{j},$$
$$\mathbf{F}_2=F_2\cos\theta_2\,\mathbf{i}+F_2\sin\theta_2\,\mathbf{j},$$
$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$
$$\mathbf{F}_n=F_n\cos\theta_n\,\mathbf{i}+F_n\sin\theta_n\,\mathbf{j}.$$

The system being in equilibrium, we have $\mathbf{F}_1+\mathbf{F}_2+\ldots+\mathbf{F}_n=0$ *i.e.* $(F_1\cos\theta_1+F_2\cos\theta_2+\ldots+F_n\cos\theta_n)\,\mathbf{i}+(F_1\sin\theta_1+F_2\sin\theta_2+\ldots+F_n\sin\theta_n)\,\mathbf{j}=0$. Equating the coefficients of **j** on either side, we get $F_1\sin\theta_1+F_2\sin\theta_2+\ldots+F_n\sin\theta_n=0$

i.e. $F_1 \frac{p}{OL_1} + F_2 \frac{p}{OL_2} + \ldots + F_n \frac{p}{OL_n} = 0$ *i.e.* $\Sigma \frac{F}{OL} = 0.$

Problem 17. *A boy runs 3 miles on a road towards east. It then turns towards north and runs 4 miles before stopping. Find the resultant distance covered by the boy.*

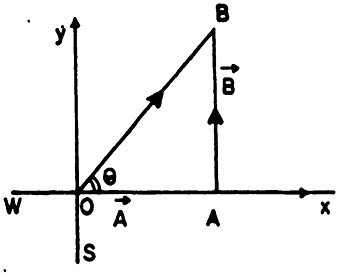


Fig. 1.18

Taking x and y-axes of co-ordinate system along east and north respectively, let **A** and **B** be the two successive displacements given by

$$\mathbf{A} = 3\mathbf{i}.$$
$$\mathbf{B} = 4\mathbf{j}.$$

So that their resultant,

$$\mathbf{r} = \mathbf{A} + \mathbf{B} = 3\mathbf{i} + 4\mathbf{j}$$

$$\therefore \quad |\mathbf{r}| = \sqrt{(3^2 + 4^2)} = \sqrt{(25)}$$

= 5 miles making an angle θ with x-axes where

$$\tan\theta = \frac{\text{coeft. of } \mathbf{j}}{\text{coeft. of } \mathbf{i}} = \frac{4}{3}.$$

Giving $\cos\theta = \frac{3}{5}$, *i.e.* $\theta = \cos^{-1}\frac{3}{5}$.

Problem 18. *A car is driven eastward for a distance of 5 miles, then northward for 3 miles and then in a direction 30° east of north for 10 miles. Draw the vector diagram and determine the total displacement of the car from its starting point.*

Taking x and y-axes towards east and north respectively, and z-axis along the vertical, let the displacements along these axes be **a**, **b**, **c** respectively. Then the resultant displacement **r** is given by

$$\mathbf{r} = \mathbf{a} + \mathbf{b} + \mathbf{c}.$$

But according to the given problem,

$$\mathbf{a} = 5\mathbf{i},\ \mathbf{b} = 3\mathbf{j}$$

and $\mathbf{c} = 10 \sin 30° \,\mathbf{i} + 10 \cos 30\mathbf{j}$

$$= 5\mathbf{i} + 5\sqrt{5}\mathbf{j}.$$

$\therefore$ $\mathbf{r} = 5\mathbf{i} + 3\mathbf{j} + 5\mathbf{i} + 5\sqrt{3}\mathbf{j}$

$$= 10\mathbf{i} + (5\sqrt{3} + 3)\,\mathbf{j}.$$

So that $|\mathbf{r}| = r = \sqrt{\{10^2 + (5\sqrt{3} + 3)^2\}}$

$$= \sqrt{(100 + 75 + 9 + 30\sqrt{3})}$$
$$= (100 + 75 + 9 + 52) = \sqrt{(236)}$$
$$= 15.35 \text{ miles}$$

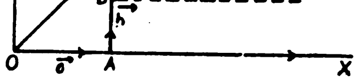


Fig. 1.19

and $\cos\theta = \frac{10}{\sqrt{(236)}}$ or $\theta = \cos^{-1}\frac{10}{\sqrt{(236)}}$.

Hence the resultant displacement has magnitude equal to 15.35 miles making an angle $\cos^{-1}\frac{10}{\sqrt{(236)}}$ north of east, *i.e.* with x-axis

Problem 19. *The projection velocity of a rocket is expressed as*

$$\mathbf{v} = 5\mathbf{i} + 7\mathbf{j} + 9\mathbf{k}$$

where **i**, **j**, **k** *are unit vectors along east, north and vertical direction respectively. Calculate the magnitude of horizontal and vertical components of the velocity. Also deduce the change in angle of projection if the vertical component is doubled.*

(*Agra, B.Sc., 1969*)

The velocity vector is given to be expressed as

$$\mathbf{v} = 5\mathbf{i} + 7\mathbf{j} + 9\mathbf{k}$$

where **i**, **j** and **k** are unit vectors along east, north and vertical directions respectively, which are taken as axes of reference.

Clearly the vertical component of velocity vector = 9**k**.

∴ the mangitude of vertical component of velocity = $|9\mathbf{k}| = 9$.

Horizontal component of velocity vector = 5**i** + 7**j**.

∴ magnitude of horizontal component of velocity

$$= \sqrt{(5^2 + 7^2)} = \sqrt{(25 + 49)} = \sqrt{(74)}.$$

In case the rocket is projected making an angle α with east, then we have

$$\cos\alpha = \frac{|5\mathbf{i}|}{|5\mathbf{i} + 7\mathbf{j} + 9\mathbf{k}|} = \frac{5}{\sqrt{(5^2 + 7^2 + 9^2)}}$$

$$= \frac{5}{\sqrt{(25 + 49 + 81)}} = \frac{5}{\sqrt{(155)}} = \frac{5}{12.45}$$

$$= \cdot 4016$$

∴ $\alpha = 66° \, 17'$ (by tables of cosine).

If the vertical component is doubled, then the velocity vector becomes

$$\mathbf{v}_1 = 5\mathbf{i} + 7\mathbf{j} + 18\mathbf{k}.$$

As such, the angle of projection of rocket with east is given by

$$\cos\beta = \frac{|5\mathbf{i}|}{|5\mathbf{i} + 7\mathbf{j} + 18\mathbf{k}|} = \frac{5}{\sqrt{(5^2 + 7^2 + 18^2)}} = \frac{5}{(398)} = \frac{5}{19 \cdot 95} = \cdot 2507$$

or $\beta = 75° \, 29'$.

Hence the change in angle = $\beta - \alpha = 75° \, 29' - 66° \, 17' = 9° \, 12'$.

1.10. LINEAR COMBINATION OF VECTORS

A vector **v** is termed as a *linear combination* of a set of *n* vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n$, if it is expressible as

$$\mathbf{v} = k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2 + k_3 \mathbf{v}_3 + \ldots + k_n \mathbf{v}_n$$

where $k_1, k_2, k_3, \ldots k_n$ are scalars.

The set of *n* vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n$ is said to be *linearly dependent* if there exists a set of *n* scalars $k_1, k_2, k_3, \ldots k_n$ such that all of them are not zero *i.e.*, at least one of them is non-zero, satisfying the relations.

$$k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2 + \ldots + k_n \mathbf{v}_n = 0$$

If all the scalars $k_1, k_n \ldots k_n$ are zero *i.e.* $k_1 = k_2 = \ldots = k_n = 0$ then the set of *n* vectors $\mathbf{v}_1, \mathbf{v}_2 \ldots \mathbf{v}_n$ combined as

$$k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2 + \ldots + k_n \mathbf{v}_n = 0$$

is said to be *linearly independent*.

Note 1. It is evident that if a set *m* ($m < n$) out of *n* vectors is linearly dependent, then the whole set containing *n* vectors is linearly dependent.

Note 2. If the set of n vectors is linearly independent, then clearly any subset of these n vectors will also be linearly independent.

Problem 20. *If a set of n (> 1) vectors is linearly dependent then at least one of these n vectors can be expressed as a linear combination of the remaining (n – 1) vectors.*

Let $\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_n$ be a set of n vectors which are linearly dependent. This set being linearly dependent, we must have a linear combination of the type

$$k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2 + \ldots + k_n \mathbf{v}_n = 0$$

where all of the scalars $k_1, k_2 \ldots, k_n$ are not zero.

Suppose $k_r \neq 0$. Then, we can write

$$\mathbf{v}_r = -\left(\frac{k_1}{k_r} \mathbf{v}_1 + \frac{k_2}{k_r} \mathbf{v}_2 + \ldots + \frac{k_n}{k_r} \mathbf{v}_n\right)$$

which can be expressed as

$$\mathbf{v}_r = k_1' \mathbf{v}_1 + k_2' \mathbf{v}_2 + \ldots + k_n' \mathbf{v}_n$$

thereby proving the proposition.

Problem 21. *Show that any set of n vectors containing the null vector is linearly dependent.*

Let $\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_n$ be a set of n vectors, of which $\mathbf{v}_n$ is a null vector *i.e.* $\mathbf{v}_n = \mathbf{0}$. Then by note 1 of § 1.10, if the set $\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_{n-1}$ of $n - 1$ vectors is linearly dependent, the set of n vectors containing these $(n - 1)$ vectors and the null vector will also be linearly dependent.

In the case when the set $\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_{n-1}$ of $(n - 1)$ vectors is linearly independent, then by definition there exists a set of $(n - 1)$ scalars $k_1, k_2 \ldots, k_{n-1}$, all of which being zero, such that

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_{n-1} \mathbf{v}_{n-1} = \mathbf{0}.$$

Assuming that $k_n \neq 0$ and $\mathbf{v}_n = \mathbf{0}$, this relation will still hold in the form

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_{n-1} \mathbf{v}_{n-1} + k_n\mathbf{v}_n = \mathbf{0}.$$

Where all the scalars $k_1, k_2 \ldots, k_n$ are not zero, as $k_n \neq 0$.

This follows that the set of n vectors $\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_n$ is linearly dependent thereby proving the proposition.

Problem 22. *Show that the set of vectors* $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ *given by*

$$\mathbf{r}_1 = 2\mathbf{a} - 3\mathbf{b} + \mathbf{c},\ \mathbf{r}_2 = 3\mathbf{a} - 5\mathbf{b} + 2\mathbf{c},\ \mathbf{r}_3 = 4\mathbf{a} - 5\mathbf{b} + \mathbf{c},$$

$\mathbf{a}, \mathbf{b}, \mathbf{c}$ *being non-zero and non-coplanar vectors, is linearly dependent.*

The vectors $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ will be linearly dependent if there exists a set of scalars k_1, k_2, k_3 not all zero, such that

$$k_1\mathbf{r}_1 + k_2\mathbf{r}_2 + k_3\mathbf{r}_3 = 0 \qquad \ldots(1)$$

i.e., $k_1\,(2\mathbf{a} - 3\mathbf{b} + \mathbf{c}) + k_2\,(3\mathbf{a} - 5\mathbf{b} + 2\mathbf{c}) + k_3\,(4\mathbf{a} - 5\mathbf{b} + \mathbf{c}) = 0$

or $(2k_1 + 3k_2 + 4k_3)\,\mathbf{a} - (3k_1 + 5k_2 + 5k_3)\,\mathbf{b} + (k_1 + 2k_2 + k_3)\,\mathbf{c} = \mathbf{0}$

$\mathbf{a}, \mathbf{b}, \mathbf{c}$ being non-zero and non-coplanar, this relation will hold only if the coefficients of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ separately vanish, *i.e.*, if

$$2k_1 + 3k_2 + 4k_3 = 0 \qquad \ldots(2)$$

$$3k_1 + 5k_2 + 5k_3 = 0 \qquad \ldots(3)$$

$$k_1 + 2k_2 + k_3 = 0 \qquad \ldots(4)$$

Multiplying (4) by 5 and then subtracting from (3) . . find

$2k_1 + 5k_2 = 0$ which is satisfied for $k_1 = 5, k_2 = -2$ (non-zero values) and then (4) yields $k_3 = -1$.

These values of k_1, k_2, k_3 also satisfy (2) and hence the relation (1) is expressible in the form

$$5\mathbf{r}_1 - 2\mathbf{r}_2 - \mathbf{r}_3 = 0 \text{ or } \mathbf{r}_3 = 5\mathbf{r}_1 - 2\mathbf{r}_2$$

showing that the set of vectors $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ is linearly dependent.

Problem 23. *Show that the set of vectors* $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ *given by*

$$\mathbf{r}_1 = \mathbf{j} - 2\mathbf{k},\ \mathbf{r}_2 = \mathbf{i} - \mathbf{j} + \mathbf{k},\ r_3 = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$$

is linearly independent.

The vectors $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ will be linearly independent if there exists a set of scalars k_1, k_2, k_3 all being zero, such that

$$k_1\mathbf{r}_1 + k_2\mathbf{r}_2 + k_3\mathbf{r}_3 = 0 \qquad ...(1)$$

i.e. $k_1(\mathbf{j} - 2\mathbf{k}) + k_2(\mathbf{i} - \mathbf{j} + \mathbf{k}) + k_3(\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = 0$

or $\quad (k_2 + k_3)\,\mathbf{i} + (k_1 - k_2 + 2k_3)\,\mathbf{j} + (-2k_1 + k_2 + k_3)\,\mathbf{k} = 0 \qquad ...(2)$

Now $\mathbf{i}, \mathbf{j}, \mathbf{k}$ being non-coplanar, this relation will hold if

$$k_2 + k_3 = 0 \qquad ...(3)$$

$$k_1 - k_2 + 2k_3 = 0 \qquad ...(4)$$

$$-2k_1 + k_2 + k_3 = 0. \qquad ...(5)$$

Solving (4) and (5),

$$\frac{k_1}{-1-2} = \frac{k_2}{-4-1} = \frac{k_3}{1-2}$$

i.e. $$\frac{k_1}{3} = \frac{k_2}{5} = \frac{k_3}{1} = \lambda \text{ (say)}$$

These equations give,

$$k_1 = 3\lambda,\ k_2 = 5\lambda,\ k_3 = \lambda$$

Substituting these values in (3) we find,

$$6\lambda = 0 \text{ i.e., } \lambda = 0$$

As such, we have

$$k_1 = 0 = k_2 = k_3$$

showing that the given set of vectors is linearly independent.

Problem 24. *Show that a necessary and sufficient condition for the vectors*

$$\mathbf{r}_1 = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k},\ \mathbf{r}_2 = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k},\ \mathbf{r}_3 = x_3\mathbf{i} + y_3\mathbf{j} + z_3\mathbf{k}$$

to be linearly independent is that the determinant

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \text{ be different from zero.}$$

The set of given vectors will be linearly independent if there exists a set of scalars k_1, k_2, k_3 all being zero, such that

$$k_1\mathbf{r}_1 + k_2\mathbf{r}_2 + k_3\mathbf{r}_3 = 0$$

i.e. $\quad k_1(x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}) + k_2(x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}) + k_3(x_3\mathbf{i} + y_3\mathbf{j} + z_3\mathbf{k}) = 0$

or $\quad (x_1k_1 + x_2k_2 + x_3k_3)\,\mathbf{i} + (y_1k_1 + y_2k_2 + y_3k_3)\,\mathbf{j} + (z_1k_1 + z_2k_2 + z_3k_3)\,\mathbf{k} = 0$

But i, j, k being non-coplanar, this result will hold only if

$$\left.\begin{aligned} x_1k_1 + x_2k_2 + x_3k_3 = 0 \\ y_1k_1 + y_2k_2 + y_3k_3 = 0 \\ z_1k_1 + z_2k_2 + z_3k_3 = 0 \end{aligned}\right\} \quad \ldots(1)$$

Elimination of k_1, k_2, k_3 from these equations with the help of a determinant yields

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = 0 \quad \ldots(2)$$

But the given set of vectors being linearly independent, we have

$$k_1 = k_2 = k_3 = 0$$

in which case equations (1) are not solvable in the form (2) showing that the given set of vectors will be linearly independent if and only if

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} \neq 0$$

otherwise the scalars k_1, k_2, k_3 will be different from zero.

1.11. PRODUCT OF TWO VECTORS

A careful observation of the ways in which two vector quantities enter into combinations in various branches of mathematics and mechanics leads us to define two well marked and distinct kinds of products, one being called *scalar or dot product* and other being called *vector or cross product*. The former yields a number (scalar) while the latter, a vector quantity. In either case the product is jointly proportional to the modules (moduli) of the two vectors.

Conventionally, the scalar or dot product of two vectors **a** and **b** is denoted by **a** · **b** or (**a**, **b**) and their vector or cross product by **a** × **b** or [**ab**].

(1) The Scalar or Dot Product of Two Vectors

Defnition. The scalar or dot product of two vectors **a** and **b**, with modules a and b respectively and their directions being inclined at an angle θ, is defined to be the real number $ab \cos \theta$, *i.e.*

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \theta.$$

Characteristics of dot product. (*i*) The dot product of two vectors **a** and **b** is independent of their order

i.e. $$\mathbf{a} \cdot \mathbf{b} = ab \cos \theta = \mathbf{b} \cdot \mathbf{a}$$

(*ii*) The dot product of two vectors **a** and **b** may be expressed as the product of two numbers, one being the length of one vector and the other resolute of the second in the direction of the first, *i.e.*

$$\mathbf{a} \cdot \mathbf{b} = (\text{length of } \mathbf{a}) \text{ times } (\text{scalar projection of } \mathbf{b} \text{ onto } \mathbf{a})$$
$$= (\text{length of } \mathbf{b}) \text{ times } (\text{scalar projection of } \mathbf{a} \text{ onto } \mathbf{b}).$$

(*iii*) If **a** · **b** = 0, then either of the two vectors is a null vector or the vectors **a** and **b** are mutually perpendicular, *i.e.*

$$\mathbf{a} = 0 \text{ or } \mathbf{b} = 0 \text{ or } \theta = \tfrac{1}{2}\pi.$$

In particular **i** · **j** = **j** · **k** = **k** · **i** = 0 ; **i**, **j**, **k** being mutually perpendicular unit vectors.

(*iv*) The vectors **a** and **b** are parallel if $\theta = 0$ or π, *i.e.*, if $\mathbf{a} \cdot \mathbf{b} = \pm ab$, where a, b are modules of **a** and **b** respectively.

(*v*) The scalar product of two equal vectors **a**, **a** is given by

$$\mathbf{a} \cdot \mathbf{a} = a \cdot a \cos 0° = a^2, \text{ since then } \theta = 0$$

In case **a** is a unit vector *i.e.*, $\mathbf{a} = \hat{\mathbf{a}}$ then $|\hat{\mathbf{a}}| = 1$ so that $\hat{\mathbf{a}} \cdot \hat{\mathbf{a}} = 1^2 = 1$

In particular $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$.

(*vi*) The scalar product of two unit vectors $\hat{\mathbf{a}}, \hat{\mathbf{b}}$ is given by

$$\hat{\mathbf{a}} \cdot \hat{\mathbf{b}} = \cos \theta \text{ since then } |\hat{\mathbf{a}}| = 1 = |\hat{\mathbf{b}}|$$

(*vii*) *The scalar product is associative i.e.* if **a**, **b** be any two vectors and m, n be any two scalars, then

$$(m\mathbf{a}) \cdot (n\mathbf{b}) = mn\,(\mathbf{a} \cdot \mathbf{b}) = mn\,\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot mn\mathbf{b} = n\mathbf{a} \cdot m\mathbf{b}$$

(*viii*) The scalar product being a number, can occur as the numerical coefficient of any vector, *e.g.* $(\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}$ represents a vector parallel to **c** and whose module is $(\mathbf{a} \cdot \mathbf{b})$ times that of **c**.

(*ix*) In the case of scalar product, the distributive law of multiplication holds *i.e.* if **a**, **b**, **c** be three vectors, then

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

Referred to Fig. 1.20, let $\vec{OA} = \mathbf{a}$, $\vec{OB} = \mathbf{b}$, $\vec{OC} = \mathbf{c}$ and projections of OB and BC on OA be respectively OM and MN, so that

$$ON = OM + MN.$$

It is also clear that

$$\vec{OC} = \vec{OB} + \vec{BC} = \mathbf{b} + \mathbf{c}$$

Now, $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \vec{OC}$ = (length of **a**) times (scalar projection of $\vec{OC}$ onto **a**)

$= a(ON)$, a being module of **a**

$= a(OM + MN)$

$= a(OM) + a\,(MN)$

$= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ by (*ii*)

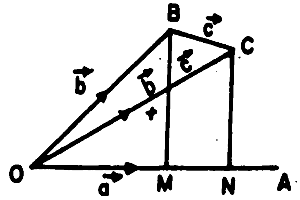


Fig. 1.20

In general, we have

$(\mathbf{a} + \mathbf{b} + \mathbf{c} \ldots) \cdot (\mathbf{l} + \mathbf{m} + \mathbf{n} \ldots) = \mathbf{a} \cdot \mathbf{l} + \mathbf{a} \cdot \mathbf{m} + \ldots + \mathbf{b} \cdot \mathbf{l} + \mathbf{b} \cdot \mathbf{m} + \ldots$ and *in particular*, $(\mathbf{a} \pm \mathbf{b}) \cdot (\mathbf{a} \pm \mathbf{b}) = (\mathbf{a} \pm \mathbf{b})^2$

$$= \mathbf{a}^2 \pm 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b}^2$$

and $$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \mathbf{a}^2 - \mathbf{b}^2.$$

(*x*) If θ be the angle between two vectors **a** and **b** whose orthogonal projections (components) in the directions of axes of x, y, z are (a_1, a_2, a_3) and (b_1, b_2, b_3) respectively and if **i**, **j**, **k** be the unit vectors along the axes, then

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$
$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k},$$

so that $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{ab} = \dfrac{(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\cdot(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})}{|(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})|\cdot|(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{j})|}$

$$= \frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{(a_1^2+a_2^2+a_3^2)}\sqrt{(b_1^2+b_2^2+b_3^2)}}.$$

(2) The Vector or Cross Product of Two Vectors

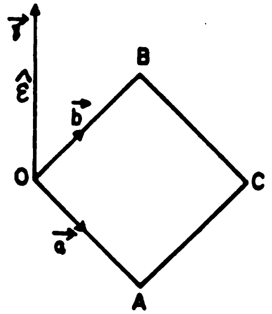


Fig. 1.21

Definition. Given two vectors **a** and **b** whose directions are inclined at an angle θ, their vector product is defined to be the vector **r**, whose module is $ab \sin\theta$ and whose direction is perpendicular to both **a** and **b**, being positive relative to a rotation from **a** to **b**, *i.e.*,

$$\mathbf{r} = \mathbf{a}\times\mathbf{b} = |\mathbf{a}|\ |\mathbf{b}|\ \sin\theta\ \hat{\varepsilon}$$
$$= ab\sin\theta\ \hat{\varepsilon}$$

where $\hat{\varepsilon}$ is a unit vector perpendicular to the plane of **a** and **b**, and has the same direction as is obtained by the motion of a right handed screw due to rotation from **a** to **b**, and a, b are the modules of **a** and **b** respectively.

Characteristics of vector product. (*i*) *The vector product is not commutative, i.e.*, by reversing the order of the factors, tne sign of the product is reversed, *e.g.*

$$\mathbf{b}\times\mathbf{a} = ba\sin(-\theta)\ \hat{\varepsilon} = -ab\sin\theta\ \hat{\varepsilon} = -\mathbf{a}\times\mathbf{b}$$

(*ii*) The magnitude of the vector product $\mathbf{a}\times\mathbf{b}$ is equal to the area of the parallelogram of which **a** and **b** are adjacent sides *i.e.*

$$|\mathbf{a}\times\mathbf{b}| = |ab\sin\theta\ \hat{\varepsilon}| = ab\sin\theta, \text{ as } |\hat{\varepsilon}| = 1$$

= OA multiplied by the perpendicular distance of OA from B.

= Area of the parallelogram $OACB$.

(*iii*) *The vector product is associative, i.e.*, if m be a scalar **a**, **b** be two vectors, then

$$(m\mathbf{a})\times\mathbf{b} = \mathbf{a}\times(m\mathbf{b}) = m(\mathbf{a}\times\mathbf{b}) = m(ab\sin\theta\ \hat{\varepsilon})$$

(*iv*) *The vectors* **a** *and* **b** *are parallel, if the angle* θ included between their directions is 0 or π *i.e.*, if $\theta = 0$ or π so that

$$\mathbf{a}\times\mathbf{b} = 0 \text{ as } \sin\theta = 0 \text{ for } \theta = 0 \text{ or } \pi$$

which follows that the vector product of two parallel vectors is a null vector.

(*v*) *The vector product of two equal vectors* **a**, **a** *is given by*

$$\mathbf{a}\times\mathbf{a} = 0.$$

Since the two vectors are equal if they are either collinear or parallel. So that the angle θ between them being 0 or π, $\sin\theta = 0$ and hecne the result follows.

In particular, if **i**, **j**, **k** are the unit vectors along the principal axes, then

$$\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = 0.$$

(*vi*) The two vectors **a** and **b** with modules a and b respectively, will be mutually perpendicular if the angle θ between their directions is 90°, so that $\sin\theta = 1$.

As such if **a**, **b** are at right angles, then $\mathbf{a}\times\mathbf{b} = ab\ \hat{\varepsilon}$, $\hat{\varepsilon}$ being a unit vector normal to the plane containing **a** and **b**.

In case a and b are unit vectors, then $|\hat{\mathbf{a}}| = 1$ and $|\hat{\mathbf{b}}| = 1$, therefore $\hat{\mathbf{a}} \times \hat{\mathbf{b}} = \hat{\varepsilon}$, which shows that the cross product of two mutually perpendicular unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ is a unit vector $\hat{\varepsilon}$ normal to the plane of $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$.

Hence, in particular if i, j, k be the unit vectors along the principal axes, then

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} = -\mathbf{j} \times \mathbf{i}$$
$$\mathbf{j} \times \mathbf{k} = \mathbf{i} = -\mathbf{k} \times \mathbf{j}$$
$$\mathbf{k} \times \mathbf{i} = \mathbf{j} = -\mathbf{i} \times \mathbf{k}$$

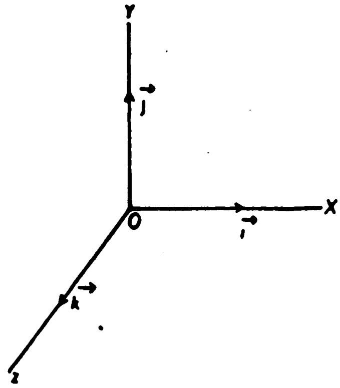


Fig. 1.22

(*vii*) The vector product of two unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ is given by

$$\hat{\mathbf{a}} \times \hat{\mathbf{b}} = \sin\theta\,\hat{\varepsilon} \text{ as } |\hat{\mathbf{a}}| = |\hat{\mathbf{b}}| = 1,$$

where θ is the angle between their directions and $\hat{\varepsilon}$ is the unit vector normal to the plane of $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$.

(*viii*) *The distributive law holds, i.e.* in case of vector product if a, b, c are three vectors, then

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}.$$

Let the components of a, b, c along the principal axes be (a_1, a_2, a_3); (b_1, b_2, b_3) and (c_1, c_2, c_3) respectively. Then if i, j, k be the unit vectors along the axes, we have

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$
$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$
$$\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$$

So that, $\mathbf{b} + \mathbf{c} = (b_1 + c_1)\mathbf{i} + (b_2 + c_2)\mathbf{j} + (b_3 + c_3)\mathbf{k}$

$$\therefore \mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times \{(b_1 + c_1)\mathbf{i} + (b_2 + c_2)\mathbf{j} + (b_3 + c_3)\mathbf{k}\}$$
$$= \{a_2(b_3 + c_3) - a_3(b_2 + c_2)\}\mathbf{i} + \{a_3(b_1 + c_1) - a_1(b_3 + c_3)\}\mathbf{j} + \{a_1(b_2 + c_2) - a_2(b_1 + c_1)\}\mathbf{k}$$

as $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ etc.

$$= \{(a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}\} + \{(a_2c_3 - a_3c_2)\mathbf{i} + (a_3c_1 - a_1c_3)\mathbf{j} + (a_1c_2 - a_2c_1)\mathbf{k}\}$$
$$= \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}.$$

[Since $\mathbf{a} \times \mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$

$$= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

Similarly $\mathbf{a} \times \mathbf{c} = (a_2c_3 - a_3c_2)\,\mathbf{i} + (a_3c_1 - a_1c_3)\mathbf{j} + (a_1c_2 - a_2c_1)\mathbf{k}$]

In general, $(\mathbf{a} + \mathbf{b} + \mathbf{c} + \ldots) \times (\mathbf{l} + \mathbf{m} + \mathbf{n} + \ldots)$

$$= \mathbf{a} \times \mathbf{l} + \mathbf{a} \times \mathbf{m} + \ldots + \mathbf{b} \times \mathbf{l} + \mathbf{b} \times \mathbf{m} + \ldots + \ldots$$

(*ix*) *Vector product in terms of components.* Consider two vectors a and b whose components are (a_1, a_2, a_3), (b_1, b_2, b_3) along the principal axes. Then if i, j, k are the unit vectors along these axes, we have

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$
$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$
$$\therefore \quad \mathbf{a}\times\mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\times(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$$
$$= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

as $\mathbf{i}\times\mathbf{j} = \mathbf{k} = -\mathbf{j}\times\mathbf{i}$ etc.

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

(Agra, 1954)

Now if θ be the angle between the directions of **a** and **b** and $\hat{\varepsilon}$, a unit vector normal to the plane of **a** and **b**, then

$$(\mathbf{a}\times\mathbf{b})^2 = (ab\sin\theta\ \hat{\varepsilon})^2 = \{(a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}\}^2$$

i.e., $a^2b^2\sin^2\theta = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$ as $\hat{\varepsilon}^2 = 1$

or $$\sin^2\theta = \frac{(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2}{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_3^2 + b_3^2)}$$

as $\mathbf{a}^2 = a^2 = a_1^2 + a_2^3 + a_3^2$ etc.

(*x*) *Cross product in terms of dot product.* By definition,

$$(\mathbf{a}\times\mathbf{b})^2 = (ab\sin\theta\ \hat{\varepsilon})^2$$
$$= a^2b^2\sin^2\theta,$$
$$= a^2b^2(1 - \cos^2\theta)$$
$$= a^2b^2 - a^2b^2\cos^2\theta$$
$$= \mathbf{a}^2\mathbf{b}^2 - (\mathbf{a}.\mathbf{b})^2, \quad \because \ \mathbf{a}^2 = a^2,\ \mathbf{b}^2 = b^2, \mathbf{a}\cdot\mathbf{b} = ab\cos\theta$$

Problem 25. *If* **a** *and* **b** *are unit vectors and* θ *is the angle between them, show that*

$$\sin\frac{\theta}{2} = \frac{1}{2}|\mathbf{a} - \mathbf{b}|.$$

We have $|\mathbf{a} - \mathbf{b}|^2 = (\mathbf{a} - \mathbf{b})^2$

$$= a^2 + b^2 - 2\mathbf{a}\cdot\mathbf{b}, \quad \because \ \mathbf{a}^2 = a^2 = 1 \text{ and } \mathbf{b}^2 = b^2 = 1$$
$$= 2 - 2\cos\theta$$
$$= 4\sin^2\theta/2,$$

i.e. $\quad |\mathbf{a} - \mathbf{b}| = 2\sin\theta/2,$

so that $\quad \sin\theta/2 = \frac{1}{2}|\mathbf{a} - \mathbf{b}|.$

Problem 26. *From the relations (Lorentz transformation equations in theory of relativity).*

$$\begin{cases} \mathbf{r}' = \mathbf{r} + \left[\dfrac{\bar{\gamma} - 1}{\mathbf{v}^2}\mathbf{v}\cdot\mathbf{r} - \bar{\gamma}t\right]\mathbf{v}, \\ t' = \bar{\gamma}\left[t - \dfrac{\mathbf{v}\cdot\mathbf{r}}{c^2}\right]. \end{cases}$$

where $\bar{\gamma} = \dfrac{c}{\sqrt{(c^2 - v^2)}}$, *prove the reciprocal relations*

$$\begin{cases} \mathbf{r} = \mathbf{r}' + \left[\dfrac{\bar{\gamma}-1}{v^2}\mathbf{v}\cdot\mathbf{r}' + \bar{\gamma}t'\right]\mathbf{v}, \\ t' = \bar{\gamma}\left[t' + \dfrac{\mathbf{v}\cdot\mathbf{r}'}{c^2}\right], \end{cases}$$

Here, $\mathbf{r}' + \left[\dfrac{\bar{\gamma}-1}{v^2}\ \mathbf{v}\cdot\mathbf{r}' + \bar{\gamma}t'\right]\mathbf{v}$

$$= \mathbf{r} + \left[\frac{\bar{\gamma}-1}{v^2}\mathbf{v}\cdot\mathbf{r} - \bar{\gamma}t\right]\mathbf{v} + \left[\frac{\bar{\gamma}-1}{v^2}\mathbf{v}\cdot\left\{\mathbf{r} + \left(\frac{\bar{\gamma}-1}{v^2}\mathbf{v}\cdot\mathbf{r} - \bar{\gamma}t\right)\mathbf{v}\right\} + \bar{\gamma}^2\left(t - \frac{\mathbf{v}\cdot\mathbf{r}}{c^2}\right)\right]\mathbf{v}$$

$$= \mathbf{r} + \frac{\gamma-1}{v^2}(\mathbf{v}\cdot\mathbf{r})\mathbf{v} - \gamma t\mathbf{v} + \frac{\gamma-1}{v^2}(\mathbf{v}\cdot\mathbf{r})\mathbf{v} + \left(\frac{\gamma-1}{v^2}\right)^2(\mathbf{v}\cdot\mathbf{r})v^2\mathbf{v}$$

$$- \frac{\gamma-1}{v^2}\gamma t v^2\mathbf{v} + \gamma^2 t\mathbf{v} - \frac{\gamma^2}{c^2}\mathbf{v}\cdot\mathbf{r}\mathbf{v}$$

$$= \mathbf{r} + 2\frac{(\gamma-1)}{v^2}(\mathbf{v}\cdot\mathbf{r})\mathbf{v} - \gamma t\mathbf{v} + \frac{(\gamma-1)^2}{v^2}(\mathbf{v}\cdot\mathbf{r})\mathbf{v} - (\gamma-1)\gamma t\mathbf{v}$$

$$+ \gamma^2 t\mathbf{v} - \frac{\mathbf{v}\cdot\mathbf{r}}{c^2 - v^2}\mathbf{v} \quad \left[\because \gamma^2 = \frac{c^2}{c^2 - v^2}\right]$$

$$= \mathbf{r} + \frac{\gamma-1}{v^2}(\mathbf{v}\cdot\mathbf{r})\ \mathbf{v}\ (2 + \gamma - 1) - \gamma t\mathbf{v} - \gamma^2 t\mathbf{v} + \gamma t\mathbf{v} + \gamma^2 t\mathbf{v} - \frac{\mathbf{v}\cdot\mathbf{r}}{c^2 - v^2}\mathbf{v}$$

$$= \mathbf{r} + \frac{\gamma^2 - 1}{v^2}(\mathbf{r}\cdot\mathbf{v})\mathbf{v} - \frac{\mathbf{v}\cdot\mathbf{r}}{c^2 - v^2}\mathbf{v}$$

$$= \mathbf{r} + \frac{1}{v^2}\left(\frac{c^2}{c^2 - v^2} - 1\right)(\mathbf{v}\cdot\mathbf{r})\ \mathbf{v} - \frac{\mathbf{v}\cdot\mathbf{r}}{c^2 - v^2}\mathbf{v}$$

$$= \mathbf{r} + \frac{\mathbf{v}\cdot\mathbf{r}}{c^2 - v^2}\ \mathbf{v} - \frac{\mathbf{v}\cdot\mathbf{r}}{c^2 - v^2}\ \mathbf{v} = \mathbf{r}.$$

Again $\bar{\gamma}\left[t' + \dfrac{\mathbf{v}\cdot\mathbf{r}'}{c^2}\right]$

$$= \bar{\gamma}\left[\bar{\gamma}\left(t - \frac{\mathbf{v}\cdot\mathbf{r}}{c^2}\right) + \frac{1}{c^2}\bar{\mathbf{v}}\cdot\left\{\mathbf{r} + \left(\frac{\bar{\gamma}-1}{v^2}\mathbf{v}\cdot\mathbf{r} - \bar{\gamma}t\right)\mathbf{v}\right\}\right]$$

$$= \bar{\gamma}^2 t - \bar{\gamma}^2\frac{\mathbf{v}\cdot\mathbf{r}}{c^2} + \bar{\gamma}\frac{\mathbf{v}\cdot\mathbf{r}}{c^2} - \frac{\bar{\gamma}}{c^2}\cdot\frac{\bar{\gamma}-1}{v^2}(\mathbf{v}\cdot\mathbf{r})v^2 - \frac{\bar{\gamma}^2 t}{c^2}v^2$$

$$= \frac{c^2 t}{c^2 - v^2} - \frac{\mathbf{v}\cdot\mathbf{r}}{c^2 - v^2} + \bar{\gamma}\frac{\mathbf{v}\cdot\mathbf{r}}{c^2} + \frac{\bar{\gamma}}{c^2}\mathbf{v}\cdot\mathbf{r} - \bar{\gamma}\frac{\mathbf{v}\cdot\mathbf{r}}{c^2} - \frac{v^2 t}{c^2 - v^2}$$

$$= \frac{(c^2 - v^2)t}{c^2 - v^2} - \frac{\mathbf{v}\cdot\mathbf{r}}{c^2 - v^2} + \frac{\mathbf{v}\cdot\mathbf{r}}{c^2 - v^2} \quad \left[\text{since } \frac{\bar{\gamma}^2}{c^2} = \frac{1}{c^2 - v^2}\right]$$

$$= t.$$

Problem 27. *What is the meaning of* $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})$ *for the case where* $\mathbf{a}^2=\mathbf{b}^2$?

Here
$$(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = \mathbf{a}^2-\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{a}-\mathbf{b}^2$$
$$= \mathbf{a}^2-\mathbf{b}^2 \text{ as } \mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$$
$$= 0 \text{ as } \mathbf{a}^2=\mathbf{b}^2.$$

This shows that either $\mathbf{a}+\mathbf{b}=0$, $\mathbf{a}-\mathbf{b}=0$ or the vectors $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$ are mutually at right angles.

In the former case when $\mathbf{a}+\mathbf{b}=0$, or $\mathbf{a}-\mathbf{b}=0$, we have $\mathbf{a}=0$ and $\mathbf{b}=0$, *i.e.* both the vectors **a** and **b** are null vectors.

Conclusively, either both the vectors **a** and **b** are null vectors or the angle between the vectors $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$ is $\frac{1}{2}\pi$.

Problem 28. *What is the unit vector perpendicular to each of the vectors* $2\mathbf{i}-\mathbf{j}+\mathbf{k}$ *and* $3\mathbf{i}+4\mathbf{j}-\mathbf{k}$? *Calculate the sine of the angle between these vectors.*

Let $\mathbf{a}=2\mathbf{i}-\mathbf{j}+\mathbf{k}$ and $\mathbf{b}=3\mathbf{i}+4\mathbf{j}-\mathbf{k}$.

If $\hat{\varepsilon}$ be a unit vector perpendicular to the plane of **a** and **b**, then since $\mathbf{a}\times\mathbf{b}$ is also a vector perpendicular to the plane of **a** and **b**, we have
$$\hat{\varepsilon}=\frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}. \qquad \ldots(1)$$

Now $\mathbf{a}\times\mathbf{b}=(2\mathbf{j}-\mathbf{j}+\mathbf{k})\times(3\mathbf{i}+4\mathbf{j}-\mathbf{k})$
$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & -1 & 1\\ 3 & 4 & -1\end{vmatrix}=-3\mathbf{i}+5\mathbf{j}+11\mathbf{k}$$

$\therefore$ $\quad |\mathbf{a}\times\mathbf{b}| = |-3\mathbf{i}+5\mathbf{j}+11\mathbf{k}| = \sqrt{(9+25+121)} = \sqrt{(155)}$.

Hence
$$\hat{\varepsilon}=\frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}=\frac{-3\mathbf{i}+5\mathbf{j}+11\mathbf{k}}{\sqrt{(155)}}.$$

Again
$$|\mathbf{a}| = |2\mathbf{i}-\mathbf{j}+\mathbf{k}| = \sqrt{(4+1+1)} = \sqrt{6}$$
$$|\mathbf{b}| = |3\mathbf{i}+4\mathbf{j}-\mathbf{k}| = \sqrt{(9+16+1)} = \sqrt{26}.$$

Thus if θ is the angle between the directions of **a** and **b**, then
$$\mathbf{a}\times\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\sin\theta\,\hat{\varepsilon}$$

i.e.
$$\sin\theta=\frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|\,\hat{\varepsilon}}=\frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}|\,|\mathbf{b}|}\text{ by (1)}$$
$$=\frac{\sqrt{(155)}}{\sqrt{6}\,\sqrt{26}}=\sqrt{\left(\frac{155}{156}\right)}.$$

Problem 29. *If* $\mathbf{a}=3\mathbf{i}+4\mathbf{j}-5\mathbf{k}$ *and* $\mathbf{b}=-\mathbf{i}+2\mathbf{j}+6\mathbf{k}$, *then calculate*

(*i*) *the module of each,*

(*ii*) *the scalar product* $\mathbf{a}\cdot\mathbf{b}$,

(*iii*) *the vector sum and difference* $\mathbf{a}+\mathbf{b}$ *and* $\mathbf{a}-\mathbf{b}$.

(*i*) We have
$$|\mathbf{a}| = |3\mathbf{i}+4\mathbf{j}-5\mathbf{k}|$$
$$= \sqrt{(9+16+25)} = 5\sqrt{2}$$

and
$$|\mathbf{b}| = |-\mathbf{i}+2\mathbf{j}+6\mathbf{k}|$$
$$= \sqrt{(1+4+36)} = \sqrt{41}.$$

(*ii*) $\mathbf{a} \cdot \mathbf{b} = (3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}) \cdot (-\mathbf{i} + 2\mathbf{j} + 6\mathbf{k})$
$= 3(-1) + 4.2 + (-5).6$
$= -3 + 8 - 30 = -25.$

(*iii*) $\mathbf{a} + \mathbf{b} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k} + (-\mathbf{i} + 2\mathbf{j} + 6\mathbf{k})$
$= 2\mathbf{i} + 6\mathbf{j} + \mathbf{k},$
$\mathbf{a} - \mathbf{b} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k} - (-\mathbf{i} + 2\mathbf{i} + 6\mathbf{k}) = 4\mathbf{i} + 2\mathbf{j} - 11\mathbf{k}.$

Problem 30. *Show that* **a** is perpendicular to **b** *if* $|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$.

We have $|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$.

Squaring both sides, we get

$$\mathbf{a}^2 + \mathbf{b}^2 + 2\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^2 + \mathbf{b}^2 - 2\mathbf{a} \cdot \mathbf{b}$$

or $$4\mathbf{a} \cdot \mathbf{b} = 0$$

or $$\mathbf{a} \cdot \mathbf{b} = 0,$$

which shows that **a** and **b** are mutually at right angles.

Problem 31. *Two particles emitting from a source have displacements* $\mathbf{r}_1 = 4\mathbf{j} + 3\mathbf{j} + 8\mathbf{k}$ *and* $\mathbf{r}_2 = 2\mathbf{i} + 10\mathbf{j} + 5\mathbf{k}$ *at any time. Find the displacement of second particle relative to first.*

Required displacement $= \mathbf{r}_2 - \mathbf{r}_1$
$= 2\mathbf{i} + 10\mathbf{j} + 5\mathbf{k} - (4\mathbf{i} + 3\mathbf{j} + 8\mathbf{k})$
$= -2\mathbf{i} + 7\mathbf{j} - 3\mathbf{k}.$

Problem 32. *Find the scalar and vector products of the vectors* **A** *and* **B**, *where* $\mathbf{A} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ *and* $\mathbf{B} = 4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$. *Also find the angle between* **A** *and* **B**.

Given $\mathbf{A} = 2\mathbf{i} + \mathbf{j} + \mathbf{k},$
$\mathbf{B} = 4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}.$

Scalar product $= \mathbf{A} \cdot \mathbf{B}$
$= (2\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$
$= 8 + 2 - 3 = 7.$

Vector product $= \mathbf{A} \times \mathbf{B}$
$= (2\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$
$= 4\mathbf{k} + 6\mathbf{j} - 4\mathbf{k} - 3\mathbf{i} + 4\mathbf{j} - 2\mathbf{i}$ ... (1)
$= -5\mathbf{i} + 10\mathbf{j}.$

According to definition of scalar proeduct,

$$\mathbf{A} \cdot \mathbf{B} = AB \cos \theta, \quad \text{...(2)}$$

where θ is the angle between **A** and **B**.

$$A = |\mathbf{A}| = \sqrt{(A_x^2 + A_y^2 + A_z^2)}$$
$$= \sqrt{(4 + 1 + 1)} = \sqrt{6},$$
$$B = |\mathbf{B}| = \sqrt{(B_x^2 + B_y^2 + B_z^2)} = \sqrt{(16 + 4 + 9)}$$
$$= \sqrt{29}.$$

Substituting values of $(\mathbf{A} \cdot \mathbf{B})$, A and B in equation (2), we get

$$7 = \sqrt{29}\sqrt{6} \cos \theta.$$

$$\therefore \quad \cos \theta = \frac{7}{\sqrt{(29 \times 6)}} = \frac{7}{\sqrt{(174)}}.$$

i.e., $$\theta = \cos^{-1} \frac{7}{\sqrt{(174)}}.$$

Problem 33. *Prove that an angle inscribed in a semi-circle is a right angle.*

Referred to Fig. 1.23 ACB is a semi-circle with AB as bounding diameter and C any point on its circumference. Let O be its centre and r the radius. Also let $\vec{AO} = \mathbf{a} = \vec{OB}$ and $\vec{OC} = \mathbf{b}$.

Now $$\vec{AC} = \vec{AO} + \vec{OC} = \mathbf{a} + \mathbf{b}$$

and $$\vec{CB} = \vec{CO} + \vec{OB} = -\mathbf{b} + \mathbf{a}.$$

$$\therefore \quad \vec{AC} \cdot \vec{CB} = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})$$
$$= \mathbf{a}^2 - \mathbf{b}^2$$
$$= AO^2 - OC^2 \text{ as } \mathbf{a}^2 = \vec{AO}^2 = AO^2 \text{ etc.}$$
$$= r^2 - r^2$$
$$= 0.$$

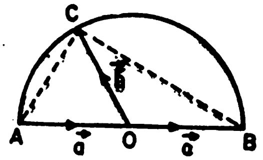


Fig. 1.23

But from the definition of dot product,

$$\cos \angle ACB = \frac{\vec{AC} \cdot \vec{CB}}{|\vec{AC}||\vec{CB}|} = 0.$$

$$\angle ACB = \tfrac{1}{2}\pi.$$

Problem 34. *Prove that the area of a triangle whose two sides are* **A** *and* **B** *is given by* $\frac{1}{2} |\mathbf{A} \times \mathbf{B}|$. *Also find the direction-cosines of normal to this area.*

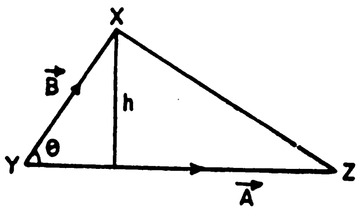


Fig. 1.24

Area of the ΔXYZ

$$= \tfrac{1}{2} A.h$$
$$= \tfrac{1}{2} AB \sin \theta$$
$$= \tfrac{1}{2} |\mathbf{A} \times \mathbf{B}|.$$

The vector area is perpendicular to the plane containing A and B. And magnitude of area

$$= \tfrac{1}{2} |\mathbf{A} \times \mathbf{B}|.$$

$$\therefore \text{Vector area} = \tfrac{1}{2}(\mathbf{A} \times \mathbf{B}) = \tfrac{1}{2}(\mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z) \times (\mathbf{i}B_x + \mathbf{j}B_y + \mathbf{k}B_z)$$
$$= \tfrac{1}{2}\mathbf{i}(A_y B_z - A_z B_y) + \tfrac{1}{2}\mathbf{j}(A_z B_x - A_x B_z) + \tfrac{1}{2}\mathbf{k}(A_x B_y - A_y B_x).$$

Thus, direction-cosines of the normal to the given area are given by

$$\cos\alpha = \frac{A_y B_z - A_z B_y}{2|\mathbf{A} \times \mathbf{B}|}, \quad \cos\beta = \frac{1}{2} \frac{A_z B_x - A_x B_z}{|\mathbf{A} \times \mathbf{B}|} \quad \text{and} \quad \cos\gamma = \frac{A_x B_y - A_y B_x}{2|\mathbf{A} \times \mathbf{B}|}$$

Note. Area of the parallelogram with sides **A** and **B** is double of the above area, *i.e.*, ($\mathbf{A} \times \mathbf{B}$).

Problem 35. *Show that* $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) + \mathbf{b} \times (\mathbf{c} + \mathbf{a}) + \mathbf{c} \times (\mathbf{a} + \mathbf{b}) = 0$.

We have $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) + \mathbf{b} \times (\mathbf{c} + \mathbf{a}) + \mathbf{c} \times (\mathbf{a} + \mathbf{b}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

$+ \mathbf{b} \times \mathbf{c} + \mathbf{b} \times \mathbf{a} + \mathbf{c} \times \mathbf{a} + \mathbf{c} \times \mathbf{b}$

$= \mathbf{a} \times \mathbf{b} - \mathbf{c} \times \mathbf{a} + \mathbf{b} \times \mathbf{c} - \mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{a} - \mathbf{b} \times \mathbf{c} = 0.$

1.12. TRIPLE PRODUCTS OF VECTORS

We have stated that the vector product of two vectors **b** and **c** is a vector quantity. So this product (**b** × **c**) may be multiplied scalarly or vectorially with a third vector **a** to give two triple products namely **a**. (**b** × **c**) and **a** × (**b** × **c**). The former being a scalar quantity is termed as *scalar triple product* and the latter being a vector quantity is called *vector triple product.*

(1) The Scalar Triple Product

Difinition. Let **a, b, c** be three vectors. Then the scalar product of any of these vectors with the vector product of the other two such as a. (**b** × **c**) is called scalar triple product of the vectors **a, b, c** and denoted by [**abc**] or [**a, b, c**]. Obviously, this type of triple product is a scalar quantity.

Note. The scalar triple product is sometimes known as Box Product.

Characteristics of Scalar Triple Product. (*i*) Geometrically interpreted as below, the scalar triple product of three vectors **a, b, c** represents the volume of a parallelopiped which has for its three coterminus edges the vectors **a, b, c.**

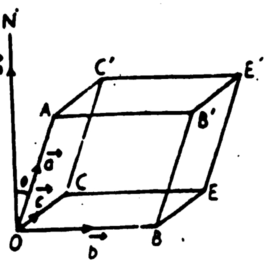


Fig. 1.25

Construct a parallelopiped with coterminus edges OA, OB and OC, such that $\vec{OA} = \mathbf{a}$, $\vec{OB} = \mathbf{b}$, $\vec{OC} = \mathbf{c}$.

Suppose that $\mathbf{n} = \mathbf{b} \times \mathbf{c}$ and its direction is ON, which is perpendicular to the plane $OBEC$ whose adjacent sides are **b** and **c**. The direction of the vector **n** is positive in the sense of rotation from OB to OC.

From the property of vector product of two vectors, it follows that $|\mathbf{n}|$ measures the area of the parallelogram $OBEC$.

Volume of the parallelopiped with coterminus edges OA, OB, OC

= (area of the parallelogram of the base $OBEC$) multiplied by the perpendicular distance of the plane $OBEC$ from the point A)

= (Area of the parallelogram $OBEC$) multiplied by (the scalar projection of OA on ON)

$= |\mathbf{n}| \, (OA \cos \theta)$, where $\angle AQN = \theta$

$= |\mathbf{n}| \; |\mathbf{a}| \cos \theta$, since $OA = |\vec{OA}| = |\mathbf{a}|$

$= \mathbf{a} \cdot \mathbf{n}$ from the definition of dot product

$= \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = [\mathbf{abc}] = V$ (say)

where V measures the volume of the parallelopiped.

The product is regarded as positive or negative according as θ is acute or obtuse. It is easy to show that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \pm V,$$

i.e., $[\mathbf{abc}] = [\mathbf{bca}] = [\mathbf{cab}] = \pm V.$

This follows that if *the cyclic order of the occurrence of the vectors* **a**, **b**, **c** *is maintained, the position of cross and dot may be interchanged without changing the value of the product.*

Again since $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ etc., therefore

$$[\mathbf{bac}] = [\mathbf{cba}] = -[\mathbf{abc}]\,] = \pm V.$$

(*ii*) If the vectors **a**, **b**, **c** are coplanar, then their scalar triple product is zero, *i.e.* $[\mathbf{abc}] = 0$.

Since the volume of the parallelopiped, so formed with coplanar vectors **a**, **b**, **c** as coterminus edges will be zero.

As such $[\mathbf{aab}] = [\mathbf{abb}] = [\mathbf{cbc}]$ etc. $= 0$.

(*iii*) The scalar triple product may be expressed in terms of components.

Let **a**, **b**, **c** be three vectors whose magnitudes in right handed system of unit vectors **i**, **j**, **k** are (a_1, a_2, a_3), (b_1, b_2, b_3) and (c_1, c_2, c_3) respectively. Then

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k},$$
$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k},$$
$$\mathbf{c} = c_1\mathbf{i} + c_1\mathbf{j} + c_3\mathbf{k},$$

We have, $\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (b_2c_3 - b_3c_2)\,\mathbf{i} + (b_3c_1 - b_1c_3)\mathbf{j} + (b_1c_2 - b_2c_1)\mathbf{k}.$

$$\therefore \quad \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \cdot \{(b_2c_3 - b_3c_2)\,\mathbf{i} + (b_3c_1 - b_1c_3)\,\mathbf{j} + (b_1c_2 - b_2c_1)\,\mathbf{k}\};$$

i.e. $$[\mathbf{abc}] = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$$
$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

In particular, $[\mathbf{ijk}] = [\mathbf{jki}] = [\mathbf{kij}] = 1$

$[\mathbf{ikj}] = [\mathbf{kji}] = [\mathbf{jik}] = -1.$

In general if the three vectors **a**, **b**, **c** are resolved in terms of three non-coplanar vectors **l**, **m**, **n**, then

$$\mathbf{a} = a_1\mathbf{l} + a_2\mathbf{m} + a_3\mathbf{n},$$
$$\mathbf{b} = b_1\mathbf{l} + b_2\mathbf{m} + b_3\mathbf{n},$$
$$\mathbf{c} = c_1\mathbf{l} + c_2\mathbf{m} + c_3\mathbf{n}.$$

We have $(\mathbf{b} \times \mathbf{c}) = (b_2c_3 - b_3c_2)\,\mathbf{m} \times \mathbf{n} + (b_3c_1 - b_1c_3)\,\mathbf{n} \times \mathbf{l} + (b_1c_2 - b_2c_1)\,\mathbf{l} \times \mathbf{m}$

$$\therefore \quad \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = a_1(b_2c_3 - b_3c_2)\,[\mathbf{lmn}] + a_2(b_3c_1 - b_1c_3)\,[\mathbf{mnl}] + a_3(b_1c_2 - b_2c_1)\,[\mathbf{nlm}]$$

$$= \begin{vmatrix} a_1 & a_2 & a_2 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} . [\mathbf{lmn}] \qquad \text{as } [\mathbf{lmn}] = [\mathbf{mnl}] = [\mathbf{nlm}].$$

(2) The Vector Triple Product.

Definition. The product of the type $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is called the vector triple product of given three vectors **a**, **b**, **c** and is expressed as

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}.$$

Suppose that $\mathbf{q} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

Then, **q** being a vector product of two vectors **a** and $(\mathbf{b} \times \mathbf{c})$ represents a vector perpendicular to both **a** and $(\mathbf{b} \times \mathbf{c})$ and therefore by the property of dot product, we have

$$\mathbf{q} \cdot \mathbf{a} = 0 \text{ and } \mathbf{q} \cdot (\mathbf{b} \times \mathbf{c}) = 0.$$

But the product $\mathbf{b} \times \mathbf{c}$ being a vector product of two vectors **b** and **c** is itself a vector perpendicular, to both **b** and **c**, *i.e.* a vector normal to the plane of **b** and **c**.

If follows that **q** lies in the plane of **b** and **c**, so that **q** is expressible in terms of **b** and **c**.

Suppose that $\mathbf{q} = s\mathbf{b} + t\mathbf{c}$, where s and t are scalars.

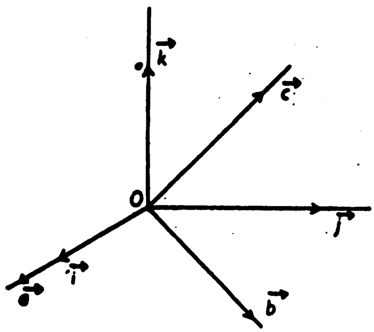


Fig. 1.26

Multiplying both sides scalarly by **a**, we get

$$\mathbf{q} \cdot \mathbf{a} = s\,(\mathbf{a} \cdot \mathbf{b}) + t\,(\mathbf{a} \cdot \mathbf{c}),$$

i.e., $\dfrac{s}{\mathbf{a} \cdot \mathbf{c}} = \dfrac{-t}{\mathbf{a} \cdot \mathbf{b}} = \lambda$ (say) (since $\mathbf{q} . \mathbf{a} = 0$)

Then

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{q} = \lambda\,[(\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}]. \quad \ldots (1)$$

In order to find λ, let us introduce an orthogonal right handed system of three unit vectors **i**, **j**, **k** such that **i** is along **a** and **j** is perpendicular to it in the plane of **a** and **b**, the direction of **k** is automatically decided because, **i**, **j**, **k** form a right handed system of vectors. Then

$$\mathbf{a} = a_1\mathbf{i},\ \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j},\ \mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}.$$

Now $\mathbf{b} \times \mathbf{c} = b_2c_3\mathbf{i} - b_1c_3\mathbf{j} + (b_1c_2 - b_2c_1)\,\mathbf{k}$

so that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (a_1b_2c_1 - a_1b_1c_2)\,\mathbf{j} - a_1b_1c_3\,\mathbf{k}$.

Also $(\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} = (a_1b_1c_1\mathbf{i} + a_1b_2c_1\mathbf{j})$ and $(\mathbf{a} \cdot \mathbf{b})\,\mathbf{c} = a_1b_1c_1\mathbf{i} + a_1b_1c_2\mathbf{j} + a_1b_1c_3\,\mathbf{k}$.

Putting these values in (1), we get, $\lambda = 1$

Substituting this value of λ in (1), we find

$$\mathbf{q} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}$$ (*Vikram, 1969*)

Characteristics of Vector Triple Product.

The vector triple product is not associative, *i.e.*,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}.$$

Since $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}$

and $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = -\,\mathbf{c} \times (\mathbf{a} \times \mathbf{b})$

$$= -\,[(\mathbf{c} \cdot \mathbf{b})\,\mathbf{a} - (\mathbf{c} \cdot \mathbf{a})\,\mathbf{b}]$$

$$= (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\,\mathbf{a};$$

which follows that the product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ represents a vector coplanar with **b** and **c** while the product $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ represents a vector lying in the plane of **a** and **b**. Hence the two products do not represent the same vector quantity, *i.e.*,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}.$$

Problem 36 (*a*). *If* $\mathbf{A} = 4\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$, $\mathbf{B} = 2\mathbf{i} - 10\mathbf{j} - 7\mathbf{k}$ *and* $\mathbf{C} = 5\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}$ *deduce the values of*

(*i*) $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$ *and* (*ii*) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$. (*Agra, 1969*)

(*iii*) *Unit vectors perpendicular to* **A** *and lying in the plane of* **B** *and* **C**.

(*b*) *Find the unit vectors which are perpendicular to vector* $2\mathbf{i}-\mathbf{j}-3\mathbf{k}$ *and lie in the plane of vector* $7\mathbf{i}-\mathbf{j}-\mathbf{k}$ *and* $\mathbf{i}+5\mathbf{j}-3\mathbf{k}$. (*Rohilkhand, 1987*)

(*a*) (*i*) Given

$$\mathbf{A} = 4\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$$
$$\mathbf{B} = 2\mathbf{i} - 10\mathbf{j} - 7\mathbf{k}$$
$$\mathbf{C} = 5\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}.$$

$$\because \quad (\mathbf{A}\times\mathbf{B}) = (4\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}) \times (2\mathbf{i} - 10\mathbf{j} - 7\mathbf{k})$$
$$= -40\mathbf{k} + 28\mathbf{j} + 10\mathbf{k} + 35\mathbf{i} + 6\mathbf{j} + 30\mathbf{i}$$
$$= 65\mathbf{i} + 34\mathbf{j} - 30\mathbf{k}.$$

$$\therefore \quad (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} = (65\mathbf{i} + 34\mathbf{j} - 30\mathbf{k})\cdot(5\mathbf{i} + 7\mathbf{j} - 4\mathbf{k})$$
$$=325 + 238 + 120$$
$$= 683.$$

(*ii*)
$$(\mathbf{B}\times\mathbf{C}) = (2\mathbf{i} - 10\mathbf{j} - 7\mathbf{k}) \times (5\mathbf{i} + 7\mathbf{j} - 4\mathbf{k})$$
$$= 14\mathbf{k} + 8\mathbf{j} + 50\mathbf{k} + 40\mathbf{i} - 35\mathbf{j} + 49\mathbf{i}$$
$$= 89\mathbf{i} - 27\mathbf{j} + 64\mathbf{k}.$$

$$\therefore \quad \mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (4\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}) \times (89\mathbf{i} - 27\mathbf{j} + 64\mathbf{k})$$
$$= -108\mathbf{k} - 256\mathbf{j} + 445\mathbf{k} - 320\mathbf{i} + 267\mathbf{j} + 81\mathbf{i}$$
$$= -239\mathbf{i} + 11\mathbf{j} + 337\mathbf{k}.$$

(*iii*) The required unit vectors are

$$\pm \frac{\mathbf{A}\times(\mathbf{B}\times\mathbf{C})}{|\mathbf{A}\times(\mathbf{B}\times\mathbf{C})|}$$
$$= \pm \frac{-239\mathbf{i} + 11\mathbf{j} + 337\mathbf{k}}{\sqrt{\{(-239)^2 + (11)^2 + (337)^2\}}}$$
$$= \pm \frac{-239\mathbf{i} + 11\mathbf{j} + 337\mathbf{k}}{\sqrt{(170811)}}$$

(b) Proceeding as in (a) (*iii*), the requied vectors are

$$\pm \frac{3\mathbf{i} - 3\mathbf{j} + \mathbf{k}}{\sqrt{(19)}}.$$

Problem 37. *Show that* $[\mathbf{a}\times\mathbf{b}, \mathbf{b}\times\mathbf{c}, \mathbf{c}\times\mathbf{a}] = [\mathbf{abc}]$

$$= \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix}$$ *by means of determinant.*

Let the components of the vectors **a**, **b**, **c** in the directions of the axes of x, y, z along which **i**, **j**, **k** are the unit vectors, be (a_1, a_2, a_3), (b_1, b_2, b_3) and (c_1, c_2, c_3) respectively. Then

$$\left.\begin{aligned} \mathbf{a} &= a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \\ \mathbf{b} &= b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} \\ \mathbf{c} &= c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k} \end{aligned}\right\}, \qquad \ldots (1)$$

so that

$$\left.\begin{aligned} &\mathbf{a}\cdot\mathbf{a} = a_1^2 + a_2^2 + a_3^2 = \Sigma a_1^2 \text{(say)},\ \mathbf{b}\cdot\mathbf{b} = \Sigma b_1^2,\ \mathbf{c}\cdot\mathbf{c} = \Sigma c_1^2 \\ &\mathbf{a}\cdot\mathbf{b} = \Sigma a_1 b_1,\ \mathbf{b}\cdot\mathbf{c} = \Sigma b_1 c_1 \text{ and } \mathbf{c}\cdot\mathbf{a} = \Sigma c_1 a_1 \end{aligned}\right\} \qquad \ldots (2)$$

Now, $[\mathbf{abc}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ from (1)

$$[\mathbf{abc}]^2 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} \Sigma a_1^2 & \Sigma a_1 b_1 & \Sigma c_1 a_1 \\ \Sigma a_1 b_1 & \Sigma b_1^2 & \Sigma b_1 c_1 \\ \Sigma c_1 a_1 & \Sigma b_1 a_1 & \Sigma c_1^2 \end{vmatrix} = \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix} \text{ from (2)}$$

Again, Let $\Delta \equiv \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = [\mathbf{abc}]$.

Then, if $A_1, B_1, C_1 \ldots$ etc. be the cofactors of $a_1, b_1, c_1 \ldots$ etc., we have

$$\left.\begin{aligned} a_1A_1 + a_2A_2 + a_3A_3 &= \Delta \\ a_1B_1 + a_2B_2 + a_3B_3 &= 0 \\ a_1C_1 + a_2C_2 + a_3C_3 &= 0 \text{ etc.} \end{aligned}\right\} \quad \ldots(3)$$

where $A_1 = b_2c_3 - b_3c_2$ etc.

Suppose that $\Delta' \equiv \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$

$$\Delta\Delta' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$= \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} \text{ from (3)}$$

$$= \Delta^3,$$

i.e. $\Delta' = \Delta^2$

or $$[\mathbf{abc}]^2 = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \quad \ldots(4)$$

But $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\mathbf{i} + (b_1a_3 - b_3a_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$

Similarly $\mathbf{b} \times \mathbf{c} = (b_2c_3 - b_3c_2)\,\mathbf{i} + (c_1b_3 - c_3b_1)\,\mathbf{j} + (b_1c_2 - b_2c_1)\,\mathbf{k}$

$\mathbf{c} \times \mathbf{a} = (c_2a_3 - c_3a_2)\,\mathbf{i} + (a_1c_3 - a_3c_1)\,\mathbf{j} + (c_1a_2 - c_2a_1)\,\mathbf{k}.$

$\therefore\ [\mathbf{a} \times \mathbf{b}, \mathbf{b} \times \mathbf{c}, \mathbf{c} \times \mathbf{a}]$

$$= \begin{vmatrix} a_2b_3 - a_3b_2 & b_1a_3 - b_3a_1 & a_1b_2 - a_2b_1 \\ b_2c_3 - b_3c_2 & c_1b_3 - c_3b_1 & b_1c_2 - b_2c_1 \\ c_2a_3 - c_3a_2 & a_1c_3 - a_3c_1 & c_1a_2 - c_2a_1 \end{vmatrix}$$

$$= \begin{vmatrix} C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \quad \text{when } A_1, B_1, C_1 \ldots\text{etc. are the cofactors in } \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$= [\mathbf{abc}]^2$ from (4).

Problem 38. *Show that* $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0.$

We have
$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}$$
$$\mathbf{b} \times (\mathbf{c} \times \mathbf{a}) = (\mathbf{b} \cdot \mathbf{a})\,\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\,\mathbf{a}$$
$$\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{c} \cdot \mathbf{b})\,\mathbf{a} - (\mathbf{c} \cdot \mathbf{a})\,\mathbf{b}.$$

Adding all together, we get
$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0.$$

Problem 39. *Find the volume of a parallelopiped whose three coterminus edges are described by the vectors* $\mathbf{i} + 2\mathbf{j}$, $4\mathbf{j}$ *and* $\mathbf{j} + 3\mathbf{k}$.

Volume of the required parallelopiped

= scalar triple product of the vectors $\mathbf{i} + 2\mathbf{j}$, $4\mathbf{j}$ and $\mathbf{j} + 3\mathbf{k}$

$= (\mathbf{i} + 2\mathbf{j}) \cdot \{4\mathbf{j} \times (\mathbf{j} + 3\mathbf{k})\}$

$= (\mathbf{i} + 2\mathbf{j}) \cdot 12\mathbf{i}$ as $\mathbf{j} \times \mathbf{j} = 0$ and $\mathbf{j} \times \mathbf{k} = \mathbf{i}$

$= 12$ $\mathbf{i} \cdot \mathbf{i} = 1$ and $\mathbf{j} \cdot \mathbf{i} = 0$.

Problem 40. *Show that the law of refraction of light passing from a medium of refractive index* μ *into one of index* μ' *is expressed by the equation* $\mu\mathbf{a} \times \mathbf{n} = \mu'\mathbf{c} \times \mathbf{n}$, *where* $\mathbf{n}, \mathbf{a}, \mathbf{c}$ *are the unit vectors perpendicular to the boundary, along the incident and along the refracted ray respectively.*

Also find the law of reflection, if $\mathbf{b}$ *be a unit vector in the direction of the reflected ray.*

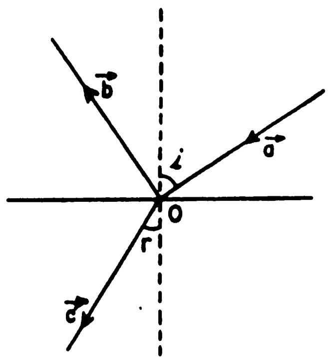


Fig. 1.27

Let i and r be the angles of incidence and refraction respectively. Then the laws of refraction are

$$\frac{\mu}{\mu'} = \frac{\sin r}{\sin i} \quad \ldots(1)$$

and $\mathbf{n}, \mathbf{a}, \mathbf{c}$ are coplanar ...(2)

Now $\mathbf{a} \times \mathbf{n} = 1\,.\,1 \sin i\ \hat{\varepsilon}$, *i.e.*,

$$\sin i = \frac{\mathbf{a} \times \mathbf{n}}{\hat{\varepsilon}}$$

where $\hat{\varepsilon}$ is a unit vector normal to the plane containing **n**, **a**, **c**.

Similarly, $\sin r = \frac{\mathbf{c} \times \mathbf{n}}{\hat{\varepsilon}}$

From (1), we have $\mu \sin i = \mu' \sin r$,

i.e., $$\mu \frac{\mathbf{a} \times \mathbf{n}}{\hat{\varepsilon}} = \mu' \frac{\mathbf{c} \times \mathbf{n}}{\hat{\varepsilon}}$$

or $$\mu \mathbf{a} \times \mathbf{n} = \mu \mathbf{c} \times \mathbf{n}.$$

Again, the angle of incidence = angle of reflection, gives

$$\mathbf{a} \times \mathbf{n} = \mathbf{b} \times \mathbf{n}.$$

Problem 41. *Decompose a vector* **r** *as a linear combination of a vector* **a**, *another vector perpendicular to* **a**, *and coplanar with* **r** *and* **a**.

We know that $\mathbf{a} \times (\mathbf{a} \times \mathbf{r})$ is a vector which is coplanar with **a** and **r** and is perpendicular to **a**. Let us therefore suppose that

$$\mathbf{r} = l\mathbf{a} + m\mathbf{a} \times (\mathbf{a} \times \mathbf{r}).$$

Premultiplying both sides of (1) scalarly and vectorially with **a**, we get

$$\mathbf{a} \cdot \mathbf{r} = l\mathbf{a} \cdot \mathbf{a}, \text{ i.e., } l = \frac{\mathbf{a} \cdot \mathbf{r}}{\mathbf{a} \cdot \mathbf{a}} \qquad \text{...(2)}$$

and $$\mathbf{a} \times \mathbf{r} = m\mathbf{a} \times [\mathbf{a} \times (\mathbf{a} \times \mathbf{r})]$$
$$= m\mathbf{a} \times [(\mathbf{a} \cdot \mathbf{r})\,\mathbf{a} - (\mathbf{a} \cdot \mathbf{a})\,\mathbf{r}] = -m\,(\mathbf{a} \cdot \mathbf{a})\,\mathbf{a} \times \mathbf{r},$$

i.e., $$m = \frac{1}{\mathbf{a} \cdot \mathbf{a}} \qquad \text{...(3)}$$

Hence from (1), (2) and (3), we have

$$\mathbf{r} = \frac{\mathbf{a} \cdot \mathbf{r}}{\mathbf{a} \cdot \mathbf{a}}\mathbf{a} - \frac{1}{\mathbf{a} \cdot \mathbf{a}}[\mathbf{a} \times (\mathbf{a} \times \mathbf{r})] \cdot$$

Problem 42. *Prove that* $[\mathbf{lmn}]\,[\mathbf{l'm'n'}] = \begin{vmatrix} \mathbf{l} \cdot \mathbf{l'} & \mathbf{l} \cdot \mathbf{m'} & \mathbf{l} \cdot \mathbf{n'} \\ \mathbf{m} \cdot \mathbf{l'} & \mathbf{m} \cdot \mathbf{m'} & \mathbf{m} \cdot \mathbf{n'} \\ \mathbf{n} \cdot \mathbf{l'} & \mathbf{n} \cdot \mathbf{m'} & \mathbf{n} \cdot \mathbf{n'} \end{vmatrix}$,

where **l**, **m**, **n**; **l′**, **m′**, **n′** *are any vectors.*

Suppose that $\mathbf{l'} \times \mathbf{m'} = \mathbf{p}$ and consider the four vectors **l**, **m**, **n**, **p** which can be connected linearly as

$$[\mathbf{lmn}]\,\mathbf{p} = [\mathbf{mnp}]\,\mathbf{l} - [\mathbf{lnp}]\,\mathbf{m} + [\mathbf{lmp}]\,\mathbf{n}. \qquad \text{...(1)}$$

Here $[\mathbf{mnp}] = \mathbf{m} \times \mathbf{n} \,.\, \mathbf{p} = (\mathbf{m} \times \mathbf{n})\,.\,(\mathbf{l'} \times \mathbf{m'}) = \begin{vmatrix} \mathbf{m} \cdot \mathbf{l'} & \mathbf{m} \cdot \mathbf{m'} \\ \mathbf{n} \cdot \mathbf{l'} & \mathbf{n} \cdot \mathbf{m'} \end{vmatrix}$,

$$[\mathbf{lnp}] = \mathbf{l} \times \mathbf{c} \,.\, \mathbf{p} = (\mathbf{l} \times \mathbf{n})\,.\,(\mathbf{l'} \times \mathbf{m'}) = \begin{vmatrix} \mathbf{l} \cdot \mathbf{l'} & \mathbf{l} \cdot \mathbf{m'} \\ \mathbf{n} \cdot \mathbf{l'} & \mathbf{n} \cdot \mathbf{m'} \end{vmatrix}$$

and $$[\mathbf{lmp}] = \mathbf{l} \times \mathbf{m} \,.\, \mathbf{p} = (\mathbf{l} \times \mathbf{m})\,.\,(\mathbf{l'} \times \mathbf{m'}) = \begin{vmatrix} \mathbf{l} \cdot \mathbf{l'} & \mathbf{l} \cdot \mathbf{m'} \\ \mathbf{m} \cdot \mathbf{l'} & \mathbf{m} \cdot \mathbf{m'} \end{vmatrix}$$

With these substitutions (1) becomes

$$[\mathbf{lmn}]\,\mathbf{p} = [\mathbf{lmn}]\,\mathbf{l'} \times \mathbf{m'} = \begin{vmatrix} \mathbf{m} \cdot \mathbf{l'} & \mathbf{m} \cdot \mathbf{m'} \\ \mathbf{n} \cdot \mathbf{l'} & \mathbf{n} \cdot \mathbf{m'} \end{vmatrix}\mathbf{l} - \begin{vmatrix} \mathbf{l} \cdot \mathbf{l'} & \mathbf{l} \cdot \mathbf{m'} \\ \mathbf{n} \cdot \mathbf{l'} & \mathbf{n} \cdot \mathbf{m'} \end{vmatrix}\mathbf{m}$$

$$+\begin{vmatrix} \mathbf{l}\cdot\mathbf{l}' & \mathbf{l}\cdot\mathbf{m}' \\ \mathbf{m}\cdot\mathbf{l}' & \mathbf{m}\cdot\mathbf{m}' \end{vmatrix}\mathbf{n}$$

Multiplying scalarly both sides by **n**′, we get

$$[\mathbf{lmn}]\ \mathbf{l}'\times\mathbf{m}'\,.\,\mathbf{n}' = \begin{vmatrix} \mathbf{m}\cdot\mathbf{l}' & \mathbf{m}\cdot\mathbf{m}' \\ \mathbf{n}\cdot\mathbf{l}' & \mathbf{n}\cdot\mathbf{m}' \end{vmatrix}\mathbf{l}\cdot\mathbf{n}' - \begin{vmatrix} \mathbf{l}\cdot\mathbf{l}' & \mathbf{l}\cdot\mathbf{m}' \\ \mathbf{n}\cdot\mathbf{l}' & \mathbf{n}\cdot\mathbf{m}' \end{vmatrix}\mathbf{m}\cdot\mathbf{n}'$$
$$+\begin{vmatrix} \mathbf{l}\cdot\mathbf{l}' & \mathbf{l}\cdot\mathbf{m}' \\ \mathbf{m}\cdot\mathbf{l}' & \mathbf{m}\cdot\mathbf{m}' \end{vmatrix}\mathbf{n}.\mathbf{n}'$$

or
$$[\mathbf{lmn}]\ [\mathbf{l'm'n'}] = \begin{vmatrix} \mathbf{l}\cdot\mathbf{l}' & \mathbf{l}\cdot\mathbf{m}' & \mathbf{l}\cdot\mathbf{n}' \\ \mathbf{m}\cdot\mathbf{l}' & \mathbf{m}\cdot\mathbf{m}' & \mathbf{m}\cdot\mathbf{n}' \\ \mathbf{n}\cdot\mathbf{l}' & \mathbf{n}\cdot\mathbf{m}' & \mathbf{n}\cdot\mathbf{n}' \end{vmatrix}$$

Note : If
$$\left.\begin{aligned} \mathbf{l} &= \mathbf{l}' = \mathbf{a}, \\ \mathbf{m} &= \mathbf{m}' = \mathbf{b}, \\ \mathbf{n} &= \mathbf{n}' = \mathbf{c}, \end{aligned}\right\}\ \text{then}\ [\mathbf{abc}]^2 = \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix}$$

Problem 43. *Prove that*

$$(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{d}) + (\mathbf{c}\times\mathbf{a})\cdot(\mathbf{b}\times\mathbf{d}) + (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = 0$$

and deduce that

$$\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \tfrac{1}{2}(\cos 2B - \cos 2A).$$

Here

$$(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{d}) + (\mathbf{c}\times\mathbf{a})\cdot(\mathbf{b}\times\mathbf{d}) + (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})$$
$$= \begin{vmatrix} \mathbf{b}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{d} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{d} \end{vmatrix} + \begin{vmatrix} \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{d} \\ \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{d} \end{vmatrix} + \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{a}\cdot\mathbf{d} \\ \mathbf{b}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix}$$
$$= (\mathbf{b}\cdot\mathbf{a})(\mathbf{c}\cdot\mathbf{d}) - (\mathbf{b}\cdot\mathbf{d})(\mathbf{c}\cdot\mathbf{a}) + (\mathbf{c}\cdot\mathbf{b})(\mathbf{a}\cdot\mathbf{d}) - (\mathbf{c}\cdot\mathbf{d})(\mathbf{a}\cdot\mathbf{b}) + (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})$$
$$- (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

= 0 as $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$ etc...

For the second part, let **a, b, c, d** be four coplanar vectors and $\hat{\varepsilon}$ be a unit vector in the direction perpendicular to the plane containing **a, b, c, d**. Let the angles between the directions of **a** and **b**, **b** and **c**, **c** and **d** be $\theta_1, \theta_2, \theta_3$ respectively.

Fig. 1.28

Since $\mathbf{b}\times\mathbf{c} = bc\sin\theta_2\,\hat{\varepsilon}$ etc., *a, b, c, d* being modules of **a, b, c, d** respectively,

$\therefore$ $(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{d}) + (\mathbf{c}\times\mathbf{a})\cdot(\mathbf{b}\times\mathbf{d})$
$+ (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = 0$ gives

$$(bc\sin\theta_2\,\hat{\varepsilon})\,.\,[cd\sin(\theta_1+\theta_3+\theta_3)\,\hat{\varepsilon}]$$
$$+ [-ca\sin(\theta_1+\theta_2)\,\hat{\varepsilon}]\,.\,[bd\sin(\theta_2+\theta_3)\,\hat{\varepsilon}] + (ab\sin\theta_1\,\hat{\varepsilon})\,.\,(cd\sin\theta_3\,\hat{\varepsilon}) = 0$$

or $\sin\theta_2\sin(\theta_1+\theta_2+\theta_3) - \sin(\theta_1+\theta_2)\sin(\theta_2+\theta_3) + \sin\theta_1\sin\theta_3 = 0$

$$[\because\ \hat{\varepsilon}\cdot\hat{\varepsilon} = 1]\cdot$$

Putting $\theta_1 = B, \theta_2 = A$ and $\theta_3 = -B$, this gives

$$\sin^2 A - \sin(A+B)\sin(A-B) - \sin^2 B = 0,$$

i.e., $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$

$$= \tfrac{1}{2}(1-\cos 2A) - \tfrac{1}{2}(1-\cos 2B)$$

$$= \tfrac{1}{2}(\cos 2B - \cos 2A).$$

Problem 44. *Prove that* $2\,(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = \begin{vmatrix} -\mathbf{a} & -\mathbf{b} & \mathbf{c} & \mathbf{d} \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}$,

where $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ *etc.*.

Since $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = [\mathbf{abd}]\,\mathbf{c} - [\mathbf{abc}]\,\mathbf{d}$

$$= \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}\mathbf{c} - \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}\mathbf{d} \qquad \ldots(1)$$

Also $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = [\mathbf{acd}]\,\mathbf{b} - [\mathbf{bcd}]\,\mathbf{a}$

$$= \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}\mathbf{b} - \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}\mathbf{a} \qquad \ldots(2)$$

Adding (1) and (2), we get

$$2\,(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = \begin{vmatrix} -\mathbf{a} & -\mathbf{b} & \mathbf{c} & \mathbf{d} \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}$$

Problem 45. *If OX, OY OZ and O'X', O'Y', O'Z' are two sets of rectangular co-ordinate axes and* $l_1, m_1, n_1;\ l_2, m_2, n_2;\ l_3, m_3, n_3$ *denote the direction cosines of the members of either set with respect to other,*

then
$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 1.$$

Let **i, j, k** ; **i′, j′, k′** be unit vectors along the two sets of axes.

Then using the adjoining scheme of transformation, we have

	i	**j**	**k**
i′	l_1	m_1	n_1
j′	l_2	m_2	n_2
k′	l_3	m_3	n_3

$$\mathbf{i}' = l_1\mathbf{i} + m_1\mathbf{j} + n_1\mathbf{k},$$
$$\mathbf{j}' = l_2\mathbf{i} + m_2\mathbf{j} + n_2\mathbf{k},$$
$$\mathbf{k}' = l_3\mathbf{i} + m_3\mathbf{j} + n_3\mathbf{k}.$$

Now $\mathbf{j}'\times\mathbf{k}' = (l_2\mathbf{i} + m_2\mathbf{j} + n_2\mathbf{k})\times(l_3\mathbf{i} + m_3\mathbf{j} + n_3\mathbf{k})$

$$= (n_2m_3 - m_2n_3)\,\mathbf{j}\times\mathbf{k} + (n_2l_3 - n_3l_2)\,\mathbf{k}\times\mathbf{i} + (l_2m_3 - l_3m_2)\,\mathbf{i}\times\mathbf{j}.$$

Multiplying scalarly by $\mathbf{i}'$, we get

$$\mathbf{i}'\cdot\mathbf{j}'\times\mathbf{k}' = (l_1\mathbf{i}+m_1\mathbf{j}+n_1\mathbf{k})\cdot[(n_2m_3-m_3n_2)\,\mathbf{j}\times\mathbf{k}+(n_2l_3-n_3l_2)\mathbf{k}\times\mathbf{i} + (l_2m_3-l_2m_2)\,\mathbf{i}\times\mathbf{j}]$$

or $$[\mathbf{i}'\mathbf{j}'\mathbf{k}'] = l_1(n_2m_3-m_3n_2)\,[\mathbf{ijk}]+m_1(n_2l_3-n_3l_2)\,[\mathbf{jki}]+n_1(l_2m_3-l_3m_2)\,[\mathbf{kij}]$$

$$=\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}[\mathbf{ijk}].$$

Since $[\mathbf{i}'\mathbf{j}'\mathbf{k}'] = [\mathbf{ijk}]$, we have $\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 1.$

1.13. PRODUCT OF FOUR VECTORS

(1) Scalar product of four vectors. If a, b, c, d are four vectors then the product of the type $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})$ is called scalar product of four vectors. In fact this being a scalar product of two vectors $(\mathbf{a}\times\mathbf{b})$ and $(\mathbf{c}\times\mathbf{d})$, is a scalar quantity. We can treat this product as a scalar triple product of three vectors **a**, **b** and $(\mathbf{c}\times\mathbf{d})$.

Since the dot and cross may be interchanged in a scalar triple product, we have

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) &= \mathbf{a}\cdot\mathbf{b}\times(\mathbf{c}\times\mathbf{d})\\ &= \mathbf{a}\cdot[(\mathbf{b}\cdot\mathbf{d})\,\mathbf{c}-(\mathbf{b}\cdot\mathbf{c})\,\mathbf{d}]\\ &= (\mathbf{a}\cdot\mathbf{c})\,(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})\,(\mathbf{b}\cdot\mathbf{c})\\ &= \begin{vmatrix}\mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c}\\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d}\end{vmatrix}\end{aligned}$$

Problem 46. *Show that*

$$(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{d})+(\mathbf{c}\times\mathbf{a})\cdot(\mathbf{b}\times\mathbf{d})+(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = 0,$$

We have $$(\mathbf{b}\times\mathbf{c})\,.\,(\mathbf{a}\times\mathbf{d}) = \begin{vmatrix}\mathbf{b}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{a}\\ \mathbf{b}\cdot\mathbf{d} & \mathbf{c}\cdot\mathbf{d}\end{vmatrix}$$

$$= (\mathbf{b}\cdot\mathbf{a})\,(\mathbf{c}\cdot\mathbf{d})-(\mathbf{b}\cdot\mathbf{d})\,(\mathbf{c}\cdot\mathbf{a}).$$

Similarly $$(\mathbf{c}\times\mathbf{a})\cdot(\mathbf{b}\times\mathbf{d}) = (\mathbf{c}\cdot\mathbf{b})\,(\mathbf{a}\cdot\mathbf{d})-(\mathbf{c}\cdot\mathbf{d})\,(\mathbf{a}\cdot\mathbf{b})$$

and $$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})\,(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})\,(\mathbf{b}\cdot\mathbf{c}).$$

Adding all together, we get

$$(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{d})+(\mathbf{c}\times\mathbf{a})\cdot(\mathbf{b}\times\mathbf{d})+(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = 0.$$

(2) Vector product of four vectors. If a, b, c, d are four vectors then the product of the type $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$ is called vector product of four vectors. The value of this vector product is a vector which is at right angles to the vectors $(\mathbf{a}\times\mathbf{b})$ and $(\mathbf{c}\times\mathbf{d})$ both, and therefore is coplanar with **a, b** and also **c, d**. Conclusively this vector is parallel to the line of intersection of a plane parallel to **a** and **b** with another plane parallel to **c** and **d**.

The value of this vector product may be obtained in two ways :

(*i*) If we put $\mathbf{c}\times\mathbf{d} = \mathbf{p}$, then

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) &= (\mathbf{a}\times\mathbf{b})\times\mathbf{p}\\ &= (\mathbf{a}\cdot\mathbf{p})\,\mathbf{b}-(\mathbf{b}\cdot\mathbf{p})\,\mathbf{a}\end{aligned}$$

$$= (\mathbf{a} \cdot \mathbf{c} \times \mathbf{d})\, \mathbf{b} - (\mathbf{b} \cdot \mathbf{c} \times \mathbf{d})\, \mathbf{a}$$
$$= [\mathbf{a}\ \mathbf{c}\ \mathbf{d}]\, \mathbf{b} - [\mathbf{b}\ \mathbf{c}\ \mathbf{d}]\, \mathbf{a}$$

(*ii*) If we put $\mathbf{a} \times \mathbf{b} = \mathbf{q}$, then

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \mathbf{q} \times (\mathbf{c} \times \mathbf{d})$$
$$= (\mathbf{q} \cdot \mathbf{d})\, \mathbf{c} - (\mathbf{q} \cdot \mathbf{c})\, \mathbf{d}$$
$$= (\mathbf{a} \times \mathbf{b} \cdot \mathbf{d})\, \mathbf{c} - (\mathbf{a} \times \mathbf{b} \cdot \mathbf{c})\, \mathbf{d}$$
$$= [\mathbf{a}\ \mathbf{b}\ \mathbf{d}]\, \mathbf{c} - [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\, \mathbf{d}$$

Linear relationship between four vectors.

The two results namely (*i*) and (*ii*), on subtraction, yield

$$0 = [\mathbf{b}\ \mathbf{c}\ \mathbf{d}]\, \mathbf{a} - [\mathbf{a}\ \mathbf{c}\ \mathbf{d}]\, \mathbf{b} + [\mathbf{a}\ \mathbf{b}\ \mathbf{d}]\, \mathbf{c} - [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\, \mathbf{d}$$

Rearranging, we get

$$\mathbf{d} = \frac{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\ \mathbf{a} + [\mathbf{d}\ \mathbf{c}\ \mathbf{a}]\ \mathbf{b} + [\mathbf{d}\ \mathbf{a}\ \mathbf{b}]\ \mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} \text{ provided } [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \neq 0.$$

This may also be expressed as

$$\mathbf{d} = \mathbf{d}.\frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}\mathbf{a} + \mathbf{d}.\ \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}\mathbf{b} + \mathbf{d}.\ \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}\mathbf{c}.$$

Note. As an alternative, the linear relationship between four non-coplanar vectors **a**, **b**, **c**, **d** may be found as follows :

Suppose $\mathbf{d} = \lambda \mathbf{a} + \mu \mathbf{b} + \nu \mathbf{c}$, where λ, μ, ν are scalars.

Multiplying both sides scalarly by $(\mathbf{b} \times \mathbf{c})$, we get

$[\mathbf{d}\ \mathbf{b}\ \mathbf{c}] = \lambda\ [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$, other terms vanishing as $[\mathbf{b}\ \mathbf{b}\ \mathbf{c}] = 0 = [\mathbf{c}\ \mathbf{b}\ \mathbf{c}]$

or
$$\lambda = \frac{[\mathbf{d}\ \mathbf{b}\ \mathbf{c}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$$

Similarly
$$\mu = \frac{[\mathbf{d}\ \mathbf{c}\ \mathbf{a}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} \text{ and } \nu = \frac{[\mathbf{d}\ \mathbf{a}\ \mathbf{b}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}.$$

Substituting the values of λ, μ, ν, we have

$$\mathbf{d} = \frac{[\mathbf{d}\ \mathbf{b}\ \mathbf{c}]\ \mathbf{a} + [\mathbf{d}\ \mathbf{c}\ \mathbf{a}]\ \mathbf{b} + [\mathbf{d}\ \mathbf{a}\ \mathbf{b}]\ \mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}.$$

Problem 47. *Prove that* $\mathbf{d} \cdot [\mathbf{a} \times \{\mathbf{b} \times (\mathbf{c} \times \mathbf{d})\}] = (\mathbf{b} \cdot \mathbf{c})\ [\mathbf{acd}]$.

Here
$$\mathbf{d} \cdot [\mathbf{a} \times \{\mathbf{b} \times (\mathbf{c} \times \mathbf{d})\}] = \mathbf{d} \cdot [\mathbf{a} \times \{(\mathbf{b} \cdot \mathbf{b})\, \mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\, \mathbf{d})\}]$$
$$= \mathbf{d} \cdot [(\mathbf{b} \cdot \mathbf{d})\, (\mathbf{a} \times \mathbf{c}) - (\mathbf{b} \cdot \mathbf{c})\, (\mathbf{a} \times \mathbf{d})]$$
$$= (\mathbf{b} \cdot \mathbf{d})\, (\mathbf{a} \times \mathbf{c} \cdot \mathbf{d}) \text{ as } \mathbf{a} \times \mathbf{d} \cdot \mathbf{d} = [\mathbf{add}] = 0$$
$$= (\mathbf{b} \cdot \mathbf{d})\ [\mathbf{acd}].$$

Problem 48. *Prove the identity*

$$\mathbf{a} \times [\mathbf{b} \times (\mathbf{c} \times \mathbf{d})] = (\mathbf{a} \times \mathbf{c})\, (\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \times \mathbf{d})\, (\mathbf{b} \cdot \mathbf{c}).$$

L.H.S. $= \mathbf{a} \times [\mathbf{b} \times (\mathbf{c} \times \mathbf{d})]$

$= \mathbf{a} \times [\mathbf{b} \cdot \mathbf{d})\, \mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\, \mathbf{d}]$

$= (\mathbf{b} \cdot \mathbf{d})\, (\mathbf{a} \times \mathbf{c}) - (\mathbf{b} \cdot \mathbf{c})\, (\mathbf{a} \times \mathbf{d})$

= R.H.S.

1.14. RECIPROCAL SYSTEM OF VECTORS

If there are two sets of non-coplanar vectors **a**, **b**, **c** and **a**′, **b**′, **c**′ such that

$$\mathbf{a}' = \frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{abc}]},\ \mathbf{b}' = \frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{abc}]},\mathbf{c}' = \frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{abc}]},$$

then **a**, **b**, **c** and **a**′, **b**′, **c**′ are said to be *Reciprocal systems* of vectors.
It is so called, because

$$\mathbf{a}\,.\,\mathbf{a}' = \mathbf{b}\,.\,\mathbf{b}' = \mathbf{c}\,.\,\mathbf{c}' = 1 \text{ since } \mathbf{a}\,.\,\frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{abc}]} = \frac{[\mathbf{abc}]}{[\mathbf{abc}]} = 1 \text{ etc.,}$$

so that $\mathbf{a}' = \mathbf{a}^{-1}, \mathbf{b}' = \mathbf{b}^{-1}, \mathbf{c}' = \mathbf{c}^{-1}$.

In fact the two systems of vectors, *i.e.* **a**, **b**, **c** and **a**′, **b**′, **c**′ are mutually reciprocal as

$$\mathbf{a} = \frac{\mathbf{b}'\times\mathbf{c}'}{[\mathbf{a}'\mathbf{b}'\mathbf{c}']},\ \mathbf{b} = \frac{\mathbf{c}'\times\mathbf{a}'}{[\mathbf{a}'\mathbf{b}'\mathbf{c}']},\ \mathbf{c} = \frac{\mathbf{a}'\times\mathbf{b}'}{[\mathbf{a}'\mathbf{b}'\mathbf{c}']}.$$

In particular if **i**, **j**, **k** be the unit vectors along the principal axes and **i**, **j**′, **k**′ their reciprocals, then

$$\mathbf{i}' = \mathbf{i}, \mathbf{j}' = \mathbf{j} \text{ and } \mathbf{k}' = \mathbf{k} \text{ as } [\mathbf{ijk}] = 1.$$

This is called *self-reciprocal system*.

Note. With these notations, the linear relationship between four vectors **a**, **b**, **c** and **d** may be expressed as

$$\mathbf{d} = (\mathbf{d}\cdot\mathbf{a}')\,\mathbf{a} + (\mathbf{d}\cdot\mathbf{b}')\,\mathbf{b} + (\mathbf{d}\cdot\mathbf{c}')\,\mathbf{c}.$$

Problem 49. *Prove that* $(\mathbf{a}\times\mathbf{a}') + (\mathbf{b}\times\mathbf{b}') + (\mathbf{c}\times\mathbf{c}') = 0$, *where* **a**, **b**, **c** *are vectors and* **a**′, **b**′, **c**′ *their reciprocals.*

We have
$$\mathbf{a}' = \frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{abc}]},\ \mathbf{b}' = \frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{abc}]},\ \mathbf{c}' = \frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{abc}]}.$$

$$\mathbf{a}\times\mathbf{a}' = \mathbf{a}\times\frac{(\mathbf{b}\times\mathbf{c})}{[\mathbf{abc}]} = \frac{1}{[\mathbf{abc}]}\left[(\mathbf{a}\cdot\mathbf{c})\ \mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\ \mathbf{c}\right]$$

Similarly
$$\mathbf{b}\times\mathbf{b}' = \frac{1}{[\mathbf{abc}]}\left[(\mathbf{b}\cdot\mathbf{a})\mathbf{c} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}\right]$$

and
$$\mathbf{c}\times\mathbf{c}' = \frac{1}{[\mathbf{abc}]}\left[(\mathbf{c}\cdot\mathbf{b})\mathbf{a} - (\mathbf{c}\cdot\mathbf{a})\mathbf{b}\right].$$

Adding all together, we get

$$(\mathbf{a}\times\mathbf{a}') + (\mathbf{b}\times\mathbf{b}') + (\mathbf{c}\times\mathbf{c}') = \frac{1}{[\mathbf{abc}]}\,[0] = 0.$$

1.15. VECTOR EQUATIONS

Here below some methods for solving the vector equations are explained with the help of examples.

Problem 50. *Solve the vector equation* $\mathbf{x}\times\mathbf{a} = \mathbf{b}$.

Given equation is $\mathbf{x}\times\ \mathbf{b} = \mathbf{b}.$ ...(1)

We know that $\mathbf{a}\times\mathbf{b}$ is a vector perpendicular to both **a** and **b**, therefore the vectors **a**, **b** and $\mathbf{a}\times\mathbf{b}$ are three non-coplanar vectors. Let us assume that the solution of the given equation is of the from $\mathbf{x} = \lambda\mathbf{a} + \mu\mathbf{b} + \nu\,(\mathbf{a}\times\mathbf{b}),$ ...(2)
where λ, μ, ν are scalars.

Since (2) is a solution of (1), therefore substituting in (1) the value of $\mathbf{x}$ from (2), we get

$$\{\lambda\mathbf{a} + \mu\mathbf{b} + \nu(\mathbf{a}\times\mathbf{b})\}\times\mathbf{a} = \mathbf{b},$$

or $$\mu(\mathbf{b}\times\mathbf{a}) + \nu\{(\mathbf{a}\times\mathbf{b})\times\mathbf{a}\} = \mathbf{b} \text{ as } \mathbf{a}\times\mathbf{a} = 0$$

or $$-\mu(\mathbf{a}\times\mathbf{b}) + \nu\{(\mathbf{a}\cdot\mathbf{a})\mathbf{b} - (\mathbf{b}\cdot\mathbf{a})\mathbf{a}\} = \mathbf{b}$$

or $$-\mu(\mathbf{a}\times\mathbf{b}) + \nu\{(a^2\mathbf{b} - (\mathbf{b}\cdot\mathbf{a})\mathbf{a}\} = \mathbf{b}, \text{ where } |\mathbf{a}| = a.$$

Equating the coefficients of like vectors on either side, we get

$$-\mu = 0, \nu a^2 = 1, -\nu\mathbf{b}\cdot\mathbf{a} = 0,$$

i.e. $$\mu = 0, \nu = 1/a^2 \text{ and } \mathbf{a}\cdot\mathbf{b} = 0 \text{ as } \nu \neq 0.$$

Substituting in (2), these values of μ and ν, the general solution of the given equation is

$$\mathbf{x} = \lambda\mathbf{a} - 1/a^2(\mathbf{a}\times\mathbf{b})$$

and the condition for the existence of this solution is

$$\mathbf{a}\cdot\mathbf{b} = 0,$$

i.e. the vectors **a** and **b** are mutually at right angles.

Problem 51. *Solve the simultaneous equations*

$$\mathbf{x}\times\mathbf{b} = \mathbf{a}\times\mathbf{b}, \mathbf{x}\cdot\mathbf{c} = 0 \textit{ provided } \mathbf{b}\cdot\mathbf{c} \neq 0.$$

The given equations are

$$\mathbf{x}\times\mathbf{b} = \mathbf{a}\times\mathbf{b} \quad \ldots(1)$$

and $$\mathbf{x}\cdot\mathbf{c} = 0. \quad \ldots(2)$$

The equation (1) can be written as

$$(\mathbf{x} - \mathbf{a})\times\mathbf{b} = \mathbf{0},$$

which follows that $(\mathbf{x} - \mathbf{a})$ and **b** are parallel, *i.e.*

$$\mathbf{x} - \mathbf{a} = t\mathbf{b}, \text{ where } t \text{ is a scalar}$$

or $$\mathbf{x} = \mathbf{a} + t\mathbf{b}, \quad \ldots(3)$$

Substituting this value of **x** in (2), we get

$$(\mathbf{a} + t\mathbf{b})\cdot\mathbf{c} = 0.$$

Giving $$t = -\frac{\mathbf{a}\cdot\mathbf{c}}{\mathbf{b}\cdot\mathbf{c}} \text{ when } \mathbf{b}\cdot\mathbf{c} \neq 0 \text{ (given).}$$

Hence the required solution is obtained by putting the value of t in (3) and that is

$$\mathbf{x} = \mathbf{a} - \frac{\mathbf{a}\cdot\mathbf{c}}{\mathbf{b}\cdot\mathbf{c}}\mathbf{b}.$$

Problem 52. *Solve the simultaneous equations*

$$s\mathbf{x} + t\mathbf{y} = \mathbf{a}, \mathbf{x}\times\mathbf{y} = \mathbf{b} \textit{ provided } \mathbf{a}\cdot\mathbf{b} = 0.$$

The given equations are

$$s\mathbf{x} + t\mathbf{y} = \mathbf{a}, \quad \ldots(1)$$

$$\mathbf{x}\times\mathbf{y} = \mathbf{b}. \quad \ldots(2)$$

Multiplying (1) vectorially by **x**, we get

$$\mathbf{x}\times(s\mathbf{x} + t\mathbf{y}) = \mathbf{x}\times\mathbf{a}$$

or $$t\mathbf{x}\times\mathbf{y} = \mathbf{x}\times\mathbf{a} \text{ as } \mathbf{x}\times\mathbf{x} = 0$$

or $$\mathbf{x} \times \mathbf{a} = t\mathbf{b} \text{ from (2).} \qquad ...(3)$$

Multiplying (3) vectorially by **a**, we have

$$\mathbf{a} \times (\mathbf{x} \times \mathbf{a}) = t\,(\mathbf{a} \times \mathbf{b})$$

or $$(\mathbf{a} \cdot \mathbf{a})\,\mathbf{x} - (\mathbf{a} \cdot \mathbf{x})\,\mathbf{a} = t\,(\mathbf{a} \times \mathbf{b})$$

or $$\mathbf{x} - \lambda\mathbf{a} = t\,\frac{(\mathbf{a} \times \mathbf{b})}{\mathbf{a}^2}$$

where λ is a scalar parameter,

i.e. $$\mathbf{x} = \lambda\mathbf{a} + \frac{1}{a^2}\,(\mathbf{a} \times \mathbf{b})$$

which is the general solution for **x**. Similarly procedure will yield the solution for **y**.

1.16. SIMPLE APPLICATIONS OF VECTORS TO MECHANICS

(1) Concurrent forces. It is found experimentally that the resultant effect of two concurrent forces is equivalent to a single force acting at the same point. The single force is represented by their vector sum. In general a system of forces acting at a point represented by the vectors $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3 \ldots \mathbf{P}_n$ is equivalent to a single resultant force **F** acting at the same point such that

$$\mathbf{F} = \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 + \ldots + \mathbf{P}_n.$$

It can be obtained by constructing a vector polygon of which

$$\vec{AB} = \mathbf{P}_1,\ \vec{BC} = \mathbf{P}_2, \ldots \vec{MN} = \mathbf{P}_n.$$

The resultant is represented by $\vec{AN} = \mathbf{F}$ which is drawn to close up the polygon opposite to that in which the sides have been drawn. The polygon does not necessarily lie in a plane as the forces, may not be coplanar.

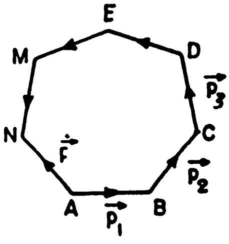


Fig. 1.29

COROLLARY. **Lami's theorem.** If three forces acting at a point be in equilibrium, then each is proportional to the sine of the angle between the other two.

In the case of three concurrent forces the closed polygon will be a triangle, the forces will be coplanar and each side is proportional to the sine of the opposite angle and hence each force will be proportional to the sine of the angle between the other two.

(2) Work done by a force. We know that a force acting on a particle does work when the displacement of the particle takes place in the direction which is not perpendicular to the direction of the force. The work done is measured by the product of the force and the resolved part of the displacement in the direction of the force. Hence if the vectors representing the force and the displacement be respectively **F** and **d** inclined at an angle θ and whose respective modules are F and d, then the work done $Fd\cos\theta = \mathbf{F} \cdot \mathbf{d}$ (by the definition of dot product).

In case **d** is perpendicular to **F**, *i.e.* $\theta = 90°$ the work done is zero.

COROLLARY. **Rate of doing work.** If a particle acted upon by a force **F** is moving with a velocity **v**, then the rate of work done, *i.e.* $\frac{dW}{dt}$ is given by

$$\frac{dW}{dt} = \mathbf{F}.\mathbf{v}.$$

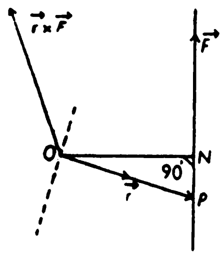


Fig. 1.30

(3) Vector moment or torque of a force. The vector moment or torque of a force about a point O is a vector quantity related to an axis through O perpendicular to the plane containing O and the line of action of the force **F**. The magnitude of the vector moment is jointly proportional to the force and the perpendicular distance ON upon the line of the force.

Take a point P on the line of action of the force. Let the position vector of P be **r**. Then the moment of the force **F** about O is represented by a vector perpendicular to the plane of **r** and **F** Since **r** × **F** is a vector perpendicular to the plane of **r** and **F**, therefore the vector representing the moment of **F** about O is **r** × **F**. Hence if **M** be the moment vector, then

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}.$$

(4) Force on a particle in a magnetic field. Let **F** be the force on a point charge in a magnetic field of intensity **B**. Then **F** is proportional to the component of **B** perpendicular to the velocity of the charge. If **v** be the velocity of the charge then the vector product **v** × **B** represents a vector normal to the plane of **v** and **B** and hence may be regarded as the component of **B** along the perpendicular to the velocity **v**. Thus if q be the charge on the particle and c the speed of light then the above relation existing between the vectors **F**, **B** and **v** is expressed as

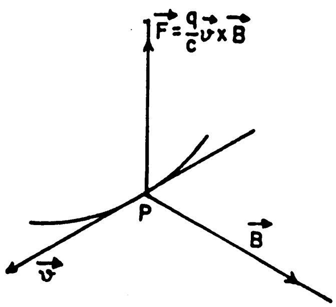


Fig. 1.31

$$\mathbf{F} = \frac{q}{c}\mathbf{v} \times \mathbf{B} \text{ in Gaussian units.}$$

or $$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \text{ in MKS units.}$$

(5) Force on a charged particle. If a particle of charge q is in an electric field of intensity **E** at rest, then force on the charged particle due to electric field $\mathbf{F}_e = q\mathbf{E}$ (in e.s.u.)

But if the charged particle is moving with velocity **v** relative to an observer, the magnetic field is produced. The force experienced by a moving charged particle dut to magnetic field $\mathbf{F}_m = q\mathbf{v} \times \mathbf{B}$ (in e.m.u.), where **B** is the intensity of magnetic field. This is known as *magnetic force.*

∴ Total force on the moving charged particle is the sum of electrostatic forces and is given by

$$\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m = q\mathbf{E} + q\,(\mathbf{v} \times \mathbf{B}),$$

or $$\mathbf{F} = q\,[\mathbf{E} + \mathbf{v} \times \mathbf{B}]. \qquad ...(1)$$

This is known as *Lorentz force.*

If **E** is in e.s.u. and **B** in gauss, the equation (1) can be written as

$$\mathbf{F} = q\left[\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c}\right], \qquad ...(2)$$

where c is speed of light in vacuum.

(6) Circular motion. Let us consider a particle moving in a circle of radius r with angular velocity ω. Let O be the centre of the circular path and A the starting point. Join O to A and draw OB perpendicular to OA. Let **i**, **j** be the unit vectors along OA and OB taken as axes of x and y respectively. Let at an instant of time t, P be the position of the particle such that its position vector referred to O as origin is **r**.

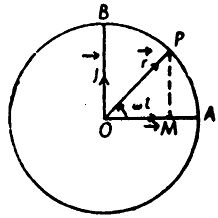


Fig. 1.32

Assuming $\angle AOP = \theta$, we have $\theta = \omega t$ ...(1)

Since the radius r of a circular orbit is constant and the unit vector $\hat{\mathbf{r}}$ rotates at a constant rate, the equation of the circular orbit can be given as

$$\mathbf{r}(t) = t\hat{\mathbf{r}}(t). \qquad \text{...(2a)}$$

Now draw PM perpendicular to OA. Then

$$\vec{OP} = \vec{OM} + \vec{MP}$$

$$= OP \cos\theta\, \mathbf{i} + OP \sin\theta\, \mathbf{j},$$

or $\quad \mathbf{r}(t) = r\hat{\mathbf{r}}(t) = r\cos\omega t\, \mathbf{i} + r\sin\omega t\, \mathbf{j}$ as $OP = \mathbf{r}$. ...(2b)

i.e. $\quad \hat{\mathbf{r}}(t) = \cos\omega t\, \mathbf{i} + \sin\omega t\, \mathbf{j}$. ...(3)

In particular if $\theta = \dfrac{\pi}{4}$, then $\hat{\mathbf{r}} = \dfrac{1}{\sqrt{2}}\mathbf{i} + \dfrac{1}{\sqrt{2}}\mathbf{j}$

and if $\quad \theta = \dfrac{\pi}{2}$ then $\mathbf{r} = \mathbf{j}$

also if $\quad \theta = 0$, then $\hat{\mathbf{r}} = \mathbf{i}$.

Now the velocity, **v** of the particle is given by (differentiation of vectors being defined in 1.17)

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = r\frac{d\hat{\mathbf{r}}}{dt} = r\left(\mathbf{i}\frac{d}{dt}\cos\omega t + \mathbf{j}\frac{d}{dt}\sin\omega t\right) \text{ from } (2b)$$

$$= \omega r(-\sin\omega t\, \mathbf{i} + \cos\omega t\, \mathbf{j}),$$

which follows that the magnitude of the velocity

$$|\mathbf{v}| = \omega\mathbf{r} \text{ say } v = \omega r. \qquad \text{...(4)}$$

Again the acceleration of the particle is given by

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$$

$$= \omega\mathbf{r}\frac{d}{dt}(-\sin\omega t\,\mathbf{i} + \cos\omega t\,\mathbf{j})$$

$$= -\omega^2 r(\cos\omega t\,\mathbf{i} + \sin\omega t\,\mathbf{j})$$

$$= -\omega^2\mathbf{r} \text{ from (3).}$$

$\therefore$ magnitude of the acceleration, say a

$$= |\mathbf{a}| = |-\omega^2\mathbf{r}|,$$

i.e., $\quad a = \omega^2 r$. ...(5)

Elimination of ω between (4) and (5) yields

$$a = \frac{v^2}{r}. \qquad \text{...(6)}$$

The acceleration given by the expression (6) is known as the **Centripetal** (*i.e.* centre seeking) acceleration.

Now we know that if f be the frequency of the particle, then

$$2\pi f = \omega, \qquad \text{...(7)}$$

and the time period T of the motion is given by

$$T = \frac{2\pi}{\omega} = \frac{1}{f} \text{ from (7).} \qquad \text{...(8)}$$

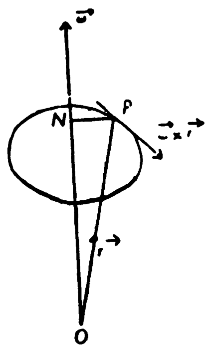


Fig. 1.33

(7) Angular velocity of a rigid body about a fixed axis. Consider the motion of a rigid body rotating about a fixed axis ON at the rate of ω radians per second. Then the angular velocity of the body is specified by the vector $\vec{\omega}$ whose module is ω and whose direction is parallel to the axis, and in the positive sense relative to the rotation.

Let O be a point on the fixed axis; P, a point fixed in the body, **r** the position vector of P referred to O as origin, and PN perpendicular to the axis of rotation. The particle at P is moving in a circular path of radius $PN = r \sin PON$ about N as centre. Its velocity vector **v** is at right angles to the plane of ω and **r** and its magnitude is given by

$$v = \omega r \sin \theta.$$

Hence $\qquad \mathbf{v} = \vec{\omega} \times \mathbf{r}.$

Problem 53. *A particle, acted on by constant forces* $6\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ *and* $3\mathbf{i} + \mathbf{j} - \mathbf{k}$ *is displaced from the point* $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ *to the point* $5\mathbf{i} + 4\mathbf{j} + \mathbf{k}$. *Find the total work done by the forces.*

Let $\qquad \mathbf{F}_1 = 4\mathbf{i} + \mathbf{j} - 3\mathbf{k}, \mathbf{F}_2 = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$

and the displacement $\qquad \mathbf{r} = 5\mathbf{i} + 4\mathbf{j} + \mathbf{k} - (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$

$$= 4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}.$$

Work done by force $\mathbf{F}_1 = \mathbf{F}_1 \cdot \mathbf{r}$

$$= (4\mathbf{i} + \mathbf{j} - 3\mathbf{k}).(4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$$

$$= 4.4 + 1.2 - 3.(-2) = 24 \text{ units.}$$

Work done by the force $\mathbf{F}_2$

$$= \mathbf{F}_2 \cdot \mathbf{r} = (3\mathbf{i} + \mathbf{j} - \mathbf{k}) \cdot (4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = 16 \text{ units.}$$

Total work done = 24 + 16 = 40 units,

Problem 54. *A rigid body is spinning with an angular velocity of 4 radians per second about an axis parallel to* $3\mathbf{j} - \mathbf{k}$ *passing through the point* $\mathbf{i} + 3\mathbf{j} - \mathbf{k}$. *Find the velocity of the particle at the point* $4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

Let **r** be the position vector of the point relative to the given point on the axis, then

$$\mathbf{r} = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k} - (\mathbf{i} + 3\mathbf{j} - \mathbf{k})$$

$$= 3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}.$$

Angular velocity of the particle is given by

$$\vec{\omega} = 4\frac{(3\mathbf{j} - \mathbf{k})}{|3\mathbf{j} - \mathbf{k}|} = \frac{4}{\sqrt{10}}(3\mathbf{j} - \mathbf{k}).$$

Hence the velocity of the particle

$$\vec{\omega} \times \mathbf{r} = \frac{2}{\sqrt{10}} (3\mathbf{j} - \mathbf{k}) \times (3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k})$$

$$= \frac{4}{\sqrt{10}} (\mathbf{i} - 3\mathbf{j} - 9\mathbf{k}).$$

Its magnitude $= \frac{4}{\sqrt{10}} |\mathbf{i} - 3\mathbf{j} - 9\mathbf{k}| = 4 \sqrt{\left(\frac{91}{10}\right)} = 12$ approx.

Problem 55. *Find the torque about the point 10***j** *of a force represented by – 3***i** *+* **j** *+ 5***k** *acting through the point 7***i** *+ 3***j** *+* **k**.

Let **F** = – 3**i** + **j** + 5**k** and **r** be the vector from the given point to the point of application of the force, then

$$\mathbf{r} = 10\mathbf{j} - (7\mathbf{i} + 3\mathbf{j} + \mathbf{k}) = -7\mathbf{i} + 7\mathbf{j} - \mathbf{k}.$$

Reqd. Torque $= \mathbf{r} \times \mathbf{F} = (-7\mathbf{i} + 7\mathbf{j} - \mathbf{k}) \times (-3\mathbf{i} + \mathbf{j} + 5\mathbf{k})$

$= 36\mathbf{i} - 38\mathbf{j} - 14\mathbf{k}.$

Problem 56. *A particle is moving in a circular orbit of radius 10 cms. If its frequency of motion is 60 cycles per sec., find the time period, veclocity and acceleration of the particle.*

Given freqency $f = 60$ cycle/sec.; radius of the orbit = 10 cms.

Let T be the time period, v the velocity and a the accerleration of the particle. Then

$$T = \frac{1}{f} = \frac{1}{60} = 0.017 \text{ sec. nearly.}$$

Now $\omega = 2\pi f = 2 \cdot \frac{22}{7} \cdot 60 = \frac{2640}{7} = 377$ radians/sec. approx.

$\therefore$ $v = \omega r = 377 \times 10 = 3.8 \times 10^3$ sec. nearly

and $a = \omega^2 r = 377 \times 377 \times 10 = 10^6$ cm/sec. nearly

Problem 57. *Calculate the force in dynes acting (1) on a proton, and (2) on an electron, in an electric field of intensity 1000 volts per cm. Given charge on electron = 4.8×10^{-10} e.s.u.*

The intensity of electric field = 1000 volts/cm.

$$= \frac{1000}{300} \text{ e.s.u. of volts/cm.}$$

$$= \frac{10}{3} \text{ stat. volts/cm.}$$

The force on the proton $= q\mathbf{E}$

$$= 4.8 \times 10^{-10} \times \frac{10}{3} = 1.6 \times 10^{-9} \text{ dynes.}$$

The force on the electron $= q\mathbf{E} = -1.6 \times 10^{-9}$ dynes.

Problem 58. *Calculate the force on the proton in dynes in a magnetic field of intensity 100 gauss directed along z–axis :*

(*i*) *when the proton moves with velocity 10^8 along x-axis.*

(*ii*) *when the proton is at rest.*

(*i*) Given **B** = 100 gauss along z-axis = 100**k**,

$\mathbf{v} = 10^8\mathbf{i}.$

$\therefore$ the force on the proton due to magnetic field

$$= \frac{q}{c} (\mathbf{v} \times \mathbf{B})$$

$$= \frac{4.8\times10^{-10}}{3\times10^{10}}\left[10^{8}\mathbf{i}\times(100\mathbf{k})\right]$$
$$= 1.6\times10^{20}[-10^{10}\mathbf{j}]$$
$$= -1.6\times10^{10}\mathbf{j} \text{ dynes,}$$

i.e., the force of magnitude 1.6×10^{10} is acting along negative direction of y-axis.

(*ii*) When the proton is at rest, there is no magnetic field and hence no force.

Problem 59. *Using a right-handed system, the electric field* **E**, *and velocity* **v** *of a particle of charge q e.s.u. are given by*

$$\mathbf{E} = 2\mathbf{i} \text{ e.s.u.}$$
$$\mathbf{B} = (3\mathbf{i} + 4\mathbf{j}) \text{ e.m.u.}$$
$$\mathbf{v} = 9\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} \text{ cm./sec.}$$

Calculate the electrostatic and magnetic force on the charge.

Electrostatic force on the charge $q = q\mathbf{E} = q.\,2\mathbf{i} = 2q\mathbf{i}$ dynes, q is in e.s.u.

Therefore electrostatic force has magnitude $2q$ along positive x-axis.

Magnetic force on charge $q = \frac{q}{c}(\mathbf{v}\times\mathbf{B})$, $\mathbf{B}$ being in e.m.u.

$$= \frac{q}{3\times10^{10}}\left[(2\mathbf{i}+3\mathbf{j}+4\mathbf{k})\times(3\mathbf{i}+4\mathbf{j})\right]$$
$$= \frac{q}{3\times10^{10}}\left[8\mathbf{k}-9\mathbf{k}+12\mathbf{j}-16\mathbf{i}\right]$$
$$= \frac{q}{3\times10^{10}}\left[-16\mathbf{i}+12\mathbf{j}-\mathbf{k}\right].$$

$\therefore$ magnitude of magnetic force $= \frac{q}{3\times10^{10}}\sqrt{\left[16^2+12^2+1^2\right]}$

$$= \frac{q}{3\times10^{10}}\left[\sqrt{401}\right]$$
$$\approx 7\times10^{-19}\,a \text{ dynes.}$$

(q is in e.s.u.)

Direction cosines of Lorentz force are

$$\left[\frac{-16}{7\times10^{-10}q}, \frac{12}{7\times10^{-10}q}, \frac{-1}{7\times10^{-10}q}\right].$$

Problem 60. *A proton is moving with velocity 10^{10} cm./sec. along z-axis in an electric field of intensity 3×10^{4} volts along x-axis and magnetic field of intensity 3000 gauss along y-axis. Calculate the magnitude and direction of total force.*

. Charge on the proton $q = 4.8\times10^{-10}$ e.s.u.

Intensity of electric field $\mathbf{E} = 3\times10^{4}\mathbf{i}$ volts

$$= \frac{3\times10^{4}}{300}\ \mathbf{i} \text{ stat - volts}$$

$\therefore$ $\mathbf{E} = 100\,\mathbf{i}$ stat-volts.

Intensity of magnetic field $\mathbf{B} = 3000\mathbf{j}$

and velocity of proton $\mathbf{v} = 10^{10}\mathbf{k}$.

$\therefore$ total force on the proton $= q\left[\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right]$

$$= 4.8 \times 10^{-10}\left[100\mathbf{i} + \frac{1}{c}\left\{10^{10}\mathbf{k} \times 3000\mathbf{j}\right\}\right]$$

$$= 4.8 \times 10^{-10}\left[100\mathbf{i} + \frac{1}{3 \times 10^{10}}\left\{-3 \times 10^{13}\mathbf{i}\right\}\right]$$

$$= 4.8 \times 10^{-10}\,[100\mathbf{i} - 1000\mathbf{i}]$$

$$= 4.8 \times 10^{-10}\,[-900\mathbf{i}]$$

$$= -4.32 \times 10^{-7}\mathbf{i} \text{ dynes.}$$

Thus the total force has magnitude 4.32×10^{-7} dynes along the negative direction of x-axis.

1.17. DIFFERENTIATION OF VECTORS

Vector Function of a Single Scalar Variable. F is a vector function of a single scalar variable t, if to each t of the range of values of t, there corresponds a vector F and is written as

$$\mathbf{F} = \mathbf{F}(t).$$

The vector F can be be expressed in components form, such as

$$\mathbf{F} = f_1(t)\,\mathbf{i} + f_2(t)\,\mathbf{j} + f_3(t)\,\mathbf{k}$$

where $f_1(t), f_2(t), f_3(t)$ are components of $\mathbf{F}(t)$ defined for the range of values of t and **i**, **j**, **k** are unit vectors in the directions of the principal axes.

Illustration: If a moving particle undergoes a displacement in any manner such that at any time t, its position is at a point P whose position vector relative to any fixed origin O is r, then the vector r is the function of scalar variable t, *i.e.*, $\mathbf{r} = \mathbf{F}(t)$.

Derivative of a vector function of a scalar variable. If $\mathbf{F}(t)$ represents a vector function of a scalar variable t, over the interval $a \le t \le b$, and if $\lim_{\delta t \to 0} \frac{\mathbf{F}(t+\delta t) - \mathbf{F}(t)}{\delta t}$ exists, then this limit is called as the *derivative or differential coefficient* of $\mathbf{F}(t)$ at t. The process of finding out differential coefficient is called *differentiation*.

By convention the method of denoting a derivative is

$$\mathbf{F}'(t) \text{ or } \frac{d}{dt}\mathbf{F}(t).$$

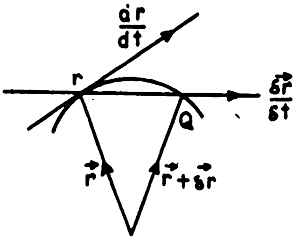


Fig. 1.34

Time-derivative of a vector. Let us suppose that vector r is a continuous single valued function of a scalar variable t, *i.e.* $\mathbf{r} = \mathbf{F}(t)$.

At an instant of time t, let P be the position of a particle whose position vector referred to a fixed origin O, be r., After an interval of time δt, let Q be the position of the moving particle along the curve $\mathbf{r} = \mathbf{F}(t)$. Assuming that an increment δt in t produces an increment $\delta \mathbf{r}$ in r, we have

$$\mathbf{r} + \delta\mathbf{r} = \mathbf{F}(t + \delta t)$$

It is apparent from Fig. 1.34 that $\overrightarrow{PQ} = \delta\mathbf{r}$.

Obviously the quotient $\frac{\delta \mathbf{r}}{\delta t}$ is a vector, since r is a vector and t is a scalar and it gives the average rate of change of r with t.

As δt tends to take zero value, the point Q moves up to coincide with P, so that the chord PQ coincides with tangent at P to the curve. Thus the vector $\frac{\delta \mathbf{r}}{\delta t}$ as $\delta t \to 0$ is along the direction of tangent at P in the sense for increasing t. The limiting value of this quotient when exists, is the derivative or differential coefficient of **r** with respect to t (time), and denoted by $\frac{d\mathbf{r}}{dt}$ or $\dot{\mathbf{r}}$ *i.e.*

$$\lim_{\delta t \to 0} \frac{\delta \mathbf{r}}{\delta t} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}.$$

But we know that velocity **v** of a particle is a vector and this is the time rate of change of the position of the particle, therefore

$$\mathbf{v} = \frac{d\mathbf{r}}{dt},$$

i.e. the first time derivative of position of a particle gives its velocity and its magnitude i.e. $v = |\mathbf{v}|$ *is known as the* **speed** *of the particle. Clearly the speed is a scalar quantity.*

Further the derivative $\frac{d\mathbf{r}}{dt}$ is also in general a function of time t, and may itself prossess a derivative, which is called the second derivative of **r** and is denoted by $\frac{d^2\mathbf{r}}{dt^2}$.

But the acceleration **a** of a particle is *a* vector and this is the time rate of change of the velocity of the particle, therefore

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2}$$

i.e. the second time-derivative of the position of a particle gives its acceleration.

Note 1. The second derivative $\frac{d^2\mathbf{r}}{dt^2}$ is also a vector function of t and hence possesses a derivative $\frac{d^3\mathbf{r}}{dt^3}$ known as the third derivative of **r**. Similarly the existence of fourth, fifth, sixth...derivatives can be stated.

Note 2. The derivative of a constant vector **c** is zero *i.e.* if **c** is a constant vector, then $\frac{d\mathbf{c}}{dt} = 0$; for, then the increment in δt producess no change in **c**.

1.18. SOME RULES FOR DIFFERENTIATION

(1) ***The derivative of the sum of two differentiable vector functions, is equal to the sum of their derivatives.***

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be two differentiable vector functions of a scalar variable t.

Suppose that $\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2$

Then if a change δt in t corresponds a change $\delta \mathbf{r}$ in **r**, $\delta \mathbf{r}_1$ in $\mathbf{r}_1$, $\delta \mathbf{r}_2$ in $\mathbf{r}_2$ we have

$$\mathbf{r} + \delta \mathbf{r} = (\mathbf{r}_1 + \delta \mathbf{r}_1) + (\mathbf{r}_2 + \delta \mathbf{r}_2)$$

i.e.

$$\delta \mathbf{r} = \delta \mathbf{r}_1 + \delta \mathbf{r}_2$$

Dividing throughout by δt and proceeding to the limit as $\delta t \to 0$, we find,

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}_1}{dt} + \frac{d\mathbf{r}}{dt}$$

This result may be extended to any number of vectors.

In particular if any vector is expressed as the sum of rectangular component vectors such as

$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors along the axes; then the derivative of $\mathbf{r}$ is given by

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$

Note. If $\mathbf{r}$ is a differentiable function of a scalar variable s and s is differentiable function of another variable t, then we can state that

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds} \cdot \frac{ds}{dt}.$$

(2) The differentiation of scalar product of two vectors. *The derivative of product of two differential vector functions, is equal to the sum of the quantities found by differentiating one of the factors and leaving first unchanged i.e.,* $\mathbf{r} = \mathbf{r}_1 \cdot \mathbf{r}_2$ *where* $\mathbf{r}_1 \cdot \mathbf{r}_2$ *are the vector functions of a scalar variable* t,

then
$$\frac{d\mathbf{r}}{dt} = \mathbf{r}_1 \cdot \frac{d\mathbf{r}_2}{dt} + \frac{d\mathbf{r}_1}{dt} \cdot \mathbf{r}_2$$

If a change δt in t corresponds the changes $\delta\mathbf{r}$ in $\mathbf{r}$, $\delta\mathbf{r}_1$ in $\mathbf{r}_1$ and $\delta\mathbf{r}_2$ in $\mathbf{r}_2$, then we have,

$$\begin{aligned} \mathbf{r} + \delta\mathbf{r} &= (\mathbf{r}_1 + \delta\mathbf{r}_1) : (\mathbf{r}_2 + \delta\mathbf{r}_2) \\ &= \mathbf{r}_1 \cdot \mathbf{r}_2 + \mathbf{r}_1 \cdot \delta\mathbf{r}_2 + \delta\mathbf{r}_1 \cdot \mathbf{r}_2 + \delta\mathbf{r}_1 \cdot \delta\mathbf{r}_2 \\ &= \mathbf{r}_1 \cdot \mathbf{r}_2 + \mathbf{r}_1 \cdot \delta\mathbf{r}_2 + \delta\mathbf{r}_1 \cdot \mathbf{r}_2, \text{ neglecting the product} \\ &\qquad \delta\mathbf{r}_1 \cdot \delta\mathbf{r}_2 \text{ as it is vanishingly small} \end{aligned}$$

or
$$\delta\mathbf{r} = \mathbf{r}_1 \cdot \delta\mathbf{r}_2 + \delta\mathbf{r}_1 \cdot \mathbf{r}_2 \text{ as } \mathbf{r} = \mathbf{r}_1 \cdot \mathbf{r}_2.$$

Dividing throughout by δt and proceeding to the limit as $\delta t \to 0$,

we get
$$\frac{d\mathbf{r}}{dt} = \mathbf{r}_1 \cdot \frac{d\mathbf{r}_2}{dt} + \frac{d\mathbf{r}_1}{dt} \cdot \mathbf{r}_2$$

Note 1. If u is a scalar function of t, and $\mathbf{r} = u\mathbf{r}_1$, then

$$\frac{d\mathbf{r}}{dt} = \frac{du}{dt}\mathbf{r}_1 + u\frac{d\mathbf{r}_1}{dt}.$$

Note 2. If $\mathbf{r} = \mathbf{r}_1 . \mathbf{r}_1$, then
$$\begin{aligned} \frac{d\mathbf{r}}{dt} &= \frac{d\mathbf{r}_1}{dt} \cdot \mathbf{r}_1 + \mathbf{r}_1 \cdot \frac{d\mathbf{r}_1}{dt} \\ &= 2\mathbf{r}_1 \cdot \frac{d\mathbf{r}_1}{dt} \text{ since dot product is commutative} \\ &= 2r_1 \frac{d\mathbf{r}_1}{dt} \text{ as } \mathbf{r}_1 \cdot \mathbf{r}_1 = \mathbf{r}^2{}_1 = r_1{}^2 \end{aligned}$$

which follows that
$$\mathbf{r}_1 \cdot \frac{d\mathbf{r}_1}{dt} = r_1 \frac{dr_1}{dt}.$$

Note 3. The necessary and sufficient condition for a vector $\mathbf{r}$ to have constant magnitude is $\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = 0$.

Since $\mathbf{r} \cdot \mathbf{r} = |\mathbf{r}|^2$, *i.e.* $2\mathbf{r}. \frac{d\mathbf{r}}{dt} = 2\,|\mathbf{r}|\frac{d}{dt}(|\mathbf{r}|)$.

Therefore $\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}=0$, if and only if $\frac{d}{dt}|\mathbf{r}|=0$, *i.e.* if $|\mathbf{r}|$ is constant.

(3) Differentiation of vector product of two vectors. Suppose that

$$\mathbf{r}=\mathbf{r}_1\times\mathbf{r}_2,$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are differentiable vector functions of a scalar variable t.

If an increment δt corresponds to increments $\delta\mathbf{r}$ in $\mathbf{r}$, $\delta\mathbf{r}_1$ in $\mathbf{r}_1$ and $\delta\mathbf{r}_2$ in $\mathbf{r}_2$, then we have

$$\mathbf{r}+\delta\mathbf{r}=(\mathbf{r}_1+\delta\mathbf{r}_1)\times(\mathbf{r}_2+\delta\mathbf{r}_2)$$

$=\mathbf{r}_1\times\mathbf{r}_2+\mathbf{r}_1\times\delta\mathbf{r}_2+\delta\mathbf{r}_1\times\mathbf{r}_2$ neglecting the product $\delta\mathbf{r}_1\times\delta\mathbf{r}_2$ as it is vanishingly small

or $\quad\delta\mathbf{r}=\mathbf{r}_1\times\delta\mathbf{r}_2+\delta\mathbf{r}_1\times\mathbf{r}_2.$

Dividing throughout by δt and proceeding to the limit $\delta t\to 0$, we have

$$\frac{d\mathbf{r}}{dt}=\mathbf{r}_1\times\frac{d\mathbf{r}_2}{dt}+\frac{d\mathbf{r}_1}{dt}\times\mathbf{r}_2.$$

Note. The necessary and sufficient condition for $\mathbf{r}$ to have constant direction is

$$\mathbf{r}\times\frac{d\mathbf{r}}{dt}=0.$$

Let r be the magnitude of $\mathbf{r}$ and $\hat{\mathbf{r}}$ be the unit vector in the direction of $\mathbf{r}$. Then

$$\mathbf{r}=r\hat{\mathbf{r}}.$$

$$\therefore\quad \frac{d\mathbf{r}}{dr}=r\frac{d\hat{\mathbf{r}}}{dt}+\frac{dr}{dt}\hat{\mathbf{r}},$$

so that

$$\mathbf{r}\times\frac{d\mathbf{r}}{dr}=\hat{r}\mathbf{r}\times\left\{\mathbf{r}\frac{d\hat{\mathbf{r}}}{dt}+\frac{dr}{dt}\hat{\mathbf{r}}\right\}$$

$$=r^2\hat{\mathbf{r}}\times\frac{d\hat{\mathbf{r}}}{dt} \qquad \ldots(i)$$

other term vanishing as $\hat{\mathbf{r}}\times\hat{\mathbf{r}}=0$.

Let us now suppose that $\mathbf{r}$ has constant direction, so that $\mathbf{r}$ is a constant vector giving,

$$\frac{d\hat{\mathbf{r}}}{dt}=0.$$

As such we have from (i)

$$\mathbf{r}\times\frac{d\mathbf{r}}{dt}=0 \text{ i.e., the condition is necessary.}$$

Now to show that the condition is sufficient, suppose that

$$\mathbf{r}\times\frac{d\mathbf{r}}{dt}=0\,,$$

So that from (i) $\quad \mathbf{r}^2\hat{\mathbf{r}}\times\frac{d\hat{\mathbf{r}}}{dt}=0 \qquad \ldots(ii)$

But $\hat{\mathbf{r}}$ is a vector having unity as its length therefore by Note 3, Rule 2 of §1.17

$$\hat{\mathbf{r}}.\frac{d\hat{\mathbf{r}}}{dt}=0. \qquad \ldots(iii)$$

From (*ii*) and (*iii*) we see that $\hat{\mathbf{r}} \neq 0$ therefore $\dfrac{d\hat{\mathbf{r}}}{dt} = 0.$

$\therefore$ $\hat{\mathbf{r}}$ is a constant vector, *i.e.* the direction of **r** remains the same.

(4) **The differentiation of a scalar triple product.** Let v be expressed as the scalar triple product of three vectors **p, q, r** as

$$\mathbf{v} = [\mathbf{pqr}]$$

$$= \mathbf{p} \cdot \mathbf{q} \times \mathbf{r} \qquad \ldots(i)$$

where **p, q, r** are the differentiable vector functions of a scalar variable t. If a change δt in t corresponds a change $\delta \mathbf{v}$ in **v**, $\delta \mathbf{p}$ in **p**, $\delta \mathbf{q}$ in **q** and $\delta \mathbf{r}$ in **r**, then we have

$$\mathbf{v} + \delta\mathbf{v} = (\mathbf{p} + \delta\mathbf{p}) \cdot \{(\mathbf{q} + \delta\mathbf{q}) \times (\mathbf{r} + \delta\mathbf{r})\}$$

$$= (\mathbf{p} + \delta\mathbf{p}) \cdot \{\mathbf{q} \times \mathbf{r} + \mathbf{q} \times \delta\mathbf{r} + \delta\mathbf{q} \times \mathbf{r} + \delta\mathbf{q} \times \delta\mathbf{r}\}$$

$$= (\mathbf{p} + \delta\mathbf{p}) \cdot (\mathbf{q} \times \mathbf{r} + \mathbf{q} \times \delta\mathbf{r} + \delta\mathbf{q} \times \mathbf{r})$$

(leaving $\delta\mathbf{q} \times \delta\mathbf{r}$ as it is vanishingly small)

$$= \mathbf{p} \cdot \mathbf{q} \times \mathbf{r} + \delta\mathbf{p} \cdot \mathbf{q} \times \mathbf{r} + \mathbf{p} \cdot \delta\mathbf{q} \times \mathbf{r} + \mathbf{p} \cdot \mathbf{q} \times \delta\mathbf{r}$$

(Leaving small value products)

or $$\delta\mathbf{v} = \delta\mathbf{p} \cdot \mathbf{q} \times \mathbf{r} + \mathbf{p} \cdot \delta\mathbf{q} \times \mathbf{r} + \mathbf{p} \cdot \mathbf{q} \times \delta\mathbf{r} \ldots \text{unsing } (i)$$

Dividing it throughout by δt and proceeding to the limit as $\delta t \to 0$, we have

$$\lim_{\delta t \to 0} \frac{\delta\mathbf{v}}{\delta t} = \lim_{\delta t \to 0} \left\{ \frac{\delta\mathbf{p}}{\delta t} \cdot \mathbf{q} \times \mathbf{r} + \mathbf{p} \cdot \frac{\delta\mathbf{q}}{\delta t} \times \mathbf{r} + \mathbf{p} \cdot \mathbf{q} \times \frac{\delta\mathbf{r}}{\delta t} \right\}$$

or $$\frac{d\mathbf{v}}{dt} = \left[\frac{d\mathbf{p}}{dt} \cdot \mathbf{q} \times \mathbf{r} + \mathbf{p} \cdot \frac{d\mathbf{q}}{dt} \times \mathbf{r} + \mathbf{p} \cdot \mathbf{q} \times \frac{d\mathbf{r}}{dt} \right]$$

$$= \left\{ \frac{d\mathbf{p}}{dt} \mathbf{qr} \right\} + \left\{ \mathbf{p} \frac{d\mathbf{q}}{dt} \mathbf{r} \right\} + \left\{ \mathbf{pq} \frac{d\mathbf{r}}{dt} \right\}$$

(5) **The differentiation of vector triple product.** Let v be expressed as the vector triple product of three vectors **p, q, r**, *i.e.*,

$$\mathbf{v} = \mathbf{p} \times (\mathbf{q} \times \mathbf{r}) \qquad (i)$$

p, q, r, being differentiable vector functions of scalar variable t. If a change δt in t corresponds a change $\delta\mathbf{p}$ in **p**, $\delta\mathbf{q}$ in **q**, and $\delta\mathbf{r}$ in **r**, then we have

$$\mathbf{v} + \delta\mathbf{v} = (\mathbf{p} + \delta\mathbf{p}) \times \{(\mathbf{q} + \delta\mathbf{q}) \times (\mathbf{r} + \delta\mathbf{r})\}$$

$$= (\mathbf{p} + \delta\mathbf{p}) \times (\mathbf{q} \times \mathbf{r} + \mathbf{q} \times \delta\mathbf{r} + \delta\mathbf{q} \times \mathbf{r} + \delta\mathbf{q} \times \delta\mathbf{r}\}$$

$$= (\mathbf{p} + \delta\mathbf{p}) \times \{\mathbf{q} \times \mathbf{r} + \mathbf{q} \times \delta\mathbf{r} + \delta\mathbf{p} \times \mathbf{r}\}$$

(neglecting product $\delta\mathbf{q} \times \delta\mathbf{r}$ as it is vanishingly small)

$$= \mathbf{p} \times (\mathbf{q} \times \mathbf{r}) + \mathbf{p} \times (\mathbf{q} \times \delta\mathbf{r}) + \mathbf{p} \times (\delta\mathbf{q} \times \mathbf{r}) + \delta\mathbf{p} \times (\mathbf{q} \times \mathbf{r})$$
$$+ \delta\mathbf{p} \times (\mathbf{q} \times \delta\mathbf{r}) + \delta\mathbf{p} \times (\delta\mathbf{q} \times \mathbf{r})$$

$$= \mathbf{p} \times (\mathbf{q} \times \mathbf{r}) + \mathbf{p} \times (\mathbf{q} \times \delta\mathbf{r}) + \mathbf{p} \times (\delta\mathbf{q} \times \mathbf{r}) + \delta\mathbf{p} \times (\mathbf{q} \times \mathbf{r})$$

(neglecting small value products)

or $$\delta\mathbf{v} = \mathbf{p} \times (\mathbf{q} \times \delta\mathbf{r}) + \mathbf{p} \times (\delta\mathbf{q} \times \mathbf{r}) + \delta\mathbf{p} \times (\mathbf{q} \times \mathbf{r}) \qquad \text{using } (i)$$

Dividing it throughout by δt and proceeding to the limit as $\delta t \to 0$, we have

$$\lim_{\delta t \to 0} \frac{\delta\mathbf{v}}{\delta t} = \lim_{\delta t \to 0} \left\{ \mathbf{p} \times \left(\mathbf{q} \times \frac{\delta\mathbf{r}}{\delta t} \right) + \mathbf{p} \times \left(\frac{\delta\mathbf{q}}{\delta t} \times \mathbf{r} \right) + \frac{\delta\mathbf{p}}{\delta t} \times (\mathbf{q} \times \mathbf{r}) \right\}$$

or $$\frac{d\mathbf{v}}{dt}=\left\{\mathbf{p}\times\left(\mathbf{q}\times\frac{d\mathbf{r}}{dt}\right)+\mathbf{q}\times\left(\frac{d\mathbf{q}}{dt}\times\mathbf{r}\right)+\frac{d\mathbf{p}}{dt}\times(\mathbf{q}\times\mathbf{r})\right\}$$

Problem 61. *If* $\mathbf{F}(t)=x(t)\,\mathbf{i}+y(t)\,\mathbf{j}+z(t)\,\mathbf{k}$, *where* x, y, z *are defferentiable functions of a scalar variable t, prove that*

$$\frac{d\mathbf{F}}{dt}=\frac{dx}{dt}\,\mathbf{i}+\frac{dy}{dt}\,\mathbf{j}+\frac{dz}{dt}\mathbf{k}.$$

By the definition of differential coefficient

$$\frac{d\mathbf{F}}{dt}=\lim_{\delta t\to 0}\frac{\mathbf{F}(t+\delta t)-\mathbf{F}(t)}{\delta t}$$

$$=\lim_{\delta t\to 0}\left\{\frac{[x(t+\delta t)\,\mathbf{i}+y(t+\delta t)\,\mathbf{j}+z(t+\delta t)\mathbf{k}]-[x(t)\mathbf{i}+y(t)\,\mathbf{j}+z(t)\mathbf{k}]}{\delta t}\right\}$$

$$=\lim_{\delta t\to 0}\left[\frac{[x(t+\delta t)-x(t)]}{\delta t}\mathbf{i}+\frac{[y(t+\delta t)-y(t)]}{\delta t}\mathbf{j}+\frac{[z(t+\delta t)-z(t)]}{\delta t}\mathbf{k}\right]$$

$$=\lim_{\delta t\to 0}\left(\frac{\delta x}{\delta t}\mathbf{i}+\frac{\delta y}{\delta t}\mathbf{j}+\frac{\delta z}{\delta t}\mathbf{k}\right)$$

$$=\frac{dx}{dt}\mathbf{i}+\frac{dy}{dt}\mathbf{j}+\frac{dz}{dt}\mathbf{k}$$

Problem 62. *If* $\mathbf{r}=t^2\mathbf{i}-t^2\mathbf{j}+(2t+1)\,\mathbf{k}$. *Find the value of*

$$\frac{d\mathbf{r}}{dt},\ \frac{d^2\mathbf{r}}{dt^2},\ \left|\frac{d\mathbf{r}}{dt}\right|,\ \left|\frac{d^2\mathbf{r}}{dt^2}\right| \text{ at } t=0.$$

(Agra, M.Sc., 1966)

Given vector **r** is the function of scalar variable t, expressed in the form

$$\mathbf{r}=t^2\mathbf{i}-t\mathbf{j}+(2t+1)\,\mathbf{k}. \quad ...(1)$$

$$\therefore \quad \frac{d\mathbf{r}}{dt}=2t\mathbf{i}-\mathbf{j}+2\mathbf{k} \quad ...(2)$$

($\because$ **i**, **j**, **k** are constant vectors)

Therefore at $t=0$

$$\frac{d\mathbf{r}}{dt}=-\mathbf{j}+2\mathbf{k}.$$

so that $$\left|\frac{d\mathbf{r}}{dt}\right|=\sqrt{[(-1)^2+(2)^2]}=\sqrt{5}.$$

Again differentiating (2), with respect to t, we get

$$\frac{d^2\mathbf{r}}{dt^2}=2\mathbf{i}$$

at $$t=0,\ \frac{d^2\mathbf{r}}{dt^2}=2\mathbf{i}.$$

$\therefore \qquad \left|\dfrac{d^2\mathbf{r}}{dt^2}\right| = |\,2\mathbf{i}\,| = 2.$

Problem 63. *If* $\mathbf{r} = \mathbf{a}\cos\omega t + \mathbf{b}\sin\omega t$, *show that*

$$\mathbf{r}\times\frac{d\mathbf{r}}{dt} = \omega\mathbf{a}\times\mathbf{b}$$

and $\dfrac{d^2\mathbf{r}}{dt^2} = -\omega^2\mathbf{r}$ **a**, **b** *being constant vectors and* ω *is also a constant.*

Given $\mathbf{r} = \mathbf{a}\cos\omega t + \mathbf{b}\sin\omega t$ ...(1)

$$\therefore \quad \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(\cos\omega t)\,\mathbf{a} + \frac{d}{dt}(\sin\omega t)\,\mathbf{b}$$

$$= -\omega\sin\omega t\,\mathbf{a} + \omega\cos\omega t\,\mathbf{b}.$$

Differentiating (2) with respect to t, we get

$$\frac{d^2\mathbf{r}}{dt^2} = -\omega\frac{d}{dt}(\sin\omega t)\mathbf{a} + \omega\frac{d}{dt}(\cot\ \omega t)\mathbf{b}$$

$$= -\omega^2\cos\omega t\,\mathbf{a} - \omega^2\sin\omega t\,\mathbf{b}$$

$$= -\omega^2[\cos\omega t\,\mathbf{a} + \sin\omega t\,\mathbf{b}]$$

$$= -\omega^2\mathbf{r} \text{ by (1)}$$

i.e. $\dfrac{d^2\mathbf{r}}{dt^2} = -\omega^2\mathbf{r}.$

Again, we have

$$\mathbf{r}\times\frac{d\mathbf{r}}{dt} = (\mathbf{a}\cos\omega t + \mathbf{b}\sin\omega t)\times(-\omega\sin\omega t\,\mathbf{a} + \omega\cos\omega t\,\mathbf{b})$$

$$= \omega\,[\mathbf{a}\cos\omega t \times -\sin\omega t\,\mathbf{a} + \mathbf{a}\times\mathbf{b}\cos^2\omega t$$
$$- \mathbf{b}\times\mathbf{a}\sin^2\omega t + \mathbf{b}\times\mathbf{b}\sin\omega t\cos\omega t]$$

$$= (\mathbf{a}\times\mathbf{b})\ \{\cos^2\omega t + \sin^2\omega t\}\ \omega$$

As $\mathbf{a}\times\mathbf{a} = \mathbf{b}\times\mathbf{b} = 0$ and $-\mathbf{b}\times\mathbf{a} = \mathbf{a}\times\mathbf{b}$

$$= (\mathbf{a}\times\mathbf{b})\ \omega,$$

i.e. $\mathbf{r}\times\dfrac{d\mathbf{r}}{dt} = \omega\,(\mathbf{a}\times\mathbf{b})$

Problem 64. *If* $\mathbf{r} = \mathbf{a}e^{\omega t} + \mathbf{B}e^{-\omega t}$, *show that*

$\dfrac{d^2\mathbf{r}}{dt^2} - \omega^2\mathbf{r} = 0$; **a**, **b** *are constant vectors and* ω *being a constant.*

Given $\quad \mathbf{r} = \mathbf{a}e^{\omega t} + \mathbf{b}e^{-\omega t}$ ...(1)

$$\therefore \quad \frac{d\mathbf{r}}{dt} = \omega\mathbf{a}e^{\omega t} + \omega\mathbf{b}e^{-\omega t}$$

Differentiating it w. r. t. t, we get

$$\frac{d^2\mathbf{r}}{dt^2} = \omega^2\mathbf{a}e^{\omega t} - \omega^2\mathbf{b}e^{-\omega t}$$

$$= \omega^2(\mathbf{a}e^{\omega t} - \mathbf{b}e^{-\omega t})$$

$$= \omega^2\mathbf{r} \text{ from (1)}$$

or $\quad \dfrac{d^2\mathbf{r}}{dt^2} - \omega^2\mathbf{r} = 0.$

Problem 65. *A particle moves along a curve whose parametric equations are* $x = e^{-t}$, $y = 2 \cos 3t$, $z = 2 \sin 3t$, *where t is the time.*

(*a*) *Determine its velocity and acceleration at any time.*

(*b*) *Find the magnitudes of velocity and acceleration at* $t = 0$.

(*a*) Since a vector **r** can be expressed in terms of rectangular components as

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

therefore, the position vector of the moving particle at any time t is

$$\mathbf{r} = e^{-t}\,\mathbf{i} + 2\cos 3t\,\mathbf{j} + 2\sin 3t\,\mathbf{k}.$$

Thus, the velocity $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = -e^{t}\,\mathbf{i} - 6\sin 3t\,\mathbf{j} + 6\cos 3t\,\mathbf{k}$

and the acceleration is $\dfrac{d\mathbf{v}}{dt} = \dfrac{d^2\mathbf{r}}{dt^2} = e^{-t}\,\mathbf{i} - 18\cos 3t\,\mathbf{j} - 18\sin 3t\,\mathbf{k}.$

(*b*) At $t = 0$, $\dfrac{d\mathbf{r}}{dt} = -\mathbf{i} + 6\mathbf{k}$ and $\dfrac{d^2\mathbf{r}}{dt^2} = \mathbf{i} - 18\mathbf{j}.$

$\therefore$ magnitude of velocity at $t = 0$ is

$$\left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{[(-1)^2 + (6)^2]} = \sqrt{[37]}\,.$$

and magnitude of acceleration at $t = 0$ is

$$\left|\frac{d^2\mathbf{r}}{dt^2}\right| = \sqrt{[(1)^2 + (-18)^2]} = \sqrt{[325]}\,.$$

Problem 66. *A particle moves along the curve* $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$, *where t is the time. Find component of its velocity and acceleration at* $t = 1$ *in the direction* $\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$.

The position vector of the moving particle at any time t is

$$\mathbf{r} = 2t^2\mathbf{i} + (t^2 - 4t)\,\mathbf{j} + (3t - 5)\,\mathbf{k}.$$

$\therefore$ velocity $= \dfrac{d\mathbf{r}}{dt} = 4t\,\mathbf{i} + (2t - 4)\,\mathbf{j} + 3\mathbf{k},$

so that $$\left[\frac{d\mathbf{r}}{dt}\right]_{\text{at } t=1} = 4\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$$

and acceleration $= \dfrac{d^2\mathbf{r}}{dt^2} = 4\mathbf{i} + 2\mathbf{j},$

so that $$\left[\frac{d^2\mathbf{r}}{dt^2}\right]_{\text{at } t=1} = 4\mathbf{i} + 2\mathbf{j}.$$

Now unit vector along $\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

$$= \frac{\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}}{\sqrt{\{(1)^2 + (-3)^2 + (2)^2\}}} = \frac{\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}}{\sqrt{(14)}}.$$

Hence, component of velocity in the direction $\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ is

$$\frac{(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k})}{\sqrt{(14)}} \cdot (4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) = \frac{16}{\sqrt{(14)}} = \tfrac{8}{7}\sqrt{(14)}$$

and component of acceleration in that direction

$$= \left(\frac{\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}}{\sqrt{(14)}}\right) \cdot (4\mathbf{i} + 2\mathbf{j})$$

$$= -\frac{2}{\sqrt{14}} = -\frac{1}{7}\sqrt{14}.$$

Problem 67. *Prove the following relations*:

(*a*) $\mathbf{v} = \mathbf{u} + \mathbf{f}t.$

(*b*) $\mathbf{s} = \mathbf{s}_0 + \mathbf{u}t + \frac{1}{2}\mathbf{f}t^2.$

(*c*) $v^2 = u^2 + 2\mathbf{f}.\mathbf{s}.$

(*a*) Velocity is defined as rate of change of displacement,

i.e. $$\mathbf{v} = \frac{d\mathbf{s}}{dt}. \quad ...(1)$$

Acceleration is defined as rate of change of veclocity, *i.e.*

$$\mathbf{f} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\frac{d\mathbf{s}}{dt}\right) = \frac{d^2\mathbf{s}}{dt^2} \quad ...(2)$$

or $$d\mathbf{v} = \mathbf{f}\, dt. \quad ...(3)$$

Let initially at $t = 0$, the velocity be $\mathbf{u}$ and after time t, let it be $\mathbf{v}$.

Intergrating (3) we get

$$\int_{\mathbf{u}}^{\mathbf{v}} d\mathbf{v} = \int_0^t \mathbf{f}\, dt$$

i.e., $$\mathbf{v} - \mathbf{u} = \mathbf{f}t.$$

$\therefore$ $$\mathbf{v} = \mathbf{u} + \mathbf{f}t$$

(*b*) We have from (*a*), $\frac{d\mathbf{s}}{dt} = \mathbf{u} + \mathbf{f}t.$

Intergrating w.r.t. to t,

$$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{f}t^2 + \mathbf{B},$$

where $\mathbf{B}$ is constant of integration.

If at $t = 0$, $\mathbf{s} = \mathbf{s}_0$ then $\mathbf{B} = \mathbf{s}_0$.

$\therefore$ $$\mathbf{s} = \mathbf{s}_0 + \mathbf{u}t + \frac{1}{2}\mathbf{f}t^2$$

(*c*) Multiplying scalarly equation (2) by $2\frac{d\mathbf{s}}{dt}$, we get

$$2\,\frac{d\mathbf{s}}{dt}\cdot\frac{d^2\mathbf{s}}{dt^2} = 2\mathbf{f}\cdot\frac{d\mathbf{s}}{dt}.$$

Integrating with respect to t, we get

$$2\int_0^t \mathbf{v}\cdot\frac{d\mathbf{v}}{dt}\, dt = \int_0^t 2\mathbf{f}\cdot\frac{d\mathbf{s}}{dt}\, dt.$$

$\therefore$ $$2\int_{\mathbf{u}}^{\mathbf{v}} \mathbf{v}\cdot d\mathbf{v} = 2\int_0^{\mathbf{s}} \mathbf{f}\cdot d\mathbf{s}$$

$\therefore$ $$[\mathbf{v}\cdot\mathbf{v}]_u^v = 2\mathbf{f}\cdot\mathbf{s}.$$

$\therefore$ $$v^2 - u^2 = 2\mathbf{f}\cdot\mathbf{s}$$

$$v^2 = u^2 + 2\mathbf{f}\cdot\mathbf{s}.$$

Problem 68. *A particle of 1 kg. moving with initial velocity* (i + 2k) *metres/sec. is acted upon by a constant force* (i + 2j – 2k) *newtons. Calculate the distance and velocity after 5 seconds. Also find the time in which the particle reaches the xy-plane.*

Mass of the particle = 1 kg.

Force acting on the particle = (i + 2j – 2k) newtons.

Acceleration $\mathbf{f} = \frac{\text{force}}{\text{mass}} = \frac{\mathbf{i}+2\mathbf{j}-2\mathbf{k}}{1} = (\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$ metres/sec.

Initial velocity $\mathbf{u} = (\mathbf{i} + 2\mathbf{k})$,

$t = 5$ seconds.

Using $\mathbf{v} = \mathbf{u} + \mathbf{f}t$ we have

Velocity after 5 seconds $= (\mathbf{i} + 2\mathbf{k}) + (\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})\, 5$

$= \mathbf{i} + 2\mathbf{k} + 5\mathbf{i} + 10\mathbf{j} - 10\mathbf{k}$

$= 6\mathbf{i} + 10\mathbf{j} - 8\mathbf{k}$

So that $v = \sqrt{6^2 + 10^2 + 8^2} = \sqrt{200}$

$= 10\sqrt{2}$ metres/sec.

Distance covered in 5 seconds,

$$\mathbf{s} = \mathbf{u}t + \tfrac{1}{2}\mathbf{f}t^2 = (\mathbf{i} + 2\mathbf{k})\, 5 + \tfrac{1}{2}(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})\, 25$$

$$= 5\mathbf{i} + 10\mathbf{k} + \tfrac{25}{2}\mathbf{i} + 25\mathbf{j} - 25\mathbf{k}$$

$$= \tfrac{35}{2}\mathbf{i} + 25\mathbf{j} - 15\mathbf{k}$$

$$= 17\cdot5\mathbf{i} + 25\mathbf{j} - 15\mathbf{k}$$

$$\therefore \quad s = \sqrt{\{(17\cdot5)^2 + (25)^2 + (15)^2\}}$$

$$= \sqrt{(306\cdot25 + 652 + 225)}$$

$$= \sqrt{(1156\cdot25)}$$

$$= 34 \text{ metres.}$$

Let t be the time when the particle reaches the xy-plane, *i.e.*, when s is equal to zero.

Using equation $\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{f}t^2$, we have

$$\mathbf{i}s_x + \mathbf{j}s_y + \mathbf{k}s_z = (\mathbf{i} + 2\mathbf{k})\, t + \tfrac{1}{2}(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})\, t^2$$

Comparing coefficients of k,

$$s_z = 2t - t^2 = 0$$

or $$t = 2 \text{ sec.}$$

Problem 69. *If* $\mathbf{r} = t^2\mathbf{i} - t\mathbf{j} + (2t + 1)\mathbf{k}$ and $\mathbf{s} = (2t - 3)\mathbf{i} + \mathbf{j} - t\mathbf{k}$, *find*

(*a*) $\frac{d}{dt}(\mathbf{r}\cdot\mathbf{s})$, (*b*) $\frac{d}{dt}(\mathbf{r}\times\mathbf{s})$,

(*c*) $\frac{d}{dt}(\mathbf{r}+\mathbf{s})$, (*d*) $\frac{d}{dt}\left(\mathbf{r}\times\frac{d\mathbf{s}}{dt}\right)_{\text{at } t=1}$

(*a*) $$\frac{d}{dt}(\mathbf{r}\cdot\mathbf{s}) = \mathbf{r}\cdot\frac{d\mathbf{s}}{dt} + \frac{d\mathbf{r}}{dt}\cdot\mathbf{s}$$

$$= \{t^2\mathbf{i} - t\mathbf{j} + (2t + 1)\,\mathbf{k}\}\cdot\{2\mathbf{j} - \mathbf{k}\}$$

$$+ \{2t\mathbf{i} - \mathbf{i} + 2\mathbf{k}\}\cdot\{(2t - 3)\,\mathbf{i} + \mathbf{j} - t\mathbf{k}\}$$

$$= 2t^2 - 2t - 1 + 4t^2 - 6t - 1 - 3t$$

$= 6t^2 - 11t - 1$

$= -6$, when $t = 1$,

i.e. $\frac{d}{dt}(\mathbf{r} \cdot \mathbf{s}) = -6$ at $t = 1$.

(*b*) $\frac{d}{dt}(\mathbf{r} \times \mathbf{s}) = \mathbf{r} \times \frac{d\mathbf{s}}{dt} + \frac{d\mathbf{r}}{dt} \times \mathbf{s}$

$= \{t^2\mathbf{i} - t\mathbf{j} + (2t+1)\mathbf{k}\} \times \{2\mathbf{i} - \mathbf{k}\}$

$+ \{2t\mathbf{i} - \mathbf{j} + 2\mathbf{k}\} \times \{(2t-3)\mathbf{i} + \mathbf{j} - t\mathbf{k}\}$

$= 4t^2\mathbf{j} + 2t\mathbf{k} + t\mathbf{i} + 2(2t+1)\mathbf{j} + 2t\mathbf{k} + 2t^2\mathbf{j}$

$+ (2t-3)\mathbf{k} + t\mathbf{i} + 2(2t-3)\mathbf{j} - 2t$

$= (2t-2)\mathbf{i} + (3t^2 + 8t - 4)\mathbf{j} + (6t-3)\mathbf{k}$

$= 0\mathbf{i} + 7\mathbf{j} + 3\mathbf{k}$ at $t = 1$.

Therefore $\frac{d}{dt}(\mathbf{r} \times \mathbf{s})_{\text{at } t=1} = 7\mathbf{j} + 3\mathbf{k}$.

(*c*) $\frac{d}{dt}(\mathbf{r} + \mathbf{s}) = \frac{d\mathbf{r}}{dt} + \frac{d\mathbf{s}}{dt}$

$= \frac{d}{dt}[t^2\mathbf{i} - t\mathbf{j} + (2t+1)\mathbf{k}] + \frac{d}{dt}[(2t-3)\mathbf{i} + \mathbf{j} - t\mathbf{k}]$

$= (2t\mathbf{i} - \mathbf{j} + 2\mathbf{k}) + (2\mathbf{i} + 0\mathbf{j} - \mathbf{k})$

$= (2t+2)\mathbf{i} - \mathbf{j} + \mathbf{k}$

$= 4\mathbf{i} - \mathbf{j} + \mathbf{k}$ at $t = 1$.

Thus $\frac{d}{dt}(\mathbf{r} + \mathbf{s}) = 4\mathbf{i} - \mathbf{j} + \mathbf{k}$ at $t = 1$.

(*d*) $\mathbf{s} = (2t-3)\mathbf{i} + \mathbf{j} - t\mathbf{k}$

$\frac{d\mathbf{s}}{dt} = 2\mathbf{i} - \mathbf{k}$.

$\therefore$ $\left(\mathbf{r} \times \frac{d\mathbf{s}}{dt}\right) = \left\{t^2\mathbf{i} - t\mathbf{j} + (2t+1)\mathbf{k}\right\} \times \{2\mathbf{i} - \mathbf{k}\}$

$= t^2\mathbf{j} + 2t\mathbf{k} + t\mathbf{i} + (2t+1)2\mathbf{j}$

$= t\mathbf{i} + (t^2 + 4t - 2)\mathbf{j} + 2t\mathbf{k}$.

$\therefore$ $\frac{d}{dt}\left(\mathbf{r} \times \frac{d\mathbf{s}}{dt}\right) = \frac{d}{dt}\{t\mathbf{i} + (t^2 + 4t + 2)\mathbf{j} + 2t\mathbf{k}\}$

$= \mathbf{i} + (2t+4)\mathbf{j} + 2\mathbf{k}$

$= (\mathbf{i} + 6\mathbf{j} + 2\mathbf{k})$ at $t = 1$.

Thus $\frac{d}{dt}\left(\mathbf{r} \times \frac{d\mathbf{s}}{dt}\right) = \mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$.

Problem 70. *Differentiate* $\frac{\mathbf{r} \times \mathbf{a}}{\mathbf{r} \cdot \mathbf{a}}$, **a** *being a constant vector.*

Let $\mathbf{v} = \frac{\mathbf{r} \times \mathbf{a}}{\mathbf{r} \cdot \mathbf{a}}$

$\therefore$ $\frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\frac{\mathbf{r} \times \mathbf{a}}{\mathbf{r} \cdot \mathbf{a}}\right)$

$$= \frac{(\mathbf{r}\cdot\mathbf{a})\,(\dot{\mathbf{r}}\times\mathbf{a})-(\mathbf{r}\times\mathbf{a})\,(\dot{\mathbf{r}}\cdot\mathbf{a})}{[\mathbf{r}\cdot\mathbf{a}]^2}$$

$$= \frac{(\dot{\mathbf{r}}\times\mathbf{a})}{(\mathbf{r}\cdot\mathbf{a})}-\frac{(\mathbf{r}\times\mathbf{a})\,(\dot{\mathbf{r}}\cdot\mathbf{a})}{[\mathbf{r}\cdot\mathbf{a}]^2} \text{ as } \frac{d\mathbf{a}}{dt}=0.$$

Problem 71. *Find the derivative of the product* $\mathbf{r}\times\mathbf{s}$ *and deduce that*

$$\frac{d}{dt}\left(\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right)=\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}. \qquad [Agra, 1965]$$

For the derivative of $\mathbf{r}\times\mathbf{s}$ see Rule (2) of §1.17

i.e.
$$\frac{d}{dt}(\mathbf{r}\times\mathbf{s})=\frac{d\mathbf{r}}{dt}\times\mathbf{s}+\mathbf{r}\times\frac{d\mathbf{s}}{dt} \qquad ...(i)$$

For second part putting $\frac{d\mathbf{r}}{dt}$ in place of s in (1), we get

$$\frac{d}{dt}\left(\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right)=\frac{d\mathbf{r}}{dt}\times\frac{d\mathbf{r}}{dt}+\mathbf{r}\times\frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right)$$

$$=\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2} \text{ as } \frac{d\mathbf{r}}{dt}\times\frac{d\mathbf{r}}{dt}=0.$$

1.19. PARTIAL DIFFERENTIATION OF VECTORS

Let **F** be a vector function depending upon more than one scalar variables, say x, y, z; then we write $\mathbf{F}=\mathbf{F}(x, y, z)$ and the partial derivative of **F** with respect to x defined as

$$\frac{\partial\mathbf{F}}{\partial x}=\lim_{\delta x\to 0}\frac{\mathbf{F}(x+\delta x, y, z)-\mathbf{F}(x, y, z)}{\delta x}$$

if this limit exists.

Similarly partial derivatives of **F** with respect to y and z can be written as

$$\frac{\partial\mathbf{F}}{\partial y}=\lim_{\delta x\to 0}\frac{\mathbf{F}(x, y+\delta y, z)-\mathbf{F}(x, y, z)}{\delta y}$$

$$\frac{\partial\mathbf{F}}{\partial z}=\lim_{\delta x\to 0}\frac{\mathbf{F}(x, y, z+\delta z)-\mathbf{F}(x, y, z)}{\delta z}$$

provided these limits exist.

Note. If $\mathbf{F}=u(x,y,z,t)\,\mathbf{i}+v(x,y,z,t)\,\mathbf{j}+w(x,y,z,t)\,\mathbf{k}$, then the partial derivatives of **F** with respect to x, y, z, t respectively, may be expressed as

$$\frac{\partial\mathbf{F}}{\partial x}=\frac{\partial u}{\partial x}\mathbf{i}+\frac{\partial v}{\partial x}\mathbf{j}+\frac{\partial w}{\partial x}\mathbf{k},$$

$$\frac{\partial\mathbf{F}}{\partial y}=\frac{\partial u}{\partial y}\mathbf{i}+\frac{\partial v}{\partial y}\mathbf{j}+\frac{\partial w}{\partial y}\mathbf{k}$$

$$\frac{\partial\mathbf{F}}{\partial z}=\frac{\partial u}{\partial z}\mathbf{i}+\frac{\partial v}{\partial z}\mathbf{j}+\frac{\partial w}{\partial z}\mathbf{k}$$

$$\frac{\partial\mathbf{F}}{\partial t}=\frac{\partial u}{\partial t}\mathbf{i}+\frac{\partial v}{\partial t}\mathbf{j}+\frac{\partial w}{\partial t}\mathbf{k}.$$

Higher order partial derivatives. The partial derivatives $\frac{\partial\mathbf{F}}{\partial x}, \frac{\partial\mathbf{F}}{\partial y}, \frac{\partial\mathbf{F}}{\partial z}...$ being themselves functions of the same set of scalar variables may again be partially differentiated, giving second order partial derivatives such as

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\right), \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y}\right), \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial z}\right), \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right), \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) \text{...etc.}$$

which are denoted by $\frac{\partial^2 F}{\partial x^2}, \frac{\partial^2 F}{\partial y^2}, \frac{\partial^2 F}{\partial z^2}, \frac{\partial^2 F}{\partial x \partial y}, \frac{\partial^2 F}{\partial y \partial x}$...etc.

Further differentiation of second order partial derivatives may give third and higher order partial derivatives.

Note 1. The two second order partial derivatives, viz.,

$$\frac{\partial^2 F}{\partial x \partial y} \text{ and } \frac{\partial^2 F}{\partial y \partial x}$$

are equal, if each of them is a continuous function, *i.e.*,

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}.$$

Note 2. If $r = F(x, y)$ and $x = f_1(t, s)$, $y = f_2(t, s)$; then we have

$$\frac{\partial r}{\partial t} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial t}$$

and

$$\frac{\partial r}{\partial s} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial s}.$$

These two results are similar to those of the results in ordinary calculus.

Total differentials. If F is a vector function of scalar variables $x, y, z...$, and we assume that the values of the variables increase from $x, y, z...$ to $x + \delta x, y + \delta y, z + \delta z...$, when the corresponding change in F is $F + \delta F$, then we write

$$F(x, y, z...,) + \delta F = F(x + \delta x, y + \delta y, z + \delta z...,)$$

or

$$\delta F = F(x + \delta x, y + \delta y, z + \delta z...,) - F(x, y, z...)$$

which may be expressed in the form,

$$\delta F = \frac{F(x+\delta x, y+\delta y, z+\delta z...) - F(x, y+\delta y, z+\delta z...,)}{\delta x}\delta x$$

$$+\frac{F(x, y+\delta y, z+\delta z...) - F(x, y, z+\delta z...,)}{\delta y}\delta y + ... + ...$$

Now if $\delta x, \delta y, \delta z...$ tend to zero then δF will also tend to zero; so that the coefficients of $\delta x, \delta y, \delta z$...in the above expression tend to the limiting values

$$\frac{\partial F(x, y, z...)}{\partial x}, \frac{\partial F(x, y, z...)}{\partial y}, \frac{\partial F(x, y, z...)}{\partial z}, ...$$

or simply to

$$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}.$$

As such the above expression can be written as

$$\delta F = \frac{\partial F}{\partial x}\delta x + \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial z}\delta z + ...$$

Proceeding to the limit when $\delta F \to 0$, we have

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial y}dz + ...$$

This gives the total differential of F for the scalar variables $x, y, z...$

The total differential $d\mathbf{F}$ is given by

$$d\mathbf{F} = du\mathbf{i} + dv\mathbf{j} + dw\mathbf{k}$$

When $\mathbf{F} = u\,(x, y, z)\,\mathbf{i} + v\,(x, y, z)\,\mathbf{j} + w\,(x, y, z)\,\mathbf{k}$

When
$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz + \ldots$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz$$

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz.$$

1.20. RULES FOR PARTIAL DIFFERENTIATION OF VECTORS

If $\mathbf{r}$ and $\mathbf{s}$ are functions of x, y, z, then

(1) $$\frac{\partial}{\partial x}(\mathbf{r}\cdot\mathbf{s}) = \mathbf{r}\cdot\frac{\partial \mathbf{s}}{\partial x} + \frac{\partial \mathbf{r}}{\partial x}.\mathbf{s}.$$

(2) $$\frac{\partial}{\partial x}(\mathbf{r}\times\mathbf{s}) = \mathbf{r}\times\frac{\partial \mathbf{s}}{\partial x} + \frac{\partial \mathbf{r}}{\partial x}\times\mathbf{s}.$$

(3) $$\frac{\partial^2}{\partial y\partial x}(\mathbf{r}\cdot\mathbf{s}) = \frac{\partial}{\partial y}\left\{\frac{\partial}{\partial x}(\mathbf{r}\cdot\mathbf{s})\right\} = \frac{\partial}{\partial y}\left\{\mathbf{r}\cdot\frac{\partial \mathbf{s}}{\partial x} + \frac{\partial \mathbf{r}}{\partial x}\cdot\mathbf{s}\right\}$$

$$= \mathbf{r}.\frac{\partial^2 \mathbf{s}}{\partial y\partial x} + \frac{\partial \mathbf{r}}{\partial y}\cdot\frac{\partial \mathbf{s}}{\partial x} + \frac{\partial \mathbf{r}}{\partial x}\cdot\frac{\partial \mathbf{s}}{\partial y} + \frac{\partial^2 \mathbf{r}}{\partial y\partial x}\cdot\mathbf{s}.$$

The proofs of these results are similar to those given in §1.18.

Problem 72. *Find* $\frac{\partial \mathbf{r}}{\partial x}, \frac{\partial \mathbf{r}}{\partial y}, \frac{\partial^2 \mathbf{r}}{\partial x^2}, \frac{\partial^2 \mathbf{r}}{\partial y^2}, \frac{\partial^2 \mathbf{r}}{\partial x\partial y}$ *for the following functions:*

(*a*) $\mathbf{r} = x\cos y\mathbf{i} + x\sin y\mathbf{j} + ae^{my}\mathbf{k}$.

(*b*) $\mathbf{r} = \frac{1}{2}a\,(x + y)\,\mathbf{i} + b/2\,(x - y)\,\mathbf{j} + \frac{1}{2}xy\mathbf{k}$.

(*c*) $\mathbf{r} = x\cos y\mathbf{i} + x\sin y\mathbf{j} + c\log\{x + \sqrt{(x^2 - c^2)}\}\mathbf{k}$.

(*a*) Given $\mathbf{r} = x\cos y\mathbf{i} + x\sin y\mathbf{j} + ae^{my}\mathbf{k}$.

$$\therefore \quad \frac{\partial \mathbf{r}}{\partial x} = \cos y\mathbf{i} + \sin y\mathbf{j}$$

$$\frac{\partial \mathbf{r}}{\partial y} = -x\sin y\mathbf{i} + x\cos y\mathbf{j} + ame^{my}\mathbf{k}$$

$$\frac{\partial^2 \mathbf{r}}{\partial x^2} = 0,$$

$$\frac{\partial^2 \mathbf{r}}{\partial y^2} = -x\cos y\mathbf{i} - x\sin y\mathbf{j} + am^2e^{my}\mathbf{k}$$

$$\frac{\partial^2 \mathbf{r}}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \mathbf{r}}{\partial y}\right) = -\sin y\mathbf{i} + \cos y\mathbf{j}.$$

(*b*) Given $$\mathbf{r} = \tfrac{1}{2}a(x + y)\,\mathbf{i} + \tfrac{1}{2}b\,(x - y)\,\mathbf{j} + \tfrac{1}{2}xy\mathbf{k}$$

$$\therefore \quad \frac{\partial \mathbf{r}}{\partial x} = \tfrac{1}{2}[a\mathbf{i} + b\mathbf{j} + y\mathbf{k}]$$

$$\frac{\partial \mathbf{r}}{\partial y} = \tfrac{1}{2}[a\mathbf{i} + b\mathbf{j} + y\mathbf{k}]$$

$$\frac{\partial^2 \mathbf{r}}{\partial x^2} = 0$$

$$\frac{\partial^2 \mathbf{r}}{\partial y^2} = \tfrac{1}{2}\mathbf{k}$$

$$\frac{\partial^2 \mathbf{r}}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \mathbf{r}}{\partial y}\right) = 0.$$

(*c*) Given $\mathbf{r} = x \cos y\mathbf{i} + x \sin y\mathbf{j} + c \log \{x + \sqrt{(x^2 - c^2)}\}\mathbf{k}$

$$\therefore \quad \frac{\partial \mathbf{r}}{\partial x} = \cos\, y\mathbf{i} + \sin\, y\mathbf{j} - c\left\{\frac{\left[1 + \frac{1}{2}\,(x^2 - c^2)^{-1/2} \cdot 2x\right]\mathbf{k}}{x + \sqrt{(x^2 - c^2)}}\right\}$$

$$= \cos y\mathbf{i} + \sin y\mathbf{j} - \frac{c}{\sqrt{(x^2 - c^2)}}\mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial y} = -x \sin\, y\mathbf{i} + x \cos\, y\mathbf{j}$$

$$\frac{\partial^2 \mathbf{r}}{\partial x^2} = \frac{1}{2}c\frac{\mathbf{k} \cdot 2x}{(x^2 - c^2)^{3/2}} = \frac{2x\mathbf{k}}{(x^2 - c^2)^{3/2}}$$

$$\frac{\partial^2 \mathbf{r}}{\partial y^2} = -x \cos\, y\mathbf{i} - x \sin\, y\mathbf{j}$$

$$\frac{\partial^2 \mathbf{r}}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \mathbf{r}}{\partial y}\right) = -\sin\, y\mathbf{i} + \cos\, y\mathbf{j}.$$

Problem 73. *If* $\mathbf{A} = x^2yz\mathbf{i} - 2xz^3\mathbf{j} + xz^3\mathbf{k}$, $\mathbf{B} = 2z\mathbf{i} + y\mathbf{j} - x^2\mathbf{k}$, *find*

$$\frac{\partial^2}{\partial x \partial y}(\mathbf{A} \times \mathbf{B}) \text{ at } (1, 0, -2).$$

We have
$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x^2yz & -2xz^3 & xz^2 \\ 2z & y & -x^2 \end{vmatrix}$$

$$= \mathbf{i}(2x^3z^3 - xyz^2) - \mathbf{j}(-x^4yz - 2xz^3) + \mathbf{k}(x^2y^2z + xz^4)$$

$$\therefore \quad \frac{\partial}{\partial y}(\mathbf{A} \times \mathbf{B}) = \mathbf{i}\,(-xz^2) - \mathbf{j}\,(-x^4z) + \mathbf{k}\,(2x^2yz)$$

and
$$\frac{\partial^2}{\partial x \partial y}(\mathbf{A} \times \mathbf{B}) = \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial y}(\mathbf{A} \times \mathbf{B})\right\} = -z^2\mathbf{i} + 4zx^3\mathbf{j} + 4xyz\mathbf{k}$$

$$= -4\mathbf{i} - 8\mathbf{j} \text{ at } (1, 0, -2)$$

Problem 74. *If* $\phi(x, y, z) = xy^2z$ *and* $\mathbf{r} = xz\mathbf{i} - xy^2\mathbf{j} + yz^2\mathbf{k}$, *find*

$$\frac{\partial^3}{\partial x^2 \partial z}(\phi\mathbf{r}) \text{ at the point } (2, -1, 1).$$

We have $\phi(\mathbf{r}) = (xy^2z)(xz\mathbf{i} - xy^2\mathbf{j} + yz^2\mathbf{k}$

$= x^2y^2z^2\mathbf{i} - x^2y^4z\mathbf{j} + xy^3z^3\mathbf{k}$

$\therefore \quad \dfrac{\partial(\phi\mathbf{r})}{\partial z} = 2x^2y^2z\mathbf{i} - x^2y^4\mathbf{j} + 3xy^3z^2\mathbf{k}$

and $\quad \dfrac{\partial^2}{\partial x \partial z}(\phi\mathbf{r}) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial \phi\mathbf{r}}{\partial z}\right) = 4xy^2z\mathbf{i} - 2xy^4\mathbf{j} + 3y^3z^2\mathbf{k}$

So that

$$\frac{\partial^3}{\partial x^2 \partial z}(\phi\mathbf{r}) = \frac{\partial}{\partial x}\left\{\frac{\partial^2(\phi\mathbf{r})}{\partial x \partial z}\right\} = 4y^2z\mathbf{i} - 2y^4\mathbf{j}$$

$= 4\mathbf{i} - 2\mathbf{j}$ at $(2, -1, 1)$

i.e. at $\quad x = 2, y = -1, z = 1.$

Problem 75. *If* **F** *depends on x, y, z, t where x, y, z depend on t, prove that*

$$\frac{d\mathbf{F}}{dt} = \frac{\partial \mathbf{F}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{F}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{F}}{\partial z}\frac{dz}{dt}.$$

Let us suppose that

$$\mathbf{F} = f_1(x, y, z, t)\,\mathbf{i} + f_2(x, y, z, t)\,\mathbf{j} + f_3(x, y, z, t)\,\mathbf{k} \quad \ldots(i)$$

So that $d\mathbf{F} = df_1\mathbf{i} + df_2\mathbf{j} + df_3\mathbf{k} \quad \ldots(ii)$

Now $\quad df_1 = \dfrac{\partial f_1}{\partial x}dx + \dfrac{\partial f_1}{\partial y}dy + \dfrac{\partial f_1}{\partial z}dz + \dfrac{\partial f_1}{\partial t}dt$

$$df_2 = \frac{\partial f_2}{\partial x}dx + \frac{\partial f_2}{\partial y}dy + \frac{\partial f_2}{\partial z}dz + \frac{\partial f_2}{\partial t}dt$$

$$df_3 = \frac{\partial f_3}{\partial x}dx + \frac{\partial f_3}{\partial y}dy + \frac{\partial f_3}{\partial z}dz + \frac{\partial f_3}{\partial t}dt$$

Putting the values of, df_1, df_2 df_2, in (*ii*) we get

$$d\mathbf{F} = \left(\frac{\partial f_1}{\partial x}dx + \frac{\partial f_1}{\partial y}dy + \frac{\partial f_1}{\partial z}dz + \frac{\partial f_1}{\partial t}dt\right)\mathbf{i} + \ldots + \ldots$$

$$= \left(\frac{\partial f_1}{\partial x}\mathbf{i} + \frac{\partial f_2}{\partial x}\mathbf{j} + \frac{\partial f_3}{\partial x}\mathbf{k}\right)dx + \left(\frac{\partial f_1}{\partial y}\mathbf{i} + \frac{\partial f_2}{\partial y}\mathbf{j} + \frac{\partial f_3}{\partial y}\mathbf{k}\right)dy$$

$$+ \left(\frac{\partial f_1}{\partial z}\mathbf{i} + \frac{\partial f_2}{\partial z}\mathbf{j} + \frac{\partial f_3}{\partial z}\mathbf{k}\right)dz + \left(\frac{\partial f_1}{\partial t}\mathbf{i} + \frac{\partial f_2}{\partial t}\mathbf{j} + \frac{\partial f_3}{\partial t}\mathbf{k}\right)dt$$

or

$$\frac{d\mathbf{F}}{dt} = \left(\frac{\partial f_1}{\partial x}\mathbf{i} + \frac{\partial f_2}{\partial x}\mathbf{j} + \frac{\partial f_3}{\partial x}\mathbf{k}\right)\frac{dx}{dt} + \left(\frac{\partial f_1}{\partial y}\mathbf{i} + \frac{\partial f_2}{\partial y}\mathbf{j} + \frac{\partial f_3}{\partial y}\mathbf{k}\right)\frac{dy}{dt}$$

$$+ \left(\frac{\partial f_1}{\partial z}\mathbf{i} + \frac{\partial f_2}{\partial z}\mathbf{j} + \frac{\partial f_3}{\partial z}\mathbf{k}\right)\frac{dz}{dt} + \left(\frac{\partial f_1}{\partial t}\mathbf{i} + \frac{\partial f_2}{\partial t}\mathbf{j} + \frac{\partial f_3}{\partial t}\mathbf{k}\right)$$

$$= \frac{\partial \mathbf{F}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{F}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{F}}{\partial z}\frac{dz}{dt} + \frac{\partial \mathbf{F}}{\partial t}.$$

Problem 76. *Prove that* $\mathbf{F} = \dfrac{e^{i\omega(t-r/c)}}{r}$ *satisfies the partial differential equation*

$$\frac{\partial^2 \mathbf{F}}{\partial r^2} + \frac{2}{r}\cdot\frac{\partial \mathbf{F}}{\partial r} = \frac{1}{c^2}\cdot\frac{\partial^2 \mathbf{F}}{\partial t^2}$$

where **a** *is a constant vector,* ω *and* c *are scalar constants and* $i = \sqrt{(-1)}$.

We are given that

$$\mathbf{F} = \mathbf{a}\frac{\left[e^{i\omega(t-r/c)}\right]}{r}$$

$$\therefore\quad \frac{\partial \mathbf{F}}{\partial r} = \mathbf{a}\frac{\left[re^{i\omega/(t-r/c)}\times(-i\omega/c) - e^{i\omega/(t-r/c)}\right]}{r^2}$$

$$= -\frac{i\omega}{cr}\mathbf{a}\,e^{i\omega/(t-r/c)} - \mathbf{a}\frac{e^{i\omega/(t-r/c)}}{r^2}$$

$$\frac{\partial^2 \mathbf{F}}{\partial r^2} = -\frac{i\omega}{c}\mathbf{a}\frac{\left[re^{i\omega/(t-r/c)}\times(-i\omega/c) - e^{i\omega/(t-r/c)}\right]}{r^2}$$

$$-\mathbf{a}\frac{\left[r^2e^{i\omega/(t-r/c)}\times(-i\omega/c) - 2e^{i\omega/(t-r/c)}\right]}{r^4}$$

$$= -\frac{\omega^2}{c^2}\mathbf{a}\frac{1}{r}e^{i\omega/(t-r/c)} + \frac{i\omega\mathbf{a}}{cr^2}e^{i\omega/(t-r/c)} + \frac{i\omega}{cr^2}e^{i\omega/(t-r/c)} + \frac{2\mathbf{a}}{r^3}e^{i\omega/(t-r/c)}$$

$$= -\frac{\omega^2}{c^2}\mathbf{a}\frac{1}{r}e^{i\omega/(t-r/c)} + \frac{2i\omega\mathbf{a}}{cr^2}e^{i\omega/(t-r/c)} + \frac{2\mathbf{a}}{r^3}e^{i\omega/(t-r/c)}$$

Again $\dfrac{\partial \mathbf{F}}{\partial t} = \dfrac{\mathbf{a}}{r}e^{i\omega/(t-r/c)}\times i\omega = \dfrac{i\omega\mathbf{a}}{r}e^{i\omega/(t-r/c)}$

and $\dfrac{\partial^2 \mathbf{F}}{\partial t^2} = \dfrac{i^2\omega^2\mathbf{a}}{r}e^{i\omega/(t-r/c)} = -\dfrac{\omega^2\mathbf{a}}{r}e^{i\omega/(t-r/c)}$

So that

$$\frac{\partial^2 \mathbf{F}}{\partial r^2} = -\frac{\omega^2\mathbf{a}}{c^2 r}e^{i\omega/(t-r/c)} + \frac{2i\omega\mathbf{a}}{cr^2}e^{i\omega/(t-r/c)} + \frac{2\mathbf{a}}{r^3}e^{i\omega/(t-r/c)} \quad \text{...(1)}$$

$$\frac{2}{r}\left(\frac{\partial \mathbf{F}}{\partial r}\right) = -\frac{2i\omega\mathbf{a}}{cr^2}e^{i\omega/(t-r/c)} - \frac{2\mathbf{a}\,e^{i\omega/(t-r/c)}}{r^3} \quad \text{...(2)}$$

and $$\frac{1}{c^2}\left(\frac{\partial^2 \mathbf{F}}{\partial t^2}\right) = -\frac{\omega\mathbf{a}}{c^2 r}e^{i\omega/(t-r/c)}, \quad \text{...(3)}$$

From (1), (2) and (3), it is evident that

$$\frac{\partial^2 \mathbf{F}}{\partial r^2} + \frac{2}{r}\,\frac{\partial \mathbf{F}}{\partial r} = \frac{1}{c^2}\,\frac{\partial^2 \mathbf{F}}{\partial t^2}$$

1.21. THE SCALAR AND VECTOR FIELDS

A physical quantity which is expressible as a continuous function and which can assume one or more definite values at each point of a region of space, is said to be a *point function* in that region and the region specifying the physical quantity is called as a *field*.

Point functions are of two kinds: *Scalar-point function* and *Vector-point function* according to the nature of the quantity concerned.

(1) Scalar-point function. At each point P of a domain or region, the function denoted by $f(P)$, or $f(x, y, z)$ is known as a *scalar-point function*. The points of the region R together with the functional values $f(P)$ constitute a *scalar-field* over R.

The examples of scalar-fields are the temperature distribution in a medium, the gravitational potential of a system of masses, and the electrostatic potential of a system of charges, etc.

(2) Vector-point function. If to each point P of a region R, there is associated a vector $\mathbf{F}(P)$, the function $\mathbf{F}(P)$ is known as a *vector-point function*, and points of R together with these vectors constitute a *vector-field* over R.

The examples of vector-fields are the velocity of a moving particle, the electrostatic, the magnetostatic, and gravitational fields, the electric intensity of force etc.

Continuity of Scalar and Vector-point Functions

(1) Scalar-point function. A scalar-point function $f(P)$ is continuous at a point $P = P_0$ if

(*i*) $f(P)$ is defined.

(*ii*) Given a number $\varepsilon > 0$ however small, there exists a positive number δ such that

$$|f(P) - f(P_0)| < \varepsilon$$

provided $|P - P_0| < \delta$, where δ depends on ε and P both.

i.e. $\delta = \delta(\varepsilon, P)$

In other words, a scalar point function $f(P)$ is said ot be *continuous* at a point P_0, if

$$Lim_{P \to P_0} f(P) = f(P_0)$$

The scalar-point function $f(P)$ is continous in a region if it is continuous at every point of the region.

(2) Vector-point function. The continuity of vector-point functions is defined similarly as in case of scalar-point functions except that f is replaced by $\mathbf{f}$ for vector-point functions.

1.22. DIRECTIONAL DERIVATIVES

If $f(x, y, z)$ be defined and differentiable at each point (x, y, z) in a certain region of space, then the partial derivative $\dfrac{\partial f}{\partial x}$ is defined as the rate of change of f at (x, y, z) in the direction of axis of x and called as the *directional derivative*.

(1) Scalar-point function. Consider a line QP drawn in a scalar-field f, the sense of it being positive from Q to P. Let $f(Q)$ and $f(P)$ be two functional values of scalar-point function f at Q and P respectively.

Then, if $\text{Lim}_{P \to Q} \dfrac{f(P) - f(Q)}{QP}$ exists, it is called *directional-derivative* of the scalar-point function at Q along QP.

(2) Vector-point function. The directional derivative for vector-point functions is defined similarly as in case of directional derivative for scalar-point functions except that f is replaced by $\mathbf{f}$ for vector-point function *i.e.* if $\text{Lim}_{P \to Q} \dfrac{\mathbf{f}(P) - \mathbf{f}(Q)}{QP}$ exists, it is called directional derivative of a vector-point function at a point along the direction of the vector.

(3) Directional derivatives of scalar-point function along coordinate axes. Referred to Fig. 1.36, we have $QP = \delta x$.

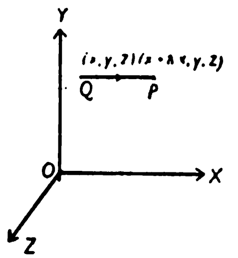


Fig. 1.36

The definition of directional derivative for scalar-point function, then leads

$$\lim_{P \to Q} \frac{f(P) - f(Q)}{QP}$$

$$= \lim_{\delta x \to 0} \frac{f(x+\delta x,\ y,\ z) - f(x,\ y,\ z)}{\delta x}$$

= partial derivative of f with respect to x

$$= \frac{\partial f}{\partial x}.$$

Proceeding similarly, we find that directional derivative at Q along the positive directions of axis of y and axis of z are $= \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z}$ respectively.

(4) Directional derivative of vector-point function along coordinate axes. Proceeding as in case of (3), we may find that directional derivative of vector-point functions in the positive directions of axes are

$$\frac{\partial \mathbf{f}}{\partial x}, \frac{\partial \mathbf{f}}{\partial f}, \frac{\partial \mathbf{f}}{\partial z} \text{ respectively.}$$

(5) Directional derivative of scalar-point function along any line. If QP be a line in the space in positive sense being from Q to P, and the direction cosines of QP are l, m, n, then the coordinates of the point P such that $QP = r$, are

$$(x + lr,\ y + mr,\ z + nr).$$

Thus the definition of directional derivative gives

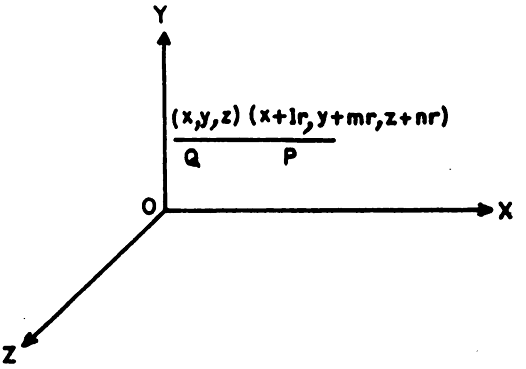


Fig. 1.37

$$\lim_{P \to Q} \frac{f(P) - f(Q)}{QP},$$

$$= \lim_{r \to 0} \frac{f(x+lr,\ y+mr,\ z+nr) - f(x,\ y,\ z)}{r} \qquad \ldots(i)$$

If $f(x, y, z)$ possesses first order continuous partial derivatives then $f(x + lr, y + mr, z + nr)$ can be expanded by Taylor's Theorem, so that we have

$$(x + lr, y + mr, z + nr) = f(x, y, z) + lr \frac{\partial f}{\partial x}(x_1, y_1, z_1)$$

$$+ mr \frac{\partial f}{\partial y}(x_1,\ y_1,\ z_1) + nr \frac{\partial f}{\partial z}(x_1,\ y_1,\ z_1) \qquad \ldots(ii)$$

where (x_1, y_1, z_1) is a point on QP.

Hence $\lim_{P \to Q} \dfrac{f(P)-f(Q)}{QP} = l\dfrac{\partial f}{\partial x}(x,\ y,\ z)$

$$+ m\frac{\partial f}{\partial y}(x,\ y,\ z) + n\frac{\partial f}{\partial z}(x,\ y,\ z) \text{ [from } (i) \text{ and } (ii)]$$

$$= l\frac{\partial f}{\partial x} + m\frac{\partial f}{\partial y} + n\frac{\partial f}{\partial z}$$

which follows that the directional derivatives along any line can be expressed in terms of those along three coordinate axes.

(6) Directional derivative for vector-point function along any line. Just as in case (5), take the direction cosines of a line as, l, m, n and define a vector-point function $\mathbf{f}$ in the region of line in terms of unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$, as

$\mathbf{f}(x, y, z) = f_1(x, y, z)\,\mathbf{i} + f_2(x, y, z)\,\mathbf{j} + f_3(x, y, z)\,\mathbf{k}$.

If $\mathbf{f}$ possesses first order continuous partial derivatives then its components f_1, f_2, f_3 will also possess first order continuous partial derivatives.

The definition of directional derivative for vector-point function, gives

$$\lim_{P \to Q} \frac{\mathbf{f}(P)-\mathbf{f}(Q)}{QP} = \mathbf{i}\frac{f_1(P)-f_1(Q)}{QP} + \mathbf{j}\frac{f_2(P)-f_2(Q)}{QP} + \mathbf{k}\frac{f_3(P)-f_3(Q)}{QP}$$

$$= \mathbf{i}\left[l\frac{\partial f_1}{\partial x} + m\frac{\partial f_1}{\partial y} + n\frac{\partial f_1}{\partial z}\right] + \mathbf{j}\left[l\frac{\partial f_2}{\partial x} + m\frac{\partial f_2}{\partial y} + n\frac{\partial f_2}{\partial z}\right]$$

$$+ \mathbf{k}\left[l\frac{\partial f_3}{\partial x} + m\frac{\partial f_3}{\partial y} + n\frac{\partial f_3}{\partial z}\right] \text{ by (5)}$$

$$= \left[\mathbf{i}\frac{\partial f_1}{\partial x} + \mathbf{j}\frac{\partial f_2}{\partial x} + \mathbf{k}\frac{\partial f_3}{\partial x}\right]l + \left[\mathbf{i}\frac{\partial f_1}{\partial y} + \mathbf{j}\frac{\partial f_2}{\partial y} + \mathbf{k}\frac{\partial f_3}{\partial y}\right]m$$

$$+ \left[\mathbf{i}\frac{\partial f_1}{\partial z} + \mathbf{j}\frac{\partial f_2}{\partial z} + \mathbf{k}\frac{\partial f_3}{\partial z}\right]n$$

$$= \frac{\partial \mathbf{f}}{\partial x}\,l + \frac{\partial \mathbf{f}}{\partial y}\,m + \frac{\partial \mathbf{f}}{\partial z}\,n$$

which is similar to the direction derivative of scalar-point function.

1.23. LEVEL SURFACES

Assuming that f is a continuous point-function, through any point P of the region considered, we can draw a surface such that, at each point on it the function has the same value as that at P. Such a surface is termed as a **level surface** *of the function.*

The examples of level surface are: isothermal surfaces and equipotential surfaces for temperature and potential respectively.

1.24. THE GRADIENT OF A SCALAR FIELD

Consider a scalar function *i.e.* a function whose value depends upon the values of co-ordinates (x, y, z). Being a scalar its value is constant at a fixed point in space.

The gradient of any scalar function ϕ is defined as

$$\text{grad } \phi = \mathbf{i}\frac{\partial \phi}{\partial x} + \mathbf{j}\frac{\partial \phi}{\partial y} + \mathbf{k}\frac{\partial \phi}{\partial z} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\phi \qquad \text{...(1)}$$

$$= \nabla \phi$$

where operator ∇ is generally known as '*del*' or '*nabla*' operator and read as 'gradient' or 'grad' in short. We have already mentioned that a scalar field is the region in which the scalar point function specifies the scalar physical quantity like temperature, electric potential, density etc. It is represented by a continuous scalar function giving the value of the quantity at each point. In scalar field all the points having same value of ϕ can be connected by means of surfaces, which are called *equal* or *level* surfaces.

Consider a co-ordinate system with axes such that any level surface lies in x-y plane while z-axis is along the normal to that level surface. Since the value of ϕ does not change along the level surface, *i.e.*,

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial y} = 0,$$

therefore $$\text{grad } \phi = \mathbf{k} \frac{\partial \phi}{\partial z} \qquad ...(2)$$

Clearly grad ϕ is directed along z-axis, *i.e.*, along the normal to the level surface. Therefore equation (2) may be written as

$$\text{grad } \phi = \frac{\partial \phi}{\partial n} \mathbf{n} \qquad ...(3)$$

where $\mathbf{n}$ is unit vector along the normal to the level surface at any point.

From equation (3) we may state, "*The magnitude of grad ϕ at any point is rate of change of function ϕ with distance along the normal to the level surface at the point and is directed along unit vector $\mathbf{n}$.*"

Note. It is to be noted that gradient of any scalar quantity is a vector.

(Agra, 1965)

Problem 77. *Prove* $\nabla r^n = n r^{n-2} \mathbf{r}$.

$$\text{L.H.S.} = \nabla r^n = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)(r^n) = \mathbf{i}\frac{\partial r^n}{\partial x} + \mathbf{j}\frac{\partial r^n}{\partial y} + \mathbf{k}\frac{\partial r^n}{\partial z}$$

$$= \mathbf{i}\, n r^{n-1}\frac{\partial r}{\partial x} + \mathbf{j}\, n r^{n-1}\frac{\partial r}{\partial y} + \mathbf{k}\, n r^{n-1}\frac{\partial r}{\partial z}$$

$$= n r^{n-1}\left[\mathbf{i}\frac{\partial r}{\partial x} + \mathbf{j}\frac{\partial r}{\partial y} + \mathbf{k}\frac{\partial r}{\partial z}\right]$$

since $r^2 = x^2 + y^2 + z^2$; $\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$.

$$= n r^{n-2}(\mathbf{i}x + \mathbf{j}y + \mathbf{k}z) = n r^{n-2}\mathbf{r}.$$

Problem 78. *If* $\mathbf{r}$ *is the position vector of a point, deduce the value of grad* $(1/r)$.

As given, $\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$.

So that $$\frac{1}{r} = \frac{1}{(x^2 + y^2 + z^2)^{1/2}}.$$

$\therefore$ grad $(1/r) = \nabla (1/r)$

$$= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\left\{\frac{1}{(x^2 + y^2 + z^2)^{1/2}}\right\}$$

$$= \mathbf{i}\frac{\partial}{\partial x}\left[\frac{1}{(x^2+y^2+z^2)^{1/2}}\right] + \mathbf{j}\frac{\partial}{\partial y}\left[\frac{1}{(x^2+y^2+z^2)^{1/2}}\right] + \mathbf{k}\frac{\partial}{\partial z}\left[\frac{1}{(x^2+y^2+z^2)^{1/2}}\right]$$

$$= \mathbf{i}\left[-\frac{1}{2}\cdot\frac{2x}{(x^2+y^2+z^2)^{3/2}}\right] + \mathbf{j}\left[-\frac{1}{2}\cdot\frac{2y}{(x^2+y^2+z^2)^{3/2}}\right] + \mathbf{k}\left[-\frac{1}{2}\cdot\frac{2z}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$= -\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}} = -\frac{\mathbf{r}}{(r^2)^{3/2}} = -\frac{\mathbf{r}}{r^3}$$

1.25. THE GRADIENT OF A SCALAR-POINT FUNCTION

If $\phi(x, y, z)$ *be defined and differentiable at each point* (x, y, z) *in a certain region of space specified as a scalar field, we have*

$$\left(l\frac{\partial\phi}{\partial x} + m\frac{\partial\phi}{\partial y} + n\frac{\partial\phi}{\partial z}\right) = (l\mathbf{i} + m\mathbf{j} + n\mathbf{k})\cdot\left(\mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}\right)$$

Its R.H.S. is the scalar product of two vectors $(l\mathbf{i} + m\mathbf{j} + n\mathbf{k})$ and $\left(\mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}\right)$, where the vector $(l\mathbf{i} + m\mathbf{j} + n\mathbf{k})$ is a unit vector along a line whose direction cosines are l, m, n and the second vector depends only on the point (x, y, z) and not on any direction. Thus we conclude that directional derivative along any line can be obtained by multiplying the vector

$$\mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z} \text{ scalarly with the unit vector } l\mathbf{i} + m\mathbf{j} + n\mathbf{k}.$$

The vector function $\mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}$ is called *the gradient of a scalar-point function* ϕ and is written as grad ϕ or $\nabla\phi$. Thus,

$$\nabla\phi = \text{grad}\,\phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z} \equiv \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}.$$

It is clear that the gradient of a scalar-point function is a vector.

In case, ϕ is a constant, grad $\phi = 0$ since $\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}$ all will be zero in that case. Its converse is also true.

1.26. THE GRADIENT OR SUM OF TWO SCALAR-POINT FUNCTIONS

If u and v are two differentiable scalar functions of x, y, z, then the gradient of their sum is given by

$$\nabla\,(u+v)=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\,\frac{\partial}{\partial x}+\mathbf{k}\frac{\partial}{\partial x}\right)(u+v)$$

$$=\mathbf{i}\,\frac{\partial}{\partial x}\,(u+v)+\mathbf{j}\,\frac{\partial}{\partial y}\,(u+v)+\mathbf{k}\,\frac{\partial}{\partial z}\,(u+v)$$

$$=\mathbf{i}\,\frac{\partial u}{\partial x}+\mathbf{i}\,\frac{\partial v}{\partial x}+\mathbf{j}\,\frac{\partial u}{\partial y}+\mathbf{j}\,\frac{\partial v}{\partial y}+\mathbf{k}\,\frac{\partial u}{\partial z}+\mathbf{k}\,\frac{\partial v}{\partial z}$$

$$=\left(\mathbf{i}\,\frac{\partial u}{\partial x}+\mathbf{j}\,\frac{\partial u}{\partial y}+\mathbf{k}\,\frac{\partial u}{\partial z}\right)+\left(\mathbf{i}\,\frac{\partial v}{\partial x}+\mathbf{j}\,\frac{\partial v}{\partial y}+\mathbf{k}\,\frac{\partial v}{\partial z}\right)$$

$$=\left(\mathbf{i}\,\frac{\partial}{\partial x}+\mathbf{j}\,\frac{\partial}{\partial y}+\mathbf{k}\,\frac{\partial}{\partial z}\right)u+\left(\mathbf{i}\,\frac{\partial}{\partial x}+\mathbf{j}\,\frac{\partial}{\partial y}+\mathbf{k}\,\frac{\partial}{\partial z}\right)v$$

$$=\nabla u+\nabla v.$$

Showing that the gradient of sum of two scalar-point functions is equal to the sum of their gradients.

This rule may be generalized for any number of scalar-point functions.

1.27. THE GRADIENT OF PRODUCT OF TWO SCALAR-POINT FUNCTIONS

If u and v be two differentiable scalar-point functions of x, y, z, then the gradient of their product is given by

$$\nabla\,(uv)=\left(\mathbf{i}\,\frac{\partial}{\partial x}+\mathbf{j}\,\frac{\partial}{\partial y}+\mathbf{k}\,\frac{\partial}{\partial z}\right)(uv)$$

$$=\mathbf{i}\frac{\partial}{\partial x}(uv)+\mathbf{j}\,\frac{\partial}{\partial y}(uv)+\mathbf{k}\,\frac{\partial}{\partial z}(uv)$$

$$=\mathbf{i}\left(u\,\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial x}\right)+\mathbf{j}\left(u\,\frac{\partial v}{\partial y}+v\frac{\partial u}{\partial y}\right)+\mathbf{k}\left(u\,\frac{\partial v}{\partial z}+v\frac{\partial u}{\partial z}\right)$$

$$=u\left[\mathbf{i}\frac{\partial v}{\partial x}+\mathbf{j}\,\frac{\partial v}{\partial y}+\mathbf{k}\,\frac{\partial v}{\partial z}\right]+v\left[\mathbf{i}\frac{\partial u}{\partial x}+\mathbf{j}\,\frac{\partial u}{\partial y}+\mathbf{k}\,\frac{\partial u}{\partial z}\right]$$

$$=u\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\,\frac{\partial}{\partial y}+\mathbf{k}\cdot\frac{\partial}{\partial z}\right)v+v\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\,\frac{\partial}{\partial y}+\mathbf{k}\,\frac{\partial}{\partial z}\right)u$$

$$=u\,\nabla\,v+v\,\nabla\,u.$$

Showing that the gradient of the product of two scalar-point functions is obtained by the same rule as is valid for derivatives of the algebraic functions.

Problem 79. (*a*) *Find* $\nabla\,\phi$ *and* $|\nabla\phi|$ *for the function.*

$$\phi=2xz^4-x^2y \text{ at the point } (2,-2,-1).$$

(*b*) *Find a unit vector perpendicular to the surface* $x^2+y^2-z^2=11$ *at the point* $(4,2,3)$. [*Madurai, 1987*]

(c) *Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 - z^2 = 3$ at the point (2, – 1, 2).*

(a) We know by the definition of grad ϕ that

$$\nabla\phi = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\phi$$

Here $\phi = 2xz^4 - x^2y$

$$\therefore \quad \nabla\phi = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)(2xz^4 - x^2y)$$

$$= \mathbf{i}\frac{\partial}{\partial x}(2xz^4 - x^2y) + \mathbf{j}\frac{\partial}{\partial x}(2xz^4 - x^2y) + \mathbf{k}\frac{\partial}{\partial x}(2xz^4 - x^2y)$$

$$= \mathbf{i}\,[2z^4 - 2xy] + \mathbf{j}\,[-x^2] + \mathbf{k}\,[8xz^3]$$

$$= \mathbf{i}\,[2(-1)^4 - 2(2)(-2)] + \mathbf{j}\,[-(+2)^2] + \mathbf{k}\,[8(2)(-1)^3] \text{ at } x = 2, y = -2, z = -1$$

$$= \mathbf{i}\,[2 + 8] + \mathbf{j}\,[-4) + \mathbf{k}\,[-16]$$

$$= 10\mathbf{i} - 4\mathbf{j} - 16\mathbf{k}.$$

And $|\nabla\phi| = \sqrt{[(10)^2 + (-4)^2 + (-16)^2]}$

$$= \sqrt{(372)} = 2\sqrt{(93)}.$$

(b) Here the level surface is $\phi = x^2 + y^2 - z^2 =$ const. and $\nabla\phi$ is perpendicular to the level surface: $\nabla\phi = 2(x\mathbf{i} + y\mathbf{j} - z\mathbf{k})$

$\Rightarrow [\nabla\phi]$ at (4, 2, 3) $= 2(4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$

$$\therefore \quad \text{required unit vector} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{2(4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})}{2\sqrt{\{(4)^2 + (2)^2 + (-3)^2\}}} = \frac{4\mathbf{j} + 2\mathbf{j} - 3\mathbf{k}}{\sqrt{29}}$$

(c) Proceeding just as in (b), the required angle say θ is given between the two surface ϕ_1, ϕ_2, as $\cos\theta = \dfrac{(\nabla\phi_1)\cdot(\nabla\phi_2)}{|\nabla\phi_1\cdot\nabla\phi_2|}$ at (2,–1, 2)

$$= \frac{8}{3\sqrt{21}}.$$

Problem 80. *If* $u = x^2z + e^{y/x}$ and $v = 2z^2y - xy^2$ *find*

(a) $\nabla(u + v)$, (b) $\nabla(uv)$ *at the point* (1, 0, –2).

We have

$$\nabla(u + v) = \nabla u + \nabla v \qquad \text{...(1)}$$

$$\nabla(uv) = u\nabla v + v\nabla u$$

Given that $u = x^2z + e^{y/x}$

$$\therefore \quad \nabla u = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \right)\left(x^2 z + e^{y/x}\right)$$

$$= \mathbf{i}\frac{\partial}{\partial x}\left(x^2 z + e^{y/x}\right) + \mathbf{j}\frac{\partial}{\partial x}\left(x^2 z + e^{y/x}\right) + \mathbf{k}\frac{\partial}{\partial z}\left(x^2 z + e^{y/x}\right)$$

$$= \mathbf{i}\left[2xz + e^{y/x} \times -\frac{y}{x^2}\right] + \mathbf{j}\left[e^{y/x} \times \frac{1}{x}\right] + \mathbf{k}\left[x^2\right]$$

$$= \mathbf{i}\left[2(1)(-2) + e^0 \times -\frac{0}{1}\right] + \mathbf{j}\left[e^0 \times \frac{1}{1}\right] + \mathbf{k}\,(1^2)$$

$$= -4\mathbf{i} + \mathbf{j} + \mathbf{k} \quad \ldots(2) \qquad \text{at } x = 1, y = 0, z = -2$$

And $\quad v = 2z^2 y - xy^2$

$$\nabla v = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \right)\left(2z^2 y - xy^2\right)$$

$$= \mathbf{i}\frac{\partial}{\partial x}\left(2z^2 y - xy^2\right) + \mathbf{j}\frac{\partial}{\partial y}\left(2z^2 y - xy^2\right) + \mathbf{k}\frac{\partial}{\partial z}\left(2z^2 y - xy^2\right)$$

$$= \mathbf{i}\,[-y^2] + \mathbf{j}\,[2z^2 - 2xy] + \mathbf{k}\,[4zy]$$

$$= \mathbf{i}\,[0] + \mathbf{j}\,[2(-2)^2] + \mathbf{k}\,[0] \quad \text{at } x = 1, y = 0, z = -2$$

$$= 8\mathbf{j}. \qquad \ldots(3)$$

Putting the values of ∇u and ∇v from (2) and (3) in (1) we get

(*a*) $\nabla(u + v) = \nabla u + \nabla_v = (-4\mathbf{i} + \mathbf{j} + \mathbf{k}) + 8\mathbf{j} = -4\mathbf{i} + 9\mathbf{j} + \mathbf{k}$

and (*b*) $\nabla(uv) = u\nabla v + v\nabla u = (x^2 z + e^{z/x})\,8\mathbf{j} + (2z^2 y + y^2 x)(-4\mathbf{i} + \mathbf{j} + \mathbf{k})$

$= (-1)\,8\mathbf{j} = -8\mathbf{j}$ at $(1, 0, -2)$.

Problem 81. *If* $A = 2x^2\mathbf{i} - 3yz\mathbf{j} + xz^2\mathbf{k}$ *and* $\phi = 2z - x^3 y$ *find*

(*a*) $A \cdot \nabla\phi$ (*b*) $A \times \nabla\phi$ *at the point* $(1, -1, 1)$.

Given $\quad \phi = 2z - x^3 y$

$$\therefore \quad \nabla\phi = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \right)\left(2z - x^3 y\right)$$

$$= \mathbf{i}\,(-3x^2 y) + \mathbf{j}\,(-x^3) + \mathbf{k}\,(2)$$

$$= 3\mathbf{i} - \mathbf{j} + 2\mathbf{k} \quad \text{at } x = 1, y = -1, z = 1$$

and $\quad A = 2x^2\mathbf{i} - 3yz\mathbf{j} + xz^2\mathbf{k}$

$$= 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} \quad \text{at } x = 1, y = -1, z = 1$$

(*a*) $\therefore \quad A \cdot \nabla\phi = (2\mathbf{i} + 3\mathbf{j} + \mathbf{k}) \cdot (3\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

$$= [6 - 3 + 2] = 5.$$

and (*b*) $\quad A \times \nabla\phi = (2\mathbf{i} + 3\mathbf{j} + \mathbf{k}) \times (3\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

$$= [-2\mathbf{k} - 4\mathbf{j} - 9\mathbf{k} + 6\mathbf{i} + 3\mathbf{j} + \mathbf{i}]$$

$$= 7\mathbf{i} - \mathbf{j} - 11\mathbf{k}.$$

Problem 82. *Prove* $\nabla\left(\dfrac{u}{v}\right) = \dfrac{v\nabla u - u\nabla v}{v^2}$, *provided* $v \neq 0$.

Given $\quad \phi = \dfrac{u}{v} = \left(uv^{-1}\right)$

$$\therefore \quad \nabla\phi = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\left(uv^{-1}\right)$$

$$= \mathbf{i}\frac{\partial}{\partial x}\left(uv^{-1}\right) + \mathbf{j}\frac{\partial}{\partial y}\left(uv^{-1}\right) + \mathbf{k}\frac{\partial}{\partial z}\left(uv^{-1}\right)$$

$$= \mathbf{i}\left[-\frac{u}{v^2}\frac{\partial v}{\partial x} + v^{-1}\frac{\partial u}{\partial x}\right] + \mathbf{j}\left[-\frac{u}{v^2}\frac{\partial v}{\partial z} + v^{-1}\frac{\partial u}{\partial z}\right] + \mathbf{k}\left[-\frac{u}{v^2}\frac{\partial v}{\partial z} + v^{-1}\frac{\partial u}{\partial z}\right]$$

$$= -\frac{u}{v^2}\left(\mathbf{i}\frac{\partial v}{\partial x} + \mathbf{j}\frac{\partial v}{\partial y} + \mathbf{k}\frac{\partial v}{\partial z}\right) + v^{-1}\left(\mathbf{i}\frac{\partial u}{\partial x} + \mathbf{j}\frac{\partial u}{\partial y} + \mathbf{k}\frac{\partial u}{\partial z}\right)$$

$$= -\frac{u}{v^2}\left[\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right]v + \frac{1}{v}\left[\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right]u$$

$$= -\frac{u}{v^2}\nabla v + \frac{1}{v}\nabla u = \frac{1}{v}\nabla u - \frac{u}{v^2}\nabla v = \frac{v\nabla u - u\nabla v}{v^2}$$

Problem 83. *If u is differentiable function of x, y, z, prove $\nabla u.d\mathbf{r} = du$.*

Given $u = u(x, y, z)$

i.e. $\quad du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$

and $\quad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

$\therefore \quad d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$.

Now $\quad \nabla u = \left(\mathbf{i}\frac{\partial u}{\partial x} + \mathbf{i}\frac{\partial u}{\partial y} + \mathbf{k}\frac{\partial u}{\partial z}\right)$.

$$\therefore \quad \nabla u \cdot d\mathbf{r} = \left(\mathbf{i}\frac{\partial u}{\partial x} + \mathbf{j}\frac{\partial u}{\partial y} + \mathbf{k}\frac{\partial u}{\partial z}\right)\cdot d\mathbf{r}$$

$$= \left(\mathbf{i}\frac{\partial u}{\partial x} + \mathbf{j}\frac{\partial u}{\partial y} + \mathbf{k}\frac{\partial u}{\partial z}\right)\cdot\left(\mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz\right)$$

$$= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = du$$

i.e. $\quad \nabla u.d\mathbf{r} = du$.

1.28. THE DIVERGENCE OF A VECTOR-POINT FUNCTION

(Agra, 1952, 65)

If $\mathbf{V}(x, y, z) = V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}$ *be a continuous differentiable vector-point function specified in a vector field, then the divergence of* $\mathbf{V}$ *is defined as:*

$$\mathbf{i}\cdot\frac{\partial \mathbf{V}}{\partial x} + \mathbf{j}\cdot\frac{\partial \mathbf{V}}{\partial y} + \mathbf{k}\cdot\frac{\partial \mathbf{V}}{\partial z},$$

and is written as $\nabla \cdot \mathbf{V}$ *or div* $\mathbf{V}$ *and read as divergence* $\mathbf{V}$.

$$\therefore \quad \nabla\cdot\mathbf{V} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot\left(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}\right)$$

$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

(Agra, 1965)

which is clearly a scalar quantity.

Note. If $\nabla \cdot \mathbf{V} = 0$ then $\mathbf{V}$ is known as *Solenoidal Vector.*

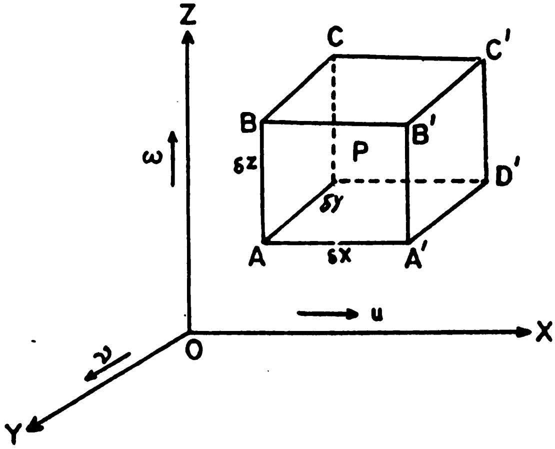


Fig. 1.38

Interpretation of div V. Let us consider the motion of a homogenous and incompressible fluid particle at an instant of time t and let u, v, w be the components of its velocity parallel to axes of x, y, z respectively.

Construct an elementary rectangular parallelopiped of fluid with sides $\delta x, \delta y, \delta z$ having $P(x, y, z)$ as the centre.

The velocity vector at P is given by

$$\mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}.$$

Now, components of velocity being the functions of x, y, z, we have

$$u = u(x, y, z),\ v = v(x, y\ z),\ w = w(x, y, z)$$

The component of velocity parallel to x-axis at a point on the face $A'D'C'B'$ will be

$$= u\left(x + \frac{\delta x}{2}, y, z\right)$$

$$= u(x, y, z) + \frac{\delta x}{2}\left[\frac{\partial u}{\partial x}\right] + \ldots \text{ (by Taylor's Theorem).}$$

The mass of the fluid passing out of the face $A'D'C'B'$ in small time δt = density of the fluid × velocity normal to the face $A'D'C'B'$ × area of face $A'D'C'B'$ × time

$$= \rho\left[u + \frac{\delta x}{2}\left(\frac{\partial u}{\partial x}\right)\right] \times \delta y \times \delta z \times \delta t$$

Similarly the mass of fluid that passes through the face $ADCB$

$$= \rho\left[u - \frac{\delta x}{2}\left(\frac{\partial u}{\partial x}\right)\right]\delta y\ \delta z\ \delta t.$$

And the mass of the fluid that passes out through the faces $ADCB$ and $A'D'C'B'$

$$= \rho\left[u + \frac{\delta x}{2}\frac{\partial u}{\partial x}\right]\delta y\ \delta z\ \delta t - \rho\left[u - \frac{\delta x}{2}\frac{\partial u}{\partial x}\right]\delta y\ \delta z\ \delta t$$

$$= \rho\frac{\partial u}{\partial x}\delta x\ \delta y\ \delta z\ \delta t.$$

If the other two pairs of faces are considered then we have masses of fluid that moves out as

$$\rho\frac{\partial v}{\partial y}\delta x\ \delta y\ \delta z\ \delta t \text{ and } \rho\frac{\partial w}{\partial z}\delta x\ \delta y\ \delta z\ \delta t.$$

The total mass of fluid that moves out the elementary parallelopiped in time δt

$$= \rho\frac{\partial u}{\partial x}\delta x\ \delta y\ \delta z\ \delta t + \rho\frac{\partial v}{\partial y}\delta x\ \delta y\ \delta z\ \delta t + \rho\frac{\partial w}{\partial z}\delta x\ \delta y\ \delta z\ \delta t$$

$$= \rho\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right]\delta x\ \delta y\ \delta z\ \delta t.$$

The volume of the elementary parallelopiped is $\delta x\ \delta y\ \delta z$ and so proceeding to the limit when $\delta x, \delta y, \delta z, \delta t$ all tend to zero we see that an amount of fluid mass per unit volume per unit time that passes through the point $P\ (x, y, z)$

$$= \rho\ \frac{\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right]\delta x\ \delta y\ \delta z\ \delta t}{\delta x\ \delta y\ \delta z\ \delta t}$$

$$= \rho\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right]$$

$$= \rho\left(\mathbf{i}\ \frac{\partial}{\partial x}+\mathbf{j}\ \frac{\partial}{\partial y}+\mathbf{k}\ \frac{\partial}{\partial z}\right)\cdot(u\mathbf{i}+v\mathbf{j}+w\mathbf{k})$$

$$= \rho\ \nabla\cdot\mathbf{V} = \rho\ \text{div}\ \mathbf{V}.$$

1.29. THE DIVERGENCE OF SUM OF TWO VECTOR FUNCTIONS

If **U** and **V** be two vector-point functions expressed as

$$\mathbf{U} = U_1\mathbf{i} + U_2\mathbf{j} + U_3\mathbf{k}$$

$$\mathbf{V} = V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}.$$

Then

$$\nabla\cdot(\mathbf{U}+\mathbf{V}) = \left(\mathbf{i}\ \frac{\partial}{\partial x}+\mathbf{j}\ \frac{\partial}{\partial y}+\mathbf{k}\ \frac{\partial}{\partial z}\right)\cdot\left[(U_1+V_1)\ \mathbf{i}+(U_2+V_2)\mathbf{j}+(U_3+V_3)\mathbf{k}\right]$$

$$= \frac{\partial}{\partial x}[U_1+V_1]+\frac{\partial}{\partial y}[U_2+V_2]+\frac{\partial}{\partial z}[U_3+V_3]$$

$$= \left(\frac{\partial U_1}{\partial x}+\frac{\partial U_2}{\partial y}+\frac{\partial U_3}{\partial z}\right)+\left(\frac{\partial V_1}{\partial x}+\frac{\partial V_2}{\partial y}+\frac{\partial V_3}{\partial z}\right)$$

$$= \left(\mathbf{i}\ \frac{\partial}{\partial x}+\mathbf{j}\ \frac{\partial}{\partial y}+\mathbf{k}\ \frac{\partial}{\partial z}\right)\cdot(U_1\mathbf{i}+U_2\mathbf{j}+U_3\mathbf{k})$$

$$+\left(\mathbf{i}\ \frac{\partial}{\partial x}+\mathbf{j}\ \frac{\partial}{\partial y}+\mathbf{k}\ \frac{\partial}{\partial z}\right)\cdot(V_1\mathbf{i}+V_2\mathbf{j}+V_3\mathbf{k})$$

$$= \nabla\cdot\mathbf{U}+\nabla\cdot\mathbf{V}.$$

$$= \text{div}\ \mathbf{U} + \text{div}\ \mathbf{V}.$$

Showing that the divergence of the sum of two vector functions is equal to the sum of their divergences.

This rule may be generalised for any number of vector functions.

1.30. THE DIVERGENCE OF PRODUCT

If the vector point function **U** is expressed as

$\mathbf{U} = U_1\mathbf{i} + U_2\mathbf{j} + U_3\mathbf{k}$ and V is a scalar point-function.

Then $\nabla\cdot(\mathbf{U}V) = \left(\mathbf{i}\ \frac{\partial}{\partial x}+\mathbf{j}\ \frac{\partial}{\partial y}+\mathbf{k}\ \frac{\partial}{\partial z}\right)\cdot\left[(U_1\mathbf{i}+U_2\mathbf{j}+U_3\mathbf{k})V\right]$

$$= \left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\cdot[VU_1\mathbf{i}+VU_2\mathbf{j}+VU_3\mathbf{k}]$$

$$= \frac{\partial}{\partial x}(VU_1)+\frac{\partial}{\partial y}(VU_2)+\frac{\partial}{\partial z}(VU_3)$$

$$= U_1\frac{\partial V}{\partial x}+V\frac{\partial U_1}{\partial x}+U_2\frac{\partial V}{\partial y}+V\frac{\partial U_2}{\partial y}+U_3\frac{\partial V}{\partial z}+V\frac{\partial U_3}{\partial z}$$

$$= \left(U_1\frac{\partial V}{\partial x}+U_2\frac{\partial V}{\partial y}+U_3\frac{\partial V}{\partial z}\right)+V\left(\frac{\partial U_1}{\partial x}+\frac{\partial U_2}{\partial y}+\frac{\partial U_3}{\partial z}\right)$$

$$= \left(\mathbf{i}\frac{\partial V}{\partial x}+\mathbf{j}\frac{\partial V}{\partial y}+\mathbf{k}\frac{\partial V}{\partial z}\right)\cdot(U_1\mathbf{i}+U_2\mathbf{j}+U_3\mathbf{k})$$

$$+V\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\cdot(U_1\mathbf{i}+U_2\mathbf{j}+U_3\mathbf{k})$$

$$= (\nabla V)\cdot\mathbf{U}+V(\nabla\cdot\mathbf{U})$$

i.e. $\quad \text{div}(\mathbf{U}V) = (\text{grad } V)\cdot\mathbf{U}+V\,\text{div }\mathbf{U}.$

Problem 84. (*a*) *If* $\mathbf{V} = x^2z\mathbf{i}-2y^2z^2\mathbf{j}+xy^2z\mathbf{k}$ *find* $\nabla\cdot\mathbf{V}$ *at the point* $(1, -1, 1)$. **(*Agra, 1961, 63*)**

(*b*) *If* $\mathbf{V} = \dfrac{x\mathbf{i}+y\mathbf{j}}{x+y}$ *find* $\nabla\cdot\mathbf{V}$

(*c*) *If* $\mathbf{V} = x\cos z\mathbf{i}+y\log x\mathbf{j}-z^2\mathbf{k}$ *evaluate* $\nabla\cdot\mathbf{V}$.

(*a*) Given $\quad \mathbf{V} = x^2z\mathbf{i}-2y^3z^2\mathbf{j}+xy^2z\mathbf{k}.$

$$\therefore\quad \nabla\cdot\mathbf{V} = \left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\cdot(x^2z\mathbf{i}-2y^3z^2\mathbf{j}+xy^2z\mathbf{k})$$

$$= \frac{\partial}{\partial x}(x^2z)+\frac{\partial}{\partial y}(-2y^2z^2)+\frac{\partial}{\partial z}(xy^2z)$$

$$= 2xz-6y^2z^2+xy^2$$

$$= 2-6+1 = -3 \text{ at } x = 1,\ y = -1,\ z = 1.$$

(*b*) Given $\quad \mathbf{V} = \dfrac{x\mathbf{i}+y\mathbf{j}}{x+y}.$

$$\nabla\cdot\mathbf{V} = \left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\cdot\left(\frac{x}{x+y}\mathbf{i}+\frac{y}{x+y}\mathbf{j}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{x+y}\right)+\frac{\partial}{\partial y}\left(\frac{y}{x+y}\right)$$

$$= \frac{(x+y)-x}{(x+y)^2}+\frac{(x+y)-y}{(x+y)^2}$$

$$= \frac{x+y}{(x+y)^2}$$

$$= \frac{1}{x+y}.$$

(c) Given $\mathbf{V} = x\cos z\,\mathbf{i} + y\log x\,\mathbf{j} - z^2\mathbf{k}$

$$\therefore \quad \nabla\cdot\mathbf{V} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot\left(x\cos z\,\mathbf{i} + y\log x\,\mathbf{j} - z^2\mathbf{k}\right)$$

$$= \frac{\partial}{\partial x}(x\cos z) + \frac{\partial}{\partial y}(y\log x) + \frac{\partial}{\partial z}(-z^2)$$

$$= \cos z + \log x - 2z.$$

Problem 85. *If $\phi = 2x^3y^2z^4$ then find*

(a) div (grad ϕ), and

(b) Show that $\nabla\cdot\nabla\phi = \nabla^2\phi$ where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

(a) Given $\phi = 2x^3y^2z^4$

$$\therefore \quad \nabla\phi = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)(2x^3y^2z^4)$$

$$= (6x^2y^2z^4)\,\mathbf{i} + (4x^3yz^4)\,\mathbf{j} + (8x^3y^2z^3)\,\mathbf{k}$$

So that
$$\nabla\cdot\nabla\phi = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot(6x^2y^2z^4\,\mathbf{i} + 4x^3yz^4\,\mathbf{j} + 8x^3y^2z^3\mathbf{k}).$$

$$= \frac{\partial}{\partial x}(6x^2y^2z^4) + \frac{\partial}{\partial y}(4x^3yz^4) + \frac{\partial}{\partial z}(8x^3y^2z^3)$$

$$= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2.$$

(b)
$$\nabla\cdot\nabla\phi = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot\left(\mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right)$$

$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = \nabla^2\phi$$

Note. $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ *is known as Laplacian operator.*

Problem 86. *If $u = 3x^2y$, $v = xz^2 - 2y$ then evaluate grad [(grad u) · (grad v)].*

As given
$$\nabla u = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)(3x^2y)$$

$$= 6xy\mathbf{i} + 3x^2\mathbf{j}$$

and
$$\nabla v = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)(xz^2 - 2y)$$

$$= z^2\mathbf{i} - 2\mathbf{j} + 2xz\mathbf{k}.$$

$$\therefore \quad \nabla u\cdot\nabla v = (6xy\mathbf{i} + 3x^2\mathbf{j})\cdot(z^2\mathbf{i} - 2\mathbf{j} + 2xz\,\mathbf{k})$$

$$= 6xyz^2 - 2x^2.$$

$\therefore \quad \nabla\,(\text{grad } u \cdot \text{grad } v) = \left[\mathbf{i}\,\dfrac{\partial}{\partial x} + \mathbf{j}\dfrac{\partial}{\partial y} + \mathbf{k}\,\dfrac{\partial}{\partial z}\right](6xyz^2 - 2x^2)$

$$= (6yz^2 - 4x)\,\mathbf{i} + (6xz^2)\,\mathbf{j} + (12xyz)\,\mathbf{k}.$$

Problem 87. *Prove* $\nabla^2(uv) = u\,\nabla^2 v + 2\,\nabla u \cdot \nabla v + v\,\nabla^2 u.$

We have $\quad \nabla^2(uv) = \nabla\,[\nabla\,(uv)]$

where, $\quad \nabla\,(uv) = \left(\mathbf{i}\,\dfrac{\partial}{\partial x} + \mathbf{j}\,\dfrac{\partial}{\partial y} + \mathbf{k}\,\dfrac{\partial}{\partial z} + \right)(uv)$

$$= \mathbf{i}\,\frac{\partial}{\partial x}(uv) + \mathbf{j}\frac{\partial}{\partial y}(uv) + \mathbf{k}\frac{\partial}{\partial z}(uv)$$

$$= \left[u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}\right]\mathbf{i} + \left[u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}\right]\mathbf{j} + \left[u\frac{\partial v}{\partial z} + v\frac{\partial u}{\partial z}\right]\mathbf{k}$$

$$= u\left[\frac{\partial v}{\partial x}\mathbf{i} + \frac{\partial v}{\partial y}\mathbf{j} + \frac{\partial v}{\partial z}\mathbf{k}\right] + v\left[\frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\,\mathbf{k}\right]$$

$$= u\,\nabla\,v + v\,\nabla\,u.$$

$\therefore \quad \nabla^2(uv) = \nabla\,(u\,\nabla\,v + v\,\nabla\,u)$

$$= \nabla\,(u\,\nabla\,v) + \nabla\,(v\,\nabla\,u)$$

$$= u\,\nabla\,(\nabla\,v) + \nabla\,v\,\nabla\,(u) + v\,\nabla\,(\nabla\,u) + \nabla\,u\,\nabla\,(v)$$

$$= u\,\nabla^2 v + \nabla\,v\,\nabla\,u + v\,\nabla^2 u + \nabla u\,\nabla v$$

$$= u\,\nabla^2 v + 2\,\nabla\,u\,\nabla\,v + v\,\nabla^2 u.$$

Problem 88. *Prove* $\nabla \cdot (u\,\nabla\,v - v\,\nabla\,u) = u\,\nabla^2 v - v\,\nabla^2 u.$

We have, $\nabla \cdot (u\,\nabla\,v) = u\,[\nabla \cdot (\nabla\,v)] + \nabla\,v \cdot \nabla\,(u)$

$$= u\,\nabla^2 v + \nabla v \cdot \nabla\,u \qquad \ldots(1)$$

and similarly $\nabla \cdot (v\,\nabla\,u) = v\,\nabla^2 u + \nabla\,u \cdot \nabla\,v.$ $\qquad \ldots(2)$

Subtracting (2) from (1), we get

$$\nabla \cdot (u\,\nabla\,v) - \nabla \cdot (v\,\nabla\,u) = u\,\nabla^2 v + \nabla\,v \cdot \nabla\,u - v\,\nabla^2 u - \nabla\,u \cdot \nabla\,v$$

$$= u\,\nabla^2 v - v\,\nabla^2 u.$$

Problem 89. *Prove that*

$$\nabla^2\left(\frac{1}{r}\right) = 0$$

where $\quad r^2 = x^2 + y^2 + z^2$ *(Agra, 1957)*

We have $\quad \nabla^2\left(\dfrac{1}{r}\right) = \left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}\right)\left[\dfrac{1}{(x^2 + y^2 + z^2)^{1/2}}\right]$

But $\quad \dfrac{\partial}{\partial x}\left[\dfrac{1}{(x^2 + y^2 + z^2)^{1/2}}\right] = -\dfrac{x}{(x^2 + y^2 + z^2)^{3/2}}$

$$\frac{\partial^2}{\partial x^2}\left[\frac{1}{(x^2 + y^2 + z^2)^{1/2}}\right] = \frac{\partial}{\partial x}\left[\frac{-x}{(x^2 + y^2 + z^2)^{3/2}}\right]$$

$$= \frac{3x^2}{(x^2+y^2+z^2)^{5/2}} - \frac{1}{(x^2+y^2+z^2)^{3/2}}$$

and similarly

$$\frac{\partial^2}{\partial y^2}\left[\frac{1}{(x^2+y^2+z^2)^{1/2}}\right] = \frac{3y^2}{(x^2+y^2+z^2)^{5/2}} - \frac{1}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial^2}{\partial z^2}\left[\frac{1}{(x^2+y^2+z^2)^{1/2}}\right] = \frac{3z^2}{(x^2+y^2+z^2)^{5/2}} - \frac{1}{(x^2+y^2+z^2)^{3/2}}$$

$$\therefore \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left[\frac{1}{(x^2+y^2+z^2)^{1/2}}\right]$$

$$= \frac{3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} - \frac{3}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{3}{(x^2+y^2+z^2)^{3/2}} - \frac{3}{(x^2+y^2+z^2)^{3/2}}$$

$$= 0$$

i.e. $$\nabla^2\left(\frac{1}{r}\right) = 0.$$

Note. The equation $\nabla^2 \phi = 0$ is known as *Laplace's equation*, and hence $\phi = \frac{1}{r}$ is a solution of this equation.

Problem 90. *Prove the following propositions:*

(a) $\mathbf{V} = 3y^4z^2\mathbf{i} + 4x^3z^2\mathbf{j} - 3x^2y^2$ *is a solenoidal vector.*

(b) $\mathbf{V} = (x+3y)\,\mathbf{i} + (y-2z)\,\mathbf{j} + (x+az)\,\mathbf{k}$ *is a solenoidal vector when* $a = -2$.

(c) $\mathbf{U} = (2x^2 + 8xy^2z)\mathbf{i} + (3x^3y - 3xy)\mathbf{j} - (4y^2z^2 + 2x^3z)\mathbf{k}$ *is not solenoidal vector. But vector* $\mathbf{V} = xyz^2\,\mathbf{U}$ *is solenoidal.*

By note on § 1.28, we know that if the divergence of a vector is zero then the vector is called *solenoidal*.

(*a*) Given $\quad \mathbf{V} = 3y^4z^2\mathbf{i} + 4x^3z^2\mathbf{j} - 3x^2y^2\mathbf{k}$.

$$\therefore \quad \nabla\cdot\mathbf{V} = \left(\mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}\right)\cdot\left(3y^4z^2\mathbf{i} + 3x^3z^2\mathbf{j} - 3x^2y^2\mathbf{k}\right)$$

$$= \frac{\partial}{\partial x}\left(y^4z^2\right) + \frac{\partial}{\partial y}\left(x^3z^2\right) + \frac{\partial}{\partial z}\left(-3x^2y^2\right)$$

$$= 0.$$

Hence the given vector V is solenoidal.

(*b*) Given $\mathbf{V} = (x + 3y)\,\mathbf{i} + (y - 2z)\,\mathbf{j} + (x + az)\,\mathbf{k}$

$$\therefore \quad \nabla \cdot \mathbf{V} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot \left[(x+3y)\,\mathbf{i} + (y-2z)\,\mathbf{j} + (x+az)\,\mathbf{k}\right]$$

$$= \frac{\partial}{\partial x}[(x+3y)] + \frac{\partial}{\partial y}[y-2z] + \frac{\partial}{\partial z}[x+az]$$

$$= 1 + 1 + a$$

In order that $\mathbf{V}$ be solenoidal vector, we should have $\nabla \cdot \mathbf{V} = 0$

i.e. $\quad a + 2 = 0 \quad$ or $\quad a = -2$

(*c*) Given $\mathbf{U} = (2x^2 + 8y^2 zx)\,\mathbf{i} + (3x^3 y - 3xy)\,\mathbf{j} - (4y^2z^2 + 2x^3 z)\,\mathbf{k}$

$$\therefore \quad \nabla \cdot \mathbf{U} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot [(2x^2 + 8y^2 zx)\,\mathbf{i} + (3x^3 y - 3xy)\,\mathbf{j} - (4y^2z^2 + 2x^3 z)\mathbf{k}]$$

$$= \frac{\partial}{\partial x}(2x^2 + 8y^2 zx) + \frac{\partial}{\partial y}(3x^3 y - 3xy) - \frac{\partial}{\partial z}(4y^2z^2 + 2x^3 z)$$

$$= 4x + 8y^2 z + 3x^3 - 3x - 8y^2 z - 2x^3 \neq 0.$$

Hence $\mathbf{U}$ is not a solenoidal vector.

But $xyz^2\,\mathbf{U} = (2x^3yz^2 + 8x^2y^3z^3)\,\mathbf{i} + (3x^4y^2z^2 - 3x^2y^2z^2)\,\mathbf{j} - (4xy^3z^4 + 2x^4yz^3)\,\mathbf{k}$

$$\therefore \quad \nabla \cdot (xyz^2\,\mathbf{U}) = \frac{\partial}{\partial x}(2x^3yz^2 + 8x^2y^3z^3) + \frac{\partial}{\partial y}(3x^4y^2z^2 - 3x^2y^2z^2) - \frac{\partial}{\partial z}(4xy^3z^4 + 2x^4yz^3)\,\mathbf{k}$$

$$= 6x^2yz^2 + 16xy^3z^3 + 6x^4yz^2 - 6x^2yz^2 - 16xy^3z^3 - 6x^4yz^2$$

$$= 0.$$

Here $xyz^2\,\mathbf{U}$ is a solenoidal vector.

1.31. THE CURL OR ROTATION OF A VECTOR POINT FUNCTION

(*Agra, 1952, 61, 63*)

Let $\mathbf{f}(x, y, z) = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$ be a continuous differentiable vector point function, then the curl of $\mathbf{f}$ or rotation of $\mathbf{f}$ is given by

$$\mathbf{i} \times \frac{\partial \mathbf{f}}{\partial x} + \mathbf{j} \times \frac{\partial \mathbf{f}}{\partial y} + \mathbf{k} \times \frac{\partial \mathbf{f}}{\partial z}$$

and is written as curl $\mathbf{f}$ or $\nabla \times \mathbf{f}$ or rot $\mathbf{f}$.

i.e., $$\nabla \times \mathbf{f} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times (f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k})$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\mathbf{k}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$ (*Agra, 1961*)

It is clear that curl $\mathbf{f}$ or rotation $\mathbf{f}$ is a vector quantity and read as del cross $\mathbf{f}$.

Note. If curl $f = 0$, f is known as *Irrotational Vector*.

Interpretation of the curl f. *If a rigid body is in motion, the curl of its linear velocity at any point gives twice its angular velocity.*

(Agra, 1954, 72; Rohilkhand, 1976)

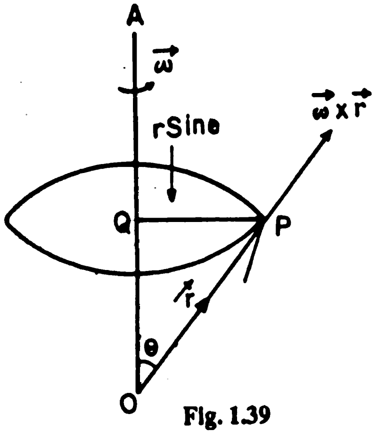


Fig. 1.39

Consider the motion of a rigid body rotating with angular velocity ω about an axis OA; O, being a fixed point in the body. Let **r** be the position vector of any point P of the body. Draw PQ perpendicular from P to the axis OA. Then,

Linear velocity V of P due to circular motion $= |\mathbf{V}|$

$$= \omega QP = \omega r \sin\theta = |\vec{\omega} \times \mathbf{r}|$$

i.e., $\mathbf{V} = \vec{\omega} \times \mathbf{r}$

where, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

and $\vec{\omega} = \omega_1 \mathbf{i} + \omega_2 \mathbf{j} + \omega_3 \mathbf{k}$.

But we know that curl $\mathbf{V} = \nabla \times \mathbf{V} = \nabla \times (\vec{\omega} \times \mathbf{r})$

$$= \nabla \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \nabla \times [(\omega_2 z - \omega_3 y)\,\mathbf{i} + (\omega_3 x - \omega_1 z)\,\mathbf{j} + (\omega_1 y - \omega_2 x)\,\mathbf{k}]$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 z - \omega_2 x \end{vmatrix}$$

$$= 2\,[\omega_1 \mathbf{i} + \omega_2 \mathbf{j} + \omega_3 \mathbf{k}]$$

$$= 2\vec{\omega}$$ Which proves the proposition.

1.32. CURL OF THE SUM OF TWO VECTOR-POINT FUNCTIONS

If **u** and **v** be two vector-point functions given by

$$\mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}$$
$$\mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$$

then $\nabla \times (\mathbf{u} + \mathbf{v}) = \nabla \times [(u_1 + v_1)\,\mathbf{i} + (u_2 + v_2)\,\mathbf{j} + (u_3 + v_3)\,\mathbf{k}]$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u_1 + v_1 & u_2 + v_2 & u_3 + v_3 \end{vmatrix}$$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}(u_3 + v_3) - \frac{\partial}{\partial z}(u_2 + v_2)\right] + \mathbf{j}\left[\frac{\partial}{\partial z}(u_1 + v_1) - \frac{\partial}{\partial x}(u_3 + v_3)\right]$$
$$+ \mathbf{k}\left[\frac{\partial}{\partial x}(u_2 + v_2) - \frac{\partial}{\partial y}(u_1 + v_1)\right]$$

$$= \mathbf{i}\left[\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right] + \mathbf{j}\left[\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}\right] + \mathbf{k}\left[\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right]$$

$$+\mathbf{i}\left[\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right]+\mathbf{j}\left[\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x}\right]+\mathbf{k}\left[\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right]$$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ u_1 & u_2 & u_3\end{vmatrix}+\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ v_1 & v_2 & v_3\end{vmatrix}$$

$= \nabla \times \mathbf{u} + \nabla \times \mathbf{v}.$ (by the definition)

i.e. curl $(u + v)$ = curl + curl v.

Hence curl of sum of two vector point functions is equal to the sum of their curls.

This result may be generalized for any number of vector-point functions.

Note. If **r** is the position vector of a variable point with respect to a fixed origin such that $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ then curl **r** = 0. *(Agra, 1967)*

$$\text{Since curl } \mathbf{r} = \left(\mathbf{i}\times\frac{\partial}{\partial x}+\mathbf{j}\times\frac{\partial}{\partial y}+\mathbf{k}\times\frac{\partial}{\partial z}\right)\times(x\mathbf{i}+y\mathbf{i}+z\mathbf{k})$$

$$=\left[\frac{\partial}{\partial y}(z)-\frac{\partial}{\partial z}(y)\right]+\mathbf{j}\left[\frac{\partial}{\partial z}(x)-\frac{\partial}{\partial x}(z)\right]+\mathbf{k}\left[\frac{\partial}{\partial x}(y)-\frac{\partial}{\partial y}(x)\right]$$

$= 0.$

1.33. CURL OF THE PRODUCT OF TWO VECTOR-POINT FUNCTIONS

We have to consider the curl of the forms $u\mathbf{v}$ and $\mathbf{u} \times \mathbf{v}$ where u is a scalar and **u**, **v** vector point functions.

Suppose, $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$

$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$

and u is a scalar point function.

$$\text{Then, curl}(u\mathbf{v}) = \nabla\times(u\mathbf{v}) = \left(\mathbf{i}\times\frac{\partial}{\partial x}+\mathbf{j}\times\frac{\partial}{\partial y}+\mathbf{k}\times\frac{\partial}{\partial z}\right)(uv_1\mathbf{i}+uv_2\mathbf{j}+uv_3\mathbf{k})$$

$$= \nabla\times(uv_1\mathbf{i}+uv_2\mathbf{j}+u\,v_3\mathbf{k})$$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ uv_1 & uv_2 & uv_3\end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}(uv_3)-\frac{\partial}{\partial z}(uv_2)\right]\mathbf{i}+\left[\frac{\partial}{\partial z}(uv_1)-\frac{\partial}{\partial x}(uv_3)\right]\mathbf{j}+\left[\frac{\partial}{\partial x}(uv_2)-\frac{\partial}{\partial y}(uv_1)\right]\mathbf{k}$$

$$=\left[u\frac{\partial v_3}{\partial y}+v_3\frac{\partial u}{\partial y}-u\frac{\partial v_3}{\partial z}-v_2\frac{\partial u}{\partial z}\right]\mathbf{i}+\left[u\frac{\partial v_1}{\partial z}+v_1\frac{\partial u}{\partial z}-u\frac{\partial v_3}{\partial x}-v_3\frac{\partial u}{\partial x}\right]\mathbf{j}$$

$$+\left[u\frac{\partial v_2}{\partial x}+v_2\frac{\partial u}{\partial x}-u\frac{\partial v_1}{\partial y}-v_1\frac{\partial u}{\partial y}\right]\mathbf{k}$$

$$=u\left[\left(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right)\mathbf{i}+\left(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x}\right)\mathbf{j}+\left(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right)\mathbf{k}\right]$$

$$+\left[\left(\frac{\partial u}{\partial y}v_3-\frac{\partial u}{\partial z}v_2\right)\mathbf{i}+\left(\frac{\partial u}{\partial z}v_1-\frac{\partial u}{\partial x}v_3\right)\mathbf{j}+\left(\frac{\partial u}{\partial x}v_2-\frac{\partial u}{\partial y}v_1\right)\mathbf{k}\right]$$

$$=u\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ v_1 & v_2 & v_3\end{vmatrix}+\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ v_1 & v_2 & v_3\end{vmatrix}$$

$= u$ curl $\mathbf{v}$ + (grad u) $\times$ $\mathbf{v}$

i.e. $\quad \nabla\times(u\mathbf{v})=u\,\nabla\times\mathbf{v}+(\nabla u)\times\mathbf{v}.$ *(Agra, 1955, 57, 61, 63, 83)*

Again curl $(\mathbf{u}\times\mathbf{v})=\nabla\times(\mathbf{u}\times\mathbf{v})$

$$=\nabla\times\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}$$

$$=\nabla\times[(v_3u_2-v_2u_3)\,\mathbf{i}+(v_1u_3-v_3u_1)\,\mathbf{j}+(v_2u_1-v_1u_2)\,\mathbf{k}]$$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ (v_3u_2-v_2u_3) & (v_1u_3-v_3u_1) & (v_2u_1-v_1u_2)\end{vmatrix}$$

$$=\mathbf{i}\left[\frac{\partial}{\partial y}(v_2u_1-v_1u_2)-\frac{\partial}{\partial z}(v_1u_3-u_1v_3)\right]+\mathbf{j}\left[\frac{\partial}{\partial z}(v_3u_2-v_2u_3)-\frac{\partial}{\partial x}(v_2u_1-v_1u_2)\right]$$

$$+\mathbf{k}\left[\frac{\partial}{\partial x}(v_1u_3-v_3u_1)-\frac{\partial}{\partial y}(v_3u_2-v_2u_3)\right]$$

$$=\mathbf{i}\left[u_1\frac{\partial v_2}{\partial y}+v_2\frac{\partial u_1}{\partial y}-u_2\frac{\partial v_1}{\partial y}-v_1\frac{\partial u_2}{\partial y}-u_3\frac{\partial v_1}{\partial z}-v_1\frac{\partial u_3}{\partial z}+u_1\frac{\partial v_3}{\partial z}+v_3\frac{\partial u_1}{\partial z}\right]$$

$$+\mathbf{j}\left[u_2\frac{\partial v_3}{\partial z}+v_3\frac{\partial u_2}{\partial z}-u_3\frac{\partial v_2}{\partial z}-v_2\frac{\partial u_3}{\partial z}-u_1\frac{\partial v_2}{\partial x}-v_2\frac{\partial u_1}{\partial x}+v_1\frac{\partial v_2}{\partial x}+u_2\frac{\partial v_1}{\partial x}\right]$$

$$+\mathbf{k}\left[u_3\frac{\partial v_1}{\partial x}+v_1\frac{\partial u_3}{\partial x}-v_3\frac{\partial u_1}{\partial x}-u_1\frac{\partial v_3}{\partial x}-u_2\frac{\partial v_3}{\partial y}-v_3\frac{\partial u_2}{\partial y}+v_2\frac{\partial u_3}{\partial y}+u_3\frac{\partial v_2}{\partial y}\right]$$

$$=(u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k})\left[\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\cdot(v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k})\right]$$

$$-(v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k})\left[\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\cdot(u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k})\right]$$

$$+\left[(v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k})\cdot\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}+\right)\right](u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k})$$

$$-\left[(u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k})\cdot\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}+\right)\right](v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k})$$

$= \mathbf{u}$ div $\mathbf{v}$ − $\mathbf{v}$ div $\mathbf{u}$ + $(\mathbf{v}\cdot\nabla)\,\mathbf{u}-(\mathbf{u}\cdot\nabla)\,\mathbf{v}$

Aliter Curl $\mathbf{f} = \mathbf{i} \times \frac{\partial \mathbf{f}}{\partial x} + \mathbf{j} \times \frac{\partial \mathbf{f}}{\partial y} + \mathbf{k} \times \frac{\partial \mathbf{f}}{\partial z} = \Sigma \mathbf{i} \times \frac{\partial \mathbf{f}}{\partial x}$.

Now Curl $(\mathbf{u} \times \mathbf{v}) = \Sigma \mathbf{i} \times \frac{\partial [\mathbf{u} \times \mathbf{v}]}{\partial x}$

$$= \Sigma \mathbf{i} \times \left[\mathbf{u} \times \frac{\partial \mathbf{v}}{\partial x} + \mathbf{v} \times \frac{\partial \mathbf{u}}{\partial x} \right]$$

$$= \Sigma \mathbf{i} \times \left(\mathbf{u} \times \frac{\partial \mathbf{v}}{\partial x} \right) + \Sigma \mathbf{i} \times \left(\mathbf{v} \times \frac{\partial \mathbf{u}}{\partial x} \right)$$

$$= \Sigma \left(\mathbf{i} \cdot \frac{\partial \mathbf{v}}{\partial x} \right) \mathbf{u} - \Sigma (\mathbf{i} \cdot \mathbf{u}) \frac{\partial \mathbf{v}}{\partial x} + \Sigma (\mathbf{i} \cdot \mathbf{v}) \frac{\partial \mathbf{u}}{\partial x} - \Sigma \left(\mathbf{i} \cdot \frac{\partial \mathbf{u}}{\partial x} \right) \mathbf{v}$$

[By vector triple product]

$$= \left[\left(\Sigma \mathbf{i} \cdot \frac{\partial \mathbf{v}}{\partial x} \right) \right] \mathbf{u} - [\Sigma (\mathbf{i} \cdot \mathbf{u})] \frac{\partial \mathbf{v}}{\partial x} + [\Sigma (\mathbf{i} \cdot \mathbf{v})] \frac{\partial \mathbf{u}}{\partial x} - \left[\Sigma \left(\mathbf{i} \cdot \frac{\partial \mathbf{u}}{\partial x} \right) \right] \mathbf{v}$$

$$= \mathbf{u} \operatorname{div} \mathbf{v} - \mathbf{v} \operatorname{div} \mathbf{u} + (\mathbf{v} \cdot \nabla) \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{v}.$$

1.34. TO EXPRESS GRADIENT OF SCALAR PRODUCT IN TERMS OF CURL

We have to show that

grad $(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \times \text{curl } \mathbf{v} + \mathbf{v} \times \text{curl } \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{u}$

We know that

$$\text{grad } (\mathbf{u} \cdot \mathbf{v}) = \Sigma \mathbf{i} \frac{\partial}{\partial x} (\mathbf{u} \cdot \mathbf{v})$$

$$= \Sigma \mathbf{i} \left[\mathbf{u} \cdot \frac{\partial \mathbf{v}}{\partial x} + \mathbf{v} \cdot \frac{\partial \mathbf{u}}{\partial x} \right]$$

$$= \Sigma \mathbf{i} \left(\mathbf{u} \cdot \frac{\partial \mathbf{v}}{\partial x} \right) + \Sigma \mathbf{i} \left(\mathbf{v} \cdot \frac{\partial \mathbf{u}}{\partial x} \right) \quad \ldots(1)$$

And $$\mathbf{u} \times \left(\mathbf{i} \times \frac{\partial \mathbf{v}}{\partial x} \right) = \left(\mathbf{u} \cdot \frac{\partial \mathbf{v}}{\partial x} \right) \mathbf{i} - (\mathbf{u} \cdot \mathbf{i}) \frac{\partial \mathbf{v}}{\partial x}$$

or $$\left(\mathbf{u} \cdot \frac{\partial \mathbf{v}}{\partial x} \right) \mathbf{i} = \left[\mathbf{u} \times \left(\mathbf{i} \times \frac{\partial \mathbf{v}}{\partial x} \right) \right] + (\mathbf{u} \cdot \mathbf{i}) \frac{\partial \mathbf{v}}{\partial x}$$

$\therefore$ $$\Sigma \left(\mathbf{u} \cdot \frac{\partial \mathbf{v}}{\partial x} \right) \mathbf{i} = \Sigma \left[\mathbf{u} \times \left(\mathbf{i} \times \frac{\partial \mathbf{v}}{\partial x} \right) \right] + \Sigma (\mathbf{u} \cdot \mathbf{i}) \frac{\partial \mathbf{v}}{\partial x}$$

$$= \mathbf{u} \times \text{curl } \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{v} \quad \ldots(2)$$

Similarly $$\Sigma \left(\mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial x} \right) \mathbf{i} = \mathbf{v} \times \text{curl } \mathbf{u} + (\mathbf{v} \cdot \nabla) \mathbf{u} \quad \ldots(3)$$

Substituting values of (2) and (3) in (1) we find,

grad $(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \times \text{curl } \mathbf{v} + (\mathbf{u} \cdot \nabla) \mathbf{v} + \mathbf{v} \times \text{curl } \mathbf{u} + (\mathbf{v} \cdot \nabla) \mathbf{u}$

$= \mathbf{u} \times \text{curl } \mathbf{v} + \mathbf{v} \times \text{curl } \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{u}.$

1.35. TO EXPRESS DIVERGENCE OF VECTOR PRODUCT IN TERMS OF CURL

We have to show that $\quad \text{div}\,(\mathbf{u} \times \mathbf{v}) = \text{curl}\,\mathbf{u} \cdot \mathbf{v} - \text{curl}\,\mathbf{v} \cdot \mathbf{u}.$

(*Meerut, 1985; Agra, 1971, 64, 61, 59*)

We know that,

$$\text{div}\,(\mathbf{u} \times \mathbf{v}) = \Sigma\,\mathbf{i} \cdot \frac{\partial}{\partial x}(\mathbf{u} \times \mathbf{v})$$

$$= \Sigma\,\mathbf{i} \cdot \left[\frac{\partial \mathbf{u}}{\partial x} \times \mathbf{v} + \mathbf{u} \times \frac{\partial \mathbf{v}}{\partial x}\right]$$

$$= \left[\Sigma\,\mathbf{i} \cdot \left(\frac{\partial \mathbf{u}}{\partial x} \times \mathbf{v}\right) + \Sigma\,\mathbf{i} \cdot \left(\mathbf{u} \times \frac{\partial \mathbf{v}}{\partial x}\right)\right]$$

$$= \Sigma\,\mathbf{i} \times \frac{\partial \mathbf{u}}{\partial x} \cdot \mathbf{v} + \Sigma\,\mathbf{i} \cdot \mathbf{u} \times \frac{\partial \mathbf{v}}{\partial x}$$

(interchanging dot and cross)

$$= \left(\Sigma\,\mathbf{i} \times \frac{\partial \mathbf{u}}{\partial x}\right) \cdot \mathbf{v} + \left(\Sigma\,\mathbf{i} \times \frac{\partial \mathbf{v}}{\partial x}\right) \cdot \mathbf{u}$$

$$= \text{curl}\,\mathbf{u} \cdot \mathbf{v} - \text{curl}\,\mathbf{v} \cdot \mathbf{u}.$$

Problem 91. *Prove the following*:

(*i*) *div grad* $\phi = \nabla^2 \phi$.

(*ii*) *curl grad* $\phi = \nabla \times (\nabla \phi) = 0$ (*Meerut, 1982, 87; Agra, 1956, 67*)

(*iii*) *div curl* $\mathbf{f} = \nabla \cdot (\nabla \times \mathbf{f}) = 0$ (*Meerut, 1982, 87; Agra, 1956, 67, 71*)

(*iv*) *curl curl* $\mathbf{f} = \nabla \times (\nabla \times \mathbf{f}) =$ *grad div* $\mathbf{f} - \nabla^2 \mathbf{f}$

(*Meerut, 1980, 81; Agra, 1959, 61, 64, 69*)

(*v*) *grad vid* $\mathbf{V} = \nabla (\nabla \cdot \mathbf{V}) =$ *curl curl* $\mathbf{V} + \nabla^2 \mathbf{V}$.

(*i*) div grad $\phi = \nabla \cdot (\nabla \phi)$

$$= \left(\mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}\right) \cdot \left(\mathbf{i}\,\frac{\partial \phi}{\partial x} + \mathbf{j}\,\frac{\partial \phi}{\partial y} + \mathbf{k}\,\frac{\partial \phi}{\partial z}\right)$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi$$

$$= \nabla^2 \phi$$

Note. This follows that

$$\text{div grad} \equiv \left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right\} \equiv \nabla^2 \quad \textit{i.e., Laplacian operator.}$$

(*ii*) curl grad $\phi = \nabla \times (\nabla \phi)$

$$= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times \left(\mathbf{i}\frac{\partial \phi}{\partial x} + \mathbf{j}\frac{\partial \phi}{\partial y} + \mathbf{k}\frac{\partial \phi}{\partial z}\right)$$

$$= \left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y} \right) \mathbf{i} + \left(\frac{\partial^2 \phi}{\partial z \partial x} - \frac{\partial^2 \phi}{\partial x \partial z} \right) \mathbf{j} + \left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x} \right) \mathbf{k}$$

$$= 0.$$

(*iii*) div curl $\mathbf{f} = \nabla \cdot (\nabla \times \mathbf{f})$.

$$= \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \cdot \left(\mathbf{i} \times \frac{\partial \mathbf{f}}{\partial x} + \mathbf{j} \times \frac{\partial \mathbf{f}}{\partial y} + \mathbf{k} \times \frac{\partial \mathbf{f}}{\partial z} \right)$$

(Taking $\mathbf{f} = f_1 \mathbf{i} + f_2 \mathbf{j} + f_3 \mathbf{k}$)

$$= \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \cdot \left[\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) \mathbf{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \mathbf{k} \right]$$

$$= \frac{\partial}{\partial x} \left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right] + \frac{\partial}{\partial y} \left[\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right] + \frac{\partial}{\partial z} \left[\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right]$$

$$= \frac{\partial^2 f_3}{\partial x \partial y} - \frac{\partial^2 f_2}{\partial x \partial z} + \frac{\partial^2 f_1}{\partial y \partial z} - \frac{\partial^2 f_3}{\partial y \partial x} + \frac{\partial^2 f_2}{\partial z \partial x} - \frac{\partial^2 f_1}{\partial z \partial y}$$

$$= 0.$$

(*iv*) curl curl $\mathbf{f} = \nabla \times (\nabla \times \mathbf{f})$

$$= \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \times \left(\mathbf{i} \times \frac{\partial \mathbf{f}}{\partial x} + \mathbf{j} \times \frac{\partial \mathbf{f}}{\partial y} + \mathbf{k} \times \frac{\partial \mathbf{f}}{\partial z} \right)$$

$$= \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \times \left[\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_2}{\partial x} \right) \mathbf{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \mathbf{k} \right]$$

$$= \left[\frac{\partial}{\partial y} \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) - \frac{\partial}{\partial z} \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) \right] \mathbf{i} + \left[\frac{\partial}{\partial z} \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) - \frac{\partial}{\partial x} \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \right] \mathbf{j} + \left[\frac{\partial}{\partial x} \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \right] \mathbf{k}$$

$$= \left[\left(\frac{\partial^2 f_2}{\partial y \partial x} + \frac{\partial^2 f_3}{\partial z \partial x} \right) \mathbf{i} - \left(\frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial z^2} \right) \mathbf{i} \right] + \left[\left(\frac{\partial^2 f_3}{\partial z \partial y} + \frac{\partial^2 f_1}{\partial x \partial y} \right) \mathbf{j} - \left(\frac{\partial^2 f_2}{\partial z^2} + \frac{\partial^2 f_2}{\partial x^2} \right) \mathbf{j} \right] + \left[\left(\frac{\partial^2 f_1}{\partial x \partial z} + \frac{\partial^2 f_2}{\partial y \partial z} \right) \mathbf{k} - \left(\frac{\partial^2 f_3}{\partial x^2} + \frac{\partial^2 f_3}{\partial y^2} \right) \mathbf{k} \right]$$

(On adding and subtracting $\frac{\partial^2 f_1}{\partial x^2}, \frac{\partial^2 f_2}{\partial y^2}, \frac{\partial^2 f_3}{\partial z^2}$ respectively in second factor of each bracket)

$$= \left[\left(\frac{\partial^2 f_1}{\partial x^2}+\frac{\partial^2 f_2}{\partial x\partial y}+\frac{\partial^2 f_3}{\partial x\partial z}\right)\mathbf{i}-\left(\frac{\partial^2 f_1}{\partial x^2}+\frac{\partial^2 f_1}{\partial y^2}+\frac{\partial^2 f_1}{\partial z^2}\right)\right] + \text{two similar terms.}$$

$$= \Sigma i\,\frac{\partial}{\partial x}\left(\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}\right)-\Sigma\left(\frac{\partial^2 f_1}{\partial x^2}+\frac{\partial^2 f_1}{\partial y^2}+\frac{\partial^2 f_1}{\partial z^2}\right)\mathbf{i}$$

$$= \text{grad div }\mathbf{f} - \nabla^2\mathbf{f}$$

Note. If div $\mathbf{f} = 0$, then grad div $\mathbf{f} = 0$ and so

curl curl $\mathbf{f} = -\nabla^2\mathbf{f}$. (*Agra, 1975*)

(*v*) grad vid $\mathbf{V} = \nabla(\nabla \cdot \mathbf{V})$.

$$= \left(\mathbf{i}\,\frac{\partial}{\partial x}+\mathbf{j}\,\frac{\partial}{\partial y}+\mathbf{k}\,\frac{\partial}{\partial z}\right)\left(\mathbf{i}\cdot\frac{\partial \mathbf{V}}{\partial x}+\mathbf{j}\cdot\frac{\partial \mathbf{V}}{\partial y}+\mathbf{k}\cdot\frac{\partial \mathbf{V}}{\partial z}\right)$$

(on taking $\mathbf{V} = V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}$)

$$= \left(\mathbf{i}\,\frac{\partial}{\partial x}+\mathbf{j}\,\frac{\partial}{\partial y}+\mathbf{k}\,\frac{\partial}{\partial z}\right)\left(\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}\right)$$

$$= \left(\frac{\partial^2 f_1}{\partial x^2}+\frac{\partial^2 f_2}{\partial x\partial y}+\frac{\partial^2 f_3}{\partial x\partial z}\right)\mathbf{i}+\left(\frac{\partial^2 f_1}{\partial y\partial x}+\frac{\partial^2 f_2}{\partial y^2}+\frac{\partial^2 f_3}{\partial y\partial z}\right)\mathbf{j}$$

$$+\left(\frac{\partial^2 f_1}{\partial z\partial x}+\frac{\partial^2 f_2}{\partial z\partial y}+\frac{\partial^2 f_3}{\partial z^2}\right)\mathbf{k}$$

$$= \left[\mathbf{i}\cdot\frac{\partial^2 \mathbf{f}}{\partial x^2}\mathbf{i}+\mathbf{i}\cdot\frac{\partial^2 \mathbf{f}}{\partial x\partial y}\mathbf{j}+\mathbf{i}\cdot\frac{\partial^2 \mathbf{f}}{\partial x\partial z}\mathbf{k}\right] + \text{two similar terms.}$$

$$= \Sigma\left(\mathbf{i}\cdot\frac{\partial^2 f}{\partial x^2}\mathbf{i}+\mathbf{i}\cdot\frac{\partial^2 f}{\partial x\partial y}\mathbf{j}+\mathbf{i}\cdot\frac{\partial^2 f}{\partial x\partial z}\mathbf{k}\right).$$

$$= \Sigma\,\frac{\partial^2 \mathbf{f}}{\partial x^2} + \nabla\times(\nabla\times\mathbf{f}) = \nabla^2\mathbf{f} + \text{curl curl }\mathbf{f}.$$

Problem 92. *Find curl of the following functions*

(*a*) $\mathbf{f} = \dfrac{x\mathbf{i}+y\mathbf{j}}{x+y}$

(*b*) $\mathbf{f} = x\cos z\mathbf{i} + y\log x\mathbf{j} - z^2\mathbf{k}$

(*c*) $\mathbf{f} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$

Let $\mathbf{f} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$; then

$$\text{curl }\mathbf{f} = \mathbf{i}\times\frac{\partial \mathbf{f}}{\partial x}+\mathbf{j}\times\frac{\partial \mathbf{f}}{\partial y}+\mathbf{k}\times\frac{\partial \mathbf{f}}{\partial z}$$

$$= \left(\mathbf{i}\,\frac{\partial}{\partial x}+\mathbf{j}\,\frac{\partial}{\partial y}+\mathbf{k}\,\frac{\partial}{\partial z}\right)\times\mathbf{f}$$

$$= \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \times (f_1 \mathbf{i} + f_2 \mathbf{j} + f_3 \mathbf{k})$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) \mathbf{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \mathbf{k}$$

(*a*) We have $\mathbf{f} = \dfrac{x}{x+y} \mathbf{i} + \dfrac{y}{x+y} \mathbf{j} + 0\mathbf{k}$

$$f_1 = \frac{x}{x+y}, \ f_2 = \frac{y}{x+y}, \ f_3 = 0$$

$\therefore$ $\dfrac{\partial f_3}{\partial y} = 0 \ ; \ \dfrac{\partial f_2}{\partial z} = 0 \ ; \ \dfrac{\partial f_1}{\partial z} = 0 \ ; \ \dfrac{\partial f_3}{\partial x} = 0$

$$\frac{\partial f_2}{\partial x} = -\frac{y}{(x+y)^2} \ ; \ \frac{\partial f_1}{\partial y} = -\frac{x}{(x+y)^2}$$

$\therefore$ $\text{curl } \mathbf{f} = 0\mathbf{i} + 0\mathbf{j} + \left[-\dfrac{y}{(x+y)^2} + \dfrac{x}{(x+y)^2} \right] \mathbf{k}$

$$= \frac{x-y}{(x+y)^2} \mathbf{k}$$

(*b*) We have $\mathbf{f} = x \cos z\mathbf{i} + y \log x\mathbf{j} - z^2\mathbf{k}$

$$f_1 = x \cos z, f_2 = y \log x, f_3 = -z^2,$$

so that $\dfrac{\partial f_3}{\partial y} = 0 \ ; \ \dfrac{\partial f_2}{\partial z} = 0 \ ; \dfrac{\partial f_1}{\partial z} = -x \sin z \ ;$

$$\frac{\partial f_3}{\partial x} = 0 \ ; \ \frac{\partial f_2}{\partial x} = \frac{y}{x} \ ; \ \frac{\partial f_1}{\partial y} = 0$$

$\therefore$ $\text{curl } \mathbf{f} = 0\mathbf{i} - x \sin z\mathbf{j} + \dfrac{y}{x}\mathbf{k}$

(*c*) We have $\mathbf{f} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$

$$f_1 = x^2, f_2 = y^2, f_3 = z^3.$$

So that $\dfrac{\partial f_3}{\partial y} = 0, \ \dfrac{\partial f_2}{\partial z} = 0, \ \dfrac{\partial f_1}{\partial z} = 0$

$$\frac{\partial f_3}{\partial x} = 0, \ \frac{\partial f_2}{\partial x} = 0, \ \frac{\partial f_1}{\partial y} = 0.$$

$\therefore$ $\text{curl } \mathbf{f} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}.$

Problem 93. *If* $u = x^2yz$, $v = xy - 3z^2$ *find*

(*a*) $\nabla\ [(\nabla u) \cdot (\nabla v)]$

(*b*) $\nabla \cdot [(\nabla u) \times (\nabla v)]$

(*c*) $\nabla \times [(\nabla u \times \nabla v)]$.

Given $u = x^2yz.$

$\therefore$ $\nabla u = \left(\mathbf{i} \dfrac{\partial}{\partial x} + \mathbf{j} \dfrac{\partial}{\partial y} + \mathbf{k} \dfrac{\partial}{\partial z} \right) (x^2 yz)$

$$= (2xyz)\, \mathbf{i} + (x^2z)\, \mathbf{j} + (x^2y)\, \mathbf{k}$$

and $$v = xy - 3z^2$$

$\therefore$ $$\nabla v = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)(xy - 3z^2)$$

$$= y\mathbf{i} + x\mathbf{j} - 6z\mathbf{k}.$$

So that $$(\nabla u)\cdot(\nabla v) = [(2xyz)\,\mathbf{i} + (x^2z)\,\mathbf{j} + (x^2z)\,\mathbf{k}]\cdot[y\mathbf{i} + x\mathbf{j} - 6z\mathbf{k}]$$
$$= 2xy^2z + x^3z - 6x^2yz \qquad \text{...(1)}$$

and $$[(\nabla u)\times(\nabla v)] = (2xyz\mathbf{i} + x^3z\mathbf{j} + x^2y\mathbf{k})\times(y\mathbf{i} + x\mathbf{j} - 6z\mathbf{k})$$
$$= (-6x^2z^2 - x^2y)\,\mathbf{i} + (x^2y^2 + 12xyz^2)\,\mathbf{j} + (2x^2yz - x^2yz)\,\mathbf{k} \qquad \text{...(2)}$$

Thus,

(*a*) $$\nabla[\nabla u\cdot\nabla v] = \left[\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial x}\right][2xy^2z + x^3z - 6x^2yz]$$
$$= [2y^2z + 3x^2z - 12xyz]\,\mathbf{i} + [4xyz - 6x^2z]\,\mathbf{j} + [2xy^2 + x^3 - 6x^2y]\,\mathbf{k}$$

(*b*) $\nabla\cdot[\nabla u\times\nabla v]$
$$= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot\left[\left(-6x^2z^2 - x^3y\right)\mathbf{i} + \left(x^2y^2 + 12xyz^2\right)\mathbf{j} + \left(2x^2yz - x^2yz\right)\right]\mathbf{k}$$
$$= \frac{\partial}{\partial x}\left(-6x^2z^2 - x^3y\right) + \frac{\partial}{\partial y}\left(x^2y^2 + 12xyz^2\right) + \frac{\partial}{\partial z}\left(2x^2yz - x^2yz\right)$$
$$= [-12xz^2 - 3x^2y + 2x^2y + 12xz^2 + 2x^2y - x^2y]$$
$$= 0.$$

(*c*) $\nabla\times[\nabla u\times\nabla v]$
$$= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial x} + \mathbf{k}\frac{\partial}{\partial z}\right)\times\left[\left(-6x^2z^2 - x^2y\right)\mathbf{i} + \left(x^2y^2 + 12xyz^2\right)\mathbf{j} + \left(2x^2yz - x^2yz\right)\right]\mathbf{k}$$
$$= \left[\frac{\partial}{\partial y}\left(2x^2yz - x^2yz\right) - \frac{\partial}{\partial z}\left(x^2y^2 + 12xyz^2\right)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}\left(6x^2z^2 - x^2y\right) - \frac{\partial}{\partial x}\left(2x^2yz - x^2yz\right)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}\left(x^2y^2 + 12xyz^2\right) - \frac{\partial}{\partial y}\left(-6x^2z^2 - x^2y\right)\right]\mathbf{k}$$
$$= [2x^2z - x^2z - 24xyz]\,\mathbf{i} + [-12x^2z - 4xyz + 2xyz]\,\mathbf{j} + [2xy^2 + 12yz^2 + x^3]\,\mathbf{k}$$
$$= (x^2z - 24xyz)\,\mathbf{i} - (12x^2z + 2xyz)\,\mathbf{j} + (2xy^2 + x^3 + 12yz^2)\mathbf{k}.$$

Problem 94. *If* $\mathbf{u} = yz^2\mathbf{i} - 3xz^2\mathbf{j} + 2xyz\mathbf{k}$ *and* $\mathbf{v} = 3x\mathbf{i} + 4z\mathbf{j} - xy\mathbf{k}$ *and* $\phi = xyz$ *find,*

(*a*) $\mathbf{u}\times(\nabla\phi)$

(*b*) $(\mathbf{u}\times\nabla)\phi$

(*c*) $(\nabla \times \mathbf{u}) \times \mathbf{v}$

(*d*) $\mathbf{v} \cdot \nabla \times \mathbf{u}$.

Given $\phi = xyz$

$$\therefore \quad \nabla\phi = \left[\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right](xyz)$$

$$= yz\mathbf{i} + zx\mathbf{j} + xy\mathbf{k}.$$

Thus,

(*a*) $\mathbf{u} \times (\nabla\phi) = (yz^2\mathbf{i} - 3xz^2\mathbf{j} + 2xyz\mathbf{k}) \times (yz\mathbf{i} + zx\mathbf{j} + xy\mathbf{k})$

$$= [-3x^2yz^2 - 2x^2yz^2]\,\mathbf{i} + [2xy^2z^2 - xy^2z^2]\,\mathbf{j} + [xyz^3 + 3xyz^3]\,\mathbf{k}$$

$$= -5x^2yz^2\mathbf{i} + xy^2z^2\mathbf{j} + 4xyz^3\mathbf{k}.$$

(*b*)
$$\mathbf{u} \times \nabla = \left(yz^2\mathbf{i} - 3xz^2\mathbf{j} + 2xyz\mathbf{k}\right) \times \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)$$

$$= \left(-3xz^2\frac{\partial}{\partial z} - 2xyz\frac{\partial}{\partial y}\right)\mathbf{i} + \left(2xyz\frac{\partial}{\partial x} - yz^2\frac{\partial}{\partial z}\right)\mathbf{j}$$

$$+ \left(yz^2\frac{\partial}{\partial y} + 3xz^2\frac{\partial}{\partial x}\right)\mathbf{k}$$

So that
$$[(\mathbf{u} \times \nabla)]\phi = \left[\left(-3xz^2\frac{\partial}{\partial z} - 2xyz\frac{\partial}{\partial y}\right)\mathbf{i}\right.$$

$$\left. + \left(2xyz\frac{\partial}{\partial x} - yz^2\frac{\partial}{\partial z}\right)\mathbf{j} + \left(yz^2\frac{\partial}{\partial y} + 3xz^2\frac{\partial}{\partial x}\right)\mathbf{k}\right](xy^2)$$

$$= (-3x^2yz^2 - 2x^2yz^2)\,\mathbf{i} + (2xy^2z^2 - xy^2z^2)\,\mathbf{j} + (xyz^3 + 3xyz^3)\,\mathbf{k}$$

$$= -5x^2yz^2\,\mathbf{i} + xy^2z^2\mathbf{j} + 4xyz^3\mathbf{k}.$$

(*c*)
$$\nabla \times \mathbf{u} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times \left(yz^2\mathbf{i} - 3xz^2\mathbf{j} + 2xyz\mathbf{k}\right)$$

$$= \left[\frac{\partial}{\partial y}(2xyz) - \frac{\partial}{\partial z}(-3xz^2)\right]\mathbf{i}$$

$$+ \left[\frac{\partial}{\partial z}(yz^2) - \frac{\partial}{\partial x}(2xyz)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(-3xz^2) - \frac{\partial}{\partial y}(yz^2)\right]\mathbf{k}$$

$$= [2xz + 6xz]\,\mathbf{i} + [2yz - 2yz]\,\mathbf{j} + [-3z^2 - z^2]\,\mathbf{k}$$

$$= 8xz\mathbf{i} + 0\mathbf{j} - 4z^2\mathbf{k},$$

$$\therefore \quad [\nabla \times \mathbf{u}] \times \mathbf{v} = (8xz\mathbf{i} + 0\mathbf{j} - 4z^2\mathbf{k}) \times (3x\mathbf{i} + 4z\mathbf{j} - xy\mathbf{k})$$

$$= 16z^2\mathbf{i} + (-12xz^2 + 8x^2yz)\,\mathbf{j} + 32xz^2\mathbf{k}$$

$$= 16z^3\mathbf{i} + (8x^2yz - 12xz^2)\,\mathbf{j} + 32xz^2\mathbf{k}.$$

(*d*) $\nabla \times \mathbf{u} = 8xz\mathbf{i} + 0\mathbf{j} - 4z^2\mathbf{k}$ [by (*c*)]

$$\therefore \quad \mathbf{v} \cdot (\nabla \times \mathbf{u}) = (3x\mathbf{i} + 4z\mathbf{j} - xy\mathbf{k}) \cdot (8xz\mathbf{i} + 0\mathbf{j} - 4z^2\mathbf{k})$$

$$= 24x^2z + 4xyz^2.$$

Problem 95. *Prove the following relations*:

(*a*) $\nabla \cdot [\nabla u \times \nabla v] = 0$.

(*b*) $\nabla \cdot [\mathbf{a} \times \mathbf{r}] = 0.$

(*c*) $\nabla \times [\mathbf{a} \times \mathbf{r}) = 2a$

(*a*) Suppose that $\nabla u = \mathbf{U}$ and $\nabla v = \mathbf{V}$.

So that $\nabla \cdot (\nabla u \times \nabla v) = \nabla \cdot (\mathbf{U} \times \mathbf{V})$

$$= \mathbf{V} \cdot (\nabla \times \mathbf{U}) - \mathbf{U} \cdot (\nabla \times \mathbf{V})$$

$$= \nabla v. (\nabla \times \nabla u) - \nabla u \cdot (\nabla \times \nabla v)$$

$$= 0 \ (\because \text{ curl grad } u = 0 \text{ etc.})$$

(*b*) If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$,

then $\nabla \cdot [\mathbf{a} \times \mathbf{r}] = \left[\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k}\right] \cdot [a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}]$

$$= \left[\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}\right] \cdot [a_2z - a_3y)\, \mathbf{i} + (a_3x - a_1z)\, \mathbf{j} + (a_1y - a_2x)\mathbf{k}]$$

$$= \frac{\partial}{\partial x} (a_2z - a_3y) + \frac{\partial}{\partial y} (a_3x - a_1z) + \frac{\partial}{\partial z} (a_1y - z_2x)$$

$$= 0.$$

(*c*) We have,

$$\nabla \times (\mathbf{a} \times \mathbf{r}) = \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}\right) \times \left[(a_2z - a_3y)\ \mathbf{i} + (a_3x - a_1z)\ \mathbf{j} + (a_1y - a_2x)\ \mathbf{k}\right]$$

$$= \left[\frac{\partial}{\partial y} (a_1y - a_2x) - \frac{\partial}{\partial z} (a_3x - a_1z)\right] \mathbf{i}$$

$$+ \left[\frac{\partial}{\partial z} (a_2z - a_3y) - \frac{\partial}{\partial x} (a_1y - a_2x)\right] \mathbf{j}$$

$$+ \left[\frac{\partial}{\partial x} (a_3x - a_1z) - \frac{\partial}{\partial y} (a_1y - a_2x)\right] \mathbf{k}$$

$$= 2a_1\mathbf{i} + 2a_2\mathbf{j} + 2a_3\mathbf{k} = 2\,(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) = 2\mathbf{a}.$$

Problem 96. *If* $\hat{\mathbf{e}}$ *is a unit vector, prove that*

(*a*) $\nabla . (\hat{\mathbf{e}} . \mathbf{r})\ \hat{\mathbf{e}} = 1.$ (*b*) $\nabla \times (\hat{\mathbf{e}} \cdot \mathbf{r})\ \hat{\mathbf{e}} = 0$

(*c*) $\nabla \cdot [\hat{\mathbf{e}} \times \mathbf{r}) \times \hat{\mathbf{e}}] = 2.$ (*d*) $\nabla \times [(\hat{\mathbf{e}} \times \mathbf{r}) \times \hat{\mathbf{e}}] = 0.$

Taking e_1, e_2, e_3 as the components of $\hat{\mathbf{e}}$ along principal axes, we have

$$\hat{\mathbf{e}} = e_1\mathbf{i} + e_2\mathbf{j} + e_3\mathbf{k} \text{ where } |\hat{\mathbf{e}}| = \sqrt{e_1^2 + e_2^2 + e_3^2} = 1 \qquad ...(1)$$

and let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ ...(2)

(*a*) We have,

$$\nabla \cdot [(\hat{\mathbf{e}} \cdot \mathbf{r})\hat{\mathbf{e}}] = \nabla \cdot [(e_1\mathbf{i} + e_2\mathbf{j} + e_3\mathbf{k}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})](e_1\mathbf{i} + e_2\mathbf{j} + e_3\mathbf{k})$$

$$= \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}\right) \cdot (e_1\mathbf{i} + e_2\mathbf{j} + e_3\mathbf{k})(e_1x + e_2y + e_3z)$$

$$= \left(e_1\frac{\partial}{\partial x} + e_2\frac{\partial}{\partial y} + e_3\frac{\partial}{\partial z}\right)(e_1x + e_2y + e_3z)$$

$$= e_1^2 + e_2^2 + e_3^2 = 1 \quad \text{by (1)}$$

(*b*) We have,

$$\nabla \times [(\hat{\mathbf{e}}\cdot\mathbf{r})\hat{\mathbf{e}}] = \left(\mathbf{i}\ \frac{\partial}{\partial x} + \mathbf{j}\ \frac{\partial}{\partial y} + \mathbf{k}\ \frac{\partial}{\partial z}\right) \times (e_1\mathbf{i} + e_2\mathbf{j} + e_3\mathbf{k})(e_1x + e_2y + e_3z)$$

as in (*a*)

$$= \mathbf{i}\ [e_2e_3 - e_2e_3] + \mathbf{j}\ [e_1e_3 - e_1e_3] + \mathbf{k}\ [e_1e_2 - e_2e_1]$$

$$= 0.$$

(*c*) We have,

$$\nabla\cdot[(\hat{\mathbf{e}}\times\mathbf{r})\times\hat{\mathbf{e}}] = \nabla\cdot[\{(e_1\mathbf{i} + e_2\mathbf{j} + e_3\mathbf{k}) \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})\} \times (e_1\mathbf{i} + e_2\mathbf{j} + e_3\mathbf{k})]$$

$$= \nabla\cdot[\{\mathbf{i}(e_2z - e_3y) + \mathbf{j}\ (e_3x - e_1z) + \mathbf{k}\ (e_1y - e_2x)\} \times (e_1\mathbf{i} + e_2\mathbf{j} + e_3\mathbf{k})]$$

$$= \left(\mathbf{i}\ \frac{\partial}{\partial x} + \mathbf{j}\ \frac{\partial}{\partial y} + \mathbf{k}\ \frac{\partial}{\partial z}\right)\cdot\left[\mathbf{i}\{e_3(e_3x - e_1z) - e_2(e_1y - e_2x)\}\right.$$
$$+ \mathbf{j}\{e_1(e_1y - e_2x) - e_3\ (e_2z - e_3y)\} + \mathbf{k}\ \{e_2(e_2z - e_3y) - e_1(e_3x - e_1z)\}]$$

$$= (e_3^2 + e_2^2) + (e_1^2 + e_3^2) + (e_2^2 + e_1^2)$$

$$= 2(e_1^2 + e_2^2 + e_3^2) = 2 \quad \text{by (1)}$$

(*d*) We have,

$$\nabla \times [(\hat{\mathbf{e}}\times\mathbf{r})\times\hat{\mathbf{e}}] = \left(\mathbf{i}\ \frac{\partial}{\partial x} + \mathbf{j}\ \frac{\partial}{\partial y} + \mathbf{k}\ \frac{\partial}{\partial z}\right) \times [\mathbf{i}\{e_3(e_3x - e_1z)$$
$$- e_2(e_1y - e_2x)\} + \mathbf{j}\{e_1(e_1y - e_2x) - e_3(e_2z - e_3y)\}$$
$$+ \mathbf{k}\ \{e_2(e_2z - e_3y) - e_1(e_3x - e_1z)\}] \quad \text{as in (c)}$$

$$= \mathbf{i}[-e_2e_3 + e_3e_2] + \mathbf{j}[-e_3e_1 + e_1e_3] + \mathbf{k}[-e_1e_2 + e_2e_1]$$

$$= 0.$$

Problem 97. *If $r = (x^2 + y^2 + z^2)^{1/2}$ and $\left(\frac{1}{r}\right)$ is a solution of Laplace's equation show that $\nabla^2\left(\frac{1}{r}\right) = 0$.* *(Agra, 1957)*

Hence or otherwise, evaluate $\nabla\times\left(\frac{\mathbf{r}}{r^3}\right)$

(*a*) Given $r = (x^2 + y^2 + z^2)^{1/2}$

and $$\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$

$$\therefore \quad \nabla^2\left(\frac{1}{r}\right) = \left(\frac{\partial}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left[\frac{1}{(x^2 + y^2 + z^2)^{1/2}}\right]$$

But $$\frac{\partial}{\partial x}\left(\frac{1}{(x^2+y^2+z^2)}\right)^{1/2}=\left[\frac{x}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{(x^2+y^2+z^2)^{1/2}}\right)=\frac{\partial}{\partial x}\left[\frac{-x}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$=\frac{3x^2}{(x^2+y^2+z^2)^{5/2}}-\frac{1}{(x^2+y^2+z^2)^{3/2}}$$

Similarly $$\frac{\partial^2}{\partial y^2}\left[\frac{1}{(x^2+y^2+z^2)^{1/2}}\right]=\frac{3y^2}{(x^2+y^2+z^2)^{5/2}}-\frac{1}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial^2}{\partial z^2}\left[\frac{1}{(x^2+y^2+z^2)^{1/2}}\right]=\frac{3z^2}{(x^2+y^2+z^2)^{5/2}}-\frac{1}{(x^2+y^2+z^2)^{3/2}}$$

$\therefore$ $$\nabla^2\left(\frac{1}{r}\right)=\frac{3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}}-\frac{3}{(x^2+y^2+z^2)^{3/2}}$$

$$=\frac{3}{(x^2+y^2+z^2)^{3/2}}-\frac{3}{(x^2+y^2+z^2)^{3/2}}=0.$$

(*b*) Suppose that $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

and given $r = (x^2 + y^2 + z^2)^{1/2}$

$$\therefore\quad \nabla\times\left(\frac{\mathbf{r}}{r^2}\right) = \nabla\times[(x^2+y^2+z^2)^{-1}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})]$$

$$= [\nabla (x^2+y^2+z^2)^{-1}]\times[x\mathbf{i}+y\mathbf{j}+z\mathbf{k}]$$

$$+ (x^2+y^2+z^2)^{-1}[\nabla\times(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})]$$

$$=\left[\nabla\left(\frac{1}{x^2+y^2+z^2}\right)\right]\times(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$$ since $\nabla\times(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=0$ by Note on § 1.32

$$=\left[\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{x^2+y^2+z^2}\right)\right]\times[x\mathbf{i}+y\mathbf{j}+z\mathbf{k}]$$

$$=\frac{-2}{(x^2+y^2+z^2)^2}[x\mathbf{i}+y\mathbf{j}+z\mathbf{k}]\times[x\mathbf{i}+y\mathbf{j}+z\mathbf{k}]$$

= 0 [Vector product of two equal vectors being zero].

Problem 98. *If* $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ *and* $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ *then prove that grad* $[(\mathbf{r}\times\mathbf{a})\cdot(\mathbf{r}\times\mathbf{b})] = (\mathbf{b}\times\mathbf{r})\times\mathbf{a} + (\mathbf{a}\times\mathbf{r})\times\mathbf{b}$.

$$\text{grad}\,[(\mathbf{r}\times\mathbf{a})\cdot(\mathbf{r}\times\mathbf{b})] = \text{grad}\,[r^2(\mathbf{a}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{r})]$$

$$= \text{grad}\,[r^2(\mathbf{a}\cdot\mathbf{b})] - \text{grad}\,[(\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{r})]$$

$$= (\mathbf{a}\cdot\mathbf{b})\,\text{grad}\,r^2 + r^2\,\text{grad}\,(\mathbf{a}\cdot\mathbf{b})$$

$$- (\mathbf{b}\cdot\mathbf{r})\,\text{grad}\,(\mathbf{a}\cdot\mathbf{r}) - (\mathbf{a}\cdot\mathbf{r})\,\text{grad}\,(\mathbf{b}\cdot\mathbf{r})$$

$= (\mathbf{a} \cdot \mathbf{b})\, 2\mathbf{r} - (\mathbf{b} \cdot \mathbf{r})\, \mathbf{a} - (\mathbf{a} \cdot \mathbf{r})\, \mathbf{b}$
$= (\mathbf{a} \cdot \mathbf{b})\, \mathbf{r} - (\mathbf{a} \cdot \mathbf{r})\, \mathbf{b} + (\mathbf{b} \cdot \mathbf{a})\, \mathbf{r} - (\mathbf{b} \cdot \mathbf{r})\, \mathbf{a}$
$= -\mathbf{a} \times (\mathbf{b} \times \mathbf{r}) - \mathbf{b} \times (\mathbf{a} \times \mathbf{r})$
$= (\mathbf{b} \times \mathbf{r}) \times \mathbf{a} + (\mathbf{a} \times \mathbf{r}) \times \mathbf{b}.$

Problem 99. *If* **r** *and r have their usual meaning show that*
(a) div $r^n\mathbf{r} = (n+3)\, r^n$, *(b) curl* $r^n\mathbf{r} = 0$. *(Agra, 1974)*

Suppose that $r^n\mathbf{r} = u\mathbf{V}$ then

$$\begin{aligned}
\text{div}\,(r^n\mathbf{r}) &= \text{div}\,(u\mathbf{V}) = u\,\text{div}\,\mathbf{V} + (\text{grad}\,u).\,\mathbf{V} \\
&= r^n\,\text{div}\,\mathbf{r} + (\text{grad}\,r^n) \cdot \mathbf{r} \\
&= r^n\,(3) + (nr^{n-2}\mathbf{r}) \cdot \mathbf{r} \\
&= 3r^n + nr^{n-2}r^2 = (n+3)\, r^n.
\end{aligned}$$

$$\begin{aligned}
(b)\ \text{curl}\,(r^n\mathbf{r}) &= \text{curl}\,(u\mathbf{V}) \\
&= (\text{grad}\,u) \times \mathbf{V} + u\,\text{curl}\,\mathbf{V} \\
&= (\text{grad}\,r^n) \times \mathbf{r} + r^n\,\text{curl}\,\mathbf{r} \\
&= (nr^{n-2}\mathbf{r}) \times \mathbf{r} + 0 \quad \because \text{curl}\,\mathbf{r} = 0 \\
&= 0 \qquad \text{as } \mathbf{r} \times \mathbf{r} = 0.
\end{aligned}$$

Problem 100. *If* $\mathbf{a} = \alpha x\mathbf{i} + \beta y\mathbf{j} + \gamma z\mathbf{k}$, *show that*
(a) $\nabla\,(\mathbf{a} \cdot \mathbf{r}) = 2\mathbf{a}$.
(b) If **r** *and r have their usual meaning prove*
curl $(\mathbf{a} \cdot \mathbf{r})\, \mathbf{a} = 0$.

(*a*) $\mathbf{a} \cdot \mathbf{r} = (\alpha x\mathbf{i} + \beta y\mathbf{j} + \gamma z\mathbf{k}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$
$= (\alpha x^2 + \beta y^2 + \gamma z^2)$

$$\text{grad}\,(\mathbf{a} \cdot \mathbf{r}) = \left(\mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}\right)(\alpha x^2 + \beta y^2 + \gamma z^2)$$
$$= (2\alpha x\mathbf{i} + 2\beta y\mathbf{j} + 2\gamma z\mathbf{k}) = 2\mathbf{a}.$$

(*b*) If $a = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

then $\mathbf{a} \cdot \mathbf{r} = (a_1x + a_2y + a_3z)$

So that $(\mathbf{a} \cdot \mathbf{r})\, \mathbf{a} = (a_1x + a_2y + a_2z)\,(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})$
$= (a_1x + a_2y + a_3z)\, a_1\mathbf{i} + (a_1x + a_2y + a_3z)\, a_2\mathbf{j}$
$+ (a_1x + a_2y + a_3z)\, a_3\mathbf{k}$

and hence

$$\text{curl}\,[(\mathbf{a} \cdot \mathbf{r})\, \mathbf{a}] = \left(\mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}\right) \times [(a_1x + a_2y + a_3z)\, a_1\mathbf{i} + \ldots + \ldots]$$
$$= \left\{\frac{\partial}{\partial y}\left[(a_1x + a_2y + a_3z)\, a_3\right] - \frac{\partial}{\partial z}\left[(a_1x + a_2y + a_3z)\, a_2\right]\right\}\mathbf{i} + \ldots + \ldots$$
$$= (a_2a_3 - a_3a_2)\,\mathbf{i} + \ldots + \ldots$$
$$= 0.$$

Problem 101. *If* u, **V**, w *are point functions and* $u\mathbf{V} = \nabla w$, *prove that*
$\mathbf{V} \cdot$ *curl* $\mathbf{V} = 0$.

We are given that

$$u\mathbf{V} = \nabla w.$$

Taking curl of both sides we have

$$\text{curl}\,(u\mathbf{V}) = \text{curl}\,(\text{grad}\, w) = 0$$

or $$(\text{grad}\, u) \times \mathbf{V} + u\, \text{curl}\, \mathbf{V} = 0 \quad \text{by Prob. 91}(ii)$$

Multiplying by V scalarly, we find

$$\mathbf{V} \cdot (\text{grad}\, u) \times \mathbf{V} + \mathbf{V} \cdot u\, \text{curl}\, \mathbf{V} = 0$$

or $$\mathbf{V} \cdot u\, \text{curl}\, \mathbf{V} = 0 \text{ as } \mathbf{V} \cdot (\text{grad}\, u) \times \mathbf{V} = 0$$

or $$\mathbf{V} \cdot \text{curl} = 0 \qquad \text{as } u \neq 0.$$

Problem 102. *If* **a** *is a constant unit vector show that*

$$\mathbf{a} \cdot [\nabla (\mathbf{V} \cdot \mathbf{a}) - \nabla \times (\mathbf{V} \times \mathbf{a})] = \nabla \cdot \mathbf{V} = div\, \mathbf{V}$$

We have $\text{grad}\,(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\,\mathbf{B} + (\mathbf{B} \cdot \nabla)\,\mathbf{A} + \mathbf{A} \times \text{curl}\,\mathbf{B} + \mathbf{B} \times \text{curl}\,\mathbf{A}$

$$\nabla (\mathbf{V} \cdot \mathbf{a}) = (\mathbf{V} \cdot \nabla)\,\mathbf{a} + (\mathbf{a} \cdot \nabla)\,\mathbf{V} + \mathbf{V} \times \text{curl}\,\mathbf{a} + \mathbf{a} \times \text{curl}\,\mathbf{V}$$

$$= (\mathbf{V} \cdot \nabla)\,\mathbf{a} + (\mathbf{a} \cdot \nabla)\,\mathbf{V} + \mathbf{a} \times \text{curl}\,\mathbf{V} \qquad \ldots(1)$$

But a being a constant vector curl a = 0.

Also we have $\text{curl}\,(\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\,\mathbf{A} - (\mathbf{A} \cdot \nabla)\,\mathbf{B} + \mathbf{A}\, \text{div}\, \mathbf{B} - \mathbf{B}\, \text{div}\, \mathbf{A}$

So that $\nabla \times (\mathbf{V} \times \mathbf{a}) = (\mathbf{a} \cdot \nabla)\,\mathbf{V} - (\mathbf{V} \cdot \nabla)\,\mathbf{a} + \mathbf{V}\, \text{div}\, \mathbf{a} - \mathbf{a}\, \text{div}\, \mathbf{V}$

$$= (\mathbf{a} \cdot \nabla)\,\mathbf{V} - (\mathbf{V} \cdot \nabla)\,\mathbf{a} - \mathbf{a}\, \text{div}\, \mathbf{V} \qquad \ldots(2)$$

$\because$ div a = 0, a being constant

Subtraction of (2) from (1) yields

$$\nabla (\mathbf{V} \cdot \mathbf{a}) - \nabla \times (\mathbf{V} \times \mathbf{a}) = 2\,(\mathbf{V} \cdot \nabla)\,\mathbf{a} + \mathbf{a} \times \text{curl}\,\mathbf{V} + \mathbf{a}\, \text{div}\, \mathbf{V}$$

Where $(\mathbf{V} \cdot \nabla)\,\mathbf{a} = \left(v_1 \dfrac{\partial}{\partial x} + v_2 \dfrac{\partial}{\partial y} + v_3 \dfrac{\partial}{\partial z} \right) \mathbf{a}$; if $\mathbf{V} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$

= 0, a being constant

Hence, $\mathbf{a} \cdot \{\nabla(\mathbf{V} \cdot \mathbf{a}) - \nabla \times (\mathbf{V} \times \mathbf{a})\} = \mathbf{a} \cdot \{\mathbf{a} \times \text{curl}\,\mathbf{V} + \mathbf{a}\, \text{div}\, \mathbf{V}\}$

$$= \mathbf{a} \cdot \mathbf{a} \times \text{curl}\,\mathbf{V} + \mathbf{a} \cdot \mathbf{a}\, \text{div}\, \mathbf{V}$$

$$= \text{div}\, \mathbf{V} \quad \because \mathbf{a} \cdot \mathbf{a} = 1, \text{ a being unit vector}$$

And $\mathbf{a} \cdot \mathbf{a} \times \text{curl}\,\mathbf{V} = \mathbf{a} \times \mathbf{a} \cdot \text{curl}\,\mathbf{V} = 0.$

Problem 103. *Prove that div* $(u \text{ grad } v) = u\, \nabla^2 v + (\text{grad } u) \cdot (\text{grad } v)$ *where u and v both are scalar point functions.*

We have $\text{div}\,(u\, \text{grad}\, v) = \text{div}\,(u\mathbf{V})$ where $\mathbf{V} = \text{grad}\, v$

$$= u\, \text{div}\, \mathbf{V} + \mathbf{V} \cdot \text{grad}\, u$$

$$= u\, \text{div}\,(\text{grad}\, v) + (\text{grad}\, v) \cdot (\text{grad}\, u)$$

$$= u\, \nabla \cdot (\nabla v) + (\text{grad}\, v) \cdot (\text{grad}\, u)$$

$$= u\, \nabla^2 v + (\text{grad}\, v) \cdot (\text{grad}\, u).$$

Problem 104. ***Prove that curl*** $(u \text{ grad } v) = \nabla u \times \nabla v = -\,curl\,(v \text{ grad } u)$ ***where u and v both are scalar point functions.***

We have

curl $(u\, grad\, v) = \text{curl}\,(u\, \mathbf{V})$ where $\mathbf{V} = \text{grad}\, v$

$$= u\, \text{curl}\, \mathbf{V} - \mathbf{V} \times \text{grad}\, u$$

$$= u\, \text{curl}\,(\text{grad}\, v) - \text{grad}\, v \times \text{grad}\, u$$

$$= 0 - \text{grad}\, v \times \text{grad}\, u.$$

$$= \text{grad}\, u \times \text{grad}\, v$$

$$= \nabla u \times \nabla v = -\,(\nabla v \times \nabla u) = -\,\text{curl}\,(v\, \text{grad}\, u).$$

Problem 105. *Prove that* $\nabla \cdot (S\, \nabla \times \mathbf{A}) = (\nabla \times \mathbf{A}) \cdot (\nabla S).$

We have $\nabla \cdot (S\, \nabla \times \mathbf{A}) = \text{div}\,(S\, \text{curl}\, \mathbf{A})$

$= \text{div}\,(S\mathbf{V})$ where $\text{curl}\,\mathbf{A} = \mathbf{V}$
$= S\,\text{div}\,\mathbf{V} + (\text{grad}\,S)\cdot\mathbf{V}$
$= S\,\text{div}\,(\text{curl}\,\mathbf{A}) + (\text{grad}\,S)\cdot(\text{curl}\,\mathbf{A})$
$= 0 + (\text{grad}\,S)\cdot(\text{curl}\,\mathbf{A})$
$= (\nabla\times\mathbf{A})\cdot(\nabla S).$

Problem 106. *Prove that*
curl $(\nabla u\times\nabla v) = \nabla u(\nabla\cdot\nabla v) - \nabla v(\nabla\cdot\nabla u) + (\nabla v\,\nabla)\,\nabla u - (\nabla u\,\nabla)\,\nabla v$
where u and v both are scalar point functions.

Suppose that $\nabla u = \mathbf{A}$ and $\nabla v = \mathbf{B}$ then,

$$\text{curl}\,(\nabla u\times\nabla v) = \text{curl}\,(\mathbf{A}\times\mathbf{B})$$
$$= \mathbf{A}\,\text{div}\,\mathbf{B} - \mathbf{B}\,\text{div}\,\mathbf{A} + (\mathbf{B}\,\text{grad})\,\mathbf{A} - (\mathbf{A}\,\text{grad})\,\mathbf{B}$$
$$= \nabla u(\nabla\cdot\nabla v) - \nabla v(\nabla\cdot\nabla u) + (\nabla v\,\nabla)\,\nabla u - (\nabla u\,\nabla)\,\nabla v.$$

Problem 107. *Prove that the values of div* **F** *and curl* **F** *are independent of the choice of rectangular axes i.e. they are invariant.* *(Lucknow, 1952, 59)*

Taking the mutual direction cosines of two sets of rectangular axes as shown in adjoining scheme and (x, y, z), (x', y', z') as corresponding coordinates, we have

$$\left.\begin{aligned}\mathbf{i}' &= l_1\mathbf{i} + m_1\mathbf{j} + n_1\mathbf{k}\\ x' &= l_1x + m_1y + n_1z \text{ etc.}\end{aligned}\right\}\quad\ldots(1)$$

$$\text{And } \left.\begin{aligned}\mathbf{i} &= l_1\mathbf{i}' + l_2\mathbf{j}' + l_3\mathbf{k}'\\ x &= l_1x' + l_2y' + l_3z' \text{ etc.}\end{aligned}\right\}\quad\ldots(2)$$

		x	y	z
		i	**j**	**k**
x'	**i**′	l_1	m_1	n_1
y'	**j**′	l_2	m_2	n_2
z'	**k**′	l_3	m_3	n_3

Also $$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial}{\partial y'}\frac{\partial y'}{\partial x} + \frac{\partial}{\partial z'}\frac{\partial z'}{\partial x} = l_1\frac{\partial}{\partial x'} + l_2\frac{\partial}{\partial y'} + l_3\frac{\partial}{\partial z'}$$

Similarly $$\frac{\partial}{\partial y} = m_1\frac{\partial}{\partial x'} + m_2\frac{\partial}{\partial y'} + m_3\frac{\partial}{\partial z'}\;;\quad \frac{\partial}{\partial z} = n_1\frac{\partial}{\partial x'} + n_2\frac{\partial}{\partial y'} + n_3\frac{\partial}{\partial z'}$$

Using $l_1^2 + l_2^2 + l_3^2 = 1$ etc. and $l_1m_1 + l_2m_2 + l_3m_3 = 0$ etc.,

We have $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} = F_1\,(l_1\mathbf{i}' + l_2\mathbf{j}' + l_3\mathbf{k}') + \ldots + \ldots$

$$= (l_1F_1 + m_1F_2 + n_1F_3)\,\mathbf{i}' + (l_2F_1 + m_2F_2 + n_2F_3)\,\mathbf{j}' + (l_3F_1 + m_3F_2 + n_3F_3)\mathbf{k}'$$

So that $$(\text{div}\,\mathbf{F})_{x',y',z'} = \frac{\partial}{\partial x'}(l_1F_1 + m_1F_2 + n_1F_3) + \frac{\partial}{\partial y'}(l_2F_1 + m_2F_2 + n_2F_3) + \frac{\partial}{\partial z'}(l_3F_1 + m_3F_2 + n_3F_3)$$

$$= \left(l_1\frac{\partial}{\partial x'} + l_2\frac{\partial}{\partial y'} + l_3\frac{\partial}{\partial z'}\right)F_1 + \ldots + \ldots$$

$$= \frac{\partial}{\partial x'}F_1 + \frac{\partial}{\partial y'}F_2 + \frac{\partial}{\partial z'}F_3 = (\text{div}\,\mathbf{F})_{x,y,z}$$

Similarly $(\text{curl}\,\mathbf{F})_{x',y',z'} = (\text{curl}\,\mathbf{F})_{x,y,z}$

Which show the required invariance.

Problem 108. *Find div grad r^m and verify that $\nabla \times \nabla r^m = 0$.*

If $r = |\mathbf{r}|$, $\mathbf{r}$ being position vector of a point, then

$$\nabla r^m = m r^{m-2} \mathbf{r}.$$

Also, div grad

$$\begin{aligned} r^m &= \nabla \cdot (\nabla r^m) \\ &= \nabla \cdot (m r^{m-2} \mathbf{r}) \\ &= m \, \nabla \cdot (r^{m-2} \mathbf{r}) \\ &= m \, [\nabla r^{m-2} \cdot \mathbf{r} + r^{m-2} \operatorname{div} \mathbf{r}] \\ &= m \, [(m-2) r^{m-1} r \cdot \mathbf{r} + r^{m-2} \cdot 3] \qquad \because \operatorname{div} \mathbf{r} = 3 \\ &= m(m-2) r^{m-4} r^2 + 3m r^{m-2} \\ &= m(m-2) r^{m-2} + 3m r^{m-2} \\ &= m(m+1) r^{m-2}. \end{aligned}$$

and curl grad

$$\begin{aligned} r^m &= \nabla \times (m r^{m-2} \mathbf{r}) \\ &= \nabla \times [m r^{m-2} x \mathbf{i} + m r^{m-2} y \mathbf{j} + m r^{m-2} z \mathbf{k}] \\ &= \left(\mathbf{i} \times \frac{\partial}{\partial x} + \mathbf{j} \times \frac{\partial}{\partial y} + \mathbf{k} \times \frac{\partial}{\partial z} \right) [m r^{m-2} x \mathbf{i} + m r^{m-2} y \mathbf{j} + m r^{m-2} z \mathbf{k}] \\ &= \left[\frac{\partial}{\partial y} (m r^{m-2} z) - \frac{\partial}{\partial z} (m r^{m-2} y) \right] \mathbf{i} + \ldots + \ldots \\ &= \left[m(m-2) \, r^{m-3} z \frac{\partial r}{\partial y} - m(m-2) \, r^{m-3} y \frac{\partial r}{\partial z} \right] \mathbf{i} + \ldots + \ldots \\ &= \left[m(m-2) \, r^{m-3} z \frac{y}{r} - m(m-2) \, r^{m-3} y \frac{z}{r} \right] \mathbf{i} + \ldots + \ldots \\ &= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = 0. \end{aligned}$$

Problem 109. *Show that* $(\mathbf{V} \cdot \nabla) \mathbf{V} = \frac{1}{2} \nabla V^2 - \mathbf{V} \times$ *curl* $\mathbf{V}$.

(Allahabad, 1958)

We have grad $(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \times \operatorname{curl} \mathbf{b} + \mathbf{b} \times \operatorname{curl} \mathbf{a} + (\mathbf{a} \cdot \nabla) \mathbf{b} + (\mathbf{b} \cdot \nabla) \mathbf{a}$

Putting $\mathbf{a} = \mathbf{b} = \mathbf{V}$, this becomes

grad $(\mathbf{V}^2) = \mathbf{V} \times \operatorname{curl} \mathbf{V} + \mathbf{V} \times \operatorname{curl} \mathbf{V} + (\mathbf{V} \cdot \nabla) \mathbf{V} + (\mathbf{V} \cdot \nabla) \mathbf{V}$

or $\quad \nabla V^2 = 2 \, \mathbf{V} \times \operatorname{curl} \mathbf{V} + 2 (\mathbf{V} \cdot \nabla) \mathbf{V}$

or $\quad (\mathbf{V} \cdot \nabla) \mathbf{V} = \frac{1}{2} \nabla V^2 - \mathbf{V} \times \operatorname{curl} \mathbf{V}$.

Problem 110. (*a*) *Show that* $\mathbf{E} = \frac{\mathbf{r}}{r^2}$ *is irrotational.*

(*b*) *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are irrotational, prove that* $\mathbf{A} \times \mathbf{B}$ *is solenoidal.*

As given, $\quad \mathbf{E} = \dfrac{\mathbf{r}}{r^2} = \dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{x^2 + y^2 + z^2}$.

In order to show that $\mathbf{E}$ is irrotational, we have to prove that

$$\nabla \times \left(\frac{\mathbf{r}}{r^2} \right) = 0.$$

Now,

$$\nabla \times \left(\frac{\mathbf{r}}{r^2}\right) = \nabla \times \left(\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{x^2 + y^2 + z^2}\right)$$

$$= \left(\mathbf{i}\ \frac{\partial}{\partial x} + \mathbf{j}\ \frac{\partial}{\partial y} + \mathbf{k}\ \frac{\partial}{\partial z}\right) \times \left[\frac{x}{x^2 + y^2 + z^2}\ \mathbf{i} + \frac{y}{x^2 + y^2 + z^2}\ \mathbf{j} + \frac{z}{x^2 + y^2 + z^2}\mathbf{k}\right]$$

$$= \left[\frac{\partial}{\partial y}\left(\frac{z}{x^2 + y^2 + z^2}\right) - \frac{\partial}{\partial z}\left(\frac{y}{x^2 + y^2 + z^2}\right)\right]\mathbf{i}$$

$$+ \left[\frac{\partial}{\partial z}\left(\frac{x}{x^2 + y^2 + z^2}\right) - \frac{\partial}{\partial x}\left(\frac{z}{x^2 + y^2 + z^2}\right)\right]\mathbf{j}$$

$$+ \left[\frac{\partial}{\partial x}\left(\frac{y}{x^2 + y^2 + z^2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{x^2 + y^2 + z^2}\right)\right]\mathbf{k}$$

$$= \left[\frac{-2zy}{x^2 + y^2 + z^2} + \frac{2zy}{x^2 + y^2 + z^2}\right]\mathbf{i} + \ldots + \ldots$$

$$= 0.$$

showing that **E** is an irrotational vector.

(*b*) If **A** and **B** are irrotational, then

$$\text{curl } \mathbf{A} = 0 \text{ and curl } \mathbf{B} = 0.$$

In order to prove that $\mathbf{A} \times \mathbf{B}$ is solenoidal, we have to show that

$$\text{div } (\mathbf{A} \times \mathbf{B}) = 0.$$

Now div $(\mathbf{A} \times \mathbf{B}) = (\text{curl } \mathbf{A}) \cdot \mathbf{B} - (\text{curl } \mathbf{B}) \cdot \mathbf{A} = 0.$

Hence $\mathbf{A} \times \mathbf{B}$ is solenoidal.

Problem 111. *Show that r^n. **r** is an irrotational vector for any value of n, but is a solenoidal only if n = – 3.* (*Agra, 1959, 79*)

We have curl $r^n\mathbf{r} = r^n \text{curl } \mathbf{r} - \mathbf{r} \times \text{grad } r^n$

$$= -\mathbf{r} \times (nr^{n-2}\mathbf{r}) \quad \text{as curl } \mathbf{r} = 0$$

$$= \mathbf{r} \times \mathbf{r}\ (nr^{n-2}) = 0.$$

showing that $r^n\mathbf{r}$ is an irrotational vector for any value of n.

Again div $r^n\mathbf{r} = r^n \text{ div } \mathbf{r} + \mathbf{r} \cdot \text{grad } r^n$

$$= 3r^n + \mathbf{r} \cdot (nr^{n-2}\mathbf{r})$$

$$= 3r^n + nr^{n-2}\, r^2 = (n + 3)r^n$$

which is zero if $n + 3 = 0$ or $n = -3$

This shows that $r^n\mathbf{r}$ is solenoidal when $n = -3$.

Problem 112. *Prove that* $\nabla \times (\nabla \times \mathbf{A}) = -\nabla^2\mathbf{A} + \nabla(\nabla \cdot \mathbf{A})$.

Take $\nabla = \mathbf{R}$

So that $\nabla \times (\nabla \times \mathbf{A}) = \mathbf{R} \times (\mathbf{R} \times \mathbf{A})$

But $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}\ (\mathbf{a} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{b})\ \mathbf{c}$

$\therefore$ $\mathbf{R} \times (\mathbf{R} \times \mathbf{A}) = \mathbf{R}\ (\mathbf{R} \cdot \mathbf{A}) - (\mathbf{R} \cdot \mathbf{R})\ \mathbf{A}$

or

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla\ (\nabla \cdot \mathbf{A}) - (\nabla \cdot \nabla)\ \mathbf{A}$$

$$= \nabla\ (\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}.$$

Problem 113. *If* $\nabla \cdot \mathbf{E} = 0$, $\nabla \cdot \mathbf{H} = 0$, $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t}$, $\nabla \times \mathbf{H} = \frac{\partial \mathbf{E}}{\partial t}$, *show that* **E** *and* **H** *satisfy* $\nabla^2 u = \frac{\partial^2 u}{\partial t^2}$.

As given,

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla \times \left(-\frac{\partial \mathbf{H}}{\partial t}\right)$$

$$= -\frac{\partial}{\partial t}(\nabla \times \mathbf{H})$$

$$= -\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{E}}{\partial t}\right) = -\frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

But by Problem 112, we have

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla^2 \mathbf{E} + \nabla(\nabla \cdot \mathbf{E}) = -\nabla^2 \mathbf{E}$$

So that $$\nabla^2 \mathbf{E} = \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

Similarly $$\nabla \times (\nabla \times \mathbf{H}) = \nabla \times \left(\frac{\partial \mathbf{E}}{\partial t}\right)$$

$$= \frac{\partial}{\partial t}(\nabla \times \mathbf{E}) = \frac{\partial}{\partial t}\left(-\frac{\partial \mathbf{H}}{\partial t}\right) = -\frac{\partial^2 \mathbf{H}}{\partial t^2}.$$

But $$\nabla \times (\nabla \times \mathbf{H}) = -\nabla^2 \mathbf{H} + \nabla(\nabla \cdot \mathbf{H}) = -\nabla^2 \mathbf{H}$$

So that $$\nabla^2 \mathbf{H} = \frac{\partial^2 \mathbf{H}}{\partial t^2}$$

i.e. E and H satisfy the equation $\nabla^2 u = \frac{\partial^2 u}{\partial t^2}$.

Problem 114. *Show that the solution to the Maxwell's equations*

$$\nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t},\ \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t},\ \nabla \cdot \mathbf{H} = 0,\ \nabla \cdot \mathbf{E} = 4\pi\rho$$ *where* ρ *is a function of* x, y, z *and* c *is the velocity of light assumed to be constant, are given by*

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t},\quad \mathbf{H} = \nabla \times \mathbf{A},$$

where **A** *and* ϕ *called the vector and scalar potentials respectively, satisfy the equations*

(1) $\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0,$ (2) $\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 4\pi\rho,$

(3) $\nabla^2 \mathbf{A} = \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2}.$

Maxwell's equations are given to be

$$\nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \quad \ldots(1)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} \qquad ...(2)$$

$$\nabla \cdot \mathbf{H} = 0 \qquad ...(3)$$

and $$\nabla \cdot \mathbf{E} = 4\pi\rho \qquad ...(4)$$

We have to show that the solutions of these equations are given by

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \qquad ...(5)$$

and $$\mathbf{H} = \nabla \times \mathbf{A} \qquad ...(6)$$

where A and ϕ are given by $$\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0, \qquad ...(7)$$

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 4\pi\rho, \qquad ...(8)$$

and $$\nabla^2 \mathbf{A} = \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2}. \qquad ...(9)$$

Putting $\mathbf{H} = \nabla \times \mathbf{A}$ from (6), we have

L.H.S. of (3) $= \nabla \cdot \nabla \times \mathbf{A}$

$= [\nabla\ \nabla\ \mathbf{A}]$

$= 0$ by the property of scalar triple product.

This shows that the equation (6) is a solution of (3).

Again putting $\mathbf{H} = \nabla \times \mathbf{A}$ in (2), we get

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \mathbf{A})$$

or $$\nabla \times \left(\mathbf{E} + \frac{1}{c}\frac{\partial A}{\partial t}\right) = 0$$ which shows that the bracketed expression is the gradient of some scalar function say ϕ and therefore,

$$\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = -\text{ grad } \phi = -\nabla\phi$$

i.e. $$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

which is the equation (5) showing that equation (5) is a solution of (2).

1.36. CURVILINEAR CO-ORDINATES

We know that the equations

$$u = f_1(x, y, z)\ ;\ v = f_2(x, y, z)\ ;\ w = f_3(x, y, z)$$

where u, v, w are parameters, represent three families of surfaces when expressed in the form

$$u = \text{const.},\ v = \text{const.},\ w = \text{const.}, \qquad ...(1)$$

where u, v, w are continuously differentiable functions defined in any region R of space.

Suppose that the three surfaces u = const., v = const., w = const., intersect in a point P of the region R. The values of u, v, w for the three surfaces intersecting at P are called the *curvilinear co-ordinates* of the point P. The three surfaces are then known as *co-ordinate surfaces*. The three surfaces intersect pairwise in three curves known as *co-ordinate curves*. Only one co-ordinate is variable on each of the co-ordinate curves. The curve on which u varies known as u-curve and similarly v-curve and w-curve are those on which v and w respectively vary. One variable is constant on each of the co-ordinate surfaces. The surface on which u is constant is known as u-surface and similarly v-surface and w-surface are those on which v and w respectively are constant.

Using the equation (1) the rectangular co-ordinates (x, y, z) and therefore the position vector **r** of any point in the region of space may be expressed in terms of curvilinear co-ordinates. Since there is a one to one correspondence between x, y, z and u, v, w the position vector **r** is a vector function of u, v, w.

Note. The loci of $u = C_1, v = C_2, w = C_3$; C_1, C_2, C_3 being constants represent the co-ordinate surfaces and the equations of the co-ordinate curves then are

$$v = C_2, w = C_3;\ w = C_3, u = C_1;\ u = C_1, v = C_2.$$

1.37. ORTHOGONAL CURVILINEAR CO-ORDINATES

A system of orthogonal curvilinear co-ordinates is one which corresponds to the points of intersection of a triply orthogonal system of three families of surfaces

$$u\,(x, y, z) = \text{const.},\ v\,(x, y, z) = \text{const.},\ w\,(x, y, z) = \text{const.}$$

which are such that, through each point P in any region R of space passes one and only one member of each family, each of the three surfaces cutting the other two orthogonally. In short the curvilinear co-ordinates u, v, w are said to be orthogonal if the co-ordinate curves are mutually perpendicular at every point $P\,(x, y, z)$ of space.

Let us suppose that $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ form a right handed system of unit vectors tangent to the co-ordiante curves u, v, w respectively at P and directed towards increasing u, v, w. Then we have

$$\left.\begin{aligned} \mathbf{e}_1 = \mathbf{e}_2 \times \mathbf{e}_3,\ \mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_1 \text{ and } \mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2 \\ \mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_2 \cdot \mathbf{e}_3 = \mathbf{e}_3 \cdot \mathbf{e}_1 = 0 \end{aligned}\right\} \quad \text{...(1)}$$

Let the arc lengths measured along the co-ordinate curves in the positive directions of u, v, w be respectively s_1, s_2, s_3. Now consider an infinitesimal parallelopiped whose diagonal is the element of arc ds along a curve tangent to PQ at P and faces coincide with planes u, v or w and length of edges are ds_1, ds_2, ds_3. Therefore,

$$ds^2 = ds_1^2 + ds_2^2 + ds_3^3. \quad \text{...(2)}$$

Let us now introduce the three numbers h_1, h_2, h_3 known as *metrical coefficients* with the property

$$\frac{ds_1}{du} = h_1,\ \frac{ds_2}{dv} = h_2,\ \frac{ds_3}{dw} = h_3$$

i.e.
$$ds_1 = h_1 du, ds_2 = h_2 dv, ds_3 = h_3 dw. \quad \text{...(3)}$$

Substituting, the values of ds_1, ds_2, ds_3 from (3) in (2) we get

$$ds^2 = h_1^2\, du^2 + h_2^2\, dv^2 + h_3^2\, dw^2. \quad \text{...(4)}$$

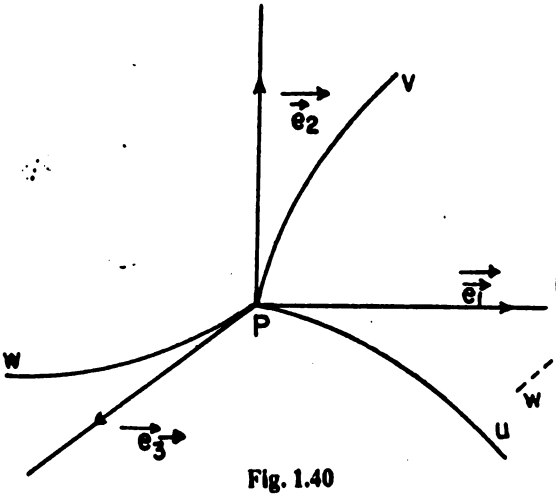


Fig. 1.40

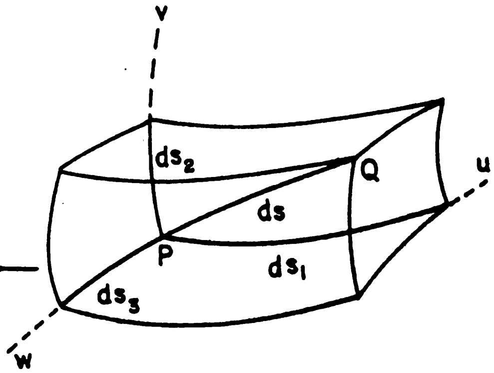


Fig. 1.41

Now if $\mathbf{r}$ be the position vector of P, referred to the origin of a rectangular co-ordinate system, the tangents to the co-ordinate curves at P are parallel to the directions of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and have the magnitudes h_1, h_2, h_3 respectively. Therefore,

$$\frac{\partial \mathbf{r}}{\partial u} = h_1 \mathbf{e}_1, \quad \frac{\partial \mathbf{r}}{\partial v} = h_2 \mathbf{e}_2, \quad \frac{\partial \mathbf{r}}{\partial w} = h_3 \mathbf{e}_3. \qquad \ldots(5)$$

These give, $\quad \dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v} = h_1 h_2 \, \mathbf{e}_1 \times \mathbf{e}_2$

$$= h_1 h_2 \mathbf{e}_3 \qquad \text{from (1)}$$

i.e. $\quad \dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v} = \dfrac{h_1 h_2}{h_3} \dfrac{\partial \mathbf{r}}{\partial w}$ from (5)

Similarly $\quad \dfrac{\partial \mathbf{r}}{\partial v} \times \dfrac{\partial \mathbf{r}}{\partial w} = \dfrac{h_2 h_3}{h_1} \dfrac{\partial \mathbf{r}}{\partial u}$,, $\qquad \ldots(6)$

and $\quad \dfrac{\partial \mathbf{r}}{\partial w} \times \dfrac{\partial \mathbf{r}}{\partial u} = \dfrac{h_3 h_1}{h_2} \dfrac{\partial \mathbf{r}}{\partial v}$,,

Also $\quad \left[\dfrac{\partial \mathbf{r}}{\partial u} \; \dfrac{\partial \mathbf{r}}{\partial v} \; \dfrac{\partial \mathbf{r}}{\partial w}\right] = \left(\dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v}\right) \cdot \dfrac{\partial \mathbf{r}}{\partial w} = h_1 h_2 \mathbf{e}_3 \cdot h_3 \mathbf{e}_3 = h_1 h_2 h_3. \qquad \ldots(7)$

1.38. CONDITION FOR ORTHOGONALITY

We have mentioned that the curvilinear co-ordinates of a point in space are determined by three continuously differentiable scalar functions

$$u = f_1(x, y, z), \; v = f_2(x, y, z), \; w = f_3(x, y, z)$$

when these functions u, v, w are not functionally related, then these equations can be solved to give x, y, z in terms of u, v, w such that

$$x = g_1(u, v, w), \; y = g_2(u, v, w), \; z = g_3(u, v, w)$$

where x, y, z are continuously differentiable functions of u, v, w.

Now the position vector $\mathbf{r}$ of any point in space, referred to the origin of rectangular axes along which the unit vectors are $\mathbf{i}, \mathbf{j}, \mathbf{k}$ is expressed as

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$= \mathbf{i}g_1(u, v, w) + \mathbf{j}g_2(u, v, w) + \mathbf{k}g_3(u, v, w)$

$= \mathbf{F}(u, v, w)$ say.

Now if $u = C_1, v = C_2, w = C_2$ where C_1, C_2, C_3 are constants, represent the co-ordinate surfaces, then the co-ordinate curves are

$$v = C_2, w = C_3;\ w = C_3, u = C_1;\ u = C_1, v = C_2$$

The co-ordinate curve $v = C_2, w = C_3$ through (C_1, C_2, C_3) is the same as

$\mathbf{r} = \mathbf{F}(u, C_2, C_3)$, u being the parameter.

Now the tangent to the curve $v = C_2, w = C_3$ is parallel to the vector $\dfrac{\partial \mathbf{r}}{\partial u}$.

Similarly the tangents to the curves $w = C_3, u = C_1$ and $u = C_1, v = C_2$ are respectively parallel to the vectors $\dfrac{\partial \mathbf{r}}{\partial v}$ and $\dfrac{\partial \mathbf{r}}{\partial w}$.

Since the dot product of two parallel vectors is zero, it therefore follows that the curvilinear co-ordiante system will be orthogonal if

$$\frac{\partial \mathbf{r}}{\partial u}\cdot\frac{\partial \mathbf{r}}{\partial v} = \frac{\partial \mathbf{r}}{\partial v}\cdot\frac{\partial \mathbf{r}}{\partial w} = \frac{\partial \mathbf{r}}{\partial w}\cdot\frac{\partial \mathbf{r}}{\partial u} = 0. \qquad \ldots(1)$$

These are the required conditions for orthogonality.

COROLLARY. The line element ds derived in relation (4) of § 1.37, may be deduced from the conditions of orthogonality.

We have $\mathbf{r} = \mathbf{F}(u, v, w)$

so that $$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u}du + \frac{\partial \mathbf{r}}{\partial v}dv + \frac{\partial \mathbf{r}}{\partial w}\cdot dw$$

$$\therefore\ d\mathbf{r}\cdot d\mathbf{r} = \left\{\frac{\partial \mathbf{r}}{\partial u}du + \frac{\partial \mathbf{r}}{\partial v}dv + \frac{\partial \mathbf{r}}{\partial w}dw\right\}\cdot\left\{\frac{\partial \mathbf{r}}{\partial u}du + \frac{\partial \mathbf{r}}{\partial v}dv + \frac{\partial \mathbf{r}}{\partial w}dw\right\}$$

$$= \left(\frac{\partial \mathbf{r}}{\partial u}\right)^2 du^2 + \left(\frac{\partial \mathbf{r}}{\partial v}\right)^2 dv^2 + \left(\frac{\partial \mathbf{r}}{\partial w}\right)^2 dw^2$$

$$+ 2\frac{\partial \mathbf{r}}{\partial u}\cdot\frac{\partial \mathbf{r}}{\partial v}du\,dv + 2\frac{\partial \mathbf{r}}{\partial v}\cdot\frac{\partial \mathbf{r}}{\partial w}dv\,dw + 2\frac{\partial \mathbf{r}}{\partial w}\cdot\frac{\partial \mathbf{r}}{\partial u}dw\,du \qquad \ldots(2)$$

Applying the conditions of orthogonality, this reduces to

$$(d\mathbf{r})^2 = \left(\frac{\partial \mathbf{r}}{\partial u}\right)^2 du^2 + \left(\frac{\partial \mathbf{r}}{\partial v}\right)^2 dv^2 + \left(\frac{\partial \mathbf{r}}{\partial w}\right)^2 dw^2 \qquad \ldots(3)$$

Putting $(d\mathbf{r})^2 = |d\mathbf{r}|^2 = ds^2$

$$\left(\frac{\partial \mathbf{r}}{\partial u}\right)^2 = \left|\frac{d\mathbf{r}}{\partial u}\right|^2 = h_1^2,\ \left(\frac{\partial \mathbf{r}}{\partial v}\right)^2 = h_2^2,\ \left(\frac{\partial \mathbf{r}}{\partial w}\right)^2 = h_3^2.$$

The relation (3) yields

$$ds^2 = h_1^2 du^2 + h_2^2 dv^2 + h_3^2 dw^2.$$

1.39. RECIPROCAL SETS OF TWO TRIADS OF MUTUALLY ORTHOGONAL VECTORS

If u, v, w be a set of curvilinear co-ordinates of a point P whose position vector is $\mathbf{r}$ with respect to the origin of a rectangular co-ordinate system, then the two sets of triads of *mutually orthogonal vectors*, ∇u, ∇v, ∇w *and* $\frac{\partial \mathbf{r}}{\partial u}, \frac{\partial \mathbf{r}}{\partial v}, \frac{\partial \mathbf{r}}{\partial w}$ ***are reciprocal to each other.***

We have shown in equation (7) of §1.37 that if $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ form a right handed system of unit vectors tangent to the co-ordinate curves u, v, w respectively at P and directed towards increasing u, v, w, then

$$\left[\frac{\partial \mathbf{r}}{\partial u}\ \frac{\partial \mathbf{r}}{\partial v}\ \frac{\partial \mathbf{r}}{\partial w}\right] = h_1 h_2 h_3. \qquad ...(1)$$

Now ∇u, ∇v, ∇w are the vectors lying along the normals to the co-ordinate surfaces which are the level surfaces of the functions u, v, w. But the curvilinear co-ordinate system is orthogonal, therefore the conditions of orthogonality stated in §1.38 when applied to the orthogonal system of vectors ∇u, ∇v, ∇w, yield

$$\nabla u \cdot \nabla v = 0,\ \nabla v \cdot \nabla w = 0,\ \nabla w \cdot \nabla u = 0. \qquad ...(2)$$

Let us now assume that the mutually orthogonal unit vectors are

$$\frac{\nabla u}{|\nabla u|}, \frac{\nabla v}{|\nabla v|}, \frac{\nabla w}{|\nabla w|} \qquad ...(3)$$

which form a right handed system.

Here $|\nabla u|$ is the directional derivative of u along the direction of the normal to the surface $u = C_1$ *i.e.* along the tangent to the curve $v = C_2$, $w = C_3$. Hence if ds_1 represents the differential of length along this curve, then we can state

$$|\nabla u| = \frac{du}{ds_1}.$$

But from the Corollary of §1.38 the line element ds_1 along the curve $v = C_2$, $w = C_3$ will be obtained by putting $dv = 0$, $dw = 0$ in $ds^2 = h_1^2 du^2 + h_2^2 dv^2 + h_3^2 dw^2$, whence we have

$$ds_1^2 = h_1^2 du^2$$

giving $$\frac{du}{ds_1} = u_1 \frac{1}{h_1},$$

so that $$|\nabla u| = \frac{1}{h_1}, \quad i.e.\ h_1 = \frac{1}{|\nabla u|}.$$

Similarly, $$h_2 = \frac{1}{|\nabla v|} \text{ and } h_3 = \frac{1}{|\nabla w|} \qquad ...(4)$$

Since, $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is a system of orthogonal unit vectors therefore (3) and (4) give

$$\frac{\nabla u}{|\nabla u|} = \frac{\nabla u}{1/h_1} = \mathbf{e}_1 \quad i.e.\ \nabla u = \frac{\mathbf{e}_1}{h_1}$$

and similarly $$\nabla v = \frac{\mathbf{e}_2}{h_2},\ \nabla w = \frac{\mathbf{e}_3}{h_3}. \qquad ...(5)$$

So that $$[\nabla u\ \nabla v\ \nabla w] = (\nabla u \times \nabla v) \cdot \nabla w$$

$$= \left(\frac{\mathbf{e}_1}{h_1} \times \frac{\mathbf{e}_2}{h_2}\right) \cdot \frac{\mathbf{e}_3}{h_3}$$

$$= \frac{\mathbf{e}_3}{h_1 h_2} \cdot \frac{\mathbf{e}_3}{h_3} \text{ as } \mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3 \text{ and } \mathbf{e}_3 \cdot \mathbf{e}_3 = 1 \qquad \ldots(6)$$

$$= \frac{1}{h_1 h_2 h_3}.$$

Multiplying (1) and (6), together, we get

$$[\nabla u \nabla v \nabla w]\left[\frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v} \cdot \frac{\partial \mathbf{r}}{\partial w}\right] = 1$$

which follows that ∇u, ∇v, ∇w form a set of vectors reciprocal to

$$\frac{\partial \mathbf{r}}{\partial u}, \frac{\partial \mathbf{r}}{\partial v}, \frac{\partial \mathbf{r}}{\partial w}.$$

1.40. GRADIENT IN TERMS OF ORTHOGONAL CURVILINEAR COORDINATES

Let $\phi(u, v, w)$ be any scalar point function given in terms of orthogonal curvilinear coordinates u, v, w.

Since u, v, w may be supposed to be the functions of rectangular Cartesian coordinates x, y, z, therefore

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial \phi}{\partial w}\frac{\partial w}{\partial x} \qquad \ldots(1)$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial \phi}{\partial w}\frac{\partial w}{\partial y} \qquad \ldots(2)$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial \phi}{\partial w}\frac{\partial w}{\partial z} \qquad \ldots(3)$$

Let us now suppose that $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit vectors along the rectangular axes x, y, z respectively and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are the mutually orthogonal unit vectors along the tangents to the coordinate curves u, v, w. Then by relation (5) of §1.39, we have

$$\nabla u = \frac{\mathbf{e}_1}{h_1}, \quad \nabla v = \frac{\mathbf{e}_2}{h_2}, \quad \nabla w = \frac{\mathbf{e}_3}{h_3} \qquad \ldots(4)$$

Now multiplying (1) by $\mathbf{i}$, (2) by $\mathbf{j}$ and (3) by $\mathbf{k}$ and then adding all together we get

$$\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\phi = \frac{\partial \phi}{\partial x}\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)u$$

$$\frac{\partial \phi}{\partial v}\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)v + \frac{\partial \phi}{\partial w}\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)w$$

i.e.,
$$\nabla \phi = \frac{\partial \phi}{\partial u}\nabla u + \frac{\partial \phi}{\partial v}\nabla v + \frac{\partial \phi}{\partial w}\nabla w,$$

$$= \frac{1}{h_1}\frac{\partial \phi}{\partial u}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial \phi}{\partial v}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial \phi}{\partial w}\mathbf{e}_3 \text{ by (4)} \qquad \ldots(5)$$

which is the required expression. It is obvious that the components of grad ϕ *i.e.* $\nabla \phi$ along the unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are respectively

$$\frac{1}{h_1}\frac{\partial \phi}{\partial u}, \frac{1}{h_2}\frac{\partial \phi}{\partial v}, \frac{1}{h_3}\frac{\partial \phi}{\partial w}.$$

(Agra, 1971)

1.41. DIVERGENCE IN TERMS OF ORTHOGONAL CURVILINEAR COORDINATES (*Agra, 1971*)

Let **F** (u, v, w) be a vector point function given in terms of orthogonal curvilinear coordinates u, v, w and let F_1, F_2, F_3 be the components of **F** along $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ the unit vectors along u, v, w axes. Then,

$$\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3$$

$$= F_1\mathbf{e}_2 \times \mathbf{e}_3 + F_2\mathbf{e}_3 \times \mathbf{e}_1 + F_3\mathbf{e}_1 \times \mathbf{e}_2 \quad \text{since } \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \text{ are mutually orthogonal vectors}$$

$$= F_1 h_2 h_3 \nabla v \times \nabla w + F_2 h_3 h_1 \nabla w \times \nabla u + F_3 h_1 h_2 \nabla u \times \nabla v \quad \text{by (5) of §1.39}$$

$$\therefore \quad \nabla \cdot \mathbf{F} = \nabla \cdot (F_1 h_2 h_3 \nabla v \times \nabla w + F_2 h_3 h_1 \nabla w \times \nabla u + F_3 h_1 h_2 \nabla u \times \nabla v).$$

$$= \nabla \cdot (F_1 h_2 h_3 \nabla v \times \nabla w) + \nabla \cdot (F_2 h_3 h_1 \nabla w \times \nabla u) + \nabla \cdot (F_3 h_1 h_2 \nabla u \times \nabla v) \quad \ldots(1)$$

By the properties of divergence and curl, we have

$$\nabla \cdot (F_1 h_2 h_3 \nabla v \times \nabla w) = F_1 h_2 h_3 \nabla \cdot (\nabla v \times \nabla w) + \nabla v \times \nabla w \cdot \nabla (F_1 h_2 h_3)$$

where $\nabla \cdot (\nabla v \times \nabla w) = \nabla w \cdot \text{curl } \nabla v - \nabla v \cdot \text{curl } \nabla w$
$= 0$

and $\nabla (F_1 h_2 h_3) = \dfrac{\partial}{\partial u}(F_1 h_2 h_3)\nabla u + \dfrac{\partial}{\partial v}(F_1 h_2 h_3)\nabla v + \dfrac{\partial}{\partial w}(F_1 h_2 h_3)\nabla w.$

So that $\nabla \cdot (F_1 h_2 h_3 \nabla v \times \nabla w)$

$$= \nabla v \times \nabla w \cdot \left\{\frac{\partial}{\partial u}(F_1 h_2 h_3)\nabla u + \frac{\partial}{\partial v}(F_1 h_2 h_3)\nabla v + \frac{\partial}{\partial w}(F_1 h_2 h_3)\nabla w\right\}$$

$= \nabla v \times \nabla w \cdot \nabla u \dfrac{\partial}{\partial u}(F_1 h_2 h_3)$ other terms vanish, by the property of scalar triple product.

$= \dfrac{\mathbf{e}_2}{h_2} \times \dfrac{\mathbf{e}_3}{h_3} \cdot \dfrac{\mathbf{e}_1}{h_1} \dfrac{\partial}{\partial u}(F_1 h_2 h_3)$ by (5) of §1.39.

$$= \frac{1}{h_1 h_2 h_3}[\mathbf{e}_2\mathbf{e}_3\mathbf{e}_1]\frac{\partial}{\partial u}(F_1 h_2 h_3)$$

$= \dfrac{1}{h_1 h_2 h_3}\dfrac{\partial}{\partial u}(F_1 h_2 h_3)$ since $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ being unit vectors $[\mathbf{e}_2\mathbf{e}_3\mathbf{e}_1] = 1$

Similarly $\nabla \cdot (F_3 h_1 h_2 \nabla w \times \nabla u) = \dfrac{1}{h_1 h_2 h_3}\dfrac{\partial}{\partial v}(F_2 h_3 h_1)$

and $\nabla \cdot (F_3 h_1 h_2 \nabla u \times \nabla v) = \dfrac{1}{h_1 h_2 h_3}\dfrac{\partial}{\partial w}(F_3 h_1 h_2)$

Substituting these values in (1) we get

$$\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u}(F_1 h_2 h_3) + \frac{\partial}{\partial v}(F_2 h_3 h_1) + \frac{\partial}{\partial w}(F_3 h_1 h_2)\right] \quad \ldots(2)$$

which gives the required expression for div **F**.

1.42. CURL IN TERMS OF ORTHOGONAL CURVILINEAR COORDINATES (*Agra, 1971, 74*)

Let **F** (u, v, w) be a vector function given in terms of orthogonal curvilinear coordinates u, v, w and let F_1, F_2, F_3 be the components of **F** along $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ the unit vectors along u, v, w axes. Then,

$$\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3$$

$$= h_1F_1\nabla u + h_2F_2\nabla v + h_3F_3\nabla w \text{ by (5) of §1.39}$$

$$\therefore \quad \nabla\times\mathbf{F} = \nabla\times\{h_1F_1\nabla u + h_2F_2\nabla v + h_3F_3\nabla w\}$$

$$= \nabla\times(h_1F_1\nabla u) + \nabla\times(h_2F_2\nabla v) + \nabla\times(h_3F_3\nabla w) \qquad \ldots(1)$$

But, we have by the properties of curl

$$\nabla\times(h_1F_1\nabla u) = \nabla(h_1F_1)\times\nabla u + h_1F_1\nabla\times\nabla u$$

$$= \nabla(h_1F_1)\times\nabla u \text{ since } \nabla\times\nabla u = 0$$

$$= \left\{\frac{\partial}{\partial u}(h_1F_1)\nabla u + \frac{\partial}{\partial v}(h_1F_1)\nabla v + \frac{\partial}{\partial w}(h_1F_1)\nabla w\right\}\times\nabla u$$

$$= \frac{\partial}{\partial v}(h_1F_1)\nabla v\times\nabla u + \frac{\partial}{\partial w}(h_1F_1)\nabla w\times\nabla u,$$

the other term vanishes

$$= \frac{1}{h_1h_2}\frac{\partial}{\partial v}(h_1F_1)\,\mathbf{e}_2\times\mathbf{e}_1 + \frac{1}{h_3h_1}\frac{\partial}{\partial w}(h_1F_1)\,\mathbf{e}_3\times\mathbf{e}_1$$

by (5) of §1.39

$$= -\frac{1}{h_1h_2}\frac{\partial}{\partial v}(h_1F_1)\,\mathbf{e}_3 + \frac{1}{h_3h_1}\frac{\partial}{\partial w}(h_1F_1)\mathbf{e}_2.$$

Similarly

$$\nabla\times(h_2F_2\nabla v) = -\frac{1}{h_3h_3}\frac{\partial}{\partial w}(h_2F_2)\,\mathbf{e}_1 + \frac{1}{h_2h_1}\frac{\partial}{\partial u}(h_1F_1)\mathbf{e}_3$$

and

$$\nabla\times(h_3F_3\nabla w) = -\frac{1}{h_3h_1}\frac{\partial}{\partial w}(h_2F_3)\,\mathbf{e}_2 + \frac{1}{h_3h_2}\frac{\partial}{\partial v}(h_3F_3)\mathbf{e}_1$$

Substituting these values in (1), we get

$$\nabla\times\mathbf{F} = \frac{1}{h_2h_3}\left[\frac{\partial}{\partial v}(h_3F_3) - \frac{\partial}{\partial w}(h_2F_2)\right]\mathbf{e}_1 + \frac{1}{h_3h_1}\left[\frac{\partial}{\partial w}(h_1F_1) - \frac{\partial}{\partial u}(h_3F_3)\right]\mathbf{e}_2$$

$$+ \frac{1}{h_1h_2}\left[\frac{\partial}{\partial u}(h_2F_2) - \frac{\partial}{\partial v}(h_1F_1)\right]\mathbf{e}_3$$

$$= \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix} \qquad \ldots(2)$$

which is the required expression for curl **F**.

This result may also be expressed as

$$\nabla\times\mathbf{F} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ h_1\mathbf{e}_1\cdot\mathbf{F} & h_2\mathbf{e}_2\cdot\mathbf{F} & h_3\mathbf{e}_3\cdot\mathbf{F} \end{vmatrix} \quad \text{since } \mathbf{e}_1\cdot\mathbf{F} = F_1 \text{ etc.}$$

1.43. LAPLACIAN (∇^2) IN TERMS OF ORTHOGONAL CURVILINEAR CO-ORDINATES

By (5) of §1.40, we have

$$\nabla \phi = \frac{1}{h_1}\frac{\partial \phi}{\partial u}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial \phi}{\partial v}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial \phi}{\partial w}\mathbf{e}_3 .$$

$$\therefore \quad \nabla^2 \phi = \nabla \cdot \nabla \phi$$

$$= \left(\mathbf{e}_1\frac{\partial}{\partial u} + \mathbf{e}_2\frac{\partial}{\partial v} + \mathbf{e}_3\frac{\partial}{\partial w}\right)\left(\frac{1}{h_1}\frac{\partial \phi}{\partial u}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial \phi}{\partial v}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial}{\partial w}\mathbf{e}_3\right)$$

$$= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u}\left(\frac{h_2 h_3}{h_1}\frac{\partial \phi}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_3 h_1}{h_2}\frac{\partial \phi}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_1 h_2}{h_3}\frac{\partial \phi}{\partial w}\right)\right]$$

which is the required expression.

1.44. EQUIVALENT EXPRESSIONS FOR $\nabla\phi$, $\nabla\cdot\mathbf{F}$ AND $\nabla\times\mathbf{F}$ IN RECTANGULAR CO-ORDINATES

In §§1.40, 1.41 and 1.42 we have derived the expressions for $\nabla\phi$, $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$ in terms of curvilinear co-ordinate system as follows:

$$\nabla \phi = \frac{1}{h_1}\frac{\partial \phi}{\partial u}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial \phi}{\partial v}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial \phi}{\partial w}\mathbf{e}_3 \qquad ...(1)$$

$$\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u}(F_1 h_2 h_3) + \frac{\partial}{\partial v}(F_2 h_3 h_1) + \frac{\partial}{\partial w}(F_3 h_1 h_2)\right] \qquad ...(2)$$

and

$$\nabla \times \mathbf{F} = \frac{1}{h_2 h_3}\left[\frac{\partial}{\partial v}(h_3 F_3) - \frac{\partial}{\partial w}(h_2 F_2)\right]\mathbf{e}_1 + \frac{1}{h_3 h_1}\left[\frac{\partial}{\partial w}(h_1 F_1) - \frac{\partial}{\partial u}(h_3 F_3)\right]\mathbf{e}_2 + \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial u}(h_2 F_2) - \frac{\partial}{\partial v}(h_1 F_1)\right]\mathbf{e}_3 \qquad ...(3)$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are unit vectors along u, v, w axes.

In order to get the equivalent expressions for these quantities in cartesian rectangular co-ordinates, if we use the transformations,

$$u = x, \; v = y, \; w = z$$

we have $\quad ds^2 = dx^2 + dy^2 + dz^2$

so that $\quad h_1 = h_2 = h_3 = 1,$

and the unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are taken as usual unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

With these substitutions, the relations (1), (2) and (3) become

$$\nabla \phi = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$$

$$\nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

and

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

1.45. CYLINDRICAL CO-ORDINATES AS A SPECIAL CURVILINEAR SYSTEM

Let P be a point in space such that its curvilinear co-ordinates are (u, v, w) and cartesian co-ordiantes are (x, y, z). Let the projection of P in the z-plane be Q whose polar co-ordinates in the plane are (r, θ). Then the *circular* or *cylindrical* co-ordinates of the point P are specified by

$u = r, v = \theta, w = z.$

These co-ordinates are transformed to cartesian co-ordinates by the help of the relations

$x = r = \cos\theta,$

$y = r\sin\theta, z = z.$

i.e. $r^2 = x^2 + y^2, \theta = \tan^{-1}\dfrac{y}{x}, z = z.$

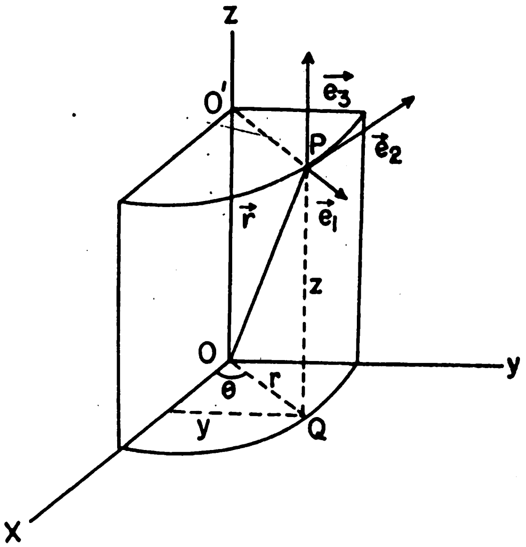


Fig. 1.42

Since x is a function of r, θ, z, we have

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta + \frac{\partial x}{\partial z}dz$$

$$= \cos\theta\, dr - r\sin\theta\, d\theta.$$

Similarly $\qquad dy = \sin\theta\, dr + r\cos\theta\, d\theta$

and $\qquad dz = dz.$

Therefore, the relation $(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$ gives

$$(ds)^2 = (\cos\theta\, dr - r\sin\theta\, d\theta)^2 + (\sin\theta\, dr + r\cos\theta\, d\theta)^2 + (dz)^2$$

$$= (dr)^2 + r^2(d\theta)^2 + (dz)^2 \qquad ...(1)$$

Also from relation (4) of §1.37, we have

$$ds^2 = h_1{}^2 du^2 + h_2{}^2 dv^2 + h_3{}^2 dw^2$$

which when transformed by the substitutions $u = r, \; v = \theta, \; w = z$, becomes

$$ds^2 = h_1{}^2 dr^2 + h_2{}^2 d\theta^2 + h_3{}^2 dz^2 \qquad ...(2)$$

Comparing (1) and (2), we get

$$h_1 = 1, \; h_2 = r, \; h_3 = 1.$$

With these substitutions, the functions $\nabla\phi$, $\nabla^2\phi$, $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$ in cylindrical coordinates become

$$\nabla\phi = \frac{\partial\phi}{\partial r}e_1 + \frac{1}{r}\frac{\partial\phi}{\partial r}e_2 + \frac{\partial\phi}{\partial z}e_3$$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} + \frac{1}{r}\frac{\partial\phi}{\partial r}$$

$$\nabla \cdot \mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_1) + \frac{1}{r}\frac{\partial F_2}{r\,\partial\theta} + \frac{\partial F_3}{\partial z} \qquad \text{(Rohilkhand, 1976, 93)}$$

$$\nabla \times \mathbf{F} = \left(\frac{1}{r}\frac{\partial F_3}{\partial\theta} - \frac{\partial F_2}{\partial z}\right)e_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial r}\right)e_2$$

$$+\left\{\frac{1}{r}\frac{\partial}{\partial r}(rF_2) - \frac{1}{r}\frac{\partial F_1}{\partial\theta}\right\}e_3.$$

(Agra, 1974; Rohilkhand, 1976)

1.46. SPHERICAL POLAR COORDINATES AS A SPECIAL CURVILINEAR SYSTEM

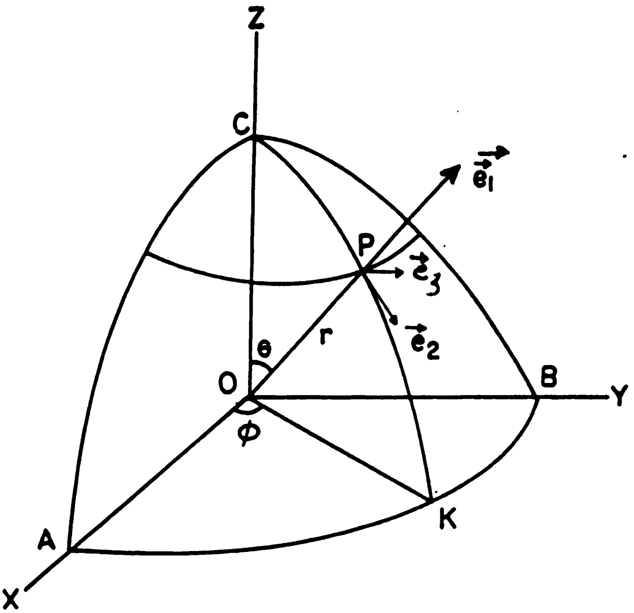


Fig. 1.43

Let P be a point in space such that its curvilinear coordinates are $(u, v, w,)$ and cartesian coordinates are (x, y, z). Then the *spherical polar* coordinates of the point P are specified by

$$u = r, \quad v = \theta, \quad w = \phi$$

where r ($= OP$) is the distance of the point P from the origin, θ is the angle between OP and the z-axis and ϕ is the angle included between the xz plane and the plane OPZ.

These coordiantes are transformed to cartesian coordinates by the help of the relations

$$x = r \sin\theta \cos\phi,$$
$$y = r \sin\theta \sin\phi,$$
$$z = r \cos\theta.$$

We have,

$$dx = \frac{\partial x}{\partial r}\, dr + \frac{\partial x}{\partial \theta}\, d\theta + \frac{\partial x}{\partial \phi}\, d\phi$$
$$= \sin\theta \cos\phi\, dr + r\cos\theta \cos\phi\, d\theta - r\sin\theta \sin\phi\, d\phi.$$

Similarly

$$dy = \sin\theta \sin\phi\, dr + r\cos\theta \sin\phi\, d\theta + r\sin\theta\cos\phi\, d\phi$$

and

$$dz = \cos\theta\, dr - r\sin\theta\, d\theta.$$

$\therefore$ $(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$ gives

$$(ds)^2 = (dr)^2 + r^2 (d\theta)^2 + r^2 \sin^2\theta\, (d\phi)^2. \qquad \ldots(1)$$

Also with these transformations, the relation (4) of §1.37 in view of (2) of §1.45 becomes

$$(ds)^2 = h_1^2 (dr)^2 + h_2^2 (d\theta)^2 + h_3^2 (d\phi)^2 \qquad \ldots(2)$$

Comparing (1) and (2) we get

$$h_1 = 1, \quad h_2 = r, \quad h_3 = r\sin\theta.$$

With these substitutions, the functions $\nabla\psi$, $\nabla^2\psi$, $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$ in spherical polar coordinates become

$$\nabla\psi = \frac{\partial\psi}{\partial r}\mathbf{e}_1 + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\mathbf{e}_2 + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\mathbf{e}_3$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial\phi^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial\psi}{\partial\theta}$$

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_1\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, F_2\right) + \frac{1}{r\sin\theta}\cdot\frac{\partial F_3}{\partial\phi}$$

(*Rohilkhand, 1976, 93*)

$$\nabla \times \mathbf{F} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\, F_3\right) - \frac{\partial F_2}{\partial\phi}\right]\mathbf{e}_1$$

$$+\frac{1}{r}\left[\frac{\partial}{\sin\theta}\frac{\partial F_1}{\partial\phi}-\frac{\partial}{\partial r}(rF_3)\right]\mathbf{e}_2+\frac{1}{r}\left[\frac{\partial}{\partial r}(rF_2)-\frac{\partial F_1}{\partial\theta}-\right]\mathbf{e}_3.$$

(*Rohilkhand, 1976*)

Problem 115. *Find an expression for ds^2 in curvilinear coordinates u, v, w. Then determine ds^2 for the special case of an orthogonal system.*

Let the position vector of a point in space be **r**, where **r** is a vector function of *u*, *v*, *w*. Then.

$$ds^2 = d\mathbf{r}\cdot d\mathbf{r} \text{ (by the assumption in §1.38)}$$

$$=\left(\frac{\partial\mathbf{r}}{\partial u}du+\frac{\partial\mathbf{r}}{\partial v}dv+\frac{\partial\mathbf{r}}{\partial w}dw\right)^2$$

$$=\left(\frac{\partial\mathbf{r}}{\partial u}\right)^2du^2+\left(\frac{\partial\mathbf{r}}{\partial v}\right)^2dv^2+\left(\frac{\partial\mathbf{r}}{\partial w}\right)^2dw^2+2\frac{\partial\mathbf{r}}{\partial u}\cdot\frac{\partial\mathbf{r}}{\partial v}du\,dv$$

$$+2\frac{\partial\mathbf{r}}{\partial v}\cdot\frac{\partial\mathbf{r}}{\partial w}dv\,dw+2\frac{\partial\mathbf{r}}{\partial w}\cdot\frac{\partial\mathbf{r}}{\partial u}dw\,du.$$

But by §1.38 the conditions of orthogonality are

$$\frac{\partial\mathbf{r}}{\partial u}\cdot\frac{\partial\mathbf{r}}{\partial v}=\frac{\partial\mathbf{r}}{\partial v}\cdot\frac{\partial\mathbf{r}}{\partial w}=\frac{\partial\mathbf{r}}{\partial w}\cdot\frac{\partial\mathbf{r}}{\partial v}=0.$$

So that the above relation becomes

$$ds^2=\left(\frac{\partial\mathbf{r}}{\partial u}\right)^2du^2+\left(\frac{\partial\mathbf{r}}{\partial v}\right)^2dv^2+\left(\frac{\partial\mathbf{r}}{\partial w}\right)^2dw^2.$$

Problem 116. *If $x = uv\cos w$, $y = uv\sin w$, $z = \frac{1}{2}(u^2 - v^2)$; find h_1, h_2, h_3 and show that $ds^2 = (u^2 + v^2)(du^2 + dv^2) + uv\,dw^2$.*

We have

$$dx=\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv+\frac{\partial x}{\partial w}dw$$

$$=u\cos w\,du+v\cos w\,dv-uv\sin w\,dw$$

$$dy=\frac{\partial y}{\partial u}du+\frac{\partial y}{\partial v}dv+\frac{\partial y}{\partial w}dw$$

$$=u\sin w\,du+v\sin w\,dv+uv\cos w\,dw$$

$$dz=\frac{\partial z}{\partial u}du+\frac{\partial z}{\partial v}dv+\frac{\partial z}{\partial w}dw$$

$$=u\,du-v\,dv+0\cdot dw.$$

$$\therefore\quad ds^2=dx^2+dy^2+dz^2$$

$$=(u\cos w\,du+v\cos w\,dv-uv\sin w\,dw)^2$$

$$+(u\sin w\,du+v\sin w\,dv+uv\cos w\,dw)^2+(u\,du-v\,dv)^2$$

$$=(u^2+v^2)\,du^2+(u^2+v^2)\,dv^2+u^2v^2\,dw^2.$$

Now comparing this relation with

$$ds^2=h_1^2du^2+h_2^2dv^2+h_3^2dw^2.$$

We get $h_1=\sqrt{(u^2+v^2)}$, $h_2=\sqrt{(u^2+v^2)}$, $h_3=uv$.

Problem 117. *If $u = 2x + 3$, $v = y - 4$, $w = z + 2$, show that u, v, w are orthogonal and find ds^2 and the metrical coefficients h_1, h_2, h_3.*

The given relations can be expressed as

$$x=\frac{u}{2}-\frac{3}{2},\quad y=v+4,\quad z=w-2.$$

Then, if **r** be the position vector of a point in space,

$$\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$$

$$= \left(\frac{u}{2} - \frac{3}{2}\right) \mathbf{i} + (v+4)\ \mathbf{j} + (w-2)\ \mathbf{k}$$

So that $\dfrac{\partial \mathbf{r}}{\partial u} = \frac{1}{2}\ \mathbf{i},\ \dfrac{\partial \mathbf{r}}{\partial v} = \mathbf{j},\ \dfrac{\partial \mathbf{r}}{\partial w} = \mathbf{k}.$

The system of coordinates u, v, w to be orthogonal, we must have

$$\frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v} = 0,\ \frac{\partial \mathbf{r}}{\partial v} \cdot \frac{\partial \mathbf{r}}{\partial w} = 0,\ \frac{\partial \mathbf{r}}{\partial w} \cdot \frac{\partial \mathbf{r}}{\partial u} = 0$$

Here $\qquad \dfrac{\partial \mathbf{r}}{\partial u} \cdot \dfrac{\partial \mathbf{r}}{\partial v} = \frac{1}{2}\ \mathbf{i} \cdot \mathbf{j} = 0.$

Similarly $\dfrac{\partial \mathbf{r}}{\partial u} \cdot \dfrac{\partial \mathbf{r}}{\partial w} = 0$ and $\dfrac{\partial \mathbf{r}}{\partial w} \cdot \dfrac{\partial \mathbf{r}}{\partial u} = 0.$

Hence u, v, w are orthogonal.

Now, to find ds^2, we have

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv + \frac{\partial x}{\partial w} dw$$

$$= \frac{1}{2} du$$

Similarly $\qquad dy = dv$, and $dz = dw$

$\therefore \qquad ds^2 = dx^2 + dy^2 + dz^2$

$$= \frac{1}{2} du^2 + dv^2 + dw^2.$$

Again to find h_1, h_2, h_3 let us compare the last relation with

$$ds^2 = h_1^2\, du^2 + h_2^2\, dv^2 + h_3^2\, dw^2.$$

Whence, we get

$$h_1 = \frac{1}{2},\ h_2 = 1,\ h_3 = 1.$$

Problem 118. *For spherical coordinates*

$$x = r \sin\theta \cos\phi,\ y = r \sin\theta \sin\phi,\ z = r\cos\theta.$$

(*a*) *Prove that the components of* $\dfrac{\partial \mathbf{r}}{\partial r}, \dfrac{\partial \mathbf{r}}{\partial \theta}, \dfrac{\partial \mathbf{r}}{\partial \phi}$ *are given by*

$$\frac{\partial \mathbf{r}}{\partial r} = \sin\theta \cos\phi\ \mathbf{i} + \sin\theta \sin\phi\ \mathbf{j} + \cos\theta\ \mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial \theta} = r\cos\theta \cos\phi\ \mathbf{i} + r\cos\theta \sin\phi\ \mathbf{j} - r\sin\theta\ \mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial \phi} = -r\sin\theta \sin\phi\ \mathbf{i} + r\sin\theta\cos\phi\ \mathbf{j}.$$

(*b*) *Verify the mutual orthogonality of* $\dfrac{\partial \mathbf{r}}{\partial r}, \dfrac{\partial \mathbf{r}}{\partial \theta}, \dfrac{\partial \mathbf{r}}{\partial \phi}$.

(*c*) *Find expressions for* $\nabla r, \nabla\theta, \nabla\phi$ *and then show that they constitute a set of vectors reciprocal to* $\dfrac{\partial \mathbf{r}}{\partial r}, \dfrac{\partial \mathbf{r}}{\partial \theta}, \dfrac{\partial \mathbf{r}}{\partial \phi}$.

If $\mathbf{r}$ be the position vector of a point in space, then

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

(*a*) We have

$$\frac{\partial \mathbf{r}}{\partial r}=\frac{\partial x}{\partial r}\,\mathbf{i}+\frac{\partial y}{\partial r}\,\mathbf{j}+\frac{\partial z}{\partial r}\,\mathbf{k}$$

$$=\sin\theta\cos\phi\,\mathbf{i}+\sin\theta\sin\phi\,\mathbf{j}+\cos\theta\,\mathbf{k}$$

and

$$\frac{\partial \mathbf{r}}{\partial \theta}=\frac{\partial x}{\partial \theta}\,\mathbf{i}+\frac{\partial y}{\partial \theta}\,\mathbf{j}+\frac{\partial z}{\partial \theta}\,\mathbf{k}$$

$$=r\cos\theta\cos\phi\,\mathbf{i}+r\cos\theta\sin\phi\,\mathbf{j}-r\sin\theta\,\mathbf{k}.$$

also

$$\frac{\partial \mathbf{r}}{\partial \phi}=\frac{\partial x}{\partial \phi}\,\mathbf{i}+\frac{\partial y}{\partial \phi}\,\mathbf{j}+\frac{\partial z}{\partial \phi}\,\mathbf{k}$$

$$=-r\sin\theta\sin\phi\,\mathbf{i}+r\sin\theta\cos\phi\,\mathbf{j}+0\cdot\mathbf{k}.$$

(*b*) Taking the values of $\frac{\partial \mathbf{r}}{\partial r}$, $\frac{\partial \mathbf{r}}{\partial \theta}$ and $\frac{\partial \mathbf{r}}{\partial \phi}$ from (*a*) we have

$$\frac{\partial \mathbf{r}}{\partial r}\cdot\frac{\partial \mathbf{r}}{\partial \theta}=(\sin\theta\cos\phi\,\mathbf{i}+\sin\theta\sin\phi\,\mathbf{j}+\cos\theta\,\mathbf{k})\cdot(r\cos\theta\cos\phi\,\mathbf{i}+r\cos\theta\sin\phi\,\mathbf{j}-r\sin\theta\,\mathbf{k})$$

$$=r\sin\theta\cos\theta\cos^2\phi+r\sin\theta\cos\theta\sin^2\phi-r\sin\theta\cos\theta$$

$$=r\sin\theta\cos\theta-r\sin\theta\cos\theta$$

$$=0.$$

Similarly $\frac{\partial \mathbf{r}}{\partial \theta}\cdot\frac{\partial \mathbf{r}}{\partial \phi}=0$ and $\frac{\partial \mathbf{r}}{\partial \phi}\cdot\frac{\partial \mathbf{r}}{\partial r}=0.$

Since all the scalar products vanish, it follows that the vectors $\frac{\partial \mathbf{r}}{\partial r}$, $\frac{\partial \mathbf{r}}{\partial \theta}$, $\frac{\partial \mathbf{r}}{\partial \phi}$ are mutually orthogonal.

(*c*) We have

$$r^2=x^2+y^2+z^2,\qquad \theta=\cos^{-1}\frac{z}{\sqrt{(x^2+y^2+z^2)}}\ \text{and}\ \phi=\tan^{-1}\frac{y}{x}.$$

$$\therefore\qquad \nabla r=\frac{\partial r}{\partial x}\,\mathbf{i}+\frac{\partial r}{\partial y}\,\mathbf{j}+\frac{\partial r}{\partial z}\,\mathbf{k}$$

$$=\frac{x}{r}\,\mathbf{i}+\frac{y}{r}\,\mathbf{j}+\frac{z}{r}\,\mathbf{k}=\sin\theta\cos\phi\,\mathbf{i}+\sin\theta\sin\phi\,\mathbf{j}+\cos\theta\,\mathbf{k}$$

$$\nabla\theta=\frac{\partial\theta}{\partial x}\,\mathbf{i}+\frac{\partial\theta}{\partial y}\,\mathbf{j}+\frac{\partial\theta}{\partial z}\,\mathbf{k}=\frac{\cos\theta\cos\phi}{r}\,\mathbf{i}+\frac{\cos\theta\sin\phi}{r}\,\mathbf{j}-\frac{\sin\theta}{r}\,\mathbf{k}$$

$$\nabla\phi=\frac{\partial\phi}{\partial x}\,\mathbf{i}+\frac{\partial\phi}{\partial y}\,\mathbf{j}+\frac{\partial\phi}{\partial z}\,\mathbf{k},$$

$$=-\frac{\sin\phi}{r\sin\theta}\,\mathbf{i}+\frac{\cos\phi}{r\sin\theta}\,\mathbf{j}+0\cdot\mathbf{k}.$$

So that $[\nabla r\ \nabla\theta\ \nabla\phi]=(\nabla r\times\nabla\theta)\cdot\nabla\phi$

$$=\left(-\frac{\sin\phi}{r}\,\mathbf{i}+\frac{\cos\phi}{r}\,\mathbf{j}+0\cdot\mathbf{k}\right)\cdot\left(-\frac{\sin\phi}{r\sin\theta}\,\mathbf{i}+\frac{\cos\phi}{r\sin\theta}\,\mathbf{j}+0\cdot\mathbf{k}\right)$$

$$=\frac{\sin^2\phi}{r^2\sin\theta}+\frac{\cos^2\phi}{r^2\sin\theta}=\frac{1}{r^2\sin\theta}.$$

Also $\left[\frac{\partial \mathbf{r}}{\partial r}, \frac{\partial \mathbf{r}}{\partial \theta}, \frac{\partial \mathbf{r}}{\partial \phi}\right] = \left(\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \phi}\right) \cdot \frac{\partial \mathbf{r}}{\partial \phi}$

$= (-r \sin \phi\, \mathbf{i} + r \cos \phi\, \mathbf{j} + 0 \cdot \mathbf{k}) \cdot (-r \sin \theta \sin \phi\, \mathbf{i} + r \sin \theta \cos \phi\, \mathbf{j})$

$= r^2 \sin \theta \sin^2 \phi + r^2 \sin \theta \cos^2 \phi$

$= r^2 \sin \theta.$

$$\therefore \quad [\nabla r \;\nabla \theta \;\nabla \phi] \left[\frac{\partial \mathbf{r}}{\partial r} \;\frac{\partial \mathbf{r}}{\partial \theta} \;\frac{\partial \mathbf{r}}{\partial \phi}\right] = 1$$

which shows that ∇r, $\nabla \theta$, $\nabla \phi$ is a set of vectors reciprocal to

$$\frac{\partial \mathbf{r}}{\partial r}, \frac{\partial \mathbf{r}}{\partial \theta}, \frac{\partial \mathbf{r}}{\partial \phi}.$$

1.47. INTEGRATION OF A VECTOR

We know that integration is the reverse process to differentiation. Thus if **F** and **r** be two vector functions such that the derivative of **F** with respect to t is equal to **r**,

i.e.
$$\frac{d\mathbf{F}}{dt} = \mathbf{r},$$

then **F** + **C** is called the *indefinite integral* or simply the *integral* of **r** with regard to t and is denoted by

$$\int \mathbf{r}\, dt = \mathbf{F} + \mathbf{C}$$

where the constant vector **C** is known as the constant of integration.

In order to apply the integration to definite problems the value of **C** may be determined from some initial or geometrical conditions.

The process of finding a vector **F** whose derivative with repect to t is equal to **r** is known as *integration* of a vector.

In general the nature of the constant of integration is the same as that of the integrand.

We thus write down the values of the following integrals :

$$\int \left(\mathbf{r} \cdot \frac{d\mathbf{s}}{dt} + \mathbf{s} \cdot \frac{d\mathbf{r}}{dt}\right) dt = \mathbf{r} \cdot \mathbf{s} + c \text{ where } c \text{ is a scalar}$$

$$\int 2\mathbf{r} \cdot \frac{d\mathbf{r}}{dt}\, dt = \int \frac{d\mathbf{r}^2}{dt}\, dt = \mathbf{r}^2 + c = \mathbf{r} \cdot \mathbf{r} + c, \; c \text{ being a scalar.}$$

$$\int 2\frac{d\mathbf{r}}{dt} \cdot \frac{d^2\mathbf{r}}{dt^2}\, dt = \frac{d\mathbf{r}}{dt} \cdot \frac{d\mathbf{r}}{dt} + c = \left(\frac{d\mathbf{r}}{dt}\right)^2 + c, \; c \text{ being a scalar}$$

$$\int \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}\, dt = \mathbf{r} \times \frac{d\mathbf{r}}{dt} + \mathbf{c}, \text{ where } \mathbf{c} \text{ is a vector.}$$

$$\int \left(\frac{1}{r} \frac{d\mathbf{r}}{dt} - \frac{dr}{dt} \frac{\mathbf{r}}{r^2}\right) dt = \frac{\mathbf{r}}{r} + \mathbf{c} = \hat{\mathbf{r}} + \mathbf{c}, \; \mathbf{c} \text{ being a vector.}$$

When **a** is a constant vector, then we have

$$\int \mathbf{a} \times \frac{d\mathbf{r}}{dt}\, dt = \mathbf{a} \times \mathbf{r} + \mathbf{c}, \; \mathbf{c} \text{ being a vector.}$$

Problem 119. *Find* **r** *from the equation* $\frac{d^2\mathbf{r}}{dt^2} = \mathbf{a}t + \mathbf{b}$, *where* **a** and **b** *are known constant vectors; given that both* **r** *and* $\frac{d\mathbf{r}}{dt}$ *vanish when* $t = 0$.

The given vector equation is

$$\frac{d^2\mathbf{r}}{dt^2} = \mathbf{a}\, t + \mathbf{b}.$$

Integrating with regard to t we get

$$\frac{d^2\mathbf{r}}{dt^2} = \mathbf{a}\,\frac{t^2}{2} + \mathbf{b}\, t + \mathbf{c}. \qquad \ldots(1)$$

Initially when $t = 0,\ \frac{d\mathbf{r}}{dt} = 0,$

$\therefore$ $\mathbf{c} = 0.$

Thus the equation (1) becomes

$$\frac{d\mathbf{r}}{dt} = a\frac{t^2}{2} + \mathbf{b}t.$$

Integrating it again with regard to t,

$$\mathbf{r} = \mathbf{a}\,\frac{t^3}{6} + \mathbf{b}\,\frac{t^2}{2} + \mathbf{d}$$

when $t = 0, \mathbf{r} = 0,$

$\therefore$ $\mathbf{d} = 0.$

Hence $\mathbf{r} = \frac{1}{6}\mathbf{a}\, t^3 + \frac{1}{2}\mathbf{b}\, t^2.$

Problem 120. *Solve for* **r** : *the equation* $\mathbf{a} \times \frac{d^2\mathbf{r}}{dt^2} = \mathbf{b}$, *where* **a** *and* **b** *are constant.*

Here $\mathbf{a} \times \frac{d^2\mathbf{r}}{dt^2} = \mathbf{b}.$

Integrating $\mathbf{a} \times \frac{d\mathbf{r}}{dt} = \mathbf{b}t + \mathbf{c}.$

Integrating again

$$\mathbf{a} \times \mathbf{r} = \frac{1}{2}\,\mathbf{b}t^2 + \mathbf{c}t + \mathbf{d}$$

where **c** and **d** are constant vectors.

Problem 121. *Given that* $\mathbf{r}(t) = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ *when* $t = 2$, *and*
$\mathbf{r}(t) = 4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ *when* $t = 3$.

Show that $\int_2^3 \mathbf{r} \cdot \frac{d\mathbf{r}}{dt}\, dt = 10.$

Here $\int_2^3 \mathbf{r} \cdot \frac{d\mathbf{r}}{dt}\, dt = \frac{1}{2}\int_2^3 2\mathbf{r} \cdot \frac{d\mathbf{r}}{dt}\, dt = \frac{1}{2}\left[\mathbf{r}^2\right]_2^3$

$$= \frac{1}{2}\left[(4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})^2 - (2\mathbf{i} - \mathbf{j} + 2\mathbf{k})^2\right]$$

$$= \frac{1}{2}\left[16 + 4 + 9 - (4 + 1 + 4)\right] = 10.$$

Problem 122. *Evaluate the integral* $\int_\Gamma \mathbf{F}\cdot d\mathbf{r}$ *where* $\mathbf{F} = c\left[-3a\sin^2\theta\cos\theta\,\mathbf{i} + a(2\sin\theta - 3\sin^2\theta)\,\mathbf{j} + b\sin 2\theta\,\mathbf{k}\right]$ *and the curve* Γ *is given by* $\mathbf{r} = a\cos\theta\,\mathbf{i} + a\sin\theta\,\mathbf{j} + b\theta\,\mathbf{k}$, *θ varying from π/4 to π/2.*

Given $\mathbf{r} = a\cos\theta\,\mathbf{i} + a\sin\theta\,\mathbf{j} + b\theta\,\mathbf{k}$.

$$\therefore \quad \frac{d\mathbf{r}}{d\theta} = -a\sin\theta\,\mathbf{i} + a\cos\theta\,\mathbf{j} + b\mathbf{k}.$$

Now

$$\int_\Gamma \mathbf{F}\cdot d\mathbf{r} = \int_{\pi/4}^{\pi/2} \mathbf{F}\cdot\frac{d\mathbf{r}}{d\theta}\,d\theta = \int_{\pi/4}^{\pi/2} c\big[-3a\sin^2\theta\cos\theta\,\mathbf{i} + a(2\sin\theta - 3\sin^3\theta)\,\mathbf{j} + b\sin 2\theta\,\mathbf{k}\big]\cdot(-a\sin\theta\,\mathbf{i} + a\cos\theta\,\mathbf{j} + b\mathbf{k})\,d\theta$$

$$= c\int_{\pi/4}^{\pi/2} \{3a^2\sin^3\theta\cos\theta + a^2(2\sin\theta - 3\sin^3\theta)\cos\theta + b^2\sin 2\theta\}\,d\theta$$

$$= c\int_{\pi/4}^{\pi/2} \{a^2(3\sin^3\theta\cos\theta + 2\sin\theta\cos\theta - 3\sin^3\theta\cos\theta) + b^2\sin 2\theta\}\,d\theta$$

$$= c(a^2 + b^2)\int_{\pi/4}^{\pi,2} \sin 2\theta\,d\theta = c(a^2 + b^2)\left[-\frac{\cos 2\theta}{2}\right]_{\pi/4}^{\pi/2}$$

$$= \frac{c}{2}\left(a^2 + b^2\right).$$

Problem 123. *Evaluate* $\int_c \mathbf{F}\cdot d\mathbf{r}$ *when* $\mathbf{F} = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$, *where c is the curve* $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, *t varying from −1 to +1.*

Equating coefficients of like vectors in

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k},$$

we get the parametric equations of the curve as

$$x = t,\ y = t^2,\ z = t^3.$$

Now from $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ we have

$$\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}.$$

$$\therefore \quad \int \mathbf{F}\cdot d\mathbf{r} = \int_{-1}^{1} \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt$$

$$= \int_{-1}^{1} (xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k})\cdot(\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k})\,dt$$

$$= \int_{-1}^{1} (xy + 2tyz + 3t^2zx)\,dt$$

$$= \int_{-1}^{1} (t^3 + 2t^6 + 3t^6)\, dt = \int_{-1}^{1} (t^3 + 5t^6)\, dt$$

$$= \left[\frac{t^4}{4} + \frac{5t^7}{7}\right]_{-1}^{1} = \frac{1}{4} - \frac{1}{4} + \frac{5}{7} - \left(-\frac{5}{7}\right) = \frac{10}{7}.$$

Problem 124. *Evaluate* $\int_1^2 (\mathbf{A} \cdot \mathbf{B} \times \mathbf{C})\, dt$ *and* $\int_1^2 \mathbf{A} \times (\mathbf{B} \times \mathbf{C})\, dt$

where $\mathbf{A} = t\mathbf{i} - 3\mathbf{j} + 2t\mathbf{k}$, $\mathbf{B} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ *and* $\mathbf{C} = 3\mathbf{i} + t\mathbf{j} - \mathbf{k}$.

We have $\mathbf{B} \times \mathbf{C} = (\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) \times (3\mathbf{i} + t\mathbf{j} - \mathbf{k})$

$$= (2 - 2t)\,\mathbf{i} + (6 + 1)\,\mathbf{j} + (t + 6)\,\mathbf{k}$$

$$= 2\,(1 - t)\,\mathbf{i} + 7\mathbf{j} + (t + 6)\,\mathbf{k}$$

so that, $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (t\mathbf{i} - 3\mathbf{j} + 2t\mathbf{k}) \cdot [2\,(1 - t)\,\mathbf{i} + 7\mathbf{j} + (t + 6)\,\mathbf{k}]$

$$= (2t - 2t^2) - 21 + 2t^2 + 12t$$

$$= 14t - 21.$$

$$\therefore \quad \int_1^2 [\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})]\, dt = \int_1^2 (14t - 21)\, dt$$

$$= \left[7t^2 - 21t\right]_1^2$$

$$= 28 - 42 - 7 + 21 = 0.$$

Again $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}\,(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}\,(\mathbf{A} \cdot \mathbf{B})$

where $\mathbf{A} \cdot \mathbf{C} = (t\mathbf{i} - 3\mathbf{j} + 2t\mathbf{k}) \cdot (3\mathbf{i} + t\mathbf{j} - \mathbf{k})$

$$= 3t - 3t - 2t = -2t.$$

So that $\mathbf{B}(\mathbf{A} \cdot \mathbf{C}) = (\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})\,(-2t)$

$$= -2t\mathbf{i} + 4t\mathbf{j} - 4t\mathbf{k}$$

and $(\mathbf{A} \cdot \mathbf{B}) = (t\mathbf{i} - 3\mathbf{j} + 2t\mathbf{k}) \cdot (\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$

$$= t + 6 + 4t$$

$$= (5t + 6).$$

So that $\mathbf{C}\,(\mathbf{A} \cdot \mathbf{B}) = (3\mathbf{i} + t\mathbf{j} - \mathbf{k})\,(5t + 6)$

$$= (15t + 18)\,\mathbf{i} + (5t^2 + 6t)\,\mathbf{j} - (5t + 6)\,\mathbf{k}.$$

Thus, $\int_1^2 \mathbf{A} \times (\mathbf{B} \times \mathbf{C})\, dt = \int_1^2 \mathbf{B}\,(\mathbf{A} \cdot \mathbf{C})\, dt - \int_1^2 \mathbf{C}\,(\mathbf{A} \cdot \mathbf{B})\, dt$

$$= \int_1^2 (-2t\mathbf{i} + 4t\mathbf{j} - 4t\mathbf{k})\, dt - \int_1^2 [(15t + 18)\,\mathbf{i} + (5t^2 + 6t)\,\mathbf{j} - (5t + 6)\,\mathbf{k}]\, dt$$

$$= \mathbf{i} \int_1^2 (-18 - 17t)\, dt + \mathbf{j} \int_1^2 (-2t - 5t^2)\, dt + \mathbf{k} \int_1^2 (t + 6)\, dt$$

$$= \mathbf{i} \left[-18t - \tfrac{17}{2}t^2\right]_1^2 + \mathbf{j} \left[-t^2 - \tfrac{5}{3}t^3\right]_1^2 + \mathbf{k} \left[\tfrac{1}{2}t^2 + 6t\right]_1^2$$

$$= \mathbf{i} \left[-36 + 18 - 34 + \tfrac{17}{2}\right] + \mathbf{j} \left[-4 + 1 - \tfrac{40}{3} + \tfrac{5}{2}\right] + \mathbf{k} \left[2 - \tfrac{1}{2} + 12 - 6\right]$$

$$= -\tfrac{87}{2}\,\mathbf{i} - \tfrac{44}{3}\,\mathbf{j} + \tfrac{15}{2}\,\mathbf{k}.$$

1.48. THE LINE INTEGRAL

The integration of a vector along a curve is known as line integral.

Suppose that **F** (**r**) is a continuous vector point function and **r** = **r** (s) is the given curve. Take any arc C of the given curve between two points A and B, for which length s of the arc has the values a and b respectively. If **t** is a unit tangent at a point of the curve, then $\mathbf{F} \cdot \mathbf{t}$ gives the resolute of **F** in the direction of the tangent. The definite integral of $\mathbf{F} \cdot \mathbf{t}$ with respect to s, between the limits a and b, is called the *line integral of the vector* **F** along the curve from A to B and is written as

$$\int_a^b \mathbf{F} \cdot \mathbf{t}\, ds = \text{Lim} \sum_a^b \mathbf{F} \cdot \mathbf{t}\, \delta s \qquad \text{...(1)}$$

But we know that $\mathbf{t} = \dfrac{d\mathbf{r}}{ds}$

∴ (1) becomes.

$$\int_A^B F \cdot d\mathbf{r} = \text{Lim} \sum_A^B \mathbf{F} \cdot \delta\mathbf{r}$$

where A and B are the end points of arc of integration, and $\delta\mathbf{r}$ is the infinitesimal vector, $\delta s t$ parallel to the tangent at the point considered.

The integration round a closed curve is denoted by the symbol $\oint$, while in mechanics this integral is known as the *circulation* of **F** about the closed curve C, being the velocity of the fluid.

Problem 125. *Compute* $I = \int (x\, dy - y\, dx)$ *over the*

(*a*) *Straight line* $y = x$ *from* $(0, 0)$ *to* $(1, 1)$

(*b*) *Parabola* $y = x^2$ *from* $(0, 0)$ *to* $(1, 1)$

(*c*) *Circle* $x = a \cos t, y = 1 + \sin t\,;\ -\dfrac{\pi}{2} \le t \le 0$

(*d*) *Integrate also round the square* $(0, 0)\,;(1, 0)\,;(1, 1)\,;(0, 1)$.

(*a*) The line integral $I = \int (x\, dy - y\, dx)$ over the line

$y = x$, $dy = dx$, is given by

$$\int (x\, dx - y\, dx) = \int (x\, dx - x\, dx) \text{ limits of } x \text{ are from 0 to 1.}$$

$$= \int_0^1 0\, dx = 0.$$

(*b*) The line integral

$$I = \int (x\, dy - y\, dx) \text{ over the parabola } y = x^2,\ dy = 2x\, dx$$

is

$$I = \int (2x^2\, dx - x^2\, dx) \text{ limits of } x \text{ being from 0 to 1}$$

$$= \int_0^1 x^2\, dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}.$$

(c) $I = \int_{-\pi/2}^{0} \left[\cos^2 t + (4 \sin t) \sin t\right] dt$ as $x = a \cos t, y = 1 + \sin t$

$= \left[\frac{\pi}{2} - 1\right]$

(d) $I = 0 + \int_0^1 dy - \int_1^0 dx + 0$ the figure being a square and integration is along straight lines.

$= 1 + 1 = 2.$

Problem 126. *Find the condition that the line integral*

$$\int_{r_1}^{r_2} \mathbf{F} \cdot d\mathbf{r} = \int_{t_1}^{t_2} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt$$

taken between the points A and B, is independent of the curve $\mathbf{r} = \mathbf{r}(t)$ *joining the points on a smooth curve.*

We observe that, if $\int \mathbf{F} \cdot d\mathbf{r}$ is independent of path in a certain region then $\oint \mathbf{F} \cdot d\mathbf{r} = 0$ for all closed curves in the region.

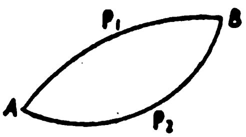


Fig. 1.44

For if AP_1BP_2A is a closed curve, then

$$\oint \mathbf{F} \cdot d\mathbf{r} = \int_{AP_1B} \mathbf{F} \cdot d\mathbf{r} + \int_{BP_2A} \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_{AP_1B} \mathbf{F} \cdot d\mathbf{r} - \int_{AP_2B} \mathbf{F} \cdot d\mathbf{r}$$

$$= 0.$$

Hence, if $\int_{r_1}^{r_2} \mathbf{F} \cdot d\mathbf{r}$ is independent of path then $\oint \mathbf{F} \cdot d\mathbf{r} = 0$.

Conversely, if $\oint \mathbf{F} \cdot d\mathbf{r} = 0$ over all closed curves of the region, then $\int \mathbf{F} \cdot d\mathbf{r}$ is the same over any two paths from A to B that do not cross.

Problem 127. *If* $d\mathbf{r}$ *is the infinitesimal vector and* $\mathbf{r} = \mathbf{r}(t)$ *is the equation of a curve, then evaluate the integrals*

(a) $\int_C \phi\, d\mathbf{r}$ (b) $\int_C \mathbf{F} \cdot d\mathbf{r}$

along the curve C where $\mathbf{F}$ *is a continuous vector and* ϕ *is a continuous point function.*

If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

then $d\mathbf{r} = \mathbf{i}\, dx + \mathbf{j}\, dy + \mathbf{k}\, dz.,$

Thus

(a) $\int_C \phi\, d\mathbf{r} = \int_A^B \phi\, (\mathbf{i}\, dx + \mathbf{j}\, dy + \mathbf{k}\, dz)$

$$= \mathbf{i} \int_{x_1}^{x_2} \phi\, dx + \mathbf{j} \int_{y_1}^{y_2} \phi\, dy + \mathbf{k} \int_{z_1}^{z_2} \phi\, dz$$

where A and B are initial and final points of the curve with coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2). Thus the integral $\int_C \phi\, d\mathbf{r}$ can be evaluated when y and z are known in terms of x for points on the curve C.

(*b*) If $\mathbf{F}(x, y, z) = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$ then as in (*a*), we find

$$\int_C \mathbf{F}\cdot d\mathbf{r} = f_1\int_C dx + f_2\int_C dy + f_3\int_C dz$$

Problem 128. *If* $\mathbf{F} = \nabla\phi$ *everywhere in a region R of space, defined by* $a_1 \le x \le a_2, b_1 \le y \le b_2, c_1 \le z \le c_2$, *where* $\phi(x, y, z)$ *is single valued and has continuous derivative in R, then show that*

(*i*) $\int_A^B \mathbf{F}\cdot d\mathbf{r}$ *is independent of path C in R joining A and B.*

(*ii*) *The converse of (i) is true*

(*iii*) $\int_A^B \mathbf{F}\cdot d\mathbf{r} = 0$ *around any closed curve C in R.*

We have

(*i*)
$$\text{grad}\,\phi = \nabla\phi = \left(\mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}\right)$$

and
$$d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$$

$$\therefore \quad \int_A^B \mathbf{F}\cdot d\mathbf{r} = \int_A^B \left(\mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}\right)\cdot(dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k})$$

$$= \int_A^B \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right)$$

$$= \int_A^B d\phi$$

$$= \phi(B) - \phi(A)$$

$$= \phi(x_2, y_2, z_2) - \phi(x_1, y_1, z_1) \qquad \ldots(1)$$

(x_1, y_1, z_1) and (x_2, y_2, z_2) being co-ordinates of A and B respectively.

It follows from (1) that the integral depends only on points A and B and not on the path joining them. This is true if only $\phi(x, y, z)$ is single valued at all points.

Note. If $\int_A^B \mathbf{F}\cdot d\mathbf{r}$ is independent of the path C joining A and B then $\mathbf{F}$ is called a *conservative field*. Thus if $\mathbf{F} = \nabla\phi$ then $\nabla\phi$ is conservative vector field and ϕ is its scalar potential.

(*ii*) Conversely, if $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of the path C joining any two points then there exists a function ϕ such that $\mathbf{F} = \nabla\phi$.

Supposing the line integral to be independent of path, we have

$$\phi(x, y, z) = \int_{(x_1, y_1, z_1)}^{(x, y, z)} \mathbf{F}\cdot d\mathbf{r}$$

$$= \int_{(x_1, y_1, z_1)}^{(x, y, z)} \mathbf{F}\cdot\frac{d\mathbf{r}}{ds}ds.$$

Differentiating we get $\dfrac{d\phi}{ds} = \mathbf{F}\cdot\dfrac{d\mathbf{r}}{ds}$ ...(2)

and also we have $\dfrac{d\phi}{ds} = \nabla\phi\cdot\dfrac{d\mathbf{r}}{ds}$. ...(3)

Subtracting (2) from (3), we find

$$(\nabla \phi - \mathbf{F}) \cdot \frac{d\mathbf{r}}{ds} = 0$$

This will hold independent of $\frac{d\mathbf{r}}{ds}$, if

$$\nabla \phi - \mathbf{F} = 0$$

or $$\mathbf{F} = \nabla \phi.$$

(*iii*) We have from (1)

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \phi(B) - \phi(A).$$

If the integration is taken round the closed curve *i.e.* when the points B and A coincide then

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^A \mathbf{F} \cdot d\mathbf{r} = \phi(A) - \phi(A)$$
$$= 0$$

or $$\oint \nabla \phi \cdot d\mathbf{r} = 0.$$

Problem 129. *If C is a simple closed curve in the xy plane not enclosing the origin. Show that*

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$$

where $$\mathbf{F} = \frac{\mathbf{i}y + \mathbf{j}x}{x^2 + y^2}$$

Given that $$\mathbf{F} = \frac{-y}{x^2 + y^2}\,\mathbf{i} + \frac{x}{x^2 + y^2}\,\mathbf{j}$$

and $$d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} \text{ when } \mathbf{r} = x\mathbf{i} + y\mathbf{j}$$

$\therefore$ $$\mathbf{F} \cdot d\mathbf{r} = \frac{-y\,dx + x\,dy}{x^2 + y^2}.$$

So that $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \frac{x\,dy - y\,dx}{x^2 + y^2}.$$

Let us change to polars by putting $x = r\cos\theta$, $y = r\sin\theta$ *i.e.*,

$$x^2 + y^2 = r^2 \text{ and } \theta = \tan^{-1}\frac{y}{x},$$

so that

$$d\theta = \frac{1}{1 + \frac{y^2}{x^2}} \cdot \frac{x\,dy - y\,dx}{x^2}$$
$$= \frac{x\,dy - y\,dx}{x^2 + y^2}$$

But the curve being closed, if there is a point P on it such that lower limit of θ at P is ϕ (say), then its upper limit will also be ϕ.

$$\therefore \quad \oint \mathbf{F}\cdot d\mathbf{r} = \int_{\phi}^{\phi} d\theta$$

$$= [\theta]_{\phi}^{\phi}$$

$$= \phi - \phi = 0.$$

Problem 130. *Show that*

$$\int_C \left(\frac{-y}{x^2+y^2}\,\mathbf{i} + \frac{x}{x^2+y^2}\,\mathbf{j} \right)\cdot d\mathbf{r} = 2\pi$$

where C is the circle $x^2 + y^2 = 1$ in the xy plane described in the counter-clockwise direction.

We have

$$\left[\frac{-y}{x^2+y^2}\,\mathbf{i} + \frac{x}{x^2+y^2}\,\mathbf{j}\right]\cdot[dx\,\mathbf{i} + dy\,\mathbf{j}] = \left[\frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy\right]$$

$$\therefore \quad \int_C \left[\frac{-y\,dx}{x^2+y^2} + \frac{x}{x^2+y^2}\,dy\right] = \int_C \left(\frac{-y\,dx + x\,dy}{x^2+y^2}\right)$$

$$= \int_0^{2\pi} d\theta \text{ changing to polars}$$

$$= 2\pi.$$

Problem 131. *(a) If $\nabla \times \mathbf{F} = \mathbf{0}$ (i.e. $\mathbf{F}$ is irrotational), prove that $\mathbf{F}$ is conservative.*

(b) If $\mathbf{F}$ is conservative field, prove that curl $\mathbf{F} = \nabla \times \mathbf{F} = \mathbf{0}$ i.e. ($\mathbf{F}$ is irrotational).

We know that if $\int_A^B \mathbf{F}\cdot d\mathbf{r}$ is independent of path C joining A and B then $\mathbf{F}$ is called a *conservative field.*

Suppose, $\mathbf{F} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$.

Then $\nabla \times \mathbf{F} = \mathbf{0}$ gives $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \mathbf{0}$

or $$\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\mathbf{k} = \mathbf{0}.$$

This will hold only if,

$$\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} = 0,\ \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} = 0,\ \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = 0$$

i.e. $$\frac{\partial f_3}{\partial y} = \frac{\partial f_2}{\partial z},\ \frac{\partial f_1}{\partial z} = \frac{\partial f_3}{\partial x},\ \frac{\partial f_2}{\partial x} = \frac{\partial f_1}{\partial y}$$

then, we have to show that $\mathbf{F} = \nabla\phi$.

Now work done $= \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (f_1\,dx + f_2\,dy + f_3\,dz)$ where C is the path from (x_1, y_1, z_1) to (x, y, z). Take in particular the path joining along straight line (x_1, y_1, z_1) to

(x, y_1, z_1) then to (x, y, z_1) and then to (x, y, z) and denote the work done along this path by $\phi(x, y, z)$. We thus have

$$\phi(x, y, z) = \int_{x_1}^{x} f_1(x, y_1, z_1)\,dx + \int_{y_1}^{y} f_2(x, y, z_1)\,dy + \int_{z_1}^{z} f_3(x, y, z)\,dz.$$

Which follows that

$$\frac{\partial \phi}{\partial z} = f_3(x, y, z)$$

$$\frac{\partial \phi}{\partial y} = f_2(x, y, z_1) + \int_{z_1}^{z} \frac{\partial f_3}{\partial y}(x, y, z)\,dz$$

$$= f_2(x, y, z_1) + \int_{z_1}^{z} \frac{\partial f_2}{\partial z}(x, y, z)\,dz$$

$$= f_2(x, y, z_1) + [f_2(x, y, z]_{z_1}^{z}$$

$$= f_2(x, y, z)$$

$$\frac{\partial \phi}{\partial x} = f_1(x, y_1, z_1) + \int_{y_1}^{y} \frac{\partial f_2}{\partial x}(x, y, z_1)\,dy + \int_{z_1}^{z} \frac{\partial f_3}{\partial x}(x, y, z)\,dz$$

$$= f_1(x, y_1, z_1) + \int_{y_1}^{y} \frac{\partial f_1}{\partial y}(x, y, z_1) + \int_{z_1}^{z} \frac{\partial f_1}{\partial z}(x, y, z)\,dz$$

$$= f_1(x, y_1, z) + [f_1(x, y, z_1)]_{y_1}^{y} + [f_1(x, y, z)]_{z_1}^{z}$$

$$= f_1(x, y, z)$$

Then $\quad \mathbf{F} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k} = \dfrac{\partial \phi}{\partial x}\,\mathbf{i} + \dfrac{\partial \phi}{\partial y}\,\mathbf{j} + \dfrac{\partial \phi}{\partial z}\,\mathbf{k} = \nabla\phi.$

(*b*) If **F** is a conservative field

$$\mathbf{F} = \nabla\phi$$

$$\therefore \quad \text{curl}\,\mathbf{F} = \nabla \times \nabla\phi = 0.$$

Problem 132. *If* $\mathbf{F} = \cos y\mathbf{i} - x\sin y\mathbf{j} - \cos z\mathbf{k}$, *show that the field is conservative.*

F is conservative if

$$\nabla \times \mathbf{F} = 0$$

Here $\quad \mathbf{F} = \cos y\mathbf{i} - x\sin y\mathbf{j} - \cos z\mathbf{k}$

$$\therefore \quad \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \cos y & -x\sin y & \cos z \end{vmatrix}$$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}(\cos z) - \frac{\partial}{\partial z}(-x\sin y)\right] + \mathbf{j}\left[\frac{\partial}{\partial z}(\cos y) - \frac{\partial}{\partial x}(\cos z)\right]$$

$$+ \mathbf{k}\left[\frac{\partial}{\partial x}(-x\sin y) - \frac{\partial}{\partial y}(\cos y)\right]$$

$$= 0\mathbf{i} + 0\mathbf{j} + (-\sin y + \sin y)\,\mathbf{k} = \mathbf{0}.$$

1.49. THE SURFACE INTEGRALS

We know that the parametric equations of a surface $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ may be combined into a single vector equation $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \mathbf{f}(u, v)$.

A surface $\mathbf{r} = \mathbf{f}(u, v)$ is termed as **smooth** if $\mathbf{f}(u, v)$ possesses continuous first order partial derivatives. In the following discussion we shall assume that the surface under consideration is smooth.

Let a smooth surface S be given by $\mathbf{F}(\mathbf{r}) = \mathbf{r} = \mathbf{f}(u, v)$. Consider S to be the two sided surface one side of which being treated as the positive side. If S is a closed surface the outer surface may be taken as positive surface. Let $\mathbf{n}$ be an outward drawn unit normal vector to any point of the positive side of S.

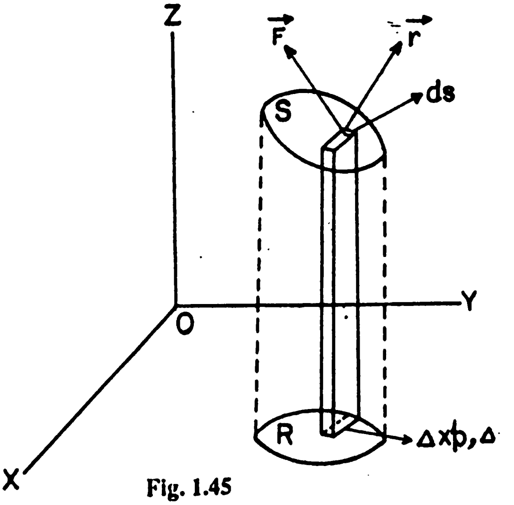


Fig. 1.45

Let us divide S into any finite number of elementary sub-surfaces and take a point (x_p, y_p, z_p) in an elementary sub-surface. Let $\mathbf{n}_p$ be the unit normal vector to this sub-surface at (x_p, y_p, z_p) drawn on the positive side of S.

If ΔS_p be the magnitude of the area of the sub-surface under consideration, then the vector area of this sub-surface S_p

$$\Delta \mathbf{S}_p = \mathbf{n}_p \Delta S_p$$

Multiplying both sides by $\mathbf{F}(x_p, y_p z_p)$. we get

$$\mathbf{F}(x_p, y_p, z_p) \cdot \Delta \mathbf{S}_p = \mathbf{F}(x_p, y_p, z_p) \cdot \mathbf{n}_p \Delta S_p$$

Consider the sum

$$\sum_{p=1}^{p=M} \mathbf{F}\left(x_p, y_p, z_p\right) \cdot \Delta \mathbf{S}_p = \sum_{p=1}^{p=M} \mathbf{F}\left(x_p, y_p, z_p\right) \cdot \mathbf{n}_p \Delta S_p$$

where summation extends as to include all sub-surfaces of S.

Take the limit of this sum as $M \to \infty$ in such a way that the largest dimension of each ΔS_p approaches to zero. This limit, if it exists, is termed as the **surface integral** of the normal component of $\mathbf{F}(\mathbf{r})$ over S and is denoted by

$$\int_S \mathbf{F}(\mathbf{r}) \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \mathbf{n}\, dS$$

or simply $\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \mathbf{n}\, dS$,

or $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$

or $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S [f_1(x, y, z)\, dy\, dz + f_2(x, y, z)\, dz\, dx + f_3(x, y, z)\, dx\, dy)]$

when $\mathbf{F} = f_1(x, y, z)\,\mathbf{i} + f_2(x, y, z)\,\mathbf{j} + f_3(x, y, z)\,\mathbf{k}$

We sometimes use surface integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ to call as **the flux of F over S.**

The notation $\oint\oint_S$ is used to indicate integration over closed surface S.

Other forms of surface integrals are:

(a) $\iint_S \phi \, dS$ (b) $\iint_S \phi \mathbf{n} \, dS$ (c) $\iint_S \mathbf{F} \times d\mathbf{S}$

where ϕ is a scalar function.

Note 1. If $\mathbf{F} = f_1 (x, y, z) \mathbf{i} + f_2 (x, y, z) \mathbf{j} + f_3 (x, y, z) \mathbf{k}$ then it can be verified that

$$\iint_S \mathbf{F} \times d\mathbf{S} = \mathbf{i} \iint_S (f_2 \, dx \, dy - f_3 \, dz \, dx)$$

$$+ \mathbf{j} \iint_S (f_3 \, dy \, dz - f_1 \, dx \, dy) + \mathbf{k} \iint_S (f_1 dz \, dx - f_2 \, dy \, dz)$$

and

$$\iint_S \phi \, \mathbf{n} \cdot dS = \iint_S \phi \, d\mathbf{S} = \mathbf{i} \iint_S \phi \, dy \, dz + \mathbf{j} \iint_S \phi dz \, dx + \mathbf{k} \iint_S \phi dx \, dy$$

Note 2. **Solenoidal vector function.** A vector point function is called **solenoidal** in a region if its flux *i.e.*, the surface integral $\int \mathbf{F} \cdot dS$ across every closed surface in the region is zero.

Problem 133. *Supposing that the surface S has projection R on the xy plane show that*

$$\iint \mathbf{F} \cdot \mathbf{n} \, dS = \iint_R \mathbf{F} \cdot \mathbf{n} \frac{dx \, dy}{|\mathbf{n} \cdot \mathbf{k}|}$$

The surface integrals, may be conveniently evaluated by expressing them as double integrals taken over the projected area of the surface S on one of the coordinate planes. This is only possible if a line perpendicular to the coordinate plane (chosen) meets the surface in only one point.

Referred to the Fig. 1.45 and § 1.49, we know that the surface integral is the limit of sum

$$\sum_{p=1}^{p=M} \mathbf{F} \left(x_p, y_p, z_p\right) \cdot \mathbf{n}_p \Delta S_p.$$

Now the projection of ΔS_p on xy plane is

$$|\mathbf{n}_p \Delta S_p| \cdot \mathbf{k} \quad \text{...(1)}$$

[As projected area = original area (cosine of angle between the normals of surface and coordinate plane)]

$$= \Delta S_p \, |\mathbf{n}_p \cdot \mathbf{k}|.$$

Also the area of projection of ΔS_p on xy plane

$$= \Delta x_p \, \Delta y_p \, (i.e. \ \delta x_p \cdot \delta y_p) \quad \text{...(2)}$$

From (1) and (2), we get, $\Delta S_p \, |\mathbf{n}_p \cdot \mathbf{k}| = \Delta x_p \, \Delta y_p$

$$\therefore \quad \Delta S_p = \frac{\Delta x_p \, \Delta y_p}{|\mathbf{n}_p \cdot \mathbf{k}|}.$$

Thus the limit of sum becomes

$$= \sum_{p=1}^{p=M} \mathbf{F} \cdot \mathbf{n}_p \frac{\Delta x_p \, \Delta y_p}{|\mathbf{n}_p \cdot \mathbf{k}|}$$

Proceeding to the limit, when $M \to 0$, Δx_p and Δy_p both $\to 0$

$$\iint_S \mathbf{F}\cdot\mathbf{n}\ dS = \iint_R \mathbf{F}\cdot\mathbf{n}\ \frac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|}.$$

Problem 134. *Evaluate*

$$\iint_S \mathbf{F}\cdot\mathbf{n}\ dS$$

where, $\mathbf{F} = 18z\mathbf{i} - 12\mathbf{j} + 3y\mathbf{k}$

S is that part of plane 2x + 3y + 6z = 12 which is located in the first octant.

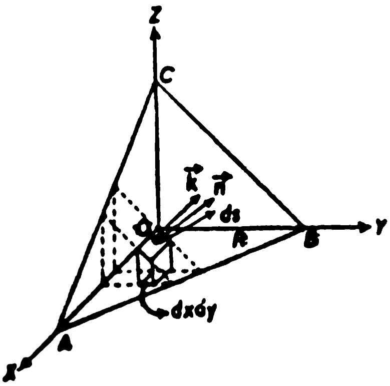


Fig. 1.46

The projection of the plane

$$2x + 3y + 6z = 12$$

on $z = 0$ plane is

$$2x + 3y = 12$$

i.e. referred to Fig. 1.46 the projection of the plane *ABC* on *xy* plane is *OAB*.

By the problem, 133 we have

$$\iint_S \mathbf{F}\cdot\mathbf{n}\ dS = \iint_R \mathbf{F}\cdot\mathbf{n}\frac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|}$$

Also we know that $\nabla\phi$ is a vector perpendicular to the surface.

$\therefore\ \phi(x, y, z) = \text{constant}.$

As such a normal vector perpendicular to the plane

$$2x + 3y + 6z = 12$$

is $$\nabla(2x+3y+6z) = \left(\mathbf{i}\ \frac{\partial}{\partial x} + \mathbf{j}\ \frac{\partial}{\partial x} + \mathbf{k}\ \frac{\partial}{\partial x}\right)(2x+3y+6z)$$

$$= 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}.$$

Unit vector along $2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$

$$= \frac{2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}}{\sqrt{(4 + 9 + 36)}}$$

$$= \frac{2}{7}\ \mathbf{i} + \frac{3}{7}\ \mathbf{j} + \frac{6}{7}\ \mathbf{k}.$$

Now $$(\mathbf{n}\cdot\mathbf{k}) = \left(\frac{2}{7}\ \mathbf{i} + \frac{3}{7}\ \mathbf{j} + \frac{6}{7}\ \mathbf{k}\right)\cdot\mathbf{k} = \frac{6}{7}.$$

Also $$\mathbf{F}\cdot\mathbf{n} = (18z\mathbf{i} - 12\mathbf{j} + 3y\mathbf{k})\cdot\left(\frac{2}{7}\ \mathbf{i} + \frac{3}{7}\ \mathbf{j} + \frac{6}{7}\ \mathbf{k}\right)$$

$$= \frac{36z - 36 + 18y}{7}.$$

But we have, $$z = \frac{12 - 2x - 3y}{6}$$

$\therefore$ $$\mathbf{F}\cdot\mathbf{n} = \frac{6\ (12 - 2x - 3y) - 36 + 18y}{7} = \frac{36 - 12x}{7}.$$

Hence $$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_R \left(\frac{36-12x}{7}\right) \frac{dx\,dy}{\frac{6}{7}}$$

$$= \iint_R (6-2x)\, dx\, dy$$

To integrate it consider the relation

$$2x + 5y = 12$$

i.e., $$y = \frac{12-2x}{3}$$

To cover the whole area BOA

y varies from 0 to $\frac{12-2x}{3}$

and x varies from 0 to 6.

$$\therefore \quad \iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \int_{x=0}^{x=6}\int_{y=0}^{y=(12-2x)/3} (6-2x)\, dx\, dy$$

$$= \int_{x=0}^{x=6} [6y - 2xy]_0^{(12-2x)/3}\, dx$$

$$= \int_{x=0}^{x=6} \left[24 - 4x - 8x + \frac{4}{3}x^2\right] dx$$

$$= \int_0^6 \left[24 - 12x + \frac{4}{3}x^2\right] dx$$

$$= \left[24x - \frac{12x^2}{2} + \frac{4}{3}\,\frac{x^3}{3}\right]_0^6$$

$$= 144 - 216 + 96 = 24.$$

Problem 135. *Evaluate* $\int_S (x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}) \cdot d\mathbf{S}$

where S is the surface of the sphere

$$x^2 + y^2 + z^2 = 1.$$

The projection of $x^2 + y^2 + z^2 = 1$

on the plane $z = 0$ is ...(1)

$$x^2 + y^2 = 1 \qquad \text{... (2)}$$

A normal vector to the surface (1) is

$$\nabla\phi = \left(\mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}\right)\left(x^2 + y^2 + z^2\right)$$

$$= 2\,[\mathbf{i}x + \mathbf{j}y + \mathbf{k}z]$$

Unit normal vector along $2\,[\mathbf{i}x + \mathbf{j}y + \mathbf{k}z]$ is given by

$$\mathbf{n} = \frac{2\mathbf{i}x + 2\mathbf{j}y + 2\mathbf{k}z}{2\sqrt{(x^2 + y^2 + z^2)}} \quad \text{where } x^2 + y^2 + z^2 = 1$$

$$= \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$$

$$\therefore \quad \mathbf{n}\cdot\mathbf{k} = (\mathbf{i}x + \mathbf{j}y + \mathbf{k}z)\cdot\mathbf{k} = z.$$

So that $\quad \mathbf{F}\cdot\mathbf{n} = (x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k})\cdot(\mathbf{i}x + \mathbf{j}y + \mathbf{k}z)$

$$= x^4 + y^4 + z^4.$$

Thus,
$$\int_S (x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k})\cdot d\mathbf{S} = 2\iint_R \mathbf{F}\cdot\mathbf{n}\,\frac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|}$$

$$= 2\iint_R (x^4 + y^4 + z^4)\,\frac{dx\,dy}{z}$$

$$= 2\iint \frac{\left[x^2 + y^4 + (1 - x^2 - y^2)^2\right]}{\sqrt{(1 - x^2 - y^2)}}\,dx\,dy$$

To cover the whole area of $x^2 + y^2 = 1$

y varies from $-\sqrt{(1-x^2)}$ to $+\sqrt{(1-x^2)}$

and $\quad x$ varies from -1 to $+1$.

$$\therefore \int_S (x^3\mathbf{j} + y^3\mathbf{j} + z^3\mathbf{k})\cdot d\mathbf{S}$$

$$= 2\int_{x=-1}^{x=1}\int_{y=-\sqrt{(1-x^2)}}^{y=+\sqrt{(1-x^2)}} \frac{x^4 + y^4 + (1 - x^2 - y^2)^2}{\sqrt{(1 - x^2 - y^2)}}\,dx\,dy$$

$$= 8\int_0^1\int_0^{\sqrt{(1-x^2)}} \left[\frac{x^4}{\sqrt{(1 - x^2 - y^2)}} + \left(-\frac{1}{2}\right)\left\{y^3(-2y)(1 - x^2 - y^2)^{-1/2}\right\} + (1 - x^2 - y^2)^{3/2}\right]dx\,dy$$

$$= 8\int_0^1 \left[x^4 \sin^{-1}\frac{y}{\sqrt{(1-x^2)}} - y^3(1 - x^2 - y^2)^{1/2} + 3\int y^2(1 - x^2 - y^2)^{1/2}\,dy + \int (1 - x^2 - y^2)^{3/2}\,dy\right]_0^{\sqrt{(1-x^2)}} dx$$

$$= 8\int_0^1 \left[\frac{\pi}{2}x^4 - 0 - \left\{y\cdot(1 - x^2 - y^2)^{3/2}\right\}_0^{\sqrt{(1-x^2)}} + \int_0^{\sqrt{(1-x^2)}} (1 - x^2 - y^2)^{3/2}\,dy + \int_0^{\sqrt{(1-x^2)}} (1 - x^2 - y^2)^{3/2}\,dy\right]dx$$

$$= 8\int_0^1 \left[\frac{\pi}{2}x^4 + 2\int_0^{\sqrt{(1-x^2)}} (1 - x^2 - y^2)^{3/2}\,dy\right]dx$$

Put $y = \sqrt{(1-x^2)}\sin\theta$

$dy = \sqrt{(1-x^2)}\cos\theta\,d\theta$

$$= 8\int_0^1 \left[\frac{\pi}{2}x^4 + 2\int_0^{\pi/2} (1 - x^2)^2\cdot\cos^4\theta\,d\theta\right)dx\Big]$$

$$= 8\int_0^1 \left[\frac{\pi}{2}x^4 + 2(1 - x^2)^2\cdot\frac{\Gamma(5/2)\Gamma(1/2)}{2\Gamma 3}\cdot\right]dx$$

$$= 8\int_0^1 \left[\frac{\pi}{2}x^4 + \left(1 + x^4 - 2x^2\right)\cdot\frac{3}{8}\pi\right] dx$$

$$= \pi \int_0^1 (4x^4 + 3 + 3x^4 - 6x^2)\, dx$$

$$= \pi \int_0^1 \left(3 - 6x^2 + 7x\right)\, dx$$

$$= \pi\left[3x - 2x^3 + \frac{5}{7}x^5\right]_0^1$$

$$= \pi\left[3 - 2 + \frac{7}{5}\right] = \frac{12}{5}\pi.$$

Problem 136. *Evaluate* $\iint_S [(x^3 - yz)\, dy\, dz - 2x^2y\, dz\, dx + z\, dx\, dy]$ *over the surface of a cube bounded by the coordinate planes and the planes*

$$x = y = z = a.$$

Here the surface is the cube in positive octant. To evaluate the given integral let us project the given surface on the three coordinate planes.

Now,

$$\iint_S \left[\left(x^3 - yz\right) dy\, dz - 2x^2y\, dz\, dx + z\, dx\, dy\right]$$

$$= \iint_S (x^3 - yz)\, dy\, dz + \iint_S - 2x^2y\, dz\, dx + \iint_S z\, dx\, dy$$

For the first integral

$$\iint_S (x^3 - yz)\, dy\, dz$$

Unit normal vector to the face $OQQ'O'$ in the outward direction is $-\mathbf{i}$.

And unit normal vector to the opposite face $ABB'A'$ in the outward direction is $+\mathbf{i}$.

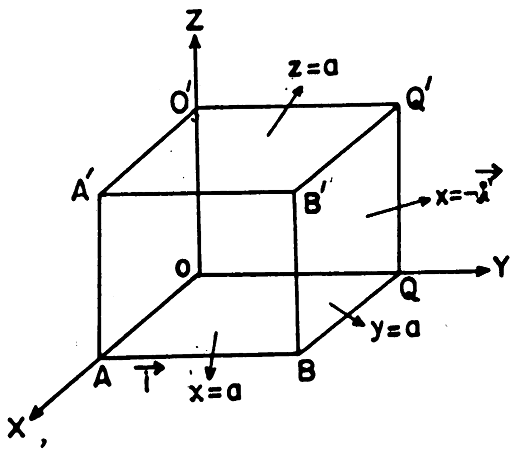


Fig. 1.47

$$\therefore \int_S (x^3 - yz)\, dy\, dz$$

$$= \iint_S (x^3 - yz)\, dy\, dz \quad \text{for } x = 0 \text{ and } x = a$$

$$= \mathbf{i}\cdot\mathbf{i}\iint_{S,\, x=0} (x^3 - yz)\, dy\, dz + \mathbf{i}\cdot\mathbf{i}\iint_{S,\, x=a} (x^3 - yz)\, dy\, dz$$

$$= -\iint_S (0 - yz)\, dy\, dz + \iint_S \left(a^3 - yz\right) dy\, dz$$

$$= \int_{y=0}^{a}\int_{z=0}^{a} yz\, dy\, dz + \int_{y=0}^{a}\int_{z=0}^{a}\left(a^3 - yz\right) dy\, dz$$

$$= \int_0^a\int_0^a a^3 dy\, dz = a^5.$$

Similarly other parts of integral are

$$-\iint_S 2x^2y\,dz\,dx = -\left[\iint \mathbf{j}\cdot(-\mathbf{j})\,2x^2y\,dz\,dx + \iint \mathbf{j}\cdot\mathbf{j}\,2x^2y\,dz\,dx\right.$$

$$y = 0 \qquad\qquad y = a$$

$$= -\int_0^a\int_0^a 2ax^2\,dx\,dz = -\tfrac{2}{3}a^5.$$

and $$\iint_S z\,dx\,dy = \iint_S \mathbf{k}.(-\mathbf{k})\,z\,dx\,dy + \iint_S (\mathbf{k}\cdot\mathbf{k})\,z\,dx\,dy$$

$$z = 0 \qquad\qquad z = a$$

$$= \int_S\int a\,dx\,dy = a^3.$$

Hence the value of the integral is $= \frac{1}{3}a^5 + a^3$.

Problem 137. *If* $\mathbf{F} = 4xz\mathbf{i} - y^2\mathbf{j} + yz\mathbf{k}$ *evaluate*

$$\iint \mathbf{F}\cdot\mathbf{n}\,dS$$

where S is the surface of the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

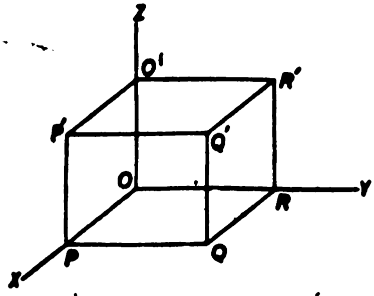


Fig. 1.48

For the face $ORR'O$

$$\mathbf{n} = -\mathbf{i}, x = 0.$$

$$\iint_{ORR'O'} \mathbf{F}\cdot\mathbf{n}\,dS$$

$$= \iint(-y^2\mathbf{j} + yz\mathbf{k})\cdot(-\mathbf{i})\,dy\,dz$$

$$= \int_0^1\int_0^1 0\;dy\,dz$$

$$= 0.$$

For the face $PQQ'P'$.

$$\mathbf{n} = \mathbf{i}, x = 1$$

$$\iint_{PQQ'P'} \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1\int_0^1 \left(4z\mathbf{i} - y^2\mathbf{j} + yz\mathbf{k}\right)\cdot\mathbf{j}\,dy\,dz$$

$$= 4\int_0^1\int_0^1 z\,dy\,dz = 2.$$

For the face $POO'P'$

$$\mathbf{n} = -\mathbf{j}, \qquad y = 0$$

$$\iint_{POO'P'} \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1\int_0^1 (4xz\mathbf{i})\cdot(-\mathbf{j})\,dx\,dz$$

$$= \int_0^1\int_0^1 0\,dx\,dz = 0.$$

For the face $QRR'Q'$

$$\mathbf{n} = \mathbf{j}, y = 1$$

$$\iint_{QRR'Q} \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1\int_0^1 (4xz\mathbf{i} - \mathbf{j} + z\mathbf{k})\cdot(\mathbf{j})\,dx\,dz$$

$$= -\int_0^1 \int_0^1 dx\ dz = -1.$$

For the face $OPQR$

$$\mathbf{n} = -\mathbf{k},\ z = 0$$

$$\iint_{OPQR} \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1\int_0^1 \left(-y^2\mathbf{j}\right)\cdot(-\mathbf{k})\ dx\,dy$$

$$= \int_0^1\int_0^1\ dx\ dy = 0.$$

For the face $O'P'Q'R'$

$$\mathbf{n} = \mathbf{k} \text{ and } z = 1$$

$$\iint_{O'P'Q'R'} \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1\int_0^1 (4x\mathbf{i} - y^2\mathbf{j} + y\mathbf{k})\cdot\mathbf{k}\ dx\ dy$$

$$= \int_0^1\int_0^1 y\ dx\ dy = \frac{1}{2}.$$

Adding all together,

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = 2 + 0 - 1 + 0 + \tfrac{1}{2} + 0 = \tfrac{3}{2}.$$

Problem 138. *If* $\mathbf{F} = y\mathbf{i} + (x - 2yz)\mathbf{j} - xy\mathbf{k}$, *evaluate*

$$\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\ dS$$

where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

Given

$$\mathbf{F} = y\mathbf{i} + (x - 2yz)\,\mathbf{j} - xy\mathbf{k}.$$

$$\nabla\times\mathbf{F} = \text{curl }\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & (x-2yz) & -xy \end{vmatrix}$$

$$= x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k}.$$

Thus to evaluate,

$$\int_S\int (x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k})\cdot\mathbf{n}\,dS = \int_R\int (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})\cdot\mathbf{n}\frac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|}$$

Normal vector

$$\nabla\phi = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\left(x^2 + y^2 + z^2\right)$$

$$= 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}.$$

Unit normal vector $\mathbf{n}$ along $2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$

$$= \frac{2(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{2\sqrt{x^2 + y^2 + z^2}}$$

$$= \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a} \qquad \because x^2 + y^2 + z^2 = a^2$$

$$(x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k})\cdot\mathbf{n} = (x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k})\cdot\left(\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}\right)$$

$$= \frac{x^2 + y^2 - 2z^2}{a}$$

$$\mathbf{n}\cdot\mathbf{k} = \left(\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}\right)\cdot\mathbf{k} = \frac{z}{a}.$$

Hence the required integral

$$I = \int_R\int \left(\frac{x^2 + y^2 - 2z^2}{a}\right)\frac{dx\,dy}{z/a}$$

$$= \int_{x=-a}^{a=+a}\int_{y=-\sqrt{(a^2-x^2)}}^{y=+\sqrt{(a^2-x^2)}} \frac{x^2 + y^2 - 2\left(a^2 - x^2 - y^2\right)}{\sqrt{\left(a^2 - x^2 - y^2\right)}}\,dx\,dy \quad \text{as } x^2 + y^2 + z^2 = a^2$$

$$= \int_{x=a}^{a}\int_{y=-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} \frac{3\left(x^2 + y^2\right) - 2a^2}{\sqrt{\left(a^2 - x^2 - y^2\right)}}\,dx\,dy \qquad \text{put } x = r\cos\theta,\ y = r\sin\theta.$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{a} \frac{3r^3 - 2a^2}{\sqrt{\left(a^2 - r^2\right)}}\,r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{a} \frac{3\left(r^2 - a^2\right) + a^2}{\sqrt{\left(r^2 - a^2\right)}}\,r\,d\theta\,dr$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{a}\left[-3r\sqrt{\left(r^2 - a^2\right)} + \frac{a^2 r}{\left(r^2 - a^2\right)}\right] d\theta\ dr$$

$$= \int_{\theta=0}^{2\pi}\left[\left(a^2 - r^2\right)^{3/2} - a^2\sqrt{\left(a^2 - r^2\right)}\right]_0^a d\theta$$

$$= \int_{\theta=0}^{2\pi}\left[a^3 - a^3\right] d\theta$$

$$= 0.$$

Problem 139. *Evaluate*

(*a*) $\int_S\int (\nabla\times\mathbf{F})\cdot\mathbf{n}\ dS$

(*b*) $\int_S\int \phi\mathbf{n}\ dS$

where $\mathbf{F} = (x + 2y)\,\mathbf{i} - 3z\mathbf{j} + x\mathbf{k}$ *and* $\phi = 4x + 3y - 2z$ *and S is the surface* $2x + y + 2z = 6$ *bounded by* $x = 0, x = 1, y = 0, y = 2$.

(*a*) Given

$$\mathbf{F} = (x + 2y)\,\mathbf{i} - 3z\,\mathbf{j} + x\mathbf{k}$$

$$\therefore \quad \nabla\times\mathbf{F} = \mathbf{i}\left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial z}(-3z)\right] + \mathbf{j}\left[\frac{\partial}{\partial z}(x + 2y) - \frac{\partial}{\partial x}(x)\right] + \mathbf{k}\left[\frac{\partial}{\partial x}(-3z) - \frac{\partial}{\partial y}(x + 2y)\right]$$

$$= 3\mathbf{i} - \mathbf{j} - 2\mathbf{k}.$$

Now normal vector to the surface $2x + y + 2z = 6$ is

$$\nabla (2x + y + 2z) = \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) (2x + y + 2z)$$

$$= 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

and unit normal vector $\mathbf{n}$ along $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

$$= \frac{2\mathbf{i} + \mathbf{j} + 2\mathbf{k}}{\sqrt{(4 + 1 + 4)}} = \frac{2\mathbf{i} + \mathbf{j} + 2\mathbf{k}}{3}$$

$\therefore$ $$(\nabla \times \mathbf{F}) \cdot \mathbf{n} = (3\mathbf{i} - \mathbf{j} - 2\mathbf{k}) \cdot \left(\frac{2\mathbf{i} + \mathbf{j} + 2\mathbf{k}}{3} \right)$$

$$= \frac{1}{3}[6 - 1 - 4] = \frac{1}{3}.$$

$$\mathbf{n} \cdot \mathbf{k} = \left[\frac{2\mathbf{i} + \mathbf{j} + 2\mathbf{k}}{3} \right] \cdot \mathbf{k} = \frac{2}{3}.$$

Projecting the given surface on the plane $z = 0$, we find

$$\int_S \int (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = \int_R \int (\nabla \times \mathbf{F}) \cdot \mathbf{n} \frac{dx\,dy}{|\mathbf{n} \cdot \mathbf{k}|}$$

$$= \int_R \int \frac{1}{3} \cdot \frac{dx\,dy}{\frac{2}{3}}$$

$$= \frac{1}{2} \int_{x=0}^{1} \int_{y=0}^{2} dx\,dy$$

$$= \frac{2}{2} = 1.$$

(*b*) $$\int_S \int \phi \mathbf{n} \, dS = \int_R \int \phi \mathbf{n} \frac{dx\,dy}{|\mathbf{n} \cdot \mathbf{k}|}$$

$$= \int_R \int (4x + 3y - 2z) \frac{(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})}{3} \frac{dx\,dy}{\frac{2}{3}}$$

$$= \frac{1}{2} \int_R \int (4x + 3y - 6 + 2x + y) (2\mathbf{i} + \mathbf{j} + \mathbf{k}) \, dx\,dy$$

$$= \frac{1}{2} \int_{x=0}^{1} \int_{y=0}^{2} [6x + 4y - 6] [2\mathbf{i} + \mathbf{j} + \mathbf{k}] \, dx\,dy$$

$$= \frac{1}{2} \int_{x=0}^{1} [12x + 8 - 12] [2\mathbf{i} + \mathbf{j} + \mathbf{k}] \, dx$$

$$= \frac{1}{2} \int_{x=0}^{1} (12x - 4) (2\mathbf{i} + \mathbf{j} + \mathbf{k}) \, dx$$

$$= \frac{1}{2} [6 - 4) [2\mathbf{i} + \mathbf{j} + \mathbf{k}] = 2\mathbf{i} + \mathbf{j} + \mathbf{k}.$$

1.50. THE VOLUME INTEGRALS

Let $\mathbf{F}(\mathbf{r})$ be a continuous vector point function and V a region enclosed by a surface

$$\mathbf{r} = \mathbf{f}(u, v).$$

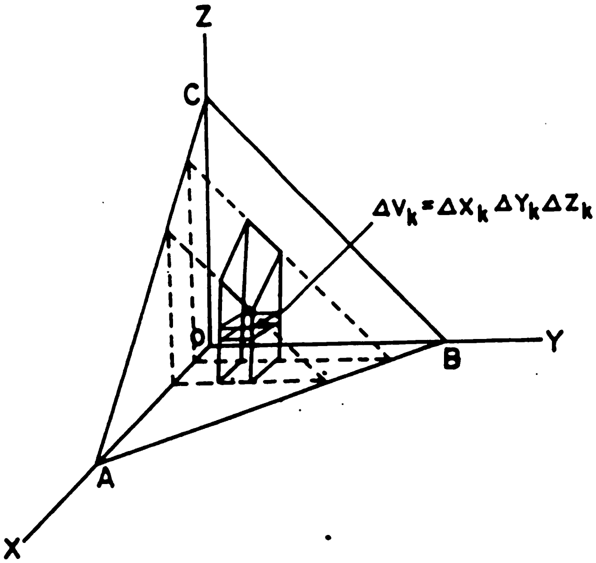


Fig. 1.49

Let us subdivide the region V into N cubes having volumes

$\Delta V_k = \Delta x_k \cdot \Delta y_k \cdot \Delta z_k$, $k = 1, 2, 3 \ldots N$

as shown in figure 1.49.

Take a point (x_k, y_k, z_k) within this cube and consider the sum

$$\sum_{k=1}^{N} \mathbf{F}\,(x_k, y_k, z_k)\ \Delta V_k \cdot x$$

taken over all possible cubes in the region. Thelimit of the sum when $N \to \infty$ in such a way that the dimensions of ΔV_k tend to zero, if exists, is denoted by the symbol $\int_V \mathbf{F}(\mathbf{r})\, dV$ or simply $\int_V \mathbf{F}\, dV$ or $\iiint_V \mathbf{F}\, dV$ and termed as the *volume integral* or *space integral.*

Its cartesian equivalence is

$$\int_V \mathbf{F}\, dV = \mathbf{i} \iiint_V F_1\, dx\, dy\, dz + \mathbf{j} \iiint_V F_2\, dx\, dy\, dz \iiint_V F_3\, dx\, dy\, dz$$

If ϕ is a continuous scalar-point function in V, then,

$$\iiint_V \phi\, dV$$

is also known as volume integral or space integral.

Problem 140. *Evaluate* $\iiint_V \mathbf{F}\, dV$

where $\mathbf{F} = 2xz\mathbf{i} - x\mathbf{j} + y^2\mathbf{k}$

and V is the region bounded by the surfaces

$$x = 0,\ y = 0,\ y = 6,\ z = x^2,\ z = 4.$$

The given solid is a parabolic cylinder with its axis parallel to y axis. The part of volume to be determined is shown in figure 1.50.

It we sub-divide the given volume into a large number of cubes and consider an elementary cube of volume δV, then the required integral is

$$\iiint_V \mathbf{F}\, dV$$

$$= \iiint_V (2xz\mathbf{i} - x\mathbf{j} + y^2\mathbf{k})\, dx\, dy\, dz\ ,$$

$$= \mathbf{i} \iiint_V 2\, xz\, dx\, dy\, dz$$

$$-\mathbf{j} \iiint_V x\, dx\, dy\, dz$$

$$+\mathbf{k} \iiint_V y^2\, dx\, dy\, dz.$$

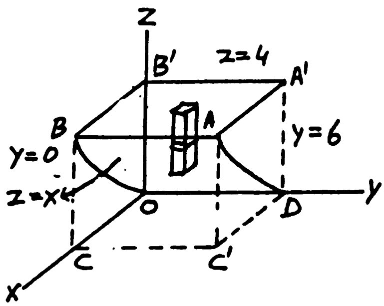


Fig. 1.50

Now to cover the whole volume, x varies from 0 to the line in which $z = x^2$ meets the plene $z = 4$ *i.e.*, x varies from 0 to $x^2 = 4$ or $x = 2$.

And y varies 0 to the plane $y = 6$ *i.e.* limits of y are from 0 to 6. Also z varies from x^2 to 4.

Thus

$$\iiint_V \mathbf{F}\, dV = 2\mathbf{i} \int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} xz\, dx\, dy\, dz - \mathbf{j} \int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} x\, dx\, dy\, dz + \mathbf{k} \int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} y^2\, dx\, dy\, dz$$

$$= 2\mathbf{i} \int_{x=0}^{2}\int_{y=0}^{6} x\left[\frac{z^2}{2}\right]_{x^2}^{4} dx\, dy - \mathbf{j} \int_{x=0}^{2}\int_{y=0}^{6} x[z]_{x^2}^{4}\, dx\, dy + \mathbf{k} \int_{x=0}^{2}\int_{y=0}^{6} y^2[z]_{x^2}^{4}\, dx\, dy$$

$$= 2\mathbf{i} \int_{x=0}^{2}\int_{y=0}^{6}\left(16x - x^5\right) dx\, dy - \mathbf{j} \int_{x=0}^{2}\int_{y=0}^{6}\left(4x - x^3\right) dx\, dy + \mathbf{k} \int_{x=0}^{2}\int_{y=0}^{6}\left(4 - x^2\right) y^2\, dx\, dy$$

$$= \mathbf{i} \int_{x=0}^{2}\left[96x - 6x^5\right] dx - \mathbf{j} \int_{x=0}^{2}\left[24x - 6x^3\right] dx + \mathbf{k} \int_{x=0}^{2} 72\left(4 - x^2\right) dx$$

$$= \mathbf{i}\left[48x - x^6\right]_0^2 - \mathbf{j}\left[12x - \tfrac{3}{2}x^4\right]_0^2 + \mathbf{k}\left[4x - \frac{x^3}{3}\right]_0^2$$

$$= 120\mathbf{i} - 24\mathbf{j} + 384\mathbf{k}.$$

Problem 141. *Let $\phi = 45x^2y$ and let V denote the closed region bounded by the planes $4x + 2y + z = 8$, $x = 0$, $y = 0$, $z = 0$ evaluate*

$$\iiint_V \phi\, dV$$

Referred to the figure 1.49 we have,

$$\iiint_V \phi\, dV = \int_V 45x^2y\, dx\, dy\, dz$$

z varies from 0 to $8 - 4x - 2y$

y varies from 0 to $4 - 2x$

x varies from 0 to 2.

$$\therefore \iiint_V \phi\, dV = \int_{x=0}^{2}\int_{y=0}^{4-2x}\int_{z=0}^{8-4x-2y} 45x^2y\, dx\, dy\, dz$$

$$= \int_{x=0}^{2}\int_{y=0}^{4-2x} 45x^2y\,(8 - 4x - 2y)\, dx\, dy$$

$$= 45 \int_{x=0}^{2} x^2\left[4y^2 - 2xy^2 - \tfrac{2}{3}y^3\right]_0^{4-2x} dx$$

$$= 45 \int_{x=0}^{2} \frac{x^2}{3}\,[12(4 - 2x)^2 - 6x(4 - 2x)^2 - 2(4 - 2x)^3]\, dx$$

$$= 45 \int_{x=0}^{2} \frac{x^2}{3} (4-2x)^2 [12-6x-2(4-2x)]\, dx$$

$$= 45 \int_{x=0}^{2} \frac{x^2}{3} (4-2x)^3\, dx$$

$$= 15 \int_0^2 x^2 [64 - 32x + 16x^2 - 8x^3]\, dx$$

$$= 15 \int_0^2 [64x^2 - 32x^3 + 16x^4 - 8x^6]\, dx$$

$$= 15 \left[\frac{64x^3}{3} - \frac{32x^4}{4} + \frac{16x^5}{5} - \frac{8x^6}{6}\right]_0^2$$

$$= 15 \left[\frac{64\times 8}{3} - 128 + \frac{16\times 32}{5} - \frac{4\times 64}{3}\right]$$

$$= [64\times 8\times 5 - 128\times 15 + 16\times 32\times 3 - 4\times 64\times 5]$$

$$= 128.$$

Problem 142. *Evaluate*

(*a*) $\iiint_V (\nabla\cdot\mathbf{F})\, dV$. (*b*) $\iiint (\nabla\times\mathbf{F}) dV$.

where V is the closed region bounded by the planes

$x = 0, y = 0, z = 0$ and $2x + 2y + z = 4$,

$\mathbf{F} = (2x^2 - 3z)\,\mathbf{i} = 2xy\mathbf{j} - 4x\mathbf{k}$.

Given $\mathbf{F} = (2x^2 - 3z)\,\mathbf{i} - 2xy\mathbf{j} - 4x\mathbf{k}$

$$\nabla.\mathbf{F} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial x} + \mathbf{k}\frac{\partial}{\partial z}\right)\left[(2x^2-3z)\mathbf{i} - 2xy\mathbf{j} - 4x\mathbf{k}\right]$$

$$= \left[\frac{\partial}{\partial x}(2x^2-3z) + \frac{\partial}{\partial y}(-2xy) + \frac{\partial}{\partial z}(-4x)\right]$$

$$= 4x - 2x$$

$$= 2x.$$

and
$$\nabla.\mathbf{F} = \mathbf{i}\left[\frac{\partial}{\partial y}(-4x) - \frac{\partial}{\partial z}(-2xy)\right] + \mathbf{j}\left[\frac{\partial}{\partial z}(2x^2-3z) - \frac{\partial}{\partial x}(-4x)\right] + \mathbf{k}\left[\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(2x^2-3z)\right]$$

$$= \mathbf{i}[0] + \mathbf{j}[-3+4] + \mathbf{k}[-2y] = \mathbf{j} - 2y\mathbf{k}.$$

Now, z varies from 0 to $4-2x-2y$

y varies from 0 to $2-x$

x varies from 0 to 2.

(*a*) $$\iiint_V (\nabla\cdot\mathbf{F})\, dV = \iiint_V 2x\; dx\; dy\; dz$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} 2x\; dx\,dy\,dz$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x} 2x(4-2x-2y)\,dx\,dy$$

$$= \int_{x=0}^{2} 2x\left[(4-2x)(2-x)-(2-x)^2\right]dx.$$

$$= \int_{x=0}^{2} 2x\left[8-4x-4x+2x^2-4+4x-x^2\right]dx$$

$$= \int_{x=0}^{2} 2x\left[4-4x+x^2\right]dx$$

$$= 2\int_{0}^{2}\left[4x-4x^2+x^3\right]dx$$

$$= 2\left[2x^2-\frac{4x^3}{3}+\frac{x^4}{4}\right]_0^2$$

$$= 2\left[8-\frac{32}{3}+4\right]=\frac{8}{3}.$$

(*b*) $$\iiint_V (\nabla\times\mathbf{F})\,dV = \iiint_V [\mathbf{j}-2y\mathbf{k}]\,dx\,dy\,dz$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} [\mathbf{j}-2y\mathbf{k}]\,dx\,dy\,dz$$

$$= \int_{0}^{2}\int_{0}^{2-x} (\mathbf{j}-2y\mathbf{k})(4-2x-2y)\,dx\,dy$$

$$= \int_{0}^{2}\left[\mathbf{j}\left\{(4-2x)y-y^2\right\}-\mathbf{k}\left\{(4-2x)y^2-\frac{4}{3}y^3\right\}\right]_0^{2-x} dx$$

$$= \int_{0}^{2}\left[\mathbf{j}\left\{(4-2x)(2-x)-(2-x)^2\right\}-\mathbf{k}\left\{(4-2x)(2-x)^2-\frac{4}{3}(2-x)^3\right\}\right]$$

$$= \int_{0}^{2}\Big[\mathbf{j}\left\{8-8x+2x^2-4-x^2+4x\right\}-\mathbf{k}\Big\{16-24x+12x^2-2x^3$$
$$-\frac{4}{3}\left(8-12x+6x^2-x^3\right)\Big\}\Big]\,dx$$

$$= \int_{0}^{2}\left[\mathbf{j}\left\{4-4x+x^2\right\}-\frac{\mathbf{k}}{3}\left\{16-24x+12x^2-2x^3\right\}\right]dx$$

$$= \left[\mathbf{j}\left\{4x-2x^2+\frac{x^3}{3}\right\}-\frac{\mathbf{k}}{3}\left\{16x-12x^2+4x^3-\frac{1}{2}x^4\right\}\right]_0^2.$$

$$= \mathbf{i}\left\{8-8+\frac{8}{3}\right\}-\frac{\mathbf{k}}{3}\{32-48+32-8\}$$

$$= \frac{8}{3}\mathbf{j}-\frac{8}{3}\mathbf{k}=\frac{8}{3}(\mathbf{j}-\mathbf{k}).$$

1.51. GAUSS' DIVERGENCE THEOREM (*Agra, 1958, 66*)

This theorem gives us a powerful device to transform the volume integral into surface integral and its statement is:

If **F** *is a continuous differentiable vector-point function and S is a closed surface enclosing a volume V, then*

$$\int_S \mathbf{F}\cdot\mathbf{n}\ dS = \int_V div\ \mathbf{F}\ dV.$$

when **n** *is the unit normal drawn outward.*

In other words, **"The normal surface integral of a function F over the boundary of a closed region is equal to the space integral of divergence of F taken throughout the enclosed space."**

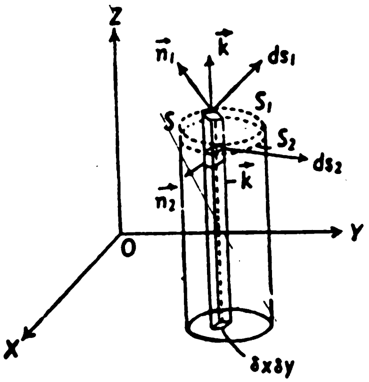


Fig. 1.51

Taking **i, j, k** as the unit vectors, along the axes of x, y and z respectively, we have

$$\mathbf{F} = F_1\ (x, y, z)\ \mathbf{i} + F_2\ \mathbf{j} + F_3\ \mathbf{k}$$

F_1, F_2, F_3 and their derivatives in any direction being assumed to be uniform, finite and continuous.

Suppose S is a closed surface such that any line parallel to the coordinate axes cuts it at the most in two points. Let the z coordinate of these two points be

$z = f_1\ (x, y)$, $z = f_2\ (x, y)$ respectively.

As such the lower and upper portions S_2 and S_1 of S are given by

$z = f_2\ (x, y)$ and $z = f_1\ (x, y)$ respectively.

Now consider the integral

$$\iiint_V \frac{\partial \mathbf{F}_3}{\partial z}\ dx\,dy\,dz = \iint_R \left[\int_{f_2}^{f_1} \frac{\partial \mathbf{F}_3}{\partial z}\ dz\right] dx\,dy$$

(R being projection of S on xy plane)

$$= \iint_R \left[\mathbf{F}_3\ (x,\ y,\ z)\right]_{f_2\ (x,y)}^{f_1\ (x,y)}\ dx\,dy$$

$$= \iint_R \left[\mathbf{F}_3\ (x,\ y,\ f_1) - \mathbf{F}_3\ (x,\ y,\ f_2)\right]\ dx\,dy$$

$$= \iint_R \mathbf{F}_3\ (x,\ y,\ f_1)\ dx\ \ dy - \int_R\int\ (x,\ y,\ f_2)\ dx\,dy$$

For S_1, we have

$$dx\ dy = \cos\gamma_1\ dS_1 = \mathbf{k}\cdot\mathbf{n}_1\ dS_1$$

where $\mathbf{n}_1$ is a unit normal vector to the surface dS_1 in outward direction.

For S_2, we have

$$dx\ dy = \cos\gamma_2\ dS_2 = -\ \mathbf{k}\cdot\mathbf{n}_2\ dS_2$$

where $\mathbf{n}_2$ is a unit normal vector to the surface dS_2 in outward direction.

$$\therefore\quad \iint_R F_3\ (x,\ y, f_1)\ dx\,dy = \iint_{S_1} F_3\ \mathbf{k}\cdot\mathbf{n}_1\ \ dS_1$$

and

$$\iint_R F_3\ (x,\ y, f_2)\ dx\,dy = -\iint_{S_2} F_3\ \mathbf{k}\cdot\mathbf{n}_2\ \ dS_2$$

So that

$$\iint_R F_3(x, y, f_1)\, dx\, dy - \iint_R F_3(x, y, f_2)\, dx\, dy$$

$$= \iint_{S_1} F_3\, \mathbf{k}\cdot\mathbf{n}_1\, dS_1 + \iint_{S_2} F_3\, \mathbf{k}\cdot\mathbf{n}_2\, dS_2$$

$$= \iint_S F_3\, \mathbf{k}\cdot(\mathbf{n}_1\, dS_1 + \mathbf{n}_2\, dS_2)$$

$$= \iint_S F_3\, \mathbf{k}\cdot\mathbf{n}\, dS; \quad [\because\ \mathbf{n}S = \mathbf{n}_1 S_1 + \mathbf{n}_2 S_2]$$

Consequently

$$\iiint_V \frac{\partial F_3}{\partial z}\, dx\, dy\, dz = \iint_S F_3\, \mathbf{k}\cdot\mathbf{n}\, dS \qquad \text{...(1)}$$

Similarly projecting S on other coordinate planes, we may find

$$\iiint_V \frac{\partial F_2}{\partial z}\, dx\, dy\, dz = \iint_S F_2\, \mathbf{j}\cdot\mathbf{n}\, dS. \qquad \text{...(2)}$$

$$\iiint_V \frac{\partial F_1}{\partial x}\, dx\, dy\, dz = \iint_S F_1\, \mathbf{j}\cdot\mathbf{n}\, dS. \qquad \text{...(3)}$$

Adding (1), (2) and (3), we get

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\, dy\, dz$$

$$= \iint_S [F_1 \mathbf{i}\cdot\mathbf{n} + F_2 \mathbf{j}\cdot\mathbf{n} + F_3\, \mathbf{k}\cdot\mathbf{n}]\, dS$$

$$\iiint_V \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot(F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k})\, dx\, dy\, dz$$

$$= \iint_S [F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}]\cdot\mathbf{n}\, dS$$

$$\iiint_V \text{div}\, \mathbf{F}\, dV = \iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_S \mathbf{F}\cdot d\mathbf{S}$$

Note. The theorem can be extended to surfaces which are such that lines parallel to the coordinate axes meet them in more than two points. This is also true for multiply connected regions.

1.52. DEDUCTIONS FROM GAUSS' THEOREM

(1) *The volume integral of the gradient of a scalar point function may be expressed in terms of the values assumed by the function at the boundary of the region i.e.*

$$\iiint_V \nabla\phi\, dV = \iint_S \phi\mathbf{n}\, dS \qquad \textit{(Agra, 1960, 56)}$$

Gauss' divergence theorem, is

$$\iiint_V \text{div}\ \mathbf{F}\, dV = \iint_S \mathbf{F}\cdot\mathbf{n}\, dS$$

Assume $\qquad \mathbf{F} = \phi\mathbf{a}$

where $\mathbf{a}$ is a constant vector. Then it becomes

$$\iiint_V \nabla.(\phi\mathbf{a})\, dV = \iint_S \phi\mathbf{a}\cdot\mathbf{n}\, dS$$

Since $\nabla.(\phi\mathbf{a}) = (\nabla\phi).\mathbf{a} = \mathbf{a}.(\nabla\phi)$ and $\phi\, \mathbf{a}.\mathbf{n} = a.(\phi\mathbf{n})$

$\therefore \qquad \iiint_V \mathbf{a}.(\nabla\phi)\ dV = \iint_S \mathbf{a}.(\phi\mathbf{n})\ dS$

or $\qquad \mathbf{a}.\iiint_V \nabla\phi\ dV = \mathbf{a}.\iint_S \phi\mathbf{n}\ dS$, a being constant

or $\qquad \iiint_V \nabla\phi\ dV = \iint_S \phi\mathbf{n}\ dS.$

(2) *The dot product of Gauss' divergence theorem is replaced by cross product i.e.*

$$\iiint_V \nabla\times\mathbf{A}\ dV = \iint_S \mathbf{n}\times\mathbf{A}\ dS = \iint_S dS\times\mathbf{A}.$$

Putting $\mathbf{F} = \mathbf{a}\times\mathbf{A}$ in Gauss' divergence theorem, we get,

$$\iiint_V \text{div}\ \mathbf{F}\ dV = \iint_S \mathbf{F}\cdot\mathbf{n}\ dS$$

where a is a constant vector,

or $\qquad \iiint_V \nabla\cdot(\mathbf{a}\times\mathbf{A})dV = \iint_S (\mathbf{a}\times\mathbf{A})\cdot\mathbf{n}\ dS.$

Since $\qquad \nabla.(\mathbf{a}\times\mathbf{A}) = -\mathbf{a}.(\nabla\times\mathbf{A})$

and $\qquad (\mathbf{a}\times\mathbf{A}).\mathbf{n} = \mathbf{a}.(\mathbf{A}\times\mathbf{n}) = (\mathbf{A}\times\mathbf{n}).\mathbf{a}.$

$\therefore \qquad -\iiint_V \mathbf{a}\cdot(\nabla\times\mathbf{A})dV = \iint_S \mathbf{a}.(\mathbf{A}\times\mathbf{n})\ dS.$

or $\qquad -\mathbf{a}.\iiint_V \nabla\times\mathbf{A}\ dV = \mathbf{a}.\iint_S \mathbf{A}\times\mathbf{n}\ dS$, a being constant

or $\qquad -\iiint_V \nabla\times\mathbf{A}\ dV = \iint_S \mathbf{A}\times\mathbf{n}\ dS$

or $\qquad \iiint_V \nabla\times\mathbf{A}\ dV = -\iint_S \mathbf{A}\times\mathbf{n}\ dS = \iint \mathbf{n}\times\mathbf{A}\ dS.$

Note. Definition of grad ϕ, *div* **F** and curl **F** can be put as

(1) $\text{grad}\ \phi = \lim_{V\to 0} \dfrac{\int_S \mathbf{n}\ \phi\ dS}{V}$

(2) $\text{div}\ \mathbf{F} = \lim_{V\to 0} \dfrac{\int_S \mathbf{n}.\mathbf{F}\ dS}{V}$

(3) $\text{curl}\ \mathbf{F} = \lim_{V\to 0} \dfrac{\int_S \mathbf{n}\times\mathbf{F}\ dS}{V}$

we here below prove (1), the other two can be proved on similar lines.

Let us take a point P enclosed in a small region of volume V bounded by a surface S. Then the first deduction of §1.52 gives

$$\int_S \mathbf{n}\ \phi\ ds = \int_V \nabla\ \phi\ dV = V\left[(\nabla\phi)_0 + \varepsilon\right]$$

where $(\nabla\phi)_0$ denotes the value of $\nabla\phi$ at P and $\varepsilon\to 0$ as $V\to 0$

$$\therefore \qquad \lim_{V\to 0} \frac{\int_S \mathbf{n}\phi\, dS}{V} = (\nabla\phi)_0 = \text{grad}\ \phi.$$

1.53. PHYSICAL INTERPRETATION OF GAUSS' DIVERGENCE THEOREM

The Gauss' Theorem can be stated as

$$\int_S \mathbf{V}.\mathbf{n}\ dS = \int_V \text{div}\ \mathbf{V}\ dV,$$

where the vector point function **V** denotes the velocity vector of an incompressible fluid of unit density and S denotes any closed surface drawn in the space of the fluid, enclosing a volume V.

Since the scalar product **V.n** represents the velocity-component at a point of the surface S in the direction of the outward drawn normal, therefore, **V.n** δS expresses the amount of fluid flowing out in unit time through the element of surface δS. As such the integral round the surface S, *i.e.*, $\int_S \mathbf{V}.\mathbf{n}\ dS$ gives the amount of fluid flowing out of the surface S in unit time. But in order to maintain the continuity of the flow the total amount of fluid flowing outwards must be continually supplied so that inside the region there are sources producing fluid.

Now the div **V** at any point represents the amount of fluid passing through that point per unit time per unit volume. So div **V** may be regarded as the source-intensity of the incompressible fluid at any point. Thus the integration round the volume V, *i.e.*, $\int_V \text{div}\ \mathbf{V}\ dV$ denotes the amount of fluid supplied by the sources inside S per unit time.

Hence the equality $\int_S \mathbf{V}.\mathbf{n}\ dS = \int_V \text{div}\ \mathbf{V}\ dV$ is justified.

i.e., the total volume per second of a moving fluid flowing out from a closed surface S is equal to the total volume per second of fluid flowing out from all volume elements in S.

1.54. GAUSS' THEOREM

If S be a closed surface and **r** *be the position vector of a point* (x, y, z) *with respect to on origin O, then*

$$\iint \frac{\mathbf{n}\cdot\mathbf{r}}{r^2}\ dS$$

is equal to 0 or 4π *according as O lies outside or inside of S.*

First case. *When origin O lies outside the closed surface S, the* divergence theorem gives

$$\iint_S \mathbf{n}.\left(\frac{\mathbf{r}}{r^3}\right) dS = \iiint_V \nabla\cdot\left(\frac{\mathbf{r}}{r^3}\right) dV$$

But
$$\nabla\cdot\left(\frac{\mathbf{r}}{r^3}\right) = \nabla\cdot\left(r^{-3}\ \mathbf{r}\right)$$
$$= \left(\nabla r^{-3}\right)\cdot\mathbf{r} + \left(r^{-3}\right)\nabla\cdot\mathbf{r}$$
$$= -3r^{-3}\mathbf{r}\cdot\mathbf{r} - \left(r^{-3}\right)(3)$$
$$= -3r^{-3} + 3r^{-3} = 0$$

i.e. $\nabla\cdot\left(\frac{\mathbf{r}}{r^3}\right)$ is zero everywhere inside V

Provided $r \neq 0$ in V *i.e.*, O is outside the closed surface S.

Then $$\iint_S \frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\, dS = 0.$$

Second case. *When O lies inside the closed surfaces.*

Here **F** is not defined at the point O within S, so we cannot apply divergence theorem. To overcome this difficulty let us surround O by a small sphere of radius a and surface S, with its centre at O and lying within S. For S_1 outward drawn normal will be directed towards O and function **F** will be continuous and differentiable at all points within S and S_1.

The divergence theorem can now be applied to the two closed surfaces S_1 and S *i.e.*

$$\int_S\int_{S_1} \frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\, dS = \iint_S \frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\, dS + \iint_{S_1} \frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\, dS = \iiint_V \nabla\cdot\left(\frac{\mathbf{r}}{r^3}\right) dV = 0$$

(O being outside the region SS_1)

$$\therefore \quad \iint_S \frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\, dS = -\iint_{S_1} \frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\, dS$$

But on S_1 ; $r = a$ and $\mathbf{n} = -\frac{\mathbf{r}}{a}$

$$\therefore \quad \frac{\mathbf{n}\cdot\mathbf{r}}{r^3} = \frac{\left(\frac{\mathbf{r}}{a}\right)\cdot\mathbf{r}}{a^3} = -\frac{\mathbf{r}\cdot\mathbf{r}}{a^4} = -\frac{a^2}{a^4} = -\frac{1}{a^2}.$$

Hence $$\iint_S \frac{\mathbf{n}\cdot\mathbf{r}}{r^3}\, dS = \iint_{S_1} \frac{1}{a^2}\, dS$$

$$= \frac{1}{a^2}\iint_S dS$$

$$= \frac{1}{a^2}\left(4\pi a^2\right)$$

$$= 4\pi.$$

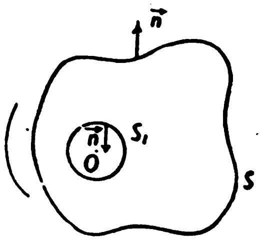


Fig. 1.52

1.55. TWO GREEN'S IDENTITIES

First identity. *If ø and ψ are scalar point functions having continuous derivatives of the second order at least, then*

$$\iiint_V \left(\phi\nabla^2\psi + \nabla\psi\cdot\nabla\phi\right) dV = \iint_S (\phi\nabla\psi)\cdot d\mathbf{S}.$$

The divergence theorem is

$$\iiint_V \nabla\cdot\mathbf{F}\, dV = \iint_S \mathbf{F}\cdot\mathbf{n}\, dS.$$

Taking $\mathbf{F} = \phi\nabla\psi$, we have

$$\iiint_V \nabla\cdot(\phi\nabla\psi)\, dV = \iint_S (\phi\nabla\psi)\cdot\mathbf{n}\, dS = \iint_V (\phi\nabla\psi)\, d\mathbf{S}.$$

But $$\nabla(\phi\nabla\psi) = \phi(\nabla\cdot\nabla\psi) + (\nabla\phi)\cdot(\nabla\psi)$$

$$= \phi\left(\nabla^2\phi\right) + (\nabla\phi)\cdot(\nabla\psi)$$

$$\therefore \quad \iiint_V \nabla\cdot(\phi\nabla\psi)\, dV = \iiint_V \left[\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi\right] dV$$

or $$\iiint_V \left[\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi\right) dV = \iint_S (\phi\nabla\psi)\cdot d\mathbf{S}$$

Second identity:

$$\iiint_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) dV = \iint_S (\phi\nabla\psi - \psi\nabla\phi)\cdot d\mathbf{S}$$

Putting $\mathbf{F}=\psi\nabla\phi$, in divergence theorem, we get

$$\iiint_V \nabla\cdot(\psi\nabla\phi)\ dV = \iint_S (\psi\nabla\phi)\cdot\mathbf{n}\, dS = \iint_S (\psi\nabla\phi)\cdot d\mathbf{S}$$

But $$\nabla\cdot(\psi\nabla\phi) = \psi(\nabla\cdot\nabla\phi) + (\nabla\psi)\cdot(\nabla\phi)$$
$$= \psi\nabla^2\phi + (\nabla\phi)\cdot(\nabla\psi)$$

$\therefore$ $$\iiint_V \nabla\cdot(\psi\nabla\phi)\ dV = \iiint_V \left[\psi\nabla^2\phi + (\nabla\phi)\cdot(\nabla\psi)\right] dV$$

And $$\iiint_V \left[\psi\nabla^2\phi + (\nabla\phi)\cdot(\nabla\psi)\right] dV = \iint_S [\psi\nabla\phi]\cdot d\mathbf{S} \quad \ldots (1)$$

Green's first identity, is

$$\iiint_V \left[\phi\nabla^2\psi + (\nabla\phi)\cdot(\nabla\psi)\right] dV = \iint_S [\phi\nabla\psi]\cdot d\mathbf{S} \quad \ldots (2)$$

Subtracting (1) from (2), we find

$$\iiint_V \left[\phi\nabla^2\psi - \psi\nabla^2\phi\right] dV = \iint_S [\phi\nabla\psi - \psi\nabla\phi]\cdot d\mathbf{S}.$$

Problem 143. *Verify divergence theorem for* $\mathbf{F} = x^2\,\mathbf{i} + y^2\,\mathbf{j} + z^2\mathbf{k}$ *taken over the cube* $0 \le x, y, z \le 1$.

The divergence theorem is

$$\iiint_V \nabla\cdot\mathbf{F}\, dV = \iint_S \mathbf{F}\cdot\mathbf{n}\ dS.$$

We have

$$\mathbf{F} = x^2\,\mathbf{i} + y^2\,\mathbf{j} + z^2\,\mathbf{k}$$

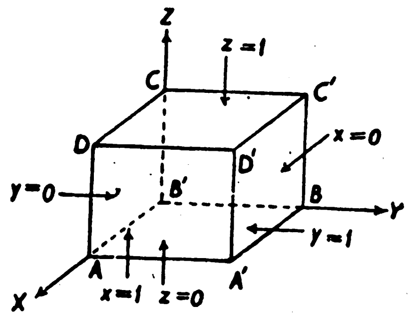


Fig. 1.53

$\therefore$ $$\nabla\cdot\mathbf{F} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot\left(x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}\right)$$
$$= 2x + 2y + 2z.$$

Now the volume integral

$$\iiint_V \nabla \cdot \mathbf{F}\, dV = \iiint_V 2(x+y+z)\, dx\, dy\, dz$$

$$= \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} 2(x+y+z)\, dx\, dy\, dz$$

$$= 2\int_{x=0}^{1}\int_{y=0}^{1}\left(x+y+\frac{1}{2}\right) dx\, dy$$

$$= 2\int_{x=0}^{1}\left(x+\frac{1}{2}+\frac{1}{2}\right) dx.$$

$$= 2\left[\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right]$$

$$= 3. \qquad \ldots (1)$$

The surface integral is contributed by six faces of the cube, *i.e.*

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_{S_1} \underset{(\text{face } x=0)}{\mathbf{F}.\mathbf{n}\, dS_1} + \iint_{S_2} \underset{(\text{face } x=1)}{\mathbf{F}.\mathbf{n}\, dS_2} + \iint_{S_3} \underset{(\text{face } y=0)}{\mathbf{F}.\mathbf{n}\, dS_3}$$

$$+ \iint_{S_4} \underset{(\text{face } y=1)}{\mathbf{F}\cdot\mathbf{n}\, dS_4} + \iint_{S_5} \underset{(\text{face } z=0)}{\mathbf{F}\cdot\mathbf{n}\, dS_5} + \iint_{S_6} \underset{(\text{face } z=1)}{\mathbf{F}\cdot\mathbf{n}\, dS_6}$$

For face S_1, $x = 0$, $\mathbf{n} = -\mathbf{i}$

$$\therefore \qquad \iint_{S_1} \mathbf{F}\cdot\mathbf{n}\, dS_1 = \iint_{S_1} -x^2\, dS_1$$

$$= \int_{y=0}^{1}\int_{z=0}^{1} 0\, dS_1 = 0$$

For face S_2, $x = 1$, $\mathbf{n} = \mathbf{i}$.

$$\therefore \qquad \iint_{S_2} \mathbf{F}\cdot\mathbf{n}\, dS_2 = \iint_{S_2} \left(x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}\right)\cdot(\mathbf{i})\, dS_2$$

$$= \iint_{S_2} x^2\, dS_2 = \iint_{S_2} dS_2$$

$$= \int_{y=0}^{1}\int_{z=0}^{1} dx\, dz = 1.$$

For face S_3, $y = 0$, $\mathbf{n} = -\mathbf{j}$,

$$\iint_{S_3} \mathbf{F}\cdot\mathbf{n}\, dS_3 = \iint_{S_3} 0\, dS_3.$$

$$= 0.$$

For face S_4, $y = 1$, $\mathbf{n} = \mathbf{j}$.

$$\iint_{S_4} \mathbf{F}\cdot\mathbf{n}\, dS_4 = \iint_{S_4} y^2\, dS_4 = \iint_{s_4} dS_4$$

$$= \int_{x=0}^{1}\int_{z=0}^{1} dx\, dz = 1.$$

For face S_5, $z = 0,\ \mathbf{n} = -\mathbf{k}$

$$\iint_{S_5} \mathbf{F}\cdot\mathbf{n}\, dS_5 = \iint_{S_5} 0\ dS_5.$$
$$= 0.$$

For face S_6, $z = 1, \mathbf{n} = \mathbf{k}$

$$\iint_{S_6} \mathbf{F}\cdot\mathbf{n}\, dS_6 = \iint_{S_6} dS_6.$$
$$= \int_{x=0}^{1}\int_{y=0}^{1} dx\, dy = 1$$

Therefore, $\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = 0 + 1 + 0 + 1 + 0 + 1.$

$$= 3. \qquad \dots (2)$$

From (1) and (2) it follows that the volume integral = Surface integral.

Problem 144. *Evaluate* $\iint_S \mathbf{r}\cdot\mathbf{n}\, dS$

where $\mathbf{r}$ *is the position vector of any point on the closed surface.*

The divergence theorem is

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iiint_V \nabla\cdot(\mathbf{F})\, dV.$$

Given $\mathbf{F} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

$$\therefore \quad \iint_S \mathbf{r}\cdot\mathbf{n}\, dS = \iiint_V \nabla\cdot(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})\, dV.$$

Now $\nabla\cdot(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \left(\mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}\right)\cdot(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

$$= 3.$$

Therefore $\iint_S \mathbf{r}\cdot\mathbf{n}\, dS = \iiint_V 3\, dV = 3V$

where V is the volume of solid bounded by closed surface S.

Problem 145. (*a*) *State and prove Gauss' theorem.*

(*b*) *If* ρ *denotes the charge density and* $\mathbf{j}$ *the current density due to the charges, show that the equation* $\frac{\partial \rho}{\partial t} + \operatorname{div}\mathbf{j} = 0$ *expresses conservation of the total charge.*

(*Agra, 1966*)

(*a*) If N be the outward flux of electrostatic intensity $\mathbf{E}$ through any closed surface S, then

$$N = \int_S \mathbf{E}\cdot d\mathbf{S} = 4\pi Q,$$

where Q denotes the total charge enclosed by the surface S.

According to the definition the outward flux of electric intensity through any closed surface is proportional to the total charge within the surface.

We know that the flux of a vector **E** across the surface element $d\mathbf{S}$ is defined as the scalar $\mathbf{E}\cdot d\mathbf{S}$. Therefore, if S be any surface, closed or open, the flux of **E** across S is $\int_S \mathbf{E}\cdot d\mathbf{S}$.

Now the electrostatic intensity or the electrostatic field vector, *i.e.*, **E** at a point due to n charges $e_1, e_2, \ldots e_n$ is defined as

$$\mathbf{E} = \sum_{i=1}^{n} \frac{e_i}{r_i^3}\mathbf{r}_i,$$

where $\mathbf{r}_i$ denotes the position vector of the point relative to the ith charge e_i.

$$\therefore \quad \int_S \mathbf{E}\cdot d\mathbf{S} = \int_S \left[\sum_{i=1}^{n} \frac{e_i \mathbf{r}_i}{r_i^3}\right]\cdot d\mathbf{S} = \sum_{i=1}^{n} e_i \int \frac{\mathbf{r}_i}{r_i^3}\cdot d\mathbf{S}$$

$$= \sum_{i=1}^{n} e_1 \int d\omega_i = \sum_{i=1}^{n} e_i \omega_i,$$

(since the solid angle $d\omega$ subtended at a point by a surface element of area dS is given by $d\omega = \frac{\mathbf{r}}{r^3}\cdot d\mathbf{S}$)

where ω_i is the solid angle subtended by the closed surface at the ith charge. But $\omega_i = 4\pi$ or 0 according as e_i is inside or outside the surface and $\Sigma e_i = Q$.

Hence $$\int_S \mathbf{E}\cdot d\mathbf{S} = 4\pi Q.$$

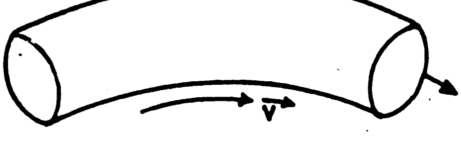


Fig. 1.54

(*b*) Suppose that a charge of density ρ is flowing with mean velocity **V**.

The charge crossing the surface element $d\mathbf{S}$ in a unit time is

$$\rho\mathbf{V}\cdot d\mathbf{S} = \mathbf{j}\cdot d\mathbf{S} \qquad \ldots(1)$$

where **j** is given to be current density vector or conduction current vector.

Now the current say i across any surface S drawn in the medium is given by $i = \int_S \mathbf{j}\cdot d\mathbf{S}$ while the total charge inside a closed surface S enclosing a volume v is given by $Q = \int_S \rho\, dv$.

