CHAPTER

8

System of Particles: Conservatiion Laws and Collisions

8.1 INTRODUCTION

In this and subsequent chapters we investigate the motion of a system of particles or body with a large number of particles. Whenever we are dealing with such situations, it is not only convenient but essential to describe the motion in the center of mass coordinates. We must understand the laws of conservation of linear momentum, angular momentum, and energy as applied to such systems. These laws will then be applied to some physical systems of interest, such as the motion of rockets and conveyor belts. We will show that the use of these laws is indispensable in the investigation of scattering or collision problems, both elastic and inelastic. Such investigations lead to an understanding of interactions between microscopic as well as macroscopic systems.

8.2 SYSTEM OF PARTICLES AND CENTER OF MASS

Whenever we are dealing with a system containing a large number of particles, it is, as said, both convenient and essential to describe the motion in the center of mass coordinates. Accordingly, let us consider a system containing N particles labeled 1, 2, . . . , N. The masses of these particles are $m_1, m_2, \ldots, m_N$ and they are located at distances $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N$ from the origin O, as shown in Fig. 8.1. The velocities of these particles are $\dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2, \ldots, \dot{\mathbf{r}}_N$ (or $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N$), while their accelerations are $\ddot{\mathbf{r}}_1, \ddot{\mathbf{r}}_2, \ldots, \ddot{\mathbf{r}}_N$ (or $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_N$), respectively. For such a system of particles, the center of mass is a point located at a distance $\mathbf{R}(X, Y, Z)$ from the origin and defined by the relation

$$(m_1 + m_2 + \cdots + m_N)\mathbf{R} = m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \cdots + m_N\mathbf{r}_N$$

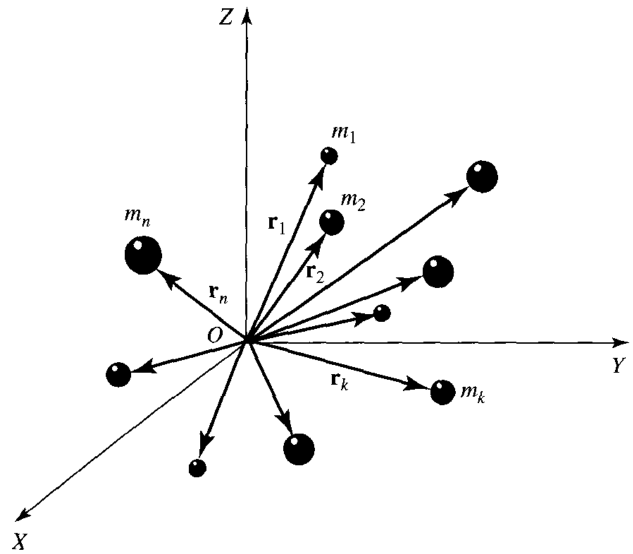


Figure 8.1 System of particles of various masses at different distances from the origin.

or

$$\sum_{k=1}^{N} m_k \mathbf{R} = \sum_{k=1}^{N} m_k \mathbf{r}_k$$

That is,

$$\mathbf{R} = \frac{\Sigma\, m_k \mathbf{r}_k}{\Sigma\, m_k} = \frac{\Sigma\, m_k \mathbf{r}_k}{M} \tag{8.1}$$

where $M = \Sigma\, m_k$ is the sum of all the masses in the system and the summation Σ is from $k = 1$ to $k = N$. In component form, we may write

$$X = \frac{1}{M} \sum m_k x_k, \qquad Y = \frac{1}{M} \sum m_k y_k, \qquad Z = \frac{1}{M} \sum m_k z_k \tag{8.2}$$

It should be clear from Eq. (8.1) that the center of mass is a *mass-weighted* average position.

The velocity $\mathbf{V}(=\dot{\mathbf{R}})$ of the center of mass can be obtained by differentiating Eq. (8.1) with respect to t; that is,

$$\mathbf{V} = \dot{\mathbf{R}} = \frac{1}{M} \sum m_k \dot{\mathbf{r}}_k \tag{8.3}$$

while the components of the velocity of the center of mass may be written as

$$V_X = \dot{X} = \frac{1}{M} \sum m_k \dot{x}_k, \qquad V_Y = \dot{Y} = \frac{1}{M} \sum m_k \dot{y}_k, \qquad V_Z = \dot{Z} = \frac{1}{M} \sum m_k \dot{z}_k \tag{8.4}$$

The acceleration $\mathbf{A}$ of the center of mass is obtained by differentiating once more; that is,

$$\mathbf{A} = \ddot{\mathbf{R}} = \frac{1}{M} \sum m_k \ddot{\mathbf{r}}_k \tag{8.5}$$

or, in component form,

$$A_X = \ddot{X} = \frac{1}{M}\sum m_k\ddot{x}_k, \qquad A_Y = \ddot{Y} = \frac{1}{M}\sum m_k\ddot{y}_k, \qquad A_Z = \ddot{Z} = \frac{1}{M}\sum m_k\ddot{z}_k \tag{8.6}$$

In the following sections, we shall find the description of motion of the center of mass coordinates and motion in the center of mass coordinate systems both interesting and useful.

We shall discuss the following three conservation laws in detail as applied to a system of particles:

1. conservation of linear momentum
2. conservation of angular momentum
3. conservation of energy

There are two approaches to this problem: (1) Newton's laws, and (2) symmetry principles.

The conservation laws are the direct consequence of the definitions made in Newtonian mechanics, that is, of Newton's second law of motion. The validity of these conservation laws holds to the extent that Newtonian mechanics provides an adequate description of nature. Furthermore, since there is no such thing as a truly isolated system, these laws can only hold approximately. But ultimately, from a modern point of view, these conservation laws are the consequence of underlying symmetries briefly discussed here and in detail in Chapter 12.

In general, a system is said to have symmetry when some characteristic in the system remains unchanged even though the system is changed in a certain respect. For example, if the system is given a linear displacement, the system remains invariant under linear displacement or translation and the system is said to have *translational symmetry*. Similarly, a system is said to have *rotational symmetry* if it remains invariant under rotation. There is a close relationship between conservation laws and symmetry principles. The conservation of linear momentum is a direct consequence of translational symmetry, that is, the *homogeneity of space*. The law of conservation of angular momentum is the consequence of rotational symmetry, that is, the *isotropy of space,* while the law of conservation of energy leads to the *homogeneity of time*. Actually, we may go a step further and state:

Any conservation law is a statement of invariance of some physical property during all physical processes.

For the time being, we shall investigate the conservation laws from the viewpoint of Newtonian mechanics.

8.3 CONSERVATION OF LINEAR MOMENTUM

For a single particle of mass m moving with velocity $\mathbf{v}$ and linear momentum $\mathbf{p}$, Newton's second law is

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \tag{8.7}$$

where $\mathbf{F}$ is the net external force acting on mass m and

$$\mathbf{p} = m\mathbf{v} \tag{8.8}$$

If m is constant and does not depend on time,

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(m\mathbf{v}) = m\frac{d\mathbf{v}}{dt} = m\mathbf{a} \tag{8.9}$$

Furthermore, if $\mathbf{F} = 0$, $\mathbf{p} =$ constant, which is the law of conservation of linear momentum for a single particle.

We now extend these ideas to a system of N particles, as shown in Fig. 8.1. Let us consider the motion of the kth particle of mass m_k, which is at a distance $\mathbf{r}_k$ from the origin, has velocity $\dot{\mathbf{r}}_k(=\mathbf{v}_k)$, and acceleration $\ddot{\mathbf{r}}_k$. The total force $\mathbf{F}_k$ acting on the kth particle is the sum of the set of two forces: (1) the sum of the external forces $\mathbf{F}_k^e$ applied to the kth particle, and (2) the sum of the internal force $\mathbf{F}_k^i$ on the kth particle by the remaining $N-1$ particles in the system. Thus the equation of motion for the kth particle, according to Newton's law, is

$$\mathbf{F}_k = \mathbf{F}_k^e + \mathbf{F}_k^i = m_k\ddot{\mathbf{r}}_k, \qquad k = 1, 2, \ldots, N \tag{8.10}$$

where

$$\mathbf{F}_k^i = \sum_{\substack{k=1\\k\neq l}}^{N} \mathbf{F}_{kl}^i \tag{8.11}$$

and $\mathbf{F}_{kl}^i$ is the force on the kth particle due to the lth particle. Because of the vector nature of Eq. (8.10), there are $3N$ simultaneous second-order differential equations to be solved. The motion of any particle k at $\mathbf{r}_k$ is obtained by solving such equations in terms of $6N$ arbitrary constants ($3N$ for the initial positions and $3N$ for initial velocities). No general methods are available for solving Eq. (8.10), which is extremely difficult to solve except in some special cases. An alternative approach is to solve these problems by using the center of mass coordinates, as will be explained later.

The momentum of the kth particle is given by

$$\mathbf{p}_k = m_k\mathbf{v}_k = m_k\dot{\mathbf{r}}_k \tag{8.12}$$

Using this, Eq. (8.10) takes the form

$$\frac{d\mathbf{p}_k}{dt} = \mathbf{F}_k = \mathbf{F}_k^e + \mathbf{F}_k^i \tag{8.13}$$

Summing on both sides over all the N particles,

$$\sum_{k=1}^{N}\frac{d\mathbf{p}_k}{dt} = \frac{d}{dt}\sum_{k=1}^{N}\mathbf{p}_k = \sum_{k=1}^{N}\mathbf{F}_k = \sum_{k=1}^{N}\mathbf{F}_k^e + \sum_{k=1}^{N}\mathbf{F}_k^i \tag{8.14}$$

Let $\mathbf{P}$ be the total linear momentum of the system of N particles and $\mathbf{F}$ be the total external force acting on the system; that is,

$$\mathbf{P} = \sum_{k=1}^{N}\mathbf{p}_k = \sum_{k=1}^{N} m_k\dot{\mathbf{r}}_k \tag{8.15}$$

and
$$\mathbf{F} = \sum_{k=1}^{N} \mathbf{F}_k^e \tag{8.16}$$

Furthermore, we shall show that the sum of all the internal forces acting on all the particles of the system is zero; that is

$$\sum_{k=1}^{N} \mathbf{F}_k^i = 0 \tag{8.17}$$

Combining Eqs. (8.15), (8.16), and (8.17) with Eq. (8.14), we obtain

$$\frac{d\mathbf{P}}{dt} = \mathbf{F} \tag{8.18}$$

This is the *momentum theorem* for a system of particles.

> ***Conservation of Linear Momentum.*** *The rate of change of total linear momentum is equal to the total external applied force; thus, if the sum of all the externally applied forces is zero, the total linear momentum* **P** *of the system will be constant.*

That is,

$$\mathbf{P} = \text{constant}, \qquad \text{if } \mathbf{F} = 0 \tag{8.19}$$

In terms of the center of mass coordinates, according to Eqs. (8.3) and (8.15),

$$\mathbf{P} = \sum_{k=1}^{N} m_k \dot{\mathbf{r}} = M\dot{\mathbf{R}} \tag{8.20}$$

which on substituting in Eq. (8.18) yields

$$M\ddot{\mathbf{R}} = \mathbf{F} \tag{8.21}$$

Equations (8.18) and (8.21) are similar in form to Newton's second law as applied to a single particle. Thus, from Eq. (8.21), we may conclude:

> *The center of mass of a system of particles moves like a single particle of mass M (total mass of the system) acted on by a single force* **F** *that is equal to the sum of all the external forces acting on the system.*

All these statements are true only if we can justify Eq. (8.17)—that is, the sum of all internal forces is zero. We now proceed to prove this by two different approaches: (1) Newton's third law, and (2) the principle of virtual work. According to Eq. (8.11),

$$\mathbf{F}_k^i = \sum_{\substack{k=1 \\ k \neq l}}^{N} \mathbf{F}_{kl}^i \tag{8.11}$$

where $\mathbf{F}^i_{kl}$ is the force exerted on the kth particle by the lth particle. According to Newton's third law, the force exerted on the kth particle due to the lth particle is equal and opposite to that exerted on l by k; that is,

$$\mathbf{F}^i_{kl} = -\mathbf{F}^i_{lk} \tag{8.22}$$

This equation is a statement of *Newton's third law in the weak form* because it simply implies that the two forces are equal and opposite, but not necessarily acting along the line joining the two particles; the *strong form* implies that their line of action should be the same. Using Eq. (8.11), the sum of all the internal forces is

$$\sum_{k=1}^{N} \mathbf{F}^i_k = \sum_{k=1}^{N} \sum_{\substack{l=1 \\ l \neq k}}^{N} \mathbf{F}^i_{kl} \tag{8.23}$$

The right side contains forces on all pairs of particles. For each pair, the total sum according to Eq. (8.22) is zero; that is $\mathbf{F}^i_{kl} + \mathbf{F}^i_{lk} = 0$. Hence the right side of Eq. (8.23) is zero, thereby proving that in Eq. (8.11) the right side is zero. That is, *the sum of all the internal forces is zero.*

In the preceding proof we had to assume that the internal forces come in pairs. We need not make this assumption if we make use of the *principle of virtual work* or *virtual displacement.* Let us assume that each particle in the system is given a small displacement $\delta\mathbf{r}$. Since each particle in the system is given the same displacement, there is no relative displacement of the system; hence no net work is done by the internal forces. No net total work is done because the internal state of the system has not changed by this virtual or imaginary displacement. The work done by the internal forces $\mathbf{F}^i_k$ in a small virtual displacement $\delta\mathbf{r}$ of the kth particle is

$$\delta W_k = \mathbf{F}^i_k \cdot \delta\mathbf{r} \tag{8.24}$$

The total work done by all the internal forces is

$$\delta W = \sum_{k=1}^{N} \delta W_k = \sum_{k=1}^{N} (\mathbf{F}^i_k \cdot \delta\mathbf{r}) = \delta\mathbf{r} \cdot \left[\sum_{k=1}^{N} \mathbf{F}^i_k\right] \tag{8.25}$$

$\delta\mathbf{r}$ has been factored out because it is the same for all the particles. If the total work done by the internal forces is zero for any displacement,

$$\delta\mathbf{r} \cdot \left[\sum_{k=1}^{N} \mathbf{F}^i_k\right] = 0$$

Since $\delta\mathbf{r}$ is not zero, we must have

$$\sum_{k=1}^{N} \mathbf{F}^i_k = \sum_{k=1}^{N} \sum_{\substack{l=1 \\ l \neq k}}^{N} \mathbf{F}^i_{kl} = 0 \tag{8.26}$$

as required.