Assuming that there are no sources or sinks inside the surface, the rate at which the charge is decreasing is

$$-\int_v \frac{\partial \rho}{\partial t}\, dv. \qquad \ldots(2)$$

Since this is due to the outward flow of charge, we have from (1) and (2),

$$-\int_v \frac{\partial \rho}{\partial t}\, dv = \int_S \mathbf{j}\cdot d\mathbf{S} = \int_v \operatorname{div}\mathbf{j}\; dv$$

by Gauss' divergence theorem §1.51

or
$$\int_V \left(\frac{\partial \rho}{\partial t} + \operatorname{div}\mathbf{j}\right) dv = 0.$$

The volume being arbitrary, the integrand must vanish identically and thus we have the equation of continuity or the equation of conservation of charge as

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\mathbf{j} = 0.$$

Problem 146. *Prove that* $\int_S \nabla\phi \times \nabla\psi \cdot d\mathbf{S} = 0.$

Put $\mathbf{F} = \nabla\phi \times \nabla\psi$ in Gauss' divergence theorem

$$\iint_S (\nabla\phi \times \nabla\psi)\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_V \nabla\cdot\mathbf{F}\; dV$$
$$= \iiint_V \nabla\cdot(\nabla\phi\times\nabla\psi)\, dV.$$

But $\nabla\cdot(\nabla\phi\times\nabla\psi) = \nabla\psi \operatorname{curl}\nabla\phi - \nabla\phi\operatorname{curl}\nabla\psi = 0.$

$$\therefore \iiint_V \nabla\cdot(\nabla\phi\times\nabla\psi)\, dV = \iiint_V [\nabla\psi\operatorname{curl}\nabla\phi - \nabla\phi\operatorname{curl}\nabla\psi]\, dV$$
$$= 0.$$

Hence $\int_S (\nabla\phi\times\nabla\psi)\cdot d\mathbf{S} = 0.$

Problem 147. (*a*) *If* $\mathbf{H} = \operatorname{curl}\mathbf{A}$, *prove that*

$\int_S \mathbf{H}\cdot\mathbf{n}\, dS = 0$ *for any closed surface S.*

(*b*) *If* **n** *is unit outward normal to any closed surface of area S show that*

$$\iiint_V \operatorname{div}\mathbf{n}\, dV = S.$$

(*a*) Given $\mathbf{H} = \operatorname{curl}\mathbf{A}$. The Divergence Theorem gives

$$\iint_S \mathbf{H}\cdot\mathbf{n}\; dS = \iiint_V \nabla\cdot\mathbf{H}\; dV,$$

But
$$\nabla\cdot\mathbf{H} = \nabla\cdot(\operatorname{curl}\mathbf{A})$$
$$= 0$$

$$\therefore \iint_S \mathbf{H}\cdot\mathbf{n}\, dS = 0.$$

(*b*) The Divergence Theorem gives

$$\iiint_V \operatorname{div}\mathbf{n}\, dV = \iint_S \mathbf{n}\cdot\mathbf{n}\, dS$$

$$= \iint_S dS = 'S.$$

Problem 148. *Prove that*

(*a*) $\int_V \mathbf{F} \cdot \nabla \phi \, dV = \int_S \phi \mathbf{F} \cdot d\mathbf{S} - \int_V \phi \nabla \cdot \mathbf{F} \, dV.$

(*b*) $\int_S \nabla \phi \cdot curl\, \mathbf{F} \, dV = \int_S (\mathbf{F} \times \nabla \phi) \, dS.$

(*a*) The divergence theorem gives

$$\int_V \mathbf{F} \cdot \nabla \phi \, dV = \int_V \nabla \cdot (\phi \mathbf{F}) \, dV - \int_V \phi \nabla \cdot \mathbf{F} \, dV$$

$$= \int_V [\phi \nabla \cdot \mathbf{F} - \mathbf{F} \cdot \nabla \phi] \, dV - \int_V \phi \nabla \cdot \mathbf{F} \, dV$$

$$= \int_V \phi \nabla \mathbf{F} \cdot dV + \int_V \mathbf{F} \cdot \nabla \phi \, dV - \int_V \phi \nabla \cdot \mathbf{F} \, dV$$

$$= \int_V \mathbf{F} \cdot \nabla \phi \, dV.$$

(*b*) Applying divergence theorem in R. H. S. of

$\int_V \nabla \phi \text{ curl } \mathbf{F} \, dV = \int_S (\mathbf{F} \times \nabla \phi) . d\mathbf{S}$, we get

$$\int_V \nabla \phi \text{ curl } \mathbf{F} \, dV = \int_V \nabla \cdot (\mathbf{F} \times \nabla \phi) \, dV$$

$$= \int_V [\nabla \phi \cdot \text{curl } \mathbf{F} - \mathbf{F} \cdot \text{curl } \nabla \phi] \, dV$$

$$= \int_V \nabla \phi \cdot \text{curl } \mathbf{F} \, dV \text{ as curl } \nabla \phi = 0.$$

Problem 149. *Show that volume enclosed by the surface S is*

$$V = \frac{1}{6} \oint_S \nabla r^2 \cdot d\mathbf{S}$$

where **r** *is the position vector to a point of dS.*

The divergence theorem gives

$$\int_S \nabla r^2 \cdot d\mathbf{S} = \int_V \nabla \cdot (\nabla r^2) \, dV$$

$$= \int_V \nabla \cdot [2r^{2-2}\mathbf{r}] \, dV$$

$$= 2 \int_V \nabla \cdot (\mathbf{r}) \, dV$$

$$= 2 \int_V 3 \, dV$$

$$= 6 \int_V dV = 6V$$

$$\therefore \quad V = \frac{1}{6} \oint_S \nabla r^2 \cdot d\mathrm{S}.$$

Problem 150. *Prove the following :*

(*a*) $\iiint_V \frac{dV}{r^2} = \iint_S \frac{\mathbf{r}\cdot\mathbf{n}}{r^2}\,dS.$

(*b*) $\iint_S r^5\mathbf{n}\ dS = \iiint_V 5r^3\mathbf{r}\ dV.$

(*a*) The divergence theorem gives

$$\iint_S \left(\frac{\mathbf{r}}{r^2}\right)\cdot\mathbf{n}\ dS = \iiint_V \nabla\cdot\left(\frac{\mathbf{r}}{r^2}\right)dV$$

$$= \iiint_V \nabla\cdot\left(\frac{1}{r^2}\mathbf{r}\right)dV$$

$$= \iiint_V \left[\frac{1}{r^2}\nabla\cdot\mathbf{r} + \nabla\frac{1}{r^2}\cdot\mathbf{r}\right]dV$$

$$= 3\iiint_V \frac{dV}{r^2} + \iiint_V -\frac{2}{r^4}\mathbf{r}\cdot\mathbf{n}\ dV$$

$$= 3\iiint_V \frac{dV}{r^2} - 2\iiint_V \frac{dV}{r^2} = \iiint_V \frac{dV}{r^2}$$

(*b*) We have

$$\iint_S r^5\mathbf{n}\ dS = \iiint_V \nabla\left(r^5\right)\ dV$$

$$= \iiint_V \left[5r^{2-2}\mathbf{r}\right]dV = 5\iiint_V r^3\mathbf{r}\ dV.$$

Problem 151. *Show that Green's second identity can be written as*

$$\iiint_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right)\,dV = \iint_S \left[\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right]dS.$$

The Green's identity is

$$\iiint_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\,dV = \iint_S [\phi\nabla\psi - \psi\nabla\phi]\cdot d\mathbf{S}.$$

$$= \iint_S (\phi\nabla\psi - \psi\nabla\phi)\cdot\mathbf{n}\ dS$$

where **n** is unit normal vector.

and $\quad \nabla\psi\cdot\mathbf{n} = \frac{\partial\psi}{\partial n}\ \nabla\phi\cdot\mathbf{n} = \frac{\partial\phi}{\partial n}.$

$$\therefore\quad \iiint_V (\phi\nabla^2\psi - \psi\,\nabla^2\phi)\,dV = \iint_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\,\frac{\partial\phi}{\partial n}\right)dS.$$

1.56. GREEN'S THEOREM IN THE PLANE

Let R be a closed region in the x · y plane bounded by a simple closed curve C and ø and ψ be two continuously differentiable functions of x and y, then Green's theorem in the plane is stated as

$$\oint_C (\psi\, dx + \phi\, dy) = \iint_R \left(\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y}\right) dx\, dy$$

where C is traversed in positive (anti-clockwise) direction.

Let the region R be bounded by simple closed curve C having the property that any straight line parallel to the axes cuts C in at most two points and let the parallels AM (say $x = a$), BN (say $x = b$); QF (say $y = c$) and DE (say $y = d$) limit the curve C in the xy plane as shown in Fig. 1.55.

Suppose the equations of the curves AQB and ADB are respectively

$$y = f_1(x) \text{ and } y = f_2(x).$$

Then, $$\iint_R \frac{\partial \psi}{\partial y} dx\, dy = \int_{x=a}^{b} \int_{y=f_1(x)}^{f_2(x)} \frac{\partial \psi}{\partial y} dx\, dy$$

$$= \int_a^b [\psi(x, y)]_{f_1(x)}^{f_2(x)} dx$$

$$= \int_a^b [\psi(x, f_2) - \psi(x, f_1)]\, dx$$

$$= -\left[\int_a^b \psi(x, f_1)\, dx - \int_b^a \psi(x, f_2)\, dx\right]$$

$$= -\oint_C \psi\, dx$$

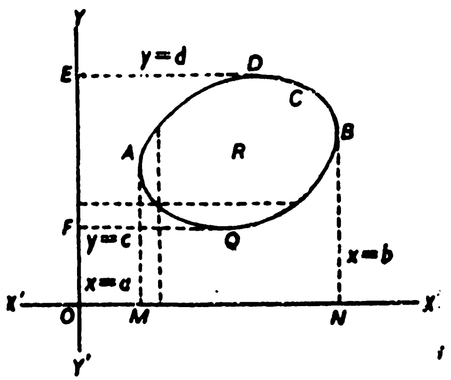


Fig. 1.55

i.e. $$\oint_C \psi\, dx = -\iint_R \frac{\partial \psi}{\partial y} dx\, dy. \quad ...(1)$$

Again if the equations to the curves DAQ and DBQ are

$x = F_1(y)$ and $x = F_2(y)$, respectively, then

$$\iint_R \frac{\partial \phi}{\partial x} dx\, dy = \int_{y=c}^{d} \int_{x=F_1(y)}^{F_2} \frac{\partial \phi}{\partial x} dy\, dx$$

$$= \int_c^d [\phi(x, y)]_{F_1(y)}^{F_2(y)} dy$$

$$= \int_c^d [\phi(F_2, y) - \phi(F_1, y)]\, dy$$

$$= \int_c^d \phi(F_2, y)\, dy + \int_d^c \phi(F_1\, y)\, dy$$

$$= \oint_C \phi\, dy$$

i.e. $$\oint_C \phi\, dy = \iint_R \frac{\partial \phi}{\partial x} dx\, dy. \quad ...(2)$$

Adding (1) and (2), we find that

$$\oint_C (\psi\ dx + \phi\ dy) = \iint_R \left(\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y}\right) dx\ dy.$$

1.57. VECTOR FORMS OF GREEN'S THEOREM IN THE PLANE

Vector treatment of Green's Theorem yields two different forms.

Form 1. *If* $\mathbf{f} = \psi\mathbf{i} + \phi\mathbf{j}$ *and* $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, *then Green's theorem takes the form*

$$\oint_C \mathbf{f} \cdot d\mathbf{r} = \iint_R (\nabla \times \mathbf{f}) \cdot \mathbf{k}\ dS.$$

Green's theorem in the plane is

$$\oint_C (\psi\ dx + \phi\ dy) = \iint_R \left(\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y}\right) dx\, dy \qquad \ldots(1)$$

Given $\quad \mathbf{r} = x\mathbf{i} + y\mathbf{j}$

i.e., $\quad d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j}$,

$$\therefore \quad \psi\ dx + \phi\ dy = (\psi\mathbf{i} + \phi\mathbf{j}) \cdot (dx\mathbf{i} + dy\mathbf{j})$$

$$= \mathbf{f} \cdot d\mathbf{r} \qquad \ldots(2)$$

and
$$\nabla \times \mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \psi & \phi & 0 \end{vmatrix} = -\frac{\partial \phi}{\partial z}\mathbf{i} + \frac{\partial \psi}{\partial z}\mathbf{j} + \left(\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y}\right)\mathbf{k}$$

so that
$$(\nabla \times \mathbf{f}) \cdot \mathbf{k} = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y} \qquad \ldots(3)$$

Substituting values from (2) and (3) in (1), we get

$$\oint_C \mathbf{f} \cdot dS = \iint_R (\nabla \times \mathbf{f}) \cdot \mathbf{k}\ dS$$

where $\quad dS = dx\, dy$ represents the elements of area.

Form 2. *If* $\mathbf{f} = \psi\mathbf{i} + \phi\mathbf{j}$, $dS = dx\ dy$, $\mathbf{g} = \mathbf{f} \times \mathbf{k}$ *and* $\mathbf{n}$ *be the outward drawn unit normal to C, then Green's theorem gives*

$$\oint_C \mathbf{g} \cdot \mathbf{n}\, ds = \iint_R \nabla \cdot \mathbf{g}\ dS,$$

where ds is an element of the curve.

Let **r** be the position vector of any point **P** and **T** be the unit tangent vector to the curve, then

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j}, \quad \text{so that } d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} \text{ and}$$

$$\mathbf{T} = \frac{d\mathbf{r}}{ds}.$$

Now **n** and **k**, both being vectors normal to the tangent at any point of the curve, the definition of cross product yields

$$\mathbf{k} \times \mathbf{n} = \mathbf{T}.$$

So that, $\psi\, dx + \phi\, dy$

$$= (\psi \mathbf{i} + \phi \mathbf{j}) \cdot (dx\mathbf{i} + dy\mathbf{j})$$
$$= \mathbf{f} \cdot d\mathbf{r}$$
$$= \mathbf{f} \cdot \frac{d\mathbf{r}}{ds} ds$$
$$= \mathbf{f} \cdot \mathbf{T}\, ds, \left(\because\ \mathbf{T} = \frac{d\mathbf{r}}{ds}\right)$$
$$= \mathbf{f} \cdot (\mathbf{k} \times \mathbf{n})\, ds$$
$$= (\mathbf{f} \times \mathbf{k}) \cdot \mathbf{n}\, ds$$

($\because$ in a scalar triple product dot and cross can be interchanged)

$$= \mathbf{g} \cdot \mathbf{n}\, ds.$$

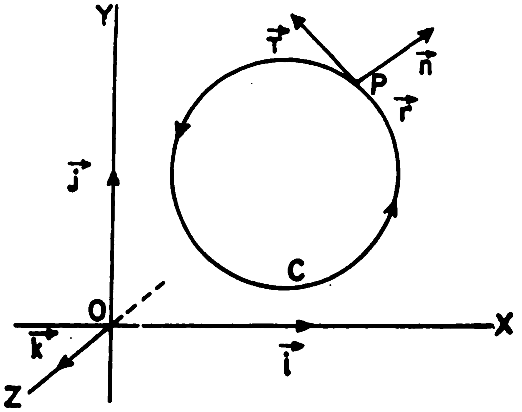


Fig. 1.56

Also $\mathbf{g} = \mathbf{f} \times \mathbf{k} + (\psi\mathbf{i} + \phi\mathbf{j}) \times \mathbf{k}$

$$= \phi\mathbf{i} - \psi\mathbf{j}$$

so that
$$\nabla \cdot \mathbf{g} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\right) \cdot (\phi\mathbf{i} - \psi\mathbf{j})$$
$$= \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y}.$$

Substituting these values in Green's theorem *i.e.*

$$\oint_C (\psi\, dx + \phi\, dy) = \iint_R \left(\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y}\right) dx\, dy$$

we get

$$\oint_C \mathbf{g} \cdot \mathbf{n}\, ds = \iint_R \nabla \cdot \mathbf{g}\, dS.$$

Note. **Physical Interpretation of Form 1.** Vector form 1 of Green's theorem is

$$\oint_C \mathbf{f} \cdot d\mathbf{r} = \iint_R (\nabla \times \mathbf{f}) \cdot \mathbf{k}\, dS.$$

Assuming that f represents a force field acting on a particle whose position vector is r, the integral $\oint_C \mathbf{f} \cdot d\mathbf{r}$ may be interpreted as expressing work done in moving the particle around the closed path C and it may be evaluated by the value of $\nabla \times \mathbf{f}$.

As a particular case if $\nabla \times \mathbf{f} = 0$ *i.e.* if $\mathbf{f} = \nabla \theta$, θ being scalar; then the integral around a closed path is zero. It follows that the work done by a particle in moving from one point of the plane to the other point of the plane is independent of the path traced in moving from one point to another in the plane. In other words this fact can be expressed by saying that the force field is **conservative.**

Conversely, if the integral around a closed path is independent of the path joining any two points in the plane *i.e.*, the integral around the closed path is zero, then $\nabla \times \mathbf{f} = 0$ where $\mathbf{f} = \psi\mathbf{i} + \phi\mathbf{j}$.

i.e.
$$-\frac{\partial \phi}{\partial z}\mathbf{i} + \frac{\partial \psi}{\partial z}\mathbf{j} + \left(\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y}\right)\mathbf{k} = 0.$$

Giving
$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}.$$

1.58. GREEN'S FORMULA

Suppose that $\psi = \frac{1}{r}$ is a scalar point function which has uniform finite and continuous derivatives upto the second order in a region V enclosed by a closed surface S.

Take a fixed point P within the region, such that r is distance from P to a variable point of the region and **r** its position vector relative to P. Now since ψ becomes infinite at P, therefore to remove this difficulty enclose P by small sphere of redius ε. Take surface of this sphere as S_1. Clearly in the region V, bounded by S and S_1, ψ is twice continuously differentiable.

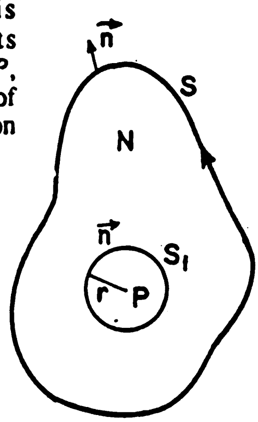


Fig. 1.57

But we know that,

$$\nabla\psi = \nabla\left(\frac{1}{r}\right) = \nabla\left[\frac{1}{\left(x^2+y^2+z^2\right)^{1/2}}\right] = -\frac{\mathbf{r}}{r^2}$$

and
$$\nabla^2\psi = \nabla\left(-\frac{\mathbf{r}}{r^2}\right) = 0.$$

Thus, applying Green's identity to the region bounded by S and S_1, we get

$$\int_V\left[-\frac{1}{r}\ \nabla^2\phi + 0\right]dv = \int_S\left[\phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial\phi}{\partial n}\right]\ dS$$

$$+\int_{S_1}\left[\phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial\phi}{\partial n}\right]dS_1 \qquad \ldots(1)$$

As the surface S_1 the direction of unit normal drawn outward from the region considered will be towards P, so that

$$\frac{\partial}{\partial n}\left(\frac{1}{r}\right) \text{ on } S_1 = \left[\nabla\left(\frac{1}{r}\right)\cdot\mathbf{n}\right]_{r=\varepsilon}$$

$$= \left[\left(-\frac{\mathbf{r}}{r^3}\right)\left(-\frac{\mathbf{r}}{r}\right)\right]_{r=\varepsilon}$$

$$= \left[\frac{(\mathbf{r})^2}{r^4}\right]_{r=\varepsilon} = \left[\frac{1}{r^2}\right]_{r=\varepsilon}$$

$$= \frac{1}{\varepsilon^2}.$$

$$\therefore \qquad \int_{S_1}\phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\ dS = \frac{1}{\varepsilon^2}\int_{S_1}\phi\ dS_1.$$

Taking to the limit as $\varepsilon\to 0$, we find

$$\text{Lim}_{\varepsilon\to 0}\int_{S_1}\phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\ dS_1 = \frac{4\pi\varepsilon^2}{\varepsilon^2}\phi\,(P) = 4\pi\ \ \phi\ (P) \qquad \ldots(2)$$

and
$$\text{Lim}_{\varepsilon\to 0}\int\ \frac{1}{r}\ \frac{\partial\phi}{\partial n}dS = 0 \qquad \ldots(3)$$

In the limiting case when $\varepsilon \to 0$, (1) yields with the help of (2) and (3),

$$\int_V -\frac{1}{r}\nabla^2 \phi \; dv = \int_S \left[\phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial \phi}{\partial n}\right] dS + 4\pi \; \phi(P)$$

Thus $$4\pi \; \phi(P) = \int_S \left[\frac{1}{r}\frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right] dS - \int_V \frac{1}{r} \; \nabla^2 \phi \; dv$$

or $$4\pi \; \phi(P) = \int_S \left[\frac{1}{r}\nabla\phi - \phi\nabla\left(\frac{1}{r}\right)\right]\cdot \mathbf{n} \; dS - \int_V \frac{1}{r} \; \nabla^2 \phi \; dv$$

$$4\pi \; \phi(P) = \int_S \left[\frac{1}{r}\nabla\phi - \phi\nabla\left(\frac{1}{r}\right)\right]\cdot d\mathbf{S} - \int_V \frac{1}{r} \; \nabla^2 \phi \; dv$$

which is known as Green's formula.

1.59. POISSON'S EQUATION WITH ITS SOLUTION

Let ϕ be a scalar point function vanishing outside a finite region, then the equation

$$\nabla^2\phi = -4\pi\rho,$$

is known as **Poisson's equation.**

Poisson's equation is $\nabla^2\phi = -4\pi\rho$ ...(1)

Green's formula is

$$4\pi \; \phi(P) = \int_S \left[\frac{1}{r}\nabla\phi - \phi\nabla\left(\frac{1}{r}\right)\right) d\mathbf{S} - \int_V \frac{1}{r}\nabla^2 \phi \; dv \quad ...(2)$$

(for a region bounded by a surface S)

$$= \int_S \left[\frac{1}{r}\nabla\phi - \phi\nabla\left(\frac{1}{r}\right)\right]\cdot d\mathbf{S} + 4\pi\int \frac{\rho}{r} \; dv \text{ by (1)} \quad ...(3)$$

In case the region V tends to infinity S also recedes to infinity.

Supposing that for large values of r, ϕ is of the form $\frac{\lambda}{r}$ where λ remains bounded, $|\nabla\phi|$ is of the form $\frac{\lambda}{r^2}$.

So that $$\int_S \left[\frac{1}{r}\nabla\phi - \phi\nabla\left(\frac{1}{r}\right)\right]\cdot d\mathbf{S} \to 0.$$

Then $$4\pi \; \phi(P) = 4\pi \int \frac{\rho}{r} dv$$

or $$\phi(P) = \int \frac{\rho}{r} dv.$$

The volume integral being carried over the whole space remains the same as the volume integral over the region outside at which ρ is zero.

Vector equivalent

If $\mathbf{f} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$.

and $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$

where $\nabla^2\mathbf{F} = -4\pi\mathbf{f}$.

Then $\mathbf{F}(P) = \int \frac{\mathbf{f}}{r} dv.$

$$\nabla^2\mathbf{F} = -4\pi\mathbf{f} \text{ is equivalent to}$$
$$\nabla^2 F_1 = -4\pi f_1;\ \nabla^2 F_2 = -4\pi f_2;\ \nabla^2 F_3 = -4\pi f_3$$

Thus for a point P of the region

$$F_1(P) = \int \frac{f_1}{r} dv;\ F_2(P) = \int \frac{f_2}{r} dv;\ F_3(P) = \int \frac{f_3}{r} dv.$$

Imposing suitable conditions on f_1, f_2, f_3, and multiplying these relations by **i**, **j**, **k** respectively and then adding, we have

$$\mathbf{i} F_1(P) + \mathbf{j} F_2(P) + \mathbf{k} F_3(P) = \int (\mathbf{i}f_1 + \mathbf{j}f_2 + \mathbf{k}f_3) \frac{1}{r} dv$$
$$\mathbf{F}(P) = \int \frac{\mathbf{f}}{r} dv.$$

1.60. LAPLACE'S EQUATION WITH ITS SOLUTION

If for a twice differentiable scalar point function ϕ, $\nabla^2\phi = 0$ *is true for every point of the region, the function* ϕ *is said to be* **harmonic** *in the region.*

The equation $\nabla^2\phi = 0 \ldots (1)$ is called Laplace's equation. Green's formula is

$$4\pi\phi(P) = \int_S \left[\frac{1}{r}\nabla\phi - \phi\nabla\left(\frac{1}{r}\right)\right]\cdot d\mathbf{S} - \int \frac{1}{r}\nabla^2\phi\, dv$$
$$= \int_S \left[\frac{1}{r}\nabla\phi - \phi\nabla\left(\frac{1}{r}\right)\right]\cdot d\mathbf{S} \text{ by (1)}$$

which follows that the harmonic function ϕ at any point within the region can be expressed in terms of the values of ϕ and $\frac{\partial\phi}{\partial n}$ at any point of the surface enclosing the region.

Problem 152. *Verify Green's theorem in the plane for*

$$\oint_C (xy + y^2)\, dx + x^2\, dy$$

where C is the closed curve of the region bounded by

$$y = x^2, \text{ and } y = x.$$

The shaded region shown in Fig. 1.58, represents the positive direction traversed by the closed region C made up of a parabola and a straight line.

Given, $\psi = (xy + y^2)$ and $\phi = x^2$

Evaluating the integral along $z = x^2$, we have,

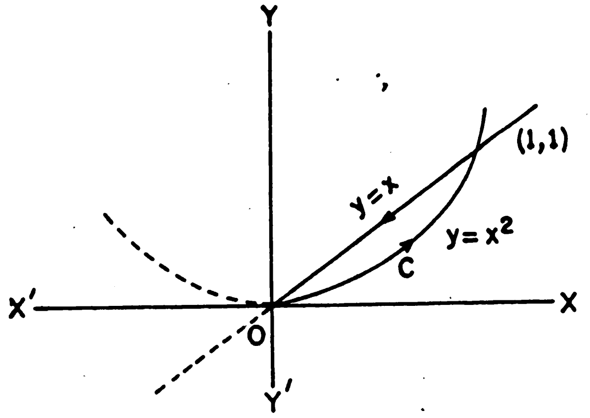


Fig. 1.58

$$\int \psi\, dx + \phi\, dy$$
$$= \int [(x \cdot x^2 + x^4)\, dx + y\, dy]$$
$$= \int_{x=0}^{1} (x^2 + x^4)\, dx + \int_{x=0}^{1} y\, dy$$
$$= \left[\frac{x^4}{4} + \frac{x^5}{5}\right]_0^1 + \left[\frac{y^2}{2}\right]_0^1$$
$$= \frac{1}{4} + \frac{1}{5} + \frac{1}{2}$$
$$= \frac{19}{20}.$$

Evaluating the line integral along $y = x$ we have

$$\int \psi \; dx + \phi \; dy = \int\left[\left(x^2 + x^2\right) dx + y^2 dy\right] = \int_{x=1}^{0} 2x^2 dx + \int_{y=1}^{0} y^2 dy$$
$$= -1.$$

$\therefore$ the required integral $= \frac{19}{20} - 1 = -\frac{1}{20}$.

Also $\frac{\partial \phi}{\partial x} = 2x$ and $\frac{\partial \psi}{\partial y} = x + 2y$.

Thus, $\iint_R \left(\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y}\right) dx\, dy$

$$= \iint_R [2x - x - 2y] dx\, dy = \int_{x=0}^{x=1}\int_{y=x^2}^{y=x} (x - 2y)\, dx\, dy$$
$$= \int_{x=0}^{1} \left[xy - y^2\right] dx = \int_{x=0}^{1}\left[x^2 - x^2 - x^3 + x^4\right] dx$$
$$= \int_{x=0}^{1}\left(x^4 - x^3\right) dx = -\frac{1}{20}.$$

It is evident that $\int \psi \; dx \; + \phi \; dy = \iint\left(\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y}\right) dx\, dy = -\frac{1}{20}$.

Thus Green's theorem is verified.

Problem 153. *Evaluate* $\oint_C [(y - \sin x)\, dx + \cos x\, dy]$ *where C is the triangle whose vertices are* $(0, 0)$; $\left(\frac{\pi}{2}, 0\right)$; $\left(\frac{\pi}{2}, 1\right)$

(a) directly, (b) by using Green's theorem in the plane.

(*a*) The line integral along OQ on which $y = 0$ and x varies from 0 to $\frac{\pi}{2} = \int_{x=0}^{\pi/2} \sin x\, dx = -1.$

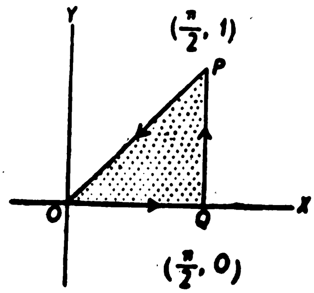


Fig. 1.59

The line integral along QP on which $x = \frac{\pi}{2}$ any y varies from 0 to 1

$$= \int_{y=0}^{1} \{(y - 1)\, 0 + 0\, dy\} = 0.$$

The line integral along PO for which $y = \frac{2x}{\pi}$ and x varies from $\frac{\pi}{2}$ to 0,

$$= \int_{\pi/2}^{0}\left\{\left(\frac{2x}{\pi} - \sin x\right)dx + \frac{2}{\pi} \cos x\, dx\right\}$$

$$= \left[\frac{x^2}{\pi} + \cos x + \frac{2}{\pi} \sin x\right]_{\pi/2}^{0} = 1 - \frac{\pi}{4} - \frac{2}{\pi}.$$

Hence the line integral along $C = -1 + 0 + 1 - \frac{\pi}{4} - \frac{2}{\pi} = -\frac{\pi}{4} - \frac{2}{\pi}$.

(*b*) In order to use Green's theorem we have

$$\psi = y - \sin x,\ \phi = \cos x,\ \frac{\partial \psi}{\partial y} = 1,\ \frac{\partial \phi}{\partial x} = -\sin x$$

$$\therefore \oint_C (\psi\, dx + \phi\, dy) = \iint \left(\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y}\right) dx\, dy = \iint_R (-y \sin x - 1)\, dx\, dy$$

$$= \int_{x=0}^{\pi/2} \int_{y=0}^{2x/\pi} [(-\sin x - 1)\, dy]\, dx = \int_{x=0}^{\pi/2} (-y \sin x - y)_0^{2x/\pi}\, dx$$

$$= \int_0^{\pi/2} \left(-\frac{2x}{\pi} \sin x - \frac{2x}{\pi}\right) dx$$

$$= \left[-\frac{2}{\pi}(-x \cos x + \sin x) - \frac{x^2}{\pi}\right]_0^{\pi/2}$$

$$= -\frac{2}{\pi} - \frac{\pi}{4}.$$

Problem 154. *Compute* $\oint \{(xy - x^2)\, dx + x^2 y\, dy\}$ *over the triangle bounded by lines* $y = 0, x = 1, y = x$ *and verify by Green's theorem.*

The line integral along OP where $y = 0$ and x varies from 0 to 1.

$$= \int_{x=0}^{1} -x^2\, dx = -\frac{1}{3}.$$

The line integral along PQ for which $x = 1$ and y varies from 0 to 1

$$= \int_{y=0}^{1} \{(y-1)\, 0 + y\, dy\} = \frac{1}{2}.$$

The line integral along QO for which $x = y$ and y varies from 1 to 0

$$= \int_1^0 \{(y^2 - y^2)\, dy + y^3\, dy\} = -\int_1^0 y^3 dy = -\frac{1}{4}.$$

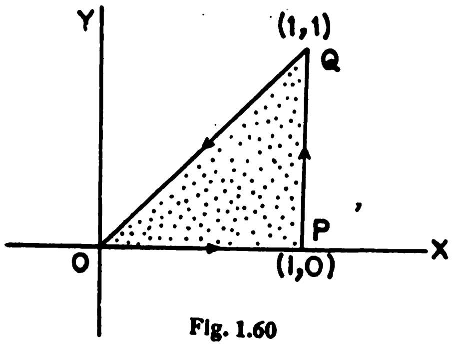


Fig. 1.60

Thus the total integral along C

$$= -\frac{1}{3} + \frac{1}{2} - \frac{1}{4} = -\frac{1}{12}.$$

Now, by Green's theorem, we have

$$\int \{(xy - x^2)\, dx + x^2 y\, dy\}$$

$$= \iint \left[\frac{\partial (x^2 y)}{\partial x} - \frac{\partial}{\partial y}(xy - x^2)\right] dx\, dy$$

$$= \int_{x=0}^{1} \int_{y=0}^{y=x} [2xy - x]\, dx\, dy$$

$$= \int_{x=0}^{1} [x^3 - x^2]\, dx = \left[\frac{x^4}{4} - \frac{x^3}{3}\right] = -\frac{1}{12}.$$

which verifies the Green's theorem.

1.61. STOKES' THEOREM IN SPACE

This states that if, **F** *is a vector function, which is uniform, finite and continuous along with its derivative in any direction, then the tangential line integral of* **F** *over any closed* *surface S bounded by a curve C is equal to the normal surface integral of curl* **F** *over S; i.e.,*

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

where **n** *is the unit normal vector at any point of S drawn in the sense in which a right handed screw would move when rotated in the sense of description of C.*

(*Agra 1956; Vikram, 1969*)

Consider a surface S such that its projections on the xy, yz, zx planes are regions bounded by simple closed curves as shown in Fig. 1.61.

Take the equation of surface as

$f(z, y, z) = 0$ *i.e.*,

$z = f_1(x, y)$

or $y = f_2(x, z)$ or $x = f_3(y, z)$

If $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$,

then we have to prove that

$$\iint_S \nabla \times (F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

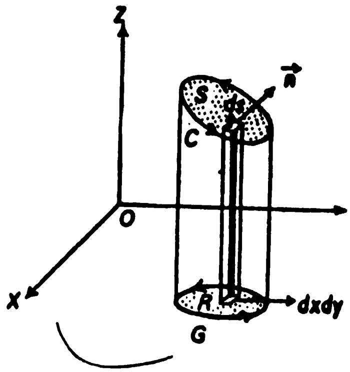


Fig. 1.61

Let us first consider,

$[\nabla \times (F_1\mathbf{i})] \cdot \mathbf{n}dS$

$$= \left[\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times F_1\mathbf{i}\right] \cdot \mathbf{n}\, dS$$

$$= \left[\frac{\partial F_1}{\partial z}\mathbf{j} - \frac{\partial F_1}{\partial y}\mathbf{k}\right] \cdot \mathbf{n}\, dS$$

$$= \left[\frac{\partial F_1}{\partial z}\mathbf{n} \cdot \mathbf{j} - \frac{\partial F_1}{\partial y}\mathbf{n} \cdot \mathbf{k}\right] dS \qquad \text{... (1)}$$

and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x\mathbf{i} + y\mathbf{j} + f_1(x, y)\mathbf{k}$

So that $\dfrac{\partial \mathbf{r}}{\partial y} = \mathbf{j} + \dfrac{\partial f_1}{\partial y}\mathbf{k}$ since $z = f_1(x, y)$.

Now $\dfrac{\partial \mathbf{r}}{\partial y}$ is perpendicular to **n** as, $\dfrac{\partial \mathbf{r}}{\partial y}$ is the tangent to the surface S.

$$\therefore \quad \mathbf{n}\frac{\partial \mathbf{r}}{\partial y} = \mathbf{n} \cdot \mathbf{j} + \frac{\partial f_1}{\partial y}\mathbf{n} \cdot \mathbf{k} = 0.$$

Giving $\mathbf{n} \cdot \mathbf{j} = -\dfrac{\partial f_1}{\partial y}\mathbf{n} \cdot \mathbf{k} = -\dfrac{\partial z}{\partial y}\mathbf{n} \cdot \mathbf{k}$.

As such (1) yields

$$[\nabla \times (\mathbf{F}_1\mathbf{i})] \cdot \mathbf{n}\, dS = -\left[\frac{\partial F_1}{\partial z} \cdot \frac{\partial z}{\partial y} + \frac{\partial F_1}{\partial y}\right] \mathbf{n} \cdot \mathbf{k}\, dS \qquad \text{...(2)}$$

But on the surface S, we have

$$F_1(x, y, z) = F_1[x, y, f_1(x, y)] = F(x, y)$$

$$\therefore \qquad \frac{\partial F_1}{\partial y}+\frac{\partial F_1}{\partial z}\cdot\frac{\partial z}{\partial y}=\frac{\partial F}{\partial y} \qquad \text{...(3)}$$

The relation (2) with the help of (3) gives

$$\left[\nabla\times(F_1\mathbf{i})\right]\cdot\mathbf{n}\ dS=-\frac{\partial F}{\partial y}\ \mathbf{n}\cdot\mathbf{k}\ dS=-\frac{\partial F}{\partial y}\ dx\,dy$$

$$\therefore \qquad \iint_S\left[\nabla\times(F_1\mathbf{i})\right]\cdot\mathbf{n}\ dS=\iint_R -\frac{\partial F}{\partial y}\ dx\,dy \qquad \text{...(4)}$$

where R is the projection of S on xy plane.

Green's theorem in plane gives

$\oint_{C_1} F\,dx=-\iint_R\frac{\partial F}{\partial y}\,dx\,dy$ where C_1 is the boundary of R.

Now at each point (x, y) of the curve C_1 the value of F being the same as that of F_1 at each point (x, y, z) of C and dx being the same for both the curves C and C_1, we conclude that

$$\oint_{C_1} F\,dx=\oint_C F_1\,dx$$

$$i.e. \qquad \oint_C F_1\,dx=-\iint_R\frac{\partial F}{\partial y}\,dx\,dy \qquad \text{...(5)}$$

The equations (4) and (5), give

$$\iint_S\left[\nabla\times(F_1\mathbf{i})\right]\cdot\mathbf{n}\ dS=\oint_C F_1\,dx \qquad \text{...(6)}$$

Similar contributions are made by the projections on the other planes, therefore

$$\iint_S\left[\nabla\times(F_2\mathbf{j})\right]\cdot\mathbf{n}\ dS=\oint_C F_2\,dy \qquad \text{...(7)}$$

$$\iint_S\left[\nabla\times(F_3\mathbf{k})\right]\cdot\mathbf{n}\ dS=\oint_C F_3\,dz \qquad \text{...(8)}$$

Adding (6), (7) and (8), we find

$$\iint_S\left[\nabla\times(F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k})\right]\cdot\mathbf{n}\ dS=\oint_C(F_1\,dx+F_2\,dy+F_3\,dz)$$

$$\iint_S(\nabla\times\mathbf{F})\cdot d\mathbf{S}=\iint_S(\nabla\times\mathbf{F})\cdot\mathbf{n}\ dS=\oint_C\mathbf{F}\cdot d\mathbf{r}.$$

Problem 155. *If ϕ is continuously differentiable scalar point function then show that*

$$\oint_C\phi\ d\mathbf{r}=\iint_S\mathbf{n}\times\nabla\phi\ dS$$

Put $\mathbf{F}=\mathbf{a}\phi$, where $\mathbf{a}$ is a constant vector, in the Stokes' theorem

$$\int_C(\mathbf{a}\phi)\cdot d\mathbf{r}=\iint_S\left[\nabla\times(\mathbf{a}\phi)\right]\cdot\mathbf{n}\,dS$$

But we know that

$$\nabla\times(\mathbf{a}\phi)=\text{grad}\ \phi\times\mathbf{a}+\phi\ \text{curl}\ \mathbf{a}=\nabla\phi\times\mathbf{a}$$

$$\therefore \quad \mathbf{a}\cdot\int_C \phi\, d\mathbf{r} = \iint_S (\nabla\phi\times\mathbf{a})\cdot\mathbf{n}\ dS$$

$$= -\iint_S \mathbf{a}\times\nabla\phi\cdot\mathbf{n}\ dS$$

$$= -\iint_S \mathbf{a}\cdot\nabla\phi\times\mathbf{n}\ dS = \mathbf{a}\cdot\iint_S \nabla\phi\times\mathbf{n}\ dS$$

Since a is a constant vector

$$\therefore \quad \int_C \phi\, d\mathbf{r} = -\iint_S \nabla\phi\times\mathbf{n}\ dS = \iint_S \mathbf{n}\times\nabla\phi\ dS.$$

Problem 156. *Show that*

$$\int_C d\mathbf{r}\times\mathbf{F} = \iint_S (\mathbf{n}\times\nabla)\times\mathbf{F}\,dS.$$

Putting $\mathbf{f} = \mathbf{a}\times\mathbf{F}$, a being a constant vector;
Stokes' theorem gives

$$\int_C (\mathbf{a}\times\mathbf{F})\cdot d\mathbf{r} = \iint_S [\nabla\times(\mathbf{a}\times\mathbf{F})]\cdot\mathbf{n}\ dS$$

But we know that

$$\nabla\times(\mathbf{a}\times\mathbf{F}) = \mathbf{a}\nabla\cdot(\mathbf{F}) - \mathbf{F}\nabla\cdot(\mathbf{a}) + (\mathbf{F}\cdot\nabla)\,\mathbf{a} - (\mathbf{a}\cdot\nabla)\,\mathbf{F}$$

$$= \mathbf{a}\nabla\cdot\mathbf{F} - (\mathbf{a}\cdot\nabla)\,\mathbf{F}$$

and $\quad [(\mathbf{a}\cdot\nabla)\mathbf{F}]\cdot\mathbf{n} = (a\cdot\nabla)\,(\mathbf{F}\cdot\mathbf{n}).$

$$\therefore \quad \int (\mathbf{a}\times\mathbf{F})\cdot d\mathbf{r} = \iint_S [\mathbf{a}\ (\nabla\cdot\mathbf{F})\ \mathbf{n}\times\mathbf{a}\cdot\nabla\ (\mathbf{F}\cdot\mathbf{n})]\ dS$$

i.e. $\quad \mathbf{a}\cdot\int \mathbf{F}\times d\mathbf{r} = \mathbf{a}\cdot\iint_S [(\nabla\cdot\mathbf{F})\ \mathbf{n} - \nabla\ (\mathbf{F}\cdot\mathbf{n})]\ dS$

(Since a is a constant vector)

$$\therefore \quad \int \mathbf{F}\times d\mathbf{r} = \iint_S [(\nabla\cdot\mathbf{F})\ \mathbf{n} - \nabla\ (\mathbf{F}\cdot\mathbf{n})]\ dS$$

$$= \iint_S -[(\mathbf{n}\times\nabla)\times\mathbf{F}]\ dS$$

or $\quad \int d\mathbf{r}\times\mathbf{F} = \iint_S [(\mathbf{n}\times\nabla)\times\mathbf{F}]\ dS.$

Note. Stokes' theorem in the plane is sometimes known as Green's Theorem in the plane.

Problem 157. *Verify Stokes' theorem for* $\mathbf{F} = (2x-y)\,\mathbf{i} - yz^2\mathbf{j} - y^2z\mathbf{k}$, *where S is the uppper half surface of the sphere* $x^2 + y^2 + z^2 = 1$ *and C is its boundary.*

(Meerut, 1980)

Stokes' theorem is

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\ dS.$$

In $z = 0$ plane the boundary C of the surface S is a circle $x^2 + y^2 = 1$.

Put $x = \cos t$, $y = \sin t$ and $z = 0$ for which $0 \le t \le 2\pi$ so that these form the parametric equations of C.

$$\therefore \quad \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C [(2x-y)\ \mathbf{i}]\cdot[dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}] = \int_C (2x-y)\ dx$$

$$= \int_{t=0}^{2\pi} [2\cos t - \sin t]\ (-\sin t\, dt)$$

$$= \int_{t=0}^{2\pi} (2 \sin t \cos t - \sin^2 t)\, dt$$

$$= \pi.$$

and
$$\nabla \times \mathbf{F} = \mathbf{i}\left[\frac{\partial}{\partial y}(-y^2 z) - \frac{\partial}{\partial z}(-yz^2)\right]$$

$$+\mathbf{j}\left[\frac{\partial}{\partial z}(2x-y) - \frac{\partial}{\partial x}(-y^2 z)\right] + \mathbf{k}\left[\frac{\partial}{\partial x}(-yz^2) - \frac{\partial}{\partial y}(2x-y)\right]$$

$$= 0\mathbf{i} + 0\mathbf{j} + \mathbf{k} = \mathbf{k}.$$

$$\therefore \quad \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_S \mathbf{k} \cdot \mathbf{n}\, dS$$

$$= \iint_R dx\, dy \text{ since } \mathbf{k} \cdot \mathbf{n}\, dS = dx\, dy$$

R being the projection of S on xy plane.

Thus, $$\iint_R dx\, dy = \int_{x=-1}^{x=+1} \int_{y=-\sqrt{(1-x^2)}}^{+\sqrt{(1-x^2)}} dx\, dy$$

$$= \int_0^1 \sqrt{(1-x^2)} = 4\frac{\pi}{4} = \pi$$

which verifies Stokes' theorem.

Problem 158. *Evaluate* $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$ *for*

$$\mathbf{F} = (y - z + 2)\,\mathbf{i} + (yz + 4)\,\mathbf{j} - xz\mathbf{k},$$

where S is the surface of the cube $x = y = z = 0$; $x = y = z = 2$ above the xy plane.

Stokes' theorem is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$$

Here the boundary C of the surface S is a square bounded by the lines $x = 0, x = 2$, $y = 0$, $y = 2$ in the xy plane. So

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{OP} \mathbf{F} \cdot d\mathbf{r} + \int_{PQ} \mathbf{F} \cdot d\mathbf{r} + \int_{QR} \mathbf{F} \cdot d\mathbf{r} + \int_{RO} \mathbf{F} \cdot d\mathbf{r}$$

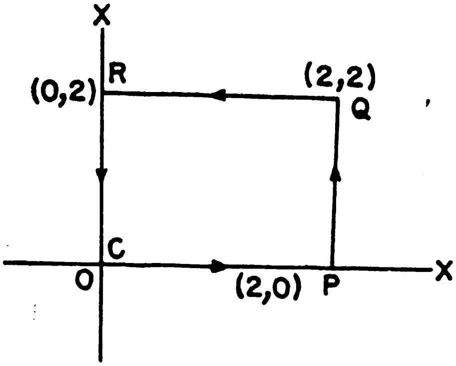


Fig. 1.62

Along, OP, $y = 0$ and x varies from 0 to 2.

$$\therefore \quad \int_{OP} \mathbf{F} \cdot d\mathbf{r} = \int [2\mathbf{i} + 4\mathbf{j}] \cdot [dx\,\mathbf{i}]$$

as $dy = 0$ and $dz = 0$

$$= 2 \int_{x=0}^{2} dx = 4.$$

Along PQ, $x = 2$ and y varies from 0 to 2 so that $dx = 0$, $dz = 0$.

$$\therefore \quad \int_{PQ} \mathbf{F} \cdot d\mathbf{r} = \int [(y + z)\,\mathbf{i} + 4\mathbf{j}] \cdot [\mathbf{j}\, dy]$$

$$= 4 \int_0^2 dy = 8.$$

Along QR, $y = 2$ and x varies from 2 to 0; so that $dy = 0$, $dz = 0$.

$\therefore$ $$\int_{QR} \mathbf{F}\cdot d\mathbf{r} = \int [4\mathbf{i}]\cdot[dx\,\mathbf{i}]$$

$$= 4\int_2^0 dx = -8.$$

Along RO, $x = 0$ and y varies from 2 to 0 so that $dx = 0$, $dz = 0$

$\therefore$ $$\int_{RO} \mathbf{F}\cdot d\mathbf{r} = \int [4\mathbf{j}]\cdot[dy\,\mathbf{j}] = 4\int_2^0 dy = -8.$$

Hence $\int_C \mathbf{F}\cdot d\mathbf{r} = 4 + 8 - 8 - 8 = -4.$

Problem 159. *Verify Stokes' theorem for the vector* $\mathbf{F} = (z, x, y)$ *taken over the half of the sphere* $x^2 + y^2 + z^2 = a^2$ *lying above xy plane.*

The projection of the surface on $z = 0$ plane is a circle $x^2 + y^2 = a^2$, of boundary C (say),

and $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$

$\therefore$ $$\int_C \mathbf{F}\cdot d\mathbf{r}$$

$$= \int_C (0\mathbf{i} + x\mathbf{j} + y\mathbf{k})\cdot(dx\mathbf{i} + dy\mathbf{j})$$

$$= \int_{y=-a}^{y=+a} x\,dy$$

$$= \int_{y=-a}^{a} \pm\sqrt{(a^2 - y^2)}\,dy$$

$$= 2\int_0^a \sqrt{(a^2 - y^2)}\,dy,$$

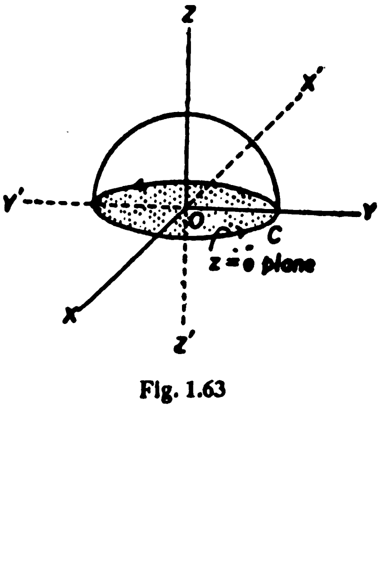


Fig. 1.63

put $x = a\sin\theta$, so that

$dx = a\cos\theta\, d\theta$

$$= 2\int_0^{\pi/2} a^2\cos^2\theta\, d\theta = \pi a^2$$

But $$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = (\mathbf{i} - \mathbf{j} + \mathbf{k})$$

$\therefore$ $$\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\, dS = \iint_S (\mathbf{i} - \mathbf{j} + \mathbf{k})\cdot\mathbf{n}\, dS$$

$$= \iint_S (\mathbf{i}\cdot\mathbf{n} - \mathbf{j}\cdot\mathbf{n} + \mathbf{k}\cdot\mathbf{n})\, dS$$

$$= \iint_S \mathbf{i}\cdot\mathbf{n}\, dS - \iint_S \mathbf{j}\cdot\mathbf{n}\, dS + \iint_S \mathbf{k}\cdot\mathbf{n}\, dS$$

$$= \iint_{R_1} dy\, dz - \iint_{R_2} dx\, dz + \iint_{R_3} dx\, dy$$

$$= \iint_{R_3} dx\, dy,$$

R_1, R_2, R_3 being projections on zy, yz, xy planes, respectively and projection on zy plane being the same as that on xz plane, the first two integrals cancel out.

Thus $$\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\ dS = \int_{x=-a}^{a}\int_{y=-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} dx\ dy$$

$$= 2\int_{-a}^{a}\sqrt{(a^2-x^2)}\ dx\ dy$$

$$= 4\int_0^a \sqrt{(a^2-x^2)}\ dx$$

$$= 4\pi\frac{a^2}{4} = \pi a^2.$$

Hence $$\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\ dS = \int_C \mathbf{F}\cdot d\mathbf{r}$$

which verifies Stokes' theorem.

Problem 160. *Prove that*

$$\int_C \mathbf{r}\times d\mathbf{r} = 2\iint_S dS$$

where S is a diaphragm enclosing a circuit C.

Putting $\mathbf{F} = \mathbf{a}\times\mathbf{r}$, where $\mathbf{a}$ is a constant vector, in Stokes' theorem, *i.e.*

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\ dS,$$

We get

$$\int_C (\mathbf{a}\times\mathbf{r})\cdot d\mathbf{r} = \iint_S \nabla\times(\mathbf{a}\times\mathbf{r})\cdot d\mathbf{S}$$

But $\nabla\times(\mathbf{a}\times\mathbf{r}) = \mathbf{a}\ \nabla\cdot\mathbf{r} - (\mathbf{a}\cdot\nabla)\ \mathbf{r} = 3\mathbf{a} - 2\mathbf{a} = 2\mathbf{a}.$

$$\therefore\quad \int_C (\mathbf{a}\cdot\mathbf{r})\times d\mathbf{r} = \iint_S 2\mathbf{a}\cdot d\mathbf{S}$$

$$\mathbf{a}\cdot\int \mathbf{r}\times d\mathbf{r} = 2\mathbf{a}\cdot\iint_S d\mathbf{S}$$

Since $\mathbf{a}$ is an arbitrary constant vector.

$$\therefore\quad \int \mathbf{r}\times d\mathbf{r} = 2\iint_S d\mathbf{S}.$$

Problem 161. *Prove that a necessary and sufficient condition that*

$$\oint \mathbf{F}\cdot d\mathbf{r} = 0$$

for every closed curve C is that $\nabla\times\mathbf{F} = 0$ *identically.* (*Rajasthan, 1978*)

The condition is sufficient :

Since if $\nabla\times\mathbf{F} = 0$

Then Stokes' theorem gives at once,

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\ dS = 0.$$

Also the condition is necessary: Since if $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ round any closed curve C, then taking $\nabla\times\mathbf{F}\neq 0$ at some point P, there will be a region with P as an interior point where $\nabla\times\mathbf{F}\neq 0$ provided $\nabla\times\mathbf{F}$ is continuous. Assuming S to be the surface contained in this region whose unit normal $\mathbf{n}$ at each point has the same direction as that of $\nabla \times \mathbf{F}$, we may express $\nabla \times \mathbf{F} = \lambda \mathbf{n}$ where λ is a positive constant. Thus Stokes' theorem gives

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \lambda \iint_S \mathbf{n} \cdot \mathbf{n}\; dS = \lambda \iint dS \neq 0$$

i.e., it yields positive contribution.

This is contrary to our hypothesis and hence $\nabla \times \mathbf{F} = 0$.

Thus necessary and sufficient condition for $\int_C \mathbf{F}. d\mathbf{r} = 0$ is that

$$\nabla \times \mathbf{F} = 0.$$

Problem 162. *Prove the following:*

(*a*) $\int_C \mathbf{r} \cdot d\mathbf{r} = 0.$

(*b*) $\int_C [\phi \nabla \phi] \cdot d\mathbf{r} = 0.$

(*c*) $\int_C [\phi \nabla \phi] \cdot d\mathbf{r} = -\int_C \psi \nabla \phi \cdot d\mathbf{r}.$

(*a*) By Stokes' theorem

$$\int_C \mathbf{r} \cdot d\mathbf{r} = \iint_S [\nabla \times \mathbf{r}] \cdot \mathbf{n}\; dS = 0 \text{ as } \nabla \times \mathbf{r} = 0.$$

(*b*) By Stokes' theorem

$$\int_C [\phi \nabla \phi] \cdot d\mathbf{r} = \iint_S [\nabla \times (\phi \nabla \phi)] \cdot \mathbf{n}\; dS.$$

But $\nabla \times (\phi\, \nabla\, \phi) = \phi\, \nabla \times (\nabla\, \phi) + (\nabla\, \phi) \times (\nabla\, \phi)$

$$= 0 + 0 = 0$$

$\therefore$ $\int (\phi \nabla \phi) \cdot d\mathbf{r} = 0.$

(*c*) By Stokes' theorem

$$\int_C [\phi \nabla \psi] \cdot d\mathbf{r} = \iint_S [\nabla \times (\phi \nabla \psi)] \cdot \mathbf{n}\; dS.$$

But $\nabla \times (\phi\, \nabla \psi) = \phi\, \nabla \times (\nabla\, \phi) + (\nabla\, \phi) \times (\nabla \psi)$

$$= \nabla\, \phi \times \nabla \psi$$

$$\therefore \quad \int_C [\phi \nabla \psi] \cdot d\mathbf{r} = \iint_S (\nabla \phi \times \nabla \psi) \cdot \mathbf{n}\; dS$$

$$= -\iint_S (\nabla \psi \times \nabla \phi) \cdot \mathbf{n}\; dS \qquad \dots (1)$$

and $-\int_C \psi \nabla \phi \cdot d\mathbf{r} = -\iint_S \nabla \times (\psi \nabla \phi) \cdot \mathbf{n}\; dS$

(by Stokes' theorem)

$$= -\iint_S \{\psi \nabla \times (\nabla \phi) + \nabla \psi \times \nabla \phi\} \cdot \mathbf{n}\; dS$$

$$= -\iint_S (\nabla \psi \times \nabla \phi) \cdot \mathbf{n}\; dS$$

as $\psi\, \nabla \times (\nabla \phi) = 0$ ... (2)

It is evident from (1) and (2),

$$\int_C \phi \nabla \psi \cdot d\mathbf{r} = -\int_C \psi \nabla \phi . d\mathbf{r}.$$

Problem 163. If $\oint \mathbf{E}\cdot d\mathbf{r} = -\dfrac{1}{C}\dfrac{\partial}{\partial t}\iint_S \mathbf{H}\cdot d\mathbf{S}$,
where S is any surface bounded by curve C, show that

$$\nabla\times\mathbf{E} = -\frac{1}{C}\frac{\partial \mathbf{H}}{\partial t}.$$

Let $$\nabla\times\mathbf{E} = -\frac{1}{C}\frac{\partial \mathbf{H}}{\partial t} \qquad \ldots (1)$$

Then Stokes' theorem yields

$$\oint_C \mathbf{E}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{E})\cdot\mathbf{n}\, dS = \iint_S (\nabla\times\mathbf{E})\cdot d\mathbf{S}$$

$$= \iint_S \left(-\frac{1}{C}\frac{\partial \mathbf{H}}{\partial t}\right)\cdot d\mathbf{S} \qquad \text{by (1)}$$

$$= -\frac{1}{C}\iint_S \frac{\partial}{\partial t}(\mathbf{H})\cdot d\mathbf{S}$$

$$= -\frac{1}{C}\frac{\partial}{\partial t}\iint_S \mathbf{H}\cdot d\mathbf{S}$$

the integral being independent of t.

Problem 164. *If the normal surface integral of a vector point function* **G** *over every open surface is equal to the tangential line integral of another function* **F** *round its boundary, prove that*

$$\mathbf{G} = curl\ \mathbf{F}.$$

We know that the normal surface integral of a vector point function G is given by

$$\iint_S \mathbf{G}\cdot d\mathbf{S}$$

where S is a surface.

And the tangential line integral of vector point function **F** is given by $\int_C \mathbf{F}\cdot d\mathbf{r}$.

Now we are given that

$$\iint_S \mathbf{G}\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r}. \qquad \ldots(1)$$

∴ Stokes' theorem, yields

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\, dS = \iint_S (\nabla\times\mathbf{F})\cdot d\mathbf{S}$$

i.e. $$\iint_S \mathbf{G}\cdot d\mathbf{S} = \iint_S (\nabla\times\mathbf{F})\cdot d\mathbf{S} \text{ by (1)}$$

which follows that

$$\mathbf{G} = \nabla\times\mathbf{F}$$

i.e. $$\mathbf{G} = \text{curl }\mathbf{F}.$$

1.62. SOME THEOREMS

THEOREM 1. The necessary and sufficient condition that a vector point function **F** *be an* **Irrotational vector function** *in a simply connected region is that curl* **F** *= 0 at every point of the region.*

The condition is necessary, since if **F** be irrotational then there exists a scalar ø such that $\mathbf{F} = \nabla\phi$

$\therefore \qquad \text{curl } \mathbf{F} = \text{curl } (\nabla \phi) = \nabla \times \nabla \phi = 0$

The condition is also sufficient, since if curl $\mathbf{F} = 0$ then it follows from Stokes' theorem that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = 0$$

showing that $\mathbf{F}$ is irrotational.

THEOREM 2. The necessary and sufficient condition that a vector point function **F** *be a* **Solenoidal vector function** *in a simply connected region is that div* $\mathbf{F} = 0$ *at every point of the region.*

The condition is necessary, since if **F** be solenoidal then at any point, we have by §1.52 (note)

$$\text{div } \mathbf{F} = \lim_{V \to 0} \iint_S \frac{\mathbf{F} \cdot d\mathbf{S}}{V} = 0$$

showing that **F** is solenoidal.

The condition is sufficient, since if div $\mathbf{F} = 0$ then Gauss' divergence theorem yields,

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_V \text{div } \mathbf{F} \, dV = 0$$

showing that the Flux $\int_S \mathbf{F} \cdot d\mathbf{S}$ across every closed surface is zero.

Note. div curl $\mathbf{F} = 0 \Rightarrow$ divergence of every curl is zero *i.e.*, curl of every function is solenoidal.

THEOREM 3. If **F** *is a continuously differentiable vector point function such that div* $\mathbf{F}=0$*, then there exists another vector point function* **f** *such that* $\mathbf{F} = \text{curl } \mathbf{f}$.

Firstly to show that **f** is any function whose curl is **F**, take a general function $\mathbf{f} + \nabla \phi$ whose curl is **F**, ϕ being continuously differentiable scalar point function.

Assuming that curl $\mathbf{f} = \mathbf{F} = \text{curl } \mathbf{g}$, we have curl $(\mathbf{g} - \mathbf{f}) = 0$

which follows that $\mathbf{g} - \mathbf{f}$ is the gradient of some scalar ϕ *i.e.*

$$\mathbf{g} - \mathbf{f} = \nabla \phi \text{ giving } \quad \mathbf{g} = \mathbf{f} + \nabla \phi$$

But if ϕ be any scalar point function, then

$$\begin{aligned} \text{curl } (\mathbf{f} + \nabla \phi) &= \text{curl } \mathbf{f} + \text{curl } \nabla \phi \\ &= \mathbf{F} + \nabla \times \nabla \phi = \mathbf{F} \end{aligned}$$

which proves the proposition.

Now to prove the main theorem, let us suppose that

$$\mathbf{F} = F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k}$$

$$\mathbf{f} = f_1 \mathbf{i} + f_2 \mathbf{j} + f_3 \mathbf{k}$$

Then, $\text{curl } \mathbf{f} = \nabla \times \mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) \mathbf{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \mathbf{k}$$

So that $\mathbf{F}$ = curl $\mathbf{f}$ gives on comparision of the coefficients of $\mathbf{i}, \mathbf{j}, \mathbf{k}$,

$$F_1 = \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, F_2 = \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \quad F_3 = \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \qquad \dots (1)$$

Also $\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0$ ... (2)

If we suppose that $f_1 = 0$, then (1) gives

$$F_2 = -\frac{\partial f_3}{\partial x} \text{ giving } f_3 = -\int_{x_0}^{x} F_2 \, dx + \phi(y, z) \qquad \dots (3)$$

and $F_3 = \frac{\partial f_2}{\partial x}$ giving $f_2 = \int_{x_0}^{x} F_3 \, dx + \psi(y, z)$, y, z being parameters ... (4)

We get from (3) and (4), $\frac{\partial f_3}{\partial y} = -\int_{x_0}^{x} \frac{\partial F_2}{\partial y} dx + \frac{\partial \phi}{\partial y}$ ... (5)

and

$$\frac{\partial f_2}{\partial z} = \int_{x_0}^{x} \frac{\partial F_3}{\partial z} dx + \frac{\partial \psi}{\partial z} \qquad \dots (6)$$

So that

$$F_1 = \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} = -\int_{x_0}^{x} \left(\frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx + \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial z}$$

$$= \int_{x_0}^{x} \frac{\partial F_1}{\partial x} dx + \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial z} \text{ with the help of (2)}$$

$$= F_1(x, y, z) - F_1(x_0, y, z) + \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial z}$$

i.e.

$$F_1(x_0, y, z) = \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial z}$$

If we now suppose that $\psi = 0$, then

$$\frac{\partial \phi}{\partial y} = F_1(x_0, y, z) \text{ gives } \phi = \int_{y_0}^{y} F_1(x_0, y, z) \, dy$$

As such we find from (3) and (4) etc.

$$f_1 = 0.$$

$$f_2 = \int_{x_0}^{x} F_3 \, dx$$

$$f_3 = -\int_{x_0}^{x} F_2 \, dx + \phi(y, z),$$

where $\phi(y, z) = \int_{y_0}^{y} F_1(x_0, y, z) \, dy$

It is evident that f_1, f_2, f_3 as determined here are the components of a vector $\mathbf{f}$ whose curl is $\mathbf{F}$.

Hence div $\mathbf{F} = 0 \Rightarrow \mathbf{F}$ = curl $\mathbf{f}$.

1.63. THE CLASSIFICATION OF VECTOR FIELDS

(*Kanpur*, *1968*; *Agra*, *1954*, *63*, *65*)

If curl $\mathbf{F}=0$ *i.e.* $\nabla \times \mathbf{F} = 0$ then $\mathbf{F}$ = grad ϕ or $\mathbf{F}$ is called as a *Lameller Field* or a *Zero Curl Field*. Also if div $\mathbf{F} = 0$ *i.e.* $\nabla \cdot \mathbf{F} = 0$ then $\mathbf{F}$ = curl $\mathbf{f}$ or $\mathbf{F}$ is called as a *Solenoidal field*. It is conventional to classify the vector fields into four:

(1) When curl $\mathbf{F} = 0$ and div $\mathbf{F} = 0$, then the first condition shows the field to be lameller or irrotational since curl $\mathbf{F} = 0 \overset{\text{implies}}{\Longrightarrow} \mathbf{F}$ = grad ϕ and in view of second condition it gives div grad $\phi = 0$ *i.e.* $\nabla^2 \phi = 0$ *i.e.* Laplace's equation showing that the field is solenoidal or incompressible. On the whole such a field is termed as a type of field, which is *irrotational motion of incompressible fluid* as shown in Fig. 1.64(*a*).

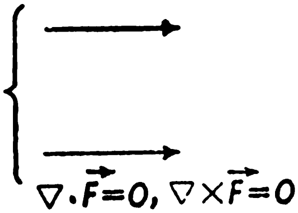


Fig. 1.64(*a*)

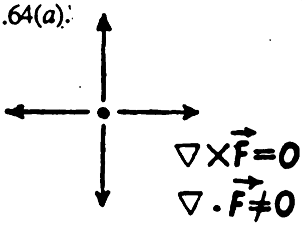


Fig. 1.64(*b*)

(2) When curl $\mathbf{F} = 0$ but div $\mathbf{F} \neq 0$. Then curl $\mathbf{F} = 0$ gives $\mathbf{F}$ = grad ϕ and in view of second condition this yields, ∇ grad $\phi \neq 0$ *i.e.*, $\nabla^2 \phi \neq 0$. So this field is termed as the type of field which is *irrotational motion of compressible fluid* as shown in Fig. 1.64(*b*).

(3) When curl $\mathbf{F} \neq 0$ but div $\mathbf{F} = 0$. Then div $\mathbf{F} = 0$ gives $\mathbf{F}$ = curl $\mathbf{f}$ which in view of first condition yields curl curl $\mathbf{f} \neq 0$ or $\nabla \times (\nabla \times \mathbf{f}) \neq 0$

i.e., grad div $\mathbf{f} - \nabla^2 \mathbf{f} \neq 0$

This shows that if $\mathbf{f}$ is solenoidal then we must have div $\mathbf{f} = 0$, so that grad div $\mathbf{f} = 0$ and as such $\nabla^2 \mathbf{f} \neq 0$. Hence such a field is termed as the type of field which is *rotational motion of incompressible fluid* as shown in Fig. 1.65.

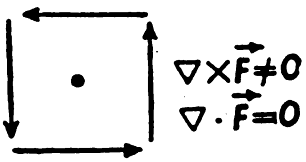


Fig. 1.65

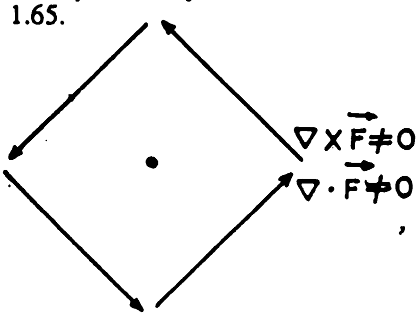


Fig. 1.66

(4) When curl $\mathbf{f} \neq 0$, also div $\mathbf{F} \neq 0$. This type of vector fields is most general and it is termed as the type of field which is *rotational motion of compressible fluid* as shown in Fig. 1.66.

In fact this field is made up of two fields namely (*i*) Lameller vector field (*i.e.* having no curl but may have div only), (*ii*) Solenoidal vector field (*i.e.* having no div but may have curl only). Mathematically.

$$\mathbf{F} = \text{grad}\ \phi + \text{curl}\ \mathbf{f}$$

So that

$$\begin{aligned} \text{div}\ \mathbf{F} &= \text{div}\ (\text{grad}\ \phi + \text{curl}\ \mathbf{f}) \\ &= \text{div grad}\ \phi \quad \because \text{div curl}\ \mathbf{f} = 0 \\ &= \nabla^2 \phi \end{aligned}$$

But div. $\mathbf{F} \neq 0$, therefore $\nabla^2 \phi \neq 0$ which determines ϕ.

Again

$$\begin{aligned} \text{curl}\ \mathbf{F} &= \text{curl}\ (\text{grad}\ \phi + \text{curl}\ \mathbf{f}) = \text{curl curl}\ \mathbf{f} \quad \because \text{curl grad}\ \phi = 0 \\ &= -\nabla^2 \mathbf{f} \end{aligned}$$

But curl $\mathbf{F} \neq 0$, $\therefore$ $\nabla^2 \mathbf{f} \neq 0$, where $\mathbf{f}$ is solenoidal vector field and this determines $\mathbf{f}$.

Such a decomposition of vector field comprising Lameller and solenoidal field is known as *Helmholtz's theorem*.

ADDITIONAL MISCELLANEOUS PROBLEMS

Problem 165. *Prove that if* $\mathbf{A}$, $\mathbf{B}$ *and* $\mathbf{C}$ *are three non-coplanar vectors, then any vector* $\mathbf{F}$ *can be put in the form* $\mathbf{F} = \alpha \mathbf{B} \times \mathbf{C} + \beta \mathbf{C} \times \mathbf{A} + \gamma \mathbf{A} \times \mathbf{B}$

Determine α, β *and* γ. *(Agra, 1971)*

Given that $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are non-coplanar vectors and we have to show that $\mathbf{B} \times \mathbf{C}$, $\mathbf{C} \times \mathbf{A}$ and $\mathbf{A} \times \mathbf{B}$ are also non-coplanar. They will be so if their scalar triple product is not zero, *i.e.*, *if* $[\mathbf{B} \times \mathbf{C}, \mathbf{C} \times \mathbf{A}, \mathbf{A} \times \mathbf{B}] \neq 0$

Now
$$\begin{aligned}[\mathbf{B} \times \mathbf{C}, \mathbf{C} \times \mathbf{A}, \mathbf{A} \times \mathbf{B}] &= (\mathbf{B} \times \mathbf{C}) \cdot (\mathbf{C} \times \mathbf{A}) \times (\mathbf{A} \times \mathbf{B}) \\ &= (\mathbf{B} \times \mathbf{C}) \cdot \{[\mathbf{CAB}]\, \mathbf{A} - [\mathbf{CAA}]\, \mathbf{B}\} \\ &= [\mathbf{BCA}]\, [\mathbf{CAB}] \quad \because [\mathbf{CAA}] = 0 \\ &= [\mathbf{ABC}]\, [\mathbf{ABC}] \\ &= [\mathbf{ABC}]^2 \qquad \ldots (1)\end{aligned}$$

But $[\mathbf{A}, \mathbf{B}, \mathbf{C}] \neq 0$ since $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are non-coplanar.

It therefore follows from (1) that $[\mathbf{B} \times \mathbf{C}, \mathbf{C} \times \mathbf{A}, \mathbf{A} \times \mathbf{B}] \neq 0$

i.e., $\mathbf{B} \times \mathbf{C}, \mathbf{C} \times \mathbf{A}, \mathbf{A} \times \mathbf{B}$ are three non-coplanar vectors and as such any vector $\mathbf{F}$ can be expressed in the form

$$\mathbf{F} = \alpha \mathbf{B} \times \mathbf{C} + \beta \mathbf{C} \times \mathbf{A} + \gamma \mathbf{A} \times \mathbf{B} \qquad \ldots (2)$$

Now to determine α, β, γ, multiply (2) scalarly by $\mathbf{A}$

$$\mathbf{A} . \mathbf{F} = \alpha\, \mathbf{A} . \mathbf{B} \times \mathbf{C}, \text{ other two scalar triple products vanish}$$

giving
$$\alpha = \frac{\mathbf{A} . \mathbf{F}}{[\mathbf{ABC}]}$$

Similarly multiplying (2) scalarly by $\mathbf{B}$ and $\mathbf{C}$ successively we find

$$\beta = \frac{\mathbf{B} . \mathbf{F}}{[\mathbf{ABC}]} \quad \text{and} \quad \gamma = \frac{\mathbf{C} . \mathbf{F}}{[\mathbf{ABC}]}$$

Problem 166. *If* $\mathbf{A}(t)$ *be a vector function of the scalar variable* t *and be of constant length, then show that* $\frac{d}{dt} \mathbf{A}(t)$ *is a vector perpendicular to* $\mathbf{A}(t)$. *(Agra, 1968)*

Given vector $\mathbf{A}$ is of constant length i.e., $|\mathbf{A}| = \text{constant}$ and is a function of t. We know that $\mathbf{A} . \mathbf{A} = \mathbf{A}^2 = |\mathbf{A}|^2 = A^2$, A being module of $\mathbf{A}$.

Differentiation gives,
$$2\mathbf{A} . \frac{d\mathbf{A}}{dt} = 2A \frac{dA}{dt}$$
$$= 0 \qquad \because \frac{dA}{dt} = 0 \text{ when } A = \text{constant}.$$

Or
$$\mathbf{A}(t) \cdot \frac{d}{dt} \mathbf{A}(t) = 0$$

which follows that $\mathbf{A}(t)$ is a vector perpendicular to $\frac{d}{dt} \mathbf{A}(t)$.

Problem 167. *Show that div* $(\nabla u \times \nabla v) = 0$ *(Agra, 1960)*

By §1.35, div $(\nabla u \times \nabla v) = \nabla v$. curl $\nabla u - \nabla u$. curl ∇v

$= \nabla v$.curl grad $u - \nabla u$. curl grad v

$= 0$ as curl grad u = curl grad $v = 0$ by Problem 91 (*ii*)

Problem 168. *Show that* $\nabla . (\mathbf{a} \times \mathbf{r}) = 0$, a *being a constant vector.* *(Agra, 1958)*

Take $\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and verify it.

Problem 169. *In the gravitational field of a mass m, the potential is given by* $-\dfrac{m}{r}$, *where r is the distance from the mass, given by* $r^2 = x^2 + y^2 + z^2$. *Obtain the components of force vector by differentiation. Find the curl of the force and show that it is zero.*

(Rohilkhand, 1977; Agra, 1955)

Here if $V = -\dfrac{m}{r}$ then $F_x = -\dfrac{\partial V}{\partial x}$ etc. and $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$.

It is easy to verify $\nabla \times \mathbf{F} = 0$.

Problem 170. *Find the Cartesian components of vector* C *which is perpendicular to the vectors* A $(2\mathbf{i} - \mathbf{j} - 4\mathbf{k})$ *and* B $(3\mathbf{i} - \mathbf{j} - \mathbf{k})$. *(Agra, 1953)*

It is easy to find the components of $\mathbf{A} \times \mathbf{B}$ which is perpendicular to A and B both.

Ans. -3, -10, 1

Problem 171. *Prove that* $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{A}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{A}) - (\mathbf{B} \cdot \mathbf{C})(\mathbf{A} \cdot \mathbf{A})$

(Agra, 1962)

Problem 172. *The rectangular components of a vector* A *are*

$$A_x = y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y}, \quad A_y = z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z}, \quad A_z = x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x}$$

Where f is given function of (x, y, z).

Express A *as vector product of two vectors and evaluate* A.r *and* A. *grad f*

(Agra, 1962)

Here $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \times \nabla f = \mathbf{r} \times \nabla f$ etc. and it is easy to show that $\mathbf{A}.\mathbf{r} = 0 = \mathbf{A}$. grad f.

Problem 173. *Establish Poisson's and Laplace's Equations.*

Gauss' theorem, for a volume distribution of density ρ, gives

$$N = \int_s \mathbf{A} \cdot \mathbf{n} \, dS = 4\pi \int_v \rho \, dv,$$

where N is the flux of the electrostatic intensity A at a point of closed surface S on which the positive unit normal is n.

With the help of divergence theorem, we find

$$\int_v (\text{div } \mathbf{A}) dv = 4\pi \int_v \rho \, dv$$

or $$\int_v (\text{div } \mathbf{A} - 4\pi\rho) \, dv = 0,$$

which is true for all volumes however small.

$\therefore$ div A $- 4\pi\rho = 0$, *i.e.*, div A $= 4\pi\rho$

or $$\nabla.\mathbf{A} = 4\pi\rho.$$

$$\therefore \nabla(\nabla A) = -4\pi\rho$$

or $$\nabla^2 A = -4\pi\rho. \qquad \textit{(Agra, 1961, 65)}$$

This is known as *Poisson's equation.*

In free space, $\rho = 0.$

$$\therefore \; \nabla^2 A = 0.$$

This is known as *Laplace's equation.*

Problem 174. *Show that curl* $\frac{a \times r}{r^3} = -\frac{a}{r^3} + \frac{3r}{r^3}$ (a.r), *where* a *is a constant vector and* $r = xi + yj + zk$.

Here, curl $\frac{a\times r}{r^3} = \nabla \times \frac{a \times r}{r^3} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times \frac{a\times r}{r^3}$

$$= i \times \frac{\partial}{\partial x}\left(\frac{a\times r}{r^3}\right) + j \times \frac{\partial}{\partial y}\left(\frac{a\times r}{r^3}\right) + k\times\frac{\partial}{\partial z}\left(\frac{a\times r}{r^3}\right) \quad \ldots (1)$$

Now $\frac{\partial}{\partial x}\left(\frac{a\times r}{r^3}\right) = -\frac{3}{r^4}\frac{\partial r}{\partial x} a \times r + \frac{1}{r^3}\left(a \times \frac{\partial r}{\partial x}\right) = -\frac{3x}{r^3} a\times r + \frac{a}{r^3} \times i$

$\because$ $r = ix + yj + zk$ gives $\frac{\partial r}{\partial x} = i$

and $r^2 = x^2 + y^2 + z^2$ gives $\frac{\partial r}{\partial x} = \frac{x}{r}$

$\therefore$ $i \times \frac{\partial}{\partial x}\left(\frac{a\times r}{r^3}\right) = -\frac{3x}{r^5}\left[i \times (a \times r)\right] + \frac{1}{r^3} i \times (a \times i)$

$$= -\frac{3x}{r^5}\left[(i\cdot r)a - (i \cdot a) r\right] + \frac{1}{r^3}\left[(i\cdot i)a - (i\cdot a)i\right]\cdot$$

$$= -\frac{3x}{r^5}\left[x a - (i \times a) r\right] + \frac{1}{r^3}\left[a - (i\cdot a) i\right]$$

$$= -\frac{3x^2}{r^5} a + \frac{3r}{r^5}(i x \cdot a) + \frac{a}{r^3} - \frac{(i\cdot a)i}{r^3}.$$

Similarly, $j \times \frac{\partial}{\partial y}\left(\frac{a\times r}{r^3}\right) = -\frac{3y^2}{r^5} a + \frac{3r}{r^5}(j y \cdot a) + \frac{a}{r^3} - \frac{(j\cdot a) j}{r^3}$

and $k \times \frac{\partial}{\partial z}\left(\frac{a\times r}{r^3}\right) = -\frac{3z^2}{r^5} a + \frac{3r}{r^5}(k z \cdot a) + \frac{a}{r^3} - \frac{(k\cdot a) k}{r^3}.$

$\therefore$ $i \times \frac{\partial}{\partial x}\left(\frac{a\times r}{r^3}\right) + j \times \frac{\partial}{\partial y}\left(\frac{a\times r}{r^3}\right) + k \times \frac{\partial}{\partial z}\left(\frac{a\times r}{r^3}\right)$

$$= -\frac{3\left(x^2+y^2+z^2\right)}{r^5} a + \frac{3r}{r^5}\{x i + y j + z k\}\cdot a + \frac{3a}{r^3} - \frac{1}{r^3}\left[(i\cdot a)\, i + (j\cdot a) j + (k \cdot a)\, k\right]$$

$= -\frac{3a}{r^3} + \frac{3r}{r^5} r\cdot a + \frac{3a}{r^3} - \frac{a}{r^3}$ since if $a = a_1 i + a_2 j + a_3 k$,

$$i\cdot a = a_1,\; j\cdot a = a_2,\; k\cdot a = a_3.$$

$$\therefore \; (i\cdot a)\, i + (j\cdot a)\, j + (k\cdot a)\, k = a_1 i + a_2 j + a_3 k = a$$

$$= -\frac{a}{r^3} + \frac{3r}{r^2}(a\cdot r). \quad \ldots (2)$$

Hence from (1) and (2).

$$\text{Curl}\; \frac{a\times r}{r^3} = -\frac{a}{r^3} + \frac{3r}{r^3}(a\cdot r).$$

Problem 175. *Give example in support that vector methods have been used to give results in a simple and elegant form.*

Ever since the development of Quaternion analysis by W.R. Hamilton and of the 'Ausdehnungslehre' by Grassman, it has been a growing feeling that new methods and ideas be applied more simply and more directly to many of the conceptions of geometry, mechanics and mathematical physics, than those long accepted. The methods of Vector Analysis are adopted on the grounds of naturalness, simplicity and directness.

As an example, Faraday the great physicist with his mind's eye visualized the lines of force emerging from the magnet and so he had a visual conception of the manner in which the electro-magnetic waves were travelling through the ether around him and so divergence and divergence theorem to him had simple meaning.

Numerous other examples can be quoted from geometry, mechanics and mathematical physics.

Problem 176. *Show that*

(*a*) *The vector product of two vectors is a vector.*

(*b*) *The gradient of a scalar function is a vector.*

(*c*) *The divergence of a vector function is a scalar.* (*Agra, 1965*)

Problem 177. *What is Green's theorem,. Use it to solve the equation.* (*Agra, 1965*)

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}=-4\pi\rho\,(x,\ y,\ z).$$

Problem 178. (*a*) *Prove that* $(\mathbf{A}\times\vec{\nabla})\times\mathbf{R}=-2\mathbf{A}$

find $(\mathbf{A}\times\vec{\nabla}).\mathbf{R}$ *where* **A** *is any vector field and* **R** *is a vector drawn from the origin to a point* $P:(x,y,z)$.

(*b*) *Find the directional derivative of the function* $\phi\,(x,y,z)=2xy+z^2$ *in the direction of the vector* $\mathbf{i}+2\mathbf{j}+2\mathbf{k}$ *at the point* $(1,-1,3)$.

(*c*) *If the divergence of a vector field* **H** *vanishes, show that it can be expressed as the curl of a vector field* **A**. (*Bombay, 1965*)

(*a*) Let $\mathbf{A}=A_x\mathbf{i}+A_y\mathbf{j}+A_z\mathbf{k}$ and $\mathbf{R}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$

then
$$\mathbf{A}\times\vec{\nabla}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\A_x&A_y&A_z\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\end{vmatrix}=\left(\frac{\partial A_y}{\partial z}-\frac{\partial A_z}{\partial y}\right)\mathbf{i}+\left(\frac{\partial A_z}{\partial x}-\frac{\partial A_x}{\partial z}\right)\mathbf{j}+\left(\frac{\partial A_x}{\partial y}-\frac{\partial A_y}{\partial x}\right)\mathbf{k}$$

So that
$$\left(\mathbf{A}\times\vec{\nabla}\right)\times\mathbf{R}=\left\{\left(\frac{\partial}{\partial z}A_y-\frac{\partial}{\partial y}A_z\right)\mathbf{i}+\left(\frac{\partial A_z}{\partial x}-\frac{\partial A_x}{\partial z}\right)\mathbf{j}+\left(\frac{\partial A_x}{\partial y}-\frac{\partial A_y}{\partial x}\right)\mathbf{k}\right\}\times(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$$
$$=-2\,(A_x\mathbf{i}+A_y\mathbf{j}+A_z\mathbf{k})=-2\mathbf{A}$$

It is also easy to show that $\left(\mathbf{A}\times\vec{\nabla}\right)\cdot\mathbf{R}=0$

(*b*) $\phi=2xy+z^2$

$$\therefore\quad \nabla\phi=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\left(2xy+z^2\right)=2y\mathbf{i}+2x\mathbf{j}+2z\mathbf{k}$$
$$=-2\,\mathbf{i}+2\,\mathbf{j}+6\,\mathbf{k}\ \text{at}\ x=1,\ y=-1,\ z=3$$

A unit vector in the direction of $\mathbf{i}+2\,\mathbf{j}+2\,\mathbf{k}$ is $\dfrac{\mathbf{i}+2\mathbf{j}+2\mathbf{k}}{\sqrt{1^2+2^2+2^2}}=\dfrac{1}{3}\,(\mathbf{i}+2\mathbf{j}+2\mathbf{k})$.

Since the directional derivative along a unit vector **u** is **u**·grad ø *i.e.*, **u**·∇ø, therefore the required directional derivative in the existing case is

$$=\frac{1}{3}(\mathbf{i}+2\mathbf{j}+2\mathbf{k})\cdot(-2\mathbf{i}+2\mathbf{j}+6\mathbf{k})$$

$$=\frac{1}{3}[-2+4+12]=\frac{14}{3}.$$

(*c*) It is easy to show that div **H** = 0 ⇒ **H** = curl **A** by Theorem (3) of §1.62.

Problem 179. (*a*) *Prove the divergence theorem of Gauss.*

(*b*) *Prove that* $\iint_S (\mathbf{N}\times\mathbf{F})\, dS = \iiint_V (\vec{\nabla}\times\mathbf{F})\, dV$

where **F** *is a vector field and* **N** *is the normal unit vector to surface* S *enclosing the volume.*

(*Bombay, 1965*)

These are well known theorems.

Problem 180. (*a*) *Prove the following by using vector methods:*

(*1*) *The medians of a triangle meet in a point of trisection of each other.*

(2) *sin* (α + β) = *sin* α *cos* β + *cos* α *sin* β

(*b*) *Prove that*

(*1*) [**A** × **B**] × [**A** × **C**] = ([**A** × **B**]· **C**) **A**

(2) $$\nabla^2=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta.\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\cdot\frac{\partial^2}{\partial\phi^2}$$

(*Nagpur, 1965; Agra, 1966*)

(*a*) – (1) See Problem 14 (*b*).

(2) Take **i**, **j** unit vectors along *OX*, *OY* two mutually perpendicular axes and consider two coplanar lines *OA* and *OB* making angles α and β with *OX*. If **â** and **b̂** be unit vectors along *OA* and *OB* respectively then

$$\hat{\mathbf{a}} = \cos\alpha\ \mathbf{i} + \sin\alpha\ \mathbf{j}$$

$$\hat{\mathbf{b}} = \cos\beta\ \mathbf{i} - \sin\beta\ \mathbf{j}$$

If $\hat{\varepsilon}$ be a unit vector perpendicular to $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ both, then

$$\hat{\mathbf{b}}\times\hat{\mathbf{a}} = 1\cdot 1\ \sin(\alpha+\beta)\ \hat{\varepsilon}$$

i.e., $\sin(\alpha+\beta)\ \hat{\varepsilon} = (\cos\beta\ \mathbf{i} - \sin\beta\ \mathbf{j}) \times (\cos\alpha\ \mathbf{i} + \sin\alpha\ \mathbf{j})$

$$= (\sin\alpha\cos\beta+\cos\alpha\sin\beta)\ \hat{\varepsilon} \qquad \because \mathbf{i}\times\mathbf{j}=\hat{\varepsilon}$$

∴ sin (α + β) = sin α cos β + cos α sin β.

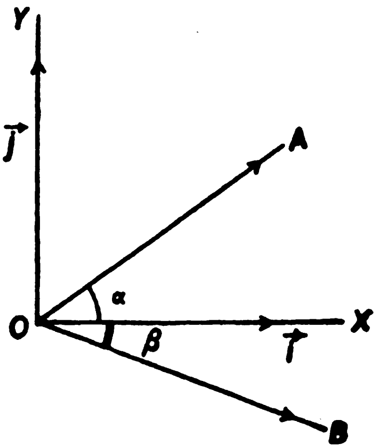


Fig. 1.67

(*b*) (1) L.H.S = [**A** × **B**] × [**A** × **C**]

= ([**A** × **B**] · **C**) **A** – ([**A** × **B**] · **A**) **C**

= ([**A** × **B**] · **C**) **A** ∵ [**A** × **B**] · **A** = [**ABA**] = 0.

(2) We have $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ in cartesian coordinates.

The transformations to polar coordinates are

$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$ ...(1)

Let us first change x, y, z to u, ϕ, z (cylindrical coordinates) by the transformation

$$x = u\cos\phi,\ y = u\sin\phi,\ z = z \qquad ...(2)$$

So that $u = \sqrt{x^2+y^2}$ and $\phi = \tan^{-1}\dfrac{y}{x}$

Also, $\dfrac{\partial u}{\partial x} = \cos\phi$, $\dfrac{\partial u}{\partial y} = \sin\phi$, $\dfrac{\partial \phi}{\partial x} = -\dfrac{\sin\phi}{r}$, $\dfrac{\partial \phi}{\partial y} = \dfrac{\cos\phi}{r}$

Now, $\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial \phi}\dfrac{\partial \phi}{\partial x} = \cos\phi\dfrac{\partial v}{\partial u} - \dfrac{\sin\phi}{r}\dfrac{\partial v}{\partial \phi}$, similarly $\dfrac{\partial v}{\partial y} = \sin\phi\dfrac{\partial v}{\partial u} + \dfrac{\cos\phi}{u}\dfrac{\partial v}{\partial \phi}$

$$\therefore \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 = \left(\frac{\partial v}{\partial u}\right)^2 + \left(\frac{1}{u}\frac{\partial v}{\partial \phi}\right)^2 \qquad ...(3)$$

Again $\dfrac{\partial^2 v}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial v}{\partial x}\right) = \left(\cos\phi\dfrac{\partial}{\partial u} - \dfrac{\sin\phi}{u}\dfrac{\partial}{\partial \phi}\right)\left(\cos\phi\dfrac{\partial_v}{\partial u} - \dfrac{\sin\phi}{u}\dfrac{\partial v}{\partial \phi}\right)$

$$= \cos\phi\left(\cos\phi\frac{\partial^2 v}{\partial u^2} - \frac{\sin\phi}{u}\frac{\partial^2 v}{\partial u\partial\phi} + \frac{\sin\phi}{u^2}\frac{\partial v}{\partial \phi}\right) - \frac{\sin\phi}{u}$$

$$\left(\cos\phi\frac{\partial^2 v}{\partial u\partial\phi} - \sin\phi\frac{\partial v}{\partial r} - \frac{\sin\phi}{u}\frac{\partial^2 v}{\partial \phi^2} - \frac{\cos\phi}{u}\frac{\partial v}{\partial \phi}\right)$$

$$= \left(\cos^2\phi\frac{\partial^2 v}{\partial u^2} - \frac{2\sin\phi\cos\phi}{u}\frac{\partial^2 v}{\partial u\partial\phi} + \frac{\sin^2\phi}{u^2}\frac{\partial^2 v}{\partial \phi^2} + \frac{\sin^2\phi}{u}\frac{\partial v}{\partial u}\right.$$

$$\left. + \frac{2\sin\phi\cos\phi}{u^2}\frac{\partial v}{\partial \phi}\right)$$

Similarly $\dfrac{\partial^2 v}{\partial y^2} = \sin^2\phi\dfrac{\partial^2 v}{\partial u^2} + \dfrac{2\sin\phi\cos\phi}{u}\dfrac{\partial^2 v}{\partial u\partial\phi} + \dfrac{\cos^2\phi}{u^2}\dfrac{\partial^2 v}{\partial \phi^2}$

$$+ \frac{\cos^2\phi}{u}\frac{\partial v}{\partial u} - \frac{2\sin\phi\cos\phi}{u^2}\frac{\partial v}{\partial \phi}$$

Adding $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = \dfrac{\partial^2 v}{\partial u^2} + \dfrac{1}{u}\dfrac{\partial v}{\partial u} + \dfrac{1}{u^2}\dfrac{\partial^2 v}{\partial \phi^2}$ ...(4)

Adding $\dfrac{\partial^2 v}{\partial z^2}$ on either side of (4),

$$\nabla^2 v = \frac{\partial^2 v}{\partial u^2} + \frac{1}{u^2}\frac{\partial^2 v}{\partial \phi^2} + \frac{\partial^2 v}{\partial z^2} + \frac{1}{u}\frac{\partial v}{\partial u} \qquad ...(5)$$

Now by putting $u = r\sin\theta$, $z = r\cos\theta$, the transformations (2) reduce to the form (1) and by applying (4), we get

$$\frac{\partial^2 v}{\partial u^2} + \frac{\partial^2 v}{\partial z^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{1}{r}\frac{\partial v}{\partial r} \qquad ...(6)$$

and

$$\frac{1}{u}\frac{\partial v}{\partial u} = \frac{1}{r\sin\theta}\left(\sin\theta\frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial \theta}\right) \qquad ...(7)$$

Substituting values from (6) and (7) in (5) we find

$$\nabla^2 v = \frac{\partial^2 v}{\partial r^2} + \frac{2}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial v}{\partial r} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v}{\partial \phi^2}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right)+\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial v}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v}{\partial\phi^2}$$

i.e. $$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Problem 181. *Evaluate the following:*

(*i*) $\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\frac{x^2}{a^2}}\, e^{ikx}\, dx$

(*ii*) $\frac{1}{(2\pi)^3}\int\int\int_{-\infty}^{+\infty} \frac{e^{-r/a}}{r}\, e^{i\mathbf{k}\cdot\mathbf{r}}\, dx\, dy\, dz \left[r=\sqrt{x^2+y^2+z^2},\ \mathbf{r}=(x,y,z)\right]$

(*Agra, 1966*)

Problem 182. (*a*) *Starting from the definition* $\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}$ *express* $\nabla^2\psi$ *in polar and cylindrical coordinates.*

(*b*) *Write the general solution of the equation* $\nabla^2\psi = 0$ *in polar coordinates.*

(*Agra, 1966*)

(See Problem 180).

Problem 183. (*a*) *Prove the following identities :*

(*i*) $\vec{\nabla}\cdot(\vec{\nabla}\times \mathbf{v})=0$, (*ii*) $\vec{\nabla}\times(\vec{\nabla}\,\phi)=0$

(*b*) *Show that if a scalar point function* ϕ *depends only on the magnitude of the position vector* **r**, *then* $\vec{\nabla}\phi = \frac{d\phi}{dr}\, \mathbf{e}_r$, $\mathbf{e}_r$ *being the unit vector in the direction of* **r**.

(*c*) *Show that* $\vec{\nabla}\times\mathbf{r} = 0$

(*Here* **v** *is a vector function of* **r** *and* $\vec{\nabla} = \mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}$ *in the usual notation*)

(*Agra, 1967*)

Problem 184. (*a*) *Define* (*i*) *The dot product and* (*ii*) *The cross product of two vectors. Show that* $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

(*b*) *If* ϕ *is any scalar and* **A** *any vector, then show that*

(*i*) *div* $\phi\,\mathbf{A} = \mathbf{A}\cdot$ *grad* $\phi + \phi$ *div* **A**

(*ii*) *curl* $\phi\,\mathbf{A} = \phi$ *curl* $\mathbf{A} - \mathbf{A}\times$ *grad* ϕ [*Vikram 1967*, (*i*) *69 also*]

Problem 185. *Explain how the Laplacian operator* ∇^2 *can be expressed in any system of orthogonal curvilinear cooordinates. Hence express it in cylindrical and spherical polar coordinates.*

(*Vikram, 1967*)

See §1.45 and §1.46.

Problem 186. *Using the Theorem of Gauss prove the following identities:*

(*i*) $\int_V \vec{\nabla}\,\phi\, d\tau = \oint_S \phi\,\mathbf{n}\, dS$, (*ii*) $\int_V \vec{\nabla}\times\mathbf{F}\, d\tau = \oint_S \mathbf{n}\times\mathbf{F}\, dS$

Here S is a closed surface enclosing the volume V and **n** *is the unit normal vector at the surface elements dS.*

(*Agra, 1968*)

Problem 187. *Explain the meanings of the operations* $\vec{\nabla}$, $\vec{\nabla}\cdot$ *and* $\vec{\nabla}\times$. *Show that*
$\vec{\nabla}\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\vec{\nabla})\,\mathbf{A} - (\mathbf{A}\cdot\vec{\nabla})\,\mathbf{B} - \mathbf{B}\,(\vec{\nabla}\cdot\mathbf{A}) + \mathbf{A}\,(\vec{\nabla}\cdot\mathbf{B})$ *(Agra, 1972)*

Meanings are clear from §1.24, §1.28 etc. For second part see §1.33.

Problem 188. (*a*) *Prove the Green's theorem*

$$\iiint_T \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right)dx = \iint_S \left(\phi\,\frac{\partial\psi}{\partial n} - \psi\,\frac{\partial\phi}{\partial n}\right)dS$$

(*b*) *A vector field is given by* $\mathbf{A} = (x^2 + xy^2)\,\mathbf{i} + (y^2 + x^2y)\,\mathbf{j}$
Show that the field is irrotational and find the scalar potential. *(Bombay, 1970)*

First part is the well known theorem and for second show $\nabla\times\mathbf{A} = 0$.

Problem 189. *Establish an expression for the components of the curl of a vector* **A** *in orthogonal curvilinear coordinates and hence obtain the radial component of curl* **A** *in spherical polar coordinates* (r, θ, ϕ). *(Agra, 1974)*

See §1.42, §1.46.

Problem 190. *Find the divergence, the gradient of the divergence and the curl of the vector* $r^n\,\hat{\mathbf{r}}$, *where* **r** *is a position vector.*

See Problem 99.

Problem 191. *If div* $\mathbf{A} = 0$, *show that curl curl* $\mathbf{A} = -\nabla^2\mathbf{A}$ *(Agra, 1975)*

Problem 192. (*a*) *Clearly explain the physical significance of divergence and curl and express them in spherical polar and circular cylindrical coordinates.*

(*b*) *If a vector function* **F** *depends on both space coordinate* $(x\ y\ z)$ *and time* t *show that*

$$d\mathbf{F} = \left(d\mathbf{r}\cdot\vec{\nabla}\right)\mathbf{F} + \frac{\delta}{\delta t}\,\mathbf{F}\,dt$$ *(Rohilkhand, 1976, 83)*

Problem 193. *Define Selonoidal and Irrotational vector fields. Show that* $r^n\,\mathbf{r}$ *is an irrotational vector for every n, but is solenoidal only if* $n = -3$, *where* **r** *is the position vector of a particle.*

(Agra, 1979)

In the gravitational field of mass m, the potential at a distance r is given by $\frac{m}{r}$.
Obtain the components of force and show that its curl is zero.

See §1.63 and Problems 111 and 169. *(Rohilkhand, 1977)*

Problem 194. *Show that the components of vector* **a** *along and perpendicular to vector* **b** *may be expressed as* $\frac{(\mathbf{a}\cdot\mathbf{b})\ \mathbf{b}}{b^2}$ and $\frac{\mathbf{b}\times(\mathbf{a}\times\mathbf{b})}{b^2}$ *respectively, where* $b = |\mathbf{b}|$.

(Rohilkhand, 1984; Agra, 84)

Problem 195. *If* ϕ *is a scalar function which is a solution of Laplace's equation* $\nabla^2\phi = 0$ *in a volume V bounded by a piecewise smooth surface S, then apply Gauss-divergence theorem to* $\mathbf{u} = \nabla\phi$ *to prove that*

$$\iint_S \mathbf{n}\cdot\nabla\phi\ dS = 0,$$

where **n** *is the unit vector normal to S.* *(Meerut, 1971)*

Hint: For $\mathbf{u} = \nabla\phi$, Gauss–divergence theorem

i.e. $$\iint_S \mathbf{u}\cdot d\mathbf{S} = \iiint_V \operatorname{div}\mathbf{u}\ dV$$

$$\Rightarrow \iint_S \nabla\phi\cdot\mathbf{n}\ dS = \iiint_V \nabla\cdot(\nabla\phi)\ dV$$

$$\Rightarrow \iint_S \mathbf{n}\cdot\nabla\phi\ dS = \iiint_V \nabla^2\phi\ dV = 0 \text{ as } \nabla^2\phi = 0.$$

CHAPTER 2

MATRICES

2.1 DEFINITIONS AND NOTATIONS

A set of mn numbers, real or complex, arranged in a rectangular array of m rows and n columns such as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix}$$

is called a **matrix** of order $m \times n$.

In other words a scheme of detached coefficients a_{ij} arranged in m rows and n columns is called a **matrix of order** m by n or an $m \times n$ **matrix** or a **matrix of type** $m \times n$.

In case $m = n$, the rectangular array becomes a square and so the matrix having number of rows and number of columns equal is called a **Square Matrix** of order n.

Any matrix obtained by deleting any number of rows and any number of columns from a given matrix is said to be a **Sub-Matrix** of the given matrix.

The mn numbers a_{ij}, $(i = 1, 2, \dots m; j = 1, 2, \dots n, i \neq j)$ constituting the $m \times n$ matrix are called its **elements** or **constituents**. The elements a_{ij} $(i = j)$ of a square matrix A are called its **diagonal elements** and their sum as trace of A denoted by

$$\text{tr. } \mathbf{A} = \sum_{i=1}^{n} a_{ii}$$

A matrix is usually denoted by capital letters like A (in clarendon type) or $[a_{ij}]$, where a_{ij} represents the (i, j)th element *i.e.*, the element in the ith row and jth column of the matrix.

Thus an $m \times n$ matrix may be expressed as

$$\mathbf{A} = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix} \begin{matrix} \text{where } 1 \le i \le m \\ \text{and } 1 \le j \le n \\ \text{but } i \neq j \\ \end{matrix}$$

We have so far used only a pair of brackets *i.e.* [ ] to denote a matrix, but a pair of parentheses *i.e.* ( ) and double bars *i.e.* ‖ ‖, are also sometimes used to indicate a matrix.

A matrix having all of its elements zero is said to be a **Null Matrix** and denoted by O *e.g.*

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A square matrix of order n having all its diagonal elements unity and zero elements everywhere else is called a **unit matrix** or an **identity matrix** and denoted by $\mathbf{I}_n$. Thus

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

It is possible that a matrix may have only a single row or a single column such as

$$\begin{bmatrix} a_1, & a_2, & \dots & a_p \end{bmatrix} \text{ and } \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_q \end{bmatrix}$$

the first one being a matrix of order $1 \times p$ is called a **row matrix** and the second one being a matrix of order $q \times 1$ is called a **column matrix**.

A single element constitutes a matrix of order 1×1.

In relation to matrices, the numbers are usually known as scalars; for they behave as operators exactly like ordinary numbers as multipliers and hence are called scalars.

ILLUSTRATIVE EXAMPLES

1. $\begin{bmatrix} 2 & 3 & -1 \\ 4 & -5 & 6 \end{bmatrix}$ is a matrix of order 2×3.

2. $\begin{bmatrix} 2 & -3 & 4 \\ 5 & 6 & -2 \\ 1 & 0 & 4 \end{bmatrix}$ is a square matrix of order 3.

3. $\begin{bmatrix} 2 & 0 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is a sub-matrix of the matrix $\begin{bmatrix} 2 & 0 & 3 & -1 & 2 \\ 4 & 5 & 6 & 7 & 8 \\ 9 & 4 & 2 & 1 & 5 \end{bmatrix}$

4. 4, 0, 6 are the diagonal elements of the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 5 & 6 \\ 2 & 0 & 3 \\ 2 & -5 & 6 \end{bmatrix}$$ whose elements are 4, 5, 6, 2, 0, 3, 2, − 5, 6

i.e. if $\begin{bmatrix} a_{ij} \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 2 & 0 & 3 \\ 2 & -5 & 6 \end{bmatrix}$ then $a_{11} = 4,\ a_{12} = 5,\ a_{13} = 6$ etc.

Also trace of A *i.e.* tr A = 4 + 0 + 6 = 10.

5. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a 3×2 null matrix.

6. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a unit matrix of order 3.

7. $[2 \quad 3 \quad -5]$ is a 1×3 row matrix.

8. $\begin{bmatrix} 2 \\ 1 \\ 5 \\ 7 \end{bmatrix}$ is a 4×1 column matrix.

2.2 EQUALITY OF MATRICES

Two matrices A *and* B *defined as*

$$A = [a_{ij}] \text{ and } B = [b_{ij}]$$

are said to be equal if both are of the same order $m \times n$ *i.e.* A *has the same number of rows and columns as* B *and each element* a_{ij} *of* A *is equal to the corresponding element* b_{ij} *of* B *i.e.* $a_{ij} = b_{ij}$ *for each pair of subscripts* i *and* j.

Hence for equality, the two matrices must be identical in every respect or broadly speaking, the two matrices are equal if and only if one is a duplicate of the other.

ILLUSTRATIVE EXAMPLES

1. If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$

then A = B if and only if

$$a_{11} = b_{11}, a_{12} = b_{12}, a_{21} = b_{21}, a_{22} = b_{22}, a_{31} = b_{31}, a_{32} = b_{32}.$$

2. The matrices $\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 5 \\ 3 & -4 & 0 \end{bmatrix}$ being of different order are not comparable for equality.

3. The matrices $\begin{bmatrix} 2 & 3 \\ 0 & 1 \\ 5 & 4 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 \\ 6 & 1 \\ 5 & 4 \end{bmatrix}$ are comparable but not equal as the element of the 2nd row and 1st column of the first matrix is not equal to the corresponding element of the second matrix.

4. The matrices $\begin{bmatrix} 2 & 1 & 5 \\ 0 & 9 & 7 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 & 5 \\ 0 & 9 & 7 \end{bmatrix}$ are equal.

5. The matrices $\begin{bmatrix} 4 & 5 & 27 \\ 6 & 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 2^2 & 5 & 3^3 \\ 3\cdot 2 & 0 & \sqrt{9} \end{bmatrix}$ are equal.

COROLLARY. **Equivalance Relation on Matrices**

If there are three matrices **A**, **B**, **C** such that they satisfy the following three properties.

(*1*) *Reflexivity* $\mathbf{A} = \mathbf{A}$

(2) *Symmetry* $\mathbf{A} = \mathbf{B}$ implies that $\mathbf{B} = \mathbf{A}$

(*3*) *Transitivity* $\mathbf{A} = \mathbf{B}$ and $\mathbf{B} = \mathbf{C}$ imply that $\mathbf{A} = \mathbf{C}$

Then the equality of matrices is said to form an equivalence relation.

2.3 ADDITION OF MATRICES

Two matrices $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ are said to be conformable for addition if they are of the same order *i.e.* they have the same number of rows and the same number of columns. The sum of the two matrices **A** and **B** is then defined as the matrix each of whose elements is the sum of corresponding elements of **A** and **B** *i.e.*

$$\mathbf{A} + \mathbf{B} = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$$

For example

$$\text{if } \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots\cdots & b_{1n} \\ b_{21} & b_{22} & \cdots\cdots & b_{2n} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ b_{m1} & b_{m2} & \cdots\cdots & b_{mn} \end{bmatrix}$$

then

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots\cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots\cdots & a_{2n}+b_{2n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots\cdots & a_{mn}+b_{mn} \end{bmatrix}$$

As another example if

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 3 \\ -4 & 1 & 5 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} -1 & 2 & 0 \\ 3 & -4 & -5 \end{bmatrix}$$

$$\text{then } \mathbf{A} + \mathbf{B} = \begin{bmatrix} 2-1 & 0+2 & 3+0 \\ -4+3 & 1-4 & 5-5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -3 & 0 \end{bmatrix}$$

COROLLARY 1. $tr\,(\mathbf{A} + \mathbf{B}) = tr\,\mathbf{A} + tr\,\mathbf{B}$

e.g. if $\mathbf{A} = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$.

then $tr\,(\mathbf{A} + \mathbf{B}) = (-1+2) + (3+1)$ *i.e.* $1 + 4 = 5 = 2 + 3 = tr\,\mathbf{A} + tr\,\mathbf{B}$

COROLLARY 2. **Subtraction.** The difference of two matrices **A** and **B** which are conformable for addition may be defined as the sum of the two matrices **A** and (–**B**) where (–**B**) is the matrix obtained by multiplying every element of **B** by –1. Thus

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$$

i.e. the elements of the difference matrix A − B are obtained by subtracting the elements of B from the corresponding elements of A.

So if $A = [a_{ij}]$ and $B = [b_{ij}]$, then

$$A - B = [a_{ij}] - [b_{ij}] = [a_{ij} - b_{ij}]$$

As an illustrative example

$$\text{if } A = \begin{bmatrix} 2 & 5 & -3 \\ 0 & 7 & -8 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & 0 & 3 \\ -2 & 0 & 5 \end{bmatrix}$$

$$\text{then } A - B = \begin{bmatrix} 2-4 & 5-0 & -3-3 \\ 0-(-2) & 7-0 & -8-5 \end{bmatrix} = \begin{bmatrix} -2 & 5 & -6 \\ 2 & 7 & -13 \end{bmatrix}$$

COROLLARY 3. Multiplication of a matrix by a number (scalar)

If A is a matrix of any order say $m \times n$ defined as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix}$$

then the addition law follows

$$A + A = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix}$$

$$\text{or } 2A = 2\begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} 2a_{11} & 2a_{12} & \cdots\cdots & 2a_{1n} \\ 2a_{21} & 2a_{22} & \cdots\cdots & 2a_{2n} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ 2a_{m1} & 2a_{m2} & \cdots\cdots & 2a_{mn} \end{bmatrix}$$

Similarly

$$2A + A = 3A = 3\begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} 3a_{11} & 3a_{12} & \cdots\cdots & 3a_{1n} \\ 3a_{21} & 3a_{22} & \cdots\cdots & 3a_{2n} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ 3a_{m1} & 3a_{m2} & \cdots\cdots & 3a_{mn} \end{bmatrix}$$

or in general if k is a number, real or complex and A is a matrix then kA the matrix obtained by multiplying every element of A by k is said to be the scalar multiple of A.

As an illustrative example if $A = \begin{bmatrix} 0 & 2 & 3 \\ -1 & 5 & 4 \end{bmatrix}$, then

$$2A = 2\begin{bmatrix} 0 & 2 & 3 \\ -1 & 5 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 6 \\ -2 & 10 & 8 \end{bmatrix}$$

$$3A = 3\begin{bmatrix} 0 & 2 & 3 \\ -1 & 5 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 9 \\ -3 & 15 & 12 \end{bmatrix}$$

It is easy to verify that

$$A + A = 2A = 5A - 3A \text{ etc.}$$

and similarly $(3 + i)A = A + A(1 + i) + A$ etc., where $i = \sqrt{-1}$

COROLLARY 4. **Linear Combination of Matrices**

The same law of addition can be applied to combine any number of matrices. Thus if A, B, C, ... K be a finite number of matrices each of order $m \times n$ (say) and α, β, γ, ... κ be scalars then

$$\alpha A + \beta B + \gamma C + \dots\dots + \kappa K = [\alpha a_{ij} + \beta b_{ij} + \gamma c_{ij} + \dots\dots + \kappa k_{ij}]$$

Where $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}]$, ..., $K = [k_{ij}]$

As an illustrative example if

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 4 & -1 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 4 & 5 \end{bmatrix}, C = \begin{bmatrix} 2 & -1 & 5 \\ 3 & 0 & 4 \end{bmatrix}$$

then

$$3A - 4B + 2C = \begin{bmatrix} 6-4+4 & 9-(-8)+(-2) & 0-12+10 \\ 12-0+6 & -3-16+0 & 6-20+8 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 15 & -2 \\ 18 & -19 & -6 \end{bmatrix}$$

COROLLARY 5. $tr.\ (\lambda A) = \lambda tr.\ A$, λ being scalar *e.g.* if $A = \begin{bmatrix} 2 & -2 \\ 0 & -1 \end{bmatrix}$ so

$\lambda A = \begin{bmatrix} 2\lambda & -2\lambda \\ 0 & -\lambda \end{bmatrix}$ clearly $tr.\ (\lambda A) = 2\lambda - \lambda = \lambda$ and $\lambda\ tr.\ A = \lambda(2-1) = \lambda$

so $tr.\ \lambda A = \lambda tr.\ A$.

2.4 PROPERTIES OF MATRIX-ADDITION

[1] The Commutative Law

If A *and* B *are two matrices of the same order say* $m \times n$, *then*

$$A + B = B + A$$

Let $A = [a_{ij}]$ and $B = [b_{ij}]$, $i = 1, 2, \dots m$

$j = 1, 2, \dots n$

We have

$$\begin{aligned} A + B &= [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}] \\ &= [b_{ij} + a_{ij}] \text{ since } b_{ij} \text{ and } a_{ij} \text{ are scalars} \\ &= [b_{ij}] + [a_{ij}] \\ &= B + A \end{aligned}$$

i.e. the commutative law of addition holds.

[2] The Associative Law

If A, B, C *are three matrices of the same order say* $m \times n$, *then*

$$(A + B) + C = A + (B + C)$$

Let $A = [a_{ij}]$, $B = [b_{ij}]$, and $C = [c_{ij}]$, $i = 1, 2, \dots m$, $j = 1, 2, \dots n$

We have $(A + B) + C = ([a_{ij}] + [b_{ij}]) + [c_{ij}]$

$$= ([a_{ij} + b_{ij}] + [c_{ij}])$$
$$= [(a_{ij} + b_{ij}) + c_{ij}]$$
$$= [a_{ij} + (b_{ij} + c_{ij})], \ a_{ij}, b_{ij}, c_{ij}, \text{ being scalars}$$
$$= [a_{ij}] + ([b_{ij} + c_{ij}])$$
$$= A + (B + C)$$

i.e. the associative law of addition holds.

[3] The Distributive Law

If A *and* B *are two matrices of the same order say* $m \times n$ *and* k *is a scalar then*

$$k(A + B) = kA + kB$$

Let $A = [a_{ij}]$ and $B = [b_{ij}]$, $i = 1, 2, \ldots m$, $j = 1, 2, \ldots n$

We have, $k(A + B) = k[a_{ij} + b_{ij}]$

$$= [k(a_{ij} + b_{ij})]$$
$$= [ka_{ij}] + [kb_{ij}]$$
$$= k[a_{ij}] + k[b_{ij}], \ k \text{ being a scalar}$$
$$= kA + kB$$

i.e. the distributive law of addition holds.

[4] Existence of Additive Identity

If A *be a matrix of any order say* $m \times n$ *and* O *a null matrix of the same order such that when it is added to* A, *leaves it unchanged*

i.e. $A + O = A$

then O *is said to be the additive identity of* A.

Its proof immediately follows from the fact that if $A = [a_{ij}]$ and O is a null matrix *i.e.* a matrix having each of its elements zero, then

$A + O = [a_{ij} + 0]$

$= [a_{ij}]$ since a zero added to any scalar leaves it unchanged.

$= A$

Because of this fact O is said to be an additive identity of A.

[5] Existence of Additive Inverse

If A *be a matrix of any order say* $m \times n$, *and there exists a matrix* –A *of the same order such that if it is added to* A *gives a null matrix* O.

i.e. $A + (-A) = O$

then (–A) *is said to be the additive inverse of* A.

Let $A = [a_{ij}]$

Then, $-A = -[a_{ij}] = [-a_{ij}]$

So that $A + (-A) = [a_{ij}] + [-a_{ij}]$

$= [a_{ij} - a_{ij}]$

$= O$

Because of this fact (–A) is said to be an additive inverse of A.

[6] **The Cancellation Law**

If A, B, C *are three matrices conformable for addition then the relation*

$$\mathbf{A} + \mathbf{B} = \mathbf{A} + \mathbf{C}$$

holds if and only if $\mathbf{B} = \mathbf{C}$

Let $\mathbf{A} = [a_{ij}]$, $\mathbf{B} = [b_{ij}]$, and $\mathbf{C} = [c_{ij}]$, $i = 1, 2, \ldots m$

$j = 1, 2, \ldots n$

Then the relation A + B = A + C follows that (i, j)th elements on either side are identically equal *i.e.*

$$a_{ij} + b_{ij} = a_{ij} + c_{ij}$$

which yields

$b_{ij} = c_{ij}$ since a_{ij}, b_{ij}, c_{ij} all are scalars.

i.e. (i, j)th element of B = (i, j)th element of C for all values of i and j.

As such B = C

Hence the relation A + B = A + C holds if and only if B = C.

Problem 1. *If* $\mathbf{A} = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \end{bmatrix}$

then find

(*i*) A + B

(*ii*) A − C

(*iii*) −2A

(*iv*) 2A + 3B − 4C

(*i*) We have

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1+3 & 2-1 & -3+2 \\ 5+4 & 0+2 & 2+5 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \end{bmatrix}$$

(*ii*) We have

$$\mathbf{A} - \mathbf{C} = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-4 & 2-1 & -3-2 \\ 5-0 & 0-3 & 2-2 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 1 & -5 \\ 5 & -3 & 0 \end{bmatrix}$$

(*iii*) We have

$$-2\mathbf{A} = -2\begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -4 & 6 \\ -10 & 0 & -4 \end{bmatrix}$$

(*iv*) We have

$$2A + 3B - 4C = 2\begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \end{bmatrix} + 3\begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \end{bmatrix} - 4\begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2+9-16 & 4-3-4 & -6+6-8 \\ 10+12-0 & 0+6-12 & 4+15-8 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & -3 & -8 \\ 22 & -6 & 11 \end{bmatrix}$$

Problem 2. *If*

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix} \text{ and } C = \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$$

Verify that $A + (B - C) = (A + B) - C$

and determine the matrix D *such that* $A + D = B$

We have

$$A + (B - C) = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} + \left\{\begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & -2 & 0 \\ 4 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -3 \\ 9 & -1 & 5 \\ 2 & 1 & 0 \end{bmatrix} \qquad \ldots(1)$$

$$(A + B) - C = \left\{\begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}\right\} - \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -3 \\ 9 & -1 & 5 \\ 2 & 1 & 0 \end{bmatrix} \qquad \ldots(2)$$

From (1) and (2) it follows that

$$A + (B - C) = (A + B) - C$$

Hence the required relation is verified.

Now given that

$$A + D = B$$

Adding (–A) to both sides, we get

$$A + D - A = B - A$$

or $\quad D + A - A = B - A$ by commutative law

or $\quad D = B - A$ since $A - A = 0$ by existence of additive inverse.

$$= \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -3 & 5 \\ -1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$$

Which is the required matrix.

2.5 MULTIPLICATION OF MATRICES

Two matrices **A** *and* **B** *are conformable for multiplication if and only if the number of columns in* **A** *is equal to the number of rows in* **B**. *The product of the two matrices* **A** *and* **B** *denoted by* **AB** *is then defined as the matrix whose elements in the ith row and jth column is the algebraic sum of the products of the elements in the ith row of* **A** *by the corresponding elements in the jth column of* **B**.

In other words the product **AB** of two matrices conformable for multiplication, is the matrix whose element in *i*th row and *j*th column is the inner or scalar product of the *i*th row of **A** by the *j*th column of **B**, while the *inner product* or *scalar product* of two numbers x and y with components $x_1, x_2, \ldots x_n$ and $y_1, y_2, \ldots y_n$ is equal to

$$x_1y_1 + x_2y_2 + \ldots\ldots\ldots + x_ny_n$$

It should be noted that inner product of two numbers with unequal numbers of components is not defined.

As an illustrative example if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

then it is clear that the two matrices are conformable for multiplication since the number of columns in **A** is equal to the number of rows in **B**.

$$\therefore AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \\ a_{41}b_{11} + a_{42}b_{21} + a_{43}b_{31} & a_{41}b_{12} + a_{42}b_{22} + a_{43}b_{32} \end{bmatrix}$$

It is worth noting that the product **BA** is not defined, since the number of columns in **B** is not equal to the number of rows in **A**.

In the product **AB**, the matrix **A** is known as *Prefactor* and **B** as *Post factor*.

As an illustration in generalised form if

$A = [a_{ij}]$, a matrix of order $m \times n$

$B = [b_{jk}]$, a matrix of order $n \times p$

then $\quad AB = C$ (say) is a matrix of order $m \times p$

i.e. C = $[c_{ik}]$ is a matrix of order $m \times p$ such that

$$c_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk}$$

$$= a_{i1}\, b_{1k} + a_{i2}\, b_{2k} + \ldots\ldots\ldots + a_{in}\, b_{nk}\ ;\ i = 1, 2, \ldots m \text{ and } j = 1, 2, \ldots p$$

Thus

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{bmatrix}$$

COROLLARY. tr (AB) = tr (BA), all matrices being sqaure of order n.
If A = $[a_{ij}]$, B = $[b_{ij}]$ then AB = C say = $[c_{ij}]$

where $$c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}$$

Let BA = D = $[d_{ij}]$ where $d_{ij} = \sum_{k=1}^{n} b_{ik}\, a_{kj}$

$$\text{Thus } tr.\ (AB) = \sum_i c_{ii} = \sum_i \left(\sum_k a_{ik} b_{ki}\right)$$

$$= \sum_{k=1}^{n}\sum_{i=1}^{n} a_{ik} b_{ki} \quad \text{(on interchanging the order of summation)}$$

$$= \sum_{k=1}^{n}\left(\sum_{i=1}^{n} b_{ki} a_{ik}\right)$$

$$= \sum_{k=1}^{n} d_{kk} = tr.\ (BA)$$

which proves the proposition.

2.6 PROPERTIES OF MATRIX-MULTIPLICATION

[1] The Commutative Law for Multiplication does not hold in general

Consider the matrices

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 3 \\ -3 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

These are conformable for multiplication and so

$$AB = \begin{bmatrix} 1+0 & 0-2 & 2-6 \\ 2+0 & 0+3 & 4+9 \\ -3+0 & 0+1 & -6+3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -4 \\ 2 & 3 & 13 \\ -3 & 1 & -3 \end{bmatrix}$$

and $$BA = \begin{bmatrix} 1+0-6 & -2+0+2 \\ 0+2-9 & 0+3+3 \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ -7 & 6 \end{bmatrix}$$

It is apparent that the product matrix AB is of order 3×3 while the product matrix BA is of order 2×2 and therefore the two product matrices are quite different *i.e.*,

$$AB \neq BA$$

This follows that the commutative law of multiplication does not hold.

Had the order of the matrices AB and BA been the same, then it would be possible that AB $\neq$ BA if every or at least one element in AB would differ from the corresponding element in BA. Though there are a few exeptions in which case the commutative law holds good. Such cases will be considered in the discussion of special matrices.

In fact for a given pair of matrices A and B it is possible that the products AB and BA may not be conformable. For example if A is of order $m \times n$ and B of order $n \times p$, then the product AB is conformable and will be of order $m \times p$ while the product BA is not conformable since number of columns in B is not equal to number of rows in A. Thus the product AB exists while BA does not.

[2] The Distributive Law for Multiplication holds good

If A, B, C *be three matrices of suitable orders such that the products* A (B + C) *and* AB, AC *are conformable then*

$$A\,(B + C) = AB + AC.$$

Suppose that

$$A = [a_{ij}] \text{ is of order } m \times n$$

$$B = [b_{jk}] \text{ is of order } n \times p$$

and

$$C = [c_{jk}] \text{ is of order } n \times p$$

Then, (B + C) is of order $n \times p$ and A is of order $m \times n$ so that A (B + C) is conformable and of order $m \times p$. Also AB and AC both are of order $m \times p$ so that the sum matrix (AB + AC) is of order $m \times p$. Hence the matrices A (B + C) and AB + AC are of the same order so that they are comparable for equality.

Now,

$$(i,\ k)\text{th element of } A\,(B + C) = \sum_{j=1}^{n} a_{ij}\,(b_{jk} + c_{jk})$$

$$= \sum_{j=1}^{n} a_{ij}\, b_{jk} + \sum_{j=1}^{n} a_{ij}\, c_{jk}$$

$= (i, k)$th element of AB + (i, k)th element of AC

$= (i, k)$th element of (AB + AC)

for all $i = 1, 2, \ldots\ldots\ldots m$ and $k = 1, 2, \ldots\ldots\ldots p$

$$\therefore\ A(B + C) = AB + AC$$

A similar procedure will show that

$$(B + C)\,A = BA + CA$$

where B, C, A are of orders $m \times n$, $m \times n$, $n \times p$ respectively.

Hence the matrix multiplication is distributive with respect to addition.

[3] The Associative Law for Multiplication holds Good

If A, B, C *be three matrices of suitable orders such that the products* (AB) C *and* A (BC) *are conformable then*

$$(AB)\,C = A\,(BC)$$

Suppose that

$$A = [a_{ij}] \text{ is of order } m \times n$$
$$B = [b_{jk}] \text{ is of order } n \times p$$
and
$$C = [c_{kl}] \text{ is of order } p \times q$$

Then, (AB) is of order $m \times p$ and C of order $p \times q$ so that (AB) C is of order $m \times q$. Also (BC) is of order $m \times q$ and A of order $m \times n$ so that A (BC) is of order $m \times q$. Hence the matrices (AB) C and A (BC) are of the same order so that they are comparable for equality.

Now, (i, k)th element of AB $= \sum_{j=1}^{n} a_{ij}\, b_{jk}$

So that, (i, l)th element of (AB) C $= \sum_{k=1}^{p}\left\{\sum_{j=1}^{n} a_{ij}\, b_{jk}\right\} c_{kl}$

$$= \sum_{k=1}^{p}\sum_{j=1}^{n} a_{ij}\, b_{jk}\, c_{kl}$$

Also, (j, l)th element of (BC) $= \sum_{k=1}^{p} b_{jk}\, c_{kl}$

So that (i, l)th element of A (BC) $= \sum_{j=1}^{n}\left\{\sum_{k=1}^{p} b_{jk}\, c_{kl}\right\} a_{ij}$

$$= \sum_{j=1}^{n}\sum_{k=1}^{p} a_{ij}\, b_{jk}\, c_{kl}$$

$$= \sum_{k=1}^{p}\sum_{j=1}^{n} a_{ij}\, b_{jk}\, c_{kl}$$

$= (i, l)$th element of (AB) C

for all $i = 1, 2, \ldots m, l = 1, 2, \ldots p$

$\therefore$ A (BC) = (AB) C

Hence the matrix multiplication is associative.

[4] *If* A *be a matrix of order* $m \times n$ *and* O *a null matrix of order* $n \times p$ *then the product* AO *is another null matrix of order* $m \times p$ *i.e.*

$$AO_{n,p} = O_{m,p}$$

Also if O be an $m \times n$ null matrix and A a matrix of order $n \times p$ then their product is a null matrix of order $m \times p$ *i.e.*

$$O_{m,n} A = O_{m,p}$$

Conclusively if A be an n-rowed square matrix and O an n-rowed null matrix, then

$$AO = OA = O$$

[5] *If the product of two matrices* A *and* B *is a null matrix then it is not essential that either of them is a null matrix i.e.*

If AB = O, it does not necessarily mean that at least one of A and B is a null-matrix. *(Meerut, 1967; Gorakhpur, 1961)*

As an illustrative example if

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

then

$$AB = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1+1 & 1-1 \\ -2+2 & 2-2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

i.e. the product of two non-zero matrices A and B is a zero matrix.

[6] *If* I *be an m-rowed unit matrix and* A *an* $m \times n$ *matrix then*

$$I_m A = A$$

Suppose that $A = [a_{ij}]$ a matrix of order $m \times n$.

Then, $I_m A$ is a matrix of order $m \times n$ and so is comparable to A.

$$\text{Now, } (i,\ j)\text{th element of } I_m A = \sum_{k=1}^{m} (i,\ k)\text{th element of } I_m \cdot a_{kj}$$

But the (i, k)th element of I_m is zero except when $k = i$, therefore the right hand sum of the above expression will have only one term different from zero and that is the ith term which is equal to

$$(i, i)\text{th element of } I_m \cdot a_{ij} = 1 \cdot a_{ij}$$
$$= a_{ij}$$

$$\therefore\ (i, j)\text{th element of } I_m A = a_{ij}$$
$$= (i, j)\text{th element of A}$$

which follows that

$$I_m A = A$$

Similarly it may be shown that if A be an $m \times n$ matrix and I_n an n-rowed unit matrix, then

$$A I_n = A$$

Note. This result shows the existence of a multiplicative identity.

[7] Positive Integral Powers of Square Matrices

If A is a square matrix of order n (say), then

$$A^2 = AA$$

and the associative law of multiplication leads

$$A^2A = (AA)\,A = A\,(AA) = AA^2$$

or

$$A^3 = AAA = A^2A = AA^2$$

In the generalised form if p, q are two positive integers, then

$$A^pA^q = (AA\ldots\ldots A, p \text{ times})\ (AA\ldots\ldots A, q \text{ times})$$
$$= AA\ldots\ldots\ldots A, (p+q) \text{ times}$$
$$= A^{p+q}$$

and

$$(A^p)^q = (AA\ldots\ldots A, p \text{ times})^q$$
$$= A^qA^q\ldots\ldots A^q, p \text{ times}$$
$$= A^{pq}$$

COROLLARY. If I is a unit matrix of any order, then

$$I^2 = I^3 = \ldots\ldots = I^p = I.$$

Problem 3. *Find the product of the matrices*

$$A = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 4 & 1 \\ -2 & 1 & 0 \\ 1 & -3 & 2 \end{bmatrix}$$

The matrix A is of the order 2 × 4 and B is of the order 4 × 3 so that they are conformable for product and the product matrix will be of the order 2 × 3.

Now,

$$AB = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 4 & 1 \\ -2 & 1 & 0 \\ 1 & -3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2\cdot2+1\cdot0+2\cdot(-2)+1\cdot1 & 2\cdot(-1)+1\cdot4+2\cdot1+1\cdot(-3) & 2\cdot0+1\cdot1+2\cdot0+1\cdot2 \\ 1\cdot2+1\cdot0+1\cdot(-2)+1\cdot1 & 1\cdot(-1)+1\cdot4+1\cdot1+1\cdot(-3) & 1\cdot0+1\cdot1+1\cdot0+1\cdot2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & 3 \end{bmatrix}.$$

Problem 4. *If* $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ *and* $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$

find AB *and show that* AB ≠ BA.

Since A is of the order 2 × 3 and B is of the order 3 × 2, therefore A and B are conformable for the products AB and BA both. AB will be of the order 2 × 2 while BA will be of the order 3 × 3.

Now,

$$AB = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot2-2\cdot4+3\cdot2 & 1\cdot3-2\cdot5+3\cdot1 \\ -4\cdot2+2\cdot4+5\cdot2 & -4\cdot3+2\cdot5+5\cdot1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$$

and $$BA = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2\cdot1-3\cdot4 & -2\cdot2+3\cdot2 & 2\cdot3+3\cdot5 \\ 4\cdot1-5\cdot4 & -4\cdot2+5\cdot2 & 4\cdot3+5\cdot5 \\ 2\cdot1-1\cdot4 & -2\cdot2+1\cdot2 & 2\cdot3+1\cdot5 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{bmatrix}$$

It is clear that AB and BA being of the different order cannot be equal

i.e. $AB \neq BA$

Problem 5. *If* $A_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ *and* $A_\beta = \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}$

so that

$$A_{\alpha+\beta} = \begin{bmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) \\ -\sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

then prove that

$$A_\alpha A_\beta = A_\beta A_\alpha = A_{\alpha+\beta}$$

We have

$$A_\alpha A_\beta = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & \cos\alpha\sin\beta + \sin\alpha\cos\beta \\ -\sin\alpha\cos\beta - \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) \\ -\sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

$$= A_{\alpha+\beta}$$

Similarly $A_\beta A_\alpha = A_{\alpha+\beta}$

Hence $A_\alpha A_\beta = A_\beta A_\alpha = A_{\alpha+\beta}$.

Problem 6. *If* $A = \begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix}$ *prove that* $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$

where n is any positive integer.

We have

$$A^2 = AA = \begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 1+2\cdot2 & -4\cdot2 \\ 2 & 1-2\cdot2 \end{bmatrix}$$

$$A^3 = A^2A = \begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -12 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} 1+2\cdot3 & -4\cdot3 \\ 3 & 1-2\cdot3 \end{bmatrix}$$

$$A^4 = A^3A = \begin{bmatrix} 7 & -12 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & -16 \\ 4 & -7 \end{bmatrix} = \begin{bmatrix} 1+2\cdot4 & -4\cdot4 \\ 4 & 1-2\cdot4 \end{bmatrix}$$

Assuming that the result is true for an integral power n, we have

$$A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$$

Multiplying both sides by A,

$$A^{n+1} = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1+2(n+1) & -4(n+1) \\ n+1 & 1-2(n+1) \end{bmatrix}$$

Thus if the law is true for A^n, it is also true for A^{n+1}. But it is true for $n = 2, 3, 4......$; hence by the method of induction the same law is true for any positive integral value n.

2.7 PARTITIONING OF MATRICES

In many cases it is found rather convenient to regard a matrix as made up of a certain number of its submatrices which may be treated as the elements of the matrix. This can be done by drawing any number of lines parallel to the rows and columns of the given matrix. The submatrices contained in rectangles so formed are then treated as the elements of the matrices. This process of dividing a given matrix into certain number of sub-matrix elements is known as the *Partitioning of matrix*.

As an illustrative example if A is a matrix defined by

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 0 & -5 \\ 0 & 3 & 6 \end{bmatrix}$$

then this matrix can be partitioned in different forms by drawing lines parallel to row or column or both.

In the first instance, we may partition it as

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 0 & -5 \\ \cdots & \cdots & \cdots \\ 0 & 3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} \text{ where } A_{11} = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 0 & -5 \end{bmatrix} \text{ and } A_{21} = [0 \quad 3 \quad 6]$$

In the second instance we may have

$$A = \begin{bmatrix} 1 & \vdots & 2 & 4 \\ -2 & \vdots & 0 & -5 \\ 0 & \vdots & 3 & 6 \end{bmatrix}$$

$$= [A_{11} A_{12}] \text{ where } A_{11} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \text{ and } A_{12} = \begin{bmatrix} 2 & 4 \\ 0 & -5 \\ 3 & 6 \end{bmatrix}$$

In the third instance we may express

$$A = \begin{bmatrix} 1 & 2 & \vdots & 4 \\ -2 & 0 & \vdots & -5 \\ \cdots & \cdots & \cdots\vdots & \cdots \\ 0 & 3 & \vdots & 6 \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ where } A_{11} = \begin{bmatrix} 1 & 2 \\ -2 & 0 \end{bmatrix}, A_{12} = \begin{bmatrix} 4 \\ -5 \end{bmatrix},$$

$$A_{21} = [0 \quad 3], A_{22} = [6]$$

As another example if we have two matrices of the same order say

$$A = \begin{bmatrix} 0 & 2 & -5 \\ 1 & 3 & 4 \\ -1 & 5 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 & 0 & 3 \\ 5 & 4 & 2 \\ 3 & -1 & 6 \end{bmatrix}$$

partitioned identically as below

$$A = \left[\begin{array}{c:cc} 0 & 2 & -5 \\ 1 & 3 & 4 \\ \hdashline -1 & 5 & 2 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

and

$$B = \left[\begin{array}{c:cc} -2 & 0 & 3 \\ 5 & 4 & 2 \\ \hdashline 3 & -1 & 6 \end{array}\right] = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

then the two matrices A and B are said to be **identically partitioned**.

The two identically partitioned matrices A and B are comparable for addition if the corresponding sub-matrices in A and B are of the same order *i.e.* the order of A_{11}, B_{11} is the same, the order of A_{12}, B_{12} is the same, the order of A_{21}, B_{21} is the same and the order of A_{22}, B_{22} is the same in the above example and then we have

$$A + B = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} + B_{11} & A_{12} + B_{12} \\ A_{21} + B_{21} & A_{22} + B_{22} \end{bmatrix}$$

2.8 PRODUCT OF MATRICES BY PARTITIONING

Let A be a matrix of order $m \times n$ and B be a matrix of order $n \times p$ so that the matrices A and B are conformable for multiplication AB. Now if the two matrices are partitioned, then *the partitioned matrices will be conformable for multiplication 'when to each partition line of* A *parallel to the columns there corresponds a partition line of* B *parallel to its rows such that the number of columns of* A *lying between two adjacent partition lines is the same as the number of rows of* B *lying between corresponding adjacent lines.'* Broadly speaking if in A there lie a partition line after third column, another line after fifth column, then in B a corresponding line should be after third row and the another line after fifth row.

It is notable that partition line in A parallel to its columns may be drawn in any arbitrary manner. Two matrices A and B partitioned in the above described manner are said to be *conformably partitioned for multiplication*, for, with such partitions the two matrices can be multiplied as usual, with sub-matrices as the elements.

As an illustrative example if A be a matrix of order 4×5 and B a matrix of order 5×6 partitioned as below :

$$A = \left[\begin{array}{cc:cc:c} 2 & 1 & 3 & -1 & 2 \\ 3 & 2 & 2 & 5 & 0 \\ \hdashline 4 & 5 & 0 & 5 & 3 \\ 0 & 2 & 6 & 7 & 4 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix} \text{ of order } 2 \times 3$$

where A_{11}	is of order	2×2
A_{12}	"	2×2
A_{13}	"	2×1
A_{21}	"	2×2
A_{22}	"	2×2
A_{23}	"	2×1

$$\text{and } B = \left[\begin{array}{ccc:ccc} 0 & 2 & 3 & 4 & -2 & 3 \\ 5 & 4 & -2 & 5 & 6 & 2 \\ \hdashline 2 & 0 & 3 & 1 & 4 & 5 \\ -2 & 3 & 4 & 5 & -6 & 0 \\ \hdashline 0 & 4 & -1 & 2 & 7 & 8 \end{array}\right] = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix} \text{ of order } 3 \times 2$$

where B_{11}	is of order	2×3
B_{12}	"	2×3
B_{21}	"	2×3
B_{22}	"	2×3
B_{31}	"	1×3
B_{32}	"	1×3

Then it is clear that A and B are conformable for multiplication. Also the partitioned matrices A and B are conformable for multiplication, for, the partitioned matrix A is of order 2×3 and the partitioned matrix B is of order 3×2.

Thus,

$$AB = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31} & A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32} \\ A_{21}B_{11} + A_{22}B_{21} + A_{23}B_{31} & A_{21}B_{12} + A_{22}B_{22} + A_{23}B_{32} \end{bmatrix}$$

$$= \left[\begin{array}{l} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 0 & 2 & 3 \\ 5 & 4 & -2 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 2 & 0 & 3 \\ -2 & 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}[0 \quad 4 \quad -1] \\ \qquad \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 4 & -2 & 3 \\ 5 & 6 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 4 & 5 \\ 5 & -6 & 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}[2 \quad 7 \quad 8] \end{array}\right.$$

$$\left.\begin{matrix}\begin{bmatrix}4&5\\0&2\end{bmatrix}\begin{bmatrix}0&2&3\\5&4&-2\end{bmatrix}+\begin{bmatrix}0&5\\6&7\end{bmatrix}\begin{bmatrix}2&0&3\\-2&3&4\end{bmatrix}+\begin{bmatrix}3\\4\end{bmatrix}\begin{bmatrix}0&4&-1\end{bmatrix}\\ \begin{bmatrix}4&5\\0&2\end{bmatrix}\begin{bmatrix}4&-2&3\\5&6&2\end{bmatrix}+\begin{bmatrix}0&5\\6&7\end{bmatrix}\begin{bmatrix}1&4&5\\5&-6&0\end{bmatrix}+\begin{bmatrix}3\\4\end{bmatrix}\begin{bmatrix}2&7&8\end{bmatrix}\end{matrix}\right]$$

$$=\left[\begin{matrix}\begin{bmatrix}5&8&4\\10&14&5\end{bmatrix}+\begin{bmatrix}8&-3&5\\-6&15&26\end{bmatrix}+\begin{bmatrix}0&8&-2\\0&0&0\end{bmatrix}\\ \begin{bmatrix}13&2&8\\22&6&13\end{bmatrix}+\begin{bmatrix}-2&18&15\\27&-22&10\end{bmatrix}+\begin{bmatrix}4&14&16\\0&0&0\end{bmatrix}\end{matrix}\right.$$

$$\left.\begin{matrix}\begin{bmatrix}25&28&2\\10&8&-4\end{bmatrix}+\begin{bmatrix}-10&15&20\\-2&21&46\end{bmatrix}+\begin{bmatrix}0&12&-3\\0&16&-4\end{bmatrix}\\ \begin{bmatrix}41&22&22\\10&12&4\end{bmatrix}+\begin{bmatrix}25&-30&0\\41&-18&30\end{bmatrix}+\begin{bmatrix}6&21&24\\8&28&32\end{bmatrix}\end{matrix}\right]$$

$$=\begin{bmatrix}\begin{bmatrix}13&13&7\\4&29&31\end{bmatrix}&\begin{bmatrix}15&34&39\\49&-16&23\end{bmatrix}\\ \begin{bmatrix}15&55&19\\8&45&38\end{bmatrix}&\begin{bmatrix}72&13&46\\59&22&66\end{bmatrix}\end{bmatrix}$$

$$=\begin{bmatrix}13&13&7&15&34&39\\4&29&31&49&-16&23\\15&55&19&72&13&46\\8&45&38&59&22&66\end{bmatrix}$$

2.9 SPECIAL MATRICES WITH THEIR PROPERTIES

While defining a matrix, we have already mentioned a few of the special types of matrices like square matrix, null matrix, row and column matrices, unit matrix etc., but in this section we have to consider almost all the special types of matrices with their properties in relations to others. Some types will be discussed in more detailed than a few others which are rarely used in different branches of mathematics, applied mathematics and mechanics.

[1] Square Matrix and Special Square Matrices

A matrix A having the number of rows and columns equal is called a **square matrix** *e.g.*

$$A=\begin{bmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\cdots&\cdots&\cdots&\cdots\\a_{n1}&a_{n2}&\cdots&a_{nn}\end{bmatrix}$$

Here A is said to be a square matrix of order n or an n-rowed square matrix or simply n-square matrix.

A matrix which is not square matrix may be called as a **rectangular matrix**, *e.g.*

$$A=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\end{bmatrix}$$

which is of order 2×3.

Determinant of a square matrix. The determinant of a square matrix A is the determinant which has got its elements all the elements of the matrix A in the same places. It is denoted by | A |. Thus if $A = [a_{ij}]$ then $| A | = | a_{ij} |$.

As another example if A be an n-square matrix given by

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

then $$|A| = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

It is easy to show that the determinant of the product matrix AB is equal to the product of the determinants of the square matrices A and B of the same order *i.e.*,

$$| AB | = | A | | B |$$

Or, in general the determinant of the product of any number of square matrices of the same order is the product of the determinants of those matrices *i.e.*

$$| ABC.........K | = | A | | B | | C |.........| K |$$

Special square matrices. While discussing the properties of matrix-multiplication we have mentioned the commutative property. In relation to square matrices A and B of the same order say n, we can say that if AB = BA then A and B *commute* and if AB = −BA then A and B *anti-commute*. It can be easily verified that an n-square matrix A commutes with itself and also with identity matrix I_n *i.e.*,

$$AA = AA$$

and $$AI_n = I_nA$$

If k is a positive integer, then the matrix A with the property $A^{k+1} = A$ is said to be *periodic*. In case k is the least positive integer such that $A^{k+1} = A$, then A is said to be of *Period k.*

In the case when $k = 1$, so that $A^2 = A$, then A is said to be *idempotent* and if p is some positive integer such that $A^p = 0$ then A is called *nilpotent*. In case p is the least positive integer such that $A^p = 0$, then A is said to be nilpotent of *index p*.

ILLUSTRATIVE EXAMPLES

1. $\begin{bmatrix} 5 & 0 & 4 \\ 2 & 3 & 9 \\ -1 & 5 & 0 \end{bmatrix}$ is a square matrix of order 3.

2. $\begin{bmatrix} 2 & 3 & 4 \\ 5 & -2 & 0 \end{bmatrix}$ is a rectangular matrix of order 2 × 3.

3. The determinant of the 3-square matrix $\begin{bmatrix} 2 & 0 & 4 \\ 5 & -2 & 1 \\ 0 & 3 & 2 \end{bmatrix}$ is $\begin{vmatrix} 2 & 0 & 4 \\ 5 & -2 & 1 \\ 0 & 3 & 2 \end{vmatrix}$

4. The matrices $\begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$ and $\begin{bmatrix} \gamma & \delta \\ \delta & \gamma \end{bmatrix}$ commute for all values of α, β, γ, δ

since

$$\begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} \gamma & \delta \\ \delta & \gamma \end{bmatrix} = \begin{bmatrix} \alpha\gamma + \beta\delta & \alpha\delta + \beta\gamma \\ \beta\gamma + \alpha\delta & \beta\delta + \alpha\gamma \end{bmatrix} = \begin{bmatrix} \gamma & \delta \\ \delta & \gamma \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$$

5. The matrix $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is idempotent, since

$$\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

6. The matrix $\begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$ is periodic of period 2, since

$$\begin{aligned} \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}^{2+1} &= \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}^2 \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix} \\ &= \begin{bmatrix} -5 & -6 & -6 \\ 9 & 10 & 9 \\ -4 & -4 & -3 \end{bmatrix}\begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix} \end{aligned}$$

7. The matrix $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent of index 3, since

$$\begin{aligned} \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}^3 &= \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}^2 \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O \end{aligned}$$

[2] Row Matrix and Column Matrix (Row and Column Vectors)

A matrix having only one row and any number (>1) of columns is said to be a *row matrix* or a *row vector e.g.*, a row matrix of order $1 \times n$ is

$[a_{11}\, a_{12} \ldots\ldots a_{1n}]$ or $[a_1\, a_2 \ldots\ldots a_n]$.

A matrix having only one column and any number (>1) of rows is said to be a *column matrix* or a *column vector e.g.*, a column matrix of order $m \times 1$ is

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ \vdots \\ a_{m1} \end{bmatrix} \text{ or } \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_m \end{bmatrix}$$

In expressing vectors as matrices, the elements of the row vector or column vector are known as the components of the vectors. Thus we can define an *n-vector* as '*an ordered n-tuple of n real or complex numbers written in a horizontal or in a vertical line.*'

Algebraic operations on vectors. If A and B be two vectors with components $a_1, a_2, \ldots\ldots a_n$ and $b_1, b_2, \ldots\ldots b_n$ respectively, then it is easy to show that

(*i*) $\mathbf{A} + \mathbf{B} = [a_1, a_2, \ldots\ldots, a_n] + [b_1, b_2, \ldots\ldots b_n]$

$= [a_1+b_1, a_2+b_2, \ldots\ldots\ldots, a_n+b_n]$

(*ii*) $k\mathbf{A} = k[a_1, a_2, \ldots\ldots\ldots a_n]$, k being a scalar

$= [ka_1, ka_2, \ldots\ldots\ldots ka_n]$

(*iii*) $\mathbf{A} \cdot \mathbf{B} = a_1b_1 + a_2b_2 + \ldots\ldots\ldots + a_nb_n$

(*iv*) $\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2 = a_1a_1 + a_2a_2 + \ldots\ldots\ldots + a_na_n$

$= |a_1|^2 + |a_2|^2 + \ldots\ldots\ldots + |a_n|^2$

(*v*) A vector will be a unit vector if $|\mathbf{A}| = 1$.

(*vi*) The vectors A and B will be orthogonal if $\mathbf{A}.\mathbf{B} = 0$.

(*vii*) The vector O [0, 0, ...0] is said to be a null vector.

(*viii*) $\mathbf{A} + \mathbf{O} = \mathbf{O} + \mathbf{A} = \mathbf{A}$

Linearly Dependent and Independent sets of vectors. A set of n vectors $\mathbf{A}_1, \mathbf{A}_2, \ldots\ldots\ldots \mathbf{A}_n$ is said to be *linearly dependent* if there exists a set of n (of which at least one is non-zero) scalars $k_1, k_2, \ldots\ldots\ldots k_n$ such that

$$k_1 \mathbf{A}_1 + k_2 \mathbf{A}_2 + \ldots\ldots\ldots + k_n \mathbf{A}_n = \mathbf{O}.$$

In the case when $k_1 = k_2 = \ldots\ldots\ldots = k_n = 0$, the set of n vectors $\mathbf{A}_1, \mathbf{A}_2, \ldots\ldots\ldots \mathbf{A}_n$ is said to be *linearly independent* if

$$k_1 \mathbf{A}_1 + k_2 \mathbf{A}_2 + \ldots\ldots\ldots + k_n \mathbf{A}_n = \mathbf{O}.$$

A vector A is known as linear *combination* of the set of vectors $\mathbf{A}_1, \mathbf{A}_2, \ldots\ldots\ldots \mathbf{A}_n$ if it is expressible as

$$\mathbf{A} = k_1 \mathbf{A}_1 + k_2 \mathbf{A}_2 + \ldots\ldots\ldots + k_n \mathbf{A}_n$$

where $k_1, k_2, \ldots\ldots\ldots k_n$ are scalars.

ILLUSTRATIVE EXAMPLES

1. [1 0 2 4] is a 1×4 row matrix (or row vector).

2. $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$ is a 3×1 column matrix (or column vector).

3. If A = [4, 6, 0, 3] and B = [2, 3, 4, 0] are two vectors, then

$$\mathbf{A} + \mathbf{B} = [4, 6, 0, 3] + [2, 3, 4, 0] = [6, 9, 4, 3]$$

$$3\mathbf{A} - 2\mathbf{B} = 3[4, 6, 0, 3] - 2[2, 3, 4, 0]$$

$$= [12, 18, 0, 9] - [4, 6, 8, 0] = [8, 12, -8, 9] \text{ etc.}$$

4. The set of vectors [1, 2, 3], [2, –2, 0] is linearly independent since $k_1[1, 2, 3] + k_2[2, -2, 0] = 0$ is equivalent to a system of equations

$$k_1 + 2k_2 = 0,\ 2k_1 - 2k_2 = 0,\ 3k_1 = 0.$$

which are satisfied only if $k_1 = k_2 = 0$.

5. The set of vectors [2, 4, 10], [3, 6, 15] is linearly dependent, since

$k_1[2, 4, 10] + k_2[3, 6, 15] = 0$ is equivalent to a system of equations

$$2k_1 + 3k_2 = 0,\ 4k_1 + 6k_2 = 0,\ 10k_1 + 15k_2 = 0$$

which are satisfied when $k_1 = 3, k_2 = -2$ *i.e.*, non-zero values of k_1 and k_2.

[3] Null Matrix

If all the elements of a matrix are zero then it is said to be a ***null matrix*** and denoted by O *e.g*.

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ is a null matrix of order } 3 \times 4.$$

The commutative law of multiplication holds good in case of a null matrix *e.g.*, if

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \text{ and } \mathbf{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

then
$$\mathbf{AO} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{OA} = \mathbf{O}$$

We have also proved that a null matrix acts as an additive identity of any matrix *i.e.*, as in the above example

$$\mathbf{A} + \mathbf{O} = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} = \mathbf{A}$$

[4] Unit Matrix or Identity Matrix

A square matrix of order n having unit elements in the Principal or leading diagonal and zero elements everywhere else, is called a *unit matrix* or *identity matrix* and denoted by $\mathbf{I}_n$. Thus

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \text{ and for } n = 4,\ \mathbf{I}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ etc.}$$

It is clear that a square matrix $\mathbf{A} = [a_{ij}]$ is the unit matrix if

$$a_{ij} = 1 \text{ for } i = j$$
$$= 0 \text{ for } i \neq j$$

It is also evident that

$$\mathbf{I}_n = \mathbf{I}_n^2 = \mathbf{I}_n^3 = \dots\dots\dots$$

The commutative law of multiplication also holds good in case of a unit matrix *e.g.*, if

$$\mathbf{A} = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 4 & 5 \\ 2 & 0 & -3 \end{bmatrix} \text{ and } \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

then
$$\mathbf{AI} = \mathbf{IA} = \mathbf{A} = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 4 & 5 \\ 2 & 0 & -3 \end{bmatrix}$$

ILLUSTRATIVE EXAMPLE

If $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$, *find* A^2 *and determine scalars a, b, s.t.*

$$I + aA + bA^2 = O$$

I, O *being unit null matrices of order two.* *(Meerut, 1970)*

Clearly $\quad A^2 = A \cdot A = \begin{bmatrix} -3 & 2 \\ -4 & -3 \end{bmatrix}$

and $\quad I + aA + bA^2 = O$

$$\Rightarrow \begin{bmatrix} 1+a-3b & 0+2a+4b \\ 0-2a-4b & 1+a-3b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 1+a-3b = 0 \\ 0+2a+4b = 0 \\ 0-2a-4b = 0 \\ 1+a-3b = 0 \end{cases}$$

$$\Rightarrow a = -\frac{2}{5}, \; b = \frac{1}{5}$$

[5] Upper and Lower Triangular Matrices, Diagonal and Scalar Matrices

A square matrix A of any order n, defined by

$$A = [a_{ij}]$$

is said to be an *upper triangular matrix* if its elements $a_{ij} = 0$ for $i > j$ where i, j are positive integers ranging over from 1 to n. Thus an upper triangular matrix of order n is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

The square matrix $A = [a_{ij}]$ of order n is said to be a *lower triangular matrix* if its elements $a_{ij} = 0$ for $i < j$ where i, j are integers such that $1 \le i \le n$, $1 \le j \le n$. Thus a lower triangular matrix of order n is

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

A square matrix of order n which is both upper and lower triangular is called a *diagonal matrix e.g.*,

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

where $a_{11}, a_{22}, a_{33}, \ldots\ldots\ldots a_{nn}$ is the *principal diagonal* and $(A)_{ij} = a_{ij}\,\delta_{ij}$.

This is denoted by diag $[a_{11}, a_{22}, a_{33}, \ldots\ldots\ldots a_{nn}]$ and the sum $a_{11} + a_{22} + a_{33}, \ldots\ldots\ldots + a_{nn}$ is called its *trace i.e., tr* $A = \Sigma a_{ij}$. Thus matrix $[a_{ij}]$ is *diagonal* if $i = j$ and *off-diagonal* or *non-diagonal* if $i \neq j$.

In a diagonal matrix if all the diagonal elements are equal to a scalar quantity say λ *i.e.*, $a_{11} = a_{22} = a_{33} = \ldots\ldots\ldots = a_{nn} = \lambda$, then the matrix is called a *scalar matrix e.g.*,

$$\begin{bmatrix} \lambda & 0 & 0 & \ldots & 0 \\ 0 & \lambda & 0 & \ldots & 0 \\ 0 & 0 & \lambda & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \lambda \end{bmatrix} \quad \textit{i.e.}\ (\mathbf{A})_{ij} = \lambda\delta_{ij}$$

It bears the name scalar matrix due to the reason that if a square matrix A of order n be multiplied by the scalar matrix of order n, then it is equivalent to multiplying the matrix A by a scalar.

In case $\lambda = 1$, the scalar matrix reduces to a *unit matrix*.

In other words an n-rowed square matrix $[a_{ij}]$ is called a *scalar matrix* when

$$a_{ij} = \lambda \text{ (some scalar) for } i = j.$$
$$= 0 \text{ for } i \neq j.$$

ILLUSTRATIVE EXAMPLES

1. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 5 \\ 0 & 0 & -2 \end{bmatrix}$ is a 3-rowed upper triangular matrix.

2. $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 5 & 0 \\ 3 & 4 & -1 \end{bmatrix}$ is a 3-rowed lower triangular matrix.

3. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a 3-rowed diagonal matrix. Its trace is $2 + 1 + 3$ *i.e.* 6.

4. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is a 3-rowed scalar matrix.

That's why? Because if we multiply it by a 3-rowed square matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & 5 \\ -2 & 5 & 4 \end{bmatrix}$$

then

$$\begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & 5 \\ -2 & 5 & 4 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 4 \\ 6 & 8 & 10 \\ -4 & 10 & 8 \end{bmatrix} = 2\begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & 5 \\ -2 & 5 & 4 \end{bmatrix} = 2\mathbf{A}$$

i.e., A has been multiplied by the scalar 2.

Problem 7(a). *Show that if* **AB** = **A** *and* **BA** = **B** *then* **A** *and* **B** *are idempotent.*

(b) *Show that if* A *and* B *are idempotent matrices, then* **A + B** *is idempotent iff*

$$\mathbf{AB} = \mathbf{BA} = 0 \qquad \textit{(Benares, 1970)}$$

(a) We have
$$\mathbf{ABA} = (\mathbf{AB})\,\mathbf{A}$$
$$= \mathbf{AA} \text{ as } \mathbf{AB} = \mathbf{A}$$
$$= \mathbf{A}^2 \qquad \ldots(1)$$

Also $ABA = A(BA)$

$= AB$ as $BA = B$

$= A$ as $AB = A$ ...(2)

From (1) and (2) it follows that

$$A^2 = A = ABA$$

Hence A is idempotent.

Again consider

$BAB = B(AB)$

$= BA$ as $AB = A$

$= B$ as $BA = B$

and also

$BAB = (BA)B$

$= BB$ as $BA = B$

$= B^2$

It is clear that $BAB = B^2 = B$

Hence B is also idempotent.

(b) A and B are idempotent

$\Rightarrow A^2 = A$ and $B^2 = B$ ...(3)

Now,

$$\begin{aligned}(A + B)^2 &= (A + B)(A + B)\\ &= A^2 + AB + BA + B^2\\ &= A + AB + BA + B \quad \text{by (3)}\\ &= A + B \iff AB + BA = 0\end{aligned}$$

$$\Rightarrow AB = BA = 0.$$

Problem 8. *If* A *and* B *are n-square matrices then show that* A *and* B *commute if and only if* $A-\lambda I$ *and* $B-\lambda I$ *commute for every scalar* λ.

If $A-\lambda I$ and $B-\lambda I$ commute, then

$$(A-\lambda I)(B-\lambda I) = (B-\lambda I)(A-\lambda I)$$

i.e. $AB-\lambda(A + B) + \lambda^2 I = BA - \lambda(B + A) + \lambda^2 I \qquad \because AI = A$ etc.

The comparison gives

$$AB = BA \text{ since } A + B = B + A$$

i.e. A and B commute.

Conversely if A and B commute then $AB = BA$.

Consider $(A-\lambda I)(B-\lambda I) = AB - \lambda(A + B) + \lambda^2 I$

$= BA - \lambda(B + A) + \lambda^2 I$

$= (B - \lambda I)(A - \lambda I)$

which follows that $A-\lambda I$ and $B-\lambda I$ commute.

Problem 9. *Derive a rule for forming the product* BA *of an* $m \times n$ *matrix* B *and* $A = diag\,(a_{11}, a_{22}, \ldots\ldots a_{nn})$.

Given $$A = \begin{bmatrix} a_{11} & 0 & \ldots & 0 \\ 0 & a_{22} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & a_{nn} \end{bmatrix}$$

and suppose that

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

then consider the product

$$\mathbf{BA} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} b_{11}a_{11} & b_{12}a_{22} & \cdots & b_{1n}a_{nn} \\ b_{21}a_{11} & b_{22}a_{22} & \cdots & b_{2n}a_{nn} \\ \cdots & \cdots & \cdots & \cdots \\ b_{m1}a_{11} & b_{m2}a_{22} & \cdots & b_{mn}a_{nn} \end{bmatrix}$$

which follows that the product **BA** of an $m \times n$ matrix **B** by a n-square diagonal matrix **A** is obtained by multiplying the first column of **B** by a_{11}, the second column of **B** by a_{22} and so on. This gives the required rule.

[6] The Transpose of a Matrix

The matrix of order $n \times m$ obtained from any matrix **A** of order $m \times n$, by interchanging its rows and columns is called the *transpose of* **A** and is denoted by **A′** or $\mathbf{A}^T$ or $\tilde{\mathbf{A}}$ (called A *tilde*). Thus if $\mathbf{A} = [a_{ij}]$, then **A′** or $\mathbf{A}^T = [a'_{ij}]$ where $a'_{ij} = a_{ji}$ *i.e.*, the (j, i)th element of **A′** is the (i, j)th element of **A**, *e.g.*, if

$$\mathbf{A} = \begin{bmatrix} 5 & 4 & -2 \\ 1 & 0 & 2 \\ 3 & 4 & -5 \\ 6 & -4 & 2 \end{bmatrix}, \text{ then } \mathbf{A}' = \begin{bmatrix} 5 & 1 & 3 & 6 \\ 4 & 0 & 4 & -4 \\ -2 & 2 & -5 & 2 \end{bmatrix}$$

Properties of the transpose of a matrix

I. *The transpose of a matrix coincides with itself, i.e., if* **A** *be a matrix then* $(\mathbf{A}')' = \mathbf{A}$.

Let $\mathbf{A} = [a_{ij}]$ be the matrix of order $m \times n$.

Then $\mathbf{A}' = [a'_{ij}]$ is the matrix of order $n \times m$.

So that $(\mathbf{A}')'$ will be the matrix of order $m \times n$.

As such **A** and $(\mathbf{A}')'$ are the matrices of the same order.

Again, (i, j)th element of **A** = (j, i)th element of **A′**.

= (i, j)th element of $(\mathbf{A}')'$

Hence $(\mathbf{A}')' = \mathbf{A}$.

II. *The determinant of the transpose of a square matrix is the same as the determinant of the matrix.*

Let $\mathbf{A} = [a_{ij}]$ be the square matrix of order n.

Then transpose of **A** *i.e.*, **A′** will be the square matrix of order n.

Now, $|A| = |a_{ij}| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$

and

$$|A'| = |a'_{ij}| = \begin{vmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$ since the interchange of rows and columns in a determinant does not change the value of the determinant.

$$= |A|$$

Hence the determinant of the transpose of a square matrix is equal to the determinant of the matrix.

III. *If k be any scalar and* A *a matrix, then*

$$(kA)' = kA'$$

Let $A = [a_{ij}]$ be the matrix of order $m \times n$. Then A' and so kA' is a matrix of order $n \times m$. Also (kA) is the matrix of order $m \times n$ and so $(kA)'$ is the matrix of order $n \times m$. Thus the matrices kA' and $(kA)'$ are of the same order.

Now,

(i, j)th element of $(kA)'$ = (j, i)th element of kA
= k times the (j, i)th element of A
= $k\, a_{ji}$
= k times the (i, j)th element of A'
= (i, j)th element of kA'

Hence $(kA)' = kA'$.

IV. *The transpose of the sum of two matrices* A *and* B *(conformable for addition) is the sum of their transposes i.e.,*

$$(A + B)' = A' + B'$$

Let $A = [a_{ij}]$ be the matrix of order $m \times n$.
and $B = [b_{ij}]$ " " $m \times n$.

Then $(A + B)$ will be the matrix of order $m \times n$.
and $(A + B)'$ " " $n \times m$.

Also A' is the matrix of order $n \times m$
and B' " " $n \times m$.

so that $(A' + B')$ is the matrix of order $n \times m$.

Thus $(A + B)'$ and $(A' + B')$ will be the matrices of the same order $n \times m$.

Again (j, i)th element of $(A + B)'$ = (i, j)th element of A + B
= $a_{ij} + b_{ij}$

and (j, i)th element of $(A' + B')$ = (j, i)th element of A' + (j, i)th element of B'

$= (i, j)$th element of **A** + (i, j)th element of **B**

$= a_{ij} + b_{ij} = (i, j)$th element of **A** + **B**

$= (j, i)$th element of $(\mathbf{A} + \mathbf{B})'$

Hence $(\mathbf{A} + \mathbf{B}) = \mathbf{A}' + \mathbf{B}'$.

V. Reversal law for a transpose. *The transpose of the product of two matrices* **A**, **B** (*conformable for multiplication*) *is the product of their transposes taken in reverse order i.e.,*

$$(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$$

Let $\mathbf{A} = [a_{ij}]$ be the matrix of order $m \times n$

and $\mathbf{B} = [b_{jk}]$ be the matrix of order $n \times p$.

Then **AB** is the matrix of order $m \times p$ and so $(\mathbf{AB})'$ is the matrix of order $p \times m$.

Also, $\mathbf{A}'$ is the matrix of order $n \times m$ and $\mathbf{B}'$ is the matrix of order $p \times n$ so that $\mathbf{B}'\mathbf{A}'$ is the matrix of order $p \times m$.

As such the order of the matrices $(\mathbf{AB})'$ and $\mathbf{B}'\mathbf{A}'$ is the same.

Again, (k, i)th element of $(\mathbf{AB})' = (i, k)$th element of **AB**

$$i.e., \quad [(\mathbf{AB})']_{ki} = (\mathbf{AB})_{ik} = \sum_j (\mathbf{A})_{ij}(\mathbf{B})_{jk} = \sum_{j=1}^{n} a_{ij} b_{jk} \qquad ...(1)$$

Now $\mathbf{A}' = [a'_{ij}]$ where $a_{ji} = a'_{ij}$

and $\mathbf{B}' = [b'_{jk}]$ where $b_{kj} = b'_{jk}$

$$\therefore \ (k, i)\text{th element of } \mathbf{B}'\mathbf{A}' = \sum_{j=1}^{n} b'_{kj} a'_{ji} = \sum_{j=1}^{n} b_{jk} a_{ij} = \sum_{j=1}^{n} a_{ij} b_{jk}$$

$= (k, i)$th element of $(\mathbf{AB})'$ from (1)

Hence $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$

Note. The general reversal law for a transpose may be stated as :

If **A**, **B**, **C**,.........**K** be any number of matrices conformable for multiplication in order, then

$$(\mathbf{ABC}...........\mathbf{JK})' = \mathbf{K}'\mathbf{J}'.........\mathbf{C}'\mathbf{B}'\mathbf{A}'.$$

[7] Symmetric and Skew-symmetric Matrices

A square matrix **A** is said to be **symmetric** if its transpose coincides with itself *i.e.*, $\mathbf{A}' = \mathbf{A}$.

In other words a square matrix $\mathbf{A} = [a_{ij}]$ is said to be symmetric if $a_{ij} = a_{ji}$ for all integral values of i and j *e.g.*, if

$$\mathbf{A} = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \text{ then } \mathbf{A}' = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

i.e., $\mathbf{A}' = \mathbf{A}$ and hence **A** is the symmetric matrix.

It is evident that the total number of independent elements in a symmetric matrix of order n is $\frac{n^2 - n}{2} + n$ *i.e.*, $\frac{1}{2} n\,(n+1)$ since all the n diagonal elements are independent and of the remaining elements $n^2 - n$ the equidistant elements from the diagonal are the same, so that the number of independent elements other than the diagonal elements is $\frac{1}{2}(n^2-n)$.

Again, a square matrix A such that $A' = -A$ is said to be **skew-symmetric.** In other words a square matrix A is skew-symmetric if $a_{ij} = -a_{ji}$ for all integral values of i and j. It follows that $a_{ii} = -a_{ii}$ for $i = j$, so that $2\,a_{ii} = 0$ *i.e.*, $a_{ii} = 0$ which shows that all the diagonal elements of a skew-symmetric matrix are zero.

As an illustrative example,

$$\text{the matrix} \begin{bmatrix} 0 & 1 & 2 & -3 \\ -1 & 0 & -4 & 5 \\ -2 & 4 & 0 & 6 \\ 3 & -5 & -6 & 0 \end{bmatrix} \text{is skew - symmetric.}$$

The number of independent elements in a skew-symmetric matrix is clearly $\frac{1}{2}(n^2-n)$.

Problem 10. *Every square matrix can be uniquely expressed as the sum of a symmetric matrix and a skew-symmetric matrix.* *(Meerut, 1975, 80)*

Let A be any square matrix. Then we have

$$A = \tfrac{1}{2}(A + A') + \tfrac{1}{2}(A - A')$$

Denoting $\frac{1}{2}(A + A')$ by P and $\frac{1}{2}(A - A')$ by Q we have

$$A = P + Q$$

Now,

$$\begin{aligned} P' &= \left\{\tfrac{1}{2}(A + A')\right\}' \\ &= \tfrac{1}{2}\{A' + (A')'\} \\ &= \tfrac{1}{2}\{A' + A\} \quad \because (A')' = A \\ &= \tfrac{1}{2}\{A + A'\} \\ &= P \end{aligned}$$

which follows that P is a symmetric matrix.

Also

$$\begin{aligned} Q' &= \left\{\tfrac{1}{2}(A - A')\right\}' \\ &= \tfrac{1}{2}\{A' - (A')'\} \\ &= \tfrac{1}{2}\{A' - A\} \\ &= -\tfrac{1}{2}\{A - A'\} \\ &= -Q \end{aligned}$$

which follows that Q is a skew-symmetric matrix.

Thus the square matrix A is expressible as the sum of a symmetric matrix P and a skew-symmetric matrix Q.

Now to show that this representation is unique, let us assume that if possible A can be expressed as

$$A = R + S$$

where R is symmetric matrix and S a skew-symmetric matrix.

Consider, $A' = (R + S)' = R' + S'$

$= R - S$ as $R = R'$ and $S' = -S$

$\therefore \frac{1}{2}(A + A') = P = R$ and $\frac{1}{2}(A - A') = Q = S$

i.e., **R** is not different from **P** and similarly **S** is not different from **Q**, showing that this representation is unique.

[8] The Conjugate of a Matrix

If the elements of a matrix A *are complex quantities, then the matrix obtained from* A, *on replacing its elements by the corresponding conjugate complex numbers, is said to be the* **conjugate matrix** *of* A *and is denoted by* $\bar{A}$ or A^* *(called* A *star).*

Thus if $A = [a_{ij}]$, then $\bar{A} = [\bar{a}_{ij}]$ where $\bar{a}_{ij}$ denotes the conjugate of a_{ij}

As an illustrative example if

$$A = \begin{bmatrix} 1-5i & -2+7i & 5 & 2i \\ -5+6i & 3-4i & -i & -2 \\ -3-5i & 4+5i & 0 & -7i \end{bmatrix}$$

then

$$\bar{A} = \begin{bmatrix} 1+5i & -2-7i & 5 & -2i \\ -5-6i & 3+4i & i & -2 \\ -3+5i & 4-5i & 0 & 7i \end{bmatrix}$$

The properties of the conjugate matrices

I. *The conjugate of the conjugate of a matrix* A *coincides with itself i.e.*, $\overline{(\bar{A})} = A$

Supposing that $A = [a_{ij}]$ is the matrix of order $m \times n$, the matrices $(\bar{A})$ and $\overline{(\bar{A})}$ will also be the matrices of the same order $m \times n$.

Now (i, j)th element of $\bar{A}$ = conjugate of the (i, j)th element of A

$= \bar{a}_{ij}$

and (i, j)th element of $\overline{(\bar{A})}$ = conjugate of the (i, j)th element of $\bar{A}$

= conjugate of $\bar{a}_{ij}$

$= a_{ij}$, since if $a_{ij} = \alpha + i\beta$, then $\bar{a}_{ij} = \alpha - i\beta$

and $(\bar{\bar{a}}_{ij}) = \alpha + i\beta = a_{ij}$

$= (i, j)$th element of A

Hence $\overline{(\bar{A})} = A$.

II. *The conjugate of the sum of two matrices* A *and* B *(conformable for addition) is the sum of their conjugate i.e.*, $(\overline{A + B}) = \bar{A} + \bar{B}$

Suppose that $A = [a_{ij}]$ and $B = [b_{ij}]$ are the matrices of the same order $m \times n$. Then the matrices $\bar{A}$ and $\bar{B}$ will also be of the same order $m \times n$ and the order of sum matrix $(A + B)$ and so that of its conjugate $(\overline{A + B})$ will be the same *i.e.*, $m \times n$

Now, $A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$

$\therefore\ \overline{(\mathbf{A}+\mathbf{B})}$ = conjugate of $[a_{ij}+b_{ij}]$

= conjugate of a_{ij} + conjugate of b_{ij}

$= \bar{a}_{ij}+\bar{b}_{ij}$

$= \overline{\mathbf{A}} + \overline{\mathbf{B}}$

Hence $\overline{(\mathbf{A}+\mathbf{B})} = \overline{\mathbf{A}} + \overline{\mathbf{B}}$.

III. *If α be a complex number and* **A** *a matrix of any order say $m \times n$, then*

$$\overline{(\alpha\mathbf{A})} = \bar{\alpha}\overline{\mathbf{A}}$$

Let $\mathbf{A} = [a_{ij}]$ be the matrix of order $m \times n$.

Then $\overline{\mathbf{A}}$ will also be the matrix of the order $m \times n$.

Now, $\alpha\mathbf{A} = \alpha[a_{ij}]$

$= [\alpha a_{ij}]$

$\therefore\ \overline{(\alpha\mathbf{A})}$ = conjugate of $[\alpha a_{ij}]$

$= [\overline{\alpha a_{ij}}]$

$= \bar{\alpha}[\bar{a}_{ij}]$

$= \bar{\alpha}\overline{\mathbf{A}}$

IV. *The conjugate of the product of two matrices* **A** *and* **B** *(conformable for multiplication) is the product of their conjugates i.e.*

$$\overline{(\mathbf{AB})} = \overline{\mathbf{A}}\,\overline{\mathbf{B}}$$

Suppose that $\mathbf{A} = [a_{ij}]$ is the matrix of order $m \times n$

and $\mathbf{B} = [b_{jk}]$ is the matrix of order $n \times p$

So that **AB** is the matrix of order $m \times p$.

The order of $\overline{(\mathbf{AB})}$ will also be $m \times p$.

Again, the order of $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$ being $m \times n$ and $n \times p$ respectively, the order of $\overline{\mathbf{A}}\,\overline{\mathbf{B}}$ will be $m \times p$. Thus the orders of the matrices $\overline{(\mathbf{AB})}$ and $\overline{\mathbf{A}}\,\overline{\mathbf{B}}$ are equal.

Now, (i, k)th element of $\overline{(\mathbf{AB})}$ = conjugate of (i, k)th element of **AB**

= conjugate of $\sum_{j=1}^{n} a_{ij}\ b_{jk}$

$= \sum_{j=1}^{n} \overline{a_{ij}b_{jk}}$

$= \sum_{j=1}^{n} \bar{a}_{ij}\bar{b}_{jk}$

= (i, k)th element of $\overline{\mathbf{A}}\,\overline{\mathbf{B}}$

Hence $\overline{(\mathbf{AB})} = \overline{\mathbf{A}}\,\overline{\mathbf{B}}$.

[9] The Conjugate Transpose or Hermitian Conjugate of a Matrix

The matrix, which is the conjugate of the transpose of a matrix A *is said to be* **conjugate transpose** *of* A *and is denoted by*

$$A^{\Theta} \text{ or } \overline{A}' \text{ or } \overline{A}^{T} \text{ or } A^{+} \text{ (called A dagger).}$$

As an illustrative example if

$$A = \begin{bmatrix} -2+3i & 3-4i & i \\ -5i & -5-3i & 4+i \end{bmatrix}$$

then
$$A' = \begin{bmatrix} -2+3i & -5i \\ 3-4i & -5-3i \\ i & 4+i \end{bmatrix}$$

So that
$$A^{\Theta} \text{ or } \overline{A}' = \begin{bmatrix} -2-3i & 5i \\ 3+4i & -5+3i \\ -i & 4-i \end{bmatrix}$$

It is easy to see that $\overline{A}' = (\overline{A})'$

The properties of conjugate transposed matrices

I. *The conjugate transpose of the conjugate transpose of a matrix* A *coincides with itself i.e.,* $(A^{\Theta})^{\Theta} = A$

we have $\quad A^{\Theta} = \overline{A}' = (\overline{A})'$

$$\therefore \quad (A^{\Theta})' = [(\overline{A})']' = \overline{A} \text{ since } (B')' = B$$

so that
$$(A^{\Theta})^{\Theta} \text{ i.e., } (\overline{A}^{\Theta})' = \overline{(\overline{A})}$$
$$= A \quad \because \overline{(\overline{A})} = A$$

II. *The conjugate transpose of the sum of two matrices* A *and* B *(conformable for addition) is the sum of their conjugate transposes i.e.,*

$$(A+B)^{\Theta} = A^{\Theta} + B^{\Theta}$$

we have
$$\begin{aligned} (A+B)^{\Theta} &= (\overline{A+B})' \\ &= (\overline{A'+B'}) \\ &= \overline{A}' + \overline{B}' \\ &= A^{\Theta} + B^{\Theta} \end{aligned}$$

III. *If* α *be a complex number and* A *a matrix of any order then*

$$(\alpha A)^{\Theta} = \overline{\alpha} A^{\Theta}$$

we have
$$\begin{aligned} (\alpha A)^{\Theta} &= (\overline{\alpha A'}) \\ &= \overline{\alpha}\overline{A}' \\ &= \overline{\alpha} A^{\Theta} \end{aligned}$$

IV. *The conjugate transpose of the product of two matrices* A *and* B *(conformable for multiplication) is the product of their conjugate transpose in reverse order i.e.*

$$(AB)^{\Theta} = B^{\Theta} A^{\Theta}$$

we have
$$(AB)^{\Theta} = (\overline{AB})' = \overline{B'A'}$$

$$= \overline{B}'\overline{A}'$$
$$= B^{\Theta}A^{\Theta}$$

[10] Hermitian and Skew-Hermitian Matrices

A square matrix $A = [a_{ij}]$ *is said to be* **Hermitian** *if* A *coincides with its conjugate transpose i.e.,* $A = A^{\Theta}(= \overline{A}')$ *or if the transpose of* A *coincides with its conjugate i.e.,* $A' = \overline{A}$

Thus $a_{ji} = \bar{a}_{ij}$ for all integral values of i and j.

So that $a_{ii} = \bar{a}_{ii}$ for $i = j$.

Which follows that every diagonal element of a Hermitian matrix is equal to its conjugate and it is only possible if all the diagonal elements are real.

As an illustrative example, the matrix

$$\begin{bmatrix} 5 & 2+3i & -i \\ 2-3i & 3 & -3-4i \\ i & -3+4i & 0 \end{bmatrix} \text{ is Hermitian.}$$

A square matrix $A=[a_{ij}]$ is said to be *Skew-Hermitian* if

$A^{\Theta} = -A$ or $A' = -\overline{A}$ *i.e.,* $\bar{a}_{ij} = -a_{ji}$ for all integral values of i and j.

Thus $\bar{a}_{ii} = -a_{ii}$ for $i = j$

i.e., $a_{ii} + \bar{a}_{ii} = 0$

Which follows that every diagonal element of a Skew-Hermitian matrix is either zero or a pure imaginary number.

As an illustrative example, the matrix

$$\begin{bmatrix} 3i & -3+4i & 4-5i \\ +3-4i & -4i & 5 \\ -4-5i & -5 & 0 \end{bmatrix} \text{ is Skew-Hermitian.}$$

Problem 11. *Every square matrix A can be uniquely expressed as the sum of a Hermitian matrix and a Skew-Hermitian matrix.*

We can express

$$A = \tfrac{1}{2}(A + A^{\Theta}) + \tfrac{1}{2}(A - A^{\Theta})$$
$$= P + Q \text{ where } P = \tfrac{1}{2}(A + A^{\Theta}),\ Q = \tfrac{1}{2}(A - A^{\Theta})$$

$$\therefore\ P^{\Theta} = \tfrac{1}{2}(A + A^{\Theta})^{\Theta}$$
$$= \tfrac{1}{2}\{A^{\Theta} + (A^{\Theta})^{\Theta}\}$$
$$= \tfrac{1}{2}(A^{\Theta} + A) \quad \because (A^{\Theta})^{\Theta} = A$$
$$= \tfrac{1}{2}(A + A^{\Theta})$$
$$= P$$

which follows that P is Hermitian.

Also

$$Q^{\Theta} = \tfrac{1}{2}(A - A^{\Theta})^{\Theta}$$
$$= \tfrac{1}{2}\{A^{\Theta} - (A^{\Theta})^{\Theta}\}$$

$$= \frac{1}{2}(A^{\Theta} - A)$$

$$= -\frac{1}{2}(A - A^{\Theta})$$

$$= -Q$$

which follows that Q is Skew-Hermitian.

Hence the square matrix A is expressible as the sum of a Hermitian and a Skew-Hermitian matrices.

In order to show that this representation is unique, let us assume if possible that there is another way of representation say

$$A = R + S$$

where R is Hermitian and S is Skew-Hermitian.

Consider $A^{\Theta} = (R + S)^{\Theta} = R^{\Theta} + S^{\Theta}$

$= R - S$ as R is Hermitian and S is Skew-Hermitian.

These give $R = \frac{1}{2}(A + A^{\Theta}) = P$

and $S = \frac{1}{2}(A - A^{\Theta}) = Q$

which show that R is not different from P and similarly S is not different from Q. Hence this representation is unique.

Problem 12. *If* A, B, C *are three matrices conformable for multiplication in the given order, then show that*

$$(ABC)' = C'B'A'$$

We have $(ABC)' = \{(AB)C\}'$

$= C'(AB)'$ by the reversal law of transpose

$= C'B'A'$

Problem 13. *If* A *and* B *be symmetric matrices, then show that* AB *is symmetric if and only if* A *and* B *commute.*

Since A and B are symmetric matrices, therefore

$$A' = A \text{ and } B' = B.$$

We have $(AB)' = B'A' = BA$ ... (1)

But AB is symmetric if and only if $(AB)' = AB$ ... (2)

From (1) and (2) it follows that AB is symmetric if

$$AB = BA$$

i.e., if A and B commute.

Conversely if A and B commute, AB = BA, then

$$(AB)' = B'A' = BA = AB.$$

Thus if A and B commute AB is symmetric.

Hence AB is symmetric if and only if A and B commute.

Problem 14. *If* A *is a m-square matrix and* P *is a matrix of order* $m \times n$ *then show that* $B = P'AP$ *will be symmetric or skew-symmetric according as* A *is symmetric or skew-symmetric.*

Assuming that A is symmetric, we have

$B' = (P'AP)' = \{P'(AP)\}' = (AP)'(P')'$ by reversl law of transposes

$= P'A'P \quad \because \quad (AP)' = P'A'$ and $(P')' = P$

$= P'AP \quad \because$ A is symmetric so that $A' = A$

Thus $(P'AP)' = P'AP$

which follows that P'AP is symmetric if A is symmetric.

Again if we assume that A is skew-symmetric *i.e.*, $A' = -A$ then

$$B' = (P'AP)' = -P'AP \text{ (proceeding as above)}$$

which follows that P'AP is skew-symmetric when A is skew-symmetric.

Problem 15. *If* **A** *and* **B** *are Hermitian, show that* **AB + BA** *is Hermitian and* **AB–BA** *is Skew-Hermitian.*

$\because$ A and B are Hermitian

$\therefore$ $A^{\Theta} = A$ and $B^{\Theta} = B$

Consider

$$\begin{aligned}(AB + BA)^{\Theta} &= (AB)^{\Theta} + (BA)^{\Theta} \\ &= B^{\Theta}A^{\Theta} + A^{\Theta}B^{\Theta} \\ &= BA + AB \\ &= AB + BA\end{aligned}$$

which follows that AB + BA is Hermitian.

Again we have

$$\begin{aligned}(AB - BA)^{\Theta} &= (AB)^{\Theta} - (BA)^{\Theta} \\ &= B^{\Theta}A^{\Theta} - A^{\Theta}B^{\Theta} \\ &= BA - AB \\ &= -(AB - BA)\end{aligned}$$

which follows that AB–BA is skew-Hermitian.

Problem 16. *Prove that every Hermitian matrix* **A** *can be expressed as* **B** + i**C** *where* **B** *is real and symmetric and* **C** *is real and skew-symmetric.*

$\because$ B and C are real,

$\therefore$ $B^{\Theta} = B'$ and $C^{\Theta} = C'$ ...(1)

(because the conjugate of a real number is itself)

Again since B is symmetric and C is skew-symmetric, we have

$$B' = B \text{ and } C' = -C \quad ...(2)$$

(1) and (2) yield,

$$B^{\Theta} = B \text{ and } C^{\Theta} = -C$$

Now assume that

$$A = B + iC$$

$$\begin{aligned}\therefore \quad A^{\Theta} &= (B + iC)^{\Theta} \\ &= B^{\Theta} - iC^{\Theta} \quad \because \text{ Conjugate of } i \text{ is } -i \\ &= B - i(-C) \\ &= B + iC = A\end{aligned}$$

which follows that A is Hermitian.

Hence the result follows.

[11] Adjugate Matrix or Adjoint of a Matrix

If $A = [a_{ij}]$ *be a square matrix of any order say n and* A_{ij} *represents the cofactor of the element* a_{ij} *in the determinant* $|A|$ *i.e.,* $|a_{ij}|$ *of the square matrix* A, *then the transpose of the matrix* $[A_{ij}]$ *is called as the* **adjugate** *or* **adjoint** *of* A *and is denoted by adj* A.

Thus if $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, then adj $A = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$

The determinant of the adj. A *i.e.*, | adj A | is said to be the adjugate determinant of A or the adjugate of | A |.

As an illustrative example if $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$, then

$$\text{cofactor of} \quad a = \begin{vmatrix} b & f \\ f & c \end{vmatrix} = bc - f^2 \text{ in } |A| = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

$$\text{"} \quad h = -\begin{vmatrix} h & f \\ g & c \end{vmatrix} = fg - ch \quad \text{"}$$

$$\text{"} \quad g = \begin{vmatrix} h & b \\ g & f \end{vmatrix} = hf - bg \quad \text{"}$$

etc.

$$\therefore \quad \text{Adj } A = \begin{bmatrix} bc - f^2 & gf - ch & hf - bg \\ gf - ch & ac - g^2 & gh - af \\ hf - bg & gh - af & ab - h^2 \end{bmatrix}$$

An important relation between a matrix A and its adjugate.

If **A** *be a square matrix of any order say n and* **I** *the unit matrix of the same order then*

$$A\,(\text{adj } A) = |A|\, I = (\text{adj } A)\, A$$

Let $A = [a_{ij}]$ be the square matrix of order n, *i.e.*,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Then if A_{ij} is the cofactor of a_{ij} for all integral values of i and j, we have

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

Since the orders of A and adj A are the same, therefore they are conformable for product. Moreover both of them being square matrices, their product is commutative.

Now we know by the properties of a determinant that if the elements of a row (column) of a determinant are multiplied by their own cofactors then the sum of the products is the value of the determinant and the sum of the products of the elements of any row (column) by the cofactors of the corresponding elements of another row (column) is zero *i.e.*, if

$$|A| = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

and $A_{11}, A_{12}, \dots\dots\dots$ are cofactors of $a_{11}, a_{12}, \dots\dots\dots$ etc., then

$$a_{11}A_{11} + a_{12}A_{12} + \dots\dots\dots + a_{1n}A_{1n} = |A| \text{ etc.}$$

and $a_{11}A_{12} + a_{11}A_{22} + \dots\dots\dots + a_{1n}A_{2n} = 0$ etc.

Applying these results, we thus have

$$A\,(\text{adj } A) = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} |A| & 0 & \dots & 0 \\ 0 & |A| & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & |A| \end{bmatrix}$$

$$= |A| \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

$$= |A|\, I.$$

Similarly it may be shown that $(\text{adj } A)\, A = |A|\, I$

Hence $A\,(\text{adj } A) = |A|\, I = (\text{adj } A)\, A$ ... (1)

which follows that multiplication of A and adj A is commutative and that their product is a scalar matrix having every diagonal element as $|A|$.

Note 1. We have shown above that

$$A\,(\text{adj } A) = \begin{bmatrix} |A| & 0 & \dots & 0 \\ 0 & |A| & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & |A| \end{bmatrix}$$

Taking determinants of either side we get

$$|A\,(\text{adj } A)| = \begin{vmatrix} |A| & 0 & \dots & 0 \\ 0 & |A| & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & |A| \end{vmatrix},$$

or $$|A|\,|\text{adj } A| = |A|^n \begin{vmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{vmatrix}$$ by the properties of determinants.

i.e., $|A|\,|\text{adj } A| = |A|^n$

which gives $|\text{adj } A| = |A|^{n-1}$ ... (2)

Note 2. If the square matrix A of order n is such that $|A| = 0$,

then $A\,(\text{adj } A) = (\text{adj } A)\, A = 0.$

Note 3. If A and B are two n-square matrices then

$$\text{adj AB} = \text{adj B adj A}$$

which may be shown as below :

Applying the result (1) we have

$$\text{AB (adj AB)} = |\text{AB}|\,\text{I} = \text{(adj AB) AB} \quad \ldots (\alpha)$$

Now
$$\begin{aligned} \text{AB adj B} \cdot \text{adj A} &= \text{A (B adj B) adj A} \\ &= \text{A }(|\text{B}|\text{I})\text{ adj A by (1)} \\ &= |\text{B}|\,(\text{A adj A}) \\ &= |\text{B}|\,(|\text{A}|\,\text{I}) \text{ by (1)} \\ &= |\text{AB}|\,\text{I} \end{aligned}$$

and
$$\begin{aligned} (\text{adj B} \cdot \text{adj A})\text{ AB} &= \text{adj B }[(\text{adj A})\text{ A}]\text{ B} \\ &= \text{adj B} \cdot |\text{A}|\,\text{I}\cdot\text{B} \\ &= |\text{A}|\,\{(\text{adj B})\text{B}\} \\ &= |\text{A}|\,|\text{B}|\,\text{I} \\ &= |\text{AB}|\,\text{I} \end{aligned}$$

$$\therefore \quad \text{AB }(\text{adj B} \cdot \text{adj A}) = (\text{AB})\text{ I} = (\text{adj B} \cdot \text{adj A})\text{ AB} \quad \ldots (\beta)$$

From (α) and (β) we conclude that

$$\text{adj AB} = \text{adj B} \cdot \text{adj A} \quad \ldots (3)$$

Note 4. The result of note 3 can be extended to the case of three n-square matrices A, B, C *i.e.*,

$$\text{adj ABC} = \text{adj C} \cdot \text{adj B} \cdot \text{adj A}.$$

It is easy to show it as below :

By result (3) we have

$$\begin{aligned} \text{adj A(BC)} &= \text{adj BC} \cdot \text{adj A} \\ &= \text{adj B} \cdot \text{adj C} \cdot \text{adj A} \text{ by (3)} \end{aligned}$$

[12] Singular and Non-singular Matrices

A square matrix $A = [a_{ij}]$ is known as *singular* matrix if its determinant $|A| = 0$ *i.e.*, $|a_{ij}| = 0$. In case $|A| \neq 0$ the matrix A is known as *non-singular* matrix.

ILLUSTRATIVE EXAMPLES

1. The matrix $A = \begin{bmatrix} 21 & 17 & 7 & 10 \\ 24 & 22 & 6 & 10 \\ 6 & 8 & 2 & 3 \\ 5 & 7 & 1 & 2 \end{bmatrix}$ is singular

since $\begin{bmatrix} 21 & 17 & 7 & 10 \\ 24 & 22 & 6 & 10 \\ 6 & 8 & 2 & 3 \\ 5 & 7 & 1 & 2 \end{bmatrix} = 0$

2. The matrix $A = \begin{bmatrix} 12 & 3 & 7 \\ 27 & 7 & 17 \\ 36 & 9 & 22 \end{bmatrix}$ is non-singular since $|A| = 3 \neq 0$.

[13] Reciprocal Matrix or Inverse of a Matrix

[*Agra, 1970; Allahabad, 65; Gorakhpur, 1963, 65*]

If there exists a square matrix **B** *of the same order as that of* **A** *such that* **AB = BA = I, I** *being a unit matrix of the same order then* **B** *is said to be the inverse of* **A** *and is denoted by* $\mathbf{A}^{-1}$. *Thus by definition,*

$$AA^{-1} = A^{-1}A = I \quad \ldots (1)$$

When the inverse of A exists, then A is said to be *Invertible.*

Inverse of a matrix in terms of its adjugate

According to the definition if A is an invertible matrix, then

$$AA^{-1} = A^{-1}A = I \quad \ldots (\alpha)$$

Also, we have from §2.9[11],

$$A\,(\text{adj } A) = (\text{adj } A)\,A = |A|\,I$$

or
$$A\left(\frac{1}{|A|}\text{adj } A\right) = \left(\frac{1}{|A|}\text{adj } A\right)A = I \quad \ldots (\beta)$$

From (α) and (β) it is obvious that

$$AA^{-1} = A\left(\frac{1}{|A|}\text{adj } A\right) \text{ and } A^{-1}A = \left(\frac{1}{|A|}\text{adj } A\right)A$$

either of which lead to

$$A^{-1} = \frac{1}{|A|}\text{adj } A \quad \ldots (2)$$

which gives the inverse of a square matrix A in terms of its adjugate.

THEOREM 1. The necessary and sufficient condition for a square matrix to be invertible is that it is non-singular.

To prove that the condition is *necessary* let us assume that A is a given square matrix which is invertible and let B be its inverse.

Then, $AB = I$ (by definition of inverse)

Taking determinant of either side we get,

$$|AB| = |I|$$

or
$$|A|\,|B| = I$$

which is only possible if neither of $|A|$ and $|B|$ is zero *i.e.*, if the matrices A and B are non-singular or if $|A| \neq 0$.

To prove that the condition is *sufficient*, let us assume that $|A| \neq 0$ and there exists a matrix B such that

$$B = \frac{1}{|A|}\text{adj } A$$

Then

$$\begin{aligned} AB &= A\frac{1}{|A|}\text{adj } A \\ &= \frac{1}{|A|}(A \text{ adj } A) \\ &= \frac{1}{|A|}|A|\,I \text{ from §2.9[11]} \\ &= I \end{aligned}$$

and similarly $BA = I$

i.e., $AB = I = BA$

which shows that A has an inverse *i.e,* A is invertible.

THEOREM 2. The inverse of a matrix is unique. *[Agra, 1970]*

Let A be an invertible (square) matrix and if possible let us assume that B and C are two inverses of A, then

$$AB = BA = I$$

$$AC = CA = I$$

Now $CAB = C(AB) = CI = C$

Also, $CAB = (CA)B = IB = B$

so that $CAB = B = C$

i.e., B is not different from C

Hence the inverse of a matrix is unique *i.e.,* there exists only one inverse matrix to a given matrix.

Properties of inverse matrices

I. The Reversal law for inverses. *If* A *and* B *are two n-square non-singular matrices (conformable for multiplication)*

then $(AB)^{-1} = B^{-1}A^{-1}$

Since A and B are non-singular square matrices, they are invertible.

Let their inverses be A^{-1} and B^{-1} respectively.

$\therefore$ $AA^{-1} = A^{-1}A = I$

and $BB^{-1} = B^{-1}B = I$

Now $|A| \neq 0, |B| \neq 0$ imply that $|AB| = |A||B| \neq 0$, which shows that AB is also invertible.

Let us now consider a matrix C given by

$$C = B^{-1}A^{-1}$$

Then,

$$\begin{aligned} C(AB) &= (B^{-1}A^{-1})(AB) \\ &= B^{-1}(A^{-1}A)B \\ &= B^{-1}(I)B \\ &= B^{-1}B \quad \because IB = B \\ &= I \end{aligned}$$

i.e., $(B^{-1}A^{-1})(AB) = I$

Similarly it can be shown that

$$(AB)(B^{-1}A^{-1}) = I$$

These results follow that $B^{-1}A^{-1}$ is the inverse of AB *i.e.,*

$$(AB)^{-1} = B^{-1}A^{-1} \quad \dots (3)$$

Note. The result can be extended to any number of square matrices which are conformable for multiplication *i.e.,*

$$(ABC\dots JK)^{-1} = K^{-1}J^{-1}\dots C^{-1}B^{-1}A^{-1}$$

II. *If* A *is a non-square matrix, then the inverse of its inverse coincides with itself i.e.,* $(A^{-1})^{-1} = A$.

We have $AA^{-1} = A^{-1}A = I$

Which can be interpreted as that A is the inverse of A^{-1} *i.e,*

$$(A^{-1})^{-1} = A \quad \dots (4)$$

III. *If* A *is a non-singular matrix, then the transpose (conjugate transpose) of an inverse is the inverse of the transpose (conjugate transpose) of* A *i.e.*

$$(A^{-1})' = (A')^{-1} \text{ and } (A^{-1})^{\Theta} = (A^{\Theta})^{-1}$$

Since A is non-singular, it is invertible and therefore

$$A\,A^{-1} = A^{-1}A = I \qquad \ldots (\gamma)$$

Taking transpose of either side and using $I' = I$, we have

$$(AA^{-1})' = (A^{-1}A)' = I$$

or $(A^{-1})' A' = A' (A^{-1}) = I$ by reversal law of transposes.

Which follows that $(A^{-1})'$ is the inverse of A' i.e.

$$(A')^{-1} = (A^{-1})' \qquad \ldots (5)$$

Again taking conjugate transpose of either side of (γ), we have

$$(AA^{-1})^{\Theta} = (A^{-1}A)^{\Theta} = I^{\Theta}$$

or $(A^{-1})^{\Theta} A^{\Theta} = A^{\Theta} (A^{-1})^{\Theta} = I \because I^{\Theta} = I$

which shows that $(A^{-1})^{\Theta}$ is the inverse or A^{Θ} i.e.

$$(A^{\Theta})^{-1} = (A^{-1})^{\Theta} \qquad \ldots (6)$$

IV. *If* A *is invertible then*

$$tr.\ (ACA^{-1}) = tr.\ C \qquad [Agra,\ 1973]$$

If $B = CA^{-1}$, then

$$tr.\ (ACA^{-1}) = tr.\ (AB)$$
$$= tr.\ (BA) \text{ by Cor. of § 2.5}$$
$$= tr.\ (CA^{-1}A)$$
$$= tr.\ CI$$
$$= tr.\ C$$

Complex Matrix Inversion

Consider a complex matrix $A + iB$ of any order $n \times n$.

Assuming that at least one of the matrices A and B is non-singular, let us put

$$(A + iB)(X + iY) = I, \text{ I being a unit matrix of order } n. \qquad \ldots (7)$$

By definition of inverse, (7) $\Rightarrow X + iY$ is the inverse of $A + iB$.

First we find A^{-1}, if it exists. We can write (7) as

$$(AX - BY) + i(AY + BX) = I$$

On equating real and imaginary parts

$$AX - BY = I \qquad \ldots (8)$$

and

$$AY + BX = 0 \qquad \ldots (9)$$

Premultiplying (9) by A^{-1} and using $A^{-1}A = I$, we find

$$A^{-1}AY + A^{-1}BX = 0 \Rightarrow IY + A^{-1}BX = 0 \Rightarrow Y = -A^{-1}BX \qquad \ldots (10)$$

$$\therefore (8) \Rightarrow AX - B(-A^{-1}BX) = I \Rightarrow (A + BA^{-1}B)\,X = I$$

By definition of inverse, it implies that X is the inverse of

$$A + BA^{-1}B \quad \text{i.e. } X = (A + BA^{-1}B)^{-1} \qquad \ldots (11)$$

so that

$$(10) \Rightarrow Y = -A^{-1}BX = -A^{-1}B\,(A + BA^{-1}B)^{-1} \qquad \ldots (12)$$

Also, if B^{-1} exists, then premultiplying (9) by B^{-1} and using $B^{-1}B = I$, we get

$$B^{-1}AY + X = 0 \Rightarrow X = -B^{-1}AY \qquad \ldots (13)$$

so that (8) $\Rightarrow -AB^{-1}AY - BY = I \Rightarrow -(AB^{-1}A + B)Y = I$

$\Rightarrow$ Y is the inverse of $-(AB^{-1}A + B)$ by def. of inverse

$$\Rightarrow Y = -(AB^{-1}A + B)^{-1} \quad \text{... (14)}$$

$$\therefore (13) \Rightarrow X = B^{-1}A(AB^{-1}A + B)^{-1} \quad \text{... (15)}$$

Remark 1. Clearly A and B both being regular (i.e. non-singular), the two expressions for X and Y are identical as we have

$X = B^{-1}A(AB^{-1}A + B)^{-1} = (A^{-1}B)^{-1}(AB^{-1}A + B)^{-1}$ by reversal law of inverses

$= \left[(AB^{-1}A + B)(A^{-1})\right]^{-1} = (A + BA^{-1}B)^{-1}$

Since $AB^{-1}A\,(A^{-1}B) = (AA^{-1})(BB^{-1})(A) = I\,I\,A = A$

Similarly two values of Y can be shown to be identical.

Remark 2. When X and Y are determined, then

$(A + iB)(X + iY) = I \Rightarrow (X + iY)$ is the inverse of $(A + iB)$

$$\Rightarrow (A + iB)^{-1} = X + iY$$

$$\Rightarrow (A + iB)^{-1} = (A + BA^{-1}B) - iA^{-1}B(A + BA^{-1}B)^{-1}$$

$$\Rightarrow (A + iB)^{-1} = B^{-1}A(AB^{-1}A + B)^{-1} - i(AB^{-1}A + B)^{-1}$$

Remark 3. In the case when A and B both are singular but A + iB is regular (i.e. non-singular), then consider the matrices $C = A + rB$ and $D = B - rA$, r being real ... (16)

Then there exists a number r.t. C or D or both become regular as is evident from the following:

$$|C| = |A + iB| = f(r) \text{ say} \quad \text{... (17)}$$

which implies, that $f(r)$ is a polynomial of degree n.

However if $f(r) = 0\ \forall r$, then we would also get $f(i) = 0$ against our assumption that (A + iB) is regular.

Now, $(C + iD) = (1 - ir)(A + iB) \Rightarrow (A + iB)^{-1} = (1 - ir)(C + iD)^{-1}$ ... (18)

where $(C + iD)^{-1}$ can be calculated as shown earlier, since it is regular.

Illustration

Compute the inverse of $M = \begin{bmatrix} 5+i & 4+2i \\ 10+3i & 8+6i \end{bmatrix}$ *(R.U. 1989)*

Here $M = A + iB = \begin{bmatrix} 5+i & 4+2i \\ 10+3i & 8+6i \end{bmatrix} \Rightarrow A = \begin{bmatrix} 5 & 4 \\ 10 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

Evidently $|A| = 0$ and $|B| = 0$ but $|A + iB| \neq 0$ i.e. A and B both are singular but A + iB is non-singular (i.e. regular). As such take a real number r (see remark 3 above) s.t.

$|C| \neq 0$ where $C = A + rB = \begin{bmatrix} 5+r & 4+2r \\ 10+3r & 8+6r \end{bmatrix}$

$\therefore |C| = 6r \neq 0$ for $r = 1$ (say) $\Rightarrow C = \begin{bmatrix} 6 & 6 \\ 13 & 14 \end{bmatrix}$ and $D = B - rA \begin{bmatrix} -4 & -2 \\ -7 & -2 \end{bmatrix}$

If X + iY is the inverse of C + iD, then $(C + iD)^{-1} = X + iY$,

where $X = (C + DC^{-1}D)^{-1}$ and $Y = -C^{-1}D(C + DC^{-1}D)^{-1}$ by (11) and (12)

Here $C^{-1} = \frac{1}{6}\begin{bmatrix} 14 & -6 \\ -13 & 6 \end{bmatrix}$, $C^{-1}D = \frac{1}{6}\begin{bmatrix} -14 & -16 \\ 10 & 14 \end{bmatrix}$ by usual inversion

So that $DC^{-1}D = \begin{bmatrix} 6 & 6 \\ 13 & 14 \end{bmatrix}$ and $C + DC^{-1}D = \begin{bmatrix} 12 & 12 \\ 26 & 28 \end{bmatrix}$

which imply $(C + DC^{-1}D) = \frac{1}{12}\begin{bmatrix} 14 & -6 \\ -13 & 6 \end{bmatrix}$ and $-C^{-1}D(C + DC^{-1}D)^{-1}\ \frac{1}{12}\begin{bmatrix} -2 & 2 \\ 7 & -4 \end{bmatrix}$

$$\therefore \qquad (C + iD)^{-1} = X + iY = \frac{1}{12}\begin{bmatrix} 14-2i & -6+2i \\ -13+7i & 6-4i \end{bmatrix}$$

giving $\quad (A + iB)^{-1} = (1 - ir)(C + iD)^{-1}$ by (18)

$$= \frac{1-i}{12}\begin{bmatrix} 14-2i & 6+2i \\ -13+7i & 6-4i \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 6-8i & -2+4i \\ -3+10i & 1-5i \end{bmatrix}$$

which is the required inverse.

Problem 17. Compute the adjoint of $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Let
$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

then
$$|A| = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

The elements of the first row of $|A|$ are 0, 1, 2 and their cofactors are

$$\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \text{ i.e., } -1,\ 8,\ -5 \text{ respectively.}$$

The elements of the second row of $|A|$ are 1, 2, 3 and their cofactors are

$$-\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix}, -\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} \text{ i.e., } 1,\ -6,\ 3 \text{ respectively.}$$

The elements of the third row of $|A|$ are 3, 1, 1 and their cofactors are

$$\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}, -\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} \text{ i.e., } -1,\ 2,\ -1 \text{ respectively.}$$

∴ The matrix having its elements as cofactors of | A | is

$$\begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix}$$

and therefore

$$\text{Adj A} = \text{transpose of} \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

Problem 18. *Verify the following:*

(*i*) *The adjoint of an identity matrix is the identity matrix.*

(*ii*) *The adjoint of a scalar matrix is a scalar matirx.*

(*iii*) *The adjoint of a diagonal matrix is a diagonal matrix.*

(*iv*) *The adjoint of a symmetric matrix is a symmetric matrix.*

(*v*) *The adjoint of the transpose of a matrix is equal to the transpose of the adjoint matrix*

(*i*) Let $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$ be an identity matrix of order $n \times n$.

If $\mathbf{A}_{ij}$ denotes the cofactors of (i, j)th element of | **I** | for all integral values of i and j ranging from 1 to n, then it is clear that

$$\begin{aligned} \mathbf{A}_{ij} &= 1 \text{ for } i = j \\ &= 0 \text{ for } i \neq j \end{aligned}$$

So that $\text{Adj } \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} = \mathbf{I}.$

(*ii*) If $\mathbf{A} = \begin{bmatrix} \lambda & 0 & \dots & 0 \\ 0 & \lambda & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda \end{bmatrix}$ be a scalar matrix of order $n \times n$, then

as in (*i*)

$$\begin{aligned} \mathbf{A}_{ij} &= \lambda^{n-1} \text{ for } i = j \\ &= 0 \text{ for } i \neq j \end{aligned}$$

So that $\text{Adj } \mathbf{A} = \begin{bmatrix} \lambda^{n-1} & 0 & \dots & 0 \\ 0 & \lambda^{n-1} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda^{n-1} \end{bmatrix}$ which is a scalar matrix.

In other words $A = \lambda I$

$\therefore$ Adj A = adj $\lambda I = \lambda^{n-1} I$ which is a scalar matrix.

(*iii*) Let $D = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ be a diagonal matrix of order 3 × 3. We shall verify the result for it, but the result can be verified for a diagonal matrix of any order in the similar way.

Then $D_{11} = bc, D_{22} = ac, D_{33} = ab$

$D_{12} = D_{13} = D_{21} = D_{23} = D_{31} = D_{32} = 0.$

Thus, adj $D = \begin{bmatrix} bc & 0 & 0 \\ 0 & ca & 0 \\ 0 & 0 & ba \end{bmatrix}$

which is a diagonal matrix.

(*iv*) If A is a symmetric matrix, then $A' = A$ and so $|A'| = |A|$.

Thus if A_{ij} denotes the cofactor of (i, j)th element in $|A|$, then

$A_{ij} = A_{ji}$ in $|A|$

As such the adjoint matrix of the symmetric matrix is also symmetric.

(*v*) Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

Then, if A_{ij} denotes the cofactor of (i, j)th element,

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

$$\therefore (\text{adj } A)' = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$$

Again, $A' = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$

So that adj $A' = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$

It is clear that (adj A)' = adj A'.

Problem 19. *If* A *is a square matrix of order n, then show that*

$$\text{adj (adj } A) = |A|^{n-2} A.$$

We have $(\text{adj } A)\, A = |A|\, I$ ...(1)

We can thus express by replacing A by adj A, as

$$\{\text{adj}(\text{adj } A)\}\text{ adj } A = |\text{adj } A|\, I$$

So that, $\{\text{adj}(\text{adj } A)\}(\text{adj } A)\, A = |\text{adj } A|\, IA$

or $\text{adj}(\text{adj } A)\, |A|\, I = |\text{adj } A|\, A$ from (1) and since $IA = A$

or $|A|\,\{\text{adj}(\text{adj } A)\} = |A|^{n-1} A \quad \because |\text{adj } A| = |A|^{n-1}$

$\therefore \quad \text{adj}(\text{adj } A) = |A|^{n-2} A$

Problem 20. *Find the inverse of the matrix*

$$(a)\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad (b)\begin{bmatrix} 1 & -1 & 3 \\ -1 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

[*Meerut, 1963, 82*]

(*a*) Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$\therefore |A| = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = 0 - 1(-1) + 1 \cdot 1$ expanding about first column

$= 2.$

Now the matrix having its elements as the cofactors of the corresponding elements of $|A|$.

$$= \begin{bmatrix} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Its transpose gives the adj A *i.e.*,

$$\text{adj } A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Now we have $\quad A^{-1} = \dfrac{1}{|A|}\text{ adj } A$

$$= \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

(*b*) Proceeding just as in (*a*), we have

$$A^{-1} = \frac{1}{|A|}\text{ adj } A$$

$$= -\frac{1}{25}\begin{bmatrix} -5 & 5 & -5 \\ 5 & -10 & -5 \\ -5 & -5 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & 0 \end{bmatrix}$$

Problem 21. *If* $D = diag\ [d_1, d_2, ...d_n], d_1, d_2, ...d_n \neq 0$, *then prove that*

$$D^{-1} = diag\ [d_1^{-1}, d_2^{-1}, ...d_n^{-1}]$$ *[Benares, 1960]*

Given
$$D = \begin{bmatrix} d_1 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ 0 & 0 & d_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & d_n \end{bmatrix}$$

So that
$$|D| = \begin{vmatrix} d_1 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ 0 & 0 & d_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & d_n \end{vmatrix} = d_1, d_2, \dots d_n \neq 0 \text{ since } d_1, d_2, \dots d_n \neq 0$$

i.e., $|D|$ is non-singular and therefore D is invertible

Now
$$\text{adj } D = \begin{bmatrix} d_2\, d_3 \dots d_n & 0 & \dots & \dots\dots 0 \\ 0 & d_1\, d_3 \dots d_n & \dots & \dots\dots 0 \\ \dots\dots & \dots\dots & \dots & \dots\dots \\ 0 & 0 & \dots & d_1\, d_2 \dots d_{n-1} \end{bmatrix}$$

$$\therefore\ D^{-1} = \frac{1}{|D|}\text{adj } D = \begin{bmatrix} \frac{1}{d_1} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{d_2} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \frac{1}{d_n} \end{bmatrix}$$

$$= \text{diag}\left[\frac{1}{d_1}, \frac{1}{d_2}, \dots\dots \frac{1}{d_n}\right]$$

$$= \text{diag}\left[d_1^{-1}, d_2^{-1}, \dots\dots d_n^{-1}\right]$$

Problem 22. *Prove that the inverse of* $\begin{bmatrix} A & O \\ B & C \end{bmatrix}$ *is* $\begin{bmatrix} A^{-1} & O \\ -C^{-1}BA^{-1} & C^{-1} \end{bmatrix}$, *where* A, C *are non-singular and hence find the inverse of the matrix*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Suppose that the inverse of the given matrix *i.e.*, $\begin{bmatrix} A & O \\ B & C \end{bmatrix}$ is $\begin{bmatrix} P & Q \\ R & S \end{bmatrix}$

where P, Q, R, S are the submatrices of the inverse matrix which has been partitioned so as to be conformable for premultiplication with the matrix $\begin{bmatrix} A & O \\ B & C \end{bmatrix}$ *i.e.*,

$$\begin{bmatrix} A & O \\ B & C \end{bmatrix}\begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix}$$

or $$\begin{bmatrix} AP & AQ \\ BP+CR & BQ+CS \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix}$$

Comparison yields,

$AP = I$ *i.e.*, $P = A^{-1}$ as A is non-singular ...(1)

$AQ = O$ *i.e.*, $Q = O$ as A is non-singular ...(2)

$BP + CR = O$ and $BQ + CS = I$

or $BA^{-1} + CR = O$ and $CS = I$ from (1) and (2) respectively

or $C^{-1}BA^{-1} + C^{-1}CR = O$ and $S = C^{-1}$ as C is non-singular

or $R = -C^{-1}BA^{-1}$ as $C^{-1}CR = IR = R$ ...(3)

and $S = C^{-1}$ ...(4)

Putting the values of P, Q, R, S the inverse of the matrix $\begin{bmatrix} A & O \\ B & C \end{bmatrix}$ is

$$\begin{bmatrix} P & Q \\ R & S \end{bmatrix} \text{ i.e., } \begin{bmatrix} A^{-1} & O \\ -C^{-1}BA^{-1} & C^{-1} \end{bmatrix}$$

Now to find the inverse of $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$, let us partition it as

$$\left[\begin{array}{cc:cc} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hdashline 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right]$$

When it is compared with $\begin{bmatrix} A & O \\ B & C \end{bmatrix}$, we have

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Now it is easy to compute $A^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$

$$C^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

As above, $\begin{bmatrix} P & Q \\ R & S \end{bmatrix}$ is the inverse of $\begin{bmatrix} A & O \\ B & C \end{bmatrix}$ when

$$P = A^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

$$Q = O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$R = -C^{-1}BA^{-1} = -\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

and $$S = C^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

Hence the inverse of $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ is $\begin{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \end{bmatrix}$

i.e., $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

[14] Unitary Matrix and Orthogonal Matrix

A square matrix A *having its elements as complex numbers is said to be* **unitary** *if*

$$A' = (\overline{A}^{-1}) \text{ or } (\overline{A})' = A^{-1} \text{ or } A^{\Theta}A = I$$

In case A is real *i.e.,* the matrix A consists of real numbers as its elements, the matrix A will be unitary if $A'A = I$ for, in this case $A^{\Theta} = A'$.

As an illustrative example the matrix

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$ is unitary, since

$$A^{\Theta} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

So that $$A^{\Theta}A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}}, & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence the matrix A is unitary.

A real unitary matrix is known as **orthogonal** *matrix i.e., a real matrix* A *is orthogonal if* $A'A = I = AA'$ *for, in this case* $A' = A^{-1}$

The determinant of an orthogonal matrix is +1 or –1.

The orthogonal matrix is said to be *proper* or *improper* according as its determinant is +1 or –1.

Since $|AA'| = |I| = 1$

i.e., $|A||A'| = 1$

or $|A|^2 = 1$ as $|A| = |A'|$

we have $|A| = \pm 1$

As an illustrative example the matrix

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \text{ is orthogonal, since}$$

$$AA' = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

and $|A| = |A'| = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \cos^2\theta + \sin^2\theta = 1$

Note: A unitary matrix of order n involves n^2 real independent parameters, whereas an orthogonal matrix of order n involves $\frac{n(n-1)}{2}$ *e.g.* the most general unitary matrix of order 2 *i.e.*

$$U = \begin{bmatrix} \cos\theta\, e^{i\alpha} & \sin\theta\, e^{i\gamma} \\ -\sin\theta\, e^{i(\beta-\gamma)} & \cos\theta\, e^{i(\beta-\alpha)} \end{bmatrix} \text{ involves four parameters } \alpha, \beta, \gamma \text{ and } \theta.$$

[15] Normal Matrix and Normal Form of Matrix

A square matrix A *is said to be* **Normal** *if* $AA^{\Theta} = A^{\Theta}A$.

The normal matrices include diagonal, real symmetric, real skew symmetric, orthogonal, Hermitian, Skew-Hermitian and unitary matrices.

As an illustrative example the unitary matrix

$$A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix} \text{ is Normal, since } AA^{\Theta} = I = A^{\Theta}A$$

The normal form of a matrix is $\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$

[16] Elementary Matrix and Elementary Operations

A square matrix of order n is said to be **elementary matrix,** *if it is obtained from a unit matrix* I_n *by subjecting it to any of the following* **elementary operations** (*transformations*):

(*i*) Interchange of any two rows (columns) to be denoted by R_{ij} (C_{ij}) for the interchange of *i*th and *j*th rows (columns) and the elementary matrix obtained may be denoted by E_{ij}.

(*ii*) Multiplication of elements of any row (column) by any non-zero scalar, to be denoted by $R_{i(\lambda)}$ $\{C_{i(\lambda)}\}$ for the multiplication of *i*th row (column) by $\lambda \neq 0$ and the elementary matrix obtained may be denoted by $E_{i(\lambda)}$.

(*iii*) Addition to the elements of any row (column) the corresponding elements of another row (column) multiplied by non-zero scalar, to be denoted by $R_{ij(\lambda)}$ $\{C_{ij(\lambda)}\}$ for the addition to *i*th row (column) of the *j*th row (column) multiplied by $\lambda \neq 0$ and the elementary matrix obtained may be denoted by $E_{ij(\lambda)}$ for row operation and by $E'_{ij(\lambda)}$ for column operation since $E'_{ij(\lambda)}$ is the transpose of E_{ij}.

It may be verified that

$$|E_{ij}| = -1$$
$$|E_{i(\lambda)}| = \lambda \neq 0$$
$$|E_{ij(\lambda)}| = |E'_{ij(\lambda)}| = 1$$

which show that no elementary matrix is singular.

As illustrative examples, the elementary matrices obtained from

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ are}$$

$$E_{12} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{2(\lambda)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } E_{23(\lambda)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{bmatrix}$$

An elementary operation is said to be the **Row operation** *or* **Column operation** *according as it is applied to rows or columns.*

By the elementary transformations or rather say that by a series of elementary row and column transformations, a matrix can be reduced to the normal form. This part will be discussed in the section of 'Rank of a matrix' in more details.

THEOREM. Every elementary row (column) transformation of a matrix can be brought about by pre-multiplication (post-multiplication) with the corresponding elementary matrix.

[Allahabad, 1966]

Consider two matrices A and B of orders $m \times n$ and $n \times p$ respectively such that

$$A = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix} \text{ and } B = \begin{bmatrix} C_1, & C_2, & \ldots\ldots, & C_p \end{bmatrix}$$

where $R_1, R_2, \ldots\ldots R_m$ are the rows of A and $C_1, C_2, \ldots\ldots C_p$ are the columns of B. Then by the 'row by column' rule for multiplication we have

$$AB = \begin{bmatrix} R_1C_1 & R_1C_2 & \ldots\ldots & R_1C_p \\ R_2C_1 & R_2C_2 & \ldots\ldots & R_2C_p \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots & \ldots\ldots\ldots \\ R_mC_1 & R_mC_2 & \ldots\ldots & R_mC_p \end{bmatrix}$$

which follows that if the rows of A are subjected to any elementary row (column) transformation, then the rows (columns) of **AB** are also subjected to the same transformation.

Conclusively every elementary row (column) transformation of the product **AB** *can be effected by subjecting the prefactor* **A** *(post factor* **B***) to the same row (column) transformation.*

We shall apply this result to the required proposition. Suppose that A is a matrix of order $m \times n$ and I an identity matrix of order m. Then it is obvious that

$$A = IA.$$

The above result can be applied to show that every elementary row transformation of the product A can be effected by subjecting the prefactor I to the same transformation *i.e.*, by pre-multiplying A by the corresponding elementary matrix.

Again taking A = AI, it can be similarly shown that every column transformation of the product A can be effected by subjecting the post factor I to the same transformation *i.e.*, by post-multiplying A by the corresponding elementary matrix.

As an illustrative example, take

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

and suppose that R_{12} operation gives

$$\mathbf{B} = \begin{bmatrix} a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11} & a_{12} & a_{13} & a_{14} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

Take the elementary matrix $\mathbf{E}_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ obtained from the identity matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ of order 3 × 3.

$$\text{Consider } \mathbf{E}_{12}\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

$$= \begin{bmatrix} a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11} & a_{12} & a_{13} & a_{14} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

$$= \mathbf{B}$$

which shows that the product B is effected by subjecting the prefactor $\mathbf{E}_{12}$ to the same row operation *i.e.*, by pre-multiplying the matrix A by the corresponding elementary matrix.

Inverses of the elementary matrices

$\mathbf{E}_{ij}$ being the elementary matrix obtained by interchanging *i*th and *j*th rows (columns) of an identity matrix I may yield back the original identity matrix I on interchanging *i*th and *j*th rows. But by the preceding theorem the interchange of *i*th and *j*th rows (columns) of $\mathbf{E}_{ij}$ may be effected by pre-multiplying (post-multiplying) by the corresponding elementary matrix $\mathbf{E}_{ij}$ *i.e.*

$$\mathbf{E}_{ij}\,\mathbf{E}_{ij} = \mathbf{I}$$

or

$$(\mathbf{E}_{ij})^{-1} = \mathbf{E}_{ij}$$

which shows that $\mathbf{E}_{ij}$ is its own inverse. Since $\mathbf{E}_{ij}^{-1}$ exists, therefore ***every elementary matrix is non-singualr.***

Again if $\mathbf{E}_{ij(\lambda)}$ is a matrix obtained by multiplying the *i*th row (column) of an identity matrix I by λ then I may be obtained back by multiplying the *i*th row (column) of $\mathbf{E}_{i(\lambda)}$ by λ^{-1}. But by the preceding theorem the multiplication of *i*th row (column) of $\mathbf{E}_{i(\lambda)}$ by $1/\lambda$ (*i.e.*, λ^{-1}) may be obtained by pre-multiplying (post-multiplying) $\mathbf{E}_{i(\lambda)}$ by the corresponding elementary matrix $\mathbf{E}_{i(\lambda^{-1})}$ *i.e.*

$E_{i(\lambda^{-1})}\ E_{i(\lambda)} = I \qquad [E_{i(\lambda)}\ E_{i(\lambda^{-1})} = I]$

so that $\qquad (E_{i(\lambda)})^{-1} = E'_{i(\lambda^{-1})}$

which follows that the inverse of $E_{i(\lambda)}$ is $E'_{i(\lambda^{-1})}$.

Further $E_{ij(\lambda)}$ being the elementary matrix obtained by adding the elements of jth row multiplied by λ to the elements of ith row of an identity matrix **I**, the original identity matrix **I** may be obtained back by adding to the elements of ith row of $E_{ij(\lambda)}$ the corresponding elements of jth row multiplied by $-\lambda$. But the same row transformation on $E_{ij(\lambda)}$ may also be effected on pre-multiplying $E_{ij(\lambda)}$ by the corresponding elementary matrix $E_{ij(-\lambda)}$ *i.e.*,

$$E_{ij(-\lambda)}\ E_{ij(\lambda)} = I$$

or
$$(E_{ij(\lambda)})^{-1} = E_{ij(-\lambda)}$$

which shows that the inverse of $E_{ij(\lambda)}$ is $E_{ij(-\lambda)}$.

The same result may be shown to hold in case of column operation.

Conclusively the inverse of an elementary matrix is also an elementary matrix of the same type.

[17] Equivalent Matrix

A matrix **B** *is called the equivalent to a matrix* **A** *if* **B** *can be obtained from* **A** *by a sequence of elementary transformations and denoted by* **B~A**.

As an illustrative example if

$$A = \begin{bmatrix} -6 & -2 & -4 & 5 \\ 3 & 4 & 5 & -1 \\ 6 & 2 & 4 & -3 \end{bmatrix}$$

then
$$A \sim \begin{bmatrix} 0 & 0 & 0 & 2 \\ 3 & 4 & 5 & -1 \\ 6 & 2 & 4 & -3 \end{bmatrix} \text{ by } R_{13(1)}$$

THEOREM 1. Two matrices **A** *and* **B** *are equivalent if and only if there exist non-singular matrices* **R** *and* **C** *such that*

$$RAC = B$$

where $R = R_1R_2.........R_n$ *and* $C = C_1C_2.........C_m$; *R's being operations affecting rows and C's those affecting columns.*

If **A** and **B** are equivalent matrices, then **B** can be obtained from **A** by a series of elementary operations. But the elementary row (column) transformation can be effected by pre-multiplying (post-multiplying) **A** by the corresponding elementary matrix, therefore if we denote the elementary row transformations by $R_1, R_2, ...R_n$ and elementary column transformations by $C_1, C_2, ...C_m$ where R_1 represents the first elementary matrix corresponding to the first elementary row transformation, etc., and similarly C_1 represents the first elementary matrix corresponding to the first elementary column transformation etc., then

$$(R_n ....R_1R_2)\ A\ (C_1C_2.........C_m) = B$$

i.e.
$$RAC = B$$

where **R** and **C** are non-singular matrices.

COROLLARY. There are three fundamental properties of the relation RAC = B:

(*i*) *Reflexivity.* Every matrix A is equivalent to itself *i.e.*, if we take

$$\mathbf{R} = \mathbf{I},\ \mathbf{C} = \mathbf{I},\ \text{then}\quad \mathbf{IAI} = \mathbf{A}$$

(*ii*) *Symmetry.* If a matrix B is equivalent to another matrix A, then A is also equivalent to B, for if

$$\mathbf{B} = \mathbf{RAC}$$

then
$$\mathbf{A} = \mathbf{R}^{-1}\mathbf{B}\mathbf{C}^{-1}$$

where R^{-1} and C^{-1} are non-singular matrices.

(*iii*) *Transitivity.* If A is equivalent to B and B is equivalent to C then A is also equivalent to C, for if

$$\mathbf{A} = \mathbf{PBQ} \text{ and } \mathbf{B} = \mathbf{RCS}$$

then
$$\mathbf{A} = \mathbf{PRCSQ} = (\mathbf{PR})\,\mathbf{C}\,(\mathbf{SQ})$$

where PR and SQ being the products of non-singular matrices, are themselves non-singular.

THEOREM 2. If a square matrix **A** *is reduced to an identity matrix* **I** *by a series of elementary row operations, then the same series of row operations applied to* **I** *yields the inverse of* **A** *i.e.,* $\mathbf{A}^{-1}$.

Suppose that $\mathbf{E}_1, \mathbf{E}_2, \ldots\ldots\ldots \mathbf{E}_r$ are the elementary matrices and I the identity matrix, such that

$$(\mathbf{E}_r \ldots\ldots\ldots \mathbf{E}_2\mathbf{E}_1)\,\mathbf{A} = \mathbf{I}$$

Then, post-multiplying both sides by $\mathbf{A}^{-1}$, we get

$$(\mathbf{E}_r \ldots\ldots\ldots \mathbf{E}_2\mathbf{E}_1)\,\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}\mathbf{A}^{-1}$$

i.e.,
$$(\mathbf{E}_r \ldots\ldots\ldots \mathbf{E}_2\mathbf{E}_1)\,\mathbf{I} = \mathbf{A}^{-1}$$

which proves the proposition.

Note. In practice, we write, A = IA

and perform a series of elementary row operations on A and prefactor I till A is reduced to I and I is reduced to B such that

$$\mathbf{I} = \mathbf{BA}$$

which follows that B is inverse of A.

As an illustrative example if $\mathbf{A} = \begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix}$,

then take

$$\overset{\mathbf{A}}{\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix}} \sim \overset{\mathbf{I}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

or
$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 9 & -11 \\ 0 & -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \text{ by } R_{21(3)} \text{ and } R_{31(-2)}$$

or $$\begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 21 \\ 0 & -1 & 4 \end{bmatrix} \sim \begin{bmatrix} -5 & 0 & 3 \\ -13 & 1 & 8 \\ -2 & 0 & 1 \end{bmatrix} \text{ by } R_{13(3)} \text{ and } R_{23(8)}$$

or $$\begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 21 \\ 0 & 0 & 25 \end{bmatrix} \sim \begin{bmatrix} -5 & 0 & 3 \\ -13 & 1 & 8 \\ -15 & 1 & 9 \end{bmatrix} \text{ by } R_{32(1)}$$

or $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 25 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{2}{5} & -\frac{3}{5} \\ -\frac{2}{5} & \frac{4}{25} & \frac{11}{25} \\ -\frac{1}{5} & 1 & 9 \end{bmatrix} \text{ by } R_{13(-2/5)} \text{ and } R_{23\left(-\frac{21}{25}\right)}$$

or $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{2}{5} & -\frac{3}{5} \\ -\frac{2}{5} & \frac{4}{25} & \frac{11}{25} \\ -\frac{3}{5} & \frac{1}{25} & \frac{9}{25} \end{bmatrix} \text{ by } R_{3\left(-\frac{1}{25}\right)}$$

This gives, $$A^{-1} = \begin{bmatrix} 1 & -\frac{2}{5} & -\frac{3}{5} \\ -\frac{2}{5} & \frac{4}{25} & \frac{11}{25} \\ -\frac{3}{5} & \frac{1}{25} & \frac{9}{25} \end{bmatrix}$$

In practice we may write the steps taken in this example as,

write. $$\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \quad \text{(the form } A = IA\text{)}$$

or $$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 9 & -11 \\ 0 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 1 & 0 \end{bmatrix} A \text{ by } R_{21(3)} \text{ and } R_{31(-2)}$$

or $$\begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 21 \\ 0 & -1 & 4 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 3 \\ -13 & 1 & 8 \\ -2 & 0 & 1 \end{bmatrix} A \text{ by } R_{13(3)} \text{ and } R_{23(8)}$$

or $$\begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 21 \\ 0 & 0 & 25 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 3 \\ -13 & 1 & 8 \\ -15 & 1 & 9 \end{bmatrix} A \text{ by } R_{32(1)}$$

or $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 25 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{2}{5} & -\frac{3}{5} \\ -\frac{2}{5} & \frac{4}{25} & \frac{11}{25} \\ -15 & 1 & 9 \end{bmatrix} A \text{ by } R_{13\left(-\frac{2}{5}\right)} \text{ and } R_{23\left(-\frac{21}{5}\right)}$$

or $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{2}{5} & -\frac{3}{5} \\ -\frac{2}{5} & \frac{4}{25} & \frac{11}{25} \\ -\frac{3}{5} & \frac{1}{25} & \frac{9}{25} \end{bmatrix}$ A by $R_{3\left(\frac{1}{25}\right)}$

which gives the required inverse.

[18] Canonical Matrix

This is a non-zero matrix in which

(i) the first few rows have non-zero elements while the elements of succeeding rows may be all zero,

(ii) the first non-zero row is unity, and

(iii) all the other elements of a column which contains the first non-zero element as unity, are zero.

As an illustrative example, the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is Canonical.

Problem 23. *Show that the matrix* $\begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix}$ *is unitary if* $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$.

Denoting the given matrix by A, it will be unitary if

$$A^{\theta}A = I$$

Consider, $A^{\theta}A = \begin{bmatrix} \alpha - i\gamma & \beta - i\delta \\ -\beta - i\delta & \alpha + i\gamma \end{bmatrix} \begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix}$

$= \begin{bmatrix} \alpha^2 + \beta^2 + \gamma^2 + \delta^2 & 0 \\ 0 & \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \end{bmatrix}$

which becomes an identity matrix if

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$$

Hence A is unitary if $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$.

Problem 24. *If* A *is a unitary matrix and* B = AP *where* P≠0, *then show that* PB^{-1} *is unitary.*

Let $C = PB^{-1}$

Then $C = P(AP)^{-1}$

$= PP^{-1}A$ by reversal law of inverses.

or $C = IA = A$

which follows that C is unitary, since A is unitary.

Problem 25. *If* A *is real skew-symmetric matrix such that* $A^2 + I = 0$, *then show that* A *is orthogonal.* *(Rohilkhand, 1982; Meerut, 69)*

Given $A^2 + I = 0$ ...(1)

∵ A is given to be real skew-symmetric,

∴ $A = -A'$

Premultiplying both sides by A, we get

$A^2 = -AA'$

or $-A^2 = AA'$ ...(2)

Adding (1) and (2) we find,

$$AA' = I$$

which follows that the matrix A is orthogonal.

Problem 26. *Reduce the matrix A to its normal form, where*

$$A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$$

We have, $A \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 1 & 2 & 3 & 1 \end{bmatrix}$ by C_{12}

$$\sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 0 & 2 & 1 & 3 \end{bmatrix} \text{ by } R_{31(-1)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 2 & 6 \\ 0 & 2 & 1 & 3 \end{bmatrix} \text{ by } C_{21(-2)} \text{ and } C_{31(2)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 2 & 1 & 3 \end{bmatrix} \text{ by } R_{2\left(\frac{1}{4}\right)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ by } R_{32(-2)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ by } C_{32\left(-\frac{1}{2}\right)} \text{ and } C_{42\left(-\frac{3}{2}\right)}$$

$$\sim \left[\begin{array}{cc:cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \end{array}\right] \text{ by partitioning}$$

$$\sim \left[\begin{array}{c:c} I_2 & O \\ \hdashline O & O \end{array}\right]$$

which gives the normal form of A.

Problem 27. *Applying elementary transformations, find the inverse of the matrix.*

$$A = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -3 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}$$

We may write, A = IA

i.e.,

$$\begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

or $\begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 2 & 1 & 4 \\ 0 & 3 & 2 & 7 \\ -1 & 1 & 2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ A by $R_{12(1)}$, $R_{24(1)}$ and $R_{34(1)}$

or $\begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 2 & 1 & 4 \\ 0 & 3 & 2 & 7 \\ 0 & 4 & 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$ A by $R_{41(1)}$

or $\begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 2 & 1 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & -1 \end{bmatrix}$ A by $R_{32(-1)}$ and $R_{42(-2)}$

or $\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & -1 \\ 4 & -3 & 0 & -3 \\ 3 & -4 & 1 & -3 \\ 1 & -1 & 0 & -1 \end{bmatrix}$ A by $R_{14(1)}$, $R_{24(4)}$ and $R_{34(3)}$

or $\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & -1 \\ 1 & 1 & -1 & 0 \\ 3 & -4 & 1 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$ A by $R_{23(-1)}$ and $R_{4(-1)}$

or $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$ A by $R_{12(-3)}$ and $R_{32(-1)}$

which follows that $A^{-1} = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

[19] Derogatory and Non-Derogatory Matrices

An n-rowed matrix is said to be derogatory or non-derogatory according as the degree of its minimal equation is less than or equal to n.

Note. If $m(x)$ is a scalar polynomial of the lowest degree with leading coefficient unity, such that $m(A) = 0$ then the polynomial $m(x)$ and the equation $m(x) = 0$ are respectively known as the minimal polynomial and the minimal equation of the matrix A. It should be committed to memory (*i*) the minimal polynomial of a matrix is unique (*ii*) the minimal polynomial of a matrix is a divisor of every polynomial that annihilates the matrix.

For example if A be a n-square matrix with each element unity, then $A^2 = nA$ and the polynmomial $x^2 - nx$ annihilates A. Hence $x^2 - nx$ is the minimal polynomial of A.

THEOREM. Every unit matrix of order ≥ 2 is derogatory.

If I be a unit matrix of order n (≥ 2), then the polynomial $m(x) = x - 1$ annihilates I so that $x-1$ is the minimal polynomial of I, since degree of $x-1$ is 1 ($<n$) therefore I is derogatory.

As an illustration $A = \begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}$ is derogatory.

Note. The minimal polynomial of a matrix is a divisor of every polynomial that annihilates the matrix.

Let $m(x)$ be a minimal polynomial of A and $h(x)$ be another polynomial that annihilates A. Then Division algorithm leads

$$h(x) = m(x)\, q(x) + r(x) \qquad \ldots (1)$$

where $r(x)$ is a zero polynomial or its degree is less than that of $m(x)$.

Put $x = A$ in (1),

$$h(A) = m(A)\, q(A) + r(A) \qquad \ldots (2)$$

$\because$ $m(x)$ and $h(x)$ both annihilate A, (2) gives

$$O = Oq(A) + r(A) \text{ i.e., } r(A) = O$$

showing that $r(x)$ also annihilates A. If $r(x) \neq 0$, then it is a non-zero polynomial of degree less than that of $m(x)$ and thereby contradicting that $m(x)$ is minimal polynomial and hence the only possibility is that $r(x)$ is zero polynomial.

Then (1) gives $h(x) = m(x)\, q(x)$

i.e., $m(x)$ is a divisor of $h(x)$.

2.10. RANK OF A MATRIX

While defining a matrix, we have already mentioned that the matrix constituted by the array of elements which are left after deleting any number of rows or columns or both of a matrix is said to be its **sub-matrix.** A square sub-matrix of a square matrix is known as a **principal sub-matrix** which may be obtained by deleting the corresponding rows and columns of the square matrix. If it is obtained by deleting only some of the last rows and the corresponding columns of the square matrix, then it is called as **leading sub-matrix.**

It is worth-noting that a square sub-matrix is not always obtained from a square matrix, but it may be obtained from any matrix after deleting certain numbers of row and columns such that the remaining matrix may have equal number of rows and columns.

The square sub-matrices of a matrix play an important role in deciding the rank of the matrix, for, the maximum order of the non-singular square sub-matrix of a matrix determines its **rank.**

We have also mentioned that the determinant of a square sub-matrix of a matrix A is said to be the **minor** of the matrix. The minor is principal or leading according as the corresponding sub-matrix is principal or leading.

As an illustrative example if $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, then 1, 2, 3, 4, 5, 6, 7, 8, 9 are the minors of order 1;

$$\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}, \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}, \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix}, \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix}, \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix}, \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}, \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}, \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix}$$

are the minors of order 2.

The sub-matrix $\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}$ is a principal sub-matrix of A.

The sub-matrix $\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$ is the leading sub-matrix of A.

It is also clear that the square sub-matrices of orders 1, 2, 3 are all non-singular, which follows that the maximum order of the non-singular square sub-matrix of A is 3 and hence the rank of A is 3.

Definition of Rank of a Matrix

Let A be a given matrix of order $m \times n$ and ρ be a natural number such that

$$\rho \leq \min (m, n)$$

where $\min (m, n) =$ the smaller of m and n for $m \neq n$

$= m = n$ for $m = n$

Let us now delete any $(m - \rho)$ rows and $(n - \rho)$ columns of the matrix A, so that the retained elements constitute a square sub-matrix of order ρ, whose determinant is a minor of the matrix A, of order ρ. Clearly there corresponds a system of minors of A to each admissible value of ρ. A definite positive integral value r of ρ, with the following two properties is known as the rank of the matrix A:

(*i*) There is at least one minor of order r, which does not vanish.

(*ii*) All the minors of order $(r + 1)$ vanish.

Conclusively the rank r of a matrix is the largest integer for which the statement "*not all minors of order r are zero*" is valid.

Thus the rank of a null-matrix is zero, the rank of a non-singular square matrix of order n is n and the rank of a singular square matrix of order n is less than n.

The rank of a matrix A is denoted by ρ (A) or Rank (A).

In determining the rank of a matrix, the following statement is often found useful :

"*If in a given matrix a certain r-rowed determinant is not zero and all the $(r + 1)$ - rowed determinants of which this r-rowed determinant is a first minor are zero, then all $(r + 1)$-rowed determinants of the matrix are zero*".

ILLUSTRATIVE EXAMPLES

1. The rank of the matrix $\begin{bmatrix} 1 & 3 & 4 & 7 \\ 2 & 4 & 5 & 8 \\ 3 & 1 & 2 & 3 \end{bmatrix}$ is 3, for the minor of order 4 is obviously zero and none of the minors of order 3 are zero.

2. The rank of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$ is 2.

3. The rank of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$ is 1.

2.11. SOME THEOREMS ON RANK

THEOREM 1. Elementary operations do not change the rank of a matrix.

Let $A = [a_{ij}]$ be the matrix of order $m \times n$ and rank r and let B be the matrix obtained from A by elementary operations. Suppose that $r' (\neq r)$ is the rank of B.

Case I. Let us first assume that B is the elementary matrix obtained from A by interchange of a pair of rows.

Suppose that B_0 is any $(r + 1)$-rowed square sub-matrix of **B**, so that $(r + 1)$ rows of the sub-matrix B_0 of **B** are also the rows of some uniquely determined sub-matrix A_0 of **A**, with the only difference that the identical rows of A_0 and B_0 may or may not occur in the same relative positions. But we know that an interchange of a pair of rows of a determinant results in the change of its sign, therefore we may have

$$|B_0| = |A_0| \text{ or } |B_0| = -|A_0|$$

Now the rank of the matrix **A** being r every $(r + 1)$-rowed minor of **A** vanishes *i.e.*, $|A_0| = 0$ and hecne $|B_0| = 0$, showing that every $(r + 1)$-rowed minor of **B** also vanishes. As such r', the rank of **B** can not exceed r, the rank of **A** *i.e.*,

$$r' \leq r$$

Moreover **A** can also be obtained from **B** by interchange of rows and therefore a similar procedure will yield that

$$r \leq r'$$

The results $r' \leq r$ and $r \leq r'$ imply that $r = r'$ *i.e.*, the interchange of a pair of rows does not alter the rank of a matrix.

It can be similarly shown that the same result holds for column operation.

Case II. Let us again assume that **B** is the elementary matrix obtained from **A** by multiplying the elements of a row by a non-zero scalar say $\lambda \neq 0$.

If $|B_0|$ is any $(r + 1)$-rowed minor of **B**, then $(r + 1)$ rows of the sub-matrix B_0 of **B** are the $(r + 1)$ rows of A_0 of **A** or one of the rows of $(r + 1)$ is multiplied by λ the corresponding row of A_0, which happens when B_0 contains the affected row. Applying the property of a determinant that the multiplication of any of its row by λ results in the multiplication of the determinants by λ, we have

$$|B_0| = |A_0| \text{ or } |B_0| = \lambda\, |A_0|$$

But the rank of **A** being r, every minor $|A_0|$ of order $r + 1$ vanishes *i.e.*, $|A| = 0$ and hence $|B_0| = 0$ showing that every $(r + 1)$-rowed minor of **B** also vanishes. As such r', the rank of **B** cannot exceed r, the rank of **A** *i.e.*,

$$r' \leq r$$

Moreover **A** can be obtained from **B** by multiplying the row of **B** (which was assumed to be multiplied by λ) by $\dfrac{1}{\lambda}$ and therefore a similar procedure will yield

$$r \leq r'$$

Thus $r' \leq r$ and $r \leq r'$ imply that $r = r'$ *i.e.*, the multiplication of the elements of a row of a matrix by a non-zero scalar λ, does not alter its rank.

It can be similarly shown that the same result holds for column operation.

Case III. Let us further assume that **B** is the elementary martrix obtained by adding to the elements of ith row of the matrix **A**, the products with any non-zero scalar λ of the corresponding elements of another say jth row of the matrix.

If $|B_0|$ is any $(r + 1)$-rowed minor of **B** corresponding to the minor $|A_0|$ of **A**, then it is apparent that if the sub-matrix A_0 does not contain the ith row of **A**, $|B_0| = |A_0|$, because by the property of a determinant, if the elements of any row are increased by the same multiple of the corresponding elements of another row, the determinant remains unchanged.

But if ith row of **A** is contained in A_0 and jth row not, then

$$|B_0| = |A_0| + \lambda\, |C_0|$$

where C_0 is a $(r + 1)$-rowed square sub-matrix of A such that all the rows of C_0 except the ith coincide with thsoe of A_0 while ith row is obtained by replacing the elements of A_0 in the ith row by λ times the corresponding elements in the jth row. Clearly $\pm |C_0|$ is a $(r + 1)$ rowed minor of **A**.

But the rank of A being r, every minor of **A** of order $r + 1$ vanishes *i.e.*, $|A_0| = 0$, $|C_0| = 0$ and hence $|B_0| = 0$ showing that every $(r + 1)$ rowed minor of **B** also vanishes. As such r' the rank of **B** cannot exceed r, the rank of A. *i.e.*,

$$r' \leq r$$

Moreover A can be obtained from **B** by an elementary transformation of the same type and therefore a similar procedure will yield

$$r \leq r'$$

Thus $r' \leq r$ and $r \leq r'$ imply that $r = r'$ *i.e.*, the addition to the elements of a row of a matrix, the products with any non-zero scalar λ of the corresponding elements of another row of the matrix does not alter its rank.

Conclusively the elementary operations do not alter the rank of a matrix.

COROLLARY 1. *Equivalent matrices have the same rank.*

If A and B are two equivalent matrices, then by the definition of equivalent matrices **B** can be obtained from A by a series of elementary operations and hence by the preceding theorem, the rank of **A** = the rank of B.

COROLLARY 2. *The rank of the transpose of a matrix is the same as that of the original matrix.* *(Meerut, 1967)*

If A is a matrix then its transpose A′ is obtained from A by interchanging its rows and columns *i.e.*, if

$$A = [a_{ij}] \text{ of order } m \times n,$$

then
$$A' = [a_{ji}] \text{ of order } n \times m$$

Let r be the rank of **A** and r' that of A′.

If **P** be a square sub-matrix of A of order r such that $|P| \neq 0$, then P' will be a sub-matrix of A′ of order r.

But $|P| = |P'|$ by the properties of determinants

$|P'| = |P| \neq 0$ as $|P| \neq 0$.

Thus by the same argument as in the above theorem

$$r' \geq r$$

Again if Q be a square sub-matrix of A of order $(r + 1)$, then Q′ will be a corresponding sub-matrix of A′ of order $r + 1$.

But the rank of **A** being r, $|Q| = 0$

and since $|Q'| = |Q|$

$\therefore$ $|Q'| = |Q| = 0$ showing that every minor of A′ of order $(r + 1)$ is zero, therefore

$$r' \leq r$$

Hence $r' \geq r$ and $r' \leq r$ imply that $r' = r$ *i.e.*, the rank of the transpose of a matrix is the same as that of the original matrix.

THEOREM 2. Every non-singular matrix of order $m \times n$ and rank r (>0) can be reduced to one of the following normal forms:

$$\begin{bmatrix} I_r & \vdots & O \\ \cdots & & \cdots \\ O & \vdots & O \end{bmatrix}, \begin{bmatrix} I_r \\ \cdots \\ O \end{bmatrix}, [I_r \vdots O], [I_r]$$

by a series of elementary operations.

Suppose that $A = [a_{ij}]$ is the given non-zero matrix of order $m \times n$ and rank r. So A being non-zero will have at least one non-zero element say $a_{kl} = \lambda \neq 0$. If we interchange the kth row with the first row and the lth column with the first column, we find a matrix B with its leading element as λ *i.e.*,

$$B = \begin{bmatrix} \lambda & b_{12} & b_{13} \ldots\ldots b_{1l} \ldots\ldots b_{1n} \\ b_{21} & b_{22} & b_{23} \ldots\ldots b_{2l} \ldots\ldots b_{2n} \\ \cdots & \cdots & \cdots \\ b_{m1} & b_{m2} & b_{m3} \ldots\ldots b_{ml} \ldots\ldots b_{mn} \end{bmatrix}$$

If we divide the elements of the first row of B by λ, we get another matrix C with its leading element unity, let it be given by

$$C = \begin{bmatrix} 1 & c_{12} & c_{13} \ldots\ldots c_{1l} \ldots\ldots c_{1n} \\ c_{21} & c_{22} & c_{23} \ldots\ldots c_{2l} \ldots\ldots c_{2n} \\ \cdots & \cdots & \cdots \\ c_{m1} & c_{m2} & c_{m3} \ldots\ldots c_{ml} \ldots\ldots c_{mn} \end{bmatrix}$$

If we now subtract the suitable multiples of the first column of C from the remaining columns and the suitable multiples of the first row of C from the remaining rows, then we get a matrix D with leading element unity and all the other elements of the first row and first column as zero *i.e.*, of the form

$$D = \begin{bmatrix} 1 & 0 & 0 \ldots\ldots\ldots\ldots 0 \\ 0 & d_{22} & d_{23} \ldots\ldots\ldots\ldots d_{2n} \\ 0 & d_{m2} & d_{m3} \ldots\ldots\ldots\ldots d_{mn} \end{bmatrix}$$

which may be expressed as

$$D = \begin{bmatrix} 1 & 0 \quad 0 \ldots\ldots 0 \\ 0 & \\ \vdots & \\ 0 & A_1 \end{bmatrix} \text{ or } \begin{bmatrix} I & \vdots & O \\ O & \vdots & A_1 \end{bmatrix} \text{ by partitioning}$$

where A_1 is a matrix of the type $(m-1) \times (n-1)$.

If $A_1 \neq 0$, then proceeding as above, we can find an elementary matrix from A_1 with leading element unity and all other elements of the first row and the first column as zero. Continuing this process r times, we shall get an elementary matrix of either of the forms.

(*i*) $\begin{bmatrix} I_r & \vdots & O \\ O & \vdots & O \end{bmatrix}$ when $r < m, n$

(*ii*) $\begin{bmatrix} I_r \\ \cdots \\ O \end{bmatrix}$ when $m = r, r < n$

(*iii*) $[I_r \vdots O]$ when $r < m, n = r$

(*iv*) $[I_r]$ when $m = n = r$.

THEOREM 3. If A *is a matrix of rank r, then there exist non-singular matrices* P *and* Q *such that*

$$PAQ = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$$

Suppose that A is a matrix of order $m \times n$ and rank r. Then as given in the preceding theorem, if $r < m, n$, A can be reduced to the normal form $\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$ by elementary operations. But by theorem 1 of §2.9, the elementary row (column) operation is effected by premultiplying (post-multiplying) the corresponding elementary matrix of suitable size, therefore corresponding to A the matrix of order $m \times n$ and rank r, there exist elementary matrices $P_1, P_2, \ldots\ldots P_s$ each of order m and $Q_1, Q_2, \ldots\ldots Q_t$ each of order n such that

$$P_s P_{s-1} \ldots\ldots P_1 A Q_1 Q_2 \ldots\ldots Q_t = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$$

Here each of the elementary matrices $P_1, P_2, \ldots$ etc, and $Q_1, Q_2, \ldots$ etc, being non-singular, their products are also non-singular and so if we take

$$P = P_1 P_2 \ldots\ldots P_s, \; Q = Q_1 Q_2 \ldots\ldots Q_t$$

then $PAQ = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$

where P is of order $m \times m$ and Q is of order $n \times n$.

THEOREM 4. If A *is a matrix of order m × n and rank r, then there exists a non-singular matrix*

(*i*) P *such that* $PA = \begin{bmatrix} G \\ O \end{bmatrix}$

where G *is a matrix of order* $r \times n$ *and rank* r *and* O *is the null matrix of order* $(m - r) \times n$.

(*ii*) Q *such that* $AQ = [HO]$

where H *is a matrix of order* $m \times r$ *and rank* r *and* O *is the null-matrix of order* $m \times (n - r)$.

By the preceding theorem there exist non-singular matrices P, Q such that $PAQ = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$

(*i*) If $Q = Q_1 Q_2 \ldots\ldots Q_t$, where $Q_1, Q_2, \ldots\ldots Q_t$ are elementary matrices, then

$$PA\, Q_1\, Q_2 \ldots\ldots Q_t = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} \qquad \ldots(1)$$

Post-multiplying the left hand side of (1) in succession by the elementary matrices $Q_t^{-1}, Q_{t-1}^{-1}, \ldots\ldots Q_2^{-1}, Q_1^{-1}$, and effecting the corresponding column operation in the right hand side of (1) while no column operation would affect the last $(m - r)$ zero rows, we shall find a relation of the form

$$PA = \begin{bmatrix} G \\ O \end{bmatrix}$$

But we know that the elementary operations, do not alter the rank and hence the rank of PA and so of G will be r, which is the rank of A.

(*ii*) Take $P = P_1 P_2 \ldots\ldots P_s$; where $P_1, P_2, \ldots\ldots P_s$ are elementary matrices, then we have as in case (*i*),

$$P_1 P_2 P_3 \ldots\ldots P_r A Q = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$$

Pre-multiplying the left hand side of (2) in succession by the elementary matrices $P_1^{-1}, P_2^{-1} \ldots\ldots P_r^{-1}$ and effecting the corresponding row operation in the right hand side of (2) while no row operation would affect the last $(n-r)$ zero columns, we shall find a relation of the form

$$AQ = [H \; O]$$

where the rank of AQ and also of H will be r, since the elementary operations do not alter the rank and the rank of A is r.

THEOREM 5. The rank of a product of two matrices **A** *and* **B** *whose ranks are r_1, r_2 respectively cannot exceed the rank of either matrix i.e., if r be the rank of product* **AB**, *then*

$$r \le r_1 \text{ and } r \le r_2. \qquad \textit{(Rohilkhand, 1984; Agra, 1968)}$$

Let **A** be the matrix of order $m \times n$ and **B** the matrix of order $n \times p$, so that the product **AB** is conformable and has order $m \times p$.

By the preceding theorem, A being the matrix of order $m \times n$ and rank r_1, there exists a non-singular matrix P which is the product of elementary matrices, such that

$$PA = \begin{bmatrix} G \\ O \end{bmatrix} \qquad \ldots(1)$$

where G is a matrix of order $r_1 \times n$ and rank r_1 and O is the null matrix of order $(m - r_1) \times n$.

Post-multiplying either side of (1) by B, we get

$$PA\,B = \begin{bmatrix} G \\ O \end{bmatrix} B \qquad \ldots(2)$$

But P being the product of elementary matrices, we have rank of (PAB) = rank of (AB) = r ...(3)

For, the elementary operations do not alter the rank.

From (2) and (3) it follows that

$$\text{rank of } \begin{bmatrix} G \\ O \end{bmatrix} B = r \qquad \ldots(4)$$

But $\begin{bmatrix} G \\ O \end{bmatrix}$ B has at most r_1 non-zero rows which arise by the multiplication of r_1 non-zero rows of G with the columns of B, therefore

$$\text{rank of } \begin{bmatrix} G \\ O \end{bmatrix} B \le r_1 \qquad \ldots(5)$$

Thus from (4) and (5) it is apparent that

$r \le r_1$ *i.e.*, the rank of (AB) $\le$ the rank of the prefactor A ...(6)

Again r = rank of (AB)

= rank of (AB)$'$ by corollary 2 of Theorem 1.

= rank of (B$'$A$'$) $\because$ (AB)$'$ = (B$'$A$'$) by reversal law of transpose

$\le$ rank of the prefactor B$'$ by (6)

$\leq$ rank of (B)

$\leq r_2$

i.e., the rank of (AB) $\leq$ the rank of the post-factor B.

THEOREM 6. The rank of a matrix does not alter by pre-multiplication (or post-multiplication) with any non-singular matrix.

Let A be be any matrix and P a non-singular matrix such that the product PA is conformable and let

$$\mathbf{B} = \mathbf{PA} \qquad ...(1)$$

Then by the preceding theorem.

$$\text{rank of } (\mathbf{B}) = \text{rank of } (\mathbf{PA}) \leq \text{rank of } (\mathbf{A}) \qquad ...(2)$$

(1) may also be written as

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{B}$$

$$\therefore \text{rank of } (\mathbf{A}) = \text{rank of } (\mathbf{P}^{-1}\mathbf{B}) \leq \text{rank of } (\mathbf{B}) \qquad ...(3)$$

From (2) and (3) it follows that

$$\text{rank } (\mathbf{A}) = \text{rank of } (\mathbf{B}).$$

Problem 28. *Find the rank of the matrix* $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$

(Agra, 1970)

Denoting the given matrix by A we have

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$$

The matrix A is of the order 3×4 and so it can have minors of order 1, 2 and 3. The minors of order 3 are

$$\begin{vmatrix} 1 & 3 & 4 \\ 3 & 9 & 12 \\ 1 & 3 & 4 \end{vmatrix}, \begin{vmatrix} 1 & 3 & 3 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 4 & 3 \\ 3 & 12 & 3 \\ 1 & 4 & 1 \end{vmatrix}, \begin{vmatrix} 3 & 4 & 3 \\ 9 & 12 & 3 \\ 3 & 4 & 1 \end{vmatrix}$$

i.e.,
$$12\begin{vmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \\ 1 & 1 & 1 \end{vmatrix}, \; 3\begin{vmatrix} 1 & 1 & 3 \\ 3 & 3 & 3 \\ 1 & 1 & 1 \end{vmatrix}, \; 4\begin{vmatrix} 1 & 1 & 3 \\ 3 & 3 & 3 \\ 1 & 1 & 1 \end{vmatrix}, \; 12\begin{vmatrix} 1 & 1 & 3 \\ 3 & 3 & 3 \\ 1 & 1 & 1 \end{vmatrix},$$

all of which vanish as they consist of at least two columns identical.

The minors of order 2 are

$$\begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix}, \begin{vmatrix} 1 & 4 \\ 3 & 12 \end{vmatrix}, \begin{vmatrix} 1 & 3 \\ 3 & 3 \end{vmatrix}, \begin{vmatrix} 3 & 4 \\ 9 & 12 \end{vmatrix}, \begin{vmatrix} 3 & 3 \\ 9 & 3 \end{vmatrix}, \begin{vmatrix} 4 & 3 \\ 12 & 3 \end{vmatrix}, \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix}, \begin{vmatrix} 1 & 4 \\ 1 & 4 \end{vmatrix}, \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix},$$

$$\begin{vmatrix} 3 & 4 \\ 3 & 4 \end{vmatrix}, \begin{vmatrix} 3 & 3 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 4 & 3 \\ 4 & 1 \end{vmatrix}, \begin{vmatrix} 3 & 9 \\ 1 & 3 \end{vmatrix}, \begin{vmatrix} 3 & 12 \\ 1 & 4 \end{vmatrix}, \begin{vmatrix} 3 & 3 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 9 & 12 \\ 3 & 4 \end{vmatrix}, \begin{vmatrix} 9 & 3 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 12 & 3 \\ 4 & 1 \end{vmatrix},$$

all of which do not vanish *e.g.*

$$\begin{vmatrix} 1 & 3 \\ 3 & 3 \end{vmatrix} = 3 - 9 = -6 \neq 0.$$

Hence the rank of A is 2

Problem 29. *Find the rank of the matrix*

$$A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ -1 & -3 & -4 & -3 \end{bmatrix}$$ *(Gorakhpur, 1965)*

We know that the elementary operations do not alter the rank or in other words the equivalent matrices have the same rank, thus since

$$A \sim \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ by } R_{31(1)}$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 3 \\ 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ by } R_{2\left(\frac{1}{3}\right)}$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ by } R_{21(-1)}$$

therefore the rank of A is clearly 1, for, the minors of equivalent matrix of orders 3 and 2, vanish.

Problem 30. *Reduce the matrix* A *to the normal form and hecne find its rank, where*

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 3 & 0 & 3 \\ 1 & -2 & -3 & -3 \\ 1 & 1 & 2 & 3 \end{bmatrix}$$ *(Delhi, 1960)*

We have $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & -2 & 0 \\ 1 & -3 & -5 & -6 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ by $C_{21(-1)}$, $C_{31(-2)}$ and $C_{41(-3)}$

or $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -3 & -5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ by $R_{21(-1)}$, $R_{31(-1)}$ and $R_{41(-1)}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 5 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ by } R_{2\left(\frac{1}{2}\right)} \text{ and } R_{3(-1)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 8 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ by } C_{32}(1) \text{ and } C_{4\left(\frac{1}{4}\right)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ by } C_{24(-3)} \text{ and } C_{3\left(\frac{1}{8}\right)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ by } C_{43(-1)}$$

$$\sim \begin{bmatrix} \mathbf{I}_3 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$$ which is the required normal form and its rank is 3.

Problem 31. *Find the non-singular matrices* **P** *and* **Q** *such that* **PAQ** *is in the normal form, where*

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix}$$

The matrix **A** being of order 4 × 3, we may write,

$$\begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{A} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

or $$\begin{bmatrix} 1 & -4 & 11 & -19 \\ 5 & 1 & 4 & -2 \\ 3 & 2 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \mathbf{A} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ by } R_{13}$$

or $$\begin{bmatrix} 1 & -4 & 11 & -19 \\ 0 & 21 & -51 & -93 \\ 0 & 14 & -34 & 62 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -5 \\ 1 & 0 & -3 \end{bmatrix} \mathbf{A} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ by } R_{21(-5)} \text{ and } R_{31(-3)}$$

or $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 21 & -51 & 93 \\ 0 & 14 & -34 & 62 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -5 \\ 1 & 0 & -3 \end{bmatrix} \mathbf{A} \begin{bmatrix} 1 & 4 & -11 & 19 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

by $C_{21(4)}$, $C_{31(-11)}$ and $C_{41(19)}$

or $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 3 & 3 \\ 0 & 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -5 \\ 1 & 0 & -3 \end{bmatrix} A \begin{bmatrix} 1 & \frac{4}{7} & \frac{11}{17} & \frac{19}{31} \\ 0 & \frac{1}{7} & 0 & 0 \\ 0 & 0 & -\frac{1}{17} & 0 \\ 0 & 0 & 0 & \frac{1}{31} \end{bmatrix}$$

by $C_{2\left(\frac{1}{7}\right)}$, $C_{3\left(-\frac{1}{17}\right)}$ and $C_{4\left(\frac{1}{31}\right)}$

or $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{3} & -\frac{5}{3} \\ \frac{1}{2} & 0 & -\frac{3}{2} \end{bmatrix} A \begin{bmatrix} 1 & \frac{4}{7} & \frac{11}{17} & \frac{19}{31} \\ 0 & \frac{1}{7} & 0 & 0 \\ 0 & 0 & -\frac{1}{17} & 0 \\ 0 & 0 & 0 & \frac{1}{31} \end{bmatrix}$$ by $R_{2\left(\frac{1}{3}\right)}$ and $R_{3\left(\frac{1}{2}\right)}$

or $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{3} & -\frac{5}{3} \\ \frac{1}{2} & 0 & -\frac{3}{2} \end{bmatrix} A \begin{bmatrix} 1 & \frac{4}{7} & \frac{9}{119} & \frac{9}{217} \\ 0 & \frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} \\ 0 & 0 & -\frac{1}{17} & 0 \\ 0 & 0 & 0 & \frac{1}{31} \end{bmatrix}$$ by $C_{32(-1)}$ and $C_{42(-1)}$

or $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{3} & -\frac{5}{3} \\ \frac{1}{2} & -\frac{1}{3} & \frac{1}{6} \end{bmatrix} A \begin{bmatrix} 1 & \frac{4}{7} & \frac{9}{119} & \frac{9}{217} \\ 0 & \frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} \\ 0 & 0 & -\frac{1}{17} & 0 \\ 0 & 0 & 0 & \frac{1}{31} \end{bmatrix}$$ by $R_{32(-1)}$

which gives the required normal form *i.e.*,

$$\begin{bmatrix} I_2 & O \\ O & O \end{bmatrix} = PAQ$$

where $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{3} & -\frac{5}{3} \\ \frac{1}{2} & -\frac{1}{3} & \frac{1}{6} \end{bmatrix}$ and $= Q \begin{bmatrix} 1 & \frac{4}{7} & \frac{9}{119} & \frac{9}{217} \\ 0 & \frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} \\ 0 & 0 & -\frac{1}{17} & 0 \\ 0 & 0 & 0 & \frac{1}{31} \end{bmatrix}$

Problem 32. *Find the rank of the product matrix* AB *when*

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$$

We have

$$AB = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix} \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence rank of (AB) = 0, since the rank of a null-matrix is zero.

2.12. SOLUTIONS OF LINEAR EQUATIONS

In this section we shall make an attempt to apply the concepts and consequences of matrices discussed in the previous sections, to find the solution of any system of linear equations, homogeneous or non-homogeneous.

It should be clearly noted that a system of n linear equations in n unknown variables has not always a solution, *e.g.* the system of equations

(*i*) $\left.\begin{matrix} 2x+3y=7 \\ 4x-5y=3 \end{matrix}\right\}$ has a single solution namely $x = 2,\ y = 1$

(*ii*) $\left.\begin{matrix} 2x-3y=1 \\ -4x+6y=2 \end{matrix}\right\}$ has no solution

and (*iii*) $\left.\begin{matrix} 3x+2y=5 \\ 6x+4y=10 \end{matrix}\right\}$ has an infinite number of solutions.

Homogeneous linear equations

Let there be a set of m homogeneous linear equations in n variables x_1, x_2, x_3 .... x_n,

$$a_{11}x_1 + a_{12}x_2 + \dots\dots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + \dots\dots + a_{2n}x_n = 0$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots\dots + a_{mn}x_n = 0$$

In contracted form this set of equations may be expressed as

$$\begin{bmatrix} a_{11} & a_{12}\dots\dots\dots a_{1n} \\ a_{21} & a_{22}\dots\dots\dots a_{2n} \\ a_{m1} & a_{m2}\dots\dots\dots a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Writing,

$$A = \begin{bmatrix} a_{11} & a_{12} \dots\dots\dots a_{1n} \\ a_{21} & a_{22} \dots\dots\dots a_{2n} \\ \dots & \dots\dots\dots\dots\dots \\ a_{m1} & a_{m2} \dots\dots\dots a_{mn} \end{bmatrix},\ X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},\ O = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where A, X, O are the matrices of orders $m \times n$, $n \times 1$ and $m \times 1$ respectively, the given set of equations is equivalent to a single matrix equation

$$AX = O \qquad \dots (1)$$

Let us assume that r is the rank of the matrix A of order $m \times n$. As regards the solution of the set of homogeneous linear equations, the following results can be summarised:

I. A system of homogeneous linear equations always have one or more solutions.

Since r cannot be greater than n, therefore,

either $\qquad r = n$ or $r < n$.

In the first case the equation (1) will have no linearly independent solutions, for in that case trival (zero) solution is the only solution, while in the second case there will be $(n - r)$ independent solutions and therefore the equation (1) will have more than one solution.

II. The number of linearly independent solutions of AX = O *is* $(n-r)$ *i.e. if we assign arbitrary values to* $(n-r)$ *of the variables, then the values of the others can be uniquely determined.*

Since the rank of A is r, it has r linearly independent columns. There is no loss of generality if we suppose that the first r columns to the left are linearly independent for, it amounts only to renaming the variable components of X.

Taking $C_1, C_2, \dots\dots\dots\dots C_r, \dots\dots\dots\dots C_n$ as column vectors of A, we can write

$A = [C_1, C_2, \dots\dots\dots\dots C_r, \dots\dots\dots\dots C_n]$ of order $1 \times n$

so that the equation AX = O is equivalent to

$$x_1 C_1 + x_2 C_2 + \dots\dots\dots\dots + x_r C_r + \dots\dots\dots\dots + x_n C_n = O \qquad \dots (2)$$

Each of the column vectors $C_{r+1}, C_{r+2} \dots\dots, C_n$ being linear combination of $C_1, C_2, \dots\dots C_r$, we have the relations of the form

$$\left.\begin{aligned} C_{r+1} &= k_{11} C_1 + k_{12} C_2 + \dots\dots\dots + k_{1r} C_r \\ C_{r+2} &= k_{12} C_1 + k_{22} C_2 + \dots\dots\dots + k_{2r} C_r \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ C_n &= k_{l1} C_1 + k_{l2} C_2 + \dots\dots\dots + k_{lr} C_r \end{aligned}\right\} \qquad \dots (3)$$

Where $\qquad l = n - r$

Now the relations (3) can be rewritten as

$$\left.\begin{aligned} & k_{11} C_1 + k_{12} C_2 + \dots\dots\dots + k_{1r} C_r - 1.\ C_{r+1} + 0.C_{r+2} + \dots + 0.\ C_n = O \\ & k_{12} C_1 + k_{22} C_2 + \dots\dots\dots + k_{2r} C_r + 0.\ C_{r+1} - 1.C_{r+2} + \dots + 0.\ C_n = O \\ & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & k_{l1} C_1 + k_{l2} C_2 + \dots\dots\dots + k_{lr} C_r + 0.\ C_{r+1} + 0.C_{r+2} + \dots\dots - 1.\ C_n = O \end{aligned}\right\} \qquad \dots (4)$$

Comparison of (2) and (4) yields

$$X_1 = \begin{pmatrix} k_{11} \\ k_{12} \\ \vdots \\ k_{1r} \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} k_{21} \\ k_{22} \\ \vdots \\ k_{2r} \\ 0 \\ -1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots\ldots X_{n-r} = \begin{pmatrix} k_{l1} \\ k_{l2} \\ \vdots \\ k_{lr} \\ 0 \\ 0 \\ \vdots \\ -1 \end{pmatrix} \qquad \ldots (5)$$

which are $(n-r)$ solutions of $AX = O$.

These $(n-r)$ solutions constitute a linearly independent set of solutions, for if we have

$$k_1 X_1 + k_2 X_2 + \ldots\ldots + k_{n-r} X_{n-r} = O \qquad \ldots (6)$$

Then a comparison of $(r+1)$th, $(r+2)$th, ...nth components on either side of (6) will give

$$k_1 = 0, k_2 = 0, \ldots\ldots, k_{n-r} = O$$

In order to show that every solution of $AX = O$ is some suitable linear combination of these $(n-r)$ solutions X_1, X_2, ..., X_{n-r}, let us construct a vector

$$X + x_{r+1} X_1 + x_{r+2} X_2 + \ldots x_n X_{n-r} \qquad \ldots (7)$$

Which is also a solution of $AX = O$, as it is the linear combination of its solutions. Obviously the last $(n-r)$ components of (7) are all zero. Let its first r components be y_1, y_2,......... y_r. Then the vector with components $y_l, y_2, \ldots\ldots\ldots y_r, 0, 0, \ldots\ldots\ldots 0$, is a solution of $AX = O$ and hence from (2), we have

$$y_1 C_1 + y_2 C_2 + \ldots\ldots\ldots + y_r C_r = O$$

But the set of vectors C_1, C_2 .... C_r is linearly independent, therefore

$$y_1 = 0, y_2 = 0\ldots\ldots\ldots, y_r = 0$$

which shows that (7) is a zero vector and thus

$$X = -x_{r+1} X_1 - x_{r+2} X_2 \ldots\ldots\ldots\ldots x_n X_{n-r}$$

Hence every solution X is a linear combination of the $n-r$ linearly independent solutions $X_1, X_2, \ldots\ldots\ldots X_{n-r}$

Note 1. In case $n = r$, there is only one solution $x_1 = x_2 = \ldots x_n = 0$ and when $n > r$, there is a unique solution other than $x_1 = x_2 = \ldots = x_n = 0$, provided that the rank of the matrix of the coefficients of the $(n-r)$ arbitrary chosen values is r.

Note 2. If $X_1, X_2, X_3, \ldots\ldots\ldots X_{n-r}$ be a set of $(n-r)$ arbitrary independent solutions of the equation $AX = O$, then its general solution is

$$X = k_1 X_1 + k_2 X_2 + \ldots\ldots\ldots + k_{n-r} X_{n-r}$$

where $k_1, k_2, \ldots\ldots\ldots k_{n-r}$ is a set of $(n-r)$ arbitrary values.

III. If the number of equations is less than the number of variables, then the solution is always other than

$$x_1 = x_2 = \ldots = x_n = 0$$

IV. If the number of equations is equal to the number of variables, a necessary and sufficient condition for solutions other than $x_1 = x_2 = \ldots = x_n = 0$ is that the determinant of the coefficients must be zero.

Non-Homogeneous Linear Equations

Let there be m non-homogeneous linear equations in variables, $x_1, x_2, \ldots\ldots\ldots x_n$, such as

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots + a_{2n}x_n = b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots + a_{mn}x_n = b_m$$

The system of given equations is equivalent to a matrix equation

$$\begin{bmatrix} a_{11} & a_{12}\ldots\ldots\ldots a_{1n} \\ a_{21} & a_{22}\ldots\ldots\ldots a_{2n} \\ \ldots & \ldots\ldots\ldots\ldots \\ a_{m1} & a_{m2}\ldots\ldots\ldots a_{n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Denoting by

$$A = \begin{bmatrix} a_{11} & a_{12}\ldots\ldots\ldots a_{1n} \\ a_{21} & a_{22}\ldots\ldots\ldots a_{2n} \\ \ldots & \ldots\ldots\ldots\ldots \\ a_{m1} & a_{m2}\ldots\ldots\ldots a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

we have the matrix equation

$$AX = B$$

This equation is said to be **consistent** *i.e.* the equation possesses a solution if the matrices A and [A, B] are of the same rank, where [A, B] denotes the matrix

$$\begin{bmatrix} a_{11} & a_{12}\ldots\ldots\ldots a_{1n} & b_1 \\ a_{21} & a_{22}\ldots\ldots\ldots a_{2n} & b_2 \\ \ldots & \ldots\ldots\ldots\ldots & \ldots \\ a_{m1} & a_{m2}\ldots\ldots\ldots a_{mn} & b_m \end{bmatrix}$$

Here the matrix A is called **Coefficient Matrix** and the matrix [A, B] is called **Augmented Matrix.**

THEOREM. The system of non-homogeneous equations AX = B *possessses a solution if*

$$\textit{Rank of } (A) = \textit{Rank of } [A, B]$$

i.e., if r and ρ be the ranks of the matrices A *and* [A, B] *respectively, then the given equations are consistent when $r = \rho$ and inconsistent when $r < \rho$.*

Supposing that $C_1, C_2, \ldots\ldots\ldots C_n$ are the column vectors of the matrix A, the equation $AX = B$ ... (1)

is equivalent to

$$[C_1, C_2 \ldots\ldots\ldots C_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = B$$

i.e., $x_1 C_1 + x_2 C_2 + \ldots\ldots + x_n C_n = B$ ... (2)

The rank of the matrix A being r, it has r linearly independent columns and so there is no loss of generality if we assume that the first r columns form a linearly independent set such that each of the remaining $n-r$ columns is a linear combination of these r columns.

Case I. If the given set of equations is consistent, then there must exist n numbers say $k_1, k_2, \ldots\ldots\ldots k_n$ such that

$$k_1 C_1 + k_2 C_2 + \ldots\ldots\ldots + k_n C_n = B \quad \ldots (3)$$

But each of the $(n-r)$ columns $C_{r+1}, C_{r+2}, \ldots\ldots\ldots C_n$ being a linear combination of first r columns $C_1, C_2 \ldots\ldots\ldots C_r$, it follows from (3) that B is also a linear combination of $C_1, C_2, \ldots\ldots\ldots C_r$. As such the greatest number of linearly independent columns of [A, B] is also r showing that r is also the rank of [A, B]. Conclusively the system of equations AX = B is consistent if rank of (A) = rank of [A, B].

Conversely if the rank of A and [A, B] are the same say r, then the greatest number of linearly independent columns of [A, B] will be r. And since the first r columns C_1, $C_2, \ldots\ldots C_r$ of [A, B] form of a linearly independent set, therefore the column B is expressible as a linear combination of the columns $C_1, C_2, \ldots\ldots\ldots C_r$. As such there exist r scalars $k_1, k_2, \ldots\ldots\ldots k_r$ such that

$$k_1 C_1 + k_2 C_2 + \ldots\ldots\ldots k_r C_r = B$$

i.e., $$k_1 C_1 + k_2 C_2 + \ldots + k_r C_r + 0\cdot C_{r+1} + 0\cdot C_{r+2} + \ldots + 0\cdot C_n = B \quad \ldots (4)$$

Comparison of (2) and (4) leads to

$$x_1 = k_1,\ x_2 = k_2, \ldots\ldots\ldots, x_r = k_r,\ x_{r+1} = 0,\ x_{r+2} = 0 \ldots\ldots, x_n = 0$$

which constitute a solution of AX = B.

Hence the given system of equations is consistent if and only if the ranks of the coefficient matrix and the augmented matrix are the same.

Case II. If rank fo (A) < rank of ([A, B]), then the equations AX = B are inconsistent as is obvious from the Case I and hence the given system of equations has no solution.

Note 3. The most general solution of AX = B is

$$X_0 + k_1 X_1 + k_2 X_2 + \ldots\ldots\ldots + k_{n-r} X_{n-r} = X_0 + Y$$

where $X_1, X_2, \ldots\ldots\ldots X_{n-r}$ is a set of $(n-r)$ solutions of AX = O *and* X_0 is any fixed solution of AX = B, also $k_1, k_2, \ldots\ldots\ldots k_{n-r}$ is any set of $(n-r)$ constants.

Here $$A(X_0 + Y) = AX_0 + AY$$
$$= B + O = B$$

So that $X_0 + Y$ is a solution of the given equation *i.e.*, AX = B.

Conversely if Z be any solution of the given equations, then

$$A(Z - X_0) = AZ - AX_0$$
$$= B - B = O$$

showing that $Z - X_0$ is a solution of AX = O.

If we write $$Z = X_0 + (Z - X_0)$$

then it follows that every solution Z of AX = B may be expressed as the sum of a fixed solution X_0 thereof and some solution $Z - X_0$ as the auxiliary equation AX = O.

Note 4. If A be a n–rowed non-singular matrix, X be an $n \times 1$ matrix, and B be an $n \times 1$ matrix, then the system of equation AX = B has a unique solution.

A being n-rowed non-singular matrix and B being a matrix of order $n \times 1$, the ranks of both A and [A, B] will be n and so the system of equations AX = B is consistent *i.e.*, possesses a solution.

If we pre-multiply either side of AX = B by A^{-1}, we have

$$A^{-1}AX = A^{-1}B$$

or $$IX = A^{-1}B$$

i.e. $$X = A^{-1}B$$

is a solution of $$AX = B$$

In order to show that this solution is unique, let us assume if possible that X_1, X_2 are two solutions of AX = B, then we have

$$AX_1 = B \text{ and } AX_2 = B$$

which follow that $$AX_1 = AX_2$$

or $$A^{-1}AX_1 = A^{-1}AX_2$$

or $$IX_1 = IX_2$$

i.e., $$X_1 = X_2$$

which shows that the solution is unique.

2.13. CRAMER'S RULE

If a matrix equation is of the form

$$AX = B \qquad \dots (1)$$

where A is a non-singular square matrix *i.e.*, $|A| \neq 0$, then premultiplying either side of (1) by A^{-1}, we have

$$A^{-1}AX = A^{-1}B$$

i.e., $$IX = A^{-1}B$$

or $$X = A^{-1}B$$

A^{-1} being unique, these are the only solutions.

As a particular case if we consider a set of three non-homogeneous linear equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

the system is equivalent to the matrix equation

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Denoting by $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

we have $$AX = B$$

Premultiplying by A^{-1}, we get as above

$$X = A^{-1}B \qquad \dots (2)$$

But we know that

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

Here
$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

and
$$\text{adj } A = \begin{vmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{vmatrix}.$$

where A_{11} etc. are the cofactors of the elements a_{11} etc. in the determinant $|A|$.

Here (2) may be expressed as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$= \frac{1}{|A|} \begin{pmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{pmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$= \frac{1}{|A|} \begin{pmatrix} b_1 \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - b_2 \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + b_3 \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ -b_1 \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + b_2 \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - b_3 \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ b_1 \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - b_2 \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + b_3 \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{pmatrix}$$

or
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix} \end{pmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

which is equivalent to

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_1 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}, \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}, \quad x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}$$

This result may be extended to the general case and thus gives the rule given by Cramer.

'For a given set of n non-homogeneous linear equations in n variables $x_1, x_2, \ldots\ldots x_n$ each x_i ($i = 1, 2, \ldots\ldots n$) is a quotient, the denominator of which is the determinant $|A|$ of the coefficient matrix A and the numerator of which is a determinant obtained from $|A|$ by substituting the column vector $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ for the ith column of $|A|$.

Problem 33. *Solve:*

$$2x + 3y - z = 0$$
$$x - y - 2z = 0$$
$$4x + y - 5z = 0$$

The given system of equations is equivalent to the matrix equation

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ 4 & 1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = O \text{ where } O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 0 & 5 & 3 \\ 1 & -1 & -2 \\ 0 & 5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = O \text{ by } R_{12(-2)} \text{ and } R_{32(-4)}$$

or

$$\begin{bmatrix} 0 & 5 & 3 \\ 1 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = O \text{ by } R_{31(-1)}$$

which is equivalent to

$$5y + 3z = 0$$
$$x - y - 2z = 0$$

giving $\quad x = \frac{7}{5}z, \; y = -\frac{3}{5}z$

which give the solution for arbitrary values of z.

The general solution is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{7}{5}z \\ -\frac{8}{5}z \\ z \end{bmatrix} = z \begin{bmatrix} \frac{7}{5} \\ -\frac{8}{5} \\ 1 \end{bmatrix} = k \begin{bmatrix} 7 \\ -3 \\ 5 \end{bmatrix} \text{ where } z = 5k \text{ is an arbitrary parameter.}$$

Problem 34. *Solve*
$$4x + 2y + z + 3u = 0$$
$$6x + iy + 4z + 7u = 0$$
$$2x + y + u = 0$$
(*Meerut, 1968*)

The given system of equations is equivalent to the matrix equation

$$\begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = O, \text{ where } O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & \frac{5}{2} & \frac{5}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = O, \text{ by } R_{21\left(-\frac{3}{2}\right)} \text{ and } R_{31\left(-\frac{1}{2}\right)}$$

or

$$\begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = O, \text{ by } R_{32\left(\frac{1}{5}\right)} \text{ and } R_{2\left(\frac{2}{5}\right)}$$

which is equivalent to

$$4x + 2y + z + 3u = 0$$
$$z + u = 0$$

giving, $z = -u, \ x = -\dfrac{y + u}{2}$

The rank of $\begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$ is 2 and number of unknown variables is 4.

$\therefore$ Number of independent solutions $= 4 - 2 = 2$

If we choose $u = k_1$ and $x = k_2$

then, $y = -2k_2 - k_1$ and $z = -k_1$

which give an infinte number of non-trivial solutions.

Problem 35. *Solve*
$$x + 2y + 3z = 0$$
$$3x + 4y + 4z = 0$$
$$7x + 10y + 12z = 0$$

The given system of equations is equivalent to the matrix equation

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = O, \text{ where } O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & -4 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = O \text{ by } R_{21(-2)} \text{ and } R_{31(-7)}$$

or $$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{5}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = O \text{ by } R_{2\left(-\frac{1}{2}\right)} \text{ and then } R_{32(4)}$$

which is equivalent to

$$x + 2y + 3z = 0$$

$$y + \frac{5}{2}z = 0, \; z = 0$$

giving $x = 0, y = 0, z = 0$.

Moreover the rank of the coefficient matrix $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$ is 3 which is the number of unknown variables and hence the system has only zero solution.

Problem 36. *State the conditions under which a system of non-homogeneous linear equations will have*

(*i*) *no solution,* (*ii*) *a unique solution, and* (*iii*) *an infinity of solutions.*

(*Meerut, 1967*)

Let the system of non-homogeneous linear equations be equivalent to the matrix equation **AX = B**

where **A** is of order $m \times n$, **X** be of order $n \times 1$ and **B** be of order $m \times 1$.

(*i*) The equation **AX = B** has no solution if

rank of (**A**) ≠ rank of ([**A, B**])

(*ii*) The equation **AX = B** has a unique solution if

rank of (**A**) = rank of [(**A, B**)]= number of unknown variables.

In particular if **A** is a square matrix, then **AX = B** will have a unique soltuion if $|\mathbf{A}| \neq 0$.

(*ii*) The equations **AX = B** will have an infinite number of solutions if rank of (**A**) = rank of ([**A, B**]) < number of unknown variables.

Problem 37. *Solve the equations by using matrix methods :*

$$x + y + z = 6$$
$$x - y + z = 2$$
$$2x + y - z = 1$$

(*Agra, 1970*)

The given system of equations is equivalent to the matrix equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

or $$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 3 \end{bmatrix} \text{ by } R_{21(-1)} \text{ and } R_{32(1)}$$

which is equivalent to
$$x + y + z = 6$$
$$-2y = -4$$
$$3x = 3$$

giving $x = 1, y = 2, z = 3$.

Problem 38. *Show that the equations*

$$\begin{aligned} x+2y-z &= 3 \\ 3x-y+2z &= 1 \\ 2x-2y+3z &= 2 \\ x-y+z &= -1 \end{aligned}$$

are consistent and solve them.

The given system of equations is equivalent to the matrix equation

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ -1 \end{bmatrix} \qquad \ldots (1)$$

Denoting the coefficient matrix by A and augmented matrix by C we have

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

Here fourth order minor of A vanishes but none of the third order of A vanishes, therefore the rank of A is 3.

Again fourth order minor of C is

$$\begin{vmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 4 & -1 & 2 & 3 \\ 4 & -2 & 3 & 1 \\ 4 & -4 & 5 & 2 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$ by adding 4th column to 1st and 3rd and subtracting 4th column from 2nd

$$= \begin{vmatrix} 4 & -1 & 3 \\ 4 & -2 & 3 \\ 4 & -4 & 5 \end{vmatrix}$$ by expanding along 4th row.

$$= 4 \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{vmatrix}$$

$$= 4 \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \end{vmatrix}$$ by subtracting 1st row from others.

$$= 4 \begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix}$$ by expanding along 1st column.

$$= 0$$

i.e., 4th order minor of C vanishes, but none of 3rd order vanishes, as may be seen easily. So the rank of C is also 3.

$\therefore$ rank of (A) = rank of (C)

i.e., rank of coefficient matrix = rank of augmented matrix.

Hence the given system of equations is *consistent.*

Now in order to solve the given system by performing $R_{21(-3)}, R_{31(-2)}$ and $R_{41(-1)}$ on (1) we have

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -6 & 5 \\ 0 & -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -8 \\ -4 \\ -4 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & \frac{5}{7} \\ 0 & 0 & -\frac{1}{7} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -8 \\ \frac{20}{7} \\ -\frac{4}{7} \end{bmatrix} \text{ by } R_{32\left(-\frac{6}{7}\right)}, R_{42\left(-\frac{3}{7}\right)}$$

or

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & \frac{5}{7} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -8 \\ \frac{20}{7} \\ 0 \end{bmatrix} \text{ by } R_{43\left(\frac{1}{5}\right)}$$

which is equivalent to

$$x + 2y - z = 3$$
$$-7y + 5z = -8$$
$$\frac{5}{7}z = \frac{20}{7}$$

giving $x = -1, y = 4, z = 4$.

Problem 39. *Solve by Cramer's rule the system of equations*

$$x + 2y + 3z = 10$$
$$2x - 3y + z = 1$$
$$3x + y - 2z = 9$$

Here $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 1 \\ 3 & 1 & -2 \end{bmatrix}$, so that $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & -3 & 1 \\ 3 & 1 & -2 \end{vmatrix} = 52.$

By Cramer's rule $x = \dfrac{\begin{vmatrix} 10 & 2 & 3 \\ 1 & -3 & 1 \\ 9 & 1 & -2 \end{vmatrix}}{|A|} = \dfrac{156}{52} = 3,$

$$y = \frac{\begin{vmatrix} 1 & 10 & 3 \\ 2 & 1 & 1 \\ 3 & 9 & -2 \end{vmatrix}}{|A|} = \frac{104}{52} = 2.$$

and
$$z = \frac{\begin{vmatrix} 1 & 2 & 10 \\ 2 & -3 & 1 \\ 3 & 1 & 9 \end{vmatrix}}{|A|} = \frac{52}{52} = 1.$$

Hence $x = 3, y = 2, z = 1$.

Aliter we have
$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & -3 & -1 \\ 3 & 1 & -2 \end{vmatrix} = 52.$$

$$A_{11} = \begin{vmatrix} -3 & 1 \\ 1 & -2 \end{vmatrix} = 5, \quad A_{21} = -\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = +7; \quad A_{31} = \begin{vmatrix} 2 & 3 \\ -3 & 1 \end{vmatrix} = 11;$$

$$A_{12} = -\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} = +7, \quad A_{22} = \begin{vmatrix} 1 & 3 \\ 3 & -2 \end{vmatrix} = -11; \quad A_{32} = -\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = +5;$$

$$A_{13} = \begin{vmatrix} 2 & -3 \\ 3 & 1 \end{vmatrix} = 11, \quad A_{23} = -\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = +5; \quad A_{33} = \begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} = -7.$$

Thus
$$\text{Adj } A = \begin{bmatrix} 5 & 7 & 11 \\ 7 & -11 & 5 \\ 11 & 5 & -7 \end{bmatrix}.$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{52} \begin{bmatrix} 5 & 7 & 11 \\ 7 & -11 & 5 \\ 11 & 5 & -7 \end{bmatrix}$$

Hence the solution of the given system of equation is given by.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{52} \begin{bmatrix} 5 & 7 & 11 \\ 7 & -11 & 5 \\ 11 & 5 & -7 \end{bmatrix} \begin{bmatrix} 10 \\ 1 \\ 9 \end{bmatrix}$$

$$= \frac{1}{52} \begin{bmatrix} 5 + 7 + 99 \\ 70 - 11 + 45 \\ 110 + 4 - 63 \end{bmatrix} = \frac{1}{52} \begin{bmatrix} 156 \\ 104 \\ 52 \end{bmatrix},$$

which gives $x = 3, y = 2, z = 1$.

2.14. CHARACTERISTIC MATRIX AND CHARACTERISTIC EQUATION OF A MATRIX

Matrix polynomials. An expression of the type

$$F(x) = A_0 + A_1 x + A_2 x^2 + \ldots\ldots\ldots + A_m x^m,$$

where $A_0, A_1, A_2, \ldots A_m$ are all square matrices of the same order, is known as *matrix polynomial* of degree m provided $A_m \neq 0$. The matrix polynomial is said to be n-rowed where $A_0, A_1, A_2, \ldots A_m$ are each of order n.

The non-zero coefficient A_m of the highest power of x is called the *leading coefficient*.

The degree of the sum of two matrix polynomials cannot exceed the degree of either polynomial.

The degree of the product of two matrix polynomials is less than or equal to the sum of the degrees of the two polynomials.

Characteristic matrix. If A be any n-rowed matrix and I be an n-rowed unit matrix, then the matrix polynomial A-xI of the first degree is said to be the *characteristic matrix* of A.

Characteristic polynomial. The determinant $|A - xI|$ is known as characteristic polynomial of the matrix A.

Characteristic equation. The equation $|A - xI| = 0$ is said to be the characteristic equation of the matrix A.

Cayley-Hamilton Theorem

Every square matrix satisfies its own characteristic equation,

i.e. if $$|A - xI| = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n = 0$$

be the characteristic equation of a square matrix A, then

$$a_0 I + a_1 A + a_2 A^2 + ... + a_n A^n = O,$$

where every x has been replaced by A *and so* $a_0 = a_0 x^0 = a_0 A^0 = a_0 I$.

Adj (A–xI) is a matrix polynomial, since the adjoint of (A–xI) is a matrix having its elements as ordinary polynomials in x.

By the property of adjugate polynomials, we have

$$(A - xI) \text{ adj } (A - xI) = |(A - xI)| \, I \qquad ... (1)$$

The relation (1) stands between three matrix polynomials namely (A – xI) adj (A–xI) and $|A - xI|I$ of which the last one is a scalar matrix polynomial. The relation (1) also shows that (A – xI) is a left factor of the scalar polynomial $|A - xI|$, so that its left functional value is O for x = A. Thus

$$|A - xI|I = O = (a_0 + a_1 x + a_2 x^2 + ... + a_n x^n) I$$

$$= a_0 I + a_1 I x + a_2 I x^2 + ... + a_n I x^n.$$

For x = A, it gives $a_0 I + a_1 A + a_2 A^2 + ... + a_n A^n = O$.

Aliter. Since $|\text{adj } C| = |C|^{n-1}$, we may suppose that

$$\text{adj } (A - xI) = B_0 + B_1 x + B_2 x^2 + ... + B_{n-1} x^{n-1},$$

then $(A - xI) \text{ adj } (A - xI) = |A - xI| \, I$ gives

$$(A - xI)(B_0 + B_1 x + B_2 x^2 + ... + B_{n-1} x^{n-1})$$

$$= (a_o + a_1 x + a_2 x^2 + ... + a_n x^n) I$$

$$= A_o I + a_1 I x + a_2 I x^2 + ... + a^n I x^n.$$

Comparing the coefficients of like powers of x,

$$AB_0 = a_0 I$$

$$AB_1 - B_0 = a_1 I,$$

$$AB_2 - B_1 = a_2 I,$$

$$... \quad ... \quad ...$$

$$-B_{n-1} = a_n I.$$

Pre-multiplying these by I, A, A^2, ... A" in order and adding,

we get $$a_0 I + a_1 A + a_2 A^2 + ... + a_n A^n = O.$$

COROLLARY. Cayley-Hamilton equation is

$$a_0 I + a_1 A + a_2 A^2 + \dots + a_n A^n = O \quad \dots (1)$$

In characteristic polynomial $|A - xI|$ it is obvious that when $n = 0$,

$$a_0 = |A| \neq 0.$$

Thus premultiplying (1), by A^{-1}, we get

$$A^{-1} = -\frac{a_1}{a_0} I - \frac{a_2}{a_0} A - \frac{a_3}{a_0} A^2 - \dots - \frac{a_n}{a_0} A^{n-1}.$$

Note. The right and left functional values of an n-rowed matrix polynomial $G(x)$ for any n-rowed matrix C are defined as

$$G_r(C) = A_0 + A_1 C + A_2 C^2 + \dots + A_k C^k,$$

$$G_l(C) = A_0 + CA_1 + C^2 A_2 + \dots + C^k A_k,$$

when

$$G(x) = A_0 + A_1 x + A_2 x^2 + \dots + A_k x^k.$$

If $G_r(C) = O$, then $C - xI$ is called a right factor of $G(x)$ and conversely and if $G_l(C) = O$, then $C - xI$ is called left factor of $G(x)$ and conversely.

Problem 40. *Find the characteristic equation of the matrix*

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix}$$

(*Meerut, 1986*)

and verify Cayley-Hamilton theorem for it. Hence or otherwise find A^{-1}.

Since

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix}$$

$$\therefore \quad A - xI = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} - x\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-x & 2 & 3 \\ 2 & -1-x & 4 \\ 3 & 1 & 1-x \end{bmatrix}$$

so that

$$|A - xI| = \begin{vmatrix} 1-x & 2 & 3 \\ 2 & -1-x & 4 \\ 3 & 1 & 1-x \end{vmatrix} = -x^3 + x^2 + 18x + 30.$$

Hence the characteristic equation is

$$-x^3 + x^2 + 18x + 30 = 0.$$

Now in order to verify Cayley-Hamilton's theorem, we have to show that

$$30I + 18A + A^2 - A^3 = O. \quad \dots (1)$$

Here

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix},$$

$$A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 3 & 14 \\ 12 & 9 & 6 \\ 8 & 6 & 14 \end{bmatrix}.$$

and $$A^3 = A^2A = \begin{bmatrix} 14 & 3 & 14 \\ 12 & 9 & 6 \\ 8 & 6 & 14 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 62 & 39 & 68 \\ 48 & 21 & 78 \\ 62 & 24 & 62 \end{bmatrix},$$

so that $30I + 18A + A^2 - A^3$

$$= 30\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 18\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 14 & 3 & 14 \\ 12 & 9 & 6 \\ 8 & 6 & 14 \end{bmatrix} - \begin{bmatrix} 62 & 39 & 68 \\ 48 & 21 & 78 \\ 62 & 24 & 62 \end{bmatrix}$$

$$= \begin{bmatrix} 30+18+14-62 & 0+36+3-39 & 0+54+14-68 \\ 0+36+12-48 & 30-18+9-21 & 0+72+6-78 \\ 0+54+8-62 & 0+12+6-24 & 30+18+14-62 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O.$$

Hence Cayley-Hamilton theorem is verified.

Now,

$$30I + 18A + A^2 - A^3 = O$$

$$\Rightarrow A^{-1} = A^{-1}I = -\tfrac{18}{30}I - \tfrac{1}{30}A + \tfrac{1}{30}A^2$$

$$\Rightarrow A^{-1} = -\frac{18}{30}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{30}\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} + \frac{1}{30}\begin{bmatrix} 14 & 3 & 14 \\ 12 & 9 & 6 \\ 8 & 6 & 14 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -\frac{5}{30} & \frac{1}{30} & \frac{11}{30} \\ \frac{10}{30} & -\frac{8}{30} & \frac{2}{30} \\ \frac{5}{30} & \frac{5}{30} & -\frac{5}{30} \end{bmatrix}$$

2.15. SUB-SPACES AND NULL SPACES

Sub-space of an n-vector V_n. A set S of vectors of V_n is said to be a sub-space of V_n if it is closed with regard to the operations of addition and scalar multiplication, *i.e.*, is $\vec{\xi}_1, \vec{\xi}_2$ be any two members of S, then $\vec{\xi}_1 + \vec{\xi}_2$ is also a member of S and when $\vec{\xi}$ be any member of S, $k\vec{\xi}$ is also a member of S, k being a scalar.

Every sub-space of V_n, being the scalar product of any vector with the scalar zero, contains the zero vector.

Any sub-space arising as a set of all linear combinations of any given set of vectors, is said to be *spanned* by the given set of vectors.

A set of vectors is called the *Basis* of a sub-space if the sub-space is spanned by the set provided the set is lineary independent.

For example if $\vec{\xi}_1, \vec{\xi}_2, \vec{\xi}_3, \ldots \vec{\xi}_n$ be a set of vectors defined as

$\vec{\xi}_1 = [1\ 0\ 0\ \ldots\ 0], \vec{\xi}_2 = [0\ 1\ 0\ \ldots\ 0], \ldots \vec{\xi}_n = [0\ 0\ 0\ \ldots\ 0\ 1],$

then this set is a basis of the vector space V_n if

$$k_1 \vec{\xi}_1 + k_2 \vec{\xi}_2 + \ldots + k_n \vec{\xi}_n = 0,$$

which is satisfied, when $k_1 = 0, k_2 = 0, k_3 = 0, \ldots k_n = 0$, so that the set is linearly independent and that any vector $\vec{\xi} = [a_1, a_2, \ldots a_n]$ of V_n may be expressed as

$$\vec{\xi} = a_1 \vec{\xi}_1 + a_2 \vec{\xi}_2 + \ldots + a_n \vec{\xi}_n.$$

Dimension of a sub-space. The number of vectors in any basis of a sub-space is said to be the dimension of the sub-space, *e.g.* dimension of V_n is n.

If we have a matrix A of order $m \times n$, then each of the m rows of A consisting of n elements is an n-vector and each of the n columns, consisting of m elements is an m-vector.

The *row space* of A is one which is spanned by the m-rows which is a sub-space of V_n and the *column space* of A is the space spanned by the n-columns which is sub-space of V_m.

The dimensions of row and column spaces are respectively known as *row rank* and *column rank* of the matrix.

Premultiplication by a non-singular matrix does not alter the row rank of a matrix.

The row rank of a matrix is the same as its rank.

The column rank of a matrix is the same as its rank.

Null space and nullity of a matrix. The sub-space generated by the vectors X, such that AX = O is said to be the *column null space* of matrix A of order $m \times n$ and rank r and its dimension $n-r$ is said to be the *column nullity* of matrix A, *i.e.*,

rank of A + column nullity of A = number of columns in the matrix A.

Similarly the sub-space of the solution of YA = O is said to be the *row null space* and its dimension $m-r$ is said to be the *row nullity* of the matrix A, *i.e.*,

rank of A + row nullity of A = number of rows in A.

In case of a square matrix the row nullity and the column nullity are equal. Thus if $\rho(A)$ denotes the rank of A, $\nu(A)$ the nullity of square matrix A and n the number of rows or columns of A, then

$$\rho(A) + \nu(A) = n.$$

Sylvester's theorem *i.e.*, **Law of nullity.**

If A *and* B *be two n-rowed square matrices, then*

$$\max. \{\nu(A), \nu(B)\} \le \nu(AB) \le \nu(A) + \nu(B),$$

where $\nu(A), \nu(B), \nu(AB)$ *stand for the nullities of* A, B *and* AB.

If $\rho(A), \rho(B), \rho(AB)$ denote the ranks of the matrices A, B, AB, then we have

$$\left.\begin{aligned} \rho(A) &= n - \nu(A) \\ \rho(B) &= n - \nu(B) \\ \rho(AB) &= n - \nu(AB) \end{aligned}\right\} \quad \ldots (1)$$

Now we know that corresponding to the matrix A of rank $\rho(A) = r$ (say), there exist non-singular matrices P and Q such that

$$\text{PAQ} = \begin{bmatrix} \text{I}_r & \text{O} \\ \text{O} & \text{O} \end{bmatrix}. \qquad \dots (2)$$

[*Note*. Here if **A** be an $m \times n$ matrix, then **P** and **Q** are $m \times$ m and $n \times n$ matrices respectively and then the form (2) is said to be *normal* form of **A** while every non-zero matrix of rank r can be reduced to the normal form by a sequence of elementary transformations.]

Premultiplying by P^{-1} and postmultiplying by Q^{-1}, (2) gives

$$\text{A} = \text{P}^{-1} \begin{bmatrix} \text{I}_r & \text{O} \\ \text{O} & \text{O} \end{bmatrix} \text{Q}^{-1}. \qquad \dots (3)$$

Consider another matrix C given by

$$\text{C} = \text{P}^{-1} \begin{bmatrix} \text{O} & \text{O} \\ \text{O} & \text{O}_{n-r} \end{bmatrix} \text{Q}^{-1}, \qquad \dots (4)$$

that
$$\text{A} + \text{C} = \text{P}^{-1} \begin{bmatrix} \text{I}_r & \text{O} \\ \text{O} & \text{O}_{n-r} \end{bmatrix} \text{Q}^{-1} = \text{P}^{-1}\,\text{Q}^{-1}.$$

It follows that A + C is a non-singular matrix.

Now $\qquad \rho\,(\text{C}) = n - \rho\,(\text{A})$

and $\qquad \rho\,(\text{A} + \text{C}) = n.$

But A and C being non-singular, we have

$$\rho\,\{(\text{A} + \text{C})\,\text{B}\} = \rho\,(\text{B}).$$

i.e.,
$$\rho\,(\text{B}) = \rho\,(\text{AB} + \text{CB})$$
$$\leq \rho\,(\text{AB}) + \rho\,(\text{CB}). \qquad \dots (5)$$

Also
$$\rho(\text{CB}) \leq \rho\,(\text{C}) = n - \rho(\text{A}). \qquad \dots (6)$$

Thus (5) and (6) yield

$$\rho\,(\text{B}) \leq \rho\,(\text{AB}) + n - \rho\,(\text{A})$$

or
$$\rho\,(\text{AB}) \geq \rho\,(\text{A}) + \rho\,(\text{B}) - n,$$

i.e., if **A**, **B** be two n-rowed square matrices, then

$$\rho\,(\text{A}) + \rho\,(\text{B}) - n \leq \rho\,(\text{AB}) \leq \min.\,[\,\rho\,(\text{A}), \rho\,(\text{B})]$$

or $n - \nu\,(\text{A}) + n - \nu\,(\text{B}) - n \leq n - \nu\,(\text{AB}) \leq \min.\,\{n - \nu\,(\text{A}), n - \nu\,(\text{B})\}$ from (1)

or
$$\nu\,(\text{A}) + \nu\,(\text{B}) \geq \nu\,(\text{AB}) \geq \max.\,\{\nu\,(\text{A}), \nu\,(\text{B})\}.$$

This proves the theorem.

Problem 41. *Find the basis of the row and column null spaces of the matrix*

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}.$$

For the column null space, let us consider the matrix equation

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \text{O} \qquad \dots (1)$$

or
$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = O$$
on subtracting the first row from 2nd and 3 times the first row from 3rd and then subtracting the new second row from the third row in the first matrix.

This is equivalent to

$$x + y - z + w = 0,$$
$$-2y + 3z - 2w = 0,$$

which yield $y = \frac{3}{2}z - w,\quad x = -\frac{1}{2}z,$

so that the general solution of (1) is

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}z \\ \frac{3}{2}z - w \\ y \\ w \end{bmatrix} = z \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} - w \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$
where z and w may be treated as two parameters

Hence the two solutions $\begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ constitute the basis of the column null space of the given matrix whose nullity thus is 2 (two).

Now for the row null space, let us consider the matrix equation

$$[xyz] \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix} = O \qquad \ldots (2)$$

or
$$[xyz] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -2 & 3 & -2 \\ 3 & -2 & 3 & -2 \end{bmatrix} = O$$
on subtracting the first column from 2nd and fourth and adding it to the third column in the second matrix.

or
$$[xyz] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 3 & -2 & 0 & 0 \end{bmatrix} = O,$$
on adding $\frac{3}{2}$ times the second column to the third and subtracting the second column from the fourth.

which is equivalent to

$$x + y + 3z = 0,$$
$$-2y - 2z = 0,$$

giving $y = -z, x = -2z,$

so that the general solution of (2) is

$$[xyz] = [-2z, -z, z] = z\,[-2, -1, 1],$$

where z may be treated as a parameter.

Hence the solution $[-2, -1, 1]$ constitutes the basis of the row null space of the given matrix whose row nullity is thus 1 (one).

The required basis of the row and null spaces of the given matrix are respectively

$$[-2\ \ -1\ \ 1]\begin{bmatrix}1 & 1 & -1 & 1\\ 1 & -1 & 2 & -1\\ 3 & 1 & 0 & 1\end{bmatrix}=O$$

and
$$\begin{bmatrix}1 & 1 & -1 & 1\\ 3 & -1 & 2 & -1\\ 1 & 1 & 0 & 1\end{bmatrix}\begin{bmatrix}-1 & 0\\ 3 & -1\\ 2 & 0\\ 0 & 1\end{bmatrix}=O$$

2.16. TRANSFORMATIONS

Linear forms. An expression of the form $\sum_{j=1}^{n} a_{ij}\ x_j$ said to be linear form of the variable x_j.

Transformation. If a_{ij} be the given constants and x_j the variables then the set of equations

$$X_i = \sum_{j=1}^{n} a_{ij}\ x_j \text{ (for } i = 1, 2 \ldots n), \qquad \ldots (1)$$

is called a *linear transformation* connecting the variables x_j and the variables X_j.

The square matrix $A \equiv [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots a_{1n}\\ a_{21} & a_{22} \ldots\ldots a_{2n}\\ \ldots & \ldots\ldots\\ a_{n1} & a_{n2} \ldots\ldots a_{nn}\end{bmatrix}$ is said to be *the matrix of transformations.*

The determinant of the matrix

$$|A| \equiv |a_{ij}| = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots a_{1n}\\ a_{21} & a_{22} \ldots\ldots a_{2n}\\ \ldots\ldots & \ldots\ldots\ldots\ldots\\ a_{n1} \ldots\ldots & \ldots\ldots a_{nn}\end{bmatrix},$$

is said to be the *determinant* or *the modulus of the transformation.*

For the sake of convenience and brevity (1) is written as

$$X = Ax \qquad \ldots (2)$$

when $|A| = 0$, the transformation is called singular and when $|A| \neq 0$, the transformation is said to be non-singular.

In case of non-singular matrix the transformation may be expressed as $x = A^{-1} X$ (on premultiplying (2) by A^{-1}).

In particular the linear transformation

$$X_1 = x_1,\ X_2 = x_2, \ \ldots\ X_n = x_n$$

whose matrix is $\begin{bmatrix} 0 & & 0......0 \\ 0 & & 1......0 \\ 0 & 0 & 0 \quad 0 \end{bmatrix} \equiv \mathbf{I}$ is said to be *the identical transformation* and its determinant is unity.

Interpretation of X = A*x* in a Vector Space. If $\mathbf{X} = \{X_1, X_2, \ldots X_n\}$ and $\mathbf{x} = \{x_1, x_2, \ldots x_n\}$ be two bases as column vectors in a vector (linear) space $\mathbf{V}_n(F)$ over the field F, then the transformation $\mathbf{X} = \mathbf{Ax}$ carries a vector $\mathbf{X}$ of $\mathbf{V}_n(F)$ into another vector $\mathbf{x}$ of the same space *i.e.* for a scalar α, it transforms αX_1 into αx_1 and for scalars α, β it transforms $\alpha X_1 + \beta X_2$ into $\alpha x_1 + \beta x_2$ etc.

When $|\mathbf{A}| \neq 0$, the correspondence between $\mathbf{X}$ and $\mathbf{x}$ is *one-one* and the transformation $\mathbf{A}^{-1}\mathbf{X} = \mathbf{x}$ assigns to each of n vectors $\mathbf{x}$ another n-vectors $\mathbf{X}$.

Resultant of two linear transformations. If the two successive transformations be $\mathbf{Y} = \mathbf{PX}$ and $\mathbf{Z} = \mathbf{QY}$, then the resultant transformation is determined by $\mathbf{Z} = \mathbf{QPX}$ which shows that the matrix of the resultant of two linear transformations, is the product of the matrices of the transformation, due regard being paid to the order of multiplication.

Evidently the rank of the product of two matrices cannot exceed the rank of either factor and if a matrix of rank r is multiplied (in any order) by a non-singular matrix, the rank of the product is also r.

Similarity Transformation. By theorem 1 of [17] on §2.9 if **A**, **B** are two non-singular matrices and ∃ two non singular matrices **P** and **Q** s.t. $\mathbf{B} = \mathbf{QAP}$ with $\mathbf{Q} = \mathbf{P}^{-1}$ so that

$$\mathbf{B} = \mathbf{P}^{-1}\,\mathbf{A}\,\mathbf{P} \qquad \ldots (3)$$

then this transformation of metrix **A** into matrix **B** is termed as *similarity transformation* and matrices **A** and **B** are known as *similar matrices*. Now,

(3) $\Rightarrow \mathbf{PBP}^{-1} = \mathbf{PP}^{-1}\,\mathbf{APP}^{-1} \Rightarrow \mathbf{IAI} = \mathbf{A}$ which is also a similarity transformation **B** into **A**.

A *matrix equation* $\mathbf{AX} = \mathbf{B}$ preserves its structure (form) under similarity transformation, since

$$\mathbf{P}^{-1}(\mathbf{AX})\,\mathbf{P} = \mathbf{P}^{-1}\,\mathbf{BP} \Rightarrow \mathbf{P}^{-1}\,\mathbf{APP}^{-1}\mathbf{XP} = \mathbf{P}^{-1}\,\mathbf{BP} \text{ as } \mathbf{PP}^{-1} = \mathbf{I}$$

$$\Rightarrow (\mathbf{P}^{-1}\,\mathbf{AP})\,(\mathbf{P}^{-1} \times \mathbf{P}) = (\mathbf{P}^{-1}\,\mathbf{BP})$$

$$\Rightarrow \mathbf{CY} = \mathbf{D} \text{ (say) by (3)} \Rightarrow \text{The form } \mathbf{AX} = \mathbf{B}.$$

ILLUSTRATIVE EXAMPLE: The trace of a matrix is invariant under similarity transformation, since by (3), we have

$T_r\,\mathbf{B} = \sum_i \mathbf{B}_{ii} = T_r\,(\mathbf{P}^{-1}\,\mathbf{AP}) = \sum_i (\mathbf{P}^{-1}\,\mathbf{AP})$, trace being sum of diagonal elements of a matrix

$$= \sum_i \sum_{j,k} P_{ij}^{-1}\,A_{jk}\,P_{ki} = \sum_{i,j,k} \left(P_{ki}\,P_{ij}^{-1}\right) A_{ij}$$

$$= \sum_{j,k} \left(PP^{-1}\right)_{kj} A_{jk} = \sum_k A_{kk} = T_r\,\mathbf{A}.$$

Unitary Transformation. If **A** be a *unitary transformation* of order $n \times n$ and **X**, **x** are column vectors of order $n \times 1$, then the linear transformation

$$X = Ax \quad \ldots (4)$$

is known as *unitary transformation.* Since

$$A^{\Theta} A = AA^{\Theta} = I, \text{ therefore}$$

$$X^{\Theta} X = (Ax)^{\Theta} (Ax) = x^{\Theta} A^{\Theta} Ax = x^{\Theta} x$$

⇒ *the norm of vectors is invariant under similarity transformation.*

In (3), if P be unitary *i.e.* $PP^{\Theta} = P^{\Theta} P = I = PP^{-1} = P^{-1}P$ or $P^{-1} = P^{\Theta}$, then the transformation $B = P^{-1} AP$ is also unitary.

Orthogonal transformation. Any transformation $x = AX$ that transforms Σx^2 into ΣX^2 is said to be an *orthogonal transformation* and the matrix A is known as *orthogonal matrix.*

If
$$A = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots a_{1n} \\ a_{21} & a_{22} \ldots\ldots a_{2n} \\ \ldots & \ldots \\ a_{n1} & a_{n2} \ldots\ldots a_{nn} \end{bmatrix}$$

and its transpose
$$A' = \begin{bmatrix} a_{11} & a_{21} \ldots\ldots a_{n1} \\ a_{12} & a_{22} \ldots\ldots a_{n2} \\ \ldots & \ldots \\ a_{1n} & a_{2n} \ldots\ldots a_{nn} \end{bmatrix}$$

then $AA' = I$ which is the necessary and sufficient condition for a square matrix X to be orthogonal,

since if $x_r = a_{r1} X_1 + a_{r2} X_2 + \ldots + a_{rn} X_n$,

then $\sum_{r=1}^{n} X_r^2 = \sum_{r=1}^{n} x_r^2 = \sum_{r=1}^{n} (a_{r1} X_1 + a_{r2} X_2 + \ldots a_{rn} X_n)^2$ gives

and
$$\left.\begin{aligned} a_{1i}^2 + a_{2i}^2 + \ldots + a_{ni}^2 = 1 \\ a_{1i}a_{1j} + a_{2i}a_{2j} + \ldots + a_{ni}a_{nj} = 0 \end{aligned}\right\}$$

for $i = 1, 2 \ldots n$, $j = 1, 2 \ldots n$ and $i \neq j$.

Now $AA' = I$ gives $|A| \, |A'| = 1$,

where $|A| = |A'|$ as interchange of rows and columns does not alter the value of the determinant.

$$\therefore \quad |A|^2 = 1, \text{ i.e., } |A| = \pm 1$$

Evidently the product of two orthogonal transformations is an orthogonal transformation. For if $x = AX$ and $X = BY$ be two orthogonal transformations then $AA' = I$, $BB' = I$.

$\therefore$ $[AB]\,[AB]' = ABB'A'$ by the law of reversal transpose

$= AIA'$

$= AA' = I.$

In (3) if P is orthogonal, then $P^{-1} = P'$ and (3) is an orthogonal transformation.

Orthonormal Set. *A set of vectors is said to be an* **orthonormal set** *of vectors if*

(*i*) *each vector of the set is a normal vector*

(*ii*) *any two vectors of the set are orthogonal*

While a complex *n*-vector X is said to be orthogonal to another complex *n*-vector Y if $(X, Y) = O$ *i.e.*, if $X^{\Theta} Y = O$.

The relation of orthogonality in the set of all complex n-vectors is *Symmetric*. The positive square root of $X^{\Theta} X$ is known as the *length of* X.

In other words using the Kronecker delta symbol $\delta_{ij} = 0$ for $i \neq j$

$= 1$ for $i = j$.

a set S of complex n–vectors $X_1, X_2, \ldots X_k$ is termed as an orthonormal set if $(X_i, X_j) = \delta_{ij}$, $i = 1, 2, \ldots k$, $j = 1, 2, \ldots k$.

THEOREM 1. Every orthonormal set of vectors is linearly independent.

Suppose that the vectors $X_1, X_2, \ldots X_k$ form an orthonormal set of vectors.

Thus (*i*) X_i is a normal vector for every i *i.e.* $X_i^{\Theta} X_j = I$

(*ii*) X_i, X_j are orthogonal vectors for every i, j such that $i \neq j$

i.e., $X_i^{\Theta} X_j = O$ for every i and j, $i \neq j$

Consider a relation $a_1 X_1 + a_2 X_2 + \ldots + a_k X_k = O$ ...(1)

where $a_1, a_2, \ldots a_k$ are scalars.

Premultiplying (1) by X_1^{Θ} and applying above condition (*i*) and (*ii*), we find

$a_1 I = O$ or $a_1 = O$ as $I \neq O$.

Similarly premultiplying (1) by $X_2^{\Theta}, X_3^{\Theta}, \ldots$ successively, we may get $a_2 = 0$, $a_3 = 0 \ldots, a_k = 0$.

As such all the scalars $a_1, a_2, \ldots a_k$ being zero, the relation (1) follows that $X_1, X_2, \ldots X_k$ form an orthonormal set of linearly independent vectors.

THEOREM 2. Show that a real matrix is unitary if and only if it is orthogonal.

If A is a real matrix, then $A^{\Theta} = A'$

$\because$ A is unitary if $A^{\Theta} A = I$ or $A'A = I$ *i.e.*, A is orthogonal.

Conversely if A is orthogonal then $A'A = I$

i.e., $A^{\Theta} A = I$, *i.e.* A is unitary.

2.17. HERMITIAN FORMS

Any expression of the forms $\sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} \; \bar{x}_i x \cdot x_j$

where $a_{ij} = \bar{a}_{ji}$ is said to be a *Hermitian form* in n-variables.

It is easy to see that $\sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} \; \bar{x}_j x_j = \bar{X}' AX$ where $A = [a_{ij}]$

$= X^{\Theta} AX$

Where X is a Column vector with components $x_1, x_2, x_3, \ldots x_n$ and X^{Θ} is a row vector with components $\bar{x}_1, \bar{x}_2, \ldots \bar{x}_n$. Also the matrix A is Hermitian as $a_{ij} = \bar{a}_{ji}$.

THEOREM 3. A Hermitian form assumes only real values.

We have $\overline{(X^{\Theta} AX)} = \bar{X}^{\Theta} \bar{A} \; \bar{X}$,

where $\overline{X^{\Theta} AX}$ being a single element matrix coincides with its transpose.

$$\therefore \; \overline{X^{\Theta} AX} = (\overline{X^{\Theta} AX})' = (\overline{X^{\Theta}} \, \overline{AX})'$$

$$= (X' \overline{AX})', \; \because \overline{X^{\Theta}} = X'$$

$$= (\bar{X})' \bar{A}' (X')' = X^{\Theta} AX, \; \because (\bar{X})' = X^{\Theta}, \bar{A}' = A \text{ and } (X')' = X$$

so that $\overline{(X^{\Theta}AX)} = X^{\Theta}AX$,

i.e., $X^{\Theta}AX$ coincides with its conjugate and therefore it is real.

Linear Transformation of a Hermitian form. Consider a Hermitian form $X^{\Theta}AX$.

Any non-singular transformation say $X = PY$, *i.e.*, $X^{\Theta} = Y^{\Theta}P^{\Theta}$ gives

$$X^{\Theta}AX = Y^{\Theta}P^{\Theta}APY = Y^{\Theta}BY \text{ (say)}$$

where $B = P^{\Theta}AP$ is the matrix of the transformation.

Now $$B^{\Theta} = (P^{\Theta}AP)^{\Theta} = P^{\Theta}A^{\Theta}(P^{\Theta})^{\Theta} = P^{\Theta}AP = B,$$

i.e., B is also Hermitian if A is Hermitian, since for a Hermitian matrix $A^{\Theta} = A$.

2.18. CHARACTERISTIC ROOTS AND VECTORS OF A MATRIX (EIGEN VALUES AND EIGEN VECTORS)

A non-zero vector X is called a *characteristic vector* of a matrix A, if there is a number λ such that $AX = \lambda X$.

Here λ is called a *characteristic root* of A corresponding to the characteristic vector X and vice versa.

Characteristic roots are often known as *Proper, Latent* of *Eigen values* and characteristic vectors are known as *Proper, Latent* or *Eigen vectors* or *invariant vectors.*

We have $AX = \lambda X = \lambda IX$, I being a unit matrix

or $$(A - \lambda I)\, X = O.$$

Since $X \neq 0$, the matrix $(A - \lambda I)$ is singular, so that

$$|(A - \lambda I)| = 0.$$

which follows that every characteristic root λ of a matrix A is a root of its characteristic equation $|A - \lambda I| = 0$.

Conversely if λ be a root of the characteristic equation $|A - \lambda I| = 0$, then the matrix equation $(A - \lambda I)\, X = 0$ possesses a non-trivial *i.e.*, non-zero solution for *X*, so that there exists a vector $X \neq 0$ such that $AX = \lambda IX = \lambda X$.

It follows that every root of the characteristic equation of a matrix is a characteristic root of the matrix.

Thus if $A = [a_{ij}]$ be an *n*-rowed square matrix and λ an indeterminate, then the characteristic equation

$$|A - \lambda I| = 0 \text{ gives } \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33}-\lambda & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0 \qquad \ldots(1)$$

which is an ordinary polynomial in λ of degree *n* and hence will give *n* values of λ on simplification. These *n* values of λ are *n* eigen values of this equation. To every eigen value there corresponds an eigen vector.

As an illustrative exmaple if

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

then its characteristic equation $|A - \lambda I| = 0$ is

$$\begin{bmatrix} 1-\lambda & 1 & 3 \\ 5 & 2-\lambda & 6 \\ -2 & -1 & -3-\lambda \end{bmatrix} = 0$$

which yields on expansion $\lambda^3 = 0$

giving $\lambda_1 = 0 = \lambda_2 = \lambda_3$ *i.e.*, all the three eigen roots are zero.

Nature of the eigen values and eigen vectors of special types of matrices.

THEOREM. 1. The eigen values of a Hermitian matrix are all real.

(Rohilkhand, 1977; Agra, 1970; Vikram, 1969)

Let λ be an eigen value of a Hermitian matrix A. Then by definition there exists a vector $X \neq 0$, such that

$$AX = \lambda X.$$

Premultiplying X^Θ we get

$$X^\Theta AX = X^\Theta \lambda X$$

$$= \lambda X^\Theta X = \lambda X^\Theta I X \quad \ldots(1)$$

so that

$$\lambda = \frac{X^\Theta AX}{X^\Theta IX}. \quad \ldots(2)$$

Also by §2.17, we have $X^\Theta AX = \overline{X^\Theta AX}$,

so that $X^\Theta AX$ is a real number and therefore in particular $X^\Theta IX$ is also a real number.

Hence from (2) it follows that λ is a real number, *i.e.*, all the eigen values of a Hermitian matrix are real.

COROLLARY. *The eigen values of a real symmetric matrix are all real.*

Hint: $\overline{A} = A$ and $A' = A \Rightarrow (\overline{A}') = \overline{A}$

$\Rightarrow A^\Theta = A \Rightarrow$ A is Hermitian

$\Rightarrow$ By Th.1, the eigen value of a Hermitian matrix are all real.

THEOREM 2. The eigen values of a skew-Hermitian matrix are purely imaginary or zero.

If A be a skew-Hermitian matrix and $AX = \lambda X$, then

$$(iA)X = (i\lambda)X.$$

But iA is Hermitian and therefore its eigen values $i\lambda$ are real. It follows that λ is either zero or purely imaginary number.

Note. In this case $\overline{X^\Theta AX} = -X^\Theta AX$.

COROLLARY. *The eigen values of a real skew-symmetric matrix are either zero or purely imaginary.* *(Meerut, 1976)*

Hint: $\overline{A} = A$ and $A' = -A \Rightarrow (\overline{A}') = -\overline{A} \Rightarrow A^\Theta = -A$

$\Rightarrow$ A is skew-Hermitian

$\Rightarrow$ By Th.2, the result follows..

THEOREM 3. The modulus of each eigen value of a unitary matrix is unity, i.e., the eigen values of a unitary form have the absolute value 1.

Let U be the unitary matrix, λ an eigen value of U and X a corresponding eigen vector; then

$$UX = \lambda X. \qquad \ldots(1)$$

Taking conjugate transpose of either side, we get

$$X^{\Theta} U^{\Theta} = \bar{\lambda} X^{\Theta} \qquad \ldots(2)$$

Post-multiplying (2) by (1), we get

$$X^{\Theta} U^{\Theta} U X = \lambda \bar{\lambda} X^{\Theta} X,$$

$\bar{\lambda}$ being complex conjugate of λ

or $\quad X^{\Theta} X = \lambda \bar{\lambda} X^{\Theta} X$ as $U^{\Theta} U = I$, U being unitary

or $\quad (1 - \lambda \bar{\lambda}) X^{\Theta} X = 0.$

Since $X \neq 0$, therefore $X^{\Theta} X \neq 0$ and hence

$$1 - \lambda \bar{\lambda} = 0, \text{ i.e., } \lambda \bar{\lambda} = |\lambda|^2 = 1,$$

i.e., the modulus of λ is unity.

COROLLARY: *The eigen values of an orthogonal matrix have the absolute unity and are real, or complex conjugate in pairs.*

Hint: $\bar{A} = A$ and $A'A = I \Rightarrow \overline{(A'A)} = \bar{I} \Rightarrow \overline{(A')}\,\bar{A} = I \Rightarrow A^{\Theta} A = I$

$\Rightarrow$ A is unitary,

THEOREM 4. Every matrix equation satisfies its own characteristic equation **(Cayley-Hamilton Theorem).**

It is easy to show that the theorem that every square matrix satisfies its own characteristic equation also holds for singul matrices that have repeated values. Its proof has already been given in §2.14.

Thus taking λ as an eigen value of the matrix A and X the corresponding eigen vector, we have

$$\begin{aligned} AX &= \lambda X, \\ A^2X &= \lambda AX = \lambda^2 X, \\ A^3X &= \lambda A^2 X = \lambda^3 X. \\ \ldots & \quad \ldots \quad \ldots \end{aligned}$$

Therefore

$$[A^n - I_{(1)} A^{n-1} + I_{(2)} A^{n-2} - \ldots + (-1)^n I_{(n)} I]\, X$$
$$= [\lambda^n - I_{(1)} \lambda^{n-1} + I_{(2)} \lambda^{n-2} - \ldots + (-1)^n I_{(n)}]\, X, \qquad \ldots(1)$$

Since $A^0 = I$

If the characteristic equation be taken as

$$|A - \lambda I| = (-1)^n [\lambda^n - I_{(1)} \lambda^{n+1} + I_{(2)} \lambda^{n-2} + \ldots + (-1)^n I_{(n)}] = 0,$$

then (1) reduces to

$$A^n - I_{(1)} A^{n-1} + I_{(2)} A^{n-2} - \ldots + (-1)^n I_{(n)} I = 0, \qquad \ldots(2)$$

which is *Cayley-Hamilton Theorem.*

COROLLARY: *To determine* A^{-1} *by using Cayley-Hamilton theorem.*

If A is non-singular matrix, then multiplying the result (2) above by A^{-1}, we get

$$A^{-1} = \frac{(-1)^{n+1}}{I_{(n)}} [A^{n-1} - I_{(1)} A^{n-2} + I_{(2)} A^{n-3} - \ldots + (-1)^n I_{(n-1)}],$$

which gives a method of finding the inverse of a matrix.

THEOREM 5. Any two given vectors corresponding to two distinct eigen values of a Hermitian matrix are orthogonal. *(Rohilkhand, 1977)*

Let X_1 and X_2 be two eigen vectors corresponding to two distinct eigen values λ_1, λ_2 of a Hermitian matrix A; then

$$AX_1 = \lambda_1 X_1, \qquad \text{...(1)}$$

$$AX_2 = \lambda_2 X_2, \qquad \text{...(2)}$$

where from theorem (1) the numbers λ_1, λ_2 are real.

Premultiplying (1) and (2) by $X^\Theta{}_2$ and $X^\Theta{}_1$ respectively

$$X_2{}^\Theta AX_1 = \lambda_1 X_2{}^\Theta X_1, \qquad \text{...(3)}$$

$$X_1{}^\Theta AX_2 = \lambda_2 X_1{}^\Theta X_2, \qquad \text{...(4)}$$

But $\quad (X_2{}^\Theta AX_1)^\Theta = X_1{}^\Theta AX_2,$

$\because$ for a Hermitian matrix $A^\Theta = A$ and also $(X_2{}^\Theta)^\Theta = X_2$.

therefore we have from (3) and (4),

$$(\lambda_1 X_2{}^\Theta X_1)^\Theta = \lambda_2 X_1{}^\Theta X_2$$

or $$\lambda_1 X_1{}^\Theta X_2 = \lambda_2 X_1{}^\Theta X_2$$

or $$(\lambda_1 - \lambda_2)\, X_1{}^\Theta X_2 = 0.$$

Since $\lambda_1 - \lambda_2 \neq 0$, otherwise the roots will not be distinct, therefore the only possibility is that $X_1{}^\Theta X_2 = 0$.

If follows that X_1, X_2 are orthogonal.

COROLLARY. *Any two eigen vectors corresponding to two distinct eigen values of a real symmetric matrix are orthogonal.* *(Agra, 1974)*

Hint: $\overline{A} = A$ and $A' = A \Rightarrow \overline{(A')} = A \Rightarrow A^\Theta = A$

$\Rightarrow \overline{A}$ is Hermitian.

THEOREM 6. Any two eigen vectors corresponding to two distinct eigen values of a unitary matrix are orthogonal.

Let X_1, X_2 be two eigen vectors corresponding to two distinct eigen values λ_1, λ_2 of unitary matrix U; then

$$UX_1 = \lambda_1 X_1 \qquad \text{...(1)}$$

$$UX_2 = \lambda_2 X_2. \qquad \text{...(2)}$$

Taking conjugate transpose of (2), we get

$$X_2{}^\Theta U^\Theta = \overline{\lambda}_2\, X_2^\Theta \qquad \text{...(3)}$$

From (1) and (3), we find

$$X_2{}^\Theta U^\Theta UX_1 = \overline{\lambda}_2 \lambda_1\, X_2{}^\Theta X$$

or $$X_2{}^\Theta X_1 = \overline{\lambda}_2 \lambda_1 X_2{}^\Theta X_1 \qquad \text{since } U^\Theta U = I$$

or $$(1 - \overline{\lambda}_2 \lambda_1)\, X_2{}^\Theta X_1 = 0. \qquad \text{...(4)}$$

But U being a unitary matrix, the modulus of each of its eigen values is unity, *i.e.*, $\lambda_2 \overline{\lambda}_2 = 1$.

So that $(1 - \lambda_2 \overline{\lambda}_1) = \lambda_2 \overline{\lambda}_2 - \overline{\lambda}_2 \lambda_1,\ \overline{\lambda}_2 (\lambda_2 - \lambda_1) \neq 0 \qquad \text{...(5)}$

From (4) and (5) it follows that $X_2{}^\Theta X_1 = 0$ *i.e.*, X_1 and X_2 are orthogonal.

THEOREM 7. The eigen vectors corresponding to distinct characteristic roots of a matrix are linearly independent.

Assuming that $X_1, X_2, \ldots\ldots, X_m$ are eigen vectors of a matrix A corresponding to distinct eigen values $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ such that $AX_i = \lambda_i X, i = 1, 2 \ldots\ldots m$, we have to show that $X_1, X_2 \ldots\ldots X_m$ are linearly independent.

Suppose that $X_1, X_2 \ldots\ldots X_m$ are not linearly independent but they are linearly dependent, then we can choose r such that $1 \le r < m$ and $X_1, X_2, \ldots\ldots X_r$ are linearly independent. but $X_1, X_2 \ldots\ldots X_r, X_{r+1}$ are linearly dependent so that there exist scalars $a_1, a_2, \ldots\ldots a_{r+1}$ not all zero, satisfying

$$a_1 X_1 + a_2 X_2 + \ldots\ldots + a_{r+1} X_{r+1} = O \qquad \ldots(1)$$

or $\quad A(a_1 X_1 + a_2 X_2 + \ldots\ldots + a_{r+1} X_{r+1}) = AO$

or $\quad a_1 AX_1 + a_2 AX_2 + \ldots\ldots + a_{r+1} AX_{r+1} = O$

or $\quad a_1(\lambda_1 X_1) + a_2(\lambda_2 X_2) + \ldots\ldots + a_{r+1}(\lambda_{r+1} X_{r+1}) = O \qquad \ldots(2)$

Multiplying (1) again by λ_{r+1} and subtracting from (2), we find,

$$a_1(\lambda_1 - \lambda_{r+1}) X_1 + a_2(\lambda_2 - \lambda_{r+1}) X_2 + \ldots\ldots + a_r(\lambda_r - \lambda_{r+1}) X_r = O \qquad \ldots(3)$$

But $X_1, X_2 \ldots\ldots X_r$ are linearly independent by hypothesis and $\lambda_1, \lambda_2, \ldots\ldots \lambda_{r+1}$ are distinct, it therefore follows from (3) that

$$a_1 = 0 = a_2 = \ldots = a_r$$

and then (1) gives $a_{r+1} X_{r+1} = O$

So that $a_{r+1} = 0$ as $X_{r+1} \neq O$

Hence $a_1 = 0 = a_2 = \ldots = a_r = a_{r+1}$

which contradicts the assumption that $a_1, a_2, \ldots a_{r+1}$ are not all zero.

It follows from this contradiction that our initial assumption is wrong and the only possibility is that $X_1, X_2, \ldots\ldots X_m$ are linearly independent.

THEOREM 8. The minimal polynomial of a matrix is a divisor of the characteristic polynomial of that matrix.

If $f(x)$ be the characteristic polynomial of a matrix A, then by Cayley-Hamilton theorem, $f(A) = 0$ so that $f(x)$ annihilates A. Thus if $m(x)$ is the minimal polynomial of A, then $m(x)$ is divisor of $f(x)$ by §2.9 [19].

THEOREM 9. Every root of the minimal equation of a matrix is also a characteristic root of that matrix.

If $f(x)$ be the characteristic polynomial of a matrix A and $m(x)$ be its minimal polynomial, then by preceding theorem $m(x)$ is a divisor of $f(x)$ and so there exists a polynomial $q(x)$ such that

$$f(x) = m(x)\, q(x) \qquad \ldots(1)$$

If λ be a root of $m(x) = 0$, then $m(\lambda) = 0 \qquad \ldots(2)$

Putting $x = \lambda$ in (1) we get, $f(\lambda) = m(\lambda)\, q(\lambda) = 0$ by (2)

which follows that λ is also a root of $f(x) = 0$

Hence λ is also a characteristic root of A.

THEOREM 10. The characteristic polynomial and hence the eigen values of similar matrices are the same. Also if X *be an eigen vector of* A *corresponding to the eigen value* λ, *then* $P^{-1}X$ *is an eigen vector of* B *corresponding to the eigen value* λ *where* $B = P^{-1}AP$.

Let A and B be two similar matrices. Then there exists an invertible matrix P such that

$$B = P^{-1} AP.$$

Consider $B - \lambda I = P^{-1} AP - \lambda I$

$= P^{-1} AP - P^{-1} (\lambda I) P \because P^{-1} (\lambda I) P = \lambda P^{-1} P = \lambda I$

$= P^{-1} (A - \lambda I) P$

$\therefore \quad |B - \lambda I| = |P^{-1}| \cdot |A - \lambda I| \cdot |P|$

$= |P^{-1}| \, |P| \cdot |A - \lambda I| \because$ scalar quantities commute

$= |P^{-1} P| \cdot |A - \lambda I| \because |CD| = |C| \, |D|$

$= |A - \lambda I| \because |P^{-1} P| = |I| = 1$

which follows that A and B have same characteristic polynomial and so they have the same eigen values.

For the second part, taking λ as one of the eigen values of A and X corresponding eigen vector, we have $AX = \lambda X$ ...(1)

$\therefore \quad B (P^{-1} X) = (P^{-1} AP) P^{-1} X = P^{-1} AX \because PP^{-1} = I$ etc.

$= P^{-1} (\lambda X)$ by (1)

$= \lambda (P^{-1} X)$

which follows that $P^{-1} X$ is an eigen vector of B corresponding to its eigen value λ.

COROLLARY 1. *The eigen values of a matrix are invariant under a similarity transformation.*

Hint : $|B - \lambda I| = 0 \Rightarrow |A - \lambda I| = 0$

$\Rightarrow \lambda$ is an eigen value of A also.

COROLLARY 2. *If X is similar to a diagonal matrix* D *then diagonal elements of* D *are the eigen values of* A.

Since D has its diagonal elements as its eigen values and D is similar to A it therefore follows that diagonal elements of D are eigen values of A.

Problem 42. *If* X *is a characteristic vector of a matrix* A *corresponding to eigen value* λ, *then show that* k X, k *being a non-zero scalar, is also an eigen vector of* A *corresponding to* λ.

As given, $X \neq 0$, and $AX = \lambda X$ ...(1)

$k \neq 0$ and $X \neq 0$ give $kX \neq 0$

Now, $A (kX) = k (AX) = k \lambda X$ by (1)

$= \lambda (kX)$

which follows that kX is an eigen vector of A corresponding to the eigen value λ.

Problem 43. *If* X *is an eigen vector of a matrix* A, *then prove that* X *cannot correspond to more than one characteristic values of* A.

If possible let us suppose that X is an eigen vector corresponding to two eigen values λ_1, λ_2 of A. Then, we have

$$AX = \lambda_1 X \text{ and } AX = \lambda_2 X$$

These give, $\lambda_1 X = \lambda_2 X$ *i.e.*, $(\lambda_1 - \lambda_2) X = O$

$\therefore \quad \lambda_1 = \lambda_2$ as $X \neq 0$.

i.e., λ_1 and λ_2 cannot be different.

Problem 44 (*a*) *Determine the eigen values and eigen vectors of the matrix*

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \text{ and}$$

(*b*) *find the eigen values and normalised vector of the matrix*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$ (*Agra, 1974*).

(*a*) The characteristic equation of A is given by $|A - \lambda I| = 0$ *i.e.*,

$$\begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix} = 0 \text{ i.e., } \lambda^2 - 7\lambda + 6 = 0 \text{ or } (\lambda - 1)(\lambda - 6) = 0$$

$\therefore$ $\lambda_1 = 1, \lambda_2 = 6$ are the eigen values of A.

Now the eigen vector X_1 of A corresponding to the eigen value $\lambda_1 = 1$ is given by the non-zero solutions of the equation $(A - 1I)\, X_1 = O$

i.e., $$\left(\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The coefficient matrix here is of rank 1 and therefore these equations have 2 – 1 *i.e.*, 1 linear independent solution.

These are equivalent to

$$4x_1 + 4x_2 = 0 \text{ and } x_1 + x_2 = 0$$

which yield, $x_1 = -x_2$

If we take $x_1 = 1$, then $x_2 = -1$ so that the eigen vector of A

$X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ corresponding to the eigen value 1.

In fact every non-zero multiple of the vector X_1 is an eigen vector of A corresponding to the eigen value 1.

Again the eigen vector X of A corresponding to the eigen value $\lambda_2 = 6$ is given by the non-zero solution of the equation $(A - 6I)\, X_2 = O$

i.e., $$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Here also the coefficient matrix being of rank 1, these equations have 2–1 *i.e.*, 1 linear independent solution.

They are equivalent to

$$-x_1 + 4x_2 = 0 \text{ and } x_1 - 4x_2 = 0$$

which are the same and satisfied by $x_1 = 4, x_2 = 1$

$\therefore$ $X_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ is an eigen vector of A corresponding to the eigen value 6. In general it may be represented by CX_2 *i.e.*, $C\begin{bmatrix} 4 \\ 1 \end{bmatrix}$. where C is any non-zero scalar.

(*b*) The eigen-value equation is

$$|A - \lambda I| = O$$

i.e., $$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$$

or $\lambda(1-\lambda)(\lambda-2)=0$

giving $\lambda = 0, 1, 2.$

Eigen vector say $\mathbf{X}_1$ corresponding to $\lambda = 0$ is given by $(\mathbf{A} - 0\,\mathbf{I})\,\mathbf{X}_1 = \mathbf{O}$

or $$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\Rightarrow x_1 = 0, x_2 + x_3 = 0, x_2 + x_3 = 0$

$\Rightarrow x_1 = 0, x_2 = -x_3 = c_1$ (say)

$\therefore\ \mathbf{X}_1 = \begin{bmatrix}0\\ c_1\\ -c_1\end{bmatrix}$ or $(0, c_1 - c_1)$

If the eigen vector to normalised to unity *i.e.*, $|\mathbf{X}_1| = 1$ or

$\sqrt{(0^2 + c_1^2 + (-c_1)^2)} = 1 \Rightarrow 2c_1^2 = 1 \Rightarrow c_1 = 1/\sqrt{2}$

$\therefore$ Normalised eigen vector

$$\mathbf{X}_1 = \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

Similarly, corresponding to $\lambda = 1, 2$, we have

$\mathbf{X}_2 = \begin{bmatrix}c_2\\ 0\\ 0\end{bmatrix}$ or $(c_2, 0, 0.)$ *i.e.* $(1, 0, 0)$ in normalised form.

and

$\mathbf{X}_3 = \begin{bmatrix}0\\ c_3\\ c_3\end{bmatrix}$ or $(0, c_3, c_3)$ *i.e.* $\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ in normalised form.

Problem 45. *Determine the eigen values and eigen vectors of the matrix*

$$A = \begin{bmatrix}3 & 1 & 4\\ 0 & 2 & 0\\ 0 & 0 & 5\end{bmatrix}$$

The characteristic equation of A is $|\mathbf{A} - \lambda\mathbf{I}| = 0$ *i.e.*,

$$\begin{bmatrix}3-\lambda & 1 & 4\\ 0 & 2-\lambda & 0\\ 0 & 0 & 5-\lambda\end{bmatrix} = 0 \ \textit{i.e.},\ (\lambda-2)(\lambda-3)(\lambda-5)=0 \quad \therefore\ \lambda_1 = 2,\ \lambda_2 = 3,\ \lambda_3 = 5$$

which are the eigen values of A.

To determine eigen vectors let us consider the eigen values one by one.

(*i*) When $\lambda_1 = 2$ the eigen vector X_1 is given by $(A - 2I) X_1 = O$

i.e.,
$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The rank of coefficient matrix being 2, the equation will have only 3 – 2 *i.e.*, 1 linearly independent solution.

These are equivalent to
$$x_1 + x_2 + 4x_3 = 0$$
$$6x_3 = 0$$
$$3x_3 = 0$$

The last two give $x_3 = 0$ and then first one gives $x_1 + x_2 = 0$.

Take $x_1 = 1$, then $x_2 = -1$ and $x_3 = 0$.

Hence $X_1 = C_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, C_1 being a scalar

(*ii*) When $\lambda_2 = 3$, the eigen vector X_2 is given by $(A - 3I) X_2 = O$

i.e.
$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which are equivalent to
$$x_2 + 4x_3 = 0$$
$$-x_2 + 6x_3 = 0$$
$$2x_3 = 0$$

giving $x_3 = 0, x_2 = 0$ and x_1 is arbitrary say $x_1 = 1$, then

$X_2 = C_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, C_2 being a scalar

(*iii*) When $\lambda_3 = 5$, the eigen vector X_3 is given by $(A - 5I) X_3 = O$

i.e.,
$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

which are equivalent to
$$-2x_1 + x_2 + 4x_3 = 0$$
$$-3x_2 + 6x_3 = 0$$

giving $x_2 = 2x_3 = \frac{2}{3}x_1$ *i.e.* $2x_1 = 3x_2 = 6x$

Take $x_3 = 1$, so that $x_2 = 2$ and $x_1 = 3$

Hence $X_3 = C_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, C_3 being a scalar.

Problem 46. *Show that the eigen values of a diagonal matrix are given by its diagonal elements.*

Let $D = \text{diag}\ [a_{11}, a_{22}, \ldots a_{nn}] = \begin{bmatrix} a_{11} & 0 & \ldots & 0 \\ 0 & a_{22} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & a_{nn} \end{bmatrix}$

Its characteristic equation is

$$|D - \lambda I| = 0 \ \ i.e. \ \begin{vmatrix} a_{11}-\lambda & 0 & \ldots & 0 \\ 0 & a_{22}-\lambda & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & a_{nn}-\lambda \end{vmatrix} = 0$$

i.e. $\quad (a_{11} - \lambda)(a_{22} - \lambda) \ldots (a_{nn} - \lambda) = 0$

giving $\lambda = a_{11}, a_{22}, \ldots a_{nn}$ which are eigen values of **D** and these are clearly the diagonal elements of **D**.

Problem 47. *Prove that any square matrix* **A** *and its transpose* **A'** *have the same eigen values.*

If $A = [a_{ij}]_{m \times m}$ then $A' = [a_{ji}]_{m \times m}$

The characteristic equations of A and A′ are

$$|A - I\lambda| = 0 \text{ and } |A' - I\lambda| = 0$$

i.e.
$$\begin{vmatrix} a_{11}-\lambda & a_{12} \ldots\ldots a_{1m} \\ a_{21} & a_{22}-\lambda \ldots\ldots a_{2m} \\ a_{m1} & a_{m2} \ldots\ldots a_{mm}-\lambda \end{vmatrix} = 0 \text{ and } \begin{vmatrix} a_{11}-\lambda & a_{21} \ldots\ldots a_{m1} \\ a_{12} & a_{22}-\lambda \ldots\ldots a_{m2} \\ \ldots & \ldots \\ a_{1m} & a_{2m} \ldots\ldots a_{mm}-\lambda \end{vmatrix} = 0$$

both the determinants have the same values as interchange of rows and columns does not alter the value of the determinant.

$$\therefore \quad |A - \lambda I| = |A' - \lambda I| = 0$$

which follows that characteristic equations of **A** and **A′** are the same and hence they have the same eigen values.

Problem 48. *If* **A** *be a non-singular matrix then prove that the eigen values of* A^{-1} *are the reciprocals of the eigen values of* **A**.

If λ be an eigen value of A and **X** the corresponding eigen vector, then we have

$$AX = \lambda X \qquad \ldots (1)$$

Pre-multiplying both sides by A^{-1} and using $A^{-1} A = I$ and $IX = X$, we get

$X = A^{-1}(\lambda X) = \lambda (A^{-1} X)$ or $A^{-1} X = \frac{1}{\lambda} X$ which in view of (1) follows that $\frac{1}{\lambda}$ is an eigen value of A^{-1} and **X** is the corresponding eigen vector.

Conversely if k is an eigen value of A^{-1}, then A being non singular gives its inverse A^{-1} to be non singular and $(A^{-1})^{-1} = A$, so that $\frac{1}{k}$ is an eigen value of A.

As such each eigen value of A^{-1} is equal to the reciprocal of some eigen value of A.

Problem 49. *If* A *is a square matrix of order n and has* $\lambda_1, \lambda_2, \ldots \lambda_n$ *as its eigen values then show that eigen values of k* A *will be kA will be* $k\lambda_1, k\lambda_2, \ldots k\lambda_n$, *k being a non-zero scalar.*

If $A = [a_{ij}]_{n \times n}$ then its characteristic equation is $|A - \lambda I| = 0$

i.e. $(\lambda-\lambda_1)(\lambda-\lambda_2)\ldots(\lambda-\lambda_n)=0$ ...(1)

The characteristic equation of kA is $|kA-\lambda I|=0$

i.e.
$$\begin{vmatrix} ka_{11}-\lambda & ka_{12} & \ldots & ka_{1n} \\ ka_{21} & ka_{22}-\lambda & \ldots & ka_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ ka_{n1} & ka_{n2} & \ldots & ka_{nn}-\lambda \end{vmatrix}=0$$

or
$$\begin{vmatrix} a_{11}-\frac{\lambda}{k}, & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22}-\frac{\lambda}{k} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn}-\frac{\lambda}{k} \end{vmatrix}=0$$

or $\left|A-\frac{\lambda}{k}I\right|=0$

i.e. $\left(\frac{\lambda}{k}-\lambda_1\right)\left(\frac{\lambda}{k}-\lambda_2\right)\ldots\left(\frac{\lambda}{k}-\lambda_n\right)=0$ by (1) on putting $\frac{\lambda}{k}$ for λ.

Hence the eigen values of kA are $k\lambda_1, k\lambda_2, \ldots k\lambda_n$.

Problem 50. *If eigen values of* A *are* $\lambda_1, \lambda_2, \ldots \lambda_n$ *then prove that those of* A^2 *will be* $\lambda_1^2, \lambda_2^2, \ldots \lambda_n^2$.

If X be the eigen vector of A corresponding to a eigen value λ of A then

$$AX = \lambda X \quad \ldots(1)$$

or $A(AX) = A(\lambda X)$

or $A^2X = \lambda(AX) = \lambda(\lambda X)$ by (1)

or $A^2X = \lambda^2 X$

which follows that λ^2 is the eigen value of A^2 corresponding to the eigen vector X.

Hence if $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigen values of A, then $\lambda_1^2, \lambda_2^2, \ldots \lambda^2_n$ are the eigen values of A^2.

Problem 51. *If* A *and* B *are two square matrices, then show that* AB *and* BA *have the same eigen values. Also show that* $A^{-1}B$ *and* BA^{-1} *have the same eigen values when* $|A| \neq 0$.

We have $BA = (A^{-1}A)BA = A^{-1}(AB)A$

So that BA and AB are similar and hence have same eigen values since similar matrices have the same eigen values.

Similarly $BA^{-1} = (AA^{-1})BA^{-1} = A(A^{-1}B)A^{-1}$

i.e. BA^{-1} and $A^{-1}B$ are similar and hence etc.

Problem 52. *Show that the eigen roots of* A^Θ *are the conjugate of the eigen roots of* A.

The characteristic equations of A and A^Θ are as given

$$|A-\lambda I|=0 \quad \text{and} \quad |A^\Theta-\bar{\lambda}I|=0$$

But $|A^\Theta-\bar{\lambda}I| = |(A-\lambda I)^\Theta| = |\overline{A-\lambda I}|$

$\because |A^\Theta| = |\overline{A'}| = |\overline{A}|$

∴ $|A^{\Theta} - \bar{\lambda}I| = 0$ iff $|\overline{A - \lambda I}| = 0$

or $|A^{\Theta} - \bar{\lambda}I| = 0$ or iff $|A - \lambda I| = 0$ since if z be a complex number then $z = 0$ iff $\bar{z} = 0$.

Thus $\bar{\lambda}$ is an eigen value of A^{Θ} iff λ is an eigen value of A.

Problem 53. *Show that the matrix* $A = \begin{vmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix}$ *is derogatory.*

The characteristic equation of A is $|A - \lambda I| = 0$

i.e., $\begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & -1-\lambda & 0 \\ 1 & 0 & -1-\lambda \end{vmatrix} = 0$ or $(\lambda - 1)(\lambda + 1)^2 = 0$

giving $\lambda = 1, -1, -1$, so that by thorem 9, the factors $x-1$ and $x+1$ both must be divisor of the minimal polynomial of A.

We have to see whether $h(x) = (x - 1)(x + 1) = x^2 - 1$ annihilates A or not.

Consider $A^2 = A \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$.

∴ $A^2 - I = 0$

Which shows that $h(x)$ annihilates A and as such $g(x)$ is the minimal polynomial of A. Also degree of $h(x)$ is less than 3 and hence A is derogatory.

Problem 54. *If* X *be an eigen vector of* $B = P^{-1}AP$ *corresponding to an eigen value* λ, *then show that* Y = PX is an eigen vector of A *corresponding to the same eigen value.*

We have $BX = \lambda X$

or $PBX = P\lambda X$ or $P(P^{-1}AP)X = P\lambda X$

or $A(PX) = \lambda(PX)$ ∵ $PP^{-1} = I$ and $IA = A$ etc.

which follows that PX is an eigen vector of A corresponding to the eigen value λ.

Since eigen vectors are often called as the *invariant eigen vectors.*

Diagonalization of matrices. Let $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$ be n distinct eigen values of a matrix A and $X_1, X_2, X_3 \ldots X_n$ be the n corresponding eigen vectors. Also let X_i be the column vector given by

$$X_i = \begin{bmatrix} X_{1i} \\ X_{2i} \\ \cdots \\ X_{ni} \end{bmatrix} \quad \ldots (1)$$

Consider a matrix E whose column vectors are the n eigen vectors such that ... (2)

$$E = \begin{bmatrix} X_{11} & X_{12} \ldots\ldots X_{1n} \\ X_{21} & X_{22} \ldots\ldots X_{2n} \\ \cdots & \cdots \\ X_{n1} & X_{n1} \ldots\ldots X_{nn} \end{bmatrix} = [X_{ij}] \text{ (say)}$$

Suppose that D is a diagonal matrix such that

$$D = \begin{bmatrix} \lambda_1 & 0......0 \\ 0 & \lambda_2......0 \\ \cdots & \cdots \\ 0 & 0......\lambda_n \end{bmatrix} = \text{diag}\,[\lambda_1, \lambda_2 ... \lambda_n] \qquad ...(3)$$

Then $$ED = \begin{bmatrix} \lambda_1 X_{11} & \lambda_2 X_{12}......\lambda_n X_{1n} \\ \lambda_1 X_{21} & \lambda_2 X_{22}......\lambda_n X_{2n} \\ \cdots & \cdots \\ \lambda_1 X_{n1} & \lambda_2 X_{n2}......\lambda_n X_{nn} \end{bmatrix} = [\lambda_j X_{ij}] \qquad ...(4)$$

(no summation over j)

$= (AX_1, AX_2, ... AX_n)$ (expressing matrix as vectors)

$= A\,(X_1, X_2, ... X_n)$

$= AE$ ...(5)

If E be a non-singular matrix, then premultiplying by E^{-1}, we get

$$E^{-1} AE = D.$$

Thus premultiplying A by E^{-1} and post-multiplying by E, we get diagonal matrix whose diagonal elements are the eigen values. *This process is called the diagonalization of the matrix* **A**.

Power of a Matrix. An integral power m of a matrix A is defined as

$$A^m = AAA \text{ ...... } m \text{ times} \qquad ...(6)$$

If A is non-singular then A^{-1} exists s.t.

$$AA^{-1} = I = A^{-1} A$$

As such a negative integral power, say $m = -n$ is defined as

$$A^{-n} = A^m = (A^{-1})^n = A^{-1} A^{-1} A^{-1} \text{ ... } m \text{ times}$$

and for $m = 0$, $A^0 = I$.

If E be a diagonalising matrix for A so that

$E^{-1} AE = D$ and $A = EDE^{-1}$, ...(7)

where D is a diagonal matrix whose diagonal elements are the eigen values of A. Thus we can write in functional form as

$$f(A) = E f(D) E^{-1} \qquad ...(8)$$

Now, (7) ⇒

$A^m = (EDE^{-1})(EDE^{-1}) \text{ ... } m \text{ times}$

$= ED^m E^{-1}$ & so $D^m = E^{-1} A^m E$ ...(9)

For $m = -n$

$A^n = A^{-m} = E\,(D^{-1})^m E^{-1}$ ...(10)

A few properties of trace in relation to eigen values of a matrix:

(i) tr AB = tr BA, (ii) tr ABCD = tr BCDA = tr CDAB,

(iii) tr (A + B) = tr A + tr B, (iv) tr (AB − BA) = tr [A, B] = 0.

ILLUSTRATIVE EXAMPLE

If $A = \begin{bmatrix} \frac{4}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{5}{3} \end{bmatrix}$, then characteristic equation

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} \frac{4}{3} - \lambda & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{5}{3} - \lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 3\lambda + 2 = 0$$

$$\Rightarrow (\lambda - 1)(\lambda - 2) = 0$$

$$\Rightarrow \lambda = 1, 2 \text{ as eigen values}$$

$\therefore$ Diagonal matrix $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

By usual method, eigen vectors of A are

$(\sqrt{2}, -1)$ and $(1, \sqrt{2})$.

$\therefore$ $E = \begin{bmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{bmatrix}$, so that $E^{-1}AE = D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ with $|A| \neq 0$

For $m = 50$ (say),

$$(9) \Rightarrow A^{50} = ED^{50}E^{-1} = \begin{bmatrix} \sqrt{2} & -1 \\ -1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2^{50} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{\sqrt{2}}{3} \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 2^{50}+2 & \left(2^{50}-1\right)\sqrt{2} \\ \left(2^{50}-1\right)\sqrt{2} & 2^{51}+1 \end{bmatrix}$$

Matrix as exponent power. Analogus to

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{\lfloor m}, \text{ we have } e^A = \sum_{m=0}^{\infty} \frac{A^m}{\lfloor m} = I + A + \frac{A^2}{\lfloor 2} + \dots \quad \dots (11)$$

For diagonal matrix D with elements $D_{ij} = \lambda_i \delta_{ij}$, we have

$$e^D = \sum_{m=d}^{\infty} \frac{D^m}{\lfloor m} \quad \dots (12)$$

with (i, j)th element s.t.

$$\left[e^D\right]_{ij} = \sum_{m=0}^{\infty} \frac{\left[D^m\right]_{ij}}{\lfloor m} = \sum_{m=1}^{\infty} \frac{(\lambda_i)^m \delta_{ij}}{\lfloor m}$$

As such

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \Rightarrow e^D = \begin{bmatrix} e^{\lambda 1} & 0 & \dots & 0 \\ 0 & e^{\lambda 2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & e^{\lambda n} \end{bmatrix} \quad \dots (13)$$

In view of (9) *i.e.* $D^m = E^{-1} A^m E$, (11) yields

$$E^{-1} e^A E = E^{-1} IE + E^{-1} AE + E^{-1} A^2 E + \dots$$
$$= I + D + D^2, + \dots = e^D \quad \dots (14)$$

So that

$$E e^D E^{-1} = e^A \quad \dots (15)$$

As such, on using $a^x = e^{x \log_e a}$, we have

$$a^A = e^{A \log_e a} = E\, e^{E(D \log_e a)E^{-1}} \quad \dots (16)$$

with

$$e^{D \log_e a} = a^D = \begin{bmatrix} a^{\lambda 1} & 0 & \dots & 0 \\ 0 & a^{\lambda 2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a^{\lambda n} \end{bmatrix} \quad \dots (17)$$

ILLUSTRATIVE EXAMPLES

If $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, then eigen values are 1, 2 *i.e.* say $\lambda_1 = 1, \lambda_2 = 2$

.Now, A being a diagonal matrix, we can write (13) as

$$e^A = \begin{bmatrix} e^{\lambda 1} & 0 \\ 0 & e^{\lambda 2} \end{bmatrix} = \begin{bmatrix} e^1 & 0 \\ 0 & e^2 \end{bmatrix} = \begin{bmatrix} e & 0 \\ 0 & e^2 \end{bmatrix}$$

Alternatively the eigen vectors corresponding to $A_1 = 1$ and $\lambda_2 = 2$ are respectively (1, 0) and (0, 1), so that

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } E^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then using (15),

$$e^A = E\, e^D\, E^{-1}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e & 0 \\ 0 & e^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e & 0 \\ 0 & e^2 \end{bmatrix}$$

A method of diagonalization in practice. Writing the characteristic equation for given matrix, the chracteristic roots can be determined. If A be an n-square matrix and its characteristic roots are $\lambda_1, \lambda_2, \ldots \lambda_n$, then the diagonal matrix is

$$A = \begin{bmatrix} \lambda_1 & 0 & 0\ldots\ldots 0 \\ 0 & \lambda_2 & 0\ldots\ldots 0 \\ \ldots & \ldots & \ldots\ldots \\ 0 & 0 & 0\ldots\ldots\lambda_n \end{bmatrix}$$

i.e., if $A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

then $|A - \lambda I| = \begin{bmatrix} \cos\theta - \lambda & -\sin\theta & 0 \\ \sin\theta & \cos\theta - \lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} = 0$

i.e., $(1-\lambda)\,(1-2\lambda\cos\theta + \lambda^2) = 0$

Characteristic roots are 1, $\dfrac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$

i.e., $\quad 1, \cos\theta \pm i\sin\theta$

i.e., $\quad 1, e^{\pm i\theta}$

$\therefore \quad \lambda_1 = e^{i\theta}, \lambda_2 = e^{-i\theta}, \lambda_3 = 1$ (say)

Hence the diagonal matrix is $\begin{bmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{-i\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Note. Besides this method we can diagonalize a square matrix

(i) by orthogonally similar matrices

(*ii*) by unitarily similar matrices

but the above method is rather convenient in practice.

THEOREM. The necessary and sufficient condition for an n-rowed matrix **A** *to be similar to a diagonal matrix is that the set of eigen vectors of* **A** *includes a set of n linearly independent vectors.*

To prove that the condition is necessary. There exists non-singular matrix **E**, such that

$$E^{-1} AE = D = \text{diag.}\ (\lambda_1, \lambda_2, \ldots \lambda_n).$$

Pre-multiplying by **E**, we get

$$AE = ED, \text{ since } EE^{-1} = I.$$

Suppose that $\quad E = [X_1\ X_2 \ldots\ldots X_n].$

$\therefore\ A[X_1\ X_2 \ldots\ldots X_n] = [X_1\ X_2 \ldots\ldots X_n]\ \text{diag.}\ (\lambda_1, \lambda_2, \ldots\ldots \lambda_n)$

or $\quad [AX_1\ AX_2 \ldots AX_n] = [\lambda_1 X_1\ \lambda_2 X_2 \ldots \lambda_n X_n].$

This is equivalent to

$$AX_1 = \lambda_1 X_1,\ AX_2 = \lambda_2 X_2, \ldots, AX_n = \lambda_n X_n.$$

so that $X_1, X_2, \ldots X_n$ are n eigen vectors of **A** whose corresponding eigen values are λ_1, $\lambda_2, \ldots\ldots, \lambda_n$. As these vectors constitute the column of a non-singular matrix, there exists a linearly independent set of n eigen vectors.

To prove that the condition is sufficient. Let $X_1, X_2, X_3, \ldots, X_n$ be a linearly independent set of n eigen vectors of the matrix **A** and let $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ be the corresponding eigen values. Then

$$AX_1 = \lambda_1 X_1, AX_2 = \lambda_2 X_2, \ldots, AX_n = \lambda_n X_n \quad \ldots (1)$$

If we write $E = [X_1, X_2, \ldots X_n]$, the system (1) is then equivalent to

$$[AX_1\ AX_2 \ldots AX_n] = [\lambda_1 X_1, \lambda_2 X_2 \ldots \lambda_n X_n]$$

or $\quad A\ [X_1\ X_2 \ldots X_n] = [X_1\ X_2 \ldots X_n]\ \text{diag.}\ (\lambda_1, \lambda_2 \ldots, \lambda_n)$

i.e., $\quad AE = ED.$

But the n-rowed matrix **E** is non-singular for its columns form a linearly independent system. Thus E^{-1} exists and hence we have

$$E^{-1} AE = D.$$

which follows that **A** is similar to a diagonal matrix.

Problem 55. *Show that the diagonalized matrix of a real symmetric matrix is orthogonal.*

If **A** be a real symmetric matrix and **D** its diagonalized matrix, then $D^{-1} AD = \text{diag}\ [\lambda_1, \lambda_2 \ldots \lambda_n]$, $\quad \ldots (1)$

where $\lambda_1, \lambda_2, \ldots \lambda_n$ are eigen values of **A**?

$\therefore\ (D^{-1} AD)' = (\text{diag}\ [\lambda_1, \lambda_2, \ldots \lambda_n])'$

or $\quad D' A' (D^{-1})' = \text{diag}\ [\lambda_1, \lambda_2 \ldots \lambda_n] = D^{-1} AD$

which follows that $D' = D^{-1}$ *i.e.*, **D** is an orthogonal matrix.

Problem 56. *Show that diagonalized matrix of a Hermitian matrix is a unitary matrix.*

If **A** be a Hermitian matrix and **D** its diagonalized matrix, then

$D^{-1} AD = \text{diag}\ [\lambda_1, \lambda_2, \ldots \lambda_n]$ and $A^{\Theta} = A \quad \ldots (1)$

where $\lambda_1, \lambda_2 \ldots \lambda_n$ are the eigen values of **A**.

$\therefore\ (D^{-1} AD)^{\Theta} = (\text{diag}\ [\lambda_1, \lambda_2, \ldots \lambda_n])^{\Theta}$

or $D^{\Theta} A^{\Theta} (D^{-1})^{\Theta} = \text{diag}\,[\lambda_1, \lambda_2, \ldots \lambda_n]$

or $D^{\Theta} A (D^{-1})^{\Theta} = D^{-1} AD$ by (1)

which follows that $D^{\Theta} = D^{-1}$ or $DD^{\Theta} = DD^{-1} = I$ *i.e.*, D is unitary.

Problem 57. *If a n-rowed square matrix* A *has n-linearly independent invariant vectors then it is similar to a diagonal matrix.*

If $X_1, X_2, \ldots X_n$ be linearly independent invariant eigen vectors of A corresponding to eigen values $\lambda_1, \lambda_2, \ldots \lambda_n$, then $AX_i = \lambda_i X_i$ for every *i*. ... (1)

Consider a square matrix B with its columns as vectors X_i such that

$$B = [X_1, X_2, \ldots X_n]$$

$$\therefore \quad AB = A\,[X_1, X_2, \ldots X_n] = [AX_1, AX_2 \ldots AX_n]$$

$$= [\lambda_1 X_1, \lambda_2 X_2 \ldots \lambda_n X_n] \text{ by (1)}$$

$$= [X_1, X_2, \ldots X_n] \begin{bmatrix} \lambda_1 & 0 \ldots\ldots 0 \\ 0 & \lambda_2 \ldots\ldots 0 \\ \ldots & \ldots \\ 0 & 0 \ldots\ldots \lambda_n \end{bmatrix}$$

$$= B\,\text{diag}\,[\lambda_1, \lambda_2, \ldots \lambda_n]$$

$\therefore\ B^{-1} AB = \text{diag}\,[\lambda_1, \lambda_2, \ldots \lambda_n]$

which follows that A is similar to diagonal matrix.

2.19. QUADRATIC FORMS AND THEIR REDUCTIONS

Quadratic Forms. A homogeneous polynomial of the second degree in any number of variables is said to be a quadratic form, *e.g.*, $ax^2 + 2hxy + by^2$, $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ are respectively the quadratic forms in 2 and 3 variables.

The general quadratic form is $\sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij}\, x_i\, x_j$ in *n* variables $x_1, x_2 \ldots x_n$ where $i = 1, 2, 3 \ldots n$ and $j = 1, 2, 3 \ldots n$ but $i \neq j$.

The coefficient of each product term is $a_{ij} + a_{ji}$, for it arises as x_{ij} as well as x_{ji}. In case $i = j$, the coefficients of square terms, *i.e.*, $x_1^2, x_2^2, \ldots x_n^2$ are $a_{11}, a_{22}, \ldots a_{nn}$.

If we now define another set of scalars, such that

$$b_{ij} = a_{ij} \text{ and } b_{ij} = b_{ji} = \frac{1}{2}(a_{ij} + b_{ij}) \text{ for } i \neq j.$$

then $$\sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij}\, x_i\, x_j = \sum_{j=1}^{n} \sum_{i=1}^{n} b_{ij}\, x_i\, x_j,$$

which follows that every quadratic form can be so adjusted that the matrix of its coefficients is symmetric.

Quadratic form as a product of matrices. If we have a quadratic form

$\sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij}\, x_i\, x_j =$ where $a_{ij} = a_{ji}$ then $\sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij}\, x_i\, x_j = X'AX$,

where A is the symmetric matrix of the given quadratic form.

We have $$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$$

Let $\quad X = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$; then $X' = [x_1, \; x_2, \ldots x_n]$.

Now $\quad \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} \; x_i \; x_j = \sum_{i=1}^{n} \left(a_{i1} \; x_1 + a_{i2} \; x_2 + \ldots + a_{in} x_n\right) \; x_i.$

From which it is obvious that the form appears as a matrix obtained as the product of the pre-factor row matrix with components

$$a_{11} \, x_1 + a_{12} \, x_2 + \ldots + a_{1n} \, x_n,$$
$$a_{21} \, x_1 + a_{22} \, x_2 + \ldots + a_{2n} \, x_n.$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{n1} \, x_1 + a_{n2} \, x_2 + \ldots + a_{nn} \, x_n$$

and the post-factor column matrix with components

$$x_1, x_2 \ldots x_n.$$

Also it is apparent that the pre-factor row matrix is the product of the pre-fector row matrix with components $x_1, x_2 \ldots x_n$ and the post-factor matrix A. Hence

$$\sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} \; x_i \; x_j = X'AX.$$

For example,

$$ax^2 + 2hxy + by^2 = [xy] \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Congruence of quadratic forms and matrices. An n-rowed square matrix B is said to be congruent to another n-rowed square matrix A, if there exists a non-singular matrix P, such that

$$B = P'AP.$$

Properties of congruence of matrices

(*i*) **Reflexivity.** *Every matrix* A *is congruent to itself as*

$$A = IAI = I'AI.$$

(*ii*) **Symmetry.** *If* A *is congruent to* B, *then* B *is also congruent to* A. If A = P' BP, then

$$B = (P')^{-1} AP^{-1} = (P^{-1})' AP^{-1}.$$

(*iii*) **Transitivity.** *If* A *is congruent to* B *and* B *to* C *then* A *is also congruent to* C.

If A = P'BP, B = Q'CQ, then

$$A = P'Q'CQP = (QP)'C\,(QP).$$

(*iv*) *The ranges of values of two congruent quadratic forms are the same.*

If the two congruent quadratic forms be X'AX and Y' BY, then there exists a non-singular matrix P, such that

$$B = P'AP.$$

To show that the sets of vlaues assumed by the two forms are the same when the vectors X and Y range over a field, let us write X = PY so that $Y = P^{-1} X$.

Suppose that $X_1'AX_1$ is a value of the first form for any vector X_1 and that $P^{-1}X_1 = Y_1$; then

$$Y_1'BY_1 = (P^{-1}X_1)'P'AP(P^{-1}X_1)$$
$$= X_1'(P')^{-1}P'APP^{-1}X_1$$
$$= X_1'AX_1, \text{ since } PP^{-1} = I \text{ etc.}$$

Congruent transformations. A pair of elementary transformations, one row and the other column, such that each of the corresponding elementary matrices is the transpose of the other, may be called an elementary congruent transformation. It may be of the following types:

(*i*) Interchange of the *i*th and *j*th rows as well as of the *i*th and *j*th columns as $E_{ij} A E_{ij}$ for $E'_{ij} = E_{ij}$.

(*ii*) Product of the *i*th row as well as the *i*th column by a non-zero scalar *c* as $E_i(c) A E_i(c)$ for $E'_i(c) = E_i(c)$.

(*iii*) Addition of *k* times the *j*th row to the *i*th row and also the addition of *k* times of the *j*th column to the *i*th column as $F_{ij}(k) A F_{ij}(k)$ for $F'_{ij}(k) = F_{ij}(k)$.

EXAMPLE. Reduce the following symmetric matrix to a diagonal form and interpret the result in terms of quadratic forms

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 2 & 3 \\ -1 & 3 & 1 \end{bmatrix}.$$

Let us write $A = I'AI$.

i.e.,
$$\begin{bmatrix} 3 & 2 & -1 \\ 2 & 2 & 3 \\ -1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In order to perform the elementary congruent transformations the prefector and the postfactor of A by the row and column parts can be subjected in two steps:

(*i*) adding $\frac{1}{3}$ times the first row to the third and $\left(-\frac{2}{3}\right)$ times the first row to the second row in L.H.S. matrix as well as in the prefector of A; and then

(*ii*) adding $\frac{1}{3}$ times of the first column to the third and $\left(-\frac{2}{3}\right)$ times the first column to the second column in L.H.S. matrix as well as in the postfactor of A.

We thus have

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & \frac{2}{3} & \frac{11}{3} \\ 0 & \frac{11}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ \frac{1}{3} & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Again performing the elementary congruent transformations by adding $\left(-\frac{11}{2}\right)$ times the second row to the third row in L.H.S. matrix as well as in the pre-factor of A and then adding $\left(-\frac{11}{2}\right)$ times the second column to the third column in L.H.S. matrix as well as in post-factor of A, we have

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & -\frac{117}{6} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ 4 & -\frac{11}{2} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{2}{3} & 4 \\ 0 & 1 & -\frac{11}{2} \\ 0 & 0 & 1 \end{bmatrix} \qquad \ldots (1)$$

Thus we get the diagonal matrix

$$\mathbf{B} = \text{diag}\left(3, \frac{2}{3}, -\frac{117}{6}\right), \qquad \ldots (2)$$

which is congruent to the given symmetric matrix.

From (1), we have

$$\mathbf{P} = \begin{bmatrix} 1 & -\frac{2}{3} & 4 \\ 0 & 1 & -\frac{11}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

so that $\mathbf{X} = \mathbf{PY}$ gives

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{2}{3} & 4 \\ 0 & 1 & -\frac{11}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= \begin{bmatrix} y_1 & -\frac{2}{3} y_2 & 4 y_3 \\ 0 & y_2 & -\frac{11}{2} y_3 \\ 0 & 0 & y_3 \end{bmatrix}$$

which is equivalent to

$$\begin{aligned} x_1 &= y_1 - \frac{2}{3} y_2 + 4 y_3 \\ x_2 &= y_2 - \frac{11}{2} y_3, \\ x_3 &= y_3 \end{aligned} \qquad \ldots (3)$$

This follows that the linear transformation (3) transforms the quadratic form

$$\mathbf{X}'\mathbf{A}\mathbf{X} = [x_1 \ x_2 \ x_3] \begin{bmatrix} 3 & 2 & -1 \\ 2 & 2 & 3 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= [x_1 \ x_2 \ x_3] \begin{bmatrix} 3x_1 + 2x_2 - x_3 \\ 2x_1 + 2x_2 + 3x_3 \\ -x_1 + 3x_2 + x_3 \end{bmatrix}$$

$$= x_1 (3x_1 + 2x_2 - x_3) + x_2 (2x_1 + 2x_2 + 3x_3) + x_3 (-x_1 + 3x_2 + x_3)$$

$$= 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1 x_2 + 6x_2 x_3 - 2x_3 x_1$$

to the diagonal form

$$Y'P'APY = 3y_1^2 + \frac{2}{3}y_2^2 - \frac{117}{6}y_3^2 \text{ by (2).}$$

Note. Corresponding to every quadratic form X′AX there exists a non-singular linear transformation X = PY, such that the form treansforms to a sum of r square terms.

$$d_1 y_1^2 + d_2 y_2^2 + \dots + d_r y_r^2$$

where $d_1, d_2 \dots d_r$ are scalars in the diagonal matrix and r is the rank of the matrix **A**.

Index and signature. X′AX be any real quadratic form of rank r, then there exists a real non-singular linear transformation say X = PY which transforms X′AX to

$$Y'P'APY = y_1^2 + y_2^2 + \dots + y_s^2 - y_{s+1}^2 - y_{s+2}^2 - \dots - y_r^2,$$

i.e., a real quadratic form is expressed as a sum and difference of the squares of the new variables, then the number of positive squares *i.e.*, s is called the *index* of the form and the difference $s - (r - s) = 2s - r$ is known as the *signature* of the form (canonical).

Definite, Semi-definite and Indefinite-real quadratic forms

If X′AX be a real quadratic form in n variables of rank r and index s, then

If $r = n$, $s = n$, the form is said to be *positive definite as* $y_1^2 + \dots + y_n^2$;

if $r = n$, $s = 0$, " " *negative definite as* $-y_1^2 - \dots - y_n^2$;

if $r < n$, $s = r$, " " *positive semi-definite as* $y_1^2 + \dots + y_r^2$;

and if $r < n$, $s = 0$, " " *negative semi-definite as* $-y_1^2 - \dots - y_r^2$

In any other case the form is called *Indefinite*.

A real symmetric matrix A is said to be definite, semi-definite or indefinite according as the corresponding real quadratic form X′AX is definite, semi-definite or indefinite.

For a definite n-rowed matrix A, there exists a real non-singular matrix P such that $P'AP = I_n$ or $-I_n$ according as A is positive or negative definite.

For a semi-definite n-rowed matrix A, there exists at real non-singular matrix P such that that $P'AP = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$ or $\begin{bmatrix} -I_r & O \\ O & O \end{bmatrix}$ according as A is positive or negative definite.

EXAMPLE. Determine the following form as definite, semi-definite or indefinite.

$$2x_1^2 + 2x_2^2 + 3x_3^2 - 4x_2x_3 - 4x_3x_1 + 2x_1x_2.$$

Let A be the real symmetric matrix corresponding to the given real quadratic form, *i.e.*,

$$\begin{aligned} X'AX &= 2x_1^2 + 2x_2^2 + 3x_3^2 - 4x_2x_3 - 4x_3x_1 + 2x_1x_2 \\ &= x_1(2x_1 + x_2 - 2x_3) + x_2(x_1 + 2x_2 - 2x_3) + x_3(-2x_1 - 2x_2 + 3x_3) \\ &= [x_1 x_2 x_3] \begin{bmatrix} 2x_1 & 2x_2 & -2x_3 \\ x_1 & 2x_2 & -2x_3 \\ -2x_1 & -2x_2 & 3x_3 \end{bmatrix} \\ &= [x_1 x_2 x_3] \begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{aligned}$$

It is clear that $A = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 3 \end{bmatrix}$

We write $A = I'AI$,

i.e., $$\begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Performing the elementary congruent transformations $R_1\left(-\frac{1}{2}\right) + R_2$; $R_1 + R_3$; $C_1\left(-\frac{1}{2}\right) + C_2$, $C_1 + C_3$, we get

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{1}{2} & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Again performing the elementary congruent transformations,

$$-\frac{2}{3} R_2 + R_3; \ -\frac{2}{3} C_2 + C_3,$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{4}{3} & -\frac{2}{3} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{1}{2} & \frac{4}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}.$$

so that the linear transformation $X = PY$, *i.e.*,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{4}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

i.e.,
$$\left.\begin{aligned} x_1 &= y_1 - \frac{1}{2} y_2 + \frac{4}{3} y_3, \\ x_2 &= \qquad y_2 - \frac{2}{3} y_3 \\ x_2 &= \qquad\qquad y_3 \end{aligned}\right\}$$

transforms the given form to the diagonal form

$$2y_1^2 + \frac{3}{2} y_2^2 + \frac{4}{3} y_3^2.$$

so that the rank is 3 and also the index is 3.

Hence the form is positive definite.

2.20. DIFFERENTIATION AND INTEGRATION OF MATRICES

Limit of a matrix. The limit of a matrix A (t) (each element of matrix being taken as function of t.) as $t \to t_0$ is a matrix A (t_0) *i.e.*,

$$\lim_{t \to t_0} A(t) = A(t_0)$$

The sequence <A (t)> converges to A if each element of A (t) converges as $t \to \infty$ to the corresponding element of A.

Derivative of a matrix. The derivative of a square matrix A (t) of order n with respect to an independent variable t is denoted by

$\frac{d}{dt}$ A (t) or by A$'$ and is defined by

$$A' = \frac{d}{dt} A(t) = \begin{pmatrix} \frac{d}{dt}a_{11} & \frac{d}{dt}a_{12} \cdots\cdots & \frac{d}{dt}a_{1n} \\ \frac{d}{dt}a_{21} & \frac{d}{dt}a_{22} \cdots\cdots & \frac{d}{dt}a_{2n} \\ \cdots & \cdots & \cdots \\ \frac{d}{dt}a_{n1} & \frac{d}{dt}a_{n2} \cdots\cdots & \frac{d}{dt}a_{nn} \end{pmatrix} = \left[\frac{d}{dt} a_{ij}(t)\right]$$

provided all the elements $a_{ij}(t)$ are differentiable.

In general the nth derivative of A (t) is defined by

$$\frac{d^n A(t)}{dt^n} = \left[\frac{d^n a_{ij}(t)}{dt^n}\right]$$

It is easy to show that

$$\frac{d}{dt}[A(t)\, B(t)] = A(t)\frac{dB(t)}{dt} + \frac{dA(t)}{dt} B(t) = A(t)\, B'(t) + A'(t)\, B(t)$$

and $\frac{d}{dt}(|A|) = \sum_{i=1}^{n} |A_i(t)|$ where $A_i(t)$ is obtained from $A(t)$ by differentiating ith row of $|A(t)|$ and n is the order of A.

Integral of a matrix. If A (t) = $[a_{ij}(t)]$, then

$$\int_b^t A(t)\, dt = \left[\int_b^t a_{ij}(t)\, dt\right]$$

Power series. If there is a square matrix with all eigen values less than 1 in absolute value, then the series $a_0 I + a_1 A + a_2 A^2 + \ldots$ is convergent. The sum of the series is denoted by f(A). Following are few convergent series for every A.

$$e^A = 1 + A + \frac{1}{\lfloor 2} A^2 + \ldots\ldots$$

$$\cos A = 1 - \frac{1}{\lfloor 2} A^2 + \frac{1}{\lfloor 4} A^2 \ldots\ldots$$

$$\sin A = A - \frac{1}{\lfloor 3} A^3 + \frac{1}{\lfloor 5} A^5 \ldots\ldots$$

and $(I - A)^{-1} = I + A + A^2 + \ldots\ldots$ converges only for all eigen values of A which are less than 1 in absolute value.

Problem 58. *For a matrix* **A** *prove that if*

$$e^{tA} = I + tA + \frac{t^2}{\lfloor 2} A^2 + \frac{t^3}{\lfloor 3} A^3 + \ldots\ldots, \text{ then}$$

$$\frac{d}{dt} e^{tA} = A e^{tA}.$$

We have $\left[\frac{d}{dt} e^{tA}\right]_{ij} = \frac{d}{dt}\left[(I)_{ij} + t(A)_{ij} + \frac{t^2}{\lfloor 2}(A^2)_{ij} + \ldots\ldots\right]$

$$= \left[(A)_{ij} + t\left(A^2\right)_{ij} + \ldots\ldots\right] = \left[A + t\,A^2 + \ldots\ldots\right]_{ij}$$

$$= \left[A\left(I + tA + \frac{t^2}{\lfloor 2} A^2 + \ldots\ldots\right)\right]_{ij} = \left(A\, e^{tA}\right)_{ij}$$

Problem 59. *Show that* $e^A = e^x \begin{bmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{bmatrix}$ when $A = \begin{bmatrix} x & x \\ x & x \end{bmatrix}$

We have $e^A \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = e^x \begin{bmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = e^x \begin{bmatrix} e^x & e^x \\ e^x & e^x \end{bmatrix} = e^{2x} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Again $e^A = I + A + \frac{1}{\lfloor 2} A^2 + \frac{1}{\lfloor 3} A^3 + \ldots\ldots = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + x \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{1}{\lfloor 2} \cdot 2x^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \ldots,$

$$\therefore\ e^A \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + x \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} + \frac{1}{\lfloor 2}\, 2x^2 \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} + \ldots\ldots$$

$$= \left[1 + 2x + 4\frac{x^2}{\lfloor 2} + \ldots\ldots\right] \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = e^{2x} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Hence the proposition.

2.21. SOLUTION OF A SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS USING DIFFERENTIATION OF MATRICES

[A] Linear differential equations of first order

Let the system be

$$\left.\begin{aligned} \frac{dx_1}{dt} &= a_{11}\, x_1 + a_{12}\, x_2 + \ldots\ldots + a_{1n}\, x_n + b_1\,(t) \\ \frac{dx_2}{dt} &= a_{21}\, x_1 + a_{22}\, x_2 + \ldots\ldots + a_{2n}\, x_n + b_2\,(t) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \frac{dx_n}{dt} &= a_{n1}\, x_1 + a_{n2}\, x_2 + \ldots\ldots + a_{nn}\, x_n + b_n\,(t) \end{aligned}\right\} \qquad \ldots(1)$$

which are equivalent to $\frac{dX}{dt} = AX + B$ ...(2)

Making substitution $X(t) = PY(t)$, (2) yields ...(3)

$$\frac{d}{dt}[PY\,(t)] = P\frac{dY}{dt} = APY + B\,(t) \quad \because \frac{dP}{dt} = 0$$

But A is similar to P if $P^{-1}AP = D$, D being Diagonal matrix

$$\therefore \frac{dY}{dt} = P^{-1} A\ PY + P^{-1} B\ (t) = DY + P^{-1} B\ (t) \qquad ...(4)$$

which is equivalent to

$$\frac{dy_1}{dt} = \lambda_1 y_1 + c_1(t),\ \frac{dy_2}{dt} = \lambda_2 y_2 + c_2\ (t)......,\ \frac{dy_n}{dt} = \lambda_n y_n + c_n(t) \qquad ...(5)$$

Their solution being given by

$$\left[y_j\ (t)\ e^{-\lambda_j(t)}\right]_0^t = \int_0^t c_j\ (t) e^{-\lambda_j\ t} dt \qquad ...(6)$$

or
$$y_j\ (t) = e^{\lambda_j t}\left[y_0(t) + \int_0^t c_j(t) e^{-\lambda_j t}\ dt\right] \text{ for } j = 1, 2, 3,......n.$$

$\therefore$ **Y**(t) and hence **X**(t) can be determined.

[B] Linear differential equations of nth order

Regarding a linear differential equation of nth order as equivalent to n linear differential equations each of order one, and writing

$$\frac{d^n}{dt^n} x\ (t) + a_1 \frac{d^{n-1}}{dt^{n-1}} x\ (t) + ...+ a_n\ x\ (t) = b\ (t) \qquad ...(7)$$

we have on making substitutions $x_1\ (t) = x\ (t),\ x_2(t) = \frac{d}{dt} x\ (t),...,$

$$x_n\ (t) = \frac{d^{n-1}}{dt^{n-1}} x\ (t)$$

$$Dx_1\ (t) = x_2(t),\ Dx_2\ (t), = x_3\ (t)......,\ Dx_{n-1}\ (t) = x_n\ (t) \text{ and}$$

$$DX_n\ (t) = -a_1\ x_n\ (t) - a_2\ x_{n-1}\ (t)...- a_n x_n\ (t) + b\ (t) \text{ where } D \equiv \frac{d}{dt}$$

the system is equivalent to Matrix-equation

$$DX\ (t) = \begin{bmatrix} Dx_1\ (t) \\ Dx_2\ (t) \\ \cdots \\ Dx_n\ (t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0......0 \\ 0 & 0 & 1......0 \\ \cdots & \cdots & \cdots \\ -a_n & -a_{n-1} & -a_{n-2}...a_n \end{bmatrix} \begin{bmatrix} x_1\ (t) \\ x_2\ (t) \\ \cdots \\ x_n\ (t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdots \\ b\ (t) \end{bmatrix}$$

$$= \mathbf{A}X\ (t) + \mathbf{B}\ (t).$$

where we have used the concept of diagonalization.

ADDITIONAL MISCELLANEOUS PROBLEMS

Problem 60. *Write short note on Hermitian Matrices.* *(Vikram, 1967)*

See §2.9. [10]

Problem 61. *What is a reciprocal matrix ? Show that a non-singular matrix* **A** $= [a_{ij}]$ *always possesses a reciprocal.*

Solve by the method of matrix theory, the linear equations.

$$x_1 + 2x_3 = 1$$
$$3x_1 - x_2 + x_3 = 2$$
$$4x_2 + 5x_3 = -1$$

(Vikram, 1967)

See 2.9 [13] Theorem 1 on it for the first part.

Now, Matrix form of the given system is AX = B *i.e.*,

$$\begin{bmatrix} 1 & 0 & 2 \\ 3 & -i & 1 \\ 0 & 4 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \text{ where } A=\begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 1 \\ 0 & 4 & 5 \end{bmatrix},\ B=\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \text{ and } X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

By $R_{21(-3)}$, $\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -5 \\ 0 & 4 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

By $R_{32(1)}$, $\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -5 \\ 0 & 3 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

which are equivalent to

$$x_1 + 2x_3 = 1$$
$$-x_2 - 5x_3 = -1$$
$$3x_2 = -2$$

Giving $x_2 = -\frac{2}{3}$, $x_3 = \frac{1}{3}$ and $x_1 = \frac{1}{3}$.

Problem 62. (*a*) *Explain with examples what you understand by a unitary matrix and a Hermitian matrix.*

(*b*) *Find the eigen values and the eigen vectors of the matrix*

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix}$$ (*Agra, 1969*)

For (*a*) see §2.9 [14] and §2.9 [10]

(*b*) Denoting the given matrix by A, its characteristic equation is

$$|A-\lambda I| = 0 \text{ i.e., } \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 2-\lambda & 3 \\ 1 & 3 & 6-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^3-9\lambda^2+9\lambda-1 = 0 \quad \text{or} \quad (\lambda-1)\left(\lambda^2-8\lambda+1\right)=0$$

Giving $\lambda = 1, 4 \pm \sqrt{15}$.

New proceeding just like in Problem 45, eigen vectors can be determined.

$$\text{Ans. } \begin{bmatrix} k \\ -k \\ -k/2 \end{bmatrix} \begin{bmatrix} \frac{5+\sqrt{15}}{10+3\sqrt{15}}\, k' \\ k' \\ \frac{5+\sqrt{15}}{\sqrt{15}+1}\, k' \end{bmatrix} \text{ etc.}$$

Problem 63. *Find the characteristic equation of the matrices*

$$(a) = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix}, \quad (b) = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix}$$

and verify Cayley Hamilton equation for them. (*Vikram, 1969*)

(*a*) Denoting the given matrix by A, its characteristic equation is

$$|A-\lambda I| = 0 \text{ i.e., } \begin{vmatrix} 1-\lambda & 2 & 3 \\ 1 & -1-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} 0 = \text{ i.e., } \lambda^3-\lambda^2-15\lambda-15=0$$

and verify that $A^3 - A^2 - 15A - 15I = 0$.

(*b*) **Ans:** $\lambda^3 - \lambda^2 - 18\lambda - 30 = 0$ as in (*a*)

Problem 64. (*a*) *Define Hermitian and orthogonal matrices. Give one example of each type.*

(*b*) *Find the matrices* C *and* C^{-1} *required to reduce the matrix* $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ *to the diagonal form by the transformation* $C^{-1}AC$.

(*Agra, 1971*)

Problem 65. *Show that the six matrices*

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \frac{1}{2}\begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}, \quad C = \frac{1}{2}\begin{bmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$$

$$D = \frac{1}{2}\begin{bmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{bmatrix}, \quad F = \frac{1}{2}\begin{bmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$$

satisfy the realtions

$$A^2 = B^2 = C^2 = E$$

$$AB = D \text{ and } AC = BA = E.$$

Problem 66. *Define eigen values and eigen vectors of a matrix. Prove that the eigen values of an orthogonal matrix are unimodular.*

Obtain the eigen values and eigen vectors of the matrix $A = \begin{bmatrix} 8 & -12 & 5 \\ 15 & -25 & 11 \\ 24 & -24 & 19 \end{bmatrix}$

(*Agra, 1972*)

Problem 67. *If the Hermitian matrices* $\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & -i \\ & 0 \end{bmatrix}, \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$

satisfy the equations

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1, \sigma_1\sigma_2 = i\sigma_3, \sigma_2\sigma_3 = i\sigma_1, \sigma_3\sigma_1 = i\sigma_2$$

$$\sigma_1\sigma_2 - \sigma_2\sigma_1 = 2i\sigma_3, \sigma_1\sigma_2 + \sigma_2\sigma_1 = 0$$

then find the eigen values of the matrices $\sigma_1, \sigma_2, \sigma_3$.

Problem 68. *Define the trace of a matrix. Prove that two matrices* A *and* B *have the same trace if* $B = T^{-1}AT$ *where* T *is a non-singular matrix.*

The eigen values of a 3 × 3 matrix H *are equal to the three cube roots of unity. Prove that* $H^3 = I$, *where* I *is a unit matrix of order 3.* (*Agra, 1973*)

Problem 69. *Show that all the matrices of Problem 65 are unitary.*

Problem 70. *Find the reciprocal of each of the matrices of Problem 65 and verify that*

$$D^{-1} = B^{-1}A^{-1}$$

Problem 71. *Find the eigen values and normalised eigen vectors of the matrix*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

(*Agra, 1974*)

Ans: Eigen values are 0, 1, 2.

Eigen vectors (0, 1, – 1), (1, 0, 0), (0, 1, 1), in normalised form are

$$\left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), (1, 0, 0), \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),$$

Problem 72. [A Significant property of diagonal Matrices].

Prove that in case of diagonal Matrices, the commutative law for multiplication holds good.

If A and B be two diagonal (square) matrices of the same order, then we have to prove that

$$AB = BA$$

Since a diagonal matrix is such that its all the elements except those in the principal diagonal are zero, we can take

$$A = [a_{ij}\,\delta_{ij}] \text{ and } B = [b_{ij}\,\delta_{ij}]$$

where $\delta_{ij} = 0$ for $i \neq j$

$= 1$ for $i = j$

$$\therefore \quad [AB]_{ik} = \sum_j [A]_{ij}\,[B]_{jk} = \sum_j a_{ij}\,\delta_{ij}\,b_{jk}\,\delta_{jk}$$

$= a_{ii}\,b_{ik}\,\delta_{ik}$ for $j = i$ and no summation over i

$= b_{ii}\,a_{ii}\,\delta ik$

$= [BA]_{ii}$ and zero for $k \neq i$

which follows that AB is also a diagonal matrix and AB = BA.

Problem 73. *If* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, *find* M^{-1}, *transpose of* M, *Hermitian Conjugate of* M *and the eigen values of* M. *(Agra, 1975)*

Problem 74. (*a*) *Find eigen value and corresponding orthonormal vector of the following matrix :*

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

(*b*) *Diagonalize the following matrix :*

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

(Rohilkhand, 1976)

Problem 75. *Show that eigen values of a Hermitian matrix are all real and its eigen vectors corresponding to two distinct eigen values are orthogonal.*

What do you mean by diagonalization of a matrix ? Show that the necessary and sufficient condition for the reduction of two matrices to the diagonal form by the same transformation is that they commute. *(Rohilkhand, 1977)*

See theorem 1 and 5 of § 2.18.

See § 2.18 after Problem 54 and also see Problem 72.

Problem 76. *If* H *is a Hermitian matrix, what kind of matrix is* e^{iH} ?

(Rohilkhand, 1992; Meerut, 1969)

Hint: H is Hermitian $\Rightarrow H^{\Theta} = H$

consider $(e^{iH})^{\Theta} \cdot e^{iH} = e^{-i}\,H^{\Theta} \cdot e^{iH} = e^{-iH}\,e^{iH} = e^{\circ} = I \Rightarrow e^{iH}$ is unitary.

Problem 77. *Show that product of two orthogonal matrices is also orthogonal.*

Hint : A, B are orthogonal $\Rightarrow AA' = I = A'A$ & $BB' = I = B'B$.

$\therefore (\mathbf{AB})'(\mathbf{AB}) = \mathbf{B'A'AB} = \mathbf{B'IB} = \mathbf{B'B} = \mathbf{I} \Rightarrow \mathbf{AB}$ is orthogonal.

Problem 78. *Show that the matrix* $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *is orthogonal.*

Problem 79. *Show that the product of two unitary matrices is also unitary.*
Proceed just as in prob. 77.

Problem 80. *Find the inverse of* $M = \begin{bmatrix} 2-i & -i \\ -2i & 1+i \end{bmatrix}$ (*R.U. 1982*)

$$\left[\textbf{Ans.} \frac{1}{2}\begin{pmatrix} 2 & -1-i \\ 2+2i & 1-3i \end{pmatrix}\right]$$

CHAPTER 3

TENSORS

3.1. INTRODUCTION

It is an established fact that a natural law gives a relation between different physical entities and its mathematical formulation is nothing but a relation between the sets of numbers representing those entities. Tensor Analysis forms that part of study which is rather suitable for the mathematical formulation of natural laws in forms which are invariant with respect to underlying frames of reference. In brief tensors are quantities obeying certain transformation laws. In wider sense a tensor formulation is very compact and good deal of clarity in its use. The tensor formulation was originated by G. Ricci and it became rather popular when Albert Einstein used it as a natural tool for the description of his general theory of relativity.

Actually tensor analysis is the generalization of vector analysis as is evident by considering a vector function $f(\mathbf{r})$ of a vector $\mathbf{r}$. This vector function is continuous at $\mathbf{r} = \mathbf{r}_0$ if,

$$\lim_{\mathbf{r}\to\mathbf{r}_0} f(\mathbf{r}) = f(\mathbf{r}_0)$$

and it is linear, if

$$f(\mathbf{r}+\mathbf{s}) = f(\mathbf{r}) + f(\mathbf{s}) \qquad \dots (1)$$

and

$$f(\lambda \mathbf{r}) = \lambda f(\mathbf{r}) \qquad \dots (2)$$

for arbitrary values of $\mathbf{r}, \mathbf{s}, \lambda$.

Now we know that as linear vector function $f(\mathbf{r})$ is completely defined only if $f(\mathbf{a}_1)$, $f(\mathbf{a}_2)$ and $f(\mathbf{a}_3)$ are given for any three non-coplanar vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$. In terms of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ as basis if we assume that

$$\mathbf{r} = x_1\,\mathbf{a}_1 + x_2\,\mathbf{a}_2 + x_3\,\mathbf{a}_3 \qquad \dots (3)$$

then we have from (1) and (2),

$$f(\mathbf{r}) = x_1 f(\mathbf{a}_1) + x_2 f(\mathbf{a}_2) + x_3 f(\mathbf{a}_3) \qquad \dots (4)$$

As such equation (3) yields

$$x_\alpha = \mathbf{r}\,.\,\mathbf{a}_\alpha\,,\ \alpha = 1, 2, 3 \qquad \dots (5)$$

Let us put $\quad f(\mathbf{a}_\alpha) = \mathbf{b}_\alpha$.

So that $\quad f(\mathbf{r}) = (\mathbf{b}_1\mathbf{a}_1 + \mathbf{b}_2\mathbf{a}_2 + \mathbf{b}_3\mathbf{a}_3).\mathbf{r}$

$$= \phi.\mathbf{r} \text{ (say)} \qquad \dots (6)$$

where the operator $\phi = \mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_2\mathbf{b}_2 + \mathbf{a}_3\mathbf{b}_3$ consists of nine components in three dimensional coordinate geometry and hence it is neither scalar nor vector quantity but is a new mathematical symbol called as the *dyadic*.

Suppose that there are two vectors **u** and **v** such that components of vector **v** are linear functions of the components of vector **u** defined as

$$\left.\begin{aligned} v_x &= a_{xx}\, u_x + a_{xy}\, u_y + a_{xz}\, u_z \\ v_y &= a_{yx}\, u_x + a_{yy}\, u_y + a_{yz}\, u_z \\ v_z &= a_{zx}\, u_x + a_{zy}\, u_y + a_{zz}\, u_z \end{aligned}\right\} \qquad \ldots (7)$$

In this way the vector **v** is placed in one-to-one correspondence with the vector **u**. The scheme of coefficients $a_{\alpha\beta}$ has thus an independent meaning if the correspondence is such that the passage from **u** to **v** is independent of the particular coordinate system in which the vectors are resolved into components. We call the coefficients $a_{\alpha\beta}$ in this case as the coefficients of a tensor.

It is observed that the nine components as mentioned above characterise the transformation of the components of one vector into those of other. The coefficients $a_{\alpha\beta}$ in general transform u_β into one of three parts of v_α.

The equations (7) are equivalent to a single vector equation,

$$\mathbf{v} = \phi \mathbf{u} \qquad \ldots (8)$$

where the operator ϕ turns **u** into **v**. It is rather graphically known as **Tensor.**

The essential part of a tensor operation is the array of coefficients like $a_{\alpha\beta}$, written in the form of a matrix such as

$$\phi = \begin{bmatrix} a_{xx} & a_{xy} & a_{xz} \\ a_{yx} & a_{yy} & a_{yz} \\ a_{zx} & a_{zy} & a_{zz} \end{bmatrix} \qquad \ldots (9)$$

As such the *dyadic* operator turns a vector **r** into the vector function $f(\mathbf{r})$ and is expressed as the sum of dyads **ab** *i.e.*

$$\phi = \sum_{\alpha} \mathbf{a}_\alpha\ \mathbf{b}_\alpha$$

In a similar way a *triadic* is expressed as the sum of the triads $\sum_{\alpha} \mathbf{a}_\alpha\ \mathbf{b}_\alpha\ \mathbf{c}_\alpha$.

Considering it as an operator that converts vector **r** into the the dyadic ϕ, we may write

$$\phi \cdot \mathbf{r} = \Sigma\, (\mathbf{a}_\alpha\ \mathbf{b}_\alpha\ \mathbf{c}_\alpha) \cdot \mathbf{r}$$

Similarly a *tetradic* is the sum of tetrads, $\Sigma\ \mathbf{a}_\alpha\ \mathbf{b}_\alpha\ \mathbf{c}_\alpha\ \mathbf{d}_\alpha$ and etc.

All such physical quantities as scalars, vectors, dyadics, triadics, tetradics etc. are collectively known as tensors of rank 1, 2, 3, 4 etc. and as such the tensors can be regarded as generalized extended form of vectors.

The examples of dyadic *i.e.* tensor of rank two are : an operator relating dielectric displacement vector with the electric vector of an electro-magnetic wave in an isotropic medium; a stress tensor relating stress and strain in an isotropic medium in which case a component of stress T is a function of every component of strain S

i.e. $$T_\alpha = \sum_{\beta=1}^{3} a_{\alpha\beta}\, S_\beta \quad \text{or} \quad \mathbf{T} = \varphi \mathbf{S} \qquad \ldots (10)$$

where φ is a nine coefficient operator in three dimensional space.

Note. The dyadic or tensor of rank two is also known as *Stress Tensor.*

An Explanatory Note on Dyads and Dyadics

We know that the gradient of a vector **f** such as

$$\nabla \mathbf{f} = \mathbf{i}f_x + \mathbf{j}f_y + \mathbf{k}f_z \quad \text{... (11)}$$

is meaningless as it consists of sum of three ordered pairs of vectors, but we sometimes take it to define as *dyadic* and the ordered vector pairs as *dyads*. In (11) we regard $\nabla \mathbf{f}$ as an operator setting up a one-one correspondence between directions **e** at a point and its directional derivative $\frac{d\mathbf{f}}{ds}$ *i.e.* $\frac{d\mathbf{f}}{ds} = \mathbf{e}.\nabla \mathbf{f}$ ... (12)

Actually the dyadic $\nabla \mathbf{f}$ replaces an infinite number of vectors $\frac{d\mathbf{f}}{ds}$, so that any sum of dyads is called as dyadic *e.g.* the dyadic

$$\mathbf{D} = \mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_2\mathbf{b}_2 + \ldots\ldots + \mathbf{a}_n\mathbf{b}_n \quad \text{... (13)}$$

is a general dyadic in which the vectors $\mathbf{a}_\alpha$ are known as *antecedents* and $\mathbf{b}_\alpha$ as *Consequents*, while the dyadic

$$\mathbf{D}_c = \mathbf{b}_1\mathbf{a}_1 + \mathbf{b}_2\mathbf{a}_2 + \ldots\ldots\ \mathbf{b}_n\mathbf{a}_n \quad \text{... (14)}$$

is said to be the conjugate of **D**, such that **D** is *symmetric* if $\mathbf{D} = \mathbf{D}_c$ and *skew* if $\mathbf{D} = -\mathbf{D}_c$.

In (11) if **f** is replaced by $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ such that

$$\mathbf{r}_x = \mathbf{i},\ \mathbf{r}_y = \mathbf{j},\ \mathbf{r}_z = \mathbf{k} \text{ then}$$

$$\nabla \mathbf{r} = \mathbf{ii} + \mathbf{jj} + \mathbf{kk} = \mathbf{I} \quad \text{... (15)}$$

where the dyadic **I** is termed as *Idemfactor* as it transforms any vector **V** into itself *i.e.*

$$\mathbf{V.I} = \mathbf{I.V} = \mathbf{V} \text{ for every } \mathbf{V}.$$

3.2. TRANSFORMATION OF COORDINATES

If we focus our attention on some point of Minkowski's four dimensional world and consider the transformation from one system of coordinates (x_1, x_2, x_3, x_4) to another system (x_1', x_2', x_3', x_4'), such that

$$x_1' = f_1(x_1, x_2, x_3, x_4) \text{ etc.}$$

then we can solve x_1, x_2, x_3, x_4 in terms of x_1', x_2', x_3', x_4' such that

$$x_1 = \phi_1(x_1', x_2', x_3', x_4') \text{ etc.}$$

and the differentials dx_1, dx_2, dx_3, dx_4 are then transformed as

$$dx_1' = \frac{\partial x_1'}{\partial x_1}dx_1 + \frac{\partial x_1'}{\partial x_2}dx_2 + \frac{\partial x_1'}{\partial x_3}dx_3 + \frac{\partial x_1'}{\partial x_4}dx_4 \text{ etc.}$$

or symbolically

$$dx_\mu' = \sum_{\alpha=1}^{4} \frac{\partial x_\mu'}{\partial x_\alpha} dx_\alpha;\ (\mu = 1, 2, 3, 4) \text{ etc.} \quad \text{... (1)}$$

3.3. THE SUMMATION CONVENTION AND KRONECKER DELTA SYMBOL

Let (x_1, x_2, x_3, x_4) and $(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3, x_4 + dx_4)$ be the coordinates of two neighbouring events considered in Minkowski's four dimensional space. Then the interval ds between these two neighbouring events in any coordinate system, is given by

$$ds^2 = g_{11}\, dx_1^2 + g_{22}dx_2^2 + g_{33}dx_3^2 + g_{44}dx_4^2 + 2g_{12}dx_1dx_2 + 2g_{13}dx_1dx_3$$
$$+ 2g_{14}dx_1dx_4 + 2g_{23}dx_2dx_3 + 2g_{24}dx_2dx_4 + 2g_{34}dx_3dx_4 \quad \ldots (1)$$

where the coefficients $g_{\mu\nu}$ ($\mu,\nu = 1, 2, 3, 4$) are functions of x_1, x_2, x_3, x_4. This follows that ds^2 is some quadratic function of the difference of coordinates.

Adopting the convention that whenever a literal suffix appears twice in a term that term is to be summed for values of the suffix 1, 2, 3, 4; the equation (1) can be written as

$$ds^2 = g_{\mu\nu}\, dx_\mu\, dx_\nu \; (\mu,\nu = 1, 2, 3, 4 \text{ and } g_{\mu\nu} = g_{\nu\mu}) \quad \ldots (2)$$

Since μ and ν each appear twice, the right hand side of (2) indicates the summation

$$\sum_{\mu=1}^{4} \sum_{\nu=1}^{4} .$$

Any literal suffix appearing twice in a term is said to be a *dummy suffix* and it may be changed freely to any other letter not already used in that term. Also two or more dummy suffixes can be interchanged *e.g.*

$$g_{\alpha\beta} \frac{\partial^2 x_\alpha}{\partial x'_\mu\, \partial x'_\nu} \frac{\partial x_\beta}{\partial x'_\lambda} = g_{\alpha\beta} \frac{\partial^2 x_\beta}{\partial x'_\mu\, \partial x'_\nu} \cdot \frac{\partial x_\alpha}{\partial x'_\lambda}$$

(by intercharging the dummy suffixes α and β and using $g_{\beta\alpha} = g_{\alpha\beta}$)

Illustration. *To prove that*

$$\frac{\partial x_\mu}{\partial x'_\alpha} \cdot \frac{\partial x'_\alpha}{\partial x_\nu} = \frac{\partial x_\mu}{\partial x_\nu} = 0 \text{ if } \mu \neq \nu$$
$$= 1 \text{ if } \mu = \nu \text{ where } \alpha = 1, 2, 3, 4.$$

Here, R.H.S. $= \dfrac{\partial x_\mu}{\partial x'_1} \dfrac{\partial x'_1}{\partial x_\nu} + \dfrac{\partial x_\mu}{\partial x'_2} \dfrac{\partial x'_2}{\partial x_\nu} + \dfrac{\partial x_\mu}{\partial x'_3} \dfrac{\partial x'_3}{\partial x_\nu} + \dfrac{\partial x_\mu}{\partial x'_4} \dfrac{\partial x'_4}{\partial x_\nu}$

$= \dfrac{\partial x_\mu}{\partial x_\nu}$

x_μ and x_ν being the coordinates of the same system, their variations are independent and so

$$dx_\mu = 0 \text{ when } \mu \neq \nu$$

and
$$dx_\mu = dx_\nu \text{ when } \mu = \nu$$

$\therefore$
$$\frac{\partial x_\mu}{\partial x'_\nu} = \frac{\partial x_\mu}{\partial x'_\alpha} \cdot \frac{\partial x'_\alpha}{\partial x_\nu} = 0 \text{ when } \mu \neq \nu,$$
$$= 1 \text{ when } \mu = \nu$$

Here the multiplier $\dfrac{\partial x_\mu}{\partial x'_\alpha} \dfrac{\partial x'_\alpha}{\partial x_\mu}$ acts as a *substitution operator.*

It is rather convenient to write

$\dfrac{\partial x_\mu}{\partial x_\nu} = \delta_{\mu\nu}$ or δ^μ_ν which is known as *Kronecker delta.*

As such the above results can be expressed as

$$\left.\begin{aligned} \delta^\mu_\nu &= 0 \text{ if } \mu \neq \nu \\ &= 1 \text{ if } \mu = \nu \end{aligned}\right\} \quad \ldots (3)$$

Thus if A (μ) be an expression involving the suffix μ, then

$$\frac{\partial x_\mu}{\partial x'_\alpha}\frac{\partial x'_\alpha}{\partial x_\nu}=A(\mu)=A(\nu) \qquad \dots(4)$$

for, the summation on the left, with respect to μ gives four terms corresponding to $\mu = 1$, 2, 3, 4; one of which will agree with ν. Denoting the other three values by σ, τ, ρ, the left hand side of (4) is

$$= 1.\,A(\nu) + 0.\,A(\sigma) + 0.\,A(\tau) + 0.\,A(\rho) \text{ by (3)}$$
$$= A(\nu)$$

i.e. $\delta^\mu_\nu\ A(\mu) = A(\nu)$ ... (5)

Evidently $\delta^\mu_\rho\ \delta^\rho_\nu = \delta^\mu_\nu$ ... (6)

and $\delta^\mu_\mu = 4$ ... (7)

for, in the latter case, $\delta^\mu_\mu = \delta^1_1 + \delta^2_2 + \delta^3_3 + \delta^4_4$

$= 1 + 1 + 1 + 1 = 4$ by (3)

3.4. TENSORS AS CLASSIFICATION OF TRANSFORMATION LAWS

We have already mentioned in §3.2 that if we consider the transformation from one system of coordinates (x_1, x_2, x_3, x_4) to another system (x_1', x_2', x_3', x_4'), then the differentials dx_1, dx_2, dx_3, dx_4 are transformed as

$$dx'_1=\frac{\partial x'_1}{\partial x_1}dx_1+\frac{\partial x'_1}{\partial x_2}dx_2+\frac{\partial x'_1}{\partial x_3}dx_3+\frac{\partial x'_1}{\partial x_4}dx_4 \text{ etc.}$$

or in short as $$dx'_\mu=\sum_{\alpha=1}^{4}\frac{\partial x'_\mu}{\partial x_\alpha}dx_\alpha \quad \mu=1,\ 2,\ 3,\ 4.$$

Any set of four quantities transformed in accordance with this law is said to be a **Contravariant Vector.** Thus if a coordinate system (A^1, A^2, A^3, A^4) transforms to the new coordinate system (A'^1, A'^2, A'^3, A'^4) where

$$A'^\mu=\sum_{\alpha=1}^{4}\frac{\partial x'_\mu}{\partial x_\alpha}A^\alpha \qquad \dots(1)$$

Then (A^1, A^2, A^3, A^4) or briefly A^μ is a contravariant vector. Hence the upper position of the suffix (which is defintely, not an exponent) is reserved to indicate contravariant vectors.

Now, if we consider an operator ϕ such that it is an invariant function of position *i.e.*, it has a fixed value at each point independent of the coordinate system used, then the four quantities

$$\frac{\partial\phi}{\partial x_1},\ \frac{\partial\phi}{\partial x_2},\ \frac{\partial\phi}{\partial x_3},\ \frac{\partial\phi}{\partial x_4} \text{ are transformed as}$$

$$\frac{\partial\phi}{\partial x'_1}=\frac{\partial x_1}{\partial x'_1}\frac{\partial\phi}{\partial x_1}+\frac{\partial x_2}{\partial x'_1}\frac{\partial\phi}{\partial x_2}+\frac{\partial x_3}{\partial x'_1}\frac{\partial\phi}{\partial x_3}+\frac{\partial x_4}{\partial x'_1}\frac{\partial\phi}{\partial x_4}, \text{ etc.}$$

or in short $$\frac{\partial\phi}{\partial x'_\mu}=\sum_{\alpha=1}^{4}\frac{\partial x_\alpha}{\partial x'_\mu},\ \frac{\partial\phi}{\partial x_\alpha}\ (\mu=1,\ 2,\ 3,\ 4).$$

Any set of four quantities transformed in accordance with this law is said to be a **Covariant Vector.**

Thus if A_μ be a covariant vector, its transformation law is

$$A'_\mu = \sum_{\alpha=1}^{4} \frac{\partial x_\alpha}{\partial x'_\mu} A_\alpha \qquad \dots (2)$$

where the lower position of the suffix indicates covariance.

Hence the relations (1) and (2) give the laws of transformation of vectors. If we denote by $A_{\mu\nu}$ a quantity with 16 components by assigning μ and ν the values 1, 2, 3, 4 independently, then a generalization of these laws yields quantities classified as

Contravariant Tensors $$A'^{\mu\nu} = \frac{\partial x'_\mu}{\partial x_\alpha} \frac{\partial x'_\nu}{\partial x_\beta} A^{\alpha\beta} \qquad \dots (3)$$

Covariant Tensors $$A'_{\mu\nu} = \frac{\partial x_\alpha}{\partial x'_\mu} \frac{\partial x_\beta}{\partial x'_\nu} A_{\alpha\beta} \qquad \dots (4)$$

Mixed Tensors $$A'^{\nu}_{\mu} = \frac{\partial x_\alpha}{\partial x'_\mu} \frac{\partial x'_\nu}{\partial x_\beta} A^{\beta}_{\alpha} \qquad \dots (5)$$

These are the tensors of *second rank.*

Similarly $A_{\mu\nu\sigma}$ has 64 components and $A_{\mu\nu\sigma\tau}$ has 256 components. Thus a tensor of higher rank is of the type

$$A'^{\tau}_{\mu\nu\sigma} = \frac{\partial x_\alpha}{\partial x'_\mu} \frac{\partial x_\beta}{\partial x'_\nu} \frac{\partial x_\lambda}{\partial x'_\sigma} \frac{\partial x'_\tau}{\partial x_\delta} A^{\delta}_{\alpha\beta\gamma} \qquad \dots (6)$$

Note. A vector is a tensor of *first rank* and an invariant or scalar is a tensor of *zero rank.*

Rank of a Tensor. The rank of a tensor is determined by the number of suffixes or indices attached to it. As a matter of fact the rank of a tensor when raised as power to the number of dimensions, yields the number of components of the tensor and hence the components of the matrix that represents the tensor. As such a tensor of rank n in four dimensional space has 4^n component. Consequently the rank of a tensor gives the number of the mode of changes of a physical quantity when passing from one system to another system which is in rotation relative to the first. It is clear from this discussion that a quantity that does not change when the axes are rotated is a tensor of *zero rank*, since the number of mode of changes is then zero. These quantities named as tensor of zero rank are scalars while the tensors of rank one are vectors.

Problem 1. (*a*) *Explain what is meant by the rank of a tensor.*

(*Agra, 1972*)

(*b*) *What is a tensor ?* (*Vikram, 1967*)

(*a*) See the note on § 3.4.

(*b*) Define as discussed in above articles or see Problem 6 discussed after.

Problem 2. *Define contravariant and covariant tensors.* (*Agra, 1967; R.U. 88*).

Contravariant tensor. If n quantities A^α ($\alpha = 1, 2, \dots n$) in a coordinate system ($x_1, x_2, \dots x_n$) are related to n other quantities A'^α ($\alpha = 1, 2, \dots n$) in another coordinate system ($x_1' x_2', \dots x_n'$) by the transformation laws

$$A'^{\mu} = \frac{\partial x'_{\mu}}{\partial x_{\alpha}} A^{\alpha} \text{ (Contravariant law)}$$

on change of the coordinate x_{β} to x'_{β} according to summation convention, then A^{α} are termed as the components of a contravariant vector or a contravariant tensor of the first rank.

Covariant tensor. If n quantities A_{α} ($\alpha = 1, 2, \ldots\ldots\ldots n$) in a coordinate system ($x_1, x_2, \ldots\ldots\ldots x_n$) are related to n other quantities A'_{α} ($\alpha = 1, 2, \ldots\ldots\ldots n$) in another coordinate system ($x'_1, x'_2, \ldots\ldots\ldots x'_n$) by the transformation laws

$$A'_{\mu} = \frac{\partial x_{\alpha}}{\partial x'_{\mu}} A_{\alpha} \text{ (Covariant law)}$$

according to summation convention, then A_{α} are termed as the components of a covariant vector or a covariant tensor of the first rank.

Problem 3. *Show that the velocity of a fluid at any point is a contravariant vector of rank one.*

(*Rohilkhand, 1988*)

Assuming that $x_{\alpha}(t)$ is the coordinate of a moving particle with the time t, we have

$$v^{\alpha} = \frac{dx_{\alpha}}{dt}$$

as the velocity of the particle.

In transformed coordinates the components of velocity are

$$v'^{\alpha} = \frac{d}{dt} x'_{\alpha}$$

But
$$v^{\alpha} = \frac{d}{dt} x_{\alpha} = \frac{\partial x'_{\alpha}}{\partial x_{\beta}} \frac{dx_{\beta}}{dt} = \frac{\partial x'_{\alpha}}{\partial x_{\beta}} v_{\beta}$$

which follows that velocity is a contravariant vector of rank one.

Problem 4. *Show that the law of transformation for a contravariant vector is transitive.*

We have
$$A'^{\mu} = \frac{\partial x'_{\mu}}{\partial x_{\alpha}} A^{\alpha}$$

Let
$$A''^{\mu} = \frac{\partial x''_{\mu}}{\partial x'_{\alpha}} A'^{\alpha}$$

$\therefore$
$$A''^{\mu} = \frac{\partial x''_{\mu}}{\partial x'_{\beta}} A'^{\beta} = \frac{\partial x''_{\mu}}{\partial x'_{\beta}} \frac{\partial x'_{\beta}}{\partial x_{\alpha}} A^{\alpha} = \frac{\partial x''_{\mu}}{\partial x_{\alpha}} A^{\alpha}$$

which shows that contravariant law is transitive.

Problem 5. *Find the components of a vector in polar coordinates whose components in cartesian co-ordinates are $\dot{x}$, $\dot{y}$ and $\ddot{x}$, $\ddot{y}$.*

As given, suppose that

$$x_1 = x, x_2 = y,$$
$$x_1' = r, x_2' = \theta$$

and (*i*) $A^1 = \dot{x}$, $A^2 = \dot{y}$

(*ii*) $A^1 = \ddot{x},\ A^2 = \ddot{y}$.

Then, we have to find C'^1, A'^2.

We have the transformations

$$x = r\cos\theta,\ y = r\sin\theta,$$

giving $$r^2 = x^2 + y^2 \text{ and } \theta = \tan^{-1}\frac{y}{x}$$

so that $$\frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial \theta}{\partial x} = -\frac{y}{r^2} \text{ and } \frac{\partial \theta}{\partial y} = \frac{x}{r^2}$$

Also $$r\dot{r} = x\dot{x} + y\dot{y} \text{ and } \dot{\theta} = \frac{x\dot{y} + y\dot{x}}{r^2}$$

(*i*) Transformation law as already defined gives

$$A'^{\mu} = \frac{\partial x'_\mu}{\partial x_\alpha} A^\alpha \quad (\alpha = 1, 2)$$

$$= \frac{\partial x'_\mu}{\partial x_1} A^1 + \frac{\partial x'_\mu}{\partial x_2} A^2$$

$$= \frac{\partial x'_\mu}{\partial x_1}\dot{x} + \frac{\partial x'_\mu}{\partial x_2}\dot{y} \text{ as } A^1 = \dot{x} \text{ and } A^2 = \dot{y}$$

∴ $$A'^1 = \frac{\partial x'_1}{\partial x_1}\dot{x} + \frac{\partial x'_1}{\partial x_2}\dot{y}$$

$$= \frac{\partial r}{\partial x}\dot{x} + \frac{\partial r}{\partial y}\dot{y} \text{ as } x_1 = x,\ x_2 = y \text{ and } x'_1 = r$$

$$= \frac{x}{r}\dot{x} + \frac{y}{r}\dot{y}$$

$$= \frac{x\dot{x} + y\dot{y}}{r} = \frac{r\dot{r}}{r} = \dot{r}$$

and $$A'^2 = \frac{\partial x'_2}{\partial x_1}\dot{x} + \frac{\partial x'_2}{\partial x_2}\dot{y}$$

$$= \frac{\partial \theta}{\partial x}\dot{x} + \frac{\partial \theta}{\partial y}\dot{y} \text{ as } x'_2 = \theta,\ x_1 = x \text{ and } x_2 = y$$

$$= -\frac{y}{r^2}\dot{x} + \frac{x}{r^2}\dot{y}$$

$$= -\frac{x\dot{x} - y\dot{x}}{r^2} = \dot{\theta},$$

(*ii*) $$A'^1 = \frac{\partial r}{\partial x}\ddot{x} + \frac{\partial r}{\partial x}\ddot{y} \text{ as in part } (i)$$

$$= \frac{x\ddot{x} + y\ddot{y}}{r}$$

But $x\dot{x} + y\dot{y} = r\dot{r}$ gives on differentiation,

$$x\ddot{x} + \dot{x}^2 + y\ddot{y} + \dot{y}^2 = r\ddot{r} + r\dot{r}^2,$$

i.e., $$x\ddot{x}+y\ddot{y}=r\ddot{r}+\dot{r}^2-\dot{x}^2-\dot{y}^2$$
$$=r\ddot{r}+\dot{r}^2-\left(\dot{r}\cos\theta-r\sin\theta\dot{\theta}\right)^2-\left(\dot{r}\sin\theta+r\cos\theta\dot{\theta}\right)^2$$
$$=r\ddot{r}+\dot{r}^2-\dot{r}^2-r^2\,\dot{\theta}^2$$
$$=r\ddot{r}-r^2\,\dot{\theta}^2$$

Thus $$A'^1=\frac{r\ddot{r}-r^2\dot{\theta}^2}{r}$$
$$=\ddot{r}-r\dot{\theta}^2$$

and $$A'^2=\frac{\partial\theta}{\partial x}\ddot{x}+\frac{\partial\theta}{\theta y}\ddot{y}$$
$$=\frac{x\ddot{y}+y\ddot{x}}{r^2}$$

But $x\dot{y}-y\dot{x}=r^2\dot{\theta}$ gives on differentiation
$$\dot{x}\dot{y}+x\ddot{y}-\dot{y}\dot{x}-y\ddot{x}=r^2\ddot{\theta}+2r\dot{r}\dot{\theta}$$
or $$x\ddot{y}-y\ddot{x}=r^2\ddot{\theta}+2r\dot{r}\dot{\theta}$$
$\therefore$ $$A'^2=\frac{r^2\ddot{\theta}+2r\dot{r}\,\dot{\theta}}{r^2}$$
$$=\ddot{\theta}+\frac{2\dot{r}\,\dot{\theta}}{r}$$

3.5. SYMMETRIC AND ANTI-SYMMETRIC TENSORS

Let a tensor be such that two contravariant or covariant indices of it can be interchanged without altering the value of the tensor, then the tensor is termed as symmetrical or symmetric in these indices.

If $A^{\mu\nu}$ and $A^{\nu\mu}$ be two contravariant tensors in a certain system of co-ordinates such that
$$A^{\mu\nu}=A^{\nu\mu}$$

Then if $A^{\mu\nu}$ and $A^{\nu\mu}$ become $A'^{\mu\nu}$ and $A'^{\nu\mu}$ in another system of coordinates, the symmetry will be maintained in this system also if $A'^{\mu\nu}=A'^{\nu\mu}$.

To show it, let us consider
$$A'^{\mu\nu}=\frac{\partial x'_\mu}{\partial x_\alpha}\frac{\partial x'_\nu}{\partial x_\beta}A^{\alpha\beta},$$
$$=\frac{\partial x'_\nu}{\partial x_\beta}\frac{\partial x'_\mu}{\partial x_\alpha}A^{\alpha\beta}\text{ (on interchanging the indices)}$$
$$=A'^{\nu\mu}\text{ as }A^{\beta\alpha}=A^{\alpha\beta}$$
which shows the symmetry in the other system also.

Similarly if we consider two covariant tensors, $A_{\mu\nu}$ and $A_{\nu\mu}$ such that $A_{\mu\nu}=A_{\nu\mu}$ in one system

Then if they become $A'_{\mu\nu}$ and $A'_{\nu\mu}$ in another system, we have

$$A'_{\mu\nu} = \frac{\partial x_\alpha}{\partial x'_\mu}, \frac{\partial x_\beta}{\beta x'_\nu} A_{\alpha\beta}$$

$$= \frac{\partial x_\beta}{\partial x'_\nu}, \frac{\partial x_\alpha}{\partial x'_\mu} A_{\beta\alpha} \text{ (on interchanging the indices)}$$

$$= A'_{\nu\mu} \text{ as } A_{\alpha\beta} = A_{\beta\alpha}.$$

In case one index is contravariant and other covariant, the symmetry cannot be easily defined. But it is notabe that Kronecker delta which is a mixed tensor is symmetrical with respect to its indices.

When $A^{\mu\nu}$ is symmetrical, we have

$$A^{11} = A^{11}, A^{22} = A^{22} \text{ etc.}$$

In all, there are $^4C_2 + 4 = {}^5C_2$ components.

As an example the components of the angular momentum of a rigid body B_μ are connected with the components of its angular velocity A_α by the relations $B_\mu = \sum_{\alpha=1}^{3} T_{\mu\alpha} A_\alpha$, where $T_{\mu\alpha}$ is the inertia tensor. This tensor is symmetric, because

$$T_{\mu\alpha} = T_{\alpha\mu} \qquad \dots (1)$$

Now it has been already mentioned that a tensor can be expressed as a matrix and the columns and rows of a matrix when interchanged, the resulting tensor is the conjugate tensor. As such the conjugate of the tensor defined in §3.1 (*a*) may be written as

$$\emptyset_c = \begin{bmatrix} a_{xx} & a_{yx} & a_{zx} \\ a_{xy} & a_{yy} & a_{zy} \\ a_{xz} & a_{yz} & a_{zz} \end{bmatrix}$$

Thus the tensor $\emptyset_c$ is the conjugate of $\emptyset$, when

$$a_{xy} = a_{yx}, a_{xz} = a_{zx}, \text{ and } a_{yz} = a_{zy} \qquad \dots (2)$$

If a single tensor satisfies this condition, it is called a **symmetric tensor.** It means that condition (1) is essential for a tensor to be symmetric. It has only six independent elements in three dimensional space and may be written as

$$\emptyset_{sym.} = \begin{bmatrix} a_{xx} & a_{xy} & a_{xz} \\ a_{yx} & a_{yy} & a_{yz} \\ a_{zx} & a_{zy} & a_{zz} \end{bmatrix},$$

The relations (2) between the components of tensors follow that any symmetrical tensor corresponds with a transformation from the principal axes to another rectangular system of axes. To express the symmetrical linear functions by graphical function, let us suppose that we have two vectors **u** and **v** such that

$$\mathbf{v} = \emptyset_{sym} \cdot \mathbf{u} \qquad \dots (3)$$

Project **u** upon **v** so that we calculate

$$\mathbf{u}.\mathbf{v} = u_x v_x + u_y v_y + u_z v_z, \qquad \dots (4)$$

where u_x, u_y, u_z and v_x, v_y, v_z are the components of **u** and **v** respectively along the principal axes.

Multiplying both sides of (3) scalarly by **u** we get

$$\mathbf{u}.\mathbf{v} = \mathbf{u}.\phi_{sym}.\mathbf{u}$$

$$= u_x v_x + u_y v_y + u_z v_z \text{ from (4)}$$

$$= u_x (a_{xx}u_x + a_{xy}u_y + a_{xz}u_z) + u_y (a_{yx}u_x + a_{yy}u_y + a_{yz}u_z) + u_z (a_{zx}u_x + a_{zy}u_y + a_{zz}u_z)$$

Say $S = a_{xx} u_x^2 + a_{yy} u_y^2 + a_{zz} u_z^2 + 2 (a_{xy} u_x u_y + a_{yz} u_y u_z + a_{zx} u_z u_x.$

where *S is a scalar point function.* ... (5)

From (5), we get

$$\frac{\partial S}{\partial u_x} = 2\left[a_{xx} u_x + a_{xy} u_y + a_{xz} u_z\right] = 2 v_x \text{ by §3.1 (7)}$$

or $$v_x = \frac{1}{2}\frac{\partial S}{\partial u_x}$$

Similarly $$v_y = \frac{1}{2}\frac{\partial S}{\partial u_y} \text{ and } v_z = \frac{1}{2}\frac{\partial S}{\partial u_z}$$

So that $\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}$; **i**, **j**, **k** being unit vectors along principal axes

$$= \frac{1}{2}\left[\frac{\partial S}{\partial u_x}\mathbf{i} + \frac{\partial S}{\partial u_y}\mathbf{j} + \frac{\partial S}{\partial u_z}\mathbf{k}\right]$$

$$= \frac{1}{2} \text{ grad } S.$$

Which shows that **v** is a vector perpendicular to the surface S = const. in the direction of the outward normal. But S = const. is an equation of the second degree in the rectangular components of **u** regarding these as coordinates defining the extremity P of the vector **u** the locus of P is a conicoid.

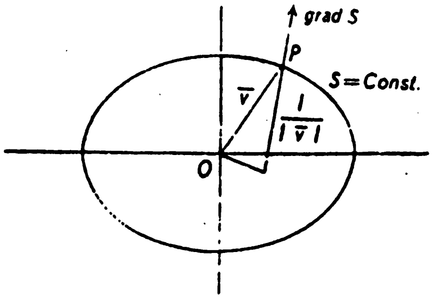


Fig. 3.1

As a particular case if $S = 1$, the surface defined under certain conditions is the *tensor ellipsoid* as shown in Fig. 3.1.

Also $\mathbf{u}.\mathbf{v} = S = 1$ = resolute of **u** in the direction of **v**.

In the direction of grade S, this resolute becomes $\dfrac{1}{|\mathbf{v}|}$

Few other examples of symmetric tensor may be given as below

$$A_{\mu\alpha\beta} = A_{\alpha\mu\beta}$$

and $$A_{\mu\alpha\beta\gamma} = A_{\mu\beta\alpha\gamma} = A_{\alpha\mu\beta\gamma} = A_{\beta\alpha\mu\gamma} = A_{\alpha\beta\mu\gamma} = A_{\beta\mu\alpha\gamma}$$

Here the first tensor is symmetric in its first two indices and the second one is symmetric in first three indices.

If a tensor is such that two contravariant or covariant indices of it when interchanged, the components of the tensor alter in sign but not in magnitude, the tensor is said to be **anti-symmetric** *or* **skew-symmetric.**

Hence if $A^{\mu\nu} = -A^{\nu\mu}$

then, $$A'^{\mu\nu} = \frac{\partial x'_\mu}{\partial x_\alpha}\frac{\partial x'_\nu}{\partial x_\beta} A^{\alpha\beta}$$

$$= -\frac{\partial x'_\nu}{\partial x_\beta}\frac{\partial x'_\mu}{\partial x_\alpha} A^{\beta\alpha}$$

$$= -A'^{\nu\mu} \text{ as } A^{\alpha\beta} = -A^{\beta\alpha}$$

Similarly if $A_{\mu\nu} = -A_{\nu\mu}$, then $A'_{\mu\nu} = -A'_{\nu\mu}$

Here $A^{12} = -A^{21}$ etc. and $A^{11} = 0 = A^{22} = A^{33}$

As such number of arithmetic components is ${}^4C_2$ only.

Evidently the components of an antisymmetric tensor satisfy the relations

$$A_{\mu\nu} + A_{\nu\mu} = 0 \text{ or } A_{\mu\nu} = -A_{\nu\mu} \quad \ldots (6)$$

which follows that the tensor changes its sign when indices are interchanged. If $\mu = \nu$, then (6) yields.

$$A_{\mu\mu} + A_{\mu\mu} = 0 \text{ or } 2A_{\mu\mu} = 0 \text{ or } A_{\mu\mu} = 0 \quad \ldots (7)$$

So in terms of coefficients, $a_{xx} = a_{yy} = a_{zz} = 0$

and $a_{xy} = -a_{yx}$, $a_{xz} = -a_{zx}$, $a_{yz} = -a_{zy}$

give the conditions for a tensor to be anti or skew-symmetric.

Thus ϕ expressed by §3.1 (9) will be antisymmetric if

$$\phi_{sk} = \begin{bmatrix} 0 & a_{xy} & a_{xz} \\ -a_{xy} & 0 & a_{yz} \\ -a_{xz} & -a_{yz} & 0 \end{bmatrix}$$

The matrix has only three components. The property of having only three components is possessed by vectors. This leads to the conclusion that an operation of ϕ_{sk} on the vector u, is exactly equivalent to the vector product of two vectors, since the final result is itself a fector v. For example consider the product of the coordinates of two points

$$A_{\mu\nu} = x_\mu \xi_\nu - \xi_\mu x_\nu \quad \ldots (8)$$

In 3-dimensional space, suffixes can have only 3 values (1, 2, 3) and so any pair of suffixes can be replaced by a single one which is not present in the pair *i.,e.*,

$$\left.\begin{aligned} A_{23} &= -A_{32} = A_1 \\ A_{12} &= -A_{21} = A_3 \\ A_{31} &= -A_{13} = A_2 \end{aligned}\right\} \quad \ldots (9)$$

Thus the anti-symmetric system may be replaced by

$\xi_{\mu\nu\sigma} = +1$ if (μ, ν, σ) is an even permutation of (1, 2, 3)

$= -1$ if " " odd " "

$= 0$ if any of the suffixes are equal.

Now equations (9) yield,

$$A_1 = \frac{1}{2}[2A_{33}],\ A_2 = \frac{1}{2}[2\ A_{13}],\ A_3 = \frac{1}{2}\ [2A_{12}]$$

i.e. $$A_1 = \frac{1}{2}\sum_{\mu,\nu=1}^{3} \xi_{\mu\nu\sigma} A_{\mu\nu} \quad \ldots (10)$$

There being only two terms in sum (10).

Few other examples of antisymmetric tensors are

$$A_{\mu\nu\sigma} = -A_{\nu\mu\sigma}$$

and
$$A_{\mu\nu\sigma\rho} = -A_{\nu\sigma\mu\rho} = A_{\sigma\mu\nu\rho} = -A_{\mu\sigma\nu\rho} = -A_{\nu\mu\sigma\rho} = -A_{\sigma\nu\mu\rho}.$$

As an illustration, *if* $A_{\mu\nu}$ *is antisymmetric tensor of second order and* B^{μ} *is a tensor of rank one, then* $A_{\mu\nu} B^{\mu} B^{\nu} = 0$, summation being taken over repeated indices.

(*Meerut, 1970*)

Interchange of dummy suffixes gives

$$A_{\mu\nu} B^{\mu} B^{\nu} = A_{\nu\mu} B^{\nu} B^{\mu} \qquad \ldots (11)$$

where $A_{\mu\nu}$ being antisymmetric *i.e.*

$$A_{\mu\nu} = -A_{\nu\mu}$$

renders

$$A_{\mu\nu} B^{\mu} B^{\nu} = -A_{\nu\mu} B^{\mu} B^{\nu}$$
$$= -A_{\nu\mu} B^{\nu} B^{\mu} \qquad \ldots (12)$$

The addition of (11) and (12) yields

$$A_{\mu\nu} B^{\mu} B^{\nu} = 0 \qquad \ldots (13)$$

3.6. INVARIANT TENSORS

It is not known about any vector which has the same components in different systems of co-ordinates, but there exist tensors of higher ranks which have the same components in all the frames of reference. These tensors are called to have the *invariant components* or *invariant tensors* in general. One of the examples of such a tensor is Kronecker symbol defined as follows:

With respect to the old frame of reference (*i.e.*, before rotation)

$$\frac{\partial x_{\mu}}{\partial x_{\nu}} = \begin{cases} 0 \text{ if } \mu \neq \nu \\ 1 \text{ if } \mu = \nu \end{cases} \text{ since } x_{\mu} \text{ is independent of } x_{\nu}$$

But
$$\frac{\partial x_{\mu}}{\partial x'_{\nu}} = \frac{\partial x_{\mu}}{\partial x'_{\alpha}} \frac{\partial x'_{\alpha}}{\partial x_{\nu}}$$

Hence
$$\frac{\partial x_{\mu}}{\partial x_{\nu}} = \frac{\partial x_{\mu}}{\partial x'_{\nu}} \frac{\partial x'_{\alpha}}{\partial x_{\nu}} = \delta^{\mu}_{\nu} \qquad \ldots (1)$$

where
$$\delta^{\mu}_{\nu} = \begin{cases} 1 \text{ if } \mu = \nu \\ 0 \text{ if } \mu \neq \nu \end{cases},$$

The symbol $\delta^{\mu}_{\nu} = \delta_{\mu\nu} = \delta^{\mu\nu} = \begin{cases} 1 \text{ if } \mu = \nu \\ 0 \text{ if } \mu \neq \nu \end{cases}$ is called as *Kronecker delta symbol*. In terms of new frame of reference (*i.e.* after the rotation), we may write

$$\delta'^{\mu}_{\nu} = \frac{\partial x'_{\mu}}{\partial x'_{\nu}} = \frac{\partial x'_{\mu}}{\partial x_{\alpha}} \frac{\partial x_{\alpha}}{\partial x'_{\nu}} = \begin{cases} 1 \text{ if } \mu = \nu \\ 0 \text{ if } \mu \neq \nu \end{cases}$$

$$= \frac{\partial x'_{\mu}}{\partial x_{\beta}} \cdot \frac{\partial x_{\alpha}}{\partial x'_{\nu}} \cdot \frac{\partial x_{\beta}}{\partial x_{\alpha}}$$

$$= \frac{\partial x'_\mu}{\partial x_\beta}\frac{\partial x_\alpha}{\partial x'_\nu}\delta^\beta_\alpha \quad \text{since } \delta^\beta_\alpha = \frac{\partial x_\beta}{\partial x_\alpha} \qquad \ldots(2)$$

(in summation convention)

Hence $\delta_\nu{}^\mu$ is invariant and transforms as mixed tensor of rank two. similarly $\delta_{\mu\nu}$ transforms as the components of covariant tensor of rank two while $\delta^{\mu\nu}$ transforms as a contravariant tensor of rank two.

Kronecker symbol can be used as *substitution multiplier.*

Since $\quad A^\mu = \frac{\partial x_\nu}{\partial x'_\alpha}\frac{\partial x'_\alpha}{\partial x_\mu}A^\nu$ (in summation convention)

$\quad = \delta^\mu_\nu A^\nu$ (by substitution of index).

Similarly $\quad A_\nu = \frac{\partial x'_\alpha}{\partial x_\nu}\frac{\partial x_\mu}{\partial x'_\alpha}A^\mu_\alpha$

$\quad = \delta^\mu_\nu A_\mu$

and $\quad \delta_{\mu\nu}\delta^{\nu\beta} = \frac{\partial x_\mu}{\partial x_\nu}\frac{\partial x_\nu}{\partial x_\beta} = \frac{\partial x_\mu}{\partial x_\beta} = \delta^\mu_\beta$

while $\quad \delta^\mu{}_\mu = \delta_1{}^1 + \delta_2{}^2 + \delta_3{}^3$ (summation convention)

$\quad = 3$ (in three-dimensional geometry)

$\quad = 4$ (in four-dimensional geometry).

Secondly, we define the generalised Kronecker delta symbol, $\delta^{\mu\nu}_{\beta\gamma}$ by

$$\delta^{\mu\nu}_{\beta\gamma} = \begin{cases} 1 \text{ if } \mu = \beta \ \ \nu = \gamma \text{ and } \mu \neq \nu \ \left(\text{as } \delta^{12}_{12} = 1\right) \\ -1 \text{ if } \mu = \gamma,\ \nu = \beta \text{ and } \mu \neq \nu \ \left(\text{as } \delta^{12}_{21} = -1\right) \\ 0 \text{ for all other combinations of indices.} \end{cases}$$

Similarly we can define $\delta^{\mu\nu\sigma}_{\beta\gamma\xi}$ as an absolute (invariant) tensor of rank six.

Conclusively if both upper and lower indices of a generalised delta consist of the same distinct numbers chosen from 1, 2, 3, the delta is 1 or −1 according as the upper indices form an even or odd permutation of the lower, in all other permutations the delta is zero. We have for example

$$\delta^{12}_{12} = 1,\ \delta^{32}_{32} = -1,\ \delta^{23}_{11} = \delta^{13}_{21} = 0,$$

$$\delta^{123}_{123} = \delta^{231}_{123} = 1,\ \delta^{213}_{123} = \delta^{321}_{123} = -1,\ \delta^{312}_{123} = 0.$$

Evaluation of the variaous possible combinations of indices shows that

$$\left.\begin{aligned} \delta^{\mu\beta}_{\beta\nu} &= -\delta^{\mu\nu}_{\nu\beta} = 2\ \delta^\mu_\beta \\ \delta^{\mu\nu}_{\beta\xi} &= \delta^\mu_\beta\ \delta^\nu_\xi - \delta^\mu_\xi\ \delta^\nu_\beta \end{aligned}\right\} \qquad \ldots(5)$$

Alternating or Permutation epsilon tensor. This tensor is also invariant component tensor of third rank and anti-symmetric in every pair of indices. Let $\epsilon_{\mu\nu\sigma}$ be such a tensor; then

$$\epsilon_{\mu\nu\sigma} = -\epsilon_{\nu\mu\sigma} = \epsilon_{\sigma\mu\nu} = -\epsilon_{\mu\sigma\nu} = \epsilon_{\nu\sigma\mu} = -\epsilon_{\sigma\nu\mu} \qquad \ldots(6)$$

But if $\mu = \nu$ then $\epsilon_{\mu\nu\sigma} = -\epsilon_{\nu\mu\sigma}$ gives $\epsilon_{\mu\mu\sigma} = -\epsilon_{\mu\mu\sigma}$

or $$\epsilon_{\mu\mu\sigma} = 0 \qquad \dots (7)$$

It is clear that whenever two indices are equal, the component is zero. Moreover a tensor of third rank in three dimensional geometry has 27 components. But in case of the tensor $\epsilon_{\mu\nu\sigma}$ only 6 components are non-vanishing. All of them have the same absolute value, 3 being positive and the rest three are negative.

So

$$\left.\begin{aligned} &\epsilon_{123}=\epsilon_{xyz}=1=\epsilon_{312}=\epsilon_{231} \\ \text{and} \quad &\epsilon_{213}=\epsilon_{321}=\epsilon_{132}=-1 \end{aligned}\right\} \qquad \dots (8)$$

while all other components are zero.

Thus all the even permutations of 1, 2 and 3 correspond to the components with value +1, while all the odd permutations correspond to −1.

The transformation law for this type of tensor in three dimensional space is given by

$$\epsilon'_{\mu\nu\sigma} = \sum_{\alpha,\beta,\gamma=1}^{3} a_{\mu\alpha} a_{\nu\beta} a_{\sigma\gamma} \ \epsilon_{\alpha\beta\gamma} = \epsilon_{\mu\nu\sigma}$$

i.e., $\epsilon_{\mu\nu\sigma}$ is invariant.

Consequently

$$\epsilon'_{\mu\nu\sigma} = \epsilon_{\mu\nu\sigma} \begin{cases} 1 \text{ when } \mu, \nu, \sigma \text{ are an even permutation} \\ -1 \text{ when } \mu, \nu, \sigma \text{ are an odd permutation,} \\ 0 \text{ when } \mu, \nu, \sigma \text{ contain two or more repeated indices.} \end{cases} \qquad \dots (9)$$

Similarly the contravariant components can also be discussed,

$$\left.\begin{aligned} &\epsilon^{123}=\epsilon^{312}=\epsilon^{231}=1, \ \epsilon^{132}=\epsilon^{213}=\epsilon^{321}=-1, \\ &\epsilon^{112}=\epsilon^{233}=\epsilon^{111}=\dots=0 \end{aligned}\right\} \qquad \dots (10)$$

Pseudo tensor. Let there be a tensor $\epsilon_{\mu\sigma\tau\rho}$ of rank 4, defined such that

$$\epsilon_{\mu\sigma\tau\rho} = \begin{cases} +1 \text{ if } \mu\sigma\tau\rho \text{ is an even permutation of } 0, 1, 2, 3 \\ -1 \text{ if } \mu\sigma\tau\rho \text{ is an odd permutation of } 0, 1, 2, 3 \\ 0 \text{ if two or more indices are equal} \end{cases} \qquad \dots (11)$$

These are termed as components of **pseudo tensor** of rank four.

In case ϕ is a scalar, the quantities $\phi\epsilon_{\mu\sigma\tau\rho}$ are called as pseudo scalars since they have only one component.

From every antisymmetric tensor $A_{\mu\sigma}$ of the second rank a pseudo tensor $A^*_{\mu\sigma}$ of the same rank can be obtained by multiplying the former with a pseudo-tensor of rank 4,

i.e. $$A^*_{\mu\sigma} = \frac{1}{2} \sum_{\alpha,\beta=0}^{3} \epsilon^{\mu\sigma\tau\rho} A_{\alpha\beta} \qquad \dots (12)$$

Thus the product of a tensor with a pseudo-tensor is a pseudo tensor. It is called dual of a given tensor.

A useful property of ϵ tensor.

ϵ tensor can be used to write the cross product of two vectors **A** and **B**.

Let $\quad \mathbf{D} = \mathbf{A} \times \mathbf{B}$, then $D_1 = A_2B_3 - A_3B_2 = \in_{123} A_2 B_3 + \in_{132} A_3B_2$;

$A_1, A_2, A_3, B_1, B_2, B_3$ being components of **A** and **B**.

$= \in_{1\nu\sigma} A_\nu B_\sigma$

Similarly $D_2 = \in_{2\nu\sigma} A_\nu B_\sigma$ (in summation convention)

and $D_3 = \in_{3\nu\sigma} A_\nu B_\sigma$

or $D_\mu = \in_{\mu\nu\sigma} A_\nu B_\sigma$ ... (13)

Evaluating various possible combinations we may have

$$\epsilon^{\mu\nu\sigma}\epsilon_{\sigma\alpha\beta} = \delta^{\mu\nu}_{\alpha\beta} = \delta^{\mu}_{\nu}\delta^{\nu}_{\beta} - \delta^{\mu}_{\beta}\ \delta^{\nu}_{\alpha}$$

Thus, if r = **C**. (**A** × **B**) = **C**.**D** for **D** = **A** × **B**.

then, $r = C_\mu D^\mu$ (in summation convention)

or $r = C_\mu{}^{\mu\nu\sigma} A_\nu A_\sigma$

$= \in^{\mu\nu\sigma} C_\mu A_\nu B_\sigma$

Similarly vector triple product of three vectors can be given as

$$\mathbf{E} = \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = \mathbf{C} \times \mathbf{D}$$

i.e.,
$$E^\mu = \in^{\mu\nu\sigma} C_\nu D_\sigma$$
$$= \in^{\mu\nu\sigma} C_\nu (\in_{\sigma\alpha\beta} A^\alpha B^\beta)$$
$$= \in^{\mu\nu\sigma} \in_{\sigma\alpha\beta} C_\nu A^\alpha B^\beta$$
$$= \delta^{\mu\nu}_{\alpha\beta}\ C_\nu\ A^\alpha B^\beta = \left(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \delta^{\mu}_{\beta}\delta^{\nu}_{\alpha}\right) C_\nu\ A^\alpha B^\beta$$

Since $\delta^\mu_\alpha\ A^\alpha = A^\mu$ etc., we get

$$E_\mu = \delta^\nu_\beta\ A^\mu C_\nu\ B^\beta - \delta^\nu_\alpha\ B^\mu C_\nu\ A^\alpha$$
$$= A^\mu\ (C_\beta B^\beta) - B^\mu\ (C_\alpha\ A^\alpha)$$

As such **E** = **A** (**C**.**B**) − **B** (**C**.**A**).

Evaluation of ∇ × (V × W) Using ∈ tensor.

Suppose, ∇ × (**V** × **W**) = ∇ × **Z**,

then $(\nabla \times \mathbf{Z})_\mu = \in^{\mu\nu\sigma} \nabla_\nu Z_\sigma = \in^{\mu\nu\sigma} \nabla_\mu (\in_{\sigma\alpha\beta} V^\alpha W^\beta)$

$$\in^{\mu\nu\sigma} \in_{\sigma\alpha\beta} \nabla_\nu V^\alpha W^\beta = \delta^{\mu\nu}_{\alpha\beta}\ (V^\alpha \nabla_\nu W^\beta + W^\beta \nabla_\nu V^\alpha)$$
$$= \left(\delta^{\mu}_{\nu}\delta^{\nu}_{\beta} - \delta^{\mu}_{\beta}\delta^{\nu}_{\alpha}\right)\left(V^\alpha\ \nabla_\nu W^\beta + W^\beta \nabla_\nu V^\alpha\right)$$
$$= V^\mu\ \nabla_\beta\ W^\beta - V^\nu\ \nabla_\nu\ W^\mu + W^\nu\ \nabla_\nu V^\mu - W^\mu\ \nabla_\alpha\ V^\alpha$$

So
$$\nabla \times \mathbf{Z} = \nabla \times (\mathbf{V} \times \mathbf{W})$$
$$= \mathbf{V}\ (\nabla . \mathbf{W}) - (\mathbf{V}.\nabla)\ \mathbf{W} + (\mathbf{W}.\nabla)\ \mathbf{V} - \mathbf{W}\ (\nabla.\mathbf{V})$$

Similarly all the vector relationships can be derived by using ∈ tensor.

Krutkov's tensor. Let us consider a tensor $A^{\mu\gamma\beta\sigma}$ of fourth rank having following properties.

(1) Antisymmetric with respect to the first pair of indices

$$A^{\mu\gamma.\beta\sigma} = -A^{\gamma\mu.\beta\sigma}$$

(2) Antisymmetric in second pair of indices

$$A^{\mu\gamma.\beta\sigma} = A^{\mu\gamma.\sigma\beta}$$

(3) Symmetric in cyclic order

$$A^{\mu\gamma.\beta\sigma} + A^{\mu\beta.\sigma\gamma} + A^{\mu\sigma.\gamma\beta} = 0$$

Then in terms of second derivatives of $A^{\mu\gamma,\beta\sigma}$ we can form a new tensor given by

$$B^{\mu\sigma}=\sum_{\gamma,\beta=0}^{3}\frac{\partial^2 A^{\mu\gamma,\beta\sigma}}{\partial x_\gamma\,\partial x_\beta} \qquad \ldots (14)$$

This tensor is called as *Krutkov's tensor*. If we differentiate equation (14) with respect to x, we have

$$\sum_{\sigma=0}^{3}\frac{\partial B^{\mu\sigma}}{\partial x_\sigma}=0. \qquad \ldots (15)$$

This is an important property of Krutkov's tensor.

Problem 6. *Define a tensor. Prove that the Kronecker symbol δ^k_i is a tensor where components are the same in every coordinate system.*

(*Rohilkhand, 1981; Agra, 1966, 70*)

We know that the tensors are quantities obeying certain transformation laws so that tensor analysis may be regarded as an indispensable part of study which is rather suitable for the mathematical formulation of natural laws which remain invariant when one coordinate system is changed to another. The rank of a tensor measures the number of the mode of changes of a physical quantity when passing from one system to another which is in rotation relative to the first. As such tensor of zero rank is a scalar quantity and the tensor of rank one is a vector quantity.

The laws of transformation of vector being defined by

$$A'^{\mu}=\sum_{\alpha=1}^{4}\frac{\partial x'_\mu}{\partial x_\alpha}A^\alpha \quad \text{(contravariant vector)}$$

and

$$A'_\mu=\sum_{\alpha=1}^{4}\frac{\partial x_\alpha}{\partial x'_\mu}A_\alpha \quad \text{(covariant vector)} \qquad (\textit{Rohilkhand 1979})$$

in Minkowski's four dimensional space, we define the tensors of rank two as follows :

Contravariant tensor : $$A'^{\mu\nu}=\frac{\partial x'_\mu}{\partial x_\alpha}\frac{\partial x'_\nu}{\partial x_\beta}A^{\alpha\beta} \qquad \ldots (1)$$

Covariant tensor : $$A'_{\mu\nu}=\frac{\partial x_\alpha}{\partial x'_\mu}\frac{\partial x_\beta}{\partial x'_\nu}A_{\alpha\beta} \qquad \ldots (2)$$

Mixed tensor : $$A'^{\nu}_{\mu}=\frac{\partial x_\alpha}{\partial x'_\mu}\frac{\partial x'_\nu}{\partial x_\beta}A_{\alpha\beta} \qquad \ldots (3)$$

Each one having 4^2 *i.e.*, 16 components.

Similarly we can define the tensors of higher ranks.

Now the Kronecker delta symbol δ_i^k is defined as

$$\delta_i^k=\frac{\partial x_k}{\partial x_i}=\frac{\partial x_k}{\partial x'_j}\frac{\partial x'_j}{\partial x_i}$$

which is easily deduced from (3) by choosing $A_\alpha{}^\beta$ to be the Kronecker delta $\delta_\alpha{}^\beta$ so that

$$A'^{\nu}_{\mu}=\frac{\partial x_\alpha}{\partial x'_\mu}\frac{\partial x'_\nu}{\partial x_\beta}\delta^\beta_\alpha=\frac{\partial x_\alpha}{\partial x'_\mu}\frac{\partial x'_\nu}{\partial x_\beta}\frac{\partial x_\beta}{\partial x_\alpha}=\frac{\partial x'_\nu}{\partial x'_\mu}=\delta^\nu_\mu$$

and now replacing μ by i, ν by j this gives

$$\delta_i^k = A_i'^k \frac{\partial x_\alpha}{\partial x_i'} \frac{\partial x_k'}{\partial x_\beta} A_\alpha^\beta = \frac{\partial x_\alpha}{\partial x_i'} \frac{\partial x_k'}{\partial x_\beta} \delta_\alpha^\beta$$

From the definition of mixed tensor, it follows that δ_i^k is a mixed tensor of order two with 16 components in 4-dimensional space.

In order to show that the components of the tensor δ_i^k are the same in every coordinate system, let us define the symbol δ_i^k as

$$\delta_i^k = 1 \text{ if } i = k$$
$$= 0 \text{ if } i \neq k$$

which is evident from $\delta_i^k = \dfrac{\partial x_k}{\partial x_i} = 0$ when $i \neq k$

$$= 1 \text{ when } i = k$$

In terms of new frame of reference (or new coordinate system), we may have

$$\delta_i'^k = \frac{\partial x_k'}{\partial x_i'} = \frac{\partial x_k'}{\partial x_j} \frac{\partial x_j}{\partial x_i'} = 1 \text{ if } i = k$$

$$= 0 \text{ if } i \neq k$$

or $$\delta_i'^k = \frac{\partial x_k'}{\partial x_l} \frac{\partial x_j}{\partial x_i'} \frac{\partial x_l}{\partial x_j} = \frac{\partial x_k'}{\partial x_l} \frac{\partial x_j}{\partial x_i'} \delta_j^l \quad \text{since } \delta_j^l = \frac{\partial x_l}{\partial x_j}$$

From which it is clear that δ_i^k is invariant and transforms as mixed tensor of rank two.

Problem 7. *Prove that Kronecker delta is a mixed tensor of rank two.*

Its solution has been given in Problem 6. (*Rohilkhand, 1983; Agra, 1968, 80, 81*)

Problem 8. *Show that symmetry properties of a tensor are invariant.*

If $A_{\lambda\mu\nu} = A_{\mu\lambda\nu}$ then we have to show that $A'_{\lambda\mu\nu} = A'_{\mu\lambda\nu}$

The definition follows:

$$A'_{\lambda\mu\nu} = \sum_{\alpha,\beta,\gamma=1}^{3} \frac{\partial x_\alpha}{\partial x_\lambda'} \frac{\partial x_\beta}{\partial x_\mu'} \frac{\partial x_\gamma}{\partial x_\nu'} A_{\alpha\beta\gamma}$$

and $$A'_{\mu\lambda\nu} = \sum_{\beta,\alpha,\gamma=1}^{3} \frac{\partial x_\beta}{\partial x_\mu'} \frac{\partial x_\alpha}{\partial x_\lambda'} \frac{\partial x_\gamma}{\partial x_\nu'} A_{\beta\alpha\gamma}$$

The given tensor having symmetrical in first two indices, we have

$$A_{\lambda\mu\nu} = A_{\mu\lambda\nu} \text{ and } A_{\alpha\beta\gamma} = A_{\beta\alpha\gamma}$$

Using this relation and comparing the two equations for $A'_{\lambda\mu\nu}$ and $A'_{\mu\lambda\nu}$ we find that both the equations are identical *i.e.* $A'_{\lambda\mu\nu} = A'_{\mu\lambda\nu}$.

Which follows that the tensor in other system is also symmetrical in first two indices. Hence the properties of symmetric tensors are invariant.

3.7. RULES WHICH GOVERN TENSOR ANALYSIS

Rule I. *The sum and difference of two tensors of the same rank result in a third tensor of the same rank. Moreover, if* $F_{\lambda\mu}$ ... and $G_{\lambda\mu}$... *are the tensors of the same rank, then* $pF_{\lambda\mu} \ldots + qG_{\lambda\mu} \ldots$ *is also a tensor of the same rank* (p, q *being numbers*).

Suppose there are two tensors $A_{\lambda\mu}$ and $B_{\lambda\mu}$, then it will be shown that $A_{\lambda\mu} + B_{\lambda\mu} = C_{\lambda\mu}$ is another tensor of the same rank.

Expressing the tensor $A_{\lambda\mu}$ in the form of a matrix, we have

$$A_{\lambda\mu} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B_{\lambda\mu} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

and $$C_{\lambda\mu} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

so that $$A_{\lambda\mu} + B_{\lambda\mu} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{bmatrix}$$

If the relations between the coefficients a's and b's be such that

$$a_{\lambda\mu} + b_{\lambda\mu} = c_{\lambda\mu}$$

then $$A_{\lambda\mu} + B_{\lambda\mu} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = C_{\lambda\mu}$$

which is a tensor of the same rank.

Similarly $$A_{\lambda\mu} - B_{\lambda\mu} = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix} = D_{\lambda\mu}$$

where $a_{\lambda\mu} - b_{\lambda\mu} = d_{\lambda\mu}$

Here $D_{\lambda\mu}$ is again a tensor of the same rank.

Further,

$$pA_{\lambda\mu} + qB_{\lambda\mu} = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} = E_{\lambda\mu}$$

where $pa_{\lambda\mu} + qb_{\lambda\mu} = e_{\lambda\mu}$

Showing that $E_{\lambda\mu}$ is also a tensor of the same rank.

The rule of addition may be generalized for any number of tensors of any rank.

Suppose there are two mixed tensors T and S of rank N, having their r indices (from λ_1 to λ_r) contravariant and s indices (from μ_1 to μ_s) covariant, then laws of their transformation may be written as

$$T'^{\alpha_1 \ldots \alpha_r}_{\beta_1 \ldots \beta_s} = \left|\frac{\partial x}{\partial x'}\right|^N \left[\frac{\partial x_{\mu_1}}{\partial x'_{\beta_1}} \cdots \frac{\partial x_{\mu_s}}{\partial x'_{\beta_s}}\right] \left[\frac{\partial x'_{\alpha_1}}{\partial x_{\lambda_1}} \cdots \frac{\partial x'_{\alpha_r}}{\partial x_{\lambda_r}}\right] T^{\lambda_1 \ldots \lambda_r}_{\mu_1 \ldots \mu_s} \quad \ldots (1)$$

$$S'^{\alpha_1 \ldots \alpha_r}_{\beta_1 \ldots \beta_s} = \left|\frac{\partial x}{\partial x'}\right|^N \left[\frac{\partial x_{\mu_1}}{\partial x'_{\beta_1}} \cdots \frac{\partial x_{\mu_s}}{\partial x'_{\beta_s}}\right] \left[\frac{\partial x'_{\alpha_1}}{\partial x_{\lambda_1}} \cdots \frac{\partial x'_{\alpha_r}}{\partial x_{\lambda_r}}\right] S^{\lambda_1 \ldots \lambda_r}_{\mu_1 \ldots \mu_s} \quad \ldots (2)$$

If the sum of two tensors $T^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_s}$ and $S^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_s}$ be a third tensor

$$U'^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_s},\ i.e.,\ U^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_s} = T^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_s} + S^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_s} \quad \ldots (3)$$

So that
$$U'^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_s} = T'^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_s} + S'^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_s}$$

$$= \left|\frac{\partial x}{\partial x'}\right|^N \left[\frac{\partial x_{\mu_1}}{\partial x'_{\beta_1}}\cdots\frac{\partial x_{\mu_s}}{\partial x'_{\beta_s}}\right]\left[\frac{\partial x'_{\alpha_1}}{\partial x_{\lambda_1}}\cdots\frac{\partial x'_{\alpha_r}}{\partial x'_{\lambda_r}}\right]\left[T^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_s} + S^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_s}\right]$$

$$= \left|\frac{\partial x}{\partial x'}\right|^N \frac{\partial x_{\mu_1}}{\partial x'_{\beta_1}}\cdots\frac{\partial x_{\mu_s}}{\partial x'_{\beta_s}}\frac{\partial x'_{\alpha_1}}{\partial x_{\lambda_1}}\cdots\frac{\partial x'_{\alpha_r}}{\partial x'_{\lambda_r}} U^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_s} \text{ by (3)} \quad \ldots (4)$$

Which is transformation equation for a tensor of rank N having r contravariant and s covariant indices and follows that the sum of two tensors of the same rank is a new tensor of the same rank.

Note. Here $\left|\frac{\partial x}{\partial x'}\right|$ is the *Jacobian of transformation* and the tensor $T^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_s}$ is known as *Relative tensor* of *weight* W. For W=0, the relative tensor becomes *Absolute tensor*, whereas for W=1, the relative tensor is known as *Tensor density*.

Rule II. *The direct product of two tensors gives a new tensor of rank equal to the sum of ranks of these tensors.*

Consider two tensors $T^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_s}$ of rank N weight W and $S^{\sigma_1\ldots\sigma_p}_{\rho_1\ldots\rho_q}$ of rank N' weight W'. Their transformation may be given as,

$$T'^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s} = \left|\frac{\partial x}{\partial x'}\right|^N \frac{\partial x_{\mu_1}}{\partial x'_{\beta_1}}\cdots\frac{\partial x_{\mu_s}}{\partial x'_{\beta_s}}\frac{\partial x'_{\alpha_1}}{\partial x_{\lambda_1}}\cdots\frac{\partial x'_{\alpha_r}}{\partial x'_{\lambda_r}} T^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_s}$$

$$S'^{\eta_1\ldots\eta_r}_{\xi_1\ldots\xi_q} = \left|\frac{\partial x}{\partial x'}\right|^{N'} \frac{\partial x_{\rho_1}}{\partial x'_{\xi_1}}\cdots\frac{\partial x_{\rho_q}}{\partial x'_{\xi_q}}\frac{\partial x'_{\eta_1}}{\partial x_{\sigma_1}}\cdots\frac{\partial x'_{\eta_p}}{\partial x'_{\sigma_p}} S^{\sigma_1\ldots\sigma_p}_{\rho_1\ldots\rho_q}$$

Then $T'^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s} S'^{\eta_1\ldots\eta_p}_{\xi_1\ldots\xi_q}$

$$= \left|\frac{\partial x}{\partial x'}\right|^{N+N'} \frac{\partial x_{\mu_1}}{\partial x'_{\beta_1}}\cdots\frac{\partial x_{\mu_s}}{\partial x'_{\beta_s}}\frac{\partial x'_{\alpha_1}}{\partial x_{\lambda_1}}\cdots\frac{\partial x'_{\alpha_r}}{\partial x_{\lambda_r}}\frac{\partial x_{\rho_1}}{\partial x'_{\xi_1}}\cdots\frac{\partial x'_{\rho_q}}{\partial x'_{\xi_q}}$$

$$, \frac{\partial x'_{\eta_1}}{\partial x_{\sigma_1}}\cdots\frac{\partial x'_{\eta_p}}{\partial x_{\sigma_p}} \times \left[T^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_s}\right]\left[S^{\sigma_1\ldots\sigma_p}_{\rho_1\ldots\rho_q}\right]$$

or
$$U'^{\alpha_1\ldots\alpha_s,\ \eta_1\ldots\eta_p}_{\beta_1\ldots\beta_s,\ \xi_1\ldots\xi_q} = \left|\frac{\partial x}{\partial x'}\right|^{N+N'} \frac{\partial x_{\mu_1}}{\partial x'_{\beta_1}}\cdots\frac{\partial x_{\mu_s}}{\partial x'_{\beta_s}}\frac{\partial x'_{\alpha_1}}{\partial x_{\lambda_1}}\cdots\cdots\frac{\partial x_{\alpha_r}}{\partial x_{\lambda_r}}$$

$$\times\frac{\partial x_{\rho_1}}{\partial x'_{\xi_1}}\cdots\frac{\partial x_{\rho_q}}{\partial x'_{\xi_q}}\frac{\partial x'_{\eta_1}}{\partial x_{\sigma_1}}\cdots\frac{\partial x'_{\eta_p}}{\partial x_{\sigma_p}} \times U^{\lambda_1\ldots\lambda_r,\ \sigma_1\ldots\sigma_p}_{\mu_1\ldots\mu_s,\ \rho_1\ldots\rho_q} \quad \ldots (5)$$

where
$$U^{\lambda_1\ldots\lambda_r,\ \sigma_1\ldots\sigma_p}_{\mu_1\ldots\mu_s,\ \rho_1\ldots\rho_q} = T^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_s} S^{\sigma_1\ldots\sigma_p}_{\rho_1\ldots\rho_q}$$

the equation (5) transforms a tensor of rank $N + N'$ and weight $W + W'$.

Note. This rule may also be stated as:

The outer product of two relative tensors is itself a relative tensor of rank and weight equal to the sum of the ranks and the sum of weights of the given relative tensors respectively.

Rule III. Contraction. *The algebraic operation by which the rank of a tensor may be lowered by 2 (or by any even number) is known as contraction.* (*Agra, 1969*)

The contraction of a tensor may be affected by adding up all the components which have equal indices in a given pair. Any two indices are converted into a pair of dummy indices.

Consider a tensor of rank 3 with one contravariant index α and two covariant indices β and γ. Then we have

$$A'^{\lambda}_{\mu\nu} = \sum_{\alpha,\beta,\gamma=0}^{3} A^{\alpha}_{\beta\lambda} \frac{\partial x_\beta}{\partial x'_\mu} \cdot \frac{\partial x_\gamma}{\partial x'_\nu} \cdot \frac{\partial x'_\lambda}{\partial x_\alpha}$$

Replacing ν by λ, we have

$$A'^{\lambda}_{\mu\nu} = \sum_{\lambda,\mu,\lambda=0}^{3} A^{\alpha}_{\beta\gamma} \frac{\partial x_\beta}{\partial x'_\mu} \cdot \frac{\partial x_\gamma}{\partial x'_\lambda} \cdot \frac{\partial x'_\lambda}{\partial x_\alpha}$$

$$= \sum_{\alpha,\beta,\gamma=0}^{3} A^{\alpha}_{\beta\gamma} \frac{\partial x_\beta}{\partial x'_\mu} \cdot \frac{\partial x_\gamma}{\partial x_\alpha}$$

But $$\frac{\partial x_\gamma}{\partial x_\alpha} = \begin{Bmatrix} 0 \text{ if } \gamma \neq \alpha \\ 1 \text{ if } \gamma = \alpha \end{Bmatrix}$$

Choosing the second condition *i.e.* if $\gamma = \alpha$, $\dfrac{\partial x_\gamma}{\partial x_\alpha} = 1$, above relation becomes

$$A'^{\lambda}_{\mu\lambda} = \sum_{\alpha,\beta=0}^{3} A^{\alpha}_{\beta\alpha} \frac{\partial x_\beta}{\partial x'_\mu}$$

i.e., $$A'_\mu = \sum_{\beta=0}^{3} \frac{\partial x_\beta}{\partial x'_\mu} A_\beta$$

Which denotes the law of transformation of tensors of rank one, *i.e.*, vectors. As a general rule we equate a certain covariant index to a contravariant index, sum on repeated indices, and obtain a new tensor of lower rank. This process is termed as **contraction.** The contraction of a tensor of rank 2 yields a scalar *i.e.*, tensor of rank zero.

Illustration. We know that the scalar product of two vectors is a scalar quantity. It follows that the scalar product of two tensors of rank one is a tensor of rank zero. As such the rank is lowered by two.

Rule IV. Extension of the rank. *The differentiation of each components of a tensor of rank n with respect to x, y, z, gives a new tensor of rank* $(n + 1)$, *e.g.*,

$$\frac{\partial A_{\lambda\mu}}{\partial x_\nu} = B_{\lambda\mu\nu} \qquad \ldots (6)$$

This rule may be proved for a simple case, where the original tensor is of rank zero, *i.e.*, a scalar say $S(x_1, x_2, x_3, t)$ where derivatives relative to the axes K are

$$\frac{\partial S}{\partial x}, \frac{\partial S}{\partial y}, \frac{\partial S}{\partial z}.$$

In system K', the scalar is $S'(x'_1, x'_2, x'_3, t')$, such that

$$S(x_1, x_2, x_3, t) = S'(x'_1, x'_2, x'_3, t')$$

So
$$\left.\begin{aligned}\frac{\partial S'}{\partial x'_1} &= \frac{\partial S}{\partial x_1}\frac{\partial x_1}{\partial x'_1} + \frac{\partial S}{\partial x_2}\frac{\partial x_2}{\partial x'_1} + \frac{\partial S}{\partial x_3}\frac{\partial x_3}{\partial x'_1}\\ \frac{\partial S'}{\partial x'_2} &= \frac{\partial S}{\partial x_1}\frac{\partial x_1}{\partial x'_2} + \frac{\partial S}{\partial x_2}\frac{\partial x_2}{\partial x'_2} + \frac{\partial S}{\partial x_3}\frac{\partial x_3}{\partial x'_2}\\ \frac{\partial S'}{\partial x'_3} &= \frac{\partial S}{\partial x_1}\frac{\partial x_1}{\partial x'_3} + \frac{\partial S}{\partial x_2}\frac{\partial x_2}{\partial x'_3} + \frac{\partial S}{\partial x_3}\frac{\partial x_3}{\partial x'_3}\end{aligned}\right\}$$

which may be written as a single equation

$$\frac{\partial S'}{\partial x'_l} = \frac{\partial x_m}{\partial x'_l}\frac{\partial S}{\partial x_m} \quad (l, m = 1, 2, 3)$$

This shows that $\frac{\partial S}{\partial x_m}$ transforms like the components of a tensor of rank one, *i.e.* vector. Thus the differentiation of the tensor of rank zero, yields a tensor of rank one. The rank of a tensor can also be extended when a tensor depends on another tensor and a differentiation is performed. For example, consider a scalar, *i.e.*, tensor of rank zero, say S depending on tensor $A_{\lambda\mu}$, so that

$$\frac{\partial S}{\partial A_{\lambda\mu}} = B_{\mu\nu}$$

is also a tensor of rank two. Thus the rank of the tensor is extended by two.

Rule V. The Quotient Law. *If $A^\lambda B_{\mu\nu}$ is a tensor for all contravariant tensors A_λ then $B_{\mu\nu}$ is also a tensor.* (*Rohilkhand, 1980, 86*)

We have
$$A'^\lambda B'_{\mu\nu} = A^\alpha B_{\beta\gamma}\frac{\partial \alpha_\beta}{\partial x'_\mu}\frac{\partial x_\gamma}{\partial x'_\nu}\frac{\partial x'_\lambda}{\partial x_\alpha}$$

$$= A'^\lambda B_{\beta\gamma}\frac{\partial x_\beta}{\partial x'_\mu}\frac{\partial x_\gamma}{\partial x'_\nu}$$

as
$$A'^\lambda = A^\alpha \frac{\partial x'_\lambda}{\partial x_\alpha} \quad \ldots (7)$$

or
$$A'^\lambda\left[B'_{\mu\nu} - B_{\mu\nu}\frac{\partial x_\alpha}{\partial x'_\mu}\frac{\partial x_\gamma}{\partial x'_\nu}\right] = 0$$

But A^λ being arbitrary, $A^\lambda \neq 0$ so that

$$B'_{\mu\nu} = B_{\mu\nu}\frac{\partial x_\beta}{\partial x'_\mu}\frac{\partial x_\gamma}{\partial x'_\nu} \quad \ldots (8)$$

which shows that $B_{\mu\nu}$ is a covariant tensor.

Note. If A be a symmetric covariant tensor of second order s.t. $|A_{\mu\nu}| = A \neq 0$, and we set,

$$A^{\mu\nu} = \frac{\text{Cofactor of } A_{\mu\nu} \text{ in } A}{A} = \frac{a_{\mu\nu}}{A} \quad \ldots (9)$$

$a_{\mu\nu}$ being cofactor of $A_{\mu\nu}$ in A, and $A_{\mu\nu}$ also being symmetric, then A and so $a_{\mu\nu}$ is symmetric. Consequently $A^{\mu\nu}$ is symmetric.

Also if B^μ be an arbitrary vector, then quotient law gives

$$B_\nu = A_{\mu\nu} B^\mu \qquad \ldots (10)$$

as an covariant vector

$$\therefore \quad B_\nu A^{\nu\sigma} = A_{\mu\nu} B^\mu A^{\mu\sigma} = A_{\mu\nu} B^\mu \frac{a_{\nu\sigma}}{A}$$

$$= \frac{A_{\mu\nu}\, a_{\nu\sigma}}{A} B^\mu = \delta^\sigma_\mu B^\mu \text{ by determinant theory.}$$

or

$$B_\nu A^{\nu\sigma} = B^\sigma \qquad \ldots (11)$$

Here (9)⟹ *symmetric contravariant tensor of rank 2*, known as **conjugate** or **reciprocal** tensor of $A_{\mu\nu}$.

Problem 9. *Show that there exists no distinction between contravariant and covariant vectors if we restrict ourselves to transformation of the type*

$$x'_\alpha = a^\alpha{}_\lambda x_\lambda + b^\alpha$$

where b^α are n constants—which do not necessarily form the components of a contravariant vector and $a_\lambda{}^\alpha$ are constant (not necessarily forming a tensor) such that

$$a^\alpha_\mu\, a^\alpha_\lambda = \delta^\mu_\lambda$$

Given

$$x'_\alpha = a^\alpha_\lambda x_\lambda + b^\alpha \qquad \ldots (1)$$

i.e.,

$$a^\alpha_\lambda x_\lambda = x'_\alpha - b^\alpha \qquad \ldots (2)$$

Multiplying (2) throughout by $a_\mu{}^\alpha$ and summing over the index α from 1 to n, we find

$$x_\mu = a^\alpha_\mu\, x'_\alpha - a^\alpha_\mu\, b^\alpha \qquad \ldots (3)$$

Now (1) and (3) yield,

$$\frac{\partial x'_\alpha}{\partial x_\beta} = a^\alpha_\beta \text{ and } \frac{\partial x_\beta}{\partial x'_\alpha} = a^\alpha_\beta$$

so that

$$\frac{\partial x'_\alpha}{\partial x_\beta} = \frac{\partial x_\beta}{\partial x'_\alpha} = a^\alpha_\beta$$

This follows that the transformation laws

$$A'^\mu = \frac{\partial x'_\mu}{\partial x_\beta} A^\beta \text{ and } A'_\mu = \frac{\partial x_\alpha}{\partial x'_\mu} A_\alpha$$

define the same type of entity without any distinction between contravariant and covariant vectors.

Problem 10. *Show that* $\dfrac{\partial A_\lambda}{\partial x_\mu}$ *is not a tensor although A_λ is a covariant tensor of rank one.*

We have $A'_\alpha = \dfrac{\partial x_\lambda}{\partial x'_\alpha} A_\lambda$ (by covariant law)

Differentiating w.r.t. x'_β we get

$$\frac{\partial A'_\alpha}{\partial x'_\beta} = \frac{\partial x_\lambda}{\partial x'_\alpha} \cdot \frac{\partial A_\lambda}{\partial x'_\beta} + \frac{\partial^2 x_\lambda}{\partial x'_\alpha\, x'_\beta} A_\lambda$$

$$= \frac{\partial x_\lambda}{\partial x'_\alpha} \frac{\partial A_\lambda}{\partial x_\mu} \cdot \frac{\partial x_\mu}{\partial x'_\beta} + \frac{\partial^2 x_\lambda}{\partial x'_\alpha \, \partial x'_\beta} A_\lambda$$

$$= \frac{\partial x_\lambda}{\partial x'_\alpha} \frac{\partial x_\mu}{\partial x'_\beta} \frac{\partial A_\lambda}{\partial x_\mu} + \frac{\partial^2 x_\lambda}{\partial x'_\beta \, \partial x'_\alpha} A_\lambda$$

The presence of the second term on the right shows that $\frac{\partial A_\lambda}{\partial x_\mu}$ does not transform as a tensor.

Problem 11. *If xy, $2y - z^2$ and xz are the components of a covariant tensor in rectangular coordinates, then find its covariant components in spherical coordinates.*

(Kanpur, 1978)

Suppose that $x = x_1$, $y = x_2$, $z = x_3$

Then the covariant components of a tensor A_λ are

$A_1 = xy = x_1 x_2$

$A_2 = 2y - z^2 = 2x_2 - x_3^2$

$A_3 = xz = x_1 x_3$

Let A'_μ be the covariant components in spherical coordinates. Then

$$x'_1 = r, x'_2 = \theta, x'_3 = \phi$$

and $$A'_\mu = \frac{\partial x_\lambda}{\partial x'_\mu} A_\lambda \qquad \dots (1)$$

Now the transformation equations are

$$x = r \sin\theta \cos\phi, \; y = r \sin\theta \sin\phi, \; z = r\cos\theta$$

which in existing case become

$$x_1 = x'_1 \sin x'_2 \cos x'_3, \; x_2 = x'_1 \sin x'_2 \sin x'_3, \; x_3 = x'_1 \cos x'_2.$$

From (1) we have

$$A'_1 = \frac{\partial x_1}{\partial x'_1} A_1 + \frac{\partial x_2}{\partial x'_1} A_2 + \frac{\partial x_3}{\partial x'_1} A_3$$

$$= (\sin x'_2 \cos x'_3)\, x_1 x_2 + \sin x'_2 \sin x'_3 \,(2x_2 - x_3^2) + \cos x'_2 . x_1 x_3$$

$$= (\sin\theta\cos\phi)(r^2 \sin^2\theta \sin\phi\cos\phi) + \sin\theta\sin\phi\,(2r\sin\theta\sin\phi - r^2\cos^2\theta) + \cos\theta\,(r^2 \sin\theta\cos\theta\cos\phi)$$

Similarly,

$$A'_2 = (r\cos\theta\cos\phi)(r^2\sin^2\theta\sin\phi\cos\phi) + (r\cos\theta\sin\phi)(2r\sin\theta\sin\phi - r^2\cos^2\theta) + (-r\sin\theta)(r^2\sin\theta\cos\theta\cos\phi)$$

and $$A'_3 = (-r\sin\theta\sin\phi)(r^2\sin^2\theta\sin\phi\cos\phi) + (r\sin\theta\cos\phi)(2r\sin\theta\sin\phi - r^2\cos^2\theta)$$

Problem 12. *If $A_{\lambda\mu}$ is a skew-symmetric tensor, show that*

$$\left(B^\mu_\nu B^\sigma_\tau + B^\mu_\tau B^\sigma_\nu\right) A_{\mu\sigma} = 0$$

$\because$ $A_{\mu\nu}$ is skew-symmetric

$\therefore$ $A_{11} = A_{22} = \dots = 0$ and $A_{12} = -A_{21}$ etc. $\qquad \dots (1)$

Now left hand expression $= B^\mu_\nu B^\sigma_\tau + B^\mu_\tau B^\sigma_\nu$

$$= \left(B_\nu^1 B_\tau^\sigma + B_\tau^1 B_\nu^\sigma\right) A_{1\sigma} + \left(B_\nu^2 B_\tau^\sigma + B_\tau^2 B_\nu^\sigma\right) A_{2\sigma} + \ldots$$

$$= \left(B_\nu^1 B_\tau^1 + B_\tau^1 B_\nu^1\right) A_{11} + \left(B_\nu^2 B_\tau^2 + B_\tau^2 B_\nu^2\right) A_{22} + \ldots$$

$$+ \left(B_\nu^1 B_\tau^2 + B_\tau^1 B_\nu^2\right) A_{12} + \left(B_\nu^2 B_\tau^1 + B_\tau^1 B_\nu^1\right) A_{21} + \ldots$$

$$= 0 + 0 \ldots + \left(B_\nu^1 B_\tau^2 + B_\tau^1 B_\nu^2\right) \{A_{12} + A_{21}\} + \ldots \text{ by (1)}$$

$$= \left(B_\nu^1 B_\tau^2 + B_\tau^1 B_\nu^2\right) \{0\} \text{ by (1)}$$

$$= 0.$$

Problem 13. *Show that every tensor can be expressed as the sum of two tensors, one of which is symmetric and other skew-symmetric in a pair of covariant or contravariant indices.* (*Rohilkhand, 1984, 87, 88; Meerut, 1980, 82, 84*)

Since $A_{\mu\nu} = \frac{1}{2}\left[A_{\mu\nu} + A_{\nu\mu}\right] + \frac{1}{2}\left[A_{\mu\nu} - A_{\nu\mu}\right]$

$= B_{\mu\nu} + C_{\mu\nu}$ (say)

where $B_{\mu\nu} = \frac{1}{2}\left[A_{\mu\nu} + A_{\nu\mu}\right] = \frac{1}{2}\left[A_{\nu\mu} + A_{\mu\nu}\right] = B_{\nu\mu}$ is symmetric

and $C_{\mu\nu} = \frac{1}{2}\left[A_{\mu\nu} - A_{\nu\mu}\right]$

$= -\frac{1}{2}\left[A_{\nu\mu} - A_{\mu\nu}\right] = -C_{\nu\mu}$ is anti-symmetric

Hence the proposition.

Problem 14. *Show that $A^\tau{}_{\mu\sigma\nu}$ is a tensor* (*Agra, 1967*)

We have $$A'^{\delta}_{\alpha\beta\gamma} = \frac{\partial x'_\delta}{\partial x_\tau} \frac{\partial x_\mu}{\partial x'_\alpha} \frac{\partial x_\sigma}{\partial x_\beta} \frac{\partial x_\nu}{\partial x'_\gamma} A^\tau_{\mu\sigma\nu}$$

which follows that $A^\tau{}_{\mu\sigma\nu}$ according to transformation law, is a mixed tensor of rank 4, contravariant of order 1 and covariant of order 3.

Problem 15. *If A^λ and B_μ are the components of a contravariant and covariant tensors of rank one, then show that $C_\mu{}^\lambda = A^\lambda B_\mu$ are the components of a mixed tensor of rank two.* (*Rohilkhand, 1979; Meerut, 1983*)

We have $A'_\lambda = A^\alpha \dfrac{\partial x'_\lambda}{\partial x_\alpha}$ and $B'_\mu = B_\beta \dfrac{\partial x_\beta}{\partial x'_\mu}$

So that $$C'^{\lambda}_{\mu} = A'^{\lambda} B'_\mu = A^\alpha B_\beta \frac{\partial x'_\lambda}{\partial x_\alpha} \frac{\partial x_\beta}{\partial x'_\mu} = A^\alpha B_\beta \frac{\partial x'_\lambda}{\partial x_\alpha} \frac{\partial x_\beta}{\partial x'_\mu}$$

which shows that $C_\mu{}^\lambda$ transforms as a mixed tensor of rank two.

Problem 16. *Show that the contracted tensor. $A^{\nu\sigma\rho}_{\lambda\mu\nu}$ is a mixed tensor.*

We have

$$A'^{\delta\xi\eta}_{\alpha\beta\gamma} = \frac{\partial x'_\delta}{\partial x_\tau} \frac{\partial x'_\xi}{\partial x_\sigma} \frac{\partial x'_\eta}{\partial x_\rho} \frac{\partial x_\lambda}{\partial x'_\alpha} \frac{\partial x_\mu}{\partial x'_\beta} \frac{\partial x_\nu}{\partial x'_\gamma} A^{\tau\sigma\rho}_{\lambda\mu\nu}$$

Setting $\delta = \gamma$ and summing we get

$$A'^{\gamma\xi\eta}_{\alpha\beta\gamma} = \frac{\partial x'_\xi}{\partial x_\sigma}\frac{\partial x'_\eta}{\partial x_\rho}\frac{\partial x_\lambda}{\partial x'_\alpha}\frac{\partial x_\mu}{\partial x'_\beta}\frac{\partial x_\nu}{\partial x_\tau} A^{\tau\sigma\rho}_{\lambda\mu\nu}$$

$$= \frac{\partial x'_\xi}{\partial x_\sigma}\frac{\partial x'_\eta}{\partial x_\rho}\frac{\partial x_\lambda}{\partial x'_\alpha}\frac{\partial x_\mu}{\partial x'_\beta}\delta^\nu_\tau A^{\tau\sigma\rho}_{\lambda\mu\nu}$$

$$= \frac{\partial x'_\xi}{\partial x_\sigma}\frac{\partial x'_\eta}{\partial x_\rho}\frac{\partial x_\lambda}{\partial x'_\alpha}\frac{\partial x_\mu}{\partial x'_\beta} A^{\nu\sigma\rho}_{\lambda\mu\nu}$$

If we compare it with the transformation law

$$A'^{\xi\eta}_{\alpha\beta} = \frac{\partial x'_\xi}{\partial x_\sigma}\frac{\partial x'_\eta}{\partial x_\rho}\frac{\partial x_\lambda}{\partial x'_\alpha}\frac{\partial x_\mu}{\partial x'_\beta} A^{\sigma\rho}_{\lambda\mu}$$

then we conclude that $A^{\nu\sigma\rho}_{\lambda\mu\nu}$ is a tensor of rank four, contravariant of rank 2 and covariant of rank 2.

Problem 17. *If the components of a tensor are zero in one co-ordinate system, then prove that the components are zero in all co-ordinate systems.* *(Rohilkhand, 1989)*

Consider the components of a tensor in the form

$$T^{\lambda_1, \lambda_2, \ldots \lambda_r}_{\mu_1, \mu_2, \ldots \mu_s}$$

where the indices $\lambda_1, \lambda_2, \ldots \lambda_r, \mu_1, \mu_2, \ldots \mu_s$ run through the integers 1, 2, ... n.

The components then transform according to the rule

$$T'^{\lambda_1, \lambda_2, \ldots \lambda_r}_{\mu_1, \mu_2, \ldots \mu_s} = \left|\frac{\partial x}{\partial x'}\right|^N \frac{\partial x'_{\lambda_1}}{\partial x_{\alpha_1}}\frac{\partial x'_{\lambda_2}}{\partial x_{\alpha_2}}\cdots\frac{\partial x'_{\lambda_r}}{\partial x_{\alpha_r}}\frac{\partial x_{\beta_1}}{\partial x'_{\mu_1}}\frac{\partial x_{\beta_2}}{\partial x'_{\mu_2}}\cdots\frac{\partial x_{\beta_s}}{\partial x'_s} T^{\alpha_1, \alpha_2, \ldots \alpha_r}_{\beta_1, \beta_2, \ldots \beta_s}$$

where $\left|\frac{\partial x}{\partial x'}\right|$ represents at Jacobian and N is the weight of a tensor field.

In this equation $T'^{\lambda_1, \lambda_2, \ldots \lambda_r}_{\mu_1, \mu_2, \ldots \mu_s}$ are the components of a tensor in K' system of reference while $T^{\alpha_1, \alpha_2, \ldots \alpha_r}_{\beta_1, \beta_2, \ldots \beta_s}$ are those in the system of reference K. Hence if $T^{\alpha_1, \alpha_2, \ldots \alpha_r}_{\beta_1, \beta_2, \ldots \beta_s}$ are zero then $T'^{\lambda_1, \lambda_2, \ldots \lambda_r}_{\mu_1, \mu_2, \ldots \mu_s}$ are also zero.

Similarly its components are zero in all the other coordinate systems.

Problem 18. *A quantity $A(\mu, \nu, \sigma, \tau)$ function of coordinates x_i transforms to another system of coordinates as*

$$A'(\alpha, \beta, \gamma, \xi) = \frac{\partial x_\mu}{\partial x'_\alpha}\frac{\partial x'_\beta}{\partial x_\nu}\frac{\partial x'_\gamma}{\partial x_\sigma}\frac{\partial x'_\xi}{\partial x_\tau} A(\mu, \nu, \sigma, \tau)$$

Is it a tensor ? If so give its order and rank.

Yes it is a tensor as it transforms according to the transformation law of mixed tensors of rank 4 having a contravariant order 3 and covariant order 1.

3.8. THE FUNDAMENTAL TENSORS *(Rohilkhand, 1980)*

In §3.3, we have already shown that the interval ds between the two neighbouring events (x_1, x_2, x_3, x_4) and $(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3, x_4 + dx_4)$ considered in Minkowski's four dimensional space, is given by

$$ds^2 = g_{\mu\nu}\, dx_\mu\, dx_\nu \quad (\mu, \nu = 1, 2, 3, 4)$$

(in summation convention by § 3.3 (2)

and $\qquad g_{\mu\nu} = g_{\nu\mu}$

A convenient way of writing it, is adopted as

$$ds^2 = g_{\mu\nu}\,(dx)^\mu\,(dx)^\nu. \qquad \text{... (1)}$$

Now ds^2 being independent of the coordinate system, is an invariant or tensor of zero rank and therefore the equation (1) shows that $g_{\mu\nu}\,(dx)^\mu$ multiplied by an arbitrary contravariant vector $(dx)^\nu$ always gives a tensor of zero rank; hence $g_{\mu\nu}\,(dx)^\mu$ is a vector since the dot product of two vectors gives a scalar *i.e.* a tensor of zero rank. Agian $g_{\mu\nu}$ multiplied by an arbitrary contravariant vector $(dx)^\mu$ always gives a vector and hence $g_{\mu\nu}$ is covariant tensor (by quotient law).

To show that $g_{\mu\nu}$ is a covariant tensor, let us assume that $g_{\mu\nu}$ becomes $g'_{\mu\nu}$ when the coordinate system becomes (x') from (x). Then, ds^2 being invariant, we have

$$ds^2 = g'_{\mu\nu}\,(dx')^\mu\,(dx')^\nu \qquad \text{... (2)}$$

From (1) and (2), it follows that

$$g'_{\mu\nu}\,(dx')^\mu\,(dx')^\nu = g_{\mu\nu}\,(dx)^\mu\,(dx)^\nu$$

i.e.,
$$g'_{\mu\nu}\,\frac{\partial x'_\mu}{\partial x_\alpha}\,\frac{\partial x'_\nu}{\partial x_\beta}\,dx_\alpha\,dx_\beta = g_{\mu\nu}\,(dx)^\mu\,(dx)^\nu$$

$$= g_{\alpha\beta}\,dx_\alpha\,dx_\beta \text{ since } dx'_\mu = \frac{\partial x'_\mu}{\partial x_\alpha}\,dx_\alpha$$

and μ, ν are dummy suffixes on the R.H.S.

The equation is true if

$$g'_{\mu\nu}\,\frac{\partial x'_\mu}{\partial x_\alpha}\,\frac{\partial x'_\nu}{\partial x_\beta} = g_{\alpha\beta} \qquad \text{... (3)}$$

Interchanging the primed and unprimed letters we get

$$g'_{\alpha\beta} = \frac{\partial x_\mu}{\partial x'_\alpha}\,\frac{\partial x_\nu}{\partial x'_\beta} = g_{\mu\nu} \qquad \text{... (4)}$$

Let us further assume that g stands for the determinant

$$g = \begin{vmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{vmatrix}, \qquad \text{... (5)}$$

and $g^{\mu\nu}$ is defined as the cofactor of $g_{\mu\nu}$ in determinant (5), divided by g, *i.e.*,

$$g^{\mu\nu} = \frac{\text{cofactor of } g_{\mu\nu} \text{ in } g}{g}, \qquad \text{... (6)}$$

Considering the inner product $g_{\mu\sigma}\,g^{\nu\sigma}$ we have,

$$g_{\mu\sigma}\,g^{\nu\sigma} = g_{\mu 1}\,g^{\nu 1} + g_{\mu 2}\,g^{\nu 2} + g_{\mu 3}\,g^{\nu 3} + g_{\mu 4}\,g^{\nu 4}$$

which with the help of assumption (6), becomes

$$g_{\mu\sigma}\, g^{\nu\sigma} = \frac{1}{g}\,[g_{\mu 1}.\ \text{cofactor of } g_{\nu 1} \text{ in } g + g_{\mu 2}.\ \text{cofactor of } g_{\nu 2} \text{ in } g$$

$$+ g_{\mu 3}.\ \text{cofactor of } g_{\nu 3} \text{ in } g + g_{\mu 4} \text{ cofactor of } g_{\nu 4} \text{ in } g]$$

$$= \frac{1}{g} \times 0 \quad i.e.\ 0 \ \text{ when } \mu \neq \nu$$

$$= \frac{1}{g} \times g \quad i.e.\ 1 \ \text{ when } \mu = \nu.$$

For, when $\mu \neq \nu$, we get a determinant with two rows indential and when $\mu = \nu$, we reproduce the determinant g divided by itself.

We write

$$\left.\begin{aligned} g_\mu^\nu &= g_{\mu\sigma}\, g^{\nu\sigma} \\ &= 0 \text{ if } \mu \neq \nu \\ &= 1 \text{ if } \mu = \nu \end{aligned}\right\} \qquad \dots (7)$$

It is therefore clear in view of §3.3, that $g_\mu{}^\nu$ has not the same property as the substitution operator or the Kronecker delta $\delta_\mu{}^\nu$.

Thus,

$$g_\mu^\nu\ A^\mu = A^\nu + 0 + 0 + 0 \ \text{ by } (7)$$

$$= A^\nu. \qquad \dots (8)$$

The equation (8) shows that $g_\mu{}^\nu$ multiplied by a contravariant vector yields a vector and hence by quotient law, $g_\mu{}^\nu$ is a tensor such that its components are the same in all coordinate systems, because if $g_\mu{}^\nu$ becomes $g'_\mu{}^\nu$ in another system of coordinates then

$$g_\mu^{\prime\nu} = \frac{\partial x_\alpha}{\partial x'_\mu} \cdot \frac{\partial x'_\nu}{\partial x_\beta} \cdot g_\alpha^\beta$$

$$= \frac{\partial x'_\nu}{\partial x_\alpha} \cdot \frac{\partial x_\alpha}{\partial x'_\mu} \quad \text{since} \ \frac{\partial x'_\nu}{\partial x_\alpha} = g_\alpha^\beta \frac{\partial x'_\nu}{\partial x_\beta}$$

$$\left.\begin{aligned} &= 0 \text{ if } \mu \neq \nu \\ &= 1 \text{ if } \mu = \nu \end{aligned}\right\} \text{by } (7)$$

Hence $g'_\mu{}^\nu = g_\mu{}^\nu$ ($\mu, \nu = 1, 2, 3, 4$)

Again $g_\nu{}^\nu$ has the same meaning as $g_\mu{}^\nu$ with $\mu = \nu$ and we have

$$g_\nu{}^\nu = g_1{}^1 + g_2{}^2 + g_3{}^3 + g_4{}^4$$

$$= 1 + 1 + 1 + 1 \text{ by } (7)$$

$$= 4. \qquad \dots (10)$$

Further, we shall show that $g^{\nu\sigma}$ is a tensor when $g_{\mu\sigma}\, g^{\nu\sigma}$ is a tensor. Multiplying the covariant vector $g_{\mu\sigma} A^\mu$ by the vector $g^{\nu\sigma}$, we have

$$g_{\mu\sigma}\, g^{\nu\sigma} A^\mu = g_\mu{}^\nu A^\mu \text{ by } (7)$$

$$= A^\nu \text{ by } (8)$$

which shows that the product is always a vector.

The tensor character of $g^{\mu\nu}$ may also be shown by denoting the covariant vector $g_{\mu\nu} A^{\nu}$ by B_{μ}. Then we have

$$g_{11} A^1 + g_{12} A^2 + g_{13} A^3 + g_{14} A^4 = B_1$$

with similar three equations for B_2, B_3 and B_4.

Solving these four equations by the method of determinants we can easily find

$$A^1 = g^{11} B_1 + g^{12} B_2 + g^{13} B_3 + g^{14} B_4$$

with similar expressions for A^2, A^3 and A^4.

So that $\quad A^{\mu} = g^{\mu\nu} B_{\nu}$

which follows by quotient law that $g^{\mu}{}_{\nu}$ is a tensor.

Hence we have defined in this article, the three fundamentsl tensors

$$g_{\mu\nu}, g_{\mu}{}^{\nu}, g^{\mu\nu}$$

of covariant, mixed and contravariant characters, respectively.

Problem 19. *Transform $ds^2 = dx^2 + dy^2 + dz^2$ into polar and cylindrical coordinates.*

The transformation equations from cartesian to polar coordinates are

$$x = r \sin\theta \cos\phi,\ y = r \sin\theta \sin\phi,\ z = r\cos\theta.$$

If a point (x_1, x_2, x_3) in cartesian becomes (x'_1, x'_2, x'_3) in polar system of coordinates, then

$$x_1 = x, x_2 = y, x_3 = z$$

and $\quad x'_1 = r, x'_2 = \theta, x'_3 = \phi.$

So that $\quad g_{11} = g_{22} = g_{33} = 1$

and $\quad g_{12} = g_{13} = g_{23} = 0.$

Now, from equation (4) of §3.8, we have

$$g'_{\alpha\beta} = \frac{\partial x_{\mu}}{\partial x'_{\alpha}} \frac{\partial x_{\nu}}{\partial x'_{\beta}} g_{\mu\nu}$$

$$\therefore \quad g'_{11} = \frac{\partial x_{\mu}}{\partial x'_1} \frac{\partial x_{\nu}}{\partial x'_1} g_{\mu\nu}$$

$$= \frac{\partial x_1}{\partial x'_1}\frac{\partial x_1}{\partial x'_1} g_{11} + \frac{\partial x_2}{\partial x'_1}\frac{\partial x_2}{\partial x'_1} g_{22} + \frac{\partial x_3}{\partial x'_1}\frac{\partial x_3}{\partial x'_1} g_{33}$$

$$= \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2$$

, as $x_1 = x$ etc. and $g_{11} = g_{22} = g_{33} = 1$

$$= \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta$$

$$= \sin^2\theta + \cos^2\theta = 1.$$

Similarly $\quad g'_{22} = \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 = r^2$

and $\quad g'_{33} = \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2 = r^2 \sin^2\theta$

Also $$g'_{12} = \frac{\partial x_\mu}{\partial x'_1}\frac{\partial x_\nu}{\partial x'_2} g_{\mu\nu}$$
$$= \frac{\partial x_1}{\partial x'_1}\frac{\partial x_1}{\partial x'_2} g_{11} + \frac{\partial x_2}{\partial x'_1}\frac{\partial x_2}{\partial x'_2} g_{22} + \frac{\partial x_3}{\partial x'_1}\frac{\partial x_3}{\partial x'_2} g_{33}$$
$$= \frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \theta}$$
$$= \sin\theta\cos\phi\,(r\cos\theta\cos\phi) + \sin\theta\sin\phi . r\cos\theta\sin\phi + \cos\theta\,(-r\sin\theta)$$
$$= r\sin\theta\cos\theta\,(\cos^2\phi + \sin^2\phi) - r\sin\theta\cos\theta$$
$$= r\sin\theta\cos\theta - r\sin\theta\cos\theta = 0.$$

Similarly, $g'_{13} = 0 = g'_{23}$.

Hence $ds^2 = g'_{\mu\nu}(dx')^\mu (dx')^\nu$ gives
$$ds^2 = g'_{11}(dx'_1)^2 + g'_{22}(dx'_2)^2 + g'_{33}(dx'_3)^2$$
$$= dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2.$$

Again the cylindrical transformations are
$$x = r\cos\theta,\ y = r\sin\theta,\ z = z,$$
so that $x_1 = x,\ x_2 = y,\ x_3 = z$

and $x'_1 = r,\ x'_2 = \theta,\ x'_3 = z.$

Giving, $g_{11} = g_{22} = g_{33} = 1.$

Also as above and since $g_{12} = g_{13} = g_{23} = 0$, we have
$$g'_{11} = 1,\ g'_{22} = r^2,\ g'_{33} = 1$$
$$g'_{12} = 0 = g'_{13} = g'_{23}.$$

Hence $$ds^2 = g'_{11}(dx'_1)^2 + g'_{22}(dx'_2)^2 + g'_{33}(dx'_3)^2$$
$$= dr^2 + r^2 d\theta^2 + dz^2.$$

Problem 20. *Show that $g_{\mu\nu}$ is a covariant tensor of the second order.*

(Rohilkhand, 1981, 87, 89)

Let ds be the distance between two neighbouring points $P(y_\alpha)$ and $Q\,(y_\alpha + dy_\alpha)$, then
$$ds^2 = \sum_{\beta=1}^{n} dy_\beta\, dy_\beta$$

But $$dy_\beta = \frac{\partial y_\beta}{\partial x_\alpha} dx_\alpha.$$

$$\therefore \quad ds^2 = \sum_{\beta=1}^{n}\left(\frac{\partial y_\beta}{\partial x_\alpha} dx_\alpha\right)\left(\frac{\partial y_\beta}{\partial x_\gamma} dx_\gamma\right)$$
$$= g_{\alpha\gamma}\, dx_\alpha\, dx_\beta$$

where $$g_{\alpha\gamma} = \sum_{\beta=1}^{n} \frac{\partial y_\beta}{\partial x_\alpha}\frac{\partial y_\beta}{\partial x_\gamma} = \frac{\partial y_\beta}{\partial x_\alpha}\frac{\partial y_\beta}{\partial x_\gamma}$$

Hence in new coordinates,
$$g'_{\xi\eta} = \frac{\partial y_\beta}{\partial x'_\xi}\frac{\partial y_\beta}{\partial x'_\eta}$$

$$= \frac{\partial y_\beta}{\partial x_\mu} \frac{\partial x_\mu}{\partial x'_\xi} \frac{\partial y_\beta}{\partial x_\nu} \frac{\partial x_\nu}{\partial x'_\eta}$$

$$= \frac{\partial x_\mu}{\partial x'_\xi} \frac{\partial x_\nu}{\partial x'_\eta} \frac{\partial y_\beta}{\partial x_\mu} \frac{\partial y_\beta}{\partial x_\nu}$$

$$= \frac{\partial x_\mu}{\partial x'_\xi} \frac{\partial x_\nu}{\partial x'_\eta} g_{\mu\nu}$$

which follows that $g_{\mu\nu}$ is covariant tensor of second order.

Problem 21. ***Show that $g_{\alpha\beta}\, dx_\alpha\, dx_\beta$ is an invariant.***

$$g'_{\alpha\beta}\, dx'_\alpha\, dx'_\beta = g_{\alpha\beta}\, dx_\alpha\, dx_\beta. \qquad \textit{(Rohilkhand 1981, 89)}$$

We have
$$g_{\mu\nu} = g'_{\alpha\beta} \frac{\partial x'_\alpha}{\partial x_\mu} \frac{\partial x'_\beta}{\partial x_\nu}$$

or
$$\left(g'_{\alpha\beta} \frac{\partial x'_\alpha}{\partial x_\mu} \frac{\partial x'_\beta}{\partial x_\nu} - g_{\mu\nu} \right) = 0$$

or
$$\left(g'_{\alpha\beta} \frac{\partial x'_\alpha}{\partial x_\mu} \frac{\partial x'_\beta}{\partial x_\nu} - g_{\mu\nu} \right) dx_\mu\, dx_\nu = 0.$$

since dx_i is arbitrary

or
$$g'_{\alpha\beta} \frac{\partial x'_\alpha}{\partial x_\mu} \frac{\partial x'_\beta}{\partial x_\nu} dx_\mu\, dx_\nu = g_{\mu\nu}\, dx_\mu\, dx_\nu$$

or
$$g'_{\alpha\beta}\, dx'_\alpha\, dx'_\beta = g_{\mu\nu}\, dx_\mu\, dx_\nu \text{ by (1) of § 3.2}$$

which shows that $g_{\alpha\beta}\, dx_\alpha\, dx_\beta$ is an invariant.

3.9. ASSOCIATED TENSORS: RAISING AND LOWERING OF SUFFIXES

The process of raising and lowering of the suffices of a tensor is employed to obtain the new tensors associated with the given tensors. In fact this process of raising and lowering the suffixes of a tensor changes a covariant suffix into a contravariant suffix and vice versa. These operations merely depend on the inner product of the given tensor with a fundamental tensor.

We define the raising of the suffix of a vector (*i.e.*, a tensor of rank one) by

$$A^\mu = g^{\mu\nu} = A_\nu \qquad \dots (1)$$

and the lowering of the suffix of a vector, by

$$A_\mu = g_{\mu\nu} A^\nu \qquad \dots (2)$$

In general for a tensor like $A^{\gamma\delta}_{\alpha\beta\mu}$, the operation of raising μ is defined as

$$A^{\gamma\delta\mu}_{\alpha\beta} = g^{\mu\nu}\ A^{\gamma\delta}_{\alpha\beta\nu} \qquad \dots (3)$$

and that of lowering μ as

$$A^{\gamma\delta}_{\alpha\beta\mu} = g_{\mu\nu}\ A^{\gamma\delta\nu}_{\alpha\beta}. \qquad \dots (4)$$

These definitions are quite consistent; for, if we first raise a suffix and then lower it, the original tensor is reproduced, *e.g.*, multiplication of (3) by $g_{\mu\sigma}$ in order to lower the suffix on the left, yields

$$g_{\mu\sigma} A^{\alpha\delta\mu}_{\alpha\beta} = g_{\mu\sigma}\, g^{\mu\nu}\ A^{\gamma\delta}_{\alpha\beta\nu}$$
$$= g^{\nu}_{\sigma} A^{\gamma\delta}_{\alpha\beta\nu} \text{ by (7) of §3.8}$$
$$= A^{\gamma\delta}_{\alpha\beta\sigma} \text{ by (8) of §3.8}$$

which verifies (4).

It is however worth notable that the raising of a suffix ν by means of $g^{\mu\nu}$ is accompanied by the substitution of μ for ν and the operation of plain substitution of μ for ν is carried by $g_{\mu}{}^{\nu}$; conclusively,

multiplication by $g^{\mu\nu}$ yields substitution with raising,

multiplication by $g_{\mu}{}^{\nu}$ yields plain substitution, and

multiplication by $g_{\mu\nu}$ yields substitution with lowering.

The operation of raising and lowering of suffixes is applicabe to a tensor of any type. For example if $A^{\alpha\beta}$ is a contravariant tensor of rank two, then

$A_{\mu}{}^{\alpha} = g_{\mu\nu} B^{\alpha\nu}$ so that second has been lowered

$A_{\nu\beta} = g_{\mu\nu} A^{\mu\beta}$ so that first has been lowered

$A_{\mu\nu} = g_{\mu\alpha}\, g_{\nu\beta}\, A^{\alpha\beta}$ so that both have been lowered.

Similarly if $B_{\alpha\beta}$ is a covariant tensor of rank two, then

$B_{r}{}^{\mu} = g^{\mu\nu} B_{\alpha\nu}$ so that second has been raised,

$B_{\beta}{}^{\nu} = g^{\mu\nu} B_{\alpha\beta}$ so that first has been raised,

$B^{\mu\nu} = g^{\mu\alpha}\, g^{\nu\beta}\, B_{\alpha\beta}$ so that both have been raised.

If two tensors are such that either of them can be obtained from the other by any combination of the operation of raising and lowering the suffixes, then these tensors are known as **associated tensors.**

Problem 22. *Show that for a rectangular system of co-ordinates the raising and lowering of a suffix leaves the components unaltered in three-dimensional space.*

We have in ordinary three-dimensional space.

$$ds^2 = dx_1{}^2 + dx_2{}^2 + dx_3{}^2$$

where $x_1 = x,\ x_2 = y,\ x_3 = z$

so that $g_{11} = g_{22} = g_{33} = 1$

and $g_{12} = g_{13} = g_{23} = 0.$

$$\therefore \qquad g = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

We thus have

$$g^{11} = \frac{\text{cofactor of } g_{11} \text{ in } g}{g} = \frac{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}{1} = 1.$$

Similarly $g^{22} = g^{33} = 1,$

which follows that

$$g_{\mu\nu} = g^{\mu\nu} = g_{\mu}{}^{\nu}$$

so that all three tensors are merely substitution operators. Hence the proposition.

Problem 23. *To show that dummy suffixes have a certain freedom of movement between the tensor-factors of an expression, prove that*

(*i*) $A_{\alpha\beta} B^{\alpha\beta} = A^{\alpha\beta} B_{\alpha\beta}$.

(*ii*) $A_{\mu\alpha} B^{\nu\alpha} = A_{\mu}{}^{\alpha} B_{\alpha}{}^{\nu}$.

(*i*) Since we have

$$A_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} A^{\alpha\beta}$$

and

$$B^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} B_{\alpha\beta}.$$

Therefore $A_{\mu\nu} B^{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} A^{\alpha\beta} g^{\mu\alpha} g^{\nu\beta} B_{\alpha\beta}$

or $A_{\mu\nu} B^{\mu\nu} = g^{\mu\alpha} g_{\mu\alpha} g^{\nu\beta} g_{\nu\beta} A^{\alpha\beta} B_{\alpha\beta} = A^{\alpha\beta} B_{\alpha\beta}$

So that $A_{\alpha\beta} B^{\alpha\beta} = A^{\alpha\beta} B_{\alpha\beta}$.

(*ii*) Since $g^{\mu\alpha} B_{\alpha}{}^{\nu} = B^{\nu\mu}$.

$$\therefore \quad g_{\mu}{}^{\alpha} g^{\mu\alpha} B_{\alpha}{}^{\nu} = g_{\mu}{}^{\alpha} B^{\nu\mu} = B^{\nu\alpha}$$

and similarly $g_{\nu\alpha} A_{\mu}{}^{\alpha} = A_{\mu\nu}$ gives

$$g_{\alpha}{}^{\nu} g_{\nu\alpha} A_{\mu}{}^{\alpha} = g_{\alpha}{}^{\nu} A_{\mu\nu} = A_{\alpha\mu}$$

$$\therefore \quad A_{\mu\alpha} B^{\nu\alpha} = g_{\alpha}{}^{\nu} g_{\nu\alpha} A_{\mu}{}^{\alpha} . g_{\mu}{}^{\alpha} g^{\mu\alpha} B_{\alpha}{}^{\nu}$$
$$= A_{\mu}{}^{\alpha} B_{\alpha}{}^{\nu}.$$

3.10. LENGTH OF A VECTOR (i.e. A TENSOR OF RANK ONE), ANGLE BETWEEN TWO VECTORS AND ORTHOGONALITY OF VECTORS

We deal in elementary vector algebra that the square of the length of the vector is its scalar-product into itself and the two vectors are mutually perpendicular if their scalar product is zero. Thus if A_{μ}, B_{μ} be two vectors, then condition of their orthogonality is

$$A_{\mu} B^{\mu} = 0. \quad \ldots (1)$$

Also if l is the length of the vector A_{μ} (or A^{μ}), then

$$A_{\mu} A^{\mu} = l^2. \quad \ldots (2)$$

In case a vector is self-perpendicular, its length will be zero, *i.e.*

$$A_{\mu} A^{\mu} = 0. \quad \ldots (3)$$

Now we have

$$ds^2 = g_{\mu\nu} (dx)^{\mu} . (dx)^{\nu}$$
$$= (g_{\mu\nu} dx^{\mu}) (dx^{\nu})$$
$$= dx_{\nu} dx^{\nu}. \quad \ldots (4)$$

$\therefore$ the displacement is self perpendicular if

$$dx_{\nu} dx^{\nu} = 0,$$

i.e., $\quad ds = 0.$

It shows that a displacement is self-perpendicular when it is along a light track $ds = 0$.

Now, if θ is the angle between the two vectors A_{μ} and B_{μ}, then

$$\cos\theta = \frac{\text{scalar product of the two vectors}}{\sqrt{(\text{length of } A_{\mu})}\ \sqrt{(\text{length of } B_{\mu})}}$$

$$= \frac{A_\mu B^\mu}{\sqrt{(A_\alpha A^\alpha)}\,\sqrt{(B_\beta B^\beta)}} = \frac{A_\mu B^\mu}{\sqrt{\{(A_\alpha A^\alpha)(B_\beta B^\beta)\}}} \quad \ldots (5)$$

3.11. METRIC TENSOR, RIEMANNIAN SPACES

We have $\qquad ds^2 = dx_\alpha\, dx^\alpha$, by § 3.10 (4) $\qquad \ldots (1)$

where dx^α with the help of transformation (1) of § 3.2 can be expressed as

$$dx^\alpha = \frac{\partial x_\alpha}{\partial x'_\mu} dx^\mu$$

Similarly $\qquad dx_\alpha = \frac{\partial x_\alpha}{\partial x'_\nu} dx^\nu$ as $dx_\nu \equiv (dx)^\nu$.

With the help of these transformations (1) can be written as

$$ds^2 = \frac{\partial x_\alpha}{\partial x'_\mu}\,\frac{\partial x_\alpha}{\partial x'_\nu}\, dx^\mu\, dx^\nu \quad \ldots (2)$$

But $\qquad ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu. \qquad \ldots (3)$

Comparison of (2) and (3) yields

$$g_{\mu\nu} = \frac{\partial x_\alpha}{\partial x'_\mu}\,\frac{\partial x_\alpha}{\partial x'_\nu} \quad \ldots (4)$$

which transforms as a tensor when a coordinate system changes from x' to x''. Therefore

$$\left.\begin{aligned} \frac{\partial x_\alpha}{\partial x''_\beta}\,\frac{\partial x_\alpha}{\partial x''_\gamma} &= \frac{\partial x_\alpha}{\partial x'_\mu}\,\frac{\partial x'_\mu}{\partial x''_\beta} \cdot \frac{\partial x_\alpha}{\partial x'_\nu}\,\frac{\partial x'_\nu}{\partial x''_\gamma} \\ \text{or} \qquad g'_{\beta\gamma} &= \frac{\partial x'_\mu}{\partial x''_\beta}\,\frac{\partial x'_\nu}{\partial x''_\gamma}\, g_{\mu\nu} \end{aligned}\right\} \quad \ldots (5)$$

Here $g_{\mu\nu}$ is a covariant symmetric tensor of rank two. It is called the **metric tensor.**

There are 'spaces' where we cannot introduce a cartesian coordinate system, *e.g.* two dimensional 'spaces' of that kind which include the surface of a sphere. Introduction of a coordinate system with latitude ϕ and longitude θ, makes it possible to express the distance between two neighbouring points on the spherical surface, in terms of their coordinate differentials, like

$$ds^2 = R^2\,(d\phi^2 + \cos^2\phi\, d\theta^2).$$

$\left(\text{On assuming equator as } \theta = 0, \text{ north pole as } \theta = \frac{\pi}{2}, \text{ south pole as } \theta = -\frac{\pi}{2}\right)$

Whenever we define a 'squared infinitesimal distance' which is an homogenous quadratic function of the coordinate differentials, the manifoid is called a **metric space** or a **Riemannian** space. In case it is possible to introduce in a Riemannian space a coordinate system with respect to which the components of the metric tensor take the values $\delta_{\mu\alpha}$ at every point, the coordinate system is a cartesian one and the space is called a **Euclidean Space.** In fact Euclidean spaces are the particular cases of Riemannian Spaces.

In all cases where the infinitesimal distance is expressed by equation (3), ds^2 being an invariant, $g_{\mu\nu}$ is always a covariant tensor.

Problem 24. *A hyper surface in a Riemannian space is given by*

$$x_i = x_i (u_1, u_2):$$

then prove that the coordinate curve of the surface is given by

$$ds^2 = h_{\mu\nu}\, du^\mu\, du^\nu$$

where
$$h_{\mu\nu} = g_{\alpha\beta} \frac{\partial x_\alpha}{\partial u_\mu} \frac{\partial x_\beta}{\partial u_\nu}$$

We have

$$ds^2 = g_{\alpha\beta}\, dx^\beta$$
$$= g_{\alpha\beta} \frac{\partial x_\alpha}{\partial x_\mu} du^\mu \frac{\partial x_\beta}{\partial x_\nu} du^\nu$$
$$= g_{\alpha\beta} \frac{\partial x_\alpha}{\partial x_\mu} \frac{\partial x_\beta}{\partial x_\nu} du^\mu\, du^\nu$$
$$= h_{\mu\nu}\, du^\mu\, du^\nu$$

Problem 25. *Surface of a sphere is a two-dimensional Riemannian space. Find its fundamental metric tensor.* (*Agra, 1974*)

If a be the fixed radius of a sphere, then its surface is given by

$ds^2 = a^2\, d\theta^2 + a^2 \sin^2\theta\, d\phi^2$, with equator as $\theta = \frac{\pi}{2}$, north pole as $\theta = 0$ and south pole as $\theta = \pi$

so that
$$g_{11} = a^2,\ g_{22} = a^2 \sin^2\theta$$
$$g_{12} = 0 = g_{21}.$$

$$\therefore \quad g = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = \begin{vmatrix} a^2 & 0 \\ 0 & a^2 \sin^2\theta \end{vmatrix}$$
$$= a^4 \sin^2\theta$$

$$g^{11} = \frac{\text{cofactor of } g_{11}}{g} = \frac{a^2 \sin^2\theta}{a^4 \sin^2\theta} = \frac{1}{a^2}$$

$$g^{22} = \frac{\text{cofactor of } g_{22}}{g}, = \frac{a^2}{a^4 \sin^2\theta} = \frac{1}{a^2 \sin^2\theta}$$

$$g^{12} = 0 = g^{21}$$

3.12. CHRISTOFFEL'S 3-INDEX SYMBOLS

Here we introduce two expressions (not tensors) known as Christoffel's symbols of the first and second kind. These will be found of great utility throughout our subsequent work.

Chirstoffel symbol of the first kind (*Rohilkhand, 1979, 89*)

$$\Gamma_{\mu\nu,\sigma} \equiv [\mu\nu, \sigma] = \frac{1}{2}\left(\frac{\partial g_{\mu\sigma}}{\partial x_\nu} + \frac{\partial g_{\nu\sigma}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\sigma}\right) \quad \ldots (1)$$

Christoffel symbol of the second kind *(Rohilkhand, 1979, 89)*

$$\Gamma^{\sigma}_{\mu\nu} \equiv \{\mu\nu,\ \sigma\} = \frac{1}{2} g^{\sigma\lambda} \left(\frac{\partial g_{\mu\lambda}}{\partial x_\nu} + \frac{\partial g_{\nu\lambda}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\lambda} \right) \qquad \ldots (2)$$

It is obvious from these expressions that

$$[\mu\nu,\ \sigma] = [\nu\mu,\ \sigma] \quad i.e \quad \Gamma_{\mu\nu,\ \sigma} = \Gamma_{\nu\mu,\ \sigma} \qquad \ldots (3)$$

and

$$\{\mu\nu,\ \sigma\} = \{\nu\mu,\ \sigma\} \quad i.e. \quad \Gamma^{\sigma}_{\mu\nu} = \Gamma^{\sigma}_{\nu\mu} \qquad \ldots (4)$$

showing that they are symmetrical with respect to μ and ν.

We also observe *the relations between two kinds*, *(Rohilkhand, 1989)*

$$\{\mu\nu,\ \sigma\} = g^{\sigma\lambda} [\mu\nu,\ \lambda] \ i.e. \quad \Gamma^{\sigma}_{\mu\nu} = g^{\sigma\lambda}\, \Gamma_{\mu\nu,\ \lambda} \qquad \ldots (5)$$

and

$$[\mu\nu,\ \sigma] = g_{\sigma\lambda} \{\mu\nu,\ \lambda\} \ i.e. \quad \Gamma_{\mu\nu,\ \sigma} = g_{\sigma\lambda}\, \Gamma^{\lambda}_{\mu\nu} \qquad \ldots (6)$$

To prove the result (5), we have from (1), on replacing σ by λ,

$$[\mu\nu,\ \lambda] = \frac{1}{2} \left(\frac{\partial g_{\mu\lambda}}{\partial x_\nu} + \frac{\partial g_{\nu\lambda}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\lambda} \right)$$

Multiplying both sides by $g^{\sigma\lambda}$, this becomes

$$g^{\sigma\lambda} [\mu\nu,\ \lambda] = \frac{1}{2} g^{\sigma\lambda} \left(\frac{\partial g_{\mu\lambda}}{\partial x_\nu} + \frac{\partial g_{\nu\lambda}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\lambda} \right)$$

$$= \{\mu\nu, \sigma\} \text{ from (2)}$$

This proves the result (5).

Again to prove the result (6), interchanging λ and σ in (2), we have

$$\{\mu\nu,\ \lambda\} = \frac{1}{2} g^{\lambda\sigma} \left(\frac{\partial g_{\mu\sigma}}{\partial x_\nu} + \frac{\partial g_{\nu\sigma}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\sigma} \right)$$

$$= g^{\lambda\sigma} [\lambda\nu, \sigma], \text{ from (1).}$$

Multiplying both sides by $g_{\sigma\lambda}$, this becomes

$$g_{\sigma\lambda} \{\mu\nu, \lambda\} = g_{\sigma\lambda}\, g^{\sigma\lambda} [\mu\nu, \sigma]$$

$$= [\mu\nu, \sigma]$$

which proves the result (6).

Now we have from (1),

$$[\mu\nu,\ \sigma] + [\sigma\nu,\ \mu] = \frac{1}{2} \left(\frac{\partial g_{\mu\sigma}}{\partial x_\nu} + \frac{\partial g_{\nu\sigma}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\sigma} \right) + \frac{1}{2} \left(\frac{\partial g_{\mu\sigma}}{\partial x_\nu} + \frac{\partial g_{\mu\nu}}{\partial x_\sigma} - \frac{\partial g_{\nu\sigma}}{\partial x_\mu} \right)$$

$$= \frac{1}{2} \left(\frac{\partial g_{\mu\sigma}}{\partial x_\nu} + \frac{\partial g_{\nu\sigma}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\sigma} + \frac{\partial g_{\mu\sigma}}{\partial x_\nu} + \frac{\partial g_{\mu\nu}}{\partial x_\sigma} - \frac{\partial g_{\nu\sigma}}{\partial x_\mu} \right)$$

$$= \frac{\partial g_{\mu\sigma}}{\partial x_\nu}, \qquad \ldots (7)$$

i.e.

$$\Gamma_{\mu\nu,\sigma} + \Gamma_{\sigma\nu,\mu} = \frac{\partial g_{\mu\sigma}}{\partial x_\nu}.$$

(Agra, 1971, 77)

Problem 26. *Find the Christoffel's symbols corresponding to*

(*a*) $ds^2 = a^2\, d\theta^2 + a^2 \sin^2\theta\, d\phi^2$. *(Rohilkhand, 1987, 90; Agra, 1974)* (*b*) $ds^2 = dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2$. (*Rohilkhand, 1979, 90*)

(*a*) We have $ds^2 = a^2\, d\theta^2 + a^2 \sin^2\theta\, d\phi^2$,

so that $g_{11} = a^2,\ g_{22} = a^2 \sin^2\theta,\ g_{12} = 0 = g_{21}$

$$\therefore \quad g = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = \begin{vmatrix} a^2 & 0 \\ 0 & a^2\sin^2\theta \end{vmatrix} = a^4 \sin^2\theta,$$

giving
$$g^{11} = \frac{\text{cofactor of } g_{11}}{g} = \frac{a^2 \sin^2\theta}{a^4 \sin^2\theta} = \frac{1}{a^2}.$$

Similarly, $g^{22} = \dfrac{1}{a^2\sin^2\theta}$ and $g^{12} = 0 = g^{21}$.

Thus Christoffel symbols of first kind are

$$[22,\ 1] = \frac{1}{2}\left(\frac{\partial g_{21}}{\partial x_2} + \frac{\partial g_{21}}{\partial x_2} - \frac{\partial g_{22}}{\partial x_1}\right), \text{ where } x_1 = \theta,\ x_2 = \phi \text{ etc.}$$

$$= \frac{1}{2}\left\{0 + 0 - \frac{\partial}{\partial\theta}\left(a^2 \sin^2\theta\right)\right\}$$

$$= -a^2 \sin\theta \cos\theta.$$

Similarly, $[12,\ 2] = \dfrac{1}{2}\dfrac{\partial g_{22}}{\partial x_1} = \dfrac{1}{2}\dfrac{\partial}{\partial\theta}\left(a^2\sin^2\theta\right) = a^2\sin\theta\ \cos\theta.$

The rest of all are zero, and the Christoffel symbols of second kind are

$$\{22,\ 1\} = g^{1\lambda}\,[22,\ \lambda]$$

$$= g^{11}\,[22,\ 1] + g^{12}\,[22,\ 1]$$

$$= \frac{1}{a^2}\left(-a^2\sin\theta\ \cos\theta\right) + 0 = -\sin\theta\ \cos\theta.$$

Similarly, $\{12,\ 2\} = g^{2\lambda}\,[12,\ \lambda]$

$$= g^{21}\,[12,\ 2] + g^{22}\,[12,\ 2]$$

$$= 0 + \frac{1}{a^2\sin^2\theta}\left(a^2 \sin\theta\ \cos\theta\right) = \cot\theta$$

and the rest of all are zero.

(*b*) We have $ds^2 = dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2$,

where $x_1 = r,\ x_2 = \theta,\ x_3 = \phi$

and $g_{11} = 1,\ g_{22} = r^2,\ g_{33} = r^2 \sin\theta,$

$g_{12} = 0 = g_{13} = \ldots$ etc.,

$$\therefore \quad g = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{vmatrix} = r^4 \sin^2\theta, \qquad \ldots (1)$$

giving
$$g^{11} = \frac{\text{cofactor of } g_{11}}{g} = \frac{\begin{vmatrix} r^2 & 0 \\ 3 & r^2\sin^2\theta \end{vmatrix}}{r^4\sin^2\theta} = \frac{r^4\sin^2\theta}{r^4\sin^2\theta} = 1.$$

Similarly $g^{22} = \dfrac{\text{cofactor of } g_{22}}{g} = \dfrac{r^2 \sin^2\theta}{r^4 \sin^2\theta} = \dfrac{1}{r^2}$,

$$g^{33} = \frac{\text{cofactor of } g_{33}}{g} = \frac{r^2}{r^4 \sin^2\theta} = \frac{1}{r^2 \sin^2\theta}$$

and the rest all are zero.

Now the Christoffel's symbols of the first kind are

$$[11, 1] = \frac{1}{2}\left(\frac{\partial g_{11}}{\partial x_1} + \frac{\partial g_{11}}{\partial x_1} - \frac{\partial g_{11}}{\partial x_1}\right) = 0.$$

Similarly $[11, 2] = 0 = [11, 3]$.

In a similar manner, it is easy to show that

$[22, 1] = -r$, $[22, 2] = 0 = [22, 3]$

$[33, 1] = -r \sin^2\theta$, $[33, 2] = -r^2 \sin\theta \cos\theta$, $[33, 3] = 0$

$[12, 1] = 0 = [21, 1]$, $[13, 1] = 0 = [31, 1]$

$[12, 2] = 0 = [21, 2]$, $[12, 3] = 0 = [21, 3]$

$[13, 1] = 0 = [31, 1]$, $[13, 2] = 0 = [31, 2]$

$[13, 3] = r \sin^2\theta = [31, 3]$, $[23, 1] = 0 = [32, 1]$

$[23, 2] = 0 = [32, 2]$, $[23, 3] = r^2 \sin\theta \cos\theta = [32, 3]$

and the Christoffel symbols of the second kind are

$\{22, 1\} = g^{1\lambda} [22, \lambda]$

$g^{11} [22, 1] + g^{12} [22, 2] + g^{13} [22, 3]$

$= -r + 0 + 0 = -r.$

Similarly $\{3, 3, 1\} = -r \sin^2\theta$, $\{11, 1\} = 0$,

$\{33, 2\} = -\sin\theta \cos\theta$, $\{13, 3\} = \frac{1}{2}$, $\{23, 3\} = \cot\theta$

and the rest all are zero.

Note. We get the **Metric tensor** in *spherical polar coordinates* as

$$g_{\mu\nu} = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{bmatrix} \text{ just as in (1)}$$

The metric tensor in *cylindrical coordinates* is given by

$$g_{\mu\nu} = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since in this case

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2$$

with $x_1 = r, x_2 = \theta, x_3 = z$

so that $g_{11} = 1, g_{22} = r^2, g_{33} = 1$

and $g_{12} = 0 = g_{13} = \ldots$ etc.

3.13. EQUATIONS OF A GEODESIC (Agra, 1963, 65, 66)

We sometimes define a geodesic as the path of shortest distance on the surface between two given points on it. Here our aim is to determine the equations of a geodesic or path between two points for which

$$\int ds \text{ is stationary.}$$

Assuming that the initial and terminal points of the path are fixed, let the path be deformed by giving every intermediate point an arbitrary infinitesimal displacement δx_σ, so that the expression

$$ds^2 = g_{\mu\nu}\, dx_\mu\, dx_\nu \text{ yields,}$$

$$2ds\,\delta(ds) = \delta(g_{\mu\nu})\, dx_\mu\, dx_\nu + g_{\mu\nu}\,\delta(dx_\mu)\, dx_\nu + g_{\mu\nu}\, dx_\mu\,\delta(dx_\nu)$$

$$= dx_\mu\, dx_\nu \frac{\partial g_{\mu\nu}}{\partial x_\sigma}\,\delta x_\sigma + g_{\mu\nu}\, dx_\nu\,\delta(dx_\mu) + g_{\mu\nu}\, dx_\mu\,\delta(dx_\nu)$$

The stationary condition is

$$\int \delta\,(ds) = 0. \qquad \dots (8)$$

Substituting the value of $\delta(ds)$ from (1) in (2), we get

$$\frac{1}{2}\int\left\{\frac{dx_\mu}{ds}\,\frac{dx_\nu}{dx}\,\frac{\partial g_{\mu\nu}}{dx_\sigma}\delta x_\sigma + g_{\mu\nu}\,\frac{dx_\nu}{ds}\frac{d}{ds}\,(\partial x_\mu) + g_{\mu\nu}\,\frac{dx_\mu}{ds}\frac{d}{ds}(\delta x_\nu)\right\} ds = 0$$

On changing the dummy suffixes in the last two terms, this becomes

$$\frac{1}{2}\int\left\{\frac{dx_\mu}{ds}\,\frac{dx_\nu}{ds}\,\frac{\partial g_{\mu\nu}}{dx_\sigma}\delta x_\sigma + \left(g_{\sigma\nu}\,\frac{dx_\nu}{ds} + g_{\mu\sigma}\frac{dx_\mu}{ds}\right)\,\frac{d}{ds}(\delta x_\sigma)\right\} ds = 0$$

or
$$\frac{1}{2}\int\left(\frac{dx_\mu}{ds}\,\frac{dx_\nu}{ds}\,\frac{\partial x_{\mu\nu}}{dx_\sigma}\delta x_\sigma\right)ds + \frac{1}{2}\int\left(g_{\sigma\nu}\,\frac{dx_\nu}{ds} + g_{\mu\sigma}\,\frac{dx_\mu}{ds}\right)\frac{d}{ds}(\delta x_\sigma)\, ds = 0$$

Integrating the second terms by parts, we have

$$\frac{1}{2}\int\frac{dx_\mu}{ds}\,\frac{dx_\nu}{ds}\,\frac{\partial g_{\mu\nu}}{dx_\sigma}\delta x_\sigma ds + \frac{1}{2}\left[\left(g_{\sigma\nu}\,\frac{dx_\nu}{ds} + g_{\mu\sigma}\,\frac{dx_\mu}{ds}\right)\delta x_\sigma\right]_0^1$$

$$-\frac{1}{2}\int\frac{d}{ds}\left(g_{\sigma\nu}\,\frac{dx_\nu}{ds} + g_{\mu\sigma}\,\frac{dx_\mu}{ds}\right)(\delta x_\sigma)\, ds = 0,$$

where the integrated part vanishes as δx_σ vanishes at both the limits.

We are, therefore left with

$$\frac{1}{2}\int\frac{dx_\mu}{ds}\,\frac{dx_\nu}{ds}\,\frac{\partial g_{\mu\nu}}{dx_\sigma}\delta x_\sigma ds - \frac{1}{2}\int\frac{d}{ds}\left(g_{\sigma\nu}\,\frac{dx_\nu}{ds} + g_{\mu\sigma}\frac{dx_\mu}{ds}\right)(\delta x_\sigma)\, ds = 0$$

or
$$\frac{1}{2}\int\left[\frac{dx_\mu}{ds}\,\frac{dx_\nu}{ds}\,\frac{\partial g_{\mu\nu}}{dx_\sigma} - \frac{d}{ds}\left(g_{\sigma\nu}\,\frac{dx_\nu}{ds} + g_{\mu\sigma}\frac{dx_\mu}{ds}\right)\right]\delta(x_\sigma)\, ds = 0.$$

This will hold for all values of the arbitrary displacement (δx_σ) at all points if

$$\frac{1}{2}\left[\frac{dx_\sigma}{ds}\,\frac{dx_\nu}{ds}\,\frac{\partial g_{\mu\nu}}{\partial x_\sigma} - \frac{d}{ds}\left(g_{\sigma\nu}\frac{dx_\nu}{ds} + g_{\mu\sigma}\frac{dx_\mu}{ds}\right)\right] = 0$$

or if $$\frac{1}{2}\frac{dx_\mu}{ds}\frac{dx_\nu}{ds}\frac{\partial g_{\mu\nu}}{\partial x_\sigma}-\frac{1}{2}\frac{\partial g_{\sigma\nu}}{ds}\frac{dx_\nu}{ds}-\frac{1}{2}g_{\sigma\nu}\frac{d^2x_\nu}{ds^2}-\frac{1}{2}\frac{\partial g_{\mu\sigma}}{ds}\frac{dx_\mu}{ds}-\frac{1}{2}g_{\mu\sigma}\frac{d^2x_\mu}{ds^2}=0$$

But we have

$$\frac{dg_{\sigma\nu}}{ds}=\frac{\partial x_{\sigma\nu}}{\partial x_\mu}\frac{dx_\mu}{ds} \text{ and } \frac{dg_{\mu\sigma}}{ds}=\frac{\partial g_{\mu\sigma}}{\partial x_\nu}\frac{dx_\nu}{ds}$$

so that the last relation becomes

$$\frac{1}{2}\frac{dx_\mu}{ds}\frac{dx_\nu}{ds}\frac{\partial g_{\mu\nu}}{\partial x_\sigma}-\frac{1}{2}\frac{\partial g_{\sigma\nu}}{\partial x_\mu}\frac{dx_\mu}{dx}\frac{dx_\nu}{ds}-\frac{1}{2}\frac{\partial g_{\mu\sigma}}{\partial x_\nu}\frac{dx_\nu}{ds}\frac{dx_\mu}{ds}-\frac{1}{2}g_{\sigma\nu}\frac{d^2x_\nu}{ds^2}-\frac{1}{2}g_{\mu\sigma}\frac{d^2x_\mu}{ds^2}=0$$

or $$\frac{1}{2}\frac{dx_\mu}{ds}\frac{dx_\nu}{ds}\left(\frac{\partial g_{\mu\nu}}{\partial x_\sigma}-\frac{\partial g_{\sigma\nu}}{\partial x_\mu}-\frac{\partial g_{\mu\sigma}}{\partial x_\nu}\right)-\frac{1}{2}\left(g_{\sigma\nu}\frac{d^2x_\nu}{ds^2}+g_{\mu\sigma}\frac{d^2x_\mu}{ds^2}\right)=0$$

Replacing the dummy suffixes μ and ν by ε in the last term, we get,

$$\frac{1}{2}\frac{dx_\mu}{ds}\frac{dx_\nu}{ds}\left(\frac{\partial g_{\mu\nu}}{\partial x_\sigma}-\frac{\partial g_{\sigma\nu}}{\partial x_\mu}-\frac{\partial g_{\mu\sigma}}{\partial x_\nu}\right)-\frac{1}{2}\left(g_{\sigma\varepsilon}\frac{d^2x_\varepsilon}{ds^2}+g_{\varepsilon\sigma}\frac{d^2x_\varepsilon}{ds^2}\right)=0.$$

or $$\frac{1}{2}\frac{dx_\mu}{ds}\frac{dx_\nu}{ds}\left(\frac{\partial g_{\mu\nu}}{\partial x_\sigma}-\frac{\partial g_{\nu\sigma}}{\partial x_\mu}-\frac{\partial g_{\mu\sigma}}{\partial x_\nu}\right)-g_{\varepsilon\sigma}\frac{d^2x_\varepsilon}{ds^2}=0.$$

Multiplying throughout by $-g^{\sigma\alpha}$, this becomes

$$\frac{1}{2}\frac{dx_\mu}{ds}\frac{dx_\nu}{ds}g^{\sigma\alpha}\left(\frac{\partial g_{\mu\sigma}}{\partial x_\nu}+\frac{\partial g_{\nu\sigma}}{\partial x_\mu}-\frac{\partial g_{\mu\nu}}{\partial x_\sigma}\right)+g^{\sigma\alpha}g_{\varepsilon\sigma}\frac{d^2x_\varepsilon}{ds^2}=0.$$

or $$g_\varepsilon^\alpha\frac{d^2x_\varepsilon}{ds^2}+\frac{dx_\mu}{ds}\frac{dx_\nu}{ds}\{\mu\nu,\ \alpha\}=0 \quad \ldots(3)$$

as $$g^{\sigma\alpha}g_{\varepsilon\alpha}=g_\varepsilon^\alpha$$

or $$\frac{d^2x_\alpha}{ds^2}+\{\mu\nu,\ \alpha\}\frac{dx_\mu}{ds}\frac{dx_\nu}{ds}=0 \quad \ldots(4)$$

which may also be written as

$$\frac{d^2x_\alpha}{ds^2}+\Gamma_{\mu\nu}^\alpha\frac{dx_\mu}{ds}\frac{dx_\nu}{ds}=0$$

since $\dfrac{d^2x_\alpha}{ds^2}=g_\varepsilon^\alpha\dfrac{d^2x_\varepsilon}{ds^2}$

which corresponding to $\alpha = 1, 2, 3, 4$ gives four equations determining a geodesic.

3.14. LAW OF TRANSFORMATION FOR CHRISTOFFEL'S SYMBOLS

(Agra, 1966, 68)

Considering the two coordinate systems x_α and x'_α in a Riemannian space the equation of geodesic line can be written as

$$\frac{d^2x_\alpha}{ds^2}+T^\alpha_{\beta\gamma}\frac{dx_\beta}{ds}\frac{dx_\gamma}{ds}=0 \qquad \ldots(1)$$

where $\Gamma_{\beta\gamma}{}^{\alpha}=\{\beta\gamma,\alpha\}$ ... (2)

and
$$\frac{d^2x'_\alpha}{ds^2}+\Gamma'^{\alpha}_{\beta\gamma}\frac{dx'_\beta}{ds}\frac{dx'_\gamma}{ds}=0$$

where $\Gamma'_{\beta\gamma}{}^{\alpha}=\{\beta\gamma,\alpha\}'$.

Now we have

$$\frac{dx'_\alpha}{ds}=\frac{dx'_\alpha}{dx_\lambda}\frac{dx_\lambda}{ds} \text{ etc.}$$

and
$$\frac{d^2x'_\alpha}{ds^2}=\frac{d}{ds}\left(\frac{dx'^\alpha}{ds}\right)=\frac{d}{ds}\left(\frac{dx'_\alpha}{\partial x_\lambda}\frac{dx_\lambda}{ds}\right)$$

$$=\frac{d}{ds}\left(\frac{\partial x'_\alpha}{\partial x_\lambda}\right)\frac{dx_\lambda}{ds}+\frac{\partial x'_\alpha}{\partial x_\lambda}\frac{d^2x_\lambda}{ds^2}$$

$$=\frac{\partial^2 x'_\alpha}{\partial x_\mu\,\partial x_\lambda}\frac{dx_\mu}{ds}\frac{dx_\lambda}{ds}+\frac{\partial x'_\alpha}{\partial x_\lambda}\frac{d^2x_\lambda}{ds^2}$$

$$=\frac{\partial^2 x'_\alpha}{\partial x_\mu\,\partial x_\nu}\frac{dx_\mu}{ds}\frac{dx_\nu}{ds}+\frac{\partial x'_\alpha}{\partial x_\lambda}\frac{d^2x_\lambda}{ds^2}$$

(replacing λ by ν in the first term)

Substituting these values in (2), we get

$$\frac{\partial x'_\alpha}{\partial x_\lambda}\frac{d^2x_\lambda}{ds^2}+\frac{\partial^2 x'_\alpha}{\partial x_\mu\,dx_\nu}\frac{dx_\mu}{ds}\frac{dx_\nu}{ds}+\Gamma'^{\alpha}_{\beta\gamma}\frac{\partial x'_\beta}{\partial x_\mu}\frac{\partial x'_\gamma}{\partial x_\nu}\frac{dx_\mu}{ds}\frac{dx_\nu}{ds}=0.$$

Multiplying by $\frac{\partial x_\lambda}{\partial x'_\alpha}$ this equation becomes

$$\frac{d^2x_\lambda}{ds^2}+\left(\Gamma^\alpha_{\beta\gamma}\frac{\partial x'_\beta}{\partial x_\mu}\frac{\partial x'_\gamma}{\partial x_\nu}\frac{\partial x_\lambda}{\partial x'_\alpha}+\frac{\partial^2 x'_\alpha}{\partial x_\mu\,\partial x_\nu}\frac{\partial x_\lambda}{\partial x'_\alpha}\right)\frac{dx_\mu}{ds}\frac{dx_\nu}{ds}=0.$$

Intercharging α,β,γ, with λ,μ,ν respectively, this becomes

$$\frac{d^2x_\alpha}{ds^2}+\left(\Gamma'^{\lambda}_{\mu\nu}\frac{\partial x'_\mu}{\partial x_\beta}\frac{\partial x'_\nu}{\partial x_\gamma}\frac{\partial x_\alpha}{\partial x'_\lambda}+\frac{\partial^2 x'_\lambda}{\partial x_\beta\,\partial x_\gamma}\frac{\partial x_\alpha}{\partial x'_\lambda}\right)\frac{dx_\beta}{ds}\frac{dx_\gamma}{ds}=0. \qquad \ldots(3)$$

Comparison of (1) and (3) yields

$$T^\alpha_{\beta\gamma}=\Gamma'^{\lambda}_{\mu\nu}\frac{\partial x'_\mu}{\partial x_\beta}\frac{\partial x'_\nu}{\partial x_\gamma}\frac{\partial x_\alpha}{\partial x'_\lambda}+\frac{\partial^2 x'_\lambda}{\partial x_\beta\,\partial x_\gamma}\frac{\partial x_\alpha}{\partial x'_\lambda}$$

This may also be written as ... (4)

$$\{\beta\gamma,\alpha\}=\{\mu\nu,\lambda\}'\frac{\partial x'_\mu}{\partial x_\beta}\frac{\partial x'_\nu}{\partial x_\gamma}\frac{\partial x_\alpha}{\partial x'_\lambda}+\frac{\partial^2 x'_\lambda}{\partial x_\beta\,\partial x_\gamma}\frac{\partial x_\alpha}{\partial x'_\lambda}$$

(Agra, 1968)

This gives the law of transformation for $\Gamma^\alpha{}_{\beta\gamma}$, *i.e.* $\{\beta\gamma, \alpha\}$. It is clear from this transformation that $\Gamma^\alpha{}_{\beta\gamma}$ are not the components of a tensor, so that $\Gamma^\alpha{}_{\beta\gamma}$ may be zero in one coordinate system, but not in all coordinate systems.

We may also arrive at the Christoffel symbols and their law of transformation by another method. We have by §3.8 (4)

$$g'_{\alpha\beta} = g_{\nu\mu} \frac{\partial x_\lambda}{\partial x'_\alpha} \frac{\partial x_\mu}{\partial x'_\beta}.$$

Differentiating it with respect to x'_γ, we have

$$\frac{\partial g'_{\alpha\beta}}{\partial x'_\alpha} = \frac{\partial g_{\lambda\mu}}{\partial x_\nu} \frac{\partial x_\nu}{\partial x'_\gamma} \frac{\partial x_\lambda}{\partial x'_\alpha} \frac{\partial x_\mu}{\partial x'_\beta} + g_{\lambda\mu} \left(\frac{\partial x_\lambda}{\partial x'_\alpha} \frac{\partial^2 x_\mu}{\partial x'_\gamma \partial x'_\beta} + \frac{\partial x_\mu}{\partial x'_\beta} \frac{\partial^2 x_\lambda}{\partial x'_\gamma \partial x'_\alpha} \right) \quad \dots (5)$$

The other two equations may be obtained from it by cyclic permutations of the indices α, β, γ and these are

$$\frac{\partial g'_{\beta\gamma}}{\partial x'_\alpha} = \frac{\partial g_{\lambda\mu}}{\partial x_\nu} \frac{\partial x_\nu}{\partial x'_\alpha} \frac{\partial x_\lambda}{\partial x_\beta} \frac{\partial x_\mu}{\partial x'_\gamma} + g_{\lambda\mu} \left(\frac{\partial x_\lambda}{\partial x'_\beta} \frac{\partial^2 x_\mu}{\partial x'_\alpha \partial x'_\gamma} + \frac{\partial x_\mu}{\partial x'_\gamma} \frac{\partial^2 x_\lambda}{\partial x'_\alpha \partial x'_\beta} \right) \quad \dots (6)$$

and
$$\frac{\partial g'_{\gamma\alpha}}{\partial x'_\beta} = \frac{\partial g_{\lambda\mu}}{\partial x_\nu} \frac{\partial x_\nu}{\partial x'_\beta} \frac{\partial x_\lambda}{\partial x'_\gamma} \frac{\partial x_\mu}{\partial x'_\alpha} + g_{\lambda\mu} \left(\frac{\partial x_\lambda}{\partial x'_\gamma} \frac{\partial^2 x_\mu}{\partial x'_\beta \partial x'_\alpha} + \frac{\partial x_\mu}{\partial x'_\alpha} \frac{\partial^2 x_\lambda}{\partial x'_\beta \partial x'_\gamma} \right) \quad \dots (7)$$

Subtracting (5) from the sum of (6) and (7) after changing dummy suffixes in their first terms and then dividing by 2, we get

$$\frac{1}{2}\left(\frac{\partial g'_{\beta\gamma}}{\partial x'_\alpha} + \frac{\partial g'_{\gamma\alpha}}{\partial x'_\beta} - \frac{\partial g'_{\alpha\beta}}{\partial x'_\gamma} \right) = \frac{\partial x_\lambda}{\partial x'_\alpha} \frac{\partial x_\mu}{\partial x'_\beta} \frac{\partial x_\nu}{\partial x'_\gamma} \cdot \frac{1}{2} \left(\frac{\partial g_{\mu\nu}}{\partial x_\lambda} + \frac{\partial g_\gamma}{\partial x_\mu} - \frac{\partial g_{\lambda\mu}}{\partial x_\nu} \right)$$
$$+ \frac{1}{2} g_{\lambda\mu} \left(\frac{\partial x_\lambda}{\partial x'_\beta} \frac{\partial^2 x_\mu}{\partial x'_\alpha \partial x'_\gamma} + \frac{\partial x_\lambda}{\partial x'_\gamma} \frac{\partial^2 x_\lambda}{\partial x'_\alpha \partial x'_\beta} + \frac{\partial x_\lambda}{\partial x'_\gamma} \frac{\partial^2 x_\mu}{\partial x'_\beta \partial x'_\alpha} \right.$$
$$\left. + \frac{\partial x_\mu}{\partial x'_\alpha} \frac{\partial^2 x_\lambda}{\partial x'_\beta \partial x'_\gamma} - \frac{\partial x_\lambda}{\partial x'_\alpha} \frac{\partial^2 x_\mu}{\partial x'_\gamma \partial x'_\beta} - \frac{\partial x_\mu}{\partial x'_\beta} \frac{\partial^2 x_\lambda}{\partial x'_\gamma \partial x'_\alpha} \right)$$

In the last term of right hand side, interchanging the dummy suffixes λ, μ wherever necessary in order to make all the second differentials of x_μ only, we get

$$\Gamma'_{\alpha\beta,\gamma} = \Gamma'_{\lambda\mu,\nu} \frac{\partial x_\lambda}{\partial x'_\alpha} \frac{\partial x_\mu}{\partial x'_\beta} \frac{\partial x_\nu}{\partial x'_\gamma} + \frac{1}{2} g_{\lambda\mu} \left(\frac{\partial x_\lambda}{\partial x'_\beta} \frac{\partial^2 x_\mu}{\partial x'_\alpha \partial x'_\gamma} + \frac{\partial x_\lambda}{\partial x'_\gamma} \frac{\partial^2 x_\mu}{\partial x'_\alpha \partial x'_\beta} \right.$$
$$\left. + \frac{\partial x_\lambda}{\partial x'_\lambda} \frac{\partial^2 x_\mu}{\partial x'_\beta \partial x'_\alpha} + \frac{\partial x_\lambda}{\partial x'_\alpha} \frac{\partial^2 x_\mu}{\partial x'_\beta \partial x'_\gamma} - \frac{\partial x_\lambda}{\partial x'_\alpha} \frac{\partial^2 x_\mu}{\partial x'_\gamma \partial x'_\beta} - \frac{\partial x_\lambda}{\partial x'_\beta} \frac{\partial^2 x_\mu}{\partial x'_\gamma \partial x'_\alpha} \right)$$
$$= \Gamma'_{\lambda\mu,\nu} \frac{\partial x_\lambda}{\partial x'_\alpha} \frac{\partial x_\mu}{\partial x'_\beta} \frac{\partial x_\nu}{\partial x'_\gamma} + g_{\lambda\mu} \frac{\partial x_\lambda}{\partial x'_\gamma} \frac{\partial^2 x_\mu}{\partial x'_\alpha \partial x'_\beta} \quad \dots (8)$$

Also we have the transformation law for the contravariant fundamental tensor as

$$g'^{\gamma\varepsilon} = g^{\rho\sigma} \frac{\partial x'_\gamma}{\partial x_\rho} \frac{\partial x'_\varepsilon}{\partial x_\sigma} \quad \dots (9)$$

Multiplying together the corresponding sides of (8) and (9), we get

$$g'^{\gamma\varepsilon}\Gamma'_{\alpha\beta,\gamma} = g^{\rho\sigma}\Gamma_{\lambda\mu,\nu}\frac{\partial x_\lambda}{\partial x'_\alpha}\frac{\partial x_\mu}{x'_\beta}\frac{\partial x_\nu}{\partial x'_\gamma}\frac{\partial x'_\gamma}{\partial x_\rho}\frac{\partial x'_\varepsilon}{\partial x_\sigma}$$

$$+g^{\rho\sigma}g_{\lambda\mu}\frac{\partial x_\lambda}{\partial x'_\gamma}\frac{\partial^2 x_\mu}{\partial x'_\alpha\,\partial x'_\beta}\frac{\partial x_\gamma}{\partial x_\rho}\frac{\partial x'_\varepsilon}{\partial x_\sigma}$$

$$=\frac{\partial x_\lambda}{\partial x'_\alpha}\frac{\partial x_\mu}{\partial x'_\beta}\frac{\partial x_\varepsilon}{\partial x_\sigma}g^{\rho\sigma}\frac{\partial x_\alpha}{\partial x'_\gamma}\frac{\partial x'_\gamma}{\partial x_\rho}\Gamma_{\lambda\mu.\nu}+g^{\rho\sigma}\frac{\partial^2 x_\mu}{\partial x'_\alpha\,\partial x'_\beta}\frac{\partial x'_\varepsilon}{\partial x_\sigma}\frac{\partial x_\lambda}{\partial x'_\gamma}\frac{\partial x'_\gamma}{\partial x_\rho}g_{\lambda\mu}$$

$$=\frac{\partial x_\lambda}{\partial x'_\alpha}\frac{\partial x_\mu}{\partial x'_\beta}\frac{\partial x'_\varepsilon}{\partial x_\sigma}g^{\rho\sigma}\Gamma_{\lambda\mu,\rho}+\frac{\partial^2 x_\mu}{\partial x'_\alpha\,\partial x'_\beta}g^{\rho\sigma}\frac{\partial x'_\varepsilon}{\partial x_\sigma}g_{\lambda\mu}$$

$$=\frac{\partial x_\lambda}{\partial x'_\alpha}\frac{\partial x_\mu}{\partial x'_\beta}\frac{\partial x'_\varepsilon}{\partial x_\sigma}\Gamma^\sigma_{\lambda\mu}+\frac{\partial^2 x_\mu}{\partial x'_\alpha\,\partial x'_\beta}g^\sigma_\mu\frac{\partial x'_\varepsilon}{\partial x_\sigma}$$

or $$\Gamma'^\varepsilon_{\alpha\beta}=\Gamma^\sigma_{\lambda\mu}\frac{\partial x_\lambda}{\partial x'_\alpha}\frac{\partial x_\mu}{\partial x'_\beta}\frac{\partial x'_\varepsilon}{\partial x_\sigma}+\frac{\partial^2 x_\mu}{\partial x'_\alpha\,\partial x'_\beta}\frac{\partial x'_\varepsilon}{\partial x_\mu}.$$

Replacing α by β, β by γ, ε by α, λ by μ, μ by ν and σ by λ this becomes

$$\Gamma'^\alpha_{\beta\gamma}=\Gamma^\lambda_{\mu\nu}\frac{\partial x_\mu}{\partial x'_\beta}\frac{\partial x_\nu}{\partial x'_\gamma}\frac{\partial x'_\alpha}{\partial x_\lambda}+\frac{\partial^2 x_\nu}{\partial x'_\beta\,\partial x'_\gamma}\frac{\partial x'_\alpha}{\partial x_\nu}.$$

Again replacing ν by λ in the last term,

$$\left.\begin{aligned}\Gamma'^\alpha_{\beta\gamma}&=\Gamma^\lambda_{\mu\nu}\frac{\partial x_\mu}{\partial x'_\beta}\frac{\partial x_\nu}{\partial x'_\gamma}\frac{\partial x'_\alpha}{\partial x_\lambda}+\frac{\partial^2 x_\lambda}{\partial x'_\beta\,\partial x'_\gamma}\frac{\partial x'_\alpha}{\partial x_\lambda},\\ \text{i.e.}\quad \{\beta\gamma,\alpha\}'&=\{\mu\nu,\lambda\}\frac{\partial x_\mu}{\partial x'_\beta}\frac{\partial x_\nu}{\partial x'_\gamma}\frac{\partial x'_\alpha}{\partial x_\lambda}+\frac{\partial^2 x_\lambda}{\partial x'_\beta\,\partial x'_\gamma}\frac{\partial x'_\alpha}{\partial x_\lambda},\end{aligned}\right\}\quad\ldots(10)$$

Interchanging primed and unprimed letters, this yields

$$\{\beta\gamma,\alpha\}=\{\mu\nu,\lambda\}'\frac{\partial x'_\mu}{\partial x_\beta}\frac{\partial x'_\nu}{\partial x_\gamma}\frac{\partial x_\alpha}{\partial x'_\lambda}+\frac{\partial^2 x'_\lambda}{\partial x_\beta\,\partial x_\gamma}\frac{\partial x_\alpha}{\partial x'_\lambda},$$

which is the same transformation as obtained in (4).

In the given Euclidean space such as,

$$ds^2=(dx_1)^2+(dx_2)^2+\ldots+(dx_n)^2,$$

$$\Gamma^\lambda_{\mu\nu}=0.$$

The equation (10), yields

$$\Gamma'^\alpha_{\beta\gamma}=\frac{\partial^2 x_\lambda}{\partial x'_\beta\,\partial x'_\gamma}\frac{\partial x'_\alpha}{\partial x_\lambda}.$$

If this new system is also cartesian, then $g_{\alpha\beta}$ = constant

or $$\Gamma'^\alpha_{\beta\gamma}=0$$

or $$\frac{\partial^2 x_\lambda}{\partial x'_\beta \partial x'_\gamma} = 0 \qquad \ldots (11)$$

and then *the symbols transform like tensor.* *(Agra, 1965, 66, 67)*

Integrating (11) twice, we get

$$x_\lambda = a_\alpha{}^\lambda x'_\alpha + b^\lambda,$$

where $a_\alpha{}^\lambda$ and b^λ are the constants of integration.

This follows that the co-ordinate transformation between two cartesian co-ordinate systems is linear.

Now inner multiplication of (10) by $\frac{\partial x_\rho}{\partial x'_\alpha}$, gives

$$\Gamma'^{\alpha}_{\beta\lambda} \frac{\partial x_\rho}{\partial x'_\alpha} = \frac{\partial x_\rho}{\partial x'_\alpha} \Gamma^\lambda_{\mu\nu} \frac{\partial x_\mu}{\partial x'_\beta} \frac{\partial x_\nu}{\partial x'_\gamma} \frac{\partial x'_\alpha}{\partial x_\lambda} + \frac{\partial x_\rho}{\partial x'_\alpha} \frac{\partial^2 x_\lambda}{\partial x'_\beta \partial x'_\gamma} \frac{\partial x'_\alpha}{\partial x_\lambda}$$

$$= \frac{\partial x_\mu}{\partial x'_\beta} \frac{\partial x_\gamma}{\partial x'_\gamma} \frac{\partial x_\rho}{\partial x'_\mu} \frac{\partial x'_\mu}{\partial x_\lambda} \Gamma^\lambda_{\mu\nu} + \frac{\partial x_\rho}{\partial x'_\alpha} \frac{\partial x'_\alpha}{\partial x_\lambda} \frac{\partial^2 x_\lambda}{\partial x'_\beta \partial x'_\gamma}$$

$$= \frac{\partial x_\mu}{\partial x'_\beta} \frac{\partial x_\nu}{\partial x'_\gamma} \Gamma^\rho_{\mu\nu} + \frac{\partial^2 x_\rho}{\partial x'_\beta \partial x'_\gamma},$$

i.e. $$\frac{\partial^2 x_\rho}{\partial x'_\beta \partial x'_\gamma} = \Gamma'^{\alpha}_{\beta\gamma} \frac{\partial x_\rho}{\partial x'_\alpha} - \Gamma^\rho_{\mu\nu} \frac{\partial x_\mu}{\partial x'_\beta} \frac{\partial x_\nu}{\partial x'_\gamma}. \qquad \ldots (12)$$

3.15. PARALLEL DISPLACEMENT OF VECTORS

Consider an absolute contravariant vector $A^\mu (x_1, x_2, x_3, \ldots\ldots\ldots x_n)$ in a cartesian co-ordinate system. Then, we have

$$A'^\mu = A^\alpha \frac{\partial x'_\mu}{\partial x_\alpha} \text{ and } A^\mu = A'^\alpha \frac{\partial x_\mu}{\partial x'_\alpha}$$

$$\therefore \quad dA'^\mu = A^\alpha \frac{\partial^2 x'_\mu}{\partial x_\beta \partial x_\alpha} \cdot \frac{\partial x_\beta}{\partial x'_\gamma} dx'_\gamma + \frac{\partial x'_\mu}{\partial x_\alpha} dA^\alpha.$$

If we now assume that the components of A^μ are constants *i.e.* $dA^\mu = 0$, then the above relation becomes

$$dA'^\mu = A^\alpha \frac{\partial^2 x'_\mu}{\partial x_\beta \partial x_\alpha} \cdot \frac{x_\beta}{\partial x'_\gamma} dx'_\gamma$$

$$= A'^\sigma \frac{\partial x_\alpha}{\partial x'_\sigma} \frac{\partial^2 x'_\mu}{\partial x_\beta \partial x_\alpha} \frac{\partial x_\beta}{\partial x'_\gamma} dx'_\gamma \text{ as } A^\sigma = A'^\sigma \frac{\partial x_\alpha}{\partial x'_\sigma}$$

$$= A'^\sigma \frac{\partial^2 x'_\mu}{\partial x_\beta \partial x_\alpha} \frac{\partial x_\beta}{\partial x'_\gamma} \frac{\partial x_\alpha}{\partial x'_\sigma} dx'_\gamma. \qquad \ldots (1)$$

From the preceding section we have the law of transformation

as $$\Gamma'^\mu_{\gamma\sigma} = \frac{\partial^2 x_\alpha}{\partial x'_\gamma \partial x'_\sigma} \cdot \frac{\partial x'^\mu}{\partial x_\alpha} + \Gamma^\sigma_{\beta\gamma} \frac{\partial x_\beta}{\partial x'_\gamma} \frac{\partial x_\gamma}{\partial x'_\sigma} \frac{\partial x'_\mu}{\partial x_\alpha}.$$

But in the cartesian co-ordinate system, we have $\Gamma^{\alpha}_{\beta\gamma} = 0$.

$$\therefore \qquad \Gamma'^{\mu}_{\gamma\sigma} = \frac{\partial^2 x'_\alpha}{\partial x'_\gamma \, \partial x'_\sigma} \frac{\partial x'_\mu}{\partial x_\alpha}.$$

Now $$\frac{\partial x'_\mu}{\partial x_\alpha} \frac{\partial x_\alpha}{\partial x'_\sigma} = \delta^{\mu}_{\sigma}.$$

Differentiating this equation partially with respect to x'_γ, we obtain

$$\frac{\partial^2 x'_\mu}{\partial x_\beta \, \partial x_\alpha} \frac{\partial x_\beta}{\partial x'_\gamma} \frac{\partial x_\alpha}{\partial x'_\sigma} = -\frac{\partial x'_\mu}{\partial x_\alpha} \frac{\partial^2 x'_\alpha}{\partial x'_\gamma \, \partial x'_\sigma}$$

$$\therefore \qquad \Gamma'^{\mu}_{\gamma\sigma} = -\frac{\partial^2 x'_\mu}{\partial x_\beta \, \partial x_\alpha} \frac{\partial x_\beta}{\partial x'_\gamma} \frac{\partial x_\alpha}{\partial x'_\sigma}.$$

With the substitution of these values in (1), we have

$$dA'^{\mu} = -A'^{\sigma} \, \Gamma'^{\mu}_{\gamma\sigma} \, dx'_\gamma \qquad \ldots (2)$$

or interchanging the primed and unprimed letters,

$$dA^{\mu} = -A^{\sigma} \, \Gamma^{\mu}_{\gamma\sigma} \, dx_\gamma.$$

Thus if A^μ is parallely displaced with respect to any Riemannian V_n along any curve, we have

$$\frac{dA^\mu}{ds} = -A^\sigma \, \Gamma^{\mu}_{\gamma\sigma} \frac{dx_\gamma}{ds}.$$

Now we write

$$\frac{\delta A^\mu}{\delta s} = \frac{dA^\mu}{ds} + A^\sigma \, \Gamma^{\mu}_{\gamma\sigma} \frac{dx_\gamma}{ds}. \qquad \ldots (3)$$

$$= -A^\sigma \, \Gamma^{\mu}_{\gamma\sigma} \frac{dx_\gamma}{ds} + A^\sigma \, \Gamma^{\mu}_{\gamma\sigma} \frac{dx_\gamma}{ds}$$

$$= 0$$

which represents a parallel displacement of a vector A^μ along the curve. Here $\dfrac{\delta A^\mu}{\delta s}$ is known as the **intrinsic derivative** of A^μ w.r.t. s.

But the equation of geodesic is,

$$\frac{d^2 x_\mu}{ds^2} + \Gamma^{\mu}_{\alpha\beta} \frac{dx_\alpha}{ds} \frac{dx_\beta}{ds} = 0$$

or $$\frac{d}{ds}\left(\frac{dx_\mu}{ds}\right) + \Gamma^{\mu}_{\alpha\beta}\left(\frac{dx_\mu}{ds}\right)\frac{dx_\beta}{ds} = 0 \qquad \ldots (4)$$

So the equation (4) represents the parallel displacement of unit tangent vector $\dfrac{dx_\mu}{ds}$ along the geodesic.

3.16. COVARIANT DERIVATIVE OF A VECTOR

(*Agra, 1959, 70*)

We have already introduced the transformation law of covariant vector as

$$\frac{\partial \phi}{dx'_\mu} = \frac{\partial x_\alpha}{\partial x'_\mu} \frac{\partial \phi}{\partial x_\alpha}, \quad \text{(by §3.4)}$$

where ϕ is an invariant function of position.

This relation follows that the derivative of an invariant is a covariant vector, but we should carefully note that the derivative of a vector is not a tensor. We shall now find certain tensors to be employed in place of ordinary derivatives of vectors.

Now consider $\frac{dx_\mu}{ds}$ when ds_μ is contravariant and ds invariant.

It is a contravariant vector and represents velocity. Thus if A_μ is a covariant vector, then the inner product $A_\mu \frac{dx_\mu}{ds}$ is invariant. As such, the rate of change of $A_\mu \frac{dx_\mu}{ds}$ per unit interval along any assigned curve must be independent of the co-ordinate system

i.e. $$\frac{d}{ds}\left(A_\mu \frac{dx_\mu}{ds}\right) \text{ is invariant.} \qquad \dots (1)$$

This leads to an assumption that we keep to the same absolute curve however the co-ordinate system is varied. This result being of no practical importance is now applied to a geodesic.

The expression (1) on performing differentiation yields

$$\frac{\partial A_\mu}{\partial x_\nu} \frac{\partial x_\nu}{ds} \cdot \frac{dx_\mu}{ds} + A_\mu \frac{d^2 x_\mu}{ds^2} \text{ is invariant along a geodesic} \qquad \dots (2)$$

The equation of a geodesic is

$$\frac{d^2 x_\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu} \frac{dx_\mu}{ds} \frac{dx_\nu}{ds} = 0$$

i.e. $$A_\mu \frac{d^2 x_\mu}{ds^2} + A_\alpha \frac{d^2 x_\alpha}{ds^2} = -A_\alpha \Gamma^\alpha_{\mu\nu} \frac{dx_\mu}{ds} \frac{dx_\nu}{ds}.$$

Its substitution in (2) gives

$$\frac{dx_\mu}{ds} \frac{dx_\nu}{ds}\left(\frac{\partial A_\mu}{\partial x_\nu} - A_\alpha \Gamma^\alpha_{\mu\nu}\right) \text{ is invariant.}$$

Since $\frac{dx_\mu}{ds}$ and $\frac{dx_\nu}{ds}$ both are contravariant vectors, their cofactor *i.e.*, $\frac{\partial A_\mu}{\partial x_\nu} - \Gamma^\alpha_{\mu\nu} A_\alpha$ is a covariant tensor of rank two. (*Agra, 1964*)

We write

$$A_{\mu,\nu} \text{ (or simply } A_{\mu\nu}) = \frac{\partial A_\mu}{\partial x_\nu} - \Gamma^\alpha_{\mu\nu} A_\alpha \qquad \dots (3)$$

and call the tensor $A_{\mu,\nu}$ the *contravariant derivative of* A_μ w.r.t. x_ν.

The result (3) may be arrived at by another method as follows :

We have

$$\Gamma'^{\rho}_{\mu\nu} \frac{\partial x_\varepsilon}{\partial x'_\rho} = \frac{\partial^2 x_\varepsilon}{\partial x'_\mu \partial x'_\nu} + \frac{\partial x_\alpha}{\partial x'_\mu} \frac{\partial x_\beta}{\partial x'_\nu} \Gamma^{\varepsilon}_{\alpha\beta} \qquad \dots (4)$$

and the transformation law of covariant vectors gives

$$A'_\mu = \frac{\partial x_\varepsilon}{\partial x'_\mu} A_\varepsilon$$

which on differentiation with respect to x'_ν yields

$$\frac{\partial A'_\mu}{\partial x'_\nu} = \frac{\partial^2 x_\varepsilon}{\partial x'_\mu \partial x'_\nu} A_\varepsilon + \frac{\partial x_\varepsilon}{\partial x'_\mu} \cdot \frac{\partial x_\xi}{\partial x'_\nu} \frac{\partial A_\varepsilon}{\partial x_\xi}$$

Applying (4) and changing the dummy suffixes in the last term, we get

$$\frac{\partial A'_\mu}{\partial x'_\nu} = \left(\Gamma'^{\rho}_{\mu\nu} \frac{\partial x_\varepsilon}{\partial x'_\rho} - \frac{\partial x_\alpha}{\partial x'_\mu} \frac{\partial x_\beta}{\partial x'_\nu} \Gamma^{\varepsilon}_{\alpha\beta} \right) A_\varepsilon + \frac{\partial x_\alpha}{\partial x'_\mu} \frac{\partial x_\beta}{\partial x'_\nu} \frac{\partial A_\alpha}{\partial x_\beta}. \qquad \dots (5)$$

Again we have from law of transformation of covariant vectors

$$A_\varepsilon \frac{\partial x_\varepsilon}{\partial x'_\rho} = A'_\rho$$

so that (5) becomes

$$\frac{\partial A'_\mu}{\partial x'_\nu} - \Gamma'^{\rho}_{\mu\nu} A'_\rho = \frac{\partial x_\alpha}{\partial x'_\mu} \frac{\partial x_\beta}{\partial x'_\nu} \left(\frac{\partial A_\alpha}{\partial x_\beta} - \Gamma^{\varepsilon}_{\alpha\beta} A_\varepsilon \right) \qquad \dots (6)$$

Showing that $\frac{\partial A_\mu}{\partial x_\nu} - \Gamma^{\rho}_{\mu\nu} A_\rho$ obeys the law of transformation of a covariant tensor.

We thus approach the result (3).

From (3), by raising a suffix we get an important associated tensor $A^{\mu}{}_{,\nu}$ as follows:

We have $A_\sigma = g_{\sigma\varepsilon} A^\varepsilon$,

so that from (3) we have

$$\begin{aligned} A_{\sigma,\nu} &= \frac{\partial A_\sigma}{\partial A_\nu} - \Gamma^{\alpha}_{\sigma\nu} A_\alpha \\ &= \frac{\partial}{\partial x_\nu} \left(g_{\sigma\varepsilon} A^\varepsilon\right) - \Gamma^{\alpha}_{\sigma\nu} \left(g_{\alpha\varepsilon} A^\varepsilon\right) \\ &= g_{\sigma\varepsilon} \frac{\partial A_\varepsilon}{\partial x_\nu} + A^\varepsilon \frac{\partial g_{\sigma\varepsilon}}{\partial x_\nu} - \Gamma_{\sigma\nu,\varepsilon} A^\varepsilon && \text{by §3.11 (6)} \\ &= g_{\sigma\varepsilon} \frac{\partial A_\varepsilon}{\partial x_\nu} + \Gamma_{\varepsilon\nu,\sigma} A^\varepsilon && \text{by §3.11 (7)} \end{aligned}$$

Multiplying throughout by $g^{\mu\sigma}$, this becomes

$$g^{\mu\sigma} A_{\sigma,\nu} = g^{\mu\sigma} \frac{\partial A^\varepsilon}{\partial x_\nu} + g^{\mu\sigma} \Gamma_{\varepsilon\nu,\sigma} A^\varepsilon$$

or

$$A^{\mu}_{\nu} = g^{\mu\sigma}_{\varepsilon} \frac{\partial A^\varepsilon}{\partial x_\nu} + \Gamma^{\mu}_{\varepsilon\nu} A^\varepsilon$$

$$= \frac{\partial A^{\mu}}{\partial x_{\nu}} + \Gamma^{\mu}_{\epsilon\nu} A^{\epsilon}. \qquad ...(7)$$

(*Agra, 1970*)

which is called the **covariant derivative of A^{μ}.** (*Rohilkhand, 1985*)

Similarly the tensors $A^{\nu}{}_{\mu}$ and $A^{\mu\nu}$ obtained from (3) and (7) by raising the suffixes are known as contravariant derivatives of A_{μ} and A^{μ}, but they are of no practical importance.

3.17. COVARIANT DERIVATIVE OF A TENSOR (*Agra, 1963*)

The covariant derivatives of a tensor of rank two are formed as

$$A_{\mu\nu,\sigma} = \frac{\partial A_{\mu\nu}}{\partial x_{\sigma}} - \Gamma^{\alpha}_{\mu\sigma} A_{\alpha\nu} - \Gamma^{\alpha}_{\nu\sigma} A_{\mu\alpha} \qquad ...(1)$$

$$A^{\nu}_{\mu\sigma} \left(\text{or } A^{\nu}_{\mu,\sigma}\right) = \frac{\partial A^{\nu}_{\mu}}{\partial x_{\sigma}} - \Gamma^{\alpha}_{\mu\sigma} A^{\nu}_{\alpha} + \Gamma^{\nu}_{\alpha\sigma} A^{\alpha}_{\mu} \qquad ...(2)$$

$$A^{\mu\nu}_{\sigma} \left(\text{or } A^{\mu\nu}_{,\sigma}\right) = \frac{\partial A^{\mu\nu}}{\partial x_{\sigma}} + \Gamma^{\mu}_{\alpha\sigma} A^{\alpha\nu} + \Gamma^{\nu}_{\alpha\sigma} A^{\mu\alpha} \qquad ...(3)$$

and a general rule for giving covariant differentiation with respect to x_{σ} may be illustrated as follows :

$$A^{\rho}_{\lambda\mu\nu\sigma} = \frac{\partial}{\partial x_{\sigma}} A^{\rho}_{\lambda\mu\nu} - \Gamma^{\alpha}_{\lambda\sigma} A^{\rho}_{\alpha\mu\nu} - \Gamma^{\alpha}_{\mu\sigma} A^{\rho}_{\lambda\alpha\nu} - \Gamma^{\alpha}_{\nu\sigma} A^{\rho}_{\lambda\mu\alpha} + \Gamma^{\rho}_{\alpha\sigma} A^{\alpha}_{\lambda\mu\nu} \qquad ...(4)$$

In order to show that the quantities on the right are actually tensors, we proceed as follows :

In place of (1) of §3.16 we may use to say that

$$\frac{d}{ds}\left(A_{\mu\nu} \frac{dx_{\mu}}{ds} \frac{dx_{\nu}}{ds}\right) \text{ is invariant}$$

i.e.,

$$\frac{dA_{\mu\nu}}{dx_{\sigma}} \frac{dx_{\sigma}}{ds} \frac{dx_{\mu}}{ds} \frac{dx_{\nu}}{ds} + A_{\mu\nu} \frac{dx_{\nu}}{ds} \frac{d^2x_{\mu}}{ds^2} + A_{\mu\nu} \frac{dx_{\mu}}{ds} \frac{d^2x_{\nu}}{ds^2}$$

is invariant along a geodesic.

But along a geodesic, we have

$$A_{\mu\nu} \frac{d^2x_{\mu}}{ds^2} = A_{\alpha\nu} \frac{d^2x_{\alpha}}{ds^2} = -A_{\alpha\nu} \Gamma^{\alpha}_{\mu\sigma} \frac{dx_{\mu}}{ds} \frac{dx_{\sigma}}{ds}$$

and

$$A_{\mu\nu} \frac{d^2x_{\nu}}{ds^2} = A_{\mu\alpha} \frac{d^2x_{\alpha}}{ds^2} = -A_{\mu\alpha} \Gamma^{\alpha}_{\nu\sigma} \frac{dx_{\nu}}{ds} \frac{dx_{\sigma}}{ds}.$$

$\therefore$

$$\left(\frac{\partial A_{\mu\nu}}{\partial x_{\sigma}} - \Gamma^{\alpha}_{\mu\sigma} A_{\alpha\nu} - \Gamma^{\alpha}_{\nu\sigma} A_{\mu\alpha}\right) \frac{dx_{\mu}}{ds} \frac{dx_{\nu}}{ds} \frac{dx_{\sigma}}{ds} \text{ is invariant,}$$

i.e.,

$$A_{\mu\nu,\sigma} \frac{dx_{\mu}}{ds} \frac{dx_{\nu}}{ds} \frac{dx_{\sigma}}{ds} \text{ is invariant}$$

which shows that $A_{\mu\nu\sigma}$ is a covariant tensor of rank three.

The results (2) and (3) may be obtained by raising the suffixes ν and μ as follows:

Since $A_{\mu\gamma} = g_{\gamma\varepsilon} A^{\varepsilon}{}_{\mu}$, therefore,

$$A_{\mu\gamma,\sigma} = \frac{\partial A_{\mu\nu}}{\partial x_\sigma} - \Gamma^{\alpha}_{\mu\sigma} A_{\alpha\gamma} - \Gamma^{\alpha}_{\gamma\sigma} A_{\mu\alpha}$$

$$= \frac{\partial}{\partial x_\sigma}\left(g_{\gamma\varepsilon} A^{\varepsilon}_{\mu}\right) - \Gamma^{\varepsilon}_{\mu\sigma} A_{\varepsilon\gamma} - \Gamma^{\alpha}_{\gamma\sigma} g_{\alpha\varepsilon} A^{\varepsilon}_{\mu}$$

$$= g_{\gamma\varepsilon}\frac{\partial A^{\varepsilon}_{\mu}}{\partial x_\sigma} + A^{\varepsilon}_{\mu} \frac{\partial g_{\gamma\varepsilon}}{\partial x_\sigma} - \Gamma^{\varepsilon}_{\mu\sigma} A_{\varepsilon\gamma} - \Gamma_{\gamma\sigma,\varepsilon} A^{\varepsilon}_{\mu}$$

$$= g_{\gamma\varepsilon}\frac{\partial A^{\varepsilon}_{\mu}}{\partial x_\sigma} - \Gamma^{\varepsilon}_{\mu\sigma} A_{\varepsilon\gamma} + \left(\frac{\partial g_{\gamma\varepsilon}}{\partial x_\sigma} - \Gamma_{\gamma\sigma,\varepsilon}\right) A^{\varepsilon}_{\mu}$$

$$= g_{\gamma\varepsilon}\frac{\partial A^{\varepsilon}_{\mu}}{\partial x_\sigma} - \Gamma^{\varepsilon}_{\mu\sigma} A_{\varepsilon\gamma} + \Gamma_{\varepsilon\sigma,\gamma} A^{\varepsilon}_{\mu} \quad \text{by (7) of §3.12.} \quad (Agra,\ 1959)$$

Multiplying throughout by $g^{\nu\gamma}$, this becomes

$$g^{\nu\gamma} A_{\mu\gamma,\sigma} = g^{\nu\gamma} g_{\gamma\varepsilon} \frac{\partial A^{\varepsilon}_{\mu}}{\partial x_\sigma} - g^{\nu\gamma} \Gamma^{\varepsilon}_{\mu\sigma} A_{\varepsilon\gamma} + g^{\nu\gamma} \Gamma_{\varepsilon\sigma,\gamma} A^{\varepsilon}_{\mu}$$

i.e.,
$$A^{\nu}_{\mu,\sigma} = \frac{\partial A^{\nu}_{\mu}}{\partial x_\sigma} - \Gamma^{\varepsilon}_{\mu\sigma} g^{\nu\gamma} A_{\varepsilon\gamma} + \Gamma^{\nu}_{\varepsilon\sigma} A^{\varepsilon}_{\mu} \quad \text{by (5) of §3.12}$$

$$= \frac{\partial A^{\nu}_{\mu}}{\partial x_\sigma} - \Gamma^{\varepsilon}_{\mu\sigma} A^{\nu}_{\varepsilon} + \Gamma^{\nu}_{\varepsilon\sigma} A^{\varepsilon}_{\mu}$$

which is the result (2).

Again, since $A_{\gamma}{}^{\nu} = g_{\gamma\varepsilon} A^{\varepsilon\nu}$, we have

$$A^{\nu}_{\gamma,\sigma} = \frac{\partial A^{\nu}_{\gamma}}{\partial x_\sigma} - \Gamma^{\alpha}_{\gamma\sigma} A^{\nu}_{\alpha} + \Gamma^{\nu}_{\alpha\sigma} A^{\alpha}_{\gamma}$$

$$= \frac{\partial}{\partial x_\sigma}\left(g_{\gamma\varepsilon} A^{\varepsilon\nu}\right) - \Gamma^{\alpha}_{\gamma\sigma} g_{\alpha\varepsilon} A^{\varepsilon\nu} + \Gamma^{\nu}_{\alpha\sigma} A^{\alpha}_{\gamma}$$

$$= g_{\gamma\varepsilon} \frac{\partial A^{\varepsilon\nu}}{\partial x_\sigma} + A^{\varepsilon\nu} \frac{\partial g_{\gamma\varepsilon}}{\partial x_\sigma} - \Gamma_{\gamma\sigma,\varepsilon} A^{\varepsilon\nu} + \Gamma^{\nu}_{\alpha\sigma} A^{\alpha}_{\gamma} \quad \text{by (6) of §3.12.}$$

or
$$A^{\nu}_{\gamma,\sigma} = g_{\gamma\varepsilon} \frac{\partial A^{\varepsilon\nu}}{\partial x_\sigma} + A^{\varepsilon\nu}\left(\frac{\partial g_{\gamma\varepsilon}}{\partial x_\sigma} - \Gamma_{\gamma\sigma,\varepsilon}\right) + \Gamma^{\nu}_{\alpha\sigma} A^{\alpha}_{\gamma}$$

$$= g_{\gamma\varepsilon} \frac{\partial A^{\varepsilon\nu}}{\partial x_\sigma} + A^{\varepsilon\nu} \Gamma_{\varepsilon\sigma,\gamma} + \Gamma^{\nu}_{\alpha\sigma} A^{\alpha}_{\gamma} \quad \text{by (7) of §3.12.}$$

Multiplying throughout by $g^{\mu\nu}$, this becomes

$$g^{\mu\gamma} A^{\nu}_{\gamma,\sigma} = g^{\mu\gamma} g_{\gamma\varepsilon} \frac{\partial A^{\varepsilon\nu}}{\partial x_\sigma} + g^{\mu\gamma} A^{\varepsilon\nu} \Gamma_{\varepsilon\sigma,\gamma} + g^{\mu\gamma} \Gamma^{\nu}_{\alpha\sigma} A^{\alpha}_{\gamma}$$

i.e.,
$$A^{\mu\nu}_{,\sigma} = \frac{\partial A^{\mu\nu}}{\partial x_\sigma} + \Gamma^{\mu}_{\varepsilon\sigma} A^{\varepsilon\nu} + \Gamma^{\nu}_{\alpha\sigma} A^{\mu\alpha}$$

$$= \frac{\partial A^{\mu\nu}}{\partial x_\sigma} + \Gamma^{\mu}_{\varepsilon\alpha} A^{\alpha\nu} + \Gamma^{\nu}_{\alpha\sigma} A^{\mu\alpha}$$

which is the result (3).

Problem 27. *Prove that $g_{\alpha\beta,\gamma} = 0$ i.e., covariant derivative of fundamental tensor vanishes identically.* *(Agra, 1963, 65, 68)*

We have by §3.17 (1).

$$g_{\alpha\beta,\gamma} = \frac{\partial g_{\alpha\beta}}{\partial x_\gamma} - \Gamma^{\mu}_{\alpha\gamma}\, g_{\mu\beta} - \Gamma^{\mu}_{\beta\gamma}\, g_{\mu\alpha}. \qquad \dots (1)$$

But
$$\Gamma^{\mu}_{\alpha\gamma} = \frac{1}{2}\, g^{\mu\beta} \left(\frac{\partial g_{\alpha\beta}}{\partial x_\gamma} + \frac{\partial g_{\beta\gamma}}{\partial x_\alpha} - \frac{\partial g_{\alpha\gamma}}{\partial x_\beta} \right)$$

so that
$$g_{\mu\beta}\, \Gamma^{\mu}_{\alpha\nu} = \frac{1}{2}\, g_{\mu\beta}\, g^{\mu\beta} \left(\frac{\partial g_{\alpha\beta}}{\partial x_\gamma} + \frac{\partial g_{\beta\gamma}}{\partial x_\alpha} - \frac{\partial g_{\alpha\gamma}}{\partial x_\beta} \right)$$

$$= \frac{1}{2} \left(\frac{\partial g_{\alpha\beta}}{\partial x_\gamma} + \frac{\partial g_{\beta\gamma}}{\partial x_\gamma} - \frac{\partial g_{\alpha\gamma}}{\partial x_\beta} \right) \text{ since } g_{\mu\beta}\, g^{\mu\beta} = 1$$

Similarly
$$g_{\mu\alpha}\, \Gamma^{\mu}_{\beta\gamma} = \frac{1}{2} \left(\frac{\partial g_{\beta\alpha}}{\partial x_\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial x_\beta} - \frac{\partial g_{\beta\gamma}}{\partial x_\alpha} \right).$$

Adding the last two relations, we get

$$g_{\mu\beta}\, \Gamma^{\mu}_{\alpha\gamma} + g_{\mu\alpha}\, \Gamma^{\mu}_{\beta\gamma} = \frac{\partial g_{\alpha\beta}}{\partial x_\gamma}.$$

Substituting this value in (1), we get

$$g_{\alpha\beta,\gamma} = g_{\mu\beta}\, \Gamma^{\mu}_{\alpha\gamma} + g_{\mu\alpha}\, \Gamma^{\mu}_{\beta\gamma} - \Gamma^{\mu}_{\alpha\gamma}\, g_{\mu\beta} - \Gamma^{\mu}_{\beta\gamma}\, g_{\mu\alpha} = 0.$$

Problem 28. *By starting with $A'_{\alpha\beta} = A_{\mu\nu} \frac{\partial x_\mu}{\partial x'_\alpha} \frac{\partial x_\nu}{\partial x'_\beta}$, verify the first result of §3.17.*

Given that

$$A'_{\alpha\beta} = A_{\mu\nu} \frac{\partial x_\mu}{\partial x'_\alpha} \frac{\partial x_\nu}{\partial x'_\beta}. \qquad \dots (1)$$

Differentiating with regard to x'_γ, we have

$$\frac{\partial A'_{\alpha\beta}}{\partial x'_\gamma} = \frac{\partial A_{\mu\nu}}{\partial x_\sigma} \frac{\partial x_\sigma}{\partial x'_\gamma} \frac{\partial x_\mu}{\partial x'_\alpha} \frac{\partial x_\nu}{\partial x'_\beta} + A_{\mu\nu} \frac{\partial^2 x_\mu}{\partial x'_\alpha\, \partial x'_\gamma} \frac{\partial x_\nu}{\partial x'_\beta} + A_{\mu\nu} \frac{\partial x_\mu}{\partial x'_\alpha} \frac{\partial^2 x_\nu}{\partial x'_\beta\, \partial x'_\gamma}$$

$$= \frac{\partial A_{\mu\nu}}{\partial x_\sigma} \frac{\partial x_\mu}{\partial x'_\alpha} \frac{\partial x_\nu}{\partial x'_\beta} \frac{\partial x_\sigma}{\partial x'_\gamma} + A_{\mu\nu} \left[\Gamma'^{\varepsilon}_{\alpha\gamma} \frac{\partial x_\mu}{\partial x'_\varepsilon} - \Gamma^{\mu}_{\xi\eta} \frac{\partial x_\xi}{\partial x'_\alpha} \frac{\partial x_\eta}{\partial x'_\gamma} \right] \frac{\partial x_\nu}{\partial x'_\beta}$$

$$+ A_{\mu\nu} \left[\Gamma'^{\varepsilon}_{\beta\gamma} \frac{\partial x_\nu}{\partial x'_\varepsilon} - \Gamma^{\nu}_{\xi\eta} \frac{\partial x_\xi}{\partial x'_\alpha} \frac{\partial x_\eta}{\partial x'_\gamma} \right] \frac{\partial x_\mu}{\partial x'_\alpha} \quad \text{by (12) §3.14.}$$

or
$$\frac{\partial A'_{\alpha\beta}}{\partial x'_\gamma} - A_{\mu\nu}\,\Gamma'^{\varepsilon}_{\alpha\gamma}\,\frac{\partial x_\mu}{\partial x'_\varepsilon}\,\frac{\partial x_\alpha}{\partial x'_\beta} - A_{\mu\nu}\,\Gamma'^{\varepsilon}_{\beta\gamma}\,\frac{\partial x_\nu}{\partial x'_\varepsilon}\,\frac{\partial x_\mu}{\partial x'_\alpha}$$
$$= \frac{\partial A_{\mu\nu}}{\partial x_\sigma}\,\frac{\partial x_\mu}{\partial x'_\alpha}\,\frac{\partial x_\nu}{\partial x'_\beta}\,\frac{\partial x_\sigma}{\partial x'_\gamma} - A_{\mu\nu}\,\Gamma^{\mu}_{\xi\eta}\,\frac{\partial x_\xi}{\partial x'_\alpha}\,\frac{\partial x_\eta}{\partial x'_\gamma}\,\frac{\partial x_\nu}{\partial x'_\beta} - A_{\mu\nu}\,\Gamma^{\nu}_{\xi\eta}\,\frac{\partial x_\xi}{\partial x'_\alpha}\,\frac{\partial x_\eta}{\partial x'_\gamma}\,\frac{\partial x_\mu}{\partial x'_\alpha}$$

or using (1), this becomes
$$\frac{\partial A'_{\alpha\beta}}{\partial x'_\gamma} - A'_{\varepsilon\nu}\,\Gamma'^{\varepsilon}_{\alpha\gamma} - A'_{\mu\tau}\,\Gamma^{\varepsilon}_{\beta\gamma} = \left(\frac{\partial A_{\mu\nu}}{\partial x_\sigma} - \Gamma^{\tau}_{\mu\sigma}\,A_{\tau\nu} - \Gamma^{\tau}_{\nu\sigma}\,A_{\mu\tau}\right)\frac{\partial x_\mu}{\partial x'_\alpha}\,\frac{\partial x_\nu}{\partial x'_\beta}\,\frac{\partial x_\sigma}{\partial x'_\gamma}$$

i.e.,
$$A'_{\alpha\beta,\,\gamma} = A_{\mu\nu,\,\sigma}\,\frac{\partial x_\mu}{\partial x'_\alpha}\,\frac{\partial x_\nu}{\partial x'_\beta}\,\frac{\partial x_\sigma}{\partial x'_\gamma}$$

showing that $A_{\mu\nu,\,\sigma}$ is a tensor of rank three and is the covariant derivative of $A_{\mu\nu}$ with respect to x_σ.

This verifies the first result of §3.17.

Problem 29. *Show that the distributive law of ordinary differentiation holds good in case of covariant differentiation of a product, i.e. to show that*

(*Rohilkhand, 1990; Agra, 1959, 63*)

(*i*) $(B_\mu C_\nu)_{,\,\sigma} = B_{\mu,\,\sigma}\,C_\nu + B_\mu\,C_{\nu,\,\sigma}$,

(*ii*) $(B^\mu C^\nu)_{,\sigma} = B^\mu{}_{,\sigma}\,C^\nu + B^\mu\,C^\nu{}_{,\sigma}$.

(*i*) R.H.S. $= B_{\mu,\,\sigma}\,C_\nu + B_\mu\,C_{\nu,\,\sigma}$

$$= \left(\frac{\partial B_\mu}{\partial x_\sigma} - \Gamma^{\alpha}_{\mu\sigma}\,B_\alpha\right) C_\nu + B_\mu\left(\frac{\partial C_\nu}{\partial x_\sigma} - \Gamma^{\alpha}_{\nu\sigma}\,C_\alpha\right)$$
$$= \frac{\partial}{\partial x_\sigma}(B_\mu\;C_\nu) - \Gamma^{\alpha}_{\mu\sigma}\,(B_\alpha\;C_\nu) - \Gamma^{\alpha}_{\nu\sigma}\,(B_\mu\;C_\alpha)$$
$$= (B_\mu C_\nu)_{,\,\sigma}.$$

(*ii*) R.H.S. $= B^\mu{}_{,\sigma}\,C^\nu + B^\mu\,C^\nu{}_{,\sigma}$

$$= \left(\frac{\partial B^\mu}{\partial x^\sigma} + \Gamma^{\mu}_{\alpha\sigma}\,B^\alpha\right)C^\nu + B^\mu\left(\frac{\partial C^\nu}{\partial x_\sigma} + \Gamma^{\nu}_{\alpha\sigma}C^\alpha\right)$$
$$= \frac{\partial}{\partial x_\sigma}(B^\mu\;C^\nu) + \Gamma^{\mu}_{\alpha\sigma}\,(B^\alpha\;C^\nu) + \Gamma^{\nu}_{\alpha\sigma}\,(B^\mu\;C^\alpha)$$
$$= (B^\mu C^\nu)_{,\sigma}.$$

Problem 30. *Prove that*.

(*a*) $g^{\mu\nu}_{,\sigma} = 0.$ (*Agra, 1959, 65*)

(*b*) $\delta^{\nu}_{\mu,\,\sigma} = 0.$ (*Agra, 1965*)

(*a*) We have
$$g_{\xi\eta}\;g^{\xi\xi} = \delta^{\xi}_{\eta}$$
$$= 0 \text{ or } 1.$$

Differentiating with respect to x_λ, this gives
$$g^{\xi\xi}\,\frac{\partial g_{\xi\eta}}{\partial x_\lambda} + g_{\xi\eta}\,\frac{\partial g^{\xi\xi}}{\partial x_\lambda} = 0$$

Multiplying throughout by $g^{\eta\rho}$, we get

$$g^{\eta\rho}\, g^{\xi\xi} \frac{\partial g_{\xi\eta}}{\partial x_\lambda} + g^{\eta\rho}\, g_{\xi\eta} \frac{\partial g^{\xi\xi}}{\partial x_\lambda} = 0$$

or
$$g^{\eta\rho}\, g^{\xi\xi}\left(\Gamma_{\xi\lambda,\eta} - \Gamma_{\lambda\eta,\xi}\right) + g^{\xi\rho} \frac{\partial g^{\xi\xi}}{\partial x_\lambda} = 0 \text{ by (7) of §3.11}$$

or
$$g^{\xi\xi}\, \Gamma^{\eta}_{\xi\lambda} + g^{\eta\rho}\, \Gamma^{\xi}_{\lambda\eta} + \frac{\partial g^{\rho\xi}}{\partial x_\lambda} = 0 \text{ by (5) of §3.11} \qquad \ldots (1)$$

Now
$$g^{\mu\nu}_{,\sigma} = \frac{\partial g^{\mu\nu}}{\partial x_\sigma} + \Gamma^{\mu}_{\alpha\sigma}\, g^{\alpha\nu} + \Gamma^{\mu}_{\alpha\sigma}\, g^{\nu\mu}$$
$$= 0 \text{ by (1).}$$

(*b*)
$$\delta^{\nu}_{\mu,\sigma} = \frac{\partial \delta^{\nu}_{\mu}}{\partial x_\sigma} - \Gamma^{\alpha}_{\mu\sigma}\, g^{\nu}_{\alpha} + \Gamma^{\nu}_{\alpha\sigma}\, \delta^{\alpha}_{\mu}$$
$$= 0 - \Gamma^{\nu}_{\mu\sigma} + \Gamma^{\nu}_{\mu\sigma}$$
$$= 0..$$

Problem 31. *Show that the covariant derivative of an invariant is the same as its ordinary derivative.* (*Rohilkhand, 1990*)

Let I be an invariant, so that IA_μ is a covariant vector.

Now the covariant derivative of IA_μ is

$$\left(IA_\mu\right)_{,\nu} = \frac{\partial\left(IA_\mu\right)}{\partial x_\nu} - \Gamma^{\alpha}_{\mu\nu}\,(IA_\alpha)$$
$$= A_\mu \frac{\partial I}{\partial x_\nu} - I\frac{\partial A_\mu}{\partial x_\nu} - \Gamma^{\alpha}_{\mu\nu}\, IA_\alpha$$
$$= A_\mu \frac{\partial I}{\partial x_\nu} + I\left(\frac{\partial A_\mu}{\partial x_\nu} - \Gamma^{\alpha}_{\mu\nu}\, A_\alpha\right)$$
$$= A_\mu \frac{\partial I}{\partial x_\nu} + IA_{\mu,\nu}$$

But from Problem 29.

$$\left(IA_\mu\right)_{,\nu} = I_{,\nu} A_\mu + IA_{\mu,\nu},$$

∴
$$I_{,\nu}\, A_\mu + I\, A_{\mu,\nu} = A_\mu \frac{\partial I}{\partial x_\nu} + IA_{\mu,\nu}$$

i.e.
$$I_{,\nu}\, A_\mu = A_\mu \frac{\partial I}{\partial x_\nu}$$

or
$$I_{,\nu} = \frac{\partial I}{\partial x_\nu}$$

which shows that the covariant derivative of an invariant is the same as its ordinary derivative.

3.18. THE CURVATURE TENSOR (RIEMANN CHRISTOFFEL TENSOR)

Consider an absolute contravariant vector A^{ε}. Its covariant derivative gives the mixed tensor.

$$A^{\varepsilon}_{,\rho} = \frac{\partial A^{\varepsilon}}{\partial x_{\rho}} + A^{\mu}\,\Gamma^{\varepsilon}_{\mu\rho} \text{ by (7) of §3.16.}$$

If this is again differentiated covariantly, then we have

$$A^{\varepsilon}_{,\rho\sigma} = \frac{\partial A^{\varepsilon}_{\rho}}{\partial x_{\sigma}} + A^{\mu}_{,\rho}\,\Gamma^{\varepsilon}_{\mu\sigma} - A^{\varepsilon}_{\mu}\,\Gamma^{\mu}_{\rho\sigma} \text{ by (2) of §3.17}$$

$$= \frac{\partial}{\partial x_{\sigma}}\left(\frac{\partial A^{\varepsilon}}{\partial x_{\rho}} + A^{\mu}\,\Gamma^{\varepsilon}_{\mu\rho}\right) + \left(\frac{\partial A^{\mu}}{\partial x_{\rho}} + A^{\beta}\,\Gamma^{\mu}_{\beta\rho}\right)\Gamma^{\varepsilon}_{\mu\sigma} - \left(\frac{\partial A^{\varepsilon}}{\partial x_{\mu}} + A^{\beta}\,\Gamma^{\varepsilon}_{\beta\mu}\right)\Gamma^{\mu}_{\rho\sigma}$$

$$= \frac{\partial^2 A^{\varepsilon}}{\partial x_{\sigma}\partial x_{\rho}} + \frac{\partial A^{\mu}}{\partial x_{\sigma}}\,\Gamma^{\varepsilon}_{\mu\rho} + A^{\mu}\frac{\partial \Gamma^{\varepsilon}_{\mu\rho}}{\partial x_{\sigma}} + \left(\frac{\partial A^{\mu}}{\partial x_{\rho}} + A^{\beta}\,\Gamma^{\mu}_{\beta\rho}\right)\Gamma^{\varepsilon}_{\mu\sigma} - \left(\frac{\partial A^{\varepsilon}}{\partial x_{\mu}} + A^{\beta}\,\Gamma^{\varepsilon}_{\beta\mu}\right)\Gamma^{\mu}_{\rho\sigma} \quad \ldots (1)$$

Similarly,

$$A^{\varepsilon}_{,\sigma\rho} = \frac{\partial^2 A^{\varepsilon}}{\partial x_{\rho}\partial x_{\sigma}} + \frac{\partial A^{\mu}}{\partial x_{\rho}}\,\Gamma^{\varepsilon}_{\mu\sigma} + A^{\mu}\frac{\partial \Gamma^{\varepsilon}_{\mu\sigma}}{\partial x_{\rho}} + \left(\frac{\partial A^{\mu}}{\partial x_{\sigma}} + A^{\beta}\,\Gamma^{\mu}_{\beta\sigma}\right)\Gamma^{\varepsilon}_{\mu\rho} - \left(\frac{\partial A^{\varepsilon}}{\partial x_{\mu}} + A^{\beta}\,\Gamma^{\varepsilon}_{\beta\mu}\right)\Gamma^{\mu}_{\sigma\rho} \quad \ldots (2)$$

Subtracting (2) from (1), after changing the dummy suffixes wherever needed, we get

$$A^{\varepsilon}_{,\rho\sigma} - A^{\varepsilon}_{,\sigma\rho} = A^{\mu}\left(\frac{\partial \Gamma^{\varepsilon}_{\mu\rho}}{\partial x_{\sigma}} - \frac{\partial \Gamma^{\varepsilon}_{\mu\sigma}}{\partial x_{\rho}} + \Gamma^{\beta}_{\mu\rho}\,\Gamma^{\varepsilon}_{\beta\sigma} - \Gamma^{\beta}_{\mu\sigma}\,\Gamma^{\varepsilon}_{\beta\rho}\right) \qquad \textit{(Agra, 1958)}$$

The quotient rule follows that the cofactor of A^{μ} must be a tensor of rank four, so that we may write

$$A^{\varepsilon}_{,\rho\sigma} - A^{\varepsilon}_{,\sigma\rho} = A^{\mu}\;B^{\varepsilon}_{\mu\rho\sigma}$$

where
$$B^{\varepsilon}_{\mu\rho\sigma} = \frac{\partial \Gamma^{\varepsilon}_{\mu\rho}}{\partial x_{\sigma}} - \frac{\partial \Gamma^{\varepsilon}_{\mu\sigma}}{\partial x_{\rho}}, + \Gamma^{\varepsilon}_{\mu\rho}\,\Gamma^{\varepsilon}_{\beta\sigma} - \Gamma^{\beta}_{\mu\sigma}\,\Gamma^{\varepsilon}_{\beta\rho}$$

This tensor is known as **curvature tensor.** On contraction this tensor yields two new tensors as follows:

(*i*)
$$B^{\sigma}_{\mu\sigma\nu} = \frac{\partial \Gamma^{\sigma}_{\mu\sigma}}{\partial x_{\nu}} - \frac{\partial \Gamma^{\sigma}_{\mu\sigma}}{\partial x_{\sigma}} + \Gamma^{\beta}_{\mu\sigma}\,\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\beta}_{\mu\nu}\,\Gamma^{\sigma}_{\beta\sigma}$$
$$= R_{\mu\nu} \text{ (say).} \quad \ldots (4)$$

This tensor is known as **Ricci tensor.**

(*ii*)
$$B^{\sigma}_{\sigma\mu\nu} = \frac{\partial \Gamma^{\sigma}_{\sigma\mu}}{\partial x_{\nu}} - \frac{\partial \Gamma^{\sigma}_{\sigma\nu}}{\partial x_{\mu}} + \Gamma^{\beta}_{\sigma\mu}\,\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\beta}_{\sigma\nu}\,\Gamma^{\sigma}_{\beta\mu}$$

$$= \frac{\partial \Gamma^\sigma_{\sigma\mu}}{\partial x_\nu} - \frac{\partial \Gamma^\sigma_{\sigma\nu}}{\partial x_\mu}$$

(the last two terms cancel when μ and ν are interchanged in either of them) ... (5)

$$= S_{\mu\nu} \text{ (say).}$$

So $$S_{\nu\mu} = B^\sigma_{\sigma\nu\mu} = \frac{\partial \Gamma^\sigma_{\sigma\nu}}{\partial x_\mu} - \frac{\partial \Gamma^\sigma_{\sigma\mu}}{\partial x_\nu}$$

$$= -\left(\frac{\partial \Gamma^\sigma_{\sigma\mu}}{\partial x_\nu} - \frac{\partial \Gamma^\sigma_{\sigma\nu}}{\partial x_\mu}\right)$$

$$= -S_{\mu\nu} \quad \text{... (6)}$$

This shows that the tensor $S_{\mu\nu}$ is anti-symmetric.

Again, we have from (4),

$$R_{\mu\nu} = \frac{\partial \Gamma^\sigma_{\mu\sigma}}{\partial x_\nu} - \frac{\partial \Gamma^\sigma_{\mu\nu}}{\partial x_\sigma} + \Gamma^\beta_{\mu\sigma}\Gamma^\sigma_{\beta\nu} - \Gamma^\beta_{\mu\nu}\Gamma^\sigma_{\beta\sigma}$$

$$= \frac{\partial \Gamma^\sigma_{\mu\sigma}}{\partial x_\nu} - \frac{\partial \Gamma^\sigma_{\sigma\nu}}{\partial x_\mu} + \Gamma^\beta_{\mu\sigma}\Gamma^\sigma_{\beta\nu} - \Gamma^\beta_{\mu\nu}\Gamma^\sigma_{\beta\sigma}$$

(intercharging σ and μ in second term)

$$= S_{\mu\nu} + \Gamma^\beta_{\mu\sigma}\Gamma^\sigma_{\beta\nu} - \Gamma^\beta_{\mu\nu}\Gamma^\sigma_{\beta\sigma}.$$

Similarly,

$$R_{\nu\mu} = -S_{\nu\mu} + \Gamma^\beta_{\nu\sigma}\Gamma^\sigma_{\beta\mu} - \Gamma^\beta_{\nu\mu}\Gamma^\sigma_{\beta\sigma}$$

$$= -S_{\nu\mu} + \Gamma^\beta_{\mu\sigma}\Gamma^\sigma_{\beta\nu} - \Gamma^\beta_{\mu\nu}\Gamma^\sigma_{\beta\sigma}$$

(interchanging μ and ν in second term)

∴ $$R_{\mu\nu} - R_{\nu\mu} = S_{\mu\nu} + S_{\nu\mu}$$

$$= S_{\mu\nu} - S_{\mu\nu} \text{ from (6)}$$

$$= 0$$

i.e. $$R_{\mu\nu} = R_{\nu\mu} \quad \text{... (7)}$$

This show that $R_{\mu\nu}$ is symmetric

3.19. RIEMANN-CHRISTOFFEL TENSOR OR COVARIANT CURVATURE TENSOR

The tensor $B_{\mu\nu\sigma\rho} = g_{\rho\varepsilon} B^\varepsilon{}_{\mu\nu\sigma}$ is known as *Riemann Christoffel* or *covariant curvature tensor.* *(Agra, 1963, 64, 65)*

We have

$$B_{\mu\nu\sigma\rho} = g_{\rho\varepsilon} B^\varepsilon{}_{\mu\nu\sigma}$$

$$= g_{\rho\varepsilon}\left(\Gamma^\alpha_{\mu\sigma}\Gamma^\varepsilon_{\alpha\nu} - \Gamma^\alpha_{\mu\nu}\Gamma^\varepsilon_{\alpha\sigma} + \frac{\partial \Gamma^\varepsilon_{\mu\sigma}}{\partial x_\nu} - \frac{\partial \Gamma^\varepsilon_{\mu\nu}}{\partial x_\sigma}\right) \text{ from (3) of § 3.18}$$

$$= \Gamma^\alpha_{\mu\sigma}\left(g_{\rho\varepsilon}\Gamma^\varepsilon_{\alpha\nu}\right) - \Gamma^\alpha_{\mu\nu}\left(g_{\rho\varepsilon}\Gamma^\varepsilon_{\alpha\sigma}\right) + \frac{\partial}{\partial x_\nu}\left(g_{\rho\varepsilon}\Gamma^\varepsilon_{\mu\sigma}\right)$$

$$-\Gamma^\varepsilon_{\mu\sigma}\frac{\partial g_{\rho\varepsilon}}{\partial x_\nu} - \frac{\partial}{\partial x_\sigma}\left(g_{\rho\varepsilon}\Gamma^\varepsilon_{\mu\nu}\right) + \Gamma^\varepsilon_{\mu\nu}\frac{\partial g_{\rho\varepsilon}}{\partial x_\sigma}$$

$$= \Gamma^{\alpha}_{\mu\sigma}\,\Gamma_{\alpha\nu,\,\rho} - \Gamma^{\alpha}_{\mu\nu}\,\Gamma_{\alpha\sigma,\,\rho} + \frac{\partial \Gamma_{\mu\sigma,\,\rho}}{\partial x_\nu} - \Gamma^{\varepsilon}_{\mu\nu}\,\frac{\partial g_{\rho\varepsilon}}{\partial x_\nu} - \frac{\partial \Gamma_{\mu\alpha,\,\rho}}{\partial x_\sigma} + \Gamma^{\varepsilon}_{\mu\nu}\,\frac{\partial g_{\rho\varepsilon}}{\partial x_\sigma}$$

$$= \Gamma^{\alpha}_{\mu\sigma}\,\Gamma_{\alpha\nu,\,\rho} - \Gamma^{\alpha}_{\mu\nu}\,\Gamma_{\alpha\sigma,\,\rho} + \frac{\partial \Gamma_{\mu\sigma,\,\rho}}{\partial x_\nu} - \frac{\partial \Gamma_{\mu\nu,\,\rho}}{\partial x_\sigma} - \Gamma^{\alpha}_{\mu\sigma}\,\frac{\partial g_{\rho\alpha}}{\partial x_\nu} + \Gamma^{\alpha}_{\mu\nu}\,\frac{\partial g_{\rho\alpha}}{\partial x_\sigma}$$

(replacing ε by α in the last two terms)

$$= \Gamma^{\alpha}_{\mu\sigma}\left(\Gamma_{\alpha\nu,\,\rho} - \frac{\partial g_{\rho\alpha}}{\partial x_\nu}\right) - \Gamma^{\alpha}_{\mu\nu}\left(\Gamma_{\alpha\sigma,\,\rho}\,\frac{\partial g_{\rho\alpha}}{\partial x_\sigma}\right)$$

$$+ \frac{\partial}{\partial x_\nu}\,\frac{1}{2}\left(\frac{\partial g_{\mu\rho}}{\partial x_\sigma} + \frac{\partial g_{\nu\rho}}{\partial x_\mu} - \frac{\partial g_{\mu\sigma}}{\partial x_\rho}\right) - \frac{\partial}{\partial x_\nu}\cdot\frac{1}{2}\left(\frac{\partial g_{\mu\rho}}{\partial x_\nu} + \frac{\partial g_{\nu\rho}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\rho}\right)$$

$$= -\Gamma^{\alpha}_{\mu\sigma}\,\Gamma_{\rho\nu,\,\alpha} + \Gamma^{\alpha}_{\mu\nu}\,\Gamma_{\rho\sigma,\,\alpha} + \frac{1}{2}\left(\frac{\partial^2 g_{\rho\sigma}}{\partial x_\mu\,\partial x_\beta} - \frac{\partial^2 g_{\mu\sigma}}{\partial x_\nu\,\partial x_\rho} - \frac{\partial^2 g_{\nu\rho}}{\partial x_\mu\,\partial x_\sigma} + \frac{\partial^2 g_{\mu\nu}}{\partial x_\sigma\,\partial x_\rho}\right)$$

(since $\Gamma_{\alpha\nu,\,\rho} = -\Gamma_{\rho\nu,\,\alpha}$ etc.)

$$= -\Gamma^{\alpha}_{\mu\sigma}\,\Gamma_{\rho\nu,\,\alpha} + \Gamma^{\alpha}_{\mu\nu}\,\Gamma_{\rho\sigma,\,\alpha} + \frac{1}{2}\left(\frac{\partial^2 g_{\rho\sigma}}{\partial x_\mu\,\partial x_\nu} + \frac{\partial^2 g_{\mu\nu}}{\partial x_\rho\,\partial x_\sigma} - \frac{\partial^2 g_{\rho\sigma}}{\partial x_\rho\,\partial x_\nu} - \frac{\partial^2 g_{\mu\nu}}{\partial x_\mu\,\partial x_\sigma}\right) \quad \ldots(1)$$

Interchanging μ and ρ, this gives

$$B_{\rho\nu\sigma\mu} = -\Gamma^{\alpha}_{\rho\sigma}\,\Gamma_{\mu\nu,\,\alpha} + \Gamma^{\alpha}_{\rho\nu}\,\Gamma_{\mu\sigma,\,\alpha} + \frac{1}{2}\left(\frac{\partial^2 g_{\mu\sigma}}{\partial x_\rho\,\partial x_\nu} - \frac{\partial^2 g_{\rho\nu}}{\partial x_\mu\,\partial x_\sigma} - \frac{\partial^2 g_{\rho\sigma}}{\partial x_\mu\,\partial x_\nu} - \frac{\partial^2 g_{\mu\nu}}{\partial x_\rho\,\partial x_\sigma}\right)$$

so that

$$\begin{aligned} B_{\mu\nu\sigma\rho} + B_{\rho\nu\sigma\mu} &= -\Gamma^{\alpha}_{\mu\sigma}\,\Gamma_{\rho\nu,\,\alpha} + \Gamma^{\alpha}_{\mu\nu}\,\Gamma_{\rho\sigma,\,\alpha} - \Gamma^{\alpha}_{\rho\sigma}\,\Gamma_{\mu\nu,\,\alpha} + \Gamma^{\alpha}_{\rho\nu}\,\Gamma_{\mu\sigma,\,\alpha} \\ &= -g^{\alpha\lambda}\,\Gamma_{\mu\sigma,\,\lambda}\,g_{\alpha\lambda}\,\Gamma^{\lambda}_{\nu\rho} + g^{\alpha\lambda}\,\Gamma_{\mu\nu,\,\lambda}\,g_{\alpha\lambda}\,\Gamma^{\lambda}_{\sigma\rho} - \Gamma^{\lambda}_{\rho\sigma}\,\Gamma_{\mu\nu,\,\lambda} + \Gamma^{\lambda}_{\rho\nu}\,\Gamma_{\mu\sigma,\,\lambda} \\ &= -\Gamma_{\mu\sigma,\,\lambda}\,\Gamma^{\lambda}_{\nu\rho} + \Gamma_{\mu\nu,\,\lambda}\,\Gamma^{\lambda}_{\sigma\rho} - \Gamma^{\lambda}_{\rho\sigma}\,\Gamma_{\mu\nu,\,\lambda} + \Gamma^{\lambda}_{\rho\nu}\,\Gamma_{\mu\sigma,\,\lambda} \\ &= 0, \end{aligned}$$

i.e. $B_{\mu\nu\sigma\rho} = -B_{\rho\nu\sigma\mu}$. ... (2)

This shows that $B_{\mu\nu\sigma\rho}$ is anti-symmetric in indices μ and ρ.

Similarly, $B_{\mu\nu\sigma\rho} = -B_{\mu\sigma\nu\rho}$ ... (3)

i.e. $B_{\mu\nu\sigma\rho}$ is anti-symmetric in indices ν and σ.

Also, it is easy to show that

$$B_{\mu\nu\sigma\rho} = B_{\rho\sigma\nu\mu}, \quad \ldots(4)$$

i.e. $B_{\mu\nu\sigma\rho}$ is symmetric in index pair (μ, ν) and (ρ, σ).

because

$$\begin{aligned} B_{\mu\nu\sigma\rho} &= -B_{\rho\nu\sigma\mu} \text{ by (2)} \\ &= -(-B_{\rho\sigma\nu\mu}) \text{ by (3)} \\ &= B_{\rho\sigma\nu\mu}. \end{aligned}$$

Thus $B_{\mu\nu\sigma\rho}$ is skew-symmetric in the first indices, skew-symmetric in the last two indices and symmetric in two pairs of indices. *(Agra, 1969)*

Further, it has the cyclic property *i.e.*

$$B_{\mu\nu\sigma\rho} + B_{\mu\sigma\rho\nu} + B_{\mu\rho\nu\sigma} = 0. \quad \ldots(5)$$

(Agra, 1968)

Writing the values of all three tensors by (1), and adding them altogether, the result follows.

We have already mentioned that a general tensor of rank four has $(4)^4$ *i.e.* 256 components. In the existing case the double anti-symmetry reduces the number to ${}^4C_2 \times {}^4C_2$ *i.e.* 36 of which 30 are paired because μ, ρ can be interchanged with ν, σ but the remaining 6 having μ, ρ as the same pair of numbers as ν, σ are without partners. As such we have 21 different components (30 paired components are quivalent to 15 different components + 6) with one furhter relation given by (5). Conclusively the Reimann-Christoffel tensor has 20 *independent components*. The scheme may be shown as:

Writing the suffixes in the order $\mu\rho\sigma\nu$, the different 21 components are

$$B_{1212}\, B_{1223}\, B_{1313}\, B_{1324}\, B_{1423}\, B_{2323}\, B_{2424}.$$
$$B_{1213}\, B_{1224}\, B_{1314}\, B_{1334}\, B_{1424}\, B_{2324}\, B_{2434},$$
$$B_{1214}\, B_{1234}\, B_{1323}\, B_{1414}\, B_{1434}\, B_{2334}\, B_{3434}.$$

With the relation $B_{1234} + B_{1342} + B_{1423} = 0$.

which reduces the number by 1.

Independent number = 21 − 1 = 20. *(Agra, 1963, 65, 76)*

Problem 32. *Prove that if $B^{\varepsilon}{}_{\mu\nu\sigma} = 0$, then space is Euclidean.*

We have

$$B^{\varepsilon}_{\mu\nu\sigma} = \frac{\partial \Gamma^{\varepsilon}_{\mu\nu}}{\partial x_{\sigma}} - \frac{\partial \Gamma^{\varepsilon}_{\mu\sigma}}{\partial x_{\nu}} + \Gamma^{\alpha}_{\mu\nu}\, \Gamma^{\varepsilon}_{\alpha\sigma} - \Gamma^{\alpha}_{\mu\sigma}\, \Gamma^{\varepsilon}_{\alpha\nu}.$$

But in Euclidean space with cartesian coordinates, we have

$$\Gamma^{\varepsilon}_{\nu\sigma}(x) = 0.$$

So, $\quad B^{\varepsilon}_{\mu\nu\sigma} = 0$ in this coordinate system.

But if $B^{\varepsilon}{}_{\mu\nu\sigma} = 0$ in one coordinate system, the components are zero in all coordinate systems.

Hence if $B^{\varepsilon}{}_{\mu\nu\sigma} = 0$, the space is Euclidean.

Problem 33. *If $R_{\mu\nu} = Kg_{\mu\nu}$, then show that $R=K$ where R is called the scalar curvature.*

Given that $R_{\mu\nu} = Kg_{\mu\nu}$.

We have
$$R = g^{\mu\nu} R_{\mu\nu}$$
$$= g^{\mu\nu} K g_{\mu\nu}$$

Hence
$$R = K \text{ as } g^{\mu\nu} g_{\mu\nu} = 1.$$

Problem 34. *Prove the* **Bianchi's Identity**

$$B^{\varepsilon}_{\mu\nu\sigma,\,\rho} + B^{\varepsilon}_{\mu\sigma\rho,\,\nu} + B^{\varepsilon}_{\mu\rho\nu,\,\sigma} = 0.$$ *(Agra, 1964, 70)*

We know that

$$B^{\varepsilon}_{\mu\nu\sigma} = \frac{\partial \Gamma^{\varepsilon}_{\mu\nu}}{\partial x_{\sigma}} - \frac{\partial \Gamma^{\varepsilon}_{\mu\sigma}}{\partial x_{\nu}} + \Gamma^{\alpha}_{\mu\nu}\, \Gamma^{\varepsilon}_{\alpha\sigma} - \Gamma^{\alpha}_{\mu\sigma}\, \Gamma^{\varepsilon}_{\alpha\nu}$$

If we differentiate this equation and evaluate at the origin of a geodesic co-ordinate system, then we have

$$B^{\varepsilon}_{\mu\nu\sigma,\,\rho} = \frac{\partial^2 \Gamma^{\varepsilon}_{\mu\nu}}{\partial x_{\rho}\, \partial x_{\sigma}} - \frac{\partial^2 \Gamma^{\varepsilon}_{\mu\sigma}}{\partial x_{\rho}\, \partial x_{\nu}}.$$

(since the Christoffel's symbols referred to the geodesic co-ordinate system vanish at O the origin, by §3.3.)

Similarly,

$$B^{\varepsilon}_{\mu\sigma\rho,\,\nu}=\frac{\partial^2\Gamma^{\varepsilon}_{\mu\sigma}}{\partial x_\nu\,\partial x_\rho}-\frac{\partial^2\Gamma^{\varepsilon}_{\mu\rho}}{\partial x_\nu\,\partial x_\sigma}$$

and
$$B^{\varepsilon}_{\mu\rho\nu,\,\sigma}=\frac{\partial^2\Gamma^{\varepsilon}_{\mu\rho}}{\partial x_\sigma\,\partial x_\nu}-\frac{\partial^2\Gamma^{\varepsilon}_{\mu\nu}}{\partial x_\sigma\,\partial x_\rho}.$$

Adding all these three equations, we get

$$B^{\varepsilon}_{\mu\nu\sigma,\,\rho}+B^{\varepsilon}_{\mu\sigma\rho,\,\nu}+B^{\varepsilon}_{\mu\rho\nu,\,\sigma}=0.$$

Problem 35. *Show that the construction of a uniform vector field is only possible when the curvature tensor vanishes.*

Or

Prove that when the Riemann-Christoffel tensor vanishes, the differential-equations $A_{\mu\nu}=\dfrac{\partial A_\mu}{\partial x_\nu}-\{\mu\nu,\,\alpha\}\,A_\alpha=0$ *are integrable.* (*Agra, 1966*)

Consider the differential equation

$$A_{\mu,\,\nu}=\frac{\partial A_\mu}{\partial x_\nu}-\Gamma^{\alpha}_{\mu\nu}\,A_\alpha=0. \qquad \dots (1)$$

This gives
$$\frac{\partial A_\mu}{\partial x_\nu}=\Gamma^{\alpha}_{\mu\nu}\,A_\alpha. \qquad \dots (2)$$

Consider the integral of L.H.S. of (2).

$$\int\frac{\partial A_\mu}{\partial x_\nu}\,dx_\nu=\int\left[\frac{\partial A_\mu}{\partial x_1}\,dx_1+\dots\right]$$
$$=\int dA_\mu=A_\mu.$$

This follows that the left hand side of (2) multiplied by x_ν is a perfect differential and as such the right hand side of (2) multiplied by dx_ν must be a perfect differential. *i.e.*,

$\Gamma^{\alpha}_{\mu\nu}\,A_\alpha\,dx_\nu$ is a perfect differential,

i.e. $\Gamma^{1}_{\mu\nu}\,A_1\,dx_\nu+\Gamma^{2}_{\mu\nu}\,A_2\,dx_\nu+\dots+\Gamma^{\alpha}_{\mu 1}\,A_\alpha\,dx_1+\Gamma^{\alpha}_{\mu 2}\,A_\alpha\,dx_2+\dots$

is a perfect differential

i.e. $\Gamma^{\alpha}_{\mu\nu}\,A_\alpha\,dx_\nu+\Gamma^{\alpha}_{\mu\sigma}\,A_\alpha\,dx_\sigma$, is a perfect differential

which is of the form $M\,dx+N\,dy=0$ and this is perfect differential if

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}\ \ i.e.\ \ \frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=0.$$

As such $\Gamma^{\alpha}_{\mu\nu}\,A_\alpha\,dx_\nu+\Gamma^{\alpha}_{\mu\sigma}\,A_\alpha\,dx_\sigma$ $\left(i.e.\ \Gamma^{\alpha}_{\mu\nu}\,A_\alpha\,dx_\nu\right)$ is perfect differential if

$$\frac{\partial}{\partial x_\sigma}\left(\Gamma^{\alpha}_{\mu\nu}\,A_\alpha\right)-\frac{\partial}{\partial x_\nu}\left(\Gamma^{\alpha}_{\mu\sigma}\,A_\alpha\right)=0.$$

or if $$A_\alpha\left(\frac{\partial\Gamma^\alpha_{\mu\nu}}{\partial x_\sigma}-\frac{\partial\Gamma^\alpha_{\mu\sigma}}{\partial x_\nu}+\Gamma^\alpha_{\mu\nu}\frac{\partial A_\alpha}{\partial x_\nu}-\Gamma^\alpha_{\mu\nu}\frac{\partial A_\alpha}{\partial x_\nu}=0.\right.$$

With the help of (2), this yields

$$A_\alpha\left(\frac{\partial\Gamma^\alpha_{\mu\nu}}{\partial x_\sigma}-\frac{\partial\Gamma^\alpha_{\mu\sigma}}{\partial x_\mu}\right)+\left(\Gamma^\alpha_{\mu\nu}\Gamma^\varepsilon_{\alpha\sigma}-\Gamma^\alpha_{\mu\sigma}\Gamma^\varepsilon_{\alpha\nu}\right)A_\varepsilon=0.$$

Changing the dummy suffix α to ε in the first term, we get

$$A_\varepsilon\left[\frac{\partial\Gamma^\varepsilon_{\mu\nu}}{\partial x_\sigma}-\frac{\partial\Gamma^\varepsilon_{\mu\rho}}{\partial x_\nu}+\Gamma^\alpha_{\mu\nu}\Gamma^\varepsilon_{\alpha\sigma}-\Gamma^\alpha_{\mu\sigma}\Gamma^\varepsilon_{\mu\nu}\right]=0$$

i.e. $$A_\varepsilon B^\varepsilon{}_{\mu\sigma\nu}=0.$$

Since A_ε is arbitrary, therefore $B^\varepsilon{}_{\mu\sigma\nu}=0$.

Hence when $B^\varepsilon{}_{\mu\sigma\nu}$ vanishes the differential dA_μ determined by (1) will be a perfect differential and as such $\int dA_\mu$ between any two points will be independent of the path of integration. Then the vector A_μ can be parallely displaced to any point thereby giving a unique result independent of the path of transfer. Displacement of such a vector gives a uniform vector field. Conclusively the construction of a uniform vector field is only possible where the curvature tensor vanishes.

3.20. SOME IMPORTANT RESULTS

We have $\quad g_{\mu\nu}\,g^{\mu\alpha}=g_\nu{}^\alpha=0\quad\text{or}\quad 1.$

Its differentiation gives

$$g^{\mu\alpha}\,dg_{\mu\nu}+g_{\mu\nu}\,dg^{\mu\alpha}=0$$

or $$g^{\mu\alpha}\,dg_{\mu\nu}=-\,g_{\mu\nu}\,dg^{\mu\alpha}.$$

Multiplying throughout by $g^{\nu\beta}$, we get

$$\begin{aligned}g^{\mu\alpha}\,g^{\nu\beta}\,dg_{\mu\nu}&=-\,g_{\mu\nu}\,g^{\nu\beta}\,dg^{\mu\alpha}\\&=-g_\mu{}^\beta\,dg^{\mu\alpha}\\&=dg^{\alpha\beta}.\end{aligned}\qquad\ldots(1)$$

Similarly, $\quad dg_{\alpha\beta}=-g_{\mu\alpha}\,g_{\nu\beta}\,dg^{\mu\nu}.\qquad\ldots(2)$

(*Agra, 1963*)

Multiplying (2) throughout by $A^{\alpha\beta}$, we get

$$\begin{aligned}A^{\alpha\beta}\,dg_{\alpha\beta}&=-\,(g_{\mu\alpha}\,g_{\nu\beta}\,A^{\alpha\beta})\,dg^{\mu\nu}\\&=-\,(g_{\mu\alpha}\,A^\alpha{}_\nu)\,dg^{\mu\nu}\\&=-\,A_{\mu\nu}\,dg^{\mu\nu}\\&=-\,A_{\alpha\beta}\,dg^{\alpha\beta}\end{aligned}$$ (replacing μ, ν by α, β respectively),

i.e. $$A^{\alpha\beta}\,dg_{\alpha\beta}=-\,A_{\alpha\beta}\,dg^{\alpha\beta}.\qquad\ldots(3)$$

(*Agra, 1967, 73, 75*)

But for any other tensor $B_{\alpha\beta}$, we have

$$A^{\alpha\beta}\,dB_{\alpha\beta}=A_{\alpha\beta}\,dB^{\alpha\beta}.$$

Now consider dg formed by taking the differential of each of $g_{\mu\nu}$ and multiplying by its cofactor $g^{\mu\nu}$ in the determinant g.

$$\frac{dg}{g} = g^{\mu\nu}\ dg_{\mu\nu}$$

$$= -g_{\mu\nu}\, dg^{\mu\nu} \text{ by (3)} \qquad \ldots (4)$$

But we have

$$\Gamma^{\sigma}_{\mu\sigma} = \frac{1}{2}\ g^{\sigma\lambda} \left\{ \frac{\partial g_{\mu\lambda}}{\partial x_\sigma} + \frac{\partial g_{\sigma\lambda}}{\partial x_\mu} - \frac{\partial g_{\mu\sigma}}{\partial x_\lambda} \right\}$$

$$= \frac{1}{2} g^{\sigma\lambda} \frac{\partial g_{\sigma\lambda}}{\partial x_\mu}, \text{ other two terms cancel, by interchange of } \sigma \text{ and } \lambda$$

$$= \frac{1}{2g}\ \frac{dg}{dx_\mu} \text{ by (4)} \qquad \ldots (5)$$

$$= \frac{\partial}{\partial x_\mu} \log \sqrt{(-g)}. \qquad \ldots (6)$$

(*Agra, 1959, 64, 68*)

Since g is always negative for real coordinates, we therefore use $\sqrt{(-g)}$.

Again
$$A^{\nu}_{.\nu} = \frac{\partial A^\nu}{\partial x_\nu} + \Gamma^{\nu}_{\mu\nu}\ A^\mu$$

$$= \frac{\partial A^\nu}{\partial x_\nu} + \frac{1}{2g}\ \frac{\partial g}{\partial x_\mu} A^\mu \quad \text{by (5)}$$

$$= \frac{\partial A^\nu}{\partial x_\nu} + \frac{1}{2g}\ \frac{\partial g}{\partial x_\nu} A^\nu \quad \text{on replacing } \mu \text{ by } \nu$$

or
$$A^{\nu}_{.\nu} = \frac{1}{\sqrt{g}}\ \frac{\partial}{\partial x_\nu} \left(\sqrt{g} A^\nu\right) \qquad \ldots (7)$$

(taking the absolute value of g).

Further, let $h_{\nu\tau} = a_{\mu\nu}\ a_{\sigma\tau}\ x_\mu\ x_\sigma$,

where $a_{\mu\nu}$ denotes constant coefficients.

Then
$$\frac{\partial h_{\nu\tau}}{\partial x_\sigma} = a_{\mu\nu}\ a_{\sigma\tau} \left(\frac{\partial x_\mu}{\partial x_\alpha}\ x_\sigma + \frac{\partial x_\sigma}{\partial x_\alpha}\ x_\mu \right)$$

$$= a_{\mu\nu}\ a_{\sigma\tau} \left(g^{\mu}_{\alpha}\ x_\sigma + g^{\sigma}_{\alpha}\ x_\mu \right) \text{ by §3.3.}$$

Repeating the process,

$$\frac{\partial^2 h_{\nu\tau}}{\partial x_\alpha\, \partial x_\beta} = a_{\mu\nu}\ a_{\sigma\tau} \left(g^{\mu}_{\alpha}\ g^{\sigma}_{\beta} + g^{\sigma}_{\alpha}\ g^{\mu}_{\beta} \right)$$

$$= a_{\alpha\nu}\ a_{\beta\tau} + a_{\beta\nu}\ a_{\alpha\tau}.$$

Here changing dummy suffixes, we have

$$\frac{\partial^2}{\partial x_\mu\, \partial x_\nu} \left(a_{\mu\nu}\ a_{\sigma\mu}\ x_\mu\ x_\sigma \right) = a_{\mu\nu}\ a_{\sigma\tau} + a_{\sigma\nu}\ a_{\mu\tau}. \qquad \ldots (8)$$

Similarly if $a_{\mu\nu\sigma}$ is symmetrical in its suffixes, we have

$$\frac{\partial^3}{\partial x_\mu\, \partial x_\nu\, \partial x_\sigma}\left(a_{\mu\nu\sigma}\ x_\mu\ x_\nu\ x_\sigma\right) = 6\,a_{\mu\nu\sigma}. \quad \text{... (9)}$$

3.21. TENSOR FORMS OF OPERATORS

[*A*] **Gradient.** If ϕ be an invariant quantity then

$$\nabla\phi = \text{grad } \phi = \phi_{,\nu} = \frac{\partial\phi}{\partial x_\nu}$$

where $\phi_{,\nu}$ is the covariant derivative of ϕ w.r.t. x_ν.

[*B*] **Divergence.** Divergence of A^μ is defined as the contraction of its covariant derivative w.r.t. x_μ *i.e.* the contraction of $A^\mu{}_{,\mu}$, so that by (7) of §3.20 we have

$$\text{div } A^\mu_{,\mu} = \frac{1}{\sqrt{g}}\ \frac{\partial}{\partial x_\mu}\left(\sqrt{g}A^\mu\right)$$

[*C*] **Curl.** The curl of A_μ is defined as

$$A_{\mu,\nu} - A_{\nu,\mu} = \frac{\partial A_\mu}{\partial x_\nu} - \frac{\partial A_\nu}{\partial x_\mu}$$

Also curl $A_{\mu,\nu} = -\epsilon^{ijk}\ A_{\mu,\nu}$

[*D*] **Laplacian.** The Laplacian of ϕ is divergence of gradient ϕ *i.e.* $\nabla^2\phi = \text{div } \phi_{,\nu}$.

But $\nabla\phi = \dfrac{\partial\phi}{\partial x_\nu}$ a covariant tensor of rank one defined as covariant derivative of ϕ, written as $\phi_{,\mu}$. The contravariant tensor of rank one associated with ϕ_μ is

$$A^\alpha = g^{\alpha\mu}\ \frac{\partial\phi}{\partial x^\mu}$$

Hence

$$\nabla^2\phi = \text{div}\left(g^{\alpha\mu}\ \frac{\partial\phi}{\partial x^\mu}\right)$$

$$= \frac{1}{\sqrt{g}}\ \frac{\partial}{\partial x_\alpha}\left(\sqrt{g}\ g^{\alpha\mu}\ \frac{\partial\phi}{\partial x^\mu}\right)$$

when g^{20}, $\sqrt{g}$ must be replaced by $\sqrt{-g}$ while the cases $g > 0$ and $g < 0$ are included in $\sqrt{|g|}$ instead of $\sqrt{g}$.

ADDITIONAL MISCELLANEOUS PROBLEMS

Problem 36. *Write the law of transformation for the tensors*

(*i*) $A^\sigma_{\mu\nu}$, (*ii*) $A^{\mu\nu}_\sigma$, (*iii*) $B^{\alpha\beta}_{\mu\nu\sigma}$, (*iv*) $B^{\mu\nu\sigma}_{\alpha\beta}$, (*v*) $C_{\mu\nu}$, (*vi*) $C^{\mu\nu}$

(*vii*) D_μ, (*viii*) D^μ.

We have,

(*i*)
$$A'^{\zeta}_{\xi\eta} = \frac{\partial x'_\zeta}{\partial x_\sigma}\ \frac{\partial x_\mu}{\partial x'_\xi}\ \frac{\partial x_\nu}{\partial x'_\eta}\ A^\sigma_{\mu\nu}$$

(*ii*)
$$A'^{\xi\eta}_\zeta = \frac{\partial x'_\xi}{\partial x_\mu}\ \frac{\partial x'_\eta}{\partial x_\nu}\ \frac{\partial x_\sigma}{\partial x'_\delta}\ A^{\mu\nu}_\sigma$$

$$(iii)\quad B'^{\gamma\delta}_{\xi\eta\zeta}=\frac{\partial x'_\gamma}{\partial x_\alpha}\frac{\partial x'_\delta}{\partial x_\beta}\frac{\partial x_\mu}{\partial x'_\xi}\frac{\partial x_\nu}{\partial x'_\eta}\frac{\partial x_\sigma}{\partial x'_\zeta}B^{\alpha\beta}_{\mu\nu\sigma}\cdot$$

$$(iv)\quad B'^{\xi\eta\zeta}_{\gamma\delta}=\frac{\partial x'_\xi}{\partial x_\mu}\frac{\partial x'_\eta}{\partial x_\nu}\frac{\partial x'_\zeta}{\partial x_\sigma}\frac{\partial x_\alpha}{\partial x'_\gamma}\frac{\partial x_\beta}{\partial x'_\delta}B^{\mu\nu\sigma}_{\alpha\beta}$$

$$(v)\quad C'_{\alpha\beta}=\frac{\partial x_\mu}{\partial x'_\alpha}\frac{\partial x_\nu}{\partial x'_\beta}C_{\mu\nu}$$

$$(vi)\quad C'^{\alpha\beta}=\frac{\partial x'_\alpha}{\partial x_\mu}\frac{\partial x'_\beta}{\partial x_\nu}C^{\mu\nu}$$

$$(vii)\quad D'_\alpha=\frac{\partial x_\mu}{\partial x'_\alpha}D_\mu \quad\text{and}\quad (viii)\ D'^{\alpha}=\frac{\partial x'_\alpha}{\partial x_\mu}D^{\mu}$$

Problem 37. *If the tensors, a_{ik}, a_{ikj}, a_{ijkl} are antisymmetric in every pair of indices, find their independent components in four dimensions.* *(Agra, 1966)*

If a_{ik} is a skew-symmetric set, then $a_{ik}=-a_{ki}$ for every pair of values i, k. In particular

$a_{11}=-a_{11}$, $a_{22}=-a_{22}$, $a_{33}=-a_{33}$, $a_{44}=-a_{44}$ giving $a_{11}=a_{22}=a_{33}=a_{44}=0$. *i.e.*, the components arising for equal values of the suffixes are all separately zero. Their number is four.

In all there are 4^2 *i.e.*, 16 components of which 4 are zero. The remaining (16–4) *i.e.*, 12 components divide themselves into pairs such that the two members of any pair are equal and opposite as $a_{12}=-a_{21}$ etc. Thus the total number of independent components of an anti-symmetric tensor a_{ik} of order 2 and dimension 4 is 6.

Similar arguments will give the independent component of a_{ikj} and a_{ikjl}.

Problem 38. *Show that the number of distinct non-vanishing components of the covariant Reimann Christoffel curvature tensor does not exceed* $\frac{1}{12}n^2\left(n^2-1\right)$.

Since $B_{\mu\nu\sigma\rho}$ is skew-symmetric in μ,ρ and ν,σ therefore the number of non-vanishing components contributed by μ,ρ is ${}^nC_2$ *i.e.*, $\frac{n(n-1)}{2}$ also those contributed by ν,σ is $\frac{1}{2}n(n-1)$. Thus the number of non-vanishing components of Riemann tensor does not exceed $\frac{n(n-1)}{2}\times\frac{n(n-1)}{2}$ *i.e.*, $\frac{n^2}{4}(n-1)^2$.

But the number of relations in Bianchi's identity *i.e.*,

$B_{\mu\nu\sigma\rho}+B_{\mu\sigma\rho\nu}+B_{\mu\rho\nu\sigma}=0$ is $n.{}^nC_3$ *i.e.*, $\frac{n^2(n-1)(n-2)}{6}$

∴ The number of distinct non-vanishing components does not exceed

$$\frac{n^2}{4}(n-1)^2-\frac{n^2(n-1)(n-2)}{6}=\frac{n^2}{12}(n-1)\left[3(n-1)-2(n-2)\right]$$

$$=\frac{n^2}{12}(n-1)(n+1)=\frac{n^2}{12}\left(n^2-1\right)$$

Problem 39. *Establish the result* $\left(A_\mu\right)_{\nu\sigma}-\left(A_\mu\right)_{\sigma\nu}=A^{\epsilon}B^{\mu}_{\epsilon\nu\sigma}$ *and prove that* $B^{\mu}_{\epsilon\nu\sigma}$ *has only twenty algebraically independent components.* *(Agra, 1958, 65, 67)*

For first part see (3) of § 3.18.

For second part see §3.19.

Problem 40. *Explain what is meant by covariant, contravariant and mixed tensors. If $B_{\nu\sigma}$ is an arbitrary covariant tensor and $A(\mu, \nu) B_{\nu\sigma} = C_{\mu\sigma}$ where $C_{\mu\sigma}$ is a tensor, then prove that $A(\mu, \nu)$ is a mixed tensor.*

Define the Christoffel three-index symbol of the second kind. Prove that it is not a tensor. Verify that $g_{\mu\nu}$ behaves as a constant in covariantt differentiation.

(Rohilkhand, 1989; Agra, 1958, 63, 65)

For definition see §3.4.

Suppose that $A(\mu, \nu) B_{\nu\sigma} = C_{\mu\sigma}$ in the coordinate system x_α.

Then in the transformed coordinate x'_α, we have

$$A'(\beta, \gamma) B'_{\gamma\xi} = C'_{\beta\xi}$$

Consider,
$$A'(\beta, \gamma) \frac{\partial x_\nu}{\partial x'_\gamma} \frac{\partial x_\sigma}{\partial x'_\xi} B_{\nu\sigma} = \frac{\partial x_\sigma}{\partial x'_\xi} \frac{\partial x_\mu}{\partial x'_\beta} C_{\mu\sigma}$$
$$= \frac{\partial x_\sigma}{\partial x'_\xi} \frac{\partial x_\mu}{\partial x'_\beta} A(\mu, \nu) B_{\nu\sigma}$$

or
$$\frac{\partial x_\sigma}{\partial x'_\xi} \left[\frac{\partial x_\nu}{\partial x'_\gamma} \frac{\partial x_\mu}{\partial x'_\xi} A'(\beta, \gamma) - \frac{\partial x_\sigma}{\partial x'_\beta} A(\mu, \nu) \right] B_{\nu\sigma} = 0 \qquad \dots (1)$$

Multiplying by $\frac{\partial x'_\xi}{\partial x_\eta}$ we have, $\frac{\partial x_\sigma}{\partial x'_\xi} \frac{\partial x'_\xi}{\partial x_\eta} = \frac{\partial x_\sigma}{\partial x_\eta} = \delta^\sigma_\eta$ so that $\delta^\sigma_\eta B_{\nu\sigma} = B_{\nu\eta}$

∴ (1) becomes

$$\left[\frac{\partial x_\nu}{\partial x'_\gamma} . A'(\beta, \gamma) - \frac{\partial x_\mu}{\partial x'_\beta} A(\mu, \nu) \right] B_{\nu\eta} = 0$$

or
$$\frac{\partial x_\nu}{\partial x'_\gamma} A'(\beta, \gamma) - \frac{\partial x_\mu}{\partial x'_\beta} A(\mu, \nu) = 0, \quad B_{\nu\eta} \text{ being arbitrary.}$$

Multiplying by $\frac{\partial x'_\eta}{\partial x_\nu}$ we get $\frac{\partial x_\nu}{\partial x'_\gamma} \frac{\partial x'_\eta}{\partial x_\nu} A'(\beta, \gamma) = \frac{\partial x_\mu}{\partial x'_\beta} \frac{\partial x'_\eta}{\partial x_\nu} A(\mu, \nu)$

i.e.,
$$\delta^\eta_\gamma A'(\beta, \gamma) = \frac{\partial x_\mu}{\partial x'_\beta} \frac{\partial x'_\eta}{\partial x_\nu} A(\mu, \nu)$$

or
$$A'(\beta, \eta) = \frac{\partial x_\mu}{\partial x'_\beta} \frac{\partial x'_\eta}{\partial x_\nu} A(\mu, \nu)$$

which follows that $A(\mu, \nu)$ is a mixed tensor.

Now in §3.12, the Christoffel's 3-index symbol of the second kind has been defined as

$$\Gamma^\sigma_{\mu\nu} = \{\mu\nu, \sigma\} = \frac{1}{2} g^{\sigma\lambda} \left(\frac{\partial g_{\mu\lambda}}{\partial x_\nu} + \frac{\partial g_{\nu\lambda}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\lambda} \right)$$

and in §3.14, we have derived the transformation law for it in equation (4) as

$$\{\beta\gamma, \alpha\} = \{\mu\nu, \lambda\}' \frac{\partial x'_\mu}{\partial x_\beta} \frac{\partial x'_\nu}{\partial x_\gamma} \frac{\partial x_\alpha}{\partial x'_\lambda} + \frac{\partial^2 x'_\lambda}{\partial x_\beta \partial x_\gamma} \frac{\partial x_\alpha}{\partial x'_\lambda}$$

which clearly shows that $\{\beta\gamma, \alpha\}$ or $\Gamma^\alpha_{\beta\gamma}$ are not the components of a tensor unless the second terms on the right are zero.

Since the covariant derivative of $g_{\mu\nu}$ is zero as is shown below, the tensor $g_{\mu\nu}$ may be treated as constant in covariant differentiation.

Covariant derivative of $g_{\mu\nu}$ is given by (1) of §3.17,

$$\partial_{\mu\nu,\sigma}=\frac{\partial g_{\mu\nu}}{\partial x_\sigma}-\Gamma^\alpha_{\mu\sigma}\, g_{\alpha\nu}-\Gamma^\alpha_{\nu\sigma}\, g_{\mu\alpha}\;.$$

But $$\Gamma^\alpha_{\mu\sigma}=\frac{1}{2}\, g^{\alpha\nu}\left(\frac{\partial g_{\mu\nu}}{\partial x_\sigma}+\frac{\partial g_{\sigma\nu}}{\partial x_\mu}-\frac{\partial g_{\mu\sigma}}{\partial x_\nu}\right).$$

So that $$g_{\alpha\nu}\Gamma^\alpha_{\mu\sigma}=\frac{1}{2}\, g_{\alpha\nu}\, g^{\alpha\nu}\left(\frac{\partial g_{\mu\nu}}{\partial x_\sigma}+\frac{\partial g_{\sigma\nu}}{\partial x_\mu}-\frac{\partial g_{\mu\sigma}}{\partial x_\nu}\right)$$

$$=\frac{1}{2}\left(\frac{\partial g_{\mu\nu}}{\partial x_\sigma}+\frac{\partial g_{\sigma\nu}}{\partial x_\mu}-\frac{\partial g_{\mu\sigma}}{\partial x_\nu}\right)\text{ as } g_{\alpha\nu}\, g^{\alpha\nu}=1$$

Similarly, $$g_{\mu\alpha}\,\Gamma^\alpha_{\nu\alpha}=\frac{1}{2}\left(\frac{\partial g_{\mu\nu}}{\partial x_\sigma}+\frac{\partial g_{\mu\sigma}}{\partial x_\nu}-\frac{\partial g_{\sigma\nu}}{\partial x_\mu}\right).$$

$\therefore$ $$-\left[\Gamma^\alpha_{\mu\sigma}\, g_{\alpha\nu}+\Gamma^\alpha_{\nu\alpha}\, g_{\mu\alpha}\right]=-\frac{\partial g_{\mu\nu}}{\partial x_\sigma}$$

Hence $$\partial_{\mu\nu,\sigma}=\frac{\partial g_{\mu\nu}}{\partial x_\sigma}-\frac{\partial g_{\mu\nu}}{\partial x_\sigma}=0.$$

Problem 41. *State the law of transformation of a mixed tensor. Show how the transformation is affected when the tensor is subjected to a Contraction.*

Calculate the Christoffel 3-index symbol of the second kind $\{\mu\sigma,\sigma\}$

Verify that the covariant derivative of the tensor $g^{\nu\alpha}$ *vanishes.* (*Agra, 1959, 65*)

For first part see (5) and (6) of §3.4 and III of §3.7.

Now we have

$$\{\mu\nu,\sigma\}=\frac{1}{2}\, g^{\alpha\lambda}\left(\frac{\partial g_{\mu\lambda}}{\partial x_\nu}+\frac{\partial g_{\nu\lambda}}{\partial x_\mu}-\frac{\partial g_{\mu\nu}}{\partial x_\lambda}\right)$$

But $\nu=\sigma$, so that

$$\{\mu\sigma,\sigma\}=\frac{1}{2}\, g^{\sigma\lambda}\left(\frac{\partial g_{\mu\lambda}}{\partial x_\sigma}+\frac{\partial g_{\sigma\lambda}}{\partial x_\mu}-\frac{\partial g_{\mu\sigma}}{\partial x_\lambda}\right)$$

$$=\frac{\partial}{\partial x_\mu}\log\sqrt{-g}\text{ by (8) of §3.20.}$$

For the last part to show that $g^{\mu\alpha}_{\ \ .\delta}=0$ see Problem 30 (*a*).

Problem 42. *Define geodesics and obtain their equations with the help of a variational principle.*

(*Agra, 1963, 65, 68*)

See §3.13.

Problem 43. *Prove that*

$$dg_{\alpha\beta}=-g_{\mu\alpha}\, g_{\nu\beta}\, dg^{\mu\nu} \qquad (\textit{Agra, 1963})$$

We have $$g_{\mu\alpha}\, g^{\mu\nu}=g_\alpha^{\ \nu}=0\text{ or }1.$$

Its differentiation gives

$$g_{\mu\alpha}\, dg^{\mu\nu}+g^{\mu\nu}\, dg_{\nu\alpha}=0$$

or $$g^{\mu\nu}\, dg_{\mu\alpha} = -g_{\mu\alpha}\, dg^{\mu\nu}$$

Multiplying throughout by $g_{\nu\beta}$, we get

$$g^{\mu\nu}\, g_{\nu\beta}\, dg_{\mu\alpha} = -g_{\mu\alpha}\, g_{\nu\beta}\, dg^{\mu\nu}$$

or $$g^{\nu}{}_{\beta}\, dg_{\mu\alpha} = -g_{\mu\alpha}\, g_{\nu\beta}\, dg^{\mu\nu}$$

or $$dg_{\alpha\beta} = -g_{\mu\alpha}\, g_{\nu\beta}\, dg^{\mu\nu}$$

Problem 44. *Show with the usual notation*

(*i*) $\dfrac{\partial g_{ij}}{\partial x_k} = [j, ik] + [i, jk]$

(*ii*) $\left\{ \begin{matrix} i \\ ik \end{matrix} \right\} = \dfrac{\partial}{\partial x_k} \log \sqrt{g}$ (*Agra, 1963, 65, 69*)

(*i*) Using (i) of §3.12, we have

$$[j, ik] = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x_k} + \frac{\partial g_{kj}}{\partial x_i} - \frac{\partial g_{ik}}{\partial x_j}\right) \quad \dots (1)$$

and $$[i, jk] = \frac{1}{2}\left(\frac{\partial g_{ji}}{\partial x_k} + \frac{\partial g_{ki}}{\partial x_j} - \frac{\partial g_{kj}}{\partial x_i}\right) = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x_k} + \frac{\partial g_{ik}}{\partial x_j} - \frac{\partial g_{kj}}{\partial x_i}\right) \quad \dots (2)$$

Adding (1) and (2) we get $[j, ik] + [i, jk] = \dfrac{\partial g_{ij}}{\partial x_k}$

(*ii*) From (4) of §3.20 we have $\dfrac{dg}{g} = -g_{\mu\nu}\, dg^{\mu\nu}$ ... (1)

So that $$\left\{ \begin{matrix} i \\ ik \end{matrix} \right\} = \{i, ik\} = \frac{1}{2}\, g^{i\lambda}\left(\frac{\partial g_{k\lambda}}{\partial x_i} + \frac{\partial g_{i\lambda}}{\partial x_k} - \frac{\partial g_{ki}}{\partial x_\lambda}\right)$$

$$= \frac{1}{2}\, g^{i\lambda}\, \frac{\partial g_{i\lambda}}{\partial x_k}$$ other two terms cancel by interchange of i and λ

$$= \frac{1}{2g}\, \frac{\partial g}{\partial x_k} \text{ by (1)}$$

$$= \frac{\partial}{\partial x_k} \log \sqrt{g} \text{ if } g \text{ is positive.}$$

Problem 45. (*a*) *Define the Kronecker and alternating tensors.*

(*b*) *Prove the following identities*

(*i*) $\delta_{ii} = 3$

(*ii*) $\delta_{ik}\ \varepsilon_{ikm} = 0$

(*iii*) $\varepsilon_{iks} \cdot \varepsilon_{mps} = \delta_{im}\, \delta_{kp} - \delta_{ip}\, \delta_{km} = 0$ (*Rohilkhand, 1983; Agra, 1969*)

(*a*) **Kronecker delta** is a notation defined by

$$\delta_{ij} \text{ or } \delta_i^j = 0 \text{ if } i \neq j$$
$$= 1 \text{ if } i = j$$

It is a mixed tensor of rank 2 as it transforms like a mixed tensor of rank two.

Alternating Tensor (or permutation tensor or Epsilon tensor) is an abstract entity of order (rank) 3 and dimension 3 such that its components are invariant for any coordinate system. It is denoted by ε_{ijk} and defined as

$$\varepsilon_{ijk} = \begin{cases} 0, \text{ if any two of } i, j, k \text{ are equal} \\ 1, \text{ if } i, j, k \text{ is a cyclic (i.e., even) permutation of } 1, 2, 3 \\ -1, \text{ if } i, j, k \text{ is an anticyclic (i.e., odd) " "} \end{cases}$$

i.e., for unequal values of the suffixes, we have

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$$

$$\varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1$$

(*b*) (*i*) In three dimensions, we have

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33}$$
$$= 1 + 1 + 1 \text{ as } \delta_{ij} = 1 \text{ for } i = j$$
$$= 3$$

(*ii*) We have

$$\delta_{ik} = 0 \text{ for } i \neq k$$
$$= 1 \text{ for } i = k$$

and $\varepsilon_{ikm} = \pm 1$ for $i \neq k \neq m$

$= 0$ for $i = k \neq m$ or $i \neq k = m$ or $k \neq i = m$

Hence in either case

$$\delta_{ik}\, \varepsilon_{ikm} = 0$$

(*iii*) We have to show that

$$\varepsilon_{iks}\, \varepsilon_{mps} = \delta_{im}\, \delta_{kp} - \delta_{ip}\, \delta_{km} = 0$$

Consider, $\varepsilon_{iks}\, \varepsilon_{mps} = \delta_{im}\, \delta_{kp} - \delta_{ip}\, \delta_{km}$ ... (1)

Either side of (1) is a tensor of order 4. In 3-dimensional space this result can be written as

$$\varepsilon_{ik1}\, \varepsilon_{mp1} + \varepsilon_{ik2}\, \varepsilon_{mp2} + \varepsilon_{ik3}\, \varepsilon_{mp3} = \delta_{im}\, \delta_{kp} - \delta_{ip}\, \delta_{km}$$

when $i = k$, we have

$$\varepsilon_{ik1} = \varepsilon_{ik2} = \varepsilon_{ik3} = 0 \text{ so that L.H.S. of (1)} = 0$$

and then $\delta_{im} = \delta_{kp} = \delta_{ip} = \delta_{km} = 0$ so that R.H.S. of (1) = 0

The same result holds when instead of $i = k$, $m = p$.

Again when $i \neq k$ and when $m \neq p$ and if the pair of unequal values of i, k is different from the pair of unequal values of m, p then also

L.H.S. = 0 = R.H.S.

As such we are left to consider the possibilities when i, k and m, p have the pairs of values 1, 2; 1, 3; 2, 3; 2, 1; 3, 1; 3, 2

Taking the first case we find

$$i = 1, k = 2, m = 1, p = 2,\ i = 1, k = 2, m = 2, p = 1$$
$$i = 2, k = 1, m = 1, p = 2,\ i = 2, k = 1, m = 2, p = 1$$

which give L.H.S. = 1 = R.H.S.; L.H.S. = − 1 = R.H.S.

L.H.S. = −1 = R.H.S.; L.H.S. = 1 = R.H.S. respectively.

In other cases also we can show similarly that

L.H.S. of (1) = R.H.S. of (1)

Problem 46. *Show* $\dfrac{\partial A_\mu}{\partial x_\nu}$ *is not a tensor even though* A_μ *is a covariant tensor of rank one, but the addition of a suitable quantity to* $\dfrac{\partial A_\mu}{\partial x_\nu}$ *causes the result to be a tensor.*

(*Agra, 1969*)

We have $A'_\alpha = \dfrac{\partial x_\mu}{\partial x'_\alpha} A_\mu$ by §3.4 (2).

Differentiating both sides w.r.t. x'_β wet get

$$\frac{\partial A'_\alpha}{\partial x'_\beta} = \frac{\partial x_\mu}{\partial x'_\alpha}\frac{\partial A_\mu}{\partial x'_\beta} + \frac{\partial^2 x_\mu}{\partial x'_\beta\,\partial x'_\alpha} A_\mu$$

$$= \frac{\partial x_\mu}{\partial x'_\alpha}\frac{\partial A_\mu}{\partial x_\nu}\frac{\partial x_\nu}{\partial x'_\beta} + \frac{\partial^2 x_\mu}{\partial x'_\beta\,\partial x'_\alpha} A_\mu$$

$$= \frac{\partial x_\mu}{\partial x'_\alpha}\frac{\partial x_\nu}{\partial x'_\beta}\cdot\frac{\partial A_\mu}{\partial x_\nu} + \frac{\partial^2 x_\mu}{\partial x'_\beta\,\partial x'_\alpha} A_\mu \qquad \ldots (1)$$

Unless the second term on the right is zero, $\dfrac{\partial A_\mu}{\partial x_\nu}$ does not transform as a tensor.

Using (12) of §3.14, we have

$$\frac{\partial^2 x_\mu}{\partial x'_\beta\,\partial x'_\alpha} = \Gamma'^{\gamma}_{\beta\alpha}\frac{\partial x_\mu}{\partial x'_\gamma} - \Gamma^{\mu}_{\sigma\tau}\frac{\partial x_\sigma}{\partial x'_\beta}\frac{\partial x_\nu}{\partial x'_\alpha}$$

Substituting in (1), we find

$$\frac{\partial A'_\alpha}{\partial x'_\beta} = \frac{\partial x_\mu}{\partial x'_\alpha}\frac{\partial A_\nu}{\partial x'_\beta}\frac{\partial A_\mu}{\partial x'_\nu} + \Gamma'^{\gamma}_{\beta\alpha}\frac{\partial x_\mu}{\partial x'_\gamma} A_\mu - \Gamma^{\mu}_{\sigma\tau}\frac{\partial x_\sigma}{\partial x'_\beta}\frac{\partial x_\tau}{\partial x'_\alpha} A_\mu$$

$$= \frac{\partial x_\mu}{\partial x'_\alpha}\frac{\partial x_\nu}{\partial x'_\beta}\frac{\partial A_\mu}{\partial x_\nu} + \Gamma'^{\gamma}_{\beta\alpha} A'_\gamma - \frac{\partial x_\mu}{\partial x'_\alpha}\frac{\partial x_\nu}{\partial x'_\beta}\Gamma^{\rho}_{\mu\nu} A_\rho$$

(by interchanging the suffixes τ, σ by μ, ν and μ by ρ in the last term)

or

$$= \frac{\partial A'_\alpha}{\partial x'_\beta} - \Gamma'^{\gamma}_{\alpha\beta} A_\gamma = \frac{\partial x_\mu}{\partial x'_\alpha}\frac{\partial x_\nu}{\partial x'_\beta}\left(\frac{\partial A_\mu}{\partial x_\nu} - \Gamma^{\rho}_{\mu\nu} A_\rho\right) \qquad \ldots (2)$$

Where $\dfrac{\partial A_\mu}{\partial x_\nu} - \Gamma^{\rho}_{\mu\nu} A_\rho$ being a covariant tensor of second rank is known to be the covariant derivative of A_μ w.r.t. x_ν and is written as

$$A_{\mu,\nu} \quad i.e., \quad A_{\mu,\nu} = \frac{\partial A_\mu}{\partial x_\nu} - \Gamma^{\rho}_{\mu\nu} A_\rho.$$

The result (2) follows that addition of a quantity in $\dfrac{\partial A_\mu}{\partial x_\nu}$ renders it to be a tensor.

Problem 47. *If $A_\sigma^{\mu\nu}$ and $B_\sigma^{\mu\nu}$ are tensors, then prove that their sum and difference are also tensors.*

We have
$$A'^{\alpha\beta}_{\gamma} = \frac{\partial x'_\alpha}{\partial x_\mu}\frac{\partial x'_\beta}{\partial x_\nu}\frac{\partial x_\sigma}{\partial x'_\gamma} A^{\mu\nu}_\sigma. \qquad \ldots (1)$$

and
$$B'^{\alpha\beta}_{\gamma} = \frac{\partial x'_\alpha}{\partial x_\mu}\frac{\partial x'_\beta}{\partial x_\nu}\frac{\partial x_\sigma}{\partial x'_\gamma} B^{\mu\nu}_\sigma \qquad \ldots (2)$$

Adding (1) and (2), we get

$$A'^{\alpha\beta}_{\gamma} + B'^{\alpha\beta}_{\gamma} = \frac{\partial x'_\alpha}{\partial x_\mu}\frac{\partial x'_\beta}{\partial x_\nu}\frac{\partial x_\sigma}{\partial x'_\gamma}\left(A^{\mu\nu}_\sigma + B^{\mu\nu}_\sigma\right)$$

which follows that $A_\sigma^{\alpha\beta}+B_\sigma^{\alpha\beta}$ is a tensor of the same rank and type as $A_\sigma^{\mu\nu}$ and $B_\sigma^{\mu\nu}$.

Subtracting (2) from (1) the second result follows.

Problem 48. *What is a tensor? Distinguish between a symmetrical and an anti-symmetrical tensor.*

If S_{ij} is symmetric and A_{ij} anti-symmetric in the indices, evaluate $A_{ij}\ S_{ji}$

See §3.5 for the first part. *(Vikram, 1967; Rohilkhand, 1976)*

Problem 49. *Distinguish between symmetric and anti-symmetric tensors. Show that the symmetry properties of a tensor are invariant.*

If A^{ij} and A_{ji} are reciprocal symmetric tensors and if u_j are components of a covariant tensor of rank one, then show that *(Vikram, 1969)*

$$A_{ij}\,u^i\,u^j = A^{ij}\,u_i\;u_j \text{ where } u^i = A^{i\alpha}\,\mu_\alpha$$

Problem 50. *Discuss the application of tensor analysis to the dynamics of a particle.*

(Agra, 1971)

Problem 51. *Explain what is meant by the rank of a tensor. Show that multiplication of tensors results in addition of their ranks and contraction reduces the rank by two.*

Given $T=\frac{1}{2}\;Mg_{mn}\;\dot{x}^m\;\dot{x}^n$ *prove that*

$$\dot{x}^j=\frac{1}{M}\;g^{jn}\left(\frac{\partial T}{\partial\dot{x}^n}\right),\; g_{mn} \text{ being the metric tensor.}$$ *(Agra, 1972)*

Here $T=\frac{1}{2}\;Mg_{mn}\;\dot{x}^m\;\dot{x}^n$ on partial differentiation w.r.t. $\dot{x}^r$ gives

$$\frac{\partial T}{\partial\dot{x}^r}=\frac{1}{2}\;Mg_{mn}\left(\dot{x}^m\;\frac{\partial\dot{x}^n}{\partial\dot{x}^r}+\dot{x}^n\;\frac{\partial\dot{x}^m}{\partial\dot{x}^r}\right)$$

g_{mn} being independent of velocity component $\dot{x}^r$

$$=\frac{1}{2}\;Mg_{mn}\;\dot{x}^m\;\frac{\partial\dot{x}^n}{\partial\dot{x}^r}+\frac{1}{2}\;Mg_{mn}\;\dot{x}^n\;\frac{\partial\dot{x}^m}{\partial\dot{x}^r}$$

$$=\frac{1}{2}\;Mg_{mn}\;\dot{x}^m\;\delta_r^n+\frac{1}{2}\;Mg_{mn}\;\dot{x}^n\;\delta_r^m$$

$$=\frac{1}{2}\;Mg_{mr}\;\dot{x}^m+\frac{1}{2}Mg_{nr}\;\dot{x}^n$$

Now, g_{mn} being symmetric, we get on replacing the indices m and n by l,

$$\frac{\partial T}{\partial\dot{x}^n}=M\gamma_{lr}\,\dot{x}^l\quad,$$

or

$$g^{rj}\;\frac{\partial T}{\partial\dot{x}^r}\;=Mg^{rj}\;g_{lr}\;\dot{x}^l$$

$$=M\;\delta_l^j\;\dot{x}^l\;\text{ as }\;g^{rj}\;g_{rl}=\delta_l^j$$

$$=M\;\dot{x}^j$$

Hence

$$\dot{x}^j=\frac{1}{M}g^{rj}\;\frac{\partial T}{\partial\dot{x}^r}$$

$$=\frac{1}{M}\;g^{jn}\;\frac{\partial T}{\partial\dot{x}^n}\;\text{ on replacing } r \text{ by } n.$$

Problem 52. *Define the intrinsic and covariant derivatives of a contravariant vector. Use the expression for the intrinsic derivative to obtain the components of the acceleration vector for the metric* $ds^2 = dr^2 + r^2\, ds$. (*Agra, 1973*)

Problem 53. *The length ds of a line element in a two dimensional surface* θ–ϕ *is given by* $ds^2 = R^2\, d\theta^2 + R^2 \sin^2\theta\, d\phi^2$, *where R is a constant. Find all components of the metric tensor* g_{ab} *and the Christoffel symbols of first kind for this surface.*

(*Rohilkhand, 1979; Agra, 1974*)

See Problems 25 and 26 (*a*).

Problem 54. *Show why the introduction of an affine connection is necessary in curved space. Find Expression for the affine connection in terms of the metric tensor?*

(*Agra, 1975*)

Hint. We should note that in case of cartesian coordinates, both kinds of Christoffel 3-index symbols vanish. Also the concept of parallel displacement being independent of the existence of a metric tensor, a space with a law of parallel displacement is known as an *affinely connected space* and $\Gamma^{\lambda}_{\mu\nu}$ the *components of an affine connection.*

Problem 55. *If* A_{ij} *is an anti-symmetric tensor and* S_{ij} *is symmetric, find whether or not any one of the following tensors is anti-symmetric, or symmetric:*

(*i*) $A_{ij} A_{ik}$ (*ii*) $A_{ij} S_{ik}$ (*iii*) $S_{ij} A_{ik}$ (*iv*) $S_{ij} S_{ik}$ (*v*) $A_{ij} S_{ik} + S_{ij} A_{ik}$ (*vi*) $A_{ij} S_{ik} - S_{ij} A_{ik}$ (*vii*) $A_{im} A_{mn} A_{nk}$ (*viii*) $A_{im} S_{mn} A_{nk}$ (*ix*) $S_{im} A_{mn} S_{nk}$ (*x*) $S_{im} S_{mn} S_{nk}$.

(*Agra, 1976*)

Problem 56. *Find the covariant and contravariant components of the acceleration vector in cylindrical and spherical coordinates.* (*Rohilkhand, 1979*)

In apherical polar coordinates, we have

$$x = r \sin\theta \cos\phi,\ y = r \sin\theta \sin\phi,\ z = r\cos\theta \quad \text{... (1)}$$

giving

$$r = \left(x^2+y^2+z^2\right)^{1/2},\quad \theta = \tan^{-1}\frac{\left(x^2+y^2\right)^{1/2}}{z},\ \phi = \tan^{-1}\frac{y}{x} \quad \text{... (2)}$$

$$\left.\begin{aligned} \therefore\quad &\frac{\partial r}{\partial x} = \sin\theta\cos\phi,\ \frac{\partial r}{\partial y} = \sin\theta\sin\phi,\ \frac{\partial r}{\partial z} = \cos\theta \\ &\frac{\partial \theta}{\partial x} = \frac{\cos\theta\cos\phi}{r},\ \frac{\partial \theta}{\partial y} = \frac{\cos\theta\sin\phi}{r},\ \frac{\partial \theta}{\partial z} = -\frac{\sin\theta}{r} \\ &\frac{\partial \phi}{\partial x} = -\frac{\sin\phi}{r\sin\theta},\ \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{r\sin\theta},\ \frac{\partial \phi}{\partial z} = 0 \end{aligned}\right\} \quad \text{... (3)}$$

Now, acceleration being a contravariant vector, its components will transform by the law

$$a'^{\mu} = \frac{\partial x'_{\mu}}{\partial x_{\alpha}} a^{\alpha} \quad (\alpha = 1, 2, 3) \quad \text{... (4)}$$

Suppose that $x_1 = x,\ x_2 = y,\ x_3 = z;\ x'_1 = r,\ x'_2 = \theta,\ x'_3 = \phi$ ...(5)

Similarly $a^1 = a_x,\ a^2 = a_y,\ a^3 = a_z;\ a'^1 = a_r,\ a'^2 = a_\theta,\ a'^3 = a_\phi$ ... (6)

We, thus have

$$a_r = \frac{\partial r}{\partial x}\frac{d^2x}{dt^2} + \frac{\partial r}{\partial y}\frac{d^2y}{dt^2} + \frac{\partial r}{\partial z}\frac{d^2z}{dt^2};\quad a_\theta = \frac{\partial \theta}{\partial x}\frac{d^2x}{dt^2} + \frac{\partial \theta}{\partial y}\frac{d^2y}{dt^2} + \frac{\partial \theta}{\partial z}\frac{d^2z}{dt^2} \quad \text{and}$$

$$a_\phi = \frac{\partial \phi}{\partial x}\frac{d^2x}{dt^2} + \frac{\partial \phi}{\partial y}\frac{d^2y}{dt^2} + \frac{\partial \phi}{\partial z}\frac{d^2z}{dt^2} \quad \text{... (7)}$$

$$\text{where } \frac{d^2x}{dt^2}=\frac{d}{dt}\left[\frac{d}{dt}(r\sin\theta\cos\phi)\right]=\frac{d}{dt}\left[\dot r\sin\theta\cos\phi+r\cos\theta\cos\phi\,\dot\theta-r\sin\theta\sin\phi\,\dot\phi\right]$$

$$=\ddot r\sin\theta\cos\phi+2\dot r\,\dot\theta\cos\theta\cos\phi-2\sin\theta\sin\phi\,\dot r\,\dot\phi-r\sin\theta\cos\phi\,\dot\theta^2$$

$$-2r\cos\theta\sin\phi\,\dot\theta\,\dot\phi+r\cos\theta\cos\phi\,\ddot\theta-r\sin\theta\cos\phi\,\dot\phi^2-r\sin\theta\sin\phi\,\ddot\phi$$

with similar expressions for $\frac{d^2y}{dt^2}$ and $\frac{d^2z}{dt^2}$

Their substitution in (7) yields with the help of (3).

$a'^1\equiv a_r=\ddot r-r\,\dot\theta^2-r\,\sin^2\theta\,\dot\phi^2$; $a'^2\equiv a_\theta=\ddot\theta-\sin\theta\cos\phi\,\dot\phi^2$ and

$$a'^3\equiv a_\phi=\dot\phi+\frac{2}{r}\,\dot r\,\dot\phi+2\cot\theta\,\dot\theta\,\dot\phi \quad \text{... (8)}$$

Note 1. By taking $\theta=\frac{\pi}{2}$, $r=\rho$ (say) so that $x=\rho\cos\phi$, $y=\rho\sin\phi$ in two-dimensional $x-y$ plane, we have the result of problem 6, such as

$$a_\rho=\ddot\rho-\rho\,\dot\phi^2,\ a_\phi=\ddot\phi+\frac{2}{\rho}\,\dot\rho\,\dot\phi \quad \text{... (9)}$$

Further to find the components of the acceleration vector in cylindrical coordinates, we have

$$x=\rho\cos\phi,\ y=\rho\sin\phi,\ z=z \quad \text{... (10)}$$

giving $a_\rho=\ddot\rho-\rho\,\dot\phi^2$, $a_\phi=\ddot\phi+\frac{2}{\rho}\,\dot\rho\,\dot\phi$ as in (9) and $a_z=\ddot z$

Note 2. Similarly covariant components may be derived.

Problem 57. *Show that in a cartesian coordinate system the contravarient and covariant components of a vector are identical.* (*Rohilkhand, 1980, 87; Benares, 1973*)

Taking (x_1, x_2, x_3) and (x'_1, x'_2, x'_3) as the coordinates of a point P relative to orthogonal cartesian coordinate systems S and S' respectively and (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3) as direction cosines of axes of x_1, x_2, x_3 respectively, we have

$$x'_1=l_1x_1+m_1x_2+n_1x_3;\ x'_2=l_2x_1+m_2x_2+n_2x_3;$$
$$x'_3=l_3x_1+m_3x_2+n_3x_3 \quad \text{... (1)}$$

$$\&\ x_1=l_1x'_1+l_2x'_2+l_3x'_3;\ x_2=m_1x'_1+m_2x'_2+m_3x'_3;$$
$$x_3=n_1x'_1+n_2x'_2+n_3x'_3 \quad \text{... (2)}$$

Transformation law of contravariant vector, *i.e.*, $A'^\mu=\sum_{\alpha=1}^{3}\frac{\partial x'_\mu}{\partial x_\alpha}A^\alpha$ in 3-dimensions leads

$$A'^1=\frac{\partial x'_1}{\partial x_1}A^1+\frac{\partial x'_1}{\partial x_2}A^2+\frac{\partial x'_1}{\partial x_3}A^3;\qquad A'^2=\frac{\partial x'_2}{\partial x_1}A^1+\frac{\partial x'_2}{\partial x_2}A^2+\frac{\partial x'_2}{\partial x_3}A^3;$$
$$A'^3=\frac{\partial x'_3}{\partial x_1}A^1+\frac{\partial x'_3}{\partial x_2}A^2+\frac{\partial x'_3}{\partial x_3}A^3 \quad \text{... (3)}$$

Which with the help of (1) become

$$A'^1=l_1A^1+m_1A^2+n_1A^3;\ A'^2=l_2A^1+m_2A^2+n_2A^3;$$
$$A'^3=l_3A^1+m_3A^2+n_3A^3 \quad \text{... (4)}$$

Again, transformation law of covariant vector *i.e.* $A'_\mu=\sum_{\alpha=1}^{3}\frac{\partial x_\alpha}{\partial x'_\mu}A_\alpha$ gives

$$A'_1=\frac{\partial x_1}{\partial x'_1}A_1+\frac{\partial x_2}{\partial x'_1}A_2+\frac{\partial x_3}{\partial x'_1}A_3;\quad A_2=\frac{\partial x_1}{\partial x'_2}A_1+\frac{\partial x_2}{\partial x'_2}A_2+\frac{\partial x_3}{\partial x'_2}A_3;$$

$$A_3' = \frac{\partial x_1}{\partial x_3'} A_1 + \frac{\partial x_2}{\partial x_3'} A_2 + \frac{\partial x_3}{\partial x_3'} A_3 \quad \ldots (5)$$

Which with the help of (2) yield

$A'_1 = l_1 A_1 + m_1 A_2 + n_1 A_3$; $A'_2 = l_2 A_1 + m_2 A_2 + n_2 A_3$;

$$A'_3 = l_3 A_1 + m_3 A_2 + n_3 A_3 \quad \ldots (6)$$

Comparison of (4) and (6) shows that there is distinction between contravariant and covaraint vectors in rctangular cartesian transformation of coordinates and hence the contravaraint and covariant components of a vector and identical.

Problem 58. (*a*) *Is force a tensor ? If so, what is its nature and rank ? Explain the reasons.*

(*b*) *Show that a vector can always be associated with an anti-symmetric tensor of rank two in three dimensional space and that such vector is not pseudo vector.*

(Rohilkhand, 1982; Meerut, 1969)

Problem 59. *Show that the following matrix respesents a tensor*

$$\begin{pmatrix} -xy & y^2 \\ x^2 & xy \end{pmatrix}, \quad \textit{(Rohilkhand, 1983)}$$

whereas $\begin{pmatrix} -xy & -y^2 \\ -x^2 & -xy \end{pmatrix}$ *is not a tensor* ***(Rohilkhand, 1984, 86)***

If $A = \begin{pmatrix} -xy & y^2 \\ x^2 & xy \end{pmatrix}$ is a tensor, then its components

$A^{11} = -xy$, $A^{12} = y^2$, $A^{21} = x^2$, $A^{22} = xy$ must obey transformation laws.

In two-dimensional plane if rectangular axes are rotated through an angle θ, keeping origin fixed, then the new coordinates (x', y') in terms of old one *i.e.* (x, y) are given by

$$x' = x \cos\theta + y \sin\theta \ ; \ y' = -x \sin\theta + y \cos\theta \quad \ldots (1)$$

Which give $\quad \dfrac{\partial x'}{\partial x} = \cos\theta, \ \dfrac{\partial x'}{\partial y} = \sin\theta, \ \dfrac{\partial y'}{\partial x} = -\sin\theta, \ \dfrac{\partial y'}{\partial y} = \cos\theta \quad \ldots(2)$

In rotated system, the transformation law yields

$$-x'y' = A'^{11} = \sum_{\alpha,\beta=1}^{2} \frac{\partial x_1'}{\partial x_\alpha} \frac{\partial x_1'}{\partial x_\beta} A^{\alpha\beta} \text{ where } x_1 = x, \ x_2 = y$$

$$= \frac{\partial x'}{\partial x}\frac{\partial x'}{\partial x} A^{11} + \frac{\partial x'}{\partial x}\frac{\partial x'}{\partial y} A^{12} + \frac{\partial x'}{\partial y}\frac{\partial x'}{\partial x} A^{21} + \frac{\partial x'}{\partial y}\frac{\partial x'}{\partial y} A^{22}$$

$$= \cos^2\theta\, A^{11} + \cos\theta \sin\theta\, A^{12} + \sin\theta\cos\theta\, A^{21} + \sin^2\theta\, A^{22}$$

In (1), if we express x, y in terms of x', y' and substitute the velues of $A^{11}, A^{12}, A^{21}, A^{22}$ in R.H.S., then we get

$$-(x\cos\theta + y\sin\theta)(-x\sin\theta + y\cos\theta) = -\cos^2\theta\, xy + \cos\theta\sin\theta\, y^2 + \sin\theta\cos\theta\, x^2 + \sin^2\theta\, xy$$

or $\quad x^2 \sin\theta\cos\theta - xy\cos^2\theta + xy\sin^2\theta + y^2\sin\theta\cos\theta = -\cos^2\theta\, xy + \cos\theta\sin\theta\, y^2 + \sin\theta\cos\theta\, x^2 + \sin^2\theta\, xy$

which is an identity showing that $A^{11} = -xy$ obeys tensor law. Similar results hold for other components. Hence A is a tensor.

Now to see whether $\begin{pmatrix} -xy & -y^2 \\ -x^2 & -xy \end{pmatrix}$ is a tensor, suppose if is a tensor with components $B^{11} = -xy$, $B^{12} = -y^2$, $B^{21} = -x^2$, $B^{22} = -xy$, then as above

$$B'^{11} = -x'y' = \sum_{\alpha,\beta=1}^{2} \frac{\partial x_1'}{\partial x_\alpha}\frac{\partial x_1'}{\partial x_\beta} B^{\alpha\beta} \text{ with } x_1 = x,\ x_2 = y$$

$$= \frac{\partial x'}{\partial x}\frac{\partial x'}{\partial x} B^{11} + \frac{\partial x'}{\partial x}\frac{\partial x'}{\partial y} B^{12} + \frac{\partial x'}{\partial y}\frac{\partial x'}{\partial x} B^{21} + \frac{\partial x'}{\partial y}\frac{\partial x'}{\partial y} B^{22}$$

$$= \cos^2\theta\, B^{11} + \cos\theta \sin\theta\, B^{12} + \sin\theta\cos\theta\, B^{21} + \sin^2\theta\, B^{22}$$

or $$-(x\cos\theta + y\sin\theta)(-x\sin\theta + y\cos\theta) = -\cos^2\theta\, xy - \cos\theta\sin\theta\, y^2 + \sin\theta\cos\theta\, x^2 - \sin^2\theta\, xy$$

or $$xy(\sin^2\theta - \cos^2\theta) + \sin\theta\cos\theta\,(x^2 + y^2) = -xy + \sin\theta\cos\theta\,(x^2 - y^2)$$

which is not satisfied. Hence components of B do not obey tensor law *i.e.* B. is not a tensor.

Problem 60. *Show that in two dimensions a skew-symmetric tensor of second rank, is a pseudoscalar.* (*Rohilkhand, 1985*)

Problem 61. *Show that the transformations of tensors form a group.* (*Benaras 1970; Kanpur 1976*)

Problem 62. *Differentiate between contravariant and covaraint tensors on the basis of transformation laws obeyed by them and hence show that the velocity of a fluid at any point is a contravariant tensor of rank one.* (*Rohilkhand, 1988*)

Problem 63. *Prove that an entity which on inner multiplication with an arbitrary tensor (Contravariant or covariant) always gives a tensor, is itself a tensor.* (*Rohilkhand, 1988*)

Problem 64. *Prove that an antisymmetric tensor of the second order can be associated with a vector in three dimensions. What is the corresponding result in four dimensions.* (*Rohilkhand, 1982; Meerut, 1969*)

Hint. Number of independent components of an anti-symmetric tensor of rank r in n-dimensional space being given by $^nC_r = \frac{\lfloor n}{\lfloor r \lfloor n-r}$ $\Rightarrow$ number of independent components of an antisymmetric tensor of rank 2 in 3-dimensional space $= {}^3C_2 = 3 \Rightarrow$ an anti-symmetric tensor of second order can be associated with a vector in 3-D. Similarly those of second order in 4-D space is $^4C_2 = 6$, but number of independent components of a vector in 4-D space is 4. Thus an anti-symmetric tensor of second order can not be associated with a vector in 4-D space.

CHAPTER 4

GROUP THEORY

4.1. INTRODUCTION TO SETS, MAPPINGS AND BINARY OPERATIONS

Set. A set is a collection of objects of any sort, having some properties in common, *e.g.* the set of all natural numbers.

The objects comprising the set are known as its *elements* or *members*.

A set is *finite* or *infinite* according as the number of its elements is finite or infinite.

The elements of a set must be *distinct* and *distinguishable*. Here *distinct* means that no element of the set is repeated and *distinguishable* means that given any object whatsoever, it is either in the set or not in the set.

The sets are denoted by braces like { }, *e.g.* {1, 2} and {1, 1, 2} which represent the same set.

Defining property of a set. Using any of the notation :, |, ∋, *s.t.* for *such that*, the defining property of a set is $\{x : P(x)$, *e.g.* a set of even numbers from 2 to 20 may be expressed as

$$\{x : x = 2n, n = 1, 2, \ldots 10\}$$

Singleton set. A set having a single element is called as singleton set *e.g.* $\{a\} = \{x : x = a\}$. As another example {0} is a singleton set having 0 as the single element.

Null set or void set or empty set. A set having no element is called an empty set and denoted by ϕ, such as $\phi = \{x : x \neq x\} = \{\ \}$.

Subset. Using the notation $\in$ for '*belong to*' and $\Rightarrow$ for '*implies*', if there are two sets A and B such that every element of A belongs to B,

i.e. $$a \in A \Rightarrow a \in B$$

the A is called a subset of B or said to be contained in B and denoted by

$$A \subset B \text{ or } B \supset A \text{ (i.e. } B \text{ contains } A).$$

Here A is *subset* of B and B is *superset* of A. *e.g.* {1, 3} is subset of {1, 2, 3} but {1, 2, 3} is superset of {1, 3}.

Equal sets. Two sets A and B are said to be equal if

$$A \subset B \text{ and } B \subset A$$

e.g. $A = \{a, b, c, d\}$ and $B = \{b, c, a, d\}$ are equal.

The Negations $a \in A, A \subset B$ and $A = B$ are $a \notin A$, $a \not\subset B$ and $A \neq B$ respectively.

Axiom of extension. Two sets A and B are equal if and only if they have the same number of elements.

Axiom of specification. If A is a set and $S(x)$ is a condition or statement then there corresponds a set B whose elements are exactly those elements x of A for which $S(x)$ is true *i.e.*

$$B = \{x \in A : S(x)\}$$

Axiom of pairing. If a and b are two objects (sets), then there exists a set A such that $a \in A$ and $b \in A$ *i.e.* $\{x \in A : x = a \text{ or } x = b\}$

Normal and abnormal sets. If a set contains itself as one of its elements, it is said to be an abnormal set, otherwise it is a normal set.

Equivalent set. Two sets A and B are said to be equivalent and denoted by $A \sim B$ if the number of elements of A is equal to the number of elements of B. Evidently two equal sets are equivalent but the converse is not true.

Proper subset. A set A is said to be the proper subset of B and denoted by $A \subset B$ (some authors use $\subseteq$ for a subset and $\subset$ for a proper subset), if every element of A is an element of B and there is atleast one element of B which is not the element of A so that $A \neq B$ *e.g.* $\{1, 3, 5\}$ is a subset of $\{1, 3, 5\}$ but it is not its proper subset while $\{1, 3, 5\}$ is a proper subset of $\{1, 3, 5, 7\}$.

Axiom of power set. The power set of a set A is the family or class of all the subsets of A and denoted by $P(A)$ *e.g.* if $A = \{1, 2, 3\}$ then

$$P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Obviously when A consists of 3 elements, $P(A)$ consists of 2^3 elements. In general if A consists of m elements, $P(A)$ will consist of 2^m elements. The power set of A is also denoted by 2^A.

Operation on Sets.

Union. The union of two sets A and B denoted by $A \cup B$ and read as 'A union B' is the set of all objects which are members of A or B (or both)

i.e. $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

The word 'or' used here gives the inclusive sense and/or.

e.g. if $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$, then $A \cup B = \{1, 2, 3, 4, 5, 6\}$

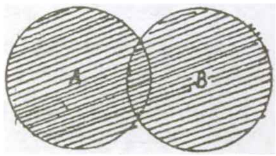


Fig. 4.1

The union of n sets $A_1, A_2 \ldots\ldots A_n$ is defined as

$$A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i = \{x : x \in A_i \text{ for some } i \text{ in the range } i = 1 \text{ to } i = n\}$$

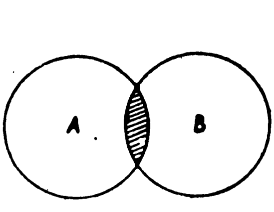


Fig. 4.2

Intersection. The intersection of two sets A and B denoted by $A \cap B$ and read as 'A intersection B' is the set of all objects which are members of both A and of B *i.e.*

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

e.g. if $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$, then $A \cap B = \{2, 4\}$

The intersection of n sets $A_1, A_2, \ldots A_n$ is defined as

$$A_1 \cap A_2 \cap \ldots \cap A_n = \bigcap_{i=1}^{n} A_i = \{x : x \in A_i \text{ for } 1 \leq i \leq n\}$$

Family or class or collection of sets. If a set consists of elements which are sets themselves, then such a set is called as 'family or class' of sets *e.g.* a set $A^* = \{\{1\}, \{1,2\}, \{1,2,3\}\}$.

Indexed family of sets. For a given index set Λ, a collection of sets such that to each member of Λ there corresponds a member of the collection of the sets, is known as indexed family of sets and written, as

$$A^* = \{A_\alpha : \alpha \in \Lambda\}$$

where $\alpha \in \Lambda$ is an index and A_α denote the indexed sets.

The arbitary union of sets $\{A_\alpha : \alpha \in \Lambda\}$ is given by $\cup \{A_\alpha : \alpha \in \Lambda\} = \{x : x \in A_\alpha$ for at least one $\alpha \in \Lambda\}$

If $\Lambda = \emptyset$, then $\cup \{A_\alpha : \alpha \in \emptyset) = \emptyset$.

The arbitrary Intersection of sets $\{A_\alpha : \alpha \in \Lambda\}$ is given by

$\cap (A_\alpha : \alpha \in \Lambda\} = \{x : x \in A_\alpha \forall \alpha \in \Lambda\}$, $\forall$ is the notation for *'for every'*.

If $\Lambda = \emptyset$ then $\cap \{A_\alpha : \alpha \in \emptyset\} = U$.

Mutually exclusive or disjoint sets. If there are two sets A and B such that $A \cap B = \emptyset$, then A and B are said to be disjoint *e.g.*, if $A = \{1, 2\}, B = \{5, 6\}$ then $A \cap B = \emptyset$.

Universal set. All the sets under consideration are assumed to be the subsets of some fixed set called as the universal set and denoted by U.

Complementary set. The complement of a set A is denoted by A' and is defined by the set of all members of the universal set U, which are not members of A *i.e.* $A' = \{x : x \in U$ and $x \notin A\}$

e.g. if $A = \{x : x < 3\}$ then $A' = \{x : x \geq 3\}$

It is notable that $A \cup A' = U$; $U' = \emptyset$: $\emptyset' = U$; $A \cap A' = \emptyset$ and $(A')' = A$.

Important properties of operations on sets

(1) Difference operation. If A and B are two sets, then the set consisting of elements which belong to A but not to B is said to be the difference of sets A and B and is denoted by $A - B$, *i.e.*

$$A - B = \{x \mid x \in A \text{ and } x \notin B).$$

For example, if $A = \{1, 2, 3, 4\}$ and $B = \{1, 4, 7\}$, then

$$A - B = \{2, 3\} \text{ and } B - A = \{7\}.$$

It is to be noted that $A - B \neq B - A$; $A - A = \emptyset$, $A - \emptyset = A$ and $A - B \subset A$.

It may also be shown that $A - B = A \cap B'$; $(A - B) \cap B = \emptyset$ and $(A - B) \cup A = A$.

The symmetric difference of A and B denoted by $A \Delta B$ is defined as $A \Delta B = (A - B) \cup (B - A)$.

(2) Commutative laws

(*i*) $A \cup B = B \cup A$.

(*ii*) $A \cap B = B \cap A$.

These can be proved as follows:

We have $x \in A \cup B \Rightarrow x \in A$ or $x \in B$

$\Rightarrow x \in B$ or $x \in A$

$\Rightarrow x \in B \cup A$.

Consequently $A \cup B \subset B \cup A$ and also $B \cup A \subset A \cup B$ (in a similar manner).

$\therefore \quad A \cup B = B \cup A$.

Similarly, if $x \in A \cap B \Rightarrow x \in A$ and $x \in B$

$\Rightarrow x \in B$ and $x \in A$

$\Rightarrow x \in B \cap A$,

so that $A \cap B \subset B \cap A$ and similarly $B \cap A \subset A \cap B$.

$\therefore \quad A \cap B = B \cap A.$

It is easy to verify that $A \cap B \subset A$ and $A \cap B \subset B$.

(3) Associative laws

(*i*) $A \cup (B \cup C) = (A \cup B) \cup C$,

(*ii*) $A \cap (B \cap C) = (A \cap B) \cap C$.

We have $x \in (A \cup B) \cup C \Rightarrow x \in (A \cup B)$ or $x \in C$

$\Rightarrow x \in A$ or $x \in B$ or $x \in C$

$\Rightarrow x \in A$ or $x \in (B \cup C)$

$\Rightarrow x \in A \cup (B \cup C)$,

so that $\quad (A \cup B) \cup C \subset A \cup (B \cup C)$

and similarly $\quad A \cup (B \cup C) \subset (A \cup B) \cup C$

$\therefore \quad A \cup (B \cup C) = (A \cup B) \cup C.$

Again $x \in (A \cap B) \cap C \Rightarrow x \in A \cap B$ and $x \in C$

$\Rightarrow (x \in A$ and $x \in B)$ and $x \in C$

$\Rightarrow x \in A$ and $(x \in B$ and $x \in C)$

$\Rightarrow x \in A$ and $x \in B \cap C$

$\Rightarrow x \in A \cap (B \cap C)$,

so that $\quad (A \cap B) \cap C \subset A \cap (B \cap C)$

and similarly $\quad A \cap (B \cap C) \subset (A \cap B) \cap C$.

$\therefore \quad A \cap (B \cap C) = (A \cap B) \cap C.$

(4) Idempotent laws

(*i*) $A \cup A = A$,

(*ii*) $A \cap A = A$.

Here $x \in A \cup A \Rightarrow x \in A$ or $x \in A$.

$\Rightarrow x \in A$.

$\therefore \quad A \cup A \subset A$ and similarly $A \subset A \cup A$, so that

$A \cup A = A.$

In a similar way it may be shown that $A \cap A = A$.

It is easy to verify that $A \cap \emptyset = \emptyset$ and $A \cap U = A$

(5) Distributive laws

(*i*) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,

(*ii*) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,

We have, $x \in A \cap (B \cup C) \Rightarrow x \in A$ and $x \in (B \cup C)$

$\Rightarrow x \in A$ and $(x \in B$ or $x \in C)$

$\Rightarrow (x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$

$\Rightarrow x \in A \cap B$ or $x \in A \cap C$

$\Rightarrow x \in (A \cap B) \cup (A \cap C)$.

Consequently $\quad A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$

and similarly $(A \cap B) \cup (A \cap C) \subset A \cap (B \cap C)$,

so that $$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Again, $x \in A \cup (B \cap C) \Rightarrow x \in A$ or $x \in B \cap C$

$\Rightarrow x \in A$ or ($x \in B$ and $x \in C$)

$\Rightarrow$ ($x \in A$ or $x \in B$) and ($x \in A$ or $x \in C$)

$\Rightarrow x \in A \cup B$ and $x \in A \cup C$

$\Rightarrow x \in (A \cup B) \cap (A \cup C)$.

Consequently $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$

and similarly $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$,

so that $$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

(6) De Morgan laws

(*i*) $A - (B \cup C) = (A - B) \cap (A - C)$

(*ii*) $A - (B \cap C) = (A - B) \cup (A - C)$.

We have $A - (B \cup C) = \{x \mid x \in A \text{ and } x \notin B \cup C\}$

$= \{x \mid x \in A \text{ and } (x \notin B \text{ and } x \notin C)\}$

$= \{x \mid (x \in A \text{ and } x \notin B) \text{ and } (x \in A \text{ and } x \notin C)\}$

$= \{x \mid x \in (A - B) \text{ and } x \in (A - C)\}$

$= \{x \mid x \in (A - B) \cap (A - C)\}$

$\therefore \quad A - (B \cup C) \subset (A - B) \cap (A - C)$.

Similarly it may be shown that $(A - B) \cap (A - C) \subset A - B \cup C$

Thus, $A - B \cup C = (A - B) \cap (A - C)$.

Similar procedure will yield $A - (B \cap C) = (A - B) \cup (A - C)$

De Morgan Laws are sometimes expressed as

$(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$.

These may be shown as follows :

$(A \cup B') = \{x : x \notin A \cup B\}$

$= \{x : x \notin A \text{ and } x \notin B\}$

$= \{x : x \in A' \text{ and } x \in B'\}$

$= \{x : x \in A' \cap B'\}$.

$\therefore \quad (A \cup B)' \subset A' \cap B'$

Similarly $A' \cap B' = \{x : x \in A' \text{ and } x \in B'\}$

$= \{x : x \notin A \text{ and } x \notin B\}$

$= \{x : x \in (A \cup B)'\}$

$\therefore \quad A' \cap B' \subset (A \cup B)'$

Hence $\quad (A \cup B)' = A' \cap B'$

Similar procedure will show that $(A \cap B)' = A' \cup B'$

Cartesian product of two sets. The product of two sets A and B is the set of all distinct ordered pairs (a, b) where $a \in A$, and $b \in B$ and denoted by $A \times B$ (read as A cross B) *i.e.*

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

e.g. if $A = \{1, 2\}$ and $B = \{p, q, r\}$, then

$$A \times B = \{(1, p), (1, q), (1, r), (2, p), (2, q), (2, r)\}$$

and $$B \times A = (p, 1), (p, 2), (q, 1), (q, 2), (r, 1), (r, 2)\}$$

It is clear that $A \times B \neq B \times A$

If either A or B is a null set, then $A \times B = \emptyset$

If the set A has m elements and B has n elements, then $A \times B$ or $B \times A$ has mn elements (ordered pairs).

The product of n sets $A_1, A_2, \ldots A_n$ is the set of all distinct ordered n-tuples $(a_1, a_2, \ldots a_n)$ where $a_1 \in A_1, a_2 \in A_2, \ldots a_n \in A_n$ and is defined as

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots a_n) : a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

Function or mapping

Let there be two non-empty sets X and Y and there is some rule or correspondence which assigns to each element $x \in X$, a unique element $y \in Y$, then this rule or correspondence is said to be a *mapping* or a *function* and denoted by f *i.e.* $f : X \to Y$ and read as 'f is a function of X to Y' or f is a mapping of X to Y'.

The set X is called the *domain* of the given function f and the set Y of all the values assumed by it is called its *Range* or *Image set*. Also Y is called the *co-domain* of f.

y is sometimes known as image of x and written as $y = f(x)$. Here $f(x)$ is read as '*image of x under the rule f*' or simply 'f of x'. The rule f is also known as *mapping* or *transformation* or *operator* and x is also known as *preimage* of y.

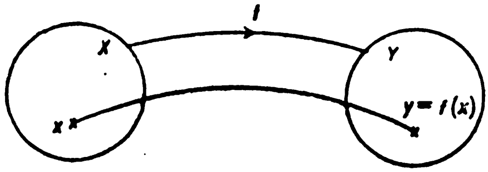


Fig. 4.3

A function whose range has a single element is said to be *constant function*.

Diagrammatical representation of $y = f(x)$ with a rule f defined by $x \to f(x)$ is shown in Fig. 4.3.

If $y = x^2$, then the rule f is $x \to x^2$

which is shown in Fig. 4.4 for positive integral values of x.

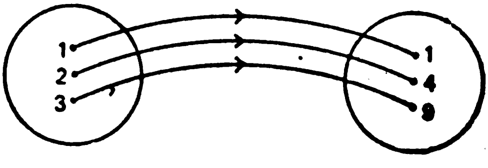


Fig. 4.4

Functions defined as sets of ordered pairs. Given two non-empty sets X and Y, a function f from X to Y is a subset of $X \times Y$ provided

(*i*) $\forall\ x \in X$, $(x, y) \in f$ for some $y \in Y$ *i.e.* $\exists$ (*there exists*) a rule f so that every element of X has image.

(*ii*) $(x, y) \in f$ and $(x, y') \in f \Rightarrow y = y'$ *i.e.* the image is unique.

The *graph* of f is defined as the subset of $X \times Y$ given by $\{[x, f(x)] : x \in X\}$ and that *range* of f as the set of all images under f given by $f[X] = \{y \in Y : y = f(x)$ for some $x \in X\} = \{f(x) : x \in X\}$.

In case $A \subset X$ then the set $\{f(x) : x \in A\}$ is known as the *image* of A under f and denoted by $f[A]$. Also if $B \subset Y$, then the set $\{x \in X : f(x) \in B\}$ is known as the *inverse image* of B under f and denoted by $f^{-1}[B]$.

Extension and restriction of a function. Given two functions f and g such that f contains the domain of g and $f(x) = g(x)\ \forall\ x$ in the domain of g, the function f is said to be the *extension* of g and g is said to be the *restriction* of f.

Real and Complex functions. If range of f consists of real numbers, f is said to be a *real function* and if its range consists of complex numbers, f is said to be a *complex function.*

Onto and Into Mappings. If the range is completely filled up, the mapping is said to be *onto* and if the range is not completely filled up then it is *Into.* In other words if $\exists$ at least one $y \in Y$ which is not an $f(x)$ for any $x \in X$, then the mapping f is said to be *Into* otherwise it is said to be *onto* or *Surjective.* The surjective function is also known as a *Surjection* or an *epimorphism.*

One-one and Many-one Mappings. Given two non-empty sets X and Y, if two different elements in X always have different images under the rule f, then f is said to be a *one-one mapping* or an *injection* or *monomorphism* of X into (onto) Y and if the two or more different elements of X have the same image under f, then f is said to be a *many-one mapping* of X into (onto) Y.

Diagrammatical representation of such functions are shown in Figures 4.5, 4.6, 4.7, 4.8.

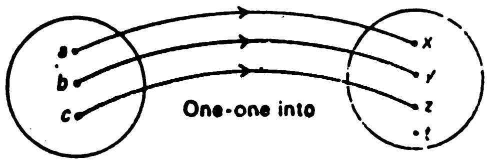


Fig. 4.5

A function which is both surjective and injective is known as *bijective i.e.* a one-one onto mapping is also known as a *bijection* and a bijection of a set X onto itself is known as **Permutation** of X.

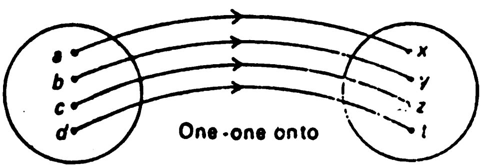


Fig. 4.6

If $f : X \to Y$, f is one-one if $x_1 \neq x_2 \Rightarrow f_1(x_1) \neq f(x_2)\ \forall\ x_1, x_2 \in X$. In case f is *into*, the rnage of f is a proper subset of Y *i.e.* $f[X] \subset Y$ and $f[X] \neq Y$.

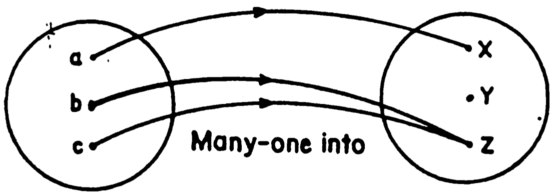


Fig. 4.7

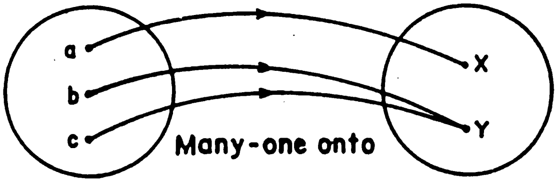


Fig. 4.8

In case f is *onto*, the range of f is equal to Y *i.e* $f[X] = Y$.

Inverse mapping. Let f represent a function (mapping) which is both *onto* and *one-one* defined as $f: X \to Y$, then its inverse mapping $f^{-1}: Y \to X$ is defined as below :

$\forall\ y \in Y$, if we find the unique element $x \in X$ s.t. $f(x) = y$ then x is defined to be $f^{-1}(y)$ *i.e.* $f^{-1}(y) = \{x : x \in X, f(x) = y\}$ which follows that $f^{-1}(y)$ is always a subset of X.

Diagrammatical representation of an inverse mapping is shown in Fig. 4.9.

One-one onto mapping is often called as *one to one correspondence*. Thus if f is a one to one correspondence between Y and X, then f^{-1} is a one to one correspondence between Y and X.

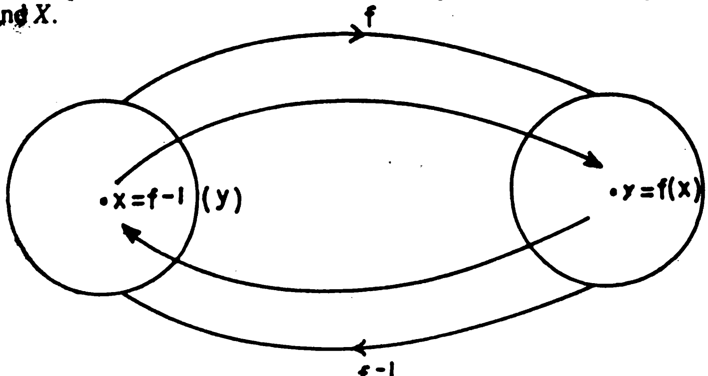


Fig. 4.9

Identity mapping. Given a non-empty set X, the identity mapping I_X on X is the mapping of X onto itself *i.e.* $I_X : X \to X$ and defined as

$$I_X(x) = x\ \forall\ x \in X$$

Equal mappings. Let there be two functions f and g defined as

$$f: X \to Y \text{ and } g: X \to Y$$

then the functions f and g are equal iff $f(x) = g(x)\ \forall\ x \in X$.

Product or Composite mapping of two mappings $f: X \to Y$ and $g: Y \to Z$ is defined to be a mapping, $gof: X \to Z$ and given by

$$(gof) = g[f(x)]\ \forall\ x \in X$$

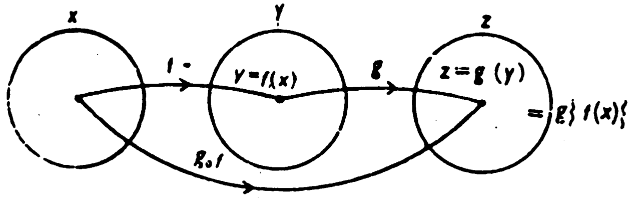


Fig. 4.10

Diagrammatical representation is shown in Fig. 4.10.

Binary operation. Binary operation over a set X is a mapping from $X \times X \to X$, *i.e.* if f denotes a binary operation over a set X, then $f : X \times X \to X$.

Binary relation or simply relation

A binary relation in a set S is a mathematical symbol denoted by R s.t. for each ordered pair $(x, y) \in S$, the statement ${}_xR_y$ is true or false in the sense that ${}_xR_y$ asserts that x is related by R to y and ${}_x\bar{R}_y$ the negation of it *i.e.* x is not related by R to y, *e.g.* if S be the set of all integers and R be a relation < (less than) defined on S, then 5<8 is true for $(5, 8) \in S$ *i.e.* ${}_5R_8$ is true while ${}_8R_5$ is false as $8 \not< 5$ which is denoted by ${}_8\bar{R}_5$.

In other words if S is a set s.t. the Cartesian product $S \times S$ involves the same set, then it is called a relation R on S. In fact the relation R on the set S is a subset of $S \times S$ *i.e.* $R \subset S \times S$.

Thus if $(x, y) \in R$, $(x, y \in S)$ then we have ${}_xR_y$.

In general a relation R from A to B between two sets A and B is a subset of $A \times B$ *i.e.* $R \subset A \times B$.

If there are m elements in A, n in B then there will be mn element in $A \times B$ and so there will be different mn relations from A to B.

Domain and range of a relation. If A and B are two sets and R is a relation from A to B, then the *domain* of R denoted by *Dom* (R) is the set of first coordinates of all the ordered pairs in R and the *range* of R denoted *Ran* (R) is the set of second coordinates of all the ordered pairs in R. Thus

Dom $(R) = \{x : (x, y) \in R$ for some $y \in B\}$

Ran $(R) = \{y : (x, y) \in R$ for some $x \in A\}$

e.g. if R be a relation in $\mathbf{Z}$ the set of natural numbers. s.t.

$$R = \{(x, y) \in \mathbf{Z} \times \mathbf{Z} : x + 2y = 10\}$$

then Dom $(R) = \{2, 4, 6, 8\}$ $\because 2 + 2.4 = 10 = 4 + 2.3 = 6 + 2.2 = 8 + 2.1$

Ran $(R) = \{4, 3, 2, 1\}$

Identity or diagonal relation. A relation R on a set A is known as identity relation or diagonal relation iff ${}_xR_y \Rightarrow x = y \ \forall \ x, y \in A$.

Compositive relation. If R be a relation from A to B and S a relation from B to C, then composite relation from A to C is denoted by SoR and is defined as the set of all ordered pairs $(x, z) \in SoR$ iff $\exists$ a $y \in B$ s.t. ${}_xR_y$ and ${}_yS_z$ *i.e.*, $(x, y) \in R$ and $(y, z) \in S$.

In other words $SoR \subset$ Dom $(R) \times$ Ran (R)

Universal relation in a set. If A is any set and R is the set $A \times A$ then R is said to be the universal relation on A.

Empty or void relation. If A is a set, then every subset of $A \times A$ is a relation on A. Since the null set ϕ is the subset of all the sets and so is of the set $A \times A$, therefore ϕ is also a relation on A. Such a relation is said to be the Empty or Void relation on A :

Inverse relation. If R be a relation from a set A to another set B, then the inverse relation of R denoted by R^{-1} is defined as the inverse relation *i.e.* R^{-1} from B to A iff

$$R^{-1} = \{y, x\} : (x, y) \in \text{R}\}$$

clearly Dom (R^{-1}) = Ran (R)

Ran (R^{-1}) = Dom (R)

So that ${}_xR_y \Leftrightarrow {}_yR_x{}^{-1}$

If $A = B$, R and R^{-1} both are the relations on A.

It is easy to verify that $(R^{-1})^{-1} = R$.

Types of Relations

I. Reflexive. A relation R on a set A is reflexive iff each member of A is R-related to itself *i.e.* ${}_xR_x \ \forall \ x \in A$ or in other-words $(x, x) \in R \ \forall \ x \in A$. Evidently a relation R on A is reflexive iff $\Delta_A \subset R$, Δ_A being identity relation. *e.g.* if A be the set of lines in a plane and R *a* relation 'parallel to' then any line $x \in A$ is parallel to itself *i.e.* ${}_xR_x \ \forall \ x \in A$ and so R is reflexive.

II. Symmetric. A relation R on a set A is symmetric iff ${}_xR_y \Rightarrow {}_yR_x ; x, y \in A$

i.e. $$(x, y) \in R \Rightarrow (y, x) \in R$$

Evidently a relation R on A is symmetric iff $R^{-1} = R$.

e.g. ${}_xR_y$ = '$(x - y)$ is even number' is symmetric since $y - x$ is also even, when $(x - y)$ is even.

III. Transitive. A relation R on a set A is transitive iff

$${}_xR_y \text{ and } {}_yR_x \Rightarrow {}_xR_z \ ; x, y, z \in A$$

i.e. $$(x, z) \in R \text{ and } (y, z) \in R \Rightarrow (x, z) \in R$$

Evidently a relation R on A is transitive iff $RoR \subset R$

e.g. the relation $x < y$ is transitive since if $x < y$ and $y < z$ then $x < z$.

IV. Anti-symmetric. A relation R on a set A is anti-symmetric iff we never have ${}_xR_y$ and ${}_yR_x$ both : $x, y \in A$ except when $x = y$

i.e. $${}_xR_y \text{ and } {}_yR_x \Rightarrow x = y \ ; x, y \in A$$

or $$(x, y) \in R \text{ and } (y, x) \in R \Rightarrow x = y \ ; x, y \in A$$

Evidently a relation R on A is anti-symmetric iff $R \cap R^{-1} \subset \Delta_A$, Δ_A being identity relation or in other words $R \cap R^{-1} = \phi$.

e.g. ${}_xR_y$ = 'x, divides y' in **Z** (set of natural numbers) is anti-symmetric since 'x divides y' and 'y divides x' $\Rightarrow x = y \ ; x, y \in$ **Z**

i.e. $${}_xR_y \text{ and } {}_yR_x \Rightarrow x = y; x, y \in \mathbf{Z}$$

Equivalencé relation. A relation R over a set S is said to be an equivalence realtion if it satifies the following properties :

(*i*) *Reflexivity, i.e.* $\forall \ x \in S$, ${}_xR_x$ or $x \sim x \ \forall \ x$ ($\sim$ called *wiggle*)

(*ii*) *Symmetry, i.e.* ${}_xR_y \Rightarrow {}_yR_x$ or $x \sim y \Rightarrow y \sim x$

(*iii*) *Transitivity, i.e.* ${}_xR_y$ and ${}_yR_x \Rightarrow {}_xR_z$ or $x \sim y$ and $y \sim z \Rightarrow x \sim z$.

Equivalence set (or class). Let R be an equivalence relation in a non-empty set S and let x be an element of S *i.e.* $x \in S$; then the elements $y \in S$ satisfying ${}_yR_x$ constitute a subset of S, known as equivalence set of x w.r.t R, *i.e.*

$$S_x \text{ or } \bar{x} \text{ or } [x] = \{y : y \in S \text{ and } {}_yR_x\}$$

The equivalence set has the following properties:

(*i*) If $z \in [x]$ then $[z] = [x]$

(*ii*) $[x] = [z]$ iff ${}_xR_z$

(*iii*) If $[x] \cap [z] \neq \phi$, then $[x] = [z]$

Partition set. Given a non-empty set S a set $P = \{X, Y, Z...\}$ of non-empty subsets of S is called a partition of S, provided

(*i*) $X \cup Y \cup Z \cup ... = S$

(*ii*) The intersection of every pair of distinct subsets of $S \in \mathbf{P}$ is the null set *e.g.* if $X, Y \in \mathbf{P}$ then either $X = Y$ or $X \cap Y = \phi$.

e.g. if $S = \{1, 2, 3, 4, 5\}$ the $\{1, 3, 5\}, \{2, 4\}$, and $\{1, 2, 3\}, \{4, 5\}$ are two different partitions of S.

Quotient set. If R be an equivalence relation defined on a non-empty set S, then the set of mutually exclusive sets in which S is partitioned w. r. t. the equivalence relation R, is called as quotient set of S for the equivalence relation R and is denoted by $\bar{S}$ or S/R.

e.g. the set $\mathbf{I}$ of all integers for the equivalence relation modulo 5 is the set $\mathbf{I}/R = \{[0], [1], [2], [3], [4]\}$.

Note. Two integers p and q are said to be *congruent modulo m* denoted by $p \equiv q \pmod m$ if $p - q$ is exactly divisible by m *i.e.* $(p - q)$ is an integral multiple of m.

A note on binary operations. A binary operation (or simply composition) usually denoted by o (or sometimes by $\cdot, *, \oplus$ etc.) is used to combine two elements of a set in order to produce another element of the set. In other words a binary operation in a set S is a function $f : S \times S \to S$. *e.g.* the binary operation 'addition' on any two elements of $\mathbf{I}$ (set of integers) gives another integer belonging to $\mathbf{I}$.

(*i*) Such an operation is *commutative* if $xoy = yox \ \forall \ x, y \in S$

(*ii*) It is *associative* if $xo\,(yoz) = (xoy)\,oz \ \forall \ x, y, z \in S$

(*iii*) It is *distributive* w.r.t. another binary operation $\oplus$ if

$$xo\,(y \oplus z) = (xoy) \oplus (xoz) \text{ (left distributive)}$$

and $(y \oplus z)\,ox = (yox) \oplus (zox)$ (right distributive) $\forall \ x, y, z \in S$.

(*iv*) It is said to have *identity element* if $\exists \ e \in S$ s.t. $xoe = eox = x \ \forall \ x \in S$

(*v*) It is *invertible* if corresponding to an $x \in S \ \exists$ a $y \in S$ s.t.

$$xoy = e = yox.$$

i.e. y is the inverse of x w.r.t. 'o' and vice versa.

(*vi*) It satisfies *concellation law* if $\forall \ x, y, z \in S$

$$xoy = xoz \Rightarrow y = z \quad \text{(left cancellation law)}$$

$$yox = zox \Rightarrow y = z \quad \text{(right cancellation law)}$$

e.g. consider two binary operations o and $\cdot$ defined on $\mathbf{I}$ (set of all integers) such that

$$xoy = x + 2y \text{ and } x \cdot y = 2xy \ \ \forall \ x, y \in \mathbf{I}.$$

Then we can verify the above laws on operations 'o' and '$\cdot$' as follows.

(*i*) We have $xoy = x + 2y \quad \forall\, x, y \in \mathbf{I}$

$$= 2y + x$$

$$= yox$$

and $x \cdot y = 2xy = 2yx \quad \forall\, x, y \in I$

$$= y \cdot x$$

i.e. commutative law holds for the operation o and

(*ii*) $xo(yoz) = xo\,(y + 2z) \;\forall\, x, y, z \in \mathbf{I}$

$$= x + 2\,(y + 2z) = x + 2y + 4z$$

and $(xoy)\,oz = (x + 2y)\,oz \;\forall\, x, y, z \in \mathbf{I}$

$$= x + 2y + 2z$$

So that $xo\,(yoz) \neq (xoy)\,oz$ *i.e.* associative law does not hold for 'o'.

Also $x \cdot (y \cdot z) = x \cdot (2yz) \quad \forall x, y, z \in \mathbf{I}$

$$= 2x\ 2yz = 4xyz$$

and $(x \cdot y) \cdot z = (2xy) \cdot z = 4xyz$

So that $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ *i.e.* '$\cdot$' is associative.

(*iii*) $xo\,(y \cdot z) = xo\,(2yz) \qquad \forall\, x, y, z \in \mathbf{I}$

$$= x + 4yz$$

$$(xoy) \cdot (xoz) = (x + 2y) \cdot (x + 2z) = 2(x + 2y)\,(x + 2z)$$

$$= 2x^2 + 4xz + 4xy + 8yz.$$

So that $xo\,(y \cdot z) \neq (xoy) \cdot (xoz)$ *i.e.* 'o' is not left distributive w.r.t. '$\cdot$'.

Also $x \cdot (yoz) = x \cdot (y + 2z) \;\forall\, x, y, z \in \mathbf{I}$

$$= 2x\,(y + 2z)$$

$$= 2xy + 4xz$$

and $(x \cdot y)o(x \cdot z) = (2xy)o(2xz)$

$$= 2xy + 4xz$$

So that $x \cdot (yoz) = (x \cdot y)o(x \cdot z)$ *i.e.* '$\cdot$' is left distributive w.r.t. 'o'

(*iv*) Assuming that $\exists$ an identity e, we have

$$xoe = eox = x \quad \forall\, x \in \mathbf{I}$$

So $xoe = x \Rightarrow x + 2e = x$

$$\Rightarrow e = 0$$

and $eox = x \Rightarrow e + 2x = x$

$$\Rightarrow e = -x$$

These imply that e is not unique and hence 'o' has no identity element.

Again if $x \cdot e = e \cdot x = x \qquad \forall\, x \in \mathbf{I}$, then

$$x \cdot e = x \Rightarrow 2xe = x$$

$$\Rightarrow e = \frac{1}{2}$$

and $e \cdot x = x \Rightarrow 2ex = x$

$$\Rightarrow e = \frac{1}{2}$$

i.e. e is unique and hence '$\cdot$' has an indentity element which is $\frac{1}{2}$.

(v) Since 'o' has no identity element, it is not invertible.

Assuming that '$\cdot$' has an inverse p, we must have

$$x \cdot p = e = p \cdot x \quad \forall x \in \mathrm{I}$$

Now $$x \cdot p = e \Rightarrow 2xp = e = \tfrac{1}{2}$$

$$\Rightarrow p = \frac{1}{4x}$$

and $$p \cdot x = e \Rightarrow 2px = \tfrac{1}{2} \Rightarrow p = \frac{1}{4x}$$

Hence '$\cdot$' is invertible and inverse of x is $\dfrac{1}{4x}$.

(vi) We have $xoy = x + 2y$ and $xoz = x + 2z \quad \forall\, x, y, z \in \mathrm{I}$

So that $xoy = xoz \Rightarrow x + 2y = x + 2z \Rightarrow y = z$ *i.e.* left cancellation law holds for 'o'

and $x \cdot y = 2xy,\ x \cdot z = 2xz \quad \forall\, x, y, z \in \mathrm{I}$

So that $x \cdot y = x \cdot z \Rightarrow 2xy = 2xz \Rightarrow y = z$ *i.e.* left cancellation law holds for '$\cdot$'

Also $yox = y + 2x$ and $zox = z + 2x \quad \forall x, y, z \in \mathrm{I}$

So $yox = zox \Rightarrow y + 2x = z + 2x \Rightarrow y = z$ *i.e.* right cancellation law holds for 'o'.

and $y \cdot x = 2yz,\ z \cdot x = 2zx \quad \forall x, y, z \in \mathrm{I}$

So $y \cdot x = z \cdot x \Rightarrow 2yz = 2zx$

$\Rightarrow y = z$ *i.e.* right cancellation law holds for '$\cdot$'

Hence cancellation law holds for o and $\cdot$ both the binary operations.

4.2. GROUPS

A group is the simplest algebraic structure found in nature wherever symmetry exists.

A group (G, o) is a system consisting of a non-empty set G such as $G = \{a, b, c, \ldots\}$ and a binary operation 'o' satisfying the following axioms :

G_1—(*Closure*). If $a \in G, b \in G$ then $aob \in G$ or in other words if $a \in G, b \in G$ then $aob = c$ (closure) where $c \in G$.

G_2—(*Associativity*). If $a, b, c \in G$ then $ao\,(boc) = (aob)oc$.

G_3—(*Existence of an identity*) If $a \in G$, then $\exists$ an identity element $e \in G$ s.t. $eoa = a \ \forall a \in G$

G_4—(*Existence of an inverse*) If $a \in G$, $\exists$ an inverse $a^{-1} \in G$ s.t. $a^{-1}\, oa = e$ where $e \in G$, being an identity element,

In addition to these four axioms if a fifth axiom of commutativity namely G_5 is also satisfied, *i.e.*

G_5—(*Commutation*). If $a, b \in G$ the $aob = boa$ so that G_3 and G_4 take the forms

G_3: $eoa = aoe = a$

G_4: $aoa^{-1} = a^{-1}\, oa = e$

then the group is said to be an *Abelian group*.

e.g. the set I (of all integers) with the binary operation 'o' taken as additive (+) is a group, for it satisfies all the four axioms

G_1—if $a, b \in \mathrm{I}$, then $a + b \in \mathrm{I}$

—if $a, b, c \in \mathrm{I}$ then $a + (b + c) = (a + b) + c$

G_3—if $a \in$ I then $\exists$ an integer zero (0) such that $0 + a = a$

G_4—if $a \in$ I, then $\exists$ an inverse $(-a)$ s.t. $-a + a = 0$ (identity element).

This group is also abelian or commutative as G_5 is also satisfied *i.e.* $G_5 - a, b \in$ I, $a + b = b + a$.

Finite and Infinite groups. A group $(G. o)$ consisting of a finite number of elements is said to be a *finite group, e.g.* the set $S = \{1, \omega, \omega^2\}$ where $\omega^3 = 1$, is a finite group under multiplication composition.

A group (G, o) consisting of a infinite number of elements is said to be an *infinite group e.g.* the set I (of all integers) is an infinite group under the addition composition.

Order of a group. The number of elements in a finite group is known as the order of the group. The infinite set is said to be of infinite order. As an example the set $\{1, -1\}$ under multiplication composition is a group of order 2.

4.3. ELEMENTARY PROPERTIES OF A GROUP

I. Uniqueness of identity *i.e. the identity element of a group (G, o) is unique.*

If possible let us assume that e and e' are two identity elcments of the group (G, o), then

$$eoa = aoe = a \;\forall\; a \in G \quad ...(1)$$

and

$$e'oa = aoe' = a \;\forall\; a \in G \quad ...(2)$$

Putting $a = e'$ in (1), we have

$$eoe' = e'oe = e' \quad ...(3)$$

Also putting $a = e$ in (2), we have

$$e'oe = eoe' = e \quad ...(4)$$

From (3) and (4) it follows that $e' = e$ *i.e.* there cannot be two identity elements for (G, o) and hence the identity element of a group is unique.

II. Uniqueness of Inverse *i.e. in a group (G, o) every element possesses a unique inverse.*

If possible let us assume that a' and a^{-1} are two inverses of a. Also let $a \in G$ and e be the identity element of G. Then we have

$$a^{-1}oa = aoa^{-1} = e \quad ...(1)$$

$$a'oa = aoa' = e \quad ...(2)$$

Post-multiplying (1) by a' we get

$$(a^{-1}oa)\,oa' = (aoa^{-1})\,oa' = eoa' = a' \quad ...(3)$$

and premultiplying (2) by a^{-1} we get

$$a^{-1}o\,(a'oa) = a^{-1}o\,(aoa') = a^{-1}oe = a^{-1} \quad ...(4)$$

But group-postulate G_2 gives

$$(a^{-1}oa)\,oa' = a^{-1}o\,(aoa')$$

$\therefore$ (3) and (4) follow that $a' = a^{-1}$ *i.e.* the inverse of an element in a group is unique.

Aliter. Taking a^{-1} and a' two inverses of $a \in G$, we have

$$aoa' = a'oa = e \text{ and } a^{-1}oa = aoa^{-1} = e$$

$$\therefore \quad a^{-1} = a^{-1}oe = a^{-1}o\,(aoa') = (a^{-1}oa)\,oa' = eoa' = a'$$

which follows the uniqueness of inverse of $a \in G$.

III. Cancellation laws *i.e., for any group (G, o), and $a, b, c \in G$, the following laws hold*

(*i*) $aob = aoc \Rightarrow b = c$ (*left cancellation law*)

(*ii*) $boa = coa \Rightarrow b = c$ (*right cancellation law*) (*i*) Taking $a^{-1} \in G$ as the inverse of $a \in G$, we have

$aob = aoc \Rightarrow a^{-1} o (aob) = a^{-1} o (aoc)$ on premultiplying by a^{-1}

$\Rightarrow (a^{-1} oa) ob = (a^{-1} oa) oc$, the composition being associative by G_2

$\Rightarrow eob = eoc$, since $\exists$ an indentity element $e \in G$ for 'o'

$\Rightarrow b = c$.

(*ii*) Again taking $a^{-1} \in G$ as the inverse of $a \in G$, we have

$boa = coa \Rightarrow (boa)oa^{-1} = (coa)oa^{-1}$ on postmultiplying by a^{-1}

$\Rightarrow bo (aoa^{-1}) = co (aoa^{-1})$ by G_2

$\Rightarrow boe = coe \quad \because \exists$ an identity element $e \in G$ for 'o'

$\Rightarrow b = c$.

Note. $aoc = cob \Rightarrow a = b$ unless the group is abelian.

IV. Uniqueness of solutions, *i.e. if $a, b \in G$, then the equations $aox = b$ and $yoa = b$ have unique solutions in G.*

If a^{-1} be the inverse of $a \in G$, then $a^{-1} \in G$ and $aoa^{-1} = e$ (identity element).

Now $a^{-1} \in G$ and $b \in G \Rightarrow a^{-1} ob \in G$

Putting $x = a^{-1} ob$ in the equation $aox = b$ we get

$$ao (a^{-1} ob) = b$$

or $$(aoa^{-1})ob = b \text{ by } G_2$$

or $$eob = b \text{ i.e. } b = b$$

which follows that $x = a^{-1} ob$ is a solution of $aox = b$.

To show that this solution is unique, let us assume that y is an element different from $a^{-1} ob$ in G s.t. it satisfies the equation $aox = b$. Then,

$$aoy = b = eob = (aoa^{-1}) ob = ao (a^{-1} ob) \text{ by } G_2.$$

So that left cancellation law yields $y = a^{-1} ob$.

As such $x = y$ *i.e.* the solution is unique.

Again, $b \in G$ and $a^{-1} \in G \Rightarrow boa^{-1} \in G$

Putting $y = boa^{-1}$ in $yoa = b$, we get

$(boa^{-1}) oa = b$ or $bo (a^{-1} oa) = b$ by G_2

or $$boe = b \text{ i.e. } b = b$$

which follows that $y = boa^{-1}$ is a solution of $yoa = b$.

To show that this solution is unique, let us assume that z is an element different from boa^{-1} in G s.t. it satisfies the equation $yoa = b$.

Then $\quad zoa = b = boe = bo (a^{-1} oa) = (boa^{-1}) oa$ by G_2.

The right cancellation law gives $z = boa^{-1}$

So that $y = z$ and hence the solution is unique.

Note. The unique solution of $xox = x$ is $x = e$ in group (G, o)

V. Inverse of the inverse is itself *i.e. if $a \in G$ then $(a^{-1})^{-1} = a$.*

Inverse law gives $(a^{-1})^{-1} oa^{-1} = e$, e being identity element in G.

Postmultiplying by a, we get

$$[(a^{-1})^{-1} oa^{-1}]oa = eoa$$

or $$(a^{-1})^{-1}o(a^{-1}oa) = a \text{ by } G_2$$

or $$(a^{-1})^{-1}oe = a \text{ by } G_4$$

or $$(a^{-1})^{-1} = a \text{ by } G_3$$

which proves the proposition.

VI. Reversal law *i.e. if* $a, b \in G$ *then* $(aob)^{-1} = b^{-1}oa^{-1}$.

Let e be the identity element in G.

Now $a \in G \Rightarrow a^{-1} \in G$ and $b \in G \Rightarrow b^{-1} \in G$

$$\therefore \quad (b^{-1}oa^{-1})\, o(aob) = b^{-1}o\,[a^{-1}\,o(aob)] \text{ by } G_2$$
$$= b^{-1}o\,[(a^{-1}oa)\,ob] \text{ by } G_2$$
$$= b^{-1}\,o\,[eob] \text{ by } G_4$$
$$= b^{-1}ob \text{ by } G_3$$
$$= e.$$

Hence by definition of inverse element of a group $b^{-1}oa^{-1}$ is the inverse of aob *i.e.*

$$(aob)^{-1} = b^{-1}\,oa^{-1}.$$

Note. The result may be generalized for any number of elements $a_1, a_2, a_3, \ldots a_n \in G$, where we have.

$$(a_1oa_2o\ldots oa_n)^{-1} = a_n^{-1}\,o\,a^{-1}{}_{n-1}\,o\ldots oa_2^{-1}\,oa_1^{-1}.$$

4.4. SOME DEFINITIONS

Semi-group. A set S with a binary operation 'o' is said to be a semi-group if it satisfies the following two axioms.

S_{G1} – (*Closure*). $a \in S, b \in S \Rightarrow aob \in S$.

S_{G2} – (*Associativity*). If $a, b, c \in S$ then $(aob)\,oc = ao\,(boc)$.

THEOREM. A semi-group (G, o) *satisfying the following postulates is a group.*

(1) G *has a left identity* e *s.t.* $eoa = a\ \forall\ a \in G$.

(2) *Every element* a *in* G *has a left inverse* a^{-1} *in* G *s.t.* $a^{-1}\,oa = e$.

Since (G, o) is a semi-group, therefore by definition it follows that

(*i*) (G, o) satisfies the closure law.

(*ii*) (G, o) satisfies the associative law.

(*iii*) a^{-1} being the left inverse of a and e the left identity we have

$$a^{-1}\,o\,(aoe) = (a^{-1}oa)\,oe \quad \text{by } (ii)$$
$$= eoe \quad \text{by postulate (2)}$$
$$= e \quad \text{by postulate (1)}$$
$$= a^{-1}\,oa \quad \text{by postulate (2)}$$

So that by left cancellation law, $aoe = a$
which follows that e is also a right identiy.

Hence the indentity element exists for the composition.

(*iv*) We have $a^{-1}\,o\,(aoa^{-1}) = (a^{-1}oa)\,oa^{-1}$ by (*ii*)

or $\quad a^{-1}o\,(aoa^{-1}) = eoa^{-1}$ by (2)

$$= a^{-1} \text{ by (1)}$$
$$= a^{-1}\,oe, \text{ since identity element exists for '}o\text{'},$$

$\therefore \quad aoa^{-1} = e$ by left cancellation law.

Which shows that a^{-1} is also a right inverse of a.

Hence every element of G has an inverse.

Since all the four group axioms are satisfied, therefore a semi-group with the given two postulates is a group.

Sub-group. *A sub-group of a group* (G, o) *is any collection of elements of* G *satisfying the axioms of* G. *In other words a non-empty subset say* H *of a group* G *is said* *to be the sub-group of G, if the binary operation 'o' in G induces a binary operation in H and the elements of H obey the group axioms.*

In other words *a* non-empty subset H of a group G is said to be a sub-group of G if it satisfies the following two axioms:

(*i*) $a, b \in H \Rightarrow aob \in H$,

(*ii*) $a \in H \Rightarrow a^{-1} \in H$

e.g., the set of even integers is a sub-group of the additive group of integers.

Proper sub-group. A sub-group of a group (G, o) other than G itself and the group consisting of the identity element alone is termed as a proper sub-group of G. *e.g.* the additive group of integers is a proper sub-group of the additive group of rational numbers.

Improper or trivial sub-groups. The group (G, o) itself and the group consisting of identity alone *i.e.* $(\{e\}, o)$ are known as trivial or improper sub-groups of (G, o).

Order of an element of a group. Let a be an element of a group (G, o) *i.e.* $a \in G$. Then the order of a is the least positive integer n s.t. $a^n = e$.

In case $\nexists$ such integer, the order of a is said to be zero or infinite. *e.g.* the order of the element -1 in the multiplicative group $\{1, -1, i, -i\}$ is 2 since $(-1)^2 = 1$, the identity element. The order of i is 4 since $i^4 = 1$.

Addition modulo m, (m being an integer). If a and b are two integers and m a positive integer, then 'addition modulo m' is denoted by $a +_m b$ and defined as $a +_m b = r$, $o \le r < m$ where r is the least positive remainder obtained on dividing the sum of a and b by m.

e.g. $12 +_3 5 = 2$ since $12 + 5 = 3\,(5) + 2$

and $-5 +_4 10 = 1$ since $-5 + 10 = 4\,(1) + 1$

Multiplication modulo p, (p being a prime). If a and b are two integers and p, a positive integer, then 'multiplication modulo p', is denoted by $a \times_p b$ and defined as

$a \times_p b = r, o \le r < p$ where r is the remainder obtained on dividing the ordinary product ab by p.

e.g. $9 \times_6 7 = 3$ since $9 \times 7 = (6)\,10 + 3$

and $-7 \times_5 8 = 4$ since $-7 \times 8 = (5)\,(-12) + 4$

Group table or composition table. It is commonly observed that a 'table' is a convenient way of either defining a binary operation in a finite set S or tabulating the effect of a binary operation in a set S. In forming a table or say a group table we arrange the elements of a group in rows and columns of a square array such that each element of the group occurs once and only once in each row or column. The composition element aob occurs at the intersection of row and column of the elements a and b of the group after the binary operation has been performed. For example consider a set

$S = \{1, 2, 3\}$ and let '·' be the binary operation in S defined by

$\therefore$ $(1, 1) \to 1, (1, 2) \to 2, (1, 3) \to 3, (2, 1) \to 2, (2, 2) \to 1, (2, 3) \to 2,$

$(3, 1) \to 3, (3, 2) \to 2, (3, 3) \to 1$

then these operations can be arranged in a table as follows:

·	1	2	3
1	1	2	3
2	2	1	2
3	3	2	1

It is clear that (i, j)th square $(i, j = 1, 2, 3)$ is the intersection of ith row (*i.e.* row labelled or faced by i) and jth column (*i.e.* column labelled or faced by j) and in this square we have put the element obtained by the binary operation '·' on (i, j) such as, $\therefore$ $(1, 2) \to 2$

As another example if $a, b \in G$ and e be an identity element of the group (G, o), such that $aoa = b$ and $aob = e$ etc., then the group table is as shown here.

o	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

Problem 1. *Show that three cube roots of unity form an abelian finite group under multiplication.*

We have the set $G = \{1, \omega, \omega^2\}$, where $\omega^3 = 1$.

	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

$\because \omega^3 = 1$

and $\omega^4 = \omega^3 \cdot \omega = \omega$

The composition table under multiplication is as shown here.

The set G forms an abelian finite group, since it satisfies all the five axioms :

G_1 – since all the elements in group table belong to G, hence closure axiom is true.

G_2 – since multiplication of complex numbers is associative, therefore G_2 is satisfied.

G_3 – since $\exists$ an identity element $1 \in G$, G_3 is satisfied.

G_4 – since the inverses of $1, \omega, \omega^2$ are respectively $1, \omega^2, \omega \in G$, G_4 is satisfied.

G_5 – commutative property is apparently satisfied since $1 \cdot \omega = \omega \cdot 1 = \omega$ etc.

Moreover the set consists of finite number of elements and hence G_1 is an abelian finite group.

Problem 2. *Show that the set of all nth roots of unity form a finite abelian group G of order n under ordinary multiplication as composition.*

By De Moivre's theorem, nth roots of unity are given by

$$(1)^{1/n} = (\cos 2\, r\pi + i \sin 2r\pi)^{1/n}$$

$$= \cos \frac{2r\pi}{n} + i \sin \frac{2r\pi}{n}, \text{ where } r = 0, 1, 2, \ldots, n-1$$

So n, nth roots of unity are

$$1, \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}, \cos 2\frac{2\pi}{n} + i \sin \frac{2.2\pi}{n}, \ldots, \cos \frac{(n-1)\cdot 2\pi}{n} + i \sin \frac{(n-1)\cdot 2\pi}{n}$$

i.e. $1, e^{1.2\pi i/n}, e^{2.2\pi i/n}, e^{3.2\pi i/n}, \ldots, e^{(n-1)\cdot 2\pi i/n}$

Now, G_1 is satisfied since the product of any two elements of the set is the element of the set such as if $a = e^{p 2\pi i/n}$, $b = e^{q \cdot 2\pi i/n} \in G$, where $0 \le p \le n-1$, $0 \le q \le n-1$, then $a.b = e^{2\pi i/n\,(p+q)}$ will belong to G if $p + q \le n - 1$. Let us assume the contrary *i.e.* $p + q > n - 1$ so that $p + q = n + m$ where $m \le n-2$ since the maximum value of $p + q$ can be $2(n-1)$ *i.e.* $2n - 2$.

$$\therefore \quad a \cdot b = e^{2\pi i/n(n+m)} = e^{2\pi i}\, e^{2\pi i m/n} = e^{2\pi i m/n}$$

$$\because \quad e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1$$

which follows that $a \cdot b \in G$ since $m \le n - 2$.

G_2 is satisfied since multiplication of complex numbers is associative.

G_3 is satisfied since there exists an identity element $e^{2\pi i \cdot 0/n} = 1$

G_4 is satisfied since $\exists$ inverse of $e^{2\pi i r/n}$ as $e^{2\pi i(n-r)/n}$ since

$$e^{2\pi i r/n}\, e^{2\pi i(n-r)/n} = e^{2\pi i n/n} = e^{2\pi i} = 1$$

G_5 is also satisfied since the multiplication of complex numbers is commutative.

Moreover the set consists of finite number of elements. Hence (G, o) is a finite abelian group.

Problem 3. *Show that the set of matrices* $A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, *where* α *is real, forms a group under multiplication.*

Let G be the set of matrices given by $A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, α being real and A_α, $A_\beta, A_\gamma \in G$. Then,

G_1 is satisfied since the product of any two matrices of the set belongs to the set, as

$$A_\alpha \cdot A_\beta = \begin{bmatrix} \cos\alpha - \sin\alpha \\ \sin\alpha \;\; \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta - \sin\beta \\ \sin\beta \;\; \cos\beta \end{bmatrix} = \begin{bmatrix} \cos(\alpha+\beta) - \sin(\alpha+\beta) \\ \sin(\alpha+\beta) \;\; \cos(\alpha+\beta) \end{bmatrix}$$

$$= A_{\alpha+\beta},\ \alpha+\beta \text{ being real.}$$

$\because$ α, β are real and so is $\alpha+\beta$, $\therefore$ $A_{\alpha+\beta} \in G$

G_2 is satisfied since $A_\alpha \cdot (A_\beta \cdot A_\gamma) = A_\alpha \cdot A_{\beta+\gamma} = A_{\alpha+\beta+\gamma} = A_{\alpha+\beta} \cdot A_\gamma = (A_\alpha \cdot A_\beta) \cdot A_\gamma$ *i.e.* the operation is associative.

G_3 is satisfied since $\exists$ an identity element $A_0 \in G$, o being real such that

$$A_o \cdot A_\alpha = A_{o+\alpha} = A_\alpha \ \forall\ A_\alpha \in G.$$

G_4 is satisfied since $\exists$ an inverse $A_{-\alpha} \in G \ \ \forall\ A_\alpha \in G$, $-\alpha$ being real as α is real, such that

$$A_{-\alpha} \cdot A_\alpha = A_{-\alpha+\alpha} = A_0 \text{ (the identity element)}$$

G_5 is also satisfied since $A_\alpha \cdot A_\beta = A_{\alpha+\beta} = A_{\beta+\alpha} = A_\beta \cdot A_\alpha$; α, β being real *i.e.* the operation is commutative.

Hence the given set of matrices forms an abelian group.

Problem 4. *If OX, OY be the two rectangular axes in the cartesian plane and* T_α *denotes the rotation of the axes through an angle* α *s.t.*

$$T_\alpha : (x, y) \to (x\cos\alpha + y\sin\alpha, -x\sin\alpha + y\cos\alpha)$$

then show that the set of these rotations w.r.t. the operation 'o' s.t. $T_\beta o T_\alpha$ *is the resultant of two such operations, forms a group.*

Let $G = \{T_\alpha : (x, y) \to (x\cos\alpha + y\sin\alpha, -x\sin\alpha + y\cos\alpha)\}$

G_1 is satisfied since if $T_\alpha \in G, T_\beta \in G$ then $T_\beta o T_\alpha \in G$ as

$T_\beta o T_\alpha (x, y) = T_\beta [T_\alpha (x, y)]$

$= x\cos(\alpha+\beta) + y\sin(\alpha+\beta), -x\sin(\alpha+\beta) + y\cos(\alpha+\beta)$ ...(1)

$= T_\alpha o\, T_\beta (x, y)$ for any (x, y) in the plane

G_2 is satisfied since if $T_\alpha, T_\beta, T_\gamma \in G$, then

$T_\gamma o (T_\beta o T_\alpha) = T_\gamma [x\cos(\alpha+\beta) + y\sin(\alpha+\beta), -x\sin(\alpha+\beta) + y\cos(\alpha+\beta)]$ by (1)

$= x\cos(\alpha+\beta+\gamma) + y\sin(\alpha+\beta+\gamma), -x\sin(\alpha+\beta+\gamma)\ y\cos(\alpha+\beta+\gamma)$

$= T_\alpha o (T_\beta o T_\gamma)$ etc..

G_3 is satisfied since $\exists$ an identity $T_0 \in G$ s.t. $T_0 o T_\alpha = T_\alpha \forall\ T_\alpha \in G$

G_4 is satisfied since $\exists$ an inverse $T_{-\alpha} \in G$ s.t. $T_{-\alpha} o T_\alpha = T_0 \forall\ T_\alpha \in G$

G_5 is satisfied by (1)

Hence the set of rotations form an abelian group.

Problem 5. *Prove that residue classes modulo m form a group w.r.t. addition of residue classes.*

If S be the set of residue classes modulo m, then we have

$$S = \{[0], [1], [2], \ldots [m-1]\}$$

G_1 is satisfied since if $[r_1], [r_2] \in S$ and $r_1 + r_2 = mk + r$ where r is the least positive remainder when $(r_1 + r_2)$ is divided by m and $o \le r < m$, then

$$[r_1] + [r_2] = [r_1 + r_2] = [r] \in S. \qquad \because\ r_1 + r_2 \equiv r \pmod m \text{ or } r_1 +_m r_2 = r$$

i.e. closure axiom is satisfied.

G_2 is satisfied since if $[r_1], [r_2], [r_3], \in S$ and $r_2 + r_3 = mk + r$ so that $r_1 + r_2 + r_3 = r_1 + mk + r = mk + (r_1 + r) = mk + r'$ (say), r' being least positive remainder when $(mk + r_1 + r)$ *i.e.* $r_1 + r_2 + r_3$ *i.e.* $r_1 + r$ is divided by m, then

$$\begin{aligned} [r_1] + ([r_2] + [r_3]) &= [r_1] + [r_2 + r_3] \\ &= [r_1] + [r] && \because\ r_1 + r_2 \equiv r \pmod m \\ &= [r_1 + r] = [r'] && \because\ r_1 + r \equiv r' \pmod m \\ &= [r_1 + r_2 + r_3] \\ &= [r_1 + r_2] + [r_3] \\ &= ([r_1] + [r_2]) + [r_3] \end{aligned}$$

i.e. addition of residue classes is associative.

G_3 is satisfied since $\exists$ an additive identity $[o] \in S$ s.t. $[o] + [r]\ \forall\ [r] \in S$

G_4 is satisfied since $\exists$ an additive inverse $[m - r] \in S$ s.t.

$$[r] + [m - r] = [m] = [o]\ \forall\ [r] \in S \text{ as } m \equiv o \pmod m.$$

Hence the set of residue classes modulo m from a group, w.r.t. addition.

Problem 6. *Prove that the non-zero residue classes modulo m (a prime integer) w.r.t. multiplication form a group.*

If S be the set of non-zero residue classes modulo m, then we have

$$S = \{[1], [2]\ [3], \ldots [r] \ldots, [m-1]\}, 0 < r \le m - 1.$$

Defining the multiplication of classes $[r_1], [r_2], [r_3] \in S$ *by* $[r_1]\ [r_2] = [r_3]$ where $o < r_1, r_2, r_3 \le m - 1$, we observe that

G_1 is satisfied since r_1, r_2 are prime to m and division of $r_1 r_2$ by m renders a non-zero remainder and so if $[r_1], [r_2] \in S$ then

$$[r_1]\ [r_2] \text{ i.e. } [r_3] \in S.$$

G_2 is satisfied since if $[r_1], [r_2], [r_3] \in S$ then taking $r_1 . r_2 = mk + r$ we have $(r_1 r_2)\ r_3 = (mk + r)\ r_3 = mkr_3 + rr_3 = p + r'$ (say) where r' is the least positive remainder when $(r_1 r_2)\ r_3$ or $r\ r_3$ is divided by m. So that

$$([r_1] \cdot [r_2])\ [r_3] = [r_1 r_2]\ [r_3] = [r]\ [r_3] \qquad \because\ r_1 r_2 \equiv r \pmod m$$

$$= [r\ r_3] = [r'] \quad \because\ r\ r_3 \equiv r' \pmod m$$

But $r_1 r_2 r_3 \equiv r' \pmod m$ $\therefore$ $[r'] = [r_1 r_2 r_3] = [r_1]\ [r_2 r_3] = [r_1]\ ([r_2]\ [r_3])$

i.e. $([r_1] \cdot [r_2]) \cdot [r_3] = [r_1]\ ([r_2]\ [r_3])$.

thereby showing that the associative law is satisfied.

G_3 is satisfied since $\exists$ an identity $[1] \in S$ s.t. $[1]\ [r] = [r]\ \ \forall\ [r] \in S$

G_4 is satisfied since $\exists$ inverse of each element, as is shown below :

Multiplying each element of S by an element $[r]$, we find

$$[1]\,[r],\ [2]\,[r], \ldots\ldots, [m-1]\,[r] \qquad \ldots(1)$$

By G_1 (closure axiom) all these $(m-1)$ elements must belong to S. Also all of them should be distinct otherwise if

$$[r_1]\,[r] = [r_2]\,[r], \qquad [r_1], [r_2] \in S$$

then left cancellation law gives $[r_1] = [r_2]$ which contradicts the hypothesis that $[r_1], [r_2]$ are distinct. Hence all the $(m-1)$ elements of (1) must be distinct and they must also be the same elements of S as already defined except that their order may be different. Conclusively in (1) there is one element which is the identity [1]. Suppose this identity element is $[r_1]\,[r] = [1]$.

Which shows that $[r_1]$ is the inverse of $[r]$.

But r_1 being arbitrary, the inverse of each element exists. Hence the non-zero residue classes modulo m w.r.t. multiplication form a group.

Problem 7. *If every element of a group (G, o) is its own inverse then show that (G, o) is abelian.*

Given that (G, o) is a group.

$\therefore$ if $a, b \in G$ then $a^{-1}, b^{-1} \in G$

also if $aob \in G$ then $(aob)^{-1} \in G$

But we have $a = a^{-1}$ and $b = b^{-1}$

As such $(aob) = (aob)^{-1} = b^{-1}\,oa^{-1} = (boa)$

i.e. (G, o) is commutative. Hence (G, o) is abelian.

Problem 8. *If (G, o) be a group and $a^2 = e$ (identity) $\forall\ a \in G$, then show that the group must be commutative.*

Given that (G, o) is a group and $a^2 = aoa = e$ also $aoa^{-1} = e$

$\therefore \quad aoa = e = aoa^{-1}$

So that left cancellation law gives $a = a^{-1}$

i.e., every element of the group is its own inverse and hence by Problem 7 it follows that the group (G, o) is commutative.

Problem 9. *Show that if a group has 3, 4 or 5 elements, then it is abelian.*

We prove the proposition for 4 elements, similar procedure can be adopted for other two.

Suppose that $G = \{e, a, b, c\}$ is a set forming the group $\{G, o\}$ where e is the identity element.

In case every element of G is its own inverse, the problem reduces to the problem 7 which has been already discussed. Consider the other case.

Let $a^{-1} = b$. Then the only alternative is that $c^{-1} = c$, so that

$aob = boa = e$ and $coc = e$ ...(1)

o	e	a	b	c
e	e	a	b	c
a	a	c	e	b
b	b	e	c	a
c	c	b	a	e

Now

$aoc \neq e$ as $c^{-1} \neq a$

$aoc \neq$ as $c \neq e$

$aoc \neq c$ as $a \neq e$

So the only alternative is that $aoc = b$.

Similar argument will give that $coa = b$

$\therefore\ aoc = coa$ ...(2)

Also

$$boc \neq e \text{ as } b^{-1} \neq c$$
$$boc \neq b \text{ as } c \neq e$$
$$boc \neq c \text{ as } b \neq e$$

leading that $boc = a$ and similarly $cob = a$

$\therefore\ boc = cob$ ...(3)

(1), (2) and (3) follow that the group (G, o) is commutative and hence it is abelian. The group table is as shown here.

Problem 10. *Show that any non-commutative group has at least six elements.*

Let (G, o) be a non-commutative group. It wll be so if it has at least one pair of non-commuting elements a and b (say).

We shall first show that a set $\{e, a, b, aob, boa\}$ having a, b non-commuting elements *i.e.* $aob \neq boa$, consists of distinct elements. Taking two at a time, there are ten possibilities leading to a contradiction of $aob \neq boa$:

(*i*) $e = a \Rightarrow aob = eob = b = boe = boa.$

(*ii*) $e = b \Rightarrow aob = aoe = a = eoa = boa$

(*iii*) $e = aob \Rightarrow aoe = eoa = (aob)\, oa = ao\, (boa)$

i.e. $e = boa$ or $aob = boa$

(*iv*) $e = boa \Rightarrow eoa = aoe = ao\, (boa) = (aob)\, oa$ *i.e.* $e = aob$ or $boa = aob$

(*v*) $a = b \Rightarrow aob = aoa = boa$

(*vi*) $a = aob \Rightarrow e = b$ thereby reducing to (*ii*)

(*vii*) $a = boa \Rightarrow e = b$ thereby reducing to (*ii*)

(*viii*) $b = aob \Rightarrow e = a$ thereby reducing to (*i*)

(*ix*) $b = boa \Rightarrow e = a$ thereby reducing to (*i*)

(*x*) $aob = boa$

Hence the elements of the set $\{e, a, b, aob, boa\}$ having (a, b) non-commuting, are all distinct.

We shall now show that at least one of the group elements aoa or $ao\, boa$ is distinct from these five namely e, a, b, aob, boa.

To show that aoa is different from each element a, b, aob, boa, we see that.

(*xi*) $aoa = a \Rightarrow a = e$ thereby reducing to (*i*)

(*xii*) $aoa = b \Rightarrow aob = ao\, (aoa) = (aoa).oa = boa$

(*xiii*) $aoa = aob \Rightarrow a = b$ thereby reducing to (*v*)

(*xiv*) $aoa = boa \Rightarrow a = b$ thereby reducing to (*v*)

These possibilities lead that either $aoa \neq e$ in which case aoa is the sixth distinct element of G or else $aoa = e$

Again we shall show that $ao\, boa$ is different from each element e, a, b, aob, boa so that it will be the sixth distinct element of G.

Obvisously $ao\, (ao\, boa) = (aoa)\, o\, (boa) = eo\, (boa) = boa.$

Now consider the case

(*xv*) $ao\, boa = e \Rightarrow boa = ao\, (ao\, boa) = aoe = o$ thereby reducing to (*vii*)

(*xvi*) $ao\, boa = a \Rightarrow aob = e$ thereby reducing to (*iii*)

(*xvii*) $ao\, boa = b \Rightarrow aob = ao\, (ao\, boa) = boa$ when $aoe = e$

(*xviii*) $ao\ boa = aob \Rightarrow a = e$ thereby reducing to (*i*)

(*xix*) $ao\ boa = boa \Rightarrow a = e$ thereby reducing to (*i*)

Conclusively a group with upto 5 elements is essentially abelian but for it to be non-abelian there should be at least six elements.

Problem 11. *Show that non-empty semi-group (G, o) forms a group if the equations $ax = b$ and $ya = b$ have unique solutions in G $\forall$ pair of elements $a, b \in G$.*

Since $ya = b$ is solvable for any $b \in G$, therefore by taking $b = a$, we find that $ya = a$ has a solution in G. Call this solution as e_1 so that $e_1 a = a$ where a is a fixed element of G.

Let $c \in G$, then $ax = c$ has a solution in G.

Thus $e_1 c = e_1 (ax) = (e_1 a) x = ax = c$

which follows that $e_1 c = c \ \forall \ c \in G$ *i.e.* e_1 is the left identity in G.

As e_1 exists in G, so $ya = e_1$ has a solution in G. Call this solution as a^{-1}. This follows that every element in G has a left inverse relative to the left identity. Hence by the theorem on §4.4, it follows that (G, o) is a group.

Problem 12. *Show that a finite-non-empty semi-group (G, o) forms a group if $ab = ac \Rightarrow b = c$ and $ba = ca \Rightarrow b = c \ \forall \ a, b, c \in G$.*

Consider a set $G = \{a_1, a_2, \ldots a_r, \ldots a_p\}$ consisting of p distinct elements. Take an element a_m and multiply it to all the elements of this group.

$$a_m a_1,\ a_m a_2, \ldots a_m a_r, \ldots a_m a_p.$$

All these elements will be distinct save possibly arranged in different order. If possible let us assume that

$$a_m a_r = a_m a_p \Rightarrow a_r = a_p$$

which contradicts the hypothesis that a_r and a_p are distinct elements of G. Thus

$G = \{a_m a_1, a_m a_2, \ldots, a_m a_r, \ldots a_m a_p\}$ consists of p distinct elements and $a_m a_1$ will be some element say a_r of G *i.e.*

$$a_m a_1 = a_r \Rightarrow ax = b \text{ has a unique solution in } G$$

Similarly we can show that

$G = \{a_1 a_m, a_2 a_m, \ldots, a_r a_m, \ldots a_p a_m\} \Rightarrow ya = b$ has a unique solution in G.

Hence by Problem 11, the semi-group (G, o) under given conditions forms a group.

Problem 13. *Show that the set of subsets of a set with the union composition is a semi-group.*

If $S_1 = \{A, B, C, \ldots\}$ be the set of subsets of a set S, then

S_{G1} is satisfied since, $A, B \in S$, and $A \subset S, B \subset S \Rightarrow A \cup B \subset S$ and $A, B \in S_1$, $\Rightarrow A \cup B \in S_1$ *i.e.* the closure law is satisfied.

S_{G2} is satisfied since if $A, B, C \in S$, then associative property of union yields, $(A \cup B) \cup C = A \cup (B \cup C)$

Hence S_1 is a semi-group (by def. § in 4.4).

Problem 14. *Show that the identity of a subgroup of a group is the same as that of the group.*

Let (H, o) be a subgroup of the group (G, o) and let e, e' be the identities of (G, o) and (H, o) respectively. Then

$$aoe' = a \ \forall \ a \in H$$

This equality will also hold in (G, o) as $a \in H \Rightarrow a \in G$.

Now if b be the inverse of $a \in G$, then we have

$$aoe' = a \Rightarrow bo\,(aoe')$$
$$\Rightarrow (boa)\,oe' = boa \text{ by } G_2 \text{ for } G$$
$$\Rightarrow eoe' = e \qquad boa = e$$
$$\Rightarrow e' = e.$$

Problem 15. *Show that the inverse of an element of a subgroup of a group is the same as the inverse of the same element regarded as an element of the group.*

Let (H, o) be a subgroup of the group (G, o) and let b_1 and b_2 be the inverses of an element a as member of H and G respectively. Also let e and e' be the identities of G and H respectively. Then by Problem 14, $e = e'$,

Now $\quad aob_1 = e' = e \Rightarrow b_2o\,(aob_1) = b_2oe$

$$\Rightarrow (b_2oa)\,ob_1 = b_2 \text{ by } G_2, G_3 \text{ for } G.$$
$$\Rightarrow eob_1 = b_2 \qquad \because\ b_2oa = e$$
$$\Rightarrow b_1 = b_2.$$

Problem 16. *Show that the necessary and sufficient conditions for a complex H to be a subgroup (H, o) of a group (G, o) are*

(i) $a_1\, b \in H \Rightarrow aob \in H\ \forall\ a, b$; and (ii) $a \in H \Rightarrow a^{-1} \in H\ \forall\ a$

(*Rohilkhand, 1976*)

The *conditions are necessary,* since (H, o) being a subgroup of (G, o) the composition in H (being also the composition in G) satisfies the closure law

i.e. $\quad a, b \in H \Rightarrow aob \in H\ \forall\ a, b$

which proves the first condition.

Also by Problem 14, the identity of H being the same as that of G and by Problem 15, the inverse of any element of H being the same as its inverse in G, we have

$$a \in H \Rightarrow a^{-1} \in H\ \forall\ a$$

which proves the second condition.

The *conditions are also sufficient,* since if the conditions (*i*) and (*ii*) hold then

G_1 is satisfied, for $\quad a, b \in H \Rightarrow aob \in H$ by condition (*i*).

G_2 is satisfied, for $\quad a, b \in H \Rightarrow aob \in H$ by (*i*) leads to

$$aob, c \in H \text{ and } a, boc \in H\ \forall\ a, b, c \in H$$

$\Rightarrow$ the same element $aoboc \in H$ *i.e.* associative law is satisfied.

G_3 is satisfied since $\quad a \in H \Rightarrow a^{-1} \in H$ by (*ii*) leads to

$$a \in H \text{ and } a^{-1} \in H \Rightarrow aoa^{-1} \in H \text{ by } (i)$$

But $aoa^{-1} = e$, (identity of G) ,

$\therefore\ e \in H$ is an identity in H, which is also identity in G, thereby showing the existence of an identity element in H.

G_4 is satisfied since from G_3 and condition (*ii*), every element of H has an inverse.

Hence (H, o) which is a sub-group of the group (G, o) satisfies all the four exioms of group.

Problem 17. *Show that a necessary and sufficient condition for a complex H to be a subgroup (H, o) of a group (G, o) is that $a \in H, b \in H \Rightarrow aob^{-1} \in H$.*

The *condition is necessary,* since when (H, o) is a subgroup of (G, o) then by condition (*ii*) of Problem 16, we have $b \in H \Rightarrow b^{-1} \in H$

Also by condition (*i*) of the Problem 16, $a, b^{-1} \in H \Rightarrow aob^{-1} \in H$.

Combining these two conditions we have $a \in H,\ b \in H \Rightarrow aob^{-1} \in H$.

The *condition is sufficient,* since if $a, b \in H \Rightarrow aob^{-1} \in H$, then we can show as below that (H, o) is a sub-group of (G, o).

The given condition yields,

$a \in H, e \in H \Rightarrow aoa^{-1} = e \in H$, e being identity of G.

This follows that G_3 is satisfied i.e. $\exists$ an identity $e \in H$.

Also $e \in H, a \in H \Rightarrow eoa^{-1} = a^{-1} \in H$

i.e. G_4 is satisfied or in other words every element in H is invertible.

As such any $b \in H \Rightarrow b^{-1} \in H$

So that $a \in H, b^{-1} \in H \Rightarrow ao\,(b^{-1})^{-1} = aob \in H$

which follows that H satisfies closure law under 'o' *i.e.* G_1 is satisfied.

Now associativity of G w.r.t 'o' immediately follows the associativity of H w.r.t. 'o' *i.e.* G_2 is satisfied.

Hence (H, o) is a group.

But (H, o) is a subset of (G, o).

Therefore (H, o) is a sub-group of (G, o).

Problem 18. *Show that the intersection of two sub-groups of a group (G, o) is a subgroup of (G, o)*

Let (H_1, o) and (H_2, o) be the two sub-groups of (G, o). then

$H_1 \cap H_2 \subset G$.

Now $a, b \in H_1 \cap H_2 \Rightarrow a, b \in H_1, a, b \in H_2$

$\Rightarrow aob \in H_1, aob \in H_2$ since H_1, H_2 being sub-groups, satisfy group axioms.

$\Rightarrow aob \in H_1 \cap H_2 \forall\, a, b \in H_1 \cap H_2$

Also $a \in H_1 \cap H_2 \Rightarrow a \in H_1$ and $a \in H_2$

$\Rightarrow a^{-1} \in H_1$ and $a^{-1} \in H_2$ since H_1, H_2 being sub-groups satisfy group axioms.

$\Rightarrow a^{-1} \in H_1 \cap H_2$.

Hence by Problem 16, $H_1 \cap H_2$ is a sub-group of G.

Problem 19. *Show that the union of two sub-groups of a group (G, o) may not be sub-group of G.*

Let (H_1, o) and (H_2, o) be the two sub-groups of (G, o) and let

$a \in H_1, b \in H_2$, so that $a, b \in H_1 \cap H_2$.

Now $a, b \in H_1 \cup H_2 \Rightarrow a \in H_1, b \in H_2 \not\Rightarrow aob \in H_1 \cup H_2$ for b may not belong to H_1.

Hence the union of two sub-groups of a group may not be sub-group of the group.

Problem 20. *Show that the set $S = \{1, i, -1, -i\}$ is a sub-group of a multiplicative group of non-zero complex numbers.*

Let $(G, \cdot)$ be a multiplicative group of non-zero complex numbers. Then $(S, \cdot)$ will be a sub-group of $(G, \cdot)$ if it satisfies both the conditions for a sub-group.

The condition (*i*) is satisfied since $1 \cdot i = i \in S$, $1 \cdot (-1) = -1 \in S$, $1 \cdot (-i) = -i \in S$, $i \cdot (-1) = -i \in S$, $i\,(-i) = 1 \in S$, $(-1) \cdot (-i) = i \in S$.

The condition (*ii*) is satisfied since the inverse of 1 is $1 \in S$, the inverse of i is $-i \in S$, the inverse of -1 is $-1 \in S$ and the inverse of $-i$ is $i \in S$.

Hence $(S, \cdot)$ is a sub-group of $(G, \cdot)$.

Problem 21. *Show that the order of every element of a group (G, o) of finite order is finite.*

If a be an element of (G, o) of finite order, then the positive integral powers of a *viz.* $a, a^2, a^3, a^4 \ldots$ will all be the members of G.

But the order of G is finite, therefore all these elements of G can not be different.

Suppose that $a^r = a^s, r > s$.

Then, $a^{r-s} = a^r o a^{-s} = a^r o a^{-r} = a^{r-r} = a^0 = e$, e being the identity in G.

If $r - s = m$, then $a^{r-s} = e \Rightarrow a^m = e$, m being a positive integer as $r > s$.

This follows that $\exists$ a positive integer m s.t. $a^m = e$.

As every set of positive integers essentially possesses a least member so the set of all those positive integers s.t. $a^m = e$ has a least member known as the order of a. But a is arbitrary and hence the order of every element of G is finite.

Problem 22. *Show that the order of any power of any element a of a group is utmost equal to the order of the element.*

Assuming that order $a = m$ and order of $(a^p) = n, p \in \mathbf{I}$ (set of integers), we have order of $a = m \Rightarrow a^m = e$, e being identity element.

$$\Rightarrow (a^m)^p = e^p$$

$$\Rightarrow a^{mp} = e$$

$$\Rightarrow (a^p)^m = e$$

$$\Rightarrow \text{order of } (a^p) \leq m$$

which proves the proposition.

Problem 23. *Show that the order of any element of a group is always equal to the order of its inverse.*

Taking the orders of a and a^{-1} as m and n respectively, we have

$$a^m = e \text{ and } (a^{-1})^n = e$$

Now a^{-1} being an exponent power of a, the Problem 22 leads to order of $(a^{-1}) \leq$ order of a *i e.* $n \leq m$.

Also since $a = (a^{-1})^{-1}$ *i.e.* a is an exponent power of a^{-1}, so by Problem 22, we have order of $a \leq$ order of (a^{-1}) *i.e.* $m \leq n$.

Hence $m \leq n$ and $n \leq m \Rightarrow m = n$.

Problem 24. *If a, b be two elements of a group (G, o) and* $ba = a^m b^n \; \forall \; a, b \in G$ *then prove that the elements* $a^m b^{n-2}$, $a^{m-2} b^n$ *and* ab^{-1} *have the same order.*

We have $\qquad (a^{-1}b)^{-1} = b^{-1}(a^{-1})^{-1} = b^{-1}a$

Since $b^{-1}a$ is the inverse of $a^{-1}b$, therefore by Problem 23, the order of $b^{-1}a$ and $a^{-1}b$ is the same.

Now $\quad a^m b^{n-2} = a^m b^n b^{-2} = (ba)\, b^{-2} \quad \because\; ba = a^m b^n$

$$= b\,(ab^{-1})\, b^{-1} \;\because\; b^{-2} = b^{-1}b^{-1}$$

But $b\,(ab^{-1})\, b^{-1}$ has the same order as that of ab^{-1} since

$$[b\,(ab^{-1})\, b^{-1}]^2 = [b\,(ab^{-1})\, b^{-1}]\,[b\,(ab^{-1})\, b^{-1}]$$

$$= [b\,(ab^{-1})]\,(b^{-1}b)\,[(ab^{-1})\, b^{-1}]$$

$$= b\,(ab^{-1})\,(e)\,(ab^{-1})\, b^{-1}$$

$$= b\,(ab^{-1})^2\, b^{-1}$$

or in general $\quad [b\,(ab^{-1})\, b^{-1}]^n = b\,(ab^{-1})^n\, b^{-1} = beb^{-1}$ if order of ab^{-1} be n

$$= bb^{-1} = e.$$

These results follow that order of ab^{-1} is the same as that of $a^m b^{n-2}$.

Again $a^{m-2}b^n = a^{-2}(a^m b^n) = a^{-2}(ba) = a^{-1}(a^{-1}b)\,a$

i.e as above, the order of $a^{-1}b$ is the same as that of $a^{m-2}b^n$.

Problem 25. *If the elements a, b and aob of a group (G, o) are each of order 2, then show that the group is abelian.*

The order of *aob* being 2, we have $(aob)^2 = e$, e is the identity in G.

$\therefore\ (aob)\,o\,(aob)\,e \Rightarrow ao\,(aob)\,o\,(aob) = aoe$

$\Rightarrow ao\,(aob)\,o\,(aob)\,ob = ao\,eob$

$\Rightarrow (aoa)\,o\,(boa)\,o\,(bob) = ao\,eob$ by associative law.

$\Rightarrow a^2o\,(boa)\,ob^2 = ao\,eob$

$\Rightarrow eo\,(boo)\,oe = ao\,eob$ since the order of a and b is 2.

$\Rightarrow boa = aob$

which proves that a and b commute and hence the group is abelian.

4.5. THE CENTRE OF A GROUP

If (G, o) be a group and H be the set of those elements $x \in G$, which commute with each element in G i.e. the set

$$H = \{x : x \in G \text{ and } aox = xoa\ \forall\ a \in G\}$$

then the set H is known as the centre *of G.*

THEOREM. The centre of G is a subgroup of (G, o).

If H be the centre of G, then we have by definition

$$H = \{x : x \in G \text{ and } aox = xoa\ \forall\ a \in G\}$$

$\therefore\ x_1, x_2 \in H \Rightarrow aox_1 = x_1oa$ and $aox_2 = x_2oa\ \forall\ a \in G$.

But $aox_1 = x_1oa = x_1o\,(x_2^{-1}ox_2)oa$, since $x_2^{-1}ox_2 = e$, the identity in H and

$x_1oeoa = x_1\,oa$

$= (x_1o\,x_2^{-1})\,o\,(x_2oa)$

$= (x_1o\,x_2^{-1})\,o\,(aox_2) \quad \because\ aox_2 = x_2oa.$

$\therefore\quad aox_1 = (x_1o\,x_2^{-1})\,o\,(aox_2) \Rightarrow (aox_1)\,ox_2^{-1} = (x_1o\,x^{-2})\,oa$

$\Rightarrow ao\,(x_1o\,x_2^{-1}) = (x_1o\,x_2^{-1})\,oa$

$\Rightarrow x_1o\ x_2^{-1}$ commutes with $a \in G$

$\Rightarrow x_1o\ x_2^{-1} \in H$

Conclusively $x_1 \in H, x_2 \in H \Rightarrow x_1\,o\,x_2^{-1} \in H$

Which follows by the definition of a sub-group that (H, o) is a sub-group of (G, o).

4.6. COSETS OR COSETS OF A SUB-GROUP

Let (G, o) be a group, (H, o) be a subgroup of (G, o) and 'a' be an element in G i.e. $a \in G$. Then the set

$aH = \{ah : h \in H\}$ *(not using the binary operation)*

i.e. the collection,

$aoH = \{aoh_1, aoh_2 \ldots, aoh_i, \ldots\},\ h_i \in H$

$= \{aox : x \in H \text{ and } a \in G\}$

is said to be the **Left Coset** *of H in G;*

and the set $Ha = \{ha : h \in H\}$

i.e. the collection, $Hoa = \{h_1oa, h_2oa, \ldots, h_ioa, \ldots\}$, $h_i \in H$

$$= \{xoa : x \in H \text{ and } a \in G\}$$

is said to be the **Right Coset** *of H in G.*

Since $eH = H = He$, therefore H is itself a coset.

If the cosets aH and bH are such that $aH \cap bH \neq \emptyset$, then $aH = bH$, hence the cosets have no element in common with H *i.e.* two cosets contain either the same elements or have no elements in common. Also the cosets do not form a group.

The number of left (or right) cosets of H in G is said to be the *Index* of H in G and denoted by $(G : H)$.

THEOREM 1. If (H, o) or simply H be a subgroup of (G, o) or simply G, then H is both a left coset and a right coset.

If e be the identity in G, then $He = eH = H$, which follows that H is both a left-coset as well as a right coset of H in G.

• *THEOREM 2. If H be a subgroup of G, then $aH = H \Leftrightarrow a \in H$.*

If e be the identity in G and so is in H, then

$$aH = H \Rightarrow ae \in H$$

i.e. $$aH = H \Rightarrow a \in H \quad \ldots(1)$$

Again, if $a \in H$ and $h \in H$ then

$$a \in H \Rightarrow ah \in H \ \forall\ h \in H$$

$$\therefore \quad aH \subset H$$

Also $a \in H \Rightarrow a^{-1} \in H$, H being a sub-group of the group G, satisfies group axioms.

$\Rightarrow a^{-1} h \in H \ \forall\ h \in H$ by closure law in H

$\Rightarrow a\,(a^{-1}h) \in H \ \forall\ h \in H$ by closure law in H

$\Rightarrow h \in aH \ \forall\ h \in H$

$\therefore \quad H \subset aH$

So $aH \subset H$ and $H \subset aH \Rightarrow aH = H$

Ultimately $$a \in H \Rightarrow aH = H \quad \ldots(2)$$

Hence $aH = H \Leftrightarrow a \in H$ by (1) and (2).

THEOREM 3. If $a, b \in G$ and $a \neq b$ then $aH = bH \Leftrightarrow a^{-1} b \in H$ where H is a sub-group of the group G.

We have,

$$aH = bH \Rightarrow a^{-1} aH = a^{-1} bH$$

$$\Rightarrow (a^{-1}a)H = (a^{-1}b)H$$

$\Rightarrow eH = (a^{-1}b)H$, e being the identity in G and so in H.

$$\Rightarrow H = (a^{-1}b)H$$

$\therefore \ aH = bH \Rightarrow a^{-1}b \in H$ by theorem 1. ...(1)

Also, if $a^{-1} b \in H$, then

$bH = e\,(bH) = (aa^{-1})\,(bH) = a(a^{-1}b)H = aH$ by theorem 1 ...(2)

(1) and (2) follow that $aH = bH \Leftrightarrow a^{-1}b \in H$.

THEOREM 4. The two left cosets aH and bH of a subgroup H of a group G are either identical or disjoint. *(Rohilkhand, 1992)*

There arise two cases:

Case I. If $aH \neq bH$, then we have to show that aH and bH are disjoint.

Let us assume if possible that $x \in aH$ and $x \in bH$.

Then $x = ay, y \in H$ and $x = bz, z \in H$

$$\therefore \quad ay = bz \Rightarrow ayz^{-1} = bzz^{-1} \Rightarrow a\,(yz^{-1}) = b(zz^{-1}) = be = b$$
$$\Rightarrow (a^{-1}a)\,(yz^{-1}) = a^{-1}b$$
$$\Rightarrow e\,(yz^{-1}) = a^{-1}b$$
$$\Rightarrow yz^{-1} = a^{-1}b$$

Thus, $yz^{-1} \in H \Rightarrow a^{-1}b \in H$.

So that by theorem 3, it follows that $aH = bH$, which contradicts the hypothesis and hence two unequal cosets cannot have any element in common *i.e.* aH and bH are disjoint.

Case II. If aH and bH are not disjoint, then we have to show that $aH = bH$.

aH and bH are not disjoint $\Rightarrow \exists$ an element common to aH and bH

$$\Rightarrow \exists h_i, h_j \text{ s.t. } ah_i = bh_j$$
$$\Rightarrow a\,(h_i h_i^{-1}) = b\, h_j h_i^{-1}$$
$$\Rightarrow a = bh_j h_i^{-1}$$
$$\Rightarrow ah = b\,(h_j h_i^{-1} h)\ \forall\ h \in H$$
$$\Rightarrow ah \in bh\ \ \forall\ h \in H$$
$$\Rightarrow aH \subset bH \qquad \ldots(1)$$

Similarly it can be shown that

$$ah_i = bh_j \Rightarrow bH \subset aH \qquad \ldots(2)$$

(1) and (2) follow that $aH = bH$ *i.e.* aH and bH are identical.

THEOREM 5. If H be a subgroup of the group G and $a \in G$ but $a \notin H$ then $\exists$ one-one mapping of H onto aH.

Taking $f : H \to aH$ defined by $f(h) = ah$, $h \in H$, we have to show that the map f is onto.

Every element of the left coset aH being of the form ah, $h \in H$, and so being the f-image of h in H, the mapping f is onto.

Again to show that f is one-one, let $h_i, h_j \in H$ s.t. $ah_i = ah_j$.

Then $ah_i = ah_j \Rightarrow h_i = h_j$ by left cancellation law.

So f is one-one.

Conclusively f is a one-one mapping of H onto aH.

Note. This theorem follows that if H be a finite sub-group, the number of elements in each of its left cosets is the same as the number of elements in H *i.e.* equal to the order of H.

THEOREM 6. **(Lagrange's theorem).** *The order of every subgroup of a finite group is a divisor of the order of the group.*

Let H be a subgroup of a finite group G. So G being finite, H is also finite.

Let m and n be the order of H and G respectively.

Since the order of H is m, therefore H consists of exactly m elements or in other words every coset aH has exactly m elements, for if $h_1, h_2, \in H$, $ah_1 = ah_2$ iff $h_1 = h_2$, hence aH has the same number of elements as H.

Now if $m = n$, the theorem is self-evident.

But if $n > m$, then G being of finite order, there are only a finite number say k, of different cosets of H in G.

Taking $H = \{h_1, h_2 \ldots, h_m\}$, if $a \in G$ but $a \notin H$ and binary operation of G being denoted multiplicatively, the distinct m elements

$$ah_1, ah_2, \ldots, ah_m \notin H \text{ but belong to } G \text{ by closure axiom.}$$

Denoting the set formed by these m elements by H *i.e.*

$$H' = \{ah_1, ah_2 \ldots, ah_m\}$$

We observe that if $H \cup H'$ is a proper subset of G then there is an element say $b \in G$ s.t. $b \notin H \cup H'$. We thus have again a set of m distinct elements

$bh_1, bh_2 \ldots bh_m$ which belong to G but not to $H \cup H'$

Denoting the set of these m elements by H'' *i.e.*

$$H' = \{bh_1, bh_2, \ldots, bh_m\}$$

and continuing this process, we see that G can be divided into k subsets each consisting of m elements.

$\therefore$ Order of G = number of elements in G

i.e. $\qquad n = k \times$ order of H

$$= km$$

which follows that the order of H is a divisor of the order of G.

COROLLARY 1. *The order of an element of a group G of finite order is a divisor of the order of the group.*

Let m be the order of the group G and $a \in G$. Then by definition,

$a^m = e$, m being least positive integer and e being identity in G.

Evidently the elements $a, a^2, a^3, \ldots a^{m-1}, a^m \in G$, are all distinct and form a sub-group of order m. Also by definition, m is the order of a.

$\therefore$ The order m of a is a divisor of the order of the group.

COROLLARY 2. *A finite group of prime order has no proper subgroup.*

Let G be a finite group of order p, where p is a prime. Then by Lagrange's theorem, the order of any sub-group of G is divisor of p. But p being prime has no divisor and hecne there is no proper sub-group of G.

COROLLARY 3. **Fermats' theorem.** *If p be a prime and 'a' a natural number not divisible by p, then*

$$a^{p-1} = 1 \pmod{p}$$

Taking the multiplicative group of non-zero residue classes modulo p and a not divisible by p, we have the equivalence class $\bar{a} \neq 0$ *i.e.* $[a] \neq 0$.

But the order of the group being $p-1$, it follows from Cor. 1, that

$$[a^{p-1}] = [1]$$

which yields $a^{p-1} = 1 \pmod{p}$.

Problem 26. *If H be a subgroup of a group G and m, n are the orders of m and n respectively then prove that $a^n = e$, e being identity in G.*

Lagrange's theorem gives $n = km$, k being some positive integer

$\therefore$ $a^n = a^{km} = (a^m)^k = e^k = e$.

Problem 27. *Find the cosets of the additive subgroup* (2I, +) *of the additive group* (I, +), I *being set of all integers.*

We have

$$\mathbf{I} = \{\ldots -3, -2, -1, 0, 1, 2, 3 \ldots\}$$

and say $H = (2\mathbf{I}, +) = \{\ldots -6, -4, -2, 0, 2, 4, 6 \ldots\}$

If $a \in \mathbf{I}$ then the coset of H in $\mathbf{I}$ corresponding to a is $2\mathbf{I} + a$ since the group being abelian, $\mathbf{I} + a = a + \mathbf{I}$

$\therefore$ $2\mathbf{I} + 0 = \{\ldots, -6, -4, -2, 0, 4, 6, \ldots\}$

$2\mathbf{I} + 1 = \{\ldots, -5, -3, -1, 1, 3, 5, 7, \ldots\}$

$2I + 2 = \{\ldots, -4, -2, 0, 2, 4, 6, 8, \ldots\} = 2I$

$2I + 3 = \{\ldots, -3, -1, 1, 3, 5, 7, 9, \ldots\} = 2I + 1$

$2I + 4 = \{\ldots, -2, 0, 2, 4, 6, 8, 10, \ldots\} = 2I$

$2I + 5 = \{\ldots, -1, 1, 3, 5, 7, 9, 11, \ldots\} = 2I + 1$ and so on.

Thus the distinct cosets of H in I are 2I and 2I + 1.

Clearly $2I \cup (2I + 1) = I$.

4.7. CYCLIC GROUPS

If a group H contains an element a s.t. it is capable of being generated by the single element a i.e. every element of H is of the form a^n for some integer n, then H is said to be a **cyclic group** *and a is known as the* **generator** *of H. We also denote $H = \{a\}$.*

Hence if H is a cyclic group, then $\exists\ a \in H, b \in H$ s.t. $a^n = b$ (in multiplicative form) or $b = na$ (in additive form) for some integer n.

Thus $H = \{a^n : n \in I\}, a \in H$, I being get of integers.

e.g. the unit circle $\{z : |z| = 1\}$ in the complex plane is a cyclic group.

Characteristics of Cyclic Group

(*i*) *Every cyclic group is abelian.*

If H be a cyclic group and a is its generator, then

$$a^m, a^n \in H\ \forall\ m, n \in I$$

$$\therefore\ a^m \, o\, a^n = a^{m+n} = a^{n+m} = a^n \, o\, a^m$$

which proves the commutative property and hence every cyclic group is abelian.

(*ii*) *The order of a cyclic group is the same as that of its generator.*

Let H be a cyclie group, a its generator and e the identity element in H. Also let n be the order of a, so that $a^n = e$

Evidently, $m \in I$ and $m < n \Rightarrow a^m \neq e$.

In case $m > n$, then if q be the quotient and r the least positive remainder when m is divided by n,

$$m = nq + r$$

So that $a^m = a^{nq+r} = a^{nq}\, o\, a^r = (a^n)^q\, o\, a^r = e^q\, o\, a^r = a^r$

where $r = 0, 1, 2, \ldots (n-1)$.

By closure axiom since $a^m \in H$, therefore n distinct elements belonging to H are a^0, $a^1, a^2, a^3, \ldots a^{n-1}$ where $a^0 = e = a^n$.

As such there are only n elements in H and hence the order of the cyclic group H is also n which is the order of its generator.

(*iii*) *The generators of a cyclic group of order n are the generators a^p where p is prime to n and $0 < p < n$.*

$$\because\ a^n = e,\ \therefore\ (a^p)^n = (a^n)^p = e^p = e$$

which shows that order of $a^p \leq n$.

Taking $s \in I$ s.t. $o < s < n$, we have ps prime to n since n is neither a factor of p, nor of s.

Let $ps = nq + r$, q being quotient and r the least positive remainder when ps is divided by n and $o \leq r \leq n - 1$.

Thus $(a^p)^s = a^{ps} = a^{nq+r} = a^{nq}\, o\, a^r = (a^n)^q\, o\, a^r = e^q\, o\, a^r = e\, o\, a^r = a^r$

where $r = 0, 1, 2, \ldots, n-1$.

It is clear that $a^r \neq e$

Hence the order of a^p is n and a^p is the generator of the group.

(iv) A subroup H' of a cyclic group H is also cyclic.

Let a be the generator of H. Given that H' is a sub-group of H. Therefore every element of H and so of H' will be of the form a^n, n being an integer..

Let m be the least positive integer s.t. $a^m \in H'$.

If m does not divide n then $\exists$ integers q (quotient) and r (remainder) s.t. $n = mq + r$ $o \leq r < m$

$\therefore \; a^n = a^{mq+r} = a^{mq} \, o \, a^r$ giving $a^r = (a^{mq})^{-1} \, oa^n$ ...(1)

But $a^m \in H'$ $\therefore$ by closure law $a^{mq} \in H'$ and so $(a^{mq})^{-1} \in H'$ since H' satisfies group axioms.

Now $a^n \in H'$ (by hypothesis)

$\therefore$ (1) yields, $a^r \in H'$ which contradicts the assumption that m is the least positive integer s.t. $a^m \in H'$

Thus the only possibility is that $r = 0$ and then $n = mq$ so that $a^n = a^{mq} = (a^m)q$.

Which follows that every element a^n of H' is of the form $(a^m)^q$ showing that a^m is the generator of H' and hence H' is cyclic.

Finite cyclic groups. *If H is a cyclic group generated by a s.t. all the powers of a are not different then $H = \{a\}$ is a finite cyclic group.*

If $n(>0)$ be the order of a, then $a^n = e$

Given any integer s $\exists$ two integers q and r s.t. $s = nq + r, o \leq r < n$.

$\therefore \; a^s = a^{nq+r} = a^{nq} \, o \, a^r = (a^n)^q \, oa^r = e^q \, oa^r = e \, oa^r = a^r$

Which follows that there are at most n distinct elements $a^1, a^2, a^3, \ldots, a^{n-1}, a^n = e$

To show that no two of these n elements are equal, let us assume if possible that $a^x = a^y, o < y < x < n$

$\therefore \; a^{x-y} = a^y \, o \, a^{-y} = a^o = e$

But $o < x - y < n$ and order of a being n, $a^{x-y} \neq e$ *i.e.* $a^x \neq a^y$.

Thus H contains exactly n (finite) distinct elements

$a^1, a^2, \ldots, a^{n-1}, a^n$.

Hence H is a finite cyclic group of order n.

Infinite cyclic groups. *If H be a cyclic group generated by a s.t. all the powers of a are distinct, then $H = \{a\}$ is an infinite cyclic group.*

Let a be the generator of H. Then all the powers of a being different the order of a is zero.

Let us assume, if possible that $a^s = a^r$ where $s > r$.

Then $a^{s-r} = a^r \, o \, a^{-r} = a^0 = e$ which contradicts the assumption that the order of a is zero.

$$\therefore \qquad a^s \neq a^r$$

i.e. H contains an infinite number of elements and hence H is an infinite cyclic group.

THEOREM 1. In an infinite cyclic group, there are exactly two distinct generators namely one generator and the other its inverse.

Let H be an infinite cyclic group and a, one of its generator. Then since $a^n = (a^{-1})^{-n}$ therefore a^{-1} is the other generator.

Also $a \neq a^{-1}$ otherwise $a = a^{-1} \Rightarrow aa^{-1} = a^2 = e$ = a finite cyclic group of order 2 which contradicts the hypothesis that the cyclic group is infinite.

To show that $\nexists$ third generator, if possible suppose that b is the third generator of H so that a and b being both generators of H, $a = b^m$ and $b = a^l$,

$\therefore \; a = (a^l)^m = a^{ml}$ ...(1)

But H being infinite cyclic group, $r \neq n \Rightarrow a^r \neq a^n$.

$\therefore$ the relation (1) is satisfied if $ml = 1, m, l$ being both integers.

It follows that either $m = +1$ or $m = -1$

i.e. either $b = a$ or $b = a^{-1}$.

So that $\nexists$ third generator of H other than a and a^{-1}.

THEOREM 2. Every subgroup of an infinite cyclic group is infinite.

Let H' be a subgroup of an infinite cyclic group H whose generator is a. Then by characteristic (*iv*) of groups, we have $H' = \{a^m\}$, m being least positive integer s.t. $a^m \in H'$

Assume, if possible that H' is finite, then $(a^m)^n = e$ for some $n > o$ which follows that a is of finite order and so H is finite which contradicts the hypothesis.

Hence H' must be an infinite cyclic subgroup of H.

Problem 28. *Show that the group formed by the set $\{1, \omega, \omega^2\}$, ω being cube root of unity i.e. $\omega^3 = 1$, is a cyclic group of order 3 with respect to multiplication.*

Here $\omega^3 = 1$ is the identity and ω is the generator as its powers generate the elements $1, \omega, \omega^2$ as tabulated:

$\cdot$	1	ω	ω
1	1	ω	ω^2
ω	ω	ω^2	$\omega^3 = 1$
ω^2	ω^2	$\omega^3 = 1$	$\omega^4 = \omega$

The group axioms are satisfied, since if

$G = \{1, \omega, \omega^2)$ w.r.t. '$\cdot$' then

G_1—$1, \omega, \omega^2 \in G$, $1 \cdot \omega$, $1 \cdot \omega^2$, $\omega \cdot \omega^2 \in G$ as $\omega^3 = 1$

G_2—$(1 \cdot \omega) \cdot \omega^2 = 1 \cdot (\omega \cdot \omega^2) = \omega \cdot \omega^2 = \omega^3 = 1$

G_3—1 is the identity element as $\omega \cdot 1 = \omega$ etc.

G_4—Inverses of $1, \omega, \omega^2$ are respectively $1, \omega^2, \omega$ as

$$1 \cdot 1 = \omega \cdot \omega^2 = \omega^2 \cdot \omega = 1 \text{ (the identity element)}$$

Hence $\{1, \omega, \omega^2\}$ is a cyclic group of order 3 with generator ω.

Problem 29. *Find all the generators of the cyclic group $\{a, a^2, a^3, a^4, a^5, a^6, a^7, a^8 = e\}$ of order 8*

Let $H = \{a, a^2, a^3, a^4, a^5, a^6, a^7, a^8 = e\}$

Since it contains all powers of a, so a is a generator.

Now $(a^3)^1 = a^3$, $(a^3)^2 = a^6$, $(a^3)^3 = a^9 = a^8 o a^1 = eoa = a$,

$(a^3)^4 = a^{12} = a^8 o a^4 = eoa^4 = a^4$, $(a^3)^5 = a^{15} = a^8 o a^7 = eoa^7 = a^7$

$(a^3)^6 = a^{18} = (a^8)^2 o a^2 = e^2 o a^2 = eoa^2 = a^2$

$(a^3)^7 = a^{21} = (a^8)^2 o a^5 = e^2 o a^5 = a^5$

$(a^3)^8 = a^{24} = (a^8)^3 = e^3 = e$

Since powers of a^3 are the elements of H so a^3 is a generator of H. Similarly a^5 and a^7 are also the generators of H.

4.8. PERMUTATION OR TRANSFORMATION

If G be a set then a one-one onto mapping $f: G \to G$ is said to be a **transformation** *or in case G is finite, f is said to be a* **permutation.**

In fact the permutation is a rearrangement of the elements of the set and the permutation groups are associated with symmetry groups introduced a bit later.

Consider a set $\{1, 2, 3\}$ with three elements. Its symmetry group or permutations may be written as

$P_1 = I$(identity) $= \begin{pmatrix} 1, & 2, & 3 \\ 1, & 2, & 3 \end{pmatrix}$, $P_2 = \begin{pmatrix} 1, & 2, & 3 \\ 2, & 3, & 1 \end{pmatrix}$, $P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

$P_4 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, $P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$, $P_6 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

If we multiply P_5 and P_6, we have

$$P_6 P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \text{ as } \begin{matrix} 1 \to 2 \to 2 \\ 2 \to 1 \to 3 \\ 3 \to 3 \to 1 \end{matrix}$$

$$= P_2$$

and $$P_5 P_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = P_3.$$

It is clear that $P_6 P_5 \neq P_5 P_6$

i.e. the *permutation multiplication is not commutative*, but it can be shown that *permutation multiplication is associative*, since

$$P_1 (P_2 P_3) = \begin{pmatrix} 1, & 2, & 3 \\ 1, & 2, & 3 \end{pmatrix}\left[\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

and $$(P_1 P_2) P_3 = \left[\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\right]\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

$\therefore P_1 (P_2 P_3) = (P_1 P_2) P_3$

To find the inverse of $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ (say) let us assume that its inverse is $\begin{pmatrix} 1 & 2 & 3 \\ x & y & z \end{pmatrix}$.

Each of the two permutations whose product is the **Identity Permutation**, $I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ *i.e.* Pre-image of every element is the same element), being called Inverse to each other, we have

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ x & y & z \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

i.e. $$\begin{pmatrix} 1 & 2 & 3 \\ z & x & y \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

Comparison gives $z = 1, x = 2, y = 3$

Hence the inverse of $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ is $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

Thus if P^{-1} be the inverse of a permutation P, then we can easily show that

(*i*) $PP^{-1} = I = P^{-1}P$

(*ii*) P^{-1} is unique

(*iii*) $(P^{-1})^{-1} = P$

(*iv*) $(PQ)^{-1} = Q^{-1}P^{-1}$ (Reversal law)

The set of all permutations of n elements

The $\underline{|n}$ *permutations on n elements (objects) form a group with respect to permutation multiplication. Such a group of all permutations of n elements is denoted by* S_n *and called the* **Symmetric group of degree n** *as it satisfies all the four group* **axioms.**

Note 1. The number of elemnts in the finite set permutated is known as the *degree of permutation.*

Note 2. The number of elements in a permutation on *n* elements is $\underline{|n}$.

A permutation which replaces n elements cyclically is said to be a **cyclic permutation.** *of degree n e.g.,* $\begin{pmatrix} 1 & 2 & 3 \ldots n-1, & n \\ 2 & 3 & 4 \ldots n, & 1 \end{pmatrix}$ *is cyclic and may be denoted by* (*1, 2, 3,...n*)

The number of elements permutated by a cycle is said to be its **length** *and the* **disjoint cycles** *are those which have no common elements.*

Every cycle can be uniquely expressed as **product of disjoint cycles**

e.g. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 2 & 5 & 7 & 6 & 1 \end{pmatrix} = (1\,4\,5\,7)\,(2\,3)\,(6)$

$= (2\,3)\,(1\,4\,5\,7)\,(6)$

where $(1\,4\,5\,7) \Rightarrow \begin{pmatrix} 1 & 4 & 5 & 7 & 2 & 3 & 6 \\ 4 & 5 & 7 & 1 & 2 & 3 & 6 \end{pmatrix}$ etc.

A cyclic permutation such as (*a, b*) *which interchanges the symbols leaving all other unchanged is called a* **Transposition.**

In other words *Transposition* is cycle of length two of the form (*a, b*) *i.e.* it is a mapping which maps each object onto itself excepting two, each of which is mapped on the other. *e.g.* (1, 2) is a transposition.

Note 3. A cycle of length one is *invariant.*

Note 4. Transposition is its own inverse, since, if (*a, b*) be the transposition $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ then $(a\,b)\,(a\,b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = I.$

Note 5. Any permutation can be resolved as the product of transpositions in infinitely many ways since every permutation can be expressed as a product of disjoint cycles and every cycle can be expressed as the product of transpositions in an infinite way, therefore the proposition follows.

Note 6. The order of transposition cannot be changed, since they may not be disjoint.

Even and odd permutations. *A permutation is said to be* **even** *or* **odd** *according as it is expressed as a product of an even or odd number of transpositions e.g. the permutation*

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 & 8 & 5 & 2 & 7 & 9 & 6 & 1 \end{pmatrix}$ is an odd permutation as it can be expressed as the product of seven transpositions such as

$(1\,3)\,(1\,8)\,(1\,6)\,(1\,7)\,(1\,9)\,(2\,4)\,(2\,5)$

and the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 5 & 2 & 1 & 3 \end{pmatrix}$ is even since it can be expressed as the product of four transpositions such as (1 6) (1 3) (1 5) (2 4).

Note 7. For any manner of expressing a given permutation as a product of transpositions, the number of transpositions is either necessarily odd or even.

Note 8. The product of two even or two odd permutations is even while the product of an even and an odd permutation is odd.

Note 9. Of the $\lfloor n$ permutations on n elements, $\frac{\lfloor n}{2}$ are even and $\frac{\lfloor n}{2}$ are odd

Alternating set or group. *The set of all even permutations of degree n is known as an* **alternating set** *or* **group** *and is denoted by A_n and symbolized as*

$A_n = \{\alpha : \alpha$ *is an even permutation on a set containing n elements*$\}$.

e.g. if $E_1, E_2, E_3, \ldots, E_s$ be the even permutations of n symbols then G_1 is satisfied, since the product of two even permutations is even and so A_n is closed.

G_2 is satisfied, since permutation composition is associative.

G_3 is satisfied, since the identity permutation considered as even permutation is also identity for even permutations.

G_4 is satisfied, since an $f \in A_n \exists f^{-1} \in A_n$ for, $f^{-1} of = fof^{-1} = I$ and f and I are even permutations so that f^{-1} is also an even permutation.

But the composition in A_n is not commutative since permutation composition is not commutative.

Hence A_n is a group which is non-abelian.

Also the set A_n contains $\frac{\lfloor n}{2}$ elements, hence (A_n, o) is a non-abelian group of order $\frac{\lfloor n}{2}$.

Permutation group. *Any group whose elements are permutations is said to be a* **permutation group.**

Any sub-group of S_n (symmetric group of degree n) is essentially a permutation group.

e.g. if $P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, $P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, $P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$, $P_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$,

$P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$, $P_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ be the six permutations on the set $\{1, 2, 3\}$, then the elements $P_1, P_2, P_3, P_4, P_5, P_6$, constitute all the elements of the symmetric group S_3 of degree 3, whose multiplication table is as shown here.

	P_1	P_2	P_3	P_4	P_5	P_6
P_1	P_1	P_2	P_3	P_4	P_5	P_6
P_2	P_2	P_1	P_5	P_6	P_3	P_4
P_3	P_3	P_4	P_1	P_2	P_6	P_5
P_4	P_4	P_3	P_6	P_5	P_1	P_3
P_5	P_5	P_6	P_2	P_1	P_4	P_2
P_6	P_6	P_5	P_4	P_3	P_2	P_1

The elements P_1, P_4, P_5 constitute one of the several subg-roups of S_3 and this is a permutation group on three symbols.

Problem 30. *Show that the cycle (1 2 3 4 5) may be expressed as a product of 3 cycles.*

We have

$$(1\,2\,3\,4\,5) = (1\,5)\,(1\,4)\,(1\,3)\,(1\,2)$$
$$= (1\,4\,5)\,(1\,3)\,(1\,2)$$

Problem 31. *Express* $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 4 & 1 & 2 & 5 \end{pmatrix}$ *as the product of disjoint cycles.*

Denoting the given permutation by f, which is the permutation of 6 positive numbers, let us determine the set of images of 1 under the successive powers of f *i.e.*

$$f(1) = 3 \text{ as } 1 \rightarrow 3$$
$$f^2 = f[f(1)] = f(3) = 4 \text{ as } 3 \rightarrow 4$$
$$f^3 = f[f^2(1)] = f(4) = 1 \text{ as } 4 \rightarrow 1$$

which follows that the first cycle is (1 3 4).

It is clear that 2 does not belong to this cycle.

We have,

$$f(2) = 6 \text{ as } 2 \rightarrow 6$$
$$f(6) = 5 \text{ and } f(5) = 2$$

i.e. another cycle is (2 6 5).

Since all the six elements are exhausted in two cycles, we have

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 4 & 1 & 2 & 5 \end{pmatrix} = (1\,3\,4)\,(2\,6\,5)$$

Problem 32. *Find all the permutations of four letters a, b, c, d which leave the expression ab + cd invariant.*

If a, b, c, d are distinct letters then the three functions

$$y_1 = ab + cd, \quad y_2 = ac + bd, \quad y_3 = ad + be$$

are distinct and there are the only functions of the given four. This follows that each of the 24 permutations on a, b, c, d replaces y_1 by y_1, y_2 or y_3 so that $\frac{1}{3} \times 24$ *i.e.* 8 of them leave y_1 invariant which may be verified by showing that I, (a, b), (cd), $(ab)\,(cd)$, $(ac)\,(bd)$, $(ad)\,(bc)$, $(a\,d\,b\,c)$, $(a\,c\,b\,d)$ leave $ab + cd$ invariant.

We have $I = \begin{pmatrix} a\,b\,c\,d \\ a\,b\,c\,d \end{pmatrix}$ where $a \rightarrow a$, $b \rightarrow b$, $c \rightarrow c$, $d \rightarrow d$ so that $ab + cd = ab + cd$

$(a, b) = \begin{pmatrix} a\,b\,c\,d \\ b\,a\,c\,d \end{pmatrix}$ where $a \rightarrow b$, $b \rightarrow a$, $c \rightarrow c$, $d \rightarrow d$ so that $ab + cd = ba + cd = ab + cd$

$(c, d) = \begin{pmatrix} a\,b\,c\,d \\ a\,b\,d\,c \end{pmatrix}$ where $a \rightarrow a$, $b \rightarrow b$, $c \rightarrow c$, $d \rightarrow c$ so that $ab + cd = ab + dc = ab + cd$

$$(a\,b)(c\,d) = \begin{pmatrix} a\,b\,c\,d \\ b\,a\,d\,c \end{pmatrix} \text{ where } \begin{array}{l} a \to b \\ b \to a \\ c \to d \\ d \to c \end{array} \text{ so that } ab + cd = ba + dc = ab + cd$$

$$(a\,c)(b\,d) = \begin{pmatrix} a\,c\,b\,d \\ c\,a\,d\,b \end{pmatrix} \text{ where } \begin{array}{l} a \to c \\ c \to a \\ b \to d \\ d \to b \end{array} \text{ so that } ab + cd = cd + ab = ab + cd$$

$$(a\,d)(b\,c) = \begin{pmatrix} a\,d\,b\,c \\ d\,a\,c\,b \end{pmatrix} \text{ where } \begin{array}{l} a \to d \\ d \to a \\ b \to c \\ c \to b \end{array} \text{ so that } ab + cd = dc + ba = ab + cd$$

$$(a\,d\,b\,c) = \begin{pmatrix} a\,d\,b\,c \\ d\,b\,c\,a \end{pmatrix} \text{ where } \begin{array}{l} a \to d \\ d \to b \\ b \to c \\ c \to a \end{array} \text{ so that } ab + cd = dc + ab = ab + cd$$

and
$$(a\;\; c\;\; b\;\; d) = \begin{pmatrix} a\,c\,b\,d \\ c\,b\,d\,a \end{pmatrix} \text{ where } \begin{array}{l} a \to c \\ c \to b \\ b \to d \\ d \to a \end{array} \text{ so that } ab + cd = cd + ba = ab + cd$$

Conclusively y_1 remains invariant by the 8 permutations mentioned above.

4.9. HOMOMORPHISM AND ISOMORPHISM OF GROUPS

(*Rohilkhand, 1989*)

Homomorphism of groups. *If* (G, o) and (G', o') *be two groups, then a mapping* $f : G \to G'$ *which retains the structure and is many one is called* **Homomorphism of the group** *G with the group G' s.t.*

$$f(aob) = f(a)\, o' f(b),\ \forall\ a, b \in G.$$

We sometimes use to say that G is homomorphic to G' and denote it by $G \simeq G'$ if $\exists$ a mapping $f : G \to G'$ s.t. $f(aob) = f(a)\, o' f(b)\ \forall\ a, b \in G$.

Properties of homomorphism

(*1*) *The group* (G', o') *is a homomorphic image of the group* (G, o)

(*2*) *The relation of homomorphism is not symmetric i.e.*

$$G \simeq G' \nRightarrow G' \simeq G$$

(*3*) *The homomorphic image of the identity of the group* (G, o) *is the identity of the group* (G', o') *i.e. if* e, e' *be the identities in* G, G' *respectively then* $f(e) = e'$.

We have $\quad a \in G \Rightarrow f(a) \in G'$

and $\quad f(aoe) = f(a)\, o' f(e)\ \forall\ a \in G$ by definition of homomorphism.

$\therefore \quad f(a)\, o'\, e = f(a) = f(aoe) = f(a)\, o' f(e)$ since $aoe = a$

and $\quad f(a)\, o'\, e' = f(a)$

So left cancellation law gives $e' = f(e)$

(4) *The homomorphic image of the inverse of any element a of a group (G, o) is the inverse of the image of a i.e.* $f(a^{-1}) = [f(a)]^{-1}\ \forall\ a \in G$

We have $a^{-1}, a \in G \Rightarrow f(a^{-1}), f(a) \in G'$

$\therefore$ $f(a^{-1})\ o'\ f(a) = f(a^{-1} o a)$, by definition of homomorphism

$= f(e) = e'$ by property (3)

But $f(a^{-1})\ o'\ f(a) = e' \Rightarrow f(a^{-1}) = [f(a)]^{-1}\ \because f(a), f(a^{-1}) \in G'$

Isomorphism of groups. *If (G, o) and (G', o') are two groups and ∃ a one-one onto mapping* $f: G \to G'$ s.t. $aob \xrightarrow{\text{mapped that}} a'\ ob'$ *where* $a \to a', b \to b', \forall\ a, b \in G$ *and* $a', b' \in G'$, *then the mapping f is called as* **Isomorphism** *and we say that G is* **isomorphic** *to G' and write* $G \cong G'$.

e.g. if G is an additive group of all integers *i.e.*

$$G = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

and G' is a multiplicative group of all positive and negative powers of an integer 2 *i.e.*

$$G' = \{2^m : m = 0, \pm 1, \pm 2 \ldots\}$$

$$= \left\{\ldots \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, 16, \ldots\right\}$$

Then we have $f(m) = 2^m$, m being an integer

and $f(m+n) = 2^{m+n} = 2^m \cdot 2^n = f(m) \cdot f(n)$, m, n being integers.

This shows that f is one-one onto and retains the group structure and hence $G \cong G'$.

Properties of isomorphism

(*i*) *The order of G = the order of G'*

(*ii*) *For isomorphic groups (G, o) and (G', o') the identity e' of G' is the image of identity e of G i.e.* $f(e) = e'$.

If $a \in G$ and $a' \in G'$ then $a' = f(a)$

$\therefore$ $f: G \to G'$ is one-one onto $\Rightarrow f(a) \in G'\ \ \forall\ a \in G$.

$\Rightarrow f(e) \in G'\ \because\ e \in G$

Now $aoe = a \Rightarrow f(aoe) = f(a)$

$\Rightarrow f(a)\ o'\ f(e) = f(a)\ o'\ e'$ by definition of isomorphism

$\Rightarrow a'\ o'\ f(e) = a'\ o'\ e'$

$\Rightarrow f(e) = e'$ by left cancellation law.

(*iii*) *For isomorphic groups (G, o) and (G', o') the image of inverse of any element a of G is the inverse of the image of a, i.e.*

$$f(a^{-1}) = [f(a)]^{-1}$$

If e, e' are identities of G, G' respectively then by property (*ii*) $f(e) = e'$

Also we have $a^{-1} o a = e = a o a^{-1}\ \forall\ a \in G$

But $a^{-1} o a = e \Rightarrow f(a^{-1} o a) = f(e)\ \forall\ a \in G$

$\Rightarrow f(a^{-1})\ o'\ f(a) = e'$ by definition of isomorphism

$\Rightarrow f(a^{-1}) = [f(a)]^{-1}$ by definition of inverse of an element in G'

(*iv*) *For isomorphic groups (G, o) and (G', o'), the order of an element* $a \in G$ *is the same as the order of its image* $a' \in G'$.

$f: G \to G'$ is one-one and onto.

If e, e' be identities in G, G' respectively, then

$$f(e) = e' \text{ and } f(aob) = f(a)\ o'\ f(b)\ \ \forall a, b \in G.$$

If n be the order of an element $a \in G$ then $a^n = e$

Also if m be the order of $f(a)$ then $[f(a)]^m = e'$

But $\quad a^n = e \Rightarrow f(a^n) = f(e)$

$\Rightarrow f(aoaoa\ldots n \text{ times}) = e'$

$\Rightarrow f(a)\,o'f(a)\,o'\ldots n$ times $= e'$ by definition of isomorphism

$\Rightarrow [f(a)]^n = e'$

$\Rightarrow$ order of $f(a) \le n$

$\Rightarrow m \le n.$

Also, $\quad [f(a)]^m = e' \Rightarrow f(a)\,o'.f(a)\,o',\ldots m$ times $= f(e)$

$\Rightarrow f(aoaoa\ldots m \text{ times}) = f(e)$ by definition of isomorphism

$\Rightarrow f(a^m) = f(e)$

$\Rightarrow a^m = e \quad \because f$ is one-one

$\Rightarrow$ order of $a \le m$

$\Rightarrow n \le m$

So that $m \le n$ and $n \le m \Rightarrow m = n$

$\Rightarrow$ order of a = order of a'.

(v) *If f is isomorphic mapping of $G \to G'$, then f^{-1} is also isomorphic.*

If f is one-one and onto then f^{-1} exists and is also one-one onto.

Also if $\quad x = f(a),\ y = f(b)$ for $a, b \in G$ and $x, y \in G'$, then

$a = f^{-1}(x),\ b = f^{-1}(y)$

But $\quad f^{-1}(xo'y) = f^{-1}[f(a)\,o'f(b)]$

$= f^{-1}[f(aob)] \because f$ is isomorphic mapping

$\Rightarrow aob \quad \because f^{-1}f(p) = p.$

$\Rightarrow f^{-1}(x)\,o\,f^{-1}(y)$

which follows that f^{-1} retains the group structure and hence f^{-1} is isomorphic.

Automorphism of groups. *An isomorphism of a group onto itself is said to be an* **automorphism** *of the group e.g. $f : G \to G'$ given by $f(a) = a^{-1}$, $a \in G$ is an automorphism iff G is an abelian group.*

In other words an automorphism f of G is a one-one transformation of G onto itself s.t. $(xy)f = (xf)(yf)\ \forall x, y \in G$

i.e. $\quad f(xy) = f(x)f(y)$

As another example the identity mapping $i : G \to G$ is an automorphism of group G.

Product of Automorphisms. If $x \leftrightarrow xf = x'$ be an automorphism of A where x' is the element of A in some order, then the mapping is automorphism and so $(xy)f = (xf)(yf) = x'y'$.

Take $x \leftrightarrow y$ another automorphism and denote z' by $z\phi$, so that

$$(xy)f\phi = [(xy)f]\phi = [(xf)(yf)]\phi = [(xf)\phi][(yf)\phi]$$
$$= [(x)f\phi][(y)f\phi]\ \forall x, y \in A$$

which shows that $f\phi$ is an automorphism of A and the mapping $f\phi$ is termed as product of automorphisms of f and ϕ.

The automorphism of a mathematical system forms a group.

The mapping $x \leftrightarrow x$ is said to be the *Identity automorphism* in the identity element of the automorphism group. So axiom G_3 is satisfied.

G_1 is satisfied since product of two automorphisms is a automorphism.

G_2 is satisfied since if we arrange the mappings

$$f : x \leftrightarrow x',\ \phi : x \leftrightarrow x'',\ \psi : x \leftrightarrow x'''$$

as $$f : x \leftrightarrow x',\ \phi : x' \leftrightarrow x'',\ \psi : x'' \leftrightarrow x'''$$

then $x'' = (x) f \phi \psi$ corresponding to x under the automorphism $f\phi\psi$ is uniquely determined whether it is obtained as $[(x)f]\ \phi\psi$ from the automorphism $f(\phi\psi)$ or as $[(x)\phi]\psi$ from $(f\phi)\psi$. Ultimately $x \leftrightarrow xf$ is an automorphism and so

$$\begin{aligned}(xy) f^{-1} &= [(xf^{-1}f)\ (yf^{-1}f)] f^{-1} \\ &= [(xf^{-1})\ (yf^{-1}) f] f^{-1} \\ &= (xf^{-1})\ (yf^{-1})\end{aligned}$$

showing that f^{-1} is an automorphism and hence G_4 is satisfied.

Conclusively the automorphisms of a mathematical system form a group.

Endomorphism of groups. *A homomorphism of a group onto itself is said to be an* **endomorphism** *of the group.*

Regular permutation group. *A permutation group to which a group G is isomorphic is said to be a* **regular permutation group.**

THEOREM 1. **Transference of group structures.**

If (G, o) is a group and G' is a set with the multiplicative composition 'o'' and if $\exists$ a one-one onto mapping $f : G \rightarrow G'$ s.t. $f(aob) = f(a)\ o' f(b)\ \forall\ a, b \in G$ then G' is also a group isomorphic to G for the given composition.

We have to show that G' is a group and $G' \cong G$.

Let $a' = f(a), b' = f(b), c' = f(c)$; $a, b, c \in G$ and $a', b', c' \in G'$ then $a'o'b' = f(a)\ o' f(b) = f(aob)$ is given.

G_1 is satisfied since
$$\begin{aligned} a', b' \in G' &\Rightarrow f(a), f(b) \in G' \\ &\Rightarrow a, b \in G \\ &\Rightarrow aob \in G \\ &\Rightarrow f(aob) \in G' \\ &\Rightarrow f(a)\ o' f(b) \in G' \\ &\Rightarrow a'o'b' \in G'\ \forall\ a', b' \in G'\end{aligned}$$

G_2 is satisfied, since
$$\begin{aligned}(a'\ o'\ b')o'\ c' &= [f(a)\ o' f(b)]\ o' f(c) \\ &= f[ao(boc)] \because \text{'o' is associative} \\ &= f(a)\ o' f(boc) \\ &= f(a)\ o'\ [f(b)\ o' f(c)] \\ &= a'\ o'\ (b'\ o'\ c')\end{aligned}$$

G_3 is satisfied, since if e be the identity in G then
$$\begin{aligned} f(e)\ o'\ a' &= [f(e)\ o' f(a)] \\ &= f(eoa) = f(a) \\ &= a' \qquad \because eoa = a\end{aligned}$$

and $$a'\ o' f(e) = f(a)\ o' f(e) = f(aoe) = f(a) = a'$$

$$\therefore \quad f(e)\ o'\ a' = a'\ o' f(e) = a'$$

G_4 is satisfied, since if $a \in G$ then $a^{-1} \in G$ so that $aoa^{-1} = e = a^{-1}oa$ and

$$\begin{aligned} aoa^{-1} = e &\Rightarrow f(aoa^{-1}) = f(e) \\ &\Rightarrow f(a)\ o' f(a^{-1}) = f(e) \\ &\Rightarrow a'\ o' f(a^{-1}) = f(e)\end{aligned}$$

also $$f(a^{-1}oa) = f(e) \Rightarrow f(a^{-1})\ o'\ f(a) = f(e) \Rightarrow f(a^{-1})\ o'\ a' = f(e)$$

$\therefore\ a' o' f(a^{-1}) = f(a^{-1})\ o'\ a' = f(e)$.

Thus $f(a^{-1})$ is the inverse of $a' \in G'$ *i.e.* $f(a^{-1}) = (a')^{-1} = [f(a)]^{-1}$

These axioms show that G' is a group.

Again $G \cong G'$ and the relation of isomorphism is symmetric

$\therefore\ G' \cong G$

THEOREM 2. The relation of isomorphism in the set of all groups is an equivalence relation.

If G be a group belonging to the set of all groups and $x \in G$, then consider a one-one onto mapping $f : G \to G$ defined by $f(x) = x\ \forall\ x \in G$.

The relation $\cong$ is reflexive, since $f(x) = f(y) \Rightarrow x = y$ *i.e.* f is one-one and $f(xy) = xy = f(x).f(y)$, operation being multiplicative.

$\therefore$ The group structure is retained and so $G \cong G\ \ \forall\ G \in S$, S being the set of all groups.

The relation $\cong$ is symmetric, since if f is isomorphism of G to G', then f is one-one onto and so f^{-1} exists s.t. f^{-1} s.t. $f^{-1} : G' \to G$

$\therefore$ By property (*v*) of isomorphism, f^{-1} is isomorphic

Thus, $G \cong G' \Rightarrow G' \cong G$.

The relation $\cong$ is transitive, since if $f : G \to G'$ and $g : G' \to G''$ be two isomorphic mappings, then composite mapping gof is also one-one onto when $gof : G \to G''$

Now $x, y \in G \Rightarrow f(x), f(y) \in G' \Rightarrow g[f(y)] \in G''$

So that $(gof)(xy) = g[f(xy)]$

$= g[f(x) f(y)]$, f being isomorphic.

$\Rightarrow g\,f(x)\ g\,f(y)$, g being isomorphic.

i.e. gof retains the group compositions and also it is one-one onto, so gof is isomorphism and maps $G \to G''$ *i.e.*

$$G \cong G',\ G' \cong G'' \Rightarrow G \cong G''$$

Hence $\cong$ is an equivalence relation.

THEOREM 3. **Cayley's Theorem.**

Every finite group G of order n (say) is isomorphic with a sub-group of symmetric group S_n

or

Every finite group G of order n is isomorphic to a permutation group (or transformation group).

Let $G = \{a_1\ a_2, \ldots a_n\}$ be a finite group of order n, with multiplicative composition and $a \in G$. Then n products

$a\,a_1, a\,a_2, \ldots, a\,a_n$ are all distinct elements of G, for if possible let us assume that $aa_i = aa_j$.

Let cancellation law give, $a_i = a_j$

But $a_i \neq a_j\ \therefore\ a\,a_i \neq a\,a_j$ so that $a\,a_1, a\,a_2, \ldots, a\,a_n$ are all distinct elements of G in some order.

$\therefore$ The mapping $f_a : G \to G$ s.t. $f_a(x) = ax$, $a \in G$, $x \in G$ is one-one and onto.

Thus $f_a = \begin{pmatrix} a_1 & a_2 \ldots\ldots a_n \\ aa_1 & aa_2 \ldots\ldots aa_n \end{pmatrix}$ is a permutation on n symbols.

Replacing a by $a_1, a_2 \ldots a_n$ in succession, we shall have x permutations $f_{a1}, f_{a2}, \ldots\ldots$, f_{an} of which no two can be equal since if $a_1, a_2 \in G$, then

$$f_{a1} = f_{a2} \Rightarrow f_{a1}(x) = f_{a2}(x) = \forall\, x \in G$$
$$\Rightarrow a_1 x = a_2 x \;\forall\, x \in G$$
$$\Rightarrow a_1 = a_2$$

Denoting the n permutations by G' *i.e.*

$$G' = \{f_a : a \in G\}$$

We have

$$f_a f_b = \begin{pmatrix} a_1 & a_2 \ldots\ldots a_n \\ aa_1 & aa_2 \ldots\ldots aa_n \end{pmatrix} \begin{pmatrix} a_1 & a_2 \ldots\ldots a_n \\ ba_1 & ba_2 \ldots\ldots ba_n \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & a_2 \ldots\ldots a_n \\ aa_1 & aa_2 \ldots\ldots aa_n \end{pmatrix} \begin{pmatrix} aa_1 & aa_2 \ldots\ldots aa_n \\ aba_1 & aba_2 \ldots\ldots aba_n \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & a_2 \ldots\ldots a_n \\ aba_1 & aba_2 \ldots\ldots aba_n \end{pmatrix} = f_{ab}$$

But $a, b \in G \Rightarrow ab \in G$ and so $f_a, f_b \in G' \Rightarrow f_{ab} \in G'$

$\therefore$ Closure axiom is satisfied.

Now to show that the set G' with the composite composition is a group isomorphic to the given group G, let us consider a mapping

$$g : G \to G' \text{ s.t. } g(a) = f_a \;\forall\, a \in G$$

Then, $g(a) = g(b) \Rightarrow f_a = f_b$

$$\Rightarrow ax = bx \;\forall\, x \in G$$
$$\Rightarrow a = b \text{ by right cancellation law.}$$

So that g is one-one and therefore G, G' consist of the same number of elements. But g being one-one mapping of G to G', g is also onto. Moreover $g(ab) = f_{ab} = f_a f_b = g(a)\, g(b)$

i.e. the group-composition is retained (preserved) by g.

Hence $G \cong G'$.

THEOREM 4. Every cyclic group of infinite order is isomorphic to the additive group of integers.

If G be an infinite cyclic group generated by a, then $G = \{a\}$ and all the powers of a are distinct.

Consider the mapping $f : G \to \text{I}$ given by $f(a^i) = i$

This mapping is onto and also one-one since $i \neq j \Rightarrow a^i \neq a^j$

$\therefore\; f(a^i \cdot a^j) = f(a^{i+j}) = i + j = f(a^i) + f(a^j)$

So that f preserves the operation and hence f is an isomorphism *i.e.*

$$(G, \cdot) \cong (\text{I}, +)$$

Problem 33. *Show that the multiplicative group $G = \{1, -1, i, -i\}$ is isomorphic to the permutation group $G' = \{I, (abcd), (ac)(bd), (adcb)\}$ on four symbols.*

Isomorhpism of G and G' will be established if we define mapping of $G \to G'$ s.t. identity element of G is mapped to identity element of G' and inverses are mapped to inverses since then the elements of same order are mapped to elements of the same order.

In G', the order of $(ac)(bd)$ is 2 and the order of each of $(abcd)$ and $(adcb)$ is 4.

Now

$[(ac)(bd)]^2 = (ac)(bd)(ac)(bd) = (ac)(ac)(bd)\cdot(bd)$, product of disjoint cycles being abelian

$= (ac)^2 (bd)^2 = II = I$ as $(ac)^2 = I$, $(bd)^2 = I$.

and $(abcd)^2 = \begin{pmatrix} a & b & c & d \\ b & c & d & a \end{pmatrix}\begin{pmatrix} a & b & c & d \\ b & c & d & a \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ c & d & a & b \end{pmatrix} = (ac)(bd)$

so that $(abcd)^4 = [(ac)(bd)]^2 = I$ as above

Similarly $(adcb)^4 = I$

We can thus define a mapping $f: G \to G'$ given by

$f(1) = I, f(-1) = (ac)(bd) = A$ (say), $f(i) = (abcd) = B$ (say), $f(-i) = (adcb) = C$ (say)

The mapping is evidently one-one and onto. The composition tables for G and G' are as shown here.

·	1	−1	i	−i
1	1	−1	i	−i
−1	−1	1	−i	i
i	i	−i	−1	1
−i	−i	i	1	−1

Table for G

·	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	A	I
C	C	B	I	A

Table for G'

Clearly in the Table of G if $1, -1, i, -i$ are replaced by I, A, B, C respectively then it transforms to the Table for G'

Hence $G \cong G'$.

Problem 34. *Find the regular permutation group isomorphic to the group $G = \{a, b, c, d\}$ with the composition table.*

·	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

Let G' be the required regular permutation group. Then by Cayley's theorem G' will consist of four permutations p_1, p_2, p_3, p_4 given by

$$p_1 = \begin{pmatrix} a & b & c & d \\ aa & ab & ac & ad \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ a & b & c & d \end{pmatrix} = I \text{ (by given composition table)}$$

$$p_2 = \begin{pmatrix} a & b & c & d \\ ba & bb & bc & bd \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ b & a & d & c \end{pmatrix} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} c & d \\ d & c \end{pmatrix} = (ab)(cd)$$

$$p_3 = \begin{pmatrix} a & b & c & d \\ ca & cb & cc & cd \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ c & d & a & b \end{pmatrix} = \begin{pmatrix} a & c \\ c & a \end{pmatrix}\begin{pmatrix} b & d \\ d & b \end{pmatrix} = (a\,c)(b\,d)$$

$$p_4 = \begin{pmatrix} a & b & c & d \\ da & db & dc & dd \end{pmatrix} = \begin{pmatrix} a' & b & c & d \\ d & c & b & a \end{pmatrix} = \begin{pmatrix} a & d \\ d & a \end{pmatrix}\begin{pmatrix} b & c \\ c & b \end{pmatrix} = (a\,d)(b\,c)$$

Hence $G' = [\{p_1, p_2, p_3, p_4\}, o]$

$= [\{I, (ab)(cd), (ac)(bd), (ad)(bc)\}, o]$

4.10. NORMAL AND CONJUGATE SUB-GROUPS

Conjugate elements. *Given a group G, an element $a \in G$ is known as the conjugate to another element $b \in G$ if $\exists$ an element $x \in G$, s.t. $a = x^{-1}bx$.*

If $a = x^{-1}bx$ then a is sometimes known as the *transform* of b by x.

Inner and Outer Automorphisms. *If a be a fixed element of a group G then the conjugation* $C_a : x \to a^{-1} xa$ *is said to be the* **Inner automorphism** *and all other automorphisms are* **outer.** *Clearly* $(a^{-1}xa)(a^{-1}ya) = a^{-1}(xy)a$ *for all x, y.*

e.g., the cyclic group of order 3 has no inner automorphism except the identity while it has outer automorphism $x \leftrightarrow x^2$

The inner automorphisms of any group G from an automorphism of G.

If C_a and C_b be two inner automorphisms, then

$$C_a \cdot C_b = C_{ab} \text{ since } b^{-1}(a^{-1}xa)\, b = (ab)^{-1} x\, (ab)$$

where
$$C_a : x \to a^{-1} xa$$
$$C_b : x \to b^{-1} xa$$

Similarly since $(a^{-1})^{-1}(a^{-1}xa)(a^{-1}) = x$

$\therefore$ the inverse of the conjugation C_a is C_a^{-1}

Properties of conjugate elements

(*i*) **Conjugacy is reflexive** *i.e. every element is conjugate with itself.*

We have $a = x^{-1} bx$; $a, b, x \in G$ ...(1)

If e be tthe identity element in G, then $e \in G$ and $e^{-1} = e$.

Thus replacing x by e in (1) we get $e^{-1} be = e^{-1} b = b$

$\therefore$ $b = e^{-1} be \Rightarrow b$ is conjugate with itself.

(*ii*) **Conjugacy is symmetric** *i.e. if a is conjugate with b, then b is conjugate with a.*

We have $a = x^{-1} bx; a, b, x \in G$

$\therefore$ $a = x^{-1} bx \Rightarrow xa = x(x^{-1} bx)$

$\Rightarrow xa = ebx$ $\qquad \because x(x^{-1}bx) = (xx^{-1})bx = ebx$

$\Rightarrow xax^{-1} = (bx)x^{-1}$ $\qquad \because eb = b$

$\Rightarrow xax^{-1} = be$ $\qquad \because (bx)x^{-1} = b(xx^{-1}) = be = b$

$\Rightarrow xax^{-1} = b$

So that

$$a = x^{-1} bx \Rightarrow b = (x^{-1})^{-1} a (x^{-1}), x^{-1} \in G$$

Which shows that b is conjugate with a.

(*iii*) **Conjugacy is transitive** *i.e. if a is conjugate with b, b is conjugate with c, then a is conjugate with c.*

We have

$$a = x^{-1} bx \text{ and } b = y^{-1}cy, a, b, c, x, y \in G$$

$\therefore a = x^{-1} bx,\ b = y^{-1} cy \Rightarrow a = x^{-1}(y^{-1}cy)\, x$

$\Rightarrow a = (x^{-1}y^{-1})\, c\, (yx)$

$\Rightarrow a = (yx)^{-1} c\, (yx)$ by reversal law of inverses

as $x, y \in G \Rightarrow xy \in G$

$\Rightarrow a$ is conjugate with c.

Note 1. *The above properties (i), (ii), (iii) when combined, show that conjugacy is an equivalence relation on G.*

(*iv*) *If a is conjugate with b and c both, then, b and c are conjugate with each other.*

We have

$$a = x^{-1}bx \text{ and } a = y^{-1} cy$$

$\therefore$ $a = x^{-1} bx, a = y^{-1} cy \Rightarrow x^{-1} bx = y^{-1} cy$

$$\Rightarrow x\,(x^{-1}bx) = x\,(y^{-1}cy)$$
$$\Rightarrow bx = (xy^{-1})\,(cy) \because\; x(x^{-1}\,bx) = (xx^{-1})\,bx = ebx = bx$$
$$\Rightarrow (bx)\,x^{-1} = (xy^{-1})\,(cy)\,x^{-1}$$
$$\Rightarrow b = (xy^{-1})\,c\,(yx^{-1}) \because\; (bx)\,x^{-1} = b(xx^{-1}) = be = b$$

Now since $x, y \in G \Rightarrow x^{-1}, y^{-1} \in G$ and so $yx^{-1}, xy^{-1} \in G$.

Hence b is conjugate with c.

Class of the group. It is observed that there may be more than two elements of a group, which are conjugate with one another. The entire set of elements $a_1, a_2, a_3, \ldots a_n$ which are conjugate with one another is termed as a *class of the group.*

If a single element of a class of a group is given then the whole class may be determined *e.g.* if the elements of a group G are $a_1(=e), a_2, a_3 \ldots, a_n$ then the class of a may be determined by forming the sequence

$$e^{-1}\,ae = a_2^{-1}\,aa_2,\; a_3^{-1}\,aa_3, \ldots, a_n^{-1}\,aa_n.$$

Evidently all the elements of this sequence being conjugate to one another form a class.

Properties of classes

(*1*) *Every element conjugate can be divided in to classes.*

(2) *Every element will appear in one and only one class.*

(*3*) *The identity element of a group being not conjugate to any other element, forms a group by itself since* $x^{-1}\,ex = e\;\forall\; x \in G$.

(*4*) *No class can be a subgroup unless it contains only the identity element e.*

(*5*) *Every element of an abelian (commutative) group being conjugate with itself since* $ax = xa \Rightarrow x^{-1}\,ax = x^{-1}\,xa$

$$\Rightarrow x^{-1}\,ax = ea = a\;\forall\; x \in G$$

the class of an abelian group consists of a single element.

(*6*) *All the elements of a class have the same order.*

If a be an element of a class then $x^{-1}\,ax$ will also be one element of it. Taking n as order of a, we have $a^n = e$, e being identity element.

$$\therefore\; (x^{-1}\,ax)^n = (x^{-1}\,ax)\,(x^{-1}\,ax)^{n-1}$$
$$= x^{-1}\,ax'.\,x^{-1}\,ax.\,(x^{-1}\,ax)^{n-2}$$
$$= a^2\,(x^{-1}ax)^{n-2}\ldots.$$
$$= a^4\,(x^{-1}\,ax)^{n-4}$$
$$= \ldots\ldots\ldots\ldots\ldots\ldots$$
$$= a^n = e$$

which follows that order of $x^{-1}\,ax$ is also n.

Hence the order of all the elements of a class is the same.

As an example consider the group G of matrices and divide the elements of this group into classes s.t. the matrices A and B belong to the same class.

We have

$$A = C^{-1}\,BC^{-1},\, A, B, C \in G.$$

$\therefore$ Trace A or $tr\,(A) = tr\,[C^{-1}\,BC^{-1}]$

$$= \sum_p \left[C^{-1}BC^{-1}\right]_{p\times p}$$

$$= \sum_p \left[\sum_{qr} (C^{-1})_{pq} (B)_{qr} (C)_{rp}\right]$$

$$= \sum_p \left[\left(\sum_q (C^{-1})_{pq}\right)\left(\sum_r (B)_{qr}\right)(C)_{rp}\right]$$

$$= \sum_q \sum_r \left[\left(\sum_p (C)_{rp} (C^{-1})_{pq}\right)(B)_{qr}\right]$$

$$= \sum_q \sum_r [CC^{-1}]_{rq} [B]_{qr}$$

$$= \sum_q \sum_r (I)_{rq} [B]_{qr}, I \text{ being unite matrix s.t.}$$

$$|I|_{rq} = 0 \text{ for } r \neq q$$
$$= 1 \text{ for } r = q$$

Hence for $r = q$, $tr\,(A) = \sum_q (B)_{pq} = tr\,(B)$

which shows that all the matrices forming a class have the same trace.

Note 2. This is a result analogus to the property (6).

Note 3. If H be a sub-group of a group G and $x \in G$, then $K = x^{-1} Hx$ is a subgroup of G.

Let $H = \{h_1, h_2, \ldots h_i \ldots\}$ and $x^{-1} h_i x$ and $x^{-1} h_j x$ be elements of the set $K \subset G$. Then to show that K is a sub-group of G, it is sufficient to show that $(x^{-1} h_i x)(x^{-1} h_j x) \in K$.

We have

$$(x^{-1} h_i x)(x^{-1} h_j x)^{-1} = (x^{-1} h_i x)(x^{-1} h_j^{-1} x)$$

by reversal law of inverses and since $(x^{-1})^{-1} = x$

$$= x^{-1} h_i (xx^{-1}) h_j^{-1} x$$
$$= x^{-1} h_i e\, h_j^{-1} x$$
$$= x^{-1} h_i h_j^{-1} x$$
$$= x^{-1} hx \because h_i, h_j^{-1} \in H \Rightarrow h_i h_j^{-1} \in H \text{ and put } h = h_i h_j^{-1}$$

Hence $x^{-1} hx \in K$.

So K is a sub-group of G.

Conjugate sub-groups. *If x, y, z, etc. be the elements of a group G i.e. x, y, z,... ∈ G, then the subgroups H, $x^{-1} Hx$, $y^{-1} Hy$, $z^{-1} Hz$,..., are known as the* **conjugate sub-groups** *of G.*

Normal Sub-groups (or Normal divisor or Invariant sub-group or Self-conjugate sub-group). *A subgroup H of a group G is said to be a* **normal sub-group** *of G if $\forall x \in G$, $x^{-1} Hx = H$ or equivalently, if $Hx = xH$ $\forall x \in G$.*

Properties of normal subgroups

(a) *If e be the identity in G, then the whole group G and {e} are normal subgroups of G* (*b*) *Every sub-group H of a commutative five group G is normal since a left coset x H is the same as the right coset Hx since*

$$x \in H \Rightarrow x^{-1} Hx = H \ \forall \ x \in G.$$

Every subgroup of an abelian group is a normal sub-group, since

$$a^{-1} xa = a^{-1} ax = x \ \forall \ a, x \in G.$$

(*c*) *The alternating group A_n is an invariant sub-group of the symmetric group S_n.*

Since if E be an element of A_n *i.e.* $E(X) = X$, then we have to show that

$$P \in S_n \Rightarrow P^{-1} EP \in A_n$$

If P is even, then $P^{-1} EP$ is even and hence is an element of A_n.

If P is odd, then $P(X) = -X$ or $P^{-1}(-X) = X$

and $(P^{-1}EP)(X) = P^{-1} E[P(X)] = P^{-1} E(-X) = P^{-1}(-X) = X$

which follows that $P^{-1} EP \in A_n$ and hence A_n is a normal sub-group.

(*d*) *The intersection of any two normal sub-groups of a group is a normal sub-group.*

If H_1, H_2, be two normal sub-groups of G and $a \in H_1 \cap H_2$, then

$$a \in H_1 \cap H_2 \Rightarrow a \in H_1, a \in H_2$$

But H_1, H_2 being normal sub-groups,

$$x^{-1} ax \in H_1 \text{ and } x^{-1} ax \in H_2 \ \forall \ x \in G$$

These imply that $x^{-1} ax \in H_1 \cap H_2$

Hence $a \in H_1 \cap H_2 \Rightarrow x^{-1} ax \in H_1 \cap H_2 \ \forall \ x \in G$

and also $H_1 \cap H_2$ is a sub-group of G

$\therefore H_1 \cap H_2$ is a normal sub-group.

Factor group or quotient group. *If H be a normal sub-group of a group G, then the group of all cosets of H in G is known as* **Factor group** *or* **quotient group** *of G by H and denoted by G/H.*

Properties of factor group

(α) The order of a factor group G/H is equal to the index of H in G.

(β) Each quotient group of an abelian group is ablelian but its converse is not true, since

$$(Hx)(Hy) = H(xy) = H(yx) = H(y) H(x)$$

and if S_3 be a symmetric group and A_3 an alternating group each of degree 3 then S_3/A_3 is an abelian group of degree 3 whereas S_3 is not abelian. The group S_3/A_3 is of order 2 and so it is abelian as every group of order 2 is abelian.

Problem 35. *If H be a subgroup of the group* (I, +), I *being set of integers, s.t.* $H = (mx : x \in \text{I})$ *where m is a fixed integer; then find the elements of the quotient group* I/*H and mention the composition table for* I/*H for m* = 5.

Clearly (I, +) is a abelian group, therefore by the properties of normal subgroups, H is a normal subgroup. The elements of I/H, which are cosets of H in I may be given as follows:

$$H + 0 = H = \{\ldots, -3m, -2m, -m, 0, m, 2m, \ldots\}$$

$$H + 1 = \{\ldots\ldots -3m + 1, -2m + 1, -m + 1, 1, m + 1, 2m + 1, \ldots\ldots\}$$

....

$$H + (m-1) = \{\ldots\ldots -2m - 1, -m - 1, -1, m - 1, 2m - 1, 3m - 1, \ldots\}$$

Then I/H has n distinct cosets as its elements.

When $m = 5$, the cosets are $H, H+1, H+2, H+3, H+4$.

The composition table is as shown here.

·	H	$H+1$	$H+2$	$H+3$	$H+4$
H	H	$H+1$	$H+2$	$H+3$	$H+4$
$H+1$	$H+1$	$H+2$	$H+3$	$H+4$	H
$H+2$	$H+2$	$H+3$	$H+4$	H	$H+1$
$H+3$	$H+3$	$H+4$	H	$H+1$	$H+2$
$H+4$	$H+4$	H	$H+1$	$H+2$	$H+3$

Problem 36. If $G = \{e, a, a^2, a^3, a^4, a^5\}$ *be a cyclic group of order 6 such that every subgroup of a cyclic group is normal, then if* $H = \{e, a^3\}$ *be a subgroup of G, find the elements of G/H and show that it is a group.*

Elements of G/H are $He = \{e, a^3\}\ e = \{e, a^3\}$

$$Ha = \{e, a^3\}\ a = \{a, a^4\}$$

$$Ha^2 = \{e, a^3\}\ a^2 = \{a^2, a^5\}$$

This is easy to show that G/H is a group and its composition table is as shown here.

Evidently G/H is a cyclic group generated by $\{a, a^4\}$.

	$\{e, a^3\}$	$\{a, a^4\}$	$\{a^2, a^5\}$
$\{e, a^3\}$	$\{e, a^3\}$	$\{e, a^4\}$	$\{a^2, a^5\}$
$\{a, a^4\}$	$\{a, a^4\}$	$\{a^2, a^5\}$	$\{e, a^3\}$
$\{a^2, a^5\}$	$\{a^2, a^5\}$	$\{e, a^3\}$	$\{a, a^4\}$

4.11. COMPLEXES AND KERNEL

Complex of a group. *A non-empty subset H of a group G is called as a* **complex** *of the group G.*

Properties of complexes. (*i*) *If Z be a complex containing the elements* a, b, c of a group G then $Z = \{a, b, c\}$

(*ii*) If $Z = \{a, b, c\}$ be a complex then $aZ = \{a^2, ab, ac\}$ etc.

(*iii*) If Z_1 and Z_2 be two complexes of a group G, then the product of Z_1, Z_2 is defined as

$$Z_1 Z_2 = \{x : x = z_1 z_2, z_1 \in Z_1, z_2 \in Z_2\}$$

Now since $z_1 \in Z_1, z_2 \in Z_2$ and $Z_1, Z_2 \subset G$

$\therefore\ z_1 z_2 = x \in G$ by closure axiom.

As such $Z_1 Z_2 \subset G$.

Which follows that $Z_1 Z_2$ is also a complex of G, obtained by multiplying every element in Z_1 with every element in Z_2.

(*iv*) *The subgroup H of a group G also gives a complex s.t.* $HH = H^2 = H$.

(*v*) *A group can be expressed as a sum of complexes.*

If $x \in G$ and $x \notin H$, H being a sub-group of G, then the complex Hx is a right coset and xH is a left coset of H in G. But cosets are not groups and they are complexes, therefore if the group G as a whole is capable of forming a complex Z which consists of all the elements of the group, then we have

$$Z = H + Hx + Hy + \ldots\ldots$$

(*vi*) *The number of complexes in a group is equal to the index of a subgroup H in G and in fact it is the order of the group divided by the order of the subgroup H.*

(vii) *The product of complexes is associative.*

Let Z_1, Z_2 and Z_3 be three complexes of a group G and let :

$z_1 \in Z_1, z_2 \in Z_2, z_3 \in Z_3$, then

$z_1 \in Z_1, z_2 \in Z_2 \Rightarrow z_1 z_2 \in Z_1 Z_2$

$\therefore \quad z_1 z_2 \in Z_1 Z_2, z_2 \in Z_3 \Rightarrow (z_1 z_2) z_3 \in (Z_1 Z_2) Z_3$

$\Rightarrow z_1 z_2 z_3 \in (Z_1 Z_2) Z_3$

But $z_1 z_2 z_3 = z_1 (z_2 z_3)$ $\qquad \therefore \quad z_1 z_2 z_3 \in Z_1 (Z_2 Z_3)$

Thus $z_1 z_2 z_3 \in (Z_1 Z_2) Z_3 \Rightarrow z_1 z_2 z_3 \in Z_1 (Z_2 Z_3)$

$\therefore \quad (Z_1 Z_2) Z_3 \subset Z_1 (Z_2 Z_3)$

Similarly $Z_1 (Z_2 Z_3) \subset (Z_1 Z_2) Z_3$

So that $(Z_1 Z_2) Z_3 = Z_1 (Z_2 Z_3)$

Inverse of complex. If Z be a complex of a group G, then its inverse is given by $Z^{-1} = \{z^{-1} : z \in Z\}$

In other words the inverse of a complex Z is the set of inverses of all elements of Z.

Properties of inverse of a complex

(1) *If $Z_1 Z_2$ be two complexes of a group G, then* $(Z_1 Z_2)^{-1} = Z_2^{-1} Z_1^{-1}$

And $x \in (Z_1 Z_2)^{-1} \Rightarrow x = (z_1 z_2)^{-1}$ for $z_1 \in Z_1, z_2 \in Z_2$

$\Rightarrow x = z_2^{-1} z_1^{-1}$ by reversal law of inverses.

$\Rightarrow x \in Z_2^{-1} Z_1^{-1}$ by definition

$\therefore \quad (Z_1 Z_2)^{-1} \subset Z_2^{-1} Z_1^{-1}$ ...(A)

Similarly if $y \in Z_2^{-1} Z_1^{-1} \Rightarrow y = z_2^{-1} z_1^{-1}$ when $z_2^{-1} \in Z_2^{-1}, z_1^{-1} \in Z_1^{-1}$

$\Rightarrow y = (z_1 z_2)^{-1}$ where $z_1 \in Z_1, z_2 \in Z_2$

$\Rightarrow y \in (Z_1 Z_2)^{-1}$ by definition

$\therefore \quad Z_2^{-1} Z_1^{-1} \subset (Z_1 Z_2)^{-1}$ ...(B)

(A) and (B) follow that $(Z_1 Z_2)^{-1} = Z_2^{-1} Z_1^{-1}$

(2) *If H be a subgroup of a group G, then $H^{-1} = H$.*

An $h^{-1} \in H^{-1} \Rightarrow h \in H$

$\Rightarrow h^{-1} \in H$, H being a group

So $H^{-1} \subset H$

Similarly an $h \in H \Rightarrow h^{-1} \in H$, H being a group

$\Rightarrow h = (h^{-1})^{-1} \in H$ by definition of inverse of a complex.

So $H \subset H^{-1}$

$\therefore \quad H^{-1} \subset H, H \subset H^{-1} \Rightarrow H^{-1} = H.$

(3) *If H, K be two subgroups of a group G, then HK is also a subgroup of G iff $HK = KH$.*

Taking $HK = KH$, we have $(HK)^{-1} = (KH)^{-1}$

$= K^{-1} H^{-1}$ by Property (1)

$= KH$ by Property (2)

$= HK \qquad \because HK = KH$

Which shows that HK is a subgroup of G.

Again taking HK as subgroup of G, we have

$(HK)^{-1} = HK$ by Property (2)

$\therefore \quad K^{-1} H^{-1} = HK$ by Property (1) *i.e.* $KH = HK$ by Property (2)

Hence the proposition.

(4) *A necessary and sufficient condition for a complex H of a group G to be a subgroup is that* $HH^{-1} = H$.

The condition is necessary since if H is a subgroup of G and $ab^{-1} \in HH^{-1}$ then

$$a \in H, b \in H \Rightarrow a \in H, b^{-1} \in H$$
$$\Rightarrow ab^{-1} \in H$$

So $ab^{-1} \in HH^{-1} \Rightarrow ab^{-1} \in H, b \in H, b^{-1} \in H^{-1}$

i.e. $HH^{-1} \subset H$

Also H is a sub-group of $G \Rightarrow$ identity $e \in H$

If $h \in H$, then $h = he = he^{-1} \in HH^{-1}, h \in H, e^{-1} \in H^{-1}$

$\therefore\ H \subset HH^{-1}$

Thus $HH^{-1} \subset H, H \subset HH^{-1} \Rightarrow HH^{-1} = H$

The condition is sufficient since if $HH^{-1} = H$, then we have

$$\therefore HH^{-1} \subset H$$

Now suppose that $a, b \in H$ so that $ab^{-1} \in HH^{-1}$

$\therefore\ HH^{-1} \subset H$ and $ab^{-1} \in HH^{-1} \Rightarrow ab^{-1} \in H$

Ultimately $a \in H, b \in H \Rightarrow ab^{-1} \in H$

Which follows that H is a sub-group as is evident from the following discussion: Taking H a subgroup of G with the same composition as in G, the identity in H and G is the same. Also $a \in H$ and $b \in H$ give $b^{-1} \in H$, H being a group.

$\therefore\ a \in H, b^{-1} \in H \Rightarrow ab^{-1} \in H$ ...(α)

Further taking H to be a non-empty subset of G s.t. $a \in H, b \in H$, and assuming that $a \in H; b \in H \Rightarrow ab^{-1} \in H$, we observe that H is non-empty and $\exists$ an $a \in H$ so that by setting $b = a$ in (α), we find

$$a \in H, a \in H \Rightarrow aa^{-1} \in H$$
$$\Rightarrow e \in H, e \text{ also being identity in } G.$$

Now $e \in H, b \in H \Rightarrow eb^{-1} \in H$ by (α) ...(β)

$$\Rightarrow b^{-1} \in H$$

$\therefore\ a \in H, b^{-1} \in H \Rightarrow a(b^{-1})^{-1} \in H \Rightarrow ab \in H$

i.e. $a \in H, b \in H \Rightarrow ab \in H$

Here (α) and (β) fulfil the requirements for H which is a complex of G, to be its subgroup

Image of a group G under a mapping f. *If* $f: G \to G'$ *be a homomorphism of a group G into a group G', then* $f(G) = \{f(x) \in G' : x \in G\}$ *is a subset of G' and is termed as the* **Image of G under f** and denoted by $Im(f)$.

Kernel of f. *If* $f: G \to G'$ *be a homomorphism of a G into G', then the subset of those elements of G which are mapped onto the identity of G' under f is said to be the* **Kernel of f** *and denoted by* $\ker(f)$ *or* $f^{-1}(e')$.

i.e. $\ker(f) = \{x \in G : f(x) = e'\}$

Propositions relating to Kernel

I. *A homomorphism* $f: G \to G'$ *is an isomorphism iff* $\ker f = \{e\}$.

Assuming that $f: G \to G'$ is an isomorphism, if $a \in \ker f$ then

$$f(a) = e' = f(e), e' \text{ being identity in } G'.$$

Now f being one-one and $a = e$, kernel of f consists of e only. Conversely if $\ker f = \{e\}$ for f to be homomorphism, and if $a, b \in G$ s.t. $f(a) = f(b)$, then

$$f(ab^{-1}) = f(a)\,f(b^{-1})$$
$$= f(a)\,[f(b)]^{-1}$$
$$= e' \qquad \because f(a) = f(b)$$

$\therefore \quad ab^{-1} \in \ker f$

or $\quad ab^{-1} = e$

or $\quad a = b$

So f is one-one and hence f is an isomorphism.

II. *If f be homomorphism of G' then $\ker(f)$ is an invariant subgroup of G.*

If $a, b \in \ker(f)$, then $f(a) = e' = f(b)$, e' being identity of G.

$\therefore \quad f(ab) = f(a)\,f(b) = e'\,e' = e'$

which implies that $ab \in \ker(f)$ *i.e.* closure axiom is satisfied.

Now $\ker(f)$ being a subset of G, associativity axiom is self-evident.

Again $f(e) = e' \Rightarrow e \in \ker(f)$, e being identity in G.

$\therefore$ There exists an identity in G.

Further if $a \in \ker(f)$ then $f(a^{-1}) = [f(a)]^{-1} = (e')^{-1} = e'$

which shows that $a^{-1} \in \ker(f)$ when $a \in \ker(f)$

This follows the existence of an inverse in G.

As such $\ker(f)$ is a subgroup of G, as $\ker(f)$ satisfies all the four group axioms.

Moreover $\ker(f)$ is an invariant subgroup of G as is shown below:

If $g \in G$ and $h \in \ker(f)$, then

$$f(g^{-1}hg) = f(g^{-1})\,f(h)\,f(g)$$
$$= [f(g)]^{-1}\,e'\,f(g) \qquad \because h \in \ker(f) \Rightarrow f(h) = e'$$
$$= [f(g)]^{-1} f(g)$$
$$= e'$$

$\therefore \quad g^{-1}hg \in \ker(f)$.

Hence $\ker(f)$ is an invariant sub-group of G.

Note 1. *It is easy to show that $Im(f)$ is a subgroup of G.*

III. *If H be a normal sub-group of a group G, then there is a homomorphism of G onto G/H.*

Let $f: G \to G/H$ be given by $f(x) = Hx\ \forall\ x \in G$

$\because\ \forall\ x \in G, \exists$ a unique coset Hx, f is a mapping.

Also the binary operation in G/H being defined by

$$(Hx)\,(Hy) = H\,(xy)$$

we have

$$f(xy) = H(xy) = (Hx)\,(Hy) = f(x)\,f(y)$$

Which follows that f is a homomorphism and it is onto since every coset $Hz \in G/H$ has z as its preimage in G.

Note 2. Natural Homomorphism. *The homomorphism $f: G \to G/H$ given by $f(x) = Hx$ is known as* **Natural Homomorphism** *or* **Canonical Homomorphism** *of G onto G/H.*

IV. *If f be a homomorphism of a group G onto a group G' with kernel k, then*

$$G/K \cong G'$$

Consider the mapping $\phi : G/K \to G'$ defined by $\phi\,(Kx) = f(x)$

Taking $Kx = Ky$, we have $xy^{-1} \in K$ and $f(xy^{-1}) = e'$, e' being identity in G' *i.e.* $f(x) f(y^{-1}) = e'$

or $f(x)\, [f(y)]^{-1} = e'$

or $f(x) = f(y)$.

This follows that ϕ is uniquely defined.

Now if $f(y) \in G'$ then Ky is the preimage of $f(y)$ in G/K under ϕ.

This follows that ϕ is onto.

Again ϕ will be one-one if $Kx = Ky$ provided $f(x) = f(y)$.

(Take an element $z = xy^{-1} \in G$ *i.e.* $zy = x$

$\therefore\ f(z) = f(xy^{-1}) = f(x) f(y^{-1})$

$= f(x)\, [f(y)]^{-1}$

$= e'$ $\qquad\qquad \because\ f(x) = f(y)$

So that $z \in K$ and $Kx = K(zy) = (Kz)y = Ky$

$\therefore\ \phi$ is one-one.

Further to show that ϕ preserves the structure, we have

$\phi(Kx)\, \phi(Ky) = f(x) f(y) = f(xy) = \phi\, [K\,(xy)] = \phi\, [(Kx)\,(Ky)]$

Hence ϕ is isomorphism and thus $G/K \cong G'$.

V. *If f is a homomorphism from the group (G, o) into the group (G', o') then the pair $(\ker f, o)$ is a normal subgroup of (G, o).*

Evidently $\ker f \neq \phi$ (non empty) since $e \in \ker f$ and $\ker f \subset G$

Now $a, b \in \ker f \Rightarrow f(a) = e', f(b) = e'$

But $f(b^{-1}) = [f(b)]^{-1} = [e']^{-1} = e'$

$\therefore\ f(aob^{-1}) = f(a) o\, [f(b)]^{-1} = e'\, oe' = e'$

$\therefore\ a, b \in \ker f \Rightarrow aob^{-1} \in \ker f$

Hence $(\ker f, o)$ is a subgroup.

Again $\forall\ a \in G$ and $h \in \ker f$, we have

$f(ao\, hoa^{-1}) = f(a)\, o\, f(h)\, o\, f(a^{-1})$

$= f(a)\, o\, f(h)\, o\, [f(a)]^{-1}$

$= f(a)\, oe'\, o\, [f(a)]^{-1}$

$= f(a)\, o\, [f(a)]^{-1}$

$= e'$

$\therefore\ \forall\, a \in G$ and $h \in \ker f \Rightarrow aohoa^{-1} \in \ker f$

Hence $(\ker f, o)$ is a normal sub-group.

Note 3. Similarly it can be shown that image of $[Im\,(f), o]$ is a sub-group of $(G'\, o')$ when f is a homomorphism of $(G\, o)$ into (G', o').

Problem 37. *If $GL\,(n, R)$ is the multiplication group of all $n \times n$ singular matrices with elements as real numbers and that G' is the multiplicative group of all non-zero real numbers, then show that the mapping $f : G \to G'$ s.t. $f(A) = |A|\ \forall\, A \in G$ is a homomorphism of G onto G' and also show that*

$\ker f = \{A \in GL\,(n, R) : |A| = e'$, the identity in $G'\}$.

Let $f : (\mathbf{C}, +) \to (\mathbf{R}^+, +)$ s.t. $f(x + iy) = x$.

We have to show that f is a homomorphism of $(\mathbf{C}, \cdot)$ onto $(\mathbf{R}, \cdot)$ and $\ker f = \{z \in \mathbf{C} : x = 0\}$ *i.e.* $\ker f$ is the imaginary y-axix.

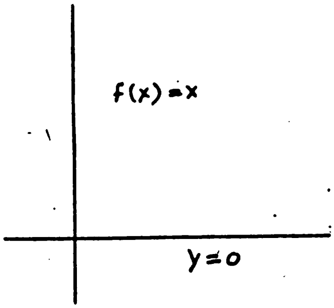


Fig. 4.11

If $z_1 = x_1 + iy_1 \in \mathbf{C}$, $z_2 = x_2 + iy_2 \in \mathbf{C}$, then

$$f(x_1 + iy_1) + f(x_2 + iy_2)$$
$$= f(x_1 + x_2) + if(y_1 + y_2)$$
$$= x_1 + x_2 \text{ by hypothesis}$$
$$= f(x_1 + iy_1) + f(x_2 + iy_2)$$

This shows that f is onto since $f(x + io) = x$ if $x \in \mathbf{R}$.

As such f is a homomorphism of $(\mathbf{C}^*, +)$ onto $(\mathbf{R}^+, +)$ and 1 is the identity element in $(\mathbf{R}^+, \cdot)$

Also ker f is given by $f(x + iy) = 0 = x$ $\forall x \in \mathbf{R}$.

i.e. ker $f = \{z \in \mathbf{C} : x = 0 = e'$, the identity in $\mathbf{R}\}$

which follows that ker f is the imaginary axis.

Problem 38. *If* $(\mathbf{R}, \cdot)$ *be a multiplicative group and* $x \in \mathbf{R}$, *then find homomorphisms and their kernels in the following mappings*

(*i*) $x \to |x|$

(*ii*) $x \to \dfrac{1}{x^2}$.

(*i*) $x \to |x| \Rightarrow x \to |x|$ and $-x \to |x|$

$\therefore$ $x \to |x|$ and $y \to |y| \Rightarrow xy \to |xy| \to |x|\,|y|$

Thus $x \to |x|$ is homomorphism.

Now $f : x \to |x| \Rightarrow f(x) = |x| \Rightarrow f(xy) = |xy| \Rightarrow f(xy) = |xy|$
$= |x|\,|y| = f(x) f(y)$

i.e. f is two-one mapping since $|f(x)| \Rightarrow f(x) = -1, 1$.

Its kernel is $|f(x)| = 1 \Rightarrow -1, +1$, *i.e.* $\{-1, 1\}$

(*ii*) Say $g : x \to \dfrac{1}{x^2} \Rightarrow g(x) = \dfrac{1}{x^2} \Rightarrow g(xy) = \dfrac{1}{(xy)^2} = \dfrac{1}{x^2y^2} = \dfrac{1}{x^2} \cdot \dfrac{1}{y^2} = g(x) \cdot g(y)$

So g is a homomorphism and it is two-one mapping since

$$g(x) = \frac{1}{x^2} \Rightarrow g(-x) = \frac{1}{x^2}$$

Now $\quad g(1) = e' \Rightarrow \dfrac{1}{x^2} = 1 \Rightarrow x = -1, 1$

$\therefore$ ker $g = \{-1, 1\}$

4.12. GROUPS OF ISOMETRIES*

Ler R be the set of real numbers and S_R be the symmetric group on R. Then $I(R)$ the *group of isometries* of R is a subgroup of S_R and defined to be the set of all elements of S_R which preserve *distance* (distance between two points $a, b \in R$ is the absolute value $|a - b|$ of $a - b$ and denoted by $d(a, b)$) and the elements of such a set are known as isometries of R.

An element $\sigma \in S_R$ is called an *Isometry* iff $d(a, b) = d(a\sigma, b\sigma)$ $\forall$ $(a, b) \in R$ with the identity mapping $i \in I(R)$, $I(R) \neq \emptyset$.

To show that $I(R)$ the group of isometries of R is a subgroup of R, suppose that $\sigma \in I(R)$ so that $\sigma^{-1} \in S_R$ as S_R is a group and $\sigma \in S_R$. Then we have to show that $\sigma^{-1} \in S_R$.

If $a, b \in R$ then σ being an isometry, $d(a\sigma^{-1}, b\sigma^{-1}) = d[(a\sigma^{-1})\sigma, (b\sigma^{-1})\sigma] = d(a, b)$

Thus $d(a, b) = d(a\sigma^{-1}, b\sigma^{-1}) \Rightarrow \sigma^{-1} \in I(R)$

If $\sigma, \tau \in I(R)$ then $\tau^{-1} \in I(R)$ and

$$d[a(\sigma\tau^{-1}), b(\sigma\tau^{-1})] = d[(a\sigma)\tau^{-1}, (b\sigma)\tau^{-1}] = d(a\sigma, b\sigma) = d(a, b)$$

So that $\sigma\tau^{-1} \in I(R)$ and hence $I(R)$ is a subgroup of S_R.

If $\sigma, \tau \in I(R)$ have the same effect on two distinct real numbers a and b i.e., $a\sigma = a\tau$ and $b\sigma = b\tau$ then $\sigma = \tau$

This version is used to describe the elements of $I(R)$.

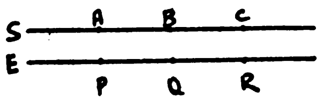
Geometrically interpreted, if $r\sigma = r + a$ where $\sigma \in I(R)$, $r \in R$ and $a\sigma = a$, then it moves the real line a units to the right and if $r\sigma = -r + a$, the real line is inverted about the origin and then moved a units to the right.

If E be the set $R^2 = R \times R$ and $(x_A, y_A) = A$, $(x_B, y_B) = B$ are two elements of E then distance $d(A, B) = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}$

$\sigma \in S_E$ the symmetric group on E is known as an isometry if for $A, B \in E$, $d(A, B) = d(A\sigma, B\sigma)$.

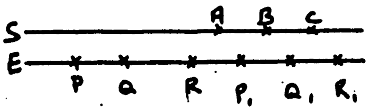
The set I f all isometries of E forms a subgroup of S_E.

Assuming the Euclidean plane E covered by an infinite rigid metal lamina S let $P, Q, R, \ldots$ be the points of E and $A, B, C, \ldots$ be the points of S initially as shown in Fig. 4.12 and after a movement as shown in Fig. 4.13.



Initially

Fig. 4.12



After a movement

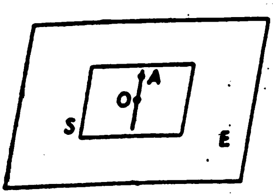
Fig. 4.13

Define $\theta : E \to E$ as an isometry given by

$$P\theta = P_1, Q\theta = Q_1, R\theta = R_{-1} \text{ where } d(A, B) = d(P_1, Q_1) = d(P, Q).$$

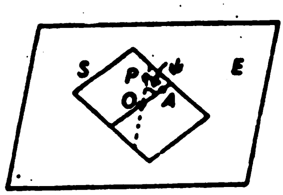
As such we have *three particular isometries.*

(*i*) **Rotation about a point.** Take a point O of S and rotate S about O through



A on the top of *P*

Fig. 4.14



A on the top of P_1

Fig. 4.15

an angle ψ. Such an isometry induced by the movement of S is the rotation about O through an angle ψ.

(*ii*) **Reflection in a line.** Choosing a line in E, turn S over this line and back to E. Such an isometry is the reflection in XY.

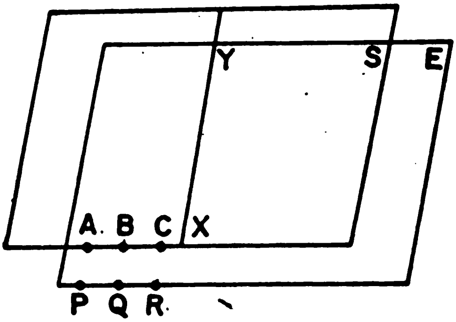


Fig. 4.16

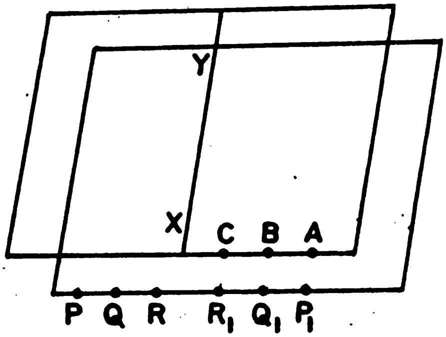


Fig. 4.17

(*iii*) **Translation.** Choosing a line XY and an isometry corresponding to a movement of S s.t. the line $X_\theta Y_\theta$ is parallel to XY, θ is the translation.

In terms of mapping. Translation is the mapping

$$\tau_{a,b} : (xy)\, \tau_{a,b} = (x + a, y + b)$$

which is isometry for each $a, b, \tau_{a,b}$ and $(\tau_{a,b})^{-1} = \tau_{-a,-b}$

Counter clockwise rotation about the origin through an angle θ is the mapping ρ_θ: $(x, y)\, \rho_\theta = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$ which is isometry for each θ, ρ_θ and $(\rho_\theta)^{-1} = \rho_{-\theta}$.

and reflection in OX is the mapping $\sigma_y : (x, y)\, \sigma_y = (x - y)$

which is isometry and $(\sigma_y)^{-1} = \sigma_y$.

Note 1. Isometries are product of reflections, translations and rotations. In other words every isometry E is expressible as the product of a reflection, a translation and a rotation.

Symmetry groups. *If S be a subset of the Euclidean plane then the set I_s of all $\sigma \in I$ s.t. $s \in S \Rightarrow s\sigma \in S$ and $t\sigma \in S \Rightarrow t \in S$, forms a sub-group of I, known as the* **symmetry group** *of S.*

Algebra of symmetries of a equilateral triangle

Case I. *Counter clockwise rotation of an equilateral triangle in its own plane about an axis through geometric centre O and perpendicular to the palne of the triangle ABC.*

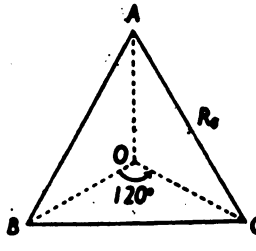


Fig. 4.18

Let us define the rotations as follows:

$R_0 : \Delta ABC \rightarrow \Delta ABC$

$R_1 : \Delta ABC \rightarrow \Delta CAB$

$R_2 : \Delta ABC \rightarrow \Delta BCA$

where R_0 is the same position of the triangle, R_1 in the counter clockwise rotation through 120° which carries A to B, B to C, C to A and R_2 is the counter clockwise rotation through 240° which carries A to C, B to A, C to B. Evidently a counter clockwise rotation through 120° is identical with a clockwise rotation through 240° and similarly a counter clockwise rotation is identical with a clockwise one through 120°. As such R_0, R_1, R_2 are the only three distinct rotations, forming a finite abelian group.

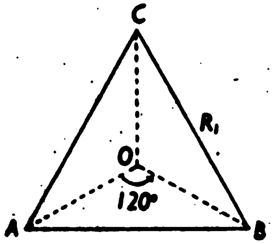


Fig. 4.19

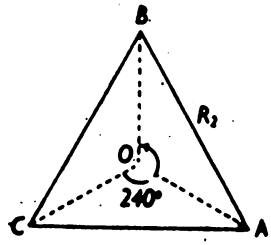


Fig. 4.20

Let $G = \{R_0, R_1, R_2\}$ with the binary operation $R_j o R_k$ being the rotation obtained by j-successive application of R_j and R_k for any two rotations R_j and R_k.

e.g. $R_1 o R_2$ represents the rotation through 120° followed by a rotation through 240° *i.e.* $R_1 o R_2$ is the rotation through 360° and hence

$$R_1 o R_2 = R_0$$

Conclusively

$$\Delta ABC \xrightarrow{R_0} \Delta ABC \xrightarrow{R_1} \Delta CAB \xrightarrow{R_2} \Delta BCA \Rightarrow$$

$$\Delta ABC \xrightarrow{R_0 o R_1} \Delta BCA$$

	R_0	R_1	R_2
R_0	R_0	R_1	R_2
R_1	R_1	R_2	R_0
R_2	R_2	R_0	R_1

which is equivalent to $\Delta ABC \xrightarrow{R_2} \Delta BCA$.

The composition table is as shown here.

This group of rotations G is abelian since, G_1 is satisfied, since every element of the table belong to S_3.

G_2 is satisfied, since $R_0 o(R_1 o R_2) = R_0 o(R_0) = R_0$

and $$(R_0 o R_1) o R_2 = R_1 o R_2 = R_0 = R_0 o(R_1 o R_2)$$

G_3 is satisfied, since R_0 is an identity in S_3.

G_4 is satisfied, since inverses of R_0, R_1, R_2 are R_0, R_2, R_1 respectively as $R_1 o R_2 = R_0$ etc.

Commutative property is also satisfied, since

$$R_0 o R_1 = R_1 = R_1 o\, R_0 \text{ etc.}$$

Case II. *Rotation of an equilateral triangle ABC about the medians AD, BE, CF.*

Let R_3, R_4, R_5 be the rotations about the medians AD, BE and CF respectively of equilateral triangle ABC, each through π. Clearly there are six coincident rotations of the triangle. There is correspondence between the group of rotations.

$$G' = \{R_0, R_1, R_2, R_3, R_4, R_5\}$$

and the symmetric group S_3 whose elements permutate A, B, C.

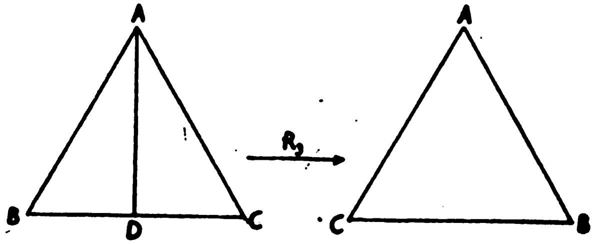


Fig. 4.21

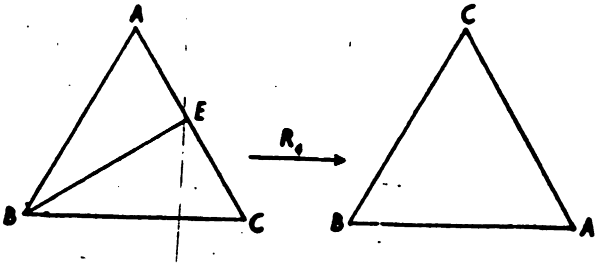


Fig. 4.22

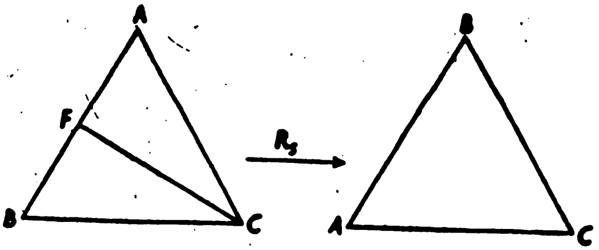


Fig. 4.23

We have,

$R_1 o R_3 = R_4 \because \underset{B\ C}{\overset{A}{\Delta}} \xrightarrow{R_1} \underset{A\ B}{\overset{C}{\Delta}} \xrightarrow{R_3} \underset{B\ A}{\overset{C}{\Delta}}$ by Fig. 4.22.

$R_1 o R_4 = R_5 \because \underset{B\ C}{\overset{A}{\Delta}} \xrightarrow{R_1} \underset{A\ B}{\overset{C}{\Delta}} \xrightarrow{R_4} \underset{A\ C}{\overset{B}{\Delta}}$ by Fig. 4.23.

$R_1 o R_5 = R_3 \because \underset{B\ C}{\overset{A}{\Delta}} \xrightarrow{R_1} \underset{A\ B}{\overset{C}{\Delta}} \xrightarrow{R_5} \underset{C\ B}{\overset{A}{\Delta}}$ by Fig. 4.21.

$R_2 o R_3 = R_5 \because \underset{B\ C}{\overset{A}{\Delta}} \xrightarrow{R_2} \underset{C\ A}{\overset{B}{\Delta}} \xrightarrow{R_3} \underset{A\ C}{\overset{B}{\Delta}}$ by Fig. 4.23.

o	R_0	R_1	R_2	R_3	R_4	R_5
R_0	R_0	R_1	R_2	R_3	R_4	R_5
R_1	R_1	R_2	R_0	R_4	R_5	R_3
R_2	R_2	R_0	R_1	R_5	R_3	R_4
R_3	R_3	R_5	R_4	R_0	R_2	R_1
R_4	R_4	R_3	R_5	R_1	R_0	R_2
R_5	R_5	R_4	R_3	R_2	R_1	R_0

Similarly

$$R_2 o R_4 = R_3,\ R_2 o R_5 = R_4 \text{ etc.}$$

In general $R_j o R_k$ for $j = 0, 1, 2, \ldots 5$ and $k = 0, 1, \ldots 5$ gives the adjoining composition table.

Clearly the set of six rotations of the equilateral triangle forms a non-abelian group.

The di-hedral group

If S be a regular polygon of $n\,(> 2)$ sides, then in any isometry of S, vertices are taken to vertices. Then the order of the symmetry group of a regular n-gon (polygon of sides $n > 2$) can be easily determined.

In this connection the following axioms are to be noted:

(1) Every regular n-gon can be circumscribed by one and only one circle.

(2) The centre of a regular n-gon S is considered onto itself by any element of I_s.

(3) If S be a regular n-gon and $\sigma \in I_s$, then vertices of S are taken onto vertices of S by σ.

The symmetry group of the regular n-gon is said to be the di-hedral group of degree n.

Determination of the orders of the di-hedral groups

Suppose the vertices of a regular n-gon S with centre O are $A_1, A_2, \ldots A_n$ in a clock wise direction. Also suppose that e_j, $1 \le j \le n$ rotates S about O in clockwise direction

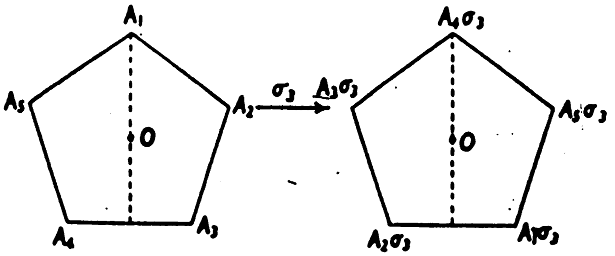


Fig. 4.24

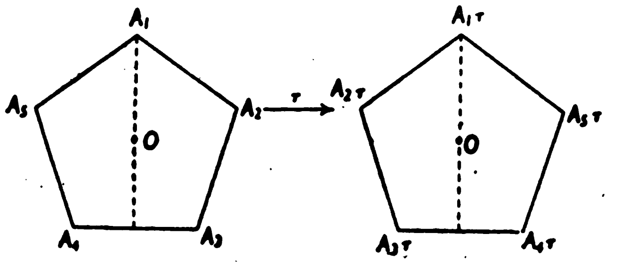


Fig. 4.25

through an angle $\dfrac{2\pi\,(j-1)}{n}$ radians *i.e.*, $\dfrac{360}{n}\,(j-1)$ degrees. So that $A_1\,\sigma_j = A_j$ etc.

For the sake of convenience we have shown here the effect of σ_3 on the regular pentagon (5 – gon) in Fig. 4.24.

Taking τ as the reflection about the line through A_1 and O, s.t.

$$A_1\tau = A_1, \ldots\; A_n\tau = A_n$$

The effect of τ on the regular pentagon is shown in Fig. 4.25.

The Fig. 4.26 shows the effect on a regular pentagon of the reflection τ followed by the rotation.

We observe tha the elements $\sigma_1, \sigma_2 \ldots, \sigma_n, \tau\sigma_1, \ldots \tau\sigma_n$ are all distinct since, of course $\sigma_j \ne \sigma_k, j \ne k$ as $A_1\sigma_j \ne A_1\sigma_k, j \ne k$.

In case $\tau\sigma_j = \sigma_k$, then $A_1\tau\sigma_j = A_1\sigma_j = A_1\sigma_k$

$\therefore\; \tau\sigma_j = \sigma_k \Rightarrow j = k$

But $\tau\sigma_j = \sigma_j \Rightarrow \tau = \sigma_1$, the identity which contradicts the hypothesis. Finally $\tau\sigma_j = \tau\sigma_k \Rightarrow \sigma_j = \sigma_k$.

Consequently there are at least $2n$ possible elements of the dihedral group of degree n, but there are no more than $2n$ since if $\sigma \in I_s$, S being regular n-gon, then there are n possibilities for $A_1\sigma$.

Since the vertices are taken to vertices, therefore $A_1\sigma$ is one of $A_1, ..., A_n$ and $A_2\sigma$

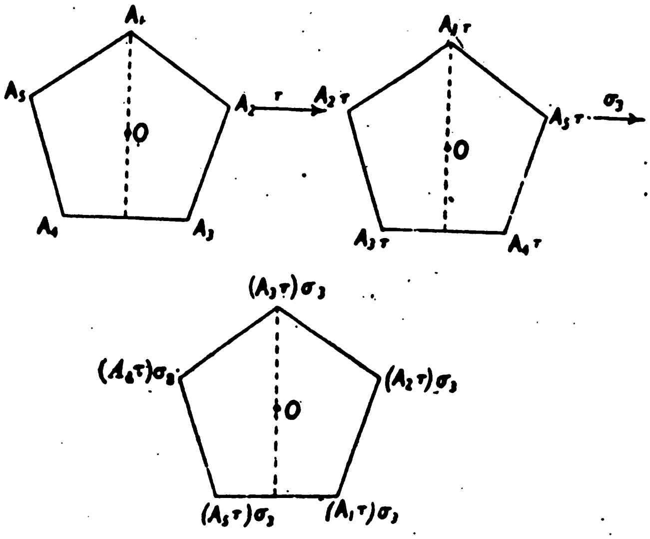


Fig. 4.26

has only two possibilities once $A_1\sigma$ has been determined as $d(A_1\sigma, A_2\sigma) = d(A_1 A_2)$ and $A_2\sigma$ must also be a vertex.

So $A_1\sigma$ and $A_2\sigma$ being once determined, $A_i\sigma$, $i = 3, 4, ...n$ may also be determined. Thus $\exists$ at most two elements $\sigma \in I_s$ which map $A_1\sigma$ to A_j. Hence there are at most $2n$ elements of I_s and so

$$|I_s| = 2n.$$

Note 2. The dihedral group of degree n is denoted by D_n.

As an example the elements of D_3 are σ_j, $\tau\sigma_j$ and $\sigma_j\,\sigma_2 = \sigma_j + 1$, $1 \le j \le 2$ and $\sigma_3\sigma_2 = \sigma_1$,

Also we have $\tau^{-1} = \tau$ and $\sigma_i\tau = \tau^2\sigma_i\tau = \tau\tau\sigma_i\tau$

Now $\tau\sigma_1\tau = \sigma_1$, σ_1 being identity

$\tau\sigma_2\tau = \sigma_3$ as $A_1\,\tau\sigma_2\tau = A_1\,\sigma_2\tau = A_2\tau = A_3$

and $A_2\,\tau\sigma_2\tau = A_3\,\sigma_2\tau = A_1\tau = A_1$

So $\sigma_3\tau = \tau\sigma_2$ and $\tau\sigma_3 = \sigma_2\tau$

The multiplication table is as shown here.

	σ_1	σ_2	σ_3	τ	$\tau\sigma_2$	$\tau\sigma_3$
σ_1	σ_1	σ_2	σ_3	τ	$\tau\sigma_2$	$\tau\sigma_3$
σ_2	σ_2	σ_3	σ_1	$\tau\sigma_3$	τ	$\tau\sigma_2$
σ_3	σ_3	σ_1	σ_2	$\tau\sigma_2$	$\tau\sigma_3$	τ
τ	τ	$\tau\sigma_2$	$\tau\sigma_3$	σ_1	σ_2	σ_3
$\tau\sigma_2$	$\tau\sigma_2$	$\tau\sigma_3$	τ	σ_3	σ_1	σ_2
$\tau\sigma_3$	$\tau\sigma_3$	τ	$\tau\sigma_2$	σ_2	σ_3	σ_1

4.13. SOME SPECIAL GROUPS WITH LINEAR OPERATORS

Vector Space or Linear Space *is an additive Abelian group L (Elements of L being called vectors) with the property that any scalar α (real or complex) and any vector* x *can be combined by the operation of scalar multiplication to yield a vector* αx *s.t.*

(*i*) $\alpha(\mathbf{x}+\mathbf{y}) = \alpha\mathbf{x}+\alpha\mathbf{y}, \mathbf{x}\in L, \mathbf{y}\in L \Rightarrow \alpha\mathbf{x}\in L$ and $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}\in L$

(*ii*) $(\alpha+\beta)\mathbf{x} = \alpha\mathbf{x}+\beta\mathbf{x}$

(*iii*) $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$

(*iv*) $1.\mathbf{x} = \mathbf{x}$

e.g. the set of $n \times n$ matrices forms a linear space.

A linear space is real or complex linear space according as the scalars are real or complex numbers.

If the linear space consists solely of the vector O with scalar multiplication defined by $\alpha \cdot O = O$ for all α, then we call it as *zero space* and denote it by $\{o\}$,

A non-empty subset M of L is said to be a *subspace* or *a linear subspace* of L if M is a linear space in its own right w.r.t. linear operations in L. In case M is a proper subset of L, when we call it a *proper subspace* of L.

Basis for Linear Space. *If S be a linear independent set of vectors in a linear space L, then ∃ a basis B for L* s.t. $S \subseteq B$ *and the* basis *for L is a linearly independent set which span the whole space L. Moreover*

$S\ (\neq \emptyset) \subset L$ is linear independent iff ⇔ each vector in the subspace $[S]$ spanned by S is uniquely expressible as a linear combination of the vectors in S *e.g.* the vector O in $[S]$ is uniquely expressible in the form

$$O = O\cdot\mathbf{x}_1 + O\cdot\mathbf{x}_2 + \ldots + O\cdot\mathbf{x}_n \text{ where } S = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$$

While the vectors in the sub space $[S]$ spanned by S are the linear combinations of the type $\mathbf{x} = \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \ldots + \alpha_n\mathbf{x}_n$; $\alpha_1, \alpha_2, \ldots \alpha_n$ are scalars.

Dimension of a linear space. *The dimension of a linear is the number of elements in its basis, e.g. L = {o} is o-dimensional and L ≠ {o} has dimension equal to the number of elements in any basis.*

A linear space is *finite-dimensional* if its dimension is *o* or a positive integer and it is *infinite-dimensional* if its dimension is not zero or a finite positive integer.

Linear transformations. If L, L' be two linear spaces with the same system of scalars, then a mapping T of L into L' is said to be a *linear transformation*

if $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ and $T(\alpha\mathbf{x}) = \alpha T(\mathbf{x})$

or equivalently if $T(\alpha\mathbf{x}+\beta\mathbf{y}) = \alpha T(\mathbf{x}) + \beta T(\mathbf{y})$.

A set of operators $T_1, T_2, \ldots$, in a linear space L forms a group if it satisfies all the four group axioms.

A linear transformation of one linear space into another is a *homomorphism* of the first space into the second, because it is a mapping which preserves the linear operations. Also it is observed that T preserves the origin (o) and negatives for

$$T(o) = T(o\cdot o) = o\cdot T(o) = o$$

and $$T(-\mathbf{x}) = T((-1)\mathbf{x}) = (-1)T(\mathbf{x}) = -T(\mathbf{x}).$$

Examples: (*i*) $T_1((\mathbf{x}_1, \mathbf{x}_2)) = (\alpha\mathbf{x}_1, \alpha\mathbf{x}_2)$, α being real, multiplies each vector in $\mathbf{R}^2$ by into self where $T_1 : \mathbf{R}^2 \to \mathbf{R}^2$

(*ii*) The mapping $D : D(p) = \dfrac{dp}{dn}$ is the linear transformation of P onto itself where the linear space P of all polynomials $p(n)$ with real coefficients is defined on $[o, 1]$

(*ii*) The mapping $I : I(f) = \int_0^1 f(x)\,dx$ is the linear transformation of C[*o*, 1] in the real linear space **R**.

Characteristics of Linear Transformations

(*i*) If T and U be two operators transforming the linear space L o L' then $(T + U)(\mathbf{x}) = T(\mathbf{x}) + U(\mathbf{x})$ ...(1)

Similarly $(\alpha T)(\mathbf{x}) = \alpha T(\mathbf{x})$ ...(2)

In nut shell if L and L' be two linear spaces with the same system of scalars, then the set of all linear transformations of L into L' is it itself a linear space w.r.t. linear operations (1) and (2).

(*ii*) If T and U be two linear transformations on L, then their *product* TU is defined by $(TU)(\mathbf{x}) = T(U\mathbf{x})$

(*iii*) If T, U, V be three linear transformations on L, then the operation is associative *i.e.* $T(UV) = (TU)V$

(*iv*) If T, U, V, be three linear transformations on L, then distributive law holds

i.e. $T(U + V) = TU + TV$

and $(T + U)V = TV + UV$

since, $((T + U)V)(\mathbf{x}) = (T + U)(V(\mathbf{x})) = T(V(\mathbf{x})) = U(V(\mathbf{x}))$

$= (TV)(\mathbf{x}) + (UV)(\mathbf{x})$

$= (TV + UV)(\mathbf{x})$.

(*v*) If T, U be two linear transformations on L and α is a scalar, then

$$\alpha(TU) = (\alpha T)U = T(\alpha U).$$

(*vi*) *Identity transformation* I is defined by $I(\mathbf{x}) = \mathbf{x}$ and it is observed that

$$I \neq o \Leftrightarrow L \neq \{o\} \text{ and } TI = IT = T$$

Also if α is a scalar, then αI is known as scalar multiplication for

$$(\alpha I)(\mathbf{x}) = \alpha I(\mathbf{x}) = \alpha \mathbf{x}.$$

(*vii*) A linear transformation T on L is *non-singular* if it is one-one and onto and *singular* otherwise.

(*viii*) If the linear transformation T on L is non-singular and one-one onto, then $\exists$ its *inverse* T^{-1} s.t. $TT^{-1} = T^{-1}T = I$.

It is easy to show that if T is non-singular then the mapping T^{-1} is also a linear transformation on L.

(*ix*) If L be the direct sum of the subspaces M and N s.t. $L = M \oplus N$, then each vector $\mathbf{z} \in L$ can be uniquely expressed as $\mathbf{z} = \mathbf{x} + \mathbf{y}$, $\mathbf{x} \in M$ and $\mathbf{y} \in N$, $\mathbf{x}$ being uniquely determined by $\mathbf{z}$, we define a mapping E of L into itself s.t. $E(\mathbf{z}) = \mathbf{x}$ and call it *projection on M along N*.

Clearly E is *idempotent* since $E^2 = E$.

Also $E^2(\mathbf{z}) = (EE)(\mathbf{z}) = E(E(\mathbf{z})) = E(\mathbf{x}) = \mathbf{x} = E(\mathbf{z})$

(*x*) *Reversal law.* If T, U be two non-singular linear transformations on L, then

$$(TU)^{-1} = U^{-1}T^{-1}$$

But $(\alpha T)^{-1} = \alpha^{-1}T^{-1}$, α being a scalar.

(*xi*) If T be a linear transformation of L to L', then we get an *isomorphic group* of operators in L' which transform $A, B, \ldots$ etc. to $A', B', \ldots$ etc s.t.

$$A' = TAT^{-1}, B = TBT^{-1}, \ldots$$

Note. If an arbitrary group G is mapped homomorphically onto a group of operators $D(G)$ in the linear space L, then the *operator group* $D(G)$ is said to be a *Representation of the group* G in the representative space L. In case n is the dimension of L, then the degree of representation is also n or in other words the *representation is n-dimensional*.

If $a, b \in G$, then $D(ab) = D(a)\,D(b)$

$$D(a^{-1}) = [D(a)]^{-1} \text{ and } D(E) = 1$$

Matrices and Linear Transformations

Linear operator T is a mapping s.t. $T : L \to L'$, L, L' being vector spaces over a field F

Let Dim. $L = n$ and its basis $B = \{x_1, x_2, \ldots x_n\}$ and let dim. $L' = m$ and its basis $B' = \{y_1, y_2, \ldots y_n\}$

$\because \forall\, x \in L$, $T(x) \in L'$, and $T : L \to L'$ $\therefore$ each $T(x)$ is expressible as a linear combination of elements of B' and in particular each $T(x_j)$, $j = 1, 2, \ldots n$ is expressible as a linear combination of m vectors in B'.

Let mn scalars $a_{ij} \in F$, $1 \le i \le m$, $1 \le j \le n$.

$$\text{So } T(x_1) = a_{11} y_1 + a_{21} y_2 + \ldots + a_{m1} y_m = \sum_{i=1}^{m} a_{i1}\, y_i$$

$$T(x_2) = a_{12} y_1 + a_{22} y_2 + \ldots + a_{m2} y_m = \sum_{i=1}^{m} a_{i2}\, y_i$$

....

$$T(x_n) = a_{1n} y_1 + a_{2n} y_2 + \ldots + a_{mn} y_n = \sum_{i=1}^{m} a_{in}\, y_i$$

or symbolically $T(x_j) = \sum_{i=1}^{m} a_{ij}\, y_i$, $j = 1, 2, \ldots, n$

So that *co-efficient matrix* of these expressions is

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} = \text{A (say)}$$

The matrix of $T : L_1 \to L_2$ (with respect to basis B, B') is the transpose of the matrix **A** *i.e.* matrix of T w.r.t. basis B, B' is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

which is written symbolically as $[T : B, B']$ or simply $[T]$

In case T is a linear opertor s.t. $T : L \to L$ *i.e.* $L = L'$, $m = n$, $B = B'$ the matrix $[a_{ij}]_{m \times n}$ will become $[a_{ij}]_{n \times n}$ and is denoted by $[T, B]$ or $[T]_B$.

$$\text{i.e. } T(x_j) = \sum_{i=1}^{n} a_{ij}\, x_i \;\; \forall\, j = 1, 2, \ldots n.$$

Conversely to find T whose matrix w.r.t. basis B is given to be $[a_{ij}]$ s.t.

$x \in V$ and $B = \{x_1, x_2 \dots x_n\}$ while $x = \sum_{j=1}^{n} \alpha_j x_j$, $\alpha_j \in F$, we have

$$T(x) = T\left(\sum_{j=1}^{n} \alpha_j x_j\right) = \sum_{j=1}^{n} \alpha_j T(x_j) = \sum_{j=1}^{n} \alpha_j \left(\sum_{j=1}^{m} \alpha_{ij} y_j\right) = \sum_{j=1}^{m} \left(\sum_{j=1}^{n} a_{ij} \alpha_j\right) y_i$$

which belongs to L' as B' its basis.

If T and U be two linear operators on a linear space L whose matrices w.r.t. a fixed basis $B = \{x_1, x_2, \dots x_n\}$ are $[T]$ and $[U]$, then

$$[T + U] = [T] + [U],\ [\alpha T] = \alpha[T] \text{ and } [TU] = [T]\,[U]$$

since if $[T] = [a_{ij}]_{n \times n}$, $U = [b_{ij}]_{n \times n}$, so that $T(x_j) = \sum_{i=1}^{n} a_{ij} x_i$, $U(x_j)$

$$= \sum_{i=1}^{n} b_{ij} x_i$$

then $[T + U](x_j) = T(x_j) + U(x_j) \qquad \forall_j = 1, 2, \dots n$

$$= \sum_{i=1}^{n} a_{ij} x_i + \sum_{i=1}^{n} b_{ij} x_i$$

$$= \sum_{i=1}^{n} \left(a_{ij} + b_{ij}\right) x_i = \sum_{i=1}^{n} c_{ij} x_i \text{ where } [c_{ij}] = [a_{ij} + b_{ij}]$$

$$\therefore\ [T + U] = [c_{ij}] = [a_{ij} + b_{ij}] = [a_{ij}] + [b_{ij}] = [T] + [U]$$

and $[\alpha T]\ (x_j) = \alpha\,[\mathrm{T}\,(x_j)] = \alpha \sum_{i=1}^{n} a_{ij} x_i = \sum_{i=1}^{n} \left(\alpha a_{ij}\right) x_i = \sum_{i=1}^{n} \left(c_{ij}\right) x_i$

$$\therefore\ [\alpha T] = [c_{ij}] = [\alpha a_{ij}] = \alpha[a_{ij}] = \alpha[T]$$

also $[TU](x_j) = T[U(x_j)] = T\left(\sum_{k=1}^{n} a_{kj} x_k\right) = \sum_{k=1}^{n} b_{kj} T(x_k)$

$$= \sum_{k=1}^{n} b_{kj} \left(\sum_{i=1}^{n} a_{ik} x_i\right)$$

$$= \sum_{i=1}^{n} \left(\sum_{k=1}^{n} a_{ik}, b_{kj}\right) x_i = \sum_{i=1}^{n} \left(c_{ij} x_i\right)$$

$$\therefore\ [TU] = [c_{ij}] = \left[\sum_{k=1}^{n} a_{ik} b_{kj}\right] = [a_{ij}]\,[b_{ij}] = [T]\,[U]$$

If $T : L \to L'$ and $U : L' \to L''$ so that $UT : L \to L''$ where L, L', L'' are finite dimensional liner spaces of dimensions n, m, p (say) and basis B, B', B'', (say) then $[UT ; B, B''] = [U ; B', B''] [T ; B, B']$

If L, L' are two finite dimensional linear spaces of dimensions n and n' and basis B and B', s.t. the function $T : L \to L'$ then its matrix w.r.t. B, B' is an **isomorphism** *between the space (L, L') and the space of all $m \times n$ matrices over the field F.*

The matrices of **identity operator** *1 and* **zero opertor 0** *on a linear space L w.r.t the basis* $B = \{x_1, x_2, \ldots x_n\}$ *are the unit and null matrices respectively, i.e.* $[I] = [\delta_{ij}]$ *where* $\delta_{ij} = 0$ *for* $i \neq j$

$= 1$ *for* $i = j$

and $[O] = [o]$

Since if T is a linear operator and $[a_{ij}]$ is the matrix of T w.r.t. basis B then

$$T(x_j) = \sum_{i=1}^{n} a_{ij} x_i$$

and $I(x_i) = x_j = \sum_{i=1}^{n} \delta_{ij} x_i = ox_1 + ox_2 + \ldots\ldots + 1 \cdot x_j + \ldots\ldots + ox_n$

$\therefore$ $[I] = [\delta_{ij}]$ is a unit matrix.

Also $O(x_j) = O = ox_1 + ox_2 + \ldots + ox_n = \sum_{i=1}^{n} O_{ij} x_i$

i.e. $[O] = [o_{ij}]$ is a null matrix.

If $[T] = [a_{ij}]$ *be the matrix of operator T on L w.r.t. basis B, then T is invertible* $\Leftrightarrow$ $[T] = [a_{ij}]$ *is invertible and in this case*

$$[T^{-1}] = [T]^{-1} = [a_{ij}]^{-1}$$

Since T is invertible, $TT^{-1} = T^{-1}T = I$

$\therefore$ $[TT^{-1}] = [T^{-1}T] = [I]$

or $[T]\,[T^{-1}] = [T^{-1}]\,[T] = [I] = \delta_{ij}$

which follows that $[T]$ is non-singular and $[T^{-1}] = [T]^{-1} = [a_{ij}]^{-1}$.

If T, U be two linear operators on a linear space L, then T and U are said to be similar if $\exists$ an invertible operator V on L s.t.

$$TV = VU \text{ i.e. } T = VUV^{-1} \text{ or } U = V^{-1}TV$$

The relation of similarity in the set of all $n \times n$ *matrices over the field F is an equivalence relation.*

Since if A and B be two $n \times n$ matrices and R is a relation of similarity then $\exists$ an invertible $n \times n$ matrix P s.t. $A = PBP^{-1}$ or $B = P^{-1}AP$.

Now R is *reflexive* since $A = IAI^{-1}$, I being an invertible unit-matrix R is *symmetric* since $A = PBP^{-1} \Rightarrow P^{-1}AP = P^{-1}(PBP^{-1})P$

$\Rightarrow P^{-1}AP = B$

$\Rightarrow B = P^{-1}A(P^{-1})^{-1} = CAC^{-1}$ where $C = P^{-1}$

$\Rightarrow B$ is similar to A and hence symmetric.

R is *transitive* since if $A = PBP^{-1}$ and $B = QCQ^{-1}$ then

$$A = P\,(QCQ^{-1})P^{-1} = (PQ)\,C\,(Q^{-1}P^{-1})$$
$$= (PQ)\,C\,(PQ)^{-1}.$$

As such a result analogus to it follows:

The relation of similarity in the set of linear operators on a linear space L is an equivalence relation.

If T is a linear operator on a vector space L and $[T]$ *be the matrix of it w.r.t. a basis B, then determinant i.e. det.* $T = \det [T]$

Linear functionals. *A linear functional f over a vector space L is a mapping which assigns to each member x ∈ L, an element f (α) which is in F (field) s.t. f is linear i.e.*

$$f(\alpha + \beta) = f(\alpha) + f(\beta),\ \alpha, \beta \in L$$
$$f(a\alpha) = af(\alpha),\ a \in F$$

or in one relation, $f(a\alpha + \beta) = af(\alpha) + f(\beta)$

If $f(\alpha) = 0\ \forall\ \alpha \in L$, then f is said be a *zero functional.*

Dual space of vector space. The set of all linear functionals f on L denoted by L^* or $V(L, F)$, F being a field, w.r.t. two compositions s.t.

$$(f_1 + f_2)\ \alpha = f_1\ (\alpha) + f_2\ (\alpha)\ \forall\ \alpha \in L$$
$$(af)\alpha = \alpha[f(\alpha)]\ \forall \alpha \in L, a \in F$$

is said to be a *dual space of L i.e.*

$$L^* = \{f: L \rightarrow F \text{ s.t. } f \text{ is linear}\}.$$

Transpose of Linear Transformation

If T be a linear operator from L to L' (over a field F) s.t.

$$\alpha \in L \Rightarrow T\ (\alpha) \in L'$$

then if $T: L \rightarrow L'$ induces a linear transformation $T^T : L'^* \rightarrow L^*$ where L^*, L'^* are dual spaces of $L\ L'$; T^T is called as *transpose* of the linear transformation T.

T^T is linear and unique and Rank (T^T) = Rank (T)

Adjoint of an Operator

If $T : L \rightarrow L$, L being a vector space over a field F, s.t. $\alpha \in L \Rightarrow T\ (\alpha) \in L$, induces a linear operator $T^* : L^* \rightarrow L^*$, L^* being dual space of L with its elements as functional on L with its elements as functional on L, then T^* on L^* is called as *adjoint* of the linear operator T on L.

T^* is linear and unique.

Its properties are : (*i*) $(T_1+T_2)^* = T_1^* + T_2^*$

(*ii*) $(kT)^* = kT^*$, k being a scalar

(*iii*) $(T_1T_2)^* = T_2^*T_1^*$ (reversal law)

(*iv*) If T is invertible, $(T^*)^{-1}=(T^{-1})^*$

(*v*) Zero and identity operators are self adjoint i.e. $O^* = O$ and $I^*=I$

(*vi*) $(T^*)^* = T^{**}=T$.

Inner Product Vector Space (*I.P.V.S.*)

A vector space with an inner product defined on it is said to be an inner product space, while the inner product of x (a_1, a_2, a_3) and y (b_1, b_2, b_3) being defined as $(x, y) = a_1b_1 + a_2 b_2 + a_3 b_3$ with the properties

$$|x| = \sqrt{a_1^2 + a_2^2 + a_3^2},\ |x| = 0 \text{ iff } a_1 = 0 = a_2 = a_3$$

$$x \text{ and } y \text{ are orthogonal if } \cos\theta = \frac{(x_1\ y)}{|x||y|} = 0 \text{ i.e. } (x, y) = 0$$

$(x, x) = 0$ and $(x, y) = (y, x)$ also $(ax + by, z) = a\ (x, z) + b\ (y, z)$

Note. *R^n (R) is an Euclidean space and C^n (C) is an unitary space. As an illustration, the set of all n × n matrices forms an inner product vector space over a field F (real or complex) if the inner product is defined as (A, B) = Trace (AB^Θ), B^Θ being transpose conjugate of B.*

Here if $B = [b_{ij}]$ then $B^\Theta = [\bar{b}_{ij}]$

So $(A+C, B) = \text{trace}\ [(A+C)\ B^\Theta]$

$= tr\,(AB^{\ominus} + CB^{\ominus})$

$= tr\,AB^{\ominus} + tr\,CB^{\ominus}$

$= (A, B) + (A, C)$

Since Trace of sum = sum of traces.

$\therefore \quad \overline{(A,\ B)} = \overline{\text{tr}\,(A,\ B)}^{\ominus}$

= Sum of diagonal elements of $AB^{\ominus}$

= Sum of conjugates of diagonal elements of $AB^{\ominus}$

= Sum of diagonal elements of $A^{\ominus}B$

= trace $A^{\ominus}B$ = tr $(BA^{\ominus}) = (B, A)$

Similarly $(kA, B) = k(A, B)$ and tr $(kA, B^{\ominus}) = k$ tr. $(AB^{\ominus}) = k\,(A, B)$ showing that the given set is an inner product vector space.

Unitary and Orthogonal Operators

If $TT^* = T^*T = I$, T is called unitary operator for complex I.P.V.S. and orthogonal for real I.P.V.S.

Normal operator

If $TT^* = T^*T$, then T is called normal operator, T^* being adjoint of T.

Characteristic Vectors and Characteristic Values

If T be a linear operator on a finite dimensional vector space $L(F)$ then a scalar $\lambda \in F$ is said to be a characteristic value of T if $\exists$ a vector $\alpha(\neq 0) \in L(F)$ s.t. $T(\alpha) = \lambda\alpha$. This non-zero vector α associated with characteristic value λ is said to be a characteristic vector of T.

In other words, roots of $|T - \lambda I| = 0$ are characteristic values of T.

If T is invertible and has characteristic root λ, then λ^{-1} is the characteristic value of T^{-1}.

If T is not invertible, then 0 is the characteristic value of T.

If $\lambda \in F$ is a characteristic value of a linear operator T on a vector space $L(F)$ then for a polynomial $p(x)$ over F, $p(\lambda)$ is a characteristic value of $p(T)$.

Hamiltonian group. *A non-commutative (i.e. non-abelian) group in which every subgroup is normal is said to be a Hamiltonian group.*

Simple group. *A simple group is one which contains no other normal subgroup except the two, one itself and the second a unit subgroup which is normal.*

Unitary groups. The set of all non-singular square matrices of order n with multiplicative compositions form a group known as a *full linear group*. Its elements are the infinite number linear transformations which change a vector into a new vector and so the order of a full linear group is infinite.

Imposing certain condition on the matrices of its transformation, we may get many subgroups of full linear group. One such type is *discrete group* obtained by excluding all matrices except those whose determinant is ± 1. The elements of a discrete group can be put into one-one correspondence with the set of positive integers.

The sub-groups of a full linear group obtained by expanding all matrices except those whose determinant is ± 1 form a *continuous group* provided its elements are non-denumerable *i.e.* uncountable. In other words a continuous group contains the elements which can be generated by continuously varying parameters in any region, known as group space. There is one-one correspondence between group-elements and points of group space. Those groups whose elements can be generated by a finite number of continuously varying parameters, are known as *finite continuous groups*.

A sub-group of a full linear group having its elements as square unitary matrices of order 2 with determinant +1 is known as *2-dimensional unimodular unitary group* or *special unitary group* denoted by SU(2) *e.g.* a matrix $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = A$ (say) will be a special unitary group if A is unitary matrix of order 2 *i.e.*, $AA^{\Theta} = I$ and secondly if $|A| = +1$.

Now $A^{\Theta} = \begin{bmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{bmatrix}$, $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$ being complex conjugate of $\alpha, \beta, \gamma, \delta$ respectively

$$\therefore \quad AA^{\Theta} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\begin{bmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{bmatrix} = \begin{bmatrix} \alpha\bar{\alpha}+\beta\bar{\beta} & \alpha\bar{\gamma}+\beta\bar{\delta} \\ \gamma\bar{\alpha}+\delta\bar{\beta} & \gamma\bar{\gamma}+\delta\bar{\delta} \end{bmatrix} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ gives}$$

$$\alpha\bar{\alpha}+\beta\bar{\beta} = 1 = \gamma\bar{\gamma}+\delta\bar{\delta} \text{ and } \alpha\bar{\gamma}+\beta\bar{\delta} = 0 = \gamma\bar{\alpha}+\delta\bar{\beta}$$

which yield, $\gamma = -\bar{\beta}$, $\delta = \bar{\alpha}$ and $\alpha\bar{\alpha}+\beta\bar{\beta} = 1$

As such a typical element of special unitary group is

$$U = \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix}; \; |U| = \alpha\bar{\alpha}+\beta\bar{\beta} = 1$$

Applying this matrix to the column vector $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that $UX = X'$

we get,
$$\begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$$

i.e.
$$\begin{bmatrix} \alpha x_1+\beta x_2 \\ -\bar{\beta}x_1+\bar{\alpha}x_2 \end{bmatrix} = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$$

or equivalently
$$\alpha x_1+\beta x_2 = x_1'$$
$$-\bar{\beta}x_1+\bar{\alpha}x_2 = x_2'$$

which transform any function of x into linear combination of x_1, x_2.

IRREDUCIBLE REPRESENTATION OF SU(2)

The Irreducible representation is defined in § 4.14. In sequence to the above special Unitary group, we have

$$Uf(x) = f(x') = f(\alpha x_1+\beta x_2, \; -\bar{\beta}x_1+\alpha x_2)$$

As such if U operates on a set of $(n+1)$ homogeneous products

$$f_m^{(n)} = x_1^{n-m}x_2^{n-m}, \quad m = 0, 1, 2, \ldots\ldots n$$

then we get
$$Uf_m^{(n)} = (\alpha x_1+\beta x_2)^m(-\bar{\beta}x_1+\bar{\alpha}x_2)^{n-m}$$
$$= \sum_{j=0}^{n} U_{mj}^{(n)} x_1^j x_2^{n-j}$$

where $U_{mj}^{(n)} = \sum_k (-1)^k \dfrac{\lfloor m \lfloor (n-m)}{\lfloor (m-k) \lfloor k \lfloor (n-m-j+k) \lfloor (j-k)} \times \alpha^k \beta^{m-k} (\bar{\alpha})^{n-m-j+k} (\bar{\beta})^{j-k}$

Character of a special unitary group SU(2). The character (see § 4.14) of a special unitary group is found out if a typical matrix by means of unitary transformation is transformed to diagonal form.

Take a unitary matrix V such that

$$V^{-1}UV = \begin{bmatrix} \gamma & 0 \\ 0 & \gamma \end{bmatrix} = U'.$$

Now $|U'| = +1$ is apparantly satisfied if $U' = \begin{bmatrix} e^{i\phi/2} & o \\ o & e^{-i\phi/2} \end{bmatrix}$

All the other matrices of the group will belong to the same class as U and U' since the class constitutes elements obtained by unitary transformation while a unitary matrix remains unitary under such a transformation.

Thus the character (defined in § 4.14) of one element of the class is given by

$$\chi^{(\frac{1}{2})} = e^{i\phi/2} + e^{-i\phi/2} \text{ by using } U'$$
$$= 2 \cos \phi$$

In general the character (defined in § 4.14) of special unitary group for any value j is given by $\chi^{(j)} = \sum_{m=-j}^{j} e^{im\phi}$

If $\chi = e^{i\phi}$, then $\chi^{(j)} = e^{-ij\phi}(1 + \chi + \chi^2 + \ldots\ldots + \chi^{2j})$

$$= e^{-ij\phi}\left(\frac{1-\chi^{2j+1}}{1-\chi}\right)$$

$$= \frac{\sin (2j+1)\frac{\phi}{2}}{\sin \frac{\phi}{2}}$$

on multiplying numerator and denominator by $i\frac{\phi}{2}$.

***n*-dimensional rotation group.** A continuous group formed from the set of all orthogonal n-dimensional matrices is said to be an *n-dimensional rotational group*. In fact this is a sub-group of a full linear group provided all elements are real unitary matrices whose determinant is +1.

e.g. if a point $P(x, y, z)$ is taken on the surface of a unit sphere and the sphere is rotated in any manner keeping its centre fixed; then the new coordinates of P say (x', y', z') related to (x, y, z) by some matrix $R(\alpha, \beta, \gamma)$ which is an element of a 3-dimensional rotation group $R^+(3)$ give a rotation factorised as product of three plane rotations described by the Eulerian angles (α, β, γ) (discussed in classical mechanics) *i.e.* $R(\alpha, \beta, \gamma) = R_z(\gamma) R_y(\beta) R_x(\alpha)$, where R_z, R_y, R_x are rotations about z, y, x axes respectively.

As an other example if we define a *2-dimensional rotation group* $R^+(2)$ as a subgroup of $R^+(3)$, then its elements are obtained by proper rotation in a plane perpendicular to a fixed axis say z-axis. Taking $R(\theta)$ as one element of this group and $T(\theta)$ an operator transforming a vector x with components x_1, x_2 to another vector x′ with components x_1', x_2' *i.e.*

$$\mathbf{x}' = T(\theta)\mathbf{x}, \quad 0 \le \theta \le 2\pi$$

such that $T(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, we have

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or equivalently, $x_1' = x_1 \cos\theta + x_2 \sin\theta$

$$x_2' = x_2 \cos\theta - x_1 \sin\theta$$

But if $R(\theta')$ is another element of the group with transformation $T(\theta')$ then

$$T(\theta)\, T(\theta') = T(\theta+\theta') = T(\theta')\, T(\theta)$$

which follows that the group is commutative *i.e.* Abelian.

Point group. The inversion operation in 3-dimensional space is given by the matrix $T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

and an identity operation I is given by the unit matrix

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Evidently $TI = T$

and $T^2 = I$ *i.e.* $T = T^{-1}$

Here T and I form a group with matrix multiplication. Such a group is said to be a *point group* since one point (say origin) remains fixed in all operations. The fixed point is sometimes known as *centre of inversion*.

Consider a point group $\{c_n\}$ with operations on a regular polygon of n sides such that there exist

(*i*) a rotation through an angle $2\pi/n$ about an n-fold axis of rotation properly.

(*ii*) a rotation through $-2\pi/n$ about an n-fold axis of rotation improperly.

(*iii*) a reflection in a plane given by σ_H, σ_V; H, V denoting Horizontal and Vertical planes.

(*iv*) an inversion.

Such operations form a point group $\{c_n\}$.

Quaternion group. If we define a group G of order 8 such that

$$G = \{1, a, a^2, a^3, b, ab, a^2b, a^3b\}$$

with the properties, $ab = ba^3$

$$b^2 = a^2$$

and $a^4 = 1$

whence $ab = ba^3$ and $b^2 = a^2 \Rightarrow ba = a^3b$

	1	a	a^2	a^3	b	ab	a^2b	a^3b
1	1	a	a^2	a^3	b	ab	a^2b	a^3b
a	a	a^2	a^3	1	ab	a^2b	a^3b	b
a^2	a^2	a^3	1	a	a^2b	a^3b	b	ab
a^3	a^3	1	a	a^2	a^3b	b	ab	a^2b
b	b	a^3b	a^2b	ab	a^2	a	1	a^3
ab	ab	b	a^3b	a^2b	a^3	a^2	a	1
a^2b	a^2b	ab	b	a^3b	1	a^3	a^2	a
a^3b	a^3b	a^2b	ab	b	a	1	a^3	a^2

Since $a^3b = a^2(ba^3) = b^3a^3 = b(a^2a^3) = ba$.

The composition table is as shown here. It is clear from this table that the group of order 8 under consideration does actually exist and defines a group. Such a group is known as ***quaternion group***. All of its subgroups are normal, though it is not abelian. Clearly a quaternion group is also a Hamiltonian group.

Torsion, Torsion-free and Mixed Groups

If G be a group such that every element of G with the identity is of infinite order then G is called as *Torsion-free group*. If G is a group such that every element of it is of finite order then G is called as *Torsion group*. Also if G is a group such that it consists of both an element of infinite order and an element (not equal to the identity) of finite order then G is called as *Mixed group*.

p-Primary group or p-group. A group G is said to be a primary group or p-group for some prime p if every element of G is of order which is a power of p. Actually a torsion group is made up of p-groups.

***p*-Prüfer group.** If Q/Z represents the additive group of rationals modulo the integers, then Q/Z is evidently a torsion group and if

$$\begin{aligned}(Q/Z)_p &= \{x+Z : x+Z \text{ of order a power of } p\}\\ &= \{x+Z : p^r x \in Z\}\\ &= \{m/p^r + Z : \text{for various integers } r \text{ and } o \le m \le p^{r-1}\}\end{aligned}$$

and $$Q/Z = \sum_{p \in II} (Q/Z)_p, \; II \text{ being set of all primes } p$$

then $(Q/Z)_p$ is said to be *p-Prüfer group* or *a group of type* P^{∞}

Clearly p-Prüfer group, $(Q/Z)_p = \bigcup_{r=1}^{\infty} C_r$ where $C_r = gp\left(\dfrac{1}{p^r} + Z\right)$ and $g \in G$ is of order $p_1^{r_1} p_2^{r_2} \ldots p_n^{r_n}$; $p_1, p_2, \ldots\ldots p_n$ being distinct primes and $r_1, r_2, \ldots\ldots r_n$ positive integers.

Since $(Q/Z)_p = \{m/p^r + Z\}$ for various integers r and $o \le m < p^{r-1}$

So $[Q/Z]_p \supset \bigcup_{r=1}^{\infty} C_r$

and $(Q/Z)_{p}. \subset \bigcup_{r=1}^{\infty} C_r.$

Which follow that $(Q/Z)_p = \bigcup_{r=1}^{\infty} C_r.$

4.14. PERMUTATIONAL REPRESENTATIONS

Generalization of Cayley's theorem *i.e. every group is isomorphic to a group of permutations.*

If G be a group and ρ a mapping; $\rho: G \to G$ s.t. $x \to xg$ $\forall x \in G$ then $g\rho$ being image of g in G under ρ, we have $g\rho: x \to xg, x \in G$

Here ρ is an isomorphism of G into a subgroup of S_G if

(*i*) $g\rho$ is a permutation of G $\forall$ $g \in G$

(*ii*) g is a homomorphism *i.e.* if $g, h \in G$, then $(gh)\rho = g\rho \cdot h\rho$

(*iii*) ρ is an isomorphism *i.e.* ρ is one-one.

(*i*) is satisfied since $g\rho$ is one-one mapping of G onto G as

$$\begin{aligned} x(g\rho) = y(g\rho) &\Rightarrow xg = yg,\ x, y \in G \\ &\Rightarrow xgg^{-1} = ygg^{-1} \\ &\Rightarrow x = y \end{aligned}$$

showing that the mapping is one-one.

Also if $x \in G$, then $(xg^{-1})g\rho = (xg^{-1})g = x$ showing that $g\rho$ is onto.

(*ii*) is satisfied since for $x \in G$ we have

$$x((gh)\rho) = x(gh) = (xg)h = [x(g\rho)](h\rho) = x(g\rho h\rho)$$

i.e. $\quad (gh)\rho = g\rho h\rho$

showing that ρ is a homomorphism.

(*iii*) is satisfied since if $g\rho = h\rho$ and $1 \in G$ is the identity element, then

$g = 1(g\rho) = 1(h\rho) = h$ showing that ρ is one-one.

For example consider a cyclic group G of order 2 s.t. $G = \{1, a\}$, where $a^2 = 1$, $1 \cdot a = a \cdot 1$, then mappings

$$1\rho: 1 \to 1, a \to a$$

and $\quad a\rho: 1 \to a, a \to 1$ shoy that ρ is an isomorphism

since $a\rho \neq 1\rho$, ρ is one-one.

Definition of a Permutational Representation

A homomorphism of a group G into the symmetric group on a set X is known as a **permutational representation** *of G on X.*

If according to Cayley's theorem ρ is the isomorphism for G, then ρ itself is a permutational representation of G on G and known as right regular representation. A mapping μ of G into the symmetric group on the set X is a permutational representation of G if

$$(gh)\mu = g\mu h\mu, \text{ for all } g, h \in G.$$

For example, if G be a diherdal group of degree 4, then G is the group of symmetries of the square. If $g \in G$ takes each vertex, of $ABCD$ to a vertex, then g being one-one, Ag, Bg, Cg, Dg are distinct vertices. Suppose that

$X = \{A, B, C, D\}$ and mapping σ_g: $x\sigma_g = xg$ $\forall x \in X$

so that $\sigma_g \in S_X$

Also if $\tau: G \to S_X$ s.t. $g\tau = \sigma_g$, then for $x \in X$, $g, h \in G$, we have $x(gh)\tau = x\sigma_{gh} = x(gh) = (xg)h = (x\sigma_g)\sigma_h = x(\sigma_g\sigma_h) = x(g\tau)(h\tau)$

giving $(gh)\tau = g\tau h\tau$

which shows that τ is a permutational representation.

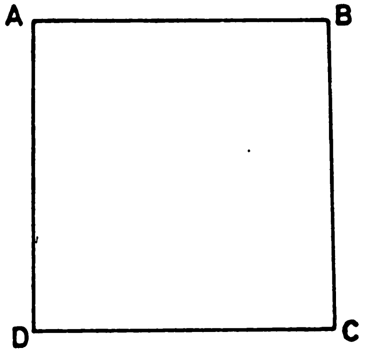


Fig. 4.27

Degree of a Representation

The number of elements in the set X gives the degree of a permutational representation or simply a representation on X. e.g. the degree of representation in the above example of dihedral group of degree 4, is four (4).

The degree of representation of the symmetric group G on $\{1, 2, 3\}$ is 6 if ρ itself is a representation of G as a permutation group on six elements

$p_1 : 1 \to 1, 2 \to 2, 3 \to 3; p_2 : 1 \to 1, 2 \to 3, 3 \to 2; p_3 : 1 \to 2, 2 \to 3, 3 \to 1;$
$p_4 : 1 \to 2, 2 \to 1, 3 \to 3; p_5 : 1 \to 3, 2 \to 2, 3 \to 1; p_6 : 1 \to 3, 2 \to 1, 3 \to 2$

So that

$p_1\rho : p_1 \to p_1, p_2 \to p_2, p_3 \to p_3, p_4 \to p_4, p_5 \to p_5, p_6 \to p_6$
$p_2\rho : p_1 \to p_2, p_2 \to p_1, p_3 \to p_5, p_4 \to p_6, p_5 \to p_3, p_6 \to p_4$
$p_3\rho : p_1 \to p_3, p_2 \to p_4, p_3 \to p_6, p_4 \to p_5, p_5 \to p_2, p_6 \to p_1$
$p_4\rho : p_1 \to p_4, p_2 \to p_3, p_3 \to p_2, p_4 \to p_1, p_5 \to p_6, p_6 \to p_5$
$p_5\rho : p_1 \to p_5, p_2 \to p_6, p_3 \to p_4, p_4 \to p_3, p_5 \to p_1, p_6 \to p_2$
$p_6\rho : p_1 \to p_6, p_2 \to p_5, p_3 \to p_1, p_4 \to p_2, p_5 \to p_4, p_6 \to p_2$

giving $(p_i\rho)(p_j\rho) = (p_i h_j)\rho$, $1 \le i, j \le 6$.

But the degree of representation of the symmetric group on $\{1, 2, 3\}$ is 3 since it is an identity isomorphism as G itself is a permutation group on $\{1, 2, 3\}$.

As another example the degree of representation of a cyclic group G of order n s.t. $G = \{1, a, a^2 ... a^{n-1}\}$, n being positive integer, is infinite.

Faithful representation, *If a representation is one-one then it is called as faithful representation.*

e.g. the representation in the above quoted examples are respectively, faithful (dihedral group); faithful; faithful; faithful.

As another example each matrix is its own faithful representation.

The number of rows and columns in a representation matrix is sometimes known as the *dimension* of the representation. As similarity transformations do not change the multiplication properties of matrices so under such transformations, the nature of representation remains invariant. As such new representations known as ***equivalent representations*** can be obtained from the given representation by means of similarity transformations.

Addition of representations. Consider m square matrices $A_1, A_2, \ldots\ldots A_m$ each of order n with their matrix representation of a group as $\mu(A_1), \mu(A_2), \ldots \mu(A_m)$ and similarly take another matrix representation consisting of m square matrices of order p s.t.

$$\mu'(A_1), \mu'(A_2), \ldots \mu'(A_m)$$

Since from the two matrix representations a single row can be found by their addition in which two representations are merely joined into one, therefore a new representation consisting of m square matrices of order $m + n$ can be obtained by adding these representations such as

$$\begin{bmatrix} \mu & \vdots & O \\ O & \vdots & \mu' \\ \leftrightarrow & & \leftrightarrow \end{bmatrix} \begin{matrix} \updownarrow n \text{ rows} \\ \\ \updownarrow p \text{ rows} \end{matrix} \quad \ldots (1)$$

$$\begin{matrix} n & & p \\ \text{columns} & & \text{columns} \end{matrix}$$

whose elements are

$$\begin{bmatrix} \mu(A_1), & O \\ O, \mu'(A_1) \end{bmatrix}, \begin{bmatrix} \mu(A_2), & O \\ O, \mu'(A_2) \end{bmatrix}, \ldots, \begin{bmatrix} \mu(A_m), & O \\ O, \mu'(A_m) \end{bmatrix}, \quad \ldots (2)$$

Calling the first representation as μ_1, second as μ_2 and their sum as μ we have

$$\mu_1 + \mu_2 = \mu \quad \ldots (3)$$

Reducible representation. A representation arising from the representation (2) by similarity transformation is called as *reducible* representation and clearly these transformations are equivalent to the representation of the form (2). Other representations for which this is not possible are termed as *irreducible representations.*

e.g. A reducible matrix can be put in the form (2), by similarity transformation by means of converting jth row and column into $j'th$ row and column. In order to effect this reducible representation take an isomorphic linear operator $T: L \to L'$, L, L' being two linear spaces and matrices, $A, B, \ldots \in L, A', B' \ldots \in L'$

We have

$$A' = TAT^{-1}, B' = TBT^{-1} \ldots \text{etc.}$$

If we choose $T_{\alpha\beta} = \delta_{\alpha\beta}$, then $(T^{-1})_{jk} = \delta_{j'k}$

and $\sum_{\beta} T_{\alpha\beta}(T^{-1})_{\beta j} = \sum_{\beta} \delta'_{\alpha\beta}\delta'_{\beta j} = \delta_{\alpha j}$

So that the similarity transformatin for this T resumes the requied renumbering such that

$$\bar{A} = T^{-1} A T$$

$$= A_{j'k} \text{ where } (\bar{A})_{jk} = \sum_{\alpha\beta} \delta_{j'\alpha} A_{\alpha\beta} \delta_{\beta j'}$$

Now we know that every non-singular matrix $\mu(A)$ is invertible and multiplication of any group element A with identity element E gives A, so the multiplication of any representation matrix $\mu(A)$ *i.e.*

$$\mu(A)\mu(E) = \mu(A) \text{ os that } \mu(E) = I, \text{ a unit matrix}$$

As such the unit matrix may be associated with the identity element of the group and we have

$$\mu(A)\mu(A^{-1}) = \mu(AA^{-1}) = \mu(E) = I$$

i.e., $\quad [\mu(A)]^{-1} = \mu(A^{-1}) \quad \ldots (4)$

In case of unitary representation *i.e.* matrices in representation being unitary, we therefore have

$$[\mu(A)^{\Theta} = \mu(A)]^{-1} \qquad \ldots(5)$$

(4) and (5) give $\mu(A^{-1}) = [\mu(A)]^{\Theta}$ ...(6)

The reduction of a representation. If basis can be found such that all the matrices $\mu(A)$ of an n-dimensional representation may be reduced to the form

$$\mu(A) = \begin{bmatrix} \mu^{(1)}(A) & R(A) \\ O & \mu^{(2)}(A) \end{bmatrix} \begin{matrix} \updownarrow m \text{ rows} \\ \updownarrow (n-m) \text{ rows} \end{matrix} \qquad \ldots(7)$$

$$\begin{matrix} \leftrightarrow & \leftrightarrow \\ m & (n-m) \\ \text{columns} & \text{columns} \end{matrix}$$

Where $\mu^{(1)}(A)$ denoted $m \times m$ matrices, $\mu^{(2)}(A)$ denote $(n-m) \times (n-m)$ matrices, O is a null matrix of $(n-m)$ rows and m columns and $R(A)$ denotes a rectangular matrix of m rows and $(n-m)$ columns, then the representation $\mu(A)$ said to be *reducible* and the procedure is said to be as method of *reduction* of the representation $\mu(A)$.

Transforming the basis in m-dimensional space of $\mu^{(1)}$, all the matrices of $\mu^{(1)}(A)$ can be brought to the form (7) *i.e.*

$$\mu^{(1)}(A) = \begin{bmatrix} \mu^{(3)}(A) & R^{(1)}(A) \\ O & \mu^{(4)}(A) \end{bmatrix} \qquad \ldots(8)$$

$\mu^{(3)}(A)$ being p-dimensional and $\mu^{(4)}(A)$ being $(n-p)$ dimensional.

Continuing this process, we may get the set of matrices $\mu^{(1)}(A)$, $\mu^{(2)}(A), \mu^{(3)}(A), \ldots \mu^{(\alpha-1)}(A), \mu^{(\alpha)}(A)$ which can not be further reduced and the dimension of irreducible representations m_j is $\sum_{j=1}^{\alpha} m_j$

In (7) if basis found is such that $R(A) = O$, then (7) becomes

$$\mu(A) = \begin{bmatrix} \mu^{(1)}(A) & O \\ O & \mu^{(2)}(A) \end{bmatrix} \qquad \ldots(9)$$

i.e. $\mu = \mu^{(1)} + \mu^{(2)}$ ...(10)

It follows that the representation $\mu(A)$ in this case is fully reducible and the reduction method is the reverse of addition.

If $\mu^{(1)}$ and $\mu^{(2)}$ are also reducible then continuing the process of reduction, the result (10) can be extended in the form

$$\mu = \mu^{(1)} + \mu^{(2)} + \mu^{(3)} + \ldots + \mu^{(\alpha)} \qquad \ldots(11)$$

The irreducible representation $\mu^{(\alpha)}$ may contain several equivalent irreducible representations which are not counted distinctly. As such a representation μ may consist of a particular irreducible representation $\mu^{(k)}$ several times *i.e.*

$$\mu = i_1 \mu^{(1)} + i_2 \mu^{(2)} + \ldots + i_k \mu^{(k)} \qquad \ldots(12)$$

$$= \sum_j i_j \mu^{(j)}, \; i_j \text{ being positive integers.}$$

As an illustration, consider a vector **x** with components x_1, x_2, x_3, and the elements of the group as operators changing **x** into a new vector **x′** with the same components in different order. Then the representation μ is a matrix s.t. $\mathbf{x}' = \mu \mathbf{x}$, rows and columns being labelled with x_1, x_2, x_3.

Taking E as identity element of the group, $\mu(E)$ is a unit matrix

$$i.e.\ \mu(E) = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Assuming that A replaces x_1 by x_3 but x_1 itself becomes x_2, that in $\mu(A)$, unity appears at the intersection of x_1th row and x_2th column. Also taking similar assumptions with B, C, D, F we have

$$\mu(A) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mu(B) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mu(C) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\mu(D) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } \mu(F) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note 1. Since in a space of l-dimensions at the most l orthogonal vectors can exist, therefore sum of squares of dimensions of all inequivalent, irreducible representations is at the most equal to the order of the group of representation *i.e.* $n_1^2 + n_2^2 + \ldots + n_s^2 \leq l$; $n_1, n_2, \ldots n_s$ being dimensions of s inequivalent irreducible representations of a group of order l.

$$\text{So} \quad \sum_{k=1}^{s} n_k^2 \leq l. \qquad \ldots(12a) \text{ (Rohilkhand, 1990)}$$

In case of all distinct irreducible representations of a group, this reduces to

$$\sum_{k=1}^{s} n_k^2 = l. \qquad \ldots(12b)$$

which is known as **Dimensionality-Theorem.**

Main Features of Reducible and Irreducible Representations

(*i*) The number of non-equivalent irreducible representations is the same as the number of classes.

(*ii*) If there are h elements in a group then the number of times the jth irreducible representation occurs in a reducible representation and given by $a_j = \frac{1}{h}\, XX_j$.

(*iii*) All the irreducible representations of an abelian group are one-dimensional.

(*iv*) A representation by non-singular matrices can be transformed into a representation by unitary matrices through a similarity transformation.

(*v*) Any matrix commutating with all the matrices of an irreducible representation is a constant matrix *i.e.* a unit matrix multiplied by a constant scalar. This feature is sometimes known as *SCHUR'S LEMMA*.

(*vi*) When a matrix A commutes with every matrix of a given representation of a group, then either A is a scalar matrix or the representation is reducible and the transformation used to diagonalize A, wholly or partly reduces the representation.

(*vii*) If there are two irreducible representations $\mu(A_1), \mu(A_2) \ldots \mu(A_h)$ and $\mu'(A_1), \mu'(A_2) \ldots \mu'(A_h)$ of dimensions d_1 and d_2 respectively and if there exists a matrix **M** with d_2 rows and d_1 columns such that $M\mu(A_j) = \mu'(A_j)\mathbf{M}$; $j = 1, 2, \ldots h$ then for $d_1 \neq d_2$, the matrix M is a null matrix whereas for $d_1 = d_2$, M is either a null matrix or a non-singular matrix.

(*viii*) The direct product of irreducible representations of two different groups is also an irreducible representation of the direct product of the groups.

Note 2. Feature (vii) can be put as:

If $\mu^{(1)}(A_j)$ *and* $\mu^{(2)}(A_j)$ *be two irreducible representations of dimensions* d_1 *and* d_2 *respectively of a group G and a matrix* **M** *of order* $d_1 \times d_2$ *satisfies*

$$\mu^{(1)}(A_j)\,\mathbf{M} = \mathbf{M}\,\mu^{(2)}(A_j) \qquad \text{...(12c)}$$

then either (*i*) $d_1 \neq d_2 \Rightarrow \mathbf{M} = \mathbf{O}$ *i.e. a null matrix* ...(12*d*)

or (*ii*) $|\mathbf{M}| \neq 0 \Rightarrow \mu^{(1)}$ *and* $\mu^{(2)}$ *are equivalent irreducible representations.*

By feature (*vii*), $\mathbf{M}\mu^{(1)}(A_j) = \mu^{(2)}(A_j)\,\mathbf{M}$ yields on taking Hermitian conjugate (transpose conjugate) of either side with the help of (6),

$\mu^{(1)}(A_j^{-1})\,\mathbf{M}^{\ominus} = \mathbf{M}^{\ominus}\mu^{(2)}(A_j^{-1})$ which post multiplied by M

$\Rightarrow \mu^{(1)}(A_j^{-1})\,\mathbf{M}^{\ominus}\mathbf{M} = \mathbf{M}^{\ominus}\mu^{(2)}(A_j^{-1})\,\mathbf{M}$

$\Rightarrow \mathbf{M}^{\ominus}\cdot\mathbf{M}\,\mu^{(1)}(A_j^{-1}) = \mu^{(2)}(A_j^{-1})\,\mathbf{M}^{\ominus}\mathbf{M}$ by associative property.

$\Rightarrow$ Matrix $\mathbf{M}^{\ominus}\mathbf{M}$ commutes with all matrices of representation

$\Rightarrow \mathbf{M}^{\ominus}\mathbf{M}$ is a unit matrix multiplied by a constant scalar by feature (*v*)

$\Rightarrow \mathbf{M}^{\ominus}\mathbf{M} = \lambda\,\mathbf{E}$ (say), E being identity element ...(12*e*)

If $d_1 = d_2 = l$ (say), then M is a square matrix, so that (12*e*) $\Rightarrow |\mathbf{M}^{\ominus}\mathbf{M}| = |\lambda \mathbf{E}| \Rightarrow |\mathbf{M}|^2 = \lambda^l$

$\therefore$ For $\lambda \neq o$, $\mathbf{M} \neq 0$, $\mathbf{M}^{-1}$ exists and or by (12*c*)

$\mu^{(2)}(A_j) = \mathbf{M}^{-1}\mu^{(1)}(A_j)\ \forall\ A_j \in G \Rightarrow \mu^{(1)}$ and $\mu^{(2)}$ are equivalent irreducible representations.

However if $\lambda = o$, then $\mathbf{M}^{\ominus}\mathbf{M} = \mathbf{O}$ yields for (j, k)th element

$$\sum_k M^{\ominus}_{jk}\cdot M_{kj} = 0 \Rightarrow \sum_k M_{jk}\, M_{kj} = \sum_k |M_{kj}|^2 = 0 \Rightarrow M_{kj} = 0 \text{ for } 1 \leq k \leq l \Rightarrow \mathbf{M} = \mathbf{O}$$

Again if $d_1 \neq d_2$ say $d_1 < d_2$ *i.e.* matrix M has d_1 columns and d_2 rows then to get a square matrix $d_1 \times d_2$ say $\mathbf{M}'$ from M, we can supplement to M by inserting $(d_2 - d_1)$ columns of zeros s.t.

$$\mathbf{M}' = \begin{bmatrix} \mathbf{M} \\ \cdots \\ \mathbf{O} \end{bmatrix} \begin{matrix} \updownarrow d_1 \\ \updownarrow d_2 - d_1 \end{matrix} \quad \text{giving } (\mathbf{M}')^{\ominus} = [\,\overset{\longleftarrow d_1 \longrightarrow}{\mathbf{M}^{\ominus}} \ \vdots\ \overset{\longleftarrow d_2 - d_1 \longrightarrow}{\mathbf{O}}\,] \updownarrow d_2$$

$\therefore (\mathbf{M}')^{\ominus}\mathbf{M}' = \mathbf{M}^{\ominus}\mathbf{M} \Rightarrow |\mathbf{M}'^{\ominus}\mathbf{M}'| = |\mathbf{M}^{\ominus}\mathbf{M}|$

$\Rightarrow |\mathbf{M}'^{\ominus}|\ |\mathbf{M}'| = \lambda^n$ so that $|\mathbf{M}'| = |\mathbf{M}^{\ominus}| = 0$

$\Rightarrow \lambda^n = o \Rightarrow \lambda = o \Rightarrow \mathbf{M}^{\ominus}\mathbf{M} = o \Rightarrow \mathbf{M} = \mathbf{O}$

Orthogonality Theorem for an Irreducible Representation

(*Rohilkhand, 1990, 92*).

Let $\mu(E), \mu(A_2), \mu(A_3)\ldots\mu(A_n)$ and $\mu'(E), \mu'(A_2),\ldots\mu'(A_n)$ *be two non-equivalent irreducible unitary representations of the same group G, then*

$$\sum_R \mu(R)_{jk}\mu'(R)_{pq} = 0 \qquad \text{...(13)}$$

holds for all elements jk and pq, where the summation extends over all group elements $E, A_2, A_3,\ldots A_h$, *E being identity element.*

Assuming the representation in unitary form as a similarity transformation always leaves multiples of the unit matrix unchanged, a matrix M commutes all the matrices $E(=A_1), A_2, \ldots A_h$ of the representation of the group of order h *i.e.*

$$A_i\text{M} = \text{M}A_i, \; i = 1, 2, 3, \ldots h \qquad \ldots(14)$$

By feature (*vii*) we have $\text{M}\mu(A_i) = \mu'(A_i)\text{M}, \; i = 1, 2, \ldots h$ ...(15)

Here (13) asserts that a matrix which satisfies (15), must be a null matrix and one which satisfies (14) must be a multiple of the identity matrix.

On account of group property of the representation all matrices of the form

$$\text{M} = \sum_R \mu'(R) X \mu(R^{-1}) \qquad \ldots(16)$$

for arbitrary l-rowed and m-columned matrix X satisfy (15). Also the group property follows that

$\sum_R \mu'(SR)XD\{S(R)^{-1}\} = \sum_R \mu'(R)XD(R^{-1}) = \text{M}$, since the same matrices appear on the left and right except in different order.

Here $\mu'(S)\text{M} = \sum_R \mu'(S)\mu'(R)XD(R)^{-1} = \sum_R \mu'(SR)X\mu'(SR^{-1})\mu(S)$

or $\qquad \mu'(S)\text{M} = \text{M}\mu'(S)$ (in concise form), ...(17)

so that by feature (*vii*) M must be a null matrix *i.e.* for arbitrary X_{ir}; $M_{pj} = \sum_{ir}\sum_R \mu'(R)_{pi}X_{ir}\mu(R^{-1})_{rj}$; while on setting all matrix elements $X_{ir} = 0$ except one number $X_{qk} = 1$, the generalized form of (15) is $\Sigma\mu'(R)_{pq}\,\mu'(R^{-1})_{kj}$ where $\mu'(R)$ and $\mu(R)$ must be irreducible, but not necessarily unitary. In case $\mu'(R^{-1})_{kj}(R)$ are unitary $\mu(R^{-1}) = [\mu(R)]^{-1}$ and hence $\mu(R^{-1}) = \mu(R)^{\Theta}$ so that (M) reduces to unitary representation.

The Character of Representation

Let μ^j with matrices $\mu^j(A)$ be a representation of a group G. Then trace of the matrix $\mu^j(A)$ *i.e.* the sum of diagonal elements of $\mu^j(A)$ is said to be the *character* of element A in the representation μ and denoted by

$$\chi^j(A) = \sum_k \{\mu^j(A)\}_k \qquad \ldots(18)$$

The character of an irreducible element is known as **simple** *and that of decomposible represenatation as* **composite.** *It is worth noting that the equivalent representation has the same set of characters, since if* μ *and* μ' *be equivalent representations with matrices* $\mu(A)$ *and* $\mu'(A)$ *of the group G, then*

$$\mu'(A) = B^{-1}\mu(A)B, \; B \text{ being some matrix.}$$

$\therefore$ $tr\,\mu'(A) = tr\,\mu(A)$, trace of a matrix being unaltered under similarty transformation.

Also the character is a class function in the group.

Since if A, B be two conjugate elements of a group G, then

$$A = U^{-1}BU,$$

and $\qquad \mu(A) = \mu(U^{-1})\mu(B)\mu(U) = \{\mu(U)\}^{-1}\mu(B)\mu(U)$

$\mu(A)$ and $\mu(B)$ being equivalent representations (as related to similarity transformations) have same set of characters. As such conjugate elements in a representation correspond to the same character and hence in describing a group by listing the character of its elements in an assumed representation, same character *i.e.* number is assigned to all elements in a given class. Thus the character is a class function in the group *i.e.* the character of a single element from each class can yield the character of the whole group.

Now if $\mu(A) = \mu_1(A) + \mu_2(A)$

then $\chi(A) = tr\,[\mu(A)] = tr\,[\mu_1(A) + \mu_2(A)] = tr\,[\mu_1(A)] + tr\,[\mu_2(A)]$

$$= \chi_1(A) + \chi_2(A) \quad ...(19)$$

Also if $\mu(A) = i_1\mu^{(1)}(A) + i_2\mu^{(2)}(A)...+ i_\alpha\mu^{(\alpha)}(A)$, by (11)

then $\chi(A) = i_1\chi^{(1)}(A) + i_{(2)}\chi^{(2)}(A) +...+ i_\alpha\chi^{(\alpha)}(A)$ ...(20)

Other Main Features of Characters

(*i*) The character of the direct product is the product of the characters *i.e.* $\chi^j(AB) = \chi^j(A)\cdot\chi^j(B)$.

(*ii*) The characters form an orthogonal system.

(*iiii*) If both of the representations $\mu(A)$ and $\mu(B)$ are of the first degree then the direct product $\mu(AB)$ is irreducible. In case both are of degree higher than one, $\mu(AB)$ is reducible.

(*iv*) Two irreducible representations are equivalent if and only if they have the same character.

Character tables. These are the devices to find characters when the complete multiplication table for a group is known. It is effected by calculating first the product of all elements in the class C_i by all elements in C_j and then arranging uniquely the resulting set of elements in classes. Evidently a given class may occur in the products many times or not at all. If $h_{ij,k}$ denotes the number of times the kth class appears, then we can write

$$C_iC_j = C_jC_i = \sum_{i=1}^{p} h_{ij,k}C_k \quad ...(21)$$

whence $\sum_{i=1}^{p} [\chi^{(i)}]^2 = g$ (say) ...(22)

where summation extends over all the numbers of classes p and g is known.

Once $h_{ij,k}$ determined, we can find characters by the use of relations

$$r_ir_j\chi^{(i)}\chi^{(j)} = \chi^{(i)}\sum_{k=1}^{p} h_{ij,k}\, r_k\chi^{(k)} \quad ...(23)$$

where r_i is the number of elements in i-th class and r_j the number of elements in jth class.

A table in which we can put all that what we have explained here, is known as a *character table.*

e.g. consider a multiplication group of matrix elements

$$E = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}, \quad A = \begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\1 & 0 & 0\end{bmatrix}, \quad B = \begin{bmatrix}0 & 0 & 1\\1 & 0 & 0\\0 & 1 & 0\end{bmatrix},$$

$$C = \begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{bmatrix}, \quad D = \begin{bmatrix}0 & 0 & 1\\0 & 1 & 0\\1 & 0 & 0\end{bmatrix}, \quad F = \begin{bmatrix}0 & 0 & 1\\1 & 0 & 0\\0 & 1 & 0\end{bmatrix},$$ then we have

$C_2 = A^2, B^2, AB, BA = 2C_1 + C_2$

$C_3 = 3C_1 + 3C_2;\ C_2C_3 = 2C_3$

other products are not required since $r_1 = 1$, $r_2 = 2$ and $r_3 = 3$.

Using (24), we therefore have

$$4\,[\chi^{(2)}]^2 = \chi^{(1)}(2\chi^{(1)} + 2\chi^{(2)})$$
$$9\,[\chi^{(3)}]^2 = \chi^{(1)}(3\chi^{(1)} + 6\chi^{(2)})$$
$$6\chi^{(2)}\chi^{(3)} = 6\chi^{(1)}\chi^{(3)}$$

Since $\chi^{(1)}$ has values, 1, 1, 2 [by (23), $g = 6$ and $p = 3$].

Solving these equations with each of 1, 1, 2 in turn, we get a table as shown here.

	C_1	C_2	C_3
δ_1	1	1	1
δ_2	1	1	−1
δ_3	2	−1	0

Here $\delta_1, \delta_2, \delta_3$ which are themselves matrices form the diagonal elements of the reducible representation.

Important Note. For detailed discussion of character tables in group theory, see Appendix C at the end of the book.

ADDITIONAL MISCELLANEOUS PROBLEMS

Problem 39. *Show that for a finite group G, every representation is equivalent to a unitary representation.*

Consider an arbitrary pair of vectors x, y s.t.

$$\{x, y\} = \sum_{A \in G} \{\mu(A)x, \mu(A)y\}, \ \mu \text{ being representation.}$$

The sum in (1) being extended over all elements A of G, for any $B \in G$, we have

$$\{\mu(B)x, \mu(B)y\} = \sum_{A \in G} \{\mu'(A)\,\mu(B)x, \mu(A)\,\mu(B)y\},$$

$$= \sum_{A \in G} \{\mu(AB)\,x, \mu(AB)y\} \qquad ...(2)$$

If A and B are fixed, AB runs through similar elements of G and so R.H.S's of (1) and (2) are identical *i.e.*

$$\{x, y\} = \{\mu(B)x, \mu(B)y\} \qquad ...(3)$$

For any operator U to be unitary, $(U_x, U_y) = (x, y) \ \forall \ x, y$.

So operators $\mu(A)$ are unitary w.r.t scalar product $\{x, y\}$.

Let u_α be a set of vectors, orthogonal w.r.t. the set of original scalar product and v_α a second set of vectors orthogonal w.r.t. a new scalar product s.t.

$$\{u_\alpha, u_\beta\} = \delta_{\alpha\beta} = \{v_\alpha, v_\beta\}$$

Take an operator T s.t. $v_\alpha = T\,u_\alpha$

So that $\quad T_x = T x_\alpha u_\alpha = x_\alpha T u_\alpha = x_\alpha v_\alpha$

$$\therefore \ \{T_x, T_y\} = x_\alpha y_\alpha = (x, y) \qquad ...(4)$$

If the equivalent representation is given by

$$\mu'(B) = T^{-1}\,\mu(B)T \qquad ...(5)$$

Then $\quad \{T^{-1}\,\mu(B)\,T_x, T^{-1}\,\mu(B)\,T_y\} = \{\mu(B)\,T_x, \mu(B)\,T_y\}$ by (4)

$= \{T_x, T_y\}$ by (3)

$= (x, y)$ by (4)

which follows that $\mu'(B)$ given by (5) is unitary and hence for finite group G, the representation can be chosen as unity.

Aliter. Let G be a group of order m represented by matrices $A_1, A_2, \ldots A_m$. A's are distinct if the representation is faithful otherwise A^s are not distinct.

Consider Hermitian Matrix H, obtained by summing over all the group elements s.t.

$$H = \sum_\alpha A_\alpha A_\alpha^\Theta \qquad ...(6)$$

Matrix H can be diagonalized as D by unitary matrix U s.t.

$$D = U^{-1} HU = \sum_\alpha U^{-1}A_\alpha A_\alpha^\Theta U$$

$$= \sum_\alpha U^{-1}A_\alpha UU^{-1}A_\alpha^\Theta U$$

$$= \sum_\alpha U^{-1}A_\alpha U(U^{-1}A_\alpha U)^\Theta$$

$$= \sum_\alpha \bar{A}_\alpha \bar{A}_\alpha^\Theta U^{-1}, (7) \because \bar{A}_\alpha = U^{-1}A_\alpha U \qquad ...(7)$$

All elements of D are real and positive since

$$D_{\beta\beta} = \sum_\alpha \sum_\gamma (\bar{A}_\alpha)_{\beta\gamma} (\bar{A}_\alpha)^\Theta_{\beta\gamma} = \sum_\alpha \sum_\gamma |(\bar{A}_\alpha)_{\beta\gamma}|^2$$

$= 0$ if $(\bar{A}_\alpha)_{\beta\gamma} = 0 \ \forall \ \gamma$ and α in which case an entry row of $\bar{A}_\alpha$ will be zero so that $|A_\alpha| = 0$ thereby contradicting the hypothesis.

Thus $D^{1/2}$ and $D^{-1/2}$ can be uniquely formed from D by taking $\pm \frac{1}{2}$ power of the diagonal elements and $D^{1/2}$ and $D^{-1/2}$ will be real if

$$(D^{1/2})^\Theta = D^{1/2} \text{ and } (D^{-1/2})^\Theta = D^{-1/2}$$

But the representation $\overline{(\bar{A}_\lambda)} = D^{-1/2} \ \bar{A}_\lambda . D^{1/2} = D^{-1/2} U^{-1} A_\lambda UD^{1/2}$ is unitary and (7) gives $I = D^{-1/2} \sum_\alpha A_\alpha A_\alpha^\Theta D^{-1/2}$

$$\therefore \ \bar{\bar{A}}_\lambda \bar{\bar{A}}_\lambda^\Theta = D^{-1/2} \ \bar{A}_\lambda D^{1/2} (D)^{-1/2} \sum_\alpha \bar{A}_\alpha \bar{A}_\alpha^\Theta D^{-1/2}) D^{1/2} \ \bar{A}_\lambda^\Theta D^{-1/2}$$

$$= D^{-1/2} \sum_\alpha \bar{A}_\lambda \bar{A}_\alpha \bar{A}_\alpha^\Theta \bar{A}_\lambda^\Theta D^{-1/2}$$

Also the group axioms for $\bar{A}_\alpha$ give that $\bar{A}_\lambda \ \bar{A}_\alpha$, $\alpha = 1, 2, ... m$ are also A_α in a different order, therefore

$$\sum_\alpha \bar{A}_\lambda \bar{A}_\alpha (\bar{A}_\lambda \bar{A}_\alpha)^\Theta = \sum_\alpha \bar{A}_\alpha \bar{A}_\alpha^\Theta$$

Thus $\quad \bar{\bar{A}}_\lambda \bar{\bar{A}}_\lambda^\Theta = D^{-1/2} \sum_\alpha \bar{A}_\lambda \ \bar{A}_\lambda^\Theta D^{-1/2} = I$

So $\quad \bar{\bar{A}}_\lambda^\Theta = (\bar{\bar{A}}_\lambda)^{-1}$

which shows that $\bar{\bar{A}}_\alpha$ is unitary.