8.4 CONSERVATION OF ANGULAR MOMENTUM

The angular momentum of a single particle is defined in terms of a cross product as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\dot{\mathbf{r}} = \mathbf{r} \times m\mathbf{v} \tag{8.27}$$

Now we extend this definition to a system of N particles. The total angular momentum $\mathbf{L}$ taken about the origin may be written as a vector sum:

$$\mathbf{L} = \sum_{k=1}^{N} (\mathbf{r}_k \times \mathbf{p}_k) = \sum_{k=1}^{N} (\mathbf{r}_k \times m_k\dot{\mathbf{r}}) \tag{8.28}$$

The total angular momentum could have been taken about any point A instead of the origin O, but in that case we must replace $\mathbf{r}_k$ by $\mathbf{r}_k - \mathbf{r}_A$, where $\mathbf{r}_A$ is the distance of point A from the origin. For simplicity, we shall use expression (8.28). Taking the time derivative of the angular momentum in Eq. (8.28) yields

$$\frac{d\mathbf{L}}{dt} = \sum_{k=1}^{N}(\dot{\mathbf{r}}_k \times m_k\dot{\mathbf{r}}_k) + \sum_{k=1}^{N}(\mathbf{r}_k \times m_k\ddot{\mathbf{r}}) \tag{8.29}$$

The first term on the right vanishes because of the definition of the cross product ($\dot{\mathbf{r}} \times m\dot{\mathbf{r}} = 0$), while $m\ddot{\mathbf{r}}$, from Eq. (8.10), is equal to the total force acting on the particle k; that is, we obtain

$$\frac{d\mathbf{L}}{dt} = \sum_{k=1}^{N}\left[\mathbf{r}_k \times \left(\mathbf{F}_k^e + \sum_{\substack{l=1 \\ l\neq k}}^{N} \mathbf{F}_{kl}^i\right)\right] = \sum_{k=1}^{N} \mathbf{r}_k \times \mathbf{F}_k^e + \sum_{k=1}^{N}\sum_{\substack{l=1 \\ l\neq k}}^{N} \mathbf{r}_k \times \mathbf{F}_{kl}^i \tag{8.30}$$

where, as before, $\mathbf{F}_k^e$ is the total external force acting on the particle k, and $\mathbf{F}_{kl}^i$ is the internal force acting on the kth particle due to the lth particle. We can prove the second term on the right to be zero if we use the *strong form of Newton's third law*; that is, the forces are equal and opposite and their line of action is the same. The second term on the right contains a sum of pairs of torques due to pairs of forces, which according to Newton's third law are equal and opposite. One such pair is

$$(\mathbf{r}_k \times \mathbf{F}_{kl}^i) + (\mathbf{r}_l \times \mathbf{F}_{lk}^i) \tag{8.31}$$

Since $\mathbf{F}_{kl}^i = -\mathbf{F}_{lk}^i$, we may write the expression in Eq. (8.31) as

$$(\mathbf{r}_k - \mathbf{r}_l) \times \mathbf{F}_{kl}^i = \mathbf{r}_{kl} \times \mathbf{F}_{kl}^i \tag{8.32}$$

(see Fig. 8.2). Expression (8.32) is zero if the internal forces are central; that is, the forces acting along the line joining the two particles cause the two particles to either attract or repel each other. Thus the second term on the right in Eq. (8.30) vanishes, and the resulting equation is

$$\frac{d\mathbf{L}}{dt} = \sum_{k=1}^{N} \mathbf{r}_k \times \mathbf{F}_k^e \tag{8.33}$$

Since $\mathbf{r}_k \times \mathbf{F}_k^e$ is the *torque* or the *moment of the external force* $\mathbf{F}_k^e$, the right side of Eq. (8.33) is the total moment (or the total torque) of all the external forces acting on the system. If we denote $\boldsymbol{\tau}_k$ to be the torque on the kth particle and $\boldsymbol{\tau}$ to be the total torque, we may write

$$\frac{d\mathbf{L}}{dt} = \sum_{k=1}^{N} \boldsymbol{\tau}_k = \sum_{k=1}^{N} \mathbf{r}_k \times \mathbf{F}_k^e \tag{8.34}$$

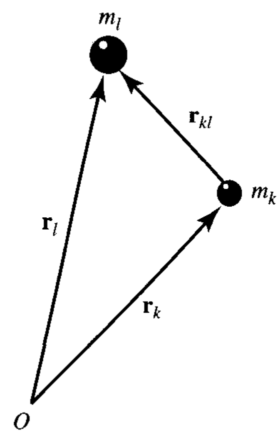


Figure 8.2 Relative distance $\mathbf{r}_{kl}$ between a pair of particles.

and
$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau} \tag{8.35}$$

which states that the time rate of change of the angular momentum of a system is equal to the total torque due to all the net external forces acting on the system. Thus we may state the following principle:

> ***Conservation of Angular Momentum.*** *For an isolated—one on which no net external forces act—the total torque τ will be zero; hence the angular momentum remains constant both in magnitude and direction.*

That is, if

$$\boldsymbol{\tau} = 0, \qquad \frac{d\mathbf{L}}{dt} = 0$$

and
$$\mathbf{L} = \sum_{k=1}^{N} \mathbf{r}_k \times m_k \mathbf{v}_k = \text{constant} \tag{8.36}$$

8.5 CONSERVATION OF ENERGY

In many situations, the total force acting on any particle in a system of particles is a function of the positions of the particles in the system. Thus the force $\mathbf{F}_k$ on the kth particle is

$$\mathbf{F}_k = \mathbf{F}_k^e + \mathbf{F}_k^i = \mathbf{F}_k(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N), \qquad \text{where } k = 1, 2, \ldots, N \tag{8.37}$$

The external forces $\mathbf{F}_k^e$ may depend on the position $\mathbf{r}_k$ of the particle k, while the internal force $\mathbf{F}_k^i$ may depend on the relative positions of the other particles relative to k, that is, $\mathbf{r}_{kl} = (\mathbf{r}_k - \mathbf{r}_l)$, and so on. If the force $\mathbf{F}_k$ satisfies the condition that

$$\nabla \times \mathbf{F}_k = \mathbf{curl}\,\mathbf{F}_k = 0 \tag{8.38}$$

there exists a potential function

$$V = V(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) \tag{8.39}$$

such that

$$F_{kx} = -\frac{\partial V}{\partial x_k}, \qquad F_{ky} = -\frac{\partial V}{\partial y_k}, \qquad F_{kz} = -\frac{\partial V}{\partial z_k}, \qquad \text{where } k = 1, 2, \ldots, N \tag{8.40}$$

Thus, under such conditions, we can derive the law of conservation of energy.

The motion of the kth particle is described by

$$m_k \ddot{\mathbf{r}}_k = m_k \dot{\mathbf{v}}_k = \mathbf{F}_k \tag{8.41}$$

which, on combining with Eq. (8.40), yields

$$m_k \frac{dv_{kx}}{dt} = -\frac{\partial V}{\partial x_k}, \qquad m_k \frac{dv_{ky}}{dt} = -\frac{\partial V}{\partial y_k}, \qquad m_k \frac{d_{kz}}{dt} = -\frac{\partial V}{\partial z_k} \tag{8.42}$$

Multiplying the first equation by v_{kx} $(=dx_k/dt)$, the second equation by v_{ky} $(=dy_k/dt)$, and the third equation by v_{kz} $(=dz_k/dt)$, and adding (using $v_k^2 = v_{kx}^2 + v_{ky}^2 + v_{kz}^2$), we get

$$\frac{d}{dt}\left(\frac{1}{2} m_k v_k^2\right) + \frac{\partial V}{\partial x_k}\frac{dx_k}{dt} + \frac{\partial V}{\partial y_k}\frac{dy_k}{dt} + \frac{\partial V}{\partial z_k}\frac{dz_k}{dt} = 0, \qquad \text{where } k = 1, 2, \ldots, N \tag{8.43a}$$

Summing over all values of k gives

$$\frac{d}{dt}\sum_{k=1}^{N}\left(\frac{1}{2} m_k v_k^2\right) + \sum_{k=1}^{N}\left(\frac{\partial V}{\partial x_k}\frac{dx_k}{dt} + \frac{\partial V}{\partial y_k}\frac{dy_k}{dt} + \frac{\partial V}{\partial z_k}\frac{dz_k}{dt}\right) = 0 \tag{8.43b}$$

where

$$\sum_{k=1}^{N}\left(\frac{1}{2} m_k v_k^2\right) = K \qquad \text{(kinetic energy)} \tag{8.44}$$

and

$$\sum_{k=1}^{N}\left(\frac{\partial V}{\partial x_k}\frac{dx_k}{dt} + \frac{\partial V}{\partial y_k}\frac{dy_k}{dt} + \frac{\partial V}{\partial z_k}\frac{dz_k}{dt}\right) = \frac{dV}{dt} \tag{8.45}$$

Hence Eq. (8.43b) takes the form

$$\frac{d}{dt}(K + V) = 0$$

or

$$K + V = E = \text{constant} \tag{8.46}$$

The total energy E, which is the sum of the kinetic and potential energy, is constant; hence Eq. (8.46) is a statement of the *law of conservation of energy* or the *energy conservation theorem.*

If the external forces are not position dependent, while the internal forces are derivable from a potential function, then the energy conservation theorem takes the form

$$\frac{d}{dt}(K + V^i) = \sum_{k=1}^{N} \mathbf{F}_k^e \cdot \dot{\mathbf{r}}_k \tag{8.47}$$

Since we have assumed that in this case the internal forces are position dependent and the corresponding potential V^i depends on the relative positions of pairs of particles, that is,

$$V_{kl}^i = V_{kl}^i(\mathbf{r}_{kl}) = V_{kl}^i(\mathbf{r}_k - \mathbf{r}_l) \tag{8.48}$$

while
$$V^i = \sum_{k=1}^{N} \sum_{l=1}^{k-1} V^i_{kl}(\mathbf{r}_{kl}) \tag{8.49}$$

we may conclude that
$$\mathbf{F}^i_k = -\hat{\mathbf{i}}\frac{\partial V^i}{\partial x_k} - \hat{\mathbf{j}}\frac{\partial V^i}{\partial y_k} - \hat{\mathbf{k}}\frac{\partial V^i}{\partial z_k} \tag{8.50}$$

It is necessary to point out that a potential function exists if the external forces are position dependent and Eq. (8.38) is satisfied. As discussed in previous chapters, that is possible only if the work done by the force between two points is independent of the path. Thus a closed system, that is, one in which no external forces act on the system, leads to the law of the conservation of energy as given by Eq. (8.46).

Suppose a system is such that it has internal frictional forces. Such frictional forces depend on the relative velocities of the particles and are not central forces. Thus the law of conservation of energy, Eq. (8.46), does not hold for such systems.

Example 8.1

Consider the following three particles of masses m1, m2, and m3 located at distances R1, R2, and R3 from the origin.

m1 = 2 kg m2 = 3 kg m3 = 4 kg

$R1 = 2t^2 \cdot i + 3t \cdot j + 4 \cdot k$ $R2 = (1 + t^2) \cdot i + (2 + 5 \cdot t) \cdot j$ $R3 = (1 + 2t^2 + 3t^3) \cdot i + (3t + 4t^2) \cdot k$

Calculate the following quantities at time t = 10 sec. **(a)** The position of the center of mass, **(b)** the velocity of the center of mass, **(c)** the linear momentum of the system, and **(d)** the kinetic energy of the system.

Solution

(a) U1 represents a unit vector matrix. R1, R2, and R3 are expressed in matrix form. R represents the position of the center of mass and may be calculated as shown.

$i := 1$ $j := 1$ $k := 1$

$t := 10$

$m1 := 2$ $m2 := 3$ $m3 := 4$

$$U1 := \begin{pmatrix} i & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & k \end{pmatrix}$$

$$R1 := \begin{bmatrix} 2 \cdot t^2 & 0 & 0 \\ 0 & 3 \cdot t & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad R2 := \begin{bmatrix} 1 + t^2 & 0 & 0 \\ 0 & 2 + 5 \cdot t & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R3 := \begin{bmatrix} 1 + 2 \cdot t^2 + 3 \cdot t^3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \cdot t + 4 \cdot t^2 \end{bmatrix}$$

$$R := \frac{m1 \cdot U1 \cdot R1 + m2 \cdot U1 \cdot R2 + m3 \cdot U1 \cdot R3}{m1 + m2 + m3} \quad R = \begin{bmatrix} 1.501 \cdot 10^3 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 192 \end{bmatrix}$$

$$|R| = 6.916 \cdot 10^6$$

(b) Differentiating R1, R2, and R3 with respect to t yield the velocities V1, V2, and V3.

$V1 = \frac{d}{dt}\left(2\cdot t^2\cdot i + 3\cdot t\cdot j + 4\cdot k\right)$ $\quad V2 = \frac{d}{dt}\left[\left(1+t^2\right)\cdot i + (2+5\cdot t)\cdot j\right]$ $\quad V3 = \frac{d}{dt}\left[\left(1+2\cdot t^2+3\cdot t^3\right)\cdot i + \left(3\cdot t + 4\cdot t^2\right)\cdot k\right]$

$V1 = 4\cdot t\cdot i + 3\cdot j$ $\quad V2 = 2\cdot t\cdot i + 5\cdot j$ $\quad V3 = 4\cdot t\cdot i + 9\cdot t^2\cdot i + 3\cdot k + 8\cdot k\cdot t$

$$V1 := \begin{pmatrix} 4\cdot t & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad V2 := \begin{pmatrix} 2\cdot t & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad V3 := \begin{bmatrix} 4\cdot t + 9\cdot t^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 + 8\cdot t \end{bmatrix}$$

The velocity V of the center of mass is

$$V := \frac{m1\cdot U1\cdot V1 + m2\cdot U1\cdot V2 + m3\cdot U1\cdot V3}{m1 + m2 + m3} \qquad V = \begin{pmatrix} 433.333 & 0 & 0 \\ 0 & 2.333 & 0 \\ 0 & 0 & 36.889 \end{pmatrix} \qquad |V| = 3.73\cdot 10^4$$

(c) The linear momentum of the system is

$$P := m1\cdot V1 + m2\cdot V2 + m3\cdot V3 \qquad P = \begin{bmatrix} 3.9\cdot 10^3 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 332 \end{bmatrix} \qquad |P| = 2.719\cdot 10^7$$

(d) The kinetic energy of the system is $K := \frac{1}{2}\cdot\left(m1\cdot V1^2 + m2\cdot V2^2 + m3\cdot V3^2\right)$

$$K = \begin{bmatrix} 1.769\cdot 10^6 & 0 & 0 \\ 0 & 46.5 & 0 \\ 0 & 0 & 1.378\cdot 10^4 \end{bmatrix} \qquad |K| = 1.134\cdot 10^{12}$$

EXERCISE 8.1 Repeat the above example for the following three masses.

m1 = 24 kg $\qquad$ m2 = 3 kg $\qquad$ m3 = 2 kg

$R1 = \left(1 + 2t^2 + 3t^3\right)\cdot j + \left(3t + 4t^2\right)\cdot k$ $\quad R2 = \left(1 + t^2\right)\cdot i + (2 + 5t)\cdot j$ $\quad R3 = 2t^2\cdot i + 3t\cdot j + 4\cdot k$

8.6 MOTION OF SYSTEMS WITH VARIABLE MASS: ROCKETS AND CONVEYOR BELTS

We will now apply the conservation laws discussed in the previous section to some particular situations. The conservation laws are applicable to any definite system of particles, which may be chosen arbitrarily by including and excluding certain parts so long as it does not exclude the forces acting on the chosen part of the system. Another restriction concerns the law of conservation of kinetic and potential energy. It holds good so long as *no* mechanical energy is converted into other forms of energy, such as heat produced by frictional forces, unless such converted amounts are taken into account.

Rocket Propulsion

Rocket technology is based on the simplest principle of conservation of linear momentum. A rocket is propelled in a forward direction by ejecting mass in a backward direction in the form of gases resulting from the combustion of fuel. Thus the forward force on the rocket is the reaction to the backward force of the ejected gases (burned-out fuel). The problem is to find the velocity of the rocket at any time after launching, or takeoff, from the ground. As shown in Fig. 8.3, at a given time t a rocket of mass m is moving with velocity $\mathbf{v}$ relative to some fixed coordinate system, say Earth. Let the velocity of the exhaust gases from the rocket be $\mathbf{u}$ with respect to the rocket: hence $\mathbf{u} + \mathbf{v}$ with respect to a fixed coordinate system. Let us say that in a time interval between t and $t + dt$ the amount of fuel exhausted is $|dm| = -dm$ (because dm is negative; hence the rate at which the fuel is exhausted is $|dm/dt| = -dm/dt$), while the mass of the rocket is $m + dm$ and its velocity $\mathbf{v} + d\mathbf{v}$.

The momentum of the system at time t is

$$\mathbf{P}(t) = m\mathbf{v} \tag{8.51}$$

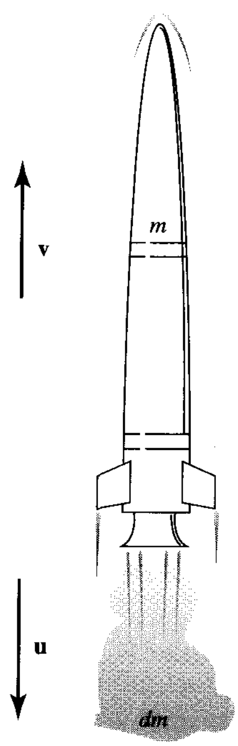


Figure 8.3 Motion of a rocket at some instant t.

and the momentum of the system at time $t + dt$ is

$$\mathbf{P}(t + dt) = \mathbf{P}_{\text{rocket}}(t + dt) + \mathbf{P}_{\text{fuel}}(t + dt) = (m + dm)(\mathbf{v} + d\mathbf{v}) + (-dm)(\mathbf{v} + \mathbf{u}) \quad (8.52)$$

The change in momentum in time interval dt is

$$d\mathbf{P} = \mathbf{P}(t + dt) - \mathbf{P}(t) \simeq m\,d\mathbf{v} - \mathbf{u}\,dm \quad (8.53)$$

where we have dropped the second-order term $dm\,d\mathbf{v}$. Since the rate of change of momentum $d\mathbf{P}/dt$ is equal to the applied external force $\mathbf{F}$, we may write Eq. (8.53) as

$$\frac{d\mathbf{P}}{dt} = \mathbf{F} = m\frac{d\mathbf{v}}{dt} - \mathbf{u}\frac{dm}{dt} \quad (8.54)$$

Note again that $\mathbf{u}$ is the velocity of the escaping gases. Equation (8.54) may be written as

$$m\frac{d\mathbf{v}}{dt} = \mathbf{u}\frac{dm}{dt} + \mathbf{F} \quad (8.55)$$

where $\mathbf{F}$ may be a gravitational force, the force of air resistance, or any other external force, $m(d\mathbf{v}/dt)$ is called the *thrust* of the rocket engine. Since dm/dt is negative, the thrust is opposite to the velocity $\mathbf{u}$ of the escaping gases. [The thrust of the rocket engine can be calculated by holding the rocket stationary and burning the fuel at the rate of dm/dt. The force $\mathbf{F}_0$ needed to hold the rocket stationary ($d\mathbf{v}/dt = 0$ and also $\mathbf{F} = 0$),

$$\mathbf{F}_0 = -\mathbf{u}\frac{dm}{dt} \quad (8.56)$$

will be the measure of the thrust.]