Problem 40. SCHUR'S LEMMA. *If a matrix commutes with all the matrices, of an irreducible representation, then show that it is a multiple of unit matrix.*

Let $A_1, A_2 ... A_m$ be the matrices in representation of a group G in unitary form and B be a matrix which commutes with all of $A_1, A_2, ... A_m$

i.e. $\quad A_\alpha B = BA_\alpha, \ \alpha = 1, 2, ... m \qquad ...(1)$

$\therefore \quad (A_\alpha B)^\Theta = (BA_\alpha)^\Theta$ (by taking transpose conjugate)

i.e. $\quad B^\Theta A_\alpha^\Theta = A_\alpha^\Theta B^\Theta \qquad ...(2)$

or $\quad A_\alpha B^\Theta A_\alpha^\Theta A_\alpha = A_\alpha A_\alpha^\Theta B^\Theta A_\alpha$

or $\quad A_\alpha B^* = B^* A_\alpha \qquad ...(3)$

$\because \quad A_\alpha A_\alpha^\Theta = A_\alpha^\Theta A_\alpha = I$, A_α being unitary, (1) and (3) follow that B and B^* both commute with all A's.

$\therefore B + B^* = H_1$ (say, $i(B - B^*) = H_2$, H_1, H_2 being Hermitian; will also commute with all A's. As such we can conclude that a matrix commutative with all the elements of a unitary representation is Hermitian.

Also assuming that B is unitary, it can be diagonalized as

$$D = U^{-1} BU$$

If A_α is unitary then $C_\alpha = U^{-1} A_\alpha U$ is also unitary and so (1) gives

$$C_\alpha D = DC_\alpha,\ \alpha = 1, 2, \ldots m$$

Equating j-kth elements on either side,

$$[C_\alpha D]_{jk} = [DC_\alpha]_{jk},$$

i.e. $$(C_\alpha)_{kj}(D_{jj}) = D_{kk}(C_\alpha)_{kj} \quad \ldots(4)$$

D being diagonal matrix.

Now $D_{jj} \neq D_{kk} \Rightarrow (C_\alpha)_{kj} = 0$ *i.e.* representation is reducible which contradicts the hypothesis and hence

$$D_{jj} = D_{kk} \ \forall\ j, k$$

which follows that D is a scalar matrix say $D = \lambda I, \lambda$ being a constant and I a unit matrix.

As such $B = UDU^{-1} = U\lambda IU^{-1} = \lambda UU^{-1} = \lambda I$

= a multiple of unit matrix.

Problem 41. *If U covers the entire unitary group then show that* $\rho(\alpha, \beta, \gamma)$ *ranges over all rotations.*

Take $$U_1 = \begin{bmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{bmatrix} \quad \ldots(1)$$

Then, $$\left.\begin{aligned} x' &= x\cos\alpha + y\sin\alpha \\ y' &= -x\sin\alpha + y\cos\alpha \\ z' &= z \end{aligned}\right\} \quad \ldots(2)$$

represent a rotation through an angle α about Z-axis. Representing it by $r' = \rho_z(\alpha) r$, r, r' being vectors with components (x, y, z) and (x', y', z') respectively, we have

$$P_r(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ \cdot\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \ldots(3)$$

Similarly $\rho_z(\theta)$ corresponds to a matrix with α replaced by θ.

Now take $$U_2 = \begin{bmatrix} \cos\beta/2 & i\sin\beta/2 \\ i\sin\beta/2 & \cos\beta/2 \end{bmatrix} \quad \ldots(4)$$

Then

$$\left.\begin{aligned} U_2^{\ominus} T_1 U_2 &= T_1 \\ U_2^{\ominus} T_2 U_2 &= \cos\beta\, T_2 + \sin\beta\, T_3 \\ U_2^{\ominus} T_3 U_2 &= -\sin\beta\, T_2 + \cos\beta\, T_3 \end{aligned}\right\} \quad \ldots(5)$$

and $r' = \rho_x(B) r$ ...(6)

whence $$\rho_x(\beta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{bmatrix} \quad \ldots(7)$$

Evidently the product of the unitary matrices $U_1(\alpha)$, $U_2(\beta)$, $U_3(\gamma)$ corresponds to the product of a rotation about z through an angle γ, about y, through β and about x through α *i.e.* corresponds to a rotation with Euler's angles α, β, γ. Hence the homomorphism is the homomorphism of the unitary group onto the entire 3-dimensional rotation group.

Problem 42. *Show that the groups of order 2 and 3 are always cyclic.*

(Rohilkhand, 1988, 92)

If E be identity and A another element of order 2-group, then $EA = A$, $AA = A^2$ also are elements of the group. Thus $A^2 = A$ or E. But $A^2 \neq A$ as A is not identity element. Hence $A^2 = E$ *i.e.* elements of the group of order two as A and $A^2 = E$ is always cyclic *i.e.* $\{A, A^2 = E\}$ is cyclic

	E	A
E	E	A
A	A	E

Again consider a *group of order 3*: one element is identity E, others are A, B *i.e.* the group is $\{E, A, B\}$ where $AA = A^2$ is an element say $A^2 =$ either E or A or B. But $A^2 \neq E$ as a subgroup or order 2 can not be obtained from a group of order 3. Also $A^2 \neq A$ as $A \neq E$ and $A^2 \neq B \Rightarrow$ group elements are E, A, A^2.

Further $A \cdot A^2 = A^3$ is also an element of the group so that $A^2 =$ either E or A or A^2. But $A^3 \neq A$ since $A^2 \neq E$ and $A^3 \neq A^2$ as $A \neq E$ or $A^3 = E$. Thus the group elements of a group of order 3 always form a cyclic group *i.e.* $\{A, A^2, A^3 = E\}$ is cyclic.

Problem 43. *Show that a group of order 4 may or may not be a cyclic group*

(Rohilkhand, 1980, 86, 88)

Consider a group of order 4 s.t. $\{A, A^2, A^3, A^4 = E\}$ which is cyclic as shown in Problem 42. Also if $A = i$, then the group is $\{i, -1, -i, 1\}$ which is clearly cyclic. But if elements are E, A, B, C then the group is Vierrer group (not cyclic).

	E	A	B	C
E	E	A	B	C
A	A	E	C	B
B	B	C	E	A
C	C	B	A	E

Here $A^2 = B^2 = C^2 = E$

$AB = BA = C, AC = CA = B, BC = CB = A$

Obviously it is an abelian group, which in practice is the group of rotations of a triangle:

E denotes no rotation or rotation through 360° or 2π, A through π about x-axis, B through π about y-axis and C through π about z-axis.

Problem 44. *If $D(A)$ and $D(B)$ denote the determinatnts of two matrices A and B of any order then show that $D(A)\,D(B) = D(AB) = D(B)\,D(A)$*

Problem 45. *If $D(A)$ is the determinant of the martrix $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ then show that*

$$B = \begin{bmatrix} \dfrac{d}{D(A)} & \dfrac{-c}{D(A)} \\ \dfrac{-b}{D(A)} & \dfrac{a}{D(A)} \end{bmatrix}$$

is the inverse of A, provided $D(A) \neq 0$, and if $D(A) = ad - bc$, then verify that $AB = BA = I$.

Problem 46. *If we define D_3 group as consisting of six matrix elements*

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}, \quad C = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix},$$

$$D = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}, \quad F = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix},$$

then form the group table and show that this group is non-abelian.

The group table is as shown here and it is easy to show that

$$AB \neq BA \text{ etc.}$$

Also note that order of each of A, B, C is 2 and tht of D, F is 3.

	E	A	B	C	D	F
E	E	A	B	C	D	F
A	A	E	D	F	B	C
B	B	F	E	D	C	A
C	C	D	F	E	A	B
D	D	C	A	B	F	E
F	F	B	C	A	E	D

Problem 47. (*a*) *Find the classes of D_2 group.*

(*b*) *Find the normal (invariant) sub group and factor group of D_2–group.*

(*a*) Since in each group identity elements form their own class and no other class contains E, therefore one class of D_3 is E.

Now to find the elements conjugate to A, we have

$E^{-1}AE = A;\ A^{-1}AA = A;\ B^{-1}AB = BAB = FB = C;\ C^{-1}AC = CAC = DC = B$

$D^{-1}AD = FAD = BD = C;\ F^{-1}AF = DAF = CF = B.$

So elements conjugate to A, are B and C and hence A, B, C form a class.

Again consider the elements conjugate to D,

$E^{-1}DE = D;\ A^{-1}DA = ADA = BA = F;\ B^{-1}DB = BDB = CB = F$

$C^{-1}DC = CDC = AC = F;\ D^{-1}DD = D;\ F^{-1}DF = DDF = FF = D.$

i.e. F is the only element conjugate to D and so D, F form a class.

Conclusively D_3 has three classes : $\mathcal{C}_1 = E$

$\mathcal{C}_2 = A, B, C$

$\mathcal{C}_3 = D, F.$

(*b*) D_3 consists of (*i*) A trivial subgroup of order one containing a single identity element (*ii*) Subgroup of order 2, (E, A) (E, A), (E, C), (*iii*) subgroup of order 3, (E, D, F) and (*iv*) subgroup of order 6 containing all elements of the group. One of these subgroups having order 3 *i.e.* $(E, D, F) = H$ is a normal subgroup, since $A^{-1}HA = (A^{-1}EA, A^{-1}DA, A^{-1}FA) = (E, ADA, AFA) = (E, BA, CA) = (E, F, D) = H$ similarly $B^{-1}HB = H$ and $C^{-1}HC = H$.

Coset of normal group H in D_3 is $AH = (AE, AD, AF) = (A, B, C)$

Similarly $BH = (B, C, A) = AH$ and $CH = (C, A, B) = AH$

As such there is only one coset (A, B, C) of H in D_3.

Order of D_3 group is 6, that of H is 3, so order of vactor group is 2. But the unit element of factor group must be normal subgroup itself and second element of it is the coset $(A, B, C) = M$ (say of the normal subgroup H in D_3 which follows from:

$$HH = (E, F, D)(E, F, D) = (E, EF, ED, FE, F^2, FD, DE, DF, D^2)$$
$$= (F, F, D, F, D, E, D, E, F)$$
$$= (E, F, D) = H,$$

similarly $HM = (A, B, C) = M$ and $M^2 = (E, F, D) = H$

∴ H and M form factor (quotient) group with H as unit element. Group table of factor group is

	H	M
H	H	M
M	M	H

Problem 48. *Defining a commutator of x and y as $x^{-1}y^{-1}xy$, $x, y \in G$ (group) and denoted by $[x, y]$ show that its inverse is also commutator.*

We have $[x, y] = x^{-1}y^{-1}xy = z$ (say), then $z^{-1} = y^{-1}x^{-1}yx = [y, x]$.

Problem 49. (*a*) *Define a group. Show that the group D_3 consisting of the symmetry elements which map an equilateral triangle onto itself is isomorphous with the permutation group of three numbers (a, b, c).* (*Rohilkhand, 1992; Agra, 1972*)

Obtain a two-dimensional representation of this group

(*b*) *Show that the group of matrices*

$$E=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\ A=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix},\ B=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix},\ C=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix},$$

$$D=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix},\ F=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}.$$

is isomorphic to D_3–group.

Hint to (*b*) : Here $AB=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}=D$ with other products s.t. there is one-one correspondence between the given group and D_3 group of matrices such that products correspond to products. Hence the result.

Problem 50. *What is meant by the matrix representation of a group. Define the character of a group element in a particular representation. Prove that all the elements of a group constituting a class have the same character.*

Obtain the independent irreducible representations of the permutational group of three elements (a, b, c) and hence the character table for the group. [*Agra, 1973*]

Problem 51. *Give example of a group which has a subgroup and construct a multiplication table for its elements.*

Problem 52. *State and prove the Orthogonality theorem for inequivalent, irreducible, unitary representations of a finite group and hence show that two inequivalent irreducible representations of a group can not have same characters and irreducible representations with equal characters are equivalent.*

[*Rohilkhand, 1977*]

See §4.14

Problem 53. *Define the character of representation of a group and show that it is a class function in the group.*

Show that the characters of the irreducible representations of a finite group form an orthogonal system in the space of group elements. (*Rohilkhand, 1978*)

See § 4.14.

Problem 54. *Prove that every subgroup of a cyclic group is cyclic.*

(*Rohilkhand; 1980*)

Problem 55. (*a*) *Define group, subgroup and class. Show that there are three classes is the group of symmetry operations of the equilateral triangles.*

(*b*) *What do you mean by reducible and irreducible representations ?*

(*Rohilkhand, 1983*)

Problem 56. *Explain the concept of 'Group Representation', Prove the orthogonality theorem is group-theory.* (*Rohilkhand, 1984*)

Problem 57. (*a*) *Define a group, a subgroup and a class.*

(*b*) *An electron is moving in the potential field of three protons located at the corners of an equilateral triangle. Show that the set of all the possible symmetry operations for this system form a group. Construct the character table of this group.*

(*Rohilkhand, 1985*)

Problem 58. *Find the classes of the following groups:*

$$E=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ A=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\ B=\begin{bmatrix}-1/2 & \sqrt{3}/2\\ \sqrt{3}/2 & 1/2\end{bmatrix},\ C=\begin{bmatrix}-1/2 & -\sqrt{3}/2\\ \sqrt{3}/2 & 1/2\end{bmatrix},$$

$$D=\begin{bmatrix}-1/2 & \sqrt{3}/2\\ -\sqrt{3}/2 & 1/2\end{bmatrix},\ F=\begin{bmatrix}-1/2 & -\sqrt{3}/2\\ \sqrt{3}/2 & -1/2\end{bmatrix}.$$

(Rohilkhand, 1986)

See Problem 46

Problem 59. *If there are s classes in a group of elements; show that there will be s different irreducible representtions, find* $d_1^2+d_2^2+...+d_s^2$. *(Rohilkhand, 1987)*

Problem 60. (*a*) *Define isomorphic and homomorphic groups. Differentiate between them and give at least two properties of each.*

(*b*) *Prove that the order of each subgroup of a finite group is a divisor of the order of the group.*

(Rohilkhand, 1989)

Problem 61. *Define character of a representation of a group and show that equivelent representation have the same set of characters.* *(Rohilkhand, 1991)*

Problem 62. *Show that covering operations of an equilateral trianlge form a group homomorphic on to the group of elements (1, –1). Find the classes, invariant subgroup and factor group of this group.*

(Rohilkhand, 1991)

Problem 63. *Obtain the characters of special unitary group SU (2).*

(Rohilkhand, 1992)

See § 4.13.

Problem 64. (*a*) *Define subgroup and class. Show that there are five classes in the group of symmetry operations of square* (D_4).

(*b*) *Show that the three-fold permutation group (ABC) is isomosphic to the point group* D_3.

(Rohilkhand, 1993)

Problem 65. (*a*) *If every element of a group G is its own inverse, then show that the group G is an abelian group.*

(*b*) *State and prove orthogonality theorem for the irreducible representations of a group.*

(Rohilkhand, 1993)

CHAPTER 5

COMPLEX VARIABLES

5.1. INTRODUCTION

Cantor, Dedekind and Weierstrass etc., extended the conception of rational numbers to a larger field known as real numbers which constitute rational as well as irrational numbers. Evidently the system of real numbers is not sufficient for all mathematical needs *e.g.* there is no real number (rational or irrational) which satisfies $x^2 + 1 = 0$. It was therefore felt necessary by Euler Gauss, Hamilton, Cauchy, Rieman and Weierstrass etc. to extend the field of real numbers to the still larger field of complex numbers. Euler for the first time introduced the symbol i with the property $i^2 = -1$ and then Gauss introduced a number of the form $\alpha + i\beta$ which satisfies every algebraic equation with real coefficients. Such a number $\alpha + i\beta$ with $i = \sqrt{-1}$ and α, β being real, is known as a *complex number.*

5.2. DEFINITIONS

Complex numbers. An ordered pair of real numbers such as (x, y) is termed as a complex number. If we write

$$z = (x, y) \text{ or } x + iy, \text{ where } i = \sqrt{-1}, \text{ then}$$

x is called the *real part* and y the *imaginary part* of the complex number z and denoted by

$$x = R_z \text{ or } R(z) \text{ or } Re(z)$$

$$y = I_z \text{ or } I(z) \text{ or } Im(z)$$

Equality of complex numbers. Two complex numbers (x, y) and (x', y') are equal iff $x = x'$ and $y = y'$.

Modulus of a complex number. If $z = x + iy$ be a complex number then its modulus (or module) is denoted by $|z|$ and given by

$$|z| = |x + iy| = +\sqrt{x^2 + y^2}$$

Evidently $\quad |z| = 0$ iff $x = 0, \ y = 0.$

5.3. OPERATION OF FUNDAMENTAL LAWS OF ALGEBRA ON COMPLEX NUMBERS

Taking three complex numbers $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, $z_3 = (x_3, y_3)$ we define the following operations:

[1] Addition. The sum of two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ (say) is defined as a complex number $z = (z_1 + z_2) = (x_1 + x_2, y_1 + y_2)$ such that its real part is the sum of real parts and imaginary part is the sum of imaginary parts of the given numbers.

(*i*) **Addition is commutative.** *i.e.* $z_1 + z_2 = z_2 + z_1$ ... (1)

Since we have $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$

$= (x_1 + x_2) + i(y_1 + y_2)$

$= (x_1 + x_2, y_1 + y_2)$

$= (x_2 + x_1, y_2 + y_1)$, all the numbers being real

$= (x_2 + iy_2) + (x_1 + iy_1)$

$= z_2 + z_1$

(*ii*) **Addition is associative.** *i.e.* $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ ... (2)

Since, we have

$z_1 + (z_2 + z_3) = x_1 + iy_1 + (x_2 + iy_2 + x_3 + iy_3)$

$= (x_1 + iy_1 + x_2 + iy_2) + x_3 + iy_3$

$= (z_1 + z_2) + z_3$

(*iii*) **There exists an additive identity** *i.e.* $z + o = z$ ... (3)

Since, we have $z + o = (x, y) + (o, o)$

$= (x + o, y + o)$

$= (x, y) = z$

(*iv*) **There exists an additive inverse** *i.e.* $z + (-z) = o$ ... (4)

Since, $z + (-z) = (x, y) + (-x, -y)$

$= (x - x, y - y)$

$= (o, o)$

$= o$

Note: If $z = (x, y)$ then $-z = (-x, -y)$ is called as *additive inverse* of z.

[2] **Subtraction.** If $z_1 = (x_1, y_1)$ then $-z_1 = (-x_1, -y_1)$ etc.

$\therefore$ $z_1 - z_2 = (x_1, y_1) + (-x_2, -y_2) = x_1 + iy_1 - x_2 - iy_2$

$= (x_1 - x_2, y_1 - y_2)$ ... (5)

[3] **Multiplication.** We have $z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$

i.e. $(x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$ ... (6)

(*i*) **Multiplication is commutative.** *i.e.* $z_1 z_2 = z_2 z_1$ ... (7)

Since, $z_1 z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$ by (6)

$= (x_2 x_1 - y_2 y_1, y_2 x_1 + y_1 x_2)$

$= (x_2 + iy_2)(x_1 + iy_1)$

$= z_2 z_1$

(*ii*) **Multiplication is associative.** *i.e.* $z_1(z_2 z_3) = (z_1 z_2) z_3 = z_1 z_2 z_3$ ... (8)

Since $z_1(z_2 z_3) = (x_1, y_1)[x_2 x_3 - y_2 y_3, x_2 y_3 + x_3 y_2]$ by (6)

$= [x_1(x_2 x_3 - y_2 y_3) - y_1(x_2 y_3 + x_3 y_2),$

$x_1(x_2 y_3 + x_3 y_2) + y_1(x_2 x_3 - y_2 y_3)]$ by (6)

$= [(x_1 x_2 - y_1 y_2) x_3 - (x_1 y_2 + x_2 y_1) y_3, (x_1 y_2 + x_2 y_1) x_3$

$+ (x_1 x_2 - y_1 y_2) y_3]$ (on rearranging)

$= [(x_1, y_1)(x_2, y_2)](x_3, y_3)$

$= (z_1 z_2) z_3$

(*iii*) **Multiplication is distributive.** *i.e.* $(z_1 + z_2)\, z_3 = z_1z_3 + z_2z_3$ ... (9)

Since $(z_1 + z_2)\, z_3 = (x_1 + x_2, y_1 + y_2)\, (x_3, y_3)$

$= [(x_1 + x_2)\, x_3 - (y_1 + y_2)\, y_3, (x_1 + x_2)\, y_3 + (y_1 + y_2)\, x_3]$ by (6)

$= [(x_1x_3 - y_1y_3) + (x_2x_3 - y_2y_3); (x_1y_3 + x_3y_1) + (x_2y_3 + x_3y_2)]$ (on arranging)

$= (x_1x_3 - y_1y_3, x_1y_3 + x_3y_1) + (x_2x_3 - y_2y_3, x_2y_3 + x_3y_2)$

$= (x_1, y_1)\, (x_3, y_3) + (x_2, y_2)\, (x_3, y_3)$

$= z_1z_3 + z_2z_3$

(*iv*) **There exists a multiplicative identity** *i.e.* $z \cdot 1 = z$ ... (10)

where $1 = (1, 0)$ is the multiplicative identity known as *unity* for the system of complex numbers.

We have $z \cdot 1 = (x, y)\, (1, o)$

$= (x, y)$

$= z$

(*v*) **There exists a multiplicative inverse** *i.e.* $zz^{-1} = 1$ ... (11)

If $z = (x, y)$, then $z^{-1} = (x, y)^{-1}$ so that we have to show that

$(x, y)\, (x, y)^{-1} = (1, 0)$

Assuming $(x, y)^{-1} = (x', y')$, this becomes

$(x, y)\, (x', y') = (1, 0)$

i.e. $(xx' - yy', xy' + yx') = (1, 0)$

which gives $xx' - yy' = 1$ (on equating real and imaginary parts)

$xy' + yx' = 0$

Solving these equations we get

$$x' = \frac{x}{x^2 + y^2} \quad y' = \frac{-y}{x^2 + y^2} \text{ provided } x^2 + y^2 \neq 0$$

Hence the complex number (x, y) has a unique multiplicative inverse $\left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2}\right)$ which is also a complex number such that (x, y) $\left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2}\right) = (1, 0)$

[4] **Division.** Consider an equation $z_1z_2 = z'$

where $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ and $z' = (x', y')$

Now $z_1z_2 = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1) = z' = (x', y')$

which gives $x_1x_2 - y_1y_2 = x'$

$x_1y_2 + x_2y_1 = y'$.

Solving $x_2 = \dfrac{y_1 y' - x_1 x'}{x_1^2 + y_1^2}$, $y_2 = \dfrac{x_1 y' - x' y_1}{x_1^2 + y_1^2}$ ...(12)

provided $x_1^2 + y_1^2 \neq 0$ i.e. $|z_1| \neq 0$

Thus we have a unique solution and $z_2 = \dfrac{z'}{z_1}$ is the quotient.

[5] **Conjugate complex numbers.** If $z = x + iy$, then $x - iy$ is said to be the conjugate of complex number z and denoted by $\bar{z}$.

Evidently $\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2$ ... (13)

$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ ... (14)

$z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$ ... (15)

$z + \bar{z} = 2x = 2R_z$ or $2R(z)$ ... (16)

$z - \bar{z} = i2y = 2iI_z$ or $2iI(z)$ ... (17)

5.4. GRAPHICAL REPRESENTATION (ARGAND DIAGRAM)

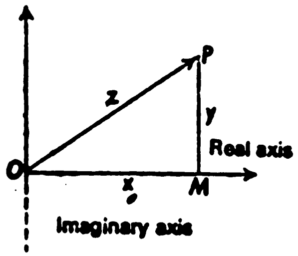


Fig. 5.1

Consider a point P in xy-plane. Let an ordered pair of values of x and y correspond to the co-ordinates of the point P. Then a complex number z may be made to correspond to the point P, where

$$z = x + iy.$$

Here z is called the complex co-ordinate of the point P.

In the adjoining figure, the x-axis is called the *real axis* or *axis of reals* and y-axis is called the *imaginary axis* or the *axis of imaginaries.*

Here $|z| = |x + iy| = \sqrt{(x^2 + y^2)}$ is the measure of length OP.

It (r, θ) be the polar co-ordinates of the point P, the polar form of the complex number z is

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

Here the number r (being taken + ive) is called the **modulus** or **absolute value** of the complex number z and θ is called the angle or **argument** of z and usually written as arg z, *i.e.*, $|z| = r$ and $\arg z = \theta$.

Now the co-ordinates of a point P' which is conjugate of z are $\bar{z} = (x, -y)$ or $(r, -\theta)$ in polars.

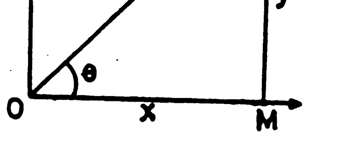


Fig. 5.2

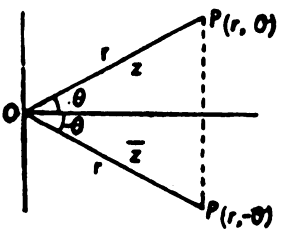


Fig. 5.3

Since $\bar{z} = r\{\cos(-\theta) + i\sin(-\theta)\}$, geometrically the points P and P' represent z and $\bar{z}$ respectively and their situations are symmetrical about the axis of reals, *i.e.* x-axis. The conjugate of z is called the **reflection** or **image** of z in the real axis.

Note 1. The plane whose points are represented by complex numbers is known as *Argand Plane* or *Argand diagram* or *Complex plane* or *Gaussian plane*.

Note 2. The complex number z representing the point (x, y) is sometimes called as *Affix* of the point (x, y).

Note 3. The *sum, difference, product* and *quotient* of complex numbers can be geometrically represented on the Argand plane as follows:

[1] **Sum.** Taking z_1 and z_2 two complex numbers represented by the points P and Q on Argand Plane and completing the parallelogram $OPRQ$, we observe that mid-points of its diagonals OR and PQ coincide, since they bisect each other *i.e.*

if $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, then mid-point of PQ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ which is also the mid-point of OR showing that co-ordinates of R are $(x_1 + x_2, y_1 + y_2)$.

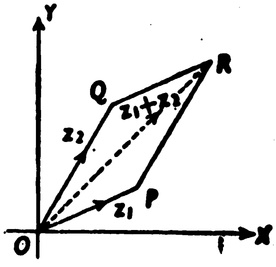


Fig. 5.4

But $z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$

$\therefore$ The sum $z_1 + z_2$ corresponds to a vector whose components are $x_1 + x_2$ and $y_1 + y_2$. As such the sum of two complex numbers z_1 and z_2 can be represented by a vector $(z_1 + z_2)$

i.e. if $\vec{OP} = z_1$, $\vec{OQ} = z_2$ then $\vec{OR} = \vec{OP} + \vec{PR} = \vec{OP} + \vec{OQ} = z_1 + z_2$.

Hence the point R on Argand Plane corresponds to the sum of two complex numbers z_1 and z_2 as shown in Fig. 5.4.

[2] **Difference.** Taking $z_1 = (x_1, y_1)$ and $z_2 = (x_2 = (x_2, y_2)$ two complex numbers represented by the points P and Q on Argand Plane and completing the parallelogram $OQPR$, we see that the point R represents the complex number $z_1 - z_2$, since $z_1 - z_2 = (x_1 - x_2, y_1 - y_2)$ being a complex number corresponds to a vector whose components are $x_1 - x_2$ and $y_1 - y_2$ and

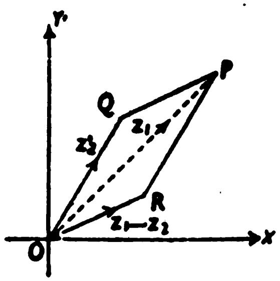


Fig. 5.5

if $\vec{OP} = z_1$, $\vec{OQ} = z_2$

then $\vec{QO} = -z_2$, so that

$$z_1 - z_2 = \vec{OP} - \vec{OQ} = \vec{OP} + \vec{QO}$$

$$= \vec{QO} + \vec{OP} = \vec{QP} = \vec{QR}$$

i.e. the difference of two complex numbers can be represented by a vector.

[3] **Product.** If $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ are two complex numbers, then $z_1 z_2 = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$

Changing to polars by putting $x_1 = r_1 \cos\theta_1, y_1 = r_1 \sin\theta_1$

$$x_2 = r_2 \cos\theta_2, y_2 = r_2 \sin\theta_2$$

where r_1, r_2 are the moduli and θ_1, θ_2 are arguments of z_1 and z_2 respectively, we have

$$z_1 z_2 = r_1 r_2 [(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i(\cos\theta_1 \sin\theta_2 + \cos\theta_2 \sin\theta_1)]$$

$$= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$$

$$\therefore \quad |z_1 z_2| = r_1 r_2 = |z_1|\,|z_2| \qquad \text{... (1)}$$

$$\text{and} \quad \arg(z_1 z_2) = \theta_1 + \theta_2 = \arg\theta_1 + \arg\theta_2 \qquad \text{... (2)}$$

i.e. the modulus of product of two complex numbers is equal to the product of their moduli and argument of the product of two complex numbers is the sum of their arguments.

In general if there are n complex numbers $z_1, z_2, ..., z_n$ with moduli $r_1, r_2, ... r_n$ and arguments $\theta_1, \theta_2 ..., \theta_n$ respectively, then repeated application of the above result yields,

$$z_1 z_2 ... z_n = r_1 r_2 ... r_n [\cos(\theta_1 + \theta_2 + ... + \theta_n) + i \sin(\theta_1 + \theta_2 + ... + \theta_n)]$$

so that $|z_1 z_2 ... z_n| = r_1 r_2 ... r_n = |z_1| \, |z_2| \, ... \, |z_n|$ .... (3)

and $\arg(z_1 z_2 ... z_n) = \theta_1 + \theta_2 ... \theta_n = \arg z_1 + \arg z_2 + ... + \arg z_n ...$ ... (4)

i.e. the modulus of the product of any number of complex quantities is equal to the product of their moduli and the argument of the product of these complex numbers is equal to the sum of their arguments.

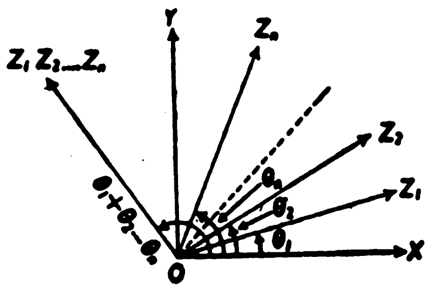


Fig. 5.6

Geometrically represented on an Argand Plane the product of n complex quantities $z_1, z_2, ..., z_n$ as shown in Fig. 5.6 follows that the length of the vector $(z_1 z_2 ... z_n)$ is the product of the lengths of the vectors $z_1, z_2, ..., z_n$ *i.e.* $|z_1 z_2 ... z_n| = |z_1| \, |z_2| ... |z_n|$ and the amplitude of $(z_1 z_2 ... z_n)$ is equal to the sum of the amplitudes of $z_1, z_2, ..., z_n$.

In a particular case when $z_1 = z_2 = ... = z_n = z$ (say), the above results may be summarised as

$z^n = r^n (\cos n\theta + i \sin n\theta)$ under the assumptions

$$r_1 = r_2 = ... = r_n = r \text{ (say)}$$

$$\theta_1 = \theta_2 = ... = \theta_n = \theta \text{ (say)}$$

i.e. $|z^n| = r^n = |z|^n$ ... (5)

and amp $z^n = n\theta = n.$ (amp z) ... (6)

Also if $r = 1$, we get the *De Moivre's theorem* for positive integral exponents such as $z^n = (\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta$ ... (7)

[4] **Quotient.** Consider two complex numbers z_1 and z_2 such that

$$z_1 = x_1 + iy_1 = r_1(\cos\theta_1 + i \sin\theta_1)$$

$$z_2 = x_2 + iy_2 = r_2(\cos\theta_2 + i \sin\theta_2)$$

The quotient of complex numbers z_1 and z_2 is given by

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i \sin\theta_1)}{r_2(\cos\theta_2 + i \sin\theta_2)} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]$$

$$\therefore \left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|} \quad ... (8)$$

$$\text{and } \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2 \quad ...(9)$$

i.e. the modulus of the quotient of two complex numbers is the quotient of their moduli and the argument of the quotient of two complex numbers is the difference of their arguments.

As a particular case defining the division as the inverse of multiplication we have

$$\frac{1}{z} = \frac{1}{r}\left[\cos(-\theta) + i\sin(-\theta)\right] = \frac{1}{r}\left[\cos\theta - i\sin\theta\right]$$

so that $\frac{1}{z^n} = z^{-n} = \frac{1}{r^n}(\cos n\theta - i\sin n\theta) = \left(\frac{1}{z}\right)^n$ ... (10)

which shows that De Moivre's theorem is valid when the exponent is any negative integer.

Geometrical representation of $\frac{z_1}{z_2}$ may be shown as below:

Let $\vec{OP}$ and $\vec{OQ}$ represent the vectors z_1 and z_2 in an Argand Plane such that $|z_1| = OP$, $|z_2| = OQ$ and arg $z_1 = \theta_1$, arg $z_2 = \theta_2$

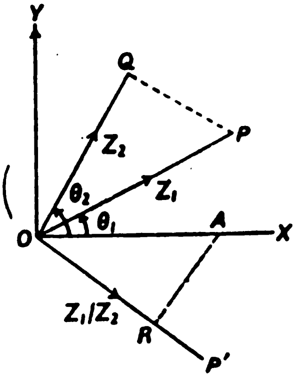
Rotate the line OP in clockwise direction through an angle θ_2 (= arg z_2) such that its new position is OP' and $\angle POP' = \theta_2$. Take $OA = 1$ (unit length) on OX and draw a line AR to meet OP' in R such that $\angle OAR = \angle OQP$.

The point R thus obtained corresponds to the quotient $\frac{z_1}{z_2}$ and it may be justified as follows:

In similar triangles OAR and OQP have

$$\frac{OR}{OA} = \frac{OP}{OQ} \text{ i.e., } OR = \frac{OP}{OQ}, \because OA = 1$$

$$= \frac{|z_1|}{|z_2|} = \left|\frac{z_1}{z_2}\right|$$



which shows that the radius vector of the point R is $\left|\frac{z_1}{z_2}\right|$.

Fig. 5.7

Also $\angle AOR = \angle POR - \angle POX = \theta_2 - \theta_1 = -(\theta_1 - \theta_2)$

i.e. vectorial angle of R is $-(\theta_1 - \theta_2)$ which, when measured in positive sense is $\theta_1 - \theta_2$.

Hence the point R represents the quotient $\frac{z_1}{z_2}$.

Note 4. **Multiplication of a Complex number by i.**

Let z be a complex number with its modulus r and amplitude θ

i.e. $z = r(\cos\theta + i\sin\theta)$

and $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$$\therefore \quad iz\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) r(\cos\theta + i\sin\theta)$$

$$= r\left[\cos\left(\frac{\pi}{2} + \theta\right) + i\sin\left(\frac{\pi}{2} + \theta\right)\right] \quad \text{... (11)}$$

which follows that iz represents a vector obtained by rotating the vector z through a right angle in the positive direction.

Note 5. **Extraction of roots.**

Suppose that $z_o^n = z$, n being positive integer ... (12)

We can express,

$$z = r(\cos\theta + i\sin\theta)$$

so that $z_o = r_o(\cos\theta_o + i\sin\theta_o)$ provided $z \neq 0$ and r_o, θ_o, are unknown.

$\therefore$ (12) gives, $r_o^n(\cos n\theta_o + i\sin n\theta_o) = r(\cos\theta + i\sin\theta)$ *by* (7)

Measuring the angles in radians, we therefore, have

$r_o^n = r$, $n\theta = \theta \pm 2m\pi$, m being zero or any positive integer which follow that r, r_o being positive, r_o is the positive nth root of r and $\theta_o = \dfrac{\theta}{n} \pm \dfrac{2m\pi}{n}$ has n distinct values for $m = 0, 1, 2, \ldots, n-1$.

As such there are n distinct solutions of (1), given by

$$z^{1/n} = z_o = r^{1/n}\left(\cos\frac{\theta + 2m\pi}{n} + i\sin\frac{\theta + 2m\pi}{n}\right), \; m = 0, 1, 2, \ldots, n-1. \quad \ldots (13)$$

which are n distinct values of $z^{1/n}$.

Here the length of each of the n vectors $z^{1/n}$ is the positive number $r^{1/n}$ and argument of one of these vectors is $\dfrac{\theta}{n}$ while the other arguments are obtained by adding multiples of $\dfrac{2\pi}{n}$ to $\dfrac{\theta}{n}$.

In particular $z = 0$, (12) has the only solution $z_o = 0$.

But $1 = \cos 0 + i\sin 0$, then nth roots of unity are given by

$$1^{1/n} = \cos\frac{2\pi m}{n} + i\sin\frac{2\pi m}{n}, \; m = 0, 1, 2, \ldots, n-1 \quad \ldots (14)$$

Taking $m = 1$, the root of unity being a complex number and denoted by ω, is given by

$$\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} \quad \ldots (15)$$

According to De Moivre's theorem, the n, nth roots of unity are given by

$$1, \omega, \omega^2, \ldots, \omega^{n-1} \quad \ldots (16)$$

Which are the vertices of a regular polygon in complex plane, of n sides inscribed in the unit circle $|z| = 1$ with one vertex at the point $z = 1$.

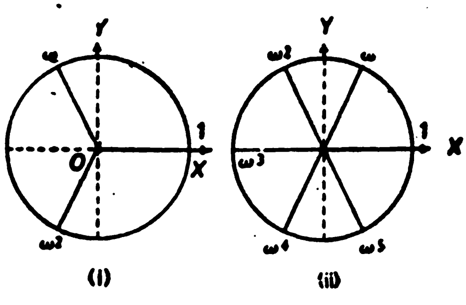


Fig. 5.8

The case (*i*) of Fig. 5.8 shows for $n = 3$ and case (*ii*) for $n = 6$.

Now if ζ is a particular nth root of z, then we have the n roots of z as

$$\zeta, \zeta\omega, \zeta\omega^2, \ldots, \zeta\omega^p, \ldots \zeta\omega^{n-1} \quad \ldots (17)$$

since ζ multiplied by ω^p implies the increment of $arg\,\zeta$ by the angle $\dfrac{2p\pi}{n}$.

If m, n be two positive integers prime to each other, then (13) and (17) yield

$$\left(z^m\right)^{1/n} = \left(r^m\right)^{1/n}\left(\cos\frac{m\theta}{n} + i\sin\frac{m\theta}{n}\right)\omega^q,\ q = 1, 2, \ldots n-1 \qquad \ldots (18)$$

$$\left(z^{1/n}\right)^m = \left(r^{1/n}\right)^m\left[\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)\omega^l\right]^m$$

$$= \left(r^m\right)^{1/n}\left(\cos\frac{\theta m}{n} + i\sin\frac{\theta m}{n}\right)\omega^{lm},\ l = 0, 1, 2, \ldots n-1 \qquad \ldots (19)$$

The two sets of n numbers will be identical if the set ω^q and ω^{lm} coincide and then n numbers in either set can be written as $z^{m/n}$ *i.e.*

$$z^{m/n} = \left(r^m\right)^{1/n}\left[\cos\left\{\frac{m}{n}(\theta + 2\pi m)\right\} + i\sin\left\{\frac{m}{n}(\theta + 2\pi m)\right\}\right]$$

$$m = 0, 1, 2, \ldots, n-1 \qquad \ldots (20)$$

we may similarly define,

$$z^{-m/n} = (z^{1/n})^{-m} = (z^{-m})^{1/n}$$

5.5. PROPERTIES OF MODULI AND ARGUMENTS AND GEOMETRY OF COMPLEX NUMBERS

[A] Properties of Moduli

(*1*) *The modulus of the product of two complex numbers is the product of their Moduli.*

If there are two numbers z_1 and z_2 defined by

$z_1 = (x_1, y_1)$ or (r_1, θ_1) in polars *i.e.* $z_1 = r_1(\cos\theta_1 + i\sin\theta_1) = r_1 e^{i\theta_1}$

$z_2 = (x_2, y_2)$ or (r_2, θ_2) in polars *i.e.* $z_2 = = r_2 e^{i\theta_2}$

Then, $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$

So that
$$|z_1 z_2| = |r_1 r_2 e^{i(\theta_1+\theta_2)}|$$
$$= r_1 r_2 |\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)|$$
$$= r_1 r_2\sqrt{\cos^2(\theta_1 + \theta_2) + \sin^2(\theta_1 + \theta_2)}$$
$$= r_1 r_2$$
$$= |z_1| . |z_2| \qquad \ldots (1)$$

(2) *The modulus of the sum of two complex numbers does never exceed the sum of their moduli.*

Let z_1 and z_2 be two complex numbers and $\bar{z}_1$ and $\bar{z}_2$ their conjugates.

We have already mentioned that $|z|^2 = z\bar{z}$.

$$\therefore |z_1 + z_2|^2 = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$$
$$= z_1\bar{z}_1 + z_2\bar{z}_2 + z_1\bar{z}_2 + z_2\bar{z}_1$$
$$= |z_1|^2 + |z_2|^2 + 2R(z_1\bar{z}_2)$$
$$\le |z_1|^2 + |z_2|^2 + 2|z_1\bar{z}_2|,$$

$\left[\because z_1\bar{z}_2 = r_1 e^{i\theta_1} . r_2 e^{-i\theta_2}\right.$

$= r_1 r_2 e^{i(\theta_1-\theta_2)}$

and $z_2\bar{z}_1 = r_1 r_2 e^{-i(\theta_1-\theta_2)}$

$\therefore z_1\bar{z}_2 + z_2\bar{z}_1$

$$\leq \{|z_1| + |z_2|\}^2 \qquad = 2r_1r_2 \cos(\theta_1 - \theta_2) = 2R(z_1\bar{z}_2).$$

$\therefore |z_1 + z_2| \leq |z_1| + |z_2|$. Also if $z = x + iy$,

$$R(z) = x$$
$$\leq \sqrt{(x^2 + y^2)}$$
$$\leq |z|.$$

Aliter. $|z_1 + z_2|^2 = |x_1 + x_2 + i(y_1 + y_2)|^2$

$$= (x_1 + x_2)^2 + (y_1 + y_2)^2 \qquad \because |A + iB|^2 = A^2 + B^2$$
$$= x_1^2 + x_2^2 + y_1^2 + y_2^2 + 2(x_1x_2 + y_1y_2)$$
$$= |z_1|^2 + |z_2|^2 + 2R(z_1\bar{z}_2)$$
$$\leq |z_1|^2 + |z_2|^2 + 2|z_1z_2|. \quad \because |\bar{z}_2| = |z_2|$$

$$|z_1 + z_1| \leq |z_1| + |z_2|. \qquad \ldots (2)$$

(3) *The modulus of difference of two complex numbers is greater than or equal to the difference of their moduli.*

Let $\quad z_1 = x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1) = r_1e^{i\theta_1}$

$\quad z_2 = x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2) = r_2e^{i\theta_2}$

Then $\quad \bar{z}_1 = r_1e^{-i\theta}$ and $\bar{z}_2 = r_2e^{-i\theta_2}$

and $\quad -(z_1\bar{z}_2 + z_2\bar{z}_1) = -\left[r_1e^{i\theta_1}.\, r_2e^{-i\theta_2}. + r_2e^{+i\theta_2}.r_1e^{-i\theta_1}\right]$

$$= -r_1r_2\left[e^{i(\theta_1-\theta_2)} + e^{-i(\theta_1-\theta_2)}\right]$$
$$= -r_1r_2\cos(\theta_1 - \theta_2) = -2R(z_1\bar{z}_2).$$

$$\therefore |z_1 - z_2|^2 = (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) \qquad \because |z|^2 = z\bar{z}$$
$$= z_1\bar{z}_1 + z_2\bar{z}_2 - (z_1\bar{z}_2 + \bar{z}_1z_2)$$
$$= |z_1|^2 + |z_2|^2 - 2R(z_1\bar{z}_2)$$

But $R(z_1\bar{z}_2) = R\{(x_1 + iy_1)(x_2 - iy_2)\} = x_1x_2 + y_1y_2$.

$$|z_1||z_2| = \sqrt{(x_1^2 + y_1^2)}\sqrt{(x_2^2 + y_2^2)}.$$

Now $\quad R(z_1\bar{z}_2) < |z_1||z_2|$,

if $\quad x_1x_2 + y_1y_2 < \sqrt{\{(x_1^2 + y_1^2)(x_2^2 + y_2^2)\}}$

or if $\quad (x_1x_2 + y_1y_2)^2 < x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + y_1^2x_2^2$

or if $\quad 2x_1x_2y_1y_2 < x_1^2y_2^2 + y_1^2x_2^2$,

which is so, since arithmetic mean of two quantities is greater than their geometric mean.

As such $\quad -R(z_1\bar{z}_2) > -|z_1||z_2|$.

Hence $\quad |z_1 - z_2|^2 \geq |z_1|^2 + |z_2|^2 - 2|z_1||z_2|$

$$\geq [|z_1| - |z_2|]^2.$$

$$\because \quad |z_1 - z_2| \geq |z_1| - |z_2|. \qquad \ldots (3)$$

[B] Properties of Arguments

(1) *The argument of the product of two complex numbers is equal to the sum of their arguments.*

Take z_1, z_2 two complex numbers with moduli r_1, r_2 and arguments θ_1, θ_2 respectively, so that

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) = r_1 e^{i\theta_1},\ z_2 = r_2 e^{i\theta_2}$$

$$\therefore\ z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)} = r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$$

giving $\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2$. ...(4)

The result may be generalized for any number.

(2) *The argument of the quotient of two complex numbers is equal to the difference of their arguments.*

Take z_1, z_2 two complex numbers with moduli r_1, r_2 and arguments θ_1, θ_2 so that $z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}$

then $$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)} = \frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right]$$

$$\therefore\ \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2 \qquad ...(5)$$

[C] Geometry of Complex Numbers

(*1*) *Arg* $\frac{z-a}{z-b}$ *gives the angle between the lines joining a to z and b to z taken in the appropriate sense.*

Take $\overrightarrow{AP} = z - a$ and $\overrightarrow{BP} = z - b$ as shown in Fig. 5.9.

$$\therefore\ \angle APB = \theta = \arg\overrightarrow{AP} - \arg\overrightarrow{BP}$$
$$= \arg(z-a) - \arg(z-b)$$
$$= \arg\frac{z-a}{z-b},\ o < \theta \le \pi, \qquad ...(6)$$

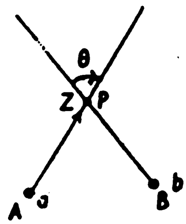


Fig. 5.9

argument being negative as shown in Fig. 5.9.

In particular if $\theta = 90°$, $\arg\frac{z-a}{z-b} = \pm\frac{\pi}{2}$ and $\frac{z-a}{z-b}$ is purely imaginary. ...(7)

(2) *Equation of a straight line joining two points* z_1 *and* z_2 *in Argand plane.* Referred to Fig 5.10, $\arg\frac{z-z_1}{z-z_2} = \pi$ or o according as z lies inside or outside the line joining A to B.

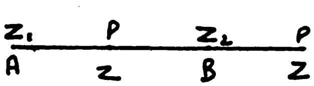


Fig. 5.10

In either case $\frac{z-z_1}{z-z_2}$ is purely real, so that

$$\frac{z-z_1}{z-z_2} = \overline{\left(\frac{z-z_1}{z-z_2}\right)} = \frac{\bar{z}-\bar{z}_1}{\bar{z}-\bar{z}_2}\quad i.e.\ z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + (z_1\bar{z}_2 - z_2\bar{z}_1) = 0 \qquad ...(8)$$

is the required equation of the straight line joining z_1 and z_2.

General Equation of a Line

$z_1 - z_2$ being purely imaginary, $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$ is also purely imaginary.

Also $z_1\bar{z}_2 - z_2\bar{z}_1$, is purely imaginary since $\overline{z_1\bar{z}_2} = \bar{z}_1 z_2$

Multiplying (8) by i, we have

$$iz(\bar{z}_1 - \bar{z}_2) - i\bar{z}(z_1 - z_2) + i(z_1\bar{z}_2 - z_2\bar{z}_1) = 0 \qquad \ldots(9)$$

each term of which now becomes real.

Setting $i(z_1\bar{z}_2 - z_2\bar{z}_1) = \lambda$, and $-i(z_1 - z_2) = \mu$; λ being real and μ a constant, so that

$$\overline{[-i(z_1 - z_2)]} = i(\bar{z}_1 - \bar{z}_2) = \bar{\mu},$$

the equation (9) becomes

$$\bar{\mu}z + \mu\bar{z} + \lambda = 0 \qquad \ldots(10)$$

where λ, μ are constants but λ real.

Note 1. In particular if $|z - z_1| = |z - z_2|$ then we have the equivalent form

$$(z - z_1)(\bar{z} - \bar{z}_1) = (z - z_2)(\bar{z} - \bar{z}_2) \text{ on using } |z|^2 = z\bar{z} \qquad \ldots(11)$$

which represents the equation of right bisector of the line joining z_1 and z_2.

Note 2. The equation (10) *i.e.* $\bar{\mu}z + \mu\bar{z} + \lambda = 0$ is the necessary and sufficient condition for z_1 to be the reflection of z_2 in the line.

(3) *Equation of a circle with centre at z_0 and redius r.*

If z is any point on the circle, then

$$|z - z_0| = r \text{ i.e. } |z - z_0|^2 = r^2$$

using $|z|^2 = z\bar{z}$, we have

$$(z - z_0)(\bar{z} - \bar{z}_0) = r^2$$

or $$z\bar{z} - z_0\bar{z} + (z_0\bar{z} - r^2) - z\bar{z}_0 = 0 \qquad \ldots(12)$$

Setting $-z_0 = \mu$, $-z_0\bar{z} - r^2 = \lambda$, this becomes

$$z\bar{z} + \bar{\mu}z + \mu\bar{z} + \lambda = 0, \lambda \text{ being real} \qquad \ldots(13)$$

General Equation of a Circle

(13) can be written as $(z + \mu)(\bar{z} + \bar{\mu}) = \mu\bar{\mu} - \lambda$

or $$|z + \mu|^2 = \mu\bar{\mu} - \lambda$$

So (13) represents the general equation of a circle if λ is real and $\mu\bar{\mu} - \lambda > 0$.

(4) *Equation of a cirle through three points z_1, z_2, z_3.*

Let A, B, C represent the points z_1, z_2, z_3 respectively. Take a point $P(z)$ on the circle.

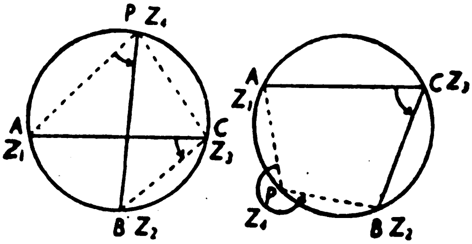


Fig. 5.11

$$\therefore \arg\frac{z_3 - z_1}{z_3 - z_2} - \arg\frac{z - z_1}{z - z_2} = 0 \text{ or } \pi$$

according as case (i) or (ii) of Fig. 5.11 exists.

$$\text{i.e. } \arg\frac{z_3 - z_1}{z_3 - z_2} \Big/ \arg\frac{z - z_1}{z - z_2} = 0 \text{ or } \pi$$

In either case $Im\left[\frac{z_3-z_1}{z_3-z_2}\Big/\frac{z-z_1}{z-z_2}\right]=0$

i.e. $\left(\frac{z_3-z_1}{z_3-z_2}\right)\Big/\left(\frac{z-z_1}{z-z_2}\right)$ is purely real

$$\therefore\quad \left(\frac{z_3-z_1}{z_3-z_2}\right)\Big/\left(\frac{z-z_1}{z-z_2}\right)=\left(\frac{\bar{z}_3-\bar{z}_1}{\bar{z}_3-\bar{z}_2}\right)\Big/\left(\frac{\bar{z}-\bar{z}_1}{\bar{z}-\bar{z}_2}\right) \qquad ...(14)$$

is the required equation.

Note 3. Condition for four points z_1, z_2, z_3, z_4 to be collinear is that

$$\frac{(z_3-z_1)\,(z_4-z_2)}{(z_3-z_2)\,(z_4-z_1)} \text{ is purely real.} \qquad ...(15)$$

(5) *Inverse points with respect to a circle.*

Two points $P(p)$, $Q(q)$ are said to be the inverse points with respect to a circle with centre $O(z)$ and radius r if

$$OP\cdot OQ=r^2 \qquad ...(16)$$

provided O, P, Q are collinear.

$\therefore\ |p-z|\ |q-z|=r^2$.

Also $\arg(p-z)=\arg(q-z)=-\arg\overline{(q-z)}$ gives

$$\arg(p-z)+\arg\overline{(q-z)}=0$$

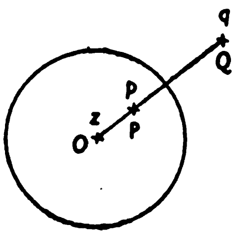


Fig. 5.12

i.e. $(p-z)\overline{(q-z)}$ is purely real say equal to r^2 so that

$$(p-z)\overline{(q-z)}=r^2 \qquad ...(17)$$

gives the required condition for p, q to be inverse points with respect to the circle with centre z and radius r.

Note 4. If $z=0$, p, q are inverse points provided

$$p\,\bar{q}=r^2 \qquad ...(18)$$

Note 5. In w plane, $w=0$ and $w=\infty$ are the inverse points with respect to the circle $|w|=1$, since if $w=w_1=0$ and $w=w_2=\infty$ are inverse points then from (18), $w_1\bar{w}_2=1$ or $\bar{w}_1w_2=\bar{1}$ *i.e.* $\bar{w}_1w_2=1$

$$\therefore\ w_2=\frac{1}{\bar{w}_1}=\frac{1}{0}=\infty$$

Note 6. In z plane, z and $\frac{1}{\bar{z}}$ are inverse points with rexpect to the unit circle $|z|=1$

Since if z_1, z_2 be inverse points then (18) gives

$$z_1\bar{z}_2=1 \text{ i.e. } z_1=\frac{1}{\bar{z}_2}=\frac{1}{\bar{z}} \text{ when } z_2=z$$

Note 7. In w plane, w and $\frac{1}{\bar{w}}$ are inverse points with respect to $|w|=1$.

Note 8. The equation $\left|\frac{z-p}{z-q}\right| = \lambda$, λ being a positive parameter, represents a family of circles for every member of which p and q are inverse points. If $\lambda = 1$, then the locus of z is the right bisector of the join of p and q.

Problem 1. *Find moduli and arguments of the following complex numbers*:

(*i*) $\frac{1-i}{1+i}$, (*ii*) $\left(\frac{2+i}{3-i}\right)^2$

(*i*) we have $\frac{1-i}{1+i} = \frac{1-i}{1+i} \times \frac{1-i}{1-i} = \frac{-2i}{2} = -i$

$$\therefore \quad \left|\frac{1-i}{1+i}\right| = |-i| = \sqrt{0^2 + (-1)^2} = 1$$

and $\arg \frac{1-i}{1+i} = \arg(-i) = -\frac{\pi}{2}$ as $-i = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)$

(*ii*) we have $\left(\frac{2+i}{3-i}\right)^2 = \frac{3-4i}{8-6i} \times \frac{8+6i}{8+6i} = \frac{-14i}{28} = -\frac{1}{2}i$

$$\therefore \quad \left|\left(\frac{2+i}{3-i}\right)^2\right| = \left|-\frac{1}{2}i\right| = \frac{1}{2}|-i| = \frac{1}{2}$$

and $\arg\left(\frac{2+i}{3-i}\right)^2 = \arg\left(-\frac{1}{2}i\right) = \frac{1}{2}\arg(-i) = \frac{1}{2}\left(-\frac{\pi}{2}\right) = -\frac{\pi}{4}$

Problem 2. *Prove that* $\arg(z) - \arg(-z) = \pm\pi$ *according as* $\arg(z)$ *is positive or negative.*

when $\arg(z)$ is positive, let $\arg(z) = \theta, -\pi < \theta < \pi$

Then $\arg(-z) = -(\pi - \theta)$ if $0 < \theta < \pi$.

$= \pi + \theta$ if $0 > \theta > -\pi$

$\therefore$ $\arg(z) - \arg(-z) = \theta + (\pi - \theta)$ or $\theta - (\pi + \theta)$ according as $\arg(z)$ is positive or negative.

$= \pi$ or $-\pi$ according as $\arg(z)$ is positive or negative.

Problem 3. *Prove that* $\arg z + \arg \bar{z} = 2n\pi$, *n being an integer including zero.*

If $z = x + iy$ then $\bar{z} = x - iy$

So $\arg z + \arg \bar{z} = \arg(z\bar{z})$

$= \arg(x+iy)(x-iy)$

$= \arg(x^2+y^2)$

$= \arg m$, where $m = x^2 + y^2$

Obviously m is real and positive.

If $m = r\cos\theta$, $0 = r\sin\theta$ then $m = r$ and $\cos\theta = 1$, $\sin\theta = 0$ so that $\theta = 2n\pi$, where n is an integer including zero.

Hence $\arg z + \arg \bar{z} = 2n\pi$, n being an integer including zero.

Problem 4. *Show that* $|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2|z_1|^2 + 2|z_2|^2$ *and deduce that* $\left|\alpha + \sqrt{\alpha^2 - \beta^2}\right| + \left|\alpha - \sqrt{\alpha^2 - \beta^2}\right| = |\alpha + \beta| + |\alpha - \beta|$, *all the numbers concerned being complex.*

We have,

$$|z_1+z_2|^2+|z_1-z_2|^2=(z_1+z_2)(\bar{z}_1+\bar{z}_2)+(z_1-z_2)(\bar{z}_1-\bar{z}_2) \quad \because |z|^2=z\bar{z}$$

$$=2z_1\bar{z}_1+2z_2\bar{z}_2$$

$$=2|z_1|^2+2|z_2|^2 \quad \ldots(1)$$

Now $\left\{\left|\alpha+\sqrt{\alpha^2-\beta^2}\right|+\left|\alpha-\sqrt{\alpha^2-\beta^2}\right|\right\}^2$

$$=\left|\alpha+\sqrt{\alpha^2-\beta^2}\right|^2+\left|\alpha-\sqrt{\alpha^2-\beta^2}\right|^2+2\left|\alpha+\sqrt{\alpha^2-\beta^2}\right|\left|\alpha-\sqrt{\alpha^2-\beta^2}\right|.$$

$$=2|\alpha|^2+2\left|\sqrt{\alpha^2-\beta^2}\right|^2+2\left|\alpha^2-(\alpha^2-\beta^2)\right| \quad \text{using (1)}$$

$$=2|\alpha|^2+2|\alpha^2-\beta^2|+2|\beta^2|$$

$$=2\{|\alpha|^2+|\beta|^2\}+2|\alpha^2-\beta^2|$$

$$=|\alpha+\beta|^2+|\alpha-\beta|^2+2|\alpha+\beta|\,|\alpha-\beta|$$

$$=\{|\alpha+\beta|+|\alpha-\beta|\}^2$$

$$\therefore \left|\alpha+\sqrt{\alpha^2-\beta^2}\right|+\left|\alpha-\sqrt{\alpha^2-\beta^2}\right|=|\alpha+\beta|+|\alpha-\beta|$$

Problem 5. *Show that an expression of the form $z^{2n}-1$ can be expressed as a product of n real quadratic factors.*

Assuming $z^{2n}-1=0$ *i.e.* $z^{2n}=1=\cos 2m\pi \pm i\sin 2m\pi$, $m=0,1,2,\ldots n$

We have $z=\cos\dfrac{2m\pi}{2n}+i\sin\dfrac{2m\pi}{2n}$ applying De Moivre's result

Here $m=0 \xrightarrow{\text{corresponds}} z=1$ and $m=n \to z=-1$

Consider $z^{2n}-1=(z^2-1)\times(2n-2)$ factors obtained by putting

$m=1,2,\ldots n-1$ in the expression $z-\left(\cos\dfrac{2m\pi}{2n}\pm i\sin\dfrac{2m\pi}{2n}\right)$

Now $\left(z-\cos\dfrac{2m\pi}{2n}-i\sin\dfrac{2m\pi}{2n}\right)\left(z-\cos\dfrac{2m\pi}{2n}+i\sin\dfrac{2m\pi}{2n}\right)$

$$=\left(z-\cos\frac{2m\pi}{2n}\right)^2+\sin^2\frac{2m\pi}{2n}$$

$$=z^2-2z\cos\frac{2m\pi}{2n}+1$$

$$\therefore z^{2n}-1=(z-1)(z+1)\left(z^2-2z\cos\frac{2\pi}{2n}+1\right)\left(z^2-2z\cos\frac{4\pi}{2n}+1\right)\ldots$$

$$\ldots\ldots\left(z^2-2z\cos\frac{2n-4}{2n}\pi+1\right)\left(z^2-2z\cos\frac{2n-2}{2n}\pi+1\right)$$

$$=(z^2-1)\left(z^2-2z\cos\frac{\pi}{n}+1\right)\left(z^2-2z\cos\frac{2\pi}{n}+1\right)\ldots\ldots\ldots$$

$$\ldots\ldots\left(z^2-2z\cos\frac{n-2}{n}\pi+1\right)\left(z^2-2z\cos\frac{n-1}{n}\pi+1\right)$$

$= (z^2 - 1) \prod_{m=1}^{n-1} \left\{ z^2 - 2z \cos \frac{m\pi}{n} + 1 \right\},$ $\prod$ denotes product of similar factors.

5.6. REGULAR FUNCTIONS

Before going into the details of Regular or Analytic functions, we first define some terms which are used frequently.

Neighbourhood of a point. Neighbourhood of a point z_0 in the Argand diagram means the set of all points z such that $|z - z_0| < \epsilon$, where ϵ is an arbitrarily chosen small positive number.

Limit point. A point z_0 is said to be a *limit point* of a set of points S in the Argand plane, if every neighbourhoood of z_0 contains a point of S other than zero.

The limit points of a set may not necessary be the points of the set. There are two types of limit points :

(*i*) **Interior points.** A limit point z_0 of the set S is said to be the *interior* or *inner point* if in the neighbourhood of z_0 there exists entirely the points of the set S.

(*ii*) **Boundary points.** A limit points z_0 is said to be the *boundary point* if all the points in the neighbourhood of z_0 do not belong to the set S.

Closed set. If all the limit points of the set belong to the set, then the set is said to be a *closed set.*

Open set. A set which consists entirely of interior points is known to be an *open set.*

Bounded and unbounded sets. A set of points is said to be *bounded* if there exists a constant number k, such that $|z| \leq k$ for all points z of the set. If there does not exist such number k the set is said to be *unbounded.*

Domain. If every pair of the points of a set of points in the Argand diagram can be joined by a polygonal arc which consists only of the points of the set, then the set of points in the Argand diagram is said to be *connex* (means connected) or *domain* or *region.*

Open domain is an open connex set of points.

Closed domain. When the boundary points of the set are also added to an open domain, it is then called a *closed domain.*

Functions of a complex variable. If $w = u + iv$ and $z = x + iy$ are two complex numbers, then w is said to be the *function* of z and written as $w = f(z)$, if to every value of z in a certain domain D, there correspond one or more values of w. If w takes only one value for each value of z in the domain D, then w is said to be *uniform* or *single-valued* function of z and if it takes more than one values for some or all values of z in the domain D, then w is known as a *many-valued* or a *multiple-valued* function of z.

Since u and v both are functions of x, y

$\therefore \quad w = f(z) = u(x, y) + iv(x, y).$

It is however notable that the path of a complex variable z is either a straight line or a curve.

Continuity. The function $f(z)$ of a complex variable z is continuous at the point z_0 if, given a positive number $\epsilon > 0$, a number δ can be so found that

$$|f(z) - f(z_0)| < \epsilon,$$

for all points z of the domain D satisfying $|z - z_0| < \delta$, where δ depends upon ϵ and also, in general, upon z_0, *i.e.*

$$\delta = \phi(\epsilon, z_0).$$

If δ is independent of z_0 or rather say that if a number $h(\epsilon)$ can be found independent of z_0 such that $|f(z) - f(z_0)| < \epsilon$ holds for every pair of points z, z_0 of the domain D for which $|z - z_0| < h$, then $f(z)$ is called *uniformly continuous* in D.

It should be noted that if a function f is continuous at $z = z_0$ i.e. if $f = u + iv$ is continuous at $z = z_0$ then it will be so iff its real and imaginary parts are separately continuous functions of x and y at the point $(x, y) = (x_0, y_0)$

Since if f is continuous at $z = z_0$ then $u(x_0, y_0)$ and $v(x_0, y_0)$ both are uniquely defined such that

$$0 \le |u(x, y) - u(x_0, y_0)| \le |f(z) - f(z_0)| \quad \ldots(1)$$

for, $\quad |f(z) - f(z_0)| = +[\{u(x, y) - u(x_0, y_0)\}^2 + \{v(x, y) - v(x_0, y_0)\}^2]^{1/2}$

as $z \to z_0$, $u(x, y) \to u(x_0, y_0)$ *i.e.* $u(x_0, y_0) = \lim\limits_{(x, y) \to (x_0, y_0)} u(x, y) \quad \ldots(2)$

This limit exists independent of the manner in which $x \to x_0, y \to y_0$. (2) shows that $u(x, y)$ is continuous at $(x_0; y_0)$.

Similarly $v(x, y)$ is continuous at (x_0, y_0).

Thus continuity of u and v for f to be continuous at $z = z_0$, is a necessary condition.

Conversely if $u(x, y)$ and $v(x, y)$ are continuous, then

$$u(x, y) \to u(x_0, y_0) \text{ and } v(x, y) \to v(x_0, y_0) \text{ as } z \to z_0$$

so that $f(z) = u(x, y) + iv(x, y) \to u(x_0, y_0) + iv(x_0, y_0) = f(z_0)$

So the condition is also sufficient.

Differentiability. If $f(z)$ be a single-valued function defined in a domain D of the Argand diagram, then $f(z)$ is said to be *differentiable* at $z = z_0$ a point of D if $\dfrac{f(z) - f(z_0)}{z - z_0}$ tends to a unique limit when $z \to z_0$, provided that z is also a point of D.

A function $f(z)$ is said to be differentiable at a point z_0, if

$\lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ exists and is a finite quantity provided by whatever path $z \to z_0$; then $f(z)$ is differentiable at $z = z_0$. The finite limit when exists is denoted by $f'(z_0)$ and termed as the differential coefficient or derivative of $f(z)$ at $z = z_0$, *i.e.*

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Precisely if for a given $\epsilon > 0$, there exists a number δ such that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta$$

i.e. writing $z - z_0 = \Delta z$ if for an $\epsilon > 0$, there exists a number δ such that $\left| \dfrac{f(z_0 + \Delta z) - f'(z_0)}{\Delta z} - f'(z_0) \right| < \epsilon$ whenever $0 < |\Delta z| < \delta$ then $f'(z_0)$ is known as the derivative of $f(z)$ at z_0.

Clearly the limiting value of $\dfrac{f(z)-f(z_0)}{z-z_0}$ is independent of the path in D along which $z \to z_0$.

Consider $f(z) = z^2$, for example, then $f'(z_0) = 2z_0$ at any point z_0

since $f'(z_0) = \lim\limits_{\Delta z \to 0} \dfrac{(z_0+\Delta z)^2 - z_0^2}{\Delta z} = \lim\limits_{\Delta z \to 0} (2z_0 + \Delta z) = 2z_0$

In view of the relation $f'(z_0) = \lim\limits_{\Delta z \to 0} \dfrac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$ whenever $f'(z_0)$ exists at any point z_0, we have

$$\lim_{\Delta z \to 0} [f(z_0+\Delta z) - f(z_0)] = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} \lim_{\Delta z \to 0} \Delta z = 0$$

which follows $\lim\limits_{z \to z_0} f(z) = f(z_0)$

i.e. f is necessarily continuous at any point z_0 where its derivative exists. But the converse is not true i.e. if a function is continuous, it is not necessarily differentiable as is evident from the following example.

Consider the function $w = |z|^2$ which is differentiable at every point. It will be shown that its derivative exists only at the point $z = 0$ and nowhere else, since

$$\frac{\Delta w}{\Delta z} = \frac{|z_0+\Delta z|^2 - |z_0|^2}{\Delta z} = \frac{(z_0+\Delta z)(\bar{z}_0 + \overline{\Delta z}) - z_0\bar{z}_0}{\Delta z}$$

$$\because |z|^2 = z\bar{z}$$

$$= \bar{z}_0 + \overline{\Delta z} + z_0 \frac{\overline{\Delta z}}{\Delta z} \qquad ...(3)$$

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \frac{dw}{dz} = \lim_{\Delta z \to 0}\left[\bar{z}_0 + \overline{\Delta z} + z_0 \frac{\overline{\Delta z}}{\Delta z}\right].$$

$$= 0 \text{ when } z_0 = 0$$

But if $z_0 \neq 0$, then taking $\theta = \arg \Delta z = \arg (z - z_0)$ we have

$$\frac{\overline{\Delta z}}{\Delta z} = \frac{e^{-i\theta}}{e^{i\theta}} = e^{-2i\theta}$$

$$= \cos 2\theta - i \sin 2\theta.$$

So that $\lim\limits_{\Delta z \to 0} \dfrac{\Delta w}{\Delta z} = \lim\limits_{\Delta z \to 0} [\bar{z}_0 + \overline{\Delta z} + z_0 (\cos 2\theta - i \sin 2\theta)]$

Here the $\lim\limits_{\Delta z \to 0} \dfrac{\overline{\Delta z}}{\Delta z}$ does not exist as $\Delta z \to 0$ in any manner, since if Δz is real, $\Delta z = \Delta x$ *i.e.* $\overline{\Delta z} = \Delta x = \Delta z$ then limit of (3) is $\bar{z} + z_0$. Also if Δz is imaginary *i.e.* $\Delta z = i\,\Delta y$ so that $\overline{\Delta z} = -\Delta z$, then limit of (3) is $\bar{z}_0 - z_0$. As such the limit does not exist when $z_0 \neq 0$ and hence $|z|^2$ has no derivative at z_0.

Analytic (or regular or holomorphic or monogesic functions)

[*Agra, 1965*]

A function $f(z)$ which is single-valued and differentiable at every point of a domain D, is said to be *regular* in the domain D.

A function may be differentiable in a domain D save possibly for a finite number of points. Such points are called *singularities* or *singular points of* $f(z)$

The necessary and sufficient conditions for $f(z)$ to be regular.

Necessary conditions. (*Rohilkhand, 1981, 84*)

If $w = f(z)$, where $w = u + iv$ and $z = x + iy$.

As such u and v both are the functions of x and y and therefore we can write $w = f(z) = u(x, y) + iv(x, y)$.

Now if $f(z) = u(x, y) + iv(x, y)$ is differentiable at a given point z, the ratio $\dfrac{f(z+\Delta z) - f(z)}{\Delta z}$ must tend to a certain finite limit as $\Delta z \to 0$ in any manner.

From the relation $z = x + iy$, we get $\Delta z = \Delta x + i\Delta y$.

If we take Δz to be wholly real, so that $\Delta y = 0$, then

$$\lim_{\Delta x \to 0} \frac{\Delta w}{\Delta x} = \lim_{\Delta x \to 0}\left[\frac{u(x+\Delta x, y) - u(x, y)}{\Delta x} + i\frac{v(x+\Delta x, y) - v(x, y)}{\Delta x}\right]$$

must exist and tend to a definite limit.

$$\therefore \qquad \frac{dw}{dx} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= u_x + iv_x \text{ (say)},$$

i.e. the partial derivatives u_x, v_x must exist at the point (x, y) and the limiting value is $u_x + iv_x$.

Similarly again if Δz be taken wholly imaginary, so that $\Delta x = 0$, we find that the partial derivatives u_y, v_y must exist at the point (x, y) and the limiting value is $v_y - iu_y$.

Since the function is differentiable, the two limits so obtained must be identical, *i.e.* $u_x + iv_x = v_y - iu_y$.

Equating real and imaginary parts, we get

$$u_x = v_y \text{ and } u_y = -v_x$$

or
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

(*Agra, 1967, 69, 71, 73*)

These two relations, which are necessary conditions for a function to be analytic, are called the *Cauchy Riemann Differential Equations*.

Sufficient conditions. The continuous single-valued function $f(z)$ is regular in a domain C if the four partial derivatives u_x, u_y, v_x, v_y exist, are continuous and satisfy the Cauchy-Riemann equations at all points of the region D.

Assuming $u_x = v_y$ and $u_y = -v_x$ and these partial derivatives are continuous, we have to show that they exist and are finite.

By the mean value theorem, we have

$$\left.\begin{aligned} f(x+\Delta x, y) - f(x, y) &= \Delta x \frac{\Delta f(x+\Delta x\,\theta,\ y)}{\Delta x}, \text{ where } 0 < \theta < 1 \\ \text{and } f(x, y+\Delta y) - f(x, y) &= \Delta y \frac{\Delta f(x, y+\theta'\Delta y)}{\Delta y}, \text{ where } 0 < \theta' < 1. \end{aligned}\right\} \quad \ldots(i)$$

Now if $w = f(z)$ and $f(z) = u(x, y) + iv(x, y)$,

$$w + \Delta w = f(z + \Delta z).$$

and
$$f(z + \Delta z) = u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y),$$

$\therefore \qquad \Delta w = f(z + \Delta z) - f(z);$

Also if $z = x + iy$, then $\Delta z = \Delta x + i\Delta y$.

Thus

$$\frac{\Delta w}{\Delta z} = \frac{f(z+\Delta z) - f(z)}{\Delta x + i\Delta y}$$

$$= \frac{u(x+\Delta x,\, y+\Delta y) + iv(x+\Delta x,\, y+\Delta y) - u(x, y) - iv(x, y).}{\Delta x + i\Delta y}$$

Now $u(x+\Delta x,\, y+\Delta y) - u(x, y)$

$$= \{u(x+\Delta x,\, y+\Delta y) - u(x+\Delta x,\, y)\} + \{u(x+\Delta x,\, y) - u(x,y)\}$$

$$= \Delta y \frac{\Delta u(x+\Delta x,\, y+\theta\,\Delta y)}{\Delta y} + \Delta x \frac{\Delta u(x+\theta'\,\Delta x,\, y)}{\Delta x}$$

[by mean value theorem as stated in (i)]

$$= \Delta x\left[\frac{\Delta u(x, y)}{\Delta x} + \epsilon_1\right] + \Delta y\left[\frac{\Delta u(x, y)}{\Delta y} + \epsilon_2\right]$$

[$\because$ if the function is continuous

$|f(z) - f(z_0)| < \epsilon$; $\therefore f(z) = f(z_0) + \epsilon_1$

when $|z - z_0| < \delta$ where $|\epsilon_1| < \epsilon$].

$$= \Delta x\left[\frac{\partial u}{\partial x} + \epsilon_1\right] + \Delta y\left[\frac{\partial u}{\partial y} + \epsilon_2\right]$$

$$\left[\because \lim_{\Delta x \to 0} \frac{\Delta u(x, y)}{\Delta x} = \frac{\partial u}{\partial x} \text{ etc.}\right]$$

Hence

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta x + i\Delta y}$$

$$= \frac{\Delta x\left(\frac{\partial u}{\partial x} + \epsilon_1\right) + \Delta y\left(\frac{\partial u}{\partial y} + \epsilon_2\right) + i\Delta x\left(\frac{\partial v}{\partial x} + \epsilon_1'\right) + i\Delta y\left(\frac{\partial v}{\partial y} + \epsilon_2'\right)}{\Delta x + i\Delta y}$$

$$= \frac{\Delta x\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial x}\right) + \Delta y\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) + \Delta x(\epsilon_1 + i\epsilon_1') + \Delta y(\epsilon_2 + i\epsilon_2')}{\Delta x + i\Delta y}$$

$$= \frac{(\Delta x + i\Delta y)\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}(\Delta x + i\Delta y) + \eta\Delta x + \eta'\Delta y}{\Delta x + i\Delta y}$$

[By applying Cauchy-Riemann's equations and putting

$\eta = \epsilon_1 + i\epsilon_1'$, $\eta' = \epsilon_2 + i\epsilon_2'$; also $i^2 = -1$].

$$\therefore \frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + \eta\frac{\Delta x}{\Delta x + i\Delta y} + \eta'\frac{\Delta y}{\Delta x + i\Delta y}.$$

Now $\left|\eta \dfrac{\Delta x}{|\Delta x + i\Delta y|}\right| = \dfrac{|\eta|\,|\Delta x|}{|\Delta x + i\Delta y|}$

$$= \frac{|\eta|\,\Delta x}{\sqrt{\{(\Delta x)^2 + (\Delta y)^2\}}}$$

$$\leq |\eta| \qquad \because \Delta x < \sqrt{\{\{\Delta x\}^2 + \{\Delta y\}^2\}}$$

$$\leq \sqrt{(\epsilon_1^2 + \epsilon_1'^2)} \quad \because \eta = \epsilon_1 + i\epsilon_1'$$

≤ 0 when $\Delta x \to 0$; ϵ_1 and $\epsilon'_1 \to 0$.

But by definition the modulus of any quantity is always + ve or zero and it is never negative.

$$\therefore \qquad \frac{\eta\Delta x}{\Delta x + i\Delta y} \to 0, \text{ when } \Delta x \to 0.$$

Similarly
$$\frac{\eta'\Delta z}{\Delta x + i\Delta y} \to 0, \text{ when } \Delta y \to 0.$$

Hence
$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x},$$

i.e., the limit exists and is finite and unique.

Therefore the sufficient conditions for the function $f(z)$ to be regular require the continuity of the four first partial derivatives of u and v.

Polar form of Cauchy-Riemann Equations

The coordinates (x, y) in terms of polar coordinates (r, θ) are given by

$$x = r\cos\theta,\ y = r\sin\theta$$

So that

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta;\ \frac{\partial r}{\partial y} = \sin\theta;\ \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r} \text{ and } \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$$

$$\therefore \frac{\partial u}{dx} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}.$$

Similarly

$$\frac{\partial u}{dy} = \frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r};\ \frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\cos\theta - \frac{\partial v}{\partial \theta}\frac{\sin\theta}{r};$$

and $\dfrac{\partial v}{\partial y} = \dfrac{\partial v}{\partial r}\sin\theta + \dfrac{\partial v}{\partial \theta}\dfrac{\cos\theta}{r}.$

Substituting these values in Cauchy-Riemann equations *i.e.*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ we get}$$

$$\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r} = \frac{\partial v}{\partial r}\sin\theta + \frac{\partial v}{\partial \theta}\frac{\cos\theta}{r} \qquad \ldots(4)$$

and
$$\frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r} = -\frac{\partial v}{\partial r}\cos\theta + \frac{\partial v}{\partial \theta}\frac{\sin\theta}{} \qquad \ldots(5)$$

Multiplying (4) by cos θ, (5) by sin θ and adding we get $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$.

Again multiplying (4) by sin θ; (5) by cos θ and subtracting, we get $\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$.

Hence Polar forms of Cauchy-Riemann conditions are

$$\frac{\partial u}{\partial r} = \frac{i}{r}\frac{\partial v}{\partial \theta},\ \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \qquad \ldots (6)$$

The derivative of the function $w = f(z)$ in the polar form is given as

$$\frac{dw}{dz} = \frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$= \frac{\partial w}{\partial r}\cos\theta - \frac{\sin\theta}{r}\frac{\partial w}{\partial \theta}$$

$$= \frac{\partial w}{\partial r}\cos\theta - \frac{\sin\theta}{r}\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right) \quad \because w = u + iv$$

$$= \frac{\partial w}{\partial r}\cos\theta - \frac{\sin\theta}{r}\left(-r\frac{\partial v}{\partial r} + ir\frac{\partial u}{\partial r}\right) \text{ using (6)}$$

$$= \frac{\partial w}{\partial r}\cos\theta - \sin\theta\left(\frac{-\partial v}{\partial r} + i\frac{\partial u}{\partial r}\right)$$

$$= (\cos\theta - i\sin\theta)\frac{\partial w}{\partial r}$$

$$= -\frac{i}{r}(\cos\theta - i\sin\theta)\frac{\partial w}{\partial \theta}$$

Condition for a Function when it Ceases to be Analytic

If $w = F(\zeta)$ and $\zeta = f(z)$, then w is said to be *function of a function* of z and we have $\frac{dw}{dz} = \frac{dw}{d\zeta}\frac{d\zeta}{dz}$, $F(\zeta)$ and $f(z)$ both being analytic.

Also if $w = f(z)$ be an analytic function of z such that corresponding to each point w_o there exists a point w_0 and $z = F(w)$ is such that to each value w_0 of w there corresponds a value z_0 of z, then the function $z = F(w)$ is said to be the *Inverse function of* $w = f(z)$. Clearly, if $f'(z_0) \neq 0$, then w_0 is a regular point of $z = F(w)$ *i.e.* z is analytic in the neighbourhood of w_0.

On account of functions being inverse we have $F'(w_0) = \frac{1}{f'(z_0)}$ *and hence the function* $z = F(w)$ *ceases to be anålytic where* $f' z) = 0$ *i.e.* $\frac{dw}{dz} = 0$ *also that* $w = f(z)$ *where* $z = F(w)$ *ceases to be regular when*

$$\frac{dw}{dz} = 0.$$

Evidently when $z = x + iy$, $w = f(z)$, we have $\frac{dw}{dz} = \frac{\partial w}{\partial x}$.

So that if $w = u + iv$, $z = f(w)$, then $\frac{dz}{dw} = \frac{\partial z}{\partial u}$.

As an illustration if $w = e^{-v}(\cos u + i \sin u)$, then $\frac{dz}{dw} = \frac{\partial z}{\partial u}$ gives

$$\frac{dz}{dw} = e^{-v}\,(-\sin u + i \cos u) = -ie^{-v}(\cos u + i \sin u) = -iz$$

$\therefore$ w ceases to be analytic when $\frac{dz}{dw} = 0$ *i.e.* $z = 0$.

To prove that if a function is regular, it is independent of $\bar{z}$ and is function of z.

$$z = x + iy,$$
$$\bar{z} = x - iy,$$

Adding and subtracting, we get

$$x = \frac{1}{2}(z + \bar{z}) \text{ and } y = \frac{1}{2i}(z - \bar{z}).$$

$$\therefore \quad \frac{\partial x}{\partial \bar{z}} = \frac{1}{2} \text{ and } \frac{\partial y}{\partial \bar{z}} = -\frac{1}{2i} \text{ etc.}$$

Now $\quad w = f(z) = u + iv.$

$$\therefore \quad \frac{\partial x}{\partial \bar{z}} = \frac{du}{d\bar{z}} + i\frac{dv}{d\bar{z}}.$$

$$= \left[\frac{\partial u}{\partial x}.\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial \bar{z}}\right] + i\left[\frac{\partial v}{\partial x}.\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}.\frac{\partial y}{\partial \bar{z}}\right]$$

$$= \frac{\partial u}{\partial x}.\frac{1}{2} - \frac{1}{2i}\frac{\partial u}{\partial y} + i\left[\frac{\partial v}{\partial x}.\frac{1}{2} - \frac{1}{2i}\frac{\partial v}{\partial y}\right]$$

$= 0$ by Cauchy-Riemann's equations.

Thus if the function is regular, it is independent of $\bar{z}$, as its differential is zero.

Laplace's Equations

Cauchy-Riemann's equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x};$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x\, \partial y} \text{ and } \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x\, \partial y}$$

Adding,
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \because \frac{\partial^2 v}{\partial x \partial y}.\frac{\partial^2 v}{\partial y \partial x}$$

i.e.,
$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad ...(ii)$$

Similarly,
$$\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \qquad ...(iii)$$

These are known as Laplace's equations, in which both u and v satisfy Laplace's equation in two dimensions.

Harmonic Functions. A function of x, y is said to be a harmonic function if it possesses continuous partial derivatives of the first and second orders and satisfies Laplace's equation.

Two harmonic functions u and v as satisfying (*ii*) and (*iii*) are known as *Conjugate harmonic functions* or simply *conjugate functions.*

Determination of Conjugate Functions

If $f(z) = u + iv$ is an analytic function such that u and v are conjugate functions then being given one of them say u, we have to determine v.

We have

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy \quad \because \; v \text{ is a function of } x, y.$$

$$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \quad \text{by cauchy-Riemann equations.}$$

The R.H.S. of this equation being of the form $Mdx + Ndy$ will be exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{where } M = -\frac{\partial u}{\partial y} \text{ and } N = \frac{\partial u}{\partial x}$$

i.e. $$\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

or $$-\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2}$$

i.e. $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

As u satisfies Laplace's equation, it is harmonic and hence its conjugate v can be found out by integrating the equation.

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy.$$

As an illustration if $u = y^3 - 3x^2y$, then $\frac{\partial u}{\partial x} = -6xy$, $\frac{\partial u}{\partial y} = 3y^2 - 3x^2$

$\frac{\partial^2 u}{\partial x^2} = -6y$, $\frac{\partial^2 u}{\partial y^2} = 6y$ so that u satisfies Laplace's equation and hence is harmonic.

$$\text{Now } v = \frac{\partial v}{\partial x}dx + \frac{\partial v}{dy}dy$$

$$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \text{ by Cauchy – Riemann equations}$$

$$= -\left(3y^2 - 3x^2\right)dx - 6xy\,dy$$

$$= -(3y^2\,dx + 6xydy) + 3x^2dx$$

Integrating $v = -3xy^2 + x^3 + c$ which is harmonic conjugate to u. Corresponding analytic function

$$f(z) = u + iv$$
$$= y^3 - 3x^2y + i(-3xy^2 + x^3 + c)$$
$$= i(x + iy)^3 + ic$$
$$= iz^3 + c'$$

Using an alternative method, $\frac{\partial u}{\partial x} = -6xy = \frac{\partial v}{\partial y}$ by Cauchy-Riemann equations.

Integrating $\frac{\partial v}{\partial x} = -6xy$ we get $v = -3xy^2 + \phi(x)$, $\phi(x)$ being arbitrary.

But $\quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ gives

$$-3y^2 + \phi'(x) = -3y^2 + 3x^2$$

i.e. $\quad \phi'(x) = 3x^2$ and so $\phi(x) = x^3 + c$

As such $\quad v = -3xy^2 + x^3 + c$

Construction of a function f(z) when one conjugate is given (due to **Milne-Thomson**)

If $z = x + iy$, $\bar{z} = x - iy$ and so $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$

$$\therefore \quad f(z) = u(x, y) + iv(x, y)$$

$$= u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$$

Treating it as a formal identity in two independent variables z and $\bar{z}$ and putting $z = \bar{z}$, we get $x = z$, $y = 0$ so that,

$$f(z) = u(z, 0) + iv(z, 0)$$

Taking $f(z) = u + iv$ to be analytic, we have

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \text{ by Cauchy-Riemann equations.}$$

Writing $\phi(x, y) \equiv \frac{\partial u}{\partial x}$, $\psi(x, y) \equiv \frac{\partial u}{\partial y}$, we have

$$f'(z) = \phi(x, y) - i\psi(x, y)$$

$$= \phi(z, 0) - i\psi(z, 0)$$

Integrating, $f(z) = \int\{\phi(z, 0) - i\psi(z, 0)\}\, dz + c$, c being arbitrary constant.

Similarly if $v(x, y)$ is given then we can find

$$f(z) = \int\{\Phi(z, 0) + i\,\Psi(z, 0)\}\, dz + C$$

where $\quad \Phi(x, y) = \frac{\partial v}{\partial y}$ and $\Psi(x, y) = \frac{\partial v}{\partial x}$

As an illustration if $u = e^x(x \cos y - y \sin y)$, then

$$\frac{\partial u}{\partial x} = e^x(x \cos y - y \sin y + \cos y) = \phi(x, y) \text{ (say)}$$

and $\frac{\partial u}{\partial y} = e^x(-x \sin y - y \cos y - \sin y) = \psi(x, y)$ (say)

So that $\phi(z, 0) = e^z(z + 1)$ and $\psi(z, 0) = 0$

$\therefore f'(z) = \phi(z, 0) - i\,\psi(z, 0)$

$= e^z(z + 1)$

Intergrating $f(z) = ze^z + c$.

Problem 6. *Prove that the function $u + iv = f(z)$ where*

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} (z \neq 0),\; f(0) = 0$$

is continuous and that the Cauchy-Riemann equations are satisfied at the origin, yet $f'(0)$ does not exist.

Here $u + iv = f(z)$

$$= \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$$

Equating real and imaginary parts on either side, we get

$$u = \frac{x^3-y^3}{x^2+y^2} \text{ and } v = \frac{x^3+y^3}{x^2+y^2} \text{ when } z \neq 0.$$

Obviously both u and v are rational and finite for all values of $z \neq 0$. Thus u and v are continuous at all those points for which $z \neq 0$. Hence $f(z)$ is continuous when $z \neq 0$.

Given that $f(0) = 0$, therefore at the origin $u = 0$, $v = 0$. Hence u and v both are continuous at the origin. As such $f(z)$ is continuous at the origin.

Conversely $f(z)$ is continuous everywhere.

Now,

$$\left(\frac{\partial u}{\partial x}\right)_{\substack{at\ x=0\\ y=0}} = \frac{\partial u(x, 0)}{\partial x} \text{ at } x = 0$$

$$= \lim_{h\to 0} \frac{u(h, 0) - u(0, 0)}{h} \because u(h, 0) = \frac{h^3-0}{h^2+0} = h \text{ etc.}$$

$$= \lim_{h\to 0} \frac{h}{h} = 1.$$

$$\left(\frac{\partial u}{\partial y}\right)_{\substack{at\ x=0\\ y=0}} = \lim_{y\to 0} \frac{\partial u(0, y)}{\partial y} = \lim_{k\to 0} \frac{u(0, k) - u(0, 0)}{k}$$

$$= \lim_{k\to 0} \frac{-k}{k} = -1 \because u(0, k) = \frac{0-k^3}{0+k^2} = -k \text{ etc.}$$

$$\left(\frac{\partial v}{\partial x}\right)_{\substack{at\ x=0\\ y=0}} = \lim_{x\to 0} \frac{\partial v(x, 0)}{\partial x}$$

$$= \lim_{k\to 0} \frac{v(h, 0) - v(0, 0)}{k} = \lim_{h\to 0} \frac{h}{h} = 1$$

and $$\left(\frac{\partial v}{\partial y}\right)_{\substack{at\ x=0\\ y=0}} = \lim_{y\to 0} \frac{\partial v(0, y)}{\partial y}$$

$$= \lim_{k\to 0} \frac{v(0, k) - v(0, 0)}{k} = \lim_{k\to 0} \frac{k}{k} = 1.$$

Thus we have found that at the origin

$$\frac{\partial u}{\partial x} = 1 = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -1 = -\frac{\partial v}{\partial x},$$

which clearly satisfy the Cauchy-Riemann equations at $z = 0$. Again differential coefficient of $f(z)$ at $z = 0$, *i.e.*

$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z - 0}$$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3(1+i)-y^3(1-i)}{(x^2+y^2)\,(x+iy)} \qquad \because z = x+iy$$

$$= \lim_{x\to 0} \frac{x^3(1+i)-m^3(1-i)x^3}{(x^2+m^2x^2)\,(x+imx)} \qquad \text{by putting } y = mx$$

$$= \frac{(1+i)-m^3(1-i)}{(1+m^2)(1+im)}$$

which is not unique, as it is different for different values of m. Therefore $f(z)$ is not continuous at $z = 0$.

Hence $f'(z)$ does not exist at the origin, *i.e.*, $z = 0$.

Problem 7. *Show that the function $f(z) = \sqrt{(|xy|)}$ is not regular at the origin although the Cauchy-Riemann equations are satisfied at that point.*

Let the function be

$$f(z) = \sqrt{(|xy|)} = u(x, y) + iv(x, y).$$

Equating real and imaginary parts, we have

$$u(x, y) = \sqrt{(|xy|)} \text{ and } v(x, y) = 0.$$

Thus,
$$\left(\frac{\partial u}{\partial x}\right)_{\text{at } \substack{x=0\\ y=0}} = \frac{\partial u(x, 0)}{\partial x} \text{ at } x = 0$$

$$= \lim_{h\to 0} \frac{u(h, 0)-u(0, 0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

Similarly at $x = 0$, $y = 0$,

$$\frac{\partial v}{\partial x} = 0, \ \frac{\partial u}{\partial y} = 0, \ \frac{\partial u}{\partial y} = 0.$$

These values clearly satisfy the Cauchy-Riemann equations.

Again $f'(0)$
$$= \lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{\substack{x=0\\ y=0}} \frac{\sqrt{(|xy|)}-0}{x+iy}$$

$$= \lim_{x\to 0} \frac{x\sqrt{|m|}}{x(1+im)} \quad \text{by putting } y = mx$$

$$= \frac{\sqrt{|m|}}{1+im},$$

which is not unique as its values are different for different values of m. So $f(z)$ is not continuous at $z = 0$. Hence it is not regular there.

Problem 8. *Prove that the function :*

$$u = x^3-3xy^2+3x^2-3y^2+1$$

satisfies Laplace's equation and determine the corresponding regular function $u + iv$

Here,
$$\frac{\partial u}{\partial x} = 3x^2-3y^2+6x, \qquad \ldots(1)$$

$$\frac{\partial^2 u}{\partial x^2} = 6x+6, \qquad \ldots(2)$$

$$\frac{\partial u}{\partial y} = -6xy-6y, \qquad ...(3)$$

$$\frac{\partial^2 u}{\partial y^2} = -6x-6. \qquad ...(4)$$

By the addition of (2) and (4) it follows that $\nabla^2 u = 0$, which clearly satisfies the Laplace's equation.

Hence u is a Harmonic function.

Now Cauchy-Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$$\therefore\ dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \quad \text{(by Cauchy-Riemann equations)}$$

$$= (6xy+6y)\,dx + (3x^2-3y^2+6x)\,dy$$

$$= (6xy\,dx + 3x^2\,dy) - 3y^2\,dy + (6y\,dx + 6x\,dx).$$

Integrating,

$v = 3x^2y - y^3 + 6xy + c$, where c is an absolute or complex constant.

Thus

$$\begin{aligned} u + iv &= x^3 - 3xy^2 + 3x^2 - 3y^2 + 1 + i[3x^2y - y^3 + 6xy + c] \\ &= (x+iy)^3 + 3(x+iy)^2 + 1 + ic \\ &= z^3 + 3z^2 + c'. \end{aligned}$$

Problem 9. *Prove that the curves u = constant,*

v = constant

intersect at right angles.

We know that the curves intersect at right angles if the tangents to them at their point of intersection are at right angles.

Differentiating partially the given equations of the curves, we get

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0,$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = 0,$$

Therefore tangents of the angles formed by the tangents drawn to the curves at their point of intersection with the real axes are respectively

$$\left(\frac{\partial y}{\partial x}\right)_1 = -\frac{\partial u}{\partial x}\Big/\frac{\partial u}{\partial y} \text{ and } \left(\frac{\partial y}{\partial x}\right)_2 = -\frac{\partial v}{\partial x}\Big/\frac{\partial v}{\partial y}.$$

Now $\left(\frac{\partial y}{\partial x}\right)_1\left(\frac{\partial y}{\partial x}\right)_2 = \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}\Big/\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y} = -1$, by Cauchy-Riemann equations

which shows that the two tangents are at right angles if the function is regular. It follows that the two curves intersect at right angles.

Problem 10. *Show that* (*a*) $\nabla^2 |u|^p = p(p-1)\ |u|^{p-2} |f'(z)|^2$.

(*b*) $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)|f(z)|^2 = 4|f'(z)|^2$ (*Rohilkhand, 1984, 87*)

(*a*) Here if u be positive $|u| = u$ and if u be negative $|-u| = u$.

Taking first u to be positive,

$$\frac{\partial}{\partial x} u^p = p.u^{p-1}\frac{\partial u}{\partial x}.$$

$$\frac{\partial^2}{\partial x^2} u^p = p(p-1)u^{p-2}\left(\frac{\partial u}{\partial x}\right)^2 + p.u^{p-1}\frac{\partial^2 u}{\partial x^2}.$$

Similarly, $\frac{\partial^2}{\partial y^2} u^p = p(p-1)u^{p-2}\left(\frac{\partial u}{\partial y}\right)^2 + p.u^{p-1}\frac{\partial^2 u}{\partial y^2}.$

Adding the last two results,

$$\nabla^2 u^p = pu^{p-1}\nabla^2 u + p(p-1)\,u^{p-2}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right],$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$

$$= p(p-1)\,u^{p-2}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]. \quad ...(1)$$

$\because \nabla^2 u = 0$ (Laplace's equation).

Now if $w = f(z) = u+iv$, then

$$f'(z) = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}.$$

$$\therefore \quad |f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2 \quad ...(2)$$

Thus from (1) and (2) it follows that

$$\nabla^2 u^p = p(p-1)\,u^{p-2}\,|f'(z)|^2.$$

Again if u be negative say u_1,

$$\therefore \quad |-u| = u_1.$$

As before, we have

$$\nabla^2 u_1^p = p(p-1)\,u_1^{p-2} = |f'(z)|^2$$

or

$$\nabla^2 |u_1|^p = p(p-1)\,|u_1|^{p-2}|f'(z)|^2.$$

(*b*) Taking the analytic function $f(z) = u+iv$, we have

$$|f(z)|^2 = u^2+v^2 \text{ and } f'(z) = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

Now, $\frac{\partial}{\partial x}u^2 = 2u\frac{\partial u}{\partial x} \Rightarrow \frac{\partial^2}{\partial x^2}u^2 = 2\left(\frac{\partial u}{\partial x}\right)^2 + 2u\frac{\partial^2 u}{\partial x^2}$

and similarly $\frac{\partial^2}{\partial y^2}u^2 = 2\left(\frac{\partial u}{\partial y}\right)^2 + 2u\frac{\partial^2 u}{\partial y^2}$

$$\therefore \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u^2 = 2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2 + 2u\left(\frac{\partial^2 u}{\partial x^2}\right)+\left(\frac{\partial^2 u}{\partial y^2}\right)\right]$$

$$= 2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right] \quad \because \nabla^2 u = 0,\ u \text{ being harmonic}$$

$$= 2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(-\frac{\partial v}{\partial x}\right)^2\right] \quad \text{by Cauchy-Riemann equations}$$

$$= 2\ |f'(z)|^2$$

Similarly, $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)v^2 = 2|f'(z)|^2$

Thus, $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(u^2+v^2) = 4\ |f'(z)|^2$

or $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$

5.7. ELEMENTARY FUNCTIONS AND MAPPING BY THEM

Exponential functions. If $z = x + iy$ and y is used as radian measure of the angle to define cos y, sin y etc, then the exponential function in terms of real valued functions is defined by

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y) \quad \text{...(1)}$$

In case z is purely real *i.e.* $y = 0$, we have $e^z = e^x$ ...(2)

and if z is purely imaginary *i.e.* $x = 0$, we have

$$e^{iy} = \cos y + i \sin y \quad \text{...(3)}$$

As such Maclaurin series representation of e^t on replacing t by iy, gives

$$\sum_{n=0}^{\infty} \frac{(iy)^n}{\lfloor n} = \sum_{n=0}^{\infty} \frac{i^{2n}y^{2n}}{\lfloor 2n} + \sum_{n=0}^{\infty} \frac{i^{2n+1}y^{2n+1}}{\lfloor 2n+1} \text{ where } \lfloor 0 = 1$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n}}{\lfloor 2n} + i\sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{\lfloor 2n+1} \quad \text{...(4)}$$

which are Maclaurin series for cos y and sin y respectively.

The Exponential function given by (1) is an *entire function* since

$$\frac{d}{dz} e^z = e^z \quad \text{...(5)}$$

Similarly $\frac{d}{dz} e^w = e^w \frac{dw}{dz}$ ...(6)

w being an analytic function of z.

Polar form of (1) is $e^z = r(\cos\theta + i \sin\theta) = re^{i\theta}$ ...(7)

where $r = e^x$, $\theta = y$.

$\therefore\ |e^z| = r = e^x$ and $\arg e^z = \theta = y$ ...(8)

Also $|e^z| > 0$ *i.e.* $e^z \neq 0$ for every value of z ...(9)

So the range of the exponential function is the entire complex plane excluding the origin where $r = 0$.

Now $e^z = -1 = \cos(\pi \pm 2m\pi) + i \sin(\pi \pm 2m\pi)$ gives

$x = 0$ and $y = \pi \pm 2m\pi,\ m = 0, 1, 2, \ldots$

If $e^{z_1} = r_1(\cos\theta_1 + i\sin\theta_1)$ so that $r_1 = e^{x_1},\ \theta_1 = y_1$

and $e^{z_2} = r_2(\cos\theta_2 + i\sin\theta_2)$ so that $r_2 = e^{x_2},\ \theta_2 = y_2$

then,

$$e^{z_1}.e^{z_2} = r_1 r_2 \{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}$$

$$= e^{x_1} e^{x_2}[\cos(y_1+y_2) + i\sin(y_1+y_2)]$$

$$= e^{x_1+x_2}.e^{i(y_1+y_2)} = e^{x_1+x_2+i(y_1+y_2)}$$

$$= e^{(x_1+iy_1)+(x_2+iy_2)} = e^{z_1+z_2} \quad \ldots(10)$$

Similarly $e^{z_1}/e^{z_2} = e^{z_1-z_2}$ ...(11)

$$\frac{1}{e^z} = e^{-z} \quad \ldots(12)$$

$$(e^z)^n = e^{nz} \quad \ldots(13)$$

n being a positive integer.

$$(e^z)^{m/n} = e^{m/n}(z + 2\pi pi) \quad \ldots(14)$$

$p = 0, 1, 2, \ldots, x-1$ and m, n are integers (+ *ve*).

Also $e^{z+2\pi i} = e^z e^{2\pi i} = e^z$ as $e^{2\pi i} = 1$ ...(15)

(15) follows that the exponential function is *periodic.*

Again $e^{\bar{z}} = \overline{(e^z)}$ ...(16)

In polar form $z = re^{i\theta}$, $\bar{z} = re^{-i\theta}$ so that

$$z_1 z_2 = r_1 r_2\, e^{i(\theta_1+\theta_2)},\quad \frac{z_1}{z_2} = \frac{r_1}{r_2}\, e^{i(\theta_1-\theta_2)} \quad \ldots(17)$$

Trigonometric Functions

$e^{iy} = \cos y + i\sin y$ and $e^{-iy} = \cos y - i\sin y$ yield

$$\cos y = \frac{e^{iy}+e^{-iy}}{2},\quad \sin y = \frac{e^{iy}-e^{-iy}}{2i} \quad \ldots(18)$$

$\sin z$ and $\cos z$ *are entire functions* as

$$\frac{d}{dz}\sin z = \cos z,\quad \frac{d}{dz}\cos z = -\sin z \text{ etc.} \quad \ldots(19)$$

Now,

$$\cos z = \cos(x+iy) = \frac{1}{2}[e^{i(x+iy)} - e^{-i(x+iy)}]$$

$$= \frac{1}{2}[e^{ix-y} + e^{-ix+y}].$$

$$= \frac{1}{2}e^{-y}(\cos x + i\sin x) + \frac{1}{2}e^{y}(\cos x - i\sin x)$$

$$= \frac{e^{y}+e^{-y}}{2}\cos x - i\,\frac{e^{y}-e^{-y}}{2}\sin x \quad \ldots(20)$$

Introducing the *hyperbolic functions* with the properties

$$\sinh z = \frac{e^z - e^{-z}}{2},\ \cosh z = \frac{e^z + e^{-z}}{2} \quad ...(21)$$

$$\frac{d}{dz}\sinh z = \cosh z,\ \frac{d}{dz}\cosh z = \sinh z \text{ etc.} \quad ...(22)$$

$$\cosh^2 z - \sinh^2 z = 1 \quad ...(23)$$

$$\cosh iz = \cosh z \text{ and } \sin iz = i \sinh z \quad ...(24)$$

the relation (20) becomes

$$\cos z = \cos(x + iy) = \cos x \cosh y - i \sin x \sinh y \quad ...(25)$$

Similarly $\sin z = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$ ...(26)

Also $\overline{\sin z} = \sin \bar{z}$ and $\overline{\cos z} = \cos \bar{z}$ ...(27)

$$\cos(z + \pi) = -\cos z,\ \sin(z + \pi) = -\sin z, \text{ etc.} \quad ...(28)$$

It is easy to show that

$$|\sin z|^2 = \sin^2 x + \sinh^2 y \quad ...(29)$$

$$|\cos z|^2 = \cos^2 x + \sinh^2 y \quad ...(30)$$

A value of z for which $f(z) = 0$ is known as a *zero* of the function f.

The real zeros of sin z and cos z are their only zeros, since

$\sin z = 0$ from (26) gives

$\sin x \cosh y = 0,\ \cos x \sinh y = 0$

x and y being real, $\cosh y \geq 1$ and $\sin x = 0$ only when $x = 0, \pm\pi, \pm 2\pi, \ldots$ and for these values of x, $\cos x \neq 0$ and thus $\sinh y = 0$ *i.e.* $y = 0$.

Also

$$\left.\begin{aligned} &\sin z = 0 \Rightarrow z = 0 \text{ or } \pm n\pi,\ n = 1, 2, 3, \ldots \\ &\cos z = 0 \Rightarrow z = \pm\frac{(2n-1)\pi}{2},\ n = 1, 2, 3, \ldots \end{aligned}\right\} \quad ...(31)$$

We may also show that

$$\cosh z = \cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y \quad ...(32)$$

$$\sinh z = \sinh x \cos y + i \cosh x \sin y \quad ...(33)$$

So that $|\sinh z|^2 = \sinh^2 x + \sin^2 y$ (34)

$$|\cosh z|^2 = \sinh^2 x + \cos^2 y \quad ...(35)$$

Sinh z and cosh z are periodic with *period* $2\pi i$.

Logarithmic Functions and Branch Points

$z = re^{i\theta}$, θ being measured in radian, gives

$$\left.\begin{aligned} \log z = \log re^{i\theta} &= \log r + i\theta \\ &= \log|z| + i \arg \theta \end{aligned}\right\} r > 0 \text{ and } -\pi < \theta < \pi \quad ...(36)$$

If $-\pi < \theta \leq \pi$, then $z = re^{i(\theta \pm 2n\pi)}$, $n = 0, 1, 2, \ldots$ ...(37)

So that $\log z = \log r + i(\theta + i2n\pi)$, $n = 0, 1, 2, \ldots$

We write log z for *principal value* of log z and Log z for its *general value.*

In (36) if we put $\log r = u$, $\theta = v$ so that

$$\frac{\partial u}{\partial r} = \frac{1}{r},\ \frac{\partial u}{\partial \theta} = 0,\ \frac{\partial v}{\partial r} = 0,\ \frac{\partial v}{\partial \theta} = 1, \text{ are all continuous functions of } z.$$

Also, $\frac{d}{dz}\log z = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$

$$= \frac{1}{re^{i\theta}} = \frac{1}{z} \qquad ...(38)$$

where $z \neq 0$ and $-\pi < \arg z < \pi$.

A *branch* F of a many-valued function f is any single-valued function which is analytic in some domain at each point of which the value $F(z)$ is one of the values $f(z)$. The equation (36) gives the *principal branch* log z. Each point of the negative real axis $\theta = \pi$ along with the origin is a singular point of the principal branch log z. Then (say) $\theta = \pi$ is said to be a *branch cut* for the principal branch and the singular point $z = 0$ common to all branch cuts for the many-valued function log z is known as a *branch point*.

Now if $w = \log z$, then $e^w = e^{\log z} = e^{(\log r + i\theta)}$

$$= e^{\log r} e^{i\theta} = re^{i\theta} = z$$

i.e. $e^{\log z} = z, z \neq 0$ ...(39)

and if $e^z = w$, $\log w = \log e^{x+iy} = \log e^x + \log e^{iy}$.

$$= x + i(y \pm 2p\pi), p = 0, 1, 2, 3, ...$$
$$= x + iy \pm 2p\pi i$$
$$= z \pm 2p\pi i$$

So $\log w = z$ when $e^z = w$ ...(40)

and $\log e^z = z$ for appropriate choice of logarithm ...(41)

Again if $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$, $r_1 > 0$, $r_2 > 0$, then

$$\log z_1 + \log z_2 = \log r_1 e^{i\theta_1} + \log r_2 e^{i\theta_2}$$
$$= \log r_1 + \log r_2 + i(\theta_1 + \theta_2)$$
$$= \log r_1 r_2 + i(\theta_1 + \theta_2)$$
$$= \log z_1 z_2 \qquad ...(42)$$

Similarly $\log z_1 - \log z_2 = \log \frac{z_1}{z_2}$ ...(43)

It is easy to verify that

$$\log z^m = m \log z \qquad ...(44)$$

$$\log z^{1/n} = \frac{1}{n} \log z \qquad ...(45)$$

$$z^{m/n} = e^{m/n \log z} \qquad ...(46)$$

In case of *complex exponents*, we define

$z^c = e^{c \log z}$, z, c being complex and $z \neq 0$ ...(47)

$$\frac{d}{dz} z^c = e^{c \log z} \cdot \frac{c}{z} = c \cdot \frac{e^{c \log z}}{e^{\log z}} = ce^{(c-1)\log z} = cz^{c-1} \qquad ...(48)$$

Also $c^z = e^{z \log c}, c \neq 0$, ...(49)

$$\frac{d}{dz} c^z = c^z \log c, \; c \neq 0 \qquad ...(50)$$

Inverse Trigonometric Functions

Defining the inverse of sine function as $w = \sin^{-1} z$, we have

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i} \quad \text{i.e. } e^{2iw} - 2ize^{iw} - 1 = 0.$$

Being quadratic in e^{iw}, this gives, $e^{iw} = iz + \sqrt{(1-z^2)}$

So that $w = \sin^{-1} z = -i \log \left\{ iz + \sqrt{1-z^2} \right\}$ ...(51)

Similarly $\cos^{-1} z = -i \log \left\{ z + \sqrt{z^2 - 1} \right\}$ ...(52)

$$\tan^{-1} z = \frac{i}{2} \log \frac{1-iz}{1+iz} = \frac{i}{2} \log \frac{1+z}{1-z} \quad ...(53)$$

So that $\dfrac{d}{dz} \sin^{-1} z = \dfrac{1}{\sqrt{1-z^2}}, \ \dfrac{d}{dz} \tan^{-1} z = \dfrac{1}{1+z^2}$ ...(54)

Also

$$\left.\begin{aligned} \sinh^{-1} z &= \log \left\{ z + \sqrt{z^2+1} \right\} \\ \cosh^{-1} z &= \log \left\{ z + \sqrt{z^2-1} \right\} \\ \tan^{-1} z &= \frac{1}{2} \log \frac{1+z}{1-z} \end{aligned}\right\} \quad ...(55)$$

Mapping. If $w = f(z)$ and corresponding to each point (x, y) in z-plane in a domain of function f, there is a point (u, v) in w-plane where $z = x + iy$ and $w = u + iv$, then this correspondence between the points of two planes is said to be a *mapping* or a *transformation* of points in the z-plane into points of the w-plane by the function f. Corresponding points or set of points are known as *images* of each other. The use of graphic terms as *translation, rotation* or *reflection* is rather convenient in mapping, *e.g.*

The mapping $w = z + c$, c being a complex constant gives the translation of every point z through the vector c *i.e.* if $z = x + iy$, $w = u + iv$, $c = c_1 + ic_2$, then the image of any point (x, y) in z-plane is the point $(x + c_1, y + c_2)$

The mapping $w = Bz$ where $B = be^{i\beta}$ and $z = re^{i\theta}$

i.e. $\qquad w = br\, e^{i(\theta + \beta)}$

maps the point (r, θ) in z-plane into a point $(br, \theta + \beta)$ into w-plane *i.e.* the mapping consists of a rotation of the radius vector of z about the origin through an angle $\beta = \arg B$ and an extension or contraction of radius vector r by $b = |B|$.

As an illustration the function $w = z^2$ maps the entire first quadrant of the z-plane, $0 \le \theta \le \pi/2$, $r \ge 0$, into the entire upper half of the w-plane.

The transformation

$$T : w = \frac{az + b}{cz + d}, \quad ...(56)$$

a, b, c, d being complex constants is termed as the *linear fractional transformation* or *bilinear transformation* or *Mobius transformation.*

Here T^{-1} *i.e.* inverse of T is $z = \dfrac{-dw + b}{cw - a}$ ...(57)

Any set of elements which satisfies all the following conditions is called a *group* :

(*i*) There is a rule of combination such that product TT' for each distinct pair T, T' of elements is an element of the set.

(*ii*) The product is associative *i.e.* $T(T'T'') = (TT')T''$

(*iii*) The set contains an identity T_0 such that $TT_0 = T_0T = T$ for each element T.

(*iv*) Each element T has an inverse T^{-1} s.t. $TT^{-1} = T^{-1}T = T_0$.

It may be shown that the set of all linear fractional transformations is a group.

If besides the above four properties a group also satisfies the commutative property then it is called as *Commutative group* or an *Abelian group.*

Problem 11. *Show that the set of complex numbers form an Abelian group under addition.*

Take three complex numbers, $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, $z_3 = (x_3, y_3)$ belonging to the set C of complex numbers. Then the addition is commutative since

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2) = (x_2 + x_1, y_2 + y_1)$$
$$= z_2 + z_1 \ \forall\ z_1, z_2 \in \mathrm{C}$$

Also $\quad z_1, z_2 \in \mathrm{C} \Rightarrow z_1 + z_2 \in \mathrm{C}$

The addition is associative since

$$\begin{aligned} z_1 + (z_2 + z_3) &= (x_1, y_1) + (x_2 + x_3, y_2 + y_3) \\ &= (x_1 + x_2 + x_3, y_1 + y_2 + y_3) \\ &= (x_1 + x_2, y_1 + y_2) + (x_3, y_3) \\ &= (z_1 + z_2) + z_3. \end{aligned}$$

There exists an additive identity $o = (o, o)$ such that

$$z + o = (x, y) + (o, o) = (x, y) = z$$

There exists an additive inverse $(-z)\ \forall\ z$ such that

$$z + (-z) = (x, y) + (-x, -y) = (o, o) \text{ the identity element.}$$

Hence the set of complex numbers form an abelian group.

Problem 12. *Show that the set of complex numbers form an abelian group under multiplication.*

It is easy to show that

$$(x_1, y_1), (x_2, y_2) \in \mathrm{C} \Rightarrow (x_1, y_1)(x_2, y_2) \text{ i.e. } (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1) \in \mathrm{C}$$
$$(x_1, y_1)\{(x_2, y_2)(x_3, y_3)\} = \{(x_1, y_1)(x_2, y_2)\}(x_3, y_3),$$
$$(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathrm{C}$$

$\exists$ multiplicative identity (1, 0) s.t. $(x, y)(1, 0) = (x, y)\ \forall\ (x, y) \in \mathrm{C}$

$\exists$ multiplicative inverse s.t. $(x, y)\left(\dfrac{x}{x^2+y^2}, \dfrac{-y}{x^2+y^2}\right) = (1, 0)$

The commutative law holds *i.e.*

$$(x_1, y_1) + (x_2, y_2) = (x_2, y_2) + (x_1, y_1).$$

Hence the given set is an abelian group.

Complex Differential Operators

If $z = x + iy$ so that $\bar{z} = x - iy$ and F be a continuous differential function, then

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x} + \frac{\partial F}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial x} = \frac{\partial F}{\partial z} + \frac{\partial F}{\partial \bar{z}} \text{ giving } \frac{\partial}{\partial x} \equiv \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \qquad \ldots(58)$$

and $$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial F}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial y} = i\frac{\partial F}{\partial z} - i\frac{\partial F}{\partial \bar{z}} \text{ giving } \frac{\partial}{\partial y} \equiv i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right) \qquad \ldots(59)$$

We then define ∇ (Del operator) $\equiv \dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y} = \dfrac{\partial}{\partial z} + \dfrac{\partial}{\partial \bar{z}} + i\left(\dfrac{\partial}{\partial z} - \dfrac{\partial}{\partial \bar{z}}\right) = 2\dfrac{\partial}{\partial \bar{z}}$

...(60)

and $\overline{\nabla}$ (Del bar) $\equiv \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} - i^2\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right) = 2\frac{\partial}{\partial z}$

Now taking $F(x, y)$ as real continuously differentiable function of scalars x, y and $A(x, y) = P(x, y) + i\,Q(x, y)$ as complex continuously differentiable function of vectors x and y, then

$$F(x, y) = F\left(\frac{z+\bar{z}}{2}, \frac{z+\bar{z}}{2i}\right) = G(z, \bar{z}) \text{ say and } A(x, y) = B(z, \bar{z}) \text{ say} \quad ...(62)$$

The *Gradient* of a real scalar function F is defined by

$$\text{grad } F = \nabla F = \frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y} = 2\frac{\partial F}{\partial \bar{z}} \quad ...(63)$$

and the gradient of a complex vector function A is defined as

$$\text{grad } A = \nabla A = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(P + iQ) = \frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} + i\left(\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right) = 2\frac{\partial B}{\partial \bar{z}} \quad ...(64)$$

Geometrically interpreted, ∇F is a vector normal to the curve $F(x, y)$ = constant, and if B is analytic function of z so that $\frac{\partial B}{\partial \bar{z}} = 0$, then gradient is also zero *i.e.* $\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}$ showing that Cauchy-Riemann equations are satisfied.

The *Divergence* of a complex vector function A is defined as

$$\text{div } A = \nabla \cdot A = Re\{\overline{\nabla} \cdot A\} = Re\left\{\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(P + iQ)\right\} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

$$= 2Re\left\{\frac{\partial B}{\partial z}\right\} \text{ where } Re \text{ denotes real part.} \quad ...(65)$$

It is notable that the divergence of a real or complex function is always a real function.

The *Curl* of a complex vector function A is defined as

$$\text{curl } A = \nabla \times A = Im(\overline{\nabla} \times A) = Im\left\{\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \times (P + iQ)\right\}$$

$$= i\left\{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\} = 2\,Im\left\{\frac{\partial B}{\partial z}\right\} \quad ...(66)$$

The *Laplacian operator* is defined as the scalar or dot product of ∇ with itself

i.e. $\nabla \cdot \nabla = \nabla^2 = Re(\overline{\nabla}\,\nabla) = R\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\,\partial \bar{z}} \quad ...(67)$$

Here below we summarize few identities involving grad, div and curl of two complex differentiable functions A_1 and A_2.

grad $(A_1 + A_2)$ = grad A_1 + grad A_2 ...(68)

div $(A_1 + A_2)$ = div A_1 + div A_2 ...(69)

curl $(A_1 + A_2)$ = curl A_1 + curl A_2 ...(70)

$$\text{grad}\,(A_1 A_2) = A_1\,(\text{grad}\,A_2) + (\text{grad}\,A_1)\,A_2 \quad ...(71)$$
$$\text{curl}\,(\text{grad}\,A) = 0, \text{ when } A \text{ is real or } Im\,(A) \text{ is harmonic} \quad ...(72)$$
$$\text{div}\,(\text{grad}\,A) = 0, \text{ when } A \text{ is imaginary or } Re\,(A) \text{ is harmonic} \quad ...(73)$$

5.8. COMPLEX INTEGRATION

It has been defined that a function of z is said to be *regular* in any domain D if the function is single-valued and differentiable at every point of the domain D. Now in order to show that a regular function possesses a second derivative, we must first of all express the function $f(z)$ of z as a contour integral round any closed contour surrounding the point z.

Let us suppose that the equations $x = \phi(t)$ and $y = \psi(t)$, where $\alpha \le t \le \beta$ represent the arc of the plane curve. Let us divide the interval (α, β) by the points $t_0, t_1, t_2, \ldots t_n$ and let these points on the curve be denoted by $P_0, P_1, P_2, \ldots P_n$. Then length of the polygonal line $P_0 P_1 P_2 \ldots P_n$ will be the sum of the lengths of the lines. $P_0 P_1$, $P_1 P_2, \ldots P_{n-1} P_n$. If $z_0, z_1, z_2, \ldots z_n$ be the points on the arc corresponding to the values t_0, $t_1, t_2, \ldots t_n$ of t, then length of the polygonal arc $P_0 P_1 P_2 \ldots P_n$

$$= \sum_{r=1}^{n} [(x_r - x_{r-1})^2 + (y_r - y_{r-1})^2].$$

The value of this sum depending upon the mode of subdivision is called the *length of an inscribed polygon*. If the arc is such that the length of all the inscribed polygons have a finite upper bound λ, the curve is known to be **Rectifiable** and λ is called the **length** of the curve.

The necessary and sufficient conditions for the arc to be rectifiable are that the functions $\phi(t)$, $\psi(t)$ must be of bounded variation in the interval (α, β). In case $\phi'(t)\ \psi'(t)$ are continuous, the curve denoted by $x = \phi(t)$, $y = \psi(t)$, when $\alpha \le t \le \beta$ is rectifiable and its length is

$$s = \int_{\alpha}^{\beta} \sqrt{\left[\{\phi'(t)\}^2 + \{\psi'(t)\}^2\right]}\, dt.$$

Riemann's definition of Integration

Defining the integration as the limit of a sum, Riemann's definition of Integration of a complex function is given as below:

Consider a function $w = f(z)$ which is continuous (but not necessarily analytic) along a curve C with end points A and B. Divide the arc C into n arcs by the points $z_0 = \alpha, z_1, z_2 \ldots, z_{r-1}, z_r, \ldots, z_{n-1}, z_n = \beta$ with z_0 being at A and z_n at B. Take points $\zeta_1, \zeta_2, \ldots, \zeta_r, \ldots \zeta_n$ such that ζ_r lies on the arc $z_{r-1} z_r$.

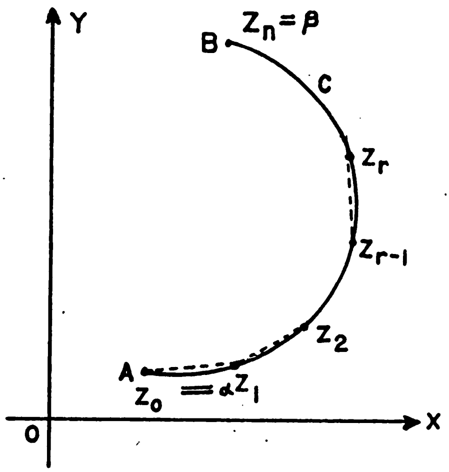


Fig. 5.13

Consider the sum $f(\zeta_1)(z_1 - z_0)$ $+ f(\zeta_2)(z_2 - z_1) + \ldots + f(\zeta_r)(z_r - z_{r-1}) + \ldots$ $f(\zeta_n)(z_n - z_{n-1})$

i.e. $$\sum_{r=1}^{n} f(\zeta_r)(z_r - z_{r-1})$$

If this sum tends to a unique limit when $n \to \infty$ *i.e.* the number of arcs of the curve C becomes indefinitely large then the definite integral of $f(z)$ along the curve C is given by

$$\int_C f(z)\, dz = \int_{AB} f(z)\, dz = \lim_{n\to\infty} \sum_{r=1}^{n} f(\zeta_r)\,(z_r - z_{r-1})$$

Note 1. If $f(z) = u + iv$, $z = x + iy = \phi(t) + i\psi(t)$ *so that* $dz = dx + idy$
$= \{\phi'(t) + i\psi'(t)\}\, dt,\ \alpha \le t \le \beta.$

We have $\int_C f(z)\, dz = \int_\alpha^\beta (u + iv)\,\{\phi'(t) + \psi'(t)\}\, dt$

$$= \int_\alpha^\beta (u + iv)\,(dx + idy) = \int_C (udx - vdy) + i(vdx + udy)$$

Note 2. If C consists of a number of arcs C_r, $\int_C f(z)\, dz = \Sigma \int_{C_r} f(z) dz.$

Continuous arc. When $\phi(t)$ and $\psi(t)$ are real continuous functions of the real variable t in the interval (α, β), the arc is known as *continuous arc.*

Multiple points. If the relations $x = \phi(t)$ and $y = \psi(t)$ are satisfied by two or more values of t in the given integral, then the point $z = (x, y)$ is called a *Multiple point.*

Jordan arc. A continuous arc having no multiple points is called a *Jordan arc.*

Regular arc of a Jordan Curve. Considering an arc of a Jordan Curve defined by the equation $z = \phi(t) + i\psi(t)$, where $\alpha \le t \le \beta$, if z be expressed as single-valued and $\phi(t)$, $\psi(t)$ as well as $\phi'(t)$, $\psi'(t)$ are continuous in the interval $\alpha \le t \le \beta$, the arc is then called *Regular arc of a Jordan Curve.*

Length of the regular Jordan arc is

$$\int_\alpha^\beta \sqrt{\left[\{\phi'(t)\}^2 + \{\psi'(t)\}^2\right]}\, dt.$$

A *Continuous Jordan Curve* is one which consists of a chain of finite number of continuous arcs.

Contour. By the word 'contour' we mean a continuous Jordan curve consisting of a chain of finite number of regular arcs. It is clear that a contour is rectifiable.

The complex integral of $f(z)$ along the regular arc L is writeen as

$$\int_L f(z) dz.$$

The integral of $f(z)$ along a contour C, which consists of a finite number of regular arcs L_r is given by

$$\int_C f(z)\, dz = \sum_r \int_{L_r} f(z)\, dz$$

Here $\int_C f(z)\, dz$ is read as integral $f(z)$ taken over the closed contour C.

Some properties in case of complex integrals are however notable:

1. $\int_C \{f(z) + \phi(z)\}\, dz = \int_C f(z)\, dz + \int_C \phi(z)\, dz.$

2. $\int_C f(z)\, dz = -\int_{-C} f(z) dz$, where $-C$ represents the direction opposite to that of C.

3. $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$, where C_1 and C_2 are two parts of C.

4. $\int_C kf(z)\,dz = k\int_C f(z)\,dz$, k being constant.

An upper bound for a complex integral. *If a function of z say f(z) is continuous on a contour L of length l and if the inequality* $|f(z)| \le M$ *is satisfied, then*

$$\left|\int_L f(z)\,dz\right| \le Ml.$$

Let the equation to the curve L be

$$x = \phi(t),\ y = \psi(t)$$

Then the length l of the curve is given by

$$l = \int\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] dt, \text{ between proper limits} \quad \ldots(1)$$

Now $\qquad z = x + iy$ gives $dz = dx + idy$

$$\therefore \qquad |dz| = |dx + idy| = \sqrt{(dx)^2 + (dy)^2}$$

So that $\int |dz| = \int \sqrt{(dx)^2 + (dy)^2}$

$$= \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\cdot dt$$

$$= l \text{ by (1)} \quad \ldots(2)$$

Now the modulus of the sum of n complex numbers being less than or equal to the sum of their moduli, we have

$$\left|\sum_{r=1}^{n} f(\zeta_r)(z_r - z_{r-1})\right| \le \sum_{r=1}^{n} |f(\zeta_r)(z_r - z_{r-1})|$$

$$\le \sum_{r=1}^{n} |f(\zeta_r)|\,|z_r - z_{r-1}|$$

Proceeding to the limit as $n \to \infty$, we have

$$\left|\int_L f(z)\,dz\right| \le \int_L |f(z)|\,|dz|$$

$$\le M\int_L |dz| \qquad \because |f(z)| \le M$$

$$\le Ml \text{ by (2)}$$

5.9. CAUCHY'S THEOREM

If f(z) is a regular function of z and if f′(z) is continuous at each point within and on a closed contour C, then $\int_C f(z)\,dz = 0$.

(Rohilkhand, 1989; Agra, 1966, 67)

(*i.e.* the integral of the function round a closed contour is zero).

Elementary Proof. Green's theorem states that if $P(x, y)$, $Q(x, y)$, $\frac{\partial Q}{\partial x}, \frac{\partial P}{\partial y}$ are all continuous functions of x and y in the domain D, then

$$\int_C (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \quad \ldots(i)$$

Let us now assume that $f(z) = u + iv$, where $z = x + iy$.

$\therefore \qquad dz = x + i\,dy.$

Substituting these values in $\int_C f(z)dz$, we get

$$\int_C f(z).dz = \int_C (u+iv)\,(dx + i\,dy)$$

$$= \int_C (udx - vdy) + i\int (vdx + udy)$$

$$= \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy + i\iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dx\,dy$$

[by result (*i*) of Green's theorem]

$$= 0 \because \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ and } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

(Cauchy-Riemann equations)

Rigorous Proof of Cauchy's Theorem

If $f(z)$ is analytic (regular) at all points within and on the closed contour, C then $\int_C f(z) = 0.$

To prove this theorem let us first consider two lemmas.

Lemma 1. If C is a closed contour, then we must have

$$\int_C dz = 0 \text{ and also } \int_C z\,dz = 0.$$

It follows from the definition of integral that

$$\int_C f(z)dz = \lim_{n\to\infty} \sum_{r=1}^{n} (z_r - z_{r-1}) f(z).$$

Taking $f(z)=1$, we have

$$\int_C dz = \lim_{n\to\infty} \sum_{r=1}^{n} \{(z_r - z_{r-1})1\}$$

$$= 0 \text{ as max. } (z_r - z_{r-1}) \to 0, \text{ when } n\to\infty.$$

Also $$\int_C z\,dz = \lim_{n\to\infty} \sum_{r=1}^{n} \{(z_r - z_{r-1})\,z\}$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \{z_{r-1}\,(z_r - z_{r-1})\} = \lim_{n\to\infty} \sum_{r=1}^{n} \{z_r (z_r - z_{r-1})\}$$

$$= \frac{1}{2}\left[\lim_{n\to\infty} \sum_{r=1}^{n} \{z_r\,(z_r - z_{r-1})\} + \lim_{n\to\infty} \sum_{r=1}^{n} \{z_{r-1}\,(z_r - z_{r-1})\}\right]$$

$$= \frac{1}{2}\lim_{n\to\infty} \sum \{(z_r + z_{r-1})\,(z_r - z_{r-1})\}$$

$$= \frac{1}{2} \lim_{n \to \infty} \sum \left(z_r^2 - z_{r-1}^2\right)$$

$= 0$ (for a closed curve).

Lemma II. (Goursat's Lemma).

Given ϵ, it is possible by suitable transversals, to divive the interior of C into a finite number of meshes, either complete squares or parts of squares, such that within each mesh there is a point z_0, such that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon,$$

i.e. $\qquad f(z) - f(z_0) = f'(z_0)(z - z_0) + \eta(z - z_0) \qquad \ldots(1)$

for all values of z in the mesh where $\eta < \epsilon$.

[*Note. Unless the coutour is a square, the sum of the meshes will not be a perfect square*].

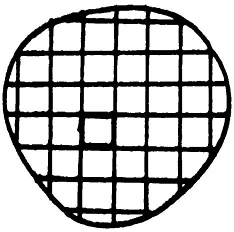


Fig. 5.14

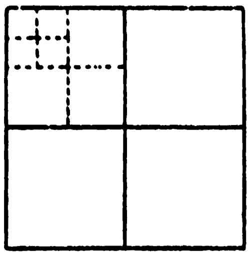


Fig. 5.15

Let us suppose that the lemma is false and however the interior of C be sub-divided. Then there will be at least one mesh for which (1) is not true. We have to show that this necessarily implies the existence of a point within or on C at which $f(z)$ is not differentiable.

Suppose that we enclose C in a large square Γ, of aera A and apply the process of repeated quadrisection. When Γ is quadrisected there is at least one of the four quarters of the square Γ for which (1) is untrue. Let it denoted by Γ_1. We quadrisect Γ_1 and take its quarter say Γ_2 for which (1) does not hold. This process is carried on indefinitely. Let the infinite sequence of squares so obtained be $\Gamma_1, \Gamma_2, \Gamma_3, \ldots \Gamma_n \ldots$ each contained in the preceding, for which the lemma is not true. Let this sequence of squares determine a limiting point ζ which clearly lies within C.

Now $f(z)$ being analytic everywhere and so at $z = \zeta$, it is differentiable at ζ and therefore, we have

$$\left| \frac{f(z) - f(\zeta)}{z - \zeta} - f'(\zeta) \right| < \epsilon,$$

i.e. $f(z) - f(\zeta) = f'(\zeta)(z - \zeta) + \eta'(z - \zeta)$, where $|z - \zeta| < \delta$ $\alpha\nu\delta$ $\eta' < \epsilon'$.

With centre ζ let us draw a circle of radius $\delta_1 < \delta$ and let

$$|z - \zeta| \le \delta_1 < \delta,$$

which contradicts our hypothesis, for, by taking ζ to be z_0, (1) is satisfied and thus it follows Goursat's lemma.

Proof of the theorem. It is obvious that some of the meshes obtained by the subdivision of the interior of C will be squares and others will not be squares. Let C_1, C_2,...C_m...be the complete squares and D_1, D_2,...D_n,...be the partial squares, then

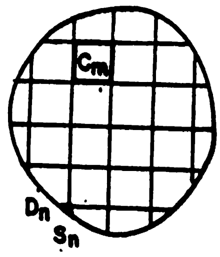


Fig. 5.16

$$\int_C f(z)\,dz = \sum \int_{C_m} f(z)\,dz + \sum \int_{D_n} f(z)\,dz \qquad \text{...(2)}$$

Also we have from (1) of the Goursat's lemma

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \eta(z-z_0), \text{ where } |\eta| < \epsilon. \qquad \text{...(3)}$$

Now

$$\int_{C_n} f(z)\,dz = \int_{C_n} \left[f(z_0) + f'(z_0)(z-z_0) + \eta(z-z_0)\right] dz$$

$$= \left[f(z_0) - f'(z_0)z_0\right] \int_{C_n} dz + f'(z_0)\int_{C_n} z\,dz + \eta\int_{C_n} (z-z_0)\,dz$$

$$= \eta\int_{C_n} (z-z_0)\,dz \text{ as } \int_{C_n} dz = \int_{C_n} z\,dz = 0 \text{ from lemma 1.}$$

$$\therefore \left|\int_{C_n} f(z)\,dz\right| \le \eta \int_{C_n} |z-z_0|\,dz$$

$\le \epsilon \int_{C_n} \sqrt{2}\,l_n \,|dz|$, side of square being l_n; and

Max $|z-z_0| = \sqrt{2}\,l_n$, where $\sqrt{2}\,l_n$ is the length of the diagonal of square.

$\le \epsilon\sqrt{2}\,l_n \int_{C_n} ds$, s being the entire perimeter of the square

$\le \epsilon\sqrt{2}\,l_n . 4l_n$

$\le 4\sqrt{2}\,\epsilon \cdot l_n^2$

$\le 4\sqrt{2}\,\epsilon \cdot A_n$, $l_n^2 = A_n$, the area of the square.

Similarly,

$$\left|\int_{D_n} f(z)\,dz\right| \le \epsilon\sqrt{2}\,l_n' \int_{D_n} ds$$

$\le \epsilon\sqrt{2}\,l_n'(4l_n + s_n)$ s_n being length of arc forming the curved boundary of D_n

$\le \epsilon\, 4\sqrt{2}\,A_n' + 2\epsilon\, s_n l_n'$, A_n' is the area of square D_n of side l_n'.

Hence (2) gives

$$\left|\int_C f(z)\,dz\right| \le \epsilon\sqrt{2}\sum_{n=1}^{\infty}\left\{4(A_n + A_n') + s_n l_n'\right\}$$

$\le \epsilon\sqrt{2}\,(4A + SL)$, S being perimeter of contour C, L is the length of a side of some square enclosing C and A the total area,

≤ 0 as $\epsilon \to 0$, S and L both being finite, so that $\int_C f(z)\,dz = 0.$

This proves the theorem.

Extension of Cauchy's Theorem

For this purpose, we define some theorems which have not yet been introduced.

Connected region. A region is known as a connected region if any two points of the region D can be connected by a curve lying wholly within the region.

Simply-connected region. A connected region is known as a simply-connected region if all the interior points of a closed curve C described in the region D, are also the points of D.

Multiply-connected region. A connected region is known as a multiply-connected region if all the points enclosed by two or more closed curves described in a region D are also the points of D.

Cross cut or simply cut. The lines drawn in a multiply-connected region, without intersecting any curve, such that the multiply-connected region is converted to a simply-connected region, are said to be cross cuts or cuts.

If the function $f(z)$ is not analytic in the whole region enclosed by a closed contour but it is analytic in the region enclosed between two closed contours then also Cauchy's integral theorem can be applied.

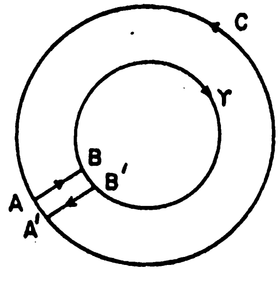


Fig. 5.17

Let the nearly equal and parallel lines AB and $A'B'$ as shown in Fig. 5.17, be used as cross-cuts by connecting the points A and B (very near to each other) on outer contour C with points A' and B' on inner contour γ. Let the simply connected contour so obtained be denoted by Γ. The function $f(z)$ being analytic in this region, the Cauchy's integral formula can be applied for this contour *i.e.*

$$\int_\Gamma f(z)\,dz = 0 \text{ which gives here.}$$

$$\int_\Gamma f(z)\,dz = \int_C f(z)\,dz + \int_{AB} f(z)dz + \int_{A'B'} f(z)dz + \int_\gamma f(z)dz = 0$$

$$= \int_C f(z)dz + \int_\gamma f(z) = 0,$$ other two integrals being equal and opposite in sense, cancel each other.

i.e. $$\int_C f(z)dz = -\int_{-\gamma} f(z)dz$$

where minus sign shows that the integral is traversed in clockwise direction.

∴ Taking the integral along γ in anticlokwise direction, we get

$$\int_C f(z)dz = \int_\gamma f(z)dz.$$

Note. In general if C be a closed curve and $C_1, C_2, C_3, \ldots, C_n$ be the other n closed curves lying inside C and $f(z)$ is analytic within these curves, then $\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \ldots + \int_{C_n} f(z)dz$ integrals being taken in anti-clockwise direction.

Moreras' Theorem as Converse of Cauchy's Theorem

If the integral $\int f(z)dz$ of a function $f(z)$ which is continuous in a region D, is zero when taken round any simple closed contour in D, then $f(z)$ is analytic in D.

Taking z_0 as a fixed point and z any variable point in D, the value of the integral.

$$\int_{z_0}^{z} f(\zeta)\, d\zeta = F(z) \text{ say} \qquad ...(4)$$

is independent of the curve joining z_0 to z and is dependent of z only:

(4) may be interpreted as:

$$F(z+h) = \int_{z_0}^{z+h} f(\zeta)\, d\zeta \qquad ...(5)$$

So that (4) and (5) give

$$F(z+h) - F(z) = \int_{z_0}^{z+h} f(\zeta)\, d\zeta - \int_{z_0}^{z} f(\zeta)\, d\zeta$$

$$= \int_{z}^{z+h} f(\zeta)\, d\zeta$$

The integration on right being independent of the curve joining z to $z + h$, may be taken along the straight line joining z to $z + h$, so that

$$\frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h}\int_{z}^{z+h} f(\zeta)\, d\zeta - \frac{1}{h} f(z) \int_{z}^{z+h} d\zeta$$

$$= \frac{1}{h}\int_{z}^{z+h} [f(\zeta) - f(z)]\, d\zeta$$

$$\therefore \left| \frac{F(z+h) - F(z)}{h} - f(z) \right| = \left| \frac{1}{h}\int_{z}^{z+h} [f(\zeta) - f(z)]\, d\zeta \right|$$

$$= \frac{1}{|h|}\int_{z}^{z+h} |f(\zeta) - f(z)|\, |d\zeta|$$

$$\le \frac{1}{|h|} \in \int_{z}^{z+h} |d\zeta| \text{ since } f(z) \text{ being continuous in } D,$$

$$|f(\zeta) - f(z)| < \in \text{ for } |\zeta - z| < \delta$$

$$\le \frac{1}{|h|} \in |h| = \in \to 0 \text{ as } h \to 0$$

$$\therefore \lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = F'(z) = f(z)$$

showing that $F(z)$ exists for all z in D i.e., $F(z)$ is analytic in D. As such $F'(z)$ and so $f(z)$ is also analytic in D, for, the derivative of an analytic function is also analytic.

5.10. CAUCHY'S INTEGRAL FORMULA

(Agra, 1974; Rohilkhand, 1986, 80)

If the function $f(z)$ is regular within and on a closed contour C and if ζ be a point within C, then

$$f(\zeta) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z - \zeta}.$$

Let us describe about the point $z = \zeta$ a small circle γ of radius δ lying entirely within C. Now consider a function

$$\phi(z) = \frac{f(z)}{z-\zeta}$$

which is regular in the region between C and γ.

By making a cross-cut joining any point of γ to any point of C by two almost equal and parallel lines, let us form a closed contour $LMCM'L'\gamma L = \Gamma$ (say) within which the function $\phi(z)$ is regular, so that by Cauchy's theorem, we get.

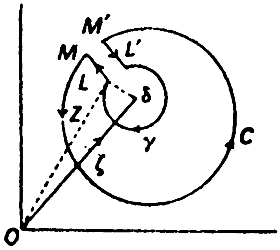


Fig. 5.18

$$\int_\Gamma \phi(z)\, dz = 0, \text{ where } \phi(z) = \frac{f(z)}{z-\zeta}.$$

The function is analytic within and on the boundary of the contour and as points M, M' are very near,

$LM = L'M'$ and $LM \,||\, L'M'$ approximately.

It follows that if the contour Γ is described in anti-clockwise direction (*i.e.*, positive sense), the cross-cut is traversed twice, once in each sense, and hence we have

$$\int_C \phi(z)dz - \int_\gamma \phi(z)dz = 0$$

or $$\int_C \phi(z)dz = \int_\gamma \phi(z)dz,$$

i.e., $$\int_C \frac{f(z)}{z-\zeta}dz = \int_\zeta \frac{f(z)}{z-\zeta}dz. \quad ...(1)$$

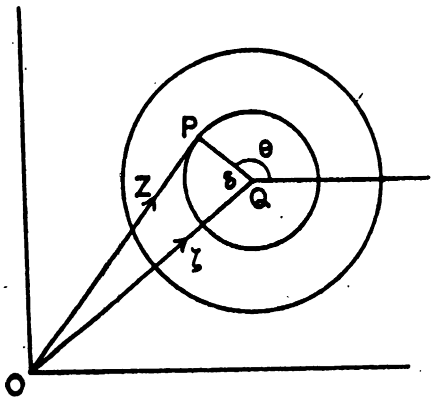


Fig. 5.19

Referring to the adjoining figure,

$$\vec{QP} = \vec{QO} + \vec{OP}$$
$$= -\zeta + z = z - \zeta.$$

$\therefore$ Complex co-ordinate of QP is

$$z - \zeta = \delta e^{i\theta}.$$

$[\because$ if $z = x + iy$,

$\therefore\ z = r(\cos\theta + i\sin\theta)$

$= re^{i\theta}]$,

where θ is the argument and δ is the magnitude of QP which is very small.

Differentiating (2) partially, we get

$$dz = i\delta e^{i\theta}\, d\theta. \quad ...(2)$$

Now (1) may be written as

$$\int_C \frac{f(z)}{z-\zeta}\, dz = \int_\gamma \frac{f(\zeta)}{z-\zeta} dz + \int_\gamma \frac{f(z)-f(\zeta)}{z-\zeta}\, dz$$

$$= f(\zeta)\int_0^{2\pi} \frac{i\delta\, e^{i\theta} d\theta}{\delta e^{i\theta}} + I \quad \text{(say)}$$

$$= 2\pi i f(\zeta) + I,$$

where $$|I| = \left| \int_\gamma \frac{f(z) - f(\zeta)}{z-\zeta} dz \right|$$

$$\leq \int_\gamma \left| \frac{f(z) - f(\zeta)}{z-\zeta} dz \right|.$$

But from the definition of continuity, the function $f(z)$ is continuous at $z = \zeta$, when

$|f(z) - f(\zeta)| < \delta_1$ if $|z - \zeta| < \delta$.

Also if $$z = x + iy,$$
$$dz = dx + i\,dy.$$

$\therefore$ $$|dz| = \sqrt{[(dx)^2 + (dy)^2]} = ds.$$

So, $|I| \leq \frac{\delta_1}{\delta} \int_\gamma ds$, where $\int_\gamma ds$ means the entire circumference of the circle γ

$$\leq \frac{\delta_1}{\delta} \times 2\pi\delta$$

$$\leq 2\pi\delta_1$$

$$\leq 0 \text{ as } \delta_1 \to 0 \text{ which is so when } \delta \to 0.$$

But we know that modulus of any quantity cannot be negative; therefore $|I| = 0$ as $\delta_1 = 0$, *i.e.*, when z coincides with ζ.

Hence $$\int_C \frac{f(z)}{z-\zeta} dz = 2\pi i f(\zeta),$$

i.e. $$f(\zeta) = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{z-\zeta}.$$ *(Agra, 1961)*

5.11. SOME RESULTS BASED ON CAUCHY'S INTEGRAL FORMULA

1. *If (z) is regular in a domain D, then its derivatives at any point $z = \zeta$ of the region D is also regular in that domain and is given by*

$$f'(\zeta) = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z-\zeta)^2}.$$

where C is any single closed contour in D surrounding the point $z = \zeta$.

(Rohilkhand, 1985)

Now, $$f'(\zeta) = \lim_{h\to 0} \frac{f(\zeta + h) - f(\zeta)}{h}$$

$$= \lim_{h\to 0} \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{h} \left\{ \frac{1}{z-\zeta-h} - \frac{1}{z-\zeta} \right\}$$

applying Cauchy's integral formula.

$$= \lim_{h\to 0} \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z-\zeta)(z-\zeta-h)}$$

$$= \frac{1}{2\pi i} \lim_{h\to 0} \int_C \frac{f(z)\,dz}{(z-\zeta)(z-\zeta-h)} = \frac{1}{2\pi i} \lim_{h\to 0} \int_C \frac{(z-\zeta-h+h) f(z)\,dz}{(z-\zeta)^2 (z-\zeta-h)}$$

$$= \frac{1}{2\pi i} \lim_{h\to 0} \int_C \left\{ \frac{1}{(z-\zeta)^2} + \frac{h}{(z-\zeta)^2\,(z-\zeta-h)} \right\} f(z)\,dz$$

$$= \frac{1}{2\pi i} \int \frac{f(z)\,dz}{(z-\zeta)^2} + \frac{1}{2\pi i} \lim_{h\to 0} \int_C \frac{hf(z)\,dz}{(z-\zeta)^2(z-\zeta-h)}$$

$$= \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z-\zeta)^2} + I \text{ (say)},$$

where $$I = \frac{1}{2\pi i} \lim_{h\to 0} \int_C \frac{hf(z)\,dz}{(z-\zeta)^2\,(z-\zeta-h)}.$$

$$\therefore \quad |I| \le \lim_{h\to 0} \frac{|h|}{2\pi} \int_C \frac{|f(z)|\,|dz|}{|(z-\zeta)|^2\,|(z-\zeta-h)|}$$

$$\le \lim_{h\to 0} \frac{|h|}{2\pi} \cdot \frac{Ml}{d^2(d-h)}$$

since $f(z)$ is regular in and on C, it is bounded, so that $|f(z)| \le M$ on C. Let d be the lower bound of the distance of ζ from C, *i.e.* $|z-\zeta| > d$ and l be the length of C.

≤ 0 as $h \to 0$

i.e. $I = 0$ as $a \to 0$

Hence $$f'(\zeta) = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z-\zeta)^2}. \qquad \ldots(1)$$

Similarly the second order derivative of $f(z)$ at $z = \zeta$ of domain D may be found as

$$f''(\zeta) = \lim_{h\to 0} \frac{f'(\zeta+h) - f'(\zeta)}{h}$$

Applying Cauchy's integral formula, in the form (1)

$$f''(\zeta) = \frac{1}{2\pi i} \lim_{h\to 0} \frac{1}{h} \left\{ \frac{1}{(z-\zeta-h)^2} - \frac{1}{(z-\zeta)^2} \right\} f(z)\,dz$$

$$= \frac{1}{2\pi i} 2 \int_C \frac{f(z)}{(z-\zeta)^3} dz + \lim_{h\to 0} h \int_C R(z)\, f(z)\,dz$$

Since $$\frac{d}{dz}\left[\frac{1}{h} \left\{ \frac{1}{(z-\zeta-h)^2} - \frac{1}{(z-\zeta)^2} \right\} \right]$$

$$= \frac{2}{(z-\zeta)^3} + h\,R(z)$$ where $R(z)$ is bounded on C so that $\int_C f(z)\,R(z)$ is finite.

$$= \frac{2}{2\pi i} \int_C \frac{f(z)}{(z-\zeta)^3}\,dz$$

$$= \frac{\underline{|2}}{2\pi i} \int_C \frac{f(z)}{(z-\zeta)^3}\,dz \quad ,$$

Proceeding similarly we can show that

$$f'''(\zeta) = \frac{\underline{|3}}{2\pi i} \int_C \frac{f(z)}{(z-\zeta)^4}\,dz \text{ etc.}$$

Here below the general result follows:

2. *If a function $f(z)$ is regular in a domain D then $f(z)$ has at any point $z = \zeta$ of the domain D, derivatives of all orders, values being given by*

$$f^{(n)}(\zeta) = \frac{\lfloor n}{2\pi i} \int_C \frac{f(z)\,dz}{(z-\zeta)^{n+1}}.$$

Let us suppose that the theorem is valid for $n = m$ and then consider

$$f^{(m+1)}(\zeta) = \lim_{h \to 0} \frac{f^{(m)}(\zeta+h) - f^{(m)}(\zeta)}{h}$$

$$= \frac{(m+1)\,!}{2\pi i} \int_C \frac{f(z)\,dz}{(z-\zeta)^{m+2}} + I \text{ by the last result,}$$

where $$I = \frac{(m+1)\,!}{2\pi i} \lim_{h\to 0} \int_C \frac{h f(z)\,dz}{(z-\zeta)^{m+2}\,(z-\zeta-h)}.$$

It is easy to show as in the previous result that

$$|I| \to 0 \text{ as } |h| \to 0.$$

$$\therefore\ f^{(m+1)}(\zeta) = \frac{(m+1)\,!}{2\pi i} \int_C \frac{f(z)\,dz}{(z-\zeta)^{m+2}},$$

which shows that the result is true for $n = m + 1$ and hence we have in general

$$f^{(n)}(\zeta) = \frac{\lfloor n}{2\pi i} \int_C \frac{f(z)\,dz}{(z-\zeta)^{n+1}}.$$

5.12. TAYLOR'S EXPANSION

(Agra, 1966, 68)

If a function $f(z)$ is regular everywhere inside a circle C whose cnetre is the point $z = a$ and radius is ρ, such that $|z-a| \le \rho$ and if ζ is a point such that $|\zeta - a| = r < \rho$,

$$\text{then } f(\zeta) = \sum_{n=0}^{\infty} a_n\,(\zeta-a)^n,$$

$$\text{where } a_n = \frac{f^{(n)}(a)}{\lfloor n}$$

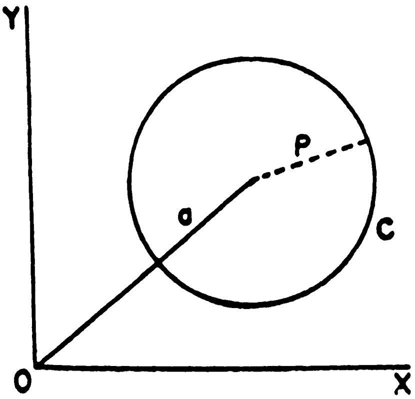


Fig. 5.20

We know that

$$\frac{1}{z-\zeta} = \frac{1}{z-a} + \frac{\zeta-a}{(z-a)(z-\zeta)}$$

$$= \frac{1}{z-a} + \frac{\zeta-a}{z-a}\left[\frac{1}{z-a} + \frac{\zeta-a}{(z-a)(z-\zeta)}\right]$$

$$= \frac{1}{z-a} + \frac{\zeta-a}{(z-a)^2} + \frac{(\zeta-a)^2}{(z-a)^2}\left[\frac{1}{z-a} + \frac{\zeta-a}{(z-a)(z-\zeta)}\right]$$

... ...

... ...

$$= \frac{1}{z-a} + \frac{\zeta-a}{(z-a)^2} + \frac{(\zeta-a)^2}{(z-a)^3} + \ldots + \frac{(\zeta-a)^n}{(z-a)^n\,(z-\zeta)}$$

Hence by the Cauchy's Integral formula, we get

$$f(\zeta) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-\zeta}\,dz$$

$$= \frac{1}{2\pi i}\int_C \left[\frac{1}{z-a} + \frac{\zeta - a}{(z-a)^2} + \ldots + \frac{(\zeta-a)^n}{(z-a)^n (z-\zeta)}\right] f(z)\, dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)\, dz}{z-a} + \frac{\zeta - a}{2\pi i}\int_C \frac{f(z)\, dz}{(z-a)^2} + \ldots + \frac{(\zeta-a)^n}{2\pi i}\int_C \frac{f(z)\, dz}{(z-a)^n (z-\zeta)}$$

$$= f(a) + (\zeta - a) f'(a) + \ldots + \frac{(\zeta-a)^{n-1} f^{n-1}(a)}{(n-1)\,!} + R_n.$$

where $R_n = \dfrac{(\zeta-a)^n}{2\pi i}\displaystyle\int_C \frac{f(z)dz}{(z-a)^n (z-\zeta)}$, the remainder after n terms in Taylor's Theorem.

Now, since $f(\zeta)$ may be written as

$$f(\zeta) = f(a + \overline{\zeta - a}).$$

$\therefore$ the expansion

$$f(\zeta) = f(a) + f'(a)(\zeta - a) + \frac{f''(a)}{\underline{|2}}(\zeta - a)^2 + \ldots$$

represents the Taylor's Theorem.

On assumptions $|\zeta - a| = r < \rho$,

$|z - a| = \rho'$ and $r < \rho' < \rho$,

as well as taking maximum value of $|f(z)|$ as M, *i.e.*, $|f(z)| \le M$ on C, we have

$$|R_n| \le \frac{|\zeta - a|^n}{2\pi}\int_C \frac{|f(z)|\,|dz|}{|z-a|^n |z-\zeta|}$$

$$\le \frac{r^n}{2\pi}\cdot\frac{M}{\rho'^{(n)}(\rho' - r)}\int_C ds \qquad \begin{aligned}[t] [\because |z-\zeta| &= |z - a - (\zeta - a)| \\ &\le |z-a| - |\zeta - a| \\ &\le [\rho' - r] \end{aligned}$$

$$\le \frac{r^n}{2\pi}\cdot\frac{M}{\rho'^n(\rho' - r)}\cdot L,$$ where L is the length of the entire circumference and hence $L = 2\pi\rho'$

$$\le \frac{r^n}{2\pi}\cdot\frac{M}{\rho'^n(\rho' - r)}\cdot 2\pi\rho'$$

$$\le \left(\frac{r}{\rho'}\right)^n \frac{M}{1 - \dfrac{r}{\rho'}},$$

≤ 0 if $n \to \infty$, since $\dfrac{r}{\rho'} < 1$.

$\therefore$ $R_n = 0$, when $n \to \infty$.

Thus we have the infinite series

$$f(\zeta) = f(a) + f'(a)(\zeta - a) + \ldots$$

$$= \sum_{n=0}^{\infty} a_n \cdot (\zeta - a)^n, \quad \text{where } a_n = \frac{f^n(a)}{\underline{|n}}$$

Note 1. Maclaurin's Expansion as a particular case of Taylor's Expansion.

In Taylor's expansion if $a = 0$, we have

$$f(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n = \sum_{n=0}^{\infty} \frac{f^n(o)}{\lfloor n} \zeta^n$$

Note 2. Here $|R_n| \to 0$ *as* $n \to \infty$, *it follows that the series* $\sum_{n=0}^{\infty} a^n(\zeta - a)^n$ *is convergent and has* $f(\zeta)$ *as its sum-function. Also if the function* $f(z)$ *is regular in the whole z-plane, the expansion is valid for all values of* ζ.

Note 3. If the function $f(\zeta)$ *has a maximum* $M(r)$ *on* $|\zeta - a| = r < \rho$, then,

$$|a_n| \le \frac{M(r)}{r^n}, \text{ where } a_n = \frac{f^n(a)}{\lfloor n}$$

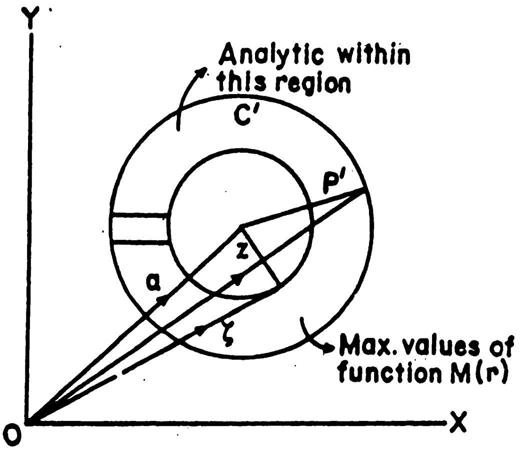


Fig. 5.21

The Taylor's expansion is

$$f(\zeta) = f(a) + f'(a)(\zeta - a) + \frac{f''(a)}{\lfloor 2}(\zeta - a)^2 + \ldots + \frac{f^n(a)}{\lfloor n}(\zeta - a)^n + \ldots$$

$$= a_0 + a_1(\zeta - a) + \ldots + a_n(\zeta - a)^n + \ldots$$

where $a_n = \dfrac{f^n(a)}{\lfloor n}$

$$= \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{(z-a)^{n+1}}.$$

$\therefore$ If C' be the circle $|\xi - a| = r$, then as in Cauchy's theorem,

$$a_n = \frac{1}{2\pi i} \int_{C'} \frac{f(z)\, dz}{(z-a)^{n+1}}.$$

$$\therefore |a_n| \le \frac{1}{2\pi} \int_{C'} \frac{|f(z)|\, |dz|}{|z-a|^{n+1}}.$$

Now if $\quad |z - a| = \rho' > r;\ \dfrac{1}{\rho'} < \dfrac{1}{r}.$

$$\therefore \quad |a_n| \le \frac{M(r)}{2\pi} \int_{C'} \frac{ds}{r^{n+1}}$$

$$\le \frac{M(r)}{2\pi \cdot r^{n+1}} . 2\pi r$$

$$\le \frac{M(r)}{r^n}.$$

Note 4. **(Liouville's theorem).** *If the function* $f(z)$ *is regular in the whole z-plane then* $f(z)$ *must be a constant provided* $|f(z)| < k$ *for all values of z.*

Since $f(z)$ is analytic for all values of z, then by Taylor's theorem, we have

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots = \sum_{n=0}^{\infty} a_n z^n,$$

where $|a_n| \le \dfrac{M(r)}{r^n}$ and $|f(z)| = M(r)$ when $|z| = r$.

As given that $|f(z)| < k$ for all values of z including ∞,

$\therefore \qquad M(r) < k.$

Hence $\qquad |a_n| \le \dfrac{k}{r^n}$

≤ 0 when $n \to \infty$ for $n<0$.

$\therefore \qquad a_n = 0$ for $n > 0$.

As such $f(z) = a_0$ is a constant.

5.13. LAURENT'S EXPANSION (Agra, 1966)

Let there be an annulus between two concentric circles C_1 and C_2 of centre $z = a$ and radii ρ_1 and ρ_2 ($\rho_1 > \rho_2$); then if $f(z)$ be regular within the annulus between C_1 and C_2, as well as on the circles, and ζ be any point of the annulus,

$$f(\zeta) = \sum_{n=0}^{\infty} a_n (\zeta - a)^n + \sum_{n=1}^{\infty} b_n (\zeta - a)^{-n},$$

where $a_n = \dfrac{1}{2\pi i} \displaystyle\int_{C_1} \dfrac{f(z)\,dz}{(z-a)^{n+1}}$ *and* $b_n = \dfrac{1}{2\pi i} \displaystyle\int_{C_2} (z-a)^{n-1} f(z)\,dz.$

(Rohilkhand, 1989)

We have proved in the Taylor's theorem that

$$\frac{1}{z-\zeta} = \frac{1}{z-a} + \frac{z-a}{(z-a)^2} + \ldots + \frac{(\zeta-a)^n}{(z-a)^n (z-\zeta)} \qquad \ldots(1)$$

Interchanging z and ζ with each other we have

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - a} + \frac{z-a}{(\zeta-a)^2} + \frac{(z-a)^2}{(\zeta-a)^3} + \ldots + \frac{(z-a)^n}{(\zeta-a)^n (\zeta - z)}$$

i.e.
$$-\frac{1}{z-\zeta} = \frac{1}{\zeta-a} + \frac{z-a}{(\zeta-a)^2} + \ldots + \frac{(z-a)^2}{(\zeta-a)^n(\zeta-z)}. \qquad \ldots(2)$$

Now, we know that

$$f(\zeta) = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{z-\zeta}$$

by Cauchy's integral.

Therefore by making a cross cut joining any point of the circle C_1 to any point of the circle C_2, we get

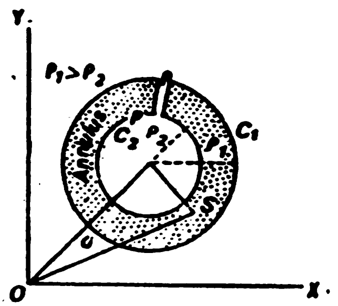


Fig. 5.22

$$f(\zeta) = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)\,dz}{z-\zeta} - \frac{1}{2\pi i} \int_{C_2} \frac{f(z)\,dz}{z-\zeta}$$

$$= \frac{1}{2\pi i} \int_{C_1} \frac{1}{z-\zeta} f(z)\,dz + \frac{1}{2\pi i} \int_{C_2} -\frac{1}{z-\zeta} f(z)\,dz.$$

Substituting the values of $\dfrac{1}{z-\zeta}$ and $-\dfrac{1}{z-\zeta}$ from (1) and (2), we get

$$f(\zeta)=\frac{1}{2\pi i}\int_{C_1}\left[\frac{1}{z-a}+\frac{\zeta-a}{(z-a)^2}+\ldots+\frac{(\zeta-a)^n}{(z-a)^n\,(\zeta-z)}\right]f(z)\,dz.$$

$$+\frac{1}{2\pi i}\int_{C_2}\left[\frac{1}{\zeta-a}+\frac{z-a}{(\zeta-a)^n}+\ldots+\frac{(z-a)^n}{(\zeta-a)^n\,(z-\zeta)}\right]f(z)\,dz$$

Here term by term integration is possible as the two series are uniformly convergent.

$$\therefore\; f(\zeta)=f(a)+(\zeta-a)f'(a)+\frac{(\zeta-a)^2}{2\,!}f''(a)+\ldots+P_n$$

$$+\frac{b_1}{\zeta-a}+\frac{b_2}{(\zeta-a)^2}+\frac{b_3}{(\zeta-a)^3}+\ldots+Q_n,$$

where
$$b_1=\frac{1}{2\pi i}\int_{C_2}f(z)\,dz,$$

$$b_2=\frac{1}{2\pi i}\int_{C_2}(z-a)\,f(z)\,dz,$$

$$\ldots\quad\ldots\quad\ldots\quad\ldots$$

$$\ldots\quad\ldots\quad\ldots\quad\ldots$$

$$b_n=\frac{1}{2\pi i}\int_{C_2}(z-a)^{n-1}\,f(z)\,dz$$

and P_n is the remainder after nth term as in the Taylor's expansion.

It is easy to show (as in Taylor's expansion) that

$$|P_n|\to 0 \text{ as } n\to\infty.$$

Also
$$Q_n=\frac{1}{(\zeta-a)^n}\cdot\frac{1}{2\pi i}\int_{C_2}\frac{(z-a)^n}{\zeta-a}f(z)\;dz.$$

$$\because\;|Q_n|\le\frac{1}{r^n}\cdot\frac{1}{2\pi}\int_{C_2}\frac{|(z-a)|^n}{|\zeta-z|}\,|f(z)|\,|dz|$$

$$\le\frac{\rho_2^n}{r^n}\,\frac{M'}{2\pi}\int_{C_2}\frac{ds}{r-\rho_2}$$

$$\le\frac{\rho_2^n}{r^n}\cdot\frac{M'}{2\pi}\cdot\frac{2\pi\rho_2}{r-\rho_2}$$

$$\le\left(\frac{\rho_2}{r}\right)^{n+1}\frac{M'}{1-\dfrac{\rho_2}{r}}\,,$$

$$\le 0 \text{ as } n\to\infty,\text{ since } \frac{\rho_2}{r}<1$$

$\because\; \rho_1>|\zeta-a|>\rho_2$
and Max. value of $|f(z)|$ is M'
Also $\zeta-z=\zeta-a-(z-a)$.
$\therefore\; |\zeta-z|>|\zeta-a|-|z-a|>r-\rho_2,$
i.e. $\dfrac{1}{|\zeta-z|}>\dfrac{1}{r-\rho_2}$

Since modulus of any quantity cannot be negative.

$\therefore\qquad Q_n=0$, when $n\to\infty$.

Hence

$$f(\zeta)=f(a)+(\zeta-a)f'(a)-\frac{(\zeta-a)^2}{\underline{|2}}f''(a)+\ldots+\frac{b_1}{\zeta-a}+\frac{b_2}{(\zeta-a)^2}+\frac{b_3}{(\zeta-a)^3}+\ldots$$

$$= \sum_{n=0}^{\infty} a_n (\zeta - a)^n + \sum_{n=1}^{\infty} b_n(\zeta - a)^{-n}. \qquad ...(3)$$

Note 1. The integrals giving the values of a_n and b_n as

$$a_n = \frac{1}{2\pi i}\int_{C_1}\frac{f(z)\,dz}{(z-a)^{n+1}} \text{ and } b_n = \frac{1}{2\pi i}\int_{C_2}\frac{f(z)\,dz}{(z-a)^{-n+1}}$$

$$= \frac{1}{2\pi i}\int_{C_2}(z-a)^{n-1} f(z)\,dz$$

are analytic everywhere in the annulus of C_1 and C_2 and therefore by Cauchy's theorem the path of integration can be taken any concentric circle C_0 lying between C_1 and C_2, so that

$$b_n = \frac{1}{2\pi i}\int_{C_0}(z-a)^{n-1} f(z)\,dz = \frac{1}{2\pi}\int_{C_0}\frac{f(z)\,dz}{(z-a)^{-n+1}} = a_{-n}$$

Thus $f(\zeta) = \sum_{-\infty}^{\infty} a_n(\zeta - a)^n$ where $a_n = \dfrac{1}{2\pi i}\displaystyle\int_{C_0}\frac{f(z)}{(z-a)^{n+1}}\,dz.$

Note 2. Zero's and Singularities of an analytic complex function.

A zero of an analytic function $f(z)$ is defined to be a value of $z = \zeta$ such that $f(\zeta) = 0$. Taylor's expansion gives

$$f(z) = f(\zeta) + (z - \zeta) f'(\zeta) + \frac{(z-\zeta)^2}{\lfloor 2} f''(\zeta) + ... + \frac{(z-\zeta)^n}{\lfloor n} f^n(\zeta) + ...$$

Here if $f(\zeta) = 0$ and $f'(\zeta) \neq 0$ then $f(z)$ is said to have a *Simple zero* at $z = \zeta$.

If $f(\zeta) = f'(\zeta) = f''(\zeta) = ... = f^{n-1}(\zeta) = 0$ and $f^{(m)}(\zeta) \neq 0$, then the point $z = \zeta$ is known as *zero of order m* of the function $f(z)$, since then

$$f(z) = \frac{(z-\zeta)^n}{\lfloor n} f^{(n)}(\zeta) + \frac{(z-\zeta)^{(n+1)}}{\lfloor n+1} f^{(n+1)}(\zeta) + ...$$

$$= (z-\zeta)^n\left[\frac{f^{(n)}(\zeta)}{\lfloor n} + \frac{(z-\zeta)}{\lfloor n+1} + ...\right]$$

$$= (z-\zeta)^n \phi(z) \qquad ...(4)$$

where $$\phi(z) = \frac{f^{(n)}(\zeta)}{\lfloor n} + \frac{(z-\zeta)}{\lfloor n+1} + ...$$

= a non-zero and analytic function at $z = \zeta$.

It follows from (4) that the point $z = \zeta$ may be defined as a *zero of order n* of the function $f(z)$ if $f(z) = (z - \zeta)^n \phi(z)$

for $f(z)$ being analytic and non-zero function at $z = \zeta$.

If we draw a small circle C about ζ as centre then within it $\phi(z) \neq 0$. Also for $z \neq \zeta$ within C, $(z-\zeta)^n \neq 0$ and hence ζ is the only zero of the function $f(z)$ within C. As such the zero point $z = \zeta$ is *isolated*. But ζ being arbitrary may be any zero point of $f(z)$ and therefore all the zeros of an analytic function $f(z)$ are isolated.

A point at which a function $f(z)$ ceases to be regular (analytic) is termed as a *singular point* of $f(z)$ and the function $f(z)$ is said to have a *singularity* at this point.

In case there is no other singularity in the neighbourhood of a singular point $z = \zeta$, the function $f(z)$ is said to have an *isolated singularity*. We can surround such a singularity by a small circle Γ with ζ as centre such that in the annulus made by Γ with a larger concentric circle C, the function $f(z)$ is analytic. Then we can expand $f(z)$ by Laurent's expansion as

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\zeta)^n + \ldots + \frac{b_1}{z-\zeta} + \frac{b_2}{(z-\zeta)^2} + \ldots + \frac{b_m}{(z-\zeta)^m} + \ldots$$

where the terms containing b's are termed as ***principal part of the expansion*** at the singularity $z = \zeta$.

There arise three possibilities:

(*i*) All the coefficients b's are zero so that the function is equal to the analytic function except at $z = \zeta$. Such a singularity is defined as ***non-essential*** or ***removable singularity*** of $f(z)$.

(*ii*) The principal part is an infinite series. Then the point $z = \zeta$ is said to be an ***essential*** or ***non-removable singularity*** of $f(z)$.

(*iii*) The principal part contains a finite number of terms *i.e.*

$$\text{the principal number part} = \frac{b_1}{z-\zeta} + \frac{b_1}{(z-\zeta)^2} + \ldots + \frac{b_m}{(z-\zeta)^m}$$

so that $b_{m+1} = b_{m+2} = \ldots = 0$

then the function $f(z)$ is said to have a ***pole*** or ***singularity of order m***. In this case, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\zeta)^n + \sum_{n=1}^{m} b_n (z-\zeta)^n$$

$$= (z-\zeta)^{-m}\left[\sum_{n=0}^{\infty} a_n (z-\zeta)^{m+n} + b_1 (z-\zeta)^{m-1} + b_2 (z-\zeta)^{m-2} + \ldots\ldots + b_m\right]$$

$$= (z-\zeta)^{-m}\phi(z) = \frac{\phi(z)}{(z-\zeta)^m}, \text{ when } \phi(z) \text{ is an analytic function at } z = \zeta$$

When $m = 1$, the pole is known as a ***simple pole***.

For a ***pole of order m*** of $z = \zeta$, the limiting value of the function is given by

$$\lim_{z \to \zeta} f(z) = \lim_{z \to \zeta} \sum_{n=1}^{m} b_n (z-\zeta)^{-n}$$

or

$$\lim_{z \to \zeta} |f(z)| = \lim_{z \to \zeta} \left| \sum_{n=1}^{m} b_n (z-\zeta)^{-n} \right|$$

$$= \lim_{z \to \zeta} \left| \frac{b_m}{(z-\zeta)^m} + \frac{b_{m-1}}{(z-\zeta)^{m-1}} + \ldots\ldots + \frac{b_1}{z-\zeta} \right|$$

$$= \lim_{z \to \zeta} \frac{1}{|z-\zeta|^m} \left| \{b_m + b_{m-1}(z-\zeta) + \ldots + b_n (z-\zeta)^{m-n} + \ldots + b_1 (z-\zeta)^{m-1}\} \right|$$

$$\geq \lim_{z \to \zeta} \frac{1}{|z-\zeta|^m} \{|b_m| + |b_{m-1}(z-\zeta)| + \ldots + |b_1 (z-\zeta)^{m-1}|\}$$

$$\geq \lim_{z \to \zeta} \frac{1}{|z-\zeta|^m} |b_m|$$

$$\geq \infty$$

i.e. $|f(z)| \to \infty$ as $z \to \zeta$.

Conclusively if $f(z)$ is analytic in a given region except at a point $z = \zeta$ and $\lim_{z \to \zeta} f(z) \to \infty$, then $f(z)$ is said to have a *pole* or *singularity* at $z = \zeta$.

For the pole of order m, we have

$$\lim_{z \to \zeta} (z-\zeta)^m f(z) = b_m = \text{constant } (\neq 0)$$

and the value of the function is given by $\dfrac{\phi(z)}{(z-\zeta)^m}$.

In case there does not exist a value of m such that $\lim_{z \to \zeta} (z-\zeta)^m f(z)$ be finite, then the point $z = \zeta$ is an essential or non-removable singularity of $f(z)$.

If an analytic function $f(z)$ has a pole of order m at the point $z = \zeta$ then $\dfrac{1}{f(z)}$ *has a zero of order m at this point.*

(*Rohilkhand, 1992*)

$z = \zeta$ will be a pole of order m of $f(z)$ if

$(z-\zeta)^m f(z)$ is analytic and non-zero at $z = \zeta$, so that

$$\phi(z) = \frac{1}{(z-\zeta)^m f(z)} \text{ is analytic or non-zero.}$$

$\therefore \dfrac{1}{f(z)} = (z-\zeta)^m \phi(z)$ follows that $\dfrac{1}{f(z)}$ has a zero of order m as $z = \zeta$.

A similar procedure will show that if $\dfrac{1}{f(z)}$ has a pole of order m then $f(z)$ will have a pole of order m.

It also follows that since zeros are isolated, the poles must be isolated.

Note 3. **The Point at infinity.** *In treating the complex variables it is convenient to regard infinity as a point. The point infinity is considered by putting* $z = \dfrac{1}{\zeta}$ *in* $f(z)$ *so that the behaviour of* $f(z)$ *at infinity is examined by the behaviour of* $f\left(\dfrac{1}{\zeta}\right)$ *at* $\zeta = 0$.

Consequently $f(z)$ is analytic or has a zero or has a simple pole or has an essential singularity etc., at infinity according as $f\left(\dfrac{1}{\zeta}\right)$ is analytic or has a zero or has a simple pole or has an essential singularity at $\zeta = 0$.

e.g. if $f\left(\dfrac{1}{\zeta}\right)$ has a zero of order m at $\zeta = 0$ then writing $\phi(\zeta)$ for $f\left(\dfrac{1}{\zeta}\right)$ its expansion at $\zeta = 0$ by Taylor's Theorem is $f\left(\dfrac{1}{\zeta}\right) = \phi(\zeta) = a_m \zeta^m + a_{m+1} \zeta^{m+1} + \ldots$ so that corresponding expansion of f(z) at $z = \infty$ is $f(z) = \frac{a_m}{z^m} + \frac{a_{m+1}}{z^{m+1}} + \ldots$ which does not contain constant terms and terms containing positive exponent powers of z and hence $f(z)$ has a zero of order m at infinity.

Similarly if $f\left(\frac{1}{\zeta}\right)$ has a pole of order m at $\zeta = 0$ then by Laurent's expansion

$$f\left(\frac{1}{\zeta}\right) = \phi(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n + \sum_{n=1}^{n} b_n \zeta^{-n}$$

So that corresponding expansion of $f(z)$ at $z = \infty$ is $f(z) = \sum_{n=1}^{m} b_m z^n + \sum_{n=0}^{\infty} a_n z^{-n}$ which has $\sum_{n=1}^{m} b_n z^n$ as its principal part at $z = \infty$ and this principal part being a finite series of positive powers of z at $z = \zeta$ shows that $f(z)$ has a pole of order m at infinity.

Problem 13. *The function $f(z)$ is regular when $|z| < R'$. Prove that if $|a| < R < R'$, $f(a) = \frac{1}{2\pi i}\int_C \frac{R^2 - a\bar{a}}{(z-a)(R^2 - z\bar{a})} f(z)\,dz$, where C is the circle $|z| = R$. Deduce Poisson's formula that if $0 < r < R$,*

$$f\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_{\theta}^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2} f\left(Re^{i\phi}\right) d\phi.$$

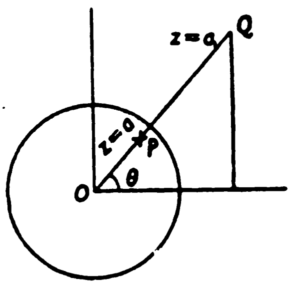


Fig. 5.23

Taking centre as origin and a line OP making an angle θ with the horizontal, let us take a point Q on OP produced, such that $OP \cdot OQ = R^2$, where $|z| = R$; then Q is called the point inverse to P w.r.t. the circle of radius R.

Let at P, $z = a$, and at Q, $z = a_1$ be the complex co-ordinates.

Now $|OP|\ |OQ| = R^2$.

$\therefore\ |OQ| = \frac{R^2}{|OP|} > R$ or $R > |OP|$.

Therefore Q must lie outside the circle.

Again $a_1 = OQ \cdot e^{i\theta} = \frac{R^2}{OP} e^{i\theta}$,

$$= \frac{R^2}{OP \cdot e^{-i\theta}} \text{ as } a = OP \cdot e^{i\theta},$$

$$= \frac{R^2}{\bar{a}} \qquad \because \text{ conjugate of } a \text{ i.e. } \bar{a} = OP \cdot e^{-i\theta}.$$

By Cauchy's theorem, $f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-a}$. ...(1)

Also the Cauchy's theorem states that if $F(z)$ is a regular function and if $F'(z)$ is continuous at each point within and on a closed contour C, then $\int_C F(z)\,dz = 0$ and thus as point Q is outside the circle and $F(z)$ is analytic, *i.e.*,

$$F(z) = \frac{f(z)}{z-a_1} \neq \infty$$

therefore
$$0 = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-a_1}. \qquad \ldots(2)$$

Subtracting (2) from (1), we get

$$f(a) = \frac{1}{2\pi i}\int_C \frac{(a-a_1)f(z)\,dz}{(z-a)(z-a_1)}$$

$$= \frac{1}{2\pi i}\int_C \frac{\left(a-\frac{R^2}{\bar{a}}\right)f(z)\,dz}{(z-a)\left(z-\frac{R^2}{\bar{a}}\right)}$$

$$= \frac{1}{2\pi i}\int_C \frac{\left(R^2-a\bar{a}\right)f(z)\,dz}{(z-a)\left(R^2-z\bar{a}\right)} \qquad \ldots(3)$$

To deduce Poisson's formula, if $0<r<R$.

Put $a = re^{i\theta}$; $\quad \therefore \bar{a} = re^{-i\theta}$,

$z = Re^{i\phi}$, i.e., $dz = iRe^{i\phi}d\phi$.

Being a circle, limits of ϕ are from 0 to 2π.

$$\therefore\ (z-a)\left(R^2-\bar{a}z\right) = \left(Re^{i\phi}-re^{i\theta}\right)\left(Re^2-re^{-i\theta}\cdot Re^{i\phi}\right)$$

$$= Re^{i\phi}\left(Re^{i\phi}-re^{i\theta}\right)\left(Re^{-i\phi}-re^{-i\theta}\right)$$

$$= Re^{i\phi}\left[R^2+r^2-2Rr\cos(\theta-\phi)\right]$$

and
$$R^2-\bar{a}a = R^2-re^{-i\theta}re^{i\theta} \qquad [\because e^{i(\theta-\phi)}+e^{-i(\theta-\phi)} = 2\cos(\theta-\phi)].$$

$$= R^2-r^2.$$

Substituting these values in result (3), we get

$$f\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\left(R^2-r^2\right)f\left(Re^{i\phi}\right)d\phi}{R^2-2Rr\cos(\theta-\phi)+r^2}$$

Problem 14. *The function $f(z)$ is regular in $|z-a|<R$. Prove that if $0<r<R$, $f'(a) = \frac{1}{\pi r}\int_0^{2\pi} P(\theta)e^{-i\theta}\,d\theta$, where $P(\theta)$ is the real part of $f(a+re^{i\theta})$.*

We know that $f'(a) = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)\,dz}{(z-a)^2}\cdot$ Put $z-a = re^{i\theta}$, $\therefore\ dz = ire^{i\theta}\,d\theta$

$$= \frac{1}{2\pi i}\int_0^{2\pi} f\left(a+re^{i\theta}\right)re^{-i\theta}\,d\theta. \qquad \ldots(1)$$

By Taylor's Theorem,

$$f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n=\sum_{n=0}^{\infty}a_n r^n e^{in\theta}$$

$$=\sum_{n=0}^{\infty}R_n e^{i\phi_n}\cdot r^n e^{in\theta},$$ where a_n is a complex quantity and therefore may be written as $a_n=R_n e^{i\phi_n}$

$$=\sum_{n=0}^{\infty}R_n r^n e^{i(n\theta+\phi_n)}.$$

Let the conjugate of $f(z)$ be $\bar{f}(z)$; then

$$\bar{f}(z)=\sum_{n=0}^{\infty}R_n r^n e^{-i(n\theta+\phi_n)}$$

$$\therefore\ \frac{1}{2\pi i}\int_C\frac{\bar{f}(z)}{(z-a)^2}\,dz=\frac{1}{2\pi i}\int_0^{2\pi}e^{-i\theta}d\theta\cdot\sum_0^{\infty}R_n r^n\cdot e^{-i(n\theta+\phi_n)}$$

$$=\frac{1}{2\pi i}\sum_0^{\infty}R_n\cdot r^n\int_0^{2\pi}e^{-i\{(n+1)\theta+\phi_n\}}d\theta$$

$$=\frac{1}{2\pi i}\sum_0^{\infty}R_n\cdot r^n\int_0^{2\pi}e^{-i\phi_n}\cdot e^{-i(n+1)\theta}\,d\theta.$$

$$=0, \qquad \ldots(2)$$

$$\left[\text{since}\left\{\frac{e^{-i(n+1)\theta}}{-i(n+1)}\right\}_0^{2\pi}=0\text{ or }\{\cos(n+1)\theta-i\sin(n+1)\theta\}_0^{2\pi}=0\right]$$

Adding (1) and (2), we have

$$f'(a)=\frac{1}{2\pi r}\int_0^{2\pi}[f(z)+\bar{f}(z)]\,e^{-i\theta}\,d\theta$$

$$=\frac{2}{2\pi r}\int_0^{2\pi}P(\theta)\,e^{-i\theta}d\theta.\ \text{Since } z+\bar{z}=2R(z),.$$

$$=\frac{1}{\pi r}\int_0^{2\pi}P(\theta)e^{-i\theta}d\theta \qquad \because f(z)+\bar{f}(z)=2Rf(z)=2P(\theta)$$

as given that $P(\theta)$ is the real part of $f(a+re^{i\theta})$.

Problem 15. *By using the integral representation of $f^n(a)$, prove that*

$$\left(\frac{x^n}{n!}\right)^2=\frac{1}{2\pi i}\int_C\frac{x^n\cdot e^{xz}}{n!\,z^{n+1}}\,dz$$

where C is any closed contour surrounding the origin. Hence prove that

$$\sum_{n=0}^{\infty}\left(\frac{x^n}{n!}\right)^2=\frac{1}{2\pi}\int_0^{2\pi}e^{2x\cos\theta}\,d\theta.$$

We know that

$$f^n(a) = \frac{n\,!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a))^{n+1}}. \quad ...(1)$$

$$\therefore\ f^n(0) = \frac{n\,!}{2\pi i}\int_C \frac{f(z)\,dz}{z^{n+1}}. \quad ...(2)$$

Let us put $f(z) = e^{xz}$.

Differentiating n times w.r.t. z, we have

$$f^n(z) = x^n e^{xz}.$$

$$f^n(0) = x^n.$$

Substituting its value in (2), we get

$$\frac{x^n}{n\,!} = \frac{1}{2\pi i}\int_C \frac{e^{xz}\,dz}{z^{n+1}}.$$

Multiplying both sides by $\frac{x^n}{n!}$, we have

$$\left(\frac{x^n}{n!}\right)^2 = \frac{1}{2\pi i}\int_C \frac{x^n}{n!}\cdot\frac{e^{xz}}{z^{n+1}}\,dz.$$

Now $$\sum_{n=0}^{\infty}\left(\frac{x^n}{n!}\right)^2 = \frac{1}{2\pi i}\int_C \frac{e^{xz}}{z}\,dz\cdot\sum_0^{\infty}\frac{x^n}{n!.z^n}$$

$$= \frac{1}{2\pi i}\int_C \frac{e^{xz}}{z}\,dz\cdot\sum_0^{\infty}\frac{(x/z)^n}{n!}$$

$$= \frac{1}{2\pi i}\int_C \frac{e^{xz}}{z}\,dz\cdot e^{x/z}$$

$$= \frac{1}{2\pi i}\int_C \frac{e^{x(z+1/z)}}{z}\,dz. \quad ...(3)$$

If we put $z = e^{i\theta}$, $dz = ie^{i\theta}d\theta$

and $\frac{1}{z} = e^{-i\theta}$, $\frac{dz}{z} = i\,d\theta$,

so that $z + \frac{1}{z} = 2\cos\theta$.

Substituting these value in (3), we find

$$\sum_0^{\infty}\left(\frac{x^n}{n!}\right)^2 = \frac{1}{2\pi}\int_0^{2\pi} e^{2x\cos\theta}\,d\theta.$$

Problem 16. *Obtain the expansion*

$$f(z) = f(a) + 2\left\{\frac{z-a}{2}f'\left(\frac{z+a}{2}\right) + \frac{(z-a)^3}{2^3\cdot 3!}f'''\left(\frac{z+a}{2}\right) + \frac{(z-a)^5}{2^5\cdot 5!}f^{(5)}\left(\frac{z+a}{2}\right) + \ldots\right\}$$

and determine its range of validity.

We know that $f(z) = f\left(\frac{z+a}{2} + \frac{z-a}{2}\right)$ and $f(a) = f\left(\frac{z+a}{2} - \frac{z-a}{2}\right)$.

Expanding them by Taylor's expansion, we get

$$f(z)=f\left(\frac{z+a}{2}\right)+\left(\frac{z-a}{2}\right)f'\left(\frac{z+a}{2}\right)+\left(\frac{z-a}{2}\right)^2\frac{1}{2\,!}f''\left(\frac{z-a}{2}\right)+\ldots \quad \ldots(1)$$

and $$f(a)=f\left(\frac{z+a}{2}\right)-\left(\frac{z-a}{2}\right)f\left(\frac{z+a}{2}\right)+\frac{1}{2\,!}\left(\frac{z-a}{2}\right)^2 f''\left(\frac{z+a}{2}\right)+\ldots \quad \ldots(2)$$

Subtracting (2) from (1),

$$f(z)-f(a)=2\left\{\frac{z-a}{2}f'\left(\frac{z+a}{2}\right)+\frac{1}{3\,!}\left(\frac{z-a}{2}\right)^3 f'''\left(\frac{z+a}{2}\right)+\ldots\right\}.$$

$$i.e.\ f(z)=f(a)+2\left\{\frac{z-a}{2}f'\left(\frac{z+a}{2}\right)+\frac{(z-a)^3}{2^3:3\,!}f'''\left(\frac{z+a}{2}\right)+\ldots\right\}$$

Range of validity or the condition for the validity of Taylor's expansion is that

$$\left|\frac{z-a}{2}\right|<\delta \text{ or } \left|z-\frac{z+a}{2}\right|<\delta,$$

from which it is obvious that z must be in the neighbourhood of the point a.

Problem 17. *Prove that* $\cosh\left(z+\frac{1}{z}\right)=a_0+\sum_1^{\infty} a_n\left(z^n+\frac{1}{z^n}\right),$

where $$a_n=\frac{1}{2\pi}\int_0^{2\pi}\cos n\theta\cosh(2\cos\theta)\,d\theta.$$

Laurent's expansion yields

$$\cosh\left(z+\frac{1}{z}\right)=a_0+\sum_1^{\infty} a_n z^n+\sum_1^{\infty} b_n z^{-n}. \quad \ldots(1)$$

Putting $\frac{1}{z}$ for z, we get

$$\cosh\left(\frac{1}{z}+z\right)=a_0+\sum_1^{\infty}\frac{a_n}{z^n}+\sum_1^{\infty} b_n z^n. \quad \ldots(2)$$

It is obvious from the two expansions *i.e.*, from (1) and (2) that $a_n=b_n$.

By Laurent's expansion, we know that

$$f(z)=\sum_0^{\infty}a_n(z-a)^n+\sum_1^{\infty}\frac{b_n}{(z-a)^n}.$$

then $$a_n=\frac{1}{2\pi i}\int_{C_1}\frac{f(z)\,dz}{(z-a)^{n+1}} \text{ and } b_n=\frac{1}{2\pi i}\int_{C_2}f(z)\,(z-a)^{n-1}\,dz.$$

As such in the existing case,

$$a_n=\frac{1}{2\pi i}\int_C\frac{\cosh(z+1/z)\,dz}{z^{n+1}} \text{ and } b_n=\frac{2}{2\pi i}\int_C\cosh\left(z+\frac{1}{z}\right)dz.z^{n-1}.$$

Putting $z=e^{i\theta}$ by taking a circle of unit radius,

$$\frac{1}{z}=e^{-i\theta}, \text{ so that } z+\frac{1}{z}=2\cos\theta$$

and $$dz = ie^{i\theta} d\theta, \frac{dz}{z} = i\, d\theta, \text{ we get}$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \frac{\cosh\left(e^{i\theta}+e^{-i\theta}\right) e^{i\theta}\, d\theta}{e^{i(n+1)\theta}}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh\left(e^{i\theta}+e^{-i\theta}\right) e^{-ni\theta}\, d\theta$$

and $$b_n = \frac{1}{2\pi}\int_0^{2\pi} \cosh\left(e^{i\theta}+e^{-i\theta}\right) e^{ni\theta}\, d\theta,$$

$\therefore$ $$2a_n = 2b_n = \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta).\ 2\cos n\theta\, d\theta,$$

i.e., $$a_n = b_n = \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta).\ \cos n\theta\, d\theta.$$

5.14. RESIDUE AND CONTOUR INTEGRATION

Definition of residue. (*Rohilkhand, 1989, 92*). The residue of a function $f(z)$ at the pole $z = a$ is defined to be the coefficient of $(z-a)^{-1}$ in the Laurent's expansion of the function $f(z)$, *i.e.*,

$$f(z) = \sum_0^{\infty} a_n (z-a)^n + \sum_1^{m} b_n (z-a)^{-n},$$

where $z = a$ is a pole of order m.

If $z = a$ be the pole of order one, then the residue is

$$b_1 = \lim_{z \to a} (z-a)\, f(z),$$

i.e., in case of a simple pole $z = a$.

$$\text{Residue} = \lim_{z \to a} (z-a) f(z).$$

Now consider the integral $b_n = \frac{1}{2\pi i}\int_{C_2} (z-a)^{n-1} f(z)\, dz$ which is the value of b_n in Laurent's expansion. Here the circle C_2 is arbitrary and may therefore be replaced by any closed contour C containing within it no other singularities except $z = a$. Thus

$$\frac{1}{2\pi i}\int_{C_2} f(z)\, dz = \frac{1}{2\pi i}\int_C f(z)\, dz,$$

where $n = 1$, *i.e.*, pole is of order one,

i.e., $$b_1 = \frac{1}{2\pi i}\int_C f(z)\, dz.$$

Hence as an alternative, the residue of a function $f(z)$ at the pole $z = a$ is equal to $\frac{1}{2\pi i}\int_C f(z)\, dz$, where C is a closed contour containing within it the only singularity $z = a$ and integration being taken round C in anti-clockwise direction, *i.e.*, the positive sense.

Similarly the residue of $f(z)$ at infinity, *i.e.*, at the point $z = \infty$ is $\frac{1}{2\pi i}\int_C f(z)\, dz$ taken round C in clockwise direction, as it is negative w.r.t. the interior of C and positive w.r.t the exterior of C, provided the value of this integral is definite.

5.15. CAUCHY'S RESIDUE THEOREM

(*Rohilkhand, 1977, 78, 79, 83, 89, 90, 93*)

If the function $f(z)$ be single-valued, continuous and regular within and on a closed contour C, except a finite number of poles (singularities) within C, then

$$\int_C f(z)\,dz = 2\pi i \Sigma R$$

where ΣR represents the sum of the residues say $R_1, R_2, R_3 \ldots R_n$ of $f(z)$ at the poles $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n$ (say) within A. (*Vikram, 1969; Rohilkhand, 1985*)

Let us draw a set of circles $\gamma_1, \gamma_2 \ldots \gamma_n$ with centres $\alpha_1, \alpha_2, \ldots \alpha_n$ and radius ρ, such that they do not intersect each other and lie entirely within the closed curve C. Then $f(z)$ is regular within the region enclosed between C and the small circles $\gamma_1, \gamma_2, \ldots \gamma_n$. The entire region C may be deformed to consist of these small circles and the polygon P. Now by Cauchy's theorem, we have

$$\int_C f(z)\,dz = \int_P f(z)\,dz + \sum_{r=1}^{n} \int_{\gamma_r} f(z)\,dz$$

But the integral round the polygon P vanishes since $f(z)$ is regular within and on the closed contour P. Therefore

$$\int_C f(z)\,dz = \sum_{r=1}^{n} \int_{\gamma_r} f(z)\,dz.$$

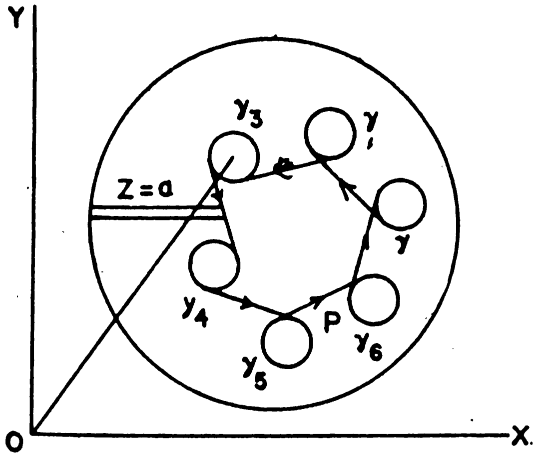


Fig. 5.24

Let us now consider $z = a$, a pole of order m; then by Laurent's expansion,

$$f(z) = \sum_{n=0}^{n} a_n (z-a)^n + \sum_{s=0}^{m} \frac{b_s}{(z-a)^s}$$

$$= \phi(z) + \sum_{1}^{m} \frac{b_s}{(z-a)^s}.$$

where $\phi(z) \left\{ = \sum_{0}^{n} a_n (z-a)^n \right\}$ is regular within and on γ_r and has no pole.

$$\therefore \int_{\gamma_r} f(z)\,dz = \int_{\gamma_r} \phi(z)\,dz + \sum_{1}^{m} b_s \int_{\gamma_r} \frac{dz}{(z-a)^s}.$$

where $\int_{\gamma_r} \phi(z)\,dz = 0$ by Cauchy's fundamental theorem.

$$\therefore \int_{\gamma_r} f(z)\,dz = \sum_{1}^{m} b_s \int_{\gamma_r} \frac{dz}{(z-a)^s}.$$

Putting $z - a = \rho e^{i\theta}$, where θ varies from 0 to 2π,

$$dz = \rho i e^{i\theta} d\theta.$$

As the point z makes a circuit which consists of the circle γ_r, therefore

$$\int_{\gamma_r} f(z)\, dz = \sum_{s=1}^{m} b_s \rho i \int_0^{2\pi} \frac{e^{i\theta} d\theta}{\rho^s e^{is\theta}}$$

$$= \sum_{1}^{m} b_s \rho^{(1-s)} i \int_0^{2\pi} e^{i(1-s)\theta}\, d\theta \cdot$$

Now $\displaystyle\int_0^{2\pi} e^{i(1-s)\theta}\, d\theta = \left[\frac{e^{i(1-s)\theta}}{i(1-s)}\right]_0^{2\pi} = 0$ if $s \neq 1$.

But if $s = 1$, all the terms will be zero except one.

$$\therefore \qquad \int_{\gamma_r} f(z)\, dz = b_1 i \int_0^{2\pi} d\theta = 2\pi i b_1,$$

where b_1 is called the residue for the function.

Let the residues for $r = 1, 2, 3 \ldots n$ be respectively $R_1, R_2, R_3, \ldots R_n$;

then $\displaystyle\int_{\gamma_1} f(z)\, dz = 2\pi i R_1,$

$$\int_{\gamma_2} f(z)\, dz = 2\pi i R_2,$$

$$\ldots \quad \ldots \quad \ldots$$

$$\ldots \quad \ldots \quad \ldots$$

$$\int_{\gamma_n} f(z)\, dz = 2\pi i R_n.$$

Hence $\displaystyle\int_C f(z)\, dz = \sum_{1}^{m} \int_{\gamma_r} f(z)\, dz$

$$= 2\pi i \Sigma R.$$

5.16. ANOTHER IMPORTANT THEOREM

Let C be the arc $\theta_1 \leq \arg(z-a) \leq \theta_2$ of the circle $|z - a| = r$ and if $\lim_{z \to a} \{(z-a) f(z)\} = b$, then

$$\lim_{z \to a} \int_C f(z)\, dz = ib\,(\theta_2 - \theta_1).$$

In order to prove it, let us assume $F(z) = (z - a) f(z)$; then

$$F(a) = \lim_{z \to a} (z-a) f(z) = b \text{ (given)}.$$

Since $f(z)$ and $(z - a)$ both are individually continuous, therefore their product $(z - a) f(z)$ must also be continuous.

For a given ϵ, $\eta(\epsilon)$ can be so found that if $|z - a| < \eta$, $|\delta| < \epsilon$, where $(z - a) f(z) = b \pm \delta$.

[That's why if $|z - a| < \eta$, $|F(z) - F(a)| = \delta < \epsilon$

i.e., $\qquad |F(z) - b| = \delta < \epsilon$.

So $\qquad F(z) = b \pm \delta \cdot$]

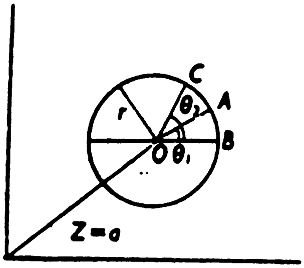


Fig. 5.25

Let us now take a contour whose centre is $z = a$ and radius is r and $f(z)$ between any two points say A and C, *i.e.*, along the curve AC; given by

$$\int_{AC} f(z)\, dz = \int_{AC} \frac{F(z)}{z-a} dz = \int_{AC} \frac{b \pm \delta}{z-a}\, dz. \qquad ...(1)$$

Let us assume that $z - a = re^{i\theta}$.

$\therefore\ dz = ire^{i\theta} d\theta$ and $\dfrac{dz}{z-a} = i\, d\theta.$

Substituting these values in (1), we get

$$\int_{AC} f(z)\, dz = i \int_{\theta_1}^{\theta_2} (b \pm \delta)\, d\theta$$

$$= ib\,(\theta_2 - \theta_1) \pm i\delta\,(\theta_2 - \theta_1).$$

Now $r \to 0$ as $z \to a$, $\because\ |F(z) - F(a)| = \delta$, $F(a) = b$ and $z - a = re^{i\theta}$

and $\quad z \to a$ as $\delta \to 0$; $\therefore\ F(z) = b \pm \delta = F(a) \pm \delta$.

Conclusively when $r \to 0$, $\delta \to 0$.

Hence $\lim\limits_{r \to 0} \int_{AC} f(z)\, dz = ib\,(\theta_2 - \theta_1)$, where $\lim\limits_{z \to a} (z-a)\, f(z) = b$.

By similar procedure we can show that

If C be the arc $\theta_1 \le \arg z \le \theta_2$ of the circle $|z| = R$ and if $\lim\limits_{r \to \infty} z f(z) \to b$ uniformly, then

$$\lim_{R \to \infty} \int_C f(z)\, dz = ib\,(\theta_2 - \theta_1).$$

Cor. *If $\theta_1 = 0$, $\theta_2 = 2\pi$, the AC covers whole the circle C (say), then*

$$\int_C f(z)\, dz = 2\pi ib.$$

5.17. COMPUTATION OF RESIDUE

(i) Residue of f(z) for a simple pole z = a,

This has been already established that

residue of $f(z)$ at $z = a$ is $= \lim\limits_{z \to a} (z - a) f(z)$.

If we put $f(z) = \dfrac{\phi(z)}{\psi(z)}$, where $\psi(z) = (z - a) F(z)$ so that $\psi(a) = 0$, then

$$\lim_{z \to a} (z-a)\, f(z) = \lim_{z \to a} \frac{(z-a)\phi(z)}{\psi(z)}$$

$$= \lim_{z \to a} \frac{(z-a)\phi(a + \overline{z-a})}{\psi(a + z - a)}$$

$$= \lim_{z \to a} \frac{(z-a)\,[\phi(a) + (z-a)\,\phi'(a) + \ldots]}{\psi(a) + (z-a)\,\psi'(a) + \ldots}$$

expanding by Taylor's theorem

$$= \lim_{z \to a} \frac{\phi(a) + (z-a)\,\phi'(a) + \ldots}{\psi'(a) + \frac{1}{\underline{|2}}\,(z-a)\,\psi''(a) \ldots} \qquad \because\ \psi(a) = 0$$

$$= \frac{\phi(a)}{\psi'(a)}.$$

Hence for a simple pole, residue at $z = a$ is $= \dfrac{\phi(a)}{\psi'(a)}$.

(*ii*) **Residue at z = a, a pole of order m.** Let $f(z)$ be of the form

$$f(z) = \frac{\phi(z)}{(z-a)^m}.$$

We have proved that Residue b_1 at $z = a$ is $\dfrac{1}{2\pi i}\int_C f(z)\, dz$.

Hence in the existing case,

$$\text{Residue} = \frac{1}{2\pi i}\int \frac{\phi(z)}{(z-a)^m}\, dz = \frac{\phi^{m-1}(a)}{\underline{|(m-1)}}.$$

by Results of Cauchy's Integral.

Hence Residue of $f(z)$ at $z = a$, a pole of order m, is

$$= \frac{\phi^{m-1}(a)}{\underline{|(m-1)}}.$$

(*iii*) **Residue of f(z) at a pole z = a of any order.** Laurent's Expansion is

$$f(z) = \sum_0^\infty a_n(z-a)^n + \sum_1^\infty \frac{b_m}{(z-a)^m}$$

$$= \sum_0^\infty a_n(z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \ldots + \frac{b_m}{(z-a)^m} + \ldots$$

Putting $z - a = t$, *i.e.* $z = a + t$, we have

$$f(a+t) = \sum_0^\infty a_n t^n + \frac{b_1}{t} + \frac{b_2}{t^2} + \ldots + \frac{b_m}{t^m} + \ldots,$$

where b_1 the residue of $f(z)$ at $z = a$ the pole of any order is the coefficient of $\dfrac{1}{t}$. Hence coefficient of $\dfrac{1}{t}$ is the residue of $f(z)$ at $z = a$, a pole of any order, in the expansion of $f(z)$ after putting $z = a + t$.

(*iv*) **Residue at infinity.** We have already defined that residue of $f(z)$ at $z = \infty$ is $\dfrac{1}{2\pi i}\int_C f(z)\, dz$ taken clockwise, (*i.e.* in negative sense) round a large circle C within which all the finite singularities are enclosed.

Putting $z = \dfrac{1}{x}$, *i.e.* $dz = -\dfrac{dx}{x^2}$ the above integral yields

$$\frac{1}{2\pi i}\int_\gamma f\left(\frac{1}{x}\right)\left(-\frac{dx}{x^2}\right),$$

taken round a small circle γ in anti-clockwise direction, (*i.e.* in positive sense).

Thus residue of $f\left(\dfrac{1}{x}\right)$ at $x = 0$ is $= \lim\limits_{x\to 0}\left[x\left\{-\dfrac{1}{x^2} f\left(\dfrac{1}{x}\right)\right\}\right]$

$$= \lim_{x \to 0}\left[-\frac{1}{x} f\left(\frac{1}{x}\right)\right]$$

if it has got a definite value.

Hence residue of $f(z)$ at $z = \infty$ is $\lim_{z \to \infty} \{-zf(z)\}$.

(v) Residue of f(z) at infinity is the negative of the coefficient of 1/z in the expansion of f(z) for values of z in the neighbourhood of z = ∞. Let $f(z)$ have a pole of order m at infinity. Then $f(1/z)$ has a pole of order m at $z = 0$. As such $f(1/z)$ can be expanded in a Laurent's series for values of z in the annulus $0 < |z| < \rho$, where ρ is small

$$f\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{m} b_n \left(\frac{1}{z}\right)^n. \quad ...(1)$$

Replacement of $1/z$ by z yields the expansion of $f(z)$ in the neighbourhood of $z = \infty$,

$$f(z) = \sum_{n=1}^{m} b_n z^n + \sum_{n=0}^{\infty} a_n z^{-n}. \quad ...(2)$$

so that the residue of $f(z)$ at infinity

$$= -\frac{1}{2\pi i}\int_C f(z)\, dz \text{ (by definition of residue)}$$

$$= -\frac{1}{2\pi i}\int_C \left[\sum_{n=1}^{m} b_n z^n + \sum_{n=0}^{\infty} a_n z^{-n}\right] dz \text{ from (2)}$$

$$= -\frac{1}{2\pi i}\int_C \frac{a_1}{z}\, dz, \text{ other integrals vanish, being of the form } \int_C \frac{dz}{z^p},\ p \neq 1$$

$$= -\frac{1}{2\pi i} a_1 \int_C \frac{dz}{z}$$

$$= -\frac{1}{2\pi i} a_1 \cdot 2\pi i \quad \because \int_C \frac{dz}{z} = 2\pi i$$

$$= -a_1$$

= the negative of the coefficient of $1/z$ in the expansion of $f(z)$ for values of z in the neighbourhood of $z = \infty$.

5.18. INTEGRATION ROUND THE UNIT CIRCLE

It is notable that in case of complex integration a curve is generally known as a contour and the process of integration along a contour is known as contour integration.

Consider $\int_0^{2\pi} \phi(\cos\theta, \sin\theta)\, d\theta$, where $\phi(\cos\theta, \sin\theta)$ is a rational function of $\sin\theta$ and $\cos\theta$.

Let us draw a circle of unit radius and take its centre as origin. Let us now consider a point P on its circumference with complex co-ordinate $z = e^{i\theta}$.

$$\therefore\ dz = ie^{i\theta} d\theta = iz\, d\theta \text{ and } \frac{1}{z} = e^{i\theta}$$

i.e.
$$d\theta = \frac{dz}{iz}.$$

Also $z+\frac{1}{z}=2\cos\theta$

and $z-\frac{1}{z}=2i\sin\theta.$

Substituting these values in the given integral,

$$\int_0^{2\pi}\phi(\cos\theta,\sin\theta)\,d\theta$$

$$=\frac{1}{i}\int_C\phi\left[\frac{z+\frac{1}{z}}{2},\frac{z-\frac{1}{z}}{2i}\right]\cdot\frac{dz}{z}$$

$$=\frac{1}{i}\int_C\frac{F(z)}{z}\,dz,$$

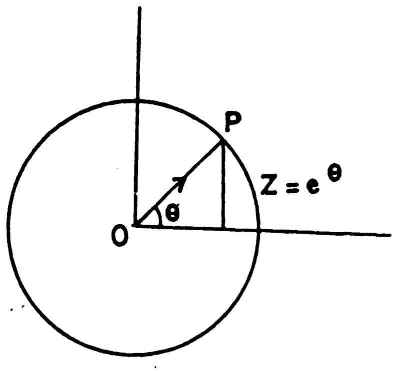


Fig. 5.26

which may be integrated, thereby giving the value of the required complex integral.

5.19. EVALUATION OF $\int_{-\infty}^{\infty} f(x)\,dx$.

Let $f(z)$ be a function such that

(*i*) $f(z)$ is analytic throughout the upper half plane except at certain points which are its poles.

(*ii*) $f(z)$ has no poles on the real ... *i.e.*, if $R\to\infty$ (R being radius of semi-circle), then it will cover entire upper half plane.

(*iii*) $zf(z)\to 0$ uniformly as $|z|\to\infty$ for $0\le\arg z\le\pi$.

(*iv*) $\int_0^{\infty} f(x)\,dx$ and $\int_{-\infty}^{0} f(x)\,dx$ both converge, then $\int_{-\infty}^{\infty} f(x)\,dx=2\pi i\Sigma R^+$,

where ΣR^+ denotes the sum of the residues of $f(z)$ at its poles in the upper half plane.

Choosing a semi-circle as contour let the origin be the centre of the semi-circle and R its radius in the upper half plane. Let the semi-circle be represented by Γ and R be chosen large enough in order that the semi-circle may include all the poles of $f(z)$.

If $f(z)$ has no poles on the boundary and we consider the closed contour C, then by residue theorem, we have

$$\int_C f(z)\,dz=\int_{-R}^{R} f(x)\,dx+\int_\Gamma f(z)\,dz,$$

since along the axis of x, $z=x$, $\therefore\ f(z)=f(x)$.

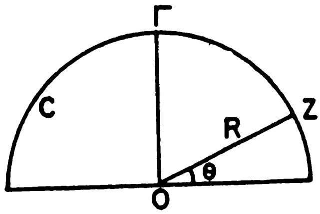


Fig. 5.27

Assuming $z=Re^{i\theta}$, $dz=iRe^{i\theta}\,d\theta=iz\,d\theta$ we have

$$\int_\Gamma f(z)\,dz=i\int_0^{\pi} f(z)\,z\,d\theta$$

$$=0 \text{ since } \left|\int_0^{\pi} f(z)\,z\,d\theta\right|\le\int_0^{\pi}|f(z)|\,|z|\,d\theta$$

$$\le 0 \text{ as } z\to\infty,\ zf(z)\to 0$$

$$=0 \text{ as modulus cannot be -ve.}$$

Hence by Cauchy's Residue theorem,

$$2\pi i\Sigma R^+ = \int_C f(z)\,dz$$

$$= \int_{-R}^{R} f(x)\ dx,\ \because \int_C f(z)\ dz = \int_{-R}^{R} f(x)\ dx$$

$$= \int_{-\infty}^{\infty} f(x)\ dx,\ \text{as } R \to \infty.$$

Hence $$\int_{-\infty}^{\infty} f(x)\ dx = 2\pi i \sum R^+.$$

Now $\int_{-\infty}^{\infty} f(x)\ dx = \int_0^{\infty} f(x)\ dx + \int_{-\infty}^{0} f(x)\ dx,$

where both the integrals on R.H.S. individually converge and hence the result.

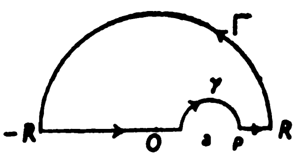


Fig. 5.28

Note 1. This theorem utilised for evaluation of the integral of the type $\int_{-\infty}^{\infty} f(x)\,dx$ by the method of contour integration, may be extended to the case in which $f(z)$ has simple poles on the real axis. In such cases, we can indent the contour by making small semi-circles in the upper half plane to cut out the simple poles on the real axis.

Suppose that $f(z)$ is a quotient function of z, whose numerator and denominator are represented by $N(z)$ and $D(z)$ respectively. Then $D(z) = 0$ gives the poles. Let us assume that $D(z) = 0$ has only one real root say $z = a$, where a is real. Let the semi-circle, indenting this pole be denoted by γ. Its centre is the point $x = a$ and let its radius be ρ.

If Γ is sufficiently large to enclose all the poles of $f(z)$ in the upper half plane, then the integral round Γ tends to zero as $R \to \infty$. We therefore have by splitting up the bound (shown in the adjoining diagram), *i.e.*, $\int_C f(z)\ dz = 2\pi i\ \Sigma R^+$ as

$$\int_{-R}^{a-\rho} f(x)\,dx + \int_\gamma f(z)\,dz + \int_{a+\rho}^{R} f(x)\ dx + \int_\Gamma f(z)\ dz = 2\pi i \Sigma R^+.$$

Here $zf(z) \to 0$ as $z \to \infty$.

$\therefore$ $\int_\Gamma f(z)\ dz \to 0$ (as shown above).

Now we have to consider $\int_\gamma f(z)\ dz.$

Fig. 5.29

Let us assume that $z = a + \rho e^{i\theta}$,

where $z = a$ if $\rho \to 0$;

then $dz = \rho i e^{i\theta} d\theta = i\,(z - a)\ d\theta.$

So $\int_\gamma f(z)\ dz = i \int_\pi^0 f(z)\ (z - a)\ d\theta$

$$\therefore \lim_{\rho \to 0} \int_\gamma f(z)\ dz = i \int_\pi^0 \lim_{z \to a} (z - a) f(z)\, d\theta$$

$$= i \sum R_0 \int_\pi^0 d\theta = -i\pi \sum R_0,$$

where R_0 is the residue of $z = a$, the simple pole on real axis.

Hence, when $R \to \infty$, we have

$$\int_{-\infty}^{a-\rho} f(x)\,dx + \int_{a+\rho}^{\infty} f(x)\,dx = 2\pi i \Sigma R^+ + i\pi \Sigma R_0,$$

where ΣR_0 denotes the sum of residues of $f(z)$ at its simple poles on the real axis, for each pole on the real axis can be treated similarly.

The left hand side of this equation is known as Principal value for dz from $-\infty$ to ∞ and denoted by P, so that

$$P\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \Sigma R^+ + i\pi\,\Sigma R_0,$$

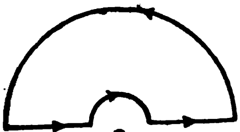


Fig. 5.30

where $\lim\limits_{\rho \to 0}\left[\int_{-\infty}^{a-\rho} f(x)\,dx + \int_{a+\rho}^{\infty} f(x)\,dx\right] = P\int_{-\infty}^{\infty} f(x)\,dx.$

Note 2. **Indented Semi-circular Contour.** *When $f(z)$ has simple poles on the real axis, we then indent the contour by making small semi-circles in the upper half-plane in order to cut out the simple poles on the real axis.*

5.20. JORDAN'S LEMMA TO EVALUATE INFINITE INTEGRALS

If Γ be a semi-circle with centre as origin and radius R and let $f(z)$ be a function such that

(*i*) $f(z)$ is analytic in the upper half plane,

(*ii*) $f(z) \to 0$ uniformly as $|z| \to \infty$ for $0 \le \arg z \le \pi$,

(*iii*) m is + ve,

then $$\int_\Gamma e^{miz} f(z) \to 0 \text{ as } R \to \infty.$$

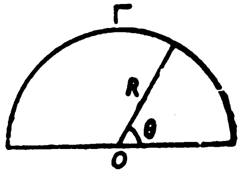


Fig. 5.31

Let us assume that $|f(z)| = \varepsilon$ when z is sufficiently large and $\varepsilon \to 0$ as $z \to \infty$.

Also, let $z = Re^{i\theta}$ and $dz = iRe^{i\theta} d\theta$

$$\therefore \left|\int_\Gamma f(z)\, e^{miz}\, dz\right| = \left|\int_0^\pi e^{imRe^{i\theta}} f(Re^{i\theta})\, iRe^{i\theta}\, d\theta\right|$$

$$\le \int_0^\pi |e^{imRe^{i\theta}}|\,|f(Re^{i\theta})|\,|i|\,R\,|e^{i\theta}|\,d\theta$$

$$\le \varepsilon\int_0^\pi |e^{imRe^{i\theta}}|\,R\,d\theta, \text{ since } |e^{i\theta}| = 1,\ |i| = 1,$$

also $|f(Re^{i\theta})| = \varepsilon$.

Now consider that $e^{imRe^{i\theta}} = e^{imR(\cos\theta + i\sin\theta)}$.

$\therefore\ |e^{imRe^{i\theta}}| = |e^{-mR(\sin\theta)}|\,|e^{imR\cos\theta}| = |e^{-mR\sin\theta}|$

$\because\ |e^{imR\cos\theta}| = |\cos(mR\cos\theta) + i\sin(mR\cos\theta)| = 1.$

$$\text{Thus}\left[\left|\int_\Gamma ze^{imz} f(z)\,dz\right| \le \varepsilon\int_0^\pi e^{-mR\sin\theta} R\,d\theta\right.$$

$$\left.\le \varepsilon\int_0^{\pi/2} e^{-mR\sin\theta} R\,d\theta + \int_{\pi/2}^{\pi} e^{-mR\sin\theta} R\,d\theta\right]$$

If we put $\theta = \pi - \phi$, where $\theta = \pi$, $\phi = 0$, $d\theta = -d\phi$, $\theta = \pi/2$, $\phi = \pi/2$.

then $$\int_{\pi/2}^{\pi} e^{-mR\sin\theta} R\, d\theta = \int_0^{\pi/2} e^{-mR\sin\phi} R\, d\phi$$

$$= \int_0^{\pi/2} e^{-mR\sin\theta} R\,\theta \quad \text{or replacing } \phi \text{ by } \theta.$$

Hence $$\left|\int_\Gamma e^{imz} f(z)\, dz\right| \le 2\varepsilon \int_0^{\pi/2} e^{-mR\sin\theta} R\, d\theta.$$

Now we know by Jordan's inequality that if $0 \le \theta \le \pi/2$.

$\frac{\sin\theta}{\theta} \ge \frac{2}{\pi}$ since $\frac{\sin\theta}{\theta}$ steadily decreases from 1 to $\frac{2}{\pi}$ as θ increases form 0 to $\pi/2$.

$$\therefore \qquad \sin\theta > \frac{2}{\pi}\theta.$$

As such $\quad e^{-mR\sin\theta} < e^{-mR(2/\pi)\theta}.$

Therefore

$$\left|\int_\Gamma e^{imz} f(z)\, dz\right| \le 2\varepsilon \int_0^{\pi/2} e^{-mR(\pi/2)\theta} R\, d\theta$$

$$\le 2\varepsilon \left[\frac{e^{-mR(\pi/2)\theta}.R}{-mR(2/\pi)}\right]_0^{\pi/2}$$

$$\le \frac{\pi\varepsilon}{m}\left[1 - e^{-mR}\right]$$

$$\le 0 \text{ when } z \to \infty \text{ and } \varepsilon \to 0.$$

Hence $\int_\Gamma e^{imz} f(z)\, dz \to 0$ as $R \to \infty$.

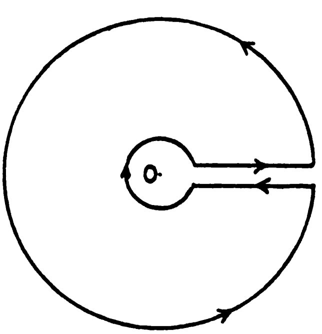


Fig. 5.32

Note 1. **Integrals involving many-vlaued functions.** *If $z^{\alpha-1}$ be many-valued function, then an integral of the type $\int_0^\infty x^{\alpha-1} f(x)\, dx$, where α is not integer can be evaluated by contour integration by taking the integral round a circle $|z| = R$ and a small circle $|z| = r$, which encloses the branch point $z = 0$ and a cross cut along the real axis from 0 to ∞, i.e. joining the ends of the two circles.*

Note 2. There is no special merit in using a particular curve as contour for a particular integration, but in fact any of circle, semi-circle, quadrant of a circle or a rectangle whichever is suitable, can be used as a contour unless otherwise stated, e.g. if the integrand is a periodic function, then rectangualr contour is rather convenient.

Problem 18. *Find the residue of (a)* $\frac{z^2}{z^2 + a^2}$ *at* $z = ia$, *(b)* $\frac{ze^{iz}}{z^4 + a^4}$ *at its poles; and*

(Rohilkhand, 1981)

(c) $\frac{z}{(z-a)(z-b)}$ *at infinity.* *(Rohilkhand, 1992)*

(*a*) As given $$f(z)=\frac{z^2}{z^2+a^2}=\frac{z^2}{(z-ia)(z+ia)}.$$

Here $z = ia$ is a pole of order one or say a simple pole.

Residue at $z = ia$ is $= \lim_{z\to ia}(z-ia)f(z)$

$$=\lim_{z\to ia}(z-ia)\cdot\frac{z^2}{(z+ia)(z-ia)}$$

$$=\lim_{z\to ia}\frac{z^2}{z+ia}=\frac{(ia)^2}{2ai}=\frac{ia}{2}.$$

Aliter 1. $f(z)=\frac{z^2}{z^2+a^2}$ is of the form $\frac{\phi(z)}{\psi(z)}$, where $\phi(z)=z^2$, $\psi(z)=z^2+a^2$.

$\therefore$ residue at $z = ia$ is $=\frac{\phi(ia)}{\psi'(ia)}$, where $\phi(ia)=(ai)^2$, $\psi'(ia)=2ai$

$$=\frac{(ai)^2}{2ai}=\frac{ai}{2}.$$

Aliter 2. Put $z = ia + t$.

$$\therefore f(ia+t)=\frac{(ia+t)^2}{(2ia+t)\,t}$$

$$=\frac{(ia)^2\left(1+\frac{t}{ia}\right)^2}{2iat\left(1+\frac{t}{2ia}\right)}=\frac{ia}{2t}\left(1+\frac{t}{ia}\right)^2\left(1+\frac{t}{2ia}\right)^{-1},$$

$$=\frac{ia}{2t}\left[1+\frac{2t}{ia}+\left(\frac{t}{ia}\right)^2\right]\left[1-\frac{t}{2ia}+\left(\frac{t}{2ia}\right)^2+\ldots\right]$$

$$=\frac{ia}{2t}\left[1+\left(\frac{2}{ia}-\frac{1}{2i}\right)t+\ldots\right]$$

Residue at $z = ia$ is $=$ coefficient of $\frac{1}{t}$ in $f(ia+t)=\frac{ia}{2}.$

(*b*) Here $f(z)=\frac{z\,e^{iz}}{z^4+a^4}$, whose poles are given by

$z^4+a^4=0 \Rightarrow z^4=-a^4=e^{i(2n-1)\pi}\cdot a^4$

$\Rightarrow z=a\,e^{i(2n-1)\pi/4}$, $n = 1, 2, 3, 4$

$\Rightarrow z=ae^{i\pi/4}, ae^{i3\pi/4}, ae^{i5\pi/4}, ae^{i7\pi/4}$

Residue of $f(z)$ at $z=ae^{i\pi/4}=\left[\frac{\phi(z)}{\psi'(z)}\right]_{z=ae^{i\pi/4}}$

$$=\left[\frac{ze^{iz}}{4z^3}\right]_{z=ae^{i\pi/4}}=\frac{ae^{i\pi/4}\left(e^{iae^{i\pi/4}}\right)}{4a^2e^{i3\pi/4}}=\frac{e^{a/\sqrt{2}(-1+i)}}{4ia^2}$$

Similarly other residues are

$$-\frac{e^{a/\sqrt{2}(1+i)}}{4ia^2},\quad \frac{e^{-a/\sqrt{2}(-1+i)}}{4ia^2},\quad \frac{e^{a/\sqrt{2}(1+i)}}{4ia^2}$$

(*c*) Here $f(z) = \dfrac{z}{(z-a)(z-b)}$

$$\underset{z\to\infty}{\text{Res}} f(z) = \underset{z\to\infty}{\text{Lim}} - zf(z)$$

$$= \underset{z\to\infty}{\text{Lim}}\left[-\frac{z^2}{(z-a)(z-b)}\right]$$

$$= \underset{z\to\infty}{\text{Lim}}\left[-\frac{1}{\left(1-\frac{a}{z}\right)\left(1-\frac{b}{z}\right)}\right] = -1$$

Problem 19. (*a*) *Find the residue of* $\dfrac{z^4}{(z-1)^4(z-2)(z-3)}$ *at* $z = 1$.

(*b*) *Find the residues of* $\dfrac{z}{(z-a)(z-b)}$ *and* $\dfrac{z^3 - z^2 + 1}{z^3}$ *at infinity.*

(*a*) Here $f(z) = \dfrac{z^4}{(z-1)^4(z-2)(z-3)}$.

To find residue at $z = 1$, which is a pole of order 4.

$$f(z) = \frac{1}{(z-1)^4}\cdot\frac{z^4}{(z-2)(z-3)} = \frac{\phi(z)}{(z-1)^4},$$

where $\phi(z) = \dfrac{z^4}{(z-2)(z-3)} = z^2 + 5z + 19 - \dfrac{16}{z-2} + \dfrac{81}{z-3}$

(breaking into partial fractions).

$$\therefore \quad \phi'(z) = 2z + 5 + \frac{16}{(z-2)^2} - \frac{81}{(z-3)^2},$$

$$\phi''(z) = 2 - \frac{32}{(z-2)^3} + \frac{162}{(z-3)^3}.$$

$$\therefore \quad \phi'''(z) = \frac{96}{(z-2)^4} - \frac{486}{(z-3)^4}.$$

Thus $\phi'''(1) = \dfrac{96}{(-1)^4} - \dfrac{486}{(-2)^4} = 96 - \dfrac{243}{8} = \dfrac{525}{8}$.

The residue at $z = 1$ is $= \dfrac{\phi'''(1)}{\lfloor 3} = \dfrac{175}{16}$.

(*b*) Both the functions are analytic at infinity.

(*i*) $$f(z) = \frac{z}{(z-a)(z-b)}$$

Residue of $f(z)$ at $z = 0 = \underset{z\to\infty}{\text{Lim}}\ \{-zf(z)\}$

$$= \lim_{z \to \infty} \left\{ -\frac{z^2}{(z-a)(z-b)} \right\}$$

$$= \lim_{z \to \infty} \left\{ \frac{1}{\left(1-\frac{a}{z}\right)\left(1-\frac{b}{z}\right)} \right\}$$

$$= -1.$$

(*ii*) $$f(z) = \frac{z^3 - z^2 + 1}{z^3}$$

Here $\lim_{z \to \infty} \{-zf(z)\}$ does not exist.

We can write,

$$f(z) = 1 - \frac{1}{z} + \frac{1}{z^3}.$$

Residue of $f(z)$ at $z = \infty$

= the negative of the coefficient of $\frac{1}{z}$ in the expansion of $f(z)$ for values of z in the neighbourhood of $z = \infty$

$= -(-1) = 1.$

Problem 20. *Apply Calculus of residues to prove that*

(*i*) $\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta} = \frac{2\pi}{\sqrt{3}}.$ (*Rohilkhand, 1989; Meerut, 1985; Vikram, 1969*)

(*ii*) $\int_0^{2\pi} \frac{\sin^2\theta \, d\theta}{a + b\cos\theta} = \frac{2\pi}{b^2}\left\{a - \sqrt{(a^2 - b^2)}\right\}$, *if* $a > b > 0$.

(*Rohilkhand, 1980, Agra, 1976, 78*)

(*iii*) $\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta - n\theta)\, d\theta = \frac{2\pi}{\lfloor n}$ *where n is a + ive integer.*

(*Rohilkhand, 1988; Meerut, 1982, 87; Agra, 1975*)

(*iv*) $\int_0^{\pi} \frac{a\, d\theta}{a^2 + \sin^2\theta} = \frac{\pi}{\sqrt{(1 + a^2)}}$, $a > 0$.

(*Rohilkhand, 1987; Meerut, 1980; Agra, 1983*)

(*v*) $\int_0^{2\pi} \frac{\cos^3 3\theta \, d\theta}{1 - 2p\cos 2\theta + p^2} = \pi . \frac{1 - p + p^2}{1 - p}$, $(0 < p < 1)$.

(*vi*) $\int_0^{2\pi} \frac{d\theta}{(5 - 4\cos\theta)^2} = \frac{40}{27}\pi$, *and* (*vii*) $\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta}\, d\theta = \frac{\pi}{6}$

(*i*) Suppose $I = \int_0^{2\pi} \frac{d\theta}{2 + \cos\theta}$. (*Agra, 1964*)

Choosing the contour as a circle of unit radius ($|z| = 1$),

$z = e^{i\theta}$; $\therefore dz = ie^{i\theta} d\theta = iz\, d\theta$,

so that $d\theta = \frac{dz}{iz}$, where θ varies from 0 to 2π.

Also when $z = e^{i\theta} = \cos\theta + i\sin\theta$.

$$\frac{1}{z} = \cos\theta - i\sin\theta.$$

Adding, $z + \frac{1}{z} = 2\cos\theta$, *i.e.* $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$.

Thus, $$I = \int_C \frac{dz}{iz\left\{2 + \frac{1}{2}\left(z + \frac{1}{z}\right)\right\}} = \frac{2}{i}\int_C \frac{dz}{z^2 + 4z + 1},$$

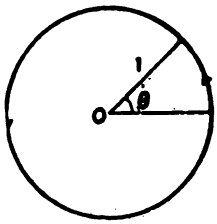


Fig. 5.33

where C denotes the unit circle $|z| = 1$.

Poles of the integrand will be given by $z^2 + 4z + 1 = 0$

or $$z = \frac{-4 \pm \sqrt{(16-4)}}{2} = -2 \pm 3,$$

of which the pole $z = -2 - \sqrt{3}$ lies outside the contour and therefore the only pole that lies within the contour is $z = -2 + \sqrt{3}$ (which is of the order one).

$\therefore$ residue at $z = -2 + \sqrt{3}$ is $\lim_{z \to -2+\sqrt{3}} \{z - (-2 + \sqrt{3})\} f(z)$ where $f(z) = \frac{2}{i(z^2 + 4z + 1)}$

$$= \frac{2}{i} \lim_{z \to -2+\sqrt{3}} \left(z + 2 - \sqrt{3}\right) \cdot \frac{1}{\left(z^2 + 4z + 1\right)}$$

$$= \frac{2}{i} \lim_{z \to -2+\sqrt{3}} \frac{z + 2 - \sqrt{3}}{\left(z + 2 - \sqrt{3}\right)\left(z + 2 + \sqrt{3}\right)}$$

$$= \frac{2}{i} \lim_{z \to -2+\sqrt{3}} \left(\frac{1}{z + 2 + \sqrt{3}}\right)$$

$$= \frac{2}{i} \cdot \frac{1}{-2 + \sqrt{3} + 2 + \sqrt{3}} = \frac{1}{i\sqrt{3}}.$$

Hence by Cauchy's residue theorem we have

$I = 2\pi i \sum R$, where $\sum R$ represents sum of the residues inside the contour

$$= 2\pi i \times \frac{1}{i\sqrt{3}} = \frac{2\pi}{\sqrt{3}}.$$

(*ii*) Suppose $I = \int_0^{2\pi} \frac{\sin^2\theta \, d\theta}{a + b\cos\theta}$

Choosing the contour C as a circle ($|z| = 1$) of unit radius,

$z = e^{i\theta}$ so that $z + \frac{1}{z} = 2\cos\theta$.

$dz = iz\,d\theta \qquad z - \frac{1}{z} = 2i\sin\theta,$

$$\therefore I = \frac{1}{i}\int_C \frac{\left\{\frac{1}{2i}\left(z - \frac{1}{z}\right)\right\}}{a + \frac{b}{2}\left(z + \frac{1}{z}\right)} \frac{dz}{z} = -\frac{1}{i}\int_C \frac{\left(z - \frac{1}{z}\right)^2}{2z\left\{2a + b\left(z + \frac{1}{z}\right)\right\}} dz$$

$$= -\frac{1}{2ib}\int_C \frac{\left(z^2 - 1\right)^2 dz}{z^2\left(2az + bz^2 + b\right)} = -\frac{1}{2ib}\int_C \frac{\left(z^2 - 1\right)^2 dz}{z^2\left(z^2 + 1 + \frac{2a}{b}z\right)}.$$

$$= -\frac{1}{2ib}\int_C \frac{(z^2-1)^2\,dz}{z^2(z-\alpha)(z-\beta)},$$

where $\alpha = \frac{-a+\sqrt{(a^2-b^2)}}{b}$ and $\beta = \frac{-a-\sqrt{(a^2-b^2)}}{b}$ are the roots of the quadratic $z^2+\frac{2a}{b}z+1=0.$

So $\qquad \alpha+\beta = -\frac{2a}{b}$, and $\alpha\beta = 1$.

As such $|\alpha|\,|\beta| = 1$, when $|\beta| > |\alpha|$

$\therefore\ |\beta| = \frac{1}{|\alpha|} > 1$, *i.e.*, $|a| < 1$ and so $z = \alpha$ is the only simple pole within the contour C. $z = \beta$ lies outside the contour. Also $z = 0$ is a pole of order two, which lies inside the contour.

Residue at $z = \alpha$ is $\lim_{z\to\alpha}(z-\alpha)\cdot\left(-\frac{1}{2ib}\right)\frac{(z^2-1)^2\,dz}{z^2(z-\alpha)(z-\beta)}$

$$= -\frac{1}{2ib}\lim_{z\to\alpha}\frac{(z^2-1)^2}{z^2(z-\beta)} = -\frac{1}{2ib}\lim_{z\to\alpha}\frac{\left(z-\frac{1}{z}\right)^2}{z-\beta}$$

$$= -\frac{1}{2ib}\frac{\left(\alpha-\frac{1}{\alpha}\right)^2}{\alpha-\beta} = \frac{(\alpha-\beta)^2}{-2ib(\alpha-\beta)} \text{ as } \alpha\beta=1,\ \therefore\ \beta=\frac{1}{\alpha}$$

$$= \frac{\alpha-\beta}{-2ib}, \text{ where } (\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta\cdot$$

$$= \frac{4a^2}{b^2} - 4 = \frac{4}{b^2}(a^2-b^2)$$

$$= \frac{1}{-2ib}\cdot\frac{2}{b}\sqrt{(a^2-b^2)}$$

$$= \frac{\sqrt{(a^2-b^2)}}{-ib^2}.$$

Again residue at $z = 0$ (a pole of order two) is the coefficient of $\frac{1}{z}$ in

$-\frac{1}{2ib}\frac{(z^2-1)^2}{z^2\left(z^2+1+\frac{2a}{b}z\right)}$, where z is a small quantity near the pole,

i.e. in $-\frac{1}{2ib}\frac{\left(1-\frac{1}{z^2}\right)^2}{\left(1+\frac{2a}{b}\cdot\frac{1}{z}+\frac{1}{z^2}\right)} = -\frac{1}{2ib}\left(1-\frac{1}{z^2}\right)^2\left(1+\frac{2a}{bz}+\frac{1}{z^2}\right)^{-1}$, *i.e.* in $-\frac{1}{2ib}\left(1-\frac{2}{z^2}+\ldots\right)\left[1-\left(\frac{2a}{bz}+\frac{1}{z^2}\right)+\ldots\right]$.

$\therefore$ residue at $z=0$ is $=-\frac{1}{2ib}\cdot\left(-\frac{2a}{b}\right)=\frac{a}{ib^2}$.

Hence by Cauchy's Residue theorem,

$I=2\pi i\Sigma R$

$$=2\pi i\left[-\frac{\sqrt{(a^2-b^2)}}{ib^2}+\frac{a}{ib^2}\right]=\frac{2\pi}{b^2}\left[a-\sqrt{(a^2-b^2)}\right].$$

(*iii*) Suppose $I=\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta-n\theta)\,\theta$.

Choosing the contour C as a circle of unit radius, $|z|=1$.

$$z=e^{i\theta},\text{ so that }\frac{1}{z}=e^{-i\theta},$$

$$dz=iz\,d\theta.$$

Now, consider,

$$J=\int_0^{2\pi} e^{\cos\theta}[\cos(\sin\theta-n\theta)+i\sin(\sin\theta-n\theta)]\,d\theta$$

$$=\int_0^{2\pi} e^{\cos\theta}\cdot e^{i(\sin\theta-n\theta)}\,.d\theta$$

or
$$J=\int_0^{2\pi} e^{\cos\theta+i\sin\theta}\cdot e^{-in\theta}\,d\theta=\int_0^{2\pi} e^{e^{i\theta}}\cdot(e^{-i\theta})^n\,d\theta,$$

where I is the real part in J.

On putting the values of $e^{i\theta}=z$ and $e^{-i\theta}=\frac{1}{z}$ etc., we have

$$J=\int_C e^z\frac{1}{z^n}\frac{dz}{iz}=\frac{1}{i}\int_C\frac{e^z}{z^{n+1}}\,dz.$$

Here the only pole inside the contour C is $z=0$ of order $(n+1)$.

Residue at $z=0$ a pole of order $(n+1)$ is $=\frac{\phi^n(0)}{\lfloor n}=\frac{1}{i\lfloor(n)}$,

, where $\phi(z)=\frac{1}{i}e^z$, $\therefore$ $\phi^n(z)=\frac{e^z}{i}$, $\phi^n(0)=\frac{1}{i}$.

Hence by Cauchy's Residue theorem,

$J=2\pi i\Sigma R$, where ΣR is the sum of residues inside C

$$=\pi i.\ \frac{1}{i\lfloor(n)}=\frac{2\pi}{\lfloor n}.$$

Therefore real part in J, *i.e.*

$$I=\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta-n\theta)\,d\theta=\frac{2\pi}{\lfloor n}.$$

(*iv*) Suppose $I = \int_0^{\pi} \dfrac{a\,d\theta}{a^2 + \sin^2\theta}$.

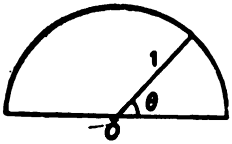


Fig. 5.34

Choosing the contour C as a semi-circle of unit radius.

$$z = e^{i\theta}, \therefore dz = iz\,d\theta$$

or $\quad d\theta = \dfrac{dz}{iz}$ and $\sin\theta = \dfrac{1}{2i}\left(z - \dfrac{1}{z}\right)$,

where θ varies from 0 to π.

$$\therefore I = \frac{a}{i}\int_C \frac{1}{a^2 + \left\{\frac{1}{2i}\left(z - \frac{1}{z}\right)\right\}^2} \cdot \frac{dz}{iz} = \frac{4a}{i}\int_C \frac{dz}{z\left\{4a^2 - z^2 - \frac{1}{z^2} + 2\right\}}$$

$$= \frac{4a}{i}\int_C \frac{z\,dz}{z^2\left(4a^2 + 2\right) - z^4 - 1} = -\frac{4a}{i}\int_C \frac{z\,dz}{-z^2\left(4a^2 + 2\right) + z^4 + 1}$$

Poles are given by $-z^2(4a^2 + 2) + z^4 + 1 = 0$.

Let us assume that

$$z^4 - z^2(4a^2 + 2) + 1 = (z^2 - \alpha^2)(z^2 - \beta^2).$$

$\therefore \alpha^2 + \beta^2 = 4a^2 + 2$ and $\alpha^2\beta^2 = 1$.

If $\alpha < 1$, then $\beta > 1$, implies that the pole within the semi-circle will be $z = \alpha$, which is the only pole inside C and $z = \beta$ will lie outside the contour.

Residue at $z = \alpha$ is

$$= \lim_{z \to \alpha} (z - \alpha)\left(-\frac{4a}{i}\right) \cdot \frac{z}{\left(z^2 - \alpha^2\right)\left(z^2 - \beta^2\right)}$$

$$= -\frac{4a}{i} \lim_{z \to \alpha} \frac{z}{(z + \alpha)\left(z^2 - \beta^2\right)}$$

$$= -\frac{4a}{i} \frac{z}{2\alpha \cdot \left(\alpha^2 - \beta^2\right)} = -\frac{2a}{i\left(\alpha^2 - \beta^2\right)}$$

Hence by Cauchy's Residue theorem,

$$I = 2\pi i \Sigma R = 2\pi i \times \frac{2a}{-i\left(\alpha^2 - \beta^2\right)}$$

$$= \frac{4\pi a}{\beta^2 - \alpha^2}, \text{ where } \beta^2 - \alpha^2 = \sqrt{\left[(\beta^2 + \alpha^2)^2 - 4\alpha^2\beta^2\right]}$$

$$= \sqrt{\left[\left(4a^2 + 2\right)^2 - 4\right]} = 4a\sqrt{\left(1 + a^2\right)}$$

$$= \frac{4\pi a}{4a\sqrt{\left(1 + a^2\right)}} = \frac{\pi}{\sqrt{\left(1 + a^2\right)}}.$$

(v) Suppose $I = \int_0^{2\pi} \frac{\cos^2 3\theta \, d\theta}{1 - 2p\cos 2\theta + p^2}$.

Choosing the contour C as a circle of unit radius,

$$z = e^{i\theta}; \quad \therefore \; dz = ie^{i\theta}\,d\theta = iz\,d\theta,$$

so that $$d\theta = \frac{dz}{iz}.$$

Also $$z^n = e^{ni\theta}; \quad \therefore \; \frac{1}{z^n} = e^{-ni\theta}.$$

As such, $2\cos 3\theta = z^3 + \frac{1}{z^3}$ and $2\cos 2\theta = z^2 + \frac{1}{z^2}$.

$$\therefore \; I = \frac{1}{i}\int_C \frac{\frac{dz}{z}\left\{\frac{1}{2}\left(z^3 + \frac{1}{z^3}\right)\right\}^2}{1 - p.\left(z^2 + \frac{1}{z^2}\right) + p^2} = \frac{1}{i}\int_C \frac{\left(z^6+1\right)^2 z^2}{4z^6\left\{\left(1+p^2\right)z^2 - pz^4 - p\right\}}\frac{dz}{z}$$

$$= -\frac{1}{4i}\int_C \frac{\left(z^6+1\right)^2 dz}{z^5\left[pz^4 - \left(1+p^2\right)z^2 + p\right]} = -\frac{1}{4i}\int_C \frac{\left(z^6+1\right)^2 dz}{4z^5\left(pz^2-1\right)\left(z^2-p\right)}.$$

Poles are $z = 0$, a pole of order five, and

$$z = \pm\sqrt{p}, \; \pm\frac{1}{\sqrt{p}}$$

If $p < 1$, then $\frac{1}{p} > 1$; so the only poles within the contour are $z = \pm\sqrt{p}$ and also $z = 0$, a pole of order five.

Residue at $z = 0$ (pole of order 5) is the coefficient of $1/z$ in $\frac{\left(z^6+1\right)^2}{z^5\left(pz^2-1\right)\left(z^2-p\right)}$

$$= \text{coeff. of } \frac{1}{z} \text{ in } \frac{\left(1 + 2z^6 + z^{12}\right)}{-4iz^5\left(1-p^2\right)}\left[\frac{p}{pz^2-1} - \frac{1}{z^2-p}\right]$$

$$= \;'' \quad '' \quad \frac{\left(1 + 2z^6 + z^{12}\right)}{-4iz^5\left(1-p^2\right)}\left[\frac{1}{p\left(1 - \frac{z^2}{p}\right)} - \frac{p}{1-pz^2}\right]$$

$$= \;'' \quad '' \quad \frac{1}{z}\text{in } \frac{1}{-4i\left(1-p^2\right)}\cdot\frac{1 + 2z^6 + z^{12}}{z^5} \times \left[\frac{1}{p}\left(1 - \frac{z^2}{p}\right)^{-1} - p\left(1 - pz^2\right)^{-1}\right]$$

$$= \;'' \quad '' \quad \frac{1}{-4i\left(1-p^2\right)}\cdot\frac{1 + 2z^6 + z^{12}}{z^5}$$

$$\times\left[\frac{1}{p}\left(1+\frac{z^2}{p}+\frac{z^4}{p^2}+\ldots\right)-p\left(1+pz^2+p^2z^4+\ldots\right)\right]$$

$$=\frac{1}{-4i\left(1-p^2\right)}\left(\frac{1}{p^3}-p^3\right)=\frac{1-p^6}{-4ip^3\left(1-p^2\right)}.$$

Residue at $z=\sqrt{p}$ is

$$=\frac{-1}{4i}\lim_{z\to\sqrt{p}}\left(z-\sqrt{p}\right)\frac{\left(z^6+1\right)^2}{z^5\left(pz^2-1\right)\left(z^2-p\right)}$$

$$=-\frac{1}{4i}\lim_{z\to\sqrt{p}}\frac{\left(z^6+1\right)^2}{z^5\left(pz^2-1\right)\left(z+\sqrt{p}\right)}$$

$$=-\frac{1}{4i}\;\frac{\left(p^3+1\right)^2}{p^{5/2}\left(p^2-1\right)\cdot 2\sqrt{p}}=\frac{\left(p^3+1\right)^2}{8ip^3\left(1-p^2\right)}.$$

Similarly, Residue at $z=-\sqrt{p}$ is $=\dfrac{\left(p^3+1\right)^2}{8ip^3\left(1-p^2\right)}.$

Hence by Cauchy's Residue theorem,

$$I=2\pi i\,\Sigma R=\frac{2\pi i}{4ip^3\left(1-p^2\right)}\left[-\left(1-p^6\right)+\frac{\left(p^3+1\right)^2}{2}+\frac{\left(p^3+1\right)^2}{2}\right]$$

$$=\frac{\pi}{2p^3\left(1-p^2\right)}\left[-1+p^6+p^6+2p^3+1\right]$$

$$=\frac{\pi}{2p^3\left(1-p^2\right)}\cdot 2p^3\left(1+p^3\right)=\pi.\frac{(1+p)\left(1-p+p^2\right)}{(1-p)\,(1+p)}$$

$$=\pi\;\frac{\left(1-p+p^2\right)}{1-p}$$

(*vi*) Suppose $I=\int_0^{2\pi}\dfrac{d\theta}{(5-4\cos\theta)^2}$

Choosing the contour C as a circle of unit radius ($|z|=1$).

$$z=e^{i\theta},\; dz=iz\,dz \text{ or } d\theta=\frac{dz}{iz}$$

Also $\dfrac{1}{z}=e^{-i\theta}$; $\quad\therefore\quad z+\dfrac{1}{z}=2\cos\theta.$

Thus, $\quad I=\dfrac{1}{i}\displaystyle\int_C\frac{dz}{z\left\{5-2\left(z+\frac{1}{z}\right)\right\}^2}=\frac{1}{i}\int_C\frac{z\,dz}{(2z^2-5z+2)^2}$

$$=\frac{1}{i}\int_C\frac{z\,dz}{(z-2)^2(2z-1)^2}$$

Poles are $z = 2$, a pole of order 2 and $z = \frac{1}{2}$, also a pole of order 2.

But the only pole that lies inside the contour is $z = \frac{1}{2}$ of order 2.

Residue at $z = \frac{1}{2}$ (Pole of order 2) is $= \frac{1}{i}\frac{\phi'\left(\frac{1}{2}\right)}{\underline{|1}} = \frac{20}{27i}$,

where $\phi(z) = \frac{z}{(z-2)^2}$, $\phi'(z) = \frac{(z-2)^2 - 2z(z-2)}{(z-2)^4}$, $\phi'\left(\frac{1}{2}\right) = \frac{\frac{9}{4}+\frac{3}{2}}{\frac{81}{16}} = \frac{20}{27}$

Hence by Cauchy's Residue theorem,

$$I = 2\pi i \Sigma R$$

$$= 2\pi i \times \frac{20}{27i} = \frac{40}{27}\pi.$$

Note. The similar procedure will show that

$$\int_0^{2\pi} \frac{d\theta}{(5+4\cos\theta)^2} = \frac{10\pi}{7}$$

and in general $\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \frac{2\pi a}{\left(a^2-b^2\right)^{3/2}}$, $a > b > 0$.

(*vii*) Suppose $I = \int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta$

Choosing the contour C as a circle of unit radius *i.e.* $|z| = 1$, we put, $z = e^{i\theta}$, $dz = iz\, d\theta$ or $d\theta = \frac{dz}{iz}$

Then $I = \int_0^{2\pi} \frac{2\cos^2\theta - 1}{5+4\cos\theta} d\theta = \int_C \frac{2(z+1/z)-1}{5+2(z+1/z)}\frac{dz}{iz} = \frac{1}{2i}\int_C \frac{z^4+1}{z^2(2z+1)(z+2)} dz$

$\therefore$ Within the contour C there are a smple pole at $z = -\frac{1}{2}$ and a pole of order 2 at $z = 0$.

Residue at $z = -\frac{1}{2}$, $= \lim_{z\to -\frac{1}{2}} \left(z+\frac{1}{2}\right) \frac{1}{2i} \frac{z^4+1}{z^2(2z+1)(z+2)} = \frac{17}{24}i$

and Residue at $z = 0$ (order 2) $=$ coefficient of $\frac{1}{z}$ in $\frac{1}{2i}\frac{z^4+1}{z^2(2z+1)(z+2)}$, z being small

$$= \text{coeffi. of } \frac{1}{z} \text{ in } \frac{1}{4i}\left(1+\frac{1}{z^4}\right)\left(1+\frac{1}{2z}\right)^{-1}\left(1+\frac{2}{z}\right)^{-1}$$

$$= \text{,, ,, } \frac{1}{4i}\left(1+\frac{1}{z^4}\right)\left(1-\frac{1}{2z}+\ldots\right)\left(1-\frac{2}{z}+\ldots\right)$$

$$= \frac{1}{4i}\left(-\frac{5}{2}\right) = -\frac{5}{8i}.$$

Hence by Cauchy's theorem.

$$I = 2\pi i \times \text{sum of residues} = 2\pi\Sigma R$$

$$= 2\pi i\left[\frac{17}{24i} - \frac{5}{8i}\right] = \frac{\pi}{6}.$$

Problem 21. *Prove by contour integration that*

(i) $\displaystyle\int_0^\infty \frac{dx}{\left(a+bx^2\right)^n} = \frac{\pi}{2^n\sqrt{b}} \cdot \frac{1.3.5...(2n-3)}{1.2.3...(n-1)} \cdot \frac{1}{a^{(2n-1)/2}}$

(ii) $\displaystyle\int_0^\infty \frac{x \sin ax}{x^2+k^2}\,dx = \frac{1}{2}\pi e^{-ak}$ *where* $a > 0$;

(iii) $\displaystyle\int_0^\infty \frac{x^6\,dx}{\left(a^4+x^4\right)^2} = \frac{3\sqrt{2\pi}}{16a}$ *where* $a > 0$; (iv) $\displaystyle\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$

(iv) $\displaystyle\int_0^\infty \frac{\cos x\,dx}{\left(x^2+a^2\right)\left(x^2+b^2\right)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$ *where* $a > b > 0$ *and*

(v) $\displaystyle\int_0^\infty \frac{\log\left(1+x^2\right)}{1+x^2}\,dx = \pi \log 2.$ *(Rohilkhand, 1989, 90, 92)*

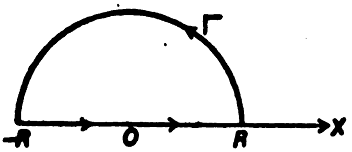


Fig. 5.35

(i) Here $f(z) = \dfrac{dz}{\left(a+bz^2\right)^n}$

Choosing the closed contour C consisting of the real axis from $-R$ to R and the upper half of a large circle $|z| = R$ represented by Γ, we have by Cauchy's residue theorem

$$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz$$

and since $\int_C f(z)\,dz = 2\pi i \Sigma R^+$, where ΣR^+ is the sum of residues,

$$\therefore \quad \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz = 2\pi i \Sigma R^+. \qquad ...(1)$$

Now poles of $f(z)$ are given by

$$\left(a+bz^2\right)^n = 0 \text{ or } b^n\left(z+i\sqrt{\frac{a}{b}}\right)^n\left(z-i\sqrt{\frac{a}{b}}\right)^n = 0.$$

i.e. $\quad z = \pm i\sqrt{\dfrac{a}{b}}$ (poles of order n).

Only the pole $z = i\sqrt{\dfrac{a}{b}}$ of order n lies within the contour.

Residue at $z = i\sqrt{\dfrac{a}{b}}$ (a pole of order n) is $= \dfrac{\phi^{n-1}\left(i\sqrt{\dfrac{a}{b}}\right)}{\underline{|(n-1)}}$

where $$\phi(z)=\frac{1}{b^n\left(z+i\sqrt{\frac{a}{b}}\right)^n},$$

$$\phi^{n-1}(z)=\frac{(-1)^{n-1}\cdot n\,(n+1)\,(n+2)\ldots(2n-2)}{b^n\left(z+i\sqrt{\frac{a}{b}}\right)^{2n-1}},$$

$$\therefore\quad \phi^{n-1}\left(i\sqrt{\frac{a}{b}}\right)=\frac{(-1)^{n-1}\cdot n\,(n+1)\,(n+2)\ldots(2n-2)}{b^n\left(2i\sqrt{\frac{a}{b}}\right)^{2n-1}},$$

Thus residue at $z=i\sqrt{\frac{a}{b}}$ (pole of order n) is

$$\frac{(-1)^{n-1}}{\lfloor(n-1)},\ \frac{1}{b^n}.\frac{n\,(n+1)\,(n+2)\ldots(2n-2)}{\left(2i\sqrt{\frac{a}{b}}\right)^{2n-1}}$$

Now $$\left|\int_\Gamma f(z)\,dz\right|=\left|\int_\Gamma\frac{dz}{(a+bz^2)^n}\right|\le\int_\Gamma\frac{|dz|}{|a+bz^2|^n},$$

which tends to zero as $z\to\infty$.

Hence when $R\to\infty$, the relation (1) yields

$$\int_{-\infty}^{\infty}\frac{dx}{(a+bx^2)^n}=2\pi i\times\frac{(-1)^{n-1}}{\lfloor(n-1)}.\frac{n(n+1)(n+2)\ldots(2n-2)}{b^n\left(2i\sqrt{\frac{a}{b}}\right)^{2n-1}}$$

$$=2\pi i\times\frac{(-1)^{n-1}(2n-2)\,!}{(n-1)\,!\,(n-1)\,!\,\sqrt{b}\left(2i\sqrt{a}\right)^{2n-1}}$$

[multiplying N^r and D^r by $(n-1)\,!$]

$$=\frac{2\pi i\times(-1)^{n-1}1.3.5\ldots(2n-3)\times2.4.6\ldots(2n-2)}{(n-1)\,!\,(n-1)\,!.\sqrt{b}\left(2i\sqrt{a}\right)^{2n-1}}$$

$$2\int_0^{\infty}\frac{dx}{(a+bx^2)^n}=\frac{2\pi}{2^{2n-1}(n-1)!}\frac{1.3.5\ldots(2n-3),\ 2^{n-1}(n-1)!}{(n-1).\sqrt{b}(a)^{n-(1/2)}}.$$

$$\therefore\quad\int_0^{\infty}\frac{dx}{(a+bx^2)^n}=\frac{\pi}{2^n\sqrt{b}}.\frac{1.3.5\ldots(2n-3)}{1.2.3.\ldots(n-1)}.\frac{1}{a^{1/2\,(2n-1)}}.$$

(*ii*) Consider $f(z)=\frac{ze^{iaz}}{z^2+k^2}$, where $\frac{z\sin az}{z^2+k^2}$ is the imaginary part of $f(z)$.

Choosing the closed contour C, which consists of real axis from $-R$ to R and upper half of a large circle $|z|=R$, represented by Γ, we have

$$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz = 2\pi i\,\Sigma R^+. \qquad ...(1)$$

Singularities of $f(z)$ are given by $z^2+k^2=0$, *i.e.* $z=\pm ik$ of which only the simple pole $z=+ik$ lies within the contour.

Residue at $z=ik$ (a simple pole) $= \lim_{z\to ik}(z-ik).\dfrac{ze^{iaz}}{(z-ik)(z+ik)}$

$$= \frac{ike^{-ak}}{2ik} = \frac{1}{2}e^{-ak}.$$

Now $\left|\int_\Gamma f(z)\,dz\right| = \left|\int_\Gamma \dfrac{ze^{iaz}\,dz}{z^2+k^2}\right| \le \int_\Gamma \dfrac{|z|\,|e^{iaz}|\,|dz|}{|z^2+k^2|} \le 0$ as $z\to\infty$

$\therefore\ \int_\Gamma f(z)\,dz = 0$, since modulus cannot be negative.

Hence (1) yields when $R\to\infty$

$$\int_{-\infty}^{\infty} \frac{xe^{iax}}{x^2+k^2} = 2\pi i\times\frac{e^{-ak}}{2} = i\pi e^{-ka}$$

or
$$2\int_0^\infty \frac{xe^{iax}}{x^2+k^2} = i\pi e^{-ak}.$$

Equating the imaginary parts on either side, we get

$$\int_0^\infty \frac{x\sin ax}{x^2+k^2}\,dx = \frac{\pi}{2}e^{-ak}.$$

(*iii*) Here $f(z) = \dfrac{z^6}{(a^4+z^4)^2}$.

$zf(z)\to 0$ as $z\to\infty$, which satisfies the condition for evaluation of an infinite integral. Poles of $f(z)$ are given by

$$z^4 = -a^4 = e^{i(2n+1)\pi}a^4$$

$\because\ e^{i(2n+1)\pi} = \cos(2n+1).\pi + i\sin(2n+1)\pi = \cos\pi = -1,$

$\therefore\ z = ae^{(2n+1)\pi i/4}$, where $n=0, 1, 2, 3$.

Thus the poles are $ae^{\pi i/4}$, $ae^{3\pi i/4}$, $ae^{5\pi i/4}$, $ae^{7\pi i/4}$ of which the first two only lie in the upper half plane (coefficient of i being +ve) which is to be chosen as a contour consisting of large semi-circle Γ, along with the real axis from $-R$ to R.

Applying the theorem of residues,

$$\int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz = 2\pi i\,\Sigma R^+.$$

where Γ represents the upper half plane.

Since $\int_\Gamma f(z)\,dz \to 0$ as $z\to\infty$.

Hence when $R\to\infty$, we have

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i\,\Sigma R^+. \qquad ...(1)$$

Now to find the residue at $z = ae^{\pi i/4} = z_1$ (say), let us put $z = ae^{\pi i/4}+t$, t being small, in the value of $f(z)$, whence we get

$$\frac{(z_1+t)^6}{\left(a^4+(z_1+t)^4\right)^2} = \frac{z_1^6+6z_1^5t+\ldots}{\left[a^4+z_1^4+4z_1^3t+6z_1^2t^2+4z_1t^3+t^4\right]^2}, \text{ where } z_1^4=-a^4$$

$$= \frac{z_1^6+6z_1^5\cdot t+\ldots}{16z_1^6t^2\left[1+\frac{6t}{4z_1}+\ldots\right]^2}$$

$$= \frac{\left(z_1^6+6z_1^5t+\ldots\right)}{16z_1^6t^2}\left[1+\frac{6t}{4z_1}\ldots\right]^{-2}$$

$$= \frac{\left(z_1^6+6z_1^5t+\ldots\right)}{16z_1^6t^2}\left[1-\frac{12t}{4z_1}\ldots\right]$$

Residue at $z = z_1$ is = coefficient of $\frac{1}{t}$ in $f(t+z_1)$

$$= \frac{6}{16z_1} - \frac{3}{16z_1} = \frac{3}{16z_1}.$$

Similarly, residue at $z = ae^{3\pi i/4} = z_2$ (say) is $\frac{3}{16z_2}$.

Hence by (1),

$$\int_{-\infty}^{\infty}\frac{x^6dx}{\left(a^4+x^4\right)^2} = 2\pi i.\frac{3}{16}\left(\frac{1}{z_1}+\frac{1}{z_2}\right)$$

$$= \frac{3\pi i}{-8a}\left[e^{-\pi i/4}+e^{-3\pi i/4}\right]$$

$$= \frac{3\pi i}{8a}\left[e^{-\pi i/4}-e^{\pi i/4}\right], \quad \therefore\ e^{-3\pi i/4} = e^{-(\pi-\pi/4)i} = e^{-\pi i}e^{\pi i/4} = -e^{\pi i/4}$$

or $$2\int_0^{\infty}\frac{x^6dx}{\left(a^4+x^4\right)^2} = -\frac{3\pi i}{8}\times 2i\sin\frac{\pi}{4} = \frac{3\pi}{4a}.\frac{1}{\sqrt{2}}$$

$$\therefore\ \int_0^{\infty}\frac{x^6dx}{\left(a^4+x^4\right)^2} = \frac{3\sqrt{2}\pi}{16a},$$

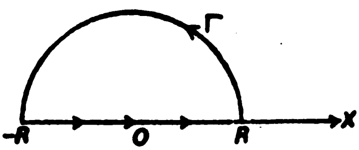


Fig. 5.36

(*iv*) Here $f(z) = \frac{1}{1+z^2}$ so that

$$zf(z) \to 0 \text{ as } z \to \infty.$$

Poles of $f(z)$ are given by $1 + z^2 = 0$ *i.e.* $z = \pm i$ of which $z = i$ lies in the upper half plane *i.e.* in the contour chosen as consisting of a large semi-circle Γ along with the real axis from $-R$ to R.

Residue of $f(z)$ at $z = i$ is $\lim_{z\to i}(z-i)\frac{1}{1+z^2} = \lim_{z\to i}\frac{1}{z+i} = \frac{1}{2i}$.

By Cauchy's residue theorem, we have

$$\int_{-R}^{R} f(x)\, dx + \int_{\Gamma} f(z)\,dz = 2\pi i \Sigma R^{+} = 2\pi i \times \frac{1}{2i} = \pi \qquad \ldots(1)$$

where $\left|\int_{\Gamma} f(z)\,dz\right| = \left|\int_{\Gamma}\frac{dz}{1+z^2}\right| \le \int_{\Gamma}\frac{|dz|}{|1+z^2|} \le \int_{\Gamma}\frac{|dz|}{|z|^2-1}$ since $|z^2+1| \ge |z|^2 - 1$

$$\le \int_0^{\pi}\frac{R\,d\theta}{R^2-1} \text{ when } z = Re^{i\theta} \text{ and } |dz| = |iRe^{i\theta}d\theta| = R\,d\theta$$

$$\le \frac{\pi R}{R^2-1} \to 0 \text{ as } R \to \infty$$

i.e. $\left|\int_{\Gamma} f(z)\, dz\right| = 0$ when $R \to \infty$ and then (1) gives

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = \pi \text{ or } 2\int_0^{\infty}\frac{dx}{1+x^2} = \pi, \text{ i.e. } \int_0^{\infty}\frac{ax}{1+x^2} = \frac{\pi}{2}$$

(v) Take $f(z) = \dfrac{e^{iz}}{(z^2+a^2)(z^2+b^2)}$

and choose the closed contour C as consisting of the upper half Γ of a large circle $|z| = R$ and the real axis $-R$ to R.

Poles of $f(z)$ are given by $(z^2+a^2)(z^2+b^2) = 0$ *i.e.* $z = \pm ia, \pm ib$ of which the only poles lying inside C are $z = ia$ and $z = ib$

Residue of $f(z)$ at $z = ai$ is $\lim\limits_{z \to ia} (z-ia) f(z)$

$$= \lim_{z\to ia}(z-ia)\frac{e^{iz}}{(z^2+a^2)(z^2+b^2)} = \frac{-e^{-a}}{(a^2-b^2)}$$

Similarly residue of $f(z)$ at $z = ib$, $= \dfrac{+e^{-b}}{2ib(a^2-b^2)}$

Hence by Cauchy's residue theorem

$$\int_{\Gamma} f(z)\,dz + \int_{-R}^{R} f(z)\,dx = 2\pi i \Sigma R^{+} = \frac{2\pi i}{2i(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

where $\left|\int_{\Gamma}\frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}dz\right| \le \int_{\Gamma}\frac{|e^{iz}|\,|dz|}{(|z|^2-a^2)(|z|^2-b^2)}$

$\because |z_1 + z_2| \ge |z_1| - |z_2|$

$$\le \int_0^{\pi}\frac{e^{-R\sin\theta}.R\,d\theta}{(R^2-a^2)(R^2-b^2)}$$

where $z = Re^{i\theta}$, and so $|dz| = R d\theta$

$$\le \frac{2R}{(R^2-a^2).(R^2-b^2)}\frac{\pi}{2R}(1-e^{-R})$$

which $\to 0$ as $R \to \infty$

∴ as $R \to \infty$, we have

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)} dx = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

Equating real and imaginary parts we get

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)} dx = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

and $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{(x^2+a^2)(x^2+b^2)} = 0.$

Note. Similar procedure will show that

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2a^3}\frac{(a+2b)}{(a+b)^2}, \quad a>0, b>0$$

and $\displaystyle\int_{-\infty}^{\infty} \frac{x^3 \sin x}{(x^2+a^2)(x^2+b^2)} dx = \frac{\pi}{(a^2-b^2)}\left(a^2e^{-a} - b^2e^{-b}\right), \quad a>0,\ b>0.$

(*vi*) take $f(z) = \dfrac{\log(z+i)}{z^2+1}$

and choose the contour C as consisting of upper half Γ of circle $|z| = R$, (R being large) and the real axis from $-R$ to R.

Poles of $f(z)$ are given by $z^2+1=0$ *i.e.* $z = \pm i$.

of which the only pole lying within the contour C is $z = i$. Residue of $f(z)$ at $z = i$ is $\lim\limits_{z\to i}(z-i).f(z)$

$$= \lim_{z\to i}\,(z-i)\frac{\log(z+i)}{z^2+1}$$

$$= \lim_{z\to i}(z-i)\,\frac{\log(z+i)}{(z-i)(z+i)} = \frac{\log(2i)}{2i}$$

$$= \frac{\log 2 + i\dfrac{\pi}{2}}{2i} \qquad \because \log i = \log e^{i\pi/2}$$

Hence by Cauchy's Residue theorem, we have

$$\int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz = 2\pi i \Sigma R^+$$

$$= 2\pi i \times \frac{1}{2i}\left(\log 2 + \frac{i\pi}{2}\right)$$

$$= \pi\left(\log 2 + \frac{i\pi}{2}\right)$$

where $\displaystyle\left|\int_{\Gamma} \frac{\log(z+i)}{z^2+1} dz\right| \le \int_{\Gamma} \frac{|\log(z+i)|}{|z^2+1|}\,|dz|$

$$\le \int_{\Gamma} \frac{|\log|z+i| + i\arg(z+i)|\,|dz|}{|z|^2-1} \qquad \text{by (36) of §5.7}$$

$$\leq \int_\Gamma \frac{\left|\log\left(|z|+1\right)+i2\pi\right||dz|}{|z|^2-1}$$

$$\because \arg(z+i) < 2\pi \text{ and } |i|=1$$

$$\leq \frac{\{\log(R+1)+2\pi\}\pi R}{R^2-1} \text{ when } z=Re^{i\theta}$$

$$\leq \left(\log\frac{(R+1)}{R+1}+\frac{2\pi}{R+1}\right)\frac{\pi}{R-1} \to 0 \text{ as } R\to\infty.$$

and hence when $R\to\infty$, we have

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} \frac{\log(x+i)}{x^2+1} dx = \pi\left(\log 2+\frac{i\pi}{2}\right)$$

or $$\int_{-\infty}^{\infty} \frac{\frac{1}{2}\log\left(1+x^2\right)+i\tan^{-1}\frac{1}{x}}{x^2+1} dx = \pi\left(\log 2+\frac{i\pi}{2}\right)$$

Equating real and imaginary parts, we get

$$\int_{-\infty}^{\infty} \frac{\frac{1}{2}\log\left(1+x^2\right)}{1+x^2} dx = \pi\log 2$$

i.e. $$\int_0^{\infty} \frac{\log\left(1+x^2\right)}{1+x^2} dx = \pi\log 2$$

and $$\int_{-\infty}^{\infty} \frac{\tan^{-1}\frac{1}{x}}{1+x^2} dx = \frac{\pi^2}{2}$$

Problem 22. *Prove by contour integration that*

$$\int_0^{\infty} \frac{\sin mx}{x} dx = \frac{\pi}{2}$$ *(Rohilkhand, 1986; Meerut, 78, 82, 84; Agra, 1963, 83)*

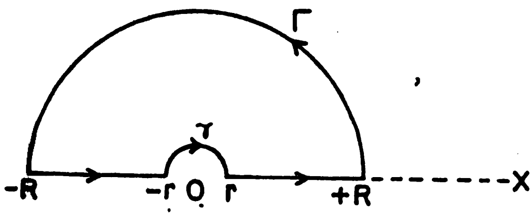


Fig. 5.37

Let us consider $\int_C f(z)\, dz$, where $f(z)=\frac{e^{imz}}{z}$ which has got a simple pole of $f(z)$ at $z=0$.

Choosing the contour C consisting of a large semi-circle $|z| = R$ indented at $z=0$, let the radius of the indentation be r. Since there is no pole within the contour, we have have by Cauchy's theorem,

$$\int_{-R}^{-r} f(x)\, dx + \int_\gamma f(z)dz + \int_r^R f(x)\, dx + \int_\Gamma f(z)dz = 0, \quad \ldots(1)$$

Residue at $z=0$ of $f(z)$ is $= \lim_{z\to 0} zf(z)$

$$= \lim_{z\to 0} z.\frac{e^{imz}}{z} = 1.$$

Now $\lim_{r\to 0}\int_\gamma f(z) = -i(\pi - 0).1 = -\pi i$ by theorem of §5.16 (–ve sign being taken as contour γ is in clockwise direction) and

$$\left|\int_\Gamma f(z)\,dz\right| \le \int_\Gamma |f(z)|\,|dz| = \int_\Gamma \frac{|e^{imz}|\,|dz|}{|z|}$$

$$\le \int_0^\pi \frac{e^{-mR\sin\theta}}{R} R\,d\theta \qquad \because\ z = Re^{i\theta},\ |dz| = R d\theta$$

$$\le 2\int_0^{\pi/2} e^{(-2mR\theta)/\pi}\,d\theta \text{ by Jordan's inequality}$$

$$\le \frac{m\pi}{R}\left(1 - e^{-mR}\right) \quad \text{which} \to 0 \text{ as } R \to \infty.$$

With these results as $R\to\infty$ and $r\to 0$, (1) reduces to

$$\int_{-\infty}^0 f(x)\,dx - \pi i + \int_0^\infty f(x)\,dx = 0$$

or
$$\int_{-\infty}^\infty \frac{e^{imx}}{x}\,dx = \pi i$$

or
$$2\int_0^\infty \frac{e^{imx}}{x}\,dx = \pi i, \text{ i.e., } \int_0^\infty \frac{e^{imx}\,dx}{x} = \frac{\pi}{2} i.$$

Equating imaginary parts on both sides, we have

$$\int_0^\infty \frac{\cos mx}{x}\,dx = \frac{\pi}{2}.$$

Note. *Equating real parts,* $\int_0^\infty \frac{\sin mx}{x}\,dx = 0.$

Problem 23. *Prove by contour integration that* $\int_0^\infty \frac{x^{\alpha-1}}{1+x} = \frac{\pi}{\sin \alpha\pi}$, $0 < \alpha < 1$, *and hence deduce that* (*Rohilkhand, 1979, 82, 93*)

$$\int_0^\infty \frac{x^{\alpha-1}}{1-x}\,dx = \pi \cot \alpha\pi,\ 0 < \alpha < 1.$$

Let us consider

$$\int_C f(z)\,dz, \text{ where } f(z) = \frac{z^{\alpha-1}}{1-z}.$$

Choosing the contour C consisting of a large semi-circle $|z| = R$ *indented at* $z = 0$, $z = 1$, let the radii of indentation be r_1 and r_2 respectively.

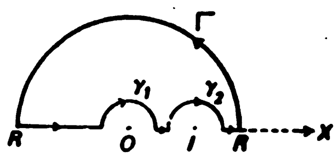


Fig. 5.38

Since there are no poles within the contour, we have

$$\int_{r_1}^{1-r_2} \frac{x^{\alpha-1}}{1-x}\,dx + \int_{1+r_2}^{R} \frac{x^{\alpha-1}}{1-x}\,dx + \int_R^{r_1} \frac{\left(xe^{\pi i}\right)^{\alpha-1}}{1-xe^{\pi i}}\,d\left(xe^{\pi i}\right) + \int_\Gamma f(z)\,dz$$
$$+\int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz = 0 \qquad \ldots(1)$$

Now residue of $f(z)$ at $z = 0$ is $= \lim_{z\to 0} z.\dfrac{z^{\alpha-1}}{1-z} = 0$ as $\alpha > 0$.

Residue of $f(z)$ at $z=1$ is $= \lim_{z\to 1}(z-1).\dfrac{z^{\alpha-1}}{1-z} = -1.$

Also $\quad \lim_{r_1\to 0}\int_{\gamma_1} f(z)\,dz = 0,$

$$\lim_{r_2\to 0}\int_{\gamma_2} f(z)\,dz = -i(\pi-0)\,(-1) = \pi i.$$

and $$\left|\int_\Gamma \frac{z^{\alpha-1}}{1-z}dz\right| \le \int_\Gamma \frac{|z|^{\alpha-1}}{|z-1|}|dz| \le \frac{R^{\alpha-1}}{R-1}\pi R = \frac{\pi R^\alpha}{R-1},$$

which $\to 0$ as $R\to\infty$, $\because\ \alpha < 1$.

With these results as $R\to\infty$ and r_1, r_2 both tending to zero, (1) reduces to

$$\int_0^1 \frac{x^{\alpha-1}}{1-x}dx + \int_0^\infty \frac{x^{\alpha-1}}{1-x}dx + \int_\infty^0 \frac{e^{\alpha\pi i}.x^{\alpha-1}}{1+x}dx + \pi i = 0$$

or $$\int_0^\infty \frac{x^{\alpha-1}}{1-x}dx - e^{\alpha\pi i}\int_0^\infty \frac{x^{\alpha-1}}{1+x}dx + \pi i = 0$$

or $$\int_0^\infty \frac{x^{\alpha-1}}{1-x}dx - (\cos\alpha\pi + i\ \sin\alpha\pi)\int_0^\infty \frac{x^{\alpha-1}}{1+x}dx = -\pi i.$$

Equating real and imaginary parts we find

$$\sin\alpha\pi\int_0^\infty \frac{x^{\alpha-1}}{1+x} = \pi,\ i.e.,\ \int_0^\infty \frac{x^{\alpha-1}}{1+x}dx = \frac{\pi}{\sin\alpha\pi}$$ *(Rohilkhand, 1990)*

and $$\int_0^\infty \frac{x^{\alpha-1}}{1-x}dx = \cos\alpha\pi\int_0^\infty \frac{x^{\alpha-1}}{1+x}dx = \cos\alpha\pi.\frac{\pi}{\sin\alpha\pi} = \pi\cot\alpha\pi.$$

Problem 24. *If $-1<p<1$ and $-\pi<\lambda<\pi$, show by contour integration that*

$$\int_0^\infty \frac{x^{-p}dx}{1+2x\cos\lambda + x^2} = \frac{\pi}{\sin p\pi}\frac{\sin p\lambda}{\sin\lambda}.$$ *(Rohilkhand, 1992)*

Let us consider $\int_C f(z)\,dz$, where $f(z) = \dfrac{z^{-p}}{1+2z\cos\lambda + z^2}$

when the contour C is chosen to be consisting of a large circle $|z| = R$, a small circle $|z| = r$, and the cross cut joining the arcs of the two circles along the real axis.

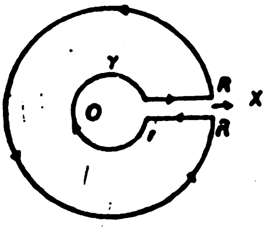


Fig. 5.39

The poles of $f(z)$ are given by

$$1 + 2z\cos\lambda + z^2 = 0$$

or $\quad (z + e^{i\lambda})(z + e^{-i\lambda}) = 0,$

giving $\quad z = -e^{i\lambda} = e^{i(\pi+\lambda)}$

and $\quad z = -e^{-i\lambda} = e^{i(\pi-\lambda)}$

Amplitudes of the point being $\pi+\lambda$, and $\pi-\lambda$ follow that $-\pi<\lambda<\pi$. As such the two amplitudes lie between 0 and 2π.

The poles within the contour are $z = -e^{i\lambda}$ and $z = -e^{-i\lambda}$.

Residue of $f(z)$ at $z = -e^{i\lambda}$ is

$$= \lim_{z \to -e^{i\lambda}} \left(z + e^{i\lambda}\right) \frac{z^{-p}}{\left(z + e^{i\lambda}\right)\left(z + e^{-i\lambda}\right)}$$

$$= (-1)^{-p} \cdot \frac{e^{-ip\lambda}}{-2i \sin\lambda} = \frac{e^{-p\pi i} . e^{-ip\lambda}}{-2i \sin\lambda}$$

Similarly, residue at $z = -e^{-i\lambda}$ is $e^{-p\pi i} \dfrac{e^{ip\lambda}}{2i \sin\lambda}$.

Sum of residues $= \dfrac{e^{-p\pi i}}{\sin\lambda} \cdot \dfrac{e^{ip\lambda} - e^{-ip\lambda}}{2i} = \dfrac{e^{-p\pi i}}{\sin\lambda} \cdot \sin p\lambda.$

Applying Cauchy's residue theorem, we have

$$\int_r^R f(x)\, dx + \int_\Gamma f(z)\, dz - \int_r^R f\left(xe^{2\pi i}\right) d\left(xe^{2\pi i}\right) - \int_\gamma f(z)\, dz$$

$$= 2\pi i \times \text{sum of the residues.} \qquad \ldots(1)$$

Here $\left|\int_\Gamma f(z)\, dz\right| \le \int_\Gamma \dfrac{|z|^{-p}\, |dz|}{\left|z + e^{i\lambda}\right| \left|z + e^{-i\lambda}\right|}$

$$\le \frac{R^{-p} \cdot 2\pi R}{(R-1)(R-1)} = \frac{2\pi}{R^{1+p}\left(1 - \frac{1}{R}\right)^2},$$

which $\to 0$ as $R \to \infty$ provided $1 + p > 0$, *i.e.* $p > -1$.

Similarly $\left|\int_\gamma f(z)\, dz\right| \le 2\pi \dfrac{r^{1-p}}{(1+r)^2}$ which $\to 0$ as $z \to 0$, if $1 - p > 0$ or $1 > p$

Hence there exists a condition $-1 < p < 1$.

With these results as $R \to \infty$ and $r \to 0$, (1) reduces to

$$\int_0^\infty f(x)\, dx - \int_0^\infty f\left(xe^{2\pi i}\right) e^{2\pi i}\, dx = 2\pi i \cdot e^{-p\pi i} \frac{\sin p\lambda}{\sin\lambda}$$

or $$\int_0^\infty \frac{x^{-p} dz}{1 + 2x \cos\lambda + x^2} - \int_0^\infty \frac{\left(xe^{2\pi i}\right)^{-p} \cdot e^{2\pi i} dx}{1 + 2xe^{2\pi i} \cos\lambda + \left(xe^{2\pi i}\right)^2} = 2\pi i e^{-p\pi i} \frac{\sin p\lambda}{\sin\lambda}$$

or $$\left(1 - e^{-2p\pi i}\right) \int_0^\infty \frac{x^{-p} dx}{1 - 2x \cos\lambda + x^2} = 2\pi i e^{-p\pi i} \frac{\sin p\lambda}{\sin\lambda} \quad \because e^{2\pi i} = 1$$

i.e. $$\int_0^\infty \frac{x^{-p} dx}{1 + 2x \cos\lambda + x^2} = \frac{2\pi i \cdot e^{-p\pi i} \sin p\lambda}{e^{-p\pi i}\left(e^{p\pi i} - e^{-p\pi i}\right) \sin\lambda} = \frac{p}{\sin p\pi} \cdot \frac{\sin p\lambda}{\sin\lambda}$$

Problem 25. *By integrating $e^{iz} z^{\alpha - 1}$ round a quadrant of a circle of radius R, prove that if $0 < \alpha < 1$,*

$$\int_0^\infty x^{\alpha-1} {\cos \atop \sin} x\, dx = \Gamma(\alpha) {\cos \atop \sin} \frac{\pi}{2}\alpha \qquad \textit{(Rohilkhand, 1983)}$$

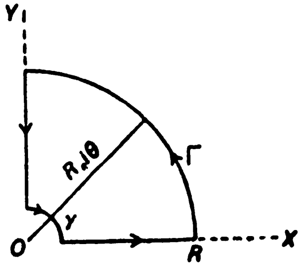


Fig. 5.40

Consider $\int_C f(z)\, dz$, where $f(z) = e^{iz} z^{\alpha-1}$.

Choosing the contour C consisting of a quadrant of a large circle $|z| = R$ indented at $z=0$, let r be the radius of indentation. By Cauchy's integral, we have

$$\int_C f(z)\, dz = 0.$$

Here $z=\infty$ is not a pole, but $z=0$ is a pole which has been excluded from the contour by indentation.

Splitting up the above contour, we get

$$\int_r^R e^{i\alpha} x^{\alpha-1} dx + \int_\Gamma e^{iz} z^{\alpha-1} dz + \int_R^r e^{i(iy)} (iy)^{\alpha-1}\, i dy + \int_\gamma e^{iz} z^{\alpha-1} dz = 0 \qquad \ldots(1)$$

Residue of $f(z)$ at $z = 0$ is $\lim_{z\to 0} z.e^{iz} z^{\alpha-1} = 0$

and $$\left|\int_\Gamma e^{iz} z^{\alpha-1} dz\right| = \left|\int_\Gamma e^{iR} e^{i\theta} \left(Re^{i\theta}\right)^{\alpha-1} iRe^{i\theta}\, d\theta\right| \text{ as } z = Re^{i\theta}$$

$$\le \int_0^{\pi/2} e^{-R\sin\theta} \left|e^{iR\cos\theta}\right| |i| \left|R^\alpha\right| \left|e^{i\alpha\theta}\right| d\theta$$

$$\le \int_0^{\pi/2} e^{-R\sin\theta} R^\alpha\, d\theta$$

$$\le \int_0^{\pi/2} e^{-2R\theta/\pi}.\, R^\alpha\, d\theta \text{ as } \frac{\sin\theta}{\theta} > \frac{2}{\pi} \text{ (Jordan's inequality)}$$

$$\le \frac{R^\alpha}{2} \cdot \frac{\pi}{R} \left[1 - e^{-R}\right] \text{ which tends to zero as } R \to \infty$$

Similarly $$\left|\int_\gamma e^{iz} z^{\alpha-1} dz\right| \le \frac{\pi}{2} \cdot \frac{r^\alpha}{r}\left(1 - e^{-r}\right) \qquad \because z = re^{i\theta}$$

$$\le \frac{\pi}{2} \cdot \frac{r^\alpha}{r} \left(r - \frac{r^2}{2!} + \ldots\right) \qquad \text{which tends to zero as } r \to 0.$$

Hence with these results as $R \to \infty$ and $r \to 0$, (1) reduces to

$$\int_0^\infty e^{ix} x^{\alpha-1}\, dx - \int_0^\infty e^{-y} i^\alpha y^{\alpha-1} dy = 0$$

or $$\int_0^\infty (\cos x + i \sin x)\, x^{\alpha-1} dx = i^\alpha \int_0^\infty e^{-y} y^{\alpha-1} dy = \left(e^{i\pi/2}\right)^\alpha \Gamma(\alpha)$$

$$\because \int_0^\infty e^{-y} y^{\alpha-1}\, dx = \Gamma(\alpha) \quad \text{and} \quad i = e^{i\pi/2}.$$

Equating real and imaginary parts, we get

$$\int_0^\infty \cos x \cdot x^{\alpha-1}\, dx = \Gamma\alpha \cos\frac{\pi}{2}\alpha \text{ and } \int_0^\infty \sin x \cdot x^{\alpha-1} dx = \Gamma\alpha \sin\frac{\pi}{2}\alpha.$$

Problem 26. *Prove that* $\int_0^\infty e^{-x^2} \cos 2ax\, dx = \frac{1}{2}\sqrt{\pi}\, e^{-a^2}$ *by integrating* e^{-z^2} *round the rectangle whose vertices are* $O, R, R+ia, ia$.

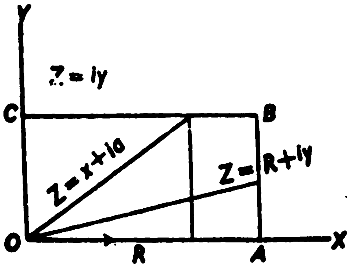


Fig. 5.41

Let A be $(R, 0)$, C be $(0, a)$ in the Argand diagram.

On OA, $z = x$; $\therefore\ dz = dx$.

On AB, $z = R + iy$; $\therefore\ dz = i\,dy$.

On BC, $z = x + ia$; $\therefore\ dz = dx$

and on OC, $z = iy$; $\therefore\ dz = i\,dy$

Now e^{-z^2} has no poles within or on this contour and if the whole contour be denoted by Γ, then by Cauchy's theorem

$$\int_{\Gamma} e^{-z^2}\,dz = 0.$$

Splitting up the contour

$$\int_0^R e^{-x^2}dx + \int_0^a e^{-(R+iy)^2} i\,dy + \int_R^0 e^{-(x+ia)^2}dx + \int_a^0 e^{-(iy)^2} i\,dy = 0. \qquad ...(1)$$

$$\text{Now } \left|\int_0^a e^{-(R+iy)^2}\, i\,dy\right| \le \int_0^a \left|e^{-(R^2+2iRy-y^2)}\right| |i|\,dy$$

$$\le \int_0^a e^{-R^2} e^{y^2} \left|e^{-2iRy}\right| dy$$

$$\le \int_0^a e^{-R^2} e^{y^2} dy \quad \because \left|e^{-2iRy}\right| = 1$$

and $|i| = 1$

which tends to zero as $R \to \infty$.

Hence when $R \to \infty$, (1) reduces to

$$\int_0^\infty e^{-x^2}dx - \int_0^\infty e^{-(x+ia)^2}dx - i\int_0^a e^{y^2}\,dy = 0$$

or $$\int_0^\infty e^{-x^2}dx - e^{a^2}\int_0^\infty e^{-x^2} e^{-2iax}\,dx - i\int_0^a e^{y^2}\,dy = 0.$$

or $$\int_0^\infty e^{-x^2}dx - e^{a^2}\int_0^\infty e^{-x^2}[\cos 2ax - i\sin 2ax]dx - i\int_0^a e^{y^2}\,dy = 0.$$

Equating real parts, $e^{a^2}\int_0^a e^{-x^2}\cos 2ax\,dx = \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

$$\therefore \int_0^a e^{-x^2}\cos 2ax\,dx = \frac{1}{2}\sqrt{\pi}\,e^{-a^2}$$

Problem 27. *Using the rectangular contour, prove the following:*

(*i*) $\int_0^\infty \frac{t^{\alpha-1}}{1+t}\,dt = \pi \operatorname{cosec} \alpha\pi;\ 0 < \alpha < 1.$ (*Rohilkhand, 1991*)

(*For $t = e^x$, Rohilkhand, 1977, 79, 82, 90*)

(*ii*) $\int_0^\infty \frac{\cosh x}{\cosh \pi x}\,dx = \frac{1}{2}\sec\frac{\alpha}{2}; -\pi < \alpha < \pi$

(*Rohilkhand, 1979, 82, 91; Agra, 1982*)

(*iii*) $\int_0^\infty \frac{\sinh \alpha x}{\cosh \pi x}\,dx = \frac{1}{2}\tan\frac{\alpha}{2};\ 0 < \alpha < \pi.$ (*Rohilkhand, 1992*) (*ii*) Given integral is

$$I = \int_0^\infty \frac{t^{\alpha-1}}{1+t}\,dt$$

Putting $t = e^x$, we get

$$I = \int_{-\infty}^{\infty} \frac{e^{\alpha x}}{e^x+1}\,dx.$$

Consider,

$$\int_C f(z)\,dz = \int_C \frac{e^{\alpha z}}{e^z+1}\,dz$$

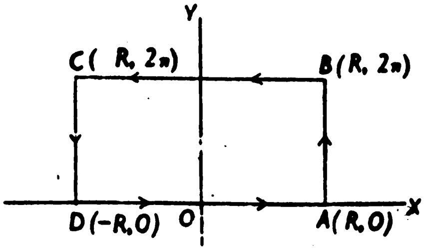


Fig. 5.42

where C is the rectangle $ABCD$ with vertices at $R, R+2\pi i, -R+2\pi i; -R$; R being positive.

Here, $f(z) = \dfrac{e^{\alpha z}}{e^z+1}$.

The poles of $f(z)$ are given by $e^z + 1 = 0$

i.e. $\qquad e^z = -1 = e^{(2n+1)\pi i}$

or $\qquad z = (2n+1)\pi i$ where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

Only the simple pole $z = \pi i$, (corresponding to $n = 0$) lies inside the contour.

$$\text{Residue of } f(z) \text{ at } z = \pi i = \left[\frac{e^{\alpha z}}{\frac{d}{dz}\left(1+e^z\right)}\right]_{z=\pi i} = \frac{e^{\alpha\pi i}}{e^{\pi i}} = -e^{\alpha\pi i}$$

as $e^{\pi i} = \cos\pi + i\sin\pi = -1$.

By Cauchy's residue theorem, we therefore have

$$\int_{-R}^{R} f(x)\,dx + \int_0^{2\pi} f(R+iy)\,i\,dy + \int_R^{-R} f(x+2\pi i)\,dx + \int_{2\pi}^{0} f(-R+iy)\,i\,dy = 2\pi i \times \text{sum of residues}$$

$$= -2\pi i e^{\alpha\pi i}. \qquad \ldots(1)$$

But

$$\left|\int_0^{2\pi} f(R+iy)\,i\,dy\right| \le \int_0^{2\pi} |f(R+iy)|\,|i\,dy|$$

$$\le \int_0^{2\pi} \left|\frac{e^{\alpha(R+iy)}}{1+e^{R+iy}}\right| . |dy| \text{ as } |i| = 1$$

$$\le \int_0^{2\pi} \left|\frac{e^{\alpha R}}{e^R-1}\right| |dy| \text{ as } \left|e^{R+iy}+1\right| \le \left|e^{R+iy}\right| - 1$$

$$\le \frac{e^{\alpha R}}{e^R-1}\cdot 2\pi.$$

$$\le \frac{2\pi}{e^{(1-\alpha)R} - e^{-\alpha R}} \text{ which} \to 0 \text{ as } R \to \infty \text{ for } 0 < \alpha < 1$$

$$\therefore \int_0^{2\pi} f(R+iy)\,i\,dy \to 0, \text{when } R \to \infty.$$

Similarly $\int_0^{2\pi} f(-R+iy)\, i\,dy \to 0$ when $R \to \infty$, provided $0 < \alpha < 1$.

Hence proceeding to the limit when $R \to \infty$, (1) gives

$$\int_{-\infty}^{\infty} f(x)\,dx + \int_{\infty}^{-\infty} f(x+2\pi i)\,dx = -2\pi i e^{\alpha\pi i}$$

or $$\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{e^x+1}\,dx - \int_{-\infty}^{\infty} \frac{e^{\alpha(x+2\pi i)}}{e^{x+2\pi i}+1}\,dx = -2\pi i\, e^{\alpha\pi i}$$

or $$\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{e^x+1}\,dx - \int_{-\infty}^{\infty} \frac{e^{2\alpha\pi i}e^{\alpha x}}{e^x+1}\,dx = -2\pi i e^{\alpha\pi i}$$

or $$\int_{-\infty}^{\infty} \frac{e^{\alpha x}\left(1-e^{2\alpha\pi i}\right)}{e^x+1}\,dx = -2\pi i e^{\alpha\pi i}$$

or $$\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{e^x+1}\,dx = \frac{-2\pi i e^{\alpha\pi i}}{1-e^{2\alpha\pi i}}$$

or $$\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{e^x+1}\,dx = \pi\cdot\frac{2i}{e^{\alpha\pi i}-e^{-\alpha\pi i}} = \frac{\pi}{\sin\alpha\pi}$$

or $$\int_0^{\infty} \frac{t^{\alpha-1}}{t+1}\,dt = \pi \operatorname{cosec} \alpha\pi$$

(*ii*) Given integral is

$$I = \int_0^{\infty} \frac{\cosh \alpha x}{\cosh \pi x}\,dx.$$

Consider

$$\int_C f(z)\,dz = \int \frac{e^{\alpha z}}{\cosh \pi z}\,dz$$

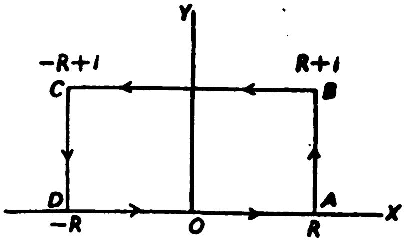


Fig. 5.43

where $$f(z) = \frac{e^{\alpha z}}{\cosh \pi z}$$

taken round the rectangular contour which consists of x-axis from $-R$ to R and y-axis from 0 to 1.

Poles of $f(z)$ are given by $\cosh \pi z = 0$.

i.e. $$\frac{e^{\pi z}+e^{-\pi z}}{2} = 0$$

or $$e^{\pi z} = -e^{-\pi z} = -\frac{1}{e^{\pi z}}$$

or $$e^{2\pi z} = -1 = e^{i(2n+1)\pi}$$

i.e. $2z = i\,(2n+1)$, where $n = 0, \pm 1, \pm 2, \ldots$

or $$z = \frac{2n+1}{2}\,i$$

Only the simple pole $z = \frac{i}{2}$ lies within the contour.

Residue of $f(z)$ at $z = \frac{i}{2} = \left[\dfrac{e^{\alpha z}}{\dfrac{d}{dz}\cosh \pi z}\right]_{z=\alpha/2}$

$$= \frac{e^{\alpha i/2}}{\pi \sinh \frac{\pi i}{2}} = \frac{e^{\alpha i/2}}{\pi i \sin \frac{\pi}{2}} = \frac{e^{\alpha i/2}}{\pi i}.$$

By Cauchy's residue theorem, we have

$$\int_{-R}^{R} f(x)\, dx + \int_0^1 f(R+iy)\, i\, dy + \int_R^{-R} f(x+i)\, dx$$

$$+\int_1^0 f(-R+iy)\, i\, dy = 2\pi i \cdot \frac{e^{\alpha i/2}}{\pi i}. \qquad \ldots(1)$$

But $$\left| \int_0^1 f(R+iy)\, i\, dy \right| \le \int_0^1 |f(R+iy)|\, |i|\, |dy|$$

$$\le \int_0^1 \left| \frac{e^{\alpha(R+iy)}}{\cosh \pi (R+iy)} \right| dy \text{ as } |i| = 1$$

$$\le \int_0^1 \frac{2e^{\alpha R} \left|e^{\alpha iy}\right| dy}{\left| e^{\pi(R+iy)} + e^{-\pi(R+iy)} \right|}$$

$$\le \int_0^1 \frac{2e^{\alpha R}}{e^{\pi R} - e^{-\pi R}}\, dy$$

$$\le \frac{2e^{\alpha R}}{e^{\pi R} - e^{-\pi R}}$$

which $\to 0$ as $R \to \infty$ when $\alpha < \pi$

$\therefore \int_0^1 f(R+iy)\, i\, dy \to 0$ as $R \to \infty$.

Similarly $\int_0^1 f(-R+iy)\, i\, dy \to 0$ as $R \to \infty$ when $\alpha < \pi$.

Hence when $R \to \infty$, (1) becomes

$$\int_{-\infty}^{\infty} f(x)\, dx + \int_{\infty}^{-\infty} f(x+i)\, dx = 2e^{i\alpha/2}$$

or $$\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{\cosh \pi x}\, dx + \int_{\infty}^{-\infty} \frac{e^{\alpha(x+i)}}{\cosh \pi (x+i)}\, dx = 2e^{i\alpha/2}.$$

But $\cosh \pi (x+i) = \cos i\pi(x+i) = \cos(\pi xi - \pi)$

$= -\cos \pi xi = -\cosh \pi x.$

$\therefore$ $$\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{\cosh \pi x}\, dx - \int_{\infty}^{-\infty} \frac{e^{\alpha(x+i)}}{\cosh \pi x}\, dx = 2e^{i\alpha/2}$$

or $$\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{\cosh \pi x}\, dx + \int_{-\infty}^{\infty} \frac{e^{\alpha x} \cdot e^{\alpha i}}{\cosh \pi x}\, dx = 2e^{i\alpha/2}$$

or $$\left(1 + e^{i\alpha}\right) \int_{-\infty}^{\infty} \frac{e^{\alpha x}}{\cosh \pi x}\, dx = 2e^{i\alpha/2}$$

or $$\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{\cosh \pi x}\, dx = \frac{2e^{i\alpha/2}}{1 + e^{\alpha i}} = \frac{2}{e^{i\alpha/2} + e^{-i\alpha/2}}$$

or $$\int_{-\infty}^{0} \frac{e^{\alpha x}}{\cosh \pi x}\, dx + \int_{0}^{\infty} \frac{e^{\alpha x}}{\cosh \pi x}\, dx = \frac{2}{2 \cos \frac{\alpha}{2}} = \sec \frac{\alpha}{2}.$$

If we replace x by $-x$ in the first integral this becomes $\int_0^{\infty} \frac{e^{-\alpha x}}{\cosh \pi x}\, dx$

$$\therefore \quad \int_0^{\infty} \frac{e^{-\alpha x}}{\cosh \pi x}\, dx + \int_0^{\infty} \frac{e^{\alpha x}}{\cosh \pi x}\, dx = \sec \frac{\alpha}{2}$$

or $$\int_0^{\infty} \frac{e^{\alpha x} + e^{-\alpha x}}{\cosh \pi x} = \sec \frac{\alpha}{2}$$

or $$2 \int_0^{\infty} \frac{\cosh \alpha x}{\cosh \pi x} = \sec \frac{\alpha}{2}$$

i.e. $$\int_0^{\infty} \frac{\cosh \alpha x}{\cosh \pi x} = \frac{1}{2} \sec \frac{\alpha}{2}$$

Aliter, Consider

$$\int_C f(z)\, dz = \int_C \frac{e^{\alpha z}}{\cosh \pi z},$$

where $$f(z) = \frac{e^{\alpha z}}{\cosh \pi z},$$

taken round the rectangular contour with vertices at

$$R,\ R + \frac{i}{2},\ -R + \frac{i}{2} \text{ and } -R,$$

indented at $z = \frac{i}{2}$, since $z = \frac{i}{2}$ is a pole of $f(z)$.

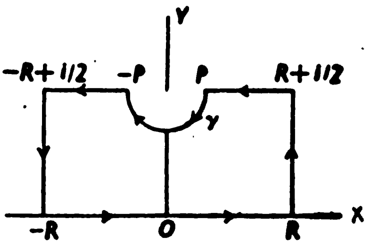


Fig. 5.44

Clearly $f(z)$ is regular within and on the contour, so that Cauchy's theorem gives

$$\int_{-R}^{R} f(x)\, dx + \int_0^{1/2} f(R + iy)\, i\, dy + \int_R^{\rho} f\left(x + \tfrac{1}{2} i\right) dx$$
$$+ \int_{\gamma} f(z)\, dz + \int_{-\rho}^{-R} f\left(x + \tfrac{1}{2} i\right) dx + \int_{1/2}^{0} f\left(-R + \tfrac{1}{2} i\right) i\, dy = 0. \quad \text{...(1)}$$

But $$\lim_{z \to \frac{1}{2} i} \left(z - \tfrac{1}{2} i\right) f(z) = \lim_{z \to \frac{1}{2} i} \frac{\left(z - \frac{1}{2} i\right) e^{\alpha z}}{\cosh \pi z}$$

$$= \frac{e^{\frac{1}{2}\alpha i}}{\pi \sinh \frac{1}{2} i \pi} = \frac{e^{\frac{1}{2}\alpha i}}{\pi i \sin \frac{1}{2} \pi} = \frac{e^{\frac{1}{2}\alpha i}}{\pi i}.$$

$$\therefore \quad \lim_{\rho \to 0} \int_{\gamma} f(z)\, dz = (-\pi i) \cdot \frac{1}{\pi i} e^{\frac{1}{2}\alpha i} = -e^{\frac{1}{2}\alpha i}$$

and $$\left| \int_0^{1/2} f(R + iy)\, i\, dy \right| \le \int_0^{1/2} \frac{e^{\alpha (R + iy)}}{\cosh \pi (R + iy)} |i|\, |dy|$$

$$\le \int_0^{1/2} \frac{e^{\alpha R}\, dy}{\cosh \pi R \cos \pi y - \sinh \pi R \sin \pi y}$$

$$\left[\because \ |\cosh \pi R \cos \pi y + i \sinh \pi R \sin \pi y| \right.$$
$$\geq \cosh \pi R \cos \pi y - \sinh \pi R \sin \pi y|$$
$$\leq \int_0^{1/2} \frac{e^{\alpha R} dy}{e^{\pi R (\cos \pi y - \sin \pi y)}}$$

writing $e^{\pi R}$ for $\cosh \pi R$ and $\sinh \pi R$ when R is large
$\to 0$ when $R \to \infty$ if $\alpha < \pi$.

$$\therefore \quad \int_0^{1/2} f(R+iy)\, i\, dy \to 0 \text{ when } R \to \infty.$$

Similarly $\int_{1/2}^{0} f(-R+iy)\, i\, dy \to 0$ when $R \to 0$ provided $\alpha < \pi$.

Hence in the limit when $R \to \infty$ and $\rho \to 0$, (1) becomes

$$\int_{-\infty}^{\infty} f(x)\, dx + \int_{\infty}^{0} f(x + \tfrac{1}{2} i)\, dx - e^{\frac{1}{2}\alpha i} + \int_0^{-\infty} f(x + \tfrac{1}{2} i) dx = 0$$

or
$$\int_{-\infty}^{\infty} f(x)\, dx - \int_{-\infty}^{\infty} f(x + \tfrac{1}{2} i)\, dx = e^{\frac{1}{2}\alpha i}$$

or
$$\int_{-\infty}^{\infty} \left[\frac{e^{\alpha x}}{\cosh \pi x} - \frac{e^{\alpha (x + \frac{1}{2} i)}}{\cosh \pi \left(x + \frac{1}{2} i\right)} \right] dx = e^{\frac{1}{2}\alpha i}$$

or
$$\int_{-\infty}^{\infty} \left[\frac{e^{\alpha x}}{\cosh \pi x} - \frac{e^{\alpha x} . e^{\frac{1}{2}\alpha i}}{i \sinh \pi x} \right] dx = \cos \tfrac{1}{2}\alpha + i \sin \tfrac{1}{2}\alpha$$

or
$$\int_{-\infty}^{\infty} \left[\frac{e^{\alpha x}}{\cosh \pi x} + \frac{i e^{\alpha x} \left(\cos \frac{1}{2}\alpha + i \sin \frac{1}{2}\alpha\right)}{\sinh \pi x} \right] dx = \cos \tfrac{1}{2}\alpha + i \sin \tfrac{1}{2}\alpha.$$

Equating real and imaginary parts, we get

$$\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{\cosh \pi x} dx - \int_{-\infty}^{\infty} \frac{e^{\alpha x} . \sin \frac{1}{2}\alpha}{\sinh \pi x} dx = \cos \tfrac{1}{2}\alpha. \qquad \ldots(2)$$

and
$$\int_{-\infty}^{\infty} \frac{e^{\alpha x} \cos \frac{1}{2}\alpha}{\sinh \pi x} dx = \sin \tfrac{1}{2}\alpha$$

or
$$\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{\sinh \pi x} = \tan \tfrac{1}{2}\alpha. \qquad \ldots(3)$$

(2) gives with the help of (3).

$$\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{\sinh \pi x} dx - \sin \tfrac{1}{2}\alpha \cdot \tan \tfrac{1}{2}\alpha = \cos \tfrac{1}{2}\alpha$$

or
$$\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{\cosh \pi x} dx = \cos \tfrac{1}{2}\alpha + \frac{\sin^2 \frac{1}{2}\alpha}{\cos \frac{1}{2}\alpha} = \sec \tfrac{1}{2}\alpha$$

or
$$\int_{-\infty}^{0} \frac{e^{\alpha x}}{\cosh \pi x} dx + \int_0^{\infty} \frac{e^{\alpha x}}{\cosh \pi x} dx = \sec \tfrac{1}{2}\alpha$$

or $$\int_0^\infty \frac{e^{-\alpha x}}{\cosh \pi x}\,dx+\int_0^\infty \frac{e^{\alpha x}}{\cosh \pi x}\,dx=\sec \tfrac{1}{2}\alpha$$

(on replacing x by $-x$ in the first integral),

i.e. $$\int_0^\infty \frac{e^{\alpha x}+e^{-\alpha x}}{\cosh \pi x}=\sec \tfrac{1}{2}\alpha$$

or $$\int_0^\infty \frac{\cosh \alpha x}{\cosh \pi x}\,dx=\tfrac{1}{2}\sec \tfrac{1}{2}\alpha.$$

(*iii*) This may be proved independently as (*ii*) or may be derived from (3) of preceding aliter as follows:

We have $$\int_{-\infty}^\infty \frac{e^{\alpha x}}{\sinh \pi x}\,dx=\tan \tfrac{1}{2}\alpha.$$

or $$\int_{-\infty}^0 \frac{e^{\alpha x}}{\sinh \pi x}\,dx+\int_0^\infty \frac{e^{\alpha x}}{\sinh \pi x}\,dx=\tan \tfrac{1}{2}\alpha$$

or $$-\int_0^\infty \frac{e^{-\alpha x}}{\sinh \pi x}+\int_0^\infty \frac{e^{\alpha x}}{\sinh \pi x}=\tan \tfrac{1}{2}\alpha$$

(on replacing x by $-x$ in the first integral)

or $$\int_0^\infty \frac{e^{\alpha x}-e^{-\alpha x}}{\sinh \pi x}=\tan \tfrac{1}{2}\alpha,\ \textit{i.e.}\ \int_0^\infty \frac{\sinh \alpha x}{\sinh \pi x}=\tfrac{1}{2}\tan \tfrac{1}{2}\alpha.$$

Problem 28. *Evaluate the following integrals by contour integration*:

(*i*) $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx.$

(*Rohilkhand, 1976, 89; Meerut, 1972, 79; Agra, 1961, 70, 75, 82*)

(*ii*) $\displaystyle\int_0^{2\pi} \frac{d\theta}{\frac{5}{4}+\sin\theta}.$ (*Agra, 1961*)

(*iii*) $\displaystyle\int_0^\infty \frac{\cos x}{x^2+a^2}\,dx.$

(*Rohilkhand, 1987; Meerut, 1981; Agra, 1962, 65, 71, 75, 80*)

(*iv*) $\displaystyle\int_{-\infty}^\infty \frac{x^2\,dx}{(x^2+a^2)^3}.$ (*Agra, 1962, 65*)

(*v*) $\displaystyle\int_0^\infty \frac{x^2}{1+x^4}\,dx.$

(*vi*) $\displaystyle\int_0^\infty \frac{1-\cos x}{x^2}.$ (*Meerut, 1968; Agra, 1964, 74*)

(*vii*) $\displaystyle\int_{-\infty}^\infty (x^2+a^2)^{-2}(x^2+b^2)^{-1}\,dx.$ (*Agra, 1966; Vikram, 1967*)

(*i*) The given integral is

$$\int_0^\infty \frac{\sin x}{x}\,dx.$$

In order to evaluate it, let us consider

$$\int_C f(z)\, dz = \int_C \frac{e^{iz}}{z}\, dz,$$

where the contour C is chosen as consisting of the real axis from $-R$ to $+R$, indented at $z = 0$ and the upper half of the semi-circle $|z| = R$ represented by Γ. Let radius of indentation be r and let it be represented by γ. Here $f(z)$ has got a simple pole at $z = 0$.

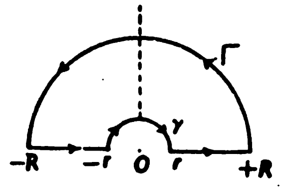


Fig. 5.45

Since there is no pole within the contour, we have by Cauchy's theorem

$$\int_C f(z)\, dz = 0,$$

$$\text{i.e., } \int_{-R}^{-r} f(x)\, dx + \int_\gamma f(z)\, dz + \int_r^R f(x)\, dx + \int_\Gamma f(z)\, dz = 0. \quad \text{...(1)}$$

Residue at $z = 0$ of $f(z)$ is $= \lim_{z=0} z f(z) = \lim_{z \to 0} z \frac{e^{iz}}{z} = 1$

Now $\lim_{r \to 0} \int_\gamma f(z)\, dz = -i\,(\pi - 0).\,1 = -\pi i$ by theorem of § 5.16 (–ve sign being taken as contour γ is in clockwise direction) and

$$\left|\int_\Gamma f(z)\, dz\right| \le \int_\Gamma |f(z)|\,|dz| = \int_\Gamma \frac{|e^{iz}|\,|dz|}{|z|}$$

$$= \int_0^\pi \frac{e^{-R \sin\theta}}{R} \cdot R\, d\theta, \quad \because z = Re^{i\theta},\ |dz| = R\, d\theta$$

$$\le 2 \int_0^{\pi/2} e^{-2R\theta/\pi} \text{ by Jordan's inequality}$$

$$\le 2.\left(\frac{-\pi}{2R}\right)\left[e^{-2R\theta/\pi}\right]_0^{\pi/2}$$

$$\le \frac{\pi}{R}\,(1 - e^{-R}), \text{ which} \to 0, \text{ as } R \to \infty.$$

With these results as $R \to \infty$ and $r \to 0$, (1) reduces to

$$\int_{-\infty}^0 f(x)\, dx - \pi i + \int_0^\infty f(x)\, dx = 0$$

or $$\int_{-\infty}^\infty f(x)\, dx = \pi i,$$

or $$2\int_{-\infty}^\infty \frac{e^{ix}}{x} dx = \pi i,$$

i.e. $$\int_0^\infty \frac{\cos x + i \sin x}{x}\, dx = \frac{\pi}{2} i.$$

Equating imaginary parts on either side, we get

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

(*ii*) Given $I = \int_0^{2\pi} \frac{d\theta}{\frac{5}{4} + \sin\theta}$.

Choosing the contour C as a circle ($|z| = 1$) of unit radius, we have

$$z = e^{i\theta}, \text{ so that } z - \frac{1}{z} = 2i \sin\theta, \text{ since } \frac{1}{z} = e^{-i\theta}$$

and $$dz = ie^{i\theta} d\theta = iz\, d\theta.$$

$$\therefore I = \frac{4}{i} \int_C \frac{1}{5 + 4\left\{\frac{1}{2i}\left(z - \frac{1}{z}\right)\right\}} \frac{dz}{z} = 4 \int_C \frac{dz}{5iz + 2(z^2 - 1)}.$$

Here $$f(z) = \frac{1}{2z^2 + 5iz - 2}.$$

Its poles are given by $2z^2 + 5iz - 2 = 0$,

i.e. $$z = \frac{-5i \pm \sqrt{(-25 + 16)}}{2} = \frac{-5i \pm 3i}{4} = -\frac{i}{2}, -2i$$

of which the pole $z = -2i$ lies outside the contour and therefore the only pole that lies within the contour is $z = -i/2$ (which is of order one).

$\therefore$ Residue at $z = -\frac{1}{2}i$ is

$$= \lim_{z \to -i/2} \left(z + \frac{1}{2}i\right) f(z) = \lim_{z \to -i/2} \left(z + \frac{1}{2}i\right) \cdot \frac{1}{\left(z + \frac{i}{2}\right)(z + 2i).2}$$

$$\lim_{z \to -i/2} \frac{1}{2(z + 2i)} = \frac{1}{2\,(-i/2 + 2i)} = \frac{1}{3i}.$$

Hence by Cauchy's theorem on residues

$I = 4 \times 4\pi i \Sigma R$, where ΣR represents the sum of residues

$$= 8\pi i.\frac{1}{3i} = \frac{8\pi}{3}.$$

(*iii*) The given integral is $\int_0^{\infty} \frac{\cos x}{x^2 + a^2}\, dx$.

Let us consider $\int_C f(z)\, dz = \int_C \frac{e^{iz}}{z^2 + a^2}\, dz = I$ (say).

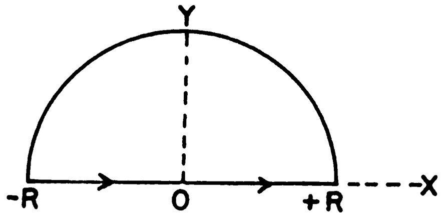


Fig. 5.46

Choosing the closed contour C, consisting of real axis from $-R$ to R and upper half of a large circle $|z| = R$, represented by Γ, we have by Cauchy's residue theorem,

$$\int_C f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_\Gamma f(z)\, dz$$

$$= 2\pi i \Sigma R^+. \qquad \ldots(1)$$

Poles of $f(z)$ are given by $z^2 + a^2 = 0$, *i.e.* $z = \pm ia$ of which only the simple pole $z = ia$ lies within the contour.

Residue at $z = +ia$ (a simple pole)

$$= \lim_{z \to ia} (z - ia) \frac{e^{iz}}{(z-ia)(z+ia)}$$

$$= \frac{e^{i^2 a}}{2ia} = \frac{e^{-a}}{2ia}.$$

Now $\left|\int_\Gamma f(z)\, dz\right| = \int_\Gamma \frac{e^{iz} dz}{z^2 + a^2}$

$$\le \int_\Gamma \frac{|e^{iz}|\,|dz|}{|z^2 + a^2|}$$

≤ 0 as $z \to \infty$.

$\therefore \int_\Gamma f(z)\, dz = 0$, since modulus cannot be negative.

Hence when $R \to \infty$, (1) reduces to

$$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \times \frac{e^{-a}}{2ia} = \frac{\pi}{a} e^{-a} = \frac{\pi}{ae^a},$$

i.e., $2\int_0^\infty \frac{(\cos x + i \sin x)}{x^2 + a^2}\, dx = \frac{\pi}{ae^a}.$

Equating real parts on either side, we get

$$\int_0^\infty \frac{\cos x}{x^2 + a^2} = \frac{\pi}{2ae^a}.$$

(*iv*) Suppose $I = \int_{-\infty}^{\infty} \frac{x^2}{\left(x^2 + a^2\right)^3} = dx.$

Here $f(z) = \frac{z^2}{\left(z^2 + a^2\right)^3},$

$zf(z) \to 0$ as $z \to \infty$, therefore the condition for the evaluation of an infinite integral is satisfied.

Let us choose the contour consisting of a semi-circle Γ of radius R, large enough to include all the poles of $f(z)$ and the part of the real axis from $-R$ to $+R$.

Poles of $f(z)$ are given by $(z^2 + a^2)^3 = 0$,

i.e., $z = \pm ia$ (each pole is of order three).

The only pole $z = \pm ai$ (of order three) lies within the contour C.

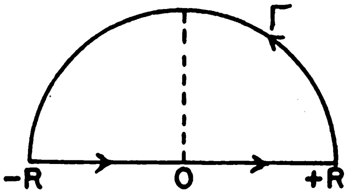


Fig. 5.47

Applying the theorem of residues,

$$\int_C f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_\Gamma f(z)\, dz = 2\pi i \Sigma R^+. \quad \text{...(1)}$$

Since $\lim_{z \to \infty} zf(z) \to 0$, we have

$$\lim_{R \to \infty} \int_C f(z)\, dz = 0.$$

Hence when $R \to \infty$, (1) becomes

$$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \Sigma R^+ \cdot \qquad \ldots(2)$$

Now residue of $f(z)$ at $z = ai$

$$= \text{coeff. of } \frac{1}{t} \text{ in } f(ai+t),\ t \text{ being small}$$

$$= \text{" "} \quad \frac{(ai+t)^2}{\left\{(ai+t)^2 + a^2\right\}^3}$$

$$= \text{" "} \quad \frac{t^2 + 2ait - a^2}{\left\{-a^2 + 2ait + t^2 + a^2\right\}^3}$$

$$= \text{" "} \quad \frac{1}{(2ait)^3}\left(t^2 + 2ait - a^2\right)\left(1 + \frac{t}{2ai}\right)^{-3}$$

$$= \text{" "} \quad \frac{-\left(t^2 + 2ait - a^2\right)}{8a^3 it^3}\left[1 - \frac{3t}{2ai} - \frac{6t^2}{4a^2} + \ldots\right]$$

$$= \text{" "} \quad \frac{-\left(t^2 + 2ait - a^2\right)}{8a^3 i}\left[\frac{1}{t^3} - \frac{3}{2ait^2} - \frac{6}{4a^2 t} + \ldots\right]$$

$$= -\frac{1}{8a^3 i}\left[1 - 3 + \tfrac{6}{4}\right] = \frac{1}{16a^3 i}.$$

Thus (2) gives

$$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \frac{1}{16a^3 i} = \frac{\pi}{8a^3},$$

i.e.,

$$I = \int_{-\infty}^{\infty} \frac{x^2\, dx}{\left(x^2 + a^2\right)^3} = \frac{\pi}{8a^3}.$$

(*v*) Suppose $I = \int_0^{\infty} \frac{x^2}{1 + x^4}\, dx \cdot$

Here $\quad f(z) = \frac{z^2}{1 + z^4} \cdot$

$zf(z) \to 0$ as $z \to \infty$, thus the condition for evaluation of an infinite integral is satisfied.

Poles of $f(z)$ are given by $1 + z^4 = 0$,

i.e. $\qquad z^4 = -1 = e^{i(2n+1)\pi}$,

so that $\qquad z = e^{i(2n+1)\pi/4}$, where n = 1, 2, 3,

i.e., the poles are $z = e^{i\pi/4}, e^{i3\pi/4}, e^{i5\pi/4}, e^{i7\pi/4}$ of which the first two only lie in the upper half plane, which is to be chosen as a contour consisting of a large semi-circle Γ and the part of the real axis from $-R$ to R.

Applying the theorem of residues,

$$\int_{-R}^{R} f(x)\, dx + \int_{\Gamma} f(z)\, dz = 2\pi i \Sigma R^+,$$

since $\int_\Gamma f(z)\,dz \to 0$ as $z \to \infty$, we have, on proceeding to the limit as $R \to \infty$,

$$\int_{-\infty}^{\infty} f(x)\ dx = 2\pi i \Sigma R^{+}. \qquad \ldots(1)$$

To find the residues at the poles, we have $f(z) = \dfrac{z^2}{1+z^4}$ which is of the form $\dfrac{\phi(z)}{\psi(z)}$, where $\phi(z) = z^2$ and $\psi(z) = 1 + z^4$.

$\therefore$ Residue at $z = e^{i\pi/4}$ is $= \dfrac{\phi\left(e^{i\pi/4}\right)}{\psi'\left(e^{i\pi/4}\right)} = \dfrac{e^{i\pi/2}}{4\left(e^{i\pi/4}\right)^3} = \frac{1}{4}\cdot e^{-\pi i/4}$

Similarly residue at $z = e^{i3\pi/4}$ is $= \frac{1}{4}\cdot e^{-3\pi i/4}$,

so that (1) gives

$$\int_{-\infty}^{\infty} f(x)\ dx = 2\pi i \tfrac{1}{4}\left[e^{-\pi i/4} + e^{-3\pi i/4}\right]$$

i.e., $2\displaystyle\int_0^{\infty} \frac{x^2\ dx}{1+x^4} = 2\pi i\cdot\frac{1}{4}\left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4} + \cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right]$

or $\quad I = \displaystyle\int_0^{\infty} \frac{x^2\ dx}{1+x^4} = \frac{\pi i}{2}\left(-2i\cdot\frac{1}{\sqrt{2}}\right) = \frac{\pi}{\sqrt{2}}.$

(*vi*) The given integral is $\displaystyle\int_0^{\infty} \frac{1-\cos x}{x^2}\,dx.$

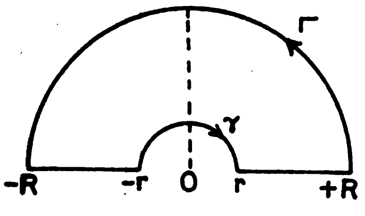


Fig. 5.48

Consider $f(z) = \dfrac{1-e^{iz}}{z^2}$.

Choose the contour C, consisting of the upper half of a circle $|z| = R$ and the real axis from $-R$ to R, indented at $z = 0$ (a pole of order 2). Let the semi-circle be denoted by Γ and the small semi-circle of indentation by γ. Let the radius of indentation be r.

By Cauchy's theorem, we have

$$\int_C f(z)\ dz = \int_{-R}^{-r} f(x)\ dx + \int_\gamma f(z)\ dz + \int_r^R f(x)\ dx + \int_\Gamma f(z)\ dz = 0. \qquad \ldots(1)$$

Now $\left|\displaystyle\int_\Gamma f(z)\ dz\right| = \left|\displaystyle\int_\Gamma \frac{1-e^{iz}}{z^2}\ dz\right|$,

$$\le \int_\Gamma \frac{\left|1-e^{iz}\right|\,|dz|}{\left|z^2\right|}$$

$$\le \int_0^{\pi} \frac{\left|1-e^{iRe^{i\theta}}\right|}{\left|R^2e^{2i\theta}\right|}\,|\,i\,|\,\left|Re^{i\theta}\right|\left|d\theta\right| \qquad \because z = Re^{i\theta}, dz = iRe^{i\theta}d\theta$$

$$\le \int_0^{\pi} \frac{1+e^{-R\sin\theta}}{R}\,d\theta, \text{ which} \to 0 \text{ as } R \to \infty.$$

$\therefore \int_\Gamma f(z)\, dz = 0$

and $\int_\gamma f(z)\, dz = \int_\pi^0 \frac{1-e^{ire^{i\theta}}}{r^2 e^{2i\theta}} ire^{i\theta}\, d\theta$ $\qquad \because z = re^{i\theta}$

$$= \int_\pi^0 \left\{1 - 1 - ire^{i\theta} - \frac{i^2 r^2}{\lfloor 2} e^{2i\theta} - \ldots\right\} \cdot \frac{i}{r} e^{-i\theta}\, d\theta$$

$$= \int_\pi^0 [1 + \text{terms containing powers of } r]\, d\theta,$$

$$\lim_{r \to 0} \int_\gamma f(z)\, dz = \int_\pi^0 d\theta = -\pi.$$

Hence when $R \to \infty$, $r \to 0$, (1) becomes

$$\int_{-\infty}^{\infty} f(x)\, dx - \pi + \int_0^{\infty} f(x)\, dx = 0,$$

i.e., $\int_{-\infty}^{\infty} f(x)\, dx = \pi$

or $\int_{-\infty}^{\infty} \frac{1-e^{ix}}{x^2} dx = \pi.$

Equating real parts on either side, we get

$$\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2} dx = \pi$$

or $$2\int_0^{\infty} \frac{1-\cos x}{x^2} = \pi,$$

i.e., $$\int_0^{\infty} \frac{1-\cos x}{x^2} = \frac{\pi}{2}.$$

(*vii*) The given integral is $\int_{-\infty}^{\infty} \left(x^2 + a^2\right)^{-2} \left(x^2 + b^2\right)^{-1} dx.$

Let $$f(x) = \frac{1}{\left(z^2 + a^2\right)^2 \left(z^2 + b^2\right)^2}$$

The poles are $z = \pm ai$ (or order 2) and $\pm bi$ (simple poles).

If we choose the contour C, consisting of the upper half of a large circle $|z| = R$ and the real axis from $-R$ to R, then the only poles lying within the contour are $+ai$ (of order 2) and $+bi$ (simple pole).

Here $zf(z) \to 0$ as $z \to \infty$.

Hence by Cauchy's theorem,

$$\int_C f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_\Gamma f(z)\, dz = 2\pi i \Sigma R^+, \qquad \ldots(1)$$

where ΣR^+ denotes the sum of the residues and Γ denotes the upper half of the semi-circle,

Now $\left| \int_\Gamma f(z)\, dz \right| = \left| \int_\Gamma \frac{dz}{\left(z^2 + a^2\right)^2 \left(z^2 + b^2\right)} \right|$

$$\leq \int_{\Gamma} \frac{|dz|}{|z^2+a^2|^2 \, |z^2+b^2|}$$

$$\leq 0 \text{ as } z \to \infty,$$

so that $\int_{\Gamma} f(z)\, dz = 0.$

$\therefore$ as $R \to \infty$, (1) reduces to

$$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \Sigma R^{+}. \qquad \ldots(2)$$

Residue at $z = ai$ (a pole of order 2)

$= \text{coeff. of } \frac{1}{t} \text{ in } f(ai+t)$

$$= \text{"} \quad \text{"} \quad \frac{1}{\left\{(t+ai)^2+a^2\right\}^2 \left\{(t+ai)^2+b^2\right\}}$$

$$= \text{"} \quad \text{"} \quad \frac{1}{t^2(2ai+t)^2 \left(b^2-a^2+2ait+t^2\right)}$$

$$= \text{"} \quad \text{"} \quad \frac{1}{-4a^2t^2\left(b^2-a^2\right)} \left\{1+\frac{t}{2ai}\right\}^{-2} \left\{1+\frac{2ait+t^2}{b^2-a^2}\right\}^{-1}$$

$$= \text{"} \quad \text{"} \quad \frac{1}{-4a^2t^2\left(b^2-a^2\right)} \left\{1-\frac{t}{ai}+\ldots\right\} \left\{1-\frac{2ait}{b^2-a^2}-\ldots\right\}$$

$$= \frac{1}{4a^2\left(b^2-a^2\right)} \left\{\frac{1}{ai}+\frac{2ai}{b^2-a^2}\right\} = \frac{b^2-3a^2}{4a^3i\left(b^2-a^2\right)^2}$$

Residue at $z = bi$ (simple pole) $= \lim_{z \to bi} (z-bi)\; f(z)$

$$= \lim_{z \to bi} (z-bi) \frac{1}{\left(z^2+a^2\right)^2 (z+bi)(z-bi)}$$

$$= \frac{1}{2bi\left(b^2-a^2\right)^2}$$

Hence from (2),

$$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \left[\frac{b^2-3a^2}{4a^3i\left(b^2-a^2\right)^2} + \frac{1}{2bi\left(b^2-a^2\right)^2}\right]$$

$$= \frac{\pi}{2a^3b\left(b^2-a^2\right)^2} \left[b^3-3a^2b+2a^3\right]$$

$$= \frac{\pi\,(b-a)^2\,(b+2a)}{2a^3b\,(b-a)^2\,(b+a)^2}$$

$$= \frac{\pi\,(b+2a)}{2a^3b\,(b+a)^2}.$$

5.21. ANALYTIC CONTINUATION

If there exist two functions $f_1(z)$ and $f_2(z)$, such that they are analytic (regular) in domains D_1 and D_2 respectively and that D_1 and D_2 have a common part, throughout which $f_1(z) = f_2(z)$, then the aggregate of values of $f_1(z)$ and $f_2(z)$ at the interior points of D_1 or D_2, can be regarded as a single regular function (say) $F(z)$. It is obvious that $F(z)$ is regular in the common part say Δ of the two domains and $F(z) = f_1(z)$ in domain D_1 and $F(z) = f_2(z)$ in domain D_2. We thus regard the function $f_2(z)$ as one, extending the domain in which $f_1(z)$ is defined and so it is called an **Analytic Continuation** of $f_1(z)$.

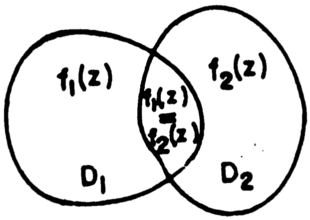


Fig. 5.49

The method of analytic continuation. Its standard method is the method of power series, which can be summarised as below.

Suppose that there is a point $P(z_0)$ in the neighbourhood of which $f(z)$ is analytic; then the function $f(z)$ can be expanded by Taylor's theorem, in a series of ascending powers of $(z - z_0)$, the coefficients of which involve the successive derivatives of $f(z)$ at z_0.

Let there be a singularity S of $f(z)$ which is nearest to P. Then a circle of centre P and radius PS is the circle of convergence within which the Taylor's expansion is valid. If we now take any point P' (not on PS) within this circle then we can find the values of $f(z)$ and all its derivatives at P', from the series by applying the method of term by term differentiation. We thus find the Taylor's series for $f(z)$ with P' as origin and this series will define a function which is regular in the circle whose centre is P. Such a circle will extend as far as the singularity of the function defined by the new series, which is nearest to P' and this may or may not be S. In either case the new cirlce of convergence may lie partly outisde the old circle and for points in the region which is included in the new circle but not in the old one, so that the new series may be utilised in defining the vlaues of $f(z)$ while the old series failed to do so.

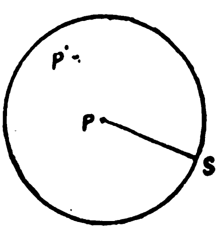


Fig. 5.50

In a similar manner, we can take another point P'' in the region for which the values of the function are known and form the Taylor's series with P'' as origin which will, in general, still further extend the domain of definition of the function and so on and so forth.

By this method of continuation, starting from a representation of a function by any one power series, we can find any number of other power series, which between them define the value of the function at all points of a domain, any point of which can be reached from P without passing through a sigularity of the function $f(z)$.

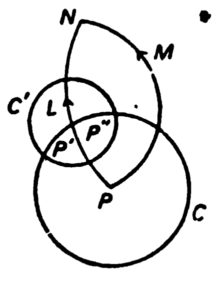


Fig. 5.51

It is easy to show that continuation by two differnt paths PLN and PMN gives the same power series provided the function is analytic and has no singularity inside the closed curve $PLNMP$. To show it, let us suppose that $S, S,' S''$,...be the power series with P, P', P'',... as origin. Then $S' = S''$ (for, each is equal to S) over a certain domain which contains P', when P'' is taken sufficiently near to P', and therefore S' will be the continuation of S''. Continuing this process we can deform the path PLN into PMN provided no singular point lies inside the path $PLNMP$.

Note. *Weierstrass defined an* **Analytic function** *of z as one power series together with all the other power series derivable from it by analytic continuation.*

An important remark. There must be at least one singularity of the analytic function on the circle of convergence C_0 of the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$.

Assuming that there is no such singularity, we can construct by the method of continuation, a function equal of $f(z)$ within C_0 and regular in a larger concentric circle Γ_0. The expansion of this function in Taylor series in powers of $(z-z_0)$ would then converge everywhere within the large circle Γ_0. But this is not possible, since the series would be the original series which has got C_0 as its circle of convergence. Let there be a point z_1 within C_0 and let C_1 be the circle of convergence of the power series,

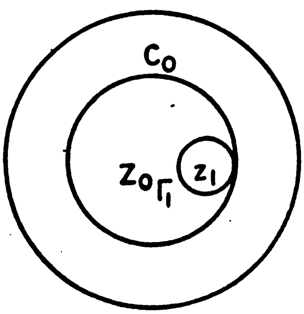


Fig. 5.52

$$\sum_{0}^{\infty} A_n(z-z_1)^n,$$

where $A_n = \dfrac{f^n(z_1)}{n!}$ (by Taylor's expansion).

Let Γ_1 be the circle of centre z_1, touching the cirlce C_0 internally. Then the new power series defined by (1) is certainly convergent within Γ_1 and has the sum $f(z)$ there.

Since the radius of C_1 cannot be less than that of Γ_1, there are these possibilities:

(*i*) C_1 has a larger radius, then Γ_1, in which case C_1 lies partly outside C_0 and the new power series provides an analytic continuation of $f(z)$. Then taking a point z_2 within C_1 and outside C_0, the process can be repeated.

(*ii*) C_0 is a natural boundary of $f(z)$. [Note. A closed curve is called a *natural boundary* of a function if the function is such that it cannot be analytically continued to any point of the same]. In this case we cannot continue $f(z)$ outside C_0 and the circle C_1 touches C_0 internally, wherever the point z lies within C_0.

(*iii*) C_1 may touch C_0 internally, though C_0 is not a natural boundary of $f(z)$. In this case the point of contact of C_0 and C_1 is a singularity of the analytic function which has been found by continuation of the original power series: for, there is necessarily one singularity on C_1 and this cannot the within C_0.

Problem 29. *Show that the two power series*

$$z+\frac{1}{2}z^2+\frac{1}{3}z^3+\ldots \qquad \ldots(1)$$

and

$$i\pi-(z-2)+\frac{1}{2}(z-2)^2-\ldots \qquad \ldots(2)$$

have no common region of convergence, but they are analytic continuations of the same function.

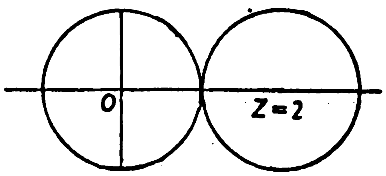


Fig. 5.53

Here the series (1) defines the function —log (1 – z), whose circle of convergence is $|z| = 1$, *i.e.* a circle with centre $z = 0$ and radius 1. The only singularity is $z = +1$.

The series (2) defines the function $i\pi - \log [1 + (z - 2)]$; where $|z - 2| \leq 1$ is a circle with centre $z = 2$ and radius 1. It touches the first circle externally as shown in the adjoining figure.

The function $i\pi - \log [1 + (z - 2)] = i\pi - \log (z - 1)$

$$= i\pi - \log [-(1 - z)]$$
$$= i\pi - \log(-1) - \log (1 - z)$$
$$= i\pi - \log e^{\pi i} - \log (1 - z)$$
$$= i\pi - \pi i - \log (1 - z)$$
$$= -\log (1 - z).$$

It is clear that the two functions defined by the given power series have no common region of convergence, but they are analytic continuations of the same function $-\log (1 - z)$.

Problem 30. *Show that the series*

$$\frac{1}{a} + \frac{z}{a^2} + \frac{z^2}{a^3} + \frac{z^3}{a^4} + \ldots$$

represents the function which can be continued analytically outside the circle of convergence.

Let the given series define a function $f(z)$ *i.e.*

$$f(z) = \frac{1}{a} + \frac{z}{a^2} + \frac{z^2}{a^3} + \frac{z^3}{a^4} + \ldots$$

$$= \frac{\frac{1}{a}}{1 - \frac{z}{a}} \left[\text{being an infinite geometric series if } \left|\frac{z}{a}\right| > 1\right]$$

$$= \frac{1}{a - z},$$

only for points within the circle fo convergence $|z| = |a|$

If we consider a series

$$\frac{1}{a-b} + \frac{z-a}{(a-b)^2} + \frac{(z-a)^2}{(a-b)^3} + \ldots \text{ where } \frac{b}{a} \text{ is real,}$$

then
$$f(z) = \frac{1}{a-b}\left[1 + \frac{z-b}{a-b} + \left(\frac{z-b}{a-b}\right)^2 + \ldots\right]$$

$$= \frac{1}{a-b}\left[1 - \frac{z-b}{a-b}\right]^{-1}$$

provided the circle of convergence say C_1 is $|z - b| = |a - b|$, when a and b both are real and positive and also $0 < b < a$, the circle of convergence C_1 of the second series touches the first circle (say) C internally at its only singularity $z = a$. If follows that there is no analytic continuation (see figure 5.54).

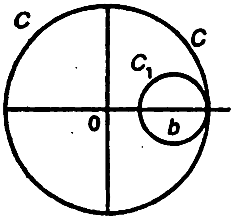


Fig. 5.54

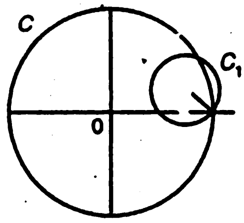
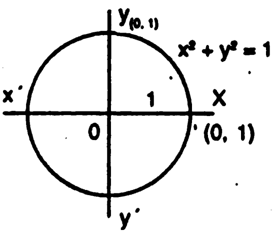


Fig. 5.55

In case $\frac{b}{a}$ is not real and $|b| < |a|$ and positive, the second series converges at points inside a circle which is partly inside and partly outside the circle $|z| = a$. In fact the two series represent the same function $\frac{1}{a-z}$ at points outside the circle $|z| = |a|$ and hence can be continued analytically as shown in Fig. 5.55.

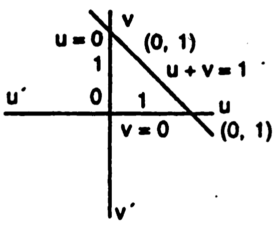
5.22. CONFORMAL MAPPING

We can set a correspondence between a domain D of x-y plane and a domain D′ of u-v plane by the transformation or mapping $w = f(z)$ i.e. $u = u(x, y)$ and $v = v(x, y)$ e.g. if $u = x^2$ and $v = y^2$, then the circular domain $x^2 + y^2 \leq 1$ in z-plane corresponds to the triangle in w-plane bounded by the lines $u = 0$, $v = 0$ and $u + v = 1$.



x – y plane (i.e. z-plane)

Fig. 5.56



u – v-plane (i.e. w-plane)

Fig. 5.57

Conformal and Isogonal transformation. If the two curves in z-plane intersect at a point z_0 (x_0, z_0) at an angle θ, then if the two corresponding curves in w-plane intersect at w_0 (u_0, v_0) at the same angle θ, where w_0 corresponds z_0, the transformation or mapping is known as **Isogonal**. In other words if only the magnitude of the angle is preserved, the mapping is Isogonal, but if the sense of rotation as well as the magnitude of the angle is preserved, the mapping is **conformal**.

Necessary and Sufficient Conditions for Conformality

I. If $f(z)$ is analytic, then the mapping is conformal

i.e. the necessary condition for conformality is that $f(z)$ must be analytic.

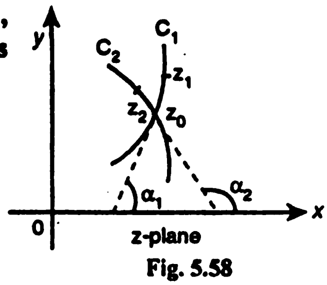


Fig. 5.58

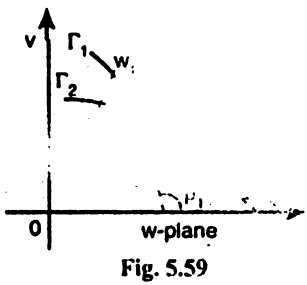


Fig. 5.59

In other words an analytic function is necessarily conformal.

Take points z_1, z_2 on curves C_1, C_2 in z-plane near to z_0 at distance r such that tangents at z, z_2 make angles, α_1 α_2 with the real axis. Let the corresponding points in w-plane be w_1, w_2 on curves Γ_1, Γ_2 near to w. Then $z_1 - z_0 = r\, e^{i\theta_1}$, $z_2 - z_0 = re^{i\theta_2}$ and if as $r \to 0$, $\theta_1 \to \alpha_1, \theta_2 \to \alpha_2$.

Also $w_1 - w_0 = \rho_1 e^{i\phi_1}$, $w_2 - w_0 = \rho_2 e^{i\phi_2}$ and if $\rho_1 \to 0, \phi \to \beta_1, \phi_2 \to \beta_2$

then $f'(z_0) \operatorname{Lim}_{z \to z_0} \dfrac{w_1 - w_0}{z_1 - z_0}$

or say, $Re^{i\lambda} = \operatorname{Lim} \dfrac{\rho_1 e^{i\phi_1}}{re^{i\theta_1}}$ or $f'(z_0) \neq 0$ and it may be written as $Re^{i\lambda}$

$$= \operatorname{Lim} \frac{\rho_1}{r} e^{i(\phi_1 - \theta_1)}$$

Equating moduli and arguments on either side, we get

$$R = \operatorname{Lim} \frac{\rho_1}{r} = |f'(z_0)| \text{ and } \lambda = \operatorname{Lim}(\phi_1 - \theta_1) = \operatorname{Lim}\phi_1 - \operatorname{Lim}\theta_1 = \beta_1 - \alpha_1$$

giving $\beta_1 = \alpha_1 + \lambda$ and similarly $\beta_2 = \alpha_2 + \lambda$

$\therefore \beta_1 - \beta_2 = \alpha_1 - \alpha_2 \Rightarrow$ The angle between the curves C_1 and C_2 in z-plane is the same as the angle between the curves Γ_1 and Γ_2 in w-plane *i.e.* the magnitude as well as the sense of rotation of the two angles in z and w planes is the same. This means the transformation is conformal.

II. If the mapping is conformal, then the function $w = f(z)$ is analytic i.e. the sufficient condition for conformality is that $f(z)$ is analytic.

Consider a pair of differentiable relations

$$u = u(x, y),\ v = v(x, y)$$

defining a transformation from z-plane (*i.e.* x-y plane) to w-plane (*i.e.* u-v plane). The transformation being of the form $w = f(z)$, where $f(z)$ is regular, is conformal.

Take the elements of lengths $d\sigma$ and ds in (u, v) and (x, y) planes respectively, so that

$$ds^2 = dx^2 + dy^2 \quad \text{...(1)}$$

and

$$d\sigma^2 = du^2 + dv^2 \quad \text{...(2)}$$

But $\partial u = \dfrac{\partial u}{\partial x} dx + \dfrac{\partial u}{\partial y} dy$ and $\partial v = \dfrac{\partial v}{\partial x} dx + \dfrac{\partial v}{\partial x} dx + \dfrac{\partial v}{\partial y} dy$

$$\therefore\ d\sigma^2 = \left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right\}(dx)^2 + 2\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right)dxdy$$

$$+ \left\{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right\}(dy)^2 \quad \text{...(2)}$$

Mapping being conformal, the ratio $d\sigma{:}ds$ is independent of direction if

$$\frac{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}{1} = \frac{\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}}{0} = \frac{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2}{1} \text{ by (1) and (2)}$$

giving $$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2=\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2=h^2 \text{ (say)} \quad (3)$$

and $$\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}=0 \quad (4)$$

Equation (3) is satisfied if $u_x\left(i.e. \frac{\partial u}{\partial x}\right)=h\cos\alpha, v_x=h\sin\alpha,$

$u_y=h\cos\beta, v_y=h\sin\beta$ and (4) is satisfied if $\alpha-\beta=\pm\frac{\pi}{2}$

Thus the correspondence is Isogonal if either

$(a) u_x=v_y, v_x=-u_y$ or $(b)\ u_x=-v_y, v_x=u_y$

Of (a) *are Cauchy - Riemann equations* and expressed as $w=f(z)$ *i.e.* $u+iv=f(x+iy)$, where $f(z)$ is regular function of z and equations (b) reduce to (a) by writing $-v$ for v i.e. by taking image formed by reflection in the real axis of the w-plane so that (b) correspond to an Isogonal but not conformal transformation.

Hence the only conformal transformation of z-plane into w-plane is of the form $w=f(z)$, where $f(z)$ is a regular function of z.

III. The case f'(z) = 0

Assuming that $f'(z_0)$ has a zero of order $(n-1)$ at the point z_0, we have in the neighbourhood of z_0 by Taylor's theorem,

$$f(z)=f(z_0)+a_n(z-z_0)^n+\ldots \text{ where } a_n=\frac{f_n(z_0)}{\underline{|n}}\neq 0$$

$$\therefore f(z)-f(z_0)=a_n(z-z_0)^n+\ldots$$

$$i.e. w_1-w_0=a_n(z_1-z_0)^n+\ldots$$

$$\text{or } \rho_1 e^{i\phi_1}=|a_n|r^n e^{i(n\theta_1+\lambda)}+\ldots \text{ where } \lambda=\arg a_n$$

$$\therefore \text{ Lim}\,\phi_1=\text{Lim}\,(n\theta_1+\lambda)=n\alpha_1+\lambda$$

Similarly, $\phi_2=n\alpha_2+\lambda$

As such the curves Γ_1, Γ_2 still have definite tangents at w_0, but the angle between the tangents is

Lim $(\phi_2-\phi_1)=n(\alpha_2-\alpha_1)\Rightarrow$ the magnitude of the angle is not preserved, but magnified.

Linear magnification $R=\text{Lim}\frac{1}{r}=0\Rightarrow$ the conformal property does not hold good at such a point where $f'(z)=0$.

Note: *The points at which* $\frac{dw}{dz}=0$ or ∞ *are called* **critical points** *of the transformation defined by* $w=f(z)$.

IV. Transformations which are Isogonal but not Conformal.

In this case the magnitude of the angles is conserved but their sign is changed such as

$$w = x - iy = \bar{z}$$

which replaces every point by its reflection in the real axis, so that angles are conserved but signs are changed. In general it is true for the transformation of the type

$w = f(\bar{z}), f(z)$ being regular.

It is a combination of two transformations

(i) $\zeta = \bar{z}$, (ii) $w = f(\zeta)$

In (i), angles are conserved but their signs are changed and in (ii) angles as well as signs are conserved. Thus $w = f(\bar{z})$ gives a transformation which is Isogonal but not conformal.

Linear or Bilinear or Mobius' Transformation

Its form is
$$w = \frac{az+b}{cz+d} \qquad \ldots (5)$$
where z, w are complex variables and a, b, c, d are complex constants.

If we write (5) in the form
$$cwz + dw - az - b = 0 \qquad \ldots (6)$$
then it is linear in z as well as w and hence it is called bilinear. It was studied by A.F. Mobius (1790–1868) as mentioned by Carathéodory and hence bears the name of Mobius.

Writing (5) as
$$w = \frac{a}{c}\frac{z + b/a}{z + d/c} \qquad \ldots (7)$$

it follows that for every value of z, there is the same value of w, if $\frac{b}{a} = \frac{d}{c}$ *i.e.* $ad - bc = 0$ ... (8)

and there correspond different values of w to different values of z if $ad - bc \neq 0$ ... (9)

Here the expression (ad–bc) is known as the **Determinant of the Transformation.**

(7) $\Rightarrow$ For every $z \neq -\frac{d}{c}$, there exists a value of w ... (10)

(5) can also be written is
$$z = \frac{dw - b}{-cw + a} = -\frac{d}{c}\frac{w - b/d}{w - a/c} \qquad \ldots (11)$$

This implies, that for every $w \neq \frac{a}{c}$, there exist a value of z ... (12)

Now, (10) and (12) $\Rightarrow$ In (5), the correspondence between w and z is one-one, except that when $c \neq 0$ for $z = -\frac{d}{c}, |w| \to \infty$ so that we may regard the point at infinity in w-plane (or extended plane) as corresponding to the point $z = -\frac{d}{c}$ in z-plane. Similarly the point at infinity in z-plane corresponds to $w = \frac{a}{c}$ in w-plane.

When $c = 0$, then (5) gives $w = \frac{a}{d}z + \frac{b}{d}$, so that for $a \neq 0$, the points at infinity in the two planes correspond.

Again if we can write (1) as

$$w = \frac{a}{c} + \frac{bc - ad}{c}\frac{1}{cz + d} \text{ with } ad - bc \neq 0 \text{ and } c \neq 0$$

or
$$w = \frac{a}{c} + \frac{bc - ad}{c^2}\frac{1}{z + \frac{d}{c}} \quad \text{... (13)}$$

then it is obtained by a superimposition of successive mappings

(*i*) $\tau = z + \frac{d}{c}$ of the type $z + \alpha$ known as **translation**

(*ii*) $\zeta = \frac{1}{\tau}$ known as **inversion**

(*iii*) $w = \frac{a}{c} + \frac{bc - ad}{c^2}\zeta$ of the type $w = \beta z$, known as **magnification and rotation.**

Note. Points which coincide with their transforms under bilinear transformation are known as **fixed points of the bilinear transformation.** Thus z is a fixed point of (5) if

$$z = \frac{az + b}{cz + d} \text{ i.e. } cz^2 + (d - a)z - b = 0 \quad \text{... (14)}$$

It will have distinct roots ζ_1, ζ_2 *if* $c \neq 0$ and the discriminant $\Delta^2 = (d - a)^2 + 4bc \neq 0$

We can take $\zeta_1 = \frac{a - d + s}{2c}, \zeta_2 = \frac{a - d - s}{2c}$

Conversely if $c \neq 0$, each of these points is mapped onto itself, but if $\Delta = 0$, there is only one fixed point.

If $c = 0$, but $d \neq 0$, we have $w = \frac{a}{d}z + \frac{b}{d}$ ... (15)

In bilinear transformation ∞ is also a fixed point and another possible fixed point is given by (15) *i.e.* by

$$(d - a)\, z = b \quad \text{... (16)}$$

Hence, if $d - a \neq 0$, the transformation has the two fixed points ∞ and $\frac{b}{d - a}$ and if $d = a$, it has only one fixed point viz. ∞.

Conclusively, the mapping or transformation of a simple analytic (regular) function $w = f(z)$ can be performed in three manners:

1. **Translation** : $w = z + \alpha$, where α is a complex constant; or $u + iv = x + iy + (p + iq)$, where $\alpha = p + iq$

$$\Rightarrow u = x + p \text{ and } v = y + q \quad \text{OR}$$
$$x = u - p \text{ and } y = v - q$$

Thus a point $P(x, y)$ in the z-plane is mapped onto the point $Q(x + p, y + q)$ i.e. the w-plane, which is merely a translation of the coordinate axes.

2. **Inversion :** $w=\frac{1}{z}$ or $Re^{i\phi}=\frac{1}{re^{i\theta}}=\frac{1}{r}e^{-i\theta}$ in polar coordinates

$$\Rightarrow R=\frac{1}{r} \text{ and } \phi=-\theta$$

Thus a point $P(r,\theta)$ in the z-plane is mapped onto the point $Q\left(\frac{1}{r},-\theta\right)$ in the w-plane, representing a reflection into real axis. As such the interior of a unit circle $|z|=1$ in the z-plane is mapped onto the exterior of the unit circle in the w-plane.

3. **Magnification and Rotation :** $w=\beta z$, where w, z are complex numbers and β is a complex constant.

Taking $w=Re^{i\phi}, z=re^{i\theta}$ and $\beta=be^{i\alpha}$, we have

$Re^{i\phi}=bre^{i(\theta+\alpha)} \Rightarrow R=br$ and $\phi=\theta+\alpha$

Thus the modulus is magnified as $R=br$ and the angle is rotated through α.

Illustrative Examples

Example 1. *To find all the mobius transformations which transform the half plane $I(z)\geq 0$ into the unit circle $|w|\leq 1$(one).*

Möbius transformation is

$$w=\frac{az+b}{cz+d}=\frac{a}{c}\frac{z+b/a}{z+d/c} \qquad \dots (1)$$

which transforms $I(z)=0$ into $|w|=1$ *i.e.* real axis in z-plane transforms into the unit circle in w-plane.

It is observed that points $w, \frac{1}{\bar{w}}$ inverse w.r.t., the unit circle in w-plane transform into points $z, \bar{z}$ symmetrical (inverse) w.r.t. the real axis in z-plane. In particular $w=0, \infty$ correspond $z=\alpha, \bar{\alpha}$ (say).

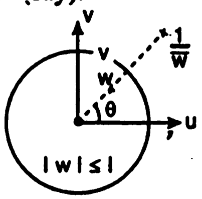


Fig. 5.60

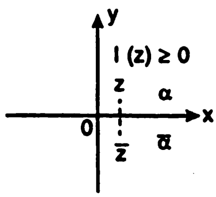


Fig. 5.61

$\therefore$ (1) $\Rightarrow -\frac{b}{a}=\alpha, -\frac{d}{c}=\bar{\alpha}$, so that (1) reduces to

$$w=\frac{a}{c}\frac{z-\alpha}{z-\bar{\alpha}} \text{ or } |w|=\left|\frac{a}{c}\right|\left|\frac{z-\alpha}{z-\bar{\alpha}}\right| \qquad \dots (2)$$

The point $z=0$ must correspond to a point in the circle $|w|=1$, so that (2) gives

$$1=\left|\frac{a}{c}\right|\left|\frac{0-\alpha}{0-\bar{\alpha}}\right| \text{ or } 1=\left|\frac{a}{c}\right|\left|\frac{\alpha}{\bar{\alpha}}\right| \text{ or } 1=\left|\frac{a}{c}\right| \text{ as } |\alpha|=|\bar{\alpha}|$$

We may write $\frac{a}{c} = e^{i\lambda}$, λ being real, then

$$(2) \Rightarrow w = \frac{z-\alpha}{z-\overline{\alpha}} e^{i\lambda} \quad \ldots (3)$$

Since $z = \alpha$ gives $w = 0$, α must be a point in the *upper half plane* i.e. $I(\alpha) > 0$. With this condition, (3) gives the required transformation.

Again, we have from (3)

$$w\overline{w} - 1 = \frac{z-\alpha}{z-\overline{\alpha}} e^{i\lambda} \cdot \frac{\overline{z}-\overline{\alpha}}{\overline{z}-\alpha} e^{-i\lambda} - 1 \quad as \ \overline{\overline{\alpha}} = \alpha$$

or
$$|w|^2 - 1 = \frac{(z-\alpha)(\overline{z}-\overline{\alpha})}{(z-\overline{\alpha})(\overline{z}-\alpha)} - 1 \text{ as } w\overline{w} = |w|^2$$

$$= \frac{z\overline{z} - z\overline{\alpha} - \overline{z}\alpha + \alpha\overline{\alpha} - z\overline{z} + z\alpha + \overline{z}\overline{\alpha} - \alpha\overline{\alpha}}{|z-\overline{\alpha}|^2} \text{ as } \overline{z} - \alpha = \overline{z - \overline{\alpha}}$$

$$= \frac{(z-\overline{z})(\alpha-\overline{\alpha})}{|z-\overline{\alpha}|^2} = \frac{2i\,Im(z) \times 2i\,Im(\alpha)}{|z-\overline{\alpha}|^2}$$

$$= -\frac{4\,Im(z)\,Im(\alpha)}{|z-\overline{\alpha}|^2} \quad \ldots (4)$$

$\therefore$ (4) $\Rightarrow |w|^2 - 1 < 0$ for $I(z) > 0$

$\Rightarrow |w| < 1$ corresponds to $I(z) > 0$

$\Rightarrow$ upper half plane in a z-plane corresponds interior of the unit circle in w-plane.

Hence (3) is the required transformation.

Example 2. *To find a Mobius transformation which maps the circle $|w| \leq 1$ into the circle $|z-1| < 1$ and maps $w = 0$, $w = 1$ respectively into* $z = \frac{1}{2}, z = 0$.

We have
$$w = \frac{az+b}{cz+d} = \frac{a}{c}\,\frac{z+\frac{b}{a}}{z+\frac{d}{c}} \quad \ldots (1)$$

We know that for a circle $|z-a| = \rho$, if one inverse point is z, then the other is $a + \frac{\rho^2}{\overline{z}-\overline{a}}$.

Here $|w| = 1$ corresponds to $|z-1| = 1$.

$\therefore$ Points $w, \frac{1}{\overline{w}}$ inverse w.r.t. the circle $|w| = 1$ correspond to points z, $1 + \frac{1}{\overline{z}-1}$ (as $a = 1$, $\rho = 1$) inverse w.r.t. the circle $|z-1| = 1$. In particular $w = 0, \infty$, correspond to $z = \frac{1}{2}, 1 + \frac{1}{\frac{1}{2}-1}$ i.e. $z = \frac{1}{2}, -1$ (since $w = 0$ for $z = \frac{1}{2}$ given)

$\therefore$ (1) gives $-\frac{b}{a}=\frac{1}{2}, \; -\frac{d}{c}=-1$

$$\text{So that (1)} \Rightarrow w=\frac{a}{c}\,\frac{z-\frac{1}{2}}{z+1} \qquad \ldots (2)$$

As $w = 1$ corresponds to $z = 0$, therefore (2) reduces to

$$1=\frac{a}{c}\,\frac{0-\frac{1}{2}}{0+1}=-\frac{a}{2c} \text{ or } \frac{a}{c}=-2, \text{ whence (2) yields}$$

$$w=-2\frac{z-\frac{1}{2}}{z+1} \text{ i.e. } w=-\frac{2z-1}{z+1}$$

which is the required transformation.

Example 3. *To show that the transformation* $w=\frac{2z+3}{z-4}$ *changes the circle* $x^2+y^2-4x=0$ *into the straight line* $4u+3=0$.

Here $\quad w=\frac{2z+3}{z-4} \Rightarrow u+iv=\frac{2(x+iy)+3}{x+iy-4}$

or $\quad u+iv=\frac{(2x+3)+i.2y}{(x-4)+iy}\times\frac{(x-4)-iy}{(x-4)-iy}=\frac{(2x^2+2y^2-5x-12)-i.11y}{(x-4)^2+y^2}$

Equating real and imaginary parts,

$$u=\frac{2x^2+2y^2-5x-12}{(x-4)^2+y^2} \ldots (1) \text{ and } v=\frac{-11y}{(x-4)^2+y^2} \qquad \ldots (2)$$

The z-curve is given as $x^2+y^2-4x=0$ or $x^2+y^2=4x \qquad \ldots (3)$

So that (1) $\Rightarrow u=\frac{3x-12}{-4x+16}=-\frac{3(x-4)}{4(x-4)}=-\frac{3}{4}$ by (3)

$\Rightarrow 4u+3=0$, which is the corresponding w-curve.

Example 4. *To show by means of transformation* $w=\left(\frac{z-ic}{z+ic}\right)^2$, *c being real, that the upper half of w-plane may be made to correspond to the interior of a certain semicircle in z-plane.*

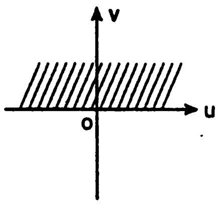
We have

$$w=\left(\frac{z-ic}{z+ic}\right)^2$$

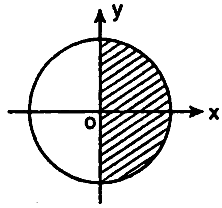
$$\text{or } u+iv=\left\{\frac{x+i(y-c)}{x+i(y+c)}\right\}^2\times\left\{\frac{x-i(y+c)}{x-i(y+c)}\right\}^2$$

$$=\left\{\frac{(x^2+y^2-c^2)^2-2icx}{x^2+(y+c)^2}\right\}^2$$

giving $u = \dfrac{(x^2 + y^2 - c^2)^2 - 4c^2x^2}{\{x^2 + (y+c)^2\}^2}$ ...(1) and $v = \dfrac{-4cx(x^2 + y^2 - c^2)}{\{x^2 + (y+c)^2\}^2}$ ...(2)



w-plane
Fig. 5.62



z-plane
Fig. 5.63

Here, (2) $\Rightarrow$ if x is + ve, v is + ve and $x^2 + y^2 - c^2$ is – ve

$\Rightarrow x > 0, v > 0, x^2 + y^2 - c^2 < 0$ or $x^2 + y^2 < c^2$

$\Rightarrow$ upper half of the w-plane corresponds to the right half interior of the circle $x^2 + y^2 = c^2$ in the z-plane

Example 5. *Under the mapping $w = z^2$, to show that the family of circles, $|w - 1| = c$ is transformed into the family of lemniscates $|z - 1||z + 1| = c$, c being a parameter.*

Here $\quad w = z^2 \Rightarrow w - 1 = z^2 - 1$ or $z^2 - 1 = w - 1$

or $\quad (z-1)(z+1) = w - 1 \Rightarrow |z-1||z+1| = |w-1|$

If $|w - 1| = c$, this gives $|z-1||z+1| = c$ i.e. it corresponds to the circle $|w-1| = c$.

Now, to show that $|z - 1||z + 1| = c$ represents a lemniscate, we have $|z-1||z+1| = c$ or $|z^2 - 1| = c$, c being real

$$z^2 - 1 = ce^{i\lambda}, \lambda \text{ being real}$$

$$\text{or } z^2 = 1 + ce^{i\lambda} = a^2 e^{2i\alpha} \text{ (say)}$$

$$\text{or } r^2 e^{2i\theta} = a^2 e^{2i\alpha} \text{ giving } r^2 = a^2 e^{-2i(\theta-\alpha)}$$

Equating real parts, $r^2 = a^2 \cos(\theta - \alpha)$, which represent a lemniscate with its axis inclined at an angle α to the real axis.

Example 6. *If $(w-1)^2 = \dfrac{4}{z}$, then show that the unit circle in the w-plane corresponds to a parabola in the z-plane and the inside of the circle to the outside of a parabola.*

Here $\quad (w-1)^2 = \dfrac{4}{z} \Rightarrow w = \dfrac{2}{\sqrt{z}} - 1 = \dfrac{2}{\sqrt{r}} e^{-i\theta/2} - 1$, where $r = e^{i\theta}$

$$\text{or } w = \frac{2}{\sqrt{r}}\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right) - 1 = \left(\frac{2}{\sqrt{r}}\cos\frac{\theta}{2} - 1\right) - i\left(\frac{2}{\sqrt{r}}\sin\frac{\theta}{2}\right)$$

$$\therefore |w| = \sqrt{\left(\frac{2}{\sqrt{r}}\cos\frac{\theta}{2} - 1\right)^2 + \left(\frac{2}{\sqrt{r}}\sin\frac{\theta}{2}\right)^2} = \sqrt{\left(\frac{4}{r} - \frac{4}{\sqrt{r}}\cos\frac{\theta}{2} + 1\right)}$$

So $|w| \leq 1$ if $\dfrac{4}{r} - \dfrac{4}{\sqrt{r}} \cos\dfrac{\theta}{2} + 1 \leq 1$ or if $r\cos^2\dfrac{\theta}{2} \geq 1$. or if $\dfrac{2}{r} \leq 1 + \cos\theta$ which follows that the unit circle $|w| = 1$ in w-plane corresponds to the parabola $\dfrac{2}{r} = 1 + \cos\theta$ in polars in z-plane and interior of the circle $|w| < 1$ corresponds to the exterior of the parabola $\dfrac{2}{r} < 1 + \cos\theta$.

Example 7. *If $w = \tan^2 \dfrac{z}{2}$, the strip in the z-plane between $x = 0$, $x = \pi/2$ is represented on the interior of the unit circle in the w-plane, cut along the real axis from $w = -1$ to $w = 0$.*

We have $w = \tan^2 \dfrac{z}{2} = \dfrac{1 - \cos z}{1 + \cos z}$

On the line $x = \pi/2, z = \dfrac{\pi}{2} + iy$,

so that
$$w = \frac{1 + i\sinh y}{1 - i\sinh y}$$

$$\therefore \; |w| = \left|\frac{1 + i\sinh y}{1 - i\sinh y}\right| = \frac{\sqrt{1 + \cosh^2 y}}{1 + \cosh^2 y} = 1$$

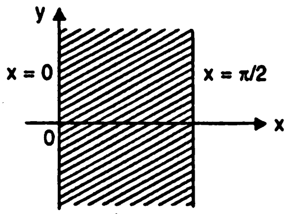


Fig. 5.64

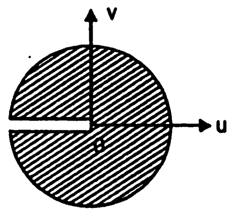


Fig. 5.65

Thus the line $x = \dfrac{\pi}{2}$ in z-plane corresponds to the unit circle $|w| = 1$ in the w-plane.

On the line $x = 0$, $z = iy$ so that

$$w = \frac{1 - \cosh z}{1 + \cosh z} = \frac{2 - e^{y} - e^{-y}}{2 + e^{y} + e^{-y}} \text{ which is real}$$

$$\text{When } y \to -\infty, w = \lim_{y \to -\infty} \frac{2 - e^{y} - e^{-y}}{2 + e^{y} + e^{-y}} = \lim_{y \to -\infty} \frac{2e^{y} - e^{2y} - 1}{2e^{y} + e^{2y} + 1} = -1$$

$$\text{When } y \to \infty, \quad w = \lim_{y \to \infty} \frac{2e^{-y} - 1 - e^{-2y}}{2e^{-y} + 1 + e^{-2y}} = -1$$

Hence as z travels along the line $x = 0$, from $y = -\infty$ to $y = +\infty$, w travels along the real axis from $w = -1$ to $w = 0$ and back again from $w = 0$ to $w = -1$ i.e. $x = 0$ in the z-plane, corresponds a cut from $u = -1$ to $u = 0$ in the w-plane.

Again, $$w = \frac{1-\cos z}{1+\cos z} = \frac{1-\cos(x+iy)}{1+\cos(x+iy)}$$

$$= \frac{1-\cos x \cosh y + i \sin x \sinh y}{1+\cos x \cosh y - i \sin x \sinh y}$$

Thus $|w| < 1$ if $\left|\frac{1-\cos x \cosh y + i \sin x \sinh y}{1+\cos x \cosh y - i \sin x \sinh y}\right| < 1$

or if $\sqrt{(1-\cos x \cosh y)^2 + \sin^2 x \sinh^2 y} < \sqrt{(1+\cos x \cosh y)^2 + \sin^2 x \sinh^2 y}$

or if $2 \cos x \cosh y > 0$

or if $\cos x > 0$ as $\cosh y$ is always positive

or $|w| < 1$ corresponds $\cos x > 0$ or $0 < x < \frac{\pi}{2}$

i.e. the strip between $x = 0$ and $x = \frac{\pi}{2}$ in the z-plane corresponds to the interior of the unit circle in the w-plane.

ADDITIONAL MISCELLANEOUS PROBLEMS

Problem 31. *Evaluate by the method of contour integration the integrals*

(*a*) $\int_0^{2\pi} \frac{d\theta}{1+\epsilon\cos\theta}$ (*Rohilkhand, 1978, Agra, 67, 89*),

(*b*) $\int_0^{2\pi} \frac{d\theta}{25-24\cos\theta}$, (*c*) $\int_0^{2\pi} \frac{\cos 2\theta \, d\theta}{5+4\cos\theta}$ (*Meerut, 1975, 78, 86; Agra, 82*)

Ans (*a*). $\frac{2\pi}{\sqrt{1-\varepsilon^2}}, 0 \le \varepsilon < 1$, (*b*) $\frac{2\pi}{7}$, (*c*) $\frac{\pi}{6}$

Proceed as in problem 20 (*i*).

Problem 32. (*a*) *Prove that the n-th derivative of a function f(z) of the complex variable z in the domain of its analyticity is given by*

$$f^{(n)}(z) = \frac{\lfloor n}{2\pi i} \oint \frac{f(z')dz'}{(z'-z)^{n+1}}$$

where the contour of integration lies within the domain.

(b) Using the above formula, show that

$$\left(\frac{d^n e^{-z}}{dz^n}\right)_{z=1} = (-1)^n e^{-1}$$ (*Agra, 1968*)

Problem 33. *Obtain the Cauchy-Riemann condition for a function f(z) to be analytic function.* (*Agra, 1969, 71*)

Problem 34. *Evaluate by the theorem of residues*

$$\int_{-\infty}^{\infty} \frac{\cos x}{a^2+x^2} dx, \ a > 0.$$

See Problem 20 (iii) **Ans.** $\frac{\pi}{ae^a}$.

Problem 35. *Apply calculus of residue to evaluate*

(*a*) $\int_0^{\infty} \frac{x^{\alpha-1}}{1+x}\, dx,\ 0<\alpha<1$ Ans. $\frac{\pi}{\sin \alpha\pi}$, (*Rohilkhand, 1979, 82*)

(*b*) $\int_0^{\infty} \frac{dx}{(x^2+1)(x^2+9)}$. Ans. $\pi/24$ (*Meerut, 1970*)

For (a) See Problem 23.

Problem 36. *State and Prove Cauchy's residue theorem. Evaluate the following integrals by the method of contour integration*

(*a*) $\int_0^{\infty} \frac{\sin x\, dx}{\sqrt{x}}$ (*Agra, 1972, 74*)

(*b*) $\int_0^{\infty} \frac{x^4\, dx}{x^6+1}$ (*Agra, 1972*)

(*a*) Take $f(z) = \frac{e^{iz}}{\sqrt{z}}$

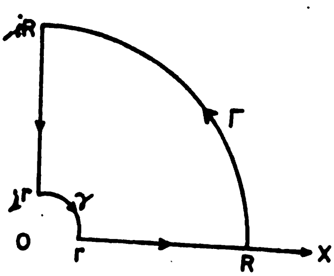


Fig. 5.66

Choose the contour C as consisting of a positive quadrant of a large circle $|z| = R$ with its bounding radii as two axes, indented at $z = 0$, as shown in Fig. 5.56. Let r be the radius of indentation at $z = 0$.

$f(z)$ has the only pole $z = 0$, of order one and it does not lie inside the contour C indented at $z = 0$.

Cauchy's theorem gives

$$\int_r^R f(x)\, dx + \int_\Gamma f(z)\, dz + \int_R^r f(iy)(i\,dy) + \int_\gamma f(z)\, dz = 0 \quad \text{...(1)}$$

Here $\left|\int_\Gamma f(z)\, dz\right| \le \int_\Gamma |f(z)|\,|dz| = \int_\Gamma \frac{|e^{iz}|\,|dz|}{|\sqrt{z}|}$

$$\le \int_0^{\pi/2} \frac{e^{-R\sin\theta}}{\sqrt{R}} R\, d\theta, \text{ taking } z = Re^{i\theta} \text{ so that } |dz| = R\, d\theta$$

$$\le \frac{R}{\sqrt{R}} \int_0^{\pi/2} e^{-2R\theta/\pi} d\theta \text{ since by Jordan's inequality}$$

$$\text{if } 0 \le \frac{\pi}{2},\ \frac{\sin\theta}{\theta} > \frac{2}{\pi}$$

$$\le \frac{-\pi}{2\sqrt{R}} \left[e^{-2R\theta/\pi}\right]_0^{\pi/2}$$

$$\le \frac{\pi}{2\sqrt{R}} \left(1-e^{-R}\right) \to 0 \text{ as } R \to \infty$$

Also residue of $f(z)$ at $z = 0$ being $= \lim_{z\to 0} \frac{ze^{iz}}{\sqrt{z}} = 0$, by §5.16 we have

$\lim_{r\to 0} \int_\gamma f(z)\, dz = -i\left(\frac{\pi}{2}-0\right)\cdot 0 = 0$, negative sign being taken as γ is traversed in clockwise sense.

Hence as $R\to\infty$, $r\to 0$, (1) reduces to

$$\int_0^{\infty} f(x)\, dx + \int_{\infty}^{0} f(iy)\, i\, dy = 0$$

i.e. $\int_0^{\infty} \frac{e^{ix}}{\sqrt{x}}\, dx = \int_0^{\infty} \frac{e^{i(iy)}}{\sqrt{iy}}\, i dy = 0$

or $\int_0^{\infty} \frac{e^{ix}}{\sqrt{x}}\, dx = \sqrt{i} \int_0^{\infty} \frac{e^{-y}}{\sqrt{y}}\, dy$. Put $\sqrt{y} = \phi$

$= 2\sqrt{i} \int_0^{\infty} e^{-\phi^2} d\phi$

$= 2\sqrt{i}\ \frac{\sqrt{\pi}}{2}$ (See Beta and Gamma functions Equation (3) of §9.3)

$= \sqrt{\pi}\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{4}\right)^{1/2}$

$= \sqrt{\pi}\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right)$

$= \sqrt{\frac{\pi}{2}}[1+i]$

Equating real and imaginary parts, we get $\int_0^{\infty} \frac{\cos x}{\sqrt{x}}\, dx = \frac{\sqrt{\pi}}{2}$ and $\int_0^{\pi/2} \frac{\sin x}{\sqrt{x}} = \frac{\sqrt{\pi}}{2}$

Note. *Similar procedure will show that* $\int_0^{\infty} \frac{\sin x^2}{x}\, dx = \frac{\pi}{4}$.

(b) Take $f(z) = \frac{z^4 dz}{z^6+1}$

and choose the contour C consisting of a large semi-circle Γ of radius R in the upper half plane and the real axis from $-R$ to R.

Poles of f(z) are given by

$z^6+1 = 0$

i.e. $z = (-1)^{\frac{1}{6}}$

$= \{\cos(2r+1)\pi + i \sin(2+1)\pi\}^{\frac{1}{6}}$

$= e^{i(2r+1)\pi/5}$ where $r = 0,1,2,3,4,5$.

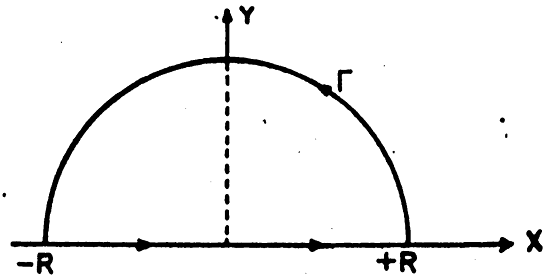


Fig. 5.67

∴ Simple poles of f(z) are

$e^{\pi i/6}, e^{\pi i/2}, e^{5\pi i/6}, e^{7\pi i/6}, e^{3\pi i/2}, e^{11\pi i/2}$

of which only the first three lie within the contour C.

Denoting by p, any one of these poles namely $e^{\pi i/6}, e^{\pi i/2}, e^{5\pi i/6}$, we have the

residue of $f(z)$ at $z = p = \left[\frac{z^4}{\frac{d}{dz}\left(z^6+1\right)}\right]_{\text{at } z=p}$ by §5.17 (i)

$= \left[\frac{z^4}{6z^5}\right]_{z=p} = \frac{1}{6p} \times \frac{p^5}{p^5}$

$= \frac{p^5}{-6}$ $\because p^6 = -1$

$\therefore$ Sum of residues $= \Sigma R^+$

$$= -\frac{1}{6}\left[e^{5\pi i/6} + e^{5\pi i/2} + e^{25\pi i/6}\right]$$

$$= -\frac{1}{6}\left[e^{5\pi i/6} + e^{2\pi i}\, e^{\pi i/2} + e^{4\pi i} e^{\pi i/6}\right]$$

$$= -\frac{1}{6}\left[e^{5\pi i/6} + e^{\pi i/2}\, e^{\pi i/6}\right] \qquad \because e^{2\pi i} = e^{4\pi i} = 1$$

$$= -\frac{1}{6}\left[\left(-\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)+i+\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)\right]$$

$$= -\frac{2i}{6} = -\frac{i}{3}.$$

Now by Cauchy's theorem,

$$\int_C f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_\Gamma f(z)\, dz = 2\pi i \Sigma R^+$$

where $\left|\int_\Gamma f(z)\, dz\right| \le \int_\Gamma |f(z)|\, |dz|$

$$\le \int_\Gamma \left|\frac{z^4}{z^6+1}\right| |dz|. \text{ Put } z = Re^{i\theta} \text{ so } |dz| = R\,d\theta$$

$$\le \int_\Gamma \frac{|z^4|\, |dz|}{|z^6|-1} \qquad \because |z^6| - 1 < |z^6 - 1|$$

$$\le \int_0^\pi \frac{R^5\, d\theta}{R^6-1} = \frac{R^5\pi}{R^6-1} \to 0 \text{ as } R \to \infty$$

Hence when $R \to \infty$, we have

$$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \times \left(\frac{-i}{3}\right) = \frac{2\pi}{3}$$

i.e. $\int_0^\infty f(x)\, dx = \int_0^\infty \frac{x^4 dx}{x^6+1} = \frac{\pi}{3}$.

Problem 37. *Derive the Cauchy-Riemann conditions for a function of a complex variable to be analytic. Test if the functions $\frac{1}{z}$ and $|z|^2$ are analytic.*

Prove that if a function is analytic on and inside a closed contour its integral over the contour must vanish. *(Agra, 1973)*

See §5.6 for the first part.

It is easy to show that $f(z) = \frac{1}{z}$ (giving $f'(z) = -\frac{1}{z^2}$, $z \neq 0$) is analytic at every point except at $z = 0$ where it is not continuous and so $f'(0)$ does not exist ($z = 0$ is a pole of $f(z)$).

The function $f(z) = |z|^2$ is not analytic at any point since $f'(z)$ exists only at the point $z = 0$ not throughout any neighbourhood. It may be shown as below

$$f(z) = |z|^2 = x^2 + y^2 \qquad \because z = x + iy$$

If $f(z) = u(x,y) + iv(x,y)$, then it is clear on comparison that

$$u(x, y) = x^2 + y^2, \; v(x, y) = 0$$

$\therefore \quad \dfrac{\partial u}{\partial x} = 2x, \; \dfrac{\partial u}{\partial y} = 2y, \; \dfrac{\partial v}{\partial x} = 0 = \dfrac{\partial v}{\partial y}$

Cauchy-Riemann equations are not satisfied except at $z = 0$.

For second part see §5.9.

Problem 38. *Defining a* **Meromorphic Function** *as one which is analytic in a region except at a finite number of poles, if $f(z)$ be an analytic function within and on a closed contour except at a finite number of poles and is not zero on C, then prove that*

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N - P$$

where N is the number of zeros and P is the number of poles inside C (pole of order m being counted m times).

Hence or otherwise deduce the **Principle of Argument** *i.e. $f(z)$ is analytic in C, then $P = 0$, and*

$$N = \frac{1}{2\pi}\Delta c \arg f(z)$$

Δ_C being the variation of $\log f(z)$ round the closed contour C.

Taking $z = a$ as a zero of order n, we have

$$f(z) = (z-a)^n \phi(z)$$

where $\phi(z)$ is analytic and not zero in C.

Thus,
$$\frac{f'(z)}{f(z)} = \frac{n}{z-z} + \frac{\phi'(z)}{\phi(z)}$$

Since $\phi(z)$ is analytic and so is $\phi'(z)$, therefore $\dfrac{\phi'(z)}{\phi(z)}$ is analytic at $z = a$. It follows that $\dfrac{f'(z)}{f(z)}$ has a simple pole at $z = a$ with its residue n.

Similarly if $z = b$ is a pole of order p then $\dfrac{f'(z)}{f(z)}$ has a simple pole at $z = b$ with its residue $-p$ since then

$$f(z) = (z-b)^{-p}\psi(z)$$

Hence by Cauchy's theorem

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \text{Sum of all residues within } C$$

$$= \Sigma n - \Sigma p$$

$$= N - P \qquad \ldots(1)$$

Now to deduce the principle of argument, $f(z)$ being analytic in C, we have $P = 0$. Then (1) reduces to

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N$$

or
$$N = \frac{1}{2\pi i}\left[\log f(z)\right]_C$$

$$= \frac{1}{2\pi i}\Delta_c \log f(z)$$ where Δ_c is given to be the variation of $\log f(z)$ round the closed contour C.

$$= \frac{1}{2\pi i}\Delta_c \{\log |f(z)| + i \arg f(z)\}$$ by (36) of §5.7.

Equating real parts on either side we get

$$N = \frac{1}{2\pi}\Delta_c \arg f(z).$$

Problem 39. (Rouche's theorem) *If $f(z)$ and $g(z)$ are analytic within and on a closed contour C and $|g(z)| < |f(z)|$ on C, then $f(z)$ and $f(z)+g(z)$ have same number of zeros inside C.*

$\because\ |g(z)| < |f(z)|$ on C. $\therefore \dfrac{|g(z)|}{|f(z)|} < 1$ on C which follows that $|f(z)| \neq 0$ otherwise this inequality will not hold.

As such $|f(z)+g(z)| > |f(z)| - |g(z)| \neq 0$ by properties of moduli.

So $|f(z)+g(z)| \neq 0$ implies that neither $f(z)$ nor $f(z)+g(z)$ has a zero on C.

Taking N and N'as number of zeros of $f(z)$ and $f(z)+g(z)$, we have by the principle of argument given in Problem 38,

$$N = \frac{1}{2\pi}\Delta_c \arg f \text{ and } N' = \frac{1}{2\pi}\Delta_c \arg(f+g)$$

So that $N'-N = \dfrac{1}{2\pi}\Delta_c \arg(f+g) - \dfrac{1}{2\pi}\Delta_c \arg f$

$$= \frac{1}{2\pi}\left[\Delta_c \arg f + \Delta_c \arg\left(1+\frac{g}{f}\right)\right] - \frac{1}{2\pi}\Delta_c \arg f$$

$$= \frac{1}{2\pi}\left[\Delta_c \arg\left(1+\frac{g}{f}\right)\right] \text{ by the properties of arguments.}$$

If $1+\dfrac{g}{f} = Q$, then $|Q-1| = \dfrac{|g|}{|f|} < 1 \quad \because |g| < |f|$

showing that Q is always an internal point of the circle with centre $Q = 1$ and radius unity. This circle being wholly to the right of imaginary axis, the join of any point of this circle to the origin makes an angle θ with x-axis such that $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$. As such putting $Q = Re^{i\theta}$ we have $\arg\left(1+\dfrac{g}{f}\right) = \arg Q = \theta$ which returns to its initial value when z describes C and hence

$$\Delta_c \arg\left(1+\frac{g}{f}\right) = 0$$

Hence, $N' - N = 0$ or $N = N'$

which proves the proposition.

Problem 40. *Evaluate the following integrals by applying the method of contour integration.*

(*i*) $\displaystyle\int_0^\infty \frac{x \sin x}{x^2+a^2}dx,\ a > 0$ Ans. $\dfrac{\pi}{2}e^{-a}$ (*Meerut, 83; Agra, 81; Kanpur, 78*)

(*ii*) $\displaystyle\int_0^\infty \frac{\sinh ax}{\sinh \pi x}dx, -\pi < a < \pi.$ Ans. $\dfrac{1}{2}\tan\dfrac{a}{2}$

(*iii*) $\displaystyle\int_0^\infty \frac{\sin mx}{x^4+a^4}dx$ Ans. $\dfrac{\pi}{2a^2}e^{-\frac{ma}{\sqrt{2}}}\left\{\sin\dfrac{ma}{\sqrt{2}}\right\}$ (*Meerut, 76*)

(iv) $\int_{-\infty}^{\infty} \frac{a \cos x + x \sin x}{x^2+a^2} dx$ Ans. $2\pi e^{-a}$ $\left\{ \text{for } a = 1, \text{ Ans. } \frac{2\pi}{e}, \text{ Meerut, 69, 77} \right\}$

$\left[\text{Hint: take } f(z) = \frac{e^{iz}}{z - ia} \right]$

(v) $\int_0^{\infty} \frac{x\, dx}{\sinh x}$ Ans. $\frac{\pi^2}{4}$

(vi) $\int_0^{\infty} \frac{a\, d\theta}{a^2 + \sin^2\theta}, \ a > 0$ Ans. $\frac{\pi}{\sqrt{1+a^2}}$ (Rohilkhand, 87)

(vii) $\int_0^{\infty} \frac{\cos 2\, ax - \cos 2\, bx}{x^2} dx, \ a \geq 0, \ b \geq 0,$ Ans. $\pi\ (b-a)$

(viii) $\int_0^{\infty} e^{-x^2 \cos 2\alpha} \cos (x^2 \sin 2\alpha)\, dx, \ \alpha \leq \frac{\pi}{4}$ Ans. $\frac{\sqrt{\pi}}{2} \cos \alpha$

and $\int_0^{\infty} e^{-x^2 \cos 2\alpha} \sin \left(x^2 \sin 2\alpha \right) dx, \ \alpha \leq \frac{\pi}{4}$ Ans. $\frac{\sqrt{\pi}}{2} \sin \alpha$

(ix) $\int_0^{\infty} e^{-x^2} \cos 2ax\, dx$ and $\int_0^{\infty} e^{-x^2} \sin 2ax\, dx$ Ans. $\frac{e^{-a^2}}{2} \sqrt{\pi}; \ e^{-a^2} \int_0^a e^{y^2} dy$

(x) $\int_0^{\infty} \frac{dx}{x^4+16}$ Ans. $\pi\sqrt{2}/256$ (Rohilkhand, 1984)

(xi) $\int_0^{\infty} \frac{dx}{\left(x^2+1\right)\left(x^2+2\right)}$ Ans. $\frac{\pi\left(\sqrt{2}+2\right)}{2\sqrt{2}\left(\sqrt{2}+1\right)^2}$ (Rohilkhand, 1984)

(xii) $\int_{-\infty}^{\infty} \frac{\sin \pi x}{x\left(1-x^2\right)} dx,$ Ans. π (Rohilkhand, 1980, 84, 86; Meerut, 80)

(xiii) $\int_0^{\infty} \frac{e^{ax}}{1+e^x} dx, \ (0 < a < 1)$ [see Problem 27 (i)] (Agra, 1974)

Problem 41. *The function w (z) is analytic in a region R, except at two simple poles A and B. Find $\oint_C w(z)\, dz$ where C is contour in the region R (i) enclosing A and B (ii) enclosing B only.*

(Agra, 1974)

Problem 42. *Evaluate the following integrals;*

(i) $\int_0^{2\pi} e^{\cos\theta} \cos (n\theta - \sin\theta)\, d\theta,$ (ii) $\int_{-\infty}^{\infty} \frac{\cos mx}{1+x^2} dx$ (Rohilkhand, 1985)

(iii) $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2} dx,$ (iv) $\int_{-\infty}^{\infty} \frac{\sin x}{x} dx$ (Agra, 1975, 80, 82; Meerut, 1972, 79)

Ans. (i) $\frac{2\pi}{\underline{|n}}$. (ii) πe^{-m}, (iii) $\frac{\pi}{2a} e^{-a}$, (iv) π

Problem 43. *Evulate the following integrals:*

(a) $\int_0^{2\pi} (a + b \cos \phi)^{-1} \sin^2\phi\, d\phi, \ (a > b > 0),$ (b) $\int_0^{2\pi} \frac{\sin^2\theta}{a + b \sin\theta} d\theta$

(Rohilkhand, 1986)

(*c*) $\int_0^\infty (1+x^2)^{-1} \log(1+x^2)\, dx$ [see Problem 21 (*vi*)] (*Agra, 1976*)

Ans. (*a*) $\frac{2\pi}{b^2}\left\{a-\sqrt{a^2-b^2}\right\}$, (*b*) $\frac{2\pi}{b}\left\{a-\sqrt{(a^2-b^2)}\right\}$, (*c*) $\pi \log 2$

Problem 44. (*a*) *Derive the Cauchy-Riemann condition for complex function to be analytic. Are the conditions sufficient?*

(*b*) *Evaluate the following integrals:*

(*i*) $\int_0^\infty \frac{\sin x}{x}\, dx$ (*ii*) $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ (*Rohilkhand, 1976, 89*)

(*iii*) $\int_{-\infty}^{\infty} \frac{\cos x + x \sin x}{1+x^2}\, dx$ (*Meerut, 1969, 77*)

Ans. (*a*) *yes*, (*b*) (*i*) $\frac{\pi}{2}$, (*ii*) π, (*iii*) $2\pi/e$.

Problem 45. *State and prove Cauchy's Residue theorem for a complex function. Explain how it is extended for the case of an isolated first order pole lying on the contour of integration. Using this theorem show that*

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\, dx = \frac{\pi}{\sin \pi a};\ \text{where } 0 < a < 1$$ (*Rohilkhand, 1977*)

see § 5.15 and Problem 27(*i*).

Problem 46. (*a*) *State and prove Cauchy's residue theorem for a complex function and using it evaluate the following integrals.*

(*i*) $\int_0^\infty \frac{\log(1+x^2)}{1+x}\, dx$ (*Rohilkhand, 1991*)

(*ii*) $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$, $a > b > 0$ (*Nagpur, 1966, 79*)

See §5.15; (*i*) see Problem 21 (*vi*); (*ii*) see Problem 31.

Ans. (*i*) $\pi \log 2$, (*ii*) $\frac{2\pi}{\sqrt{(a^2-b^2)}}$

(*b*) Evaluate the following

(*i*) $\int_0^\infty \frac{\sin mx}{x}\, dx$, $m > 0$. Ans. $\frac{\pi}{2}$ (*Rohikhand, 1992; Agra, 83; Meerut, 1978, 82*)

(*ii*) $\int_0^\infty \frac{\log x}{(1+x^2)^2}$, Ans. $-\frac{\pi}{4}$ (*Rohilkhand, 1983*)

Problem 47. *Obtain the necessary conditions for the function $f(z)$ to be analytic at a point. Show that the real and imaginary parts of an analytic function satisfy Laplace's equation.*

(*Rohilkhand, 1981*)

Problem 48. (*a*) *Which of the following are analytic functions of complex variable* z : (*i*) $|z|$ (*ii*) $Re(z)$, (*iii*) $e^{\sin z}$

(*b*) *The function $f(z)$ has a pole of order n at $z = z_0$. Show that the function $f'(z)/f(z)$ has a simple pole at z_0. What is the residue ?*

(*c*) *Explain the function* $W(z) = \frac{1}{(z-1)(z-2)}$ *between the annular region* $z = 1$ *and* $z = 2$.

(*Rohilkhand 1981*)

(*a*) (*i*) $z = x + iy \Rightarrow |z| = \sqrt{(x^2+y^2)}$

Given $f(z) = u + iv = |z| = \sqrt{(x^2+y^2)}$ gives on equating real and imaginary parts,

$u = \sqrt{(x^2+y^2)}$, $v = 0$.

Here $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ do not exist $\Rightarrow |z|$ is not analytic.

(*ii*) $f(z) = Re\ (z) = Re\ (x + iy) = x = u + iv \Rightarrow u = x, v = 0 \Rightarrow \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ do not exist $\Rightarrow Re(z)$ is not analytic.

(*iii*) $f(z) = u + iv = e^{\sin z} = e^{\sin (x+iy)} \Rightarrow u + iv = e^{\sin x \cosh y}$ [cos (cos x sinh y) + i sin (cos x sinh y)] from which, u *and* v can be found and then verified that Cauchy-Riemann equations are not satisfied and hence $e^{\sin z}$ is not analytic.

(*b*) Say $f(z) = \frac{\phi(z)}{(z-z_0)^n} = \phi(z)(z-z_0)^{-n}$ ...(1)

$\therefore\ f'(z) = (z - z_0)^{-n}\ [\phi'(z) - n\phi(z)(z - z_0)^{-1}]$

$f''(z) = (z-z_0)^{-n}\left[\phi''(z) - 2n(z-z_0)\quad \phi'(z) + n\ (n+1)(z-z_0)^{-2}\right]$

So $\psi(z) = \frac{f''(z)}{f'(z)}$

$$= \left[\frac{\left\{(z-z_0)^2\ \phi''(z) - 2n\,(z-z_0)\phi'(z) + n\ (n+1)\phi(z)\right\}\Big/\left\{(z-z_0)\phi'(z) - n\phi(z)\right\}}{z-z_0}\right]$$

$\Rightarrow \psi(z)$ has a simple pole at $z = z_0$.

$$\underset{z=z_0}{\text{Res}}\ \phi(z) = \lim_{z \to z_0}\ (z-z_0)\psi(z) = \frac{n(n+1)\ \phi(z)}{-n\,\phi(z_0)} = -(n+1)$$

(*c*) $W(z) = \frac{1}{(z-1)\ (z-2)} = \frac{1}{z-2} - \frac{1}{z-1} = -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$ for $1 < z < 2$.

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$

Problem 49. *State and prove Cauchy's theorem for the function of a complex variable. Evaluate* $\int_{-\infty}^{\infty}\frac{\cos px - \cos qx}{x^2}dx$ *where p and q are real constants that may be positive or negative.*

(*Rohilkhand, 1982*)

Ans: $-(p - q)\ \pi$

Hint: Taking $f(z) = \frac{e^{ipz} - e^{iqz}}{z^2} \Rightarrow z = 0$ (double pole) and choosing a semi-circular contour, the result follows.

Problem 50. *Find the poles and residues at the poles for the following functions :* $\frac{z}{\cos z}$ and $\frac{z+1}{z^2-2z}$ (*Rohilkhand, 1982*) (*i*) $f(z) = \dfrac{z}{\cos z} \Rightarrow$ poles are given by $\cos z = 0$

i.e. $z = (2n-1)\dfrac{\pi}{2}$, $n = 1, 2, 3$ are simple poles.

Residue of $f(z)$ at $z = (2n-1)\dfrac{\pi}{2} = \left[\dfrac{\phi(z)}{\psi'(z)}\right]_{z=(2n-1)\pi/2}$

$$= \left[\frac{z}{-\sin z}\right]_{z=(2n-1)\pi/2} = \frac{(2n-1)\pi/2}{-\sin(2n-1)\pi/2} = -(2n-1)\pi/2, \quad n = 1, 2, 3$$

(*ii*) $f(z) = \dfrac{z+1}{z^2-2z} = \dfrac{z+1}{z(z-2)} \Rightarrow$ Poles are given by $z(z-2) = 0$ *i.e.* $z = 0, 2$

Residue of $f(z)$ at $z = 0 = \lim\limits_{z \to 0}(z-0)\, f(z) = -1/2$

Residue of $f(z)$ at $z = 2 = \lim\limits_{z \to 2}(z-2) f(z) = 3/2$

Problem 51. *Evaluate* $\displaystyle\int_0^\infty \frac{\log n}{\left(1+x^2\right)^2} dx$ (*Rohilkhand, 1983, 85*)

Ans. $-\pi/4$

Problem 52. (*a*) *What conditions are satisfied by an analytic function of a complex variable ?*

(*b*) *Test whether the following are analytic in the finite plane* (*i*) $\sin z$, (*ii*) $|z|$, (*iii*) z^2, (*iv*) $|z|^2$, (*v*) $\dfrac{1}{z}$, (*vi*) *log z.*

(*i*) $f(z) = u + iv = \sin z = \sin(x+iy) = \sin x \cosh y + \cos x \sinh y$

$\Rightarrow u = \sin x \cosh y,\ v = \cos x \sinh y$

$$\Rightarrow \frac{\partial u}{\partial x} = \cos x \cosh y,\ \frac{\partial u}{\partial y} = \sin x \sinh y,\ \frac{\partial v}{\partial x} = -\sin x \sinh y,\ \frac{\partial v}{\partial y} = \cos x \cosh y$$

which satisfy Cauchy-Riemann equations and hence sin z is analytic.

(*ii*) See Prob. 48 (*a*) (*i*) (*iii*) and (*iv*) are easy to test.

(*v*) $f(z) = u + iv = \dfrac{1}{z} = \dfrac{1}{x+iy} = \dfrac{x-iy}{x^2+y^2} \Rightarrow u = \dfrac{x}{x^2+y^2} = \left(x^2+y^2\right)^{-1}$ *and*

$$v = -y(x^2+y^2)^{-1} \Rightarrow \frac{\partial u}{\partial x} = -\frac{x^2-y^2}{\left(x^2+y^2\right)^2},\ \frac{\partial u}{\partial y} = -\frac{2xy}{\left(x^2+y^2\right)^2},$$

$$\frac{\partial v}{\partial x} = \frac{2xy}{\left(x^2+y^2\right)^2},\ \frac{\partial v}{\partial y} = -\frac{x^2-y^2}{\left(x^2+y^2\right)^2}$$

which satisfy Cauchy-Riemann equations and hence $\dfrac{1}{z}$ is analytic.

(*vi*) $f(z) = u + iv = \log z = \log re^{i\theta} = \log r + i\theta \Rightarrow u = \log r$,

$$v = \theta \Rightarrow \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \ \&\ \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} = -r\frac{\partial v}{\partial r}$$

are not satisfied and hence log z is not analytic.

Problem 53. *Apply Calculus of residues to prove that*

(*a*) $\int_0^{\infty} \frac{dx}{\sqrt{x}(1+x)} = \pi$ (*b*) $\int_0^{\infty} \frac{dx}{x^2+4} = \frac{\pi}{8}$ (*Rohilkhand, 1988*)

Problem 54. *What do you mean by an analytic function of complex variable? Derive necessary and sufficient conditions for a function to be analytic. If* $u(x, y) = x^2 - y^2$ *is the real part of an analytic function* $f(z) = u + iv$, *find* v. (*Rohilkhand, 1991*)

Ans. $v = 2xy$.

Problem 55. (*a*) *State Cauchy-Riemann equations for an analytic function and show that if* $f(z) = u + iv$ *is an analytic function and* $F = u_i + v_j$ *is a vector, then div* $F = 0$ *and curl* $F = 0$ *are equivalent to Cauchy-Riemann equations.*

(*b*) *Distinguish between the situation when:*

(*i*) *A Taylor's series, and*

(*ii*) *A Laurent series expansion of a function of a complex variable z is valid about a point* $z = a$. *Define residue at a pole.* (*Rohilkhand, 1993*)

Problem 56. *Discuss the application of the transformation* $w = \frac{iz+1}{z+i}$ *to the areas in the z-plane which are respectively inside and outside the unit circle with its centre at the origin.* (*R.U., 1985*)

Problem 57. *By the transformation* $w = z^2$, *show that the circles* $|z - a| = c$; *a, c being real, in z-plane correspond to the Limacon in the w-plane.* (*R.U., 1992*)

Problem 58. *Show that the relation* $w = \tan^2\left(\frac{\pi\sqrt{z}}{4}\right)$ *transforms the interior of the unit circle* $|w| = 1$ *into z-space lying within the parabola.* (*R.U., 1987*)

Problem 59. *If* $z = \frac{4aw\cot\alpha}{1 + 2w\cot\alpha - w^2}$ $(0 < \alpha < \frac{\pi}{4})$, *show that when w describes a unit circle, z describes twice over an arc of a certain circle subtending an angle* 4α *at the centre.* (*R.U., 1996*)

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CHAPTER 6

BETA, GAMMA AND ERROR FUNCTIONS

6.1. DEFINITIONS

Under the study of Definite Integrals, we come across two very important integrals known as Eulerian Integrals which are of the type

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad \text{and} \quad \int_0^\infty e^{-x}x^{n-1}\,dx,$$

where the quantities m and n are supposed to be positive. These integrals are fundamental and hold an important place that they are widely applied in different branches of mathematical analysis like mechanics, physics etc.

The first Eulerian integral is generally known as *Beta Function* and defined as $\beta(m, n)$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \text{ where m and n are positive.}$$

The second Eulerian integral is known as *Gamma Function* and is defined as

$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx$, where n is positive. (*Agra, 1961*)

Note : Weierstrass (1815-1897) defined the Gamma function as

$$\frac{1}{\Gamma(n)} = ne^{\gamma n}\prod_{m=1}^{\infty}\left[\left(1+\frac{n}{m}\right)e^{-n/m}\right]$$

where γ is known as *Euler's* or *Mascheroni's* constant and defined as

$$\gamma = \lim_{m\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\left(\frac{1}{m}-\log_e m\right)\right)$$

$$= \lim_{m\to\infty}(u_m - \log_e m) \text{ with } u_m = \sum_{p=1}^{m}\frac{1}{p} \text{ and } \gamma = 0.5772157 \text{ nearly}$$

6.2. FUNDAMENTAL PROPERTY OF GAMMA FUNCTIONS

$$\Gamma(n+1) = n\Gamma(n)$$

In order to prove this relation, let us consider the integral

$$\int_0^\infty e^{-x}x^n\,dx = \Gamma(n+1)$$

Integrating it by parts taking e^{-x} as second function, we get

$$\int_0^\infty e^{-x}x^n dx = \left[-e^{-x}x^n\right]_0^\infty - \int_0^\infty e^{-x}x^{n-1}dx$$

$$= n\int_0^\infty e^{-x}x^{n-1}dx,$$

$$\therefore \Gamma(n+1) = n\Gamma(n) \quad \ldots(1)$$

since $\frac{x^n}{e^x}$ vanishes for both the limits as $\lim_{x\to 0}\frac{x^n}{e^x} = 0$ and $\lim_{x\to\infty}\frac{x^n}{e^x} = \lim_{x\to\infty}\frac{x^n}{1+x+\frac{x^2}{\lfloor 2}+\ldots} = 0$

From (1) it is evident that if the value of $\Gamma(n)$ is known for n between two successive positive integers, then the value Γn for any positive value of n can be determined by the successive application of (1).

Now (1) can be written as

$$\Gamma(n) = \frac{\Gamma(n+1)}{n} \qquad \ldots(2)$$

If $-1 < n < 0$ then (2) gives Γn, since $n + 1$ is positive. As such the value of Γn may be determined if $-2 < n < -1$ since then $\Gamma(n + 1)$ on the R.H.S. of (2) is known. Similarly Γn may be determined when $-3 < n < -2$ and so on so forth.

Hence $\Gamma n = \int_0^{\infty} e^{-x} x^{n-1}\, dx = \frac{\Gamma(n+1)}{n}$ define Γn completely for all value of n except $n = 0, -1, -2, -3\ldots$

Now replacing n by $n - 1$ in (1) we get

$$\Gamma_n = (n-1)\,\Gamma(n-1)$$

Similarly $\Gamma(n - 1) = (n - 2)\,\Gamma(n - 2)$ etc.

Hence (1) yields

$$\Gamma(n + 1) = n(n - 1)(n - 2)\ldots 3.2.1\,\Gamma(1)$$

But by definition $\Gamma(1) = \int_0^{\infty} e^{-x} dx = \left[-e^{-x}\right]_0^{\infty} = 1$

$\therefore\ \Gamma(n + 1) = n(n - 1)(n - 2)\ldots 3.2.1 = \lfloor(n)$. provided n is a positive integer ...(3)

Putting $n = 0$ in (3) we get

$$\Gamma(1) = \lfloor 0 = 1 \qquad \because\ \lfloor 0 = 1$$

$$\therefore\qquad \Gamma(1) = 1 \qquad \ldots(4)$$

Also if we put $n = 0$ in (2), then we find

$$\Gamma(0) = \frac{\Gamma(1)}{0} = \infty \qquad \ldots(5)$$

By repeated application of (2), it may be shown that the gamma function becomes infinite when x is zero or any negative integer *i.e.*

$$\Gamma(-n) = \infty \qquad \ldots(6)$$

when $n = 0$ or a positive integer.

But the function has finite value for negative values of n which are not integer.

Note. *Gauss's Pi function in terms of gamma function is defined by*

$$\Pi(n) = \Gamma(n + 1) = \lfloor n \qquad \ldots(7)$$

6.3. THE VALUE OF $\Gamma\left(\frac{1}{2}\right)$ AND GRAPH OF THE GAMMA FUNCTION

We have by definition

$$\Gamma n = \int_0^{\infty} e^{-x} x^{n-1} dx,\ n > 0$$

Putting $x = \phi^2$ *i.e.* $dx = 2\phi\, d\phi$, we get

$$\Gamma(n) = 2\int_0^\infty \phi^{2n-1} e^{-\phi^2}\, d\phi$$

when $n = \frac{1}{2}$, this yields,

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-\phi^2}\, d\phi \quad ...(1)$$

Suppose $I = \int_0^\infty e^{-\phi^2}\, d\phi$

Putting $\phi = \lambda\psi$ so that $d\phi = \lambda\, d\psi$

We have $I = \int_0^\infty e^{-\lambda^2\psi^2} \cdot \lambda\, d\psi$

Multiplying both sides by $e^{-\lambda^2}$, we find

$$I \cdot e^{-\lambda^2} = \int_0^\infty e^{-\lambda^2(1+\psi^2)}\, \lambda\, d\psi$$

Integrating both sides w.r.t λ within the limits 0 to ∞,

$$I\int_0^\infty e^{-\lambda^2}\, d\lambda = \int_0^\infty \int_0^\infty e^{-\lambda^2(1+\psi^2)}\, \lambda\, d\lambda\, d\psi$$

or

$$I\int_0^\infty e^{-\phi^2}\, d\phi = \int_0^\infty \left[-\frac{1}{2}\cdot\frac{e^{-\lambda^2(1+\psi^2)}}{1+\psi^2}\right]_0^\infty d\psi$$

$$I^2 = \frac{1}{2}\int_0^\infty \frac{d\psi}{1+\psi^2} = \frac{1}{2}\left[\tan^{-1}\psi\right]_0^\infty = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$$

$\therefore$

$$I = \int_0^\infty e^{-\phi^2}\, d\phi = \frac{\sqrt{\pi}}{2} \quad ...(2)$$

From (1) and (2), we get

$$\Gamma\left(\frac{1}{2}\right) = 2.\ \frac{\sqrt{\pi}}{2} = \sqrt{\pi} \quad ...(3)$$

Now putting $n = -\frac{1}{2}$ in (2) of § 6.2, we find

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\frac{1}{2}}{-\frac{1}{2}} = -2\sqrt{\pi} \text{ by (3)}, \quad ...(4)$$

Similarly $\Gamma\left(-\frac{3}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}\right)}{-\frac{3}{2}} = -\frac{2}{3}\left(-2\sqrt{\pi}\right) = 4\ \frac{\sqrt{\pi}}{3}$ etc. ...(5)

The graph of Γn may be shown as below under the definition that the function becomes continuous function of n except when $n = 0$ or any negative integer.

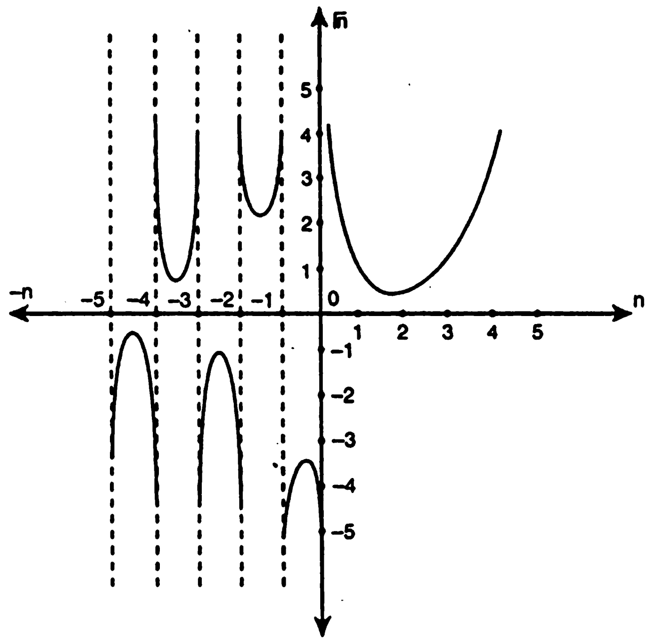


Fig. 6.1

6.4. TRANSFORMATION OF GAMMA FUNCTION

By definition

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx \qquad \ldots(1)$$

Putting $x = \lambda y$, $dx = \lambda dy$ in (1), we get

$$\Gamma(n) = \int_0^\infty e^{-\lambda y} \lambda^n \cdot y^{n-1} dy$$

or

$$\frac{\Gamma(n)}{\lambda^n} = \int_0^\infty e^{-\lambda y} y^{n-1} dy \qquad \ldots(2)$$

If we put $e^{-x} = y$ in (1), then we get

$$\Gamma(n) = -\int_1^0 y\left(\log\frac{1}{y}\right)^{n-1} \frac{1}{y} dy = \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy \qquad \ldots(3)$$

$$\left[\because x = \log\frac{1}{y} \text{ and } dx = \frac{1}{1/y}\left(-\frac{1}{y^2}\right)dy = -\frac{1}{y}dy\right]$$

Again if we write $x = y^{1/n}$ in (1), we get

$$\Gamma(n) = \frac{1}{n}\int_0^\infty e^{-y^{1/n}} (y)^{(n-1)/n} . (y)^{(1-n)/n} dy \left[\because dx = \frac{1}{n} y^{(1-n)/n} dy\right]$$

$$= \frac{1}{n}\int_0^\infty e^{-y^{1/n}} dy,$$

$$\therefore n\Gamma(n) = \Gamma(n+1) = \int_0^\infty e^{-y^{1/n}} dy \qquad ...(4)$$

COROLLARY. If we replace n by $\frac{1}{2}$ in (4), we find

$$\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-y^2} dy,$$

which is the same as (1) of § 6.3.

Note : As studied by Legendre, Houvar , Schlömilch etc., the **Incomplete Gamma functions** are defined as

$$\gamma(n, x) = \int_0^x t^{n-1} e^{-t} dt \qquad ...(1)$$

and
$$\Gamma(n,x) = \int_x^\infty t^{n-1} e^{-t} dt \qquad ...(2)$$

Evidently,
$$\gamma(n,x) + \Gamma(n,x) = \int_0^\infty t^{n-1} e^{-t} dt$$
$$= \Gamma(n) \qquad ...(3)$$

6.5. TO SHOW THAT $\beta(m,n) = \beta(n,m)$

By definition

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$$

Replacing x by $1-x$, we get

$$\beta(m,n) = \int_1^0 (1-x)^{m-1}\{1-(1-x)\}^{n-1}(-dx)$$
$$= \int_0^1 (x)^{n-1}(1-x)^{m-1} dx$$
$$= \beta(n,m)$$

6.6. DIFFERENT FORMS OF BETA FUNCTION

Substituting $\frac{y}{1+y}$ for x, we have

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$$

$$\left[\because dx = \frac{(1+y)-y}{(1+y)^2} dy = \frac{1}{(1+y)^2} dy \text{ and } 1-x = \frac{1}{1+y}\right]$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m-1}} \cdot \frac{1}{(1+y)^{n-1}} \cdot \frac{dy}{(1+y)^2}$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy \qquad ...(1)$$

Also, since $\beta(m,n)=\beta(n,m)$,

$$\therefore \quad \beta(m,n)=\int_0^\infty \frac{y^{n-1}}{(1+y)^{n+m}}\,dy \qquad ...(2)$$

Note : The Incomplete Beta function is defined as

$$\beta_x(m,n)=\int_0^x t^{m-1}(1-t)^{n-1}dt \qquad ...(3)$$

where $m, n > 1$ and $0 \le x \le 1$.

6.7. TO FIND THE RELATION BETWEEN BETA AND GAMMA FUNCTIONS

$$\beta(m,n)=\frac{\Gamma m\Gamma n}{\Gamma(m+n)}.$$ *(Agra, 1962, 65, 70, 71, 72, 74)*

From equation (2) of § 6.4, we have

$$\frac{\Gamma m}{\lambda^m}=\int_0^\infty e^{-\lambda x}x^{m-1}dx,$$

i.e., $\Gamma m=\int_0^\infty \lambda^m e^{-\lambda x}x^{m-1}dx$

Multiplying both sides by $e^{-\lambda}\lambda^{n-1}$ and integrating w.r.t. λ within the limits 0 to ∞, we get

$$\Gamma m\int_0^\infty e^{-\lambda}.\lambda^{n-1}d\lambda=\int_0^\infty\left[\int_0^\infty e^{-\lambda(1+x)}\lambda^{m+n-1}\,d\lambda\right]x^{m-1}dx$$

or $$\Gamma m\Gamma n=\int_0^\infty \frac{\Gamma(m+n)}{(1+x)^{m+n}}.x^{m-1}dx \text{ by eqn. (2) of § 6.4}$$

$$=\Gamma(m+n)\,\beta(m,n) \text{ by § 6.5}$$

$$\therefore \quad \beta(m,n)=\frac{\Gamma m\Gamma n}{\Gamma(m+n)}$$

COROLLARY. If we put $m+n=1$ in

$$\Gamma m\Gamma n=\int_0^\infty (m+n).\frac{x^{m-1}}{(1+x)^{m+n}}dx,$$

we have $\Gamma m\Gamma(1-m)=\int_0^\infty \frac{x^{m-1}}{1+x}dx, [\because \Gamma 1=1]$

$$=\frac{\pi}{\sin m\pi} \qquad (Agra, 1964)$$

$$\left[\because \int_0^\infty \frac{x^{2m}}{1+x^{2n}}dx=\frac{\pi}{2n\sin\frac{2m+1}{2n}\pi} \text{ when } m \text{ and } n \text{ are +ve integers and } n>m \text{ as will be shown in § 6.9}\right]$$

Replacing m by $\frac{1}{2}$, we get

$$\Gamma\frac{1}{2}\Gamma\frac{1}{2}=\pi \text{ or } \Gamma\frac{1}{2}=\sqrt{\pi}$$ (*Nagpur*, 1965)

Independent Proof for $\Gamma m\Gamma(1-m)=\frac{\pi}{\sin m\pi}$

We have $\beta(m,n)=\frac{\Gamma m\Gamma n}{\Gamma(m+n)} \Rightarrow \Gamma m\Gamma n=\Gamma(m+n)\,\beta(m,n)$

which, on putting $m+n=1$, reduces to

$$\Gamma m\Gamma(1-m)=\beta(m,1-m) \text{ as } \Gamma 1=1$$

$$=\int_0^1 x^{m-1}(1-x)^{-m}\,dx \text{ by def.}$$

$$=\int_0^1 \frac{x^{m-1}}{(1-x)^m}\,dx=\int_0^\infty \frac{x^{m-1}}{1+x}\,dx \quad \text{on replacing } \frac{x}{1-x} \text{ by } x$$

Consider $I=\int \frac{z^{m-1}}{1+z}\,dz$ along the contour as shown in the adjoining diagram, having the only pole inside the contour given by $1+z=0 \Rightarrow z=-1=e^{\pi i}$

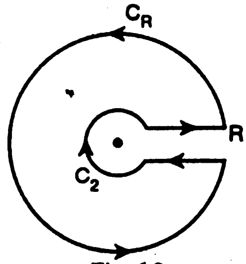


Fig. 6.2

Residue of $f(z)=\frac{z^{m-1}}{1+z}$ at $z=e^{\pi i}=\lim_{z\to -1}\{z-(-1)\}f(z)$

$$=\lim_{z\to e^{\pi i}}\frac{(z+1)z^{m-1}}{1+z}=e^{i\pi(m-1)}$$

Applying Cauchy's Residue theorem of complex analysis,

$$\int_r^R \frac{x^{m-1}}{1+x}\,dx+\int_{C_R} f(z)\,dz+\int_R^r \frac{(xe^{2\pi i})^{m-1}}{1+xe^{2\pi i}}\,d(xe^{2\pi i})+\int_{Cr} f(z)\,dz=2\pi i\times e^{i\pi}(m-1)$$

Proceeding to the limit as $R\to\infty$, $r\to 0$, we find

$$\left|\int_{CR}\frac{z^{m-1}}{1+z}\,dz\right|\le\int_{CR}\frac{|z|^{m-1}}{|z+1|}|dz|$$

Put $z=\text{Re}^{i\theta}$ so that $dz=i\,\text{Re}^{i\theta}\,d\theta$ and $|z|=R$, $|dz|=Rd\theta$ also θ varies from 0 to 2π

$$\le\frac{2\pi R^m}{R-1} \text{ as } |1+z|\ge|z|-1 \text{ and } \int_{Cr}|dz|=2\pi R$$

As $R\to\infty$ and $0<m<1$, $\frac{2\pi R^m}{R-1}\to 0$, so $\lim_{r\to 0}\int_{CR}\frac{z^{m-1}}{1+z}\,dz\to 0$

Similarly $\lim_{r\to 0}\int_{Cr}\frac{z^{m-1}}{1+z}\,dz<\frac{2\pi r^m}{1-r}\to 0$ for $m>0$

As such we are left with

$$\int_0^\infty \frac{x^{m-1}}{1+x}dx - e^{2\pi im}\int_0^\infty \frac{x^{m-1}}{1+x}dx = -2\pi i e^{\pi im} \text{ or } e^{-\pi i} = -1 \text{ and } e^{2\pi i} = 1$$

or
$$(1-e^{2\pi im})\int_0^\infty \frac{x^{m-1}}{1+x}dx = -2\pi i e^{\pi im}$$

or
$$\int_0^\infty \frac{x^{m-1}}{1+x}dx = \frac{2\pi i e^{\pi im}}{e^{2\pi im}-1} = \frac{2\pi i}{e^{\pi im}-e^{-\pi im}} = \frac{\pi}{\sin m\pi}$$

Hence $\Gamma\, m\, \Gamma\,(1-\ m) = \int_0^\infty \frac{x^{m-1}}{1+x}dx^{-1} = \frac{\pi}{\sin m\pi}$

6.8. REDUCTION OF DEFINITE INTEGRALS TO GAMMA FUNCTIONS

[1] To show that

$$\int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}dy = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}$$

We know that

$$\int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}dy = \int_0^1 \frac{x^{m-1}}{(1+y)^{m+n}}dy + \int_1^\infty \frac{y^{m-1}}{(1+y)^{m+n}}dy$$

Substituting $\frac{1}{x}$ for y in the second integral on R.H.S., we get

$$\int_1^\infty \frac{y^{m-1}}{(1+y)^{m-n}}dy = \int_1^0 \frac{\left(\frac{1}{x}\right)^{m-1}}{\left(1+\frac{1}{x}\right)^{m+n}}\left(-\frac{1}{x^2}\right)dx, \because\ y = \frac{1}{x}, dy = -\frac{1}{x^2}dx$$

$$= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}dx$$

$$\therefore\ \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}dy = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}dx$$

[$\because$ change of variable does not change the value of integral]

or
$$\beta(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}$$

[2] If we substitute $\frac{ay}{b}$ for x, we get

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx = a^m b^n \int_0^\infty \frac{y^{m-1}}{(ay+b)^{m+n}}dy$$

Since $\int_0^\infty \frac{x^{m-1}dx}{(1+x)^{m+n}} = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}$ by §§ 6.6 and 6.7

$$\therefore a^m b^n \int_0^\infty \frac{y^{m-1}dy}{(ay+b)^{m+n}} = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}$$

or

$$\int_0^\infty \frac{y^{m-1}dy}{(ay+b)^{m+n}} = \frac{\Gamma m \Gamma n}{a^m b^n \Gamma(m+n)}$$

COROLLARY. Substituting $y = \tan^2\theta$, this relation transforms to

$$\int_0^{\pi/2} \frac{\sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}} = \frac{\Gamma m \Gamma n}{2a^m b^n \Gamma(m+n)}$$

[3] If we put $x = \sin^2\theta$, we get

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx = 2\int_0^{\pi/2} \sin^{2(m-1)}\theta\cos^{2(n-1)}\theta, \sin\theta\cos\theta\, d\theta$$

$[\because dx = 2\sin\theta\cos\theta\, d\theta]$

or

$$\beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)} = 2\int_0^{\pi/2} [\sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta$$

$$\therefore \int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta = \frac{\Gamma m \Gamma n}{2\Gamma(m+n)}$$

COROLLARY. Replacing $2m-1$ by p and $2n-1$ by q, this relation reduces to

$$\int_0^{\pi/2} \sin^p\theta\cos^q\theta\, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

Putting $p = 0$ and $q = 0$ in succession, we get

$$\int_0^{\pi/2} (\cos\theta)^q\, d\theta = \frac{\Gamma\left(\frac{q+1}{2}\right)\sqrt{\pi}}{2\Gamma\left(\frac{q}{2}+1\right)} \text{ and } \int_0^{\pi/2} (\sin\theta)^p\, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)}{2\Gamma\left(\frac{p}{2}+1\right)} . \frac{\sqrt{\pi}}{2}$$

[4] By Putting $x = \sin^2\theta$ in $\frac{\Gamma m \Gamma n}{\Gamma(m+n)}$ ' $= \int_0^1 x^{m-1}(1-x)^{n-1}d\theta$

we have just proved that

$$\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta = \frac{\Gamma m \Gamma n}{2\Gamma(m+n)} \qquad ...(1)$$

Now if we put $2n = 1$, we have

$$\int_0^{\pi/2} \sin^{2m-1}\theta\, d\theta = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma m}{\Gamma\left(m+\frac{1}{2}\right)} \quad ...(2)$$

Again putting $m = n$ in (1), we find

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2m-1}\theta\, d\theta = \frac{(\Gamma m)^2}{2\Gamma(2m)}$$

or
$$\frac{\{\Gamma(m)\}^2}{2\Gamma(2m)} = \frac{1}{2^{2m-1}} \int_0^{\pi} \sin^{2m-1} 2\theta\, d\theta \quad [\because 2\sin\theta\cos\theta = \sin 2\theta]$$

$$= \frac{1}{2^{2m}} \int_0^{\pi/2} \sin^{2m-1}\phi\, d\phi \text{ put } 2\theta = \phi, d\theta = \frac{1}{2} d\phi$$

$$= \frac{1}{2^{2m}} \cdot 2\int_0^{\pi/2} \sin^{2m-1}\phi\, d\phi \quad [\because \sin(\pi - \phi) = \sin\phi$$

(Prop. of definite integral)]

$$\therefore \int_0^{\pi/2} \sin^{2m-1}\theta\, d\theta = \frac{2^{2m-2}(\Gamma m)^2}{\Gamma(2m)} \text{ (replacing } \phi \text{ by } \theta). \quad ...(3)$$

From (2) and (3) it is obvious that

$$\frac{2^{2m-2}(\Gamma m)^2}{\Gamma(2m)} = \frac{\Gamma m}{\Gamma\left(m+\frac{1}{2}\right)} \frac{\sqrt{\pi}}{2}$$

or
$$\Gamma m \Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma(2m)$$

This may also be put in the form

$$\Gamma(2m) = \frac{2^{2m-1}}{\sqrt{\pi}} \Gamma m \Gamma\left(m+\frac{1}{2}\right)$$

(Agra, 1962, 65)

which is known as **Legendre's Duplication formula.**

6.9. MISCELLANEOUS PROPOSITIONS

[1] Infinite and Improper Integrals: Their Principal and General values.

Riemann introduced the definition of definite integral $\int_a^b f(x)dx$ under two assumptions:

(*i*) Integral $f(x)$ is bounded and

(*ii*) The range of integration *i.e.* (a, b) is finite

Cauchy modified this definition to include the cases :

(*i*) The range of integration is infinite, *e.g.*

$$\int_1^{\infty} \frac{dx}{x^{2/3}}, \int_{-\infty}^{\infty} \frac{dx}{1+x^2}, \text{ and}$$

(*ii*) the integrand becomes infinite, *i.e.* the function is unbounded in the range *e.g.*, $\int_{-1}^{1} \frac{dx}{x^{2/3}}$ (unbounded at $x = 0$).

The former is known as *Infinite integral of first kind* and the latter as *infinite integral of second kind.* But by convention we are accustomed of speaking the first type as **infinite integral** and the second kind as **Improper integral.**

If there is an integral $\int_a^b f(x)\, dx$, where $f(x)$ becomes infinite at $x = c$ $(a < c < b)$, then in order to exclude c, the point $x = c$ may be enclosed in a small neighbourhood $(c - \mu\varepsilon, c + \nu\varepsilon)$, μ and ν being any arbitrary constants, so that

$$\int_a^b f(x)\, dx = \lim_{\varepsilon \to 0}\left[\int_a^{c-\mu\varepsilon} f(x)\, dx + \int_{c+\nu\varepsilon}^{b} f(x)\, dx\right].$$

This limit is known to be the **General value** of the given integral. In case $\mu = \nu$, the value of this limit is said to be the **Principal value** of the integral.

The integrals having infinite limits may be expressd as follows :

$$\int_a^{\infty} f(x)\, dx = \lim_{\varepsilon \to 0} \int_a^{1/\varepsilon} f(x)\, dx,$$

$$\int_{-\infty}^{b} f(x)\, dx = \lim_{\varepsilon \to 0} \int_{-1/\varepsilon}^{b} f(x)\, dx,$$

$$\int_{-\infty}^{\infty} f(x)\, dx = \lim_{\varepsilon \to 0} \int_{-1/\mu\varepsilon}^{1/\nu\varepsilon} f(x)\, dx \text{ etc.}$$

[2] Evaluation of the Integrals of the Type $\int_{-\infty}^{\infty} \frac{f(x)}{F(x)}\, dx$,

where $\frac{f(x)}{F(x)}$ is rational algebraic function such that the degree of $f(x)$ is at least two lower than that of $F(x)$ and all the roots of the equation $F(x) = 0$ are imaginary.

Since imaginary roots occur in pairs and all the roots of $F(x) = 0$ are imaginary, it follows that $F(x)$ must be of an even number in degree say $2n$. Further it follows that $F(x)$ cannot become zero for any real value of x and consequently the integrand $\frac{f(x)}{F(x)}$ is always finite for all real values of x.

Let us suppose that any pair of imaginary roots is $a \pm ib$, so that the factors of $F(x)$ corresponding to these roots are $(x - a - ib)$ and $(x - a + ib)$. Let the corresponding partial fractions be of the form

$$\frac{A - iB}{x - a - ib} \text{ and } \frac{A + iB}{x - a + ib}.$$

Then $$\frac{A - iB}{(x-a) - ib} + \frac{A + iB}{(x-a) + ib} = \frac{2A\,(x-a) + 2Bb}{(x-a)^2 + b^2}.$$

Now, $$\int_{-\infty}^{\infty} \frac{2A(x-a) + 2Bb}{(x-a)^2 + b^2}\, dx$$

$$= \lim_{\varepsilon \to 0} \left[\int_{-1/\mu\varepsilon}^{1/\nu\varepsilon} \frac{2A\,(x-a)}{(x-a)^2+b^2}\,dx + \int_{-1/\mu\varepsilon}^{1/\nu\varepsilon} \frac{2Bb}{(x-a)^2+b^2}\,dx \right]$$

$$= \lim_{\varepsilon \to 0} \left[A \log \left\{ (x-a)^2 + b^2 \right\} + 2B \tan^{-1} \frac{x-a}{b} \right]_{-1/\mu\varepsilon}^{1/\nu\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \left[A \log \frac{(1-a\nu\varepsilon)^2 + b^2\nu^2\varepsilon^2}{(-1-a\mu\varepsilon)^2 + b^2\mu^2\varepsilon^2} \cdot \frac{\mu^2}{\nu^2} + 2B \left\{ \tan^{-1}\left(\frac{\frac{1}{\nu\varepsilon} - a}{b} \right) + \tan^{-1}\left(\frac{\frac{1}{\mu\varepsilon} + a}{b} \right) \right\} \right]$$

$$= A \log \frac{\mu^2}{\nu^2} + 2\pi B = 2\left(A \log \frac{\mu}{\nu} + \pi B \right). \qquad \ldots(1)$$

As we have already supposed that $F(x)$ is of degree $2n$, so it will consist of n pairs of imaginary roots. Let these pairs be $a_1 \pm ib_1, a_2 \pm ib_2, \ldots a_n \pm ib_n$ and let the corresponding constants in the partial fractions be $(A_1, B_1), (A_2, B_2), \ldots (A_n, B_n)$; then we have

$$\int_{-\infty}^{\infty} \frac{f(x)}{F(x)}\,dx = \int_{-\infty}^{\infty} \frac{2A_1\,(x-a_1) + 2B_1b_1}{(x-a_1)^2 + b_1^2}\,dx + \int_{-\infty}^{\infty} \frac{2A_2\,(x-a_2) + 2B_2b_2}{(x-a_2)^2 + b_2^2}\,dx + \ldots + \int_{-\infty}^{\infty} \frac{2A_n\,(x-a_n) + 2B_nb_n}{(x-a_n)^2 + b_n^2}\,dx$$

$$= 2\left[A_1 \log \frac{\mu}{\nu} + \pi B_1 \right] + 2\left[A_2 \log \frac{\mu}{\nu} + \pi B_2 \right] + \ldots + 2\left[A_n \log \frac{\mu}{\nu} + \pi B_n \right]$$

from (1)

$$= 2\,(A_1 + A_2 + \ldots + A_n) \log \frac{\mu}{\nu} + 2\pi\,(B_1 + B_2 + \ldots + B_n). \qquad \ldots(2)$$

Also $\dfrac{f(x)}{F(x)} = \dfrac{2A_1\,(x-a_1) + 2B_1b_1}{(x-a_1)^2 + b_1^2} + \ldots + \dfrac{2A_n\,(x-a_n) + 2B_nb_n}{(x-a_n)^2 + b_n^2}$.

$$\therefore f(x) = \frac{2A_1\,(x-a_1) + 2B_1b_1}{(x-a_1)^2 + b_1^2} \cdot F(x) + \ldots$$

Since $(x - a_1)^2 + b_1^2$ is one of the factors of $F(x) = 0$, therefore $F(x)$ is completely divisible by $(x - a_1)^2 + b_1^2$ etc...

$\therefore f(x) = \{2A_1\,(x - a_1) + 2B_1b_1\}\ [x^{2n-2} + \ldots] + \ldots$

But $f(x)$ being at most of degree $2n - 2$, on equating the coefficients of x^{2n-1} on either side, we get

$$0 = 2\,(A_1 + A_2 + \ldots A_n).$$

Hence from (2), we get

$$\int_{-\infty}^{\infty} \frac{f(x)}{F(x)}\,dx = 2\pi(B_1 + B_2 + \ldots + B_n).$$

Applications of this result.

(*i*) **Evaluation of** $\int_{-\infty}^{\infty} \frac{x^{2m}\, dx}{1+x^{2n}}$, where m and n are positive integers and $n > m$.

Here $1 + x^{2n} = 0$ gives $x = (-1)^{1/2n}$, *i.e.*, all the roots imaginary.

If α be a root of the equation $x^{2n} + 1 = 0$, then $\alpha = (-1)^{1/2n}$.

Putting $-1 = k\cos\theta$ and $0 = k\sin\theta$ so that $k = 1$, $\theta = \pi$, we get

$\alpha = [\cos(2r\pi+\pi) + i\sin(2r\pi+\pi)]^{1/2n}$ where $r = 0, 1, 2, \ldots$

$$= \cos\frac{2r+1}{2n}\pi + i\sin\frac{2r+1}{2n}\cdot\pi \text{ by De Moivre's theorem}$$

where $r = 0, 1, 2, \ldots, 2n-1$

Since imaginary roots occur in pairs, α is of the form

$$\alpha = \cos\frac{2r+1}{2n}\pi \pm i\sin\frac{2r+1}{2n}\pi, \text{ where } r = 0, 1, 2, \ldots, (n-1).$$

Now α being one of the roots of $x^{2n}+1 = 0$, $(x-\alpha)$ is one of the factors of $(x^{2n}+1)$.

Let $\frac{A-Bi}{x-\alpha}$ be the partial fraction corresponding to any root α of the type $\cos\frac{2r+1}{2n}\pi + i\sin\frac{2r+1}{2n}\pi$. Then

$$\frac{A-Bi}{x-\alpha} = \frac{\alpha^{2m}}{2n\alpha^{2n-1}(x-\alpha)}$$

[**Note.** *If* $(x-a)$ *is a factor of* $F(x)$ *in* $\frac{f(x)}{F(x)}$, *then the partial fraction corresponding to* $(x-a)$ *is* $\frac{f(a)}{F'(a)\cdot(x-a)}$].

$$\therefore\ A-Bi = \frac{\alpha^{2m+1}}{2n\cdot\alpha^{2n}} = \frac{\alpha^{2m+1}}{-2n}, \text{ since } \alpha^{2n} = -1, \quad \alpha \text{ being root of } x^{2n}+1 = 0$$

$$= -\frac{1}{2n}\left[\cos\frac{2r+1}{2n}\pi + i\sin\frac{2r+1}{2n}\cdot\pi\right]^{2m+1}$$

$$= -\frac{1}{2n}\left[\cos(2r+1)\frac{(2m+1)}{2n}\pi + i\sin(2r+1)\frac{(2m+1)}{2n}\pi\right]$$

$$= -\frac{1}{2n}[\cos(2r+1)\theta + i\sin(2r+1)\theta], \text{ where } \theta = \frac{2m+1}{2n}\pi.$$

Equating imaginary parts on either side, we get

$$B = \frac{1}{2n}\sin(2r+1)\theta \text{ where } r = 0, 1, 2, \ldots, (n-1).$$

Hence if $B_1, B_2, \ldots, B_n$ be the constants corresponding to $r = 0, 1, 2, \ldots (n-1)$, then

$$\int_{-\infty}^{\infty} \frac{x^{2m}}{1+x^{2n}}dx = 2\pi\,[B_1 + B_2 + \ldots + B_n]$$

i.e., $$\int_{-\infty}^{\infty} \frac{x^{2m}}{1+x^{2n}}dx = \frac{\pi}{n}[\sin\theta + \sin 3\theta + \ldots + \sin(2n-1)\theta]$$

$$= \frac{\pi}{4n \sin\theta} [2 \sin^2\theta + 2 \sin\theta \sin 3\theta + \ldots + 2 \sin\theta \sin (2n-1)\theta]$$

$$= \frac{\pi}{4n \sin\theta} [(1 - \cos 2\theta) + (\cos 2\theta - \cos 4\theta) + \ldots + \{\cos (2n-2)\theta - \cos 2n\theta\}$$

$$= \frac{\pi}{4n \sin\theta} [1 - \cos 2n\theta] = \frac{\pi}{4n \sin\theta} \cdot 2 \sin^2 n\theta$$

$$= \frac{\pi \sin^2 n\theta}{2n \sin\theta}.$$

But $\sin n\theta = \sin \frac{2m+1}{2}\pi = \pm 1$, so that $\sin^2 n\theta = 1$.

Hence $\int_0^\infty \frac{x^{2m}\,dx}{1+x^{2n}} = \frac{\pi}{2n \sin\theta} = \frac{\pi}{2n} \operatorname{cosec} \frac{2m+1}{2n}\pi.$

(*ii*) **Evaluation of** $\int_0^\infty \frac{x^{2m}}{1-x^{2n}}\,dx$, where m and n are positive integers and $n > m$.

Here the equation $1 - x^{2n} = 0$ has got only two real roots ± 1, corresponding to the factors $(1-x)$ and $(1+x)$.

Now $(x-1)$ corresponding to the partial fraction

$$-\frac{(1)^{2m}}{-2n(+1)^{2n-1}(x-1)},\ i.e.,\ \frac{1}{-2n(x-1)}$$

and $(x+1)$ corresponds to the partial fraction

$$\frac{(1)^{2m}}{-2n(-1)^{2n-1}(x+1)},\ i.e.,\ \frac{1}{2n(x+1)}.$$

$$\therefore \frac{1}{2n(x+1)} - \frac{1}{2n(x-1)} = \frac{1}{2n}\left[\frac{x-1-x-1}{x^2-1}\right] = \frac{1}{n(1-x^2)}.$$

But $\int_0^\infty \frac{dx}{1-x^2} = \int_0^1 \frac{dx}{1-x^2} + \int_1^\infty \frac{dx}{1-x^2}.$

If we put $x = \frac{1}{z}$ in the last integral, we have

$$\int_0^\infty \frac{dx}{1-x^2} = -\int_1^0 \frac{(1/z^2)\,dz}{1-(1/z)^2} = \int_0^1 \frac{dz}{z^2-1} = -\int_0^1 \frac{dx}{1-x^2}$$

(by the properties of definite integrals),

so that $\int_0^\infty \frac{dx}{1-x^2} = \int_0^1 \frac{dx}{1-x^2} - \int_0^1 \frac{dx}{1-x^2} = 0.$

It follows that the part of the definite integral corresponding to real roots of $1 - x^{2n} = 0$ vanishes and as such we have to consider only the integral corresponding to $(n-1)$ conjugate pairs of imaginary roots.

Now $1 - x^{2n} = 0$ gives $x = (1)^{1/2n}$

or $x = (\cos 2r\pi + i \sin 2r\pi)^{1/2n}$ on putting $1 = k\cos\theta$, $0 = k \sin\theta$,

so that $k = 1$, $\theta = 0$,

$$= \cos \frac{2r\pi}{2n} \pm i \sin \frac{2r\pi}{2n}$$

Leaving the roots corresponding to $r = 0$ and $r = n$, which are real, let α be any of the $(n-1)$ pairs of imaginary roots, so that

$$\alpha = \cos\frac{2r\pi}{2n} + i \sin\frac{2r\pi}{2n} \quad \text{where } r = 1, 2, 3, \ldots, (n-1).$$

If $\frac{A - Bi}{x - \alpha}$ be the partial fraction corresponding to any root α, then

$$A - Bi = \frac{\alpha^{2m}}{-2n\alpha^{2n-1}} = -\frac{1}{2n}\alpha^{2m+1} \qquad \text{since } \alpha^{2n} = 1$$

$$= -\frac{1}{2n}\left[\cos \frac{2r\pi}{2n} + i \sin \frac{2r\pi}{2n}\right]^{2m+1}$$

$$= -\frac{1}{2n}\left[\cos \frac{2r(2m+1)\pi}{2n} + i \sin \frac{2r(2m+1)\pi}{2n}\right]$$

$$= -\frac{1}{2n}[\cos 2r\theta + i \sin 2r\theta] \quad \text{where } \theta = \frac{2m+1}{2n}\pi$$

and $\qquad r = 1, 2, 3, \ldots, (n-1).$

Equating imaginary parts on either side, we get

$$B = \frac{1}{2n} \sin 2r\theta.$$

Hence if $B_1, B_2, \ldots, B_n$ be the constants corresponding to $r = 1, 2, \ldots, (n-1)$, then

$$\int_0^\infty \frac{x^{2m}}{1 - x^{2n}}\, dx = \pi\, [B_1 + B_2 + \ldots + B_n]$$

$$= \frac{\pi}{2n}\, [\sin 2\theta + \sin 4\theta + \ldots + \sin (2n-2)\,\theta]$$

$$= \frac{\pi}{4n \sin \theta}\, (2 \sin \theta \sin 2\theta + 2 \sin \theta \sin 4\theta + \ldots + 2 \sin \theta \sin (2n-2)\,\theta]$$

$$= \frac{\pi}{4n \sin \theta}\, [(\cos \theta - \cos 3\theta) + (\cos 3\theta - \cos 5\theta) + \ldots + \{\cos (2n-3)\theta - \cos (2n-1)\theta\}]$$

$$= \frac{\pi}{4n \sin \theta}\, [\cos \theta - \cos (2n-1)\theta]$$

$$= \frac{\pi}{4n \sin \theta} \cdot 2 \sin n\theta \cdot \sin (n-1)\,\theta.$$

But $\sin (n-1)\,\theta = \sin (n-1) \frac{2m+1}{2n}\pi$

$$= \sin \left[\frac{2m+1}{2}\pi - \frac{2m+1}{2n}\pi\right] = \pm \cos\frac{2m+1}{2n}\pi$$

and $\qquad \sin n\theta = \frac{2m+1}{2}\pi = \pm 1.$

Hence $\int_0^\infty \frac{x^{2m}}{2-x^{2n}}dx = \frac{\pi}{2n}\cot\frac{2m+1}{2n}\pi.$

Deductions. We have proved

$$\int_0^\infty \frac{x^{2m}}{1+x^{2n}}dx = \frac{\pi}{2n}\operatorname{cosec}\frac{2m+1}{2n}\pi \qquad ...(1)$$

and
$$\int_0^\infty \frac{x^{2m}}{1-x^{2n}}dx = \frac{\pi}{2n}\cot\frac{2m+1}{2n}\pi \qquad ...(2)$$

If we put $x^{2n} = z$ and then $\frac{2m+1}{2n} = \alpha$, we can easily show that (1) and (2) give respectively

$$\int_0^\infty \frac{z^{\alpha-1}dz}{1+z} = \pi \operatorname{cosec}\pi\alpha \text{ and } \int_0^\infty \frac{z^{\alpha-1}dz}{1-z} = \pi\cot\pi\alpha.$$

[*iii*] **Evaluation of** $\Gamma\frac{1}{n}\Gamma\frac{2}{n}\Gamma\frac{3}{n}...\Gamma\left(\frac{n-1}{n}\right)$, where n is a positive integer.

Let
$$P = \Gamma\frac{1}{n}\Gamma\frac{2}{n}\Gamma\frac{3}{n}...\Gamma\left(\frac{n-1}{n}\right)$$
$$= \Gamma\frac{1}{n}\Gamma\frac{2}{n}\Gamma\frac{3}{n}...\Gamma\left(1-\frac{1}{n}\right). \qquad ...(1)$$

By reversing the order of the factors, this may be written as

$$P = \Gamma\left(1-\frac{1}{n}\right)\Gamma\left(1-\frac{2}{n}\right)\Gamma\left(1-\frac{3}{n}\right)...\Gamma\frac{2}{n}\Gamma\frac{1}{n}$$
$$= \Gamma\left(1-\frac{1}{n}\right)\Gamma\left(1-\frac{2}{n}\right)\Gamma\left(1-\frac{3}{n}\right)...\Gamma\left(1-\frac{n-2}{n}\right)\Gamma\left(1-\frac{n-1}{n}\right). \qquad ...(2)$$

Multiplying (1) and (2), we get

$$P^2 = \left\{\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)\right\}\left\{\Gamma\left(\frac{2}{n}\right)\Gamma\left(1-\frac{2}{n}\right)\right\}...\left\{\Gamma\left(\frac{n-1}{n}\right)\Gamma\left(1-\frac{n-1}{n}\right)\right\}$$

$$= \frac{\pi}{\sin\frac{\pi}{n}}\cdot\frac{\pi}{\sin\frac{2\pi}{n}}...\frac{\pi}{\sin\frac{n-1}{n}\pi}, \text{ since } \Gamma m\,\Gamma(1-m) = \frac{\pi}{\sin n\pi} \qquad \text{(by §6.7)}$$

$$= \frac{\pi^{n-1}}{\sin\frac{\pi}{n}\sin\frac{2\pi}{n}...\sin\frac{n-1}{n}\pi}, \qquad ...(3)$$

Suppose that $D = \sin\frac{\pi}{n}\sin\frac{2\pi}{n}...\sin\frac{n-1}{n}\pi.$

In order to calculate the value of D, let us consider the equation $1 - x^{2n} = 0$, *i.e.*, $x = (1)^{1/2n}$

$$= (\cos 2r\pi + i\sin 2r\pi)^{1/2n}$$
$$= \cos\frac{2r\pi}{2n} + i\sin\frac{2r\pi}{2n}, \qquad \text{where } r = 0, 1, 2, ...(2n-1),$$

so that we have the identity

$$1-x^{2n}=(1-x)(1+x)\left\{x-\left(\cos\frac{\pi}{n}+i\sin\frac{\pi}{n}\right)\right\}\left\{x-\left(\cos\frac{\pi}{n}-i\sin\frac{\pi}{n}\right)\right\}$$

$$\times\left\{x-\left(\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}\right)\right\}\left\{x-\left(\cos\frac{2\pi}{n}-i\sin\frac{2\pi}{n}\right)\right\}\ldots$$

$$\left\{x-\left(\cos\frac{n-1}{n}\pi+i\sin\frac{n-1}{n}\pi\right)\right\}\left\{x-\left(\cos\frac{n-1}{n}\pi-i\sin\frac{n-1}{n}\pi\right)\right\},$$

i.e. $$\frac{1-x^{2n}}{1-x^2}=\left(1-2x\cos\frac{\pi}{n}+x^2\right)\left(1-2x\cos^2\frac{\pi}{n}+x^2\right)\ldots$$

$$\ldots\left(1-2x\cos\frac{n-1}{n}\pi+x^2\right)$$

Putting $x=1$ and $x=-1$ in turn in the above identity, we have

$$n=\left(2-2\cos\frac{\pi}{n}\right)\left(2-2\cos\frac{2\pi}{n}\right)\ldots\left(2-2n\cos\frac{n-1}{n}\pi\right)$$

$$=\left(2\sin\frac{\pi}{2n}\right)^2\left(2\sin\frac{2\pi}{2n}\right)^2\ldots\left(2\sin\frac{n-1}{n}\pi\right)^2 \qquad \ldots(4)$$

and $$n=\left(2+2\cos\frac{\pi}{n}\right)\left(2+2\cos\frac{2\pi}{n}\right)\ldots\left(2+2\cos\frac{n-1}{n}\pi\right)$$

$$=\left(2\cos\frac{\pi}{2n}\right)^2\left(2\cos\frac{2\pi}{2n}\right)^2\ldots\left(2\cos\frac{n-1}{2n}\pi\right)^2. \qquad \ldots(5)$$

$$\left[\text{Since } \lim_{x\to1}\frac{1-x^2}{1-x^2}=\lim_{x\to1}\frac{-2nx^{2n-1}}{-2n}=n, \text{ also } \lim_{x\to-1}\frac{1-x^{2n}}{1-x^2}=n\right]$$

Multiplying (4) and (5) we get

$$n^2=2^{2n-2}\sin^2\frac{\pi}{n}\sin^2\frac{2\pi}{n}\ldots\sin^2\frac{n-1}{n}\pi=2^{2(n-1)}\cdot D^2$$

so that $$D=\frac{n}{2^{n-1}}.$$

[Note. *The value of D may be calculated by using the identity*

$$\frac{\sin n\theta}{\sin\theta}=2^{n-1}\sin\left(\theta+\frac{\pi}{n}\right)\sin\left(\theta+\frac{2\pi}{n}\right)\ldots\sin\left(\theta+\frac{n-1}{n}\pi\right)$$

If we put $\theta=0$, $\lim_{\theta\to0}\frac{\sin n\theta}{\sin\theta}=\lim_{\theta\to0}\frac{n\cos n\theta}{\cos\theta}=n,$

so that $n=2^{n-1}\cdot\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\sin\frac{n-1}{n}\pi$]

Thus we have from (3)

$$P^2=\frac{\pi^{n-1}}{n}.2^{n-1}$$

i.e. $P=\Gamma\frac{1}{n}\Gamma\frac{2}{n}\ldots\Gamma\left(\frac{n-1}{n}\right)=\frac{(2\pi)^{(n-1)/2}}{\sqrt{n}}$, which is the required value.

Problem 1. *Show that* $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}dx}{(a+x)^{m+n}} = \frac{\Gamma m \Gamma n}{a^n (1+a)^m \Gamma(m+n)}$.

Let $I = \int_0^1 \frac{x^{m-1}(1-x)^{n-1}dx}{(a+x)^{m+n}}$

Put $\frac{x(1+a)}{(a+x)} = y$, so that $x = \frac{ay}{1+a-y}$.

and $(1+a)\cdot\frac{a+x-x}{(a+x)^2}dx = dy$, *i.e.* $dx = \frac{(a+x)^2}{a(1+a)}dy$.

As such $1-x = 1-\frac{ay}{1+a-y} = \frac{1+a-y-ay}{1+a-y} = \frac{(1+a)(1-y)}{1+a-y}$.

Also $a+x = a+\frac{ay}{1+a-y} = \frac{a+a^2-ay+ay}{1+a-y} = \frac{a(1+a)}{1+a-y}$.

and therefore, $dx = \frac{a^2(1+a)^2}{(1+a-y)^2 a(1+a)}dy = \frac{a(1+a)}{(1+a-y)^2}dy$.

Thus

$$I = \int_0^1 \frac{a^{m-1}y^{m-1}(1+a)^{n-1}(1-y)^{n-1}(1+a-y)^{m+n}a(1+a)}{(1+a-y)^{m-1}(1+a-y)^{n+1}a^{m+n}(1+a)^{m+n}(1+a-y)^2}dy$$

$$= \frac{1}{a^n(1+a)^m}\int_0^1 y^{m-1}(1-y)^{n-1}dy$$

or $I = \frac{1}{a^n(1+a)^m}\beta(m,n) = \frac{1}{a^n(1+a)^m}\frac{\Gamma m\Gamma n}{\Gamma(m+n)}$.

Problem 2. *Show that* $2^n\Gamma(n+\frac{1}{2}) = 1.3.5...(2n-1)\sqrt{\pi}$. *(Agra, 1966)*

We know that $\Gamma(n+1) = n\Gamma n$.

$$\therefore \Gamma\left(n+\tfrac{1}{2}\right) = \left(n-\tfrac{1}{2}\right)\Gamma\left(n-\tfrac{1}{2}\right)$$

$$= \left(n-\tfrac{1}{2}\right)\left(n-\tfrac{3}{2}\right)\Gamma\left(n-\tfrac{3}{2}\right)$$

... ...

$$= \left(n-\tfrac{1}{2}\right)\left(n-\tfrac{3}{2}\right)\left(n-\tfrac{5}{2}\right)\left(n-\tfrac{7}{2}\right)...\tfrac{5}{2}.\tfrac{3}{2}.\tfrac{1}{2}\Gamma\tfrac{1}{2}$$

$$= \left(n-\tfrac{1}{2}\right)\left(n-\tfrac{3}{2}\right)\left(n-\tfrac{5}{2}\right)...\tfrac{5}{2}.\tfrac{3}{2}.\tfrac{1}{2}\sqrt{\pi}$$

$$= \frac{\sqrt{\pi}}{2^n}(2n-1)(2n-3)...5.3.1$$

or $2^n\Gamma(n+\frac{1}{2}) = 1.3.5...(2n-3).(2n-1)\sqrt{\pi}$.

Problem 3. *Show that* $\int_0^1 \frac{dx}{\sqrt{(1-x^n)}} = \frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}\cdot\frac{\sqrt{\pi}}{n}$. *(Agra, 1972)*

Suppose $I = \int_0^1 \frac{dx}{\sqrt{(1-x^n)}}$.

Put $\quad x^n = \sin^2\theta$, i.e. $x = \sin^{2/n}\theta$.

$\therefore \quad dx = \frac{2}{n}\sin^{(2-n)/n}\theta \cos\theta \, d\theta.$

Thus $\quad I = \frac{2}{n}\int_0^{\pi/2}\frac{\sin^{(2-n)/n}\theta \cos\theta \, d\theta}{\cos\theta}.$

$$= \frac{2}{n}\int_0^{\pi/2}\sin^{(2-n)/n}\theta \, d\theta$$

$$= \frac{2}{n}\cdot\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{n}\right)}{2\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)} = \frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}\cdot\frac{\sqrt{\pi}}{n}.$$

Problem 4. *Prove the relation*

$$\int_0^{\pi/2}\frac{d\theta}{\sqrt{(\sin\theta)}}\times\int_0^{\pi/2}\sqrt{(\sin\theta)}\, d\theta = \pi.$$

$$\text{L.H.S.} = \int_0^{\pi/2}(\sin\theta)^{-1/2}d\theta \times \int_0^{\pi/2}(\sin\theta)^{1/2}d\theta$$

$$= \frac{\Gamma\frac{1}{2}\Gamma\frac{1}{4}}{2\Gamma\frac{3}{4}}\times\frac{\Gamma\frac{1}{2}\Gamma\frac{3}{4}}{2\Gamma\frac{5}{4}}$$

$$= \frac{\pi}{4}\cdot\frac{\Gamma\frac{1}{4}}{\frac{1}{4}\Gamma\frac{1}{4}} = \pi \qquad \left[\because \Gamma\frac{5}{4} = \frac{1}{4}\Gamma\frac{1}{4}\right]$$

Problem 5. *Prove that* $\int_0^{\pi/2}\sqrt{(\tan\theta)}\, d\theta = \frac{\Gamma\frac{1}{4}\Gamma\frac{3}{4}}{2}$ *(Nagpur, 1965)*

$$\text{L.H.S.} = \int_0^{\pi/2}(\sin\theta)^{1/2}\times(\cos\theta)^{-1/2}\, d\theta$$

$$= \frac{2\Gamma\frac{3}{4}\Gamma\frac{1}{4}}{2\Gamma 1} = \frac{\Gamma\frac{1}{4}\,\Gamma\left(1-\frac{1}{4}\right)}{2} = \frac{1}{2}\cdot\frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{\sqrt{2}}$$

Problem. 6. *Express the definite integral*

$$\int_0^{\pi/2}\frac{d\theta}{\sqrt{\left(1-k^2\sin^2\theta\right)}}$$

in the form of series, where $k < 1$.

$$I = \int_0^{\pi/2}\frac{d\theta}{\sqrt{\left(1-k^2\sin^2\theta\right)}} = \int_0^{\pi/2}\left(1-k^2\sin^2\theta\right)^{-1/2}d\theta.$$

Expanding it by Binomial Theorem for any index,

$$I = \int_0^{\pi/2}\left[1+\frac{1}{2}k^2\sin^2\theta+\frac{1.3}{2.4}k^4\sin^4\theta+\frac{1.3.5}{2.4.6}k^6\sin^6\theta+\ldots\right]d\theta$$

$$= [\theta]_0^{\pi/2} + \frac{1}{2}k^2 \cdot \frac{1}{2} \cdot \frac{\pi}{2} + \frac{1.3}{2.4}k^4 \cdot \frac{1.3}{2.4} \cdot \frac{\pi}{2} + \frac{1.3.5}{2.4.6}k^6 \cdot \frac{1.3.5}{2.4.6} \cdot \frac{\pi}{2} + \ldots$$

$$= \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1.3}{2.4}\right)^2 k^4 + \left(\frac{1.3.5}{2.4.6}\right)^2 k^6 + \ldots\right]$$

Problem 7. *Prove* $\int_0^1 \frac{35x^3}{32\sqrt{(1-x)}}dx = 1$ *(Nagpur, 1965)*

Suppose $I = \int_0^1 \frac{35x^3}{32\sqrt{(1-x)}}dx$ put $x = \sin^2\theta$ $\therefore\ dx = 2\sin\theta\cos\theta\, d\theta$

$$= \int_0^{\pi/2} \frac{35\ \sin^6\theta . 2\sin\theta\cos\theta\ d\theta}{32\ \cos\theta}$$

$$= \frac{35}{16}\int_0^{\pi/2} \sin^7\theta\ d\theta$$

$$= \frac{35}{16} \cdot \frac{\Gamma 4 \Gamma \frac{1}{2}}{2\Gamma \frac{9}{2}} = \frac{35}{16} \cdot \frac{3.2.1\sqrt{\pi}}{2 . \frac{7}{2} . \frac{5}{2} . \frac{3}{2} . \frac{1}{2}\sqrt{\pi}} = 1.$$

Problem 8. *Prove that* $\int_0^1 \frac{x^2\, dx}{\left(1-x^4\right)^{1/2}} \times \int_0^1 \frac{dx}{\left(1+x^4\right)^{1/2}} = \frac{\pi}{4\sqrt{2}}.$

Let $I_1 = \int_0^1 \frac{x^2\, dx}{\left(1-x^4\right)^{1/2}}$ and $I_2 = \int_0^1 \frac{dx}{\left(1+x^4\right)^{1/2}}.$

If we put $x^2 = \sin\theta$ in I_1, *i.e.* $2x\, dx = \cos\theta\, d\theta$, we have

$$I_1 = \frac{1}{2}\int_0^{\pi/2} \frac{\sqrt{(\sin\theta)} \cdot \cos\theta\ d\theta}{\cos\theta}$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{1/2}\theta\ d\theta = \frac{1}{2} \cdot \frac{\Gamma\frac{3}{4}\Gamma\frac{1}{2}}{2\Gamma\frac{5}{4}} = \frac{1}{4} \cdot \frac{\Gamma\frac{3}{4}\Gamma\frac{1}{2}}{\frac{1}{4}\Gamma\frac{1}{4}} = \frac{\Gamma\frac{3}{4}\Gamma\frac{1}{2}}{\Gamma\frac{1}{4}}.$$

Again putting $x^2 = \tan\phi$ in I_2, *i.e.* $2x\, dx = \sec^2\phi\, d\phi$, we get

$$I_2 = \frac{1}{2}\int_0^{\pi/4} \frac{\sec^2\phi\ d\phi}{\sqrt{(\tan\phi)} \cdot \sec\phi} = \frac{1}{2}\int_0^{\pi/4} \frac{\sqrt{2}\, d\phi}{\sqrt{(2\sin\phi\cos\phi)}}$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/4} \frac{d\phi}{\sqrt{(\sin 2\phi)}}$$ put $2\phi = \psi$, *i.e.* $2\ d\phi = d\psi$

$$= \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \sin^{-1/2}\psi\ d\psi$$

$$= \frac{1}{2\sqrt{2}} \cdot \frac{\Gamma\frac{1}{4}\Gamma\frac{1}{2}}{2\Gamma\frac{3}{4}} = \frac{\Gamma\frac{1}{4}\Gamma\frac{1}{2}}{4\sqrt{2}\Gamma\frac{3}{4}}.$$

Hence $I_1 \times I_2 = \dfrac{\Gamma\frac{3}{4}\Gamma\frac{1}{2}}{\Gamma\frac{1}{4}} \times \dfrac{\Gamma\frac{1}{4}\Gamma\frac{1}{2}}{4\sqrt{2}\Gamma\frac{3}{4}}$.

$$= \frac{\left(\Gamma\frac{1}{2}\right)^2}{4\sqrt{2}} = \frac{\pi}{4\sqrt{2}}$$

Problem 9. *Show that*

$$\int_0^1 \frac{x^n dx}{\sqrt{(1-x^2)}} = \frac{1.3.5\ldots(n-1)}{2.4.6\ldots n}\cdot\frac{\pi}{2} \text{ or } \frac{2.4.6\ldots(n-1)}{1.3.5\ldots n}$$

according as n is even or odd positive integer.

L.H.S.= $\displaystyle\int_0^1 \frac{x^n dx}{\sqrt{(1-x^2)}}$ put $x = \sin\theta,\ dx = \cos\theta\, d\theta$

$$= \int_0^{\pi/2} \frac{\sin^n\theta \cdot \cos\theta\, d\theta}{\cos\theta}$$

$$= \int_0^{\pi/2} \sin^n\theta\, d\theta = \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\frac{1}{2}}{2\Gamma\left(\frac{n+2}{2}\right)}$$

$$= \frac{\frac{n-1}{2}\cdot\frac{n-3}{2}\ldots\frac{3}{2}\frac{1}{2}\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{2.\frac{n}{2}\cdot\left(\frac{n}{2}-1\right)\ldots 2.1}$$

or
$$= \frac{\frac{n-1}{2}\cdot\frac{n-3}{2}\ldots 2.1.\Gamma\frac{1}{2}}{2.\frac{n}{2}\left(\frac{n-2}{2}\right)\left(\frac{n-4}{2}\right)\ldots\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}$$
according as n is even or odd

$$= \frac{1.3.5\ldots(n-3)\,(n-1)}{2^{n/2}\cdot 1.2.3\ldots\frac{n}{2}}\cdot\frac{\pi}{2} \text{ or } \frac{1.2\ldots\frac{n-1}{2}.2^{(n-1)/2}}{1.3.5\ldots n}$$

according as n is even or odd

$$= \frac{1.3.5\ldots(n-1)}{2.4.6\ldots n}\cdot\frac{\pi}{2} \text{ or } \frac{2.4.6\ldots(n-1)}{1.3.5\ldots n}$$
according as n is even or odd.

Problem 10. *Prove that* $\displaystyle\int_0^1 \left(1-x^n\right)^{1/n} dx = \frac{1}{n}\,\frac{[\Gamma(1/n)]^2}{2\Gamma(2/n)}$.

Let $I = \displaystyle\int_0^1 \left(1-x^n\right)^{1/n} dx$, put $x^n = \sin^2\theta$, *i.e.*, $x = \sin^{2/n}\theta$,

so that $dx = \dfrac{2}{n}\sin^{(2-n)/2}\theta\, \cos\theta\, d\theta$

$$= \frac{2}{n}\int_0^{\pi/2} \cos^{(2/n)+1}\theta \ \sin^{(2/n)-1}\theta \ d\theta$$

$$= \frac{2}{n}\cdot\frac{\Gamma\left(\frac{1}{n}+1\right)\Gamma\left(\frac{1}{n}\right)}{2\Gamma\left(\frac{2}{n}+1\right)}$$

$$= \frac{1}{n}\cdot\frac{1/n\ \Gamma(1/n)\,\Gamma(1/n)}{2/n\ \Gamma(2/n)} = \frac{1}{n}\cdot\frac{\{\Gamma(1/n)\}^2}{\Gamma(2/n)}.$$

Problem 11. *Show that* $\int_0^1 x^{m-1}\left(1-x^a\right)^n dx = \frac{1}{a}\,\frac{\lfloor n\,\Gamma(m/a)}{\Gamma\left(\frac{m}{a}+n+1\right)}$

L.H.S. $= \int_0^1 x^{m-1}\left(1-x^a\right)^n dx$ put $x^a = \sin^2\theta$, *i.e.* $x = \sin^{2/a}\theta$,

so that $dx = \frac{2}{a}\sin^{(2/a)-1}\theta\ \cos\theta\ d\theta$

$$= \frac{2}{a}\int_0^{\pi/2} \sin^{2(m-1)/a}\theta.\cos^{2n}\theta.\ \sin^{(2/a)-1}\theta\ \cos\theta\ d\theta$$

$$= \frac{2}{a}\int_0^{\pi/2} \sin^{(2m/a)-1}\theta.\cos^{(2n+1)}\theta\ d\theta$$

$$= \frac{2}{a}\cdot\frac{\Gamma(m/a)\,\Gamma(n+1)}{2\Gamma\left(\frac{m}{a}+n+1\right)}$$

$$= \frac{2}{a}\cdot\frac{\Gamma(m/a)\lfloor n}{2\Gamma\left(\frac{m}{a}+n+1\right)} = \frac{1}{a}\cdot\frac{\lfloor n\,\Gamma(m/a)}{\Gamma\left(\frac{m}{a}+n+1\right)}$$

Problem 12. *The equation of motion of a particle moving from rest, towards a centre of attraction point, situated at distance 'a' cms. apart from it, is given by*

$$\frac{d^2x}{dt^2}+\frac{k}{x}=0,\quad \textit{where } k \textit{ is a constant.}$$

Applying your knowledge of gamma functions, show that it will reach the centre of attraction point in a time given by

$$T = a\sqrt{\left(\frac{\pi}{2k}\right)}$$ (*Nagpur, 1965*)

The given equation of motion is

$$\frac{d^2x}{dt^2}+\frac{k}{x}=0 \quad ,$$

or
$$\frac{d^2x}{dt^2} = -\frac{k}{x}.$$

Multiplying both sides by $2\frac{dx}{dt}dt$, this equation may be written as

$$2\frac{d^2x}{dt^2}\frac{dx}{dt}dt = -\frac{2k}{x}dx.$$

Integrating, $\left(\dfrac{dx}{dt}\right)^2 = -2k\log x + A$, A being a constant.

Initially when $x = a$, $\dfrac{dx}{dt} = 0$; $\therefore A = 2k\log a$

We thus have

$$\left(\frac{dx}{dt}\right)^2 = -2k\log x + 2k\log a = 2k\log\frac{a}{x}$$

or
$$\frac{dx}{dt} = \sqrt{(2k)}\sqrt{\left(\log\frac{a}{x}\right)},$$

so that
$$\int_0^T dt = \frac{1}{\sqrt{(2k)}}\int_0^a\left(\log\frac{a}{x}\right)^{-1/2}dx$$

or
$$T = \frac{a}{\sqrt{(2k)}}\int_0^a\left(\log\frac{a}{x}\right)^{-1/2}dx$$

put $\log\dfrac{a}{x} = p$ or $x = ae^{-p}$

$\therefore dx = -ae^{-p}\,dp$

when $x = 0$, $p = \infty$

and when $x = a$, $p = 0$

$$= -\frac{1}{\sqrt{(2k)}}\int_\infty^0 (p)^{-1/2}e^{-p}dp$$

$$= \frac{a}{\sqrt{(2k)}}\int_0^\infty p^{(1/2)-1}e^{-p}dp$$

$$= \frac{a}{\sqrt{(2k)}}\Gamma\left(\frac{1}{2}\right) \text{ by the definition of Gamma function}$$

$$= \frac{a}{\sqrt{(2k)}}\sqrt{\pi}$$

$$= a\sqrt{\left(\frac{\pi}{2k}\right)}$$

6.10. ERROR FUNCTION OR PROBABILITY INTEGRAL

The ***error function*** denoted by $erf(x)$ is defined as

$$Erf(x) \text{ or } erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \qquad ...(1)$$

which arises in the solution of certain partial differential equations.

The definition (1) follows the properties of error function as

$$erf(-x) = -erf(x)' \qquad ...(2)$$

$$erf(0) = 0 \qquad ...(3)$$

$$erf(\infty) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}dt = \frac{2}{\sqrt{\pi}}.\frac{\sqrt{\pi}}{2} \; by \S\ 6.3.\ (2)$$
$$= 1 \qquad ...(4)$$

and
$$erf(ix) = \frac{2i}{\sqrt{\pi}}\int_0^x e^{t^2}dt \text{ when } i = \sqrt{-1} \qquad ...(5)$$

The **complementary error function** denoted by *erfc* (*x*) is defined as

$$erfc(x) \text{ or } Erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \qquad ...(6)$$

whereas the **General Error function** $E_n(x)$ is defined as

$$E_n(x) = \frac{1}{\Gamma\left(\frac{n+1}{n}\right)} \int_0^\infty e^{-t^n} dt \qquad ...(7)$$

On adding (1) and (6), we get

$$erf\,(x) + erfc\,(x) = \frac{2}{\sqrt{\pi}} \left[\int_0^x e^{-t^2} dt + \int_x^\infty e^{-t^2} dt \right]$$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2}\, dt = \frac{2}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} \text{ by 6.3 (2)}$$

$$= 1 \qquad ...(8)$$

which is the resul (2) defined as $erf(\infty) = 1$

$$(8) \Rightarrow erf(x) = 1 - erfc(x) \text{ or } erfc\,(x) = 1 - erf(x) \qquad ...(9)$$

For small values of *x*, the integrand in the right of (1), can be expanded as

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x \left[1 - t^2 + \frac{t^4}{\lfloor 2} - ... + \frac{(-1)^n t^{2n}}{\lfloor n} + ... \right] dt$$

$$= \frac{2}{\sqrt{\pi}} \left[x - \frac{x^3}{3 \lfloor 1} + \frac{x^5}{5 \lfloor 2} - ... + \frac{(-1)^n x^{2n+1}}{(2n+1) \lfloor n} + ... \right] \qquad ...(10)$$

the bracketed power series on the right being term by term integrable.