Let us consider a special case of Eq. (8.55) that prevails when $\mathbf{F} = 0$, that is, when no gravitational force or air resistance is present, which may be the case when the rocket is far in outer space. Equation (8.55) for $\mathbf{F} = 0$ is

$$m\frac{d\mathbf{v}}{dt} = \mathbf{u}\frac{dm}{dt} \quad (8.57)$$

Multiplying both sides by dt/m and integrating,

$$\int_{v_0}^{v} d\mathbf{v} = \mathbf{u}\int_{m_0}^{m}\frac{dm}{m}$$

$$\mathbf{v} - \mathbf{v}_0 = \mathbf{u}\ln m\Big|_{m_0}^{m}$$

Since $m_0 > m$, it is preferable to write

$$\mathbf{v} = \mathbf{v}_0 - \mathbf{u}\ln\frac{m_0}{m} \quad (8.58)$$

which states that the change in velocity $\mathbf{v} - \mathbf{v}_0$, or the final velocity $\mathbf{v}$, depends on two factors. A large value of $\mathbf{v}$ results from (1) large values of $\mathbf{u}$, the velocity of the exhaust gases, and

(2) large values of m_0/m, where m_0 is the initial mass of the rocket and its fuel, while m is the final mass when all the fuel has been used up. The final velocity is independent of the rate of burning fuel. Large values of m_0/m mean that we have a large fuel-to-payload ratio. To increase the value of m_0/m by large amounts, *staged rockets* are used for launching satellites and spacecrafts.

Near Earth's surface, we cannot neglect the force of gravitational pull. Thus, substituting $\mathbf{F} = m\mathbf{g}$ in Eq. (8.55), we obtain

$$m\frac{d\mathbf{v}}{dt} = \mathbf{u}\frac{dm}{dt} + m\mathbf{g} \qquad \textbf{(8.59)}$$

which on rearranging and integrating,

$$\int_{v_0}^{v} d\mathbf{v} = \mathbf{u}\int_{m_0}^{m} \frac{1}{m}\,dm + \mathbf{g}\int_0^t dt$$

results in

$$\mathbf{v} = \mathbf{v}_0 - \mathbf{u}\ln\frac{m_0}{m} + \mathbf{g}t \qquad \textbf{(8.60)}$$

Assuming that at $t = 0$, $\mathbf{v}_0 = 0$, and since $\mathbf{u}$ is opposite to $\mathbf{v}$, we may write Eq. (8.60) in scalar form as

$$v = u\ln\frac{m_0}{m} - gt \qquad \textbf{(8.61)}$$

Initially, the rocket thrust must be large enough to overcome gravitational force $m_0\mathbf{g}$. Subsequently, the preceding equations will describe the motion of the rocket.

A Conveyor Belt

Consider the conveyor belt shown in Fig. 8.4. We are interested in calculating the force $\mathbf{F}$ needed to keep the conveyor belt moving with horizontal uniform speed v, while sand or some other material is continuously dropping on the belt from a stationary hopper at a rate dm/dt. Let M be the mass of the belt and m be the mass of the sand on the belt. The total momentum of the system, the belt, and the sand on the belt is

$$p = (m + M)v \qquad \textbf{(8.62)}$$

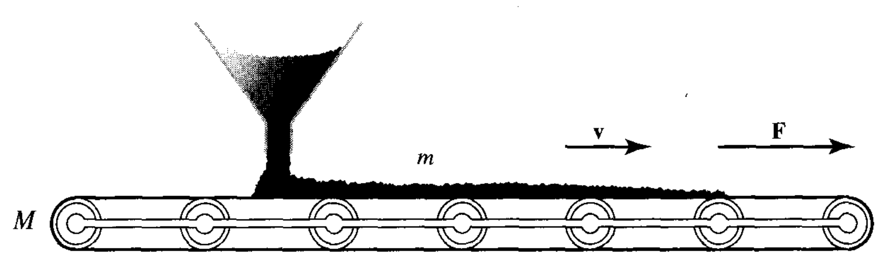


Figure 8.4 Conveyor belt.

Thus, according to the linear momentum theorem, since M and v are constants while m is changing,

$$F = \frac{dp}{dt} = v\frac{dm}{dt} \tag{8.63}$$

where F is the force applied to the belt. The power that must be supplied by the force to keep the belt moving with uniform speed v is

$$\text{Power} = P = Fv = v^2\frac{dm}{dt} = \frac{d}{dt}mv^2 = 2\frac{d}{dt}\left(\frac{1}{2}mv^2\right)$$

$$= 2\frac{d}{dt}\left(\frac{1}{2}(m+M)v^2\right) = 2\frac{dK}{dt} \tag{8.64}$$

That is, the power needed is twice the rate at which the kinetic energy is increasing. This implies that the law of conservation of mechanical energy does not apply here. The missing power is used up in doing work against the friction force, as explained next.[1]

When sand hits the belt, it must accelerate from zero speed to the belt speed over a short distance, during which some sliding must occur between the belt and the sand. To an observer at rest on the belt, the falling sand would appear to have a horizontal motion with speed v in the opposite direction to that of the belt. The belt exerts a horizontal force dF_f on the sand of mass dm to change its speed from $-v$ to 0. It does not matter whether the acceleration time is 1 s or 1/100 s; the power developed by the frictional forces between the belt and the sand is exactly one-half the power supplied.

Example 8.2

A spherical raindrop falling through fog or mist accumulates mass due to condensation at a rate proportional to its cross-sectional area and velocity. **(a)** Calculate the acceleration of the raindrop in terms of its radius and velocity. The raindrop starts from rest and has almost zero size. **(b)** Suppose a raindrop falling from a height of 3000 m has a radius of 1 mm and a speed of 10 m/sec when it reaches the surface of Earth. Calculate the time it takes to reach the surface.

Solution

(a) For a spherical raindrop of radius r and density ρ, the cross-sectional area A and mass m are as shown.

$$A = \pi \cdot r^2 \qquad m = \rho \cdot \frac{4 \cdot \pi \cdot r^3}{3}$$

If k is the constant of proportionality, the rate at which the raindrop gains mass is

$$\frac{d}{dt}m = k \cdot \pi \cdot r^2 \cdot v \qquad \text{(i)}$$

[1]Arom Mu-Shiang Mu, *The Physics Teacher.* April 1986.

The initial momentum pi of a particle of mass m moving with velocity v is

$$pi = m \cdot v$$

The final momentum pf after the drop has gained mass dm and velocity dv, is

$$pf = (m + dm)\cdot(v + dv)$$

or

$$pf = m\cdot v + m\cdot dv + dm\cdot v + dm\cdot dv$$

pf after neglecting the product dm dv is

$$pf = m\cdot v + v\cdot dm + m\cdot dv$$

The change in the momentum dp is

$$dp = pf - pi = v\cdot dm + m\cdot dv \quad \text{(ii)}$$

The change in the momentum is also

$$dp = F\cdot dt = m\cdot g\cdot dt \quad \text{(iii)}$$

Combining the two equations

$$v\cdot dm + m\cdot dv = m\cdot g\cdot dt$$

Substituting for m and dm, rearranging, and noting that a = dv/dt, gives

$$\frac{d}{dt}v = g - \left(\frac{v}{m}\right)\cdot\left(k\cdot\pi\cdot r^2\cdot v\right)$$

$$a = g - \frac{v^2}{m}\cdot k\cdot\pi\cdot r^2 \qquad \text{or} \qquad a = g - \frac{3}{4}\cdot\frac{v^2}{(\rho\cdot r)}\cdot k$$

Substituting for a = dv/dt and v = dy/dt or writing y in terms of double integral, we can solve for y.

$$\frac{d^2}{dt^2}y = \left[g - \frac{3}{4}\cdot\frac{v^2}{(\rho\cdot r)}\cdot k\right]$$

$$y = \int\int\left[g - \frac{3}{4}\cdot\frac{v^2}{(\rho\cdot r)}\right]dt\,dt = \frac{1}{2}\cdot t^2\cdot g - \frac{3}{8}\cdot t^2\cdot\frac{v^2}{(\rho\cdot r)}$$

The resulting equation for y is

$$y = \frac{1}{8}\cdot t^2\cdot\frac{\left(4\cdot g\cdot\rho\cdot r - 3\cdot v^2\right)}{(\rho\cdot r)} \quad \text{(iv)}$$

Solving the above equation for t gives two roots. The positive root yields the expression for t given by Eq. (v).

$$\begin{bmatrix} -2\cdot\sqrt{\rho}\cdot\dfrac{\sqrt{r}}{\sqrt{4\cdot g\cdot\rho\cdot r - 3\cdot v^2\cdot k}}\cdot\sqrt{y}\cdot\sqrt{2} \\ 2\cdot\sqrt{\rho}\cdot\dfrac{\sqrt{r}}{\sqrt{4\cdot g\cdot\rho\cdot r - 3\cdot v^2\cdot k}}\cdot\sqrt{y}\cdot\sqrt{2} \end{bmatrix}$$

(a) What is the significance of the negative root?

(b) Calculate the value of v and a as a function of time t.

$$t = 2\cdot\sqrt{\rho}\cdot\frac{\sqrt{r}}{\sqrt{4\cdot g\cdot\rho\cdot r - 3\cdot v^2\cdot k}}\cdot\sqrt{y}\cdot\sqrt{2} \quad \text{(v)}$$

Given the values for the raindrop, calculate the time t it takes to hit the surface. (As a check, the value of y is also calculated by using Eq. (iv) and equals 3000 m.).

$$r := .001\cdot m \qquad v := 10\cdot\frac{m}{sec} \qquad k := .005$$

$$\rho := 1.1 \qquad y := -3000\cdot m \qquad g := 9.8\cdot\frac{m}{sec^2}$$

$$t := 2\cdot\sqrt{\rho}\cdot\frac{\sqrt{r}}{\sqrt{4\cdot g\cdot\rho\cdot r - 3\cdot v^2\cdot k}}\cdot\sqrt{y}\cdot\sqrt{2} \qquad t = 4.257\cdot sec$$

(c) Calculate the value of v and a just before the raindrop hits the ground.

$$y := \frac{1}{2}\cdot t^2\cdot g - \frac{3}{8}\cdot t^2\cdot\frac{v^2}{(\rho\cdot r)}\cdot k \qquad y = -3\cdot 10^3 \cdot m$$

EXERCISE 8.2 Repeat the above example for a raindrop that falls from a height of 5000 m and has a radius of 2 mm and a speed of 20 m/sec when it reaches the surface of Earth. Calculate the time it takes to reach Earth's surface. The rest of the constants have the same values as in the above example.

Example 8.3

Consider a one-stage rocket, assuming constant g, having only a vertical thrust, a constant rate of change of mass k, and a constant exhaust velocity v0. **(a)** Graph the velocity and the altitude as a function of time. **(b)** Calculate the final velocity v and the altitude z.

Solution

After defining constant k and substituting for a and k, we may write the equation for the rocket as

mi = initial mass

mf = final mass

$$\frac{d}{dt}m = -k \qquad m\cdot a = \left(\frac{d}{dt}v\right)\cdot m = k\cdot v0 - m\cdot g$$

$$v = -v0\cdot\int_{mi}^{mf}\frac{1}{m}\,dm + \frac{g}{k}\cdot\int_{mi}^{mf}1\,dm$$

Solving for v, after integration and simplification, gives

$$v = \frac{(-v0\cdot k\cdot \ln(mf) + v0\cdot k\cdot \ln(mi) + g\cdot mf - g\cdot mi)}{k}$$

$$v = -v0\cdot \ln(mf) + v0\cdot \ln(mi) + \frac{1}{k}\cdot g\cdot mf - \frac{1}{k}\cdot g\cdot mi$$

Using the definition v = dz/dt, we can write an expression for v, which on integration yields the value of z.

$$z = \int_{mi}^{mf} -v0\cdot \ln(mf) + v0\cdot \ln(mi) + \frac{1}{k}\cdot g\cdot mf - \frac{1}{k}\cdot g\cdot mi\,dm$$

$$z = (-v0\cdot k\cdot \ln(mf) + v0\cdot k\cdot \ln(mi) + g\cdot mf - g\cdot mi)\cdot\frac{(mf - mi)}{k}$$

Using these expressions and the values of mi, mf, v0, and k, we can calculate the final values of z and v.

$$mi := 60000 \qquad mf := 45000 \qquad v0 := 6000 \qquad k := 150 \qquad g := 9.8$$

$$z := (-v0\cdot k\cdot \ln(mf) + v0\cdot k\cdot \ln(mi) + g\cdot mf - g\cdot mi)\cdot\frac{(mf - mi)}{k}$$

$$z = -1.119\cdot 10^7$$

$$v := -v0\cdot \ln(mf) + v0\cdot \ln(mi) + \frac{1}{k}\cdot g\cdot mf - \frac{1}{k}\cdot g\cdot mi$$

$$v = 746.092$$

We graph the results.

Looking at the graph and the data, answer the following questions.

(a) Why do the values of z and v have opposite signs?

(b) What effect does the negative value of g have on the value of z and v?

$v0 := 6000 \quad mi := 1000 \quad k := 150 \quad g := -9.8$

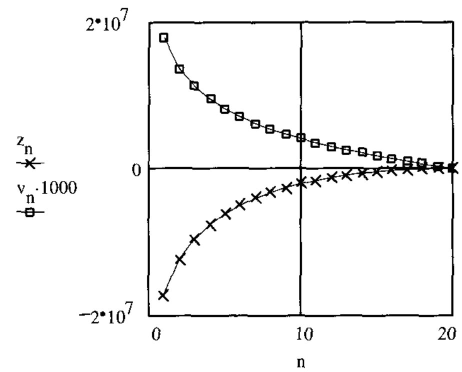
$n := 1..20 \quad mf_n := n \cdot \frac{mi}{20} \quad m_n := \frac{mf_n}{mi}$

$$z_n := \left(-v0 \cdot k \cdot \ln(mf_n) + v0 \cdot k \cdot \ln(mi) + g \cdot mf_n - g \cdot mi\right) \cdot \frac{(mf_n - mi)}{k}$$

$$z_n := \left(v0 \cdot k \cdot \ln\left(\frac{mi}{mf_n}\right) + g \cdot mf_n - g \cdot mi\right) \cdot \frac{(mf_n - mi)}{k}$$

$$v_n := -v0 \cdot \ln(mf_n) + v0 \cdot \ln(mi) + \frac{1}{k} \cdot g \cdot mf_n - \frac{1}{k} \cdot g \cdot mi$$

$z_1 = -1.713 \cdot 10^7$	$v_1 = 1.804 \cdot 10^4$
$z_4 = -7.767 \cdot 10^6$	$v_4 = 9.709 \cdot 10^3$
$z_8 = -3.322 \cdot 10^6$	$v_8 = 5.537 \cdot 10^3$
$z_{12} = -1.236 \cdot 10^6$	$v_{12} = 3.091 \cdot 10^3$
$z_{16} = -2.704 \cdot 10^5$	$v_{16} = 1.352 \cdot 10^3$
$z_{20} = 0$	$v_{20} = 0$
$m_{20} = 1$	$mf_{20} = 1 \cdot 10^3$



EXERCISE 8.3 Show that the maximum height achieved by the rocket discussed in the example is

$$z_{\max} = \frac{v_0^2(\ln R)^2}{2g} - v_0 t_b\left(\frac{\ln R}{1 - R} - 1\right)$$

(*Hint*: After burnout, the rocket will continue to climb without power; $mgh = \frac{1}{2}mv_b^2$.)

8.7 ELASTIC COLLISIONS AND CONSERVATION LAWS

When two or more objects come close enough (with or without any physical contact) so that there is some sort of interaction between them, with or without the presence of external forces, we say that a collision has taken place between the objects. After a collision, the velocities of the colliding objects may or may not be the same as before the collision. Very often we are interested in describing the nature of the interaction (or the type of force) between microscopic particles. If the particles are incident on a target, the paths and energies of the interacting particles will change. By measuring the energies and the angular distributions of these scattered particles, we can gain information about their structure and the nature of the forces involved.

By applying conservation laws, many details of a collision can be predicted without knowing much about the nature of the interaction or force. Collisions may be divided into two broad categories: *elastic collisions,* in which both linear momentum and kinetic energy are conserved, and *inelastic collisions,* in which conservation of linear momentum holds good, but kinetic energy is not conserved. Thus, if $\mathbf{P}_i$ and K_i are the initial linear momentum and kinetic energy before collision, while $\mathbf{P}_f$ and K_f are the final linear momentum and kinetic energy after collision, then

$$\text{For elastic collisions:} \quad \mathbf{P}_i = \mathbf{P}_f \quad \text{and} \quad K_i = K_f \tag{8.65}$$

$$\text{For inelastic collisions:} \quad \mathbf{P}_i = \mathbf{P}_f \quad \text{and} \quad K_i \neq K_f \tag{8.66}$$

In this section, we limit our discussion to elastic collisions.

Let us consider an elastic collision between two objects, as shown in Fig. 8.5. An object of mass m_1 moving with a velocity $\mathbf{v}_{1i}$, called the *incident particle,* strikes an object of mass m_2 at rest, called the *target particle,* both being along the X-axis. (Nothing is lost in generality by assuming one of the masses to be at rest. If both masses were moving, we could view the collision from a reference frame that is moving with the same velocity as that of one of the masses, say m_2. In that frame of reference, m_2 will be at rest.) After collision, mass m_1 is moving with velocity $\mathbf{v}_{1f}$, making an angle θ with the X-axis, and mass m_2 is moving with a velocity $\mathbf{v}_{2f}$, making an angle ϕ with the X-axis, as shown in Fig. 8.5(b). Remember that if $\mathbf{v}_{1f}$ is in the XY-plane, then $\mathbf{v}_{2f}$ also must be in the same XY-plane. This is because if $\mathbf{v}_{2f}$ is not in the XY-plane there will be a component of velocity, after collision, in the Z direction; but this cannot happen because there was no Z component of velocity before the collision, hence leading to nonconservation of linear momentum.

The conservation of linear momentum and energy requires

$$\mathbf{P}_i = \mathbf{P}_f \quad \text{and} \quad K_i = K_f \tag{8.67}$$

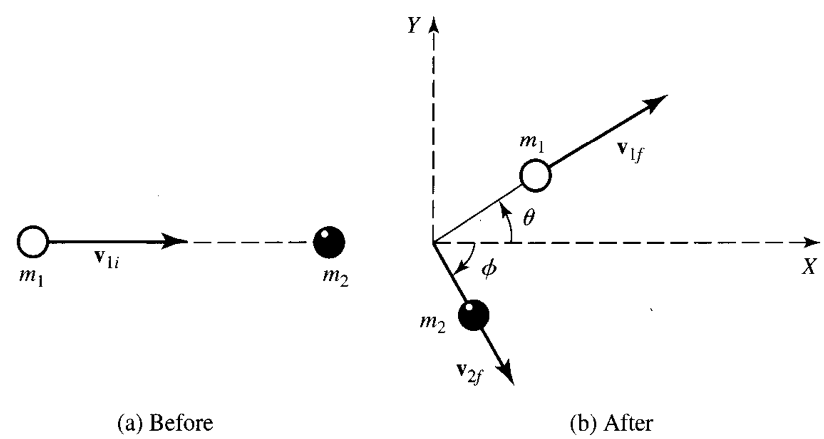


Figure 8.5 Elastic collision between two objects: (a) before, and (b) after collision.

where $\mathbf{P}_i$ and $\mathbf{P}_f$ are the initial and final linear momenta, while K_i and K_f are the initial and final kinetic energies. That is,

$$\mathbf{p}_{1i} + \mathbf{p}_{2i} = \mathbf{p}_{1f} + \mathbf{p}_{2f} \tag{8.68}$$

and

$$K_{1i} + K_{2i} = K_{1f} + K_{2f} \tag{8.69}$$

where

$$\mathbf{p}_{1i} = m_1\mathbf{v}_{1i}, \qquad \mathbf{p}_{2i} = 0, \qquad \mathbf{p}_{1f} = m_1\mathbf{v}_{1f}, \qquad \mathbf{p}_{2f} = m_2\mathbf{v}_{2f}$$
$$K_{1i} = \tfrac{1}{2}m_1v_{1i}^2, \qquad K_{2i} = 0, \qquad K_{1f} = \tfrac{1}{2}m_1v_{1f}^2, \qquad K_{2f} = \tfrac{1}{2}m_2v_{2f}^2$$

Using these and writing Eq. (8.68) in component form along the X- and Y-axes with the help of Fig. 8.5, we obtain

$$m_1v_{1i} = m_1v_{1f}\cos\theta + m_2v_{2f}\cos\phi \tag{8.70}$$

$$0 = m_1v_{1f}\sin\theta - m_2v_{2f}\sin\phi \tag{8.71}$$

and, from Eq. (8.69), we get

$$\tfrac{1}{2}m_1v_{1i}^2 = \tfrac{1}{2}m_1v_{1f}^2 + \tfrac{1}{2}m_2v_{2f}^2 \tag{8.72}$$

In most situations, m_1, m_2, and v_{1i} are known, while v_{1f}, v_{2f}, θ, and ϕ are the unknown quantities. Thus we have three equations [(8.70), (8.71), (8.72)] and four unknowns. We can eliminate one of the four unknowns, say ϕ, and find the relations between the other three, v_{1f}, v_{2f}, and θ. We may write Eqs. (8.70) and (8.71) as

$$m_1v_{1i} - m_1v_{1f}\cos\theta = m_2v_{2f}\cos\phi$$
$$m_1v_{1f}\sin\theta = m_2v_{2f}\sin\phi$$

Squaring and adding these equations and dividing by m_1^2 yield

$$v_{1i}^2 + v_{1f}^2 - 2v_{1i}v_{1f}\cos\theta = \left(\frac{m_2}{m_1}\right)^2 v_{2f}^2 \tag{8.73}$$

while from Eq. (8.72) we obtain

$$v_{2f}^2 = \frac{m_1}{m_2}(v_{1i}^2 - v_{1f}^2) \tag{8.74}$$

Substituting for v_{2f}^2 from Eq. (8.74) into Eq. (8.73) resulting in a quadratic equation in v_{1f}/v_{1i}, which when solved gives

$$\frac{v_{1f}}{v_{1i}} = \frac{m_1}{m_1 + m_2}\left[\cos\theta \pm \sqrt{\cos^2\theta - \left(\frac{m_1^2 - m_2^2}{m_1^2}\right)}\right] \tag{8.75}$$

This equation reveals a great deal of information about elastic collisions. In the following discussion we must keep in mind that the quantity under the radical sign cannot be negative because that would yield complex value for v_{1f}, which is physically meaningless.

Case (a) $\theta = 0$: These are collisions in one dimension; that is, they correspond to a *head-on* collision. Substituting $\theta = 0$ in Eq. (8.75) yields

$$\frac{v_{1f}}{v_{1f}} = 1 \quad \text{or} \quad \frac{v_{1f}}{v_{1f}} = \frac{m_1 - m_2}{m_1 + m_2} \tag{8.76}$$

Substituting these in Eq. (8.74) yields

$$v_{2f} = 0, \qquad \text{if } \frac{v_{1f}}{v_{1f}} = 1 \tag{8.77}$$

which corresponds to no collision; and

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1f} \tag{8.78}$$

if

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} \tag{8.79}$$

Thus Eqs. (8.78) and (8.79) represent head-on collisions, that is, collisions in one direction. Let us consider a few special cases of these two equations for head-on collisions.

(i) Suppose $m_1 = m_2$. Equations (8.78) and (8.79) give

$$v_{1f} = 0 \quad \text{and} \quad v_{2f} = v_{1i} \tag{8.80}$$

That is, the incident particle comes to a stop, while the target particle starts moving with the velocity of the incident particle.

(ii) If $m_1 \ll m_2$, we get

$$v_{1f} \simeq -v_{1i} \quad \text{and} \quad v_{2f} \simeq 0 \tag{8.81}$$

That is, the incident particle is reflected back with the same speed, while the target particle hardly moves.

(iii) If $m_1 \gg m_2$, we get

$$v_{1f} \simeq v_{1i} \quad \text{and} \quad v_{2f} \simeq 2v_{1i} \tag{8.82}$$

That is, the incident particle keeps moving as if nothing happened, while the target particle takes off with twice the velocity of the incident particle.

All the preceding situations are illustrated in Fig. 8.6.

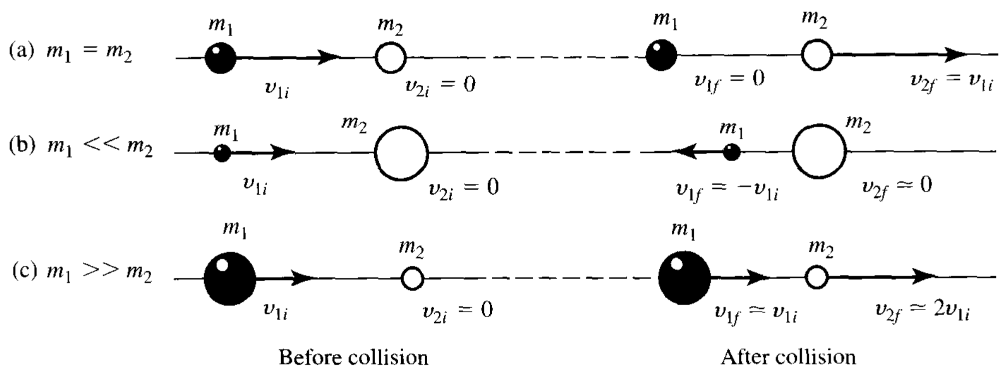


Figure 8.6 One-dimensional elastic collision between two objects.

Case (b) $m_1 > m_2$: For v_{1f} to be real, the quantity under the radical sign must be positive; that is,

$$\cos^2\theta \geqslant \frac{m_1^2 - m_2^2}{m_1^2} \tag{8.83}$$

Furthermore, the quantity under the radical sign will be zero (minimum), say for $\theta = \theta_m$, which according to Eq. (8.83) is

$$\cos^2\theta_m = \frac{m_1^2 - m_2^2}{m_1^2} = 1 - \frac{m_2^2}{m_1^2}, \qquad 0 \leqslant \theta_m \leqslant \frac{\pi}{2} \tag{8.84}$$

The scattering angle θ must be less than θ_m, because, if $\theta > \theta_m$ and $\pi/2 \leqslant \theta \leqslant \pi$, the quantity under the radical sign will be negative. Thus θ_m represents the maximum angle $= \theta_{\max}$; hence (because cos θ decreases with increasing θ)

$$\theta \leqslant \theta_{\max} \quad \text{and} \quad 0 < \theta_{\max} < \frac{\pi}{2} \tag{8.85}$$

Figure 8.7 shows the plot of maximum scattering angle $\theta_{\max}$ versus m_2/m_1. Note that if $m_1 \gg m_2$, the scattering angle will be very small (a very large mass can hardly be expected to be deflected by a small mass at rest). Furthermore, for $\theta < \theta_{\max}$, there will be two values of v_{1f}/v_{1i}; the larger value corresponds to be *glancing* collision, whereas the smaller value corresponds to a head-on collision.

Case (c) $m_1 < m_2$: For this case there is no restriction on the value of the scattering angle, which can be anywhere from 0 to π. A situation in which θ is greater than $\pi/2$ is called *back-scattering*. If $\theta = 0$, $v_{1f}/v_{1i} = 1$, which corresponds to no collisions. If $\theta = 0$ and $\phi = 0$, we get [as in case (a)]

$$\frac{v_{1f}}{v_{1i}} = \frac{m_1 - m_2}{m_1 + m_2} \quad \text{and} \quad \frac{v_{2f}}{v_{1i}} = \frac{2m_1}{m_1 + m_2} \tag{8.86}$$

Figure 8.7

Consider an elastic collision between a particle of mass m1 moving with velocity v1 and a particle of mass m2 at rest. The graph of the scattering angle θ as a function of the mass ratio m = m2/m1 where m1 > m2 is shown below.

According to Eq. (8.84) $\cos(\theta)^2 = 1 - m^2$ and $0 < \theta < \frac{\pi}{2}$ where $m = \frac{m2}{m1}$

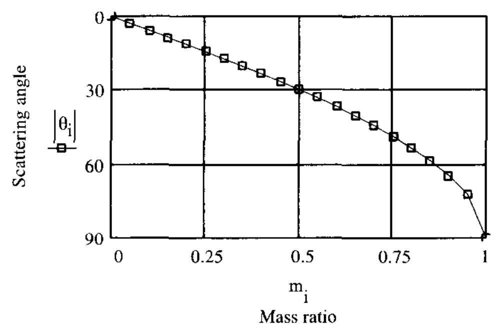
Solving for θ gives $\cos(\theta) = \sqrt{1 - m^2}$ $\quad \operatorname{acos}\left(\sqrt{1 - m^2}\right) = \theta$

Solve for N = 20 different values of m_i resulting in 20 different values of θ_i. $\quad N := 20 \quad i := 0..N$

$$m_i := \frac{i}{20} \qquad \theta_i := -\operatorname{acos}\left[\sqrt{1 - (m_i)^2}\right]\cdot\frac{360}{2\cdot\pi}$$

(a) When m is about 0, that is, m1 >> m2, the value of the scattering angle θ_i is 0 degree.

(b) When $m_1 = m_2$, and $m_i = 1$, the scattering angle is maximum, that is, θmax = 90 degree = π/2.



We can also show that for head-on collisions

$$\frac{m_1}{m_2} = \frac{2K_{1i}}{K_{2f}} - 1 \pm \left[\left(\frac{2K_{1i}}{K_{2f}} - 1\right) - 1\right]^{1/2} \tag{8.87}$$

Case (d) $m_1 = m_2$: By multiplying Eq. (8.70) by cos θ and Eq. (8.71) by sin θ and adding, we get

$$v_{1i}\cos\theta = v_{1f} + v_{2f}\cos(\theta + \phi) \tag{8.88}$$

Since $m_1 = m_2$, Eq. (8.75) yields

$$v_{1f} = v_{1i}\cos\theta \tag{8.89}$$

From Eqs. (8.88) and (8.89), we obtain

$$\cos(\theta + \phi) = 0 \quad \text{or} \quad \theta + \phi = \frac{\pi}{2} \tag{8.90}$$

That is, the two particles leave at right angles to each other. The example of such a collision is observed on a pool table when a cue ball is seen to leave the struck ball at a right angle.

8.8 INELASTIC COLLISIONS

In many situations in both the microscopic and macroscopic worlds, the kinetic energy of the system before collision is not the same as after collision; that is, kinetic energy is not conserved. For example, atoms, molecules, and nuclei possess internal kinetic and potential energies. When such particles collide, kinetic energy may be absorbed or released. Collisions in which the final kinetic energy of the system is less than the initial kinetic energy (that is, energy is absorbed by the system), are called *endoergic* or *first kind reactions* or *collisions*. Collisions in which the final kinetic energy is more than the initial kinetic energy (that is, energy is released), are called *exoergic* or *second kind reactions* or *collisions*. Thus if the initial kinetic energy is K_i and the final kinetic energy is K_f, the *disintegration energy* Q of the reaction is defined as

$$Q = K_f - K_i \tag{8.91}$$

If $Q > 0$ exoergic, inelastic second kind **(8.92a)**

If $Q < 0$ endoergic, inelastic first kind **(8.92b)**

If $Q = 0$ elastic collision **(8.92c)**

In all of these cases, the law of the conservation of linear momentum holds good. The law of conservation of energy will hold good only if all internal energies, as well as any other energy, such as heat by friction, are taken into account. Furthermore, in inelastic collisions the nature of the particles after collision may be completely different from those before collision.