Now, $$\int_x^\infty e^{-t^2} dt = -\frac{1}{2} \int_x^\infty \frac{1}{t} d(e^{-t^2}) dt$$

$$= -\frac{1}{2} \left[\left\{ \frac{1}{t} e^{-t2} \right\}_x^\infty - \int_x^\infty \left(-\frac{1}{t^2} \right) e^{-t^2} dt \right], \text{on integrating by parts}$$

$$= \frac{e^{-x^2}}{2x} - \frac{1}{2} \int_x^\infty \frac{1}{t^2} e^{-t^2} dt$$

$$= \frac{e^{-x^2}}{2x} - \frac{1}{2} \int_x^\infty - \frac{1}{t^2} \frac{d(e^{-t^2})}{-2t} dt$$

$$= \frac{e^{-x^2}}{2x} + \frac{1}{2^2} \int_x^\infty \frac{1}{t^3} d(e^{-t^2}) dt$$

$$= \frac{e^{-x^2}}{2x} + \frac{1}{2^2} \left[\left\{ \frac{1}{t^3} e^{-t^2} \right\}_x^\infty \left(-\frac{3}{t^4} \right) e^{-t^2} dt \right] \text{by parts}$$

$$= \frac{e^{-x^2}}{2x} - \frac{e^{-x^2}}{2^2 x^3} + \frac{1.3}{2^2} \int_x^\infty \frac{e^{-t^2}}{t^4} dt$$

$$= \frac{e^{-x^2}}{2x}\left[1-\frac{1}{2x^2}+\frac{1.3}{2^2x^4}\ldots+\frac{(-1)^n 1.3\ldots(2n-1)}{2^n x^{2n}}\right]$$

$$+\frac{(-1)^{n+1}1.3\ldots(2n+1)}{2^{n+1}}\int_x^{\infty}\frac{e^{-t^2}}{t^{2n+2}}dt \qquad \ldots(11)$$

Thus,

$$(9) \Rightarrow erf(x) = 1 - erf(x) = 1-\frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-t^2}dt \text{ by}(6)$$

or

$$erf(x) \approx 1-\frac{e^{-x^2}}{x\sqrt{\pi}}\left[1+\sum_{n=1}^{\infty}(-1)^n\frac{1.3.5\ldots(2n-1)}{(2x^2)^n}\right] \qquad \ldots(12)$$

which renders asymptotic expansion of error function.
For large values of x, the relation (12) reduoes to

$$erf(x) = 1-\frac{e^{x^2}}{x\sqrt{\pi}} \qquad \ldots(13)$$

$$\text{If } x \to 0, (1) \Rightarrow erf(x) = \frac{2}{\sqrt{\pi}}\int_0^0 e^{-t^2}dt = 0 \qquad \ldots(14)$$

$$\text{If } x \to \infty, (1) \Rightarrow erf(\infty) = \frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-t^2}dt = \frac{2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} = 1 \text{ which is}(4)$$

Also $erf(-x) = \frac{2}{\sqrt{\pi}}\int_0^{-x} e^{-t^2}dt = \frac{2}{\sqrt{\pi}} = \int_0^x e^{-t^2}(-dt)$ on replacing t by $-t$

or

$$erf(-x) = -\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt = -erf(x) \qquad \ldots(15)$$

Again, $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \le \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-t^2}dt \text{ as } x \to \infty$

$$\le \frac{2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} = 1 \text{ i.e. } erf(x) \le 1 \Rightarrow |erf(-x)| \le 1$$

Similarly for $n < 0$, $erf(-x) \le 1$
Thus $|\, erf(-x)\,| \le 1$ for every value of n. ...(16)

6.11. FACTORIAL FUNCTION

We are familiar with the factorial notation for positive integral n,

$$\lfloor n \text{ or } n! = n(n-1)\ldots 3.2.1 \qquad \ldots(1)$$

Pochhammer introduced the *factorial function* for $n \ge 1$ as

$$(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1) \qquad \ldots(2)$$

which is expressed in a compact form as $(\alpha)_n = \prod_{j=1}^{n}(\alpha+j-1)$ ...(3)

Obviously, for $\alpha \ne 0, (\alpha)_0 = 1$ and $(1)_n = 1(1+1)\ldots(1+n-1) = 1.2\ldots n = \lfloor n$

...(4)

Using the Gamma function i.e. $\Gamma(n+1) = n\Gamma(n)$, we have

$$\begin{aligned}\Gamma(\alpha+n) &= (\alpha+n-1)\Gamma(\alpha+n-1)\\ &= (\alpha+n-1)(\alpha+n-2)\Gamma(\alpha+n-2)\\ &= (\alpha+n-1)(\alpha+n-2)(\alpha+n-3)\Gamma(\alpha+n-3)\\ &= \ldots\ldots\\ &= (\alpha+n-1)(\alpha+n-2)(\alpha+n-3)\ldots(\alpha+1)\alpha\Gamma(\alpha)\\ &= \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-2)(\alpha+n-1)\Gamma(\alpha)\\ &= (\alpha)_n\Gamma(\alpha) \Rightarrow (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\end{aligned} \quad \ldots(5)$$

If m be a positive integer and n is a non-negative integer then,

$$(\alpha)_{mn} = \alpha(\alpha+1)(\alpha+2)\ldots.(\alpha+mn-1) \text{ by } (2)$$

$$= (m)^{mn}\left[\left(\frac{\alpha}{m}\right)\left(\frac{\alpha+1}{m}\right)\ldots\left(\frac{\alpha+mn-1}{m}\right)\right]\times\left[\left(\frac{\alpha}{m}+1\right)\left(\frac{\alpha+1}{m}+1\right)\ldots\left(\frac{\alpha+m-1}{m}+1\right)\right]\times$$

$$\left[\left(\frac{\alpha}{m}+n-1\right)\left(\frac{\alpha+1}{m}+n-1\right)\ldots\left(\frac{\alpha+m-1}{m}+n-1\right)\right]$$

$$= m^{mn}\left[\left(\frac{\alpha}{m}\right)\left(\frac{\alpha}{m}+1\right)\ldots\left(\frac{\alpha}{m}+n-1\right)\right]\times\left[\left(\frac{\alpha+1}{m}\right)\left(\frac{\alpha+1}{m}+1\right)\ldots\left(\frac{\alpha+1}{m}+n-1\right)\right]\times\ldots$$

$$\times\left[\left(\frac{\alpha+m-1}{m}\right)\left(\frac{\alpha+m-1}{m}+1\right)\ldots\left(\frac{\alpha+m-1}{m}+n-1\right)\right]$$

$$= m^{mn}\left(\frac{\alpha}{m}\right)\left(\frac{\alpha+1}{m}\right)_n\ldots\left(\frac{\alpha+m-1}{m}\right)_n \text{ by (2)} \quad \ldots(6)$$

For $m = 2$, (6) $\Rightarrow (\alpha)_{2n} = 2^{2n}\left(\frac{\alpha}{2}\right)_n\left(\frac{\alpha+1}{2}\right)_n$ ...(7)

Illustrations:

[1] For $0 \le m \le n, (\alpha)_{n-m} = \dfrac{(-1)^m(\alpha)_n}{(1-\alpha-n)_m}$

Here, $(\alpha)_{n-m} = \alpha(\alpha+1)\ldots(\alpha+n-m-1)$ by def.

$$= \frac{\alpha(\alpha+1)\ldots(\alpha+n-m-1)\times(\alpha+n-m)\ldots(\alpha+n-1)}{(\alpha+n-m)(\alpha+n-m+1)\ldots(\alpha+n-1)}$$

$$= \frac{(\alpha)_n}{(-1)^m(1-\alpha-n)(2-\alpha-n)\ldots(m-\alpha-n)}\times\frac{(-1)^m}{(-1)^m}$$

$$= \frac{(-1)^m(\alpha)_n}{(1-\alpha-n)_m} \quad \ldots(8)$$

For $\alpha = 1, (8) \Rightarrow (1)_{n-m} = \dfrac{(-1)^m(1)_n}{(1-1-n)_n} = \dfrac{(-1)^m \lfloor n}{(-n)_m}$ ...(9)

Also $(1)_{n-m} = 1(1+1)(1+2)\ldots(1+n-m-1) = \lfloor n-m$ ..(10)

$$\therefore \text{ (9) and (10)} \Rightarrow (1)_{n-m} = \lfloor (n-m) = \frac{(-1)^m \lfloor n}{(-n)_m} \qquad \text{...(11)}$$

[2] *If* α *is not an integer, then* $\dfrac{\Gamma(1-\alpha-n)}{\Gamma(1-\alpha)} = \dfrac{(-1)^n}{(\alpha)_n}$

Here,

$$\frac{\Gamma(1-\alpha-n)}{\Gamma(1-\alpha)} = \frac{\Gamma(1-\alpha-n-1)(1-\alpha-n-2)\ldots}{(1-\alpha-1)(1-\alpha-2)\ldots(1-\alpha-n)(1-\alpha-n-1)(1-\alpha-n-2)\ldots}$$

$$= \frac{(-1)^{\alpha+n}(\alpha+n)(\alpha+n+1)\ldots}{(-1)^{\alpha}\,\alpha(\alpha+1)\ldots(\alpha+n-1)(\alpha+n)\ldots(\alpha+n+1)}$$

$$= \frac{(-1)^n}{\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1)} = \frac{(-1)^n}{(\alpha)_n} \qquad \text{...(12)}$$

[3] $(\alpha+m)_n = \dfrac{(\alpha)_{n+m}}{(\alpha)_m}$

Here $(\alpha)_{n+m} = \alpha(\alpha+1)\ldots(\alpha+m-1)(\alpha+m)(\alpha+m+1)\ldots(\alpha+m+n-1)$

$$= (\alpha)_m(\alpha+m)_n \Rightarrow (\alpha+m)_n = \frac{(\alpha)_{n+m}}{(\alpha)_m} \qquad \text{...(13)}$$

Gauss Multiplication Theorem

We have by (6), for a positive integer q and a non-negative integer n,

$$(\alpha)_{qn} = q^{qn}\left(\frac{\alpha}{q}\right)_n\left(\frac{\alpha+1}{q}\right)_n\ldots\left(\frac{\alpha+q-1}{q}\right)_n$$

$$= q^{qn}\prod_{k=1}^{q}\left(\frac{\alpha+k-1}{q}\right)_n \qquad \text{...(14)}$$

Using (5), we get

$$\frac{\Gamma(\alpha+qn)}{\Gamma(\alpha)} = q^{qn}\prod_{k=1}^{q}\frac{\Gamma\left(\dfrac{\alpha+k-1}{q}+n\right)}{\Gamma\left(\dfrac{\alpha+k-1}{q}\right)}$$

For $\alpha = qm$, this yields,

$$\frac{\Gamma(qm)}{\prod_{k=1}^{q}\Gamma\left(m+\dfrac{k-1}{q}\right)} = \frac{\Gamma(qm+qn)}{q^{qn}\prod_{k=1}^{q}\Gamma\left(m+n+\dfrac{k-1}{q}\right)}$$

Now, proceeding to the limit as $n \to \infty$, the L.H.S. being independent of n, we find

$$\frac{\Gamma(qm)}{\prod_{k=1}^{q}\Gamma\left(m+\dfrac{k-1}{q}\right)} = \lim_{n\to\infty}\frac{\Gamma(qm+qn)}{q^{qn}\prod_{k=1}^{q}\Gamma\left(m+n+\dfrac{k-1}{q}\right)}$$

$$= \lim_{n\to\infty}\left[\frac{\Gamma(qm+qn)}{\lfloor\underline{(qn-1)}\,(qn)^{qm}}\;\frac{(qn)^{qm}\,\Gamma(qn-1)}{q^{qn}}\prod_{k=1}^{q}\frac{\lfloor\underline{(n-1)}\;n^{m+(k-1)/q}}{\Gamma\left(m+n+\dfrac{k-1}{q}\right)}\cdot\frac{1}{\lfloor\underline{(n-1)}\;n^{m+(k-1)/q}}\right]$$

$$= \lim_{n\to\infty}\frac{(qn)^{qm}\lfloor\underline{(qn-1)}}{q^{qn}}\prod_{k=1}^{q}\frac{1}{\lfloor\underline{(n-1)}\;n^{m+(k-1)/q}}$$

$$\left[\because\ \lim_{n\to\infty}\frac{\Gamma(qm+qn)}{\lfloor\underline{(qn-1)}\,(qn)^{qm}}\to 1 \text{ and } \lim_{n\to\infty}\frac{\lfloor\underline{(n-1)}\;n^{m+(k-1)/q}}{\Gamma\left(m+n+\dfrac{k-1}{q}\right)}\to 1\right]$$

$$= \lim_{n\to\infty}\frac{(qn)^{qm}\lfloor\underline{(qn-1)}}{q^{qn}\left\{\lfloor\underline{(n-1)}\right\}^{q}\cdot n^{qm+(q-1)/2}} \text{ as } \prod_{k=1}^{q}\frac{1}{n^{m+(k-1)/q}}=\frac{1}{n^{q\{m+(k-1)q/2\}}}$$

$$= \lim_{n\to\infty}\frac{(qn)^{qm}\lfloor\underline{(qn-1)}}{q^{qn}\left\{\lfloor\underline{(n-1)}\right\}^{q}\cdot n^{(q-1)/2}}$$

$= K$ (say), being independent of m ...(15)

For determining K, suppose that $m=\dfrac{1}{q}$ so that (15) gives

$$\frac{1}{qK}=\prod_{k=1}^{q}\Gamma\left(\frac{k}{q}\right)=\prod_{k=1}^{q-1}\Gamma\left(\frac{q-k}{q}\right)$$

$$=\left[\because\ \prod_{k=1}^{q}\Gamma\left(\frac{1}{q}+\frac{k-1}{q}\right)=\prod_{k=1}^{q}\Gamma\left(\frac{k}{q}\right)=\frac{1}{q}.\frac{2}{q}.\frac{3}{q}\ldots\frac{q-2}{q}.\frac{q-1}{q}.\frac{q}{q}=\frac{1}{q}.\frac{2}{q}\ldots\frac{q-1}{q}\right.$$

$$\left.=\prod_{k=1}^{q}\Gamma\left(\frac{k}{q}\right) \text{ or } \prod_{k=1}^{q-1}\Gamma\left(\frac{q-k}{q}\right)\right]$$

$$\therefore\ \frac{1}{qK}\cdot\frac{1}{qK}=\prod_{k=1}^{q-1}\Gamma\left(\frac{k}{q}\right)\prod_{k=1}^{q-1}\Gamma\left(\frac{q-k}{q}\right)$$

or $$\frac{1}{q^2K^2}=\prod_{k=1}^{q-1}\Gamma\left(\frac{k}{q}\right)\Gamma\left(1-\frac{k}{q}\right)=\prod_{k=1}^{q-1}\frac{\pi}{\sin\dfrac{\pi k}{q}} \text{ by 6.7(cor.)}$$

or $q^2K^2\pi^{q-1}=\displaystyle\prod_{k=1}^{q-1}\sin\frac{\pi k}{q}=\frac{q}{2^{q-1}}$ as shown below.

Note. Giving $K=\dfrac{1}{q^{1/2}(2\pi)^{(q-1)/2}}$ $\therefore$ (15) $\Rightarrow$ $$\frac{\Gamma(qm)}{q^{qm}\displaystyle\prod_{k=1}^{q}\Gamma\left(m+\frac{k-1}{q}\right)}=\frac{1}{q^{1/2}(2\pi)^{(q-1)/2}}$$

i.e. $$\Gamma(qm) = q^{qm-1/2}(2\pi)^{-(q-1)/2}\prod_{k=1}^{q}\Gamma\left(m+\frac{k-1}{q}\right) \quad ...(16)$$

which is known as **Gauss multiplication theorem.**

In (16), if we put $q = 2$, this yields to **Legendre's Duplication formula** as

$$\Gamma(2m) = 2^{2m-1}\pi^{-1/2}\Gamma(m)\Gamma\left(m+\frac{1}{2}\right) \quad ...(17)$$

Since $$\prod_{k=1}^{2}\Gamma\left(m+\frac{k-1}{2}\right) = \Gamma\, m\, \Gamma\left(m+\frac{1}{2}\right)$$

Note. *To show* $\prod_{k=1}^{q-1}\sin\frac{\pi k}{q} = \frac{q}{2^{q-1}}$.

If α be the q^{th} root of unity, then

$$\alpha = (1)^{1/q} = e^{2\pi i/q}$$

Also, for all x, we have

$$x^q - 1 = (x-1)\prod_{k=1}^{q-1}(x-\alpha^k)$$

which gives on differentiation, w.r.t., u

$$q.x^{q-1} = \prod_{k=1}^{q-1}(x-\alpha^k) + (x-1)P(x), \; P(x) \text{ being a polynomial in } x$$

For $x = 1$, this yields

$$q = \prod_{k=1}^{q-1}(1-\alpha^k) = \prod_{k=1}^{q-1}(1-e^{2\pi ik/q})$$

$$= \prod_{k=1}^{q-1}\frac{e^{-\pi ik/q} - e^{\pi ik/q}}{e^{-\pi ik/q}} = \prod_{k=1}^{q-1}(-1)e^{\pi ik/q}.2i\sin\frac{\pi k}{q}$$

$$= (-2i)^{q-1}\exp\left(i\pi[\{1+2+...+(q-1)\}/q]\right)\prod_{k=1}^{q-1}\sin\frac{\pi k}{q}$$

$$= (-2i)^{q-1}\exp\left\{\frac{i\pi}{2}(q-1)\right\}\prod_{k=1}^{q-1}\sin\frac{\pi k}{q}$$

$$= (-2i)^{q-1}(i)^{q-1}\prod_{k=1}^{q-1}\sin\frac{\pi k}{q}, \text{ as } \exp\left(\frac{i\pi}{2}\right) = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$$

$$= 2^{q-1}\prod_{k=1}^{q-1}\sin\frac{\pi k}{q}$$

giving $$\prod_{k=1}^{q-1}\sin\frac{\pi k}{q} = \frac{q}{2^{q-1}} \quad ...(18)$$

6.12. SOME ALLIED FUNCTIONS

[A] Whittakar Function. Whittakar defined a function $W_{K,m}(z)$ in the manner

$$W_{k,m}(z)=\frac{e^{-z/2}z^k}{\Gamma\left(\frac{1}{2}-k+m\right)}\int_0^\infty t^{-k-\frac{1}{2}+m}\cdot\left(1+\frac{t}{z}\right)^{-k-\frac{1}{2}+m}\cdot e^{-t}dt \quad ...(1)$$

for all values of k and m, and all values of z except negative values, whereas this integral suffices to define $W_{k,m}$ in the critical cases when $m+\frac{1}{2}-k$ is a positive integer. It is notable that various functions used in applied mathematics are expressible by means of Whittakar function *e.g.*, the error function,

$$erfc(x)=\frac{e^{-x^2/2}}{\sqrt{\pi x}}W_{-\frac{1}{4},\frac{1}{4}}(x^2) \quad ...(2)$$

since, (1) gives on taking $k=-\frac{1}{4}, m=\frac{1}{4}$ and $z=x^2$

$$W_{-\frac{1}{4},\frac{1}{4}}(x^2)=\frac{e^{-x^2/2}(x^2)^{-1/4}}{\Gamma\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{4}\right)}\int_0^\infty t^{\frac{1}{4}-\frac{1}{2}+\frac{1}{4}}\left(1+\frac{t}{x^2}\right)^{-\frac{1}{4}-\frac{1}{2}+\frac{1}{4}}\cdot e^{-t}dt$$

$$=e^{-x^2/2}\cdot x^{-1/2}\int_0^\infty\left(1+\frac{t}{x^2}\right)^{-1/2}e^{-t}\,dt$$

$$=2x^{3/2}e^{-x^2/2}\int_1^\infty e^{x^2(1-p^2)}\,dp$$

on putting $t=x^2(p^2-1)$ *i.e.* $dt=2px^2dp$

$$=2x^{1/2}e^{x^2/2}\int_x^\infty e^{-q^2}dq \text{ on putting } p=\frac{q}{x} \text{ i.e. } dp=\frac{dq}{x}$$

$$=2\sqrt{x}e^{x^2/2}\int_x^\infty e^{-t^2}dt \quad \text{on replacing by } t$$

$$=2\sqrt{x}e^{x^2/2}\frac{\sqrt{\pi}}{2}erfc(x) \text{ by (8) of § 6.10}$$

so that $erfc(x)=\dfrac{e^{-x^2/2}}{\sqrt{\pi x}}W_{-\frac{1}{4},\frac{1}{4}}(x^2)$

As another example, the Incomplete gamma function $\gamma(n,x)$ can be expressed as Whittakar function in the form

$$\gamma(n,x)=\Gamma(n)-e^{-x/2}x^{(n-1)/2}W_{\frac{1}{2}(n-1),\frac{n}{2}}(x) \quad ...(3)$$

Since, (1) gives on taking $k=\frac{1}{2}(n-1), m=\frac{n}{2}, z=x$,

$$W_{\frac{1}{2}(n-1),\frac{n}{2}}(x)=\frac{e^{-x/2}x^{(n-1)/2}}{\Gamma(1)}\int_0^\infty\left(1+\frac{t}{x}\right)^{n-1}e^{-t}dt$$

$$=e^{-x/2}x^{(n-1)/2}\int_x^\infty\left(\frac{p}{x}\right)^{n-1}e^{-(p-x)}dp \text{ on putting } t=p-x \text{ i.e. } dt=dp$$

$$= e^{x/2}x^{-(n-1)/2}\int_x^\infty p^{n-1}e^{-p}dp$$

$$= e^{x/2}x^{-(n-1)/2}\left[\int_0^\infty t^{n-1}e^{-t}dt - \int_0^x t^{n-1}e^{-t}dt\right]$$

$$= e^{x/2}x^{-(n-1)/2}\left[\Gamma(n) - \gamma(n,x)\right] \text{ by def}$$

$$\text{or } \gamma(n,x) = \Gamma(n) - e^{-x/2}x^{(n-1)/2}\,W_{\frac{1}{2}(n-1),\frac{n}{2}}(x)$$

[B] The Logarithmic Integral function

As discussed by Euler, Laguerre, Stieltjes etc., the Logaithmic Integral function is defined by the equation

$$l_i(x) = \int_0^x \frac{dt}{\log t} \text{ with} |\arg(-\log x)| < \pi \qquad ...(4)$$

[C] The Exponential Integral function. It is defined as

$$E_1(x) = \int_x^\infty \frac{e^{-t}}{t}dt, x > 0 \text{ and } E_i(x) = \int_{-\infty}^x \frac{e^t}{t}dt \qquad ...(5)$$

Obviously, $E_1(x) = -E_i(-x)$ ...(6)

[D] The sine and cosine integrals

The *sine integral* is defined as

$$S_i(x) = \int_0^x \frac{\sin t}{t}dt \text{ and } s_i(x) = \int_\infty^x \frac{\sin t}{t}dt \qquad ...(7)$$

The *Cosine integral* is defined as

$$C_i(x) = \int_\infty^x \frac{\cos t}{t}dt \qquad ...(8)$$

[E] The Fresnel's Integrals

The integrals arise in wave-motion problems and defined as

$$S(x) = \int_0^x \sin\frac{\pi}{2}t^2 dt \qquad ...(9)$$

$$C(x) = \int_0^x \cos\frac{\pi}{2}t^2 dt \qquad ...(10)$$

Evidently, $S(\infty) = C(\infty) = \frac{1}{2}$ ...(11)

and $C(x) + iS(x) = \frac{1+i}{2} erf\left\{\frac{\sqrt{\pi}}{2}(1-i)x\right\}$ ...(12)

[F] The Elliptic Integrals:

The Elliptic Integrals of the first, second and third kind are successively defined as

$$F(k,\phi) = \int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}, 0 < k < 1 \qquad ...(13)$$

$$E(k,\phi) = \int_0^\phi \sqrt{1-k^2\sin^2\theta}\,d\theta, 0 < k < 1 \qquad ...(14)$$

$$\Pi(k,\phi,a) = \int_0^\phi \frac{d\theta}{\sqrt{(1-k^2\sin^2\theta)(1+a^2\sin^2\theta)}}, 0 < k < 1, a \neq k \qquad ...(15)$$

[G] Complete Elliptic Integrals

The complete Elliptic integrals of the first, second and third kind are successively obtained by replacing ϕ by $\frac{\pi}{2}$ in (13), (14) and (15) respectively and these are

$$K(k)=\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \qquad ...(16)$$

$$E(k)=\int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}\,d\theta \qquad ...(17)$$

$$\Pi(k,a)=\int_0^{\pi/2}\frac{d\theta}{\sqrt{(1-k^2\sin^2\theta)(1+a^2\sin^2\theta)}} \qquad ...(18)$$

Some Illustrations:

(1) $E_t(x)=-E_i(x)$

We have $\quad E_1(x)=\int_x^{\infty}\frac{e^{-t}}{t}dt=\int_{-x}^{-\infty}\frac{e^p}{-p}(-dp)$ on putting $-t=p$

$$=\int_{-\infty}^{-x}\frac{e^p}{p}dp=E_i(-x)$$

(2) $l_i(x)=E_i(\log x)=-E_1(-\log x)$

We have

$$E_i(\log x)=\int_{-\infty}^{\log x}\frac{e^t}{t}dt$$

$$=\int_0^x\frac{p}{\log p}\cdot\frac{dp}{p}\text{ on putting } t=\log p \text{ i.e. } p=e^t \text{ so that } dt=\frac{dp}{p}$$

$$=\int_0^x\frac{dp}{\log p}=l_i(x)$$

and $\quad -E_1(-\log x)=-\int_{-\log x}^{\infty}\frac{e^{-t}}{t}dt$, put $-\log p=t$ i.e. $p=e^{-t}$ so that $dt=\frac{-dp}{p}$

$$=-\int_x^0\frac{p}{-\log p}\left(\frac{-dp}{p}\right)=\int_0^x\frac{dp}{\log p}=l_i(x)$$

(3) $\int_0^{\infty}\frac{dt}{\sqrt{1+t^4}}=K\left(\frac{1}{\sqrt{2}}\right)$

We have $\int_0^{\infty}\frac{dt}{\sqrt{1+t^4}}=\int_0^1\frac{dt}{\sqrt{1+t^4}}+\int_1^{\infty}\frac{dt}{\sqrt{1+t^4}}$. Replace t by $\frac{1}{t}$ in the second integral

$$=\int_0^1\frac{dt}{\sqrt{1+t^4}}+\int_0^1\frac{dt}{\sqrt{1+t^4}}$$

$$=2\int_0^1\frac{dt}{\sqrt{1+t^4}},\text{ put } t=\tan\frac{\theta}{2}$$

$$=\int_0^{\pi/2}\frac{d\theta}{\sqrt{\sin^4\frac{\theta}{2}+\cos^4\frac{\theta}{2}}}=\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}=K\left(\frac{1}{\sqrt{2}}\right)$$

ADDITIONAL MISCELLANEOUS PROBLEMS

Problem 13. *Prove the following* (*i*) $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$ (*Agra, 1970*)

(*ii*) $\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta=\frac{1}{2}\Gamma\frac{1}{4}\Gamma\frac{3}{4}$ and (*iii*) $\int_0^1\frac{35x^3}{32\sqrt{1-x}}\,dx=1$ (*Nagpur, 1965*)

Problem 14. *Express the product* 2.5...,8(3*n*–1) *in terms of the gamma function.* (*Agra, 1966*)

Using $\Gamma(n+1)=n\,\Gamma(n)$ we have

$$\Gamma\left(n+\frac{2}{3}\right)=\left(n-\frac{1}{3}\right)\Gamma\left(n-\frac{1}{3}\right)=\left(n-\frac{1}{3}\right)\Gamma\left(n-\frac{4}{3}\right)\Gamma\left(n-\frac{7}{3}\right)=...$$

$$=\left(n-\frac{1}{3}\right)\left(n-\frac{4}{3}\right)......\frac{8}{3}\cdot\frac{5}{3}\cdot\frac{2}{3}\Gamma\frac{2}{3}$$

Hence $2.5.8...(3n-1)=3^n\,\Gamma\left(n+\frac{2}{3}\right)\Big/\Gamma\frac{2}{3}$

Problem 15. (*a*) *Evaluate the following integrals*

(*i*) $\int_0^\infty\frac{\cos(xy)}{x^2+b^2}\,dx$. (*ii*) $\int_0^\infty e^{-by}\cos(xy)\,dy,\ b>0.$

(*b*) *From the above results show that*

$$\int_0^\infty\cos(xy)\,\cos(xy')\,dx=\frac{\pi}{2}\delta(y-y')$$

where $y>0$, $y'>0$ *and* $\delta(y-y')$ *being the Dirac delta function.* (*Agra, 1968*)

(*a*) (*i*) We have $\int_0^\infty 2ze^{-(b^2+x^2)z^2}\,dz=-\frac{1}{b^2+x^2}\left[e^{-(b^2+x^2)z^2}\right]_0^\infty=\frac{1}{b^2+x^2}$

Multiplying both sides by cos *xy* and integrating within limits from $x=0$ to ∞, we get

$$\int_0^\infty\frac{\cos xy}{b^2+x^2}\,dx=\int_0^\infty\int_0^\infty\cos xy\cdot 2ze^{-(b^2+x^2)z^2}\,dz\,dx=\int_0^\infty 2ze^{-b^2z^2}\left[\int_0^\infty e^{-x^2z^2}\cos xy\,dx\right]dz.$$

Here $I=\int_0^\infty e^{-x^2z^2}\cos xy\,dx$ gives $\frac{dI}{dy}=-\int_0^\infty e^{-x^2z^2}x\sin xy\,dx$

$$=\left[\frac{e^{-x^2z^2}\sin xy}{2z^2}\right]_0^\infty-\frac{y}{2z^2}\int_0^\infty e^{-x^2z^2}\cos xy\,dx,\ \text{(integrating by parts)}$$

$$=0-\frac{y}{2z^2}I$$

$$\therefore\ \frac{dI}{I}=-\int\frac{y}{2z^2}\,dy$$

Integrating $\log I=-\frac{y^2}{4z^2}+\log A$ *i.e.* $I=Ae^{-y^2/4z^2}$ whence A can be determined by putting $y=0$ so that $A=[I]_{y=0}=\left[\int_0^\infty e^{-x^2z^2}\cos xy\,dx\right]_{y=0}$

$$=\int_0^\infty e^{-x^2z^2}\,dx=\frac{\sqrt{\pi}}{2z}$$

As such $I=\int_0^\infty e^{-x^2z^2}\cos xy\,dx=\frac{\sqrt{\pi}}{2z}e^{-y^2/4z^2}$

Thus $\int_0^\infty \frac{\cos xy}{b^2+x^2}dx=\int_0^\infty 2\cdot ze^{-b^2z^2}\cdot\frac{\sqrt{\pi}}{2z}e^{-y^2/4z^2}dz=\sqrt{\pi}\int_0^\infty e^{\left(e^{-b^2z^2}+\frac{y^2}{4z^2}\right)}dz$

$$=\sqrt{\pi}\int_0^\infty e^{-b^2\left(z^2+\frac{y^2}{4b^2z^2}\right)}dz=\sqrt{\pi}\,J \text{ (say)}$$

Now consider $J=\int_0^\infty e^{-b^2\left(z^2+\frac{y^2}{4b^2z^2}\right)}dz$

so that $\frac{dJ}{dy}=-\frac{1}{2}\int_0^\infty e^{-b^2\left(z^2+\frac{y^2}{4b^2z^2}\right)}\frac{y}{z^2}dz\cdot$ Put $\frac{y}{2bz}=p$ so that $dp=-\frac{y}{2bz^2}dz$.

$$=+b\int_\infty^0 e^{-b^2\left(p^2+\frac{y^2}{4b^2p^2}\right)}dp=-b\int_0^\infty e^{-b^2\left(z^2+\frac{y^2}{4b^2z^2}\right)}dz.$$

(on replacing p by z)

$$=-bJ$$

$\therefore\ \frac{dJ}{J}=-b\,dy$ which gives on integration $J=Be^{-by}$ where B is determined by putting $y=0$,

so that $B=[J]_{y=0}=\int_0^\infty e^{-b^2z^2}dz=\frac{\sqrt{\pi}}{2b}$ and hence $J=\frac{\sqrt{\pi}}{2b}e^{-by}$

Thus $\frac{\cos xy}{b^2+x^2}dx=\sqrt{\pi}\,J=\sqrt{\pi}\cdot\frac{\sqrt{\pi}}{2b}e^{-by}=\frac{\pi}{2b}e^{-by}$

(*ii*) (say) $I=\int_0^\infty e^{-by}\cos xy\,dy=\left[-\frac{e^{-by}}{b}\cos xy\right]_0^\infty-\frac{x}{b}\int_0^\infty e^{-by}\sin xy\,dy$

(integrating by parts)

$$=\frac{1}{b}-\frac{x}{a}\left[-\frac{e^{-by}}{b}\sin xy\right]_0^\infty-\frac{x^2}{b^2}\int_0^\infty e^{-by}\cos xy\,dy=\frac{1}{b}-0-\frac{x^2}{b^2}I$$

so that $I\left[1+\frac{x^2}{b^2}\right]=\frac{1}{b}$ or $I=\int_0^\infty e^{-by}\cos xy\,dy=\frac{b}{x^2+b^2}$.

(*b*) This follows from the definition of Dirac delta function given in §8.10.

Problem 16. (*a*) *Evaluate the following integrals*

(*i*) $\int_{-\infty}^\infty e^{-\alpha|x|+ikx}dx,\ \alpha>0$ *and* $|x|$ *is the absolute value of* x

(*ii*) $\frac{1}{\beta}\int_0^\infty\left(x^\beta-1\right)e^{-\alpha x}dx$

(*b*) *In the second integral given above, take the limit* $\beta\to 0$ *and hence evaluate integral* $\int_0^\infty \log x.e^{-\alpha x}dx$

(*Use the approximations* $\alpha^\beta\cong 1+\beta\log\alpha$ *and* $\Gamma(1+\beta)\cong 1+\xi\beta,\ \xi=-0.577218$ *for small* β.

(*Agra, 1967*)

(*i*) Say $I = \int_{-\infty}^{\infty} e^{-\alpha|x|+ikx}\,dx = \int_{-\infty}^{\infty} e^{-\alpha|x|}(\cos kx + i\sin kx)\,dx$

$$= \int_{-\infty}^{\infty} e^{-\alpha|x|}\cos kx\,dx + i\int_{-\infty}^{\infty} e^{-\alpha|x|}\sin kx\,dx = I_1 + iI_2 \text{ (say)} \quad ...(1)$$

Now $|x|$ is absolutely positive if x is negatively large or positively large and α is always positive also the values of $\cos kx$ and $\sin kx$ are always finite lying between -1 and $+1$ and hence both the integrals I_1 and I_2 being convergent exist.

We have $I_2 = \int_{-\infty}^{0} e^{-\alpha|x|}\sin kx\,dx + \int_{0}^{\infty} e^{-\alpha|x|}\sin kx\,dx = I_{21} + I_{22}$ (say)

Replacing x by $-x$, $I_{21} = -\int_0^{\infty} e^{-\alpha|x|}\sin kx\,dx$.

So that $I_2 = I_{21} + I_{22} = -\int_0^{\infty} e^{-\alpha|x|}\sin kx\,dx + \int_0^{\infty} e^{-\alpha|x|}\sin kx\,dx = 0$

Hence $I = I_1 = \int_{-\infty}^{\infty} e^{-\alpha|x|}\cos kx\,dx = 2\int_0^{\infty} e^{-\alpha|x|}\cos kx\,dx$,

the integrand being even function of x

$$= 2\int_0^{\infty} e^{-\alpha x}\cos kx\,dx = \frac{2b}{x^2+b^2} \text{ by Problem 15 } (a-ii)$$

(*ii*) We have $\frac{1}{\beta}\int_0^{\infty}\left(x^{\beta}-1\right)e^{-\alpha x}dx = \frac{1}{\beta}\int_0^{\infty} e^{-\alpha x}x^{(\beta+1)-1}\,dx - \frac{1}{\beta}\int_0^{\infty} e^{-\alpha x}dx$

$$= \frac{1}{\beta}\left[\frac{\Gamma(\beta+1)}{\alpha^{\beta+1}} - \frac{1}{\alpha}\right] \quad \because \int_0^{\infty} e^{-\alpha x}x^{\beta-1}dx = \frac{\Gamma\beta}{\alpha\beta} \text{ and for } \beta = 1, \text{ this gives } \int_0^{\infty} e^{-\alpha x}dx = \frac{1}{\alpha}$$

$$= \frac{1}{\alpha}\left[\frac{\Gamma(\beta+1)}{\beta\alpha^{\beta}} - \frac{1}{\beta}\right]$$

(*b*) In (*ii*) we have shown that $\frac{1}{\beta}\int_0^{\infty}\left(x^{\beta}-1\right)e^{-\alpha x}dx = \frac{1}{\alpha}\left[\frac{\Gamma(\beta+1)}{\beta\alpha^{\beta}} - \frac{1}{\beta}\right]$

$$= \frac{1}{\alpha}\left[\frac{\Gamma(\beta+1)-\alpha^{\beta}}{\beta\alpha^{\beta}}\right]$$

when $\beta \to 0$, both sides of this relation are of the form $\frac{0}{0}$ and hence taking limit as $\beta \to 0$, we have

$$\frac{\int_0^{\infty} \lim_{\beta\to 0}\frac{d}{d\beta}\left(x^{\beta}-1\right)e^{-\alpha x}dx}{\lim_{\beta\to 0}\frac{d}{d\beta}(\beta)} = \lim_{\beta\to 0}\left[\frac{(1+\xi\beta)-(1+\beta\log\alpha)}{\beta(1+\beta\log\alpha)}\right]\cdot\frac{1}{\alpha},$$

or $\int_0^{\infty}\log x\,e^{-\alpha x}dx = \lim_{\beta\to 0}\frac{1}{\alpha}\left[\frac{\beta(\xi-\log\alpha)}{\beta(1+\beta\log\alpha)}\right] = \lim_{\beta\to 0}\left[\frac{\xi-\log\alpha}{1+\beta\log\alpha}\cdot\frac{1}{\alpha}\right]$

$$= \frac{\xi-\log\alpha}{\alpha}$$

Problem 17. *Show by integrating by parts that*

$$\int_0^{n} e^{-n^2}dn = \frac{\sqrt{\pi}}{2} - \frac{e^{-x^2}}{2x}\left(1 - \frac{1}{2x^2} + \frac{1\times 3}{2^2x^4}\right) + \frac{1\times 3\times 5}{2^3}\int_x^{\infty}\frac{e^{-n^2}}{n^6}dx \qquad \text{(Agra, 1971)}$$

Show how this expression may be used to compute the value of erf (x) for large values of x.

Problem 18. *Define the error function erf (x).* *(Agra, 1973)*

Problem 19. *Define Gamma and Beta functions.* (*Agra, 1974*)

Problem 20. *If* $\Gamma z = \int_0^\infty e^{-t} t^{z-1}\, dt$, *evaluate*

$$\int_0^{\pi/2} \cos^{m-1} x \sin^{n-1} x\, dx \text{ in terms of gamma function.}$$ (*Agra, 1975*)

Problem 21. *Define Beta and Gamma functions. Show that*

(*a*) $\Gamma(n+1) = n\Gamma n$ (*b*) $\beta(m,n) = \dfrac{\Gamma m \Gamma n}{\Gamma(m+n)}$

(*c*) $\Gamma\frac{1}{2} = \sqrt{\pi}$ (*d*) $\Gamma\frac{1}{4}\Gamma\frac{3}{4} = \pi 2^{1/2}$ (*Agra, 1976*)

Problem 22. (*a*) *Derive the recurrence relation* $\Gamma(z+1) = z\Gamma z$ *from Euler's integral*

$$\Gamma_z = \int_0^\infty e^{-t} t^{z-1}\, dt;\ R.P.\ (z) > 0$$

(*b*) *Show by means of beta function that*

$$\int_t^z \frac{dx}{(z-x)^{1-\alpha}(x-t)^\alpha} = \frac{\pi}{\sin \pi\alpha};\ 0 < \alpha < 1$$ (*Rohilkhand, 1976*)

(*a*) See. §6.2.

(*b*) Let $I = \int_t^z \frac{dx}{(z-x)^{1-\alpha}(x-t)^\alpha} = \int_t^z (x-t)^{-\alpha}\cdot(z-x)^{\alpha-1}\, dx$

Put $x = t\sin^2\theta + z\cos^2\theta$, so that $dx = -2(z-t)\sin\theta\cos\theta\, d\theta$

$x - t = (z-t)\cos^2\theta$ and $z - x = (z-t)\sin^2\theta$

Also when $x = z$, $\sin^2\theta = 0 \Rightarrow \theta = 0$ and $x = t$, $\cos^2\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$. With these substitutions we have

$$I = -\int_{\pi/2}^0 (z-t)^{-\alpha}\cos^{-2\alpha}\theta\cdot(z-t)^{\alpha-1}\sin^{2\alpha-2}\theta\cdot 2(z-t)\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2\alpha-1}\theta\cos^{1-2\alpha}\theta\, d\theta$$

$$= 2\cdot\frac{\Gamma\alpha\Gamma\{1-\alpha)\}}{2\Gamma\{\alpha+(1-\alpha)\}} \text{ by property of Gamma function}$$

$$= \frac{\Gamma\alpha\Gamma(1-\alpha)}{\Gamma\{\alpha+(1-\alpha)\}} = \beta(\alpha, 1-\alpha)\ \because\ \beta(m,n) = \frac{\Gamma m\Gamma n}{\Gamma(m+n)}$$

$$= \int_0^\infty \frac{y^{\alpha-1}}{1+y}\, dy \text{ by (2) of §6.5.}$$

But we have $\int_0^\infty \frac{y^{2m}}{1+y^{2n}} = \frac{\pi}{2n}\operatorname{cosec}\frac{2m+1}{2n}\pi$ by §6.8.

On putting $n = \frac{1}{2}$, $m = \frac{\alpha-1}{2}$, this yields

$$I = \int_0^\infty \frac{y^{\alpha-1}}{1+y}\, dy = \pi\operatorname{cosec}\pi\alpha = \frac{\pi}{\sin\pi\alpha} \text{ when } 0 < \alpha < 1.$$

Allter. We have, by definition of Beta function,

$$\beta(m,n) = \int_0^1 y^{m-1}(1-y)^{n-1}\, dy$$

Put $y = \frac{x-t}{z-t}$ so that $dy = \frac{dx}{z-t}$

$$\therefore \quad \beta(m,n) = \int_t^z \frac{(x-t)^{m-1}}{(z-t)^{m-1}} \cdot \frac{(z-x)^{n-1}}{(z-t)^{n-1}} \cdot \frac{dx}{z-t}$$

$$= \frac{1}{(z-t)^{m+n-1}} \int_t^z (x-t)^{m-1} (z-x)^{n-1} \, dx$$

or $\int_t^z (z-x)^{n-1} (x-t)^{m-1} \, dx = (z-t)^{m+n-1} \beta(m,n)$

If we put $n = \alpha$, $m = 1 - \alpha$, this yields

$$\int_t^z \frac{dx}{(z-x)^{1-\alpha} (x-t)^{\alpha}} = \beta(\alpha, 1-\alpha)$$

$$= \frac{\Gamma\alpha\Gamma(1-\alpha)}{\Gamma(\alpha+1-\alpha)} = \Gamma\alpha \; \Gamma(1-\alpha).$$

$$= \frac{\pi}{\sin \pi\alpha} \qquad \because \; \Gamma n \Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

+ β

CHAPTER 7

DIFFERENTIAL EQUATIONS

(ORDINARY AND PARTIAL WITH FINITE DIFFERENCE *i.e.* INTERPOLATION)

7.1. INTRODUCTION

In mathematics we call the changing entities as *variables* and the rate of change of one variable with respect to another as a *derivative*. Equations expressing a relationship among these variables and their derivatives are known as *Differential equations*. In other words differential equations originate wherever a universal law is expressed by means of variables and their derivatives.

A relationship between two variables x and y is expressed as $f(x, y) = 0$ and read as "f of x, y equals zero" or as "a function of x, y equals zero". A function difining y as a function of x in the form $f(x, y) = 0$ is said to be an *implicit function* of x. A special class of constant functions is referred as *elementary functions*. An equation involving x, the function $f(x)$ which defines a function of x and one or more of its derivatives is called an *ordinary differential equation*. In other words an ordinary differential equation involves only one independent variable say x such as $\frac{dy}{dx} = \frac{1}{x}$ while a *partial differential equation* is one which involves two or more independent variables say x *and* y such as $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = x^2$. The *order* of a differential equation is the order of the highest derivative involved in that equation *e.g.* $\frac{d^3y}{dx^3} = \frac{2}{x^3}$ is of order 3.

If $y = f(x)$ defines y as a function of x on any interval, then $f(x)$ is said to be an *explicit solution* or simply a *solution* of an ordinary differential equation involving $x, f(x)$ and its derivatives. Whenever it can be shown that an implicit function does satisfy a given differential equation on any intérval then $f(x, y) = 0$ is said to be an *implicit solution* of the differential equation.

The solution of a differential equation of order n contains n arbitrary constants say c_1, $c_2, \ldots c_n$ and called an *n-parameter family of solutions*, referring to the constants as parameters.

An n-parameter family of solutions of an nth order differential equation is called its *general solution* and the function obtained by giving a definite set of values to the constants $c_1, c_2, \ldots c_n$ in the family is said to be the *Particular Integral* of the differential equation. There are differential equations which have solutions not obtainable from the n-parameter family whatsoever values are given to the constants, such solutions are called *Singular solutions*.

Then n conditions which enable us to find the values of n arbitrary constants in an n-parameter family if given in terms of one value of the independent variable, are called *Initial conditions* or *boundary conditions*.

The reader of this book is presumed to have passed through the degree level where the methods of solving ordinary differential equations of first order are discussed in details. In the next section we discuss some methods of solving ordinary differential equations of second order which are of physical interest.

7.2. SOLUTION IN SIMPLE CASES OF ORDINARY DIFFERENTIAL EQUATIONS OF SECOND ORDER

Generally an ordinary equation of second order is of the from

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X$$

where P, Q, X are functions of x only.

I. Linear Equations with Constant Coefficients.

The general solution is found by usual methods of finding the complementary function and particular integral of the equation. Although the student is presumed to have a sound knowledge of the methods to be employed for finding the complementary function and particular integral, but still then, we summarize them as below:

Let there be a differential equation of the type

$$\frac{d^2y}{dx^2} + p_1\frac{dy}{dx} + p_2 y = X$$

where p_1, p_2 are constants and X is a function of x.

In terms of the D operator, it may be written

$$\left(D^2 + p_1 D + p_2\right)y = X \text{ where } D \text{ stands for } \frac{d}{dx} \text{ i.e., } D \equiv \frac{d}{dx}$$

or we may write thus, $f(D)\,y = X$.

To find the complementary function (C.F). The X is removed and replaced by zero. Then an auxiliary equation is written either by putting $y = e^{mx}$ whence, we get

$$m^2 + p_1 m + p_2 = 0,$$

or simply writing $D^2 + p_1 D + p_2 = 0$ *i.e.*, $f(D) = 0$.

In either case, we get the roots of the quadratic.

Case I. If the roots are of the type m_1 and m_2 (real and distinct) C.F. is

$$C_1 e^{m_1 x} + C_2 e^{m_2 x}.$$

Case II. If $m_1 = m_2$ *i.e.*, both the roots are real and equal, C.F. is

$$(C_1 + C_2 x)e^{m_1 x}.$$

Case III. If the roots are imaginary *i.e.*, of the type $\alpha \pm i\beta$, C.F. is

$$e^{\alpha x}[C_1 \cos \beta x + C_2 \sin \beta x] \quad \text{or} \quad C_1 e^{\alpha x} \cos(\beta x + C_2).$$

Case IV. If the roots are of the type $\alpha \pm \sqrt{\beta}$, C.F. is

$$C_1 e^{\alpha x} \cosh\left(\sqrt{\beta}\, x + C_2\right).$$

Note. *The number of the arbitrary constants will be the same as the order of the equation.*

To find the Particular Integral (P.I.)

We have $\text{P.I.} = \dfrac{X}{f(D)}$ which for $f(D) = D - \alpha \Rightarrow \dfrac{X}{D-\alpha} = e^{\alpha x}\int Xe^{-\alpha x}\,dx.$

Case I. If $X = e^{ax}$ where a is any constant.

$$\text{P.I.} = \frac{e^{ax}}{f(D)} = \frac{e^{ax}}{f(a)} \text{ if } f(a) \neq 0.$$

Case II. If $X = x^m$, where m is a positive integer ·

$$\text{P.I.} = \frac{x^m}{f(D)} = [f(D)]^{-1}x^m.$$

Expand $[f(D)]^{-1}$ binomially upto mth power of D and then operate x^m on every term.

Case III. If $X = \sin ax$ or $\cos ax$.

$$\text{P.I.} = \frac{\sin ax \text{ or } \cos ax}{f(D^2)} = \frac{\sin ax \text{ or } \cos ax}{f(-a^2)} \text{ provided } f(-a^2) \neq 0.$$

In case $f(-a^2) = 0$, $\dfrac{\sin ax}{f(D^2)}$ = Imaginary part of $\dfrac{e^{iax}}{f(D^2)}$

and $\dfrac{\cos ax}{f(D^2)}$ = Real part of $\dfrac{e^{iax}}{f(D^2)}$, which is case I.

Case IV. If $X = e^{ax}V$, where V is any function of x, then

$$\text{P.I.} = \frac{e^{ax}V}{f(D)} = e^{ax}\cdot\frac{1}{f(D+a)}\cdot V\cdot$$

Case V. If $X = x\cdot V$, where V is any function of x, then

$$\text{P.I.} = \frac{xV}{f(D)} = x\frac{1}{f(D)}\cdot V - \frac{1}{f(D)}f'(D)\frac{1}{f(D)}V$$

$$= \left\{x - \frac{1}{f(D)}f'(D)\right\}\frac{1}{f(D)}V$$

Hence *general solution* = C.F. + P.I.

Problem 1. *Solve* $\dfrac{d^2y}{dx^2} - y = x\ \sin x + (1+x^2)\ e^x$

or $$(D^2 - 1)\,y = x\sin x + (1 + x^2)\,e^x$$

Now its complementary function and particular integral may be found thus :

For complementary function, the auxiliary equation is

$$D^2 - 1 = 0 \text{ or } D = \pm 1.$$

$\therefore$ C.F. is $C_1e^x + C_2e^{-x}$.

Particular integral $= \dfrac{x\sin x}{D^2-1} + \dfrac{(1+x^2)\,e^x}{D^2-1}$

$= \text{imaginary part in } \dfrac{xe^{ix}}{D^2-1} + e^x\dfrac{1}{(D+1)^2-1}(1+x^2)$

$= \text{imaginary part in } e^{ix}\cdot\dfrac{1}{(D+i)^2-1}\cdot x + e^x\cdot\dfrac{1}{D^2+2D}\ (1+x^2)$

$= \text{imaginary part in } e^{ix} \dfrac{1}{D^2 + 2iD - 2} x + \dfrac{e^x \cdot 1}{2D\left(1 + \dfrac{D}{2}\right)} \left(1 + x^2\right)$

$= \text{imaginary part in } \dfrac{e^{ix}}{-2}\left[1 - \dfrac{2iD + D^2}{2}\right]^{-1} x + \dfrac{e^x}{2} \cdot D^{-1} \left\{1 + \dfrac{D}{2}\right\}^{-1} \cdot \left(1 + x^2\right)$

$= \text{imaginary part in } \dfrac{e^{ix}}{-2}\left[1 + \dfrac{2iD + D^2}{2} + \ldots\right] x + \dfrac{e^x}{2} D^{-1}\left[1 - \dfrac{D}{2} + \dfrac{D^2}{4} - \ldots\right]\left(1 + x^2\right)$

$= \text{imaginary part in } \dfrac{e^{ix}}{-2}[x + i] + \dfrac{e^x}{2} D^{-1} \left[1 + x^2 - x + \tfrac{1}{2}\right]$

$= \text{imaginary part in } \left(\dfrac{\cos x + i \sin x}{-2}\right) (x + i) + \dfrac{e^x}{2} \int \left(x^2 - x + \tfrac{3}{2}\right) dx$

$= -\tfrac{1}{2} (x \sin x + \cos x) + \dfrac{e^x}{2}\left[\dfrac{x^3}{3} - \dfrac{x^2}{2} + \dfrac{3}{2} x\right]$

Hence the general solution is

$$y = C_1 e^x + C_2 e^{-x} - \tfrac{1}{2} (x \sin x + \cos x) + \frac{xe^x}{12} (2x^2 - 3x + 9).$$

II. Linear Equations with Variable Coefficients (Homogeneous Linear Equations).

Consider, $p_0 x^2 \dfrac{d^2y}{dx^2} + p_1 x \dfrac{dy}{dx} + p_2 y = X.$

Put $x = e^z$, *i.e.*, $z = \log x$; $\therefore \dfrac{dz}{dx} = \dfrac{1}{x}$.

Then
$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz},$$
$$\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d}{dz}\left(\frac{1}{x}\frac{dy}{dz}\right) = \frac{1}{x^2}\frac{d^2y}{dz^2} - \frac{1}{x^3}\frac{dx}{dz}\frac{dy}{dz}$$
$$= \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right).$$

If we put $\dfrac{d}{dz} \equiv D$, we have

$$x\frac{dy}{dx} = Dy, \; x^2\frac{d^2y}{dx^2} = D\,(D-1)\,y, \ldots, x^n\frac{d^ny}{dx^n} = D(D-1)\ldots(D-n+1)$$

Now substituting these values in (1), we get

$$p_0 D\,(D - 1)\, y + p_1 Dy + p_2 y = X,$$

which may be solved by the method employed in I.

Problem 2. *Solve* $x^2 \dfrac{d^2y}{dx^2} - x \dfrac{dy}{dx} + y = 2 \log x.$

Put $x = e^z$ and denote $\dfrac{d}{dz}$ by D; we have

$$D(D-1)\,y - Dy + y = 2z \text{ or } (D^2 - 2D + 1)\,y = 2z.$$

Auxiliary equation is

$$D^2 - 2D + 1 = 0$$

or $$(D-1)^2 = 0; \quad \therefore\ D = 1 \text{(repeated twice)}.$$

$$\text{C.F.} = (c_1 + c_2 z)\, e^z.$$

$$\text{P.I.} = \frac{2z}{D^2 - 2D + 1} = 2\left[1 - (2D - D^2)\right]^{-1} \cdot z$$

$$= 2(1 + 2D \ldots)\, z = 2\,(z + 2) = 2z + 4.$$

$\therefore$ General solution is $y = (c_1 + c_2 z)e^z + 2z + 4$

$$= (c_1 + c_2 \log x)\, x + 2 \log x + 4.$$

Note. *Any equation of the type*

$$(a+bx)^n \frac{d^n y}{dx^n} + P_1\,(a+bx)^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \ldots + P_{n-1}\,(a+bx)\frac{dy}{dx} + P_n y = F(x)$$

can be reduced to the homogeneous linear form by putting $z = ax + b$ or this can be solved by putting $ax + b = e^z$ as above.

III. Exact Differential Equations and Equations of other Special types.

The equations of the type

$$P_0 \frac{d^2 y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = 0,$$

where P_0, P_1 and P_2 are the functions of x, is said to be exact if

$$P_2 - P_1' + P_0'' = 0$$

or in general an equation of order n (say),

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + P_n y = Q(x)$$

is exact if $\quad P_n - P'_{n-1} + \ldots + (-1)^n P_0^{(n)} = 0,$

where $P', P'', \ldots P^{(n)}$ are the successive derivatives of P.

In case the equation is exact, its first integral is

$$P_0 y_{n-1} + (P_1 - P'_0)\, y_{n-2} + (P_2 - P'_1 + P_0'')\, y_{n-2} + \ldots$$

$$\ldots + \{P_{n-1} - P'_{n-2} + \ldots + (-1)^n P_0^{n-1}\}\, y = \int Q(x) + C,$$

where y_n stands for $\frac{d^n y}{dx^n}$ etc.

Problem 3. *Solve* $\quad \sin^2 x \frac{d^2 y}{dx^2} = 2y$

or $$\frac{d^2 y}{dx^2} - 2y \operatorname{cosec}^2 x = 0$$

or $$\cot x \frac{d^2 y}{dx^2} - 2y \cot x \operatorname{cosec}^2 x = 0,$$

which is exact and hence its first integral is

$$\cot x \frac{dy}{dx} + \operatorname{cosec}^2 x = c_1$$

or
$$\frac{dy}{dx} + \frac{1}{\sin x \cos x} y = c_1 \tan x,$$

which is a linear differential equation of the first order.

$$\text{Integrating factor} = e^{\int \frac{dx}{\sin x \cos x}} = e^{\int 2 \operatorname{cosec} 2x \, dx}$$
$$= e^{\log \tan x} = \tan x.$$

$$\therefore \text{ The solution is } y \tan x = \int c_1 \tan^2 x \, dx + c_2$$
$$= c_1 \int \left(\sec^2 x - 1\right) dx + c_2$$
$$= c_1 (\tan x - x) + c_2.$$

Note 1. *Sometimes the equation becomes exact by multiplying an integrating factor x^m, where m can be found by applying the condition of exactness.*

Note 2. *Equations of the form* $\frac{d^2y}{dx^2} = f(y)$ *can be integrated on multiplying by* $2\frac{dy}{dx}$, *whence we get*

$$\left(\frac{dy}{dx}\right)^2 = 2\int f(y)\, dy + c_1$$

or
$$\frac{dy}{dx} = \sqrt{\left\{c_1 + 2\int f(y)\, dy\right\}}$$

which may further be integrated by any of the standard methods.

Note 3. *Equations not containing x directly can be integrated by putting* $\frac{dy}{dx} = p$.

Problem 4. $\frac{d^2y}{dx^2} + \frac{dy}{dx} + \left(\frac{dy}{dx}\right)^3 = 0.$

Put $\frac{dy}{dx} = p; \quad \therefore \quad \frac{d^2y}{dx^2} = p\frac{dp}{dy}.$

So we get $p\frac{dp}{dy} + p + p^3 = 0.$

or
$$\frac{dp}{1+p^2} = -dy.$$

Integrating $\tan^{-1} p = c - y$, *i.e.*, $\frac{dy}{dx} = p = \tan(c - y)$, which can further be integrated by standard methods.

Similarly equations not containing y can be integrated by putting

$$\frac{dy}{dx} = p \left(\because \frac{d^2y}{dx^2} = \frac{dp}{dx} = p\frac{dp}{dy} \text{ etc}\right).$$

Note 4. *Equations in which y appears in only two derivatives whose orders differ by unity can be integrated as follows :*

Problem 5. $a\dfrac{d^2y}{dx^2}=\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{1/2}$

Put $\dfrac{dy}{dx}=q$; $\therefore$ $\dfrac{d^2y}{dx^2}=\dfrac{dq}{dx}$. We have

$$a\frac{dq}{dx}=\left(1+q^2\right)^{1/2}$$

or
$$\frac{dq}{\sqrt{(1+q^2)}}=\frac{dx}{a}.$$

Integrating, $\quad \sinh^{-1} q=\dfrac{x}{a}+c_1.$

$$\therefore \quad q=\frac{dy}{dx}=\sinh\left(\frac{x}{a}+c_1\right).$$

Integrating again, $y=a\cosh\left(\dfrac{x}{a}+c_1\right)+c_2.$

IV. The Complete Solutions in Terms of a Known Integral.
Let $y=y_1$ be a known integral in the complementary function of

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=X. \qquad \ldots(1)$$

Put $y=vy_1$; so that $\dfrac{dy}{dx}=y_1\dfrac{dv}{dx}+v\dfrac{dy_1}{dx}$

and
$$\frac{d^2y}{dx^2}=y_1\frac{d^2v}{dx^2}+2\frac{dv}{dx}\cdot\frac{dy_1}{dx}+v\frac{d^2y_1}{dx^2}.$$

With these substitutions, (1) gives

$$y_1\left(\frac{d^2v}{dx^2}\right)+\frac{dv}{dx}\left(2\frac{dy_1}{dx}+Py_1\right)+v\left(\frac{d^2y_1}{dx^2}+P\frac{dy_1}{dx}+Qy_1\right)=X$$

or
$$\frac{d^2v}{dx^2}+\left(P+\frac{2}{y_1}\frac{dy_1}{dx}\right)\frac{dv}{dx}=\frac{X}{y_1}, \qquad \ldots(2)$$

since
$$\frac{d^2y_1}{dx^2}+P\frac{dy_1}{dx}+Qy_1=0 \text{ by hypothesis.}$$

Putting $\dfrac{dv}{dx}=p$, (2) becomes

$$\frac{dp}{dx}+\left(P+\frac{2}{y_1}\frac{dy_1}{dx}\right)p=\frac{X}{y_1}, \qquad \ldots(3)$$

which is a linear equation of first order and hence its integrating factor

$$=e^{\int\left(P+\frac{2}{y_1}\frac{dy_1}{dx}\right)dx}=e^{\int P\,dx+2\log y_1}=y_1^2e^{\int P\,dx}$$

Hence solution of (3) in p is

$$p\cdot y_1^2 e^{\int P dx} = \int\left[y_1^2\cdot\frac{X}{y_1}e^{\int P dx}\right]dx + c_1$$

or $$p = \frac{dv}{dx} = \frac{c_1 e^{-\int P dx}}{y_1^2} + \frac{e^{-\int P dx}}{y_1^2}\int\left(y_1\cdot X\cdot e^{\int P dx)}\, dx\right)$$

Integrating,

$$v = c_2 + c_1\int\frac{e^{-\int P dx}}{y_1^2}\,dx + \int\frac{e^{-\int P dx}}{y_1^2}\int\left(y_1\cdot X\cdot e^{\int P dx}(dx)^2\right)$$

Hence the required solution of (1) is

$$y = vy_1 = c_2 y_1 + c_1 y_1\int\frac{e^{-P\int dx}}{y_1^2}dx + y_1\int\frac{e^{-P\int dx}}{y_1^2}\times\int\left(y_1\cdot X\cdot e^{\int P dx}\right)(dx)^2.$$

Problem 6. *Solve* $x^2\frac{d^2y}{dx^2} - \left(x^2 + 2x\right)\frac{dy}{dx} + (x+2)\; y = x^2\cdot e^x.$

Obviously $y = x$ is a part of the complementary.

Putting $y = vx$; the given equation gives

$$\frac{d^2v}{dx^2} + \left(\frac{2}{x} - \frac{x^2+2x}{x^2}\right)\frac{dv}{dx} = \frac{x\cdot e^x}{x}.$$

Now put $\frac{dv}{dx} = p$; then we get

$$\frac{dp}{dx} - p = e^x$$

Being a linear equation of first order in p,

Integrating factor $= e^{-\int dx} = e^{-x}$.

Thus $$p\cdot e^{-x} = \int e^x\cdot e^{-x}\,dx + C_1$$

$$= x + C_1$$

or $$p = \frac{dv}{dx} = \left(x + C_1\right)e^x.$$

$\therefore$ Required solution is $y = vx = (x^2 - x + C_1 x)\, e^x + C_2 x$.

Note. *It will be helpful to find that*

$y = x$ *is a part of the complementary*, if $P + Qx = 0,$

$y = x^2$ " " " $2 + 2Px + Qx^2 = 0,$

$y = e^x$ " " " $P + Q + 1 = 0,$

$y = e^{-x}$ " " " $1 - P + Q = 0,$

$y = e^{ax}$ " " " $1 + \frac{P}{a} + \frac{Q}{a^2} = 0$ etc.

V. Transformation of Equation

(i) By Changing the Dependent Variable

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=X. \qquad ..(4)$$

Putting $y=vy_1$ this becomes

$$\frac{d^2y}{dx^2}+\left(P+\frac{2}{y_1}\frac{dy_1}{dy}\right)\frac{dv}{dx}+\frac{v}{y_1}\left(\frac{d^2y_1}{dx^2}+P\frac{dy_1}{dx}+Qy_1\right)=\frac{X}{y_1} \qquad ..(5)$$

Choosing y_1 such that $\frac{d^2y_1}{dx^2}+P\frac{dy_1}{dx}+Qy_1=0,$

(5) reduces to $\qquad \frac{d^2y}{dx^2}+P_1\frac{dy}{dx}=\frac{X}{y_1}, \qquad ...(6)$

where $\qquad P_1=P+\frac{2}{y_1}\frac{dy_1}{dx}$, giving $y_1=e^{-\frac{1}{2}\int P\,dx}$ for $P_1=0$

Now equation (6) may be solved as in previous method.

Problem 7. $\qquad \frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}=n^2y.$

Here $\qquad P_1=\frac{2}{x}.$

Take $\qquad y_1=e^{-\frac{1}{2}\int P\,dx}=e^{-\frac{1}{2}\int\frac{2}{x}dx}=e^{-\log x}=1/x.$

Now put $y=vy_1$ whence the given equation becomes

$$\frac{d^2v}{dx^2}-n^2v=0.$$

Its solution is $v=C_1e^{nx}+C_2e^{-nx}$.

The required solution is $y=vy_1=\frac{1}{x}\left(C_1e^{nx}+C_2e^{-nx}\right)$

(ii) By Changing the Independent Variable

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=X. \qquad ...(7)$$

We know that

$$\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx} \text{ and } \frac{d^2y}{dx^2}=\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2+\frac{dy}{dz}\cdot\frac{d^2z}{dx^2}$$

Substituting these values in (7), we get

$$\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2+\frac{dy}{dz}\cdot\frac{d^2z}{dx^2}+P\frac{dy}{dz}\cdot\frac{dz}{dx}+Qy=X$$

or

$$\frac{d^2y}{dz^2}+\frac{\frac{d^2z}{dx^2}+P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}\cdot\frac{dy}{dz}+\frac{Q}{\left(\frac{dz}{dx}\right)^2}y=\frac{X}{\left(\frac{dz}{dx}\right)^2}$$

or $$\frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+Q_1y=X_1, \qquad \ldots(8)$$

where $$P_1=\frac{\frac{d^2z}{dx^2}+P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2},\quad Q_1=\frac{Q}{\left(\frac{dz}{dx}\right)^2}\text{ and }X_1=\frac{X}{\left(\frac{dz}{dx}\right)^2}$$

If z be chosen such that $\frac{d^2z}{dx^2}+P\frac{dz}{dx}=0,$

i.e., $$z=\int e^{-\int P\,dx}\,dx,$$

then the given equation changes into

$$\frac{d^2y}{dz^2}+Q_1y=X_1,$$

which can be solved if Q_1 is a constant or a constant multiplied by $\frac{1}{z^2}\cdot$

Again if z be chosen that Q_1 be a^2 (a constant), then

$$a^2\left(\frac{dz}{dx}\right)^2=Q$$

i.e., $$az=\int\sqrt{Q}\,dx.$$

With this substitution (7) reduces to

$$\frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+a^2y=X,$$

which can be solved if P_1 is a constant.

Problem 8. *Solve* $\frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}+\frac{a^4}{x^4}y=0.$

We can choose z such that

$$\left(\frac{dz}{dx}\right)^2=Q=\frac{a^2}{x^4},$$

giving $$z=+\frac{a}{x}\cdot$$

Now changing the independent variable from x to z when $z=\frac{a}{x}$, we have

$$P_1=\frac{\left(\frac{d^2z}{dx^2}+P\frac{dz}{dx}\right)}{\left(\frac{dz}{dx}\right)^2}=0.$$

The given equation reduces to

$$\frac{d^2y}{dz^2}+y=0.$$

Its solution is $y = C_1 \cos z + C_2 \sin z$.

Required solution is $y = C_1 \cos \dfrac{a}{x} + C_2 \sin \dfrac{a}{x}$.

7.3. SIMPLE PROBLEMS FROM PHYSICS

Problem 9. *Solve* $L\dfrac{di}{dt} + \dfrac{\int i\,dt}{C} = 0$, *the differential equation which means that the self-induction and capacity in a circuit neutralize each other. Determine the constants in such a way that I is the maximum current and i = 0 when t = 0.*

Given $$L\frac{di}{dt} + \frac{\int i\,dt}{C} = 0.$$

or $$\frac{di}{dt} + \frac{1}{LC}\int i\,dt = 0.$$

Differentiating, we get $\dfrac{d^2 i}{dt^2} + \dfrac{i}{LC} = 0$.

Its auxiliary equation is $\left(D^2 + \dfrac{1}{LC}\right) = 0$, when $D \equiv \dfrac{d}{dt}$

or $$D = \pm\sqrt{(-1)}\ \frac{1}{\sqrt{(LC)}}.$$

Solution is $i = A \cos \dfrac{t}{\sqrt{(LC)}} + B \sin \dfrac{t}{\sqrt{(LC)}}$, ...(1)

where A and B are two arbitrary constants.

Applying the given conditions, when $i = 0$, $t = 0$, (1) gives

$$0 = A.$$

Again $i = I$, when $\dfrac{di}{dt} = 0$.

$$\because \frac{di}{dt} = -\frac{A}{\sqrt{(LC)}} \sin\frac{t}{\sqrt{(LC)}} + \frac{B}{\sqrt{(LC)}} \cos\frac{t}{\sqrt{(LC)}}$$

For maximum or minimum of i, $\dfrac{di}{dt} = 0$ gives

$$\tan\frac{t}{\sqrt{(LC)}} = \frac{B}{A},$$

$$= \infty \text{ when } A = 0$$

$$= \tan\frac{\pi}{2}.$$

$$\therefore t = \frac{\pi}{2}\sqrt{(LC)}.$$

Putting in (1), $i = I$ when $t = \dfrac{\pi}{2}\sqrt{(LC)}$, we get

$$I = A \cos \frac{\pi}{2} + B \sin \frac{\pi}{2},$$

giving $\quad B = I$

Hence putting the values of the constants A and B in (1), the required solution is

$$i = I \sin \frac{t}{\sqrt{(LC)}}.$$

Problem 10. *The relation between the small horizontal deflection θ of a bar magnet under the action of the earth's magnetic field is*

$$A \frac{d^2\theta}{dt^2} + MH\,\theta = 0,$$

when A is the moment of inertia of the magnet about the axis, M the magnetic moment of the magnet and H the horizontal component of the intensity of the field due to the Earth. Find the time of a complete vibration.

$$\left[\text{Ans.} \quad T = 2\pi\sqrt{\left(\frac{A}{MH}\right)}\right]$$

Problem 11. *In the case of the simple pendulum of length l, the equation connecting the acceleration due to gravity and the angle θ through which the pendulum swings is $l \dfrac{d^2\theta}{dt^2} + g\theta = 0$ where θ is small.*

Determine the time of an oscillation.

$$\left[\text{Ans.} \quad T = 2\pi\sqrt{\left(\frac{l}{g}\right)}\right]$$

Problem 12. *Solve* $\dfrac{d^2i}{dt^2} + \dfrac{R}{L}\dfrac{di}{dt} + \dfrac{i}{LC} = 0$, *where* $R^2C = 4L$.

[**Ans.** $i = e^{-(Rt/2L)} \cdot (C_1 + C_2 t)$.

Problem 13: *The differential equation for a circuit containing resistance R, self-inductance L and capacitance C, in terms of current i and the time t is*

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = \frac{1}{L}\frac{d}{dt}E(t)$$

$E(t)$ being the electromotive force which is the sum of voltage drops in a closed circuit according to Kirchoff's second law. Find the current i and interpret the result physically.

Given equation is $\quad \dfrac{d^2i}{dt^2} + \dfrac{R}{L}\dfrac{di}{dt} + \dfrac{i}{LC} = \dfrac{1}{L}\dfrac{d}{dt}E(t)$ ...(1)

Assume, $E(t) = F \sin(\omega t + \beta)$

So that $\quad \dfrac{d}{dt}E(t) = F\omega \cos(\omega t + \beta)$

Then (1) becomes, $\dfrac{d^2i}{dt^2} + \dfrac{R}{L}\dfrac{di}{dt} + \dfrac{1}{CL}i = \dfrac{F\omega}{L}\cos(\omega t + \beta)$ ...(2)

The solution may easily be found as

$$i = A e^{-\frac{R}{2L}t} \sin\left(\frac{\sqrt{(4CL-R^2C^2)}}{2CL}t+\epsilon\right)$$

$$+F\omega C\left[\frac{R\omega C \sin(\omega t+\beta)+\left(1-CL\omega^2\right)\cos(\omega t+\beta)}{(R\omega C)^2+\left(1-CL\omega^2\right)^2}\right]$$

Putting $\sin\alpha = \dfrac{1-CL\omega^2}{\sqrt{(R\omega C)^2+\left(1-CL\omega^2\right)^2}}$

and $\cos\alpha = \dfrac{R\omega C}{\sqrt{(R\omega C)^2+\left(1-CL\omega^2\right)^2}}$ ...(3)

The second term on the R.H.S, of (3)

$$= \frac{F\omega C}{\sqrt{(R\omega C)^2+\left(1-CL\omega^2\right)^2}}\left[\sin(\omega t+\beta)\cos\alpha+\cos(\omega t+\beta)\sin\alpha\right]$$

$$= \frac{F\omega C}{\sqrt{(R\omega C)^2+\left(1-CL\omega^2\right)^2}}\sin(\omega t+\beta+\alpha)$$

Hence the solution is

$$i = Ae^{-\frac{R}{2L}t}\sin\left(\frac{\sqrt{4CL-R^2C^2}}{2CL}t+\epsilon\right)+\frac{F\omega C}{\sqrt{(R\omega C)^2+\left(1-CL\omega^2\right)^2}}\sin(\omega t+\beta+\alpha)$$

...(4)'

The current in the circuit consists of two parts namely a *damped harmonic motion* due to the first part and a *simple harmonic motion* dut to the second part. The first part consists of the *damping factor* $e^{-\frac{R}{2L}t}$ and may be called as *transient current* while the second part as *steady state current*. The function $E(t)$ or $\frac{d}{dt}E(t)$ is called the *input* of the system and the solution as *output* of the system.

Amplitude of steady state current is

$$A_m = \frac{F\omega C}{\sqrt{(R\omega C)^2+\left(1-CL\omega^2\right)^2}} = \frac{F}{\sqrt{R^2+\left(\frac{1}{\omega C}-L\omega\right)^2}}$$

whose denominator is called as *Impedence* Z of the circuit *i.e.*

$$Z = \sqrt{R^2+\left(\frac{1}{\omega C}-L\omega\right)^2}$$

So that $\dfrac{dZ}{d\omega}=0$ gives $2\left(\dfrac{1}{\omega C}-L\omega\right)\left(-\dfrac{1}{C\omega^2}-L\right)=0$

i.e. $$\omega^2 = \frac{1}{CL} \text{ or } \omega = \frac{1}{\sqrt{CL}}$$

showing that the impedence Z of the current is minimum for $\omega = \frac{1}{\sqrt{CL}}$ and then the amplitude A_m will be maximum and in such case we say that the electromotive force is in *Resonance* with the circuit.

Putting $\omega = \frac{1}{\sqrt{CL}}$ in (5) we have Max $A_m = \frac{F}{R}$

i.e. Max $A_m \propto \frac{1}{R}$ when Resonance occurs.

Consequently R should be sufficiently large to prevent a circuit breakdown.

Problem 14. *A particle weighing 8 lbs distant 12 ft. apart from a fixed point O is given a velocity of 16 ft/sec in a direction perpendicular to x-axis. The particle is attracted to the fixed point by a force F whose magnitude is inversely proportional to the cube of its distance from O. Taking constant of proportionality as 10, find the distance of the particle from O as a function of time t.*

Let (r, θ) be the polar coordinates of the position of the particle at time t. Then we have

$$F_r \propto \frac{1}{r^3} \text{ i.e. } F_r = -\frac{10}{r^3} = ma_r \quad \text{...(1)}$$

a_r being component of acceleration along r *i.e.* radial axis and

$$F_\theta = 0 = m\, a_\theta \quad \text{...(2)}$$

a_θ being component of acceleration in a direction perpendicular to radial axis

Given $mg = 8$ so that $m = \frac{8}{32} = \frac{1}{4}$.

Since the particle subject to a central force moves in a plane, we have

$$a_r = \ddot{r} - r\,\dot{\theta}^2 \text{ and } a_\theta = 2\dot{r}\,\dot{\theta} + r\,\ddot{\theta}$$

$$\therefore \text{ (1) and (2) give } \frac{1}{4}\left(\ddot{r} - r\,\dot{\theta}^2\right) = -\frac{10}{r^3} \text{ i.e. } \ddot{r} - r\,\dot{\theta}^2 = -\frac{40}{r^3} \quad \text{...(3)}$$

and $$\frac{1}{4}\left(2\dot{r}\,\dot{\theta} + r\ddot{\theta}\right) = 0 \quad \text{...(4)}$$

(4) multiplied by $2r$ is equivalent to $\frac{d}{dt}\left(r^2\,\dot{\theta}\right) = 0$.

Integrating $r^2\,\dot{\theta} = A$, A being constant of integration.

Initially when $t = 0$, $r = 12$, $v_\theta = 16$ (velocity component perpendicular to r *i.e.* x-axis since when $\theta = 0$, r is along x-axis).

Also $v_\theta = r\,\dot{\theta} = 16$, so that $r^2\dot{\theta} = A$ gives $A = r\,(r\,\dot{\theta}) = 12 \times 16 = 192$

$$\therefore \quad \dot{\theta} = \frac{A}{r^2} = \frac{192}{r^2} \quad \text{...(5)}$$

From (5) and (3), we have

$$\ddot{r} = \frac{(192)^2 - 40}{r^3} = \frac{36824}{r^3} \quad \text{...(6)}$$

Put $u = \frac{dr}{dt} = \dot{r}$ so that $\dot{u} = \frac{36824}{r^3}$ ...(7)

Multiplying both sides of (7) by $2u$, we get

$$2u\,\dot{u} = 2 \times \frac{36824}{r^3} u = 2 \times \frac{36824}{r^3} \dot{r}$$

or $d(u^2) = 2 \times \frac{36824}{r^3} dr$ which gives on integration, $u^2 = -\frac{36824}{r^2} + B$

Initially when $t = 0$, $r = 12$, $u = \dot{r} = 0$, so that $B = \frac{36824}{144}$

$$\therefore\ u^2 = \dot{r}^2 = -\ 36824\left(\frac{1}{r^2} - \frac{1}{144}\right) = 36824\left(\frac{r^2 - 144}{144\,r^2}\right)$$

or $$\left(\frac{dr}{dt}\right)^2 = \frac{144\,r^2}{36824\,(r^2 - 144)}$$

i.e. $dt = \frac{12r}{191.9\sqrt{r^2 - 144}} dr$ which yields on integration

$$t = \frac{1}{15.99}\sqrt{r^2 - 144} + C$$

Initially when $t = 0$, $r = 12$ $\therefore\ C = 0$.

Hence $(15.99t)^2 = r^2 - 144$

or $r^2 = 144 + 255.68t^2$ *i.e.* $r = \sqrt{144 + 255.68\,t^2}$

7.4. PARTIAL DIFFERENTIAL EQUATIONS

These are the equations containing one or more partial derivatives and are concerned with at least two independent variables. The order of a partial differential equation is the order of its highest derivative appearing in the equation:

[A] The Partial Differential Equation of the first Order. Let a relation

$$\phi(x, y, z, a, b) = 0 \qquad ...(1)$$

be derived from the partial differential equation

$$F(x, y, z, p, q) = 0 \qquad ...(2)$$

where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$.

The solution (1) consisting of as many arbitrary constants as the number of independent variables is called the *Complete Integral* of (2). If we give particular values to a and b in (1), then it becomes *Particular Integral.*

Since the envelope of all the surfaces given by (1) is touched at each of its points by some one of these surfaces, the coordinates of any point on the envelope with p and q belonging to the envelope at that point must satisfy (2). Hence the relation found by eliminating a and b between $\phi\ (x, y, z, a, b) = 0$, $\frac{d\phi}{da} = 0$, $\frac{d\phi}{db} = 0$

is called the *Singular Integral.*

If $b = f(a)$ then (1) becomes $\phi[x, y, z, a, f(a)] = 0$.

The elimination of $f(a)$ between this equation and $\frac{d\phi}{da} = 0$ gives the *General Integral.*

Methods of Solution

(*i*) **Lagrange's Method.**

Lagrange's equation is of the form $Pp + Qq = R$ ...(3)

P, Q, R being functions of x, y, z.

If $u = f(x, y, z) = a$ satisfies (3), then we get on differentiating (3) partially w.r.t. x and y,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\, p = 0 \text{ and } \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\, q = 0$$

giving $p = -\frac{\partial u}{\partial x}\Big/\frac{\partial u}{\partial z},\quad q = -\frac{\partial u}{\partial y}\Big/\frac{\partial u}{\partial z}$

so that (3) yields, $P\frac{\partial u}{\partial x} + Q\,\frac{\partial u}{\partial y} + R\frac{\partial u}{\partial z} = 0$ ...(4)

Obviously $u = a$ satisfies (4) and hence comparing (3) and (4) we get equations known as *Lagrange's Subsidiary equations, i.e.*

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \text{...(5)}$$

which are also satisfied by $u = a$.

If $v = b$ be another integral of (5), then $\phi\,(u, v) = 0$ or $u = \phi(v)$, ϕ, being arbitrary function, is an integral of (3).

Problem 15. *Solve* $(y^3x - 2x^4)\, p + (2y^4 - x^3y)\, q = 9z\,(x^3 - y^3)$

Lagrange's subsidiary equations are

$$\frac{dx}{y^3x - 2x^4} = \frac{dy}{2y^4 - x^3y} = \frac{dz}{9z\left(x^3 - y^3\right)} \qquad \text{...(1)}$$

First two fractions give $\frac{dy}{dx} = \frac{2y^4 - x^3y}{y^3x - 2x^4}$

which being a homogeneous equation may be solved by putting $y = vx$ whence may we get

$$\frac{dx}{x} = \frac{v^3 - 2}{v(v+1)\left(v^2 - v + 1\right)}\, dv = \left(-\frac{2}{v} + \frac{1}{v+1} + \frac{2v-1}{v^2 - v + 1}\right) dv$$

Integrating $\log x + \log A = -2\log v + \log(v+1) + \log(v^2 - v + 1)$

i.e. $$Ax = \frac{(v+1)\left(v^2 - v + 1\right)}{v^2} = \left(\frac{y}{x} + 1\right)\left(\frac{y^2}{x^2} - \frac{y}{x} - 1\right)\Big/\frac{y^2}{x^2}$$

or $$Ax^2y^2 = x^3 + y^3 \quad i.e. \quad \frac{x^2y^2}{x^3 + y^3} = A$$

Again from (1) we have

$$\frac{\frac{dx}{x}}{y^3 - 2x^3} = \frac{\frac{dy}{y}}{2y^3 - x^3} = \frac{dz}{9z\left(x^3 - y^3\right)} = \frac{\frac{dx}{x} + \frac{dy}{y}}{-3\left(x^3 - y^3\right)}$$

The last two fractions give $\frac{dx}{x} + \frac{dy}{y} = -\frac{dz}{-3z}$

Integrating $x^3y^3 = \frac{1}{Bz}$ or $z = \frac{1}{x^3y^3}\,\phi\left(\frac{x^3+y^3}{x^2y^2}\right)$ by taking

$$\frac{1}{B} = \phi\left(\frac{1}{A}\right)$$

or
$$z = \frac{1}{x^3y^3}\phi\left(\frac{x}{y^2} + \frac{y}{x^2}\right)$$

which is the required solution.

(*ii*) Standard Methods:

Standard I. Equations involving p and q only as $F(p, q) = 0$ ...(6)

have their complete integrals $z = ax + by + c$ ...(7)

where a and b are connected by the relation $F(a, b) = 0$.

Problem 16. *Solve* $3p^2 - 2q^2 = 4pq$.

Its solution is $z = ax + by + c$ provided $3a^2 - 2b^2 = 4ab$

i.e.,
$$b = \frac{-4a \pm \sqrt{16a^2 + 24a^2}}{4} = a\left(-1 \pm \frac{\sqrt{10}}{2}\right)$$

Hence the complete integral is

$$z = a\left[x + y\left(-1 \pm \frac{\sqrt{10}}{2}\right)\right] + c$$

Standard II. The equation analogous to Clairaut's form

$\left(i.e.,\ y = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right) \text{ which has its solution as } y = xc + f(c)\right)$, such that

$$z = px + qy + f(p, q)$$

has for its complete integral, $z = ax + by + f(a, b)$.

Problem 17. *Solve* $z = px + qy - 2\sqrt{pq}$

Its complete integral is $z = ax + by - 2\sqrt{ab}$

Standard III. The equations of the form $F(z, p, q) = 0$ are solved by putting $q = ap$, (a being an arbitrary constant) and changing p into $\frac{dz}{dX}$ where $X = x + ay$ and then solving the resulting ordinary differential equations between z and X.

Problem 18. *Solve* $q(p^2z + q^2) = 4$.

Putting $q = ap$ where $p = \frac{dz}{dX}$ and $X = x + ay$, we have

$p = \frac{dz}{dX} = \pm\frac{1}{3\sqrt{z + a^2}}$ *i.e.*, $dX = \frac{3}{2}\left(z + a^2\right)^{\frac{1}{2}} dz$, taking +ve sign

Integrating $X + b = (z + a^2)^{3/2}$ or $(x + ay + b)^2 = (z + a^2)^3$

Standard IV. Equations of the form $f_1(x, p) = f_1(y, q)$ are solved by putting $f_1(x, p) = f_2(y, q) = a$ (an arbitrary constant).

These equations give p and q which when substituted in $dz = pdx + qdy$ give the complete integral.

Problem 10. *Solve* $q = 2yp^2$.

We have $p^2 = \dfrac{q}{2y} = a^2$ (say).

When $p^2 = a^2$, $p = \dfrac{dz}{dx} = a$ we have $z = ax +$ constant

and when $q = \dfrac{dz}{dy} = 2a^2y$ we have $z = a^2y^2 +$ constant.

The complete integral is $z = ax + a^2y^2 + b$.

(*iii*) **Charpit's Method.** Let the partial differential equation be

$$F(x, y, z, p, q) = 0 \qquad \ldots(8)$$

Since z depends upon x and y both, therefore

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = p\,dx + q\,dy \qquad \ldots(9)$$

Forming Charpit's auxiliary equations (proof is not required)

$$\frac{dp}{\dfrac{\partial F}{\partial x} + p\dfrac{\partial F}{\partial z}} = \frac{dq}{\dfrac{\partial F}{\partial y} + q\dfrac{\partial F}{\partial z}} = \frac{dz}{-p\dfrac{\partial F}{\partial p} - q\dfrac{\partial F}{\partial q}} = \frac{dx}{-\dfrac{\partial F}{\partial p}} = \frac{dy}{-\dfrac{\partial F}{\partial q}}$$

We may find a relation $f(p, q) = 0 \qquad \ldots(10)$

between p and q. (8) and (10) will yield p and q which when substituted in (9) give the required solution.

Problem 20. *Solve* $2zx - px^2 - 2qxy + pq = 0$

Charpit's auxiliary equations are

$$\frac{dp}{2z - 2qy} = \frac{dq}{0} = \frac{dz}{px^2 + 2xyq - 2pq} = \frac{dx}{x^2 - q} = \frac{dy}{2xy - p}$$

Whence $dq = 0$ gives $q = a$ (constant)

Putting $q = a$ in the given equation we get $p = \dfrac{2x(z - ay)}{x^2 - a}$.

Now substituting values of p and q in $dz = p\,dx + q\,dy$ we have

$$dz = \frac{2x(z - ay)}{x^2 - a}dx + a\,dy$$

or

$$\frac{dx - a\,dy}{z - ay} = \frac{2x}{x^2 - a}dx$$

which gives on integration, $z - ay = c(x^2 - a)$ *i.e.*, $z = ay + c(x^2 - a)$

[B] Partial Differential Equations of the Second and Higher Orders

Such an equation of second order is of the form

$$Rr + Ss + Tt = V$$

where $r = \dfrac{\partial^2 z}{\partial x^2}$, $s = \dfrac{\partial^2 z}{\partial x \partial y}$, $t = \dfrac{\partial^2 z}{\partial y^2}$ and R, S, T, V are functions of x, y, z, p, q,

where $p = \dfrac{\partial z}{\partial x}$, $q = \dfrac{\partial z}{\partial y}$.

The complete solution of such equation will contain two arbitrary functions as $z = f(x + ay) + \phi(x - ay)$.

Methods of Solution:

(*i*) **By Inspection.** Method is clear from the following Problem.

Problem 21. *Solve* $ar = xy$ *i.e.,* $a \dfrac{\partial^2 z}{\partial x^2} = xy$.

Integrating with regard to x, $a \dfrac{\partial z}{\partial x} = \dfrac{x^2}{2} y + \phi(y)$, constant of integration with regard to x being possibly a function of y.

Integrating again with regard to x,

$$az = \int \frac{x^2}{2} ydy + \int \phi(y)\, dx + \text{const.}$$

$$= \frac{x^3 y}{6} + x\phi(y) + \psi(y).$$

(*ii*) **Monge's Method.**

The equation is $Rr + Ss + Tt = V$ ...(1)

Total differentials of p and q being

$$dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy = rdx + sdy$$

and $$dp = \frac{\partial q}{\partial x} dx + \frac{\partial q}{\partial y} dy = sdx + tdy$$

We have $r = \dfrac{dp - sdy}{dx}$, $t = \dfrac{dq - sdx}{dy}$.

Substituting these values of r and t in (1) we may get

$(Rdpdy + Tdpdx - Vdxdy) - s(Rdy^2 - Sdxdy + Tdx^2) = 0$ ...(2)

If any relation between x, y, z, p, q makes each of the bracketed expressions in (2) vanish, this relation will also satisfy (2). Hence from (2) we have so-called *Monge's subsidiary equations* as

$$Rdy^2 - Sdxdy + Tdx^2 = 0 \qquad ...(3)$$

$$Rdpdy + Tdqdx - Vdxdy = 0 \qquad ...(4)$$

Let (3) resolve into two linear equations

$$dy - m_1 dx = 0 \qquad ...(5)$$

$$dy - m_2 dx = 0 \qquad ...(6)$$

Combining (5) with (4) and with $dz = pdx + qdy$ if necessary, we may get two integrals $u_1 = a$, $v_1 = b$ giving an *intermediary integral* $u_1 = f_1(v_1)$, f_1 being an arbitrary function.

Similarly from (6) and (4) alongwith $dz = pdx + qdy$, we may find another intermediary integral $u_2 = f_2(v_2)$, f_2 being arbitrary.

These two intermediary integrals can yield p and q which when substituted in $dz = pdx + qdy$ will yield the complete integral on integration.

In case $m_1 = m_2$ either of the intermediary integral may be integrated to give the complete integral.

Problem 22. *Solve $r + (a+b)s + abt = xy$.*

Putting $r = \dfrac{dp - sdy}{dx}$ and $t = \dfrac{dq - sdx}{dy}$ in the given equation we get Monge's subsidiary equations as $dy^2 - (a+b) + dxdy + abdx^2 = 0$ ...(1)

$$dpdy + abdqdx - xydxdy = 0 \quad ...(2)$$

(1) yields, $dy - adx = 0$ and $dy - bdx = 0$

Integrating them, $y - ax = c_1$ and $y - bx = c_2$

Comparing these with (2) we get $adp + abdq - ax(c_1 + ax)dx = 0$

$$bdp + abdq - bx(c_2 + ax)dx = 0$$

Their integration yields $p + bq - c_1\dfrac{x^2}{2} - \dfrac{ax^3}{3} = k_1$

$$p + aq - c_2\frac{x^2}{2} - \frac{bx^3}{3} = k_2$$

or
$$p + bq - (y - ax)\frac{x^2}{2} - \frac{ax^3}{3} = \phi_1(c_1) = \phi_1(y - ax) \quad ...(3)$$

$$p + aq - (y - bx)\frac{x^2}{2} - \frac{bx^3}{3} = \phi_2(c_2) = \phi_2(y - bx) \quad ...(4)$$

Solving (3) and (4),

$$p = \frac{1}{a-b}\left[\frac{yx^2}{2}(a-b) - \left(a^2 - b^2\right)\frac{x^3}{6} + a\phi_1(y - ax) - b\phi_2(y - bx)\right]$$

and $q = \dfrac{1}{a-b}\left[\dfrac{x^3}{6}(a-b) - \phi_1(y-ax) + \phi_2(y-bx)\right]$

Substituting these values of p and q in $dz = p\,dx + q\,dy$, we find

$$dz = \left[\frac{yx^2}{2} - (a+b)\frac{x^3}{6} + \frac{a\phi_1\,(y-ax)}{a-b} - \frac{b\phi_2\,(y-bx)}{a-b}\right]dx$$

$$+\left[\frac{x^3}{6} - \frac{\phi_1(y-ax)}{a-b} + \frac{\phi_2\,(y-bx)}{a-b}\right]dy$$

$$= -(a+b)\frac{x^3}{6}dx + \frac{3x^2y\,dx + x^3dy}{6} - \frac{1}{a-b}[\phi_1(y-ax)(dy - adx)]$$

$$+ \frac{1}{a-b}[\phi_2\,(y-bx)\,(dy - bdx).]$$

Integrating, $z = -(a+b)\dfrac{x^4}{24} + \dfrac{yx^3}{6} + \psi_1(y - ax) + \psi_2(y - bx)$

(*iii*) **Monge's Method of Integrating** $Rr + Ss + Tt + U(rt - s^2) = V, R, S, T, U, V$, being functions of x, y, z, p, q.

Putting $r = \dfrac{dp - sdy}{dx}$ and $t = \dfrac{dq - sdx}{dy}$ in the given equation we get

$(Rdp\ dy + Tdq\ dx + Udp\ dq - Vdx\ dy) - s\,(Rdy^2 - Sdx\ dy + Tdx^2$
$+ Udp\ dx + Udq\ dy) = 0$

Say $\qquad N - sM = 0$

Consider,

$$M + \lambda N = Rdy^2 + Tdx^2 - (S + \lambda V)\,dxdy + Udpdx + Udqdy$$
$$+ Rdpdy + \lambda Rdpdy + \lambda Tdqdx + \lambda Udpdq$$
$$\equiv (Ady + Bdx + Cdp)\,(Edy + Fdx + Gdq) \text{ (say)}$$

Then equating the coefficients of $dy^2, dx^2, dpdq$ we get

$$AE = R,\ BF = T,\ GC = \lambda U$$

Also taking $A = R, E = 1, B = kT, F = \frac{1}{k}, C = mU, G = \frac{\lambda}{m}$ and equating the coefficients of the other five terms, we may find

$$kT + \frac{R}{k} = -(S + \lambda V) \quad \text{...(14)}$$

$$\frac{\lambda R}{m} = U \quad \text{...(15)}$$

$$\frac{kT\lambda}{m} = \lambda T, \quad \text{...(16)}$$

$$mU = \lambda R \quad \text{...(17)}$$

$$\frac{mU}{k} = U \quad \text{...(18)}$$

From (18), $m = k$ which satisfies (16).

From (15) or (17) $m = \frac{\lambda R}{U} = k$ and hence from (14),

$$\lambda^2 (RT + UV) + \lambda US + U^2 = 0 \quad \text{...(19)}$$

So if λ is a root of (19), the required factors of $M + \lambda N$ are

$$\left(R\,dy + \lambda\frac{RT}{U}dx + \lambda\ Rdp\right)\left(dy + \frac{U}{\lambda R}dx + \frac{U}{R}dq\right)$$

i.e. $\frac{R}{U}\ (Udy + \lambda Tdx + \lambda Udp)\ \frac{1}{\lambda R}\ (\lambda Rdy + Udx + \lambda Udq)$

we thus obtain integrals from the linear equations

$$Udy + \lambda Tdx + \lambda Udp = 0 \quad \text{...(20)}$$

$$\lambda Rdy + Udx + \lambda Udq = 0 \quad \text{...(21)}$$

If $u_1 = f_1(v_1)$ and $u_2 = f_2(v_2)$ be the two intermediary integrals so obtained for finding p and q and substituting in $dz = pdx + qdy$ we get the required solution after integration.

Problem 23. *Solve* $z\,(1 + q^2)r - 2pqzs + z\,(1 + p^2)t - z^2(s^2 - rt) + 1 + p^2 + q^2 = 0.$

Here $R = z(1 + q^2), S = -2pqz, T = (1 + p^2)z, U = z^2V, V = -(1 + p^2 + q^2)$

$\therefore$ quadratic in λ is $(RT + UV)\lambda^2 + \lambda US + U^2 = 0$ *i.e.* $(pd\lambda - z)^2 = 0$ giving $\lambda = \frac{z}{pq}$.

Now the system of intermediary integrals is

$$Udy + \lambda Tdx + \lambda Udp = 0,\ Udx + \lambda Rdy + \lambda Udq = 0$$

i.e., $\quad pqdy + (1 + p^2)\,dx + zdp = 0, \quad \text{...(1)}$

$\quad pq\,dx + (1 + q^2)\,dx + zdq = 0 \quad \text{...(2)}$

Also $\quad dz = pdx + qdy \quad \text{...(3)}$

So (1) can be written as $dx + p(pdx + qdy) + zdp = 0$

or $dx + pdz + zdp = 0$ giving $x + pz = A$

Similarly from (2), $y + qz = B$

So that $p = \frac{A-x}{z}$ and $q = \frac{B-y}{z}$

Putting in $dz = pdx + qdy$ and integrating we get the required solution.

$$dz = \frac{A-x}{z}dx + \frac{B-y}{z}dy \text{ or } -zdz = (A-x)(-dx) + (B-y)(-dy)$$

$$-\frac{z^2}{2} = \frac{(A-x)^2}{2} + \frac{(B-y)^2}{2} + \text{const. } i.e.\ z^2 + (x-A)^2 + (y-B)^2 = C^2$$

[C] General Linear Partial Differential Equations of an order higher than the first.

Such equations are of the form

$$A_0\frac{\partial^n z}{dx^n} + A_1\frac{\partial^n z}{\partial x^{n-1}dy} + \ldots + A_n\frac{\partial^n z}{\partial y^n} + B_0\frac{\partial^{n-1} z}{\partial x^{n-1}} + \ldots$$

$$+ M\frac{\partial z}{\partial x} + N\frac{\partial z}{\partial y} + Pz = f(x,y)$$

or $[A_0D^n + A_1D^{n-1}D' + \ldots + A_nD'^n + B_0D^{n-1} + \ldots + MD + ND' + P]z = f(x,y)$

where $D \equiv \frac{\partial}{\partial x}$ and $D' \equiv \frac{\partial}{\partial y}$

i.e. $F(D, D')z = f(x, y)$.

Its complete solution consists of two parts : (*i*) *Complementary Function* (*C.F.*) (*ii*) *Particular Integral* (*P.I.*).

The complementary function is obtained from $F(D,D') = 0$.

Methods of Solution.

(*i*) **Complementary function of a homogeneous partial differential equation with constant coefficients.**

Such an equation is of the form

$(A_0D^n + A_1D^{n-1}D' + A_2D^{n-2}D'^2 + \ldots + A_nD'^n)z = f(x, y)$

Taking the trial solution as $z = \phi(y + mx)$, so that $D^n(z)$

$= m^n\phi^{(n)}(y + mx)$, $D'^n(z) = \phi^{(n)}(y + mx)$

and in general $D^rD'^s = m^r\phi^{(r+s)}(y + mx)$

The *auxiliary equation* of the given equation becomes

$A_0m^n + A_1m^{n-1} + \ldots + A_n = 0$

If the n roots given by it be $m_1, m_2, \ldots, m_n$, then the required complementary function is $z = \phi_1(y + m_1x) + \phi_2(y + m_2x) + \ldots + \phi_n(y + mx)$

Problem 24. *Solve* $\frac{\partial^2 z}{\partial x^2} - a^2\frac{\partial^2 z}{\partial y^2} = 0$ *i.e.* $\left(D^2 - a^2D'^2\right)z = 0.$

Putting $z = \phi(z + mx)$, the auxiliary equation is $m^2 - a^2 = 0$ giving $m = \pm a$

Hence the solution is $z = \phi(y + ax) + \psi(y - ax)$

Note. *If an equation has repeated roots such as* $(D - mD')^2 z = 0$, *the solution is* $z = x\phi(y + mx) + \psi(y + mx)$

If a root is repeated thrice then $z = x^2\phi(y + mx) + x\psi(y + mx) + \chi(y + mx)$ and so on.

(ii) Particular integral of homogeneous equations.

Let the equation be $F(D, D')\, z = \phi(x, y)$. Then

$$P.I. = \frac{1}{F(D, D')}\,\phi(x, y).$$

Case I. If $\phi(x, y) = e^{ax+by}$ then $\dfrac{1}{F(D, D')}e^{ax+by} = \dfrac{1}{F(a, b)}e^{ax+by}$

provided $F(a, b) \neq 0$.

Case II. If $\phi(x, y) = \sin ax$ or $\cos ax$ then express $F(D, D')$ as $F(D^2, DD', D'^2)$ and write

$$\frac{\sin ax \text{ or } \cos ax}{F\left(D^2, DD', D'^2\right)} = \frac{\sin ax \text{ or } \cos ax}{F\left(-a^2, -ab, -b^2\right)} \text{ provided } F(-a^2, -ab, -b^2) \neq 0$$

Case III. If $\phi(x, y) = x^m y^n$, then $\dfrac{1}{F(D, D')}x^m y^n = \left[F(D, D')\right]^{-1} x^m y^n$, expanding binomially and operating $x^m y^n$ on every term.

Case IV. If $\phi(x, y) = e^{ax+by}V$ which may also arise in case of failures of cases I and II.

$$\frac{1}{F(D, D')}e^{ax+by}V = e^{ax+by}\frac{1}{F(D + a, D' + b)}V$$ which reduces to any of the above cases.

Problem 25. *Find the Particular Integral of* $(D^2 - 2DD' + D'^2)z = 12xy$.

We have, $P.I. = \dfrac{1}{F\left(D^2 - 2DD' + D'^2\right)}\, 12xy$

$$= 12\frac{1}{(D - D')}xy = \frac{12}{D^2}\left(1 - \frac{D'}{D}\right)^{-1} xy$$

$$= \frac{12}{D^2}\left(1 + \frac{2D'}{D} + \ldots\right)xy = 12 \cdot \frac{1}{D^2}\left(xy + \frac{2}{D}x\right)$$

$$= 12\left[\frac{1}{D^2}xy + \frac{2}{D^3}x\right] = 12\left[\frac{x^3 y}{6} + \frac{x^4}{12}\right] = 2x^3 y + x^4.$$

Complete Integral is $z = x\phi(y + x) + \psi(y + x) + 2x^3 y + x^4$.

COROLLARY 1. In case $\phi(x, y) = f(ax + by)$ and $F(D, D')$ is a homogeneous function of D, D' of degree n. Then

$$D^n f(ax + by) = a^n f^n(ax + by)$$

$$D'^n f(ax + by) = b^n f^n(ax + by)$$

So that $F(D, D')f(ax + by) = F(a, b) f^n(ax + by)$

$$\therefore \quad \frac{f^n(ax + by)}{F(D, D')} = \frac{f(ax + by)}{F(a, b)} \text{ when } F(a, b) \neq 0$$

Hence $\dfrac{f(ax+by)}{F(D, D')}$ may be evaluated by integrating $f(ax + by)$, n times with regard to $(ax + by)$ and then dividing by $F(a, b)$.

Problem 26. *Solve* $\dfrac{\partial^2 V}{\partial x^2}+\dfrac{\partial^2 V}{\partial y^2}=12(x+z)$ *i.e.* $\left(D^2+D'^2\right)V=12(x+y)$.

Auxiliary equation is $m^2 + 1 = 0$ *i.e.* $m = \pm i$ so that C.F. is $\phi(y + ix) + \psi(y - ix)$

and $P.I.=\dfrac{12(x+y)}{D^2+D'^2}=\dfrac{12\cdot\dfrac{(x+y)^2}{2.3}}{1^2+1^2}=(x+y)^2$

Hence the solution is $V = (x + y)^3 + \phi(y + ix) + \psi(y - ix)$.

COROLLARY 2. In case, method of Cor. 1 fails *i.e.* $F(a, b) = 0$, then consider

$$(D - mD')z = p - mq = x^r\psi(y + mx) \quad \text{...(22)}$$

Lagrange's subsidiary equations are $\dfrac{dx}{1}=\dfrac{dy}{-m}=\dfrac{dx}{x^r\ \psi(y+mx)}$

of which first two fractions, give $dy + mdx = 0$ *i.e.* $y + mx$ = const. = c (say). From first and third fractions, we have

$$dx=\frac{dz}{z^r\,\phi(y+mx)}=\frac{dz}{x^r\,\psi(c)} \text{ or } dz=x^r\,\psi(c)\,dx$$

Integrating $\quad z=\dfrac{x^{r+1}}{r+1}\psi(c)=\dfrac{x^{r+1}}{r+1}\psi(y+mx)$

Thus (22) yields, $(D-mD')\dfrac{x^{r+1}}{r+1}\psi(y+mx)=x^r\,\psi(y+mx)$.

i.e.
$$\frac{1}{D-mD'}x^r\,\psi(y+mx)=\frac{x^{r+1}}{r+1}\psi(y+mx)$$

Hence
$$\frac{1}{(D-mD')^n}\psi(y+mx)=\frac{1}{(D-mD')^{n-1}}\frac{x}{1}\psi(y+mx)$$
$$=\frac{1}{(D-mD')^{n-2}}\frac{x^2}{1\cdot 2}(y+mx)$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$=\frac{x^n}{\lfloor n}\psi(y+mx)$$

Problem 27. *Solve* $(D^2 - 6DD' + 9D'^2)z = 6x + 2y$. *i.e.* $(D - 3D')^2z = 6x + 2y$. *C.F. is clearly,* $x\phi(y + 3x) + \psi(y + 3x)$

$$P.I.=\frac{1}{(D-3D')^2}\cdot(6x+2y)=\frac{2}{(D-3D')^2}\cdot(y+3x)$$
$$=2\cdot\frac{x^2}{\lfloor 2}(y+3x)=x^2(y+3x).$$

Solution is $z = x^2(3x + y) + x\phi(y + 3x) + \psi(y + 3x)$.

COROLLARY 3. General rule.

Consider $(D - mD')\, z = p - mq = f(x, y)$

Lagrange's subsidiary equatons are $\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{f(x, y)}$

First two fractions give $y + mx = c$

and then first and third fractions yield $dz = f(x, y)\, dx = f(x, c - mx)\, dx$

$\therefore \qquad z = \int f(x, c - mx)\, dx + \text{const.}$

Hence, $\frac{f(x, y)}{D - mD'} = \int f(x, c - mx)\, dx$, where c is replaced by $y + mx$ after integration.

Problem 28. *Solve* $(2D^2 - DD' - 3D'^2)z = 5e^{x-y}$ *i.e.,* $(2D - 3D')(D + D')z = 5e^{x-y}$.

Clearly C.F. is $\phi(y-x)+(2y+3x)$

$$P.I. = \frac{1}{(D + D')(2D - 3D')} 5e^{x-y} = \frac{1}{D + D'} \frac{5e^{x-y}}{2 - 3(-1)} \text{ by Cor. 1.}$$

$$= \frac{1}{(D + D')} e^{x-y} = \int e^{x-(c+x)}\, dx = \int e^{-c}\, dx = xe^{-c} = xe^{x-y}$$

Solution is $z = xe^{x-y} + \phi(y - x) + \psi(2y + 3x)$.

(iii) **Non-homogeneous linear equations (complementary function)**

Consider $(D - mD' - n)\, z = 0$ *i.e.,* $p - mq = nz$

Lagrange's subsidiary equations are $\frac{dx}{1} = \frac{dx}{-m} = \frac{dx}{nz}$

First two fractions give $y + mx = \text{const.}$

First and third fractions give $n\, dx = \frac{dz}{z}$ *i.e.,* $nx = \log z - \log k$ or $z = k\, e^{nx}$.

Hence the integral of given equation is $z = e^{nx}\, \phi(y + mx)$.

Note 1. *If factors are repeated say* $(D - mD' - n)^2 z = 0$, *then*

$$z = xe^{nx}\, \phi(y + mx) + e^{nx}\, \psi(y + mx).$$

Note 2. *The Particular Integral is obtained by the methods already discussed.*

Problem 29. *Solve*

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial z}{\partial x}\frac{\partial z}{\partial y} - 2\frac{\partial^2 z}{\partial y^2} + 2\frac{\partial z}{\partial y} + 2\frac{\partial z}{\partial x} = e^{2x+3y} + \sin(2x + y) + xy.$$

Given equation is $(D^2 - DD' + 2D'^2 + 2D' + 2D)z = e^{2x+3y} + \sin(2x + y) + xy$

or $\qquad (D + D')(D - 2D' + 2)\, z = e^{2x+3y} + \sin(2x + y) + xy.$

C.F. is $\phi(y - x) + e^{2x}\psi(y + 2x)$ and by usual methods

$$P.I. = -\frac{1}{10}e^{2x+3y} - \frac{1}{6}\cos(2x + y) + \frac{x}{24}(6xy + 9x - 2x^2 - 6y - 12)$$

Hence the solution.

(iv) **Equations Reducible to Linear Form.**

Consider an equation all of whose terms are of the form

$$Ax^m y^m = \frac{\partial^{m+n} z}{\partial x^m \partial y^n}$$

Put $x = e^u$ *i.e.* $u = \log x$ and $y = e^v$ *i.e.* $v = \log y$, then it is easy to verify that if $D \equiv \frac{\partial}{\partial u} \equiv x\frac{\partial}{\partial x}$; $D' \equiv \frac{\partial}{\partial v} \equiv y\frac{\partial}{\partial y}$,

then $x^2\frac{\partial}{\partial x^2} = D(D-1)$, $x^3\frac{\partial^3}{\partial x^3} = D(D-1)(D-2)$ etc.

$$y^2\frac{\partial^2}{\partial y^2} = D'(D'-1),\ y^3\frac{\partial^3}{\partial y^3} = D'(D'-1)(D'-2) \text{ etc.}$$

or in general $x^m y^n \frac{\partial^{m+n}}{\partial x^m \partial y^n} = x^m \frac{\partial^m}{\partial x^m} \cdot y^n \frac{\partial^n}{\partial y^n}$

$$= D(D-1)\ldots(D-m+1).\, D'(D'-1)\ldots(D'-n+1).$$

So that the given equation will reduce to $F(D, D')\, z = V$ which can be integrated by usual methods.

Problem 30. *Solve* $x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} - y\frac{\partial z}{\partial y} + x\frac{\partial z}{\partial x} = 0.$

Putting $u = \log x$, $v = \log y$ with $D \equiv \frac{\partial}{\partial u}$, $D' \equiv \frac{\partial}{\partial v}$, this becomes

$$\{D(D-1) - D'(D'-1) - D' + D\}z = 0 \text{ i.e. } (D^2 - D'^2)\, z = 0$$

or

$$(D - D')(D + D')\, z = 0$$

So that solution is $z = f(v+u) + F(v-u) = f(\log xy) + F\left(\log \frac{y}{x}\right)$

$$= \phi(xy) + \psi\left(\frac{y}{x}\right).$$

(v) Equations in which Linear Factors of $F(D, D') = 0$ are not possible. Consider $(D^2 - D')\, z = 0$.

Assume $z = Ae^{\alpha x + \beta y}$, giving $D'z = A\beta e^{\alpha x+\beta y}$ and $D^2 z = A\alpha^2 e^{\alpha x+\beta y}$

So that the given equation, yields $\alpha^2 - \beta = 0$ *i.e.* $\alpha^2 = \beta$

Hence the Complementary function is $z = Ae^{\alpha x + \alpha^2 y} = Ae^{\alpha(x+\alpha y)}$ or in general, $z = \Sigma A e^{\alpha(x+\alpha y)}$

7.5. SPECIAL TYPES OF DIFFERENTIAL EQUATIONS ARISING IN PHYSICS

[A] Hyperbolic, Parabolic and Elliptical Equations

Consider $f_1(x, y, z)\frac{\partial^2 z}{\partial x^2} + f_2(x, y, z)\frac{\partial^2 z}{\partial x \partial y} + f_3(x, y, z)\frac{\partial^2 z}{\partial y^2} +$

$$F(x, y, z, p, q) = 0 \qquad \ldots(1)$$

Writing it as $Rr + Ss + Tt + V = 0$,

where $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x\, dy}$, and $t = \frac{\partial^2 z}{\partial y^2}$ etc.

(*i*) if $S^2 > 4RT$, the equation (1) is said to be *Hyperbolic*

(*ii*) if $S^2 = 4RT$, ,, ,, *Parabolic*

(*iii*) if $S^2 < 4RT$, ,, ,, *Elliptic*.

e.g. $\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2}$, *i.e.* $r = t$ isa hyperbolic equation

$\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y}$, *i.e.* $r = q$ isa parabolic equaion

and $\frac{\partial^2 z}{\partial x^2} = -\frac{\partial^2 z}{\partial y^2}$, *i.e.* $r = -t$ isan elliptic equation.

[B] Diffusion Equation or Fourier Equation of Heart Flow or Time Dependent Equation

A Parabolic equation of the type $\Delta^2 V = \frac{1}{k}\frac{\partial V}{\partial t}$

is said to be a *diffusion equation*. However if p be the source density, then $\nabla^2 V = \rho$ is known *Poisson's equation*.

Here $\nabla^2 = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ (in 3 – D); k is the diffusivity and $V(x, y, z, t)$ is the temperature at any point (x, y, z) of a solid at time t.

One-dimensional diffusion equation is $\frac{\partial^2 z}{\partial x^2} = \frac{1}{k}\frac{\partial V}{\partial t}$

Two-dimensional diffusion equation is $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{1}{k}\frac{\partial V}{\partial t}$

Three-dimensional diffusion equation is $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{k}\frac{\partial V}{\partial t}$

COROLLARY. In steady state heat flow, (1) yields $\nabla^2 V = 0$ ...(3)

Here $\nabla^2 V \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$ is known as *Laplaces' equation*.

Its cylindrical form is $\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = 0$

and spherical polar form is

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

[C] Wave Equation

A hyperbolic equation of the type $\Box^2 V = 0$ ...(4)

where $\Box^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$, c being a constant, is said to be a *wave equation*.

Here $\Box^2$ is the *D' Alembertian* which is a four dimensional analogue of the Laplacian in Minkowski's four-dimensional space.

One-dimensional wave equation is $\dfrac{\partial^2 V}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2 V}{\partial t^2}$

Two-dimensional wave equation is $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = \dfrac{1}{c^2}\dfrac{\partial^2 V}{\partial t^2}$

Three-dimensional wave equation is $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 z}{\partial z^2} = \dfrac{1}{c^2}\dfrac{\partial^2 V}{\partial t^2}$

Note. If V be the time dependent space part of the solution of either diffusion or wave equation, then

$$\nabla^2 V \pm k^2 V = 0$$

or

$$(\nabla^2 \pm k^2)V = 0$$

is known as *Helmholtz equation.*

However, in wave mechanics if a system consists of a particle of mass m, in a field of fixed energy E, potential every V then the wave function ψ is given by the *Schröedinger's equation* (or *Time-independent Schröedinger's equation*)

$$\nabla^2\psi + \frac{2m}{\hbar^2}(E - V)\psi = 0$$

where $\hbar = \dfrac{h}{2\pi} = \dfrac{6.6253 \times 10^{-27}}{2\pi}$ erg.-sec.

$= 1.0544 \times 10^{-27}$ erg.-sec.

h being Planck's constant.

This equation if generalized in view of equation given by *E.* Schröedinger in 1926 as

$$\frac{-\hbar}{2m}\nabla^2\psi + \frac{-\hbar}{i}\frac{\partial\psi}{\partial t}x = 0 \text{ yields}$$

$$\frac{-\hbar}{2m}\nabla^2\psi + V\psi = -\frac{\hbar}{i}\frac{\partial\psi}{\partial t}$$

Methods of Solution

(*i*) **Separation of Variables**

The equations introduced in this section may conveniently be solved by the method of separation of variables. This method is illustrated here.

Consider $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{1}{k}\dfrac{\partial z}{\partial t}$, where $z = z(x, t)$ ...(5)

Suppose that $z = X(x)T(t)$ ...(6)

$X(x)$ being function of x only and $T(t)$ function of t only.

We have from (6), $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial^2 X}{\partial x^2} T$ and $\dfrac{\partial z}{\partial t} = X\dfrac{dT}{dt}$

Their substitutions in (5) yield $\dfrac{1}{X}\dfrac{\partial^2 X}{\partial x^2} = \dfrac{1}{kT}\dfrac{dT}{dt}$

Assuming that L.H.S. of (7) is independent of t and R.H.S. independent of x, each side of (7) can be equated to some constant known as *Constant of separation*.

According as this constant is negative or positive, there arise two cases:

Case I. Let the constant of separation be $-n^2$. Then (7) gives

$$\frac{1}{X}\frac{d^2X}{dx^2} = -n^2 \text{ i.e. } \frac{d^2X}{dx^2} + n^2X = 0 \qquad ...(8)$$

and

$$\frac{1}{kT}\frac{dT}{dt} = -n^2 \text{ i.e. } \frac{dT}{dt} + n^2kT = 0 \qquad ...(9)$$

(8) gives on integration $X = a\cos(nx + \alpha)$ a and α being arbitrary constants and (9) gives on integration $T = be^{-kn^2t}$, b being an arbitrary constant.

The two solutions when combined in accordance with (6) i.e. $z = XT$, yield

$$z = A\cos(nx + \alpha)e^{-kn^2t} \qquad ...(10)$$

where $A(= ab)$, n and α are constants and they can be determined by the boundary conditions.

Case II. Let the constant of separation be n^2. Then (7) gives

$$\frac{d^2X}{dx^2} - n^2X = 0 \text{ and } \frac{dT}{dt} - n^2kT = 0$$

which give on integration, $X = a\cosh(nx + \alpha)$ and $T = be^{kn^2t}$

So that the combined solution of (5) is $z = A\cosh(nx + \alpha)e^{kn^2t}$ ...(11)

The solutions (10) and (11) can be expressed as

$$z = A_n\cos(nx + \alpha_n)e^{-kn^2t} \qquad ...(12)$$

$$z = A_n\cosh(nx + \alpha_n)e^{kn^2t} \qquad ...(13)$$

where the constant A_n corresponds to a particular value of n.

Since these are the solutions of (5) for all n, therefore summing over all values of n, the solution of (5) is

$$z(x, t) = \sum_{n=0}^{\infty} A_n \cos(nx + \alpha_n) e^{-kn^2 t} \quad ...(14)$$

or

$$z(x, t) = \sum_{n=0}^{\infty} A_n \cosh(nx + \alpha_n) e^{kn^2 t} \quad ...(15)$$

according as the constant of separation is negative or positive.

Problem 31. *Solve* $\frac{\partial^2 z}{\partial x^2} = \frac{1}{k}\frac{\partial z}{\partial t}$ *with the boundary conditions,* $z = 0$ *when* $x = 0$ *and* $x = l$ *for all values of* t.

Using (10), the solution is $z(x, t) = A\cos(nx + \alpha)e^{-kn^2 t}$

Initially $z = 0$ when $x = 0$ gives $z(0, t) = A\cos\alpha\, e^{-kn^2 t} = 0$

and $x = l$ for all t gives $z(l, t) = A\cos(nl + \alpha)\, e^{-kn^2 t} = 0$

So that $A\cos\alpha = 0$ and $A\cos nl\cos\alpha - A\sin nl\sin\alpha = 0$.

Which yield $\alpha = \frac{\pi}{2}$ and $n = \frac{\pi}{l}, \frac{2\pi}{l}, \frac{3\pi}{l}, ...$

With $n = \frac{\pi}{l}$, the solution is $z(x, t) = A\sin nx e^{-kn^2 t}$

Note. *Here* $n = \frac{\pi}{l}, \frac{2\pi}{l}, \frac{3\pi}{l}, ...$ *are called* **eigen values.**

(*ii*) D'Alembert's Method.

Consider the equation of vibrating cords

$$\frac{\partial^2 u}{\partial t^2} = h^2 \frac{\partial^2 u}{\partial x^2} \quad ...(16)$$

which is also known as *D'Alembert's equation.*

Suppose that $v = x + ht$

$$\left.\begin{aligned} v &= x + ht \\ w &= x - ht \end{aligned}\right\} \quad ...(17)$$

So that $\frac{\partial v}{\partial x} = \frac{\partial w}{\partial x} = 1,\ \frac{\partial v}{\partial t} = h,\ \frac{\partial w}{\partial t} = -h$

and $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial x} = \frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}$ *i.e.* $\frac{\partial}{\partial x} = \frac{\partial}{\partial v} + \frac{\partial}{\partial w}$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\frac{\partial}{\partial v} + \frac{\partial}{\partial w}\right)\left(\frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}\right) = \frac{\partial^2 u}{\partial v^2} + 2\frac{\partial^2 u}{\partial v\,\partial w} + \frac{\partial^2 u}{\partial w^2} \quad ...(18)$$

Also $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial t} + \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial t} = h\left(\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w}\right)$ *i.e.* $\frac{\partial}{\partial t} = h\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial w}\right)$

$$\therefore \frac{\partial^2 u}{\partial t^2} = h^2\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial w}\right)\left(\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w}\right) = h^2\left(\frac{\partial^2 u}{\partial v^2} - 2\frac{\partial^2 u}{\partial v\,\partial w} + \frac{\partial^2 u}{\partial w^2}\right) \quad ...(19)$$

(16) yields with the help of (18) and (19), $\dfrac{\partial^2 u}{\partial v\, \partial w} = 0$.

Integrating with regard to w, we get $\dfrac{\partial u}{\partial v}$ = const. $= f(v)$ (say).

Integrating it again with regard to v, $u = \int f(v)\, dv$ + const.

$$= \phi(v) + \psi(w) \text{ (say)}$$

i.e. $$u(x, t) = \phi(x + ht) + \psi(x - ht) \quad \text{...(20)}$$

Physically interpreted (20) represents two plane waves travelling in opposite directions with same period.

Aliter. Writing $D_x \equiv \dfrac{\partial}{\partial x}$, $D_x^2 \equiv \dfrac{\partial^2}{\partial x^2}$, (16) becomes

$$\frac{\partial^2 u}{\partial t^2} = (h D_x)^2 u \quad \text{...(21)}$$

Treating (20) as an ordinary differential equation with constant coefficients such as

$$\frac{\partial^2 u}{\partial t^2} = a^2 u \quad \text{...(22)}$$

where $$a = h D_x \quad \text{...(23)}$$

We have the solution of (22) as $u = A_1 e^{at} + A_2 e^{-at}$ ...(24)

where A_1, A_2 are arbitrary constants.

Now (22) is formally satisfied by

$$u = e^{hD_x t}\phi(x) + e^{-hD_x t}\psi(x) \quad \text{...(25)}$$

Since (22) has been integrated with regard to t and so the arbitrary constants A_1, A_2 appearing in (24) can be replaced by arbitrary functions $\phi(x)$ and $\psi(x)$ of x. Now we know that Taylor's expansion is

$$f(x + h) = f(x) + \frac{h}{\lfloor 1}f'(x) + \frac{h^2}{\lfloor 2}f''(x) + \ldots + \frac{h^n}{\lfloor n}f^n(x) + \ldots$$

Using the symbolic operator $D_x^n = \dfrac{d^n}{dx^n}$ for $n = 1, 2, 3\ldots\ldots$

so that $D_x^r f(x) = \dfrac{d^r}{dx^r} f(x)$

This becomes $$f(x + h) = f(x) + \frac{h}{\lfloor 1} D_x f(x) + \frac{h}{\lfloor 2} D_x^2 f(x) + \ldots + \frac{h^n}{\lfloor n} D_x^n f(x) + \ldots$$

$$= \left(1 + \frac{hD_x}{\lfloor 1} + \frac{h^2 D_x^2}{\lfloor 2} + \ldots + \frac{h^n D_x^n}{\lfloor n} + \ldots\right) f(x)$$

$$= e^{hD_x} \cdot f(x) \quad \text{...(26)}$$

Using (26), and replacing h by ht and by $-ht$ in succession, (25) yields

$$u(x, t) = \phi(x + ht) + \phi(x - ht)$$

which is the same as (20).

7.6. FINITE DIFFERENCES

If $f(x)$ be a function of an independent variable x, then in calculus we study the change in $f(x)$ corresponding to an infinitely small change in x and denote this change by δx or dx or Δx. Here we shall use the last notation.

The change Δx in x may be constant for all values of x or may vary for different values of x. Taking $\Delta x = h$ (a constant) the increment in function $f(x)$ will be $f(x + \Delta x) - f(x)$ which is represented by $\Delta f(x)$ *i.e.* the increment Δx in x corresponds an increment $\Delta f(x)$ in $f(x)$. This $\Delta f(x)$ is called as the *first difference* of $f(x)$ and the interval h as the interval of differencing. In other words the *first difference* of a function $f(x)$ denoted by $\Delta f(x)$ and read as 'Delta f of x' is defined as

$$\Delta f(x) = f(x + h) - f(x) \qquad \ldots(1)$$

where h is a fixed constant. In fact it is the difference in values of the function for two neighbouring values of x, h units apart.

The *second difference* $\Delta^2 f(x)$ is defined as the difference of the first difference of $f(x)$ for two neighbouring values of x, h units apart and written as

$$\Delta^2 f(x) = \Delta[\Delta f(x)] = \Delta[f(x + h) - f(x)] = \Delta f(x + h) - \Delta f(x) \qquad \ldots(2)$$

Similarly the *third difference* $\Delta^3 f(x)$ is the difference of the second difference of $f(x)$ for two neighbouring values of x, h units apart *i.e.*

$$\Delta^3 f(x) = \Delta[\Delta^2 f(x)] \qquad \ldots(3)$$

and in general the *nth difference* is $\Delta^n f(x) = \Delta[\Delta^{n-1} f(x)]$, $n = 1, 2, \ldots$ ...(4)

In finding $\Delta f(x)$ by (1) it is clear that $f(x)$ is shifted to $f(x + h)$ and then $f(x)$ is subtracted from $f(x + h)$. The operation of shifting $f(x)$ to $f(x + h)$ is sometimes denoted by $Ef(x)$ *i.e.*,

$$f(x + h) = Ef(x) \qquad \ldots(5)$$

As such $f(x + 2h) = Ef(x + h) = E\{Ef(x)\} = E^2 f(x)$ ...(6)

$$f(x + 3h) = Ef(x + 2h) = E\{E^2 f(x)\} = E^3 (fx) \qquad \ldots(7)$$

and in general $f(x + nh) = E^n f(x)$ ...(8)

It should be noted that Δ^n and E^n do not bear n as an exponent power but they simply represent the n times operations.

Relation between Δ and E.

$$\because \quad \Delta f(x) = f(x + h) - f(x)$$

$$= Ef(x) - f(x)$$

or $\quad Ef(x) = \Delta f(x) + f(x) = (\Delta + 1) f(x)$,

so $\quad E \equiv \Delta + 1.$ ...(9)

i.e. operators E and Δ are symbolically connected with this identity.

Difference table. The successive differences $\Delta f(x)$, $\Delta^2 f(x) \ldots \Delta^r f(x)$ etc. of $f(x)$ for different values of x are easily calculated from a table, known as *difference table*, by subtraction. In any difference table the independent variable (say x) is called *argument* and the corresponding value of the function [as $f(x)$] is said to be the *entry*. A difference table can be constructed as shown on next page.

DIFFERENCE TABLE

Amount x	Entry $f(x)$	First difference $\Delta f(x)$	Second difference $\Delta^2 f(x)$	Third difference $\Delta^3 f(x)$	Fourth difference $\Delta^4 f(x)$
a	$f(a)$				
		$\Delta f(a) = f(a+h) - f(a)$			
$a+h$	$f(a+h)$		$\Delta^2 f(a)$		
		$\Delta f(a+h)$ $= f(a+2h) - f(a+h)$		$\Delta^3 f(a)$	
$a+2h$	$f(a+2h)$		$\Delta^2 f(a+h)$		$\Delta^4 f(a)$
		$\Delta f(a+2h)$ $= f(a+3h) - f(a+2h)$		$\Delta^3 f(a+h)$	
$a+3h$	$f(a+3h)$		$\Delta^2 f(a+2h)$		
		$\Delta f(a+3h)$			
$a+4h$	$f(a+4h)$				

Clearly each of the $\Delta f(x)$, $\Delta^2 f(x)$, or $\Delta^3 f(x)$ etc. is obtained by subtracting the corresponding value from its succeeding value *i.e.*

$$\Delta^r f(x+kh) = \Delta^{r-1} \Delta f(x+kh)$$
$$= \Delta^{r-1} \left\{ f(x+\overline{k+1}h) - f(x+kh) \right\}$$

or
$$\Delta^r f(x+kh) = \Delta^{r-1} f(x+\overline{k+1}h) - \Delta^{r-1} f(x+kh) \quad \ldots(10)$$

for every value of k and r.

Giving different values to k and r, we get different results of the difference table.

In difference table its first given term *i.e.* $f(a)$ is called the leading term and the differences $\Delta f(a)$, $\Delta^2 f(a)$, $\Delta^3 f(a)$ etc. are called leading differences of $f(a)$.

Difference of any order can be expressed in terms of the function alone.

(*i*) $\Delta f(a) = f(a+h) - f(a)$. ...(11)

(*ii*) $f(a) = \Delta\{\Delta f(a)\} = \Delta\{f(a+h) - f(a)\} = \Delta f(a+h) - \Delta f(a)$
$= [f(a+2h) - f(a+h)] - [f(a+h) - f(a)]$
$= f(a+2h) - 2f(a+h) + f(a)$. ...(12)

(*iii*) $\Delta^3 f(a) = \Delta\{(\Delta^2 f(a))\} = \Delta\{f(a+2h) - 2f(a+h) + f(a)\}$
$= \Delta f(a+2h) - 2\Delta f(a+h) + \Delta f(a)$
$= \{f(a+3h) - f(a+2h)\} - 2\{f(a+2h) - f(a+h)\} + \{f(a+h) - f(a)\}$.

or $\Delta^3 f(a) = f(a+3h) - 3f(a+2h) + 3f(a+h) - f(a)$ ...(13)

Properties of the operators Δ and E.

(*i*) If a is a constant $\Delta a = 0$ and $Ea = a$. ...(14)

(*ii*) $\Delta\{af(x) = a\,\Delta f(x)$ and $E\,\{af(x)\} = a\,Ef(x)$ ...(15)

[$\because$ $E\{af(x)\} = af(x+h) - aEf(x)$].

(*iii*) $\Delta^r \Delta^s f(x) = \Delta^{r+s} f(x)$ and $E^r E^s f(x) = E^{r+s} f(x)$ ...(16)

[$\because$ $\Delta^r \Delta^s f(x) = (\Delta\Delta\ldots r \text{ times})\ (\Delta\Delta\ldots s \text{ times})\ f(x)$
$= (\Delta\Delta\ldots r+s \text{ times}) f(x) = \Delta^{r+s} f(x)$].

(iv) $\Delta\{f(x) + \phi(x)\} = \Delta f(x) + \Delta\,\phi(x)$

and $$E\{f(x) + \phi(x)\} = Ef(x) + E\phi(x) \qquad \ldots(17)$$

$[\because \; E\{f(x) + \phi(x)\} = f(x+h) + \phi(x+h) = Ef(x) + E\phi(x)]$.

Formulation for *nth* difference

If we call $y_0, y_1, y_2, \ldots y_n$ the values of $f(x)$ corresponding to $f(x_0), f(x_0 + h), \ldots f(x_0 + nh)$ then by (1) we have

$$\Delta y_0 = y_1 - y_0 \qquad \ldots(18)$$

$$\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0$$

$$= (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0 \qquad \ldots(19)$$

$$\Delta^3 y_0 = \Delta(\Delta^2 y_0) = \Delta(y_2 - 2y_1 + y_0) = \Delta y_2 - 2\Delta y_1 + \Delta y_0$$

$$= (y_3 - y_2) - 2(y_2 - y_1) + (y_1 - y_0)$$

$$= y_3 - 3y_2 + 3y_1 - y_0 \qquad \ldots(20)$$

Similarly $\Delta^4 y_0 = y_4 - 4y_3 + 6y_2 - 4y_1 + y_0$ ...(21)

and so on.

Clearly, the coefficients in the expansion of $\Delta^n y_0$ will be the same as the coefficients in the expansion of $(x-1)^n$ *i.e.* $1, {}^nC_1, {}^nC_2, {}^nC_3, \ldots {}^nC_{n-1}, 1$ and therefore it will follow by the induction method that

$$D^n y_0 = \sum_{k=0}^{n} \frac{(-1)^k \lfloor n}{\lfloor k \; \lfloor n-k}\, y_{n-k} \text{ or } \sum_{k=0}^{n} (-1)^k \binom{n}{k} y_{n-k} \qquad \ldots(22)$$

where $\Delta^0 y_0 = y_0$ and in terms of $f(x)$ it becomes

$$D^n f(x_0) = \sum_{k=0}^{n} \frac{(-1)^k \lfloor n}{\lfloor k \; \lfloor n-k} f[(x_0 + (n-k)h] \qquad \ldots(23)$$

Problem 32. *Evaluate*

(a) $\Delta\, x(x+1)(x+2)(x+3)$

(b) $\Delta^n (ab^{cx})$ (c) $\Delta^2 \left[\dfrac{5x+12}{x^2+5x+6}\right]$

(d) $\Delta^n (e^{a+bx})$ (e) $\Delta^n (ax^n + bx^{n-1})$

(f) $\Delta^n \left(\dfrac{1}{x}\right)$ (g) $\Delta\,(x!)$

(a) $\Delta\, x(x+1)(x+2)(x+3)$

$$= (x+1)(x+2)(x-3)(x+4) - x(x+1)(x+2)(x+3)$$

$$= (x+1)(x+2)(x+3)\{(x+4) - x\}$$

$$= 4(x+1)(x+2)(x+3).$$

(b) $\Delta^n (ab^{cx}) - \Delta^{n-1} (\Delta ab^{cx})$.

Now $\Delta(ab^{cx}) = ab^{c(x+1)} - ab^{cx} = ab^{cx}\{b^c - 1\}$.

$\therefore \quad \Delta^2 (ab^{cx}) = \Delta(ab^{cx})(b^c - 1) = (b^c - 1)\,\Delta(ab^{cx})$

$$= (b^c - 1)^2 (ab^{cx}) \text{ and so on.}$$

$\therefore \quad \Delta^n (ab^{cx}) = (b^c - 1)^n (ab^{cx})$

(c) $$\Delta^2\left[\frac{5x+12}{x^2+5x+6}\right]=\Delta^2\left[\frac{2(x+3)+3(x+2)}{(x+2)(x+3)}\right]$$

$$=\Delta^2\left[\frac{2}{(x+2)}+\frac{3}{(x+3)}\right]$$

$$=\Delta\cdot\Delta\left[\frac{2}{(x+2)}+\frac{3}{(x+3)}\right].$$

Also $$\Delta\left[\frac{2}{(x+2)}+\frac{3}{(x+3)}\right]=\left[\frac{2}{(x+3)}-\frac{2}{(x+2)}\right]+\left[\frac{3}{(x+4)}-\frac{3}{(x+3)}\right]$$

$$=\text{either }\frac{-2}{(x+2)(x+3)}-\frac{3}{(x+4)(x+3)}\text{ or }\frac{3}{(x+4)}-\frac{1}{(x+3)}-\frac{2}{(x+2)} \qquad ...(1)$$

So that

$$\Delta^2\left[\frac{5x+12}{x^2+5x+6}\right]=\Delta\left[-\frac{2}{(x+2)(x+3)}-\frac{3}{(x+3)(x+4)}\right]$$

$$=-2\left[\frac{1}{(x+3)(x+4)}-\frac{1}{(x+2)(x+3)}\right]$$

$$-3\left[\frac{1}{(x+4)(x+5)}-\frac{1}{(x+3)(x+4)}\right]$$

$$=-2\left[-\frac{(x+2)-(x+4)}{(x+2)(x+3)(x+4)}\right]$$

$$-3\left[\frac{(x+3)-(x+5)}{(x+3)(x+4)(x+5)}\right]$$

or $$\Delta^2\left[\frac{5x+12}{x^2+5x+6}\right]=\frac{4}{(x+2)(x+3)(x+4)}+\frac{6}{(x+3)(x+4)(x+5)}$$

(d) $\Delta^n\left(e^{a+bx}\right)=\Delta^{n-1}\left(\Delta e^{a+bx}\right).$

$\because\ \Delta e^{a+bx}=e^{a+b(x+1)}-e^{a+bx}=e^{a+bx}\left\{e^b-1\right\}$

$\therefore\ \Delta^2 e^{a+bx}=(e^b-1)^2\,e^{a+bx}$ and so on.

Hence $\Delta^n\ e^{a+bx}=\left(e^b-1\right)^n e^{a+bx}.$

(e) $\Delta^n\left(ax^n+bx^{n-1}\right)=\Delta^n\left(ax^n\right)+\Delta^n\left(bx^{n-1}\right).$

But $\Delta^n\left(bx^{n-1}\right)=0$ (as bx^{n-1} is a polynomial of $(n-1)$ degree only)

$\therefore\ \Delta^n(ax^n)=a\,\Delta^{n-1}(\Delta x^n)=a\,\Delta^{n-1}\{(x+1)^n-x^n\}$

$=a\,\Delta^{n-1}\{\Delta x^{n-1}+$ terms with powers less than $(n-1)$ in $x\}$

$=a\,\Delta^{n-1}\,nx^{n-1}=a\,\Delta^{n-2}\cdot n\,\{(n-1)\,x^{n-2}+$ terms with powers less than $(n-2)$ in $x\}$

$=a\,\Delta^{n-3}\,n\,(n-1)\,(n-2)\,x^{n-3}$ and so on.

Ultimately, $\Delta^n(ax^n)=a\,(n\,!).$

(*f*) $\Delta^n \left(\frac{1}{x}\right) = \Delta^{n-1} \cdot \Delta \left(\frac{1}{x}\right)$.

$$\because \quad \Delta \left(\frac{1}{x}\right) = \frac{1}{x+1} - \frac{1}{x} = \frac{(-1)}{x(x+1)}$$

$$\therefore \quad \Delta^2 \left(\frac{1}{x}\right) = \frac{(-1)^2}{x(x+1)(x+2)} \text{ and so on.}$$

$$\Delta^n \left(\frac{1}{x}\right) = \frac{(-1)^n}{x(x+1)(x+2)\ldots(x+n)} - \frac{(-1)^n}{(x+1)^{(n+1)}}$$

$$= (-1)^n \left[(x+n) - (n+1)\right]^{(n+1)} = (-1)^n \left[x-1\right]^{(n+1)}$$

(*g*) $\Delta \lfloor(x) = \lfloor(x+1) - \lfloor x = \lfloor x \{(x+1) - 1\} = x \lfloor(x)$.

7.7. POLYNOMIAL INTERPOLATION

The process of estimating the value of the function for any intermediate value of the variable with the help of certain given values of the function corresponding to a number of variable values, is called *Interpolation.*

In other words if a function $y = f(x)$ is known for values $x_1, x_2, x_3, \ldots, x_n$ as $f(x_1)$, $f(x_2), \ldots, f(x_n)$, then the process of finding the value of $f(x)$ for some other value of x (lying between the values, $x_1, x_2, \ldots, x_n$) is called *Interpolation.*

When we estimate $f(x)$ for some such variable value which lies outside the given values, the process is called *extrapolation.*

If the exact form of the function $f(x)$ is not known, or, known in a complicated form, then to solve the problem of interpolation we assume that the function $f(x)$ can always be taken as a polynomial in x. Only after making this assumption the calculus of finite differences can be applied to obtain the value of function $f(x)$, for any unknown value of variable x. Moreover the polynomials are the simplest functions.

Validity of this assumption of replacing $f(x)$ by a polynomial function lies in an important theorem due to Weierstrass, which states that if $f(x)$ is continuous between x_1 and x_n, then it can be replaced by a Polynomial of suitable degree in that interval with as small an error as we please.

The degree of polynomial depends upon the number of given values of the function. In general if n values are given, we can fit a polynomial of degree $(n-1)$, no matter whatever be the exact degree of the polynomial.

This limitation of fitting a polynomial of degree $(n-1)$ only, when n values are known, creates two problems:

(*i*) If $f(x)$ is of a degree higher than $n-1$, then we shall interpolate the values of $f(x)$ only approximately.

(*ii*) If $f(x)$ is not actually of polynomial form but of a different form like exponential etc., then also the interpolated values will not be exactly the same. Such functions as exponential can be expanded upto infinity, but we can assume them only of degree $(n-1)$, if n values are given.

7.8. INTERPOLATION WITH EQUAL INTERVALS,

(*i.e.*, when function-values are given at equidistant intervals).

[1] Newton-Gregory Formula.

If n is a positive integer, then

$$f(x+nh) = f(x) + {}^nC_1\,\Delta^2 f(x) + {}^nC_2\,\Delta^2 f(x) + {}^nC_3\,\Delta^3 f(x) + \ldots + {}^nC_n\,\Delta^n f(x).$$

We have $f(x+nh) = E^n f(x) = (1+\Delta)^n f(x)$

$$= \{1 + {}^nC_1\Delta + {}^nC_2\,\Delta^2 + \ldots + {}^nC_n\Delta^n\}\, f(x)$$
$$= f(x) + {}^nC_1\Delta\, f(x) + {}^nC_2\Delta^2 f(x) + \ldots + {}^nC_n\, \Delta^n f(x) \qquad \ldots(1)$$

which is Newton-Gregory formula.

Here we get $f(x+nh)$ in terms of $f(x)$ and its leading differences.

Note 1. *This formula can be applied, when there are n equidistant terms, and*

(*i*) *we want to obtain an intermediate value.*

(*ii*) *out of those one value is missing which we want to obtain,*

(*iii*) *out of those r values are missing which we want to determine from the $(n-r)$ known values.*

Note 2. *The Gregory's expansion is convergent. But its nature of convergence may abolish if n becomes either a negative quantity or a fractional one, and the formula may take a divergent shape. So the results obtained will be very much approximate and sometimes inaccurate also.*

If the function remains a polynomial one, then Newton's advancing difference formula, where $f(x)$ is expressed in terms of $f(0)$ and its leading differences, can be used for all values of x, positive or negative, integral or fractional. Also this formula gives highly approximate results if $f(x)$ is of any other form than a polynomial.

[2] Newton's Advancing Difference Formula. By (8) of §7.6, taking $h=1$ we get,

$$f(x) = E^x f(0)$$
$$= (1+\Delta)^x f(0) = \{1 + {}^xC_1\,\Delta + {}^xC_2\,\Delta^2 + \ldots\}\, f(0)$$

or
$$f(x) = f(0) + {}^xC_1\,\Delta f(0) + {}^xC_2\,\Delta^2 f(0) + \ldots \qquad \ldots(2)$$

which is Newton's advancing difference formula.

So long as $f(x)$ is a polynomial in x, this binomial expansion is valid for all values of x if starting value of variable is 0 and interval of differencing is unity.

[This formula can also be deduced from the Gregory formula by putting $a = 0$ and $h = 1$].

Problem 33. *Show that Newton's formula*

$$u_x = u_0 + x_{(1)}\,\Delta u_0 + x_{(2)}\,\Delta^3 u_0 + x_{(3)}\,\Delta^3 u_0 + \ldots$$

can be put into the form

$$u_x = u_0 + x\Delta u_0 - xa\,\Delta^2 u_0 + xab\,\Delta^3 u_0 - xabc\,\Delta^4 u_0 + \ldots$$

where $a = 1 - \frac{1}{2}(x+1)$, $b = 1 - \frac{1}{3}(x+1)$, $c = 1 - \frac{1}{4}(x+1)$, *etc.*

Hence show that the successive coefficients converge slowly and tend eventually to numerical equality.

$\because \quad a = 1 - \frac{1}{2}(x+1)$, $\qquad \therefore \quad -a = \frac{1}{2}(x+1) - 1 = \frac{1}{2}(x-1)$

Also $b = 1 - \frac{1}{3}(x+1)$, $\qquad \therefore \quad -b = \frac{1}{3}(x+1) - 1 = \frac{1}{3}(x-2)$

Similarly $\qquad -c = \frac{1}{4}(x+1) - 1 = \frac{1}{4}(x-3)$

$\qquad -d = \frac{1}{5}(x+1) - 1 = \frac{1}{5}(x-4)$, etc.

Given
$$u_x = u_0 + \frac{x}{1}\,\Delta u_0 + \frac{x(x-1)}{1.2}\,\Delta^2 u_0 + \frac{x(x-1)(x-2)}{1.2.3}\,\Delta^3 u_0 + \frac{(x-1)(x-2)(x-3)}{2.3.4}\,\Delta^4 u_0 + \ldots$$

or
$$u_x = u_0 + x\Delta u_0 + x\left(\frac{x-1}{2}\right)\Delta^2 u_0 + x\left(\frac{x-1}{2}\right)\left(\frac{x-2}{3}\right)\Delta^3 u_0 + x\left(\frac{x-1}{2}\right)\left(\frac{x-2}{3}\right)\left(\frac{x-3}{4}\right)\Delta^4 u_0 + \ldots$$

$$= u_0 + x\Delta u_0 + (x-a)\Delta^2 u_0 + x(-a)(-b)\Delta^3 u_0 + x(-a)(-b)(-c)\Delta^4 u_0 + \ldots$$

or $u_x = u_0 + x\Delta u_0 - xa\Delta^2 u_0 + xab\Delta^3 u_0 - xabc\Delta^4 u_0 + xabcd\Delta^5 u_0 - \ldots$

It is clear from the values of a, b, c and d etc. that successive coefficients in the expression converge.

Problem 34. *Given the values sin 45° = .7071, sin 50° = .7660, sin 55° = ·8192, sin 60° = .8660, find sin 52°.*

θ	sin θ	Δ sin θ	Δ² sin θ	Δ² sin θ
45	·7071			
		·0589		
50	·7660		—·0057	
		·0532		—·0007
55	·8192		—·0068	
		·0468		
60	·8660			

$$\sin 52° = \sin(45 + 5 \times 1.4) = E^{1.4}\sin 45° = (1+\Delta)^{1.4}\sin 45°$$

$$= \sin 45° + 1.4\,\Delta\sin 45° + \frac{(1.4)(.4)}{2\,!}\Delta^2 \sin 45° + \frac{(1.4)(.4)(1.4-2)}{3\,!}\Delta^2 \sin 45°$$

$$= ·7071 + (1.4)(·0589) + (·7)(·4)(-·0057) + (·7)(·4)(-·2)(-·0007)$$

$$= ·7071 + ·08246 - ·001596 + 0.0000392$$

$$= ·7880032 = 0·7880 \text{ approx.}$$

Problem 35. *Given log x for x = 310, 320, 330, 340, 350 and 360 according to the following table. Find the value of log 3375.*

DIFFERENCE TABLE

x	$\log x$	$\Delta \log x$	$\Delta^2 \log x$	$\Delta^3 \log x$	$\Delta^4 \log x$	$\Delta^5 \log x$
310	2·4913617					
		.0137883				
320	2·5051500		–.0004244			
		.0133639		.0000255		
330	2.5185139		–.0003989		–.0000025	
		.0129642		.0000230		0
340	2.5314781		–.0003751		–.0000025	
		.0125891		.0000205		
350	2.5440680		–.0003546			
		.0122345				
360	2.5563025					

We shall first evaluate log 337.5.

$\because \quad \log 337.5 = \log(310 + 10 \times 2.75) = E^{27.5} \log 310$

$$= \log 310 + 2.75\,\Delta \log 310 + \frac{2.75\,(1.75)}{2\,!}\Delta^2 \log 310$$

$$+ \frac{(2.75)\,(1.75)\,(.75)}{3\,!}\Delta^3 \log 310$$

$$+ \frac{(2.75)\,(1.75)\,(.75)\,(-.25)}{4\,!}\Delta^4 \log 310$$

(taking approximation upto five places)

$$\log 337.5 = 2.49136 + 2.75(.01379) + \frac{2.75\,(1.75)}{2}(-.00042)$$

$$+ \frac{(2.75)\,(1.75)\,(.75)}{6}(.00003) \text{ \{leaving other terms\}}$$

$$= 2.49136 + 2.75(.01379) - (2.75)(1.75)(.00021) + (1.375)(1.75)(.75)(.00003)$$

$$= 2.49136 + .03792 - .00101 + .00005 = 2.52832$$

Now log 3375 = log 10 × 337.5 = log 10 + log 337.5 = 1 + 2.52832.

i.e., log 3375 = 3.52832.

Problem 36. *The following table is given :*

x	0	1	2	3	4
$f(x)$	3	6	11	18	27

What is the form of the function f(x) ?

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	3			
		3		
1	6		2	
		5		0
2	11		2	
		7		0
3	18		2	
		9		
4	27			

$\because f(x) = f(0) + {}^xC_1\,\Delta f(0) + {}^xC_2\,\Delta^2 f(0)$ {as third differences are zero}

$$= 3 + x \cdot 3 + \frac{x(x-1)}{\lfloor 2} \cdot 2$$

or $f(x) = 3 + 3x + x(x-1) = 3 + 3x + x^2 - x$

or $f(x) = x^2 + 2x + 3.$

Problem 37. *Given* $f(0) = 1, f(1) + f(2) = 10$ *and* $f(3) + f(4) + f(5) = 65$; *find* $f(4)$.

Since we are given three relations, so we can assume $f(x)$ a polynomial of second degree.

Suppose $\quad f(x) = A + Bx + Cx^2.$

$\therefore \quad f(0) = 1$ gives $\quad A = 1.$ ...(1)

and $f(1) + f(2) = 10$ gives $(A + B + C) + (A + 2B + 4C) = 10$

or $\quad 2A + 3B + 5C = 10$

or $\quad 3B + 5C = 8$ ...(2)

and $f(3) + f(4) + f(5) = 65$ gives

$$(A + 3B + 9C) + (A + 4B + 16C) + (A + 5B + 25C) = 65$$

or $\quad 12B + 50C = 65 - 3 = 62.$ ...(3)

(2) and (3) give $C = 1$ and $B = 1$.

$\therefore \quad f(x) = x^2 + x + 1.$

Hence $f(4) = 21$.

[3] Equidistant Terms with Terms Missing.

In general if p out of n equidistant values of the functions are unknown, *i.e.*, only $(n - p)$ function-values are known, then we shall assume the function of degree $(n - p - 1)$. Thus $(n - p)$th difference of the function will be zero for every value of the variable. Equating these differences to zero, we may have as many equations as we like (for finding out missing terms) by giving different values to x.

Problem 38. *Given* $f(1) = 386, f(3) = 530, f(5) = 810$, *find* $f(2)$ *and* $f(4)$.

Since only three values of the function are known, so $f(x)$ may be assumed of second degree. Thus $\Delta^3 f(x) = 0$ for every value of x.

$\Delta^3 f(1) = 0$ gives $(E - 1)^3 f(1) = 0$

or $\quad f(4) - 3f(3) + 3f(2) - f(1) = 0$

or $\quad f(4) + 3f(2) = 3f(3) + f(1) = 1976.$ ...(1)

$\Delta^3 f(2) = 0$ gives $f(5) - 3f(4) + 3f(3) - f(2) = 0$

or $\quad 3f(4) + f(2) = f(5) + 3f(3) = 810 + 1590$

$\quad = 2400.$ ...(2)

Solving (1) and (2), we get

$$f(2) = 441, \; f(4) = 653.$$

Problem 39. *Estimate the missing terms in the following table where* $f(x) = 3^x$.

x	1	2	3	4	5	6
$f(x)$	3	9	—	81	—	729

Explain why $f(3)$ *differs from* 3^3 *and* $f(5)$ *from* 3^5.

As only four values are given so $\Delta^4 f(x) = 0$ for every x.

As above, $\Delta^4 f(1)$ gives

$$f(5) - 4f(4) + 6f(3) - 4f(2) + f(1) = 0 \quad ...(1)$$

and $\Delta^4 f(2)$ gives

$$f(6) - 4f(5) + 6f(4) - 4f(3) + f(2) = 0 \quad ...(2)$$

Substituting the given values in (1) and (2), we get

$$f(5) + 6f(3) = 357 \quad (3)$$

$$f(5)+f(3)=306 \qquad \ldots(4)$$

Solving (3) and (4), $f(3) = 10.2$ and $f(5) = 295.8$.

When estimating $f(3)$ and $f(5)$, we have assumed here that the function $f(x)$ is a polynomial of degree 3, whereas it is an exponential function 3^x. That is why $f(3)$ and $f(5)$ are differing from 3^3 and 3^5, *i.e.* from 27 and 243 respectively.

7.9. INTERPOLATION WITH UNEQUAL INTERVALS

In §7.8 we have assumed that values of $f(x)$ are given at equidistant values of x, say for a, $a+h$, $a+2h$ etc, but it may happen that the values of $f(x)$ are not given at equal distances but are given at points say $x_0, x_1, x_2, x_3, \ldots, x_n$, where (x_1-x_0), (x_2-x_1), (x_3-x_2), ...,(x_n-x_{n-1}) are not necessarily equal. We define divided differences to explain the interpolation formulae in such cases.

[1] Divided Differences

Suppose $f(x_0), f(x_1), \ldots f(x_n)$ are the values of $f(x)$ corresponding to values $x_0, x_1, \ldots, x_n$ which are not necessarily equally spaced. Then

(*i*) $\dfrac{f(x_1)-f(x_0)}{x_1-x_0}, \dfrac{f(x_2)-f(x_1)}{x_2-x_1}, \ldots \dfrac{f(x_n)-f(x_{n-1})}{x_n-x_{n-1}}$ etc. are called *first divided differences* of x for the variable-values (x_1, x_0) $(x_2, x_1), \ldots, (x_n, x_{n-1})$ respectively. These differences may be denoted as $f(x_1, x_0), f(x_2, x_1), \ldots, f(x_n, x_{n-1})$.

(*ii*) Similarly *second divided differences* are

$$\frac{f(x_2,\ x_1)-f(x_1,\ x_0)}{x_2-x_0},\ \frac{f(x_3,\ x_2)-f(x_2,\ x_1)}{x_3-x_1} \text{ etc.}$$

denoted by $f(x_2, x_1, x_0), f(x_3, x_2, x_1)$ etc.

(*iii*) And similarly *third divided differences* are

$$\frac{f(x_3,\ x_2,\ x_1)-f(x_2,\ x_1,\ x_0)}{x_3-x_0}$$

denoted by $f(x_3, x_2, x_1, x_0)$ etc.

[2] The Divided Differences remain Uaffected by Changing the Permutations of their Variable Values.

$$\because f(x_1,\ x_0)=\frac{f(x_1)-f(x_0)}{x_1-x_0}=\frac{f(x_1)}{(x_1-x_0)}+\frac{f(x_0)}{(x_0-x_1)}$$

$$=\frac{f(x_0)-f(x_1)}{x_0-x_1}=f(x_0,\ x_1).$$

Similarly $f(x_2,\ x_1,\ x_0)=\dfrac{f(x_2,\ x_1)-f(x_1,\ x_0)}{(x_2-x_0)}$

$$=\frac{\left\{\dfrac{f(x_2)}{(x_2-x_1)}+\dfrac{f(x_1)}{(x_1-x_2)}\right\}-\left\{\dfrac{f(x_1)}{(x_1-x_0)}+\dfrac{f(x_0)}{(x_0-x_1)}\right\}}{x_2-x_0}$$

$$=\frac{f(x_2)}{(x_2-x_0)(x_2-x_1)}+\frac{f(x_1)}{(x_1-x_0)(x_1-x_2)}+\frac{f(x_0)}{(x_0-x_1)(x_0-x_2)}$$

$= f(x_0, x_1, x_2) = f(x_1, x_0, x_2)$ etc.

The nth divided difference is sometimes written as:

$$\underset{x_1, x_2, \ldots x_n}{\overset{n}{\Delta}} f(x_0) = f(x_0, x_1, \ldots x_n)$$

$$= \sum \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2) + \ldots (x_0 - x_n)}$$

Σ is the summation of all x's.

[3] The nth divided differences of a polynomial of the nth degree are also constant. If $a_0 x^n$ is the leading term of the polynomial then nth divided difference will be a_0.

Problem 40. *Find $f(2, 4, 9, 10)$ where $f(x) = x^4 + x^2 + 1$.*

$\therefore \underset{4, 9, 10}{\Delta} \Delta^3 f(2)$ or $f(2, 4, 9, 10) = 25$. [See Table below]

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
2	21			
		$\frac{273-21}{4-2} = 126$		
			$\frac{1274-126}{9-2} = 164$	
4	273			
		$\frac{6643-273}{9-4} = 1274$	$= \underset{4, 9}{\Delta^2} f(2)$	$\frac{364-164}{10-2} = 25$
9	6643		$\frac{3458-1274}{10-4} = 364$	$= \underset{4, 9, 10}{\Delta^2} f(2)$
		$\frac{10101-6643}{10-9} = 3458$	$= \underset{9, 10}{\Delta^2} f(4)$	
10	10101			

[4] Newton's Divided Difference Formula

Suppose $f(x_0), f(x_1), \ldots f(x_n)$ are the $(n + 1)$ values corresponding to values $x_0, x_1 \ldots x_n$. With these $(n + 1)$ values we can fit a polynomial of degree n. Thus nth divided difference $f(x_0, x_1, x_2, \ldots x_n)$ = constant and $(n + 1)$th divided difference $f(x, x_0, x_1, \ldots x_n) = 0$, where x is some value other than $x_0, x_1, \ldots, x_n$.

$$\because f(xx_0, x_1, \ldots x_n) = \frac{f(xx_0, x_1, \ldots x_{n-1}) - f(x_0, x_1, \ldots x_n)}{x - x_n} = 0$$

$$\therefore f(xx_0, x_1, \ldots x_{n-1}) = f(x_0, x_1, \ldots x_n)$$

or $$\frac{f(xx_0, x_1, \ldots x_{n-2}) - f(x_0, x_1, \ldots x_{n-1})}{x - x_{n-1}} = f(x_0, x_1, \ldots x_n)$$

or $$f(xx_0, x_1, \ldots x_{n-2}) = f(x_0, x_1, \ldots x_{n-1}) + (x - x_{n-1}) f(x_0, x_1, \ldots x_n)$$

$$= \frac{f(xx_0 \ldots x_{n-3}) - f(x_0, x_1, \ldots x_{n-2})}{x - x_{n-2}}$$

$$f(xx_0, x_1, \ldots x_{n-3}) = f(x_0, x_1, \ldots x_{n-2}) + (x - x_{n-2}) f(x_0, x_1, \ldots x_{n-1}) + (x - x_{n-2})(x - x_{n-1}) f(x_0, x_1, \ldots x_n)$$

$$= \frac{f(xx_0 \ldots x_{n-4}) - f(x_0, x_1 \ldots x_n)}{x - x_{n-3}}$$

or $$f(xx_0, x_1, \ldots x_{n-4}) = f(x_0, x_1, \ldots x_{n-3}) + (x - x_{n-3}) f(x_0, x_1, \ldots x_{n-2}) + (x - x_{n-3})(x - x_{n-2}) f(x_0, x_1, \ldots x_{n-1}) + (x - x_{n-3})(x - x_{n-2}) \times (x - x_{n-1}) f(x_0, x_1, \ldots x_n),$$

and so on.

The last but one step will be

$$f(x, x_0) = f(x_0, x_1) + (x - x_1) f(x_0, x_1, x_2) + (x - x_1)(x - x_2) f(x_0, x_2, x_3) + \ldots + (x - x_1)(x - x_2) \ldots (x - x_{n-1}) f(x_0, x_1, \ldots, x_n)$$

$$= \frac{f(x) - f(x_0)}{x - x_0}$$

or $$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + \ldots + (x - x_0) \ldots (x - x_{n-1}) f(x_0 \ldots x_n)$$

or $$f(x) = f(x_0) + (x - x_0) \underset{x_1}{\Delta} f(x_0) + (x - x_0)(x - x_1) \underset{x_1, x_2}{\Delta^2} f(x_0) + \ldots + \ldots + (x - x_0)(x - x_1) \ldots (x - x_{n-2}) \underset{x_1, x_2, \ldots x_{n-1}}{\Delta^{n-1}} f(x_0) + (x - x_0)(x - x_1) \ldots (x - x_{n-1}) \underset{x_1, x_2, \ldots x_n}{\Delta^n} f(x_0)$$

which is Newton's divided difference formula.

[5] Newton's Gregory Formula as a Particular Case of Newton's divided Difference Formula.

In the above Newton's divided difference formula, if we put $x = x_0 + nh$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, $x_3 = x_0 + 3h$ etc., we get the various terms as under:

$$(x - x_0) f(x_0, x_1) = nh \frac{f(x_1) - f(x_0)}{x_1 - x_0} = nh \frac{f(x_0 + h) - f(x_0)}{h}$$

$$= nh \frac{\Delta f(x_0)}{h} = n \Delta f(x_0) = {}^nC_1 \Delta f(x_0)$$

$$(x - x_0)(x - x_1) f(x_0, x_1, x_2) = (nh) (\overline{n - 1} h) \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

$$= (nh)(\overline{n - 1} h) \frac{\dfrac{\Delta f(x_1) - \Delta f(x_0)}{h}}{2h}$$

$$= \frac{n(n-1)}{\underline{|2}} \Delta \{ f(x_0 + h) - f(x_0) \}$$

$$= \frac{n(n-1)}{\underline{|2}} \Delta^2 f(x_0) = {}^nC_2 \Delta^2 f(x_0).$$

Similarly,

$$(x - x_0)(x - x_1)(x_1 - x_2) f(x_0, x_1, x_2, x_3) = \frac{n(n-1)(n-2)}{(\underline{|3})} \Delta^3 f(x_0),$$

and finally, $(x - x_0)(x - x_1)\ldots(x - x_{n-1}) f(x_0, x_1, x_2, \ldots, x_n)$

$$= \frac{n(n-1)(n-2)\ldots(n-\overline{n-1})}{(\underline{|n})} \Delta^n f(x_0)$$

$$= {}^nC_n \Delta^n f(x_0).$$

Substituting these values in the Newton's divided difference formula, we get

$$f(x_0 + nh) = f(x_0) + {}^nC_1 \Delta f(x_0) + {}^nC_2 \Delta^2 f(x_0) + \ldots + {}^nC_n \Delta^n f(x_0).$$

Problem 41. *Find the polynomial of the lowest possible degree which assumes the values 1245, 33, 5, 9 and 1335 when x has the values –4, –1, 0, 2, and 5 respectively. Also obtain the value of the polynomial at the abscissa 1.*

We have to find a polynomial satisfied by (–4, 1245), (–1, 33), (0, 5), (2, 9) and (5, 1335) and the value for '1' is to be interpolated. The difference table is given here.

$$\therefore \quad f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3)$$
$$+ (x - x_0)(x - x_1)(x - x_2)(x - x_3) f(x_0, x_1, x_2, x_3, x_4)$$
$$= 1245 - (x + 4)\, 404 + (x + 4)(x + 1)\, 94 - (x + 4)(x + 1)\, x.14$$
$$+ (x + 4)(x + 1)\, x\, (x - 2)\, 3$$

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
–4	1245				
		$\frac{1245-33}{-4-(-1)} = -404$			
–1	33		$\frac{-404-(-28)}{-4-0} = 94$		
		$\frac{33-5}{-1-(0)} = -28$		$\frac{94-10}{-4-2} = -14$	
0	5		$\frac{-28-2}{-1-2} = 10$		$\frac{-14-13}{-4-5} = 3$
		$\frac{5-9}{0-2} = 2$		$\frac{10-88}{-1-5} = 13$	
2	9		$\frac{2-442}{0-5} = 80$		
		$\frac{9-1335}{2-5} = 442$			
5	1335				

$$f(x) = 5 - 14x + 6x^2 - 5x^3 + 3x^4.$$

when $x = 1, f(x) = 5 - 14 + 6 - 5 + 3 = -5$

[6] Lagrange's Formula.

Suppose $(n+1)$ values of $f(x)$ are known as $f(x_0), f(x_1), \ldots, f(x_n)$ at points $x_0, x_1, \ldots, x_n$, which are not necessarily equally spaced. Thus $f(x)$ can be taken as a polynomial of degree n and one of the possible forms of $f(x)$ can be taken us under

$$f(x) = A_0(x - x_1)(x - x_2)\ldots(x - x_n) + A_1(x - x_0)(x - x_2)\ldots(x - x_n)$$
$$+ A_2(x - x_0)(x - x_1)(x - x_3)\ldots(x - x_n) + \ldots + A_n(x - x_0)\ldots(x - x_{n-1}).$$

At the point $x = x_0, f(x) = f(x_0)$. Putting $x = x_0$ and $f(x) = f(x_0)$, we get

$$A_0 = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)}.$$

Also at the point $x = x_1, f(x) = f(x_1)$, this substitution gives us

$$A_1 = \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)}.$$

Similarly,
$$A_2 = \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)\ldots(x_2 - x_n)}.$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$A_n = \frac{f(x_n)}{(x_n - x_0)(x_n - x_1)\ldots(x_n - x_{n-1})}.$$

With these values, we get

$$f(x) = \frac{(x - x_1)(x - x_2)\ldots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)} f(x_0) + \frac{(x - x_0)(x - x_2)\ldots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)} f(x_1) + \ldots$$
$$+ \frac{(x - x_0)(x - x_1)(x - x_2)\ldots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)(x_n - x_2)\ldots(x_n - x_{n-1})} f(x_n).$$

which is **Lagrange's Formula.**

Problem 42. *By means of Lagrange's formula, prove that*

(*i*) $y_1 = y_3 - .3\,(y_5 - y_{-3}) + .2\,(y_3 - y_{-5})$ *approximately.*

(*ii*) $y_0 = \frac{1}{2}(y_1 + y_{-1}) - \frac{1}{8}\left[\frac{1}{2}(y_3 - y_1) - \frac{1}{2}(y_{-1} - y_{-3})\right]$ *approximately.*

(*i*) Here we are given y_{-5}, y_{-3}, y_3 and y_5 and y_1 is to be determined.

$$\therefore\ y_1 = \frac{(1+3)(1-3)(1-5)}{(-5+3)(-5-3)(-5-5)} y_{-5} + \frac{(1+5)(1-3)(1-5)}{(-3+5)(-3-3)(-3-5)} y_{-3}$$
$$+ \frac{(1+5)(1+3)(1-5)}{(3+5)(3+3)(3-5)} y_3 + \frac{(1+5)(1+3)(1-3)}{(5+5)(5+3)(5-3)} y_5$$

or $$y_1 = \frac{4(-2)(-4)}{(-2)(-8)(-10)} y_{-5} + \frac{(6)(-2)(-4)}{(2)(-6)(-8)} y_{-3} + \frac{(6)(4)(-4)}{(8)(6)(-2)} y_3 + \frac{(6)(4)(-2)}{(10)(8)(2)} y_5$$

or $$y_1 = -\tfrac{2}{10} y_{-5} + \tfrac{1}{2} y_{-3} + y_3 - \tfrac{3}{10} y_5 - .2\,y_{-5} + .5\,y_{-3} - .3\,y_5$$
$$= y_3 - .3(y_5 - y_{-3}) + .2(y_{-3} - y_{-5}).$$

(*ii*) Here we are given the values of y_{-3}, y_{-1}, y_1 and y_3 and y_0 is to be determined.

$$y_0 = \frac{(0+1)(0-1)(0-3)}{(-3+1)(-3-1)(-3-3)} y_{-3} + \frac{(0+3)(0-1)(0-3)}{(-1+3)(-1-1)(-1-3)} y_{-1} + \frac{(+3)(+1)(-3)}{(1+3)(1+1)(1-3)} y_1 + \frac{(+3)(+1)(-1)}{(3+3)(3+1)(3-1)} y_3$$

or $$y_0 = -\tfrac{3}{48} y_{-3} + \tfrac{9}{16} y_{-1} + \tfrac{9}{16} y_1 - \tfrac{3}{48} y_3$$

$$= \tfrac{8}{16}(y_1 + y_{-1}) - \tfrac{1}{16}\left[(y_3 - y_1) - (y_{-1} - y_{-3})\right]$$

or $$y_0 = \tfrac{1}{2}(y_1 + y_{-1}) - \tfrac{1}{8}\left[\tfrac{1}{2}(y_3 - y_1) - \tfrac{1}{2}(y_{-1} - y_{-3})\right].$$

7.10. CENTRAL DIFFERENCE FORMULAE

When applying Newton's Gregory formula with $(n + 1)$ known values of the variable, we assumed the function as nth degree polynomial, for finding out any intermediate value, but actually the polynomial happens to be different from nth degree, so Newton's formula gives an approximate result only. For better results central difference formulae are introduced.

Suppose function $f(x)$ takes the values $f(0), f(1), f(2), f(3), f(4), f(5), f(6), f(7), \ldots,$ equally spaced having unit interval.

If we shift our origin to the 4th point then $f(x)$ will become $f(-3), f(-2), f(-1), f(0), f(1), f(2), f(3) \ldots.$

[1] Gauss' Forward Formula.

Newton's advancing difference formula (in which interval is unity and zero is the starting value of the variable) is

$$f(x) = f(0) + {}^xC_1 \Delta f(0) + {}^xC_2 \Delta^2 f(0) + {}^xC_3 \Delta^3 f(0) + \ldots + {}^xC_r \Delta^r f(0)$$

as $\Delta^2 f(0) = \Delta^3 f(-1) + \Delta^2 f(-1)$ $[\because \Delta^r f(-1) = \Delta^{r-1} \{f(0) - f(-1)\}]$

and $\Delta^3 f(0) = \Delta^4 f(-1) + \Delta^3 f(-1).$

Similarly, $\Delta^4 f(0) = \Delta^5 f(-1) + \Delta^4 f(-1) + \Delta^3 f(-1)$

$$= \Delta^4 f(-2) + \Delta^3 f(-2),$$

$$\Delta^4 f(-1) = \Delta^5 f(-2) + \Delta^4 f(-2) \text{ etc.}$$

In general $\Delta^r f(p) = \Delta^{r+1} f(p-1) + \Delta^r (p-1)$, for every integral values of r and p.

Substituting these transformations in advancing difference formula we get

$$f(x) = f(0) + {}^xC_1 \Delta f(0) + {}^xC_2 \Delta^2 f(0) + {}^xC_3 \Delta^3 f(0) + {}^xC_4 \Delta^4 f(0) + {}^xC_5 \Delta^5 f(0) + \ldots$$

$$= f(0) + {}^xC_1 \Delta f(0) + {}^xC_2 \{\Delta^2 f(-1) + \Delta^3 f(-1)\} + {}^xC_3 \{\Delta^3 f(-1) + \Delta^4 f(-1)\}, + {}^xC_4 \{\Delta^4 f(-1) + \Delta^5 f(-1)\} + {}^xC_6 \{\Delta^5 f(-1) + \Delta^6 f(-1)\}\} \text{ and so on}$$

or $$f(x) = f(0) + {}^xC_1 \Delta f(0) + {}^xC_2 \Delta^2 f(-1) + ({}^xC_2 + {}^xC_3) \Delta^3 f(-1) + ({}^xC_3 + {}^xC_4) \Delta^4 f(-1) + ({}^xC_4 + {}^xC_5) \Delta^5 f(-1) + ({}^xC_5 + {}^xC_6) \Delta^6 f(-1) \text{ and so on}$$

or $$f(x) = f(0) + {}^xC_1 \Delta f(0) + {}^xC_2 \Delta^2 f(-1) + {}^{x+1}C_3 \Delta^3 f(-1) + {}^{x+1}C_4 \Delta^4 f(-1) + {}^{x+1}C_5 \Delta^5 f(-1) + {}^{x+1}C_6 \Delta^6 f(-1) + \ldots \text{and so on,}$$

(Now we have to transform Δ's of $f(-1)$ into Δ's of $f(-2)$ after 4th term, and then Δ's of $f(-2)$ into Δ's of $f(-3)$ after 6th term and then after 8th term Δ's of $f(-3)$ into Δ's of $f(-4)$ and so on).

$$\therefore f(x) = f(0) + {}^{x}C_1\,\Delta f(0) + {}^{x}C_2\,\Delta^2 f(-1) + {}^{x+1}C_3\Delta^3 f(-1) + {}^{x+1}C_4\,\{\Delta^4 f(-2) + \Delta^5 f(-2)\} + {}^{x+1}C_5\,\{\Delta^5 f(-2 + \Delta^6 f(-2)\} + {}^{x+1}C_6\,\{\Delta^6 f(-2) + \Delta^7 f(-2)\} + \ldots$$

or $$f(x) = f(0) + {}^{x}C_1\,\Delta f(0) + {}^{x}C_2\Delta^2 f(-1) + {}^{x+1}C_3\Delta^3 f(-1) + {}^{x+1}C_4\Delta^4 f(-2) + \{{}^{x+1}C_4 + {}^{x+1}C_5\}\,\Delta^5 f(-2) + \{{}^{x+1}C_5 + {}^{x+1}C_6\}\,\Delta^6 f(-2) + \{{}^{x+1}C_6 + {}^{x+1}C_7\}\,\Delta^7 f(-2) + \ldots$$

or $$f(x) = f(0) + {}^{x}C_1\,\Delta f(0) + {}^{x}C_2\Delta^2 f(-1) + {}^{x+1}C_3\Delta^3 f(-1) + {}^{x+1}C_4\Delta^4 f(-2) + {}^{x+2}C_5\Delta^5 f(-2) + {}^{x+2}C_6\Delta^6 f(-2) + {}^{x+2}C_7\Delta^7 f(-2) + \ldots$$

or $$f(x) = f(0) + {}^{x}C_1\Delta f(0) + {}^{x}C_2\Delta^2 f(-1) + {}^{x+1}C_3\Delta^3 f(-1) + {}^{x+1}C_4\Delta^4 f(-2) + {}^{x+2}C_5C_5\Delta^5 f(-2) + {}^{x+2}C_6\Delta^6 f(-3) + {}^{x+3}C_7\Delta^7 f(+3) + \ldots$$

$$\{\because \Delta^6 f(-2) = \Delta^7 f(-3) + \Delta^6 f(-3)\}$$

which is **Gauss' forward formula.**

If origin is not 0 and interval is not unity then, in general,

$$f(a + xh) = f(a) + {}^{x}C_1\Delta f(a) + {}^{x}C_2\Delta^2 f(a-h) + {}^{x+1}C_3\Delta^3 f(a-h) + {}^{x+1}C_4\Delta^4 f(a-2h) + {}^{x+2}C_5\Delta^5 f(a-2h) + \ldots$$

This formula contains the odd differences just below the central line through $f(0)$ and even differences on the central line. The following table will make it clear:

[2] A central difference table can be prepared as follows:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$	$\Delta^6 f(x)$
–3	$f(-3)$						
		$\Delta f(-3)$					
–2	$f(-2)$		$\Delta^2 f(-3)$				
		$\Delta f(-2)$		$\Delta^3 f(-3)$			
–1	$f(-1)$		$\Delta^2 f(-2)$		$\Delta^4 f(-3)$		
		$\Delta f(-1)$		$\Delta^3 f(-2)$		$\Delta^5 f(-3)$	
0	$f(0)$		$\Delta^2 f(-1)$		$\Delta^4 f(-2)$		$\Delta^6 f(-3)$
		$\Delta f(0)$		$\Delta^4 f(-1)$		$\Delta^5 f(-2)$	
1	$f(1)$		$\Delta^2 f(0)$		$\Delta^4 f(-1)$		$\Delta^6 f(-2)$
		$\Delta f(1)$		$\Delta^3 f(0)$		$\Delta^5 f(-1)$	
2	$f(2)$		$\Delta^2 f(1)$		$\Delta^4 f(0)$		
		$\Delta f(2)$		$\Delta^3 f(1)$			
3	$f(3)$		$\Delta^2 f(2)$				
		$\Delta f(3)$					
4	$f(4)$						

[3] Gauss' Backward Formula.

Here Newton's advancing difference formula is so transformed as to have a formula involving odd differences just about the central line through $f(0)$ and even differences as before *i.e.* on the line.

$\therefore\ f(x) = f(0) + {}^xC_1\Delta f(0) + {}^xC_2\Delta^2 f(0) + {}^xC_3\Delta^3 f(0) + {}^xC_4\Delta^4 f(0) + \ldots$

$= f(0) + {}^xC_1\{\Delta f(-1) + \Delta^2 f(-1)\} + {}^xC_2\{\Delta^2 f(-1) + \Delta^3 f(-1)\} + {}^xC_3\{\Delta^3 f(-1) + \Delta^4 f(-1)\} + \ldots$

$= f(0) + {}^xC_1\Delta f(-1) + ({}^xC_1 + {}^xC_2)\Delta^2 f(-1) + ({}^xC_2 + {}^xC_3)\Delta^3 f(-1) + ({}^xC_3 + {}^xC_4)\Delta^4 f(-1) + ({}^xC_4 + {}^xC_5)\Delta^5 f(-1) + \ldots$

or $f(x) = f(0) + {}^xC_1\Delta f(-1) + {}^{x+1}C_2\Delta^2 f(-1) + {}^{x+1}C_3\Delta^3 f(-1) + {}^{x+1}C_4\Delta^4 f(-1) + {}^{x+1}C_5\Delta^5 f(-1) + \ldots$

or $f(x) = f(0)\ {}^xC_1\Delta f(-1) + {}^{x+1}C_2\Delta^2 f(-1) + {}^{x+1}C_3\{\Delta^3 f(-2) + \Delta^4 f(-1)\} + {}^{x+1}C_4\{\Delta^4 f(-2) + \Delta^5 f(-2)\} + \ldots$

or $f(x) = f(0) + {}^xC_1\Delta f(-1) + {}^{x+1}C_2\Delta^2 f(-1) + {}^{x+1}C_3\Delta^3 f(-2) + {}^{x+2}C_4\Delta^4 f(-2)$.

Thus the backward formula is

$$f(x) = f(0) + {}^xC_1\Delta f(-1) + {}^{x+1}C_2\Delta^2 f(-1) + {}^{x+1}C_3\Delta^3 f(-2) + {}^{x+2}C_4\Delta^4 f(-2) + {}^{x+2}C_5\Delta^5 f(-3) + \ldots$$

In case origin is a and interval of differencing not unity, then $f(a - nh) = f\{a + n(-h)\}$ (taking h negative)

$\therefore\ \Delta f(a) = f(a + h) - f(a) = f(a - h) - f(a) = -\Delta f(a - h)$.

$\Delta^2 f(a - h) = f(a + h) - 2f(a) + f(a - h)$

$= f(a - h) - 2f(a) + f(a + h) = \Delta^2 f(a - h)$.

Thus even differences remain as such whereas odd ones change their signs.

$$\therefore\ f(a - nh) = f(a) - {}^nC_1\Delta f(a - h) + {}^nC_2\Delta^2 f(a - h) - {}^{(n+1)}C_3\Delta^3 f(a - 2h) + {}^{(n+1)}C_4\Delta^4 f(a - 2h) - {}^{(n+1)}C_5\Delta^5 f(a - 3h) + \ldots$$

[4] Newton's Stirling Formula.

This formula is obtained directly from forward formulae.

$$\because\ f(a + nh) = f(a) + {}^nC_1\Delta f(a) + {}^nC_2\Delta^2 f(a - h) + {}^{n+1}C_3\Delta^3 f(a - h) + {}^{n+1}C_4\Delta^4 f(a - 2h) + {}^{n+2}C_5\Delta^5 f(a - 2h) + {}^{n+2}C_6\Delta^6 f(a - 3h) + \ldots$$

$$= f(a) + n\Delta f(a) + \frac{n^2 - n}{2}\Delta^2 f(a - h) + \frac{n(n^2 - 1^2)}{\underline{|3}}\Delta^2 f(a - h) + \frac{(n+1)n(n-1)(n-2)}{\underline{|4}}\Delta^4 f(a - 2h) + \frac{(n^2 - 2^2)(n^2 - 1^2)n}{\underline{|5}}\Delta^5 f(a - 2h) + \ldots$$

$$= f(a) + n\frac{2\Delta f(a) - \Delta^2 f(a-h)}{2} + \frac{n^2}{2}\Delta^5 f(a - h) + \frac{n(n^2 - 1^2)}{\underline{|3}}\ \frac{2\Delta^3 f(a-h) - \Delta^4 f(a-2h)}{2} + \frac{n^2(n^2 - 1^2)}{\underline{|4}}\Delta^4 f(a - 2h)$$

$$+ \frac{n(n^2 - 1^2)(n^2 - 2^2)}{\underline{|5}}\ \frac{2\Delta^5 f(a-2h) - \Delta^6 f(a-3h)}{2} + \frac{n^2(n^2 - 1^2)(n^2 - 2^2)}{\underline{|6}}\Delta^6 f(a - 3h) + \ldots$$

$$\therefore f(a+nh) = f(a) + n\,\frac{\Delta f(a) + \Delta f(a-h)}{2} + \frac{n^2}{2}\,\Delta^2 f(a-h)$$

$$+\frac{n\left(n^2-1^2\right)}{\lfloor 3}\,\frac{\Delta^3 f(a-h) + \Delta^3 f(a-2h)}{2} + \frac{n^2\left(n^2-1^2\right)}{\lfloor 4}\,\Delta^4 f(a-2h)$$

$$+\frac{n\left(n^2-1^2\right)\left(n^2-2^2\right)}{\lfloor 5}\,\frac{\Delta^5 f(a-2h) + \Delta^5 f(a-3h)}{2}$$

$$+\frac{n^2\left(n^2-1^2\right)\left(n^2-2^2\right)}{\lfloor 6}\,\Delta^6 f(a-3h) + \ldots$$

which is Stirling's formula.

If $a = 0$, $h = 1$, and $n = x$, we get the formula as

$$f(x) = f(0) + \frac{x}{\lfloor 1}\,\frac{\Delta f(0) + \Delta f(-1)}{2} + \frac{x^2}{\lfloor 2}\,\Delta^2 f(-1)$$

$$+\frac{x\left(x^2-1^2\right)}{\lfloor 3}\,\frac{\Delta^3 f(-1) + \Delta^2 f(-2)}{2} + \frac{x^2\left(x^2-1^2\right)}{\lfloor 4}\,\Delta^4 f(-2)$$

$$+\frac{x\left(x^2-1^2\right)\left(x^2-2^2\right)}{\lfloor 5} \times \frac{\Delta^5 f(-2) + \Delta^5 f(-3)}{2} + \ldots$$

The central difference table shows that this formula involves the mean of the odd differences above and below the central line and even differences on the central line. This gives very good results in successive approximations as it decreases or increases uniformly.

[5] Bessel's Formula

Gauss backward formula is

$$f(x) = f(0) + {}^{x}C_1 \Delta f(-1) + {}^{x+1}C_2 \Delta^2 f(-1) + {}^{x+1}C_3 \Delta^3 f(-2) + {}^{x+2}C_4 \Delta^4 f(-2) + \ldots \quad \ldots(1)$$

And forward formula is

$$f(x) = f(0) + {}^{x}C_1 \Delta f(0) + {}^{x}C_2 \Delta^2 f(-1) + {}^{x+1}C_3 \Delta^3 f(-1) + {}^{x+1}C_4 \Delta^4 f(-2) + \ldots \quad \ldots(2)$$

Shifting the origin to 1 in backward formula (1) *i.e.* at 4 in the original data, we get

$$f(x) = f(1) + {}^{x-1}C_1 \Delta f(0) + {}^{x}C_2 \Delta^2 f(0) + {}^{x}C_3 \Delta^3 f(-1) + {}^{x+1}C_3 \Delta^4 f(-1) + \ldots \quad \ldots(3)$$

Taking mean of (2) and (3), we get

$$f(x) = \tfrac{1}{2}\,[f(0) + f(1)] + \tfrac{1}{2}\,[{}^{x}C_1 + {}^{x-1}C_1]\,\Delta f(0) + xC_2\,\frac{\Delta^2 f(0) + \Delta^2 f(-1)}{2}$$

$$+\tfrac{1}{2}\left[{}^{x+1}C_3 + {}^{x}C_3\right]\Delta^3 f(-1) + {}^{x+1}C_4\,\frac{\Delta^4 f(-1) + \Delta^4 f(-2)}{2}$$

$$+\tfrac{1}{2}\left[{}^{x+1}C_5 + {}^{x+2}C_5\right]\Delta^5 f(-2) + \ldots$$

or $$f(x) = \tfrac{1}{2}\,[f(0) + f(1)] + \left(x - \tfrac{1}{2}\right)\Delta f(0) + \frac{x(x-1)}{\lfloor 2}\,\frac{\Delta^2 f(0) + \Delta^2 f(-1)}{2}$$

$$+\frac{\left(x-\frac{1}{2}\right)x(x-1)}{\lfloor 3}\,\Delta^3 f(-1) + \frac{(x+1)\,x\,(x-1)\,(x-2)}{\lfloor 4}$$

$$\times\frac{\Delta^4 f(1-)+\Delta^4 f(-2)}{\lfloor 2}+\frac{(x-\frac{1}{2})\ (x-1)\ (x-2)}{\lfloor 5}\times\Delta^5 f(-2)+\ldots$$

which is **Bessel's formula.**

If we repalce $x-\frac{1}{2}$ by t, *i.e.* x by $t+\frac{1}{2}$, we get

$$f\left(t+\tfrac{1}{2}\right)=\tfrac{1}{2}\,[f(0)+f(1)]+t\,\Delta f(0)+\frac{t^2-\left(\frac{1}{2}\right)^2}{\lfloor 2}\;\frac{\Delta^2 f(0)+\Delta^2 f(-1)}{2}$$

$$+\frac{t\left\{t^2-\left(\frac{1}{2}\right)^2\right\}}{\lfloor 3}\Delta^3 f(-1)+\frac{\left\{t^2-\left(\frac{1}{2}\right)^2\right\}\left\{t^2-\left(\frac{3}{2}\right)^2\right\}}{\lfloor 4}\tfrac{1}{2}\left\{\Delta^4 f(-1)+\Delta^4 f(-2)\right\}$$

$$+\frac{t\left\{t^2-\left(\frac{1}{2}\right)^2\right\}\left\{t^2-\left(\frac{3}{2}\right)^2\right\}}{\lfloor 5}\;\Delta^5 f(-2)+\ldots$$

which is rather convenient shape of Bessel's formula for practical purposes {but only if $x>\frac{1}{2}$}.

Here we have alternately mean of the differences falling on both the sides of central line and differences lying upon central line.

[6] Laplace-Everett's Formula.

Gauss forward formula is

$$f(x)=f(0)+x\,\Delta f(0)+{}^{x}C_2\Delta^2 f(-1)+{}^{x+1}C_3\Delta^3 f(-1)+{}^{x+1}C_4\Delta^4 f(-2)$$
$$+{}^{x+2}C_5\Delta^5 f(-2)+\ldots \qquad \ldots(4)$$

$$\because \qquad \Delta^{2k+1}f(a)=\Delta^{2k}f(a+h)-\Delta^{2k}f(a).$$

With the help of this relation, eliminating odd differences from (4), we get

$$f(x)=f(0)+{}^{x}C_1\,\{f(1)-f(0)\}+{}^{x}C_2\Delta^2 f(-1)+{}^{x+1}C_3\,\{\Delta^2 f(0)$$
$$-\Delta^2 f(-1)\}+{}^{x+1}C_4\Delta^4 f(-2)+{}^{x+2}C_5\,\{\Delta^4 f(-1)-\Delta^4 f(-2)\}+{}^{x+2}C_6 f(-3)+\ldots$$

or $$f(x)=(1-x)f(0)+{}^{x}C_1 f(1)+({}^{x}C_2-{}^{x+1}C_3)\,\Delta^2 f(-1)+{}^{x+1}C_3\Delta^2 f(0)$$
$$+({}^{x+1}C_4-{}^{x+2}C_5)\,\Delta^4 f(-2)+{}^{x+2}C_5\Delta^4 f(-1)+({}^{x+2}C_6-{}^{x+3}C_7)\,\Delta^6 f(-3)$$
$$+{}^{x+3}C_7\Delta^6 f(-2)+({}^{x+3}C_8-{}^{x+4}C_9)\,\Delta^8 f(-4)+\ldots$$

or $$f(x)=(1-x)\,f(0)+xf(1)+\left[\frac{x^{(2)}}{\lfloor 2}-\frac{(x+1)^{(3)}}{\lfloor 3}\right]\Delta^2 f(-1)+\frac{(x+1)^{(3)}}{\lfloor 3}\Delta^2 f(0)$$
$$+\left(\frac{(x+1)^{(4)}}{\lfloor 4}-\frac{(x+2)^{(5)}}{\lfloor 5}\right)\Delta^4 f(-2)+\frac{(x+2)^{(5)}}{\lfloor 15}\Delta^4 f(-1)+\ldots$$

or $$f(x)=\left[(1-x)\,f(0)+\left\{\frac{x^{(2)}}{\lfloor 2}+\frac{(x+1)^{(3)}}{\lfloor 3}\right\}\Delta^2 f(-1)\right.$$
$$\left.+\left\{\frac{(x+1)^{(4)}}{\lfloor 4}-\frac{(x+2)^{(5)}}{\lfloor 5}\right\}\Delta^4 f(-2)+\ldots\right]$$

$$+\left[xf(1)+\frac{(x+1)^{(3)}}{\lfloor 3}\Delta^2 f(-1)+\frac{(x+2)^{(5)}}{\lfloor 5}\Delta^4 f(-1)+\ldots\right]$$

If $(1-x)=t$ in the 1st part of the formula, then we get

$$f(x)=\left[tf(0)+\frac{t(t^2-1^2)}{\lfloor 3}\ \Delta^2 f(-1)+\frac{t(t^2-1^2)(t^2-2^2)}{\lfloor 5}\ \Delta^4 f(-2)+\ldots\right]$$

$$+\left[xf(1)+\frac{x(x^2-1^2)}{\lfloor 5}.\Delta^2 f(0)+\frac{x(x^2-1^2)(x^2-2^2)}{\lfloor 5}\ \Delta^4 f(-1)+\ldots\right]$$

which is the **Everett's formula.**

Problem 43. *Given $u_0, u_1, u_2, u_3, u_4, u_5$ (fifth differences constant), prove that $u_{2\frac{1}{2}}=\frac{1}{2}c+\frac{25\ (c-b)+3\ (a-c)}{256}$ by any central difference formula, where $a=u_0+u_5$, $b=u_1+u_4$, $c=u_2+u_3$.*

Bessel's formula is

$$u_{\left(t+\frac{1}{2}\right)}=\frac{1}{2}\left[u_0+u_1\right]+t\,\Delta u_0+\frac{t^2-\left(\frac{1}{2}\right)^2}{\lfloor 2}\ \frac{\Delta^2 u_0+\Delta^2 u_{-1}}{2}+\frac{t\left\{t^2-\left(\frac{1}{2}\right)^2\right\}}{\lfloor 3}\ \Delta^3 u_{-1}$$

$$+\frac{\left\{t^2-\left(\frac{1}{2}\right)^2\right\}\left\{t^2-\left(\frac{3}{2}\right)^2\right\}}{\lfloor 4}\ \frac{\Delta^4 u_{-1}+\Delta^4 u_{-2}}{2}+\frac{t\left\{t^2-\left(\frac{1}{2}\right)^2\right\}\left\{t^2-\left(\frac{3}{2}\right)^2\right\}}{\lfloor 5}\ \Delta^5 u_{-2},$$

(fifth differences constant means that differences above fifth are zero).

∴ Putting $t=0$ in the formula

$$u_{\frac{1}{2}}=\frac{1}{2}\left[u_0+u_1\right]-\frac{1}{8}\frac{\Delta^2 u_0+\Delta^2 u_{-1}}{2}+\frac{\left(-\frac{1}{4}\right)\left(-\frac{9}{4}\right)}{24}\ \frac{\Delta^4 u_{-1}+\Delta^4 u_{-2}}{2}$$

$$=\frac{1}{2}\left(u_0+u_1\right)-\frac{1}{8}\frac{\Delta^2 u_0+\Delta^2 u_{-1}}{2}+\frac{3}{128}\ \frac{\Delta^4 u_{-1}+\Delta^4 u_{-2}}{2}$$

$$=\frac{1}{2}\left(u_0+u_1\right)-\frac{1}{16}\left\{\overline{u_2-2u_1+u_0}+\overline{u_1-2u_0+u_{-1}}\right\}$$

$$+\frac{9}{256}\left\{\overline{u_3-4u_2+6u_1-4u_0+u_{-1}}+\overline{u_2-4u_1+6u_0-4u_{-1}+u_{-2}}\right\}$$

$$=\frac{1}{2}\left(u_0+u_1\right)-\frac{1}{16}\left\{u_2-u_1-u_0+u_{-1}\right\}+\frac{3}{256}\left\{u_3-3u_2+2u_1+2u_0-3u_{-1}+u_{-2}\right\}$$

or $$u_{\frac{1}{2}}=\frac{1}{2}\left(u_0+u_1\right)+\frac{1}{256}\left\{3(u_3+u_{-2})-9(u_2+u_{-1})+6(u_1+u_0)\right.$$

$$\left.+16\left(u_0+u_1\right)-16\left(u_2+u_{-1}\right)\right\}.$$

Shifting the origin to –2, we get

$$u_{2\frac{1}{2}}=\frac{1}{2}\left(u_2+u_3\right)+\frac{1}{256}\left\{3\left(u_5+u_0\right)-9\left(u_4+u_1\right)+6\left(u_3+u_2\right)\right.$$

$$\left.+16\left(u_2+u_3\right)+6\left(u_4+u_1\right)\right\}$$

$$=\frac{1}{2}c+\frac{1}{256}\left\{3a-9b+6c+16c-16b\right\}$$

or $$u_{2\frac{1}{2}} = \frac{1}{2}c + \frac{1}{256}\{3a + 22c - 25b\}$$

or $$u_{2\frac{1}{2}} = \frac{1}{2}c + \frac{25(c-b) + 3(a-c)}{256}.$$

7.11. APPROXIMATE INTEGRATION

The approximate integration or numerical quadrature deals with the evaluation of approximate area under a curve by determining the value of a definite integral with the help of *a given set of numerical values of the function under integration.* For this purpose we first approximate the function into the form of a polynomial applying some formula of interpolation. The degree of the polynomial depends upon the number of known values of the function. With n known values the function can be supposed as a polynomial of degree $(n - 1)$. Now approximating the form of the function, we can integrate it within desired limits.

[1] Trapezoidal Rule

Suppose there are two values of the function $f(x)$, $f(0)$ and $f(1)$, represented by H and K on the curve. There can be unlimited curves passing through these two points; the simplest of them will be one degree polynomial, *i.e.* a straight line, say $f(x) = A + Bx$.

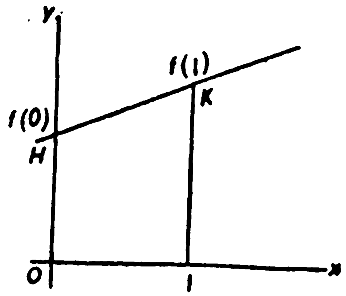


Fig. 7.1

As such area under the curve between $\{0, f(0)\}$, $\{1, f(1)\}$ and the axis of x is

$$= \int_0^1 f(x)\,dx = \int_0^1 (A + Bx)\,dx$$

$$= \left[Ax + \frac{Bx^2}{2}\right]_0^1 = A + \frac{B}{2}. \qquad ...(1)$$

Suppose $$\int_0^1 f(x)\,dx = lf(0) + mf(1) \qquad ...(2)$$

$$= l(A) + m(A + B) \qquad [\text{by } f(x) = A + Bx]$$

$$= (l + m)A + mB. \qquad ...(3)$$

Comparing (1) and (3), we get

$$l + m = 1,\ m = \tfrac{1}{2},$$

$$\therefore \qquad l = \tfrac{1}{2},$$

Substituting these values of l and m in (2), we have

$$\int_0^1 f(x)\,dx = \tfrac{1}{2}[f(0) + f(1)].$$

[2] Effect of Changing Origin and Scale upon Trapezoidal Rule.

Suppose $x = \dfrac{z-a}{h}$ {where a and h are constant and z is a new variate}.

So that $z = a + xh$, giving $dz = h\,dx$ or $dx = \dfrac{dz}{h}$.

When $x = 0$, $z = a$ when $x = 1$, $z = a + h$.

Also we have $\int_0^1 f(x)\, dx = \frac{1}{2}[f(0) + (1)]$.

$$\therefore \quad \int_a^{a+h} f(z)\, \frac{dz}{h} = \frac{1}{2}[f(a) + f(a+h)].$$

or
$$\int_a^{a+h} f(z)\, dz = \frac{h}{2}\,[f(a) + f(a+h)] = \int_a^{a+h} f(x)\, dx.$$

[3] Extension of Trapezoidal Rule.

We have by taking n intervals each of length h,

$$\int_a^{a+nh} f(x)\, dx = \int_a^{a+h} f(x)\, dx + \int_{a+h}^{a+2h} + \int_{a+2h}^{a+3h} + \ldots + \int_{a+\overline{n-1}h}^{a+nh} f(x)\, dx$$

$$= h\,[\tfrac{1}{2}\ \{f(a) + f(a+nh)\} + \{f(a+h) + f(a+2h) + \ldots$$

$$+\{f(a+\overline{n-1}h) + f(a+nh)\}]$$

= distance between two consecutive ordinates

× [mean of the first and the last ordinates

+ sum of all the intermediate terms].

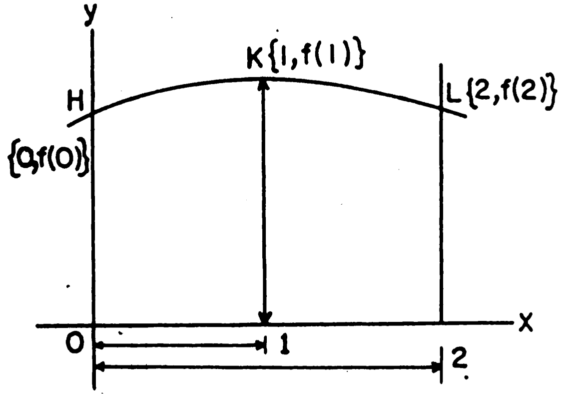


Fig. 7.2

[4] Simpson's 1/3 Rule.

Consider three equidistant points $H\ \{0, f(0)\}$, $K\ (1, f(1))$ and $L\ \{2, f(2)\}$, where the function $f(x)$ takes values $f(0), f(1)$ and $f(2)$ respectively. There are unlimited number of curves to pass through these three points H, K and L and second degree curve will be the simplest one.

Suppose

$$f(x) = A + Bx + Cx^2.$$

$$\therefore \quad f(0) = A, \qquad \ldots(4)$$

$$f(1) = A + B + C \qquad \ldots(5)$$

and
$$f(2) = A + 2B + 4C. \qquad \ldots(6)$$

The area between the curve HKL and the axis of X is

$$\int_0^2 f(x)\, dx = \int_0^2 (A + Bx + Cx^2)\, dx$$

$$= \left[Ax + \tfrac{1}{2}Bx^2 + \tfrac{1}{3}\, Cx^3\right]_0^2 = 2A + 2B + \tfrac{8}{3}\, C \qquad \ldots(7)$$

Suppose $\int_0^2 f(x)dx = lf(0) + mf(1) + nf(2)$

$$= lA + m\ (A + B + C) + n\ (A + 2B + 4C)$$

[by (4), (5) and (6)]

$$= A(l + m + n) + B\ (m + 2n) + C\ (m + 4n) \qquad \ldots(8)$$

Comparing (7) and (8), we get

$$\left.\begin{aligned} l + m + n &= 2 \\ m + 2n &= 2 \\ m + 4n &= \tfrac{8}{3} \end{aligned}\right\}, \quad \therefore \quad l = \tfrac{1}{3} m = \tfrac{4}{3} n = \tfrac{1}{3}.$$

So that $\int_0^2 f(x)\,dx = \frac{1}{3}f(0) + \frac{4}{3}f(1) + \frac{1}{3}f(2)$

i.e. $\int_0^2 f(x)\,dx = \frac{1}{3}\{f(0) + 4f(1) + f(2)\}$ ...(9)

which is Simpson's one-third rule.

Aliter. $\int_0^2 f(x)\,dx = \int_0^2 \left\{f(0) + {}^xC_1\Delta f(0) + {}^xC_2\Delta^2 f(0)\right\}dx$

$$= \int_0^2 \left[f(0) + x\left\{\Delta f(0) - \tfrac{1}{2}\Delta^2 f(0)\right\} + \tfrac{1}{2}x^2\Delta^2 f(0)\right]dx$$

$$= \left[xf(0) + \frac{x^2}{2}\left\{\Delta f(0) - \tfrac{1}{2}\Delta^2 f(0)\right\} + \frac{x^3}{6}\Delta^2 f(0)\right]_0^2$$

$$= 2f(0) + 2\Delta f(0) + \tfrac{1}{3}\Delta^2 f(0)$$

$$= 2f(0) + 2\{f(1) - f(0)\} + \tfrac{1}{3}\{f(2) - f\,2(1) + f(0)\}$$

or $\int_0^2 f(x)\,dx = \frac{1}{3}\{f(0) + 4f(1) + f(2)\}$.

[5] Effect of Changing Unit and Origin on Simpson's $\frac{1}{3}$ Rule.

Suppose $x = \dfrac{z-a}{h}$; $\therefore$ $dx = \dfrac{dz}{h}$.

Thus $\int_0^2 f(x)\,dx = \frac{1}{3}\{f(0) + 4f(1) + f(2)\}$, changes to

$$\int_0^{a+2h} f(z)\,dz = \frac{h}{3}\{f(a) + 4f(a+h) + f(a+2h)\}.$$

In particular. $\int_{-1}^{1} f(x)\,dx = \frac{1}{3}\{f(-1) + 4f(0) + f(1)\}$ ...(10)

Also when $a = 0$ this yields

$$\int_0^{2h} f(z)\,dz = \frac{h}{3}\{f(0) + 4f(h) + f(2h)\} \qquad ...(11)$$

[6] Extension of Simpson's $\frac{1}{3}$ rule (for all even values of n).

We have $\int_a^{a+nh} f(x)\,dx = \int_a^{a+2h} f(x)\,dx + \int_{a+2h}^{a+4h} f(x)\,dx + \ldots + \int_{a+\overline{n-2}h}^{a+nh} f(x)\,dx$

$$= \frac{h}{3}[\{f(a) + 4f(a+h) + f(a+2h)\} + \{f(a+2h) + 4f(a+3h) + f(a+4h)\}$$

$$+\ldots+\{f(a+\overline{n-2}h) + 4f(a+\overline{n-1}h) + f(a+nh)\}]$$

$$= \frac{h}{3}[\{f(a) + f(a+nh)\} + 4\{f(a+h) + f(a+3h) + \ldots + f(a+\overline{n-1}h)\}$$

$$+ 2\{f(a+2h) + f(a+4h)\} + \ldots + f(a+\overline{n-2}h)\}] \qquad ...(12)$$

= (one-third of the interval) {(sum of the 1st and the last terms)
+ 4 (sum of the odd number of terms) + 2 (sum of the even number of terms)}.

- [7] **Simpson's $\frac{3}{8}$th rule.**

Consider four points $\{0, f(0)\}$, $\{1, f(1)\}$, $\{2, f(2)\}$ and $\{3, f(3)\}$ instead of three as taken in [4]

Then Required area $= \int_0^3 f(x)\ dx.$ ...(13)

For the sake of convenience let us double the unit of interval and then shift the origin to 3. By this transformation we have the four points $\{-3, f(-3)\}$, $\{-1, f(-1)\}$, $\{1, f(1)\}$ and $\{3, f(3)\}$ as given and so we have to find out the value of the integral $\int_{-3}^3 f(x)\ dx$., where $f(x) = A + Bx + Cx^2 + Dx^3$, a third degree polynomial with four given values.

Suppose $\int_{-3}^3 f(x)\ dx = m\ \{f(-3) + f(3)\} + n\ \{f(-1) + f(1)\}$

or $\int_{-3}^3 \{A + Bx + Cx^2 + Dx^3\}\ dx = m\ \{2A + 18C\} + n\ \{2A + 2C\}$

$\{\because f(-3) = A - 3B + 9C - 27D \text{ etc.}\}$

or $\left[Ax + \frac{Bx^2}{2} + \frac{Cx^3}{3} + \frac{Dx^4}{4}\right]_{-3}^3 = A\ (2m + 2n) + C\ (18m + 2n)$

or $6A + 18C = (2m + 2n)\ A + (18m + 2n)C.$

$\therefore\ (2m + 2n) = 6$ and $18m + 2n = 18,$

giving $16m = 12$ or $m = \frac{3}{4}$

and $2n = 6 - 2m = 6 - \frac{3}{2} = \frac{9}{2}$ or $n = \frac{9}{4}.$

Substituting these values of m and n in (13) we get

$$\int_{-3}^3 f(x)\ dx = \frac{3}{4}\ \{f(-3) + f(3)\} + \frac{9}{4}\ \{f(-1) + f(1)\}$$

or $\int_{-3}^3 f(x)\ dx = \frac{3}{8}\ [2 \cdot \{f(-3) + f(3)\} + 6\ \{f(-1) + f(1)\}]$ ...(14)

which is the form of the Simpson's $\frac{3}{8}$th rule.

Shifting the origin to -3, we get

$$\int_0^6 f(x)\ dx = \tfrac{3}{8}[2\ \{f(0) + f(6)\} + 6\{f(2) + f(4)\}] \quad \text{...(15)}$$

which is another form of the formula.

Making the interval half of the present one which is 2, *i.e.*, making the interval unity, we have

$$\int_0^3 f(x)\ dx = \tfrac{3}{8}\ [\{f0) + f(3)\} + 3\ \{f(1) + f(2)\}] \quad \text{...(16)}$$

Note. *Like the $\frac{1}{3}$rd rule, the $\frac{3}{8}$th rule can also be obtained by evaluating the integral.*

$$\int_0^3 f(x)\ dx = \int_0^3 E^x f(0)\ dx = \int_0^3 (1 + \Delta)^x\ f(0)\ dx$$

$$= \int_0^3 \left\{1 + x\Delta + \frac{x(x-1)}{\lfloor 2}\Delta^2 + \frac{x(x-1)\ (x-2)}{\lfloor 3}\Delta^3\right\} f(0)\, dx$$

$$= \int_0^3 \left\{ 1 + x\Delta + \left(\frac{x^2}{2} - \frac{x}{2} \right) \Delta^2 + \left(\frac{x^3}{6} - \frac{x^2}{2} + \frac{x}{3} \right) \Delta^3 \right\} f(0)\ dx$$

$$= \left[x + \frac{x^2}{2} \Delta + \left(\frac{x^3}{6} - \frac{x^2}{4} \right) \Delta^2 + \left(\frac{x^4}{24} - \frac{x^3}{6} + \frac{x^2}{6} \right) \Delta^3 \right]_0^3 f(0)$$

$$= f(0) + \frac{9}{2} \Delta f(0) + \left(\frac{9}{2} - \frac{9}{4} \right) \Delta^2 f(0) + \frac{27}{8} - \frac{9}{2} + \frac{9}{6} \Big) \Delta^3 f(0)$$

$$= 3f(0) + \frac{9}{2} \{f(1) - f(0)\} + \frac{9}{4} \{f(2) - 2f(1) + f(0)\}$$

$$+ \frac{9}{24} \{f(3) - 3f(2) + 3f(1) - f(0)\}$$

$$= \left(3 - \frac{9}{2} + \frac{9}{4} - \frac{9}{24} \right) f(0) + \left\{ \frac{9}{2} - \frac{9}{2} + \frac{9}{8} \right\} f(1) + \left\{ \frac{9}{4} - \frac{9}{8} \right\} (2) + \frac{3}{8} f(3)$$

$$= \frac{3}{8} [\{f(0) + f(3)\} + 3\{f(1) + f(2)\}].$$

[8] Effect of Changing Origin and Scale upon $\frac{3}{8}$th Rule.

We have $\int_0^3 f(x)\ dx = \frac{3}{8} [\{f(0) + f(3)\} + 3\ \{f(1) + f(2)\}]$.

Suppose $\frac{z-a}{h} = x$, so that $dz = hdx$

$$\therefore \int_0^{a+3h} f(z)\ dz = \frac{3h}{8} [f(a) + f(a+3h)\} + 3\{f(a+h) + f(a+2h)\}] \qquad \ldots(17)$$

[9] Extension of Simpson's $\frac{3}{8}$th rule. (When n is a multiple of 3.)

$$\int_a^{a+nh} f(x)\ dx = \int_a^{a+3h} f(x)\ dx + \int_{a+3h}^{a+6h} f(x)\ dx + \int_{a+6h}^{a+9h} f(x)\ dx + \ldots \int_{a+\overline{n-3}h}^{a+nh} f(x)\ dx$$

$$= \frac{3h}{8} [\{f(a) + f(a+nh)\} + 2\ \{f(a+3h) + f(a+6h) + \ldots + (a + \overline{n-3h})\}$$

$$+3\ \{f(a+h) + f(a+2h) + f(a+4h) + f(a+5h) + \ldots\}] \qquad \ldots(18)$$

$$= \frac{3h}{8}\ [\text{(sum of the first and last terms)}$$

+2 (sum of 4th, 7th, 10th etc. terms)

+ 3 {sum of the remaining 2nd, 3rd, 5th, 6th, 8th, 9th, etc. terms}]

[10] Weddle's Rule.

If $f(x)$ is given for certain equidistant values of x say $x_0, x_0 + h, x_0 + 2h, \ldots$ and the range (a, b) is divided into n equal parts each of width h *i.e.* $b - a = nh$ and if $x_0 = a, x_1 = x_0 + h, \ldots x_n = a + nh = b$, assuming that $(n + 1)$ ordinates $y_0, y_1, \ldots y_n$ are equidistant then by a charge of scale, $u = \frac{x - x_0}{h}, dx = hdu$, the *general quadrature* formula is

$$\int_a^b y\, dx = \int_{x_0}^{x_0+nh} y_x\, dx = \int_0^n y_{x_0+hu} \cdot hdu = h \int_0^n \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{\lfloor 2} \Delta^2 y_0 + \ldots \right] du$$

$$= h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{\lfloor 2} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{\lfloor 3} + \ldots \text{upto } (n+1) \text{ terms}\right] \quad \ldots(19)$$

Putting $n = 6$, and neglecting all differences above the sixth, we find

$$\int_{x_0}^{x_0+6h} y\,dx = h\left[6y_0 + 18\Delta y_0 + 27\Delta^2 y_0 + 24\Delta^3 y_0 + \frac{123}{10}\Delta^4 y_0 + \frac{33}{10}\Delta^5 y_0 + \frac{41}{140}\Delta^6 y_0\right]$$

Replacing the last term by $\frac{3}{10}\Delta^6 y_0$ and neglecting the error $\frac{h}{140}\Delta^6 y_0$ made, we have

$$\int_{x_0}^{x_0+6h} y\,dx = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$$

Similarly,

$$\int_{x_0+6h}^{x_0+12h} y\,dx = \frac{3h}{10}[y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}]$$

....

$$\int_{x_0+(n-6)h}^{x_0+nh} y\,dx = \frac{3h}{10}[y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n]$$

n being multiple of 6.

Adding all together

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + \ldots] \quad \ldots(20)$$

which is Weddle's rule.

Note. *The trapezoidal rule, Simpson's $\frac{1}{3}$rd rule and Simpson's $\frac{3}{8}$th rule can successively be obtained by putting $n = 1$, $n = 2$ and $n = 3$ respectively in (19).*

Problem 44. *Evaluate the following:*

(*i*) $\int_0^1 \frac{x^2}{1+x^3}\,dx$ *and hence find the value of* $\log_e 2$.

(*ii*) $\int_2^{10} \frac{dx}{1+x}$ *by dividing the range into eight equal parts.*

(*i*) We have $\int_0^1 f(x)\,dx = \frac{1}{2}\cdot\frac{1}{3}\left\{f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right\}$. ...(1)

$$\therefore \int_0^1 \frac{x^2}{1+x^3}\,dx = \frac{1}{2}\cdot\frac{1}{3}\left\{\frac{0}{1+0} + 4.\frac{\left(\frac{1}{2}\right)^2}{1+\left(\frac{1}{2}\right)^3} + \frac{1^2}{1+1^3}\right\}$$

$$= \frac{1}{6}\left\{0 + \frac{8}{9} + \frac{1}{2}\right\} = \frac{25}{108} \quad \ldots(2)$$

Now putting $x^3 = t$ in the given integral we get

$$\int_0^1 \frac{x^2}{1+x^3}\,dx = \frac{1}{3}\int_0^1 \frac{dt}{1+t} = \frac{1}{3}\{\log_e(1+t)\}_0^1 = \frac{1}{3}\log_e 2 \quad \ldots(3)$$

$\therefore$ from (2) and (3), $\log_e 2 = 3 \times \frac{25}{108} = \frac{25}{36} = .6944$.

(*ii*) Dividing the range into eight parts by means of 9 points of the x-values as 2, 3, 4, 5, 6, 7, 8, 9, 10, the extended form of Simpson's $\frac{1}{3}rd$ rule gives.

$$\int_2^{10} f(x)\,dx = \tfrac{1}{3}[\{f(2)+f(10)\} + 2\{f(4)+f(6)+f(8)+4\{f(3)+f(5)$$
$$+f(7)+f(9)\}] \qquad ...(4)$$

But $f(x) = \dfrac{1}{1+x}$.

$\therefore$ $f(2) = .333, f(3) = .25, f(4) = .20, f(5) = .166, f(6) = .143,$
$f(7) = .125, f(8) = .111, f(9) = .10, f(10) = .091.$

Substituting these values in (4) we get

$$\int_2^{10} f(x)dx = \tfrac{1}{3}[\{.333+.091\} + 2\{.20+.143+.111\} + 4\ \{.25 + .166 + .125 + .10\}]$$
$$= \tfrac{1}{3}\ [.422 + .908 + 2.564] = 1.298.$$

Hence $\displaystyle\int_2^{10} \frac{dx}{1+x} = 1.298.$

Problem 45. *If u_x is a function whose fifth differences are contant, $\int_{-1}^{1} u_x\,dx$ can be expressed in the form $pu_{-\alpha} + qu_0 + pu_\alpha$. Find the values of p, q and α.*

Use this formula, after making necessary changes in the origin and scale, to find the value of $\log_e 2$ to four places of decimals from the equation $\int_0^1 \frac{1}{1+x}dx = \log_e 2$.

(*i*) As fifth differences of the function are constant, so u_x must be of fifth degree such as

$$u_x = a + bx + cx^2 + dx^3 + ex^4 + fx^5.$$

$$\therefore \quad \int_{-1}^{1} u_x\,dx = \int_{-1}^{1}\left\{a + bx + cx^2 + dx^3 + ex^4 + fx^5\right\}dx$$
$$= 2\left\{a + \frac{c}{3} + \frac{e}{5}\right\}. \qquad ...(1)$$

Now $pu_{-\alpha} + qu_0 + pu_\alpha = 2p\{a + \alpha^2 c + \alpha^4 e\} + qa.$ ...(2)

But $\displaystyle\int_{-1}^{1} u_x\,dx = pu_{-\alpha} + qu_0 + pu_\alpha.$

Now equating the coefficients of a, c and e in (1) and (2), we get $2 = 2p + q$...(3), $\frac{2}{3} = 2p\alpha^2$ ...(4) and $\frac{2}{5} = 2p\alpha^4$ ...(5)

$$\therefore \quad \alpha^2 = \frac{2p\alpha^4}{2p\alpha^2} = \frac{\frac{2}{5}}{\frac{2}{3}} = \frac{3}{5}.$$

Substituting this value of α^2 in (4), we get $p = \frac{5}{9}$.

And then substituting this value of p in (3), we get $q = \frac{8}{9}$.

Thus $\displaystyle\int_{-1}^{1} u_x\,dx = \tfrac{1}{9}\{5(u_\alpha + u_{-\alpha}) + 8u_0\},$ where $\alpha^2 = \frac{3}{5}$.

(*ii*) $\because$ $\int_{-1}^{1} u_x\,dx = \frac{1}{9}\{5(u_\alpha + u_{-\alpha}) + 8u_0\}$.

$\therefore$ $\int_0^2 u_x\,dx = \frac{1}{9}\{5(u_{1+\alpha} + u_{1-\alpha}) + 8u_1\}$ (shifting the origin to -1)

$\therefore$ $\int_0^1 u_x\,dx = \frac{1}{18}\{5u_{(1+\alpha)/2} + u_{(1-\alpha)/2} + 8u_{1/2}\}$. ...(6)

(Dividing the unit by 2)

$\because$ $u_x = \frac{1}{1+x}$, $\therefore$ $u_{1/2} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$

and $u_{(1+\alpha)/2} + u_{(1-\alpha)/2}$ $\frac{1}{1+\frac{1+\alpha}{2}} + \frac{1}{1+\frac{1-\alpha}{2}} = \frac{12}{9-\alpha^2} = \frac{12}{9-\frac{8}{5}} = \frac{10}{7}$.

Substituting these values in (6), we get

$$\int_0^1 \frac{1}{1+x}\,dx = \frac{1}{18}\left\{5 \times \frac{10}{7} + \frac{8\times 2}{3}\right\} = \frac{131}{189} =.6931.$$

But $\int_0^1 \frac{1}{1+x}dx = \log_e 2$, $\therefore$ $\log_e 2 = .6931$.

Problem 46. *Calculate* $\int_0^{\pi/2} \sin x\,dx$ *by*

(*i*) *trapezoidal rule*

(*ii*) *Simpson's rule* (*using 11 ordinates*)

Dividing $\left(0, \frac{\pi}{2}\right)$ into 10 equal parts so that $h = \frac{\pi}{20}$ we have from trigonometric tables.

$y_1 = \sin 0 = 0.0000$

$y_2 = \sin \frac{\pi}{20} = 0.1564$

$y_3 = \sin \frac{\pi}{10} = 0.3090$

$y_4 = \sin \frac{3\pi}{10} = 0.4540$

$y_5 = \sin \frac{\pi}{5} = 0.5878$

$y_6 = \sin \frac{\pi}{4} = 0.7071$

$y_7 = \sin \frac{3\pi}{10} = 0.8090$

$y_8 = \sin \frac{7\pi}{20} = 0.8910$

$y_9 = \sin \frac{2\pi}{5} = 0.9511$

$y_{10} = \sin \frac{9\pi}{20} = 0.9877$

$y_{11} = \sin \frac{\pi}{2} = 1.0000$

(*i*) Required value of the integral

$$= h\left\{\tfrac{1}{2}(y_1 + y_{11}) + (y_2 + y_3 + \ldots + y_{10})\right\}$$

$$= \frac{\pi}{20}\{0.5 + 5.8531\} = 0.9981.$$

(*ii*) Required value of the integral

$$=\frac{h}{3}\left[y_1+y_{11}+2(y_3+y_5+y_7+y_9)+4(y_2+y_4+y_6+y_8+y_{10})\right]$$

$$=\frac{\pi}{60}\,[1+5.3138+12.7848] \;=\; 19.0986\times 0.0524 \;=\; 1.0006.$$

Problem 47. *Use Weddle's rule to find the value of* $\int_4^{5.2} \log_e x\, dx$

Total interval = 5.2 − .4 = 1.2

Dividing 1.2 into six equal parts, $h = 0.2$.

By Table we have

x	4.0	4.2	4.4	4.6	4.8	5.0	5.2
$\log_e x$	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487

Required value of the integral

$$=(0.2)\times\frac{3}{10}[(1.3863+5)(1.4351)+1.4816+6(1.5261)$$

$$+1.5686+5(1.6094)+2(1.6487)$$

$$=1.8278 \text{ approx.}$$

[11] Picard's Method

Consider $\dfrac{dy}{dx}=f(x,\ y)$ ...(21)

Suppose it is required to find its solution under the condition $y = y_0$ when $x = x_0$.

Integration of (1) gives $y=\int f(x,\ y)\ dx + C$

When $x=x_0,\ y=y_0 \quad \therefore\ C=y_0-\int f(x_0,\ y)\ dx$

So that $y=y_0+\int f(x,\ y)\,dx-\int f(x_0,\ y)\ dx$

$$=y_0+\int_{x_0}^{x} f(x,\ y)\ dx \qquad \text{...(22)}$$

Now (22) is integrated by successive approximation.

The first approximation is found by putting $y = y_0$ in the integrand of (22), whence we get

$$y_1=y_0+\int_{x_0}^{x} f(x,\ y_0)\ dx\ ,$$

The second approximation is obtained by replacing y by y_1 in R.H.S. of (22).

i.e. $\quad y_2=y_0+\int_{x_0}^{x} f(x,\ y_1)\ dx$

and so on.

In general the nth approximation is

$$y_n=y_0+\int_{x_0}^{x} f(x,\ y_{n-1})\,dx.$$

Problem 48. *Solve* $\frac{dy}{dx} = x + y^2$ *where* $y = 0$ *when* $x = 0$.

Here $x_0 = 0, y_0 = 0$ and $f(x, y) = x + y^2$

$\therefore$ First approximation is $y_1 = y_0 + \int_{x_0}^{x} (x, y_0)\,dx = 0 + \int_0^x x\,dx = \frac{x^2}{2}$

$$\because f(x, y_0) = x + 0^2 = x$$

Second approximation is

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx,\ f(x, y_1) = x + \left(\frac{x^2}{2}\right)^2 = x + \frac{x^4}{4}$$

$$= 0 + \int_0^x \left(x + \frac{x^4}{4}\right) dx = \frac{x^2}{2} + \frac{x^5}{20}.$$

Third approximation is

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\,dx,\ f(x, y_2) = x + \left(\frac{x^2}{2} + \frac{x^5}{20}\right)^2$$

$$= x + \frac{x^4}{4} + \frac{x^7}{20} + \frac{x^{10}}{400}$$

$$= 0 + \int_0^x \left(x + \frac{x^4}{4} + \frac{x^7}{20} + \frac{x^{10}}{400}\right) dx$$

$$= \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{x^{11}}{4400}.$$

Which gives fairly good approximation.

[12] Extension of Picard's Method.

Consider simultaneous equations

$$\frac{dy}{dx} = f(x, y, z),\ \frac{dy}{dx} = g(x, y, z) \qquad ...(23)$$

subject to the conditions $y = y_0, z = z_0$ when $x = x_0$

The first approximations of (23) are

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0, z_0)\,dx,\quad z = z_0 + \int_{x_0}^{x} g(x, y_0, z_0)\,dx \qquad ...(24)$$

In general the nth approximations are

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1}, z_{n-1})\,dx,\quad z_n = z_0 + \int_{x_0}^{x} g(x, y_{n-1}, z_{n-1})\,dx$$

Problem 49. *Solve* $\frac{dy}{dx} = x + z,\ \frac{dz}{dx} = x - y^2$ *where* $y = 2,\ z = 1$ *when* $x = 0$.

Here $x_0 = 0, y_0 = 2, z_0 = 1, f(x, y, z) = x + z$ and $g(x, y, z) = x - y^2$

$\therefore$ First approximations are $y_1 = 2 + \int_0^x (x + 1)\,dx = 2 + x + \frac{x^2}{2}$

$$z_1 = 1 + \int_0^x (x - 4)\,dx = 1 - 4x + \frac{x^2}{2}.$$

Similarly second and third approximations can be evaluated.

[13] Runge-Kutta Method.

Consider $\dfrac{dy}{dx} = f(x, y),\ y = y_0$ when $x = x_0$

Assume that $y = F(x)$. Then Taylor's theorem yields

$$F(x_0 + h) = f(x_0) + h F'(x_0) + \frac{h^2}{\underline{|2}} F''(x_0) + \frac{h^2}{\underline{|3}} F'''(x_0) + \ldots \qquad \ldots(25)$$

Here $F'(x) = \dfrac{dy}{dx} = f(x, y) = f$ (say).

Take $p = \dfrac{\partial f}{\partial x},\ q = \dfrac{\partial f}{\partial y},\ r = \dfrac{\partial^2 f}{\partial y^2},\ s = \dfrac{\partial^2 f}{\partial x \partial y},\ t = \dfrac{\partial^2 f}{\partial y^2}.$

At $y = y_0$ when $x = x_0$, let $p = p_0,\ q = q_0,\ r = r_0,\ s = s_0,\ t = t_0$.

$$\text{Now } F''(x) = \frac{df}{dx} = \left(\frac{\partial}{\partial x} + \frac{dy}{dx}\frac{\partial}{\partial y}\right) f = p + fq \qquad \because\ df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$F'''(x) = \frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right) = \left(\frac{\partial}{\partial x} + \frac{dy}{dx}\frac{\partial}{\partial y}\right)(p + fq)$$

$$= \frac{\partial p}{\partial x} + q\frac{\partial f}{dx} + f\frac{\partial q}{\partial x} + \frac{dy}{dx}\frac{\partial p}{\partial y} + \frac{dy}{dx} q \frac{\partial f}{\partial y} + \frac{dy}{dx} f \frac{\partial q}{\partial y}$$

$$= r + pq + fs + f(s + q^2 + ft)$$

$$\text{Hence } F(x_0 + h) - F(x_0) = hf_0 + \frac{h^2}{2}(p_0 + f_0 q_0) + \frac{h^3}{6}(r_0 + 2 f_0 s_0$$

$$+ f_0^2 t_0 + p_0 q_0 + f_0 q_0^2) + \ldots \qquad \ldots(26)$$

Here the first term represents the first approximation *i.e.*,

$$F(x_0 + h) - F(x_0) = hf_0 \text{ i.e. } y = y_0 + hf(x_0, y_0) \qquad \ldots(27)$$

The second approximation may be taken as

$$y - y_0 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{1}{2} hf_0\right) = k_1 \text{ (say)} \qquad \ldots(28)$$

whence by Taylor's expansion, we have

$$f\left(x_0 + \frac{h}{2},\ y_0 + \frac{hf_0}{2}\right) = f_0 + \frac{1}{2} hp_0 + \frac{1}{2} hf_0 q_0 + \frac{1}{\underline{|2}}\left(\frac{1}{4} h^2 r_0 + \frac{1}{2} h^2 f_0 s_0 + \frac{1}{4} h^2 f_0^2 t_0\right)$$

$$\text{giving } k_1 = hf_0 + \tfrac{1}{2} h^2 (p_0 + f_0 q_0) + \tfrac{1}{8} h^3 (r_0 + 2 f_0 s_0 + f_0^2 t_0) \qquad \ldots(29)$$

From (26) it is observed that the difference in the two values of k_1 found by (28) and (29) is in the coefficient of h^3.

In order to find the extra terms in h^3, Runge-Kutta replaces

$hf(x_0 + h,\ y_0 + hf_0)$ by $k''' = hf(x_0 + h,\ y_0 + k'')$

where $k'' = hf(x_0 + h,\ y_0 + hf_0)$

The modified formula therefore becomes $\frac{1}{6}\{k' + 4k_1 + k'''\}$

where $k' = hf_0 = \frac{2}{3} k_1 + \frac{1}{3} k_2 = k_1 + \frac{1}{3}(k_2 - k_1)$ and $k_2 = \frac{1}{2}(k' + k''')$.

Hence this method is applied in the sequence:

$$k' hf_0\ ;\ k'' = hf(x_0 + h,\ y_0 + k');\ k''' = hf(x_0 + h,\ y_0 + k'')$$

$k_1 = hf\left(x_0+\frac{h}{2},\ y_0+\frac{1}{2}k'\right)$; $k_2=\frac{1}{2}(k'+k''')$ and $k = k_1+\frac{1}{3}(k_2-k_1)$.

Problem 50. *Apply Runge-Kutta method to find y when x = 0.3 from* $\frac{dy}{dx} = x+y^2$ *where* $y = 0$ *when* $x = 0$.

Here $x_0 = 0, y_0 = 0, h = 0.3, f(x, y) = x + y^2$ and $f_0 = f(x_0, y_0) = 0$.

$\therefore\ k' = hf_0 = 0$

$k'' = hf(x_0+h, y_0+k') = 0.3\times f\{0.3, 0\} = 0.3\times 0.3 = 0.09$

$k''' = hf(x_0+h, y_0+k'') = 0.3\times f\{0.3, 0.09\} = 0.3\times\{0.3+0.0081\} = 0.0924.$

$$k_1 = hf\left(x_0+\frac{h}{2},\ y_0+\frac{k'}{2}\right) = 0.3\times f\{0.15, 0\} = 0.3\times 0.15 = 0.045$$

$$k_2 = \tfrac{1}{2}(k'+k''') = \tfrac{1}{2}\times 0.0924 = 0.0462$$

$$k = k_1+\tfrac{1}{3}(k_2-k_1) = 0.045+0.0004 = 0.0454.$$

7.12. PERTURBATION METHOD

[A] First Order Differential Equations.

When dealing with physical problems, we frequently encounter a differential equation of the type

$$\frac{dy}{dx}+y^2 = 0 \text{ with } y(1) = 1 \text{ (say)} \qquad ...(1)$$

whch has been disturbed by a small effect. We thus modify (1) as

$$\frac{dy}{dx}+y^2 = \epsilon x,\ y(1) = 1 \qquad ...(2)$$

where ϵ is arbitrarily small.

We have now to determine by how much the solution of (1) has been changed on account of the presence of the disturbing factor ϵx. This change is known as a *Perturbation.*

Suppose $y_0(x)$ is a solution of (1) satisfying $y(1) = 1$ and assume the solution of (2) as

$$y(x) = y_0(x)+p(x),\ p(x) \text{ being perturbation} \qquad ...(3)$$

Expanding $y(x)$ in a series in powers of ϵ, such that

$$y(x) = y_0(x)+\epsilon y_1(x)+\epsilon^2 y_2(x)+\epsilon^3 y_3(x)+... \qquad ...(4)$$

and comparing (3) and (4) we find

$$p(x) = \epsilon y_1(x)+\epsilon^2 y_2(x)+\epsilon^3 y_3(x)+... \qquad ...(5)$$

Here the first term $\epsilon y_1(x)$ is known as *first order perturbation*, second term $\epsilon^2 y_2(x)$ as *second order perturbation* etc.

With the substitution of (4), (2) yields

$$y_0'+\epsilon y_1'+\epsilon^3 y_3'+\epsilon^2 y_2'+...+(y_0+\epsilon y_1+\epsilon^2 y_2+\epsilon^3 y_3+...)^2 = \epsilon x \qquad ...(6)$$

or $$(y_0'+y_0^2)+(y_1'+2y_0y_1)\,\epsilon+(y_2'+2y_0y_2+y_1^2)\,\epsilon^2+... = \epsilon x \qquad ...(7)$$

Equating like powers of ϵ on either side, we get

$$y_0'+y_0^2 = 0,\ y_1'+2y_0y_1 = x,\ y_2'+2y_0y_2+y_1^2 = 0, \text{ etc.} \qquad ...(8)$$

Solving these equations in succession, we may find $y_1(x), y_2(x), \ldots y_3(x) \ldots$ in (4) and each of these functions satisfies the initial condition $y(1) = 1$. But y_0 is a solution of (1) and hence we have

$$y_0(1) = 1, y_1(1) = 0, y_2(1) = 0 \quad \ldots(9)$$

Now in order to solve (1), substitute (4) in (2) to give

$$(y_0' + y_0^2) + (y_0' + 2y_0y_1)\in + (y_2' + 2y_0y_2 + y_1^2)\in^2 + \ldots = \in x \quad \ldots(10)$$

Equating coefficients of $\in^0, \in, \in^2$, on either side of (10), we get

$$y_0' + y_0^2 = 0, \ y_1^2 + 2y_0y_1 = x, \quad y_2' + 2y_0y_2 + y_1^2 = 0 \quad \ldots(11)$$

of which first equation gives for $y_0(1) = 1$, $y_0 = \frac{1}{x}$, and then second one gives $y_1' + \frac{2}{x}y_1 = x$ which with $y_1(1) = 0$, yields,

$$y_1 = \frac{1}{4}\left(x^2 - \frac{1}{x^2}\right)$$

With these values the third equation of (11) gives

$$y_2' + \frac{2}{x}y = -\frac{1}{16}\left(x^4 - 2 + \frac{1}{x^4}\right)$$

which with $y_2(1) = 0$ yields $y_2 = -\frac{1}{16}\left(\frac{x^5}{7} - \frac{2x}{3} - \frac{1}{x^3}\right) - \frac{2}{21x^3}$.

With these substitutions (4) yields

$$y = \frac{1}{x} + \frac{\in}{4}\left(x^2 - \frac{1}{x^2}\right) - \frac{\in^2}{336}\left(3x^5 - 14x - \frac{21}{x^3} + \frac{32}{x^2}\right) \quad \ldots(12)$$

This follows that the solution of (1) is $\frac{1}{x}$; but where the disturbing function $\in x$ is present, the first and second order perturbation terms are respectively the second and third terms in (12).

[B] Second Order Differential Equations.

Consider an equation $$\frac{d^2y}{dx^2} + y = 0 \quad \ldots(13)$$

With $y(0) = 0, y'(0) = 1$. ...(14)

(13) is the differential equation of S.H.M.

Taking the disturbing function as $-2\in(y')^2$, $\in$ being small, (13) can be modified as

$$y'' + y = -2\in(y')^2 \quad \ldots(15)$$

with $y(0) = 0, y'(0) = 1$.

If $y_0(x)$ is a solution of (13), then suppose that

$$y = y_0 + \in y_1 + \in^2 y_2 + \ldots \quad \ldots(16)$$

giving $y'' = y_0'' + \in y_1'' + \in^2 y_2'' + \ldots$

So that (16) will satisfy (14) under the assumptions

$$y_0(0) = 0, y_1(0) = 0, y_2(0) = 0, \ldots; \ y_0'(0) = 1, \ y_1'(0) = 0, \ y_2'(0) = 0 \ldots \quad \ldots(17)$$

Substituting (16) in (13) and using the terms upto $\in^2$, we find

$$y_0'' + \in_1 y_1'' + \in^2 y_2'' + y_0 + \in y_1 + \in^2 y_2 = -2\in[(y_0')^2 + \in^2 (y_1')^2$$

$$+ \in^4 (y_2') + 2\in y_0' y_1' + 2\in^2 y_0' y_2' + 2\in^3 y_1' y_2'] \qquad ...(18)$$

Collecting the coefficients of like powers of $\in$, we have,

$$(y_0'' + y_0) + (y_1'' + y_1)\in + (y_2'' + y_2)\in^2 = -2(y_0')^2 \in - 4 y_0' y_1' \in^2 \qquad ...(19)$$

Equating the coefficients of $\in^0, \in, \in^2$ on either side, we find

$$y_0'' + y_0 = 0,\ y_1'' + y_1 = -2\ (y_0')^2,\ y_2'' + y_2 = -4\ y_0' y_1' \qquad ...(20)$$

of which the first equation in view of $y_0(0) = 0$, $y_0'(0) = 1$ gives a solution

$$y_0 = \sin x,\quad y_0' = \cos x \qquad ...(21)$$

and then the second equation gives $y_1'' + y_1 = -2\ \cos^2 x$ ...(22)

So that a general solution of (22) is

$$y_1 = A_1 \sin x + A_2 \cos x - \tfrac{4}{3}\sin^2 x - \tfrac{2}{3}\cos^2 x;$$

$$y'_1 = A_1 \cos x - A_2 \sin x - \tfrac{4}{3} \sin x \cos x \qquad ...(23)$$

$\therefore$ A particular solution of (23) satisfying $y_1(0) = 0$, $y_1'(0) = 0$, is

$$y_1 = \tfrac{2}{3} \cos x - \tfrac{4}{3}\sin^2 x - \tfrac{2}{3}\cos^2 x;$$

$$y_1' = -\tfrac{2}{3}\sin x - \tfrac{8}{3}\sin x \cos x + \tfrac{4}{3}\sin x \cos x = -\tfrac{2}{3}\sin x - \tfrac{4}{3}\sin x \cos x \qquad ...(24)$$

With these substitutions in the third equation of (20) [*i.e.*, y_0 from (21) and y_1 from (24)], yields

$$y_2'' + y_2 = -4\left(-\tfrac{2}{3}\sin x \cos x - \tfrac{4}{3}\sin x \cos^2 x\right)$$

$$= \tfrac{8}{3}\sin x \cos x + \tfrac{16}{3}\sin x - \tfrac{16}{3}\sin^3 x \qquad ...(25)$$

Its Complementary function is

$$y_c = A_1 \sin x + A_2 \cos x \qquad ...(26)$$

While a particular solution;

$$\left.\begin{array}{l} \text{of } y_2'' + y_2 = \tfrac{8}{3} \sin x \cos x \text{ is } y_2 = -\tfrac{8}{9} \sin x \cos x \\ \text{of } y_2'' + y_2 = \tfrac{16}{3} \sin x \text{ is } y_2 = -\tfrac{8}{3} \cos x \\ \text{of } y_2'' + y_2 = \tfrac{16}{3} \sin^3 x \text{ is } y_2 = \tfrac{1}{6} \sin 3x - 2x \cos x \end{array}\right\} \qquad ...(27)$$

Hence with the help of (26) and (27), a general solution of (25) is

$$\left.\begin{array}{l} y_2 = A_1 \sin x + A_2 \cos x - \tfrac{8}{9}\sin x \cos x - \tfrac{2}{3}x \cos x - \tfrac{1}{6}\sin 3x, \\ y_2' = A_1 \cos x - A_2 \sin x + \tfrac{8}{9}\left(\sin^2 x - \cos^2 x\right) + \tfrac{2}{3}(x \sin x - \cos x) - \tfrac{1}{2}\cos 3x \end{array}\right\} \qquad ...(28)$$

Using the initial conditions of (17), (28) yields,

$$A_2 = 0,\ A_1 = \tfrac{8}{9} + \tfrac{2}{3} + \tfrac{1}{2} = \tfrac{37}{18} \qquad ...(29)$$

Hence using (28) and (29), a particular solution of (25) satisfying $y_2(0) = 0$, $y_2'(0) = 0$ is

$$y_2 = \tfrac{37}{18}\sin x - \tfrac{8}{9}\sin x \cos x - \tfrac{2}{3}\sin x \cos x - \tfrac{1}{6}\sin 3x \qquad ...(30)$$

Making substitutions for y_0, y_1 and y_2 from (21), (24) and (30) to (16) we find the solution as

$$y = \sin x + \epsilon\left(\tfrac{2}{3}\cos x - \tfrac{4}{3}\sin^2 x - \tfrac{2}{3}\cos^2 x\right)$$
$$+ \epsilon^2\left(\tfrac{37}{18}\sin x - \tfrac{8}{9}\sin x\cos x - \tfrac{2}{5}\cos x - \tfrac{1}{3}\sin 3x\right) \qquad ...(31)$$

This follows that in the absence of disturbing function the solution of (15) is $\sin x$, while in the presence of disturbing function $-2\epsilon\,(y')^2$, the first and second order perturbation terms are respectively the second and third terms in (31).

ADDITIONAL MISCELLENEOUS PROBLEMS

Problem 51. *Transform the equation* $\Delta^2\psi = \dfrac{1}{c^2}\dfrac{\partial\psi}{\partial t^2}$ *to spherical coordinates. If* ψ *depends on r and t only, show that the equation can be written in the form*

$$\frac{\partial^2}{\partial r^2}(r\psi) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(r\psi)$$

Hence show that the general solution is of the form

$$\psi(r, t) = \frac{1}{r}\{f(r-ct) + g(r+ct)\}$$

Explain the physical meaning of this solution. *(Bombay, 1965)*

Hint. Use the transformations $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

Problem 52. *Obtain Newton's formula for interpolation. Using this or some other formula find the value of the derivative at a point.*

The following table gives the corresponding values of x and y

x	*3*	*4*	*5*	*6*	*7*	*8*
y	*205*	*240*	*259*	*262*	*250*	*224*

Find the value of x for which y is maximum. *(Bombay, 1970)*

Problem 53. *Find the general solution of the equation* $\dfrac{d^2x}{dt^2} + k^2x = \phi(t)$, *where k is a real constant and the function* $\phi(t)$ *is a given function.* *(Rohilkhand, 1980)*

Hint. We have $(D^2 + k^2)\,x = \phi(t)$ with $D \equiv \dfrac{d}{dt}$

For C.F. $D^2 + k^2 = 0 \Rightarrow D = \pm ik$

$\therefore \qquad C.F. = Ae^{ikt} + Be^{-ikt}$

and

$$P.I. = \frac{\phi(t)}{D^2 + k^2} = \frac{\phi(t)}{(D - ik)(D + ik)} = \frac{1}{2ik}\left[\frac{\phi(t)}{D - ik} - \frac{\phi(t)}{D + ik}\right]$$

$$= \frac{1}{2ik}\left[e^{ikt}\int e^{-ikt}\,\phi(t)\,dt - e^{-ikt}\int e^{ikt}\,\phi(t)\,dt\right]$$

Hence the solution is $x = C.F. + P.I.$

Problem 54. *Find the solution of one-dimensional Time-independent Schröedinger's wave equation*

$$\frac{d^2\psi}{dx^2}+\frac{2m}{\hbar^2}(E-V)\psi=0 \qquad ...(1)$$

when the potential energy function V(x) is given by $V(x)=\begin{cases}0 \text{ for } x<0\\ V_0 \text{ for } x>0\end{cases}$

For $x<0$, if we take $k_0^2=\frac{2mE}{\hbar^2}$, then (1) yields $\frac{d^2\psi}{dx^2}+k_0^2\psi=0$ ...(2)

or $(D^2+k_0^2)\psi=0$, where $D\equiv\frac{d}{dx}$

Here $D^2+k_0^2=0\Rightarrow D=\pm i k_0$, as $\psi\neq 0$. Hence the solution of (2) is

$$\psi=Ae^{ik_0x}+Be^{-ik_0x} \text{ for } x<0 \qquad ...(3)$$

where A, B are courtants of integration.

Again for $x>0$, if we take $k^2=\frac{2m(E-V_0)}{\hbar^2}$, then (1) gives

$$\frac{d^2\psi}{dx^2}+k^2\psi=0 \qquad ...(4)$$

or $$(D^2+k^2)\psi=0 \text{ with } D\equiv\frac{d}{dx}$$

So that $D^2+k^2=0\Rightarrow D=\pm i\,k$ *as* $\psi\neq 0$

∴ The solution of (4) is $\psi=Ce^{ikx}+De^{-ikx}$ ...(5)

where C, D are constants of integration.

In the case when $x=0$, then (4) and (5) render $\psi=0=A+B=C+D$ ...(6)

Similarly for $\frac{d\psi}{dx}=0$, the equation (4) and (5) give $k_0(A-B)=k(C-D)$ ...(7)

Solving (6) and (7), we get

$$C=\frac{2k_0}{k_0+k}A-\frac{k_0-k}{k_0+k}D,\; B=\frac{k_0-k}{k_0+k}A+\frac{2k_0}{k_0+k}D \qquad ...(8)$$

Substituting these values in (3) and (5), we get the general solution as

$$\psi=A\psi_1+D\psi_2 \qquad ...(9)$$

where $$\psi_1=\begin{cases}e^{ik_0x}+\frac{k_0-k}{k_0+k}e^{-ik_0x} & (x<0)\\ \frac{2k_0}{k_0+k}e^{ikx} & (x>0)\end{cases} \qquad ...(10)$$

and $$\psi_2=\begin{cases}\frac{2k}{k_0+k}e^{-ik_0x} & (x<0)\\ e^{-ikx}-\frac{k_0-k}{k_0+k}e^{ikx} & (x>0)\end{cases} \qquad +B \qquad ...(11)$$

CHAPTER 8

HARMONICS

(With Special Functions)

8.1. INTRODUCTION

In the previous chapter on Differential Equations we have already mentioned that an equation of the type

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0 \text{ or in terms of 'del' operator } \nabla \text{ (nabla)}, \ \nabla^2 V=0 \qquad ...(1)$$

is known as *Laplace's equation.* It frequently appears in applied mathematics when discussions are made on mechanics, sound, eletricity, heat etc, wherever the theory of potential is involved *e.g.* if V be the Newtonian potential due to an attracting mass at a point (x, y, z) not forming a part of the mass itself then V satisfies (1). Similarly if V be the electric potential at (x, y, z) where the electric density is zero, then V satisfies (1). Moreover if a body is in a state of equilibrium as to temperature, V being the temperature at (x, y, z),

$$\frac{dV}{dt}=0 \text{ and } V \text{ satisfies (1).}$$

The equation (1) can be transformed to spherical polar form by using the transformation $x=r\sin\theta\cos\phi,\ y=r\sin\theta\sin\phi,\ z=r\cos\theta$

so that $r^2=x^2+y^2+z^2,\ \phi=\tan^{-1}\dfrac{y}{x},\ \theta=\tan^{-1}\dfrac{\sqrt{x^2+y^2}}{z}$

and $\dfrac{\partial r}{\partial x}=\dfrac{x}{r}=\sin\theta\cos\phi,\ \dfrac{\partial r}{\partial y}=\dfrac{y}{r}=\sin\theta\sin\phi,\ \dfrac{\partial z}{\partial r}=\dfrac{z}{r}=\cos\theta,$

$$\frac{\partial\theta}{\partial x}=\frac{\cos\theta\cos\phi}{r},\ \frac{\partial\theta}{\partial y}=\frac{\cos\theta\sin\phi}{r};\ \frac{\partial\theta}{\partial z}=-\frac{\sin\theta}{r},$$

$$\frac{\partial\phi}{\partial x}=-\frac{\sin\phi}{r\sin\theta},\ \frac{\partial\phi}{\partial y}=\frac{\cos\phi}{r\sin\theta},\ \frac{\partial\phi}{\partial z}=0$$

whence $$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial x}+\frac{\partial u}{\partial\phi}\frac{\partial\phi}{\partial x}=\frac{\partial u}{\partial r}\sin\theta\cos\phi$$

$$+\frac{\partial u}{\partial\theta}\frac{\cos\theta\ \cos\phi}{r}+\frac{\partial u}{\partial\phi}\left(-\frac{\sin\phi}{r\sin\theta}\right)$$

giving $$\frac{\partial}{\partial x}=\sin\theta\cos\phi\frac{\partial}{\partial r}+\frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta}-\frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\cdot$$

$$\therefore \quad \frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) = \left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$

$$\left(\sin\theta\cos\phi\frac{\partial V}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial V}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial V}{\partial\phi}\right)$$

$$= \sin^2\theta\cos^2\phi\frac{\partial^2 V}{\partial r^2} + \frac{2\sin\theta\cos\theta\cos^2\phi}{r}\frac{\partial^2 V}{\partial r\partial\theta}$$

$$- \frac{2\sin\theta\cos\theta\cos^2\phi}{r^2}\frac{\partial V}{\partial\theta} - \frac{2\sin\phi\cos\phi}{r}\frac{\partial^2 V}{\partial r\,\partial\phi}$$

$$+ \frac{\sin\phi\cos\phi}{r^2}\frac{\partial V}{\partial\phi} + \frac{\cos^2\theta\cos^2\phi}{r}\frac{\partial V}{\partial r}$$

$$+ \frac{\cos^2\theta\cos^2\phi}{r^2}\frac{\partial^2 V}{\partial\theta^2} - \frac{2\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2 V}{\partial\theta\partial\phi}$$

$$+ \frac{\cos^2\theta\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial V}{\partial\phi} + \frac{\sin^2\phi}{r}\frac{\partial V}{\partial r} + \frac{\cos\theta\sin^2\phi}{r^2\sin\theta}\frac{\partial V}{\partial\theta}$$

$$+ \frac{\sin^2\phi}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} + \frac{\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial V}{\partial\phi}$$

with similar expression for $\frac{\partial^2 V}{\partial y^2}$ by replacing ϕ by $\frac{\pi}{2} + \phi$ in this expression

$$\text{and} \quad \frac{\partial^2 V}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial z}\right) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial\theta}\right)$$

$$= \cos^2\theta\frac{\partial^2 V}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 V}{\partial r\,\partial\theta} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial V}{\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial V}{\partial r}$$

$$+ \frac{\sin^2\theta}{r^2}\frac{\partial^2 V}{\partial r^2}$$

With these substitutions, the spherical polar form of (1) is

$$\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial V}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0 \quad \text{...(2)}$$

Which may be put as

$$\frac{1}{r^2}\left\{\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}\right\} = 0 \quad \text{...(3)}$$

Also if $\mu = \cos\theta$, it takes the form

$$r\frac{\partial^2(Vr)}{\partial r^2} + \frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial V}{\partial\mu}\right\} + \frac{1}{1-\mu^2}\frac{\partial^2 V}{\partial\phi^2} = 0 \quad \text{...(4)}$$

The subject matter of *Spherical Harmonics* is partly concerned with the development of the functions which will satisfy this equation.

A homogeneous rational integral algebraic function of (x, y, z) of degree n in the form $r^n f(\theta, \phi)$ in spherical polar coordinates which is a value of V satisfying (1) is said to be a *solid spherical Harmonics* of nth degree and the function $f(\theta, \phi)$ is said to be a *surface spherical Harmonics* of nth degree. Since Laplace employed these spherical Harmonics in determining V, these are also known as ***Laplace's coefficients.***

In case V is independent of ϕ *i.e.* $\frac{\partial V}{\partial \phi} = 0$, and so $\frac{\partial^2 V}{\partial \phi^2} = 0$, then (3) reduces to

$$r\frac{\partial^2(rV)}{\partial r^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0 \qquad ...(5)$$

If we put $V = r^n P$, P being a function of θ only and then change the independent variable θ by the transformation $\mu = \cos\theta$, the equation (5) yields

$$\frac{d}{\partial\mu}\left(1-\mu^2\right)\frac{dP}{d\mu} + n(n+1)\ P = 0 \qquad ...(6)$$

which is known as Legendre's equation and will be discussed a bit later.

A function satisfying (5) or (6) is said to be a *Surface Zonal Harmonics*. A special class of zonal harmonics is sometimes known as *Legendrean Coefficients*.

There are equations belonging to such a form which cannot be solved by any of the methods discussed in the previous chapter then we claim to find a convergent series arranged according to powers of the independent variable, which will approximately express the value of the dependant variable. For this purpose the series solution of linear differential equations will be discussed in the next section and then in the subsequent sections the series integration method will be used to discuss the solutions of very important equations that often occur in investigations in applied mathematics such as the equations of Riccati, Bessel, Legendre, Hermite, Laguerre and the Hypergeometric series.

Note. Bessel's functions are known as **Cylindrical Harmonics.**

8.2. METHODS OF INTEGRATION IN SERIES

[A] Power Series Solutions of Linear Differential Equations.

We know that a linear differential equation is one in which both the dependent variables and their derivatives are of the first degree and these do not occur as products of dependent variables and/or their derivatives.

A series of the form

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + ... \qquad ...(1)$$

where $a_0, a_1, a_2, ..., x_0$ are constants and x is a variable, is said to be a *power series*, and it may converge

(*i*) only for the single value $x = x_0$

(*ii*) absolutely for $|x - x_0| < \in$ *i.e.* for values of x in the neighbourhood of x_0 and diverge for $|x - x_0| < \in$ while at the end points $x_0 \pm \in$, it may either converge or diverge.

(*iii*) absolutely for all x *i.e.* for $-\infty < x < \infty$.

The set of values of x for which the power series converges is said to be the *Interval of Convergence* and denoted by I.

In case power series converges on an interval $I : |x - x_0| < R$, R being a positive constant, then the power series defines a function $f(x)$ which is continuous for each x in I.

If $f(x)$ be a function defined by a power series *i.e.*

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 ... \text{ for } I : |x - x_0| < R \qquad ...(2)$$

then the power series obtained by differentiating each term of (2) defines the derivative of $f(x)$ on the same interval I *i.e.*

$$f'(x) = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + ... \text{for } I : |x - x_0| < R \qquad ...(3)$$

We can thus define the successive derivative $f''(x), f'''(x)$...and so on.

It is clear that $f(x_0) = a_0, f'(x_0) = a_1, \dfrac{f''(x_0)}{\lfloor 2} = a_2 \ldots$ etc.

As such (2) yields

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{\lfloor 2}(x - x_0)^2 + \ldots\ldots$$

$$+ \frac{f^n(x_0)}{\lfloor n}(x - x_0)^n + \ldots; |x - x_0| < R \qquad \ldots(4)$$

which is *Taylor's series expansion* in powers of $(x - x_0)$.

If $x_0 = 0$ this becomes

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{\lfloor 2}x^2 + \ldots + \frac{f^n(0)}{\lfloor n}x^n + \ldots, \; |x| < R \qquad \ldots(5)$$

which is *Maclaurin's series expansion* in power of x.

Two functions $f(x)$ and $g(x)$ defined by power series such that

$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots, \; |x - x_0| < R$

$g(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)^2 + \ldots \; |x - x_0| < R$

will be equal if and only if $a_0 = b_0, a_1 = b_1, a_2 = b_2$ etc.

A function $f(x)$ is said to be *analytic* or *regular* at a point $x = x_0$ if it can be expanded as a Taylor's series expansion in powers of $(x - x_0)$ for every x in the neighbourhood of x_0 and it is *analytic on the interval I* if it is analytic at every point of the interval.

Now consider a linear differential equation with variable coefficients, such that

$$y^{(n)} - f_{n-1}(x)\, y^{(n-1)} + \ldots + f_1(x)\, y' + f_0(x)\, y = Q(x) \qquad \ldots(6)$$

Where $y^{(n)} = \dfrac{d^n y}{dx^n}$ *etc...., then a sufficient condition for the existence of a power series solution of* (6) *is that each function* $f_0(x), f_1(x), \ldots f_{n-1}(x), Q(x)$ *is analytic at* $x = x_0$ *and the solution which is unique must satisfy the n initial conditions*

$$y(x_0) = a_0, y'(x_0) = a_1, \ldots y^{(n-1)}(x_0) = a_{n-1}.$$

There are two methods for solving such equations by series integrations.

(1) *First Series Method* in which the series solution is found by successive differentiation.

(2) *Second Series Metheod* in which the series solution is obtained by undetermined coefficients.

We explain these methods applying on the following Problems.

Problem 1. *Apply Power-series method to find the solution of the following linear equation*:

$$\frac{d^2y}{dx^2} - (x+1)\frac{dy}{dx} + x^2 y = x$$

i.e. $y'' - (x+1)y' + x^2 y = x$ *with the initial conditions* $y(0) = 1, y'(0) = 1$

We have $y'' - (x+1)y' + x^2 y = x \qquad \ldots(1)$

Its comparison with (6) of §8.2 gives

$$f_0(x) = x^2, f_1(x) = -x - 1, Q(x) = x.$$

All these functions being polynomials, the series solution of (1) is valid for all x, since a polynomial is a finite series and therefore the functions $f_0(x), f_1(x), Q(x)$ each being polynomial, every solution of (1) has a Taylor series expansion valid for all x.

First Method. Suppose we seek a solution in the form of Maclaurin series. Then replacing $f(x)$ by $y(x)$ in (5) of §8.2, we get

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{\underline{|2}}x^2 + \frac{y'''(0)}{\underline{|3}}x^3 + \frac{y^{(4)}(0)}{\underline{|4}}x^4 + \ldots \quad \ldots(2)$$

Initially when $x = 0, y = 1, y' = 1$, so that (1) gives $y''(0) = 1$ ...(3).

We thus know the values $y(0), y'(0), y''(0)$ and in order to find the values of succeeding coefficients, let us take successive derivatives of (1) and evaluate them at $x = 0, y = 1$. Differentiating (1) w.r.t. x, we get

$$y''' - (x + 1)y'' - y' + x^2y' + 2xy = 1 \quad \ldots(4)$$

which yields $y'''(0) = 3$ ...(5)

when $x = 0, y = 1, y' = 1, y'' = 1$.

Again differentiating (4) w.r.t x, we find

$$y^{(4)} - (x + 1)y''' - 2y'' + x^2y'' + 4xy' + 2y = 0.$$

Which yields $y^{(4)}(0) = 3$ ...(6)

when $x = 0, y = y' = y'' = 1, y''' = 3$.

Similarly $y^{(5)}(0), y^{6}(0), \ldots$ etc. can be evaluated.

Substituting these values in (2) we get the required series solution as

$$y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{8} + \ldots, -\infty < x < \infty \quad \ldots(7)$$

Second Method. Suppose the series solution in powers of x has the form

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \ldots \quad \ldots(8)$$

Since $y(x)$ is valid for all x, therefore its two successive derivatives will also be valid for all x. The two successive derivatives of (8) are

$$y'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots \quad \ldots(9)$$

$$y''(x) = 2a_2 + 6a_3x + 12a_4x^2 + \ldots \quad \ldots(10)$$

Substituting the values of (8), (9) and (10) in (1) we get

$$2a_2 + 6a_3x + 12a_4x^2 + \ldots - (x + 1)(a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots)$$
$$+ x^2(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots) = x$$

i.e., $(2a_2 - a_1) + (6a_3 - 2a_2 - a_1 - 1)x + (12a_4 - 3a_3 - 2a_2 + a_0)x^2 \ldots = 0$.

This being an identity in x, the coefficients of powers of x vanish separately

i.e. $2a_2 - a_1 = 0$ i.e. $a_2 = \frac{a_1}{2}$,

$$\left.\begin{aligned} 6a_3 - 2a_2 - a_1 - 1 = 0 \text{ i.e., } a_3 &= \frac{2a_2 + a_1 + 1}{6} \\ 12a_4 - 3a_3 - 2a_2 + a_0 = 0 \text{ i.e., } a_4 &= \frac{3a_3 + 2a_2 - a_0}{12} \end{aligned}\right\} \quad \ldots(11)$$

Comparison of (8) with Taylor series i.e. (4) of §8.2 yields, $f(x_0) = y(0) = a_0 = 1$, $f'(x_0) = y'(0) = a_1 = 1$.

As such (11) give $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{2}$, $a_4 = \frac{1}{8}$.

Substituting these values in (8) we get the required series solution as

$$y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{2} + \frac{x^4}{8} \ldots, -\infty < x < \infty.$$

Problem 2. *Find by power series methods a particular solution of the linear differential equation:*

$$y''' + \frac{1}{x}y' - \frac{1}{x^2}y = 0,\ x \neq 0 \text{ and } y(1) = 1,\ y'(1) = 0,\ y''(1) = 1$$

Here $f_0(x) = -\frac{1}{x^2}$, $f_1(x) = \frac{1}{x}$, $Q(x) = 0$ etc.

Ans. $y(x) = 1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} - \frac{(x-1)^4}{8} + \frac{(x-1)^5}{15} - \ldots 0 < x < 2$

[B] Ordinary Points and Singularities of a Linear Differential Equation.

Consider a linear differential equation of the form

$$y^{(n)} + F_{n-1}(x)\, y^{(n-1)} + \ldots + F_1(x)\, y' + F_0(x) y = Q(x) \qquad \ldots(7)$$

A point $x = x_0$ is said to be an *ordinary point* of (7) if each function $F_0, F_1, \ldots F_{n-1}$ and Q is analytic at $x = x_0$ *i.e.* each function has a Taylor series expansion in powers of $(x - x_0)$ valid in the neighbourhood of x_0.

Whenever $x = x_0$ is an ordinary point then (7) has a solution which is also analytic at $x = x_0$ *i.e.*, the solution has a Taylor series representation in powers of $(x - x_0)$ in the neighbourhood of x_0.

A point $x = x_0$ is said to be a *singularity* of (7) if one or more of the functions $F_0(x) \ldots, F_{n-1}(x)$, $Q(x)$ are not analytic at $x = x_0$.

If we now consider a second order linear equation

$$y'' + F_1(x)y' + F_2(x)\, y = 0 \qquad \ldots(8)$$

F_1, F_2 being continuous functions of x, then if $x = x_0$ is a singularity of (8) and the product functions $F_1(x)(x - x_0)$, and $F_2(x)(x - x_0)^2$ both are analytic at $x = x_0$, then the point $x = x_0$ is said to be a ***regular singularity***. In case one or both of the product functions are not analytic at $x = x_0$ then the point $x = x_0$ is said to be an *irregular singularity*.

[C] Frobenius Method for Solving a Homogeneous Linear Differential Equation.

Consider,

$$y'' + F_1(x)y' + F_2(x)y = 0 \qquad \ldots(9)$$

In case (9) has an irregular singularity at $x = x_0$ then it is too difficult to find the series solution of (9) here, but if it has a regular singularity at $x = x_0$ then the series solution of (9) can be found out in the neighbourhood of x_0. For the purpose of finding the solution in later case, Frobenius introduced a series solution *i.e.*,

$$y = (x - x_0)^m\, [a_0 + a_1 (x - x_0) + a_2(x - x_0)^2 + a_3 (x - x_0)^3 + \ldots],\ a_0 \neq 0 \qquad \ldots(10)$$

which is known as a *Frobenius Series.*

When $m = 0$, (10) reduces to the usual Taylor series and hence Taylor's series is a special case of Frobenius series.

In case m is negative or has non-integral positive values then (10) is not a Taylor series.

Assuming that x_0 is a regular singularity of (9) and F_1 or F_2 or both are not analytic at $x = x_0$ but $(x - x_0) F_1$ and $(x - x_0)^2 F_2$ both are analytic at $x = x_0$, it follows that F_1 has $(x - x_0)$ in its denominator and/or F_2 has $(x - x_0)^2$ in its denominator, thereby showing that either $F_1(x) = f_1(x)/(x - x_0)$ or $F_2(x) = f_2(x)/(x - x_0)^2$ or both exists. In either case if (9) is multiplied by $(x - x_0)^2$, it will transform to the form

$$(x - x_0)^2 y'' + (x - x_0) f_1(x)\, y' + f_2(x)\, y = 0 \qquad ...(11)$$

where $f_1(x)$ and $f_2(x)$ both now become analytic at $x = x_0$.

Existence of Frobenius Series Solution.

If x_0 is a regular singularity of (11), then it has at least one Frobenius series solution of the form (10) if it is valid in the common interval of convergence of $f_1(x)$ and $f_2(x)$ of (11) except possibly for $x = x_0$.

Without loss of generality, taking $x_0 = 0$, (11) reduces to

$$x^2 y'' + x f_1(x) y' + f_2(x)\, y = 0 \qquad ...(12)$$

where $f_1(x)$ and $f_2(x)$ are analytic at $x = 0$ and therefore each function has a Taylor series expansion in powers of x valid in the neighbourhood of $x = 0$. Let us assume

$$f_1(x) = b_0 + b_1 x + b_2 x^2 + ... \qquad ....(13)$$

$$f_2(x) = c_0 + c_1 x + c_2 x^2 \cdots \qquad ...(14)$$

For $x_0 = 0$, Frobenius series *i.e.*, (10) and its next two derivatives become

$$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + ... + a_n x^{m+n} + ...,\ a_0 \neq 0, \qquad ...(15)$$

$$y' = a_0 m x^{m-1} + a_1 (m + 1)\, x^m + a_2 (m + 2)\, x^{m+1} + ... + a_n (m + n)\, x^{m+n-1} + ... \qquad ...(16)$$

$$y'' = a_0 m(m - 1)\, x^{m-2} + a_1 m\, (m + 1)\, x^{m-1} + a_2\, (m + 1)\, (m + 2) x^m$$
$$+ ... + a_n (m + n - 1)\, (m + n)\, x^{m+n-2} + ... \qquad ...(17)$$

$y(x)$ will be a solution of (12) if (13), (14), (15), (16), (17) identically satisfy (12) *i.e.*, if

$$x^2[a_0 m(m - 1)x^{m-2} + a_1 m(m + 1)\, x^{m-1} + ... + a_n (m + n - 1)\, (m + n)\, x^{m+n-2} + ...]$$
$$+ x[b_0 + b_1 x + b_2 x^2 + ...)\, [a_0 m x^{m-1} + a_1 (m + 1) x^m + ... + a_n\, (m + n)\, x^{m+n-1} + ...]$$
$$+ (c_0 + c_1 x + c_2 x^2 + ...)\, [a_0 x^m + a_1 x^{m+1} + ... + a_n x^{m+n} + ...) \equiv 0 \qquad ...(18)$$

If we expand (18) and collect the coefficients of like power of x, then we get

$$a_0\, [m(m - 1) + b_0 m + c_0] x^m + [a_1\, \{(m + 1) m + b_0\, (m + 1) + c_0\}$$
$$+ a_0 (b_1 m + c_1)] x^{m+1} + ... [a_n \{(m + n)\, (m + n - 1) + b_0 (m + n) + c_0\} + a_{n-1}$$
$$\{b_1 (m + n - 1) + c_1\} + a_{n-2}\, \{b_2\, (m + n - 2) + c_2\} + ... + a_0\, (b_n{}^m + c_n)]\, x^{m+n} \equiv 0$$
$$...(19)$$

This relation being an identity, each of the coefficients of x^k, $k = m, m + 1, ..., m + n, ...$ is zero. But since $a_0 \neq 0$, therefore the first term in (19) gives

$$m(m - 1) + b_0 m + c_0 = 0\ , \qquad ...(20)$$

which is known as *Indicial Equation*. This is quadratic in m and hence has two roots say m_1 and m_2. There arise three cases according as $m_1 \neq m_2$ and $m_1 - m_2$ = not an integer; $m_1 \neq m_2$ but $m_1 - m_2$ = *an* integer; $m_1 = m_2$.

Case I. *When m_1, m_2 are distinct and their difference is not an integer.*

Let us first put $m = m_1$ in all the coefficients of (19) except the first one and then equate each equal to zero. We are thus capable of finding $a_1, a_2, ... a_n, ...$ in terms of a_0. Substituting these values of $a_1, a_2, ... a_n$, in (15) we get the solution $y(x)$.

With similar procedure taking $m = m_2$, we may get another solution.

Problem 3. *Solve* $2x^2y'' - xy' + (1 - x^2)y = 0$.

Let the series solution be

$$y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots + a_nx^{m+n} + \ldots, \; a_0 \neq 0 \qquad \ldots(1)$$

so that $y' = ma_0x^{m-1} + (m + 1)a_1x^m + (m + 2)\, a_2x^{m+1} + \ldots + (m + n)\, a_nx^{m+n-1} + \ldots$

$$y'' = m(m-1)a_0x^{m-2} + (m+1)m\, a_1x^{m-1} + \ldots (m+n)\,(m+n-1)\, a_n x^{m+n-2} + \ldots$$

Substituting these values in the given equation, we get an identity

$$(m-1)\,(2m-1)\, a_0x^m + m(2m+1)\, a_1x^{m+1} + \{(m+1)\,(2m+3)\, a_2 - a_0\}$$
$$x^{m+2} + \ldots + \{(m+n-1)\,(2m+2n-1)a_n - a_{n-2}\}x^{m+n} = 0 \qquad \ldots(2)$$

Since $a_0 \neq 0$, the coefficient of first term vanishes if $x = 1, \frac{1}{2}$ *i.e.*, roots of the indicial equation are 1 and $\frac{1}{2}$ which are distinct and not differ by an integer.

The coefficients of x^{m+n} equated to zero gives the *recursion formula* ;

$$a_n = \frac{1}{(m+n-1)\,(2m+2n-1)}\, a_{n-2} \text{ for } n \geq 2 \qquad \ldots(3)$$

since all terms except the first two in (2) vanish if $a_2, a_3, \ldots$ satisfy (3)

The second term of (2) gives $a_1 = 0$ since $m(2m + 1) \neq 0$ for $m = 1, \frac{1}{2}$.

When $m = 1$, (3) gives $a_n = \dfrac{1}{n(2n+1)} a_{n-2}$.

Putting $n = 2, 4, 6, 8, \ldots$ we get (since $a_1 = a_3 = a_5 \ldots = 0$)

$$a_2 = \frac{1}{2.}a_0$$

$$a_4 = \frac{1}{4\cdot9}a_2 = \frac{1}{2\cdot5\cdot4\cdot9}a_0 = \frac{1}{2\cdot4\cdot5\cdot9}a_0$$

$$a_6 = \frac{1}{6\cdot13}a_4 = \frac{1}{2\cdot4\cdot6\cdot5\cdot9\cdot13}a_0 \text{ etc.}$$

Hence from (1), say $y_1 = a_0x\left[1 + \dfrac{x^2}{2\cdot5} + \dfrac{x^6}{2\cdot4\cdot5\cdot9} + \dfrac{x^6}{2\cdot4\cdot6\cdot5\cdot9\cdot13} + \ldots\right]$

Again when $m = \dfrac{1}{2}$, (3) gives $a_n = \dfrac{1}{\left(n - \frac{1}{2}\right)\cdot 2n} a_{n-2}$

Here also $a_1 = a_3 = a_5 = \ldots = 0$

and

$$a_2 = \frac{1}{\frac{3}{2}\cdot4}a_0 = \frac{1}{2\cdot3}a_0 ,$$

$$a_4 = \frac{1}{2\cdot4\cdot3\cdot7}a_0$$

$$a_6 = \frac{1}{2\cdot4\cdot6\cdot3\cdot7\cdot11}a_0 \text{ etc.}$$

So that from (1), say $y_2 = a_0\sqrt{x}\left[1 + \dfrac{x^2}{2\cdot3} + \dfrac{x^6}{2\cdot4\cdot3\cdot7} + \dfrac{x^6}{2\cdot4\cdot6\cdot3\cdot7\cdot11} + \ldots\right]$

The complete solution is

$$y = Ay_1 + By_2$$

$$= Ax\left[1+\frac{x^2}{2\cdot 5}+\frac{x^4}{2\cdot 4\cdot 5\cdot 9}+\dots\right]+B\sqrt{x}\left[1+\frac{x^2}{2\cdot 3}+\frac{x^4}{2\cdot 4\cdot 3\cdot 7}+\dots\right]$$

where a_0 has been incorporated into the arbitrary constants A and B.

Problem 4. *Solve* $x^2y'' + x\left(x+\frac{1}{2}\right)y' - \left(x^2+\frac{1}{2}\right)y = 0$

Ans. $y = Ax\left[1-\frac{2}{5}x+\frac{9}{35}x^2-\frac{82}{945}x^3+\dots\right]+Bx^{-1/2}\left[1-x+\frac{3}{2}x^2-\frac{13}{18}x^3+\dots\right]$

Problem 5. *Solve* $4xy'' + 2(1-x)y' - y = 0.$

Ans. $y = A\left(1+\frac{1}{2}x+\frac{1}{2^2\cdot \lfloor 2}x^2+\frac{1}{2^2\cdot \lfloor 3}x^3+\dots\right)$

$$+B\sqrt{x}\left[1+\frac{x}{1\cdot 3}+\frac{x^2}{1\cdot 3\cdot 5}+\frac{x^3}{1\cdot 3\cdot 5\cdot 7}+\dots\right]$$

Case II. ***When $m_1 \neq m_2$ and their difference is an integer.***

If m_1, m_2 differ by a non-zero integer, we can write them as m and $m + N$ where N is a positive integer. Then the indicial equation (20) can be written as

$$(m+N)(m+N-1) + b_0(m+N) + c_0 = 0 \qquad \dots(21)$$

since $m + N$ is a root of (20).

Comparison of L.H.S. of (21) with the coefficient of a_n in (19) shows that both of them are exactly the same except that n is replaced by N. This follows that the use of smaller root m in (19) to find a's, will make each coefficient of x^k zero and so we shall be stopped at the term in which a_N appears and whose coefficient is zero. Consequently the equation for a_N can not be solved in terms of previous a's unless accidentally the remaining terms also add to zero. In such cases, the equation will be satisfied for all arbitrary values of a_N. We may then continue to find values of $a_{N+1}, a_{N+2}, \dots$ in terms of a_0 and a_N.

Now there arise two possibilities:

Possibility (1). *In (19), the coefficient of a_N is zero and the remaining terms in the coefficient of x^{m+N} also add to zero.* Then the larger root $m + N$ will determine a set of values of a's in terms of a_0 and the smaller root m will determine two sets of values of a's; one in terms of a_0 and the other in terms of a_N, whose linear combination will yield the general solution.

Problem 6. *Solve* $x^2y'' + xy' + \left(x^2-\frac{1}{2^2}\right)y = 0$, *(which is Bessel equation of index $\frac{1}{2}$).*

Clearly $x = 0$ is a regular singularity of the equation

$$x^2y''+xy'+\left(x^2-\frac{1}{2^2}\right)y=0 \qquad \dots(1)$$

Let its Frobenius series solution be

$$y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \dots \qquad \dots(2)$$

Derive y' and y'' and put these values in (1). Proceeding just like in Problem 3, we shall find, the indicial equation as

$$m(m-1)+m-\frac{1}{4}=0 \text{ i.e. } m^2-\frac{1}{4}=0 \quad \ldots(3)$$

Giving $m=\pm\frac{1}{2}$ which differ by an integer unity.

The indicial equation may also be found by comparing (1) with (12) of § 8.2 [C] whence we get $f_1(x)=1, f_2(x)=-\frac{1}{4}+x^2$

So that $f_1(x) = 1 = b_0+b_1x+b_2x^2+\ldots$ which is (13) of § 8.2 [C]

and $f_2(x) = -\frac{1}{4}+x^2 = c_0+c_1x+c_2x^2+\ldots$ which is (14) of § 8.2 [C]

Comparison gives, $b_0=1$, $c_0=\frac{-1}{4}$, $c_1=0$, $c_2=1$, remaining b's and c's being zero.

Substituting these values in the indicial equation (20) we get (3) giving $m=-\frac{1}{2},\frac{1}{2}$ so that $N=\frac{1}{2}-\left(-\frac{1}{2}\right)=1$.

∴ Taking smaller root $-\frac{1}{2}$ and setting the second coefficient of (19) to zero, we find $a_1\left[\frac{1}{2}\left(-\frac{1}{2}\right)+\frac{1}{2}-\frac{1}{4}\right]+a_0(0)=0$ i.e. $0\cdot a_1+0\cdot a_0=0$.

Since the coefficient of a_1 is zero and the other term in its coefficient is also zero, therefore we have the first possibility. Thus a_0 and a_1 are arbitrary.

In case of $m=-\frac{1}{2}$, the recursion formula by setting the coefficient of x^{m+n} as zero is $a_n\left[\left(-\frac{1}{2}+n\right)\left(-\frac{3}{2}+n\right)+\left(-\frac{1}{2}+n\right)-\frac{1}{4}\right]+a_{n-1}(0)+a_{n-2}(1)=0$

i.e. $$a_n=-\frac{a_{n-2}}{n^2-n}, \text{ for } n\geq 2$$

which yields,

$$a_2=-\frac{1}{2}a_0 \qquad \text{and} \qquad a_3=-\frac{1}{6}a_1$$

$$a_4=-\frac{1}{12}a_2=\frac{1}{24}a_0 \qquad a_5=-\frac{1}{20}a_3=\frac{1}{120}a_1$$

$$a_6=-\frac{1}{30}a_4=-\frac{1}{720}a_0 \qquad , \qquad a_7=-\frac{1}{42}a_5=\frac{1}{5040}a_1$$

etc... etc...

Substituting these values in (2), we get the solution for $m=-\frac{1}{2}$,

$$y=a_0x^{-1/2}\left(1-\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{720}x^6+\ldots\right)+a_1x^{1/2}$$

$$\left(1-\frac{x^2}{6}+\frac{x^4}{120}-\frac{x^6}{5040}+\ldots\right)$$

Possibility (2). *In (19) the coefficient of* a_N *is zero and the remaining terms in the coefficient of* x^{m+N} *do not add to zero.* Then the larger root $m+N$ will determine a set of values of a's in terms of a_0. In this case there will be only one Frobenius series solution of (12) and a second independent solution will be of the form.

$$y_2(x) = u(x) - b_N y_l(x) \log x, n > 0 \qquad ...(22)$$

where N is the positive integral difference betwen m_l and m_2 ; $y_l(x)$ is a Frobenius series solution of (12) with the larger root $m+N$ and $u(x)$ is a Frobenius series of the form.

$$u(x) = x^m(b_0 + b_l x + b_2 x^2 + ...) \qquad ...(23)$$

where m is the smaller root of (2). If we substitute (22) and (23) with their necessary derivatives in (12), we shall find that $y_2(x)$ is a solution of it provided.

$$x^2 u'' + xf_l u' + f_2 u = b_N [2xy_1' + (f_l - 1) y_1] \qquad ...(24)$$

Substituting for u, u', u'', y, and y'_1 in it, we get the values of b's in (23). These solutions are known as *Logarithmic solutions*.

Problem 7. *Solve* $x^2y'' - x(2-x)y' + (2+x^2)y = 0.$ ...(1)

If we divide (1) by x^2, then we observe that $x = 0$ is a regular singularity.

Let the Frobenius series solution be

$$y = x^m(a_0 + a_1 x + a_2 x^2 + ...) \qquad ...(2)$$

Comparing it with (12), $f_1(x) = -2 + x, f_2(x) = 2 + x^2$ ...(3)

So that (13) and (14) give on comparison

$$b_0 = -2, b_1 = 1; c_0 = 2, c_1 = 0, c_2 = 1. \qquad ...(4)$$

the remaining b's and c's being zero.

Hence from (20), the indicial equation is

$$m(m-1) - 2m + 2 = 0 \text{ i.e. } m^2 - 3m + 2 = 0 \qquad ...(5)$$

giving $m = 1, 2$, *i.e.* $m_l = 1, m_2 = 2$ which differ by $N = 1$.

Taking $m = 1$ and setting the second coefficient of (19) as zero, we have

$$a_1(2 - 4 + 2) + a_0(1 + 0) = 0 \text{ i.e. } 0.a_1 + a_0 = 0 \qquad ...(6)$$

So that $a_0 \neq 0$ but the coefficient of $a_N = a_1$ is zero and hence $m = 1$ cannot give a solution.

Now taking $m = 2$ and setting the second coefficient of (19) equal to zero, we find

$$a_1(3.2 - 2.3 + 2) + 2a_0 = 0 \quad i.e. \quad a_1 = -a_0 \qquad ...(7)$$

Again by setting the coefficient of x^{m+n} of (19) equal to zero when $m = 2$, the recursion formula is

$$a_n[(2+n)(1+n) - 2(2+n) + 2] + (1+n)a_{n-1} + a_{n-2} = 0$$

$$i.e. \ (n^2 + n)a_n = -(n+1)a_{n-1} - a_{n-2} \text{ for } n \geq 2 \qquad ...(8)$$

When $n = 2$, (8) gives $6a_2 = -3a_1 - a_0 = 3a_0 - a_0$ by (7)

$$i.e. \quad a_2 = \frac{a_0}{3} \qquad ...(9)$$

When $n = 3$, (8) gives $12a_3 = -4a_2 - a_1 = -\frac{4}{3}a_0 + a_0$ by (7) and (9)

$$i.e. \quad a_3 = -\frac{a_0}{36} \text{ and so on.}$$

Hence from (2), the solution is

$$y_1 = a_0 x^2 \left[1 - x + \frac{x^2}{2} - \frac{x^3}{36} + \ldots\right] \quad \ldots(10)$$

A second solution of (1) will have the form

$$y_2(x) = u(x) - b_1 y_1(x) \log x,\ x > 0 \text{ as } N = 1, \quad \ldots(11)$$

where $u(x) = x\,(b_0 + b_1 x + b_2 x^2 + \ldots\ldots)$.

Case III. *When* $m_1 = m_2$.

In this case only one Frobenius series solution is possible.

Problem 8. *Solve* $x^2 y'' + xy' + x^2 y = 0$ *(which is Bessel equation for index 0).*

Given equation is $\quad x^2 y'' + xy' + x^2 y = 0 \quad \ldots(1)$

Dividing it by x^2 we observe that $x = 0$ is a regular singularity of it.

Let the series solution of it be

$$y = x^m (a_0 + a_1 x + a_2 x^2 + \ldots) \quad \ldots(2)$$

Comparing it with (12) of §8.2 [*C*], we find

$$f_1(x) = 1, f_2(x) = x^2$$

So that (13) and (14) of §8.2 [*C*] give on comparison

$$b_0 = 1, c_0 = 0, c_1 = 0, c_2 = 1 \quad \ldots(3)$$

other b's and c's being zero.

The indicial equation (20) becomes here

$$m(m-1) + m = 0 \quad i.e.\ m^2 = 0 \text{ giving } m = 0, 0 \quad \ldots(4)$$

Taking $m = 0$ and setting the second coefficient in (19) equal to zero, we have

$$a_1(1+0) + a_0(0) = 0 \quad i.e.\ a_1 = 0 \quad \ldots(5)$$

Again setting the coefficient of x^{m+n} in (19) equal to zero, we find the recursion formula $a_n\,[n(n-1) + (1)(n)] + a_{n-1}(0) + a_{n-2}(1) = 0$

i.e.

$$a_n = -\frac{a_n - 2}{n^2}, \quad n \geq 2 \quad \ldots(6)$$

So that

$$a_2 = -\frac{1}{2} a_0$$

$$a_3 = -\frac{1}{3^2} a_1 = 0 = a_5 = u_7 = a_9 = \ldots$$

$$a_4 = -\frac{1}{4^2} a_2 = \frac{1}{2^2 \cdot 4^2} a_0$$

$$a_6 = -\frac{1}{6^2} a_4 = -\frac{1}{2^2.4^2.6^2} a_0 \text{ etc.}$$

Hence from (2), a solution is

$$y = (\text{say})\, y_1 = a_0\left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \ldots\right) \quad \ldots(7)$$

A second solution may be found by (22) with $N = 0$ as

$$y = c_1 y_1 + c_2 \left(\frac{x^2}{2^2} - \frac{1+\frac{1}{2}}{2^2.4^2} x^2 + \frac{1+\frac{1}{2}+\frac{1}{3}}{2^2.4^2.6^2} x^6 + \ldots + y_1 \log x\right),\ x > 0.$$

Another Method. If the indicial equation has equal roots say $m = \alpha, \alpha$, the two independent solutions are obtained by putting $m = \alpha$ in y and $\frac{\partial y}{\partial m}$ both. The second solution will always consist of the product of the first solution or a numerical multiple of it and $\log_e x$, added to another series.

Problem 9. *Solve* (*i*) $x(1 - x^2) y'' + (1 - 3x^2) y' - xy = 0$ ...(1)

(*ii*) $x(1-x)y''+(1-5x)y'-4y=0$...(1*a*) (*Rohilkhand, 1985*)

(*i*) Let the series solution be $y = \sum_{0}^{\infty} a_n x^{m+n}$ ...(2)

Then substituting for y, y', y'' from (2) to (1), and equating to zero the coefficient of x^m we get the indicial equation as $m^2 = 0$ *i.e.* $m = 0, 0$ if $a_0 \neq 0$

Also equating to zero the coefficient of x^{m+n}, we shall find the recursion formula

$$a_n = \frac{(m+n-1)^2}{(m+n)^2} a_{n-2} \text{ for } n \geq 2 \text{ which yields when } m = 0,$$

$$a_n = \frac{(n-1)^2}{n^2} a_{n-2} \text{ for } n \geq 2 \quad \text{...(3)}$$

While the coefficient of x^{m+1} equated to zero will give $a_1 = 0$

$$\therefore \quad a_2 = \frac{1^2}{2^2} a_0 \text{ and } a_3 = \frac{2^2}{3^2} a_1 = 0 = a_5 = a_7 = \ldots$$

$$a_4 = \frac{3^2}{4^2} a_2 = \frac{1^2.3^2}{2^2.4^2} a_0 \text{ and } a_6 = \frac{1^2.3^2.5^2}{2^2.4^2.6^2} a_0 \text{ etc.}$$

Hence a solution is

$$y_1 = a_0 \left[1 + \frac{1^2}{2^2} x^2 + \frac{1^2.3^2}{2^2.4^2} x^4 + \ldots\right] \quad \text{...(4)}$$

But from $a_n = \frac{(m+n-1)^2}{(m+n)^2} a_{n-2}$ we find

$$y = a_0 x^m \left[1 + \left(\frac{m+1}{m+2}\right)^2 x^2 + \left(\frac{m+1}{m+2}\right)^2 \left(\frac{m+3}{m+4}\right)^2 x^4 + \ldots\right] \quad \text{...(5)}$$

If it is substituted in L.H.S. of (1), we get a single term $a_0 m^2 x^{m-1}$ $(m \neq 0)$

Its partial differential coefficient w.r.t. m is

$2a_0 mx^{m-1} + a_0 m^2 x^{m-1} \log x$ which is zero when $m = 0$.

It means $\frac{\partial}{\partial m}\left[x(1-x^2)\frac{d^2}{dx^2} + (1-3x^2)\frac{d}{dx} - x\right] y = 2a_0 mx^{m-1} + a_0 m^2 x^{m-1} \log x$

i.e. $\left\{x(1-x^2)\frac{d^2}{dx^2} + (1-3x^2)\frac{d}{dx} - x\right\}\frac{\partial y}{\partial m} = 2a_0 mx^{m-1} + a_0 m^2 x^{m-1} \log x$

(the differential operators being commutative)

So $\frac{\partial y}{\partial m}$ is another solution and hence from (5)

$$\frac{\partial y}{\partial m} = y \log x + a_0 x^m \left[2\left(\frac{m+1}{m+2}\right) - \frac{1}{(m+2)^2} x^2 + \ldots\right]$$

When $m = 0$, $y = y_1 = a_0\left[1 + \frac{1^2}{2^2}x^2 + \frac{1^2.3^2}{2^2.4^2}x^4 + \ldots\right]$

and so $\frac{\partial y}{\partial m} = y_1 \log x + a_0 \left[\frac{1}{2^2}x^2 + \frac{2\cdot 3}{2^2.4^2}\cdot\frac{7}{4}x^4 + \ldots\right]$(when $m = 0$) ...(6)

When $m = 0$ let in (5) $a_0 = a_1$ and in (6) $a_0 = b$, then

$$y_1 = a\left(1 + \frac{1^2}{2^2}x^2 + \frac{1^2\cdot 3^2}{2^2\cdot 4^2}x^4 + \ldots\right) = au \text{ (say)}$$

$$\frac{\partial y}{\partial m} = bu \log x + b\left\{\frac{1}{4}x^2 + \frac{21}{128}x^4 + \ldots\right\} = bv \text{ (say)}$$

Therefore the complete primitive is

y = au + bv

$$= (a + b \log x)\left[1 + \frac{1^2}{2^2}x^2 + \frac{1^2\cdot 3^2}{2^2\cdot 4^2}x^4 + \ldots\right] + b\left[\frac{x^2}{4} + \frac{21x^4}{128} + \ldots\right]$$

(*ii*) Taking the series solution of (1*a*) as $y = \sum_{n=0}^{\infty} a_n x^{m+n}$

so that $\frac{dy}{dx} = \sum_{n=0}^{\infty}(m+n)a_n x^{m+n-1}$, $\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty}(m+n)(m+n-1)a_n x^{m+n-2}$

we have by (1*a*), $\sum_{n=0}^{\infty}\left[-(m+n+2)^2 x^{m+n} + (m+n)^2 x^{m+n-1}\right]a_n = 0$

Which renders on equating to zero the coefficients of x^{m-1}, x^m, x^{m+n} respectively,

$m^2 a_0 = 0 \Rightarrow m = 0, 0$ as $a_0 \neq 0$

$$-(m+2)^2 a_0 + (m+1)^2 a_1 = 0 \Rightarrow a_1 = \left(\frac{m+2}{m+1}\right)^2 a_0$$

$$-(m+n+2)^2 a_n + (m+n+1)^2 a_{n+1} = 0 \Rightarrow a_{n+1} = \left(\frac{m+n+2}{m+n+1}\right)^2 a_n$$

For $n = 1, 2, 3, \ldots$ the last one gives

$$a_2 = \left(\frac{m+3}{m+2}\right)^2 a_1 = \left(\frac{m+3}{m+1}\right)^2 a_0,$$

$$a_3 = \left(\frac{m+4}{m+1}\right)^2 a_0 \text{ etc.}$$

Hence the series solution is

$$y\, a_0\, x^m = \left[1 + \left(\frac{m+2}{m+1}\right)^2 x + \left(\frac{m+3}{m+1}\right)^2 x^2 + \ldots\right]$$

For $m = 0$, $y = a_0\,[1 + 2^2x + 3^2x^2 + \ldots]$

Proceeding just as in (*i*), the second solution of (1*a*) is

$$\frac{\partial y}{\partial m} = a_0 x^m \log \left[1 + \left(\frac{m+2}{m+1}\right)^2 x + \left(\frac{m+3}{m+1}\right)^2 x^2 + \dots\right]$$

$$+ a_0 x^m \left[2\left(\frac{m+2}{m+1}\right)\left\{\frac{1}{m+1} - \frac{m+3}{(m+1)^2}\right\}x + 2\left(\frac{m+3}{m+1}\right)\left\{\frac{1}{m+1} - \frac{m+3}{(m+1)^2}\right\}x^2 + \dots\right]$$

$$= y \log x + a_0 x^m \left[2\left(\frac{m+2}{m+1}\right)\left\{\frac{1}{m+1} - \frac{m+2}{(m+1)^2}\right\}x^2\right.$$

$$\left. + 2\left(\frac{m+3}{m+1}\right)\left\{\frac{1}{m+1} - \frac{m+3}{(m+1)^2}\right\}x^2 + \dots\right]$$

Putting $m = 0$ and replacing a_0 by a and b respectively with two solutions, we have

$$y = a\,[1 + 2^2 x + 3^2 x^2 + \dots] = au \text{ (say)}$$

$$\left(\frac{\partial y}{\partial m}\right)_{m=0} = bu \log x + b\,[1.2\,(1-2)x + 2.3(1-3)x^2 + \dots$$

$$= bu \log x - 2b\,(1.2x + 2.3x^2 + 3.4x^3 + \dots)$$

$$= bv \text{ (say)}$$

Hence the complete solution is $y = au + bv$ *i.e.*

$$y = (a + b \log x)\,[1 + 2x + 3x^2 + \dots] - 2b\,[1.2x + 2.3x^2 + \dots]$$

Problem 10. *Solve* $xy'' + y' + xy = 0$.

Ans. $y = (a + b \log x)\left[1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 . 4^2} - \frac{x^6}{2^2 . 4^2 . 6^2} + \dots\right]$

$$+ b\left[\frac{x^2}{2^2} - \frac{x^4}{2^2 . 4^2}\left(1 + \frac{1}{2}\right) + \frac{x^6}{2^2 . 4^2 . 6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right)\dots\right]$$

Problem 11. *Solve* $x^2(x+1)\,y'' + x\,(x+1)\,y' - y = 0$.

Ans. $y = ax\left(1 - \frac{x}{3} + \frac{x^2}{6} - \frac{x^3}{10} + \dots\right) + bx^{-1}\,(1 - x)$.

Problem 12. *Solve in series* $xy'' + 2y' + xy = 0$.

Ans. $y = a\left[1 - \frac{x^2}{\lfloor 3} + \frac{x^4}{\lfloor 5}\dots\right] + bx^{-1}\left[1 - \frac{x^2}{2} + \frac{x^4}{4}\dots\right]$

[D] Particular Integral (*P.I.*)

Consider $$x^4 y'' + xy' + y = \frac{1}{x} \qquad \dots(1)$$

Complementary function is the solution of $x^4 y'' + xy' + y = 0$ and will be obtained as above to give

$$\text{C.F.} = a_0\left(1 - \frac{1}{\lfloor 3}x^{-2} - \frac{1}{\lfloor 5}x^{-4}\dots\right) + b_0\left(x - \frac{1}{x}\right)$$

To find P.I. put $y = C_0 x^m$ in (1), so that $m(m-1)\,C_0 x^{m+2} = x^{-1}$ which yields

$$m + 2 = -1 \text{ and } m(m-1)C_0 = 1 \text{ i.e. } m = -3 \text{ and then } C_0 = \frac{1}{12}.$$

With $m = -3$, find the recursion formula by the usual method, and this is here

$$C_n = \frac{2n}{(2n+3)(2n+4)} C_{n-1}$$

giving $C_1 = \frac{1}{5\cdot 6} C_0,\ C_2 = \frac{2\cdot 4}{5\cdot 6\cdot 7\cdot 8} C_0,\ C_3 = \frac{2\cdot 4\cdot 6}{5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10} C_0$ etc...

$$\text{Hence P.I.} = \frac{x^{-2}}{12}\left(1 + \frac{2}{5\cdot 6}x^{-2} + \frac{2\cdot 4}{5\cdot 6\cdot 7\cdot 8}x^{-4} + \dots\right)$$

$$= 2x^{-3}\left(\frac{1}{\lfloor 4} + \frac{2}{\lfloor 6}x^{-2} + \frac{2\cdot 4}{\lfloor 8}x^{-4} + \dots\right)$$

Thus the complete integral is $y =$ C.F. + P.I.

8.3. LEGENDRE DIFFERENTIAL EQUATION, FUNCTIONS AND POLYNOMIALS WITH PROPERTIES

[A] Legendre's Differential Equation. *(Agra, 1961, 63)*

This equation is of the form

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0. \qquad \dots(1)$$

The equation of such type can be solved in series of ascending or descending powers of x. Suppose, we have to integrate it in a series of descending powers of x. There is no singularity at $x = 0$, so the solution of the equation can be obtained in the form of a series developed about $x = 0$.

Let us assume the solution of the given equation in the form of series.

$$y = \sum_{r=0}^{\infty} a_r x^{k-r}$$

$$\therefore \quad \frac{dy}{dx} = \sum_{r=0}^{\infty} (k-r)\, a_r x^{k-r-1}$$

and
$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} (k-r)(k-r-1)\, a_r x^{k-r-2}.$$

Substituting these values in (1), we have

$$\sum_{r=0}^{\infty} [(1-x^2)(k-r)(k-r-1)x^{k-r-2} - 2x(k-r)x^{k-r-1} + n(n+1)x^{k-r}]a_r \equiv 0$$

or
$$\sum_{r=0}^{\infty} [(k-r)(k-r-1)x^{k-r-2} + \{n(n+1) - 2(k-r) - (k-r)(k-r-1)\}x^{k-r}a_r \equiv 0$$

or
$$\sum_{r=0}^{\infty} [(k-r)(k-r-1)x^{k-r-2} + \{n(n+1) - (k-r)(k-r+1)\}x^{k-r}]a_r \equiv 0. \qquad \dots(2)$$

The relation (2) is an identity and therefore the coefficients of various powers of x can be equated to zero.

Let us first equate the coefficient of x^k the highest power of x {by putting $r = 0$ in (2)} to zero; then we get

$$a_0\{n(n+1) - k(k+1)\} = 0,$$

where a_0 being the coefficient of the first term of the series cannot be zero, *i.e.* $a_0 \neq 0$ and thus

$$n(n+1) - k(k+1) = 0$$

or $$n^2 + n - k^2 - k = 0$$

or $$n^2 - k^2 + (n-k) = 0$$

or $$(n-k)(n-k+1) = 0$$

which gives $$k = n \text{ or } -n-1 \qquad \text{...(3)}$$

Again equating the coefficient of x^{k-1} to zero, by putting $r = 1$ in (2), we have

$$\{n(n+1) - (k-1)k\}\, a_1 = 0. \qquad \text{...(4)}$$

From (3), $\{n(n+1) - k(k-1)\} \neq 0$

and therefore $a_1 = 0$.

Let us now equate the coefficient of x^{k-r}, the general term in (2), to zero,

$$(k-r+2)(k-r+1)\, a_{r-2} + \{n(n+1) - (k-r)(k-r+1)\}\, a_r = 0$$

or $$a_r = -\frac{(k-r+2)(k-r+1)}{n(n+1) - (k-r)(k-r+1)} a_{r-2}. \qquad \text{...(5)}$$

Putting $k = n$, the recurrence formula is

$$a_r = -\frac{(n-r+2)(n-r+1)}{n^2 + n - (n-r)(n-r+1)} a_{r-2}$$

$$= -\frac{(n-r+2)(n-r+1)}{n^2 + n - n^2 + nr - n + nr - r^2 + r} a_{r-2}$$

$$= -\frac{(n-r+2)(n-r+1)}{r(2n-r+1)} a_{r-2}. \qquad \text{...(6)}$$

Again putting $k = -n-1$ in (5), we have the recurrence formula as

$$a_r = -\frac{(-n-r+1)(-n-r)}{n^2 + n - (-n-r-1)(-n-r)} a_{r-2}$$

$$= -\frac{(n+r-1)(n+r)}{n^2 + n - (n+r+1)(n+r)} a_{r-2}$$

$$= \frac{(n+r-1)(n+r)}{r(2n+r+1)} a_{r-2}. \qquad \text{...(7)}$$

Case I. When $k = n$, we have by putting $r = 2, 3, \ldots$ in (6),

$$a_2 = -\frac{n(n-1)}{2(2n-1)} a_0$$

$$a_3 = -\frac{(n-2)(n-2)}{3(2n-2)} a_1,$$

$$= 0 \text{ since } a_1 = 0.$$

Similarly $a_5, a_7, a_9 \ldots$etc. all the a's having odd suffixes are zero.

$$a_4 = -\frac{(n-2)(n-3)}{4(2n-3)} a_2 = \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4(2n-1)(2n-3)} a_0 \text{ (by putting value of } a_2 \text{ etc.)}$$

In general, $$a_{2r} = (-1)^r \frac{n(n-1)(n-2)(n-3)\ldots(n-2r+1)}{2 \cdot 4 - 2r(2n-1)(2n-3)\ldots(2n-2r+1)} a_0$$

(by putting value of a_2 etc.)

Hence the series solution when $k = n$, is

$$y = a_0\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\,(2n-1)(2n-3)}x^{n-4}\ldots\right] \quad \ldots(8)$$

where a_0 is an arbitrary constant and is equal to

$$\frac{1\cdot 3\cdot 5\ldots(2n-1)}{\lfloor n} \quad \ldots(9)$$

where n is a positive integer.

This solution of Legendre's equation is known as $P_n(x)$, *i.e.*

$$P_n(x) = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{\lfloor n}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\,(2n-1)(2n-3)}x^{n-4} - \ldots\right] \quad \ldots(10)$$

Case II. When $k = -n-1$, we have by putting $r = 2, 3, \ldots$ in (7),

$$a_2 = \frac{(n+1)(n+2)}{2(2n+3)}a_0.$$

Now a_3 will contain a_1 and hence is zero. As such $a_5, a_7, a_9\ldots$ all are zero.

$$a_4 = \frac{(n+3)(n+4)}{4(2n+5)}a_2$$

$$= \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4\,(2n+3)(2n+5)}a_0; \text{ by putting value of } a_2 \text{ etc.}$$

In general, $a_{2r} = \dfrac{(n+1)\ldots(n+2r)}{2\cdot 4\ldots 2r\,(2n+3)\ldots(2n+2r+1)}a_0.$

Hence the series solution is

$$y = a_0\left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4\,(2n+3)(2n+5)}x^{-n-5} + \ldots\right] \quad \ldots(11)$$

With $\quad a_0 = \dfrac{\lfloor n}{1\cdot 3\cdot 5\ldots(2n+1)}$ $\quad \ldots(12)$

this solution is known as $Q_n(x)$.

$$\therefore \quad Q_n(x) = \frac{\lfloor n}{1\cdot 3\cdot 5\cdot(2n+1)}\left[x^{-n-1} + \frac{(n+1)(n+2)}{2\cdot(2n+3)}x^{-n-3}\right.$$

$$\left. + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4(2n+3)(2n+5)}x^{-n-5} + \ldots\right] \quad \ldots(13)$$

The most general solution of the Legendre's equation is

$$y = AP_n(x) + BQ_n(x). \quad \ldots(14)$$

where A and B are arbitrary constants.

Note 1. *If the infinite series as the solution of a given differential equation is reduced into a finite series, then the solution is called as polynomial. $P_n(x)$ gives Legendre's polynomials or zonal harmonics or Legendre coefficients (n being a positive integer) of first kind and $Q_n(x)$ gives Legendre's polynomials of the second kind for positive integral n.*

Note 2. *The importance of a solution lies in its convergence and the convergency may be seen by ratio test e.g.,* $\lim_{n\to 0}\left|\frac{u_{n+2}}{u_n}\right| = \frac{1}{x^2}$ *from (8) or (11).*

Thus the series (8) or (11) will be convergent if $|x| > 1$ *i.e.* the above solutions for Legendre equation are not convergent in the interval $-1 < x < 1$. In order to find the convergent solution of (1) we seek for solution in descending powers of x.

Suppose a series solution of (1) is

$$y = \sum_{r=0}^{\infty} a_r x^{k+r}, \quad a_0 \neq 0 \qquad ...(15)$$

So that $y' = \sum_{r=0}^{\infty} (k+r) a_r x^{k+r-1}$

$$y'' = \sum_{r=0}^{\infty} (k+r)(k+r-1)\, a_r x^{k+r-2}$$

Substituting these values of y, y' and y'' in (1) we get the identity

$$\sum_{r=0}^{\infty}\left[(k+r)(k+r-1)x^{k+r-2} - (k+r-n)(k+r+n+1)x^{k+r}\right]a_r \equiv 0 \qquad ...(16)$$

Equating to zero the coefficient of x^{k-2} (when $r = 0$) the first term in (16) under the assumption $a_0 \neq 0$, yields $k(k-1) = 0$ giving $k = 0, 1$.

Now equating the coefficient of second term *i.e.* x^{k+1} to zero, we have $a_1 (k+1) k = 0$ giving $a_1 \neq 0$ for $k = -1$ while a_1 may or may not be zero for $k = 0$.

Equating the coefficient of general term *i.e.* x^{k+r} to zero, we find the recursion formula

$$a_{r+2} = \frac{(k+r-n)(k+r+n+1)}{(k+r+2)(k+r+1)} a_r \qquad ...(17)$$

Putting $k = 0$, (17) gives $a_{r+2} = \frac{(r-n)(r+n+1)}{(r+2)(r+1)} a_r$ ...(18)

And putting $k = 1$, (17) gives $a_{r+2} = -\frac{(1+r-n)(2+r+n)}{(3+r)(2+r)} a_r$ ...(19)

Case I. When $k = 0$, we have by putting $r = 0, 1, 2, 3, 4, 5, \ldots$ in (18)

$$a_2 = -\frac{n(n+1)}{\lfloor 2} a_0 \quad \text{and} \quad a_3 = -\frac{(n-1)(n+2)}{\lfloor 3} a_1$$

$$a_4 = \frac{n(n-2)(n+1)(n+3)}{\lfloor 4} a_0 \quad \text{and} \quad a_5 = \frac{(n-1)(n-3)(n+2)(n+4)}{\lfloor 5} a_1$$

and in general $a_{2r} = (-1)^r \frac{n(n-2)\ldots(n-2r+1)}{\lfloor 2r} a_0$

$$a_{2r+1} = \frac{(-1)^r (n-1)(n-3)\ldots(n-2r+1)(n+2)\ldots(n+2r)}{\lfloor 2r+1} a$$

Hence the series solution for $k = 0$, is

$$y = a_0\left[1 - \frac{n(n+1)}{\lfloor 2}x^2 + \frac{n(n-2)(n+1)(n+3)}{\lfloor 4}x^4 + \ldots\right]$$

$$+ a_1 x\left[1 - \frac{(n-1)(n+2)}{\lfloor 3}x^2 + \frac{(n-1)(n-3)(n+2)(n+4)}{\lfloor 5}x^4 \ldots\right] \quad \ldots(19)$$

Case II. When $k = 1$, we have from (19),

$$a_2 = -\frac{(n-1)(n+2)}{\lfloor 3}a_0 \quad \text{and} \quad a_1 = a_3 = a_5 = a_{2r+1} = \ldots = 0$$

$$a_4 = \frac{(n-1)(n-3)(n+2)(n+4)}{\lfloor 5}a_0$$

$$a_{2r} = \frac{(-1)^r (n-1)(n-3)\ldots(n-2r+1)(n+2)\ldots(n+2r)}{\lfloor 2r+1}a_0$$

Hence the series solution for $k = 1$ is

$$y = a_0 x\left[1 - \frac{(n-1)(n+2)}{\lfloor 3}x^2 + \frac{(n-1)(n-3)(n+2)(n+4)}{\lfloor 5}x^4 + \ldots\right] \quad \ldots(20)$$

The solution (20) is included in (19) in the coefficient of $a_1$except that $a_1$is to be replaced by a_0. Hence setting $a_1 = 0$ for $k = 0$ also, the solution (19) reduces to

$$y = a_0\left[1 - \frac{n(n+1)}{\lfloor 2}x^2 + \frac{n(n-2)(n+1)(n+3)}{\lfloor 4}x^4 + \ldots\right] \quad \ldots(21)$$

It may be shown by ratio test that the solutions (20) and (21) are convergent in the interval $-1 < x < 1$.

Calling the solution (21) as $S_n(x)$ and (20) as $T_n(x)$, the general solution of Legendre equation in ascending powers of x is

$$y = A\ S_n(x) + B\ T_n(x). \quad \ldots(22)$$

where A and B are arbitrary constants.

[B] Legendre Polynomials.

If we put $n = 2r$ (say) *i.e.* if n be taken as an even positive integer then (21) gives the Legendre polynomial as

$$y = a_0\left[1 - \frac{n(n+1)}{\lfloor 2}x^2 + \ldots + (-1)^{n/2}\frac{n(n-2)\ldots4.2\ (n+1)(n+3)\ldots(2n-1)}{\lfloor n}x^n\right]$$

...(23)

While (8) gives

$$y = a_0 x^n\left[1 - \frac{n(n-1)}{2(2n-1)}x^{-2} + \ldots + (-1)^{n/2}\frac{\lfloor n}{n(n-2)\ldots2(n+1)(n+3)\ldots(2n-1)}x^{-n}\right]$$

...(24)

(23) and (24) will be identical if (24) is multiplied by

$$(-1)^{n/2}\frac{n(n-2)\ldots4\cdot2\ (n+1)(n+3)\ldots(2n-1)}{\lfloor n}$$

and then the solutions (8), (21) will become identical.

Again if we take, n as an odd negative integer then (21) is identical with (11). Also if n is an odd positive integer then (20) reduces to

$$y = a_0\left[x - \frac{(n-1)(n+2)}{\lfloor 3}x^3 + \ldots + (-1)^{(n-1)/2}\frac{(n-1)(n-3)\ldots 2\,(n+2)\ldots(2n-1)}{\lfloor n}x^n\right] \quad \ldots(25)$$

and (8) reduces to

$$y = a_0\, x^n\left[1 - \frac{n(n-1)}{2(2n-1)}x^{-2} + \ldots + (-1)^{(n-1)/2}\frac{\lfloor n}{2.4\ldots(n-1)\,(n-3)\ldots(2n-1)}x^{-n+1}\right] \quad \ldots(26)$$

which becomes identical when multiplied by the coefficient of last term in (26).

Further if n is an even negative integer, (20) and (11) become identical.

These discussions follow the conclusions :

(*i*) For integral values for n, the solutions (8) and (11) have great utility of them.

(*ii*) For positive integral n, (8) is a polynomial but (11) is an infinite series and the complete integral is a linear combination of them,

(*iii*) For negative integral n, (8) is an infinite series and (11) is a polynomial.

(*iv*) For positive integral n, in (9) or (10) we have chosen

$$a_0 = \frac{1.3.5\ldots(2n+1)}{\lfloor n}.$$

(*v*) For negative integral n, in (12) or (13) we have chosen

$$a_0 = \frac{\lfloor n}{1.3.5\ldots\,(2n+1)}.$$

(*vi*) For positive integral n, the polynomial $P_n(x)$ has the expansion given by (10) ending with term free from x *i.e.*

$$P_n(x) = \sum_{r=0}(-1)^r\frac{1.3.5\ldots(2n-1)}{\lfloor n}\,\frac{n(n-1)\ldots(n-2r+1)}{2.4\ldots 2r(2n-1)\,(2n-3)\ldots(2n-2r+1)}\,x^{n-2r}$$

$$= \sum_{n=0}^{n/2}(-1)^r\frac{1.3.5\ldots(2n-2r-1)}{2^r\lfloor r\lfloor n-2r}x^{n-2r} \quad \ldots(27)$$

or more concisely

$$P_n(x) = \sum_{n=0}^{n/2}(-1)^r\frac{\lfloor 2n-2r}{2^n\lfloor r\lfloor n-r\lfloor n-2r}x^{2n-r} \quad \ldots(28)$$

Evaluation of the values of $P_0(\mu)$, $P_1(\mu)$, $P_2(\mu)$, $P_3(\mu)$, $P_4(\mu)$, $P_5(\mu)$, $P_6(\mu)$ *and* $P_7(\mu)$, *etc.*

We have

$$P_n(\mu) = \frac{1.3.5\ldots(2n-1)}{n\,!}\left[\mu^n - \frac{n(n-1)}{2(2n-1)}\mu^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.\,(2n-1)\,2n-3)}\mu^{n-4} - \ldots\right]$$

$$= \frac{(2n)\,!}{(2.4.6\ldots 2n)\,n\,!}\left[\mu^n - \frac{n(n-1)}{2(2n-1)}\mu^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.\,(2n-1)(2n-3)}\mu^{n-4} - \ldots\right]$$

$$= \frac{(2n)\,!}{2^n\cdot n\,!\,n\,!}\left[\mu^n - \frac{n(n-1)}{2(2n-1)}\mu^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.\,(2n-1)(2n-3)}\mu^{n-4} - \ldots\right]$$

Putting $n = 0, 1, 2, 3, 4, 5, 6, 7$, etc., we have

$$P_0(\mu) = 1,$$

$$P_1(\mu) = \frac{2\,!}{2\cdot 1\,!\cdot 1\cdot !}\,[\mu], \text{ other terms vanishing}$$

$$= \mu,$$

$$\left.\begin{aligned}
P_2(\mu) &= \frac{4\,!}{2^2\cdot 2\,!\cdot 2\,!}\,[\mu^2 - \tfrac{1}{3}] \text{ other terms vanishing}\\
&= \frac{4\cdot 3\cdot 2}{2\cdot 2\cdot 2\cdot 2}\;\frac{3\mu^2-1}{3} = \frac{3\mu^2-1}{2}.\\
P_3(\mu) &= \frac{6\,!}{2^3\cdot 3\,!\,3\cdot !}\left[\mu^2 - \frac{3\cdot 2}{3\cdot 5}\,\mu\right] = \frac{5\mu^3-3\mu}{2}.\\
\text{Similarly } P_4(\mu) &= \tfrac{1}{8}\,(35\mu^4 - 30\mu^2 + 3),\\
P_5(\mu) &= \tfrac{1}{8}\,(63\mu^5 - 70\mu^3 + 15\mu),\\
P_6(\mu) &= \tfrac{1}{16}\,(231\mu^6 - 315\mu^4 + 105\mu^2 - 5)\\
P_7(\mu) &= \tfrac{1}{16}(429\mu^7 - 693\mu^5 + 315\mu^3 - 35\mu) \text{ etc.}
\end{aligned}\right\} \quad \ldots(29)$$

[C] Generating Functions for P_n (x)

P_n (μ) is the coefficient of h^n in $(1-2\mu h+h^2)^{-1/2}$

We have

$$(1 - 2\mu h + h^2)^{-1/2} = [1 - h\,(2\mu - h)]^{-1/2}$$

$$= 1 + \tfrac{1}{2}h\,(2\mu - h) + \frac{1.3}{2.4}h^2(2\mu - h^2)^2$$

$$+ \frac{1.3.5}{2.4.6}h^5(2\mu - h)^3 + \ldots + \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}h^n(2\mu - h)^n + \ldots$$

(by Binomial expansion)

The coefficient of h^n in the expansion

$$= \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}(2\mu)^n + \frac{1.3.5\ldots(2n-3)}{2.4.6\ldots(2n-2)}(2\mu)^{n-2}\cdot{}^{n-1}C_1$$

$$+ \frac{1.3.5\ldots(2n-5)}{2.4.6\ldots(2n-4)}(2\mu)^{n-4}\cdot{}^{n-2}C_2 - \ldots$$

$$= \frac{1.3.5\ldots(2n-1)}{n\,!}\left[\mu^n - \frac{n\,(n-1)}{2.\,(2n-1)}\mu^{n-2} + \frac{n\,(n-1)\,(n-2)\,(n-3)}{2.2.\,2\,!\,(2n-1)\,(2n-4)}\mu^{n-4} \ldots\right],$$

$$= \frac{1.3.5\ldots(2n-1)}{n\,!}\left[\mu^n - \frac{n\,(n-1)}{2.\,(2n-1)}\mu^{n-2} + \frac{n\,(n-1)\,(n-2)\,(n-3)}{2.4\,(2n-1)\,(2n-3)}\mu^{n-4} + \ldots\right],$$

which is $P_n(\mu)$.

Hence $$\sum_{n=0}^{\infty} h^n\,P_n(\mu) = (1 - 2\mu h + h^2)^{-1/2}. \quad \ldots(30)$$

(Agra, 1967)

COROLLARY 1. *$P_n(1)$ is the coefficient of h^n in $(1-2h+h^2)^{-1/2}$ i.e., in $(1-h)^{-1}$.*

(*Rohilkhand, 1982*)

As such $P_n(1)$ = coeff. of h^n in $\{1+h+h^2+...+h^n+...\}=1$, which is however a distinguishing property of Legendre's Polynomials.

COROLLARY 2. Putting $\mu=\cos\theta$, we have

$$\sum_{n=0}^{\infty} h^n P_n(\cos\ \theta)=\{1-2(\cos\theta)h+h^2\}^{-1/2}$$

$$=\{1-(e^{i\theta}+e^{-i\theta})h+e^{-i\theta}\cdot e^{-i\theta}h^2\}^{-1/2}$$

$$=(1-he^{i\theta})^{-1/2}\ (1-he^{-i\theta})^{-1/2}$$

$$=\left[1+\tfrac{1}{2}he^{i\theta}+\frac{1.3}{2.4}h^2e^{2i\theta}+\frac{1.3.5}{2.4.6}h^3e^{3i\theta}+...+\frac{1.3...(2n-1)}{2.4...2n}h^ne^{in\theta}+...\right]$$

$$\times\left[1+\tfrac{1}{2}he^{-i\theta}+\frac{1.3}{2.4}h^2e^{-2i\theta}+\frac{1.3.5}{2.4.6}h^3e^{-3i\theta}+...+\frac{1.3.5...(2n-1)}{2.4...2n}h^ne^{-ni\theta}+...\right]$$

Equating coefficients of h^n on either side,

$$P_n(\cos\theta)=\frac{1.3.5...(2n-1)}{2.4.6...2n}\left(e^{ni\theta}+e^{-ni\theta}\right)+\frac{1}{2}\cdot\frac{1.3.5...(2n-3)}{2.4.6...(2n-2)}$$

$$\left\{e^{(n-2)i\theta}+e^{-(n-2)\ i\theta}\right\}+\frac{1.2}{2.4}\cdot\frac{1.3.5...(2n-5)}{2.4.6...(2n-4)}\left\{e^{(n-4)i\theta}+e^{-(n-4)i\theta}\right\}+...$$

$$=2.\frac{1.3.5...\ (2n-1)}{2.4...2n}\left[\cos n\theta+\frac{1}{2}\cdot\frac{2}{2n-1}\cos\ (n-2)\ \theta\right.$$

$$\left.+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{2n\ (2n-2)}{(2n-1)\ (2n\ -3)}\cos\ (n-4)\theta+...\right.$$

$$=2.\frac{1.3.5...\ (2n-1)}{2.4.6...2n}\left[\cos\ n\theta+\frac{n}{2n-1}\cos\ (n-2)\theta\right.$$

$$\left.+\frac{1.3}{2\ !}\cdot\frac{n}{(2n-1)\ (2n-3)}\cos\ (n-4)\theta+...\right]. \quad ...(31)$$

COROLLARY 3. We know that

$$(1-2h\cos\theta+h^2)^{-1/2}=\{(1-h)^2+2h\ (1-\cos\ \theta)\}^{-1/2}$$

$$=\left\{(1-h)^2+4h\ \sin^2\frac{\theta}{2}\right\}^{-1/2}$$

$$=\frac{1}{1-h}\left\{1-\frac{(-1)\ 4h\ \sin^2\ \theta/2}{(1-h)^2}\right\}^{-1/2}$$

$$=\frac{1}{1-h}\left[1+\frac{1}{2}\left\{-\frac{4h\ \sin^2\theta/2}{(1-h)^2}\right\}+\frac{1.3}{2.4}\left\{-\frac{4h\ \sin^2\theta/2}{(1-h)^2}\right\}^2+...\right.$$

$$\left....+\frac{1.3.5...(2r-1)}{2.4...2r}\left\{-\frac{4h\ \sin^2\ \theta/2}{(1-h)^2}\right\}^r+...\right]$$

$$=\frac{1}{1-h}+\frac{1}{2}\left[\left\{-\frac{4h\ \sin^2\theta/2}{(1-h)^2}\right\}\frac{1}{1-h}+...\right.$$

$$\ldots+\frac{1.3.5\ldots(2r-1)\,(-1)^r}{2.4\ldots2r}\cdot\frac{4^r h^r\,\sin^{2r}\theta/2}{(1-h)^{2r+1}}+\ldots$$

$$=\frac{1}{1-h}+\sum_{r=1}^{\infty}(-1)^r\frac{1.3.5\ldots(2r-1)\,2^r.2^r h^r\,\sin^{2r}\theta/2}{2\cdot4\ldots2r\,(1-h)^{2r+1}}.$$

Equating coefficients of h^n on either side,

$$P_n(\cos\theta)=1+\sum_{r=1}^{\infty}(-1)^r\frac{1.3.5\ldots(2r-1)}{2.4\ldots2r}.2^r\cdot2^r\sin^{2r}\frac{\theta}{2}$$

$$\times\ \frac{(2r+1)\,(2r+2)\ldots\,[(2r+1)+(n-r-1)]}{(n-1)\,!}$$

$$\left[\begin{array}{l}\text{since coeff. of } h^{n-1} \text{ in } (1-h)^{-(2r+1)} \text{ is}\\ \dfrac{(2r+1)\,(2r+2)\ldots(2r+1+n-r-1)}{(n-r)\,!}\ \text{etc.}\end{array}\right]$$

$$\therefore\ P_n(\cos\theta)=1+\sum_{r=1}^{\infty}(-1)^r\frac{1.3.5\ldots\,(2r-1)}{2.4\ldots2r}\cdot\frac{(n+r)\,!}{(2r)\,!\,(n-r)\,!}\cdot2^r\cdot2^r\sin^{2r}\frac{\theta}{2}$$

$$=1+\sum_{r=1}^{\infty}(-1)^r\frac{(2r)\,!\,(n-r)\,!\,(n-r+1)\,(n-r+2)\ldots(n+r)}{2^r.\,r\,!.\,2^r\,r\,!.\,(2r)\,!\,(n-r)\,!}\times2^r\cdot2^r\sin^{2r}\frac{\theta}{2}$$

$$=1+\sum_{r=1}^{\infty}(-1)^r\frac{(n-r+1)\,(n-r+2)\ldots(n+r)}{r\,!\,r\,!}\sin^{2r}\frac{\theta}{2}$$

$$=1-\frac{n(n+1)}{1\,!\,1\,!}\sin^2\frac{\theta}{2}+\frac{(n-1)\,n\,(n+1)\,(n+2)}{2\,!\,2\,!}\sin^4\frac{\theta}{2}$$

$$-\frac{(n-2)\,(n-1)\,n(n+1)\,(n+2)\,(n+3)}{3\,!\,3\,!}\sin^6\frac{\theta}{2}+\ldots$$

$$=1+\frac{(-n)\,(n+1)}{1.1}\sin^2\frac{\theta}{2}+\frac{(-n)\,(-n+1)\,(n+1)\,(n+2)}{1.2.1\,(1+1)}\left(\sin^2\frac{\theta}{2}\right)^2$$

$$+\frac{(-n)\,(-n+1)\,(-n+2)\,(n+1)\,(n+2)\,(n+3)}{1.2.3.1\,(1+1)\,(1+2)}\left(\sin^2\frac{\theta}{2}\right)^3+\ldots$$

$$=F\left(-n,\ n+1,\ 1,\ \sin^2\frac{\theta}{2}\right)\qquad\ldots(32)$$

Replacing θ by $\theta+\pi$.

$$P_n(-\cos\theta)=(-1)^n\,P_n\,(\cos\theta)=F\left(-n,\ n+1,\ 1,\ \cos^2\frac{\theta}{2}\right)\qquad\ldots(33)$$

i.e. $P_n\,(\cos\theta)=(-1)^n\cdot F\,(-n,\,n+1,\,1,\,\cos^2\theta/2)$.

[C] Integral for $P_n\,(\mu)$.

I. $P_n(\mu)=\dfrac{1}{\pi}\displaystyle\int_0^{\pi}\left\{\mu\mp\sqrt{(\mu^2-1)}\ \cos\phi\right\}^n d\phi.$ *(Vikram, 1962; Rohilkhand, 1986)*

It is easy to show that

$$\int_0^\pi \frac{d\phi}{a \pm b\cos\phi} = \frac{\pi}{\sqrt{(a^2 - b^2)}}, \text{ when } a^2 > b^2.$$

Let us put $a = 1 - \mu h$, $b = h\sqrt{(\mu^2 - 1)}$, so that

$$a^2 - b^2 = 1 - 2\mu h + h^2.$$

Then $\dfrac{\pi}{(1 - 2\mu h + h^2)^{-1/2}} = \displaystyle\int_0^\pi \frac{d\phi}{1 - h\mu \pm h\sqrt{(\mu^2 - 1)}\cos\phi}$

or $$(1 - 2\mu h + h^2)^{-1/2} = \frac{1}{\pi}\int_0^\pi \left[1 - h\left\{\mu \mp \sqrt{(\mu^2 - 1)}\cos\phi\right\}\right]^{-1} d\phi$$

$$= \frac{1}{\pi}\int_0^\pi \left[\sum_{n=0}^{\infty} h^n \left\{\mu \mp \sqrt{(\mu^2 - 1)}\cos\phi\right\}^n\right] d\phi$$

where h is sufficiently small

$$= \frac{1}{\pi}\sum_{n=0}^{\infty} h^n \int_0^\pi \left[\mu \mp \sqrt{(\mu^2 - 1)}\cos\phi\right]^n d\phi.$$

Equating the coefficient of h^n on either side, we have

$$P_n(\mu) = \frac{1}{\pi}\int_0^\pi \left\{\mu \mp \sqrt{(\mu^2 - 1)}\cos\phi\right\}^n d\phi, \qquad \ldots(34)$$

which is known as **first of the Laplace's Integrals.**

COROLLARY 4. If we put $\mu = \cos\theta$.

$$P_n(\cos\theta) = \frac{1}{\pi}\int_0^\pi (\cos\theta \mp i\sin\theta\cos\phi)^n d\phi. \qquad \ldots(35)$$

COROLLARY 5. $P_n(\mu) = \dfrac{1}{\pi}\displaystyle\int_0^\pi \frac{d\phi}{\left\{\mu \pm \sqrt{(\mu^2 - 1)}\cos\phi\right\}^{n+1}}.$

We have as above

$$\int_0^\pi \frac{d\phi}{a - b\cos\phi} = \frac{\pi}{\sqrt{(a^2 - b^2)}}, \text{ where } a^2 > b^2.$$

Putting $a = \mu h - 1$, $b = \mp h\sqrt{(\mu^2 - 1)}$, we get

$$\frac{\pi}{\sqrt{(1 - 2\mu h + h^2)}} = \int_0^\pi \frac{d\phi}{h\left\{\mu \pm \sqrt{(\mu^2 - 1)}\cos\phi\right\} - 1}$$

or $$\frac{\pi}{h\sqrt{\left(1 - \frac{2\mu}{h} + \frac{1}{h^2}\right)}} = \int_0^\pi \frac{d\phi}{h\left\{\mp\sqrt{(\mu^2 - 1)}\cos\phi\right\}\left[1 - \frac{1}{\{\mu \pm \sqrt{(\mu^2 - 1)}\cos\phi\}}\right]}$$

or $$\sum_{n=0}^{\infty}\frac{1}{h^{n+1}}P_n(\mu) = \frac{1}{\pi}\int_0^{\pi}\left[\frac{\left\{1-\dfrac{1}{h\left\{\mu\pm\sqrt{(\mu^2-1)}\cos\phi\right\}}\right\}}{h\left\{\mu\pm\sqrt{(\mu^2-1)}\cos\phi\right\}}\right]d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[\sum_{n=0}^{\infty}\frac{1}{h^{n+1}\left\{\mu\pm\sqrt{(\mu^2-1)}\cos\phi\right\}^{n+1}}\right]d\phi$$

$$= \frac{1}{\pi}\sum_{n=0}^{\pi}\frac{1}{h^{n+1}\left\{\mu\pm\sqrt{(\mu^2-1)}\cos\phi\right\}^{n+1}}$$

Equating coefficients of $\frac{1}{h^{n+1}}$ on either side, we get

$$P_n(\mu) = \frac{1}{\pi}\int_0^{\pi}\frac{d\phi}{\left\{\mu\pm\sqrt{(\mu^2-1)}\cos\phi\right\}^{n+1}} \qquad \text{...(36)}$$

which is known as **Second of the Laplace's Integrals.**

COROLLARY 6. It is obvious from the above two Laplace's integrals that

$$P_n(\mu) = P_{-(n+1)}(\mu). \qquad \text{..:(37)}$$

[D] Recurrence Formulae for $P_n(\mu)$.

I. $nP_n(\mu) = (2n-1)\,\mu\,P_{n-1}(\mu) - (n-1)\,P_{n-2}(\mu)$.

(Rohilkhand, 1983, 84, 86; Agra, 1953, 61, 74; Vikram, 1963)

Suppose $V = (1-2\mu h + h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h^n P_n(\mu)$

or $$V^2(1-\mu h + h^2) = 1.$$

Differentiating w.r.t. h.

$$2V.\frac{dV}{dh}(1-2\mu h + h^2) + V^2(2h-2\mu) = 0$$

or $$\frac{dV}{dh}(1-2\mu h + h^2) + V(h-\mu) = 0$$

or $$(1-2\mu h + h^2)\sum_{n=0}^{\infty} nh^{n-1}P_n(\mu) + (h-\mu)\sum_{n=0}^{\infty} h^n P_n(\mu) = 0$$

$$\left[\because \frac{dV}{dh} = \sum_{n=0}^{\infty} nh^{n-1}P_n(\mu)\right]$$

The coefficient of h^{n-1} equated to zero gives

$$nP_n(\mu) - 2\mu(n-1)P_{n-1}(\mu) + (n-2)P_{n-2}(\mu) + P_{n-2}(\mu) - \mu P_{n-1}(\mu) = 0$$

or $$nP_n(\mu) = (2n-1)\mu P_{n-1}(\mu) - (n-1)P_{n-2}(\mu). \qquad \text{...(38)}$$

Note. *If $n-1=m$, then this result reduces to*

$(m+1)P_{m+1}(\mu)=(2m+1)\mu P_m(\mu)-mP_{m-1}(\mu)$ ...(39)

(*Agra 65; Vikram, 63*)

II. $(\mu^2-1)\dfrac{dP_n(\mu)}{d\mu}=n\{\mu P_n(\mu)-P_{n-1}(\mu)\}$

$$=-(n+1)\{\mu P_n(\mu)-P_{n+1}(\mu)\}.$$

We have

$\mu P_n(\mu)-P_{n+1}(\mu)$

$$=\frac{\mu}{\pi}\int_0^\pi\left\{\mu+\sqrt{(\mu^2-1)}\cos\phi\right\}^n d\phi-\frac{1}{\pi}\int_0^\pi\left\{\mu+\sqrt{(\mu^2-1)}\cos\phi\right\}^{n-1} d\phi$$

$$=\frac{1}{\pi}\int_0^\pi\left\{\mu+\sqrt{(\mu^2-1)}\cos\phi\right\}^{n-1}\left[\mu\left\{\mu+\sqrt{(\mu^2-1)}\cos\phi\right\}-1\right]d\phi$$

$$=\frac{\mu^2-1}{\pi}\int_0^\pi\left\{\mu^2+\sqrt{(\mu^2-1)}\cos\phi\right\}^{n-1}\left[1+\frac{\mu\cos\phi}{\sqrt{(\mu^2-1)}}\right]d\phi$$

$$=\frac{\mu^2-1}{\pi}\frac{d}{d\mu}\int_0^\pi\frac{\left\{\mu+\sqrt{(\mu^2-1)}\cos\phi\right\}^n}{n}d\phi$$

$$=\frac{\mu^2-1}{n}\frac{d}{d\mu}P_n(\mu).$$

$\therefore\quad (\mu^2-1)\dfrac{dP_n(\mu)}{d\mu}=n\{\mu P_n(\mu)-P_{n-1}(\mu)\}$ ...(40)

(*Agra, 1965*)

Replacing n by $(-n-1)$, as $P_n(\mu)=P_{-(n+1)}(\mu)$ and as such $P_{-(n+2)}\mu=P_{n+1}(\mu)$, we get

$$(\mu^2-1)\frac{dP_n(\mu)}{d\mu}=-(n+1)\{\mu P_n(\mu)-P_{n+1}(\mu)\} \qquad ...(41)$$

III. By Legendre's equation, we have

$$(\mu^2-1)\frac{d^2P_n(\mu)}{d\mu^2}+2\mu\frac{dP_n(\mu)}{d\mu}-n(n+1)P_n(\mu)=0.$$

This may be written as

$$\frac{d}{d\mu}\left\{(\mu^2-1)\frac{dP_n(\mu)}{d\mu}\right\}=n(n+1)P_n(\mu),$$

or $\quad \dfrac{d}{d\mu}\left[n\{\mu P_n(\mu)-P_{n-1}(\mu)\}\right]=n(n+1)P_n(\mu)$ from (40)

or $\quad \mu\dfrac{dP_n(\mu)}{d\mu}+P_n(\mu)-\dfrac{dP_{n-1}(\mu)}{d\mu}=(n+1)P_n(\mu)$

or $\quad \mu\dfrac{dP_n(\mu)}{d\mu}-\dfrac{dP_{n-1}(\mu)}{d\mu}=nP_n(\mu).$ ...(41*a*)

(*Rohilkhand 1980, 87; Agra, 1954; Vikram, 62*)

IV. Replacing n by $-(n+1)$ and applying $P_{-n-1}(\mu) = P_n(\mu)$, $P_{-n-2}(\mu) = P_{n+1}(\mu)$ etc., in (41a) we get

$$-\mu \frac{dP_n(\mu)}{d\mu} + \frac{dP_{n+1}(\mu)}{d\mu} = (n+1)\ P_n(\mu). \qquad \text{...(42)}$$

Addition of (41) and (42) yields

$$(2n+1) P_n(\mu) = \frac{d}{d\mu} P_{n+1}(\mu) - \frac{d}{d\mu} P_{n-1}(\mu). \qquad \text{...(43)}$$

(Rohilkhand, 1984, 89; Agra, 1952, 55, 61, 74; Vikram, 62, 63)

Now (43) may be written as

$$\frac{d}{d\mu} P_{n+1}(\mu) = (2n+1) P_n(\mu) + \frac{d}{d\mu} P_{n-1}(\mu).$$

Replacing n by $(n-1)$, we get

$$\frac{d}{d\mu} P_n(\mu) = (2n-1) P_{n-1}(\mu) + \frac{d}{d\mu} P_{n-2}(\mu).$$

$$= (2n-1) P_{n-1}(\mu) + (2n-5) P_{n-3}(\mu) + \frac{d}{d\mu} P_{n-4}(\mu). \qquad \text{...(44)}$$

$$\left[\begin{array}{l}\text{since by putting } n-2 \text{ for } n, \\ \dfrac{dP_{n-2}(\mu)}{d\mu} = (2n-5) P_{n-3}(\mu) + \dfrac{d}{d\mu} P_{n-4}(\mu)\end{array}\right].$$

The repeated application of this replacement will give

$$\frac{d}{d\mu} P_n(\mu) = (2n-1) P_{n-1}(\mu) + (2n-5) P_{n-3}(\mu) + (2n-9) P_{n-5}(\mu) + \ldots$$

ending with $3P_1(\mu)$ or $P_0(\mu)$ according as n is even or odd. ...(48)

This is known as *Christoffel's Expansion.*

Again multiplying (40) by $(n+1)$, (41) by n and adding, we have

$$(2n+1)\ (\mu^2 - 1) \frac{dP_n(\mu)}{d\mu} = n(n+1)\ \{P_{n+1}(\mu) - P_{n-1}(\mu)\} \qquad \text{...(49)}$$

[E] Some Important Results.

$[e_1]$. $P_n(-\mu) = (-1)^n P_n(\mu)$. ...(50)

(Agra, 1967)

We know that

$$P_n(\mu) = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{\lfloor n}\left[\mu^n - \frac{n(n-1)}{2 \cdot (2n-1)} \mu^{n-2} + \ldots\right]$$

$$= \frac{\lfloor (2n)}{2 \cdot 4 \cdot 6 \ldots 2n \lfloor n}\left[\mu^n - \frac{n(n-1)}{2\ (2n-1)} \cdot \mu^{n-2} + \ldots\right]$$

$$= \frac{\lfloor (2n)}{2^n \lfloor n}\left[\mu^n - \frac{n(n-1)}{2\ (2n-1)} \cdot \mu^{n-2} + \ldots\right]$$

$$\therefore\ P_n(-\mu) = \frac{\lfloor (2n)}{2^n \lfloor n} - \left[(-\mu)^n - \frac{n(n-1)}{2\ (2n-1)} (-\mu)^{n-2} + \ldots\right]$$

$$P_n(-\mu)=\frac{\lfloor(2n)}{2^n\lfloor n}\left[(-1)^n\mu^n-\frac{n(n-1)}{2(2n-1)}(-1)^{n-2}\mu^{n-2}+\ldots\right]$$

$$=\frac{\lfloor(2n)}{2^n\lfloor n}\left[(-1)^n\mu^n-\frac{n(n-1)(-1)^n}{2(2n-1)(-1)^2}\mu^{n-2}+\ldots\right]$$

$$=(-1)^n\cdot\frac{\lfloor(2n)}{2^n\lfloor n}\left[\mu^n-\frac{n(n-1)}{2(2n-1)}\mu^{n-2}+\ldots\right]$$

$$=(-1)^n P_n(\mu).$$

Note. *If $n=2m$, this gives $P_{2m}(-\mu)=P_{2m}(\mu)$.* (*Agra, 1965*)

If $n=2m+1$, we have

$$P_{2m+1}(-\mu)=-P_{2m+1}(\mu). \qquad \ldots(51)$$

(*Agra, 65*)

[e_2] Rodrigue's formula

$$P_n(\mu)=\frac{1}{2^n\lfloor n}\left(\frac{d}{d\mu}\right)^n(\mu^2-1)^n. \qquad \ldots(52)$$

(*Rodrigue's formula*)

(*Rohilkhand, 1986; Agra, 1956, 60, 63; Vikram, 63, 64*)

Let $y=(\mu^2-1)^n$.

$$\therefore\quad \frac{dy}{d\mu}=n(\mu^2-1)^{n-1}\cdot 2\mu=\frac{n\cdot y\cdot 2\mu}{\mu^2-1},$$

i.e.

$$(\mu^2-1)\frac{dy}{d\mu}=2ny\mu.$$

Differentiating this $(n+1)$ times by Leibnitz's theorem, we get

$$(\mu^2-1)\frac{d^{n+2}y}{d\mu^{n+2}}+{}^{n+1}C_1(2\mu)\cdot\frac{d^{n+1}y}{d\mu^{n+1}}+{}^{n+1}C_2(2)\cdot\frac{d^ny}{d\mu^n}$$

$$=2n\left[\mu\frac{d^{n+1}y}{d\mu^{n+1}}+{}^{n+1}C_1(1)\frac{d^ny}{d\mu^n}\right]$$

or

$$(\mu^2-1)\frac{dy^{n+2}}{d\mu^{n+2}}+2\mu\frac{d^{n+1}y}{d\mu^{n+1}}-n(n+1)\frac{d^ny}{d\mu^n}=0$$

Putting $\dfrac{d^ny}{d\mu^n}=z$, we have

$$(\mu^2-1)\frac{d^2z}{d\mu^2}+2\mu\frac{dz}{d\mu}-n(n+1)z=0,$$

which is Legendre's equation, one of whose solution is z and hence it is satisfied by Cz or $C\dfrac{d^ny}{d\mu^n}$, where C is an arbitrary constant.

Hence $P_n(\mu)=C\dfrac{d^ny}{d\mu^n}=C\left(\dfrac{d}{d\mu}\right)^n(\mu^2-1)^n$

$$= C\left(\frac{d}{d\mu}\right)^n (\mu-1)^n (\mu+1)^n$$

$$= C\left[\lfloor n \; (\mu+1)^n + \text{ terms containing } (\mu-1) \text{ as one of the factors}\right]$$

on differentiating n times by Leibnitz's theorem.

$\therefore \; P_n(1) = C\cdot 2^n \lfloor n$, where $P_n(1)=1$; $\therefore \; C = \frac{1}{2^n \lfloor n}$

Hence $P_n(\mu) = \frac{1}{2^n \lfloor n}\left(\frac{d}{d\mu}\right)^n (\mu^2-1)^n$.

[e_3] *Series solution of Hypergeometric equation. i.e., to integrate*

$$x(1-x)\frac{d^2y}{dx^2} + \{\gamma - (\alpha+\beta+1)x\}\frac{dy}{dx} - \alpha\beta\gamma = 0 \quad \ldots(53)$$

in series of ascending power of x, and to show that its complete primitive is

$AF(\alpha, \beta, \gamma, x) + Bx^{1-\gamma} F(\alpha+1-\gamma, \beta+1-\gamma\, 2-\gamma, x)$

where A and B are arbitrary constants and $F(\alpha, \beta, \gamma, x)$ *stands for the series*

$$1 + \frac{\alpha\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)(\beta+1)}{1\cdot2\cdot\gamma\cdot(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot2\cdot3\cdot\gamma(\gamma+1)(\gamma+2)}x^3 + \ldots \quad \ldots(54)$$

Suppose $y = \sum_{r=0}^{\infty} a_r x^{k+r}$.

$$\therefore \quad \frac{dy}{dx} = \sum_{r=0}^{\infty} a_r(k+r)\, x^{k+r-1},$$

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (k+r)(k+r-1)\, x^{k+r-2}.$$

Substituting these values in the given hypergeometric equation, we get

$$\sum_{r=0}^{\infty}[x(1-x)(k+r)(k+r-1)x^{k+r-2} + \{\gamma - (\alpha+\beta+1)x\}$$

$$\times (k+r)x^{k+r-1} - \alpha B x^{k+r}]a_r \equiv 0$$

or $$\sum_{r=0}^{\infty}[\{(k+r)(k+r-1) + \gamma(k+r)\}x^{k+r-1} - \{(k+r)(k+r-1)$$

$$+ (\alpha+\beta)+1)(k+1) + \alpha\beta\}x^{k+r}]a_r \equiv 0 \quad \ldots(55)$$

Being an identity, let us equate the coefficients of various powers of x to zero.

Firstly equating to zero the coeff. of x^{k-1} by putting $r=0$, we get

$$\{k(k-1) + \gamma k\}a_0 = 0.$$

$\therefore \; k(k-1) + \gamma k = 0$, since $a \neq 0$ being the coeff. of first term, or $k^2 - k + \gamma k = 0$ or $k(k-1+\gamma) = 0$,

which gives $k = 0, 1-\gamma$. ...(56)

Now equating to zero the coeff. of general term, *i.e.*, x^{k+r} in identity (55) we have

$$\{(k+r)(k+r+1)+\gamma(k+r+1)\}a_{r+1}-\{(k+r)(k+r-1)+(\alpha+\beta+1)(k+r)+\alpha\beta\}a_r=0.$$

$$\therefore\ a_{r+1}=\frac{(k+r)^2+(k+r)(\alpha+\beta)+\alpha\beta}{(k+r+1)(k+r)+\gamma(k+r+1)}a_r$$

$$=\frac{(k+r)^2+\alpha(k+r)+\beta(k+r)+\alpha\beta}{(k+r+1)+\gamma(k+r\ \ +\gamma)}a_r$$

$$=\frac{(k+r+\beta)(k+r+\alpha)}{(k+r+1)(k+r+\gamma)}a_r \qquad \text{...(57)}$$

Case I. ***When*** $k=0$.

$$a_{r+1}=\frac{(\alpha+r)(\beta+r)}{(r+1)(r+\gamma)}a_r. \qquad \text{...(58)}$$

Putting $r=0,1,2,3$...successively.

$$a_1=\frac{\alpha\cdot\beta}{1\cdot\gamma}a_0,$$

$$a_2=\frac{(\alpha+1)(\beta+1)}{2\cdot(\gamma+1)}a_1=\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2\gamma(\gamma+1)(\gamma+2)}a_0,$$

$$a_3=\frac{(\alpha+2)(\beta+2)}{3\cdot(\gamma+2)}a_2=\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}a_0$$

... ... etc.

Hence the series solution is

$$y=a_0\left[1+\frac{\alpha\cdot\beta}{1\cdot\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}x^2+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot2\cdot3\cdot\gamma(\gamma+1)(\gamma+2)}x^3+\ldots\right]$$

$$=a_0F(\alpha,\beta,\gamma,x) \qquad \text{...(59)}$$

Case II. ***When*** $k=1-\gamma$.

$$a_{r+1}=\frac{(\alpha+r+1-\gamma)(\beta+r+1-\gamma)}{(2-\gamma+r)(r+1)} \qquad \text{...(60)}$$

Putting $r=0,1,2,3,$... successively,

$$a_1=\frac{(\alpha+1-\gamma)(\beta+1-\gamma)}{(2-\gamma)\cdot1}a_0,$$

$$a_2=\frac{(\alpha+1-\gamma+1)(\beta+1-\gamma+1)}{(2-\gamma+1)\cdot2}a_1$$

$$=\frac{(\alpha+1-\gamma)(\alpha+1-\gamma+1)(\beta+1-\gamma)(\beta+1-\gamma+1)}{(2-\gamma)(2-\gamma+1)\cdot1\cdot2}a_0$$

... ... etc.

Hence the series solution is

$$y = \alpha_0\left[x^{1-\gamma} + \frac{(\alpha+1-\gamma)(\beta+1-\gamma)}{(2-\gamma)\cdot 1}x^{1-\gamma+1} + \frac{(\alpha+1-\gamma)(\alpha+1-\gamma+1)(\beta+1-\gamma)(\beta+1-\gamma+1)}{(2-\gamma)(2-\gamma+1)\cdot 1\cdot 2}x^{1-\gamma+2} + \ldots\right]$$

$$= a_0 x^{1-\gamma} F(\alpha + 1 - \gamma, \beta + 1 - \gamma, 2 - \gamma, x). \qquad \ldots(61)$$

Hence the complete primitive is

$$y = AF(\alpha, \beta, \gamma, x) + Bx^{1-\gamma} F(\alpha + 1 - \gamma, \beta + 1 - \gamma, 2 - \gamma, x) \qquad \ldots(62)$$

where A and B are arbitrary constants.

Problem 13. *Show that*

$$P_{2m+1}(0) = 0 \text{ and } P_{2m}(0) = (-1)^m \frac{1.3.5\ldots(2m-1)}{2.4.6\ldots2m}$$

We know that

$$P_n(\mu) = \frac{1.3.5\ldots(2n-1)}{\lfloor n}\left[\mu^n - \frac{n(n-1)}{2(2n-1)}\mu^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}\mu^{n-4} + \ldots\right]$$

$$\therefore P_{2m+1}(\mu) = \frac{1.3.5\ldots\{2(2m+1)-1\}}{\lfloor(2m+1)} \times \left[\mu^{2m+1} - \frac{(2m+1)(2m+1-1)}{2\{2(2m+1)-1\}}\mu^{2m-1} + \ldots\right]$$

so that $P_{2m+1}(0) = \dfrac{1.3.5\ldots(4m+1)}{\lfloor(2m+1)}[0] = 0.$

Again, we have

$$\left(1 - 2\mu h + h^2\right)^{-1/2} = \sum_{r=0}^{\infty} h^r P_r(\mu)$$

$$= P_0(\mu) + hP_1(\mu) + h^2P_2(\mu) + \ldots h^{2m} P_{2m}(\mu) + \ldots$$

Putting $\mu = 0$, we get

$$P_0(0) + hP_2(0) + h^2P_2(0) + \ldots + h^{2m}P_{2m}(0)$$

$$= (1 + h^2)^{-1/2}$$

$$= \{1 - (-h^2)\}^{-1/2}$$

$$= 1 + \tfrac{1}{2}\left(-h^2\right) + \frac{\frac{1}{2}\cdot\frac{3}{2}\left(-h^2\right)^2}{\lfloor 2}, + \ldots + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdots\left(\frac{2r-1}{2}\right)}{\lfloor r}\left(-h^2\right)^r + \ldots$$

expanding by Binomial Theorem.

Equating the coefficients of x^{2m} on either side.

$$P_{2m}(0) = (-1)^m \frac{1.3.5\ldots(2m-1)}{2^m\cdot\lfloor m} = (-1)^m \cdot \frac{1.3.5\ldots(2m-1)}{2.4.6\ldots2m}.$$

Problem 14. *Show that the Legendre's equation*

$$\left(1-\mu^2\right)\frac{d^2y}{d\mu^2} - 2\mu\frac{dy}{d\mu} + n(n+1)y = 0$$

changes into the hypergeometric form

$$x(1-x)\frac{d^2y}{dx^2} + \left(\frac{1}{2} - \frac{3x}{2}\right)\frac{dy}{dx} + \frac{n(n+1)}{4}y = 0.$$

Hence show by comparison that its complete primitive is

$$AF\left(-\frac{n}{2}, \frac{n+1}{2}, \tfrac{1}{2}, \mu^2\right) + B\mu\, F\left(-\frac{n-1}{2}, \frac{n+2}{2}, \frac{3}{4}, \mu^2\right).$$

Given $\mu^2 = x$; $\therefore\ 2\mu = \dfrac{dx}{d\mu}$.

Now $$\frac{dy}{d\mu} = \frac{dy}{dx}\cdot\frac{dx}{d\mu} = 2\mu\cdot\frac{dy}{dx}.$$

$$\frac{d^2y}{d\mu^2} = \frac{d}{d\mu}\left(2\mu\frac{dy}{dx}\right)$$

$$= 2\,\frac{dy}{dx} + 2\mu\frac{d^2y}{dx^2}\frac{dx}{d\mu}$$

$$= 2\,\frac{dy}{dx} + 4\mu^2\frac{d^2y}{dx^2}$$

Substituting these values in the given Legendre's equation, we get

$$(1-x)\left(4x\frac{d^2y}{dx^2} + 2\,\frac{dy}{dx}\right) - 4x\,\frac{dy}{dx} + n(n+1)\,y = 0$$

or $$x(1-x)\frac{d^2y}{dx^2} + \frac{1-x-2x}{2}\frac{dy}{dx} + \frac{n(n+1)}{4}y = 0$$

or $$x(1-x)\frac{d^2y}{dx^2} + \left(\frac{1}{2} - \frac{3x}{2}\right)\frac{dy}{dx} + \frac{n(n+1)}{4}y = 0.$$

Again, we have already proved in $E(e_3)$ of § 8.3 that the complete primitive of the hypergeometric series

$$x(1-x)\frac{d^2y}{dx^2} + [\gamma - (\alpha+\beta+1)]\frac{dy}{dx} - \alpha\beta\, y = 0$$

is $$AF(\alpha, \beta, \gamma, x) + Bx^{1-\gamma}F(\alpha+1-\gamma, \beta+1-\gamma, 2-\gamma, x).$$

In the existing case, by comparison, we have

$$\gamma = \frac{1}{2},\ \alpha+\beta+1 = \frac{3}{2},\ \alpha\beta = -\frac{n(n+1)}{4},$$

i.e. $$\gamma = \frac{1}{2},\ \alpha+\beta = \tfrac{1}{2},\ \alpha\beta = -\frac{n(n+1)}{4}$$

Now $$(\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta$$

$$= \frac{1}{4} + n(n+1)$$

$$= \frac{1}{4} + n^2 + n = \left(n + \frac{1}{2}\right)^2$$

$\therefore$ $$\alpha - \beta = n + \frac{1}{2} \text{ and } \alpha+\beta = \frac{1}{2}$$

Solving $$\alpha = \frac{n+1}{2},\ \beta = -\frac{n}{2}.$$

Hence by comparison the complete primitive of the transformed hypergeometric series is

$$AF\left(\frac{n+1}{2}, \frac{-n}{2}, \frac{1}{2}, \mu^2\right) + B\mu F\left(\frac{n+2}{2}, -\frac{n-1}{2}, \frac{3}{2}, \mu^2\right)$$

Problem 15. *Show that*

$$P_n(\cos\theta) = \cos^{2n}\frac{\theta}{2} F\left(-n, -n, 1, -\tan^2\frac{\theta}{2}\right)$$

We know from the first Lapalce's Integral that

$$P_n(\mu) = \frac{1}{\pi}\int_0^{\pi}\left\{\mu \pm \sqrt{(\mu^2-1)}\cos\phi\right\}^n d\phi.$$

$\therefore P_n(\cos\theta)$

$$= \frac{1}{\pi}\int_0^{\pi} (\cos\theta + i\sin\theta\cos\phi)^n d\phi \text{ (taking positive sign)}$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\left(\frac{e^{i\phi}+e^{i\phi}}{2}\right)\right]^n d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[\cos^2\frac{\theta}{2} + i\sin\frac{\theta}{2}\cos\frac{\theta}{2}e^{i\phi} + i^2\sin^2\frac{\theta}{2}e^{i\phi}\cdot e^{-i\phi} + i\sin\frac{\theta}{2}\cos\frac{\theta}{2}e^{-i\phi}\right]^n d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[\cos\frac{\theta}{2}\left\{\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}e^{i\phi}\right\} + i\sin\frac{\theta}{2}e^{-i\phi}\left\{\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}e^{i\phi}\right\}\right]^n d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}e^{i\phi}\right)\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}e^{-i\phi}\right)\right]^n d\phi$$

$$= \frac{1}{\pi}\cos^{2n}\frac{\theta}{2}\int_0^{\pi}\left(1 + i\tan\frac{\theta}{2}e^{i\phi}\right)^n\left(1 + i\tan\frac{\theta}{2}e^{-i\phi}\right)^n d\phi$$

$$= \frac{1}{\pi}\cos^{2n}\frac{\theta}{2}\int_0^{\pi}\left[\left\{1 + ni\tan\frac{\theta}{2}e^{i\phi} - \frac{n(n-1)}{\lfloor 2}\tan^2\frac{\theta}{2}e^{2i\phi} + \ldots\right\}\right.$$

$$\left.\times\left\{1 + ni\tan\frac{\theta}{2}e^{-i\phi} - \frac{n(n-1)}{\lfloor 2}\tan^2\frac{\theta}{2}e^{-i\phi} + \ldots\right\}\right]d\phi.$$

Here the diagonal product will give cosines of multiples of ϕ and when they will be taken in groups, they vanish in limits of integration. We have only to consider the column products, *i.e.*

$$P_n(\cos\theta) = \frac{1}{\pi}\cos^{2n}\frac{\theta}{2}\int_0^{\pi}\left[1 - \tan^2\frac{\theta}{2} + \frac{n^2(n-1)^2}{\lfloor 2\lfloor 2}\tan^4\frac{\theta}{2} - \ldots\right]d\phi$$

$$= \frac{1}{\pi}\cos^2\frac{\theta}{2}\left[\pi - \pi n^2\tan^2\frac{\theta}{2} + \pi\cdot\frac{n^2(n-1)^2}{\lfloor 2\lfloor 2}\tan^4\frac{\theta}{2} - \ldots\right]$$

$$= \cos^{2n}\frac{\theta}{2}\left[1 + \frac{(-n)(-n)}{1\cdot 1}\left(-\tan^2\frac{\theta}{2}\right)\right.$$

$$+\frac{(-n)(-n+1)(-n)(-n+1)}{1\cdot 2\cdot 1\cdot (1+1)}\left(-\tan^2\frac{\theta}{2}\right)^2-\ldots\Bigg]$$

$$=\cos^{2n}\frac{\theta}{2}\,F\left(-n,\ -n,\ 1,\ -\tan^2\frac{\theta}{2}\right)$$

Another form of Problem 15.

Show that

$$P_n(\cos\theta)=\cos^n\theta\, F\left(-\frac{n}{2},\ \frac{1}{2}-\frac{n}{2},\ 1,\ -\tan^2\theta\right)$$

The first of Laplace's integrals gives

$$P_n(\cos\theta)=\frac{1}{\pi}\int_0^\pi(\cos\theta+i\sin\theta\cos\phi)^n d\phi$$

$$=\frac{1}{\pi}\cos^n\theta\int_0^\pi(1+i\tan\theta\ \cos\phi)^n d\phi$$

$$=\frac{1}{\pi}\cos^n\theta\int_0^\pi\Bigg\{1+ni\tan\theta\cos\phi-\frac{n(n-1)}{\lfloor 2}\times$$

$$\tan^2\theta\cos^2\phi-\frac{n(n-1)(n-2)}{\lfloor 3}\tan^3\theta\cos^3\phi+\ldots\Bigg\}d\phi.$$

In limits from 0 to $\pi/2$ all the cosine terms having odd powers vanish.

$$\therefore P_n(\cos\theta)=\frac{2}{\pi}\cos^n\theta\int_0^{\pi/2}\left[1-\frac{n(n-1)}{\lfloor 2}\tan^2\theta\ \cos^2\phi+\ldots\right]d\phi$$

$$=\frac{2}{\pi}\cos^n\phi\Bigg[\frac{\pi}{2}-\frac{n(n-1)}{\lfloor 2}\tan^2\theta\frac{1}{2}\cdot\frac{\pi}{2}$$

$$+\frac{n(n-1)(n-2)(n-3)}{\lfloor 4}\tan^4\theta\cdot\frac{3\cdot 1\cdot\pi}{4\cdot 1\cdot 2}+\ldots\Bigg]$$

$$=\cos^n\theta\Bigg[1+\frac{\left(-\frac{1}{2}n\right)\left(-\frac{1}{2}n+\frac{1}{2}\right)}{1\cdot 1}\left(-\tan^2\theta\right)$$

$$+\frac{\left(-\frac{1}{2}n\right)\left(-\frac{1}{2}n+1\right)\left(-\frac{1}{2}n+\frac{1}{2}\right)\left(-\frac{1}{2}n+\frac{3}{2}\right)}{1\cdot 1}\left(-\tan^2\theta\right)^2+\ldots\Bigg]$$

$$=\cos^n\theta\, F\left[-\frac{1}{2}n,\ \frac{1}{2}-\frac{1}{2}n,\ 1,\ -\tan^2\theta\right].$$

[F] Orthogonal Properties of Legendre's Polynomials of the First Kind

To show that

$$\int_{-1}^{1}P_m(\mu)P_n(\mu)d\mu=0 \text{ unless } m=n$$

$$=\frac{2}{2n+1}\ \text{if } m=n$$

or $$= \frac{2}{2n+1} \delta_{m,n}$$ *in Kronecker delta symbol m, n being positive integers.*

(Agra, 1963; Kurukshetra, 1965; Rohilkhand, 1980, 81, 93)

Rodrigue's formula is

$$P_n(\mu) = \frac{1}{2^n \lfloor n} \left(\frac{d}{d\mu}\right)^n (\mu^2 - 1)^n.$$

With its applications, we have

$$\int_{-1}^{1} P_m(\mu)\, P_n(\mu)\, d\mu$$

$$= \frac{1}{2^{m+n} \lfloor m \lfloor n} \int_{-1}^{1} \left(\frac{d}{d\mu}\right)^m (\mu^2-1)^m \cdot \left(\frac{d}{d\mu}\right)^n (\mu^2-1)^n d\mu$$

$$= \frac{1}{2^{m+n} \lfloor m \lfloor n} \left[\left\{ \left(\frac{d}{d\mu}\right)^m (\mu^2-1)^m \cdot \left(\frac{d}{d\mu}\right)^{n-1} (\mu^2-1)^n \right\}_{-1}^{1} \right.$$

$$\left. - \int_{-1}^{1} \left(\frac{d}{d\mu}\right)^{m+1} (\mu^2-1)^m \cdot \left(\frac{d}{d\mu}\right)^{n-1} (\mu^2-1)^n d\mu \right] \text{ (integrating by parts)}$$

$$= -\frac{1}{2^{m+n} \lfloor m \lfloor n} \left[\int_{-1}^{1} \left(\frac{d}{d\mu}\right)^{m+1} (\mu^2-1)^m \cdot \left(\frac{d}{d\mu}\right)^{n-1} (\mu^2-1)^n d\mu \right]$$

(the first term vanishing for both the limits.)

Continuing the process of integration by parts m times on R.H.S., we get

$$\int_{-1}^{1} P_m(\mu) P_n(\mu)\, d\mu = \frac{(-1)^m}{2^{m+n} \lfloor m \lfloor n} \int_{-1}^{1} \left(\frac{d}{d\mu}\right)^{2m} (\mu^2-1)^m \left(\frac{d}{d\mu}\right)^{n-m} (\mu^2-1)^n d\mu$$

Here $$\left(\frac{d}{d\mu}\right)^{2m} (\mu^2-1)^m = \frac{d^{2m} (\mu^2-1)^m}{d\mu^{2m}} = \lfloor (2m)$$

$$\therefore \int_{-1}^{1} P_m(\mu)\, P_n(\mu)\, d\mu = \frac{(-1)^m \lfloor 2m}{2^{m+n} \lfloor m \lfloor n} \int_{-1}^{1} \left(\frac{d}{d\mu}\right)^{n-m} (\mu^2-1)^n d\mu \quad \text{...(63)}$$

$$= \frac{(-1)^m \lfloor 2m}{2^{m+n} \lfloor m \lfloor n} \left[\left(\frac{d}{d\mu}\right)^{n-m-1} (\mu^2-1)^n \right]_{-1}^{1}$$

$$= \frac{(-1)^m \lfloor 2m}{2^{m+n} \lfloor m \lfloor n} [0] = 0 \text{ when } m \neq n \text{ and } n > m. \quad \text{...(64)}$$

Again when $m = n$, from (63), we have

$$\int_{-1}^{1} P_n^2(\mu)\, d\mu = \frac{(-1)^n \lfloor 2n}{2^n \lfloor n} \int_{-1}^{1} (\mu^2-1)^n d\mu$$

or $$\int_{-1}^{1} P_n^2(\mu)\, d\mu = \frac{(-1)^m\, |\underline{2n}}{2^n\, |\underline{n}} (-1)^n \int_{-1}^{1} \left(1-\mu^2\right)^n d\mu$$

$$= \frac{1.3.5\ldots(2n-1)}{2^n |\underline{n}} \int_{-1}^{1} \left(1-\mu^2\right)^n d\mu \quad \text{put } \mu = \cos\theta,\ d\mu = -\sin\theta\, d\theta$$

$$= -\frac{1.3.5\ldots(2n-1)}{2^n |\underline{n}} \int_{\pi}^{0} \sin^{2n+1}\theta\, d\theta$$

$$= \frac{2 \times 1.3.5\ldots.(2n-1)}{2^n |\underline{n}} \int_{0}^{\pi/2} \sin^{2n+1}\theta\, d\theta$$

$$= 2.\frac{1.3.5\ldots(2n-1)}{2^n |\underline{n}} \times \frac{2n\,(2n-2)\ldots 2}{(2n+1)\,(2n-1)\ldots 3.1.}$$

$$= 2.\frac{2^n |\underline{n}}{2^n |\underline{n}\,(2n+1)} + \frac{2}{2n+1}. \qquad \ldots(65)$$

Problem 16. *Show that* $\int_{-1}^{1} \left\{\frac{dP_n(\mu)}{d\mu}\right\}^2 d\mu = n(n+1)$.

We have proved that

$$\frac{dP_n(\mu)}{d\mu} = (2n-1)\,P_{n-1}(\mu) + (2n-5)\,P_{n-3}(\mu) + \ldots \text{ ending with } 3P_1(\mu) \text{ or } P_0(\mu)$$

according as n is even or odd.

$$\therefore \int_{-1}^{1} \left\{\frac{dP_n(\mu)}{d\mu}\right\}^2 d\mu = \int_{-1}^{1} [(2n-1)\,P_{n-1}(\mu) + (2n-5)\,P_{n-3}(\mu) + \ldots \text{ ending with}$$

$$3P_1(\mu) \text{ or } P_0(\mu) \text{ according as } n \text{ is even or odd}]^2\, d\mu.$$

Here on the R.H.S. the product terms vanish in limits and only the square terms are left.

$$\therefore \int_{-1}^{1} \left\{\frac{dP_n(\mu)}{d\mu}\right\}^2 d\mu = (2n-1)^2 \frac{2}{2\,(n-1)+1} + (2n-5)^2 \cdot \frac{2}{2\,(n-3)+1} + \ldots$$

ending with $3^2 \cdot \frac{2}{2+1}$ or $1^2 \cdot \frac{2}{2 \times 0 + 1}$ according as n is even or odd

$$= 2[(2n-1) + (2n-5) + (2n-9) + \ldots + 3] \text{ when } n \text{ is even} \quad \ldots(1)$$

$$= 2[(2n-1) + (2n-5) + \ldots + 1]. \text{ when } n \text{ is odd.} \quad \ldots(2)$$

Now we know that in an arithmetical progression having a as its first term, l the last term, d the common difference and N the total number of terms,

$$l = a + (N-1)\,d, \text{ i.e. } N = \frac{l-a}{d} + 1.$$

In (1), $$N = \frac{2n-1-3}{4} + 1 = \frac{n}{2}.$$

$$\therefore \text{ sum} = 2.\ \frac{N}{2}[a+l] = \frac{2n}{4}[3 + 2n - 1] = n\,(n+1).$$

In (2), $$N = \frac{2n-1-1}{4} + 1 = \frac{n+1}{2}.$$

$$\therefore \text{ sum} = 2\cdot\left(\frac{n+1}{4}\right)\{1+2n-1\} = n(n+1).$$

We see that the sum in two cases is the same whether n is even or odd.

Hence $$\int_{-1}^{1}\left\{\frac{dP_n(\mu)}{d\mu}\right\}^2 d\mu = n(n+1).$$

Problem 17. *Show that* $\int_{-1}^{1}(1-x^2)\left\{\frac{dP_n(x)}{dx}\right\}^2 dx = \frac{2n(n+1)}{2n+1}$.

From recurrence formula (45) and (44), we have

$$(1-x^2)\frac{dP_n(x)}{dx} = \frac{n(n+1)}{2n+1}\{P_{n-1}(x) - P_{n+1}(x)\}, \qquad \text{...(1)}$$

and $\frac{dP_n(x)}{dx} = (2n-1)P_{n-1}(x) + (2n-5)P_{n-3}(x) + (2n-9)P_{n-5}(x)\ldots$ending within $3P_1(x)$ or $P_0(x)$ according as n is even or odd. ...(2)

Multiplying (1) and (2) together and integrating within the limits -1 and 1, we get

$$\int_{-1}^{1}(1-x)^2\left\{\frac{dP_n(x)}{dx}\right\}^2 dx = \int_{-1}^{1}\left[\frac{n(n+1)}{2n+1}\{P_{n-1}(x) - P_{n+1}(x)\}\{(2n-1)P_{n-1}(x) + (2n-3)P_{n-3}(x)\ldots\}\right]dx = 0$$

or $$\int_{-1}^{1}(1-x^2)\left\{\frac{dP_n(x)}{dx}\right\}^2 dx = \frac{n(n+1)}{2n+1}(2n-1)\int_{-1}^{1}P_{n-1}^2(x)\,dx \quad \text{other terms vanishing}$$

$$= \frac{n(n+1)(2n-1)}{2n+1}\cdot\frac{2}{2(n-1)+1}$$

$$= \frac{2n(n+1)}{2n+1}$$

Problem 18. *If (u, ϕ, z) and (r, θ, ϕ) be the cylindrical and polar co-ordinates of the same point and if $\mu = \cos\theta$, show that*

$$P_n(\mu) = (-1)^n\cdot\frac{r^{n+1}}{\underline{|n}}\left(\frac{\partial}{\partial z}\right)^n\cdot\left(\frac{1}{r}\right)$$

Here $$r = \sqrt{(x^2+y^2+z^2)} = (u^2+z^2)^{1/2}$$

$$\therefore \quad \frac{1}{r} = (u^2+r^2)^{-1/2} = \phi(u, z) \text{ say,}$$

so that, by Taylor's theorem, we have

$$\phi(u, z-k) = \{u^2+(z-k)^2\}^{-1/2}$$

$$= \phi(u, z) - k\frac{\partial}{\partial z}\phi(u, z) + \frac{k^2}{2!}\left(\frac{\partial}{\partial z}\right)^2\phi(u, z)\ldots+(-1)^n\frac{k^n}{n!}\left(\frac{\partial}{\partial z}\right)^n\phi(u, z)+\ldots \qquad \text{...(1)}$$

Also, $$\phi(u, z-k) = \{u^2+(z-k)^2\}^{-1/2}$$
$$= \{u^2+z^2-2zk+k^2\}^{-1/2}$$
$$= \{r^2-2r\cos\theta.\,k+k^2)^{-1/2}$$

$$=\left(r^2\right)^{-1/2}\left[1-2\cos\theta\cdot\frac{k}{r}+\frac{k^2}{r^2}\right]^{-1/2}$$

$$=\frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{k}{r}\right)^n P_n(\cos\theta). \quad ...(2)$$

Equating the coefficients of k^n in (1) and (2), we get

$$\frac{1}{r}\cdot\frac{1}{r^n}\,P_n(\cos\theta)=\frac{(-1)^n}{n\,!}\left(\frac{\partial}{\partial z}\right)^n\phi(u,\,z)$$

or

$$P_n(\cos\theta)=(-1)^n\cdot\frac{r^{n+1}}{n\,!}\left(\frac{\partial}{\partial z}\right)^n\cdot\frac{1}{r}.$$

Note. ***Relations between polar and cartesian co-ordinates:***

$$x=r\sin\theta\cos\phi,\ y=r\sin\theta\sin\phi,\ z=r\cos\theta.$$

Relations between cylindrical and cartesian co-ordinates:

$$x=u\cos\phi,\ y=u\sin\phi,\ z=z.$$

[G] The Associated Legendre Polynomials

Laplace's equations in spherical co-ordinates may be expressed as

$$r\frac{\partial^2(rV)}{\partial r^2}+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0. \quad ...(66)$$

Assuming that V is a product of single-valued function, let

$$V=R\cdot\Theta\cdot\Phi;$$

where R is the function of r only, Θ is the function of θ only and Φ is the function of ϕ only.

Then $$\frac{\partial V}{\partial r}=\Theta\cdot\Phi\frac{dR}{dr};\ \frac{\partial^2 V}{\partial r^2}=\Theta\cdot\Phi\,\frac{d^2R}{dr^2},$$

$$\frac{\partial V}{\partial\theta}=R\cdot\Phi\,\frac{\partial\Theta}{\partial\theta}$$

and $$\frac{\partial V}{\partial\phi}=R\cdot\Theta\,\frac{d\Phi}{\partial\phi};\ \frac{\partial^2 V}{\partial\phi^2}=R\cdot\Theta\cdot\frac{d^2\Phi}{\partial\phi^2}.$$

Substituting these values in (66), we get

$$r\Theta\cdot\Phi\frac{d^2}{dr^2}(rR)+\frac{1}{\sin\theta}R\cdot\Phi\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{\sin^2\theta}\cdot R\cdot\Theta\frac{d^2\Phi}{d\phi^2}=0$$

or $$\frac{r}{R}\frac{d^2(rR)}{dr^2}+\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right)+\frac{1}{\Phi\sin^2\theta}\cdot\frac{d^2\Phi}{d\phi^2}=0 \quad ...(67)$$

(on dividng throughout by $R\cdot\Theta\cdot\Phi\cdot$)

or $$\frac{r\sin^2\theta}{R}\frac{d^2(rR)}{dr^2}+\frac{\sin\theta}{\Theta}\cdot\frac{d\left(\sin\theta\,\frac{d\Theta}{d\theta}\right)}{d\theta}=-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} \quad ...(68)$$

Consider the equation (67) whose R.H.S. contains ϕ, but the L.H.S. does not. It follows that $-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}$ = constant (say) n^2, so that

$$\frac{d^2\Phi}{d\phi^2} + n^2\Phi = 0. \qquad \ldots(69)$$

The roots of this equation being $\pm\, in$, its solution is

$$\Phi = A \cos n\phi + B \sin n\phi. \qquad \ldots(70)$$

Now the equation (67) by the subsitution $-\dfrac{1}{\Phi}\dfrac{d^2\Phi}{d\phi^2} = n^2$ becomes

$$\frac{r}{R}\frac{d^2(rR)}{dr^2} + \frac{1}{\Theta \sin\theta}\frac{d\left(\sin\theta \dfrac{d\Theta}{d\theta}\right)}{d\theta} - \frac{n^2}{\sin^2\theta} = 0. \qquad \ldots(71)$$

It is obvious that the first term of the equation (71) does not contain θ and the second and third terms do not contain r. It follows that the first term *i.e.* $\dfrac{r}{R}\dfrac{d^2(rR)}{dr^2}$ must be constant.

Let $\dfrac{r}{R}\dfrac{d^2(rR)}{dr^2} = m\,(m+1). \qquad \ldots(72)$

[Its solution may easily be shown to be $R = Cr^m + Dr^{-m-1}$.]

Putting this value in (71), we get

$$m(m+1) + \frac{1}{\Theta\sin\theta}\cdot\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{n^2}{\sin^2\theta} = 0$$

i.e.
$$\frac{1}{\sin\theta}\frac{d\left(\sin\theta\dfrac{d\Theta}{d\theta}\right)}{d\theta} + \left[m(m+1) - \frac{n^2}{\sin^2\theta}\right]\Theta = 0. \qquad \ldots(73)$$

Taking $\Theta(\cos\theta) = z(x)$, *i.e.* replacing Θ by z and $\cos\theta$ by x, (73) becomes

$$\left(1-x^2\right)\frac{d^2z}{dx^2} - 2x\frac{dz}{dx} + \left[m\,(m+1) - \frac{n^2}{1-x^2}\right]z = 0. \qquad \ldots(74)$$

If we now put $z = (1-x^2)^{n/2}\, y$.

i.e
$$\frac{dz}{dx} = -nx\left(1-x^2\right)^{(n/2)-1} y + \left(1-x^2\right)^{n/2}\frac{dy}{dx}$$

and
$$\frac{dz}{dx^2} = -2nx\left(1-x^2\right)^{(n/2)-1}\frac{dy}{dx} - ny\left[\left(1-x^2\right)^{(n/2)-1} - (n-2).x^2(1-x^2)^{(n/2)-2}\right] + (1-x^2)^{n/2}\frac{d^2y}{dx^2},$$

then the equation (74) transforms to

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - 2\,(n+1)\,x\frac{dy}{dx} + [m\,(m+1) - n\,(n+1)]\,y = 0. \qquad \ldots(75)$$

In order to find the series solution of this differential equation, let us suppose that its series solution is

$$y = \Sigma\, a_k x^k,$$

so that $\dfrac{dy}{dx} = \sum k a_k . x^{k-1}$ and $\dfrac{d^2y}{dx^2} = \sum k\,(k-1)\,a_k\cdot x^{k-2}$.

With these substitutions, (10) gives

$$\sum [(1-x^2)\, k\,(k-1)\, x^{k-2} - 2\,(n+1)\, xkx^{k-1} + \{m\,(m+1) - n\,(n+1)\}\, x^k)\, a_k = 0$$

or $$\sum [k\,(k-1)\, x^{k-2} + \{m\,(m+1) - n\,(n+1) - 2k\,(n+1) - k\,(k-1)\}]\, x^k\, a_k = 0.$$

This is an identical equation. We therefore, on equating the coefficients of x^k on either side, get

$$\{(k+2)\,(k+2-1)\, a_{k+2} + \{m\,(m+1) - n\,(n+1) - 2k\,(n+1) - k\,(k-1)\}a_k = 0$$

or $$(k+1)\,(k+2)\, a_{k+2} + (m^2 - n^2 + m - n - 2nk - k - k)a_k = 0$$

or $$(k+1)\,(k+2)\, a_{k+2} + (m-n-k)\,(m+n+1+k)\, a_k = 0$$

or $$a_k = -\frac{(k+1)\,(k+2)}{(m-n-k)\,(m+n+1+k)}\, a_{k+2}. \qquad \ldots(76)$$

If $a_k = 0$, then it is obvious that $a_{k-2} = a_{k-4} = a_{k-6} = \ldots 0$.

But $a_k = 0$ if $k = -1$ or $k = -2$.

Now $a_{k+2} = -\dfrac{(m-n-k)\,(m+n+1+k)}{(k+1)\,(k+2)}\, a_k$ gives the sequence of coefficients

$$a_2 = -\frac{(m-n)\,(m+n+1)}{2\,!}\, a_0$$

$$a_4 = -\frac{(m-n-2)\,(m+n+3)}{3 \cdot 4}\, a_2$$

$$= \frac{(m-n)\,(m-n-2)\,(m+n+1)\,(m+n+3)}{4\,!}\, a_0$$

... ...etc.

and $$a_3 = -\frac{(m-n-1)\,(m+n+2)}{3\,!}\, a_0$$

$$a_5 = -\frac{(m-n-3)\,(m+n+4)}{4 \cdot 5}\, a_0$$

$$= \frac{(m-n-1)\,(m-n-3)\,(m+n+2)\,(m+n+4)}{5\,!}\, a_0$$

Taking a_0 and a_1 as unity, since they are arbitrary, we have the two series solutions of (74) as

$$P_m^n(x) = (1-x^2)^{n/2} \left[x - \frac{(m-n)\,(m+n+1)}{\lfloor 2}\, x^2 + \frac{(m-n)\,(m-n-2)\,(m+n+1)\,(m+n+3)}{\lfloor 4}\, x^4 + \ldots \right]$$ *(Rohilkhand 1990)*

and $$Q_m^n\,(x) = (1-x^2)^{n/2} \left[x - \frac{(m-n-1)\,(m+n+2)}{\lfloor 3}\, x^3 + \frac{(m-n-1)\,(m-n-3)\,(m+n+2)\,(m+n+4)}{\lfloor 5}\, x^5 - \ldots \right];$$

where $P_m^n(x)$ and $Q_m^n(x)$ are called as *associated functions* of the nth order and mth degree.

In case when $(m-n)$ is a positive integer, $P_m^n(x)$ or $Q_m^n(x)$ will terminate involving x^{m-n} and in that case

$$z = (1-x^2)^{n/2}\left[x^{m-n} - \frac{(m-n)(m-n-1)}{2\cdot(2m-1)}x^{m-n-2}\right.$$
$$\left. + \frac{(m-n)(m-n-1)(m-n-2)(m-n-3)}{2\cdot 4\cdot(2m-1)(2m-3)}x^{m-n-4} - \ldots\right] \quad \ldots(77)$$

where the bracketed expression ends with a term involving x^0 if $(m-n)$ is even and x if $(m-n)$ is odd.

If m and n both are integers, then

$$z = \frac{2^m \lfloor m \lfloor m-n}{\lfloor (2m)} P_m^n(x).$$

With the substitution of this value of z along with $x = \cos\theta$ (77) becomes

$$P_m^n(x) = \frac{(2m)\sin^n\theta}{2^m \lfloor m \lfloor (m-n)}\left[x^{m-n} - \frac{(m-n)(m-n-1)}{2.\ (2m-1)}x^{m-n-2}\right.$$
$$\left. + \frac{(m-n)(m-n-1)(m-n-2)(m-n-3)}{2\cdot 4\cdot (2m-1)(2m-3)}x^{m-n-4}\ldots\right] \quad \ldots(78)$$

where the bracketed expression ends with a term involving x^0 if $(m-n)$ is even and x if $(m-n)$ is odd.

Further if we differentiate (75) w.r.t. x, then we get

$$(1-x^2)\frac{d^3y}{dx^3} - [2x + 2(n+1)x]\frac{d^2y}{dx^2} + [-2(n+1) + m(m+1) - n(n+1)]\frac{dy}{dx} = 0$$

i.e. $$(1-x^2)\frac{d^3y}{dx^3} - 2(n+2)x\frac{d^2y}{dx^2} + [m(m+1) - (n+1)(n+2)]\frac{dy}{dx} = 0. \quad \ldots(79)$$

Clearly if y satisfies (75) for n, then $\frac{dy}{dx}$ satisfies (79) for $n+1$. Thus for $n = 0$, y becomes identical with $P_m(x)$ and so $\frac{d^n}{dx^n}P_m(x)$ satisfies (75) for $n = n$ and the solution of (74) is

$$P_m^n(x) = (1-x^2)^{n/2}\frac{d^nP_m(x)}{dx^n}. \quad \ldots(80)$$

These functions are called the *associated Legendre Polynomials* or *associated harmonics.*

Note. *The associated Legendre functions of the second kind are*

$$Q_m^n(x) = (1-x^2)^{n/2}\frac{d^nQ_m(x)}{dx^n}. \quad \ldots(81)$$

Now (80) yields on putting $n = 0$,

$$P_m^n(x) = P_m(x) \quad \ldots(82)$$

From (80) the values of associated Legendre polynomials can be found with the help of (79) as

$$\left.\begin{aligned} P_1^1 &= (1-x^2)^{1/2} = \sin\theta \text{ when } x = \cos\theta \\ P_2^1 &= 3x(1-x^2)^{1/2} = 3\cos\theta\sin\theta \text{ when } x = \cos\theta \\ P_2^2 &= \frac{3}{2}(5x^2-1)(1-x^2)^{1/2} = \frac{3}{2}(5\cos^2\theta - 1)\sin\theta \end{aligned}\right\} \text{etc.} \qquad \ldots(83)$$

A generating function for associated Legendre polynomials may be derived from that of $P_n(x)$, in the form

$$\sum_{k=0}^{\infty} P_{n+k}^{n}(\mu)\,h^k = \frac{\lfloor 2n \,(1-\mu^2)^{n/2}}{2^n \lfloor n \,(1-2hx+h^2)^{n+1}} \qquad \ldots(84)$$

Now there being two indices in Associated Legendre polynomials, several recurrence formulae can be derived, but we summarize here below a few of them. We have from (43)

$$(2n+1)\,P_n(\mu) = \frac{d}{d\mu}P_{n+1}(\mu) - \frac{d}{d\mu}P_{n-1}(\mu)$$

Its differentiation, m times w.r.t. μ yields

$$(2n+1)\frac{d^m}{dx^m}P_n(\mu) = \frac{d^{m+1}}{d\mu^{m+1}}P_{n+1}(\mu) - \frac{d^{m+1}}{d\mu^{m+1}}P_{n-1}(\mu)$$

Multiplying throughout by $(1-\mu^2)^{(m-1)/2}$, it becomes with the help of (80)

$$(2n+1)(1-\mu^2)^{1/2}\,P_n^m(\mu) = P_{n+1}^{m+1}(\mu) - P_{n-2}^{m+1}(\mu) \qquad \ldots(85)$$

(*Rohilkhand, 1990*)

$$= (n+m)(n+m-1)\,P_{n-1}^{m-1}(\mu) - (n-m+1)(n-m+2)\,P_{n-1}^{m-1}(\mu) \qquad \ldots(86)$$

and with similar few results

$$(2n+1)\mu\,P_n^m(\mu) = (n+m)\,P_{n-1}^m(\mu) + (n-m+1)P_{n+1}^m(\mu) \qquad \ldots(87)$$

$$P_n^{m-1}(\mu) - \frac{2m\mu}{(1-x^2)^{1/2}}P_n^m(\mu) + [n(n+1) - m(m-1)]\,P_n^{m-1}(\mu) = 0 \qquad \ldots(88)$$

$$(1-\mu^2)^{1/2}\left(P_n^m\right) = \frac{1}{2}P_n^{m+1}(\mu) - \frac{1}{2}(n+m)(n-m+1)P_n^{m-1}(\mu) \qquad \ldots(89)$$

Further using Rodrigue's formula, orthogonality of Legendre associated functions may be shown as

$$\int_{-1}^{1} P_p^m(\mu)\,P_q^m(\mu)\,d\mu = 0 \quad \text{or} \quad \frac{2}{2q+1}\frac{\lfloor q+m}{\lfloor q-m} \qquad \ldots(90)$$

according as $q \neq p$ or $q = p$.

If it is transformed by the transformation $\mu = \cos\theta$, it becomes

$$\left.\begin{aligned} \int_0^{\pi} P_p^m(\cos\theta)\,P_q^m(\cos\theta)\sin\theta\,d\theta &= 0 \text{ when } q \neq p \\ &= \frac{2}{2q+1}\frac{\lfloor q+m}{\lfloor q-m} \text{ when } q = p \end{aligned}\right\} \qquad \ldots(91)$$

or

$$= \frac{2}{2q+1}\frac{\lfloor q+m}{\lfloor q-m}\delta_{q,\,p} \text{ in Kronecker delta symbol.}$$

(*Rohilkhand, 1990*)

If we now define a function

$$\bar{P}_l^m(\cos\theta) = \left(\sqrt{\frac{2l+1}{2}\,\frac{\lfloor l-m}{\lfloor l+m}}\right) P_l^m(\cos\theta), \quad -l \le m \le l \qquad ...(92)$$

then $\int_{-1}^{1} \bar{P}_l^m(\mu)\bar{P}_{l'}^m(\mu)\,d\mu = \delta_{l,l'}$. ...(93)

So that $\bar{P}_l^m$ is orthogonal w.r.t. θ.

Also we introduce a function $\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}$

then $\int_0^{2\pi} \Phi_{m_1}^*(\phi)\,\Phi_{m_2}(\phi)\,d\phi = \frac{1}{2\pi}\int_0^{2\pi} e^{-im_1\phi} e^{im_2\phi}\,d\phi = \delta_{m_1, m_2}$ ...(94)

Showing that $\Phi_m(\phi)$ is an orthonormal function w.r.t. azimuthal angle.

As such the spherical harmonics

$$Y_l^m(\theta, \phi) = \bar{P}_l^m(\cos\theta)\,\Phi_m(\phi) = \sqrt{\frac{2l+1}{4\pi}\left(\frac{\lfloor l-m}{\lfloor l+m}\right)}\, P_l^m(\cos\theta)\, e^{im\phi} \qquad ...(95)$$

The spherical harmonics $Y_l^m(\theta, \phi)$ are a complete orthonormal set of functions *i.e.*

$$\left(Y_{l'}^{m'}, Y_l^m\right) = \delta_{ll'}, \delta_{mm'} \qquad ...(96)$$

We can also show that $P_l^{-m}(\mu) = (-1)^m \frac{\lfloor l-m}{\lfloor l+m} P_l^m(\mu)$ ...(97)

$$Q_l^m = \text{constant} \times r^l\, Y_l^m(\theta, \phi) \qquad ...(98)$$

$$Y_l^{-m} = (-1)^m Y_l^{m*} \qquad ...(99)$$

Also few of the spherical Harmonics are

$$\left.\begin{aligned}
Y_0^0 &= \frac{1}{\sqrt{4\pi}},\quad Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta,\\
Y_1^1 &= -\sqrt{\frac{3}{8\pi}}\, e^{i\phi}\sin\theta\\
Y_2^0 &= \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1),\\
Y_2^1 &= -\sqrt{\frac{15}{8\pi}}\, e^{i\phi}\sin\theta\,\cos\theta\\
Y_2^2 &= \sqrt{\frac{15}{32\pi}}\, e^{2i\phi}\sin^2\theta \text{ etc.}
\end{aligned}\right\} \qquad ...(100)$$

Addition theorem of spherical Harmonics is

$$Y_l(0, \phi) = \sqrt{\frac{2l+1}{\mu\pi}}\left(Y_l^0, Y_l\right) = \frac{2l+1}{4\pi}\left(P_l(\cos\theta), Y_l\right) \qquad ...(101)$$

[H] Legendre's function of second kind i.e. Neumann's Integral for $Q_n(x)$ where $x > 1$.

We have from (13)

$$Q_n(x) = \frac{\lfloor n}{1\cdot 3\cdot 5\ldots(2n+1)}\left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4\cdot(2n+3)\,2n+5)}x^{-n-5}+\ldots\right]$$

$$= \frac{2^n \lfloor n}{\lfloor 2n+1}\sum_{r=0}^{\infty}\frac{\lfloor n+2r\cdot x^{-(n+2r+1)}}{2r\lfloor r\,(2n+3)(2n+5)\ldots(2n+2r-1)} \qquad \ldots(102)$$

Neumann's formula for Legendre's function is defined as

$$\int_{-1}^{1}\frac{P_n(x)\,dx}{y-x} = 2\,Q_n(y) \qquad \ldots(103)$$

where $|y| > 1$ and n is a positive integer.

The recurrence formulae for $Q_n(x)$ can be summarised without proof as they have have no utility in physics:

I. $(n+1)\,Q_n(x) - (2n-1)x\,Q_{n-1}(x) + (n-1)\,Q_{n-2}(x) = 0$ ...(104)

II. $Q'_{n+1} - Q'_{n-1} = (2n+1)\,Q_n$ ...(105)

III. $(x^2-1)\,Q'_n(x) = \dfrac{n(n+1)}{2n+1}\left[Q_{n+1}(x) - Q_{n-1}(x)\right]$ ...(106)

IV. $$\left.\begin{aligned} P_n(x)Q'_n(x) - Q_n(x)P'_n(x) &= \frac{-1}{x^2-1} \\ Q_n(x) = P_n(x)\int_x^{\infty}\frac{dx}{(x^2-1)\,P_n^2(x)} \end{aligned}\right\} \qquad \ldots(107)$$

V. $P_nQ_{n-1} - Q_nP_{n-1} = \dfrac{1}{n}$ ...(108)

VI. $P_nQ_{n-2} - Q_nP_{n-2} = \dfrac{2n-1}{n(n-1)x}$ ...(109)

Relations between $P_n(x)$ and $Q_n(x)$ are

$$(1-x^2)\,\{Q_n(x)\,P'_n(x) - P_n(x)\,Q'_n(x)\} = \text{const.} \qquad \ldots(110)$$

where n is a positive integer,

and $$Q_n(x) = P_n(x)\int_x^{\infty}\frac{dx}{(x^2-1)\,\{P_n(x)\}^2}, \qquad \ldots(111)$$

8.4. BESSEL'S EQUATION, FUNCTIONS AND POLYNOMIALS

[A] Bessel's Differential Equation. *(Agra, 1961, 66, 74)*

This equation is of the form

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0. \qquad \ldots(1)$$

There is singularity at $x = 0$, and this is non-essential or removable singularity and hence the given equation may be solved by the method of series integration as allowed by Fusch-theorem.

In order to integrate it in a series of ascending powers of x, let us assume that its series solution is

$$y = \sum_{r=0}^{\infty} a_r x^{k+r}$$

$$\therefore \quad \frac{dy}{dx} = \sum_{r=0}^{\infty} a_r\,(k+r)\,x^{k+r-1},$$

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r(k+r)\,(k+r-1)\;x^{k+r-2}.$$

Substituting these values in (1), we get

$$\sum_{r=0}^{\infty}\left[(k+r)\,(k+r-1)x^{k+r-2} + \frac{1}{x}(k+r)\,x^{k+r-1} + \left(1-\frac{n^2}{x^2}\right)x^{k+1}\right]a_r \equiv 0$$

or $$\sum_{r=0}^{\infty}\left[\{(k+r)\,(k+r-1) + (k+r) - n^2\}\,x^{k+r-2} + x^{k+r}\right]a_r \equiv 0$$

or $$\sum_{r=0}^{\infty}\left[\{(k+r)^2 - n^2\}\,x^{k+r-2} + x^{k+r}\right]a_r \equiv 0. \qquad \text{...(2)}$$

The relaton (2) being an identity, let us equate the coefficients of various powers of x to zero.

Equating to zero the coefficient of lowest power of x, *i.e.*, x^{k-2} by putting $r = 0$ in (2), we have

$$(k^2 - n^2)\,a_0 = 0.$$

Being the coefficient of first term, $a_0 \neq 0$.

$\therefore \quad k^2 - n^2 = 0$, *i.e.*, $k = \pm n$. ...(3)

Now equating to zero the coefficient of x^{k-1} by putting $r = 1$ in (2), we get

$$\{(k+1)^2 - n^2\}a_1 = 0.$$

But from (3), $(k+1)^2 - n^2 \neq 0$; $\therefore\ a_1 = 0$. ...(4)

Equating to zero the coefficient of general term, *i.e.* x^{k+r} in (2), we find

$$\{(k+r+2)^2 - n^2\}\,a_{r+2} + a_r = 0$$

or $$a_{r+2} = -\frac{a_r}{(k+r+2-n)\,(k+r+2+n)}. \qquad \text{...(5)}$$

Case I. When $k = +n$. by putting $r = 0, 1, 2, \ldots$ in (5), we get

$$a_2 = -\frac{a_0}{2\,(2n+2)}$$

and $a_1 = a_3 = a_5 \ldots = 0$.

$$a_4 = -\frac{a_2}{4\,(2n+4)} = \frac{a_0}{2\cdot 4\,(2n+2)\,(2n+4)},$$

$$a_6 = -\frac{a_4}{6\,(2n+6)} = -\frac{a^2}{2\cdot 4\cdot 6\,(2n+2)\,(2n+4)\,(2n+6)}$$

... ...

$$a_{2r} = \frac{(-1)^r a_0}{2\cdot 4\cdot 6\ldots 2r\cdot (2n+2)(2n+4)\ldots(2n+2r)}.$$

Hence the series solution is

$$y = a_0\left[x^n - \frac{x^{n+2}}{2(2n+2)} + \frac{x^{n+1}}{2\cdot 4(2n+2)(2n+4)} - \ldots\right]$$

$$= a_0 x^n\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2\cdot 4(2n+2)(2n+4)} - \ldots \right.$$

$$\left.\ldots \frac{(-1)^r x^{2r}}{2\cdot 4\ldots 2r(2n+2)\ldots(2n+2r)} + \ldots\right]$$

$$= a_0 x^n \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{2^r (r)!\cdot 2^r (n+1)\ldots(n+r)} \qquad \ldots(6)$$

If $a_0 = \frac{1}{2^n \Gamma(n+1)}$, this solution is called as $J_n(x)$.

Thus $J_n(x) = \frac{x^n}{2^n\Gamma(n+1)}\cdot\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}}{2^{2r}(r)!(n+1)(n+2)\ldots(n+r)}$

$$= \sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r}\cdot\frac{1}{r!\,\Gamma(n+r+1)} \qquad \ldots(7)$$

Case II. When $k = -n$.

The series solution is obtained by replacing n by $-n$ in the value of $J_n(x)$, whence, we get

$$J_{-n}(x) = \sum_{r=0}^{\infty}(-1)^r\cdot\left(\frac{x}{2}\right)^{-n+2r}\cdot\frac{1}{r!\,\Gamma(-n+r+1)}. \qquad \ldots(8)$$

The complete primitive of Bessel's equation is

$$A\,J_n(x) + B\,J_{-n}(x),$$ *(Agra, 1961)*

where n is not an integer, A, B being two arbitrary constants.

COROLLARY. Bessel's equation for $n = 0$ is

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0.$$ *(Nagpur, 1965)*

Its series solution by the same substitution $y = \sum_{r=0}^{\infty} a_r x^{k+r}$ (as above) is obtained to be

$$y = a_0\left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} - \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \ldots\right).$$

If $a_0 = 1$, this solution is denoted by $J_0(x)$, *i.e.*

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} - \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \ldots \qquad \ldots(9)$$

where $J_0(x)$ is called *Bessel function of zeroeth order*.

In fact $J_0(x)$ is that solution of Bessel's equation for $n = 0$, which is equal to unity for $x = 0$.

Note. *$J_n(x)$ is called Bessel's function of the first kind of order n.*

[B] Generating Function for $J_n(x)$, i.e. to show that

$$e^{(x/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} t^n J_n(x).$$

We know that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$= \sum_{r=0}^{\infty} \frac{x^r}{r!}$$

$$\therefore \quad e^{xt/2} = \sum_{r=0}^{\infty} \frac{x^r t^r}{2^r r!} \qquad \ldots(10)$$

Similarly, $e^{-x/2t} = \sum_{s=0}^{\infty} \frac{(-1)^s x^s}{2^s t^s s!}$. ...(11)

Multiplying (10) and (11), we get

$$e^{x/2\,(t-1/t)} = \sum_{s=0}^{\infty} \frac{x^r t^r}{2^r r!} \times \sum_{s=0}^{\infty} \frac{(-1)^s x^s}{2^s t^s s!}.$$

In order to find the (t^n)th term, we should replace r by $n + s$ and then coefficient of t^n is

$$\sum_{s=0}^{\infty} \frac{x^{n+s}}{2^{n+s}(n+s)!} \times \frac{-(1)^s x^s}{2^s s!} = \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{n+2s} \frac{1}{(n+s)!\,s!} = J_n(x) \qquad \ldots(12)$$

Again the coefficient of t^{-n} is obtained by putting $s = n + r$ and then coefficient of

$$t^{-n} = \sum_{r=0}^{\infty} \frac{x^r}{2^r r!} \times \frac{(-1)^{n+r} x^{n+r}}{2^{n+r}(n+r)!}.$$

$$= (-1)^n \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{r!\,(n+r)!}.$$

$$= (-1)^n \cdot J_n(x)$$

$$= J_{-n}(x) \qquad \ldots(13)$$

since $J_{-n}(x) = (-1)^n J_n(x)$, where n is a positive integer.

It may be shown as below:

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{-n+2r} \cdot \frac{1}{r!\,\Gamma(-n+r+1)},$$

which tends to zero if $-n + r + 1 = 0$, *i.e.* $r = n - 1$ ($\because \ \Gamma 0 = \infty$).

Hence all the terms upto nth, vanish and therefore the limit $r = 0$ may be changed to $r = n$.

$$\therefore\ J_{-n}(x) = \sum_{r=n}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{-n+2r} \cdot \frac{1}{r!\,\Gamma(-n+r+1)}.$$

Now putting $r = n + s$, where s is a positive integer, we have

$$J_{-n}(x) = \sum_{s=0}^{\infty} (-1)^{n+s} \left(\frac{x}{2}\right)^{n+2s} \cdot \frac{1}{(n+s)!\,\Gamma(-n+s+n+1)}$$

$$= \sum_{s=0}^{\infty} (-1)^{n+s} \left(\frac{x}{2}\right)^{n+2s} \frac{1}{(n+s)!\,\Gamma(s+1)}$$

or $$J_{-n}(x) = (-1)^n \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{n+2s} \frac{1}{s!\,\Gamma(n+s+1)}$$

$$= (-1)^n J_n(x). \qquad \ldots(14)$$

(Agra, 1961)

Hence from (12) and (13), we have

$$e^{x/2(t-1/t)} = \sum_{n=-\infty}^{\infty} t^n J_n(x). \qquad \ldots(15)$$

COROLLARY. Putting $t = e^{i\phi}$ and $\frac{1}{t} = e^{-i\phi}$, we get

$$e^{ix(e^{i\phi}-e^{-i\phi})/2i} = \sum_{n=-\infty}^{\infty} e^{ni\phi} J_n(x)$$

or $e^{ix\sin\phi} = J_0(x) + \{J_1(x)\, e^{i\phi} + J_{-1}(x)\, e^{-i\phi}\} + \{J_2(x)\, e^{2i\phi} + J_{-2}(x)\, e^{-2i\phi}\} + \ldots$

or $\cos(x \sin\phi) + i \sin(x \sin\phi) = J_0(x) + J_1(x)\,\{e^{i\phi} - e^{-i\phi}\} + J_2(x)\,\{e^{2i\phi} + e^{-2i\phi}\} + \ldots$

[since $J_n(x) = J_{-n}(x)$ when n is even]

$= J_0(x) + 2i \sin\phi\, J_1(x) + 2\cos 2\phi\, J_2(x) + \ldots$

Equating real and imaginary parts, we get

$$\cos(x \sin\phi) = J_0(x) + 2\cos 2\phi\, J_2(x) + 2\cos 4\phi\, J_4(x) + \ldots \qquad \ldots(16)$$

and $$\sin(x \sin\phi) = 2J_1(x) \sin\phi + 2J_3(x) \sin 3\phi + 2J_5(x) \sin 5\phi + \ldots \qquad \ldots(17)$$

Replacing ϕ by $\frac{\pi}{2} - \phi$, we have, from (16) and (17)

$$\cos(x \cos\phi) = J_0(x) - 2\cos 2\phi\, J_2(x) + 2\cos 4\phi\, J_4(x) \ldots \qquad \ldots(18)$$

and $$\sin(x \cos\phi) = 2\cos\phi\, J_1(x) - 2\cos 3\phi\, J_3(x) + 2\cos 5\phi\, J_5(x) \ldots \qquad \ldots(19)$$

Note. Results (18) and (19) are known as Jacobi's series.

[C] Integrals for $J_0(x)$ and $J_n(x)$.

I. $J_0(x) = \frac{1}{\pi} \int_0^{\pi} \cos(x \sin\phi)\, d\phi.$

We have by (16),

$\cos(x \sin\phi) = J_0(x) + 2\cos 2\phi\, J_2(x) + 2\cos 4\phi\, J_4(x) + \ldots$

If we integrate both the sides of this relation with respect to ϕ from the limits 0 to ϕ, then we see that all the integrals except first of the R.H.S. vanish, thereby giving

$$\int_0^{\pi} \cos (x \sin \phi)\, d\phi = J_0(x) \int_0^{\pi} d\phi = \pi J_0(x). \quad \text{...(20)}$$

II. $J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos (n\phi - x \sin \phi)\, d\phi.$

We have already proved that

$$\cos (x \sin \phi) = J_0(x) + 2 \cos 2\phi\, J_2(x) + 2 \cos 4\phi\, J_4(x) + \ldots \quad \text{...(21)}$$

and $\sin (x \sin \phi) = 2J_1(x) \sin \phi + 2 \sin 3\phi\, J_3(x) + 2 \sin 5\phi\, J_5(x) + \ldots$ ...(22)

If we multiply (21) by $\cos n\phi$, (22) by $\sin n\phi$ and integrate between the limits 0 to π, we have

$$\int_0^{\pi} \cos (x \sin \phi) \cos n\phi\, d\phi = 0 \text{ or } \pi J_n(x) \quad \text{...(23)}$$

according as n is odd or even

and $$\int_0^{\pi} \sin (x \sin \phi) \sin n\phi\, d\phi = \pi J_n(x) \text{ or } 0 \quad \text{...(24)}$$

according as n is odd or even.

Adding (23) and (24), we find

$$\int_0^{\pi} [\cos (x \sin \phi) \cos n\phi + \sin (x \sin \phi) \sin n\phi]\, d\phi = \pi J_n(x)$$

whether n is odd or even

or $$\int_0^{\pi} \cos (n\phi - x \sin \phi) d\phi = \pi J_n(x),$$

i.e. $$J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos (n\phi - x \sin \phi) d\phi \cdot \quad \text{...(25)}$$

III. $$J_n(x) = \frac{1}{\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)} \left(\frac{x}{2}\right)^n \int_0^{\pi} \cos (x \sin \phi) \cos^{2n}\phi\, d\phi \quad \text{...(26)}$$

(Rohilkhand, 1987)

If we expand $\cos (x \sin \phi)$ in the powers of $x \sin \phi$, the general terms is

$$(-1)^r \frac{x^{2r}}{(2r)!} \sin^{2r}\phi.$$

General term of R.H.S. of (26)

$$= \frac{2}{\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)} \left(\frac{x}{2}\right)^n (-1)^r \int_0^{\pi} \frac{x^{2r}}{(2r)!} \sin^{2r}\phi \cos^{2n}\phi\, d\phi, \quad \text{....(27)}$$

where $\int_0^{\pi} \sin^{2r}\phi \cos^{2n} d\phi$

$$= 2\int_0^{\pi/2} \sin^{2r}\phi \cos^{2n}\phi\, d\phi$$
$$= \int_0^1 t^r (1-t)^n \frac{dt}{t^{1/2} (1-t)^{1/2}}$$

Put $\sin^2 \phi = t$,
$\therefore\ 2 \sin \phi \cos \phi\, d\phi = t$
or $d\phi = \frac{dt}{2\sqrt{t(1-t)}}$

$$= \int_0^1 t^{(2r-1)/2} (1-t)^{(2n-1)/2} \, dt$$

$$= \beta\left(\frac{2r+1}{2}, \frac{2n+1}{2}\right)$$

$$= \frac{\Gamma\left(\frac{2r+1}{2}\right)\Gamma\left(\frac{2n+1}{2}\right)}{\Gamma\left\{\left(\frac{2r+1}{2}\right)+\left(\frac{2n+1}{2}\right)\right\}} \qquad \because \beta(m, n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}$$

$$= \frac{\frac{2r-1}{2}\cdot\frac{2r-3}{2}\cdots\frac{1}{2}\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+r+1)}$$

Substituting this value in R.H.S. of (27) we have

$$\frac{2}{\sqrt{\pi}\,\Gamma\left(n\frac{1}{2}\right)}\left(\frac{x}{2}\right)^n (-1)^r \int_0^\pi \frac{x^{2r}}{(2r)!} \sin^{2r}\phi \cos^{2n}\phi \, d\phi$$

$$= \frac{2}{\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)}\left(\frac{x}{2}\right)^n (-1)^r \frac{x^2r}{(2r)!} \times \frac{\frac{2r-1}{2}\cdot\frac{2r-3}{2}\cdots\frac{1}{2}\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+r+1)}$$

$$= 2\left(\frac{x}{2}\right)^n (-1)^r \cdot \frac{x^{2r}}{\Gamma(n+r+1)} \cdot \frac{(2r-1)(2r-3)\ldots 1}{2^r 2r(2r-1)(2r-2)\ldots 1}$$

$$= \left(\frac{x}{2}\right)^{n+2r} (-1)^r \cdot \frac{1}{r!\,\Gamma(n+r+1)}$$

$$= (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{r!\,(n+r)!} = \text{general term in } J_n(x).$$

Hence $$J_n(x) = \frac{1}{\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)}\left(\frac{x}{2}\right)^n \int_0^\pi \cos(x \sin\phi) \cos^{2n}\phi \, d\phi. \qquad \ldots(28)$$

[D] Recurrence Formulae for $J_n(x)$.

I. We know that

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^n \frac{1}{r!\,(n+r)!}\left(\frac{x}{2}\right)^{n+2r},$$

where n is a positive integer.

Differentiating it w.r.t x, we get

$$J_n'(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,(n+r)!} (n+2r)\left(\frac{x}{2}\right)^{n+2r-1} \cdot \frac{1}{2}.$$

Multiplying both sides by x, we have

$$xJ_n'(x) = \sum_{r=0}^{\infty} (-1)^r \frac{n+2r}{r!\,(n+r)!} \cdot \left(\frac{x}{2}\right)^{n+2r}$$

$$= n\sum_{r=0}^{\infty}(-1)^r \frac{n}{r!\,(n+r)!}\cdot\left(\frac{x}{2}\right)^{n+2r} + x\sum_{r=0}^{\infty}(-1)^r\cdot\frac{2r}{r!\,(n+r)!}\cdot\left(\frac{x}{2}\right)^{n+2r-1}$$

or $$x\cdot J_n'(x) = nJ_n(x) + x\sum_{r=1}^{\infty}(-1)^r\frac{n}{(r-1)!\,(n+r)!}\cdot\left(\frac{x}{2}\right)^{n-1+2r}$$

[since on R.H.S the second term vanishes for $r = 1$ and hence limit of $r = 0$ may be repalced by $r = 1$]

Putting $r - 1 = s$, we have

$$x\cdot J_n'(x) = nJ_n(x) + x\sum_{s=0}^{\infty}(-1)^{s+1}\frac{1}{s!\,(n+1+s)!}\cdot\left(\frac{x}{2}\right)^{n+2s-1}$$

$$= nJ_n(x) - xJ_{n+1}(x).$$ *(Rohilkhand, 1980, 89)* ...(29)

II. Again $xJ_n'(x) = \sum_{r=0}^{\infty}(-1)^r\frac{(n+2r)}{r!\,(n+r)!}\left(\frac{x}{2}\right)^{n+2r-1}\frac{x}{2}$ may be written as

$$xJ_n'(x) = \sum_{r=0}^{\infty}(-1)^r\frac{-n+2\,(n+r)}{r!\,(n+r)!}\left(\frac{x}{2}\right)^{n+2r}$$

$$= -n\sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\,(n+r)!}\cdot\left(\frac{x}{2}\right)^{n+2r} + x\sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\,(n+r-1)!}\cdot\left(\frac{x}{2}\right)^{n-1+2r}$$

$$= -nJ_n(x) + xJ_{n-1}(x).$$ *(Agra, 1962, 64)* ...(30)

Sum and difference of (29) and (30) give

III. $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$ ...(31)

IV. $2nJ_n(x) = x\,\{J_{n+1}(x) + J_{n-1}(x)\}$. ...(32)

(Rohilkhand, 1978, 83, 85, 90; Agra, 1964)

V. $$\frac{d}{dx}\left\{x^nJ_n(x)\right\} = x^nJ_{n-1}(x).$$ ...(33)

Here $$\frac{d}{dx}\{x^nJ_n(x)\} = nx^{n-1}J_n(x) + x^nJ_n'(x)$$

$$= x^{n-1}\{nJ_n(x) + xJ_n'(x)\}$$

$$= x^{n-1}\{nJ_n(x) - nJ_n(x) + xJ_{n-1}(x)\}$$ by (30)

$$= x^nJ_{n-1}(x).$$

VI. Similarly it is easy to show that

$$\frac{d}{dx}\{x^{-n}J_n(x)\} = -x^{-n}J_{n+1}(x).$$ ...(34)

Problem 19. *Show that* $y = \frac{1}{\pi}\int_0^{\pi}\cos(x\cos\phi)\,d\phi$ *satisfies the differential equation* $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0$ *and that y is no other than* $J_0(x)$.

Given $$y = \frac{1}{\pi}\int_0^{\pi}\cos(x\cos\phi)\,d\phi$$ ...(1)

If we differentiate it w.r.t. x under the sign of integration, we find

$$\frac{dy}{dx}=\frac{1}{\pi}\int_0^{\pi}-\cos\phi\sin(x\cos\phi)\,d\phi \qquad ...(2)$$

and

$$\frac{d^2y}{dx^2}=\frac{1}{\pi}\int_0^{\pi}\cos^2\phi\cos(x\cos\phi)\,d\phi. \qquad ...(3)$$

Now from (2), we have

$$\frac{dy}{dx}=-\frac{1}{\pi}\Big[\{\sin(x\cos\phi\ \sin\phi\}_0^{\pi}$$

$$-\int_0^{\pi}-x\sin\phi\cos(x\cos\phi)\cdot\sin\phi\,d\phi\Big] \text{ (integrating by parts)}$$

$$=-\frac{x}{\pi}\int_0^{\pi}\sin^2\phi\cos(x\cos\phi)\,d\phi$$

$$=-\frac{x}{\pi}\int_0^{\pi}(1-\cos^2\phi)\cos(x\cos\phi)\,d\phi$$

$$=-\frac{x}{\pi}\int_0^{\pi}\cos(x\cos\phi)\,d\phi+\frac{x}{\pi}\int_0^{\pi}\cos^2\phi\cos(x\cos\phi)\,d\phi$$

$$=-xy-x\frac{d^2y}{dx^2}\text{ from (1) and (3)}$$

or

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+y=0.$$

Hence $y=\frac{1}{\pi}\int_0^{\pi}\cos(x\cos\phi)\,d\phi$ satisfies $\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+y=0$,

which is the Bessel's equation for $n=0$.

Since $y=1$ when $x=0$, therefore y is no other than $J_0(x)$, as $J_0(x)$ being the solution of Bessel's equation is unity for $x=0$.

Problem 20. *Prove that (a)* $J_{1/2}(x)=\sqrt{\left(\frac{2}{\pi x}\right)}\sin x.$

[Agra, 1962, 66; Kanpur, 1968, 89]

and (b) $J_{-1/2}(x)=\sqrt{\left(\frac{2}{\pi x}\right)}\cos x$ *(Rohilkhand, 80, 89; Agra, 1981)*

(*a*) We know that

$$J_n(x)=\frac{x^n}{2^n\Gamma(n+1)}\left[1-\frac{x^2}{2\cdot2\,(n+1)}+\frac{x^4}{2\cdot4\cdot2\,(n+1)\,(n+2)}-...\right].$$

$$\therefore\ J_{1/2}(x)=\left(\frac{x}{2}\right)^{1/2}\frac{1}{\Gamma(\frac{3}{2})}\cdot\left[1-\frac{x^2}{2\cdot2\cdot\frac{3}{2}}+\frac{x^4}{2\cdot4\cdot2\ \frac{3}{2}\cdot\frac{5}{2}}-...\right]\text{ for } n=\frac{1}{2}$$

$$=\sqrt{\left(\frac{x}{2}\right)}\cdot\frac{1}{\frac{1}{2}\sqrt{\pi}}\left[1-\frac{x^2}{2\cdot3}+\frac{x^4}{2\cdot3\cdot4\cdot5}-...\right]$$

$$= \sqrt{\left(\frac{2x}{\pi}\right)} \cdot \frac{1}{x}\left[x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} - \dots\right]$$

$$= \sqrt{\left(\frac{2}{\pi x}\right)} \sin x.$$

(*b*) Proceed just as in (*a*) for $n = -1/2$.

Problem 21. *Show that (i)* $J_0'(x) = -J_1(x)$ *and (ii)* $2J_0''(x) = J_2(x) - J_0(x)$.

From recurrence formula (29) we have

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x).$$

Putting $n = 0$, we get $J_0'(x) = -J_1(x)$,
which proves the first result.

Now differentiating it and multiplying throughout by 2, we get

$$2J_0''(x) = -2J_1'(x)$$

or $$2J_0''(x) = -[J_0(x) - J_2(x)]$$ by recurrence formula III.

$$= J_2(x) - J_0(x).$$

Problem 22. *Prove that* $J_{n+3} + J_{n+5} = \frac{2}{x}(n+4)\,J_{n+4}$.

From recurrence formula IV, we have

$$2nJ_n = x(J_{n-1} + J_{n+1}).$$

Replacing n by $n + 4$, we get

$$\frac{2}{x}(n+4)\,J_{n+4} = J_{n+3} + J_{n+5}.$$

Problem 23. *Prove that*

$$\cos x = J_0(x) - 2J_2(x) + 2J_4(x)\dots$$

$$\sin x = 2J_1(x) - 2J_3(x) + 2J_5(x)\dots$$

We know that

$$\cos(x\cos\phi) = J_0(x) - 2\cos 2\phi\, J_2(x) + 2\cos 4\phi\, J_4(x)\dots$$

and $$\sin(x\sin\phi) = 2\cos\phi\, J_1(x) - 2\cos 3\phi\, J_3(x) + 2\cos 5\phi\, J_5(x)\dots$$

Putting $\phi = 0$, we get

$$\cos x = J_0(x) - 2J_2(x) + 2J_4(x)\dots$$

$$\sin x = 2J_1(x) - 2J_3(x) + 2J_5(x)\dots$$

Problem 24. *Establish the relation*

$$J_n(x)\,J_{-n}'(x) - J_n'(x)\,J_{-n}(x) = -\frac{2\sin n\pi}{\pi x}.$$

We know that $J_n(x)$ and $J_{-n}(x)$ are the two solutions of Bessel's equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0. \qquad \dots(1)$$

Hence, if $y = J_n(x)$, $\frac{dy}{dx} = J_n'(x)$ and $\frac{d^2y}{dx^2} = J_n''(x)$, we have from (1)

$$J_n''(x) + \frac{1}{x}J_n'(x) + \left(1 - \frac{n^2}{x^2}\right)J_n(x) = 0. \qquad \dots(2)$$

Similarly putting $y = J_{-n}(x)$, we get from (1)

$$J_{-n}''(x) + \frac{1}{x} J_{-n}'(x) + \left(1 - \frac{n^2}{x^2}\right) J_{-n}(x) = 0. \qquad \ldots(3)$$

Multiplying (2) by $J_{-n}(x)$, (3) by $J_n(x)$, and then subtracting (3) from (2), we have

$$J_n''(x) J_{-n}(x) - J_{-n}''(x) J_n(x) + \frac{1}{x} \{J_n'(x) J_{-n}(x) - J_{-n}'(x) J_n(x)\} = 0.$$

Put $z = J_n'(x) J_{-n}(x) - J_{-n}'(x) J_n(x)$.

$$\therefore \ z' = J_n''(x) J_{-n}(x) + J_n'(x) J_{-n}'(x) - J_{-n}'(x) J_n'(x) - J_{-n}''(x) J_n(x)$$

$$= J_n''(x) J_{-n}(x) - J''_{-n}(x) J_n(x).$$

Thus $\quad z' + \frac{z}{x} = 0$ or $\frac{z'}{z} = -\frac{1}{x}.$

Integrating, $\log z = -\log x + \log C$, where C is some arbitrary constant

$$= \log \frac{C}{x}.$$

$$\therefore \quad z = \frac{C}{x}.$$

or $J_n'(x) J_{-n}(x) - J_{-n}'(x) J_n(x) = \frac{C}{x}.$

Equating the coefficients of $\frac{1}{x}$ on either side, we get

$$\frac{1}{2^n \Gamma(n+1)} \cdot \frac{1}{2^{-n}\Gamma(-n+1)} \{n - (-n)\} = C$$

or $$C = \frac{2n}{\Gamma(n+1)\,\Gamma(-n+1)} = \frac{1}{\Gamma n\,\Gamma(1-n)}$$

$$= \frac{2}{\pi / \sin n\pi} = \frac{2 \sin n\pi}{\pi}; \quad \because \Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi} \quad \text{(Gamma functions).}$$

Hence $J_n'(x)\ J_{-n}(x) - J_{-n}'(x)\ J_n(x) = \frac{2 \sin n\pi}{\pi}.$

[E] Orthogonal Properties of Bessel's Polynomials.

To prove that $\int_0^a J_n(\mu r) J_n(\mu' r)\, r\, dr = 0$ where μ and μ' are different roots of $J_n(\mu a) = 0$.

Since $J_n(x)$ is a solution of Bessel's equation,

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right) y = 0 \qquad \ldots(35)$$

therefore putting $x = \mu r$ and calling $y = u$ in (35) we get

$$\frac{1}{\mu^2} \frac{d^2u}{dr^2} + \frac{1}{\mu r} \cdot \frac{1}{\mu} \frac{du}{dr} + \left(1 - \frac{n}{\mu^2 r^2}\right) u = 0,$$

$$\left[\because \frac{dy}{dx} = \frac{du}{dx} = \frac{du}{dr} \cdot \frac{dr}{dx} = \frac{1}{\mu} \cdot \frac{du}{dr} \text{ and } \frac{d^2y}{dx^2} = \frac{1}{\mu} \frac{d}{dr}\left(\frac{1}{\mu} \frac{du}{dr}\right) = \frac{1}{\mu^2} \cdot \frac{d^2u}{dr^2}\right]$$

Multiplying throughout by $\mu^2 r^2$.

$$r^2\frac{d^2u}{dr^2}+r\frac{du}{dr}+\left(\mu^2r^2-n^2\right)u=0. \qquad ...(36)$$

Similarly putting $x=\mu' r$ and calling $y=v$ in (36), we have

$$r^2\frac{d^2v}{dr^2}+r\frac{dv}{dr}+\left(\mu'^2r^2-n^2\right)v=0. \qquad ...(37)$$

If we multiply (36) by $\frac{v}{r}$, (37) by $\frac{u}{r}$ and subtract

$$r(vu''-ruv'')+(vu'-uv')+(\mu^2-\mu'^2)\,ruv=0,$$

where $u'=\frac{du}{dr}$ and $v'=\frac{dv}{dr}$ etc.

or $$\frac{d}{dr}\{r(vu'-uv')\}+(\mu^2-\mu'^2))\}\,ruv=0, \qquad ...(38)$$

where $u=J_n(\mu r)$ and $v=J_n(\mu' r)$.

Integrating (38) w.r.t. r between the limits 0 and a, we get

$$\left[r\left\{J_n(\mu r)J_n'(\mu' r)\mu'-J_n(\mu' r)J_n'(\mu r)\mu\right\}\right]_0^a-\int_0^a\left(\mu^2-\mu'^2\right)J_n(\mu r)J_n(\mu' r)\,r\,dr=0.$$

The first term vanishes for both the limits since

$$J_n(\mu a)=0,\ J_n(\mu' a)=0.$$

Hence $$\int_0^a(\mu^2-\mu'^2)\,J_n(\mu r)\,J_n(\mu' r)\,r\,dr=0,$$

i.e., $$\int_0^a J_n(\mu r)\,J_n(\mu' r)\,r\,dr=0 \qquad \text{as } \mu^2-\mu'^2\neq 0. \qquad ...(39)$$

[F] Bessel's Functions of the Second Kind (Neumann functions).

Bessel's equation of zero order is

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+y=0. \qquad ...(40)$$

One of its solution is $J_0(x)$ so that $J_0''+\frac{1}{x}J_0'+J_0=0$

Let the other solution of (40) be

$$y=u\,J_0(x)+v \qquad u,\ v \text{ being functions of } x.$$

Then $y'=uJ_0'+u'J_0+v'$

$$y''=uJ_0''+2u'J_0'+J_0u''+v''$$

Substituting these values in (40) and using $J_0''+\frac{1}{x}J_0'+J_0=0$,

we get $$\left(v''+\frac{1}{x}v'+v\right)+J_0\left(u''+\frac{1}{x}u'\right)+2u'J_0'=0$$

Choosing u such that $u''+\frac{1}{x}u'=0$ *i.e.* $\frac{u''}{u'}=-\frac{1}{x}$.

Integration gives $\log u' = -\log x$ *i.e.* $u' = \frac{1}{x}$

giving $u = \log x$

then the above transformed equation becomes

$$v'' + \frac{1}{x} v' + v + \frac{2}{x} J_0' = 0$$

or $$v'' + \frac{1}{x} v' + v = \frac{2}{x} J_1 \qquad \because J_0' = -J_1 \text{ by problem 21.}$$

But the recurrence formula $2n J_n = x(J_{n-1} + J_{n+1})$ gives on replacing n by $n + 1$

$$2(n+1) J_{n+1} = x(J_n + J_{n+2})$$

i.e. $$\frac{x}{2} J_n = (n+1) J_{n+1} - \frac{1}{2} x J_{n+2}.$$

Replacing n by $n + 2$, this gives

$$\frac{x}{2} J_{n+2} = (n+3) - \frac{x}{2} J_{n+4}$$

i.e. $$\frac{x}{2} J_n = (n+1) J_{n+1} - (n+3) J_{n+3} + \frac{x}{2} J_{n+4}$$

Repeating the process we find

$$\frac{x}{2} J_n = (n+1) J_{n+1} - (n+3) J_{n+3} + (n+5) J_{n+5} \ldots$$

that if $n = 1$, $\frac{x}{2} J_1 = 2J_2 - 4J_4 + 6J_6 \ldots$

Thus $$v'' + \frac{1}{x} v' + v = \frac{4}{x^2} [2J_2 - 4J_4 + 6J_6 \ldots$$

$$= \frac{4}{x^2} \sum (-1)^{n/2-1} nJ_{n-1} \quad n \text{ being even integer}$$

$$= \sum \frac{n^2}{x^2} \cdot \frac{4(-1)^{n/2-1}}{n} J_n(x)$$

But $\frac{4(-1)^{n/2-1}}{n} J_n(x)$ is the particular solution of

$$y'' + \frac{1}{x} y' + y = \frac{4(-1)^{n/2-1}}{n} J_n(x) \cdot \frac{n^2}{x^2}.$$

Thus $v = \sum \frac{4(-1)^{n/2-1}}{n} J_n(x)$, (by comparison).

Hence the solution of (40) is

$$y = J_0(x) \log x + \sum \frac{4}{n} (-1)^{n/2-1} J_n = Y_0 \text{ (say)}$$

But Neumann gave the solution of (40) as

$$N_0(x) = J_0(x) \log x - \sum_{p=1}^{\infty} (-1)^p \frac{\left(\frac{x}{2}\right)^{2p}}{(\lfloor p)^2} \left[1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{p}\right] \qquad \ldots(41)$$

where $N_0(x)$ is said to be *Neumann functions of the second kind* of order zero.

Weber gave the solution of (40) as

$$Y_0(x)=\frac{2}{\pi}\left[\log\frac{x}{2}+\gamma\right]J_0(x)-\frac{2}{\pi}\sum_{p=1}^{\infty}(-1)^p\frac{\left(\frac{1}{2}x\right)^{2p}}{(\lfloor p)^2}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{p}\right), \qquad \ldots(42)$$

where γ is Euler's constant defined as

$$\gamma=\lim_{m\to\infty}\left[1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots+1/m-\log m\right]. \qquad \ldots(43)$$

From (41) and (42) it is easy to show that

$$Y_0(x)=\frac{2}{\pi}\{N_0(x)-(\log 2-\gamma)J_0(x)\}. \qquad \ldots(44)$$

The complete primitive of (40) is

$$y=AJ_0(x)+BY_0(x) \qquad \ldots(45)$$

In case of Bessel's equation of nth order, *i.e.*

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left[1-\frac{n^2}{x^2}\right]y=0, \qquad \ldots(46)$$

the complete primitive is

$$y=A'J_n(x)+B'Y_n(x), \qquad \text{(Rohilkhand, 1984)} \qquad \ldots(47)$$

where $Y_n(x)=\dfrac{2}{\pi}\left[\gamma+\log\dfrac{x}{2}\right]J_n(x)-\dfrac{1}{\pi}\displaystyle\sum_{p=0}^{n-1}\frac{\Gamma(n-p)}{\lfloor p}\left[\frac{2}{x}\right]^{n-2p}$

$$-\frac{1}{\pi}\sum_{p=0}^{\infty}(-1)^p\frac{\left(\frac{1}{2}x\right)^{n+2p}}{\lfloor p\,\Gamma(n+p+1)}\left\{1+\frac{1}{2}+\frac{1}{3}+\ldots\frac{1}{p}+1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n+p}\right\}, \qquad \ldots(48)$$

when n is integral.

But if n is not an integer, then

$$Y_n=\frac{J_n(x)\cos n\pi-J_{-n}(x)}{\sin n\pi}. \qquad \ldots(49)$$

[G] Bessel's Functions of the third kind (Hankel functions).

These are defined as $H_n^{(1)}(x)=J_n(x)+iY_n(x)$. $\ldots(50)$

$$H_n^{(2)}(x)=J_n(x)-iY_n(x). \qquad \ldots(51)$$

Thus addition and subtraction of (50) and (51) give

$$J_n(x)=\tfrac{1}{2}\left\{H_n^1(x)+H_n^{(2)}(x)\right\}$$

and

$$Y_n(x)=-\tfrac{1}{2}i\left\{H_n^{(1)}(x)-H_n^{(2)}(x)\right\}$$

[H] The Modified Bessel's Functions.

Bessel's differential equation of nth order is

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{n^2}{x^2}\right)y=0. \qquad \ldots(52)$$

Put $x=iz$, so that $\dfrac{dx}{dz}=i$, *i.e.*, $\dfrac{dz}{dx}=\dfrac{1}{i}$.

$$\therefore \qquad \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{1}{i}\frac{dy}{dz}$$

and $$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{i}\frac{dy}{dx}\right)=\frac{d^2y}{dz^2}\cdot\frac{dz}{dx}=\frac{1}{i^2}\frac{d^2y}{dz^2}=-\frac{d^2y}{dz^2}.$$

With these substitutions, (52) becomes

$$\frac{d^2y}{dz^2}+\frac{1}{iz}\cdot\frac{1}{i}\frac{dy}{dz}+\left\{1-\frac{n^2}{(iz)^2}\right\}y=0,$$

i.e., $$\frac{d^2y}{dz^2}+\frac{1}{z}\frac{dy}{dz}-\left(1+\frac{n^2}{z^2}\right)y=0. \qquad ...(53)$$

Since the complete primitive of (52) is $y = AJ_n(x) + BY_n(x)$, therefore the solution of (53) is obtained by putting $x = iz$ and that is

$$y = AJ_n(iz) + BY_n(iz). \qquad ...(54)$$

Now we know that

$$J_n(x)=\sum_{r=0}^{\infty}(-1)^r\frac{(x/2)^{n+2r}}{\lfloor r\,\lfloor(n+r)}$$

Putting $x = iz$, this becomes

$$J_n(iz)=\sum_{r=0}^{\infty}(-1)^r\frac{(iz)^{n+2r}}{2^{n+2r}\lfloor r\,\lfloor(n+r)}$$

$$=i^n\sum_{r=0}^{\infty}\frac{z^{n+2r}}{2^{n+2r}\lfloor r\,\lfloor(n+r)}$$

$\therefore (i^{2r}) = (i^2)^r = (-1)^r$ and therefore $(-1)^r\cdot i^{2r} = (-1)^{2r} = 1$

or $$i^{-n}J_n(iz)=\sum_{r=0}^{\infty}\frac{z^{n+2r}}{2^{n+2r}\lfloor r\,\lfloor(n+r)} \qquad ...(55)$$

Here $i^{-n}J_n(iz)$ is denoted by $I_n(z)$, which is known as the *modified function of the first kind* of order n. Thus replacing z by x, we get

$$I_n(x)=i^{-n}J_n(ix)=\sum_{r=0}^{\infty}\frac{x^{n+2r}}{2^{n+2r}\lfloor r\,\Gamma\lfloor(n+r+1)} \qquad ...(56)$$

$\because \lfloor(n+r) = \Gamma(n+r+1).$

If we put $n = 0$, its particular case is found as

$$I_0(x)=\sum_{r=0}^{\infty}\frac{(x/2)^{2r}}{(\lfloor r)^2}$$

$$=1+\left(\frac{x}{2}\right)^2+\frac{(x/2)^4}{(\lfloor 2)^2}+\frac{(x/2)^6}{(\lfloor 3)^2}+... \qquad ...(57)$$

when n is integral, *the modified function of the second kind*, *i.e.* $K_n(x)$ is related with the modified function as

$$K_n(x)=\tfrac{1}{2}\sum_{r=0}^{n-1}(-1)^r\frac{\lfloor(n-r-1)}{\lfloor r}\left(\frac{2}{x}\right)^{n-2r}$$

$$+(-1)^{n+1}\frac{1}{2}\sum_{r=0}^{\infty}\frac{(x/2)^{n+2r}}{\lfloor r \lfloor (n+r)}\left[\log\frac{x}{2}-\frac{1}{2}\{\psi(r+1)+\psi(n+r+1)\}\right] \quad ...(58)$$

where $\psi(r+1)=\left(1+\frac{1}{2}+\frac{1}{3}+...+1/r\right)-\gamma$ ...(59)

and $\psi(n+r+1)=\left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n+r}\right)-\gamma$. ...(60)

But if n is not an integer, then

$$K_n(x)=\frac{\pi}{2}\frac{[I_{-n}(x)-I_n(x)]}{\sin n\pi} \quad ...(61)$$

In particular when n is an integer, we have

$$K_0(x)=-\left\{\gamma+\log\frac{x}{2}\right\}I_0(x)+\sum_{r=1}^{\infty}\frac{\left(\frac{1}{2}x\right)^{2r}}{(\lfloor r)^2}\left\{1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{r}\right\} \quad ...(62)$$

Hence the complete primitive of the equation of the type

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}-\left(1-\frac{n^2}{x^2}\right)y=0$$

is $y=AI_n(x)+BK_n(x)$. ...(63)

[1] The Ber, Bei, Ker, Kei Functions.

Suppose we have an equation of the form

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}-ip^2xy=0. \quad ...(64)$$

This may be written as

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}-\left(p\sqrt{i}\right)^2y=0.$$

Obviously this is a modified Bessel equation of the order zero and therefore its solution is

$$y=AI_0(xp\sqrt{i})+BK_0(xp\sqrt{i}). \quad ...(65)$$

In order to express this solution into real and imaginary parts, we introduce four new functions, namely, Ber (*i.e.* Bessel real); Bei (*ie.* Bessel imaginary); Ker and Kei are their analogues.

These are defined as

$$I_0(x\sqrt{i})=\text{Ber}(x)+i\,\text{Bei}(x), \quad ...(66)$$

$$K_0(x\sqrt{i})=\text{Ker}(x)+i\,\text{Kei}(x), \quad ...(67)$$

where $\text{Ber}(x)=\frac{\left(\frac{1}{2}x\right)^4}{(\lfloor 2)^2}+\frac{\left(\frac{1}{2}x\right)^8}{(\lfloor 4)^2}$ ...(68)

and $\text{Bei}(x)=\left(\frac{x}{2}\right)^2-\frac{\left(\frac{1}{2}x\right)^6}{(\lfloor 3)^2}+\frac{\left(\frac{1}{2}x\right)^{10}}{(\lfloor 5)^2}-...$ ...(69)

Problem 25. *Derive Bessel's equation from that of Legendre.* *(Rohilkhand, 1993)*

We know that Legendre's equation is

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0 \quad ...(1)$$

Differentiating it N times, we have by Leibnitz's theorem,

$$(1-x^2)\frac{d^{N+2}y}{dx^{N+2}}+N(-2x)\frac{d^{N+1}y}{dx^{N+1}}+\frac{N(N-1)}{1\cdot 2}(-2)\cdot\frac{d^Ny}{dx^N}-2x\frac{d^{N+1}y}{dx^{N+1}}$$
$$+N\cdot(-2)\cdot\frac{d^Ny}{dx^N}+n(n+1)\cdot\frac{d^Ny}{dx^N}=0,$$

i.e.,
$$(1-x^2)\frac{d^{N+2}y}{dx^{N+2}}-2(N+1)x\frac{d^{N+1}y}{dx^{N+1}}+\{n(n+1)-N(N+1)\}\frac{d^Ny}{dx^N}=0. \quad ...(2)$$

If we put $z=\dfrac{d^Ny}{dx^N}$, the equation (2) becomes

$$(1-x^2)\frac{d^2z}{dx^2}-2(N+1)x\frac{dz}{dx}+\{n(n+1)-N(N+1)\}z=0. \quad ...(3)$$

Now put

$$Y=(1-x^2)^{\frac{1}{2}N}z \quad i.e.\ z=Y(1-x^2)^{-\frac{1}{2}N}$$

so that
$$\frac{dz}{dx}=(1-x^2)^{-\frac{1}{2}N}\frac{dY}{dx}+Y\left(-\frac{N}{2}\right)(1-x^2)^{-\frac{1}{2}N-1}(-2x)$$
$$=(1-x^2)^{-\frac{1}{2}N}\frac{dY}{dx}+Nx(1-x^2)^{-\frac{1}{2}N-1}Y$$

and
$$\frac{d^2z}{dx^2}=(1-x^2)^{-\frac{1}{2}N}\frac{d^2Y}{dx^2}+2Nx(1-x^2)^{-N/2-1}\frac{dY}{dx}$$
$$+N\left\{(1-x^2)^{-\frac{1}{2}N-1}+x(1-x^2)^{-\frac{1}{2}N-2}\left(-\frac{N}{2}-1\right)(-2x)\right\}Y$$

or

$$\frac{d^2z}{dx^2}=(1-x^2)^{-\frac{1}{2}N}\frac{d^2Y}{dx^2}+2Nx(1-x^2)^{-\frac{1}{2}N-1}\frac{dY}{dx}+N\{1+(N+1)x^2\}(1-x^2)^{-\frac{1}{2}N-2}Y.$$

Substituting these values in (3), the transformed equation is

$$(1-x^2)\frac{d^2Y}{dx^2}+\{(2Nx-2(N+1)x\}\frac{dY}{dx}+\left[n(n+1)-N(N+1)-\frac{2Nx^2(N+1)}{1-x^2}\right.$$
$$\left.+\frac{N\{1+(N+1)x^2\}}{1-x^2}\right]Y=0 \quad \left[\text{on dividing throughout by } (1-x^2)^{-\frac{1}{2}N}\right]$$

or
$$(1-x^2)\frac{d^2Y}{dx^2}-2x\frac{dY}{dx}+\left\{n(n+1)-\frac{N}{(1-x^2)}\right\}y=0 \quad ...(4)$$

Now in order to change the independent variable, put

$$X^2 = n^2(1-x^2)\ ,\ i.e.\ 1-x^2 = \frac{X^2}{n^2}.$$

and $X = n\sqrt{(1-x^2)}$ gives $\dfrac{dX}{dx} = \dfrac{-nx}{\sqrt{(1-x^2)}}$,

so that $$\frac{dY}{dx} = \frac{dY}{dX}\cdot\frac{dX}{dx} = \frac{-nx}{\sqrt{(1-x^2)}}\frac{dY}{dX}$$

and $$\frac{d^2Y}{dx^2} = \frac{d}{dx}\left(\frac{dY}{dx}\right) = \frac{d}{dx}\left(\frac{-nx}{\sqrt{(1-x^2)}}\frac{dY}{dX}\right)$$

or $$\frac{d^2Y}{dx^2} = -n\left[\left\{\frac{\sqrt{(1-x^2)}+\dfrac{x^2}{\sqrt{(1-x^2)}}}{1-x^2}\right\}\frac{dY}{dX} + \frac{x}{\sqrt{(1-x^2)}}\cdot\frac{d^2Y}{dX^2}\cdot\frac{dX}{dx}\right]$$

$$= -n\,\frac{1}{(1-x^2)^{3/2}}\cdot\frac{dY}{dX} - \frac{nx^2}{1-x^2}\,\frac{d^2Y}{dX^2}\Bigg]$$

With these substitutions (4) becomes

$$n^2x^2\,\frac{d^2Y}{dX^2} + \left[\frac{-n}{\sqrt{(1-x^2)}} + \frac{2nx^2}{\sqrt{(1-x^2)}}\right]\frac{dY}{dX} + \left\{n\,(n+1) - \frac{N^2}{1-x^2}\right\} = 0$$

$$i.e.\ \left(1-\frac{X^2}{n^2}\right)\frac{d^2Y}{dX^2} + \frac{(2x^2-1)}{X}\frac{dY}{dX} + \left\{1+\frac{1}{n}-\frac{N^2}{X^2}\right\}Y = 0 \quad \because X^2 = n^2(1-x^2)$$

or $$\left(1-\frac{X^2}{n^2}\right)\frac{d^2Y}{dX^2} + \left(1-\frac{2x^2}{n^2}\right)\frac{1}{X}\frac{dY}{dx} + \left(1+\frac{1}{n}-\frac{N^2}{X^2}\right)Y = 0.$$

Proceeding to the limit when $n \to \infty$, we have

$$\frac{d^2Y}{dX^2} + \frac{1}{X}\frac{dY}{dX} + \left(1-\frac{N^2}{X^2}\right)Y = 0$$

which is clearly the Bessel's differential equation.

Problem 26. *Integrate in series the Bessel's equation of zeroeth order.*

(*Nagpur, 1965*)

Bessel's equation is

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1-\frac{n^2}{x^2}\right)y = 0.$$

If $n = 0$ this equation reduces to

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0, \qquad ...(1)$$

which is known as Bessel's differential equation of zeroeth order.

In order to find the series solution of (1), let us assume that its series solution is of the form

$$y=\sum_{r=0}^{\infty} a_r x^{k+r},$$

so that $$\frac{dy}{dx}=\sum_{r=0}^{\infty} a_r\,(k+r)\,x^{k+r-1}$$

and $$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty} a_r\,(k+r)\,(k+r-1)\,x^{k+r-2}.$$

Putting these values in (1), we get

$$\sum_{r=0}^{\infty} a_r(k+r)\,(k+r-1)x^{k+r-2}+\sum_{r=0}^{\infty} a_r(k+r)x^{k+r-2}+\sum_{r=0}^{\infty} a_r x^{k+r}=0$$

or $$\sum_{r=0}^{\infty}[\{(k+r)\,(k+r-1)+k+r\}x^{k+r-2}+x^{k+r}]a_r=0 \qquad ...(2)$$

The equaton (2) being an identity, we can equate the coefficients of various powers of x to zero.

Equating to zero the coefficient of lowest power of x, *i.e.*, x^{k-2} by putting $r=0$ in (2), we have

$$\{k\,(k-1)+k\,\}a_0=0,\ i.e.,\ k^2\,a_0=0.$$

Being the coefficient of first term $a_0\neq 0$ and therefore

$$k=0. \qquad ...(3)$$

Now equating to zero the coefficient of x^{k-1} by putting $r=1$ in (2), we get

$$\{(k+1)\,k+k+1\}\,a_1=0,\ i.e.,\ (k+1)^2\,a_1=0,$$

which gives $a_1=0$ since $(k+1)^2\neq 0$ by virtue of (3).

Equating to zero the coefficient of general term, *i.e.*, x^{k+r} in (2), we get

$$\{(k+r+2)\,(k+r+1)+k+r+2\}\,a_{r+2}+a_r=0,$$

i.e., $$(k+r+2)^2\,a_{r+2}+a_r=0$$

or $$a_{r+2}=-\frac{a_r}{(r+2)^2}, \qquad ...(4)$$

since by (3) $k=0$.

Now $a_1=0$ and putting $r=1, 3, 5,...$ in (4), we have

$$a_3=-\tfrac{1}{9}a_1=0.$$

Similarly, $a_5=0$, $a_7=0$,

i.e. $a_1=a_3=a_5=a_7=\ldots=0.$

Again putting $r=0, 2, 4, \ldots$ in (4), we get

$$a_2=-\frac{a_0}{2^2},\quad a_4=-\frac{1}{4^2}a^2=\frac{1}{2^2\cdot 4^2}a_0,$$

$$a_6=-\frac{a_0}{6^2\cdot 4^2\cdot 2^2}\,a_8=-\frac{a_0}{8^2\cdot 6^2\cdot 4^2\cdot 2^2}\ \text{etc...}$$

Substituting these values in $y = \sum_{r=0}^{\infty} a_r x^{k+r}$, we get the required series solution as

$$y = a_0 \left[1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots \right]$$

Note. *This solution is denoted by $J_0(x)$ when $a_0 = 1$.*

Problem 27. *Solve the equation*

$$x^2 \frac{d^2y}{dx^2} + x \frac{d^2y}{dx^2} + \left(x^2 - \frac{1}{9} \right) y = 0. \qquad \textit{(Agra, 1964)}$$

The given equation may be written as

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{\frac{1}{9}}{x^2} \right) y = 0 \qquad \text{(on dividing throughout by } x^2\text{).}$$

This is the same differential equation as discussed in §11.4 [*A*] under the head Bessel's differential equation. Here $n^2 = \frac{1}{9}$.

Problem 28. *Solve the differential equation*

$$\frac{d^2\psi}{dx^2} + (E - x^2)\,\psi = 0 \text{ such that } \psi \to 0 \text{ as } |x| \to \infty. \qquad \textit{(Agra, 1962)}$$

The given differential equation is

$$\frac{d^2\psi}{dx^2} + (E - x^2)\psi = 0 \qquad \ldots(1)$$

Put $\psi = v e^{-x^2/2}$

so that $\dfrac{d\psi}{dx} = e^{-x^2/2}\dfrac{dv}{dx} - vxe^{-x^2/2}$

and $\dfrac{d^2\psi}{dx^2} = e^{-x^2/2}\dfrac{d^2v}{dx^2} - 2xe^{-x^2/2}\dfrac{dv}{dx} - ve^{-x^2/2} + vx^2 e^{-x^2/2}.$

Substituting these values in (1) and dividing throughout by $e^{-x^2/2}$ we get

$$\frac{d^2v}{dx^2} - 2x\frac{dv}{dv} + (E-1)v = 0. \qquad \ldots(2)$$

Let its series solution be

$$v = \sum_{r=0}^{\infty} a_r x^{k-r}, \qquad \ldots(3)$$

then $\dfrac{dv}{dx} = \sum_{r=0}^{\infty} a_r (k-r) x^{k-r-1}$

and $\dfrac{d^2v}{dx^2} = \sum_{r=0}^{\infty} a_r (k-r)(k-r-1)\, x^{k-r-2}$

Substituting these values in (3), we get

$$\sum_{r=0}^{\infty} a_r (k-r)(k-r-1) x^{k-r-2} - 2\sum_{r=0}^{\infty} a_r (k-r) x^{k-r} + (E-1)\sum_{r=0}^{\infty} a_r x^{k-r} = 0.$$

or $$\sum_{r=0}^{\infty} [\{(k-r)(k-r-1)\} x^{k-r-2} + \{(E-1) - 2(k-r)\} x^{k-r}] a_r = 0. \quad ...(4)$$

The relation (4) being an identity, the coefficients of various powers of x can be equated to zero.

Let us first equate the coefficient of x^k the highest power of x [by putting $r = 0$ in (2)], to zero; then we get

$$(E-1-2k)\, a_0 = 0$$

This gives $$k = \frac{E-1}{2}, \quad ...(5)$$

since $a_0 \neq 0$, being the coefficient of first term of the series.

Now equating the coefficient of x^{k-1} to zero, by putting $r = 1$ in (4), we get

$$\{E - 1 - 2(k-1) a_1\} = 0.$$

Here $E - 1 - 2(k-1) \neq 0$ by virtue of (5) and therefore, we have

$$a_1 = 0.$$

Further equating to zero the coefficient of x^{k-r}, the general term, in (4), we have

$$(k-r+2)(k-r+1) a_{r-2} + \{(E-1) - 2(k-r)\} a_r = 0$$

or $$a_r = \frac{(k-r+2)(k-r+1)}{E-1-2(k-r)} a_{r-2}$$

$$= \frac{\left(\frac{E-1}{2} - r + 2\right)\left(\frac{E-1}{2} - r + 1\right)}{E-1-2\left(\frac{E-1}{2} - r\right)} a_{r-2} \quad \text{from (5)}$$

$$= -\frac{(E+3-2r)(E+1-2r)}{8r} a_{r-2}.$$

Putting $r = 2, 3, 4, 5, 6$ ...., we get

$$a_2 = -\frac{(E-1)(E-3)}{8.2} a_0.$$

$$a_3 = -\frac{(E-3)(E-5)}{8.3} a_1 = 0 \text{ as } a_1 = 0.$$

Similarly $a_5 = a_7 = a_9 = \ldots = 0$.

and $$a_4 = -\frac{(E-5)(E-7)}{8.4} a_2$$

$$= \frac{(E-1)(E-3)(E-5)(E-7)}{8^2 \cdot 2 \cdot 4} a_0,$$

$$a_6 = -\frac{(E-9)(E-11)}{8 \cdot 6} a_4$$

$$= -\frac{(E-1)(E-3)(E-5)(E-3)(E-9)(E-11)}{8^3 \cdot 2 \cdot 4 \cdot 6} a_0 \text{ etc.}$$

Substituting the values of these coefficients in (3), we get

$$v = a_0\left[x^{(E-1)/2} - \frac{(E-1)(E-3)}{8 \cdot 2} x^{(E-5)/2} + \frac{(E-1)(E-3)(E-5)(E-7)}{8 \cdot 2 \cdot 4} x^{(E-9)/2} - \ldots\right]$$

Putting this value of v in $\psi = ve^{-x^2/2}$, the series solution of the given differential equation (1) is

$$\psi = a_0 e^{-x^2/2}\left[x^{(E-1)/2} - \frac{(E-1)(E-3)}{8\cdot 2}x^{(E-5)/2} + \frac{(E-1)(E-3)(E-5)(E-7)}{8^2\cdot 2\cdot 4}x^{(E-9)/2} - \ldots\right]$$

Problem 29. *When calculating the dependence of the current density upon the distance ρ from the axis, we come across a scalar equation of the following form in cylindrical co-ordinates:*

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\sigma}\frac{\partial u}{\partial \rho} = \frac{4\pi\sigma\mu}{c^2}\frac{\partial u}{\partial t}.$$

Find a solution for u which is a periodic function of the time.

Suppose that $u = \phi(\rho)e^{iwt}$ is a solution; then

$$\frac{\partial u}{\partial \rho} = \frac{d\phi}{d\rho}e^{iwt}$$

and
$$\frac{\partial^2 u}{\partial \rho^2} = \frac{d^2\phi}{d\rho^2}e^{iwt}$$

also
$$\frac{\partial u}{\partial t} = i\omega\phi(\rho)e^{iwt}.$$

Substituting these values in the given equation, we get

$$\frac{d^2\phi}{d\rho^2} + \frac{1}{\rho}\frac{d\phi}{d\rho} = i\omega\frac{4\pi\sigma\mu}{c^2}\phi$$

or
$$\frac{d^2\phi}{d\rho^2} + \frac{1}{\rho}\frac{d\phi}{d\rho} + k^2\phi = 0, \qquad \ldots(1)$$

where $k^2 = -\frac{4\pi\rho i\omega}{c^2}$.

Now in order to change the independent variable, let us put

$$x = k\rho, \text{ so that } \frac{dx}{d\rho} = k.$$

∴
$$\frac{d\phi}{d\rho} = \frac{d\phi}{dx}\cdot\frac{dx}{d\rho} = k\frac{d\phi}{dx}$$

and
$$\frac{d^2\phi}{d\rho^2} = \frac{d}{d\rho}\left(\frac{d\phi}{d\rho}\right) = \frac{d}{d\rho}\left(k\frac{d\phi}{dx}\right) = k\frac{d^2\phi}{dx^2}\cdot\frac{dx}{d\rho} = k^2\frac{d^2\phi}{dx^2}.$$

With these substitutions (1) reduces to

$$\frac{d^2\phi}{dx^2} + \frac{1}{x}\frac{d\phi}{dx} + \phi = 0,$$

which is the Bessel's differential equation of zeroeth order, as we have already discussed in Problem 26. Its solution is

$$\phi = a_0 J_0(x) = a_0\left[1 - \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} - \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \ldots\right].$$

Hence $u = a_0 J_0(x)\cdot e^{i\omega t}$, where a_0 is the amplitude factor

$$= a_0 J_0 (k\rho)\, e^{i\omega t}$$

$$= a_0 J_0 \left\{\sqrt{(-i)}\sqrt{(4\pi\sigma\mu\omega)}\cdot\frac{\rho}{c}\right\} e^{iwt}$$

$$= a_0\left[\text{ber}\left\{\sqrt{(4\pi\sigma\mu\omega)}\cdot\frac{\rho}{c}\right\} + i\ \text{bei}\left\{\sqrt{(4\pi\sigma\mu\omega)}\,\frac{\rho}{c}\right\}\right] e^{iwt}$$

which is the required solution.

Note. *Ber and Bei functions.*

The real and imaginary parts of the function $J_0\left(x\sqrt{-i}\right)$, are known as 'ber (x)' and 'bei (x)', i.e.,

$$R\left[J_0\, x\,\sqrt{-i}\right] = \text{ber}\ x,$$

$$I\left[J_0\, x\,\sqrt{-i}\right] = \text{bei}\ x.$$

Problem 30. *Find the integeral of the differential equation of cylindrical wave*

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial u}{\partial r} = \frac{1}{c^2}\,\frac{\partial^2 u}{\partial t^2}.$$

Suppose that $u = v(r)\ e^{iwt}$.

$$\therefore\qquad \frac{\partial u}{\partial r} = \frac{dv}{dr} e^{iwt} \text{ and } \frac{\partial^2 u}{\partial r^2} = \frac{d^2 v}{dr^2} e^{iwt}.$$

Also
$$\frac{\partial u}{\partial t} = i\,\omega\, v\,(r) e^{iwt}$$

and
$$\frac{\partial^2 u}{\partial t^2} = -\omega^2 v\,(r) e^{iwt}.$$

Substituting these values in the given wave equation, we get

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{dv}{dr} + \frac{\omega^2}{c^2} v = 0. \qquad \ldots(1)$$

Now put $r = \frac{cx}{\omega}$, so that $\frac{dx}{dr} = \frac{\omega}{c}$.

$$\therefore\qquad \frac{dv}{dr} = \frac{dv}{dx}\cdot\frac{dx}{dr} = \frac{\omega}{c}\frac{dv}{dx}$$

and
$$\frac{d^2 v}{dr^2} = \frac{d}{dr}\left(\frac{\omega}{c}\frac{dv}{dx}\right) = \frac{\omega^2}{c^2}\frac{d^2 v}{dx^2}.$$

With these substitutions the equation (1) transforms to

$$\frac{d^2 v}{dx^2} + \frac{1}{x}\frac{dv}{dx} + v = 0,$$

which is the Bessel's differential equation of zeroeth order and therefore its solution is

$v = a_0\, J_0\,(x)$ (as in the previous problem).

Hence the required solution is

$$u = a_0 J_0\left(\frac{r\,\omega}{c}\right) e^{iwt}$$

Problem 31. *Prove* (*a*) $\sqrt{\left(\frac{\pi x}{2}\right)} \cdot J_{3/2}(x) = \frac{\sin x}{x} - \cos\ x.$ (*Agra, 1966*)

and (*b*) $\sqrt{\left(\frac{\pi x}{2}\right)} J_{5/2}(x) = \frac{3 - x^2}{x^2} \sin\ x - \frac{3}{x} \cos\ x.$ (*Rohilkhand, 1989*)

(*a*) We have

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{\lfloor\underline{s}\ \lfloor\underline{(n+s)}} \cdot \left(\frac{x}{2}\right)^{n+2s}$$

$$= \frac{x^n}{2^n \lfloor\underline{n}} - \frac{x^{n+2}}{2^{n+2}\ \lfloor\underline{1}\ \lfloor\underline{(n+1)}} + \frac{x^{n+1}}{2^{n+1}\ \lfloor\underline{2}\ \lfloor\underline{(n+2)}} - \ldots \quad \text{since } \lfloor\underline{0} = 1$$

$$= \frac{x^n}{2^n \lfloor\underline{n}} \left[1 - \frac{x^2}{2^2 \cdot 1 \cdot (n+1)} + \frac{x^4}{2^4 \cdot 1 \cdot 2\ (n+1)\ (n+2)} - \ldots\right]$$

$$= \frac{x^n}{2^n \Gamma(n+1)} \left[1 - \frac{x^2}{2 \cdot (2n+2)} + \frac{x^4}{2 \cdot 4\ (2n+2)\ (2n+4)} - \ldots\right],$$

$$\because \Gamma(n+1) = \lfloor\underline{n}$$

Putting $n = \frac{3}{2}$ on either side, we get

$$J_{3/2}(x) = \frac{x^{3/2}}{2^{3/2} \Gamma \frac{5}{2}} \left[1 - \frac{x^2}{2 \cdot 5} + \frac{x^4}{2 \cdot 4 \cdot 5 \cdot 7} - \frac{x^6}{2 \cdot 4 \cdot 6 \cdot 5 \cdot 7 \cdot 9} + \ldots\right]$$

$$= \frac{x^{3/2}}{2^{3/2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi}} \left[1 - \frac{x^2}{2 \cdot 5} + \frac{x^4}{2 \cdot 4 \cdot 5 \cdot 7} - \frac{x^6}{2 \cdot 4 \cdot 6 \cdot 5 \cdot 7 \cdot 9} + \ldots\right]$$

i.e.
$$\sqrt{\left(\frac{\pi x}{2}\right)} J_{3/2}(x) = \frac{1}{3}\left[x^2 - \frac{x^4}{2\ 5} + \frac{x^6}{2 \cdot 4 \cdot 5 \cdot 7} - \frac{x^8}{2 \cdot 4 \cdot 6 \cdot 5 \cdot 7 \cdot 9} + \ldots\right]$$

$$= \frac{x^2}{3} - \frac{x^4}{2 \cdot 3 \cdot 5} + \frac{x^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 7} - \frac{x^8}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 9} + \ldots$$

$$= \frac{2x^2}{\lfloor\underline{3}} - \frac{4x^4}{\lfloor\underline{5}} + \frac{6x^6}{\lfloor\underline{7}} - \frac{8x^8}{\lfloor\underline{9}} + \ldots$$

$$= \frac{(3-1)x^2}{\lfloor\underline{3}} - \frac{(5-1)x^4}{\lfloor\underline{5}} + \frac{(7-1)x^6}{\lfloor\underline{7}} - \frac{(9-1)x^8}{\lfloor\underline{9}} + \ldots$$

$$= \frac{x^2}{\lfloor\underline{2}} - \frac{x^2}{\lfloor\underline{3}} - \frac{x^4}{\lfloor\underline{4}} + \frac{x^4}{\lfloor\underline{5}} + \frac{x^6}{\lfloor\underline{6}} - \frac{x^6}{\lfloor\underline{7}} - \ldots$$

$$= -\frac{x^2}{\lfloor\underline{3}} + \frac{x^4}{\lfloor\underline{5}} - \frac{x^6}{\lfloor\underline{7}} + \ldots + \frac{x^2}{\lfloor\underline{2}} - \frac{x^4}{\lfloor\underline{4}} + \frac{x^6}{\lfloor\underline{6}} - \ldots$$

$$= \frac{1}{x}\left[x - \frac{x^3}{\lfloor\underline{3}} + \frac{x^5}{\lfloor\underline{5}} - \frac{x^7}{\lfloor\underline{7}} + \ldots\right] - \left[1 - \frac{x^2}{\lfloor\underline{2}} + \frac{x^4}{\lfloor\underline{4}} - \frac{x^6}{\lfloor\underline{6}} + \ldots\right]$$

$$= \frac{1}{x} \sin\ x - \cos\ x.$$

(*b*) For $n = 3/2$, (32) (Rec. rel IV) of § 8.4 [D] *i.e.*, $J_{n+1} = \frac{2n}{x} J_n - J_{n-1}$ gives

$$J_{5/2} = \frac{3}{x} J_{3/2} - J_{1/2} = \sqrt{\frac{2}{\pi x}} \left[\frac{3}{x} \left(-\cos x + \frac{1}{x} \sin x \right) - \sin x \right] \text{ by Prob (20a) \& 31(a)}$$

$$= \sqrt{\left(\frac{2}{\pi x}\right)} \left[\frac{3 - x^2}{x} \sin x - \frac{3}{x} \cos x \right].$$

Problem 32. *Prove that*

(*a*) $$\frac{J_{n+1}(x)}{J_n(x)} = \frac{x/2}{(n+1)-} \frac{(x/2)^2}{(n+2)-} \frac{(x/2)^3}{(n+3)- \ldots}$$

(*b*) *Define spherical Bessel function $J_n(x)$ of first kind and show that*

(*i*) $J_{n-1}(x) + J_{n+1}(x) = \frac{2n+1}{x} J_n(x)$ (*ii*) $\int_{-\infty}^{\infty} J_n(x) J_m(x) = \frac{\pi}{2n+1} \delta_{m,n}$

(*Rohilkhand, 1990*)

(*c*) *Starting from relation* $e^{x(t-1/t)/2} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$, *deduce that*

$$J_n(x+y) = \sum_{k=-\infty}^{\infty} J_k(x) J_{n-k}(y). \qquad \textit{(Rohilkhand, 1990).}$$

(*a*) We have the recurrence relation.

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$$

or

$$J_{n-1}(x) = \frac{2n}{x} J_n(x) - J_{n+1}(x).$$

Replacing n by $(n + 1)$, this becomes

$$J_n(x) = \frac{2(n+1)}{x} J_{n+1}(x) - J_{n+2}(x). \qquad \ldots(1)$$

Now $$\frac{J_{n+1}(x)}{J_n(x)} = \frac{1}{\frac{J_n(x)}{J_{n+1}(x)}} = \frac{1}{\frac{\frac{2(n+1)}{x} J_{n+1}(x) - J_{n+2}(x)}{J_{n+1}(x)}} \text{ from (1)}$$

$$= \frac{1}{\frac{2(n+1)}{x} - \frac{J_{n+2}(x)}{J_{n+1}(x)}} = \frac{1}{\frac{2(n+1)}{x} - \frac{1}{\frac{J_{n+1}(x)}{J_{n+2}(x)}}}$$

$$= \frac{1}{\frac{2(n+1)}{x} - \frac{1}{\frac{\frac{2(n+2)}{x} J_{n+2}(x) - J_{n+3}(x)}{J_{n+3}(x)}}} \text{ by (1)}$$

$$= \cfrac{1}{\cfrac{2(n+1)}{x} - \cfrac{1}{\cfrac{2(n+2)}{x} - \cfrac{1}{\cfrac{J_{n+2}(x)}{J_{n+3}(x)}}}}$$

$$= \cfrac{1}{\cfrac{2(n+1)}{x} - \cfrac{1}{\cfrac{2(n+2)}{x} - \cfrac{1}{\cfrac{2(n+3)}{x} - \dots}}}$$

$$= \cfrac{x/2}{(n+1) - \cfrac{x/2}{\cfrac{2(n+2)}{x} - \cfrac{1}{\cfrac{2(n+3)}{x} - \dots}}}$$

by repeated application of (1)

$$= \cfrac{x/2}{(n+1) - \cfrac{(x/2)^2}{(n+2) - \cfrac{x/2}{2(n+3)/x} - \dots}}$$

$$= \cfrac{x/2}{(n+1) - \cfrac{(x/2)^2}{(n+2) - \cfrac{(x/2)^3}{(n+3)} - \dots}}$$

(*b*) We have the recurrence relation, $J_{n+1}(x) = \dfrac{2n}{x} J_n(x) - J_{n-1}(x)$, which yields on replacing n by $m - \frac{1}{2}$;

$$J_{m+1/2}(x) = \frac{2\left(m - \frac{1}{2}\right)}{x} J_{m-1/2}(x) - J_{m-3/2}(x) = A_1(x)\, J_{m-1/2}(x) + B_1(x) J_{m-3/2}(x) \text{ (say)} \dots \quad \dots(2)$$

Replacing m by $(m - 1)$,

$$(2) \Rightarrow J_{m-1/2}(x) = \frac{2\left(m - \frac{3}{2}\right)}{x} J_{m-3/2}(x) - J_{m-5/2}(x),$$

so that $(2) \Rightarrow J_{m+1/2}(x) = \dfrac{2\left(m - \frac{1}{2}\right)}{x}\left[\dfrac{2\left(m - \frac{3}{2}\right)}{x} J_{n-3/2}(x) - J_{m-5/2}(x)\right]$

$$= A_2(x) J_{m-3/2}(x) + B_2(x) J_{m-5/2}(x)$$

Proceeding in this manner, we find

$$J_{m+1/2}(x) = A_m(x)\, J_{1/2}(x) + B_m(x)\, J_{-1/2}(x)$$

$$= A_m(x) \sqrt{\frac{2}{\pi x}} \sin x + B_m(x) \sqrt{\frac{2}{\pi x}} \cos x \qquad \text{by Prob. 20.}$$

$$= C_m(x) \sin x + D_m \cos x \text{ (say)} \quad \dots(3)$$

Here $J_{m+1/2}(x)$ is known as *spherical Bessel Function of First Kind* and it is the solution of the equation of the form

$$r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}+\left[\mu^2r^2-m\,(m+1)\right]R=0 \qquad ...(4)$$

which reduces to Bessel's equation, on setting $R(\mu r)=\dfrac{Z(\mu r)}{(\mu r)^{1/2}}$ ...(5)

i.e. $$r^2\frac{d^2Z}{dr^2}+r\frac{dZ}{dr}+\left[\mu^2r^2-\left(m+\tfrac{1}{2}\right)^2\right]Z=0$$

or $$\frac{d^2Z}{dr^2}+\frac{1}{r}\frac{dZ}{dr}+\left[\mu^2-\frac{\left(m+\frac{1}{2}\right)^2}{r^2}\right]Z=0 \qquad ...(6)$$

whose solution is Z (μx), so that the solution of (4) is

$$R\,(\mu r)=\frac{Z_{m+1/2}\,(\mu r)}{(\mu r)^{1/2}} \qquad ...(7)$$

which are the *Spherical Bessel functions* say $J_m(x)$ related to the Bessel functions as

$$J_m(x)=\sqrt{\frac{\pi}{2x}}\,J_{m+1/2}(x) \qquad ...(8)$$

(*i*) The recurrence-relation $J_{n-1}(x)+J_{n+1}(x)=\dfrac{2n}{x}J_n(x)$ yields on replacing n by $n+\frac{1}{2}$;

$$J_{n-1/2}\,(x)+J_{n+3/2}\,(x)=\frac{2n+1}{x}J_{n+1/2}(x)$$

or $$\sqrt{\frac{\pi}{2x}}\,J_{n-1/2}(x)+\sqrt{\frac{\pi}{2x}}\,J_{n+3/2}(x)=\frac{2n+1}{x}\sqrt{\frac{\pi}{2x}}\,J_{n+1/2}(x)$$

$$\Rightarrow J_{(n-1)}\,(x)+J_{(n+1)}(x)=\frac{2n+1}{x}J_n(x) \quad \text{by (8)}$$

(*ii*) For $x=kr$, the spherical Bessel functions $J_n\,(x)$ and $J_m(x)$ satisfy

$$x^2\,\frac{d^2J_n(x)}{dx^2}+2x\,\frac{dJ_n(x)}{dx}+\left[x^2-n\,(n+1)\right]J_n(x)=0 \qquad ...(9)$$

and $$x^2\frac{d^2J_m(x)}{dx^2}+2x\,\frac{dJ_m(x)}{dx}+\left[x^2-m(m+1)\right]J_m(x)=0 \qquad ...(10)$$

Multiplying (9) by $J_m(x)$, (10) by $J_n(x)$ and subtracting, we get

$$[n\,(n+1)-m\,(m+1)]J_nJ_m=x^2\left[\frac{d^2J_n}{dx^2}J_m-J_n\frac{d^2J_m}{dx}\right]-2x\left[\frac{dJ_n}{dx}J_m-J_n\,\frac{dJ_n}{dx}\right]$$

$$=\frac{d}{dx}\left[x^2\left(\frac{dJ_n}{dx}J_m-J_n\frac{dJ_n}{dx}\right)\right]$$

which on integrating *w.r.t.* x from 0 to ∞, gives

$$[n(n+1)-m\,(m+1)]\int_0^\infty J_n(x)\,J_m(x)\,dx=\left[x^2\left(\frac{dJ_n}{dx}\right)J_m-J_n\left(\frac{dJ_m}{dx}\right)\right]_0^\infty \qquad ...(11)$$

Here R.H.S.$\to 0$ as $x\to 0$; m, n being non-negative integers.

For $x\to\infty$, $J_n(x)\to \frac{1}{2}\sin\left(x-\frac{n\pi}{2}\right), \frac{dJ_n}{dx}\to\frac{1}{x}\cos\left(x-\frac{n\pi}{2}\right)$

and so $x^2\left(\frac{dJ_n}{dx}J_m - J_n\frac{dJ_m}{dx}\right)\to\cos\left(x-\frac{n\pi}{2}\right)\sin\left(n-\frac{m\pi}{2}\right)$

$$-\sin\left(x-\frac{n\pi}{2}\right)\cos\left(x-\frac{m\pi}{2}\right)+f(x^{-1})+\sin\left(\frac{n-m}{2}\pi\right)+f(x^{-1})$$

$$(11)\Rightarrow \int_0^\infty J_n(x)J_m(x)\ dx = \frac{\sin (n-m)\pi/2}{n(n+1)-m(m+1)} \qquad ...(12)$$

where R.H.S. $\to 0\ \forall\ (n-m)$.

If $(n-m)$ is odd, so is J_m and other is even as $J_P(x) = (-1)^p, J_p(-x)$. Thus the product $J_n(x)\,J_m(x)$ is odd and so the integerand vanishes, so that the lower limit may be extended to $-\infty$

$$i.e.\ \int_{-\infty}^0 J_n(x)J_m(x)\ dx = 0 \quad \text{if } m\neq n, m, n\geq 0. \qquad ...(13)$$

Which is *orthogonal property of spherical Bessel functions.*

$$\text{For } m=n, \text{ we can show that } \int_{-\infty}^{\infty}[J_n(x)]^2\ dx = \frac{\pi}{2n+1} \qquad ...(14)$$

$$\therefore\ (13)\ \&\ (14)\Rightarrow \int_{-\infty}^{\infty} J_n(x)J_m(x)\ dx = \frac{\pi}{2n+1}\delta_{m,n} \qquad ...(15)$$

where the functions $\sqrt{\left(\frac{2n+1}{\pi}\right)}J_n(x)$ form a *Complete Orthonormal set.*

(c) We have

$$\sum_{r=-\infty}^{\infty}J_n(x)\ t^n = e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} \Rightarrow \sum_{n=-\infty}^{\infty}J_n(x+y)t^n = e^{\left(\frac{x+y}{2}\right)\left(t-\frac{1}{t}\right)}$$

$$= e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}.e^{\frac{y}{2}\left(t-\frac{1}{t}\right)}$$

$$= \sum_{k=-\infty}^{\infty}t^k J_k(x)\sum_{j=-\infty}^{\infty}t^j J_j(y)$$

$$= \sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}J_k(x)\ J_j(y)\ t^{k+j}$$

Replacing j by $n-k$ and equating coefficients of t^n on either side, we get

$$J_n(x+y) = \sum_{k=-\infty}^{\infty}J_k(x)\ J_{n-k}(y) \text{ for any integer } n.$$

Note: For positive integeral n, we can write it

$$J_n(x+y) = \sum_{-\infty}^{-1}J_k(x)\ J_{n-k}(y) + \sum_{k=0}^{n}J_k(x)\ J_{n-k}(y) + \sum_{k=n+1}^{\infty}J_k(x)\ J_{n*k}(y)$$

$$= \sum_{k=1}^{\infty} (-1)^k J_k(x)\ J_{n+k}(y) + \sum_{k=0}^{n} J_k(x)\ J_{n-k}(y) + \sum_{k=1}^{\infty} (-1)^k\ J_{n+k}(x)\ J_k(y)$$

as $J_n(x) = (-1)^n J_{-n}(x)$

$$= \sum_{k=0}^{\infty} J_k(x)\ J_{n-k}(y) + \sum_{k=1}^{\infty} (-1)^k [J_k(x)\ J_{n+k}(y) + J_{n+k}(x)\ J_k(y)]$$

8.5. HYPERGEOMETRIC OR GAUSS EQUATION AND FUNCTIONS

[A] Hypergeometric or Gauss' Differential Equation.

This equation is of the form

$$x(1-x)\ \frac{d^2y}{dx^2} + \{\gamma - (\alpha + \beta + 1)x\}\ \frac{dy}{dx} - \alpha\beta\ y = 0 \qquad ...(1)$$

where α, β, γ are parametric constants.

Here $x = 0$, $x = 1$ and $x = \infty$ are the singularities, since on dividing (1) by $x(x-1)$ we observe that coefficients of $\frac{dy}{dx}$ and y become infinite when $x = 0$, 1 or ∞. Thus we can integrate (1) in series about $x = 0$ or $x = 1$ or $x = \infty$. We therefore discuss the series integration in three cases.

Case [a_1]. When $x = 0$, then taking the series solution of (1) as

$$y = \sum_{r=0}^{\infty} a_r\ x^{k+r} \qquad ...(2)$$

We have already discussed the solution in §8.3 $E[e_3]$ and obtained the solution as

$$y = AF\ (\alpha, \beta, \gamma, x) + Bx^{1-\gamma} F(\alpha + 1 - \gamma, \beta + 1 - \gamma, 2 - \gamma, x) \qquad ...(3)$$

where A and B are arbitrary constants,

and $F(\alpha, \beta, \gamma, x) = 1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha+1)\ (\beta+1)}{1 \cdot 2 \cdot \gamma \cdot (\gamma+1)} x^2$

$$+ \frac{\alpha(\alpha+1)\ (\alpha+2)\ \beta\ (\beta+1)\ (\beta+2)}{1 \cdot 2 \cdot 3 \cdot \gamma\ (\gamma+1)\ (\gamma+2)} x^3 + ...$$

Case [a_2]. When $x = 1$, is the singularity, then the series solution is obtained by developing the series about $x = 1$, by making a substitution

$$X = 1 - x \text{ in (1)} \qquad ...(4)$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dX}\frac{dX}{dx} = -\frac{dy}{dX}, \text{ since by (1) } \frac{dX}{dx} = -1$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dX}\left(-\frac{dy}{dX}\right)\frac{dX}{dx} = \frac{d^2y}{dX^2}$$

Substituting these values in (1) and arranging we get

$$(X^2 - X)\ \frac{d^2y}{dX^2} + [(1 + \alpha + \beta)X + (\gamma - 1 - \alpha - \beta)]\frac{dy}{dX} + \alpha\beta\ y = 0. \qquad ...(5)$$

which is similar to (1) except that γ is repalced by $1 + \alpha + \beta - \gamma$ and x by $1 - x$ and hence by (3) the solution in this case becomes,

$$y = AF\ (\alpha, \beta, 1 + \alpha + \beta - \gamma, 1 - x) + B(1 - x)^{\gamma-\alpha-\beta}$$
$$F(\gamma - \alpha, \gamma - \beta, \gamma - \alpha - \beta + 1, 1 - x) \qquad ...(6)$$

Case [a₃]. When $x = \infty$ gives a singularity, then the series solution of (1) is obtained by developing the series about $x = \infty$, by making a substitution

$$x = \frac{1}{X} \text{ in (1)} \qquad \ldots(7)$$

So that $\frac{dy}{dx} = \frac{dy}{dX}\frac{dX}{dx} = -\frac{1}{x^2}\frac{dy}{dX} = -X^2\frac{dy}{dX}$ etc.

$\therefore$ (1) becomes

$$X^2(1-X)\frac{d^2y}{dX^2} + \left[2X(1-X) - (1+\alpha+\beta)X + \gamma X^2\right]\frac{dy}{dX} + \alpha\beta\, y = 0 \qquad \ldots(8)$$

Let its series solution be

$$y = \sum_{r=0}^{\infty} a_r X^{k+r},\; a_0 \neq 0 \qquad \ldots(9)$$

So that $\frac{dy}{dX} = \sum_{r=0}^{\infty} a_r (k+r) X^{k+r-1}$

$$\frac{d^2y}{dX^2} = \sum_{r=0}^{\infty} a_r (k+r)(k+r-1) X^{k+r-2}$$

Substituting these values in (8) we get the identity

$$\sum_{r=0}^{\infty} [\{(k+r)(k+r-1) + (1-\alpha-\beta)(k+r) + \alpha\beta\} X^{k+r}$$
$$- \{(k+r)(k+r-1) + (2-\gamma)(k+r)\} X^{k+r+1}]\, a_r \equiv 0 \qquad \ldots(10)$$

Equating to zero the coefficient of X^k (the first term) in (10), we get

$$[k(k-1) + (1-\alpha-\beta)k + \alpha\beta]\, a_0 = 0$$

$\because a_0 \neq 0,\ \therefore k(k-1) + (1-\alpha-\beta)k + \alpha\beta = 0$

or $k^2 - (\alpha+\beta)k + \alpha\beta = 0$

or $(k-\alpha)(k-\beta) = 0$ giving $k = \alpha, \beta$ ...(11)

Again equating to zero the coefficient of x^{k+r} in (10), we find the recurrence relation as

$$\{k+r)(k+r-1) + (1-\alpha-\beta)(k+r) + \alpha\beta\}\, a_r - \{(k+r-1)(k+r-2)$$
$$+ (2-\gamma)(k+r-1)\}\, a_{r-1} = 0$$

i.e. $(k+r-\alpha)(k+r-\beta)\, a_r - (k+r-1)(k+r-\gamma)\, a_{r-1} = 0$

or $$a_r = \frac{(k+r-1)(k+r-\gamma)}{(k+r-\alpha)(k+r-\beta)}\, a_{r-1} \qquad \ldots(12)$$

when $k = \alpha$, (10) gives $a_r = \frac{(\alpha+r-1)(\alpha+r-\gamma)}{r(\alpha+r-\beta)}\, a_{r-1}$

So that $a_1 = \frac{\alpha(\alpha+1-\gamma)}{1\cdot(\alpha+1-\beta)}\, a_0$

$$a_2 = \frac{\alpha(\alpha+1)\cdot(\alpha+1-\gamma)(\alpha+2-\gamma)}{1\cdot 2\cdot(\alpha+1-\beta)(\alpha+2-\beta)}\, a_0$$

$$a_3 = \frac{\alpha\,(\alpha+1)\,(\alpha+2)\cdot(\alpha+1-\gamma)\,(\alpha+2-\gamma)\,(\alpha+3-\gamma)}{1\cdot 2\cdot\,(\alpha+1-\beta)\,(\alpha+2-\beta)\,(\alpha+3-\gamma)}\,a_0 \text{ etc.}$$

$\therefore$ Solution is $y = a_0 X^{\alpha} F(\alpha, \alpha+1-\gamma, \alpha+1-\beta, X)$

$$= a_0 x^{-\alpha} F\left(\alpha,\ \alpha+1-\gamma,\ \alpha+1-\beta,\ \frac{1}{x}\right) \quad \text{...(13)}$$

and similarly when $k = \beta$, the solution is

$$y = a_0 x^{-\beta} F\left[\beta,\ \beta+1-\gamma,\ \beta+1-\alpha,\ \frac{1}{x}\right] \quad \text{...(14)}$$

Hence the complete integral is

$$y = Ax^{-\alpha}F(\alpha, \alpha+1-\gamma, \alpha+1-\beta, \frac{1}{x} + Bx^{-\beta}F\left[\beta,\ \beta+1-\gamma,\ \beta+1-\alpha,\ \frac{1}{x}\right] \quad \text{...(15)}$$

where A and B are arbitrary constants.

We thus have found all the three possible solutions of Hypergeometric or Gauss equation, *i.e.*,

(*i*) for $x = 0$, exponents are $0, 1-\gamma$ by (3)

(*ii*) for $x = \infty$, exponents are $-\alpha, -\beta$ by (13)

(*iii*) for $x = 1$, exponents are $0, \gamma-\alpha-\beta$ by (6)

These results may be shown by a scheme as follows:

$$y = P\begin{bmatrix} 0 & \infty & 1 & \\ 0 & \alpha & 0 & x \\ 1-\gamma & \beta & \gamma-\alpha-\beta & \end{bmatrix} \quad \text{...(16)}$$

where the R.H.S. is said to be the *Riemann-P Function of the equation.*

In symbolic form, $F(\alpha, \beta, \gamma; x) = F\begin{bmatrix} \alpha, & \beta & \\ & & x \\ & \gamma & \end{bmatrix}$ ...(17)

which is known as *Hypergeometric function.*

We also denote $F(\alpha, \beta, \gamma, x)$ by $\sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k \underline{|k}} x^k$ ...(18)

[B] Particular Cases of Hypergeometric Series.

(*i*) we have $(1+x)^n = 1 + nx + \frac{n\,(n-1)}{1\cdot 2}x^2 + \frac{n\,(n-1)\,(n-2)}{1\cdot 2\cdot 3}x^3 + \ldots$

$$= F(-n,\ 1,\ 1,\ -x)$$

(*ii*) $\log\,(1+x) = x - \frac{1}{2}x^2 + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

$$= x\left[1 - \frac{1}{2}x + \frac{1}{3}x^2 - \frac{1}{4}x^3 + \ldots\right]$$

$$= x\,F\,(1,\ 1,\ 2,\ -x)$$

(iii) $\sin^{-1} x = x + \frac{x^3}{\lfloor 3} + \frac{3^2 \cdot x^5}{\lfloor 5} + \frac{3^2 \cdot 5^2}{\lfloor 7} x^7 + \ldots$

$$= x\left[1 + \frac{\frac{1}{2}\cdot\frac{1}{2}}{1\cdot\frac{3}{2}} x^2 + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}}{1\cdot 2\cdot\frac{3}{2}\cdot\frac{5}{2}} x^4 + \ldots\right]$$

$$= xF\left[\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, x^2\right]$$

(iv) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$

$$= x\left[1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \ldots\right]$$

$$= x\left[1 + \frac{1\cdot\frac{1}{2}}{1\cdot\frac{3}{2}}\left(-x^2\right) + \frac{1\cdot 2\cdot\frac{1}{2}\cdot\frac{3}{2}}{1\cdot 2\cdot\frac{3}{2}\cdot\frac{5}{2}}\left(-x^2\right)^2 + \ldots\right]$$

$$= xF\left[1, \tfrac{1}{2}, \tfrac{3}{2}, -x^2\right]$$

(v) $e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots$

$$= \lim_{n\to\infty}\left\{1 + \frac{n\cdot 1\, x}{1\cdot 1\, n} + \frac{n\,(n+1)\, 1\cdot 2}{1\cdot 2\cdot 1\cdot 2}\left[\frac{x}{n}\right]^2 + \ldots\right\}$$

$$= \lim_{n\to\infty} F\left[n,\ 1, 1,\ \frac{x}{n}\right].$$

[C] Simple Properties of Hypergeometric Function.

(1) Symmetry property. The value of a hypergeometric function does not change by the interchange of parameters α and β.

$$\because\ F(\alpha, \beta, \gamma, x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k\,(\beta)_k}{(\gamma)_k\,\lfloor k} x^k$$

$$= \sum_{k=0}^{\infty} \frac{(\beta)_k\,(\alpha)_k}{(\gamma)_k\,\lfloor k} x^k$$

$$= F(\beta, \alpha, \gamma, x)$$

$$\therefore\quad F\begin{bmatrix}\alpha, & \beta & \\ & & x\\ & \gamma & \end{bmatrix} = F\begin{bmatrix}\beta, & \alpha & \\ & & x\\ & \gamma & \end{bmatrix}, \qquad \ldots(19)$$

(2) Differentiation of Hypergeometric Functions. We have

$$F(\alpha, \beta, \gamma, x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k\,(\beta)_k}{(\gamma)_k\,\lfloor k} x^k$$

Its differentiation w.r.t. x, yields,

$$\frac{d}{dx}(F(\alpha, \beta, \gamma, x)) = \sum_{k=0}^{\infty} k\frac{(\alpha)_k(\beta)_k}{k\,\lfloor k-1\,(\gamma)_k} x^{k-1} = \sum_{k=1}^{\infty} \frac{(\alpha)_k(\beta)_k}{\lfloor k-1\,(\gamma)_k} x^{k-1}$$

$\because$ it vanishes for $k = 0$

$$= \sum_{p=0}^{\infty} \frac{(\alpha)_{p+1}\,(\beta)_{p+1}}{\lfloor p\,(\gamma)_{p+1}} \text{ on putting } k-1=p.$$

$$= \sum_{p=0}^{\infty} \frac{\alpha\cdot(\alpha+1)_p \cdot \beta\,(\beta+1)_p}{\lfloor p\,\gamma(\gamma+1)_p} \quad \left[\begin{array}{ll} \because (\alpha)_{p+1} & = \alpha(\alpha+1)\,(\alpha+2)\ldots(\alpha+p). \\ & = \alpha[(\alpha+1)\,(\alpha+2),\ldots(\alpha+1+p-1)] \\ & = \alpha\,(\alpha+1)_p \text{ etc.} \end{array}\right]$$

$$= \frac{\alpha\beta}{\gamma} \sum_{p=0}^{\infty} \frac{(\alpha+1)_p\,(\beta+1)_p}{\lfloor p\,(\gamma+1)_p}$$

$$= \frac{\alpha\beta}{\gamma} F(\alpha+1, \beta+1, \gamma+1, x) \qquad \ldots(20)$$

$$\frac{d^2}{dx^2} F(\alpha, \beta, \gamma, x) = \frac{d}{dx}\left[\frac{\alpha\beta}{\gamma} F(\alpha+1, \beta+1, \gamma+1, x)\right]$$

$$= \frac{\alpha\beta}{\gamma} \frac{d}{dx} F(\alpha+1, \beta+1, \gamma+1, x)$$

$$= \frac{\alpha\beta}{\gamma} \cdot \frac{(\alpha+1)\,(\beta+1)}{\gamma+1} F(\alpha+1+1, \beta+1+1, \gamma+1+1, x)$$

by applying (20)

$$= \frac{\alpha(\alpha+1)\,\beta(\beta+1)}{\gamma\,(\gamma+1)} F(\alpha+2, \beta+2, \gamma+2, x) \qquad \ldots(21)$$

Repeating the process m times, we may have

$$\frac{d^m}{dx^m} F(\alpha, \beta, \gamma, x) = \frac{\alpha\,(\alpha+1)\ldots(\alpha+m-1)\cdot\beta\,(\beta+1)\ldots(\beta+m-1)}{\gamma\,(\gamma+1)\ldots(\gamma+m-1)}$$

$$F\,(\alpha+m, \beta+m, \gamma+m, x)$$

$$= \frac{(\alpha)_m(\beta)_m}{(\gamma)_m} F\,(\alpha+m, \beta+m, \gamma+m, x) \qquad \ldots(22)$$

In symbolic form (22) can be written as

$$\frac{d^m}{dx^m} F\left[\begin{matrix}\alpha, & \beta & \\ & \gamma & \end{matrix}\; x\right] = \frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\left[\begin{matrix}\alpha+m, \beta+m & \\ \gamma+m & \end{matrix}\; x\right]$$

COROLLARY 1. When $x = 0$,

$$F\,(\alpha, \beta, \gamma, 0) = \lim_{x\to 0} \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{\lfloor k\,(\gamma)_k} x^k$$

$$= \lim_{x\to 0}\left[1 + \frac{\alpha\cdot\beta}{\gamma}x + \frac{\alpha(\alpha+1)\,\beta\,(\beta+1)}{1\cdot 2\cdot\gamma\,(\gamma+1)}x^2 + \ldots\right]$$

$= 1$, since all the terms except the first vanish.

Similarly $F(\alpha+1, \beta+1, \gamma+1, 0) = 1$

Hence from (18), it follows that

$$\left[\frac{d}{dx}F(\alpha,\beta,\gamma,x)\right]_{x=0}=\frac{\alpha\beta}{\gamma}. \qquad \text{...(23)}$$

CORLLLARY 2. Had the parameter α been a negative integer say $-N$, then we should have

$$F(-N,\beta,\gamma,x)=\sum_{k=0}^{N}\frac{(-N)_k(\beta)_k}{\lfloor k\,(\gamma)_k}x^k \qquad \text{...(24)}$$

since it vanishes for $k = N, N+1, N+2,...$etc.

Similarly if β were a negative integer say $-M$, then

$$F(\alpha,-M,\gamma,x)=\sum_{k=0}^{M}\frac{(\alpha)_k(-M)_k}{\lfloor k\,(\gamma)_k}x^k \qquad \text{...(25)}$$

In case $\alpha = -N$ and $\gamma = -(N+M)$, N, M being positive integers, we have

$$F(-N,\beta,-N-M,x)=\sum_{k=0}^{\infty}\frac{(-N)_k(\beta)_k}{(-N-M)_k\lfloor k}x^k \qquad \text{...(26)}$$

where $(-N)_k=(-N)(-N+1)...(-N+k-1)=(-1)^k\dfrac{\lfloor N}{\lfloor N-k}$

Similarly $(-N-M)_k=(-1)^k\dfrac{\lfloor N+M}{\lfloor N+M-k}$

So that $\dfrac{(-N)_k}{(-N-M)_k}=\dfrac{\lfloor N}{\lfloor N-k}\times\dfrac{\lfloor N+M-k}{\lfloor N+M}=\dfrac{\lfloor N}{\lfloor N+M}$

$$[(N+M-k)(N+M-k-1)...(N-k+1)]$$

$$=\left[\left(1-\frac{k}{N+M}\right)\left(1-\frac{k}{N+M-1}\right)...\left(1-\frac{k}{N+1}\right)\right]$$

Hence (26) yields

$$F(-N,\beta,-N-M,x)=\sum_{k=0}^{\infty}\left[\left(1-\frac{k}{N+M}\right)\left(1-\frac{k}{N+M-1}\right)...\left(1-\frac{k}{N+1}\right)\right]\frac{(\beta)_k}{\lfloor k}x^k \qquad \text{...(27)}$$

Here it is notable that on the R.H.S. the terms do not vanish for $k = 0, 1, 2,...N$ and they vanish for $k = N+1, N+2,..., N+M$; but the terms do not vanish for $k = N+M+1, N+M+2,...$, since the term corresponding to $k = N+M+1$ is

$$\left(1-\frac{N+M+1}{N+M}\right)...\left(1-\frac{N+M+1}{N+1}\right)\cdot\frac{(\beta)_{N+M+1}}{\lfloor N+M+1}x^{N+M+1}$$

Conclusively the series which stopped for $\alpha = -N$ or $\beta = -M$ at Nth and Mth terms respectively, starts again when $\alpha = -N$ and $\gamma = -(N+M)$ or likewise when $\beta = -M$ and $\gamma = -(N+M)$.

(3) Integral Formula for the Hypergeometric Functions.

We have $F(\alpha,\beta,\gamma,x)=\displaystyle\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{\lfloor k\,(\gamma)_k}x^k$

where $(\alpha)_k = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+k-1) = \dfrac{\lfloor\alpha+k-1}{\lfloor\alpha-1} = \dfrac{\Gamma(\alpha+k)}{\Gamma(\alpha)}$

Similarly $(\beta)_k = \dfrac{\Gamma(\beta+k)}{\Gamma\beta}$, $(\gamma)_k = \dfrac{\Gamma(\gamma+k)}{\Gamma\gamma}$

Also we have $B(m,n) = \dfrac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$ (Beta function)

$$\therefore \frac{(\beta)_k}{(\gamma)_k} = \frac{\Gamma(\beta+k)}{\Gamma\beta} \times \frac{\Gamma\gamma}{\Gamma(\gamma+k)} = \frac{\Gamma\gamma}{\Gamma\beta} \cdot \frac{\Gamma(\beta+k)}{\Gamma(\gamma+k)} \times \frac{\Gamma(\gamma-\beta)}{\Gamma(\gamma-\beta)}$$

$$= \frac{\Gamma\gamma}{\Gamma\beta\Gamma(\gamma-\beta)} \cdot \frac{\Gamma(\beta+k)\,\Gamma(\gamma-\beta)}{\Gamma(\gamma+k)}$$

$$= \frac{\Gamma\gamma}{\Gamma\beta\Gamma(\gamma-\beta)} \frac{\Gamma(\beta+k)\,\Gamma(\gamma-\beta)}{\Gamma\{(\beta+k)+(\gamma-\beta)\}} = \frac{\Gamma\gamma}{\Gamma\beta\Gamma(\gamma-\beta)} \cdot B(\gamma-\beta,\, \beta+k)$$

$$= \frac{\Gamma\gamma}{\Gamma\beta\Gamma(\gamma-\beta)} \int_0^1 (1-t)^{\gamma-\beta-1} t^{\beta+k-1}\, dt \quad \because B(P,Q) = \int_0^1 (1-x)^{P-1} x^{Q-1}\, dx$$

$$= \frac{1}{B(\beta,\gamma-\beta)} \int_0^1 (1-t)^{\gamma-\beta-1} t^{\beta+k-1}\, dt$$

With these substitutions, we therefore, have

$$F(\alpha,\beta,\gamma,x) = \frac{1}{B(\beta,\gamma-\beta)} \sum_{k=0}^{\infty} \frac{(\alpha)_k}{\lfloor k} \cdot x^k \int_0^1 (1-t)^{\gamma-\beta-1} t^{\beta+k-1}\, dt$$

$$= \frac{1}{B(\beta,\gamma-\beta)} \int_0^1 (1-t)^{\gamma-\beta-1} t^{\beta-1} \left\{ \sum_{k=0}^{\infty} \frac{(\alpha)_k}{\lfloor k} (xt)^k \right\} dt$$

(on interchanging the order of integration and summation)

$$= \frac{1}{B(\beta,\gamma-\beta)} \int_0^1 (1-t)^{\gamma-\beta-1} t^{\beta-1} (1-xt)^{-\alpha}\, dt$$

$$\because \sum_{k=0}^{\infty} \frac{(\alpha)_k}{\lfloor k} (xt)^k = 1 + \frac{\alpha\,(xt)}{\lfloor 1} + \frac{\alpha\,(\alpha+1)}{\lfloor 2}(xt)^2 + \ldots = (1-xt)^{-\alpha}.$$

We thus get the integral formula for hypergeometric series as

$$F(\alpha,\beta,\gamma,x) = \frac{1}{B(\beta,\gamma-\beta)} \int_0^1 (1-t)^{\gamma-\beta-1} t^{\beta-1} (1-xt)^{-\alpha}\, dt \qquad \ldots(28)$$

which is valid for $|x| < 1$ and $\gamma > \beta > 0$.

COROLLARY 3. *Gauss Theorem* or *Gauss Formula*

If we put $x = 1$ in (25), we have

$$F(\alpha,\beta,\gamma,1) = \frac{1}{B(\beta,\gamma-\beta)} \int_0^1 (1-t)^{\gamma-\beta-1} t^{\beta-1} (1-t)^{-\alpha}\, dt$$

$$= \frac{1}{B(\beta,\gamma-\beta)} \int_0^1 (1-t)^{\gamma-\beta-\alpha-1} t^{\beta-1} dt = \frac{B(\beta,\gamma-\alpha-\beta)}{B(\beta,\gamma-\beta)} \qquad \ldots(29)$$

Using $B(P, Q) = \dfrac{\Gamma P \Gamma Q}{\Gamma(P+Q)}$, (27) yields

$$F(\alpha, \beta, \gamma, 1) = \frac{\Gamma\gamma\ \Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\ \Gamma(\gamma-\beta)} \qquad \text{...(30)}$$

which is known as Gauss' theorem.

COROLLARY 4. **Vandermond's theorem.**

In (30) if we put $\alpha = -n$, we get

$$F(-n, \beta, \gamma, -1) = \frac{\Gamma\gamma\ \Gamma(\gamma-\beta+n)}{\Gamma(\gamma+n)\ \Gamma(\gamma-\beta)}$$

$$= \frac{1.2\ldots(\gamma-1).\ 1.2\ldots(\gamma-\beta+n-1)}{1.2\ldots(\gamma+n-1).\ 1.2\ldots(\gamma-\beta-1)}$$

$$= \frac{(\gamma-\beta)(\gamma-\beta+1)\ldots(\gamma-\beta+n-1)}{\gamma(\gamma+1)\ldots(\gamma+n-1)}$$

$$= \frac{(\gamma-\beta)_n}{(\gamma)_n} \qquad \text{...(31)}$$

This is known as *Vandermond's theorem.*

COROLLARY 5. **Kummer's theorem.**

In (27) if we put $x = -1$ and $\gamma = \beta - \alpha + 1$, we get

$$F(\alpha, \beta, \beta-\alpha+1, -1) = \frac{1}{B(\beta, \gamma-\beta)} \int_0^1 \left(1-t^2\right)^{-\alpha} t^{\beta-1}\, dt$$

Put $t^2 = y$ so that $2t\, dt = dy$

$$= \frac{\Gamma(\beta-\alpha+1)}{\Gamma\beta\Gamma(\gamma-\beta)} \int_0^1 (1-y)^{-\alpha} y^{(\beta-1)/2} \frac{dy}{2\sqrt{y}}$$

$$= \frac{1}{2} \frac{\Gamma(\beta-\alpha+1)}{\Gamma\beta\Gamma(\gamma-\beta)} \int_0^1 (1-y)^{-\alpha} y^{\beta/2-1}\, dy$$

$$= \frac{1}{2} \cdot \frac{\Gamma(\beta-\alpha+1)}{\Gamma\beta\Gamma(\gamma-\beta)} B\left(\frac{\beta}{2}, 1-\alpha\right)$$

$$= \frac{1}{2} \cdot \frac{\Gamma(\beta-\alpha+1)}{\Gamma\beta\Gamma(1-\alpha)} \cdot \frac{\Gamma\frac{\beta}{2}\Gamma(1-\alpha)}{\Gamma\left(\frac{\beta}{2}+1-\alpha\right)}$$

$\because\ \gamma = \beta - \alpha + 1$ gives, $\gamma - \beta = 1 - \alpha$

$$= \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta+1)} \cdot \frac{\Gamma\left(\frac{\beta}{2}+1\right)}{\Gamma\left(1-\alpha+\frac{\beta}{2}\right)} \qquad \text{...(32)}$$

(on dividing and multiplying by β)

Which is known as *Kummer's theorem.*

[D] Linear Relationships of Hypergeometric Functions.

If we put $1 - t = p$ in (29) we get

$$F(\alpha, \beta, \gamma, x) = \frac{1}{B(\beta, \gamma-\beta)} \int_0^1 p^{\gamma-\beta-1}(1-p)^{\beta-1}\{1 - x(1-p)\}^{-\alpha}\, dp$$

$$= \frac{(1-x)^{-\alpha}}{B(\beta, \gamma-\beta)} \int_0^1 (1-p)^{\beta-1} p^{\gamma-\beta-1} \left\{1 - \frac{xp}{x-1}\right\}^{-\alpha} dp$$

or $F(\alpha, \beta, \gamma, x) = \dfrac{(1-x)^{-\alpha}}{B(\beta, \gamma-\beta)} B(\gamma-\beta, \beta)\, F\left(\alpha, \beta-\gamma, \gamma, \dfrac{x}{x-1}\right)$ by using (28)

$$= (1-x)^{-\alpha} F\left(\alpha, \gamma-\beta, \gamma, \frac{x}{x-1}\right) \qquad \ldots(33)$$

$\because$ by symmetry property $B(\beta, \gamma-\beta) = B(\gamma-\beta, \beta)$.

Similarly it may be shown that

$$F(\alpha, \beta, \gamma, x) = (1-x)^{-\beta} F\left(\gamma-\alpha, -\beta, \gamma, \frac{x}{x-1}\right) \qquad \ldots(34)$$

With $x = \dfrac{1}{2}$, (33) yields

$$F\left(\alpha, \beta, \gamma, \frac{1}{2}\right) = 2^{\alpha} F(\alpha, \gamma-\beta, \gamma, -1) \qquad \ldots(35)$$

If $\beta = 1 - \alpha$, this gives

$$F\left(\alpha, 1-\alpha, \gamma, \frac{1}{2}\right) = 2^{\alpha} F(\alpha, \gamma+\alpha-1, \gamma, -1)\cdot$$

$$= \frac{2^{\alpha}\Gamma\gamma\,\Gamma\left(\dfrac{\gamma+\alpha-1}{2}+1\right)}{\Gamma(\gamma+\alpha)\,\Gamma\left(1-\alpha+\dfrac{\gamma+\alpha-1}{2}\right)} \text{ using (32) } \textit{i.e.} \text{ Kummer's theorem}$$

$$= \frac{2^{\alpha}\Gamma\gamma\,\Gamma\left(\dfrac{\gamma}{2}+\dfrac{1}{2}+\dfrac{\alpha}{2}\right)}{\Gamma(\gamma+\alpha)\,\Gamma\left(\dfrac{1}{2}-\dfrac{\alpha}{2}+\dfrac{\gamma}{2}\right)}$$

$$= \frac{2^{\alpha}\,\Gamma\left(\dfrac{\gamma}{2}\right)\cdot\dfrac{\gamma}{2}\left(\dfrac{\gamma}{2}+1\right)\left(\dfrac{\gamma}{2}+2\right)\ldots(\gamma-1)\cdot\Gamma\left(\dfrac{\gamma}{2}+\dfrac{1}{2}\right)\cdot\left(\dfrac{\gamma}{2}+\dfrac{1}{2}\right)\ldots\left(\dfrac{\gamma}{2}-\dfrac{1}{2}+\dfrac{\alpha}{2}\right)}{\Gamma\left(\dfrac{\gamma}{2}+\dfrac{\alpha}{2}\right)\left(\dfrac{\gamma}{2}+\dfrac{\alpha}{2}\right)\ldots(\gamma+\alpha-1)\,\Gamma\left(\dfrac{1}{2}-\dfrac{\alpha}{2}+\dfrac{\gamma}{2}\right)}$$

$$= \frac{2^{\alpha}\,\Gamma\left(\dfrac{\gamma}{2}\right)\cdot\dfrac{1}{2^{\gamma/2}}\gamma\,(\gamma+2)\ldots(2\gamma-2)\cdot\dfrac{1}{2^{\alpha/2}}\Gamma\left(\dfrac{\gamma}{2}+\dfrac{1}{2}\right)(\gamma+1)\,(\gamma+3)\ldots(\gamma-1+\alpha)}{\dfrac{1}{2^{\gamma/2-\alpha/2}}\Gamma\left(\dfrac{\gamma}{2}+\dfrac{\alpha}{2}\right)\cdot(\gamma+\alpha)\,(\gamma+\alpha+2)\ldots(2\gamma+2\alpha-2)\Gamma\left(\dfrac{1}{2}-\dfrac{\alpha}{2}+\dfrac{\gamma}{2}\right)}$$

$$= \frac{\Gamma\left(\dfrac{\gamma}{2}\right)\Gamma\left(\dfrac{\gamma+1}{2}\right)}{\Gamma\left(\dfrac{\gamma+\alpha}{2}\right)\Gamma\left(\dfrac{1-\alpha+\gamma}{2}\right)} \qquad \ldots(36)$$

(on simplifying)

Further, we have shown the solution of the hypergeometric series for $x = 0$ as

$$y = AF(\alpha, \beta, \gamma, x) + Bx^{1-\gamma}F(\alpha + 1 - \gamma, \beta + 1 - \gamma, 2 - \gamma, x) \text{ by (3)}$$

Which is convergent for $|x| \leq 1$ *i.e.*, in the interval $(-1, 1)$,
and the solution for $x = 1$ by (6) is

$$y = AF(\alpha, \beta, 1 + \alpha + \beta - \gamma, 1 - x) + B(1 - x)^{\gamma-\alpha-\beta} F(\gamma - \alpha, \gamma - \beta, \gamma - \alpha - \beta + 1, 1 - x)$$

which is convergent for $|1 - x| \leq 1$ *i.e.*, in the interval $(0, 2)$

Clearly the solution (3) and (6) are convergent in the common interval (0, 1) in which there exist a linear relationship between different hypergeometric functions.

Let the relation be

$$F(\alpha, \beta, \gamma, x) = AF(\alpha, \beta, 1 + \alpha + \beta - \gamma, 1 - x) + B(1 - x)^{\gamma-\alpha-\beta} F(\gamma - \alpha, \gamma - \beta, \gamma - \alpha - \beta + 1, 1 - x) \qquad \ldots(37)$$

If we put $x = 0$ in (37) we get by Cor. 1 of § 8.5 (C),

$$F(\alpha, \beta, \gamma, 0) = 1 = AF(\alpha, \beta, 1 + \alpha + \beta - \gamma, 1) + BF(\gamma - \alpha, \gamma - \beta, \gamma - \alpha - \beta + 1, 1)$$

$$= A\frac{\Gamma(1+\alpha+\beta-\gamma)\,\Gamma(1-\gamma)}{\Gamma(1+\beta-\gamma)\,\Gamma(1+\alpha-\gamma)} + B\frac{\Gamma(\gamma-\alpha-\beta+1)\,\Gamma(1-\gamma)}{\Gamma(1-\beta)\,\Gamma(1-\alpha)} \qquad \ldots(38)$$

by Gauss theorem (29)

Putting again, $x = 1$ in (37) we get with the help of Gauss theorem,

$$F(\alpha, \beta, \gamma, 1) = AF(\alpha, \beta, 1 + \alpha + \beta - \gamma, 0) = \frac{\Gamma\gamma\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}$$

i.e. $A = \dfrac{\Gamma\gamma\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}$ since by Cor. 1 of § 8.5 (C) it may be shown that

$$F(\alpha, \beta, 1 + \alpha + \beta - \gamma, 0) = 1.$$

Substituting this value of A in (38), we find

$$1 = \frac{\Gamma\gamma\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)} \cdot \frac{\Gamma(1+\alpha+\beta-\gamma)\,\Gamma(1-\alpha)}{\Gamma(1+\beta-\gamma)\,\Gamma(1+\alpha-\gamma)} + B\frac{\Gamma(\gamma-\alpha-\beta+1)\,\Gamma(1-\gamma)}{\Gamma(1-\beta)\,\Gamma(1-\alpha)}$$

$$= \frac{\sin\pi(\gamma-\alpha)\sin\pi(\gamma-\beta)}{\sin\pi\gamma\sin(\gamma-\alpha-\beta)} + B\frac{\Gamma(1-\gamma)\,\Gamma(1+\gamma-\alpha-\beta)}{\Gamma(1-\alpha)\,\Gamma(1-\beta)}$$

by using $\Gamma p\,\Gamma(1-p) = \dfrac{\pi}{\sin p\pi}$

$$\text{or } B = \frac{\Gamma(1-\alpha)\,\Gamma(1-\beta)}{\Gamma(1-\gamma)\,\Gamma(1+\gamma-\alpha-\beta)}\left[\frac{\sin\pi\gamma\sin\pi(\gamma-\alpha-\beta) - \sin\pi(\gamma-\alpha)\sin\pi(\gamma-\beta)}{\sin\pi\gamma\,\sin\pi(\gamma-\alpha-\beta)}\right]$$

$$= \frac{\Gamma(1-\alpha)\,\Gamma(1-\beta)\,\Gamma(\alpha+\beta-\gamma)}{\Gamma(1-\gamma)\cdot\dfrac{\pi}{\sin\pi(\alpha+\beta-\gamma)}} \cdot \frac{-\sin\pi\alpha\sin\pi\beta}{\sin\pi\gamma\sin\pi(\gamma-\alpha-\beta)}$$

$$= \frac{\Gamma(1-\alpha)\,\Gamma(1-\beta)\,\Gamma(\alpha+\beta-\gamma)}{\Gamma(1-\gamma)\;\pi} \cdot \frac{\dfrac{\pi}{\Gamma\alpha\,\Gamma(1-\alpha)} \cdot \dfrac{\pi}{\Gamma\beta\;\Gamma(1-\beta)}}{\dfrac{\pi}{\Gamma\gamma\;\Gamma(1-\gamma)}}$$

by using $\Gamma p\, \Gamma(1-p) = \dfrac{\pi}{\sin p\pi}$

$$= \frac{\Gamma\gamma\ \Gamma(\alpha+\beta-\gamma)}{\Gamma\alpha\,\Gamma\beta}$$

Now (37) yields with the substitutions for A and B

$$F(\alpha, \beta, \gamma, x) = \frac{\Gamma\gamma\ \Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)} F(\alpha, \beta, 1+\alpha+\beta-\gamma, 1-x)$$

$$+\frac{\Gamma\gamma\Gamma(\alpha+\beta-\gamma)}{\Gamma\alpha\ \Gamma\beta} x^{\gamma-\alpha-\beta}\ F(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1, 1-x)$$

...(39)

which is the required relationship.

If now replace x by $\dfrac{1}{x}$ in (39), we get

$$F\left(\alpha, \beta, \gamma, \frac{1}{x}\right) = \frac{\Gamma\gamma\ \Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\alpha-\beta)} F\left(\alpha, \beta, 1+\alpha+\beta-\gamma, 1-\frac{1}{x}\right)$$

$$+\frac{\Gamma\gamma\Gamma(\alpha+\beta-\gamma)}{\Gamma\alpha\,\Gamma\beta}\left(1-\frac{1}{x}\right)^{\gamma-\beta-\alpha} F\left(\gamma-\alpha, \gamma-\beta, x-\alpha-\beta+1, 1-\frac{1}{x}\right)$$

But from (33) we have

$$F\left(\alpha, \beta, \gamma, 1-\frac{1}{x}\right) = \left(\frac{1}{x}\right)^{-\alpha} F(\alpha, \gamma-\beta, \gamma, 1-x)$$

$$= x^{\alpha} F(\alpha, \gamma-\beta, \gamma, 1-x),$$

$$\therefore\ F\left(\alpha, \beta, \gamma, \frac{1}{x}\right) = \frac{\Gamma\gamma\ \Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)} x^{\alpha} F(\alpha, \alpha-\gamma+1, \alpha+\beta-\gamma+1, 1-x)$$

$$+\frac{\Gamma\gamma\Gamma(\alpha+\beta-\gamma)}{\Gamma\alpha\,\Gamma\beta} x^{\beta}\ (x-1)^{\gamma-\alpha-\beta}$$

$$F = (\gamma-\alpha, 1-\alpha, \gamma-\alpha-\beta+1, 1-x) \qquad ...(40)$$

where $1 < x < 2$ and $1 > y > \alpha+\beta$.

[E] Various Representations in Terms of Hypergeometric Functions.

$[e_1]$ If we put $x = (r_2 - r_1)\,u + r_1$ i.e. $u = \dfrac{x - r_1}{r_2 - r_1}$ ...(41)

i.e. $\dfrac{du}{dx} = \dfrac{1}{r_2 - r_1}$

So that $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \dfrac{1}{r_2 - r_1}\dfrac{dy}{du}$ and $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{1}{r_2 - r_1}\dfrac{dy}{du}\right)$

$$= \frac{1}{(r_2 - r_1)^2}\frac{d^2y}{du^2}, \quad r_1 \neq r_2$$

in the equation $(x - r_1)(x - r_2)\dfrac{d^2y}{dx^2} + \dfrac{b}{a}(x - r_3)\dfrac{dy}{dx} + \dfrac{c}{a}y = 0,$ ...(42)

Then we find the Gauss equation or hypergeometric equation,

$$u\,(1-u)\frac{d^2y}{du^2}+\left[\frac{b}{a}\frac{r_1-r_3}{r_1-r_2}-\frac{b}{a}u\right]\frac{dy}{du}-\frac{c}{a}y=0 \qquad \text{...(43)}$$

Its series solution by usual method or comparison with §8.3 $E(e_3)$, is

$$y=AF\,(\alpha,\beta,\gamma,u)+B\,u^{1-\gamma}F\,(\alpha-\gamma+1,\ \beta-\gamma+1,\ 2-\gamma,\ u) \qquad \text{...(44)}$$

where $\gamma=\dfrac{b\,(r_1-r_3)}{a\,(r_1-r_2)},\ \alpha+\beta+1=\dfrac{b}{a}$ and $\alpha\beta=\dfrac{c}{a}$ ...(45)

$$\therefore\ y=AF\left(\alpha,\ \beta,\ \gamma;\ \frac{x-r_1}{r_2-r_1}\right)+B\left(\frac{x-r_1}{r_2-r_1}\right)^{1-\gamma}F\left(\alpha-\gamma+1,\ \beta-\gamma+1,\ 2-\gamma,\ \frac{x-r_1}{r_2-r_1}\right) \qquad \text{...(46)}$$

COROLLARY. Tschebycheff's equation is

$$(x-1)\,(x+1)\frac{d^2y}{dx^2}+x\,\frac{dy}{dx}-n^2y=0 \qquad \text{...(47)}$$

Comparing it with (42), we get

$r_1=1,\ r_2=-1,\ r_3=0,\ a=1,\ b=1,\ c=-n^2,\ x=-2u+1,\ \alpha=n,\ \beta=-n,\ \gamma=\frac{1}{2}$

and hence by (46), the solution is

$$y=AF\left(n,\ -n;\ \tfrac{1}{2},\ \frac{1-x}{2}\right)+B\left(\frac{1-x}{2}\right)^{1/2}F\left(n+\frac{1}{2},\ -n+\frac{1}{2},\ \frac{3}{2};\ \frac{1-x}{2}\right) \qquad \text{...(48)}$$

with the help of (48) and § 8.3 $E(e_3)$, the *Tschebycheff's Polynomials* may be found as

$$\left.\begin{aligned}&T_0(x)=1,\ T_1(x)=x,\ T_2(x)=2x^2-1,\\&T_3(x)=4x^3-3x,\ T_4(x)=8x^4-8x^2+1\end{aligned}\right\} \qquad \text{...(49)}$$

$[e_2]$ Legendre's Polynomials have already been discussed in problems on Legendre's Polynomials.

$[e_3]$ **Elliptical integrals** of the first and second kind are

$$k\,(x)=\int_0^{\pi/2}\frac{d\theta}{(1-x^2\sin^2\phi)^{1/2}} \qquad \text{...(50)}$$

and $$E\,(x)=\int_0^{\pi/2}(1-x^2\sin^2\phi)^{1/2}\,d\phi \qquad \text{...(51)}$$

respectively.

We have $k\,(x)=\int_0^{\pi/2}(1-x^2\sin^2\phi)^{-1/2}\,d\phi$

$$=\int_0^{\pi/2}(1+\tfrac{1}{2}\cdot x^2\sin^2\phi+\frac{\frac{1}{2}\left(\frac{1}{2}+1\right)}{\underline{|2}}x^4\sin^4\phi+.........]\,d\phi$$

$$=\int_0^{\pi/2}\sum_{k=0}^{\infty}\frac{\frac{1}{2}\left(\frac{1}{2}+1\right)...\left(\frac{1}{2}+k-1\right)}{\underline{|k}}x^{2k}\sin^{2k}\phi\,d\phi$$

$$=\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k x^{2k}}{\underline{|k}}\int_0^{\pi/2}\sin^{2k}\phi\,d\phi$$

$$= \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k x^{2k}}{\lfloor k} \cdot \frac{\Gamma\left(2k+\frac{1}{2}\right)}{\Gamma(k+1)} \frac{\sqrt{\pi}}{2} \quad \text{by Gamma integrals}$$

$$= \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k x^{2k}}{\lfloor k} \cdot \frac{\left(\frac{1}{2}\right)_k}{\lfloor k} \frac{\pi}{2}$$

$$= \frac{\pi}{2} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_k}{(1)_k \lfloor k} \left(x^2\right)^k \quad \text{since } \lfloor k = 1.2.......(k-1) = (1)_k.$$

$$= \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}, 1;\ x^2\right) \qquad ...(52)$$

We can similarly show that

$$E(x) = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}, 1;\ x^2\right) \qquad ...(53)$$

Problem 33. *Solve in series*

$$x(1-x)y'' + 4(1-x)y' - 2y = 0.$$

Comparing it with the hypergeometric series we get

$$\alpha + \beta + 1 = 4, \gamma = 4, \alpha\beta = 2$$

i.e. $\alpha = 1, \beta = 2, \gamma = 4$ or $\alpha = 2, \beta = 1, \gamma = 4$

For the first choice the solution is $y_1 = F(1, 2, 4, x)$

$= F(2, 1, 4, x)$ by symmetry property

$$= 1 + \frac{x}{2} + \frac{3x^2}{10} + \frac{x^3}{5} + \frac{x^4}{7} + ...$$

For the second choice the solution is $y_2 = x^{-3} F(-2, -1, -2, x)$

$= x^{-3}(1-x)$ since the third term vanishes as one of $\alpha - \gamma + 2$ or $\beta - \gamma + 2$ is zero. Moreover fourth term has zero for its denominator as $\gamma = 4$

Hence the complete solution is

$$y = AF(1, 2, 4, x) + B\frac{1-x}{x^3}.$$

Problem 34. *Solve in Series*

$$(x - x^2)\frac{d^2y}{dx^2} + \left(\frac{3}{2} - 2x\right)\frac{dy}{dx} - \frac{1}{4}y = 0.$$

Comparing with Gauss equation, we have

$$\alpha + \beta + 1 = 2,\ \alpha\beta = \frac{1}{4},\ \gamma = \frac{3}{2} \text{ giving } \alpha = \beta = \frac{1}{2},\ \gamma = \frac{3}{2}.$$

So the solution is

$$y = AF(\alpha, \beta, \gamma, x) + Bx^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, x)$$

$$= AF\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, x\right) + Bx^{-\frac{1}{2}} F\left(0, 0, \frac{1}{2}, x\right)$$

$$= AF\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, x\right) + Bx^{-\frac{1}{2}} \text{ since } F\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, x\right)$$

$$= 1 + \frac{x}{6} + \frac{3x^2}{40} + \frac{5x^3}{112} + \ldots \text{and } F\left(0, 0, \frac{1}{2}, x\right) = 1.$$

Problem 35. *Transform $y'' + n^2y = 0$ to hypergeometric form by the substitution $u = \sin^2 x$ and prove that*

(*i*) $\cos nx = F\left(\frac{n}{2}, -\frac{n}{2}, \frac{1}{2}, \sin^2 x\right)$

(*ii*) $\sin nx = n \sin x \; F\left(\frac{1}{2} - \frac{1}{2}n, \frac{1}{2} + \frac{1}{2}n, \frac{3}{2}; \sin^2 x\right)$

for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

Given equation is $\quad y'' + n^2 y = 0 \qquad \ldots(1)$

Also given that $\quad u = \sin^2 x \qquad \ldots(2)$

which gives $\frac{du}{dx} = 2 \sin x \cos x = \sin 2x$

So that $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \sin 2x \frac{dy}{du}$ and $\frac{d^2y}{dx^2} = 2\cos 2x \frac{dy}{du} + \sin^2 2x \frac{d^2y}{du^2}$

Substituting these values in (1),

$$4 \sin^2 x \cos^2 x \frac{d^2y}{du^2} + 2(1 - 2\sin^2 x)\frac{dy}{du} + n^2 y = 0$$

or

$$u(1-u)\frac{d^2y}{du^2} + \left(\frac{1}{2} - u\right)\frac{dy}{du} + \frac{n^2}{4}y = 0 \qquad \ldots(3)$$

which is hypergeometric form, with $\gamma = \frac{1}{2}$, $\alpha + \beta + 1 = 1$, $\alpha\beta = -\frac{n^2}{4}$.

i.e. $\alpha = \frac{n}{2}$, $\beta = -\frac{n}{2}$, $r = \frac{1}{2}$ and hence the solution is

$$y = AF\left(\frac{n}{2}, -\frac{n}{2}, \frac{1}{2}, u\right) + Bu^{1/2} F\left(\frac{1+n}{2}, \frac{1-n}{2}, \frac{3}{2}, u\right)$$

$$= AF\left(\frac{n}{2}, -\frac{n}{2}, \frac{1}{2}, \sin^2 x\right) + B \sin x F\left(\frac{1+n}{2}, \frac{1-n}{2}, \frac{3}{2}, \sin^2 x\right) \qquad \ldots(4)$$

where $|\sin x| < 1$ *i.e.* $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

But $\sin nx$ and $\cos nx$ are the solutions of (1) and so of (3). We can therefore take from (4).

$$\sin nx = AF\left(\frac{n}{2}, -\frac{n}{2}, \frac{1}{2}, \sin^2 x\right) + B \sin x F\left(\frac{1+n}{2}, \frac{1-n}{2}, \frac{3}{2}, \sin^2 x\right) \qquad \ldots(5)$$

When $x = 0$, $A = 0$ and so $\frac{\sin nx}{\sin x} = BF\left(\frac{1+n}{2}, \frac{1-n}{2}, \frac{3}{2}, \sin^2 x\right)$ (by 5)

Which follows that when $x = 0$, $B = n$ and hence

$$\sin nx = n \sin x F\left(\frac{1-n}{2}, \frac{1+n}{2}, \frac{3}{2}, \sin^2 x\right) \qquad \ldots(6)$$

Also, $\cos nx = AF\left(\frac{n}{2}, -\frac{n}{2}, \frac{1}{2}, u\right) + B\sqrt{u}\,F\left(\frac{1+n}{2}, \frac{1-n}{2}, \frac{3}{2}, u\right)$

where $u = \sin^2 x$ ...(7)

when $x = 0, A = 1$

and differentiating (7) w.r.t. x, we find

$$-n \sin nx = A(-n^2)\, F\left(\frac{n+2}{2}, \frac{2-n}{2}, \frac{3}{2}, u\right)\frac{du}{dx} + B\sqrt{u}\,\frac{1-n^2}{6}$$

$$F\left(\frac{3+n}{2}, \frac{3-n}{2}, \frac{5}{2}, u\right)\frac{du}{dx} + BF\left(\frac{1+n}{2}, \frac{1-n}{2}, \frac{3}{2}, u\right)\frac{1}{2\sqrt{u}}\frac{du}{dx}$$

by §8.5 [*C*] (2).

Where $u = \sin^2 x$ and $\frac{du}{dx} = \sin 2x$

When $x = 0, B = 0$

So that $\cos nx = F\left(\frac{n}{2}, -\frac{n}{2}, \frac{1}{2}, \sin^2 x\right)$ ...(8)

8.6. CONFLUENT HYPERGEOMETRIC EQUATIONS AND FUNCTIONS

[A] Confluent Hypergeometric Differential Equation.

We have $x(1-x)\frac{d^2y}{dx^2} + \{\gamma - (\alpha + \beta + 1)\, x\}\frac{dy}{dx} - \alpha\beta y = 0.$ ...(1)

Replacing x by $\frac{x}{\beta}$ we have

$$x\left(1 - \frac{x}{\beta}\right)\frac{d^2y}{dx^2} + \left\{\gamma - \left(1 + \frac{\alpha+1}{\beta}\right)x\right\}\frac{dy}{dx} - \alpha y = 0 \quad ...(2)$$

Its solution as compared with the hypergeometric series, is

$$F\left(\alpha, \beta, \gamma, \frac{x}{\beta}\right).$$

Hence if we let $\beta \to \infty$; then (2) reduces to

$$x\frac{d^2y}{dx^2} + (\gamma - x)\frac{dy}{dx} - \alpha y = 0 \quad ...(3)$$

Whose solution is $\lim\limits_{\beta \to \infty} F\left(\alpha, \beta, \gamma, \frac{x}{\beta}\right)$ ...(4)

We call the equation (3) as *the confluent hyper-geometric differential equation.*

Now since $(\beta)_k = \beta(\beta+1)(\beta+2)...(\beta+k-1)$, therefore

$$\lim_{\beta \to \infty} \frac{(\beta)_k}{\beta^k} = \lim_{\beta \to \infty}\left(1 + \frac{1}{\beta}\right)\left(1 + \frac{2}{\beta}\right)...\left(1 + \frac{k-1}{\beta}\right) = 1$$

As such (4) may be represented as

$$\underset{\beta\to\infty}{\text{Lim}}\, F\left(\alpha, \beta, \gamma, \frac{x}{\beta}\right) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\gamma)_k} \cdot \frac{x^k}{\lfloor k} = F(\alpha, \gamma, x) \text{ (say)} \quad \ldots(5)$$

Here $F(\alpha, \gamma, x)$ or sometimes, ${}_1F_1(\alpha, \gamma, x)$ showing that there is one parameter of the type α and one of the type γ, is the solution of (3) and is said to be the *confluent hypergeometric function.*

Note. *Since equation (3) has a removable (non-essential) singularity at $x = 0$ so its solution may be developed directly by series method about $x = 0$, choosing the series as* $y = \sum_{r=0}^{\infty} a_r x^{k+r}$.

Substituting for y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ in (3) and equating the coefficient of first and the general term (*i.e.* x^{k+r}) to zero, it is easy to find that indicial equation $k(k+\gamma-1) = 0$ giving $k = 0, 1-\gamma$ when $a_0 \neq 0$ ...(6)

and the recurrence relation as $a_{r+1} = \frac{k+r+\alpha}{(k+r+1)(k+r+\gamma)} a_r$ ...(7)

which for $k = 0$ gives

$$y = a_0 \left[1 + \frac{\alpha}{\gamma} x + \frac{\alpha(\alpha+1)\, x^2}{1\cdot 2\cdot \gamma\cdot (\gamma+1)} x^2 + \ldots\right]$$

$$= a_0 \sum_{k=0}^{\infty} \frac{(\alpha)_k}{\lfloor k\, (\gamma)_k} x^k = a_0 F(\alpha, \gamma, x) \quad \ldots(8)$$

and for $k = 1-\gamma$ gives

$$y = a_0{}^{1-\gamma}\left[1 + \frac{\alpha'}{\gamma'} x + \frac{\alpha'(\alpha'+1)}{1\cdot 2\, \gamma'(\gamma'+1)} x^2 + \ldots\right]$$

where $\alpha' = \alpha - \gamma + 1$ and $\gamma' = 2 - \gamma$

$$= a_0 x^{1-\gamma} \sum_{k=0}^{\infty} \frac{(\alpha')_k}{\lfloor k\, (\gamma')_k} x^k = a_0 x^{1-\gamma} F(\alpha', \gamma', x)$$

$$= a_0 x^{1-\gamma} F(\alpha x - 1, 2-\gamma, x) \quad \ldots(9)$$

where $F(\alpha - \gamma + 1, 2 - \gamma, x)$ is said to be the confluent hypergeometric function of the second kind.

The general solution is therefore,

$$y = AF(\alpha, \gamma, x) + Bx^{1-\gamma} F(a - \gamma + 1, 2 - \gamma, x) \quad \ldots(10)$$

It is valid for $\gamma > 0$.

By ratio test

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(\alpha)_{n+1}\, \lfloor n}{(\alpha)_n\, \lfloor n+1} \cdot \frac{(\gamma)_n}{(\gamma)_{n+1}}\, x\right| = \left|\frac{\alpha + n}{(\gamma+n)(n+1)}\, x\right|$$

which $\to 0$ as $n \to \infty$ *i.e.* $\left|\frac{u_{n+1}}{u_n}\right| < 1$ for all x, it is obvious that the series is convergent.

[B] Simple Properties of Confluent Hypergeometric Functions.

[1] Differentiation of Confluent Hypergeometric Functions i.e.

$$\frac{d}{dx}F(\alpha,\gamma,x)=\frac{\alpha}{\gamma}F(\alpha+1,\gamma+1,x) \quad \ldots(11)$$

which may be similarly shown as in case of hypergeometric functions.

$$\frac{d^2}{dx^2}F(\alpha,\gamma,x)=\frac{\alpha}{\gamma}\left(\frac{\alpha+1}{\gamma+1}\right)\cdot F(\alpha+2,\gamma+2,x) \quad \ldots(12)$$

and in general

$$\frac{d^m}{dx^m}F(\alpha,\gamma,x)=\frac{(\alpha)_m}{(\gamma)_m}F(\alpha+m,\gamma+m,x) \quad \ldots(13)$$

When $x=0$, $F(\alpha,\gamma,x)=1$ ...(14)

$$\left[\frac{d}{dx}F(\alpha,\beta,x)\right]_{x=0}=\frac{\alpha}{\gamma} \quad \ldots(15)$$

and

$$\left[\frac{d^m}{dx^m}F(\alpha,\gamma,x)\right]_{x=0}=\frac{(\alpha)_m}{(\gamma)_m} \quad \ldots(16)$$

In case α is a negative integer, the series stops after a certain number of terms but when γ is also negative then the series restarts after a certain number of terms, as shown in case of hypergeometric functions.

[2] Integral Formulae for the Confluent Hypergeometric Function.

We have $$\frac{(\alpha)_k}{(\gamma)_k}=\frac{1}{B(\alpha,\gamma-\alpha)}\int_0^1(1-t)^{\gamma-\alpha-1}t^{\alpha+k-1}\,dt \quad \ldots(17)$$

Just in §8.5 [C-3]

$$\therefore\ F(\alpha,\gamma,x)=\sum_{k=0}^{\infty}\frac{(\alpha)_k x^k}{\lfloor k\,(\gamma)_k}$$

$$=\sum_{k=0}^{\infty}\frac{x^k}{\lfloor k}\cdot\frac{1}{B(\alpha,\gamma-\alpha)}\int_0^1(1-t)^{\gamma-\alpha-1}t^{\alpha+k-1}\,dt$$

$$=\frac{1}{B(\alpha,\gamma-\alpha)}\int_0^1(1-t)^{\gamma-\alpha-1}t^{\alpha-1}\sum_{k=0}^{\infty}\frac{(xt)^k}{\lfloor k}$$

$$=\frac{1}{B(\alpha,\gamma-\alpha)}\int_0^1 e^{xt}t^{\alpha-1}(1-t)^{\gamma-\alpha-1}\,dt$$

$$\because\ \sum_{k=0}^{\infty}\frac{(xt)^k}{\lfloor k}=1+\frac{xt}{\lfloor 1}+\frac{x^2t^2}{\lfloor 2}+\ldots=e^{xt}$$

$$=\frac{\Gamma\gamma}{\Gamma\alpha\ \Gamma(\gamma-\alpha)}\int_0^1 e^{xt}t^{\alpha-1}(1-t)^{\gamma-\alpha-1}\,dt \quad \ldots(18)$$

using $B(l,m)=\dfrac{\Gamma l\,\Gamma m}{\Gamma(l+m)}$

which is the integral formula for the confluent hypergeometric function.

COROLLARY 1. In (18) if we put $t = 1 - p$, then

$$F(\alpha, \gamma, x) = \frac{\Gamma\gamma}{\Gamma\alpha\,\Gamma(\gamma-\alpha)}\int_0^1 e^{x(1-p)}(1-p)^{\alpha-1}p^{\gamma-\alpha-1}\,dp$$

$$= \frac{\Gamma\gamma}{\Gamma\alpha\,\Gamma(\gamma-\alpha)}e^x\int_0^1 e^{-xp}p^{\gamma-\alpha-1}(1-p)^{\alpha-1}\,dp$$

$$= \frac{\Gamma\gamma}{\Gamma\alpha\,\Gamma(\gamma-\alpha)}e^x\frac{\Gamma(\gamma-\alpha)\,\Gamma\alpha}{\Gamma\gamma}\;F(\gamma-\alpha, \gamma, -x) \text{ using (18)}$$

$$= e^x F(\gamma-\alpha, \gamma, -x) \qquad ...(19)$$

COROLLARY 2. In (18) if we put $x = 0$, then we find

$$F(\alpha, \gamma, 0) = \frac{\Gamma\gamma}{\Gamma\alpha\,\Gamma(\gamma-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}\,dt$$

$$= \frac{\Gamma\gamma}{\Gamma\alpha\,\Gamma(\gamma-\alpha)}B(\alpha, \gamma-\alpha)$$

$$= \frac{\gamma}{\Gamma\alpha\,\Gamma(\gamma-\alpha)}\cdot\frac{\Gamma\alpha\,\Gamma(\gamma-\alpha)}{\Gamma\gamma}$$

$$= 1$$

i.e. $F(\alpha, \gamma, 0) = 1$ ...(20)

[C] Various Representations in terms of Confluent Hypergeometric Functions.

[c_1] Elementary Functions.

We have $F(\alpha, \gamma, x) = \sum_{k=0}^{\infty}\frac{(\alpha)_k}{\lfloor k\;(\gamma)_k}x^k$

Putting $\gamma = \alpha$, we have $F(\alpha, \alpha, x) = \sum_{k=0}^{\infty}\frac{x^k}{\lfloor k} = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + ...$

$$= e^x \qquad ...(21)$$

Similarly,

$$F(\alpha+1, \alpha, x) = \left(1 + \frac{x}{\alpha}\right)e^x \qquad ...(22)$$

$$F(1, 2, x) = \sum_{k=0}^{\infty}\frac{(1)_k x^k}{(2)_k \lfloor k} = \sum_{k=0}^{\infty}\frac{x^k}{\lfloor k+1}$$

$$= \frac{1}{x}\left[x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + ...\right]$$

$$= \frac{e^x - 1}{x} \qquad ...(23)$$

$$F(-2, 1, x) = 1 - 2x + \frac{x^2}{2}$$

$$F(n, n+1, -x) = n\int_0^x t^{n-1}e^{-t}\,dt \qquad ...(24)$$

[c_2] **Error Function.**

We have

$$erf\,(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-n^2} dn \quad (\textit{see Ch. VI})$$

$$= \frac{2}{\sqrt{\pi}} \int_0^x \sum_{k=0}^{\infty} (-1)^k \frac{n^{2k}}{\lfloor k} dn$$

$$= \frac{2}{\sqrt{\pi}} \int_0^x \sum_{k=0}^{\infty} \frac{(-1)^k}{\lfloor k} \int_0^x n^{2k} dn$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{\lfloor k} \frac{x^{2k+1}}{(2k+1)}$$

$$= x \sum_{k=0}^{\infty} \frac{(-1)^k}{\lfloor k} \frac{x^{2k}}{(2k+1)}$$

$$= \frac{2x}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-x^2)^k}{\lfloor k\,(2k+1)}$$

$$= \frac{2x}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k (-x^2)^k}{\lfloor k \left(\frac{3}{2}\right)_k}$$

$$\because \quad 2k+1 = \frac{(2k+1)(2k-1)(2k-3)\ldots 1}{(2k-1)(2k-3)\ldots 1}$$

$$= \frac{\frac{1}{2}\cdot\frac{3}{2}\ldots\frac{2k-1}{2}\cdot\frac{2k+1}{2}}{\left(\frac{1}{2}\cdot\frac{3}{2}\ldots\frac{2k-1}{2}\cdot\frac{2k-3}{2}\right)\cdot\frac{1}{2}}$$

$$= \frac{\frac{3}{2}\left(\frac{3}{2}+1\right)\ldots\frac{2k+1}{2}}{\frac{1}{2}\left(\frac{1}{2}+1\right)\ldots\frac{2k-1}{2}} = \frac{\left(\frac{3}{2}\right)_k}{\left(\frac{1}{2}\right)_k}$$

$$= \frac{2x}{\sqrt{\pi}} F\left(\frac{1}{2}, \frac{3}{2}, -x^2\right) \qquad \ldots(25)$$

[c_3] **Fresnel's integrals.** The first and second kinds are respectively,

$$C(x) = \frac{1}{2\sqrt{2}}\left[e^{\pi i/4} \phi\left(\frac{\sqrt{\pi}}{\sqrt{2}} x e^{-\pi i/4}\right) + e^{-\pi i/4} \phi\left(\frac{\sqrt{\pi}}{\sqrt{2}} x e^{\pi i/4}\right)\right] \qquad \ldots(26)$$

and

$$S\,(x) = \frac{1}{2i\sqrt{2}}\left[e^{\pi i/4} \phi\ \sqrt{\left(\frac{\pi}{2}\right)} x e^{-\pi i/4} - e^{-\pi i/4} \phi\ \sqrt{\frac{\pi}{2}}\, e^{\pi i/4}\right] \qquad \ldots(27)$$

where $\phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-n^2} dx$

by $[c_2]$, $\phi\left\{\sqrt{\frac{\pi}{2}}\, xe^{-\pi i/4}\right\} = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{\pi/2}\, xe^{-\pi i/4}} e^{-n^2}\, dn.$

$$= \sqrt{2}\, xe^{-\pi i/4} \sum_{k=4}^{\infty} \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{2}\pi i x^2\right)_k}{\left(\frac{3}{2}\right)_k \lfloor k}$$

$$= \sqrt{2}\, xe^{-\pi i/4}\, F\left(\frac{1}{2}, \frac{3}{2}, -\frac{\pi i x^2}{2}\right)$$

Similarly $\phi\left\{\sqrt{\frac{\pi}{2}}\, xe^{\pi i/4}\right\} = \sqrt{2x}\, e^{\pi i/4}\, F\left(\frac{1}{2}, \frac{3}{2}, \frac{\pi i x^2}{2}\right)$

Hence $$C(x) = \frac{x}{2}\left[F\left(\frac{1}{2}, \frac{3}{2}, \frac{\pi i x^2}{2}\right) + F\left(\frac{1}{2}, \frac{3}{2}, \frac{-\pi i x^2}{2}\right)\right] \quad ...(28)$$

and $$S(x) = \frac{x}{2i}\left[F\left(\frac{1}{2}, \frac{3}{2}, \frac{\pi i x^2}{2}\right) - F\left(\frac{1}{2}, \frac{3}{2}, \frac{-\pi i x^2}{2}\right)\right] \quad ...(29)$$

$[c_4]$ Bessel (Cylindrical) Functions.

We have $$J_n(x) = \frac{\left[\frac{x}{2}\right]^n}{\Gamma\frac{1}{2}\, \Gamma\left[n+\frac{1}{2}\right]} \int_{-1}^{1} (1-t^2)^{n-\frac{1}{2}} e^{ixt}\, dt$$

Put $2p = 1 + t$, $$J_n(x) = \frac{2^{2n}\left[\frac{x}{2}\right]^n e^{-in}}{\Gamma\frac{1}{2}\, \Gamma\left[n+\frac{1}{2}\right]} \int_0^1 e^{2xip}\, p^{n-\frac{1}{2}} (1-p)^{n-\frac{1}{2}}\, dp$$

$$= \frac{\left[\frac{x}{2}\right]^n}{\Gamma(n+1)} e^{-ix}\, F\left[n+\frac{1}{2}, 2n+1, 2ix\right] \quad ...(30)$$

Since by putting $\alpha = n + \frac{1}{2}, \gamma = 2n + 1$ and x for $2ix$ in

$$F(\alpha, \gamma, x) = \frac{\Gamma\gamma}{\Gamma\alpha\, \Gamma(\gamma-\alpha)} \int_0^1 e^{xt}\, t^{\alpha-1}(1-t)^{\gamma-\alpha-1}\, dt,$$

we get

$$F\left[n+\frac{1}{2}, 2n+1, 2ix\right] = \frac{\Gamma(2n+1)}{\Gamma\left[n+\frac{1}{2}\right]\Gamma\left[n+\frac{1}{2}\right]} \int_0^1 e^{2ixt}\, t^{n-\frac{1}{2}}(1-t)^{n-\frac{1}{2}}\, dt$$

Problem 36. *Solve in series the* **Whittakar Equation**

$$\frac{d^2W}{dx^2}+\left\{-\frac{1}{4}+\frac{k}{x}+\frac{\frac{1}{4}-m^2}{x^2}\right\}W(x)=0$$

and represent its functions in terms of confluent hypergeometric functions.

Taking $W(x)=x^{\gamma/2}e^{-x/2}y(x)$, $k=\frac{\gamma}{2}-\alpha,\ m=-\frac{1}{2}+\frac{\gamma}{2}$, the equation reduces to

$$x\frac{d^2y}{dx^2}+(\gamma-x)\frac{dy}{dx}-\alpha y=0$$

Which is confluent hypergeometric equation and hence its functions by putting $\alpha=\frac{1}{2}-k+m$ and $\gamma=1+2m$ in $F(\alpha,\gamma,x)$ and $x^{\gamma-1}F(1+\alpha-\gamma,2-\gamma,x)$ are $F(\frac{1}{2}-k+m,1+2m,x)$ and $x^{-2m}F(\frac{1}{2}-k-m,1-2m,x)$ respectively.

Thus the solution of given equation is

$$y=AF\left[\frac{1}{2}-k+m,\ 1+2m,\ x\right]+Bx^{-2m}F\left[\frac{1}{2}-k-m,\ 1-2m,\ x\right]$$

where A and B are arbitrary constants.

Also Whittakar's functions are

$$W_{k,m}(x)=x^{\frac{1}{2}+m}e^{-x/2}F\left(\frac{1}{2}-k+m,\ 1+2m,\ x\right)$$

and $$W_{k,-m}(x)=x^{\frac{1}{2}-m}e^{-x/2}F\left[\frac{1}{2}-k-m,\ 1-2m,\ x\right]$$

8.7. HERMITE EQUATION, FUNCTIONS AND POLYNOMIALS

[A] Hermite's Differential Equation.

This equation is of the form

$$\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2vy=0 \qquad ...(1)$$

where v is a parameter.

Suppose its series solution is

$$y=\sum_{r=0}^{\infty}a_r x^{k+r},\quad a_0\neq 0 \qquad ...(2)$$

So that $\frac{dy}{dx}=\sum_{r=0}^{\infty}a_r(k+r)x^{k+r-1}$,

and $$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty}a_r(k+r)(k+r-1)x^{k+r-2}$$

Substituting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get the identity

$$\sum_{r=0}^{\infty}[(k+r)(k+r-1)x^{k+r-2}-2(k+r-v)x^{k+r}]a_r\equiv 0 \qquad ...(3)$$

Equating the coefficient of the first term (*i.e.* x^{k-2}) (by putting $r = 0$), to zero, we get

$$a_0 k(k-1) = 0 \text{ giving } k = 0, 1 \text{ as } a_0 \neq 0 \qquad \ldots(4)$$

Now equating to zero the coefficient of second term (*i.e.* x^{k-1}) in (3) we get

$a_1 k(k+1) = 0$ *i.e.* $a_1 = 0$ when $k = -1$ and a_1 may or may not be zero when $k = 0$.

Also equating the coefficient of x^{k+r} to zero, we find

$$a_{r+2}(k+r+2)(k+r+1) - 2a_r(k+r-\nu) = 0$$

giving the recurrence relation

$$a_{r+2} = \frac{2(k+r-\nu)}{(k+r+2)(k+r+1)} a_r \qquad \ldots(5)$$

when $k = 0$, (5) becomes $a_{r+2} = \dfrac{2(r-\nu)}{(r+2)(r+1)} a_r \qquad \ldots(6)$

and when $k = 1$, (5) becomes $a_{r+2} = \dfrac{2(1+r-\nu)}{(r+3)(r+2)} a_r \qquad \ldots(7)$

Case I. When $k = 0$, putting $r = 0, 1, 2, 3, \ldots$ in (6) we have

$$a_2 = -\frac{2}{\underline{\lfloor 2}} \nu a_0; \quad a_3 = -\frac{2(\nu-1)}{\underline{\lfloor 3}} a_1$$

$$a_4 = -\frac{2^2 \nu(\nu-2)}{\underline{\lfloor 4}} a_0; \quad a_5 = \frac{2^2(\nu-1)(\nu-3)}{\underline{\lfloor 5}} a_1$$

....

$$a_{2r} = \frac{(-2)^r \nu(\nu-2)\ldots(\nu-2r+2)}{\underline{\lfloor 2r}} a_0; \quad a_{2r+1} = \frac{(-2)^r(\nu-1)(\nu-3)\ldots(\nu-2r+1)}{\underline{\lfloor 2r+1}} a_1$$

Now if $a_1 = 0$, then $a_3 = a_5 = a_7 = a_{2r+1} = \ldots = 0$.

But if $a_1 \neq 0$, then (2) gives for $k = 0$, $y = \sum_{r=0}^{\infty} a_r x^r$

$$\begin{aligned}
\text{i.e. } y &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \\
&= a_0 + a_2 x^2 + a_4 x^4 + \ldots + a_1 x + a_3 x^3 + a_5 x^5 + \ldots \\
&= a_0\left[1 - \frac{2\nu}{\underline{\lfloor 2}} x^2 + \frac{2^2 \nu(\nu-2)}{\underline{\lfloor 4}} x^4 - \ldots + (-1)^r \frac{2^r}{\underline{\lfloor 2r}} \nu(\nu-2)\ldots(\nu-2r+2) x^{2r} + \ldots\right] \\
&\quad + a_1 x\left[1 - \frac{2(\nu-1)}{\underline{\lfloor 3}} x^2 + \frac{2^2(\nu-1)(\nu-3)}{\underline{\lfloor 5}} - \ldots\right. \\
&\quad \left. + (-1)^r \frac{2^r}{\underline{\lfloor 2r+1}} (\nu-1)(\nu-3)\ldots(\nu-2r+1) x^{2r} + \ldots\right] \qquad \ldots(8) \\
&= a_0\left[1 + \sum_{r=1}^{\infty} \frac{(-1)^r 2^r}{\underline{\lfloor 2r}} \nu(\nu-2)\ldots(\nu-2r+2) x^{2r}\right] \\
&\quad + a_0\left[x + \sum_{r=1}^{\infty} \frac{(-1)^r 2^r}{2r+1} (\nu-1)(\nu-3)\ldots(\nu-2r+2) x^{2r+1}\right] \qquad \ldots(9)
\end{aligned}$$

Case II. When $k = 1$, then $a_1 = 0$ and so by putting $r = 0, 1, 2, 3, \ldots$ in (7) we find

$$a_2 = -\frac{2\,(\nu-1)}{\lfloor 3}a_0$$

$$a_4 = \frac{2^2\,(\nu-1)\,(\nu-3)}{\lfloor 5}a_0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{2r} = (-1)^r\frac{2^r\,(\nu-1)\,(\nu-3)\ldots(\nu-2r+1)}{\lfloor 2r+1}a_0$$

Hence the solution is

$$= a_0 x\left[1 - \frac{2(\nu-1)}{\lfloor 3}x^2 + \frac{2^2(\nu-1)(\nu-3)}{\lfloor 5}x^4 \ldots + \frac{(-1)^r 2^r(\nu-1)(\nu-3)\ldots(\nu-2r+1)}{\lfloor 2r+1}x^{2r} + \ldots\right. \qquad \ldots(10)$$

clearly the solution (10) is included in the second part of (8) except that a_0 is replaced by a_1 and hence in order that the Hermite equation may have two independent solutions, a_1 must be zero, even if $k = 0$ and then (8) reduces to

$$y = a_0\left[1 - \frac{2\nu}{\lfloor 2}x^2 + \frac{2^2\nu(\nu-2)}{\lfloor 4}x^4 - \ldots\right.$$

$$\left. + (-1)^r\,\frac{2^r}{\lfloor 2r}\,\nu(\nu-2)\ldots(\nu-2r+2)\;x^{2r} + \ldots\right] \qquad \ldots(11)$$

The complete integral of (1) is then given by

$$y = A\left[1 - \frac{2\nu}{\lfloor 2}\,x^2 + \frac{2^2\,\nu(\nu-2)}{\lfloor 4}x^4 - \ldots\right] + B\left[1 - \frac{2(\nu-1)}{\lfloor 3}x^2\right.$$

$$\left. + \frac{2^2\,(\nu-1)\,(\nu-3)}{\lfloor 5}x^4 \ldots\right] \qquad \ldots(12)$$

where A and B are arbitrary constants.

[B] Hermite Polynomials.

The Hermite polynomial $H_n(x)$ is defined as

$$f(x,\,t) = e^{2tx-t^2} = \sum_{n=0}^{\infty} H_n\,(x)\,\frac{t^n}{\lfloor n}\,, \qquad \ldots(13)$$

For all integral values of n and all real values of x (13) can be written as

$$f(x,\,t) = e^{x^2}\,e^{-(x-t))^2} = \frac{H_0\,(x)}{\lfloor 0} + \frac{H_1\,(x)}{\lfloor 1}\,t + \frac{H_2(x)}{\lfloor 2}\,t^2 + \ldots + \frac{H_n\,(x)}{\lfloor n}t^n + \ldots$$

So that
$$\left[\frac{\partial^n f(x,\,t)}{\partial f^n}\right]_{t=0} = \frac{H_n\,(x)}{\lfloor n}\,\lfloor n = H_n(x)$$

$$= \left[\frac{\partial^n}{\partial t^n}e^{-(x-t)^2}\right]_{t=0} e^{x^2} \qquad \ldots(14)$$

If we put $x - t = p$ i.e. $t = x - p = 0$ then for $t = 0, x = p$

and $\frac{\partial}{\partial t} = -\frac{\partial}{\partial p}$ so that $\frac{\partial^n}{\partial t^n}\left\{e^{-(x-t)^2}\right\} = (-1)^n \frac{\partial^n}{\partial p^n} e^{-p^2}$

$$\therefore \left[\frac{\partial^n}{\partial t^n} e^{-(x-t)^2}\right]_{t=0} = (-1)^n \frac{\partial^n}{\partial x^n} e^{-x^2} = (-1)^n \frac{d^n}{dx^n} e^{-x^2} \qquad \text{...(15)}$$

From (14) and (15), we therefore have

$$H_n(x) = e^{x^2} (-1)^n \frac{d^n}{dx^n}\left(e^{-x^2}\right) \qquad \text{...(16)}$$

From (16) we can calculate Hermite polynomials of various degrees such as

$$\left.\begin{array}{ll} H_0(x) = 1 & H_4(x) = 16x^4 - 48x^2 + 12 \\ H_1(x) = 2x & H_5(x) = 32x^5 - 160x^3 + 120x \\ H_2(x) = 4x^2 - 2 & H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120 \\ H_3(x) = 8x^3 - 12x & H_7(x) = 128x^7 - 1344x^5 + 3360x^3 \\ & \qquad -1680x \end{array}\right\} \qquad \text{...(17)}$$

[C] Hermite Polynomial in Terms of Confluent Hypergeometric Functions.

We have $\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{\lfloor n} = e^{2tx - t^2} = e^{2tx} \cdot e^{-t^2}$

$$= \left[1 + 2tx + \frac{(2x)^2}{\lfloor 2} t^2 + \ldots + \frac{(2x)^n}{\lfloor n} t^n + \ldots\right] \times \left[1 - \frac{t^2}{\lfloor 1} + \frac{(t^2)^2}{\lfloor 2} - \ldots + \frac{(-1)^n t^{2n}}{\lfloor n} + \ldots\right]$$

Equating the coefficients of $\frac{t^n}{\lfloor n}$ on either we get

$$H_n(x) = (2x)^n - \frac{n(n-1)}{\lfloor 2}(2x)^{n-2} + \frac{n(n-1)(n-2)(n-2)(n-3)}{\lfloor 2}(2x)^{n-4} + \ldots$$

$$= (2x)^n F\left(-\frac{n}{2}, \frac{1}{2} - \frac{n}{2}, -\frac{1}{x^2}\right) \qquad \text{...(18)}$$

Aliter, we have $\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{\lfloor n} = e^{2tx - t^2} = e^{2tx} e^{-t^2}$

$$= \sum_{n=0}^{\infty} \frac{(2x)^n t^n}{\lfloor n} \times \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{\lfloor n}$$

$$\therefore H_n(x) = \sum_{k=0}^{\infty} \sum_{k=0}^{n/2} (-1)^k \frac{\lfloor n \,(2x)^{n-2k}}{\lfloor k \,\lfloor n-2k} \qquad \text{...(19)}$$

Even Hermite polynomials are, therefore

$$H_{2n}(x)=\sum_{k=0}^{n}(-1)^k\frac{\lfloor 2n}{\lfloor k\lfloor 2n-2k}\cdot(2x)^{2n-2k}$$

$$=\lfloor 2n\ (-1)^n\sum_{k=0}^{n}\frac{(-1)^k\ (2x)^{2k}}{\lfloor n-k\lfloor 2k}=(-1)^n\frac{\lfloor 2n}{\lfloor n}\sum_{k=0}^{n}(-n)_k\ \frac{(2x)^{2k}}{\lfloor 2k}$$

$$=(-1)^n\frac{\lfloor 2n}{\lfloor n}\sum_{k=0}^{\infty}\frac{(-n)_k\ (x^2)^k}{\left(\frac{1}{2}\right)_k\ \lfloor k}\quad\because\quad \lfloor 2k=2^{2k}\left(\frac{1}{2}\right)_k\ \lfloor k$$

$$=(-1)^n\frac{\lfloor 2n}{\lfloor n}F\left(-n,\ \frac{1}{2};\ x^2\right)\qquad\ldots(20)$$

Similarly odd Hermite polynomials are

$$H_{2n+1}(x)=\sum_{k=0}^{(2n+1)/2}\frac{(-1)^k\ \lfloor 2n+1\,(2x)^{2n+1-2k}}{\lfloor k\lfloor 2n+1-2k}$$

$$=(-1)^n\frac{\lfloor 2n+1}{\lfloor n}2x\sum_{k=0}^{n}\frac{(-n)_k x^{2k}}{\left(\frac{3}{2}\right)_k\lfloor k}$$

$$=(-1)^n\ \frac{\lfloor 2n+1}{\lfloor n}2xF\left(-n,\ \frac{3}{2};\ x^2\right)\qquad\ldots(21)$$

[D] Recurrence formulae for $H_n(x)$ and to show that $H_n(x)$ is a solution of Hermite Equation.

Hermite equation is $y''-2xy'+2ny=0$ for integral values taking $\nu=n$.

Also, $$e^{2xt-t^2}=\sum_{n=0}^{\infty}\frac{H_n(x)\ t^n}{\lfloor n}\qquad\ldots(22)$$

I. Differentiating partially w.r.t. x, we have

$$2t\ e^{-2xt-t^2}=\sum_{n=0}^{\infty}H_n'(x)\ \frac{t^n}{\lfloor n}$$

i.e. $$2t\sum_{n=0}^{\infty}\frac{H_n(x)t^n}{\lfloor n}=\sum_{n=0}^{\infty}H_n'(x)\frac{t^n}{\lfloor n}$$

which yields on equating the coefficients of $\frac{t^n}{\lfloor n}$ on either side,

$$2\ \frac{\lfloor n}{\lfloor n-1}\ H_{n-1}(x)=H_n'\ (x)$$

i.e. $\quad 2n\,H_{n-1}(x)=H_n'(x)\qquad\ldots(23)$

II. Differentiating partially w.r.t. 't', both sides of (22) we get

$$2\ (x-t)\ e^{2xt-t^2}=\sum_{n=1}^{\infty}H_n(x)\ \frac{t^{n-1}}{\lfloor n-1}\quad\because n=0 \text{ corresponds to the vanishing of R.H.S}$$

or $2x\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{\lfloor n} - 2\sum_{n=0}^{\infty} H_n(x)\frac{t^{n+1}}{\lfloor n} = \sum_{n=1}^{\infty} H_n(x)\frac{t^{n-1}}{\lfloor n-1}$

Equating the coefficients of t^n on either side we find

$$2x\frac{H_n(x)}{\lfloor n} - 2\frac{H_{n-1}(x)}{\lfloor n-1} = \frac{H_{n+1}(x)}{\lfloor n}$$

i.e. $2x H_n(x) = 2nH_{n-1}(x) + H_{n+1}(x)$ ...(24)

III. Eliminating $H_{n-1}(x)$ from (23) and (24) we get

$$2x H_n(x) = H_n'(x) + H_{n+1}(x)$$

or $H_n'(x) = 2x H_n(x) - H_{n+1}(x)$ ...(25)

IV. Differentiating (25) w.r.t. x we find

$$H_n''(x) = 2x H_n'(x) + 2H_n(x) - H'_{n+1}(x)$$

Putting $H'_{n+1}(x) = 2(n+1) H_n(x)$ obtained from (23) on replacing n by $n+1$: we have

$$H_n''(x) = 2xH_n'(x) + 2H_n(x) - 2(n+1)H_n(x)$$

or $H''_n(x) - 2x H_n'(x) + 2n H_n(x) = 0$ ...(26)

which clearly follows that $y = H_n(x)$ is a solution of Hermite equation.

Note. e^{2tx-t^2} *is known as* **Generating function of Hermite** *Polynomial.*

Probiem 37. *Prove that*

(*i*) $H_{2m}(0) = (-1)^m 2^{2m}\left(\frac{1}{2}\right)_m$

(*ii*) $H'_{2m+1}(0) = (-1)^m 2^{2m+1}\left(\frac{3}{2}\right)_m$

(*iii*) $H_{2m+1}(0) = 0$

(*iv*) $H'_{2m}(0) = 0$

(*v*) $\frac{d^m}{dx^m}\{H_n(x)\} = \frac{2^m\lfloor n}{\lfloor n-m} H_{n-m}(x)$, for $m < n$

(*i*) Even Hermite polynomials are

$$H_{2m}(x) = \sum_{k=0}^{m}\frac{(-1)^k \lfloor 2m\,(2x)^{2m-2k}}{\lfloor k\lfloor 2m-2k}$$

$$\therefore\ H_{2m}(0) = \frac{(-1)^m\lfloor 2m}{\lfloor m} = (-1)^m\frac{2m(2m-1)-3.2.1}{m(m-1)\ldots 3.2.1}$$

$$= (-1)^m 2^{2m}\frac{\lfloor m}{\lfloor m}\,\frac{1}{2}\left(\frac{1}{2}+1\right)\ldots\left(\frac{1}{2}+m-1\right) = (-1)^m\ 2^{2m}\left(\frac{1}{2}\right)_m.$$

(*ii*) From recurrence relation I, we have on replacing n by $2m+1$,

$$H'_{2m+1}(x) = 2(2m+1) H_{2m}(x)$$

$$\therefore\ H'_{2m+1}(0) = 2(2m+1) H_{2m}(0) = 2(2m+1)\cdot(-1)^m 2^{2m}\left(\frac{1}{2}\right)_m$$

by part (*i*)

$$= (2m+1)\,(-1)^m\, 2^{2m+1} \left[\frac{(2m-1)\cdot(2m-3)\ldots 3.1.}{2^m}\right]$$

$$= (-1)^m (2m+1) \left[\frac{3}{2}\cdot\left(\frac{3}{2}+1\right)\ldots\left(\frac{3}{2}+m-1\right)\right]$$

$$= (-1)^m\, 2^{2m+1} \left(\frac{3}{2}\right)_m.$$

(*iii*) Odd Hermite Polynomials are

$$H_{2m+1}(x) = \sum_{k=0}^{2m+1/2} \frac{(-1)^k \,\underline{|2m+1}\,(2x)^{2m+1-2k}}{\underline{|k}\;(2m+1-2k)}$$

$\therefore\; H_{2m+1}(0) = 0$, since all terms containing x become zero.

(*iv*) From recurrence relation I, we have

$$H'_{2m}(x) = 2(2m)\, H_{2m-1}(x)$$

$$\therefore\; H'_{2m}(0) = 4mH_{2m-1}(0)$$

$$= 0 \text{ by } (iii)$$

(*v*) From recurrence relation I, we have

$$H'_n(x) = 2n\, H_{n-1}(x) \qquad \ldots(i)$$

i.e. $\dfrac{d}{dx}\{H_n(x)\} = 2n\, H_{n-1}(x)$

$$\therefore\; \frac{d^2}{dx^2}\{H_n(x)\} = 2n\,\frac{d}{dx}\{H_{n-1}(x)\} = 2n\cdot 2(n-1)\; H_{n-2}(x) \text{ by using (1)}$$

$$= 2^2 n\,(n-1)\, H_{n-2}(x).$$

Similarly $\dfrac{d^3}{dx^3}\{H_n(x)\} = 2^3 n\,(n-1)\,(n-2)\; H_{n-3}(x)$

Proceeding similarly m times we find

$$\frac{d^m}{dx^m}\{H_n(x)\} = 2^m\; n(n-1)\ldots(n-m+1)\; H_{n-m}(x) \text{ where } m < n$$

$$= \frac{2^m}{\underline{|n-m}}\; H_{n-m}(x)$$

[E] Hermite Functions.

An equation closely related to Hermite equation is

$$\frac{d^2\psi}{dx^2} + (\lambda - x^2)\;\psi = 0 \qquad \ldots(27)$$

If we change the dependent variable ψ to y by the substitution

$$\psi = e^{-x^2/2} y \qquad \ldots(28)$$

So that $\dfrac{d\psi}{dx} = e^{-x^2/2}\dfrac{dy}{dx} - y\,e^{-x^2/2}.\,x$

and $\dfrac{d^2\psi}{dx^2} = e^{-x^2/2}\dfrac{d^2y}{dx^2} - 2x\,e^{-x^2/2}\dfrac{dy}{dx} - \left(e^{-x^2/2} - x^2 e^{-x^2/2}\right)y$

We get from (1)

$$y'' - 2xy' + (\lambda - 1)\,y = 0 \qquad ...(29)$$

If we put $\lambda - 1 = 2v$, then (29) reduces to Hermite equation *i.e.*

$$y'' - 2xy' + 2vy = 0$$

It therefore follows that the general solution of (27) is given by

$$\psi = e^{-x^2/2} y$$

where y is given by (12) of § 8.7.

Thus if the parameter λ be of the form $1 + 2n$, n being a positive integer, then the solution of (27) will be a constant multiple of the function ψ_n defined by

$$\psi_n(x) = e^{-x^2/2} H_n\,(x) \qquad ...(30)$$

where $H_n\,(x)$ is the Hermite polynomial of degree n.

Here the function $\psi_n(x)$ is said to be the *Hermite function of order n*.

Recurrence Relations for $\psi_n\,(x)$;

Differentiating (30) w.r.t. x, we have

$$\psi_n'(x) = e^{-x^2/2} H_n'(x) - e^{-x^2/2} x\, H_n(x)$$

$$= 2ne^{-x^2/2} H_{n-1}(x) - xe^{-x^2/2} H_n(x)$$

$$\because\ H_n'(x) = 2n\ \ H_{n-1}(x) \text{ by (23)}$$

$$= 2n\,\psi_{n-1}\,(x) - x\,\psi_n(x) \text{ using (30)}$$

$$\therefore\ 2n\,\psi_{n-1}\,(x) = x\,\psi_n\,(x) + \psi'_n\,(x) \qquad ...(31)$$

Also from (24), $2x\,H_n\,(x) = 2n\,H_{n-1}(x) + H_{n+1}(x)$

which may be expressed by using (30), as

$$2xe^{-x^2/2} H_n(x) = 2ne^{-x^2/2}\ (x)\ +\ e^{-x^2/2} H_{n+1}(x)$$

i.e.

$$2x\,\psi_n\,(x) = 2n\,\psi_{n-1}\,(x) + \psi_{n+1}\,(x) \qquad ...(32)$$

Eliminating $2n\,\psi_{n-1}\,(x)$ from (31) and (32) we find

$$x\,\psi_n\,(x) + \psi'_n\,(x) = 2x\,\psi_n\,(x) - \psi_{n+1}\,(x)$$

i.e.

$$\psi'_n\,(x) = x\,\psi_n(x) - \psi_{n+1}\,(x) \qquad ...(33)$$

[F] Orthogonal Properties of Hermite Polynomials.

Now since $H_n\,(x)$ is a solution of Hermite equation, we have

$$H''_n(x) - 2x\,H'_n\,(x) + 2n\ \ H'_n\,(x) = 0 \text{ by (26)}$$

If we put $y = e^{-x^2/2} H_n(x)$ *i.e.* $H_n\,(x) = ye^{x^2/2}$

So that $H'_n\,(x) = y'e^{x^2/2} + xy\,e^{x^2/2}$,

and $\quad H_n''(x) = y''x^{x^2/2} + 2\,xy'\ \ e^{x^2/2} + y\left(1 + x^2\right) e^{x^2/2}$

then we get $\quad y'' + (1 - x^2 + 2n)\,y = 0 \qquad ...(34)$

Since $y = e^{-x^2/2}\ H_n(x) = \psi_n(x)$ by (30), it therefore follows that $\psi_n(n)$ satisfies (34) and hence

$$\psi''_n + (2n + 1 - x^2)\,\psi_n = 0 \qquad ...(35)$$

for a function ψ_m, this relation is

$$\psi''_m + (2m + 1 - x^2)\,\psi_m = 0 \qquad ...(36)$$

Multiplying (35) by ψ_m; (36) by ψ_n and subtracting we get

$$2(m-n)\,\psi_m\psi_n = \psi_m\psi''_n - \psi_n\psi''_m \qquad ..(37)$$

Integrating over $(-\infty, \infty)$, we have

$$2\,(m-n)\int_{-\infty}^{\infty}\psi_m\ \psi_n\ dx = \int_{-\infty}^{\infty}(\psi_m\ \psi''_n - \psi_n\ \psi''_m)\ dx$$

$$= \left[\psi_m\psi'_n - \psi_n\psi'_m\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}(\psi'_m\psi'_n - \psi'_n\psi'_m)\ dx$$

(on integrating by parts)

$= 0 \quad \because\ \psi_n(x) \to 0$ as $|x| \to \infty$ for all positive integral values of n.

or $\displaystyle\int_{-\infty}^{\infty}\psi_m\psi_n dx = 0$ if $m \neq n$

Symbolically $\displaystyle I_{m,n} = \int_{-\infty}^{\infty}\psi_m\psi_n\ dx = \int_{-\infty}^{\infty}e^{-x^2}\ H_m(x)\ H_n(x)\ dx = 0$

when $m \neq n$ ...(38)

In particular $\quad L_{n-1,\,n+1} = 0$ ...(39)

Now from (32) we have $2x\psi_n(x) = 2n\psi_{n-1}(x) + \psi_{n+1}(x)$

$$\therefore\ \int_{-\infty}^{\infty}2x\,\psi_n(x)\,\psi_{n-1}\ dx = 2n\int_{-\infty}^{\infty}\psi_{n-1}(x)\,\psi_{n-1}\ (x)\ dx$$

$\because\ \displaystyle\int_{-\infty}^{\infty}\psi_{n-1}\ \psi_{n+1}\ dx = 0$ by (30)

$$= 2n\,I_{n-1,\,n-1} \qquad ...(40)$$

Also $\psi_n(x) = e^{-x^2/2}H_n(x)$

$$= (-1)^n e^{x^2/2}\ \frac{d^n}{dx^n}\left(e^{-x^2}\right) \text{ by (16)}$$

Thus (40) gives

$$-\int_{-\infty}^{\infty}2x\,e^{x^2/2}\frac{d^n}{dx^n}\left(e^{-x^2/2}\right)\frac{d^{n-1}}{dx^{n-1}}\left(e^{-x^2}\right)\ dx = 2n\ I_{n-1,\,n-1}$$

or $\displaystyle 2n\,I_{n-1,\,n-1} = -\int_{-\infty}^{\infty}d\left(e^{x^2}\right)\frac{d^n}{dx^n}\left(e^{-x^2}\right)\frac{d^{n-1}}{dx^{n-1}}\left(e^{-x^2}\right)\ dx$

$$= \left[e^{-x^2}\ \frac{d^n}{dx^n}\left(e^{-x^2}\right)\cdot\frac{d^{n-1}}{dx^{n-1}}\left(e^{-x^2}\right)\right]_{-\infty}^{\infty}$$

$$+\int_{-\infty}^{\infty}e^{-x^2}\left\{\frac{d^n}{dx^n}\left(e^{-x^2}\right)\frac{d^n}{dx^n}\left(e^{-x^2}\right) + \frac{d^{n+1}}{dx^{n+1}}\left(e^{-x^2}\right)\cdot\frac{d^{n-1}}{dx^{n-1}}\left(e^{-x^2}\right)\right\}\ dx$$

(on integrating by parts)

$= 0 + I_{n,\,n} + I_{n+1,\,n-1}$

$= I_{n,\,n}$ by (39)

$$\therefore\quad I_{n,\,n} = 2n\,I_{n-1,\,n-1} \qquad ...(41)$$

Applying (41), repeatedly we have

$I_{n,\,n} = 2n\ I_{n-1,\,n-1} = 2n\ \ 2(n-1)\,I_{n-2,\,n-2}$

$= 2^2 n\ (n-1)\cdot 2(n-2)\,I_{n-3,\,n-3}$

$= 2^3 n\,(n-1)\,(n-2)\,I_{n-3,\,n-3}$

$= \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots$

$= 2^n n(n-1)(n-2)\ldots 3.2.1.\, I_{0,0}$

where $I_{0,0} = \int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$ (See Beta and Gamma functions)

$$\therefore \quad I_{n,n} = 2^n \lfloor n \sqrt{\pi} \qquad \ldots(42)$$

Combining the two results (38) and (42); we have in terms of Kronecker delta symbol

$$I_{m,n} = \int_{-\infty}^{\infty} e^{-x^2} H_m(x)\, H_n(x)\, dx = 2^n \lfloor n \sqrt{\pi}\ \delta_{m,n} \qquad \ldots(43)$$

where $\delta_{m,n} = 0$ when $m \neq n$

$= 1$ when $m = n$.

(43) may also be written as

$$I_{m,n} = \int_{-\infty}^{\infty} \psi_m(x)\, \psi_n(x)\, dx = \int_{-\infty}^{\infty} e^{-x^2} H_m(x)\, H_n(x)\, dx$$

$$= 2^n \lfloor n \sqrt{\pi}\ \delta_{m,n} \qquad \ldots(44)$$

Again $2x\, \psi_n(x) = 2n\, \psi_{n-1}(x)\, \psi_{n+1}(x)$ gives

$$\int_{-\infty}^{\infty} x\, \psi_m(x)\, \psi_n(x)\, dx = nI_{m,\, n-1} + \tfrac{1}{2} I_{m,\, n+1}$$

$$= 0 \text{ for } m \neq n = 1$$

and $\int_{-\infty}^{\infty} x\psi_n(x)\, \psi_{n+1}(x)\, dx = n\, I_{n+1,\, n-1} + \frac{1}{2} I_{n+1,\, n+1}$

$$= \tfrac{1}{2}\, 2^{n+1} \lfloor n+1\ \sqrt{\pi} \text{ as above}$$

$$= 2^n \lfloor (n+1)\ \sqrt{\pi} \text{ for } m = n$$

$$\text{Hence } \int_{-\infty}^{\infty} x\, \psi_m(x) \psi_n(x) dx = 2^n \lfloor n+1\ \sqrt{\pi}\ \delta_{m,n} \qquad \ldots(45)$$

Further $2n\, \psi_{n-1}(x) = x\, \psi_n(x) + \psi'_n(x)$ gives

$$\int_{-\infty}^{\infty} \psi_m(x)\, \psi'_n(x)\, dx = 2n \int_{-\infty}^{\infty} \psi_m(x)\, \psi_{n-1}(x)\, dx - \int_{-\infty}^{\infty} x\psi_m(x)\, \psi_n(x)\, dx$$

$$= 0 \text{ if } m \neq, n = 1$$

and $= 2n\, I_{n-1,\, n-1} - 2^{n-1} \lfloor n\ \sqrt{\pi}$ if $m = n = 1$

$$= 2^n \lfloor n\ \sqrt{\pi} - 2^{n-1} \lfloor n\ \sqrt{\pi} = 2^{n-1} \lfloor n\ \sqrt{\pi}$$

$$\text{Hence } \int_{-\infty}^{\infty} \psi_m(x)\, \psi'_n(x)\, dx = 2^{n-1} \lfloor n\ \sqrt{\pi}\ \delta_{m,n} \qquad \ldots(46)$$

In the last if we take $m = n + 1$, then

$$\int_{-\infty}^{\infty} \psi_m(x)\, \psi'_n(x)\, dx = 2n \int_{-\infty}^{\infty} \psi_{n+1}(x)\, \psi_{n-1}(x)\, dx - \int_{-\infty}^{\infty} x\, \psi_{n+1}(x)\, \psi_n(x)\, dx$$

$$= -2^n \lfloor n+1 \sqrt{\pi}.$$

Problem 38. *Prove that* $H_n(-x) = (-1)^n H_n(x)$.

We have $\sum_{n=0}^{\infty} \frac{H_n(x)\, t^n}{\lfloor n} = e^{2tx - t^2} = e^{2tx} e^{-t^2} = \sum_{n=0}^{\infty} \frac{(2x)^n t^n}{\lfloor n} \times \sum_{n=0}^{\infty} \frac{(-1) t^{2n}}{\lfloor n}$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n/2} \frac{(-1)^k (2x)^{n-2k}}{\lfloor k \lfloor n-2k}.$$

Equating coefficient of $\frac{t^n}{\lfloor n}$ on either side, we get

$$H_n(x) = \sum_{k=0}^{n/2} \frac{(-1)^k \lfloor n\, (2x)^{n-2k}}{\lfloor k \lfloor n-2k}$$

Replacing x by $-x$ we get

$$H_n(-x) = \sum_{k=0}^{n/2} \frac{(-1)^k \lfloor n\, (-2x)^{n-2k}}{\lfloor k \lfloor n-2k}$$

$$= \sum_{k-0}^{n/2} \frac{(-1)^k (-1)^{n-2k} \lfloor n\, (2x)^{n-2k}}{\lfloor k \lfloor n-2k}$$

$$= (-1)^n \sum_{k=0}^{n/2} \frac{(-1)^k \lfloor n\, (2x)^{n-2k}}{\lfloor k \lfloor n-2k}$$

$$= (-1)^n H_n(x)$$

Problem 39. *Prove*

$$\int_{-\infty}^{\infty} xe^{-x^2} H_n(x) H_m(x)\, dx = \sqrt{\pi}\left[2^{n-1} \lfloor n\, \delta_{m,\, n-1} + 2^n \lfloor n+1\, \delta_{n+1,\, m}\right].$$

Integrating by parts we have

$$\int_{-\infty}^{\infty} xe^{-x^2} H_n(x) H_m(x) dx = \int -\frac{1}{2} e^{-x^2} H_n(x)\ H_m(x)\ dx\Big]_{-\infty}^{\infty}$$

$$+ \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2} \frac{d}{dx}\{H_n(x) H_m(x)\} dx$$

$$= 0 + \int_{-\infty}^{\infty} e^{-x^2}\{H'_n(x)\ H_m(x) + H_n(x)\ H'_m(x)\}\ dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2}[2n\, H_{n-1}(x)\ H_m(x) + 2m\, H_n(x)\ H_{m-1}(x)] dx \qquad \text{by (24)}$$

$$= n\int_{-\infty}^{\infty} e^{-x^2} H_{n-1}(x)\ H_m(x)\ dx + m\int_{-\infty}^{\infty} e^{-x^2} H_n(x)\ H_{m-1}(x)\ dx$$

$$= n\sqrt{\pi}\ 2^{n-1} \lfloor n-1\ \delta_{m,\, n-1} + m\sqrt{\pi}\ 2^n \lfloor n\ \delta_{n,\, m-1}$$

(by orthogonal properties)

$$= \sqrt{\pi}[2^{n-1} \lfloor n\, \delta_{m,\, n-1} + 2^n \lfloor n+1\, \delta_{n+1,\, m}] \quad \because \delta_{n,\, m-1} = \delta_{n+1,\, m}.$$

Problem 40. *Verify,* $P_n(x) = \frac{2}{\sqrt{\pi}\lfloor n} \int_0^{\infty} t^n e^{-t^2} H_n(xt)\ dt.$

Problem 41. *Show that if m is an integer,* $\int_{-\infty}^{\infty} x^m e^{-x^2} H_n(x)\ dx = 0.$

8.8. LAGUERRE EQUATION AND POLYNOMIAL WITH PROPERTIES

[A] Laguerre's Differential Equation.

This equation is of the form

$$xy'' + (1 - x)\, y' + vy = 0 \qquad ...(1)$$

Dividing by x, it is observed that $x = 0$ is a regular singularity of (1) and hence it has a series solution. Let its series solution be

$$y = \sum_{r=0}^{\infty} a_r\, x^{k+r} \qquad a_0 \neq 0 \qquad ...(2)$$

$$\therefore \qquad y' = \sum_{r=0}^{\infty} a_r (k + r)\, x^{k+r-1}$$

and $\quad y'' = \sum_{r=0}^{\infty} a_r (k + r)\,(k + r - 1)\, x^{k+r-2}$

Substituting these values in (1), we get the identity

$$\sum_{r=0}^{\infty} \left[(k + r)^2 x^{k+r-1} - (k + r - v)\, x^{k+r}\right] a_r = 0 \qquad ...(3)$$

Equating to zero the coeffcient of x^{k-1} (the first and the lowest term); we get

$$k^2 = 0 \text{ i.e. } k = 0 \text{ as } a_0 \neq 0 \qquad ...(4)$$

Again equating to zero the coefficient of x^{k+r} in (3) we get

$$(k + r + 1)^2\, a_{r+1} - (k + r - v)\, a_r = 0$$

which gives the recurrence relation

$$a_{r+1} = \frac{k + r - v}{(k + r + 1)^2}\, a_r \qquad ...(5)$$

For $k = 0$, this yields, $a_{r+1} = \dfrac{r - v}{(r + 1)^2}\, a_r$

$\therefore\ a_1 = -va_0 = (-1)\, va_0$

$$a_2 = \frac{1 - y}{2^2} = (-1)^2 \frac{v(v - 1)}{(\lfloor 2)^2}\, a_0$$

Similarly $a_3 = (-1)^3 \dfrac{v(v - 1)\,(v - 2)}{(\lfloor 3)^2}\, a_0$

... ...

$$a_r = (-1)^r \frac{v(v - 1)\ldots(v - r + 1)}{(\lfloor r)^2}\, a_0$$

Hence the solution is

$$y = a_0 \left[1 - vx + \frac{v(v - 1)}{(\lfloor 2)^2} x^2 - \ldots + (-1)^r \frac{v\,(v - 1)\ldots(v - r + 1)}{(\lfloor r)^2} x^r + \ldots\right] \qquad ...(6)$$

$$= a_0 \sum_{k=0}^{\infty} \frac{(-1)^k \lfloor v}{(\lfloor k)^2 \lfloor v - k} \qquad ...(7)$$

In case v is a positive integer put $v = n$, so that Laguerre equation becomes

$xy'' + (1 - x)y' + ny = 0$ for positive integeral n. ...(8)

When $v = n$ (a positive integer) and $a_0 = \lfloor n$ then solution for (8) is said to be the Laguerre polynomial of degree n and denoted by $L_n(x)$ i.e.

$$L_n(x) = (-1)^n\left[x^n - \frac{n^2}{\lfloor 1}x^{n-1} + \frac{n^2(n-1)^2}{\lfloor 2}x^{n-2} + \ldots + \ldots (-1)^n \lfloor n\right] \quad \ldots(9)$$

Then the solution of Laguerre equation for v to be a positive integer is

$$y = AL_n(x) \quad \ldots(10)$$

From (9), it is easy to show that

$$\left.\begin{array}{ll} L_n(0) = \lfloor n, & L_2(x) = x^2 - 4x + 2 \\ L_0(x) = 1, & L_3(x) = -x^3 + 9x^2 - 18x + 6 \\ L_1(x) = 1 - x, & L_4(x) = x^4 - 16x^3 + 72x^2 - 96x + 48 \end{array}\right\} \quad \ldots(11)$$

Also $L_n(x)$ being the solution of (8), we should have

$$xL_n''(x) + (1 - x)L_n'(x) + nL_n(x) = 0 \quad \ldots(12)$$

[B] Laguerre Polynomials with their Representation in Terms of Confluent Hypergeometric Series.

The Laguerre Polynomials $L_n(x)$ are given by the relation

$$(1 - t)\sum_{n=0}^{\infty} \frac{L_n(x)}{\lfloor n} t^n = e^{\frac{-xt}{1-t}} \quad \ldots(13)$$

where n is a positive integer and x is a positive real number.

(13) can be written as

$$\sum_{n=0}^{\infty} \frac{L_n(x)\, t^n}{\lfloor n} = \frac{1}{1-t} e^{\frac{-xt}{1-t}}$$

$$= \frac{1}{1-t}\left[1 - \frac{xt}{1-t} + \frac{x^2 t^2}{\lfloor 2\,(1-t)^2} - \ldots + \frac{(-1)^k x^k t^k}{\lfloor k\,(1-t)^k} + \ldots\right]$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^k t^k}{\lfloor k\,(1-t)^{k+1}}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^k t^k}{\lfloor k}(1-t)^{-(k+1)}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{\lfloor k} t^k \left[1 + (k+1)t + \frac{(k+1)(k+2)}{\lfloor 2} t^2 + \ldots \right.$$

$$\left. \ldots + \frac{(k+1)(k+2)\ldots(k+l)}{\lfloor l} t^l + \ldots\right]$$

$$= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^k (k+1)_l}{\lfloor k\, \lfloor l} x^k t^{k+l} \quad \text{where } (k+1)_l = \frac{\Gamma(k+1+l)}{\Gamma(k+1)}$$

Equating the coefficients of t^n on either side (coefficient of t^n on R.H.S. being obtained by putting $l = n - k$), we get

$$\frac{L_n(x)}{\lfloor n} = \sum_{k=0}^{\infty} \frac{(-1)^k \; (k+1)_{n-k}}{\lfloor r \; \lfloor n-k} x^k \qquad \ldots(14)$$

Here $(k+1\;)_{n-k} = \dfrac{\Gamma(k+1+n-k)}{\Gamma(k+1)} + \dfrac{\Gamma(n+1)}{\Gamma(k+1)} = \dfrac{\lfloor n}{\lfloor k}$

$$\frac{(-1)^k}{\lfloor n-k)} = \frac{(-1)^k \; n\,(n-1)\ldots(n-k+1)}{\lfloor n}$$

$$= \frac{(-n)\;\;(-n+1)\;(-n+2)\ldots(-n+k-1)}{\lfloor n} = \frac{(-n)_k}{\lfloor n}$$

$\therefore$ (14) yields

$$L_n(x) = \lfloor n \sum_{k=0}^{\infty} \frac{(-n)_k}{\lfloor n} \cdot \frac{\lfloor n}{(\lfloor k)^2} x^k = \lfloor n \sum_{k=0}^{\infty} \frac{(-n)_k}{(\lfloor k)^2} x^k$$

$$= \lfloor n \left[1 + \frac{(-n)}{\lfloor 1 \; \lfloor 1} x + \frac{(-n)\;(-n+1)}{\lfloor 2 \; \lfloor 2} x^2 + \frac{(-n)\;(-n+1)\;(-n+2)}{\lfloor 3 \; \lfloor 3} x^3 + \ldots \right]$$

$$= \lfloor n \; F(-n,\; 1;\; x\,) \qquad \ldots(15)$$

From which it follows that $L_n(x)$ is a polynomial of degree n in x and that the coefficient of x^n is $(-1)^n$.

[C] Recurrence formulae for Laguerre Polynomials. The generating function for Laguerre Polynomial is

$$e^{\frac{-xt}{1-t}} = (1-t) \sum_{n=0}^{\infty} \frac{L_n(x)}{\lfloor n} t^n \text{ by (13)} \qquad \ldots(16)$$

I. Differentiating w.r.t. 't' it gives

$$-\frac{x}{(1-t)^2} e^{-\frac{xt}{1-t}} = (1-t) \sum_{n=0}^{\infty} \frac{L_n(x)\; t^{n-1}}{\lfloor n-1} - \sum_{n=0}^{\infty} \frac{L_n(x)\; t^n}{\lfloor n}$$

Using (16), we have

$$-\frac{x}{(1-t)^2}(1-t) \sum_{n=0}^{\infty} \frac{L_n(x)\; t^n}{\lfloor n} = (1-t) \sum_{n=0}^{\infty} \frac{L_n(x)\; t^{n-1}}{\lfloor n-1} - \sum_{n=0}^{\infty} \frac{L_n(x) t^n}{\lfloor n}$$

or $x \displaystyle\sum_{n=0}^{\infty} \frac{L_n(x)\; t^n}{\lfloor n} + (1-t)^2 \sum_{n=0}^{\infty} \frac{L_n(x)\; t^{n-1}}{\lfloor n-1} - (1-t) \sum_{n=0}^{\infty} \frac{L_n(x)\; t^n}{\lfloor n} = 0$

Equating to zero the coefficient of t^n, we find

$$x\,\frac{L_n(x)}{\lfloor n} + \frac{L_{n+1}(x)}{\lfloor n} - 2\frac{L_n(x)}{\lfloor n-1} + \frac{L_{n-1}(x)}{\lfloor n-2} - \frac{L_n(x)}{\lfloor n} + \frac{L_{n-1}(x)}{\lfloor n-1} = 0$$

i.e. $\quad L_{n+1}(x\,) + (x - 2n - 1)L_n(x\,) + n^2 L_{n-1}(x\,) = 0 \qquad \ldots(17)$

II. Again differentiating (16) w.r.t. x we get

$$-\left(\frac{t}{1-t}\right) e^{\frac{-xt}{1-t}} = (1-t) \sum_{n=0}^{\infty} \frac{L'_n(x)\; t^n}{\lfloor n}$$

or $-\left(\frac{t}{1-t}\right)(1-t)\sum_{n=0}^{\infty}\frac{L_n(x)}{\lfloor n}t^n=(1-t)\sum_{n=0}^{\infty}\frac{L'_n(x)}{\lfloor n}t^n$ by (16)

or $t\sum_{n=0}^{\infty}\frac{L_n(x)\,t^n}{\lfloor n}+(1-t)\sum_{n=0}^{\infty}\frac{L'_n(x)\,t^n}{\lfloor n}=0$

Equating to zero the coefficient of t^n, we get

$$\frac{L'_n(x)}{\lfloor n}-\frac{L'_{n-1}(x)}{\lfloor n-1}+\frac{L_n(x)}{\lfloor n-1}=0$$

i.e. $L'_n(x)-n\,L'_{n-1}(x)+n\,L_{n-1}(x)=0$ ...(18)

III. Now differentiating (17) w.r.t. x, we find

$$L'_{n+1}(x)-(x-2n-1)L'_n(x)+L_n(x)+n^2L'_{n-1}(x)=0.$$

Differentiating it again w.r.t. x, we have

$$L''_{n+1}(x)+(x-2n-1)L_n''(x)+2L'_n(x)+n^2L''_{n-1}(x)=0$$

Replacing n by $n+1$, this yields

$$L''_{n+2}(x)+(x-2n-3)L''_{n-1}(x)+(n+1)^2L_n''(x)+2L'_{n+1}(x)=0 \quad ...(a)$$

Whence from (18),

$$L_n'(x)=n\{L'_{n-1}(x)-L_{n-1}(x)\}$$

or $L'_{n+1}(x)=(n+1)\{L'_n(x)-L_n(x)\}$ (on replacing n by $n+1$) (b)

$\therefore\ L''_{n+1}(x)=(n+1)\{L_n''(x)-L_n'(x)\}$ (on differentiating) (c)

or $L''_{n+2}(x)=(n+2)\{L''_{n+1}(x)-L'_{n+1}(x)\}$ on replacing n by $n+1$)

Thus we have from (a)

$$(n+2)\{L''_{n+1}(x)-L'_{n+1}(x)\}+(x-2n-3)L''_{n+1}(x)+(n+1)^2L_n''(x)+2L'_{n+1}(x)=0$$

or $(x-n-1)L''_{n+1}(x)-n\,L'_{n+1}(x)+(n+1)^2L_n''(x)=0$

Eliminating $L''_{n+1}(x)$ and L'_{n+1} by (b) and (c), we get

$x\,L''_n(x)+(1-x)L'_n(x)+n\,L_n(x)=0$ ...(19)

which clearly shows that $y=AL_n(x)$ is a solution of Laguerre equation.

[D] Differentiation formula for Laguerre Polynomial
(Analogue of Rodrigues formula)

We have

$$(1-t)^{-1}e^{\left(1-\frac{1}{1-t}\right)x}=\sum_{n=0}^{\infty}\frac{L_n(x)t^n}{\lfloor n} \text{ by (13)}$$

Differentiating it w.r.t. 't' n times by Leibnitz theorem for succesive differentiation, we have

$$e^x\frac{d^n}{dt^n}\left[(1-t)^{-1}e^{-\frac{x}{1-t}}\right]=L_n(x)+L_{n+1}(x)t+\ldots \quad ...(20)$$

since all terms upto the term containing t^{n-1} vanish when differentiated n times.

Now $\frac{d}{dt}\left[(1-t)^{-1}e^{-\frac{x}{1-t}}\right]=\frac{1-x-t}{(1-t)^3}e^{-\frac{x}{1-t}}$

So that

$$\lim_{t\to 0}\frac{d}{dt}\left[(1-t)^{-1}e^{-\frac{x}{1-t}}\right]=(1-x)e^{-x}=\frac{d}{dx}(xe^{-x})$$

Similarly

$$\lim_{t\to 0}\frac{d^2}{dt^2}\left[(1-t)^{-1}e^{-\frac{x}{1-t}}\right]=\frac{d^2}{dx^2}(x^2e^{-x})$$

And in general

$$\lim_{t\to 0}\frac{d^n}{dt^n}\left[(1-t)^{-1}e^{-\frac{x}{1-t}}\right]=\frac{d^n}{dx^n}(x^ne^{-x})$$

Hence proceeding to the limit as $t\to 0$, (20) yields

$$e^n\frac{d^n}{dx^n}\left(x^ne^{-x}\right)=L_n(x) \qquad \text{...(21)}$$

[E] Integral Property of Laguerre Polynomials.

Laguerre equation is $xy''+(1-x)y'+ny=0$

for positive integral n. ...(22)

$$\left.\begin{aligned}\text{If we take } y_n &= \frac{1}{2\pi i}\oint\frac{t^{-n-1}}{1-t}e^{-\frac{xt}{1-t}}dt\\ \text{So that } y_n' &= \frac{1}{2\pi i}\oint\frac{t^{-n}}{(1-t)^2}e^{-\frac{xt}{1-t}}dt\\ \text{and } y_n'' &= \frac{1}{2\pi i}\oint\frac{t^{-n+1}}{(1-t)^3}e^{-\frac{xt}{1-t}}dt\end{aligned}\right\} \qquad \text{...(23)}$$

Then Laguerre Polynomials being the solution of (22), its L.H.S. becomes with the substitutions of (23),

$$\frac{1}{2\pi i}\oint\left[\frac{xt^2}{(1-t)^2}-\frac{(1-x)\,t}{1-t}+n\right]\frac{t^{-n-1}}{1-t}e^{-\frac{xt}{1-t}}dt$$

$$i.e.\ -\frac{1}{2\pi i}\oint\frac{d}{dt}\left[\left|\frac{t^{-n}}{1-t}\right|e^{-\frac{xt}{1-t}}\right]dt$$

which should be zero for the contour being closed and hence L.H.S. of (22) vanishes for

$$y=\frac{1}{2\pi i}\oint\frac{t^{-n-1}}{1-t}e^{-\frac{xt}{1-t}}dt \qquad \text{...(24)}$$

It means, for this value y is a solution of (22) and therefore we may express $L_n(x)$ which is an already established solution of (22), as

$L_n(x)=Ay_n(x)$ A being an arbitrary constant. ...(25)

Now for $x = 0$, we get from (9) and (23), $L_n(0) = \lfloor n$ ...(26)

and $$y_n(0) = \frac{1}{2\pi i} \oint \frac{t^{-n-1}}{1-t} dt$$

$$= \frac{1}{2\pi i} \times 2\pi i \text{ by contour integration}$$

$$= 1 \qquad ...(27)$$

So (25) gives for $x = 0$, with the help of (26) and (27)

$L_n(0) = A y_n(0)$ *i.e.*, $\lfloor n = A$.

Hence from (25),

$$L_n(x) = \lfloor n \, y_n(x)$$

$$= \frac{\lfloor n}{2\pi i} \oint \frac{t^{-n-1}}{1-t} e^{-\frac{xt}{1-t}} dt \text{ by (24)} \qquad ...(28)$$

Now by contour integration, y_n = coefficient of t^n in $(1-t)^{-1} e^{-\frac{xt}{1-t}}$

Hence $$(1-t)^{-1} e^{\frac{-xt}{1-t}} = \sum_{n=0}^{\infty} y_n t^n = \sum_{n=0}^{\infty} L_n(x) \frac{t^n}{\lfloor n} \qquad ...(29)$$

which is generating function of Laguerre Polynomials.

[F] Orthogonal Properties of Laguerre Polynomials.

The Laguerre polynomials themselves do not form an orthonormal set since the Laguerre equation is not self-adjoint. We, therefore, introduce a function

$$\phi_n(x) = \frac{1}{\lfloor n} e^{-x/2} L_n(x) \qquad ...(30)$$

and then will show that ϕ's form an orthonormal set *i.e.*

$$\int_0^{\infty} \phi_m(x) \phi_n(x) \, dx = \int_0^{\infty} e^{-x} \frac{L_m(x)}{\lfloor m} \frac{L_n(x)}{\lfloor n} dx = \delta_{m,n} \qquad ...(31)$$

Over the interval $0 \le x \le \infty$, when $\delta_{m,n} = 0$ for $m \ne n$

$= 1$ for $m = n$.

Since $L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x})$ by (21), therefore, we have

$$\int_0^{\infty} e^{-x} x^m L_n(x) \, dx = \int_0^{\infty} e^{-x} x^m e^x \frac{d^n}{dx^n}(x^n e^{-x}) \, dx$$

$$= \int_0^{\infty} x^m \frac{d^n}{dx^n}(x^n e^{-x}) \, dx$$

Integrating the R.H.S. by parts, we get

$$\int_0^{\infty} e^{-x} x^m L_n(x) \, dx = \left[x^m \frac{d^{n-1}}{dx^{n-1}} x^n e^{-x} \right]_0^{\infty} - \int_0^{\infty} m x^{m-1} \frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x}) \, dx$$

$$= (-1) m \int_0^{\infty} x^{m-1} \frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x}) \, dx$$

$$= (-1)^2 m(m-1)\int_0^\infty x^{m-2} \frac{d^{n-2}}{dx^{n-2}}(x^n e^{-x})\, dx$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= (-1)^m \lfloor m \int_0^\infty \frac{d^{n-m}}{dx^{n-m}}(x^n e^{-x})\, dx$$

$$= 0 \text{ if } n > m.$$

Similarly, $\int_0^\infty e^{-x} x^n L_m(x)\, dx = 0$ for $m < n$

Now $L_m(x)$ being a polynomial of degree m in x and $L_n(x)$ that of degree n in x, we have

$$\int_0^\infty e^{-x} L_m(x)\, L_n(x) dx = 0 \text{ if } m \neq n.$$

i.e. $$\int_0^\infty e^{-x} \frac{L_m(x)}{\lfloor m} \frac{L_n(x)}{\lfloor n} dx = 0 \text{ if } m \neq n \quad \ldots(32)$$

In case $m = n$, then the term of degree n in $L_n(x)$ is $(-1)^n x^n$,

$$\therefore \int_0^\infty e^{-x}\{L_n(x)\}^2 dx = (-1)^n \int_0^\infty e^{-x} x^n L_n(x)\, dx$$

$$= (-1)^n \int_0^\infty e^{-x} x^n e^x \frac{d^n}{dx^n}(x^n e^{-x})\, dx$$

$$= (-1)^n \int_0^\infty x^n \frac{d^n}{dx^n}(x^n e^{-x}) dx$$

$$= (-1)^{2n} \lfloor n \int_0^\infty x^n e^{-x}\, dx \text{ (on integrating by parts } n \text{ times)}$$

$$= (\lfloor n)^2$$

or $$\int_0^\infty e^{-x/2} \frac{L_n(x)}{L^n} e^{-x/2} \frac{L_n(x)}{L^n} dx = 1 \quad \ldots(33)$$

Combining (32) and (33) we have

$$\int_0^\infty \phi_m(x)\, \phi_n(x)\, dx = \int_0^\infty e^{-x/2} \frac{L_m(x)}{\lfloor m} e^{-x/2} \frac{L_n(x)}{\lfloor n} dx = \delta_{m,\,n}$$

Note. $\phi_n(x)$ *satisfies the equation*

$$x\phi''_n(x) + \phi'_n(x) + \left(n + \frac{1}{2} - \frac{x}{4}\right)\phi_n(x) = 0$$

[G] The associated Laguerre Polynomials and Functions.

For positive integral n, Laguerre differential equation is

$$xy_2 + (1-x)y_1 + ny = 0 \text{ where } y_1 = \frac{dy}{dx} \text{ and } y_2 = \frac{d^2y}{dx^2}.$$

Differentiating it m times by Leibnitz theorem, we get

$$xy_{m+2} + {}^mC_1\, y_{m+1} + (1-x)\, y_{m+1} + {}^mC_1\, y_m (-1) + ny_m = 0$$

or $$xy_{m+2} + (m-1-x)\, y_{m+1} + (n-m)\, y_m = 0$$

Writing $y_m = D^m y = D^m L_n(x)$, for $n \geq m$, this can be expressed as

$$x D^2 (y_m) + (m+1-x) D (y_m) + (n-m) y_m = 0 \quad \ldots(34)$$

From (34) it follows that

$$y_m = D^m L_n(x) = \frac{d^m}{dx^m} L_n(x) \equiv L_n^m(x) \text{ (say)} \qquad ...(35)$$

is the solution of

$$xy'' + (m+1-x)\,y' + (n-m)\,y = 0 \qquad ...(36)$$

where m is an integer such that $m \geq 0$.

The polynomial $L_n^m(x)$ introduced here is said to be the *Associated Laguerre Polynomials* of degree $n - m$.

It may be shown that in terms of confluent hypergeometric functions.

$$L_n^m(x) = \frac{(-1)^m\,(\lfloor n)^2}{\lfloor m\,\lfloor n-m} F(-n+m,\ m+1,\ x),\ n \geq m \qquad ...(37)$$

or

$$= (m+1)_k\, F(-n, m+1, -x) \qquad ...(38)$$

From $L_n(x) = e^x D^n (xn\, e^{-x}),\ D \equiv \dfrac{d}{dx}$

we may infer that

$$L_n^m(x) = D^m\{e^x D^n(x^n e^{-x})\} = \frac{e^x x^{-m} d^n\,(e^{-x} x^{n+m})}{dx^n} \qquad ...(39)$$

From $e^{-\frac{xt}{1-t}} = (1-t)\sum_{n=0}^{\infty} \dfrac{L_n(x)\,t^n}{\lfloor n}$, we may conclude that the generating function for Associated Laguerre polynomials is

$$(-1)^m t^m e^{-\frac{xt}{1-t}} = (1-t)^{m+1} \sum_{n=m}^{\infty} L_n^m(x) \frac{t^n}{\lfloor n} \qquad ...(40)$$

Note 1. *The generalized Laguerre Polynomials are defined by*

$$L_n^m(x) = \frac{\lfloor m+n}{\lfloor m\,\lfloor n} F(-n,\ m+1,\ x) \qquad ...(41)$$

where n is a positive integer or zero.

This is a solution of $xy'' + (m+1-x)\,y' + ny = 0$ ...(42)

Note 2. *The associated Laguerre equation (36) is not self-adjoint and hence does not form orthonormal system, but it can be made so by multiplying the weighing function $e^{-x} x^m$. We thus define the integral*

$$I_{p,n} = \int_0^{\infty} y_{n,m}\, y_{p,m}\, x^q\, dx \qquad ...(43)$$

For $q = 1$, where $y_{n,m} = e^{x/2} x^{(q-1)/2} L_n^m(x)$ ...(44)

known as *associated Laguerre function* is the solution of

$$xy'' + 2y' + \left[n - \frac{q-1}{2} - \frac{4}{x}\frac{q^2-1}{4x}\right] y = 0$$

which is obtained by putting

$$y = e^{-x/2} x^{(q-1)/2} v \text{ i.e. } v = y\, e^{x/2} e^{(q-1)/2} \text{ in} \qquad ...(45)$$

$$xv'' + (q+1-x)\,v' + (n-q)\,v = 0 \text{ (analogous to (36)}, \ v = L_n^m(x) \qquad ...(46)$$

we have

$$y_{n,m} = e^{-x/2} x^{(m-1)/2} L_n^m(x) \text{ and } y_{p,m} = e^{-x/2} x^{(m-1)/2} L_p^m(x)$$

So that $I_{p,n} = \int_0^\infty e^{-x} x^{m-1} L_n^m(x)\, L_p^m(x) x^q\, dx$ ...(47)

Problem 42. *Show that* $L_n(2x) = \lfloor n \sum_{m=0}^{\infty} \frac{2^{n-m}(-1)^m}{\lfloor m \lfloor n-m} L_{n-1}(x),\ n > m$

where $L_n(x) = \frac{1}{\lfloor n} e^x D^n (x^n e^{-x})$.

We have $L_0(x) = 1$

$$L_1(x) = \frac{1}{\lfloor n}(1-x) = 1-x$$

Replacing x by $2x$, $L_1(2x) = 1 - 2x$

$$= 2(1-x) - 1 = 2L_1 x - 2^0 \frac{\lfloor 1}{\lfloor 1} L_0(x)$$

$$= \lfloor 1 \sum_{m=0}^{1} \frac{2^{1-m}(-1)^m}{\lfloor m \lfloor 1-m} L_{1-m}(x)$$

Also, $L_2(2x) = \frac{1}{\lfloor 2}(2 - 8x + 4x^2) = 1 - 4x + 2x^2$

$$= 4\left(1 - 2x + \frac{x^2}{2}\right) - 4(1-x) + 1$$

$$= 2^2 L_2(x) - 2\frac{\lfloor 2}{\lfloor 1 \lfloor 1} L_1(x) + \frac{\lfloor 2}{\lfloor 0 \lfloor 2} L_0(x)$$

$$= \lfloor 2 \sum_{m=0}^{2} \frac{2^{2-m}(-1)^m}{\lfloor m \lfloor 2-m} L_{2-m}(x)$$

and $L_3(3x) = \frac{1}{\lfloor 3}(6 - 36x + 36x^2 - 8x^3) = 1 - 6x + 6x^2 - \frac{4}{3}x^3$

$$= 2^3\left(1 - 3x + \frac{3}{2}x^2 - \frac{x^3}{6}\right) - 12\left(1 - 2x + \frac{x^2}{2}\right) + 2.3(1-x) - 1$$

$$= 2^3 L_3(x) - \frac{2^2 \lfloor 3}{\lfloor 1 \lfloor 2} L_2(x) + 2\frac{\lfloor 3}{\lfloor 2 \lfloor 1} L_1(x) - 2^0 \frac{\lfloor 3}{\lfloor 3} L_0(x)$$

$$= \lfloor 3 \sum_{m=0}^{3} \frac{2^{3-m}(-1)^m}{\lfloor m \lfloor 3-m} L_{3-m}(x)$$

Similarly $L_4(2x) = \lfloor 4 \sum_{m=0}^{4} \frac{2^{4-m}(-1)^m}{\lfloor m \lfloor 4-m} L_{4-m}(x)$

We can thus generalize the result as

$$L_n(2x) = \lfloor n \sum_{m=0}^{n} \frac{2^{n-m}(-1)^m}{\lfloor m \lfloor n-m} L_{n-1}(x)$$

Problem 43. *Prove the following:*

(*i*) $\frac{d}{dx} L_n^m(x) = L_n^{m+1}(x)$

(*ii*) $L_n^m(x) - n L_{n-1}^m(x) + n L_{n-1}^{m-1}(x) = 0.$

(*i*) We have $L_n^m(x) = D^m L_n(x)$

$\therefore D\left[L_n^m(x)\right] = D^{m+1} L_n(x) = L_n^{m+1}(x)$

(*ii*) L.H.S. $= D^m \{e^x D^n (x^n e^{-x}) - n D^m \{e^x D^{n-1}(x^{n-1} e^{-x})\}$

$+ n D^{m-1} \{e^x D^{n-1}(x^{n-1} e^{-x})\}$

$= D^{m-1} \{e^x D^n (x^n e^{-x}) + e^x D^{n+1}(x^n e^{-x})\} - n D^{m-1} \{e^x D^{n-1}(x^{n-1} e^{-x})$

$+ e^x D^n (x^{n-1} e^{-x})\} + n D^{m-1} \{e^x D^{n-1}(x^{n-1} e^{-x})\}$

$= D^{m-1} \{e^x D^n (x^n e^{-x}) + e^x D^n (-x^n e^{-x} + n x^{n-1} e^{-x})\}$

$- n D^{m-1} \{e^x D^{n-1}(x^{n-1} e^{-x}) + e^x D^n (x^{n-1} e^{-x})\} + n D^{m-1} \{e^x D^{n-1}(x^{n-1} e^{-x})\}$

$= n D^{m-1} \{e^x D^n (x^{n-1} e^{-x})\} - n D^{m-1} \{e^x D^n (x^{n-1} e^{-x})\}$

$= 0.$

Problem 44. *Prove the following* :

(*i*) $L_3^1(x) = -18 + 18x - 3x^2$

(*ii*) $L_4^2(x) = 144 - 96x + 12x^2$

(*iii*) $L_4^4(x) = 24.$

Problem 45. *Show that* $L_n^m(2x) = \lfloor m+n \sum_{k=0}^{n} \frac{2^{n-k}(-1)^k}{\lfloor(m+n-k)\lfloor k} L_{n-k}^m(x)$

Problem 46. *Prove the following* :

(*i*) $L_n^m(x) - L_{n-1}^m(x) = L_n^{m-1}(x)$

(*ii*) $\frac{d}{dx} L_n^m(x) = -L_{n-1}^{m+1}(x).$

(*iii*) $n L_n^m(x) = (2n+m-1) L_{n-1}^m(x) - (n+m-1) L_{n-2}^m(x)$

Problem 47. *Establish the completeness of Laguerre functions.*

We have considered the sequence of functions $\phi_1(x), \phi_2(x) \ldots \phi_n(x)$ such that

$$\int_a^b \phi_m(x)\, \phi_n(x)\, dx = 0 \text{ for } m \neq n$$

Then we call the functions $\phi_k(x)$, $k = 1, 2, 3, \ldots$ to form an *orthogonal sequence* for the interval (a, b).

In addition if $\int_a^b \{\phi_n(x)\}^2 dx = 1$ for all values of n, then we call the functions as to form an *orthonormal set.*

In case there is no integrable function $\psi(x)\ (\neq 0)$ such that

$$\int_a^b \psi(x)\, \phi_n(x)\, dx = 0 \text{ for all values of } n,$$

then we say that the sequence is a *complete orthogonal sequence.*

Problem 48. *Find a series solution of*

(*i*) $xy'' + (1 + x)y' + y = 0,$ **Ans.** $y = e^{-x} L_0(x) = e^{-x}$

(*ii*) $xy'' + (1 + x)y' + 2y = 0,$ **Ans.** $y = e^{-x} L_1(x) = e^{-x}(1 - x)$

(*iii*) $xy'' + (1 + x)y' + \frac{3}{2}y = 0,$ **Ans.** $y = e^{-x} L_{1/2}(x)$

(*iv*) $y'' + \left(\frac{1}{4x^2} + \frac{1}{2x} - \frac{1}{4}\right)y = 0$ **Ans.** $y = e^{-x/2} x^{1/2} L_0(x) = e^{-x/2} x^{1/2}$

Problem 49. *If a function f(x) defined in* $(0, \infty)$ *is expressed as*

$$f(x) = C_0 L_0(x) + C_1 L_1(x) + C_2 L_2(x) + \ldots \quad \ldots(1)$$

then show that $C_k = \dfrac{\int_0^\infty e^{-x} L_k(x)\, f(x)\, dx}{\int_0^\infty e^{-x}[L_k(x)]^2\, dx}$ *for* $k = 0, 1, 2, \ldots$

Hint. Integrating over 0 to ∞ after multiplying (1) by $e^{-x} L_k(x)$ and using orthogonal properties *i.e.* $\int_0^\infty e^{-x} L_m(x) L_n(x)\, dx = 0, m \neq n$ the result is obtained.

Problem 50. *Show that the Laguerre and Hermite Polynomials are related by*

$$H_{2m}(x) = (-1)^m\, 2^{2m}\, \underline{|m}\, L_m^{-1/2}(x^2)$$

and $H_{2m+1}(x) = (-1)^m 2^{2m+1} \underline{|m}\, x L_m^{1/2}(x^2)$.

8.9. RICCATI's DIFFERENTIAL EQUATION

In general form it is $x\dfrac{dy}{dx} - ay + by^2 = cx^n$ ...(1)

But the particular form $\dfrac{dy}{dx} + by^2 = cx^m$ ...(2)

is commonly known as Riccati's equation.

Considering first (1) and changing the independent variable x to z by the transformation $z = x^a$ and dependent variable y to u by the transformation $y = uz$, it becomes

$$\frac{du}{dz} + \frac{b}{a}u^2 = \frac{c}{a} z^{\frac{n}{a} - 2} \quad \ldots(3)$$

which is of the form (2)

Now considering (2) and making a substitution $y = \dfrac{u}{x}$, it becomes

$$x\frac{du}{dx} - u + bu^2 = cx^n \text{ where } m = n - 2 \quad \ldots(4)$$

which is of the form (1).

There arise two cases:

I. *If* $n = 2a$, *(1) can be integrated in finite series*

Put $y = ux^a$ in (1) which reduces to

$$x^{a+1}\frac{du}{dx} + bx^{2a}u^2 = cx^n$$

or $x^{1-a}\dfrac{du}{dx} + bu^2 = cx^{n-2a} = c$ for $n = 2a$

$$\therefore\ x^{1-a}\frac{du}{dx}=c-bu^2$$

or $\dfrac{du}{c-bu^2}=x^{a-1}\,dx$

which is integrable as variables are separated.

II. *If* $\dfrac{n\pm 2a}{2n}$ *is a positive integer, (1) is integrable in finite terms.*

Put $y=A+\dfrac{x^n}{y_1}$ in (1) which reduces to

$$-aA+bA^2+(n-a+2bA)\frac{x^n}{y_1}+b\frac{x^{2n}}{y_1^2}-\frac{x^{n+1}}{y_1^2}\frac{dy_1}{dx}=cx^n$$

Choosing A such that $-aA+bA^2=0$ *i.e.* $A=a/b$ so that substitution becomes $y=\dfrac{a}{b}+\dfrac{x^n}{y_1}$ and the reduced equation is

$$x\frac{dy_1}{dx}-(a+n)\,y_1+cy_1^2=bx^n \qquad ...(5)$$

which retains the form (1) except that a has been replaced by $a+n$;b. by c and c by b.

Making a second substitution $y_1=\dfrac{a+n}{c}+\dfrac{x^n}{y_2}$, the last equation reduces to

$$x\frac{dy_2}{dx}-(a+2n)\,y_2+by_2^2=cx^n \qquad ...(6)$$

Hence, making such k transformations we find,

$$x\frac{dy_k}{dx}-(a+kn)\,y_k+cy_k^2=bx^n \text{ when } k \text{ is odd} \qquad ...(7)$$

and $x\dfrac{dy_k}{dx}-(a+kn)\,y_k+cy_k^2=bx^n$ when k is even ...(8)

In either case the equation is integrable in finite terms by 1 when $n=2(a+kn)$ *i.e.* if $\dfrac{n-2a}{2n}$ is a positive integer.

Now with the choice $A=0$; the first substitution $y=\dfrac{x^n}{y_1}$ reduces (1) to

$$x\frac{dy_1}{dx}-(n-a)\,y_1+y_1^2=bx^n \qquad ...(9)$$

which is the same as (5) except that the sign of a is reversed.

The second substitution $y=\dfrac{n-a}{c}+\dfrac{x^n}{y_2}$ yields

$$x\frac{dy_2}{dx}-(2n-a)\,y_2+by_2^2=cx^n \qquad ...(10)$$

Making such k transformations, we have

$$x\frac{dy_k}{dx}-(kn-a)\,y_k+cy_k^2=bx^n \text{ when } k \text{ is odd} \qquad ...(11)$$

and $x\dfrac{dy_k}{dx}-(kn-a)\,y_k+by_k^2=cx^n$ when k is even ...(12)

In either case the equation is integrable in finite terms if

$n=2(kn-a)$ *i.e.* if $\dfrac{n+2a}{2n}$ is a positive integer.

But since $m=n-2$ is the condition for integrability of (1), we have

$$m+2\pm 2=2k(m+2)$$

Taking negative sign, $m=-\dfrac{4k}{2k-1}$

and taking positive sign, $m=\dfrac{4(k-1)}{2k-1}=\dfrac{4k'}{2k'+1}$ when $k'=k-1$.

Hence Riccati's equation is integrable in finite terms if $m=-\dfrac{4k}{2k\pm 1}$ *where k is zero or a positive integer. The integration is carried by the substitution* $y=y_1+\dfrac{1}{v}$, *where* y_1 *is supposed to be a known particular integral of the equation, whence the transformed equation becomes linear.*

In fact Riccati's equation and its general form are particular cases of the equation

$$\frac{dy}{dx}=P+Qy+Ry^2$$

where P, Q, R are functions of x, since the substitution

$y=-\dfrac{1}{R}\dfrac{1}{u}\dfrac{du}{dx}$ reduces it to the form

$$\frac{d^2u}{dx^2}-\left(Q+\frac{1}{R}\frac{dR}{dx}\right)\frac{du}{dx}+PRu=0$$

which is integrable.

CORROLARY. *Relation between Riccati's and Bessel's Equations.*

Riccati's equation is $\dfrac{dy}{dx}+by^2=cx^m$ ...(13)

which is non-linear first order equation.

Put $by=\dfrac{1}{v}\dfrac{dv}{dx}$ so that $b\dfrac{dy}{dx}=\dfrac{1}{v}\dfrac{d^2v}{dx^2}-\dfrac{1}{v^2}\left(\dfrac{dv}{dx}\right)^2$

then (13) becomes

$$\frac{d^2v}{dx^2}-bcvx^m=0 \qquad ...(14)$$

Taking $bc=a^2$, yields

$$\frac{d^2v}{dx^2}-a^2x^mv=0 \qquad ...(15)$$

when b and c have same sign (in case exponential functions occur in y)

and $$\frac{d^2v}{dx^2}+a^2x^mv=0 \qquad ...(16)$$

when b and c have opposite sign (in case circular functions occur only)

Changing x to z by $qz = x^q$ when $q = \frac{1}{2}\ m+1 = \frac{1}{n}$ (say)

The reduced equation is

$$\frac{d^2v}{dx^2} - \frac{n-1}{z}\ \frac{dv}{dz} - bcv = 0 \qquad ...(17)$$

which is integrable in finite form if

$$\frac{1}{n} = \frac{1}{2}m+1 = 1 - \frac{2k}{2k \pm 1} = \frac{\pm 1}{2k \pm 1}$$

i.e. if n is an odd integer (17) can be written as by putting $n - 1 = 2p$,

$$\frac{d^2v}{dz^2} - \frac{2p}{z}\frac{du}{dz} - bc = 0 \qquad ...(18)$$

which is integrable in a finite form when p is an integer.

Reducing it to normal form by the substitution $vz^{-p} = w$, we have

$$\frac{d^2w}{dz^2} - bcw = \frac{p(p+1)}{z^2}\ w \qquad ...(19)$$

If we further reduce it by the substitution $w = t\sqrt{z}$, we find

$$z^2\ \frac{d^2t}{dz^2} + z\frac{dt}{dz} + \left[(-bc)z^2 - \left(p + \frac{1}{2}\right)^2\right] t = 0$$

or $$\frac{d^2t}{dz^2} + \frac{1}{z}\ \frac{dt}{dz} + \left[(-bc) - \frac{\left(p + \frac{1}{2}\right)^2}{z^2}\right] t = 0 \qquad ...(20)$$

which is clearly Bessel's equation with solution

$$t = A\ J_{p+1/2}\ \{z(-bc)^{1/2}\} + B\ J_{-(p+1/2)}\ \{z(-bc)^{1/2}\}$$

Problem 51. *Solve* $\frac{dy}{dx} = \cos x - y \sin x + y^2$

Since $y = \sin x$ is a particular solution of the given equation, we therefore, have $y_1 = \sin x$.

Put $y = \sin x + \frac{1}{v}$ *i.e.* $\frac{dy}{dx} = \cos x - \frac{1}{v^2}\frac{dv}{dx}$,

The reduced equation is

$$\frac{dv}{dx} + v \sin x = -1$$

which is linear in v and hence its integrating factor

$$= e^{\int \sin x\ dx} = A - e^{-\cos x}$$

Thus the solutions is $v.e^{-\cos x} = A - \int e^{-\cos x}\ dx$.

or $$v = A\,e^{\cos x} - e^{\cos x} \int e^{-\cos x}\ dx$$

i.e. $y = \sin x + \left[A e^{\cos x} - e^{\cos x} \int e^{-\cos x}\, dx\right]$

or $\dfrac{1}{y - \sin x} = A e^{\cos x} - e^{\cos x} \int e^{-\cos x}\, dx.$

8.10. THE DIRAC-DELTA FUNCTION WITH ITS FORMAL PROPERTIES

Consider a function $\delta(x)$ which is zero everywhere except at $x = 0$ and tends to ∞ in such a manner that $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$ ...(1)

with
$$\left.\begin{aligned} \delta(t) &= 0 \text{ if } t \neq 0 \\ &= \infty \text{ if } t = 0 \end{aligned}\right\} \quad \text{...(2)}$$

This is known as *Dirac-delta function* and used in mathematical physics wherever functions exist with non-zero values in very short interval, *e.g.* an impulsive force acting for a short while is defined as $\delta(x - \xi)$ by $\lim_{a \to 0} C e^{-(x-\xi)^2/a}$

where the constant $C(a)$ is chosen such that $\int_{-\infty}^{\infty} \delta(x - \xi)\, dx = 1$

and hence using the mean value theorem of integral calculus, we have

$$\int_{-\infty}^{\infty} f(x)\, \delta(x - \xi)\, dx = f(\xi).$$

Let us again consider a function

$$\left.\begin{aligned} \delta_a(x) &= \frac{1}{2a}, \quad -a < x < a \\ &= 0, \quad |x| > a \end{aligned}\right\} \quad \text{...(3)}$$

$$\begin{aligned} \text{Then, } \int_{-\infty}^{\infty} \delta_a(x)\, dx &= \int_{-\infty}^{-a} \delta_a(x)\, dx + \int_{-a}^{a} \delta_a(x)\, dx + \int_{a}^{\infty} \delta_a(x)\, dx \\ &= 0 + \int_{-a}^{a} \frac{1}{2a}\, dx + 0 = \frac{1}{2a}\, [a - (-a)] \\ &= 1 \end{aligned} \quad \text{...(4)}$$

In case $f(x)$ is integrable in the interval $(-a, a)$, then from mean value theorem,

$$\int_{-\infty}^{\infty} f(x)\, \delta_a(x)\, dx = \frac{1}{2a} \int_{-a}^{a} f(x)\, dx = f(\theta a);\ |\theta| \leq 1$$

Let us now define $\delta(x) = \lim_{a \to 0} \delta_a(x)$

As such (3) and (4) yield

$$\delta(x) = 0, \text{ when } x \neq 0 \quad \text{...(5)}$$

and
$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1 \quad \text{...(6)}$$

which define Dirac-delta function.

Further, since we have

$$\int_{-a}^{\infty} f(x)\, \delta_a(x)\, dx = \frac{1}{2a} \int_{-\infty}^{a} f(x)\, dx = f(\theta a),\ |\theta| \leq 1$$

$$\therefore \int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0). \qquad ...(7)$$

which by change of variable, reduces to

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a) \qquad ...(8)$$

or symbolically, $f(x)\,\delta(x-a) = f(a)\,\delta(x-a)$ ...(9)

In case $f(x) = x$, (9) yields, $x\delta x = 0$ ...(10)

In a similar manner we can show

$$\delta(-x) = \delta x \qquad ...(11)$$

$$\delta(ax) = \frac{1}{a}\delta(x),\ a>0 \qquad ...(12)$$

$$\delta\left(a^2 - x^2\right) = \frac{1}{2a}\{\delta(x-a)+\delta(x+a)\},\ a>0 \qquad ...(13)$$

Now assuming that $\delta'(x)$ *i.e.*, differential of $\delta(x)$ exists and regarding $\delta\,(x)$ and $\delta'(x)$ both as ordinary functions in the rule for integrating by parts, we have

$$\int_{-\infty}^{\infty} f(x)\,\delta'(x)dx = [f(x)\,\delta(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\,\delta(x)\,dx$$

$$= 0 - f'(0) \text{ by (7)}$$

$$= -f'(0) \qquad ...(14)$$

If $\delta^{(n)}$ be the nth derivative of $\delta(x)$, then similarly we find on repeating this process n times,

$$\int_{-\infty}^{\infty} f(x)\,\delta^{(n)}\,(x)dx = (-1)^n f^{(n)}(0) \qquad ...(15)$$

Problem 52. *Show that the function* $\delta(x) = \lim_{\varepsilon\to 0} \frac{\sin(2\pi \in x)}{\pi x}$ *is a Dirac-delta function.*

We have $\delta(x) = \lim_{\varepsilon\to 0} \frac{\sin(2\pi \in x)}{\pi x}$

$\therefore\ \delta(x) = 0$, when $x \neq 0$. ...(1)

and $\int_{-\infty}^{\infty} \frac{\sin(2\pi \in x)}{\pi x}\,dx = 2\int_{0}^{\infty} \frac{\sin(2\pi \in x)}{\pi x}\,dx$, the function being even.

$$= \frac{2}{\pi}\cdot\frac{\pi}{2} = 1 \qquad ...(2)$$

It follows from (1) and (2) that the given fucntion is Dirac-delta function.

8.11. RIEMANN-ZETA FUNCTION

If $s = \sigma + i\,t$ where σ and t are real and if $\in > 0$, then the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^2} \qquad ...(1)$$

is a uniformly convergent series of analytic functions in a domain in which $\sigma \geq 1 + \in$ and hence the series is an analytic function of s in that domain. This function $\zeta(s)$ is called as *Riemman-zeta-function*.

The *generalized zeta function* for $\sigma \geq 1 + \epsilon$ is defined as

$$\zeta(s, a) = \sum_{n=1}^{\infty} \frac{1}{(a+n)^s} \quad \ldots(2)$$

where a is a constant.

If $0 < a \leq 1$ and $\arg(a + n) = 0$, then $\zeta(s, 1) = \zeta(s)$. ...(3)

The expression of $\zeta(s, a)$ as an infinite integral, when $\sigma \geq 1 + \epsilon$ and $\arg x = 0$, is

$$\zeta(s, a) = \frac{1}{\Gamma s} \int_0^{\infty} \frac{x^{s-1} e^{-ax}}{1 - e^{-x}} dx \quad \ldots(4)$$

The expression of $\xi(s, a)$ as a contour integral is

$$\zeta(s, a) = -\frac{\Gamma(1 - s)}{2\pi i} \int_{\infty}^{(0+)} \frac{(-z)^{s-1} e^{-az}}{1 - e^{-z}} dz \quad \ldots(5)$$

Riemann's relation between $\zeta(s)$ and $\zeta(1 - s)$ is

$$2^{1-s} \Gamma(s)\, \zeta(s) \cos\left(\tfrac{1}{2} s\pi\right) = \pi^s \zeta(1 - s) \quad \ldots(6)$$

Hermite's formula for $\zeta(s, a)$ is

$$\zeta(s, a) = \tfrac{1}{2} a^{-s} + \frac{a^{1-s}}{s-1} + 2 \int_{0}^{\infty} \left(a^2 + y^2\right)^{-\frac{1}{2}s} \left\{\sin\left(s \text{ arc} \tan\frac{y}{a}\right)\right\} \frac{dy}{e^{2\pi y} - 1} \quad \ldots(7)$$

$$\therefore \qquad \zeta(o, a) = \tfrac{1}{2} - a \quad \ldots(8)$$

and

$$\left\{\frac{d}{ds} \zeta(s, a)\right\}_{s=0} = \log \Gamma(a) - \tfrac{1}{2} \log(2\pi) \quad \ldots(9)$$

$$\zeta'(o) = -\tfrac{1}{2} \log(2\pi) \quad \ldots(10)$$

ADDITIONAL MISCELLANEOUS PROBLEMS

Problem 53. *Solve the equation $x^2 y'' + xy' + (x^2 - n^2)\, y = 0$, n = constant, in the neighbourhood of the point $x = 0$. What are the singular points of the equation?*

(Bombay, 1965)

Problem 54. *Prove that if the functions $P_l(x)$ are defined by the equation*

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{l=0}^{\infty} P_l(x)\, t^l$$

then $P_l(x)$ is a polynomial of l^{th} degree, satisfying the following relations :

(a) $\int_{-1}^{1} P_l(x)\, P_{l'}(x)\, dx = \frac{2}{2l+1} \delta_{ll'}$ *(Rohilkhand, 1981)*

(b) $(l+1) P_{l+1}(x) - (2l+1)\, x P_l(x) + l P_{l-1}(x) = 0$ *(Bombay, 1965)*

Problem 55. *Define Bessel functions and obtain the series solution of the Bessel differential equation for $n = 0$ either by the use of operators or otherwise.*

Show that $\left[J_{\frac{1}{2}}(x)\right]^2 + \left[J_{-\frac{1}{2}}(x)\right]^2 = \frac{2}{\pi x}$ *(Nagpur, 1965)*

Problem 56. *What are Legendre Polynomials ? Show that*

(i) $(m + 1) P_{m+1}(x) = (2m + 1)x P_m(x) - m P_{m-1}(x)$

(ii) $(x^2 - 1) P'_m(x) = mx\, P_m(x) - m\, P_{m-1}(x)$, *where* $P'_m(x) = \frac{d}{dx} P_m(x)$

(iii) $P_{2m}(-x) = P_{2m}(x)$

(iv) $P_{2m+1}(-x) = -P_{2m+1}(x)$ *(Agra, 1965)*

Problem 57. *Write the differential equation satisfied by Bessel's function of order n.*

Express the following Bessel functions in terms of trigonometric functions :

(i) $J_{1/2}(x)$, (ii) $J_{-1/2}(x)$; (iii) $J_{3/2}(x)$, (iv) $J_{-3/2}(x)$ *(Agra, 1966; Vikram, 67)*

Problem 58. *The generating function of Legendre Polynomial is*

$$T(w, s) = (1 - 2ws + s^2)^{-1/2} = \sum_{l=0}^{\infty} s^l P_l(w)$$

Using this prove the following properties of Legendre polynomials

(i) $\int_{-1}^{1} P_m(w)\, P_n(w)\, dw = \frac{2}{2m+1} \delta_{mn}$

(ii) $P_m(-w) = (-1)^m P_m(w)$

(iii) $\sum_{l=0}^{\infty} (2l+1) P_l(w) = 0$ *for* $w \neq \pm 1$ *(Agra, 1967)*

Problem 59. *Show that* $4x^2 - 2$ *is a polynomial solution of the differential equation.*

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2\alpha y = 0$$

where α *is a positive integer.* *(Vikram, 1967)*

Hint. Given equation is Hermite differential equation and show that

$$H_2(x) = 4x^2 - 2.$$

Problem 60. *Show that* $\int_0^{\infty} (\cos xy)(\cos xy')\, dx = \frac{1}{2}\pi\delta(y - y')$

with $y > 0$, $y' > 0$ *and* $\delta(y - y')$ *being the Dirac-delta function.* *(Agra, 1968)*

Problem 61. *The generating function for functions of integral orders is*

$$G(x, h) = e^{x/2(h - h^{-1})} = \sum_{n=-\infty}^{\infty} h^n J_n(x)$$

(a) *Show by direct substitution that* $G(x, h)$ *satisfies the equation*

$$x^2 \frac{\partial^2 G}{\partial x^2} + \frac{\partial G}{\partial x} + x^2 G - \left(h \frac{\partial}{\partial h}\right)^2 G = 0$$

use this result to show that $J_n(x)$ *satisfies Bessel's differential equation,*

$$x^2 \frac{\partial^2 y}{\partial x^2} + x\frac{\partial y}{\partial x} + \left(x^2 - n^2\right) y = 0$$

(b) *In the generating function, putting* $h = e^{i\theta}$, *show that*

$$\cos(x \sin\theta) = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x) \cos 2n\theta$$

$$\sin(x \sin\theta) = \sum_{n=0}^{\infty} J_{2n+1}(x) \sin(2n+1)\theta$$ *(Agra, 1968)*

Problem 62. *Solve by series integration, the equation*

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \lambda(\lambda+1)y = 0$$

where λ is an integer. Discuss the nature of the solution for $\lambda = 1, 2,$ (*Vikram, 1969*)

Problem 63. *Write short note on Bessel's functions.* (*Vikram, 1969*)

Problem 64. *Solve the equation*

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

where n is a positive integer. If $P_n(x)$ is the Polynomial solution, prove that these functions form an orthogonal system. (*Agra, 1969*)

Problem 65. (*a*) *Solve the equation in series*

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - \frac{1}{4}\right)y = 0$$

(*b*) *Show that* $\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = 0$ *unless* $m = n$

$$= \frac{2}{2n+1} \text{ if } m = n$$

m, n being positive integers. (*Rohilkhand, 1980, 81; Agra, 1970*)

(*c*) *Using Rodrigne's formula, prove that*

$$\int_{-1}^{1} x^m P_n(x) = \frac{2^{n+1}(\lfloor n)^2}{\lfloor 2n+1}\,\delta_{mn}.$$ (*Rohilkhand, 1987*)

Problem 66. *Obtain the solution of the Bessel's differential equation*

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)\,y = 0$$

in the form of a power series. Discuss this solution in the neighbourhood of $x = 0$.

Show that $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)\ (n = 1, 2, 3, \ldots\ldots)$ (*Bombay, 1970*)

Problem 67. (*a*) *Show that Bessel functions defined by* $e^{x/2(t-1/t)} = \sum_{-\infty}^{\infty} J_n(x)\, t^n$ *have the integral representation*

$$J_n(x) = \frac{1}{2\pi}\int_0^{2\pi} \cos(x\sin\theta - n\theta)\, d\theta$$ (*Agra, 1973*)

(*b*) If α *and* β *are the roots of the equation* $J_0(x) = 0$, *show that*

$$\int_0^1 xJ_0(\alpha x)\, J_0(\beta x)\, dx = \frac{1}{2}\delta_{\alpha\beta}\, J_1^2(\alpha)$$ (*Rohilkhand, 1982; Agra, 1971*)

Hint for (*b*) $J_n(x)$ is the solution of Bessel's equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0 \qquad \ldots(1)$$

Take two Bessel's functions of the first kind s.t.

$$u = J_n(\alpha x),\quad v = J_n(\beta x) \qquad \ldots(2)$$

Replacing x by αx and y by u in (1), we get

$$x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} + \left(\alpha^2 x^2 - n^2\right)u = 0 \qquad \ldots(3)$$

Similarly replacing x by βx and y by v in (1), we get

$$x^2\frac{d^2v}{dx^2}+x\frac{dv}{dv}+\left(\beta^2 x^2 - n^2\right)v=0 \quad ...(4)$$

Multiplying (3) by $\frac{v}{x}$, (4) by $\frac{u}{x}$ and subtracting

$$x\left[v\frac{d^2u}{dx^2}-u\frac{d^2v}{dx^2}\right]+\left(v\frac{du}{dx}-u\frac{dv}{dx}\right)+\left(\alpha^2-\beta^2\right)xuv=0$$

or $$\frac{d}{dx}\left[x\left(v\frac{du}{dx}-u\frac{dv}{dx}\right)\right]+\left(\alpha^2-\beta^2\right)xuv=0$$

$$\frac{d}{dx}\left[x\left\{J_n(\beta x)\frac{d}{dx}J_n(\alpha x)-J_n(\alpha x)\frac{d}{dx}J_n(\beta x)\right\}\right]$$

$$+(\alpha^2-\beta^2)\,xJ_n(\alpha x)\,J_n(\beta x) = 0 \text{ by } (2) \quad ...(5)$$

Integrating *w.r.t.* x within limits 0 and 1,

$$\left[xJ_n(\beta x)\frac{d}{dx}J_n(\alpha x)-xJ_n(\alpha x)\frac{d}{dx}J_n(\beta x)\right]_0^1$$

$$+\int_0^1(\alpha^2-\beta^2)\,x\,J_n(\alpha x)\,J_n(\beta x)\,dx = 0 \quad ...(6)$$

when α, β are distinct roots of $J_n(\mu)=0$; $J_n(\alpha)=0$, $J_n(\beta)=0$ and $J_n(0)$ are all finite; the first term on L.H.S. of (6) vanishes for both the limits, giving

$$\left(\alpha^2-\beta^2\right)\int_0^1 x\,J_n(\alpha x)\,J_n(\beta x)\,dx = 0 \text{ or } \int_0^1 xJ_n(\alpha x)\,J_n(\beta x)\,dx = 0 \quad ...(7)$$

as $\alpha \neq \beta$

Again, when $\alpha=\beta$, then (6) gives

$$\int_0^1 xJ_n(\alpha x)\,J_n(\alpha x)\,dx$$

$$=\lim_{\beta\to\alpha}\frac{\left[xJ_n(\beta x)\frac{d}{dx}J_n(\alpha x)-xJ_n(\alpha x)\frac{d}{dx}J_n(\beta x)\right]_0^1}{\beta^2-\alpha^2} \quad ...(8)$$

with $J_n(\alpha)=0$ and $\beta\to\alpha$, R.H.S. $\to \lim_{\beta\to\alpha}\frac{\left[xJ_n(\beta x)\frac{d}{dx}J_n(\alpha x)\right]_0^1}{\beta^2-\alpha^2}$,

Rec-relation

$$\frac{d}{dx}\left[x^{-n}J_n(x)\right]=-x^{-n}J_{n+1}(x) \text{ gives} \quad ...(9)$$

$$x^{-n}\frac{d}{dx}J_n(x)-nx^{-n-1}J_n(x)=-x^{-n}J_{n+1}(x)$$

or $$x\frac{d}{dx}J_n(x) = nJ_n(x)-xJ_{n+1}(x) \quad ...(10)$$

or $$\alpha x\frac{d}{d(\alpha x)}J_n(\alpha x)=nJ_n(\alpha x)-\alpha xJ_{n+1}(\alpha x), \text{ replacing } x \text{ by } \alpha x$$

or $$x\frac{d}{dx}J_n(\alpha x)=xJ_n(\alpha x)-\alpha xJ_{n+1}(\alpha x) \quad ...(11)$$

With its help, R.H.S. of (8) yields

$$\lim_{\beta\to\alpha} \frac{\left[J_n(\beta x)\left\{ nJ_n(\alpha x) - \alpha x J_{n+1}(\alpha x)\right\}\right]_0^1}{\beta^2 - \alpha^2}$$

$$= \lim_{\beta\to\alpha} \frac{\left[-\alpha x J_n(\beta x)\, J_{n+1}(\alpha x)\right]_0^1}{\beta^2 - \alpha^2}$$

$$= \lim_{\beta\to\alpha} \frac{-\alpha\, J_n(\beta)\, J_{n+1}(\alpha)}{\beta^2 - \alpha^2} \quad \text{as } J_n(0) = 0 \text{ for } n = 1, 2, 3, \ldots$$

$$= \lim_{\beta\to 0} \frac{\frac{\partial}{\partial\beta}\left[x J_n(\beta)\, J_{n+1}(\alpha)\right]}{\frac{\partial}{\partial\beta}(\beta^2 - \alpha^2)} = \lim_{\beta\to 0} \frac{-\alpha \frac{\partial}{\partial\beta} J_n(\beta)\, J_{n+1}(\alpha)}{2\beta} \qquad \ldots(12)$$

Replacing x by β in (10), we get

$$\beta \frac{\partial}{\partial\beta} J_n(\beta) = nJ_n(\beta) - \beta J_{n+1}(\beta)$$

i.e.

$$\frac{\partial}{\partial\beta} J_n(\beta) = \frac{1}{\beta}\left[nJ_n(\beta) - \beta J_{n+1}(\beta)\right]$$

So that (12) *i.e.* R.H.S. of (8) becomes

$$\lim_{\beta\to\alpha} \frac{-\alpha}{2\beta}\cdot\frac{1}{\beta}\left[nJ_n(\beta) - \beta J_{n+1}(\beta)\right] J_{n+1}(\alpha)$$

$$= -\frac{\alpha}{2\alpha^2}\left[nJ_n(\alpha) - \alpha J_{n+1}(\alpha)\right] J_{n+1}(\alpha)$$

With its help (8) renders

$$\int_0^1 x J_n^2(\alpha x)\, dx = \frac{1}{2} J_{n+1}^2(\alpha) \qquad \ldots(13)$$

which is normalization condition for Bessel's functions.

Combining (7) and (13), we get

$$\int_0^1 x\, J_n^2(\alpha x)\, J_n(\beta x)\, dx = \frac{1}{2} J_{n+1}^2(\alpha)\, \delta_{\alpha\beta} \qquad \ldots(14)$$

For $n = 0$, this reduces to

$$\int_0^1 x\, J_0(\alpha x)\, J_0(\beta x)\, dx = \frac{1}{2}\delta_{\alpha\beta}\, J_1^2(\alpha)$$

Problem 68. *Explain the significance of spherical harmonics and discuss some general properties of harmonic functions.* *(Rohilkhand, 1981; Agra, 1971)*

Problem 69. *Prove that the polynomial solution $P_n(x)$ of the Legendre equation*

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)\,y = 0$$

satisfies the orthogonality condition $\int_{-1}^{1} P_n(x)\, P_m(x)\, dx = \frac{2}{2n+1}\,\delta_{nm}$.

(Rohilkhand, 1981)

Show that this property allows for the expansion of an arbitrary function $f(x)$ into a series of Legendre polynomials provided that $f(x)$ is sectionally continuous in $-1 \le x \le 1$ and $\frac{df(x)}{dx}$ is sectionally continuous in $-1 < x < 1$. *(Agra, 1972)*

Problem 70. *Prove that* $[J_0(x)]^2 + 2[J_1(x)]^2 + 2[J_2(x)]^2 + \ldots\ldots = 1.$

(Agra, 1973)

Problem 71. *Prove that the function* $y = \dfrac{d^n}{dx^n}\left(x^2 - 1\right)^n$ *satisfies Legendre's differential equation*

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)\,y = 0$$

Hence obtain Rodrigue's formula for Legendre-polynomials $P_n(x)$.

Using this formula prove that

$$\int_{-1}^{1} x^m P_n(x)\,dx = 0 \text{ for } m < n.$$

(Rohilkhand, 1985; Agra, 1973)

Problem 72. *Show that* $\displaystyle\int_{-1}^{1} x\,P_n(x)\,P_{n-1}(x)\,dx = \frac{2n}{4n^2 - 1}$

(Rohilkhand, 1983, 84, 89; Agra, 1974)

Recurrence relation

$$P_n = (2n-1)\,xP_{n-1} - (n-1)\,P_{n-2} \text{ gives}$$

$$x\,P_{n-1} = \frac{1}{2n-1}\left[nP_n + (n-1)\,P_{n-2}\right] \qquad \ldots(1)$$

$$\therefore \quad \int_{-1}^{1} xP_n(x)\,P_{n-1}(x)\,dx = \int_{-1}^{1} P_n(x)\left\{x\,P_{n-1}(x)\right\}dx$$

$$= \int_{-1}^{1} P_n(x)\left[\frac{1}{2n-1}\left\{nP_n + (n-1)\,P_{n-2}\right\}\right]dx \text{ by (1)}$$

$$= \frac{1}{2n-1}\left[n\int_{-1}^{1}\left[P_n(x)\right]^2 dx + (n-1)\int_{-1}^{1} P_n(x)\,P_{n-2}(x)\,dx\right]$$

$$= \frac{1}{2n-1}\left[n\cdot\frac{2}{2n+1}\,\delta_{mn} + (n-1)\cdot\frac{2}{2n+1}\,\delta_{n,\,n-2}\right]$$

by orthogonal property of Legendre's polynomials.

$$= \frac{1}{2n-1}\cdot\frac{2n}{2n+1} = \frac{2n}{4n^2-1} \text{ as } \delta_{mn} = 1 \;\&\; \delta_{n,\,n-2} = 0$$

Problem 73. *Express the electrostatic potential at a point P in space, due to two equal but opposite point charges such that the distance between them is small compared to their distance from P, in terms of Legendre-polynomials.* *(Agra, 1975)*

We know that the potential at a point $P(x, y, z)$ distant r from O (origin), and ρ from z-axis such that OP makes an angle θ with z-axis, due to a charge q at $z = \zeta$ is given by

$$\varphi(r, \theta) = \frac{q}{4\pi\varepsilon}\cdot\frac{1}{\rho}, \; \varepsilon \text{ being inductive capacity.}$$

Here ρ is given by

$$\rho = (r^2 + \zeta^2 - 2r\,\zeta\cos\theta)^{1/2}$$

$$\text{or} \quad \frac{1}{\rho} = \frac{1}{\zeta}\left[1 + \left(\frac{r}{\zeta}\right)^2 - 2\frac{r}{\zeta}\cos\theta\right]^{-1/2}$$

$$= \frac{1}{\zeta}\left[1 + \frac{r}{\zeta}\cos\theta + \left(\frac{r}{\zeta}\right)^2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + \ldots\right]^{-1/2}$$

on expanding by Binomial theorem under assumption $\left|\left(\frac{r}{\zeta}\right)^2 - 2\frac{r}{\zeta}\cos\theta\right| < 1$

$$= \frac{1}{\zeta}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{r}{\zeta}\right)^n,\ r < \zeta$$

where coefficients of $\frac{r}{\zeta}$ being polynomials in $\cos\theta$ are termed as Legendre polynomials, such that

$$P_0(\cos\theta) = 1,\ P_1(\cos\theta) = \cos\theta,$$

$$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1) = \frac{1}{4}(3\cos 2\theta + 1)$$

$$P_3(\cos\theta) = \frac{1}{2}(5\cos^3\theta - 3\cos\theta) = \frac{1}{8}(5\cos 3\theta + 3\cos\theta) \text{ etc.}$$

In case $r > \zeta$, we have on interchanging r and ζ

$$\frac{1}{\rho} = \frac{1}{r}\left[1 + \left(\frac{\zeta}{r}\right)^2 - 2\frac{\zeta}{r}\cos\theta\right]^{-1/2}$$

$$= \frac{1}{r}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{\zeta}{r}\right)^n$$

Let us now consider the potential at P due to a dipole consisting of a point charge $+q$ at a point $z = d$ on the x-axis and an equal but opposite charge $-q$ at O, the origin, such that the distance of P is very large as compared to d.

In view of the last relation, the potential at a distant point P is given by

$$\phi = \frac{q}{4\pi\varepsilon}\left(\frac{1}{\rho} - \frac{1}{r}\right)$$

$$= \frac{q}{4\pi\varepsilon}\cdot\frac{1}{r}\left[\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{d}{r}\right)^n - 1\right]$$

$$= \frac{q}{4\pi\varepsilon}\cdot\frac{1}{r}\left[1 + \frac{d}{r}\cos\theta + \left(\frac{d}{r}\right)^2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + \ldots - 1\right]$$

$$= \frac{qd}{4\pi\varepsilon}\cdot\frac{\cos\theta}{r^2}, \text{ neglecting higher order terms as } d << r.$$

Problem 74. *For Bessel functon $J_n(x)$, find out a and b, where*

$$\frac{d}{dx}J_n(x) = a\,J_{n-1}'(x) + b\,J_{n+1}(x) \qquad \textit{(Agra, 1975)}$$

Hint. Compare with the recurrence relation

$$J_{n-1}(x) - J_{n+1}(x) = 2\frac{d}{dx}J_n(x).$$

Ans. $a = \frac{1}{2},\ b = -\frac{1}{2}$.

Problem 75. *Solve the differential equation*

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + l(l+1)\,y = 0$$

where l is a positive integer. Discuss the orthogonal properties of the functions obtained as solution. *(Rohilkhand, 1987; Agra, 1975, 1976)*

Problem 76. (*a*) *For Bessel functon* $J_n(x)$ *prove that*

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta \qquad \textit{(Rohilkhand, 1988)}$$

and hence show that $J_0(x) = \frac{2}{\pi}\int_0^1 \frac{\cos xt}{\sqrt{1-t^2}}\, dt$

(*b*) *Show that the recurrence relation*

$$J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

following directly from differentiation of

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta \qquad \textit{(Rohilkhand, 1976)}$$

Hint. For (*a*): In a section it has been proved that

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta$$

Put $\theta = 0$, $J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)\, d\theta$

$$= \frac{2}{\pi}\int_0^{\pi/2} \cos(x\sin\theta)\, d\theta \text{ (being given function of } \theta)$$

Now put $\sin\theta = t$ so that $d\theta = \frac{dt}{\cos\theta} = \frac{dt}{\sqrt{1-t^2}}$

Hence $J_0(x) = \frac{2}{\pi}\int_0^1 \frac{\cos xt}{\sqrt{1-t^2}}\, dt.$

Problem 77. (*a*) *Derive the Rodrigue's formula for the Legendre polynomial.*
(*b*) *Using Rodrigue's formula prove that*

$$\int_{-1}^{1} x^m P_n(x)\, dx = 0 \text{ if } m < n, \text{ and so } \int_{-1}^{1} x^2 P_5(x)\, dx = 0$$

(Rohilkhand, 1976, 78, 89)

Hint. For (*b*) Using Rodrigue's formula, we have

$$\int_{-1}^{1} x^m P_n(x)\, dx = \frac{1}{2^n \lfloor n} \int_{-1}^{1} x^m \left(\frac{d}{dx}\right)^n (x^2-1)^n\, dx$$

On integrating R.H.S. by parts repeatedly, the first term always vanishes and we are left with

$$\int_{-1}^{1} x^m P_n(x)\, dx = \frac{(-1)^m \lfloor m}{2^n \lfloor n} \int_{-1}^{1} \left(\frac{d}{dx}\right)^{n-m} (x^2-1)^n\, dx$$

$$= \frac{(-1)^m \lfloor m}{2^n \lfloor n}\left[\left(\frac{d}{dx}\right)^{n-m-1} (x^2-1)^n\right]_{-1}^{1} = 0 \text{ for } m < n.$$

Problem 78. *Show that the coefficient of* t^n *in the expansion of*

$e^{i(t-t^{-1})x/2}$ equals $\frac{1}{\pi}\int_0^{\pi} \cos(n\phi - x\sin\phi)\, d\phi$ *(Agra, 1976)*

Hint: The coefficient of t^n in $e^{i(t-t^{-1})x/2}$ is $P_n(x)$ which is also equal to

$\frac{1}{\pi}\int_0^{\pi} \cos(n\phi - x\sin\phi)\, d\phi$ and hence the proposition..

Problem 79. *Show that two independent solutions of Bessel's differential equation* $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2) y = 0$; *cannot be obtained by series integration method. Explain how the second sotution is obtained in this case. If* $J_0(\rho\lambda)$ *is Bessel's function of zero order and* $z > 0$, *show that*

$$\int_0^\infty e^{-z\lambda} J_0(\rho\lambda)\, d\lambda = \frac{1}{\sqrt{\rho^2 + z^2}}, \quad (\rho, z, \geq 0).$$ (*Rohilkhand, 1977, 82, 87*)

To show $\int_0^\infty e^{-z\lambda} J_0(\rho\alpha)\, d\lambda = \frac{1}{\sqrt{(\rho^2 + z^2)}}$

We have by §8.4 (18),

$$\cos(x \sin\theta) = J_0 + 2J_2 \cos 2\theta + \ldots + 2J_{2m} \cos 2m\,\theta + \ldots \qquad \ldots(1)$$

$$\therefore \int_0^\pi \cos(x \sin\theta)\, d\theta = \int_0^\pi J_0\, d\theta + \int_0^\pi 2J_2 \cos 2\theta\, d\theta + \ldots$$

$$\ldots + \int_0^\pi 2\, J_{2m} \cos 2m\,\theta\, d\theta + \ldots$$

$$= J_0 \cdot \pi + 0 + 0 + \ldots\ldots + 0 +$$

$$\therefore J_0(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin\theta)\, d\theta \qquad \ldots(2)$$

Now,

$$\int_0^\infty e^{-z\lambda} J_0(\rho\lambda)\, d\lambda = \int_0^\infty e^{-z\lambda} \left[\frac{1}{\pi} \int_0^\pi \cos(\rho\lambda \sin\theta)\, d\theta\right]$$

$$= \frac{1}{\pi} \int_0^\pi \left[\int_0^\infty e^{-z\lambda} \cos(\rho\lambda \sin\theta)\, d\lambda\right] d\theta$$

$$= \frac{1}{\pi} \int_0^\pi \left[\int_0^\infty e^{-z\lambda} \left(\frac{e^{i\rho\lambda \sin\theta} + e^{-i\rho\lambda \sin\theta}}{2}\right) d\lambda\right] d\theta$$

$$= \frac{1}{2\pi} \int_0^\pi \left[\int_0^\infty \left\{e^{-(z + i\rho \sin\theta)\lambda} + e^{-(z - i\rho \sin\theta)\lambda}\right\} d\lambda\right] d\theta$$

$$= \frac{1}{2\pi} \int_0^\pi \left[\frac{e^{-(z + i\rho \sin\theta)\lambda}}{-(z + i\rho \sin\theta)} + \frac{e^{-(z - i\rho \sin\theta)\lambda}}{-(z - i\rho \sin\theta}\right]_0^\infty d\theta$$

$$= \frac{1}{2\pi} \int_0^\pi \left[\frac{1}{z + i\rho \sin\theta} + \frac{1}{z - i\rho \sin\theta}\right] d\theta$$

$$= \frac{1}{2\pi} \int_0^\pi \frac{2z}{z^2 + \rho^2 \sin^2\theta}\, d\theta = \frac{2z}{2\pi} \cdot 2 \int_0^{\pi/2} \frac{d\theta}{z^2 + \rho^2 \sin^2\theta}$$

$$= \frac{2z}{\pi} \int_0^{\pi/2} \frac{\operatorname{cosec}^2\theta}{z^2 \operatorname{cosec}^2\theta + \rho^2}\, d\theta$$

$$= \frac{2z}{\pi} \int_0^{\pi/2} \frac{\operatorname{cosec}^2\theta\, d\theta}{(z^2 + \rho^2) + z^2 \cot^2\theta}\, d\theta$$

$$= \frac{2z}{\pi} \left[\frac{1}{z\sqrt{(z^2 + \rho^2)}} \cot^{-1} \frac{z \cot\theta}{\sqrt{(z^2 + \rho^2)}}\right]_0^{\pi/2} \quad \text{for } \lambda, \rho, \geq 0$$

$$= \frac{2}{\sqrt{\rho^2+z^2}}\left[\cot^{-1}0-\cot^{-1}\infty\right]$$

$$= \frac{2}{\pi\sqrt{(\rho^2+z^2)}}\cdot\frac{\pi}{2} = \frac{1}{\sqrt{(\rho^2+z^2)}}$$

Problem 80. *Define associated Legendre' polynomials and prove their Orthogonality condition.*

If $P_n^m(x)$ are the Associated Legendre' polynomials, show that

$$P_n^{-m}(x) = (-1)^m \frac{\lfloor n-m}{\lfloor n+m} P_n^m(x). \qquad \textit{(Rohilkhand, 1977)}$$

Problem 81. *Show that $J_n(x)$, the Bessel's function of first kind of order n, is the coefficient of t^n in the expansion of the function $e^{x/2(t-1/t)}$*

(Rohilkhand, 1978, 82).

Hint: See §8.4 [*B*]

Problem 82. *Derive Rodrigue's formula for the differential form of Legendie's polynomials from a set of orthogonal functions in the interval (–1, 1).*

(Rohilkhand, 1978)

Or, show that the Legendre polynomial $P_n(x)$ may be derived in the following differential form: $P_n(x) = \frac{1}{2^n \lfloor n}\frac{d^n}{dx^n}(x^2-1)^n$, and hence prove the Legendre polynomials form an orthogonal set in the interval $-1 \le x \le 1$.

Hint: See the hint on Problem 77 which is exactly the same.

Problem 83. *Solving Bessel's differential equation by the series integration method, derive the expression for Bessel's function of first kind and discuss how the second independent solution of this equation be obtained when its constant is integer.* *(Rohilkhand, 1979)*

Hint: See §8.4 [*A*] and [*F*].

Problem 84. *Give the differential equaton satisfied by Legendre polynomials $P_n(x)$.*

(Rohilkhand, 1980, 82)

Hint: See equation (1) of § 8.3 [*A*].

Problem 85. *Express* $\frac{d}{dx}P_{n+1}(x) - x\frac{d}{dx}P_n(x)$ in terms of $P_n(x)$.

(Rohilkhand, 1980)

Hint: See § 8.3 [*D*] III.

Problem 86. *Show that the Bessel functions behave like trigonometic function with damped amptitude.* *(Rohilkhand 1980)*

Problem 87. *Show that the Bessel diffferential equation:*

$$y'' + \frac{1}{\rho}y' + \left(1-\frac{v^2}{\rho^2}\right)y = 0, \ (v > 0) \textit{ has two solutions, one regular at } \rho = 0$$

and the other singular. Prove that of v is an integer, one solution is a constant multiple of the other.

(Rohilkhand, 1981, 87)

Problem 88. *If $P_l(x)$ is a solution of the equation:*

$\left(1-x^2\right)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + l(l+1)y = 0$, *then prove that* $\left(1-x^2\right)^{m/2}\frac{d^m P_l(x)}{dx^m}$ *is a solution of equation*

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+\left[l\,(l+1)-\frac{m^2}{(1-x^2)}\right]y=0,\ (l,\ m\ \textit{are positive integers})$$

(Rohilkhand, 1981)

Problem 89. *If n is a positive integer, prove that*

$\int_{-1}^{1} P_n(x)\left\{1-2xz+z^2\right\}^{-1/2} dx = \frac{2z^n}{2n+1}$, *and hence making use of Rodrigue's formula, deduce that*

$$\int_{-1}^{1}\left(1-x^2\right)^n\left(1-2xz+z^2\right)^{-n-1/2} dx = \frac{2^{2n+1}\,(\lfloor n)^2}{\lfloor(2n+1}$$

where $P_n(x)$ is Legrendre's polynomials *(Rohilkhand, 1982)*

Rodrigue's formula is $P_n(x) = \frac{1}{2^n\lfloor n}\frac{d^n}{dx^n}(x^2-1)^n$ ...(1)

Now,

$$\left(1-2xz+z^2\right)^{-1/2}=\sum_m z^m P_m(x)$$

$$\Rightarrow \int_{-1}^{1} P_n(x)\left(1-2xz+z^2\right)^{-1/2} dx = \sum_m z^m \int_{-1}^{1} P_n(x)\,P_m(n)\,dx$$

$$=\sum_m^{\infty} z^m\frac{2\delta_{mn}}{2m+1} = \frac{2z^n}{2n+1}$$

$$\Rightarrow \int_{-1}^{1}\frac{1}{2^n\lfloor n}\left\{\frac{d^n}{dx^n}\left(x^2-1\right)^n\right\}\left(1-2xz+z^2\right)^{-1/2} dx = \frac{2z^n}{2n+1} \text{ by (1)}$$

$$\Rightarrow \int_{-1}^{1}\left\{\frac{d^n}{dx^n}\left(x^2-1\right)^n\right\}\left(12xz+z^2\right)^{-1/2} dx = \frac{2^n\lfloor n\cdot 2z^n}{2n+1}$$

which renders on integrating by parts

$$\left[\left(1-2xz+z^2\right)^{-1/2}\frac{d^{n-1}}{dx^{n-1}}\left(x^2-1\right)^n\right]_{-1}^{1}-\int_{-1}^{1}\left(-\frac{1}{2}\right)\left(1-2xz+z^2\right)^{-3/2}$$

$$(-2z)\frac{d^{n-1}}{dx^{n-1}}\left(x^2-1\right)^n dx = \frac{2^{n+1}\lfloor n\cdot z^n}{2n+1}$$

or $$(-1)^{-1} z\int_{-1}^{1}(1-2xz+z^2)^{-3/2}\frac{d^{n-1}}{dx^{n-1}}\left(x^2-1\right)^n dx = \frac{2^{n+1}\lfloor n\, z^n}{2n+1}$$

which on integreting by parts again yields

$$(-1)^2 z\left[\left(1-2xz+z^2\right)^{-3/2}\frac{d^{n-2}}{dx^{n-2}}\left(x^2-1\right)\right\}_{-1}^{1}$$

$$-\int_{-1}^{1}\left(-\frac{3}{2}\right)\left(1-2xz+z^2\right)^{-5/2}(-2z)\frac{d^{n-2}}{dx^{n-2}}\left(x^2-1\right)^n dx\right]=\frac{2^{n+1}\lfloor n\, z^n}{2n+1}$$

$$\Rightarrow (-1)^2 z^2\int_{-1}^{1} 1.3\,(1-2xz+z^2)^{-5/2}\frac{d^{n-2}}{dx^{n-2}}(x^2-1)\,dx = \frac{2^{n+1}\lfloor n\, z^n}{2n+1}$$

Continuing this process of integration by parts, we reach

$$(-1)^n z^n\int_{-1}^{1} 1\cdot 2\cdot 3\cdots(2n-1)\,(1-2xz+z^2)^{-(2n+1)/2}\cdot(x^2-1)^n dx = \frac{2^{n+1}\lfloor n\, z^n}{2n+1}$$

or $(-1)^n z^n \int_{-1}^{1}(x^2-1)^n(1-2xz+z^2)^{-n-\frac{1}{2}}dx = \dfrac{2^{n+1}\lfloor n\, z^n}{1\cdot3\cdot5\ldots(2n-1)\,(2n-1)\,(2n+1)}$

$$= \frac{(2\cdot4\cdot6\ldots2n)\,2^{n+1}\lfloor n}{1\cdot2\ldots(2n+1)\,(2n))\,(2n+1)}$$

$$= \frac{2^n(1\cdot2\cdot3\ldots n)\,2^{n+1}\lfloor n}{\lfloor 2n+1} = \frac{2^n\lfloor n\, 2^{n+1}\lfloor n}{\lfloor 2n+1}$$

$$\Rightarrow \int_{-1}^{1}(1-x^2)^n(1-2xz+z^2)^{-n-\frac{1}{2}}dx = \frac{2^{2n+1}(\lfloor n)^2}{\lfloor 2n+1}$$

Problem 90. (*a*) *Prove that a general solution of the differential equation:* $x^2y'' + xy' + (x^2 - n^2)\,y = 0$, *for all value of n is* $y(x) = C_1 J_n(x) + C_2 Y_n(x)$; *where* $J_n(x)$ *and* $Y_n(x)$ *are the Bessel's functions of the first kind and second kind respectivly* C_1 *and* C_2 *being constants.*

(*b*) *Show that* $\dfrac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$. (*Rohilkhand, 1983, 84, 85, 89*)

Problem 91. *Prove* $(n+1)\,P_n(x) = \dfrac{d}{dx}P_{n+1}(x) - \dfrac{d}{dx}P_{n-1}(x)$ (*Rohilkhand, 1983*)

Problem 92. *Prove* $\int_{-1}^{1}\mu^2 P_2(\mu)\,d\mu = \dfrac{4}{15}$. (*Rohilkhand, 1985*)

Problem 93. *Find the solution of the differential equation:*

$$x\frac{d^2y}{dx^2} + (1+n-x)\frac{dy}{dx} + my = 0 \qquad \ldots(1)$$ (*Rohilkhand, 1985*)

Laguerre's equation of order $m+n$ *is given by*

$$x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + (m+n)\,y = 0 \qquad \ldots(2)$$

If z be a solution of (2), *then* $y = z$ *satisfies* (2)

i.e.
$$x\frac{d^2z}{dx^2} + (1-x)\frac{dz}{dx} + (m+n)z = 0$$

which on differentiating n times w.r.t. x by Leibnitz's, theorem gives

$$\left\{x\frac{d^{n+2}z}{dx^{n+2}} + {}^nC\cdot1\cdot\frac{d^{n+1}z}{dx^{n+1}}\right\} + \left\{(1-x)\frac{d^{n+1}z}{dx^{n+1}} + {}^nC_1\,(-1)\frac{d^nz}{dx^n}\right\} + (m+n)\frac{d^nz}{dx^2} = 0$$

or
$$x\frac{d^{n+2}}{dx^{n+2}} + (n+1-x)\frac{d^{n+1}z}{dx^{n+1}} + m\frac{d^nz}{dx^n} = 0$$

or
$$x\frac{d^2}{dx^2}\left(\frac{d^nz}{dx^n}\right) + (1+n-x)\frac{d}{dx}\left(\frac{d^nz}{dx^n}\right) + m\frac{d^nz}{dx^n} = 0$$

In view of (1), it shows that $\dfrac{d^nz}{dx^n}$ is a solution of (1).

Since $L_{m+n}(x)$ is the solution of (2), therefore $\dfrac{d^n}{dx^n}L_{m+n}(x)$ is the solution of (1).

Also $(-1)^n \dfrac{d^n}{dx^n}L_{m+n}(x)$ is a solution of (1), since $(-1)^n$ is a constant.

As such $L_m^n(x)$ is a solution of (1) as we define $L_m^{n\prime}(x) = (-1)^n \dfrac{d^n}{dx^n} L_{m+n}(x)$.

Problem 94. (*a*) *Find the solution of the following differential equation:*

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + 4(x^2-n^2)\,y = 0,$$

(*b*) *If* α *and* β *are different roots of equation* $J_n(\mu) = 0$, *then show that*

$\int_0^1 x J_n(\alpha x)\, J_n(\beta x)\, dx = \frac{1}{2} J_{n+1}^2(\alpha)\, \delta(\alpha\beta)$, *where symbols have usual meanings.*

(*Rohilkhand, 1986*)

Problem 95. *If a and b are distinct positive roots of the equation* $J_n(\alpha) = 0$, where $J_n(\alpha)$ *is Bessel's function of nth order, prove that* $\int_0^1 x J_n(ax)\, J_n(bx)\, dx = 0$

(*Rohilkhand, 1988*)

Problem 96. *Prove that Legendre's Polynomial* $P_n(x)$ *is the coefficient of* z^n in the expansion of $Q = \{1 - 2xz + z^2\}^{-1/2}$ *in ascending powers of z, provided that* $-1 \le x \le 1$ and $|z| < 1$. *Hence or otherwise show that*

$P_2(\cos\theta) = \frac{1}{4}(3\cos 2\theta + 1)$ and $P_3(\cos\theta) = \frac{1}{8}(5\cos 3\theta + \cos\theta)$ (*Rohilkhand, 1988*)

Problem 97. *Prove that* $v = P_n^m(x) = (1-x^2)^{m/2} \dfrac{d^m}{dx^m} P_n(x)$ *satisfies the equation:*

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left\{n(n+1) - \frac{m^2}{1-x^2}\right\} y = 0.$$ (*Rohilkhand, 1988*)

Problem 98. *Show that if* $R_m(x)$ *is a polynomial of degree* $m < n$, *we have*

$$\int_{-1}^1 P_n(x) R_m(x)\, dx = 0$$ (*Rohilkhand, 1988*)

Problem 99. *Show that* $3x^2 + 5x^3 = P_0(x) + 3P_1(x) + 2P_2(x) + 2P_3(x)$.

(*Rohilkhand, 1988*)

Problem 100. *Starting from the relation* $J_n(x) = \sum_{s=0}^{\infty} \dfrac{(-1)^s}{\pi(s)\,\pi(n+s)} \left(\dfrac{x}{2}\right)^{n+2s}$ *symbols having their usual meanings, deduce that*

$J_{1/2}(x) = \sqrt{\dfrac{2}{\pi x}} \sin x$; $J_{-1/2}(x) = \sqrt{\dfrac{2}{\pi x}} \cos x$ and $\dfrac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$.

Using these results prove that $J_{5/2}(x) = \sqrt{\dfrac{2}{\pi x}} \left(\dfrac{3-x^2}{x^2} \sin x - \dfrac{3}{x}\cos x\right)$

(*Rohilkhand, 1989*)

Problem 101. *Prove that an arbitrary function* $F(x)$ *of x can be expanded in a series of Legendre's polynomicals in the form* $F(x) = \sum_{r=0}^{\infty} a_r\, \beta_r(x)$. *If* $F(x) = R_m(x)$, *where*

$R_m(x)$ is a polynomial of degree m in x, show that $R_m(x)=\sum_{r=0}^{m} a_r \beta_r(x)$. *If* $m < n$, *use the expansion to prove that* $\int_{-1}^{1}\beta_n(x)R_m(x)dx=0$.

(*Rohilkhand, 1989*)

Problem 102. *Obtain the Legendre series for the function f(x) given by*

$$f(x)=\begin{cases}0, & -1<x<0\\ 1, & 0<x<1\end{cases}$$ (*Rohilkhand, 1990*)

Problem 103. *If* (u, ϕ, z) and (r, θ, ϕ) *are the cylindrical and polor coordinates of the same point and if* $\mu=\cos\theta$, *then show their*

$$P_n(\mu)=(-1)^n\frac{r^{n+1}}{\underline{|n}}\frac{\partial^n}{\partial z^n}\left(\frac{1}{r}\right).$$

Problem 104. *If the electronic charge of hydrogen atom is distributed about the proton as origin in accordance with distribution function* $P(\rho)=-k\cdot\rho^{2m+2}\cdot e^{-\rho}[L_{n+l}^{2m+1}(\rho)]^2$. *n, m being total and angular quantum numbers designating the state of the atom, k a constant for given state* $\rho=\frac{2r}{na_0}$, *r, the radius vector,* a_0 *being fundamental constant (known as first Bohr radius), then calculate mean values of* r^{-1} *and r for the distribution.*

Hint: Mean value of r^{-1} *i.e.* $\frac{1}{r}$ is given by

$$\overline{r^{-1}}=\int_0^\infty\frac{P(\rho)}{r}d\rho\Big/\int_0^\infty P(\rho)d\rho=\frac{2}{na_0}\int_0^\infty\frac{P(\rho)}{\rho}d\rho\Big/\int_0^\infty P(\rho)d\rho$$

Here $N'=k\int_0^\infty\rho^{2m+1}\cdot e^{-\rho}\left[L_{n+1}^{2m+1}(\rho)\right]^2d\rho$, L being Legendre function

$$=k\int_0^\infty e^{-\rho}\cdot\rho^{l+q-1}L_{n+1}^{l}(\rho)\,L_{n+1}^{l}(\rho)d\rho \text{ for } l=2m+1$$

$$=k\,I_{n+1,\,n+1}=\frac{k\left(\underline{|n+m}\right)^3}{\underline{|n+m-l}}=I_1\,(say)\ for\ q=1$$

and $D'=k\int_0^\infty e^{-\rho}\cdot e^{2m+2}L_{n+1}^{2m+1}(\rho)\,L_{n+m}^{2m+1}(\rho)d\rho$

$$=k\int_0^\infty e^{-\rho}\cdot\rho^{l+q-1}L_{n+m}^{l}(\rho)\,L_{n+m}^{l}(\rho)d\rho \text{ for } l=2m+1$$

$$=k\,I_{n+m,n+m}=k\frac{\left(\underline{|n+m}\right)^3}{\underline{|n+m-l}}n=I_2\text{ (say) for } q=2$$

$$\therefore\quad \overline{r^{-1}}=\frac{2}{na_0}\cdot\frac{I_1}{I_2}=\frac{1}{a_0n^2}=a_0^{-1}\ if\ n=1\text{ and } m=0$$

Similarly $\bar{r}=\frac{3}{2}a_0$ *for* $n=1, m=0$ with the choice $q=2$ and $q=3$

CHAPTER 9

FOURIER'S SERIES, INTEGRALS AND TRANSFORMS

9.1. DEFINITION AND EXPANSION OF A FUNCTION OF x.

A Fourier series is a representation employed to express a periodic function $f(x)$ defined in an interval say $(-\pi, \pi)$ a linear relation between the sines and cosines of the same period, *viz.*

$$f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \ldots + a_n \cos nx + \ldots$$
$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots + b_n \sin nx + \ldots$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx. \quad \ldots(1)$$

In order to determine the values of the coefficients a_0, a_n and b_n, let us first integrate both the sides of (1) between the limits $-\pi$ and π, whence we get

$$\int_{-\pi}^{\pi} f(x)\ dx = a_0 \int_{-\pi}^{\pi} dx \text{ other integrals vanishing,}$$

i.e.
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(v)\ dv,$$

replacing x by v to distinguish from $f(x)$.

Again multiplying both sides of (1) by $\cos nx$ and integrating between the limits $-\pi$ and π, we get

$$\int_{-\pi}^{\pi} f(x) \cos nx\, dx = a_n \int_{-\pi}^{\pi} \cos^2 nx\, dx$$

other integrals vanishing

$$= \frac{a_n}{2}\int_{-\pi}^{\pi} (1 + \cos 2nx)\ dx = \frac{a_n}{2} \cdot 2\pi$$

$$\therefore\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(v) \cos nv\ dv.$$

Further multiplying both sides of (1) by $\sin nx$ and integrating between the limits $-\pi$ and π, we get as above

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(v) \sin nv\, dv.$$

Hence

$$f(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(v)\ dv+\frac{1}{\pi}\sum_{n=1}^{\infty}\cos nx\int_{-\pi}^{\pi}f(v)\cos nv\ dv$$

$$+\frac{1}{\pi}\sum_{n=1}^{\infty}\sin nx\int_{-\pi}^{\pi}f(v)\sin nv\,dv \qquad ...(2)$$

The expansion on R.H.S. of (2) is known as *Fourier's series for f(x)* and a_0, a_n, b_n etc, are known as *Fourier's constants* for $f(x)$.

Now (2) may also be written as

$$f(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(v)\ dx+\frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi}f(v)\cos n\ (x-v)\ dv \qquad ...(3)$$

which is valid for $-\pi\le x\le\pi$

Deductions from (2)

(*i*) If $f(x)$ be an odd function of x, *i.e.* if $f(-x)=-f(x)$, then

$$\int_{-\pi}^{\pi}f(v)dv=0.$$

Also

$$\int_{-\pi}^{\pi}\cos nvf(v)\ dv=0$$

and

$$\int_{-\pi}^{\pi}\sin nv\ f(v)\ dv=2\int_{0}^{\pi}\sin nvf(v)\ dv. \qquad ...(4)$$

As such $f(x)=\dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\sin nx\int_{0}^{\pi}\sin nv\ f(v)\ dv.$

(*ii*) If $f(x)$ be an even function of x, *i.e.*, $f(-x)=f(x)$,

$$f(x)=\frac{1}{\pi}\int_{0}^{\pi}f(v)\ dv+\frac{2}{\pi}\sum_{n=1}^{\infty}\cos nx\int_{0}^{\pi}\cos nv\ f(v)\ dv \qquad ...(5)$$

COROLLARY 1. To find a cosine series for $f(x)$ when $0\le x\le\pi$, let us assume that

$$f(x)=a_0+\sum_{n=0}^{\infty}a_n\cos nx. \qquad ...(6)$$

Integrating both sides from 0 to π, we have

$$\int_{0}^{\pi}f(x)\ dx=\int_{0}^{\pi}a_0\ dx \text{ other integrals vanishing}$$

$$=a_0\pi$$

$$\therefore \qquad a_0=\frac{1}{\pi}\int_{0}^{\pi}f(v)\ dv.$$

Again multiplying both sides of (6) by $\cos nx$ and integrating from 0 to π, we get

$$\int_{0}^{\pi}\cos nx\,f(x)\,dx=a_n\int_{0}^{\pi}\cos^2 nx\,dx \quad \text{other integrals vanishing}$$

$$=\frac{a_n}{2}\int_{0}^{\pi}(1+\cos 2nx)\ dx$$

$$=\frac{a_n}{2}\cdot\pi$$

$\therefore \qquad a_n = \dfrac{2}{\pi}\int_0^{\pi} \cos nv\, f(v)\, dv.$

Hence

$$f(x) = \frac{1}{\pi}\int_0^{\pi} f(v)\, dv + \frac{2}{\pi}\sum_{n=1}^{\infty} \cos nx \int_0^{\pi} \cos nv\; f(v)\, dv \qquad ...(7)$$

COROLLARY 2. To find a sine series for $f(x)$ when $0 \le x \le \pi$. *(Agra, 1961)*

Let us assume that

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

Multiplying both sides by $\sin nx$ and integrating from 0 to π, we have

$$\int_0^{\pi} f(x) \sin nx\, dx = \tfrac{1}{2} b_n \int_0^{\pi} dx, \quad \text{other integrals vanishing}$$

$$= \frac{\pi}{2} \cdot b_n,$$

i.e. $\qquad b_n = \dfrac{2}{\pi}\int_0^{\pi} \sin nv\; f(v)\, dv.$

Hence $\qquad f(x) = \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty} \sin nx \int_0^{\pi} f(v)\; nv\, dv. \qquad ...(8)$

Important Remark

We generally take the Fourier series expansion in the interval $(-\pi, \pi)$ with period 2π, as

$$f(x) = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots$$
$$+ (a_n \cos nx + b_n \sin nx) + \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots$$
$$= \frac{a_0}{2} \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad ...(9)$$

where $\qquad a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\, dx$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx \qquad ...(10)$$

and $\qquad b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x) \sin nx\, dx$

In case the interval is $(-l, l)$ with period $2l$, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}\right) \qquad ...(11)$$

when $\quad a_n = \dfrac{1}{l}\displaystyle\int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} dx$ and $b_n = \dfrac{1}{l}\displaystyle\int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} dx \qquad ...(12)$

[These are obtained by replacing x by $\dfrac{\pi x}{l}$ in (9) and (10)]

9.2. DIRICHLET'S CONDITIONS *(Rohilkhand, 1991)*

A function $f(x)$ is said to satisfy the Dirichlet's condition in any interval (a, b) in which the function is defined, if it is subjected to either of the conditions.

(*i*) $f(x)$ is bounded in (a, b) *i.e.*, there axists an upper bound M such that $|f(x)| \le M$ for the values of x in (a, b) and the interval (a, b) can be divided into a finite number of open sub-intervals in each of which the funcitons $f(x)$ is monotonic.

(*ii*) $f(x)$ has a finite number of points of infinite discontinuity in the interval (a, b), but when the arbitrary small neighbourhoods of these points are excluded then $f(x)$ remains bounded in the deleted interval and the interval can be divided into a finite number of open sub-intervals in each of which $f(x)$ is monotonic. Also the infinite integral $\int_0^b f(x)\, dx$ is to be absolutely convergent.

Here below we clarify some terms used in these conditions:

Monotonic Functions. A function $f(x)$ defined in (a, b) is said to be monotonically increasing if $x_1, x_2 \in (a, b), x_2 > x_1 \Rightarrow f(x_2) \ge f(x_1)$, strictly monotonically increasing if $x_1, x_2 \in (a, b), x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$ monotonically decreasing if $x_1, x_2 \in (a, b), x_2 < x_1 \Rightarrow f(x_2) \le f(x_1)$ and strictly monotonically decreasing if $x_1, x_2 \in (a, b), x_2 < x_1 \Rightarrow f(x_2) < f(x_1)$.

If $f(x)$ be monotonically increasing in $[a, b]$, then for any point c such that $a < c < b$, $f(c-0)$ and $f(c+0)$ both exist.

Similarly, if $f(x)$ be monotonically decreasing in $[a, b]$ then for $a < c < b$, $f(c-0)$ and $f(c+0)$ both exist.

A function $f(x)$ tends to the limit l as $x \to c$ *i.e.*, $\lim_{x \to c} f(x) = l$, if for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - l| < \epsilon$. In case x is any point of (a, b), then it satisfies the condition $0 < |x - c| < \delta$.

Right handed and left handed limits.

i.e. $\lim_{x \to c+0} f(x)$ and $\lim_{x \to (c-0)} f(x)$

When a function $f(x)$ tends to l as x tends to c through values greater than c if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$|f(x) - l| < \epsilon$ and x is any point of (a, b) satisfying the condition $c < x < c + \delta$, then we say that the right handed limit exists and write $f(c+0) = \lim_{x \to (c-0)} f(x) = l$.

Similarly, when $f(x)$ tends to l as x tends to c through values smaller than c if to each $\epsilon > 0$ there corresponds $\delta > 0$ such that

$$x \in (a, b) \cap (c - \delta, c) \Rightarrow f(x) \in (l - \epsilon, l + \epsilon)$$

then we say that the left handed limit exists and write

$$f(c-0) = \lim_{x \to (c-0)} f(x) = l.$$

If $f(c+0) = f(c-0) = l$ *i.e.* $\lim_{x \to c+0} f(x) = \lim_{x \to c-0} f(x) = l$,

then we say that $\lim_{x \to c} f(x)$ exists at $x = c$.

If c is a limit point of (a, b) then $f(x)$ is said to be **continuous** at c if and only if $\lim_{x \to c} f(x)$ exists and equals $f(c)$ *i.e.*, if

$$\lim_{x \to c+0} f(x) = \lim_{x \to c-0} f(x) = f(c).$$

A function $f(x)$ is said to be continuous in an interval (a, b) if it is continuous at every point of the interval.

Discontinuities. A function $f(x)$ is said to be discontinuous at a point $x = c$ of its domain if $f(x)$ is not continuous at c. The discontinuity of $f(x)$ at c arises in either of the ways:

(*i*) $\lim_{x \to c} f(x)$ eixsts but $\lim_{x \to c} f(x) \neq f(c)$

(*ii*) $\lim_{x \to c} f(x)$ does not exist.

In first case the discontinuity is said to be of the *first kind* or a simple discontinuity while in the second case it is of the *second kind.*

Actually we classify the discontinuities of five types as follows:

(1) **Removable discontinuity.** When $f(c + 0)$ and $f(c - 0)$ both exist and are equal but differ from $f(c)$ *i.e.*, $f(c + 0) = f(c - 0) \neq f(c)$.

(2) **Discontinuity of first kind or ordinary discontinuity.** When $f(c + 0)$ and $f(c - 0)$ both exist and are finite but have different values while $f(c)$ may or may not be equal to either or them.

(3) **Discontinuity of the second kind.** When either $f(c + 0)$ or $f(c - 0)$ or both do not exist.

(4) **Mixed discontinuity.** When only one of $f(c + 0)$ and $f(c - 0)$ exists and the other does not exist.

(5) **Infinite discontinuity.** When either of $f(c + 0)$ and $f(c - 0)$ or $f(c)$ or both are infinite *e.g.* if $f(x) = \frac{1}{x}$

$$f(0 + 0) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} \frac{1}{h} = \infty$$

$$f(0 - 0) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} \left(\frac{1}{-h}\right) = -\infty$$

and $$f(0) = \frac{1}{0} = \infty$$

Hence $f(x)$ has an infinite discontinuity at $x = 0$.

Note. *We denote by* $[a, b]$ *closed interval,* (a, b) *open interval,* $[a, b)$ *semi-closed i.e., closed on the left and open on the right and similarly* $(a, b]$ *open on the left and closed on the right.*

9.3. ASSUMPTIONS FOR THE VALIDITY OF FOURIER'S SERIES EXPANSION, WITH ALLIED THEOREMS

The Fourier's series expansion and the determination of Fourier's constants is valid under the following assumptions:

(*i*) The expansion of $f(x)$ in a series of sines and cosines of integral multiplies of x is possible in the given interval.

(*ii*) The given function $f(x)$ is single-valued, continuous and integrable in the given range.

(*iii*) The series $a_0 + \sum_{1}^{\infty} (a_n \text{ con } nx + b_n \sin nx)$ is term by term integrable *i.e.*, the series is uniformly convergent in the interval $(-\pi, \pi)$.

(*iv*) The given function $f(x)$ given in the interval $(-\pi, \pi)$ satisfies Dirichlet's conditions so that the sum on the R.H.S. of (2) §9.1, has a limit as $n \to \infty$ and is equal to $f(x)$ at any point x $(-\pi < x < \pi)$, $f(x)$ being continuous and it is equal to $\frac{1}{2}[f(x+0) + f(x-0)]$ when there is an ordinary discontinuity at the point; also it is equal to $\frac{1}{2}[f(-\pi+0) + f(\pi-0)]$ at $x = \pm\pi$ when the limits $f(\pi-0)$ and $f(-\pi+0)$ exist.

THEOREM 1. If a function $f(x)$ is bounded and integrable in $[0, a]$, $a > 0$ and is monotonic in some interval $[0, c]$, c being positive and less than a i.e., $0 < c < a$, then

$$\lim_{n\to\infty} \int_0^a f(x) \frac{\sin nx}{x} dx = f(+0) \int_0^\infty \frac{\sin x}{x} dx \text{ where } f(+0) = f(0+0).$$

Suppose that $f(+0) = 0$ or without affecting the result, $f(0) = 0$ and take a positive number $h < c$.

We have by second mean value theorem that if $f(x)$ is continuous in $[a, a+h]$, derivable in $(a, a+h)$ then there exists at least one number θ between 0 and 1 such that

$$f(a+h) - f(a) = hf'(a+\theta h),\ 0 < \theta < 1 \text{ i.e., } \theta \in (0, 1)$$

Thus there exists $h' \in [0, h]$, such that

$$\int_0^h f(x) \frac{\sin nx}{x} dx = f(0) \int_0^{h'} \frac{\sin nx}{x} dx + f(h) \int_{h'}^h \frac{\sin nx}{x} dx$$

$$= f(h) \int_{h'}^h \frac{\sin nx}{x} dx \quad \text{since } f(0) = 0$$

$$= f(h) \int_{nh'}^{nh} \frac{\sin v}{v} dv \text{ on putting } nx = v$$

$$= f(h) \int_{nh'}^{nh} \frac{\sin x}{x} dx \text{ (changing the variable } v \text{ to } x) \qquad \ldots(1)$$

Now $\int_0^\infty \frac{\sin x}{x} dx$ being convergent, there exists some $k > 0$ such that $\left|\int_0^t \frac{\sin x}{x} dx\right| \le k$, for every $t \ge \theta$.

$$\text{i.e. } \left|\int_{nh'}^{nh} \frac{\sin x}{x} dx\right| = \left|\int_0^{nh} \frac{\sin x}{x} dx - \int_0^{nh'} \frac{\sin x}{x} dx\right| \le 2k \qquad \ldots(2)$$

Also we have $f(h) \to 0$ as $h \to (0+0)$ ...(3)

from (1), (2) and (3) we conclude that given an $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\left| \int_0^h f(x) \frac{\sin nx}{x} dx \right| \le 2k\,|f(h)| < \frac{\epsilon}{2} \text{ for } 0 < h \le \delta.$$

$$\text{But } \int_0^a f(x) \frac{\sin nx}{x} dx = \int_0^\delta f(x) \frac{\sin nx}{x} dx + \int_\delta^a f(x) \frac{\sin nx}{x} dx$$

where the second integral on the right tends to 0 as $n \to \infty$ and there exists a positive integer m such that for every $n \ge m$, we have

$$\left|\int_0^a f(x) \frac{\sin nx}{x} dx\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$$\text{Hence } \lim_{n\to\infty} \int_0^a f(x) \frac{\sin nx}{x} dx = f(+0) \int_0^\infty \frac{\sin x}{x} dx$$

In a general case replacing $f(x)$ by $(f(x) - f(0))$ we have

$$\lim_{n\to\infty} \int_0^a [f(x) - f(+0)] \frac{\sin nx}{x} dx = 0$$

But $\int_0^a \frac{\sin nx}{x} dx = \int_0^{na} \frac{\sin y}{y} dy \to \int_0^\infty \frac{\sin y}{y} dy$ as $n \to \infty$

Hence $\lim_{n\to\infty} \int_0^a f(x) \frac{\sin nx}{x} dx = f(+0) \int_0^\infty \frac{\sin x}{x} dx.$

Note. *The theorem is also true when $0 < a < \pi$.*

THEOREM 2. If $f(x)$ is bouned and integrable in $(-\pi, \pi)$ and monotonic in $[-c, 0)$ and $(0, c)$ where c is some positive number less than π, then

$$\frac{1}{2} a_n + \sum_{n=1}^{\infty} a_n = \frac{f(+0) + f(-0)}{2}$$

where $f(+0) = f(0+0)$ and $f(-0) = f(0-0)$.

We have

$$\frac{1}{2} a_0 + \sum_{n=1}^{m} a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx + \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sum_{n=1}^{m} \cos nx\, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \left\{1 + 2 \sum_{n=1}^{m} \cos nx\right\} dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x). \frac{\sin\left(m+\frac{1}{2}\right)x}{\sin \frac{1}{2}} dx \text{ (by summation)}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{0} f(x) \frac{\sin\left(m+\frac{1}{2}\right)x}{\sin \frac{x}{2}} dx + \frac{1}{2\pi} \int_0^{\pi} f(x) \frac{\sin\left(m+\frac{1}{2}\right)x}{\sin \frac{x}{2}} dx$$

Replacing x by $-x$ in the first integral on the right we have

$$\frac{1}{2} a_0 + \sum_{n=1}^{m} a_n = \frac{1}{2\pi} \int_0^{\pi} f(-x) \frac{\sin\left(m+\frac{1}{2}\right)x}{\sin \frac{x}{2}} dx + \frac{1}{2\pi} \int_0^{\pi} f(x) \frac{\sin\left(m+\frac{1}{2}\right)x}{\sin \frac{x}{2}} dx$$

$$= \frac{1}{2\pi} \cdot 2 \int_0^{\pi/2} f(-2x) \frac{\sin (2m+1)\, x}{\sin x} dx + \frac{1}{2\pi} \cdot 2 \int_0^{\pi/2} f(2x) \frac{\sin (2m+1)\, x}{\sin x} dx \text{ (on replacing } x \text{ by } 2x)$$

$$= \frac{1}{\pi} [f(-0) + f(+0)] \int_0^\infty \frac{\sin x}{x} dx \quad \text{as } m \to \infty \quad \text{by theorem 1.}$$

If we take $f(x) = 1$ for every value of x, then we find

$$\frac{1}{2} a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} 1.\, dx = 1 \text{ and } a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos nx\, dx = 0.$$

Hence we get

$$1 = \frac{1+1}{\pi} \int_{-\infty}^{\infty} \frac{\sin x}{x} dx \qquad i.e. \int_0^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}$$

So that $\frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n = \frac{f(+0) + f(-0)}{2}$.

THEOREM 3. If $f(x)$ is bounded and integrable in $[-\pi, \pi]$ and if it is possible to divide $[-\pi, \pi]$ into finite number of open sub-intervals, in each of which $f(x)$ is monotonic, then

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos n\xi + b_n \sin n\xi) = \tfrac{1}{2} [f(\xi - 0) + f(\xi + 0)] \text{ for } -\pi < \xi < \pi$$

$$= \tfrac{1}{2} [f(\pi - 0) + f(-\pi + 0) \text{ for } \xi = \pm\pi.$$

In order to prove it, let us first prove a lemma.

Lemma. *If $f(x)$ is bounded and integrable in every interval and is periodic with 2π as its period, then $\int_{-\pi}^{\pi} f(x)\, dx = \int_{-\pi}^{\pi} f(a+x)\, dx$, a being any number whatsoever.*

Putting $a + x = y$, we have

$$\int_{-\pi}^{\pi} f(a+x) dx = \int_{a-\pi}^{a+\pi} f(y)\, dy$$

$$= \int_{a-\pi}^{-\pi} f(y)\, dy + \int_{-\pi}^{\pi} f(y)\, dy + \int_{\pi}^{a+\pi} f(y)\, dy$$

Again putting $y = z - 2\pi$, we have

$$\int_{a-\pi}^{-\pi} f(y)\, dy = \int_{a+\pi}^{\pi} f(z - 2\pi)\, dz = -\int_{\pi}^{a+\pi} f(z)\, dz = -\int_{\pi}^{a+\pi} f(y)\, dy$$

$$\therefore \quad \int_{-\pi}^{\pi} f(a+x)\, dx = \int_{-\pi}^{\pi} f(y)\, dy \qquad \text{...(1)}$$

Now we come to the main theorem. We have

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{m} (a_n \cos n\xi + b_n \sin n\xi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$+ \sum_{n=1}^{m} \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, (\cos nx \cos n\xi + \sin nx \sin n\xi)\, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \left[1 + 2 \sum_{n=1}^{m} \cos n(x - \xi)\right] dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+\xi) \left[1 + 2 \sum_{n=1}^{m} \cos nx\right] dx \text{ by Lemma}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+\xi)\, \frac{\sin\left(m + \frac{1}{2}\right)x}{\sin \frac{x}{2}}\, dx \text{ (by summation)}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} f(x+\xi)\,\frac{\sin\left(m+\frac{1}{2}\right)x}{\sin\frac{x}{2}}\,dx + \frac{1}{2\pi}\int_{0}^{\pi} f(x+\xi)\,\frac{\sin\left(m+\frac{1}{2}\right)x}{\sin\frac{x}{2}}\,dx$$

$$= \frac{1}{2\pi}\int_{0}^{\pi} f(-x+\xi)\,\frac{\sin\left(m+\frac{1}{2}\right)x}{\sin\frac{x}{2}}\,dx + \frac{1}{2\pi}\int_{0}^{\pi} f(x+\xi)\,\frac{\sin\left(m+\frac{1}{2}\right)x}{\sin\frac{x}{2}}\,dx$$

(on replacing x by $-x$ in the first integral)

$$= \frac{1}{\pi}\int_{0}^{\pi/2} f(-2x+\xi)\,\frac{\sin\left(m+\frac{1}{2}\right)x}{\sin\frac{x}{2}}\,dx + \frac{1}{\pi}\int_{0}^{\pi/2} f(2x+\xi)\,\frac{\sin\left(m+\frac{1}{2}\right)x}{\sin\frac{x}{2}}$$

(on replacing x by $2x$)

$$\therefore\ \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos n\xi + b_n\sin n\xi) = \frac{1}{\pi}\,\{f(\xi-0)+f(\xi+0)\}\int_0^{\infty}\frac{\sin x}{x}\,dx$$

as $m \to \infty$ by theorem 1.

$$= \frac{1}{\pi}\,\{f(\xi-0)+f(\xi+0)\}\frac{\pi}{2} \text{ as in theorem 2.}$$

$$= \frac{1}{2}\,[f(\xi+0)+f(\xi-0)] \text{ when } m \to \infty.$$

we can thus restate this result as follows :

If $f(x)$ is bounded and integrable in $[-\pi, \pi]$ and if it is possible to divide $[-\pi, \pi]$ into a finite number of open subintervals in each of which $f(x)$ is monotonic, then the Fourier series corresponding to $f(x)$ converges for every x and if S be the sum function of the series, then

$S(x) = \frac{1}{2}\,[f(x+0)+f(x-0)]$ for every x in $(-\pi, \pi)$

$S(x) = \frac{1}{2}\,[f(\pi-0)+f(-\pi+0)]$ for every $x = \pm\pi$

and $S(x+2\pi) = S(x)$.

Note 1. *Here the relation $S(x+2\pi) = S(x)$ is very useful in determining the value of the sum function at a point which does not lie in the interval $[-\pi, \pi]$*

Note 2. *If $x = \xi$ be a point of continuity of $f(x)$, then*

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos n\xi + b_n\sin n\xi) = \frac{f(\xi+0)+f(\xi-0)}{2}$$

$$= \frac{f(\xi)+f(\xi)}{2} = f(\xi)$$

Conclusively if $f(x)$ satisfies the conditions of the theorem 3 in the interval $[-\pi, \pi]$, then the sum of the Fourier series corresponding to $f(x)$ is actually $f(x)$ at all such points x of $[-\pi, \pi]$, $f(x)$ being continuous and at points x, the points of discontinuity, the sum of the Fourier series is $\frac{1}{2}\,[f(x+0)+f(x-0)]$.

Note 3. Half range 'series.

If $f(x)$ satisfies the conditions of theorem 3 in $[0, \pi]$, then the sum of the sine series $\Sigma\, b_n \sin nx$ where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx$ is equal to $\frac{1}{2}\,[f(x+0) + f(x-0)]$ at every point x s.t. $0 < x < \pi$ and is zero when $x = 0$ or π.

Define an odd function $F(x)$ in $[-\pi, \pi]$ which is identical with $f(x)$ in $[0, \pi]$.

Thus, $F(x) = f(x)$ in $[0, \pi]$ and $F(x) = -F(-x) = -f(-x)$ in $[-\pi, 0]$ so that $F(x)$ satisfies the condition of theorem 3 in $[-\pi, \pi]$ provided $f(x)$ does so in $[0, \pi]$. Hence the sum of the series $\Sigma\, b_n \sin nx$ with

$$b_n = \frac{2}{\pi}\int_0^{\pi} F(x) \sin nx\, dx \text{ is } \frac{1}{2}\,[F(x+0) + F(x-0)]$$

$$= \frac{1}{2}\,[f(x+0) + f(x-0)] \text{ at every point } x \text{ in } (0, \pi)$$

and this sum is $= \frac{1}{2}\,[F(+0) + F(-0)] = 0$ for F odd at $x = 0$ or π.

Similarly if $f(x)$ satisfies the conditions of theorem 3 in $[0, \pi]$, then the sum of the series $\frac{1}{2}\, a_0 + \Sigma\, a_n \cos nx$ where $an = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\, dx$ is $\frac{1}{2}\,[f(x+0) + f(x-0)]$ at every point x in $(0, \pi)$ and is equal to $f(+0)$ for $x = 0$ and $f(\pi - 0)$ for $x = \pi$.

Note 4. Interval $[0, 2\pi]$.

If $f(x)$ satisfies the conditions of the theorem 3 in $[0, 2\pi]$ then the sum of the series $\frac{1}{2}\, a_0 + \Sigma\, (a_n \cos nx + b_n \sin x)$ where

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\, dx,\quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\, dx,$$

is $\frac{1}{2}\,[f(x+0) + f(x-0)]$ at every point x in $(0, 2\pi)$ and is $\frac{1}{2}\,[f(2\pi - 0) + f(+0)]$ at $x = 0$ or 2π.

Also it is periodic with period 2π.

Writing $x = y + \pi$ so that y varies in $[-\pi, \pi]$ as x varies in $[0, 2\pi]$, let us assume that $f(x) = f(y + \pi) = F(y)$, whence F satisfies conditions of theorem 3 ' ı $[-\pi, \pi]$. As such the sum of the series

$$\frac{1}{2}\,\alpha_0 + \Sigma\, (\alpha_n \cos ny + \beta_n \sin ny) \text{ where}$$

$$\alpha_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(y) \cos ny\, dy \text{ and } \beta_n = \int_{-\pi}^{\pi} F(y) \sin ny\, dy,$$

is $\frac{1}{2}\,[F(y+0) + F(y-0)]$ for $-\pi < y < \pi$ and

is $\frac{1}{2}\,[F(\pi - 0) + F(-\pi + 0)$ at $y = \pm\pi$, being periodic with period 2π.

Changing the variable by the substitution $x = y + \pi$, we have

$$\alpha_n = \frac{1}{\pi}\int_0^{2\pi} F(x-\pi) \cos n(x-\pi)\, dx = \frac{(-1)^n}{\pi}\int_0^{2\pi} f(x) \cos nx\, dx$$

$$\beta_n = \frac{1}{\pi}\int_0^{2\pi} F(x-\pi) \sin n(x-\pi)\, dx = \frac{(-1)^n}{\pi}\int_0^{2\pi} f(x) \sin x\, dx$$

and $\cos ny = (-1)^n \cos nx$ and $\sin ny = (-1)^n \sin nx$

So that $\frac{1}{2}[F(y+0)+F(y-0)] = \frac{1}{2}[f(y+\pi+0)+f(y+\pi-0)]$

$$= \frac{1}{2}[f(x)+0)+f(x-0)]$$

and $\frac{1}{2}[F(\pi-0)+F(-\pi+0)] = \frac{1}{2}[f(2\pi-0)+f(+0)]$

Note 5. Interval $[-l, l]$, *l being a real number.*

If $f(x)$ satisfies the conditions of theorem 3 in $[-l, l]$, then the sum function of the series $\frac{1}{2}a_0 + \Sigma\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$

$$\text{where } a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx \text{ and } b_n = \frac{1}{l}\int_{-l}^{l} f(x)\frac{\sin n\pi x}{l}\,dx$$

is $\frac{1}{2}[f(x+0)+f(x-0)]$ *for* $-l<x<l$

and is $\frac{1}{2}[f(l-0)+f(-l+0)$ *for* $x=\pm l$, *being periodic with period* $2l$.

Putting $y=\frac{\pi x}{l}$ and considering the function $F(x)$ such that

$$f(x) = f\left(\frac{ly}{\pi}\right) = F(y)$$

so that y varies in $[-\pi, \pi]$ as x varies in $[-l, l]$, we can prove the proposition.

Note 6. Interval [a, b] *in general.*

If $f(x)$ satisfies the condition of theorem 3 in $[a, b]$, then the sum of the series

$$\frac{1}{2}a_0 + \sum\left(a_n\cos\frac{2n\pi x}{b-a} + b_n\sin\frac{2n\pi x}{b-a}\right)$$

$$\text{where } a_n = \frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,dx,\quad b_n = \frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi x}{b-a}\,dx$$

is $\frac{1}{2}[f(x)+0)+f(x-0)]$ *for* $a<x<b$ *and is* $\frac{1}{2}[f(a+0)+f(b-0)]$ *for* $x=a$ *or* b, *being periodic with period* $b-a$.

The result is obvious by putting $y=\frac{2\pi x}{b-a}+\frac{b+a}{b-a}\pi$ so that y varies in $[-\pi, \pi]$ as x varies in $[a, b]$, (the transformation is obtained from $y=Lx+M$ such that $y=-\pi$ when $x=a$ and $y=\pi$ when $x=b$, by determining the constants L, M).

9.4. COMPLEX REPRESENTATION OF A FOURIER'S SERIES

We have $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$ ...(1)

where $a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{-inx}\,dx$ and $a_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{inx}\,dx$ ...(2)

Here a_n and a_{-n} are said to be *conjugate imaginaries.*

In case we consider a function $f(t)$ which is periodic with a period $T=\frac{2\pi}{\omega}$ then we can write

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{in\omega t},\ f(t) \text{ being defined in } (-\infty, \infty) \qquad ...(3)$$

Where $\omega = \dfrac{2\pi}{T}$ and $f(t+T) = f(t)$ ...(4)

Now R.H.S. of (3) being real, the coefficients of the series on the R.H.S. of (3) must be such that no imaginary terms occur.

Integrating (3) over 0 to T we have

$$\int_0^{2\pi/\omega} f(t)\, dt = \int_0^{2\pi/\omega}\left(\sum_{n=-\infty}^{\infty} a_n e^{in\omega t}\right) dt = \sum_{n=-\infty}^{\infty} a_n \int_0^{2\pi/\omega} e^{in\omega t}\, dt, \quad ...(5)$$

(under the assumption that term by term integration is permissible).

$$= 0 \text{ when } n \neq 0 \quad \because \int_0^{2\pi/\omega} \frac{1}{in\omega}\left[e^{in\omega t}\right]_0^{2\pi/\omega} = \frac{1}{in\omega}\left(e^{i2n\pi} - 1\right) = 0$$

$$= T \text{ when } n = 0 \quad \because \int_0^{2\pi/\omega} dt = \frac{2\pi}{\omega} = T$$

$\therefore$ (5) reduces to $\int_0^T f(t)\, dt = a_0 T$ giving $a_0 = \frac{1}{T}\int_0^T f(t)\, dt = \overline{f(t)}$ (say) ...(6)

where $\overline{f(t)}$ denotes the mean value of $f(t)$.

Now multiplying (3) by $e^{-in\omega t}$ and integrating over 0 to T, we have

$\int_0^T f(t)\, e^{-in\omega t}\, dt = a_n T$, other terms being equal to zero.

This gives $$a_n = \frac{1}{T}\int_0^T f(t)\, e^{-in\omega t}\, dt \quad ...(7)$$

Replacing n by $-n$ in (7) we get $a_{-n} = \dfrac{1}{T}\int_0^T f(t)\, e^{in\omega t}\, dt$ ...(8)

From (7) and (8) we conclude that $a_{-n} = \bar{a}_n$. ...(9)

In order to find the usual real form of the Fourier series, (3) can be expressed as

$$f(t) = \sum_{n=-\infty}^{-1} a_n e^{in\omega t} + a_0 + \sum_{n=1}^{\infty} a_n e^{in\omega t}$$

$$= \sum_{n=\infty}^{1} a_{-n} e^{-in\omega t} + a_0 + \sum_{n=1}^{\infty} a_n e^{in\omega t} \quad \text{(on replacing } n \text{ by } -n \text{ in the first term)}$$

or $$f(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n e^{in\omega t} + a_{-n} e^{-in\omega t}\right) dt \quad ...(10)$$

$$= a_0 + \sum_{n=1}^{\infty}(a_n + a_{-n})\cos n\omega t + \sum_{n=1}^{\infty} i(a_n - a_{-n})\sin n\omega t$$

$\because e^{in\omega t} = \cos n\omega t + i \sin n\omega t$ and $e^{-in\omega t} = \cos n\omega t - i \sin n\omega t$

If we put $a_n + a_{-n} = \alpha_n$, $t\,(a_n - a_{-n}) = \beta_n$ and $\alpha_0 = 2a_0$

then we find

$$f(t) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \alpha_n \cos n\omega t + \sum_{n=1}^{\infty} \beta_n \sin n\omega t \quad ...(11)$$

which is the same form as (1) of §9.1 and we can thus determine the coefficients α_n and β_n as

$$\alpha_n = a_n + a_{-n} = \frac{1}{T}\int_0^T f(t)\left\{e^{-in\omega t} + e^{in\omega t}\right\} dt$$

$$= \frac{2}{T}\int_0^T f(t)\cos n\omega t\, dt \qquad \ldots(12)$$

and $$\beta_n = i(a_n - a_{-n}) = \frac{1}{T}\int_0^T f(t)\, i\left\{e^{-in\omega t} - e^{in\omega t}\right\} dt$$

$$= \frac{1}{T}\int_0^T f(t)\,\frac{1}{i}\left\{e^{in\omega t} - e^{-in\omega t}\right\} dt \because i^2 = -1$$

$$= \frac{2}{T}\int_0^T f(t)\sin n\omega t \qquad \ldots(13)$$

The introduction of the term $\frac{\alpha_0}{2}$ in (11) enables (12) to give general term α_n applicable for α_0 as well. In either of the forms real or complex the constant term of Fourier series is always equal to the mean value of the function.

Important Note. Students are advised to commit to memory the following two integrals:

(i) $$\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{(a^2+b^2)}\{a\sin bx - b\cos bx\}$$

$$= \frac{e^{ax}}{r}\sin(bx - \alpha),\text{ where } r = \sqrt{a^2+b^2} \text{ and } \tan\alpha = \frac{b}{a}.$$

(ii) $$\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}\{a\cos bx + b\sin bx\}$$

$$= \frac{e^{ax}}{r}\cos(bx - \alpha),\text{ where } r = \sqrt{a^2+b^2},\ \tan\alpha = \frac{b}{a}.$$

Problem 1. *Obtain Fourier's series for the expansion of $f(x) = x\sin x$ in the interval $-\pi < x < \pi$. Hence deduce that*

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \ldots$$ *(Vikram, 1969)*

Here $x\sin x$ is an even function of x and we have already shown that when $-\pi \le x \le \pi$ and $f(x)$ is an even function of x, the Fourier's series is

$$f(x) = \frac{1}{\pi}\int_0^\pi f(v)\, dv + \frac{2}{\pi}\sum_{n=1}^\infty \cos nx \int_0^\pi f(v)\cos nv\, dv,$$

where $f(x) = x\sin x$; $\therefore f(v) = v\sin v$.

Thus,

$$x\sin x = \frac{1}{\pi}\int_0^\pi v\sin v\, dv + \frac{2}{\pi}\sum_{n=1}^\infty \cos nx \int_0^\pi v\sin v\cos nv\, dv.$$

Now $$\int_0^\pi v\sin v\, dv = \{-v\cos v\}_0^\pi + \{\sin v\}_0^\pi = \pi$$

and $$\int_0^\pi v\sin v\cos nv\, dv = \frac{1}{2}\int_0^\pi v\{\sin(n+1)v - \sin(n-1)v\}\, dv$$

$$= \frac{1}{2}\left[\left\{-v\frac{\cos(n+1)v}{n+1}\right\}_0^{\pi} + \left\{\frac{\sin(n+1)v}{(n+1)^2}\right\}_0^{\pi}\right.$$

$$\left. - \left\{-v\frac{\cos(n-1)v}{n-1}\right\}_0^{\pi} - \left\{\frac{\sin(n-1)v}{(n-1)^2}\right\}_0^{\pi}\right]$$

$$= \frac{1}{2}\left[-\pi\frac{\cos(n+1)\pi}{n+1} + \pi\frac{\cos(n-1)\pi}{n-1}\right]$$

$$= \frac{1}{2}\left[\frac{\pi\cos n\pi}{n+1} - \frac{\pi\cos n\pi}{n-1}\right], \; n \text{ being an integer.}$$

Hence $x \sin x = 1 - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos n\pi \cos nx}{n^2-1}$

Corresponding to $n = 1$, we have

$$\int_0^{\pi} v \sin v \cos v \, dv = \frac{1}{2}\int_0^{\pi} v \sin 2v\, dv$$

$$= \frac{1}{2}\left[\left\{-\frac{v\cos 2v}{2}\right\}_0^{\pi} + \left\{\frac{\sin 2v}{4}\right\}_0^{\pi}\right] = -\frac{\pi}{4}.$$

Hence

$$x \sin x = 1 - \frac{2}{\pi}\left[-\frac{\pi}{4}\cos x + \sum_{n=1}^{\infty}\frac{\cos n\pi \cos nx}{n^2-1}\right]$$

$$= 1 - 2\left[-\frac{\cos x}{4} + \frac{\cos 2x}{1\cdot 3} - \frac{\cos 3x}{2\cdot 4} + \frac{\cos 4x}{3\cdot 5} - \ldots\right].$$

Putting $x = \pi/2$, we get

$$\frac{\pi}{2} = 1 - 2\left[-\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} - \frac{1}{5\cdot 7} + \ldots\right]$$

or

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots$$

Problem 2. *Find a series of sines and cosines of multiples of x, which will represent $x + x^2$ in the interval $-\pi < x < \pi$.* *(Agra, 1969)*

Deduce that $\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$ *(Agra, 1962)*

We have

$$x + x^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}(v + v^2)\, dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\cos nx \int_{-\pi}^{\pi}(v + v^2)\cos nv \, dv$$

$$+ \frac{1}{\pi}\sum_{n=1}^{\infty}\sin nx \int_{-\pi}^{\pi}(v + v^2)\sin nv \, dv.$$

Now $\int_{-\pi}^{\pi}(v + v^2)\, dv = \left[\frac{v^2}{2} + \frac{v^3}{3}\right]_{-\pi}^{\pi} = \frac{2\pi^3}{3}.$

$$\int_{-\pi}^{\pi}(v+v^2)\cos nv\,dv=\left\{\frac{v\sin nv}{n}\right\}_{-\pi}^{\pi}+\left\{\frac{\cos nv}{n^2}\right\}_{-\pi}^{\pi}$$

$$+\left\{\frac{v^2\sin nv}{n}\right\}_{-\pi}^{\pi}-2\int_{-\pi}^{\pi}\frac{v\sin nv}{n}dv$$

$$=-2\left[\left\{-\frac{\cos nv}{n^2}\right\}_{-\pi}^{\pi}+\left\{\frac{\sin nv}{n^3}\right\}_{-\pi}^{\pi}\right]$$

$$=\frac{4\pi\cos n\pi}{n^2}$$

and $\int_{-\pi}^{\pi}(v+v^2)\sin nv\,dv=\left\{-\frac{v\sin nv}{n}\right\}_{-\pi}^{\pi}+\left\{\frac{\sin nv}{n^2}\right\}_{-\pi}^{\pi}$

$$-\left\{\frac{v^2\cos nv}{n}\right\}_{-\pi}^{\pi}+2\int_{-\pi}^{\pi}v\cos nv\,dv$$

$$=-\frac{2\pi\cos n\pi}{n}.$$

Hence the series is

$$x+x^2=\frac{1}{2\pi}\frac{2\pi^3}{3}+4\sum_{n=1}^{\infty}\frac{\cos nx\cos n\pi}{n^2}-2\sum_{n=1}^{\infty}\frac{\sin nx\cos n\pi}{n}$$

$$=\frac{\pi^2}{3}+4\left[-\frac{\cos x}{1^2}+\frac{\cos 2x}{2^2}-\frac{\cos 3x}{3^2}+\ldots\right]-2\left[-\frac{\sin x}{1}+\frac{\sin 2x}{2}-\frac{\sin 3x}{3}+\ldots\right]$$

$$=\frac{\pi^2}{3}-4\left(\cos x-\frac{1}{2^2}\cos 2x+\frac{1}{3^2}\cos 3x+\ldots\right)+2\left(\sin x-\tfrac{1}{2}\sin 2x+\tfrac{1}{3}\sin 3x\ldots\right).$$

Second part. We have

$$x+x^2=\frac{\pi^2}{3}-4\left[\cos x-\frac{1}{2^2}\cos 2x+\frac{1}{3^2}\cos 3x+\ldots\right]$$

$$+2\left[\sin x-\tfrac{1}{2}\sin 2x+\tfrac{1}{3}\sin 3x+\ldots\right]$$

At extremum π and $-\pi$, the sum of series

$$=f(\pi)=\tfrac{1}{2}[f(-\pi+0)+f(+\pi-0]$$

$$=\tfrac{1}{2}\left[-\pi+\pi^2+\pi+\pi^2\right]=\pi^2.$$

Putting $x=\pi$ in the above series, we have

$$f(\pi)=\pi^2=\frac{\pi^2}{3}+4\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right]$$

or $$\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots$$

or $$\frac{\pi^2}{6}=\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

Problem 3. *Find the series of sines and cosines of multiples of x which represents* $f(x)$ in the interval $-\pi < x < \pi$, *where*

$$f(x) = 0, \text{ when } -\pi < x \le 0,$$
$$= \frac{\pi x}{4}, \text{ when } 0 < x < \pi$$

and hence deduce $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$ *(Rohilkhand, 1991; Meerut, 1982, 83)*

Let $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$,

where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{2\pi}\left[\int_0^{\pi} f(x)\, dx + \int_{-\pi}^{0} f(x)\, dx\right]$

$$= \frac{1}{2\pi}\left[\frac{\pi x^2}{8}\right]_0^{\pi} = \frac{\pi^2}{16},$$

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} f(x) \cos nx\, dx + \int_{-\pi}^{0} f(x) \cos nx\, dx\right]$$

$$= \frac{1}{4n^2}[1 - \cos n\pi] = 0 \text{ for } n \text{ even}$$

$$= \frac{1}{2n^2} \text{ for } n \text{ odd.}$$

Similarly $b_n = -\frac{(-1)^n \pi}{4n}$.

So $f(x) = \frac{\pi^2}{16} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{4n^2}\cos nx - \frac{(-1)^n \pi}{4n}\sin nx\right]$

$$\therefore \frac{\pi x}{4} = \frac{\pi^2}{16} + \left(-\frac{1}{2}\cos x + \frac{\pi}{4}\sin x\right) - \frac{\pi}{4\cdot 2}\sin 2x\left(-\frac{1}{2\cdot 3^2}\cos 3x + \frac{\pi}{4\cdot 3}\sin 3x \ldots\right]$$

Now $f(\pi) = \frac{1}{2}[f(-\pi) + f(\pi)] = \frac{1}{2}\left(0 + \frac{\pi^2}{4}\right) = \frac{\pi^2}{8}$.

So $f(\pi) = \frac{\pi^2}{8} = \frac{\pi^2}{16} + \frac{1}{2}\left(1 + \frac{1}{5} + \frac{1}{5^2} + \ldots\right)$

or $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$

Problem 4. *Find Fourier's series for* $f(x)$ *in the interval* $(-\pi, \pi)$ *where*

$$f(x) = \pi + x \text{ when } -\pi < x < 0,$$
$$f(x) = \pi - x \text{ when } 0 < x < \pi.$$

Hint. Here $a_0 = \frac{\pi}{2}$, $a_n = 0$ for n even

$$= \frac{4}{\pi n^2} \text{ for } n \text{ odd,}$$

$$b_n = 0.$$

So $\quad f(x) = \frac{\pi}{2} + \frac{4}{\pi}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right)$

Problem 5. *Obtain the Fourier series for a function $f(x)$, where*

$$f(x) = \cos x \text{ for } 0 \le x \le \pi$$

$$f(x) = -\cos x \text{ for } -\pi \le x \le 0.$$

Let $f(x) = a_0 + \sum (a_n \cos nx + b_n \sin nx)$.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx = 0,$$

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx = 0,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx = \frac{4n}{\pi(n^2-1)}\left[1+(-1)^n\right]$$

$$= 0 \text{ for } n \text{ odd}$$

$$= \frac{8n}{\pi(n^2-1)} \text{ for } n \text{ even.}$$

$$\therefore \quad f(x) = \frac{8}{\pi}\left[\frac{2}{1\cdot 3}\sin 2x + \frac{4}{3\cdot 5}\sin 4x \ldots\right].$$

Problem 6. *Find the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ using Fourier series.*

Let $x^2 = \sum_{n=1}^{n=\infty} (a_n \cos nx + b_n \sin nx)$,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\, dx = \frac{\pi^2}{3},$$

and $\quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\, dx = (-1)^n \frac{4}{n^2}$

$$b_n = 0.$$

Hence $x^2 = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2}\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \ldots\right]$

$$\therefore \quad 0 = \frac{\pi^2}{3} - 4\left[1 - \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right]$$

$$= \frac{\pi^2}{3} - 4\left[\left\{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right\} - \frac{2}{2^2} - \frac{2}{4^2} \ldots\right]$$

$$= \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{1}{n^2} + 4\left(\frac{2}{2^2}\right)\sum\frac{1}{n^2} = \frac{\pi^2}{3} - 2\sum\frac{1}{n^2}.$$

Hence $2\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{3}$

or $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}.$

Problem 7. *Prove that* $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$, *using the Fourier's series.*

$$\text{L.H.S.} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$$

$$=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots+\frac{1}{2^2}+\frac{1}{4^2}+\ldots-\frac{1}{2^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$=\sum_{n=1}^{\infty}\frac{1}{n^2}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{3}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{3}{4}\frac{\pi^2}{6}=\frac{\pi^2}{8}$$

$\left[\text{since } \sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6} \text{ by Problem 6.}\right]$

Problem 8. *Using Fourier's series prove that*

$$\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^2}{90}$$

Hint. Take the function $f(x)=x^4-2\pi^2x^2$ and proceed similarly as in Problem 6.

Problem 9. *Find a function of x which is equal to kx, when x lies between 0 and l/2 and is k (l – x) when x lies between l/c and l.*

Given that $f(x)=kx$ when $0\le x\le l/2$,

$f(x)=k(l-x)$ when $l/2\le x\le l$.

We know that, when $0\le y\le\pi$,

$$f(y)=\frac{1}{\pi}\int_0^{\pi}f(u)\,du+\frac{2}{\pi}\sum_{n=1}^{\infty}\cos ny\int_0^{y}\cos nu\, f(u)\,du.$$

Putting $y=\frac{\pi x}{l}$ when $0\le x\le l$, and $u=\frac{\pi v}{l}$, *i.e.*, $du=\frac{\pi}{l}dv.$

we get

$$f\left(\frac{\pi x}{l}\right)=\frac{1}{l}\int_0^{l}f\left(\frac{\pi v}{l}\right)dv+\frac{2}{l}\sum_{n=1}^{\infty}\cos\frac{n\pi x}{l}\int_0^{l}f\left(\frac{\pi v}{l}\right)\cos\frac{n\pi v}{l}dv.$$

This may be expressed as

$$\phi(x)=\frac{1}{l}\int_0^{1}\phi(v)\,dv+2\sum_{n=1}^{\infty}\cos\frac{n\pi x}{l}\int_0^{l}\phi(v)\cos\frac{n\pi v}{l}dv.$$

Now $\int_0^{l}\phi(v)\,dv=\int_0^{l/2}\phi(v)\,dv+\int_{l/2}^{l}\phi(v)\,dv$

$$=\int_0^{l/2}kv\,dv+\int_{l/2}^{l}k(l-v)\,dv$$

$$= \frac{k}{2}\frac{l^2}{2} + k\left[lv - \frac{v^2}{2}\right]_{l/2}^{l}$$

$$= \frac{kl^2}{8} + kl^2 - \frac{kl^2}{2} - \frac{kl^2}{2} + \frac{kl^2}{8} = \frac{kl^2}{4}$$

and $\int_0^l \phi(v)\cos\frac{n\pi v}{l}\,dv$

$$= \int_0^{l/2} kv\cos\frac{n\pi v}{l}\,dv + \int_{l/2}^{l} k(l-v)\cos\frac{n\pi v}{l}\,dv$$

$$= \frac{kl}{n\pi}\left\{v\sin\frac{n\pi v}{l}\right\}_0^{ll} + \frac{kl^2}{n^2\pi^2}\left\{\cos\frac{n\pi v}{l}\right\}_0^{ll} + kl\frac{l}{n\pi}\left\{\sin\frac{n\pi v}{l}\right\}_{l/2}^{l}$$

$$- \frac{kl}{n\pi}\left\{v\sin\frac{n\pi v}{l}\right\}_{ll}^{l} - \frac{kl^2}{n^2\pi^2}\left\{\cos\frac{n\pi v}{l}\right\}_{l/2}^{l}$$

$$= \frac{kl^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{kl^2}{n^2\pi^2}\left[\cos\frac{n\pi}{2} - 1\right] - \frac{kl^2}{n\pi}\sin\frac{n\pi}{2} + \frac{kl^2}{2n\pi}\sin\frac{n\pi}{2}$$

$$- \frac{kl^2}{n^2\pi^2}\left[\cos n\pi - \cos\frac{n\pi}{2}\right]$$

$$= \frac{kl^2}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - \cos n\pi - 1\right]$$

$$= -\frac{4kl^2}{n^2\pi^2} \quad \text{for } n = 4m + 2 \text{ and } 0 \text{ for other values of } n$$

Hence if n is of the form $4m + 2$.

$$\phi(x) = \frac{1}{l}\frac{kl^2}{.4} + \frac{2}{l}\sum_{n=1}^{\infty}\cos\frac{\pi x}{l}\left(-\frac{4kl^2}{n^2\pi^2}\right)$$

$$= \frac{kl}{4} - \frac{8kl}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l}$$

$$= \frac{kl}{4} - \frac{8kl}{\pi^2}\left[\frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{6^2}\cos\frac{6\pi x}{l} + \frac{1}{10^2}\cos\frac{10\pi x}{l} + \ldots\right]$$

[when $n = 4m + 2$, where $m = 0, 1, 2, \ldots$; $\therefore\ n = 2, 6, 10\ldots$].

Problem 10. *If* $f(x) = x$ *for* $0 \le x \le \frac{\pi}{2}$.

$$f(x) = \pi - x \quad \textit{for } \frac{\pi}{2} \le x \le \pi,$$

express this function by a sine series and also by a cosine series.

(*i*) **To Express as a Sine Series.** We know that

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\sin \pi x\int_0^{\pi} f(v)\sin nv\,dv$$

Here

$$\int_0^{\pi} f(v)\sin nv\,dv$$

$$= \int_0^{\pi/2} v\sin nv\,dv + \int_{\pi/2}^{\pi}(\pi - v)\sin nv\,dv$$

$$= \left[-v\frac{\cos nv}{n} + \frac{\sin nv}{n^2}\right]_0^{\pi/2} + \left[-(\pi - v)\frac{\cos nv}{n} - \frac{\sin nv}{n^2}\right]_{\pi/2}^{\pi}$$

$$= -\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2} + \frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}$$

$$= \frac{2}{n^2}\sin\frac{n\pi}{2}.$$

Hence $f(x) = \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin nx$

$$= \frac{4}{\pi}\left[\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \ldots\right].$$

(*ii*) **To Express as a Cosine Series.** We know that

$$f(x) = \frac{1}{\pi}\int_0^{\pi} f(v)\,dv + \frac{2}{\pi}\sum_{n=1}^{\infty}\cos nx\int_0^{\pi} f(v)\cos nv\,dv.$$

Now $\displaystyle\int_0^{\pi} f(v)\,dv = \int_0^{\pi/2} v\,dv + \int_{\pi/2}^{\pi}(\pi - v)\,dv$

$$= \left[\frac{v^2}{2}\right]_0^{\pi/2} + \left[\pi v - \frac{v^2}{2}\right]_{\pi/2}^{\pi} = \frac{\pi^2}{8} + \frac{\pi^2}{2} - \frac{3\pi^2}{8} = \frac{\pi^2}{4}$$

and $\displaystyle\int_0^{\pi}\cos nv\,f(v)\,dv = \int_0^{\pi/2} v\cos nv\,dv + \int_{\pi/2}^{\pi}(\pi - v)\cos nv\,dv$

$$= \left\{\frac{v\sin nv}{n}\right\}_0^{\pi/2} + \left\{\frac{\cos nv}{n^2}\right\}_0^{\pi/2} + \frac{\pi}{n}\{\sin nv\}_{\pi/2}^{\pi}$$

$$-\left[\frac{v\sin nv}{n} + \frac{\cos nv}{n^2}\right]_{\pi/2}^{\pi}$$

$$= \frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\left(\frac{n\pi}{2} - 1\right) - \frac{\pi}{n}\sin\frac{n\pi}{2} + \frac{\pi}{2n}\sin\frac{n\pi}{2}$$

$$-\frac{\cos n\pi}{n^2} + \frac{1}{n^2}\cos\frac{n\pi}{2}$$

$$= \frac{1}{n^2}\left\{1 + \cos n\pi - 2\cos\frac{n\pi}{2}\right\}$$

$= -\dfrac{4}{n^2}$, if n is of the for $4m + 2$ and is zero for other values of n.

Hence the cosine series when n is of the form $4m+2$ is

$$f(x)=\frac{\pi}{4}-\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$$

$$=\frac{\pi}{4}-\frac{8}{\pi}\left\{\frac{1}{2^2}\cos 2x+\frac{1}{6^2}\cos 6x+\ldots\right\} \text{ by putting } n=2,6,10, \text{ etc.}$$

Problem 11. *Find a series of sines of multiples of x which represents x in the interval $(0,\pi)$. Hence deduce that*

$$\frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$$

Draw graph of the function.

Let $x=a_0+\sum_{n=1}^{\infty}a_n\cos nx$.

$$\left.\begin{aligned} a_0&=\frac{1}{\pi}\int_0^{\pi}x\,dx=\frac{\pi}{2}\\ a_n&=\frac{2}{\pi}\int_{\pi}^{\pi}x\cos nx\,dx=0 \text{ for } n \text{ even}\\ &=-4/(n\pi^2) \text{ for } n \text{ odd.}\end{aligned}\right\} \text{for cosine series}$$

Hence the coefficients of cosines of even multiples are zero.

Now $\quad x=\frac{\pi}{2}-\frac{4}{\pi}\left[\cos x+\frac{1}{3^2}\cos 3x+\ldots\right]$.

Putting $x=0$, we have

$$0=\frac{\pi}{2}-\frac{4}{\pi}\left[1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right]$$

or

$$\frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots$$

Graph. In the interval $(0,\pi)$, the line $y=x$ gives the curves represented by the series. Hence $f(x)$ represented by the above series contains cosine terms only. So this is an even function and therefore the curve is symmetrical about the axis of y along which $f(x)$ is plotted. 2π is the period of the series, hence the portion between π to $-\pi$, repeats indefinitely on both the sides and the sum is continuous for all values of x.

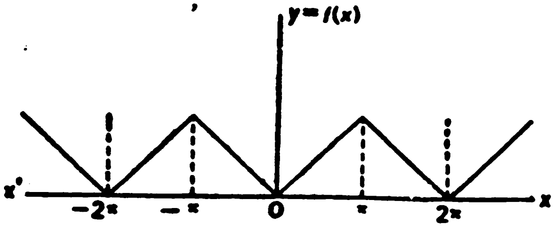


Fig. 9·1

In fact the graph of the sum of n terms of Fourier series for $f(x)$ approximates to the graph of $f(x)$ the greater the value of n is, the closer is the approximation. Retaining upto three terms of the Fourier series *i.e.*

$$y=\frac{\pi}{2}-\frac{\pi}{4}\left\{\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\ldots\right\}$$

the graph is as shown below:

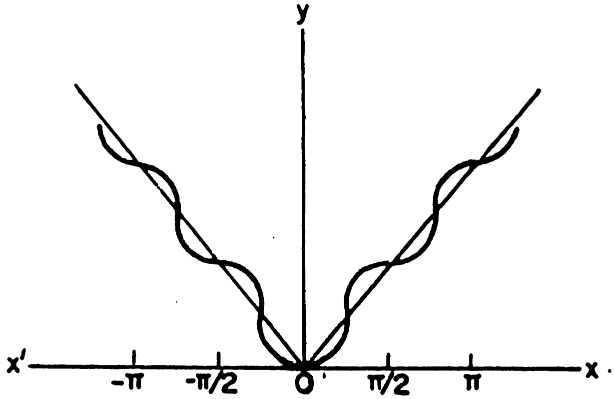


Fig. 9.2

Problem 12. *Find the series of sines of multiples of x which represents x in the interval $\pi \geq x \geq 0$. Show by a graph the nature of the series.*

Here $$x=\sum_{n=1}^{\infty} b_n \sin nx,$$

where $$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \quad \text{[for sine series]}$$

$$=\frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$=\frac{2}{\pi}\left[-x\frac{\cos nx}{n}+\frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$=-\frac{2}{\pi}\cdot\pi\frac{1}{n}\cos n\pi=-\frac{2}{n}(-1)^n$$

$\therefore$ $$x=2\left[\sin x-\frac{\sin 2x}{2}+\tfrac{1}{3}\sin 3x+\ldots\right]$$

The sum is discontinuous at $x=\pi$.

Graph. The curve is symmetrical about the origin. For unrestricted values of x, the series represented between $(-\pi, \pi)$ repeats indefinitely in both the directions. The points $\pm\pi, \pm 3\pi$ ... are points of discontinuity.

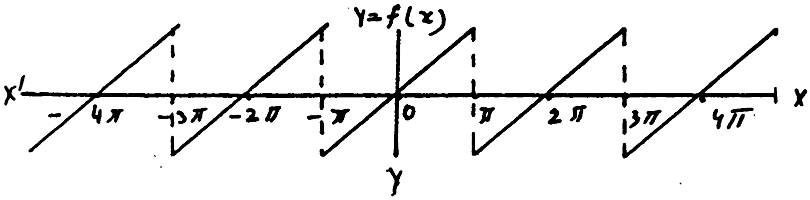


Fig. 9.3

Problem 13. *Find the Fourier series for the periodic function f(x) defined by*

$$f(x) = -\pi \text{ if } -\pi < x < 0$$
$$f(x) = x \quad \text{if } 0 < x < \pi.$$

Hence prove that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

Plot the graph for the series.

Let $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

Here $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} -\pi\, dx + \int_0^{\pi} x\, dx\right]$$

$$= -\frac{\pi}{4}.$$

And $a_n = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -\pi \cos nx\, dx + \int_0^{\pi} x \cos nx\, dx\right]$

$$= \frac{1}{\pi n^2}[\cos n\pi - 1] = 0 \text{ if } n \text{ is even}$$

$$\text{and} = \frac{-2}{\pi n^2} \text{ if } n \text{ is odd.}$$

$$b_n = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -\pi \sin nx\, dx + \int_0^{\pi} x \sin nx\, dx\right]$$

$$= \frac{1}{n}[1 - 2\cos n\pi] = -\frac{1}{n} \text{ for } n \text{ even}$$

$$\text{and} = \frac{3}{n} \text{ for } n \text{ odd.}$$

$\therefore$ $$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\cos x - \frac{2}{\pi}\frac{\cos 3x}{3^2} - \frac{2}{\pi}\frac{\cos 5x}{5^2} \dots$$

$$+3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{\sin 4x}{4} + \dots$$

One discontinuity occurs at $x = 0$.

Hence $f(0) = \frac{f(0^+) + f(0^-)}{2} = -\frac{\pi}{2}$ sin $f(0^+) = 0$ and $f(0^-) = -\pi$.

Putting $x = 0$ in the series, we have

$$f(0) = -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right] = -\frac{\pi}{2}.$$

Hence $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

Graph. Discontinuities occur at $x = \pm\pi$ as follows:

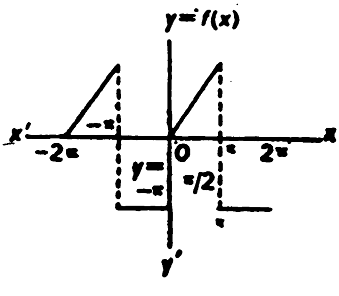


Fig. 9·4

Problem 14. *Find a series of sines and consies of multiples of x which will represent* $\frac{\pi}{2\sinh\pi}e^x$ *in the interval* $-\pi < x < \pi$. *Find the sum of the series for* $x = \pm\pi$.

We have $f(x) = \frac{\pi}{2\sinh\pi}e^x = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ ...(1)

Here $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\pi}{2\sinh\pi}e^x\, dx = \frac{1}{2\pi}\cdot\frac{\pi}{2\sinh\pi}\left[e^x\right]_{-\pi}^{\pi}$

$= \frac{1}{4\sinh\pi}\left[e^{\pi} - e^{-\pi}\right] = \frac{1}{4\sinh\pi}\cdot 2\sinh\pi = \frac{1}{2}$ ...(2)

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx,\ n \geq 1$

$= \frac{1}{2\sinh\pi}\cdot\int_{-\pi}^{\pi} e^x \cos nx\, dx$

$= \frac{1}{2\sinh\pi}\left[\frac{ex}{1+n^2}(\cos nx + n \sin nx)\right]_{-\pi}^{\pi}$

$= \frac{\left(e^{\pi} - e^{-\pi}\right)\cos n\pi}{\left(1+n^2\right)\cdot 2\sinh\pi}$

$= \frac{2\sin h\pi \cos nx}{\left(1+n^2\right)\cdot 2\sinh\pi} = \frac{\cos nx}{1+n^2} = \frac{(-1)^2}{1+n^2}$ ...(3)

Similarly $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx = \frac{1}{2\sinh\pi}\int_{-\pi}^{\pi} e^x \sin nx\, dx$

$= \frac{1}{2\sinh\pi}\left[\frac{e^x}{1+n^2}(\sin nx - n\cos nx)\right]_{-\pi}^{\pi}$

$= \frac{1}{2\sinh\pi}\cdot\frac{1}{1+n^2}(-n)(-1)^n(e^{\pi} - e^{-\pi})$

$= -(-1)^n\ \frac{n}{1+n^2}$ ...(4)

Substituting the values of a_0, a_n, b_n from (2), (3), (4) in (1) we get

$$f(x) = \frac{\pi e^x}{2\sinh\pi} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos nx - n\sin nx)$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{1+n^2} + \sum_{n=1}^{\infty} \frac{n(-1)^{n+1}}{1+n^2}\sin nx$$

$$= \frac{1}{2} + \left(-\frac{1}{1+1^2}\cos x + \frac{1}{1+2^2}\cos 2x - \frac{1}{1+3^2}\cos 3x + \ldots\right)$$

$$+\left(\frac{1}{1+1^2}\sin x - \frac{2}{1+2^2}\sin 2x + \frac{3}{1+3^2}\sin 3x \ldots\right) \text{ when } -\pi < x < \pi$$

Now when $x = \pm\pi$, the sum of the series $= \frac{1}{2}[f(-\pi+0) + f(\pi-0)]$

$$= \frac{1}{2}\left[\frac{\pi}{2\sinh\pi}e^{-\pi} + \frac{\pi}{\sinh\pi}e^{\pi}\right]$$

$$= \frac{1}{2}\cdot\frac{\pi}{2\sinh\pi}\cdot\left(e^{-\pi} + e^{\pi}\right)$$

$$= \frac{1}{2}\frac{\pi}{2\sinh\pi}\cdot 2\cosh\pi = \frac{\pi}{2}\coth\pi$$

Problem 15. *Find a Fourier's series of $f(x)$ in the interval $[-l, l]$ where*

$$f(x) = \frac{l}{4} \quad \text{for} -l \le x \le -\frac{l}{2}$$

$$= \frac{x^2}{l} \quad \text{for} -\frac{l}{2} \le x \le \frac{l}{2}$$

$$= \frac{l}{4} \quad \text{for} \quad \frac{l}{2} \le x \le l$$

and draw the graph of the function represented by the series.

Since the function $f(x)$ has the same value for positive and negative values of x in $[-l, l]$, it therefore follows that the function $f(x)$ is even and so the Fourier's series for $f(x)$ will be purely a cosine series *i.e.*

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where $a_0 = \frac{1}{l}\int_0^l f(x)\,dx = \frac{1}{l}\int_0^{l/2}\frac{x^2}{l}dx + \frac{1}{l}\int_{l/2}^{l}\frac{l}{4}\,dx = \frac{l^3}{24l} + \frac{l}{8} = \frac{l}{6}$

and $a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx = \frac{2}{l}\int_0^{l/2}\frac{x^2}{l}\cos\frac{n\pi x}{l}dx + \frac{2}{l}\int_{l/2}^{l}\frac{l}{4}\cos\frac{n\pi x}{l}dx$

$$= \frac{2l}{n^2\pi^2}\left[\cos\frac{n\pi}{2} - \frac{2}{n\pi}\sin\frac{n\pi}{2}\right]$$

$$\therefore f(x) = \frac{l}{6} + \frac{2l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\cos\frac{n\pi}{2} - \frac{2}{n\pi}\sin\frac{n\pi}{2}\right)\cos\frac{n\pi x}{l}$$

Clearly the shape of the curve between $\left(-l, -\frac{l}{2}\right)$ and $\left(\frac{l}{2}, l\right)$ is given by $y = \frac{l}{4}$ which represents a straight line parallel to the axis of x at a distance $\frac{l}{4}$ from x axis.

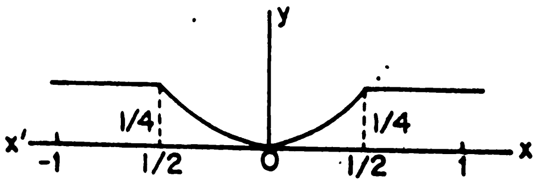


Fig. 9.5

Also the shape of the curve between $\left(-\frac{l}{2}, \frac{l}{2}\right)$ is given by $y = \frac{x^2}{l}$ or $x^2 = ly$ which represents a parabola whose latus rectum is l and the axis of the parabola is y-axis. Hence the shape of curve is as shown in Fig. 9.5.

Note. *If the values of x are not restricted between $(-l, l)$ then the part of the curve from $-l$ to l will be repeated indefinitely in both directions.*

Problem 16. *If the function defined by $y = x^2$ from 0 to $\frac{\pi}{2}$ and by $y = 0$ from $\frac{\pi}{2}$ to π be represented by a series of sines of multiples of x, show that the coefficient of sin nx is*

$$\left(\frac{4}{\pi n^3} - \frac{\pi}{2n}\right) \cos\frac{n\pi}{2} + \frac{2}{n^2} \sin\frac{n\pi}{2} - \frac{4}{\pi n^3}.$$

To what values does the series converge at the point $x = \frac{\pi}{2}$?

Sketch the graph of the function represented by the series for values of x not restricted to lie between 0 and π : and also indicate the graph of the cosine series which represents the same function in the interval 0 to π.

We have $\quad f(x) = x^2$ for $0 < x < \frac{\pi}{2}$

$= 0$ for $\frac{\pi}{2} < x < \pi$.

For a purely sine series in $(0, \pi)$, the function must be odd in $(-\pi, \pi)$ so that $f(x)$ in $(-\pi, 0)$ is defined such that $f(-x) = -f(x)$ in the form

$$y = -x^2 \quad \text{for } -\frac{\pi}{2} < x < 0$$

$$= 0 \quad \text{for } -\pi < x < -\frac{\pi}{2}$$

and then $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$, where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \; dx = \frac{2}{\pi}\int_0^{\pi/2} x^2 \sin nx \, dx$$

$$=\frac{2}{\pi}\left[\left(-\frac{\cos nx}{n}\cdot x^2\right)_0^{\pi/2}+\int_0^{\pi/2}\frac{\cos nx}{n}\cdot 2x\,dx\right]$$

$$=\frac{2}{\pi}\left[-\frac{\pi^2}{4n}\cos\frac{n\pi}{2}+\left(\frac{2x\sin nx}{n^2}\right)_0^{\pi/2}-\frac{2}{n^2}\int_0^{\pi/2}\sin nx\,dx\right]$$

$$=\frac{2}{\pi}\left[-\frac{\pi^2}{4n}\cos\frac{n\pi}{2}+\frac{\pi\sin n\pi/2}{n^2}+\frac{2}{n^2}\left(\frac{\cos nx}{n}\right)_0^{\pi/2}\right]$$

$$=\frac{2}{\pi}\left[-\frac{\pi^2}{4n}\cos\frac{n\pi}{2}+\frac{\pi}{n^2}\sin\frac{n\pi}{2}+\frac{2}{n^2}\cos\frac{n\pi}{2}-\frac{2}{n^3}\right]$$

$$=\left(\frac{4}{\pi n^3}-\frac{\pi}{2n}\right)\cos\frac{n\pi}{2}+\frac{2}{n^2}\sin\frac{n\pi}{2}-\frac{4}{\pi n^3}$$

which is the required coefficient of $\sin nx$.

Had we supposed $y = x^2$ in $\left(-\frac{\pi}{2}, 0\right)$ and $y = 0$ in $\left(-\pi, -\frac{\pi}{2}\right)$, the form of the Fourier's series would have been purely a cosine series.

Now, the sum of the series at $x = \frac{\pi}{2}$ sin $= \frac{1}{2}\,[f(x+0)+f(x-0)]$

$$=\frac{1}{2}\left[f\left(\frac{\pi}{2}+0\right)+f\left(\frac{\pi}{2}-0\right)\right]$$

$$=\frac{1}{2}\left[0+\frac{\pi^2}{4}\right]=\frac{\pi^2}{8}.$$

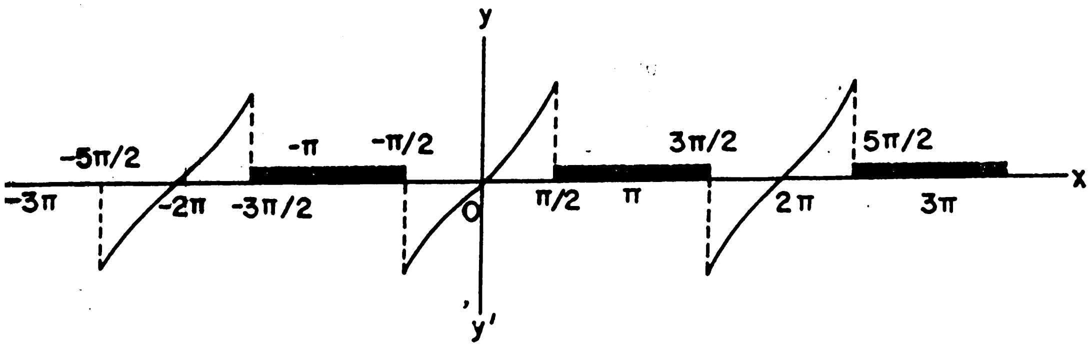


Fig. 9.6

The shape of the curve between $\left(-\pi, -\frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \pi\right)$ is given by $y = 0$ *i.e.*, x – axis.

The shape of the curve between $\left(0, \frac{\pi}{2}\right)$ is given by $y = x^2$ which is a parabola whose latus rectum is unity and axis is the y-axis. Also the shape of the curve between

$\left(-\frac{\pi}{2}, 0\right)$ is given by $y = -x^2$ which is a similar parabola as in $\left(0, \frac{\pi}{2}\right)$ except that it has its concavity towards negative y-axis. The shape is as shown in Fig. 9.6.

The period of Fourier's series for $f(x)$ being 2π, the part of the curve lying between $(-\pi, \pi)$ is repeated indefinitely in both the directions.

Again for a purely cosine series in $(-\pi, 0)$ the function $f(x)$ must be such that $f(-x) = f(x)$ *i.e.*, $f(x)$ in $(-\pi, 0)$ is defined such that,

$$y = x^2 \text{ for } -\frac{\pi}{2} < x < 0$$

$$= 0 \text{ for } -\pi < x < -\frac{\pi}{2}$$

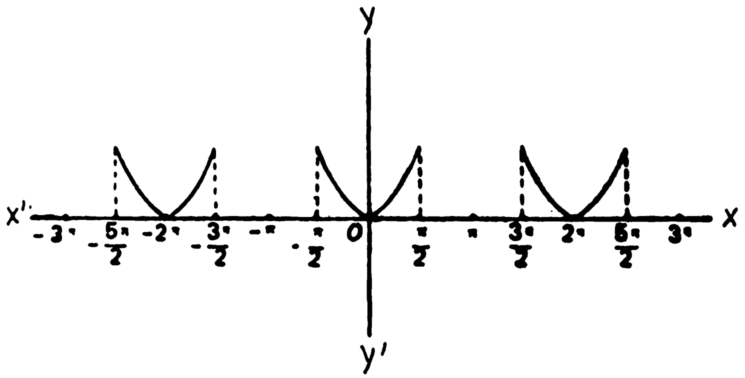


Fig. 9.7

As such the only difference here will be between $\left(-\frac{\pi}{2}, 0\right)$ where the shape of the curve will be a parabola given by $y = x^2$ with its concavity towards the positive end of y-axis. The shape is as shown in Fig. 9.7.

9.5. PARSEVAL'S IDENTITY FOR FOURIER SERIES

If the Fourier series for $f(x)$ converges uniformly in $(-l, l)$, then

$$\frac{1}{l}\int_{-l}^{l} \{f(x)\}^2 \, dx = \frac{a_0^2}{2} + \sum \left(a_n^2 + b_n^2\right)$$

where a_0, a_n, b_n are Fourier's constants.

We have $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}\right)$ ...(1)

Multiplying both sides of (1) by $f(x)$ and integrating term by term from $-l$ to l, we get

$$\int_{-l}^{l} \{f(x)\}^2 \, dx = \frac{a_0}{2} \int_{-l}^{l} f(x) \, dx + \sum_{n=1}^{\infty} \left\{a_n \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} \, dx + b_n \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} \, dx\right\}$$

$$= \frac{a_0^2}{2}\, l + l \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right),$$ by using Fourier coefficients so that

$$\int_{-l}^{l} f(x)\, dx = l\, a_0,\ \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l}\, dx = l\, a_n \text{ and}$$

$$\int_{-l}^{l} f(x) \sin \frac{n\pi x}{l}\, dx = l\, b_n.$$

$$\therefore\ \frac{1}{l}\int_{-l}^{l} \{f(x)\}^2\, dx = \frac{a_0^2}{2} + \sum\left(a_n^2 + b_n^2\right) \qquad \ldots(2)$$

COROLLARY. *Riemann's theorem.* An important cosequence of Parseval's identity is Riemann's theorem *i.e.*,

$$\lim_{n\to\infty} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l}\, dx = 0,\ \lim_{n\to\infty} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l}\, dx = 0 \qquad \ldots(3)$$

9.6. FOURIER'S INTEGRAL

If $f(x)$ satisfies the Dirichlet's condition in $-\pi \le x \le \pi$, and $\int_{-\infty}^{\infty} f(x)\, dx$ converges *i.e.*, is integrable in $-\infty < x < \infty$, then we have the Fourier series expansion for $f(x)$ as

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx, \qquad \ldots(1)$$

where
$$\left.\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx. \end{aligned}\right\} \qquad \ldots(2)$$
and

This function of x may be developed into a trigonometric series for all values of x between $x = -c$ and $x = c$ by putting

$$z = \frac{\pi}{c} x, \text{ where } z = -\pi \text{ when } x = -c$$

and
$$z = \pi \text{ when } x = c,$$

i.e., $$f(x) = f\left(\frac{c}{\pi} z\right)$$

Then the series (1) may be developed in terms of z as

$$f\left(\frac{c}{\pi} z\right) = a_0 + a_1 \cos z + a_2 \cos 2z + a_3 \cos 3z + \ldots$$

$$+ b_1 \sin z + b_2 \sin 2z + b_3 \sin 3z + \ldots \qquad \ldots(3)$$

where $$\left.\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(\frac{c}{\pi}z\right)dz,\\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{c}{\pi}z\right)\cos nz\,dz,\\ \text{and}\quad b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{c}{\pi}z\right)\sin nz\,dz. \end{aligned}\right\} \quad \ldots(4)$$

Now if we replace z by $\frac{\pi}{c}x$, then (3) becomes

$$f(x) = a_0 + a_1\cos\frac{\pi x}{c} + a_2\cos\frac{2\pi x}{c} + a_3\cos\frac{3\pi x}{c} + \ldots$$
$$+ b_1\sin\frac{\pi x}{c} + b_2\sin\frac{2\pi x}{c} + b_3\sin\frac{3\pi x}{c} + \quad \ldots(5)$$

Its coefficients being the same as those of (3) is therefore valid from $x = c$ to $x = c$, where

$$a_0 = \frac{1}{2\pi}\int_{-c}^{c} f(x)\cdot\frac{\pi}{c}\,dx = \frac{1}{2c}\int_{-c}^{c} f(x)\,dx \quad \because \text{ when } z = \frac{\pi}{c}x,\ dz = \frac{\pi}{c}dx$$

$$\left.\begin{aligned} &= \frac{1}{2c}\int_{-c}^{c} f(t)\,dt \text{(say)}\\ a_n &= \frac{1}{\pi}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}\cdot\frac{\pi}{c}\,dx\\ &= \frac{1}{c}\int_{-c}^{c} f(t)\cos\frac{n\pi t}{c}\,dt\\ \text{and}\quad b_n &= \frac{1}{\pi}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}\cdot\frac{\pi}{c}\,dx\\ &= \frac{1}{2c}\int_{-c}^{c} f(t)\sin\frac{n\pi t}{c}\,dt. \end{aligned}\right\} \quad \ldots(6)$$

Now if we substitute the values of the coefficients $a_0, a_1, a_2, a_3 \ldots b_1, b_2, b_3 \ldots$ given by (6) in (5), then we get

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)\,dt + \frac{1}{c}\int_{-c}^{c} f(t)\cos\frac{\pi x}{c}\,dt + \frac{1}{c}\int_{-c}^{c} f(t)\cos\frac{2\pi t}{c}\cos\frac{2\pi x}{c}\,dx + \ldots$$
$$+ \frac{1}{c}\int_{-c}^{c} f(t)\sin\frac{\pi t}{c}\sin\frac{\pi x}{c}\,dt + \frac{1}{c}\int_{-c}^{c} f(t)\sin\frac{2\pi t}{c}\sin\frac{2\pi x}{c}\,dt + \ldots$$
$$= \frac{1}{c}\int_{-c}^{c} f(t)\left[\frac{1}{2} + \cos\frac{\pi t}{c}\cos\frac{\pi x}{c} + \cos\frac{2\pi t}{c}\cos\frac{2\pi x}{c} + \ldots\right.$$
$$\left. + \sin\frac{\pi t}{c}\sin\frac{\pi x}{c} + \sin\frac{2\pi t}{c}\sin\frac{2\pi x}{c} + \ldots\right]dt$$
$$= \frac{1}{c}\int_{-c}^{c} f(t)\left[\frac{1}{2} + \left\{\cos\frac{\pi t}{c}\cos\frac{\pi x}{c} + \sin\frac{\pi t}{c}\sin\frac{\pi x}{c}\right\}\right.$$
$$\left. + \left\{\cos\frac{2\pi t}{c}\cos\frac{2\pi x}{c} + \sin\frac{2\pi t}{c}\sin\frac{2\pi x}{c}\right\} + \ldots\right]dt$$

$$= \frac{1}{c}\int_{-c}^{c} f(t)\left[\frac{1}{2} + \cos\frac{\pi}{c}(x-t) + \cos\frac{2\pi}{c}(x-t) + \ldots\right]dt$$

$$= \frac{1}{2c}\int_{-c}^{c} f(t)\left[1 + 2\cos\frac{\pi}{c}(x-t) + 2\cos\frac{2\pi}{c}(x-t) + \ldots\right]dt$$

$$= \frac{1}{2\pi}\int_{-c}^{c} f(t)\left[\frac{\pi}{c}\cos\frac{0\pi}{c}(x-t) + \frac{\pi}{c}\cos\frac{2\pi}{c}(x-t)\right.$$

$$\left. + \frac{\pi}{c}\cos\left(\frac{-\pi}{c}\right)(x-t) + \frac{\pi}{c}\cos\frac{2\pi}{c}(x-t) + \frac{\pi}{c}\cos\left(\frac{-2\pi}{c}\right)(x-t) + \ldots\right]dt$$

since $2\cos\phi = \cos\phi + \cos(-\phi)$

$$= \frac{1}{2\pi}\int_{-c}^{c} f(t)\left[\ldots\frac{\pi}{c}\cos\left(-\frac{n\pi}{c}\right)(x-t) + \ldots + \frac{\pi}{c}\cos\left(-\frac{2\pi}{c}\right)(x-t) + \frac{\pi}{c}\cos\left(-\frac{\pi}{c}\right)(x-t)\right.$$

$$+ \frac{\pi}{c}\cos\frac{0.\pi}{c}(x-t) + \frac{\pi}{c}\cos\frac{\pi}{c}(x-t) + \frac{\pi}{c}\cos\frac{2\pi}{c}(x-t) + \ldots$$

$$\left. + \frac{\pi}{c}\cos\frac{n\pi}{c}(x-t) + \ldots\right]dt$$

$$= \frac{1}{2\pi}\int_{-c}^{c} f(t)\left[\lim_{n\to\infty}\sum_{r=-n}^{r=n}\frac{\pi}{c}\cos\frac{r\pi}{c}(x-t)\right]dt$$

$$= \frac{1}{2\pi}\int_{-c}^{c} f(t)\left[\lim_{n\to\infty}\sum_{r=-n}^{r=n}\frac{1}{c/\pi}\cos\frac{r}{c/\pi}(x-t)\right]dt.$$

If c becomes indefinitely large *i.e.* as $c \to \infty$, $\frac{c}{\pi} \to \infty$, we have

$$\lim_{c\to\infty}\sum_{r=-\infty}^{\infty}\frac{1}{c/\pi}\cos\frac{r}{c/\pi}(x-t) = \int_{-\infty}^{\infty}\cos u(x-t)\,du$$

(by the definition of integral as the limit of a sum).

Hence $f(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} f(t)\,dt\int_{-\infty}^{\infty}\cos u(x-t)\,du.$ ...(7)

This double integral is known as *Fourier's Integral* and holds if x is a point of continuity of $f(x)$.

Aliter. We have $f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right)$ ...(8)

where $a_n = \dfrac{1}{l}\displaystyle\int_{-l}^{l} f(u)\cos\frac{n\pi u}{l}\,du$ and $b_n = \dfrac{1}{l}\displaystyle\int_{-l}^{l} f(u)\sin\frac{n\pi u}{l}\,du$ ...(9)

so that

$$a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l} = \frac{1}{\pi}\int_{-l}^{l} f(u)\cos\frac{n\pi}{l}(u-x)\,dx$$

and $\dfrac{a_0}{2} = \dfrac{1}{2l}\displaystyle\int_{-l}^{l} f(u)\,du$

∴ (8) gives

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(u)\ du + \frac{1}{l}\sum_{n=1}^{\infty} f(u)\ \cos\frac{n\pi}{l}(u-x)\ du \qquad ...(10)$$

Assuming that $\int_{-\infty}^{\infty} |f(u)| du$ converges, the first term on R.H.S. of (10) approaches zero as $l \to \infty$ and hence (10) yields

$$f(x) = \lim_{l\to\infty} \frac{1}{l}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} f(u)\ \cos\frac{n\pi}{l}(u-x)\ dx \qquad ...(11)$$

Puttting $\frac{\pi}{l} = \Delta t$, (11) can be written as

$$f(x) = \lim_{\Delta t\to 0} \sum_{n=1}^{\infty} \Delta t\, F(n\,\Delta t) \qquad ...(12)$$

where $F(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x) \cos t\,(u-x)\ du$ ...(13)

Thus (12) gives

$$f(x) = \int_0^{\infty} F(t)\,dt = \frac{1}{\pi}\int_0^{\infty} dt \int_{-\infty}^{\infty} f(u)\cos t(u-x)\ du \qquad ...(14)$$

Which is Fourier's Integral formula.

Note. *The complex form of Fourier's integral is*

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{itx}\ dt \int_{-\infty}^{\infty} f(u)\ e^{-itu}\ du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\ e^{it(x-u)}\ du\,dt \qquad ...(15)$$

9.7. DIFFERENT FORMS OF FOURIER'S INTEGRALS

Fourier's Integral is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\ dt \int_{-\infty}^{\infty} \cos u\,(x-t)\ du \qquad ...(1)$$

Here $\int_{-\infty}^{\infty} \cos u\,(x-t)\ du = \int_{-\infty}^{0} \cos u\,(x-t)\ dt + \int_{0}^{\infty} \cos u\,(x-t)\ dt$

Say $I = I_1 + I_2$

Replacing u by $-u$ in I_1, we get

$I_1 = -\int_{\infty}^{0} \cos u\,(x-t)\ du = \int_{0}^{\infty} \cos u\,(x-t)\ du = I_2.$

Here $I = \int_{-\infty}^{\infty} \cos u\,(x-t)\ du = 2\int_{0}^{\infty} \cos u\,(x-t)\ du.$

Therefore (1) becomes

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\ dt \int_{0}^{\infty} \cos u\,(x-t)\ du. \qquad ...(2)$$

Now as the limits of integration in (2) do not involve the variables u or t, the order of integration may be changed, *i.e.*

$$f(x)=\frac{1}{\pi}\int_0^\infty du \int_{-\infty}^\infty f(t)\cos u(x-t)\, dt. \qquad ...(3)$$

(*Agra, 1964*)

If $f(x)$ be an odd function of x, *i.e.* $f(-x)=-f(x)$, then

$$\int_{-\infty}^\infty f(t)\cos u(x-t)\, dt=\int_{-\infty}^0 f(t)\cos u(x-t)\, dt+\int_0^\infty f(t)\cos u(x-t)\, dt.$$

Replacing t by $-t$, we have

$$\int_{-\infty}^0 f(t)\cos u(x-t)\, dt=-\int_\infty^0 f(-t)\cos u(x+t)\, dt$$

$$=-\int_0^\infty f(t)\cos u(x+t)\, dt \quad \because f(-t)=-f(t).$$

Thus $\int_{-\infty}^\infty f(t)\cos u(x-t)\, dt$

$$=-\int_0^\infty f(t)\cos u(x+t)\, dt+\int_0^\infty f(t)\cos u(x-t)\, dt$$

$$=\int_0^\infty f(t)\,[\cos u(x-t)-\cos u(x+t)]\, dt$$

$$=\int_0^\infty f(t)\sin ux\sin ut\, dt$$

Substituting it in (3), we get

$$f(x)=\frac{2}{\pi}\int_0^\infty du\int_0^\infty f(t)\sin ut\sin ux\, dt. \qquad ...(4)$$

Changing the order of integration this may be written as

$$f(x)=\frac{2}{\pi}\int_0^\infty f(t)\, dt\int_0^\infty \sin ut\sin ux\, du. \qquad ...(5)$$

Again if $f(x)$ be an even function of (x), *i.e.*, $f(-x)=f(x)$, we have

$$\int_{-\infty}^\infty f(t)\cos u(x-t)\, dt=\int_{-\infty}^0 f(t)\cos u(x-t)\, dt+\int_0^\infty f(t)\cos u(x-t)\, dt.$$

Replacing t by $-t$ in the first integral on the right, we get

$$\int_{-\infty}^\infty f(t)\cos u(x-t)\, dt=-\int_\infty^0 f(-t)\cos u(x+t)\, dt+\int_0^\infty f(t)\cos u(x-t)\, dt$$

$$=\int_0^\infty f(t)\cos u(x+t)\, dt+\int_0^\infty f(t)\cos u(x-t)\, dt$$

$$\because f(-t)=f(t)$$

$$=\int_0^\infty f(t)[\cos u(x+t)+\cos u(x-t)]\, dt$$

$$=2\int_0^\infty f(t)\cos ut\cos ux\, dt$$

Its substitution in (3) yields,

$$f(x)=\frac{2}{\pi}\int_0^\infty du\int_0^\infty f(t)\cos ut\cos ux\, dt \qquad ...(6)$$

$$= \frac{2}{\pi}\int_0^\infty f(t)\, dt \int_0^\infty \cos ut \cos ux\, du \quad \text{(On changing the order of integration)}$$

Problem 17. *Show that the sum function of the Integral formula is* $\frac{1}{2}[f(x+0) + f(x-0)]$ *corresponding to the function* $f(x)$ *in the interval* $0 < x < l$.

By Weierstrass test, since $\left|\int_{-\infty}^{\infty} f(u) \cos t(x-u)\, du\right| \le \int_{-\infty}^{\infty} |f(u)|\, du$ (which converges), the integral $\int_{-\infty}^{\infty} f(u) \cos t(x-u)\, du$ converges absolutely and uniformly for all values of t. Thus by reversing the order of integration, we have,

$$\frac{1}{\pi}\int_{t=0}^{l} dt \int_{u=-\infty}^{\infty} f(u) \cos t(x-u)\, du = \frac{1}{\pi}\int_{u=-\infty}^{\infty} f(u)\, du \int_{t=0}^{l} \cos t(x-u)\, dt$$

$$= \frac{1}{\pi}\int_{u=-\infty}^{\infty} f(u) \frac{\sin l(u-x)}{u-x} du$$

$$= \frac{1}{\pi}\int_{p=-\infty}^{\infty} f(x+p) \frac{\sin lp}{p} dp \quad \text{when } u = x+p$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x+p)\frac{\sin lp}{p} dp + \frac{1}{\pi}\int_0^\infty f(x+p)\frac{\sin lp}{p} dp$$

$$= \frac{1}{2}[f(x+0) f(x-0)] \textit{ when } l \to \infty \text{ by theorem 3 of § 9.3.}$$

Problem 18. *Show that* $\int_0^\infty \frac{\cos ux}{u^2+1} = \frac{\pi}{2} e^{-x},\ x \ge 0$.

Putting $f(x) = e^{-x}$ in the Fourier's Integral form (6) of §9.7, we have

$$f(x) = e^{-x} = \frac{2}{\pi}\int_0^\infty du \int_0^\infty e^{-t} \cos ut \cos ux\, dt = \frac{2}{\pi}\int_0^\infty \cos ux\, du \int_0^\infty e^{-t} \cos ut\, dt.$$

$$= \frac{2}{\pi}\int_0^\infty \cos ux\, du \left[\frac{e^{-t}}{1+u^2}(-\cos ut + u \sin ut)\right]_0^\infty$$

$$= \frac{2}{\pi}\int_0^\infty \cos \frac{\cos ux}{1+u^2} du$$

$$\therefore \quad \int_0^\infty \frac{\cos ux}{1+u^2} du = \frac{\pi}{2} e^{-x}$$

Note. *If* $x = 0$, *this result reduces to* $\int_0^\infty \frac{du}{1+u^2} = \frac{\pi}{2}$.

9.8. A REMARK ON CONVERGENCE OF FOURIER'S SERIES

The Fourier constants in real form by (12) and (13) of §9.4 are

$$\alpha_n = \frac{2}{T}\int_0^T f(t) \cos n\omega t\, dt \quad \text{and } \beta_n = \frac{2}{T}\int_0^T f(t) \sin n\omega t\, dt \qquad \text{...(1)}$$

It is plausible from (1) that α_n and β_n must diminish indefinitely as n increases since the more rapid is the fluctuation in sign of $\cos n\omega x$ and $\sin n\omega x$, the more complete is the cancelling of the various elements of integrals (1) which are known as *Riemann-Lebesgue theorem*. Stokes has formulated rather definite results, according to which if $f(t)$ satisfies Dirichlet's conditions under the careful vigilence whether in particular cases discontinuities of $f(t)$ or its derivatives are introduced at the terminal points of the various segments, then

(i) if $f(t)$ has a finite number of isolated discontinuities in a period the coefficients converge to zero ultimately, like the sequence

$$\left\{\frac{1}{m}\right\} = 1,\ \frac{1}{2},\ -\frac{1}{3},\ldots\ \frac{1}{n},\ldots \qquad \ldots(2)$$

(ii) if $f(t)$ is everywhere continuous while its first derivative $f(t)$ possesses a finite number of isolated discontinuities then the coefficients converge like the sequence

$$\left\{\frac{1}{m^2}\right\} = 1,\ \frac{1}{2^2},\ \frac{1}{3^2},\ldots\ \frac{1}{n^2},\ldots \qquad \ldots(3)$$

(iii) if $f(t)$ and $f'(t)$ are continuous but the second derivative $f''(t)$ is discontinuous at isolated points, the coefficients converge like the sequence

$$\left\{\frac{1}{m^3}\right\} = 1,\ \frac{1}{2^3},\ \frac{1}{3^3},\ldots\ldots\ \frac{1}{n^3},\ldots\ldots \qquad \ldots(4)$$

(iv) if in general, $f(t)$ and its derivatives upto $(n-1)$ order are continuous but nth derivative in a period has a finite number of isolated discontinuities, the coefficients converge like the sequence,

$$\left\{\frac{1}{m^{n+1}}\right\} = 1,\ \frac{1}{2^{n+1}},\ \frac{1}{3^{n+1}},\ldots\ldots,\ \frac{1}{n^{n+1}},\ldots\ldots \qquad \ldots(5)$$

The above statements can be demonstrated as follows:

Integrating by parts, we have from (1),

$$\beta_n = \frac{2}{T}\int_0^T f(t)\sin n\omega t\,dt = -\frac{1}{n}\left[\frac{f(t)\cos n\omega t}{\pi}\right]_0^T + \frac{1}{\pi n}\int_0^T f'(t)\cos n\omega t\,dt \qquad \ldots(6)$$

$$\because\ T = \frac{2\pi}{\omega}$$

The first term on R.H.S. of (6) vanishes in either case (i) there are no points of discontinuity of $f(t)$ in the range from $t=0$ to $t=T$, (ii) there is no discontinuity of $f(t)$ at $t=0$ or T. In case there is a discontinuity, there exists an upper limit M (say) to the coefficient of $\frac{1}{n}$ for all n. The second term on R.H.S. of (6) vanishes as $n\to\infty$ due to fluctuations of $\cos n\omega t$. Hence β_n is comparable with M/n.

When there is no discontinuity in $f(t)$, we have

$$\beta_n = \frac{1}{\pi n}\int_0^T f'(t)\cos n\omega t\ dt \qquad \ldots(7)$$

Integrating again by parts, we find,

$$\beta_n = \frac{1}{\pi^2}\left[\frac{f'(t)\sin n\omega t}{\pi\omega}\right]_0^T - \frac{1}{\pi n^2\omega}\int_0^T f(t)\sin n\omega t\,dt \qquad \ldots(8)$$

If $f'(t)$ has discontinuities, take M as the upper limit of the coefficient of $\frac{1}{n^2}$ as discussed above, then β_n is comparable with M/n^2, since second term on (8) vanishes due to fluctuations of $\sin n\omega t$.

As regards the convergence of a Fourier series, the above statements are very useful and so one should see initially how well or how poorly the series will converge.

Differentiation of a Fourier series makes the convergence poorer while Integration increases its rate of convergence. Conclusively if a Fourier series has been differentiated until it converges as $\frac{1}{n}$, it is no further differentiable.

9.9. PHYSICAL APPLICATIONS OF FOURIER'S SERIES

[1] Furier Series involving Phase Angles

By (11) of §9.4 we have

$$f(t)=\frac{\alpha_0}{2}+\sum_{n=1}^{\infty}\alpha_n\cos n\omega t+\sum_{n=1}^{\infty}\beta_n \sin n\omega t \qquad ...(1)$$

where $\alpha_n=\frac{2}{T}\int_0^T f(t)\cos n\omega t\,dt,\ \beta_n=\frac{2}{T}\int_0^T f(t)\sin n\omega t\,dt$ and $T=\frac{2\pi}{\omega}$ (period) ...(2)

Let $\alpha_n \cos n\omega t+\beta_n \sin n\omega t=\gamma_n\cos(n\omega t-\phi_n)$, ϕ_n being the phase angles.

$$=\gamma_n\cos n\omega t\cos\phi_n+\gamma_n\sin n\omega t\sin\phi_n \qquad ...(3)$$

Equating coefficients of $\cos n\omega t$ and $\sin n\omega t$ on either side of (3), we get

$$\alpha_n=\gamma_n\cos\phi_n \text{ and } \beta_n=\gamma_n\sin\phi_n \qquad ...(4)$$

which give $\gamma_n=\sqrt{\alpha_n^2+\beta_n^2}$ and $\phi_n=\tan^{-1}\frac{\beta_n}{\alpha_n}$ ...(5)

Hence the series (1) takes the form

$$f(t)=\frac{\alpha_0}{2}+\sum_{n=1}^{\infty}\gamma_n\cos(n\omega t-\phi_n) \qquad ...(6)$$

or

$$f(t)=\frac{\alpha_0}{2}+\sum_{n=1}^{\infty}\gamma_n\sin\left(n\omega t+\frac{\pi}{2}-\phi_n\right) \qquad ...(7)$$

[2] Effective Values and the Average of a Product.

When dealing with the problems in electrical-circuit theory and in the theory of mechanical vibrations, we require to find the root mean square or effective value of a periodic function. In terms of complex Fourier-series expansion, a periodic function $f(t)$ is given by

$$f(t)=\sum_{n=-\infty}^{\infty}a_n e^{in\omega t} \quad \text{where } T=\frac{2\pi}{\omega}\text{ (period)} \qquad ...(8)$$

The root-mean square or effective value of the function say f, over a period T is given by

$$f_E^2=\frac{1}{T}\int_0^T f^2(t)\,dt$$

$$=\frac{1}{T}\int_0^T\left[\sum_{n=-\infty}^{\infty}a_n e^{in\omega t}\sum_{m=-\infty}^{\infty}a_m e^{im\omega t}\right]dt$$

$$=\frac{1}{T}\int_0^T\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}a_n a_m e^{i(n+m)\omega t}\,dt$$

$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_n a_m \int_0^{2\pi/\omega} e^{i(n+m)\omega t} dt \qquad \ldots(9)$$

(on carrying term by term integration)

$$\left.\begin{aligned} \text{But } \int_0^{2\pi/\omega} e^{ip\omega t} dt = \left[\frac{e^{ip\omega t}}{ip\omega}\right]_0^{2\pi/\omega} &= 0 \text{ for integral } m \neq 0 \\ &= \frac{2\pi}{\omega} = T \text{ for } m = 0 \end{aligned}\right\} \qquad \ldots(10)$$

which follows that all the integrals in (9) vanish except those for $m = -n$.

$$\therefore \text{(9) reduces to } f_E^2 = \frac{1}{T} \sum_{n=-\infty}^{\infty} a_n a_{-n} T = \sum_{n=-\infty}^{\infty} a_n a_{-n} \qquad \ldots(11)$$

Regarding a_{-n} as the conjugate of a_n, the quantity in summation of R.H.S. of (11) is the square of the magnitude of a_n. Also the summation over negative values of n yields the same result as summation over positive values of n. Thus (11) reduces to

$$f_E^0 = 2\sum_{n=1}^{\infty} |a_n|^2 + a_0^2 \text{ where } a_0 \text{ corresponds } n = 0 \qquad \ldots(12)$$

Again, to find the average value over a product of two periodic functions with the same period T (say), let us assume two functions

$$f_1(t) = \sum_{n=-\infty}^{\infty} a_n e^{in\omega t}$$

and
$$\left. f_2(t) = \sum_{m=-\infty}^{\infty} b_m e^{im\omega t} \text{ with } T = \frac{2\pi}{\omega} \right\} \qquad \ldots(13)$$

Then average of the product

$$= \frac{1}{T} \int_0^T f_1(t)\, f_2(t)\, dt$$

$$= \frac{1}{T} \int_0^T \left[\sum_{n=-\infty}^{\infty} a_n e^{in\omega t} \sum_{m=-\infty}^{\infty} b_m e^{im\omega t} \right] dt$$

$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_n b_m \int_0^T e^{i(n+m)\omega t} dt$$

(on interchanging the order of integration, and summation)

$$= \sum_{n=-\infty}^{\infty} a_n b_{-n} \quad \text{[evaluated just as in (11)]} \qquad \ldots(14)$$

[3] Thermal State. *Let us first consider the Fourier's problem of the permanent state of temperatures in a thin rectangular plate of breadth π and of infinite length, whose faces are impervious to heat. Suppose we have to find the temperature at any point of the plate assuming that the two long edges of the plate are kept at the constant temperature zero, that one of the short edges say the base of the plate is kept at the temperature unity and that the temperatures of the points in the plate decrease indefinitely as we recede from the base.*

We know from the Analytical Theory of heat that the change of temperature at different points of a solid when heat flows within the solid, is given by

$$\frac{\partial u}{\partial t}=a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right).$$

where t represents the time, u the temperature at any point (x, y, z) of the solid and a is a constant.

In case of permanent state of temperature in a thin rectangular plate this reduces to

$$0=a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right).$$

i.e. taking the base of the plate as x-axis and one end of the base as origin, the temperature u of any point is given by

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, \quad \text{...(15)}$$

provided, $u = 0$ when $x = 0$. ...(16) $u = 0$ when $x = \pi$, ...(17)

$u = 0$ when $y = \infty$, ...(18) $u = 1$ when $y = 0$ ...(19)

Let us suppose that $u = A^{\alpha y+\beta x}$ is a solution of (15); then we have

$$\frac{\partial^2 u}{\partial x^2}=A\beta^2\cdot e^{\alpha y+\beta x} \text{ and } \frac{\partial^2 u}{\partial y^2}=A\alpha^2\cdot e^{\alpha y+\beta x}.$$

Substituting these values in (15), we get

$$\alpha^2+\beta^2=0 \text{ or } \beta=\pm i\,\alpha.$$

Thus we get $u = Ae^{\alpha y\pm i\alpha x}$, where A and α are two constants,

i.e $\quad u = Ae^{\alpha y}\, e^{i\alpha x}$ and $u = Ae^{\alpha y}e^{-i\alpha x}$.

Adding, $\quad 2u = Ae^{\alpha x}\,(e^{i\alpha y}+e^{-i\alpha x})$

$\quad = Ae^{\alpha y}\,.\,2\cos\alpha x.$

$\therefore \quad u = Ae^{\alpha y}\cos\alpha x.$ ...(20)

This solution may also be adjusted in the form

$$u = Be^{\alpha y}\,\sin\alpha x. \quad \text{...(21)}$$

The solution (21) satisfies the conditions

$u = 0$ when $x = 0$, $u = 0$ when $x = \pi$, α being an integer.

Also $u = 0$, when $y = \infty$, α being negative.

Hence u may be expressed as a sum of terms of the form $Ce^{-ny}\,\sin nx$, n being a positive integer, which satisfies the conditions (16), (17) and (18).

$$\therefore\ u = C_1e^{-y}\sin x + C_2\,e^{-2y}\,\sin 2x + C_3e^{-3y}\sin 3x + \quad \text{...(22)}$$

Again to satisfy the condition (19), if we put $y = 0$ in (15), we get

$$u = C_1\sin x + C_2\sin 2x + C_3\sin 3x + \ldots \quad \text{...(23)}$$

If the series on R.H.S. of (23) be developed to represent unity the condition (19) is satisfied. For this we have to find the suitable values for the coefficients C_1, C_2, C_3...etc.

Consider the Fourier's series defined by $f(x) = 1$ for all values of x lying between 0 and π.

Here $f(x) \equiv C_1\sin x + C_2\sin 2x + C_3\sin\ 3x + \ldots,$

where $C_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi}\sin nx\,dx$, $\because f(x) = 1$; also $f(x)$ is an odd function of x as $f(-x) = -f(x)$.

Now $\int_0^{\pi} \sin nx \, dx = \left[-\frac{\cos nx}{x} \right]_0^{\pi} = \frac{1}{n}\left[1 - (-1)^n\right]$

$= 0$ if n is even

and $= \frac{2}{\pi}$ if n is odd.

Hence

$$f(x) = \sum_{n=1}^{\infty} C_n \sin nx$$

$$= \frac{2}{\pi} \sum_{1}^{\infty} \frac{2}{n} \sin nx, \text{ when } n \text{ is odd and zero when } n \text{ is even}$$

$$1 = \frac{4}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots \right]$$, n being odd which gives the values of the constants $C_1, C_2, C_3 \ldots$ in order that the condition (19) is satisfied.

Substituting values of constants given by $C_n = \frac{4}{n\pi}$ (n being odd) in (22), the required solution to give the temperature at any point is

$$u = \frac{4}{\pi}\left[e^{-y} \sin x + \tfrac{1}{3} e^{-3y} \sin 3x + \tfrac{1}{5} e^{-5y} \sin 5x + \ldots \right]. \quad \ldots(24)$$

Note. *Similarly we can find the potential function at any point of a long, thin rectangular conducting sheet of breadth π through which an electric current flows, supposing that the two long edges are kept at potential zero and one short edge at potential unity.*

Deductions.

(*i*) If the temperature of the base of the plate be a given function of x, while the other conditions remain unaltered.

We have already shown that

$$f(x) = \sum_{n=1}^{\infty} C_n \sin nx,$$

where $$C_n = \frac{2}{\pi} \int_0^{\pi} f(v) \sin nv \, dv.$$

Thus we have form (22)

$$u = \frac{2}{\pi} \sum_{n=1}^{\infty} \left[e^{-ny} \sin nx \int_0^{\pi} f(v) \sin nv \, dv \right]. \quad \ldots(25)$$

(*ii*) Assuming the temperature of the base as unity and the breadth of the plate as π, we have proved that

$$u = \frac{4}{\pi}\left[e^{-y} \sin x + \tfrac{1}{3} e^{-3y} \sin 3x + \tfrac{1}{5} e^{-5y} \sin 5y + \ldots \right]. \quad \ldots(26)$$

In order to sum the series on R.H.S., let us consider

$$\log(1+z) = z - \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3 - \tfrac{1}{4}z^4 + \ldots \text{ when } |z| < 1$$

and $$\log(1-z) = -z - \tfrac{1}{2}z^2 - \tfrac{1}{3}z^3 - \tfrac{1}{4}z^4 - \ldots \text{ when } |z| < 1.$$

$$\therefore \tfrac{1}{2}[\log(1+z) - \log(1-z)] = z + \frac{z^3}{3} + \frac{z^5}{5} + \ldots \text{ when } |z| < 1. \quad \ldots(27)$$

Now z being a complex quantity, may be expressed as

$$z = r(\cos\theta + i\sin\theta).$$

Then,

$$\log(1+z) = \log[1 + r(\cos\theta + i\sin\theta)]$$

$$= \frac{1}{2}\log\left[(1+r\cos\theta)^2 + r^2\sin^2\theta\right] + i\tan^{-1}\frac{r\sin\theta}{1+r\cos\theta}$$

$$\left[\because \log(X+iY) = \frac{1}{2}\log\left(X^2+Y^2\right) + i\tan^{-1}\frac{Y}{X}\right]$$

$$= \frac{1}{2}\log\left[1+2r\cos\theta + r^2\right] + i\tan^{-1}\frac{r\sin\theta}{1+r\cos\theta}.$$

Similarly,

$$\log(1-z) = \frac{1}{2}\log\left[1-2r\cos\theta + r^2\right] - i\tan^{-1}\frac{r\sin\theta}{1-r\cos\theta}.$$

$$\therefore \frac{1}{2}[\log(1+z) - \log(1-z)]$$

$$= \frac{1}{2}\left[\log\frac{1+2r\cos\theta+r^2}{1-2r\cos\theta+r^2} + i\left\{\tan^{-1}\frac{r\sin\theta}{1+r\cos\theta} + \tan^{-1}\frac{r\sin\theta}{1-r\cos\theta}\right\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\log\frac{1+2r\cos\theta+r^2}{1-2r\cos\theta+r^2} + i\tan^{-1}\frac{2r\sin\theta}{1-r}\right]$$

$$\left[\because \tan^{-1}a + \tan^{-1}b = \tan^{-1}\frac{a+b}{1-ab}\right].$$

Hence from (27), we have

$$\frac{1}{2}\left[\frac{1}{2}\log\frac{1+2r\cos\theta+r^2}{1-2r\cos\theta+r^2} + i\tan^{-1}\frac{2r\sin\theta}{1-r^2}\right]$$

$$= r(\cos\theta + i\sin\theta) + \frac{r^3(\cos\theta + i\sin\theta)^3}{3} + \ldots$$

$$= r(\cos\theta + i\sin\theta) + \frac{r^3(\cos 3\theta + i\sin 3\theta)}{3} + \ldots$$

Equating real and imaginary parts, we get

$$\frac{1}{4}\log\frac{1+2r\cos\theta+r^2}{1-2r\cos\theta+r^2} = r\cos\theta + \frac{r^3\cos 3\theta}{3} + \frac{r^5\cos 5\theta}{5} + \ldots \quad \ldots(28)$$

and $$\frac{1}{2}\tan^{-1}\frac{2r\sin\theta}{1-r^2} = r\sin\theta + \frac{r^3\sin 3\theta}{3} + \frac{r^5\cos 5\theta}{5} + \ldots \quad \ldots(29)$$

(28) and (29) are valid for all values of θ provided $r < 1$.

Also e^{-y} is less than one if y is positive.

Hence replacing r by e^{-y}, (29) gives

$$\frac{e^{-y}}{1}\sin x + \frac{e^{-3y}}{3}\sin 3x + \frac{e^{-5y}}{5}\sin 5x + \ldots - \frac{1}{2}\tan^{-1}\frac{2e^{-y}\sin x}{1-e^{-2y}}.$$

$$= \frac{1}{2} \tan^{-1} \frac{2 \sin x}{e^y - e^{-y}}$$

$$= \frac{1}{2} \tan^{-1} \frac{\sin x}{\sinh y}.$$

Substituting the value of this series in (18), we get

$$u = \frac{2}{\pi} \tan^{-1} \frac{\sin x}{\sinh y}. \qquad \text{...(30)}$$

For Isothermal lines, u = constant

i.e. $\frac{2}{\pi} \tan^{-1} \frac{\sin x}{\sinh y} = a$ (say)

or $$\frac{\sin x}{\sinh y} = \tan \frac{a\pi}{2} \qquad \text{...(31)}$$

Again

$$\frac{1}{4} \log \frac{1 + 2r \cos\theta + r^2}{1 - 2r \cos\theta + r^2} = \frac{1}{4} \log \frac{1 + 2e^{-y} \cos x + e^{-2y}}{1 - 2e^{-y} \cos x + e^{-2y}}.$$

replacing r by e^{-y} and θ by x

$$= \frac{1}{4} \log \frac{\frac{e^y + e^{-y}}{2} + \cos x}{\frac{e^y + e^{-y}}{2} - \cos x} = \frac{1}{4} \log \frac{\cosh y + \cos x}{\cosh y - \cos x}$$

For the lines of flow,

$$\frac{1}{\pi} \log \frac{\cosh y + \cos x}{\cosh y - \cos x} = b \text{ (a constant)}$$

or $$\frac{\cosh y + \cos x}{\cosh y - \cos x} = e^{\pi b}. \qquad \text{...(32)}$$

[since $u_1 = \frac{1}{4} \log \frac{\cosh y + \cos x}{\cosh y - \cos x}$ is the solution for the problem if the isothermal lines are the lines of flow of the existing problem and conversely the lines of flow are the isothermal lines of the present problem.]

[4] Transverse Vibrations of a String

We now consider the problem of the transverse vibrations of stretched string fastened at the ends. Suppose that the string is initially distorted into some given curve and then allowed to swing. Let the length of the string be l and the equation of the curve to which the string is distorted initially be y = f(x) with reference to the position of equilibrium of the string as x-axis and one of the ends of the origin.

In acoustics, such vibrations are given by

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}. \qquad \text{...(33)}$$

We have to get an expression for y, which is the solution of the equation (33) subject to the conditions

$$y = 0 \text{ when } x = 0. \qquad \text{...(34)}$$

$$y = 0 \text{ when } x = l, \quad \text{...(35)}$$

$$y = f(x) \text{ when } t = 0, \quad \text{...(36)}$$

and $$\frac{\partial y}{\partial t} = 0 \text{ when } t = 0. \quad \text{...(37)}$$

Let $y = Ae^{\alpha x+\beta t}$ be the solution of (33),

$$\therefore \quad \frac{\partial^2 y}{\partial t^2} = A^2\beta e^{\alpha x+\beta t} \text{ and } \frac{\partial^2 y}{\partial x^2} = A\alpha^2 e^{\alpha x+\beta t}.$$

Substituting these values in (33), we get

$$\beta^2 = a^2\alpha^2,$$

giving $$\beta = \pm a\alpha$$

$$\therefore \quad y = Ae^{\alpha x \pm a\alpha t} \quad \text{...(38)}$$

is a solution of (33).

Here if we replace α by $+\alpha i$ and $-\alpha i$ in succession

$$y = Ae^{(x \pm at)\alpha i},$$

$$y = Ae^{-(x \pm at)\alpha i}.$$

Adding and then dividing by 2, we have

$$y = A \cos \alpha (x \pm at). \quad \text{...(39)}$$

This may further be expressed as

$$y = B \sin \alpha (x \pm at). \quad \text{...(40)}$$

From (38),

$y = A \cos \alpha (x + at) = A \{\cos \alpha x \cos \alpha a - \sin \alpha x \sin \alpha at\}$

and $y = A \cos \alpha (x - at) = A \{\cos \alpha x \cos \alpha at + \sin \alpha x \sin \alpha at\}$.

Writing y successively equal to half the sum and difference of these values, we can write

$$y = A \cos \alpha x \cos \alpha at,$$

$$y = A \sin ax \sin \alpha at.$$

Similarly from (40),

$$y = B \sin \alpha x \cos \alpha at,$$

$$y = B \cos \alpha x \sin \alpha at.$$

Out of these four values of y, if we take

$$y = B \sin \alpha x \cos \alpha at,$$

it is obvious that this satisfies the conditions (34) and (37) and this may be made to satisfy (35) by putting $\alpha = \dfrac{n\pi}{l}$ when n is any integer.

Hence we can express

$$y = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{i} \cos \frac{n\pi at}{l},$$

i.e., $$y = b_1 \sin \frac{\pi x}{l} \cos \frac{\pi at}{l} + b_2 \sin \frac{2\pi x}{l} \cos \frac{2\pi at}{l}$$

$$+ b_3 \sin \frac{3\pi x}{l} \cos \frac{3\pi at}{l} + \ldots \quad \text{...(41)}$$

The relation (41) satisfies the conditions (34), (35) and (37). In order that this may satisfy the condition (11) also, let us put $t = 0$ in (41), whence we get

$$y = b_1 \sin \frac{\pi x}{l} + b_2 \sin \frac{2\pi x}{l} + b_3 \sin \frac{3\pi x}{l} + \ldots \qquad \ldots(42)$$

If we can however develop $f(x)$ into a series of the form (42), then a comparison will give the values of the coefficients b_1, b_2, b_3 etc., which when substituted in (41), will give a solution of the partial differential equation (33) under the conditions mentioned above.

Now consider the Fourier's series defined by

$$f(x) = b_1 \sin \frac{\pi x}{l} + b_2 \sin \frac{2\pi x}{l} + b_3 \sin \frac{3\pi x}{l} + \ldots$$

for all values of x lying between 0 and l.

Then $\quad b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$

$= \frac{2}{l} \int_0^l f(v) \sin \frac{n\pi v}{l}\, dv$ (replacing x by v in order to discriminate).

Substituting this values of b_n in

$$y = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n a \pi t}{l},$$

the complete solution of the differential equation (8) may be given as

$$y = \frac{2}{l} \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} \cos \frac{n\pi a t}{l} \int_0^l f(v) \sin \frac{n\pi v}{l}\, dv. \qquad \ldots(43)$$

Deduction. If the shape of the curve into which the string is distorted be given by $y = f(x) = b \sin \frac{n\pi x}{l}$, then

$$f(v) = b \sin \frac{n\pi v}{l}.$$

Thus $\int_0^l f(v) \sin \frac{n\pi v}{l}\, dv = b \int_0^l \sin^2 \frac{n\pi v}{l}\, dv$

$$= \frac{b}{2} \int_0^l \left(1 - \cos \frac{2n\pi v}{l}\right) dv$$

$$= \frac{b}{2} \left[v - \frac{l}{2n\pi} \sin \frac{2n\pi v}{l}\right]_0^l = b\, \frac{l}{2}.$$

Hence the solution (43) reduces to

$$y = b \sin \frac{n\pi x}{l} \cos \frac{n\pi a t}{l}. \qquad \ldots(44)$$

Note. *For more applications see the next chapter.*

9.10. THE FOURIER TRANSFORMS

[A] Fourier Sine Transforms. They can be subdivided in two namely the infinite Fourier sine transform and the Finite Fourier sine transforms.

$[a_1]$ *The Infinite Fourier sine Transform* of a function $F(x)$ of x such that $0 < x < \infty$ is denoted by $f_s(n)$, n being a positive integer and is defined as

$$f_s(n) = \int_0^{\infty} F(x) \sin nx\, dx \qquad \ldots(1)$$

Here $F(x)$ is called as the *Inverse Fourier sine transform of* $f_s(n)$ and defined as

$$F(x) = \frac{2}{\pi}\int_0^\infty f_s(n)\sin nx\,dx \qquad \ldots(2)$$

Thus if $f_s(n) = f_s[F(x)]$, then $F(x) = f_s^{-1}[f_s(n)]$.(3)

where f is the symbol for Fourier transform and f^{-1} for its inverse.

Problem 19. *Find the sine transform of e^{-x}.*

We have

$$f_s(n) = \int_0^\infty e^{-x}\sin nx\,dx = \left[\frac{e^{-x}}{1+n^2}(-\sin nx - n\cos nx)\right]_0^\infty = \frac{n}{1+n^2}.$$

Problem 20. *Find the inverse sine transform of $e^{-\lambda n}$.*

We have

$$f_s^{-1}\left[e^{-\lambda n}\right] = \frac{2}{\pi}\int_0^\infty e^{-\lambda n}\sin nx\,dn = \frac{2}{\pi}\left[\frac{e^{-\lambda n}}{\lambda^2+x^2}(-\lambda\sin nx - x\cos nx)\right]_0^\infty$$

$$= \frac{2}{\pi}\cdot\frac{x}{\lambda^2+x^2}.$$

[illegible]: *The Finite Fourier sine transform* of a function $F(x)$ of x such that $0 < x < l$ is denoted by $f_s(n)$, n being a positive integer and is defined as

$$f_s(n) = \int_0^l F(x)\sin\frac{n\pi x}{l}\,dx \qquad \ldots(4)$$

In case $l = \pi$, this becomes

$$f_s(n) = \int_0^\pi F(x)\sin nx\,dx \qquad \ldots(5)$$

and the inversion formula is

$$F(x) = \frac{2}{\pi}\sum_{n=1}^\infty f_s(n)\sin nx \qquad \ldots(6)$$

whence a_n is the coefficient of sin nx in the expansion of $F(x)$ in a sine series and is given by

$$a_n = \frac{2}{\pi}\int_0^\pi F(x)\sin nx\,dx$$

$$= \frac{2}{\pi}f_s(n) \text{ by (5)} \qquad \ldots(7)$$

Problem 21. *Find the Fourier sine transform of $F(x) = x$ such that $0 < x < 2$.*

We have $f_s(n) = \int_0^2 F(x)\sin\frac{n\pi x}{2}\,dx$ $\quad\because l = 2$ in the existing case.

$$= \int_0^2 x\sin\frac{n\pi x}{2}\,dx$$

$$= \left[x.\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right]_0^2 + \int_0^2\frac{2}{n\pi}\cos\frac{n\pi x}{2}\,dx \text{ (on integrating by parts)}$$

$$= \left[\frac{-2x}{n\pi} \cos \frac{n\pi x}{2} + \frac{4}{n^2\pi^2} \sin \frac{n\pi x}{2} \right]_0^2 = -\frac{4}{n\pi} \cos n\pi.$$

[B] Fourier Cosine Transforms. They can also be subdivided into two namely Infinite and finite cosine transforms.

[b_1] *The Infinite Fourier Cosine Transform* of $F(x)$ for $0 < x < \infty$, is defined as

$$f_c(n) = \int_0^\infty F(x) \cos nx \, dx, \; n \text{ being a positive integer.} \quad ...(8)$$

Here the function $F(x)$ is called as the *Inverse cosine transform* of $f_c(n)$ and is defined as

$$F(x) = \frac{2}{\pi} \int_0^\infty f_c(n) \cos nx \, dx \quad ...(9)$$

Thus if $f_c(n) = f_c[F, (x)]$, then $F(x) = f^{-1}{}_c[f_c(n)]$ ...(10)

Problem 22. *Find the cosine transform of $x^n e^{-ax}$.*

We have $\int_0^\infty e^{-ax} \cos nx \, dx = \frac{a}{a^2 + n^2}$ and $f_c(n) = \int_0^\infty x^n e^{-ax} \cos nx \, dx$

Differentiating the first relation n times w.r.t. 'a' we find

$$\int_0^\infty x^n e^{-ax} \cos nx \, dx = (-1)^n \frac{d^n}{da^n} \left(\frac{a}{a^2 + n^2} \right)$$

$$= \frac{\underline{|n} \cos \left\{ (n+1) \tan^{-1} \frac{n}{a} \right\}}{\left(a^2 + n^2\right)^{(n+1)/2}} \text{ by usual method.}$$

Hence
$$f_c(n) = \frac{\underline{|n} \cos \left\{ (n+1) \tan^{-1} \frac{n}{a} \right\}}{\left(a^2 + n^2\right)^{(n+1)/2}}$$

Problem 23. *Find $f^{-1}{}_c \{e^{-\lambda n}\}$*

We have $f_c^{-1} \left\{ e^{-\lambda n} \right\} = \frac{2}{\pi} \int_0^\infty e^{-\lambda n} \cos nx \, dx$

$$= \frac{2}{\pi} \left[\frac{e^{-\lambda n}}{\lambda^2 + x^2} (-\lambda \cos nx + x \sin nx) \right]_0^\infty = \frac{2}{\lambda^2 + x^2}$$

[b_2] *The Finite Fourier cosine transform* of $F(x)$ for $0 < x < l$ is defined as

$$f_c(n) = \int_0^l F(x) \cos \frac{n\pi x}{l} \, dx \quad ...(11)$$

when $l = \pi$, this becomes

$$f_c(n) = \int_0^\pi F(x) \cos nx \, dx \quad ...(12)$$

and the inversion formula is

$$F(x) = \frac{1}{\pi} f_c(0) + \frac{2}{\pi} \sum_{n=1}^\infty f_c(n) \cos nx \quad ...(13)$$

when $f_c(0) = \int_0^{\pi} F(x)\ dx$ ...(14)

Also b_n the coefficient of cos nx in the expansion of $F(x)$ in a cosine series is given by

$$b_n = \frac{2}{\pi}\int_0^{\pi} F(x) \cos nx\ dx = \frac{2}{\pi} f_c(n) \text{ by (12)} \qquad ...(15)$$

Problem 24. *Find the finite Fourier cosine transform of x.*

We have $f_c(n) = \int_0^{\pi} x \cos nx\, dx$

$$= \left[\frac{x \sin nx}{n}\right]_0^{\pi} - \frac{1}{n}\int_0^{\pi} \sin nx\, dx \text{ (on integrating by parts)}$$

$$= 0 - \frac{1}{n}\left[\frac{\cos nx}{-n}\right]_0^{\pi}$$

$$= \frac{1}{n^2}\left\{(-1)^n - 1\right\},\ n = 1, 2, 3, ...$$

But if $n = 0$, $f_c(0) = \int_0^{\pi} x\ dx = \left[\frac{x^2}{2}\right]_0^{\pi} = \frac{\pi^2}{2}$.

Note. *On the next page are tabulated some useful Fourier sine and cosine transforms in a concise form.*

[C] The Complex Fourier Transforms.

The Complex Fourier Transform of a function $F(x)$ for $-\infty < x < \infty$, is defined as

$$f(n) = \int_{-\infty}^{\infty} F(x)\ e^{inx}\ dx \qquad ...(16)$$

where e^{inx} is said to be the *Kernel* of the transform.

The inversion formula is $F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(n) e^{-inx}\, dn$ ...(17)

Problem 25. *Find the Fourier Complex Transform of*

$$F(x) = \begin{cases} 1 - x^2, & |x| < 1 \\ 0, & |x| < 1 \end{cases}$$

We have $f(n) = \int_{-1}^{1} (1 - x^2)\ e^{inx}\ dx = \left[(1 - x^2)\frac{e^{inx}}{in} + \frac{2}{in}\int_{-1}^{1} x\, e^{inx}\ dx\right]$

(on integrating by parts)

$$= 0 + \frac{2}{in}\left[\frac{xe^{inx}}{in}\right]_{-1}^{1} - \frac{2}{(in)^2}\int_{-1}^{1} e^{inx}\, dx$$

$$= \frac{2}{-n^2}\left[e^{in} + e^{-in}\right] + \frac{2}{n^2}\left[\frac{e^{inx}}{in}\right]_{-1}^{1} = -\frac{2}{n^2}\left(e^{in} + e^{-in}\right) + \frac{2}{in^3}\left(e^{in} - e^{-in}\right)$$

$$= -\frac{4}{n^2} \cos n + \frac{4}{n^3} \sin n = -\frac{4}{n^3}(n \cos n - \sin n).$$

Infinite Fourier Sine Transforms

$$f_s(n) = \int_0^\infty F(x) \sin nx\,dx, \quad F(x) = \frac{2}{\pi}\int_0^\infty f_s(n) \sin x\,dn$$

$F(x)$	$f_c(n)$
e^{-x}	$n/(1+n^2)$
$xe^{-x^2/2}$	$\sqrt{\pi/2}\; ne^{-n^2/2}$
$\dfrac{\sin x}{x}$	$\dfrac{1}{2}\log\left\lvert\dfrac{1+n}{1-n}\right\rvert$
$\left.\begin{matrix} x(1-x^2)2, & 0<x<1, \nu>-1 \\ 0. & x>1\end{matrix}\right\}$	$\sqrt{\pi}\; 2^{\nu-1/2}\Gamma(\nu+1)\; n^{-1/2-\nu} J_{\nu+3/2}(n)$
$x^{\mu-1}, 0<\mu<1$	$\sqrt{\mu}\;\sin\left(\dfrac{\mu x}{2}\right) n^{-\mu}$
$x^{\nu'} e^{-\mu_x}$	$\dfrac{2^\nu \mu^\nu\,(\Gamma\nu)\, n}{(\mu^2+n^2)^{\nu+1}}$
$\left.\begin{matrix} 0, & 0<x<a \\ \{x^2-a^2\}^{-1/2}, & x>a\end{matrix}\right\}$	$J_0(an)$

Finite Fourier Sine Transforms

$$f_s(n) = \int_0^\pi F(x)\sin nx\,dx, \quad F(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} f_s(n) \sin nx$$

$F(x)$	$f_s(n)$
1	$\dfrac{1}{n}\left[1+(-1)^{n+1}\right]$
x	$(-1)^{n+1}\cdot \pi/n$
$\left.\begin{matrix} x, & 0<x\le\pi/2 \\ \pi-x, & \pi/2\le x<\pi\end{matrix}\right\}\ldots\ldots$	$\dfrac{2}{n^2}\sin\dfrac{n\pi}{2}$
$1-\dfrac{x}{\pi}$	$\dfrac{1}{n}$
$x^2\ldots\ldots$	$\dfrac{\pi^2}{n}(-1)^{n-1}-\dfrac{2}{n^3}\left[1-(-1)^n\right]$
$x(\pi-x)$	$\dfrac{2}{n^3}\left[1-(-1)^n\right]$
$x^3\ldots$	$(-1)^n\pi[6/n^3-\pi^2/n]$
$x(\pi^2-x^2)$	$(-1)^{n+1}\,6\pi/n^3$
$e^{\lambda x}\ldots\ldots$	$\dfrac{n}{n^2+\lambda^2}\left[1+(-1)^n e^{\lambda\pi}\right]$
$\cos\lambda x$	$\dfrac{n}{n^2-\lambda^2}\left[1-(-1)^n \cos\lambda\pi\right]$
$\cos mx,\ m=1,2,3,\ldots\ldots$	$\begin{cases}\dfrac{n}{n^2-m^2}\left[1-(-1)^{n+m}\right], & n\ne m\\ 0, & n=m\end{cases}$
$\sin mx,\ m=1,2,3,\ldots\ldots$	$\begin{cases}0, & n\ne m\\ \pi/2, & n=m\end{cases}$
$\dfrac{\sin\lambda(\lambda-x)}{\sin\lambda\pi}$	$\dfrac{n}{n^2-\lambda^2},\ \lambda\ne 0,1,2,\ldots$
$\dfrac{\sinh\lambda(\pi-x)}{\sinh\lambda\pi}$	$\dfrac{n}{\lambda^2+n^2}$

Infinite Fourier Cosine Transforms $f_c(n) = \int_0^\infty F(x) \cos nx\,dx,\; F(x) = \frac{2}{\pi}\int_0^\infty f_c(n) \cos nx\,dx$		*Finite Fourier Cosine Transforms* $f_c(n) = \int_0^\pi F(x) \cos nx\,dx,\; F(x) = \frac{1}{\pi} f_c(o) + \sum_{n=1}^{\infty} f_c(n) \cos nx$	
$F(x)$	$f_c(n)$	$F(x)$	$f_c(n)$
$\left.\begin{matrix} 1, & 0 < x < a \\ 0, & x > a \end{matrix}\right\}$......	$\frac{\sin na}{n}$	1	$\begin{cases} \pi, & n = 0 \\ 0, & n = 1, 2, 3, \ldots \end{cases}$
$x^{\mu-1}, 0 < \mu < 1$	$\sqrt{\mu}\; n^{-\mu} \sin \frac{\mu\pi}{2}$	$\left.\begin{matrix} 1, & 0 < x < \pi/2 \\ -1, & \pi/2 < x < \pi \end{matrix}\right\}$......	$\begin{cases} 0, & n = 0 \\ 2/\pi \sin n\pi/2, & n = 1, 2, 3, \ldots \end{cases}$
e^{-x}......	$\frac{1}{1+n^2}$	x	$\begin{cases} \pi^2/2, & n = 0 \\ 1/n^2 \left[(-1)^n - 1\right], & n = 1,2,3,\ldots \end{cases}$
Sech $x\pi$	$\sqrt{\frac{\pi}{2}} \frac{1}{1+n^4}$	x^2......	$\begin{cases} \pi^3/3, & n = 0 \\ 2\pi/n^2 (-1)^n, & n = 1,2,3,\ldots \end{cases}$
e^{-x^2}......	$\sqrt{\frac{\pi}{2}} e^{-n^2/2}$	x^3	$\begin{cases} \pi^4/4, & n = 0 \\ 3\pi^2/n^2(-1)^n + 6/n^4\left[(-1)^n - 1\right], & n = 1, 2, 3\ldots\ldots \end{cases}$
$\left.\begin{matrix} \cos x, & 0 < x < a \\ 0, & x > a \end{matrix}\right\}$	$\frac{1}{2}\left[\sin \frac{a(1-n)}{1-n} + \sin \frac{a(1+n)}{1+n}\right]$	$\left(1 - \frac{x}{\pi}\right)^2$......	$\begin{cases} \pi/3, & n = 0 \\ 2/\pi n^2, & n = 1,2,3,\ldots\ldots \end{cases}$
$\sin \frac{x^2}{2}$......	$\sqrt{\frac{\pi}{2}}\left[\cos \frac{n^2}{2} - \sin \frac{n^2}{2}\right]$	$e^{\lambda x}$	$\frac{\lambda}{\lambda^2 + n^2}\left[(-1)^n e^{\lambda x} - 1\right]$
$\cos \frac{x^2}{2}$	$\sqrt{\frac{\pi}{2}}\left[\cos \frac{n^2}{2} + \sin \frac{n^2}{2}\right]$	$\sin \lambda x$......	$\frac{\lambda}{n^2 - \lambda^2}\left[(-1)^n \cos \lambda\pi - 1\right], \; n \neq \lambda$
$\left.\begin{matrix} \left(1 - x^2\right)^\nu, & 0 < x < 1 \\ 0, & x > 1 \text{ and } \nu > -3/2 \end{matrix}\right\}$......	$\sqrt{\pi}\; 2^{\nu-1/2}\; \Gamma(\nu+1) \cdot n^{-\nu-1/2}\; J_{\nu+1/2}(n)$	$\sin mx; m = 1, 2, 3, \ldots\ldots$	$\begin{cases} 0, & n = m \\ m/(n^2 - m^2)\left[(-1)^{n+m} - 1\right], & n \neq m \end{cases}$
		$\frac{\cos \lambda(\pi - x)}{\sinh \pi\lambda}$	$\frac{\lambda}{\lambda^2 + n^2}$

Problem 26. *Find the Complex Fourier transform of $e^{-|x|}$ and then invert it.*

(Rohilkhand, 1982, 86)

We have $f(n) = \int_{-\infty}^{\infty} e^{-|x|}\, e^{inx}\, dx = \int_{-\infty}^{0} e^{(1+in)x}\, dx + \int_{0}^{\infty} e^{-(1-in)x}\, dx$

$$= \frac{1}{1+in} + \frac{1}{1-in} = \frac{2}{1+n^2}$$

So that the inversion formula gives,

$$F(n) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2}{1+n^2} e^{inx}\, dx = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-inx}}{1+n^2}\, dx$$ which may be integrated by contour Integration.

Note. *Several other* **Complex Fourier Transforms** *have been tabulated on the next page.*

[D] Parseval's Identity for Fourier Integrals.

It is stated as $\int_{-\infty}^{\infty} |F(x)|^2\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(n)|^2\, dn$ ...(18)

where $f(n)$ is the Fourier transform of $F(x)$.

Problem 27. *Find the Fourier transform of*

$$F(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$$

(Agra, 1982; Kanpur, 1970)

Hence or otherwise evaluate $\int_{0}^{\infty} \frac{\sin^2 nx}{n^2}\, dx$.

We have $f(n) = \int_{-\infty}^{\infty} F(x)\, e^{-inx}\, dx = \int_{-a}^{a} 1 \cdot e^{-inx}\, dx$

$$\left[\frac{e^{-inx}}{-in}\right]_{-a}^{a} = \frac{e^{+ina} - e^{-ina}}{in} = \frac{2\sin na}{n} \quad \text{for } n \neq 0$$

For $n = 0$, $f(n) = 2 \lim_{n\to 0} \frac{\sin na}{n} = 2 \lim_{n\to 0} \frac{1}{n}\left[na - \frac{1}{\lfloor 3}n^3a^3 + \ldots\right] = 2a$

Now using Parseval's identity, we find

$$\int_{-a}^{a} 1^2 \cdot dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4\sin^2 na}{n^2}\, dn \text{ when } n \neq 0$$

i.e. $$\frac{1}{2\pi}\cdot 2\int_{0}^{\infty} \frac{4\sin^2 na}{n^2}\, dn = [x]_{-a}^{a} = 2a$$

$\therefore$ $$\int_{0}^{\infty} \frac{\sin^2 nx}{n^2}\, dx = \frac{\pi x}{2}.$$

[E] Relation between the Fourier Transform of the Derivatives of a Function.

If $f(n)$ be the Fourier transform of $F(x)$, then we have to express the Fourier transform of the function $\frac{d^m F}{dx^m}$ in terms of $f(n)$.

$F(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-inx}\,dx$	$f(n)=\int_{-\infty}^{\infty} F(x)\, e^{inx}\,dx$	$F(x)$	$f(n)$
$\frac{1}{\lvert x\rvert}$	$\frac{1}{\lvert n\rvert}$	$\frac{\sinh ax}{\sinh \pi x}, -\pi < a < \pi$	$\frac{\sin a}{\cosh n + \cos a}$
$\left.\begin{matrix} e^{i\omega x}, & a < x < b \\ 0, & x < a,\ x > b \end{matrix}\right\}$......	$i\,\frac{e^{ia(\omega+n)} - e^{ib(\omega+n)}}{n}$	$\frac{\cosh ax}{\cosh \pi x}, -\pi < a < \pi$	$2\,\frac{\cos a/2 \cosh a/2}{\cosh n + \cos a}$
$\left.\begin{matrix} e^{-\lambda x + i\omega x}, & x > 0 \\ 0, & x < 0 \end{matrix}\right\}$......	$\frac{i}{\omega + n + i\lambda}$	$\frac{\sin\{b(a^2+x^2)^{1/2}\}}{(a^2+x^2)^{1/2}}$......	$\left\{\begin{matrix} 0, \lvert n\rvert > b \\ \pi J_0\left[a\sqrt{b^2-n^2}\right], \lvert n\rvert < b \end{matrix}\right.$
$\frac{e^{-\lambda\lvert x\rvert}}{\lvert x\rvert^{1/2}}$......	$\frac{\left[\left(\lambda^2+n^2\right)^{1/2}+\lambda\right]^{1/2}}{\left(\lambda^2+n^2\right)^{1/2}}$	$\left[\begin{matrix} \frac{\cosh\left\{b\left(a^2-x^2\right)^{1/2}\right\}}{\left(a^2-x^2\right)^{1/2}}, & \lvert x\rvert < a \\ 0, & \lvert x\rvert > a \end{matrix}\right]$	$\pi J_0\left[a\sqrt{n^2+b^2}\right]$
$e^{-\lambda x^2},\ R(\lambda) > 0$	$\sqrt{\frac{\pi}{\lambda}}, e^{-n^2/4\lambda}$	$\begin{matrix} \frac{\cos\left\{b\left(a^2-x^2\right)^{1/2}\right\}}{\sqrt{a^2-x^2}}, & \lvert x\rvert < a \\ 0, & \lvert x\rvert > a \end{matrix}$	$\pi J_0\left[a\sqrt{n^2-b^2}\right]$
$\sin \lambda x^2$...	$\sqrt{\frac{\pi}{\lambda}}\sin\left(\frac{n^2}{4\lambda}+\frac{\pi}{4}\right)$		
$\cos \lambda x^2$	$\sqrt{\frac{\pi}{\lambda}}\cos\left(\frac{n^2}{4\lambda}-\frac{\pi}{4}\right)$	$\left.\begin{matrix} P_\nu(x), & \lvert x\rvert < 1 \\ 0, & \lvert x\rvert > 1 \end{matrix}\right\}$......	$i^\nu \sqrt{2}\, J_{\nu+1/2}(n)$
$\left.\begin{matrix} \left(a^2-x^2\right)^{-1/2} & \lvert x\rvert < a \\ 0, & \lvert x\rvert > a \end{matrix}\right\}$......	$\pi J_0(an)$		

We have by the definition of Fourier-transform,

$$f\left[\frac{d^m F}{dx^m}\right] = \int_{-\infty}^{\infty} \frac{d^m F}{dx^m} e^{inx}\,dx = f^m(n) \text{ (say)} \qquad ...(19)$$

So that $f^m(n) = \int_{-\infty}^{\infty} \frac{d^m F}{dx^m} e^{inx}\,dx = \left[\frac{d^{m-1} F}{dx^{m-1}} e^{inx}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (in) e^{inx} \frac{d^{m-1} F}{dx^{m-1}}\,dx$

(on integrating by parts)

$$= -in \int_{-\infty}^{\infty} \frac{d^{m-1} F}{dx^{m-1}} d^{inx}\,dx,$$

under the assumption $\frac{d^{m-1} F}{dx^{m-1}} \to 0$ as $|x| \to \infty$

$$= -in\; f^{m-1}(n) \text{ by (19)} \qquad ...(20)$$

Repeating the same process under the assumption $\frac{d^r F}{dx^r} \to 0$

as $|x| \to \infty$, $r = 1, 2, 3, \ldots (m-1)$

we get after $(m-1)$ operations, $f^m(n) = (-in)^m f(n)$ ...(21)

which follows that the Fourier transform of $\frac{d^m F}{dx^m}$ *is* $(-in)^m$ *times the Fourier transform of* $F(x)$ *subject to the condition that* $\frac{d^r F}{dx^r} \to 0$ *when* $|x| \to \infty$, *for* $r = 1, 2, 3, \ldots (m-1)$.

By similar procedure we can find a relation between the sine and cosine Fourier transforms of the derivatives of a function, such as

$$f_c^m(n) = \int_0^{\infty} \frac{d^m F}{dx^m} \cos nx\,dx = \left[\frac{d^{m-1} F}{dx^{m-1}} \cos nx\right]_0^{\infty} + n \int_0^{\infty} \frac{d^{m-1} F}{dx^{m-1}} \sin nx\,dn$$

integrating by parts

$$= -\alpha_{n-1} + f_s^{m-1}(n) \qquad ...(22)$$

Under the assumptions,

$$\frac{d^{m-1} F}{dx^{m-1}} \to 0 \text{ as } x \to \infty \text{ and } \frac{d^{m-1} F}{dx^{m-1}} \to \alpha_{n-1} \text{ as } x \to 0.$$

Similarly, integrating, $f_s^m(n) = \int_0^{\infty} \frac{d^m F}{dx^m} \sin nx\,dx$

$$= -n f_c^{m-1}(n) \qquad ...(23)$$

(22) and (23) yield,

$$f_c^m(n) = -\alpha_{n-1} - n^2 f_c^{m-2}(n) \qquad ...(24)$$

Repeating the procedure $f_c^m(n)$ may be expressed as the sum of α's and either $f_c(n)$ or $f_c'(n)$ or $f_c'(n)$ or $f_c'(n)$. $f_c(n)$ will occur when x is odd and in that case we can write $\alpha_0 + n f_s(n)$ in place of $f_c'(n)$. We thus have

$$f_c^{2m}(n) = -\sum_{r=0}^{m-1} (-1)^r \alpha_{2m-2r-1}\, n^{2r} + (-1)^m n^{2m} f_c(n) \qquad ...(25)$$

and $$f_c^{2m+1}(n) = -\sum_{r=0}^{m}(-1)^r \alpha_{2m-2r} n^{2r} + (-1)^m n^{2m+1} f_s(n) \qquad \ldots(26)$$

Similar procedure with the help of (22) and (23), will yield

$$f_s^m(n) = n\alpha_{n-2} - n^2 f_s^{m-2}(n) \qquad \ldots(27)$$

$$f_s^{2m}(n) = -\sum_{r=1}^{m}(-1)^r n^{2r-1}\alpha_{2n-2r} + (-1)^{m+1} n^{2m} f_s(n) \qquad \ldots(28)$$

and $$f_s^{2m+1}(n) = -\sum_{r=1}^{m}(-1)^r n^{2r-1}\alpha_{2n-2r+1} + (-1)^{m+1} n^{2m+1} f_s(n) \qquad \ldots(29)$$

Note 1. *The following results are easily deducible*

(i) $\int_0^\infty \frac{d^2 F}{dx^2} \cos nx\, dx = -n^2 f_c(n)$

(ii) $\int_0^\infty \frac{d^4 F}{dx^4} \cos nx\, dx = n^4 f_c(n)$

} *when* $x = 0,\ \frac{dF}{dx} = \frac{d^3F}{dx^3} = 0$ ...(30)

(iii) $\int_0^\infty \frac{d^2 F}{dx^2} \sin nx\, dx = n^2 f_s(n)$

(iv) $\int_0^\infty \frac{d^4 F}{dx^4} \sin nx\, dx = n^4 f_c(n)$

} *when* $x = 0,\ F = \frac{d^2F}{dx^2} = 0$ ...(31)

(v) $$\int_0^\infty \frac{\partial F}{\partial t} \sin nx\, dx = \frac{\partial}{\partial t}\int_0^\infty F \sin nx\, dx = \frac{\partial f}{\partial t} \qquad \ldots(32)$$

Note 2. *In case the transforms are finite, then consider.*

$$\int_0^\pi \frac{\partial F}{\partial x} \sin nx\, dx = [F(x) \sin nx]_0^\pi - n\int_0^\pi F(x) \cos nx\, dx, \text{ integrating by parts}$$

$$= -n f_c(n) \qquad \ldots(33)$$

under the assumption that $F(0)$ and $F(\pi)$ both are finite.

Similarly, $$\int_0^\pi \frac{\partial F}{\partial x} \cos nx\, dx = [F(x) \cos nx]_0^\pi + n\int_0^\pi F(x) \sin nx\, dx$$

$$= (-1)^n F(\pi) - F(0) + n f_s(n) \qquad \ldots(34)$$

Assuming that $F(x) \to 0$ at $x = \pi$ and at $x = 0$, (34) reduces to

$$\int_0^\pi \frac{\partial F}{\partial x} \cos nx\, dx = n f_s(n) \qquad \ldots(35)$$

and (33) reduces to

$$\int_0^\pi \frac{\partial^2 F}{\partial x^2} \sin nx\, dx = -n\int_0^\pi \frac{\partial F}{\partial x} \cos nx\, dx$$

$$= n[(-1)^{n+1} F(\pi) + F(0)] - n^2 f_s(n) \text{ by (34)} \qquad \ldots(36)$$

If $F(0) = F(\pi) = 0$, then (36) yields,

$$\int_0^\pi \frac{\partial^2 F}{\partial x^2} \sin nx\, dx = -n^2 f_s(n) \qquad \ldots(37)$$

Similarly (34) yields

$$\int_0^{\pi} \frac{\partial^2 F}{\partial x^2} \cos nx\, dx = (-1)^n F'(\pi) - F'(0) - n^2 f_c(n) \qquad ...(38)$$

In case $\frac{\partial^2 F}{\partial x^2}$ vanishes at $x = 0$ and at $x = \pi$, it is easy to see that (31) gives

$$\int_0^{\pi} \frac{\partial^4 F}{\partial x^4} \sin nx\, dx = -n^2 \int \frac{\partial^2 F}{\partial x^2} \sin nx\, dx = n^4 f_s(n) \qquad ...(39)$$

and when $\frac{\partial F}{\partial x}, \frac{\partial^3 F}{\partial x^3}$ vanish at $x = 0$ and at $x = \pi$, (38) gives

$$\int_0^{\pi} \frac{\partial^2 F}{\partial x^2} \cos nx\, dx = -n^2 f_c(n) \qquad ...(40)$$

So that $\int_0^{\pi} \frac{\partial^4 F}{\partial x^4} \cos nx\, dx = n^4 f_c(n)$ ...(41)

Problem 28. *Determine the function F such that*

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0,\ 0 < x < \pi,$$

with the boundary condition $F = 0$ *when* $x = 0$ *and* $x = \pi$
$= 0$ *when* $y = 0$
$= F_0$ *(const.) when* $y = \pi$.

F being given to be zero when $x = 0$ and $x = \pi$, we have to use the finite sine transform *i.e* $f(n) = \int_0^{\pi} F(x) \sin nx\, dx$

Applying it to the given differential equation we have

$$\int_0^{\pi} \frac{\partial^2 F}{\partial x^2} \sin nx\, dx + \int_0^{\pi} \frac{\partial^2 F}{\partial y^2} \sin nx\, dx = 0$$

with the condition, $f = 0$ when $y = 0$ and $f = \int_0^{\pi} F_0 \sin nx\, dx$ when $y = \pi$

By (37) we have, $\int_0^{\pi} \frac{\partial^2 F}{\partial x^2} \sin nx\, dx = -n^2 f$

$$\therefore \quad -n^2 f + \frac{\partial^2 f}{\partial y^2} = 0 \text{ where } \frac{\partial^2 f}{\partial y^2} = \int_0^{\pi} \frac{\partial^2 F}{\partial y^2} \sin nx\, dx,$$

or $\frac{\partial^2 f}{\partial y^2} - n^2 f = 0.$

Its general solution is $f = A \sinh ny$

But $f = F_0 \int_0^{\pi} \sin nx\, dx$ when $y = \pi$

$$= F_0 \left[-\frac{\cos nx}{n}\right]_0^{\pi} = 0 \text{ when } n \text{ is even}$$

$$= -2 F_0/n \text{ when } n \text{ is odd.}$$

So that considering the two solutions for f we conclude $f = 0$ when n is even

and $$f = \frac{2F_0}{n} \operatorname{cosech} n\pi \sinh ny \text{ when } n \text{ is odd.}$$

Hence the inversion formula will give on replacing n by $2m+1$,

$$F = \frac{4F_0}{\pi} \sum_{m=0}^{\infty} (2m+1)^{-1} \operatorname{cosech} (2m+1)\pi \sinh (2m+1)y \sin (2m+1)x$$

[F] Multiple Fourier Transforms.

If $F(x, y)$ be a function of two variables x and y, then assuming it to be the function x only its fourier transform $\phi(n, y)$ is given by

$$\phi(n, y) = \int_{-\infty}^{\infty} F(x, y)\, e^{inx}\, dx \qquad \text{...(42)}$$

Now if $f(n, l)$ be the Fourier complex transform of $\phi(n, y)$ which is regarded as function of y only then

$$f(n, l) = \int_{-\infty}^{\infty} \phi(n, y)\, e^{ily}\, dy \qquad \text{...(43)}$$

These two results when combined, give

$$f(n, l) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, e^{i(nx+ly)}\, dx\, dy \qquad \text{...(44)}$$

and the inversion formula is

$$f(x, y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(n, l)\, e^{-i(nx+ly)}\, dn\, dl \qquad \text{...(45)}$$

Similarly in case of three variables x, y, z, we have

$$f(n, l, m) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y, z)\, e^{i(nx+ly+mz)}\, dx\, dy\, dz \qquad \text{...(44A)}$$

and $$f(x, y, z) = \frac{1}{8\pi^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(n, l, m)\, e^{-i(nx+ly+mz)}\, dn\, dl\, dm \qquad \text{...(45A)}$$

Note 1. *The result may be generalized for any number of variables.*

Note 2. *In case the Fourier transforms are finite such that $F(x, y)$ is a function of two independent variables x, y where $0 \le x \le \pi$ and $0 \le y \le \pi$, then the sine transform of $F(x, y)$ is given by*

$$f_s(n, l) = \int_0^{\pi}\int_0^{\pi} F(x, y) \sin nx \sin ly\, dx\, dy \qquad \text{...(46)}$$

and the inversion formula is

$$F(x, y) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\sum_{l=1}^{\infty} f_s(n, l) \sin nx \sin ly \qquad \text{...(47)}$$

[G] Convolution or Faltung Theorem for Fourier Transforms.

If $F(x)$ and $G(x)$ are two functions such that $-\infty < x < \infty$ then their Faltung or Convolution F^*G is defined as

$$H(x) = F * G = \int_{-\infty}^{\infty} F(n)\, G(x-n)\, dn \qquad \text{...(48)}$$

It is worth noting that the Fourier Transform of the Convolution of $F(x)$ and $G(x)$ is the product of their Fourier transforms *i.e.*

$$f[F^*G] = f[F]\, f[G] \qquad \text{...(49)}$$

Since $f[F^*G] = \int_{-\infty}^{\infty} H(x)\, e^{-inx}\, dx$ by definition

$$= \int_{-\infty}^{\infty} F(x)\, e^{-inx}\, dx \int_{-\infty}^{\infty} G(x)\, e^{-inx}\, dx$$

$$= f[F].f[G].$$

[H] Evaluation of Integrals with the help of Fourier Inversion Theorem.

Let $I_1 = \int_0^{\infty} e^{-ax} \cos nx\, dx$ and $I_2 = \int_0^{\infty} e^{-ax} \sin nx\, dx$.

Integrating by parts, we have

$$I_1 = \left[-\frac{1}{a} e^{-ax} \cos nx \right]_0^{\infty} - \frac{n}{a} \int_0^{\infty} e^{-ax} \sin nx\, dx \Bigg] = \frac{1}{a} - \frac{n}{a} I_2.$$

Similarly $I_2 = \frac{n}{a} I_1$

These give on solving $I_1 = \frac{a}{a^2 + n^2}$ and $I_2 = \frac{n}{a^2 + n^2}$

Thus taking $F(x) = e^{-ax}$, its sine and cosine Fourier transforms are $\frac{a}{a^2 + n^2}$ and $\frac{n}{a^2 + n^2}$ respectively, so that the inversion formula gives

$$e^{-ax} = \frac{2}{\pi} \int_0^{\infty} \frac{a}{a^2 + n^2} \cos nx\, dn \qquad ...(50)$$

and $$e^{-ax} = \frac{2}{\pi} \int_0^{\infty} \frac{a}{a^2 + n^2} \sin nx\, dn \qquad ...(51)$$

i.e. $$\int_0^{\infty} \frac{\cos nx}{a^2 + n^2}\, dn = \frac{\pi}{2a} e^{-ax} \text{ and } \int_0^{\infty} \frac{n \sin nx}{a^2 + n^2}\, dn = \frac{\pi}{2} e^{-ax} \qquad ....(52)$$

9.11. APPLICATIONS OF FOURIER TRANSFORMS TO BOUNDARY VALUE PROBLEMS

Problem 29. *Find the finite Fourier sine and cosine transform of* $\frac{\partial^2 U}{\partial x^2}$ *where* U *is a function of* x *and* t *for* $0 < x < l,\ t > 0$.

We have $f_s \left\{ \frac{\partial U}{\partial x} \right\} = \int_0^l \frac{\partial U}{\partial x} \sin \frac{n\pi x}{l}\, dx$, by definition

$$= \left[U(x, t) \sin \frac{n\pi x}{l} \right]_0^l - \frac{n\pi}{l} \int_0^l U(x, t) \cos \frac{n\pi x}{l}\, dx,$$

integrating by parts

$$= 0 - \frac{n\pi}{l} f_c(U) = -\frac{n\pi}{l} f_c\{U\} \qquad ...(1)$$

Also $f_c\left\{\frac{\partial U}{\partial x}\right\}=\int_0^l \frac{\partial U}{\partial x}\cos\frac{n\pi x}{l}\,dx=\left[U(x,t)\cos\frac{n\pi x}{l}\right]_0^l$

$$+\frac{n\pi}{l}\int_0^l U(x,t)\sin\frac{n\pi x}{l}\,dx$$

$$=U(l,t)\cos n\pi-U(0,t)+\frac{n\pi}{l}f_s\{U\} \qquad ...(2)$$

Replacing U by $\frac{\partial U}{\partial x}$ in (1), we get

$$f_s\left\{\frac{\partial^2 U}{\partial x^2}\right\}=-\frac{n\pi}{l}f_c\left\{\frac{\partial U}{\partial x}\right\}$$

$$=-\frac{n\pi}{l}\left[U(l,t)\cos n\pi-U(0,t)+\frac{n\pi}{l}f_s\{U\}\right] \text{ by (2)}$$

$$=-\frac{n\pi}{l}U(l,t)\cos n\pi+\frac{n\pi}{l}U(0,t)-n^2\frac{\pi^2}{l^2}f_s\{U\}$$

Again replacing U by $\frac{\partial U}{\partial x}$ in (2) we find

$$f_c\left\{\frac{\partial^2 U}{\partial x^2}\right\}=\frac{\partial U(l,t)}{\partial x}\cos n\pi-\frac{\partial U(0,t)}{\partial x}+\frac{n\pi}{l}f_s\left\{\frac{\partial U}{\partial x}\right\}$$

$$=\frac{\partial U(l,t)}{\partial x}\cos n\pi-\frac{\partial U(0,t)}{\partial x}-\frac{n^2\pi^2}{l^2}f_c\{U\} \text{ by (1)}$$

Problem 30. *Solve* $\frac{\partial U}{\partial t}=\frac{\partial^2 U}{\partial x^2}$, $x>0$, $t>0$ *subject to the conditions*

$$U(0,t)=0,\ U(x,0)=\begin{cases}1, & 0<x<1\\ 0, & x\ge 1\end{cases} \quad \textit{and } U(x,t) \textit{ is bounded.}$$

Taking the Fourier sine transform of both sides of the given equation, we get

$$\int_0^\infty \frac{\partial U}{\partial t}\sin nx\,dx=\int_0^\infty \frac{\partial^2 U}{\partial x^2}\sin nx\,dx \qquad ...(1)$$

Denoting $\int_0^\infty U(x,t)\sin nx\,dx$ by $u=u(n,t)$ we have

$$\frac{du}{dt}=\int_0^\infty \frac{\partial U(x,t)}{\partial t}\sin nx\,dx=\int_0^\infty \frac{\partial^2 U}{\partial x^2}\sin nx\,dx \text{ by (1)}$$

$$=\left[\sin nx\frac{\partial U}{\partial x}\right]_0^\infty+n\int_0^\infty \cos nx\frac{\partial U}{\partial x}\,dx, \text{ on integrating by parts}$$

$$=0-n\left[\ |\cos nx\,U|_0^\infty+n\int_0^\infty \sin nx\cdot U\,dx\right], \text{ under the assumption,}$$

$$\frac{\partial U}{\partial x}\to 0 \text{ as } x\to\infty$$

$$=n\,U(0,t)-n^2u, \text{ under the assumption } U\to 0 \text{ as } x\to\infty \qquad ...(2)$$

Now $u(n, t) = \int_0^\infty U(x, t) \sin nx \, dx$ gives

$$u(n, 0) = \int_0^\infty U(x, 0) \sin nx \, dx = \int_0^1 1 . \sin nx \, dx \text{ by the condition for } U(x, 0)$$

$$= \left[-\frac{\cos nx}{n}\right]_0^1 = \frac{1-\cos n}{n} \quad \ldots(3)$$

Using the condition $U(0, t) = 0$, (2) yields $\frac{du}{dt} = -n^2 u$ or $\frac{du}{u} = -n^2 \, dt$.

Integrating $\log u = -n^2 t + \log A$, A being constant of integration

$\therefore \; u = Ae^{-n^2 t}$.

Initially when $t = 0$, $u(n, t) = u(n, 0) = \frac{1-\cos n}{n}$ by (3); so that

$$A = \frac{1-\cos n}{n}$$

Hence $u(n, t) = \frac{1-\cos n}{n} e^{-n^2 t}$

Applying the inversion formula for Fourier sine transform, we have

$$U(x, t) = \frac{2}{\pi} \int_0^\infty u(n, t) \sin nx \, dx = \frac{2}{\pi} \int_0^\infty \frac{1-\cos n}{n} e^{-n^2 t} \sin nx \, dx$$

which gives the required solution, physically interpreted as the temperature at any point x at any time t in a solid $x > 0$.

Problem 31. *Use finite Fourier transforms to solve* $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$,

$U(0, t) = 0, U(\pi, t) = 0, U(x, 0) = 2x$ *where* $0 < x < \pi, t > 0$.

Give physical interpretation of the problem.

Taking the finite Fourier sine transform of both sides of the given differential equation, we have

$$\int_0^\pi \frac{\partial U}{\partial t} \sin nx \, dx = \int_0^\pi \frac{\partial^2 U}{\partial x^2} \sin nx \, dx \quad \ldots(1)$$

Denoting by $u = u(n, t) = \int_0^\pi U(x, t) \sin nx \, dx$, ...(2)

we find $\int \frac{du}{dt} = \int_0^\pi \frac{\partial U}{\partial t} \sin nx \, dx = \int_0^\pi \frac{\partial^2 U}{\partial x^2} \sin nx \, dx$ by (1)

$$= \left[\sin nx \frac{\partial U}{\partial x}\right]_0^\pi - n \int_0^\pi \cos nx \frac{\partial U}{\partial x} \, dx, \text{ integrating by parts}$$

$$= 0 - n \int_0^\pi \cos nx \frac{\partial U}{\partial x} \, dx \qquad \because U(0, t) = U(\pi, t) = 0$$

$$= -n \left[\cos nx \cdot U(x, t)\right]_0^\pi - n^2 \int_0^\pi \sin nx \, U(x, t) \, dx$$

$= -n^2 u$, since the first integral vanishes for $U(0, t) = U(\pi, t) = 0$.

Its solution is, $u = Ae^{-n^2 t}$, A being constant of integration.

Now from (2), $u(n, 0) = \int_0^{\pi} U(x, 0) \sin nx \, dx = \int_0^{\pi} 2x \sin nx \, dx$

$\because U(x, 0) = 2x$

$$= 2\left[-x\frac{\cos nx}{n}\right]_0^{\pi} + \frac{2}{n}\int_0^{\pi} \cos nx \, dx,$$

$$= -\frac{2\pi}{n}\cos n\pi + \frac{2}{n^2}[\sin nx]_0^{\pi} = -\frac{2\pi}{n}\cos n\pi$$

When $t = 0$, $u = -\frac{2\pi}{n}\cos n\pi$ applied to $u = Ae^{-n^2 t}$ gives

$$A = -\frac{2\pi}{n}\cos n\pi$$

$$\therefore \quad u = -\frac{2\pi}{n}\cos n\pi \, e^{-n^2 t}.$$

Applying the inversion formula for finite Fourier sine transform, we have

$$U(x, t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(-\frac{2\pi}{n}\cos n\pi \, e^{-n^2 t}\right)\sin nx.$$

For physical interpretation, $U(x, t)$ may be regarded as the temperature at any point x at an instant of time t in a solid bounded by the planes $x = 0$ and $x = \pi$. The boundary conditions $U(0, t) = 0$ and $U(\pi, t) = 0$ give the zero temperature at the ends while $U(x, 0) = 2x$ represents that the initial temperature is a function of x.

Problem 32. *Determine the displacements $Y(x, t)$ in a horizontal string stretched from the origin to the point $(\pi, 0)$ when the motion is due to the weight of the string alone. The string may be taken to be initially at rest in the position $Y = 0$.*

Taking the axes of reference as shown in Fig 9.8, and the string being released from the rest in position $Y = 0$, the boundary value problem is

$$\frac{\partial^2 Y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} + g \qquad \ldots(1)$$

where $0 < x < \pi$, $t > 0$ and

$a^2 = \frac{\text{tension}}{\text{linear density}}$ with the boundary conditions.

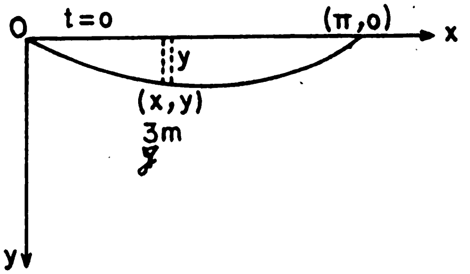


Fig. 9.8

$$Y = 0 = \frac{\partial Y}{\partial t} \text{ when } t = 0 \text{ and } Y = 0 \text{ when } x = 0 \text{ or } \pi. \qquad \ldots(2)$$

Taking finite Fourier sine transform of both sides of (1), we get

$$\int_0^{\pi} \frac{\partial^2 Y}{\partial t^2}\sin nx \, dx = a^2\int_0^{\pi}\frac{\partial^2 Y}{\partial x^2}\sin nx \, dx + g\int_0^{\pi}\sin nx \, dx \qquad \ldots(3)$$

Suppose that $y(n, t) = \int_0^{\pi} Y(x, t)\sin nx \, dx$, then

$$\frac{d^2 y}{dt^2} = -a^2 n^2 y + g\left[\frac{1 + (-1)^{n+1}}{n}\right] \text{ by (37) of §9.10 [E]}$$

$$= -a^2n^2y + g\left[\frac{1-(-1)^n}{n}\right] \qquad \ldots(4)$$

With transformed conditions (2) as $y = \dfrac{dy}{dt} = 0$ when $t = 0$. (4) may be written as

$$\left(D^2 + a^2n^2\right)y = \frac{g}{n}\left[1-(-1)^n\right] \text{ where } D \equiv \frac{d}{dt}.$$

Its complementary function is $A \cos nat + B \sin nat$

and particular integral is $\dfrac{g}{n}\left[1-(-1)^n\right]\dfrac{1}{a^2n^2}\left[1+\left(\dfrac{D}{an}\right)^2\right].1$ *i.e.* $\dfrac{g}{a^2n^3}\left[1-(-1)^n\right]$

So the complete integral of (4) is

$$y = A \cos nat + B \sin nat + \frac{g}{a^2n^3}\left[1-(-1)^n\right]$$

$$\therefore \quad \frac{dy}{dt} = -Ana \sin nat + Bna \cos nat$$

Applying the conditions, $y = 0 = \dfrac{dy}{dt}$ when $t = 0$, we get

$$A = -\frac{g}{a^2n^3}\left[1-(-1)^n\right] \text{ and } B = 0$$

Hence the solution of (4) is

$$Y = \frac{g}{a^2n^3}\left[1-(-1)^n\right](1-\cos ant)$$

Using the inversion formula, we find

$$y = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{g}{a^2n^3}\left[1-(-1)^n\right](1-\cos nat)\sin nx$$

$$= \frac{2g}{\pi a^2}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^3}(1-\cos nat)\sin nx$$

$$= \frac{2g}{\pi a^2}\sum_{n=1}^{\infty}\frac{x(\pi-x)}{2}(1-\cos nat)\sin nx, \text{ if we apply}$$

$$f^{-1}\left\{\frac{1-(-1)^n}{n^2}\right\} = \frac{x(\pi-x)}{2}$$

Problem 33. *Let $V(x, y)$ denote the electrostatic potential in a region bound[illegible] the planes $x = 0, x = \pi$ and $y = 0$ in which there is a uniform distribution of spa[illegible] of density $\dfrac{h}{4\pi}$. If the planes $x = 0$ and $y = 0$ are kept at potential zero, the [illegible] another fixed potential $V = 1$ is finite as $y \to \infty$, then determine V.*

We know that the electrostatic potential function $V(x, y)$ satisfies Poisson's equation which is

$$\nabla^2 V = -4\pi\rho \text{ or } \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}$$

$$= -4\pi\rho \text{ in two dimensions.}$$

Here $\rho = \frac{h}{4\pi}$, the volume density of charge.

$$\therefore \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = -h \text{ where}$$

$0 < x < \pi, y > 0$ ...(1)

with the boundary conditions

$V(0, y) = 0$, $V(\pi, y) = 1$ for $y > 0$

and $V(x, 0) = 0$ for $0 < x < \pi$

also $V(x, y)$ is fnite for $y > 0$, $0 \le x \le \pi$.

...(2)

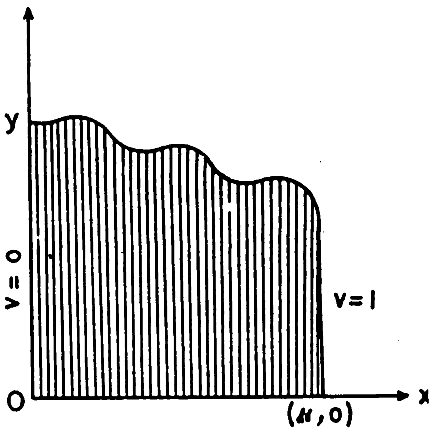


Fig. 9.9

Taking the finite Fourier sine transform of both sides of (1), we get

$$\int_0^\pi \frac{\partial^2 V}{\partial x^2} \sin nx\, dx + \int_0^\pi \frac{\partial^2 V}{\partial x^2} \sin nx\, dx = -h \int_0^\pi \sin nx\, dx \qquad ...(3)$$

Assuming that $v = \int_0^\pi V \sin nx\, dx$, (3) gives

$$-n^2 v + \frac{d^2 v}{dy^2} = n(-1)^n - hf_s\{1\} \qquad ...(4)$$

since $V = 0$ at $x = 0$ and $V = 1$ at $x = \pi$ so that $v = 0$ when $y = 0$ and x is finite.

The complete integral of (4) is

$$v = Ae^{-ny} + Be^{ny} + \frac{n(-1)^n - hf_s\{1\}}{-n^2}$$

But v is finite for $y \to \infty$, $\therefore B = 0$

Also when $y = 0$ $v = 0$ $\therefore$ $A = \frac{n(-1)^n - hf_s\{1\}}{n^2}$

Hence the solution of (4) is

$$v = \frac{hf_s\{1\} - n(-1)^n}{n^2}(1 - e^{ny})$$

Using the inversion formula, we find

$$V = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{hf_s\{1\} - n(-1)^n}{n^2} \sin nx.$$

ADDITIONAL MISCELLENEOUS PROBLEMS

Problem 34. *Expand the function $f(t) = kt$ in the interval $-T/2 < t < T/2$ in a Complex Fourier series. Plot the function defined by the series outside this range.*

Problem 35. *Show that if $f(t) = \phi(-t)$ for $-T/2 < t < 0$ and $f(t) = \phi(t)$ for $0 < t < T/2$ then the real Fourier series for $f(t)$ contains no sine terms.*

Problem 36. *Show that if $f(t) = -\phi(-t)$ for $-T/2 < t < 0$ and $f(t) = \phi(t)$ for $0 < t < T/2$ then the real Fourier series for $f(t)$ contains no cosines terms.*

Problem 37. *Expand the function of period 12 defined as follows in the interval $-6 < t < 6$,*

$$
\begin{aligned}
(t) &= 0 \text{ for } -6 \le t \le -3 \\
&= t+3 \text{ for } -3 \le t \le 0 \\
&= 3-t \text{ for } 0 < t \le 3 \\
&= 0 \text{ for } 3 < t \le 6
\end{aligned}
$$

Plot the function.

Problem 38. *Using Fourier cosine integral, show that for $x > 0$,*

$$e^{-bx} = \frac{2b}{\pi} \int_0^\infty \frac{\cos ux}{u^2+b^2}\, dx.$$ (*Meerut, 1979*)

Consider $f(x) = e^{-bx}$, $x > 0$ s.t.

$f(-x) = f(x)$ for b > 0 *i.e.* as even function of x.

By Fourier's integral

$$f(x) = \frac{2}{\pi}\int_0^\infty f(t)\, dt \int_0^\infty \cos ut \cos ux\, dx$$

$$= \frac{2}{\pi}\int_0^\infty \left(e^{-bt}\cos ut\, dt\right) \cos ux\, dx \qquad ...(1)$$

where $$\int_0^\infty e^{-bt}\cos ut\, dt = \left[\frac{e^{-bt}}{b^2+u^2}(u \sin ut - b\cos ut)\right]_0^\infty$$

$$= \frac{b}{b^2+u^2} \qquad ...(2)$$

$\therefore$ $$(1) \Rightarrow f(x) = e^{-bx} = \frac{2}{\pi}\int_0^\infty \frac{b}{b^2+u^2} \cos ux\, dx$$

$$\Rightarrow e^{-bx} = \frac{2b}{\pi}\int_0^\infty \frac{\cos ux}{b^2+u^2}.$$

Problem 39. (*a*) *State the Dirichlet conditions for the expansion of a function $f(x)$ in a Fourier series. Obtain the Fourier expansion of the function*

$$f(x) = \begin{cases} t, & 0 < t < 2 \\ 0, & 2 < t < 4 \end{cases}$$

(*b*) *State the Fourier Integral theorem. Show that if*

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-ixt}\, dt, \text{ then } f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)\, e^{ixt}\, dt$$

Obtain $f(t)$ for the function $g(x)$ defined by

$$
\begin{aligned}
g(x) &= 0, x < 0 \\
&= 1, x > 0
\end{aligned}
$$ (*Bombay, 1965*)

Problem 40. *Expand the periodic function*

$$f(x) = \begin{cases} -1 \text{ for } -\pi < x < 0 \\ 1 \text{ for } 0 < x < \pi \end{cases}$$

in Fourier series, and show that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = \frac{\pi}{4}$ (*Rohilkhand, 1985; Agra, 1965*)

Problem 41. *Obtain a Fourier series expansion of the periodic function $f(t)$, the period of which is T and the form of which within the first period i.e., for $0 \le t \le T$ is given by $f(t) = t(T - t)$*

Obtain the form of the Fourier coefficients for large n to order $\left(\frac{1}{n^2}\right)$. (Agra, 1967)

Problem 42. *Obtain the Fourier series expansions of the following periodic functions, each with period T and having the following forms during the first period i.e. $0 \le t \le T$;*

$$(i)\ f(t) = \sin \frac{\pi t}{T}$$

$$(ii)\ f(t) = \left(\frac{t^2}{T^2}\right)$$

which one has the more rapidly convergent expansion. (Agra, 1968)

Problem 43. *Discuss briefly the role of Fourier's series in mathematical Physics.* (Vikram, 1969)

Problem 44. (*a*) *State the conditions under which a function can be expanded in the form of a Fourier series.*

(*b*) *Express the following function in a Fourier series*

$$F(x) = x + x^2;\ -\pi < x < \pi.$$ (Agra, 1969)

Problem 45. (*a*) *State the Dirichlet conditions for the Fourier expansion of a periodic function. From the Fourier series obtain the Fourier Integral formula*

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(\xi)\cos[\xi - x)\, d\xi\, dx \quad (-\infty < x < \infty)$$

(*b*) *Let $f(\xi) = 1$ for $0 < \xi < 1$ and $f(\xi) = 0$ for $\xi > 1$*

Use the above formula to show that

$$\int_0^{\infty} \frac{\sin x \cos \alpha\xi}{\alpha}\, d\alpha = \frac{\pi}{4} \text{ when } \xi = 1.$$ (Bombay, 1970)

Problem 46. (*a*) *Find the Fourier-cosine-series for the function $f(x) = x$ in the range $0 < x < \pi$.*

(*b*) *Using the Fourier-sine-integral evaluate $\int_0^{\infty} \frac{\sin x}{x}\, dx$.* (Agra, 1971)

Problem 47. *If V is the Temperature at time t and k the diffusivity of the material, find V from the partial differential equation $\frac{\partial V}{\partial t} = k\frac{\partial^2 V}{\partial x^2}$, $x > 1$, $t > 0$*

With the boundary conditions $V = V_0$ for $x = 0$, $t > 0$

$V = 0$ for $t = 0$, $x > 0$

$$\left[\text{Ans. } V = V_0\left[1 - \frac{2}{\pi}\int_0^{\infty} e^{-n^2 kt} \sin nx \frac{dn}{n}\right]\right]$$

Problem 48. *Determine the solution of the equation $\frac{\partial^4 V}{\partial x^4} + \frac{\partial^2 V}{\partial y^2} = 0$, $-\infty < x < \infty$, $y \ge 0$ satisfying the conditions*

(*i*) *V and its partial derivatives tend to zero as $x \to \pm\infty$*

(*ii*) $V = F(x)$, $\frac{\partial V}{\partial y} = 0$ when $y = 0$

$$\left[\text{Ans. } V(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\cos(n^2 y)\, e^{-inx}\, dn\right]$$

Problem 49. *Solve the wave equation* $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$ *for a stretched string fixed by rigid supports at the ends* $x = 0$ *and* $x = l$ *if at* $t = 0$, $\frac{\partial y}{\partial t} = 0$ *and* $y\ (x, t) = y_0(x)$.

(*Agra, 1974*)

Problem 50. *Find the Fourier transform of the Gaussian Probability function:*

$f(x) = Ne^{-\alpha x^2}$ (*N*, α *are constants*). (*Agra, 1974*)

Fourier transform of the function is

$$\bar{f}(n) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-inx}\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} Ne^{-\alpha x^2}\cdot e^{-inx}\, dx$$

$$= \frac{N}{2\pi}\int_{-\infty}^{\infty} e^{-(\alpha x^2+inx^2)}\, dx = \frac{N}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha\left(x^2+\frac{inx}{\alpha}\right)}\, dx$$

$$= \frac{N}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha\left\{x^2+\frac{in}{\alpha}x+\left(\frac{in}{2\alpha}\right)^2\right\}}\, e^{\alpha\left(\frac{in}{2\alpha}\right)^2}\, dx$$

$$= \frac{N}{2\pi}\, e^{-\frac{n^2}{4\alpha}}\int_{-\infty}^{\infty} e^{-\alpha\left(x+\frac{in}{2\alpha}\right)^2}\, dx, \text{ Put } x + \frac{in}{2\alpha} = y,$$

$$= \frac{N}{2\pi}\, e^{-\frac{n^2}{4\alpha}}\int_{-\infty}^{\infty} e^{-\alpha y^2}\, dy = \frac{N}{2\pi}\, e^{-n^2/4\alpha}\cdot\frac{\sqrt{\pi}}{\sqrt{\alpha}}$$

$$= \frac{N}{2\sqrt{\pi\alpha}}\, e^{-n^2/4\alpha}$$

Problem 51. *Find the Fourier transform of* (*i*) 7, (*ii*) $e^{-x/a}$, (*iii*) $\frac{1}{x^2+a^2}$

(*Rohilkhand, 1980*)

Ans.

Problem 52. *Find the Complex Fourier transform of*

(*i*) $e^{-|t|}$, (*ii*) $\frac{e^{-r/a}}{r}$, (*iii*) e^{-r^2/a^2}, *where 'a' is a constant and* $r = \left(x^2+y^2+z^2\right)^{1/2}$

(*Rohilkhand, 1982, 86*)

Hint to (*i*). See Prob. 26.

Hint to (*iii*). Given $f(x, y, z) = e^{-r^2/a^2} = e^{-(x^2+y^2+z^2)/a^2}$

Its Fourier Transform by (44A) of §9.10 is given by

$$f(n, l, m) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2+z^2)/a^2}\, e^{i(nx+ly+mz)}\, dx\, dy\, dz$$

$$= \int_{-\infty}^{\infty} e^{-x^2/a^2} e^{inx} dx \int_{-\infty}^{\infty} e^{-y^2/a^2} e^{ily} dy \int_{-\infty}^{\infty} e^{-z^2/a^2} e^{imz} dz \qquad \ldots(1)$$

Here $\int_{-\infty}^{\infty} e^{-x^2/a^2} e^{inx} dx = \int_{-\infty}^{\infty} e^{-(x^2-ina^2x)/a^2} dx$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{a^2}\left\{x^2-ina^2x+\left(\frac{ia^2n}{2}\right)^2\right\}}\, e^{\frac{1}{a^2}\left(\frac{ia^2n}{2}\right)^2}\, dx$$

$$= e^{-\frac{a^2n^2}{4}} \int_{-\infty}^{\infty} e^{-\left(x-\frac{ia^2n}{2}\right)^2 / a^2} dx, \text{ Put } x-\frac{ia^2n}{2} = p, \quad dx = dp$$

$$= e^{-\frac{a^2n^2}{4}} \int_{-\infty}^{\infty} e^{-p^2/a^2} dp$$

$$= e^{-\frac{a^2n^2}{4}} \sqrt{\left(\frac{\pi}{1/a^2}\right)} \text{ as } \int_{-\infty}^{\infty} e^{-\alpha p^2} dp = \sqrt{\frac{\pi}{\alpha}}$$

$$= e^{-\frac{a^2n^2}{4}} a\sqrt{\pi}$$

Similarly

$$\int_{-\infty}^{\infty} e^{-y^2/a^2} e^{ily} dy = e^{-a^2l^2/4} \cdot a\sqrt{\pi} \ \& \ \int_{-\infty}^{\infty} e^{-z^2/a^2} e^{imz} dz = e^{-a^2m^2/4} \cdot a\sqrt{\pi}$$

Hence (*i*) gives $f(n, l, m) = e^{-a^2n^2/4} \cdot a\sqrt{\pi} \cdot e^{-a^2l^2/4} a\sqrt{\pi} \cdot e^{-a^2m^2/4} a\sqrt{\pi}$

$$= a^3 (\pi)^{3/2} e^{-(n^2+l^2+m^2)a^2/4}$$

Problem 53. *If $F_1(k)$ and $F_2(k)$ are Fourier transforms of $f_1(x)$ and $f_2(x)$ respectively, show that the Fourier Transform of $f_1(x) f_2(x)$ is given by*

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_1(k') F_2(k-k') dk'$$

(Rohilkhand, 1983)

Problem 54. *Find the Fourier Transform of*

$$f(x) = \begin{cases} 1/\epsilon, & |x| \le \epsilon \\ 0, & |x| > \epsilon \end{cases}$$

Determine the limit of this transform as $\epsilon \to 0$ and discuss the result;

(Rohilkhand, 1983)

Hint: We have $\bar{f}(n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-inx} dx = \frac{1}{2\pi} \int_{-\epsilon}^{\epsilon} \frac{1}{\epsilon} e^{-inx} dx$

$$= \frac{1}{2\pi} \frac{1}{\epsilon} \left[\frac{e^{-inx}}{-in}\right]_{-\epsilon}^{\epsilon} = \frac{1}{2\pi} \cdot \frac{1}{\epsilon} \cdot \frac{e^{in\epsilon} - e^{-in\epsilon}}{in}$$

$$= \frac{1}{2\pi} \cdot \frac{2}{n\epsilon} n\epsilon = \frac{1}{\pi} \frac{\sin n\epsilon}{n\epsilon}$$

which is the required Fourier transform.

Now, $\lim_{\epsilon \to 0} \bar{f}(n) = \lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\sin n\epsilon}{n\epsilon} = \frac{1}{\pi}$, whereas the function $\to \infty$ as $n \to 0$.

Problem 55. *What is meant by Fourier trnasform?* *(Rohilkhand, 1986)*

Problem 56. *Apply Fourier series solution method to solve the wave equation $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2}$ for a stretched string fastened to fixed supports at its two ends and initially plucked at its mid-point, giving it initial displacement h. If initial velocity be zero at all points of the string, prove that the displacement δ at a point distant α from a fixed end is given by*

$$\delta = \frac{2h}{\pi^2} \left[\sin\frac{\pi\alpha}{l} \cos\frac{\pi ct}{l} - \frac{1}{9} \sin\frac{3\pi\alpha}{l} \cos\frac{3\pi ct}{l} + \ldots\right]$$

where *l denotes the length of the string.* *(Rohilkhand, 1989)*

CHAPTER 10

THE LAPLACE TRANSFORMS

10.1. INTRODUCTION

The Laplace transform which is a part of the new-growing topic known as operational calculus is easily and effectively applicable to the boundary value problems of differential equations arising in physics, mathematics and engineering. The subject was mainly originated in the work of Heaviside who found it useful to solve the equation of electromagnetic theory in the end of nineteenth century.

10.2. DEFINITION OF THE INTEGRAL TRANSFORM

All such transforms as Laplace transform, Fourier-transform and Hankel transform are included in the term Integral transform and we define it as follows:

If there is a known function $K(\alpha, x)$ of two variables α and x such that the integral

$$\int_0^\infty K(\alpha, x)\, F(x)\, dx \qquad \ldots(1)$$

is convergent, then the integral (1) is termed as the Integral transform of the function $F(x)$ and is denoted by $\bar{F}(x)$ or $T\{F(x)\}$ i.e.

$$\bar{F}(x) = T\{F(x)\} = \int_0^\infty K(\alpha, x)\, F(x)\, dx \qquad \ldots(2)$$

The function $K(\alpha, x)$ introduced here is sometimes known as the *Kernel* of the transformation and α is a parameter (real or complex) independent of x.

10.3. DEDUCTION OF THE DEFINITION OF THE LAPLACE TRANSFORM FROM THAT OF THE INTEGRAL TRANSFORM

In §10.2 we have defined the Integral transform of $F(x)$ as

$$T\{F(x) = \int_0^\infty K(\alpha, x)\, F(x)\, dx \qquad \ldots(1)$$

where $K(\alpha, x)$ is the Kernel of the transformation.

If we take the Kernel,

$$\left.\begin{aligned} K(\alpha, x) = K(s, t) &= 0 \text{ for } t < 0 \\ &= e^{-st} \text{ for } t \geq 0 \end{aligned}\right\} \qquad \ldots(2)$$

then the transform

$$T\{F(x)\} = \int_0^\infty e^{-st} F(t)\, dt \text{ for } t \geq 0 \qquad \ldots(3)$$

is known as the *Laplace transform*.

10.4. DEFINITION OF THE LAPLACE TRANSFORM

If $F(t)$ be a function of t defined for all positive values of t (*i.e.* $t \geq 0$), then the Laplace transform of $F(t)$ denoted by $L\{F(t)\}$ or $\bar{F}(s)$ or $f(s)$ is defined by the expression

$$L\{F(t)\} = \bar{F}(t) = f(s) = \int_0^\infty e^{-st} F(t)\, dt \qquad \ldots(1)$$

where s is a parameter (real or complex).

If the integral $\int_0^\infty e^{-st} F(t) dt$ converges for some value of s, then the Laplace transform of $F(t)$ is said to exist, otherwise it does not exist.

Problem 1. *Find the Laplace transform of the following functions:*

(*i*) $F(t) = 1$

(*ii*) $F(t) = t$

(*iii*) $F(t) = t^n, n = 0, 1, 2, 3, \ldots\ldots\ldots$ (*Meerut, 1986*)

By definition of Laplace transform, we have

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt \qquad \ldots(1)$$

(*i*) when $F(t) = 1$, (1) becomes

$$L\{1\} = \int_0^\infty e^{-st}\, 1\, dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s},\ s > 0$$

(*ii*) when $F(t) = t$, (1) gives

$$L\{t\} = \int_0^\infty e^{-st} \cdot t\, dt$$

$$= \left[\frac{e^{-st}}{-s} \cdot t\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\, dt, \text{ (integrating by parts).}$$

$$= 0 + \frac{1}{s} \int_0^\infty e^{-st}\, dt$$

$$= \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s^2},\ s > 0$$

(*iii*) when $F(t) = t^n$, the transform (1) reduces to

$$L\{t^n\} = \int_0^\infty e^{-st}\, t^n\, dt$$

$$= \left[\frac{e^{-st}}{-s} \cdot t^n\right]_0^\infty + \frac{n}{s} \int_0^\infty e^{-st} \cdot t^{n-1}\, dt, \text{ (integrating by parts).}$$

$$= 0 + \frac{n}{s}\left[\frac{e^{-st}}{-s}\, t^{n-1}\right]_0^\infty + \frac{n(n-1)}{s^2} \int_0^\infty e^{-st} \cdot t^{n-2}\, dt, \text{ (integrating by parts)}$$

$$= \frac{n(n-1)(n-2)}{s^3} \int_0^\infty e^{-st} \cdot t^{n-3}\, dt$$ (repeating the process of integration by parts, whence first integral vanishes for both the limits)

$= \ldots\ldots\ldots\ldots\ldots\ldots$

$$= \frac{n(n-1)(n-2)\ldots\ldots\ldots 3\cdot 2\cdot 1}{s^n}\int_0^\infty e^{-st}\cdot t^0\,dt \text{ where } t^0 = 1$$

$$= \frac{\lfloor n}{s^n}\cdot\frac{1}{s} = \frac{\lfloor n}{s^{n+1}},\ s > 0. \text{ or } \frac{\Gamma(n+1)}{s^{n+1}},\ s > 0$$

Problem 2. *Find the Laplace transform of e^{at}.* (*Meerut, 1971*)

Here $F(t) = e^{at}$

Hence by definition of Laplace transform, we have

$$L\{e^{at}\} = \int_0^\infty e^{-st}\cdot e^{at}\,dt$$

$$= \int_0^\infty e^{-(s-a)t}\,dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$= \frac{1}{s-a},\ s > a \text{ for which the integral converges,}$$

otherwise it diverges for $s \le a$.

Problem 3. *Find the Laplace transform of sin at and cos at.*

We have, by definition of Laplace transform (*Meerut, 1971*)

$$L\{F(t)\} = \int_0^\infty e^{-st}F(t)\,dt$$

when $F(t) = \sin at$, then

$$L\{\sin at\} = \int_0^\infty e^{-st}\sin at\,dt$$

$$= \left[\frac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at)\right]_0^\infty$$

$$\left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

$$= \frac{a}{s^2+a^2},\ s > 0$$

Again when $F(t) = \cos at$, then

$$L\{\cos at\} = \int_0^\infty e^{-st}\cos at\,dt)$$

$$= \left[\frac{e^{-st}}{s^2+a^2}(-s\cos at + a\sin at)\right]_0^\infty$$

$$\left[\because \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right]$$

$$= \frac{s}{s^2 + a^2}, \; s > 0$$

Problem 4. *Find the Laplace transform of sinh at and cosh at.*

We have, by definition of Laplace transform

$$L\{\sinh at\} = \int_0^\infty e^{-st} \sinh at \, dt$$

$$= \frac{1}{2} \int_0^\infty e^{-st}(e^{at} - e^{-at}) \, dt$$

$$= \frac{1}{2} \left[\int_0^\infty e^{-st} \cdot e^{at} dt - \int_0^\infty e^{-st} \cdot e^{-at} dt \right]$$

$$= \frac{1}{2} \left[\frac{1}{s-a} - \frac{1}{s+a} \right] \text{ by Problem 2}$$

$$= \frac{a}{s^2 - a^2}, \; s > |a|$$

and

$$L\{\cosh at\} = \int_0^\infty e^{-st} \cosh at \, dt$$

$$= \frac{1}{2} \int e^{-st} (e^{at} + e^{-at}) \, at$$

$$= \frac{1}{2} \left[\frac{1}{s-a} + \frac{1}{s+a} \right] \text{ by Problem 2}$$

$$= \frac{s}{s^2 - a^2}, \; s > |a|$$

Problem 5. *Find the Laplace transform of the following functions:*

(*i*) $F(t) = t \sin at$

(*ii*) $F(t) = t \cos at$

By definition, the Laplace transform of a function $F(t)$ is given by

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t) \, dt$$

(*i*) when $F(t) = t \sin at$, we have

$$L\{t \sin at\} = \int_0^\infty e^{-st} \cdot t \sin at \, dt$$

$$= \left[t \cdot \frac{e^{-st}}{s^2 + a^2} (-s \sin at - a \cos at) \right]_0^\infty$$

$$- \int_0^\infty \frac{e^{-st}}{s^2 + a^2} (-s \sin at - a \cos at) \, dt$$

[on integrating by parts, treating t as first function and $e^{-st} \sin at$ as the second function and using the result

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx),$$

while the first integral vanishes for both the limits]

$$= \frac{s}{s^2+a^2}\int_0^\infty e^{-st}\, at \sin dt + \frac{a}{s^2+a^2}\int_0^\infty e^{-st}\cos at\, dt$$

$$= \frac{s}{s^2+a^2}\cdot\frac{a}{s^2+a^2} + \frac{a}{s^2+a^2}\cdot\frac{s}{s^2+a^2} \text{ by Problem 3 for } s>0$$

$$= \frac{2as}{(s^2+a^2)^2},\ s>0.$$

(*ii*) when $F(t) = t\cos at$, we have

$$L\{t \cos at\} = \int_0^\infty e^{-st}\cdot t\cos at\, dt$$

On R.H.S., integrating by parts treating t as first function and $e^{-st}\cos at$ as second function and using the result $\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$, while the first integral vanishes for both the limits, we are left with

$$L\{t\cos at\} = \frac{s}{s^2+a^2}\int_0^\infty e^{-st}\cos at\, dt - \frac{a}{s^2+a^2}\int_0^\infty e^{-st}\sin at\, dt \quad \text{(as in (i))}$$

$$= \frac{s}{s^2+a^2}\cdot\frac{s}{s^2+a^2} - \frac{a}{s^2+a^2}\cdot\frac{a}{s^2+a^2} \text{ by Problem 3 for } s>0$$

$$= \frac{s^2-a^2}{(s^2+a^2)^2},\ s>0.$$

Problem 6. *Find the Laplace transform of t^a, where a is positive but not necessarily an integer.*

Hint: Proceed just like in Problem 1 (*iii*) and get the result $\frac{\Gamma(a+1)}{s^{a+1}}$, $s>0$ since if a is not an integer, then $\underline{|a}$ is not defined.

Problem 7. *Find the Laplace transform of e^{-at}*

Ans. $\frac{1}{s+a}$ (replace a by $-a$ in Problem 2)

Problem 8. *Find the Laplace transform of the following functions.*

(*i*) $F(t) = e^{at}\sin bt$ **Ans.** $b/\{(s-a)^2+b^2\}$

(*ii*) $F(t) = e^{at}\cos bt$. **Ans.** $(s-a)/\{(s-a)^2+b^2\}$ (*Meerut, 1981, 85, 86*)

Hint. $L\{e^{at}\sin bt\} = \int_0^\infty e^{-st}\cdot e^{at}\sin bt\, dt$

$$= \int_0^\infty e^{-(s-a)t}\sin bt\, dt$$

$$= \left[\frac{e^{-(s-a)t}}{(s-a)^2+b^2}\{-(s-a)\sin bt - b\cos bt\}\right]_0^\infty$$

$$= \frac{b}{(s-a)^2+b^2} \text{ etc.}$$

Note. *The results so far derived alongwith a few more can be tabulated as following :*

$F(t)$	$L\{F(t)\} = f(s) = \bar{F}(s)$
1	$1/s,\ s > 0$
t	$1/s^2,\ s > 0$
$t^n,\ n = 0, 1, 2\ldots$	$\lfloor n/s^{n+1}$ or $\Gamma(n+1)/s^{n+1},\ s > 0$
$t^a,\ a > 0$ but not necessary an integer	$\Gamma(a+1)/s^{a+1},\ a > 0$
$e^{at};\ e^{-at}$	$\frac{1}{s-a};\ \frac{1}{s+a},\ s > a$
$\sin at$	$a/(s^2 + a^2),\ s > 0$
$\cos at$	$s/(s^2 + a^2),\ s > 0$
$\sinh at$	$a/(s^2 - a^2),\ s > 0$
$\cosh at$	$s/(s^2 - a^2),\ s > 0$
$t \sin at$	$2as/(s^2 + a^2)^2,\ s > 0$
$t \cos at$	$(s^2 - a^2)/(s^2 + a^2)^2$
$e^{at} \sin bt$	$b/\{(s-a)^2 + b^2\}$
$e^{at} \cos bt$	$(s-a)/\{(s-a)^2 + b^2\}$
$t^{n-1} e^{at},\ n > 0$	$\lfloor n\ /\ (s-a)^n$
$J_0(at)$ and $t J_0(at)$	$1/(s^2 + a^2)$ and $s/(s^2 + a^2)^{3/2}$

10.5. FUNCTIONS OF EXPONENTIAL ORDER

A function $F(t)$ is said to be the function of exponential order m as $t \to \infty$, when for a given positive integer N, there exists real constants $M > 0$ and m such that

$$\left| e^{-mt} F(t) \right| < M$$

or

$$|F(t)| < M e^{mt} \quad \text{for } t > N$$

As an illustrative example the function $F(t) = t^2$ is of exponential order 3 since $|t^2| = t^2 < e^{3t}$ for all $t > 0$ because

$$\lim_{t\to\infty} e^{-mt} \cdot t^2 = \lim_{t\to\infty} \frac{t^2}{e^{mt}}$$

$$= \lim_{t\to\infty} \frac{t^2}{1 + mt + \frac{m^2 t^2}{\lfloor 2} + \frac{m^3 t^3}{\lfloor 3} + \ldots}$$

$$= \lim_{t\to\infty} \frac{1}{\frac{1}{t^2} + \frac{m}{t} + \frac{m^2}{\lfloor 2} + \frac{m^3 t}{\lfloor 3} + \ldots}$$

$$= 0.$$

Problem 9. *Show that function t^3 [$= F(t)$] is of exponential order as $t \to \infty$.*

Take m as some fixed positive value, and consider

$$\lim_{t\to\infty} (e^{-mt} t^3) = \lim_{t\to\infty} \frac{t^3}{e^{mt}}$$

$$\lim_{t\to\infty} \frac{3t^2}{m\, e^{mt}}$$ (on differentiating numerator and denominator w.r.t. 't' by L' Hospital's rule)

$$= \lim_{t\to\infty} \frac{6t}{m^2 e^{mt}}$$ (by L' Hospital's rule)

$$= \lim_{t\to\infty} \frac{6t}{m^3 e^{mt}}$$ (,, ,,)

$$= 0.$$

This follows that $|t^3| < e^{mt}$ for all $t > 0$

$\therefore$ t^3 is of exponential order given by

$t^3 = O(e^{mt})$, $t \to \infty$ for any fixed positive value of m.

Note. *Here the notation 'O'* stands for *'of the order of'*.

Problem 10. *Show that the function $F(t) = e^{t^3}$ is not of exponential order as $t \to \infty$.*

Taking m as some fixed positive value, we have

$$\lim_{t\to\infty} e^{-mt} F(t) = \lim_{t\to\infty} e^{-mt} e^{t^3}$$

$$= \lim_{t\to\infty} e^{t(t^2 - m)}$$

$= \infty$ for all values of m.

Hence it is not possible to find a number M such that

$e^{t^3} < M e^{mt}$ for all values of m.

Thus the given function is not of exponential order as $t \to \infty$.

Problem 11. *Show that the function $F(t) = t^n$ for $n = 1, 2, 3, \ldots$ is of exponential order as $t \to \infty$.*

Hint. Repeated application of L' Hospital's rule gives

$$\lim_{t\to\infty} e^{-mt} \cdot t^n = \lim_{t\to\infty} \frac{t}{e^{mt}} = \lim_{t\to\infty} \frac{\lfloor n}{m^n e^{mt}} = 0$$

and so $t^n = O(e^{mt})$, $t \to \infty$ etc...

Problem 12. *Show that the function* $F(t) = e^{t^2}$ *is not of exponential order as* $t \to \infty$.

Note 2. *Actually, functions of exponential order do not grow in absolute value more rapidly than* Me^{mt} as *t increases but in practice there is no such restriction since M and m may be as large as we desire. For example the bounded functions like sin at or cos at are of exponential order.*

10.6. PIECEWISE OR SECTIONAL CONTINUITY

Given a closed interval $[a, b]$ *a function* $F(t)$ *is called* **piecewise continuous** *or* **sectionally continuous** *in* $[a, b]$, *i.e.* $a \le t \le b$, *if the interval can be divided into a finite number of sub-intervals such that in each of these sub-intervals, the function remains continuous and possesses finite right and left hand limits.*

In other words, the function $F(t)$ is sectionally or piecewise continuous in the closed interval $[a, b]$, if the closed interval $[a, b]$ can be divided into a finite number of sub-intervals $c \le t \le d$ such that

(*i*) $F(t)$ is continuous in the open interval $c < t < d$,

(*ii*) $F(t)$ approaches a limit as t approaches each end-point within interval *i.e.*

$\lim_{t \to c+0} F(t)$ and $\lim_{t \to c-0} F(t)$ both exist.

For example, referring to the Fig. 10.1 it is evident that the function $F(t)$ is continuous in the open intervals (a, t_1), (t_1, t_2), (t_2, t_3), etc., *i.e.*

$a < t < t_1$; $t_1 < t < t_2$; $t_2 < t < t_3$ etc,

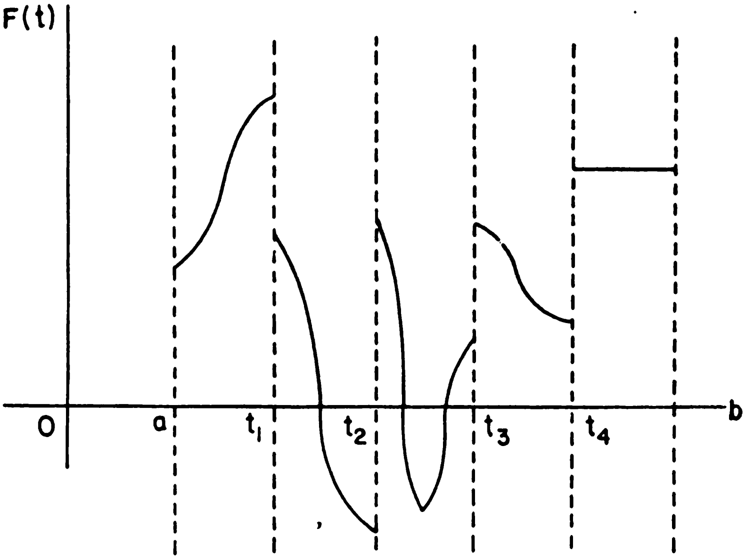


Fig. 10.1

The right hand and left hand limits at t_1 exist and are given by

$$\lim_{\epsilon \to 0} F(t_1 + \epsilon) = F(t_1 + 0) = F(t_1 +)$$

and

$$\lim_{\epsilon \to 0} F(t_1 - \epsilon) = F(t_1 - 0) = F(t_1 -)$$

where ϵ is positive.

Similar is the case for points t_2, t_3 and t_4.

10.7. A FUNCTION OF CLASS A

If a function $F(t)$ is sectionally continuous over every finite interval in the range $t \geq 0$ and is of exponential order as $t \to \infty$, then the function is termed as '*a function of class A*'

10.8. SUFFICIENT CONDITIONS FOR THE EXISTENCE OF LAPLACE TRANSFORMS

If a function $F(t)$ is piecewise or sectionally continuous in every finite interval $0 \leq t \leq N$ and is of exponential order m for $t > N$ then its Laplace transform $L\{F(t)\}$ i.e. $f(s)$ exists for all $s > m$.

or

If a function $F(t)$ is of class A, then the Laplace transform of $F(t)$ i.e., $L\{F(t)\}$ exists.

or

If a function $F(t)$ is sectionally continuous on every finite interval in the range $t \geq 0$ and satisfies the condition

$$|F(t)| \leq Me^{mt}$$

for al $t \geq 0$ and for some constants m and M, then $L\{F(t)\}$ i.e. $f(s)$ exists for all $s > m$.

We have, for any positive integer N,

$$L\{F(t) = \int_0^\infty e^{st} F(t)\, dt$$

$$= \int_0^N e^{-st} F(t)\, dt + \int_N^\infty e^{-st} F(t)\, dt$$

$$= I_1 + I_2 \text{ (say)} \quad \ldots(1)$$

Since $F(t)$ is sectionally continuous in every finite interval $0 \leq t \leq N$, it therefore, follows that the integral I_1 exists.

Also $F(t)$ being of exponential order m for $t > N$, I_2 exists, since

$$|I_2| = \left|\int_N^\infty e^{-st} F(t)\, dt\right| \leq \int_N^\infty \left|e^{-st} F(t)\right| dt$$

$$\leq \int_0^\infty e^{-st} |F(t)|\, dt$$

$$\leq \int_0^\infty e^{-st} \cdot Me^{mt}\, dt$$ as $F(t)$ is of exponential order m and so $|F(t)| \leq Me^{mt}$,

$$\leq \int_0^\infty e^{-(s-m)\, t} M\, dt$$

$$\therefore |I_2| \leq \frac{M}{s-m}$$ for $s > m$. (Here R.H.S. $\to 0$ *as* $s \to \infty$)

COROLLARY 1. From (1), it therefore follows that $L\{F(t)\}$ exists for $s > m$. ...(2)

If the function $F(t)$ is real valued in $L\,F(t)\} = \int_0^\infty e^{-st} F(t)\, dt = f(s)$ *such that (i) $F(t)$ is R-integrable (i.e. has Riemann-integral) and (ii) $F(t)$ satisfies, $|F(t)| \leq M^{mo^t}$ for* *all $t > N$; M, m_0, N being positive constants, then $\int_0^\infty e^{-st} F(t)\, dt$ is convergent (absolutely) for $m > m_0$, s being a complex number.*

Supposing, $s = m + in$, we have

$$\left| e^{-st} F(t) \right| = \left| e^{-(m+in)t} F(t) \right|$$

$$= e^{-mt} \left| F(t) \right| \quad \because \left| e^{-nit} \right| = 1 \text{ for every } t \geq N$$

$$\leq e^{-mt} \cdot M e^{m_0 t}$$

$$\leq M e^{-(m-m_0)t}$$

Hence, if the integral $\int_0^\infty e^{-(m-m_0)t}\, dt$ is convergent, then

$$\int_0^\infty e^{-st} F(t)\, dt \text{ for } m > m_0, \text{ is absolutely convergent} \qquad \text{...(3)}$$

COROLLARY 2. $L\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\, dt$ *converges, for all s greater than some fixed value s_0 of s.*

In corollary 1, if we replace m_0 by s_0 and m by s then the proposition follows.

COROLLARY 3. *If the function $F(t)$ is of class A and $L\{F(t)\} = f(s)$, then* $\lim_{s \to \infty} f(s) = 0$

The function $F(t)$ being of class A is bounded over the range $0 \leq t \leq N$, so that $|F(t)| < M_1$ (say) for $m \leq t \leq N$.

Also $F(t)$ being of exponential order m, we have

$|F(t)| < M_2 e^{mt}$ for $t \geq N$

Let us assume that $M = \text{Max.} \{M_1, M_2\}$

and $\qquad p = \text{Max.} \{m, 0\}$

Thus, $|F(t)| < M e^{pt}$ for $t \geq 0$

i.e. $$\int_0^\infty e^{-st} F(t)\, dt < M \int_0^\infty e^{-st} e^{pt}\, dt = M \int_0^\infty e^{-(s-p)}\, dt = \frac{M}{s-p},\ s > p$$

Here $\dfrac{M}{s-p} \to 0$ as $s \to \infty$.

Hence $$\lim_{s \to \infty} \int_0^\infty e^{-st} F(t)\, dt = 0$$

i.e. $$\lim_{s \to \infty} f(s) = 0. \qquad \text{...(4)}$$

Note. The conditions mentioned in this article for the existence of Laplace transforms are only sufficient but not necessary as is evident from the following example:

Let $$F(t) = \frac{1}{\sqrt{t}} = t^{-\frac{1}{2}}$$

Obviously the function $F(t) = t^{-1/2}$ is not sectionally continuous in every finite interval in the range $t \geq 0$, for

$$F(t) \to \infty \text{ as } t \to 0.$$

But $t^{-1/2}$ is integrable from 0 to any positive value N.

Also $t^{-1/2}$ is of exponential order, since

$|F(t)|$ *i.e.* $|t^{-1/2}| < Me^{mt}$ for all $t > 0$ with $M = 1$ and $m = 0$ as $t^{-1/2} \to 0$ when $t \to \infty$.

As such the Laplace transform of $t^{-1/2}$ exists and may be evaluated as follows:

$$L\{F(t)\} = L\{t^{-1/2}\} = \int_0^\infty e^{-st}\, t^{-1/2}\, dt \text{ for } s > 0, \qquad \text{Put } t^{-1/2} = n$$

$$= 2\int_0^\infty e^{-sx^2}\, dx,\ s > 0 \qquad \therefore\ -\tfrac{1}{2} t^{-1/2}\, dt = dx$$

Again put $s^{1/2} x = y$ so that $s^{1/2} dx = dy$

$$\therefore\ L\{t^{-1/2}\} = 2s^{-1/2} \int_0^\infty e^{-y^2}\, dy,\ s > 0$$

$$= 2s^{-1/2} \cdot \tfrac{1}{2}\sqrt{\pi} \qquad \because \int_0^\infty e^{-y^2}\, dy = \tfrac{1}{2}\sqrt{\pi}$$

$$= \sqrt{\frac{\pi}{s}} \text{ for } s > 0, \text{ even if } t^{-1/2} \to \infty \text{ as } t \to 0$$

i.e. the Laplace transform of $t^{-1/2}$ exists even if the function is not sectionally continuous.

As an Aliter, the integral $\int_0^\infty e^{-st}\, t^{-1/2}\, dt,\ s > 0$ can be evaluated by a single substitution $\sqrt{st} = x$ *i.e.* $\frac{dt}{\sqrt{t}} = \frac{2}{\sqrt{s}}\, dx$ whence we have

$$L\{t^{-1/2}\} = \frac{2}{\sqrt{s}} \cdot \int_0^\infty e^{-x^2}\, dx = \frac{2}{\sqrt{s}} \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\frac{\pi}{s}} \qquad \ldots(5)$$

Problem 13. *Show that the Laplace transform of the function*

$$F(t) = t^n,\ 0 > n > -1$$

exists, although it is not a function of the class A.

Here the function $F(t) = t^n$ is not sectionally continuous and so is not of the class A, in every finite interval in the range $t \geq 0$, since

$$t^n \to 0 \text{ as } t \to 0 \text{ for } 0 > n > -1$$

Hence the function has infinite discontinuity at $t = 0$

$$\text{Also, } \lim_{t \to \infty}\{e^{-mt} F(t)\} = \lim_{t \to \infty}\left(\frac{t^n}{e^{mt}}\right)$$

$$= \lim_{t \to \infty}\left(\frac{nt^{n-1}}{me^{mt}}\right) \text{ by } L' \text{ Hospital rule}$$

$$= \lim_{t \to \infty} \frac{n^2\, t^{n-2}}{m^2\, e^{mt}} \qquad ''$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \lim_{t \to \infty} \frac{\lfloor n}{m^n\, e^{mt}} \quad \text{by repeated application of } L' \text{ Hospital rule}$$

$$= 0$$

which follows that $F(t) = t^n$ is of exponential order.

[Or this may be argued thus, $t^n \to 0$ as $t \to 0$ so that t^n is of exponential order with $M = 1$ and $m = 0$, for, $t^n < Me^{mt}$].

Moreover, the function t^n $(0 > n > -1)$ is integrable from 0 to any positive number N.

Further, $L\{F(t)\} = \int_0^\infty e^{-st} \cdot t^n \, dt$ Put $st = x$

$$= \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s} \qquad i.e. \; dt = \frac{dx}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty x^n e^{-x} \, dx$$

$$= \frac{\Gamma(n+1)}{s^{n+1}}$$ by the definition of gamma function.

Hence the Laplace transform of t^n exists even if it is not a function of the class A.

10.9. SOME PROPERTIES OF LAPLACE TRANSFORMS

[A] Linearity Property

A Laplace transform $L\{F(t)\}$ is said to be linear if for every pair of function $F_1(t)$ and $F_2(t)$ and for every pair of constants C_1 and C_2, we have

$$L\{C_1 F_1(t) + C_2 F_2(t)\} = C_1 L\{F_1(t)\} + C_2 L\{F_2(t)\}$$
$$= C_1 f_1(s) + C_2 f_2(s)$$

where $f_1(s)$ and $f_2(s)$ are linear transforms of $F_1(t)$ and $F_2(t)$ respectively

We have $L\{F_1(t)\} = f_1(s) = \int_0^\infty e^{-st} F_1(t) \, dt$

and $L\{F_2(t)\} = f_2(s) = \int_0^\infty e^{-st} F_2(t) \, dt$

So that $L\{C_1 F_1(t)\} = C_1 f_1(s) = \int_0^\infty e^{-st} C_1 F_1(t) \, dt = C_1 L\{F_1(t)\}$

and $L\{C_2 F_2(t)\} = C_2 f_2(s) = \int_0^\infty e^{-st} C_2 F_2(t) \, dt = C_2 L\{F_2(t)\}$

$\therefore$ $L\{C_1 F_1(t) + C_2 F_2(t)\} = \int_0^\infty e^{-st} \{C_1 F_1(t) + C_2 F_2(t)\} \, dt$ by definition

$$= \int_0^\infty e^{-st} C_1 F_1(t) \, dt + \int_0^\infty e^{-st} C_2 F_2(t) \, dt$$
$$= C_1 L\{F_1(t)\} + C_2 L\{F_2(t)\}$$
$$= C_1 f_1(s) + C_2 f_2(s) \qquad \text{...(1)}$$

The result may be generalized for any number of functions and for the same number of arbitrary constants *i.e.*,

$$L\left\{\sum_{r=1}^{n} C_r F_r(t)\right\} = \sum_{r=1}^{n} C_r L\{F_r(t)\}. \qquad \text{...(2)}$$

Problem 14. *Find Laplace Transform of $4e^{5t} + 6t^3 - 4\cos 3t + 3\sin 4t$.*

Applying the linearity property, we have

$$L\{4e^{5t} + 6t^3 - 4\cos 3t + 3\sin 4t\}$$
$$= 4L\{e^{5t}\} + 6L\{t^3\} - 4L\{\cos 3t\} + 3L\{\sin 4t\}$$

$$= 4\left(\frac{1}{s-5}\right) + 6\left(\frac{\lfloor 3}{s^4}\right) - 4\left(\frac{s}{s^2+9}\right) + 3\left(\frac{4}{s^2+16}\right)$$

$$= \frac{4}{s-5} + \frac{36}{s^4} - \frac{4s}{s^2+9} + \frac{12}{s^2+16}$$

[B] First Translation (or Shifting) Property

If $f(s)$ be the Laplace transform of $F(t)$, then the Laplace transform of $e^{at} F(t)$ is $f(s-a)$, where a is any real or complex number i.e., if

$L\{F(t)\} = f(s)$, then $\quad L\{e^{at} F(t)\} = f(s-a)$.

Given, $L\{F(t)\} = \int_0^{\infty} e^{-st} F(t)\, dt = f(s)$

$$\therefore\ L\{e^{at} F(t)\} = \int_0^{\infty} e^{-st} e^{at} F(t)\, dt$$

$$= \int_0^{\infty} e^{-(s-a)t} F(t)\, dt$$

$$= \int_0^{\infty} e^{-ut} F(t)\, dt \text{ by putting } u = s - a$$

$$= f(u)$$

$$= f(s-a) \qquad \ldots(3)$$

Problem 15. *Find the Laplace transform of $e^{-2t} \sin 3t$.*

We have $L\{\sin 3t\} = \dfrac{3}{s^2+9}$

$$\therefore\ L\{e^{-2t} \sin 3t\} = \frac{3}{(s+2)^2 + 9} = \frac{3}{s^2 + 4s + 13}.$$

[C] Second Translaton (or Shifting) Property

If $L\{F(t)\} = f(s)$ and $G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$

Then $L\{G(t)\} = e^{-as} f(s)$.

We have $L\{G(t)\} = \int_0^{\infty} e^{-st} G(t)\, dt$

$$= \int_0^{a} e^{-st} G(t)\, dt + \int_a^{\infty} e^{-st} G(t)\, dt$$

$$= \int_0^{a} e^{-st} \cdot 0\, dt + \int_a^{\infty} e^{-st} F(t-a)\, dt$$

$$= \int_a^{\infty} e^{-st} \cdot F(t-a)\, dt$$

$$= \int_0^{\infty} e^{-s(u+a)} F(u)\, du, \text{ by taking } u = t - a \text{ i.e. } du = dt.$$

when $t = a$, $u = 0$ and when $t = \infty$, $u = \infty$.

$$= e^{-sa} \int_0^{\infty} e^{-su} F(u)\, du$$

$$= e^{-sa} f(s) \qquad \ldots(4)$$

Problem 16. *Find the Laplace transform of F (t), where*

$$F(t)=\begin{cases}\cos\left(t-\dfrac{2\pi}{3}\right), & t>\dfrac{2\pi}{3}\\ 0, & t<\dfrac{2\pi}{3}\end{cases}$$

We have, $L\{F(t)\}=\int_0^\infty e^{-st}F(t)\,dt$

$$=\int_0^{2\pi/3} e^{-st}\cdot 0.\,dt+\int_{2\pi/3}^\infty e^{-st}\cos\left(t-\frac{2\pi}{3}\right)dt$$

$$=\int_0^\infty e^{-s\left(u+\frac{2\pi}{3}\right)}\cdot\cos u\,du \text{ by taking } u=t-\frac{2\pi}{3}$$

$$=e^{-\frac{2\pi s}{3}}L\{\cos u\}$$

$$=e^{-\frac{2\pi s}{3}}\frac{s}{s^2+1}$$

[D] The Change of Scale Property

If $L\{F(t)=f(s)$, *then* $L\{F(at)\}=\frac{1}{a}f\left(\frac{s}{a}\right)$ ...(5)

We have $L\{F(t)\}=\int_0^\infty e^{-st}F(t)\,dt=f(s)$

$\therefore$ $L\{F(at)\}=\int_0^\infty e^{-st}F(at)\,dt$ (on replacing t by at)

$$=\int_0^\infty e^{-su/a}F(u)\frac{du}{a}$$ by taking $at=u$

$$=\frac{1}{a}\int_0^\infty e^{-pu}F(u)\,du$$ where $p=\frac{s}{a}\cdot$

$$=\frac{1}{a}\int_0^\infty e^{-pu}F(u)\,dt$$ (replacing u by t)

$$=\frac{1}{a}f(p)$$

$$=\frac{1}{a}f\left(\frac{s}{a}\right) \qquad \because p=\frac{s}{a}\cdot$$

Problem 17. *Find the Laplace transform of cos 5t.*

We have, $L\{\cos t\}=\frac{s}{s^2+1}$, $s>0$

$\therefore$ $L\{\cos 5t\}=\frac{1}{5}\frac{s/5}{(s/5)^2+1}=\frac{s}{s^2+25}\cdot$

[E] Laplace Transform of Derivatives

If F (t) is continuous for $t\ge 0$ and of exponential order as $t\to\infty$ while F′ (t) is sectionally continuous i.e., F′ (t) is of class A for $t\ge 0$, and if $L\{F(t)\}=f(s)$, then $L\{F'(t)\}=sf(s)-F(0)$.

In general if $L\{F(t)\} = f(s)$ and $F(t), F'(t), F''(t), \ldots F^{(n-1)}(t)$ are continuous for $t \geq 0$ and of exponential order as $t \to \infty$ while $F^{(n)}(t)$ is sectionally continuous for $t \geq 0$, then

$$L\{F^{(n)}(t)\} = s^n f(s) - \sum_{r=0}^{n-1} s^{n-1-r} F^{(r)}(0)$$

$$= s^n f(s) - s^{n-1} F(0) - s^{n-2} F'(0) \ldots - sF^{(n-2)}(0) - F^{(n-1)}(0)$$

Since $L\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\, dt$

$\therefore L\{F'(t)\} = \int_0^\infty e^{-st} F'(t)\, dt = \left[e^{-st} F(t)\right]_0^\infty + s\int_0^\infty e^{-st} F(t)\, dt$, integrating by parts

$= 0 - F(0) + sf(s)$

$= sf(s) - F(0)$ ...(6)

Applying the result (6), we have

$L\{F''(t)\} = sL\{F'(t)\} - F'(0)$

$= s\{sf(s) - F(0)\} - F'(0)$ by (6)

$= s^2 f(s) - sF(0) - F'(0)$ ...(7)

Similarly $L\{F'''(t)\} = s^3 f(s) - s^2 F(0) - sF'(0) - F''(o)$ ...(8)

Generalizing it, we find

$$\left.\begin{aligned} L\{F^{(n)}(t)\} &= s^n f(s) - s^{n-1} F(0) - s^{n-2} F'(0) \ldots - sF^{(n-2)}(0) - F^{(n-1)}(0) \\ &= s^n f(s) - \sum_{r=0}^{n-1} s^{n-1-r} F^{(r)}(0) \end{aligned}\right\} \quad \ldots(9)$$

Problem 18. *Find the Laplace transform of $F'(t)$ when $F(t) = e^{3t}$*

Given $F(t) = e^{3t}$, $\therefore F(0) = 1$ and $F'(t) = 3e^{3t}$

As such $L\{e^{3t}\} = sL\{e^{3t}\} - 1$ by (6) above

$$= \frac{s}{s-3} - 1 = \frac{3}{s-3}.$$

Aliter. $L\{F'(t)\} = L\{3e^{3t}\} = 3L\{e^{3t}\} = \dfrac{3}{s-3}$.

[F] Derivatives of Laplace Transforms

If the function $F(t)$ is sectionally continuous for $t \geq 0$ and if $L\{F(t)\} = f(s)$, then $f'(s) = L\{-tF(t)\}$

We have $f(s) = \int_0^\infty e^{-st} F(t)\, dt$.

Differentiating either side w.r.t. 's' we get

$f'(s) = \int_0^\infty (-t)\, e^{-st} F(t)\, dt = \int_0^\infty e^{-st} \{-t\, F(t)\}\, dt$

$= L\{-tF(t)\}$ ...(10)

In general if $F(t)$ is sectionally continuous for $t \geq 0$ and if $L\{F(t)\} = f(s)$, then

$f^{(n)}(s) = L\{(-t)^n F(t)\}$ ...(11)

where $f^{(n)}(s) = \dfrac{d^n}{ds^n} f(s)$ *for all integral values of n.*

We may state it as

$$L\{t^n F(t)\} = (-1)^n f^n(s) = (-1)^n \frac{d^n}{ds^n} f(s) \qquad \text{...(12)}$$

Problem 19. *Find the Laplace transform of* $t^3 e^t$.

Since $L\{e^t\} = f(s) = \dfrac{1}{s-a}$

$$\therefore \; L\{t^3 e^t\} = (-1)^3 \frac{d^3}{ds^3}\left(\frac{1}{s-a}\right) = (-1)^3 \times \frac{(-1)(-2)(-3)}{(s-1)^4} = \frac{6}{(s-1)^4}$$

[G] Laplace Transform of Integrals

(*i*) *If* $L\{F(t)\} = f(s)$, *then* $L\left\{\int_0^t F(u)\, du\right\} = \dfrac{f(s)}{s}$ ...(13)

Let $G(t) = \int_0^t F(u)\, du$

Then $G'(t) = \dfrac{d}{dt}\left[\int_0^t F(u)\, du\right] = F(t)$ and $G(0) = \int_0^0 F(u)\, du = 0.$

Applying the property [*E*], we have

$$L\{G'(t)\} = sL\{G(t)\} - G(0)$$

i.e. $L\{F(t)\} = sL\{G(t)\} - 0$ or $f(s) = sL\left\{\int_0^t F(u)\, du\right\}$

i.e. $L\left\{\int_0^t F(u)\, du\right\} = \dfrac{f(s)}{s}$

(*ii*) *If* $L\{F(t)\} = f(s)$ then $L\left\{\dfrac{F(t)}{t}\right\} = \int_s^\infty f(u)\, du$ ...(14)

Let $G(t) = \dfrac{F(t)}{t}$, so that $F(t) = tG(t)$

$\therefore \; L\{F(t)\} = L\{tG(t)\}$ (on taking Laplace transform)

$= (-1)\dfrac{d}{ds} L\{G(t)\}$ by property [*F*]

i.e $-f(s) = \dfrac{d}{ds} L\{G(t)\}$

Integrating both sides with regard to *s*, we get

$$-\int_\infty^s f(s)\, ds = L\{G(t)\},$$

i.e., $L\{G(t)\} = \int_s^\infty f(u)\, du$, on the assumption that $\lim_{s\to\infty} L\{G(s)\} \to 0$.

Problem 20. *Find the Laplace transform of* (*a*) $\int_0^\infty \dfrac{\sin t}{t} dt$, (*b*) $\dfrac{\sin at}{t}$, *and determine if transform of* $\dfrac{\cos at}{t}$ *exists or not.* (*Meerut, 83; Kanpur, 70*)

(*a*) We have $L\{\sin t\} = \dfrac{1}{s^2+1} = f(s)$ (say) and $$L\left\{\frac{\sin t}{t}\right\}=\int_s^\infty \frac{du}{u^2+1} \text{ by (14)}, \because f(u)=\frac{1}{u^2+1}$$

$$=\left[\tan^{-1} u\right]_s^\infty = \frac{\pi}{2}-\tan^{-1} s$$

$$=\cot^{-1} s \quad \because \tan^{-1} s+\cot^{-1} s=\frac{\pi}{2}$$

$$=\tan^{-1}\frac{1}{s}.$$

Hence by (13) of property [*G*],

$$L\left\{\int_0^\infty \frac{\sin t}{t}\,dt\right\}=\frac{1}{s}\tan^{-1}\frac{1}{s}.$$

(*b*) We have

$$L\left\{\frac{\sin at}{t}\right\}=\int_s^\infty \frac{du}{u^2+a^2} \text{ by (14) above}$$

$$=\left[\tan^{-1}\frac{u}{a}\right]_s^\infty=\frac{\pi}{2}-\tan^{-1}\frac{s}{a}=\cot^{-1}\frac{s}{a}$$

Again,

$$L\left\{\frac{\cos at}{t}\right\}=\int_s^\infty \frac{u\,du}{u^2+a^2}=\tfrac{1}{2}\left[\log\left(u^2+a^2\right)\right]_s^\infty$$

$$=\tfrac{1}{2}\left[\lim_{u\to\infty}\log\left(u^2+a^2\right)-\log\left(s^2+a^2\right)\right]$$

whence $\lim_{u\to\infty}\log\left(u^2+a^2\right)\to\infty$, so limit does not exist and hence the Laplace transform of $\frac{\cos at}{t}$ does not exist.

[H] Periodic Functions

If $F(t)$ is a periodic function with period $T>0$, so that

$$F(t+T)=F(t), \text{ then } L\{F(t)\}=\frac{\int_0^T e^{-st}F(t)\,dt}{1-e^{-sT}} \qquad \text{...(15)}$$

We have $L\{F(t)\}=\int_0^\infty e^{-st}F(t)\,dt$

$$=\int_0^T e^{-st}F(t)\,dt+\int_T^{2T} e^{-st}F(t)\,dt+\int_{2T}^{3T} e^{-st}F(t)\,dt+\ldots$$

$$=\sum_{n=0}^{\infty}\int_{nT}^{(n+1)T} e^{-st}F(t)\,dt$$

If we put $t=u+nT$, then $F(u+nT)=F(u) \because F(t+T)=F(t)$ (given). Thus,

$$L\{F(t)\}=\sum_{n=0}^{\infty} e^{-s(u+nT)}F(u)\,du$$

or $$L\{F(t)\}=\sum_{n=0}^{\infty} e^{-snT}\int_s^T e^{-su}F(u)\,du$$

$$= \left(1 + e^{-sT} + e^{-2sT} + \ldots\right) \int_0^T e^{-su} F(u)\, du$$

$$= \frac{1}{1-e^{-sT}} \int_0^T e^{-su} F(u)\, du \quad \because \frac{1}{1-e^{-sT}} = (1-e^{-sT})^{-1}$$

$$= 1 + e^{-sT} + e^{-2sT} + \ldots \text{ when } |e^{-sT}| < 1$$

$$= \frac{\int_0^T e^{-st} F(t)\, dt}{1-e^{-st}} \text{ (replacing } u \text{ by } t\text{)}$$

ILLUSTRATION

Consider a half wave rectifier function, given by

$$F(t) = \begin{cases} \sin \omega t, & 0 < t < \frac{\pi}{\omega} \\ 0, & \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$$

with period $\frac{2\pi}{\omega}$, then

$$L\{F(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} F(t)\, dt$$

$$= \frac{1}{1-e^{-sT}} \left[\int_0^{T/2} e^{-st} \sin \omega t\, dt + \int_{T/2}^T e^{-st} \cdot 0\, dt\right]$$

i.e. $$f(s) = \frac{1}{1-e^{-sT}} \int_0^{\pi/\omega} e^{-st} \sin \omega t\, dt \quad \text{as } T = \frac{2\pi}{\omega} \Rightarrow \frac{T}{2} = \frac{\pi}{\omega}$$

$$= \frac{1}{1-e^{-sT}} \left[\frac{e^{-st}}{s^2+\omega^2} (-s \sin \omega t - \omega \cos \omega t)\right]_0^{\pi/\omega}$$

$$= \frac{\omega}{\left(s^2+\omega^2\right)(1-e^{-\pi s/\omega})} \text{ (on simplification).}$$

Problem 21. *Find the Laplace transform of F (t) when F (t) is a periodic function with period 2π, such that*

$$F(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$$

We have $$L\{F(t)\} = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} F(t)\, dt$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\int_0^{\pi} e^{-st} \sin t\, dt + \int_{\pi}^{2\pi} e^{-st} \cdot 0 \cdot dt\right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{-st}}{s^2+1} (-\sin t - \cos t)\right]_0^{\pi} + 0$$

$$\because \int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx)$$

$$= \frac{1}{1-e^{-2\pi s}} \frac{e^{-\pi s} + 1}{s^2+1}$$

$$= \frac{1}{\left(1 - e^{-\pi s}\right)\left(1 + e^{-\pi s}\right)} \cdot \frac{1 + e^{-\pi s}}{1 + s^2}$$

$$= \frac{1}{\left(1 - e^{-\pi s}\right)\left(1 + s^2\right)}$$

[I] Initial Value Theorem

If $L\{F(t)\} = f(s)$ *then* $\lim_{t \to 0} F(t) = \lim_{s \to \infty} sf(s)$ ...(16)

We have $L\{F'(t)\} = sf(s) - F(0)$ by property [E]

i.e. $\int_0^\infty e^{-st} F'(t)\, dt = sf(s) - F(0)$

Taking the limit as $t \to \infty$,

$$\lim_{s \to 0} \int_0^\infty e^{-st} F'(t)\, dt = \lim_{s \to \infty} sf(s) - F(0)$$

or $\lim_{s \to \infty} sf(s) = F(0) + \int_0^\infty \left(\lim_{s \to \infty} e^{-st}\right) F'(t)\, dt$

$= F(0) + 0 \qquad \because \lim_{s \to \infty} e^{-st} = 0$

$= \lim_{t \to 0} F(t)$

Problem 22. *Verify the initial value theorem for the function* $F(t) = e^{-3t}$.

$\because \quad F(t) = e^{-3t}$

$\therefore f(s) = L\{F(t)\} = L\{e^{-3t}\} = \dfrac{1}{s+3}$

Now $\lim_{t \to 0} F(t)\} = \lim_{t \to 0} e^{-3t} = 1$

and $\lim_{s \to \infty} sf(s) = \lim_{s \to \infty} \dfrac{s}{s+3} = 1.$

Hence $\lim_{t \to 0} F(t) = \lim_{s \to \infty} sf(s)$.

[J] Final-Value Theorem

If $L\{F(t)\} = f(s)$, then $\lim_{s \to 0} sf(s) = \lim_{t \to \infty} F(t)$ ...(17)

We have $L\{F'(t)\} = sf(s) - F(0)$ by Prop. [E]

i.e. $\int_0^\infty e^{-st} F'(t)\, dt = sf(s) - F(0)$

Taking the limit as $s \to 0$,

$$\lim_{s \to 0} \int_0^\infty e^{-st} F'(t)\, dt = \lim_{s \to 0} sf(s) - F(0)$$

or $\lim_{s \to 0} sf(s) = F(0) + \lim_{s \to 0} \int_0^\infty e^{-st} F'(t)\, dt$

$= F(0) + \int_0^\infty \left(\lim_{s \to 0} e^{-st}\right) F'(t)\, dt$

$= F(0) + \int_0^\infty 1 \cdot F'(t)\, dt$

$$= F(0) + \int_0^\infty \frac{d}{dt} F(t)\, dt$$
$$= F(0) + [F(t)]_0^\infty$$
$$= F(0) + \underset{t\to\infty}{\text{Lim}} F(t) - F(0)$$
$$= \underset{t\to\infty}{\text{Lim}} F(t).$$

Problem 23. *Verify the Final-value theorem for the function*
$$F(t) = e^{-2t}.$$

We have $F(t) = e^{-2t}$ so that $f(s) = L\{F(t)\} = L\{e^{-2t}\} = \dfrac{1}{s+2}$

$$\therefore \quad \underset{t\to\infty}{\text{Lim}} F(t) = \underset{t\to 0}{\text{Lim}}\, e^{-2t} = 0$$

and $$\underset{s\to 0}{\text{Lim}}\, s f(s) = \underset{t\to 0}{\text{Lim}} \frac{s}{s+2} = 0$$

Hence $\underset{t\to\infty}{\text{Lim}} F(t) = \underset{s\to\infty}{\text{Lim}}\, s f(s).$

[K] Behaviour of $f(s)$ as $s\to 0$ and $s \to \infty$

We have $L\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\, dt$

when $s \to 0$, $f(0) = \int_0^\infty F(t)\, dt$ ...(18)

and when $s\to\infty$, $\underset{s\to\infty}{\text{Lim}} f(s) = \int_0^\infty 0 \,.F(t)\, dt = 0$ ...(19)

[L] Relationship with Fourier Transform

Let us define a function $F(t)$ such that

$$F(t) = \begin{cases} e^{-xt} f(t) & ; t > 0 \\ 0 & ; t < 0 \end{cases}$$

Then $f[F(t)] = \int_{-\infty}^\infty e^{int} F(t)\, dt$ by definition of complex Fourier transform

$$= \int_{-\infty}^0 e^{int} \cdot 0\, dt + \int_0^\infty e^{int} e^{-xt} f(t)\, dt$$
$$= \int_0^\infty e^{-(x-in)t} f(t)\, dt$$
$$= \int_0^\infty e^{-st} f(t)\, dt \qquad \text{by taking } x - in = s$$

$= L\{f(t)\}$ by definition of Laplace transform

Hence $f[F(t)] = L\{f(t)\}$ ...(20)

10.10. SOME METHODS FOR FINDING LAPLACE TRANSFORMS

[1] Direct method. This is based on the definition of Laplace transforms given in §10.4, *e.g.*

$$L\{(t^2+1)^2\} = L\{t^4 + 2t^2 + 1\} = L\{t^4\} + 2L\{t^2\} + L\{1\}$$
$$= \int_0^\infty e^{-st} t^4 \cdot dt + 2\int_0^\infty e^{-st} \cdot t^2\, dt + \int_0^\infty e^{-st} \cdot 1 \cdot dt$$

$$= \frac{\lfloor 4}{s^5} + 2 \cdot \frac{\lfloor 2}{s^3} + \frac{1}{s} = \frac{24 + 4s^2 + s^4}{s^5}$$

[2] Series-expansion method. If the function $F(t)$ is expressible as a Power series *e.g.*

$$F(t) = a_0 + a_1 t + a_2 t^2 + \ldots = \sum_{n=0}^{\infty} a_n t^n$$

then the Laplace transform of $F(t)$ is obtained by taking Laplace transforms of each term in the series *e.g.*

$$L\{\sin\sqrt{t}\} = L\left\{\sqrt{t} - \frac{(\sqrt{t})^3}{\lfloor 3} + \frac{(\sqrt{t})^5}{\lfloor 5} - \frac{(\sqrt{t})^7}{\lfloor 7} + \ldots\right\}$$

$$= L\{t^{1/2}\} - \frac{1}{\lfloor 3} L\{t^{3/2}\} + \frac{1}{\lfloor 5} L\{t^{5/2}\} - \frac{1}{\lfloor 7} L\{t^{7/2}\} + \ldots\ldots$$

$$= \frac{\Gamma 3/2}{s^{3/2}} - \frac{\Gamma 5/2}{\lfloor 3\, s^{5/2}} + \frac{\Gamma 7/2}{\lfloor 5\, s^{7/2}} - \frac{\Gamma 9/2}{\lfloor 7\, s^{9/2}} + \ldots\ldots$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{1}{2^2 s} + \frac{1}{\lfloor 2}\left(\frac{1}{2^2 s}\right)^2 - \frac{1}{\lfloor 3}\left(\frac{1}{2^2 s}\right)^3 + \ldots\ldots\ldots\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[e^{-\frac{1}{2^2 s}}\right] = \frac{e^{-\frac{1}{4s}}\sqrt{\pi}}{2s^{3/2}} \qquad \ldots(2)$$

[3] Method of differential equations. If a differential equation satisfied by the function $F(t)$ can be determined, then its Laplace transform may be evaluated by using the properties of Laplace transforms *e.g.*

if $F(t) = \sin\sqrt{t}$ then $F'(t) = \frac{1}{2\sqrt{t}}\cos\sqrt{t}$

and $F''(t) = \frac{1}{2}\left[-\frac{1}{2t^{3/2}}\cos\sqrt{t} - \frac{1}{2t}\sin\sqrt{t}\right] = \frac{1}{2t}\left[-F'(t) - \frac{1}{2}F(t)\right]$

i.e. $4t F''(t) + 2F'(t) + F(t) = 0$ ...(3)

which is clearly satisfied by $F(t) = \sin\sqrt{t}$

Assuming that $L\{F(t)\} = L\{\sin\sqrt{t}\} = f(s)$, we have on using the properties [*E*] and [*F*] of §10.9,

$$L\{4t F''(t)\} = 4L\{t F''(t)\}$$

$$= -4\frac{d}{ds}\left[s^2 f(s) - sF(0) - F(0)\right]$$

$$= -8s f(s) - 4s^2 f'(s) - 4F(o)$$

$$L\{2F'(t)\} = 2Lf(F'(t)) = 2[sf(s) - F(o)]$$

$$= 2sf(s) - 2F(o)$$

and $L\{F(t)\} = f(s)$

∴ Taking Laplace transform of (3), we get

$$L\{4t F''(t)\} + L(2F'(t)) + L\{F(t)\} = 0$$

i.e. $-8sf(s) - 4s^2 f'(s) - 4F(o) + 2sf(s) - 2F(o) + f(s) = 0$

or $\quad 4s^2 f'(s) + (6s-1) f(s) = 0$ since $F(o) = \sin\sqrt{o} = 0$

or $\quad \dfrac{f'(s)}{f(s)} = \dfrac{1-6s}{4s^2} = \dfrac{1}{4s^2} - \dfrac{3}{2s}$

Integrating with regard to 's', we find

$$\log f(s) = \log C - \frac{1}{4s} - \frac{3}{2}\log s, \quad C \text{ being constant of integration}$$

$$= \log \frac{C}{s^{3/2}} e^{-\frac{1}{4s}}$$

i.e. $$f(s) = \frac{C e^{-\frac{1}{4s}}}{s^{3/2}} \qquad \text{...(4)}$$

Now to determine C, apply the limits of initial-value theorem *i.e.* when $t \to 0$, $s \to \infty$.

For t small, $\sin\sqrt{t} = \sqrt{t}$ so that $L\{\sqrt{t}\} = \dfrac{\Gamma 3/2}{s^{3/2}} = \dfrac{\sqrt{\pi}}{2s^{3/2}}$

and for s large, $f(s) = \dfrac{C}{s^{3/2}}$ as $\lim\limits_{s\to\infty} e^{-\frac{1}{4s}} = 1$

$\therefore \quad L\{F(t) = f(s)\}$ gives for $t \to 0$, $s \to \infty$,

$$\frac{\sqrt{\pi}}{2s^{3/2}} = \frac{C}{s^{3/2}} \quad i.e.\ C = \frac{\sqrt{\pi}}{2}.$$

Hence $$f(s) = \frac{\sqrt{\pi}\, e^{-\frac{1}{4s}}}{2s^{3/2}}. \qquad \text{...(5)}$$

[4] **Method of differentiation with respect to a parameter.** This method is based on differentiation of Laplace transform of a known function w.r.t. a parameter, *e.g.* if

$F(t) = t \sin at$ then we have to find $L\{t \sin at\}$

Consider $L\{\cos at\} = \displaystyle\int_0^\infty e^{-st} \cos at\, dt = \frac{s}{s^2+a^2}$

Differentiating w.r.t. a, we get

$$\frac{d}{da}\int_0^\infty e^{-st} \cos at\, dt = \frac{d}{da}\left(\frac{s}{s^2+a^2}\right)$$

i.e. $$\int_0^\infty e^{-st} (-t \sin at)\, dt = \frac{-2as}{(s^2+a^2)^2}$$

or $$+L\{t \sin at\} = \frac{+2as}{(s^2+a^2)^2} \qquad \text{...(6)}$$

.. .. THE LAPLACE TRANSFORMS OF SOME SPECIAL TIONS

function. Euler's Gamma function is defined as

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx, \quad n > 0.$$

Its important properties have already been discussed in chapter on Beta and Gamma functions, but a few of them are mentioned here.

$$\Gamma(n+1) = n\Gamma n \text{ for } n > 0 \text{ and } \Gamma n = \frac{\Gamma(n+1)}{n} \text{ for } n < 0$$

$\Gamma(n+1) = \lfloor n$ for positive integral values of n.

$$\Gamma n\ \Gamma(n-1) = \frac{\pi}{\sin n\pi},\ 0 < n < 1,\ \Gamma\tfrac{1}{2} = \sqrt{\pi}$$

For large n, $\Gamma(n+1) \approx \sqrt{2\pi n}\ n^n e^{-n}$ (Stirling formula)

Now we have $L\{t^n\} = \int_0^\infty e^{-st}\, t^n\, dt$

Put $st = u$ i.e. $t = \frac{u}{s}$ and $dt = \frac{du}{s}$

$$\therefore\ L\{t^n\} = \frac{1}{s^{n+1}}\int_0^\infty e^{-u}\ u^n\ du = \frac{\Gamma(n+1)}{s^{n+1}} \qquad \ldots(1)$$

(by the definition of Gamma function)

If we now put $n = -\frac{1}{2}$,

$$L\{t^{-1/2}\} = \frac{\Gamma\frac{1}{2}}{s^{1/2}} = \sqrt{\frac{\pi}{s}} \qquad \ldots(2)$$

[2] Bessel Functions. Bessel function of order n is defined as

$$J_n(t) = \frac{t}{2^n\Gamma(n+1)}\left\{1 - \frac{t^2}{2(2n+2)} + \frac{t^4}{2.4\ (2n+2)\ (2n+4)}\ldots\ldots\right\}$$

which satisfies Bessel's defferential equation

$$y''(t) + \frac{1}{t}\ y'(t) + \left(1 - \frac{n^2}{t^2}\right) y(t) = 0$$

or $$t^2 J_n''(t) + t\, J_n'(t) + (t^2 - n^2)\, J_n(t) = 0$$

Some important properties are:

$J_{-n}(t) = (-1)^n J_n(t)$, n being positive integral

$J_n(it) = i^{-n} J_n(t)$, J_n being modified Bessel function of order n.

$$J_{n+1}(t) = \frac{2n}{t}\ J_n(t) - J_{n-1}(t)$$

$$\frac{d}{dt}\ [t_n J_n(t)] = t^n J_{n-1}(t) \text{ which becomes } J'_0(t) = -J_1(t) \text{ for } n = 0$$

$$e^{-t\left(x-\frac{1}{x}\right)/2} = \sum_{n=-\infty}^{\infty} J_n(x)\ x^n$$

known as generating function for the Bessel functions.

$J_0(t)$ is called Bessel function of order zero and has for its expansion

$$J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2\cdot 4^2} - \frac{t^6}{2^2\cdot 4^2\cdot 6^2} + \ldots\ldots\ldots$$

$$\therefore\ L\{J_0(t)\} = L\{1\} - L\left\{\frac{t^2}{2^2}\right\} + L\left\{\frac{t^4}{2^2 \cdot 4^2}\right\}\ldots\ldots\ldots$$

$$= \frac{1}{s} - \frac{\lfloor 2}{2^2 s^3} + \frac{\lfloor 4}{2^2 \cdot 4^2\, s^5}\ldots\ldots\ldots$$

$$= \frac{1}{s}\left[1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1\cdot 3}{2\cdot 4}\left(\frac{1}{s^4}\right)\ldots\ldots\ldots\right]$$

$$= \frac{1}{s}\left(1 + \frac{1}{s}\right)^{-1/2} = \frac{1}{\sqrt{s^2+1}} \qquad \ldots(3)$$

Similarly $L\{J_1(t)\} = 1 - s/\sqrt{s^2+1}$ ...(4)

Aliter. $J_0(t)$ satisfies the equation

$$t' J''_0(t) + J_0'(t) + t\,J_0(t) = 0$$

$$\therefore\ L\{tJ_0''(t)\} + L(J_0'(t)\} + L\{tJ_0(t)\} = 0$$

Taking $L\{J_0(t)\} = f(s)$ and using properties [*E*] and [*F*] of §10.9.

$$-\frac{d}{ds}\left\{s^2 f(s) - sF(o) - F'(o)\right\} + \{sf(s) - F(o)\} - \frac{d}{ds} f(s) = 0$$

where $F(t) = J_0(t)$ gives $F(o) = 1$ and $F'(o) = 0$

$$\therefore\ -2s\,f(s) - s^2 f'(s) + 1 + s\,f(s) - 1 - f'(s) = 0$$

or $\quad sf(s) + (s^2+1) f'(s) = 0$

i.e. $\quad \dfrac{f'(s)}{f(s)} = \dfrac{-s}{s^2+1} = -\dfrac{1}{2}\cdot\dfrac{2s}{s^2+1}.$

Integrating with regard to '*s*'

$\log f(s) = -\frac{1}{2}\log(s^2+1) + \log C$, C being constant of integration

$$\therefore\ f(s) = \frac{C}{\sqrt{s^2+1}}$$

Applying initial value theorem, we have

$$\lim_{s\to\infty} sf(s) = \lim_{t\to\infty} F(t)$$

i.e. $\displaystyle\lim_{s\to\infty}\frac{sC}{\sqrt{s^2+1}} = \lim_{t\to 0} J_0(t)$ which gives $C = 1$

Hence $f(s) = \dfrac{1}{\sqrt{s^2+1}}$ *i.e.* $L\{J_0(t)\} = \dfrac{1}{\sqrt{s^2+1}}$

Now using the change of scalar property, we have

$$L\{J_0(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right) \text{ where } f(s) = L\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$$

so that $f\left(\dfrac{s}{a}\right) = \dfrac{1}{\sqrt{\left(\dfrac{s}{a}\right)^2 + 1}} = \dfrac{1}{\sqrt{s^2+a^2}}$

Hence $\quad L\{J_0(at)\} = \dfrac{1}{\sqrt{s^2+a^2}}$ ...(5)

Further to deduce $L\{t J_0(at)\}$, using the property [F] of §10.9, we get

$$L\{t J_0(at)\} = -\frac{d}{ds}\left[L\{J_0(at)\}\right] = -\frac{d}{ds}\left(\frac{1}{\sqrt{s^2+a^2}}\right) = \frac{s}{\left(s^2+a^2\right)^{3/2}} \quad \ldots(6)$$

Similarly $L\{t J_1(t)\} = \dfrac{1}{\left(s^2+1\right)^{3/2}}$ ...(7)

[3] The error function and its complement

The *error function* of a variable t denoted by erf (t) or $E_r(t)$ is defined as

$$\text{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2}\,dx = E_r(t)$$

and the *complement of the error function* denoted by erf $C(t)$ is defined by

$$\text{erf}\,C(t) = 1 - \text{erf}(t) = 1 - \frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2}\,dx = \frac{2}{\sqrt{\pi}}\int_t^\infty e^{-x^2}\,dx.$$

It is notable that

$$\lim_{t\to 0} \text{erf}(t) = 0 \text{ and } \lim_{t\to\infty} \text{erf}(t) = 1.$$

$$\text{Thus erf}\sqrt{t} = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}\,dx = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} \left\{1 - x^2 + \frac{x^4}{\underline{\lvert 2}} - \frac{x^6}{\underline{\lvert 3}} + \ldots\right\} dx$$

$$= \frac{2}{\sqrt{\pi}}\left[t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5\,\underline{\lvert 2}} - \frac{t^{7/2}}{7\,\underline{\lvert 3}} + \ldots\right]$$

$$\therefore\ L\{\text{erf}\sqrt{t}\} = \frac{2}{\sqrt{\pi}} L\left\{t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5\,\underline{\lvert 2}} - \frac{t^{7/2}}{7\,\underline{\lvert 3}} + \ldots\right\}$$

$$= \frac{2}{\sqrt{\pi}}\left[\frac{\Gamma 3/2}{s^{3/2}} - \frac{\Gamma 5/2}{s^{5/2}} + \frac{\Gamma 7/2}{s^{7/2}} - \frac{\Gamma 9/2}{s^{9/2}} + \ldots\right]$$

$$= \frac{1}{s^{3/2}} - \frac{1}{2}\cdot\frac{1}{s^{5/2}} + \frac{1\cdot 3}{2\cdot 4}\,\frac{1}{s^{7/2}} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\,\frac{1}{s^{9/2}} + \ldots$$

$$= \frac{1}{s^{3/2}}\left(1 + \frac{1}{s}\right)^{-1/2} = \frac{1}{s\sqrt{s+1}} \quad \ldots(8)$$

[4] The sine, cosine and exponential integrals

The sine integral is defined as $S_i(t) = \displaystyle\int_0^t \frac{\sin x}{x}\,dx$

and the cosine integral is defined as $C_i(t) = \displaystyle\int_0^t \frac{\cos x}{x}\,dx$

Also the exponential integral is defined as $E_i(t) = \displaystyle\int_t^\infty \frac{e^{-x}}{x}\,dx$

$$\text{We have } S_i(t) = \int_0^t \frac{\sin x}{x}\,dx = \int_0^t \frac{1}{x}\left(x - \frac{x^3}{\underline{\lvert 3}} + \frac{x^5}{\underline{\lvert 5}} - \frac{x^7}{\underline{\lvert 7}} + \ldots\right) dx$$

$$= \int_0^t \left\{1 - \frac{x^2}{\underline{\lvert 3}} + \frac{x^4}{\underline{\lvert 5}} - \frac{x^6}{\underline{\lvert 7}} + \ldots\right\} dx$$

$$= t - \frac{t^2}{3\lfloor 3} + \frac{t^5}{5\lfloor 5} - \frac{t^7}{7\lfloor 7} + \ldots$$

$$\therefore\ L\{S_i(t)\} = L\left\{t - \frac{t^3}{3\lfloor 3} + \frac{t^5}{5\lfloor 5} - \frac{t^7}{7\lfloor 7} + \ldots\right\}$$

$$= \frac{1}{s^2} - \frac{3}{3\lfloor 3}\cdot\frac{\lfloor 3}{s^4} + \frac{1}{5\lfloor 5}\cdot\frac{\lfloor 5}{s^6} - \frac{1}{7\lfloor 7}\cdot\frac{\lfloor 7}{s^8} + \ldots$$

$$= \frac{1}{s}\left[\frac{1/s}{1} - \frac{(1/s)^3}{3} + \frac{(1/s)^5}{5} - \frac{(1/s)^7}{7} + \ldots\right]$$

$$= \frac{1}{s}\tan^{-1}\frac{1}{s} \qquad \ldots(9)$$

Similarly $L\{C_i(t)\} = \frac{1}{2s}\log\left(s^2+1\right)$ ...(10)

and $L\{E_i(t)\} = L\left\{\int_t^\infty \frac{e^{-x}}{x}\,dx\right.$; Put $x = ty$ i.e. $\frac{dx}{x} = \frac{dy}{y}$

(on logarithmic differentiation)

$$= L\int_1^\infty \frac{e^{-ty}}{y}\,dy$$

$$= L\int_0^\infty e^{-st}\left\{\int_1^\infty \frac{e^{-ty}}{y}\,dy\right\}dt \text{ by definition of Laplace transform}$$

$$= \int_1^\infty \frac{1}{y}\left\{\int_0^\infty e^{-(s+y)\,t}\,dt\right\}dy, \text{ by changing the order of integration}$$

$$= \int_1^\infty \frac{1}{y}\cdot\frac{1}{s+y}\,dy$$

$$= \int_1^\infty \frac{1}{s}\left[\frac{1}{y} - \frac{1}{s+y}\right]dy = \frac{1}{s}\left[\log y - \log(s+y)\right]_1^\infty$$

$$= \frac{1}{s}\left[-\log\left(\frac{s}{y}+1\right)\right]_1^\infty$$

$$= \frac{1}{s}\log(s+1) \qquad \ldots(11)$$

[5] Heaviside unit function or unit step function

Unit step function is defined as

$$U(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a. \end{cases}$$

$$\therefore\ L\{U(t-a)\} = \int_0^\infty e^{-st}\,U(t-a)\,dt$$

$$= \int_0^a e^{-st}\cdot 0\,dt + \int_a^\infty e^{-st}\cdot 1\cdot dt = \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$= \frac{1}{s}e^{-as} \qquad \ldots(12)$$

[6] Dirac-delta function or unit impulse function

A function $\delta(t)$ such that

$$\lim_{\varepsilon \to 0} F_\delta(t) = \delta(t) \text{ where } F_\delta(t) = \begin{cases} 1/\epsilon, & 0 \le t \le \epsilon \\ 0, & t > \epsilon \end{cases}$$

and $\int_0^\infty F_\delta(t)\, dt = 1$

With the properties:

(*i*) (*a*) $\int_0^\infty \delta(t)\, dt = 1$

(*ii*) $\int_0^\infty \delta(t)\, G(t) = G(0)$ for any continuous function $G(t)$

(*iii*) $\int_0^\infty \delta(t-a)\, G(t)\, dt = G(a)$ for any continuous function $G(t)$ is called as unit impulse function or Dirac-delta function.

We have $L\{\delta(t)\} = \int_0^\infty e^{-st} \lim_{\varepsilon \to 0} F_\delta(t)\, dt$ which is not defined since $\lim_{\varepsilon \to 0} F_\delta(t)$ does not exist and hence it is useful to consider $\delta(t) = \lim_{\varepsilon \to 0} F_\delta(t)$ to be such that $L\{\delta(t)\} = 1$. ...(13)

$$\text{Also } L\{F_\delta(t)\} = \int_0^\infty e^{-st} F_\delta(t)\, dt = \int_0^\varepsilon e^{-st} \cdot \frac{1}{\epsilon}\, dt + \int_\varepsilon^\infty e^{-st} \cdot 0 \cdot dt$$

$$= \frac{1}{\epsilon} \int_0^\varepsilon e^{-st}\, dt = \frac{1 - e^{-\varepsilon s}}{\epsilon s} \quad \text{...(14)}$$

[7] **Null functions.** A null function $N(t)$ for all $t > 0$ is defined as

$$\int_0^t N(x)\, dx = 0$$

e.g. if $F(t) = \begin{cases} 1, & t = 1 \\ 0, & \text{otherwise,} \end{cases}$ then it is a null function, since $\int_0^t F(x)\, dx = 0$ for all $t > 0$. As such the Laplace transform of a null function is zero *i.e.* $L\{N(t)\} = 0$ ...(15)

10.12. EVALUATION OF INTEGRALS WITH THE HELP OF LAPLACE TRANSFORMS

By definiton, we have

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt = f(s) \quad \text{...(1)}$$

Assuming that the integral is convergent and proceeding to the limit $s \to 0$, this reduces to

$$\int_0^\infty F(t)\, dt = f(0) \quad \text{...(2)}$$

(1) and (2) are sometimes used to evaluate integrals.

Problen 24. *Evaluate the following integrals*

(*a*) $\int_0^\infty t^2 e^{-t} \sin t\, dt$

(*b*) $\int_0^\infty J_0(t)\, dt$

(c) $\int_0^\infty e^{-t} \text{ erf } \sqrt{t} \, dt.$

(a) We have $L\{\sin t\} = \dfrac{1}{s^2+1}$

$\therefore \; L\{t^2 \sin t\} = (-1)^2 \dfrac{d^2}{ds^2} L\{\sin t\}$ by Prop. [F] of §10.9

$$= \frac{d^2}{ds^2}\left(\frac{1}{s^2+1}\right) = \frac{d}{dx}\left(\frac{-2s}{(s^2+1)^2}\right) = -\frac{2(1-s^2)}{(1+s^2)^3}$$

So that $\int_0^\infty e^{-st} \cdot t^2 \sin t \, dt = -\dfrac{2(1-s^2)}{(1+s^2)^3}$.

Putting $s = 1$, we find $\int_0^\infty e^{-t} t^2 \sin t \, dt = 0$

(b) We have $L\{J_0(t)\} = \dfrac{1}{\sqrt{s^2+1}}$ by (3) of §10.11

i.e. $\int_0^\infty e^{-st} J_0(t) \, dt = \dfrac{1}{\sqrt{s^2+1}}$

Proceeding to the limit as $s \to 0$, we get

$$\int_0^\infty J_0(t) \, dt = 1.$$

(c) We have, $L\left\{\text{erf } \sqrt{t}\right\}$ by (8) of §10.11

i.e. $\int_0^\infty e^{-st} \text{ erf } \sqrt{t} \, dt = \dfrac{1}{s\sqrt{s+1}}$.

Proceeding to the limit $s \to 1$, we find

$$\int_0^\infty e^{-x} \text{ erf } \sqrt{t} \, dt = \frac{1}{\sqrt{2}}.$$

10.13. LAPLACE TRANSFORM OF THE LAPLACE TRANSFORM

We have $L\{F(t)\} = \int_0^\infty e^{-st} F(t) \, dt = f(s)$

$\therefore \; L[L\{F(t)\}] = L\left\{\int_0^\infty e^{-st} F(t) \, dt\right\}$

i.e. $L\{f(s)\} = \int_0^\infty e^{-ps} \, ds \int_0^\infty e^{-st} F(t) \, dt$ ...(1)

Here the area of integration being the whole positive quadrant, we can change the order of integration and hence we get

$$L[L\{F(t)\}] = \int_0^\infty F(t) \, dt \int_0^\infty e^{-s(t+p)} \, ds$$

$$= \int_0^\infty F(t) \, dt . \left[\frac{e^{-s(t+p)}}{t+p}\right]_0^\infty$$

$$= \int_0^{\infty} \frac{F(t)}{t+p} \qquad ...(2)$$

10.14. THE INVERSE LAPLACE TRANSFORM

If $f(s)$ be the Lapalce transform of a function $F(t)$ *i.e.* $L\{F(t)\} = f(s)$ then $F(t)$ is said to be an *Inverse Laplace transform* of $f(s)$ and written as

$$F(t) = L^{-1}\{f(s)\}] \qquad ...(1)$$

Here L^{-1} is known as the *Inverse Laplace transformation operator*

e.g. if $L\{e^{2t}\} = \frac{1}{s-2}$ then $e^{2t} = L^{-1}\left\{\frac{1}{s-2}\right\}$.

Uriqueness of Inverse Laplace transform.

If $N(t)$ be a null function then $L\{N(t)\} = 0$

and $L\{F(t)\} = f(s)$

$\therefore \quad L\{F(t) + N(t)\} = L\{F(t)\} + L\{N(t)\}$
$= f(s)$

So that $L^{-1}\{f(s)\} = F(t)$

and also $L^{-1}\{f(s)\} = F(t) + N(t)$.

Showing that we can have two different functions with the same Laplace transform and hence the inverse Laplace transform of a function is not unique. It is, however, unique if we do not allow null function which in general, does not arise in cases of physical interest. This result is known as *Lerch's theorem* which states:

If $F(t)$ is sectionally continuous function in every finite interval $0 \le t \le N$ and is of exponential order for $t < N$ such that $L\{F(t)\} = f(s)$ then the inverse Laplace transform of $f(s)$ i.e. $L^{-1}\{f(s)\} = F(t)$ is unique.

Conclusively, a given function $f(s)$ cannot have more than one inverse transform $F(t)$ that is continuous for each positive t. But a function $f(s)$ may not have a continuous inverse transform *e.g.* the Laplace transform of $s^{-1} e^{-ks}$, is the step function which is discontinuous.

Note. *Some inverse transforms can be tabulated as follows:*

$f(s)$	$L^{-1}\{f(s)\} = F(t)$	$f(s)$	$L^{-1}\{f(s)\} = F(t)$
$\frac{1}{s}$, $s > 0$	1	$2s/(s^2+a^2)^2$	$t \sin at/a$
$\frac{1}{s^2}$, $s > 0$......	t	$(s^2-a^2)/(s^2+a^2)$	$t \cos at$
$\frac{1}{s^{n+1}}$, $n = 0, 1, 2, 3, ...$	$t^n/\lfloor n$	$1/\{(s-a)^2+b^2\}$	$e^{at} \sin bt/b$
$1/(s-a)$......	e^{at}	$(s-a)/\{(s-a)^2+b^2\}$	$e^{at} \cos bt$
$1/(s^2+a^2)$, $s > 0$	$\frac{\sin at}{a}$	$1/\sqrt{s^2+a^2}$	$J_0(at)$
$s/(s^2+a^2)$, $s > 0$......	$\cos at$	$s/(s^2+a^2)^{3/2}$	$t J_0(at)$
$1/(s^2-a^2)$, $s > \lvert a \rvert$	$\frac{\sinh at}{a}$	$1/(s-a)^n$	$\frac{t^{n-1} e^{at}}{\lfloor n}$, $n > 0$
$s/(s^2-a^2)$, $s > \lvert a \rvert$	$\cosh at$		

10.15. SOME PROPERTIES OF INVERSE LAPLACE TRANSFORM

[A] Linearity Property

If the Laplace transform of $F_1(t)$ and $F_2(t)$ be $f_1(s)$ and $f_2(s)$ respectively and C_1, C_2 are constants then

$$L^{-1}\{C_1 f_1(s) + C_2 f_2(s)\} = C_1 L^{-1}\{f_1(s)\} + C_2 L^{-1}\{f_2(s)\} = C_1 F_1(t) + C_2 F_2(t) \quad \ldots (1)$$

By Prop. [*A*] of §10.9, we have

$$L\{C_1 F_1(t) + C_2 F_2(t)\} = C_1 L\{F_1(t)\} + C_2 L\{F_2(t)\} = C_1 f_1(s) + C_2 f_2(s)$$

$$\therefore \; C_1 F_1(t) + C_2 F_2(t) = L^{-1}\{C_1 f_1(s) + C_2 f_2(s)\}$$

i.e $L^{-1}\{C_1 f_1(s) + C_2 f_2(s)\} = C_1 F_1(t) + C_2 F_2(t) = C_1 L^{-1}\{f_1(s)\} + C_2 L^{-1}\{f_2(s)\}$

Problem 25. *Find* $L^{-1}\left\{\dfrac{2(s-a)}{(s-a)^2+b^2} + \dfrac{8-6s}{16s^2+9} + \dfrac{24-30\sqrt{s}}{s^4}\right\}$

$$L^{-1}\left\{\frac{2(s-a)}{(s-a)^2+b^2} + \frac{8-6s}{16s^2+9} + \frac{24-30\sqrt{s}}{s^4}\right\} = L^{-1}\left\{\frac{2(s-a)}{(s-a)^2+s^2}\right\}$$

$$+ L^{-1}\left\{\frac{8-6s}{16s^2+9}\right\} + L^{-1}\left\{\frac{24-30\sqrt{s}}{s^4}\right\}$$

$$= 2L^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{s^2+9/16}\right\} - \frac{3}{8}L^{-1}\left\{\frac{s}{s^2+9/16}\right\}$$

$$+24L^{-1}\left\{\frac{1}{s^4}\right\} - 30L^{-1}\left\{\frac{1}{s^{7/2}}\right\}$$

$$= 2e^{at}\cos bt + \frac{1}{2}\sin\frac{3t}{4} - \frac{3}{8}\cos\frac{3t}{4} + 24\frac{t^3}{\lfloor 3} - 30\frac{t^{5/2}}{\Gamma 7/2}$$

$$= 2e^{at}\cos bt + \frac{1}{2}\sin\frac{3t}{4} - \frac{3}{8}\cos\frac{3t}{4} + 4t^3 - \frac{16}{\sqrt{\pi}}t^{5/2}$$

[B] First Translation (or Shifting) Property

If $L^{-1}\{f(s)\} = F(t)$, *then* $L^{-1}\{f(s-a)\} = e^{at}F(t)$.

We have, $f(s) = \int_0^\infty e^{-st} F(t)\, dt = L\{F(t)\}$

$$\therefore f(s-a) = \int_0^\infty e^{-(s-a)t} F(t)\, dt = \int_0^\infty e^{-st}.e^{at} F(t)\, dt$$

$$= L\{e^{at}F(t)\}$$

i.e. $L^{-1}\{f(s-a)\} = e^{at}F(t)$ ...(2)

(2) may also be expressed as

$$L^{-1}\{f(s)\} = e^{-at}L^{-1}\{f(s-a)\} \quad \ldots (3)$$

Which follows that the substitution of $s-a$ for s in the transform corresponds to the multiplication of the object function $F(t)$ by the function e^{at}.

We may illustrate it as follows:

$\therefore \; \dfrac{\lfloor m}{s^{m+1}} = L\{t^m\}$, where $m = 1, 2, \ldots$ and $s > 0$

$$\therefore \quad \frac{\lfloor m}{(s-a)^{m+1}} = L\{t^m e^{at}\}, s > a$$

Also, $\because \; L\{\cos bt\} = \dfrac{s}{s^2+b^2}$

$$\therefore \; L\{e^{-at}\cos bt\} = \frac{s+a}{(a+a)^2+b^2}, \quad s > -a.$$

Problem 26. *Find* (*a*) $L^{-1}\left\{\dfrac{1}{\sqrt{2s+5}}\right\}$ (*b*) $L^{-1}\left\{\dfrac{s+1}{s^2+6s+25}\right\}$

(*a*) We have $L^{-1}\left\{\dfrac{1}{\sqrt{2s+5}}\right\} = \dfrac{2}{\sqrt{2}} L^{-1} \cdot \dfrac{1}{(s+5/2)^{1/2}}$

$$= \frac{1}{\sqrt{2}} e^{-5t/2} \cdot \frac{t^{-1/2}}{\Gamma(1/2)} = \frac{1}{\sqrt{2\pi}} t^{-1/2} e^{st/2}$$

(*b*) We have $L^{-1}\left\{\dfrac{s+1}{s^2+6s+25}\right\} = L^{-1}\left\{\dfrac{s+3-2}{(s+3)^2+4^2}\right\}$

$$= L^{-1}\left\{\frac{s+3}{(s+3)^2+4^2}\right\} - 2L^{-1}\left\{\frac{1}{(s+3)^2+4^2}\right\}$$

$$= e^{-3t}\cos 4t - 2e^{3t}\frac{\sin 4t}{4}$$

$$= e^{-3t}\left[\cos 4t - \frac{1}{2}\sin 4t\right].$$

[C] Second Translation (or Shifting) Property

If $L^{-1}\{f(s)\} = F(t)$, *then* $L^{-1}\{e^{-as}f(s)\} = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$ ...(4)

Let $e^{-as}f(s) = L\{G(t)\}$, then $L^{-1}\{e^{-as}f(s)\} = G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$

So that $L\{G(t)\} = \int_0^\infty e^{-st}G(t)\,dt = \int_0^a e^{-st}\cdot 0\cdot dt + \int_a^\infty e^{-st}\cdot F(t-a)\,dt$

$= \int_0^\infty e^{-st}F(t-a)\,dt$ Put $t-a=u$ so that $dt = du$

$$= \int_0^\infty e^{-s(u+a)}F(u)\,du = e^{-as}\int_0^\infty e^{-su}F(u)\,du$$

$= e^{-as}\int_0^\infty e^{-st}F(t)\,dt$, on replacing u by t.

$$= e^{-as}L\{F(t)\} = e^{-as}f(s)$$

$\therefore \quad G(t) = L^{-1}\{e^{-as}f(s)\}$

Note. In terms of Heaviside unit step function $G(t)$ can be expressed as $F(t-a).U(t-a)$.

Problem 27. *Find* $L^{-1}\left\{\dfrac{s\,e^{-2\pi s/3}}{s^2+9}\right\}$

$$\because \quad L^{-1}\left\{\frac{s}{s^2+9}\right\}=\cos 3t$$

$$\therefore \quad L^{-1}\left\{\frac{s\,e^{-2\pi s/3}}{s^2+9}\right\}=\begin{cases}\cos 3\left(t-\dfrac{2\pi}{3}\right), & t>\dfrac{2\pi}{3}\\ 0, & t<\dfrac{2\pi}{3}\end{cases}$$

$$\doteq\begin{cases}\cos 3t, & t>\dfrac{2\pi}{3}\\ 0, & t<\dfrac{2\pi}{3}\end{cases}$$

$$=\cos 3t.\ U\left(t-\frac{2\pi}{3}\right)$$

[D] Change of Scale Property

If $L^{-1}\{f(s)\}=F(t)$, *then* $L^{-1}\{f(as)\}=\dfrac{1}{a}F\left(\dfrac{t}{a}\right)\cdot\ a>0$ ...(5)

$$\because \ f(s)=\int_0^\infty e^{-st}F(t)\,dt, \qquad \therefore \ f(as)=\int_0^\infty e^{-ast}F(t)\,dt$$

Put $at=u$ so that $dt=\dfrac{du}{a}$

$$\therefore \quad f(as)=\int_0^\infty e^{-us}F\left(\frac{u}{a}\right)\cdot\frac{du}{a}=\frac{1}{a}\int_0^\infty e^{-su}F\left(\frac{u}{a}\right)du$$

$$=\frac{1}{a}\int_0^\infty e^{-st}F\left(\frac{t}{a}\right)dt \text{ (on replacing } u \text{ by } t)$$

$$=\frac{1}{a}L\left\{F\left(\frac{t}{a}\right)\right\}=L\left\{\frac{1}{a}F\left(\frac{t}{a}\right)\right\}$$

$$\therefore \quad L^{-1}\{f(as)\}=\frac{1}{a}F\left(\frac{t}{a}\right).$$

Problem 28. *If* $L^{-1}\left\{\dfrac{e^{-1/s}}{s^{1/2}}\right\}=\dfrac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$, *find* $L^{-1}\left\{\dfrac{e^{-a/s}}{s^{1/2}}\right\}$, $a>0$

$$\because \ L^{-1}\left\{\frac{e^{-1/s}}{s^{1/2}}\right\}=\frac{\cos 2\sqrt{t}}{\sqrt{\pi t}}, \quad \therefore \ L^{-1}\left\{\frac{e^{-1/cs}}{(cs)^{1/2}}\right\}=\frac{1}{c}\cdot\frac{\cos 2\sqrt{t/c}}{\sqrt{\pi t/c}}$$

or
$$\frac{1}{\sqrt{c}}L^{-1}\left\{\frac{e^{-1/cs}}{s^{1/2}}\right\}=\frac{1}{\sqrt{c}}\frac{\cos 2\sqrt{t/c}}{\sqrt{\pi t}}$$

Putting $c=\dfrac{1}{a}$, we find

$$L^{-1}\left\{\frac{e^{-a/s}}{s^{1/2}}\right\}=\frac{\cos 2\sqrt{at}}{\sqrt{\pi t}}.$$

[E] Inverse Laplace Transform of Derivatives

If $L^{-1}\{f(s)\} = F(t)$, *then* $L^{-1}\{f^{(n)}(s)\} = (-1)^n \left\{\dfrac{d^n}{ds^n} f(s)\right\}$

$$= (-1)^n t^n F(t) \qquad \ldots(6)$$

By Prop. [*F*] of §10.9, we have

$L\{t^{(n)} F(t)\} = (-1)^n f^{(n)}(s)$, $\therefore$ $t^n F(t) = L^{-1}\{(-1)^n f^{(n)}(s)\} = (-1)^n L^{-1}\{f^{(n)}(s)\}$

or $L^{-1}\{f^{(n)}(s)\} = (-1)^n t^n F(t) = (-1)^{-n} \cdot \dfrac{(-1)^{2n}}{(-1)^{2n}} F(t) \because (-1)^{2n} = 1$

$$= (-1)^n t^n F(t) = (-1)^n \left\{\frac{d^n}{ds^n} f(s)\right\}$$

Problem 29. *Find* $L^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\}$

Consider $\dfrac{d}{ds}\left\{\dfrac{1}{s^2+a^2}\right\} = \dfrac{-2s}{(s^2+a^2)^2}$ *i.e.* $\dfrac{s}{(s^2+a^2)^2} = -\dfrac{1}{2}\dfrac{d}{ds}\left(\dfrac{1}{s^2+a^2}\right)$

$$\therefore \quad L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = -\tfrac{1}{2} L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+a^2}\right)\right\}$$

$$= \frac{t \sin at}{2a} \quad \text{since } L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{\sin at}{a}$$

[F] Multiplication by s^n

If $L^{-1}\{f(s)\} = F(t)$ and $F(0) = 0$, then $L^{-1}\{sf(s)\} = F'(t)$ $\ldots(7)$

By Prop. [*E*] of §10.9, $L\{F'(t)\} = sf(s) - F(0) = sf(s) \because F(o) = 0$

$\therefore F'(t) = L^{-1}\{sf(s)\}$.

In case $F(0) \neq 0$, then $L^{-1}\{sf(s) - F(0)\} = F'(t)$

or $L^{-1}\{sf(s)\} = F'(t) + L^{-1}\{F(0)\} = F'(t) + F(0)\delta(t)$

where $\delta(t)$ is the Dirac-delta function.

Problem 30. *Find* $L^{-1}\left\{\dfrac{1}{(s^2+1)^2}\right\}$ *when* $L^{-1}\left\{\dfrac{s}{(s^2+1)^2}\right\} = \dfrac{1}{2} t \sin t$.

We have, $L^{-1}\left\{s \cdot \dfrac{s}{(s^2+1)^2}\right\} = \dfrac{d}{dt}\left(\dfrac{1}{2} t \sin t\right)$

or $L^{-1}\left\{\dfrac{s^2+1-1}{(s^2+1)^2}\right\} = \dfrac{1}{2}(t \cos t + \sin t)$

or $L^{-1}\left\{\dfrac{1}{(s^2+1)}\right\} - L^{-1}\left\{\dfrac{1}{(s^2+1)^2}\right\} = \dfrac{1}{2}(t \cos t + \sin t)$

or $L^{-1}\left\{\dfrac{1}{(s^2+1)^2}\right\} = L^{-1}\left\{\dfrac{1}{(s^2+1)}\right\} - \dfrac{1}{2}(t \cos t + \sin t)$

$$= \sin t - \tfrac{1}{2}(t \cos t + \sin t) = \tfrac{1}{2}(\sin t - t \cos t)$$

[G] Division by Powers of s

If $L^{-1}\{f(s)\} = F(t)$, then (*i*) $L^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t F(u)\, du$ ...(8)

(*ii*) $L^{-1}\left\{\frac{f(s)}{s^2}\right\} = \int_0^t \int_0^v F(u)\, du\, dv$ ...(9)

(*i*) Let $G(t) = \int_0^t F(u)\, du = L^{-1}\left\{\frac{f(s)}{s}\right\}$, then $G'(t) = F(t),\ G(o) = 0$

$\therefore L\{G'(t)\} = sL\{G(t)\} - G(0) = sL\{G(t)\}$

$= f(s) \because L^{-1}\left\{\frac{f(s)}{s}\right\} = G(t)$ yields $sL\{G(t)\} = f(s)$.

Hence using the Prop. [*G*] of §10.9,

$L\{G(t)\} = \frac{f(s)}{s}$ or $L^{-1}\left\{\frac{f(s)}{s}\right\} = G(t) = \int_0^t F(u)\, du.$

(*ii*) Let $G(t) = \int_0^t \int_0^v F(u)\, du\, dv = L^{-1}\left\{\frac{f(s)}{s^2}\right\}$, then $s^2 L\{G(t)\} = f(s)$

Also $G'(t) = \int_0^t F(u)\, du$, $G''(t) = F(t)$ and $G(o) = G'(o) = 0$

so that $L\{G''(t)\} = s^2 L\{G(t)\} - sG(o) - G'(o) = s^2 L\{G(t)\} = f(s)$

$\therefore L\{G(t)\} = \frac{f(s)}{s^2}$ or $L^{-1}\left\{\frac{f(s)}{s^2}\right\} = G(t) = \int_0^t \int_0^v F(u)\, du\, dv$

This may be written as $L^{-1}\left\{\frac{f(s)}{s^2}\right\} = \int_0^t \int_0^t F(t)\, (dt)^2$ ...(10)

or in general $L^{-1}\left\{\frac{f(s)}{s^n}\right\} = \int_0^t \int_0^t \cdots \int_0^t F(t) \cdot (dt)^n$ ...(11)

Problem 31. *Find* $L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\}$

$\because L^{-1}\left\{\frac{1}{(s^2+1)}\right\} = \sin t,\ \therefore L^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \int_0^t \sin u\, du = 1 - \cos t$

and hence $L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \int_0^t (1 - \cos u)\, du = t - \sin t.$

[H] The Convolution Theorem or the Convolution Property

If $L^{-1}\{f(s)\} = F(t)$ *and* $L^{-1}\{g(s)\} = G(t)$, *then*

$$L^{-1}\{f(s)\, g(s)\} = \int_0^t F(u)\, G(t-u)\, du = F^* G \quad \text{...(12)}$$

where F^*G *is known as* **Convolution** *or* **Faltung** *of F and G and this convolution is commutative i.e.* $F^* G = G^* F$.

The result (12) follows from $f(s)\, g(s) = L\left\{\int_0^t F(u)\, G(t-u)\, du \ldots\right\}$ ...(13)

which may be proved as below:

$\because f(s) = L\{F(t)\}$ and $g(s) = L\{G(t)\}$

$\therefore f(s)\cdot g(s) = L\{F(t)\}\cdot L\{G(t)\}$

$$= \int_0^\infty e^{-sw} F(w)\,dw \int_0^\infty e^{-sv} G(v)\,dv$$

$$= \int_0^\infty \int_0^\infty e^{-s(w+v)} F(w)\,G(v)\,dw\,dv$$

$$= \iint_R e^{-s(w+v)} F(w)\,G(v)\,dw\,dv \qquad \ldots(14)$$

where R represents the positive quadrant of the plane over which the double integration extends.

In order to transform the double integration from the region $R(w, v)$ to another region $S(x, y)$, let us make the substitutions

$$w = y,\ v = x - y$$

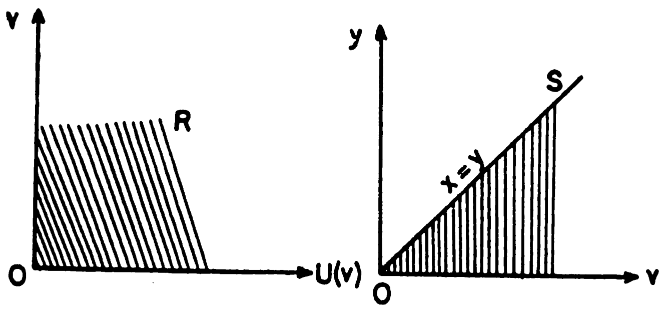


Fig. 10.2

whose mappings are shown in Fig. 10.2

When $w = 0, y = 0, w = \infty, y = \infty$

and when $v = 0, x - y = 0, v = \infty, (x - y) = \infty$

The Jacobian of the transformation is

$$\frac{\partial(w, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1 \text{ giving } dw\,dv = dx\,dy.$$

Thus (14) transforms to

$$f(s)\,g(s) = \iint_S e^{-sx} F(y)\,G(x-y)\,dx\,dy$$

$$= \int_0^\infty e^{-sx}\left[\int_0^x F(y)\,G(x-y)\,dy\right]dx$$

$$= \int_0^\infty e^{-st} \left[\int_0^t F(v)\ G(t-v)\ du\right] dt$$ (on replacing the variables y by u and x by t).

$$= L\left\{\int_0^t F(v)\ G(t-v)\ du\right\} = L\{F^* G\}$$

Hence $L^{-1}\{f(s)\ g(s)\} = \int_0^t F(u)\ G(t-u)\ du = F^* G$

Now to show that the operation $F^* G$ is commutative, we have

$$F^* G = \int_0^t F(u)\ G(t-u)\ du = \int_0^t F(t-z)\ G(z)\ dz, \text{ when } t-u=z$$

$$= \int_0^t G(z)\ F(t-z)\ dz = G^* F. \qquad ...(15)$$

Thus this theorem may be restated as

If $f(s)$ and $g(s)$ are the transforms of two functions $F(t)$ and $G(t)$ that are sectionally continuous on each interval $0 \le t \le u$, then the transform of the convolution $F(t)^* G(t)$ exists and it is $f(s)\ g(s)$.

e.g. if $F(t) = t$ and $G(t) = e^{at}$ satisfy these conditions, then

$$L^{-1}\left\{\frac{1}{s^2}\cdot\frac{1}{s-a}\right\} = t^* e^{at} = \int_0^t u e^{a(t-u)}\ du = e^{at}\int_0^t u e^{-au}\ du$$

$$= \frac{1}{a^2}(e^{at} - at - 1).$$

COROLLARY. *If $f(s) = L\{F(t)\}$, then* $\int_0^t\int_0^v F(u)\ du\, dv = \int_0^t (t-u) F(u)\ du$ ...(16)

By convolution theorem,

$$L\left\{\int_0^t (t-u)\ F(u)\ dv\right\} = L(t)\ L\{F(t)\} = \frac{f(s)}{s^2}$$

$$\because\ L(t) = \frac{1}{s^2} \text{ and } L\{F(t)\} = f(s)$$

Hence by (9),

$$\int_0^t (t-u)\ F(u)\ du = L^{-1}\left\{\frac{f(s)}{s^2}\right\} = \int_0^t\int_0^v F(v)\ du\, dv$$

which may also be expressed as

$$\int_0^t\int_0^t F(t)\ (dt)^2 = \int_0^t (t-u)\ F(u)\ du \qquad ...(17)$$

or in general $\int_0^t\int_0^t ....\int_0^t F(t)\ (dt)^n = \int_0^t \frac{(t-u)^{n-1}}{\underline{|n-1}} F(u)\ du$ ...(18)

Problem 32. *Using the convolution theorem, evaluate*

$$L^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\}$$

$$\because\ L^{-1}\left\{\frac{1}{s^2}\right\} = t \text{ and } L^{-1}\left\{\frac{1}{(s+1)^2}\right\} = te^{-t}$$

$$\because L^{-1}\left\{\frac{1}{s^2 (s+1)^2}\right\}=\int_0^t (ue^{-u})(t-u)\,du = \int_0^t (ut-u^2)\,e^{-u}\,du$$

$$=\left[-(ut-u^2)\,e^{-u}\right]_0^t+\int_0^t (t-2u)\,e^{-u}\,du, \text{ integrating by parts}$$

$$=0+\int_0^t (t-2u)\,e^{-u}\,du=\left[-(t-2u)\,e^{-u}\right]_0^t-2\int_0^t e^{-u}du$$

$$=te^{-t}+t-2\left[-e^{-u}\right]_0^t=t\,e^{-t}\,t+2e^{-t}-2$$

$$=(t+2)\,e^{-t}+t-2.$$

10.16. EVALUATION OF SOME INTEGRALS BY INVERSE LAPLACE TRANSFORMS

[1] The Beta Function. Which is defined as

$$\beta(m,\ n)=\int_0^1 x^{m-1}\,(1-x)^{n-1}dx \text{ for } m>0,\ n>0.$$

The result $\beta(m,n)=\dfrac{\Gamma m\Gamma n}{\Gamma(m+n)}=\displaystyle\int_0^1 x^{m-1}\,(1-x)^{n-1}\,dx$ can be exhibited by the help of inverse Laplace transform as shown below.

Let $G(t) = \displaystyle\int_0^t x^{m-1}\,(t-x)^{n-1}\,dx$

By convolution theorem,

$$L\,\{G(t)\}=L\,\{t^{m-1}\}\cdot L\,\{t^{n-1}\}=\frac{\Gamma m}{s^m}\cdot\frac{\Gamma n}{s^n}=\frac{\Gamma m\Gamma n}{s^{m+n}}$$

$$\therefore\quad G(t)=L^{-1}\left\{\frac{\Gamma m\Gamma n}{s^{m+n}}\right\}=\Gamma m\Gamma n\,L^{-1}\left\{\frac{1}{s^{m+n}}\right\}=\Gamma m\Gamma n\frac{t^{m+n-1}}{\Gamma(m+n)}$$

i.e. $$\int_0^t x^{m-1}\,(t-x)^{n-1}\,dx=\frac{\Gamma m\Gamma n}{\Gamma(m+n)}t^{m+n-1} \quad\ldots(1)$$

Putting $t=1$, this reduces to

$$\int_0^t x^{m-1}\,(t-x)^{n-1}\,dx=\frac{\Gamma m\Gamma n}{\Gamma(m+n)}=\beta(m,\ n) \quad\ldots(2)$$

Problem 33. *Show that* $\displaystyle\int_0^{\pi/2} sin^{2m-1}\,\theta\,cos^{2n-1}\,\theta\,d\theta=\frac{\Gamma m\Gamma n}{2\Gamma(m+n)}$

$$=\frac{1}{2}\beta(m,\ n).$$

We have $\beta(m,\ n)=\displaystyle\int_0^1 x^{m-1}\,(1-x)^{n-1}dx=\frac{\Gamma m\Gamma n}{\Gamma(m+n)}$

Putting $x=\sin^2\theta$, this gives

$$\beta(m,\ n)=2\int_0^{\pi/2}\sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta=\frac{\Gamma m\Gamma n}{\Gamma(m+n)}$$

i.e. $$\int_0^{\pi/2}\sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta=\frac{\Gamma m\Gamma n}{2\Gamma(m+n)}=\frac{1}{2}\beta(m,\ n)$$

[2] Evaluation of $\int_0^t J_0(u)\, J_0(t-u)du$

Let $G(t)=\int_0^t J_0(u)\; J_0(t-u)\; du$

By convolution theorem,

$$L\{G(t)\}=L\{J_0(t)\}\cdot L\{J_0(t)\}=\frac{1}{\sqrt{s^2+1}}\cdot\frac{1}{\sqrt{s^2+1}}=\frac{1}{s^2+1}$$

So that $G(t)=L^{-1}\left(\frac{1}{s^2+1}\right)$

i.e. $$\int_0^t J_0(u)\; J_0(t-u)\; du=\sin t \qquad \text{...(3)}$$

[3] Evaluation of $\frac{1}{\pi}\int_{-1}^1 e^{itw}\,(1-w^2)^{-1/2}\, dw$

$\because \dfrac{1}{\sqrt{s^2+1}}=\dfrac{1}{\sqrt{s+i}}\cdot\dfrac{1}{\sqrt{s-i}}$ and $L\{J_0(t)\}=\dfrac{1}{\sqrt{s^2+1}}$

$\therefore$ by convolution theorem

$$J_0(t)=L^{-1}\left\{\frac{1}{\sqrt{s^2+1}}\right\}=L^{-1}\left\{\frac{1}{\sqrt{s+i}}\cdot\frac{1}{\sqrt{s-i}}\right\}$$

$$=\int_0^t\frac{u^{-1/2}\,e^{-iu}}{\sqrt{\pi}}\cdot\frac{(t-u)^{-1/2}\,e^{i(t-u)}}{\sqrt{\pi}}du \quad \because\; L^{-1}\left\{\frac{1}{\sqrt{s+a}}\right\}=\frac{t^{-1/2}\,e^{-at}}{\sqrt{\pi}}$$

$$=\frac{1}{\pi}\int_0^t e^{i(t-2u)}\; v^{-1/2}\,(t-u)^{-1/2}\, du$$

$$=\frac{1}{\pi}\int_0^t e^{it(1-2v)}\; v^{-1/2}\;(1-v)^{-1/2}\, dv \text{ by putting } u=vt$$

$$=\frac{1}{\pi}\int_{-1}^1 e^{itw}\;(1-w^2)^{-1/2}\, dw \text{ by putting } 1-2v=w$$

Hence $$\frac{1}{\pi}\int_{-1}^1 e^{itw}\,(1-w^2)^{-1/2}\,dw=J_0(t) \qquad \text{...(4)}$$

(4) Evaluation of $\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}dx.$

$$\because\; L^{-1}\left\{\frac{1}{\sqrt{s}}\right\}=\frac{1}{\sqrt{\pi t}}, \quad \therefore\; L^{-1}\left\{\frac{1}{\sqrt{s+1}}\right\}=\frac{e^{-t}}{\sqrt{\pi t}}$$

then by convolution theorem,

$$L^{-1}\left\{\frac{1}{s\sqrt{s+1}}\right\}=\int_0^t 1.\;\frac{e^{-u}}{\sqrt{\pi u}}du$$

$$=\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}dx \text{ by putting } \sqrt{u}=x \text{ i.e. } \frac{1}{\sqrt{u}}\,du=2\,dx$$

$$=\operatorname{erf}\sqrt{t}$$

Hence $\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}dx = L^{-1}\left\{\frac{1}{s\sqrt{s+1}}\right\} = \text{erf}\,\sqrt{t}$ ...(5)

Problem 34. *Show that* $\int_0^\infty \cos x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$.