Let us consider an inelastic collision between a particle of mass m_1 moving with velocity $\mathbf{v}_{1i}$ with a particle of mass m_2 at rest, as shown in Fig. 8.8. The collision between these two par-

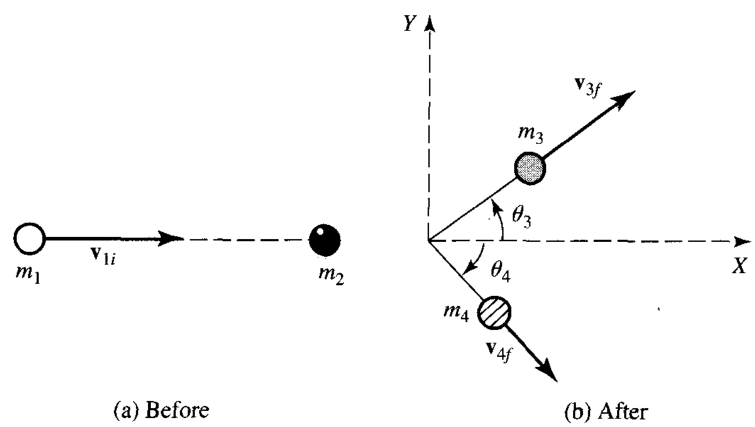


Figure 8.8 Inelastic collision between two particles: (a) before, and (b) after collision.

ticles results in two new particles of mass m_3 and m_4 moving with velocities $\mathbf{v}_{3f}$ and $\mathbf{v}_{4f}$, making angles θ_3 and θ_4 with the initial direction of the velocity of the incident particle m_1, which is the X-axis. Let K_1, K_2 (=0 in this case), K_3, and K_4, be the kinetic energies of particles m_1, m_2, m_3, and m_4, respectively, and Q the disintegration energy. From the laws of conservation of momentum and kinetic energy, we may write

$$m_1 v_{1i} = m_3 v_{3f} \cos\theta_3 + m_4 v_{4f} \cos\theta_4 \tag{8.93}$$

$$0 = m_3 v_{3f} \sin\theta_3 - m_4 v_{4f} \sin\theta_4 \tag{8.94}$$

and

$$K_1 + Q = K_3 + K_4 \tag{8.95}$$

θ_4 can be eliminated from Eqs. (8.93) and (8.94) by rearranging, squaring, and adding, resulting in

$$(m_4 v_{4f})^2 = (m_1 v_{1i})^2 + (m_3 v_{3f})^2 - 2 m_1 m_3 v_{1i} v_{3f} \cos\theta_3 \tag{8.96}$$

Combining Eqs. (8.95) and (8.96) and using the relations

$$K_1 = \tfrac{1}{2} m_1 v_{1i}^2, \qquad K_3 = \tfrac{1}{2} m_3 v_{3f}^2, \qquad K_4 = \tfrac{1}{2} m_4 v_{4f}^2$$

we may obtain the following value for Q:

$$Q = K_3 + K_4 - K_1 = K_3\left(1 + \frac{m_3}{m_4}\right) - K_1\left(1 - \frac{m_1}{m_4}\right) - 2\left(\frac{m_1 m_3 K_1 K_3}{m_4^2}\right)^{1/2} \cos\theta_3 \tag{8.97}$$

Thus, when a particle of mass m_1 and known velocity $\mathbf{v}_{1i}$ collides with mass m_2, Eq. (8.97) allows us to calculate the value of Q by measuring θ_3 and the velocity $\mathbf{v}_{3f}$ of a particle of mass m_3 with additional knowledge of mass m_4. Note that we have eliminated the quantity $\mathbf{v}_{4f}$ because it is usually, especially in nuclear reactions, very hard to measure this quantity.

Consider an inelastic collision in one dimension between two objects. Such collisions are always endoergic, as we will show now. Suppose an object of mass m_1 moving with velocity v_1 collides with an object of mass m_2 at rest, the two objects stick together after the collision (such as a bullet striking a piece of wood and becoming embedded), and now move together with velocity v_2. Thus, according to the law of conservation of momentum,

$$m_1 v_1 = (m_1 + m_2) v_2$$

or

$$v_2 = \frac{m_1 v_1}{m_1 + m_2} \tag{8.98}$$

Kinetic energy is not conserved in this case; hence

$$Q = K_f - K_i = \tfrac{1}{2}(m_1 + m_2) v_2^2 - \tfrac{1}{2} m_1 v_1^2$$

Substituting for v_2 from Eqs. (8.98), we obtain

$$Q = K_1 \frac{-m_2}{m_1 + m_2} \tag{8.99}$$

which is a negative quantity; hence the collision is endoergic. The amount of energy changed into heat is equal to $|Q|$. In our discussion we have assumed that no rotational energies are involved. Equation (8.99) also requires that the minimum kinetic energy K_1 needed to start an endoergic reaction be greater than $|Q|$ by a factor $1 + m_1/m_2$. Thus the minimum energy is called the *threshold energy.*

$$(K_1)_{\text{thres}} = \left(1 + \frac{m_1}{m_2}\right)|Q| \tag{8.100}$$

For endoergic reactions in general, K_1 must be $\geq (K_1)_{\text{thres}}$.

Finally, let us define another commonly used term regarding collisions, the *coefficient of restitution.* Consider a head-on elastic collision between two masses, as shown in Fig. 8.9. The laws of conservation of momentum and energy require that

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \tag{8.101}$$

$$\tfrac{1}{2} m_1 v_{1i}^2 + \tfrac{1}{2} m_2 v_{2i}^2 = \tfrac{1}{2} m_1 v_{1f}^2 + \tfrac{1}{2} m_2 v_{2f}^2 \tag{8.102}$$

Solving these two equations yields

$$v_{2f} - v_{1f} = v_{1i} - v_{2i} \tag{8.103a}$$

$$(v_{\text{rel}})_f = -(v_{\text{rel}})_i \tag{8.103b}$$

$$\text{Speed of recession} = \text{speed of approach}$$

This results states that *the ratio of the relative velocity after collision to the relative velocity before collision between the two bodies in a head-on collision is constant.* This may be written as

$$v_{2f} - v_{1f} = e(v_{1i} - v_{2i}) \tag{8.104}$$

where e is called the *coefficient of restitution.* As is obvious, if the collision is elastic, $e = 1$, while for a perfect inelastic collision (in which two bodies move as one after collision) $e = 0$. For other inelastic collisions, e varies between 0 and 1.

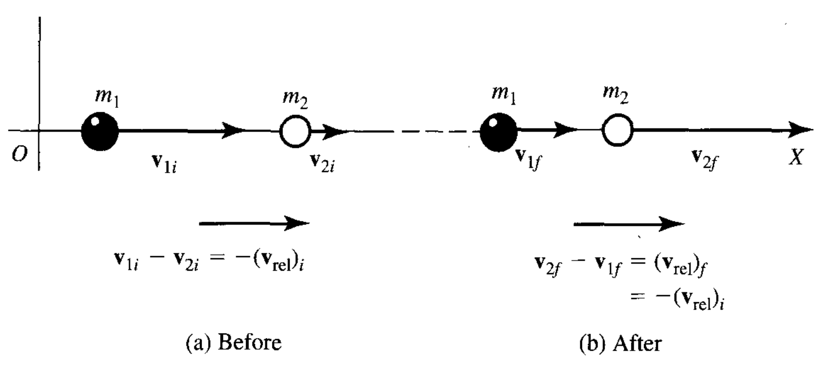


Figure 8.9 One-dimensional elastic collision between two masses, m_1 and m_2, showing that $(\mathbf{v}_{\text{rel}})_f = -(\mathbf{v}_{\text{rel}})_i$.

8.9 TWO-BODY PROBLEM IN CENTER-OF-MASS COORDINATE SYSTEM

In many situations we find it quite convenient and useful to describe the motion of a system consisting of two bodies as observed in a center-of-mass coordinate system (CMCS, or CM system) instead of in a laboratory coordinate system (LCS, or LAB system). Furthermore, we shall describe the collisions between two objects as viewed from a CMCS. The advantage of using a CMCS is that under special circumstances the two-body problem can be reduced to two single-body problems described as (1) the motion of the center of mass, and (2) the relative motion (that is, the motion of either particle with respect to the other). The CMCS was discussed in Chapter 7, but we shall consider further details of the system in this chapter.

Let us consider a system consisting of two bodies of mass m_1 and m_2 at distances $\mathbf{r}_1$ and $\mathbf{r}_2$ from the origin O, as shown in Fig. 8.10. Let $\mathbf{F}_1^e$ and $\mathbf{F}_2^e$ be the external forces acting on m_1 and m_2, respectively, while $\mathbf{F}_{12}^i$ is the internal force acting on body m_1 due to m_2, and $\mathbf{F}_{21}^i$ the internal force acting on m_2 due to m_1. According to Newton's third law, force $\mathbf{f}$ may be defined as

$$\mathbf{F}_{12}^i = -\mathbf{F}_{21}^i = \mathbf{f} \tag{8.105}$$

while the total external force acting on the system is

$$\mathbf{F} = \mathbf{F}_1^e + \mathbf{F}_2^e \tag{8.106}$$

According to Newton's second law, the motion of the two bodies in the LAB system may be written as

$$m_1\ddot{\mathbf{r}}_1 = \mathbf{F}_1^e + \mathbf{F}_{12}^i \tag{8.107}$$

$$m_2\ddot{\mathbf{r}}_2 = \mathbf{F}_2^e + \mathbf{F}_{21}^i \tag{8.108}$$

To change now from a LAB system to a CM system, we use the following relations, which were discussed in Chapter 7. The *center-of-mass coordinate* $\mathbf{R}$ is given by (from Section 7.2)

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \tag{8.109}$$

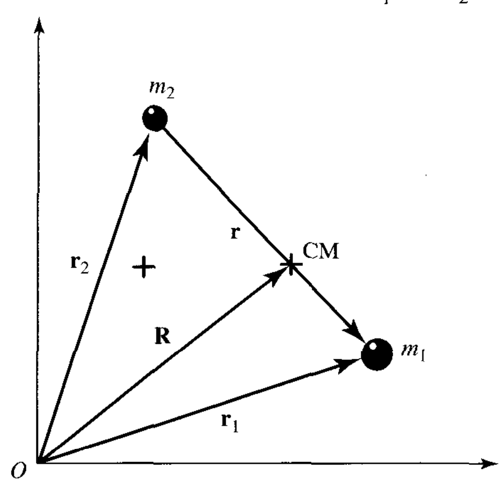


Figure 8.10 Center of mass and relative motion for a system consisting of two particles.

and the *relative coordinate* **r** is given by

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \tag{8.110}$$

while the reverse transformations are given by

$$\mathbf{r}_1 = \mathbf{R} + \frac{m_2}{m_1 + m_2}\mathbf{r} \tag{8.111}$$

$$\mathbf{r}_2 = \mathbf{R} - \frac{m_1}{m_1 + m_2}\mathbf{r} \tag{8.112}$$

We want to rewrite the equations of motion of the two bodies m_1 and m_2 in terms of the CM coordinates **R** and the relative coordinates **r**. To do this, we first add Eqs. (8.107) and (8.108): that is,

$$m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = \mathbf{F}_1^e + \mathbf{F}_2^e + \mathbf{F}_{12}^i + \mathbf{F}_{21}^i$$

Using Eqs. (8.105) , (8.106), and (8.109), the preceding equation may be written as

$$(m_1 + m_2)\ddot{\mathbf{R}} = \mathbf{F}$$

or

$$M\ddot{\mathbf{R}} = \mathbf{F} \tag{8.113}$$

where $M = m_1 + m_2$ is the total mass and **F** is the total external force acting on the system. This is first of the two equations we are looking for.

Now multiply Eq. (8.107) by m_2 and Eq. (8.108) by m_1 and subtract:

$$m_1m_2(\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2) = m_2\mathbf{F}_1^e - m_1\mathbf{F}_2^e + m_2\mathbf{F}_{12}^i - m_1\mathbf{F}_{21}^i$$

Using the result given in Eq. (8.105), we may write this equation as

$$m_1m_2(\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2) = m_1m_2\left(\frac{\mathbf{F}_1^e}{m_1} - \frac{\mathbf{F}_2^e}{m_2}\right) + (m_1 + m_2)\mathbf{f} \tag{8.114}$$

Let us consider a special case in which either

$$\mathbf{F}_1^e = \mathbf{F}_2^e = 0 \tag{8.115}$$

or

$$\frac{\mathbf{F}_1^e}{m_1} = \frac{\mathbf{F}_2^e}{m_2} \tag{8.116}$$

That is, the external forces acting on the objects are proportional to their masses; hence Eq. (8.114) may be written as

$$m_1m_2(\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2) = (m_1 + m_2)\mathbf{f} \tag{8.117}$$

Introducing the quantity *reduced mass* μ, defined as

$$\mu = \frac{m_1m_2}{m_1 + m_2} \tag{8.118}$$

and using Eq. (8.110), $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, we may write Eq. (8.117) as

$$\mu\ddot{\mathbf{r}} = \mathbf{f} \tag{8.119}$$

Thus Eqs. (8.113) and (8.119) are the two required equations. Equation (8.113) is the familiar equation for the motion of the center of mass according to which mass M is acted on by the total external force $\mathbf{F}$, producing an acceleration $\ddot{\mathbf{R}}$, while Eq. (8.119) is the equation of motion of mass μ acted on by an internal force $\mathbf{f} = \mathbf{F}^i_{12}$, producing an acceleration $\ddot{\mathbf{r}}$. Equation (8.119) may also be described as the motion of particle of mass μ at the position of m_1 as viewed from the position of m_2, assuming m_2 to be at rest.

We may also write an expression for the linear momentum $\mathbf{P}$, angular momentum $\mathbf{L}$, and total kinetic energy K in terms of CM coordinates. Using Eqs. (8.109) through (8.112), we may write the center-of-mass velocity $\mathbf{V}$ as

$$\mathbf{V} = \dot{\mathbf{R}} = \frac{m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2}{m_1 + m_2} = \frac{m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2}{M} \tag{8.120}$$

and the relative velocity $\mathbf{v}$ as

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2 \tag{8.121}$$

And for the inverse transformation, we may write

$$\mathbf{v}_1 = \dot{\mathbf{r}}_1 = \dot{\mathbf{R}} + \frac{m_2}{m_1 + m_2}\dot{\mathbf{r}} = \dot{\mathbf{R}} + \frac{\mu}{m_1}\dot{\mathbf{r}} \tag{8.122}$$

$$\mathbf{v}_2 = \dot{\mathbf{r}}_2 = \dot{\mathbf{R}} - \frac{m_1}{m_1 + m_2}\dot{\mathbf{r}} = \dot{\mathbf{R}} - \frac{\mu}{m_2}\dot{\mathbf{r}} \tag{8.123}$$

The total linear momentum of the system is

$$\mathbf{P} = m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2 = M\dot{\mathbf{R}} \tag{8.124}$$

and the total angular momentum $\mathbf{L}$ of the system is

$$\mathbf{L} = m_1(\mathbf{r}_1 \times \dot{\mathbf{r}}_1) + m_2(\mathbf{r}_2 \times \dot{\mathbf{r}}_2) \tag{8.125}$$

Substituting for $\dot{\mathbf{r}}_1$ and $\dot{\mathbf{r}}_2$ from Eqs. (8.122) and (8.123), we obtain

$$\mathbf{L} = M(\mathbf{R} \times \dot{\mathbf{R}}) + \mu(\mathbf{r} \times \dot{\mathbf{r}})$$

or

$$\mathbf{L} = M(\mathbf{R} \times \mathbf{V}) + \mu(\mathbf{r} \times \mathbf{v}) \tag{8.126}$$

while the total kinetic energy K is given by

$$K = \tfrac{1}{2}m_1\dot{r}_1^2 + \tfrac{1}{2}m_2\dot{r}_2^2 \tag{8.127}$$

Substituting for $\dot{r}_1$ and $\dot{r}_2$, we get

$$K = \tfrac{1}{2}M\dot{R}^2 + \tfrac{1}{2}\mu v^2 \tag{8.128}$$

or

$$K = \tfrac{1}{2} MV^2 + \tfrac{1}{2} \mu v^2 \qquad \textbf{(8.129)}$$

This equation states that the kinetic energy of a system is equal to the sum of the kinetic energy of mass M moving with velocity V of the center of mass (kinetic energy of the center of mass) and the kinetic energy of the reduced mass μ moving with a relative velocity v (kinetic energy of relative motion).

8.10 COLLISIONS IN CENTER-OF-MASS COORDINATE SYSTEM

In previous sections, we discussed elastic and inelastic collisions between two objects from the point of view of an observer at rest with respect to the coordinates fixed in a laboratory coordinate system (LCS). In many circumstances, it is convenient to make observations from a coordinate system that is moving with respect to the LCS. One such coordinate system commonly used is the center-of-mass coordinate system (CMCS) as discussed previously. Collisions are observed by an observer at the center of mass, hence moving with the same velocity as the center of mass. We start with the discussion of elastic collisions between two objects as observed from the center of mass.

Suppose at a given instant a particle of mass m_1 at x_1 is moving with velocity v_{1i}, while a particle of mass m_2 at x_2 is at rest, as shown in Fig. 8.11. The center of mass x_c is given by

$$(m_1 + m_2)x_c = m_1x_1 + m_2x_2 \qquad \textbf{(8.130)}$$

while the velocity of the center of mass obtained by differentiating Eq. (8.130) is

$$(m_1 + m_2)v_c = m_1\dot{x}_1 + m_2\dot{x}_2 \qquad \textbf{(8.131)}$$

where $v_c = dx_c/dt$, while for the situation shown in Fig. 8.11, $\dot{x}_1 = v_{1i}$ and $\dot{x}_2 = 0$. Thus the velocity of the center-of-mass v_c with respect to the LCS is given by

$$v_c = \frac{m_1 v_{1i}}{m_1 + m_2} = \frac{\mu}{m_2} v_{1i} \qquad \textbf{(8.132)}$$

where μ is the reduced mass.

Let the collision between m_1 and m_2 be observed by an observer moving with velocity v_c of the center of mass; that is, the observer is in the CMCS. The velocities of mass m_1 and m_2

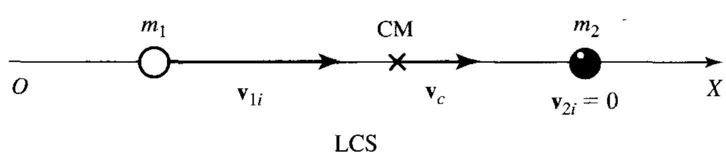


Figure 8.11 Velocity of m_1 and m_2 and their center of mass in the laboratory coordinate system (LCS).

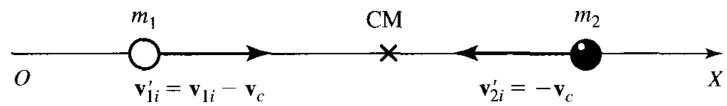


Figure 8.12 Motion of particles m_1 and m_2 in the center-of-mass coordinate system (CMCS).

with respect to the CMCS are v'_{1i} and v'_{2i} (prime indicates that the quantity is described in the CMCS):

$$v'_{1i} = v_{1i} - v_c = v_{1i} - \frac{m_1}{m_1 + m_2} v_{1i} = \frac{m_2}{m_1 + m_2} v_{1i} = \frac{\mu}{m_1} v_{1i} \tag{8.133}$$

$$v'_{2i} = v_{2i} - v_c = 0 - v_c = -\frac{m_1}{m_1 + m_2} v_{1i} = -\frac{\mu}{m_2} v_{1i} \tag{8.134}$$

Figure 8.12 shows the motion of these two particles with respect to the CMCS. The corresponding momentum of each particle before collision in the CMCS is

$$p'_{1i} = m_1 v'_{1i} = \frac{m_1 m_2}{m_1 + m_2} v_{1i} \tag{8.135}$$

$$p'_{2i} = m_2 v'_{2i} = -\frac{m_1 m_2}{m_1 + m_2} v_{1i} \tag{8.136}$$

Thus the total linear momentum of the system in the CMCS before collision is

$$P'_1 = p'_{1i} + p'_{2i} = \frac{m_1 m_2}{m_1 + m_2} v_{1i} - \frac{m_1 m_2}{m_1 + m_2} v_{1i} = 0 \tag{8.137}$$

That the total linear momentum before collision is zero is one of the most important characteristics of the CMCS. This implies that to conserve linear momentum the total linear momentum in the CMCS after collision must also be zero. That is, *as viewed from the CMCS, two particles of mass m_1 and m_2 approach each other in a straight line and, after collision, recede from each other in a straight line with the same initial velocities,* as shown in Fig. 8.13(a). The line joining the receding particles can make any angle θ_c (in CMCS), as shown. For the sake of comparison, Fig. 8.13(b) shows the collision as viewed from the LCS.

We may now look at the following problem. First, how do we get back from the CMCS to the LCS? Second, what is the relation between the angles the particles make after collision with their initial direction in both the LCS and the CMCS?

In the CMCS, the final velocity and direction of the particles after collision are shown in Fig. 8.13(a). To find the final velocities of the particles in the LCS, we may reverse the procedure used for changing from the LCS to the CMCS. This is achieved by adding to the final velocities v'_{1f} $(=v_{1i} - v_c)$ and v'_{2f} $(=v_c)$, the velocity v_c of the center of mass shown in Fig. 8.14. Thus the velocity $\mathbf{v}_{1f}$ and $\mathbf{v}_{2f}$ of m_1 and m_2, respectively, in the LCS are

$$\mathbf{v}_{1f} = \mathbf{v}'_{1f} + \mathbf{v}_c \tag{8.138}$$

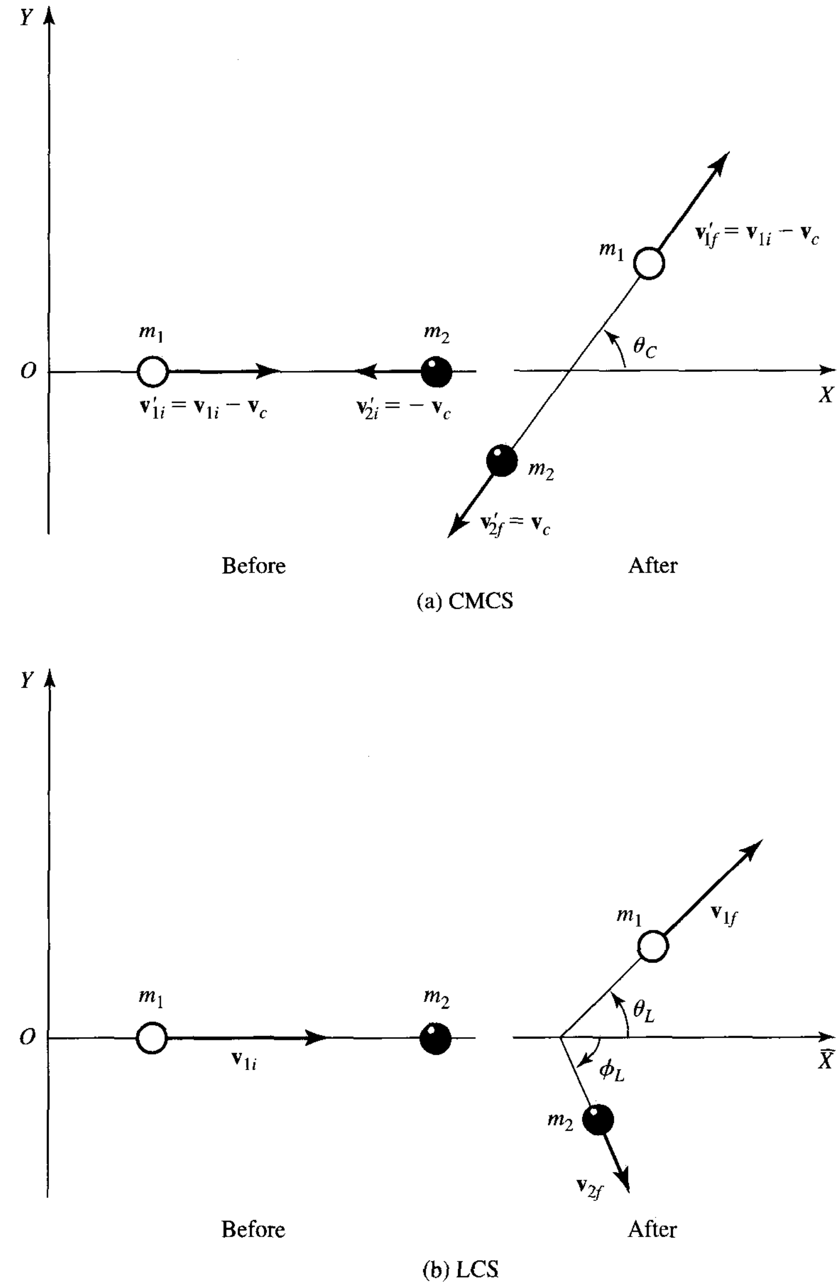


Figure 8.13 Collision between two particles of mass m_1 and m_2 as viewed from (a) the CMCS and (b) the LCS.

$$\mathbf{v}_{2f} = \mathbf{v}'_{2f} + \mathbf{v}_c \tag{8.139}$$

With the help of Fig. 8.14, we can find the relation between angles θ_L and ϕ_L in the LCS and θ_C in the CMCS. For example, let us consider Eq. (8.138) and the top half of Fig. 8.14. Resolving into components, Eq. (8.138) may be written as

$$v_{1f} \cos \theta_L = v_c + v'_{1f} \cos \theta_C \tag{8.140}$$

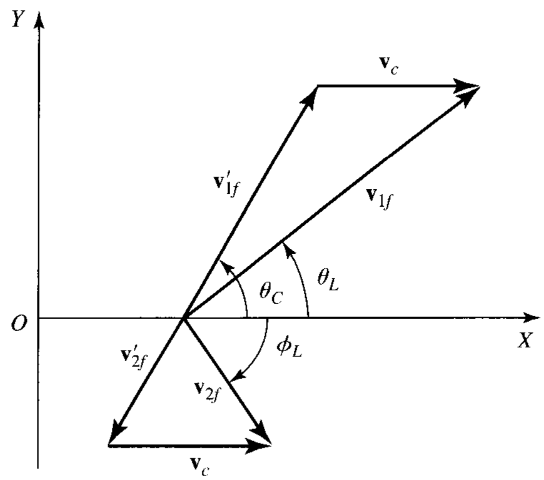


Figure 8.14 Relation between angles θ_L and ϕ_L in the LCS and θ_C in the CMCS after collision.

$$v_{1f} \sin \theta_L = v'_{1f} \sin \theta_C \tag{8.141}$$

Dividing one by the other,

$$\tan \theta_L = \frac{v'_{1f} \sin \theta_C}{v_c + v'_{1f} \cos \theta_C} = \frac{\sin \theta_C}{(v_c/v'_{1f}) + \cos \theta_C} \tag{8.142}$$

or

$$\tan \theta_L = \frac{\sin \theta_C}{\gamma + \cos \theta_C} \tag{8.143}$$

where

$$\gamma = \frac{v_c}{v'_{1f}} = \frac{\text{velocity of the center of mass in LCS}}{\text{velocity of } m_1 \text{ after collision in CMCS}} \tag{8.144}$$

The values of v_c and v'_{1f} are given by Eqs. (8.132) and (8.133). From Eq. (8.132)

$$v_c = \frac{m_1}{m_1 + m_2} v_{1i} = \frac{\mu}{m_2} v_{1i} \tag{8.145}$$

where μ is the reduced mass and v_{1i} is the initial relative velocity $(= v_{1i} - v_{2i} = v_{1i} - 0 = v_{1i})$. v'_{1f} $(= v'_{1i})$, the final relative velocity, from Eq. (8.133) is equal to

$$v'_{1f} = \frac{m_2}{m_1 + m_2} v_{1f} = \frac{\mu}{m_1} v_{1f} \tag{8.146}$$

Thus combining the preceding three equations (and noting that final velocities are equal to initial velocities in the CMCS), we get

$$\gamma = \frac{v_c}{v'_{1f}} = \frac{m_1 v_{1i}}{m_2 v_{1f}} \tag{8.147}$$

For inelastic collisions, $v_{1i} \neq v_{1f}$, Eq. (8.143) becomes

$$\tan \theta_L = \frac{\sin \theta_C}{(m_1 v_{1i}/m_2 v_{1f}) + \cos \theta_C}, \qquad \textit{inelastic collisions} \tag{8.148}$$

For elastic collisions, $v_{1i} = v_{1f}$, and Eq. (8.148) takes the form

$$\tan \theta_L = \frac{\sin \theta_C}{(m_1/m_2) + \cos \theta_C}, \qquad \textit{elastic collisions} \tag{8.149}$$

Let us consider some special cases of Eq. (8.149) for elastic collisions.

Case (a): If $m_1 = m_2$, as is the case in collisions between neutrons and protons, we may write Eq. (8.149) as

$$\tan \theta_L = \frac{\sin \theta_C}{1 + \cos \theta_C} = \frac{2 \sin(\theta_C/2) \cos(\theta_C/2)}{2 \cos^2(\theta_C/2)} = \tan \frac{\theta_C}{2} \tag{8.150}$$

That is,

$$\theta_L = \frac{\theta_C}{2} \tag{8.151}$$

Since, in the CMCS, θ_C may have any value of between 0 and π, θ_L can have a maximum value of $\pi/2$, in agreement with the previous discussion.

Case (b): If $m_2 \gg m_1$, we may write Eq. (8.149) as

$$\tan \theta_L \simeq \frac{\sin \theta_C}{\cos \theta_C} = \tan \theta_C \tag{8.152}$$

That is,

$$\theta_L \simeq \theta_C \tag{8.153}$$

which states that, for heavy targets, the scattering angle in the LCS is the same as in the CMCS.

Case (c): If $m_1 > m_2$, the incident particle is heavier than the target particle. In this case, θ_L must be very small, no matter whatever the value of θ_C. This correctly corresponds to the situation in Eq. (8.85), where it was noted that θ_L cannot be larger than a certain maximum value θ_{max}.

8.11 AN INVERSE-SQUARE REPULSIVE FORCE: RUTHERFORD SCATTERING

Most of our efforts have been devoted to the study of motion of particles in an attractive inverse square force field. There is an important class of physical applications in which the motion of the particle is in an inverse-square repulsive force field. Such situations involve the deflection or scattering of fast-moving atomic particles such as protons and alpha particles by positively

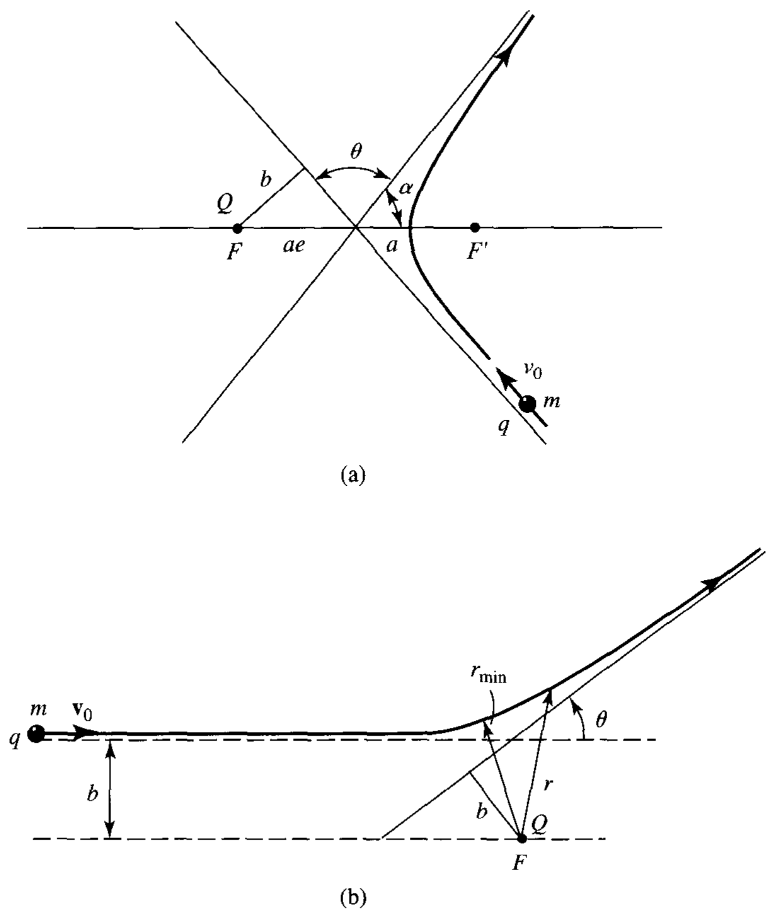


Figure 8.15 (a) Hyperbolic trajectory of a positively charged particle q in the field of a positively charged force center Q, that is, in a repulsive force field. Note that θ is the scattering angle and b is the impact parameter. (b) Same as in (a), but also showing the relation between r, $r_{\min}$, b, and θ.

charged nuclei. Paths of such scattered particles are hyperbolic. The first such experiments involving the scattering of alpha particles by nuclei were carried out by Geiger and Marsden (Rutherford's students) and analyzed by Rutherford and will be discussed at some length.

As shown in Fig. 8.15(a) and (b), a positively charged particle of charge q, mass m, and velocity v_0 is incident on a target nucleus of positive charge Q and mass M at rest. The inverse-square repulsive force between the two particles is

$$F = k\frac{Qq}{r^2} = \frac{K}{r^2} \tag{8.154}$$

where $k = 8.99 \times 10^9$ N-m^2/C^2 and $K = kQq$ is positive; hence F is a repulsive force.

In the particular case of alpha particle scattering by nuclei, $q = 2e$ and $Q = Ze$, where Z is the atomic number of the nuclei and e is the charge of the electron. Since $e = 1.6 \times 10^{-19}$ C

Figure 8.15(c)

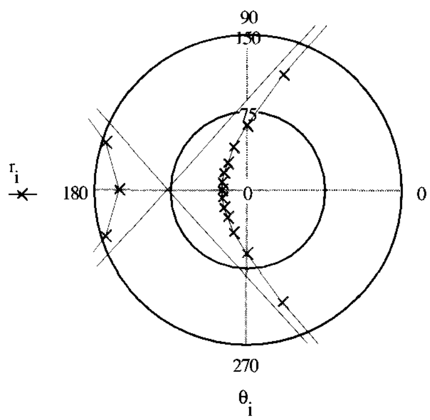
The path of the scattered particle is shown using the numerical values for e and a. The calculations for different parameters are given below.