Let $G(t) = \int_0^\infty \cos tx^2\,dx$, then taking Laplace transform,

$$L\{G(t)\} = \int_0^\infty e^{-st}\,dt \int_0^\infty \cos tx^2\,dx = \int_0^\infty dx \int_0^\infty e^{-st} \cos tx^2\,dx$$

$$= \int_0^\infty L\{\cos tx^2\}\,dx = \int_0^\infty \frac{s}{s^2+x^4}\,dx$$

If we put $x^2 = s\tan\theta$ or $x = \sqrt{s}\,\sqrt{\tan\theta}$, then we get

$$L\{G(t)\} = \frac{1}{2\sqrt{s}}\int_0^{\pi/2} \frac{d\theta}{\sqrt{\tan\theta}} = \frac{1}{2\sqrt{s}}\int_0^{\pi/2} \sin^{-1/2}\theta\,\cos^{+1/2}\theta\,d\theta$$

$$= \frac{1}{2\sqrt{s}}\,\frac{\Gamma\frac{1}{4}\Gamma\frac{3}{4}}{2\Gamma 1} \text{ by Problem 33.}$$

$$= \frac{1}{2\sqrt{s}}\,\frac{\pi}{2\sin\frac{\pi}{4}} \text{ using } \Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi},\ 0 < n < 1$$

$$= \frac{1}{2\sqrt{s}}\cdot\frac{\pi\sqrt{2}}{2} = \frac{\pi\sqrt{2}}{4\sqrt{s}}.$$

Using inverse transform,

$$G(t) = \int_0^\infty \cos tx^2\,dx = \frac{\pi\sqrt{2}}{4}L^{-1}\left\{\frac{1}{\sqrt{s}}\right\} = \frac{\pi\sqrt{2}}{4}\cdot\frac{t^{-1/2}}{\sqrt{\pi}} = \frac{\sqrt{2\pi}}{4}\,t^{-1/2}$$

Putting $t = 1$, we get

$$\int_0^\infty \cos x^2\,dx = \frac{\sqrt{2\pi}}{4} = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

Problem 35. *Show that* $\int_0^\infty \cos e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}$

Take $G(t) = \int_0^\infty e^{-x^2}$, so that

$$L\{G(t) = \int_0^\infty \frac{dx}{s+x^2} = \left[\frac{1}{\sqrt{s}}\tan^{-1}\frac{x}{\sqrt{s}}\right]_0^\infty = \frac{\pi}{2\sqrt{s}}$$

Taking inverse,

$$G(t) = \int_0^\infty e^{-tx^2}\,dx = \frac{\pi}{2}\,\frac{t^{-1/2}}{\sqrt{\pi}} = \frac{1}{2}\sqrt{\pi}\,t^{-1/2}$$

Putting $t = 1$, $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

10.17. SOME METHODS FOR FINDING INVERSE LAPLACE TRANSFORM

[1] Partial Fraction-method

Consider a rational fraction $\frac{N(s)}{D(s)}$ where $N(s)$ and $D(s)$ are polynomials in s with no common factor and the degree of $N(s)$ is lower than degree of $D(s)$.

Let
$$f(s)=\frac{N(s)}{D(s)} \qquad \ldots(1)$$

Its inverse Laplace transform can be determined whenever the elementary factors of $D(s)$ are known.

(*i*) Taking first the case when $D(s)$ has no repeated factors, we can write

$$D(s)=(s-\alpha_1)(s-\alpha_2)\ldots(s-\alpha_n)$$

where all $\alpha_r'^s$ are distinct and coefficient of s^n being assumed unity without any loss of generality. Here $D(s)$ has been taken as a polynomial of degree n so that $r=1,2,3,\ldots n$.

By theory of partial fractions, we have

$$f(s)=\frac{N(s)}{D(s)}=\frac{N(s)}{(s-\alpha_1)(s-\alpha_2)\ldots(s-\alpha_n)}$$

$$=\frac{A_1}{s-\alpha_1}+\frac{A_2}{s-\alpha_2}+\ldots+\frac{A_r}{s-\alpha_r}+\ldots+\frac{A_n}{s-\alpha_n}$$

Multiplying both sides by $(s-\alpha_r)$ and taking the limit as $s\to\alpha_r$, we have

$$A_r=\lim_{s\to\alpha_r}\frac{N(s)\cdot(s-\alpha_r)}{D(s)}=N(\alpha_r).\lim_{s\to\alpha_r}\frac{s-\alpha_r}{D(s)}$$

$$=N(\alpha_r)\cdot\lim_{s\to\alpha_r}\frac{1}{D'(s)} \text{ by L'Hospital's rule}$$

$$=\frac{N(\alpha_r)}{D'(\alpha_r)}$$

Thus
$$f(s)=\frac{N(s)}{D(s)}=\frac{N(\alpha_1)}{D'(\alpha)(s-\alpha_1)}+\ldots+\frac{N(\alpha_r)}{D'(\alpha_r)(s-\alpha_r)}+\ldots+\frac{N(\alpha_n)}{D'(\alpha_n)(s-\alpha_n)}$$

$$=\sum_{r=1}^{n}\frac{N(\alpha_r)}{D'(\alpha_r)}\cdot\frac{1}{s-\alpha_r} \qquad \ldots(2)$$

where $D'(\alpha_r)=(s-\alpha_1)(s-\alpha_2)\ldots(s-\alpha_{r-1})(s-\alpha_{r+1})\ldots(s-\alpha_n)$ at $s=\alpha_r$.

If $f(t)$ be the inverse Laplace transform of $f(s)$, then

$$F(t)=L^{-1}\{f(s)\}=L^{-1}\left\{\frac{N(s)}{D(s)}\right\}=\left[\sum_{r=1}^{n}\frac{N(\alpha_r)}{D'(\alpha_r)}\right]L^{-1}\left\{\frac{1}{s-\alpha_r}\right\}$$

$$=\sum_{r=1}^{n}\frac{N(\alpha_r)}{D'(\alpha_r)}e^{\alpha_r t} \qquad \ldots(3)$$

which is known as *Heaviside Expansion formula*.

(*ii*) Now taking The case when $D(s)$ has repeated linear factors, we can write

$$f(s) = \frac{N(s)}{D(s)} = \frac{\phi(s)}{(s-\alpha)^{n+1}} \qquad \ldots(4)$$

where $\phi(s)$ is the quotient of polynomials $N(s)$ and the one obtained by removing the factor $(s-\alpha)^{n+1}$ from $D(s)$. Then, we have

$$\frac{\phi(s)}{(s-\alpha)^{n+1}} = \frac{A_0}{s-\alpha} + \frac{A_1}{(s-\alpha)^2} + \ldots + \frac{A_r}{(s-\alpha)^{r+1}} + \ldots + \frac{A_n}{(s-\alpha)^{n+1}} + g(s)$$

where $g(s)$ denotes the sum of partial fractions corresponding to the other factors of $D(s)$.

Multiplying both sides by $(s-\alpha)^{n+1}$ we get

$$\phi(s) = A_0(s-\alpha)^n + A_1(s-\alpha)^{n-1} + \ldots + A_r(s-\alpha)^{n-r} + \ldots$$
$$\ldots + A_n + g(s)(s-\alpha)^{n+1}$$

Taking the limit as $s \to \alpha$, $A_n = \phi(\alpha)$ and so we have on differentiating both sides $(n-r)$ times w.r.t. s,

$$\phi^{(n-r)}(\alpha) = \lfloor n-r \, A_r \text{ i.e. } A_r = \frac{\phi^{(n-r)}(\alpha)}{\lfloor n-r}$$

Hence from (4)

$$f(s) = \sum_{r=0}^{n} \frac{\phi^{(n-r)}(\alpha)}{\lfloor n-1} \cdot \frac{1}{(s-\alpha)^{r+1}} + g(s) \qquad \ldots(5)$$

$$\because \phi^{(0)}(\alpha) = \phi(\alpha)$$

$\therefore$ Taking the inverse transform, we have

$$F(t) = L^{-1}\{f(s)\} = \sum_{r=0}^{n} \frac{\phi^{(n-r)}(\alpha)}{\lfloor n-r} \cdot \frac{t^r e^{\alpha t}}{\lfloor r} + G(t) \qquad \ldots(6)$$

by using the relations $L\{e^{\lambda t}F(t)\} = f(s-\lambda)$ and $L\{t^{r+1}\} = \Gamma(r+1)\, s^{-r+1}$

Also here, $L^{-1}\{g(s)\} = G(t)$

Applying Leibnitz theorem for differentiation of product of two functions, (6) can be written as

$$F(t) = \frac{1}{\lfloor n} \left\{ \frac{\partial^n}{\partial s^n} \phi(s)\, e^{st} \right\}_{s=\alpha} + G(t) \qquad \ldots(7)$$

(*iii*) Taking the case when $D(s)$ has non-repeated quadratic factors of the form $(s-\alpha)^2 + \beta^2$, $s > 0$ and α, β being real numbers, then we can write

$$f(s) = \frac{N(s)}{D(s)} = \frac{\phi(s)}{(s-\alpha)^2 + \beta^2} = \frac{A_1 s + A_2}{(s-\alpha)^2 + \beta^2} + g(s) \qquad \ldots(8)$$

where $g(s)$ denotes the sum of partial fractions corresponding to the other factors of $D(s)$ and A_1, A_2 are real constants. Then we have

$$\phi(\alpha + i\beta) = \lim_{s \to \alpha + i\beta} \phi(s) = (\alpha + i\beta) A_1 + A_2 = \phi_1 + i\phi_2 \text{ (say)}$$

i.e., $(\alpha A_1 + A_2) + i\beta A_1 = \phi_1 + i\phi_2$, ϕ_1, ϕ_2 being real and imaginary parts of ϕ.

Equating real and imaginary parts,

$$\phi_1 = \alpha A_1 + A_2 \text{ and } \phi_2 = \beta A_1 \text{ giving } A_1 = \frac{\phi_2}{\beta} \text{ and } A_2 = \phi_1 - \frac{\alpha}{\beta}\phi_2.$$

Thus (8) reduces to

$$f(s) = \frac{1}{\beta} \cdot \frac{(s-\alpha)\phi_2 + \beta\phi_1}{(s-\alpha)^2 + \beta^2} + g(s) \qquad \ldots(9)$$

Taking the inverse transform, we have

$$F(t) = L^{-1}\{f(s)\} = \frac{1}{\beta}(\phi_2 \cos\beta t + \phi_1 \sin\beta t)\, e^{\alpha t} + G(t) \qquad \ldots(10)$$

where $\quad G(t) = L^{-1}\{g(s)\}$.

(*iv*) Lastly taking the case when $D(s)$ has the square of a quadratic function of the form $\{(s-\alpha)^2 + \beta^2\}^2$, then we can write,

$$f(s) = \frac{N(s)}{D(s)} = \frac{\phi(s)}{\left\{(s-\alpha)^2+\beta^2\right\}^2} = \frac{A_1 s + A_2}{(s-\alpha)^2+\beta^2} + \frac{A_3 s + A_4}{\left\{(s-\alpha)^2+\beta^2\right\}^2} + g(s) \qquad \ldots(11)$$

where $G(s)$ denotes the sum of partial fractions corresponding to the other factors of $D(s)$ and A_1, A_2, A_3, A_4 are real constants. Then we have

$$\phi(s) = (A_1 s + A_2)\{(s-\alpha)^2 + \beta^2\} + (A_3 s + A_4) + g(s)\{(s-\alpha)^2 + \beta^2\}^2$$

So that $\phi(\alpha + i\beta) = \lim_{s\to\alpha+i\beta} \phi(s) = A_3(\alpha + i\beta) + A_4 = \phi_1 + i\phi_2$ (say)

Giving $A_3\alpha + A_4 = \phi_1$ and $\beta A_3 = \phi_2$ which yield

$$A_3 = \frac{\phi_2}{\beta} \text{ and } A_4 = \frac{1}{\beta}(\beta\phi_1 - \alpha\phi_2)$$

Also $\phi'(s) = A_1[(s-\alpha)^2 + \beta^2] + 2(A_1 s + A_2)(s-\alpha) + \frac{d}{ds}[g(s)\{(s-\alpha)^2+\beta^2\}]$

so that $\phi'(\alpha + i\beta) = 2i\beta\,[A_1(\alpha + i\beta) + A_2] = \phi_1' + i\phi_2'$ (say)

giving $-2\beta^2 A_1 = \phi_1'$ and $2\alpha\beta A_1 + 2\beta A_2 = \phi_2'$

which yield $A_1 = -\frac{1}{2\beta^2}\phi'$ and $A_2 = \frac{\beta\phi_2' + \alpha\phi_1'}{2\beta^2}$.

Hence (11) reduces to

$$f(s) = \frac{1}{2\beta^2}\frac{(-\phi_1' s + \beta\phi_2' + \alpha\phi_1')}{(s-\alpha)^2+\beta^2} + \frac{1}{\beta}\frac{(\phi_2 s + \beta\phi_1 - \alpha\phi_2)}{\beta\left[(s-\alpha)^2+\beta^2\right]^2} + g(s) \qquad \ldots(12)$$

Taking inverse Laplace transform, we get —

$$F(t) = L^{-1}\{f(s)\} = \frac{1}{2\beta^2} e^{\alpha t}\,[\,|\phi(\alpha + i\beta)|\,\{\sin(\beta t + \psi) - \beta t\cos(\beta t + \psi)\}$$

$$-\beta\{\,|\phi'(\alpha + i\beta)|\,\cos(\beta t + \psi')\}] \qquad \ldots(13)$$

where $|\phi(\alpha + i\beta)| = \sqrt{\phi_1^2 + \phi_2^2}$ and $|\phi'(\alpha + i\beta)| = \sqrt{\phi_1'^2 + \phi_2'^2}$

giving $\quad \psi = \tan^{-1}\frac{\phi_2}{\phi_1}$ and $\psi' = \tan^{-1}\frac{\phi_2'}{\phi_1'}$.

As a generalization if $D(s)$ has $(n + 1)$ power of the quadratic factors $\{(s-\alpha)^2 + \beta^2\}$ then corresponding term in $F(t)$ will be of the form $t^r e^{\alpha t}\sin\beta t$ and $t^r e^{\alpha t}\cos\beta t$ where $r = 0, 1, 2, \ldots n$.

Problem 36. *Find the inverse Laplace transform of the following functions*

(i) $\dfrac{1}{(s+1)(s-2)}$, (ii) $\dfrac{3s+1}{(s-1)(s^2+1)}$,

(iii) $\dfrac{1}{(s-2)(s+2)^2}$, (iv) $\dfrac{4s}{(s-2)(s^2+4)}$.

(i) Here $f(s) = \dfrac{1}{(s+1)(s-2)} = \dfrac{1}{3(s+1)} - \dfrac{1}{3(s-2)}$

$\therefore\ L^{-}\{f(s)\} = \frac{1}{2}\left[e^{-t} - e^{2t}\right].$

(ii) Here $f(s) = \dfrac{3s+1}{(s-1)\left(s^2+1\right)} = \dfrac{3s+1}{(s-1)(s+i)(s-i)} = \dfrac{N(s)}{D(s)}$ (say)

Then $N(s) = 3s + 1$, $D(s) = s^3 - s^2 + s - 1$ so that $D'(s) = 3s^2 - 2s + 1$ and zeros $\alpha_1 = 1$, $\alpha_2 = i$, $\alpha_3 = -i$

Using Heaviside's expansion formula, we have

$$L^{-1}\{f(s)\} = \frac{N(1)}{D'(1)}e^t + \frac{N(i)}{D'(i)}e^{it} + \frac{N(-i)}{D'(-i)}e^{-it}$$

$$= \frac{4e^t}{2} + \frac{3i+1}{-(2+2i)}e^{it} + \frac{-3i+1}{-2+2i}e^{-it}$$

$$= 2e^t - \frac{(3i+1)(1-i)}{2(1+i)(1-i)}e^{it} + \frac{(3i-1)(1+i)}{2(1-i)(1+i)}e^{-it}$$

$$= 2e^t - \tfrac{1}{2}(i+2)e^{it} + \tfrac{1}{2}(i-2)e^{-it}$$

$$= 2e^t - \tfrac{1}{2i}\left(e^{it} - e^{-it}\right) - \left(e^{it} + e^{-it}\right)$$

$$= 2e^t - \sin t - 2\cos t.$$

(iii) Here $f(s) = \dfrac{1}{(s-2)(s+2)^2}$.

The term in the inverse transform of $f(s)$ corresponding to $(s-2)$ is $\dfrac{1}{16}e^{2t}$ and corresponding to $(s+2)^2$ we have $\phi(s) = (s-2)^{-1}$ which yields, $\phi(-2) = -\dfrac{1}{4}$ and $\phi'(s) = -\dfrac{1}{(s-2)^2}$ yields $\phi'(-2) = -\dfrac{1}{16}$

so that by (5) of § 10.17,

$$L^{-1}\{f(s)\} = \frac{1}{16}e^{2t} + \frac{\phi'(-2)\,e^{-2t}}{\lfloor 1\ \lfloor 1} + \frac{\phi(-2)te^{-2t}}{\lfloor 0\ \lfloor 0}$$

$$= \frac{1}{16}e^{2t} - \frac{1}{16}e^{-2t} - \frac{1}{4}te^{-2t}$$

$$= \frac{1}{16}\left[e^{2t} - e^{-2t} - 4te^{-2t}\right]$$

(iv) Here $f(s) = \dfrac{4s}{(s-2)\left(s^2+4\right)}$.

The term in the inverse transform corresponding to $(s-2)$ is e^{2t} and corresponding to s^2+4 we have $\phi(s)=\dfrac{4s}{s-2}$, which yields $\phi(2i)=2(1-i)$ so that its real and imaginary parts are $\phi_1=2$, $\phi_2=-2$.

Also comparing with (9) of § 10.17, we have $\alpha=0$ and $\beta=2$

$$\text{Hence } L^{-1}\{f(s)\}=e^{2t}+\tfrac{1}{2}L^{-1}\left\{\frac{-2s}{s^2+4}+\frac{4}{s^2+4}\right\}$$
$$=e^{2t}+\tfrac{1}{2}[-2\cos 2t+2\sin 2t]$$
$$=e^{2t}+(\sin 2t-\cos 2t)=e^{2t}+\sqrt{2}\sin\left(2t-\frac{\pi}{4}\right).$$

[2] Series Expansion Method

If $f(s)$ has a series expansion in inverse powers of s such as

$$f(s)=\frac{a_0}{s}+\frac{a_1}{s^2}+\frac{a_2}{s^3}+\ldots \qquad \ldots(14)$$

then under suitable conditions, we can invert term by term to find

$$F(t)=a_0+a_1t+\frac{a_2t^2}{\lfloor 2}+\ldots \qquad \ldots(15)$$

Problem 37. *Find* $L^{-1}\left\{\dfrac{e^{-\frac{1}{s}}}{s}\right\}$

$$\text{We have } L^{-1}\left\{\frac{e^{-1/s}}{s}\right\}=L^{-1}\left\{\frac{1}{s}\left(1-\frac{1}{s}+\frac{1}{\lfloor 2\cdot s^2}-\frac{1}{\lfloor 3\cdot s^3}+\ldots\right)\right\}$$
$$=L^{-1}\left\{\frac{1}{s}\right\}-L^{-1}\left\{\frac{1}{s^2}\right\}-\frac{1}{\lfloor 2}L^{-1}\left\{\frac{1}{s^3}\right\}-\frac{1}{\lfloor 3}L^{-1}\left\{\frac{1}{s^4}\right\}+\ldots$$
$$=1-t+\frac{t^2}{(\lfloor 2)^2}-\frac{t^3}{(\lfloor 3)^2}+\ldots$$
$$=1-\frac{(2\sqrt{t})^2}{2^2}+\frac{(2\sqrt{t})^4}{2^2\cdot 4^2}-\frac{(2\sqrt{t})^6}{2^2\cdot 4^2\cdot 6^2}+\ldots$$
$$=J_0(2\sqrt{t}).$$

[3] Method of Differential Equations

By solving differential equations, the inverse Laplace transforms of some functions can be found as is evident from the following Problem.

Problem 38. *Find* $L^{-1}\left\{e^{-\sqrt{s}}\right\}$

Let $y(s)=e^{-\sqrt{s}}$ and $L\{Y(t)\}=y(s)\cdot$

So that $L^{-1}\{y(s)\}=L^{-1}\left\{e^{-\sqrt{s}}\right\}=Y(t)$

$$\text{Also } \frac{dy}{ds}=y'=-\frac{e^{-\sqrt{s}}}{2\sqrt{s}} \text{ and } \frac{d^2y}{ds^2}=y''=\frac{e^{-\sqrt{s}}}{4s}+\frac{e^{-\sqrt{s}}}{4s^{3/2}}$$

or $\qquad 4sy'' = e^{-\sqrt{s}} + \dfrac{e^{-\sqrt{s}}}{\sqrt{s}} = y - 2y'$

$\therefore \qquad 4sy'' + 2y' - y = 0 \qquad \ldots(1)$

which is the differential equation satisfied by $y = e^{-\sqrt{s}}$.

Now $y = L\{Y\}$ gives $y' = L\{-tY\}$ and $y'' = L\{(-1)^2 t^2 Y\} = L\{t^2 Y\}$

So that $sy'' = L\left\{\dfrac{d}{dt}(t^2 Y)\right\} = L\{t^2 Y' + 2tY\}$

Thus (1) reduces to

$4L(t^2 Y' + 2tY) + 2L\{-tY\} - L\{Y\} = 0$

or $\qquad L\{4t^2 Y' + (6t-1)Y\} = 0$ or $4t^2 Y' + (6t-1)Y = L^{-1}\{0\} = 0$

i.e. $4t^2 \dfrac{dY}{dt} + (6t-1)Y = 0$ which gives on separating the variables,

$\dfrac{dY}{Y} + \dfrac{6t-1}{4t^2}\, dt = 0$ or $\dfrac{dY}{Y} + \left(\dfrac{3}{2t} - \dfrac{1}{4t^2}\right) dt = 0$

or $\qquad \dfrac{dY}{Y} = -\left(\dfrac{3}{2t} - \dfrac{1}{4t^2}\right) dt$

Integrating, $\log Y = -\dfrac{3}{2}\log t - \dfrac{1}{4t} + \log C$, C being a constant of integration

$\therefore \qquad Y = C\dfrac{e^{-1/4t}}{t^{3/2}}$ or $tY = \dfrac{C}{\sqrt{t}}\, e^{-1/4t} \qquad \ldots(2)$

Now $L\{tY\} = -\dfrac{d}{ds}L\{y\} = -\dfrac{d}{ds}\left(e^{-\sqrt{s}}\right) = \dfrac{e^{-\sqrt{s}}}{2\sqrt{s}}$

Applying the final value Theorem, *i.e.* when $t \to \infty$, $s \to 0$, we have from (2)

$tY = \dfrac{C}{\sqrt{t}} e^{-1/4t} \to \dfrac{C}{\sqrt{t}}$ so that $L\{tY\} \to L\left\{\dfrac{C}{\sqrt{t}}\right\} = C\sqrt{\pi}/\sqrt{s}$

and $L\{tY\} = \dfrac{e^{-\sqrt{s}}}{2\sqrt{s}} \to \dfrac{1}{2\sqrt{s}}$, whence $\dfrac{C\sqrt{\pi}}{\sqrt{s}} = \dfrac{2}{2\sqrt{s}}$ gives $C = \dfrac{1}{2\sqrt{\pi}}$.

Hence the solution of (1) is $Y = \dfrac{1}{2\sqrt{\pi}\, t^{3/2}} \cdot e^{-1/4t}$

i.e. $L^{-1}\left\{e^{-\sqrt{s}}\right\} = e^{-1/4t}/2\sqrt{\pi}\, t^{3/2}$.

[4] Method of Differentiation with Respect to a Parameter

This method is illustrated with the help of following Problem.

Problem 39. *Find* $L^{-1}\left\{\dfrac{1}{(s^2+a^2)^{3/2}}\right\}$

We have $L(J_0(at)) = \dfrac{1}{\sqrt{s^2+a^2}}$.

Differentiating w.r.t. the parameter 'a' we get

$$\frac{d}{da} L\{J_0(at)\} = \frac{d}{da}\left[\frac{1}{\sqrt{s^2+a^2}}\right] \text{ or } L\left\{\frac{d}{da} J_0(at)\right\} = \frac{-a}{(s^2+a^2)^{3/2}}$$

or $$L\{t J_0'(at)\} = \frac{-a}{(s^2+a^2)^{3/2}} \text{ i.e. } L^{-1}\left\{\frac{1}{(s^2+a^2)^{3/2}}\right\} = -\frac{t}{a} J_0'(at)$$

$$= \frac{t}{a} J_1(at) \quad \because J_0'(x) = -J_1(x)$$

10.18. APPLICATIONS OF LAPLACE TRANSFORMS TO DIFFERENTIAL EQUATIONS

The Laplace transforms can be successfully applied to solve

(*i*) ordinary differential equations with constant coefficients such as

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = F(t), \ a, \ b \text{ being constants.}$$

(*ii*) ordinary differential equations with variable coefficients such as

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + y = F(t)$$

(*iii*) simultaneous ordinary differential equations such as

$$\frac{d^2y}{dt^2} + a\frac{dy}{dx} + b\frac{dx}{dt} = F_1(t) \text{ and } c\frac{dy}{dt} + \frac{d^2x}{dt^2} + d\frac{dx}{dt} = F^2(t);$$

a, b, c, d being constants.

(*iv*) partial differential equations such as

$$\frac{\partial^2 y}{\partial t^2} + a\frac{\partial^2 y}{\partial x^2} + y = F(x, t)$$

(*v*) integral equations *i.e.* the equation containing a dependent variable under an integral sign, such as

$$Y(t) = F(t) + \int_a^b K(u, t)Y(u)\,du, \ a, b \text{ being constants or functions of } t.$$

Here $F(t)$ and the Kernal $K(u, t)$ are known and $Y(t)$ is to be determined.

We explain these applications with the help of following problems.

Problem 40. *Solve* $x''(t) + 4x'(t) + 4x(t) = 4e^{-2t}$; $x(0) = -1, x'(0) = 4$ *verify that your solution satisfies the differential equation and the boundary conditions.*

Let $L\{x(t)\} = f(s)$ ...(1)

Then $L\{x'(t)\} = sf(s) - F(0)$ where $F(0) = x(0) = -1$

$= sf(s) + 1$ ...(2)

and $L(x''(t)\} = s^2 f(s) - s\,F(0) - F'(0)$, where $F(0) = x(0) = -1$ and $F'(0) = x'(0) = 4$

$= s^2 f(s) + s - 4$ ...(3)

Taking Laplace transform of the given equation, we get

$L\{x''(t)\} + 4L\{x'(t)\} + 4L\{x(t)\} = 4L\{e^{-2t}\}$

or $s^2 f(s) + s - 4 + 4sf(s) + 4 + 4f(s) = \dfrac{4}{s+2}$ by (1), (2) and (3) or $(s^2+4s+4)f(s) = \dfrac{4}{s+2} - s$ i.e. $f(s) = \dfrac{4}{(s+2)^3} - \dfrac{s}{(s+2)^2}$.

$$= \frac{4}{(s+2)^3} + \frac{2}{(s+2)^2} - \frac{1}{s+2} \quad \because \ s = (s+2) - 2$$

Taking inverse Laplace transform of both sides, we find

$$L^{-1}\{f(s)\} = 4L^{-1}\left\{\frac{1}{(s+2)^3}\right\} + 2L^{-1}\left\{\frac{1}{(s+2)^2}\right\} - L^{-1}\left\{\frac{1}{(s+2)}\right\}$$

i.e. $$x(t) = \frac{4(-1)^2}{2} L^{-1}\left\{\frac{d^2}{dx^2}\left(\frac{1}{s+2}\right)\right\} + 2\,(-1)\,L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s+2}\right)\right\} - e^{-2t}$$

$$= 2(-1)^2 \cdot (-1)^2 t^2 e^{-2t} + 2(-1)(-1)\, te^{-2t} - e^{-2t} = e^{-2t}(2t^2 + 2t - 1)$$

Hence the required solution is

$$x(t) = e^{-2t}(2t^2 + 2t - 1) \qquad \text{...(4)}$$

Now to verify the result, we have from (4)

$$x'(t) = -2e^{-2t}(2t^2+2t-1) + e^{-2t}(4t+2) = e^{-2t}(-4t^2 - 4t + 2 + 4t + 2)$$
$$= 4e^{-2t}(-t^2 + 1) \qquad \text{...(5)}$$

and $x''(t) = -8e^{-2t}[-t^2 + 1] + 4e^{-2t}(-2t) = 8e^{-2t}(t^2 - t - 1)$

So that $x''(t) + 4x'(t) + 4x(t) = 8e^{-2t}(t^2 - t - 1) + 16e^{-2t}(-t^2 + 1)$
$+ 4e^{-2t}(2t^2 + 2t - 1)$

$$= 4e^{-2t}[2t^2 - 2t - 2 - 4t^2 + 4 + 2t^2 + 2t - 1] = 4e^{-2t} \qquad \text{...(6)}$$

which is the same as given and hence the result is verified.

Again from (4), we have $x(0) = -1$ and from (5), $x'(0) = 4$ which clearly satisfy the boundary conditions.

Problem 41. *Solve the differential equation* $ty''(t) + y'(t) + ty(t) = 0$.
Under the condition that $y(0) = 1$ *and* $y(t)$ *is bounded.*

Let $L\{y(t)\} = f(s)$ so that $L^{-1}\{f(s)\} = y(t)$ ...(1)

Now $L(y'(t) = sf(s) - F(0)$, where $F(0) = y(0) = 1$

$$= sf(s) - 1 \qquad \text{...(2)}$$

and $L\{y''(t)\} = s^2 f(s) - s\,F(0) - F'(0)$ where $F(0) = y(0) = 1$

and $F'(0) = y'(0) = k$ (const.) say

$$= s^2 f(s) - s - k.$$

Taking Laplace transform of the given equation,

$L\{ty''(t)\} + L\{y'(t)\} + L\{ty(t)\} =,0$

or $$-\frac{d}{ds} L\{y''(t)\} + L\{y'(t)\} - \frac{d}{ds} L\{y(t)\} = 0$$

or $$-\frac{d}{ds}\{s^2 f(s) - s - k\} + sf(s) - 1 - \frac{d}{ds}\{f(s)\} = 0 \text{ by (1) and (2)}$$

or $$-2sf(s) - s^2 f'(s) + 1 + sf(s) - 1 - f'(s) = 0$$

or $$-sf(s) - (s^2+1) f'(s) = 0$$

or $$\frac{f'(s)}{f(s)} = -\frac{s}{s^2+1} = -\frac{1}{2}\cdot\frac{2s}{s^2+1}.$$

Integrating, $\log f(s) = -\frac{1}{2}\log(s^2+1) + \log C$, C being a constant of integration

$$\therefore \qquad f(s) = \frac{C}{\sqrt{s^2+1}} \qquad ...(3)$$

Taking inverse Laplace transform, we have

$$L^{-1}\{f(s)\} = C\,L^{-1}\left\{\frac{1}{\sqrt{s^2+1}}\right\} \text{ or } y(t) = C\,J_0(t)$$

Initially when $t = 0$, $y(t) = y(0) = 1$, so that $1 = CJ_0(0) = C \because J_0(0) = 1$.

Hence the required solution is $y(t) = J_0(t)$.

Problem 42. *Solve* $X'' + Y' + 3X = 15e^{-t}$

$$Y'' - 4X' + 3Y = 15 \sin 2t.$$

Subject to $X(0) = 35$, $X'(0) = -48$, $Y(0) = 27$, $Y'(0) = -55$.

Let $L(X) = x$ and $L(Y) = y$ so that $L^{-1}\{x\} = X$ and $L^{-1}\{y\} = Y$.

Also $L\{X'\} = s\,L\{X\} - X(0) = sx - 35$; $L\{X''\} = s^2L\{X\} - sX(0) - X'(0)$

$$= s^2x - 35s + 48$$

and $L\{Y'\} = sL\{Y\} - Y(0) = sy - 27$; $L\{Y''\} = s^2L\{Y\} - sY(0) - Y'(0) = s^2y - 27s + 55$

$\therefore$ Taking Laplace transforms of the given equations, we have

$$s^2x - 35s + 48 + sy - 27 + 3x = \frac{15}{s+1}$$

and $$s^2y - 27s + 55 - 4(sx - 35) + 3y = \frac{30}{s^2+4}$$

or $$\left(s^2+3\right)x + sy - \left(35s - 21 + \frac{15}{s+1}\right) = 0$$

and $$-4sx + \left(s^2+3\right)y - \left(27s - 195 + \frac{30}{s^2+4}\right) = 0$$

Solving $$\frac{x}{-s\left(27s - 195 + \frac{30}{s^2+4}\right) + \left(s^2+3\right)\left(35s - 21 + \frac{15}{s+1}\right)}$$

$$= \frac{y}{4s\left(35s - 21 + \frac{15}{s+1}\right) + \left(s^2+3\right)\left(27s - 195 + \frac{30}{s^2+4}\right)} = \frac{1}{\left(s^2+3\right)^2 + 4s^2}$$

So that $$x = \frac{1}{s^4 + 10s^2 + 9}\left[-27s^2 + 195s - \frac{30s}{s^2+4} + 35s^3 - 21s^2 + \frac{15s^2}{s+1} + 105s - 63 + \frac{45}{s+1}\right]$$

$$= \frac{1}{\left(s^2+1\right)\left(s^2+9\right)(s+1)\left(s^2+4\right)}\Big[\left(35s^3 - 48s^2 + 300s - 63\right)$$

$$(s+1)(s^2+4) - 30s(s+1) + 15(s^2+3)(s^2+4)\Big]$$

$$= \frac{35s^3 - 48s^2 - 300s - 63}{(s^2+1)(s^2+9)} + \frac{15(s^2+3)}{(s+1)(s^2+1)(s^2+9)} - \frac{30s}{(s^2+1)(s^2+4)(s^2+9)}$$

$$= \frac{26s-15}{8(s^2+1)} + \frac{15s-369}{8(s^2+9)} + \frac{3}{s+1} - \frac{15}{8}\cdot\frac{s-1}{s^2+1} - \frac{9}{8}\cdot\frac{s-1}{s^2+9} - \frac{10s}{(s^2+1)} + \frac{2s}{s^2+4} - \frac{6s}{8(s^2+9)}$$

(by resolving into partial fractions)

$$= \frac{30s}{s^2+1} - \frac{45}{s^2+9} + \frac{3}{s+1} + \frac{2s}{s^2+4} \quad ...(1)$$

Similarly $y = \dfrac{30s}{s^2+9} - \dfrac{60}{s^2+1} - \dfrac{3}{s+1} + \dfrac{2}{s^2+4}$ ...(2)

Taking inverse Laplace transforms of (1) and (2) we get

$$L^{-1}\{x\} = X = 30\cos t - 45\cdot\frac{\sin 3t}{3} + 3e^{-t} + 2\cos 2t$$

$$= 30\cos t - 15\sin 3t + 3e^{-1} + 2\cos 2t$$

and $L^{-1}\{y\} = Y = 30\cos 3t - 60\sin t - 3e^{-t} + \dfrac{2\sin 2t}{2}$.

$$= 30\cos 3t - 60\sin t - 3e^{-t} + \sin 2t$$

which are the required solutions.

Problem 43. *Find the solution of* $\dfrac{\partial^2 V}{\partial x^2} - \dfrac{\partial^2 V}{\partial t^2} = xt$ *satisfying the conditions* $V = \dfrac{\partial V}{\partial t} = 0$ *where* $t = 0$.

Let $L\{V\} = v$ so that $L^{-1}\{v\} = V$, then

$$L\left\{\frac{\partial^2 V}{\partial t^2}\right\} = s^2 L\{V\} - sV(0) - V'(0) = s^2v - 0 - 0 = s^2 v$$ taking t as principal variable

and $L\left\{\dfrac{\partial^2 V}{\partial x^2}\right\} = \dfrac{d^2}{dx^2}L\{V\} = \dfrac{d^2 v}{dx^2}$, taking x as the secondary variable.

Also $L\{xt\} = \dfrac{x}{s^2}$.

∴ Taking Laplace transform of the given equation, we have

$$\frac{d^2v}{dx^2} - s^2 v = \frac{x}{s^2} \text{ i.e. } (D^2 - s^2)\, v = \frac{x}{s^2} \text{ where } D \equiv \frac{d}{dx} \quad ...(1)$$

Its complementary function is $C_1 e^{sx} + C_2 e^{-sx}$

$D^2 - s^2 = 0$ gives $D = \pm s$

and particular integral $= \frac{x/s^2}{D^2 - s^2} = \frac{1}{-s^2}\left[1 - \left(\frac{D}{s^2}\right)^2\right]^{-1} \frac{x}{s^2}$

$$= -\frac{1}{s^2}\left[1 + \frac{D^2}{s^2} + \ldots\right] \cdot \frac{x}{s^2} = -\frac{x}{s^4}$$

$\therefore$ The complete integral of (1) is

$$v = C_1 e^{sx} + C_2 e^{-sx} - \frac{x}{s^4} \qquad \ldots(2)$$

Now $V = 0$ gives $L\{V\} = v = 0$ for all values of x, therefore $C_1 = 0$, otherwise $v \to \infty$ as $x \to \infty$.

As such (2) reduces to $v = C_2 e^{-sx} - \frac{x}{s^4}$ $\qquad \ldots(3)$

But when $x = 0, v = 0 \therefore C_2 = 0$

Hence (3) becomes $v = -\frac{x}{s^4}$

Taking inverse Laplace transform of both sides, we find

$L^{-1}\{v\} = L^{-1}\left\{-\frac{x}{s^4}\right\}$ i.e. $V = -\frac{xt^3}{\lfloor 3} = -\frac{xt^3}{6}$ which is the required solution.

Problem 44. *Solve the following integral equation and verify your solution.*

$$F(t) = 1 + 2\int_0^t F(t-x) e^{-2x}\, dx.$$

Taking Laplace transform of both sides, we get

$$L\{F(t)\} = L\{1\} + 2L\left\{\int_0^t F(t-x) e^{-2x}\, dx\right\}$$

Assuming $L\{F(t)\} = f(s)$ so that $L\{1\} = \frac{1}{s}$

and $L\left\{\int_0^t F(t-x) e^{-2x}\, dx = \frac{f(s)}{s+2}\right.$, the last equation becomes,

$$f(s) = \frac{1}{s} + \frac{2f(s)}{s+2}$$

or $\left(1 - \frac{2}{s+2}\right) f(s) = \frac{1}{s}$ or $\frac{1}{s+2} f(s) = \frac{1}{s}$ $= \frac{1}{s} \cdot \frac{s+2}{s} = \frac{1}{s} + \frac{2}{s^2}$

Taking inverse Laplace transform of both sides, we find

$L^{-1}\{f(s)\} = L^{-1}\left\{\frac{1}{s}\right\} + 2l^{-1}\left\{\frac{1}{s^2}\right\}$ i.e. $F(t) = 1 + 2t$ which is the required solution.

In order to verify the solution, putting

$$F(t) = 1 + 2t \text{ i.e. } F(t-x) = 1 + 2(t-x),$$

we have, R.H.S. of the given equation $= 1 + 2\int_0^t \{1 + 2(t-x)\} e^{-2x}\, dx$

$$= 1 + 2\left[\int_0^t e^{-2x}\,dx + 2t\int_0^t e^{-2x}\,dx - 2\int_0^t x e^{-2x}\,dx\right]$$

$$= 1 - e^{-2t} + 1 - 2te^{-2t} + 2t + 2te^{-2t} + e^{-2t} - 1 = 1 + 2t = F(t) = \text{L.H.S.}$$

Hence the solution is verified.

10.19. APPLICATIONS OF LAPLACE TRANSFOMS TO BOUNDARY VALUE PROBLEMS

In view of §10.9 (*E*), it can be concluded that Laplace transforms are capable of transforming the differential equations with constant coefficients into algebraic equations in the form of transformed function. Since various problems used in physics are mathematically formulated in the shape of ordinary and partial differential equations involving one or more unknown functions together with initial or boundary conditions fitting the physical situation, therefore, Laplace transform can be successfully applied to such boundary value problems.

Problem 45. *A particle of mass 3 gms moves on the x-axis and is attracted towards a fixed point in its path with a force whose numerical value is 12x. Assuming that the particle is initially at rest at x = 5, determine the position of the particle at any time t, (a) when there is no other force, (b) when there is a damping force whose numerical value is 6 times the instantaneous velocity.*

(*a*) Let *P* be the particle and *O* the fixed point. Since the particle is attracted towards *O*, therefore when the particle is on the right of *O* *i.e.* in the direction $\overrightarrow{OP}$ then the net force is $-12x$ (being in the direction of $\overrightarrow{PO}$) and smilarly when the particle is moving along $\overrightarrow{PO}$, the force being in the direction of *OP* is again $-12x$. Hence applying Newton's second law of motion is mass × acceleration = acting force, the equation of motion is $3\dfrac{d^2x}{dt^2} = -12x$

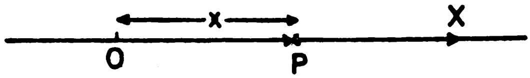


Fig. 10.3

or $$\frac{d^2x}{dt^2} + 4x = 0 \qquad \ldots(1)$$

Let $L\{x\} = \xi$ so that $L^{-1}(\xi) = x$

$$\therefore \quad L\left\{\frac{d^2x}{dt^2}\right\} = s^2 L\{x\} - sx(0) - x'(0) \text{ where } x(0) = 5 \text{ and } x'(0) = 0$$
(boundary conditions)

$$= s^2\xi - 5s$$

Taking Laplace transform of (1), we therefore get

$$s^2\xi - 5s + 4\xi = 0 \text{ or } (s^2 + 4)\,\xi = 5s \text{ or } \xi = \frac{5s}{s^2+4}$$

$\therefore\ L^{-1}\{\xi\} = 5L^{-1}\left\{\dfrac{s}{s^2+4}\right\}$ *i.e.* $x = 5\cos 2t$, which gives the position of the particle.

(*b*) In this case the equation of motion becomes

$$3\frac{d^2x}{dt^2} = -12x - 6\frac{dx}{dt} \text{ i.e., } \frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 4x = 0 \qquad \ldots(2)$$

with the initial conditions $x(0) = 5, x'(0) = 0$

Taking Laplace transform of (2), $L\left\{\frac{d^2x}{dt^2}\right\} + 2L\left\{\frac{dx}{dt}\right\} + 4L\{x\} = 0$

or $s^2\xi - 5s + 2(s\xi - 5) + 4\xi = 0$ when $L\{x\} = \xi$

or $(s^2 + 2s + 4)\xi = 5s + 10$ or $\xi = \frac{5s+10}{(s+2)^2} = \frac{5}{s+2}$

$\therefore L^{-1}\{\xi\} = x = 5e^{-2t}$ which is the required solution.

Problem 46. *An inductor of H henrys and a capacitor of C farads are in series with a generator of E volts. At t = 0 the charge on the capacitor and current in the circuit are zero. Find the charge on the capacitor at any time t > 0 if E = E_0, a constant.*

If Q be the instantaneous charge and I the current at any time t, then $I = \frac{dQ}{dt}$ ...(1)

Now voltage drop across the inductor $= H\frac{dI}{dt} = H\frac{d^2Q}{dt^2}$ by (1)

Voltage drop across the capacitor $= \frac{Q}{C}$ and the voltage drop across the generator = − voltage rise = $-E$.

Thus, according to Kirchoff's law, the algebraic sum of voltage drops (potential drops) around any closed circuit being zero, the equation determining Q is

$$H\frac{d^2Q}{dt^2} + \frac{Q}{C} - E = 0 \text{ or } \frac{d^2Q}{dt^2} + \frac{Q}{HC} = \frac{E}{H} \qquad \ldots(2)$$

with the initial (boundary) conditions $Q(0) = 0, I(0) = 0, Q'(0) = 0$ and $E = E_0$ for $t > 0$

Assuming that $L\{Q\} = q$, we have

$$L\left\{\frac{d^2Q}{dt^2}\right\} = s^2L\{Q\} - sQ(0) - Q'(0) = s^2q - 0 - 0 = s^2q$$

$\therefore$ Laplace transform of (2) gives $L\left\{\frac{d^2Q}{dt^2}\right\} = \frac{1}{HC}L\{Q\} = \frac{E_0}{H}L\{1\}$

$$\text{i.e. } s^2q + \frac{q}{HC} = \frac{E_0}{Hs} \text{ or } \left(s^2 + \frac{1}{HC}\right)q = \frac{E_0}{Hs} \text{ or } q = \frac{E_0/H}{s\left(s^2 + \frac{1}{HC}\right)}$$

$$\text{or } q = \frac{E_0C}{s} - \frac{CE_0s}{s^2 + \frac{1}{HC}} = CE_0\left\{\frac{1}{s} - \frac{s}{s^2 + \frac{1}{HC}}\right\}$$

Taking inverse Laplace transform, we have

$$Q = L^{-1}\{q\} = CE_0\left[L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{s}{s^2 + \frac{1}{HC}}\right\}\right] = CE_0\left[1 - \cos\frac{t}{\sqrt{HC}}\right]$$

which gives the required charge on the capacitor.

Problem 47. *A beam which is clamped at its ends $x = 0$ and $x = 1$ carries a uniform load w per unit length. Show that deflection at any point is $Y(x) = \frac{wx^2(l-x)^2}{24EI}$, where E is Young's modulus of elasticity for the beam and I is the moment of Inertia of a cross-section of the beam about the axis.*

The transverse deflection $Y(x)$ at any point x is given by the differential equation

$$\frac{d^4Y}{dx^4} = \frac{w}{EI},\ 0 < x < l \qquad \ldots(1)$$

with the initial conditions, $y(0) = 0$, $y'(0) = 0$ and $y(l) = 0$, $y'(l) = 0$...

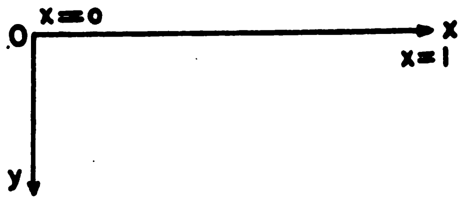


Fig. 10.4

Assuming that $L\{Y\} = y$ i.e. $L^{-1}\{Y\} = Y$, we have

$$L\left\{\frac{d^4Y}{dx^4}\right\} = s^4 L\{Y\} - s^3 Y'(0) - s^2 Y''(0) - sy''(0) - Y'''(0).$$

$$= s^4 y - 0 - 0 - sC_1 - C_2 \text{ under the assumption that } Y''(0) = C_1$$

and $\qquad Y'''(0) = C_2.$

$\therefore$ Taking Laplace transform of (1), we have

$$L\left\{\frac{d^4Y}{dx^4}\right\} = \frac{w}{EI}L\{1\} \text{ i.e., } s^4 y - sC_1 - C_2 = \frac{w}{EIs} \text{ or } y = \frac{C_1}{s^3} + \frac{C_2}{s^4} + \frac{w}{EIs^5}$$

Taking inverse Laplace transform, we find

$$Y = L^{-1}\{y\} = C_1\frac{x^2}{\lfloor 2} + C_2\cdot\frac{x^3}{\lfloor 3} + \frac{w}{EI}\frac{x^4}{\lfloor 4} = \frac{C_1 x^2}{2} + \frac{C_2 x^3}{6} + \frac{wx^4}{24EI} \therefore \qquad \ldots(2)$$

Applying the initial condition that $Y(x) = 0$ when $x = l$ and $Y'(x) = 0$ when $x = l$, (2) gives

$$0 = \frac{C_1 l^2}{2} + \frac{C_2 l^3}{6} + \frac{wl^4}{24EI} \qquad \ldots(3)$$

and $\qquad Y' = C_1 x + \frac{C_2 x^2}{2} + \frac{wx^3}{6EI}$ gives $0 = C_1 l + \frac{C_2 l^2}{2} + \frac{wl^3}{6EI} \qquad \ldots(4)$

Solving (3) and (4); $C_1 = \frac{wl^2}{12EI}$ and $C_2 = -\frac{wl}{2EI}$ which when substituted in (2) give

$$Y = \frac{wl^2x^2}{24EI} - \frac{wlx^3}{24EI} + \frac{wx^4}{24EI} = \frac{wx^2}{24EI}\left[l^2 - 2lx + x^2\right] = \frac{wx^2(l-x)^2}{24EI}.$$

Problem 48. *A semi-infinite solid $x > 0$ is initially at temperature zero. At time $t = 0$, a constant temperature $U_0 > 0$ is applied and maintained at the face $x = 0$. Find the temperature at any point of the soild at any tme $t > 0$.*

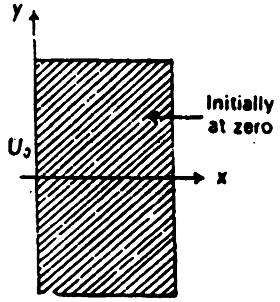


Fig. 10.5

Let $U(x, t)$ be the temperature of the soild at any point x and at any time t. Then one-dimensional heat conduction equation gives

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{k}\frac{\partial U}{\partial t}, \quad x > 0,\ t > 0 \qquad \text{...(1)}$$

where k is the diffusivity defined by $\frac{k}{C\rho}$ where k is the thermal conductivity, C the specific heat and ρ the volume density. The boundary conditions are

(*i*) $U(x, 0) = 0$ (*ii*) $U(0, t) = U_0$ (given) and (*iii*) U is bounded *i.e.*,

$$|U(x, t)| < M.$$

Assuming that $L\{U(x, t)\} = u(x, s)\}$ *i.e.* $L^{-1}\{u(x, s)\} = U(x, t)$, we have

$$L\left\{\frac{\partial U}{\partial t}\right\} = sL\{U\} - U(x, 0) = su \text{ and } L\left\{\frac{\partial^2 U}{\partial x^2}\right\} = \frac{d^2}{dx^2}[L\{U\}] = \frac{d^2 u}{dx^2}$$

$\therefore$ Taking Laplace transform of (1)

$$\frac{d^2u}{dx^2} = \frac{1}{k} su \text{ or } \frac{d^2u}{dx^2} - \frac{s}{k}u = 0 \qquad \text{...(2)}$$

with boundary condition $u(o, s) = L\{U(o, t)\} = U_0 L\{1\} = \frac{U_0}{s}$ ...(3)

and $u = u(x, s)$ is required to be bounded.

The solution of (2) is $u(x, s) = C_1 e^{x\sqrt{s/k}} + C_2 e^{-x\sqrt{s/k}}$ ...(4)

In order that u be bounded (finite) as $x \to \infty$, we must choose $C_1 = 0$ otherwise the temperature will be infinite when $x \to \infty$.

Thus (4) reduces to $u(x, s) = C_2 e^{-x\sqrt{s/k}}$

But from (3) $u(x, s) = \frac{U_0}{s}$ when $x = 0$, therefore $C_2 = \frac{U_0}{s}$

Hence the solution of (2) is $u(x, s) = \frac{U_0}{s} e^{-x\sqrt{s/k}}$

Taking inverse Laplace transform, we find

$$U(x, t) = U_0 \operatorname{erf} C\left(\frac{x}{2\sqrt{kt}}\right) = U_0\left\{1 - \frac{2}{\sqrt{\pi}}\int_0^{x/2\sqrt{kt}} e^{-u^2} du\right\}$$

which gives the required temperature.

Problem 49. *An infinitely long string having one end at $x = 0$ is initally at rest on the x-axis. The end $x = 0$ undergoes a periodic transverse displacement given by $A_0 \sin \omega t$, $t > 0$. Find the displacement of any point on the string at any time t.*

Let $Y(x, t)$ be the transverse displacement of the string at any point x and at any time t. Then the transverse displacement is given by

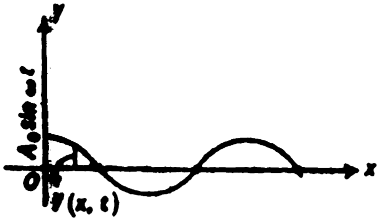


Fig. 10.6

$$\frac{\partial^2 Y}{\partial t^2} = a^2 \frac{\partial^2 Y}{\partial x^2}, \ x > 0, \ t > 0 \qquad ...(1)$$

where $a^2 = \frac{T}{\rho}$, T being the constant tension in the string and ρ the constant linear density. The boundary conditions are

(i) $Y(x, o) = 0$, (ii) $\frac{\partial Y}{\partial t}(x, o) = 0$,

(iii) $Y(o, t) = A_0 \sin \omega t$, and

(iv) the displacement is bounded i.e.,

$$|Y(x, t)| < M.$$

Assuming that $L\{Y(x, t)\} = y(x, s)$ i.e., $L^{-1}\{Y(x, s)\} = Y(x, t)$ we have

$$L\left\{\frac{\partial^2 Y}{\partial t^2}\right\} = s^2 L\{Y\} - s\,Y(x, o) - \frac{\partial Y}{\partial t}(x, o) = s^2 y - o - o$$

and $$L\left\{\frac{\partial^2 Y}{\partial x^2}\right\} = \frac{d^2}{dx^2} L\{Y\} = \frac{d^2 y}{dx^2}.$$

∴ Taking Laplace transform of (1), we get

$$s^2 y = a^2 \frac{d^2 y}{dx^2} \text{ i.e., } \frac{d^2 y}{dx^2} - \frac{s^2}{a^2} y = 0 \qquad ...(2)$$

with the boundary condition $Y(o, t) = L\{Y(o, t)\} = L\{A_0 \sin \omega t\}$

$$= \frac{A_0 \omega}{s^2 + \omega^2} \text{ and } y(x, s) \text{ is bounded.} \qquad ...(3)$$

The solution of (2) is $y(x, s) = C_1 e^{-sx/a} + C_2 e^{-sx/a}$ ...(4)

But $y(x, s)$ being bounded when $x \to \infty$, $C_1 = 0$ otherwise the displacement will be infinite when $x \to \infty$.

∴ (4) reduces to $y(x, s) = C_2 e^{-sx/a}$

Applying (3), we find $C_2 = \frac{A_0 \omega}{s^2 + \omega^2}$, therefore the solution of (2) is

$$y(x, s) = \frac{A_0 \omega}{s^2 + \omega^2} e^{-sx/a}$$

Taking inverse Laplace transform, we get

$$y(x, t) = \begin{cases} A_0 \sin \omega (t - x/a), & t > x/a \\ o, & t < x/a \end{cases}$$

which shows that a point of the string is at rest until $t = x/a$ and there after it undergoes motion identical with that of the end $x = 0$ but lags behind it in time by the amout x/a.

10.20. THE COMPLEX INVERSION FORMULA OR INVERSION THEOREM

IF $L\{F(t)\} = f(s)$, where $F(t)$ has a continuous derivative and is of exponential order, then

$$F(t) = \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} f(s)\, ds, \ t > 0. \qquad ...(1)$$

In the previous chapter we have introduced that if $\phi(t)$ be a function of t having continuous derivative such that $\int_{-\infty}^{\infty} \phi(t)\, dt$ is absolutely convergent, then the Fourier's integral is

$$\phi(t) = \frac{1}{\pi}\int_0^{\infty} dx \int_{-\infty}^{\infty} \phi(u) \cos x(t-u)\, du \qquad \text{...(2)}$$

But $\cos x(t-u)$ being an even function of x and $\sin x(t-u)$ an odd function of x, we can expres (2) as

$$\phi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \phi(u) \cos x(t-u)\, du$$

and
$$0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \phi(u) \sin x(t-u)\, du$$

or
$$\phi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \phi(u)\, [\cos xt \cos xu + \sin xt \sin xu]\, du \qquad \text{...(3)}$$

and
$$0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \phi(u)\, [\sin xt \cos xu - \cos xt \sin xu]\, du \qquad \text{...(4)}$$

Multiplying (4) by i and adding to (3) we get

$$\phi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \phi(u)\, [(\cos xt \cos xu + \sin xt \sin xu] + i(\sin xt \cos xu - \cos xt \sin xu)]\, du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \phi(u)\, [(\cos xt + i \sin xt)(\cos xu - i \sin xu)]\, du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \phi(u) \cdot e^{ixt} e^{-ixu}\, du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixt}\, dx \int_{-\infty}^{\infty} \phi(u) e^{isu}\, du$$

Assuming that $\phi(t) = e^{-\gamma t} F(t)$ for $t > 0$ and $\phi(t) = 0$ for $t < 0$ so that the Integral theorem is also satisfied, we have for $t > 0$

$$e^{-\gamma t} F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixt}\, dx \int_{-\infty}^{\infty} e^{-\gamma u} F(u)\, e^{-ixu}\, du,\ \gamma > t$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixt}\, dx \int_{-\infty}^{\infty} e^{-(\gamma+ix)u} F(u)\, u$$

$\therefore\ e^{-\gamma t} F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixt}\, dx f(\gamma + ix)$, put $\gamma + ix = s$ so that $ds = i dx$

$$= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{(s-\gamma)t} f(s)\, ds = \frac{e^{-\gamma t}}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} f(s)\, ds$$

i.e.
$$F(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} f(s)\, ds.$$

Aliter. By Cauchy's integral formula, mentioned in chapter on complex variables, we have

$$f(s) = \frac{1}{2\pi i}\int_C \frac{f(z)\, dz}{z-s}.$$

Assuming that $f(z)$ is regular on the right of $x = \gamma$ and distorting the contour C and integrating from $\gamma + i\infty$ to $\gamma - i\infty$, then to the right of $\gamma - i\infty$ to $\gamma + i\infty$ and back to $\gamma + i\infty$, we get,

$$f(s) = \frac{1}{2\pi i}\int_{\gamma+i\infty}^{\gamma-i\infty}\frac{f(z)\,dz}{z-s}$$

$$= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{f(z)\,dz}{s-z} \quad \ldots(5)$$

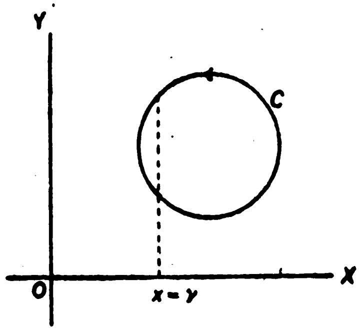


Fig. 10.7

Now since γ must be smaller than the real part of s *i.e.* $\gamma < R(s)$ and assuming that $L^{-1}\{f(s)\} = F(t)$, we get on taking Laplace transform of (5),

$$L^{-1}\{f(s)\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f(z)\,dz \cdot L^{-1}\left\{\frac{1}{s-z}\right\}$$

i.e., $$F(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f(z)\,e^{zt}\,dz \quad \because L\{e^{zt}\} = \int_0^\infty e^{-st}e^{zt}dt$$

$$= \frac{1}{s-z} \text{ if } R(s) > R(z)$$

$$= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f(s)\,e^{st}\,ds \text{ (on replacing } z \text{ by } s)$$

= sum of residues of $e^{st}f(s)$ at the poles of $f(s)$

= Σ (residues of $e^{st}f(s)$ at poles of $f(s)$) ...(6)

This gives the use of *residue theorem* in finding Inverse Laplace Transform.

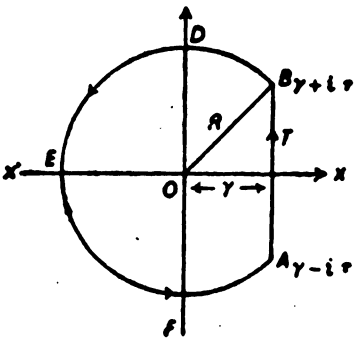


Fig. 10.8

Note. *Referred to Fig. 10.8, if arc BDEFA =* Γ and

$T = \sqrt{R^2 - r^2}$, then

$$F(t) = \lim_{R\to\infty}\frac{1}{2\pi i}\int_{\gamma-iT}^{\gamma+iT} e^{st} f(s)\,ds$$

or $$F(t) = \lim_{R\to\infty}\left\{\frac{1}{2\pi i}\oint_C e^{st} f(s)\,ds - \frac{1}{2\pi i}\int e^{st}f(s)\,ds\right\}$$

where C is the contour of Fig. 10.8 and known as *Bromwich Contour.*

Problem 50. *Evaluate* $L^{-1}\left\{\frac{1}{(s-1)(s-2)^2}\right\}$ *by residue-method.*

We have $$L^{-1}\left\{\frac{1}{(s+1)(s-2)^2}\right\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{st}\,ds}{(s+1)(s-2)^2}$$

$$= \frac{1}{2\pi i}\oint_C\frac{e^{st}\,ds}{(s+1)(s-2)^2} = \text{sum of residues of } \frac{e^{st}}{(s+1)(s-2)^2}$$

at poles $s = -1$ (simple pole) and $s = 2$ (pole of order 2).

But residue of $\frac{e^{st}}{(s+1)(s-2)^2}$ *at* $s=1$ is $\underset{s\to -1}{\text{Lim}}\ (s+1)\ f(s)$ where

$f(s)=\frac{e^{st}}{(s+1)(s-2)^2}$; $\qquad = \underset{s\to -1}{\text{Lim}} \frac{e^{st}}{(s-2)^2}=\frac{1}{9}e^{-t}$

and residue of $\frac{e^{st}}{(s+1)(s-2)^2}$ at $s=2$ (order two)

$$= \underset{s\to 2}{\text{Lim}} \frac{1}{\lfloor 1} \frac{d}{ds}\left[(s-2)^2\cdot\frac{e^{st}}{(s+1)(s-2)^2}\right] = \underset{s\to 2}{\text{Lim}} \frac{d}{ds}\left[\frac{e^{st}}{(s+1)}\right]$$

$$= \underset{s\to 2}{\text{Lim}} \frac{\{(s+1)\ t-1\}}{(s+1)^2} e^{st} = \frac{1}{3}te^{2t} - \frac{1}{9}e^{2t}$$

Hence $L^{-1}\left\{\frac{1}{(s+1)(s-1)^2}\right\} = \frac{1}{9}e^{-t}+\frac{1}{3}te^{2t}-\frac{1}{9}e^{2t}.$

Problem 51. *Solve the following differential equation* $\frac{d^2X}{dt^2}+\lambda^2 X=0.$

Given equation is $\qquad \frac{d^2X}{dt^2}+\lambda^2 X=0 \qquad$ ...(1)

Assuming $L\{X(t)\}=x(s), X(0)=c_1$ and $X'(0)=c_2$, we have

$$L\left\{\frac{d^2X}{dt^2}\right\}=s^2 L\{X\}-sX(0)-X'(0)=s^2x-sc_1-c_2.$$

$\therefore$ taking Laplace transform of (1), we find

$$L\left\{\frac{d^2X}{dt^2}\right\}\lambda^2 L\{X\}=0 \ i.e.\ s^2x-sc_1-c_2+\lambda^2x=0 \text{ or } (s^2+\lambda^2)x=sc_1+c_2$$

or $$x=\frac{sc_1+c_2}{s^2+\lambda^2}=\frac{sc_1+c_2}{(s+i\lambda)(s-i\lambda)}=\frac{\frac{c_1}{2}+\frac{ic_2}{\lambda}}{s+i\lambda}+\frac{\frac{c_1}{2}-\frac{ic_2}{\lambda}}{s-i\lambda}$$

$$=\frac{A_1}{s+i\lambda}+\frac{A_2}{s-i\lambda} \text{ where } A_1=\frac{c_1}{2}+\frac{ic_2}{\lambda} \text{ and } A_2=\frac{c_1}{2}-\frac{ic_2}{\lambda}$$

Applying Laplace inversion theorem, we get

$$X=\frac{A_2}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}\frac{e^{st}}{s+i\lambda}ds+\frac{A_2}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{st}}{s-i\lambda}ds=A_1I_1+A_2I_2 \text{ (say)}$$

Here $I_1=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{st}}{s+i\lambda}ds.$ Put $s+i\lambda=S$ *i.e.* $ds=dS$

$$=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{(S-i\lambda)t}}{S}dS=e^{-\lambda t}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{St}\,dS}{S}$$

$$=e^{-i\lambda}\text{ (Residue of } \frac{e^{St}}{S} \text{ at } S=0)=e^{-i\lambda t}\underset{S\to 0}{\text{Lim}}\ S\cdot\frac{e^{St}}{S}=e^{-i\lambda t}$$

Similarly $I_2=e^{i\lambda t}$.

Hence $X=A_1e^{-i\lambda t}+A_2e^{i\lambda t}$ which is the required solution.

Problem 52. *An infinitely long circular cylinder of unit radius has a constant initial temperature U_0. At $t = 0$, a temperature of 0^0C is applied to the surface and is maintained. Find the temperature at any point of the cylinder at any time t.*

Take any point P of the cylinder. Taking z-axis along the axis of the cylinder and x, y axes as shown in Fig. 10.9, let (r, ϕ, z) be the cylindrical coordinates of P. Now the axis of z being coincident with the axis of the cylinder, the temperature is independent of ϕ and z and as such it may be taken as $U(r, t)$.

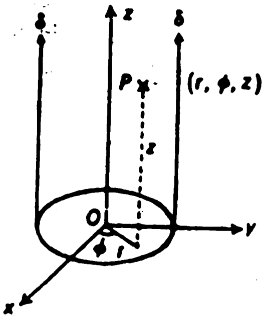


Fig. 10.9

The temperature at any point of the cylinder is given by

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}\right),\ 0 < r < 1 \qquad ...(1)$$

with the boundary conditions (*i*) $U(1, t) = 0$, (*ii*) $U(r, o) = U_0$ and (*iii*) the temperature is finite (bounded) *i.e.* $|U(r, t)| < M$.

The equation (1) is equivalent to

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} \qquad ...(2)$$

where t is to be replaced by kt since $\frac{1}{k}\frac{\partial U}{\partial t} = \frac{\partial U}{\partial (kt)}$

Assuming that $L\{U(r, t)\} = u(r, s)$, we have

$$L\left\{\frac{\partial U}{\partial r}\right\} = \frac{d}{dr}L\{U\} = \frac{du}{dr},\ L\left\{\frac{\partial^2 U}{\partial r^2}\right\} = \frac{d^2}{dr^2}L\{U\} = \frac{d^2 u}{dr^2}$$

and $$L\left\{\frac{\partial U}{\partial t}\right\} = sL\{U\} - U(r, o) = su - U_0$$

$\therefore$ taking Laplace transform of (2), we get

$$su(r, s) - U_0 = \frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr}\ i.e. \frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - su = -U_0 \qquad ...(3)$$

with the boundary condition, $u(1, s) = 0$ and $u(r, s)$ is bounded.

In order to find the general solution of (3), let us take two steps.

First step. The complimentary function of (3) is given by

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - su = 0 \text{ or } \frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + (si^2) = 0 \because\ i^2 = -1$$

or $\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \left(i\sqrt{s}\right)^2 u = 0$, which is modified Bessel's equation of order zero and may also be expressed as

$$\frac{1}{\left(i\sqrt{s}\right)^2}\frac{d^2u}{dr^2} + \frac{1}{\left(i\sqrt{s}\right)^2}\cdot\frac{1}{r}\cdot\frac{du}{dr} + u = 0 \text{ or } \frac{d^2u}{d\left(i\sqrt{s}r\right)^2} + \frac{1}{i\sqrt{s}r}\frac{du}{d\left(i\sqrt{s}r\right)} + u = 0,$$

or $\frac{d^2u}{dp^2} + \frac{1}{p}\frac{du}{dp} + u = 0$ when $i\sqrt{s}\,r = p$.

Being Bessel equation of zero order, its solution is

$u = a_0 J_0 (p)$ *i.e.* $u(r, s) = a_0 J_0 \left(i\sqrt{s}\, r\right)$, a_0 being arbitrary constant ...(4)

and this is the complementary function of (3).

Second step. The particular integral of (3) is given by

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - su = -U_0.$$

As discussed in the Chapter of differential equations, its reduced equation on changing the independent variable r to z by the substitution,

$$z = \int e^{-\int \frac{1}{r} dx} dr = \int \frac{1}{\sqrt{r}} dr = 2\sqrt{r}, \text{ becomes}$$

$$\frac{d^2u}{dz^2} + Q_1 u = X_1 \text{ where } Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{-s}{\frac{1}{r}} = -rs \text{ and } X_1 = \frac{X}{\left(\frac{dz}{dx}\right)^2} = -U_0 \cdot r$$

or $$\frac{d^2u}{dz^2} - sru = -U_0 r \text{ or } (D^2 - sr)\, u = -U_0\, s \text{ where } D = \frac{d}{dz}$$

$$\therefore \text{ Particular integral} = \frac{-U_0 r}{D^2 - rs} = \frac{U_0}{s}\left\{1 - \frac{D^2}{rs}\right\}^{-1} 1 = \frac{U_0}{s}\left\{1 + \frac{D^2}{rs} + \ldots\right\} \cdot 1 = \frac{U_0}{s}$$

Hence the complete integral of (3) is

$$u(r, s) = a_0 J_0 \left(i\sqrt{s}\, r\right) + \frac{U_0}{s}.$$

Initially when $r = 1$, $u(r, s) = u(1, s) = 0$ gives $a_0 = \dfrac{-U_0}{s J_0\left(i\sqrt{s}\right)}$

Thus $$u(r, s) = \frac{U_0}{s} - \frac{U_0 J_0\left(i\sqrt{s}\, r\right)}{s J_0\left(i\sqrt{s}\right)}.$$

Now using complex inversion formula, we find

$$U(r, t) = U_0 - \frac{U_0}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{st} J_0\left(i\sqrt{s}\, r\right)}{s J_0\left(i\sqrt{s}\right)} ds \qquad \because L^{-1}\left\{\frac{1}{s}\right\} = 1 \qquad ...(5)$$

Here $$\frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{st} J_0\left(i\sqrt{s}\, r\right)}{s J_0\left(i\sqrt{s}\right)} ds = \text{sum of residues of } \frac{e^{st} J_0\left(i\sqrt{s}\, r\right)}{s J_0\left(i\sqrt{s}\right)}$$

at the poles given by $J_0(i\sqrt{s}) = 0$

Now $J_0(i\sqrt{s})$ has simple poles at the points where

$$i\sqrt{s} = \lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n, \ldots \text{ and at } s = 0$$

$$= \lambda_n \text{ where } n = 1, 2, 3, \ldots n, \ldots \text{and at } s = 0$$

i.e. $s = -\lambda_n^2$ where $n = 1, 2, 3, \ldots n$ and at $s = 0$

Residue of $\dfrac{e^{st} J_0\left(i\sqrt{s}\, r\right)}{s J_0\left(i\sqrt{s}\right)}$ at $s = 0$ is $= \lim\limits_{s \to 0} (s - o) \cdot \dfrac{e^{st} J_0\left(i\sqrt{s}\, r\right)}{s J_0\left(i\sqrt{s}\right)} = \dfrac{e^0 J_0(o)}{J_0(o)} = 1$

and Residue of $\dfrac{e^{st} J_0\left(i\sqrt{s}\, r\right)}{sJ_0\left(i\sqrt{s}\right)}$ at $s = -\lambda_n^2$ is $= \lim\limits_{s\to-\lambda_n^2} \left(s+\lambda_n^2\right)\cdot\dfrac{e^{st} J_0\left(i\sqrt{s}\, r\right)}{sJ_0\left(i\sqrt{s}\right)}$

$$= \lim_{s\to-\lambda_n^2} \frac{e^{st} J_0(i\sqrt{s}\, r)}{s} \lim_{s\to-\lambda_n^2} \frac{s+\lambda_n^2}{J_0(i\sqrt{s})}$$

$$= e^{-\lambda_n^2 \cdot t}\frac{J_0(\lambda_n r)}{-\lambda_n^2} \lim_{s\to-\lambda_n^2} \frac{s+\lambda_n^2}{J_0(i\sqrt{s})} \left(\text{form } \frac{0}{0}\right)$$

$$= \frac{e^{-\lambda_n^2 \cdot t} J_0(\lambda_n r)}{-\lambda_n^2} \lim_{s\to-\lambda_n^2} \frac{1}{J'_0(i\sqrt{s})(i/2\sqrt{s})} \quad \text{(differentiating num and denom. w.r.t. s.)}$$

$$= \frac{e^{-\lambda_n^2 \cdot t} J_0(\lambda_n r)}{-\lambda_n^2}. \qquad \because \frac{i}{2\sqrt{s}} = \frac{i^2}{2i\sqrt{s}} = \frac{-1}{2\lambda n}$$

$$= \frac{2e^{-\lambda_n^2 \cdot t} J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n)} \qquad \because J'_0(x) = -J_1(x)$$

Hence (5) yields,

$$U(r,\ t) = U_0 - U_0\left\{1 - \sum_{n=1}^{\infty} \frac{2e^{-\lambda_n^2 t} J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n)}\right\} = 2U_0 \sum_{n=1}^{\infty} \frac{e^{-\lambda_n^2 t} J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n)}$$

Replacing t by kt, the required solution is

$$U(r,\ t) = 2U_0 \sum_{n=1}^{\infty} \frac{2e^{-kt\lambda_n^2} J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n)}$$

ADDITIONAL MISCELLANEOUS PROBLEMS

Problem 53. *Solve* $\dfrac{d^2x}{dt^2} + \omega^2 x = \cos \omega t,\ t > 0$, *subject to the initial conditions*

$$x = x_0 \text{ and } \frac{dx}{dt} = x_1 \text{ when } t = 0.$$

Ans. $x = x_0 \cos \omega t + x_l \dfrac{\sin \omega t}{\omega} + \dfrac{t}{2\omega} \sin \omega t.$

Problem 54. *Solve* $3\dfrac{dx_1}{dt} + 2x_1 + \dfrac{dx_2}{dt} = 1,\ t \geq 0$ *and* $\dfrac{dx_1}{dt} + 4\dfrac{dx_2}{dt} + 3x_2 = 0$, *subject to the initial conditions* $x_1 = 0 = x_2$ *when* $t = 0$.

Ans. $x_1 = \dfrac{1}{2} - \dfrac{1}{5}e^{-t} - \dfrac{3}{10}e^{-6t/11}$ *and* $x_2 = \dfrac{1}{5}\left(e^{-t} - e^{-6t/11}\right)$.

Problem 55. *A constant electromotive force E is applied at $t = 0$ to an electrical circuit consisting of an inductance L, resistance R and capacitance C in series. The initial values of the current i and the charge (q) on the capacitor are 0. Find the the current.*

Hint: The current is given by $L\dfrac{di}{dt} + R_i + \dfrac{q}{c} = E$ *where* $\dfrac{dq}{dt} = i$.

Ans. $i = \frac{E}{\omega L} e^{-at} \sin \omega t$ if $\omega^2 > 0$

$= \frac{E}{L} te^{-at}$ if $\omega^2 = 0$ $\Bigg\}$ where $a = \frac{R}{2L}$ and $\omega^2 = \frac{1}{LC} - \frac{R^2}{4L^2}$

$= \frac{E}{kL} e^{-at} \sinh kt$ if $\omega^2 < 0$ where $k^2 = -\omega^2$

Problem 56. *Solve Bessel's equation* $\frac{d^2 z}{du^2} + \frac{1}{u}\frac{dz}{du} + \left(1 - \frac{n^2}{u^2}\right) z = 0.$

Ans. $z = J_n(u)$.

Problem 57. *Show that the solution of* $\frac{d^2 y}{dt^2} + a^2 y = F(t)$ *subject to the initial conditions* $y = 0 = y'$ *when* $t = 0$, *is* $y = \frac{1}{a}\int_0^t F(u) \sin a(t-u)\, du$.

Problem 58. *A particle of mass m can perform small oscillations about a position of equilibrium under a restoring force* mn^2 *times the displacement. It is started from rest by a constant force* F_0 *which acts for a time T and then ceases. Show that the amplitude of the subsequent oscillation is* $(2F_0/mn^2) \sin (nT/2)$.

Problem 59. *If* L_n *be the Laguerre Polynomial given by* $L_n(x) = \frac{e^x}{\lfloor n} \frac{d^n}{dx^n}(e^{-x} x^n)$

then show $L\{L_n(x)\} = \frac{(s-1)^n}{s^{n+1}}$.

Problem 60. *Prove the following:*

(i) $L\left\{\frac{2}{t}\sinh t\right\} = \log \frac{s+1}{s-1}$, (ii) $L\left\{\int_0^t J_0(\mu)\, J_0(t-\mu)\, d\mu\right\} = \frac{1}{s^2+1}$.

(ii) $L^{-1}\left\{\frac{f(s)}{\sinh(\mu s)}\right\} = 2\sum_{k=0}^{\infty} F(t - 2k\cdot\mu - \mu)\, \phi(t - 2k\mu - \mu)$.

Problem 61. (*a*) *Find the Laplace transforms of the functions* e^{-at}, $\cos(bt + a)$, $(t^2 + 1)$

(*b*) *Solve by means of Laplace transforms*

$z'' + 2z' + \int_0^t y\, dt = t;\ z'' + 2z' + y = \sin 2t;\ z(0) = -1;\ z'(0) = 1$ (*Bombay, 1965*)

Problem 62. (*a*) *Define Laplace transform of a function* $f(t)$, *obtain the Laplace transforms for* (*i*) t^n, (*ii*) e^{-at}, (*iii*) $\sin \omega t$, (*iv*) te^{-at}, (*v*) $\int_0^t f(t)\, dt$.

(*b*) *Solve the following differential equation:* $\frac{d^2 x}{dt^2} + 4\frac{dx}{dt} + 13x = 0$

using Laplace transforms, given that $x = 3$ *and* $\frac{dx}{dt} = 6$ *when* $t = 0$. (*Bombay, 1970*)

Problem 63. *Use Laplace transform to solve*<

(*a*) $\frac{d^2 x}{dt^2} + qy = 0$ *given that* $y(0) = 0$, $y(0) = 2$; (*Meerut, 1980*)

(*b*) $\frac{d^2 x}{dt^2} - 2\frac{dx}{dt} + 2x$, *given that* $x_0 = x_1 = 0$; (*Meerut, 1982*)

[Ans: $x(t) = e^t \cos t$]

CHAPTER 11

HANKEL TRANSFORMS

11.1. INTRODUCTION

In §7.5 [*B*], we have mentioned the form of Laplace's equation as

$$\nabla^2 V \equiv \frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0 \qquad ...(1)$$

This can be transformed to cylindrical coordinates (u, ϕ, z) by using the transformation $x = u\cos\phi$, $y = u\sin\phi$, $z = z$ so that $u^2 = x^2+y^2$ and $\tan\phi = \frac{y}{x}$

Giving $\frac{\partial u}{\partial x}=\frac{x}{u}=\cos\phi$, $\frac{\partial u}{\partial y}=\frac{y}{u}=\sin\phi$, $\frac{\partial \phi}{\partial x}=-\frac{\sin\phi}{u}$ and $\frac{\partial \phi}{\partial y}=\frac{\cos\phi}{u}$

$$\therefore \quad \frac{\partial V}{\partial x}=\frac{\partial V}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial V}{\partial \phi}\frac{\partial \phi}{\partial x}+\frac{\partial V}{\partial z}\frac{\partial z}{\partial x}=\frac{\partial V}{\partial u}\cos\phi+\frac{\partial V}{\partial \phi}\left(-\frac{\sin\phi}{u}\right)$$

$$i.e.\quad \frac{\partial}{\partial x} \equiv \cos\phi\,\frac{\partial}{\partial u}-\frac{\sin\phi}{u}\,\frac{\partial}{\partial \phi}$$

and
$$\frac{\partial^2 V}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right)=\left(\cos\phi\,\frac{\partial}{\partial u}-\frac{\sin\phi}{u}\frac{\partial}{\partial \phi}\right)\left(\cos\phi\,\frac{\partial V}{\partial u}-\frac{\sin\phi}{u}\frac{\partial V}{\partial \phi}\right)$$

$$=\cos^2\phi\,\frac{\partial^2 V}{\partial u^2}-\frac{2\sin\phi\cos\phi}{u}\frac{\partial^2 V}{\partial u\partial\phi}+\frac{2\sin\phi\cos\phi}{u^2}\frac{\partial V}{\partial \phi}+\frac{\sin^2\phi}{u}\frac{\partial V}{\partial u}+\frac{\sin^2\phi}{u^2}\frac{\partial^2 V}{\partial \phi^2}$$

similarly
$$\frac{\partial^2 V}{\partial y^2}=\sin^2\phi\,\frac{\partial^2 V}{\partial u^2}+\frac{2\sin\phi\cos\phi}{u}\frac{\partial^2 V}{\partial u\partial\phi}-\frac{2\sin\phi\cos\phi}{u^2}\frac{\partial V}{\partial \phi}+\frac{\cos^2\phi}{u}\frac{\partial V}{\partial u}+\frac{\cos^2\phi}{u^2}\frac{\partial^2 V}{\partial \phi}$$

so that
$$\nabla^2 V=\frac{\partial^2 V}{\partial u^2}+\frac{1}{u}\frac{\partial V}{\partial u}+\frac{1}{u^2}\frac{\partial^2 V}{\partial z^2}=0 \qquad ...(2)$$

is the cylindrical form of (1).

In order to solve it by the method of separation of variables discussed in §.7.5 we put

$V = U\,\Phi Z$ ...(3)

where $U = U(u)$, $\Phi=\Phi(\phi)$, $Z = Z(z)$

We have

$$\frac{\partial V}{\partial u}=\Phi Z\,\frac{\partial U}{\partial u};\ \frac{\partial^2 V}{\partial u^2}=\Phi Z\,\frac{\partial^2 U}{\partial u^2},\ \frac{\partial^2 V}{\partial \phi^2}=UZ\,\frac{\partial^2 \Phi}{\partial \phi^2}\ \text{and}\ \frac{\partial^2 V}{\partial z^2}=U\,\Phi\,\frac{\partial^2 Z}{\partial z^2}$$

Their substitution in (2) yields,

$$\Phi Z\,\frac{\partial^2 U}{\partial u^2}+\frac{1}{u}\frac{\partial U}{\partial u}\Phi Z+\frac{1}{u^2}UZ\,\frac{\partial^2 \Phi}{\partial \phi^2}+U\,\Phi\frac{\partial^2 Z}{\partial z^2}=0 \qquad \text{...(4)}$$

Dividing throughout by $U\,\Phi\,Z$, we find

$$\frac{1}{U}\left(\frac{\partial^2 U}{\partial u^2}+\frac{1}{u}\frac{\partial U}{\partial u}\right)+\frac{1}{u^2}\frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2}+\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}=0 \qquad \text{...(5)}$$

The first three terms being independent of z, so must also be the last *i.e.* $\frac{\partial^2 Z}{\partial z^2}=CZ$, C being a constant. ...(6)

Again the first two and the last terms in (5) being independent of ϕ, so must also be the third one and hence $\frac{\partial^2 \Phi}{\partial \phi^2}=D\Phi$, D being a constant ...(7)

As such (5) becomes

$$u^2\,\frac{\partial^2 U}{\partial u^2}+u\,\frac{\partial U}{\partial u}+\ (D+Cu^2)\ U=0 \qquad \text{...(8)}$$

If we put $C=k^2, D=-n^2$, (6) and (7) become $\frac{\partial^2 Z}{\partial z^2}=k^2 Z$ and $\frac{\partial^2 \Phi}{\partial \phi^2}=-n^2\,\Phi$ whose solutions are $Z=A_1\,e^{kz}+B_1\,e^{-kz}$ and $\Phi=A_2\cos n\phi+B_2\sin n\,\phi$ respectively,

and (8) reduces to $u^2\frac{\partial^2 U}{\partial u^2}+u\frac{\partial U}{\partial u}+(k^2u^2-n^2)U=0$

With $v=ku$, this becomes, $v^2\,\frac{\partial^2 U}{\partial v^2}+v\,\frac{\partial U}{\partial v}+\ (v^2-n^2)\ U=0$

which is Bessel's differential equation of order n and its solutions are known as *cylindrical functions* or Bessel's functions of order n.

Hence if $J_n\,(v)$ be a Bessel function of order n, then the solution of Laplace's equation is $V\ =\ U\Phi Z\ =J_n\,(ku)\ e^{\pm in\phi}\ e^{\pm kz}$ ...(10)

which is known as a *Cylindrical Harmonic* and this harmonic is symmetrical about z-axis if $n=0$.

In §8.4 we have already discussed the Bessel's equation in the form

$$\frac{d^2y}{dx^2}+\frac{1}{x}\cdot\frac{dy}{dx}+\left(1-\ \frac{n^2}{x^2}\right)y=0 \qquad \text{...(11)}$$

which is the same form as (9).

Its solution is $y=AJ_n\,(x)+BJ_{-n}\,(x)$ ...(12)

A, B being arbitrary constants; and $J_n\,(x)$ is known as ***Bessel's function of first kind*** of order n given by

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{\lfloor r \; \Gamma(n+r-1)} \quad ...(13)$$

and $$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{-n+2r}}{\lfloor r \; \Gamma(-n+r-1)}$$

such that $J_n(x) = (-1)^n J_{-n}(x)$ ...(14).

The series on the right of (12) is absolutely convergent for all values of x. In case n is an integer, the second independent solution of (11) may be shown to be

$$G_n(x) = \frac{\pi}{2} \text{cosec} \left[J_{-n}(x) - e^{-in\pi} J_n(x)\right] \quad ...(15)$$

where $G_n(x)$ is known as *Bessel function of the second kind of order n.*

In this case the general solution of (11) is

$$y = AJ_n(x) + BG_n(x) \quad ...(16)$$

Also, we have the recurrence relations for Bessel's functions of the first kind

$$xJ_n' = nJ_n - xJ_{n+1} = xJ_{n-1} - nJ_n \quad ...(17)$$

$$2J_n' = J_{n-1} - J_{n+1} \quad ...(18)$$

$$2nJ_n = x\{J_{n-1} + J_{n+1}\} \quad ...(19)$$

$$\frac{d}{dx}(x^{-n}J_n) = -x^{-n} J_{n+1} \quad ...(20)$$

$$\frac{d}{dx}(x^n J_n) = x^n J_{n-1} \quad ...(21)$$

Here (21) provides the definite integral

$$\int_0^{\mu} x^n J_{n-1}(x)\, dx = \left[x^n J_n(x)\right]_0^{\mu} = \mu^n J_n(\mu),\; n > 0 \quad ...(22)$$

If we put $n = 1$ and make the substitutions $x = \xi r, \mu = \xi a$ in the integral on the left of (22), we find

$$\int_0^a r J_0(\xi r)\, dr = \frac{a}{\xi} J_1(a\xi) \quad ...(23)$$

which is frequently used in the application of the theory of *Hankel transforms* to special functions.

Also we may write

$$\int_0^a r^3 J_0(\xi r)\, dr = \int_0^a r^2 \frac{1}{\xi} \frac{\partial}{\partial r}[rJ_1(\xi r)]\, dr = \left[\frac{r^3}{\xi} J_1(\xi r)\right]_0^a - \frac{2}{\xi}\int_0^a r^2 J_1(\xi r)\, dr$$

Now putting on the right of (22), $n = 2, x = \xi r, \mu = \xi a$ and using (19) with $n = 1$, we may find $$\int_0^a r^2 J_0(\xi r)\, dr = \frac{a^2}{\xi^2}\left[2J_0\; \xi a) + \left(a\xi - \frac{4}{a\xi} J_1(\xi a)\right)\right] \quad ...(24)$$

Subtracting (24) from (23) multiplied by a^2, we find

$$\int_0^a r\,(a^2-r^2)\,J_0(\xi r)\,dr=\frac{4a}{\xi^3}J_1(\xi a)-\frac{2a^2}{\xi^2}J_0(\xi a) \quad ...(25)$$

A few more integrals involving Bessel functions may be referred as

$$\int_0^\infty e^{-ax}J_0(\xi x)\,dx=(a^2+\xi^2)^{-1/2} \quad \text{(Rohilkhand, 1982, 87)} \quad ...(26)$$

$$\int_0^\infty e^{-ax}J_1(\xi x)\,dx=\frac{1}{\xi}-\frac{a}{\xi\sqrt{(a^2+\xi^2)}} \quad ...(27)$$

$$\int_0^\infty xe^{-ax}J_0(\xi x)\,dx=a\,(a^2+\xi^2)^{-3/2} \quad ...(28)$$

$$\int_0^\infty xe^{-ax}J_1(\xi x)\,dx=\xi\,(a^2+\xi^2)^{-3/2} \quad ...(29)$$

$$\int_0^\infty e^{-ax}J_1(\xi x)\frac{dx}{x}=\frac{(a^2+\xi^2)^{1/2}-a}{\xi} \quad ...(30)$$

Further if n is half an odd integer we have

$$J_n(x)=\frac{x^n}{2^n\,\Gamma(n+1)}\left[1-\frac{x^2}{\lfloor 1\;2^2\,(n+1)}+\frac{x^4}{\lfloor 2\;2^2\cdot 4^2\,(n+1)\,(n+2)}\right]$$

which gives for $n=\frac{1}{2}$ and $n=-\frac{1}{2}$ ...(31)

$$J_{1/2}(x)=\sqrt{\frac{2}{x\pi}}\sin x \text{ and } J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cos x \quad ...(32)$$

11.2. DEFINITION OF INFINITE HANKEL TRANSFORM

If $J_n(px)$ be the Bessel function of the first kind of order n, then the Hankel transform of a function $f(x)$, $(0<x<\infty)$ denoted by $\bar{f}(p)$ is defined as

$$\bar{f}(p)=\int_0^\infty f(x)\cdot xJ_n(px)\,dx \quad ...(1)$$

Here $xJ_n(px)$ is the *Kernel* of the transformation as referred in § 10.2.

We sometimes write (1) as

$$H\{f(x)\}=\bar{f}(p)=\int_0^\infty f(x)\cdot xJ_n(px)\,dx \quad ...(2)$$

Problem 1. *Taking $xJ_0(px)$ as the Kernel of the transformation, find the Hankel transform of the following functions :*

(i) $f(x)=\dfrac{e^{-ax}}{x}$, (ii) $f(x)=e^{-ax}$, (iii) $f(x)=\begin{cases}1, & 0<x<a,\ n=0\\ 0, & x>a,\ n=0\end{cases}$

(iv) $f(x)=\begin{cases}a^2-x^2, & 0<x<a,\ n=0\\ 0, & x>a,\ n=0\end{cases}$

(*i*) We have $H\{f(x)\} = \bar{f}(p) = \int_0^\infty \frac{e^{-ax}}{x} \cdot x J_0(px)\,dx$

$$= \int_0^\infty e^{-ax} J_0(px)\,dx = (a^2 + p^2)^{-1/2} \text{ by (26) of §11.1.}$$

(*ii*) $\bar{f}(p) = \int_0^\infty e^{-ax} x J_0(px)\,dx = \frac{a}{(a^2 + p^2)^{-3/2}}$ by (28) of §11.1.

Aliter. $\bar{f}(p) = \int_0^\infty e^{-ax} \cdot x J_0(px)\,dx = \left[-\frac{1}{a} e^{-x} \cdot x J_0(px)\right]_0^\infty$

$$+\frac{1}{a}\int_0^\infty e^{-ax} \frac{d}{dx}\{x J_0(px)\}\,dx \text{ (on integrating by parts)}$$

$$= 0 + \frac{1}{a}\int_0^\infty e^{-ax} \{J_0(px) + x J'_0(px)\}\,dx$$

$$= \frac{1}{a}\int_0^\infty e^{-ax} J_0(px)\,dx + \frac{1}{a}\int_0^\infty e^{-ax} x J'_0(px)\,dx$$

But we have $x J_n'(x) = n J_n(x) - x J_{n+1}(x)$ by (17) of §11.1.
Writing $x = 0$ and replacing x by px, we get $J_0'(xp) = -J_1(xp)$,

$$\therefore \quad \bar{f}(p) = \frac{1}{a}\int_0^\infty e^{-ax} J_0(px)\,dx - \frac{1}{a}\int_0^\infty e^{-ax} x J_1(px)\,p\,dx$$

$$= \frac{1}{a\left(a^2 + p^2\right)^{3/2}} - \frac{1}{a} \cdot \frac{p^2}{a\left(a^2 + p^2\right)^{3/2}} \text{ by (26) and (27) of §11.1}$$

$$= \frac{a}{\left(a^2 + p^2\right)^{3/2}}$$

(*iii*) $H\{f(x)\} = \bar{f}(p) = \int_0^\infty f(x) \cdot x J_0(px)\,dx = \int_0^a 1 . x J_0(px)\,dx + \int_a^\infty 0 . x J_0(px)\,dx$

$$= \frac{a}{p} J_1(ap) \text{ by (23) of §11.1}$$

(*iv*) $H\{f(x)\} = \bar{f}(p) = \int_0^\infty f(x) x \cdot J_0(px)\,dx$

$$= \int_0^a \left(a^2 - x^2\right) x J_0(px)\,dx + \int_a^\infty 0 \cdot x J_0(px)\,dx$$

$$= \frac{4a}{p^2} J_1(pa) - \frac{2a^2}{p^2} J_0(pa) \text{ by (25) of §11.1.}$$

Note. On the next page we have tabulated some useful Hankel transforms.

11.3. INVERSE FORMULA FOR HANKEL TRANSFORM

If $\bar{f}(p)$ be the Hankel transform of the function $f(r)$ for $-\infty < x < \infty$,

i.e., $\bar{f}(p) = \int_{-\infty}^\infty f(r) \cdot r J_n(pr)\,dr$ ...(1)

Then $f(r) = \int_{-\infty}^\infty \bar{f}(p) \cdot p J_n(pr)\,dp$ ...(2)

is said to be the inversion formula for the Hankel transform $\bar{f}(p)$ and we may write
$f(r) = H^{-1}\{\bar{f}(p)\}$ ...(3)

$f(x)$	n	$\bar{f}(p)$	$f(x)$	n	$\bar{f}(p)$
$\begin{cases} x^n, & 0<x<a \\ 0, & x>a \end{cases}$	>-1	$\frac{a^{n+1}}{p} J_{n+1}(pa)$	$\frac{e^{-ax}}{x}$	1	$\frac{1}{p} - \frac{p}{p(p^2+a^2)^{1/2}}$
$\begin{cases} 1, & 0<x<a \\ 0, & x>a \end{cases}$	0	$\frac{a}{p} J_1(ap)$	e^{-ax}	1	$p(p^2+a^2)^{-3/2}$
$\begin{cases} a^2-x^2, & 0<x<a \\ 0, & x>a \end{cases}$	0	$\frac{4a}{p^2} J_1(pa) - \frac{2a^2}{p^2} J_0(pa)$	$\frac{\sin ax}{a}$	0	$\begin{cases} 0, & p>a \\ (a^2-p^2)^{-1/2}, & 0<p<a \end{cases}$
x^{m-1}	>-1	$\frac{2m\Gamma\left(\frac{1}{2}+\frac{1}{2}\left(m+\frac{1}{2}n\right)\right)}{p^{m+1}\Gamma\left(\frac{1}{2}-\frac{1}{2}m+\frac{1}{2}n\right)}$	$\frac{\sin ax}{a}$	1	$\begin{cases} \frac{a}{\left(p^2-a^2\right)^{1/2}}, & p>a \\ 0, & p<a \end{cases}$
$x^{-2}e^{-ax}$	1	$\frac{\sqrt{p^2+a^2}-a}{p}$	$\frac{\sin x^2}{x^2}$	0	$\begin{cases} \sin^{-1}\frac{1}{p}, & p>1 \\ \frac{\pi}{2}, & p<1 \end{cases}$
e^{-ax}	0	$a(p^2+a^2)^{-3/2}$	$\frac{a}{\left(a^2+x^2\right)^{3/2}}$	0	e^{-ap}
$x^n e^{-qx^2}$	>-1	$\frac{p}{(2q)^{n+1}} e^{-p^2/4q}$	$x^{m-2} e^{-ax^2}$	>-1	$\frac{p^n \Gamma\left(\frac{n}{2}+\frac{m}{2}\right)}{2^{n+1} p^{m/2+n/2} \Gamma(1+n)} \times$ $F\left\{\frac{n}{2}+\frac{m}{2}, n+1, -\frac{p^2}{4a}\right\}$ in terms of confluent hypergeometric function.
$\frac{e^{-ax}}{x}$	0	$\left(p^2+a^2\right)^{-1/2}$			

In chapter 9, we have introduced that if $\bar{f}(s)$ be the complex Fourier transform of $f(x)$ in $-\infty < x < \infty$, *i.e.*, $\bar{f}(s) = \int_{-\infty}^{\infty} f(x) \cdot e^{isx}\, dx$

Then we have $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}(s)\, e^{-isx}\, dx.$

In §9.10 [*F*] dealing with multiple Fourier transform, we have considered the functions of two variables such as

$$\bar{f}(s,t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(s,\ y)\, e^{i(sx+ty)}\, dx\, dy \qquad ...(4)$$

then $$f(x,y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{f}(s,\ t)\, e^{i(sx+ty)}\, ds\, dt \qquad ...(5)$$

If we put $x = r\cos\theta$, $y = r\sin\theta$, $s = p\cos\alpha$, $t = p\sin\alpha$, then (4) and (5) transform to $$\bar{f}(p,\alpha) = \frac{1}{4\pi^2} \int_0^{\infty} r\, dr \int_0^{2\pi} f(r,\theta)\, e^{ipr\cos(\theta-\alpha)}\, d\theta \qquad ...(6)$$

and $$f(r,\theta) = \frac{1}{4\pi^2} \int_0^{\infty} p\, dp \int_0^{2\pi} \bar{f}(p,\alpha)\, e^{ipr\cos(\theta-\alpha)}\, d\alpha \qquad ...(7)$$

Taking $f(r,\theta) = f(r)\, e^{-in\theta}$, (6) reduces to

$$\bar{f}(p,\alpha) = \int_0^{\infty} f(r)\, r\, dr \int_0^{2\pi} e^{i\{-n\theta + pr\cos(\theta-\alpha)\}}\, d\theta$$

If we now make a substitution, $\phi = \alpha - \theta - \frac{\pi}{2}$, then it becomes

$$\bar{f}(p,\alpha) = \int_0^{\infty} f(r)\, r\, dr \int_0^{2\pi} e^{i\{n(\phi+\pi/2-\alpha) + pr\cos(\phi+\pi/2)\}}\, d\phi$$

$$= \int_0^{\infty} f(r)\, r\, dr \cdot e^{in(\pi/2-\alpha)} \int_0^{2\pi} e^{i(n\phi - pr\sin\phi)}\, d\phi$$

$$= \int_0^{\infty} f(r)\, r\, dr \cdot 2\pi\, e^{in(\pi/2-\alpha)}\, J_n(pr)\, dr$$

$$\because\ J_n(pr) = \frac{1}{2\pi} \int_0^{2\pi} e^{i(n\phi - pr\sin\phi)}\, d\phi$$

$$= 2\pi e^{in(\pi/2-\alpha)} \int_0^{\infty} f(r) \cdot r\, J_n(pr)\, dr$$

$$= 2\pi e^{in(\pi/2-\alpha)}\, \bar{f}(p) \qquad ...(8)$$

Again taking $f(r,\theta) = f(r)\, e^{-in\theta}$, and using (8), (7) yields

$$f(r)\, e^{-in\theta} = \frac{1}{2\pi} \int_0^{\infty} p\, dp \int_0^{2\pi} 2\pi, e^{in(\pi/2-\alpha)}\, \bar{f}(p) e^{-ipr\cos(\theta-\alpha)}\, d\alpha$$

$$= \frac{1}{2\pi} \int_0^{\infty} p\, \bar{f}(p)\, dp \int_0^{2\pi} e^{i\{n(\pi/2-\alpha) - pr\cos(\theta-\alpha)\}}\, d\alpha$$

Putting $\psi = \theta - \alpha + \frac{\pi}{2}$, this becomes

$$f(r)\, e^{-in\theta} = \int_0^{\infty} p\bar{f}(p)\, dp \int_0^{2\pi} e^{i\{n(\psi-\theta) - pr\cos(\pi/2-\psi)\}}\, d\psi$$

$$= \frac{1}{2\pi}\int_0^\infty p\bar{f}(p)\, dp\, e^{-in\theta}\int_0^{2\pi} e^{i(n\psi - pr\sin\psi)\}}\, d\psi$$

i.e. $f(r) = \frac{1}{2\pi}\int_0^\infty p\bar{f}(p)\, dp \cdot 2\pi\, J_n(pr)\, dp$

$$= \int_0^\infty \bar{f}(p)\, p\, J_n(pr)\, dp$$

which is required inversion formula.

Problem 2. *Find $H^{-1}\{p^2 e^{-ap}\}$ when $n = 1$.*

We have $H^{-1}\{p^2 e^{-ap}\} = \int_0^\infty p^2 e^{-ap} \cdot p\, J_1(px)\, dp$

$$= \int_0^\infty e^{-ap} J_1(px)\frac{dp}{p} = \frac{(a^2 + x^2)^{1/2} - a}{x}$$

11.4. PARSEVAL'S THEOREM FOR HANKEL TRANSFORMS

If $\bar{f}(p)$ and $\bar{g}(p)$ be respectively the Hankel transform of $f(x)$ and $g(x)$ then

$$\int_0^\infty x f(x)\, g(x)\, dx = \int_0^\infty p\bar{f}(p)\, \bar{g}(p)\, dp. \quad \text{...(1)}$$

We have $\int_0^\infty p\bar{f}(p)\, \bar{g}(p)\, dp = \int_0^\infty p\bar{f}(p)\, dp \int_0^\infty x\, g(x)\, J_n(px)\, dx$

(on substituting $\bar{g}(p) = \int_0^\infty g(x)\, x J_n(px)\, dx,\ n \geq -\frac{1}{2}$)

$$= \int_0^\infty x\, g(x)\, dx \int_0^\infty p\bar{f}(p)\, J_n(px)\, dx$$

(changing the order of integration)

$$= \int_0^\infty x\, g(x)\, f(x) \text{ by §11.3.}$$

11.5. LINEARITY PROPERTY

If $f(x)$ and $g(x)$ are two functions and C_1, C_2 two constants, then

$$H\{C_1 f(x) + C_2 g(x)\} = C_1 H\{f(x)\} + C_2 H\{g(x)\} \quad \text{...(1)}$$

We have $H\{C_1 f(x) + C_2 g(x)\} = \int_0^\infty \{C_1 f(x) + C_2\, g(x)\} \cdot x J_n(px)\, dx$

$$= C_1 \int_0^\infty f(x) \cdot x J_n(px)\, dx + C_2 \int_0^\infty g(fx) \cdot x J_n(px)\, dx$$

$$= C_1 H\{f(x)\} + C_2 H\{g(x)\}.$$

This result can be extended to any number of functions.

11.6. HANKEL TRANSFORM OF THE DERIVATIVES OF A FUNCTION

If $\bar{f}_n(p)$ be the Hankel transform of order n of the function $f(x)$ *i.e.*

$\bar{f}_n(p) = \int_0^\infty x f(x)\, J_n(px)$, then the Hankel transform of $\frac{df}{dx}$ is

$$\bar{f}'_n(p) = \int_0^\infty x \frac{df}{dx} J_n(px)\, dx \qquad \text{...(1)}$$

$$= \left[x f(x)\, J_n(px)\right]_0^\infty - \int_0^\infty f(x) \frac{d}{dx} \{x J_n(px)\}\, dx \qquad \text{...(2)}$$

(on integrating by parts)

$$= -\int_0^\infty f(x)\, \{J_n(px) + x J'_n(px)\}\, dx \qquad \text{...(3)}$$

under the assumption that $x\, f(x) \to 0$ and $x \to 0$ or $x \to \infty$

$$= -\int_0^\infty f(x)\, \{(1-n)\, J_n(px)\}\, dx - \int_0^\infty px f(x)\, J_{n-1}(px)\, dx$$

($\because x J_n(px) = px\, J_n(px) - n\, J_n(px)$ by (17) of §11.1)

$$= (n-1) \int_0^\infty f(x)\, J_n(px)\, dx - p \bar{f}_{n-1}(p)$$

$$= (n-1)\, I - p f_{n-1}(p) \text{ (say)...4 where } I = \int_0^\infty f(x)\, J_n(px)\, dx$$

The recurrence relation,

$$J_{n-1}(x) - \frac{2n}{x} J_n(x) + J_{n+1}(x) = 0 \text{ gives}$$

$$J_{n-1}(px) - \frac{2n}{px} J_n(px) + J_{n+1}(px) = 0 \quad \text{on replacing } x \text{ by } px$$

i.e. $2n\, J_n(x) = px\, J_{n-1}(x) + px\, J_{n+1}(x)$

so that
$$2nI = 2n \int_0^\infty f(x)\, J_n(px)\, dx$$

$$= p\left[\int_0^\infty x f(x)\, J_{n-1}(px)\, dx + \int_0^\infty x f(x)\, J_{n+1}(px)\, dx\right]$$

$$= p f_{n-1}(p) + p f_{n+1}(p) \qquad \text{...(5)}$$

Hence (5) reduces to

$$\bar{f}_n(p) = \frac{n-1}{2n} p \bar{f}_{n-1}(p) + \frac{n-1}{2n} p f_{n+1}(p) - p f_{n-1}\, p$$

$$= -p\left[\frac{n+1}{2n} f_{n-1}(p) - \frac{n-1}{2n} f_{n+1}(p)\right] \qquad \text{...(6)}$$

which gives the required Hankel transform of $\dfrac{df}{dx}$

Replacing n by $(n-1)$ and $(n+1)$ in succession (6) yields,

$$\bar{f}'_{n-1}(p) = -p\left[\frac{n}{2(n-1)} \bar{f}_{n-2}(p) - \frac{n-2}{2(n-1)} \bar{f}_n(p)\right]$$

and
$$\bar{f}'_{n+1}(p) = -p\left[\frac{n+2}{2(n+1)} \bar{f}_n(p) - \frac{n}{2(n+1)} \bar{f}_{n+2}(p)\right]$$

Using these results and replacng f by f' in (6) we find

$$\bar{f}''_n(p) = -p\left[\frac{n+1}{2n} \bar{f}'_{n-1}(p) - \frac{n-1}{2n} \bar{f}_{n+2}(p)\right]$$

$$= \frac{p^2}{4}\left[\frac{n+1}{n-1}\bar{f}_{n-2}(p) - 2\frac{n^2-3}{n^2-1}\bar{f}_n(p) + \frac{n-1}{n+1}\bar{f}_{n+2}(p)\right] \quad ...(7)$$

COROLLARY. Putting $n = 1, 2, 3$, successively in (6) we have

$$\bar{f}_1(p) = -p\bar{f}_0(p) \quad ...(8)$$

$$\bar{f}_2(p) = -p\left[\frac{3}{4}\bar{f}_1(p) - \tfrac{1}{4}\bar{f}_3(p)\right] \quad ...(9)$$

$$\bar{f}'_3(p) = -p\left[\frac{2}{3}\bar{f}_2(p) - \tfrac{1}{3}\bar{f}_4(p)\right] \quad ...(10)$$

11.7. HANKEL TRANSFORMS OF $\frac{d^2f}{dx^2}$, $\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx}$ and $\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx} - \frac{n^2}{x^2}f$ UNDER CERTAIN CONDITIONS

We have $H\left\{\frac{d^2f}{dx^2}\right\} = \int_0^\infty \frac{d^2f}{dx^2}\cdot xJ_n(px)\,dx = \left[\frac{df}{dx}xJ_n(px)\right]_0^\infty$

$$- \int_0^\infty \frac{df}{dx}\cdot\frac{d}{dx}\{xJ_n(px)\}\,dx.$$

Assuming that $xf(x) \to 0$ as $x \to 0, x \to \infty$, we have

$$H\left\{\frac{d^2f}{dx^2}\right\} = -\int_0^\infty \frac{df}{dx}\left[J_n(px) + pxJ'_n(px)\right]dx$$

$$= \int_0^\infty x\frac{d^2f}{dx^2}\cdot J_n(px)\,dx$$

or $$\int_0^\infty x\left(\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx}\right)J_n(px)dx = -p\int_0^\infty \frac{df}{dx}\cdot xJ'_n(px)\,dx$$

$$= -p\left|f(x)\,xJ'_n(px)\right|_0^\infty - \int_0^\infty f(x)\frac{d}{dx}\{xJ'_n(px)\}\,dx$$

$$= p\int_0^\infty f(x)\frac{d}{dx}\{xJ'_n(px)\}dx \quad ...(1)$$

$\because xf(x) \to 0$ as $x \to 0$ or ∞.

But $J_n(px)$ satisfies Bessel's differential equation *i.e.*

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) + \left(1 - \frac{n^2}{x^2}\right)xy = 0,$$

$$\therefore \quad \frac{d}{dx}\left(x\frac{dy}{dx}J_n(x)\right) + \left(1 - \frac{n^2}{x^2}\right)x\cdot J_n(x) = 0,$$

or $$\frac{1}{p}\frac{d}{dx}\left[px\cdot J'_n(px)\right] = -\left(p^2 - \frac{n^2}{x^2}\right)\frac{x}{p^2}J_n(px),$$

on replacing x by px

or $$\frac{d}{dx}[xJ'_n(px)] = -\left(p^2 - \frac{n^2}{x^2}\right)\frac{x}{p}J_n(px)$$

As such (1) reduces to

$$\int_0^\infty \left(\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx}\right)\cdot xJ_n(px)\,dx = -p\int_0^\infty f(x)\left(p^2 - \frac{n^2}{x^2}\right)\frac{x}{p}J_n(px)\,dx$$

or $$\int_0^\infty \left(\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx} - \frac{n^2}{x^2}f\right)\cdot xJ_n(px)\,dx = -p^2\int_0^\infty f(x)\cdot xJ_n(px)\,dx$$

$$= -p^2\bar{f}_n(p) \quad \text{by §11.6} \qquad \ldots(2)$$

where $\bar{f}_n(p)$ is the Hankel transform of order n of $f(x)$.

COROLLARY 1. If we put $n = 0$ in (2), we get

$$\int_0^\infty x\left(\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx}\right)J_0(px)\mathrm{dx} = -p^2\bar{f}_0(p) \qquad \ldots(3)$$

where $\bar{f}_0(p)$ is the Hankel Transform of Zeroeth order.

COROLLARY 2. If we put $n = 1$ in (2), we find

$$\int_0^\infty \left(\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx} - \frac{f}{x^2}\right)xJ_1(px) = -p^2\bar{f}_1(p) \qquad \ldots(4)$$

or $$\int_0^\infty x\frac{df}{dx}J_1(px) = -p\bar{f}(p) \qquad \ldots(5)$$

where $$\bar{f}(p) = \int_0^\infty xf(x)J_0(px)\,dx.$$

Problem 3. *Find* $H\left\{\frac{\partial f}{\partial x}\right\}$ *when* $f = \frac{e^{-ax}}{x}$ *and* $n = 1$.

We have

$$H\left\{\frac{\partial f}{\partial x}\right\} = \int_0^\infty x\frac{df}{dx}J_1(px)\,dx = -p\bar{f}_0(p) \text{ by (8) of §11.6.}$$

$$= -p\int_0^\infty xf(x)\,J_0(px)\,dx = -p\int_0^\infty e^{-ax}J_0(px)\,dx = \frac{-ap}{\left(a^2+p^2\right)^{3/2}}$$

Problem 4. *Find* $H\left\{\frac{d^2f}{dt^2}\right\}$ *when* $f = f(x, t)$.

We have $$H\left\{\frac{d^2f}{dt^2}\right\} = \int_0^\infty x\frac{\partial^2 f}{\partial t^2}J_n(px)\,dx = \frac{\partial^2}{\partial t^2}\int_0^\infty xf(x, t)\,J_n(px)\,dx$$

$$= \frac{d^2}{dt^2}\bar{f}(p, t).$$

Problem 5. *Evaluate* $\int_0^\infty \left(\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx}\right)\cdot xJ_0(px)\,dx$, *when* $f(x) = e^{-ax}$.

We have by (3) of §11.7.

$$\int_0^\infty x\left(\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}\right)J_0(px)dx = -p^2\bar{f}_0(p)$$

$$= -p^2\int_0^\infty e^{-ax}J_0(px)dx = \frac{-ap^2}{\left(a^2+p^2\right)^{3/2}}.$$

11.8. APPLICATIONS TO BOUNDARY VALUE PROBLEMS

Whenever there is symmetry about an axis in a problem, then the use of polar coordinates is advised. Using Hankel transform a variable ranging from 0 to ∞ can be excluded.

Problem 6. *Find the potential $V(r, z)$ of a field due to a flat circular disc of unit radius with centre at origin and axis along z-axis, satisfying the differential equation* $\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}+\frac{\partial^2 V}{\partial z^2}=0,\ 0\le r\le\infty,\ z\ge 0$ *and the boundary conditions:*

(i) $V = V_0$ *when* $z = 0, 0\le r<1$

(ii) $\frac{\partial V}{\partial z}=0$ *when* $z=0,\ r>1.$

Taking the Hankel transform of the given equations for $n = 0$, we have

$$\int_0^\infty\left(\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}\right)rJ_0(pr)dr = -\int_0^\infty\frac{\partial^2 V}{\partial z^2}rJ_0(pr)dr$$

or
$$-p^2\bar{V} = -\frac{d^2\bar{V}}{dz^2}\quad\text{where }\bar{V}=\int_0^\infty V\cdot rJ_0(pr)dr \qquad ...(1)$$

i.e.,
$$\frac{d^2\bar{V}}{dz^2}-p^2\bar{V}=0\quad\text{whose solution is }\bar{V}(p,z)=Ae^{+pz}+Be^{-pz} \qquad ...(2)$$

Now since $V\to 0$ as $z\to\infty$ and so $\bar{V}\to 0$ as $z\to\infty$, therefore $A = 0$. As such (2) reduces to $\bar{V}(p, z) = Be^{-pz}$.

But B being independent of z may be taken as $B(p)$ *i.e.* function of p alone.

$\therefore \bar{V}(p, z) = B(p)e^{-pz}$

Applying inversion formula,

$$V(r, z)=\int_0^\infty B(p)\,e^{-pz}\cdot pJ_0(pr)dp \qquad ...(3)$$

Applying the conditions when $z = 0$,

$$V=\int_0^\infty p\cdot B(p)\,J_0(pr)dp = V_0,\ 0\le r<1 \qquad ...(4)$$

and
$$\left(\frac{\partial V}{\partial z}\right)_{z=0}=\int_0^\infty -p^2 B(pr)J_0(pr)dp = 0\ r>1 \qquad ...(5)$$

Comparing (4) and (5) with the integrals

$$\int_0^\infty J_0(pr)\frac{\sin p}{p}dp=\frac{\pi}{2},\ 0\le r<1 \text{ and}$$

$$\int_0^\infty J_0(pr)\sin p\,dp = 0,\ r>0 \text{ respectively, then we find}$$

$$B(p) = \frac{2}{\pi} V_0 \frac{\sin p}{p}.$$

Hence the required solution is

$$V(r,\ z) = \frac{2V_0}{\pi} \int_0^\infty e^{-pz} \cdot \frac{\sin p}{p} J_0(pr)\,dp.$$

Problem 7. *The magnetic potential V for a circular disc of radius a and strength ω, magnetised parallel to its axis, satisfying Laplace's equation is equal to 2πω on the disc itself and vanishes at exterior points in the plane of the disc. Show that at the point (r, z), $z > 0,\ V = 2\pi\omega \int_0^\infty e^{-pz} J_0(pr) J_1(pa)\,dp$.*

The magnetic potential V satisfies the Laplace's equation

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial z^2} = 0,\ \ 0 < r < \infty \qquad \text{...(1)}$$

With the boundary condition

$$V = 2\pi\omega,\ 0 \le r < a,\ z = 0 \text{ and } V = 0,\ r > a,\ z = 0.$$

Hankel transform of (1) for $n = 0$ gives

$$\int_0^\infty \left(\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r}\right) rJ_0(pr)\,dr + \int_0^\infty \frac{\partial^2 V}{\partial z^2} \cdot rJ_0(pr)\,dr = 0.$$

or $$-p^2 \bar{V} + \frac{d^2\bar{V}}{dz^2} = 0 \quad \text{where } \bar{V} = \int_0^\infty V \cdot rJ_0(pr)\,dr \qquad \text{...(2)}$$

and $$[\bar{V}]_{z=0} = \int_0^a [V]_{z=0} \cdot rJ_0(pr)\,dr + \int_a^\infty [V]_{z=0}\, rJ_0(pr)\,dr$$

$$= \int_0^a 2\pi\omega.\ rJ_0(pr)\,dr + 0 = 2\pi\omega \int_0^a \frac{1}{p}\frac{d}{dr}(rJ_1(pr))\,dr$$

$$= \frac{2\pi\omega a}{p} J_1(pa) \qquad \text{...(3)}$$

Solution of (2) is $\bar{V} = Ae^{pz} + Be^{-pz}$ ...(4)

But $\bar{V} \to 0$ as $z \to \infty$ $\therefore$ $A = 0$ and so (4) reduces to $\bar{V} = Be^{-pz}$ ...(5)

From (3) and (5), we have $[\bar{V}]_{z=0} = B = \dfrac{2\pi\omega a}{p} J_1(pa)$

Hence $\bar{V} = \dfrac{2\pi\omega a}{p} J_1(pa)\, e^{-pz}$

Applying inversion formula we find

$$V(r,\ z) = 2\pi\omega a \int_0^\infty e^{-pz} J_0(pr)\ J_1(pr)\ dp.$$

11.9. FINITE HANKEL TRANSFORMS

If $f(r)$ be a function satisfying Dirichlet's conditions (given in §9.2) in the interval (0, a), then its finite Hankel transform is defined as

$$\bar{f}(p_i) = \int_0^a r f(r)\ J_n(p_i r)\ dr \qquad \text{...(1)}$$

where p_i is a positive root of the transcendental equation $J_n(p_i a) = 0$ ...(2)

Also at any point in (0, a) where f (x) is continuous, we have

$$f(r) = \frac{2}{a^2}\sum_i \bar{f}(p_i)\, \frac{J_n(p_i r)}{\{J'_n(p_i a)\}^2} \quad ...(3)$$

Note 1. *For the sake of convenience the upper limit a is generally converted to unity (1) by suitable transformation.*

Note 2. *All the roots of* $J_n(\lambda_i) = 0$ *are real and unrepeated (except* $\lambda_i = 0$*). It is more convenient to take the sum over all the positive roots* λ_i*. As such when n = 0, (3) can be easily transformed by substitution* $p_i = \frac{\lambda_i}{a}$*. In case a = 1, the equation is automatically reduced to* $J_0(p_i) = 0$.

Note 3. *When n = 0 and a = 1, then* $J_0'(x) = -J_1(x)$ ...(4)

so that the inversion formula (3) becomes

$$f(r) = 2\sum_i \bar{f}(p_i)\, \frac{J_0(p_i r)}{\{J_1(p_i)\}^2} \quad ...(5)$$

where p_i *are roots of* $J_0(p_i) = 0$ ...(6)

Note 4. *If f (r) is represented by* **generalized Fourier—Bessel series**

$$f(r) = \sum_i C_i J_n(p_i r),\ 0 \le r \le a \quad ...(7)$$

then the coefficient C_i *is given by*

$$C_i = \frac{2}{a^2 J_{n+1}^2(p_i a)} \int_0^a r f(r)\, J_n(p_i r)\, dr$$

$$= \frac{2\bar{f}(p_i)}{a^2\left[J_{n+1}(p_i a)\right]^2} = \frac{2\bar{f}(p_i)}{a^2\left[J'_n(p_i a)\right]^2} \text{ using (4)} \quad ...(8)$$

Hence $$f(r) = \frac{2}{a^2}\sum_i \bar{f}(p_i)\, \frac{J_n(p_i r)}{\left[J'_n(p_i a)\right]^2} \quad ...(9)$$

In case a = 1, this becomes

$$f(r) = 2\sum_i \bar{f}(p_i)\, \frac{J_n(p_i r)}{J_n^2(p_i)} \quad ...(10)$$

Summations being taken over $J_n(p_i) = 0$.

Note 5. *f (r) is the inverse Hankel transform of* $\bar{f}(p_i)$.

Note 6. $r J_n(p_i r)$ *is the Kernel of the transform.*

Another forms of Hankel Transform

[1] When, the range of variation does not include the origin *i.e.* $f(r)$ satisfies Dirichlets' condition in $0 < b \le r \le a$, then the finite Hankel transform is defined as

$$\bar{f}(p_i) = \int_a^b r f(r)\left[J_n(p_i r) Y_n(p_i a) - Y_n(p_i a) J_n(p_i a)\right] dr \quad ...(11)$$

where Y_n is the Bessel function of order n of second kind and p_i is a root of the equation

$$J_n(p_i a)\, Y_n(p_i b) - J_n(p_i b)\, Y_n(p_i a) = 0 \quad ...(12)$$

Also then at each point of (a, b) where $f(r)$ is continuous we have the inversion formula

$$f(r)=\sum_i \frac{2p_i^2 J_n^2(p_i a)\ \bar{f}(p_i)}{J_n^2(p_i b)-J_n^2(p_i a)}\left[J_n(p_i x)\ Y_n(p_i b)-J_n(p_i b)\ Y_n(p_i a)\right] \quad ...(13)$$

where the summation extends over all the positive roots of (12).

[2] If $f(r)$ satisfies Dirichlets' conditions in the closed interval [0, 1] *i.e.* $0 \le r \le 1$, and its finite Hankel transform is defined as

$$\bar{f}(p_i)=\int_0^1 r f(r)\ J_n(p_i r)\ dr \quad ...(14)$$

in which p_i is a root of the transcendental equation

$$p J_n'(p)+h J_n(p)=0 \quad ...(15)$$

then at each point of [0, 1] where $f(r)$ is continuous, we have the inversion formula

$$\bar{f}(r)=2\sum_p \frac{p^2 \bar{f}(p)}{h^2+p^2-n^2}\ \frac{J_n(pr)}{J_n^2(p)} \quad ...(16)$$

where the summation extends over all the positive roots of (15).

If $n=0$, (16) reduces to

$$f(r)=2\sum_p \frac{p^2 \bar{f}(p)}{h^2+p^2}\cdot\frac{J_0(pr)}{J_0^2(p)} \quad ...(17)$$

and (15) reduces to $h J_0(p)=pJ_1(p)\quad \because J_0'(p)=-J_1(p)$ ...(18)

Also using (15), (16) may be written as

$$f(r)=2\sum_p \frac{\bar{f}(p)}{1+\dfrac{p^2-n^2}{h^2}}\cdot\frac{J_n(pr)}{\dfrac{h^2 J_n^2(p)}{p^2}} \quad ...(19)$$

or

$$f(r)=2\sum_p \frac{\bar{f}(p)}{1+\dfrac{p^2-n^2}{h^2}}\cdot\frac{J_n(pr)}{J'_n(p)} \quad ...(20)$$

where the summation extends over all the roots of the equation

$$J_n(p)=\frac{p J'_n(p)}{h} \quad ...(21)$$

Evidently if $h \to \infty$, then (20) and (21) reduce to (3) and (2) where $a=1$ and hence we may conclude that (3) and (2) are limiting case of (20) and (21) respectively.

Problem 8. *Find $H_n\{r^n\}$ taking $r J_n(pr)$ as the Kernel of transform.*

$$\because \quad \frac{d}{dx}\left[x^n J_n(x)\right]=x^n J_{n-1}(x)$$

$$\therefore \quad \int_0^b x^n J_{n-1}(x)\ dx=\left[x^n J_n(x)\right]_0^b=b^n J_n(b),\ n>0$$

$$\because x^n J_n(x) \to 0 \text{ as } x \to 0$$

Replacing n by $(n+1)$, $\int_0^b x^{n+1} J_n(x)\ dx=b^{n+1} J_{n+1}(b)$

Writing pr for x and pa for b; $\int_0^a r^n \cdot r J_n(pr)\, dr = \frac{a^{n+1}}{p} J_{n+1}(pa)$.

$$\therefore \quad \bar{H}_n\{r^n\} = \frac{a^{n+1}}{p} J_{n+1}(ap) = \frac{1}{p} J_{n+1}(p) \text{ for } a = 1.$$

Problem 9. *Find the Hankel transform of* $\frac{J_n(\alpha x)}{J_n(\alpha)}$, *taking* $xJ_n(px)$ *as the Kernel of the transform.*

$J_n(\alpha x)$ and $J_n(px)$ being the solutions of Bessel's equation, we have

$$x^2 \frac{d^2}{dx^2} J_n(\alpha x) + x \frac{d}{dx} J_n(\alpha x) + (\alpha^2 x^2 - n^2) J_n(\alpha x) = 0 \quad \ldots(1)$$

$$x^2 \frac{d^2}{dx^2} J_n(px) + x \frac{d}{dx} J_n(px) + (p^2 x^2 - n^2) J_n(px) = 0 \quad \ldots(2)$$

Multiplying (1) by $J_n(px)$, (2) by $J_n(\alpha x)$ and subtracting, we get

$$(\alpha^2 - p^2)\, x J_n(\alpha x) J_n(px) = \frac{d}{dx}\left[x\left\{J_n(\alpha x)\frac{d}{dx} J_n(px) - J_n(px)\frac{d}{dx} J_n(\alpha x)\right\}\right]$$

Integrating with regard to x from 0 to 1 and using $J_n(p) = 0$, we find

$$(\alpha^2 - p^2)\int_0^1 x J_n(\alpha x)\, J_n(px)\, dx = p\, J_n(\alpha)\, J'_n(p)$$

$$\therefore \quad \int_0^1 \frac{J_n(\alpha x)}{J_n(\alpha)} x\, J_n(px)\, dx = \frac{p}{\alpha^2 - p^2} J_n'(p)$$

$$i.e. \quad H_n\left\{\frac{J_n(\alpha x)}{J_n(\alpha)}\right\} = \frac{p}{\alpha^2 - p^2} J_n'(p)$$

Note. *For $n = 0$, using $J_1(p) = -J_0'(p)$, this yields*

$$H_n\left\{\frac{J_n(\alpha x)}{J_n(\alpha)}\right\} = -\frac{p}{\alpha^2 - p^2} J_1(p).$$

Here below we tabulate a few of finite Hankel transforms.

$f(x) = \frac{2}{a^2}\sum_i \bar{f}(p_i)\frac{J_n(p_i x)}{\lvert J'(p_i a)^2 \rvert}$	n	$\bar{f}_i(p_i) = \int_0^a f(x)\, J_n(px)\, dx$
x^n	> -1	$\frac{a^{n+1}}{p_i} J_{n+1}(p_i a)$
c	0	$\frac{ac}{p_i} J_1(pa)$
$a^2 - x^2$	0	$\frac{4a}{p_i^2} J_1(p_i a)$
$\frac{J_r(xa)}{J_r(\alpha a)}$	> -1	$\frac{p_i \alpha}{a^2 - p^2} J'_n(p\alpha)$
$\frac{J_0(ax)}{J_0(a\alpha)}$	0	$\frac{p_i \alpha}{a^2 - p_i^2} J_1(p_i \alpha)$

Problem 10. *Find $f(x)$ if $H\{f(x)\} = \frac{c}{p} J_1(pa)$, p being positive root of $J_0(pa)$.*

Inversion formula gives $f(x) = \frac{2}{a^2} \sum_i \bar{f}(p) \frac{J_n(px)}{[J'_n(pa)]^2}$

where summation extends over all positive roots of $J_n(pa)$.

$$\therefore \quad f(x) = \frac{2}{a} \sum_i \frac{c}{p} \frac{J_1(pa)\, J_0(px)}{\{J'_0(pa)\}^2} = \frac{2}{a} \sum_i \frac{c}{p} \cdot \frac{J_1(pa)\, J_0(px)}{\{J_1(pa)\}^2}$$

$$\because J_0'(x) = -J_1(x)$$

$$= \frac{2}{a^2} \sum_i \frac{c}{p} \frac{J_n(px)}{J_1(pa)}.$$

11.10. FINITE HANKEL TRANSFORM OF $\frac{df}{dx}$, p being root of $J_n(p) = 0$.

$$H_n\left\{\frac{df}{dx}\right\} = \int_0^1 \frac{df}{dx} \cdot x J_n(px)\, dx = \left[f(x) \cdot x J_n(px)\right]_0^1 - \int_0^1 f(x) \frac{d}{dx}\{x J_n(px)\}\, dx$$

$$= -\int_0^1 f(x) \frac{d}{dx}\{x J_n(px)\}\, dx \qquad \ldots(1)$$

Now we have the recurrence relations

$2 J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$ and $2n\, J_n(x) = x\,[J_{n-1}(x) + J_{n+1}(x)]$

which on replacing x by px, become

$J_n'(px) = \frac{1}{2}[J_{n-1}(px) - J_{n+1}(px)]$ and $J_n(px) = \frac{px}{2n}[J_{n-1}(px) + J_{n+1}(px)]$

Substituting them in $\frac{d}{dx}\{x J_n(px)\} = xp\, J_n'(px) + J_n(px)$ we get

$$\frac{d}{dx}\{x J_n(px)\} = \frac{px}{2}\left[J_{n-1}(px) - J_{n+1}(px)\right] + \frac{px}{2n}\left[J_{n-1}(px) + J_{n+1}(px)\right]$$

$$= \frac{p}{2n}\left[(n+1)\, x J_{n-1}(px) - (n-1)\, J_{n+1}(px)\right]$$

So that (1) yields

$$H_n\left\{\frac{df}{dx}\right\} = \frac{p}{2n} \int_0^1 [(n-1)\, x f(x)\, J_{n+1} - (n+1)\, x\, f(x)\, J_{n-1}]\, dx$$

$$= \frac{p}{2n}\left[(n-1)\, H_{n+1}\{f\} - (n+1)\, H_{n-1}\{f\}\right] \qquad \ldots(2)$$

where $H_n\{f\}$ denotes the Hankel transform of order n.

Note. *For $n = 1$, (2) yields* $H_1\left\{\frac{df}{dx}\right\} = -p\, H_0\{f(x)\}$ $\qquad \ldots(3)$

11.11. FINITE HANKEL TRANSFORM OF $\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx}$.

Case 1. When p is a root of $J_n(p) = 0$.

We have $H_n\left\{\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}\right\}=\int_0^1 x\left\{\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}\right\}J_n(px)\,dx$

$$=\int_0^1 x\,\frac{d^2f}{dx^2}J_n(px)\,dx+\int_0^1\frac{df}{dx}J_n(px)\,dx$$

$$=H_n\left\{\frac{d^2f}{dx^2}\right\}+\int_0^1\frac{df}{dx}J_n(px)\,dx \qquad ...(1)$$

In result (2) of §11.1, replacing f by $\frac{df}{dx}$, we get

$$H_n\left\{\frac{d^2f}{dx^2}\right\}=\frac{p}{2n}\left[+(n-1)H_{n+1}\left\{\frac{df}{dx}\right\}-(n+1)\cdot H_{n-1}\left\{\frac{df}{dx}\right\}\right] \qquad ...(2)$$

Now replacing x by px in the recurrence relation

$2n\,J_n(x)=x\,[J_{n-1}(x)+J_{n+1}(x)]$

we have $J_n(px)=\frac{px}{2n}\,[J_{n-1}(x)+J_{n+1}(px)].$

So that $\int_0^1\frac{df}{dx}J_n(px)\,dx=\int_0^1\frac{df}{dx}\cdot\frac{px}{2n}[J_{n-1}(x)+J_{n+1}(px)]\,dx$

$$=\frac{p}{2n}\int_0^1 x\frac{df}{dx}J_{n-1}(px)\,dx+\frac{p}{2n}\int_0^1 x\,\frac{df}{dx}J_{n+1}(px)\,dx$$

$$=\frac{p}{2n}\left[H_{n-1}\left\{\frac{df}{dx}\right\}+H_{n+1}\left\{\frac{df}{dx}\right\}\right] \qquad ...(3)$$

With the help of (2) and (3); (1) yields,

$$H_n\left\{\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}\right\}=\frac{p}{2}\left[-H_{n-1}\left\{\frac{df}{dx}\right\}+H_{n+1}\left\{\frac{df}{dx}\right\}\right] \qquad ...(4)$$

Case II. When p is a root of $p\,J_n'(p)+h\,J_n(p)=0$ ...(5)

Taking $n=0$, (5) becomes $p\,J_0'(p)+hJ_0(p)=0$ ...(6)

we have $H_0\left\{\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}\right\}=\int_0^1\left(\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}\right)x\,J_0(px)\,dx$

$$=\int_0^1\frac{d^2f}{dx^2}x\,J_0(px)\,dx+\int_0^1\frac{df}{dx}\cdot J_0(px)\,dx$$

$$=\left[\frac{df}{dx}\cdot x\,J_0(px)\right]_0^1-\int_0^1\frac{df}{dx}\cdot\frac{d}{dx}\{x\,J_0(px)\}\,dx+\int_0^1\frac{df}{dx}\cdot J_0(px)\,dx$$

$$=\left[\frac{df}{dx}\right]_{x=1}J_0(p)-\int_0^1\frac{df}{dx}\cdot\{xp\,J'_0(px)+J_0(px)\}\,dx+\int_0^1\frac{df}{dx}J_0(px)\,dx$$

$$=\left[\frac{df}{dx}\right]_{x=1}J_0(p)-p\int_0^1\frac{df}{dx}x\,J'_0(px)\,dx$$

$$=\left[\frac{df}{dx}\right]_{x=1}J_0(p)-p\left[f(x)\cdot xJ'_0(px)\right]_0^1+p\int_0^1 f(x)\cdot\frac{d}{dx}\{x\,J'_0(px)\}\,dx$$

$$= \left[\frac{df}{dx}\right]_{x=1} J_0(p) - pf(1)\; J_0'(p) + p\int_0^1 f(x)\cdot\{xpJ_0''(px) + J_0'(px)\}\, dx$$

But $J_0(x)$ being the solution of Bessel's equation

$$x^2 \frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0, \text{ we have}$$

$$x J_0''(x) + J_0'(x) + x J_0(x) = 0.$$

Replacing x by px, $px\, J_0''(px) + J_0'(px) = -px\; J_0(px)$

$$\therefore\quad H_0\left\{\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx}\right\} = \left[\frac{df}{dx}\right]_{x=1} J_0(p) - pf(1)\left\{-\frac{hJ_0(p)}{p}\right\}$$

$$-p^2 \int_0^1 xf(x)\; J_0(px)\; dx \text{ by (6)}$$

$$= \left\{\left[\frac{df}{dx}\right]_{x=1} + hf(1)\right\} J_0(p) - p^2 H_0\{f(x)\}$$

$$= \left[\frac{df}{dx} + hf(x)\right]_{x=1} J_0(p) - p^2 H_0\{f(x)\} \qquad \text{...(7)}$$

If boundary condition be such that $\frac{df}{dx} + h\,f(x) = 0$ when $x = 1$, then we get

$$H_0\left\{\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx}\right\} = -p^2 H_0\{f(x)\} \qquad \text{...(8)}$$

Case III. When p is a root of $J_n(pa)\, Y_n(pb) - J_n(pb)\, Y_n(pa) = 0$ ...(9)

Integrating by parts, the following, we have

$$\int_a^b x\,[J_n(px)\; Y_n(bp) - J_n(bp)\; Y_n(xp)]\left(\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx}\right) dx$$

$$= -p\left\{xf(x)\,[J_n'(xp)\; Y_n(bp) - J_n(bp)\; Y_n'(xp)] + p\int_a^b xf(x)\,\frac{d}{dx}\right.$$

$$[J_n'(xp)\, Y_n(bp) - J_n(bp)\, Y_n'(xp)] + f(x)\,[J_n'(xp)\, Y_n(bp) - J_n(bp)$$

$$\left. Y_n'(xp)]\right\} dx \qquad \text{...(10)}$$

It is easy to show that

$$J_n'(pa)\; Y_n(bp) - Y_n'(ap)\; J_n(pb) = \frac{1}{pa}\frac{J_n(pb)}{J_n(pa)} \qquad \text{...(12)}$$

With the help of (11), (10) yields,

$$H_n\left\{\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx}\right\} = \int_0^1\left(\frac{n^2}{x^2} - p^2\right) xf(x)\,[J_n(px)\; Y_n(bp) - J_n(bp)$$

$$Y_n(xp)]\; dx$$

or $$H_n\left\{\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx} - \frac{n^2}{x^2}\right\} = \frac{J_n(pb)}{J_n(pa)} f(a) - f(b) - p^2 H_n\{f(x)\} \qquad \text{...(12)}$$

Note 1. *If boundary condition is such that* $f(a) = 0 = f(b)$, *then* (12) *yields*

$$H_n \left\{ \frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx} - \frac{n^2}{x^2} \right\} = -p^2 H_n \{f(x)\} \qquad ...(13)$$

Note 2. *If* $n = 0$ *then* (13) *reduces to*

$$H_0 \left\{ \frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx} \right\} = -p^2 H_0 \{f(x)\} \qquad ...(14)$$

which is the same as (8).

11.12. FINITE HANKEL TRANSFORM OF $\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx} - \frac{n^2}{x^2} f(x)$, p being a root of $J_n(pa) = 0$.

We have,

$$H_n \left\{ \frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx} - \frac{n^2}{x^2} f(x) \right\} = \int_0^a \left[\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx} - \frac{n^2}{x^2} f(x) \right] x J_n(px)\, dx$$

$$= \int_0^a \frac{d^2f}{dx} \cdot x J_n(px)\, dx + \int_0^a \frac{df}{dx} J_n(px)\, dx - n^2 \int_0^a \frac{1}{x} f(x)\, J_n(px)\, dx$$

$$= \left[\frac{df}{dx} \cdot x J_n(px) \right]_0^a - \int_0^a \frac{df}{dx} \cdot \frac{d}{dx} \{x J_n(px)\}\, dx + \int_0^a \frac{df}{dx} \cdot J_n(px)\, dx$$

$$- x^2 \int_0^a \frac{1}{x} f(x)\, J_n(px)\, dx$$

$$= -\int_0^a \frac{df}{dx} \{px J_n'(px) + J_n(px)\}\, dx + \int_0^a \frac{df}{dx} \cdot J_n(px)\, dx - n^2$$

$$\int_0^a \frac{1}{x} f(x)\, J_n(px)\, dx \quad \because J_n(pa) = 0$$

$$= -p \int_0^a \frac{df}{dx} \cdot x J_n'(px)\, dx - n^2 \int_0^a \frac{1}{x} f(x)\, J_n(px)\, dx$$

$$= -p \left[f(x)\, x J_n'(px) \right]_0^a + p \int_0^a f(x) \cdot \frac{d}{dx} \{x J_n'(x)\}\, dx$$

$$- n^2 \int_0^a \frac{1}{x} f(x)\, J_n(px)\, dx$$

$$= -pa\, f(a)\, J_n'(ap) \;\; p \int_0^a f(x) \cdot \{px J_n''(px) + J_n'(px)\}\, dx$$

$$- n^2 \int_0^a \frac{1}{x} f(x)\, J_n(px)\, dx$$

But $J_n(x)$ being the solution of Bessel's equation,

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)\, y = 0,$$

we have $\quad x^2 J_n''(x) + x J_n'(x) + (x^2 - n^2) J_n(x) = 0$

Replacing x by px, this gives

$$p^2 x^2 J_n''(px) + px J_n'(px) + (p^2x^2 - n^2) J_n(px) = 0$$

i.e., $px J_n''(px) + J_n'(px) = px\left(\frac{n^2}{p^2x^2}-1\right)J_n(px)$

$$\therefore \quad H_n\left\{\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}-\frac{n^2}{x^2}f(x)\right\} = -paf(a)\ J_n'(ap) + p\int_0^a f(x)$$

$$\left\{px\left(\frac{n^2}{p^2x^2}-1\right)J_n(px)\right\}dx - n^2\int_0^a \frac{1}{x} f(x)\ J_n(px)\ dx$$

$$= -paf(a)\ J_n'(ap) - p^2\int_0^a f(x)\ xJ_n(px)\ dx$$

$$= -paf(a)\ J_n'(ap) - p^2 H_n\{f(x)\} \qquad ...(1)$$

Note 1. *If $a = 1$, (1) becomes*

$$H_n\left\{\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}-\frac{n^2}{x^2}f(x)\right\} = -pf(1)\ J_n'(p) - p^2 H_n\ \{f(x)\} \qquad ...(2)$$

Note 2. *When $n = 0$, (2) reduces to*

$$H_0\left\{\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}\right\} = -pf(1)\ J_0'(p) - p^2 H_0\ \{f(x)\}$$

$$= pf(1)J_1(p) - p^2 H_0\{f(x) \qquad \because J_0'(p) = -J_1(p) \qquad ...(3)$$

ADDITIONAL MISCELLANEOUS PROBLEMS

Problem 11. *Find the Finite Hankel transform $\phi(\rho)$ where*

$$\phi(\rho) = \frac{2^{1+n-m}}{\Gamma(m-n)}\rho^n(1-\rho^2)^{m-n-1}$$

We have $H_n\{\phi(\rho)\} = \dfrac{2^{1+n-m}}{\Gamma(m-n)}\displaystyle\int_0^1 \rho^{n+1}(1-\rho)^{m-n-1} J_n(p\rho)\ d\rho$

If we expand Bessel function $J_n(p\rho)$ in ascending powers of $p\rho$ and integrate term by term, we get

$$H_n\{\phi(\rho)\} = \frac{2^{1+n-m}}{\Gamma(m-n)}\sum_{s=0}^{\infty}\int_0^1 \rho^{n+1}(1-\rho^2)^{m+n-1}\ \frac{(-1)^s(p\rho)^{n+2s}}{2^{n+2s}\lfloor s\ \Gamma(n+s+1)}d\rho$$

$$= \frac{1}{\Gamma(m-n)}\sum_{s=0}^{\infty}\frac{p^{n+2s}}{2^{m+2s}}\int_0^1 t^{n+s}(1-t)^{m-n-1}\ \frac{(-1)^s\ dt}{\lfloor s\ \Gamma(n+s+1)}, \quad \text{where } \rho^2 = t$$

$$= \sum_{s=0}^{\infty}\frac{(-1)^s\ p^{m+2s}}{2^{m+2s}\ \Gamma(m+s+1)}\frac{p^{n-m}}{\lfloor s} = p^{n-m}.J_m(p).$$

Problem 12. *Find the finite Hankel transform of $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)-\frac{n^2V}{r}$, where $V = 0$ when $r = 0$ and $V = V_1$ when $r = 1$.*

We have $\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial V}{\partial r}\right)-\dfrac{n^2V}{r} = \dfrac{\partial^2V}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial V}{\partial r}-\dfrac{n^2V}{r^2} = f(r)$ (say)

Hence using (2) of §11.12, we have

$$H\{f(r)\} = -pf(1)J_n'(p) - p^2 H_n\{f(r)\} \text{ when } f(r) = V \text{ and } f(0) = 0, f(1) = V_1$$
$$= -pV_1 J_n'(p) - p^2 H_n V$$

Problem 13. *Solve* $\dfrac{\partial^2 V}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial V}{\partial r} = \dfrac{1}{k}\dfrac{\partial V}{\partial t}$, $0 \le r < 1$, $t > 0$

where $\dfrac{\partial V}{\partial r} + hV = 0$, *when* $r = 1, t > 0$ *and* $V = 1$ *when* $t = 0$, $0 \le r < 1$; h, k *being constants.*

Assuming that $\bar{V} = \int_0^1 Vr\, J_0(pr)\, dr$ ...(1)

where p is a root of $p\, J_0'(p) + h\, J_0(p) = 0$ ...(2)

and multiplying the given equation by $rJ_0(pr)$ and then integrating with regard to r from 0 to 1, we have

$$\int_0^1 \left(\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r}\right) rJ_0(pr)\, dr = \frac{1}{k}\int_0^1 \frac{\partial V}{\partial r}\, rJ_0(pr)\, dr = \frac{1}{k}\frac{d\bar{V}}{dt}.$$

$$\frac{1}{k}\frac{d\bar{V}}{dt} = \left\{\frac{\partial V}{\partial r} rJ_0(pr)\right\}_0^1 - \int_0^1 \frac{\partial V}{\partial r}\left\{J_0(pr) + rpJ_0(pr)\right\} rp + \int_0^1 \frac{\partial V}{\partial r} J_0'(pr)\, dr\Big\}$$

$$= \left[\frac{\partial V}{\partial r} rJ_0(pr)\right]_0^1 - p\int_0^1 r\frac{\partial V}{\partial r} J_0'(pr)\, dr$$

$$= J_0(p)\left[\frac{\partial V}{\partial r}\right]_{r=1} - p\left[\left| VJ_0'(pr)\right|_0^1 - \int_0^1 V\{J_0'(pr) + pr\, J_0''(p)\}\, dr\right]$$

$$= J_0(p)\left[\frac{\partial V}{\partial r}\right]_{r=1} - p[V]_{r=1} J_0'(p) - p^2\int_0^1 V_r J_0(pr)\, dr \qquad \because J_0(pr) \text{ satisfies Bessel's equation.}$$

$$= J_0(p)\left[\frac{\partial V}{\partial r} + hV\right]_{r=1} \qquad \because pJ_0'(p) = -hJ_0(p)$$

$$= -p^2\bar{V}$$

i.e., $\dfrac{d\bar{V}}{dt} = -kp^2\bar{V}$ where $\bar{V} = \int_0^1 r\, J_0(pr)\, dr = \dfrac{J_1(p)}{p}$ when $t = 0$,

$\because V = 1$ when $t = 0$, $0 \le r < 1$

$\therefore$ Its solution is $\bar{V} = \dfrac{J_1(p)}{p} e^{-kp^2 t}$

Applying inversion formula, we find

$$V = 2\sum_p e^{-kp^2 t}\frac{pJ_1(p)\, J_0(pr)}{h^2 + p^2 J_0^2(p)}$$

where summation extends over all positive roots of (2).

Problem 14. *Solve* $\dfrac{\partial^2 f}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial f}{\partial r} = \dfrac{1}{k}\dfrac{\partial f}{\partial t}$, $0 \le r < 1$, $t > 0$

where $f = f_0$ *when* $r = 1, t > 0$ *and* $f = 0$ *when* $t = 0$, $0 \le r < 1$.

Hint. Take $\bar{f}(p) = \int_0^1 f \cdot r\, J_0(pr)\, dr$, p being root of $J_0(p) = 0$ and proceed like in Problem 13.

Ans. $f = 2f_0 \sum_p (1 - e^{-kp^2 t})\dfrac{J_0(pr)}{pJ_1(p)}.$

Problem 15. *Viscous fluid is contained between two infinitely long concentric circular cylinders of radi a and b. The inner cylinder is kept at rest and outer cylinder suddenly starts rotating with uniform angular velocity ω. Find the velocity v of the fluid if the equation of motion is*

$$\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}=\frac{1}{\nu}\frac{\partial v}{\partial t},\quad a<r<b,\ t>0$$

ν being Kinematic viscosity.

Hint. Take $f^2(p)=\int_a^b f(r)\cdot r B_1(pr)\,dr,\ b>a$ where

$B_1(pr)=J_1(pr)\,Y_1(pa)-Y_1(pr)\,J_1(pa)$, $Y_1(pr)$ being Bessel's function of second kind of order one, and p is a positive root of

$$J_1(pb)\,Y_1(pa)=Y_1(pb)\,J_1(pa).$$

Multiplying the given equation by $rB_1(pr)$ and integrating w.r.t. 'r' from a to b with boundary conditions $v=b\omega$ when $r=b$

$v=0$ when $r=a$

$v=0$ when $t=0$.

Ans. $$v=\pi b\omega\sum_p \frac{1-e^{-\nu p^2 t}}{J_1^2(pa)-J_1^2(pb)}\,J_1(pa)\,J_1(pb)\,B_1(pr).$$

Problem 16. *If α and β are the roots of the equation $J_0(x)=0$, show that*

$$\int_0^1 x J_0(\alpha x)\,J_0(\beta x)=\frac{1}{2}\delta_{\alpha\beta}\,J_1^2(\alpha)$$ *(Agra, 1971)*

$$=0 \text{ for } \alpha\neq\beta$$ *(Rohilkhand, 1982)*

Problem 17. *If α and β are different roots of equation $J_n(\mu)=0$,* then show that

$$\int_0^1 x J_n(\alpha x)\,J_n\,\beta x)\,dx=\tfrac{1}{2}\,J_{n+1}^2(\alpha)\,\delta_{\alpha\beta}$$

where symbols have usual meanings. *(Rohilkhand, 1986)*

CHAPTER 12

DIFFUSION, WAVE AND LAPLACE'S EQUATIONS

12.1. INTRODUCTION

In 7.5, dealing with special types of differential equations we have already mentioned diffusion equation (or Fourier equation of heat flow), Laplace's equations as particular cases of steady-state heat flow and wave equations in one, two and three dimensions along with the two methods of solutions namely (*i*) method of separation of variables and (*ii*) D' Alembert's mehod. Now in subsequent chapters namely, 8th, 9th, 10th and 11th chapters we have discussed various methods of solution which may be successfully applied to solve different types of equations encountered in physics, mechanics and applied mathematics. In the present chapter we shall try to make a systematic study of one, two and three dimensional diffusion, Laplace's and wave equations with boundary values problems while a few of them have been already solved by the methods of transforms.

12.2. DIFFUSION EQUATION OR FOURIER EQUATION OF HEAT FLOW

(*Agra, 1974*)

Assuming that the temperature at any point (x, y, z) *of a solid at time* t *is* $u(x, y, z, t)$, *the thermal conductivity of the solid is* K, *the density of the solid is* ρ *and specific heat is* σ, *the heat equation*

$$\frac{\partial u}{\partial t} = h^2 \nabla^2 u \qquad ...(1)$$

where $h^2 = \dfrac{K}{\rho\sigma} = k$ *(say),* k *being known as* diffusivity, *is said to be the equation of diffusion or the Fourier equation of heat flow.*

We know that heat flows from points at hegher temperature to the points at lower temperature and the rate of decrease of temperature at any point varies with the direction. In other words the amount of heat say ΔH crossing an element of surface ΔS in Δt seconds is proportional to the greatest rate of derease of the temperature u *i.e.*

$$\Delta H = K \Delta S \, \Delta t \left| \frac{du}{dt} \right| \qquad ...(2)$$

If v be the velocity of heat flow given by

$$\mathbf{v} = -K \text{ grad } u = -K \cdot \nabla u \qquad ...(3)$$

Here $u(x, y, z\ t)$ is the temperature of the solid at (x, y, z) at an instant of time t and K the thermal conductivity of the solid is a positive constant in cal./cm–sec °C units.

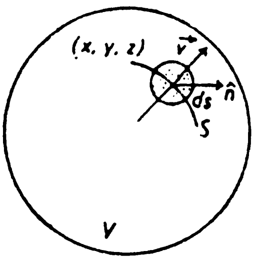


Fig. 12.1

Let S be the surface of an arbitrary volume V of the solid. Then the total flux of heat flow across S per unit time is given by

$$H = \iint_S (-K\nabla u).\ \hat{\mathbf{n}}\ dS \qquad ...(4)$$

where $\hat{\mathbf{n}}$ is the positive outward drawn normal vector to the element dS and the negative sign shows that increase of temperature with the increase of x so that $\frac{\partial u}{\partial x}$ is positive and heat flows towards negative x from points of higher temperature to those of lower temperature, thereby rendering the flux to be negative.

Now applying Gauss's divergence theorem according to which if V be the volume bounded by a closed surface S and A be a vector function of position with continuous derivative, we have the quantity of heat entering S per unit time as

$$\iint_S (K\nabla u)\ \hat{\mathbf{n}}\ dS = \iiint_V \nabla\cdot(K\nabla u)\ dV \qquad ...(5)$$

i.e.,
$$\iiint_V \nabla\cdot\mathbf{A}\, dV = \iint_S \mathbf{A}\cdot\hat{\mathbf{n}}\, dS = \oiint_S \mathbf{A}\cdot d\mathbf{S} \qquad ...(6)$$

Taking volume element $dv = dx\,dy\,dz$, the heat contained in

$$V = \iiint_V \sigma\rho u\, dV.$$

∴ The time rate of increase of heat is given by

$$\frac{\partial}{\partial t}\iiint_V \sigma\rho u\, dV = \iiint_V \sigma\rho\ \frac{\partial u}{\partial t}\ dV \qquad ...(7)$$

Equating R.H.S's of (5) and (7), we find

$$\iiint_V \left[\sigma\rho\frac{\partial u}{\partial t} - \nabla\cdot(K\nabla u)\right] dV = 0 \qquad ...(8)$$

But V being arbitrary and the integrand being assumed to be continuous the realtion (8) will be identically zero for every point if

$$\sigma\rho\frac{\partial u}{\partial t} = \nabla\cdot(K\nabla u)$$

or
$$\frac{\partial u}{\partial t} = \frac{K}{\sigma\rho}\nabla\cdot\nabla u = h^2\ \nabla^2 u = k\nabla^2 u \text{ where } h^2 = k = \frac{K}{\sigma\rho}.$$

or
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{k}\frac{\partial u}{\partial t} = \frac{1}{h^2}\frac{\partial u}{\partial t} \qquad ...(9)$$

This is three-dimensional diffusion equation.

COROLLARY 1. If the temperature within a substance be assumed to be independent of z *i.e.*, there being no heat flow in direction of z, then (9) reduces to

$$\frac{\partial u}{\partial t} = h^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad ...(10)$$

which is known as two dimensional diffusion equation or the equation for two-dimensional flow parallel to $x-y$ plane.

COROLLARY 2. Putting $\frac{\partial u}{\partial y}=\frac{\partial u}{\partial z}=0$ (9) we get $\frac{\partial u}{\partial t}=h^2\,\frac{\partial^2 u}{\partial x^2}$ ...(11)

which is the equation for the one-dimensional flow of heat along a bar.

COROLLARY 3. For steady-state heat flow, u is independent of time *i.e.*, $\frac{\partial u}{\partial t}=0$ and hence (9) reduces to

$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0 \qquad ...(12)$$

which is known as three-dimensional *Lapiace's equation.*

12.3. ONE-DIMENSIONAL DIFFUSION EQUATION

$$\frac{\partial u}{\partial t}=h^2\frac{\partial^2 u}{\partial x^2}=k\,\frac{\partial^2 u}{\partial x^2}$$

[*A*] Independent derivation of $\frac{\partial u}{\partial t}=k\,\frac{\partial^2 u}{\partial x^2}$.

Consider one-dimensional flow of electricity in a long insulated cable and specify the current i and voltage E at any time in the cable by x-coordinate and time-variable t.

The potential drop E in a line-element δx of length at any point x is given by $-\delta E=$ $iR\delta x+L\delta x\,\frac{\partial i}{\partial t}$ ...(1)

where R and L aer respectively resistance and inductance per unit length.

If C and G be respectively capacitance to earth and conductance per unit length, then we have

$$-\delta i = GE\,\delta x + C\,\delta x\,\frac{\partial E}{\partial t} \qquad ...(2)$$

Rewriting (1) and (4), $\frac{\partial E}{\partial x}+Ri+L\,\frac{\partial i}{\partial t}=0$ ...(3)

and $\frac{\partial i}{\partial x}+GE+C\,\frac{\partial E}{\partial t}=0$ ...(4)

Differentiating (3) wr.r.t. x and (4) w.r.t. t we have

$$\frac{\partial^2 E}{\partial x^2}+R\,\frac{\partial i}{\partial x}+L\frac{\partial^2 i}{\partial x\,\partial t}=0 \qquad ...(5)$$

and

$$\frac{\partial^2 i}{\partial x\,\partial t}+G\,\frac{\partial E}{\partial t}+C\frac{\partial^2 E}{\partial t^2}=0 \qquad ...(6)$$

Eliminating $\frac{\partial^2 i}{\partial x\,\partial t}$ from (5) and (6) we get

$$\frac{\partial^2 E}{\partial x^2}=CL\,\frac{\partial^2 E}{\partial t^2}+LG\,\frac{\partial E}{\partial t}-R\,\frac{\partial i}{\partial x} \qquad ...(7)$$

Again eliminating $\frac{\partial i}{\partial x}$ from (4) and (7) we find

$$\frac{\partial^2 E}{\partial x^2} = CL\,\frac{\partial^2 E}{\partial t^2} + (CR+GL)\,\frac{\partial E}{\partial t} + RGE \qquad ...(8)$$

Differentiation of (3) w.r.t. 't' and (4) w.r.t. 'x' yields

$$\frac{\partial^2 E}{\partial x \partial t} + R\,\frac{\partial i}{\partial t} + L\frac{\partial^2 i}{\partial t^2} = 0 \qquad ...(9)$$

and
$$\frac{\partial^2 i}{\partial x^2} + G\frac{\partial E}{\partial x} + C\frac{\partial^2 E}{\partial x \partial t} = 0 \qquad ...(10)$$

Elimination of $\frac{\partial E}{\partial x}$ and $\frac{\partial^2 E}{\partial x \partial t}$ from (3), (9) and (10) gives

$$\frac{\partial^2 i}{\partial x^2} = CL\,\frac{\partial^2 i}{\partial t^2} + (CR + GL)\frac{\partial i}{\partial t} + RGi \qquad ...(11)$$

(7) and (11) follow that E and i satisfy a second order partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = CL\,\frac{\partial^2 u}{\partial t^2} + (CR + GL)\frac{\partial u}{\partial t} + RGu \qquad ...(12)$$

which is known as *telegraphy equation.*

If the leakage to the ground is small then $G = 0 = L$ and hence (12) reduces to $\frac{\partial^2 u}{\partial x^2} = CR\,\frac{\partial u}{\partial t} = \frac{1}{k}\frac{\partial u}{\partial t}$ where $k = \frac{1}{CR}\cdot$

which is one-dimensional diffusion equation.

[B] Solution of $\frac{\partial u}{\partial t} = h^2\,\frac{\partial^2 u}{\partial x^2}$ or $u_t = h^2 u_{xx}$.

The solution of this equation by the method of separation of variables has already been discussed in §7.5. Here below we discuss the solution in different conditions.

[b_1] (Both the ends of a bar at temperature zero).

If both the ends of a bar of length l are at temperature zero and the initial temperature is to be prescribed function $F(x)$ in the bar, then find the temperature at a subsequent time t.

One-dimensional heat equation is $\frac{\partial u}{\partial t} = h^2\,\frac{\partial^2 u}{\partial x^2}$ ...(1)

we have to find a function $u(x, t)$ satisfying (1) with the boundary conditions $u(0, t) = u(l, t) = 0$, $t \geq 0$, l being the length of bar ...(2)

and $u(x, 0) = F(x), 0 < x < l$ ...(3)

In order to apply the method of separation of variables, let us assume that

$u(x, t) = X(x)\,T(t)$, X and T being respectively the functions of x and t alone.

So that $\frac{\partial u}{\partial t} = X\,\frac{dT}{dt}$ and $\frac{\partial^2 u}{\partial x^2} = T\frac{d^2 X}{dx^2}\cdot$

Their substitution in (1) gives

$$\frac{1}{X}\,\frac{d^2 X}{dx^2} = \frac{1}{h^2 T}\,\frac{dT}{dt} \qquad ...(4)$$

The L.H.S. and R.H.S. of (4) are constants because of variables being separated and hence we can write $\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{h^2T}\frac{dT}{dt} = -\lambda^2$ (constant of separation).

Here $\frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2$ *i.e.*, $\frac{d^2X}{dx^2} + \lambda^2 X = 0$ gives $X = A\cos\lambda x + B\sin\lambda x$ ...(5)

and $\frac{1}{h^2T}\frac{dT}{dt} = -\lambda^2$ *i.e.*, $\frac{dT}{dt} + h^2\lambda^2 T = 0$ gives $T = Ce^{-\lambda^2 h^2 t}$ ...(6)

In view of condition (2) *i.e.*, $u = 0$ at $x = 0$ or (5) gives $A = 0$ and λ be chosen such that $\sin\lambda l = 0$ *i.e.*, $\lambda = \frac{n\pi}{l}$, n being an integer.

Hence the solution (1) *i.e.*, $u = XT$ takes the form

$$u = B\sin\frac{n\pi}{l}x\, e^{-\frac{n^2\pi^2h^2}{l}t} \qquad ...(7)$$

Summing over for all values of n, this becomes

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{l}x\, e^{-\frac{n^2\pi^2h^2}{l}t} \qquad ...(8)$$

Applying condition (3) *i.e.*, $u(x, 0) = F(x)$ at $t = 0$ we have

$$F(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{l}x \text{ for } 0 < x < l \qquad ...(9)$$

So that $$B_n = \frac{2}{l}\int_0^l F(x)\sin\frac{n\pi x}{l}\,dx \qquad ...(10)$$

which is obtained by multiplying (9) by $\sin\frac{n\pi x}{l}$ and then integrating from $x = 0$ to $x = l$.

Hence the required solution is

$$u(x, t) = \frac{2}{l}\sum_{n=1}^{\infty} e^{-\frac{n^2\pi^2h^2}{l}t}\sin\frac{n\pi}{l}x\int_0^l F(u)\sin\frac{n\pi}{l}u\,du \qquad ...(11)$$

Deduction: (Insulated faces)

If instead of the ends of a bar of length l having kept at temperature zero, they are impervious to heat and the initial temperature is the prescribed function F(x) in the bar, then to find the temperature at a subsequent time t, we have the boundary conditions

$$\frac{\partial u}{\partial x} = 0 \text{ at } x = 0 \text{ or } l \text{ for all } t \qquad ...(12)$$

$$u(x, 0) = F(x),\ 0 < x < l \qquad ...(13)$$

Then the solution follows from (5) as

$$u = A\cos\lambda x + B\sin\lambda x$$

which in view of (12) requires $B = 0$ and $\sin\lambda x = 0$ *i.e.*, $\lambda = \frac{n\pi}{l}$, $n = 0, 1, 2, 3...$

So that the general solution of the one-dimensional diffusion equation will be of the form

$$u(x, t) = B_0 + \sum_{n=1}^{\infty} A_n e^{-\frac{n^2\pi^2h^2}{l}t} \cos \frac{n\pi x}{l} \qquad ...(14)$$

where B_0 corresponds to $n = 0$.

By (13), this yields, $F(x) = u(x, 0) = B_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l}$ ...(15)

from which we can easily find the coefficients

$$A_n = \frac{2}{l}\int_0^l F(x) \cos\frac{n\pi x}{l}\, dx \qquad ...(16)$$

and $B_0 = \frac{1}{2} A_0$

Note 1. *The temperature in a slab having initial temperature $F(x)$ and the faces $x = 0$, $x = \pi$ thermally insulated is given by*

$$u(x, t) = B_0 + \sum_{n=1}^{\infty} B_n e^{-n^2h^2t} \cos nx. \qquad ...(18)$$

where $$A_n = \frac{2}{\pi}\int_0^\pi f(x) \cos nx\, dx \qquad ...(19)$$

and $$B_0 = \frac{1}{2}A_0 = \frac{1}{\pi}\int_0^\pi F(x)\, dx \qquad ...(20)$$

Note 2. *The temperature in a slab having initial temperature $F(x)$ and the faces $x = 0$, $x = l$ thermally insulated is given by*

$$u(x, t) = \frac{1}{l}\int_0^l F(x)\, dx + \frac{2}{l}\sum_{n=1}^{\infty} e^{-\frac{n^2\pi^2h^2}{l}t} \cos \frac{n\pi x}{l}\int_0^l F(x) \frac{n\pi x}{l} dx \qquad ...(21)$$

[b$_2$] (One end of a bar at temperature u_0 and other at zero temperature).

If a bar of length l is a at temperature v_0 such that one of its ends $x = 0$ is kept at zero temperature and the other end $x = l$ is kept at temperature u_0, then find the temperature at any point x of the bar at an instant of time $t > 0$.

or

A rod of length l and thermal conductivity h^2 is maintained at a uniform temperature v_0. At $t = 0$ the end $x = 0$ is suddenly cooled to 0°C by application of ice and the end $x = l$ is heated to the temperature u_0 by applying steam, the rod being insulated along its length so that no heat can transfer from the sides. Find the temperature of the rod at any point at any time.

The equation is $\frac{\partial u}{\partial t} = h^2 \frac{\partial^2 u}{\partial x^2}$, $0 < x < l$, $t > 0$ ...(1)

With boundary conditions $u(0, t) = 0$, $u(l, t) = u_0$ for all t ...(2)

and $u(x, 0) = v_0$ ...(3)

Let the solution of (1) be $u(x, t) = X(x)\, T(t)$ ...(4)

where X is a function of x alone and T is a function of t alone.

Substituting from (4) $\frac{\partial u}{\partial t} = X\frac{dT}{dt}$ and $\frac{\partial^2 u}{\partial x^2} = T\frac{d^2X}{dx^2}$, in (1) we get $\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{h^2T}\frac{dT}{dt}$ where variables are separated and hence terms on either side are constants.

Now there arise three possibilities:

[1] $\frac{d^2X}{dx^2} = 0$, $\frac{dT}{dt} = 0$ whence the solution is $X = Ax + B, T = C$ ...(5)

[2] $\frac{d^2X}{dx^2} = \lambda^2 x$, $\frac{dT}{dt} = h^2\lambda^2 T$, the solution being $X = Ae^{\lambda x} + Be^{-\lambda x}$, $T = Ce^{h^2\lambda^2 t}$ ...(6)

[3] $\frac{d^2X}{dx^2} = -\lambda^2 x$, $\frac{dT}{dt} = -h^2\lambda^2 T$, the solution being $X = A\cos\lambda x + B\sin\lambda x$, $T = Ce^{-h^2\ 2}$. ...(7)

The combined solution in any of the three cases is $u = XT$. But $u = XT$ increases indefinitely with time t so possibility [2] is ruled out since then $u \to 0$ as $t \to \infty$. Conclusively the possibilities [1] and [3] determine the solution of (1) in the form

$$u(x, t) = u_S(x) + u_T(x, t) \quad ...(8)$$

where $u_S(x)$ is the temperature distribution after a long interval of time when there exists steady state of temperature and $u_T(x, t)$ is the transient effects which die down when the time passes. Consequently there exists uniform temperature after one and $x = 0$ being kept at zero temperature and the end $x = l$ at $u = u_0$ so that

$$u_S(x) = \frac{u_0}{l}x, \text{ whence (8) yields } u(x, t) = \frac{u_0}{l}x + u_T(x, t) \quad ...(9)$$

with boundary conditions $u_T(0, t) = 0 = u_T(l, t)$ by (2) ...(10)

and $$u_T(x, 0) = v_0 - \frac{u_0}{l}x \text{ by (3)} \quad ...(11)$$

Hence the possibility [3] *i.e.*, the solution (7) reduces to

$$u_T(x, t) = (A\cos\lambda x + B\sin\lambda x)\, e^{-\lambda^2 h^2 t} \quad ...(12)$$

whence in view of (2), this requires $A = 0$ and $\sin\lambda l = 0$ *i.e.*, $\lambda = \frac{n\pi}{l}$, n being an integer.

We thus obtain a solution

$$u_T(x, t) = \sum_{n=1}^{\infty} B_n\, e^{-\frac{n^2\pi^2h^2}{l^2}t} \sin\frac{n\pi}{l}x \quad ...(13)$$

In view of (11), this gives $u_T(x, 0) = v_0 - \frac{u_0}{l}x = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{l}x$.

$$\therefore \quad B_n = \frac{2}{l}\int_0^l \left(v_0 - \frac{u_0}{l}x\right)\sin\frac{n\pi}{l}x\,dx$$

$$= \frac{2}{n\pi}\,[v_0 - (-1)^n (v_0 - u_0)] \text{ (on integrating by parts)}$$

Hence the general solution of (1) with the help of (9) and (13) is

$$u(x, t) = \frac{u_0}{l}x + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[v_0 - (-1)^n (v_0 - u_0)\right]e^{-\frac{n^2\pi^2h^2}{l^2}t}\sin\frac{n\pi}{l}x \qquad ...(14)$$

which gives temperature at any point x of the bar at any time $t > 0$.

Note. If we set $v_0 = 0$, then (14) takes the from

$$u(x, t) = \frac{u_0}{l}\left[x + \frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(-1)^n e^{-\frac{n^2\pi^2h^2}{l^2}t}\sin\frac{n\pi}{l}x\right] \qquad ...(15)$$

[b_3] **(Temperature in an infinite bar)**

If an infinite bar of small cross-section is insulated such that there is no transfer of heat at the surface and the temperature of the bar at $t = 0$ is given by an arbitrary function $F(x)$ of x (taking the bar along x-axis), then find the temperature of the rod at any point of the bar at any time t.

The boundary value problem is $\dfrac{\partial u}{\partial t} = h^2\dfrac{\partial^2 u}{\partial x^2}$ ...(1)

With initial condition, $u(x, 0) = F(x), -\infty < x < \infty$ ...(2)

Let the solution be $u(x, t) = X(x)\,T(t)$ ...(3)

whence (1) gives $\dfrac{1}{X}\dfrac{d^2X}{dx^2} = \dfrac{1}{h^2T}\dfrac{dT}{dt} = -\lambda^2$ (say) ...(4)

Then the solution of (1) is

$$u(x, t) = XT = (A\cos\lambda x + B\sin\lambda x)\,e^{-h^2\lambda^2 t} \qquad ...(5)$$

Here the arbitrary constants A and B being periodic may be taken as $A = A(\lambda), B = B(\lambda)$ and due to the linearity and homogeneity of the heat equation we may write

$$u(x, t) = \int_0^{\infty} u(x, t, \lambda)\,d\lambda = \int_0^{\infty} e^{-h^2\lambda^2 t}\,[A(\lambda)\cos\lambda x + B(\lambda)\sin(\lambda x)]d\lambda.$$

The condition (2) claims that ...(6)

$$u(x, 0) = F(x) = \int_0^{\infty}[A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x]\,d\lambda$$

In view of Fourier's integrals we have

$$A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty}F(\mu)\cos(\mu\lambda)\,d\mu \text{ and } B(\lambda) = \frac{1}{\pi}\int_0^{\infty}F(\mu)\sin(\lambda\mu)\,d\mu$$

so that $u(x, o) = \dfrac{1}{\pi}\displaystyle\int_0^{\infty}\left[\int_{-\infty}^{\infty}F(\mu)\cos\lambda(x-\mu)d\mu\right]d\lambda$

As such (6) takes the form

$$u(x, t) = \frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{\infty}F(\mu)\cos\lambda(x-\mu)\,e^{-h^2\lambda^2 t}\,d\mu\right]d\lambda$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}F(\mu)\left[\int_0^{\infty}e^{-h^2\lambda^2 t}\cos\lambda(x-\mu)\,d\lambda\right]d\mu \qquad ...(7)$$

But we know that $\int_0^{\infty} e^{-x^2} \cos 2bx\, dx = \sqrt{\pi}\, e^{-b^2/2}$

So that $\int_0^{\infty} e^{-h^2\lambda^2 t} \cos \lambda\,(x-\mu)\, d\lambda = \dfrac{\sqrt{\pi}}{2h\sqrt{t}}\, e^{-\left(\frac{x-\mu}{2h\sqrt{t}}\right)^2}$

$$= \frac{\sqrt{\pi}}{2h\sqrt{t}}\, e^{-\frac{(x-\mu)^2}{4h^2 t}}$$

Hence (7) gives

$$u(x, t) = \frac{1}{2h\sqrt{\pi t}} \int_{-\infty}^{\infty} F(\mu)\, e^{-\frac{(x-\mu)^2}{4h^2 t}}\, d\mu \qquad \ldots(8)$$

which gives the required temperature at any point at any time.

Problem 1. *Solve* $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$ *where* $u = 0$ *for* $t = \infty$ *and* $x = 0$ *or* l.

Taking $u(x, t) = X(x)\, T(t)$, the solution of the given equation is

$$X = A \cos \lambda x + B \sin \lambda x, \qquad T = Ce^{-\lambda^2 t} \qquad \text{by §12.3 } [b_1]$$

with boundary conditions, $u(0, t) = 0$ and $u(x, \infty) = 0$.

Hence putting $h = 1$ in (8) of §12.3 $[b_1]$ the required solution is

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}\, e^{-\frac{n^2\pi^2}{l^2} t}$$

Problem 2. *Solve* $\dfrac{\partial \theta}{\partial t} = h^2 \dfrac{\partial^2 \theta}{dx^2}$ *under the boundary conditions*

$$\theta(0, t) = \theta(l, t) = 0,\ t > 0 \qquad \ldots(1)$$

and $\qquad \theta(x, 0) = x,\ 0 < x < l, \qquad \ldots(2)$

l being the length of the bar.

Proceeding just like in §12.3 $[b_1]$, we get the required solution on putting $\theta(x, 0) = F(x) = x$ in (8) of §12.3 $[b_1]$,

$$\theta(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}\, e^{-n^2\pi^2 h^2/l^2 t}$$

where

$$B_n = \frac{2}{l}\int_0^l F(x) \sin \frac{n\pi x}{l}\, dx = \frac{2}{l}\int_0^l x \sin \frac{n\pi}{l} x\, dx$$

$$= -\frac{2l}{n\pi} \cos n\pi,$$

$$= \begin{cases} \dfrac{2l}{n\pi} & \text{when } n \text{ is odd} \\ \dfrac{-2l}{n\pi} & \text{when } n \text{ is even} \end{cases}$$

Hence

$$\theta(x, t) = \frac{2l}{\pi}\left[e^{-\pi^2 h^2 t/l^2} \sin \frac{\pi}{l} - \tfrac{1}{2} e^{-2^2\pi^2 h^2 t/l^2} \sin \frac{2\pi}{l} x + \tfrac{1}{3} e^{-3^2\pi^2 h^2 t/l^2} \sin \frac{3\pi}{l} x \ldots\right]$$

Problem 3. *Find the temperature u (x, t) in a bar of length l, perfectly insulated, and whose ends are kept at temperature zero while the initial temperature is given by*

$$F(x)=\begin{cases} x, & 0<x<l/2 \\ l-x, & \frac{l}{2}<x<l. \end{cases}$$

The boundary value problem is $\frac{\partial u}{\partial t}=h^2\frac{\partial^2 u}{\partial x^2}\cdot$

With conditions $u(0,t)=u(l,t)=0$ and $u(x,0)=$

$$F(x)=\begin{cases} x, & o<x<l/2 \\ l-x, & \frac{l}{2}<x<l. \end{cases}$$

Hence by (8) of §12.3 [b_1] the required solution is

$$u(x,t)=\sum_{n=1}^{\infty} B_n\ F(x)\sin\frac{n\pi x}{l}dx$$

where $B_n=\frac{2}{l}\int_0^l F(x)\sin\frac{\pi nx}{l}\,dx=\frac{2}{l}\int_0^{l/2} x\sin\frac{n\pi}{l}x\,dx+\int_{l/2}^{l}(l-x)\sin\frac{n\pi x}{l}\,dx$

$$=\frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}=\begin{cases} \frac{4l}{n^2\pi^2} & \text{for } n=1,5,9,\ldots \\ 0 & \text{for } n=2,4,6,\ldots \\ -\frac{4l}{n^2\pi^2} & \text{for } n=3,7,11,\ldots \end{cases}$$

Hence the solution is

$$u(x,t)=\frac{4l}{\pi^2}\left[-\frac{1}{1^2}\sin\frac{\pi x}{l}e^{-k^2\pi^2 t/l^2}-\frac{1}{3^2}\sin\frac{3\pi x}{l}e^{-3^2k^2\pi^2 t/l^2}+\ldots\ldots\right]$$

Note. *Had we considered the case of slab with its ends x = 0 and n = l maintained at temperature zero and inital temperature being*

$$F(x)=\begin{cases} T_0, & o<x<l/2 \\ o, & l/2<x<l \end{cases}$$

then we should have

$$B_n=\frac{2}{l}\int_0^l F(x)\sin\frac{n\pi x}{l}\,dx=\frac{2T_0}{l}\int_0^l \sin\frac{n\pi x}{l}dx$$

$$=\frac{4T_0}{n\pi}\sin^2\frac{n\pi}{4}$$

and the solution would be

$$u(x,t)=\frac{4T_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin^2\frac{n\pi}{4}e^{-n^2\pi^2k^2t/l^2}\sin\frac{n\pi x}{l}.$$

Problem 4. *Solve* $\frac{\partial u}{\partial t}=h^2\frac{\partial^2 u}{\partial x^2}$, $0<x<\pi$, $t<0$, *under the boundary conditions*

$u_x(0,t)=0=u_x(\pi,t)$ *and* $u_x(x,0)=\sin x$

By (14) of §12.3 [b_1], we have

$$u(x,\ t) = B_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l} e^{-n^2\pi^2k^2t/l^2} \text{ where } B_0 = \frac{A_0}{2} \text{ and } l = \pi$$

$$= \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos nx\, e^{-n^2k^2 t}$$

where $$A_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\, dx = \begin{cases} o, \text{ when } n \text{ is odd} \\ -\dfrac{4}{\pi\left(4m^2-1\right)}, \text{ when } n = 2m \end{cases}$$

and $A_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} \sin x\, dx = \dfrac{4}{\pi}$.

Hence the required solution is

$$u(x,\ t) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{m=1}^{\infty}\left(4m^2 - 1\right)^{-1} e^{-4m^2k^2t} \cos 2mx.$$

Problem 5. *The face $x = 0$ of a slab is maintained at temperature zero and heat is supplied at constant rate at the face $x = \pi$, so that $\dfrac{\partial u}{\partial x} = \mu$ when $x = \pi$. If the initial temperature is zero, show that*

$$u(x,\ t) = \mu x + \sum_{j=1}^{\infty} \frac{(-1)^j}{\left(j-\frac{1}{2}\right)^2} \sin\left(j-\tfrac{1}{2}\right) x\, e^{-\left(j-\frac{1}{2}\right)^2 t}$$

where the unit of time is so chosen that $k = 1$.

Taking $u\ (x,\ t)$ as the temperature of the slab, the boundary value problem is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ 0 < x < \pi,\ t > 0. \quad \text{...(1)}$$

with condition $u(0,\ t) = 0$ ...(2)

$u(x,\ 0) = 0$ ...(3)

and $\dfrac{\partial}{\partial x} u(\pi,\ t) = \mu$ ...(4)

Applying the method of separatic¬ of variables, the solutions of the given equation are

(*i*) $u = (Ae^{\lambda x} + Be^{-\lambda x})\ e^{\lambda^2 t}$

(*ii*) $u = A_1 + B_1 x$

(*iii*) $u = (A_2 \cos \pi x + B_2 \sin \pi x)\ e^{-\lambda^2 t}$

according as the constant of variation is λ^2 or $-\lambda^2$.

Here (*i*) is inadmissible as $u \to \infty$ when $t \to \infty$.

(*ii*) alone is inadequate to give the complete solution and hence the complete solution is given by (*ii*) and (*iii*) jointly *i.e.* $$u(x, t) = u_S(x) + u_T(x, t) \qquad ...(5)$$

where $u_S(x)$ is the temperature distribution after a long period of time when the slab has reached the steady state of the temperature distribution and $u_T(x, t)$ denotes the transient effects which die down with the passage of time.

From (*ii*) $u_S(x) = A_1 + B_1 x$ ...(6)

and from (*iii*) $u_T(x, t) = (A_2 \cos \lambda x + B_2 \sin \lambda x)\, e^{-\lambda^2 t}$ ...(7)

Applying (2), (6) gives $A_1 = 0$ and by (4), (6) gives $\mu = B_1$

so that $u_S = \mu x$ ...(8)

Thus with the help of (7) and (8), (5) reduces to

$$u(x, t) = \mu x + (A_2 \cos \lambda x + B_2 \sin \lambda x)\, e^{-\lambda^2 t} \qquad ...(9)$$

Applying (2), *i.e.* $u(0, t) = 0$, we get $A_2 = 0$ ...(10)

Applying (4) *i.e* $\mu = \dfrac{\partial}{\partial x} u(\pi, t)$,

we have $(\mu + \lambda B_2 \cos \lambda\pi)\, e^{-\lambda^2 t} = \mu$

i.e. $$\cos \lambda\pi = 0 \text{ giving } \lambda\pi = (2j - 1)\frac{\pi}{2} \; i.e. \; \lambda = j - \frac{1}{2} \qquad ...(11)$$

As such $u_T(x, t) = B_j \sin\left(j - \frac{1}{2}\right)x \cdot e^{\left(j - \frac{1}{2}\right)^2 t}$ where we have set $B_j = B_2$. Summing over all j, the general solution is

$$u_T(x, t) = \sum_{j=1}^{\infty} B_j \sin\left(j - \tfrac{1}{2}\right)x\, e^{-\left(j - \frac{1}{2}\right)^2 t} \qquad ...(12)$$

Hence from (5)

$$u(x, t) = \mu x + \sum_{j=1}^{\infty} B_j \sin\left(j - \tfrac{1}{2}\right)x\, e^{-\left(j - \frac{1}{2}\right)^2 t} \qquad ...(13)$$

Applying the condition (3), $0 = \mu x + \sum\limits_{j=1}^{\infty} B_j \sin\left(j - \frac{1}{2}\right)x$

i.e. $$-\mu x = \sum_{j=1}^{\infty} B_j \sin\left(j - \tfrac{1}{2}\right)x \text{ so that } B_j = \frac{1}{\pi}\int_j^{\pi} (-\mu x) \sin\left(j - \tfrac{1}{2}\right)x\, dx$$

$$= \frac{2\mu}{\pi} \frac{(-1)^j}{\left(j - \frac{1}{2}\right)^2}$$

Hence (13) reduces to

$$u(x, t) = \mu x + \frac{2\mu}{\pi} \sum_{j=1}^{\infty} \frac{(-1)^j}{\left(j - \frac{1}{2}\right)^2} \sin\left(j - \tfrac{1}{2}\right) x\, e^{-\left(j - \frac{1}{2}\right)^2 t}$$

which is the required relation.

12.4. TWO-DIMENSIONAL DIFFUSION EQUATION

i.e. $$\frac{\partial u}{\partial t} = h^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

Consider a thin rectangular plate whose surface is impervious to heat flow and which has an arbitrary function of temperature F(x, y) at t = 0. Its four edges say x = 0, x = a, y = 0, y = b are kept at zero temperature. We have to determine the subsequent temperature at a point of the plate as t increases *(Rohilkhand, 1993)*

The boundary value problem is $\frac{\partial u}{\partial t} = h^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$ ...(1)

Subject to the conditions for all t, (*i*) $u(0, y, t) = 0$, (*ii*) $u(a, y, t) = 0$, (*iii*) $u(x, 0, t) = 0$, (*iv*) $u(x, b, t) = 0$ and the initial condition (*v*) $u(x, y, 0) = F(x, y)$

In order to apply the method of separation of variables, let us assume that

$$u(x, y, t) = X(x)\, Y(y)\, T(t) \qquad ...(2)$$

where X is a function of x alone, Y is a function of y alone and T is a function of t alone. From (2) we have

$$\frac{\partial u}{\partial t} = XY \frac{dT}{dt},\ \frac{\partial^2 u}{\partial x^2} = YT \frac{d^2 X}{dx^2} \text{ and } \frac{\partial^2 u}{\partial y^2} = XT \frac{d^2 Y}{dy^2}$$

Substituting them in (1), we find

$$\frac{1}{h^2 T} \frac{dT}{dt} = \frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2}, \text{ after dividing by } XYT \qquad ...(3)$$

In (3), the variables being separated, we can assume

$$\frac{1}{X} \frac{d^2 X}{dx^2} = -\lambda_1^2;\ \frac{1}{Y} \frac{d^2 Y}{dy^2} = -\lambda_2^2 \text{ and } \frac{1}{h^2 T} \frac{dT}{dt} = -\lambda^2$$

so that $$\lambda^2 = \lambda_1^2 + \lambda_2^2 \qquad ...(4)$$

The general solutions of (4) are

$X = A \cos \lambda_1 x + B \sin \lambda_1 x$; $Y = C \cos \lambda_2 y + D \sin \lambda_2 y$:

$$T = E e^{-\lambda^2 h^2 t} \qquad ...(5)$$

So that with the help of (5), (2) gives the solution of (1) in the form $u(x, y, t) = (A \cos \lambda_1 x + B \sin \lambda_1 x)(C \cos \lambda_2 y + D \sin \lambda_2 y)\, e^{-\lambda^2 h^2 t}$ ...(6)

In view of condition (*i*), $0 = u(o, y, t) = A(C \cos \lambda_2 y + D \sin \lambda_2 y)\, e^{-\lambda^2 h^2 t}$, giving $A = 0$.

In view of condition (*ii*), we claim $\sin \lambda_1 x = 0$ i.e. $\lambda_1 = \frac{m\pi}{a}$, m being an integer.

Similarly applying conditions (*iii*) and (*iv*) to (6), we get $C = 0$ and $\lambda^2 = \frac{n\pi}{b}$, n being an integer. As such (5) takes the form

$$u_{mn}(x, y, t) = B_{mn} e^{-\lambda^2{}_{mn} h^2 t} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y, \text{ where}$$

$$\lambda^2 = \lambda_{mn}^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right).$$

Summing over all the possible values of m and n, the general solution is

$$u(x, y, t) = \sum_{m,\, n=1}^{\infty} B_{mn}\, e^{-\lambda_{mn}^2 h^2 t} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \qquad \ldots(7)$$

where B_{mn} are arbitrary constants to be determined by the condition (v)

i.e. $$F(x, y) = u(x, y, o) = \sum_{m,\, n=1}^{\infty} B_{mn} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \qquad \ldots(8)$$

Multiplying both sides of (8) by $\sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y\, dx\, dy$ and integrating with regard to x from 0 to a and with regard to y from 0 to b we get on using orthogonality properties of the sines,

$$B_{mn} = \frac{4}{ab} \int_0^a \int_0^b F(x, y) \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y\, dx\, dy \qquad \ldots(9)$$

which gives the arbitrary constants of (7).

Problem 6. *A rectangular plate bounded by the lines $x = 0, y = 0, x = a, y = b$ has an initial distribution of temperature given by $F(x, y) = B \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$. The edges are maintained at zero temperature and the plane faces are impervious to heat. Find the temperature at any point at any time.*

By (7) of §12.4, the general solution is

$$u(x, y, t) = \sum_{m,\, n=1}^{\infty} B_{mn}\, e^{-\lambda_{mn}^2 h^2 t} \left\{ \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \right\}$$

where $B_{mn} = \frac{4}{ab} \int_0^a \int_0^b F(x, y) \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y\, dx\, dy$ by (9) of §12.4.

$$= \frac{4B}{ab} \int_0^a \int_0^b \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \sin \frac{m\pi x}{a} \sin \frac{n\pi v}{b}\, dx\, dy$$

$$\because\ F(x, y) = B \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$$

$$\therefore\quad B_{m1} = \frac{4B}{ab} \int_0^a \frac{b}{2} \sin \frac{\pi x}{a} \sin \frac{m\pi x}{a}\, dx$$

$$\because\ \int_0^b \sin \frac{\pi y}{b} \sin \frac{n\pi y}{b}\, dy = \begin{cases} 0, & \text{for } n = 2, 3, 4, \ldots \\ b/2, & \text{for } n = 1. \end{cases}$$

So that $B_{11} = B$.

Also $\lambda_{11}^2 = \pi^2 \left(\frac{1}{a^2} + \frac{1}{b^2} \right)$

Hence the solution is

$$u(x, y, t) = B_{11}\, e^{-\lambda_{mn}^2 h^2 t} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$$

$$= B\, e^{-h^2 \pi^2 \left(\frac{1}{a^2} + \frac{1}{b^2} \right)} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}.$$

Problem 7. *A semi-infinite plate having width π has its faces insulated. The semi-infinite edges are kept at 0°C while the infinite edge is maintained at 100°C. Assuming that the initial temperature is 0°C, find the temperature at any point at any time.*

Taking the diffusivity *i.e.* $h^2 = 1$, the boundary value probelm is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad \text{...(1)}$$

with conditions (*i*) $u(o, y, t) = 0$

(*ii*) $u(\pi, y, t) = 0$, (*iii*) $u(x, y, o) = 0$;

(*iv*) $u(x, o, t) = 100$ and

(*v*) $|u(x, y, t)| < M$

where $0 < x < \pi, y > 0, t > 0$.

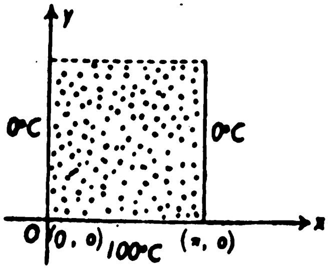


Fig. 12.2.

Taking Laplace transform of (1) and assuming $L\{u(x, y, t)\} = U(x, y, t)$,

We have $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = sU$ by using condition (*iii*) ...(2)

But the finite Fourier sine transform of a function $F(x)$, $0 < x < l$ is defined as

$$f_s(x) = \int_0^l F(x) \sin \frac{n\pi x}{l} \, dx, \; n \text{ being an integer} \quad \text{...(3)}$$

Multiplying (2) by $\sin nx$ and integrating from 0 to π, we get

$$\int_0^\pi \frac{\partial^2 U}{\partial x^2} \sin nx \, dx + \int_0^\pi \frac{\partial^2 U}{\partial y^2} \sin nx \, dx = \int_0^\pi sU \sin nx \, dx$$

Setting $\bar{U} = \int_0^\pi U \sin nx \, dx$, this becomes

$$-n^2 \bar{U} + nU(\pi, y, s) \cos n\pi + nU(o, y, s) + \frac{d^2 \bar{U}}{dy^2} = s\bar{U} \quad \text{...(4)}$$

But from the Laplace transforms of conditions (*i*) and (*ii*) we have

$$U(o, y, s) = 0, \; U(\pi, y, s) = 0$$

$\therefore$ (4) reduces to $\frac{d^2 \bar{U}}{dy^2} - (n^2 + s) \bar{U} = 0$ ...(5)

Its solution is $\bar{U} = A e^{y\sqrt{n^2+s}} + B e^{-y\sqrt{n^2+s}}$ ...(6)

By condition (*v*), $\bar{U}$ being bounded, as $y \to \infty$, we have $A = 0$ so that (6) yields,

$\bar{U} = Be^{-y\sqrt{n^2+s}}$ ...(7)

Applying condition (*iv*),

$$\bar{U}(n, o, s) = \int_0^\pi \frac{100}{s} \sin nx \, dx = \frac{100}{s}\left(\frac{1 - \cos n\pi}{n}\right) \quad \text{...(8)}$$

In (7) if we put $y = 0$, we get with the help of (8),

$$B = \bar{U} = \frac{100}{s}\left(1 - \frac{\cos n\pi}{n}\right)$$

Hence $\bar{U} = \frac{100}{s} \frac{(1-\cos n\pi)}{n} e^{-y\sqrt{n^2+s}}$.

Applying Fourier sine inversion formula, we find

$$U = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{100}{s}\left(\frac{1-\cos n\pi}{n}\right) e^{-y\sqrt{n^2+s}} \sin nx \qquad \ldots(9)$$

Now we have $$L^{-1}\left\{e^{-y\sqrt{s}}\right\} = \frac{y}{2\sqrt{\pi t^3}} e^{-\frac{y^2}{4t}}$$

so that $$L^{-1}\left\{e^{-y\sqrt{s+n^2}}\right\} = \frac{y}{2\sqrt{\pi t^3}} e^{-\frac{y^2}{4t}} e^{-n^2 t}$$

Thus $$L^{-1}\left\{\frac{e^{-y\sqrt{s+n^2}}}{s}\right\} = \int_0^t \frac{y}{2\sqrt{\pi t^3}} e^{-y^2/4v} e^{-n^2 v} dv$$

$$= \frac{2}{\sqrt{\pi}} \int_{y/2\sqrt{t}}^{\infty} e^{-(p^2 + n^2 y^2/4p^2)} dp \quad \text{where } p^2 = \frac{y^2}{4v}$$

Hence taking the inverse Laplace transform of (9) term by term, we get

$$u(x, y, t) = \frac{400}{\pi^{3/2}} \sum_{n=1}^{\infty} \left(\frac{1-\cos n\pi}{n}\right) \sin nx \int_{y/2\sqrt{t}}^{\infty} e^{-(p^2 + n^2 y^2/4p^2)} dp.$$

12.5. THREE-DIMENSIONAL DIFFUSION EQUATION

We have $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{h^2}\frac{\partial u}{\partial t} \qquad \ldots(1)$$

where $u = u(x, y, z, t)$.

Let, $u(x, y, z, t) = X(x)\, Y(y)\, Z(z)\, T(t) \qquad \ldots(2)$

where X, Y, Z, T being respectivley the functions of x, y, z, t alone.

From (2) we have $$\frac{\partial u}{\partial t} = XYZ\frac{dT}{dt}, \quad \frac{\partial^2 u}{\partial x^2} = YZT\frac{d^2X}{dx^2}, \quad \frac{\partial^2 u}{\partial y^2} = XZT\frac{d^2u}{dy^2}$$

$$\text{and } \frac{\partial^2 u}{\partial z^2} = XYT\frac{d^2Z}{dz^2}$$

Their substitution in (1) yields

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{y}\frac{d^2Y}{dy^2} + \frac{1}{z}\frac{d^2Z}{dz^2} = \frac{1}{h^2T}\frac{dT}{dt}$$

$= -\lambda^2$ (say) as variables are separable.

Now taking $$\frac{1}{X}\frac{d^2X}{dx^2} = -\lambda_1^2, \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = -\lambda_2^2, \quad \frac{d^2Z}{dz^2} = -\lambda_3^2$$

so that $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \lambda^2$, we get the solutions

$$X = A_1 \cos \lambda_1 x + B_1 \sin \lambda_1 x = a\cos(\lambda_1 x + \alpha_{\lambda_1})$$

Similarly $Y = b\cos(\lambda_2 x + \alpha_{\lambda_2})$, $Z = c$ and $(\lambda_3 x + \alpha_{\lambda_3})$ and

$$T = de^{-\lambda^2 k^2 t} = de^{-(\lambda_1^2+\lambda_2^2+\lambda_3^2)\lambda^2 t}$$

Hence for all values of t, the general solution of (11) is

$$u(x,\ y,\ z,\ t) = \sum_{\lambda_1=0}^{\infty}\sum_{\lambda_2=0}^{\infty}\sum_{\lambda_3=0}^{\infty} A_{\lambda_1\lambda_2\lambda_3}\cos\left(\lambda_1 x + \alpha_{\lambda_1}\right)$$

$$\cos\left(\lambda_2 x + \alpha_{\lambda_2}\right)\cos\left(\lambda_3 x + \alpha_{\lambda_3}\right)e^{-k^2(\lambda_1^2+\lambda_2^2\lambda_3^2)t}$$

12.6. LAPLACE'S EQUATION

In §12.2 (cor 3) we have already derived the Cartesian form of three-dimensional Laplace's equation as a particular case of steady heat flow in the form.

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad \ldots(1)$$

In cylindrical coordinates (r, θ, z), it is as shown in §11.1,

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\ \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad \ldots(2)$$

and in Polar spherical coordinates (r, θ, ϕ), it is as shown in §8.1,

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0 \qquad \ldots(3)$$

Two-dimensional Cartesian form of Laplace's equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \ldots(4)$$

Taking u as independent of z, the two-dimensional Laplace's equation in cylindrical coordinates is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad \ldots(5)$$

and in Polar coordinates (r, θ) it resumes the same form as (5).

One-dimensional Laplace's equation is $\dfrac{\partial^2 u}{\partial x^2} = 0.$ ...(6)

Its solution being easy and straight has no points of worth consideration and hence we shall consider only two and three-dimensional Laplace equations.

12.7. TWO-DIMENSIONAL LAPLACE'S EQUATION (STEADY FLOW OF HEAT)

[A] Solution of two-dimensional Laplace's equation in Cartesian coordinates

We have $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0,$ ...(1)

(taking temperature as independent of time).

This can be solved either by the method of separation of variables or by the application of integral transforms as is evident from the following problems.

Problem 8. *Determine the steady state temperature distribution in a thin plate bounded by the lines $x = 0$, $x = l$, $y = 0$ and $y = \infty$, assuming that heat cannot escape from either surface of the plate, the edges $x = 0$ and $x = l$ being kept at a temperatue zero and also the lower edge $y = 0$ is kept at temperature $F(x)$ and the edge $y = \infty$ at temperature zero.*

The boundary value problem is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad ...(1)$$

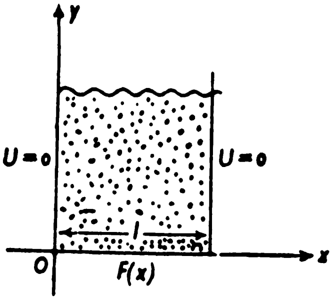


Fig. 12.3

With conditions

(i) $u(0, y) = 0$, (ii) $u(l, y) = 0$

(iii) $u(x, 0) = F(x)$, and

(iv) $u(x, \infty) = 0$.

In order to apply the method of separation of variables, assume that $u(x, y) = X(x)\, Y(y)$ ...(2)

So that $\dfrac{\partial^2 u}{\partial x^2} = Y\dfrac{d^2X}{dx^2}$ and

$$\frac{\partial^2 u}{\partial y^2} = X\frac{d^2Y}{dy^2}$$

$\therefore$ (1) gives $\dfrac{1}{X}\dfrac{d^2X}{dx^2} = -\dfrac{1}{Y}\dfrac{d^2Y}{dy^2} = -\lambda^2$ (say) as variables are separated

Here $\dfrac{1}{X}\dfrac{d^2X}{dx^2} = -\lambda^2$ *i.e.* $\dfrac{d^2X}{dx^2} + \lambda^2 X = 0$ gives

$$X = A\cos\lambda x + B\sin\lambda x \qquad ...(3)$$

and $\dfrac{1}{Y}\dfrac{d^2Y}{dx^2} = \lambda^2$ *i.e.* $\dfrac{d^2Y}{dy^2} - \lambda^2 Y = 0$ gives $Y = Ce^{\lambda y} + De^{-\lambda y}$ ...(4)

As such a solution of (1) is

$$u(x, y) = XY = (A\cos\lambda x + B\sin\lambda x)(Ce^{+\lambda y} + De^{-\lambda y}) \qquad ...(5)$$

Applying condition (iv) we have $C = 0$ and applying (i) $A = 0$, so that (5) takes the form

$$u(x, y) = B\sin\lambda x . e^{-\lambda y} \qquad ...(6)$$

But condition (ii) yields, $\sin\lambda l = 0$

i.e. $\lambda = \dfrac{n\pi}{l}$, n being an integer.

Hence for all distinct n, the general solution of (1) is

$$u(x, y) = \sum_{n=0}^{\infty} B_n \sin\frac{n\pi}{l}\, e^{-n\pi y/l} \qquad ...(7)$$

which gives the required temperature in the thin plate, where

$$B_n = \frac{2}{l}\int_0^l F(x)\sin\frac{n\pi x}{l}dx \text{ and } F(x) = u(x, \infty) = \sum_{n=0}^{\infty} B_n \sin\frac{n\pi x}{l}$$

Problem 9. Temperature distribution in a finite plate.

Find the steady state temperature distribution of a thin rectangular plate bounded by the lines $x = 0, x = l, y = 0, y = b$ assuming that the edges $x = 0, x = l$ and $y = 0$ are maintained at temperature zero while the edge $y = b$ is maintained at temperature $F(x)$.

The boundary value problem is $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ ...(1)

with the conditions (*i*) $u(0, y) = 0$, (*ii*) $u(l, y) = 0$; (*iii*) $u(x, 0) = 0$ and (*iv*) $u(x\ b) = F(x)$.

Proceeding just like in Problem 8, we get the general solution of (1) as

$$u(x, y) = \sum_{n=1}^{\infty}\left[B_n e^{n\pi y/l} + C_n e^{-n\pi y/l}\right]\sin\frac{n\pi x}{l} \qquad ...(2)$$

In view of condition (*i*), $C_n = -B_n$ so that (2) reduces to

$$u(x, y) = \sum_{n=1}^{\infty} B_n\left[e^{n\pi y/l} - e^{-n\pi y/l}\right]\sin\frac{n\pi x}{l}$$

$$= \sum_{n=1}^{\infty} D_n \sinh\frac{n\pi y}{l}\sin\frac{n\pi y}{l} \text{ on setting } D_n = 2B_n$$

By condition (*iv*), $F(x) = \sum_{n=1}^{\infty} D_n \sinh\dfrac{n\pi b}{l}\sin\dfrac{n\pi x}{l}$ so that

$$D_n \sinh\frac{n\pi b}{l} = \frac{2}{l}\int_0^l F(x)\sin\frac{n\pi x}{l}\,dx.$$

Hence the solution is

$$u(x, y) = \frac{2}{l}\sum_{n=1}^{\infty} \sinh\frac{n\pi y}{l}\,\text{cosech}\,\frac{n\pi b}{l}\sin\frac{n\pi x}{l}\int_0^l F(x)\sin\frac{n\pi x}{l}\,dx.$$

Problem 10. (Insulated at one side). *Determine the steady state temperature in a rectangular plate of length a and width b with sides maintained at temperature zero while the lower end is kept at temperature $F(x)$ and upper one insulated.*

The boundary value problem is $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ ...(1)

with conditions (*i*) $u(0, y) = 0$, (*ii*) $u(a, y) = 0$, (*iii*) $u(x, 0) = F(x)$ and (*iv*) $u(x, b) = 0$

Proceeding just like in Problem 8, we have

$$u(x, y) = \sum_{n=1}^{\infty}\left[C_n \cosh\frac{n\pi y}{a} + D_n \sinh\frac{n\pi y}{a}\right]\sin\frac{n\pi x}{a} \qquad ...(2)$$

In view of (*iii*) we have

$$F(x) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{a} \text{ so that } C_n = \frac{2}{a}\int_0^a F(x)\sin\frac{n\pi x}{a}\,dx \text{ and}$$

in view of (*iv*), $0 = u_y(x, b) = \sum_{n=1}^{\infty}\dfrac{n\pi}{a}\left(C_n \cosh\dfrac{n\pi b}{a} + D_n \sinh\dfrac{n\pi b}{a}\right)\sin\dfrac{n\pi x}{a}$,

giving $D_n = -C_n \tanh\dfrac{n\pi b}{a}$

Hence $u(x, y) = \frac{2}{a}\sum_{n=1}^{\infty}\left(\cosh\frac{n\pi y}{a} - \tanh\frac{n\pi y}{a}\sin\frac{n\pi y}{a}\right)\sin\frac{n\pi x}{a}$

$$\int_0^a F(x)\sin\frac{n\pi x}{a}\,dx.$$

Problem 11. *Heat flows in a semi-infinite rectangular plate, the end $x = 0$ being kept at temperature $T\,^\circ C$ and the edges $y = a$ at temperature zero, then show that the temperature at any point (x, y) is given by*

$$u(x, y) = \frac{4T}{\pi}\sum_{r=1}^{\infty}\frac{1}{2r+1}\sin\frac{(2r+1)\pi y}{a}e^{-(2r+1)\pi x/a}$$

The boundary value problem is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ ...(1)

with conditions (*i*) $u = 0$ when $y = 0$, (*ii*) $u = 0$ when $y = a$, (*iii*) $u = T$ when $x = 0$.

The solution by usual method is

$$u(x, y) = (A\cos ny + B\sin ny)\,e^{-nx}$$

In view of (*i*), $0 = Ae^{-nx}$ *i.e.* $A = 0$.

In view of (*ii*), $0 = B\sin na\, e^{-nx}$ giving $\sin na = 0$ *i.e.*, $na = (2r+1)\pi$.

Hence the general solution is

$$u(x, y) = \sum_{r=1}^{\infty} B_n \sin(2r+1)\frac{\pi y}{a}e^{-(2r+1)\pi x/a} \quad ...(2)$$

In view of (*iii*), $T = \sum_{r=1}^{\infty} B_n \sin(2r+1)\frac{\pi y}{a}$ so that

$$B_n = \frac{2}{a}\int_0^a T\sin(2r+1)\frac{\pi y}{a}dy = \frac{4T}{(2r+1)\pi}$$

Hence $u(x, y) = \frac{4T}{\pi}\sum_{r=1}^{\infty}\frac{1}{2r+1}\sin\frac{(2r+1)\pi y}{a}e^{-(2r+1)\pi x/a}$

Problem 12. *A square plate has its faces and its edges $x = 0$ and $x = \pi$ $(0 < y < \pi)$ insulated. Its edges $y = 0$ and $y = \pi$ are kept at temperatures zero and $f(x)$ respectively. Show that the formula for its steady temperature is*

$$u(x, y) = \frac{1}{2\pi}a_0 y + \sum_{n=1}^{\infty} a_n \frac{\sinh ny}{\sinh n\pi}\cos nx,$$

$$\text{where } a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx,\quad n = 0, 1, 2, \ldots$$

Hint. The boundary value problem is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ with conditions

(*i*) $u_x(0, y) = 0$, (*ii*) $u_x(\pi, y) = 0$, (*iii*) $u(x, 0) = 0$, (*iv*) $u(x, \pi) = f(x)$.

Apply method of separation of variables.

Problem 13. *If $u(x, y)$ denotes the electrostatic potential in a region bounded by the planes $x = 0, x = \pi$ and $y = 0$ in which there is a uniform distribution of space charge of density $\frac{h}{4\pi}$. If the planes $x = 0$ and $y = 0$, are kept at potential zero, the plane $x = \pi$ at an other fixed potential $u = 1$ and u is finite as $y \to \infty$, then find u.*

The function $u(x, y)$ satisfies Poisson equation $\nabla^2 u = -4\pi\rho$ in two dimensions, where $\rho = \frac{h}{4\pi}$ and hence the boundary value problem is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -h \; (0 < x < \pi, \; y > 0) \quad ...(1)$$

with conditions $u = 0$ when $x = 0$, $u = 1$ when $x = \pi$ ...(2)

and $u = 0$ when $y = 0$, $(0 < x < \pi)$ and $u < M$ $(0 \le x \le \pi, y > 0)$ ...(3)

where M is some constant.

Using finite Fourier transform, (1) gives

$$\int_0^{\pi} \frac{\partial^2 u}{\partial x^2} \sin \lambda x \, dx + \int_0^{\pi} \frac{\partial^2 u}{\partial y^2} \sin \lambda x \, dx = -h \int_0^{\pi} \sin \lambda x \, dx$$

or

$$\frac{d^2 \bar{u}}{dy^2} - p^2 \bar{u} = p(-1)^p - h F_s \{1\}, \quad ...(4)$$

where

$$\bar{u} = \int_0^{\pi} u \sin \lambda x \, dx$$

(under the conditions $u = 0$ at $x = 0$ and $u = 1$ at $x = \pi$).

Finite Fourier transform of (3) gives $\bar{u} = 0$ when $y = 0$ and $|\bar{u}| < M\pi$ ...(5)

$\therefore$ Solution of (4) is $\bar{u} = Ae^{-py} + Be^{py} + \dfrac{p(-1)^p - hF_s(1)}{-p^2}$

Since $\bar{u}$ is finite when $y \to \infty$ $\therefore B = 0$.

Also $y = 0$, $\bar{u} = 0$ gives $A = \dfrac{p(-1)^p - hF_s\{1\}}{p^2}$

Hence $\bar{u} = \dfrac{hF_s\{1\} - p(-1)^p}{p^2}(1 - e^{pv})$

Applying the inversion formula for finite Fourier sine transform, we find

$$u(x, y) = \frac{2}{\pi} \sum \bar{u} \sin nx.$$

Problem 14. *Solve* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ *for* $0 < x < \pi$, $0 < \pi$, $0 < y < \pi$ *under the boundary conditions*

$$u(x, 0) = x^2, \; u(x, \pi) = 0, \; u_x(0, y) = \frac{\partial}{\partial x} u_x(0, y) = 0 = u_x(\pi, y).$$

Ans. $u(x, y) = \dfrac{\pi}{3}(\pi - y) + 4\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sinh n(\pi - y) \cos nx}{n^2 \sinh n\pi}$

[B] Solution of two-dimensional Laplace's equation in cylindrical (or Polar) coordinates.

The Laplace's equation in this case is $\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}=0$ ...(1)

Assume $u(r, \theta)=R(r)\,\Theta(\theta)$

So that $\dfrac{\partial^2 u}{\partial r^2}=\Theta\dfrac{\partial^2 R}{\partial r^2}$, $\dfrac{\partial u}{\partial r}=\Theta\dfrac{dR}{dr}$ and $\dfrac{\partial^2 u}{\partial \theta^2}=R\,\dfrac{d^2\Theta}{d\theta^2}$.

$\therefore$ (1) gives $\Theta\,\dfrac{d^2R}{dr^2}+\dfrac{1}{r}\Theta\dfrac{dR}{dr}+\dfrac{1}{r^2}\dfrac{Rd^2\Theta}{d\theta^2}=0$ ...(2)

or $$\frac{1}{R}\left(r^2\,\frac{d^2R}{dr^2}+r\frac{dR}{dr}\right)=-\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}=n^2 \text{ (say)} \qquad ...(3)$$

as variables are separated.

Here $-\dfrac{1}{\Theta}\dfrac{d^2\Theta}{d\theta^2}=n^2$ *i.e.*, $\dfrac{d^2\Theta}{d\theta^2}+n^2\,\Theta=0$ gives $\Theta=A\cos n\theta+B\sin n\theta$ ...(4)

and $$\frac{1}{R}\left(r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}\right)=n^2 \text{ i.e., } r^2\,\frac{d^2R}{dr^2}+\frac{dR}{dr}-n^2R=0$$

being homogeneous, takes the form $\dfrac{d^2R}{ds^2}-n^2R=0$ on putting $r=e^s$ and then its solution is

$$R=Ce^{ns}+De^{-ns} \text{ i.e., } R=C\,r^n+Dr^{-n} \qquad ...(5)$$

Taking $n=0$ we have from (3)

$$\frac{d^2\Theta}{d\theta^2}=0 \text{ giving } \Theta=A_0\theta+B_0 \qquad ...(6)$$

and $r^2\,\dfrac{d^2R}{dr^2}+r\,\dfrac{dR}{dr}=0$ or $\dfrac{d^2R}{ds^2}=0$ (when $r=e^s$) giving $R=C_0s+D_0=C_0\log r+D_0$ ...(7)

The solution of Laplace's equation in cylindrical coordinates when u is independent of z is known as *Circular Harmonics* and n is the degree of the harmonic. Hence the Circular Harmonics of degree zero are given by

$$u_0=(A\theta+B)\,(C\log r+D) \text{ by (6) and (7)}$$

and those of degree n are given by

$$u_n=(A_n\cos n\theta+B_n\sin n\theta)\,(C_nr^n+D_n\,r^{-n}) \text{ by (4) and (5).}$$

The general single-valued solution of (1) for all possible n may be written as

$$u=A_0\log r+\sum_{n=1}^{\infty}(A_n\cos n\theta+B_n\sin n\theta)\,(C_n\,r^n+D_n\,r^{-n})+C_0 \qquad ...(8)$$

where A_0, A_n, B_n, C_n, D_n and C_0 all are arbitrary constants.

Problem 15. *For a semi-circular plate of radius a with boundary diameter at 0° C and surface at 100°C, show that the temperature distribution is given by*

$$u=\frac{400}{\pi}\sum_{n=1}^{\infty}\frac{r^{2n-1}}{2n-1}\frac{\sin(2n-1)\,\theta}{a^{2n-1}}$$ *(Rohilkhand, 1980)*

The boundary value problem, $\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}=0$ ...(1)

Its solution by (8) of § 12.7 [*B*] is

$$u = A_0 \log r + \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta)(C_n r^n + D_n r^{-n}) + C_0 \quad ...(2)$$

But temperature being finite at $r = 0$, (2) should not contain terms of log r and r^{-n} and this will be so if $A_0 = 0 = D_n$.

Moreover at $r = 0$, u being zero, we shoud have $C_0 = 0$

Hence (2) reduces to $u = \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta) C_n r^n$

$$= \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) r^n \text{ taking } a_n = A_n C_n \text{ etc.}$$

Now a being the radius of the sphere and assuming $u = u(a)$ at $r = a$, we have

$$u(a) = \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) a^n \quad ...(3)$$

where $a_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} \dfrac{u(a)}{a^n} \cos n\theta \, d\theta = \dfrac{200}{\pi a^n}\displaystyle\int_0^{\pi} \cos n\theta \, d\theta = 0$ and

$$b_n = \frac{2}{\pi}\int_0^{\pi} \frac{u(a)}{a^n} \sin n\theta \, d\theta$$

i.e., $b_n = \dfrac{200}{\pi a^n}\displaystyle\int_0^{\pi} \sin n\theta \, d\theta = 0$ when n is even and $\dfrac{400}{n\pi a^n}$ when n is odd.

Hence (3) reduces to

$$u = \frac{400}{\pi}\sum_{n=1}^{\infty} \frac{r^{2n-1}}{2n-1} \frac{\sin(2n-1)\theta}{a^{2n-1}} \text{ for } n \text{ odd.}$$

Problem 16. *Determine the steady state temperature at any point of a semi-circular metal plate of radius a whose circumference is maintained to a given temperature of T° whereas the base is kept at zero temperature*

$\because u = u(r, \theta)$, the boundary value problem is

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0 \quad ...(1)$$

with conditions (*i*) $u = 0$ when $\theta = 0$ for $0 \le r < a$

(*ii*) u is finite when $r \to 0$

(*iii*) $u = T$ when $r = a$ for $0 < \theta < \pi$

Solution of (1) by (8) of §12.7 [*B*] is

$$u = A_0 \log r + \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta)(C_n r^n + D_n R^{-n}) + C_0 \quad ...(2)$$

In view of condition (*ii*), u being finite, (2) must not contain terms of log r and r^{-n} and this will be so when $A_0 = 0 = D_n$. Thus (2) reduces to

$$u = \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)\, r^n + C_0 \text{ when } a_n = A_n C_n,\ b_n = B_n C_n \qquad \text{...(3)}$$

In view of condition (*i*) $C_0 = 0$ and hence (3) yields

$$u = \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)\, r^n \qquad \text{...(4)}$$

By condition (*iii*), this gives, $T = \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)\, a^n$ from which we find

$$a_n = \frac{2}{\pi}\int_0^{\pi} \frac{T}{a^n} \cos n\theta \, d\theta = \frac{2T}{\pi a^n}\int_0^{\pi} \cos n\theta = 0 \text{ and}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \frac{T}{a^n} \sin n\theta \, d\theta = \frac{2T}{\pi a^n}\int_0^{\pi} \sin n\theta \, d\theta = \frac{2T}{n\pi a^n}(1 - \cos n\pi)$$

Hence (4) gives the required solution as

$$u = \frac{2T}{n\pi a^n} - \sum_{n=1}^{\infty} (1 - \cos n\pi)\, r^n \sin n\theta .$$

Problem 17. *A long cylinder is made of two halves, the upper half is at the temperature T_1 and the lower half at the temperature T_2. Find the distribution of temperature inside the cylinder.*

Taking the axis of cylinder along z-axis, there is symmetry along z-axis and hence z-axis has no effect on the distribution of temperature. At the centre where $r = 0$, we have u = finite. The boundary value problem is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad \text{...(1)}$$

Its solution is (by (8) of §12.7[*B*])

$$u = A_0 \log r + \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta)(C_n r^n + D_n r^{-n}) + C_0 \qquad \text{...(2)}$$

u being finite, we have to eliminate the terms of $\log r$ and r^{-n} so that $A_0 = 0 = D_n$

$$\therefore \text{ (2) becomes } u = \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) + C_0 \qquad \text{...(3)}$$

where $a_n = A_n C_n$ etc.

Suppose that $u = F(\theta)$ at $r = R$ (say), then (3) gives

$$f(\theta) = C_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)\, R^n \qquad \text{...(4)}$$

This gives $C_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, d\theta$

$$a_n = \frac{1}{R^n \pi}\int_0^{2\pi} f(\theta) \cos n\theta \, d\theta \text{ and } b_n = \frac{1}{R^n \pi}\int_0^{2\pi} f(\theta) \sin n\theta \, d\theta$$

But we are given that

$f(\theta) = T_1$ for $\pi > \theta > 0$ (upper half)

and $f(\theta) = T_2$ for $2\pi > \theta > \pi$ (lower half)

$$\therefore\ C_0 = \frac{1}{2\pi}\left[\int_0^{\pi} T_1\, d\theta + \int_{\pi}^{2\pi} T_2 d\theta\right] = \frac{T_1 + T_2}{2}$$

$$a_n = \frac{1}{\pi R^n}\left[\int_0^{\pi} T_1 \cos n\theta\ d\theta + \int_{\pi}^{2\pi} T_2 \cos n\theta\ d\theta\right] = 0$$

and $$b_n = \frac{1}{\pi R^n}\left[\int_0^{\pi} T_1 \sin n\theta\ d\theta + \int_{\pi}^{2\pi} T_2 \sin n\theta\ d\theta\right]$$

$$= \frac{1}{\pi R^n}\left[\int_0^{\pi} \frac{T_1 + T_2}{n}(1 - \cos n\pi)\right] = \begin{cases} 2/n\pi R^n\ (T_1 - T_2) \text{ for } n \text{ odd} \\ 0, \text{ for n even} \end{cases}$$

Hence the solution (3) reduces to

$$u = \frac{T_1 + T_2}{2} + \sum_{n=1}^{\infty} \frac{2}{n\ \pi\ R^n}\ (T_1 - T_2) \sin n\theta \cdot r^n \text{ where } n = 1, 3, 5, \ldots$$

$$= \frac{T_1 + T_2}{2} + \frac{2}{\pi R}(T_1 - T_2) \sin\theta - r + \frac{2}{3\pi R^3}(T_1 - T_2) \sin 3\theta \cdot r^2 + \ldots$$

which gives the required distribution of temperature.

Problem 18. *If u is a function of r and θ satisfying*

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad \ldots(1)$$

within the region of the plane bounded by $r = a,\ r = 0,\ \theta = 0,\ \theta = \frac{\pi}{2}$ *and also satisfying the boundary conditions* $u = 0$ *when* $\theta = 0$, $u = 0$ *when* $\theta = \frac{\pi}{2}$, $u = 0$ *when* $r = b$ *and* $u = \theta\left(\frac{\pi}{2} - \theta\right)$ *when* $r = 0$ *then show that*

$$u = \frac{16}{\pi}\sum_{n=1}^{\infty} \frac{(r/b)^{4n-2}(b/r)^{4n-2}}{(a/b)^{4n-2} - (b/a)^{4n-2}} \frac{\sin(4n-2)}{(4n-2)}.$$

Fig. 12.4

The solution of (1) with the help of (4) and (5) of §12.7 [B] is $u(r, \theta) = (A \cos m\theta + B \sin m\theta)$ is $(Cr^m + Dr^{-m})$ ...(2)

where we have taken m^2 as constant of separation. Applying the boundary condition $u = 0$, when $\theta = 0$, (2) gives $0 = A\,(Cr^m + Dr^{-m})$ *i.e.* $A = 0$

$\therefore$ (2) becomes $u(r, \theta) = (C'\, r^m + D'\, r^{-m}) \sin m\theta$ ...(3)

where $BC = C'$ and $BD = D'$.

The condition $u = 0$ when $\theta = 0$, gives $0 = (C'r^m + D'r^{-m}) \sin \frac{m\pi}{2}$ *i.e.* $\sin \frac{m\pi}{2} = 0$

or $\frac{m\pi}{2} = (2n - 1)\pi$ giving $m = 4n - 2$.

Also the condition $u = 0$ when $r = b$ gives $0 = (C'r^m + D'r^{-m}) \sin m\theta$ *i.e.* $C'b^m + D'b^{-m} = 0$ which yields with $m = 4n - 2$, $D' = -C'b^{2m} = -C'b^{8n-4}$.

As such (3) reduces to $u = C' [r^{4n-2} - b^{8n-4} r^{-(4n-2)}] \sin (4n - 2) \theta$

Considering all possible n, the general solution becomes

$$u = \sum_{n=1}^{\infty} C'_n [r^{4n-2} - b^{8n-4} r^{-(4n-2)}] \sin (4n - 2) \theta \qquad \ldots(4)$$

Applying the condition $u = \theta\left(\frac{\pi}{2} - \theta\right)$ when $r = a$, (4) yields

$$\theta\left(\frac{\pi}{2} - \theta\right) = \sum_{n=1}^{\infty} C'_n\left[a^{4n-2} - b^{8n-4}\, a^{-(4n-2)}\right] \sin (4n - 2)\theta$$

So that $C'_n\left(a^{4n-2} - \dfrac{b^{8n-4}}{a^{4n-2}}\right) = \dfrac{4}{\pi}\displaystyle\int_0^{\pi/2} \theta\left(\frac{\pi}{2} - \theta\right) \sin (4n - 2)\theta \, d\theta$

$$= \frac{4}{\pi} \cdot \frac{4}{(4n-2)^3}.$$

Giving $C'_n = \dfrac{16}{\pi(4n-2)^3} \cdot \dfrac{a^{4n-2}}{a^{8n-4} - b^{8n-4}}.$

Hence (4) reduces to

$$u = \sum_{n=1}^{\infty} \frac{16}{\pi} \sum_{n=1}^{\infty} \frac{\left\{r^{4n-2} - b^{8n-4} r^{-(4n-2)}\right\} a^{4n-2}}{a^{8n-4} - b^{8n-4}} \; \frac{\sin (4n - 2)\, \theta}{(4n-2)^3}$$

$$= \frac{16}{\pi} \sum_{n=1}^{\infty} \frac{\left(\frac{r}{b}\right)^{4n-2} - \left(\frac{b}{r}\right)^{4n-2}}{\left(\frac{a}{b}\right)^{4n-2} - \left(\frac{b}{a}\right)^{4n-2}} \cdot \frac{\sin (4n - 2)\theta}{(4i-2)^3}$$

12.8. THREE-DIMENSIONAL LAPLACE'S EQUATION

[A] Solution of three-dimensional Laplace-equation in Cartesian coordinates.

The equation is $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$ ...(1)

Suppose $u = X(x)Y(y)\,Z(z)$ ...(2)

Then (1) yields, $\dfrac{1}{X}\dfrac{d^2X}{\partial x^2} + \dfrac{1}{Y}\dfrac{d^2Y}{dy^2} + \dfrac{1}{Z}\dfrac{d^2Z}{dz^2} = 0$ ...(3)

This relation being of form $F_1(X) + F_2(Y) + F_3(Z) = 0$ will be true only if F_1, F_2, F_3 are constant functions since x, y, z and so X, Y, Z are independent functions. We therefore take constants $-n^2, -m^2, + p^2$ such that $p^2 = m^2 + n^2$ and

$$\frac{1}{X}\frac{d^2X}{dx^2} = -n^2 \text{ i.e } \frac{d^2X}{dx^2} + n^2 X = 0 \text{ giving } X = A \cos nx + B \sin nx \qquad \ldots(4)$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -m^2 \text{ i.e } \frac{d^2Y}{dy^2} + m^2Y = 0 \text{ giving } Y = C\cos my + D\sin xy \qquad ...(5)$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = p^2 \text{ i.e } \frac{d^2Z}{dx^2} - p^2Z = 0 \text{ giving } Z = E\,e^{pz} + F\,e^{-pz} \qquad ...(6)$$

As such the combined solution of (1) is

$$u = (A\cos nx + B\sin nx)(C\cos my + D\sin my)(E\,e^{pz} + Fe^{-pz}) \qquad ...(7)$$

where $p^2 = m^2 + n^2$.

As an alternative this may be taken as

$$u = (Ae^{nx} + B\,e^{-nx})(C\,e^{my} + D\,e^{-my})[E\cos pz + F\sin pz] \qquad ...(8)$$

Note. *We can easily verify that Laplace's equation* $\nabla^2 u = 0$ *is satisfied by*

$$u = \frac{l}{\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}} \qquad ...(9)$$

where l is a constant and (a, b, c) are coordinates of a fixed point.

[B] Solution of three-dimensional Laplace-equation in cylindrical coordinates *(Rohilkhand, 1981, 88)*

We have $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial\theta^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$ ....(10)

Suppose that $u(r, \theta, z) = R(r)\,\Theta(\theta)\,Z(z)$. ...(11)

Then (10) yields $\dfrac{1}{R}\dfrac{d^2R}{dr^2} + \dfrac{r}{R}\dfrac{dR}{dr} + \dfrac{1}{r^2\Theta} = \dfrac{d^2\Theta}{d\theta^2} + \dfrac{1}{Z}\dfrac{d^2Z}{dz^2} = 0.$ ...(12)

As variable are separated, we may take

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = \lambda^2 \text{ and } \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = -\mu^2 \text{ i.e. } \frac{d^2Z}{dz^2} - \lambda^2 Z = 0$$

and $\dfrac{d^2\Theta}{d\theta^2} + \mu^2\Theta = 0$ ...(13)

These yield $Z = e^{\pm\lambda z}$ and $\Theta = e^{\pm i\mu\theta}$ ...(14)

or in other words the solutions of (13) are

$$Z = A_1 e^{\lambda z} + B_1 e^{-\lambda z},\ \Theta = A_2\cos\mu\theta + B_2\sin\mu\theta \qquad ...(15)$$

Also then (12) reduces to

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(\lambda^2 - \frac{\mu^2}{r^2}\right)R = 0 \qquad ...(16)$$

which is Bessel's equation and takes the form

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{\mu^2}{x^2}\right)R = 0, \text{ on putting } \lambda r = x \qquad ...(17)$$

Its general solutions are

$$R = A_3 J_\mu(\lambda r) + B_3 J_{-\mu}(\lambda r), \text{ for fractional } \mu \qquad ...(18)$$

and $R = A_3 J_\mu(\lambda r) + B_3 Y_\mu(\lambda r)$, for integral μ ...(19)

As such the solutions for (10), with the help of (15), (18) and (19) are

$$u(r, \theta, z) = (A_1 e^{\lambda z} + B_1 e^{-\lambda z})(A_2\cos\mu\theta + B_2\sin\mu\theta)[A_3 J_\mu(\lambda r) + B_3 J_{-\mu}(\lambda r)] \qquad ...(20)$$

and $u(r, \theta, z) = (A_1 e^{\lambda z} + B_1 e^{-\lambda z})(A_2 \cos \mu\theta + B_2 \sin \mu\theta)[A_3 J_\mu(\lambda r) + B_3 Y_\mu(\lambda r)]$ ...(21)

Note 1. *the general solution of (16) may be written as*

$$R = A_{\lambda\mu} J_\mu(\lambda r) + B_{\lambda\mu} Y_\mu(\lambda r) \quad ...(22)$$

where $A_{\lambda\mu}$ and $B_{\lambda\mu}$ are constants.

Since $Y_\mu(\lambda r) \to \infty$ as $r \to 0$, therefore in a physical problem if μ is finite along the line $r = 0$, then we must have $B_{\lambda\mu} = 0$ and hence the solution of (10) may be written as

$$u = \sum_\lambda \sum_\mu A_{\lambda\mu} J_\mu(\lambda r)\, e^{\pm\lambda z \pm i\mu\theta} \quad ...(23)$$

Note 2. *Trying the superposition, the general solution of (10) may be written as*

$$u(r, \theta, z) = \sum_{\mu=0}^{\infty} J_\mu(\lambda r)[e^{\lambda z}(A_\mu \cos \mu\theta + B_\mu \sin \mu\theta) + e^{-\lambda z}(C_\mu \cos \mu\theta + D_\mu \sin \mu\theta)] \quad ...(24)$$

Note 3. *Ina problem if there is symmetry about z-axis, then we may take $\mu = 0$ and the solution will be*

$$u(r, \theta, z) = \sum_\lambda A_\lambda J_0(\lambda r) e^{\pm\lambda z} \quad ...(25)$$

Note 4. *If in a problem of symmetry about z-axis, $u \to 0$ as $r \to 0$ and $z \to \infty$, then the solution is of the form* $u(r, \theta, z) = \sum_\lambda A_\lambda J_0(\lambda r) e^{-\lambda z}$ ...(26)

[C] Solution of three-dimensional Laplace-equation in spherical Polar coordinates *(Rohilkhand, 1981, 86, 93)*

We have $$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cos\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0 \quad ...(27)$$

or equivalently, $$r^2 \frac{\partial^2 u}{\partial r^2} + 2r\frac{\partial u}{\partial r} + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{\sin^2\theta} + \frac{\partial^2 u}{\partial \phi^2} = 0 \quad ...(28)$$

Suppose $u(r, \theta, \phi) = R(r)\,\Theta(\theta)\,\Phi(\phi)$ ...(29)

Then (27) and (28) yield on dividing by $R\Theta\Phi$

$$\frac{1}{R}\frac{d^2R}{dr^2} + \frac{2}{rR}\frac{dR}{dr} + \frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2} + \frac{\cos\theta}{r^2\Theta}\frac{d\Theta}{d\theta} + \frac{1}{r\sin^2\theta}\cdot\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0 \quad ...(30)$$

and $$\left[\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{2r}{dR}\frac{dR}{dr} + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)\right]\sin^2\theta = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \lambda^2 \text{(say)} \quad ...(31)$$

Considering (31), $$-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \lambda^2 \; i.e. \frac{d^2\Phi}{d\phi^2} + A^2\Phi = 0 \text{ gives } \Phi = C\,e^{\pm i\lambda\phi} \quad ...(32)$$

and $$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{\lambda^2}{\sin^2\theta} = n(n+1) \textit{ (say)},$$

gives $$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - n(n+1)R = 0 \quad ...(33)$$

and $\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left[n(n+1)-\frac{\lambda^2}{\sin^2\theta}\right]\Theta=0$ ...(34)

Considering (30), if we write $\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=-\lambda^2$ and $\frac{r^2}{R}\frac{d^2R}{dr^2}+\frac{2r}{R}\frac{dR}{dr}=n(n+1)$, then we have

$\Phi = C\,e^{\pm i\lambda\phi}$ which is (32); $r^2\frac{dR}{dr^2}+\frac{2rdR}{dr}-n(n+1)R=0$ which is (33) and $\frac{d^2\Theta}{d\theta^2}+\cot\theta\frac{d\Theta}{d\theta}+\left\{n(n+1)-\frac{\lambda^2}{\sin^2\theta}\right\}=0$ which is (34).

Now the equation (33) being homogeneous if we put $r=e^s$, then it reduces to $\{D(D-1)+2D-n(n+1)\}R=0$ where $D\equiv\frac{d}{ds}$

or $(D-n)(D+n+1)R=0$ giving $R=Ae^{ns}+Be^{-(n+1)s}=Ar^n+Br^{-n-1}$ ...(35)

Again if we put $\cos\theta=\mu$ in (3) then since

$$\frac{d\Theta}{d\theta}=\frac{d\Theta}{d\mu}\frac{d\mu}{d\theta}=-\sin\theta\frac{d\Theta}{d\mu}\; i.e.\;\frac{1}{\sin\theta}\frac{d}{d\theta}\equiv-\frac{d}{d\mu},$$

we have $\frac{d}{d\mu}\left\{(1-\mu^2)\frac{d\Theta}{d\mu}\right\}+\left\{n(n+1)-\frac{\lambda^2}{1-\mu^2}\right\}\Theta=0$

$$i.e.\;(1-\mu^2)\frac{d^2\Theta}{d\mu^2}-2\mu\frac{d\Theta}{d\mu}+\left\{n(n+1)-\frac{\lambda^2}{1-\mu^2}\right\}\Theta=0 \quad ...(36)$$

which is Legendre's associated equation and hence has the solution

$$\Theta=A\,P_n^\lambda(\mu)+B\,Q_n^\lambda(\mu)=A\,P_n^\lambda(\cos\theta)+B\,Q_n^\lambda(\cos\theta) \quad ...(37)$$

In other words if we take $\Theta=\Theta(\cos\theta)$ from associated Legendre equation, then the solution of (27) is of the form $(Ar^n+Br^{-n-1})\,\Theta(\cos\theta)\,e^{\pm i\lambda\phi}$

So that summing over for all n and trying superposition, the general solution of (27) may be written as

$$u(r,\theta,\phi)=\sum_{n=0}^{\infty}\left(A_nr^n+\frac{B_n}{r^{n+1}}\right)\Theta(\cos\theta)\,e^{\pm i\lambda\phi} \quad ...(38)$$

Note 1. *If $\lambda=0$, then (36) reduces to* $(1-\mu^2)\frac{d^2\Theta}{d\mu^2}-2\mu\frac{d\Theta}{d\mu}+n(n+1)\Theta=0$

which is Legendre's equation and hence we have for integral n,

$$\Theta=P_n(\mu)=P_n(\cos\theta)=\frac{1}{2^n\underline{|n}}\frac{d^n}{d\mu^n}(\mu^2-1)^n$$

and also $\Theta=Q_n(\mu)=Q_n(\cos\theta)=\frac{1}{2}P_n(\mu)\log\frac{\mu+1}{\mu-1}-\sum_{j=0}^{p}\frac{2n-4j-1}{(2j+1)(n-j)}P_{n-2j-1}(\mu)$

where $p=\frac{1}{2}(n-1)$ or $\frac{1}{2}n-1$ according as n in odd or even:

Thus $\Theta = C_n P_n(\mu) + D_n Q_n(\mu)$ so that $u = \sum_{n=0}^{p}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right)$

$$\{C_n P_n(\cos\theta) + D_n Q_n(\cos\theta)\, e^{-\lambda\phi} \quad ...(39)$$

In case $D_n = 0$ under specified boundary conditions, then

$$\Theta = C_n P_n(\cos\theta)$$

Hence the solution is $u = \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right)P_n(\cos\theta)\cdot e^{-i\lambda\phi}$ ...(40)

Note 2. *If there is axial symmetry about z-axis, then u depends only on r and θ and so (27) reduces to*

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = \theta \quad ...(41)$$

Its solution by putting $\phi = 0$ in (40), is

$$u(r,\theta) = \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right)\{C_n P_n(\cos\theta) + D_n Q_n(\cos\theta)\} \quad ...(42)$$

In case $D_n = 0$ uder specified conditions, then

$$u(r,\theta) = \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right)P_n(\cos\theta) \quad ...(43)$$

Problem 19. *If the surface S of a sphere of radius a is kept at a fixed distribution of electric potential $u = F(\theta)$, then find the potential u at all points in space which is assumed to be free of further charge.*

In this case $\frac{\partial^2 \mu}{\partial \phi^2}$ being zero, we have the constant of separation *i.e.* $\lambda = 0$ and hence equation (36) of § 12.8[*C*], reduces to

$$(1-\mu^2)\frac{d^2\Theta}{d\mu^2} - 2\mu\frac{d\Theta}{d\mu} + n(n+1)\Theta = 0 \quad ...(1)$$

Being Legendre's equation the Legendre's polynomial $P_n(\mu) = P_n(\cos\theta)$ is the solution of (1) *i.e.* we have

$$u = \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right)P_n(\cos\theta) \quad ...(2)$$

Now to determine the potential u, we consider the problem in two cases :

Case I. *Outside the sphere.* Since the potential at infinity vanishes *i.e.* $\lim_{r\to\infty} u = 0$, the boundary condition requires that any positive power of r should not be present in the solution (2), thereby giving $A_n = 0$ that (2) reduces to

$$u = \sum_{n=0}^{\infty}\frac{B_n}{r^{n+1}}P_n(\cos\theta) \quad ...(3)$$

Assuming that $u = F(\theta)$ when $r = a$, (3) gives

$$F(\theta) = f(\cos\theta)\text{ (say)} = \sum_{n=0}^{\infty}\frac{B_n}{a^{n+1}}P_n(\cos\theta) \quad ...(4)$$

If we replace $\cos\theta$ by u in (4), we get $a^{n+1}f(u) = \sum_{n=0}^{\infty} B_n P_n(u)$

so that $a^{n+1}\int_{-1}^{1} f(u)\ P_n(u)\ du = \int_{-1}^{1} \sum_n B_n P_n(u).\ P_n(u)\ du = B_n \int_{-1}^{1} P_n^2(u)\ du$

$$= \frac{2B_n}{2n+1} \text{ giving } B_n = \frac{2n+1}{2} a^{n+1} \int_0^{\pi} F(\theta)\ P_n(\cos\theta)\ \sin\theta\, d\theta \qquad \ldots(5)$$

on setting $u = \cos\theta$

Putting this value of B_n in (3), we get the required potential outside the sphere *i.e.*

$$u = \sum_{n=0}^{\infty} \frac{2n+1}{2} \cdot \left(\frac{a}{r}\right)^{n+1} P_n(\cos\theta) \int_0^{\pi} F(\theta)\ P_n(\cos\theta) \sin\theta\ d\theta \qquad \ldots(6)$$

Case II. *Inside the sphere*. Since the potential inside the sphere can not be infinite, therefore the general solution must not contain any negative power of r, thereby giving $B_n = 0$, so that (2) reduces to

$$u = \sum_{n=0}^{\infty} A_n\, r^n\, P_n\ (\cos\theta),\ r < a \qquad \ldots(7)$$

when $r = a$, $u = F(\theta)$, therefore (7) gives $F(\theta) = \sum_{n=0}^{\infty} A_n\ a^n\ P_n\ (\cos\theta)$

from which we have as in case *I*,

$$A_n = \frac{2n+1}{a^n} \int_0^{\pi} F(\theta)\ P_n(\cos\theta) \sin\ \theta d\theta \qquad \ldots(8)$$

Substituting (8) in (7) we get the required potential inside the sphere.

Problem 20. *Find a solution of the equation*

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial\phi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\phi}{\partial\theta}\right) = 0 \qquad \ldots(1)$$

in the form $\phi = f(r) \cos\theta$, *given that (i)* $-\dfrac{\partial\phi}{\partial r} = u\cos\theta$ *when* $r = a$ *and*

(ii) $-\dfrac{\partial\phi}{\partial r} = 0$ *when* $r = \infty$.

We have $\phi = f(r)\cos\theta$

$$\therefore\ \frac{\partial\phi}{\partial r} = f'(r)\cos\theta,\ \frac{\partial\phi}{\partial\theta} = -f(r)\sin\theta.$$

Their substitution in (1) gives

$$\frac{\partial}{\partial r}\{r^2 f'(r)\cos\theta\} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\{-\sin\theta\, f(r)\sin\theta\} = 0$$

i.e., $$r^2 f''(r) + 2r f'(r) - 2f(r) = 0 \qquad \ldots(2)$$

This equation being homogeneous, can be reduced to the form

$\{D(D-1) + 2D - 2\} f(r) = 0$ by putting $r = e^s$ and $D \equiv \dfrac{d}{ds}$

or $(D^2 + D - 2) f(r) = 0$ or $(D-1)(D+2) f(r) = 0$

$\therefore\ f(r) = A\, e^s + B\, e^{-2s} = Ar + \frac{B}{r^2}$ so that $\phi = Ar\cos\theta + \frac{B}{r^2}\cos\theta$ ...(3)

(3) gives $\frac{\partial\phi}{\partial r} = A\cos\theta - \frac{2B}{r^3}\cos\theta\cdot$ ...(3)

Applying the condition (*i*) $-u\cos\theta = A\cos\theta - \frac{2B}{a^3}\cos\theta\cdot$ ...(4)

and applying (*ii*) $0 = A\cos\theta$ *i.e.*, $A = 0$ and then from (4) $B = \frac{a^3 u}{2}$

Hence (3) yields $\phi = \frac{1}{2}\frac{a^3 u}{r^3}\cos\theta.$

Problem 21. *Find the permanent temperature within a solid sphere of radius unit when one half of the surface of the sphere is kept at constant temperature 0°C and the other half of the surface at 1°C.* (*Rohilkhand, 1977*)

The distribution being symmetrical about z-axis, we have

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) = 0$$

i.e.,
$$r^2\frac{\partial^2\phi}{\partial r^2} + 2r\frac{\partial\phi}{\partial r} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) = 0 \quad ...(1)$$

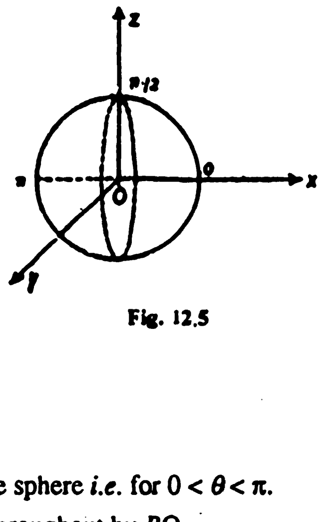


Fig. 12.5

with boundary conditions

(*i*) $\phi = 1$ for $0 < \theta < \frac{\pi}{2}$ and (*ii*) $\phi = 0$ for $\frac{\pi}{2} < \theta < \pi$

under the consideration of distribution for upper half of the sphere *i.e.* for $0 < \theta < \pi$.

Assuming that $\phi = R(r)\,\Theta(\theta)$, (1) yields on dividing throughout by $R\Theta$,

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{2r}{R}\frac{dR}{dr} = -\frac{1}{\Theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = 0 \quad ...(2)$$

or separating the variables and taking λ^2 as constant of separation,

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{2r}{R}\frac{dR}{dr} = \frac{1}{\Theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = \lambda^2 \text{ (say)}$$

so that
$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - \lambda^2 R = 0 \quad ...(3)$$

and
$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \lambda^2\Theta = 0 \quad ...(4)$$

Taking $\lambda^2 = n(n+1)$, (3) yields $r^2 \dfrac{d^2R}{dr^2} + 2r \dfrac{dR}{dr} - n(n+1) R = 0$ which being a homogeneous equation can be solved by substitution $r = e^s$, to give the solution $R = Ar^n + \dfrac{B}{r^{n+1}}$.

Also taking $\cos\theta = \mu$, (4) yields, $\dfrac{\partial}{\partial\mu}\left\{(1-\mu^2)\dfrac{d\Theta}{d\mu}\right\} + n(n+1)\,\Theta = 0$ which is Legendre's equation and hence $P_n(\mu)$ is a solution of it *i.e.*, $\Theta = P_n(\mu) = P_n(\cos\theta)$

Combining the two solutions we have for all n

$$\phi = \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right) P_n(\cos\theta) \qquad \ldots(5)$$

Now the temperature at the centre being finite it is required that $B_n = 0$

$$\therefore \text{ (5) reduces to } \phi = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\mu) \qquad \ldots(6)$$

But by orthogonal properties of Legendre's polynomials, we have

$$\int_{-1}^{1} P_n^2(\mu) = \frac{2}{2n+1}$$

Also $\qquad r = 1$ gives $\phi = \sum A_n P_n(\mu) \qquad \ldots(7)$

Multiplying (7) by $P_n(\mu)$ and integrating with regard to μ from -1 to 1, we have

$$A_n = \frac{2n+1}{2}\int_{-1}^{1}\phi\; P_n(\mu)\; d\mu = \frac{2n+1}{2}\int_0^{\pi}\phi\; P_n(\cos\theta)\sin\theta\; d\theta \text{ when } \mu = \cos\theta$$

$$= \frac{2n+1}{2}\left[\int_0^{\pi/2}\phi\, P_n(\cos\theta)\sin\theta\; d\theta + \int_{\pi/2}^{\pi}\phi\, P_n(\cos\theta)\sin\theta\; d\theta\right]$$

$$= \frac{2n+1}{2}\int_0^{\pi/2} P_n(\cos\theta)\sin\theta\; d\theta$$

$$\because \phi = 1 \text{ for } 0 < \theta < \pi/2 \text{ and } \phi = 0 \text{ for } \pi/2 < \theta < \pi.$$

$$= \frac{2n+1}{2}\int_0^1 P_n(\mu)\; d\mu \text{ giving } A_0 = \frac{1}{2}\int_0^1 1.\; d\mu = \frac{1}{2}$$

and $\qquad A_1 = \dfrac{3}{2}\displaystyle\int_0^1 P_1(\mu)\; d\mu = \dfrac{3}{2}\int_0^1 \mu\, d\mu = \dfrac{3}{4}$

$$A_2 = \frac{5}{2}\int_0^1 P_2(\mu)\; d\mu = \frac{5}{2}\int_0^1 \frac{3\mu^2 - 1}{2}\; d\mu = \frac{5}{4}\left[\mu^2 - \mu\right]_0^1 = 0$$

$$A_3 = \frac{7}{2}\int_0^1 P_3(\mu)\; d\mu = \frac{7}{4}\int_0^1 (5\mu^2 - \mu)\; d\mu = \frac{7}{4}\left[\frac{5}{4}\mu^2 - \frac{3}{2}\mu^2\right]_0^1 = -\frac{7}{16} \text{ etc.}$$

Hence $\phi = \dfrac{1}{2} + \dfrac{3}{4}\cdot r\, P_1(\cos\theta) - \dfrac{7}{16}\, r^3\, P_3(\cos\theta) + \ldots$

12.9. GENERAL PROPERTIES OF HARMONIC FUNCTIONS

(Agra, 1971)

We know that functions satisfying Laplace's differential equation are said to be the *Harmonic functions*. Now to discuss general properties of such functions, let us consider a vector point function A and a scalar point function u satisfying Laplace's equation *i.e.*, $\nabla^2 u = 0$ ...(1)

such that $$\mathbf{A} = \nabla u \qquad ...(2)$$

$\therefore \quad \nabla \cdot \mathbf{A} = \nabla \cdot (\nabla u) = \nabla^2 u = 0$ ...(3)

But Gauss' divergence theorem gives $\iint_S \mathbf{A} \cdot d\mathbf{s} = \iiint_V (\nabla \cdot \mathbf{A})\, du$ ...(4)

which with the help of (3) yields $\iint_S \mathbf{A} \cdot d\mathbf{s} = \iint_S (\nabla u) \cdot d\mathbf{s} = 0$ ...(5)

If we take curl of both sides of (2), we get $\nabla \times \mathbf{A} = \nabla \times \nabla u = 0$ ...(6)

But Stoke's theorem for a vector field A is

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = 0 \qquad ...(7)$$

where integral being taken over the closed curve C bounding the open surface S.

(7) with the help of (4) reduces to $\oint_C (\nabla u) \cdot d\mathbf{r} = 0$ ...(8)

From (5) and (8) certain important properties of harmonic functions can be deduced. Applying Green's theorem *i.e.*

$$\iiint_V \left(v \nabla^2 w - w \nabla^2 v\right) du = \iint_S (v \nabla w - w \nabla v) \cdot d\mathbf{s} \qquad ...(9)$$

we can easily exhibit that if $\nabla^2 u = 0$ in a region bounded by a sphere of radius r then the value of u say u_0 at the centre of the sphere is given by

$$u_0 = \frac{1}{4\pi r^2} \iint_S u\, ds \qquad ...(10)$$

where integral is taken over the surface of the sphere.

These results may be categorically stated as:

(*i*) From (10), the average value of a harmonic function on the surface of a sphere in which it has no singularity *i.e.*, the points where the function becomes infinite, is equal to its value at the centre of the sphere.

(*ii*) From (5), it follows that a harmonic function having no singularity in a given region cannot have a maximum or minimum value in the region.

(*iii*) From (*ii*) we conclude that a harmonic function without singularity within a region and being constant everywhere on the bounding surface of the region, has the same constant value everywhere inside the region.

(*iv*) Two harmonic functions having identical values on a closed contour and having no singularity within the contour, are identical throughout the region bounded by the contour.

(*v*) From Green's theorem it follows that if the normal derivative of a harmonic function is zero on a closed surface within which there is no singularity, the function is constant.

(*vi*) It follows from (*v*) that if two harmonic functions have the same normal derivative on a closed surface within which there are no singularities, they differ at most by an additive constant.

12.10. THE WAVE EQUATIONS

[A] Derivation of one-dimensional wave equation

Consider a flexible string of length l tightly stretched between two points $x = 0$ and $x = l$ on x-aixs, with its ends at these ends. If the string is set into small transerve vibration, the displacement say $u(x, t)$ from the x-axis of any point x of the string at any time t is given by $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ where $c^2 = \frac{T}{\rho}$, T being tension and ρ the linear density,

The equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ ...(1)

is known as one-dimensional wave equation.

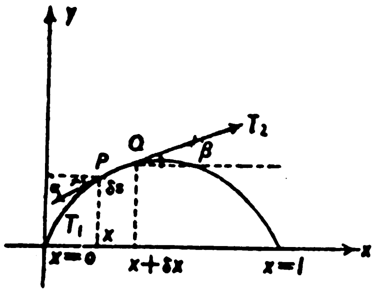


Fig. 12.6

Let the string (assumed to be perfectly flexible) of length l tightly stretched between the points $x = 0$ and $x = l$ on x-axis be distorted and then at a certain instant of time say $t = 0$, it is released and allowed to vibrate. To determine its deflection (displacement from x-axis) at any point x at any time t, let us take the following assumptions:

(*i*) The string is uniform *i.e.* its mass m per unit length is constant.

(*ii*) The string is perfectly elastic and so offers no resistance to any bending.

(*iii*) The tension T is so large that the action of gravitational force on the string is negligible.

(*iv*) The motion of the string is a small transverse vibration in a vertical plane *i.e.* each particle of the string moves strictly in the vertical plane so that the deflection and slope (gradient) at any point of the string are very small in absolute value.

Consider the motion of an element PQ of length δs of the string. The string being perfectly elastic the tensions T_1 at P and T_2 at Q are tangential to the curve of the string. Let T_1 and T_2 make angle α and β respectively with the horizontal.

There being no motion in the horizontal direction, we have

$$T_1 \cos\alpha = T_2 \cos\beta = T \text{ (say)} = \text{constant} \quad \text{...(2)}$$

Mass of the element PQ is $\rho\delta s$. By Newton's second law of motion we therefore have

$$T_2 \sin\beta - T_1 \sin\alpha = (\rho\delta s) . \frac{\partial^2 u}{\partial t^2}, \quad \text{...(3)}$$

$\frac{\partial^2 u}{\partial t^2}$ being upward acceleration of PQ

Using (1), (2) yields $\frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \frac{\rho\delta s}{T} . \frac{\partial^2 u}{\partial t^2}$

i.e.
$$\tan\beta - \tan\alpha = \frac{\rho\delta s}{T}\frac{\partial^2 u}{\partial t^2} \quad \text{...(4)}$$

Replacing δs by δx since the gradient of the curve is very small, (4) gives

$$\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x = \frac{\rho\delta s}{T}\frac{\partial^2 u}{\partial t^2} \quad \text{...(5)}$$

since $\tan\alpha$ and $\tan\beta$ are slopes at x and $x + \delta x$ respectively.

or
$$\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}-\left(\frac{\partial u}{\partial x}\right)_x}{\delta x}=\frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

i.e.
$$\frac{u_x(x+\delta x,\ t)-u_x(x,\ t)}{\delta x}=\frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}.$$

Proceeding to the limit as $\delta x \to 0$, we get

$$\frac{\partial^2 u}{\partial x^2}=\frac{\rho}{T}\frac{\partial^2 u}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \text{ where } \frac{1}{c^2}=\frac{\rho}{T}.$$

Note 1. $c^2=\frac{T}{\rho}$ *reveals that the constant* $\frac{T}{\rho}$ *is positive.*

Note 2. *Since u is dependent of x and t both, therefore we have used the partial derivative* $\frac{\partial^2 u}{\partial x^2}$ *and* $\frac{\partial^2 u}{\partial t^2}$.

Note 3. *If a force F(x, t) per unit of mass acts in the u-direction along the string, in addition to the tension of the string, then*

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}+F.$$

[B] Derivation of two-dimensional wave equation

In case of a rectangular membrane, the two-dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2}=c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \qquad ...(6)$$

Consider the motion of a stretched membrane supposed to be stretched and fixed along its entire boundary in the x-y plane. Let us take the following assumptions:

(*i*) The membrane is homogeneous *i.e.* mass (say) ρ per unit area is constant.

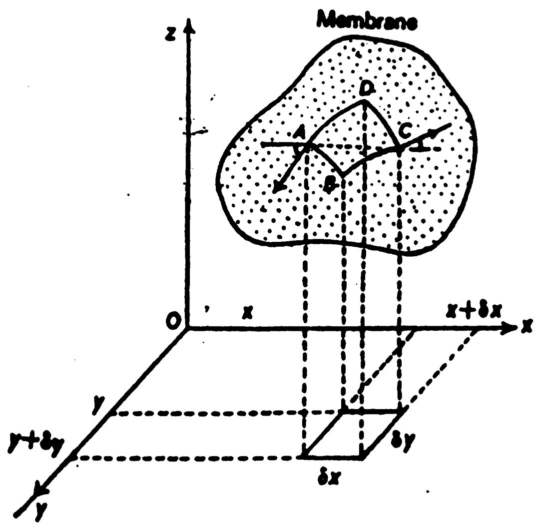


Fig. 12.7

(*ii*) The membrane is perfectly flexible and so thin that it offers no resistance to any bending.

(*iii*) The tension T per unit length caused by the stretching of the membrane is invariant during the motion and retains the same value at each of its points and in all the directions.

(*vi*) The deflection $u(x, y, t)$ of the membrane during the motion is negligible as compared to the size of the membrane. Also all the angles of inclination are small.

Consider the motion of an element $ABCD$ of the membrane. Let its area be $\delta x\, \delta y$. T being the tension per unit length, the force acting on the edges are $T\delta x$ and $T\delta y$ approximately. Also the membrane being perfectly flexible, the tensions $T\delta x$ and $T\delta y$ are tangential to the membrane. Let α, β be the inclinations of these tensions with the horizontal. Then the horizontal components of the forces at one pair of opposite edges are $T\delta y \cos \alpha$ and $T\delta y \cos \beta$. When α and β are very small $\cos \alpha \to 1$ and $\cos \beta \to 1$ so that $T\delta y \cos \alpha \to T\delta y$ and $T\delta y \cos \beta \to T\delta y$ *i.e.* the horizontal components of the forces at opposite edges are nearly equal and hence the motion of the particles of the membrane in horizontal direction is negligibly small. As such we assume that every particle of the membrane moves vertically.

The resultant vertical force $= T\delta y \sin \beta - T\delta y \sin \alpha.$

$$= T\delta y\,(\tan \beta - \tan \alpha)$$

$\because$ α, β being small $\sin \alpha = \alpha = \tan \alpha$ and $\sin \beta = \beta = \tan \beta$.

$$= T\delta y[u_x(x + \delta x, y_1) - u_x(x, y_2)] \qquad \ldots(7)$$

where u_x denotes the partial derivative of u w.r.t. x and y_1, y_2 are the values of y between y and $y + \delta y$.

Similarly, the resultant vertical force acting on the other two edges

$$= T\delta x\,[u_y(x_1, y + \delta y) - u_y(x_2, y)] \qquad \ldots(8)$$

where u_y denotes the partial derivative of u w.r.t. y and x_1, x_2 are the values of x between x and $x + \delta x$.

By Newton's second law of motion, we have

$$\text{Total vertical force on the element} = \rho\,\delta x\,\delta y\,\frac{\partial^2 u}{\partial t^2}$$

i.e. $T\delta y\,[u_x(x + \delta x, y_1) - u_x(x, y_2)] + T\delta x\,[u_y(x_1, y + \delta y) - u_y(x_2, y)]$

$$= \rho\,\delta x\,\delta y\,\frac{\partial^2 u}{\partial t^2}$$

where $\dfrac{\partial^2 u}{\partial t^2}$ is the acceleration of the element.

$$\text{Thus } \frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\left[\frac{u_x(x + \delta x, y_1) - u_x(x, y_2)}{\delta x}\right] + \frac{T}{\rho}\left[\frac{u_y(x_1, y + \delta y) - u(x_2, y)}{\delta y}\right]$$

Proceeding to the limit as $\delta x \to 0$ and $\delta y \to 0$, we have

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\left[u_{xx} + u_{yy}\right] = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = c^2\,\nabla^2 u \text{ where } \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \qquad \ldots(9)$$

Note 1. *If* $u = v(x, y)\, e^{iwt}$, (9) yields $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} + k^2 v = 0$ *where* $k^2 = \left(\dfrac{w}{c}\right)^2$ ...(10)

Note 2. *The three-dimensional wave equation is*

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = c^2 \nabla^2 u \qquad ...(11)$$

where $\nabla^2 \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial}{\partial z^2}$.

12.11. GREEN'S FUNCTIONS FOR THE WAVE EQUATION

The wave equation is

$$\nabla^2 \psi \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \psi = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} \qquad ...(1)$$

Also written as,

$$\Box^2 \psi \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^3}\right) \psi = 0 \qquad ...(2)$$

If its solution be of the form

$$\psi(x, y, z, t) = \Psi(x, y, z)\, e^{\pm ic\lambda t} \qquad ...(3)$$

Then (1) gives, $\nabla^2 \Psi + \lambda^2 \Psi = 0$ ...(4)

which known as *Space form of the wave equation* or *Helmholtz's equation.*

Taking $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ as the position vector of a point (x, y, z) and $\mathbf{r}' = x'\,\mathbf{i} + y'\,\mathbf{j} + z'\,\mathbf{k}$ as the position vector of an isolated point (x', y', z'), the *Green's function* $G(\mathbf{r}, \mathbf{r}')$ is defined as

$$G(\mathbf{r}, \mathbf{r}') = H(\mathbf{r}, \mathbf{r}') + \frac{1}{|\mathbf{r}' - \mathbf{r}|} \qquad ...(5)$$

where $H(\mathbf{r}, \mathbf{r}')$ satisfies $\left(\dfrac{\partial^2}{\partial x'^2} + \dfrac{\partial^2}{\partial y'^2} + \dfrac{\partial^2}{\partial z'^2}\right) H(\mathbf{r}, \mathbf{r}') = 0$ ...(6)

Using *Green's formula i.e.* $\psi(\mathbf{r}) = \dfrac{1}{4\pi}\displaystyle\int_S \left\{\frac{1}{|\mathbf{r}' - \mathbf{r}|}\frac{\partial \psi(\mathbf{r}')}{\partial n} - \psi(\mathbf{r}')\frac{\partial}{\partial n}\frac{1}{|\mathbf{r}' - \mathbf{r}|}\right\} dS'$ ...(7)

it may be shown that

$$\psi(\mathbf{r}) = \frac{1}{4\pi}\int_S \left\{G(\mathbf{r}, \mathbf{r}')\frac{\partial \psi(\mathbf{r}')}{\partial n} - \psi(\mathbf{r}')\frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n}\right\} dS' \qquad ...(8)$$

where $\mathbf{n}$ is the unit outward drawn normal to the surface S.

Now we claim that the solution of space form of the wave equation under certain boundary conditions can be made to depend on the determination of the appropriate Green's function. Let us assume that $G(\mathbf{r}, \mathbf{r}')$ satisfies the equation

$$\left(\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2}\right)G(\mathbf{r},\mathbf{r}')+\lambda^2 G(\mathbf{r},\mathbf{r}')=0 \quad ...(9)$$

under the assumption that $G(\mathbf{r},\mathbf{r}')$ is finite and continuous w.r.t. either their variables x, y, z or x', y', z' for the points $\mathbf{r}$, $\mathbf{r}'$ belonging to a region V bounded by a closed surface S except in $\mathbf{r}$ – neighbourhood where there is a singularity of the type

$$\frac{e^{-i\lambda|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \text{ as } |\mathbf{r}'|\to|\mathbf{r}| \quad ...(10)$$

Now $\Psi(\mathbf{r})$ being the solution of (4) and its partial derivatives of the first and second orders being continuous within the volume V on the closed surface S we have

$$\frac{1}{4\pi}\int_S \left\{\frac{e^{i\lambda|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial\Psi(\mathbf{r}')}{\partial n}-\Psi(\mathbf{r}')\frac{\partial}{\partial n}e^{i\lambda|\mathbf{r}-\mathbf{r}'|}.\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right\}dS'$$

$$=\begin{cases}\Psi(\mathbf{r}), & \text{if } \mathbf{r} \text{ lies inside } V\\ 0, & \text{if } \mathbf{r}' \text{ does not lie inside } V\end{cases} \quad ...(11)$$

Using (10), we therefore have

$$\Psi(\mathbf{r})=\frac{1}{4\pi}\int_S\left\{G(\mathbf{r},\mathbf{r}')\frac{\partial\Psi(\mathbf{r}')}{\partial n}-\Psi(\mathbf{r}')\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n}\right\}dS' \quad ...(12)$$

Taking $G(\mathbf{r},\mathbf{r}')$ such that it satisfies the boundary condition

$$G_1(\mathbf{r},\mathbf{r}')=0 \quad ...(13)$$

whereas the point $\mathbf{r}'$ lies on the surface S, then (12) reduces to

$$\Psi(\mathbf{r})=\frac{1}{4\pi}\int_S\Psi(\mathbf{r}')\frac{\partial G_1(\mathbf{r},\mathbf{r}')}{\partial n}dS' \quad ...(14)$$

which gives Ψ at any pont $\mathbf{r}$ within S.

Again if $G_2(\mathbf{r},\mathbf{r}')$ is such a functin satisfying $\frac{G_2(\mathbf{r},\mathbf{r}')}{\partial n}=0$

for $\mathbf{r}'$ lying inside S we have ...(15)

$$\Psi(\mathbf{r})=\frac{1}{4\pi}\int_S\frac{\partial\psi(\mathbf{r}')}{\partial n}G_2(\mathbf{r},\mathbf{r}')\,dS' \quad ...(16)$$

which gives Ψ at any point within S provided $\frac{\partial\Psi}{\partial n}$ is known at every point of S

COROLLARY. *Green's function for Diffusion equation:*

The diffusion equation is $\frac{\partial u}{\partial t}=h^2\nabla^2 u$ ...(17)

Let $u(\mathbf{r},t)$ be a solution of it: Then for a volume V enclosed by a surface S, the boundary condition is $u(\mathbf{r}\,t)=\phi(\mathbf{r},t)$ ...(18)
when $\mathbf{r}$ lies inside S.

The Initial condition is $u(\mathbf{r},o)=f(\mathbf{r})$ when $\mathbf{r}$ lies inside V ...(19)

If we define Green's function $G(\mathbf{r},\mathbf{r}',t-t')$, $t>t'$

such that $$\frac{\partial G}{\partial t}=h^2\nabla^2 G \quad ...(20)$$

With boundary condition $G(\mathbf{r},\mathbf{r}',t-t')=0$ when $\mathbf{r}'$ lies inside S ...(21)
and initial condition $\lim_{t\to t'} G\to 0$ ...(22)

at all the points of V except at the point $\mathbf{r}$ where G takes the form

$$\frac{e^{-\frac{|\mathbf{r}-\mathbf{r}'|^2}{4h^2(t-t')}}}{8\left[\pi h^2(t-t')\right]^{3/2}} \qquad \ldots(23)$$

Now G being a function of t and hence of $(t-t')$ only, (20) is equivalent to

$$\frac{\partial G}{\partial t'} + h^2 \nabla^2 G = 0 \qquad \ldots(24)$$

Physically interpreted $G(\mathbf{r}, \mathbf{r}', t-t')$ is the temperature at any point $\mathbf{r}'$ at time t due to an instantaneous point source of unit strength generated at time t' of the point $\mathbf{r}$. Initially, the temperature of the solid is zero and the surface is kept at zero temperature.

Equations (17) and (18) being valid for $t' < t$, can be rewritten as

$$\frac{\partial u}{\partial t'} = h^2 \nabla^2 u, t' < t \qquad \ldots(25)$$

and $\qquad u(\mathbf{r}', t) = \phi(\mathbf{r}', t)$ when $\mathbf{r}'$ lies inside S $\qquad \ldots(26)$

Equations (24) and (25) yield,

$$\frac{\partial}{\partial t'}(uG) = u\frac{\partial G}{\partial t'} + G\frac{\partial u}{\partial t'} = h^2\left[G\nabla^2 u - u\nabla^2 G\right]$$

so that for an arbitrary small $\epsilon > 0$, we find

$$\int_0^{t-\epsilon}\left\{\int_V \frac{\partial}{\partial t'}(uG)\, dv'\right\}dt' = h^2 \int_0^{t-\epsilon}\left\{\int_V \left[G\nabla^2 u - u\nabla^2 G\right]dv'\right\}dt' \qquad \ldots(27)$$

or, changing the order of integration,

$$\int_V (uG)_{t'=t-\epsilon}\, dv' - \int_V (uG)_{t'=0}\, dv'$$

$$= u(\mathbf{r}, t)\int_V \left[G(\mathbf{r}, \mathbf{r}'), t-t'\right]_{t'=t-\epsilon} dv' - \int_V G(\mathbf{r}, \mathbf{r}', t)\, f(\mathbf{r}')\, dv'$$

By (23), for $G(\mathbf{r}, \mathbf{r}', t-t')$ we have $\int_V \left[G(\mathbf{r}, \mathbf{r}', t-t'\right]_{t'=t-0} dv' = 1$

So that when $\epsilon \to 0$, L.H.S. of (27)

$$= u(\mathbf{r}, t) - \int_V f(\mathbf{r}')\, G(\mathbf{r}, \mathbf{r}', t)\, dv'$$

Hence applying Green's theorem to the R.H.S. of (27) and using (18) and (21) we may find

$-h^2\int_0^t dt' \int_S \phi(\mathbf{r}', t)\frac{\partial G}{\partial n}\, dS'$ in limit when $\epsilon \to 0$ and $\frac{\partial G}{\partial n}$ denoting the derivative of G along outward drawn normal to the surface S.

We shall ultimately find,

$$u(\mathbf{r}, t) = \int_V f(\mathbf{r}')G(\mathbf{r}, \mathbf{r}', t)\, dv' - h^2\int_0^t dt' \int_S \phi(\mathbf{r}', t)\, \frac{\partial G}{\partial n}\, dS' \qquad \ldots(28)$$

which gives the solution of (17) with boundary conditions (18) and (19).

12.12. HOMOGENEOUS AND INHOMOGENEOUS WAVE EQUATIONS

In the next Chapter we shall discuss Maxwell's electromagnetic field equations in the form

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{E}}{\partial t} \quad \text{...(1)}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \text{...(2)}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \text{...(3)}$$

and

$$\nabla \cdot \mathbf{D} = \rho \quad \text{...(4)}$$

In addition to these equations, we have few more relations in a homogeneous isotropic medium.

$$\mathbf{D} = k\mathbf{E} \quad \text{...(5)}$$

$$\mathbf{B} = \mu \mathbf{H} \quad \text{...(6)}$$

and

$$\mathbf{J} = \sigma \mathbf{E} \quad \text{...(7)}$$

The method of integration to be used here for electrodynamical equations actually leads us to homogeneous wave equation as shown below. For the purpose of their integration, introduce a vector a known as *magnetic vector potential* such that

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \text{...(8)}$$

(1) and (8) yield $\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A}) = -\nabla \times \frac{\partial \mathbf{A}}{\partial t}$ ...(9)

(on changing the order of time and space derivatives).

We can write (9) as $\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0$ ...(10)

which follows that $\left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right)$ is an irrotational vector and hence it is expressible as the gradient of a scalar point function such that

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \phi, \ \phi \text{ being a scalar potential} \quad \text{...(11)}$$

or

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi \quad \text{...(12)}$$

Multiply (2) by μ and using (6), we have

$$\nabla \times \mathbf{B} = \mu \mathbf{J} + \mu \frac{\partial \mathbf{D}}{\partial t} \quad \text{...(13)}$$

But $\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ ...(14)

$\therefore$ (13) gives $\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu \mathbf{J} + \mu \frac{\partial \mathbf{D}}{\partial t}$ ...(15)

Differentiation of (12) w.r.t. 't' yields

$$\frac{\partial \mathbf{E}}{\partial t} = -\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla \frac{\partial \phi}{\partial t} \quad \text{...(16)}$$

Elimination of $\frac{\partial \mathbf{E}}{\partial t}$ from (15) and (16) with the help of (5) gives

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu \mathbf{J} + \mu\kappa\left(-\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla\frac{\partial \phi}{\partial t}\right) \quad ...(17)$$

or

$$-\nabla^2 \mathbf{A} = \mu \mathbf{J} - \mu\kappa\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla\left(\nabla \cdot \mathbf{A}\mu\kappa\frac{\partial \phi}{\partial t}\right) \quad ...(18)$$

It follows from (18) that curl of **A** is specified by its divergence but div **A** is not specified. But to find **A** uniquely, curl **A** and div **A** both should be specified and hence let us assume that

$$\nabla \cdot \mathbf{A} = -\mu\kappa\frac{\partial \phi}{\partial t} \quad ...(19)$$

So that (18) yields $\nabla^2 \mathbf{A} - \mu\kappa\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}$ ...(20)

Also (4) with the help of (5) gives $\nabla \cdot \mathbf{E} = \frac{\rho}{\kappa}$ ...(21)

which with the help of (12) becomes $\nabla \cdot \left(-\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi\right) = \frac{\rho}{\kappa}$ ...(22)

or

$$-\frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) - \nabla^2 \phi = \frac{\rho}{\kappa} \quad ...(23)$$

Elimination of ∇ **A** from (19) and (23), yields

$$\nabla^2 \phi - \mu\kappa\frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\kappa} \quad ...(24)$$

If we put $c = \frac{1}{\sqrt{\mu\kappa}}$, (20) and (24) reduce to

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J} \quad ...(25)$$

and $\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\kappa}$ ...(26)

which have got the same form and known as *Inhomogeneous wave equations* or *Lorentz's equations* and they lead to the conclusion that magnetic vector potential **A** and scalar potential ϕ are propagated in accordance with a equation of the form

$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = -f(x, y, z, t)$ which is claimed to solve ...(27)

with intial conditions $u = 0$ and $\frac{\partial u}{\partial t} = 0$ at $t = 0$ ...(28)

In order to use the method of Laplace transform, assume that

$L\{u(x, y, z, t)\} = U(x, y, z, s)$ and $L\{f(x, y, z, t)\} = F(x, y, z, s)$ ...(29)

Taking Laplace transform of (27), we get $\nabla^2 U - \frac{s^2}{c^2} = -F$ ...(30)

It we put $\kappa^2 = -\frac{s^2}{c^2}\phi$, $\kappa = \frac{s}{c}i$, $i = \sqrt{-1}$, then it becomes

$$\nabla^2 U + \kappa^2 U + F = 0 \qquad ...(31)$$

which is *Helmholtz's equation.*

In particular case (31) can abe taken as $\nabla^2 U_0 + \kappa^2 U_0 = 0$ ...(32)

which is the standard form of Helmholtz's equation and its particular solution is

$$U_0 = \frac{e^{\pm i\kappa r}}{r} \qquad ...(33)$$

where r is the distance from a point and U_0 is determined at another point.

Using this particular solution, w can find the general solution of (31) as

$$U(x, y, z, s) = \frac{1}{4\pi}\iiint F(x_1, y_1, z_1) e^{\pm i\kappa r} \cdot \frac{1}{r} dv \qquad ...(34)$$

where $r = \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}$ and $dv = dx\,dy\,dz$ ...(35)

It may be verified that (34) satisfies (31).

Now substituting $\kappa = \frac{s}{c} i$ (35) becomes

$$U(x, y, z, s) = \frac{1}{4\pi}\iiint \frac{F(x_1, y_1, z_1)}{r} e^{\pm s/cr} dv \qquad ...(36)$$

Taking inverse Laplace transform of (36) we find the solution of inhomogeneous wave equation (27) as

$$u(x, y, z, s) = \frac{1}{4\pi}\iiint \frac{f\left(x, y, z, t - \frac{r}{c}\right)}{r} dv \qquad ...(37)$$

[Since we define the Laplace transform of $F(t)$ as $f(s) = L\{F(t)\}$

$= \int_0^\infty e^{-st} F(t)\, dt$. Under the condition that definite integral of $F(t)$ exists and $F(t) = 0$ for $t < 0$. Also we define the inverse transform $L^{-1}\{f(s)\} = F(t)$ and $L\left\{\frac{d^2F}{dt^2}\right\} = s^2 f - sF'(0) - s^2F(0)$ where

$F'(0)$ is $\frac{dF}{dt}$ evaluated at $t = 0$ and

$$L^{-1}\{e^{as} f(s)\} = \begin{bmatrix} o, & t < a \\ F(t-a), & t > a. \end{bmatrix} \text{ Also } L\left\{\frac{\partial^2 u}{\partial t^2}\right\} = s^2 U\Bigg]$$

The equation (37) shows that the effects in variation of $F(x_1, y_1, z_1, t)$ do not approach the point (x, y, z) unless the time t is retarded by r/c.

As such we can write the solutions of (25) and (26) as

$$A = \frac{\mu}{4\pi}\iiint \frac{\left(J(x_1, y_1, z_1, t) - \frac{r}{c}\right)}{r} dv \qquad ...(38)$$

and $$\phi = \frac{1}{4\pi\kappa}\iiint \frac{\rho\left(x_1,\ y_1,\ z_1,\ t-\frac{r}{c}\right)}{r}dv \qquad ...(39)$$

These give *retarded potentials* of electro-dynamics.

12.13. THEORY OF WAVE GUIDES

Here we have to discuss the propagation of electromagnetic waves travelling in the longitudinal direction in a homogeneous isotropic medium filling the interior of a metal tube of infinite length, under the assumptions

(*i*) The tube has a uniform corss-section.

(*ii*) The tube is placed straight along x-axis.

(*iii*) The conductivity of the tube is infinite.

(*iv*) The medium is devoid of free charges.

(*v*) x-y plane is the plane of cross-section of the tube.

(*vi*) x-axis is along the wave guide.

Taking E_0, H_0, σ, κ and μ as electric intensity, magnetic intensity, conductivity, electric inductive capacity and magnetic inductive capacity respectively, we can write the fundamental Maxwell's equations in the forms

$$\nabla \times \mathbf{E}_0 = -\mu\frac{\partial \mathbf{H}_0}{\partial t}, \qquad ...(1)$$

$$\nabla \times \mathbf{H}_0 = \sigma\mathbf{E}_0 + \kappa\frac{\partial \mathbf{E}_0}{\partial t}. \qquad ...(2)$$

$$\nabla \cdot \mathbf{E}_0 = 0 \qquad ...(3)$$

and $$\nabla \cdot \mathbf{H}_0 = 0 \qquad ...(4)$$

In order to discuss the possible oscillations propagating inside the wave guide, we can take $\mathbf{E}_0$ and $\mathbf{H}_0$ of the form $e^{i\omega t}$ such that

$$\mathbf{E}_0 = \mathbf{E}\ e^{i(\omega t - ax)} \qquad ...(5)$$

and $$\mathbf{H}_0 = \mathbf{H}\ e^{i(\omega t - ax)} \qquad ...(6)$$

The frequency of oscillation being given by

$= \frac{\omega}{2\pi} \cdot a$ is know as the propagation constant.

Now we have $$\mathbf{E} = \mathbf{i}\ E_x + \mathbf{j}\ E_y = \mathbf{k}\ E_z \qquad ...(7)$$

and $$\mathbf{H} = \mathbf{i}\ H_x + \mathbf{j}\ H_y + \mathbf{k}\ H_z \qquad ...(8)$$

If we substitute for $\mathbf{E}_0$ and $\mathbf{H}_0$ from (5) and (6) into (1) and (2) we get the Cartesian components as

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -i\omega\mu H_x;\ \frac{\partial E_x}{\partial z} + aE_x = -i\omega\mu H_y;$$

$$aE_y + \frac{\partial E_x}{\partial y} = i\omega\mu H_z \qquad ...(9)$$

and $$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = (\sigma + i\omega\kappa)E_x;\ \frac{\partial H_x}{\partial z} + aH_z = (\sigma + i\omega\kappa)E_y;$$

$$-aH_y - \frac{\partial H_x}{\partial y} = (\sigma + i\omega\kappa)E_z \qquad ...(10)$$

It is observed that there are two types of waves namely (*i*) *TE* (Transverse electric) or *H* waves and (*ii*) *TM* (transverse magnetic) of *E* waves, which exist independently and satisfy equations (9) and (10).

Case I. *TE or H waves are* characterized by

$$E_x = 0 \text{ and } H_x \neq 0 \qquad \text{...(11)}$$

which follows that in the direction of propagation, the *electric field* has no component while the *magnetic field* has a component.

If we put $E_x = 0$, (9) and (10) yield

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = i\omega\mu H_x;\ aE_z = -i\omega\mu H_y;\ aE_y = i\omega\mu H_z \qquad \text{...(12)}$$

and
$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = 0;\ \frac{\partial H_x}{\partial z} + aH_z = (\sigma + i\omega\kappa)E_y;$$

$$-aH_y - \frac{\partial H_x}{\partial y} = (\sigma + i\omega\kappa)E_z \qquad \text{...(13)}$$

Elimination of H_y, H_z, E_y and E_z yields

$$\frac{\partial^2 H_x}{\partial z^2} + \frac{\partial^2 H_x}{\partial y^2} = -\left[a^2 - (\sigma + i\omega\kappa)i\omega\mu\right]H_x \qquad \text{...(14)}$$

But in an electric region inside the wave guide $\sigma << \omega\kappa$ so that (14) reduces to

$$\frac{\partial^2 H_x}{\partial z^2} + \frac{\partial^2 H_x}{\partial y^2} = -\left(a^2 + \omega^2\mu\kappa\right)H_x = -K^2 H_x \qquad \text{...(15)}$$

where
$$K^2 = a^2 + \omega^2\mu\kappa$$

Hence magnetic intensity H_x can be determined under given boundary conditions.

Now from (12) and (13) we can derive

$$H_y = -\frac{a}{\kappa^2}\frac{\partial H_x}{\partial y};\ H_z = -\frac{a}{\kappa^2}\frac{\partial H_x}{\partial z};\ E_y = \frac{i\omega\mu}{a}H_z;$$

$$E_z = -\frac{i\omega\mu}{a}H_y \qquad \text{...(16)}$$

Thus **E** and **H** can be determined if H_x is known.

In case the surface of the metallic wave guide is a perfect conductor, then the tangential component of **E** vanishes and for a *rectangular wave guide* with its sides parallel *y* and *z* axes, $E_z = 0 = E_y$ at the surface of the wave guide. As such it follows from (16) that

$$\frac{\partial H_x}{\partial y} = 0 = \frac{\partial H_x}{\partial z}.$$

If *n* be the normal to the surface then at the surface of a wave guide of any cross-section, we have

$$\frac{\partial H_x}{\partial n} = 0.$$

Taking general coordinate system, we can write (15) as

$$\nabla^2_{y,z} H_x + \kappa^2 H_x = 0 \qquad \text{...(17)}$$

where $\nabla^2_{y,z} \equiv \frac{\partial^2}{\partial y^2}\frac{\partial^2}{\partial z^2}$ subject to $E_z = -\frac{i\omega\mu}{a}H_y$.

Its solution therefore gives the possible value of K and hence the value of 'a' the constant of propagation such as

$$a = \sqrt{K^2 - \omega^2\mu\kappa} \quad ...(18)$$

These are imaginary value of 'a' which lead to possible wave propagation along wave guide while for real 'a', the wave is rapidly attenuated as it proceeds along x-axis of the wave guide.

Case II. *TM or E waves* are characterised by $H_x = 0$ and $E_x \neq 0$ ...(19)

Hence (9) and (10) for $H_x = 0$ yield

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = 0;\ \frac{\partial E_x}{\partial z} + aE_z = -i\omega\mu H_y;$$

$$aE_y + \frac{\partial E_x}{\partial y} = i\omega\mu H_z \quad ...(20)$$

and $$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = (\sigma + i\omega\kappa)E_x;\ aH_z = (\sigma + i\omega\kappa)E_y;\ -aH_y = (\sigma + i\omega\kappa)E_z \quad ...(21)$$

Elimination of E_z, E_y, H_y and H_z yields

$$\frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = -\left(a^2 + \omega^2\mu\kappa\right)E_x = -\kappa^2 E_x \quad ...(22)$$

For general coordinate system, this can be written as

$$\nabla^2_{y,z}E_x + \kappa^2 E_x = 0$$

where $$\nabla^2_{y,z} = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad ...(23)$$

In case of the surface of a perfectly conducting wave guide E has no tangential component at the surface, thereby giving $E_x = 0$ at the surface of the wave guide.

The determination of the possible values of K leads to the possible value of 'a' the propagation constant.

From (19) and (20) E_z, E_y, H_y and H_z can be determined.

12.14. SOLUTION OF ONE-DIMENSIONAL WAVE EQUATION

The equation is $$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \quad \textit{(Rohilkhand, 1980)} ...(1)$$

Its solution by *D' Alemberts'* method has already been given in §7.5. Here we solve it by the method of separation of variables.

Assume $u(x, t) = X(x)\,T(t)$ ...(2)

where X is a function of x along and T that of t alone.

$$\therefore\quad \frac{\partial^2 u}{\partial x^2} = T\frac{\partial^2 X}{\partial x^2} \text{ and } \frac{\partial^2 u}{\partial t^2} = X\frac{\partial^2 T}{\partial t^2}$$

which when substituted in (1) give

$$X\frac{d^2T}{dt^2} = c^2 T\frac{d^2X}{dx^2} \quad i.e.,\quad \frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2T}\frac{d^2T}{dt^2}$$

on dividing throughout by XTc^2.

As variables are separated, taking λ as constant of separation, we have

$$\frac{1}{X}\frac{d^2X}{dx^2}=\frac{1}{c^2T}\frac{d^2T}{dt^2}=\lambda \text{ giving } \frac{d^2X}{dx^2}=\lambda X \text{ and } \frac{d^2T}{dt^2}=c^2\lambda T \quad ...(3)$$

There arise three possibilities:

(*i*) $\lambda = 0$, so that by (3) $\frac{d^2X}{dx^2}=0,\ \frac{d^2T}{dt^2}=0$

giving $X = Ax + B,\ T = Ct + D$ ...(4)

(*ii*) $\lambda = \mu^2$, " " $\frac{d^2X}{dx^2}-\mu^2X=0,\ \frac{d^2T}{dt^2}-\mu^2c^2T=0$ giving

$X = Ae^{\mu x}+Be^{-\mu x},\ T = Ce^{\mu ct}+De^{-\mu ct}$

(*iii*) $\lambda = -\mu^2$, " " $\frac{d^2X}{dx^2}+\mu^2X=0,\ \frac{d^2T}{dt^2}+\mu^2c^2T=0$ giving

$X = A\cos\mu x + B\sin\mu x;$

$T = C\cos\mu ct + D\sin\mu ct$ ...(6)

If we impose the boundary conditions

$$u(o,t)=0,\ u(l,t)=0 \text{ for all } t, \quad ...(7)$$

and the initial condition

$$u(x,o)=F(x);\ \left(\frac{\partial u}{\partial t}\right)_{t=0}=g(x) \quad ...(8)$$

then (7) asserts that $u(o,t)=X(o)\,T(t)=0$ and $u(l,t)=X(l)\,T(t)=0$
which imply that either $T(t)=0$ or $X(o)=0$ and $X(l)=0$

Thus from (4) when $x = 0$, we have $B = 0$ and $X(l) = 0 = Al + B$ then gives $A = 0$.

Also from (5) when $x = 0$, we have

$X(o)=0=A+B$ and $X(l)=0=Ae^{\mu l}+Be^{-\mu l}$ giving $A = B = 0$.

In either case $A = B = 0$ give $X(x) = 0$, so that the solutions (4) and (5) fail to give that solution of (1) and it is the solution (6) which is periodic in time and is capable of giving a solution of (1).

Combining the two solutions of (6) we have a general solution of (1) as

$$u(x,t)=(A\cos\mu x+B\sin\mu x)(C\cos\mu ct+D\sin\mu ct) \quad ...(9)$$

Now to determine the constants A, B and μ, we adjust them so as (9) satisfies (7) *i.e.*,

$u(o,t)=0=A\{(\cos\mu ct+D\sin\mu ct)\}=0$ giving $A=0$

and $u(l,t)=0=(o+B\sin\mu l)(C\cos\mu ct+D\sin\mu ct)$ holds for

$\mu l = n\pi$ *i.e.*, $\mu=\frac{n\pi}{l}$, n being a positive integer.

Hence the solution of (1) satisfying the boundary conditions (7), may be written as

$$u_n(x,t)=\left(C_n\cos\frac{n\pi ct}{l}+D_n\sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l} \quad ...(10)$$

Now applying the initial condition (8), (10) yields

$$u(x, o) = C_n \sin \frac{n\pi x}{l} = F(x)$$

and $$\left(\frac{\partial u}{\partial t}\right)_{t=0} = \left[-\frac{n\pi c}{l} C_n \sin \frac{n\pi ct}{l} + \frac{n\pi c}{l} D_n \cos \frac{n\pi ct}{l}\right]_{t=0} \sin \frac{n\pi x}{l}$$

$$= \frac{n\pi c}{l} D_n \sin \frac{n\pi x}{l} = g(x)$$

It is notable that a mere single term as solution will not satisfy $u(x, o)$ and $\left(\frac{\partial u}{\partial t}\right)_{t=0}$

In fact the solution (2) is linear and homogeneous and hence it indicates that the sum of any number of distinct solutions of (1) is also a solution of (1). As such the required solution of (1) in place of (10) may be taken as

$$u(x, t) = \sum_{n=1}^{\infty} \left(C_n \cos \frac{n\pi ct}{l} + D_n \sin \frac{n\pi ct}{l}\right) \sin \frac{nx\pi}{l} \qquad ...(11)$$

where $C_n \sin \frac{n\pi x}{l} = F(x)$ and $\frac{n\pi c}{l} D_n \sin \frac{n\pi x}{l} = g(x)$.

Of course, the solution (11) satisfies (7) and hence together with (8), it provides

$$u(x, o) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l} = F(x) \text{ and } \left(\frac{\partial u}{\partial t}\right)_{t=0}$$

$$= \sum_{n=1}^{\infty} \frac{n\pi c}{l} D_n \sin \frac{n\pi x}{l} = g(x) \qquad ...(12)$$

The R.H.S.'s of (12) being Fourier expansion, we have

$$C_n = \frac{2}{l} \int_0^l F(x) \sin \frac{n\pi x}{l} dx \text{ and } \frac{n\pi c}{l} D_n = \frac{2}{l} \int_0^l g(x) \sin \frac{n\pi x}{l} dx \qquad ...(13)$$

Hence (11) gives the required solution of (1), for all values of C_n and D_n given by (13) satisfying (7) and (8).

COROLLARY 1. In (8) we assume $g(x) = 0$, the initial velocity, then (13) yields $D_n = 0$ and hence (12) reduces to

$$u(x, t) = \sum_{n=1}^{\infty} C_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} = \frac{1}{2} \sum_{n=1}^{\infty} C_n \left[\sin \frac{n\pi}{l}(x - ct) + \sin \frac{n\pi}{l}(x + ct)\right]$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} C_n \sin \frac{n\pi}{l}(x - ct) + \frac{1}{2} \sum_{n=1}^{\infty} C_n \sin \frac{n\pi}{l}(x + ct) \qquad ...(14)$$

Thus replacing x by $x - ct$ and $x + ct$ successively in (11) we find two series

$$\sum_{n=1}^{\infty} C_n \sin \frac{n\pi}{l}(x - ct) \text{ and } \sum_{n=1}^{\infty} C_n \sin \frac{n\pi}{l}(x + ct)$$

we may therefore conclude that

$$u(x, t) = \frac{1}{2} [f(x - ct) + f(x + ct)] \qquad ...(15)$$

Which is the solution of wave equation (1), where f is the odd periodic extension of F with period $2l$.

COROLLARY 2. If we put $\lambda_n = \frac{n\pi c}{l}$ then the functions given by (10) are termed as the *Eigen functions* or *characteristic functions* and the values $\lambda_n = \frac{n\pi c}{l}$ are know as *Eigen Values* or *Characteristic Values* of the vibrating string and the set

$\Lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n)$ is known as the *Spectrum*.

We also observe that u_n represents a harmonic motion with frequency $\frac{\lambda_n}{2\pi} = \frac{nc}{2l}$ cycles per unit time. We call this motion as the *nth normal mode* of the string. In case $n = 1$, the normal mode is caled as the *fundamental mode* while the normal modes for $n = 2, 3, 4, \ldots$ are called as *Overtone*.

Note. *In §7.5. while discussing the D' Alemberts' method for solving one-dimensional wave equation of the type*

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \ldots(1)$$

We have found a solution of it in the form

$$u(x, t) = \phi(x + ct) + \psi(x - ct) \qquad \ldots(2)$$

We require the verification of the boundary condition

$$u(o, t) = 0, \; u(l, t) = 0 \qquad \ldots(3)$$

and the initial conditions $u(x, o) = f(x)$ ...(4)

and
$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x) \qquad \ldots(5)$$

Obviously
$$u'(x, t) = \frac{\partial u}{\partial t} = c\phi'(x + ct) - c\psi(x - ct) \qquad \ldots(6)$$

Applying (4) and (5) to (2) and (6) we get

$$u(x, o) = \phi(x) + \psi(x) = F(x) \qquad \ldots(7)$$

and
$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = c\{\phi'(x) - \psi'(x)\} = 0 \qquad \ldots(8)$$

Assuming $g(x) = 0$ in particular (8) yields

$\phi'(x) = \psi'(x)$ giving on integration $\phi(x) = \psi(x) + \lambda$ ...(9)

So that (7) and (9) render

$$\phi(x) = \tfrac{1}{2}[F(x) - \lambda] \qquad \ldots(10)$$

and
$$\psi(x) = \tfrac{1}{2}[F(x) + \lambda] \qquad \ldots(11)$$

whence with the help of (10) and (11), (2) yields

$$u(x, t) = \tfrac{1}{2}[F(x + ct) + F(x - ct)] \qquad \ldots(12)$$

which reduces to

$$u(o, t) = \tfrac{1}{2}[F(ct) + F(-ct) = 0$$

and
$$u(l, t) = \tfrac{1}{2}[F(l + ct) + F(l - ct) = 0 \qquad \ldots(13)$$

by the use of (3) and (4).

If follows from (13) that the function F is odd and periodic with period $(2l)$ and hence (12) is the solution of (1). Physically interpreted (2) represents two plane waves travelling in oppositive directions with the same period.

Problem 22. *A string is stretched between two fixed points (0, 0) and (1, 0) and released at rest from the positions $u = \lambda \sin \pi x$. Show that the formula for its subsequent displacement $u(x, t)$ is given by $u(x, t) = \lambda \cos (c\pi t) \sin (\pi x)$, c^2 being diffusivity.*

(Agra, 1972)

The boundary value problem is $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ ...(1)

with boundary conditions $u(o, t) = 0$ and $u(1, t) = 0$ ...(2)

and initial conditions $u(x, 0) = \lambda \sin \pi x = 0$ and $\left(\dfrac{\partial u}{\partial t}\right)_{t=0} = 0$ ...(3)

By (11) of §12.14, we therefore have

$$u(x, t) = \sum_{n=1}^{\infty} C_n \cos (n\pi ct) \sin (n\pi x) \text{ where } C_n = 2\int_0^1 \lambda \sin \pi x \sin n\pi x\, dx$$

It is obvious that $C_n = 0$ for $n = 2, 3, \ldots$ but $C_1 = \lambda \int_0^1 \sin^2 \pi x\, dx' = \lambda$

Hence $u(x, t) = \lambda \cos (\pi ct) \sin (\pi x)$.

Problem 23. *Show that the deflection of vibrating string of length π (its ends being fixed and $c^2 = 1$), corresponding to zero initial velocity and initial deflection $F(x) = \lambda (\sin x - \sin 2x)$ is given by $u(x, t) = \lambda (\cos t \sin x - \cos 2t \sin 2x)$.*

The boundary value problem is $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$ (as $c^2 = 1$) ...(1)

with conditions $g(x) = 0$ and $F(x) = \lambda.(\sin x - \sin 2x)$.

Hence by (11) of §12.14 we have (as $D_n = 0$)

$$u(x, t) = \sum_{n=1}^{\infty} C_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} = \sum_{n=1}^{\infty} C_n \cos nt \sin nx \because c = 1 \text{ and } \pi = l$$

$$\text{where } C_n = \frac{2}{\pi}\int_0^{\pi} F(x) \sin \frac{n\pi x}{l} dx = \frac{2}{\pi}\int_0^{\pi} \lambda(\sin x - \sin 2x) \sin nx\, dx$$

$$= \frac{2\lambda}{\pi}\int_0^{\pi} \sin x \sin nx\, dx - \frac{2\lambda}{\pi}\int_0^{\pi} \sin 2x \sin nx\, dx$$

Clearly $C_n = 0$ for $n = 3, 4, 5, \ldots$ and $C_1 = \lambda$, $C_2 = \lambda$.

Hence the required deflection of the vibrating string is given by

$u(x, t) = C_1 \cos t \sin x + C_2 \cos 2t \sin 2x = \lambda (\sin x - \sin 2x)$

which verifies the assertion.

Problem 24. *Solve the wave equation $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ if the string of length $2a$ is originally plucked at the middle point by giving it an initial displacement d from the mean position.*

The boundary value problem is $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ with initial conditions:

$$F(x) = u(x, o) = \begin{cases} \frac{d}{a}, & 0 \le x < a \\ \frac{d}{a}(2a - x), & a \le x \le 2a. \end{cases}$$

Also initial velocity being zero *i.e.*, $g(x) = 0$ we have $D_n = 0$

$\therefore$ By (11) of §12.14, $u(x, t) = \sum_{n=1}^{\infty} C_n \cos\frac{n\pi ct}{2a} \sin\frac{n\pi x}{2a}$.

where $C_n = \frac{2}{l}\int_0^l F(x) \sin\frac{n\pi x}{l} dx = \frac{2}{2a}\int_0^{2a} F(u) \sin\frac{n\pi u}{l} du$

$$= \frac{1}{a}\int_0^a \frac{d}{a} u \sin\frac{n\pi u}{2a} du + \frac{1}{a}\int_a^{2a} (2a - u) \sin\frac{n\pi u}{2a} du = \frac{8d}{\pi^2}\frac{1}{n^2} \sin\frac{n\pi}{2}$$

which vanishes for $n = 2, 4, 6, \ldots$

Hence $u(x, t) = \frac{8d}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} \sin\frac{n\pi}{2} \cos\frac{n\pi ct}{2a} \sin\frac{n\pi x}{2a}$

$$= \frac{8d}{\pi^2}\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{(2r-1)^2} \cos\frac{(2r-1)\pi ct}{2a} \sin\frac{(2r-1)\pi x}{2a}.$$

Problem 25. *A string is stretched between two fixed points (o, o) and (l, o) and released at rest from the deflection given by*

$$F(x) = \begin{cases} \frac{2\lambda}{l}x, & 0 < x < \frac{1}{2} \\ \frac{2\lambda}{l}(l - x), & \frac{l}{2} < x < l \end{cases}$$

Show that the deflection of the string at any time t is given by

$$u(x, t) = \frac{8\lambda}{\pi^2}\sum_{n=1}^{\infty} \sin\frac{n\pi}{2} \cos\frac{n\pi ct}{l} \sin\frac{n\pi x}{l}.$$

Put $a = \frac{l}{2}$ in the previous problem.

Problem 26. *The points of trisection of a string are pulled aside through a distance d on opposite sides of the equilibrium-position and the string is released from rest. Show that the displacement of the string at any subsequent time is given by*

$$u(x, t) = \frac{9d}{\pi^2}\sum_{r=1}^{\infty}\frac{1}{r^2} \sin\frac{2r\pi}{3} \sin\frac{2r\pi}{3a} x \cos\frac{2r\pi ct}{3a}.$$

Also show that the mid-point of the string always remains at rest.

Consider OB as equilibrium-position of the string of length $3a$ (say), and C, D are points of trisection, which are pulled through a distance d as opposite sides and released.

The boundary value problem is

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \quad \ldots(1)$$

Using the conceptions of coordinate geometry, the equation of line OP is

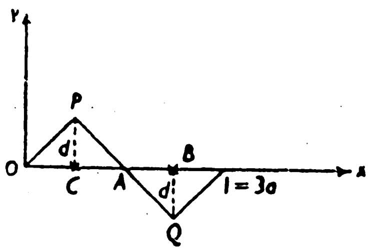


Fig. 12.8

$$y = \frac{d-0}{a-0}(x-0) \text{ i.e., } y = \frac{d}{a}x$$

the equation of PQ is $y - d = \dfrac{-d-d}{2a-a}(x-a)$ *i.e.* $y = \dfrac{d(3a-2x)}{a}$ and the equation of QB is $y-(-d) = \dfrac{0-(-d)}{3a-2a}(x-2a)$ *i.e.,* $y = \dfrac{d(x-3a)}{a}$.

Hence the initial deflection is stated as

$$F(x) = \begin{cases} \dfrac{d}{a}x, & 0 \le x \le a \\ \dfrac{d}{a}(3a-2x), & a \le x \le 2a \\ \dfrac{d}{a}(x-3a), & 2a \le x \le 3a \end{cases}$$

and the initial velocity $g(x) = 0$ so that $D_n = 0$ in (11) of §12.14

∴ we have from (B) of §12.14

$$C_n = \frac{2}{l}\int_0^l F(x)\sin\frac{n\pi x}{l}\,dx = \frac{2}{3a}\frac{d}{a}\left[\int_0^a x\sin\frac{n\pi x}{3a}dx + \int_a^{2a}(3a-2x)\sin\frac{n\pi x}{3a}dx + \int_{2a}^{3a}(x-3a)\sin\frac{n\pi x}{3a}dx\right]$$

$$= \frac{18d}{n^2\pi^2}\{1+(-1)^n\}\sin\frac{n\pi}{3}$$ on integrating by parts and simplifying.

Obviously $C_n = 0$ for $x = 1, 3, 5, 7, \ldots$ *i.e.* being odd

and for even n, $C_n = \dfrac{18d}{n^2\pi^2}\cdot 2\sin\dfrac{n\pi}{3} = \dfrac{36d}{n^2\pi^2}\sin\dfrac{n\pi}{3}$.

Hence by (11) of §12.4, the solution is

$$u(x,t) = \frac{36}{\pi^2}\sum_{r=1}^{\infty}\frac{1}{(2r)^2}\sin\frac{2r\pi}{3}\sin\frac{2r\pi}{3a}x\cos\frac{2r\pi ct}{3a}$$

$$= \frac{9d}{\pi^2}\sum_{r=1}^{\infty}\frac{1}{r^2}\sin\frac{2r\pi}{3}\sin\frac{2r\pi}{3a}x\cos\frac{2r\pi ct}{3a}$$

If we set $x = \dfrac{3a}{2}$, this result reduces to

$$u(x,t) = 0 \text{ since } \sin\frac{2r\pi}{3a}\cdot x = \sin r\pi = 0 \text{ for each } r.$$

This follows that the mid-point of the string always remains at rest.

12.15. SQUARE WAVE *(Agra, 1961)*

In case of a square wave the displacement x is constant $= c$(say) for certain interval of time and zero for the next same interval and so on. This wave is as shown in Fig. 12.9.

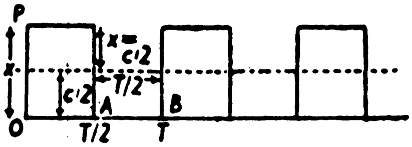


Fig. 12.9

Thus for OA, $x = f(x) = c$ from $t = 0$ to $t = T/2$
and for AB, $x = f(x) = 0$ from $t = T/2$ to T ...(1)

Taking time-axis of co-ordinates through the lowest point of the displacement curve we have by Fourier theorem.

$$F(x) = a_0 + \sum_{r=1}^{\infty} a_r \cos rnt + \sum_{r=1}^{\infty} b_r \sin rnt. \quad ...(2)$$

Here a_0 shows that distance between the axis of coordinates and the axis of the wave and $a_0 = \frac{1}{T}\int_0^T f(x)\, dt$, ...(3)

$$a_r = \frac{2}{T}\int_0^T f(x) \cos (r.\, nt)\, dt, \quad ...(4)$$

$$b_r = \frac{2}{T}\int_0^T f(x) \sin (r.\, nt)\, dt. \quad ...(5)$$

Thus we have

$$a_0 = \frac{1}{T}\int_0^T f(x)\, dt = \frac{1}{T}\left[\int_0^{T/2} f(x)\, dt + \int_{T/2}^T f(x)\, dt\right]$$

$$= \frac{1}{T}\int_0^{T/2} c\, dt = \frac{c}{2}.$$

But the axis of displacement curve being the line $x = c/2$ shown by dotted line in Fig. 12.9, we have

$$a_r = \frac{2}{T}\int\left[\int_0^{T/2} f(x) \cos rnt\, dt + \int_{T/2}^T f(x) \cos rnt\, dt\right]$$

$$= \frac{2}{T}\int_0^{T/2} c \cos rnt\, dt \Bigg] = 0.$$

and $$b_r = \frac{2}{T}\left[\int_0^{T/2} c \sin rnt\, dt\right] = \frac{c}{\pi r}\,[1 - \cos r\pi] = 0 \text{ if } r \text{ is even}$$

$$= \frac{2c}{\pi} \text{ if } r \text{ is odd.}$$

As such the even terms of the sine series disappear and we are left with odd terms only, so that

$$x = f(x) = \frac{c}{2} + \frac{2c}{\pi}\left[\sin nt + \frac{\sin 3nt}{2} + \frac{1}{5}\sin 5nt + ... + \frac{1}{r}\sin rnt + ...\right]$$

This series represents a *square wave*. If we take first three terms, then curve is shown by thick lines in Fig. 12.10.

In case we take large number of terms, then the curve is as shown in Fig. 12.11.

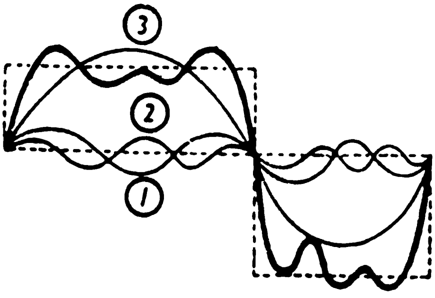


Fig. 12.10

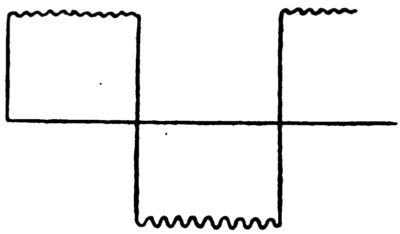


Fig. 12.11

12.16. SOLUTION OF TWO-DIMENSIONAL WAVE EQUATION

[A] Vibrations of a Rectangular Membrane

Consider the oscillation of a uniform rectangular membrane for which due to uniformity T and ρ are constants, T being tension in dynes per cm. length of the edge and ρ the density in gm. per cm² of the membrane. The free oscillations of it are given by

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad \ldots(1)$$

where $c = \sqrt{\dfrac{T}{\rho}}$ = constant.

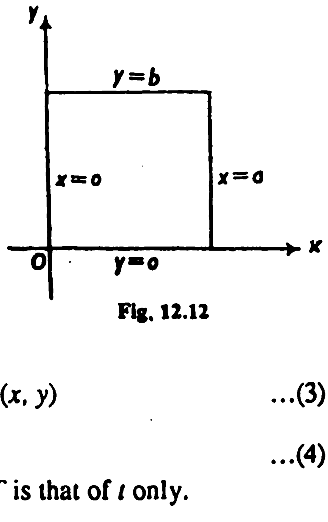


Fig. 12.12

With boundary conditions

$u = 0$ for $x = 0, x = a, y = 0, y = b$

i.e. for $u = u\,(x, y, t)$, the conditions are

$u(o, y, t) = 0,\ u(a, y, t) = 0,\ u(x, o, t) = 0,$
$u(x, b, t) = 0 \qquad \ldots(2)$

Also the initial displacement and initial velocity are given by

$$u(x, y, o) = F(x, y) \text{ and } u_t(x, y, o) = \left[\frac{\partial}{\partial t}u(x, y, t)\right]_{t=0} = g(x, y) \qquad \ldots(3)$$

Assume that $u(x, y, t) = X(x)\,Y(y)\,T(t)$ $\qquad \ldots(4)$

where X is the function of x only, Y is that of y only and T is that of t only.

Then (1) yields $\dfrac{1}{X}\dfrac{d^2X}{dx^2} + \dfrac{1}{Y}\dfrac{d^2Y}{dy^2} = \dfrac{1}{c^2T}\dfrac{d^2T}{dt^2}$ $\qquad \ldots(5)$

Here variables have been separated and hecne choice of the constant of separation gives way to three possibilities:

(*i*) $\dfrac{1}{X}\dfrac{d^2X}{dx^2} = 0, \dfrac{1}{Y}\dfrac{d^2Y}{dy^2} = 0, \dfrac{1}{c^2T}\dfrac{d^2T}{dt^2} = 0$ giving $X = A_1 x + B_1,$

$$Y = A_2 y + B_2,\ T = A_3 t + B_3 \qquad \ldots(6)$$

(*ii*) $\dfrac{1}{X}\dfrac{d^2X}{dx^2} = \lambda_1^2, \dfrac{1}{Y}\dfrac{d^2Y}{dy^2} = \lambda_2^2, \dfrac{1}{c^2T}\dfrac{d^2T}{dt^2} = \lambda^2$ so that $\lambda^2 = (\lambda_1^2 + \lambda_2^2)\,c^2$, giving

$$X = A_1 e^{\lambda_1 x} + B_1 e^{-\lambda_1 x},\ Y = A_2 e^{\lambda_2 y} + B_2 e^{-\lambda_2 y},\ T = A_3 e^{c\lambda t} + B_3 e^{-c\lambda t} \qquad \ldots(7)$$

(iii) $\frac{1}{X}\frac{d^2X}{dx^2} = -\lambda_1^2, \frac{1}{Y}\frac{d^2Y}{dy^2} = -\lambda_2^2, \frac{1}{c^2T}\frac{d^2T}{dt^2} = -\lambda^2$

so that $\lambda^2 = (\lambda_1^2 + \lambda_2^2)\, c^2$ giving

$$X = A_1 \cos \lambda_1 x + B_1 \sin \lambda_1 x,\ Y = A_2 \cos \lambda_2 y + B_2 \sin \lambda_2 y$$

$$T = A_3 \cos \lambda ct + B_3 \sin \lambda ct \qquad \ldots(8)$$

In view of 2nd and 4th conditions of (2), we conclude that

$$X(o) = X(a) = 0,\ Y(o) = 0 = Y(b),$$

thereby rendering the solutions (5) and (6) unable to give the solution of (1).

As regards (7), the function $X(x)$ is a linear combination of $\sin \lambda_1 x$ and $\cos \lambda_1 x$ in which cosine is to be rejected in view of the condition $u = 0$ at $x = 0$ *i.e.* $u(o, y, t) = 0$ and hence $X(o) = 0$. We are left with $X(x) = B_1 \sin \lambda_1 x = 0$ as $u(a, y, t) = 0$ gives $X(a) = 0$ so that $\sin \lambda_1 x = 0 = \sin m\pi$, m being a positive integer. Thus we get

$$\lambda_1 = \frac{m\pi}{a} \text{ and similarly } Y = \sin \lambda_2 y = 0 \text{ yields}$$

$$\lambda_2 = \frac{n\pi}{b} \text{ for integral } n.$$

Consequently the required solution of (1) takes the form

$$\left(A \cos \lambda_{mn} t + B \sin \lambda_{mn} t\right) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \qquad \ldots(9)$$

where A and B are constants and $\lambda_{mn} = \lambda$ is given by

$$\lambda_{mn}^2 = \left(\frac{m\pi c}{a}\right)^2 + \left(\frac{n\pi c}{b}\right)^2 \qquad \ldots(10)$$

Such quantities are termed as *eigen values* or *characteristic values* of vibrating membranes.

Trying superposition and accounting for different constants A and B for each choice of m and n, we can write the solution as

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(A_{mn} \cos \lambda_{mn} t + B_{mn} \sin \lambda_{mn} t\right) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \qquad \ldots(11)$$

Now applying the condition (3), *i.e.* for $t = 0$, $u(x, y, o) = F(x, y)$, we get

$$F(x, y) = u(x, y, o) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \qquad \ldots(12)$$

which is known as the *Double Fourier Sine Series of* $F(x, y)$.

Multiplying (11) by $\sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$, and integrating over the rectangle

$x = o, x = a, y = 0, y = b$ we find the value of A_{mn} *i.e.*,

$$A_{mn} = \frac{4}{ab}\int_0^a \int_0^b F(x, y) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dxdy \qquad \ldots(13)$$

Also differentiating (10) w.r.t. 't' and applying the condition (3)

i.e. for $t = 0$, $\frac{\partial}{\partial t} u(x, y, t) = g(x, y)$, we get

$$B_{mn} = \frac{4}{ab\lambda_{mn}} \int_0^a \int_0^b g(x, y) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} dxdy \qquad ...(14)$$

Obviously the general term of (11) is a periodic function of time with period $\frac{2\pi}{\lambda_{mn}}$ and having the frequency

$$\frac{\lambda_{mn}}{2\pi} = \frac{c}{2}\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^{1/2} \qquad ...(15)$$

which are termed as *characteristic frequencies* or eigen *frequencies* and the associated oscillations given by (8) are termed as *Modes*.

The *fundamental mode* is the mode of the lowest frequency obtained by putting $m = n = 1$ in (8).

COROLLARY. If we assume that the initial velocity $g(x, y) = 0$ so that by (3), $u_t(x, y, o) = 0$ then clearly $B_{mn} = 0$ and hence the solution of (1) reduces to the form

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \cos \lambda_{mn} t \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \qquad ...(16)$$

where $$A_{mn} = \frac{4}{ab} \int_0^a \int_0^b F(x, y) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} dxdy \qquad ...(17)$$

and $$\lambda_{mn} = \pi^2 c^2 \left[\frac{m^2}{a^2} + \frac{n^2}{b^2}\right] \qquad \text{(Rohilkhand, 1988)} \quad ...(18)$$

Note. ***Distinction between the behaviour of vibrating strings and membranes.***

The main difference is that for every eigen frequency of vibration of a string there exists a corresponding mode such that the string is divided into equal parts by fixed posited nodes, whereas for an oscillating membrane with a given characteristic property, there exists points on the membrane which are at rest and they constitute *nodal lines*. Sine the shape of the nodal lines for a given frequency is not the same, therefore for a given eigen frequency there may be more than one mode *e.g.* for a rectangular membrane with $a = b$, by (14), the frequency is given by

$$\lambda_{mn} = \frac{c\pi}{a}\sqrt{m^2 + n^2} = \kappa\sqrt{m^2 + n^2} \text{ where } \kappa = \pi c / a \qquad ...(19)$$

and by (10) the fundamental mode is $u_{11} = (A_{11} \cos \lambda_{11} t + B_{11} \sin \lambda_{11} t)$

$$\sin\frac{\pi x}{a} \sin\frac{\pi y}{a} \text{ where } \lambda_{11} = a\sqrt{2}\,\cdot$$

But u_{11} being zero for all t only when $x = 0, y = 0, x = a, y = a$, therefore in the interior of the membrane for this frequency, there does not exist a nodal line.

In case $m = 1, n = 2$ and $m = 2, n = 1$, we find two modes

$$u_{12} = (A_{12} \cos \lambda_{12} t + B_{12} \sin \lambda_{12} t) \sin\frac{\pi x}{a} \sin\frac{2\pi y}{a} \text{ and } u_{21}$$

$$= (A_{21} \cos \lambda_{21} t + B_{21} \sin \lambda_{21} t) \sin\frac{2\pi x}{a} \sin\frac{\pi y}{a} \qquad ...(20)$$

having the same frequency since $\lambda_{12} = \lambda_{21} = a\sqrt{5}$ for $y = \frac{a}{2}$,

$u_{12} = 0$ and for $x = \frac{a}{2}$, $u_{21} = 0$

We can thus show the existence of oscillations with the same frequency but different nodal lines.

[B] Vibrations of a circular membrane. (Bessel's functions)

If we transform $\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$ by the substitutions

$$x = r\cos\theta, y = r\sin\theta$$

we get the polar form $\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)$ ...(1)

where $u(r, \theta, t)$ is the deflection of the membrane.

The equation (1) is known as the *wave equation for a circular membrane*, if the boundary is the circle $r = a$ so that the boundary condition is

$$u(a, \theta, t) = 0, -\pi \le \theta \le 0, t \ge 0 \quad ...(2)$$

Let us take the initial conditions for $0 \le r \le a, -\pi \le \theta \le 0$, as

$$u(r, \theta, 0) = F(r, \theta) \quad ...(3)$$

and

$$u_t(r, \theta, 0) = \left[\frac{\partial u}{\partial t}(r, \theta, t)\right]_{t=0} = g(r, \theta) \quad ...(4)$$

Assume that $u(r, \theta, t) = R(r)\Theta(\theta)\,T(t)$ ...(5)

where R is the function of r alone, Θ that of θ alone and T that of t alone.

Then (1) yields

$$\frac{1}{c^2 T}\frac{d^2 T}{dt^2} = \frac{1}{R}\frac{d^2 R}{dr^2} + \frac{1}{rR}\frac{dR}{dr} + \frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2} = -\lambda^2 \text{ (say) as}$$

variables are separated.

The terms in it being constant we can take

$$\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = -m^2.$$

As such we have the ordinary differential equations for R, Θ and T as

$$\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(\lambda^2 - \frac{m^2}{r^2}\right)R = 0 \quad ...(6)$$

$$\frac{d^2\Theta}{d\theta^2} + m^2\Theta = 0 \quad ...(7)$$

$$\frac{d^2 T}{dt^2} + c^2\lambda T = 0 \quad ...(8)$$

The solution of (7) is of the form $\Theta = De^{\pm im\theta}$ ...(9)

$m = 0, 1, 2, ...$

D being a constant.

Also putting $s = \lambda r$, (6) becomes

$$\frac{d^2R}{ds^2} + \frac{1}{s}\frac{dR}{ds} + \left(1 - \frac{m^2}{s^2}\right)R = 0 \qquad ...(10)$$

which is Bessel's equation and hence its general solution is

$$R(r) = K_1 J_m(s) + K_2 J_{-m}(s) = K_1 J_m(\lambda r) + K_2 J_{-m}(\lambda r) \qquad ...(11)$$

K_1, K_2 being constants.

But as $r \to 0$, $J_{-m} \to \infty$ thereby contradicting the hypothesis that deflection of the membrane is always finite. Thus to avoid the term containing J_{-m} choose $K_2 = 0$, so that (11) reduces to $R(r) = K_1 J_m(\lambda r)$ ...(12)

The boundary condition (2) requires that

$$R(a) = K_1 J_m(\lambda a) = 0 \text{ i.e. } J_m(\lambda a) = 0 \qquad ...(13)$$

Assuming $\lambda_1, \lambda_2, \lambda_3, \ldots$ as positive roots of (13), the general solution of (8) is

$$T = A \cos c\lambda t + B \sin \lambda ct \qquad ...(14)$$

With the help of (9), (12) and (14), the required oscillation has the form

$$[A \cos \lambda ct + B \sin \lambda ct]\, e^{\pm im\theta} J_m(\lambda r) \qquad ...(15)$$

Trying superposition and using distinct values of constants A and B for each choice of m and n, the general solution of (1) may be taken as

$$u(r, \theta, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{A_{mn} \cos\left(c\lambda_{mn} t\right) + B_{mn} \sin\left(c\lambda_{mn} t\right)\right\} . e^{\pm im\theta} J_m(\lambda_{mn} r) \qquad ...(16)$$

which satisfies the conditions (2), (3) and (4).

In case the equation (1) is radially symmetrical *i.e.* the solution is independent of θ, we have from (16) by putting $m = 0$.

$$u(r, t) = \sum_{n=1}^{\infty} \{A_n \cos c\lambda_n t + B_n \sin c\lambda_n t\} J_0(\lambda_n r) \qquad ...(17)$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the positive roots of $J_0(\lambda a) = 0$

Here (17) gives the general solution of (1).

In case $t = 0$, the initial condition (3) gives

$$u(r, o) = F(r) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r) \qquad ...(18)$$

since $F(r, T)$ reduces to $F(r)$ when independent of θ.

Here A_n is the coefficient of Fourier-Bessel series determined by

$$A_n = \frac{2}{a^2 J_1^2(\lambda_n a)} \int_0^a F(r) J_0(\lambda_n r) r\,dr \qquad ...(19)$$

Also the condition (4) requires,

$$u_t(r, o) = \sum_{n=1}^{\infty} c\lambda_n B_n J_0(\lambda_n r) = g(r),$$ g being independent of θ and hence B_n are determined by

$$B_n = \frac{2}{c\lambda_n a^2 J_1^2(\lambda_n a)} \int_0^a g(r) J_0(\lambda_n r) r\,dr \qquad ...(20)$$

COROLLARY. If we take $g(r, \theta) = u_t(r, \theta, o) = 0$ then (17) yields on putting $t = 0$.

$$u(r, t) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r) \cos(\lambda_n ct) \qquad \textit{(Rohilkhand, 1985)} \qquad ...(21)$$

where $F(r) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r)$ and the coefficient A_n of Fourier Bessel series are given by

$$A_n = \frac{2}{a^2 J_1^2(\lambda_n a)} \int_0^a F(r) J_0(\lambda_n r) r dr \qquad ...(22)$$

The equation (21) may be expressed by means of modes and eigen frequencies as

$$\frac{\lambda_n}{2\pi} = \frac{c\lambda_n}{2\pi} = \frac{c\lambda_n}{2\pi a}$$

and the fundamental mode is given by $J_0(\lambda_1 r) \cos(\lambda_1 ct)$.

Note. *The generalized Fourier-Bessel series is*

$$f(x) = \sum_{m=1}^{\infty} C_m J_n(\lambda_{mn})$$

where $\lambda_{mn} = \frac{\alpha_{mn}}{a}$, *n fixed and* $m = 1, 2, 3...$

also $C_m = \frac{1}{a^2 J_{n+1}(\alpha_{mn})} \int_0^a x f(x) \; J_n(\lambda_{mn} x) dx, \; m = 1, 2, 3...$

Problem 27. *Find the deflection $u(x, y, t)$ of the square membrane with $a = b = 1$ and $c = 1$ under the conditions that the initial velocity is zero and the initial deflection is*

$$F(x, y) = A \sin \pi x \sin 2\pi y$$

The boundary value problem is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] \text{ where } c^2 = 1 \qquad ...(1)$$

with boundary conditions $u = 0$ for $x = 0, \; x = 1, \; y = 0, \; y = 1$ ...(2)

and the initial conditions $u(x, y, 0) = F(x, y) = A \sin \pi x \sin 2\pi y$ ...(3)

and $u_t(x, y, 0) = g(x, y) = 0$ ...(4)

By (10) of §12.16, we therefore have the solution as

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \lambda_{mn} t \sin m\pi x \sin n\pi y$$

where $\lambda_{mn}^2 = \pi^2(m^2 + n^2)$ and $A_{mn} = 4A \int_0^1 \int_0^1 \sin \pi x \sin m\pi x \sin 2\pi y \sin n\pi y \, dx \, dy$

Clearly $A_{mn} = 0$ where n is odd.

Also $A_{m2} = 4A \int_0^1 \int_0^1 \sin \pi x \sin m\pi x \sin^2 2\pi y \, dx \, dy$

$$= 2A\int_0^1 \sin \pi x \sin m\pi x \, dx$$

$= 0$ where n is given

and $\quad A_{12} = 2A\int_0^1 \sin^2 \pi x \, dx = A.$

Hence the required solution is

$$u(x, y, t) = A_{12} \cos \lambda_{12} t \sin \pi x \sin 2\pi y$$

$$= A \cos \sqrt{5}\, t \sin \pi x \sin 2\pi y$$

which gives the required deflection of the membrane.

12.17. SOLUTION OF THREE-DIMENSIONAL WAVE EQUATION

The equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad ...(1)$$

Suppose that we have to solve it under the conditions

(i) $\dfrac{\partial u}{\partial x} = 0$ when $x = 0,\ x = a$

(ii) $\dfrac{\partial u}{\partial y} = 0$ when $y = 0,\ y = a$

(iii) $\dfrac{\partial u}{\partial z} = 0$ when $z = 0,\ z = a$

(iv) $u \neq 0$ at $t = 0$.

Assume $\quad u = X(x)\, Y(y)\, Z(z)\, T(t) \qquad ...(2)$

Then (1) yields

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = \frac{1}{c^2T}\frac{d^2T}{dt^2} \qquad ...(3)$$

Here variables have been separated. Hence each of the four terms in this equation must be constant say

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\lambda_1^2; \frac{1}{Y}\frac{d^2Y}{dy^2} = -\lambda_2^2; \frac{1}{Z}\frac{d^2Z}{dz^2} = -\lambda_3^2$$

and $\quad \dfrac{1}{c^2T}\dfrac{d^2T}{dt^2} = -\lambda^2$ so that $\lambda^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$.

Then we get

$$X = A_1 \cos(\lambda_1 x + \alpha_1),\ Y = A_2 \cos(\lambda_2 y + \alpha_2,)\ Z = A_3 \cos(\lambda_3 z + \alpha_3)$$

and $\quad T = B \cos(\lambda ct + \alpha)$

where $A_1, A_2, A_3, B, \alpha_1, \alpha_2, \alpha_3$ and α all are arbitrary constants.

Hence the solution of (1) is

$$u = \Sigma A \cos(\lambda_1 x + \alpha_1) \cos(\lambda_2 y + \alpha_2) \cos(\lambda_3 z + \alpha_3) \cos(\lambda ct + \alpha) \qquad ...(4)$$

However we can take the solution of (1) in the form

$$u = Ce^{\pm i(\lambda_1 x + \lambda_2 y + \lambda_3 z + \lambda ct)}$$

where $\quad \lambda^2 = \lambda_1{}^2 + \lambda_2{}^2 + \lambda_3{}^2 \qquad ...(5)$

Actually, $e^{\pm i\lambda_1 x} = A_1 \cos \lambda_1 x \pm A_2 \sin \lambda_1 x$

and $\dfrac{\partial u}{\partial x} = 0$ when $x = 0$ implies $A_2 = 0$ etc.

Then $u = C \cos \lambda_1 x \cos \lambda_2 y \cos \lambda_3 z \cos \lambda ct$ ...(6)

The result (6) may be deduced from (4) since when

$$x = 0, \sin \alpha_1 x = 0 \text{ etc.}$$

If we now define $\dfrac{\partial u}{\partial x}$ as the gradient in the direction of the normal of each face we see that the boundary condition (*iv*) holds at $t = 0$.

Also $\dfrac{\partial u}{\partial x} = 0$ when $x = 0, y = 0, z = 0$ and the boundary condition (*i*) requires that $\sin \lambda_1 a \cos \lambda_2 y \cos \lambda_3 z \cos \lambda ct = 0$

i.e., $\sin \lambda_1 a = 0$ or $\lambda_1 a = N_1 \pi$ *i.e.* $\lambda_1 = \dfrac{N_1 \pi}{a}$, N_1 being an integer.

Similarly $\lambda_2 = \dfrac{N_2 \pi}{a}$, $\lambda_3 = \dfrac{N_3 \pi}{a}$ and $\lambda = \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}$

$$= \frac{\pi}{a}\sqrt{\left(N_1^2 + N_2^2 + N_3^2\right)}$$

As such (6) reduces to

$$u = C \cos \frac{N_1 \pi}{a} x \cos \frac{N_2 \pi}{a} y \cos \frac{N_3 \pi}{a} z \cos \left\{\left(N_1^2 + N_2^2 + N_3^2\right)^{1/2} \frac{\pi ct}{a}\right\}$$

Hence the general solution of (1) is

$$u = \sum_{N_1 = 1}^{\infty} \sum_{N_2 = 1}^{\infty} \sum_{N_3 = 1}^{\infty} C_{N_1 N_2 N_3} \cos \frac{N_1 \pi}{a} x \cos \frac{N_2 \pi}{a} y \cos \frac{N_3 \pi}{a} z \cos \left\{\left(N_1^2 + N_2^2 + N_3^2\right)^{1/2} \frac{\pi ct}{a}\right\}$$

...(6)

Note 1. *Cylindrical form of three-dimensional were equation is*

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad ...(7)$$

Assuming $u = R(r). \Theta(\theta). Z(z). T(t)$, (7) yields

$$\frac{1}{R}\left(\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right) + \frac{1}{r^2 \Theta}\frac{d^2 \Theta}{d\theta^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} = \frac{1}{c^2 T}\frac{d^2 T}{dt^2},$$

where variables are separated.

Taking $\dfrac{1}{\Theta}\dfrac{d^2 \Theta}{d\theta^2} = -\mu^2$, $\dfrac{1}{Z}\dfrac{d^2 Z}{dz^2} = -\nu^2$, $\dfrac{1}{T}\dfrac{d^2 T}{dt^2} = -c^2 \lambda^2$, ...(8)

λ^2, μ^2, ν^2 being constants, we have

$$\frac{1}{R}\left(\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right) - \frac{\mu^2}{r^2} = \nu^2 - \lambda^2 = -\alpha^2 \text{ (say) giving}$$

$$\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(\alpha^2 - \frac{\mu^2}{r^2}\right) R = 0 \quad ...(9)$$

which is Bessel's equation of order μ and argument αr.

∴ Solutions of (8) and (9) are

$$\Theta = A_1 \cos \mu\theta + B_1 \sin \mu\theta, Z = A_2 \cos vz + B_2 \sin vz$$

$$T = A_3 \cos \lambda ct + B_3 \sin \lambda ct, R = A_4 J_\mu(\alpha r) + B_4 Y_\mu(\alpha r)$$

Hence the general solution of (7), is

$$u = \Sigma A\, J_\mu(\alpha r) \cos \mu\theta \cos vz \cos \lambda ct \quad \text{...(10)}$$

where $\alpha^2 = \lambda^2 - v^2$ and the term $Y_\mu(\alpha r)$ may be included if required by the conditions of the problem.

In case of axial symmetry, u is independent of θ *i.e.*, $\mu = 0$ and hence $u = \Sigma A J_0(\alpha r) \cos vz \cos \lambda ct$ ...(11)

Also if u is independent of z, then $v = 0$ and $\alpha = \lambda$, hence

$$u = \Sigma A J_0(\lambda r) \cos \lambda ct. \quad \text{...(12)}$$

Note 2. ***Polar spherical form of three-dimensional wave equation is***

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = -\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad \text{...(13)}$$

Assuming $u = R(r)\Theta(\theta)\Phi(\phi)T(t)$, (13) yields

$$\frac{1}{R}\frac{d^2R}{dr^2} + \frac{2}{rR}\frac{dR}{dr} + \frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2} + \frac{\cot\theta}{r^2\Theta}\frac{d\Theta}{d\theta} + \frac{1}{r^2 \sin^2\theta}\frac{1}{\Phi}\cdot\frac{d^2\Phi}{d\phi^2} = \frac{1}{c^2 T}\frac{d^2T}{dt^2} \quad \text{...(14)}$$

As variables are separated, we can take

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -m^2, \frac{1}{c^2 T}\frac{d^2T}{dt^2} = -p^2 \quad \text{...(15)}$$

and $\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} + \frac{\cot\theta}{\Theta}\frac{d\Theta}{d\theta} = \frac{m^2}{\sin^2\theta} - n(n+1)$ ...(16)

where m, n, p are constants.

(14) reduces with the hlep of (15) and (16), to

$$\frac{1}{R}\frac{d^2R}{dr^2} + \frac{2}{rR}\frac{dR}{dr} = \frac{n(n+1)}{r^2} - p^2 \quad \text{...(17)}$$

Solutions of (15) are $\Phi = \cos m\phi + i \sin m\phi$ or $e^{\pm im\phi}$ and $T = \cos pct + i \sin pct$ or $e^{\cdot \kappa pt}$

Also equation (16) can be written as

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left\{n(n+1) - \frac{m^2}{\sin^2\theta}\right\}\Theta = 0 \quad \text{...(18)}$$

Putting $\cos\theta = \mu$, so that $\frac{1}{\sin\theta}\frac{d}{d\theta} = -\frac{d}{d\mu}$, (18) yields

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{d\Theta}{d\mu}\right\} + \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\}\Theta = 0$$

$$(1-\mu^2)\frac{d^2\Theta}{d\mu^2} - 2\mu\frac{d\Theta}{d\mu} + \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\}\Theta = 0 \quad \text{...(19)}$$

which is associated Legendre's equation, having the solution

$$\Theta = AP_n^m(\mu) + BQ_n^m(\mu) = AP_n^m(\cos\theta) + BQ_n^m(\cos\theta) \quad \text{...(20)}$$

If we neglect Q_n^m we can write $\Theta = AP_n^m(\cos\theta)$ ...(21)

$$\Theta = AP_n^m(\mu) + BQ_n^m(\mu) = AP_n^m(\cos\theta) + BQ_n^m(\cos\theta) \qquad ...(20)$$

If we neglect Q_n^m we can write $\Theta = AP_n^m(\cos\theta)$ ...(21)

COROLLARY 1

Schröedinger equation as a spherical symmetry

The Schröedinger equation for the energy states of a particle of mass m, moving in central, spherically symmetric field of force, such that its potential energy merely depends upon the distance r of the particle from the centre of force, is given by

$$\nabla^2\psi + \frac{2m}{\hbar^2}[E - V(r)]\psi = 0 \qquad ...(1)$$

where
$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Since the potential function $V(r)$ is independent of θ and ϕ when we use the transformation of cartesian (x, y, z) coordinats to spherical polar (r, θ, ϕ) coordinates, namely

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta \qquad ...(2)$$

Therefore, proceeding first as in § 8.1, we get the spherical polar form of (1) as

$$\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right] + \frac{2m}{\hbar^2}[E - V(r)]\psi = 0 \qquad ...(3)$$

To solve it, we can use the method of separation of variables, by assuming

$$\psi = R(r)S(\theta,\phi) \qquad ...(4)$$

where R is a function of r only and S is a function of θ, ϕ, but independent of r. As such (3) transforms to

$$\frac{1}{R}\left[\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{2m}{\hbar^2}[E - V(r)]r^2R\right] = -\frac{1}{S}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 S}{\partial\phi^2}\right]$$

$$= K(\text{say}) \qquad ...(5)$$

Since L.H.S. depends upon r while R.H.S. is independent of r, therefor each member can be taken as a constant K.

Now, the equation (5) can be separated as

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{2m}{\hbar^2}[E - V(r)]r^2R = K$$

i.e.
$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left[\frac{2m}{\hbar^2}\{E - V(r)\} - \frac{K}{r^2}\right]R = 0 \qquad ...(6)$$

and
$$-\frac{1}{S}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 S}{\partial\phi^2}\right] = K$$

i.e. $$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 S}{\partial\phi^2}=-KS \quad ...(7)$$

Here the relation (7) being independent of the energy E and potential energy $V(r)$, can be separated again by assuming

$$S(\theta,\phi)=\Theta(\theta)\,\Phi(\phi) \quad ...(8)$$

where Θ is a function of θ only and ϕ is a function of ϕ

Thus (7) transforms to

$$\frac{1}{\Theta}\left[\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)\right]+K\sin^2\theta=-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=l^2\text{(say)} \quad ...(9)$$

where

$$-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=l^2 \Rightarrow \frac{d^2\Phi}{d\phi^2}+l^2\Phi=0$$

$$\Rightarrow (D^2+l^2)\Phi=0, \text{ where } D=\frac{d}{d\phi}$$

$$\Rightarrow D=\pm il \text{ as } \Phi\neq 0$$

So its solution is

$$\Phi=e^{il\phi} \text{ or } \cos l\phi+i\sin l\phi \quad ...(10)$$

Again $\frac{1}{\Theta}\left\{\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)\right\}+K\sin^2\theta=l^2$

or $$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left\{n(n+1)-\frac{l^2}{\sin^2\theta}\right\}\Theta=0 \quad ...(11)$$

on taking $k=n(n+1)$

Now if we put

$$\cos\theta=\mu \text{ i.e. } \frac{1}{\sin\theta}\frac{d}{d\theta}\equiv-\frac{d}{d\mu}$$

then the equation (11) reduces to

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{d\Theta}{d\mu}\right\}+\left\{n(n+1)-\frac{l^2}{1-\mu^2}\right\}\Theta=0$$

or $$(1-\mu^2)\frac{d^2\Theta}{d\mu^2}-2\mu\frac{d\Theta}{d\mu}+\left\{n(n+1)-\frac{l^2}{1-\mu^2}\right\}\Theta=0 \quad ...(12)$$

Which is associated Legendre equation, and has for its solution as

$$\Theta=AP_n^m(\mu)+BQ_n^m(\mu)=AP_n^m(\cos\theta)+BQ_n^m(\cos\theta) \quad ...(13)$$

However, if we neglect Q_n^m, then we can write

$$\Theta=AP_n^m(\cos\theta) \quad ...(14)$$

Where A is a constant to be determined by initial conditions.

COROLLARY 2

One-dimensional Harmonic oscillation

If a particle undergoes linear simple harmonic oscillation, then the restoring force F is proportional to the displacement x i.e.

$$F \propto x \text{ giving } F = -kx \qquad ...(1)$$

where k is a positive constant and the negative sign implies that the motion is always constrained to its mean position.

But if m be the mass of the particle, then by Newton's second law of motion,

$$m\frac{d^2x}{dt^2} = -kx \text{ or } \frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \qquad ...(2)$$

$$\Rightarrow \left(D + \frac{k}{m}\right)x \text{ for } D = \frac{d}{dt}$$

$$\Rightarrow D = -\frac{k}{m}$$

Hence the solutions of (2) can be taken as

$$x = A\sin(Bt + C), \qquad ...(3)$$

where A, B, C are constants.

Now the motion being periodic, if ν_0 be the frequency of oscillations, then we can assume $B = 2\pi\nu_0$, so that (3) yields

$$x = A\sin(2\pi\nu_0 t + C) \qquad ...(4)$$

giving $$\frac{d^2x}{dt^2} = -4\pi^2\nu_0 . A\sin(2\pi\nu_0 t + c) = -4\pi^2\nu_0 x \qquad ...(5)$$

with the help of (4) and (5), (2) renders

$$k = 4\pi^2\nu_0^2 m \qquad ...(6)$$

Now, the potential energy V of the harmonic oscillator is given by

$$V = -\int F dx = \int kx.dx = \frac{1}{2}kx^2 + V_0, \qquad ...(7)$$

where V_0 the constant of integration may be assumed as the initial potential energy, such that when $x = 0$, $V_0 = 0$

$$\therefore (7) \Rightarrow V = \frac{1}{2}kx^2 = 2\pi^2\nu_0^2 mx^2 \text{ by (6)}$$

Using Schrödinger's equation $\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - V)\psi = 0$, the wave equation for one-dimensional harmonic oscillator becomes

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar}(E - \frac{1}{2}kx^2)\psi = 0 \qquad ...(8)$$

where k is given by (6)

We can rewrite (8) as

$$\frac{d^2\psi}{dx^2} + 2\sqrt{\frac{m}{\hbar^2}}\sqrt{\frac{mk}{\hbar^2}}\left[\frac{E}{\sqrt{k}} - \frac{1}{2}kx^2\right]\psi = 0$$

ADDITIONAL MISCELLANEOUS PROBLEMS

Problem 28. *Solve* $\dfrac{\partial^2 u}{\partial x^2} = 0$ *subject to the conditions* $u = \lambda^2$ *when* $x = 0$ $u = 1$ *when* $x = 1$

Ans. $u = x(1 - \lambda^2) + \lambda^2$ when $\dfrac{\partial u}{\partial x} = \phi(\lambda)$.

Problem 29. *Transform the equation* $\nabla^2\psi = \dfrac{1}{c^2}\dfrac{\partial^2\psi}{\partial t^2}$ *to spherical polar coordinates. If* ψ *depends on* r *and* t *only, show that the equation can be written in the form* $\dfrac{\partial^2}{\partial r^2}\Psi(r, t) = \dfrac{1}{c^2}\dfrac{\partial^2\Psi(r, t)}{\partial t^2}$. *Hence show that the general solution is of the form*

$$\Psi(r, t) = \frac{1}{2}\{f(r - ct) + g(r + ct)\}$$

Explain the physical meaning of this solution. (*Meerut, 1971; Bombay, 1965*)

We have $$\frac{\partial^2\Psi}{\partial t^2} = c^2\frac{\partial^2\Psi}{\partial r^2},\ -\infty < r < \infty,\ t > 0 \qquad ...(1)$$

with boundary condition $\psi(r, o) = f(r)$ and $\dfrac{\partial\Psi}{\partial t} = 0$ at $t = 0$

$$(1) \Rightarrow F\left\{\frac{\partial^2\Psi}{\partial t^2}\right\} = c^2 F\left\{\frac{\partial^2\Psi}{\partial r^2}\right\}$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{\partial^2\Psi}{\partial t^2}e^{inr}\,dr = c^2\int_{-\infty}^{\infty}\frac{\partial^2\Psi}{\partial t^2}e^{inr}\,dr$$

$$\Rightarrow \frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty}\Psi e^{inr}\,dr = c^2(in)^2\bar{\Psi}(n,t)$$

$$\Rightarrow \frac{\partial^2}{\partial t^2}\bar{\Psi}(n,t) + c^2 n^2\bar{\Psi}(n,t) = 0 \qquad ...(2)$$

Where $\bar{\Psi}(n, t)$ is Fourier transform of $y(x, t)$.

Solution of (2) is $\bar{\Psi} = A\cos cnt + B\sin cnt$ ...(3)

Boundary condition $\dfrac{\partial\Psi}{\partial t} = 0$ at $t = 0 \Rightarrow F\left\{\dfrac{\partial\Psi}{\partial t}\right\} = \displaystyle\int_{-\infty}^{\infty}\frac{\partial\Psi}{\partial t}e^{inr}\,dr = 0$

$$\Rightarrow \frac{\partial}{\partial t}\int_{-\infty}^{\infty}\Psi e^{inr}\,dr = 0 \Rightarrow \frac{\partial\bar{\Psi}}{\partial t} = 0 \text{ at } t = 0 \qquad ...(4)$$

By (4) and $\left[\dfrac{d\Psi}{\partial t}\right]_{t=0} = cn[-A\sin cnt + B\cos cnt]$, (3) gives $B = 0$ and hence $\bar{\Psi} = A\cos cnt$

Initial condition $\Psi(r, o) = f(r) \Rightarrow F[\Psi(r, o)] = F\{f(r)\}$

i.e. $$\bar{\Psi}(n, o) = \int_{-\infty}^{\infty} f(r)\, e^{-inr}\,dr = \bar{f}(n) \qquad ...(6)$$

where $f(n)$ is Fourier tansform of $f(r)$ *i.e.*

$$f(r) = \frac{1}{2}\int_{-\infty}^{\infty}\bar{f}(n)\, e^{-inr}\,dr \qquad ...(7)$$

So that using (6)

(5) $\Rightarrow \left[\bar{\Psi}\right]_{t=0} = A = \bar{f}(n)$

$$\Rightarrow \bar{\Psi} = F\{\Psi(r, t)\} = \bar{f}(n)\cos cnt \quad ...(8)$$

Inverse Fourier transform follows:

$$\Psi(r, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\Psi} e^{-inr}\, dn = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(n)\cos cnt\, e^{-inr}\, dn$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(n)\left[\frac{e^{inct} + e^{-inct}}{2}\right] e^{-inr}\, dr$$

$$= \frac{1}{2}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(n)\, e^{in(r+ct)}\, dt + \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(n)\, e^{in(r-ct)} dr\right]$$

$$= \frac{1}{2}\, [f(r + ct) + f(r - ct)]$$

Problem 30. *Establish the equation for heat conduction* $k\nabla^2 T = \frac{\partial T}{\partial t}$.

A slender rod whose curved surface is perfectly insulated stretches from $x = 0$ *to* $x = \infty$. *Find the temperature in the rod as a function of x and t if the left end of the rod is maintained at the constant temperature 0°C and if initially the temperature along the rod is given by* $T(x, 0) = f(x)$. *(Bombay, 1965)*

Problem 31. *Solve the partial differential equation for the vibrations of a square elastic membrane fixed at the edges,* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$.

The length of each side of the square is L, c is the velocity of elastic waves and u(x, y, z) is the displacement of the point (x, y) at time t normal to the plane of the membrane.

Obtain the lowest frequency of vibration if c = 10,000 metres per sec. and $L = \sqrt{2}$ *metres.*

(Agra, 1967)

Problem 32. *Solve the differential equation* $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} + \lambda^2 \psi = 0$ *in which* λ *is a constant and the function* ψ *exists only in the region defined by* $0 \le x \le l_1;\ 0 \le y \le l_2;\ 0 \le z \le l_3$

(Vikram, 1967)

Problem 33. *Solve the two-dimensional Lapalce's equation* $\nabla^2 \phi = 0$ *in the first quadrant i.e.,* $0 \le x \le \infty$ *and* $0 \le y < \infty$, *with the boundary condition that on the x and y axes* $\phi = \phi$.

(Agra, 1968; Vikram, 1969)

Problem 34. *Express* ∇^2 *in cylindrical and spherical polar coordinates.*

(Vikram, 1969)

Problem 35. *Write down the general wave equation. Solve it for a rectangular elastic membrane under uniform tension and fixed at the boundaries. What are the different modes under which it can vibrate?* *(Agra, 1969)*

Problem–36. *Obtain the solution for a guided electromegnetic wave through an infinitely long cavity of rectangular cross-section and discuss the limitations on the propagation wave length.* *(Agra, 1969, 1970)*

Problem 37. *Obtain the partial differential equation for the heat conduction in a homogeneous medium.*

i.e. four faces of an infinitely long rectangular prism bounded by the planes $x = 0$, $x = a$, $y = 0$ and $y = b$ are kept at temperature zero. If the initial temperature distribution is $\theta_0(x, y)$, find an expression for the temperature at a point in terms of x, y and the time t,

(*Bombay, 1970*)

Problem 38. *What is meant by 'circular membrane'? Find a general expression for the same.*

(*Bombay, 1970*)

Problem 39. *Find an expression for the velocity of transverse waves along a stretched membrane.*

A circular membrane is under a uniform tension. Considering the wave equation, discuss its vibrations.

Problem 40. *Deduce the Fourier equation for one-dimensional flow of heat along a solid bar. If one end of the bar is periodically heated and cooled and the bar is covered with heat insulating material, find the temperature changes along any point of the bar.*

(*Agra, 1970*)

Problem 41. *Obtain the axially symmetrical solutions of the three-dimensional Laplace's equation.*

(*Agra, 1970, 74*)

Problem 42. *Calculate the potential due to gravitating ring of uniform density at any point of its axis.*

(*Agra, 1970*)

Problem 43. (*a*) *Explain the significance of spherical harmonics and discuss some general properties of harmonic functions.*

(*b*) *Determine the potential outside and inside a spherical surface which is kept at a fixed distribution of electrical potential of the form $V = F(\theta)$. It is assumed that the space inside and outside the surface is free of charges.* (*Agra, 1971*)

Problem 44. *Solve the one-dimensional wave equation $\frac{\partial^2 f}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$ for a string of length l fixed at both ends. The boundary conditions are*

$$\{f(x, t)\}_{t=0} = \frac{2hx}{l}, \quad 0 \le x < \frac{l}{2}$$

$$= \frac{2h}{l}(l - x), \quad \frac{l}{2} \le x < l.$$

Discuss the nature of the solution. (*Agra, 1972*)

Problem 45. *Write Laplace's equation in the spherical polar coordinates. Solve the equation by the mthod of separation of variables in the case of cylindrical symmetry. Use this solution to obtain the gravitational potential of a uniform ring of radius a and mass m. Assume the ring to be made of thin wire.*

(*Agra, 1972, 74*)

Problem 46. *Solve the heat diffusion equation $h^2\nabla^2\theta = \frac{\partial\theta}{\partial t}$ for a semi-infinite plate bounded by the lines $x = 0$, $x = \infty$; $y = 0$ and $y = b$. θ is the temperature, t the time and h is constant. The boundary conditions are prescribed by*

$$\left.\begin{aligned} \theta(x, y, t) &= \theta(y) \text{ for } x = 0 \\ \theta(x, y, t) &= 0 \text{ for } y = 0 \text{ and } y = b \end{aligned}\right\} \quad 0 \le t \le \infty.$$

(*Agra, 1973*)

Problem 47. *Define the magnetic vector potential* **A** *and show that any electromagnetic field can be derived from this potential* **A** *and a scalar potential ϕ. Prove that the two potentials satisfy inhomogeneous wave equations of the same form.*

(*Agra, 1973*)

Problem 48. *Write Laplace's equation in cartesian, cylindrical and spherical coordinates. Solve Laplace's equation for the spherically symmetric case.*

(Agra, 1975, 1976)

Hint. In addition to the solution of Laplace's equation discussed in this chapter, see relevant sections alongwith Problem 180 of Ch. 1.

Problem 49. *Solve the wave equation for electromagnetic waves in a cylindrical wave guide. Discuss the solution, its significance and applications.* *(Agra, 1975)*

The electromagnetic wave equation is

$$\nabla^2\psi-\mu\varepsilon\frac{\partial^2\psi}{\partial t^2}-\mu\sigma\frac{\partial\mu}{\partial t}=0 \qquad \ldots(1)$$

where μ is permeability, σ is conductivity and ε is inductive capacity. By §1.45, in cylindrical coordinates (r, θ, z), we have

$$\nabla^2\psi=\frac{\partial^2\psi}{\partial t^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}+\frac{\partial^2\psi}{\partial z^2}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}+\frac{\partial^2\psi}{\partial z^2} \qquad \ldots(2)$$

$\therefore$ (1) can be written as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}+\frac{\partial^2\psi}{\partial z^2}-\mu\varepsilon\frac{\partial^2\psi}{\partial r^2}-\mu\sigma\frac{\partial\psi}{\partial t}=0 \qquad \ldots(3)$$

The elementary harmonic solutions of this equation may be expressed in the form

$$\psi=F(r,\theta)\,e^{\pm ihz-i\omega t} \qquad \ldots(4)$$

where $F(r, \theta)$ is a solution of

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F_1}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2}+\left(k^2-h^2\right)F=0 \qquad \ldots(5)$$

which may be easily separated by writing

$$F(r,\theta)=F_1(r)\,F_2(\theta)$$

where $F_1(r)$ and $F_2(\theta)$ are arbitrary solutions of the ordinary differential equations

$$r\frac{d}{dr}\left(r\frac{dF_1}{dr}\right)+\left\{\left(k^2-h^2\right)r^2-n^2\right\}F_1=0. \qquad \ldots(7)$$

and

$$\frac{d^2F}{d\theta^2}+n^2F_2=0 \qquad \ldots(8)$$

where n and h are separation constants and may be chosen in accordance with the physical requirements.

The solutions of equation (7) which is satisfied by radial function $F_1(r)$ being Bessel's functions which are generally reserved for that particular solution $J_n\left(r\sqrt{k^2-h^2}\right)$ which is finite on the axis $r=0$, we use the name *circular cylinder function* for any particular solution of (7) and denote it by $F_1=Z_n\left(r\sqrt{k^2-h^2}\right)$, where n is the order of the function with argument $r\sqrt{k^2-h^2}$.

As such the particular solutions of (1), being periodic in t and θ, may be constructed from elementary waves of the form

$$\psi_m=e^{im\theta}Z_m\left(r\sqrt{k^2-h^2}\right)e^{\pm ihz-i\omega t}$$

where the propagation constant h is a complex number.

It follows from (9) that an explicit expression for h in terms of frequency ω and medium-constants can only be found by observing the behaviour of ψ over a cylinder r = constant or on a plane z = constant.

Now to study the properies of the function $Z_n\left(r\sqrt{k^2-h^2}\right)$, putting $\lambda = r\sqrt{k^2-h^2}$, we observe that $Z_n(\lambda)$ satisfies the equation

$$\frac{d^2Z_n}{d\lambda^2}+\frac{1}{\lambda}\frac{dZ_n}{d\lambda}+\left(1-\frac{n^2}{\lambda^2}\right)Z_n=0 \qquad ...(10)$$

which is characterized by a regular singularity at $\lambda = 0$ and an essential singularity at $\lambda = \infty$.

Clearly the cylinder function or the Bessel function of first kind *i.e.*, $J_n(\lambda)$ is a particular solution of (10) and it is finite at $\lambda = 0$. As such Bessel function may be expanded in a series of ascending powers of λ and this series is convergent, for all finite values of the argument λ.

$$\text{Here } J_n(\lambda)=\sum_{s=0}^{\infty}\frac{(-1)^s}{\underline{|s}\,\Gamma(n+s+1)}\left(\frac{\lambda}{2}\right)^{n+2s} \qquad ...(11)$$

Another solution in terms of Bessel function of the second kind defined by

$$N_n(\lambda)=\frac{1}{\sin n\pi}\left[J_n(\lambda)\cos n\pi - J_{-n}(\lambda)\right] \qquad ...(12)$$

may also be obtained.

Problem 50. *Distinguish clearly between the phase and the group velocities. Find the relation between them.*

(Agra, 1975, 1976)

It is generally observed that the notion of phase velocity is only applicable to fields which are periodic in space *i.e.* , the fields representing wave trains of infinite duration. Denoting by $\psi(z, t)$ the state of medium, such that

$$\phi(z, t) = A\, e^{ikz-i\omega t} \qquad ...(1)$$

the surfaces of constant phase of state may be defined by

$$kz - \omega t = \text{constant}$$

and the velocity of propagation of these surfaces is given by

$$v=\frac{dz}{dt}=\frac{\omega}{k} \text{ by (2)} \qquad ...(3)$$

where v is termed as *phase velocity*.

Now to discuss the concept of group velocity; consider first the superposition of two harmonic waves differing very sligthly in frequency and wave number, such that

$$\psi_1 = \cos(kz - \omega t) \qquad ...(4)$$

$$\psi_2 = \cos[(k+\delta k)z - (\omega+\delta\omega)t] \qquad ...(5)$$

The resultant of (4) and (5) is

$$\psi = \psi_1 + \psi_2 = \cos(kz-\omega t) + \cos[(k+\delta k)z - (\omega+\delta\omega)t]$$

$$= 2\cos\tfrac{1}{2}(z\delta k - t\delta\omega)\cos[(k+\tfrac{1}{2}\delta k)z - (\omega + \tfrac{1}{2}\varepsilon\omega)t] \qquad ...(6)$$

which is an expression for the phenomenon of '*beats*'.

It follows from (6) that the field oscillates at a frequency negligibly different from ω, with its effective amplitude

$$a = 2\cos\frac{1}{2}(z\delta k - t\delta\omega) \qquad \ldots(7)$$

varying slowly between the algebraic sum of amplitudes of the component-waves and zero.

Due to constructive and destructive interference, the field distribution along time and space axes forms periodically repeated '*beats*' or '*groups*', whence the surfaces over which the group amplitude 'a' is constant are given by

$$z\delta k - t\delta\omega = \text{constant}$$

yielding the *group velocity* as

$$u = \frac{z}{t} = \frac{\delta\omega}{\delta k} \qquad \ldots(8)$$

which follows that group velocity is the ratio of the difference of frequency to the difference of wave number.

In case the medium is non-dispersive, $\delta k = \frac{1}{v}\delta\omega$, so that the group velocity coincides with the phase velocity v. Of course, in a dispersive medium the group velocity and the phase velocity are quite different.

Conclusively the group velocity u differs from the phase velocity v in a dispersive medium, and if the dispersion is normal then $u < v$ while if the dispersion is anomalous then $u > v$. Also in the neighbourhood of an absorption band the group velocity u may become infinte or even negative.

Problem 51. *Find the solution of the equation*

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = -\phi(x, y, z)$$

where $\phi(x, y, z)$ is a known function. Discuss the uniqueness of the solution.

(*Agra, 1976*)

Hint. For its solution see §1.59 where ρ has been regarded as a function of x, y, z.

Problem 52. *Obtain the differential equation for the vibrations of Sonometer string under tension, solve the equation and give the expression for the velocity of the waves in the string.*

(*Agra, 1976*)

Problem 53. (*a*) *Discuss the modes of vibrations of a circular membrane.*

(*b*) *Solve the equation,*

$$\frac{\partial V}{\partial t} = h^2\frac{\partial^2 u}{\partial x^2}$$

subject to initial conditions $v = F(x)$ at $t = 0$ (*Rohilkhand, 1976*)

Problem 54. *Using the methd of separation of variables, solve Laplace equation to find the temperature inside and outside a sphere when its boundary is at the finite temperature $\psi = f(\theta)$*

Find out the permanent temperature within a solid sphere of radius unity when one half of the surface of the sphere is kept at constant temperature 0°C and the other half of its surface is at 1°C

(*Rohilkhand, 1977, 90*)

See §12.6, 12.7 and 12.8 (see Problem 21).

Problem 55. *Derive the differential equation for the vibrations of a circular membrane and by solving it show that the allowed angular frequencies of vibrations are determined by the equation* $J_n\left[a\left(\frac{m\omega^2}{T}\right)^{1/2}\right] = 0,\ n = 0, 1, 2, \ldots$

where 'a' is the radius of the membrane, m is the mass per unit area of the membrane *and* T *is the tension of unit length.*

(*Rohilkhand, 1978, 90, 92*)

Problem 56. *Discuss the method of separation of variables for solving two-dimensional Laplace equation in spherical coordinate system and hence find the distribution of the electro-static potential produced by conducting ring carrying a total electric charge.*

(*Rohilkhand, 1978*)

See §12.7[B].

Problem 57. *Solving wave equation explain how the displacements at different points of a trensverely vibrating stetched string are obtained.*

If a string is plucked at a distance equal to one-fourth of its length from one end by the amount 'a' find the displacement of the string.

(*Rohilkhand, 1979, 91*)

Hint: See § 12.10. [*A*] and the problem 24.

Problem 58. *Find the solution of Poisson equation in term of Green's function. Using Fourier transform show that the Green's function satisfying the non-homogeneous equation* $(\nabla_1^2 + k_0^2)\, G(\mathbf{r}_1, \mathbf{r}_2) = -\,\delta(\mathbf{r}_1, \mathbf{r}_2)$ is $G(\mathbf{r}_1, \mathbf{r}_2) = \dfrac{1}{(2\lambda)^3} \displaystyle\int e^{i\mathbf{k}\cdot} \dfrac{(\mathbf{r}_1 - \mathbf{r}_2)}{k^2 - k_0^2}\, d\mathbf{k}.$

(*Rohilkhand, 1979, 87, 90*)

Hint: See §12.11

Problem 59. *Transform the two-dimensional eqaution of heat conduction to palar coordinates.*

(*Rohilkhand, 1980*)

Hint: See § 12.6

Problem 60. (*a*) *Write a short note on the uses of Green's functions.* (*b*) *Find the complete Green's function required to solve the equation :* $\dfrac{d^2y}{dx^2} + k^2 y = f(x),\ 0 \le n \le L$ *with the boundary conditions* $y(o) = y(L) = 0$

(*Rohilkhand, 1981, 84*)

Problem 61. (*a*) *Solve the Laplace equation is Sperical polar coordinates* (*b*) *Discuss the properties of spherical Harmonics.* (*Rohilkhand, 1981, 86*)

Problem 62. *What is Green's function? Solve the differential equation for the motion of a simple harmonic oscillator under the influence of a given time-dependent external force using the technique of Green's function.*

(*Rohilkhand, 1982*)

Problem 63. *Obtain the solution for a vibrating string obeying the equation* $\dfrac{\partial^2 u(x,t)}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2 u(x,t)}{\partial t^2},\ -l < x < l,$ *with the boundary conditions:*

$$u(-l, t) = u(l, t) = 0 \text{ for all } t$$

$$u(x, 0) = \begin{cases} k(l - x), & 0 < x < l \\ k(l + x), & -l < x < 0 \end{cases}$$

(*Rohilkhand, 1983, 84, 87*)

Problem 64. (*a*) *Find the solution of Poisson equation using Green's function technique.*

(*b*) *Find the solution of the equation* $\nabla^2\psi = Ar + B$, *inside a sphere* $r < a$, *if the boundary condition* $\psi\,|_{r=a} = 0$ *is satisfied on the sphere.*

(*Rohilkhand, 1983*)

Problem 65. *Use Green's function to obtain the general solution of the differential equation:*

$L(y) = \dfrac{d^2y}{dx^2} + k^2y = x$ *in the range* $0 \le x \le 1$. (*Rohilkhand, 1984*)

Problem 66. *Show that the differential equation for the vibration (in z-plane) of a circular membrane in (x-y plane) is given by* $\dfrac{\partial^2 z}{\partial t^2} = c^2\left(\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2}\right)$ *where* $c = T/\rho$, ρ *being the surface density and T is the uniform tension acting on the membrane.*

Hint: See §12.10 [*B*] (*Rohilkhand, 1985*)

Problem 67. (*a*) *If a string is plucked at its mid-point by the displacement h, find the displacement at other point.*

(*b*) *Find the characteristic frequency of sound vibrations in a rectangular box of sides a, b and c.* (*Rohilkhand, 1992*).

Problem 68. If S_n *and* S_m *are zonal spherical harmonics, then prove that*

$$\iint_V S_m S_n \, dS = 0, \; n \ne m. \qquad (\textit{Meerut, 1977})$$

Hint: Green's theorem is

$$\iiint_V (\psi\nabla^2\phi - \phi\nabla^2\psi)\, dV = \iint_S (\psi\nabla\phi - \phi\nabla\psi)\cdot d\mathbf{S} \qquad ...(1)$$

Setting $\psi = r^m S_m$, $\phi = r^n S_n$ with $\nabla^2\psi = 0$, $\nabla^2\phi = 0$. we have

L.H.S. of (1) = 0.

Consider the surface of a unit shpere in Green's theorem, $(\nabla\psi)_s = \dfrac{\partial}{\partial r}(r^m S_m) = mr^{m-1} S_m = m S_m$ at $r = 1$ and $(\nabla\phi)_s = nS_n$ also $\psi = S_m$, $\phi = S_n$ at $r = 1$. Hence $(\nabla\psi)_s$ and $(\nabla\phi)_s$ being directed normal to the sphere,

$$\nabla\psi \cdot d\mathbf{S} = \nabla\psi dS, \; \nabla\phi \cdot d\mathbf{S} = \nabla\phi\, dS$$

$$\therefore (1) \Rightarrow \iint_S (S_m \cdot n dS_n - S_n \cdot m\, dS_m) = 0 \Rightarrow (n-m)\iint_S S_m S_n\, dS = 0$$

$$\Rightarrow \iint S_m S_n \, dS = 0 \text{ for } n \ne m.$$

Problem 69. (*a*) *A transversely vibrating string of length l is stretched between two points A and B the initial displacement of each point of the string is zero, and the initial velocity at a distance x from A is kx (l − x). Find the displacement of the string at any subsequent time.*

(*b*) *Define Green function. Find the solution of the equation* $\nabla^2\phi = -\rho(r)$ *using Green's functions.* (*Rohilkhand, 1993*)

Problem 70. (*a*) *Discuss the propagation of electro-magnetic waves is a rectangular wave guide and derive an expression for cut-off frequency.*

(*b*) *Define replection coefficient and 'voltage standing wave ratio' for a two-conductor transmission line and obtain a relation between them.* (*Rohilkhand, 1993*)

+B

CHAPTER 13

MAXWELL'S ELECTROMAGNETIC FIELD EQUATIONS

13.1. INTRODUCTION

It was due to Clark Maxwell that a change in the electric displacement vector was supposed to be equivalent to an electric current known as *displacement current*. The total flow obtained by the resultant of *displacement current* and *conduction current* (which arises out of the actual motion of electrons or electric charges) was supposed to be such that the total flow into any surface would invariably be zero.

In fact the equation of electromagnetic field have not yet been finally established, but we have used the fact that when there is a flow of steady current, then

$$\text{Curl } \mathbf{H} = 4\pi \mathbf{j} \quad \ldots(1)$$

where $\dot{\mathbf{H}}$ is the negative field and $\mathbf{j}$ is the current vector. In case of those electromagnetic fields where the flow of current is unsteady, the equation (1) is insufficient and requires modification for unsteady flow of currents.

Since we know that div curl $\mathbf{H} = 0$ ...(2)

$\therefore$ (1) yields div $\mathbf{j} = 0$ ...(3)

which is always true in case of steady flow of currents.

But in case of unsteady flow of currents when generally div $\mathbf{j} \neq 0$, the equation of continuity of charge taken in mixed units, is given by

$$\text{div } \mathbf{j} = -\frac{1}{c}\frac{\partial \rho}{\partial t} = -\frac{1}{4\pi c}\left(\frac{\partial}{\partial t} \text{ div } \mathbf{D}\right) \text{ where div } \mathbf{D} = 4\pi\rho \quad \ldots(4)$$

$$\text{div}\left(\mathbf{j} + \frac{1}{4\pi c}\frac{\partial \mathbf{D}}{\partial t}\right) = 0 \quad \ldots(5)$$

$$\text{div } \mathbf{J} = 0 \text{ where } \mathbf{J} = \mathbf{j} + \frac{1}{4\pi c}\frac{\partial \mathbf{D}}{\partial t} \quad \ldots(6)$$

Here ρ is the free charge density at a scalar point, $\mathbf{D}$ is the Maxwell's electric displacement vector and $\mathbf{J}$ is called *Maxwell's total current.*

Since the operators div and $\frac{\partial}{\partial t}$ are commutative, it therefore follows that $\mathbf{J}$ is a quantity whose divergence is always zero and in case of steady current flow $\mathbf{J}$ reduces to $\mathbf{j}$. Conclusively in (1), j should be replaced by $\mathbf{J}$ in order that the result (1) will be valid for steady as well as unsteady current flows. We call $\mathbf{j}$ as *conduction current* and the part $\frac{1}{4\pi c}\frac{\partial \mathbf{D}}{\partial t}$ of $\mathbf{J}$ as *displacement current.*

If $\mathbf{P}$ be the *polarisation* of the medium, then we have

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}, \text{ E being electric field} \qquad ...(7)$$

As such (6) yields $\mathbf{J} = \mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{P}}{\partial t} + \frac{1}{4\pi c}\frac{\partial \mathbf{E}}{\partial t}$ ...(8)

The part of $\frac{1}{4\pi c}\frac{\partial \mathbf{D}}{\partial t}$ coming from $\frac{\partial \mathbf{P}}{\partial t}$ is known as *Polarisation current* and that from $\frac{1}{4\pi c}\frac{\partial \mathbf{E}}{\partial t}$ the *ether displacement current* or *displacement current in vaccum.*

The modification of (1) supported by experiments and the electromagnetic theory of light based on Maxwell's electromagnetic field equations, is given a sound theoretical basis by replacing $\mathbf{j}$ by $\mathbf{J}$ in (1) whence we get

$$\text{Curl } \mathbf{H} = 4\pi\mathbf{J} = 4\pi\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} \qquad ...(9)$$

where $\mathbf{j}$ and $\mathbf{H}$ are measured in e.m.u. and $\mathbf{D}$ in e.s.u

and $c = 3 \times 10^{10}$ cm/sec.

Here the term $\frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$ assumes importance only if $\frac{\partial \mathbf{D}}{\partial t}$ is small as compared to c, otherwise it may be neglected if $\frac{\partial \mathbf{D}}{\partial t}$ is small as compared to c. Those unsteady states in which the last term in R.H.S. of (9) may be neglected are known as *quasi-unsteady states.* Dealing with very high frequencies as compared to that of light, the displacement current has considerable significance.

If we now introduce $\mathbf{B}$ as magnetic induction of the medium, given by

$$\mathbf{B} = \mu\mathbf{H} \qquad ...(10)$$

then (9) reduces to curl $\mathbf{B} = 4\pi\mu\mathbf{j} + \frac{\mu}{c}\frac{\partial \mathbf{D}}{\partial t}$ ...(11)

Using Gauss's, divergence theorem, it follows from (5),

$$\oint\left(\mathbf{j} + \frac{1}{4\pi c}\frac{\partial \mathbf{D}}{\partial t}\right)\cdot d\mathbf{S} = 0 \qquad ...(12)$$

13.2. MAXWELL'S EQUATIONS FOR ELECTROMAGNETIC FIELD

As already discussed in §13.1, taking Gaussian units or the mixed system of units such that the quantities $\mathbf{D}$, $\mathbf{E}$, ρ with allied quantities are measured in e.s.u. and the quantities $\mathbf{B}$, $\mathbf{H}$, $\mathbf{j}$ with allied quantities in e.m.u, the differential or local form of the electromagnetic field equations of Maxwell in accordance with the elementary laws of electricity and magnetism laid by Gâuss, Faraday, Ampere and others can be summarised as follows:

Divergence equations
$$\text{div } \mathbf{D} = \nabla \cdot \mathbf{D} = 4\pi\rho \qquad ...(1)$$
$$\text{div } \mathbf{B} = \nabla \cdot \mathbf{B} = 0 \qquad ...(2)$$

Circuital equations
$$\text{curl } \mathbf{H} = \nabla \times \mathbf{H} = 4\pi\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} \qquad ...(3)$$
$$\text{curl } \mathbf{E} = \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad ...(4)$$

where the equation (4) is the generalized law of induction.

These equations are considered in association of the following *subsidiary relations* or macroscopic constitutive equations

$$\mathbf{D} = K\mathbf{E} \quad ...(5)$$

$$\mathbf{B} = \mu\mathbf{H} \quad ...(6)$$

$$\mathbf{j} = \sigma\mathbf{E} \quad ...(7)$$

where K is the dielectric constant, μ is the permeability of the medium and σ is the conductivity of the medium. Here $\mathbf{j}, \sigma, \mathbf{E}$ are measured in the mixed units, but if $\mathbf{j}$ is in e.m.u. and σ, $\mathbf{E}$ in e.s.u. then we have $\mathbf{j} = \frac{\sigma\mathbf{E}}{c}$, also if $\mathbf{j}, \sigma$ are in e.m.u. and $\mathbf{E}$ in e.s.u, then $\mathbf{j} = c\sigma\mathbf{E}$.

In addition to the above, few other quantities such as the scalar potential ϕ and magnetic vector potential $\mathbf{A}$ defined by

$$\mathbf{B} = \text{curl } \mathbf{A} \quad ...(8)$$

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi \quad ...(9)$$

are also required.

Since (8) does not provide the unique definition of $\mathbf{A}$, therefore we supplement it by div $\mathbf{A} = 0$ (in steady states) and

$$\text{div } \mathbf{A} + \frac{\mu}{c} K \frac{\partial \phi}{\partial t} = 0 \quad ...(10)$$

Here ϕ is in e.s.u. and $\mathbf{A}$ in e.m.u.

13.3. TO SHOW $\nabla \cdot \mathbf{D} = 4\pi\rho$.

We know that the molecules of dielectric substance are formed of charged entities, such as atomic nucleii and electrons and hence when dielectric substances are inserted in the field of electromagnetic waves, the charged entities are displaced in opposite directions of their mean position *i.e.*, the dielectric is said to be polarised. Such polarised charges form an electric dipole. Let $\mathbf{r}$ be the separation between positive and negative charges in each dipole and δq the magnitude of each displaced charge; then the electric moment say $d\mathbf{p}$ of dipole being $\mathbf{r}\,\delta q$ is defined over the elementary volume δV as

$$d\mathbf{p} = \int_{\delta V} \mathbf{r}\, dq \quad ...(1)$$

Let us introduce the polarisation vector $\mathbf{P}$ as the resultant of the electric dipole moment of molecules per unit volume *i.e.*,

$$\mathbf{P} = \lim_{\delta V \to 0} \frac{d\mathbf{p}}{dV} \quad ...(2)$$

Now representing the charge density of the charge caused by polarisation of the dielectric, by ρ_P, the charge carried out of the surface S enclosing the volume V in the electric field of $\mathbf{E}$, is given by $-\int_V \rho_P\, dV$. Also the net charge diverging out of the elementary volume δV in this dielectric being div $\mathbf{P}.\delta V$, the charge carried out of the volume V is $\int_V$ div $\mathbf{P}\, dV = 0$. We thus have

$$\int_V \text{div } \mathbf{P}\, dV = -\int_V \rho_P\, dV \quad ...(3)$$

Using Gauss's divergence theorem, this yields,

$$\int_S \mathbf{P}\cdot\mathbf{n}\; dS = -\int_V \rho_P\, dV \qquad ...(4)$$

where n is the unit outward drawn normal to the surface element δS.

But according to Guass's theorem of electrostatics, the total normal outward electric field flux across any arbitrary closed surface is 4π time the charge enclosed by it *i.e*,

$$\int_S \mathbf{E}\cdot\mathbf{n}\; dS = 4\pi.\ \text{(total charge enclosed in volume } V) \qquad ...(5)$$

Now, there are two types of charges inside V.

(*i*) the free charge of density ρ and (*ii*) the polarised charge of density ρ_P.

$$\therefore\ \int_S \mathbf{E}\cdot\mathbf{n}\; dS = 4\pi\left[\int_V \rho\, dV + \int_V \rho_P\, dV\right]$$

$$= 4\pi\left[\int_V \rho\, dV - \int_V \mathbf{P}\cdot\mathbf{n}\; dS\right] \text{by (4)}$$

or $$\int_S (\mathbf{E}+4\pi\mathbf{P})\cdot\mathbf{n}\; dS = 4\pi\int_V \rho\, dV$$

or $$\int_S D\cdot\mathbf{n}\; dS = 4\pi\int_V \rho\, dV \text{by (7) of §13.1} \qquad ...(6)$$

Hence by Gauss's divergence theorem $\int_V \text{div}\,\mathbf{D}\; dV = \int_V 4\pi\rho\, dV$ which yields ...(7)

$$\text{div}\,\mathbf{D} = 4\pi\rho. \qquad ...(8)$$

Note 1. *In case of an ordinary dielectric i.e. a dielectric free from charge,* $\rho = 0$ *and hence div* $\mathbf{D} = \nabla\cdot\mathbf{D} = 0$ ...(9)

Note 2. *By (7) of §13.1, we have* $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} = \mathbf{E}\left[1 + 4\pi\frac{\mathbf{P}}{\mathbf{E}}\right]$

which follows that $\mathbf{D} = K\mathbf{E}$ ...(10)

where $K = 1 + 4\pi\,\frac{\mathbf{P}}{\mathbf{E}}$ and known as dielectric constant of the medium.

13.4. TO SHOW DIV B = 0

Since the lines of force are either closed or go off to infinity, therefore the surface integral of magnetic induction B over a closed surface is zero, so that (6) of §13.3 reduces here

$$\int_S \mathbf{B}\cdot\mathbf{n}\; dS = 0 \text{ or div } \mathbf{B} = \nabla\cdot\mathbf{B} = 0 \qquad ...(1)$$

Note. Since $\mathbf{B} = \mu\mathbf{H}$ $\therefore$ (11) yields $\nabla\cdot\mathbf{H} = 0$ ...(2)

13.5. TO SHOW CURL E $= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$.

According to Faraday's law of electromagnetic induction the induced E.M.F. around a closed circuit is negative times the rate of change of total flux ϕ of the magnetic induction B through the circuit. Induced E.M.F.

$$= -\frac{1}{c}\frac{\partial\phi}{\partial t} \qquad ...(1)$$

Also if E be the electric field in the direction $d\mathbf{l}$, then

$$\text{E.M.F.} = \Sigma \mathbf{E}.d\mathbf{l} \qquad ...(2)$$

If we consider an elementary rectangle $PQRS$ with sides δy and δz as shown in Fig. 13.1, then around it we have

$$\text{E.M.F.} = \frac{\text{work}}{\text{charge}} = \Sigma \mathbf{E} \cdot d\mathbf{l}$$

$$= [E_z(y+dy) - E_z(y)]\,dz - [E_y(z+dz) - E_y(z)]dy$$

$$= \frac{\partial E_z}{\partial y} dy\,dz - \frac{\partial E_y}{\partial z} dy\,dz \qquad \text{...(3)}$$

$$\because E_z(y+dy) = E_z(y) + \frac{\partial E_z}{\partial y} dy$$

by Taylor's theorem

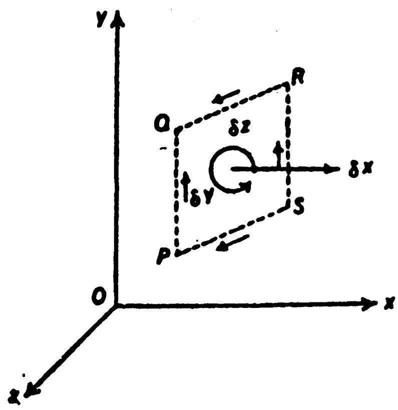


Fig. 13.1

$$\text{Also } \Sigma\, \mathbf{E} \cdot d\mathbf{l} = -\frac{1}{c}\frac{d\phi_x}{dt} = -\frac{1}{c}\frac{\partial B_x}{dt} dy\,dz \qquad \text{...(4)}$$

Hence from (3) and (4) equating the two values of E.M.F., we have

$$-\frac{1}{c}\frac{\partial B_x}{\partial t} dy\,dz = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = \text{curl}_x \mathbf{E}$$

$$\text{Similarly } -\frac{1}{c}\frac{\partial B_y}{\partial t} = \text{curl}_y \mathbf{E} \text{ and } -\frac{1}{c}\frac{\partial B_z}{\partial t} = \text{curl}_z \mathbf{E}$$

where B_x, B_y, B_z are components of **B** along principal axes.

Combining the last three relations we can write

$$-\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \text{curl } \mathbf{E} \text{ i.e., } \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad \text{...(5)}$$

Note. In case of free space when $\mu = 1$ and $\mathbf{B} = \mathbf{H}$, (5) yields

$$\text{Curl } \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} \qquad \text{...(6)}$$

13.6. PHYSICAL INTERPRETATION OF MAXWELL'S EQUATIONS

Consider div $\mathbf{D} = 4\pi\rho$, which may be written as

$$\int \text{div } \mathbf{D}\, dV = 4\pi\int \rho\, dV \text{ i.e., } \int_S \mathbf{D}\cdot d\mathbf{S} = 4\pi\int_V \rho\, dV$$ by Gauss' divergence theorem.

It follows, **Gauss's theorem** *i.e., the flux of the displacement vector* **D** *across any closed surface S is directly porportional to the total electric charge in V.*

The equation div $\mathbf{B} = 0$, may be written as

$$\int_S \mathbf{B}.d\mathbf{S} = 0, \text{ by Gauss's divergence theorem}$$

which follows: **Magnetic flux theorem** *i.e., the total normal magnetic induction across any closed surface S is zero*

Also the equation curl $\mathbf{H} = 4\pi\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$ can be expressed by Stoke's theorem, as

$$\int_S \text{curl } \mathbf{H}.\, d\mathbf{S} = 4\pi \iint \left(\mathbf{j} + \frac{1}{4\pi c}\frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S} \text{ or } \int_C \mathbf{H}.d\mathbf{l} = 4\pi \iint \left(\mathbf{j} + \frac{1}{4\pi c}\frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S}$$

which is **Ampere's generalized circuital relation** *i.e., the work done in carrying a unit pole round a closed circuit C is equal to 4π times the total current in the circuit.*

Lastly the equation curl $\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$ can be written by the help of Stoke's theorem, as

$$\int \text{curl } \mathbf{E}.d\mathbf{S} = \int \left(-\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \right).d\mathbf{S} \text{ or } \int_C \mathbf{E}.d\mathbf{l} = -\frac{1}{c}\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

which is **Generalized law of electromagnetic induction** *i.e., the total electromagnetic force around a closed circuit C is equal to* $-\frac{1}{c}$ *times the rate of change of magnetic induction through C.*

13.7. DECAYING OF FREE CHARGE

Consider curl $\mathbf{H} = 4\pi\mathbf{j} + \frac{\partial \mathbf{D}}{\partial t}$ and $\mathbf{j} = \sigma\mathbf{E}$ in e.s.u. or e.m.u. throughout.

Elimination of j yields, curl $\mathbf{H} = 4\pi\sigma\mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = -\frac{4\pi\sigma\mathbf{D}}{k} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$ $\because \mathbf{D} = K\mathbf{E}$

$$\therefore \text{div curl } \mathbf{H} = \text{div}\left(\frac{4\pi\sigma\mathbf{D}}{K} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} \right) = 0$$

which with the help of $\mathbf{D} = 4\pi\rho$, gives $\frac{4\pi\sigma\rho}{K} + \frac{\partial \rho}{\partial t} = 0$ or $\frac{d\rho}{\rho} = -\frac{4\pi\sigma}{K}dt$

It yields on integration, $\rho = \rho_0 e^{-t/T}$ ...(1)

where ρ_0 is the charge at any point $t = 0$ and $T = -\frac{K}{4\pi\sigma}$, T being known as *time of relaxation.*

It follows from (1) that the charge dies out exponentially at a rate quite independent of any other electromagnetic phenomena taking place simultaneously. In most ordinary case we take $\rho = 0$.

13.8. POYNTING VECTOR

Since the electromagnetic effects in a field are completely attributed to charges at rest or in motion and a charge moving in a field experiences a force so that the work is done on the moving charge by the forces of the field, we therefore claim to determine the rate of change of energy on account of the interaction of the field and the charges in it.

Let **E** be the electric field and **B** the magnetic induction at a point in the field occupied by a charge moving with velocity **v**. Also let ρ be the charge density and dV an element of volume at the point under consideration. Then the force on a charge $\rho\, dV$ (charge on the elementary volume), is $\left[\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right] \rho\, dV$.

The rate of work done on this charge $= \left[\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c}\right] . \mathbf{v}\ \rho\ dV$

$= (\mathbf{E}.\mathbf{v})\ \rho\ dV \because (\mathbf{v} \times \mathbf{B}).\ \mathbf{v} = 0.$

$\therefore$ the total rate of work done by the forces of the field in the region under consideration.

$$= \int_V (\mathbf{E}.\mathbf{v})\ \rho dV = c \int_V (\mathbf{E}.\mathbf{j}) dV \text{ in mixed units, since } \rho \mathbf{v} = \mathbf{j} c$$

$$= \int_V \mathbf{E}. \left(\frac{1}{4\pi} \text{ curl } \mathbf{H} - \frac{1}{4\pi c} \frac{\partial \mathbf{D}}{\partial t}\right) dV$$

$\because$ by Maxwell's equations. $\mathbf{j} = \frac{1}{4\pi} \text{ curl } \mathbf{H} - \frac{1}{4\pi c} \cdot \frac{\partial \mathbf{D}}{\partial t}$

$$= \frac{c}{4\pi} \int_V (\mathbf{E}.\ \text{curl } \mathbf{H}) dV \ - \frac{1}{4\pi} \int_V \left(\mathbf{E}. \frac{\partial \mathbf{D}}{\partial t}\right) dV$$

$$= \frac{c}{4\pi} \int_V \{\text{div } (\mathbf{H} \times \mathbf{E}) + \mathbf{H}.\ \text{curl } \mathbf{E}\}\, dV - \frac{1}{4\pi} \int_V \left(\mathbf{E}. \frac{\partial \mathbf{D}}{\partial t}\right) dV$$

$\because$ $\mathbf{E}.\ \text{curl } \mathbf{H} = \text{div } (\mathbf{H} \times \mathbf{E}) + \mathbf{H}.\ \text{curl } \mathbf{E}$

$$= \frac{c}{4\pi} \int_V \text{div } (\mathbf{H} \times \mathbf{E}) dV + \frac{c}{4\pi} \int_V (\mathbf{H} \cdot \text{ curl } \mathbf{E}) dV - \frac{1}{4\pi} \int_V \left(\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t}\right) dV$$

Hence using the Gauss's divergence theorem and Maxwell's equation curl $\mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}$, we have the rate of change of energy $= \frac{c}{4\pi} \int_s (\mathbf{H} \times \mathbf{E}).\ dS - \frac{1}{4\pi} \int_V$ $\left(\mathbf{H}. \frac{\partial \mathbf{B}}{\partial t} + \mathbf{E}. \frac{\partial \mathbf{D}}{\partial t}\right) dV$ ...(1)

Regarding μ and K as independent of time, we can write (1) as

$$-\frac{\partial}{\partial t} \int_V \frac{1}{8\pi} (\mathbf{B}.\mathbf{H} + \mathbf{E}.\mathbf{D}) dV = \frac{c}{4\pi} \int_S (\mathbf{E} \times \mathbf{X}).\ dS + \text{Rate at which field is doing work}$$

$\because \frac{\partial}{\partial t} (\mathbf{B}.\mathbf{H}) = 2\mathbf{H}. \frac{\partial \mathbf{B}}{\partial t}$ and $\frac{\partial}{\partial t} (\mathbf{E}.\mathbf{D}) = 2\mathbf{E}. \frac{\partial \mathbf{D}}{\partial t}$

We conclude from (2) that the electromagnetic energy of the field per unit volume must be taken as $\frac{1}{8\pi} (\mathbf{B}.\mathbf{H} + \mathbf{D}.\mathbf{E})$ ...(3)

Then the equation (2) expresses the principle of conservation of energy if the first term on the right of (2) represents a rate of flow of radiation-energy across S, the surface enclosing V.

The vector (say) $\vec{\Pi} = \frac{c}{4\pi} (\mathbf{E} \times \mathbf{H})$ ...(4)

is known as *Radiant vector* or *Poynting vector* and it gives the rate at which the energy is radiated across unit area.

It is notable that $\vec{\Pi}$ is normal to the plane of E and H and it is not unique since $\vec{\Pi}$ + A also satisfies (2) because $\nabla \cdot \mathbf{A} = 0$. Also the flux of Π across a closed surfacd is only singnificant.

13.9. POYNTING THEOREM

The rate at which the electric energy in any given region is increasing is equal to the Integral $\frac{c}{4\pi}\iint [l(\gamma Y-\beta Z)+m(\alpha Z-\gamma X)+n(\beta X-\alpha Y)]\,dS$ *taken over the boundary of the origin, l, m, n being the direction cosines of the inward normal to the surface element dS; α, β, γ, being the components of magnetic force and X, Y, Z being the components of electric force.*

Assuming that the electromagnetic energy is not confined to the regions occupied by electric charges, but magnets and currents spread over the whole space, the magnetic kinetic energy T and electric potential energy W of an isotropic medium are given by

$$T=\frac{1}{8\pi}\iiint \mu\left(\alpha^2+\beta^2+\gamma^2\right)\,dx\,dy\,dz \qquad \ldots(1)$$

$$W=\frac{1}{8\pi}\iiint K\left(X^2+Y^2+Z^2\right)\,dx\,dy\,dz \qquad \ldots(2)$$

Assuming that the energy is localized in a medium, the total energy in any closed region is,

$$T+W=\iiint\left\{\frac{K}{8\pi}\left(X^2+Y^2+Z^2\right)+\frac{\mu}{8\pi}\left(\alpha^2+\beta^2+\gamma^2\right)\right\}dx\,dy\,dz \qquad \ldots(3)$$

Differentiating it and replacing $\mu\alpha$, by α, KX by $4\pi f$, etc., we get

$$\frac{d}{dt}[T+W]=\iiint\left[\left(X\frac{\partial f}{\partial t}+Y\frac{\partial g}{\partial t}+Z\frac{\partial h}{\partial t}\right)+\frac{1}{4\pi}\left(\alpha\frac{d\alpha}{dt}+\beta\frac{d\beta}{dt}+\gamma\cdot\frac{d\gamma}{dt}\right)\right]dx\,dy\,dz$$

$$=\frac{c}{4\pi}\iiint\left[X\left(\frac{\partial\gamma}{\partial y}-\frac{\partial\beta}{\partial z}\right)+\ldots-\alpha\left(\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z}\right)-\ldots\ldots\right]dx\,dy\,dz$$

$$-c\iiint(uX+vY+wZ)\,dx\,dy\,dz \qquad \ldots(4)$$

where $$-\frac{1}{c}\frac{d\alpha}{dt}=\frac{\partial Z}{\partial t}-\frac{\partial Y}{dz}\text{ etc.,}$$

and $$4\pi\left(u+\frac{1}{c}\frac{\partial f}{\partial t}\right)=\frac{\partial\gamma}{\partial y}-\frac{\partial\beta}{\partial z}\text{ etc.}$$

Here u, v, w are components of ordinary current at any point, which is produced by the moving electric charges.

The last term on the right of (4) gives exactly the rate at which work is done or the energy is dissipated by the flow of currents so that the first part of this must represent the rate at which energy flows into the region from outside.

Hence using Green's theorem, the first part on the right of (4), yields

$$\frac{c}{4\pi}\iiint\left[\frac{\partial}{\partial x}(Z\beta-Y\gamma)+\frac{\partial}{\partial y}(Y\gamma-Z\alpha)+\frac{\partial}{\partial z}(Y\alpha-X\beta)\right]dx\,dy\,dz$$

$$=-\frac{c}{4\pi}\iint[l(Z\beta-Y\gamma)+m(Y\gamma-Z\alpha)+n(Y\alpha-X\beta)]\,dS.$$

Problem 1. *If* **Z** *be a vector satisfying the equation*

$$\nabla^2\mathbf{Z}=\frac{4\pi\sigma\mu}{c}\frac{\partial\mathbf{Z}}{\partial t}+\frac{K\mu}{c^2}\frac{\partial^2\mathbf{Z}}{\partial t^2} \qquad \ldots(1)$$

then show that the field may be defined by the equations:

$$A = 4\pi\sigma\mu Z + \frac{K\mu}{c}\frac{\partial Z}{\partial t} \quad ...(2)$$

$$\phi = -\text{div } Z, \quad ...(3)$$

$$E = \text{curl curl } Z, \quad ...(4)$$

$$B = \text{curl } A \quad ...(5)$$

We have from (2) and (3) $\text{div } A + 4\pi\sigma\mu\phi + \frac{K\mu}{c}\frac{\partial \phi}{\partial t} = 0$

which is one of the Maxwell's equations.

Again from (4), we have $E = \text{grad div } Z - \nabla^2 Z$

$$= -\text{grad } \phi - \frac{\partial}{\partial t}\left(\frac{4\pi\sigma\mu}{c} Z + \frac{K\mu}{c^2}\frac{\partial^2 Z}{\partial t^2}\right) \text{ by (1) and (3)}$$

$$= -\text{grad } \phi - \frac{1}{c}\frac{\partial A}{\partial t}$$

which is another Maxwell's equation.

Also curl B = curl (curl A) by (5)

$$= \text{curl curl}\left(4\pi\sigma\mu Z + \frac{K\mu}{c}\frac{\partial Z}{\partial t}\right) \text{by (2)}$$

$$= 4\pi\sigma\mu \ (\text{curl curl } Z) + \left(\frac{K\mu}{c}\right)\frac{\partial}{\partial t}(\text{curl curl } Z)$$

$$= 4\pi\sigma\mu \ E + \frac{K\mu}{c}\frac{\partial E}{\partial t} \text{ by (4)}$$

$$= 4\pi\mu j + \frac{\mu}{c}\frac{\partial D}{\partial t} \qquad \because \sigma E = j \text{ and } kE = D$$

This is also one of the Maxwell's equations.

Hence the given equations define the field under consideration.

Problem 2. *Initially the electric and magnetic fields in free space are $E_y(x, 0) = \sin ax$, $H_z(x, 0) = \sin ax$. Find E_y as function of time.*

We have discussed in the previous chapter that the most general solution of one-dimensional wave equation is

$$E_y(x, t) = f(x + ct) + g(x - ct) \quad ...(1)$$

$$E_z(x, t) = -f(x + ct) + g(x - ct) \quad ...(2)$$

The negative sign being taken since E and H the directions of propagation form a right handed system. This is also justified on applying Maxwell's equation curl $E = -\frac{1}{c}\frac{\partial H}{\partial t}$.

At time $t = 0$, we have $E_y, (0, t) = \sin ax = f(x) + g(x)$ by (1)

$E_z\ (0, t) = \sin ax = -f(x) + g(x)$ by (2)

solving $f(x) = 0$, $g(x) = \sin ax$.

Hence $E_y(x, t) = \sin a(x - ct)$ is the required solution.

Problem 3. *Show that a plane polarised electromagnetic wave with* H + $ke^{pt+ n.r.}$, *satisfies the equation* H = 0 *only if* n *is perpendicular to* K.

Take, $\mathbf{H} = H_1\mathbf{i} + H_2\mathbf{j} + H_3\mathbf{k}$

$$\mathbf{K} = K_1\mathbf{i} + K_2\mathbf{j} + K_3\mathbf{k}$$
$$\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$$
and
$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$\therefore \mathbf{n}\,.\,\mathbf{r} = n_1x + n_2y + n_3z \text{ and } \frac{\partial H_1}{\partial x} = \frac{\partial}{\partial x}\{K_1\, e^{i(pt+i\,\mathbf{n}.\mathbf{r})} = n_1K_1\, e^{i(pt+\mathbf{n}.\mathbf{r})}$$

and
$$\text{div } \mathbf{H} = \left(\mathbf{i}.\frac{\partial}{\partial x} + \mathbf{j}.\frac{\partial}{\partial y} + \mathbf{k}.\frac{\partial}{\partial z}\right)(H_1\mathbf{i} + H_2\mathbf{j} + H_3\mathbf{k})$$
$$= \frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} + \frac{\partial H_3}{\partial z} = e^{i(pt+\mathbf{n}.\mathbf{r})}\{n_1K_1 + n_2K_2 + n_3K_3\}$$
$$= \mathbf{n}.\mathbf{K}\; e^{i(pt+\mathbf{n}.\mathbf{r})}$$

div $\mathbf{H} = 0 \Rightarrow \mathbf{n}.\mathbf{K} = 0$ *i.e.*, vector **n** is $\perp^r$ to **K**.

ADDITIONAL MISCELLANEOUS PROBLEMS

Problem 4. *Show that Maxwell's equations for free space are satisfied by*

$$\mathbf{H} = \frac{1}{c}\frac{\partial}{\partial t}(\textit{grand}\ \phi \times \mathbf{k});\ \mathbf{E} = -\mathbf{k}\left(\frac{1}{c^2}\right)\frac{\partial^2\phi}{\partial t^2} + \frac{\partial}{\partial z}\ (\textit{grad}\ \phi)$$

where k is a unit vector along z-axis and ϕ *satisfies* $\nabla^2\phi = \frac{1}{c}\frac{\partial^2\phi}{\partial t^2}$.

Problem 5. *If* **k** *is a constant vector and* ϕ *is a scalar function of position and the time, show that* $c^2\,\mathbf{E} = c^2\,(\mathbf{k}.\nabla)\,\nabla\phi - \mathbf{k}\ddot{\phi}$; $c\,\mathbf{H} = \frac{\partial}{\partial t}\,(\nabla\phi \times \mathbf{k})$ *satisfy Maxwell's equations in free space provided that* $\ddot{\phi} = c^2\nabla^2\phi$.

CHAPTER 14

SPECIAL THEORY OF RELATIVITY

1.41. INTRODUCTION

We know that all the physical laws deal with the characteristics of certain objects in space in the course of time, while the location of an event or the position of a body is described by a suitable frame of reference which constitutes a conceptual framework rigidly connected with some material body or a well defined point. Since all the bodies cannot become suitable reference systems, therefore the choice of the reference body always played an important role in the development of science. The heliocentric frame of reference introduced by Galileo was not accepted until the time of Newton who gave a rather comprehensive presentation of it, with the help of his pail experiment. He filled a pail with water and twisted the rope supporting the pail, around its axis, so that the water gradually came into rotation with its plane surface shaped into a paraboloid. Ultimately when the water gained the same speed of rotation as the pail he stopped the pail so that it gradually and eventually came to rest to resume the shape of a plane. Evidently the surface of water is not influenced by the state of motion of the pail, but the deviation from a plane increases with the deviation from the particular state of motion. This whole process is based on the frame of reference connected with the pail while the angular velocity of the pail is related to a more suitable frame of reference say earth.

There are frames of reference known as *inertial system* with respect to which the law of inertial takes its familiar form *i.e.* in absence of forces, the space coordinates of a mass point are linear functions of time, whereas there exists another class of frames of reference in which the space coordinates are not linear functions of time. According to laws of mechanics all the frames of reference which are inertial systems are equivalent for describing the nature and for formulating its laws. Such a notion leads to the '**Principle of Relativity**'.

The development of Maxwell's electromagnetic field equations was found apparently incompatible with the principle of relativity, according to which electromagnetic waves in empty space should propagate with a constant velocity $c = 3 \times 10^{10}\,cm/sec$. But it was not found true with respect to both of the two different intertial systems moving relatively to each other. Consequently several experiments failed to find such a frame reference with respect to which the speed of electromagnetic radiation would be constant in all directions so that ***absolute rest, absolute motion*** etc., might be defined and with respect to which Earth's motion could be determined. One such an experiment is due to Michelson and Morley.

Lorentz conceived the existence of one privileged frame of reference which could not be demonstrated experimentally and he had to introduce several assumptions in the support of his theory. Finally Albert Einstein recognised that the revised fundamental concepts about space and time would be capable of resolving the impasse between theory and experiment. This is that revised concept of Einstein, which is now known as

Special theory of relativity and which establishes the fundamental equivalence of all interital systems. According to Einstein there is nothing like *absolute motion*, but all motions are relative and hence the physical laws are independent of the motion of observer. Einstein also ruled out the concept of '*absolute time*' and time veries from one inertial system ot another intertial system. He also ruled out the concept of '*absolute space*', since the time is not absolute and therefore the distance between two points measured in two intertial systems cannot be absolute as is evident from the **Galilean Transformation.**

Suppose S, S' are two frames of reference, one at rest and the other moving with uniform velocity v. O, O' ae two observers at the origins of S and S' respectively, observing the same event at P whose coordinates are (x, y, z, t) w.r.t. O and (x', y', y', t') w.r.t. O'; S, S' being supposed to be parallel to each other and their origins being coincident at $t = 0, t' = 0$. There arise two cases:

Case I. Let the frame S' have the velocity v w.r.t. S in the direction of X' only. Then O' will have velocity v along X'-axis only so that both X and X' axes coincide and consequently the two systems can be combined to each other by the equations

$$x' = x - vt, \; y' = y, \; z' = z, \; t' = t \qquad ...(1)$$

Case II. Let the frame S' have the velocity v along any straight line in any direction. Then if v_x, v_y, v_z be the components of v along x, y, z, axes, the two systems

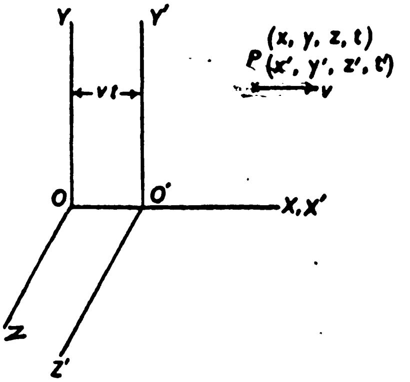


Fig. 14.1

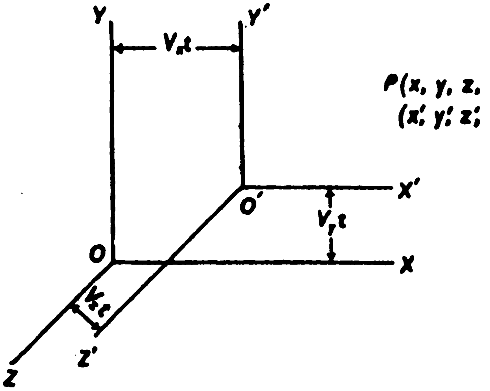


Fig. 14.2

can be related by the equations

$$x' = x - v_x t, \; y' = y - v_y t, \; z' = z - v_z t, \; t' = t \qquad ...(2)$$

(1) and (2) constitute the **Galilean Transformations.**

In either case the distance being expressed in terms of ~~time, which~~ is not absolute, it is clear that the space is not absolute.

The theory of relativity is studied under two heads:

(1) **Special theory of relativity.** Which deals with inertial systems *i.e.* systems moving with uniform velocity.

(2) **General theory of relativity.** Which deals with non-inertial systems *i.e.* systems moving with accelerated velocity.

In this chapter we have only to deal with special theory of relativity.

14.2. BASIC POSTULATES OF SPECIAL THEORY OF RELATIVITY

(*Agra, 1969*)

There are two basic postulates of special theory of relativity:

I. The principle of relativity. *The natural physical laws preserve or retain their form for all inertial frames i.e. they retain the same form relative to all observers in a state of relative uniform motion.*

This postualte is an extension of the conclusion drawn from Newtonian Mechanics *i.e.* the velocity is not absolute but it is relative as is evident from the failure of Michelson-Morley experiment performed to determine the velocity of earth through ether.

II. The principle of constancy of velocity of light. *The velocity of light in vacuum is independent of the velocity of observer or the velocity of the source i.e. it has the same value in all inertial frames.*

This postualte is not true according to Galilean transformations, but is verified experimentally that the velocity of light calculated by any means remains constant. It is this postualte, which draws a demarcation line between the classical theory and the theory of relativity given by Einstein. Of course, the constancy of velocity of light requires the following axioms as to introduce the transformation laws:

(*i*) The velocity of light c must have the same value in all inertial frames.

(*ii*) The transformations should be linear and approaching to Galilean transformations for low velocities *i.e.*, $v << c$.

(*iii*) The transformation laws should be independent of 'absolute time' and 'absolute space' notions.

14.3. LORENTZ TRANSFORMATIONS

(*Agra, 1962, 67; Kanpur, 69*)

H.A. Lorentz introduced transformation equations relating the observations of position and time taken by two observers in two different inertial frames, in order to satisfy the above axioms.

Take S, S' two inertial frames of reference such that S' is moving with uniform velocity v along X'-axis relative to S. Let O, O' be two observers in two systems, situated at their origins. If we consider the two sets of axes X, Y, Z and X', Y', Z' parallel, then the choice of the origins of the two systems falls in taking their origins coincident at $t = 0$, $t' = 0$. It is also convenient to take X, X' axes of two systems coincident, so that the velocity of S' is permanently along X axis. Let the two observers O, O' observe the same event at P whose coordinates are (x, y, z, t) w.r.t. S and (x', y', z', t') w.r.t. S' (*see* Fig. 14.1).

With these assumptions clearly, points which are at rest relative to S' will move with speed v relative to S in X-direction. Particularly the point at $x' = 0$ will move with speed v in X-direction *i.e.* $x' = 0$ will be identical with $x = vt$, so that the first of our transformation equation is

$$x' = \alpha(x - vt) \qquad \ldots(1)$$

where α is a constant to be determined later and vt is the distance traversed by S' in time t along X-axis.

Now velocity of S' being along X-axis only, it follows from symmetry that

$$y' = y \text{ and } z' = z \qquad \ldots(2)$$

This set of equations is not complete unless we formulate an equation connecting t' the time measure in S', with the space and time coordinates *i.e.* x, y, z, t in S. Due to homogeneity of space and time, t' must linearly depend on t, x, y, z but for reasons of symmetry we assume that t' does not depend on y and z, otherwise two clocks in S'-system in $x' = 0$ plane would appear to disagree as observed from S and hence we take

$$t' = \beta t + \gamma x \quad ...(3)$$

where β and γ are not merely constants but are functions of v and are to be determined along with α.

Let us now assume that the event P is a light signal, produced when both t and t' are zero and when origins of two systems coincide. Also let the light pulse produced at $t = 0$ spread out as a growing sphere such that the radius of wave front so produced grows with speed c. But the velocity of light in both the systems being the same *i.e.* c in all directions, the progress of the spherical wave front is described by either of the following equations:

$$x^2 + y^2 + z^2 = c^2t^2 \quad ...(4)$$

$$x'^2 + y'^2 + z'^2 = c^2t'^2 \quad ...(5)$$

Substituting values of x', y', z', t' from (1), (2) and (3) to (5), we get

$$\alpha^2 (x - vt)^2 + y^2 + z^2 = c^2(\beta t + \gamma x)^2$$

or $$(\alpha^2 - c^2\gamma^2)x^2 + y^2 + z^2 - 2xt\,(\alpha^2 v + c^2\beta\gamma) = (c^2\beta^2 - \alpha^2v^2)t^2 \quad ...(6)$$

The equations (4) and (6) representing the same motion are identical and hence comparison of various coefficients, yields

$$\alpha^2 - c^2\gamma^2 = 1 \quad ...(7)$$

$$\alpha^2 v + c^2\,\beta\gamma = 0 \quad ...(8)$$

$$c^2\beta^2 - \alpha^2v^2 = c^2 \quad ...(9)$$

Multiplying (7) by v^2 and then subtracting (8) from it, we have

$$v\,(1 + c^2\gamma^2) + c^2\beta\gamma = 0 \quad ...(10)$$

Again multiplying (7) by v^2 and adding it ot (9), we get

$$-\,v^2(1 + c^2\gamma^2) + c^2\beta^2 = c^2 \quad ...(11)$$

Now multiplying (10) by v and adding it to (11), we find

$$\beta^2 - 1 = -\,v\beta\gamma \text{ i.e. } \gamma = \frac{1-\beta^2}{v\beta} \quad ...(12)$$

Elimination of γ between (10) and (12) gives

$$v\left\{1 + c^2\left(\frac{\beta^2-1}{v\beta}\right)^2\right\} + c^2\left(\frac{1-\beta^2}{v}\right) = 0$$

or $$v^2\beta^2 + c^2(\beta^2 - 1)^2 + c^2\beta^2(1 - \beta^2) = 0$$

or $$\beta^2\left[v^2 + c^2 - 2c^2\right] + c^2 = 0 \text{ giving } \beta^2 = \frac{c^2}{c^2 - v^2} = \frac{1}{1 - \frac{v^2}{c^2}} \quad ...(13)$$

With the help of (13), (9) yields (on taking positive roots only)

$$\alpha^2 = \frac{c^2}{c^2 - v^2} = \beta^2 \text{ so that } \alpha = \beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad ...(14)$$

In view of (14), (8) gives

$$\gamma = \frac{-\alpha^2 v}{c^2\beta} = \frac{-\alpha v}{c^2} = \frac{-\beta v}{c^2} \quad ...(15)$$

From (12), (14) and (15), we get

$$\gamma = \frac{1-\beta^2}{v\beta} = -\frac{\beta v}{c^2} = -\frac{v}{c^2\sqrt{1-\frac{v^2}{c^2}}} \quad ...(16)$$

Also by (8),

$$\alpha^2 = \frac{-c\beta^2\gamma}{v} = \beta^2 = \frac{1}{1-\frac{v^2}{c^2}} \quad ...(17)$$

In view of these results, the transformation equations (1), (2), (3) become

$$x' = \frac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}},\ y' = y,\ z' = z \text{ and } t' = \frac{t - \frac{vx}{c^2}}{1-\frac{v^2}{c^2}} \quad ...(18)$$

These are known as **Lorentz transformation equations.**

Inversely, we have

$$x = \frac{x' + vt'}{\sqrt{1-\frac{v^2}{c^2}}},\ y = y',\ z = z' \text{ and } t = \frac{t' + \frac{vx'}{c^2}}{1-\frac{v^2}{c^2}} \quad ...(19)$$

These are known as **Lorentz inverse transformation equations.**

A look at the transformations (18) and (19) reveals that S has the relative velocity $-v$ w.r.t. S'. This conclusion is not trivial since neither the unit length nor the unit time is directly comparable in S and S'.

In case v is very small then $v/c \to 0$, so that by (14), $\beta \to 1$ and then Lorentz transformations reduce to the Galilean transformations *i.e.*

$$x' = x - vt,\ y' = y,\ z' = z,\ t' = t \quad ...(20)$$

If $v > c$, the Lorentz transformations give imaginary values of x' and t' leading to an absurdity and hence v cannot be greater than c.

Again if $v << c$ so that $\frac{v^2}{c^2}$ is negligible, the Lorentz transformations take the form

$$x' = x - vt,\ y' = y,\ z' = z \text{ and } t' = t - \frac{vx}{c^2} \quad ...(21)$$

In view of the transformations $y' = y$ and $z' = z$ so that

$y^2 + z^2 = y'^2 + z'^2$, (4) and (5) yield

$$x^2 - c^2t^2 = x'^2 - c^2t'^2 \quad ...(22)$$

We may also write

$$x^2 + (ict)^2 = x'^2 + (ict')^2 \text{ where } i = \sqrt{-1} \quad ...(23)$$

Now using the conceptions of four-dimensional geometry and imaginary rotations, we can determine a necessary connection between two sets of coordinates (x_1, x_2, x_3, x_4) and (x_1', x_2', x_3', x'_4).

Setting $x = x_1$, $x' = x_1'$ and $ict = x_4$, $ict' = x_4'$, (23) can be written as

$$x_1^2 + x_4^2 = x_1'^2 + x_4'^2 \quad ...(24)$$

Geometrically interpreted if x_1, x_4 and x_1', x_4' form two rectangular sets of coordinates axes, with the same origin, then L.H.S. and R.H.S. of (24) separately give the distances of the points (x_1, x_4) and (x_1', x_4') from the same origin. Consequenctly either the two sets of axes coincide or are inclined at an angle θ (say) with each other. In later case we can express

$$x_1' = x_1 \cos\theta + x_4 \sin\theta, \; x_4' = -x_1 \sin\theta + x_4 \cos\theta \qquad ...(25)$$

Of course, as is evident from (24) and (25) a circle in original set retains its shape in the rotated set and the coordinates in two sets are connected by circular functions sin θ and cos θ.

If we conceive of an object at rest in S, moving with velocity v w.r.t. S' then we can write $\frac{dx_1}{dt'} = 0$ and $\frac{dx}{dt} = v$, so that

$$\frac{dx_1'}{dx_4'} = \frac{dx_1'}{dx_4} \Big/ \frac{dx_4'}{dx_4} = \frac{\frac{dx_1}{dx_4}\cos\theta + \sin\theta}{-\frac{dx_1}{dx_4}\sin\theta + \cos\theta} = 0$$

or $$\frac{dx_1}{dx_4} = -\tan\theta \qquad ...(26)$$

But when $x = x_1$ and $x_4 = ict$, $\frac{dx}{dt} = v$ gives

$$\frac{dx_1}{dx_4} = \frac{dx_1}{dt} \cdot \frac{dt}{dx_4} = v . \frac{1}{ic} = -\frac{iv}{c} \qquad ...(27)$$

$\therefore$ (26) and (27) give $\tan\theta = \frac{iv}{c}$ ...(28)

Accordingly $$\frac{\sin\theta}{iv} = \frac{\cos\theta}{c} = \frac{\sqrt{\cos^2\theta + \sin^2\theta}}{\sqrt{c^2 + (iv)^2}} = \frac{1}{c\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\therefore \quad \sin\theta = \frac{iv}{\sqrt{1 - \frac{v^2}{c^2}}} \text{ and } \cos\theta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

The substitution of these values in (25) yields Lorentz transformation equations.

The coordinats x and t in (22) may also be related witht x', t' by hyperbolic functions such as

$$x' = x \cosh\zeta - ct \sinh\zeta; \; ct' = -x \sinh\zeta + ct \cosh\zeta \qquad ...(29)$$

COROLLARY. The above transformations have been derived in a particular case when the frame S' is moving with uniform velocity v along X-axis. We can generalize these results by taking the motion of S' w.r.t. S, along any straight line, so that its velocity

$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$$

where v_x, v_y, v_z are the components of $\mathbf{v}$ along X, Y, Z axes respectively.

Taking $\mathbf{r} = x\mathbf{j} + y\mathbf{j} + z\mathbf{k}$ as the position vector of an event its resolved parts along and perpendicular to $\mathbf{v}$ may be given by $\frac{(\mathbf{r}\cdot\mathbf{v})\mathbf{v}}{v^2}$ and $\mathbf{r} - \frac{(\mathbf{r}\cdot\mathbf{v})\mathbf{v}}{v^2}$.

Denoting them by P and Q respectively so that $\mathbf{r} = \mathbf{P} + \mathbf{Q}$ and applying the transformations of (18), we have

$$\mathbf{Q}' = \mathbf{Q} = \mathbf{r} - \frac{(\mathbf{v}\cdot\mathbf{r})\mathbf{v}}{v^2} \qquad ...(30)$$

and
$$\mathbf{P}' = \frac{\mathbf{P} - \mathbf{v}t}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{(\mathbf{r}\cdot\mathbf{v})\mathbf{v} - \mathbf{v}t}{\sqrt{1-\frac{v^2}{c^2}}} \qquad ...(31)$$

Thus $\mathbf{r}' = \mathbf{P}' + \mathbf{Q}' = \mathbf{r} + \frac{(\mathbf{r}\cdot\mathbf{v})\mathbf{v}}{v^2}(\gamma - 1) + \gamma \mathbf{v}t$ ...(32)

where
$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}.$$

Also we have the transformation

$$t' = \gamma\left(t - \frac{\mathbf{r}\cdot\mathbf{v}}{c^2}\right) \qquad ...(33)$$

(32) and (33) give the general Lorentz transformations without rotation. Their inverse transformations may be obtained by replacing v by – v whence we have

$$\mathbf{r} = \mathbf{r}' + \frac{\mathbf{r}'\cdot\mathbf{v}}{v^2}(\gamma - 1) + \gamma\mathbf{v}t' \text{ and } t = \gamma\left(t' + \frac{\mathbf{r}'\cdot\mathbf{v}}{c^2}\right) \qquad ...(34)$$

Problem 1. *State the fundamental postulates of special theory of relativity and deduce the Lorentz transformations.* *(Agra, 1954, 56, 62, 67)*

§§14.2, 14.3 constitute the answer.

Problem 2. *Show that Lorentz transformations* $x' = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}(x - vt)$, $y' = y$ $z' = z$ *and* $t = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}\left(t - \frac{vx}{c^2}\right)$ *may be obtained from the two basic postulates of the special theory of relativity.* *(Agra, 1960)*

This is the same as Problem 1.

Problem 3. *Show that* $x^2 + y^2 + z^2 - c^2t^2$ *is invariant.*

If (x, y, z, t) and (x', y', z', t) be the coordinates of the same event observed by two observes in frames S and S' where S' is moving with velocity v relative to S, then we have to show that

$$x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2$$

Lorentz transformations are

$$x' = \beta(x - vt),\ y' = y,\ z' = z \text{ and } t' = \beta\left(t - \frac{vx}{c^2}\right)$$

where
$$\beta = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

We thus have

$$x'^2 + y'^2 + z'^2 - c^2 t'^2 = \beta^2 (x - vt)^2 + y^2 + z^2 - c^2 \beta^2 \left(t - \frac{vx}{c}\right)^2$$

$$= \beta^2 \left[x^2 + v^2 t^2 - 2vxt - c^2\left(t^2 + \frac{v^2 x^2}{c^4} - \frac{2vtx}{c^2}\right)\right] + y^2 + z^2$$

$$= \beta^2 \left(1 - \frac{v^2}{c^2}\right) x^2 - c^2 t^2 \left(1 - \frac{v^2}{c^2}\right) \beta^2 + y^2 + z^2$$

$$= x^2 - c^2 t^2 + y^2 + z^2 \qquad \because \ \beta = \frac{1}{1 - \frac{v^2}{c^2}}$$

$$= x^2 + y^2 + z^2 - c^2 t^2$$

which shows the invariance of $x^2 + y^2 + z^2 - c^2 t^2$.

Problem 4. *Derive Lorentz inverse transformations.*

Lorentz transformations are

$$x' = \beta\,(x - vt),\ y' = y,\ z' = z,\ t' = \beta\left(t - \frac{vx}{c^2}\right)$$

where $\beta = \dfrac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}}$.

Solving them for x, y, z, we find

$$x = \frac{x'}{\beta} + v\left(t' + \frac{vx}{c^2}\right) = \frac{1}{\beta}(x' + vt') + \frac{v^2 x}{c^2}$$

or

$$\left(1 - \frac{v^2}{c^2}\right) x = \frac{1}{\beta}(x' + vt') \ i.e.\ x = \beta(x' + vt')$$

As such, $\quad t' = \beta\left(t - \dfrac{vx}{c^2}\right)$ gives

$$t = \frac{t'}{\beta} + \frac{v}{c^2}\beta\,(x' + vt') = \frac{t'}{\beta}\left(1 + \frac{v^2}{c^2}\beta^2\right) + \frac{vx'}{c^2}\beta$$

$$= \frac{t'}{\beta}\left[\frac{1 - \frac{v^2}{c^2} + \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}\right] + \frac{vx'}{c^2}\beta = \frac{t'}{\beta\left(1 - \frac{v^2}{c^2}\right)} + \frac{vx'}{c^2}\beta$$

$$= \frac{t'}{\beta}\beta^2 + \frac{vx'}{c^2}\beta = \beta\left(t' + \frac{vx'}{c^2}\right)$$

Hence Lorentz inverse transformations are

$$x = \beta\,(x' + vt'),\ y = y',\ z = z',\ t = \beta\left(t' + \frac{vx'}{c^2}\right)$$

14.4. THE KINEMATIC EFFECTS OF THE LORENTZ TRANSFORMATION

[A] Lorentz-Fitzgerald Contraction *(Agra, 1969)*

Consider two frames of reference S, S' such that S' is moving with velocity v along x-axis relative to S. Then Lorentz inverse transformations are

$$x = \beta\,(x' + vt'),\ y = y',\ z = z',\ t = \beta\left(t' + \frac{vx'}{c^2}\right) \qquad ...(1)$$

where $\beta = \dfrac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}}$.

Suppose that there is a rod of length l placed parallel to Y-axis in system S, which is at rest. Then if y_1, y_2 be the Y-coordinates of the end points of the rod, we have

$$l = y_2 - y_1$$

But from (1), $\quad y'_2 - y'_1 = y_2 - y_1 = l$

which follows that the length of the rod remains unaltered when measured by an observer in S'. Conclusively the lengths perpendicular to the direction of motion are unaltered.

Further suppose that the rod of length l is placed parallel to X-axis in system S, which is at rest. Let x_1, x_2 be the X-coordinates of the end points of rod, measured at the same instant t in S so that $t_1 = t_2 = t$ and $l = x_2 - x_1$. Also let x'_1, x'_2 be the X-coordinates of the two ends of the rod, measured by an observer in system S' at the same instant t', so that $t_1' = t'_2 = t'$. If l' be the length of the rod in the system S', then $l' = x'_2 - x'_1$.

Now, $l = x_2 - x_1 = \beta\,(x'_2 + vt'_2) - \beta(x'_1 + vt'_1)$ by (1)

$$= \beta(x'_2 - x'_1) + v\beta(t'_2 - t'_1)$$

$$= \beta l' = \frac{l'}{\sqrt{1 - \dfrac{v^2}{c^2}}} \qquad \because t'_1 = t'_2$$

$$\therefore\quad l' = l\sqrt{1 - \frac{v^2}{c^2}} = l\left\{1 - \frac{v^2}{c^2}\right\}^{1/2} = l\left\{1 - \frac{1}{2}\cdot\frac{v^2}{c^2} + \ldots\right\}$$

which follows that $l' < l$ and $l' = l\left(1 - \dfrac{v^2}{c^2}\right)^{1/2}$ ...(2)

Here l is the length of the rod for an observer at rest w.r.t. the rod while to an observer in S' which is in relative motion w.r.t. the rod, the rod appears to be of length l'. Actually speaking to the first observer the rod appears to be at rest while to the second observer it appears to be in motion, the length of the rod being measured in the direction of motion. Clearly the length of the rod appears to be longest in the reference system w.r.t. which it is at rest. The length l of the rod measured by an observer which is at rest w.r.t. the rod is said to be the *proper length* of the rod.

Thus Lorentz-Fitzgerald *contraction* follows:

Every rigid body seems to be of maximum dimensions when at rest relative to the observer while its dimensions appear to be contracted in the direction of relative motion *by the factor* $\left(1-\frac{v^2}{c^2}\right)^{1/2}$ *, when it moves with velocity v relative to the observer, whereas its dimensions perpendicular to the direction of motion are unaffected.*

Consequently the length is not an absolute quantity. An interesting consequence of this fact is that a moving sphere seems to be a prolate ellipsoid. In case $v = c$, from (2) $l' = 0$ *i.e.*, the rod appears to be of length zero to an observer in system S' moving with velocity of light relative to S. Also if $v > c$, l' is imaginary leading to an absurdity. Hence $v \not> c$ *i.e.*, $v < c$ always.

Problem 5. *Determine three-dimensional volume element dx dy dz under Lorentz transformations.*

We have by Lorentz-Fitzgerald contraction

$$dx' = \left(1-\frac{v^2}{c^2}\right)^{1/2} dx,\ dy' = dy,\ dz' = dz$$

So that $dx'.dy'\,dz' = \left(1-\frac{v^2}{c^2}\right)^{1/2} dx\,dy\,dz.$

Problem 6. *A rod of length 100 cm. is placed in a satellite moving with velocity 0.8c relative to a laboratory. Determine its length as observed by an observer (a) in the satellite (b) in the laboratory.*

(*a*) The observer being in the satellite is at rest relative to the rod and hence the length of the rod measured by him is 100 cm.

(*b*) The length of the rod as measured by an observer in the laboratory is given by

$$l' = l\sqrt{1-\frac{v^2}{c^2}} \text{ where } l = 100 \text{ cm},\ v = 0.8c$$

$$\therefore\ l' = 100\sqrt{1-\left(\frac{0.8c}{c}\right)^2} = 100\ \sqrt{1-0.64} = 100 \times 0.6 = 60 \text{ cm}.$$

Problem 7. *The length of a rocketship is 100 metres on the ground. When it is in flight its length appears to be 99 metres to an observer on the ground. Determine its speed.*

We have $$l' = l\sqrt{1-\frac{v^2}{c^2}}$$

where $l = 100$ metres $= 10^4$ cm; $l' = 99$ metres $= 99 \times 10^2$ cm.

$$\therefore\ \left(1-\frac{v^2}{c^2}\right) = \left(\frac{l'}{l}\right)^2 = \left(\frac{99\times10^2}{10^4}\right)^2 = (.99)^2$$

So that $\frac{v^2}{c^2} = 1-(.99)^2 = (1-0.99)(1+0.99) = 0.01 \times 1.99 = 199 \times 10^{-4}$

$$\therefore\quad v = c(199 \times 10^{-4})^{1/2} = \left(\sqrt{199}\times10^{-2}\right)\times3\times10^{10}$$

$$= 3\ \sqrt{199} \times 10^8 = 3 \times 14.1 \times 10^8 = 4.23 \times 10^9 \text{ cm./sec.}$$

Problem 8. *Determine the length of a rod (proper length 100 cm) moving with a velocity of 0.8c in a direction inclined at an angle of 60° ti its own length.*

Taking **i**, **j** as unit vectors along X and Y axes respectively, the proper length l of the rod moving with velocity $0.8c$ along X-axis is given by

$$\mathbf{l} = l \cos 60° \, \mathbf{i} + l \sin 60° \, \mathbf{j} = \frac{l}{2}\mathbf{i} + l\sqrt{\frac{3}{2}} \, \mathbf{j}.$$

Now the motion being along X-axis, the contraction will take place along X-axis and the component of length along Y-axis will be unaffected.

Thus if l_x' and l_y' be the components of length of the rod when in motion along X-axis, we find

$$l'_x = l_x \sqrt{1 - \frac{v^2}{c^2}} = l \cos 60° \sqrt{1 - \left(\frac{.8c}{c}\right)^2} = \frac{l}{2} \times 0.6 = 0.3\, l$$

and $$l'_y = l \sin 60° = \frac{l\sqrt{3}}{2} = 0.866 l.$$

$\therefore$ Length of the moving rod $= \sqrt{l_x'^2 + l_y'^2}$

$$= l\sqrt{(0.3)^2 + (0.866)^2} = l\sqrt{.09 + 0.75} \text{ approx.}$$

$$= l\sqrt{0.84} = 0.92\ l \text{ approx.}$$

$$= 100 \times 92 = 92 \text{ cm.}$$

[B] Time-dilation or Apparent Retardation of Clocks

Consider two frames of reference S, S' such that S' is moving with velocity v along X-axis relative to S. Conceive of a clock placed in S at $x = x_1$, giving a signal at time t_1, such that t'_1 is the time measured by an observer in S' corresponding to it. Then Lorentz transformation gives

$$t_1' = \beta\left(t_1 - \frac{vx_1}{c^2}\right) \text{ where } \beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad ...(3)$$

Suppose that the clock gives another signal at time t_2 in S, such that its corresponding time in S' is t'_2. Then

$$t_2' = \beta\left(t_2 - \frac{vx_1}{c^2}\right) \quad ...(4)$$

Putting $t_2 - t_1 = t$ and $t'_2 - t'_1 = t'$ and then subtracting (3) from (4), we get $t'_2 - t'_1 = \beta\,(t_2 - t_1)$

i.e. $$t' = \beta t = \frac{t}{\sqrt{1 - \frac{v^2}{c^2}}} = t\left\{1 - \frac{v^2}{c^2}\right\}^{-1/2} = t\left\{1 + \frac{1}{2}\cdot\frac{v^2}{c^2} + \ldots\right\}$$

So that $t' > t$ and $$t = t'\left(1 - \frac{v^2}{c^2}\right)^{1/2} \quad ...(5)$$

which follows that the time inteveral t appears to be dilated or lengthened by the factor $\left(1 - \frac{v^2}{c^2}\right)^{-1/2}$ to the moving observer.

If we reverse the process *i.e.* put the clock at a point x'_1 in S', giving signals at times t'_1, t'_2 while the corresponding times in S are t_1 and t_2 respectively then by Lorentz inverse transformation, we have

$$t_1' = \beta\left(t_1 + \frac{vx_1'}{c^2}\right) \text{ and } t_2' = \beta\left(t_2 + \frac{vx_1'}{c^2}\right)$$

So that $t' = t'_2 - t'_1 \beta(t_2 - t_1) = \beta t$ implying $t' > t$ as above.

It follows that the time interval t' appears to be dilated by the factor $\left(1 - \frac{v^2}{c^2}\right)^{-1/2}$ to an observer moving with velocity v relative to S'.

Conclusively a moving clock runs more slowly than a stationary one i.e., every clock seems to go at its fastest rate when it is at rate relative to the observer and if it moves w.r.t. the observer with velocity v, its rate appears to go at its slowest rate by the factor

$$\sqrt{1 - \frac{v^2}{c^2}}$$

It should be noted that the time recorded by a clock moving with a given system is said to be the **proper time** for that system.

Problem 9. *Show that four-dimensional volume element dx dy dz dt is invariant under Lorentz transformation.* *(Agra, 1967, 70)*

Lorentz-Fitzgerald contraction gives

$$dx' = \left(1 - \frac{v^2}{c^2}\right)^{1/2} dx,\ dy' = dy,\ dz' = dz$$

Time-dilation gives $dt' = \dfrac{dt}{\sqrt{1 - \frac{v^2}{c^2}}}$.

So that $dx'\, dy'\, dz'\, dt' = dx\, dy\, dz\, dt$.

Which shows the required invariance.

Problem 10. *Write a short note on* **Clock Paradox.**

We have already discussed in time-dilation that according to an observer in S, a clock in S' is going slow while from the point of view of S', it is a clock in S which is moving fast and hence when the observer comes back from S' to S, he finds just reversed phenomena. We use to call it as a Clock Paradox.

To be more explanatory an observer posited in either frame of reference finds that the rate of clock in other system is slow. There is complete reciprocity between the two frames S and S'.

Problem 11. *In the laboratory the life-time of a particle moving with speed 2.8 ×10¹⁰ cm./sec. is found to be 2.5 × 10⁻⁷ sec.. Calculate the proper life-time of the particle.* *(Agra, 1969)*

Time-dilation gives $t' = \dfrac{t}{\sqrt{1 - \frac{v^2}{c^2}}}$

where $t' = 2.5 \times 10^{-7}$ sec., $v = 2.8 \times 10^{10}$ cm/sec.,
$c = 3 \times 10^{10}$ cm/sec.

$$\therefore\ t = t'\sqrt{1-\frac{v^2}{c^2}} = 2.5\times10^{-7}\sqrt{1-\left(\frac{2.8\times10^{10}}{3\times10^{10}}\right)}$$

$$= 2.5\times10^{-7}\sqrt{1-(0.93)^2} = 2.5\times10^{-7}\times\sqrt{1-0.87} \text{ approx.}$$

$$= 2.5\times10^{-7}\times .36 = 9\times10^{-8}\text{ sec.}$$

[*C*] Relativity of Simultaneity *(Agra, 1964, 66, 67)*

Consider two frames of reference S, S' such that S' is moving with velocity v along X-axis relative to S. Let the two events in S with space and time coordinates (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2) respectively, occur simultaneously, so that

$$x_1 \neq x_2 \text{ but } t_1 = t_2$$

Let t'_1, t'_2 be the corresponding times in S'. Then by Lorentz transformations we have

$$t_1' = \beta\left(t_1 - \frac{vx_1}{c^2}\right) \text{ and } t_2' = \beta\left(t_2 - \frac{vx_2}{c^2}\right)$$

where
$$\beta = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\therefore\quad t_2' - t'_1 = \beta(t_2 - t_1) + \beta\frac{v}{c^2}(x_1 - x_2)$$

$$= \frac{\beta v}{c^2}(x_1 - x_2) \quad \because\ t_1 = t_2$$

In view of $x_1 \neq x_2$, this follows that $t'_1 \neq t'_2$ *i.e.* the same two events are not simultaneous in S'.

Conclusively two simultaneous events at different points, for an observer at rest in S remain no longer simultaneous to an observer in S' which is moving with velocity v along X-axis relative to S.

In case $x_1 = x_2$, then however the events are simultaneous since than $t'_2 = t'_1$.

[*D*] Relativistic Composition of Velocities *(Agra, 1964, 65, 68, 71)*

Consider two frames of reference S, S' such that S' is moving with velocity v along X-axis relative to S which is at rest. Let the space and time coordinates of a moving particle P be (x, y, z, t) and (x', y', z', t') w.r.t. S and S' respectively. Also let u_x, u_y, u_z and u'_x, u'_y, u'_z be the velocity components of P w.r.t S and S' respectively. Then

$$u_x = \frac{dx}{dt},\quad u_y = \frac{dy}{dt},\quad u_z = \frac{dz}{dt}$$

and
$$u_x' = \frac{dx'}{dt'},\quad u_y' = \frac{dy'}{dt'},\quad u_z' = \frac{dz'}{dt'}$$

Lorentz inverse transformations are

$$x = \beta(x' + vt'),\ y = y',\ z = z' \text{ and } t = \beta\left(t' + \frac{vx'}{c^2}\right)$$

where
$$\beta = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}},$$

Their differentiations give

$$dx = \beta(dx' + v\,dt'),\ dy = dy',\ dz = dz';$$

$$dt = \beta\left(dt' + \frac{v}{c^2}\,dx'\right), \text{ so that}$$

$$u_x = \frac{dx}{dt} = \frac{\beta(dx' + v\,dt')}{\beta\left(dt' + \frac{v}{c^2}dx'\right)} = \frac{\left(\frac{dx'}{dt'} + v\right)}{\left(1 + \frac{v}{c^2}\frac{dx'}{dt'}\right)} = \frac{v'_x + v}{1 + \frac{v}{c^2}u'_x} \quad \ldots(6)$$

$$u_y = \frac{dy}{dt} = \frac{dy'}{\beta\left(dt' + \frac{v}{c^2}dx'\right)} = \frac{\frac{dx'}{dt'}}{\beta\left(1 + \frac{v}{c^2}\frac{dx'}{dt'}\right)}$$

$$= \frac{u'_y}{1 + \frac{v}{c^2}u'_x} \quad \ldots(7)$$

$$u_x = \frac{dz}{dt} = \frac{dz'}{\beta\left(d't + \frac{v}{c^2}dx'\right)} = \frac{\frac{dz'}{dt'}}{\beta\left(1 + \frac{v}{c^2}\frac{dx'}{dt'}\right)}$$

$$= \frac{u'_z}{1 + \frac{v}{c^2}u'_x} \quad \ldots(8)$$

(6), (7), (8) give the velocity components u_x, u_y, u_z as the composition of velocities (u'_x, u'_y, u'_z) and $(v, 0, 0)$.

COROLLARY 1. The relativistic law of addition of two velocities in the same direction say X-axis, is given by putting $u'_x = u'$, $u'_y = 0$, $u'_z = 0$ in the above components whence we get

$$u_x = \frac{u' + v}{1 + \frac{u'v}{c^2}} \quad \ldots(9)$$

COROLLARY 2. The velocity of light is an absolute constant, quite independent of the motion of the reference system.

Putting $u' = c$ in (9), we get

$$u_x = \frac{c + v}{1 + \frac{cv}{c^2}} = \frac{(c + v)}{c + v} = c \quad \ldots(10)$$

which follows that the velocity of light c remains unchanged if compounded with a velocity smaller than c.

COROLLARY 3. The resultant of two velocities each being less than c, is also less than c.

Take $u' = c - p$, $v = c - q$ where $p, q > 0$ and p, q are constants. Then (9) yields

$$u_x = \frac{c-p+c-q}{1+\frac{(c-p)(c-q)}{c^2}} = \frac{2c-p-q}{1+\frac{c^2-(p+q)c+pq}{c^2}}$$

$$= c\left[\frac{2c-p-q}{\frac{2c^2-(p+q)c+pq}{c}}\right] = c\left[\frac{2c-(p+q)}{2c-(p+q)+\frac{pq}{c}}\right]$$

which follows that $u_x < c \quad \because \; 2c-(p+q) < 2c-(p+q)+\frac{pq}{c}$

for $p, q > 0$ and $\frac{pq}{c} > 0$.

COROLLARY 4. The addition of any velocity to c the velocity of light results in c.

Taking $u' = c$ and $v = c$, (9) gives

$$u_x = \frac{c+c}{1+\frac{c \cdot c}{c^2}} = \frac{2c}{2} = c.$$

COROLLARY 5. The velocity of light *i.e.* is known as **fundamental velocity** due to the following facts.

(*i*) c is constant in all directions.

(*ii*) c remains the same for all observers irrespective of the velocities of the source and the observer.

(*iii*) c is invariant for any two systems related by Lorentz transformations.

(*iv*) c remains unchanged by adding any velocity to it, which is less than c.

(*v*) No velocity can be greater than c.

Problem 12. *Deduce* **Fresnel's coefficient of drag** *from Lorentz transformations (or in particular from relativistic velocity composition law)*

(*Agra, 1958, 66*)

Assuming that the frame S' is water moving with velocity v along X-axis relative to another inertial frame S which is at rest, and taking μ as the refractive index of water, the velocity of a light particle (photon) relative to water (*i.e.* S')

is $\qquad \frac{c}{\mu} = u'$ (say) towards X-axis.

If u_x be the velocity of the light particle w.r.t. S, then we have by relativistic velocity addition law

$$u_x = \frac{\frac{c}{\mu}+v}{1+\frac{c}{\mu}\cdot\frac{v}{c^2}} = \left(\frac{c}{\mu}+v\right)\left(1+\frac{v}{c\mu}\right)^{-1}$$

$$= \left(\frac{c}{\mu}+v\right)\left\{1-\frac{v}{\mu c}+\frac{v^2}{\mu^2c^2}-\dots\right\}$$

$$= \left(\frac{c}{\mu}+v\right)\left(1-\frac{v}{\mu c}\right), \text{ neglecting squares of } \frac{v}{c} \;\because\; v << c$$

$$= -\frac{c}{\mu} + v - \frac{v}{\mu^2}, \text{ neglecting } \frac{v^2}{\mu c} \text{ as } v << c$$

$$= \frac{c}{\mu} + v\left(1 - \frac{1}{\mu^2}\right) = u' + v\left(1 - \frac{1}{\mu^2}\right)$$

which follows that the velocity u' is increased as observed from S due to drag of water or system S' moving with velocity v by an amount $v\left(1 - \frac{1}{\mu^2}\right)$

$$\therefore \text{ Coefficient of drag} = \frac{v\left(1 - \frac{1}{\mu^2}\right)}{v} = 1 - \frac{1}{\mu^2}$$

which is known as *Fresnel's coefficient of drag.*

Problem 13. *If a photon traverses the path in such a say that it moves in $X' - Y'$ plane making an angle ϕ with X-axis of system S' then prove that for the frame S, $u_x^2 + u_y^2 = c^2$, where S' is moving with velocity v relative to S.*

If in a system S', $u'_y = c \sin \phi$ and $u'_x = c \cos \phi$ then show that in frame S, $u_x^2 + u_y^2 = c^2$ where S' is moving with velocity v relative to S.

We have

$$u_x = \frac{u'_x + v}{1 + \frac{v}{c^2}u'_x} = \frac{c \cos \phi + v}{1 + \frac{v}{c}\cos \phi}$$

and

$$u_y = \frac{u'_y}{\beta\left(1 + \frac{v}{c^2}u'_x\right)} = \frac{c \sin \phi \sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{v}{c}\cos \phi}$$

Squaring and adding these expressions, we find

$$u_x^2 + u_y^2 = \frac{1}{\left(1 + \frac{v}{c}\cos \phi\right)^2}\left[(c \cos \phi + v)^2 + c^2 \sin^2\phi\left(1 - \frac{v^2}{c^2}\right)\right]$$

$$= \frac{1}{\left(1 + \frac{v}{c}\cos \phi\right)^2}\left[c^2 + v^2 + 2cv \cos \phi - v^2 \sin^2\phi\right]$$

$$\because \sin^2 \phi + \cos^2 \phi = 1$$

$$= \frac{1}{\left(1 + \frac{v}{c}\cos \phi\right)^2}\left[c^2 + 2cv\cos\phi + v^2 \cos^2\phi\right] \because 1 - \sin^2 \phi = \cos^2 \phi$$

$$= \frac{c^2\left(1 + \frac{v}{c}\cos \phi\right)^2}{\left(1 + \frac{v}{c}\cos \phi\right)^2} = c^2.$$

[E] Relativistic Composition of Accelerations

Consider two frames of reference S, S' such that S' is moving with velocity v along X-axis relative to S which is at rest. Let the space and time coordinates of a moving particle P be (x, y, z, t) and (x', y', z', t') w.r.t. S and S' respectively. Also let u_x, u_y, u_z; a_x, a_y, a_z and u_x', u_y', u_z'; a_x', a_y', a_z' be the velocity and acceleration components of P w.r.t. S and S' respectively. Then

$$a_x = \frac{du_x}{dt},\ a_y = \frac{du_y}{dt},\ a_z = \frac{du_z}{dt} \text{ and } a_x' = \frac{du_x'}{dt'},\ a_y' = \frac{du_y'}{dt'},\ a_z' = \frac{du_z'}{dt'}$$

Lorentz inverse transformations are

$$x = \beta(x' + vt'),\ y = y',\ z = z',\ t = \beta\left(t' + \frac{vx'}{c^2}\right) \text{where } \beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

These give on differentiation,

$$dx = \beta(dx' + v\,dt'),\ dy = dy',\ dz = dz',\ dt = \beta\left(dt' + \frac{v}{c^2}\,dx'\right) \text{ giving}$$

$$\frac{dt}{dt'} = \beta\left(1 + \frac{v}{c^2}u_x'\right), \text{ so that by (6), (7) and (8), we have}$$

$$a_x = \frac{d}{dt}u_x = \frac{d}{dt}\left[\frac{u_x' + v}{1 + \frac{v}{c^2}u_x'}\right] = \frac{d}{dt'}\left[\frac{u_x' + v}{1 + \frac{v}{c^2}u_x'}\right]\frac{dt'}{dt}$$

$$= \frac{\frac{du_x'}{dt'}\left(1 + \frac{v}{c^2}u_x'\right) - (v + u_x')\frac{v}{c^2}\frac{du_x'}{dt'}}{\left(1 + \frac{v}{c^2}u_x'\right)^2} \cdot \frac{1}{\beta\left(1 + \frac{v}{c^2}u_x'\right)}$$

$$= \frac{\frac{du_x'}{dt}\left(1 - \frac{v^2}{c^2}\right)}{\beta\left(1 + \frac{v}{c^2}u_x'\right)^3} = \frac{\frac{du_x'}{dt'}}{\left(1 + \frac{v}{c}u_x'\right)^3\beta^3} \qquad \text{...(10)}$$

Similarly,

$$a_y = \frac{1}{\beta^2}\left[\frac{\frac{du_y'}{dt'}}{\left(1 + \frac{v}{c^2}u_x'\right)^2} - \frac{\frac{v}{c^2}\frac{du_x'}{dt'}u_y'}{\left(1 + \frac{v}{c^2}u_x'\right)}\right] \qquad \text{...(11)}$$

$$a_z = \frac{1}{\beta^2}\left[\frac{\frac{du_z'}{dt'}}{\left(1 + \frac{v}{c^2}u_x'\right)^2} - \frac{\frac{v}{c^2}\frac{du_x'}{dt'}u_z'}{\left(1 + \frac{v}{c^2}u_x'\right)}\right]. \qquad \text{...(12)}$$

Equations (10), (11), (12), give the acceleration components a_x, a_y, a_z in terms of a_x', a_y', a_z' as observed in system S'. Evidently the components of acceleration in S' are constants, but the components of acceleration in S are not constant, in general owing to the fact that they contain velocity components along with acceleration components.

COROLLARY 1. In particular if the particle is instantaneously at rest relative to frame S', then $u'_x = 0\ u'_y = u'_z$, so that (10), (11) and (12) reduce to

$$a_x = \left(1 - \frac{v^2}{c^2}\right)^{3/2} a'_x,\quad a_y = \left(1 - \frac{v^2}{c^2}\right) a'_y,\quad a_z = \left(1 - \frac{v^2}{c^2}\right) a'_z \qquad \text{...(13)}$$

which follows that unlike Galilean transformations, the acceleration is different in two inertial frames. This is due to the relativity of space and time.

COROLLARY 2. *Sets of instantaneous direction cosines of the motion of the particle P in frames S and S'.*

Let l, m, n and l', m', n' be the instantaneous direction cosines of the motion of P in frames S and S' respectively. Then we have

$$l = \frac{u_x}{\sqrt{u_x^2 + u_y^2 + u_z^2}},\quad m = \frac{u_y}{\sqrt{u_x^2 + u_y^2 + u_z^2}},\quad n = \frac{u_z}{\sqrt{u_x^2 + u_y^2 + u_z^2}} \qquad \text{...(14)}$$

and

$$l' = \frac{u'_x}{\sqrt{u_x'^2 + u_y'^2 + u_z'^2}},\quad m = \frac{u'_y}{\sqrt{u_x'^2 + u_y'^2 + u_z'^2}},\quad n' = \frac{u'_z}{\sqrt{u_x'^2 + u_y'^2 + u_z'^2}} \qquad \text{...(15)}$$

Using Lorentz transformations, we have

$$l = \frac{\dfrac{u'_x + v}{1 + \dfrac{v}{c^2}u'_x}}{\dfrac{1}{1 + \dfrac{v}{c}u'_x}\sqrt{(u'_x + v)^2 + \dfrac{1}{\beta^2}(u'_y + u_z'^2)}}$$

$$= \frac{u'_x + v}{\sqrt{u_x'^2 + 2u'_x v + v^2 + \left(1 - \dfrac{v^2}{c^2}\right)(u_y'^2 + u_z'^2)}}$$

$$= \frac{u'_x + v}{\sqrt{(u_x'^2 + u_y'^2 + u_z'^2) + 2u'_x v + \dfrac{v^2}{c^2}(c^2 - u_y'^2 - u_z'^2)}}$$

Dividing numerator denominator of R.H.S. by

$\sqrt{u_x'^2 + u_y'^2 + u_z'^2}$ and using (15), we get

$$l = \frac{l' + \dfrac{v}{\sqrt{u_x'^2 + u_y'^2 + u_z'^2}}}{\left[1 + \dfrac{2l'v}{\sqrt{u_x'^2 + u_y'^2 + u_z'^2}} + \dfrac{v^2}{c^2}\left(\dfrac{c^2}{u_x'^2 + u_y'^2 + u_z'^2} - m'^2 - n'^2\right)\right]^{1/2}}$$

Similarly m and n can be expressed.

Problem 14. *A particle P instantaneously at rest in frame S' experiences an acceleration* $\mathbf{a}' = 3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$. *Determine the acceleration measured from an observer in frame S when S' is moving with velocity .98c relative to S along X-axis.*

Let a_x, a_y, a_z and a'_x, a'_y, a'_z be the components of $\mathbf{a}'$ in systems S and S' respectively. Then as we are given,

$$a'_x = 3, \quad a'_y = 4, \quad a'_z = 12$$

By Cor. 1, $a_x = \left(1 - \frac{v^2}{c^2}\right)a'_x = \left\{1 - \left(\frac{.98c}{c}\right)^2\right\} \cdot 3 = 0.024$ (on simplification)

$$a_y = \left(1 - \frac{v^2}{c^2}\right)a'_y = \left\{1 - \left(\frac{.98c}{c}\right)^2\right\} \cdot 4 = 0.1584 \quad ,,$$

$$a_z = \left(1 - \frac{v^2}{c^2}\right)a'_z = \left\{1 - \left(\frac{.98c}{c}\right)^2\right\} \cdot 12 = 0.4752 \quad ,,$$

Hence $\mathbf{u} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$

$= 0.024\ \mathbf{i} + 0.1584\ \mathbf{j} + 0.4752\ \mathbf{k}$.

14.5. ENERGY-MOMENTUM RELATION

If a particle of mass m is moving with velocity v then its momentum p in classical Machanics is given by $p = mv$ ...(1)

and the *law of conservation of momentum* states (*i*) the mass of a moving body is the same as that of the stationary body and (*ii*) the total momentum of a body remains unaltered unless an external force is applied. The first hypothesis does not stand true when examined by the help of Lorentz transformations, whereas the Lorentz invariance of law of conservation of momentum states that the mass of a moving body does not remain constant but it changes with velocity and given by

$$m = \frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}} \qquad ...(2)$$

(*Agra, 1969, 70, 71*)

where m_0 is the mass of the particle when at rest and m is its mass when moving with velocity u.

In order to prove (2), consider two frame S, S' such that S' is moving with velocity v along X-axis relative to S. Suppose that m_1 is the mass of the particle moving with velocity u_1 in S along X-axis and its corresponding mass and velocity in S' are m'_1 and u'_1 respectively.

$$\text{Let } \beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}};\ \beta_1 = \frac{1}{\sqrt{1 - \frac{u_1^2}{c^2}}} \text{ and } \beta'_1 = \frac{1}{\sqrt{1 - \frac{u_1'^2}{c^2}}} \qquad ...(3)$$

Velocity-composition law gives

$$u_1 = \frac{u'_1 + v}{1 + \frac{v}{c^2}u'_1} \text{ giving } u'_1 = \frac{u_1 - v}{1 - \frac{v}{c^2}u_1} \qquad ...(4)$$

$$\text{So that } \beta'_1 u'_1 = \frac{u_1 - v}{\left(1 - \frac{u_1'^2}{c^2}\right)^{1/2}\left(1 - \frac{v}{c^2}u_1\right)}$$

$$= \frac{u_1 - v}{\left\{1 - \frac{(u_1 - v)^2}{c^2\left(1 - \frac{v}{c^2}u_1\right)^2}\right\}^{1/2} \cdot \left(1 - \frac{v}{c^2}u_1\right)} \text{ by (4)}$$

$$= \frac{(u_1 - v)c}{\left[c^2\left(1 - \frac{v}{c^2}u_1\right)^2 - (u_1 - v)2\right]^{1/2}}$$

$$= \frac{(u_1 - v)c}{\left[c^2\left\{1 + \frac{v^2}{c^4}u_1^2 - \frac{2vu_1}{c^2}\right\} - \left(u_1^2 + v^2 - 2u_1u\right)\right]^{1/2}}$$

$$= \frac{(u_1 - v)}{\frac{1}{c}\left[c^2 + \frac{v^2}{c^2}u_1^2 - 2vu_1 - u_1^2 - v^2 + 2vu_1\right]^{1/2}}$$

$$= \frac{u_1 - v}{\left[1 - \frac{u_1^2 + v^2}{c^2} + \frac{u_1^2 v^2}{c^4}\right]^{1/2}} = \frac{u_1 - v}{\left[\left(1 - \frac{v^2}{c^2}\right)\left(1 - \frac{u_1^2}{c^2}\right)\right]^{1/2}}$$

$$= \beta\beta_1(u_1 - v) \text{ by (3)}$$

or $$\frac{\beta_1' u_1'}{\beta_1} = \beta(u_1 - v) \qquad ...(5)$$

Assume that there are a number of such particles moving along X-axis and their masses and momentums being invariant in S, so that

$$\Sigma m_1 = \text{const.}, \ \Sigma m_1 u_1 = \text{const.} \qquad ...(6)$$

But β and v being the same for every particle, (6) yield

$$\Sigma m_1 \beta v = \text{const.}, \ \Sigma m_1 u_1 \beta = \text{const.}$$

which on subtraction give $\Sigma m_1 \beta(u_1 - v) = \text{const.}$

or $$\Sigma m_1 \frac{\beta_1' u_1'}{\beta_1} = \text{const. by (5)} \qquad ...(7)$$

In frame S', applying law of conservation of momentum, we get

$$\Sigma m'_1 u'_1 = \text{const.} \qquad ...(8)$$

Comparison of (7) and (8) renders $\frac{m_1 \beta_1'}{\beta_1} = m_1'$

or $\frac{m_1}{\beta_1} = \frac{m_1'}{\beta_1'} = m_0$ (say) as absolute constant giving

$$m_1 = \frac{m_0}{\sqrt{1 - \frac{u_1^2}{c_2}}} \text{ and } m_1' = \frac{m_0}{\sqrt{1 - \frac{u_1'^2}{c_2}}} \text{ with the help of (3).}$$

We conclude from these results that for a body of mass m moving with velocity u relative to that system

$$m = \frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}}$$

which is the result (2) showing that the mass of a body increases with the increase of velocity.

When $u = 0$, (2) gives $m = m_0$ *i.e.*, mass of the particle at rest is m_0 and hence m_0 is known as the *rest mass.*

In case $u << c$ *i.e.*, $\frac{u}{c} << 1$ then also $m = m_0$.

Now we establish the *mass-energy relation i.e.,*

$$E = mc^2 \qquad \text{...(9)}$$

(*Agra, 1969, 71*)

We know that the increase in energy of the particle by applying a force F is given in terms of work *i.e., if* dT be the increment in Kinetic energy T due to an increase dr in the displacement r, then

$$dT = F \cdot dr = F \cdot \frac{dr}{dt} \cdot dt = Fv\, dt \qquad \text{...(10)}$$

$$\because v = \frac{dr}{dt} \text{ (velocity)}$$

But the force being defined as rate of change of momentum we have

$$F = \frac{d}{dt}(mv) \text{ of } F\, dt = d(mv) \qquad \text{...(11)}$$

with the help of (11), (10) yields

$$dT = v F\, dt = v \cdot d(mv) = vd\left(\frac{m_0 v}{1 - \frac{v^2}{c^2}}\right) \text{ by (2)}$$

$$= m_0 v \left[\sqrt{1 - \frac{v^2}{c^2}} + \frac{v^2}{c^2\sqrt{1 - \frac{v^2}{c^2}}}\right] \frac{dv}{\left(1 - \frac{v^2}{c^2}\right)} \text{ (on differentiation)}$$

$$= \frac{m_0 v\, dv}{c^2\left(1 - \frac{v^2}{c^2}\right)^{3/2}} \text{ i.e., } c^2\, dm = \frac{m_0 v\, dv}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}} \qquad \text{...(13)}$$

Comparing (12) and (13), we find

$$dT = c^2 dm \qquad \text{...(14)}$$

If we consider the body initially at rest whence its rest mass is m_0 and on applying this force as it acquires velocity so its mass increases and becomes say $m = m_0 + dm$. Now the total Kinetic energy acquired by the body is given by

$$T = \int dT = \int_{m_0}^{m} c^2\, dm \text{ by (4)}$$
$$= c^2 [m - m_0]$$

or $$T + m_0 c^2 = mc^2 \qquad ...(15)$$

Total energy E = Kinetic energy of moving body + energy at rest

i.e., $E = c^2 (m - m_0) + m_0 c^2 = mc^2$

which establishes the relation (9).

In the last we prove the *energy-momentum relation* or *Einstein relation*:

$$E^2 = c^2p^2 + m_0{}^2c^4 \qquad ...(16)$$

where E is the energy and p is the momentum of a particle of rest mass m_0.

Using (1) and (9), we have

$$E^2 - c^2p^2 = (mc^2)^2 - c^2(mv)^2$$
$$= m^2c^4 - m^2 c^2 v^2$$
$$= \left(\frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}\right)^2 c^4 - \left(\frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}\right)^2 c^2 v^2$$

by (2) on replacing v by u in the existing case.

$$= \frac{m_0^2 c^2}{1 - \frac{v^2}{c^2}} \left(c^2 - v^2\right) = m_0^2\, c^4$$

Hence $E^2 = c^2p^2 + m_0{}^2c^4$

which establishes the relation (16).

Problem 15. *Formulate the energy-momentum transformation in the space time of special relativity.* *(Agra, 1975)*

Consider two system S and S' such that S' is moving with velocity v along X-axis. If m and m' are the masses of a body in S and S' respectively, such that it is moving with velocities u (u_x, u_y, u_z) and $u'(u'_x, u'_y, u'_z)$ respectively. Then we have

$$m = \frac{m_0}{\sqrt{1 - \frac{u^2}{c_2}}},\ m' = \frac{m_0}{\sqrt{1 - \frac{u'^2}{c^2}}} \qquad ...(1)$$

where m_0 is the rest mass of the body.

Here $u^2 = u_x{}^2 + u_y{}^2 + u_z{}^2$ and $u'^2 = u'_x{}^2 + u'_y{}^2 + u'_z{}^2 \qquad ...(2)$

The velocity-composition law gives

$$u'_x = \frac{u_x - v}{1 - \frac{v}{c^2}u_x},\ u'_y = \frac{u_y\left(1 - \frac{v^2}{c^2}\right)^{1/2}}{1 - \frac{v}{c^2}u_x},\ u'_z = \frac{u_z\left(1 - \frac{v^2}{c^2}\right)^{1/2}}{1 - \frac{v}{c^2}u_x} \qquad ...(3)$$

Now from (1),

$$\frac{m}{m'} = \left(\frac{1-\frac{u'^2}{c^2}}{1-\frac{u^2}{c^2}}\right)^{1/2}$$

where $1-\frac{u'^2}{c^2} = 1-\frac{1}{c^2}\left(u_x'^2 + u_y'^2 + u_z'^2\right)$

$$= 1-\left[(u_x - v)^2 + u_y^2\left(1-\frac{v^2}{c^2}\right) + u_z^2\left(1-\frac{v^2}{c^2}\right)\right]\frac{1}{\left(1-\frac{v}{c^2}u_x\right)^2 c^2} \text{ by (3)}$$

$$= \frac{c^2\left(1-\frac{u^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{c^2\left(1-\frac{v}{c^2}u_x\right)^2} \text{ (on simplification)}$$

So that $\left(\frac{1-\frac{v'^2}{c^2}}{1-\frac{u^2}{c}}\right)^{1/2} = \frac{\left(1-\frac{v^2}{c^2}\right)^{1/2}}{1-\frac{v}{c^2}u_x}$

$$\therefore \frac{m}{m'} = \frac{\left(1-\frac{v^2}{c^2}\right)^{1/2}}{1-\frac{v}{c^2}u_x} \text{ i.e., } m' = m\frac{\left(1-\frac{v^2}{c^2}u_x\right)}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} = m\beta\left(1-\frac{v}{c^2}u_x\right) \quad \text{...(4)}$$

where $\beta = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$.

Also from (3),

$$u_x' = \frac{u_x - v}{1-\frac{v}{c^2}u_x}, \quad u_y' = \frac{u_y}{\beta\left(1-\frac{v}{c^2}u_x\right)}, \quad u_z' = \frac{u_z}{\beta\left(1-\frac{v}{c^2}u_x\right)} \quad \text{...(5)}$$

Let p_x, p_y, p_z and p'_x, p'_y, p'_z be the components of momentum p of the particle, acquired due to velocity in frames S and S' respectively.

Then, $p_x = mu_x, p_y = mu_y, p_z = mu_z$ in frame S,

and $p'_x = m'u'_x = (mu_x - mv)\beta$ by (4) and (5)

$$= \left(p_x - \frac{v}{c^2}E\right)\beta \qquad \because E = mc^2$$

Similarly, $p'_y = m'u'_y = mu_y = p_y$

and $p'_z = m'u_z = mu_z = p_z$

Hence if E' corresponds to E, in frame S', then

$E' = m'c^2 = \beta\,(mc^2 - mv\,u_x)$ by (4)

$= \beta\,(E - vp_x)$

Hence the required transformations are

$$p'_x = \beta\left(p_x - \frac{vE}{c^2}\right);\ p'_y = p_y,\ p'_z = p'_z = p_z,\ E' = \beta(E - vp_x)$$

where $$\beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Problem 16. *Show that* $p^2 - \frac{E^2}{c^2}$ *is Lorentz invariant.*

Using notations of Problem 15, we have

$$p^2 = p_x^2 + p_y^2 + p_z^2 \text{ and } p'^2 = p_x'^2 + p_y'^2 + p_z'^2$$

$$\therefore\ p'^2 - \frac{E'^2}{c^2} = p_x'^2 + p_y'^2 + p_z'^2 - \frac{1}{c^2}\beta\,(E - vp_x)^2 \quad \because\ E' = \beta(E - vp_x)$$

$$= \beta^2\left(p_x - \frac{vE}{c^2}\right)^2 + p_y^2 + p_z^2 - \beta^2(E - vp_x)^2 \cdot \frac{1}{c^2}$$

$$= \beta^2\left[-\frac{1}{c^2}\left(1 - \frac{v^2}{c^2}\right)E^2 + p_x^2\left(1 - \frac{v^2}{c^2}\right)\right] + p_y^2 + p_z^2$$

$$= p_x^2 + p_y^2 + p_z^2 - \frac{E^2}{c^2} \qquad \because\ \beta^2\left(1 - \frac{v^2}{c^2}\right) = 1$$

$$= p^2 - \frac{E^2}{c^2}$$

which proves the required invariance.

Problem 17. *Discuss* **space-like** *and* **time-like** *intervals.*

(Agra, 1966, 67, 70, 75)

If S, S' be two frames of reference such that S' is moving with velocity v along X-axis relative to S, then Lorentz transformations are

$$x' = \beta(x - vt),\ y' = y,\ z' = z,\ t' = \beta\left(t - \frac{vx}{c^2}\right)$$

where $$\beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad ...(1)$$

Take two events with coordinates (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2) in S, then we have

$$s_{12}^2 = -\left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right] + c^2(t_2 - t_1)^2 \qquad ...(2)$$

[Since in Minkowski's four-dimensional space, the interval between two events (x, y, z, x_4) and $(x + dx, y + dy, z + dz, x_4 + dx_4)$ is given by $ds^2 = dx^2 + dy^2 + dz^2 + dx_4^2$.

On setting $x_4 = ict$ where we use to say the coordiantes of an event

(x, y, z, ict) as (x, y, z, t), we have

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2$$

For the system S', (2) becomes

$$s_{12}'^2 = -[(x'_2 - x'_1)^2 + (y'_2 - y'_1)^2 + (z'_2 - z'_1)^2] + c^2(t'_2 - t'_1)^2$$

$$= -[\beta^2\{(x_2 - x_1) - v(t_2 - t_1)\}^2 + (y'_2 - y'_1)^2] + c^2(z'_2 - z'_1)^2$$

$$+ c^2\beta^2\left\{(t_2 - t_1) - \frac{v}{c^2}(x_2 - x_1)\right\}^2$$

$$= -[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2] + c^2(t_2 - t_1)^2$$

$$\left[\text{on simplification and using } \beta^2\left(1 - \frac{v^2}{c^2}\right) = 1\right]$$

$$= s_{12}^2 \text{ by (2)}$$

$$\therefore \quad s'_{12} = s_{12} \qquad \ldots(3)$$

which follows that the interval s_{12} is Lorentz invariant.

Now if $s_{12} = 0$, the interval given by (2) is known as *singular* interval and then we have

$$-[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2] + c^2(t_2 - t_1)^2 = 0$$

which in the form of elements dx, dy, dz, dt, reduces to

$$-[dx^2 + dy^2 + dz^2] + c^2dt^2 = 0$$

This is termed as the equation of a *null cone.*

Now if we assume that the two events occur at the same point in S' and the first event occurs after the second one, so that

$$x'_2 = x'_1,\ y'_2 = y'_1,\ z'_2 = z'_1;\ t'_2 > t'_1$$

then, we have $\quad s'_{12} = c^2(t'_2 - t'_1)^2 > 0$

or $\quad s_{12}' > 0$

But $s_{12} = s_{12}'$ by (3), therefore $s_{12} > 0$ *i.e.* the interval s_{12} is real. Such intervals which are real are known as *time like intervals*, because s_{12}' contains only time component.

Hence the condition for time-like interval is

$$c^2(t_2 - t_1)^2 > (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.$$

Conclusively if an interval is time like, then it corresponds to a frame of reference in which the interval between two events is real.

Again if two events in S' occur at the same time so that $t'_1 = t'_2$ then we have

$$S_{12}'^2 = -[(x'_2 - x'_1)^2 + (y'_2 - y'_1)^2 + (z'_2 - z'_1)^2] < 0$$

i.e. $s_{12}^2 < 0$ and hence s_{12} is imaginary.

Such imaginary intervals are known as *space like intervals*, because S_{12}' contains only space coordinates.

Hence the condition for space like intervals is

$$c^2(t_2 - t_1)^2 < (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

Conclusively if an interval is space like, then it corresponds to a frame of reference in which the interval between two events is imaginary and the two events occur at the same instant of time.

Problem 18. *The rest of an electron is 9×10^{-28} gm, what will be its mass if it were moving with velocity 0.8 c.* (*Kanpur, 1968*)

Given $v = 0.8c$ so that $\frac{v^2}{c^2} = \left(\frac{0.8c}{c}\right) = 0.64.$

Also given that $m_0 = 9 \times 10^{-28}$ gm.

$$\therefore \quad m = \frac{m_0}{\left(1 - \frac{v}{c^2}\right)^{1/2}} = \frac{9 \times 10^{-28}}{(1 - 0.64)^{1/2}}$$

$$= \frac{9 \times 10^{-28}}{0.6} = 15 \times 10^{-28} \text{ gm.}$$

Problem 19. *Find the velocity that an electron must be given so that its momentum is 10 times its rest mass times the speed of light. What is the energy at this speed.* (*Agra, 1970*)

We have

$$p = 10 m_0 c = mv = \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}$$

or
$$1 - \frac{v^2}{c^2} = \frac{v^2}{100c^2}$$

i.e.
$$\left(1 + \frac{1}{100}\right)\frac{v^2}{c^2} = 1 \quad \text{or} \quad \frac{v^2}{c^2} = \frac{100}{101}$$

or
$$v = c\sqrt{\frac{100}{101}} = 0.995c$$

or
$$v = 0.995 \times 3 \times 10^{10} \text{ cm/sec.}$$
$$= 2.985 \times 10^{10} \text{cm./sec.}$$

which is the required velocity.

Again, $m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{9 \times 10^{-28}}{\left(1 - \frac{100}{101}\right)^{1/2}} = 90.36 \times 10^{-28}$gm.

Also $E = mc^2 = 90.36 \times 10^{-28} \times (3 \times 10^{10})^2 = 8.13 \times 10^{-6}$ ergs.

ADDITIONAL MISCELLANEOUS PROBLEMS

Problem 20. *Apply the law of relativistic composition of velocities to derive the elementary formula for aberration of light.* (*Agra, 1964; Banaras, 1970*)

[**Hint.** *Taking θ, θ' as angle between X-axis and the direction of motion of a light signal, in S, S' take $u_x = c \cos \theta$, $u_y = -c \sin \theta$, $u_z = 0$ $u'_x = c \cos \theta'$, $u'_y = -c \sin \theta'$,*

$u'_z = 0$ and use Lorentz transformation to prove $\tan\theta' = \dfrac{\tan\theta\left(1-\dfrac{v^2}{c^2}\right)^{1/2}}{1+\dfrac{v}{c}\sec\theta}$ *which in classical treatment takes the shape* $\delta\theta = \dfrac{v}{c}\sin\theta$]

Problem 21. *Give relativistic treatment of Doppler's effect and distinguish it from the classial treatment.*

(Kanpur, 1971; Agra, 1965)

Problem 22. *How much electric energy could theoretically be obtained by annihilation of 1 gm. of matter.*

(Agra, 1961)

[**Ans.** 5.885×10^{32} *e.v.*]

Problem 23. *What is the annual loss in the mass of the sum, if approximately 2 calories of radiated energy are received by each square cm. of the earth's surface per minute ?*

(Agra, 1954)

[*Take distance of sun from the earth* 150×10^6 *k.m.*] [**Ans.** 6.915×10^{16} *kgm/year*]

Problem 24. *Suppose that total mass of 1 kg is transformed into energy, how large is this energy in kilo-watt hours.*

(Agra, 1959)

[**Ans.** 2.5×10^{10} *K.W.H.*]

Problem 25. *Caluculate the Kinetic energy of an electron moving with a velocity of 0.98 times the velocity of light in the laboratory system* *(Vikram, 1967)*

Ans. 3.25×10^{-6} *ergs.*1

CHAPTER 15

STATISTICAL PROBABILITY

15.1. INTRODUCTION

While reading newspapers and magazines in our daily life, we come across the word 'statistics', the use of which is not new, but its use in the present meaning is older than the year 1839 when the American Statistical Association was founded. The words 'statist', satistics' and 'statistical', seem to be derived more or less indirectly from the Latin word '*status;* means a political state'.

The dictionary meaning of statistics is 'numerical data collected sytematically', or 'the science of collecting and interpreting such information'. According to its conventional use the word statistics may be defined as follows :

"By 'statistics' we mean quantitative data affected to a marked extent by multiplicity of causes, the 'statistical methods' mean elucidation of quantitative data affected by a multiplicity of causes and 'theory of statistics' means the exposition of statistical methods."

As a historical account Mr W. Hooper, M.D. (1770) while translating 'The Elements of Universal Erudition' written by Baron J.F. Von Bielfeld, defines the word 'statistics' as "the science that teaches us what is the political arrangement of all modern states of the known world". The German philosopher E. A. W. Zimmermuann (1787) defines the word '*statistik*' (statistics) in the preface to 'A Political Survey of the Present State of Europe' as "that branch of the political knowledge, which has for its object the actual and relative power of the several modern states, the power arising from their natural advantages, the industry and civilisation of their inhabitants and the wisdom of their Government...."

In early years statistics was supposed to be the science of kings, used for the purpose of administration while in later stage it was considered as necessarily a branch of economics. Now-a-days meanings of statistics are interpreted in different ways by different classes. To a *layman* 'statistics' is nothing but a collection of figures and a *Statistician* is no more than a computer who always counts the number of things. To an *economist* the field of statistics lies in quantitative analysis. To a *physicist* statistics is a probability distribution which forms the basis of the theory of errors.

The 'statistics' which was primarily regarded as a branch of economics has now become so popular and its application has become so wide that no branch of human knowledge escapes its approach. The prediction of H.G. Wells that "statistical thinking will one day be as necessary for efficient citizenship as the ability to read and write" now seems to become exactly true. In fact the statistical knowledge is going to become an unavoidable part of general education which provides the student according to George W. Snedecor "an awareness of, and harmony with, the statistical content of the society" as mentioned in his article 'A Proposed Basic Course in Statistics'.

Today the naturalists, the biologists, the astronomers, the administrators, the businessmen, the economists, the chemists, the physicists, the photographers, all make frequent use of statistical methods of which the *probability*, because of the nature of statistical data and models, is the fundamental tool in statistical theory. Regarding probability as an idealization of the proportion of times that a certain result will occur in repeated trials of experiment, a probability model is the type of mathematical model. Consequently an astronomer uses statistical methods in making predictions about eclipses, a biologist utilizes them to generalize the laws of variations and heredity, a meteorologist uses them for weather forecasts, regarding temperature pressure and rainfall, etc. Due to the wide scope of statistics, it is almost impossible for any statistician to be expert in all branches.

The *statistical probability* can be successfully employed in finding the possible configurations of the birthdays of people in a year, in classifying the accidents according to the weekdays, in firing to get the number of hits at a number of targets, in sampling to classify people according to age or profession, in irradiation in biology when the cells in retina of the eye are exposed to light, in cosmic ray experiments to find the number of particles hitting the Geiger counters, in an elevator to find the different arrangements of discharging the passengers, in dice-rolling to find the possibie outcomes of throw, in chemistry when a long chain poylmer reacts with oxygen, in theory of photographic emulsions to find whether a grain reacts if it is hit by a certain number of quanta and in finding the possible distributions of misprints in a certain number of pages of a book.

The dictionary meaning of *probability* is *'likelihood'* or *'anything that has appearance of truth.* It seems that it has been derived from the Latin word *'probare'* meaning 'to prove'. For instance if we draw 99 balls out of an urn containing 100 balls and it is per chance that all the 99 balls are of green colour, then it is always possible or probable that the remaining one ball may be of some other colour say red, white or black. Though the uniformity of the colour of 99 balls is unable to confirm that the remaining ball is also green in colour unless we are told that all the hundred balls are of the same colour, but the conclusion that the balls are all green is not based on certainty rather than it is based on 'likelihood' or 'probability'.

In dealing with the mathematical theory of probability we give a numerical measure to the probability. For example if we toss a coin, then the probability of falling the head or the tail up is equal, or mathematically speaking, the probability of falling the head or the tail up each is half *i.e.*, $\frac{1}{2}$. As another example, the probability of drawing a heart from a pack of cards is $\frac{1}{4}$ as there are four colours (heart, diamond, spade and club) in all.

In fact the probability plays the same role in mathematics as the mass in mechanics *e.g.*, the motion of the planetary system can be discussed without knowing their individual masses. But in statistics we are concerned much with physical or statistical probabilities which do not refer to judgements but the possible outcomes of a conceptual experiment

Initially R.A. Fisher and R. Von Mises developed the statistical or empirical attitude towards probability.

15.2. DEFINITIONS OF PROBABILITY

Based on classical concepts, we give two definitions of probability.

[*A*] **The mathematical or 'a priori' probability.** If there are q number of exhaustive, mutually exclusive and equally likely cases of an event and suppose that p of them are favourable to the happenings of an event A under the given set of conditions, then the mathematical probability of the event A is defined as

$$P(A) = \frac{p}{q}.$$

We sometimes put this definition in the words 'the odds in favour of A are p to $(q - p)$' or 'the odds against A are $(q - p)$ to p'. More precisely if we assume that the odds in favour of the event A are m to n (or n to m against A), the probability of happening the event A is defined as

$$P(A) = \frac{m}{m+n}.$$

e.g. The probability of drawing a white ball from a bag containing 3 white and 4 red balls is $\frac{3}{3+4}$ *i.e.* $\frac{3}{7}$.

Note 1. *The word 'exhaustive' used in the definition assures the happening of an event either in favour or against and rules out the possibility of happening of neither (in favour or against) in any trial. The word, 'mutually exclusive' is a safeguard against the probability of two simultaneous happening in a trial, e.g., in tossing a coin, the head and tail cannot fall together, but falling of either excludes the other. The word 'equally likely' means equally probable, i.e., no happening is biased or partially bound to occur.*

[*B*] **The statistical or empirical or 'a posteriori' probability.** If in a large number of trials performed under the same conditions, the limit of the ratio of the number of happenings of an event A to the total number of trials is unique and finite when the number of trials tends to infinity, then this limit measures the probability of the happening of the event A.

Thus if in a large number of trials performed under the same set of conditions, p is the probability of happening of an event A and q that of its failure, then the probability of its happening in the next trial is $\frac{p}{p+q}$, being assumed to determine the empirical probability that there is no known information relative to the probability of the happening of the event other than the past trials.