$N := 20 \qquad i := 0..N$

$a := 50 \qquad e := 1.5$

$\theta_i := \frac{2\cdot\pi}{N}\cdot i \qquad r_i := \frac{a\cdot(e^2 - 1)}{1 - e\cdot\cos(\theta_i)}$

θ = scattering angle = $\pi - 2\alpha$
$\alpha = \pi - \theta/2$
b = impact parameter
r0 = the distance of closest approach



$\alpha := \mathrm{acos}\left(\frac{1}{e}\right) \qquad \theta := 2\cdot\mathrm{atan}\left(\frac{1}{\sqrt{e^2 - 1}}\right)$

$\theta 1 := \pi - 2\cdot\alpha \qquad r_{20} = -125$

$\alpha = 0.841\cdot\mathrm{rad} \qquad \theta = 1.459\cdot\mathrm{rad} \qquad \theta 1 = 1.459\cdot\mathrm{rad} \qquad b := r_{20}\cdot\sin(\theta) \qquad r0 := \min(r)$

$\alpha = 48.19\cdot\mathrm{deg} \qquad \theta = 83.621\cdot\mathrm{deg} \qquad \theta 1 = 83.621\cdot\mathrm{deg} \qquad b = -124.226 \qquad r0 = -292.705$

(C = coulomb),

$$K = kQq = 2kZe^2 = (4.6 \times 10^{-28}\ \text{N-m}^2)Z \tag{8.155}$$

K being positive. Equation (7.106) for the eccentricity e,

$$e = \sqrt{1 + \frac{2EL^2}{mK^2}} \tag{7.106}$$

suggests that $e > 1$; hence the trajectory of an incident alpha particle will be hyperbolic, as shown in Fig. 8.15(a). This is the negative branch trajectory of the hyperbola. The repulsive force center is at F. The scattering angle θ, the angle between the two asymptotes, is

$$\theta = \pi - 2\alpha \tag{8.156}$$

Therefore,

$$\tan\frac{\theta}{2} = \tan\left(\frac{\pi}{2} - \alpha\right) = \cot\alpha \tag{8.157}$$

In the equation for a hyperbola (from Chapter 7) given by

$$r = \frac{a(e^2 - 1)}{1 - e\cos\theta} \tag{7.113}$$

and for the particle at infinity, $r = \infty$, $\theta = \alpha$, the above equation yields

$$\cos \alpha = \frac{1}{e} \tag{8.158}$$

where e is the eccentricity. Thus, combining Eqs. (8.157) and (8.158),

$$\tan \frac{\theta}{2} = \cot \alpha = \frac{\cos \alpha}{\sin \alpha} = \frac{1/e}{(1 - 1/e^2)^{1/2}}$$

$$\tan \frac{\theta}{2} = \frac{1}{\sqrt{e^2 - 1}} \tag{8.159}$$

Substituting for e^2 from Eq. (7.106), we obtain

$$\tan \frac{\theta}{2} = \sqrt{\frac{mK^2}{2EL^2}} \tag{8.160}$$

Referring to Fig. 8.15(b), when the alpha particle is at infinity, its potential energy is $V = K/r = K/\infty = 0$; hence the total energy E is all kinetic and is given by

$$E = \tfrac{1}{2} m v_0^2 \tag{8.161}$$

Since the alpha particle is headed toward the force center F, in the absence of any force it will not be deflected but will pass the force center at a distance b. This distance b by which the particle misses the force center is called the *impact parameter* for the collision. Also, the angular momentum of the particle is

$$L = m v_0 b \tag{8.162}$$

which will remain constant during all its motion due to the law of conservation of angular momentum. Substituting for E and L from Eqs. (8.161) and (8.162) into Eq. (8.160),

$$\tan \frac{\theta}{2} = \sqrt{\frac{mK^2}{2(\frac{1}{2} m v_0^2)(m v_0 b)^2}}$$

Hence

$$\tan \frac{\theta}{2} = \frac{K}{m v_0^2 b} \tag{8.163}$$

where $K = kQq$. Relation (8.163) may be written as

$$b = \frac{K}{m v_0^2} \cot \frac{\theta}{2} \tag{8.164}$$

or

$$\theta = 2 \operatorname{arccot}\left[\left(\frac{m v_0^2}{K}\right) b\right] \tag{8.165}$$

The scattering angle θ can be measured experimentally; hence the impact parameter b can be calculated from Eq. (8.165). Equation (8.164) also states that as b increases θ decreases—that is, the smaller the impact parameter, the larger the scattering angle. In actual practice, we measure the number of particle $N(\theta)$ scattered at different angles. Hence we must find a way to eliminate b in Eq. (8.165) and find the relation between $N(\theta)$ and θ. This leads us to the concept of a cross section, as discussed next.

A typical experiment setup is shown in Fig. 8.16. A beam of charged particles coming from source S is incident on a thin foil that is the target. The particles are scattered in different directions after colliding with the target nuclei. Suppose the particles with impact parameter b are deflected through an angle θ; then those particles with an impact parameter $b + db$ will be deflected through an angle $\theta + d\theta$, where $d\theta$ is negative, as shown in Fig. 8.17. Suppose there are N particles incident on the target foil and the foil contains n nuclei per unit area; that is, there are n scattering centers per unit area. (The foil is considered thin enough so that nuclei do not hide one behind the other.) Thus the number of alpha particles dN that will be scattered through an angle θ and $\theta + d\theta$ is proportional to the scattering centers n and the number of incident particles N; that is,

$$dN = nN\,d\sigma \tag{8.166}$$

where $d\sigma$ is defined as the *cross section* for scattering through an angle θ and $\theta + d\theta$. $d\sigma$ can be thought of as the effective area surrounding each scattering center, which the incident particle must hit in order to be scattered. Thus the total sensitive area for scattering in a unit target area is $n\,d\sigma$; hence the justification for Eq. (8.166). [Note that if the incident particles have im-

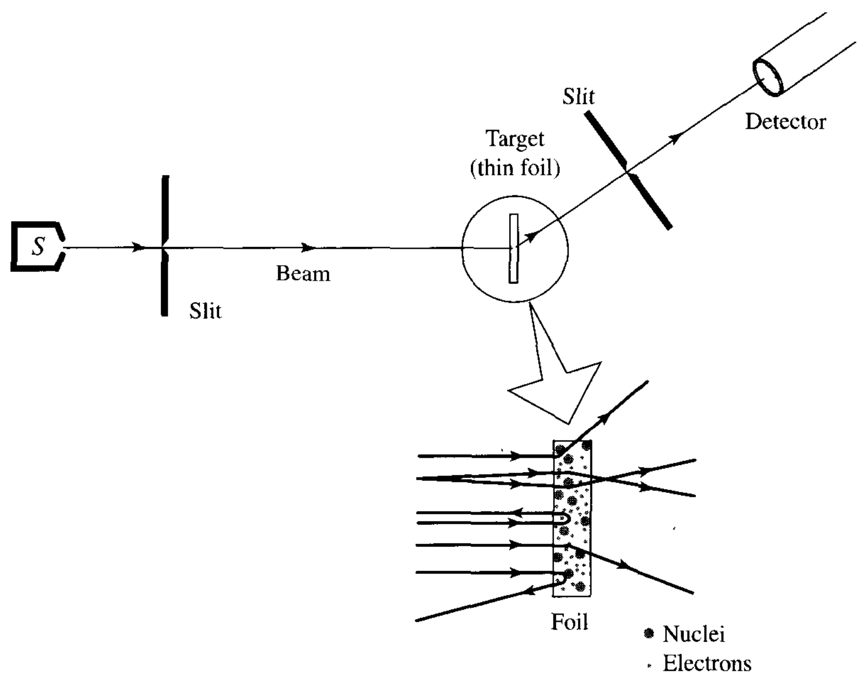


Figure 8.16 Typical experimental setup for investigating the scattering of charged particles from a target of thin foil.

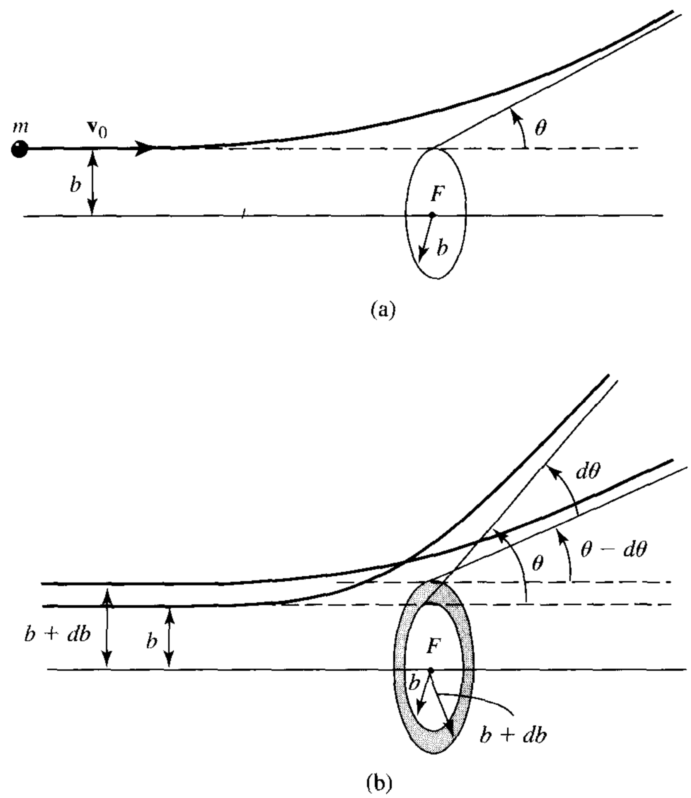


Figure 8.17 (a) A particle with impact parameter b is scattered through an angle θ. (b) Particles with impact parameters between b and $b + db$ are scattered through angles between θ and $\theta - d\theta$.

pact parameters between 0 and b the particles will be scattered through an angle θ or greater than θ. The cross section in this cases is σ and is equal to the area of a disk of radius b in Fig. 8.17(a) with the center at F:

$$\sigma = \pi b^2 \tag{8.167}$$

hence

$$d\sigma = 2\pi b \, db \tag{8.168}$$

as we shall see next.]

Referring to Fig. 8.17(b), the incident particles approaching the scattering center F have an impact parameter between b and $b + db$. These particles will be scattered through an angle between θ and $\theta - d\theta$ if they hit an area of a ring around F of inner radius b and outer radius $b + db$. Thus the area of the ring is the cross-sectional area $d\sigma$; that is,

$$d\sigma = 2\pi b \, db \tag{8.168}$$

We can express b and db in terms of θ and $d\theta$ by using Eq. (8.164), according to which

$$b = \frac{K}{mv_0^2} \cos\frac{\theta}{2} \tag{8.164}$$

and, differentiating this, we get

$$db = -\frac{K}{2mv_0^2} \frac{1}{\sin^2(\theta/2)} \, d\theta \tag{8.169}$$

We may also use Eq. (8.164) to write b as

$$b = \frac{K}{2mv_0^2} \frac{\sin\theta}{\sin^2(\theta/2)} \tag{8.170}$$

Substituting for db and b from Eqs. (8.169) and (8.170) into Eq. (8.168), we get, after omitting the negative sign,

$$d\sigma = 2\pi \left(\frac{K}{2mv_0^2}\right)^2 \frac{\sin\theta}{\sin^4(\theta/2)} d\theta \tag{8.171}$$

Remembering $K = kQq$, we get

$$d\sigma = 2\pi \left(\frac{kQq}{2mv_0^2}\right)^2 \frac{\sin\theta}{\sin^4(\theta/2)} d\theta \tag{8.172}$$

which is the *Rutherford scattering formula.* $d\sigma$ can be measured experimentally by using Eq. (8.166) and can be compared with the theoretical value calculated by using Eq. (8.172).

Rutherford used the derived formula to make an interpretation of his experiment on the scattering of alpha particles ($q = 2e$) by target nuclei ($Q = Ze$) in the form of thin foils. Expression (8.172) held good as long as the perihelion distance ($a + ae$) was larger than 10^{-14} m. From this he concluded that the positive charge of the nucleus must be concentrated in a sphere with a radius of less than 10^{-14} m. The incident alpha particle can come closest to the nucleus for an impact parameter of $b = 0$. This will result in a minimum distance of the perihelion; and at this distance all the kinetic energy of the incident alpha particle is changed into potential energy, and the particle starts turning back. Thus

$$K = V = \frac{kQq}{r_{\min}} \tag{8.173}$$

The use of Eq. (8.173) can give some idea about the magnitude of the nuclear radius. Deviations from the Rutherford scattering formula will occur if the kinetic energy K of the incident particle is greater than the minimum potential energy at a distance $r_{\min}$. From such observations, Rutherford concluded that the nuclear radius was 10^{-14} m.

In the preceding discussion, it was assumed that the target was heavy as compared to the incident particle and hence was assumed to be at rest during the collision. If the target nucleus is not heavy, the nucleus itself will move during the collision, as shown in Fig. 8.18. The diffi-

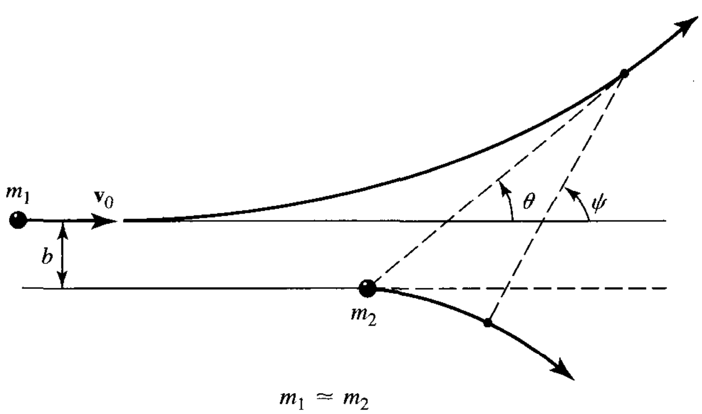


Figure 8.18 Scattering by a target of mass m_2, which is almost equal to the incident particle of mass m_1; that is, $m_1 \simeq m_2$.

culty can be overcome by considering the collision in the CM coordinate system. The final result can be obtained by replacing m by the reduced mass μ [$= mM/(M + m)$] and θ by θ_C in Eq. (8.171); that is,

$$d\sigma = 2\pi\left(\frac{K}{2\mu v_0^2}\right)^2 \frac{\sin\theta_C}{\sin^4(\theta_C/2)}\, d\theta_C \qquad \textbf{(8.174)}$$

In the case where $m_1 = m_2$, we have shown, Eq. (8.151), $\theta_C = 2\theta_L = 2\theta$; hence

$$d\sigma = 4\pi\left(\frac{K}{2\mu v_0^2}\right)^2 \frac{\sin 2\theta}{\sin^4\theta}\, d\theta \qquad \textbf{(8.175)}$$

PROBLEMS

8.1. Find the center of mass, the velocity of the center of mass, the linear momentum, and the kinetic energy of the following system:

$$m_1 = 1\text{ kg},\quad \mathbf{r}_1 = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}},\quad \mathbf{v}_1 = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$$

$$m_2 = 2\text{ kg},\quad \mathbf{r}_2 = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}},\quad \mathbf{v}_2 = 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$$

8.2. Consider the following three particles:

$$m_1 = 1\text{ kg},\quad \mathbf{r}_1 = 2t^2\hat{\mathbf{i}} + 3t\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$$

$$m_2 = 3\text{ kg},\quad \mathbf{r}_2 = (1 + t^2)\hat{\mathbf{i}} + (2 + 5t)\hat{\mathbf{j}}$$

$$m_3 = 5\text{ kg},\quad \mathbf{r}_3 = (1 + 2t^2)\hat{\mathbf{i}} + 4t^2\hat{\mathbf{k}}$$

Calculate the following at $t = 0$ and $t = 10$ s.
(a) The position of the center of mass, **(b)** the velocity of the center of mass, **(c)** the linear momentum, and **(d)** the kinetic energy of the system.

8.3. Find the velocity and acceleration of the center of mass of a system consisting of the following two objects at $t = 0$ and $t = 10$ s.

$$m_1 = 2\text{ kg},\quad \mathbf{r}_1 = 2\hat{\mathbf{i}} + 3t\hat{\mathbf{j}} + 4t^2\hat{\mathbf{k}}$$

$$m_2 = 4\text{ kg},\quad \mathbf{r}_2 = t^2\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 6t^3\hat{\mathbf{k}}$$

8.4. A projectile of mass m is fired with a velocity of 50 m/s at an angle of 60° with the horizontal. At the top (maximum height), it explodes into two fragments, creating an additional energy E, with the result that one fragment is observed to be moving directly upward. What is the direction of the other fragment? Calculate the velocity of both fragments.

8.5. A projectile of mass M ($=m_1 + m_2$) is fired with velocity v making an angle θ with the horizontal. At the top it explodes into two masses, m_1 and m_2, creating an additional energy E. Show that the two fragments strike the ground at a distance apart equal to

$$\frac{v\sin\theta}{g}\left[2E\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\right]^{1/2}$$

8.6. If a projectile explodes at the top (maximum height) with an additional energy E, under what circumstances will one of the fragments land at the starting position?

8.7. A fire boat draws water from a bay through a vertical inlet and sprays it out at a rate of 10 m/s. The diameter of the nozzle of the fire hose is 20 cm. Calculate the horizontal force from the propellers necessary to keep the boat stationary. The density of water is 1020 kg/m^3.

8.8. A bucket of 0.5 kg is placed on a spring scale and water is added to it from a height of 2 m at a rate of 5 ml/s. Find the scale reading as a function of time.

8.9. A chain of length L and mass M is held vertically so that the bottom of the chain just touches the horizontal table top, as shown in Fig. P8.9. If the upper end of the chain is released, determine the force on the table top, as the function of the length of the chain above the table top, while it is falling.

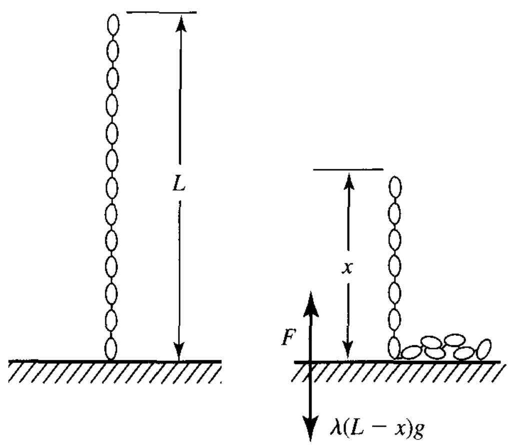


Figure P8.9

8.10. For the falling chain in Fig. P.8.10, show that when all of the chain clears the table the speed of the chain is

$$v = \sqrt{(g/L)(L^2 - a^2)}$$

where $y = a$, when $t = 0$.

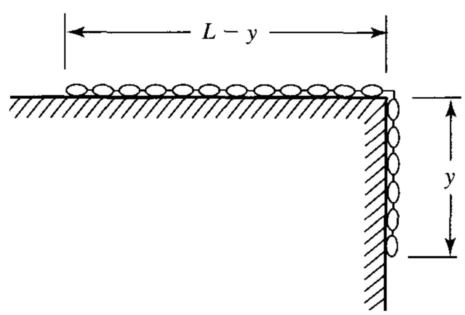


Figure P8.10

8.11. A raindrop as it falls through fog or mist collects mass at a uniform rate. The drop starts from rest with zero radius and remains spherical at all times. Show that the acceleration with which it falls is $g/7$.

8.12. A raindrop of initial mass m_0 is falling under the influence of gravity. Due to condensation from fog or mist, the mass of the drop increases at a rate directly proportional to its instantaneous mass and velocity. Show that eventually the speed of the drop becomes constant. Derive an expression for this terminal velocity. Graph the velocity versus time.

8.13. Consider a spherical raindrop of initial mass m_0 falling through fog or mist. Due to condensation, the raindrop increases in mass at a rate proportional to its mass and velocity. In addition to the force of gravity, the force of friction is present, which is proportional to the velocity and mass of the drop; that is, $F_f = -kmv$. Calculate the velocity of the drop as a function of time. Graph v versus t.

8.14. A raindrop of initial mass m_0 falls vertically through fog or mist. Due to condensation, the mass of the raindrop increases linearly with time; that is, $m = m_0 + \lambda t$. The frictional drag force on the mass m is proportional to its mass and velocity; that is, $F_f = -kmv$. Calculate the velocity of the raindrop as a function of time, assuming the presence of gravitational force and frictional drag. Graph v versus t.

8.15. Calculate the thrust of a test jet engine if it takes in air at a rate of 100 kg/s and exhausts it at a speed of 500 m/s.

8.16. A rocket has an initial mass of 60,000 kg, and the speed of the burned exhaust gases is 6000 m/s. What should be the minimum mass flow rate of the gases to ensure life-off from the surface of Earth?

8.17. A rocket of 60,000-kg mass is burning gases at a rate of 150 kg/s, and the speed of the exhaust gases is 6000 m/s. If the rocket is fired vertically upward from the surface of Earth, what will be its height and speed after 45,000 kg of fuel is expended? Graph the velocity and height as a function of time.

8.18. A rocket propulsion type of car has a mass m_0 without fuel, and its fuel has mass m. The ejecting fuel has a velocity V with respect to the rocket, and the fuel burns at a rate of $k = dm/dt$. Find the acceleration and velocity as a function of time and the velocity when all the fuel has burned out. Graph a and v versus t.

8.19. As a rocket ascends it loses mass at a rate proportional to its instantaneous mass; that is, $dm/dt = bm$, where b is a constant. The motion of the rocket is retarded by air resistance proportional to its velocity; that is, $F_f = -kv$, where k is constant. Find the velocity of the rocket as a function of time. Graph and discuss the outstanding features.

8.20. During the first second of its flight, a rocket exhausts $\frac{1}{50}$ of its mass with a velocity of 2000 m/s. Calculate the acceleration of the rocket. If the rocket exhausts at a constant rate, will it be possible to attain a constant acceleration?

8.21. A rocket of mass $M + m$, where m is the mass of the fuel, rises vertically and ejects gases at a rate of q and with an exhaust velocity of u. Calculate the velocity and the acceleration as a function of time and graph them for the values given next. The initial mass is 4×10^4 kg, $q = 600$ kg/s, and $u = 2000$ m/s. If the fuel burns out in 50 s, calculate the acceleration at $t = 0$ s, 20 s, 40 s, and 50 s.

8.22. A rocket has a mass of m_0 and a mass ratio of R, burns at a rate of dm/dt, and has an exhaust velocity of v_0. Find how long after ignition of the engines it will take the rocket to lift off from the ground. Calculate for the case in which m_0 is 5×10^4 kg, R is 3, the burning rate is 120 kg/s, and the exhaust velocity is 1000 m/s.

8.23. A lunar landing craft is hovering over the Moon's surface. One-third of its mass is fuel, while the exhaust velocity u is 1200 m/s. How long will it take before the craft runs out of fuel? Assume that the acceleration due to gravity on the surface of the Moon is one-sixth of that on Earth.

8.24. Suppose a two-stage rocket starts with a mass m_i. At the end of the first stage, the mass of the rocket is m_1. Before the second stage engines are ignited, some of the mass is discarded, and the starting mass is m_2. The final mass when the engines of the second stage are shut down is m_f. Assuming that the exhaust velocity in both stages is v_0, find the terminal velocity of the second stage.

8.25. An empty truck of mass M starts from rest under an applied force F. At the same time, coal begins to drop into the truck at a rate of $b = dm/dt$. What is the speed of the truck when a mass m of the coal has fallen in?

8.26. An open truck is traveling at a constant speed of 90 km/h and is collecting water from a rainstorm. If it picks up 50 kg of water over a distance of 1000 m, calculate the force and the power required to maintain a constant speed.

8.27. A freight car of mass m contains a mass of coal m. At $t = 0$, a force F is applied. As the car starts rolling, the coal starts dropping at a rate of $b = dm/dt$. What is the speed of the car when all the coal has dropped out?

8.28. A chute discharges sand at the rate of 500 kg/min onto a conveyor belt that is inclined at an angle of 12° to the horizontal and is moving at a rate of 4 m/s. The sand falls at a speed of 5 m/s. Calculate the force necessary to keep the belt moving at a constant speed.

8.29. Consider a conveyor belt inclined at an angle θ from the horizontal so that the belt forms an inclined plane. At the bottom end of the belt, material, deposited at a rate of dm/dt, travels a distance l and then is taken off the upper end of the incline. Calculate the power needed to keep the belt moving at a steady speed v.

8.30. Consider a conveyor belt inclined at an angle θ from the horizontal so that the belt forms an inclined plane. At the top end of the belt, material is deposited at a rate of dm/dt, travels a distance l, and then falls off the lower end of the incline. Assuming a constant force of friction f, calculate the steady speed of the belt.

8.31. Derive Eqs. (8.86) and (8.87).

8.32. In Fig. 8.6(a) if $v_{2i} \neq 0$, show that after the collision $v_{1f} = v_{2i}$ and $v_{2f} = v_{1i}$.

8.33. A neutron of mass m_1 moving with velocity v collides with an atomic nucleus of mass m_2 at rest. Calculate the maximum fractional loss in kinetic energy of the neutron if the atomic nucleus is **(a)** hydrogen, **(b)** carbon, **(c)** iron, and **(d)** lead.

8.34. A particle of mass m_1 and velocity v_{1i} collides with a particle of mass m_2 moving with velocity v_{2i} exactly in the opposite direction. If, after collision, mass m_1 leaves at an angle θ_1 with the initial direction, what is the value of v_{1f}?

8.35. A particle of mass m_1 moving with velocity v_0 collides elasticity with a particle of mass m_2 at rest. At what scattering angle will be momentum of the mass m_1 be half its initial value? What are the restrictions in terms of m_1/m_2?

8.36. A billiard ball of mass m collides with an identical ball at rest. After collision, the two balls leave at angles $\pm\theta$ with the initial direction. Prove that for this to happen the two balls have a rotational kinetic energy of $[1 - (\cos^{-2}\theta)/2]K_i$, where K_i is the initial kinetic energy. Assume that there are no frictional losses in energy.

8.37. Consider a perfect elastic collision between two balls, one of mass m and the other of unknown mass, each moving with a speed v_0 but in opposite directions. After collision, the ball of unknown mass comes to rest. Calculate the unknown mass and the velocity of the ball of mass m.

8.38. A ball of mass m with energy E strikes a ball of mass M at rest. After collision, the ball of mass m is scattered at an angle of 90° from its original direction. Calculate the energy of mass M after collision.

8.39. A particle of mass m_1 moving with velocity v_1 collides with a particle of mass m_2 moving with a velocity v_2, both having the same initial kinetic energy. Find the conditions in terms of v_1/v_2 and m_1/m_2 so that mass m_1 is at rest after collision.

8.40. A particle of mass m moving with velocity v_0 collides with a mass M moving in the opposite direction. After collision, the mass m has velocity $v_0/2$ and moves at right angles to the initial direction, while mass M moves in a direction making an angle of 30° with the initial path of m. Find the ratio m/M.

8.41. A particle of mass m_1 has a head-on collision with a particle of mass m_2 at rest. If the coefficient of restitution is e, calculate the energy loss in this collision.

8.42. A ball of mass m is dropped from a height h onto a horizontal surface. Show that the vertical height through which the ball rises before it stops rebounding is $h(1 + e^2)/(1 - e^2)$, where e is the coefficient of restitution.

8.43. Show that the loss in kinetic energy when two objects collide is $\frac{1}{2}\mu V^2(1 - e^2)$, where μ is the reduced mass, V is the relative speed before collision, and e is the coefficient of restitution.

8.44. A particle of mass m_1 moving at right angles to mass m_2 collides as shown in Fig. P8.44. Calculate the velocity of each particle after collision, assuming that the coefficient of restitution is 0.4, m_1 = 3 kg, m_2 = 2 kg, v_{1i} = 2 m/s, and v_{2i} = 3 m/s.

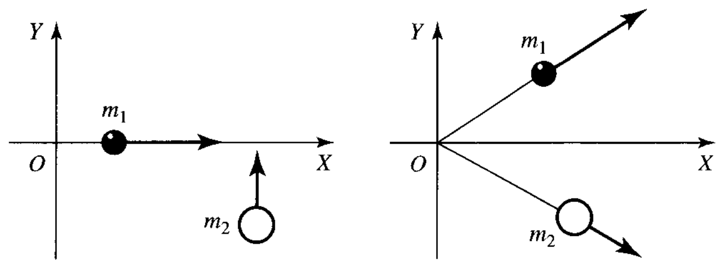


Figure P8.44

8.45. Consider the situation shown in Fig. P8.45. Ball A of mass $2m$ is raised to a height of h so that its string makes an angle of 45° with the vertical, and it is then let go. To what height will ball B of mass m rise if the coefficient of restitution is 0.5?

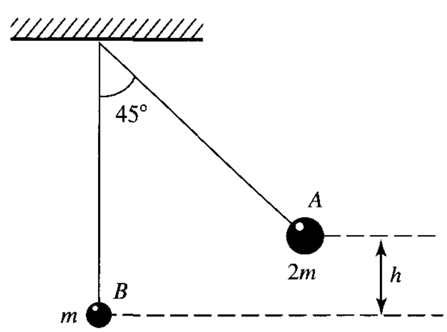


Figure P8.45

8.46. A ball of mass m moving downward with a velocity of $\mathbf{v}$ and making an angle θ with the horizontal strikes a flat surface and rebounds at angle ϕ, as shown in Fig. P8.46. Calculate the velocity of the ball, angle ϕ, and the change in the kinetic energy. Assume that the surface is smooth and the coefficient of restitution is e.

Figure P8.46

8.47. A ball of 1-kg mass moving with a speed of 2 m/s strikes a wooden bar of 2-kg mass moving to the right, with a center-of-mass velocity of 1.5 m/s, as shown in Fig. P8.47. If the coefficient of restitution is 0.4, and the plane is which this collision takes place is smooth, calculate the following quantities just after collision: **(a)** velocity of the ball, and **(b)** linear velocity and angular velocity of the bar.

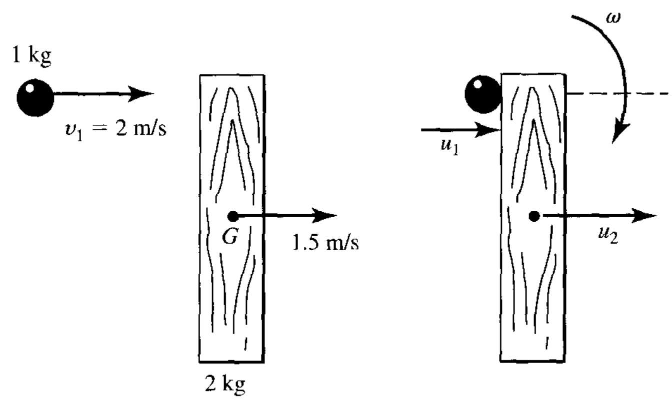


Figure P8.47

8.48. A neutron in a nuclear reactor moving with an initial speed of 120 m/s collides with a deuteron (heavy hydrogen in which the nucleus is made of a proton and a neutron) at rest. The neutron is scattered at an angle of 30°. Calculate the recoil angle for the deuteron and the speed of both the neutron and deuteron after the collision. Draw a diagram showing this collision in the CMCS and the corresponding angles in the CMCS.

8.49. Repeat Problem 8.48 if the deuteron is replaced by a carbon atom with a mass of 12 u.

8.50. An alpha particle of mass 4 u moving with a velocity 2000 m/s collides with a carbon atom of mass 12 u at rest. The alpha particle is scattered through an angle of 30°. Considering the collision to be perfectly elastic, calculate the velocities of both particles after collision and the scattering angle of the recoiling carbon. Describe this collision in the CMCS.

8.51. Derive an expression for the Rutherford scattering cross section in terms of the recoil angle.

8.52. Obtain an expression for the Rutherford scattering cross section for the case in which the mass of the incident particle is very large compared to the mass of the target particle.

8.53. Somewhere in outer space a star of mass m moving with velocity v_0 is headed toward a star of mass $2m$ at rest. The impact parameter in this case is b. Calculate the speeds and the direction of the two stars.

8.54. Show that the differential scattering cross section of mass m from a fixed force center

$$\mathbf{F} = \frac{K}{r^3}\,\hat{\mathbf{r}}$$

is given by

$$\sigma(\theta) = \frac{k\pi^2(\pi - \theta)}{mv_0^2\theta^2(2\pi - \theta)^2 \sin\theta}$$

8.55. A spaceship of mass m moving with velocity $\mathbf{v}_0$ approaches the Moon ($M \gg m$). The distance of closest approach is b, and the velocity $\mathbf{v}_0$ is perpendicular to the orbital velocity $\mathbf{V}$ of the Moon. Show that if the spaceship passes behind the Moon it gains kinetic energy as it leaves the Moon.

8.56. Obtain an expression for the differential scattering cross section in the CMCS for the case where the target particle is much heavier than the incident particle.

8.57. A particle of mass m moving with velocity v_0 collides with a particle of mass m at rest. M is scattered through an angle θ_C in the CM system. What is the final velocity of m in the LCS? Calculate the fractional loss of kinetic energy of m.

SUGGESTIONS FOR FURTHER READING

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*The asterisk indicates works of an advanced nature.