In other words, if an event A happens on pN occasions when a large number N is taken out of a series of trials, then the probability $P(A)$ of the event A is p defined as

$$P(A) = \lim_{N \to \infty} \frac{pN}{N} = p.$$

Precisely if m is the number of times in which the event A occurs in a series of n trials, then $P(A) = \lim_{n \to \infty} \frac{m}{n}$.

Note 2. If p is the probability of happening of an event A, i.e., $P(A) = p$ *and q that of not happening of that event denoted by* $P(\bar{A})$ *is given by* $P(\bar{A}) = q = 1 - p$, *so that*

$$P(A) + P(\bar{A}) = 1.$$

Conclusions. (*i*) The probability $P(A)$ of an event A lies between 0 and 1, *i.e.*,

$$0 \le P(A) \le 1.$$

(*ii*) The probability of an impossible event is zero, *i.e.*,

$$P(0) = 0.$$

(*iii*) The probability of a certain event E is one, *i.e.*,

$$P(E) = 1.$$

Problem 1. *Discuss the meaning and scope of statistics and show how it can be applied to social and physical sciences.*

If seems that the word 'statistics' was derived from the Latin word '*status*' which means a political state. The dictionary gives the meaning of statistics as 'numerical data collected systematically' or the science of collecting and interpreting such information.' According to its popular use, by statistics we mean quantitative data affected to a marked extent by multiplicity of causes.

Primarily statistics was supposed to be the science of kings used for the purpose of administration, but later on it was regarded as a branch of economics. Nowadays the meaning of statistics are interpreted in different ways by different classes, *e.g.*, for a layman statistics is nothing but a collection of figures, to an economist the field of statistics lies in quantitative analysis and to a physicist statistics is a probability distribution which forms the basis of the theory of errors.

Statistics, which was accepted for some time as necessarily a branch of economics has now become so popular and its application so wide that no branch of human knowledge escapes its approach. Today the naturalists, the biologists, the astronomers, the administrators, the businessmen, the economists, the chemists, the physicists, the photographers, all make frequent use of statistical methods of which probability is the fundamental tool. An astronomer uses statistical methods in making predictions about eclipses, a biologist utilizes them to generalize the laws of variations and heredity, a meteorologist uses them for weather forecasts, regarding temperature pressure and rainfall, etc.

So far as the applications of statistics to social and physical sciences are concerned, the statistical probability can be successfully employed in finding certain inferences in social and physical fields. Dealing with probability, we are able to show that the distribution of r balls in n cells has n^r different ordered samples with replacement of size r and without replacement it has ${}^nP_r = \frac{n\,!}{(n-r)\,!}$ different ordered samples of size r. Thus to find the possible configuration of birthday of people in a year, take r the number of people and n the number of days in a year; to classify accidents according to the week days, take r the number of accidents and n the number of days in a week; to classify people according to age or profession, take a group of r people and n will be the number of classes; to observe irradiation in biology when the cells in the retina of the eye are exposed to light r is the number of light particles and n the number of cells; in cosmic ray experiments, r is the number of particles hitting the Geiger counters and n is the number of counters function; in an elevator to find the possible arrangements of discharging the passengers, r is the number of passengers and n is the number of floors; in dice throw r is the number of dice and n is the number of faces *i.e.* six; in tossing coins r is the number of coins and $n = 2$; in coupon collecting r is the number of coupons collected and n are the kinds of coupons; in chemistry when a long chain polymer reacts with oxygen, r is the number of the reacting oxygen molecules and n is the number of polymer chains, etc.

Problem 2. *From a pack of 52 cards two are drawn at random; find the chance that one is a knave and other a queen.*

Total number of ways of drawing 2 cards from 52 cards = ${}^{52}C_2$.

The required cards a knave and a queen appear in different four colours, therefore each card can be drawn in 4 different ways. But the two events happen simultaneously and hence the number of favourable ways = 4 × 4 = 16.

$$\therefore \text{ required probability} = \frac{16}{^{52}C_2} = \frac{16}{\frac{52.51}{1.2}} = \frac{8}{663}.$$

Problem 3. *What is the chance that a leap year, selected at random, will contain 53 Sundays ?*

There are 366 days in a leap year. Dividing 366 by 7, the number of days of a week, we conclude that the leap year consists of 52 complete weeks and 2 days more. These two days can be combined in 7 different ways as under :

(*i*) Sunday and Monday, (*ii*) Monday and Tuesday (*iii*) Tuesday and Wednesday, (*iv*) Wednesday and Thursday, (*v*) Thursday and Friday, (*vi*) Friday and Saturday, (*vii*) Saturday and Sunday.

Of these seven combinations only (*i*) and (*vii*) are favourable so that the required chance = $\frac{2}{7}$.

Problem 4. *From a bag containing 4 white and 5 black balls a man draws 3 at random; what are the odds against these being all black?*

Total number of ways in which 3 balls can be drawn = $^9C_3$.

Number of ways in which 3 black balls can be drawn = $^5C_3$.

$$\text{Required chance} = \frac{^5C_3}{^9C_3} = \frac{5.4.3.}{1.2.3.}\Big/\frac{9.8.7}{1.2.3.} = \frac{5}{42}.$$

Problem 5. *A card is drawn from an ordinary pack and a gambler bets that it is a spade or an ace. What are the odds against his winning this bet?*

Number of ways in which a card can be drawn from 52 cards = $^{52}C_1 = 52$.

There are four aces so that the number of ways in which a card can be an ace is $^4C_1 = 4$.

Now there are 13 spade cards of which one is an ace. Out of the remaining 12 spade cards, a spade card can be drawn in $^{12}C_1$, *i.e.* 12 number of ways.

$\therefore$ The number of ways in which the drawn card may be a spade or an ace = 12 + 4 = 16.

$$\text{Hence the required probability} = \frac{16}{52} = \frac{4}{13} = \frac{4}{9+4},$$

i.e. the odds against the gambler's winning are as 9 to 4.

Problem 6. *The chance of an event happening is the square of the chance of a second event but the odds against the first are the cube of the odds against the second. Find the chance of each.*

Let the chance of the happenings of the first and the second events be p and p' respectively. Then according to the first condition, we have

$$p = p'^2. \qquad \text{...(1)}$$

According to the second condition, we have

$$\frac{1-p}{p} = \left(\frac{1-p'}{p'}\right)^3. \qquad \text{...(2)}$$

Substituting the value of p from (1) to (2), we get

$$\frac{1-p'^2}{p'^2} = \left(\frac{1-p'}{p'}\right)^3$$

or $$\frac{(1-p')(1+p')}{p'^2} = \frac{(1-p')(1-2p'+p'^2)}{p'^2}$$

or $$p'(1+p') = 1 - 2p' + p'^2$$

or $$p' + p'^2 = 1 - 2p' + p'^2$$

or $$3p' = 1, \text{ i.e., } p' = \tfrac{1}{3}$$

and then from (1) $$p = \tfrac{1}{9}.$$

Problem 7. *Three cards are drawn at random from an ordinary pack. Find the chance that they are a king, a queen and a knave.*

Total number of ways in which 3 cards can be drawn from 52 cards = ${}^{52}C_3$.

The pack or cards consists of 4 kings, 4 queens and 4 knaves and therefore each of a king, a queen and a knave can be drawn in ${}^4C_1$, *i.e.* 4 ways. But all the three events happen together, hence the number of ways in which a king, a queen and a knave can be drawn

$$= 4 \times 4 \times 4 = 64.$$

$\therefore$ required probability $= \dfrac{64}{{}^{52}C_3} = \dfrac{64.1.2.3}{52.51.50} = \dfrac{16}{5525}$.

Problem 8. *Eight letters, to each of which corresponds an envelope, are placed in the envelopes at random. What is the probability that all letters are not placed in the right envelopes.*

Total number of ways in which 8 letters can be placed in 8 envelopes = 8!.

Also there is only one way in which all the letters are placed in their right envelopes.

$\therefore$ Probability that all the letters are placed in the right envelopes $= \dfrac{1}{8!}$.

Hence the required probability that all the letters are not placed in their right envelopes $= 1 - \dfrac{1}{8\,!}$.

Problem 9. *A and B stand in a ring with 10 other persons. If the arrangement of the persons is at random, find the chance that there are exactly 3 persons between A and B.*

In a ring 12 persons can stand in 11 ! ways and 3 persons between A and B can be selected in ${}^{10}C_3$ ways. A and B interchange their positions in 2 ! ways. Also 3 persons between A and B can stand in 3 ! ways and the other 7 persons in 7 ! ways.

$$\therefore \text{ Number of favourable ways } = 2\,!\,3\,!\,7\,!\,{}^{10}C_3$$

$$= 2\,!\,3\,!\,7\,!,\frac{10\,!}{7\,!\,3\,!}$$

$$= 2\,!\,10\,!.$$

Required probability $= \dfrac{2\,!\,10\,!}{11\,!} = \dfrac{2}{11}$.

Problem 10. *A number is chosen from each of two sets*

(1, 2, 3, 4, 5, 6, 7, 8, 9); (1, 2, 3, 4, 5, 6, 7, 8, 9).

If p_1 denotes the probability that the sum of the two numbers be 10 and p_2 the probability that their sum is 8, find $p_1 + p_2$.

Each set consists of 9 numbers and hence the total number of ways of choosing one from each = ${}^9C_1 \times {}^9C_1 = 81$.

A sum of 10 may be found in 9 different ways, like (1, 9), (2, 8), (3, 7), (4, 6), (5, 5), (6, 4), (7, 3), (8, 2), (9, 1) so that

$$p_1 = \frac{9}{81} = \frac{1}{9}.$$

Similarly a sum of 8 can be found in 7 ways so that $p_2 = \frac{7}{81}$.

$$\therefore \qquad p_1 + p_2 = \frac{1}{9} + \frac{7}{81} = \frac{16}{81}.$$

15.3. EVENTS

A collection of all possible outcomes of an experiment is said to be an **event.**

If such a collection contains the outcome of an event, then that event is said to have occurred.

An event is said to be *simple* or *compound* according as it cannot or can be decomposed.

e.g. if we toss a coin, it will turn up either a head or a tail. Thus there are only two possible outcomes giving a simple event.

If we throw a pair of dice, then to have sum of 5 is a compound event as a sum of 5 can be obtained as (1, 4), (2, 3), (3, 2), (4, 1). Thus this compound event consists of 4 elementary events.

Mutually exclusive events. Two or more events are said to be mutually exclusive if the happening or occurrence of any one of them excludes the happening of the others.

For example, if we toss a coin and it falls with head up, then the falling of the head up excludes the simultaneous happening of the tail up *i.e.*, the two events of the falling head and tail up with a coin cannot happen together, but the happening of one excludes the happening of the other. So the two events are mutually exclusive.

Compound events (Joint occurrence). The simultaneous occurrence of two or more events in connection with each other is said to be a compound event.

For example, if we have an urn containing 100 balls of different colours say red and green and suppose 60 are red and 40 are green. If then it is proposed to draw 10 balls each of red colour, it is a simple event. But if it is proposed to draw first 20 balls of red colour and then 10 balls of green colour, then it is a compound event.

Dependent and independent events. Two or more events are said to be *dependent* or *independent* according as the occurrence of one does or does not affect the occurrence of the other or others. The dependent events are sometimes known as *contingent*.

For example, if from an urn containing 10 balls, it is proposed to draw 2 balls, then if a ball is drawn and it is not replaced unless the second ball is drawn, the event of the drawing of the second ball is dependent on that of the first. But if first ball is replaced and then the second ball is drawn, the event of drawing the second ball is independent.

15.4. THEOREM OF TOTAL PROBABILITY, i.e. ADDITIVE LAW OF PROBABILITY

If there are n mutually exclusive events $A_1, A_2, A_3, \ldots A_n$ whose probabilities are $P(A_1)$, $P(A_2)$, $P(A_3), \ldots P(A_n)$ respectively, then the probability that one of them will happen is the sum of their separate probabilities, i.e.,

$P(A_1 + A_2 + A_3 + \ldots + A_n) = P(A_1) + P(A_2) + P(A_3) + \ldots + P(A_n)$, where $P(A_1 + A_2 + \ldots + A_n)$ denotes the probability of occurrence of at least one of the n events $A_1, A_2, A_3, \ldots A_n$.

Suppose there are N total number of exhaustive, mutually exclusive and equally likely cases of which $m_1, m_2, \ldots m_n$ are favourable to the events $A_1, A_2, \ldots A_n$ respectively. Then the total number of cases favourable to either A_1 or A_2 or A_3 or...or A_n is $m_1 + m_2 + m_3 + \ldots + m_n$ so that the probability of happening of at least one of these events is

$$P(A_1 + A_2 + A_3 + \ldots + A_n) = \frac{m_1 + m_2 + m_3 + \ldots + m_n}{N}$$

$$= \frac{m_1}{N} + \frac{m_2}{N} + \ldots + \frac{m_n}{N}$$

$$= P(A_1) + P(A_2) + P(A_3) + \ldots + P(A_n).$$

Conclusions. (*i*) In case an event A is comprised by n mutually exclusive forms $A_1, A_2, \ldots A_n$, *i.e.*

$$A = A_1 + A_2 + \ldots + A_n,$$

then probability A, *i.e.* $P(A)$ is the sum of the probabilities of $A_1, A_2, \ldots A_n$ separately, *i.e.*,

$$P(A) = P(A_1) + P(A_2) + \ldots + P(A_n).$$

(*i*) in case the n mutually exclusive events are exhaustive also, so that there is certainty of happening of at least one, *i.e.*, $P(A_1 + A_2 + \ldots + A_n) = 1$, we have

$$P(A_1) + P(A_2) + \ldots + P(A_n) = 1.$$

Note. We generally use the following notations:

$P(A)$ denotes the probability for an event A to happen,

$P(\bar{A})$	"	"	"	"	A not to happen,
$P(A+B)$	"	"	"	"	occurrence of at least one of the events A and B
$P(AB)$	"	"	"		occurrence of both the events A and B,
$P(A\bar{B})$	"	"	"		happening of A and not of B,
$P(\bar{A}B)$	"	"	"	"	" B " A,
$P(\overline{AB})$	"	"	"		happening of neither of A and B,

If A and B are two events such that AB and $A\bar{B}$ are two exhaustive and mutually exclusive forms in which A can occur, then we have

$$P(A) = P(AB) + P(A\bar{B}) \text{ from conclusion } (ii)$$

Similarly, $$P(B) = P(BA) + P(B\bar{A})$$

$$= P(AB) + P(\bar{A}B).$$

so that $$P(A) + P(B) = P(AB) + \{(A\bar{B}) + P(\bar{A}B) + P(AB)\}.$$

But from the above theorem of total probability, we can write

$$P(A+B) = P(A\bar{B}) + P(\bar{A}B) + P(AB),$$

i.e., probability that at least one of A and B happens = the sum of probabilities that A happens B not, B happens A not, and A, B both happen.

Thus, $$P(A) + P(B) = P(AB) + P(A+B)$$

or $$P(A+B) = P(A) + P(B) - P(AB). \quad \ldots(1)$$

Generalization of this result. *To prove that the formula for the probability of occurrence of at least one of the n give events* $A_1, A_2, \ldots A_n$ *is*

$$P(A_1 + A_2 + \ldots + A_n) = S_1 - S_2 + S_3 - \ldots + (-1)^{n-1} S_n$$

where S_r *Stands for the sum of probabilities of simultaneous occurrence of exactly r of n events, the summation extending over all possible combinations.* (*Agra, 57, 62*)

For two mutually exclusive events A_1 and A_2 the result (1) gives $P(A_1 + A_2) = P(A_1) + P(A_2) - P(A_1 A_2)$.

Similarly,

$$\begin{aligned} P(A_1 + A_2 + A_3) &= P(A_1) + P(A_2 + A_3) - P\{A_1(A_2 + A_3)\} \\ &= P(A_1) + P(A_2) + P(A_3) - P(A_2 A_3) - P(A_1 A_2 + A_1 A_3) \\ &= P(A_1) + P(A_2) + P(A_3) - P(A_2 A_3) \\ &\qquad - P(A_1 A_2) - P(A_1 A_3) + P(A_1 A_2 A_3) \\ &= \sum_{i=1}^{3} P(A_i) - \sum_{\substack{i,j=1 \\ i \neq j}}^{3} P(A_i A_j) + P(A_1 A_2 A_3). \qquad \ldots(2) \end{aligned}$$

Thus it follows from induction method that if there are n events $A_1, A_2, A_3, \ldots A_n$, then the generalization of the result (2) gives

$$P(A_1 + A_2 + \ldots + A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{\substack{i,j=1 \\ i \neq j}}^{3} P(A_i A_j) + \sum_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{3} P(A_i A_j A_k)$$

$$- \ldots + (-1)^{n-1} P(A_1 A_2 \ldots A_n) \qquad \ldots(3)$$

Denoting by $S_1, S_2, \ldots, S_n$ the sum of the probabilities of simultaneous occurrence of exactly 1, 2,..., n of the n events, the summation extending over all possible combinations, we have

$$P(A_1 + A_2 + \ldots + A_n) = S_1 - S_2 + S_3 - \ldots + (-1)^{n-1} S_n.$$

COROLLARY. In case the events $A_1, A_2, \ldots, A_n$ are mutually exclusive, then

$$P(A_i A_j) = 0,$$
$$P(A_i A_j A_k) = 0,$$

....

$$P(A_1 A_2 \ldots A_n) = 0,$$

so that (3) reduces to the theorem of total probability, *i.e.*,

$$P(A_1 + A_2 + \ldots + A_n) = \sum_{i=1}^{n} P(A_i) = P(A_1) + P(A_2) + \ldots + P(A_n).$$

Problem 11. *If the probability of a horse A winning a race is* $\frac{1}{5}$ *and the probability of a horse B winning the same race is* $\frac{1}{6}$, *what is the probability that one of the horses wins.*

Let p_1 and p_2 be the probabilities of A and B respectively; then

$$P_1 = \tfrac{1}{5} \text{ and } p_2 = \tfrac{1}{6}.$$

The two events being mutually exclusive, the probability that one of them wins

$$= P_1 + p_2$$
$$= \tfrac{1}{5} + \tfrac{1}{6} = \tfrac{11}{30}.$$

Problem 12. *Six cards are drawn at random from a pack of 52 cards. What is the probability that 3 will be red and 3 black?*

Total number of ways of drawing 6 cards out of 52 = ${}^{52}C_6$.

There are 26 red and 26 black cards; therefore the number of ways in which 3 red and 3 black cards can be drawn

$$= {}^{26}C_3 \times {}^{26}C_3.$$

$$\therefore \text{ Required probability } = \frac{{}^{26}C_3 \times {}^{26}C_3}{{}^{52}C_6}$$

$$= \frac{\dfrac{26.25.24}{1.2.3} \times \dfrac{26.25.24}{1.2.3}}{\dfrac{52.51.50.49.48.47}{1.2.3.4.5.6}} = \frac{13000}{39151}$$

Problem 13. *The first twelve letters of the alphabet are written down at random. What is the probability that there are four letters between the letters A and B?*

Denoting the positions of letters as

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12,

when A is kept at 1, B should be placed at 6 to have four letters in between

"	2	"	"	7	"
"	3	"	"	8	"
"	4	"	"	9	"
"	5	"	"	10	"
"	6	"	"	11	"
"	7	"	"	12	"

Thus A and B can be placed in 7 ways so as to have four letters in between. Also A and B can interchange their positions in 2! ways. The four letters between A and B can be chosen in ${}^{10}C_4$ ways out of remaining 10 letters when A and B have already been placed. Moreover, these four letters can be arranged in 4! ways and the remaining 6 letters in 6! ways.

$\therefore$ Number of favourable ways = $7.2!.\,{}^{10}C_4\,4!\,6!$

and total number of arrangements in which 12 letters can be put = 12!

$$\text{Hence the required probabilities } = \frac{7.2!.\,{}^{10}C_4\;4!\;6!}{12!}$$

$$= \frac{7.2!\cdot 10!\cdot 4!\cdot 6!}{4!\cdot 6!\cdot 12!} = \frac{14}{12.11}$$

$$= \frac{7}{66}.$$

Problem 14. *One of two events must happen; given that the chance of the one is two-thrids that of the other, find the odds in favour of the other.*

Let p_1, p_2 be the probabilities of the two events and suppose that x is the chance of happening of either, say

$$p_1 = x \text{ so that } p_2 = \tfrac{2}{3}x.$$

But we have $p_1 + p_2 = 1$ for a sure event,

i.e., $x + \frac{2}{3}x = 1$ or $\frac{5}{3}x = 1$ giving $x = \frac{3}{5}$.

$\therefore \quad p_1 = \frac{3}{5}$ and as such $p_2 = \frac{2}{3} \cdot \frac{3}{5} = \frac{2}{5}$.

Thus odds in favour of the other $= \frac{2}{5} : \frac{3}{5}$, *i.e.*, 2 : 3.

15.5. THEOREM OF COMPOUND PROBABILITY OR MULTIPLICATIVE LAW OF PROBABILITY

If there are two events A and B, probabilities of their happening being P(A) and P(B) respectively, then the probability P(AB) of the simultaneous occurrence of the events A and B is equal to the probability of A multiplied by the conditional probability of B (i.e., the probability of B when A has occurred) or the probability of B multiplied by the conditional probability of A, i.e.,

$$P(AB) = P(A)\,P(B/A)$$
$$= P(B)\,P(A/B),$$

where P(B/A) denotes conditional probability of B and P(A/B) that of A and that if the two events are independent, then the theorem of compound probability is

$$P(AB) = P(A)\,P(B).$$

Suppose there are N total number of mutually exclusive and equally likely cases of which m are favourable to A. Let m_1 be the number of cases favourable to A and B both, while m_1 included in m. Thus

$$P(B/A) = \frac{m_1}{m} \text{ and } P(A) = \frac{m}{N}.$$

Now $P(AB)$ = the probability of happening of A and B both

$$= \frac{m_1}{N} = \frac{m_1}{m} \cdot \frac{m}{N}$$
$$= P(A)\,P(B/A). \qquad \ldots(1)$$

The interchange of A and B will yield a similar result

$$P(AB) = P(BA) = P(B)\,(A/B). \qquad \ldots(2)$$

In case the two events A and B are independent, *i.e.*, the occurrence of one does not affect the other, $P(B/A)$ is the same as $P(B)$ and $P(A/B)$ is the same as $P(A)$, so that the results (1) and (2) both become $P(AB) = P(A)\,P(B)$.

Generalization. The result (3) may be generalized as, if there are $A_1, A_2, \ldots A_n$, n mutually independent events, then the compound probability is given by

$$P(A_1, A_2, A_3 \ldots A_n) = P(A_1)\,P(A_2) \ldots P(A_n) \qquad \ldots(4)$$

In case of n mutually exclusive events $A_1, A_2, \ldots A_n$, the result (1) may be generalised as

$$P(A_1A_2 \ldots A_n) = P(A_1)\,P(A_2/A_1)\,P(A_3/A_1A_2) \ldots P(A_n/A_1A_3 \ldots A_{n-1}).$$

Conclusions. (*i*) If p be the chance that an event will happen in one trial, the chance that it will happen in any assigned succession of r trials is p^r; for in this case

$$P(A_1) = P(A_2) = \ldots = P(A_r) = p$$

$\therefore$ required probability $= P(A_1)\,P(A_2) \ldots P(A_r)$

$$= p.p \ldots r \text{ times} = p^r.$$

(*ii*) If $p_1, p_2, p_3, \ldots p_n$ are the probabilities that n events happen, then probability that all the events fail is

$$(1 - p_1)(1 - p_2)(1 - p_3) \ldots (1 - p_n)$$

Hence the chance that at least one of these events happens

$$= 1 - (1 - p_1)(1 - p_2) \ldots (1 - p_n)$$ (*IAS, 51*)

Problem 15. *In shuffling a pack of cards three are accidentally dropped; find the chance that the missing cards should be from different suits:*

The pack consists of 52 cards.

The chance of dropping a card $= \frac{{}^{52}C_1}{{}^{52}C_1} = 1.$

When one card is dropped, there remain 51 cards of which 39 cards are of suits different from that of dropped one. Thus the chance of dropping a card of different suit in second draw

$$= \frac{{}^{39}C_1}{{}^{51}C_1} = \frac{39}{51}.$$

When two cards are dropped, there remain 40 cards of which 26 cards are of suits different from thsoe of dropped cards. Thus the chance of dropping a card of different suit in third draw.

$$\frac{{}^{26}C_1}{{}^{50}C_1} = \frac{26}{50}.$$

The events being dependent, the required chance

$$= 1 \times \frac{39}{51} \times \frac{26}{50} = \frac{169}{425}.$$

Problem 16. *The face cards (three from each suit) are removed from a full pack. Out of the 40 remaining cards, 4 are drawn at random:*

(*a*) *What is the probability that they belong to different suit?*

(*b*) *What is the probability that the 4 cards drawn belong to different suits and different denominations.*

Having removed 12 face cards, the remaining 40 consist of 10 cards of each suit :

(*a*) Chance of drawing a card in first draw $= \frac{{}^{40}C_1}{{}^{40}C_1} = 1.$

Having drawn 1 card, there remain 39 cards of which 30 are of suits different from the drawn one.

∴ Chance of drawing a card of different suit in second draw

$$= \frac{{}^{30}C_1}{{}^{39}C_1} = \frac{30}{39}.$$

Having drawn two cards, there remain 38 cards of which 20 are of suits different from the drawn cards.

∴ Chance of drawing a card in third draw $= \frac{{}^{20}C_1}{{}^{38}C_1} = \frac{20}{38}$

Having drawn three cards, there remain 37 cards of which 10 are of suits different from drawn cards.

∴ Chance of drawing a card in fourth draw $= \frac{{}^{10}C_1}{{}^{37}C_1} = \frac{10}{37}.$

All the events being dependent, the required probability

$$= 1 \times \frac{30}{39} \times \frac{20}{38} \times \frac{10}{37} = \frac{1000}{9139}.$$

(*b*) Chance of drawing a card in first draw $= \frac{^{40}C_1}{^{40}C_1} = 1.$

Having drawn one card, there remain 39 cards of which 9 cards are of the same suit and 3 of the same denomination (value) so that 27 cards out of 39 are such that they are of different colours and different denominations from the drawn one.

$\therefore$ Chance of drawing a card in second draw $= \frac{^{27}C_1}{^{39}C_1} = \frac{27}{39}.$

Similarly chance of drawing a card in third draw $= \frac{^{16}C_1}{^{38}C_1} = \frac{16}{38}$

and chance of drawing a card in fourth draw $= \frac{^{7}C_1}{^{37}C_1} = \frac{7}{37}.$

Problem 17. *From a pack of cards two cards are drawn, the first being repalced before the second is drawn, Find the probability that the first is a diamond and the second is a king.*

Let A denote the event of drawing a diamond and B denote the event of drawing a king in the second draw, when the first card has been replaced. Then

$P(A)$ = Probability of drawing a diamond

$$= \frac{^{13}C_1}{^{52}C_1} = \frac{1}{4}.$$

$P(B)$ = Probability of drawing a king

$$= \frac{^{4}C_1}{^{52}C_1} = \frac{1}{13}.$$

The two events being independent, we have

$$P(AB) = P(A)\,P(B) = \frac{1}{4} \times \frac{1}{13} = \frac{1}{52}.$$

Problem 18. *Find the chance of throwing a 6 at least once in two throws of a single die.*

Let A denote the event of throwing a six in the first throw and B that in the second throw. Then probability of throwing a 6 at least once in two throws may be represented by $P(A+B)$.

Now $P(A)$ = Prob. of throwing a 6 in first throw.

$$= \tfrac{1}{6}$$

and $P(B)$ = Prob. of throwing a 6 in second throw

$$= \tfrac{1}{6}.$$

But we have

$$P(A+B) = P(A) + P(B) - P(AB)$$

where $P(AB) = P(A)\,P(B)$, A, B being independent events

$$\therefore \quad P(A+B) = \tfrac{1}{6} + \tfrac{1}{6} - \tfrac{1}{6}\cdot\tfrac{1}{6}$$

$$= \tfrac{1}{3} - \tfrac{1}{36} = \tfrac{11}{36}.$$

Problem 19. *A coin is tossed three times. Find the probability of getting head and tail alternately.*

Let $P(A)$ and $P(B)$ represent the probability of getting head and tail respectively. Then

$$P(A) = \frac{1}{2} = P(B).$$

The alternate occurring, of head and tail may happen in two ways :

(*i*) staring with head,

(*ii*) starting with tail.

In case of first, the probability of the event

$$= P(A)\,P(B)\,P(A)$$
$$= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$$
$$= \frac{1}{8}.$$

In second case the probability of the event

$$= P(B)\,P(A)\,P(B)$$
$$= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$$
$$= \frac{1}{8}.$$

But the two events being mutually exclusive, the total probability of happening any one of them $= \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$.

Problem 20. *Four persons are chosen at random from a group containing 3 men, 2 women and 4 children. Show that the chance that exactly two of them will be children is* $\frac{10}{21}$.

Given	Men		Women		Children	Total no. of persons
	3	+	2	+	4	=9.

4 persons out of 9 can be selected in ${}^9C_4$ ways,

2 children out of 4 can be selected in ${}^4C_2$ ways.

When 2 children have to be selected, we are left with 5 persons (3 men and 2 women). To make the company of four including the two already selected children we can choose 2 persons out of 5 in ${}^5C_2$ ways.

$\therefore$ Number of ways of selecting 4 persons $= {}^4C_2 \times {}^5C_2$, the two events being independent.

$$\text{Hence the required probability} = \frac{{}^4C_2 \times {}^5C_2}{{}^9C_4} = \frac{10}{21}.$$

Problem 21. *In a bag there are 6 balls of which 3 ae white and 3 are black. They are drawn successively without replacement. What is the chance that the colours alternate?*

Suppose the first drawn ball is white. Then probability of its drawing is ${}^3C_1/{}^6C_1 = \frac{3}{6}$. In the second draw, probability of the ball being black is then ${}^3C_1/{}^5C_1 = \frac{3}{5}$ (because there remain only 5 balls after first draw). Thus

probability of white ball in third draw $= \dfrac{{}^2C_1}{{}^4C_1} = \dfrac{2}{4}$.

" black " fourth " $= \dfrac{{}^2C_1}{{}^3C_1} = \dfrac{2}{3}$.

" white " fifth " $\dfrac{{}^1C_1}{{}^2C_1}=\frac{1}{2}$,

" black " sixth " $=\dfrac{{}^1C_1}{{}^1C_1}=\frac{1}{1}$.

$\therefore$ Total probability of this event $=\frac{3}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{1}{1}=\frac{1}{20}$.

Similarly the probability of the event with a start of black ball $=\frac{3}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{1}{1}=\frac{1}{20}$.

Hence the two events being mutually exclusive, the probability of happening of any one of these events $=\frac{1}{20}+\frac{1}{20}=\frac{1}{10}$.

Problem 22. *Three groups of children contain 3 girls and 1 boy, 2 girls and 2 boys, 1 girl and 3 boys. One child is selected at random from each group. Show that the chance that the three selected cosist of 1 girl and 2 boys is* $\frac{13}{32}$.

One girl and 2 boys can be selected in three different ways as discussed below:

probability of selecting girl from 1st and boys from 2nd and 3rd groups $=\frac{3}{4}\times\frac{2}{4}\times\frac{3}{4}=\frac{9}{32}$,

probability of selecting girl from 2nd and boys from 1st and 3rd groups $=\frac{2}{4}\times\frac{1}{4}\times\frac{3}{4}=\frac{3}{32}$,

probability of selecting girl from 3rd and boys from 1st and 2nd groups $=\frac{1}{4}\times\frac{1}{4}\times\frac{2}{4}=\frac{1}{32}$.

All these events being mutually exclusive, the required probability $=\frac{9}{32}+\frac{3}{32}+\frac{1}{32}=\frac{13}{32}$.

Problem 23. *A can bit a target 2 times in 5 shots; B, 2 times in 5 shots; C, 3 times in 4 shots. They fire a volley. What is the probability that 2 shots hit ?*

There arise three cases, probabilities of which are as follows:

(*i*) Probability that *A* hits, *B* hits and *C* does not hit $=\frac{3}{5}\cdot\frac{2}{5}\cdot\frac{1}{4}=\frac{6}{100}$,

(*ii*) Probability that *A* hits, *B* does not hit, *C* hits $=\frac{3}{5}\cdot\frac{3}{5}\cdot\frac{3}{4}=\frac{27}{100}$.

(*iii*) Probability that *A* does not hit, *B* hits, *C* hits $=\frac{2}{5}\cdot\frac{2}{5}\cdot\frac{3}{4}=\frac{12}{100}$.

All these events being mutually exclusive, the required probability $=\frac{6}{100}+\frac{27}{100}+\frac{12}{100}=\frac{45}{100}=\frac{9}{20}$.

15.6. PROBABILITY: A NEW STANDPOINT

[A] Sample space or outcome space. *A set consisting of the elementary events as its elements is said to be a sample space. It is generally denoted by S.*

A sample space provides a mathematical model of an ideal experiment in the sense that every conceivable outcome of the experiment is completely described by one and only one sample point.

In fact an element in S is known as *sample point* or *sample* and an event *e.g.* event A is a subset of the sample space S. The event $A = \{a\}$ consisting of a single sample point $a \in S$ is said to be an *elementary event*. The null sets ϕ and S itself are events. The null set ϕ is said to be an *impossible event* while S is said to be a *sure* or *certain event*.

[B] **Correspondence between sets and events.** Let there be two events A and B. Then

(*i*) $A \cup B$ denotes an event which occurs *iff* (if and only if) A occurs or B occurs (or both occur).

(*ii*) $A \cap B$ denotes an event which occurs iff A occurs and B occurs.

(*iii*) The complementary event of A denoted by A' or A' is an event which occurs iff A does not occur.

$B - A$ will be the event consisting of all points not contained in the event A but contained in B, *i.e.*,

$$B - A = B \cap A'$$

$$\therefore \quad A - A' = \phi = A \cap A'.$$

Also $$A \cup A' = S.$$

A complementary event A' is always *mutually exclusive* and *exhaustive*.

(*iv*) Two events A and B are called *mutually exclusive* or *disjoint* if $A \cap B = \phi$.

Problem 24. *A coin is tossed and it is observed whether a head or tail is up. Describe the suitable sample space of the experiment.*

Denoting by H the event in which the coin turns up head and by T, the event in which the coin turns up tail, the sample space of the experiment consists of only two elements *i.e.*, $S = \{H, T\}$.

Problem 25. *If a die is thrown and the number of spots on the uppermost face is observed, describe the suitable sample space for this experiment.*

The spots on the six faces of a die are 1, 2, 3, 4, 5, 6. When the die is thrown any of the six numbers will appear on the uppermost face; therefore

$$S = \{1, 2, 3, 4, 5, 6\}.$$

Problem 26. *Let there be a pack of cards. Describe the suitable sample space for drawing a red card.*

There are 52 cards in all out of which 26 are red and other 26 are black. Denoting by $e_1, e_2, e_3, \ldots, e_{26}$ the events of drawing a red card, we have

$$S = \{e_1, e_2, \ldots, e_{26}\}.$$

Problem 27. *There is a box containing 4 chits numbered 1, 2, 3, 4. Describe the sample space of drawing two different numbered chits one after another.*

Every outcome will be an ordered pair such that $1 \le x \le 4$, $1 \le y \le 4$ but $x \ne y$ where x denotes the number of first drawn chit and y that of the second.

Possible outcomes may be tabulated thus:

x \ y	1	2	3	4
1	×	1, 2	1, 3	1, 4
2	2, 1	×	2, 3	2, 4
3	3, 1	3, 2	×	3, 4
4	4, 1	4, 2	4, 3	×

$\therefore$ the sample space consists of 12 ordered pairs.

Hence, $S = \{(x, y) ; 1 \le x \le 4, 1 \le y \le 4 \text{ and } x \ne y\}$.

Problem 28. *An urn contains three red and two white balls. Two balls are drawn and their colour is noted. Set up the sample space of this experiment.*

Let the red ball be numbered as R_1, R_2, R_3 and white balls as W_1, W_2.

Denoting the event of drawing ball R_1 in first draw and R_2 in second draw by R_1R_2, the sample space consists of 20 outcomes as given below :

$$S = \{R_1R_2, R_1R_3, R_1W_1, R_1W_2, R_2R_1, R_2R_3, R_2W_1, R_2W_2, R_3R_1, R_3R_2, R_3W_1, R_3W_2, W_1R_1, W_1R_2, W_1R_3, W_1W_2, W_2R_1, W_2R_2, W_2W_3, W_2W_1\}.$$

[C] The modern concept of probability. It is a well known fact that the concept of probability developed from evil habits of games of chance or gambling used in France in 17th century. In this connection the French nobleman and gambler Chevalier de méré consulted the well known mathemafician Blaise Pascal (1623-1662) who began to think over the problem how and to what degree of accuracy a gambler can be assured of his chance of success. Pascal solved the problem of de mere and had a correspondence with Pierre de Fermat (1601-1665) who became interested in this and other similar problems.

The phenomena occurring in nature of any secrets can be either *Deterministic* or *Probabilistic*, *e.g.*, if a train moves at the rate of 20 km./hr., it is deterministic that it will travel 100 km. in 5 hours, but if coin is tossed, then it is probabilistic to say that the chance of each either 'heat up' or 'tail up' is equal.

Actually the theory of probability deals with the things likely to occur and their chance or probable values. It is a measure of degree of uncertainty rather than accuracy.

The concepts of probability are connected with the events or occurrences and the Repeated trials.

[D] Probability of an event. If in a series of n trails all made under the same conditions an event A is observed m times, then m is said to be the frequency of success and the ratio $\frac{m}{n}$ is said to be the *relative frequency* of success.

The *probability* of the event A is defined to be the limit p of the ratio $\frac{m}{n}$ when n tends to infinity (if it exists), *i.e.*,

$$P(A) = p = \lim_{n \to \infty} \frac{m}{n}.$$

This is sometimes known as the *frequency definition of probability*. In other words, if the event A consists of r clear events out of n, then

$$P(A) = \frac{r}{n} = \frac{\text{Number of elementary events in } A}{\text{Number of elementary events in sample space } S} = \frac{N(A)}{N(S)}.$$

In terms of sample space S associated with Σ the class of events if P be a real-valued function defined on Σ, then P is said to be the *probability function* and $P(A)$ the *probability* of the event A which satisfies the following conditions:

(*i*) For every event A, $0 \le P(A) \le 1$.

(*ii*) $P(S) = 1$ *i.e.* $\sum_{i=1}^{n} P(A_i) = 1$

where $S = \{A_1\} \cup \{A_2\} \cup \ldots \cup \{A_n\}$.

If $S = \phi$ (a null set), then $\sum_{i=1}^{n} P(A_i) = 0$ *i.e.* $P(\phi) = 0$.

(*iii*) If A and B are two mutually exclusive (*i.e.* disjoint) events *i.e.* $A \cap B = \phi$, then $P(A \cup B) = P(A) + P(B)$.

(*iv*) If $A_1, A_2, A_3, \ldots, A_n$ are n mutually exclusive events then $P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$.

[E] Some theorems. *THEOREM* **1. The law of total probability.**

If A and B are two mutually exclusive events, then

$$P(A \cup B) = P(A) + P(B)$$

or in general if $A_1, A_2, A_3, \ldots, A_n$ *are n mutually exclusive events, then*

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n).$$

Out of m mutually exclusive and equally likely cases, let a correspond to the occurrence of event A and b corresponds to the occurrence of event B, so that the event $A \cup B$ occurs in $(a + b)$ cases due to the nature of the events A and B. Then,

$$P(A) = \frac{a}{m}, \quad P(B) = \frac{b}{m}$$

and

$$P(A \cup B) = \frac{a+b}{m}$$

$$= \frac{a}{m} + \frac{b}{m}$$

$$= P(A) + P(B). \qquad \ldots(1)$$

To generalize the theorem by mathematical induction, since the theorem is true for $n = 2$, let us assume that it is true for any positive integral value n *i.e.*

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_1) + \ldots + P(A_n). \qquad \ldots(2)$$

Let $A_1, A_2, A_3, \ldots, A_n, A_{n+1}$ be $(n + 1)$ mutually exclusive events, then

$$P(A_1 \cup A_2 \cup \ldots \cup A_n \cup A_{n+1}) = P(A_1 \cup A_2 \cup \ldots \cup A_n) + P(A_{n+1}) \text{ by (1)}$$

$$= P(A_1) + P(A_2) + \ldots + P(A_n) + P(A_{n+1}) \text{ by (2)}$$

This follows that the theorem is true for $n + 1$ and hence the theorem is true for any integral value n.

THEOREM 2. For any two events A and B, the probability that either A or B or both occur is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

$A \cup B$ can be decomposed into two mutually exclusive events $A - B$ and B *i.e.*,

$A \cup B = (A - B) \cup B$.

Then from theorem 1, we have

$$P(A \cup B) = P[(A - B) \cup B]$$

$$= P(A - B) + P(B)$$

Now A can be decomposed into two mutually exclusive events $A - B$ and $A \cap B$ *i.e.*

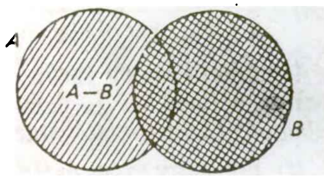


Fig. 15.1

$$A = (A - B) \cup (A \cap B),$$

so that $P(A) = P(A - B) + P(A \cap B)$ by theorem 1

or $P(A - B) = P(A) - P(A \cap B)$.

Thus $P(A \cup B) = P(A) - P(A \cap B) + P(B)$

$$= P(A) + P(B) - P(A \cap B). \quad \text{...(1)}$$

Aliter. $A \cup B = A \cup (B - A \cap B)$ where A and $B - A \cap B$ are disjoint.

$\therefore \quad P(A \cap B) = P(A) + P[B - A \cap B]$

we have $B = [A \cap B] \cup [B - A \cap B]$ where $A \cap B$ and $B - A \cap B$ are disjoint.

$\therefore \quad P(B) = P(A \cap B) + P[B - A \cap B]$.

Hence $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

COROLLARY. For any events A, B, C applying this theorem twice we may easily get

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \quad \text{...(2)}$$

THEOREM 3. Let A be an event then its complementary event $\bar{A}$ will be disjoint to A, then $P(A) = 1 - P(\bar{A})$.

Since A, $\bar{A}$ are mutually exclusive events,

$$P(A \cup \bar{A}) = P(A) + P(\bar{A}).$$

But we know that $P(S) = 1 = P(A) + P(\bar{A})$

giving $\quad P(A) = 1 - P(\bar{A})$.

THEOREM 4. If A and B are two events such that $A \subset B$, then

$$P(A) \le P(B).$$

Since $A \subset B$, therefore B can be decomposed into two mutually exclusive events A and $B - A$ *i.e.*,

$$B = A \cup (B - A),$$

so that by theorem 1,

$$P(B) = P(A) + P(B - A)$$

$\therefore P(B) \ge P(A)$ as $0 \le P(B - A) \le 1$.

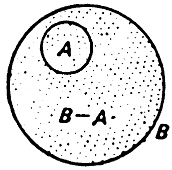


Fig. 15.2

THEOREM 5. If A and B are events, then

$$P(A \cup B) \le P(A) + P(B).$$

By theorem 2, we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$\therefore \; P(A \cup B) \le P(A) + P(B)$ as $0 \le P(A \cap B) \le 1$.

COROLLARY. This result may be generalised in case of n events $A_1, A_2, A_3, \ldots, A_n$ as

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) \le P(A_1) + P(A_2) + \ldots + P(A_n).$$

[F] Borel-field. We assume that a family B of certain subsets of S (outcome space) is given, which will be called events and that this family satisfies the following axioms :

B_1. The outcome space S and the empty set ϕ are in B *i.e.*

$$S \in B, \; \phi \in B.$$

B_2. If each set of finite or countable sequence $a_1, a_2, \ldots, a_i, \ldots$ is in B, then their union and intersection are in B *i.e.*,

$$A_i \in B \text{ for } i = 1, 2, \ldots, \Rightarrow \cup_i A_i \in B$$

$$A_i \in B \text{ for } i = 1, 2, \ldots, \Rightarrow \cap_i A_i \in B.$$

B_3. $A \in B \Rightarrow \bar{A} = S - A \in B$.

The family B satisfying these three axioms is defined to be *Borel-field B* on the outcome space S.

[G] Probability measure of a Borel-field. Given a outcome space S and a Borel-field B, we consider a set-function $P(A)$ on B *i.e.* a rule which ascribes to every $A \in B$ are number $P(A)$, then this $P(A)$ is called as *probability measure* on B provided it satisfies the following axioms :

P_1. $\forall$ (for every) $A \in B, P(A) \geq 0$ *i.e.* non-negative real number.

P_2. $P(S) = 1$

P_3. If $A_1 A_2, \ldots, A_j, \ldots$ is a finite or countable sequence of mutually exclusive events, then

$$P(\cup_j A_j) = \sum_j P(A_j)$$

Illustration. $P(A) + P(\bar{A}) = 1$ and $A \cup \bar{A} = S$.

Axiom P_3 gives $P(A \cup \bar{A}) = P(S) = 1$

" P_2 gives $0 \leq P(A) \leq 1$ from P_1.

Also $P(\phi) = 0$, $S \cup \phi = S$, $P(S) + P(\phi) = P(S)$ *i.e.*, $P(\phi) = 0$.

Problem 29. *Let A and B be events with* $P(A \cup B) = \frac{7}{8}$ $P(A \cap B) = \frac{1}{4}$ *and* $P(\bar{A}) = \frac{5}{8}$. *Find* $P(A)$, $P(B)$ *and* $P(A \cap \bar{B})$.

We have $P(A) = 1 - P(\bar{A})$

$$= 1 - \frac{5}{8} = \frac{3}{8}.$$

Again we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

i.e.,
$$\frac{7}{8} = \frac{3}{8} + P(B) - \frac{1}{4}.$$

i.e.,
$$P(B) = \frac{7}{8} - \frac{3}{8} + \frac{1}{4} = \frac{3}{4}.$$

Further we have $A - B = A \cap \bar{B}$, so that

$$P(A \cap \bar{B}) = P(A - B)$$

i.e.,
$$P(A - B) = P(A) - P(A \cap B) \text{ by theorem 2}$$

$$= \frac{3}{8} - \frac{1}{4} = \frac{1}{8}.$$

$\therefore$
$$P(A \cap \bar{B}) = P(A - B) = \frac{1}{8}.$$

Problem 30. *In an experiment of tossing two dice, if A denotes the event that the sum of the spots on uppermost faces is 7, find* $P(A)$.

A seven can be found in following 6 ways

(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1).

While two dice can be thrown in 6^2 *i.e.*, 36 ways.

$\therefore$ chance of throwing a 7 with two dice $= \frac{6}{36} = \frac{1}{6}$.

If $e_1, e_2, e_3, e_4, e_5, e_6$ denote the events of throwing a seven as (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1) respectively, then the event $A = \{e_1, e_2, e_3, e_4, e_5, e_6\}$, so that $P(A) = P(e_1) + P(e_2) + P(e_3) + P(e_4) + P(e_5) + P(e_6)$

But
$$P(e_1) = P(e_2) = \ldots = P(e_6) = \frac{1}{36}.$$

$\therefore$
$$P(A) = \frac{1}{36} + \frac{1}{26} + \ldots \text{times.}$$

$$= \frac{1}{6}.$$

Problem 31. *For any two events A and B prove that*

$$P[(A \cap \bar{B}) \cup (B \cap \bar{A})] = P(A) + P(B) - 2P(A \cap B)$$

A can be decomposed into two mutually exclusive events $A \cap \bar{B}$ and $A \cap B$ *i.e.*,

$$A = (A \cap \bar{B}) \cup (A \cap B)$$

Similarly

$$B = (\bar{A} \cap B) \cup (A \cap B)$$

$$\therefore \quad P(A) = P(A \cap \bar{B}) + P(A \cap B)$$

and $\quad P(B) = P(\bar{A} \cap B) + P(A \cap B).$

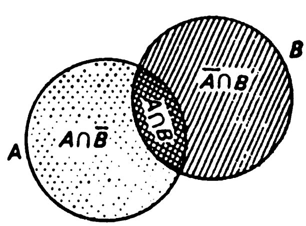


Fig. 15.3

Adding the last two results

$$P(A) + P(B) = P(A \cap \bar{B}) + P(\bar{A} \cap B) + 2P(A \cap B)$$

$$= P[(A \cap \bar{B}) \cup (B \cap \bar{A})] + 2P(A \cap B)$$

as $A \cap \bar{B}$ and $B \cap \bar{A}$ are mutually exclusive events,

$$\therefore P[A \cap \bar{B}) \cup (B \cap \bar{A})] = P(A) + P(B) - 2P(A \cap B).$$

Problem 32. *From a pack of cards one card is selected at random. What is the probability that the card is a spade, an honour or a deuce ?*

Denoting by A, B, C respectively the events of drawing a spade, an honour and a deuce, we have

$$P(A) = \frac{{}^{13}C_1}{{}^{52}C_1} = \frac{13}{52}, \quad P(B) = \frac{{}^{12}C_1}{{}^{52}C_1} = \frac{12}{52}, \quad P(C) = \frac{{}^{4}C_1}{{}^{52}C_1} = \frac{4}{52}$$

$$P(A \cap B) = \frac{{}^{3}C_1}{{}^{52}C_1} = \frac{3}{52}, \quad P(A \cap C) = \frac{{}^{1}C_1}{{}^{52}C_1} = \frac{1}{52}, \quad P(B \cap C) = 0 \text{ and}$$

$$P(A \cap B \cap C) = 0.$$

Required probability

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

$$= \frac{13}{52} + \frac{12}{52} + \frac{4}{52} - \frac{3}{52} - \frac{1}{52} - 0 + 0 = \frac{25}{32}.$$

Problem 33. *An integer is chosen at random from the first 200 positive integers. What is the probability that the integer chosen is divible by 6 or by 8?*

Let A denote the event that an integer selected is divisible by 6 and B denote the event that an integer selected is divisible by 8. Then the sample space is

$$S = \{1, 2, 3, \ldots, 200\}.$$

$$P(A) = \frac{33}{200}, \quad P(B) = \frac{25}{200}, \text{ and } P(A \cap B) = \frac{8}{200}.$$

Required probability $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{33}{200} + \frac{25}{200} - \frac{8}{200}$$

$$= \frac{50}{200} = \frac{1}{4}.$$

Problem 34. *A coint and a die are thrown together. Find the chance of throwing a head and 5 or a tail and 6.*

Let A denote an event of throwing a head and 5;

and let B " " " a tail and 6.

The two events being mutually exclusive, we have

$$P(A \cup B) = P(A) + P(B).$$

The sample space consists of 12 points as follows:

$$S = \{(H, 1), (H, 2), \dots (H, 6); (T, 1), (T, 2), \dots (T, 6)\}.$$

For the event $A = \{(H, 5)\}$, there is one sample point, so that $P(A) = \frac{1}{12}$ and for the event $B = \{(T, 6)\}$, there is also one sample point, so that $P(B) = \frac{1}{12}$.

Hence $P(A \cup B) = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$.

Problem 35. *A has 3 shares in a lottery in which there are 3 prizes and 6 blanks and B has 1 share in lottery in which there is one prizeg and two black. Show that A's chance of success is to B's as 16 to 7.*

We have to show that $\dfrac{P(A)}{P(B)} = \frac{16}{7}$.

We have $P(A) + (\bar{A}) = 1$, $\bar{A}$ being the event that A fails to win a prize.

Sample points $= {}^9C_3$.

Event points $= {}^6C_3$.

$$\therefore\ P(\bar{A}) = \frac{{}^6C_3}{{}^9C_3} = \frac{5}{21}, \text{ so that } P(A) = 1\frac{5}{21} = \frac{16}{21}.$$

$$\text{Similarly } P(\bar{B}) = \frac{{}^2C_1}{{}^3C_1} = \frac{2}{3}, \text{ so that } P(B) = 1 - \frac{2}{3} = \frac{1}{3}.$$

$$\text{Required ratio} = \frac{P(A)}{P(B)} = \frac{16}{21}\Big/\frac{1}{3} = \frac{16}{7}.$$

Aliter: Let

$A_1 = A$ wins prize when 1 share wins but other two fail,

$A_2 =$ " " 2 shares win " one fails,

$A_3 =$ " " 3 " " 0 fail.

$$\text{Then } P(A_1) = \frac{{}^3C_1 \times {}^6C_2}{{}^9C_3},\ P(A_2) = \frac{{}^3C_2 \times {}^6C_1}{{}^9C_3} \text{ and } P(A_3) = \frac{{}^3C_3 \times {}^6C_0}{{}^9C_3}$$

$$\therefore\ P(A) = P(A_1) + P(A_2) + P(A_3)$$

$$= \frac{{}^3C_1 \times {}^6C_2 + {}^3C_2 \times {}^6C_1 + {}^3C_3 \times {}^6C_0}{{}^9C_3}$$

$$= \frac{16}{21}.$$

Similarly $P(B)$ can be found to be $\frac{1}{3}$ and hence the required ratio.

[H] Conditional probability. *If A and B are two dependent events in the sample space S, then the conditional probability of A given B (i.e. the probability of occurring of the event A on the assumption that B has already occurred) is defined as*

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \qquad \dots(1)$$

provided $P(B) \neq 0$.

Similarly the conditional probability of B given A is defined as

$$P(B/A) = \frac{P(B \cap A)}{P(A)} \quad ...(2)$$

provided $P(A) \neq 0$.

Note. *In first case if* $P(B) = 0$ *or in second case* $P(A) = 0$, *the conditional probability is undefined.*

[I] The law of compound probability. In the previous article we have de... the conditional probability of A given B as

$$P(B/A) = \frac{P(A \cap B)}{P(B)}.$$

Multiplying both sides by $P(B)$, this becomes

$$P(B)\,P(A/B) = P(A \cap B),$$

i.e. $P(A \cap B) = P(B)\,P(A/B)$.

Similarly $P(B \cap A) = P(A)\,P(B/A)$.

But $B \cap A = A \cap B$.

$\therefore$ $P(B \cap A) = P(A \cap B)$,

so that $P(A \cap B) = P(A).\,P(B/A) = P(B).P(A/B)$.

This gives the law of compound probability.

COROLLARY 1. In case of three events A, B, C, this law becomes

$$P(A \cap B \cap C) = P(A).P(B/A).P(C/A \cap B).$$

COROLLARY 2. In case of $n(\geq 2)$ events $A_1, A_2, A_3, \ldots A_n$ for which $P(A_1 \cap A_2 \cap \ldots \cap A_n) \not< 0$, then the compound probability

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1).P(A_2/A_1).P(A_3/A_1 \cap A_2) \ldots P[A_n/A_1 \cap A_2 \cap \ldots \cap A_{n-1}].$$

This may be proved by the method of induction as follows :

Denoting by S_n the statement given above and denoting by N the set of those integers n for which S_n is true, we have by the method of induction for $N > 1$,

(*i*) $2 \in N$ for S_2 is the statement which is true,

(*ii*) $k \in N$, k being assumed an integer > 1.

If we prove that $k + 1 \in N$, then the statement becomes universally true.

For S_k the statement is

$$P(A_1 \cap A_2 \cap \ldots \cap A_k) = P(A_1)\,P(A_2/A_1) \ldots P(A_k/A_1 \cap A_2 \cap \ldots \cap A_{k-1}). \quad ...(1)$$

If may be verified by using the definition of conditional probability and properties of set-intersection, that

$$\frac{P(A_1 \cap A_2 \cap \ldots \cap A_{k+1})}{P(A_1 \cap A_2 \cap \ldots \cap A_k)} = P(A_{k+1}/A_1 \cap A_2 \cap \ldots \cap A_k). \quad ...(2)$$

Multiplying (1) and (2) together, we get

$$P(A_1 \cap A_2 \cap \ldots \cap A_{k+1}) = P(A_1).P(A_2/A_1) \ldots P(A_{k+1}/A_1 \cap A_2 \cap \ldots \cap A_k),$$

which shows that if $k \in N$, then $k + 1 \in N$, *i.e.* the statement is true for all values of k and hence the result holds good for any integral value n.

COROLLARY 3. Since B can be decomposed into mutually exclusive events $A \cap B$ and $\bar{A} \cap B$ *i.e.*,

$$B = (A \cap B) \cup (\bar{A} \cap B)$$

we have, $P(B) = P[(A \cap B) \cup (\bar{A} \cap B)]$

$$= P(A \cap B) + P(\bar{A} \cap B) \text{ as } P(E \cup F) = P(E) + P(F)$$

$$= P(A).P(B/A) + P(\bar{A})\ P(B/\bar{A})$$

Problem 36. *Prove the following cases assuming that in each case the conditional probabilities are defined.*

(*i*) *For any event A,* $0 \le P(A/B) \le 1$, *where* $P(B) > 0$, *B being another event.*

(*ii*) *For a certain event S,* $P(S/B) = 1$, *where* $P(B) > 0$;

(*iii*) $P(A_1 \cup A_2/B) = P(A_1/B) + P(A_2/B) - P(A_1 \cap A_2/B)$.

But A_1 *and* A_2 *are mutually exclusive and* $P(B) > 0$,

$$P(A_1 \cup A_2/B) = P(A_1/B) + P(A_2/B)$$

or in general if $A_1, A_2, \ldots A_n$ *is a sequence of n mutually exclusive events, then*

$P(A_1 \cup A_2 \cup \ldots \cup A_n/B) = (A_1/B) + P(A_2/B) + \ldots + P(A_n/B)$

(*iv*) *If A and B are mutually exclusive and* $P(A \cup B) \neq 0$; *then*

$$P(A/A \cup B) = \frac{P(A)}{P(A) + P(B)}.$$

(*i*) We have $A \cap B \subset B$, so that B can be decomposed into two mutually exclusive events $A \cap B$ and $B - A \cap B$ *i.e.*

$$B = (A \cap B) \cup (B - A \cap B)$$

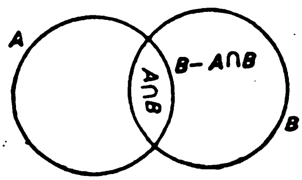


Fig. 15.4

$\therefore$ $P(B) = P(A \cap B) + P(B - A \cap B)$

$\ge P(A \cap B)$ as $P(B - A \cap B) \ge 0$

or $P(A \cap B) \le P(B)$.

Thus, $P(A/B) = \dfrac{P(A \cap B)}{P(B)} \le 1$

Also $P(A/B) \ge 0$.

$\therefore$ $0 \le P(A/B) \le 1$.

(*ii*) We have $S \cap B = B$, S being a certain event

i.e., $P(S \cap B) = P(B)$.

$$\therefore \quad P(S/B) = \frac{P(S \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1.$$

(*iii*) We have

$$P(A_1 \cup A_2/B) = \frac{P[(A_1 \cup A_2) \cap B]}{P(B)}$$

$$= \frac{P[(A_1 \cap B) \cup (A_2 \cap B)]}{P(B)}$$

$$= \frac{P(A_1 \cap B) + P(A_2 \cap B) - P[(A_1 \cap B) \cap (A_2 \cap B)]}{P(B)}$$

$$= \frac{P(B).P(A_1/B) + P(B).P(A_2/B) - P[(A_1 \cap A_2) \cap B]}{P(B)}$$

or $P(A_1 \cup A_2/B) = P(A_1/B) + P(A_2/B) - \dfrac{P(B).P(A_1 \cap A_2/B)}{P(B)}$

$= P(A_1/B) + P(A_2/B) - P(A_1 \cap A_2/B)$.

In case A_1 and A_2 are mutually exclusive events, then $A_1 \cap B$ and $A_2 \cap B$ are also mutually exclusive, so that

$$P[(A_1 \cup A_2) \cap B] = P[(A_1 \cap B) \cup (A_2 \cap B)]$$
$$= P(A_1 \cap B) + P(A_2 \cap B)$$

$$\therefore \quad P(A_1 \cup A_2/B) = \frac{P[(A_1 \cup A_2) \cap B]}{P(B)}$$
$$= \frac{P(A_1 \cap B) + P(A_2 \cap B)}{P(B)}$$
$$= \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)}$$
$$= P(A_1/B) + P(A_2/B).$$

Again if $A_1, A_2, \ldots, A_n$ are mutually exclusive events, then $A_1 \cap B, A_2 \cap B, \ldots, A_n \cap B$ are also mutually exclusive, so that

$$P[(A_1 \cup A_2 \cup \ldots \cup A_n) \cap B] = P[(A_1 \cap B) \cup (A_2 \cap B) \cup \ldots \cup (A_n \cap B)]$$
$$= P(A_1 \cap B) + P(A_2 \cap B) + \ldots + P(A_n \cap B)$$

$$\therefore P(A_1 \cup A_2 \cup \ldots \cup A_n/B) = \frac{P[(A_1 \cup A_2 \cup \ldots A_n) \cap B]}{P(B)}$$
$$= \frac{P(A_1 \cap B) + P(A_2 \cap B) + \ldots + P(A_n \cap B)}{P(B)}$$
$$= \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)} + \ldots + \frac{P(A_n \cap B)}{P(B)}$$
$$= P(A_1/B) + P(A_2/B) + \ldots + P(A_n/B).$$

(*iv*) Since A and B are mutually exclusive events, we have

$$A \cup B = \phi$$

and $$A \cap (A \cup B) = (A \cup A) \cap (A \cup B)$$
$$= A \cap \phi = A$$

so that $$P[A \cap (A \cup B)] = P(A)$$

and also $$P(A \cup B) = P(A) + P(B)$$

Hence $$P(A/A \cup B) = \frac{P[A \cap (A \cup B)]}{P(A \cup B)}$$
$$= \frac{P(A)}{P(A) + P(B)}$$

Problem 37. *If A and B are two events such that $P(A) = \frac{3}{8}$, $P(B) = \frac{5}{8}$ and $P(A \cup B) = \frac{3}{4}$, find $P(A/B)$ and $P(B/A)$.*

We have $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

i.e., $$\tfrac{3}{4} = \tfrac{3}{8} + \tfrac{5}{8} - P(A \cap B)$$

giving $P(A \cap B) = \frac{1}{4}\cdot$

Now $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{5}{8}} = \frac{2}{5}$

and $P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{\frac{1}{4}}{\frac{5}{8}} = \frac{2}{5}$ as $B \cap A = A \cap B$.

Problem 38. *If A and B are two events such that* $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{4}$ *and* $P(A \cup B) = \frac{1}{2}$, *find* $P(A/A)$, $P(B/A)$, $P(A \cap \bar{B})$ *and* $P(A/\bar{B})$.

We have $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

i.e., $\frac{1}{2} = \frac{1}{3} + \frac{1}{4} - P(A \cap B)$ giving $P(A \cap B) = \frac{1}{12}$

$$\therefore \quad P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{12}}{\frac{1}{4}} = \frac{1}{3}$$

$$P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{\frac{1}{12}}{\frac{1}{3}} = \frac{1}{4}.$$

Now, $P(\bar{B}) = 1 - P(B) = 1 - \frac{1}{4} = \frac{3}{4}$.

A can be decomposed into two mutually exclusive events $A \cap B$ and $A \cap \bar{B}$, so that

$$A = (A \cap B) \cup (A \cup \bar{B})$$

$$\therefore \quad P(A) = P[(A \cap B) \cup (A \cap \bar{B})]$$

$$= P(A \cap B) + P(A \cap \bar{B}),$$

so that $P(A \cap \bar{B}) = P(A) - P(A \cap B)$

$$= \frac{1}{3} - \frac{1}{12} = \frac{1}{4}$$

also $P(A/\bar{B}) = \frac{P(A \cap \bar{B})}{P(\bar{B})} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}.$

Problem 39. *A bag contains 3 black and 4 white balls. Two balls are drawn at random one at a time without replacement.*

(i) What is the probability that the second ball selected is white?

(ii) What is the conditional probability that the first ball selected is white if the second ball is known to be white?

Let W_1, W_2 denote the events of drawing a white ball in first and second draw, then

$$P(W_1) = \frac{{}^2C_1}{{}^7C_1} = \frac{4}{7}, \quad P(W_2) = \frac{{}^4C_1}{{}^7C_1} = \frac{4}{7}$$

$$\therefore \quad P(\overline{W_1}) = 1 - P(W_1) = 1 - \frac{4}{7} = \frac{3}{7}$$

$$P(W_2/W_1) = \frac{{}^3C_1}{{}^6C_1} = \frac{3}{6} = \frac{1}{2}$$

$$P(W_2/\overline{W_1}) = \frac{{}^4C_1}{{}^6C_1} = \frac{4}{6} = \frac{2}{3}\cdot$$

Also $P(W_2/W_1) = \frac{P(W_2 \cap W_1)}{P(W_1)}$

i.e., $$\tfrac{1}{2}=\frac{P(W_2\cap W_1)}{\tfrac{4}{7}}$$

or $$P(W_2\cap W_1)=\tfrac{4}{7}\cdot\tfrac{1}{2}=\tfrac{2}{7}\cdot$$

(*i*) $$P(W_2)=P\left[(W_1\cap W_2)\cup(\overline{W}_1\cap W_2)\right]$$
$$=P(W_1\cap W_2)+P(\overline{W}_1\cap W_2)$$
$$=P(W_1).P(W_2/W_1)+P(\overline{W}_1).P(W_2/\overline{W}_1)$$
$$=\tfrac{4}{7}\cdot\tfrac{1}{2}+\tfrac{3}{7}\cdot\tfrac{2}{3}=\tfrac{2}{7}+\tfrac{2}{7}=\tfrac{4}{7}\cdot$$

(*ii*) $$P(W_1/W_2)=\frac{P(W_1\cap W_2)}{P(W_2)}$$
$$=\frac{\tfrac{2}{7}}{\tfrac{4}{7}}=\tfrac{1}{2}\cdot$$

[J] **Partition of a set.** Let there be a set A;

$$A=\{A_1,A_2,\ldots,A_n\}.$$

The, the partition of the set A is the set $\{A_1,A_2,\ldots,A_n\}$ provided

(*i*) $A_j\subseteq A$ for $j=1,2,\ldots,n$ *i.e.*, $A_1,A_2,\ldots A_n$ ae subsets of A (*i.e.* they are inclusive).

(*ii*) $A_j\cap A_k=\phi$ for $j=1,2,\ldots,n$, $k=1,2,\ldots,n$ and $j\neq k$ *i.e.* they are disjoint.

(*iii*) $A_1\cup A_2\cup\ldots\cup A_n=A$ *i.e.* they are exhaustive.

We can see that every element of A is a member of one and only one of the subsets in the partition.

Further we can see that if S be a sample, $S=\{A,\overline{A}\}$,

where $$A=\{A\cap\overline{B},A\cap B\}$$

and $$A\cup B=\{A\cap\overline{B},A\cap B,\overline{A}\cap B\}$$

then $$S=\{A,\overline{A}\}=\{A\cap\overline{B},A\cap B,\ \overline{A}\cap B,A\cap\overline{B}\}$$

Illustration. Suppose there is a pack of cards, then

$$S=(52\text{ outcomes})$$
$$=\{A_S,A_H,A_D,A_C\}.$$

where S,H,D,C stand for spade, heart, diamond and club respectively and $A_S=A_H=A_D=A_C=13$.

It is clear that

(*i*) $A_j\subseteq S$ for $j=S,H,D,C$, *i.e.* A_s,A_H,A_D,A_C are subsets of S.

(*ii*) $A_j\cap A_k=\phi$ for $j=S,H,D,C$, $k=S,H,D,C$, but, $j\neq k$, *i.e.*, they are disjoint.

(*iii*) $A_S\cup A_H\cup A_D\cup A_C=S$ *i.e.* they are exhaustive.

All the three conditions being satisfied the set $\{A_S,A_H,A_D A_C\}$ represents the partition of S.

[K] **Baye's theorem.** In order to prove the theorem given by Thomas Baye, let us first introduce a Lemma, required for its proof.

Lemma. *Let $\{A_1, A_2, \ldots, A_n\}$ be a partition of the sample space S and suppose that each of the events $A_1, A_2, \ldots, A_n$ has non-zero probability, i.e., $P(A_j) > 0$ for $j = 1, 2, \ldots, n$. Then for any event A, we have*

$$P(A) = P(A_1).P(A \mid A_1) + P(A_2).P(A \mid A_2) + \ldots + P(A_n).P(A \mid A_n)$$

$$= \sum_{j=1}^{n} P(A_j)\, P(A \mid A_j).$$

Its proof. As $A_1, A_2, \ldots, A_n$ are partitions of S, therefore

$$\{A \cap A_1, A \cap A_2, \ldots, A \cap A_n\}$$

will represent the partition of A.

Thus $\quad A = (A \cap A_1) \cup (A \cap A_2) \cup \ldots \cup (A \cap A_n).$

so that $\quad P(A) = P(A \cap A_1) + P(A \cap A_2) + \ldots + P(A \cap A_n)$

$$= \sum_{j=1}^{n} P(A \cap A_j)$$

Applying the result of the conditional probability, *i.e.*,

$$P(A \mid A_j) = \frac{P(A \cap A_j)}{P(A_j)} \text{ giving } P(A \cap A_j)$$

$$= P(A_j).\, P(A \mid A_j)$$

we have $\quad P(A) = \sum_{j=1}^{n} P(A_j).\, P(A \mid A_j), \qquad \ldots(1)$

which proves the lemma.

Statement of Baye's theorem. *If an event A can occur only if one of the mutually exclusive events $A_1, A_2, \ldots, A_n$, i.e., $A \subset \bigcup_{k=1}^{n} A_k$, $A_k \cap A_j = \phi$ when $k \neq j$ and suppose we are given the probabilities $P(A_k)$, $k = 1, 2, \ldots, n$ and the conditional probabilities $P(P \mid A_k)$, then we are required to find the probability of A_k when it is given that A has already occurred and $P(A) > 0$ for each integer k $(1 \leq k \leq n)$ then Baye's formula is*

$$P(A_k \mid A) = \frac{P(A_k)\, P(A \mid A_k)}{\sum_{j=1}^{n} P(A_j)\, P(A \mid A_j)}.$$

Proof. By the definition of conditional probability, we have

$$P(A_k \mid A) = \frac{P(A \cap A_k)}{P(A)}$$

i.e., $\quad P(A \cap A_k) = P(A)\, P(A_k) \mid A)$

$$= P(A_k)\, P(A \mid A_k)$$

and by (1). $\quad P(A) = \sum_{j=1}^{n} P(A_j).P(A \mid A_j).$

$$\therefore \quad P(A_k|A) = \frac{P(A_k).P(A|A_k)}{\sum_{j=1}^{n} P(A_j).P(A|A_j)}.$$

Problem 40. *There are three coins, identical in appearance, one of which is ideal and the other two biased with probabilities $\frac{1}{3}$ and $\frac{2}{3}$ respectively for a head. One coin is taken at random and tossed twice. If a head appears both the times, what is the probability that the ideal coin was chosen.*

Let A_1, A_2, A_3 denote the events of choosing the 1st (ideal) 2nd and 3rd coins respectively. Then

$$P(A_1) = P(A_2) = P(A_3) = \tfrac{1}{3}.$$

Let A be the event of obtaining 2 heads in two tosses of the selected coin.

Probability of getting a head in a toss $= \frac{1}{2}$.

$$\therefore \quad P(A|A_1) = \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{4}.$$

Probability of turning head up with 2nd coin is $\frac{1}{3}$ and that with 3rd coin is $\frac{2}{3}$; therefore in two tosses,

$$P(A|A_2) = \left(\tfrac{1}{3}\right)^2 = \tfrac{1}{9} \text{ and } P(A|A_3) = \left(\tfrac{2}{3}\right)^2 = \tfrac{4}{9}.$$

Using Baye's formula,

$$P(A_1|A) = \frac{P(A_1)P(A|A_1)}{\sum_{j=1}^{3} P(A_j).P(A|A_j)}$$

$$= \frac{\frac{1}{3}\cdot\frac{1}{4}}{\frac{1}{3}\cdot\frac{1}{4}+\frac{1}{3}\cdot\frac{1}{9}+\frac{1}{3}\cdot\frac{4}{9}}$$

$$= \frac{\frac{1}{12}}{\frac{1}{12}+\frac{1}{27}+\frac{4}{27}} = \frac{\frac{1}{12}}{\frac{29}{108}} = \frac{1}{12}\times\frac{108}{29} = \frac{9}{29}.$$

Problem 41. *In a bolt factory, machines A, B, C manufacture respectively 25, 35 and 40% of the total. Of their output, 5, 4 and 2% are defective bolts. A bolt is drawn random from the procedure and is found defective. What are the probabilities that it was manufactured by machine A, B or C?*

Here $P(A) = \frac{25}{100}$, $P(B) = \frac{35}{100}$, $P(C) = \frac{40}{100}$.

Let E denote the event of drawing a defective bolt.

$$P(E|A) = \tfrac{5}{100}, \quad P(E|B) = \tfrac{4}{100}, \quad P(E|C) = \tfrac{2}{100}.$$

Using Baye's formula

$$P(A|E) = \frac{P(A)\ P(E|A)}{P(A)\ P(E|A) + P(B)\ P(E|B) + P(C).P(E|C)}$$

$$= \frac{\frac{25}{100}\times\frac{5}{100}}{\frac{25}{100}\times\frac{5}{100}+\frac{35}{100}\times\frac{4}{100}+\frac{40}{100}\times\frac{2}{100}}$$

$$= \frac{125}{125+140+80} = \frac{125}{345}.$$

Similarly $P(B \mid E) = \frac{140}{345}$ and $P(C \mid E) = \frac{80}{345}$.

[L] Independent events. *The two events A and B are said to be stochastically independent if and only if $P(A \mid B) = P(A)$, i.e., the happening of the event B does not affect the happening of A.*

We have

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{ and } P(B|A) = \frac{P(B \cap A)}{P(A)}$$

The two events are independent to each other when

$$P(A \mid B) = P(A), P(A) > 0,$$
$$P(B \mid A) = P(B), P(B) > 0.$$

Thus the two events A and B are stochastically independent if and only if

$$P(A \cap B) = P(A)\, P(B).$$

In general, $m(m > 2)$ events $A_1, A_2, \ldots, A_m$ are independent if

$$P(A_1 \cap A_2 \cap \ldots \cap A_m) = P(A_1).\, P(A_2) \ldots P(A_m).$$

Problem 42. *If A and B are two independent events in a sample space S, then prove that*

(*i*) $\bar{A}$ *and* $\bar{B}$ *are independent,*

(*ii*) $\bar{A}$ *and B are independent,*

(*iii*) *A and* $\bar{B}$ *are independent,*

(*iv*) $P(A \cup B) = 1 - P(\bar{A})\, P(\bar{B})$.

(*i*) We have $P(\bar{A} \cup \bar{B}) = 1 - P(A \cup B)$

$$= 1 - [P(A) + P(B) - P(A \cap B)]$$
$$= 1 - P(A) - P(B) + P(A).P(B)$$

as A and B are independent

$$= [1 - P(A)]\, [1 - P(B)]$$
$$= P(\bar{A}).P(\bar{B})$$

showing that $\bar{A}$ and $\bar{B}$ are independent.

(*ii*) We have $P(\bar{A} \cap \bar{B}) = P(B) - P(A \cap B)$

$$= P(B) - P(A).P(B)$$
$$= P(B)\, [1 - P(A)]$$
$$= P(B).P(\bar{A})$$

showing that $\bar{A}$ and B are independent.

(*iii*) We have $A = A \cap \bar{B} \cup A \cap B.$

$$\therefore \quad P(A) = P(A \cap \bar{B}) + P(A \cap B)$$
$$= P(A \cap \bar{B}) + P(A).P(B)$$

or $$P(A \cap \bar{B}) = P(A)\, [1 - P(B)]$$
$$= P(A) \cdot (\bar{B})$$

showing that A and $\bar{B}$ are independent.

(*iv*) We have $(A \cup B) \cup (\bar{A} \cup \bar{B}) = S$.

$\therefore\ P(A \cup B) + P(\bar{A} \cap \bar{B}) = P(S) = 1$

or $\quad P(A \cup B) = 1 - P(\bar{A} \cap \bar{B})$

$= 1 - P(\bar{A}).P(\bar{B})$ as $\bar{A}$ and $\bar{B}$ and independent by (*i*).

Problem 43. *From a pack of cards, if the event of drawing a spade card is denoted by A and that of drawing an honour card by B, then show that A and B are independent events.*

We have
$$P(A) = \frac{^{13}C_1}{^{52}C_1} = \frac{1}{4},$$

$$P(B) = \frac{^{12}C_1}{^{52}C_1} = \frac{12}{52} = \frac{3}{13},$$

$$P(A \cap B) = \frac{^{3}C_1}{^{52}C_1} = \frac{3}{52}.$$

Then
$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{3}{52}}{\frac{3}{13}}$$

$$= \frac{1}{4} = P(A).$$

Since $P(A \mid B) = P(A)$, the events A and B are independent.

15.7. REPEATED TRIALS: BINOMIAL AND MULTINOMIAL EXPANSIONS .

[A] Binomial theorem. *If the probability of the happening of an event in one trial is known, then it is required to find the probability of its happening once, twice, thrice...exactly in n trials.*

Let p be the probability of the happening of the event in a single trial and let q be the probability of its failure such that $q = 1 - p$. Suppose we have to find the probability of exactly r successes in n trials.

There are n trials in all. Out of these n trials r can be taken in ^nC_r ways.

Now the chance that the event happens in r trials and fails in the remaining $(n-r)$ trails $= p.p...r$ times $\times\ q.q...(n-r)$ time $= P^r q^{n-r}$.

$\therefore$ the chance of exactly r successes in n trials $= {^nC_r} p^r q^{n-r}$.

Putting $r = 1, 2, 3, ...$ in succession we get the probabilities of exactly 1, 2, 3,...successes.

But we know that $^nC_r p^r q^{n-r}$ is $(r+1)$th term in the binomial expansion of $(q+p)^n$; *hence the probability that the event will happen exactly r times in n trials is the* $(r+1)$*th term in the binomial expansion of* $(q+p)^n$.

[B] Multinomial theorem. *If there are n dice with faces marked from 1 to f and these are thrown at random, then we have to find the chance that the sum of the numbers exhibited on the uppermost faces is equal to p.*

Since a die has f faces, if can fall in f ways. Similarly the second die can fall in f ways. Thus the two dice can fall together in $f \times f$ *i.e.*, f^2 ways. Similarly we can show that when the n dice fall together, they can fall in f^n ways.

Now the number of ways in which the numbers thrown will have p for their sum is equal to the coefficient of x^p in the expansion of

$$(x^1 + x^2 + x^3 + \ldots + x^f)^n$$

because this coefficient arises out of the different ways in which n of the indices 1, 2, 3,..., f can be taken so as to form p by addition.

Hence the required probability

$$= \frac{\text{Coeft. of } x^p \text{ in } \left(x^1 + x^2 + \ldots + x^f\right)^n}{f^n}$$

$$= \frac{\text{Coeft. of } x^p \text{ in } x^n\left[\dfrac{1-x^f}{1-x}\right]^n}{f^n}$$

$$= \frac{\text{Coeft. of } x^{p-n} \text{ in } \left(1-x^f\right)^n (1-x)^{-n}}{f^n}$$

Problem 44. *What is the chance that a person with two dice (the faces of each being numbered 1 to 6) will throw aces exactly 4 times in 6 trials.*

Probability of throwing an ace with a die $= \frac{1}{6}$.

$\therefore$ probability of throwing an ace with two dice in a single throw

$$= \tfrac{1}{6} \times \tfrac{1}{6} = \tfrac{1}{36}.$$

so that the probability of not throwing an ace in a throw

$$= 1 - \tfrac{1}{36} = \tfrac{35}{36}.$$

The required chance of throwing exactly four aces in 6 trials is (4 + 1)th *i.e.*, 5th term in expansion of $\left(\frac{35}{36} + \frac{1}{36}\right)^6$.

$\therefore$ the required chance $= {}^6C_4 \left(\frac{35}{36}\right)^2 \left(\frac{1}{36}\right)^4$.

Problem 45. *An experiment succeeds twice as often as it fails. Find the chance that in the next six trials there will be at least 4 successes.*

Let p the probability of success and q that of failure. Then

$$p = 2q \text{ and } p + q = 1$$

which give $\quad p^1 = \frac{2}{3},\ q = \frac{1}{3}.$

The probabilities of 4, 5 and 6 succeses are the 5th, 6th and 7th terms in the expansion of $\left(\frac{1}{3} + \frac{2}{3}\right)^6$ *i.e.*, they are ${}^6C_4 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^4, {}^6C_5 \left(\frac{1}{3}\right) \cdot \left(\frac{2}{3}\right)^5$ and ${}^6C_6 \left(\frac{2}{3}\right)^6$.

The three events being mutually exclusive, the required chance

$$= {}^6C_4 \left(\tfrac{1}{3}\right)^2 \left(\tfrac{2}{3}\right)^4 + {}^6C_5 \left(\tfrac{1}{3}\right) \left(\tfrac{2}{3}\right)^5 + {}^6C_6 \left(\tfrac{1}{3}\right)^6$$

$$= \frac{2^4}{729}(15 + 6 \times 2 + 4) = \frac{496}{729}.$$

Problem 46. *An ordinary six-faced die is thrown 4 times. What are the probabilities of obtaining 4, 3, 2, 1, 0 aces ?*

Let p be the chance of obtaining an ace in a single throw and q that of not obtaining an ace. Then

$p = \frac{1}{6}$ and $q = 1 - \frac{1}{6} = \frac{5}{6}$.

When a die is thrown 4 times, the chances of 4, 3, 2, 1, 0 aces are the 5th, 4th, 3rd, 2nd and 1st terms in the binomial expansion of $(q+p)^4$ *i.e.*, $\left(\frac{5}{6}+\frac{1}{6}\right)^4$.

$\therefore$ Probability of 4 aces $= {}^4C_4 \left(\frac{1}{6}\right)^4 = \frac{1}{1296}$.

" 3 " $= {}^4C_3 \left(\frac{5}{6}\right)\left(\frac{1}{6}\right)^3 = \frac{20}{1296}$.

" 2 " $= {}^4C_2 \left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right)^2 = \frac{150}{1296}$.

" 1 " $= {}^4C_1 \left(\frac{5}{6}\right)^3\frac{1}{6} = \frac{500}{1296}$.

" 3 " $= {}^4C_0 \left(\frac{5}{6}\right)^4 = \frac{625}{1296}$.

Problem 47. *If m things are distributed among a men and b women, show that the chance that the number of things received by men is odd, is* $\frac{1}{2}\frac{(b+a)^m-(b-a)^m}{(b+a)^m}$.

The probability that a men get a thing $= \dfrac{a}{a+b}$

" " b women " $= \dfrac{b}{a+b}$.

If out of m things 'a' men get only one thing and 'b' women get the remaining things, then the probability for men is 2nd term in the expansion of $\left(\dfrac{b}{a+b}+\dfrac{a}{a+b}\right)^m$

i.e., it is ${}^mC_1\left(\dfrac{b}{a+b}\right)^{m-1}\dfrac{a}{a+b}$.

Similarly the probabilities that 'a' men get three things, five things, ... are respectively

$${}^mC_3\left(\frac{b}{a+b}\right)^{m-3}\left(\frac{b}{a+b}\right)^3, {}^mC_5\left(\frac{b}{a+b}\right)^{m-5}\left(\frac{a}{a+b}\right)^5, \ldots$$

Hence the probability that the number of things received by men is odd, is

$$= {}^mC_1\left(\frac{b}{a+b}\right)^{m-1}\frac{a}{a+b} + {}^mC_3\left(\frac{b}{a+b}\right)^{m-3}\left(\frac{a}{a+b}\right)^3 + {}^mC_5\left(\frac{a}{a+b}\right)^{m-5}\left(\frac{a}{a+b}\right)^5 + \ldots$$

$$= \frac{1}{(a+b)^m}\left[{}^mC_1\, ab^{m-1} + {}^mC_3\, a^3b^{m-3} + \ldots\right]$$

$$= \frac{1}{2}\cdot\frac{(b+a)^m-(b-a)^m}{(b+a)^m}.$$

Problem 48. *If n biscuits be distributed at random among N beggars, what is the chance that a particular beggar receives r(< n) biscuits.*

The total number of ways in which n biscuits can be distributed among N beggars $= N^n$.

The sepecified beggar getting r biscuits in ${}^nC_r$ ways, the remaining $(n-r)$ biscuits among $(N-1)$ beggars can be distributed in $(N-1)^{n-r}$ ways

$\therefore$ Number of favourable ways $= {}^nC_r(N-1)^{n-r}$.

Hence the required probability $= \dfrac{{}^nC_r(N-1)^{n-r}}{N^n}$.

Problem 49. *A, B and C in order toss a coin. The first one to throw a head wins. What are their respective chances of winning? Assume that the game may continue indefinitely.*

Probability of throwing a head with a coin $= \frac{1}{2}$.

$\therefore$ Probability of not throwing a head with a coin

$$= 1 - \tfrac{1}{2} = \tfrac{1}{2}$$

A will have to win if he throws head in 1st. 4th, 7th,...throws. The probabilities of these events are given by $\frac{1}{2}$, $\left(\frac{1}{2}\right)^3 \cdot \frac{1}{2}$, $\left(\frac{1}{2}\right)^6 \cdot \frac{1}{2}$,...respectively.

$$\therefore A\text{'s chance of winning} = \tfrac{1}{2} + \left(\tfrac{1}{2}\right)^3 \cdot \tfrac{1}{2} + \left(\tfrac{1}{2}\right)^6 \tfrac{1}{2} + \ldots\infty$$

$$= \frac{\frac{1}{2}}{1-\left(\frac{1}{2}\right)^3} = \frac{4}{7}$$

B will have to win if he throws head in 2nd, 5th, 8th,..., throws.

$$\therefore B\text{'s chance of winning} = \left(\tfrac{1}{2}\right).\tfrac{1}{2} + \left(\tfrac{1}{2}\right)^4.\tfrac{1}{2} + \left(\tfrac{1}{2}\right)^7.\tfrac{1}{2} + \ldots\infty$$

$$= \frac{\frac{1}{2}\cdot\frac{1}{2}}{1-\left(\frac{1}{2}\right)^3} = \frac{2}{7}.$$

Similarly C will have to win if he throws head in 3rd, 6th, 9th,...throws.

$$\therefore C\text{'s chance of winning} = \left(\tfrac{1}{2}\right)^2.\tfrac{1}{2} + \left(\tfrac{1}{2}\right)^5.\tfrac{1}{2} + \left(\tfrac{1}{2}\right)^8.\tfrac{1}{2} + \ldots\infty$$

$$= \frac{\left(\frac{1}{2}\right)^2\cdot\frac{1}{2}}{1-\left(\frac{1}{2}\right)^3} = \frac{1}{7}.$$

Hence the chances of A, B, C are $\frac{4}{7}$, $\frac{2}{7}$ and $\frac{1}{7}$ respectively.

Problem 50. *A, B, C, and D cut a pack of cards successively in the order mentioned. What are their respective chances of first cutting a spade?*

The chance of cutting a spade $= = \frac{1}{5}\frac{3}{2} = \frac{1}{4}$.

$\therefore$ the chance of not cutting a spade $= 1 - \frac{1}{4} = \frac{3}{4}$.

A wins if he cuts a spade in 1st, 5th, 9th,...throws.

$\therefore A$'s chance of cutting a spade

$$= \tfrac{1}{4} + \left(\tfrac{3}{4}\right)^4.\tfrac{1}{4} + \left(\tfrac{3}{4}\right)^8.\tfrac{1}{4} + \ldots\infty$$

$$= \frac{\frac{1}{4}}{1-\left(\frac{3}{4}\right)^4} = \frac{64}{175}.$$

B wins if he cuts a spade in 2nd, 6th, 10th,...throws.

$\therefore$ B's chance of cutting a spade

$$=\frac{3}{4}\cdot\frac{1}{4}+\left(\frac{3}{4}\right)^5\cdot\frac{1}{4}+\left(\frac{3}{4}\right)^5\cdot\frac{1}{4}+\ldots\infty$$

$$=\frac{\frac{3}{4}\cdot\frac{1}{4}}{1-\left(\frac{3}{4}\right)^4}=\frac{48}{175}.$$

Similarly C's chance of cutting a spade

$$=\left(\frac{3}{4}\right)^2\cdot\frac{1}{4}+\left(\frac{3}{4}\right)^6\cdot\frac{1}{4}+\left(\frac{3}{4}\right)^{10}\cdot\frac{1}{4}+\ldots\infty$$

$$=\frac{\left(\frac{3}{4}\right)^2\cdot\frac{1}{4}}{1-\left(\frac{3}{4}\right)^4}=\frac{36}{175}.$$

and D's chance of cutting a spade

$$=\left(\frac{3}{4}\right)^3\cdot\frac{1}{4}+\left(\frac{3}{4}\right)^7\cdot\frac{1}{4}+\left(\frac{3}{4}\right)^{11}\cdot\frac{1}{4}+\ldots$$

$$=\frac{\left(\frac{3}{4}\right)^3\cdot\frac{1}{4}}{1-\left(\frac{3}{4}\right)^4}=\frac{27}{175}.$$

Problem 51. *Find the chance of throwing 10 exactly in one throw with three dice.*

Three dice can be thrown in 6^3, *i.e.* 216 ways.

The number of ways of getting 10 by throwing 3 dice

$= \text{coeft. of } x^{10} \text{ in } (x + x^2 + \ldots + x^6)^3$

$= \text{coeft. of } x^{10} \text{ in } \dfrac{x^3(1-x^6)^3}{(1-x)^3}$

$= \text{coeft. of } x^{10} \text{ in } x^3(1 - 3x^6 + 3x^{12} - x^{18})(1-x)^{-3}$

$= \text{coeft. of } x^{10} \text{ in } x^3(1 - 3x^6 + 3x^{12} - x^{18})(1 + 3x + 6x^2 - 10x^3 + 15x^4 + 21x^5 + 24x^6 + 36x^7 + \ldots)$

$= 36 - 9 = 27.$

$\therefore$ the required probability $= \frac{27}{216} = \frac{1}{8}$.

Aliter. Favourable number of ways may be found out as below.

Dice: 1st	2nd	3rd	Number of ways
6	2	2	$\frac{3\,!}{2\,!}=3$
6	3	1	$3\,!=6$
5	3	2	$3\,!=6$
5	4	1	$3\,!=6$
4	3	3	$\frac{3\,!}{2\,!}=3$
4	4	2	$\frac{3\,!}{2\,!}=3$
			Total 27

$\therefore$ required chance $= \frac{27}{216} = \frac{1}{8}$.

Problem 52. *Four dice are thrown, what is that probability that the sum of the numbers appearing on the dice is 18?*

4 dice can be thrown in 6^4 ways.

Favourable number of ways to give a sum of 18 with 4 dice

$= \text{coeft. of } x^{18} \text{ in } (x + x^2 + \ldots + x^6)^4$

$= \quad " \quad " \quad x^4\left(\frac{1-x^6}{1-x}\right)^4$

$= \quad " \quad " \quad x^4(1 - x^6)(1 - x)^{-4}$

$= \quad " \quad " \quad x^4(1 - 4x^6 + 6x^{12} \ldots)(1 + 4x + 10x^2 + \ldots + 165x^8 + \ldots + 680x^{14} + \ldots)$

$= 680 - 660 + 60 = 80.$

$\therefore$ required probability $= \frac{80}{6^4} = \frac{5}{81}$.

Problem 53. *Find the chance of throwing 10 with 4 dice.*

Proceeding just as above, the required probability is

$$\frac{104}{6^4} = \frac{13}{162}$$

Problem 54. *Determine the probability of throwing more than 8 with 5 perfectly symmetrical dice.*

Total number of ways in which 3 dice can be thrown $= 6^3 = 216$.

The favourable number of getting the sum as 3, 4, 5, 6, 7, 8 will be equal to the sum of coefficients of $x^3, x^4, x^5, x^6, x^7, x^8$ in the expansion of $(x + x^2 + \ldots + x^6)^3$,

i.e., of $x^2(1 - x^6)^3(1 - x)^{-3}$,

i.e., of $x^3(1 - 3x^6 + \ldots)(1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5 + \ldots)$

$= 1 + 3 + 6 + 10 + 15 + 21 = 56.$

$\therefore$ The probability of getting the sum $\leq 8 = \frac{56}{6^3} = \frac{7}{27}$.

Hence the probability of getting the sum $> 8 = 1 - \frac{7}{27} = \frac{20}{27}$.

Problem 55. *Counters marked 1, 2, 3 are placed in a bag; one is withdrawn and replaced three times. What is the chance of obtaining a total of 6?*

Total number of ways of drawing a counter three times $= 3^3 = 27$

Favourable number of getting a sum of 6

$= \text{the coeft. of } x^6 \text{ in } (x + x^2 + x^3)^3.$

$= \quad " \quad " \quad x^3(1 - x^3)^3(1' - x)^{-3}$

$= \quad " \quad " \quad x^3(1 - 3x^3 + \ldots)(1 + 3x + 6x^2 + 10x^3 + \ldots)$

$= 10 - 3 = 7.$

$\therefore$ Required chance $= \frac{7}{27}$.

Problem 56. *Nine cards are drawn at random from a set of cards. Each card is marked with one of the numbers 1, 0 or – 1 and it is equally likely that any of the three numbers will be drawn. Find the chance that the sum of the numbers drawn is zero.*

The total number of ways in which 9 cards can be drawn $= 3^9$.

The favourable number of ways of getting a sum of zero

$= \text{the coeft. of } x^0 \text{ in } (x^{-1} + x^0 + x^1)^9$

$= \quad " \quad " \quad (x^{-1} + 1 + x)^9$

$= \quad " \quad " \quad \frac{1}{x^9}(1 + x + x^2)^9$

$= \quad " \quad " \quad \frac{1}{x^9}(1 + x^3)^9 \, (1 + x)^{-9}$

$= \quad " \quad " \quad \frac{1}{x^9}(1 - 9x^3 + 36x^6 - 84x^9 + \ldots) \times (1 + 9x$

$+ 45x^2 + 165x^3 + \ldots + 3003x^6 + \ldots + 24310x^9 + \ldots)$

$= 24310 - 9 \times 3003 + 165 \times 36 - 84$

$= 3139$

$\therefore$ Required chance $= \frac{3139}{(3)^9}$.

15.8. MEASURES OF CENTRAL TENDENCY

[A] Arithmetic mean or simply mean. *Let $x_1, x_2, x_3, \ldots, x_n$ be the n values of a variate (or variable) x; then their arithmetic mean denoted by m or M or $\bar{x}$s defined to be*

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$$

or using the sigma (Σ) notation,

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} \text{ or simply } \frac{\Sigma x}{n} \qquad \ldots(1)$$

In case the 'weights' $w_1, w_2, w_3, \ldots, w_n$ are attached to the n variate-values $x_1, x_2, x_3, \ldots x_n$ respectively, then the '*weighted mean*' is defined to be

$$\bar{x} = \frac{w_1x_1 + w_2x_2 + \ldots + w_nx_n}{w_1 + w_2 + w_3 + \ldots + w_n} = \frac{\sum_{i=1}^{n} w_ix_i}{\sum_{i=1}^{n} w_i} = \frac{\Sigma wx}{\Sigma w} \qquad \ldots(2)$$

In other words if the value x_1 occurs f_1 times, x_2 occurs f_2 times, and so on, then

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + \ldots + f_nx_n}{f_1 + f_2 + \ldots + f_n} = \frac{\sum_{i=1}^{n} f_ix_i}{\sum_{i=1}^{n} w_i} = \frac{\Sigma fx}{\Sigma f} \qquad \ldots(3)$$

or

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{n} f_ix_i = \frac{\Sigma fx}{N} \qquad \ldots(4)$$

where $N = f_1 + f_2 + \ldots + f_n$ = total frequency. ...(5)

If we assume that A *is any assumed mean,* whose deviation from the variate x is ξ, then if M is the arithmetic mean, we have

$$\xi = x - A \text{ and } M = \frac{1}{N} \Sigma fx \text{ by (4).}$$

Consider $\frac{1}{N} \Sigma f \xi = \frac{1}{N} \Sigma f(x - A)$

$$= \frac{1}{N} \Sigma fx - \frac{1}{N} \Sigma f \cdot A$$

$$= M - A \text{ since from (5) } \Sigma f = N.$$

$$\therefore \quad M = A + \frac{1}{N} \Sigma f \xi \qquad ...(6)$$

This result used as a short cut method for finding the arithmetic mean, often facilitates calculation.

Again if h be the width of equal class-intervals in a frequency table and u be the new variate defined as

$$u = \frac{x - A}{h} \text{ i.e., } x = A + uh.$$

then, we have

$$\Sigma fx = \Sigma f(A + uh)$$

$$= \Sigma f \cdot A + h\, \Sigma fu$$

Dividing either side by Σf, we find

$$\frac{\Sigma f(x)}{\Sigma f} = A + h \frac{\Sigma fu}{\Sigma f}$$

i.e. $$\bar{x} \text{ or } M = A + h\bar{u} \qquad ...(7)$$

where $u = \dfrac{\Sigma fu}{\Sigma f}$, the arithmetic mean for values of the variate u.

Properties of the Arithmetic Mean

I. *The algebraic sum of the deviations of the variate-values from their mean is zero.*

Let ξ be the deviation of the variate x from the mean $\bar{x}$, then the sum of the deviations of the variate-values from the mean

$$= \Sigma f(x - \bar{x})$$

$$= \Sigma fx - \Sigma f\bar{x}$$

$$= N \cdot \bar{x} - N\bar{x} \text{ from (4) and (5)}$$

$$= 0.$$

II. *If $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_r$ are the arithmetic means of r distributions with respective frequencies $N_1, N_2, \ldots N_r$, then the mean, $\bar{x}$ of the whole distribution with total frequency $N = \sum_{i=1}^{r} N_i$ is given by*

$$\bar{x} = \frac{1}{N} \sum_{i=1}^{r} N_i \bar{x}_i.$$

Let f_{ij} denote the frequency of the observation x_{ij} of the ith distribution, so that $N_i = \sum_{j=1}^{n} f_{ij}$; then

$$\bar{x}_i = \frac{\sum_{j=1}^{n} f_{ij} x_{ij}}{\sum_{j=1}^{n} f_{ij}} = \frac{\sum_{j} f_{ij} x_{ij}}{N_i}$$

Now $\bar{x}$ being the mean of the whole distribution, we have

$$\bar{x}_i = \frac{\sum\sum_{ij} f_{ij} x_{ij}}{\sum\sum_{ij} f_{ij}} = \frac{\sum_i N_i \cdot \bar{x}_i}{\sum_i N_i} = \frac{1}{N} \sum_{i=1}^{r} N_i \bar{x}_i .$$

Problem 57. *The following table gives the population of males at different age groups of the U.K. and India at the time of the census of 1931.*

Age group (years)	*U.K. (in lakhs)*	*India (in lakhs)*
0-5	18	214
5-10	19	258
10-15	20	222
15-20	18	157
20-25	16	145
25-30	14	161
30-40	27	257
40-50	25	184
50-60	19	120
above 60	17	100

compare the average age of males in the two countries and account for the difference, if any.

Take the assumed mean as 27.5.

Age group (years)	*Mid-values (x)*	*Deviation from assumed mean 27.5 (ξ)*	*U.K.*		*India*	
			Population in lakhs (f_1)	$f_1 \xi$	*Population in lakhs (f_2)*	$f_2 \xi$
0-5	2.5	−25.0	18	−450	214	−5350.0
10-15	7.5	−20.0	19	−380	258	−5160.0
10-15	12.5	−15.0	20	−300	222	−3330.0
15-20	17.5	−10.0	18	−180	157	−1570.0
20-25	22.5	−5.0	16	−80	145	−725.0
25-30	27.5	0	14	0	161	0
30-40	35.0	7.5	27	202.5	257	1927.5
40-50	45.0	17.5	25	437.5	180	3220.0
50-60	55.0	27.5	19	422.5	124	3300.0
above 60	65.0	37.5	17	637.5	100	3750.0
Total			$193 = N_1 = \Sigma f_1$	$410 = \Sigma f_1 \xi$	$1818 = N_2 = \Sigma f_2$	$-3937.5 = \Sigma f_1 \xi$

For U.K., average age of the people

$$= A + \frac{\Sigma f_1 \xi}{\Sigma f_1} = 27.5 + \frac{410}{198} = 27.5 + 2.12 = 29.62.$$

For India, average age of the people

$$= A + \frac{\Sigma f_2 \xi}{\Sigma f_2} = 27.5 - \frac{3937 \cdot 5}{1818} = 27.5 - 2.17 = 25.33.$$

These averages show that the average age of the people of U.K. is higher than that of the people in India.

Problem 58. *If a variate x is expressed as a linear function of two variates u and v in the form $x = au + bv$, show that*

$$\bar{x} = a\bar{u} + b\bar{v}.$$

Let the number of variables for each u and v be n. Then

$$x_i = au_i + bv_i,$$

so that

$$\sum_{i=1}^{n} x_i = a \sum_{i=1}^{n} u_i + b \sum_{i=1}^{n} v_i$$

or

$$\frac{\Sigma x_i}{n} = a\frac{\Sigma u_i}{n} + b\frac{\Sigma v_i}{n}$$

i.e.,

$$\bar{x} = a\bar{u} + b\bar{v}.$$

Problem 59. *Show that if $\bar{x}$ is the arithmetic mean of the values x_i, $i = 1, 2, \ldots, n$, then*

$$\sum_{i=1}^{n} f_i (x_i - \bar{x})^2 = \sum_{i=1}^{n} f_i x_i - N\bar{x}^2$$

where $N = \sum_{i=1}^{n} f_i$.

We have

$$\frac{\sum_i f_i x_i}{\sum f_i} = \frac{\sum_i f_i x_i}{N} \qquad \ldots(1)$$

Now

$$\sum_i f_i (x_i - \bar{x})^2 = \sum_i f_i (x_i^2 + \bar{x}^2 - 2x_i\bar{x})$$

$$= \sum_i f_i x_i^2 + \bar{x}^2 \sum_i f_i - 2\bar{x} \sum_i f_i x_i$$

$$= \sum_i f_i x_i^2 + \bar{x}^2 \cdot N - 2\bar{x} \cdot N\bar{x} \text{ from (1)}$$

$$= \sum_i f_i x_i^2 - N\bar{x}^2.$$

Problem 60. *A distribution consists of 3 components with frequencies 45, 40, 65 having their means 2, 2.5 and 2 respectively. Prove that the mean of the combined distribution is 2.13 approximately.*

Using the property II of arithmetic mean, the required mean is given by

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{3} N_i \bar{x}_i.$$

where $N = 45 + 40 + 65 = 150.$

and $\sum_{i=1}^{3} N_i \bar{x}_i = N_1\bar{x}_1 + N_2\bar{x}_2 + N_2\bar{x}_3$

$$= 45 \times 2 + 40 \times 2.5 + 65 \times 2$$
$$= 90 + 100 + 130$$
$$= 320.$$

$\therefore \quad \bar{x} = \frac{320}{150} = \frac{32}{15} = 2.13$ approx.

Problem 61. *Supposing the frequencies of values 0, 1, 2, ... n of a variable to be given by the terms of the binomial series*

$$q^n, \frac{n}{1}\cdot q^{n-1}p, \frac{n(n-1)}{1.2}q^{n-1}p^2, \ldots \; p^n,$$

where $p + q = 1$, find the mean.

Let M be the required mean; then

$$M = \frac{\Sigma fx}{\Sigma f} = \frac{0\cdot q^n + 1\cdot {}^nC_1 q^{n-1}p + 2\cdot {}^nC_2 q^{n-2}\cdot p^2 + \ldots n\cdot p^n}{q^n + {}^nC_1 q^{n-1}p + {}^nC_2 q^{n-2}p^2 + \ldots + np^n}$$

$$= \frac{1\cdot nC_1 q^{n-1}p + 2\cdot {}^nC_2 q^{n-2}p^2 + \ldots + np^n}{(q+p)^n}$$

But $(q+p)^n = 1$ as $q + p = 1$ (given)

$\therefore \quad M = {}^nC_1 q^{n-1}p + {}^nC_2 q^{n-2}p^2 + \ldots + np^n.$ ...(1)

Now, we have

$$(q+p)^n = q^n + {}^nC_1 q^{n-1}p + {}^nC_2 q^{n-2}p^2 + \ldots + p^n.$$

Differentiating both sides w.r.t. p and then multiplying throughout by p we get

$$n(q+p)^{n-1}p = {}^nC_1 q^{n-1}p + {}^nC_2 q^{n-2}p^2\cdot 2 + \ldots + np^n$$
$$= M \text{ from (1).}$$

Thus $M = np$ as $p + q = 1$.

Problem 62. *Show that the arithmetic mean of the series 1, 2, 4, 8, 16, ...2^n is* $\frac{2^{n+1}-1}{n+1}$

$$\text{Required mean} = \frac{1 + 2 + 2^2 + \ldots + 2^n}{n+1},$$
$$= \frac{2^{n+1}-1}{n+1}$$

Problem 63. *Show that the weighted arithmetic mean of first n natural numbers whose weights are equal to the corresponding numbers is equal to $\frac{1}{3}(2n+1)$.*

If $\bar{x}$ be the required arithmetic mean, then

$$\bar{x} = \frac{1\cdot 1 + 2\cdot 2 + 3\cdot 3 + \ldots + n\cdot n}{1 + 2 + 3 + \ldots + n}$$

$$= \frac{\Sigma n^2}{\Sigma n} = \frac{n(n+1)(2n+1)}{6} \bigg/ \frac{n(n+1)}{2}$$

$$= \frac{2n+1}{3}.$$

Problem 64. *The arithmetic mean of n numbers of a series is $\bar{x}$. The sum of the first $(n-1)$ terms is k. Show that the nth number is $n\bar{x} - k$.*

Let t_n be the nth term of the given series; then

$$\bar{x} = \frac{k + t_n}{n}$$

giving $$t_n = n\bar{x} - k.$$

[B] Geometric mean *Let $x_1, x_2, x_3, \ldots, x_n$ be the n values of a variable x; then geometric mean denoted by G is defined to be*

$$\left.\begin{aligned} G &= (x_1 \cdot x_2 x_3 \ldots x_n)^{1/n} \\ \text{or} \quad \log G &= \frac{\sum_{i=1}^{n} \log x_i}{n} = \frac{\Sigma \log x}{n} \end{aligned}\right\} \quad \ldots(1)$$

If the value x_1 occurs f_1 times, x_2 occurs f_2 times and so on, then

$$\left.\begin{aligned} G &= (x_1^{f_1} \cdot x_2^{f_2} \cdot \ldots \cdot x_n^{f_n})^{1/N} \text{ where } N = f_1 + f_2 + \ldots + f_n \\ \text{or} \quad \log G &= \frac{\sum_{i=1}^{n} f_i \log x_i}{n} = \frac{\sum f \log x}{n} \end{aligned}\right\} \quad \ldots(2)$$

which show that the logarithm of the geometric mean of a series of values is the arithmetic mean of their logarithms.

Properties of Geometric Mean

I. *The geometric mean of a series is less than its arithmetic mean, i.e., if. A be the arithmetic mean and G the geometric mean of a series, then*

$$A > G$$

We prove the result for a series having two numbers only and the result may be extended to any number.

Let x_1, x_2 be the values of a variate x. Then

$$A = \frac{x_1 + x_2}{2} \text{ and } G = \sqrt{(x_1 x_2)}.$$

Consider $$A - G = \frac{x_1 + x_2}{2} - \sqrt{(x_1 x_2)}.$$

$$= \tfrac{1}{2}\left[\sqrt{x_1} - \sqrt{x_2}\right]^2$$

$$= \text{a + ve quantity as } \left(\sqrt{x_1} - \sqrt{x_1}\right)^2 > 0$$

showing that $A > G$.

II. *If $G_1, G_2, \ldots, G_r$ be the geometric means of r distributions with respective frequencies $N_1, N_2, \ldots, N_r$, then the geometric mean G of the whole distribution with total frequency $N = \sum_{i=1}^{r} N_i$ is given by*

$$N \log G = N_1 \log G_1 + N_2 \log G_2 + \ldots + N_r \log G_r$$

i.e.,
$$\log G = \frac{1}{N} \sum_{i=1}^{r} N_i \log G_i.$$

Let f_{ij} denote the frequency of the observation x_{ij} of the ith distribution so that $N_i = \sum_{j=1}^{n} f_{ij}$, then

$$\log G_i = \frac{\sum_{j=1}^{n} f_{ij} \log G_{ij}}{\sum_{j=1}^{n} f_{ij}} = \frac{\sum_j f_{ij} \log G_{ij}}{N_i}.$$

But G being the geometric mean of the whole distribution, we have

$$\log G = \frac{\sum_i \sum_j f_{ij} \log G_{ij}}{\sum_i \sum_j f_{ij}} = \frac{\sum_i N_i \log G_i}{\sum_i N_i}$$

$$= \frac{1}{N} \sum_{i=1}^{r} N_i \log G_i.$$

III. *If G_1, G_2 are the geometric means of two series of observations and G the geometric mean of the ratios of corresponding observations, then G is equal to the ratio of their geometric means,*

i.e.,
$$G = \frac{G_1}{G_2}.$$

Let x_1, x_2, be the two variates corresponding to the two series of observations with frequency n each, and let x be the ratio of the two variates; then we have

$$x = \frac{x_1}{x_2},$$

so that
$$\log x = \log x_1 - \log x_2.$$

$\therefore$
$$\Sigma \log x = \Sigma \log x_1 - \Sigma \log x_2.$$

Dividing throughout by n.

$$\frac{\Sigma \log x}{n} = \frac{\Sigma \log x_1}{n} - \frac{\Sigma \log x_2}{n}$$

i.e.,
$$\log G = \log G_1 - \log G_2$$

giving
$$G = \frac{G_1}{G_2}$$

IV. *The geometric mean G of the product of r sets of observations with geometric means G_1, G_2 respectively, is the product of the geometric means of the component series, i.e.,*

$$G = G_1 G_2 \ldots G_r.$$

Let $x_1, x_2, \ldots x_r$ be the variates corresponding to r sets of observations and x their product, *i.e.*,

$$x = x_1 x_2 \ldots x_r;$$

then $$\log x = \log x_1 + \log x_2 + \ldots + \log x_r$$

or $$\Sigma \log x = \Sigma \log x_1 + \Sigma \log x_2 + \ldots + \Sigma \log x_r$$

or $$\frac{\Sigma \log x}{n} = \frac{\Sigma \log x_1}{n} + \frac{\Sigma \log x_2}{n} + \ldots + \frac{\Sigma \log x_r}{n}$$

i.e., $$\log G = \log G_1 + \log G_2 + \ldots + \log G_r$$

giving $$G = G_1 \cdot G_2 \ldots G_r.$$

Problem 65. *Fnd the geometric mean of the series.*

$$1, 2, 4, 8, 16 \ldots 2^n.$$

$$G = \{1.2.2^2.2^4 \ldots 2^n\}^{1/(n+1)}$$

$$= \{2^{1+2+3+\ldots+n}\}^{1/(n+1)}$$

$$= \{(2)^{n(n+1)/2}\}^{1/(n+1)} = 2^{n/2}.$$

[C] Harmonic mean. *Let $x_1, x_2, x_3, \ldots, x_n$ be the n values of a variate x; their harmonic mean denoted by H is defined to be the reciprocal of the arithmetic mean of their reciprocals, i.e.,*

$$\frac{1}{H} = \frac{\frac{1}{x_1} + \frac{1}{x_2} + \ldots \frac{1}{x_n}}{n} = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{1}{x_i}\right) = \frac{1}{n} \Sigma \frac{1}{x} \qquad \ldots(1)$$

or $$H = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}} = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}} = \frac{n}{\Sigma \frac{1}{x}}.$$

If the value x_1 occurs f_1 times, x_2 occurs f_2 times and so on, then

$$\frac{1}{H} = \frac{\frac{f_1}{x_1} + \frac{f_2}{x_2} + \ldots + \frac{f_n}{x_n}}{N} \text{ where } N = \sum_{i=1}^{n} f_i$$

$$= \frac{\sum_{i=1}^{n} \left(f_i \times \frac{1}{x_i}\right)}{\sum_{i=1}^{n} f_i} = \frac{\Sigma f \cdot \frac{1}{x}}{N} \qquad \ldots(2)$$

Property of harmonic mean. *The harmonic mean of a series of quantities is less than the geometric mean of the same quantities and a fortiori (Latin phrase, means 'with stronger or greater reason') less than the arithmetic mean of the same quantities, i.e., if A, G, H be the arithmetic, geometric, and harmonic means of a series, then $A > G > H$.*

In property *I* of geometric mean, we have already proved for two quantities x_1, x_2 that $A > G$.

Now consider $G - H = \sqrt{(x_1 x_2)} - \frac{2x_1x_2}{x_1 + x_2}$, as $H = \frac{2}{\frac{1}{x_1} + \frac{1}{x_2}}$

$$= \frac{\sqrt{(x_1x_2)}}{x_1 + x_2}\left[x_1 + x_2 - 2\sqrt{(x_1x_2)}\right]$$

or
$$G - H = \frac{\sqrt{(x_1x_2)}}{x_1 + x_2}\left\{\sqrt{(x_1)} - \sqrt{(x_2)}\right\}^2$$

$\therefore\ G > H$; also $A > G$

$\therefore\ A > G > H.$

This result can be extended to any number of values of x.

Problem 66. *A variate takes values $a, ar, ar^2, \ldots, ar^{n-1}$ each with frequency unity. Show that the A.M., A is $\frac{a(1 - r^n)}{n(1 - r)}$, the G.M., G is $ar^{(n-1)/2}$ and the H.M., H is $\frac{a.n(1 - r)r^{n-1}}{1 - r^n}$. Prove that $AH = G^2$. Prove also that $A > G > H$ unless $n = 1$ when all the three means coincide.*

We have

$$A = \frac{a + ar + ar^2 + \ldots + ar^{n-1}}{n}$$

$$= \frac{a}{n}\left[1 + r + r^2 + \ldots + r^{n-1}\right]$$

$$= \frac{a}{n} \cdot \frac{1 - r^n}{1 - r} \quad \text{if } r < 1.$$

$$G = (a.\, ar.\, ar^2 \ldots ar^{n-1})^{1/n}$$

$$= \left[a^n.\, r^{1+2+\ldots+(n-1)}\right]^{1/n}$$

$$= a.\, \left\{r^{(n-1)/2.n}\right\}^{1/n} = ar^{(n-1)/2}$$

and
$$H = \frac{n}{\frac{1}{a} + \frac{1}{ar} + \frac{1}{ar^2} + \ldots + \frac{1}{ar^{n-1}}}$$

$$= \frac{n}{\frac{1}{a}\left[1 + \frac{1}{r} + \frac{1}{r^2} + \ldots + \frac{1}{r^{n-1}}\right]},$$

$$= \frac{an}{\left(\frac{1}{r^n} - 1\right) \Big/ \left(\frac{1}{r} - 1\right)} \quad \text{if } r < 1,\ i.e., \frac{1}{r} > 1$$

$$= \frac{an(1 - r)\, r^{n-1}}{1 - r^n}$$

Now
$$AH = \frac{a(1 - r^n)}{n(1 - r)} \times \frac{an(1 - r)\, r^{n-1}}{1 - r^n}$$

$$= a^2 r^{n-1} = \{ar^{(n-1)/2}\}^2$$
$$= G^2.$$

Again to prove $A > G > H$, consider

$$A - G = \frac{a}{n}\frac{1-r^n}{1-r} - ar^{(n-1)/2}$$
$$= \frac{a}{n}\left(1 + r + r^2 + \ldots + r^{n-1}\right) - ar^{(n-1)/2}$$
$$= \frac{a}{n}\left[\left(1 + r + r^2 + \ldots + r^{n-1}\right) - n \cdot r^{(n-1)/2}\right]$$
$$= \frac{a}{n}\left[\left(1 - r^{(n-1)/2}\right) + \left(r - r^{(n-1)/2}\right) + \left(r^2 - r^{(n-1)/2}\right) + \left(r^{(n-1)} - r^{(n-1)/2}\right)\right]$$
$$= \frac{a}{n}\left[\left(1 - r^{(n-1)/2}\right) + r\left(1 - r^{(n-3)/2}\right) + r^2\left(1 - r^{(n-5)/2}\right) + \ldots + r^{n-1}\left(1 - r^{(1-n)/2}\right)\right]$$

= a + ve quantity, since every factor in the bracket of R.H.S. expression is + ve when $r < 1$.

$\therefore A > G.$

But if $n = 1$, $A - G = 0$, *i.e.*, $A = G$.

$$\text{Also, } G - H = ar^{(n-1)/2} - \frac{an(1-r)\, r^{n-1}}{1 - r^n}$$
$$= ar^{(n-1)/2} - \frac{anr^{n-1}}{1 + r + r^2 + \ldots + r^{n-1}}$$

$$\text{as } \frac{1-r^n}{1-r} = 1 + r + r^2 + \ldots + r^{n-1}$$

$$= \frac{ar^{(n-1)/2}}{1 + r + r^2 + \ldots + r^{n-1}}\left[1 + r + r^2 + \ldots + r^{n-1} - nr^{(n-1)/2}\right]$$
$$= \frac{ar^{(n-1)/2}}{1 + r + r^2 + \ldots + r^{n-1}}\left[1 - r^{(n-1)/2} + r\left(1 - r^{(n-3)/2}\right) + \ldots + r^{n-1}\left(1 - r^{(1-n)/2}\right)\right]$$

as above

= a + ve quantity as above.

$\therefore G > H.$

But if $n = 1$, $G - H = 0$, *i.e.*, $G = H$.

Hence $A > G > H$ unless $n = 1$ in which case $A = G = H$, *i.e.*, all the three means coincide.

Problem 67. *Find the average rates of (a) motion in the case of a person who rides the first mile at 10 m.p.h., the next mile at 8 m.p.h. and the third mile at 6 m.p.h.; (b) increase in population which in the first decade has increased 20% in the next 25% and in the third 44%.*

(*a*) For this case, the harmonic mean being the suitable average, we have

$$H = \frac{3}{\frac{1}{10} + \frac{1}{8} + \frac{1}{6}} = \frac{3 \times 120}{47} = \frac{360}{47} = 7.66 \text{ m.p.h. approx}$$

(*b*) For this case, the geometric mean being the suitable average, we have

$$G = (20 \times 25 \times 44)^{1/3}.$$

$$\therefore \quad \log G = \tfrac{1}{3}\,[\log 20 + \log 25 + \log 44]$$

$$= \tfrac{1}{3}\,[1.3010 + 1.3979 + 1.6435]$$

$$= \tfrac{1}{3} \times 4.3424 = 1.4475 = \log 28.02.$$

$$\therefore \quad G = 28.02\%.$$

[D] Median and quartiles. The *median* is defined as the middle-most or central value of the variate when the variate-values are arranged in ascending or descending order of magnitude *i.e.*, it is the value of the variate for which greater and smaller values occur with equal frequency or in other words the total frequency above and below this value is divided into two equal halves.

In case the total frequency n is an odd number, then the value of $\left(\frac{n+1}{2}\right)$th item gives the median, but if it is an even number then $\left(\frac{n}{2}\right)$th and $\left(\frac{n}{2}+1\right)$th are the central items so that their arithmetic mean gives the median.

For grouped data, the median is formulated as

$$\text{Median} = l_1 + \frac{\frac{N}{2} - f}{f} \times h$$

where l_1 = the lower limit of the median class.
N = the total frequency,
f = the frequency of the median class.
f_1 = the cumulative frequency before entering the median class.
h = the size of the class-interval of the median class.

For a continuous series, the median is formulated as

$$\text{Median} = l_1 + \frac{l_2 - l_1}{f}\,(m - f_1)$$

where l_1 = the lower limit of the median-class,
l_2 = the upper limit of the median-class,
f = the frequency of the median-class,
f_1 = the cumulative frequency before entering the median-class,
m = the size of the middle item.

The *quartiles* or *partition values* are defined to be the values of the variate which divide the total frequency into a number of equal parts. *Quartiles, deciles* and *percentiles* are the worth-considerable among partition values.

The *quartiles* are the values of the variate that divide the total frequency into four equal parts, 1st, 2nd, 3rd quartiles are denoted by Q_1, Q_2, Q_3 respectively, while Q_2 is the median. We have

$$Q_i = l_1 + \frac{\left(\frac{iN}{4} - f_1\right)}{f} \times h, \quad i = 1, 2, 3$$

where l_1 = the lower limit of the class,
N = the total frequency,
h = the size of the class,
f_1 = the cumulative frequency upto and including the class preceding the class in which the particular quartile lies.
f = the frequency of this class.

$$\text{Inter-quartile range} = Q_3 - Q_1.$$

The *deciles* are the values of the variate which divide the total frequency into ten equal parts and given by

$$D_i = l_1 + \frac{\left(\frac{jN}{10}\right) - f_1}{f} \times h, \quad j = 1, 2, \ldots, 9$$

the other quantities having the same meanings as in quartiles.

The *Percentiles* are the values of the variate which divide the total frequency into hundred equal parts and given by

$$P_k = l_1 + \frac{\left(\frac{kN}{100} - f_1\right)}{f} \times h, \quad k = 1, 2, \ldots 99$$

the other quantities having the same meanings as above.

Problem 68. *Show that for J-shaped distribution with the maximum frequency towards the lower values of the variate, the median is nearer to Q_1, than Q_3.*

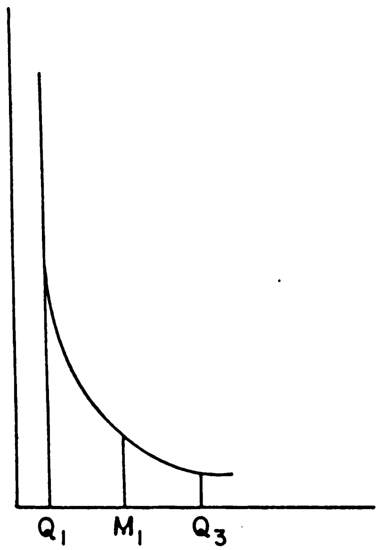


Fig. 15.5(*a*)

In the adjoining diagram, let M_i stand for the median; then we have to show that M_i is nearer to Q_1 then to Q_3 *i.e.*

$$M_i - Q_1 < Q_3 - M_i$$

Let $y = f(x)$ be the equation of the curve (J-shaped) under consideration and Q_1, Q_3, M_i locate the positions of first quartile, third quartile and median respectively.

Now according to the definition of quartiles, the area between Q_1 and median is equal to the area between Median and Q_3 and each is equal to $\frac{1}{4}$ th of the area of the entire curve *i.e.*,

$$\text{Area} = \int_{Q_1}^{M_i} f(x)\, dx = \tfrac{1}{4} \text{th of the entire area}$$

$$= \int_{M_i}^{Q_3} f(x)\, dx.$$

The curve $y = f(x)$ being continuous, we have

$$f(x) > f(M_i) \text{ for } Q_1 \le x \le M_i$$

and $$f(x) < f(M_i) \text{ for } M_i \le x \le Q_3.$$

Also $f(x)$ is positive as y is positive.

$$\therefore \quad f(M_i)\int_{Q_1}^{M_i} dx < \int_{Q_1}^{M_i} f(x)\, dx < f(M_i)\int_{M_i}^{Q_3} dx,$$

i.e. $\quad f(M_i)(M_i - Q_1) < f(M_i)(Q_3 - M_i)$

or $\quad M_i - Q_1 < Q_3 - M_i$

which shows that M_i is nearer to Q_1 than to Q_3.

Problem 69. *Determine the quartiles and the median for the following table.*

Income			*No. of persons*	*Cumulative frequency*
Below	Rs. 30		69	69
Between	Rs. 30 and below	Rs. 40	167	236
,,	Rs. 40 ,, ,,	Rs. 50	207	443
,,	Rs. 50 ,, ,,	Rs. 60	65	508
,,	Rs. 60 ,, ,,	Rs. 70	58	566
,,	Rs. 70 ,, ,,	Rs. 80	27	593
,,	Rs. 80 and over		10	603
			Total 603	

Median = size of the $\frac{603+1}{2}$ th, *i.e.* 302nd item

= between Rs 40 and below Rs 50.

Applying the formula,

Median $l_1 + \frac{\frac{N}{2} - f_1}{f} \times h$, where $l_1 = 40, f_1 = 236, f = 207$, h 10 and $N = 603$, we have

$$\text{Median} = 40 + \frac{\frac{603}{2} - 236}{207} \times 10$$

$$= 40 + \frac{301 \cdot 52 - 36}{207} \times 10$$

$$= 40 + \frac{65 \cdot 5 \times 10}{207} = 40 + \frac{655}{207}.$$

$$= 40 + 3.2 = 43.2 \text{ approx.}$$

Also, $\quad Q_1 = l_1 + \frac{\frac{N}{4} - f_1}{f} \times h$, where $l_1 = 30,\ f_1 = 69,\ f = 167$,

$h = 10, N = 603$ and $N/4$ lies in second group

$$= 30 + \frac{\frac{603}{4} - 69}{167} \times 10$$

$$= 30 + \frac{151 - 69}{167} \times 10, \text{ taking } \frac{N}{4} = 151 \text{ approx.}$$

$$= 30 + \tfrac{820}{167} = 30 + 4.9 \text{ approx.}$$

$$= 34.9$$

and $$Q_3 = l_1 + \frac{\frac{3N}{4} - f_1}{f} \times h, \text{ where } l_1 = 50, f_1 = 443, f = 65, h = 10,$$

$N = 603$ and $\frac{3}{4} N$ lies in 4th group

$$= 50 + \frac{\frac{3}{4} \times 603 - 443}{65} \times 10$$

$$= 50 + \frac{452 - 443}{65} \times 10 \text{ approx.}$$

$$= 50 + \tfrac{90}{65} = 50 + 1.7 \text{ approx.}$$

$$= 51.7.$$

[E] **Mode (or Modal value).** *The* **mode** *is that variate-value of a distribution for which the frequency is maximum.*

It may be found by following three methods:

(*i*) **By inspection.** When the measures of all the items are given in a frequency table, then the mode is the size of the item which occurs most frequently or in a frequency curve it is situated on the x-axis at the positon of greatest ordinate, *i.e.* the peak of the curve.

(*ii*) **By grouping.** The items are grouped and regrouped until the point of greatest frequency is unaltered by adjustments of grouping. The mode is then situated at the smaller group common to each of the larger groups. Such a grouping is affected by writing the actual frequencies before their respective size, adding the frequencies in two's leaving out the first frequency, adding in three's leaving out the first frequencies and then adding in three's leaving out the first two frequencies.

(*iii*) **By using the formula** (in a continuous series).

$$\text{Mode} = l_1 + \frac{f_2}{f_1 + f_2} \times h.$$

where f_1 = the lower limit of the modal-class (*i.e.* class having maximum frequency),

f_1 = the frequency of the class preceding the modal class,

f_2 = the frequency of the class following the modal class,

h = the size of the modal class.

Note. *Sometimes the formula used is*

$$Mode = l_1 + \frac{f - f_1}{2f - f_1 - f_2} \times h$$

where l_1 = the lower limit of the modal class (*i.e.* class having maximum frequency),

f_1 = the frequency of the class preceding the modal class,

f_2 = the frequency of the class following the modal class,

h = the size of the modal class.

f = the maximum frequency.

Problem 70. *Evaluate the values of mean, mode and median for the following grouped cumulative data:*

No. of days absent	*No. of students (F)*
Less than 5	29
,, 10	124
,, 15	349
,, 20	442
,, 25	478
,, 30	487
,, 35	493
,, 40	497
,, 50	500

These results can be tabulated as follows:

No. of days absent (class)	*Mid-value (x)*	*No. of students (frequency) (f)*	*Cumulative frequency*	*Deviation from assumed mean (225) (ξ)*	*fξ*
0-5	2.5	29	29	–20	–580
5-10	7.5	95	124	–15	–1425
10-15	12.5	225	349	–10	–2250
15-20	17.5	93	442	–5	–465
20-25	22.5	36	478	0	0
25-30	27.5	9	487	+5	45
30-35	32.5	6	493	+10	60
35-40	37.5	4	497	+15	60
40-50	45	3	500	+22.5	67.5
	Total	$500 = \Sigma f = N$			$-4488.5 = \Sigma f\xi$

$\therefore$ Mean $= A + \dfrac{1}{N}\Sigma f\xi$

$= 22.5 + \dfrac{-4488.5}{500} = 22.5 - 8.977$

$= 13.523$ *i.e.* 13.5 approx.

Median $=$ A.M. of sizes of $\frac{500}{2}$th and $\left(\frac{500}{2}+1\right)$th items

$=$ A.M. of sizes of 250th and 251st items

$= 12.5$ as both items lie in the same group.

This gives a rough value.

We calculate it by using the formula

$$\text{Median} = l_1 + \frac{\frac{N}{2} - f_1}{f} \times h$$

where $l_1 = 10$, $\frac{N}{2} = 250$, $f_1 = 124$, $f = 225$, $h = 5$.

$$\therefore \text{ Median} = 10 + \frac{250 - 124}{225} \times 5 = 10 + \frac{630}{225}$$

$$= 10 + 2.8 = 12.8.$$

This method gives rather accurate value.

By inspection, the value of the mode is 12.5 which is the size of the items of maximum frequency.

By grouping method this can be calculated as follows:

Size of the item (mid-value)	*Frequency I*	*II*	*III*	*IV*	*V*	*VI*
2.5	29	124		349		
7.5	95		320		413	
12.5	225	318				354
17.5	93		129	138		
22.5	36	45			51	
27.5	9		15			19
32.5	6	10		13		
37.5	4		7			
45	3					

Analysis table:

Columns	*Size of item having* maximum frequency
I	12.5
II	12.5, 17.5
III	7.5, 12.5
IV	2.5, 7.5, 12.5
V	7.5, 12.5, 17.5
VI	12.5, 17.5, 22.5

Here

12.5 occurs 6 items

17.5 ,, 3 ,,

7.5 ,, 3 ,,

2.5 ,, 1 ,,

Since 12.5 occurs maximum number of times, hence

Mode = 12.5.

By using the formula,

$$\text{mode} = l_1 + \frac{f_2}{f_1 + f_2} \times h$$

where $l_1 = 10, f_1 = 95, f_2 = 93, h = 5$

we have
$$\text{mode} = 10 + \frac{93}{95 + 93} \times 5$$
$$= 10 + \frac{465}{188} = 10 + 2.47$$
$$= 12.47.$$

Using the other formula,

$$\text{mode} = l_1 + \frac{f - f_1}{2f - f_1 - f_2} \times h$$

where $l_1 = 10, f_1 = 95, f_2 = 93, f = 225, h = 5,$

we have
$$\text{mode} = 10 + \frac{225 - 95}{2 \times 225 - 95 - 93} \times 5$$
$$= 10 + \frac{130 \times 5}{250 - 188} = 10 + \frac{650}{62} = 20.5 \text{ approx.}$$

which gives very high result as compared to the previous results.

[F] Empirical relation between mean, median and mode. There exists an approximate relation between mean, median and mode for a moderately asymmetrical distribution and this is

$$\text{mode} = \text{mean} - 3\,(\text{mean} - \text{median}).$$

15.9. MEASURES OF DISPERSION (OR VARIATION)

The measurement of the scatter of the size of the items of a series about the average is said to be a measure of **variation,** *or* **scatter** *or* **spread** *or* **dispersion.**

(*Agra, 1975*)

[A] The range. This measure of dispersion known as range is the simplest possible measure to compute and the easiest to understand, but it is the least useful and informative.

The **range** *is the difference between the greatest and the least values in the series.*

If x_g and x_l denote the greatest and the least measurements of a series, then

$$\text{Its range} = x_g - x_l$$

and the coefficient of the range or the scatter or simply the ratio of the range is defined to be the ratio $\dfrac{x_g - x_l}{x_g + x_l}$.

Problem 71. *Find the range and coefficient of range for the following set of observations:*

10, 15, 20, 37, 58, 60, 90.

$$\text{Range} = x_g - x_l = 90 - 10 = 80.$$

$$\text{Coefficient of range} = \frac{x_g - x_l}{x_g + x_l} = \frac{90 - 10}{90 + 10} = \frac{80}{100} = .8.$$

[B] **The quartile deviation or the semi-interquartile range.** We have introduced the interquartile range as the difference of upper and lower quartiles, *i.e.*, Interquartile range = $Q_3 - Q_1$.

Half of this difference is said to be **semi-interquartile range,** *i.e.*

$$\text{Semi-interquartile range} = \frac{Q_3 - Q_1}{2}.$$

The **quartile deviation** or **semi-interquartile range** is a better measure of dispersion than the range. Its coefficient is defined by

$$\text{Coefficient of quartile deviation} = \frac{\frac{Q_3 - O_1}{2}}{\frac{Q_3 + Q_1}{2}} = \frac{Q_3 - Q_1}{Q_3 + Q_1}.$$

Note. *The difference between the ninth and first deciles are similarly called as* **interdecile range** *which contains 80% of the total frequency while inter-quartile range contains 50% of the total frequency. Thus the ranges give a fairly good idea of the scatteredness of the distribution and are commonly used in elementary descriptive statistics.*

Problem 72. *Calculate quartile deviaton for the following data:*

From size (acres)	*No. of Farms*
0–40	394
41–80	461
81–120	391
121–160	334
161–200	169
201–240	113
241 and over	148

The given data maybe arranged in a cumulative frequency table as follows:

Farm size (acres) (x)	*No. of Farms (frequency)* (f)	*Cumulative frequency*
0–40	394	394
41–80	461	855
81–120	391	1246
121–160	334	1580
161–200	169	1749
201–240	113	1862
241 and over	148	2010

$\because \quad \frac{1}{4}N = \frac{2010}{4} = 502.5.$

$\therefore \quad Q_1$ lies in the class (41-80).

Thus, $Q_1 = l_1 + \frac{\frac{1}{4}N - f_1}{f} \times h$ where $l_1 = 41,\ f_1 = 394,\ f = 461,\ h = 40$

and $\frac{1}{4}N = \frac{2010}{4} = 502.5$

$$= 41 + \frac{502.5 - 394}{461} \times 40$$

$$= 41 + \frac{108.5 \times 40}{461} = 41 + \frac{4340}{461} = 41 + 9.4 = 50.4$$

Again $\frac{3}{4}N = 3 \times 502.5 = 1507.5.$

$\therefore$ Q_3 lies in the class (121 –160).

Thus $Q_3 = l_1 + \frac{\frac{3}{4}N - f_1}{f} \times h$, where $l_1 = 121,\ f_1 = 1246,\ f = 334.$

$h = 40,\ \frac{3}{4}N = 1507.5$

$$= 121 + \frac{1507.5 - 1246}{334} \times 40$$

$$= 121 + \frac{261.5 \times 40}{334} = 121 + \frac{10460}{334}$$

$$= 121 + 31.02$$

$$= 152.02.$$

Quartile deviation $= \frac{1}{2}(Q_3 - Q_1)$

$$= \frac{1}{2}(152.02 - 50.4)$$

$$= \frac{1}{2} \times 101.62 = 50.81.$$

[C] The mean (or average) deviation. *The mean deviation is defined as the sum of the absolute values of the deviations from an average (median mode or arithmetic mean) divided by the number of items i.e.,* if x_i $(i = 1, 2, 3, \ldots, n)$ are the variate-values of x with frequencies f_i $(i = 1, 2, 3, \ldots, n)$ such that $N = \sum_i f_i$ and M be the average (mean, median or mode), then

Mean deviation $= \frac{1}{N} \sum_i f_i \,|x_i - M|$ or simply $\frac{\Sigma f\,|x - M|}{N}$.

If we denote the difference $x - M$ by ξ, then

$$\text{Mean deviation} = \frac{\Sigma f|\xi|}{N}.$$

Problem 73. *Find the mean deviations from the median and the mean of the following data:*

Size of items:	4	6	8	10	12	14	16
Frequency:	2	4	5	3	2	1	4

These data in tabulated form are as follows :

Size (x)	Frequency f	Cumulative frequency	$f.x$	Deviations from median i.e., 8 $\lvert\xi_1\rvert$	Deviations from mean i.e., 9.7 $\lvert\xi_2\rvert$	$f\lvert\xi_1\rvert$	$f\lvert\xi_2\rvert$
4	2	2	8	4	5.7	8	11.4
6	4	6	24	2	3.7	8	14.8
8	5	11	40	0	1.7	0	8.5
10	3	14	30	2	0.3	6	0.9
12	2	16	24	4	2.3	8	4.6
14	1	17	14	6	4.3	6	4.3
16	4	21	64	8	6.3	22	25.2
Total	$21 = N$		$204 = \Sigma fx$			$68 = \Sigma f\lvert\xi_1\rvert$	$69.7 = \Sigma f\lvert\xi_2\rvert$

$$\text{Median} = \text{size of } \left(\frac{N+1}{2}\right)\text{th item}$$

$$= \text{size of } \frac{21+1}{2}\text{th } i.e., \text{ 11th item} = 8$$

$$\text{and mean } = \frac{\Sigma fx}{N} = \frac{204}{21} = 9.7 \text{ approx.}$$

$$\therefore \text{ Mean deviation from the median} = \frac{\Sigma f\lvert\xi_1\rvert}{N} = \frac{68}{21} = 3.238$$

$$\text{and mean deviation from the mean} = \frac{\Sigma f\lvert\xi_2\rvert}{N} = \frac{69.7}{21} = 3.319$$

Note. Median coefficient of dispersion

$$= \frac{\textit{mean deviation from the median}}{\textit{median}}$$

$$= \frac{3.238}{8} = 0.40475$$

and mean coefficient of dispersion

$$= \frac{\textit{Mean deviation from the mean}}{\textit{mean}}$$

$$= \frac{3.319}{9.7} = 0.34 \text{ approx.}$$

[D] The standard deviation. *We define the standard deviation of a variate x as the square root of the arithmetic mean of the squares of all deviations of x from the arithmetic mean of the observations and denote it by σ.* *(Agra, 1975)*

Thus if M be the arithmetic mean of variate values $x_1, x_2, \ldots x_n$ with frequencies $f_1, f_2, \ldots f_n$ so that $N = \sum_i f_i$, then

$$\sigma = \sqrt{\left(\frac{1}{N}\sum_i f_i(x_i - M)^2\right)} \text{ or simply } \sqrt{\left(\frac{1}{N}\Sigma f(x - M)^2\right)}$$

giving $$\sigma^2 = \frac{1}{N}\Sigma f(x - M)^2.$$

The quantity σ^2 i.e., the square of the standard deviation is termed as **variance** *and denoted by μ_2, which is the second moment about the mean.* *(Agra, 1975)*

$$\therefore \quad \mu_2 = \sigma^2 = \frac{1}{N}\Sigma f(x - M)^2$$

The ratio $\frac{\sigma}{M} \times 100$ is known as the **coefficient of variation.**

When the deviations of x are measured from an assumed mean A, then the square root of the arithmetic mean of the squares of all deviations from x is termed as the **root-mean deviation** and denoted by s *i.e.,*.

$$s = \sqrt{\left(\frac{1}{N}\sum_i f_i(x_i - A)^2\right)} \text{ or simply } \sqrt{\left[\frac{1}{N}\Sigma f(x - A)^2\right]}$$

The square of this quantity s *i.e.,* s^2 is termed as **mean sqaure deviation** and denoted by μ'_2 which is the second moment about the assumed mean A, *i.e.,*

$$\mu'_2 = s^2 = \frac{1}{N}\Sigma f(x - A)^2.$$

Relation between standard and root-mean square deviations.

Let M be the mean and A the assumed mean.

Also let $M - A = d$ and $x - A = \xi$; then

$x - M = x - A + A - M = (x - A) - (M - A) = \xi - d$

Now, $\sigma^2 = \frac{1}{N}\Sigma f(x - M)^2 = \frac{1}{N}\Sigma f(\xi - d)^2$ as $x - M = \xi - d$

$$= \frac{1}{N}\Sigma f(\xi^2 - 2\xi d + d^2) = \frac{1}{N}\Sigma f\xi^2 - 2d.\frac{1}{N}\Sigma f\xi + d^2\frac{1}{N}\Sigma f$$

$$= \frac{1}{N}\Sigma f(x - A)^2 - 2d.\frac{1}{N}\Sigma f(x - A) + d^2 \text{ as } \Sigma f = N$$

$$= s^2 - 2d\left(\frac{\Sigma fx}{N} - A\frac{\Sigma f}{N}\right) + d^2 \text{ as } s^2 = \frac{1}{N}\Sigma f(x - A)^2$$

$$= s^2 - 2d(M - A) + d^2 = s^2 - 2d.d + d^2 = s^2 - d^2$$

i.e., $s^2 = \sigma^2 + d^2$ or $\mu'_2 = \mu_2 + d^2$,

which gives relation between s and σ or between μ'_2 and μ_2.

Calculation of standard deviation.

(*i*) By short-cut method.

We have, $\sigma^2 = \frac{1}{N}\Sigma f(x - M)^2 = \frac{1}{N}\Sigma f\{(x - A) - (M - A)\}^2$

$$= \frac{1}{N}\Sigma f\{\xi - (M - A)\}^2 \text{ where } \xi = x - A$$

$$= \frac{1}{N} \Sigma f\{\xi^2 - 2(M-A).\,\xi + (M-A)^2\}$$

$$= \frac{1}{N} \Sigma f\xi^2 - 2(M-A).\,\frac{1}{N}\Sigma f\xi + 2\,(M-A)^2 \frac{1}{N}\Sigma f$$

$$= \frac{1}{N} \Sigma f\xi^2 - 2d.\frac{1}{N}\Sigma f\xi + d^2 \text{ as } \Sigma f = N.$$

But $d^2 = s^2 - \sigma^2 = \frac{1}{N}\{\Sigma f(x-A)^2 - \Sigma f(x-M)^2\}$

$$= \frac{1}{N}[\Sigma f\{(x-A)^2 - (x-M)^2\}] \text{ for the same distribution.}$$

$$= \frac{1}{N}[\Sigma f(x-A+x-M)\,(x-A-x+M)\}]$$

$$= \frac{1}{N}\Sigma f\{(\xi + \xi - d).d\} \text{ as } M-A = d \text{ and } x-M = \xi - d$$

$$= \frac{1}{N}\Sigma f\{2\xi d - d^2\} = 2d.\,\frac{1}{N}\Sigma f\xi - d^2.\,\frac{1}{N}\Sigma f$$

or $\quad 2d^2 = 2d.\,\frac{\Sigma f\xi}{N}$ as $\Sigma f = N$

giving $d = M - A = \frac{\Sigma f\xi}{N}$.

Hence $\sigma^2 = \frac{1}{N}\Sigma f\xi^2 - 2\left(\frac{\Sigma f\xi}{N}\right)^2 + \left(\frac{\Sigma f\xi}{N}\right)^2 = \frac{1}{N}\Sigma f\xi^2 - \left(\frac{\Sigma f\xi}{N}\right)^2$

i.e. $$\sigma = \sqrt{\left\{\frac{1}{N}\Sigma f\xi^2 - \left(\frac{\Sigma f\xi}{N}\right)^2\right\}}.$$

(*ii*) **By step deviation method.** Taking $u = \frac{x-A}{h} = \frac{\xi}{h}$, *i.e.*, $\xi = hu$, we have

$$\sigma^2 = \frac{1}{N}\Sigma f\xi^2 - \left(\frac{\Sigma f\xi}{N}\right)^2 = \frac{1}{N}\Sigma fh^2u^2 - \left(\frac{\Sigma fhu}{N}\right)^2$$

$$= h^2\left[\frac{1}{N}\Sigma fu^2 - \left(\frac{\Sigma fu}{N}\right)^2\right],$$

or $\quad \sigma_x^2 = h^2\sigma_u^2$,

i.e., $\quad \sigma_x = h\sigma_u$.

Properties of the standard deviation. *The standard deviations of two sets containing N_1 and N_2 members are σ_1 and σ_2 respectively, being measured from their respective means M_1 and M_2. If the two sets are grouped together as one set of $N_1 + N_2$ members, then the standard deviation σ, of this set, measured from its mean is given by*

$$\sigma^2 = \frac{N_1\sigma^2 + N_2\sigma^2}{N_1 + N_2} + \frac{N_1N_2}{(N_1+N_2)^2}(M_1 - M_2)^2.$$

If s_1^2 and s_2^2 be the mean square deviations of the two sets, s^2 be the mean square deviation of the combined set and A be the assumed mean: then

$$s^2 = \frac{1}{N_1 + N_2} \sum_{i=1}^{N_1+N_2} f(x_i - A)^2$$

$$= \frac{1}{N_1 + N_2}\left[\sum_{i=1}^{N_1} f(x_i - A)^2 + \sum_{i=N_1+1}^{N_1+N_2} f(x_i - A)^2\right]$$

$$= \frac{1}{N_1 + N_2}\left[N_1.\frac{1}{N_1}\sum_{i=1}^{N_1} f(x_i - A)^2 + N_2.\ \frac{1}{N_2}\sum_{j=1}^{N_2}(x_j - A)^2\right]$$

$$= \frac{1}{N_1 + N_2}\left[N_1.s_1^2 + N_2.s_2^2\right]$$

$$= \frac{1}{N_1 + N_2}\ [N_1(\sigma_1^2 + d_1^2) + N_2(\sigma_2^2 + d_2^2)] \text{ as } s^2 = \sigma^2 + d^2$$

$$= \frac{N_1\sigma_1^2 + N_2\sigma_2^2}{N_1 + N_2} + \frac{N_1 d_1^2 + N_2 d_2^2}{N_1 + N_2}$$

$$= \frac{N_1\sigma_1^2 + N_2\sigma_2^2}{N_1 + N_2} + \frac{N_1(M_1 - A)^2 + N_2(M_2 - A)^2}{N_1 + N_2} \text{ as } d = M - A.$$

If the assumed mean A be regarded as the mean M of the combined set, then by the properties of arithmetic mean, we have

$$A = M = \frac{N_1M_1 + N_2M_2}{N_1 + N_2}$$

and then s becomes identical with σ, *i.e* $s = \sigma$.

$$\therefore \quad \sigma^2 = \frac{N_1\sigma_1^2 + N_2\sigma_2^2}{N_1 + N_2} + \frac{1}{N_1 + N_2}\left[N_1\left\{M_1 - \frac{N_1M_1 + N_2M_2}{N_1 + N_2}\right\}^2 + N_2\left\{M_2 - \frac{N_1M_1 + N_2M_2}{N_1 + N_2}\right\}^2\right]$$

$$= \frac{N_1\sigma_1^2 + N_2\sigma_2^2}{N_1 + N_2} + \frac{1}{N_1 + N_2}\left[\frac{N_1.N_2^2(M_1 - M_2)^2}{(N_1 + N_2)^2} + \frac{N_2.N_1^2(M_2 - M_1)^2}{(N_1 + N_2)^2}\right]$$

$$= \frac{N_1\sigma_1^2 + N_2\sigma_2^2}{N_1 + N_2} + \frac{N_1N_2}{N_1 + N_2}(M_1 - M_2)^2.\frac{(N_2 + N_1)}{(N_1 + N_2)^2}$$

$$= \frac{N_1\sigma_1^2 + N_2\sigma_2^2}{N_1 + N_2} + \frac{N_1N_2}{(N_1 + N_2)^2}(M_1 - M_2)^2. \qquad \ldots(1)$$

which is the required result.

This result can be extended to a combined series of observations consisting of r component series containing $N_1, N_2, \ldots, N_r$ members with standard deviations $\sigma_1, \sigma_2, \ldots, \sigma_r$ and means deviating from the mean of the combined series by $d_1, d_2, \ldots, d_r$ whence the standard deviation σ of the whole series with component series having means $M_1, M_2, \ldots, M_r$ respectively, is given by

$$\sigma^2 = \frac{N_1\sigma_1^2 + N_2\sigma_2^2 + \ldots + N_r^2}{N_1 + N_2 + \ldots + N_r} + \frac{1}{(N_1 + N_2 + \ldots + N_r)^2}$$
$$\times \{N_1N_2(M_1 - M_2)^2 + N_2N_3(M_2 - M_3)^2 + \ldots$$
$$\ldots + N_{r-1}N_r(M_{r-1} - M_r)^2\} \qquad \ldots(2)$$

Using Σ notation, the resutl (2) can be expressed as

$$\sigma_2 = \frac{\sum_{i=1}^{r} N_i\sigma_i^2}{\sum_{i=1}^{r} N_i} + \frac{1}{\left(\sum_{i=1}^{r} N_i\right)^2} \sum_{\substack{j,\,k=1\\ j\neq k}}^{r} N_jN_k\left(M_j - M_k\right)^2. \qquad \ldots(3)$$

In case $M_1 = M_2 = M_3 = \ldots,$

$$\sigma^2 = \frac{N_1\sigma_1^2 + N_2\sigma_2^2 + \ldots + N_r\sigma_r^2}{N_1 + N_2 + \ldots + N_r} \qquad \ldots(4)$$

Problem 74. *Show that if the variable takes the values 0, 1, 2,..., n with frequentcies given by the terms of the binomial series $q^n, {}^nC_1 q^{n-1} p, {}^nC_2 q^{n-2}p^2, \ldots, p^n$ where $p + q = 1$, then the mean square deviation is $n^2p^2 + npq$ and the variance is npq.*

By Problem 61, the mean M from such a distribution is given by

$$M = np$$

Now $s^2 = \dfrac{\Sigma f \xi^2}{\Sigma f}$ where $\xi = x - A$, A being assumed mean.

Assuming that $A = 0$, we have

$$s^2 = q^n.0^2 + {}^nC_1.q^{n-1}p.1^2 + {}^nC_2q^{n-2}.p^2.2^2 + \ldots + p^n.n^2 \qquad \ldots(1)$$

But $(q + p)^n = q^n + {}^nC_1q^{n-1} p + {}^nC_2q^{n-2}p^2 + \ldots + p^n$

Differentiating w.r.t. p and multiplying both sides by p, we find

$$n(q + p)^{n-1} p = {}^nC_1q^{n-1}.p + {}^nC_2.q^{n-2}.2p^2 + \ldots np^n$$

Again differentiating w.r.t. p and multiplying throughout by p, we have

$$p[np(n-1)(q + p)^{n-2} + n(q + p)^{n-1}]$$
$$= {}^nC_1.q^{n-1}p.1^2 + {}^nC_1 q^{n-2}p^2.2^2 + \ldots + p^n.n^2 = s^2 \text{ by (1)}$$

$\therefore \quad s^2 = p[n(n-1)p + n] = np[np - p + 1]$

$= np(np + q)$ as $q + p = 1$ gives $1 - p = q$

$= n^2p^2 + npq.$

Also $\sigma^2 = s^2 - d^2 = n^2p^2 + npq - (np)^2$ as $M = d = np$ when $A = 0$

$= npq.$

Problem 75. *Show that if deviations are small compared with the mean M so that $\left(\frac{x}{M}\right)^3$ and higher powers of $\frac{x}{M}$ may be neglected,*

(*i*) $\quad G = M\left(1 - \dfrac{\sigma^2}{2M^2}\right)$

(*ii*) $\quad M^2 - G^2 = \sigma^2.$

(*iii*) $H = M\left(1 - \frac{\sigma^2}{M^2}\right)$.

(*iv*) $MH = G^2$

(*v*) $H + M = 2G$.

(*vi*) $\text{Mean } \sqrt{x} = \sqrt{M}\left\{1 - \frac{\sigma^2}{8M^2}\right\}$.

where σ is the standard deviation, A, G, M are respectively arithmetic, geometric and harmonic means of the variate X.

We have $X - M = x$ so that $X = M + x$

Let N be the total frequency of the distribution. Then

(*i*) $\because G = \left(X_1^{f_1} \cdot X_2^{f_2} \cdot \ldots \cdot X_n^{f_n}\right)^{1/N}$ where $N = \Sigma f$.

$(X_1 f_1, X_2 f_2, \ldots, X_n f_n)^{1/N}$ where $N = \Sigma f$.

$\therefore \log G = -\frac{1}{N} \Sigma f \log X = \frac{1}{N} \Sigma f \log (M + x)$ by (1)

$= \frac{1}{N} \Sigma f \log M\left(1 + \frac{x}{M}\right) = \frac{1}{N}\left[\Sigma f \cdot \log M + \Sigma f \log\left(1 + \frac{x}{M}\right)\right]$

$= \frac{1}{N} \Sigma f \log M + \frac{1}{N} \Sigma f\left\{\frac{x}{M} - \frac{x^2}{2M^2} + \ldots\right\}$ expanding the log-function

$= \log M . \frac{\Sigma f}{N} = \frac{1}{N} \Sigma f\left(\frac{x}{M} - \frac{x^2}{2M^2}\right)$ neglecting higher powers of x/M

$= \log M + \left[\frac{1}{M}\frac{\Sigma fx}{N} - \frac{1}{2M^2}\frac{\Sigma fx^2}{N}\right] = \log M - \frac{1}{2M^2} . \sigma^2$

as $\sigma^2 = \frac{\Sigma fx^2}{N}$ and $\Sigma fx = 0$ by first property of A.M.

$\therefore \log \frac{G}{M} = -\frac{\sigma^2}{2M^2}$ *i.e.* $\frac{G}{M} = e^{-\sigma^2/2M^2}$ ...(2)

$= \left(1 - \frac{\sigma^2}{2M^2} + \ldots\right)$

or $G = M\left(1 - \frac{\sigma^2}{2M^2}\right)$ neglecting higher order terms.

(*ii*) By (2), $G^2 = M^2 e^{-\sigma^2/M^2}$

$= M^2\left\{1 - \frac{\sigma^2}{M^2}\right\}$ neglecting higher order terms

$= M^2 - \sigma^2$ *i.e.* $M^2 - G^2 = \sigma^2$

(*iii*) We have $\frac{1}{H} = \frac{1}{N}\Sigma f\left(\frac{1}{X}\right) = \frac{1}{N}\Sigma f\left(\frac{1}{M+x}\right)$

$$= \frac{1}{N}\Sigma f\left\{\frac{1}{M\left(1+\frac{x}{M}\right)}\right\} = \frac{1}{N}\cdot\frac{1}{M}\Sigma f\left(1+\frac{x}{M}\right)^{-1}$$

$$= \frac{1}{M}\cdot\frac{1}{M}\Sigma f\left(1-\frac{x}{M}+\frac{x^2}{M^2}\right) \text{ neglecting higher order terms}$$

$$= \frac{1}{M}\cdot\left[\frac{\Sigma f}{N}-\frac{1}{M}\cdot\frac{\Sigma fx}{N}+\frac{1}{M^2}\frac{\Sigma fx^2}{N}\right]$$

$$\doteq \frac{1}{M}\left[1+\frac{\sigma^2}{M^2}\right] \text{as } \Sigma fx = 0 \text{ and } \frac{\Sigma fx^2}{N} = \sigma^2,$$

i.e., $H = M\left[1+\frac{\sigma^2}{M^2}\right]^{-1} = M\left[1-\frac{\sigma^2}{M^2}\right]$ neglecting higher order terms.

(*iv*) From the above relation $H = M\left(1-\frac{\sigma^2}{M^2}\right)$, we have

$$MH = M^2\left(1-\frac{\sigma^2}{M^2}\right) = M^2 - \sigma^2 = G^2 \text{ by } (ii)$$

(*v*) We have $M + H = M + M\left(1-\frac{\sigma^2}{M^2}\right) = 2M - \frac{\sigma^2}{M}$

$$= 2M\left[1-\frac{\sigma^2}{2M^2}\right] = 2G \text{ by } (i)$$

(*vi*) Mean $\sqrt{X} = \frac{1}{N}\Sigma f\sqrt{X} = \frac{1}{N}\Sigma f(M+x)$

$$= \frac{1}{N}\Sigma f\left\{\sqrt{(M)}\left(1+\frac{x}{M}\right)^{1/2}\right\}$$

$$= \sqrt{M}\cdot\frac{1}{N}\cdot\left(1+\frac{x}{2M}-\frac{x^2}{8M^2}\right)\left(\text{neglecting higher powers of } \frac{x}{M}\right)$$

$$= \sqrt{M}.\left[\frac{\Sigma f}{N}+\frac{1}{2M}\frac{\Sigma fx}{N}-\frac{1}{8M^2}\frac{\Sigma fx^2}{N}\right]$$

$$= \sqrt{M}\left[1-\frac{\sigma^2}{8M^2}\right] \text{as } \Sigma fx = 0 \text{ and } \frac{\Sigma fx^2}{N} = \sigma^2.$$

Problem 76. *Show that for any discrete distribution the standard deviation is not less than the mean deviation from the mean.*

With usual notation we have to show that

$$\sqrt{\left\{\frac{1}{N}\Sigma f(x-M)^2\right\}} \geq \left\{\frac{1}{N}\Sigma f|x-M|\right\}.$$

i.e.,
$$\frac{1}{N}\ \Sigma f(x-M)^2 \geq \left\{\frac{1}{N}\Sigma f|x-M|\right\}^2,$$

or $$N\Sigma f\xi^2 \geq (\Sigma f\ |\xi|)^2 \qquad \text{where } \xi = x - M$$

or $$(f_1+f_2+\ldots+f_n)(f_1\xi_1^2+f_2\xi_2^2+\ldots+f_n\xi_n^2) \geq (f_1\ |\xi_1|\ +f_2\ |\xi_2|\ +\ldots+f_n\ |\xi_n|\ )^2$$

or $$f_1f_2(\xi_1^2+\xi_2^2)+\ldots \geq 2f_1f_2\xi_1\xi_2+\ldots$$

or $$f_1f_2(\xi_1-\xi_2)^2+\ldots \geq 0,$$

which is true, since being the sum of perfect squares, the L.H.S. is never negative. Hence the proposition.

Problem 77. *From a sample of n observations, the arithmetic mean and variance are calculated. It is then found that one of the values, x_1, is in error and should be replaced by x_1', show that the adjustment to the variance to correct this error is*

$$\frac{1}{N}(x_1'-x_1)\left(x_1'+x_1-\frac{x_1'-x_1+2T}{n}\right),$$

where T is the total of the original results.

If V be the variance then V being the sqaure of the standard deviation, we have

$$V = \frac{x_1^2+x_1^2+\ldots+x_n^2}{n}-\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)^2$$

$$= \frac{x_1^2+x_1^2+\ldots+x_n^2}{n}-\left(\frac{T}{n}\right)^2=\frac{1}{n}\left[\sum_{i=1}^{n}x_i^2-\frac{T^2}{n}\right]$$

If V' be the corrected variance, then we have

$$V' = \frac{x_1'^2+x_2^2+\ldots+x_n^2}{n}-\left(\frac{x_1'+x_2+\ldots+x_n}{n}\right)^2$$

$$= \frac{x_1'^2+\left(\Sigma x_i^2-x_1^2\right)}{n}-\left(\frac{x_1'+T-x_1}{n}\right)^2$$

$$= \frac{1}{n}\left[\left(\Sigma x_i^2+x_1'^2-x_1^2\right)-\frac{\left(T-x_1+x_1'^2\right)^2}{n}\right]$$

The required adjustment to correct the error is = $V'-V$

$$= \frac{1}{n}\left[\left\{\left(\Sigma x_i^2+x_1'^2-x_1^2\right)-\Sigma x_i^2\right\}-\frac{1}{n}\left\{(T-x_1+x_1')^2-T^2\right\}\right]$$

$$= \frac{1}{n}\left[\left(x_1'^2-x_1^2\right)-\frac{1}{n}\left\{(-x_1+x_1')^2+2T(-x_1+x_1')\right\}\right]$$

$$= \frac{1}{n}\left[\left(x_1'^2-x_1^2\right)-\frac{(x_1'-x_1)}{n}\{x_1'-x_1+2T\}\right]$$

$$= \frac{1}{n}\left[(x_1'-x_1)\left(x_1'+x_1-\frac{x_1'-x_1+T}{n}\right)\right].$$

Problem 78. *A distribution consists of three components with frequencies 200, 250, 300, having means of 25, 10 and 15, standard deviations, of 3, 4 and 5 respectively. Show that the mean of the combined distribution is 16 and its standard deviation is 7.2 approximately. Find also C.V.*

The mean M of the combined distribution, is given by

$$M = \frac{1}{N}\sum_{i=1}^{3} N_i\bar{x}_i = \frac{N_1\bar{x}_1 + N_2\bar{x}_2 + N_3\bar{x}_3}{N_1 + N_2 + N_3}$$

where $N_1 = 200, N_2 = 250, N_3 = 300, \bar{x}_1 = 25, \bar{x}_2 = 10, \bar{x}_3 = 15.$

$$\therefore \quad M = \frac{200\times 25 + 200\times 10 + 300\times 15}{200 + 250 + 300} = \frac{5000 + 2500 + 4500}{750}$$

$$= \frac{12000}{750} = 16.$$

Now, the standard deviation σ of the combined distribution is given by

$$\sigma^2 = \frac{N_1\sigma_1^2 + N_2\sigma_2^2 + N_3\sigma_3^2}{N_1 + N_2 + N_3} + \frac{1}{(N_1 + N_2 + N_3)^2}\{N_1N_2(M_1 - M_2)^2 + N_2N_3(M_2 - M_3)^2 + N_3N_1(M_3 - M_1)^2\}$$

where $N_1 = 200, N_2 = 250, N_3 = 300,$

$\sigma_1 = 3, \sigma_2 = 4, \sigma_3 = 5$

$M_1 = 25, M_2 = 10, M_3 = 15.$

$$\therefore \quad \sigma^2 = \frac{200\times 9 + 250\times 16 + 300\times 25}{750} + \frac{1}{(750)^2}\{50000\times 225 + 75000\times 25 + 6000\times 100\}$$

$$= \frac{1800 + 4000 + 7500}{750} + \frac{1}{750\times 750}[11250000 + 1875000 + 6000000]$$

$$= \frac{1330}{750} + \frac{1}{750\times 750}\times 19125000$$

$$= \frac{1330}{75} + \frac{191250}{75\times 75} = \frac{266}{15} + \frac{510}{15} = \frac{776}{15} = 51.73$$

giving $\sigma = 7.2$ approx.

The coefficient of variation (C.V.) $= \dfrac{\sigma}{M}\times 100$

$$= \frac{7.2}{16}\times 100 = 45$$

Problem 79. *The first of two samples has 100 items with mean 15 and standard deviation 3. If the whole group has 250 items with mean 15.6 and standard deviation $\sqrt{(13.44)}$, find the standard deviation of the second group.*

We have $N_1 = 100, N_2 = 250 - 100 = 150, M_1 = 15, \sigma_1 = 3,$

$M = 15.6, \sigma = \sqrt{13.44}$

$$\sigma^2 = \frac{N_1\sigma_1^2 + N_2\sigma_2^2}{N_1 + N_2} + \frac{N_1N_2}{(N_1 + N_2)^2}(M_1 - M_2)^2$$

$$13.44 = \frac{100\times 9 + 150\times\sigma_2^2}{250} + \frac{100\times 150}{(250)^2}(15 - M_2)^2. \quad ...(1)$$

Also $\quad M = \dfrac{N_1M_1 + N_2M_2}{N_1 + N_2}$

i.e. $15.6 = \dfrac{100 \times 15 + 150 \times M_2}{250}$

$150M_2 = 250 \times 15.6 - 1500 = 2400$ giving $M_2 = 16$

Substituting this value of M_2 in (1), we get

$$13.44 = \frac{900 + 150\sigma_2^2}{250} + \frac{6}{25} \times (15-16)^2$$

$$= \frac{900 + 150\sigma_2^2}{250} + (.24)$$

or $900 + 150\sigma_2^2 = 250\ [13.44 - .24] = 250 \times 13.20 = 3300$

or $150\sigma_2^2 = 2400$ *i.e.* $\sigma_2^2 = 16$ or $\sigma_2 = 4$.

Problem 80. *Calculate (a) the quartile, (b) the mean, and (c) the standard deviation wages from the following data :*

Weekly wage in dollars – 35 – 36 – 37 – 38 – 39 – 40 – 41 – 42

No. of wage-earners 14 20 42 54 45 18 7

Weekly wage in dollars class	*Mid-value x*	*Frequency f*	*Cumulative frequency*	*Deviation from assumed mean = 38. 5 (ξ)*	ξ^2	$f\xi$	$f\xi^2$
35 – 36	35.5	14	14	–3	9	–42	126
36 – 37	36.5	20	34	–2	4	–40	80
37 – 38	37.6	42	76	–1	1	–42	42
38 – 39	38.5	54	130	0	0	0	0
39 – 40	39.5	45	175	1	1	45	45
40 – 41	40.5	18	193	2	4	18	72
41 – 42	41.5	7	200	3	9	7	63
Total		200 = Σf = N				–54 = $\Sigma f\xi$	428 = $\Sigma f\xi^2$

$\because$ $\frac{1}{4}N = \frac{200}{2} = 50$ $\therefore$ Q_1 lies in group 37 – 38

and $\frac{3}{4}N = 150$, $\therefore$ Q_3 lies in group 39 – 40

Q_2 = Median = Mean of the sizes of $\left\{\frac{200}{2}\text{th and } \left(\frac{200}{2}+1\right)\text{th}\right\}$ items

$= 38.5$

$$Q_1 = l_1 + \frac{\frac{1}{2}N - f_1}{f} \times h \text{ where } l_1 = 37,\ \tfrac{1}{4}N = 50, f_1 = 34, f = 42,\ h = 1$$

$$= 37 + \frac{50-34}{42} \times 41$$

$$= 37 + \frac{5}{21} = 37 + .38$$

$$= 37.38$$

$$Q_3 = l_1 + \frac{\frac{3}{4}N - f_1}{f} \times h \text{ where } l_1 = 39,\ \frac{3N}{4} = 150, f_1 = 45,\ h = 1$$

$$= 39 + \frac{150 - 130}{45} \times 1$$

$$= 39 + \tfrac{4}{9} = 39 + .44 = 39.44.$$

Mean $= A + \frac{1}{N} \Sigma f\xi$ when; $A = 38.5$,

$$= 38.5 + \frac{-54}{200}$$

$$= 38.5 - .27 = 38.23$$

$$\sigma = \left[\frac{1}{N}\Sigma f \xi^2 - \left(\frac{\Sigma f \xi}{N}\right)^2\right]$$

$$= \sqrt{\left\{\frac{428}{200} - \left(\frac{54}{200}\right)^2\right\}} = \sqrt{(2.14 - .0729)} = \sqrt{(2.0671)} = 1.4 \text{ approx.}$$

15.10. MOMENTS

For a distribution having variate-values $x_1, x_2, \ldots x_n$ with frequencies $f_1, f_2, \ldots f_n$ respectively such that $N = \sum_{i=1}^{n} f_i$ (or simply Σf), the rth moment about any point (or assumed mean A) is defined as

$$\mu_r' = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - A)^r \text{ or simply } \frac{1}{N}\Sigma f(x - A)^r, \qquad \ldots(1)$$

where μ_r' represents the moment about any point A (other than the mean).

As such rth moment can be regarded as the arithmetic mean of the variate $(X - A)^r$.

If we now consider the rth moment about the mean M of the distribution then it is denoted by μ_r and given by

$$\mu_r = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - M)^r, \text{ i.e. } \frac{1}{N}\Sigma f(x - M)^r. \qquad \ldots(2)$$

The definitions (1) and (2) give that

$$\mu_0' = \mu_0 = \frac{1}{N}\Sigma f = 1. \qquad \ldots(3)$$

$$\mu_1' = \frac{1}{N}\Sigma f(x - A)$$

$$= \frac{1}{N}\Sigma fx - A\frac{\Sigma f}{N}, \text{ where } \Sigma f = N \text{ and } \frac{\Sigma fx}{N} = M$$

$$\left.\begin{aligned} &= M - A = d \text{ (say)} \\ \text{Where } A = 0, \mu_1' &= M = \bar{x} \end{aligned}\right\} \qquad \ldots(4)$$

$$\mu_1 = \frac{1}{N}\Sigma f(x - M) = \frac{1}{N}\Sigma f - M.\frac{1}{N}\Sigma f = M - M = 0 \qquad \ldots(5)$$

$$\mu_2' = \frac{1}{N}\Sigma f(x - A)^2$$

$$= \frac{1}{N} \Sigma f\xi^2, \text{ when } x - A = \xi$$

$$= s^2 = \sigma^2 + d^2. \quad ...(6)$$

$$\mu_2 = \frac{1}{N} \Sigma f(x - M)^2 = d^2. \quad ...(7)$$

Thus, $\mu_2' - \mu_2 = s^2 - \sigma^2 = d^2$. ...(8)

[A] Moments about the mean in terms of moments about any point and conversely.

Taking $x - A = \xi$ and $M - A = d$, we have

$$x - M = (x - A) - (M - A) = \xi - d$$

so that $\mu_r = \frac{1}{N} \Sigma f(x - M)^r = \frac{1}{N} \Sigma f(\xi - d)^r$

$$= \frac{1}{N} [\Sigma f(\xi^r - {}^rC_1 \xi^{r-1} d + {}^rC_2 \xi^{r-2} d^2 - \ldots + (-d)^r)]$$

$$= \frac{1}{N} \Sigma f\xi^r - d.{}^rC_1 \frac{1}{N} \Sigma f\xi^{r-1} + d^2.{}^rC_2 \frac{1}{N} \Sigma f\xi^{r-2} - \ldots$$

$$+ (-1)^{r-1}.d^{r-1}.{}^rC_{r-1} \frac{1}{N} \Sigma f\xi + (-1)^r d^r \frac{1}{N} \Sigma f$$

$$= \mu_r' - {}^rC_1 d\mu'_{r-1} + {}^rC_2 d^2.\mu'_{r-2} - \ldots + (-1)^{r-1} r\, d^{r-1} \mu_1' + (-1)^r.d^r \quad ...(1)$$

But we have $d = \mu_1'$ by (4)

$$\therefore \mu_r = \mu_r' - {}^rC_1 \mu_1' \mu'_{r-1} + {}^rC_2 \mu_1'^2 \mu'_{r-2} - \ldots + (-1)^{r-1}(r - 1)\mu_1'^r \quad ...(2)$$

Putting $r = 2, 3, 4, \ldots$ successively, we get

$$\mu_2 = \mu_2' - \mu_1'^2 = \mu_2' - d^2 \quad ...(3)$$

$$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2\mu_1'^2\mu_1' = \mu_3' - 3d\mu_2' + 2d^3 \quad ...(4)$$

$$\left.\begin{aligned} \mu_4 &= \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1' \\ &= \mu_4' - 4d\mu_3' + 6d^2\mu_2' - 3d^4 \end{aligned}\right\}, \quad ...(5)$$

... ... etc.

Conversely, $\mu'_r = \frac{1}{N} \Sigma f(x - A)^r = \frac{1}{N} \Sigma f(x - M + M - A)^r$

$$= \frac{1}{N} \Sigma f(X + d)^r, \text{ when } X = x - M$$

$$= \frac{1}{N} [\Sigma fX^r + {}^rC_1 d \Sigma fX^{r-1} + {}^rC_2 d^2 \Sigma fX^{r-2} + \ldots + d\ \Sigma f]$$

$$= \mu_r + {}^rC_1 \mu_{r-1} d + {}^rC_2 \mu_{r-2} d^2 + \ldots + d^r. \quad ...(6)$$

Putting $r = 2, 3, 4, \ldots$ successively, we get

$$\mu_2' = \mu_2 + d^2 \text{ as } \mu_1 = 0, \quad ...(7)$$

$$\mu_3' = \mu_3 + 3d\mu_2 + d^3, \quad ...(8)$$

$$\mu_4' = \mu_4 + 4d\mu_3 + 6d^2\mu_2 + d^4, \quad ...(9)$$

... ... etc.

[B] Change of origin and scale of moments. If we introduce a new variate u related to x, such that $u = \frac{x - A}{h}$, where h is the unit or scale of moments, then

$$x - A = hu \text{ and } \bar{x} - A = h\,\bar{u}, \qquad \ldots(1)$$

where $\bar{x}$ and $\bar{u}$ denote the means of variates x and u respectively.

Now, $$\mu_r = \frac{1}{N} \Sigma f(x - \bar{x})^r$$
$$= \frac{1}{N} \Sigma f\{(A + hu) - (A + h\,\bar{u})\}^r \text{ by (1)}$$
$$= \frac{h^r}{N} \Sigma f(u - \bar{u})^r. \qquad \ldots(2)$$

Also $$\mu_r' = \frac{1}{N} \Sigma f(x - A)^r = \frac{1}{N} \Sigma f(hu)^r = \frac{h^r}{N} \Sigma f u^r. \qquad \ldots(3)$$

Here (2) and (3) show that the rth moment of the variate x is h^r times the corresponding moment of the variate x.

If $h = 1$, then (2) becomes $\mu_r = \frac{1}{N} \Sigma f(u - \bar{u})^r$

which follows that "*a change of origin does not affect the moments about the mean.*"

Again if we take $A = \bar{x}$ and $h = \sigma_x$ then the distribution of x has zero mean and unit variance and $\frac{x - \bar{x}}{\sigma_x}$ is said to be the standard form of the variate.

[C] Sheppard's corrections for moments of grouped data. The approximation of assuming the frequencies to be concentrated at the mid-values of class-intervals in a grouped frequency distribution is corrected in case of moments by W.F. Sheppard as follows :

$$\mu_2 \text{ (corrected)} = \mu_2 - \frac{h^2}{12}$$
$$\mu_3 \text{ (corrected)} = \mu_3$$
$$\mu_4 \text{ (corrected)} = \mu_4 - \tfrac{1}{2} h^2 \mu_2 + \tfrac{7}{240} h^4$$

where h is the width of the class-interval.

These results may be used in cases where the frequency distribution is continuous and the distribution tapers off to zero in both directions.

Problem 81. *The first three moments of a distribution about the values 2 of the variable are 1, 16 and – 40. Show that the mean is 3, the variance 15 and $\mu_3 = -86$. Also show that the first three moments about $x = 0$ are 3, 24 and 76.*

Moments about $A = 2$, are $\mu_1' = 1$, $\mu_2' = 16$, $\mu_3' = -40$

We have $\mu_1' = \frac{1}{N} \Sigma f(x - A)$, *i.e.* $1 = \frac{1}{N} \Sigma fx - 2$ as $\Sigma f = N$

$$\frac{1}{N} \Sigma fx = \text{mean} = 3$$

$\therefore$ $$\mu_2 = \sigma^2 = \mu_2' - \mu_1'^2 = 16 - 1 = 15$$

and $$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2\mu'^3$$
$$= -40 - 3 \times 1 \times 16 + 2.1^3 = -86.$$

Again when $A = 0$, let the three moments about $x = 0$ be v_1', v_2', v_3', then

$$v_1' = \frac{1}{N} \Sigma f(x - 0) = \frac{1}{N} \Sigma fx = \text{mean} = 3$$

$$\nu_2' = \mu_2 + d^2.$$

Here $\mu_2 = 15$ and $d = M - A - 3$

$\therefore \quad \nu_2' = 15 + 9 = 24.$

Also $\nu_3' = \mu_3 + 3d\mu_2 + d^3$

$= -86 + 3 \times 3 \times 15 + (3)^3 = 76.$

[D] Factorial and absolute moments. About the origin *i.e.* $x = 0$, of a distribution, the rth *factorial moment* is defined as

$$\mu'_{(r)} = \frac{1}{N}\sum_i f_i x_i^{(r)} \text{ or simply } \frac{1}{N} \Sigma f x^{(r)} \qquad ...(1)$$

where $N = \sum_i f_i = \Sigma f$ and $x^{(r)} = x(x-1)(x-2)...(x-r+1)$

and the rth *absolute moment* is defined as

$$\nu_r = \frac{1}{N} \sum_i f_i \ |x_i|^r \text{ or simply } \frac{1}{N} \Sigma f |x|^r \qquad ...(2)$$

where $\quad N = \sum_i f_i = \Sigma f.$

[E] Moments for bivariate distribution. If there are two measurable characters say x and y corresponding to each member of the population *e.g.* heights and weights of students in the survey of the poplulation of a class of students, then each member has a number of pair of values like $(x_1, y_1), (x_2, y_2), ..., (x_i, y_i), ...$ .The total assemblage of such pairs of values of x and y along with their frequencies constitutes what is called the *Bivariate frequency distribution* of x and y.

If $f_i\ (i = 1, 2, ..., n)$ be the frequency of the pair (x_i, y_i), $i = 1, 2, ..., n$, then the moment about the origin of the distribution, of the order r, s with regard to x and y respectively (*i.e.* order r in x and order s in y) is defined as

$$\mu_{rs}' = \frac{1}{N} \sum_{i=1}^{n} f_i x_i^r y_i^s = \frac{1}{N} \Sigma f x^r y^s \qquad ...(1)$$

where $\quad N = \sum_{i=1}^{n} f_i = \Sigma f.$

Thus,

$$\left.\begin{aligned} \mu'_{10} &= \frac{1}{N}\sum_i f_i x_i = \bar{x}, \text{ the mean for } x \\ \mu'_{01} &= \frac{1}{N}\sum_i f_i y_i = y, \text{ the mean for } y \end{aligned}\right\} \qquad ...(2)$$

$$\left.\begin{aligned} \mu'_{20} &= \frac{1}{N}\sum_i f_i x_i^2 = {}_x\mu'_2 \\ \mu'_{02} &= \frac{1}{N}\sum_i f_i y_i^2 = {}_y\mu'_2 \end{aligned}\right\} \qquad ...(3)$$

$$\mu'_{11} = \frac{1}{N}\sum_i f_i x_i y_i \qquad ...(4)$$

In a similar manner the moment μ_{rs} about the means $\bar{x}$, $\bar{y}$ may be defined as

$$\mu'_{rs} = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^r (y_i - \bar{y})^s \qquad \ldots(5)$$

So that $\mu_{10} = \mu_{01} = 0$

$$\mu_{11} = \frac{1}{N}\sum_i f_i(x_i - \bar{x})(y_i - \bar{y})$$

$$= \frac{1}{N}\sum_i f_i x_i y_i - \bar{x}.\frac{1}{N}\sum_i f_i y_i - \bar{y}.\frac{1}{N}\sum_i f_i x_i + \bar{x}\bar{y}$$

$$= \mu_{11}' - \bar{x}\bar{y} - \bar{x}\bar{y} + \bar{x}\bar{y} = \mu'_{11} - \bar{x}\bar{y} \qquad \ldots(6)$$

The quantity μ_{11} is known as the **Covariance** between the variates x and y and denoted by Cov (x, y), *i.e.*

$$\text{Cov}(x, y) = \mu_{11} = \mu_{11}' - \bar{x}\bar{y} \qquad \ldots(7)$$

Also

$$\left.\begin{aligned} \mu_{20} &= \sigma_x^2 = \frac{1}{N}\sum_i f_i(x_i - \bar{x})^2 = \mu'_{20} - \bar{x}^2 \\ \mu_{02} &= \sigma_y^2 = \frac{1}{N}\sum_i f_i(y_i - \bar{y})^2 = \mu'_{02} - \bar{y}^2 \end{aligned}\right\} \qquad \ldots(8)$$

Problem 82. *Show that the first four factorial moments are related to ordinary moments by the relations*

$$\mu_{(1)}' = \mu_1' = \bar{x}$$

$$\mu_{(2)}' = \mu_2' - \bar{x}$$

$$\mu_{(3)}' = \mu_3' - 3\mu_2' + 2\bar{x}$$

$$\mu_{(4)}' = \mu_4' - 6\mu_3' + 11\mu'_2 - 6\bar{x}$$

For assumed mean $A = 0$, $\mu_1' =$ mean $= \bar{x}$ by (4) of §15.10

Now $\mu'_{(r)} = \frac{1}{N}\sum_i f_i x_i^{(r)}$ gives

$$\mu_{(1)'} = \frac{1}{N}\Sigma f_i x_i^{(1)} = \frac{1}{N}\Sigma f_i x_i = \mu_1' = \bar{x}$$

$$\mu_{(2)}' = \frac{1}{N}\sum_i f_i x_i^{(2)} = \frac{1}{N}\sum_i f_i x_i(x_i - 1)$$

$$= \frac{1}{N}\sum_i f_i x_i^2 - \frac{1}{N}\sum_i f_i x_i = \mu_2' - \mu_1' = \mu_2' - \bar{x}$$

$$\mu_{(3)}' = \frac{1}{N}\sum_i f_i x_i^{(3)} = \frac{1}{N}\sum_i f_i x_i(x_i - 1)(x_i - 2)$$

$$= \frac{1}{N}\sum_i f_i x_i^3 - 2.\frac{1}{N}\sum_i f_i x_i^2 + 2.\frac{1}{N}\sum_i f_i x_i$$

$$= \mu_3' - 3\mu_2' + 2\bar{x}$$

Similarly, $\mu_{(4)}' = \mu_4' - 6\mu_3' + 11\mu_2' - 6\bar{x}$.

Problem 83. *Show that, for bivariate distributions, the moments about the mean are invariant for the change of origin but not for the change of scale.*

Let the origin for x and y be transferred to A and A' respectively such that the unit of scale for both remains the same say h. Then,

$$x = A + hu \text{ so that } \bar{x} = A + h\bar{u}$$
$$y = A' + hv \qquad \bar{y} = A' + h\bar{v}$$

u, v being the new variates.

$$\therefore \quad \mu_{11} = \frac{1}{N}\sum_i f_i(x_i - \bar{x})(y_i - \bar{y}) \text{ when } N = \Sigma f_i$$

$$= \frac{h^2}{N}\sum_i f_i(u_i - \bar{u})(v_i - \bar{v})$$

$$= h^2\left[\frac{1}{N}\sum_i f\mu_i v_i - \frac{1}{N}\bar{u}\sum_i f_i v_i - \bar{v}.\frac{1}{N}\sum_i f\bar{\mu} + \bar{u}\bar{v}\frac{1}{N}\Sigma f_i\right]$$

$$= h^2\left[\frac{1}{N}\sum_i f\mu_i v_i - \bar{u}\,\bar{v} - \bar{u}\bar{v} + \bar{u}\bar{v}\right] = h^2\left[\frac{1}{N}\sum_i f\mu_i v_i - \bar{u}\bar{v}\right]$$

$h = 1$, yields, $\quad \mu_{11} = \frac{1}{N}\sum_i f\mu_i v_i - \bar{u}\bar{v}$

$$\text{By } [E], \mu_{11} = \frac{1}{N}\sum_i f_i x_i y_i - \bar{x}\bar{y} = \frac{1}{N}\sum_i f_i u_i v_i - \bar{u}\bar{v}.$$

Conclusively the moment of order 1 in x and order 1 in y about the mean is invariant for the change of origin but not for the change of scale. The similar results may be shown to hold for other moments also.

[F] β and γ coefficients of Karl Pearson. It is observed that in statistical work, there are certain quantities calculated from moments about the mean, which are found very useful. Such quantities are β and γ coefficients introduced by Karl Pearson. He defined β-coefficients as follows:

$$\beta_1 = \frac{\beta_3^2}{\beta_2^3} = \frac{\mu_3^2}{\sigma^6}, \; \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{\mu_4}{\sigma^4} \qquad ...(1)$$

The other two coefficients γ_1 and γ_2 are defined as

$$\gamma_1 = +\sqrt{\beta_1} = \frac{\mu_3}{\sigma^3}; \; \gamma_2 = B_2 - 3 = \frac{\mu_4 - 3\mu_2^2}{\mu_2^2} = \frac{\mu_4 - 3\sigma^4}{\sigma^4} \qquad ...(2)$$

All these coefficients are pure numbers since μ_n has the dimension n. Their importance lies in giving information regarding the shape of the curve obtained from the frequency distribution.

β_1 gives a measure of departure from symmetry *i.e.* of *Skewness.*

β_2 or its derivative γ_2 gives a measure of flatness or peakedness of a distribution and known as *Kurtosis.*

[G] Cumulants. T.N. Thiele introduced a set of quantities similar to moments, which have a comparatively little theoretical and practical importance. Such quantities are known as cumulants. Their definition constitutes a complicated mathematical structure and hence for the elementary purposes, we mention the first four cumulants as follows:

$$\left.\begin{aligned}\kappa_1 &= \mu_1' \\ \kappa_2 &= \mu_2' - \mu_1'^2 \\ \kappa_3 &= \mu_3' - 3\mu_1'\mu_2' + 2\mu_1'^3 \\ \kappa_4 &= \mu_4' - 4\mu_1'\mu_3' - 3\mu_2'^2 + 12\mu_1'^2\mu_2' - 6\mu_1'^4\end{aligned}\right\} \quad ...(1)$$

As a particular case, about the mean, they are defined as

$$\kappa_1 = 0,\ \kappa_2 = \mu_2,\ \kappa_3 = \mu_3,\ \kappa_4 = \mu_4 - 3\mu_2^2 \quad ...(2)$$

15.11. EXPECTATION

[A] Univariate probability distribution. If a variate x assumes $x_1, x_2, ..., x_n$ (denoted by x_i for $i = 1, 2, ... n$) values, with their respective probabilities $p_1, p_2, ... p_n$ (denoted by p_i for $i = 1, 2, ... n$) corresponding to the n exhaustive and mutually exclusive cases which may result from a trial, then since the values taken by x depend on chance, x is called a *chance* or *random variable* (variate) or *stochastic variate* (a greek term) and the set of values x_i together with their probabilities p_i constitutes what is called the *univariate probability distribution* of the variate of that trial. In such cases since the variate takes a finite or enumerably infinite set of values (*i.e.* discrete set of values), the distribution is sometimes known as *Discrete Probability distribution.*

[B] Mathematical expectation or expected values.

Suppose $\phi(x)$ is a function of variate x such that it takes values $\phi(x_i)$ $(i = 1, 2, ... n)$ i.e. $\phi(x_1), \phi(x_2), ..., \phi(x_n)$ when x takes values x_i $(i = 1, 2, ... n)$ with probabilities p_i $(i = 1, 2, ... n)$, the **expected** *or* **proable value** *of $\phi(x)$ denoted by $E\{\phi(x)\}$ is defined as*

$E\{\phi(x)\}$ or simply $E(\phi) = p_1\phi(x_1) + p_2\phi(x_2) + ... + p_n\phi(x_n)$

$$= \sum_{i=1}^{n} p_i \phi(x_i) \text{ where } \sum_{i=1}^{n} p_i = 1 \quad ...(1)$$

Assuming $\phi(x) = x^r$, we have

$$E(x^r) = p_1 x_1{}^r + p_2 x_2{}^r + ... + p_n x_n{}^r = \sum_{i=1}^{n} p_i x_i^r \quad ...(2)$$

which is said to be the rth *moment of the discrete probability distribution* about $x = 0$ and denoted by μ_r' *i.e.*

$$\mu_r' = E(x^r) = \sum_i p_i x_i{}^r. \quad ...(3)$$

This gives $\mu_1' = E(x) = \sum_i p_i x_i = \bar{x}$ by (4) of §15.10 ...(4)

which is said to define the *expected values* or the *expectation* of the variate x.

Again the rth moment of the probability distribution about the mean $\bar{x}$ is defined as

$$\mu_r = E(x_i - \bar{x})^r = \sum_i p_i (x_i - \bar{x})^r, \quad ...(5)$$

so that μ_2, the variance of the distribution of x denoted by Var (x) is given by

$$\mu_2 = \text{Var}(x) = E\{x - E(x)\}^2 \text{ as } \bar{x} = E(x) \text{ from (4).} \quad ...(6)$$

Problem 84. *For a stochastic variable x, prove that*

$$\text{Variance} = E(x^2) - \{E(x)\}^2$$

and deduce that $E(x^2) \geq \{E(x)\}^2$.

We have, $\text{Var}(x) = E\{x - E(x)\}^2$

$$= \sum_{i=1}^{n} p_i (x_i - \bar{x})^2 \text{ where } \bar{x} = E(x) \text{ and using (5) of [B]}$$

$$= \sum_{i} p_i(x_i^2 - 2\,\bar{x}x_i + \bar{x}^2)$$

$$= \sum_{i} p_i x_i^2 - 2\,\bar{x} \sum_{i} p_i x_i + \bar{x}^2 \text{ as } \Sigma p_i = 1$$

$$= E(x^2) - 2\,\bar{x}\,E(x) + \bar{x}^2$$

$$= E(x^2) - 2\,\{E(x)\}^2 + \{E(x)\}^2 \text{ as } \bar{x} = E(x)$$

$$= E(x^2) - \{E(x)\}^2$$

Thus, $\quad \text{Var}(x) = \sum_{i} p_i (x_i - \bar{x})^2 = E(x^2) - \{E(x)\}^2$

But $\quad \sum_{i} p_i (x_i - \bar{x})^2 \geq 0$ as p_i are positive

i.e. $\quad E(x^2) - \{E(x)\}^2 \geq 0$

or $\quad E(x^2) \geq \{E(x)\}^2$.

Problem 85. *If a pair of fair dice is tossed and x denotes the sum of the numbers on the two dice, then find the expectation of x.*

The x may be at least 2 and at the most 12.

Let $x_1, x_2, \ldots, x_{11}$ be the values of x corresponding to $x = 2, 3, \ldots, 12$ respectively and let $p_1, p_2, \ldots p_{11}$, denote their probabilities, then, the probabilities with the variate-values may be tabulated as below :

x :	2	3	4	5	6	7	8	9	10	11	12
p :	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Required expectation $E(x)$ is given by

$$E(x) = p_1x_1 + p_2x_2 + \ldots + p_{11}x_{11}$$

$$= \frac{1}{36}.2 + \frac{2}{36}.3 + \frac{3}{36}.4 + \frac{4}{36}.5 + \frac{5}{36}.6 + \frac{6}{36}.7 + \frac{5}{36}.8 + \frac{4}{36}.9 + \frac{3}{36}.10 + \frac{2}{36}.11 + \frac{1}{36}.12$$

$$= \frac{2+6+12+20+30+42+40+36+30+22+12}{36,} = \frac{252}{36} = 7$$

Problem 86. *If a random variable x takes the values* $x_k = \dfrac{(-1)^k 2^k}{k}$, $k = 1, 2, \ldots$ *with probabilities* $p_k = \dfrac{1}{2^k}$, *find* $E(x)$.

We have $E(x) = p_1x_1 + p_2x_2 + p_3x_3 + \ldots$

$$= \tfrac{1}{2}\cdot(-2) + \frac{1}{2^2}\cdot\left(\frac{2^2}{2}\right) + \frac{1}{2^3}\left(-\frac{2^3}{3}\right) + \ldots$$

$$= -1 + \tfrac{1}{2} - \tfrac{1}{2} + \ldots = -\left[1 - \tfrac{1}{2} + \tfrac{1}{3} \ldots\right]$$
$$= -\log(1+1) = -\log 2.$$

Problem 87. *Show that if x is a stochastic variate and a is a constant,*

(*i*) $E(a) = a$

(*ii*) $E(ax) = aE(x)$

(*iii*) $\text{Var}(ax) = a^2 \text{Var}(x)$.

It follows from the definition that

(*i*) $E(a) = \sum_i p_i \cdot a_i$

$= a\Sigma p_i$, a being constant a_i for $i = 1, 2, 3, \ldots$

$= a$ as $\sum_i p_i = 1$.

(*ii*) $E(ax) = \sum_i p_i(ax_i)$, a being a constant

$$= a \sum_i p_i x_i = a\,E(x)$$

(*iii*) $E(ax) \sum p_i(ax_i), = a \sum_i p_i x_i = aE(x)$ ...(1)

and $\text{Var}(ax) = E\{ax - E(ax)\}^2 = E\{ax - a\,E(x)\}^2$ from (1)

$= a^2 E\{x - E(x)\}^2 = a^2 \text{Var}(x)$.

[C] **Mathematical expectation of a sum of stochastic variates.** *The expectation of two stochastic variates is equal to the sum of their expectations.*

If x any y are the two stochastic or random variates, then we have to show that $E(x + y) = E(x) + E(y)$.

Let the variate x take m values x_i $(i = 1, 2, \ldots m)$ with probabilities p_i $(i = 1, 2, \ldots m)$ and y take n values $y_j (j = 1, 2, \ldots n)$ with probabilities p_j' $(j = 1, 2, \ldots n)$. As such the sum $x + y$ is a stochastic variate taking mn values $x_i + y_j$ $(i = 1, 2, \ldots m, j = 1, 2, \ldots n)$ with probabilities p_{ij} (say) denoting the probabilities of value x_i of x and at the same time, of value y_j of y. Thus if x assumes a definite value x_i, y assumes one of the values y_1, $y_2 \ldots y_n$ so that the sum $\sum_{j=1}^{n} p_{ij}$ represents the probability p_i of x taking the value x_i *i.e.*

$$\sum_{j=1}^{n} p_{ij} = p_i. \quad \ldots(1)$$

Similarly $\sum_{i=1}^{m} p_{ij}$ represents the probability p_j' of y taking the value y_j, *i.e.*, $\sum_{i=1}^{m} p_{ij}$ $= p_j'$. ...(2)

Now, $E(x + y) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij}(x_i + y_j)$ by definition

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij}x_i + \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij}y_j$$

$$= \sum_{i=1}^{m} \left\{ x_i \left(\sum_{j=1}^{n} p_{ij} \right) \right\} + \sum_{j=1}^{n} \left\{ y_j \left(\sum_{i=1}^{m} p_{ij} \right) \right\}$$

$$= \sum_{i=1}^{m} x_i p_i + \sum_{j=1}^{n} y_j p_i' \text{ by (1) and (2)}$$

$$= E(x) + E(y).$$

This result can be extended to any number n of variates *i.e.* if $x, y, z, \ldots$ be n stochastic variates, then the expectation of their sum is equal to the sum of their expectations, *i.e.*,

$$E(x + y + z + \ldots) = E(x) + E(y) + E(z) + \ldots \qquad \ldots(4)$$

Since $E(x + y + z + \ldots) = E(x + \overline{y + z + \ldots})$

$= E(x) + E(Y + z + \ldots)$ by (3)

$= E(x) + E(y + \overline{z + \ldots})$

$= E(x) + E(y) + E(- + \ldots)$ by (3)

$\ldots \quad \ldots \quad \ldots \quad \ldots$

$= E(x) + E(y) + E(z) + \ldots$

Problem 88. *If n fair dice are tossed and x denotes the sum of the numbers on the n dice, then find the expectation of x.*

Denoting by x_i, the number of ith dice, we have

$$x = x_1 + x_2 + \ldots + x_n,$$

so that $E(x) = E(x_1 + x_2 + \ldots + x_n)$

$= E(x_1) + E(x_2) + \ldots + E(x_n)$ by (4) of [C].

But for the ith dice, the variate being the number of points on the dice its values are 1, 2, 3, 4, 5, 6 each having the probability $\frac{1}{6}$, therefore

$$E(x_i) = \tfrac{1}{6}.1 + \tfrac{1}{6}.2 + \tfrac{1}{6}.3 + \tfrac{1}{6}.4 + \tfrac{1}{6}.5 + \tfrac{1}{6}.6 = \tfrac{21}{6} = \tfrac{7}{2}.$$

$\therefore E(x_1) = E(x_2) = \ldots = E(x_n) = \frac{7}{2}$.

Hence $E(x) = \dfrac{7}{2} + \dfrac{7}{2} + \dfrac{7}{2} + \ldots n$ terms $= \dfrac{7n}{2}$.

Problem 89. *If an unbiased coin is tossed n times, find the mathematical expectation of the number of heads in all the n tosses.*

Let x denote the total number of heads in all the n tosses and let x_i be the variate-value which takes 1 when at the ith toss head turns up and takes the value zero when it is not so. Then,

$$x = x_1 + x_2 + \ldots + x_n,$$

so that $\quad E(x) = E(x_1 + x_2 + \ldots + x_n)$

$= E(x_1) + E(x_2) + \ldots + E(x_n)$

Now x_i takes the values 1, 0 with probabilties $\frac{1}{2}$, $\frac{1}{2}$ respectively.

$\therefore \quad E(x_i) = 1.\frac{1}{2} + 0.\frac{1}{2} = \frac{1}{2}$, *i.e.*, $E(x_1) = E(x_2) = \ldots = E(x_n) = \frac{1}{2}$.

Thus $E(x)\ \frac{1}{2} + \frac{1}{2} + \ldots$ n times $= \frac{n}{2}$.

Problem 90. *Thirteen cards are drawn simultaneously from a deck of 52. If aces count 1, face cards 10 and others according to denomination find the expectation of the total score on 13 cards.*

If x_i is the number corresponding to the ith card, then x_i takes values 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 10, 10, 10 each having the probability $\frac{1}{13}$. Hence

$$E(x_i) = \frac{1}{13}\,[1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 10 + 10 + 10] = \frac{85}{13}.$$

$\therefore$ required expectation

$$E(x) = E(x_1) + E(x_2) + \ldots + E(x_{13})$$
$$= \frac{85}{13} + \frac{85}{13} + \ldots 13 \text{ times} = 85.$$

Problem 91. *A box contains a white and b black balls; c balls are drawn. Find the expectation of the number of white balls drawn.*

Let s be the number of white balls among the c drawn. Then defining a variate x_i, such that

$$x_i = 1 \text{ if } i\text{th ball drawn is white, } i = 1, 2, \ldots, c$$
$$= 0 \text{ of } i\text{th ball drawn is black.}$$

We have $s = x_1 + x_2 + \ldots + x_c$

so that $E(s) = E(x_1 + x_2 + \ldots + x_c)$

$$= E(x_1) + E(x_2) + \ldots + E(x_c).$$

But $E(x_i) = 1.\dfrac{a}{a+b} + 0.\dfrac{b}{a+b} = \dfrac{a}{a+b}$,

since the probability of $x_i = 1$ is $\dfrac{a}{a+b}$ and that of $x_i = 0$ is $\dfrac{b}{a+b}$.

Hence $E(s) = \dfrac{a}{a+b} + \dfrac{a}{a+b} + \ldots c$ times $= \dfrac{ac}{a+b}$.

Problem 92. *A box contains 2^n tickets among which ${}^cC_i$ tickets bear the number $i (i = 0, 1, 2, \ldots, n)$. A group of m tickets is drawn. Let S denote the sum of their numbers. Find E(S).*

Let the variates $x_1, x_2, \ldots x_m$ represent the numbers on the first, second,...mth tickets.

Now, x_i being a stochastic variate assume values ,0 1, 2,...n and have probabilities $\dfrac{{}^nC_0}{2^n}, \dfrac{{}^nC_1}{2^n}, \ldots \dfrac{{}^nC_n}{2^n}$, respectively, so that

$$E(x_i) = \frac{1}{2^n}[1.\,{}^nC_1 + 2.\,{}^nC_2 + \ldots + n.\,{}^nC_n]$$

$$= \frac{n}{2^n}\,[1 + {}^{n-1}C_1 + {}^{n-1}C_2 + \ldots + {}^{n-1}C_{n-1}]$$

$$= \frac{n}{2^n}\,(1+1)^{n-1} = \frac{n}{2^n}.\,2^{n-1} = \frac{n}{2}.$$

$\therefore$ $S = x_1 + x_2 + \ldots + x_m$ gives

$$E(S) = E(x_1 + x_2 + \ldots + x_m)$$

$$= E(x_1) + E(x_2) + \ldots + E(x_m) = \sum_{i=1}^{m} E(x_i) = m.\frac{n}{2} = \frac{mn}{2}$$

[D] Mathematical expectation of a product of independent stochastic variates. Two stochastic variates are said to be *independent* if the probability assumed by either of the variates does not depend upon the value taken by the other.

The expectation of the product of two independent stochastic variates is the product of their expectations, i.e. if x, y, be two independent stochastic variates, then

$$E(xy) = E(x)\,E(y).$$

Suppose the variate x assumes the m values $x_1, x_2, \ldots x_m$ and y assumes the n values $y_1, y_2, \ldots, y_n$. Let the probabilities of $x = x_i$ and $y = y_j$ be p_i and p_j' respectively. Now the variates x and y being independent, the compound probability of x taking the value x_i and y taking $y_j = p_i p_j'$.

Now, $$E(xy) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i p_j' x_i y_j \text{ by definition}$$

$$= \sum_{i=1}^{m} x_i p_i \sum_{j=1}^{n} p_j' y_j = E(x).E(y) \qquad \ldots(1)$$

This result can be extended to any number n of variates $x, y, z, \ldots$ as

$$E(xyz\ldots) = E(x)\,E(y)\,E(z)\ldots, \qquad \ldots(2)$$

where $x, y, z, \ldots$ are n independent stochastic variates.

Problem 93. *Find the expected value of the product of points on n dice tossed all together.*

Denoting by x_i the number of points on ith dice, we have

$$E(x_1 x_2 \ldots x_n) = E(x_1)\,E(x_2)\ldots E(x_n).$$

But $E(x_i) = \frac{7}{2}$ by Problem 89.

$\therefore\ E(x_1 x_2 \ldots x_n) = \frac{7}{2}.\frac{7}{2}\ldots n \text{ times} = \left(\frac{7}{2}\right)^n.$

[E] Covariance. *The* **covariance** *between two variates x and y whose expected values (or means) are $\bar{x}$ and $\bar{y}$, is defined as*

$$\left.\begin{aligned} \text{cov}(x, y) &= E[(x - \bar{x})(y - \bar{y})] \\ &= E[\{x - E(x)\}\{y - E(y)\}] \end{aligned}\right\} \qquad \ldots(1)$$

i.e. the covariance of two variates x and y is the expectation of the product of their deviations from their means.

Problem 94. *Show that the covariance of two independent variates is always zero.*

Let x and y be two independent variates with their expected values (or means) $\bar{x}$ and $\bar{y}$ respectively. Then we have

$$\text{cov}(x, y) = E\{(x - \bar{x})(y - \bar{y})\}$$

$= E(x - \bar{x}).E(y - \bar{y})$, the variates and so their means being independent

$$= \{E(x) - E(\bar{x})\}.\{E(y) - E(\bar{y})\}$$

$$= (\bar{x} - \bar{x})(\bar{y} - \bar{y}) = 0.$$

Problem 95. *Show that the converse of Problem 94 is not true, i.e. if the covariance of two variates is zero, it is not necessary that they are independent.*

Let us introduce two variates u and v with same variance, such that

$$x = u + v, \qquad \bar{x} = \bar{u} + \bar{v},$$
$$y = u - v \quad \text{so that} \quad \bar{y} = \bar{u} - \bar{v},$$

Now $\text{cov}\,(x, y) = E\{(x - \bar{x})\,(y - \bar{y})\}$

$$= E\{(u - \bar{u} + v - \bar{v})\,(u - \bar{u} - v\ \bar{v})\}$$

or $$\text{cov}\,(x, y) = E\{(\bar{u} - u)^2 - (v - \bar{v})^2\}$$
$$= E(u - \bar{u})^2 - E(v - \bar{v})^2$$
$$= \text{var}\,(u) - \text{var}\,(v)$$
$$= 0, \text{ since } u \text{ and } v \text{ have the same variance.}$$

Here though $\text{cov}\,(x, y) = 0$, but x and y are not necessarily independent; for, if u and v are taken as the number of points on two dice, then their sum and difference, *i.e.*, $u + v = x$ and $u - v = y$ are either both even or both odd and thus are dependent to each other. Hence the converse of problem 94 is not true.

Problem 96. *Show that* $\text{cov}\,(x, y) = E(xy) - E(x)\,E(y)$.

$$\text{cov}\,(x, y) = E\{(x - \bar{x})\,(y - \bar{y})\} = E(xy - \bar{x}y - x\bar{y} + \bar{x}\,\bar{y})$$
$$= E(xy) - \bar{x}E(y) - \bar{y}E(x) + \bar{x}\,\bar{y} \quad \text{as } E(\bar{x}\,\bar{y}) = \bar{x}\,\bar{y}$$
$$= E(xy) - \bar{x}\,\bar{y} - \bar{y}\,\bar{x} + \bar{x}\,\bar{y} = E(xy) - \bar{x}\,\bar{y}$$
$$= E(xy) - E(x)\,E(y).$$

Problem 97. *Prove the following :*

(*i*) *cov* $(x + a, y + b) = \text{cov}\,(x, y)$,

(*ii*) *cov* $(ax, by) = ab\ \text{cov}\,(x, y)$.

(*iii*) $\text{Cov}\left(\dfrac{x - \bar{x}}{\sigma_x}, \dfrac{y - \bar{y}}{\sigma_y}\right) = \dfrac{1}{\sigma_x\,\sigma_y}\ \text{cov}\,(x, y)$,

where a and b are any numbers.

(*i*) $\text{cov}\,(x + a, y + b) = E[\{(x + a) - E(x + a)\} - E(y + b)\}]$

$$= E[\{x - E(x)\}\ \{y - E(y)\}] \text{ as } E(a) = a \text{ etc.}$$
$$= E\{(x - \bar{x})\,(y - \bar{y})\}$$
$$= \text{cov}\,(x, y).$$

(*ii*) We have $\text{cov}\,(ax, by) = E[\{ax - E(ax)\}\ \{by - E(by)\}]$

$$= E[\{ax - aE(x)\}\ \{by - bE(y)\}]$$
$$= abE\{(x - \bar{x})\,(y - \bar{y})\}$$
$$= ab\ \text{cov}\,(x, y).$$

(*iii*) We have

$$\text{cov}\left(\frac{x - \bar{x}}{\sigma_x}, \frac{y - \bar{y}}{\sigma_y}\right) = \frac{1}{\sigma_x\,\sigma_y}\ \text{cov}\ (x - \bar{x},\ y - \bar{y})$$

$$= \frac{1}{\sigma_x\,\sigma_y}.\ E[\{(x - \bar{x}) - E(x - \bar{x})\}\ \{(y - \bar{y}) - E(y - \bar{y})\}]$$

$$= \frac{1}{\sigma_x \sigma_y} E[(x-\bar{x})(y-\bar{y})]$$

$$= \frac{1}{\sigma_x \sigma_y} \text{cov}(x, y).$$

Problem 98. *Prove that cov $(x, x) =$ var (x).*

We have cov $(x, x) = E[(x-\bar{x})(x-\bar{x})] = E(x-\bar{x})^2$

$= E\{x - E(x)\}^2 =$ var (x).

[F] Variance of n variates. *The variance of two independent variates x and y is equal to the sum of their variances i.e.,*

$$var\,(x+y) = var\,(x) + var\,(y).$$

We have var $(x+y) = E[(x+y) - E(x+y)]^2$

$= E[(x+y) - (\bar{x} + \bar{y})]^2$

$= E[(x-\bar{x})^2 + (y-\bar{y})^2 + 2(x-\bar{x})(y-\bar{y})]$

$= E(x-\bar{x})^2 + E(y-\bar{y})^2 + 2E(x-\bar{x})(y-\bar{y})$

$=$ var (x) + var (y) + cov (x, y) ...(1)

But by Problem 93, the covariance of two independent variates is zero, *i.e.*, cov $(x, y) = 0$.

$\therefore$ var $(x+y) =$ var (x) + var (y). ...(2)

COROLLARY 1. It is easy to show that

var $(x-y) =$ var (x) − var (y) ...(3)

COROLLARY 2. If $x_1, x_2, \ldots x_n$ be n independent variates then the result (2) can be extended as

var $(x_1 + x_2 + \ldots + x_n) =$ var (x_1) + var (x_2) +...+ var (x_n),

i.e., the variance of the sum of any number of independent variates is equal to the sum of their variances.

COROLLARY 3. *If $x_1, x_2, \ldots x_n$ be n random variates with finite variances $\sigma_1^2, \sigma_2^2, \ldots \sigma_n^2$, then the variance of a variate u defined as*

$$u = a_1x_1 + a_2x_2 + \ldots + a_nx_n,$$

$a_1, a_2 \ldots a_n$ being constants, can be found out.

We have $E(u) = E[a_1x_1 + a_2x_2 + \ldots + a_nx_n]$

$= E(a_1x_1) + E(a_2x_2) + \ldots + E(a_nx_n)$

$= a_1 E(x_1) + a_2E(x_2) + \ldots + a_nE(x_n)$,

so that

$u - E(u) = a_1[x_1 - E(x_1)] + a_2[x_2 - E(x_2)] + \ldots + a_n[x_n - E(x_n)]$.

Squaring both sides,

$[u - E(u)]^2 = a_1^2[x_1 - E(x_1)]^2 + a_2^2[x_2 - E(x_2)]^2 + \ldots$

$\ldots + a_n^2[x_n - E(x_n)]^2 + 2a_1a_2[x_1 - E(x_1)][x_2 - E(x_2)] + \ldots$

$\ldots + 2a_{n-1}a_n\{x_{n-1} - E(x_{n-1})\}\{x_n - E(x_n)\}$.

Taking expected values, we get

$E[u - E(u)]^2 = a_1^2 E\{x_1 - E(x_1)\}^2 + \ldots + a_n^2 E\{x_n - E(x_n)\}^2$

$+ 2a_1a_2E[\{x_1 - E(x_1)\}\{x_2 - E(x_2)\}] +$

$\ldots + 2a_{n-1}a_n[\{x_{n-1} - E(x_{n-1})\}\{x_n - E(x_n)\}]$, *i.e.*, $\text{var}(u) = a_1^2 \text{var}(x_1) + a_2^2 \text{var}(x_2) + \ldots + a_n^2 \text{var}(x_n)$

$$+ 2a_1a_2 \text{cov}(x_1, x_2) \ldots + 2a_{n-1}a_n \text{cov}(x_{n-1}, x_n)$$

$$= \sum_{i=1}^{n} a_i^2 \text{var}(x_i) + 2 \sum_{\substack{i,j=1 \\ i \neq j}}^{n} a_i a_j \text{ cov}(x_i, x_j). \quad \ldots(4)$$

Deductions

(*i*) If $a_1 = a_2 = 1, a_3 = a_4 = \ldots = a_n = 0$, (4) reduces to (1), *i.e.*,

$$\text{var}(x_1 + x_2) = \text{var}(x_1) + \text{var}(x_2) + 2 \text{cov}(x_1, x_2). \quad \ldots(5)$$

(*ii*) If $a_1 = 1, a_2 = -1, a_3 = a_4 = \ldots = a_n = 0$, (4) reduces to

$$\text{var}(x_1 - x_2) = \text{var}(x_1) + \text{var}(x_2) - 2 \text{cov}(x_1, x_2). \quad \ldots(6)$$

(*iii*) If x_1 and x_2 are independent variates, then $\text{cov}(x_1, x_2) = 0$ so that results (5) and (6) reduce to (2) and (3), *i.e.*,

$$\text{var}(x_1 \pm x_2) = \text{var}(x_1) \pm \text{var}(x_2). \quad \ldots(7)$$

(*iv*) If $a_1 = a_2 = \ldots = a_n = \frac{1}{n}$ so that $x = \bar{x}$, and if x_i's be the independent variates and also if $\text{var}(x_i) = \sigma^2$, then (4) reduces to

$$\text{var}(\bar{x}) = \frac{\sigma^2}{n^2} + \frac{\sigma^2}{n^2} + \ldots n \text{ times} = \frac{\sigma^2}{n} \quad \ldots(8)$$

Problem 99. *An urn contains pN white and qN black balls, the total number of balls being N, p + q = 1. Balls are drawn one by one (without being returned to the urn) until a certain number n of balls is reached. What is the dispersion of the number of white balls drawn?*

Let x_i ($i = 1, 2, 3, \ldots, n$) be n variates such that

$x_i = 1$ if the *i*th ball drawn in white

$= 0$ if the *i*th ball drawn is black.

Then, $E(x_i) = p.\,1 + q.0 = p.$

If m be the number of white balls drawn, then

$$m = x_1 + x_2 + \ldots + x_n$$

$$\therefore \quad E(m) = E(x_1 + x_2 + \ldots + x_n) = E(x_1) + E(x_2) + \ldots + E(x_n)$$

$$= p + p + \ldots n \text{ terms} = np;$$

and the dispersion of the number of white balls drawn is

$$\text{var}(m) = E[\{m - E(m)\}^2] = E\{(m - np)^2\}$$

$$= E[\{x_1 + x_2 + \ldots + x_n) - np\}^2]$$

$$= E[(x_1 + x_2 + \ldots + x_n)^2 - 2np(x_1 + x_2 + \ldots + x_n) + n^2p^2]$$

$$= E[(x_1 + x_2 + \ldots + x_n)^2] - 2np(x_1 + x_2 + \ldots + x_n) + n^2p^2]$$

$$= E[x_1^2 + x_2^2 + \ldots + x_n^2) + 2(x_1x_2 + x_1x_3 + \ldots + x_ix_j + \ldots)] - 2np.\, np + n^2p^2$$

$$= \sum_{i=1}^{n} E(x_i^2) + 2 \sum_{\substack{i,j=1 \\ i \neq j}}^{n} E(x_ix_j) - n^2p^2 \quad \ldots(1)$$

wh^ $E(x_i^2) = p.\,1 + q.0 = p$, as the possible values of x_i^2 are 1 and 0 with probabilities

$P(x_i x_j) = P(x_i).\ (x_j/x_i)$ as $x_i x_j$ assumes values 1 and 0 according as ith and jth balls are both white or not. The probability to attain the value 1 is given by $P(x_i x_j)$ which is the product of probability of x_i and the conditional probability of x_j when x_i has already happened.

Here $P(x_i) = p$ and $P(x_j \mid x_i) = \dfrac{pN-1}{N-1}$

so that $$P(x_i x_j) = p.\ \frac{pN-1}{N-1}$$

and therefore $$E(x_i x_j) = p.\ \frac{pN-1}{N-1}\cdot 1 + 0 = \frac{p(Np-1)}{N-1}.$$

Thus $\sum_{i=1}^{n} E\left(x_i^2\right) = p + p + \ldots n \text{ times} = np$

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{n} E\left(x_i x_j\right) = \frac{p(pN-1)}{N-1} + \ldots + {}^nC_2 \text{ times}$$

$$= \frac{n(n-1)}{1.2}\,\frac{p(pN-1)}{N-1}.$$

With these values (1) yields

$$\text{var}\,(m) = np + 2.\frac{(n-1)}{1.2}\cdot\frac{p(pN-1)}{N-1} - n^2p^2$$

$$= \frac{np}{N-1}[(N-1) + (n-1)\ (pN-1) - np\ (N-1)]$$

$$= \frac{np}{N-1}\ [N\{1 + p(n-1) - np\} + \{-1 - (n-1) + np\}]$$

$$= \frac{np}{N-1}\ [N(1-p) - n(1-p)]$$

$$= \frac{np}{N-1}\ (1-p)\ (N-n) = \frac{npq\ (N-n)}{N-1} \text{ as } 1 - p = q.$$

15.12. CONTINUOUS UNIVARIATE PROBABILITY DISTRIBUTION

The distributions considered so far are the discrete distributions in which the variate assumes a finite or enumerably infinite set of values, but there are the distributions in which a variate like height or weight can take a non-enumerably infinite set of values in a given interval $a \le x \le b$. Such variates are said to be *continuous variates* and then probability distributions are known as *continuous probability distributions* or simply *continuous distributions* :

[A] Probability density function. If $f(x)$ is a continuous function of x defining the probability distribution of the stochastic variate x such that the probability of the value of the variate lies in the infinitesimal interval $(x - \frac{1}{2}dx,\ x + \frac{1}{2}dx)$ and is expressible in the form $f(x)\,dx$ or symbolically

$$P(x - \tfrac{1}{2}dx \le X \le x + \tfrac{1}{2}dx) = f(x)dx$$

then the function $f(x)$ is said to be the *probability density function* or simply *density function* and $f(x)\,dx$ is known as *probability differential.*

The continuous curve given by $y = f(x)$ is said to be the *probability density curve* or simply *probability curve*. In case this curve is symmetrical, the distribution is said to be symmetrical.

Though the range or interval of the variate may be finite or infinite, but it is convenient to consider it always infinite event if it is finite whence the density function outside the given range may be assumed zero. Suppose that X assumes values in the interval (a, b) and let its density function there be $\phi(x)$; then its distribution is that of a variate with density function defined as below

$$\begin{aligned} f(x) &= 0 \quad \text{for } x < a, \\ &= \phi(x) \text{ for } a \le x \le b \\ &= 0 \quad \text{for } x > b. \end{aligned}$$

The density function $f(x)$ satisfies the following two properties :

(*i*) $f(x) \ge 0$ for every x; as negative probability is meaningless

(*ii*) $\int_{-\infty}^{\infty} f(x)\, dx = 1$, *i.e.*, the probability of a sure event is unity. ...(1)

The probability that the variate X falls in an interval (a_1, b_1) is given by

$$P(a_1 \le X \le b_1) = \int_{a_1}^{b_1} f(x)\, dx.$$

Note. 1. The geometrical interpretation of property (ii) of density function namely $\int_{-\infty}^{\infty} f(x)\, dx = 1$ *is that the total area under the curve is unity. In case it is not unity, it can be so by multiplying a suitable constant e.g.*

$$\int_{-\infty}^{\infty} x\,(1 - x)\, dx = \tfrac{1}{6}.$$

Multiplying both sides by 6, this gives

$$\int_{-\infty}^{\infty} 6x\,(1 - x)\, dx = 1.$$

Thus the probability density function may be so defined

$$\begin{aligned} f(x) &= 0 \text{ for } x < 0 \\ &= 6x\,(1 - x) \text{ for } 0 \le x \le 1 \\ &= 0 \text{ for } x > 1. \end{aligned}$$

Note. 2. The variate is said to have a rectangular distribution of probability when $f(x)$ is constant throughout.

[B] Cumulative distribution function. Let the function $F(x)$ be the probability that the value of the variate X is $\le x$, *i.e.*

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\, dx,$$

such that $$F(b) = P(X \le b) = \int_{-\infty}^{b} f(x)\, dx$$

and $$F(b) - F(a) = P(a \le X \le b) = \int_{a}^{b} f(x)\, dx.$$

$F(x)$ is said to be the *cumulative distribution function* of x or simply the *function*. It has the following properties :

$x) = f(x) \ge 0$, so that $F(x)$ is non-decreasing function.

$-\infty) = 0.$

(*iii*) $F(\infty) = 1$.

Note. Density function f(x) is related to distribution F(x) by f(x) = F'(x).

Problem 100. *If f(x) has probability density kx^2, $0 < x < 1$, determine k and find the probability that $\frac{1}{3} < x < \frac{1}{2}$.*

Since $f(x)$ has probability density kx^2, therefore

$$\int_0^1 f(x)\, dx = 1,$$

i.e.,
$$\int_0^1 kx^2\, dx = 1,$$

or
$$\left[k\frac{x^3}{3}\right]_0^1 = 1 \text{ giving } k = 3, \text{ slo that } f(x) = 3x^2.$$

Again, $P\left(\frac{1}{3} < x < \frac{1}{2}\right) = \int_{1/3}^{1/2} f(x)\, dx = \int_{1/3}^{1/2} 3x^2\, dx = 3\left[\frac{x^3}{3}\right]_{1/3}^{1/2}$

$$= \frac{1}{8} - \frac{1}{27} = \frac{19}{216}.$$

Problem 101. *If a function f(x) of x is defined as follows :*

$$f(x) = 0 \text{ for } x < 2$$
$$= \frac{1}{18}(3 + 2x) \text{ for } 2 \le x \le 4$$
$$= 0 \text{ for } x > 4,$$

then show that it is a density function and find the probability that a variate having this density will be within the interval $2 \le x \le 3$.

In order that $f(x)$ is a density function, it must satisfy the two conditions :

(*i*) $f(x) \ge 0$ for every x.

(*ii*) $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

Here (*i*) follows from the definition of $f(x)$

Now for (*ii*), consider

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{2} f(x)\, dx + \int_2^4 f(x)\, dx + \int_4^{\infty} f(x)\, dx$$

$$= \int_{-\infty}^{2} 0.dx + \int_2^4 \frac{3+2x}{18}\, dx + \int_4^{\infty} 0.dx$$

$$= 0 + \frac{1}{18}\left[3x + x^2\right]_2^4 + 0$$

$$= \frac{1}{18}[3(4-2) + (16-4)] = \frac{1}{18}(6+12) = 1$$

which satisfies the second condition.

Hence the function $f(x)$ is a density function.

Now, $P(2 \le x \le 3) = \int_2^3 f(x)\, dx = \int_2^3 \frac{3+2x}{18}\, dx$

$$= \frac{1}{18}\left[3x + x^2\right]_2^3 = \frac{1}{18}[3+5] = \frac{8}{18} = \frac{4}{9}$$

Problem 102. *Verify that the following is a distribution function,*

$$F(x) = \begin{cases} 0, & x < -a \\ \frac{1}{2}\left(\frac{a}{x}+1\right), & -a \le x \le a \\ 1, & x > a. \end{cases}$$

The function $F(x)$ to be a density function, must satisfy the following conditions.

(*i*) $F(x) \ge 0$,

(*ii*) $F(-\infty) = 0$,

(*iii*) $F(+\infty) = 1$.

Here the first condition is obviously satisfied, *i.e.*, $F(x) \ge 0$ from the given definition of $F(x)$.

From $F(x) = 0$ for $x < -a$, it is clear that $F(-\infty) = 0$ and from $F(x) = 1$ for $x > a$, it follows that $F(\infty) = 1$.

Problem 103. *Supposing the life in hours of a certain kind of radio tube has the density* $f(x) = \frac{100}{x^2}$ *when* $x \ge 100$ *and zero, when* $x < 100$*, what is the porbability that none of three such tubes in a given radio set will have to be replaced during the first 150 hours of operation ? What is the probability that all three of the original tubes will have been replaced during the first 150 hours.*

We have $$f(x) = \frac{100}{x^2}, \quad x \ge 100$$
$$= 0, \quad x < 100.$$

The function $f(x)$ will be a probability density function if

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

i.e., if $$\int_{-\infty}^{100} f(x)\, dx + \int_{100}^{\infty} f(x)\, dx = 1$$

or if $$\int_{-\infty}^{100} 0.dx + \int_{100}^{\infty} \frac{100}{x^2} dx = 1$$

or if $$0 + 100\left[-\frac{1}{x}\right]_{100}^{\infty} = 1$$

which is so and hence $f(x)$ is a density function.

Now $$P(100 \le x \le 150) = \int_{100}^{150} \frac{100}{x^2} dx$$
$$= 100\left[-\frac{1}{x}\right]_{100}^{150} = 100\left[\frac{1}{100} - \frac{1}{150}\right]$$
$$= 100.\frac{50}{100 \times 150} = \frac{1}{3}.$$

(since the probability of failure during the first 100 hours is zero).

$\therefore$ Probability that none of the three bulbs shall have to be replaced $= \frac{1}{3}.\frac{1}{3}.\frac{1}{3} = \frac{1}{27}$.

Again, the probability that one tubes fails during the first 15C hours

$$= 1 - \tfrac{1}{3} = \tfrac{2}{3}$$

Hence the probability that all three tubes fail during the first 150 hours = $\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{8}{27}$.

Problem 104. *A bomb-plane carrying three bomes flies directly above a rail load track. If a bomb falls within 40 feet of the track, the track will be sufficiently damaged to disrupt traffic. With a certain bomb sight, the density of the points of impact of a bomb is*

$$f(x) = \frac{100+x}{10{,}000}, -100 \le x \le 0$$

$$= \frac{100-x}{10{,}000}, \quad 0 < x \le 100$$

$$= 0, \text{ elsewhere.}$$

If all three bombs are used what is the probability that the track will be damaged where x represents the vertical deviation from the aiming point which is the track in this case.

Since a bomb falls within 40 feet of the track, it therefore follows that in order to disrupt traffic the track will be damaged when the bomb falls within 40 feet either side of the track. Thus the probability of one bomb damaging the track is given by

$$P(-40 < x < 40) = \int_{-40}^{40} f(x)\, dx$$

$$= \int_{-40}^{0} f(x)\, dx + \int_{0}^{40} f(x)\, dx$$

$$= \int_{-40}^{0} \frac{100+x}{10{,}000}\, dx + \int_{0}^{40} \frac{100-x}{10{,}000}\, dx$$

$$= \frac{1}{10{,}000}\left[\left|100x + \frac{x^2}{2}\right|_{-40}^{0} + \left|100x - \frac{x^2}{2}\right|_{0}^{40}\right]$$

$$= \frac{1}{10{,}000}[+4{,}000 - 800 + 4{,}000 - 800] = \frac{6400}{10{,}000} = \frac{16}{25}.$$

So that the probability that the track is not damaged by one bomb

$$= 1 - \tfrac{16}{25} = \tfrac{9}{25}.$$

∴ The probability that the track is not damaged by any of three bombs = $\left(\frac{9}{25}\right)^3$

Hence the probability that the track is damaged by at least one bomb = $1 - \left(\frac{9}{25}\right)^3$ $= 1 - \frac{729}{15625} = \frac{14896}{15625}$.

[C] Mean, median, mode, moments etc. for a continuou[illegible] univariate probability distribution.

(*i*) **The Mathematical Expectation.** The expected value [illegible] lue) of any function $\phi(x)$ of a random variate x having probability densi[illegible] $f(x)$ [illegible] cumulative distribution function $F(x)$, is given by

$$E[\phi(x)] = \int_{-\infty}^{\infty} \phi(x)\, f(x)\, dx = \int_{-\infty}^{\infty} \phi(x)\, dF(x) \qquad \ldots(1)$$

(ii) The Arithmetic Mean

The mean (arithmetic mean) is obtained from (1) by putting $\phi(x) = x$ whence

$$M = \bar{x} = E(x) = \mu_1' = \int_{-\infty}^{\infty} x f(x)\, dx \qquad ...(2)$$

which represents the x-coordinate of the centre of gravity of the area of the curve bounded by x-axis.

(iii) The Geometric Mean

The geometric mean G is defined as

$$\log G = \int_{-\infty}^{\infty} \log x f(x)\, dx = E(\log x) \qquad ...(3)$$

(iv) The Harmonic Mean

The harmonic mean H is defined as

$$\frac{1}{H} = \int_{-\infty}^{\infty} \frac{1}{x} f(x)\, dx = E\left(\frac{1}{x}\right) \qquad ...(4)$$

(v) The Median

The median say 'a' being that value of a variate x which divides the total frequency into two equal parts *i.e.*,

$$P(x \le a) = P(x \ge a)$$

is defined as

$$\int_{-\infty}^{a} f(x)\, dx = \int_{a}^{\infty} f(x)\, dx = \tfrac{1}{2} \int_{-\infty}^{\infty} (x)\, dx = \tfrac{1}{2} \qquad ...(5)$$

(vi) The Quartiles

The lower quartile Q_1 is defined as

$$\int_{-\infty}^{Q_1} f(x)\, dx = \tfrac{1}{4} \qquad ...(6)$$

and the upper quartile Q_3 is defined as

$$\int_{Q_3}^{\infty} f(x)\, dx = \tfrac{1}{4}. \qquad ...(7)$$

(viii) The Mode

The mode or modal value being that value of a variate x for which the probability density $f(x)$ is maximum, is defined as

$$f'(x) = 0 \text{ and } f''(x) < 0, \qquad ...(8)$$

provided that the values of x given by $f'(x) = 0$ be within the range of the variate x.

(viii) The Mean Deviation

The mean deviation from the mean $(\bar{x})$ is defined as

$$\text{Mean deviation} = \int_{-\infty}^{\infty} |x - \bar{x}| . f(x)\, dx \qquad ...(9)$$

(ix) The Moments

The rth moment about any arbitrary point 'a' is given by

$$\mu_r' = \int_{-\infty}^{\infty} (x - a)^r f(x)\, dx \qquad ...(10)$$

and the rth moment about the mean $\bar{x}$ is defined as

$$\mu_r = \int_{-\infty}^{\infty} (x-\bar{x})^r f(x)\,dx. \qquad \ldots(11)$$

Problem 105. *If $f(x) = e^{-x}$, $0 \le x \le \infty$ show that it is a probability density function and hence evaluate mean μ_2, μ_3, μ_4.*

It is clear that $f(x) \ge 0$ for every x

$$\text{and} \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$

$$= 0 + \int_0^{\infty} -e^{-x}\,dx = 0 + \left[-e^{-x}\right]_0^{\infty} = 1.$$

Both the conditions being satisfied, $f(x)$ is a probability density function.

Now mean $= \int_0^{\infty} xe^{-x}\,dx = \left[-xe^{-x}\right]_0^{\infty} + \int_0^{\infty} e^{-x}\,dx$ (integrating by parts)

$$= 0 + \left[-e^{x}\right]_0^{\infty} = 1. \qquad \ldots(1)$$

$$\mu_2 = \int_0^{\infty} (x-1)^2 e^{-x} dx$$

$$= \left[-(x-1)^2\, e^{-x}\right]_0^{\infty} + 2\int_0^{\infty} (x-1)\, e^{-x}\,dx \text{ (integrating by parts)}$$

$$= 1 + 2\left[\left\{-(x-1)e^{-x}\right\}_0^{\infty} + \int_0^{\infty} e^{-x}\,dx\right]$$

$$= 1 + 2\,[0 - 1 + 1] = 1 \text{ from (1)} \qquad \ldots(2)$$

$$\mu_3 = \int_0^{\infty} (x-1)^3 e^{-x}\,dx$$

$$\mu_3 = \left[-(x-1)^3\, e^{-x}\right]_0^{\infty} + 3\int_0^{\infty} (x-1)^2\, e^{-x}\,dx$$

$$= -1 + 3 = 2 \text{ from (2)} \qquad \ldots(3)$$

and

$$\mu_4 = \int_0^{\infty} (x-1)^4 e^{-x}\,dx$$

$$= \left[-(x-1)^4\, e^{-x}\right]_0^{\infty} + 4\int_0^{\infty} (x-1)^3\, e^{-x}\,dx$$

$$= 1 + 4\,(2) = 9. \qquad \text{from (3)}$$

Problem 106. *A frequency function in the range $(-3, 3)$ is defined by*

$$y = \tfrac{1}{16}(3+x)^2,\; -3 < x < -1,$$

$$y = \tfrac{1}{16}(6-2x^2),\; -1 < x < 1,$$

and

$$y = \tfrac{1}{16}(3-x)^2,\quad 1 < x < 3.$$

Find the mean and the standard deviation of the distribution.

We have first to test whether the given function $y = f(x)$ (say) is a density function and it will be so if $\int_{-3}^{3} f(x)\,dx = 1$.

Now $\int_{-3}^{3} f(x)\,dx = \int_{-3}^{-1}\frac{1}{16}(3+x)^2\,dx + \int_{-1}^{1}\frac{1}{16}\left(6-2x^2\right)dx + \int_{1}^{3}\frac{1}{16}(3-x)^2\,dx$

$$= \frac{1}{16}\left[\frac{(3+x)^3}{3}\right]_{-3}^{-1} + \frac{1}{16}\left[6x - \tfrac{2}{3}x^3\right]_{-1}^{1} + \frac{1}{16}\left[\frac{(3+x)^3}{-3}\right]_{1}^{3}$$

$$= \tfrac{1}{6} + \tfrac{2}{3} + \tfrac{1}{6} = 1.$$

Hence $f(x)$ is a density function.

Now, mean $= \mu_1' = \int_{-3}^{3}(x-0).f(x)\,dx$ (about the origin)

$$= \frac{1}{16}\left[\int_{-3}^{-1} x.(3+x)^2\,dx + \int_{-1}^{1} x.(6-2x)^2\,dx + \int_{1}^{3} x.(3-x)^2\,dx\right]$$

$$= \frac{1}{16}\left\{\left[\tfrac{9}{2}x^2 + 2x^3 + \frac{x^4}{4}\right]_{-3}^{-1} + \left[3x^2 - \frac{x^4}{2}\right]_{-1}^{1} + \left[\tfrac{9}{2}x^2 - 2x^3 + \frac{x^4}{4}\right]_{1}^{3}\right\}$$

$$= \tfrac{1}{16}\left\{\left(\tfrac{9}{2} - 2 + \tfrac{1}{4}\right) - \left(\tfrac{81}{2} - 54 + \tfrac{81}{4}\right)\right\} + (0-0)$$
$$+ \left\{\left(\tfrac{81}{2} - 54 + \tfrac{81}{4}\right) - \left(\tfrac{9}{2} - 2 + \tfrac{1}{4}\right)\right\}$$

$$= 0$$

and $$\mu_2' = \frac{1}{16}\left[\int_{-3}^{-1} x^2(3+x)^2\,dx + \int_{-1}^{1} x^2\left(6-2x^2\right)dx + \int_{1}^{3} x^2.(3-x)^2\,dx\right]$$

= 1 (integrating and simplifying).

$\therefore$ standard deviation $\mu_2 = \mu_2' - \mu_1'^2 = 1.$

15.13. TCHEBYCHEFF'S INEQUALITY

If x be a random variate with expected value $E(x) = \bar{x}$ (i.e. mean) and standard deviation σ, then for any positive number ϵ we have the inequality

$P(x < \epsilon^2\bar{x}) \geq 1 - \frac{1}{\epsilon^2}$ (known as Tchebycheff's lemma) and the probability that the chosen value does not differ from the mean by more than $\epsilon\sigma$ does not exceed $\frac{1}{\epsilon^2}$ is represented by Tchebycheff's inequality as

$$P(|x - \bar{x}| \geq \epsilon\sigma) \leq \frac{1}{\epsilon^2}.$$

If x be a non-negative variate, then we have

$$\bar{x} = \int_0^{\infty} xf(x)\,dx$$

$$= \int_0^{\epsilon^2\bar{x}} xf(x)\,dx + \int_{\epsilon^2\bar{x}}^{\infty} xf(x)\,dx$$

$$\geq \int_{\epsilon^2\bar{x}}^{\infty} xf(x)\,dx \qquad \because \int_0^{\epsilon^2\bar{x}} xf(x)\,dx \geq 0$$

$$\geq \epsilon^2\bar{x}\int_{\epsilon^2\bar{x}}^{\infty} f(x)\,dx = \epsilon^2\bar{x}\,P\left(x \geq \epsilon^2\bar{x}\right)$$

i.e., $\bar{x} \geq \epsilon^2 \bar{x} P(x \geq \epsilon^2 \bar{x})$

or $\frac{1}{\epsilon^2} \geq P(x \geq \epsilon^2 \bar{x}) = 1 - P(x < \epsilon^2 \bar{x})$

$$\therefore \quad P(x < \epsilon^2 \bar{x}) \geq 1 - \frac{1}{\epsilon^2} \qquad ...(1)$$

which is Tchebycheff's lemma.

If we replace x by $(x - \bar{x})^2$, then $\bar{x}$ {*i.e.* $E(x)$} will be replaced by variance of x *i.e.* σ^2. As such (1) becomes

$$P\{(x - \bar{x})^2 < \epsilon^2 \sigma^2\} \geq 1 - \frac{1}{\epsilon^2}$$

or
$$P(|x - \bar{x}| < \epsilon \sigma) \geq 1 - \frac{1}{\epsilon^2} \qquad ...(2)$$

or
$$1 - P(|x - \bar{x}| < \epsilon \sigma) \leq \frac{1}{\epsilon^2}$$

or
$$P(|x - \bar{x}| \geq \epsilon \sigma) \leq \frac{1}{\epsilon^2}. \qquad ...(3)$$

This proves Tchebycheff's inequality, which restricts the upper limit of the probability.

COROLLARY 1. Law of large numbers.

If we put in (2),

$$x = \frac{x_1 + x_2 + \ldots + x_n}{n}, \quad \bar{x} = \frac{\bar{x}_1 + \bar{x}_1 + \ldots + \bar{x}_n}{n} \text{ and } \epsilon \sqrt{\left(\frac{E_n}{n^2}\right)} = \lambda,$$

where λ is an arbitrary positive number and E_n is the mathematical expectation of the variate,

$$u = (x_1 + x_2 + \ldots + x_n - \bar{x}_1 - \bar{x}_2 - \ldots - \bar{x}_n)^2.$$

Then, we have

$$P\left[\left|\frac{x_1 + x_2 + \ldots + x_n}{n} - \frac{\bar{x}_1 + \bar{x}_2 + \ldots + \bar{x}_n}{n}\right| < \lambda\right] \geq 1 - \frac{E_n}{n^2 \lambda^2}$$

$$\geq 1 - \delta \text{ provided } \frac{E_n}{n^2 \lambda^2} < \delta.$$

This gives the *law of large numbers*, stated as below:

Under the assumption that the probability approaches unity or certainty as near as we please, it is expected that the mean (arithmetic) of the values actually taken by n random variates will differ from the mean of their expectations or less than any arbitrary positive small number when the number of variates is sufficiently large and

$$\frac{E_n}{n^2} \to 0 \text{ as } n \to \infty.$$

COROLLARY 2. Bernoulli's theorem

For given positive numbers λ and δ, however small, if m be the number of successes in n independent trials, constant probability of success being p, then a number N depending on λ and δ, can be found such that the probability in order that the inequality $\left|\frac{m}{n} - p\right| < \lambda$ holds good, is greater than $1 - \delta$, provided $n > N$.

Let us associate with trials, 1, 2, 3,...n random (stochastic) variates $x_1, x_2, \ldots, x_n$ such that

$$x_i = 1 \text{ when } i\text{-th trial is a success}$$
$$= 0, \text{ when } i\text{-th trial is a failure,}$$

then $m = x_1 + x_2 + \ldots + x_n$.

Now trials being independent, the variates $x_1, x_2, \ldots x_n$ are also independent.

$\therefore$ $E(x_i) = 1.p + q.0 = p$ where $p = 1 - q$

$E(m) = 1 + 1 + \ldots n$ times $= np$

$E(x_i^2) = p.1 + q.0 = p$ as the possible values of x_i^2 are 1 and 0 with probabilities p and q.

$$\begin{aligned}\text{Var } (x_i) &= E[\{x_i - E(x_i)\}^2] \\ &= E[(x_i - p)^2] = E[x_i^2 - 2px_i + p^2] \\ &= \Sigma(x_i^2) - 2p\Sigma(x_i) + p^2 \\ &= p - 2p^2 + p^2 = p - p^2 \\ &= p(1-p) = pq \text{ as } p + q = 1\end{aligned}$$

and

$$\text{var}(m) = \sum_{i=1}^{n} \text{var}(x_i)$$
$$= \sum_{i=1}^{n} pq = npq.$$

As such $E\left(\frac{m}{n}\right) = \frac{1}{n}E(m) = p$ [*See* Problem 87 (*ii*)]

$\text{Var}\left(\frac{m}{n}\right) = \frac{1}{n^2}\text{ Var }(m) = \frac{pq}{n}$ by Problem 87 (*iii*) $= \sigma^2$ (say)

Also let $\lambda = \epsilon\sigma$.

Then an application of form (2) of Tchebycheff's inequality for $\frac{m}{n}$ variates, yields

$$P\left(\left|\frac{m}{n} - p\right| < \lambda\right) \geq 1 - \frac{\sigma^2}{\lambda^2} = 1 - \frac{pq}{n\lambda^2}$$
$$\geq 1 - \delta \text{ when } n > N = \frac{pq}{\delta\lambda^2}.$$

This completes Bernoulli's theorem.

Problem 107. *For the number of points x on a dice, prove that Tchebycheff's inequality gives*

$$P[\,|x - E(x)| > 2.5\sigma] < .47$$

while the actual probability is nearly zero.

$\because$ $E(x) = \frac{1}{6}.1 + \frac{1}{6}.2 + \frac{1}{6}.3 + \frac{1}{6}.4 + \frac{1}{6}.5 + \frac{1}{6}.6$ by Problem 88

$$= \frac{21}{6} = \frac{7}{2} = 3.5$$

$\therefore$ $\text{Var}(x) = \sigma^2 = E[\{x - E(x)\}^2) = E(x^2) - \{E(x)\}^2$

$$= \frac{1}{6}\left[1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2\right] - \left(\frac{7}{2}\right)^2$$

$$= \tfrac{1}{6}[1 + 4 + 9 + 16 + 25 + 36] - \tfrac{49}{4}$$

$$= \frac{91}{6} - \frac{49}{4} = \frac{182 - 147}{12} = \frac{35}{12}$$

$$= 2.9167$$

$\therefore$ $\sigma = 1.7$ approx.

The maximum deviation of x from $E(x)$ is $6 - 3.5 = 2.5 = \frac{3}{2}\sigma$. The probability of greater deviation is zero, whereas Tchebycheff's inequality asserts that this probability is smaller than 0.47.

Problem 108. *Let x_i assume values i and $-i$ with equal probabilities, show that the law of large numbers cannot be applied to variables $x_1, x_2, x_3, \ldots x_n$. But if x_i assumes values i^α and $-i^\alpha$, the law of large numbers can be applied to $x_1, x_2, \ldots x_n$ provided $\alpha < \frac{1}{2}$.*

We have $\quad P(x_i = i) = \frac{1}{2} = P(x_i = -i)$

$\therefore$ $\quad E(x_i) = \frac{1}{2}.i + \frac{1}{2}(-i) = 0, i = 1, 2, \ldots n$

and $\quad E(x_i^2) = \frac{1}{2}i^2 + \frac{1}{2}i^2 = i^2, i = 1, 2\ldots$

$$E(x) = E(x_1 + x_2 + \ldots + x_n)$$

$$= E(x_1) + E(x_2) + \ldots + E(x_n)$$

$$= 0 + 0 + \ldots + 0 = 0$$

$$E_n = E(x_1 + x_2 + \ldots x_n)^2 = \sum_{i=1}^{n} E\left(x_i^2\right)$$

$$= \sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \ldots + n^2$$

$$= \frac{1}{6}n(n+1)(2n+1)$$

so that $\quad \dfrac{E_n}{n^2} = \dfrac{1}{\sigma n}(n+1)(2n+1)$

$\therefore$ $\lim\limits_{n \to \infty} \dfrac{E_n}{n^2} = \infty \neq 0$ and hence the law of large numbers is not applicable to this case.

Further when x_i assumes values i^α, $-i^\alpha$, we have

$E(x_i) = \frac{1}{2}.i^\alpha + \frac{1}{2}(-i^\alpha) = 0$ as $P(x_i = i^\alpha) = \frac{1}{2} = P(x_i = -i^\alpha)$

$E(x_i^2) = \frac{1}{2}.i^{2\alpha} + \frac{1}{2}.i^{2\alpha} = i^{2\alpha}, i = 1, 2, \ldots, n$

So that $E_n = \sum\limits_{i=1}^{n} E\left(x_i^2\right) = 1^{2\alpha} + 2^{2\alpha} + 3^{2\alpha} + \ldots + n^{2\alpha}$

$$\sim \frac{n^{2\alpha+1}}{2\alpha+1} \text{ by Euler's summation formula}$$

$\therefore \quad \dfrac{E_n}{n^2} = \dfrac{n^{2\alpha-1}}{2\alpha+1}$

i.e., $\lim\limits_{n\to\infty} \dfrac{E_n}{n^2} = \lim\limits_{n\to\infty} \dfrac{n^{2\alpha-1}}{2\alpha+1} = 0$ iff $2\alpha - 1 < 0$ *i.e.*, $\alpha < \frac{1}{2}$.

Hence in this case the law of large numbers holds only if $\alpha < \frac{1}{2}$.

15.14. CHARACTERISTIC FUNCTION

The characteristic function for a continuous probability distribution having $f(x)$ as its density function is defined as

$$\phi(t) = E\left(e^{itx}\right) = \int_{-\infty}^{\infty} e^{itx} f(x)\, dx = \int_{-\infty}^{\infty} e^{itx} dF(x),$$

where t is a parameter.

$\phi(t)$ has the properties:

(*i*) $\phi(t)$ is continuous in t,

(*ii*) $\phi(t)$ is defined in every finite t interval,

(*iii*) $\phi(0) = 1$,

(*iv*) $\phi(t)$ and $\phi(-t)$ are conjugate quantities,

(*v*) $|\phi(t)| \le \int_{-\infty}^{\infty} |e^{itx}|\, f(x)\, dx \le 1 = \phi(0)$.

15.15. FOURIER'S INVERSION THEOREM

If $F(x)$ and $\phi(t)$ denote the distribution and characteristic functions respectively of a random variate x, then

$$F(x) - F(0) = \lim_{t\to\infty} \frac{1}{2\pi} \int_{-t}^{t} \frac{1 - e^{-itx}}{it} \phi(t)\, dt$$

$$= P_r\left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - e^{ixt}}{it} \phi(t)\, dt\right],$$

where P_r denotes the principal value of the integeral.

In case $F(x)$ is everywhere continuous and $dF(x) = F(x)\, dx$, we can express

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi(t)\, dt, \qquad \text{...(1)}$$

which is equivalent to Fourier's integral

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} e^{-it(x-y)} f(y)\, dy$$

and can be easily deduced from (1) by the substitution

$$f(t) = \int_{-\infty}^{\infty} e^{-ity} f(y)\, dy. \qquad \text{...(2)}$$

Problem 109. *Show that the distribution*

$$dF = \frac{1}{\pi}\left(\frac{1 - \cos x}{x^2}\right) dx, \quad -\infty \le x \le \infty,$$

has the characteristic function

$$\phi(t) = 1 - |t|, \quad |t| \le 1,$$
$$= 0, \quad |t| > 1,$$

It is eivdent from calculus of residues that

$$\int_0^{\infty}\frac{1-\cos x}{x^2}dx=\frac{\pi}{2} \text{ or } \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1-\cos x}{x^2}dx=1.$$

$$\therefore f(x)=\frac{1}{\pi}\cdot\frac{1-\cos x}{x^2}, -\infty\le x\le\infty.$$

$\therefore$ the characteristic function is given by

$$\phi(t)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1-\cos x}{x^2}e^{itx}dx$$

$$=\frac{2}{\pi}\int_0^{\infty}\frac{1-\cos x}{x^2}\cos tx\,dx$$

$$=1-|t| \quad |t|\le 1$$

$$=0 \quad |t|>1.$$

Problem 110. *Show that the distribution for which the characteristic function* $e^{-|t|}$ *has the density function*

$$f(x)=\frac{1}{\pi}\cdot\frac{dx}{1+x^2}, -\infty\le x\le\infty.$$

We have $f(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}e^{-|t|}e^{-itx}dt$

$$=\frac{1}{2\pi}\left[\int_{-\infty}^{0}e^{t}e^{-itx}dt+\int_0^{\infty}e^{-t}e^{-itx}dt\right]$$

$$=\frac{1}{2\pi}\left[\int_0^{\infty}e^{-t}e^{itx}dt+\int_0^{\infty}e^{-t}e^{-itx}dt\right] \text{ by replacing}$$

t by $-t$ in the first integral or putting $t=-z$ etc.

$$=\frac{1}{2\pi}\int_0^{\infty}e^{-t}\left(e^{itx}+e^{-itx}\right)dt$$

$$=\frac{1}{\pi}\int_0^{\infty}e^{-t}\cos tx\,dx$$

$$=\frac{1}{\pi}\left[-e^{-t}\cos tx\right]_0^{\infty}-\frac{x}{\pi}\int_0^{\infty}e^{-t}\sin tx\,dx \text{ (integrating by parts)}$$

$$=\frac{1}{\pi}+\frac{x}{\pi}\left[e^{-t}\sin tx\right]_0^{\infty}-\frac{x^2}{\pi}\int_0^{\infty}e^{-t}\cos tx\,dx$$

$$=\frac{1}{\pi}-x^2f(x)$$

giving $$f(x)=\frac{1}{\pi}\cdot\frac{1}{1+x^2}, -\infty\le x\le\infty.$$

15.16. THEORETICAL DISTRIBUTIONS

Having started with certain general hypotheses, if it is possible to deduce mathematically the frequency distributions of certain populations or universes, then such distributions are said to be *theoretical distributions*, which constitute three types of distributions.

(*i*) **Binomial distribution.** It was discovered by James Bernoulli ((1654–1705) in the year 1700 and was first published in 1713, eight years after the death of Bernoulli.

(*ii*) **Poisson distribution.** It was discovered by the French mathematician and physicist Siméon Denis Poisson (1781-1840) who published it in 1837.

(*iii*) **Normal distribution.** Though it was first discovered by De Moivre as early as 1733, but associated with the names of the distinguished French mathematician Pierre Simon, Marquis de Laplace (1749-1827) and the German mathematician and physicist Carl Friedrich Gauss (1777-1855) who discussed it independently at the close of 18th and the beginning of 19th centuries.

Here below we shall discuss these three distributions taking one by one.

[A] The Binomial Distribution

If we toss a coin which is a uniform, homogeneous circular disc, then nothing is biased to make it to tend to fall more often on the one side than on the other. It is therefore expected that in a series of throws the coin will fall heads-up and tails-up an approximately equal number of times and so the chance of throwing heads or tails with a coin is $\frac{1}{2}$. Similarly the chance of throwing an ace with a fair die is $\frac{1}{6}$. Instead of considering the particular instances we generally use to say that the chance of success of an event is p and chance of its failure is q such that $p + q = 1$. Assuming the events in a number of trials to be independent, the chances p and q may be supposed to remain constant throughout.

If we take a number of sets of n trials and count the number of successes in each set, then there will be some sets with no success, some with one success, some with two successes, some with three successes and so on. The classification of the sets according to the number of successes which they contain, will give us a frequency distribution.

Suppose there are N sets of n trails in which the chance of the success and failure are respectively p and q. We have to find the frequencies of 0, 1, 2, 3,...successes in cases of one event, two events, three events and so on.

When $n = 1$, *i.e.*, in case of a single event, out of N sets of 1 trial each we expect Np successes and Nq failures.

When $n = 2$, *i.e.*, there are N sets of two events or N sets of two trials each, the event which has taken place once is repeated again. We have Nq failure of the first event or trial and the events being independent among these Nq there will be $Nq \times q$ failures and $Nq \times p$ successes of the second event on the average. Similarly among the Np successes of the first event there are $Np \times p$ successes and $Np \times q$ failures of the second event on the average. Hence there are in total Nq^2 failures of both events. $2Nqp$ cases of the two events with one success and one failure and Np^2 successes of both events. Thus the frequencies of 0, 1, 2 successes are respectively

$$Nq^2, 2Nqp, Np^2$$

When $n = 3$ *i.e.*, there are N sets of three events, we see that of the Nq^2 cases in which the first two events were failures, we have $Nq^2 \times q$ a third failure and $Nq^2 \times p$ one success, of the $2Nqp$ cases, $2Npq^2$ will give two failures and one success and $2Np^2q$ one failure and two successes and of the Np^2 cases, Np^2q will give one failure and two successes and Np^3 a third success. Hence the frequencies of 0, 1, 2, 3 successes are respectively

$$Nq^3, 3Nq^2p, 3Nqp^2, Np^3.$$

From the foregoing discussions we conclude that the frequencies 0, 1, 2,... successes are given

for one event by the binomial expansion of $N(q + p)$

for two " " " " " $N(q + p)^2$

for three " " " " " $N(q + p)^3$.

In general for N sets of n-events (trials) the frequencies of 0, 1, 2, 3,...successes are given by the successive terms in the binomial expansion of $N(q+p)^n$ *i.e.*,

$$N\left[q^n + nq^{n-1} + \frac{n(n-1)}{1.2}q^{n-2}p^2 + \frac{n(n-1)(n-2)}{1.2.3}q^{n-3}q^3 + \ldots\right].$$

This is called the ***Binomial Frequency Distribution*** or simply ***Binomial Distribution*** and the quantities n, p(or q) are said to be *parameters* of Binomial distribution.

Characteristics of Binomial Distribution

(1) Its general form depends on parameters p, q and n.

(2) The probability that there are r successes in n independent trials is given by ${}^nC_r p^r q^{n-r}$ and hence in all the N sets, this is given by $N.{}^nC_r p^r q^{n-r}$.

(3) The numerical coefficients of the binomial expansion can be found by Pascal's triangle.

Exponent power of $(p+q)$ i.e. n	*Coefficients of terms*											*Sum of coefficients*
1	1	1										2
2	1	2	1									4
3	1	3	3	1								8
4	1	4	6	4	1							16
5	1	5	10	10	5	1						32
...	...	...	...	...	...	...	...	...	...	...	...	...
10	1	10	45	120	210	252	210	120	45	10	1	1024
11	1	11	55	...	...	...	...	...	...	...	...	
etc.												

(4) It is chiefly applied when the population being sampled is infinite so that 'p' remains unchanged by sampling.

(5) It can be applied to finite populations also if they are not too small.

(6) It is used under the conditions :

(*i*) The variable is discrete.

(*ii*) A dichotomy (*i.e.* process of classification of collected individuals into two classes according to whether they do or do not possess a particular attribute) exists.

(*iii*) Statistical independence is assumed.

(*iv*) The exponent power n is finite and small.

(*v*) For symmetrical distribution $p = q$ and for asymmetrical $p \neq q$.

Constants of the Binomial Distribution

Let us take an arbitrary origin at 0 (zero) successes so that the successive deviations are 0, 1, 2, 3,...n.

1. The Mean.

We have

the mean = μ_1' (about the origin)

$$= \sum_{r=0}^{n} {}^nC_r . p^r q^{n-r} . r$$

$= (q.0) + ({}^nC_1 q^{n-1}p.1) + ({}^nC_2 q^{n-2}p^2.2) + \ldots + (p^n.n)$
$= p\,[nq^{n-1} + n(n-1)\,q^{n-2}\,p + \ldots + np^{n-1}]$
$= np\,[q^{n-1} + (n-1)\,q^{n-2}p + \ldots + n^{n-1}]$
$= np\,(q+p)^{n-1}$
$= np$ since $q + p = 1$.

2. The Variance and Standard Deviation

We have, μ_2' (about the origin)

$$= \sum_0^n {}^nC_r.\, p^r q^{n-r}.r^2$$

$$= (q^n.0) + ({}^nC_1 q^{n-1}p.1^2) + ({}^nC_2\, q^{n-2}p^2.2^2) + \ldots + (p^n.n^2)$$

or $$\mu_2' = np\left[q^{n-1} + 2(n-1)\,q^{n-2}p + \frac{3(n-1)\,(n-2)}{2}q^{n-3}p^2 + \ldots + np^{n-1}\right]$$

$$= np\,[(n-1)\,p + 1]$$

since the bracketed expression is the first moment of $(q+p)^{n-1}$ about origin – 1 and hence is equal to $(n-1)\,p + 1$.

As an *alternative*,

$$\mu_2' = \sum_0^n {}^nC_r p^r q^{n-r}.r^2 = \sum_0^n {}^nC_r p^r q^{n-r}[r\,(r-1) + r]$$

$$= n(n-1)\,p^2\Sigma^{n-2}C_{r-2}p^{r-2}q^{n-r} + np\Sigma^{n-1}C_{r-1}p^{r-1}q^{n-r}$$
$$= n(n-1)\,p^2(p+q)^{n-2} + np\,(p+q)^{n-1}$$
$$= n(n-1)p^2 + np \text{ as } p + q = 1$$
$$= np[(n-1)\,p + 1].$$

$\therefore$ The variance $= \sigma^2 = \mu_2 = \mu_2' - \mu_1'^2$

$$= np[(n-1)p + 1] - (np)^2$$
$$= np\,(1-p)$$
$$= nqp,$$

so that standard deviation $= \sigma = \sqrt{\mu_2} = \sqrt{(nqp)}$.

3. Third Moments about the Origin and about the Mean

$$\mu_3' = \sum_0^n {}^nC_r p^r q^{n-r}.r^3$$

$$= \sum_0^n {}^nC_r p^r q^{n-r}\,\{r\,(r-1)\,(r-2) + 3r\,(r-1) + r\}$$

$$= n(n-1)\,(n-2).p^3\Sigma^{n-3}C_{r-3}p^{r-3}q^{n-r}$$
$$+ 3n(n-1)\,p^2\Sigma^{n-2}C_{r-2}p^{r-2}q^{r-r} + np\Sigma^{n-1}\,C_{r-1} + p^{r-1}\,q^{n-r}$$
$$= n(n-1)\,(n-2)p^3\,(p+q)^{n-3} + 3n(n-1)p^2\,(p+q)^{n-2} + np\,(p+q)^{n-1}$$
$$= n(n-1)\,(n-2)p^3 + 3n(n-1)p^2 + np \text{ as } p + q = 1$$
$$= np\{(n-1)\,(n-2)p^2 + 3\,(n-1)p + 1\}$$

and $$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$
$$= npq\,(q-p).$$

4. Fourth Moments about the Origin and about the Mean

$$\mu_4' = \sum_0^n {}^nC_r p^r q^{n-r}.r^4$$

$$= \sum_0^n {}^nC_r p^r q^{n-r}.\{r(r-1)(r-2)(r-3) + 6r(r-1)(r-2) + 7r(r-1) + r\}$$

$$= np\,\{(n-1)(n-2)(n-3)p^3 + 6(n-1)(n-2)p^2 + 7(n-1)p + 1\} \text{ as above}$$

and $\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^3$

$$= 3p^2q^2n^2 + pqn\,(1 - 6qp).$$

Problem 111. *A perfect cubic die is thrown a large number of times in sets of 8. The occurrence of a 5 or a 6 is called a success. In what proportion of the sets would you expect 3 successes.*

Number of faces in a die = 6.

a '5' + a '6' = 2 successes.

$\therefore\ p = \frac{2}{6} = \frac{1}{3}$, so that $q = 1 - p = \frac{2}{3}$ and $n = 8$.

The binomial distribution is therefore $N\left(\frac{2}{3} + \frac{1}{3}\right)^8$.

Probability of 3 successes in one set of 8 = ${}^8C_3 p^3 q^5$

$$= \frac{8.7.6}{1.2.3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^5 = \frac{8 \times 7 \times 32}{81 \times 81}.$$

$\therefore$ Probability of 3 successes in 100 sets $= \frac{8 \times 7 \times 32}{81 \times 81} \times 100$

$$= \frac{179200}{6561} = 27.31\%.$$

Problem 112. *An irregular six-faced die is thrown and the expectation that in 10 throws it will give 5 even number is twice the expectation that it will give 5 even numbers is twice the expectation that it will give four even numbers. How many times in 10,000 sets of 10 throws would you expect it to give no even numbers ?*

If p is the expectation of getting an even number then the probability of 5 even numbers in 10 throws of a die is twice the probability of 4 even numbers in 10 throws of the die, *i.e.*

$${}^{10}C_5 p^5 q^5 = 2 \times {}^{10}C_4 p^4 q^6,$$

$$\frac{3}{5}p = q = 1 - p \text{ giving } p = \frac{5}{8} \text{ and } q = \frac{3}{8}.$$

$\therefore$ Frequency for no even number in 10,000 sets of 10 throws = $10{,}003.q^{10} = 10{,}000 \left(\frac{3}{8}\right)^{10}$ = 1 nearly.

Problem 113. *Show the results of throwing 12 dice, 4096 times, a throw of 4, 5 or 6 being called a success.*

Find the expected frequencies and compare the actual mean with those of the expected distribution. Calculate the standard deviation.

Probability of success of falling 4, 5 or 6 $= \frac{3}{6} = \frac{1}{2} = p$(say).

$\therefore\ q = 1 - p = \frac{1}{2}.$

Success	*Frequency*	*Success*	*Frequency*
0	–	7	847
1	7	8	·536
2	60	9	257
3	198	10	71
4	430	11	11
5	731	12	–
6	948		
			Total 4,096

Binomial distribution is $4096\left(\frac{1}{2}+\frac{1}{2}\right)^{12}$

Frequency of 0 successes $= 4096\left(\frac{1}{2}\right)^{12} = 1,$

" 1 success $= 4096.\ {}^{12}C_1 q^{11}.p = 12\times\dfrac{4096}{2^{12}} = 12.$

" 2 successes $= 4096\ {}^{12}C_2 q^{10}p^2 = 66$

" 3 successes $= {}^{12}C_3 q^9 p^3 = 220$

" 4 successes $= 4096\ {}^{12}C_4.q^8P^4 = 495$...etc.

Now, expected mean $= np = 12.\frac{1}{2} = 6,$

$$\sigma^2 = nqp = 12.\tfrac{1}{2}.\tfrac{1}{2} = 3.$$

$\therefore$ standard deviation $\sigma = \sqrt{3} = 1.732.$

Problem 114. *If a coin is tossed N times, where N is very large even number, show that the probability of getting exactly $\frac{1}{2}N - p$ haads and $\frac{1}{2}N + p$ tails is approximately*

$$\left(\frac{2}{\pi N}\right)^{12} e^{-2p^2/N}$$

Since the coin is unbiased $p = q = \frac{1}{2}$.

$$\therefore \text{ Required probability} = {}^N C_{\frac{1}{2}N-p}\, p^{\frac{n}{2}-p} q^{\frac{n}{2}+p}$$

$$= \frac{N!}{\left(\frac{1}{2}N-p\right)!\left(\frac{1}{2}N+p\right)!}\cdot\frac{1}{2^N} \text{ as } p = q = \tfrac{1}{2}$$

Applying *Stirling's theorem i.e.,*

$$n! = \sqrt{(2\pi n)}.n^n e^{-n} \text{ approx.}$$

$$= e^{-n}.n^{n+\frac{1}{2}}\sqrt{(2\pi)},\ n \text{ being large,}$$

we have the required probability

$$= \frac{\sqrt{(2\pi)}.N^{N+\frac{1}{2}}e^{-N}}{\left[\sqrt{(2\pi)}\right]^2 \left(\frac{1}{2}N-p\right)^{\frac{1}{2}N-p+\frac{1}{2}}e^{-\left(\frac{1}{2}N-p\right)}\left(\frac{1}{2}N+p\right)^{\frac{1}{2}N+p+\frac{1}{2}}e^{-\left(\frac{1}{2}N+p\right)}} \cdot \frac{1}{2^N}$$

$$= \frac{1}{\sqrt{(2\pi)}} \cdot \frac{N^{N+\frac{1}{2}}}{\left(\frac{1}{2}N\right)^{N+1}\left(1-\frac{2p}{N}\right)^{\frac{1}{2}N-p+\frac{1}{2}}\left(1+\frac{2p}{N}\right)^{\frac{1}{2}N+p+\frac{1}{2}}} \cdot \frac{1}{2^N}$$

$$= \frac{2}{\sqrt{(2\pi)}\ N\frac{1}{2}}\left(1+\frac{2p}{N}\right)^{\frac{1}{2}N-p+\frac{1}{2}}\left(1-\frac{2p}{N}\right)^{\frac{1}{2}N+p+1}$$

$$= \left(\frac{2}{\pi N}\right)^{\frac{1}{2}}\left(1-\frac{4p^2}{N^2}\right)^{N/2} \text{ neglecting other terms}$$

$$= \left(\frac{2}{\pi N}\right)^{\frac{1}{2}} e^{-2p^2/N} \text{ approx. when } N \text{ is large, for then}$$

$$\lim_{N\to\infty}\left(1-\frac{4\pi^2}{N^2}\right)^{N/2} = \lim_{N\to\infty}\left(1-\frac{2p^2/N}{\frac{1}{2}N}\right) - e^{-2p^2/N}$$

[B] Limiting form of the binomial distribution when n is large.

We know that for large values of n, each term of the binomial becomes small, but we consider the sum of the terms lying within certain ranges. It is observed that as n increases and becomes infinitely large, the curve representing the binomial distribution becomes smooth and continuous and approaches the *normal curve* which is a symmetrical curve such that the maximum frequencies are clustered around the mean and the deviation below the mean are equal in number and magnitude to the deviations above the mean. Thus normal curve is a standardized and special case of the binomial and the distribution giving this particular curve is said to be the *normal distribution*. We consider it in two cases :

Case I. *Normal distribution as a limiting case of binomial distribution where $p = q$.*

When $p = q = \frac{1}{2}$, the terms of the series $N\left(\frac{1}{2}+\frac{1}{2}\right)^n$ are

$$N\left(\tfrac{1}{2}\right)^n\left[1+n+\frac{n(n-1)}{1.2}+\frac{n(n-1)(n-2)}{1.2.3}+\ldots\right]$$

The frequency of m successes is

$$N\left(\tfrac{1}{2}\right)^n . {}^nC_m \ i.e.,\ N\left(\tfrac{1}{2}\right)^n \frac{n!}{m!\,(n-m)!}$$

and the frequency of $(m+1)$ successes in $N\left(\frac{1}{2}\right)^n . {}^nC_{m+1}$

$$= N\left(\tfrac{1}{2}\right)^n \frac{n!}{(m+1)!\,(n-m-1)!}$$

$$= \left\{N\left(\tfrac{1}{2}\right)^n \frac{n!}{m!(n-m!)}\right\}\frac{n-m}{m+1}$$

$$= \left(\frac{n-m}{m+1}\right)\text{th times the frequency of } m \text{ successes.}$$

So that the frequency of $(m+1)$ successes is greater than that of m successes if

$$\frac{n-m}{m+1} > 1$$

or if $\quad n - m > m + 1$

or if $\quad m < \dfrac{n-1}{2}$

For the sake of convenience let us suppose that $n = 2k$; then the frequency of k successes say $y_0 = N\left(\frac{1}{2}\right)^{2k} \dfrac{(2k)!}{k!k!}$ and the frequency of $(k+x)$ successes $y_x = N\left(\frac{1}{2}\right)^{2k} \dfrac{(2k)!}{\lfloor(k+x) \lfloor(k-x)}$.

$$\therefore \frac{y_x}{y_0} = \frac{k!k!}{(k+x)!\,(k-x)!} = \frac{k!k(k-1)\,(k-2)\ldots(k-x+1)\,(k-x)!}{k!\,(k+1)\,(k+2)\ldots(k+x).(k-x)!}$$

$$= \frac{k^x\left(1-\frac{1}{k}\right)\left(1-\frac{2}{k}\right)\ldots\left(1-\frac{x-1}{k}\right)}{k^x\left(1+\frac{1}{k}\right)\left(1+\frac{2}{k}\right)\ldots\left(1+\frac{x-1}{k}\right)\left(1+\frac{x}{k}\right)}$$

or $\log \dfrac{y_x}{y_0} = \log\left(1-\dfrac{1}{k}\right) + \log\left(1-\dfrac{2}{k}\right) + \ldots \log\left(1-\dfrac{x-1}{k}\right)$

$$-\log\left(1+\frac{1}{k}\right) - \log\left(1+\frac{2}{k}\right) - \ldots - \log\left(1+\frac{x-1}{k}\right) - \log\left(1-\frac{x}{k}\right).$$

Supposing k to be very large as compared with x so that $\left(\dfrac{x}{k}\right)^2$ may be neglected, expanding the logarithmic functions as $\log(1+z) = z - \dfrac{z^2}{2} + \dfrac{z^3}{3} \ldots$ and $\log(1-z) = -z - \dfrac{z^2}{2} - \dfrac{z^3}{3}\;\dfrac{z^2}{2} - \dfrac{z^3}{3} \ldots$ and then neglecting the squares of $\dfrac{x}{k}$ *i.e.*, $\left(\dfrac{x}{k}\right)^2$, we find

$$\log \frac{y_x}{y_0} = -\frac{1}{k} - \frac{2}{k} - \frac{3}{k} - \ldots - \frac{x-1}{k} - \frac{1}{k} - \frac{2}{k} - \ldots - \frac{x-1}{x} - \frac{x}{k}$$

$$= -\frac{2}{k}(1 + 2 + 3 + \ldots + \overline{x-1}) - \frac{x}{k}$$

$$= -\frac{2}{k} \cdot \frac{x(x-1)}{2} - \frac{x}{k}$$

$$= -\frac{x^2}{k}$$

or $\quad \dfrac{y_x}{y_0} = e^{-x^2/k}$

i.e., $\quad y_x = y_0 e^{-x^2/k}$.

But for a binomial distribution with $p = q = \frac{1}{2}$, $n = 2k$, we have

$$\sigma^2 = npq = 2k.\tfrac{1}{2}.\tfrac{1}{2} = \tfrac{1}{2}k \text{ i.e., } k = 2\sigma^2$$

Hence $\quad y_x = y_0 e^{-\left(x^2/2\sigma^2\right)}$

which gives the *normal distribution* and the equation $y = y_0 e^{-x^2/2\sigma^2}$ represents *normal curve.*

Case II. *Normal distribution as a limiting case of binomial distribution $p \neq q$ but $p \approx q$.*

The frequency of m successes is

$$N.{}^nC_m q^{n-m} p^m = N.\frac{n!}{m!(n-m)!}.q^{n-m}p^m$$

and the frequency of $(m+1)$ successes is $N.{}^NC_{m+1} q^{n-m-1} p^{m+1}$

$$= \frac{n!}{(m+1)!\,(n-m-1)!} q^{n-m-1} p^{m+1}$$

$$= \left\{N.\ \frac{n!}{m!\,(n-m)!} q^{n-m} p^m\right\}.\frac{n-m}{m+1}\cdot\frac{p}{q}$$

$$= \left(\frac{n-m}{m+1}.\frac{p}{q}\right)\text{th times the frequency of } m \text{ successes.}$$

So that the frequency of $(m+1)$ successes is greater than that of m successes if $\frac{n-m}{m+1}\cdot\frac{p}{q} > 1$ or if $(n-m)\,p > (m+1)\,q$ *i.e.* if $m < n\,p - q$.

Assuming that np is a whole number, since there is no loss of generality as n ultimately tends to infinity, the frequency of np successes may be taken as maximum frequency.

The frequency of np successes (say)

$$y_0 = N.\frac{n!}{np!\ (n-np)!}.q^{n-np}p^{np}$$

$$= N.\frac{n!}{np!\ nq!} q^{nq} p^{np} \text{ as } 1-p = q$$

and the frequency of $(np + x)$ successes (say)

$$y_x = N.\frac{n!}{(np\ + x)!\ (np - x)!} q^{nq-x} p^{np+x}$$

$$= N.\frac{n!}{np!\ nq!} q^{nq}\, p^{np} \text{ as } 1-p = q$$

$$\therefore \quad \frac{y_x}{y_0} = \frac{(np)!\ (nq)!}{(np)\ + x)!\ (nq - x)!} q^{-x} p^x.$$

Applying the Stirling's theorem that when n is large

$$n! = \sqrt{(2\pi n)}\ n^n e^{-n} = e^{-n} n^{n+1/2}\sqrt{(2\pi)} \text{ approx.}$$

we have

$$\frac{y_x}{y_0} = \frac{e^{-np}(np)^{np+1/2}\ \sqrt{(2\pi)} \times e^{-nq}\ (nq)^{nq+1/2}\ \sqrt{(2\pi)}}{e^{-(np+x)}(np+x)^{np+x+1/2}\sqrt{(2\pi)} \times e^{-(nq-x)}(nq-x)^{nq-x+1/2}\ \sqrt{(2\pi)}}\cdot\left(\frac{np}{nq}\right)^x$$

$$= \frac{(np)^{np+1/2}\,(nq)^{nq+1/2}}{(np)^{np+x+1/2}\left\{1+\dfrac{x}{np}\right\}^{np+x+1/2} \times \left\{1-\dfrac{x}{nq}\right\}^{nq-x+1/2}}\left(\frac{np}{nq}\right)^x$$

$$= \frac{1}{\left(1+\dfrac{x}{np}\right)^{np+x+1/2}\left(1-\dfrac{x}{nq}\right)^{nq-x+1/2}}$$

or $$\log\frac{y_x}{y_0} = -\left(np+x+\tfrac{1}{2}\right)\log\left(1+\frac{x}{np}\right) - \left(nq-x+\tfrac{1}{2}\right)\log\left(1-\frac{x}{nq}\right)$$

$$= \left(np+x+\tfrac{1}{2}\right)\left[\frac{x}{np}-\frac{x^2}{2n^2p^2}+\ldots\right] + \left(nq-x+\tfrac{1}{2}\right)\left[\frac{x}{nq}+\frac{x^2}{2n^2q^2}+\ldots\right]$$

$$= -x + \frac{x^2}{2np} - \frac{x^2}{np} - \frac{x}{2np} + x + \frac{x}{2nq} - \frac{x^2}{nq} + \frac{x}{2nq}$$

neglecting the terms containing $\frac{1}{n^2}$

$$= \frac{x^2}{n}\left\{\frac{1}{2p}-\frac{1}{p}+\frac{1}{2q}-\frac{1}{q}\right\} - \frac{x}{2n}\left\{\frac{1}{p}-\frac{1}{q}\right\}$$

$$= \frac{x^2(p+q)}{2npq} - \frac{x(q-p)}{2npq} = -\frac{x^2}{2npq} - \frac{(q-p)}{2npq}x \text{ as } p+q=1.$$

Since q and p both are less than 1 and very nearly equal, therefore $\frac{q-p}{2npq}$ can be neglected and then we are left with

$$\log\frac{y_x}{y_0} = -\frac{x^2}{2npq} = -\frac{x^2}{2\sigma^2} \text{ as } \sigma^2 = npq$$

or $$\frac{y_x}{y_0} = e^{-x^2/2\sigma^2}$$

i.e., $$y_x = y_0 e^{-x^2/2\sigma^2}$$

Note. The curve given by $y = y_0 e^{-x^2/2\sigma^2}$ *is said to be the normal curve.*

A normal curve *is symmetrical about the point* $x = 0$ *where the ordinate has its maximum value. In a normal curve, the mean, the median and mode coincide.*

[C] The Normal Distribution

We have just introduced that the equation to the normal curve is

$$y = y_0\, e^{-x^2/2\sigma^2} \qquad \ldots(1)$$

Its area $= \int_{-\infty}^{\infty} y_0 e^{-x^2/2\sigma^2}\,dx$

$$= 2y_0 \int_0^{\infty} e^{-x^2/2\sigma^2}\,dx, \text{ being symmetrical}$$

$$= 2y_0 \cdot \frac{\sqrt{\pi}}{2 \cdot \frac{1}{\sqrt{2}\sigma}} \text{ as } \int_0^{\infty} e^{-a^2x^2}\, dx = \frac{\sqrt{\pi}}{2a} \text{ if } a > 0$$

$$= \sigma y_0 \sqrt{(2\pi)} \qquad \ldots(2)$$

$= 2.506627\, y_0\sigma$ by putting the value of $\sqrt{(2\pi)}$.

In order to make (1), the normal probability curve, the value of y_0 be so determined that the total frequency is one *i.e.*, the area of the normal curve is unity and this will be so if from (2)

$$\sigma y_0 \sqrt{(2\pi)} = 1$$

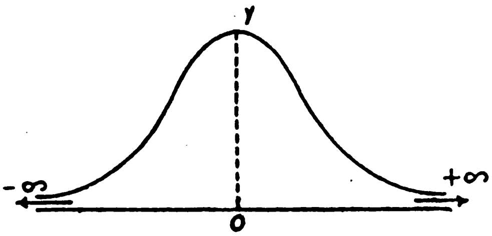


Fig. 15.53

i.e., $$y_0 = \frac{1}{\sigma\sqrt{(2\pi)}} \qquad \ldots(3)$$

Substituting this value of y_0 in (1), the standard form of the normal curve becomes

$$y = \frac{1}{\sigma\sqrt{(2\pi)}} e^{-x^2/2\sigma^2} \qquad \ldots(4)$$

which is the *normal distribution*.

In deriving the form (1) of normal curve, we have taken the mean at the origin, but if however we take another point as the origin such that the excess of the mean over the arbitrary origins is m, the form of the normal curve is

$$y = \frac{1}{\sigma\sqrt{(2\pi)}} e^{-(x-m)^2/2\sigma^2} \qquad \ldots(5)$$

which represents the standard form of the normal curve with origin at $(m, 0)$.

Physical conditions leading a normal curve

(*i*) The casual forces that affect individual events are mutally independent.

(*ii*) The casual forces of equal magnitude are very large in number.

(*iii*) The casual force operate in such a way that the maximum frequencies are clustered around the mean value thereby giving a symmetrical curve. Conclusively the deviations below the mean are equal in number and magnitude to those above the mean.

(*iv*) The normal distribution can be used as an error distribution by inquiring what law of distribution errors of observation should obey in order to make the mean of a set of measurements the most probable value of the 'true' magnitude.

Hence according to Gauss, if we call the '**precision**' h, such that $h^2 = \frac{1}{2\sigma^2}$, the form (4) becomes

$$y = \frac{h}{\sqrt{\pi}} e^{-h^2 x^2} \quad \ldots(6)$$

It is clear form (6), that as h increases, the normal curve would become narrower and as such h is a measure of closeness of the mass of observations to the true value.

Definition of a normal distribution. *A normal distribution is a continuous distribution given by*

$$y = \frac{1}{\sigma\sqrt{(2\pi)}} . e^{-\frac{1}{2}[(x-m)/\sigma]^2},$$

where x is a continuous normal variate distributed with proability density function f(x) $= \frac{1}{\sigma\sqrt{(2\pi)}} e^{-\frac{1}{2}[(x-m)/\sigma]^2}$, *with mean m and standard deviation* σ.

Properties of a Normal Distribution

(1) When $p = q$ or $p \approx q$ (*i.e.*, p is very nearly equal to q), the distribution fitted is symmetrical. But a normal curve is distinguished from the other symmetrical curves in a markable point that a normal curve is symmetrical not only with regard to skewness as are all symmetrical curves; but it is also symmetrical with regard to peakedness (*i.e.*, kurtosis).

(2) The normal curve is a mathematical abstraction, not found in practical work, but used to describe the form of distribution that would be obtained by some continuous data in very large numbers.

(3) The normal curves are based upon regular variation and uniformity conjoined so that no single force playing on each item of the distribution is dominating.

(4) In a normal distribution, items differing from the mean (or median or mode which coincide in a symmetrical distribution) by the same amount in either direction occur with the same frequency. As such above and below the mean at equal distances, there are same number of measurements.

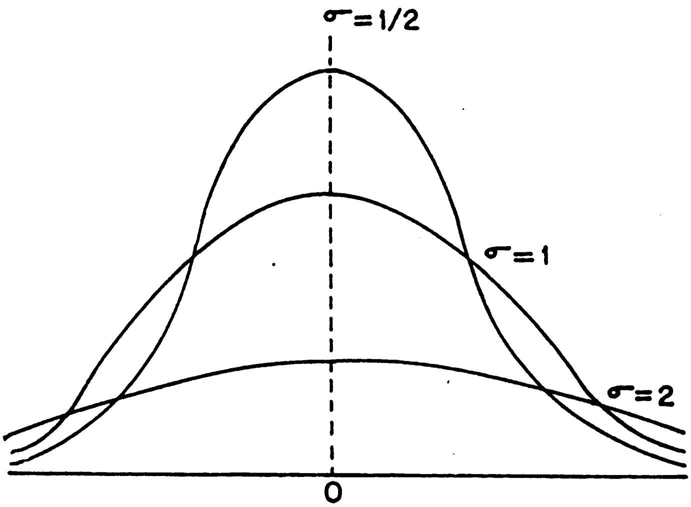


Fig. 15.6

(5) Normal curve is a single peaked.

(6) Normal curve is asymptotic to the horizontal base as y decreases rapidly when x increases numerically.

(7) The mean, median and mode coincide and lower and upper quartiles are equidistant from the median.

(8) The curve can be completely specified by mean (*i.e.*, origin of x) and the standard deviation along with the value of y_0 found as in equation (3).

(9) The points of inflexion of the normal curve are obtained by putting $\frac{d^2y}{dx^2}=0$ provided $\frac{d^3y}{dx^3}\neq 0$ for these points.

These are found to be

$$x=\pm\sigma,\ y=\frac{1}{\sigma\sqrt{(2\pi)}}e^{-1/2}.$$

Note. The points of inflexion are the points where the curvature changes its direction.

Constants of Normal Distribution

1. The Mean

Mean = μ_1' (about the origin)

$$=\frac{1}{\sigma\sqrt{(2\pi)}}\int_{-\infty}^{\infty}x.e^{-\left[(x-m)^2/2\sigma^2\right]}dx \text{ Put } \frac{x-m}{\sqrt{2}\sigma}=z,\ i.e.\ dx=\sqrt{2}\sigma\,dz$$

$$=\frac{1}{\sigma\sqrt{(2\pi)}}\int_{-\infty}^{\infty}(m+\sqrt{2}.\sigma z)e^{-z^2}\,dz.\ \sqrt{2}\sigma$$

$$=\frac{2m}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-z^2}\,dz,$$ the second integral vanishes, being an odd function of z

$$=\frac{2m}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{2}=m.$$

2. The Standard Deviation and Variance

We have μ_2 (about the origin)

$$=\frac{1}{\sigma\sqrt{(2\pi)}}\int_{-\infty}^{\infty}x^2.e^{-\left[(x-m)^2/2\sigma^2\right]}dx$$

or $$\mu_2'=\frac{1}{\sigma\sqrt{(2\pi)}}\int_{-\infty}^{\infty}\left(m+\sqrt{2}\sigma z\right)^2e^{-z^2}\ \sqrt{2}\sigma\ dz \text{ Put } \frac{x-m}{\sqrt{2}\sigma}=z\ i.e.\ dx=\sqrt{2}\sigma dz$$

$$=\frac{m^2}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-z^2}\,dz+\frac{2\sqrt{2}\sigma}{\sqrt{\pi}}\int_{-\infty}^{\infty}z.e^{-z^2}dz+\frac{2\sigma^2}{\sqrt{\pi}}\int_{-\infty}^{\infty}z^2.e^{-z^2}\,dz$$

$$=\frac{2m^2}{\sqrt{\pi}}\int_{0}^{\infty}e^{-z^2}\,dz+0+\frac{4\sigma^2}{\sqrt{\pi}}\int_{0}^{\infty}z^2e^{-z^2}\,dz$$

the second integral vanishes by the property of definite integrals

$$=\frac{2m^2}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{2}+\frac{4\sigma^2}{\sqrt{\pi}}\int_{0}^{\infty}z^2e^{-z^2}\,dz$$

$$=m^2-\frac{2\sigma^2}{\sqrt{\pi}}\int_{0}^{\infty}z(-2z)e^{-z^2}dz$$

$$=m^2-\frac{2\sigma^2}{\sqrt{\pi}}\left[\left(ze^{-z^2}\right)_0^{\infty}-\int_0^{\infty}e^{-z^2}dz\right]$$ integrating by parts

$$=m^2+\frac{2\sigma^2}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{2}=m^2+\sigma^2.$$

$\therefore$ The variance $\mu_2=\mu_2'-\mu_1{}^2$

$$=m^2+\sigma^2-m^2=\sigma^2$$

$\therefore$ Standard deviation $=\sqrt{\mu_2}=\sigma.$

3. **The Mean Deviation from the Mean**

The mean deviation about the mean

$$m = \int_{-\infty}^{\infty} |x-m| \cdot \frac{1}{\sigma\sqrt{(2\pi)}} e^{-\frac{1}{2}[(x-m)/\sigma]^2} dx$$

$$= \frac{\sigma\sqrt{2}}{\sigma\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \left|\sqrt{2}\sigma z\right| . e^{-z^2} dz \qquad \text{Put } \frac{x-m}{\sqrt{2}\sigma} = z$$

$$= \sigma\sqrt{\frac{2}{\pi}}\left[\int_{-\infty}^{0} -ze^{-z^2} dz + \int_{0}^{\infty} ze^{-z^2} dz\right], \text{ i.e., } dx = \sqrt{2}\sigma\, dz$$

$$= \sigma\sqrt{\frac{2}{\pi}}\left\{\left[\frac{e^{-z^2}}{2}\right]_{-\infty}^{0} + \left[\frac{e^{-z^2}}{-2}\right]_{0}^{\infty}\right\}$$

$$= \sigma\sqrt{\frac{2}{\pi}}\left\{\tfrac{1}{2} + \tfrac{1}{2}\right\} = \sigma\sqrt{\frac{2}{\pi}} = 0.7979\ \sigma = \tfrac{4}{5}\sigma \text{ approx.}$$

4. **Moments about the Mean**

Let us first consider the odd moments about the mean—

$$\mu_{2n+1} = \int_{-\infty}^{\infty} (x-m)^{2n+1} \frac{1}{\sigma\sqrt{(2\pi)}} e^{-(x-m)^2/2\sigma^2} dx.$$

$$\text{Put } \frac{x-m}{\sqrt{2}\sigma} = z, \text{ i.e., } dx = \sqrt{2}\sigma\, dz$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left(\sqrt{2}\sigma z\right)^{2n+1} e^{-z^2} dz$$

$= 0$ being an odd function of z, by the properties of definite integral.

$\therefore\ \mu_3 = \mu_5 = \mu_7 = \ldots = 0$

i.e., all odd moments about the mean are zero.

Let us now consider the even moments about the mean

$$\mu_{2n} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-m)^{2n} . e^{-(x-m)^2/\sqrt{2}\sigma} dx \quad \text{Put } \frac{x-m}{\sqrt{2}\sigma} = z,\ dx = \sqrt{2}\sigma\, dz$$

$$= \frac{2^n\sigma^{2n}}{\sqrt{\pi}} \int_{-\infty}^{\infty} z^{2n} . e^{-z^2} dz$$

$$= \frac{2^{n+1}.\sigma^{2n}}{\sqrt{\pi}} \int_{0}^{\infty} e^{-z^2} z^{2n-1}\, z dz \quad \text{Put } z^2 = t,\ 2z\, dz = dt$$

$$= \frac{2^n\sigma^{2n}}{\sqrt{\pi}} \int_{0}^{\infty} e^{-t} t^{(2n-1)/2} dt = \frac{2^n\sigma^{2n}}{\sqrt{\pi}} \int_{0}^{\infty} e^{-t} . t^{(n+1/2)-1} dt$$

$$= \frac{2^n\sigma^{2n}}{\sqrt{\pi}} \Gamma\left(n + \tfrac{1}{2}\right) \text{ by the definition of Gamma integral.}$$

In particular, $\mu_k = \dfrac{2^{k/2}.\sigma^k}{\sqrt{\pi}} \Gamma\left(\dfrac{k+1}{2}\right)$ when $k = 2n$

Now we have $\mu_2 = \dfrac{2\sigma^2}{\sqrt{\pi}} \Gamma \tfrac{3}{2} = \dfrac{2\sigma^2}{\sqrt{\pi}} . \tfrac{1}{2}\sqrt{\pi} = \sigma^2,$

$$\mu_4 = \frac{2^2\sigma^4}{\sqrt{\pi}}\Gamma\tfrac{5}{2} = \frac{4\sigma^4}{\sqrt{\pi}}.\tfrac{3}{2}.\tfrac{1}{2}\sqrt{\pi} = 3\sigma^4.$$

5. The Normal Probability Integral or Error Functions

It has been shown that the total area of the normal curve being unity, it is given by

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}$$

or putting $\frac{1}{2\sigma^2} = h^2$, this becomes

$$y = \frac{h}{\sqrt{\pi}} e^{-h^2x^2}$$

where h is known as *precision*.

Thus the probability that a deviation lies between x and $-x$ is

$$P = \frac{h}{\sqrt{\pi}} \int_{-x}^{x} e^{-h^2x^2} dx$$

$$= \frac{2h}{\sqrt{\pi}} \int_{0}^{x} e^{-h^2x^2} dx$$

$$= \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-h^2x^2} h\, dx$$

or $$\phi(hx) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-h^2x^2} h\, dx$$

so that $$\phi(y) = \frac{2}{\sqrt{\pi}} \int_{0}^{y} e^{-y^2} dy$$

which is known as *error function* or the *probability integral*.

Note. *The probable error λ or quartile deviation is defined to be the error such that the chance of an error lying within the limits $m - \lambda$ and $m + \lambda$ is exactly the same as chance of an error lying outside these limits i.e.*

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{m-\lambda}^{m+\lambda} e^{-\frac{1}{2}[(x-m)/\sigma]^2} dx = \tfrac{1}{2}. \text{ Put } \frac{x-m}{\sigma} = z$$

or $$\frac{1}{\sqrt{2\pi}} \int_{0}^{\lambda/\sigma} e^{-z^2/2} dz = \tfrac{1}{4}$$

whence from table $\frac{\lambda}{\sigma} = 0.6745$ *i.e.* $\lambda = 0.6745\sigma = \frac{2}{3}\sigma$ approx.

$\therefore$ $Q_1 = m - \frac{2}{3}\sigma$ and $Q_2 = m + \frac{2}{3}\sigma$.

TABLE 1—AREA UNDER THE NORMAL CURVE

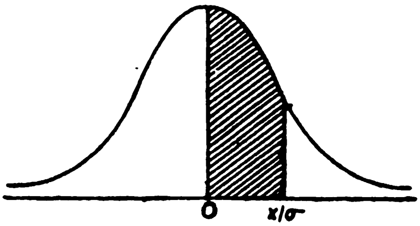


Fig. 15.7

Area being measured from mean (x =0) to distance x/σ.

x/σ	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0754
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1.700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1960	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2258	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2518	.2549
0.7	.2580	.2612	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2966	.2956	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4529	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4626	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.5857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.8493	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.3932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952

AREA UNDER THE NORMAL CURVE (contd).

x/σ	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4983	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.49865	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.49903	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.499313	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.499517	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.499663	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.499767	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998
3.6	.499841	.4998	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.7	.499892	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.8	.499928	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.9	.499952	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000
4.0	.499968	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000

Table for ordinates. Normal curve is given by

$$y = \frac{1}{\sigma\sqrt{(2\pi)}} e^{-x^2/2\sigma^2} \qquad \ldots(1)$$

So the tables are to be prepared to give the values of $\frac{1}{\sqrt{(2\pi)}} e^{-x^2/2\sigma^2}$, the origin of x being taken at the origin whence division of these values by σ will yield the ordinates y as required by (1).

TABLE II—ORDINATES OF THE STANDARD NORMAL CURVE

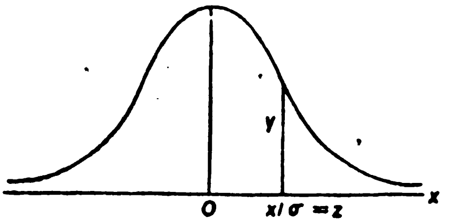


Fig. 15.8

Ordinates (y) being obtained on dividing by σ, the values of $\frac{1}{\sqrt{(2\pi)}} e^{-x^2/2\sigma^2}$ given by the following table, origin of x being at the mean.

$z = x/\sigma$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.3989	.3989	.3989	.3988	.3986	.3984	.3982	.3980	.3977	.3973
0.1	.3970	.3965	.3961	.3956	.3951	.3945	.3939	.3932	.3925	.3918

ORDINATES OF THE STANDARD NORMAL CURVE (Contd.)

$z = x/\sigma$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.2	.3910	.3902	.3894	.3885	.3876	.3867	.3857	.3847	.3836	.3825
0.3	.3814	.3802	.3790	.3778	.3765	.3752	.3739	.3725	.3712	.3697
0.4	.3683	.3668	.3653	.3637	.3621	.3605	.3589	.3772	.3555	.3538
0.5	.3521	.3503	.3485	.3467	.3448	.3429	.3410	.3391	.3372	.3352
0.6	.3332	.3312	.3292	.3271	.3251	.3230	.3209	.3187	.3166	.3144
0.7	.3123	.3101	.3079	.3056	.3034	.3011	.2989	.2966	.2943	.2920
0.8	.2897	.2874	.2850	.2827	.2803	.2780	.2756	.2732	.2709	.2685
0.9	.2661	.2637	.2613	.2589	.2565	.2541	.2516	.2492	.2468	.2444
1.0	.2420	.2396	.2371	.2347	.2323	.2299	.2275	.2251	.2227	.2203
1.1	.2179	.2155	.2131	.2107	.2083	.2059	.2036	.2012	.1989	.1965
1.2	.1942	.1919	.1895	.1872	.1849	.1826	.1804	.1781	.1758	.1736
1.3	.1714	.1691	.1669	.1647	.1626	.1604	.1582	.1561	.1539	.1518
1.4	.1497	.1476	.1456	.1435	.1415	.1394	.1354	.1354	.1334	.1315
1.5	.1295	.1276	.1257	.1238	.1219	.1200	.1182	.1163	.1145	.1127
1.6	.1109	.1092	.1074	.1057	.1040	.1023	.1006	.0989	.0973	.0957
1.7	.0940	.0925	.0909	.0893	.0878	.0863	.0848	.0833	.0818	.0804
1.8	.0790	.0775	.0761	.0748	.0734	.0721	.0707	.0694	.0681	.0669
1.9	.0656	.0644	.0632	.0620	.0608	.0596	.0584	.0573	.0562	.0551
2.0	.0540	.0529	.0519	.0508	.0498	.0488	.0478	.0468	.0459	.0449
2.1	.0440	.0431	.0422	.0413	.0404	.0396	.0387	.0379	.0371	.0363
2.2	.0355	.0347	.0339	.0332	.0325	.0317	.0310	.0303	.0297	.0290
2.3	.0283	.0277	.0270	.0264	.0258	.0252	.0246	.0241	.0235	.0229
2.4	.0224	.0219	.0213	.0208	.0203	.0198	.0194	.0189	.0184	.0180
2.5	.0175	.0171	.0167	.0163	.0158	.1054	.0151	.0147	.0143	.0139
2.6	.0136	.0132	.0129	.0126	.0122	.0119	.0116	.0113	.0110	.0107
2.7	.0104	.0101	.0099	.0096	.0093	.0091	.9988	.0086	.0084	.0081
2.8	.0079	.0077	.0075	.0073	.0071	.0069	.0067	.0065	.0063	.0061
2.9	.0060	.0058	.0056	.0055	.0053	.0051	.0050	.0048	.0047	.0046
3.0	.0044	.0043	.0042	.0040	.0039	.0038	.0037	.0036	.0035	.0034
3.1	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026	.0025	.0025
3.2	.0024	.0023	.0022	.0022	.0021	.0020	.0020	.0019	.0018	.0018
3.3	.0017	.0017	.0016	.0016	.0015	.0015	.0014	.0014	.0013	.0013
3.4	.0012	.0012	.0012	.0011	.0011	.0011	.0010	.0010	.0009	.0009
3.5	.0009	.0008	.0008	.0008	.0008	.0007	.0007	.0007	.0007	.0006
3.6	.0006	.0006	.0006	.0005	.0005	.0005	.0005	.0005	.0005	.0004
3.7	.0004	.0004	.0004	.0004	.0004	.0004	.0003	.0003	.0003	.0003
3.8	.0003	.0003	.0003	.0003	.0003	.0002	.0002	.0002	.0002	.0002
3.9	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0001	.0001

Proportions of items included within $\pm\sigma, \pm 2\sigma, \pm 3\sigma$ of the mean in normal curve. The total area of a normal curve being treated as unity, the probability corresponding to any interval in the range of the variate is measured by the area under the curve within that interval given by Table 1. Hence if m is the mean of the normal distribution, then P, the probability from m to any value x of the variate is given by

$$P = \frac{1}{\sigma\sqrt{(2\pi)}}\int_m^x e^{-(x-m)^2/2\sigma^2}dx \text{ Put } \frac{x-m}{\sigma} = z \text{ i.e., } dx = \sigma\, dz$$

$$= \frac{1}{\sqrt{(2\pi)}}\int_0^x e^{-z^2/2}dz$$

This value P is known as the *Probability integral* or *Error function*. Thus

$$P[m-\sigma < x < m+\sigma] = \frac{1}{\sigma\sqrt{(2\pi)}}\int_{m-\sigma}^{m+\sigma} e^{-\frac{1}{2}[(x-m)/\sigma]^2}dx$$

Put $\frac{x-m}{\sigma} = z,\ dx = \sigma\, dz$

$$\therefore\ P[-1 < z < 1] = \frac{1}{\sqrt{(2\pi)}}\int_{-1}^{1} e^{-z^2/2}dz$$

$$= \frac{2}{\sqrt{(2\pi)}}\int_0^1 e^{-z^2/2}dz$$

$= 2 \times .3413$ since from table I, for $z = 1$,

$$\frac{1}{\sqrt{(2\pi)}}\int_0^1 e^{-z^2/2}dz = .3413$$

$= .6826$

which follows that 68.26% of the items in the normal distribution fall between the range $\pm\sigma$ of the mean.

Now

$$P[m-2\sigma < x < m+2\sigma] = \frac{1}{\sigma\sqrt{(2\pi)}}\int_{m-2\sigma}^{m+2\sigma} e^{-\frac{1}{2}[(x-m)/\sigma]^2}$$

Put $\frac{x-m}{\sigma} = z$, i.e., $dx = \sigma\, dz$

$$= \frac{1}{\sqrt{(2\pi)}}\int_{-2}^{2} e^{-z^2/2}dz$$

$$= \frac{2}{\sqrt{(2\pi)}}\int_0^2 e^{-z^2/2}dz$$

$= 2 \times .4772$ from Table I for $z = 2$

$= .9544.$

So that 95.44% of the items in the normal distribution fall within the range $\pm 2\sigma$ of the mean.

Again

$$P[m-3\sigma < x < m+3\sigma] = \frac{1}{\sigma\sqrt{(2\pi)}}\int_{m-3\sigma}^{m+3\sigma} e^{-\frac{1}{2}[(x-m)/\sigma]^2}dx$$

Put $\frac{x-m}{\sigma} = z$, $\quad\therefore dx = \sigma\, dz$

$$= \frac{1}{\sqrt{(2\pi)}} \int_{-3}^{3} e^{-z^2/2}\, dx$$

$$= \frac{2}{\sqrt{(2\pi)}} \int_{0}^{3} e^{-z^2/2}\, dz$$

$$= 2 \times .49865$$

$$= .99720,$$

i.e., 99.72% of the items in the normal distribution fall within the range ± 3σ of the mean.

Conclusively a normal curve can be used to find

(*i*) The number of cases at any given distance from the mean;

(*ii*) the number of cases lying within certain range of values in the distribution;

(*iii*) the probability that a case selected at random lies above or below a given point.

Problem 115. *If n is large and neither of p and q is too close to zero, then show that the binomial distribution can be closely approximated by a normal distribution with standardized variable given by* $z = \dfrac{x - np}{\sqrt{(npq)}}$, *where x is the binomial variate with mean np and standard deviation* $\sqrt{(npq)}$.

From the given properties of x, we conclude that z is a variate with mean zero and variance unity. Also as x goes from 0 to n, z takes values $\dfrac{-np}{\sqrt{(npq)}}$ to $\dfrac{nq}{\sqrt{(npq)}}$ so that the total range of z is

$$\frac{nq}{\sqrt{(npq)}} - \left(\frac{-nq}{\sqrt{(npq)}}\right) = \frac{n(p+q)}{\sqrt{(npq)}} = \frac{n}{\sqrt{(npq)}},\ i.e.,\ z \text{ jumps } \frac{1}{\sqrt{(npq)}} \text{ at each stage.}$$

Now
$$\frac{1}{\sqrt{(npq)}} \to 0 \text{ as} \to \infty$$

$$\frac{nq}{\sqrt{(npq)}} \to +\infty \text{ as } n \to \infty,\ p, q \neq 0$$

$$\frac{-np}{\sqrt{(npq)}} \to -\infty \text{ as } n \to \infty,\ p, q \neq 0$$

As such the distribution of z is a continuous distribution from $-\infty$ to $+\infty$, with mean zero and variance unity.

If $P(n, x)$ denotes the probability for the variate taking the value x, then

$$P(n, x) = {}^nC_x p^x q^{n-x}$$

$$= \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

Applying Stirling's theorem *i.e.*, $n! = \sqrt{(2\pi)}.e^{-n} n^{n+1/2}$, we get

$$P_n = \lim_{n\to\infty} P(n, x) = \lim_{n\to\infty} \frac{\sqrt{(2\pi)}.e^{-n}.n^{n+1/2}.p^x p^{n-x}}{\sqrt{(2\pi)}.e^{-x}.x^{x+1/2}.\sqrt{(2\pi)}.e^{-(n-x)}\,(n-x)^{n-x+1/2}}$$

$$= \lim_{n\to\infty}\left(\frac{1}{B}\cdot\frac{1}{\sqrt{(2npq)}}\right)$$

where $$B = \left(\frac{x}{np}\right)^{x+1/2} \left(\frac{n-x}{nq}\right)^{n-x+1/2} \quad ...(1)$$

Now given substitution is $z = \dfrac{x - np}{\sqrt{(npq)}}$

or $$x = np + z\sqrt{(npq)}$$

$$\left.\begin{aligned} \text{i.e., } \frac{x}{np} = 1 + z\sqrt{\left(\frac{q}{np}\right)} \text{ and } \frac{n-x}{nq} &= \frac{1}{nq}[n - np - z\sqrt{(npq)}] \\ &= \frac{1}{np}\left[nq - z\sqrt{(npq)}\right] \\ &= 1 - z\sqrt{\left(\frac{p}{nq}\right)} \end{aligned}\right\} \quad ...(2)$$

Taking logarithms of both sides of (1), we have

$$\log B = \left(x + \tfrac{1}{2}\right)\log\frac{x}{np} + \left(n - x + \tfrac{1}{2}\right)\log\frac{n-x}{nq}.$$

Making substitutions from (2), this becomes

$$\log B = \left[np + z\sqrt{(npq)}\right]\log\left[1 + z\sqrt{\left(\frac{q}{np}\right)}\right]$$

$$+\left[nq - z\sqrt{(npq)}\right]\log\left[1 - z\sqrt{\left(\frac{p}{nq}\right)}\right]$$

$$= \left[np + z\sqrt{(npq)}\right]\left\{z\sqrt{\left(\frac{q}{np}\right)} - \frac{1}{2}\frac{z^2 q}{np} + \ldots\right\}$$

$$+\left[nq - z\sqrt{(npq)}\right]\left\{-z\sqrt{\left(\frac{p}{nq}\right)} - \frac{1}{2}\frac{z^2 p}{np} + \ldots\right\}$$

$$= \frac{z}{2\sqrt{n}}\left(\sqrt{\frac{q}{p}} - \sqrt{\frac{p}{q}}\right) + \frac{z^2}{2} - \frac{z^2}{4n}\left(\frac{q}{p} + \frac{p}{q}\right) + \text{terms}$$

containing higher power of $(1/n)$

when $n \to \infty$, $\log B \to \dfrac{z^2}{2}$ i.e. $B \to e^{+z^2/2}$

Now as x takes integral values, z jumps through $\dfrac{1}{\sqrt{(npq)}}$ i.e., $(npq)^{-1/2}$ so that increment in z i.e., $dz = (npq)^{-1/2}$ when $n \to \infty$. Thus, if dP represents the probability for the variate z to lie within the range $z - \frac{1}{2}dz$ to $z + \frac{1}{2}dz$, then

$$dP = \frac{1}{\sqrt{(2\pi)}} e^{-z^2/2} dz, \quad -\infty \le z \le \infty$$

which is the required normal distribution for z.

Hence $f(z)=\frac{1}{\sqrt{(2\pi)}}e^{-z^2/2}$ as $dP=f(z)\,dz$.

If m be the mean and σ the standard deviation of the normal distribution, then we can repalce z by $\frac{x-m}{\sigma}$ and dz by $\frac{1}{\sigma}\,dx$, whence we have

$$dP=\frac{1}{\sigma\sqrt{(2\pi)}}e^{-\frac{1}{2}[(x-m)/\sigma]^2}dx$$

giving $$f(x)=\frac{1}{\sigma\sqrt{(2\pi)}}e^{-\frac{1}{2}[(x-m)/\sigma]^2}$$

so that $y=f(x)$ *i.e.* $y=\frac{1}{\sigma\sqrt{(2\pi)}}e^{-\frac{1}{2}[(x-m)/\sigma]^2}$

gives the normal probability curve.

Problem 116. *If skulls are classified A, B, C according as the length, breadth idex as under 75, between 75 and 80 or over 80 find approximately (assuming that the distribution is normal) the mean and standard deviation of series in which A are 58 per cent, B are 38 per cent and C are 4 per cent being given that if*

$$f(t)=\frac{1}{\sqrt{(2\pi)}}\int_0^t e^{-(x^2/2)}\,dx,$$

then $f(0.20)=0.08$ *and* $f(1.75)=0.46$.

Let m be the mean and σ the standard deviation for the given distribution.

As given the area between $t=0$ and $t=0.20$ is 0.08, so that the area to the left of this ordinate is $0.5+0.08=0.58$ which corresponds to $x=75$.

$$\therefore\quad \frac{75-m}{\sigma}=0.20 \qquad ...(1)$$

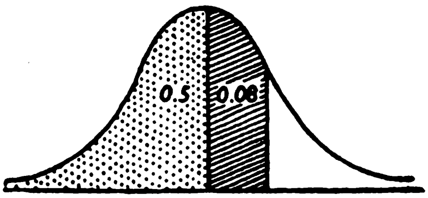


Fig. 15.9

Also the area to the right of the ordinate at $x=80$ is 0.04, so that the area to the left of this ordinate is $1-0.04=0.96$ *i.e.* the area from 0 to $t=\frac{80-m}{\sigma}$ is $0.96-0.5=0.46$, which corresponds to $t=1.75$.

$$\therefore\quad \frac{80-m}{\sigma}=1.75. \qquad ...(2)$$

(1) and (2) are

$$0.20\sigma=75-m,\ 1.75\sigma=80-m.$$

Subtracting $1.55\sigma=5$, *i.e.*, $\sigma=\frac{5}{1.55.}=3.2$ approx.

and then $$m=75-0.20\times 3.2$$
$$=75-0.64=74.4 \text{ approx.}$$

Problem 117. *In a normal distribution 31 per cent of the items are under 45 and 8 per cent are over 64. Find the mean and standard deviation of the distribution.*

Let m be the mean and σ the standard deviation of the distribution. Since 31 per cent of the items are under 45; therefore 19 per cent of the items lie between 45 and m. Thus if

$$\phi(t) = \frac{1}{\sqrt{(2\pi)}} e^{-t^2/2} \qquad \text{where } t = \frac{x-m}{\sigma},$$

then $$\int_{(45-m)\sigma}^{0} \phi(t)\,dt = .19, \text{ i.e.} \int_0^{(m-45)/\sigma} \phi(t)\,dt = .19.$$

From Table I, we have $\int_0^{0.5} \phi(t)\,dt = .1915.$

So $$\frac{m-45}{\sigma} = .5 \text{ approx.} \qquad ...(1)$$

Again since 8 per cent of the items are over 64, therefore 42 per cent lie between m and 64, so that $\int_0^{(64-m)\sigma} \phi(t)\,dt = .42 = \int_0^{1.4} \phi(t)\,dt$ nearly from Table I.

$$\therefore \quad \frac{64-m}{\sigma} = 1.4. \qquad ...(2)$$

Solving (1) and (2), we get $m = 50$, $\sigma = 10$.

Problem 118. *If x is a normal variate with mean $\bar{x}$ and S.D. σ, find the mean and variance of variate y defined by*

$$y = \tfrac{1}{2}\left(\frac{x-\bar{x}}{2}\right)^2.$$

For the normal variate x, we have

$$f(x)\,dx = \frac{1}{\sigma\sqrt{(2\pi)}} e^{-\frac{1}{2}\{(x-\bar{x})/\sigma\}^2} dx. \qquad ...(1)$$

If $y = \frac{1}{2}\left(\frac{x-\bar{x}}{\sigma}\right)^2$, *i.e.*, $dy = \frac{x-\bar{x}}{\sigma} \cdot \frac{dx}{\sigma}$ or $dx = \frac{\sigma}{\sqrt{(2y)}} dy$,

then (1) becomes

$$f(y)\,dy = \frac{1}{\sigma\sqrt{(2\pi)}} e^{-y} . \frac{\sigma}{\sqrt{(2y)}}\,dy = \frac{1}{2\sqrt{\pi}} y^{-1/2} e^{-y} dy,$$

so that

mean $= \mu_1'$ (about the origin), for a normal distribution

$$= 2\int_0^\infty y f(y)\,dy = \frac{1}{\sqrt{\pi}} \int_0^\infty y^{1/2} e^{-y} dy$$

$$= \frac{1}{\sqrt{\pi}} \cdot \Gamma\tfrac{3}{2} \text{ (by gamma integrals)} = \frac{1}{\sqrt{\pi}} . \tfrac{1}{2}\sqrt{\pi} = \tfrac{1}{2}$$

and $$\mu_2' = 2\int_0^\infty y^2 f(y)\,dy$$

$$= \frac{1}{\sqrt{\pi}} \int_0^\infty y^{3/2} e^{-y}\,dy = \frac{1}{\sqrt{\pi}} \Gamma\tfrac{5}{2} = \frac{1}{\sqrt{\pi}} \cdot \tfrac{3}{2}\tfrac{1}{2}\sqrt{\pi} = \tfrac{3}{4};$$

$$\therefore \quad \sigma^2 = \mu_2 = \mu_2' - \mu_1'^2 = \tfrac{3}{4} - \tfrac{1}{4} = \tfrac{1}{2} = \text{variance.}$$

Problem 119. *If a normal distribution is grouped in intervals of total frequency N and S is the sum of squares of the frequencies, an estimate of the standard deviation σ is given by* $\frac{N^2}{2S\sqrt{\pi}}$.

The normal frequency distribution with mean m and standard deviation σ is given by

$$dF = f(x)\,dx = y\,dx = \frac{N}{\sigma\sqrt{(2\pi)}} e^{-\frac{1}{2}[(x-m)/\sigma]^2}$$

where the ordinate of the normal curve is

$$y = \frac{N}{\sigma\sqrt{(2\pi)}} e^{-\frac{1}{2}[(x-m)/\sigma]^2}.$$

$\therefore$ square of the ordinate is

$$y^2 = \frac{N^2}{\sigma^2 . 2\pi} \cdot e^{-[(x-m)/\sigma]^2}.$$

as such the distribution function for y^2 is

$$dP = y^2\,dx = \frac{N^2}{2\pi\sigma^2} \cdot e^{-[(x-m)/\sigma]^2} dx$$

$$= \frac{N^2}{2\sigma\sqrt{\pi}} \cdot \frac{1}{\left(\frac{\sigma}{\sqrt{2}}\right)\sqrt{(2\pi)}} \cdot e^{-\frac{1}{2}(x-m)^2/(\sigma/\sqrt{2})^2} dx,$$

which gives a normal distribution with S.D. $\sigma/\sqrt{2}$ and total frequency $N^2/\left(2\sigma\sqrt{\pi}\right)$. Hence taking S as equivalent to this value, we have

$$S = \frac{N^2}{2\sigma\sqrt{\pi}}, \quad i.e. \quad \sigma = \frac{N^2}{2S\sqrt{\pi}}.$$

Problem 120. *The local authorities in a certain city instal 2000 lamps in the streets of the city. If the lamps have an average life of 1000 burning hours with a standard deviation of 200 hours, (i) what number of lamps might be expected to fail in first 700 burning hours, and (ii) after what period of burning hours would we expect that 10 per cent of the lamps would have failed? Assume that the lives of the lamps are normally distributed. You are given that*

$$F(1.50) = 0.933,\ F(1.28) = .900 \text{ where } F(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{(2\pi)}} e^{-z^2/2} dz$$

(*i*) We have the average life of the lamp *i.e.*, m = 1000 hour and standard deviation σ = 200 hours.

$\because$ The normal distribution being symmetrical, the area of the normal curve to the left of $x = 700$ is equal to that to the right of $x = (2000 - 700) = 1300$.

$$\therefore \qquad z = \frac{x - m}{\sigma} = \frac{1300 - 1000}{200} = 1.50$$

and area to the left of $x = 1300$ is $= \dfrac{1}{\sqrt{(2\pi)}} \displaystyle\int_{-\infty}^{z} e^{-z^2/2} dz = F(z)$

where $z = 1.50$ and also $F(1.50) = 0.933$ which gives the probability of a value of x lying to the left of $x = 1300$ or right of $x = 700$.

$\therefore$ probability of its failure = 1 − .933 = 067.

So that the number of lamps expected to fail in first 700 hours

= 2000 × . 067 = 134.

(*ii*) The failure of 10 per cent of lamps gives that a value of x is to be so found that the arc of the standard normal curve to the left of it is $\frac{10}{100}$ *i.e.*, 0.1. But the area to the left of $z = 1.28$ is .900, hence by symmetry in the distribution of a normal curve it follows that the area to the left of $z = -1.28$ is equal to the area to the right of $z = 1.28$, *i.e.*, 0.1.

Hence $$z = -1.28 = \frac{x-m}{\sigma} = \frac{x-1000}{200}$$

giving $x = 1000 - 256 = 744$.

Hence after 744 hours, it is expected that 10 per cent of the lamps fail.

[D] The Poisson Distribution

We have shown in the preceding sections that the normal curve is the limit to the binomial, whether p is or is not equal to q provided that n becomes sufficiently large in order to make $(q-p)$ negligibly small as compared to $\sqrt{(npq)}$. Now we have to consider the limit to the same series if p (or q) becomes sufficiently small and n is increased sufficiently to keep the product np (or nq) a finite number say m *i.e.*, $np = m$.

The probability of r successes in the binomial distribution is given by the $(r+1)$th term in the binomial expansion $(q+p)^n$, *i.e.*,

$$P(r) = {}^nC_r p^r q^{n-r} = \frac{n!}{r!(n-r)} p^r q^{n-r}$$

$$= \frac{n!}{r!(n-r)}\left(\frac{m}{n}\right)^r\left(1-\frac{m}{n}\right)^{n-r}$$

$$\left(\because\ np = m,\ i.e.,\ p = \frac{m}{n} \text{ and so } q = 1-p = 1-\frac{m}{n} \text{ as } p+q=1\right)$$

$$= \frac{m^r}{r!}\left(1-\frac{m}{n}\right)\cdot\frac{n}{(n-r)!\cdot n^n\left(1-\frac{m}{n}\right)^r}$$

Here $$\lim_{n\to\infty}\left(1-\frac{m}{n}\right)^n = \lim_{n\to\infty}\left\{\left(1-\frac{m}{n}\right)^{-n/m}\right\}^{-m} = e^{-m}$$

and applying Stirling's approximation *i.e.*, $n! = n^n e^{-n}\sqrt{(2\pi n)}$, when n is large,

$$P(r) = \frac{m^r e^{-m}}{r!}\cdot\frac{n^n e^{-n}\sqrt{(2\pi n)}}{(n-r)^{n-r}e^{-(n-r)}\sqrt{\{2\pi\ (n-r)\}\times n^r\left(1-\frac{m}{n}\right)^r}}$$

$$= \frac{m^r e^{-m} e^{-r}}{r!}\left(1-\frac{r}{n}\right)^{-n}\left(1-\frac{r}{n}\right)^{r-\frac{1}{2}}\left(1-\frac{m}{n}\right)^{-r}$$

$$= \frac{m^r e^{-m} e^{-r}}{r!} \because \lim_{n\to\infty}\left(1-\frac{r}{n}\right)^{-n} \lim_{n\to\infty} = \left\{\left(1-\frac{r}{n}\right)^{-n/r}\right\}^r = e^r;$$

$$\lim_{n\to\infty}\left(1-\frac{r}{n}\right)^{r-\frac{1}{2}} = 1 \text{ and } \lim_{n\to\infty}\left(1-\frac{m}{n}\right)^{-r} = 1$$

$$= \frac{m^r e^{-m}}{r}$$

Hence successive terms in the expansion of $(q + p)^n$ are

$$e^{-m}, e^{-m}.\, m, e^{-m}\frac{m^2}{2!}, e^{-m}.\frac{m^3}{3!}, \text{ etc.}$$

and the limiting value of $(q + p)^n$ is

$$e^{-m}\left\{1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \ldots\right\}$$

This expression is known as *Poisson's distribution* or *Poisson's Exponential limit.*

Note 1. *The limiting form of N $(q + p)^n$ is*

$$N.e^{-m}\left\{1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \ldots\ldots\right\}$$

Note 2. *The quantity m introduced in Poisson distribution is said to be be the parameter of Poisson distribution.*

Note. 3. Characteristics of the Poisson distribution.

(i) This is the limiting form of binomial distribution when n is large and p (or q) is small.

(ii) Here p or q is very close to zero or unity, but if p is close to zero, the distribution is J-shaped or unimodal.

(iii) Since it consists of a single parameter m, the entire distribution can therefore be obtained by knowing the mean only.

Note. 4. *Some examples of Poisson distribition are :*

(i) The number of defective screws per box of 100 screws.

(ii) The number of typographical errors per page in typed material.

(iii) The number of cars passing through a certain street in time t.

Constans of the Poisson Distribution

Assume that origin is located at the first term of the distribution, so that the values of the deviation from the assumed origin are 0, 1, 2,......

1. The Mean

$$\text{Mean} = \mu_1' \text{ (about the origin) } \sum_0^{\infty} e^{-m}.\frac{m^r}{r!}.r$$

$$= e^{-m}\left[0 + m + \left(\frac{m^2}{2!}\times 2\right) + \left(\frac{m^3}{3!}\times 3\right) + \ldots\right]$$

$$= me^{-m}\left(1 + \frac{m}{1!} + \frac{m^2}{2!} + \ldots\right) = me^{-m}.e^{m} = m$$

2. The Variance

We have μ_2' (about the origin)

$$= \sum_0^{\infty} e^{-m}\frac{m^r}{2!}.r^2$$

$$= e^{-m}\left[0 + m + \left(\frac{m^2}{2!}\times 2^2\right) + \left(\frac{m^3}{3!}\times 3^2\right) + \ldots\right]$$

$$= me^{-m}\left[1+\left(\frac{m}{1!}\times 2\right)+\left(\frac{m^2}{2!}\times 3\right)+\ldots\right]$$

$$= me^{-m}\left[1+\frac{m}{1!}(1+1)+\frac{m^2}{2!}(2+1)+\ldots\right]$$

$$= me^{-m}\left[\left(1+\frac{m}{1!}+\frac{m^2}{2!}+\ldots\right)+\left(m+\frac{m^2}{1!}+\frac{m^3}{2!}+\ldots\right)\right]$$

$$= me^{-m}\left[e^m + m\left(1+\frac{m}{1!}+\frac{m^2}{2!}+\ldots\right)\right]$$

$$= me^{-m}\left[e^m + me^m\right] = m(m+1)$$

$\therefore$ variance $= \mu_2 = \sigma^2 = \mu_2' - \mu_1'^2 = m^2 + m - m^2 = m$

so that standard deviation, $\sigma = \sqrt{m}$.

3. The Moments

μ_3' (about the origin)

$$= \Sigma\, e^{-m}\frac{m^r}{r!}.r^3$$

$$= e^{-m}\Sigma m^r.\left\{\frac{r(r-1)(r-2)+3r(r-1)+r}{r!}\right\}$$

$$= e^{-m}m^3\Sigma\frac{m^{r-3}}{(r-3)!} + e^{-m}.3m^2\Sigma\frac{m^{r-2}}{(r-2)!} + e^{-m}.m\Sigma\frac{m^{r-1}}{(r-1)!}$$

or $\quad \mu_3' = m^3 + 3m^2 + m$

so that $\quad \mu_3 = \mu_3' - 3\mu_2'.\mu_1' + 2\mu_1'^3$

$$= m^3 + 3m^2 + m - 3m^2 - m^3 + 2m^3 = m$$

and $\quad \mu_4' = \Sigma e^{-m}.\frac{m^r}{r!}.r^4$

$$= e^{-m}\Sigma m^r.\left\{\frac{r(r-1)(r-2)(r-3)+6r(r-1)(r-2)+7(r-1)r+r}{r!}\right\}$$

$$= e^{-m}.m^4\Sigma\frac{m^{r-4}}{(r-4)!} + e^{-m}.6m^3\Sigma\frac{m^{r-3}}{(r-3)!} + e^{-m}.7m\Sigma\frac{m^{r-2}}{(r-2)!} + e^{-m}.m\Sigma\frac{m^{r-1}}{(r-1)!}$$

$$= m^4 + 6m^3 + 7m^2 + m$$

so that $\quad \mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

$$= m^4 + 6m^3 + 7m^2 - 4m(m^3 + 3m^2 + m) + 6m^2(m^2 + m) - 3m^4$$

$$= 3m^2 + m.$$

Problem 121. *Define a Poisson random variable and give some physical situations illustrating it. Find out its mean and variance.*

If X and Y are independently distributed as Poisson random variates with parameters λ and μ respectively find the probability distribution of X + Y.

We know that a random variable is a function defined as a sample space. A random variable x assuming values 0, 1, 2,..., r,...with probabilities

$$e^{-m},\ me^{-m}, \frac{m^2}{2!}e^{-m},\ldots,\frac{m^r e^{-m}}{r!},\ldots$$

i.e., $$P(x=r)=\frac{m^r e^{-m}}{r!},\ r=0,1,2,\ldots$$

is said to be a *Poisson random variable.*

In order to illustrate it with the help of a physical situation, let us consider a random variate X denoting the number of calls during time t at a telephone switch board. Let us assume that the calls are independent and the probability of a call in time dt is λ idt.

Denoting by $P_x(t)$ and $P_x(t+\delta t)$, the chance of x calls in times t and $t+\delta t$ respectively we have two mutually exclusive possibilities, (*i*) there are x-calls in t and no call in time dt, (*ii*) there are $(x-1)$ calls in t and one call in dt, neglecting the possibility of more than one call in dt, for it would be of order 2 (*i.e.*, dt^2) and higher.

Thus, $$P_x(t+\delta t)=P_x(t).\{1-\lambda\,\delta t\}+P_{x-1}(t)\,\lambda\,\delta t$$
$$=P_x(t)-P_x(t).\lambda\,\delta t+P_{x-1}(t).\lambda\,\delta t$$

or $$P_x(t+\delta t)-P_x(t)=\{P_{x-1}(t)-P_x(t)\}.\lambda\,\delta t$$

or $$\frac{P_x(t+\delta t)-P_x(t)}{\delta t}=\lambda\left[P_{x-1}(t)-P_x(t)\right].$$

Proceeding to the limit as $\delta t\to 0$, we have

$$\lim_{\delta t\to 0}\frac{P_x(t+\delta t)-P_x(t)}{\delta t}=\lim_{\delta t\to 0}\lambda\left[P_{x-1}(t)-P_x(t)\right]$$

i.e., $$\frac{dP_x(t)}{dt}=\lambda\left[P_{x-1}(t)-P_x(t)\right].$$

Putting $$P_x(t)=\frac{(\lambda t)^x}{x!}.f(t)$$

so that $$P_0(t)=f(t)\text{ and }P_0(0)=F(0)=1, \quad \ldots(1)$$

We have $$\frac{d}{dt}\left[\frac{(\lambda t)^x}{x!}f(t)\right]=\lambda\left[\frac{(\lambda t)^{x-1}}{(x-1)!}f(t)-\frac{(\lambda t)^x}{x!}f(t)\right]$$

or $$\frac{(\lambda t)^x}{x!}f'(t)+\frac{\lambda^x}{x!}f(t).xt^{x-1}=\frac{\lambda^x t^{x-1}}{(x-1)!}f(t)-\frac{\lambda t}{x}.\frac{\lambda^x t^{x-1}}{(x-1)!}f(t).$$

Diving throughout by $\frac{\lambda^x t^{x-1}}{(x-1)!}f(t)$, this reduces to

$$\frac{\lambda t}{x}.\frac{f'(t)}{f(t)}+1=1-\frac{\lambda t}{x}\ \text{ or }\ \frac{f'(t)}{f(t)}=-\lambda.$$

Integrating with regard to t, this yields

$\log f(t)=-\lambda t+A$, A being constant of integration.

Initially when $t=0$, $f(0)=1$, from (1), giving $A=0$

$\therefore \quad \log f(t)=-\lambda t$ or $f(t)=e^{-\lambda t}$

or $$\frac{(\lambda t)^x}{x!}f(t)=P_x(t)=e^{-\lambda t}.\frac{(\lambda t)^x}{x!}$$

where $$\sum_{x=0}^{\infty}P_x(t)=e^{-\lambda t}\sum_{x=0}^{\infty}\frac{(\lambda t)^x}{x!}=e^{-\lambda t}.e^{\lambda t}=1.$$

It follows that the number of calls in a fixed time t is a Poisson variate with parameter λt.

For the second part, we have

$$P(X=x)=e^{-\lambda}.\frac{\lambda^x}{x!},\quad P(Y=y)=e^{-\mu}\frac{\mu^y}{y!}$$

so that $P(X=x, Y=y)=P(X=x).P(Y=y)$

$$=e^{-\lambda}\frac{\lambda^x}{x!}.e^{-\mu}\frac{\mu^y}{y!}=e^{-(\lambda+\mu)}\frac{\lambda^x\mu^y}{\lfloor x \lfloor y}$$

Since the variables take values 0, 1, 2, 3,...; let us find the Probability that their sum *i.e.* $S+Y$ takes values r so that $y=r-x$. Summing for all values of x from 0 to r, we have

$$P(x+y=r)=e^{-(\lambda+\mu)}\sum_{x=0}^{r}\frac{\lambda^x\mu^{r-x}}{\lfloor x \lfloor r-x}=e^{-(\lambda+\mu)}.\frac{\mu^r}{\lfloor r}\sum_{x=0}^{r}\frac{\lfloor r}{\lfloor x \lfloor r-x}\left(\frac{\lambda}{\mu}\right)^x$$

$$=e^{-(\lambda+\mu)}.\frac{\mu^r}{\lfloor r}\sum_{x=0}^{r}{}^rC_x\left(\frac{\lambda}{\mu}\right)^x$$

$$=e^{-(\lambda+\mu)}.\frac{\mu^r}{\lfloor r}\left[{}^rC_0+{}^rC_1.\frac{\lambda}{\mu}+{}^rC_2\left(\frac{\lambda}{\mu}\right)^2+\ldots+{}^rC_r\left(\frac{\lambda}{\mu}\right)^r\right]$$

$$=e^{-(\lambda+\mu)}.\frac{\mu^r}{\lfloor r}\left(1+\frac{\lambda}{\mu}\right)^r=e^{-(\lambda+\mu)}.\frac{(\lambda+\mu)^r}{\lfloor x+y}\quad\because r=x+y$$

which is a Poisson distribution with mean $(\lambda+\mu)$.

Problem 122. *find the probability that at most 5 defective fuses will be found in a box of 200 fuses if experience shows that 2 per cent of such fuses are defective.*

Here $n=200,\ q=\frac{2}{100}=.02$, so that $m=nq=4$.

Hence $P(x\le 5)=\sum_{x=0}^{5}e^{-4}.\frac{4^x}{x!}$

$$=e^{-4}\left[1+4+\frac{4^2}{2!}+\frac{4^3}{3!}+\frac{4^4}{4!}+\frac{4^5}{5!}\right]$$

$$=e^{-4}\left[1+4+8+\frac{32}{3}+\frac{32}{3}+\frac{128}{15}\right]$$

$$=e^{-4}.\frac{643}{15}=0.0183\times\frac{643}{15}=.785\text{ approx.}\quad\left[\because e^{-4}=0.0183\right]$$

Problem 123. *Fit a Poissons's distribution to the set of observations :*

Deaths	*0*	*1*	*2*	*3*	*4*
Frequency	*122*	*60*	*15*	*2*	*1*

and calculate theoretical frequencies.

We have, Mean $=\frac{0\times122+1\times60+2\times15+3\times2+4\times1}{122+60+15+2+1}$

$$=\frac{100}{200}=0.5$$

$\therefore \quad e^{-0.5} = 1 - (0.5) + \frac{1}{2}(0.5)^2 - \frac{1}{6}(0.5)^3 + \ldots$

$= 1 - 0.5 + 0.125 - 0.0208 + \ldots = .61$ approx.

The theoretical frequency of r deaths is given by

$$Ne^{-m}.\frac{m^r}{r!} = 200.\ e^{-0.5}.\frac{(0.5)^r}{r!} = 122.\ \frac{(0.5)^r}{r!}$$

which gives for $r = 0, 1, 2, 3, 4$ the theoretical frequencies as 122, 61, 15, 2 and 0 respectively.

Problem 124. *In a certain factory turning out razor blades, there is a small chance $\frac{1}{500}$ for any blade to be defective. The blades are supplied in packets of 10. Use Poisson distribution to calculate the approximate number of packets containing no defective, one defective and two defective blades respectively in a consignment of 10,000 packets, given that $e^{-.02} = 0.9802$.*

Here $N = 10{,}000$, $m = np = 10 \times \frac{1}{500} = 0.02$ and $e^{-.02} = .9802$.

$\therefore$ Required frequencies are given by $Ne^{-m}, Ne^{-m}.m, Ne^{-m}.\dfrac{m^2}{2!}$

i.e., $10{,}000 \times .9802;\ 10{,}000 \times .9802 \times 0.02;\ 10{,}000 \times .9802 \times \dfrac{(02)^2}{2}$

i.e., 9802; 196, 2 packets.

Problem 125. *If x is a Poissonian variate with mean m,*

(*a*) *What would be the expectation of $e^{-kx}.\ kx$ where k is a constant.*

(*b*) *Show that, expectation of $(e^{-kx}) = e^{-m\left(1-e^{-k}\right)}$.*

(*a*) We have, $E(kxe^{-kx}) = \sum_{x=0}^{\infty} kx.e^{-kx}.\dfrac{e^{-m}m^x}{x!}$

$$= mke^{-(m+k)}\sum_{x=0}^{\infty} e^{-k(x-1)}\frac{m^{x-1}}{(x-1)!}$$

$$= mke^{-(m+k)}.\sum_{x=0}^{\infty}\frac{\left(me^{-k}\right)^{x-1}}{(x-1)!}$$

$$= mke^{-(m+k)}e^{me^{-k}} = mke^{m\left(e^{-k}-1\right)-k}$$

and $E(e^{-kx}) = \sum_{x=0}^{\infty} e^{-kx}\dfrac{e^{-m}m^x}{x!} = e^{-m}\sum_{x=0}^{\infty}\dfrac{\left(me^{-k}\right)^x}{x!}$

$$= e^{-m}.e^{me^{-k}} = e^{-m\left(1-e^{-k}\right)}$$

15.17. OTHER DISTRIBUTIONS

(A) Casual distribution. This is a distribution with one parameter (say u) like Poisson distribution and found ccasually in exact sciences. In this type of distribution the variate takes with certainty one value, *i.e.* $x = u$, such that $P(x = u)) = 1$,

$$P(x = t) = 0 \text{ when } t \neq u,$$

$$P(x \le t) = 0 \text{ when } t < u,$$
$$= 1 \text{ when } t \ge u.$$

Problem 126. *Show that the moment generating function for a casual distribution is e^{ut}, i.e., $M_0(t) = e^{ut}$.*

We have $M_0(t) = \sum_{u=0}^{\infty} P(x = u)\ e^{ut}$

$$= \sum_{u=0}^{\infty} e^{ut} \text{ as } P(x = u) = 1$$

$$= e^{ut} \text{ as } P(x = t) = 0, t \ne u.$$

[B] Rectangular or uniform distribution. A distribution in which the variate takes values $x_1, x_2, \ldots, x_n$ (or $1, 2, \ldots, n$ in particular) each having the same probability $1/n$, is said to be a rectangular or uniform distribution. Thus,

$$P(x = x_i) = \frac{1}{n}, \ i = 1, 2, \ldots n.$$

$$\therefore \qquad \sum_{i=1}^{n} P(x = x_i) = \frac{1}{n} + \frac{1}{n} + \ldots n \text{ times } = 1$$

In tossing a coin or throwing a die, this type of distribution appears.

Problem 127. *If $f(x)$ be the probability law of continuous random variable x in the interval $a \le x \le b$, so that $f(x) = 0$ for $x < a$ and $x > b$ and $\int_a^b f(x) = dx = 1$, show that the varible $P = \int_a^x f(x)\,dx$ has a rectangular distribution.*

Given $P = \int_a^x f(x)\,dx = P(X \le x)$, so that $dP = f(x)\,dx$.

Assuming $u = F(x)$, where $F'(x) = f(x)$, we have

$$dP = f(x)dx = f(x)\frac{dx}{du}du.$$

$$= f(x)\frac{1}{f(x)}\ du = du, \ 0 \le u \le 1.$$

Since u is a distribution with unit range, therefore $u = P$ is a rectangular distribution.

Problem 128. *A variate x has uniform distribution over the unit interval. Find the function of x having the distribution*

$$dP = e^{-x}\,dx, \ 0 \le x \le \infty.$$

Here $\qquad dP = e^{-x}\,dx.$

Suppose $u = F(x)$ where $F'(x) = f(x) = e^{-x}$

$$= \int_0^x e^{-x}dx$$

$$= \left[-e^{-x}\right]_0^x = 1 - e^{-x}$$

giving $\qquad x = \log_e \frac{1}{1-u}.$

[C] The negative binomial distribution. In a succession of Bernoulli trial (Binomial distribution) let $P(r)$ denote the probability that exactly $r + k$ ($k > 0$), trials are required to produce k successes. This will so happen when the last trial *i.e.*, $(r + k)$th trials is a success with probability p and the previous $(r + k - 1)$ trials must have $(k - 1)$ successes with probability ${}^{r+k-1}C_{k-1}p^{k-1}q^r$, where $q = 1 - p$,

$\therefore$ $P(r)$ = prob. of $(k - 1)$ successes in $(x + k - 1)$ trails

× probability of $(x + k)$th success

$$= {}^{r+k-1}C_{k-1}p^{k-1}.q^r.p$$

$$= {}^{r+k-1}C_{k-1}p^kq^r \qquad ...(1)$$

$$= \frac{p^k(k + r - 1)\ (k + r - 2)...\{k + r + 1 - (r + 1)\}}{r!}q^r$$

$$= \frac{p^k(k + r - 1)\ (k + r - 2)...(k + 1).k}{r!}q^r$$

$$= p^k(-1)^r\frac{(-k)\ (-k - 1)...(-k - r + 1)}{r!}q^r$$

$$= p^k.\ (-1)^r.\ {}^{-k}C_rq^r$$

$$= {}^{-k}C_rp^k(-q)^r, \qquad ...(2)$$

whence

$$\sum_0^{\infty} P(r) = p^k \sum_{r=0}^{\infty} {}^{-k}C_r\ (-q)^r$$

$$= p^k[1 - q]^{-k} = p^kp^{-k} = 1. \qquad ...(3)$$

The distributions given by (1) and (2) for $k \geq 0$ even of k is not an integer are known as negative *binomial distribution.*

Deduction

(1) Pascal's distribution. The distribution

$$P(r) = {}^{-k}C_rp^k(-q)^r \qquad ...(4)$$

when regarded as one having two parameters p and k is said to be the Pascal-distribution.

(2) Geometric distribution. In (1) if we put $k = 1$, this becomes

$$P(r) = {}^rC_0pq^r \qquad ...(5)$$

$$= q^rp,\ r = 0, 1, 2,... \text{ and } q = 1 - p.$$

This is known as geometrical distribution, which can also be written directly by considering that r is the number of failures preceding the first success in a Bernoulli-sequence of trials, probability of success being p, so that r may be regarded as random variate with probability distribution

$$P(r) = q^r.p,\ r = 0, 1, 2,... \text{and } q = 1 - p,$$

(3) Polya's distribution. If we put $k = \dfrac{1}{\beta}$, $p = \dfrac{1}{1 + \beta\mu}$, $q = \dfrac{\beta\mu}{1 + \beta\mu}$ in (1), we get

$$P(r) = \frac{(1 + \beta)\ (1 + 2\beta)...[1 + (r - 1)\ \beta]}{r!}\left(\frac{1}{1 + \beta\mu}\right)^{1/\beta}\left(\frac{\mu}{1 + \beta\mu}\right)^r, \qquad ...(6)$$

which is known as Polya's distribution with two parameters β and μ.

(4) Second form of geometric distribution. In Polya's distribution given by (6), if we put $\beta = 1$, we get

$$P(r) = \frac{1}{1+\mu}\left(\frac{\mu}{1+\mu}\right)^r. \qquad \ldots(7)$$

(5) General term in the binomial expansion of $(Q-P)^k$. Putting $P\frac{1}{Q}$, $q = \frac{P}{Q}$, so that $Q - P = 1$ as $p + q = 1$, (1) gives

$$P(r) = Q^{-k.r+k-1}C_{k-1}\left(\frac{P}{Q}\right)^r$$

$$= Q^{-k}.^{-k}C_r\left(\frac{-P}{Q.}\right)^r,$$ $r = 0, 1, 2,\ldots$ for the form (2), which gives the general term of $(Q-P)^{-k}$.

[D] Hypergeometric distribution. From an urn containing m white and n red balls if r balls are drawn at a time without replacement, then the probability that x out of r are white is given by

$$P(x) = \frac{{}^mC_x\,{}^nC_{r-x}}{{}^{m+n}C_r},\quad n = 0, 1, 2\ldots r,\; r \le m,\; r \le n \qquad \ldots(1)$$

where $\sum_{x=0}^{r} P(x) = 1$ since $\sum_{x=0}^{r} {}^mC_x\,{}^nC_{r-x} = {}^{m+n}C_r$

which is evident by equating the coefficients of x^r on either side of $(1+x)^m(x+1)^n = (1+x)^{m+n}$.

The distribution given by (1) is said to be the hypergeometric distribution.

Another form of hypergeometric distribution. From an urn containing Np white and Nq red balls such that $p + q = 1$, if n balls containing x red and $(n-x)$ white are drawn, then their probability is given by

$$f(x) = \frac{{}^{Np}C_x\,{}^{Nq}C_{n-x}}{{}^NC_n} \qquad \ldots(2)$$

where $f(x)$ represents a probability density function known as hypergeometric function.

As above it is evident that $\sum_{x=0}^{Np} f(x) = 1$.

Consider $\lim_{N\to\infty} \frac{{}^{Np}C_x\,{}^{Nq}C_{n-x}}{{}^NC_n}$,

$$= \lim_{N\to\infty} \frac{(Np)!}{x!\,(Np-x)!}\cdot\frac{(Nq)!}{(n-x)!(Nq-n-x)!}\cdot\frac{n!(N-n)!}{N!}$$

$$= \lim_{N\to\infty} \frac{n!}{x!\,(n-x)!}\cdot\frac{\{Np\,(Np-1)\ldots(Np-x+1)\}\;Nq(Nq-1)\ldots\ldots(Nq-n+x+1)\}}{N(N-1)\ldots(N-n+1)}$$

$$= \lim_{N \to \infty} \frac{{}^nC_x\left\{p\left(p-\frac{1}{N}\right)\left(p-\frac{2}{N}\right)\ldots\left(p-\frac{x-1}{N}\right)\right\} \times\left\{q\left(q-\frac{1}{N}\right)\left(q-\frac{2}{N}\right)\ldots\left(q-\frac{n-x-1}{N}\right)\right\}}{\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\ldots\left(1-\frac{n-1}{N}\right)}$$

$$= {}^nC_x p^x q^{n-x}$$

which is binomial frequency distribution.

Problem 129. *Find the mean and variance of hypergeometric distribution.*

Case I. Using the form (1), we have

$$\text{Mean} = E(x) = \sum_{x=0}^{r} x.P(x) = \sum_{x=0}^{r} \frac{x.{}^mC_x\,{}^nC_{r-x}}{{}^{m+n}C_r}$$

$$= \frac{m\sum_{x=1}^{r} {}^{m-1}C_{x-1}\,{}^nC_{r-x}}{{}^{m+n}C_r} = \frac{m}{{}^{m+n}C_r}\,{}^{m-1}C_y\;{}^nC_{r-y-1} \text{ (if } x-1=y)$$

$$= \frac{m}{{}^{m+n}C_r}.\,{}^{m-1+n}C_{r-1}$$

$$= \frac{m(r!)\,(m+n-r)!}{(m+n)!} \times \frac{(m-1+n)!}{(r-1)\,!(m+n-r)!} = \frac{mr}{m+n}$$

and $\mu_2' = E(x^2) = \sum_{x=0}^{r} x^2 P(x) = \sum_{x=0}^{r} x(x-1)\;P(x) + \sum_{x=0}^{r} xP(x) \quad \because\; x^2 = \{x(x-1)+x\}$

$$= \frac{m(m-1).r.(r-1)}{(m+n)\,(m+n-1)} + \frac{mr}{m+n} \qquad \text{(solving as above.)}$$

$$\therefore\; \mu_2 = \mu_2' - \mu_1'^2 = \frac{m(m-1)\;r\,(r-1)}{(m+n)\,(m+n-1)} + \frac{mr}{m+n} - \left(\frac{mr}{m+n}\right)^2$$

i.e., $\text{variance} = \dfrac{mnr\,(m+n-r)}{(m+n)^2\,(m+n-1)}$ (simplifying)

Case II. Using the form (2), we have

$$\text{Mean} = \mu_1' = \sum_{x=0}^{n}\left\{x\frac{{}^{Np}C_x\,{}^{Nq}C_{n-x}}{{}^NC_n}\right\} = \sum_{x=1}^{n}\left\{\frac{x.{}^{Np}C_x\,{}^{Nq}C_{n-x}}{{}^NC_n}\right\}$$

$$= Np\sum_{x=0}^{n}\left\{{}^{Np-1}C_{x-1}\,{}^{Nq}C_{x-1}/{}^NC_n\right\}$$

$$= Np.\,{}^{Np+Nq-1}C_{n-1}/{}^NC_n = np \qquad \text{(simplifying)}$$

and $$\mu_2' = \sum_{x=0}^{n} x^2 f(x) = \sum_{x=0}^{n}\{x(x-1)+x\}\,f(x)$$

$$= \frac{np(n-1)(Np-1)}{N-1} \qquad \text{(on simplification)}$$

$$\therefore \qquad \mu_2 = \mu_2' - \mu_1'^2 \text{ gives } \mu_2 = \frac{n(N-n)pq}{N-1}.$$

[E] Multinomial distribution. If there are balls of r colours in proportions given by $p_1, p_2 \ldots, p_r$ such that $p_1 + p_2 + \ldots + p_r = 1$, then the probability of drawing x_1 balls of the first colour, x_2 of the second colour and so on without replacement is given by

$$f(x_1, x_2, \ldots x_n) = \frac{{}^{Np_1}C_{x_1}\, {}^{Np_2}C_{x_2} \ldots {}^{Np_r}C_{x_r}}{{}^{N}C_n}$$

where $n = x_1 + x_2 + \ldots + x_n$ and $x_1 \le Np_1, x_2 \le Np_2, \ldots, x_r \le Np_r$.

If the drawings are with replacement, then the distribution given by

$$f(x_1, x_2, \ldots x_n) = \frac{n!}{x_1! x_2! \ldots x_r!} p_1^{x_1} p_2^{x_2} \ldots p_r^{x_r}$$

is said to be the multinomial distribution.

Problem 130. *From an urn containing 7 black and 3 white balls, 5 balls are drawn with replacement; find the frequency function for the number of black balls obtained.*

Let x balls drawn be black out of 5 drawings.

Then
$$f(x) = \frac{5!}{x!(5-x)!} \left(\frac{{}^7C_1}{{}^{10}C_1} \right)^x \left(\frac{{}^3C_1}{{}^{10}C_1} \right)^{5-x}$$

$$= \frac{5!}{x!(5-x)!} \left(\frac{7}{10} \right)^x \left(\frac{3}{10} \right)^{5-x}.$$

[F] Cauchy distribution. This distribution is given by

$$dP = \frac{1}{\pi} \cdot \frac{dx}{1+(x-\mu)^2}, -\infty < x < \infty.$$

Problem 131. *Find the mean of Cauchy distribution.*

We have,

$$\text{Mean} = \mu_1' = \frac{1}{\mu} \int_{-\infty}^{\infty} \frac{x\,dx}{1+(x-\mu)^2}$$

$$= \frac{1}{\mu} \int_{-\infty}^{\infty} \frac{(x-\mu)\,dx}{1+(x-\mu)^2} + \frac{\mu}{\pi} \int_{-\infty}^{\infty} \frac{dx}{1+(x-\mu)^2}$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(x-\mu)\,dx}{1+(x-\mu)^2} + \frac{\mu}{\pi} \left[\tan^{-1}(x-\mu) \right]_{-\infty}^{\infty}$$

$$= \mu + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(x-\mu)\,dx}{1+(x-\mu)^2} \qquad \text{Put } x - \mu = y, dx = dy$$

$$= \mu + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y\,dx}{1+y^2} = \mu + \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{-1/\varepsilon}^{1/\varepsilon} \frac{y\,dx}{1+y^2}$$

$$= \mu + \frac{1}{2\pi} \underset{\varepsilon \to 0}{\text{Lim}} \left[\log \left(1 + y^2\right) \right]_{-1/\varepsilon}^{1/\varepsilon} = \mu + \frac{1}{2\pi} \underset{\varepsilon \to 0}{\text{Lim}} \log \frac{1 + 1/\epsilon^2}{1 + 1/\epsilon^2}$$

$$= \mu + \frac{1}{2\pi} \underset{\varepsilon \to 0}{\text{Lim}} \log 1 = \mu \quad \because \log 1 = 0.$$

15.18. THE PRINCIPLE OF LEAST SQUARES

In tackling a number of problems, there arise a situation in which the values of some physical quantities involved in mathematical theory of observations are not direclty known but we have some functions involving such quantities, then we are required to find the best estimates of these quantities. Such an estimation is done by the method known as *the principle of least squares* which was originally use by Gauss in 1795 and became polular after publishing in 1805 by Legendre.

Let us suppose that there are n independent linear equations in m unknown quantities (variables) $x_1, x_2, x_3, \ldots x_m$ such that

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1m}x_m &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2m}x_m &= b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots + a_{nm}x_m &= b_n \end{aligned}\right\} \quad \ldots(1)$$

where a' s and b's are constants.

In case $n = m$, we can find in general a unique set of values which satisfy the above system of equations. But when $n > m$ *i.e.* number of equations is greater than the number of variable, then there exists no such solution, while in the case when $n < m$, then the *best* or *most plausible* values of the variables can be determined by the principle of least squares according to which these values of unknowns are those which render S a minimum, where

$$S = R_1^2 + R_2^2 + \ldots + R_n^2 = \sum_{\alpha=1}^{n} R_\alpha^2 \quad \ldots(2)$$

$R_1, R_2, \ldots\ldots R_n$ being the *residuals* or *errors* given by

$$R_\alpha = a_{\alpha 1} + x_1 + a_{\alpha 2} x_2 + \ldots\ldots + a_{\alpha m} x_m - b_\alpha \quad \ldots(3)$$

The notion of maxima and minima in differential calculus leads us that S will be a minimum if $\frac{\partial S}{\partial x_\alpha} = 0, \alpha = 1, 2, \ldots m$

i.e.
$$\frac{\partial S}{\partial x_1} = \frac{\partial S}{\partial x_2} = \ldots\ldots = \frac{\partial S}{\partial x_m} = 0 \quad \ldots(4)$$

provided $\frac{\partial^2 S}{\partial x_\alpha^2} > 0$ for every $\alpha = 1, 2, \ldots\ldots, m$.

Note 1. Equations (4) are known as **normal equations.**

Problem 132. *Find the most plausible values of x, y, z from the following equations:*

$x - y + 2z = 3$, $3x + 2y - 5z = 5$, $4x + y + 4z = 21$, $-x + 3y + 3z = 14$.

Here $R_1 = x - y + 2z - 3$; $R_2 = 3x + 2y - 5z - 5$

$R_3 = 4x + y + 4z - 21$; $R_4 = -x + 3y + 3z - 14$ are the residuals.

$$\therefore \quad S = (x - y + 2z - 3)^2 + (3x + 2y - 5z - 5)^2 + (4x + y + 4z - 21)^2 + (-x + 3y + 3z - 14)^2$$

Normal equations given by $\frac{\partial S}{\partial x}=0$, $\frac{\partial S}{\partial y}=0$ and $\frac{\partial S}{\partial z}=0$ are on simplification

$$27x+6y=88,\ 6x+15y+z=70,\ -y+54z=107$$

which give on solving, $x=2.47$, $y=3.55$ and $z=1.92$.

These are the most plausible values

Note 2. *Here it may be shown that* $\frac{\partial^2 S}{\partial x^2}, \frac{\partial^2 S}{\partial y^2}, \frac{\partial^2 S}{\partial z^2}$ *all are positive for these values of* x, y, z.

Curve-fitting by method of least squares

The principle of least squares enables us to fit a polynomial of any degree *i.e.*, to get a close functional relation between the variables x and y. Besides a polynomial, this relationship may be any of algebraic, exponential or logarithmic.

Assuming that we are given n paired observations (x_1, y_1), (x_2, y_2) ......(x_n, y_n), we have to fit a polynomial of degree p such as

$$y=a_0+a_1x+a_2x^2+......+a_px^p \quad ...(5)$$

where constants a's can be determined by the method of least squares.

Taking (5) to be true, the estimated values of y for (x_1, y_1) is $a_0+a_1x_1+a_2x_1^2+......+a_px_1^p$ whereas the observed value is y_1, so that the error of estimation say E_1 is

$$E_1=(y_1-a_0-a_1x_1-a_2x_1^2......-a_px_1^p)$$

Similarly if $E_2, E_3,...., E_n$ be the errors of estimation for the other paired observations, then we have

$$E_2=y_2-a_0-a_1x_2-a_2x_2^2......-a^px_2^p$$

....

$$E_n=y_n-a_0-a_1x_n-a_2x_n^2......-a_px_n^p$$

$\therefore$ S, the sum of squares of the errors of estimation is

$$S=E_1^2+E_2^2+......+E_n^2=\sum_{\alpha=1}^{n} E_\alpha^2$$

$$=\Sigma(y-a_0-a_1x-a_2{}^2......-a_px^p)$$

The normal equations given by

$\frac{\partial S}{\partial a_0}=0$, $\frac{\partial S}{\partial a_1}=0$, $\frac{\partial S}{\partial a_2}=0$......, $\frac{\partial S}{\partial a_n}=0$ are on simplification

$$\left.\begin{aligned}\Sigma y &= na_0+a_1\Sigma x+a_2\Sigma x^2+......+a_p\Sigma x^p\\ \Sigma xy &= a_0\Sigma x+a_1\Sigma x^2+a_2\Sigma x^3+......+a_p\Sigma x^{p+1}\\ \Sigma x^2y &= a_0\Sigma x^2+a_1\Sigma x^3+a_2\Sigma x^4+......+a_p\Sigma x^{p+2}\\ &\cdots\cdots\cdots\cdots\cdots\cdots\\ \Sigma x^py &= a_0\Sigma x^p+a_1\Sigma x^{p+1}+a_2\Sigma x^{p+2}+...+a_p\Sigma x^{2p}\end{aligned}\right\} \quad ...(6)$$

These $(p+1)$ normal equations can uniquely determine the $(p+1)$ constants $a_0, a_1, a_2,...a_p$.

COROLLARY 1. In case the polynomial (5) is of degree 1 *i.e.*, it is a *straight line*, the normal equations are (on putting $p=1$)

$$\left.\begin{aligned}\Sigma y &= na_0 + a_1 \Sigma x \\ \Sigma xy &= a_0 \Sigma x + a_1 \Sigma x^2\end{aligned}\right\} \quad ...(7)$$

COROLLARY 2. In case the polynomial (5) is of degree 2 *i.e.* it is a *parabola* the normal equations (on putting $p = 2$) are

$$\left.\begin{aligned}\Sigma y &= na_0 + a_1 \Sigma x + a_2 \Sigma x^2 \\ \Sigma xy &= a_0 \Sigma x + a_1 \Sigma x^2 + a_2 \Sigma x^3 \\ \Sigma x^2 y &= a_0 \Sigma x^2 + a_1 \Sigma x^3 + a_2 \Sigma x^4\end{aligned}\right\}$$

Note 3. In order to ease the calculations a change of scale and origin may be suggested by putting $u = u = \frac{x - A}{h}$, $v = \frac{y - B}{k}$.

Problem 133 *Fit a straight line to the following data regarding x as the independent variable* -

x :	0	1	2	3	4
y :	0	1.8	3.3	4.5	6.3

We have to fit a straight line say $y = a_0 + a_1 x$ (1)

Normal equations are $\Sigma y = na_0 + a_1 \Sigma x$, $\Sigma xy = a_0 \Sigma x + a_1 \Sigma x^2$ ...(2)

As given,

x :	0	1	2	3	4	$\therefore \Sigma x = 10$
y :	0	1.8	3.3	4.5	6.3	$\therefore \Sigma y = 16.9$
xy :	0	1.8	6.6	13.5	25.2	$\therefore \Sigma xy = 47.1$
x^2 :	0	1	4	9	16	$\therefore \Sigma x^2 = 30$

Substituting these values in (2), we get

$16.9 = 5a_0 + 10a_1$; $47.1 = 10a_0 + 30a_1$

which give on solving $a_0 = 0.72$, $a_1 = 1.33$.

Their substitution in (1) yields the required straight line as

$y = 0.72 + 1.33\, x$.

Problem 134. *Fit a second degree parabola to the following data* :

x :	1	2	3	4	5
y :	1090	1220	139C	1625	1915

Using the change of scale and origin as $u = x - 3$, $y = \frac{y - 1450}{5}$, let the required parabola be $v = a_0 + a_1 u + a_2 v^2$ ...(1)

The normal equations are

$\Sigma v = na_0 + a_1 \Sigma u + a_2 \Sigma u^2$, $\Sigma uv = a_0 \Sigma u + a_1 \Sigma u^2 + a_2 \Sigma u^3$ and

$\Sigma u^2 v = a_0 \Sigma u^2 + a_1 \Sigma u^3 + a_2 \Sigma u^4$ ...(2)

Now,

u :	−2	−1	0	1	2	$\therefore \Sigma u = 0$
v :	−72	−46	−12	35	93	$\therefore \Sigma v = -2$
uv :	144	46	0	35	186	$\therefore \Sigma uv = 411$
u^2 :	4	1	0	1	4	$\therefore \Sigma u^2 = 10$
$u^2 v$:	−288	−46	0	35	372	$\therefore \Sigma u^2 v = 73$
u^3 :	−8	−1	0	1	8	$\therefore \Sigma u^3 = 0$
u^4 :	16	1	0	1	16	$\because \Sigma u^4 = 34$

Substituting these values in (2), we get

$$-2 = 5a + 10a_2;\ 411 = 10a_1;\ 73 = 10a_0 + 34a_2$$

which give on solving $a_0 = -11.4$, $a_1 = 41.1$, $a_2 = 5.5$.

Hence (1) gives $v = -11.4 + 41.1u + 5.5u^2$

i.e., $$\frac{y-1450}{5} = -11.4 + 41.1(x-3) + 5.5(x-3)^2$$

or $$y = 1024 + 40.5x + 27.5x^2.$$

which is the required parabola.

Problem 135. *Fit the curve $y = ae^{bx}$ to the following data, when $e = 2.71828$*

x:	0	2	4
y:	5.012	10	31.62

Equation to the curve is $y = ae^{bx}$

Taking logarithms, $\log_{10} y = \log_{10} a + (b \log_{10} e)x$

say $$y = A + Bx \quad ...(1)$$

Normal equations are

$$\Sigma Y = nA + B\Sigma x,\ \Sigma xY = A\Sigma x + B\Sigma x^2 \quad ...(2)$$

Now,					
x:	0	2	4	So that Σx	$= 6$
y:	5.012	10	31.62		
$Y = \log_{10} y$:	0.7	1.0	1.5	(by log table) ΣY	$= 3.2$
x^2:	0	4	16	Σx^2	$= 20$
xY:	0	2	6	ΣxY	$= 8$

$\therefore$ (2) give $3.2 = 3A + 6B$, $8 = 6A + 20B$

Solving $A = 0.666$, $B = 0.2$.

So that $a = (10)^A = (10)^{0.666} = 4.642$ and $b = \dfrac{B}{\log_{10} e} = 0.46$ approx. (by log table)

Hence the required equation (1) yields

$$y = 4.642\, e^{0.46x}.$$

15.19. CORRELATION AND REGRESSION

The two variables x and y are said to be correlated when they are so related or linked that a change in one is accompanied by a change in other such that an increase in one brings an increase or decrease in the other. In case an increase in one is accompanied by an increase in the other, then the two variables are said to be *positively correlated e.g.* valume and temperature of a gas. But if an increase in one is linked with the decrease in other, then the two variables are said to be *negatively correlated e.g.* pressure and volume of a gas.

In 1890, Karl Pearson defined the '*Product moment correlation coefficient*' or simply '*coefficient of correlation*' between the two variables x and y such that

$$r_{xy} = \frac{E(x-\bar{x})(y-\bar{y})}{\sqrt{E(x-\bar{x})^2}\sqrt{E(y-\bar{y})^2}} = \frac{\text{Cov}(x,y)}{\sqrt{\text{Var}(x)}\sqrt{\text{Var}(y)}} = \frac{\mu_{11}}{\sigma_x\, \sigma_y} \quad ...(1)$$

since $E(x-\bar{x})^2$ gives us a measure for the variation in x and $E(y-\bar{y}^2)^2$ gives that in y while $E(x-\bar{x})(y-\bar{y})$ gives us the measure for simultaneous variation in x and y.

Also cov $(x, y) = E(x-\bar{x})(y-\bar{y}) = \mu_{11}$ and σ_x and σ_y are the standard deviations for x and y respectively.

Properties of r, the Coefficient of Correlation

I. It is purely a number (being ratio) and hence has no unit of measurement.

II. It is independent of the origin and scale.

Suppose that we introduce two variables u and v such that $\bar{v}$

$$u = \frac{x-a}{h}, \quad v = \frac{y-b}{k} \quad \text{...(2)}$$

where (a, b) is the new origin instead of (0, 0).

(2) can be written as $x = a + hu$, $y = b + kv$ giving

$$\bar{x} = a + h\bar{u}, \quad \bar{y} = b + k\bar{v}$$

where u, v are respectively the means of the u and v variates.

Thus, Cov $(x, y) = E(x - \bar{x})(y - \bar{y})$ by §15.11

$$= E\{(a + hu) - (a + h\bar{u})\}\{(b + kv) - (b + k\bar{v})$$

$$= E(hu - h\bar{u})(kv - k\bar{v}) = hk\, E(u - \bar{u})(v - \bar{v})$$

$$\text{Var}(x) = E(x - \bar{x})^2 = E\{(a + hu) - (a + h\bar{u})\}^2$$

$$= (hu - h\bar{u})^2 = h^2 E(u - \bar{u})^2 = h^2 \text{Var}(u)$$

Similarly Var $(y) = k^2 E(v - \bar{v})^2 = k^2$ Var (v)

Hence from (1),

$$r_{xy} = \frac{\text{Cov}(x, y)}{\sqrt{\text{Var}(x)}\sqrt{\text{Var}(y)}} = \frac{hk}{\sqrt{h^2k^2}} = \frac{E(u - \bar{u})(v - \bar{v})}{\sqrt{\text{Var}(u)}\sqrt{\text{Var}(v)}}$$

$$= \frac{hk}{\sqrt{h^2k^2}} r_{uv}$$

In case h and k both are positive or both negative, we get

$$r_{xy} = r_{uv} \quad \text{...(3)}$$

showing that r_{uv} is independent of a, b (*i.e.* change of origin) and of h, k (*i.e.* change of scale).

But if h and k are of opposite sign, then we have $r_{xy} = -r_{uv}$.

III. When x and y are independent *i.e.* uncorrelated then $r_{xy} = 0$

$$\because \quad r_{xy} = \frac{\text{Cov}(x, y)}{\sigma_x \sigma_y} = \frac{E(x - \bar{x})(y - \bar{y})}{\sqrt{E(x - \bar{x})^2}\sqrt{E(y - \bar{y})^2}}$$

But x, y bieng independent, $E(x, y) = E(x) E(y)$ by §15.11.

So that $E(x - \bar{x})(y - \bar{y}) = E(x - \bar{x}) E(y - \bar{y})$

$$= \{E(x) - E(\bar{x})\}\{E(y) - E(\bar{y})\}$$

$$= (\bar{x} - \bar{x})(\bar{y} - \bar{y})$$

$$r_{xy} = 0. \quad \text{...(4)}$$

Note. Its converse is not true i.e. if two variates are uncorrelated, it is not necessary that they are independent too, e.g. if x is a variate with constant density function say

$f(x) = \frac{1}{4}$, $-1 \le x \le 1$ and $y = x^2$, then we have

$$E(x) = \int_{-1}^{1} xf(x)\,dx = \frac{1}{4}\int_{-1}^{1} x\,dx = \frac{1}{4}\left[\frac{x^2}{2}\right]_{-1}^{1} = 0$$

$\therefore \quad E(x).\, E(y) = 0.\, E(y) = 0.$

Also $E(xy) = E(x^3) = \int_{-1}^{1} x^3.f(x)dx = \frac{1}{4}\int_{-1}^{1} x^3\,dx = \frac{1}{4}\left[\frac{x^2}{2}\right]_{-1}^{1}$

$$= 0$$

So that Cov $(x, y) = E(x - \bar{x})(y - \bar{y}) = E(xy) - E(x)E(y) = 0 - 0 = 0$ showing that x and y are uncorrelated but y and x are not independent as is obvious by the relation $y = x^2$.

IV. Limits for the coefficient of correlation are $-1 \le r \le 1$.

Let us assume that,

$$X_i = \frac{x_i - \bar{x}}{\sigma_x} \text{ so that } \sum_{i=1}^{n} X_i^2 = \frac{1}{\sigma_x^2}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{n\sigma_x^2}{\sigma_x^2} = n \quad \text{...(5)}$$

$$\text{and } Y_i = \frac{y_i - \bar{y}}{\sigma_x} \text{ so that } \sum_{i=1}^{n} Y_i^2 = \frac{1}{\sigma_y^2}\sum_{i=1}^{n}(y_i - \bar{y})^2 = \frac{u\sigma_y^2}{\sigma_y^2} = n \quad \text{...(6)}$$

$$\therefore\ r = \frac{E(x-\bar{x})(y-\bar{y})}{\sigma_x\sigma_y} = \frac{1}{n\sigma_x\sigma_y}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$$

$$= \frac{1}{n}\sum_{i=1}^{n} X_iY_i \quad \text{...(7)}$$

Now, since a perfect squared quantity is never negative, therefore

$$\sum_{i=1}^{n}(X_i + Y_i)^2 \ge 0 \text{ i.e. } \frac{1}{n}\sum_{i=1}^{n}(X_i + Y_i)^2 \ge 0$$

or $\quad \frac{1}{n}\sum_{i=1}^{n} X^2 + \frac{1}{n}\Sigma Y_i^2 + \frac{2}{n}\ \Sigma X_iY_i \ge 0$

or $\quad 1 + 1 + 2r \ge 0$ by (5), (6) and (7)

giving $\quad 2r \ge -2$ *i.e.* $r \ge -1$ ...(8)

Again, $\quad \sum_{i=1}^{n}(X_i - Y_i)^2 \ge 0$ and hence $\frac{1}{n}\sum_{i=1}^{n}(X_i - Y_i)^2 \ge 0$

$$\therefore\ \frac{1}{n}\sum_{i=1}^{n} X_i^2\ X_i^2 + \frac{1}{n}\sum_{i=1}^{n} Y_i^2 - \frac{2}{n}\sum_{i=1}^{n} X_iY_i \ge 0$$

i.e. $\quad 1 + 1 - 2r \ge 0$ by (5), (6) and (7)

giving $\quad 2 - 2r \ge 0$ or $2r \le 2$ *i.e.* $r \le 1$ ...(9)

Combining (8) and (9), $\quad -1 \le r \le 1$ ...(10)

Various other Forms of r

If α and β be the deviations of the two variates x and y from their respective means and σ_x, σ_y be the standard deviations of these series, then we define

$$r = \frac{\Sigma\ \alpha\beta}{n\sigma_x\sigma_y} = \frac{p}{\sigma_x\sigma_y} \text{ where } p = \frac{\Sigma\ \alpha\beta}{n} \quad \text{...(11)}$$

Let M_x, M_y be the true means; A_x, A_y be the assumed means for the two series and ξ, η be the deviations from the assumed means *i.e.*

$$\alpha = x - M_x, \beta = y - M_y$$
$$\xi = x - A_x, \eta = y - A_y$$

Also let $d_x = M_x - A_x$ and $d_y = M_y - A_y$.

Then, $\xi = x - A_x = x - M_x + M_x - A_x = \alpha + dx$ and similarly $\eta = \beta + dy$

$\therefore \quad \Sigma\xi = \Sigma\alpha + \Sigma dx = nd_x \because \Sigma\alpha = 0$ by Problem I of Arithmetic-mean

giving $\quad d_x = \dfrac{\Sigma\xi}{n}$ and similarly $d_y = \dfrac{\Sigma\eta}{n}$

Now, $\Sigma\xi\eta = \Sigma(\alpha + d_x)(\beta + d_y) = \Sigma\alpha\beta + d_y\Sigma x + d_x\Sigma y + \Sigma d_x d_y$

$= \Sigma\alpha\beta + \eta d_x d_y \quad \because \quad \Sigma\alpha = 0 = \Sigma\beta$

$$\therefore \quad \Sigma\alpha\beta = \Sigma\xi\eta - nd_x d_y = \Sigma\xi\eta - \frac{1}{n}\,\Sigma\xi\Sigma\eta$$

and $\quad \sigma_x = \sqrt{\dfrac{\Sigma\xi^2}{n} - \left(\dfrac{\Sigma\xi}{n}\right)^2}$ and $\sigma_y = \sqrt{\dfrac{\Sigma\eta^2}{n} - \left(\dfrac{\Sigma\eta}{n}\right)^2}$ by §15.9 [*D*]

With these substitutions (11) yields

$$r = \frac{\Sigma\xi\eta - \frac{1}{n}\,\Sigma\xi\Sigma\eta}{\sqrt{\Sigma\xi^2 - \frac{1}{n}(\Sigma\xi)^2}\,\sqrt{\Sigma\eta^2 - \frac{1}{n}(\Sigma\eta)^2}} \qquad \ldots(12)$$

Further if we use the change of scale and origin such that

$$u = \frac{x - A_x}{h} = \frac{\xi}{h} \text{ and } v = \frac{y - A_y}{k} = \frac{\eta}{k}, \text{ then (12) reduces to}$$

$$r = \frac{\Sigma uv - \frac{1}{n}\,\Sigma u\,\Sigma v}{\sqrt{\Sigma u^2 - \frac{1}{n}(\Sigma u)^2}\,\sqrt{\Sigma v^2 - \frac{1}{n}(\Sigma v)^2}} \qquad \ldots(13)$$

which for a bivariate frequency distribution yields

$$r = \frac{\Sigma fuv - \dfrac{\Sigma u\,\Sigma v}{\Sigma f}}{\sqrt{\Sigma fu^2 - \frac{1}{\Sigma f}(\Sigma fu)^2}\,\sqrt{\Sigma fv^2 - \frac{1}{\Sigma f}(\Sigma fv)^2}} \qquad \ldots(14)$$

where f is the frequency of a particular rectangle in the correlation table, whose mention is not needed in the present volume.

Problem 136. *x and y are two variates with variance σ_x^2 and σ_y^2 respectively and r is the coefficient of correlation between them. If*

$$u = x + ky, \; v = x + \frac{\sigma_x}{\sigma_y}y,$$

find the value of k so that u and v may be uncorrelated.

$\because \quad u$ and v are uncorrelated, Cov $(u, v) = 0$.

$\therefore\quad 0 = \text{Cov}(u, v) = E(u - \bar{u})(v - \bar{v})$

$$= E(x + ky - \overline{\bar{x} + k\bar{y}})\left(x + \frac{\sigma_x}{\sigma_y}y - \overline{\bar{x} + \frac{\sigma_x}{\sigma_y}\bar{y}}\right)$$

$$= E[(x - \bar{x}) + k(y - \bar{y})]\left\{(x - \bar{x}) + \frac{\sigma_x}{\sigma_y}(y - \bar{y})\right\}$$

$$= E(x - \bar{x})^2 + kE(x - \bar{x})(y - \bar{y}) + \frac{\sigma_x}{\sigma_y}E(x - \bar{x})(y - \bar{y}) + k\frac{\sigma_x}{\sigma_y}E(y - \bar{y})^2$$

i.e., $\sigma_x^2 + \left(k + \frac{\sigma_x}{\sigma_y}\right) r\sigma_x\sigma_y + k\frac{\sigma_x}{\sigma_y}\sigma_y^2 = 0$

or $k(-\sigma_x\sigma_y - r\sigma_x\sigma_y) = \sigma_x^2 + r\sigma_x^2$

or $-k\sigma_x\sigma_y(1 + r) = \sigma_x^2(1 + r)$ giving $k = -\frac{\sigma_x}{\sigma_y}$.

Problem 137. *Show that the coefficient of correlation r between two variables x and y is given by*

$$r\ (\sigma_x^2 + \sigma_y^2 - \sigma^2_{x-y})/2\sigma_x\sigma_y,$$

where σ_x^2, σ_y^2 *and* σ^2_{x-y} *are the variances of x, y and x – y respectively.*

We have $\sigma^2_{x-y} = \sigma_x^2 + \sigma_y^2 - 2\,\text{Cov}(x, y)$.

$\therefore\quad \text{Cov}(x, y) = \{\sigma_x^2 + \sigma_y^2 - \sigma^2_{x-y}\}/2.$

$$\therefore\quad r = \frac{\text{Cov}(x, y)}{\sigma_x\sigma_y} = \frac{(\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2)}{2\sigma_x\sigma_y}.$$

Probable Error of r

According to Secrist, 'the probable error of the correlation coefficient is an amount which if added to and subtracted from the mean correlation coefficient, produces amounts within which the chances are even that a coefficient of correlation from a series selected at random will fall'.

Since in a normal distribution $m \pm .6745\,\sigma$ covers 50 per cent of the total area, therefore *Probable Error* (P.E.) is 0.6745 times the *Standard Error* (S.E.) defined as

S.E. of $r = \frac{1 - r^2}{\sqrt{N}}$, ($N$ being total number of observation) ...(15)

$\therefore$ P.E. of $r = 0.6745\ (\text{S.E.}) = 0.6745\,\frac{1 - r^2}{\sqrt{N}}$ ...(16)

The significance of correlation is decided as follows :

If $r <$ P.E., there is no correlation
But if $r < 6$ P.E., there is correlation ...(17)

Actually the two limits within which the coefficient of correlation always lies, are

$r \pm$ P.E. ...(18)

Rank Correlation

Suppose a group of n individuals is ranked according to two characters or attributes A and B (say) such as $x_1, x_2 \ldots x_n$ and $y_1, y_2, \ldots, y_n$ respectively where n ranks for each variable ranges from 1 to n. Then using the notations introduced in this section,

we have $$M_x = M_y = \frac{1+2+\ldots+n}{n} = \frac{n(n+1)}{2.n} = \frac{n+1}{2}$$

So that $$\sigma_x = \sqrt{\frac{1}{n}\Sigma(x - M_x)^2} = \sqrt{\frac{1}{n}\Sigma\left(x - \frac{n+1}{2}\right)^2}$$

$$= \sqrt{\frac{1}{n}\left\{\Sigma x^2 - (n+1)\,\Sigma x + \Sigma\left(\frac{n+1}{2}\right)^2\right\}}$$

Here $\Sigma x^2 = 1^2 + 2^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

and $$\Sigma x = 1 + 2 + \ldots + n = \frac{n}{2}(n+1)$$

$$\therefore \quad \sigma_x = \sqrt{\frac{1}{n}\left\{\frac{n(n+1)(2n+1)}{6} - \frac{n}{2}(n+1)^2 + n\left(\frac{n+1}{2}\right)^2\right\}} = \sqrt{\frac{n^2-1}{12}}.$$

Similarly $$\sigma_y = \sqrt{\frac{n^2-1}{12}}$$

Let $d = \alpha - \beta$, then

$$\Sigma d^2 = \Sigma(\alpha-\beta)^2 = \Sigma\alpha^2 + \Sigma\beta^2 - 2\,\Sigma\,\alpha\beta$$

$$= \Sigma(x - M_x)^2 + \Sigma(y - M_y)^2 - 2\,\Sigma\,\alpha\beta$$

$$= \frac{n(n^2-1)}{12} + \frac{n(n^2-1)}{12} - 2\,\Sigma\,\alpha\beta \text{ solving as above}$$

giving $$\Sigma\,\alpha\beta = \frac{1}{12}\,n(n^2-1) - \tfrac{1}{2}\,\Sigma d^2.$$

Hence by (11), we have

$$r = 1 - \frac{6\Sigma d^2}{n(n^2-1)} \qquad \ldots(19)$$

where $$d = \alpha - \beta = \left(x - \frac{n+1}{2}\right) - \left(y - \frac{n+1}{2}\right) = x - y \qquad \ldots(20)$$

Problem 138. *Ten students got the following percentage of marks in Principles of Economics and Statistics*

Students :	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Marks in Economics :	*78*	*36*	*98*	*25*	*75*	*82*	*90*	*62*	*65*	*39*
Marks in Statistics :	*84*	*51*	*91*	*60*	*68*	*62*	*86*	*58*	*53*	*47*

Calculate rank correlation coefficient.

Denoting by x and y the ranks in Economics and Statistics respectively, we have

Students :	1	2	3	4	5	6	7	8	9	10	Total
x :	4	9	1	10	5	3	2	7	6	8	
y :	3	9	1	6	4	5	2	7	8	10	
$\lvert x-y \rvert = \lvert d \rvert$:	1	0	0	4	1	2	0	0	2	2	
d^2 :	1	0	0	16	1	4	0	0	4	4	30

Putting $\Sigma d^2 = 30$ and $n = 10$ in (19), we find

$$r = 1 - \frac{6 \times 30}{10\,(100-1)} = 1 - \frac{2}{11} = \frac{9}{11} = .82 \text{ approx.}$$

Regression and Lines of Regression

Assuming that there exists an association or relationship between two variables x and y for which n paird values $(x_1, y_1), (x_2, y_2)\ldots\ldots (x_n, y_n)$ are observed, if we plot these points on a graph paper with proper choice of scale then these points are found more or less concentrated round a curve termed as *curve of regression* and the relationship is said to be expressed by means of *curvilinear regression*. In case the curve is a straight line, then it is known as the *line of regression* and the regression is known to be linear. In fact the line of regression gives the best fit or best estimate in the least square sense to an assigned probability distribution.

Whenever such a straight line falls to the choice that sum of squares of deviations parallel to y-axis is minimum then it is called *the line of regression of y on x* which gives the best estimate of y for any assigned value of x. Similarly if the sum of squares of deviations parallel to x-axis is minimum, then the line is called the *line of regression of x on y* which gives the best estimate of x for given y.

Let the form of the line of best fit of y for given x be $y = ax + b$ and let $\bar{x}$, $\bar{y}$ be the means of two series.

Also let $R_1, R_2, \ldots\ldots, R_n$ be the residuals for the points $(x_1, y_1), (x_2, y_2), \ldots\ldots (x_n, y_n)$ with frequencies $f_1, f_2, \ldots, f_n$ respectively, then we have $R_i = f_i\,(y_i - ax_i - b)$, so that

$$S = \sum_{i=1}^{n} f_i\,(y_i - ax_i - b).$$

Applying the principle of least squares, S will be a minimum

if $\dfrac{\partial S}{\partial a} = 0,\ \dfrac{\partial S}{\partial b} = 0$ *i.e.* if $\Sigma f_i x_i\,(y_i - ax_i - b) = 0$;

$$\Sigma f_i\,(y_i - ax_i - b) = 0$$

where $\Sigma f_i = N$, $\Sigma f_i x_i = N\bar{x}$, $\Sigma f_i y_i = \bar{N}\, N\bar{y}$, $\Sigma f_i x_i^2 = N(\sigma_x^2 + \bar{x}^2)$

$\Sigma f_i y_i^2 = N(\sigma_y^2 + \bar{y}^2)$ and $\Sigma f_i x_i y_i = N(\mu_{11} + \bar{x}\bar{y})$ by §15.10 (*E*).

$$\therefore\ \bar{x}\bar{y} + \mu_{11} = a\left(\bar{x}^2 + \sigma_x^2\right) + b\bar{x} \text{ and } \bar{y} = a\bar{x} + b.$$

Solving these, we get $a = \dfrac{\mu_{11}}{\sigma_x^2}$, $b = \bar{y} - \dfrac{\mu_{11}}{\sigma_x^2}\bar{x}$.

Putting these values in $y = ax + b$, the line of regression of y on x is

$$y - \bar{y} = \frac{\mu_{11}}{\sigma_x^2}(x - \bar{x}) \qquad \ldots(21)$$

Similarly the line of regression of x on y is

$$x - \bar{x} = \frac{\mu_{11}}{\sigma_y^2}(y - \bar{y}) \qquad ...(22)$$

Here the coefficients $\frac{\mu_{11}}{\sigma_x^2}$ and $\frac{\mu_{11}}{\sigma_y^2}$ are known as the *regression coefficients* of y on x and of x on y respectively and denoted by

$$b_{yx} = \frac{\mu_{11}}{\sigma_x^2},\ b_{xy} = \frac{\mu_{11}}{\sigma_y^2} \qquad ...(23)$$

In view of (1), we have $\mu_{11} = r\sigma_x\sigma_y$

$$\therefore \quad b_{yx} = r\frac{\sigma_y}{\sigma_x} \text{ and } b_{xy} = \frac{\sigma_x}{\sigma_y} \qquad ...(24)$$

Clearly $b_{yx}.b_{xy} = r^2$ ...(25)

which shows that the correlation-coefficient r is the geometric mean between the regression coefficients.

Using the change of scale and origin such that

$$u = \frac{x-a}{h} \text{ and } v = \frac{y-b}{k}, \text{ we may have}$$

$$b_{yx} = \frac{k}{h}\, b_{vu} \text{ and } b_{xy} = \frac{k}{h} b_{uv} \text{ whence } \sigma_x = h\sigma_u \text{ etc.} \qquad ...(26)$$

Problem 139. *Show that θ, the acute angle between two lines of regression, is given by*

$$\tan\theta = \frac{1-r^2}{r}\frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}$$

Interpret the case when $r = 0$, $r = \pm 1$.

$\because$ Regression line of y on x is $y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$

Its slope $= r\frac{\sigma_y}{\sigma_x} = m_1$ (say)

And regression line of x on y is $x - \bar{x} = r\frac{\sigma_x}{r\sigma_y}(y - \bar{y})$.

Its slope $= \frac{\sigma_y}{r\sigma_x} = m_2$ (say)

$$\therefore \quad \tan\theta = \pm\frac{m_1 - m_2}{1 + m_1 m_2} = \pm\frac{r\frac{\sigma_y}{\sigma_x} \sim \frac{\sigma_y}{r\sigma_x}}{1 + r\frac{\sigma_y}{\sigma_x}\cdot\frac{\sigma_y}{r\sigma_x}} = \pm\frac{r^2\sigma_x\sigma_y \sim \sigma_x\sigma_y}{r\sigma_x^2 + r\sigma_y^2}$$

For θ, acute only, $\tan\theta = +\dfrac{\sigma_x\sigma_y(r^2 \sim 1)}{(\sigma_x^2+\sigma_y^2)r} = \dfrac{r^2 \sim 1}{r}\,\dfrac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}$

or $\tan\theta = \dfrac{1-r^2}{r}\,\dfrac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}$ {$\because r^2 \le 1$, hence $1-r^2$ is also positive}.

Case I. When $r = 0$, $\tan\theta = \infty$, $\therefore \theta = 90°$, *i.e.*, both regression lines are mutually perpendicular. Thus estimated value of x (or y) will be the same for all values of y(or x).

Case II. When $r = \pm 1$, $\tan 0 = 0$. $\therefore \theta = 0$ or π, *i.e.*, both regression lies coincide and so there is perfect correlation (either positive when $r = 1$ or negative when $r = -1$) between the variables involved.

Problem 140. *Given the following data, find what will be (a) the height of a policeman whose weight is 200 lbs., (b) the weight of a policeman who is 5 ft. tall.*

Average height = 68 inches, average weight 150 lbs.

Coefficient of correlation between height and weight = .6.

S.D. of heights = 2.5 inches.

S.D. of weights = 20 lbs.

Suppose variable x indicates height (in inches) and y indicates weight (in lbs). Then we have

$\bar{x} = 68$ inches, $\bar{y} = 150$ lbs., $\sigma_x = 2.5$ inches, $\sigma_y = 20$ lbs., $r_{xy} = .6$.

Now, we want to estimate y for $x = 5$ ft. or for 60 inches and x for $y = 200$ lbs.

(*i*) $\because$ Regression line of y on x is $y - \bar{y} = r\dfrac{\sigma_y}{\sigma_y}(x - \bar{x})$

$$\therefore \quad y - 150 = .6 \times \frac{20}{2.5}(x - 68).$$

or $$y - 150 = 4.8\,(x - 68). \quad \ldots(1)$$

(*ii*) $\because$ Regression line of x on y is $x - \bar{x} = r\dfrac{\sigma_x}{\sigma_y}(y - \bar{y})$

$$\therefore \quad x - 68 = .6\,\frac{2.5}{20}\,(y - 150)$$

or $$x - 68 = .75\,(y - 150). \quad \ldots(2)$$

(*a*) Since (1) is the regression line of y on x so we shall estimate y.

Putting $x = 60$ in (1), we get $y = 150 + 4.8\,(60 - 68)$

$= 150 - 38.4 = 111.6$ lbs.

Thus policeman of height 5 feet will have 112.6 lbs. weight.

(*b*) Since (2) is the regression line of x on y so we shall estimate x.

Putting $y = 200$ in (2), we get $x = 68 + .75\,(200 - 150)$

$= 68 + .75\,(50) = 105.5$ inches.

15.20. SAMPLING DISTRIBUTIONS

A *Population* or *Universe* is defined as the collection or class or aggregate of objects or a set of results of an operation. It is *finite* if the number of objects is finite and it is *infinite* if the number of objects is infinite.

A *sample* is defined as a part of the population selected from it in order to study its properties. The sample is to be *random* when the selection is made at random. A unit of sample or a sampling unit included in the sample is known as *sample unit.* The difference between a value of the population and its estimate derived from sample is said to be a *sampling error,* which may arise due to human bias or due to some other reasons. Actually the sampling procedure is based on the theory of probability as every sampling unit has a definite probability of being included in the sample.

A *simple random sampling* is one in which every sampling unit has an equal chance of being included in the samples *e.g.* if N be the size of a population, then ${}^NC_n$ samples of size n are possible so that each sample has a chance $\frac{1}{{}^NC_n}$ of being selected. Also the probability of some rth unit being selected at rth draw is

$$\frac{N-1}{N}\cdot\frac{N-2}{N-1}\cdots\cdots\frac{N-(r-1)}{N-(r-1)+1}\cdot\frac{1}{N-(r-1)}=\frac{1}{N}.$$

Properties of random sampling

I. *The sample mean is an unbiased estimate of population mean.* Taking X_1, $X_2, \ldots X_n$ as the population values and $x_1, x_2, x_n, \ldots\ldots$ as the sample values, we have

$$\text{Population mean } x = \frac{1}{n}\sum_{i=1}^{N} X_i$$

$$\text{and sample mean} \quad \bar{X} = \frac{1}{N}\sum_{i=1}^{n} x_i \qquad \ldots(1)$$

Let the population total $\sum_{i=1}^{N} X_i = X$ and the sample total

$$\sum_{i=1}^{n} x_i = x \qquad \ldots(2)$$

Then we have to prove that $E(\bar{x}) = \bar{X}$ ...(3)

$$\text{Now, } E(\bar{x}) = E\left\{\frac{1}{n}(x_1 + x_2 + \ldots + x_n)\right\} = \frac{1}{n}E(x_1 + x_2 + \ldots + x_n)$$

$$= \frac{1}{n}\{E(x_1) + E(x_2) + \ldots + E(x_n)\} \text{ by (4) of §15.11 (C).} \qquad \ldots(4)$$

Since x_i can be either $X_1, X_2, \ldots X_n$ each with probability $\frac{1}{N}$, therefore

$$E(x_i) = \frac{X_1}{N} + \frac{X_2}{N} + \ldots + \frac{X_n}{N} = \sum_{i=1}^{N}\frac{X_i}{N} = \bar{X} \text{ by (1)}$$

But $\bar{X}$ being a term free from index i,

$\therefore \quad E(x_1) = E(x_2) = \ldots E(x_n) = \bar{X}$

Thus (4) gives, $E(\bar{x}) = \frac{1}{n}\,[\bar{X} + \bar{X} + \ldots n \text{ terms}] = \frac{1}{n}.n\bar{X} = \bar{X}$ which proves (3).

COROLLARY 1. we have $E(N\bar{x}) = NE(\bar{x}) = N\bar{X} = x$ by (1) and (2) ...(5)

This follows that $N\bar{x}$ is an unbiased estimate of X and we denote it by

$$\hat{X} = N\bar{x} \qquad \ldots(6)$$

II. *The variance of the sample mean $\bar{x}$ from a simple random sample is*

$$Var\,(\bar{x}) = E\left(\bar{x} - \bar{X}\right)^2 = \frac{S^2}{n}\frac{N-n}{N} \qquad \ldots(7)$$

where
$$S^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(X_i - \bar{X}\right)^2 \qquad \ldots(8)$$

We have
$$\text{Var}\,(\bar{x}) = E\{\bar{x} - E(\bar{x})\}^2 = \left(\bar{x} - \bar{X}\right)^2$$

$$= E\left\{\frac{1}{n}\left(x_1 + x_2 + \ldots + x_n\right) - \bar{X}\right\}^2$$

$$= \frac{1}{n^2}E\left\{\left(x_1 + x_2 + \ldots + x_n\right) - n\bar{X}\right\}^2$$

$$= \frac{1}{n^2}\left[\left\{E\left(x_1 - \bar{X}\right)^2 + E\left(x_2 - \bar{X}\right)^2 + \ldots + E\left(x_n - \bar{X}\right)^2\right\}\right.$$

$$\left. + 2\sum_{i=1}^{n-1}\sum_{\substack{j=2 \\ i<j}}^{n} E\left(x_i - \bar{X}\right)\left(x_j - \bar{X}\right)\right] \qquad \ldots(9)$$

But $E\left(x_i - \bar{X}\right)^2 = \frac{1}{N}\left(X_1 - \bar{X}\right)^2 + \frac{1}{N}\left(X_2 - \bar{X}\right)^2 + \ldots + \frac{1}{N}\left(N_N - \bar{X}\right)$ as in I.

$$= \frac{1}{N}\sum_{i=1}^{N}\left(x_i - \bar{X}\right)^2 = \frac{1}{N}(N-1)S^2 \text{ by (8)}$$

and $E\{(x_i - \bar{X})(x_j - \bar{X})\}$

$$= \frac{1}{N}\cdot\frac{1}{N-1}\sum_{i=1}^{N-1}\sum_{\substack{j=2 \\ i<j}}^{N}\left(X_i - \bar{X}\right)\left(X_j - \bar{X}\right)$$

so that
$$\sum_{i=1}^{n-1}\sum_{\substack{j=2 \\ i<j}}^{n} E\left\{\left(x_i - \bar{X}\right)\left(x_j - \bar{X}\right)\right\}$$

$$= \frac{n(n-1)}{N(N-1)}\sum_{i=1}^{N-1}\sum_{\substack{j=2 \\ i<j}}^{N}\left(X_i - \bar{X}\right)\left(X_j - \bar{X}\right)$$

Hence (9) becomes

$$\text{Var}(\bar{x}) = \frac{1}{n}\left\{n.\ \frac{N-1}{N}S^2 + \frac{2n(n-1)}{N(N-1)}\sum_{i=1}^{N-1}\sum_{\substack{j=2\\i<j}}^{N}(X_i-\bar{X})(X_j-\bar{X})\right\} \quad ...(10)$$

But $N\bar{X} = \sum_{i=1}^{N} X_i$ gives $(X_1-\bar{X})+(X_2-\bar{X})+...+(X_n-\bar{X})=0$

or $\{(X_1-\bar{X})+(X_2-\bar{X})+...+(X_n-\bar{X})^2=0$

or $\sum_{i=1}^{N}(X_i-X)^2 + 2\sum_{i=1}^{N-1}\sum_{\substack{j=2\\i<j}}^{N}(X_i-\bar{X})(X_j-\bar{X})=0$

i.e. $2\sum_{i=1}^{N-1}\sum_{\substack{j=2\\i<j}}^{N}(x_i-\bar{X})(X_j-\bar{X}) = -\sum_{i=1}^{N}(X_i-\bar{X})^2 = -(N-1)S^2$

With its substitution, (10) yields

$$\text{Var}\ (\bar{x}) = \frac{1}{n^2}\left[n\frac{N-1}{N}S^2 - \frac{n(n-1)}{N(N-1)}(N-1)S^2\right] = \frac{N-n}{Nn}S^2$$

which proves the result (7)

COROLLARY 2. With the notation of Cor. 1, we have

$$\text{Var}\ (\hat{X}) = \text{Var}\ (N\bar{x}) = N^2\ \text{Var}\ (\bar{x}) = \frac{N(N-n)}{n}S^2 \quad ...(11)$$

COROLLARY 3. By (7), $\sigma_{\bar{x}}^2 = \frac{N-n}{N}\cdot\frac{S^2}{n}$

gives

$$\sigma_{\bar{x}} = \frac{S}{\sqrt{n}}\sqrt{\frac{N-n}{N}} \quad ...(12)$$

which gives standard error of $\bar{x}$.

COROLLARY 4. When $N \to \infty$ *i.e.*, the population becomes to be infinite, then

$$\lim_{N\to\infty}\frac{N-n}{N} = \lim_{N\to\infty}\left(1-\frac{n}{N}\right) = 1 \text{ and } \lim_{N\to\infty}\sqrt{\frac{N-n}{N}}$$

$$= \lim_{N\to\infty}\sqrt{1-\frac{n}{N}} = 1.$$

$\therefore$ by (7) or (12), $\text{Var}\ (\bar{x}) = \frac{S^2}{n}$ *i.e.*, $\sigma_{\bar{x}} = \frac{S}{\sqrt{n}}$ ...(13)

COROLLARY 5. When $N \to \infty$, then $N-1 \approx N$.

So that $\frac{1}{N-1}\sum_{i=1}^{N}(X_i-\bar{X})^2 \approx \frac{1}{N}\sum_{i=1}^{N}(X_i-\bar{X})^2$ *i.e.*, $S^2 \approx \sigma^2$ or $S \approx \sigma$

Hence by (13), $\sigma_{\bar{x}} = \dfrac{\sigma}{\sqrt{n}}$ ...(14)

which gives the standard error of $\bar{x}$, the sample mean.

III. *Taking s^2 and S^2, as the sample variance and population variance respectively, s^2 is an unbiased estimate of S^2 i.e. $E(s^2) = S^2$.*

We have, $s^2 = \dfrac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$ and $S^2 = \dfrac{1}{N-1} \sum_{i=1}^{N} (X_i - \bar{X})^2$

$$\therefore \quad s^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left\{ (x_i - \bar{X}) - (\bar{x} - \bar{X}) \right\}^2 = \frac{1}{n-1} \left\{ \sum_{i=1}^{n} (x_i - \bar{X})^2 + n(\bar{x} - \bar{X})^2 - 2(\bar{x} - \bar{X}) \sum_{i=1}^{n} (x_i - \bar{X}) \right\}$$

$$= \frac{1}{n-1} \left\{ \sum_{i=1}^{n} (x_i - \bar{X})^2 + n(\bar{x} - \bar{X})^2 - 2(\bar{x} - \bar{X}).n(\bar{x} - \bar{X}) \right\}$$

$$= \frac{1}{n-1} \left\{ \sum_{i=1}^{n} (x_i - \bar{X})^2 - n(\bar{x} - \bar{X})^2 \right\}$$

so that $$E(s^2) = \frac{1}{n-1} E\left\{ \sum_{i=1}^{n} (x_i - \bar{X})^2 - n(\bar{x} - \bar{X})^2 \right\}$$

$$= \frac{1}{n-1} \left\{ \sum_{i=1}^{n} E(x_i - \bar{X})^2 - nE(\bar{x} - \bar{X})^2 \right\}$$

$$= \frac{1}{n-1} \left\{ n \sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{N} - n \text{ var } (\bar{x}) \right\}$$

$$\because E(x_i - \bar{X})^2 = \sum_{i=1}^{n} \frac{1}{N} (X_i - \bar{X})^2 \quad \forall i$$

$$= \frac{1}{n-1} \left[\frac{n}{N} \cdot (N-1) S^2 - n \cdot \frac{N-n}{\sqrt{n}} S^2 \right] \text{ by II}$$

$$= S^2 \qquad ...(15)$$

COROLLARY 6. Unbiased estimate of the variance of $\bar{x}$ is denoted by $v(\bar{x})$ and given as $v(\bar{x}) = \dfrac{N-n}{N} . \dfrac{s^2}{n}$ ...(16)

In limit $v(\bar{x}) = \dfrac{s^2}{n}$ ...(17)

Similarly unbiased estimate of the variance of $\hat{X}$ denoted by $v(\hat{X})$ is given as

$$v(\hat{X}) = \frac{N(N-n)}{n} s^2 \qquad ...(18)$$

Defference between the means of two large samples

Let there be two random samples of size n_1 and n_2 respectively, taken from the same population of standard deviation σ. Assuming that the samples are independent, the standard error say ϵ of the difference of their means is given by

$$\epsilon^2 = \epsilon_1^2 + \epsilon_2^2 \quad \text{...(19)}$$

where ϵ_1, ϵ_2 are the standard errors of the means of two samples given by

$$\epsilon_1^2 = \frac{\sigma^2}{n_1} \text{ and } \epsilon_2^2 = \frac{\sigma^2}{n_2}$$

Hence
$$\epsilon^2 = \sigma^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right) \quad \text{...(20)}$$

COROLLARY 7. In case the samples are drawn from two different populations with standard deviations σ_1, and σ_2 respectively, then

(20) yields
$$\epsilon^2 = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2} \quad \text{...(21)}$$

COROLLARY 8. The values $\bar{x} \mp 1.96\frac{\sigma}{\sqrt{n}}$ are known as 95 per cent *Fiducial limits* or *confidence limits* for the mean of the population from which the sample has been taken.

Problem 141. *Assuming that N is large, show that the error in writing* $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$ *is approximately* $\frac{50(n-1)}{N}$ *per cent of the value of* $\sigma_{\bar{x}}$.

$$\text{Reqd. error} = \left\{\frac{\sigma}{\sqrt{n}} - \frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}\right\} \Big/ \frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}$$

$$= \sqrt{\frac{N-1}{N-n}} - 1 = \left\{1 + \frac{n-1}{N-n}\right\}^{1/2} - 1$$

$$= 1 + \frac{1}{2}\cdot\frac{n-1}{N-n} - \frac{1.3}{2.4}\left(\frac{n-1}{N-n}\right)^2 + \ldots - 1$$

$$= \frac{n-1}{2(N-n)} \text{ approx} \approx \frac{n-1}{2N}$$

$$\therefore \text{ Percentage of error} = \frac{n-1}{2N}.100 = \frac{n-1}{N}.50 \text{ approx.}$$

15.21. THEORY OF ERRORS

We know that an *error* is a quantity which must be added to the true value in order to get the observed value in performing a physical experiment *i.e.*

$$\text{Observed value} = \text{True value} + \text{error} \quad \text{...(1)}$$

or $\quad$ Observed value − error = True value

which is generally expressed by saying that

$$\text{Observed value} + \text{correction} = \text{True value} \quad \text{...(2)}$$

It means that a *correction* is the error with reversed sign.

In testing a statistical hypothesis, there arise two types of possible errors : (1) by rejecting the hypothesis when we ought to accept it *i.e.* when it is true and (2) by accepting the hypothesis when we ought to reject it *i.e.* when it is false. On the other hand in order to understand what we mean by true value of a physical quantity we first estimate the true values from observations (measurements), then try to seek what certainty can be attributed to these estimates and finally compare estimates found from different sets of observations.

Since any number formed empirically by expressing it in units of the smallest possible unit say ϵ that can be measured by a physical apparatus, is an integer, therefore any real number lying in an interval of length ϵ may be termed as *true value of the quantity e.g.*, probable values or probabilities may be regarded as the true values of the relative frequencies. Since no measuring instrument is perfect and also there arise several other disturbing factors, every observation is liable to be a random variable as different observations are capable of yielding different results. We thus conclude that all such observations made are full of errors and deviate from the true values to a certain degree of accuracy. We may classify errors into three types : (1) *Coarse* or *gross errors*, (2) *systematic errors* and (3) *random* or *statistical errors*. The coarse errors are mainly caused by the mishandling of the apparatus due to carelessness of an observer. The systematic errors are caused in a certain direction due to one or more reasons governed by a definite rule. Consequently in a repeated set of observations made under constant conditions, the same systematic errors are bound to occur. Such errors can be rectified by knowing the governing rules of these errors. The **Random Errors** are the uncertainties not showing any irregularities and having equal chances for positive and negative values for such errors.

In order to find the best estimated values for an observed quantity, let us assume that t_1 and t_2 are two independent observations for an unknown physical quantity t and $\phi(t_1, t_2)$ is the best estimate for t. If t_1, t_2 are increased by an amount α, then it amy be assumed that the estimate $\phi(t_1, t_2)$ is also increased by α *i.e.*

$$\phi(t_1 + \alpha, t_2 + \alpha) = \phi(t_1, t_2) + \alpha \qquad \ldots(3)$$

Also assuming that multiplication of t_1, t_2 by a constant β, results in multiplication of the estimate by β we have

$$\phi(\beta t_1, \beta t_2) = \beta\phi(t_1, t_2) \qquad \ldots(4)$$

Again the two observations being made under the same conditions, it is immaterial which observation yields t, and which one t_2, we have

$$\phi(t_1, t_2) = \phi(t_2, t_1) \qquad \ldots(5)$$

Now keeping t_1, t_2 fixed and setting $\alpha = -t_2$, (3) gives

$$\phi(t_1 - t_2, 0) = \phi(t_1, t_2) - t_2 \text{ i.e. } \phi(t_1, t_2) = \phi(t_1 - t_2, 0) + t_2 \qquad \ldots(6)$$

with its substitution (4) yields

$$\phi(\beta t_1, \beta t_2) = \beta\phi(t_1 - t_2, 0) + \beta t_2 = \beta t_2 + \phi\{\beta(t_1 - t_2), 0\} \text{ by (4)}$$

or $\beta\phi(t_1, t_2) = \beta t_2 + \phi\{\beta(t_1 - t_2), 0\}$ by (4) ...(7)

If we now assume that $\beta = \dfrac{1}{t_1 - t_2}$ and $t_1 \neq$ then (7) becomes

$$\frac{1}{t_1 - t_2}\phi(t_1, t_2) = \frac{t_2}{t_1 - t_2} + (1, 0)$$

giving $$t_2 + (t_1 - t_2)\,\phi(1, 0) = \phi(t_1, t_2) \qquad \ldots(8)$$

Proceeding similarly, we can find

$$t_1 + (t_2 - t_1)\,\phi\,(1, 0) = \phi\,(t_2, t_1) \qquad ...(9)$$

In view of (5), (8) and (9) yield

$$t_2 + (t_1 - t_2)\,\phi\,(1, 0) = t_1 + (t_2 - t_1)\,\phi\,(1, 0) \text{ giving } \phi\,(1, 0) = \tfrac{1}{2}$$

$$\therefore \text{ (8) gives } t_2 + \tfrac{1}{2}\,(t_1 - t_2) = \phi\,(t_1, t_2) \text{ i.e. } \phi\,(t_1, t_2) = \tfrac{1}{2}\,(t_1 + t_2) \qquad ...(10)$$

Had we taken $t_1 = t_2$, (6) would have given

$$\phi(t_1, t_1) = t_1 + \phi(0, 0) \qquad ...(11)$$

with the choice $\beta = 0$, (8) would give $\phi\,(0, 0)$ ...(12)

So that in view of (12), (11) gives $\phi\,(t_1, t_1) = t_1 = \frac{1}{2}(t_1 + t_1)$ which is the same result as (10).

Now we know that the best estimate of the true value say v made by n observation $t_1, t_2 \ldots t_n$ is defined as

$$m \approx v = \bar{t} = \frac{t_1 + t_2 + \ldots + t_n}{n} = \frac{[t]}{n} \qquad ...(13)$$

where m is the expected values of t and $[t] = \sum_{i=1}^{n} t_i$.

Now we define, true error as $x_i = t_i - v,\ i = 1, 2 \ldots\ldots n$ ...(14)

and best errors as $\epsilon_i = t_1 - t,\ i = 1, 2 \ldots\ldots n$ ...(15)

Further to calculate the probability of best estimate to lie within an assigned range of true values take $f(x_i)$ as the probability density of the random variable x_i, $i = 1, 2, \ldots n$. By the theorem of compound probability, we have

$$f(x_1, x_2) = f(x_1)\,f(x_2) \text{ and } f(x_1, x_2, x_3) = f(x_1).\,f(x_2).\,f(x_3) \qquad ...(16)$$

where x_1, x_2, x_3 etc. are errors made in first, second and third etc. experiments.

Thus true value of observed quantity i.e. $v = \frac{1}{3}\,(t_1 + t_2 + t_3)$ ...(7)

in case of three observations only.

As such $f(x_1, x_2, x_3) = f(x_1).\,f(x_2).\,f(x_3)$

$= f(t_1 - v).\,f(t_2 - v).\,f(t_3 - v)$ is maximum and so its logarithm will also be maximum i.e.

$\log f(t_1 - v) + \log f(t_2 - v) + \log\,(t_3 - v)$ is maximum.

For maximum, its differential w.r.t. v must be equated to zero i.e.,

$$\frac{f'(t_1 - v)}{f(t_1 - v)} + \frac{f'(t_2 - v)}{f(t_2 - v)} + \frac{f'(t_3 - v)}{f(t_3 - v)} = 0 \qquad ...(18)$$

Setting $F(x) = \dfrac{f'(x)}{f(x)}$, we have $F(x_1) = F(t_1 - v) = \dfrac{f'(t_1 - v)}{f(t_1 - v)}$

and similarly $F(x_2) = \dfrac{f'(t_2 - v)}{f(t_2 - v)},\ F(x_3) = \dfrac{f'(t_3 - v)}{f(t_3 - v)}$

$$\therefore \text{ (18) gives } F(x_1) + F(x_2) + F(x_3) = 0 \qquad ...(19)$$

which is true when (17) is true *i.e.* $3v = t_1 + t_2 + t_3$

or $\quad (t_1 - v) + (t_2 - v) + (t_3 - v) = 0$ or $x_1 + x_2 + x_3 = 0$ ...(20)

In case of two variables x_1 and x_2; (9) and (20) reduce to

$$F(x_1) + F(x_2) = 0 \text{ provided } x_1 + x_2 = 0 \qquad ...(21)$$

In case of a single variable x_1; this condition, reduces to

$$F(x_1) = 0 \text{ when } x_1 = 0 \qquad ...(22)$$

Setting $x_1 = x_3$ and $x_2 = -x_3$ in (21), we find

$$F(x_3) = -F(-x_3) \qquad ...(23)$$

So that (19) yields,

$$F(x_1) + F(x_2) = -F(x_3) = F(-x_3) \text{ by (23)}$$

$$= F(x_1 + x_2) \text{ by (20)} \qquad ...(24)$$

Partial Differentiation of (24) w.r.t. x_1 and x_2 in succession gives

$$F'(x_1) = F'(x_1 + x_2) \text{ and } F'(x_2) = F'(x_1 + x_2)$$

giving $\quad F'(x_1) = F'(x_2)$

Treating x_2 as constant, $F'(x_1) = F'(x_1 + x_2)$ will only hold if

$F'(x_1) = \text{constant} = k'(\text{say})$, so that

$$F(x_1) = k'x_1 = \frac{f'(x_1)}{f(x)} \text{ implying } \frac{f'(x)}{f(x)} = F(x) = kx$$

which gives on integration $f(x) = A\, e^{\frac{1}{2}kx^2}$, A being constant of integration.

Now $f(x)$ being probability density function, we have by §15.12 (A),

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 = A\int_{-\infty}^{\infty} e^{\frac{1}{2}kx^2}\,dx$$

Setting $k = -2h^2$, we have $1 = A\int_{-\infty}^{\infty} e^{-h^2x^2}dx$ Put $h^2x^2 = \frac{p^2}{2}$ *i.e.* $dx = \frac{1}{\sqrt{2}\,h}dp$

or $$1 = \frac{A}{h\sqrt{2}}\int_{-\infty}^{\infty} e^{-p^2/2}dp$$

$$= \frac{A}{h\sqrt{2}}\sqrt{2\pi} \text{ i.e. } A = \frac{h}{\sqrt{\pi}}.$$

Hence $$f(x) = \frac{h}{\sqrt{\pi}}e^{-h^2x^2} \qquad ...(25)$$

which is called *Gaussian law of error*, and asserts that the probability for an error x to be $p_1 < \sqrt{2}\,hx < p_2$ is given by

$$\int_{p_1/\sqrt{2}h}^{p_2/\sqrt{2}h} \frac{h}{\sqrt{\pi}}e^{-h^2x^2}dx \qquad ...(26)$$

with $p = \sqrt{2}\,hx$, this yields $\frac{1}{\sqrt{2\pi}}\int_{p_1}^{p_2} e^{-p^2/2}\,dp = \phi(p_2) - \phi(p_1)$ ...(27)

where $$\phi(p) = \frac{1}{\sqrt{2\pi}}\int_0^p e^{-p^2/2}\,dp.$$

Here h is called *precision constant* and it measures the accuracy of the observer. An error with probability $\frac{1}{2}$ is known as the *probable error* and may be determined by the value of x given by

$$\frac{h}{\sqrt{\pi}} e^{-h^2 x^2} = \frac{1}{2} \text{ i.e. } x = \frac{0.4769}{h} \qquad \ldots(28)$$

This renders the most probable error.

We define the *absolute error* $E(|x|)$ as

$$E(|x|) = \int_{-\infty}^{\infty} (x)\ f|x|dx = \frac{2h}{\sqrt{\pi}} \int_0^{\infty} x e^{-h^2 x^2} dx = \frac{1}{h\sqrt{\pi}}$$

$$= \frac{0.5642}{h} \qquad \ldots(29)$$

and the *mean square error* $E(x^2)$ as

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x) dx = \frac{2h}{\sqrt{\pi}} \int_0^{\infty} x^2 e^{-h^2 x^2} dx = \frac{1}{2h^2} = \frac{0.5}{h^2} \qquad \ldots(30)$$

Lastly to discuss the effect of increasing the number of observations n, we have

$$x_i = t_i - v \text{ giving } \bar{x} = \bar{t} - v \qquad \ldots(31)$$

where $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$ and $\bar{t} = \frac{1}{n} \sum_{i=1}^{n} t_i.$

Conclusively, the error in the mean is the mean of the errors. Also the positive and negative errors tending to cancel in computing $\sum x_i$, as n the number of observations increases, the value of error decreases.

ADDITIONAL MISCELLANEOUS PROBLEMS

Problem 142. *State and Prove the normal law of errors and find an expression for the measure of precision and the probable error of the arithmetic mean.*

A cylinder has a lenght "l" cm., which is measurable with a probable error ± a and has a radius "a" cm., which is measurable with a probable error ± b. Find the area of its curved surface and determine the probable error of this value. *(Nagpur, 1965)*

Problem 143. *Derive the normal law of errors and calculate the probable error of an observation.*

Problem 144. *(a) Write short notes on the binomial, the Poissonian and normal distributions.*

(b) Derive the normal law of errors from first principles and discuss some of its applications.

Problem 145. *(a) Define Poisson's distribution and discuss its importance in Physics.*

(b) Calculate the mean and the standard deviation for the Poisson distribution.

(Agra, 1970)

Problem 146. *Write a comprehensive note on the theory of errors.* *(Agra, 1971)*

Problem 147. *If the probability function P(x) of events x is given by the Gaussian function*

$$P(x) = Ae^{-kx^2}$$

prove that $k = \dfrac{1}{2\sigma^2}$ *and* $A = \dfrac{1}{\sigma\sqrt{2\pi}}$ *where* σ *is the standard deviation.*

Assuming the errors in a series of observations to have the Gaussian distribution obtain expressions for the probable error of a single observation and that in the arithmetic mean of the observations. (*Agra, 1972*)

Problem 148. *Define the error function erf* (x) *and obtain a convergent expansion for it in ascending powers of* x *correct upto* x^5. (*See Appendix B*)

Use the approximate expression to calculate the percentage of molecules of an ideal gas which have speeds less than $\dfrac{1}{2\sqrt{b}}$. *The speed distribution of the molecules is given to be* $\dfrac{dN_c}{N} = \dfrac{2}{\sqrt{\pi}} b^3 c^2 e^{-bc^2}\ dc$; $\dfrac{dN_c}{N}$ *being the fraction of molecules having speeds in the range* c *to* $+ dc$. *Given* $\pi = 3.4$ *and* $e^{1/4} = 1.28$. (*Agra, 1973*)

Problem 149. (*a*) *Explain what you mean by a binomial distribution. Find its mean and standard deviation.*

(*b*) *The mean number of particles emitted in one second by a radioactive sample is 4.5. Write an expression for the probability of just 3 particles being emitted in a particular second.* (*Agra, 1974*)

Problem 150. *Explain the meanings of the following terms in the theory of errors: dispersion, variance, standard deviation, regression and correlation.* (*Agra, 1975*)

Problem 151. (*a*) *Explain normal distribution and derive an expression for the same.*

(*b*) *Explain the Principle of least squares.* (*Rohilkhand, 1976*)

APPENDICES

[A] SOME FORMULATED RESULTS IN BASIC MATHEMATICS

(a) Laws of Exponent Powers

$$x^a . x^b = x^{a+b};\ x^a . y^a = (xy)^a;\ \left(x^a\right)^b = x^{ab}; \frac{x^a}{x^b} = x^{a-b};\ x^{a/b} = \sqrt[b]{x^a},$$

$$x^0 = 1.\ x^{-a} = \frac{1}{x^a} \text{ provided } x \neq 0.$$

(b) Factors and Expansions

Taking n and r both as positive integers, we have

$$^nC_r = \left(\frac{n}{r}\right) = \frac{\lfloor\underline{n}}{\lfloor\underline{r}\lfloor\underline{n-r}} \text{ and } \lfloor\underline{r} = r(r-1)\,(r-2)\ldots 3.2.1.$$

$$^nC_0 = 1 = {^nC_n};\ {^nC_r} = {^nC_{n-r}} \qquad \lfloor\underline{1} = \lfloor\underline{0} = 1 \text{ and } \lfloor\underline{-r} = \infty$$

$$(x+y)^n = x^n + {^nC_1}\, x^{n-1} y + {^nC_2}\, x^{n-2} y^2 + \ldots + {^nC_n} y^n$$

$$= \sum_{r=0}^{n} {^nC_r} x^{n-r} y^r$$

$x^n + y^n = (x+y)\,(x^{n-1} - x^{n-2} + \ldots\ldots + y^{n-1})$, n being an odd integer

$x^n + y^n = (x+y)\,(x^{n-1} - x^{n-2} y + \ldots\ldots - y^{n-1})$, n being an even integer

$x^n - y^n = (x-y)\,(x^{n-1} + x^{n-2} y + \ldots\ldots + y^{n-1})$.

$$(x+y+z+\ldots\ldots)^n = \sum_{\text{over all } r_i} \frac{\lfloor\underline{n}}{\lfloor\underline{r_1}\lfloor\underline{r_2}\lfloor\underline{r_3}\ldots} x^{r_1} y^{r_2} z^{r_3} \ldots\ldots \text{ where } \Sigma r_i = n.$$

In particular:

$$x^2 - y^2 = (x-y)\,(x+y);\ x^3 \pm y^3 = (x \pm y)\,(x^2 \mp xy + y^2)$$

(c) Summations

$$\Sigma n = 1 + 2 + 3 + \ldots\ldots + n = \frac{n(n+1)}{2}$$

$$\Sigma n^2 = 1^2 + 2^2 + \ldots\ldots + n^2 = \frac{n(n+1)\,(2n+1)}{6}$$

$$\Sigma n^2 = \frac{n^2(n+1)^2}{4},\ \Sigma n^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$$

$$\Sigma n^p = 1^p + 2^p + \ldots\ldots + n^p = \frac{n^{p+1}}{p+1} + \frac{n^p}{2} + \frac{1}{2}\,{^pC_1} B_1 n^{p-1}$$

$$-\frac{1}{4}\,{^pC_3} B_2 n^{n-3} + \frac{1}{6}\,{^pC_5} B_3\, n^{p-5} + \ldots$$

where B's are Bernoulli's number and the series ends with n when p is even and with n^2 when p is odd. The Bernoulli's numbers are

$$B_1 = \frac{1}{6},\ B_2 = \frac{1}{30}\ B_3 = \frac{1}{42},\ B_4 = \frac{1}{30},\ B_5 = \frac{5}{66},\ B_6 = \frac{691}{2730},\ B_7 = \frac{7}{6} \text{ etc...}$$

Taking a s the first term, d common difference and l the last term of an arithmetical progression, the sum

$$S = a + (a + d) + (a + 2d) + \ldots\ldots + \{a + (n-1)d\}$$

$$= \frac{n}{2}\,[2a + (n-1)d] = \frac{n}{2}[a + l]$$

where $l = a + (n-1)d = n$th term of the series.

If a be the first term and r the common ratio of a geometrical progression, then the sum

$$S = a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a\left(r^n - 1\right)}{r - 1} \text{ if } r > 1$$

$$= \frac{a\left(1 - r^n\right)}{1 - r} \text{ if } r < 1$$

$$= \frac{a}{1 - r} \text{ if } n \text{ is infinite.}$$

(d) Ratio and Proportions

If $\frac{p}{q} = \frac{r}{s}$ then $\frac{p+q}{q} = \frac{r+s}{s}$ and $\frac{p-q}{q} = \frac{r-s}{s}$.

Also $\frac{p-q}{p+q} = \frac{r-s}{r+s}$ (componendo and dividendo)

In general if $\frac{p}{q} = \frac{r}{s} = \frac{t}{u} = \ldots\ldots = k$ then

$$k = \frac{p + r + t + \ldots}{q + s + u + \ldots} = \frac{xp + yr + zt + \ldots}{xq + ys + zu + \ldots}$$

(e) Logarithms

We express $10^2 = 100$ by saying $\log_{10} 100 = 2$ and read 'logarithm of 100 to the base 10 is 2:

In general, taking the base of the logarithm as e, we express $e^x = y$ by saying 'logarithm of y to the base 'e' is x' *i.e.* $\log_e y = x$

$$\log_e x + \log_e y = \log_e xy;\ \log_e x - \log_e y = \log_e \frac{x}{y}$$

$$\log_e x^m = m \log_x x;\ \log_e x^{-n} = \log_e \frac{1}{x^n} = -n \log_e x$$

$$\log_a x = \log_b x \log_a b = \frac{\log_b x}{\log_b a} \text{ (change of base)}$$

(f) Special Series

(*i*) *Binomial series* for $|x| < 1$ and for any n, positive, negative, integral or fractional,

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)}{1.2} x^2 \pm \frac{n(n-1)(n-2)}{1.2.3} x^3 + \ldots\ldots$$

$$(1 \pm x)^n = 1 \mp nx + \frac{n(n+1)}{1.2} x^2 \mp \frac{n(n+1)(n+2)}{1.2.3} x^3 + \ldots\ldots$$

In particular,

$$(1 \pm x)^{-1} = 1 \mp x + x^2 \mp x^3 + x^4 \mp x^5 + \ldots\ldots$$

(*iii*) *Exponential series*

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \ldots\ldots = 2.71828\ldots\ldots$$

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots\ldots$$

$$e^{-x} = \lim_{n \to \infty} \left(1 - \frac{x}{n}\right)^n = 1 - x + \frac{x^2}{\lfloor 2} - \frac{x^3}{\lfloor 3} + \ldots\ldots$$

$$a^x = + x \log_e a + \frac{(x \log_e a)^2}{\lfloor 2} + \ldots\ldots$$

(*iii*) *Logarithmic series*

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots \qquad |x| < 1$$

$$\log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots \qquad |x| < 1$$

(*iv*) *Other series*

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$$

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots$$

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots$$

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\ldots$$

$$\frac{\pi^2}{24} = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \ldots\ldots$$

(*v*) *Taylor's series*

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{\lfloor 2} f''(a) + \ldots + \frac{h^n}{\lfloor n} f^n(a) + \ldots$$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{\lfloor 2}f''(a) + \ldots + \frac{(x-a)^n}{\lfloor n}f^n(a) + \ldots$$

(*vi*) *Maclaurin's series*

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2}f''(0) + \ldots + \frac{x^n}{\lfloor n}f^n(0) + \ldots\ldots$$

(g) **Stirling's formula**

For very large values of n,

$$\lfloor n \approx \sqrt{2\pi n}.n^n e^{-n} \text{ approx.} = e^{-n} n^{n+1/2}\sqrt{2\pi}$$

$$\log \lfloor n = \left(n + \tfrac{1}{2}\right)\log n - n \log e + \log \sqrt{2\pi}$$

where $\pi = 3.14159\ldots$ and $e = 2.71828\ldots$

(h) **Trigonometric functions**

(*i*) In a right-angled triangle ABC where $B = 90°$, and $\angle BAC = \theta$ (say) AB is the base, BC is the perpendicular in respect to θ and AC is the hypotenuse.

$$\sin\theta = \frac{\text{perpendicular}}{\text{hypotenuse}} = \frac{BC}{AC};\ \cos\theta = \frac{\text{base}}{\text{hypotenuse}} = \frac{AB}{AC}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{BC}{AB},\ \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{AB}{BC}$$

$$\sec\theta = \frac{1}{\cos\theta} \text{ and } \operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

(*ii*) Signs of these functions in different quadrants

Quadrant	sin	cos	tan	cot	sec	cosec
I	+	+	+	+	+	+
II	+	–	–	–	–	+
III	–	–	+	+	–	–
IV	–	+	–	–	+	–

(*iii*) Values of these functions (sine and cosine only others can be calculated)

	0°	30°	45°	60°	90°	180°	270°	360°
Sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	–1	0
Cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	–1	0	1

$\sin(90° \mp \theta) = \cos\theta;\ \cos(90° \mp \theta) = \pm\sin\theta$

$\sin(180° \mp \theta) = \pm\sin\theta;\ \cos(180° \mp \theta) = -\cos\theta$

$\sin(360° \mp \theta) = \mp\sin\theta;\ \cos(360° \mp \theta) = \cos\theta$

$\sin^2\theta + \cos^2\theta = 1;\ 1 + \tan^2\theta = \sec^2\theta;\ 1 + \cot^2\theta = \operatorname{cosec}^2\theta$

$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$

$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$

$\sin 2\theta = 2\sin\theta\cos\theta;\ \cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$

$$\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}; \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}; \tan(\theta \pm \phi) = \frac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta\tan\phi}$$

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta, \cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

$$\sin\theta + \sin\phi = 2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}; \sin\theta - \sin\phi = 2\cos\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}; \cos\theta + \cos\phi = 2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2};$$

$$\cos\theta - \cos\phi = 2\sin\frac{\theta+\phi}{2}\sin\frac{\phi-\theta}{2}.$$

(*iv*) $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \operatorname{cosec}^{-1}\frac{1}{x}$

$$\cos^{-1}x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{\sqrt{1-x^2}}{x} = \sec^{-1}\frac{1}{x}$$

$$\tan^{-1}x = \sin^{-1}\frac{x}{\sqrt{1+x^2}} = \cos^{-1}\frac{1}{\sqrt{1+x^2}} = \cot^{-1}\frac{1}{x}$$

$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}.$$

(*v*) *Hyperbolic functions*

$$\sin h\,x = \frac{e^x - e^{-x}}{2}, \cos h\frac{e^x + e^{-x}}{2}, \tan h\,x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\sin h\,x}{\cos h\,x}$$

$$\cot h\,x = \frac{1}{\tan h\,x}; \sec h\,x = \frac{1}{\cos h\,x}. \operatorname{cosec} h\,x = \frac{1}{\sin h\,x}$$

$$\sin h^{-1}x = \log\left\{x + \sqrt{x^2+1}\right\}; \cos h^{-1}x = \log\left\{x + \sqrt{x^2+1}\right\}$$

$$\tan h^{-1}x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right).$$

$\cosh^2 x - \sinh^2 x = 1$; $\operatorname{sech}^2 x + \tanh^2 x = 1$,

$\sinh 2x = 2\sinh x\cosh x$;

$\cosh 2x = \cosh^2 x + \sinh^2 x = 1 + 2\sinh^2 x = 2\cosh^2 x - 1$

$\sin ix = i\sinh x$; $\cos ix = \cosh x$

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for all values of n.

(*vi*) *Trigonometric series*

$$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \ldots\ldots$$

$$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \ldots\ldots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots\ldots$$

$$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + \ldots\ldots |x| < 1$$

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 \ldots\ldots \qquad |x| \le 1$$

$$\sinh x = x + \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 3} + \ldots\ldots$$

$$\cosh x = 1 + \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} + \ldots\ldots$$

$$\tanh x = x - \frac{x^3}{3} + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \ldots\ldots$$

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\ldots |x| < 1$$

(i) Analytical Geometry

If Cartesian coordinates of a point P be (x, y) and its polar coordinates be (r, θ), where x is called abscissa, y is called ordinate, r is called radius vector and θ is called vectorial angle then,

$$x = r\cos\theta, \qquad y = r\sin\theta$$

or

$$r = \sqrt{x^2 + y^2} \text{ and } \theta = \tan^{-1}\frac{y}{x}$$

Distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The slope (gradient) of the line is given by $\frac{y_2 - y_1}{x_2 - x_1}$ and denoted by m.

If m_1, m_2 be the gradients of two straight lines, the angle between them is

$$\tan^{-1}\frac{m_1 \sim m_2}{1 + m_1 m_2}.$$

The two lines will be perpendicular or parallel according as,

$$m_1 m_2 = -1 \text{ and } m_1 = m_2.$$

Rectangular transformation (1) New axes parallel to old axes : If (x, y) and (x', y') respectively be the coordinates of a point w.r.t. old and new axes and (h, k) be the coordinates of the new origin w.r.t. old axes, then

$$x = x' + h, \qquad y = y' + k$$

(2) New axes rotated through an angle θ about the origin from old axes : If (x, y) and (x', y') respectively be the coordinates of a point w.r.t. old and new axes, then

$$x = x'\cos\theta - y'\sin\theta,\ y = x'\sin\theta + y'\cos\theta$$

or conversely,

$$x' = x\cos\theta + y\sin\theta,\ y' = -x\sin\theta + y\cos\theta$$

(j) Differential Calculus

$d(ax) = a\,dx$, a being a constant, $d(a) = 0$

$d(2x - 3y + z) = 2dx - 3dy + dz$

$d(xy) = xdy + ydx$

$$d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$$

$d(x^n) = nx^{n-1}\,dx;\ d(x^y) = yx^{y-1}\,dx + x^y(\log_e x)\,dy$

$d(e^x) = e^x dx;\ d(d^{\pm ax}) = \pm\, ae^{\pm ax}\,dx$

$d(a^x) = (a^x \log_e a)\,dx;\ d(x^x) = x^x(1 + \log_e x)\,dx$

$$d(\log_e x) = \frac{1}{x}dx;\ d(\log_a x) = \frac{1}{x}\log_a x\,dx = \frac{dx}{x\log_e a}$$

$d(\sin x) = \cos x\,dx;\ d(\cos x) = -\sin x\,dx$

$d(\tan x) = \sec^2 x\,dx;\ d(\cot x) = -\operatorname{cosec}^2 x\,dx$

$d(\sec x) = \sec x \tan x\,dx;\ d(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x\,dx$

$d(\sinh x) = \cosh x\,dx;\ d(\cosh x) = \sinh x\,dx$

$d(\tanh x) = \operatorname{sech}^2 x\,dx;\ d(\coth x) = -\operatorname{cosech}^2 x\,dx$

$d(\operatorname{sech} x) = \operatorname{sech} x \tanh x\,dx;\ d(\operatorname{cosech} x) = -\operatorname{cosech} x \coth x\,dx$

$$d(\sin^{-1} x) = \frac{dx}{\sqrt{1-x^2}};\ d(\cos^{-1} x) = -\frac{dx}{\sqrt{1-x^2}}$$

$$d(\tan^{-1} x) = \frac{dx}{1+x^2};\ d(\cot^{-1} x) = -\frac{dx}{1+x^2}$$

$$d(\sec^{-1} x) = \frac{dx}{x\sqrt{x^2+1}};\ d(\operatorname{cosec}^{-1} x) = -\frac{dx}{x\sqrt{x^2-1}}$$

$$d(\sinh^{-1} x) = \frac{dx}{\sqrt{x^2+1}};\ d(\cosh^{-1} x) = \frac{dx}{\sqrt{x^2-1}}$$

$$d(\tanh^{-1} x) = \frac{dx}{1-x^2};\ d(\coth^{-1} x) = -\frac{dx}{x^2-1}$$

$$d(\operatorname{sech}^{-1} x) = \frac{dx}{-x\sqrt{(1-x^2)}};\ d(\operatorname{cosech}^{-1} x) = -\frac{dx}{x\sqrt{(x^2+1)}}.$$

Taking $D \equiv \dfrac{d}{dx}$, we have

$D^n(ax+b)^m = m(m-1)(m-2)\ldots\ldots(m-n+1)a^n(ax+b)^{m-n}$

$D^n(ax+b)^{-1} = (-1)^n \lfloor n\ a^n(ax+b)^{-n-1}$

$D^n e^{ax} = a^n e^{ax};\ D^n a^x = (\log a)^n a^x$

$$D^n \log(ax+b) = \frac{(-1)^{n-1}\lfloor n-1 . a^n}{(ax+b)^n}$$

$$D^n \sin(ax+b) = a^n \sin\left(ax + b + \frac{n\pi}{2}\right)$$

$$D^n \cos(ax+b) = a^n \cos\left(ax + b + \frac{n\pi}{2}\right)$$

$$D^n\left\{e^{ax}\begin{matrix}\sin\\ \cos\end{matrix}(bx+c)\right\}=r^n e^{ax}\begin{matrix}\sin\\ \cos\end{matrix}(bx+c+n\phi)$$

where $r=\sqrt{a^2+b^2}$ and $\phi=\tan^{-1}\dfrac{b}{a}$

$$D^n(uv)=(D^n u).v+nC_1 D^{n-1}uDv+nC_2D^{n-2}u.\,D^2v+\ldots$$
$$+\,{}^nC_rD^{n-r}u.\,D^rv+\ldots+u.\,D^nv$$

(k) Integral Calculus

$$\int x^n dx=\frac{x^{n+1}}{n+1},\ n\neq -1;\ \int\frac{1}{x}dx=\log x;\ \int e^x dx=e^x;\ \int a^x dx=\frac{a^x}{\log a}$$

$$\int\sin x\,dx=-\cos x;\ \int\cos x\,dx=\sin x;\ \int\tan x\,dx=-\log\cos x$$

$$\int\cot x\,dx=\log\sin x;\int\sec x\,dx=\log\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)=\log(\sec x+\tan x)$$

$$\int\operatorname{cosec} x\,dx=\log\tan\frac{x}{2};\int\frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\frac{x}{a};\ \int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log\frac{x-a}{x+a};$$

$$\int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a};\int\frac{dx}{\sqrt{a^2+x^2}}=\sin^{-1}\frac{x}{a}=\log\left\{x+\sqrt{x^2+a^2}\right\}$$

$$\int\frac{dx}{\sqrt{x^2-a^2}}=\cos h^{-1}\frac{x}{a}=\log\left\{x+\sqrt{x^2-a^2}\right\};\ \int\frac{dx}{x\sqrt{x^2-a^2}}=\frac{1}{a}\sec^{-1}\frac{x}{a}$$

$$\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

$$\int\sqrt{a^2+x^2}\,dx=\frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\sin h^{-1}\frac{x}{a}$$

$$\int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\cos h^{-1}\frac{x}{a}$$

$$\int e^{ax}\sin bx\,dx=\frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left(bx-\tan^{-1}\frac{b}{a}\right)$$

$$\int e^{ax}\cos bx\,dx=\frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left(bx-\tan^{-1}\frac{b}{a}\right)$$

$$\int\sin h\,x\,dx=\cos h\,x;\ \int\cos h\,x\,dx=\sin h\,x;\ \int\sec h^2x\,dx=\tan h\,x$$

$$\int\operatorname{cosec} h^2x\,dx=-\cot h\,x$$

$$\int(ax+b)^n\,dx=\frac{(ax+b)^{n+1}}{(n+1)a};\ \int\frac{dx}{ax+b}=\frac{1}{a}\log(ax+b)$$

$$\int e^{ax}dx=\frac{1}{a}e^{ax};\int\sin(ax+b)\,dx=-\frac{1}{a}\cos(ax+b)$$

$$\int\sin^n dx=-\frac{\sin^{n-1}ax\cos ax}{na}+\frac{n-1}{n}\int\sin^{n-2}ax\,dx,\ n \text{ positive integer}$$

$$\int \frac{dx}{\sin ax} = \frac{1}{a} \log \tan \frac{ax}{2} = \frac{1}{a} \log (\operatorname{cosec} ax - \cot ax)$$

$$\int \frac{dx}{1 \pm \sin ax} = \mp \frac{1}{a} \tan\left(\frac{\pi}{4} \mp \frac{ax}{2}\right)$$

$$\int \sin ax \sin bx\, dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)}, \quad a^2 \neq b^2$$

$$\int \cos^n x\, dx = \frac{\cos^{n-1} ax \sin ax}{na} + \frac{n-1}{n} \int \cos^{n-2} ax\, dx, \ n \text{ positive integer}$$

$$\int \frac{dx}{\cos ax} = \frac{1}{a} \log \tan\left(\frac{ax}{2} + \frac{\pi}{4}\right) = \frac{1}{a} \log (\tan ax + \sec ax)$$

$$\int \frac{dx}{1+\cos ax} = \frac{1}{a} \tan \frac{ax}{2}, \int \frac{dx}{1-\cos ax} = -\frac{1}{a} \cos \frac{ax}{2}$$

$$\int \sin^m ax \cos^n ax\, dx = -\frac{\sin^{m-1} ax \cos^{n+1} ax}{a(m+n)} + \frac{m-1}{m+n} \int \sin^{m-2} ax \cos^n ax dx; \ m, n > 0$$

$$= \frac{\sin^{m+1} ax \cos^n ax}{a(m+n)} + \frac{n-1}{m+n} \int \sin^m as \cos^{n-2} as\, dx; \ m, n > 0$$

$$\int \tan^n ax\, dx = \frac{1}{a(n-1)} \tan^{n-1} ax - \int \tan^{n-2} ax\, dx, \text{ integral } n > 1$$

$$\int \cot^n ax\, dx = -\frac{1}{a(n-1)} \cot^{n-1} ax - \int \cot^{n-2} ax\, dx, \text{ integral } n > 1$$

$$\int \sec^n ax\, dx = \frac{1}{a(n-1)} \frac{\sin ax}{\cos^{n-1} ax} + \frac{n-2}{n-1} \int \sec^{n-2} ax\, dx, \text{ integral } n > 1$$

$$\int \operatorname{cosec}^n ax\, dx = -\frac{1}{a(n-1)} \frac{\cos ax}{\sin^{n-1} ax} + \frac{n-2}{n-1} \int \operatorname{cosec}^{n-2} ax\, dx, \text{ integral } n > 1$$

$$\int x^n e^{ax} dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax} dx, \text{ positive } n$$

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$$

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx)$$

$$\int \log ax\, dx = x \log ax - x; \int \frac{dx}{x \log ax} = \log (\log ax)$$

$$\int x^x (\log ax)^m dx = \frac{x^{n+1}}{n+1} (\log ax)^m - \frac{m}{n+1} \int x^n ((\log ax)^{m-1} dx$$

$$\int e^{ax} \log bx\, dx = \frac{1}{a} e^{ax} \log bx - \frac{1}{a} \int \frac{x^{ax}}{x} dx$$

$$\int x^m(ax+b)^n dx = \frac{1}{a(m+n+1)}\left[x^m(ax+b)^{n+1} - mb\int x^{m-1}(ax+b)^n dx\right]$$

$$= \frac{1}{m+n+1}\left[x^{m+1}(ax+b)^n + nb\int x^m(ax+b)^{n-1}dx\right], m>0,\ m+n+1\neq 0$$

$$\int \frac{dx}{(ax+b)(cx+d)} = \frac{1}{bc-ad}\log\frac{cx+d}{ax+b},\ bc-ad\neq 0$$

$$\int \frac{dx}{(cx+d)\sqrt{ax+b}} = \frac{2}{\sqrt{c}\sqrt{ad-bc}}\tan^{-1}\frac{\sqrt{c(ax+b)}}{ad-bc},\ c>0, ad>bc$$

Definite Integrals

$$\int_0^\infty \frac{a\,dx}{a^2+x^2} = \begin{cases} \pi/2, & a>0 \\ 0 & a=0 \\ -\pi/2, & a<0 \end{cases}$$

$$\int_0^\infty x^{n-1}e^{-x}dx = \int\left[\log\frac{1}{x}\right]^{n-1}dx = \sqrt{n}$$

$$\int_0^\infty x^{m-1}(1-x)^{n-1}dx = \int_0^\infty \frac{x^{m-1}dx}{(1+x)^{m+n}} = B(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$$

$$\int_0^\infty \frac{x^{n-1}dx}{1+x} = \frac{\pi}{\sin n\pi};\ 0<n<1$$

$$\int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2},\ a>0$$

$$\int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2},\ a>0$$

$$\int_0^\pi \frac{dx}{a+b\cos x} = \frac{\pi}{\sqrt{a^2-b^2}},\ a>b>0$$

$$\int_0^\infty \frac{\cos ax\,dx}{1+x^2} = \frac{\pi}{2}e^{-a},\ a>0 \text{ and } \frac{\pi}{2}e^{a},\ a<0$$

$$\int_0^\infty \frac{\cos x}{\sqrt{x}}dx = \int_0^\infty \frac{\sin x\,dx}{\sqrt{x}} = \sqrt{\frac{\pi}{2}};\ \int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{\pi}{2ab}$$

$$\int_0^\infty e^{-ax}\cosh bx\,dx = \frac{a}{a^2-b^2},\ a>0,\ a^2\neq b^2$$

$$\int_0^\infty e^{-ax}\sinh bx\,dx = \frac{b}{a^2-b^2},\ a>0,\ a^2\neq b^2$$

$$\int_0^\infty e^{-x}\log x\,dx = \int_0^\infty\left(\frac{1}{1-e^{-x}} - \frac{1}{x}\right)e^{-x}dx$$

$$= \int_0^\infty \frac{1}{x}\left(\frac{1}{1+x} - e^{-x}\right)dx$$

$$= \int_0^1 \left(\frac{1 - e^{-x} - e^{-1/x}}{x} \right) dx = \gamma = .5772152$$

where γ is Euler's constant given by

$$r = \lim_{t \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{t} - \log t \right)$$

(l) Elliptic Integrals

Elliptic integrals of first, second and third kinds are respectively

$$K(k, \phi) = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}};\ 0 < k < 1$$

$$E(k, \phi) = \int_0^{\phi} \sqrt{1 - k^2 \sin^2 \theta}\ d\theta;\ 0 < k < 1$$

$$\Pi(k, \phi, a) = \int_0^{\phi} \frac{d\theta}{\left(\sqrt{1 - k^2 \sin^2 \theta}\right)\left(1 + a^2 \sin^2 \theta\right)},\ 0 < k < 1, a \neq k$$

If ϕ, the upper limit is replaced by $\frac{\pi}{2}$ these become complete elliptic integrals:

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta}\ d\theta$$

$$\Pi(k, a) = \int_0^{\pi/2} \frac{d\theta}{\left(\sqrt{1 - k^2 \sin^2 \theta}\right)\left(1 + a^2 \sin^2 \theta\right)}$$

(m) Probability functions

If $\phi(x)$ be a normal function *s.t.* $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

and $\alpha = \int_{-x}^{x} \phi(x)\, dx$ *i.e.* are under $\phi(x)$ from $-x$ to x, then $\frac{1}{2}(1 + \alpha) = \int_{-\infty}^{x} \phi(x)\, dx$

Second derivative if $\phi(x)$ *i.e.* $\phi^{(2)}(x) = (x^2 - 1)\,\phi(x)$

Third " " *i.e.* $\phi^{(3)}(x) = (3x - x^3)\,\phi(x)$

Fourth " " *i.e.* $\phi^{(4)}(x) = (x^4 - 6x^2 + 3)\,\phi(x)$

Now if $(p + q)^n = \sum_{r=0}^{n} {}^nC_r p^{n-r} q^r,\ p + q = 1$, then

$$\sum_{r=a}^{b} {}^nC_r p^{n-r} q^r = \int_{x_1}^{x_2} \phi(x)\, dx + \left[\frac{q - p}{6\sigma} \phi^{(2)}(x) + \frac{1}{24}\left(\frac{1}{\sigma^2} - \frac{6}{n} \right) \phi^{(3)}(x) \right]_{x_1}^{x_2} \text{ approx.}$$

where $x_1 = \frac{\left(a - \frac{1}{2} - nq\right)}{\sigma};\ x_2 = \frac{\left(b + \frac{1}{2} - nq\right)}{\sigma},\ \alpha = \sqrt{npq}$ and $a \leq r \leq b$.

$$\sum_{r=0}^{r} {}^nC_r p^{n-r} q^r = \int_{x}^{\infty} \phi(x)\, dx + \frac{q - p}{6\sigma} \phi^{(2)}(x) - \frac{1}{24}\left(\frac{1}{\sigma^2} - \frac{6}{n} \right) \phi^{(3)}(x) \text{ approx.}$$

where $x = \dfrac{\left(n - r - \frac{1}{2} - np\right)}{\sigma}$

$$\sum_{r=r}^{n} {}^nC_r p^{n-r} q^r = \int_x^\infty \phi(x)\, dx - \frac{q-p}{6\sigma}\phi^{(2)}(x) - \frac{1}{24}\left(\frac{1}{\sigma^2} - \frac{6}{n}\right)\phi^{(3)}(x) \text{ approx.}$$

where $x = \dfrac{\left(r - \frac{1}{2} - nq\right)}{\sigma}$.

[B] Asymptotic Expansion of Error-Function

The *Error function* is defined as

$$\text{erf } x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2} dt \quad ...(1) \text{ where } \int_0^\infty e^{-t^2} dt = \frac{1}{2}\sqrt{\pi} \quad ...(2)$$

(1) and (2) yield $\text{erf } \infty = \lim_{x\to\infty} \text{erf } x = 1$ ...(3)

Using Maclaurin series of e^{-x} after integrating term by term, for small values of $|x|$, $\text{erf } x = \dfrac{2}{\sqrt{\pi}}\left(x - \dfrac{x^3}{3\lfloor 1} + \dfrac{x^5}{5\lfloor 2} - \dfrac{x^5}{7\lfloor 3} + \ldots\right)$ ...(4)

For large values of x, the asymptotic expansion of erf x given below is used.

The *Complementary error function* is defined as

$$\text{erfc } x = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2} dt \quad ...(5)$$

The relation between error and complementary error functions is given by

$$\text{erfc } x = 1 - \text{erf } x$$

or

$$\text{erf } x = 1 - \text{erfc } x \quad ...(6)$$

Now to find the asymptotic expansion of erf x, let us put $t^2 = p$ *i.e.* $dt = \dfrac{1}{2\sqrt{p}} dp$ so that we have

$$\text{erfc } x = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2} dt = \frac{1}{\sqrt{\pi}}\int_{x^2}^\infty e^{-p}.p^{-\frac{1}{2}} dp$$

If we go on integrating the R.H.S. by parts, then we shall have after n trials, an integral of the form

$$\text{(say) } I_n(x) = \int_{x^2}^\infty e^{-p} p^{-\frac{2n+1}{2}} dp, \quad n = 0, 1, 2... \quad ...(7)$$

whence $\text{erfc } x = \dfrac{1}{\sqrt{\pi}} I_0(x)$ ...(8)

Now integrating R.H.S. of (7) by parts, we get

$$I_n(x) = \left[p^{-\frac{2n+1}{2}}\left(-e^{-p}\right)\right]_{x^2}^\infty - \frac{2n+1}{2}\int_{x^2}^\infty p^{-\frac{2n+3}{2}} e^{-p} dp$$

$$= \frac{1}{x^{2n+1}} e^{-x^2} - \frac{2n+1}{2} I_{n+1}(x), \quad n = 0, 1, 2, \ldots\ldots$$

Multiplying both sides by e^{x^2}, we get the recurrence relation

$$e^{x^2} I_n(x) = \frac{1}{x^{2n+1}} - \frac{2n+1}{2} e^{x^2} I_{n+1}(x) \qquad ...(9)$$

Applying this result repeatedly, we find

$$e^{x^2} I_0(x) = \frac{1}{x} - \frac{1}{2} e^{x^2} I_1(x) \text{ (for } n = 0)$$

$$= \frac{1}{x} - \frac{1}{2x^3} + \frac{1}{2} \cdot \frac{3}{2} e^{x^2} I_2(x) \text{ (for } n = 1)$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$= \left\{ \frac{1}{x} - \frac{1}{2x^3} + \frac{1.3}{2^2 x^5} - \ldots + (-1)^{n-1} \frac{1.3.5 \ldots (2n-3)}{2^{n-1} x^{2n-1}} \right\}$$

$$+ (-1)^n \frac{1.3.5 \ldots (2n-1)}{2^n} e^{x^2} I_n(x) \qquad ...(10)$$

In order to show that this series is an asymptotic expansion, consider,

$$e^{x^2} I_0(x) \sim \frac{1}{x} - \frac{1}{2x^3} + \frac{1.3}{2^2 x^5} \ldots \qquad ...(11)$$

and denote $T_{2n-1} = \frac{1}{x} - \frac{1}{2x^3} + \frac{1.3}{2^2 x^5} - \ldots + (-1)^{n-1} \frac{1.3.5 \ldots (2n-3)}{2^{n-1} x^{2n-1}}$ ...(12)

As such (10) can be rewritten as

$$\left(e^{x^2} I_0(x) - T_{2n-1} \right) = (-1)^n \frac{1.3.5 \ldots (2n-1)}{2^n} e^{x^2} I_n(x)$$

Multiplying both sides by x^{2n-1} and putting

$f_n = (-1)^n \dfrac{1.3.5 \ldots (2n-1)}{2^n}$, the last relation yields

$$\left(e^{x^2} I_0(x) - T_{2n-1} \right) x^{2n-1} = f_n e^{x^2} I_n(x) \qquad ...(13)$$

The series in (11) will be an asymptotic expansion, if we can show that for each fixed $n = 1, 2, \ldots f_n \to 0$ as $x \to \infty$.

Now in (7), we have

$$\frac{1}{p^{2n+1/2}} \leq \frac{1}{x^{2n+1}} \text{ for all } p \geq x^2$$

So that, $I_n(x) = \int_{x^2}^{\infty} \frac{e^{-p}}{p^{2n+1/2}} dp < \frac{1}{x^{2n+1}} \int_{x^2}^{\infty} e^{-p} dp$

$$= \frac{e^{-x^2}}{x^{2n+1}} \qquad ...(14)$$

Thus R.H.S. of (13) *i.e.*

$$|f_n| e^{x^2} x^{2n-1} I_n(x) < \frac{|f_n|}{x^2} \to 0 \text{ as } x \to \infty.$$

Hence the series on the right hand side of (10) is an asymptotic expansion of the function $e^{x^2} I_0(x)$.

Thus erf $x = 1 - \text{erfc}\, x = 1 - \frac{I_0(x)}{\sqrt{\pi}}$ renders the required asymptotic expansion of the error function as

$$\text{erf } x \sim 1 - \frac{1}{\sqrt{\pi}} e^{-x^2}\left(\frac{1}{x} - \frac{1}{2x^3} + \frac{1.3}{2^2 x^5} - \frac{1.3.5}{2^3 x^7} + \ldots\right) \quad \ldots(15)$$

For large values of x, this reduces to

$$\text{erf } x \approx 1 - \frac{1}{\sqrt{\pi}} \frac{e^{-x^2}}{x} \quad \ldots(16)$$

Note 1. $\frac{1.3.5\ldots(2n-1)}{2^n \sqrt{\pi}} |I_n(x)|$ *is known as the* **absolute value of the error.**

Note 2. *Here ~ read as asymptotically equal.*

[C] Character Tables in Group Theory

Denoting a reducible or irreducible representation of a group $G = \{E, A, B, \ldots\}$ by Γ, its *character* χ may be defined as the set of traces of all the matrices of representation Γ *i.e.*

$$\chi(A) = \sum_i \Gamma_{ii}(A) \quad \ldots(1)$$

where Γ_{ii} are the elements of the matrix corresponding to A, the matrix of representation.

It is notable that the character is the same as the representation in case of one-dimensional representation and also the characters are the same in case of representations of conjugate elements since the trace of a matrix under a similarity transformation is invariant.

Taking A and B as conjugate elements there may be found an element C such that

$$A = C^{-1} BC \quad \ldots(2)$$

which yields on taking traces on either side

$$\Gamma(A) = \Gamma(C^{-1})\, \Gamma(B)\, \Gamma(C) \quad \ldots(3)$$

But the cyclic property of traces gives for any three matrices P, Q, R,

$$tr\,(PQR) = tr\,(QRP) = tr\,(RPQ) \quad \ldots(4)$$

$$\therefore \quad tr\,\Gamma(A)) = tr\,(\Gamma(B) \Rightarrow \chi(A) = \chi(B) \quad \ldots(5)$$

Now if we denote an irreducible representation of the group G by $\Gamma^{(i)}$ and assume that an irreducible representation may occur two or more times in reduction of a reducible representation Γ, then the matrices of representation Γ are the direct sum of the matrices of the components of irreducible representations *i.e.*

$$\Gamma = \sum_i a_i \Gamma^{(i)} \quad \ldots(6)$$

where a_i's are non-negative integers.

Taking traces of either side of (6), for A, the matrix of representation,

$$tr\,\Gamma(A) = \sum_i a_i\, tr\,\Gamma^{(i)}_{(A)} \quad \forall A \in G$$

which in view of (5) yields

$$\chi(A) = \sum_i a_i \chi^{(i)}(A) \quad \ldots(7)$$

Now if we call $\Gamma_{jk}^{(i)}$ as the representation vectors in the group space, then the orthogonality theorem for an irreducible representation in the present notation can be stated as

$$\sum_{A \in G} \Gamma_{jk}^{(i)}(A)\Gamma_{j'k'}^{(i')}\left(A^{-1}\right) = 0$$

i.e.,
$$\sum_{A \in G} \Gamma_{jk}^{(i)}(A)\Gamma_{k'j'}^{(i')^*}(A) = 0 \qquad \text{...(8)}$$

which means that the product of (j, k)th element of the irreducible representation $\Gamma^{(i)}$ with the complex conjugate of the (k', j')th element of the irreducible representation $\Gamma^{(i')}$ summed over all the group elements is zero.

Also taking g as the order of the group G and m as the dimension of the representation, in case of the elements of a single unitary irreducible representation, we have

$$\sum_{A \in G} \Gamma_{jk}^{(i)}(A)\, \Gamma_{j'k'}^{(i)}\left(A^{-1}\right) = \frac{g}{m}\, \delta_{jk'}\, \delta_{kj'}$$

i.e.
$$\sum_{A \in G} \Gamma_{jk}^{(i)}(A)\, \Gamma_{k'j'}^{(i')^*}(A) = \frac{g}{m}\, \delta_{jk'}\, \delta_{kj'} \qquad \text{...(9)}$$

Combining (8) and (9) we can write

$$\sum_{A \in G} \Gamma_{jk}^{(i)}(A)\, \Gamma_{k'j'}^{(i')^*}(A) = \frac{g}{m}\, \delta_{ii'}\, \delta_{jk'}\, \delta_{kj'} \qquad \text{...(10)}$$

In order to transform (10) into an orthogonality relation for the characters of irreducible representation, putting $j = k$ and $j' = k'$ and summing over j and j' we get with the hlep of (1),

$$\sum_{A \in G} \chi^{(i)}(A)\chi^{(i')^*}(A) = \frac{g}{m}\delta_{ii'} m = g\,\delta_{ii'} \qquad \text{...(11)}$$

where $\chi^{(i)}(A)$ represents the character of the element A in the irreducible representation $\Gamma^{(i)}$

Now multiplying both sides of (7) by $\chi^{(i')^*}(A)$ and summing over all the elements of G, we find

$$\sum_{A \in G} \chi'^*(A)\ \chi(A) = \sum_i a_i \sum_{A \in G} \chi^{(i')^*}\chi^{(i)}(A) = a_i\ g \text{ by (11)}$$

givin $\quad {}_{i'} = \dfrac{1}{g} \displaystyle\sum_{A \in G} \chi^{(i')^*}(A)\chi(A)$

or replacing i' by i, this becomes

$$a_i = \frac{1}{g} \sum_{A \in G} \chi^{(i)^*}(A)\chi(A) \qquad \text{...(12)}$$

which gives coefficient a_i as introduced in (6).

It should be noted that the characters of the irreducible representations are known as *Primitive* or *simple character* whereas those of reducible ones are known as *compound character*. Actually a compound character may be decomposed into simple characters.

Again multiplying (7) by its complex conjugate *i.e.*, $\chi^*(A)$ and summing over all the elements of G and then dividing *by* g, we get

$$\frac{1}{g}\sum_{A\varepsilon G}\chi^*(A)\chi(A)=\frac{1}{g}\sum_{i,j}a_i^*a_j\sum_{A\varepsilon G}\chi^{(i)*}(A)\chi^{(j)}(A)$$

$$=\sum_i |a_i|^2 \text{ by (12)} \qquad ...(13)$$

If the R.H.S. of (13) is unity for any representation Γ, then it follows that all a_i's except one must be zero. Let $a_k \neq 0$, then it must be unity *i.e.* $a_k = 1$, in which case Γ must be equivalent (identical) with the irreducible representation $\Gamma^{(k)}$ and hence for a representation to be irreducible the *criterion* (*i.e.*, the necessary and sufficient condition) is

$$\sum_{A\varepsilon G}\chi^*(A)\chi(A)=g \qquad ...(14)$$

Now to discuss some character tables, let us use the following notations.

Let the entire group consist of s classes say $c_1, c_2, c_3, ...c_s$ with elements $h_1, h_2, h_3, ..., h_s$ then clearly

$$g = h_1 + h_2 + ...h_s.$$

Also, take

E = Identity operation

C_n = rotation about an axis of symmetry through an angle equal to $\frac{2\pi}{n}$.

C_2', C_2'' = rotation through 180° about two axes perpendicular to C_n.

σv = reflection in a plane of symmetry, containing the principal axis *i.e.* the axis having the largest n value in C_n.

σ_d = reflection in a plane of symmetry, containing the principal axis and bisecting the angle between C_2' and C_2''.

σ_h = reflection in a plane perpendicular to the principal axis.

i = inversion in a centre of symmetry.

s_n = rotation about an axis by $\frac{2\pi}{n}$, followed by a reflection in a plane perpendicular to the axis of rotation.

Here $\sigma_d^2 = \sigma_d . \sigma_d$ and $C_n^3 = C_n . C_n . C_n$ etc...

Let R, T denote the irreducible representations of the rotational and translational coordinates respectively in addition to x, y, z coordinates. The reflections are also denoted by m, with subscripts if necessary, in places of σv (vertical reflection), σ_h (horizontal reflection) and σ_d (diagonal reflection).

To clearly understand these notations, we give two particular examples.

[1] Symmetry Transformations of an Equilateral Triangle.

Let A, B, C be the vertices of an equilateral traingle with D, E, F as mid-points of sides BC, CA, AB respectively and O the centroid.

To discuss the operations (transformations) of the triangle such as rotations reflections, inversion, translation etc., leaving the triangle invariant, if we take z-axis as an axis through O normal to the plane of triangle, then rotations through $\frac{2\pi}{3}$ about z-axis and its multiples leave the triangle invariant.

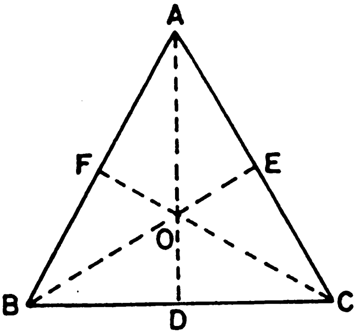


Fig. C. 1

The six symmetry operations of an equilateral triangle are:

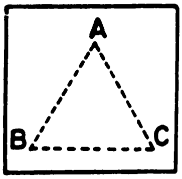


Fig.C.2

E→identity operation.

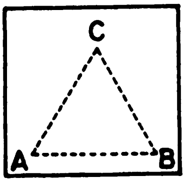


Fig.C.3

C_3→anticlockwise rotation through $\frac{2\pi}{3}$ about z- axis.

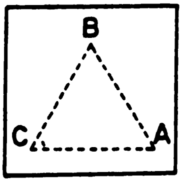


Fig. C.4

C_3^2→anticlockwise rotation through $\frac{4\pi}{3}$ about z- axis.

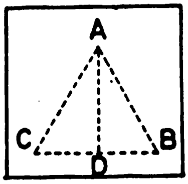


Fig. C.5

m_1→ reflection in the vertical plane passing through AD

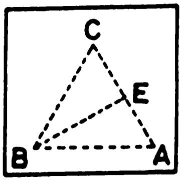


Fig.C.6

m_2→reflection in the vertical plane passing through BE.

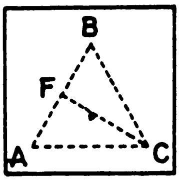


Fig. C.7

m_3→reflection in the vertical plane passing through CF.

It is easy to see that the operation $C_3 m_1$ is the same as that of m_3 *i.e.* $C_3 m_1 = m_3$ etc. Also since $(C_3)^{-1} = C_3^2$ gives $C_3(C_3)^{-1} = C_3.C_3^2$ *i.e.* $E = C_3.\ C_3^2$ therefore inverse of C_3 is C_3^2 and vice versa. E is the identity element.

Such a set being a group of order six is denoted by C_{3v} and known as the *symmetry group of an equilateral triangle.*

Second operation	First operation → E	C_3	C_3^2	m_1	m_2	m_3
E	E	C_3	C_3^2	m_1	m_2	m_3
C_3^2	C_3^2	E	C_3	m_2	m_3	m_1
C_3	C_3	C_3^2	E	m_3	m_1	m_2
m_1	m_1	m_3	m_2	E	C_3	C_3^2
m_2	m_2	m_1	m_3	C_3^2	E	C_3
m_3	m_3	m_2	m_1	C_3	C_3^2	E

The ordering of rows and colums being immaterial, we have chosen an ordering such that principal diagonal contains the identity element E only and this can be effected by choosing an order such that an element in the first column (second operation) is inverse of the corresponding element in first row (*i.e.* first operation).

Now if $G = \{E, A, B, C, \ldots\}$ be a finite group of order 'g' with identity element E and $T = \{T(E), T(A), T(B), \ldots\}$ be a set of square matrices all of the same order such that

$$T(A)\ T(B) = T(AB) \qquad \ldots(15)$$

then there arise two cases : (1) All the matrices of the set T are distinct in which case G and T are isomorphic to each other and so the representation generated by the elements of T is known as a *faithful representation* of G, (2) when the elements of T are not all distinct, C and T being homomorphic to each other, the representation is known as an *unfaithful representation* of G.

An idendity representation which is actually an unfaithful one is the simplest representation obtained by associating unity (which is a square matrix of order one) with every element of the group *e.g.* for the group C_{3v}, we have

Element :	E	C_3	C_3^2	m_1	m_2	m_3	...(16)
Representation :	1	1	1	1	1	1	

It should be noted that every group has at least one faithful representation.

Now to prepare the *character table*, for group C_{3v}, we are required to have the following points in consideration :

Note I. *If C be the total number of irreducible representations of a finite group* $G = \{E, A, B, \ldots\}$ *of order g and dimension* l_i *for* $i = 1, 2, 3 \ldots c$ *then* $\sum_{i=1}^{c} l_i^2 \leq g$. ...(17)

Note II. *The character being a function of the classes just as a representation is a function of the group elements, we have number of irreducible representation of G ≤ number of classes of G* ...(17 *a*)

Note III. *Taking* n_k *as the number of elements in a class,* C_k *of group G, and* $\phi^{(i)}$ *as the character vectors in the class space, the orthogonality of characters yields for different classes,*

$$\sum_{i=1}^{c} \phi_k^{(i)*}\phi_l^{(i)} = \frac{g}{n_k}\delta_{kl} \quad ...(18)$$

whereas the product of two classes is defined as

$$\mathcal{C}_i\mathcal{C}_j = \sum_k a_{ijk}\mathcal{C}_k \quad ...(19)$$

a_{ijk} being non-negative integers.

Denoting by P_i^{α} the resulting matrix obtained by adding the matrices representing the elements of $\mathcal{C}_i$, in the irreducible representation $\Gamma^{(\alpha)}$, we have

$$P_i^{\alpha} = \sum_{A \in \mathcal{C}_i} \Gamma^{(\alpha)}(A) \quad ...(20)$$

Constructing the matrices for all the classes of G in a similar way, we have the **normalization condition** *for any $B \varepsilon G$.*

$$[\Gamma^{(\alpha)}(B)]^{-1} P_i^{(\alpha)} \Gamma(B) = \sum_{A \in \mathcal{C}_i} [\Gamma^{(\alpha)}(B)]^{-1}(A)\Gamma^{(\alpha)}(B)$$

$$= \sum_{A \in \mathcal{C}_i} \Gamma^{(\alpha)}\left(B^{-1}AB\right) = \sum_{A \in \mathcal{C}_j} \Gamma^{(\alpha)}(A) \quad ...(21)$$

Note IV. Conjugate elements and classes. *If A, B and C be elements of a group such that $A^{-1}BA = C$, then B and C are said to be* **conjugate elements** *and the operation is said to be a* **similarity transformation** *of B by A Clearly $ACA^{-1} = B$* ...(22)

If we split the elements of a group into sets such that all the elements of set are conjugate to each other, but no two elements of different sets are conjugate to each other, then such sets of elements are said to be the **conjugacy class** *or simply* **classes of a group.**

How to prepare a character table for a group say C_{3v}.

1. According to the preceding note, we conclude that:

(E) Constitutes a class by itself in any group, since for any element A of the group, $A^{-1}EA = E$.

Also (C_3, C_3^2) is a class of C_{3v} since $m_1^{-1}C_3m_1 = C_3^2$ as $m_1^{-1} = m_1$ renders, $(m_1^{-1}C_3)m_1 = (m_1C_3)m_1 = m_3m_1 = C_3^2$ from the table.

Similarly (m_1, m_2, m_3) is a class of C_{3v}.

Hence the classes of C_{3v} are (E), (C_3, C_3^2,) (m_1, m_2, m_3) say, $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ respectively. Then $\mathcal{C}_1 = (E)$; $\mathcal{C}_2 = (C_3, C_3^2) = 2C_3$ (say) and $\mathcal{C}_3 = (m_1, m_2, m_3) = 3\sigma_v$ (say).

2. Since C_{3v} has three classes, it must have three irreducible representations say $\Gamma^{(1)}$, $\Gamma^{(2)}$, $\Gamma^{(3)}$ whose dimensions may be taken as l_1, l_2, l_3 respectively.

In view of (17), here $g = 6$ since C_{3v} consists of 6 elements,

$\therefore\ l_1^2 + l_2^2 + l_3^2 = 6.$

With integral l_i, the only possible solution is that two of l_i's are unity while the remaining one is 2 say $l_1 = l_2 = 1$ and $l_3 = 2$.

3. In view of (16), the first row is obtained by writing unity for character of each class. The first column is obtained in view of the facts that the matrix of E in any

representation is a unit matrix having its trace or character as l_i which is the dimension of the representaion *viz.* $l_1 = 1 = l_2$, $l_3 = 2$.

4. The character table for C_{3v}.

Characters ↓		Classes → $\mathscr{C}_1$ (E)	$\mathscr{C}_2$ (C_3, C_3^2) *i.e.* $2C_3$	$\mathscr{C}_3$ (m_1, m_2, m_3) *i.e.* $3\sigma_v$
$\Gamma^{(1)} \rightarrow$	$\chi^{(1)}$	1	1	1
$\Gamma^{(2)} \rightarrow$	$\chi^{(2)}$	1	1	-1
$\Gamma^{(3)} \rightarrow$	$\chi^{(3)}$	2	- 1	0

Explanation. The Character being identical with one-dimensional representation, for $\Gamma^{(1)}$ and $\Gamma^{(2)}$ the characters themselves must satisfy the multiplication table so that for elements whose square equals E such as m_1, m_2, m_3 the ony allowed characters are $\pm$ 1.

Now the elements in the same class having the same character as the character of (C_3, C_3^2) in $\Gamma^{(2)}$ can be determined.

Also since for C_3, C_3^2 with $(C_3)^3 = (C_3^2)^3 = E$, the one-dimensional representation could be the power of $i = \sqrt{-1}$, for, if n is the order of an element A so that $A^n = E$, then its only one-dimensional representation can be powers of $e^{2\pi i/n}$ as these are only numbers whose nth power yields unity because such numbers are *unitary* as their inverses are equal to their complex conjugate. Again since $(C_3)^2 = (C_3^2)^2 = C_3^2$, $\chi(C^3)$ and χ (C_3^2) can only be $\pm$ 1 for $\Gamma^{(2)}$. Now the normalization condition (21) can only be achieved by taking + 1 in $\Gamma^{(2)}$. Finally the third row is determined by using the orthogonality relations (18) for the columns.

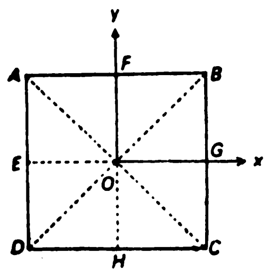
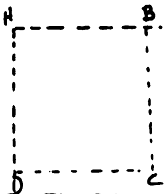


Fig. C.8

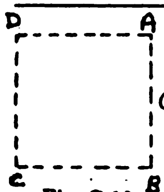
(2) Symmetry Transformations of a Square.

Let A, B, C, D be the vertices of a square with E, F, G, H as mid-points of sides DA, AB, BC and CD respectively and O the centroid. Take OG as x-axis, OF as y-axis and an axis through O normal to the plane of the square as z-axis. Then the eight symmetry operations of the square are as follows:



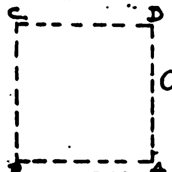
$E \rightarrow$ identity operation

Fig. C.9



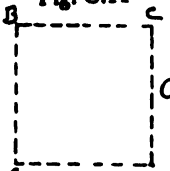
$C_4 \rightarrow$ clockwise rotation through 90° about z-aixs.

Fig. C.10



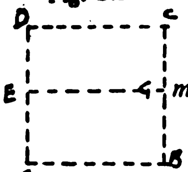
$C_4^2 \rightarrow$ clockwise rotation through 180° about z-axis

Fig. C.11



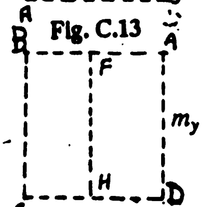
$C_4^3 \rightarrow$ clockwise rotation through 270° about z-axis

Fig. C.12



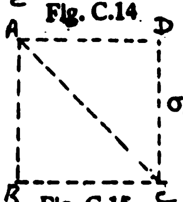
$m_x \rightarrow$ reflection in the vertical plane passing through EG *i.e.* x-axis

Fig. C.13



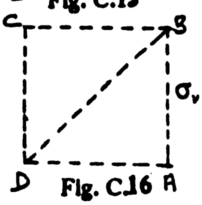
$m_y \rightarrow$ reflection in the vertical plane passing through FH *i.e.* y-axis

Fig. C.14



$\sigma_u \rightarrow$ reflection in the vertical plane passing through AC

Fig. C.15



$\sigma_v \rightarrow$ reflection in the vertical plane passing through BD

Fig. C.16

It is easy to see that $\sigma_u C_4 = m_x$ etc.

Also $(C_4)^{-1} = C_4^3$ *i.e.* $C_4 C_4^3 = C_4^3 C_4 = E$ *i.e.* C_4 is the inverse of C_4^3 etc.

We also have $C_4 m_x = \sigma_u$, $\sigma_v C_4^3 = m_x$, $\sigma_u \sigma_v = C_4^2$ etc.

Group multiplication table for C_{4v}.

Second operation ↓	First operation → E	C_4	C_4^2	C_4^3	m_x	m_y	σ_u	σ_v
E	E	C_4	C_4^2	C_4^3	m_x	m_y	σ_u	σ_v
C_4^3	C_4^3	E	C_4	C_4^2	σ_v	σ_u	m_x	m_y
C_4^2	C_4^2	C_4^3	E	C_4	m_y	m_x	σ_v	σ_u
C_4	C_4	C_4^2	C_4^3	E	σ_u	σ_v	m_y	m_x
m_x	m_x	σ_v	m_y	σ_u	E	C_4^2	C_4^3	C_4
m_y	m_y	σ_u	m_x	σ_v	C_4^2	E	C_4	C_4^3
σ_u	σ_u	m_x	σ_v	m_y	C_4	C_4^3	E	C_4^2
σ_v	σ_v	m_y	σ_x	m_x	C_4^3	C_4	C_4^2	E

In view of (22), the five classes of C_{4v} are

(E), $(C_4)^2$, (C_4, C_3), (m_x, m_y), (σ_u, σ_v)

(since $C_4^{-1}\, m_x\, C_4 = m_y$ as $(C_4^{-1}\, m_x)\, C_4 = C_4^3 m_x) C_4 = \sigma_u C_4 = m_y$ etc).

These classes can also be justified by the following simple rules :

1. Since rotations through different angles must be put in different classes, C_4 and C_4^2 belong to different classes.

2. Since rotations through an angle in clockwise and anticlockwise sense about an axis belong to a class only when there exists a transformation in the group, such that it reverses the sense of the coordinate system, therefore C_4 and C_4^3 are put in the same class, for m_x or σ_u operation changes the sense of the coordinate system.

3. Since rotations through the same angle about two different axes or reflection in two different planes belong to the same class only when there exists some element of group, which is capable of bringing the two axes or planes into each other, therefore m_x, m_y belong to the same class but σ_u, m_x do not belong to the same class, for in the former case σ_u can bring the line EF into the line FH.

The character table for C_{4v}

Characters ↓	classes → $\mathscr{C}_1$ (E)	$\mathscr{C}_2$ (C_4^2)	$\mathscr{C}_3$ (C_4, C_4^3) i.e., $2\, C_4$	$\mathscr{C}_4$ (m_x, m_y) i.e. $2m$	$\mathscr{C}_5$ (σ_u, σ_v) (i.e., 2σ)
$\Gamma^{(1)} \to \chi^{(1)}$	1	1	1	1	1
$\Gamma^{(2)} \to \chi^{(2)}$	1	1	1	−1	−1
$\Gamma^{(3)} \to \chi^{(3)}$	1	1	−1	1	−1
$\Gamma^{(4)} \to \chi^{(4)}$	1	1	−1	−1	1
$\Gamma^{(5)} \to \chi^{(5)}$	2	−2	0	0	0

Explanation. Since C_{4v} has five classes, it must have five irreducible representations by 17(*a*) say $\Gamma^{(1)}, \Gamma^{(2)}, \Gamma^{(3)}, \Gamma^{(4)}, \Gamma^{(5)}$, whose dimensions are l_1, l_2, l_3, l_4, l_5 respectively, such that by (17)

$$l_1^2 + l_2^2 + l_3^2 + l_4^2 + l_5^2 = 8,$$

since C_{4v} consists of 8 elements *i.e.* $g = 8$

With integral l_i, the only possible solution is that $l_1 = l_2 = l_3 = l_4 = 1$ and $l_5 = 2$. Now in view of (16), the first row is obtained by writing unity for character of each class and the first column is obtained by the fact that the matrix of E in any representation is a unit matrix having its trace or character as l_i, the dimension of representation *viz* $l_1 = l_2 = l_3 = l_4 = 1, l_5 = 2$.

Now the character being identical with one-dimensional representation for $\Gamma^{(1)}$ through $\Gamma^{(4)}$ the characters must themselves satisfy the multiplication table so that for elements whose square equals E such as C_4^2, m_x, σ_u, the characters to be allowed are ± 1. But since $m_x m_y = C_4^2$ or $\sigma_u \sigma_v = C_4^2$, therefore m_x and m_y both are represented by + 1 or – 1, because elements in the same class have the same characters and hence the characters of C_4^2 in $\Gamma^{(2)}, \Gamma^{(3)}, \Gamma^{(4)}$ are determined. Also since for C_4 and C_4^3, with $(C_4)^4 = (C_4^3)^4 = E$, the one-dimensional representation could be the power of $i = \sqrt{-1}$. Further $(C_4)^2 = (C_4^3)^2 = C_4^2$ imply that $\phi(C_4^3)$ can only be $\pm$ 1 for $\Gamma^{(2)}, \Gamma^{(3)}, \Gamma^{(4)}$. Now the normalization condition (21) can be achieved by taking + 1 for one of the classes $\mathscr{C}_3$, $\mathscr{C}_4$, $\mathscr{C}_5$ in representations $\Gamma^{(2)}, \Gamma^{(3)}, \Gamma^{(4)}$ and – 1 for the remaining ones. We thus determine characters for the first four representations and the fifth one can be obtained in view of (18) for the columns.

Character tables for some Symmetry Point Groups

C_1	E
	1

C_2	E	C_2
	1	1
	1	–1

C_{2v}	E	C_2	σ_u	σ_v
	1	1	1	1
	1	1	–1	–1
	1	–1	1	–1
	1	–1	–1	1

C_3	E	C_3	C_3^2	$\omega = e^{\frac{2\pi i}{3}}$
	1	1	1	
	1	ω	ω^2	
	1	ω^2	ω	

C_4	E	C_4	C_4^2	C_4^3
	1	1	1	1
	1	–1	1	–1
	1	$-i$	–1	i
	1	i	–1	$-i$

D_2	E	$C_2(x)$	$C_2(y)$	$C_2(z)$
	1	1	1	1
	1	1	−1	−1
	1	−1	1	−1
	1	−1	−1	1

D_3	E	$2C_3$	$3C_2'$
	1	1	1
	1	1	−1
	2	−1	0

D_4	E	C_4^2	$2C_4$	$2C_2'$	$2C_2''$
	1	1	1	1	1
	1	1	1	−1	−1
	1	1	−1	1	−1
	1	1	−1	−1	1
	2	−2	0	0	0

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