C H A P T E R

15

Vibrating Strings and Fluids

15.1 INTRODUCTION

This chapter is a continuation of the study of mechanics of continuous media such as strings and fluids (gases and liquids). Because a large number of particles are involved, it is cumbersome to apply the laws of mechanics and investigate the resultant motion. Some simplifying assumptions must be made and an overall picture of the motion obtained. We shall divide our discussion into three parts. First, we investigate transverse vibrations of strings in one dimension. To start, we shall consider a simple case but then generalize it by the methods of Lagrange formulation. Second, we study sound waves, that is, longitudinal waves in a gaseous medium. In both of these cases, the main problem involves setting up a wave equation describing the given situation, followed by solving these differential equations by applying appropriate boundary conditions.

Third, we investigate fluids at rest and in motion. We close the chapter by investigating the motion of fluids in the presence of frictional forces (viscous forces).

15.2 VIBRATING STRING

We investigate the propagation of waves along vibrating strings. Our discussion is divided in two parts: the equation of motion and the general solution (normal modes of vibration).

Equation of Motion

Consider a homogeneous string of length L that is fixed at both ends: $x = 0$ and $x = L$. The string has a linear density (mass per unit length) μ and is under tension T throughout the string. The string is in equilibrium along the X-axis, as shown in Fig. 15.1(a). We are interested in investi-

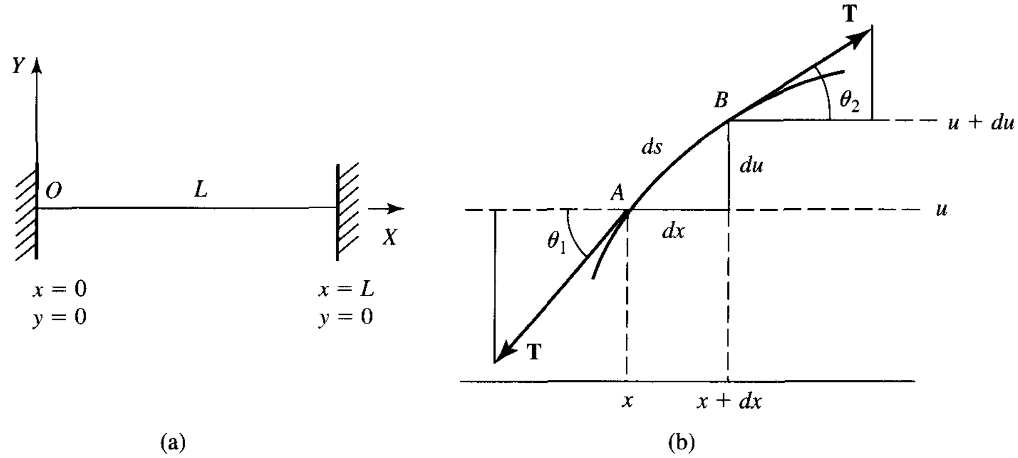


Figure 15.1 **(a)** A string of length L is horizontal when in equilibrium. **(b)** A small portion ds of a string under a small displacement results in transverse vibrations.

gating the motion of such a string following an initial lateral displacement from its equilibrium position. Also, the displacements of the string are not large enough to change the tension T appreciably. Furthermore, we assume that the force due to gravity $(=\mu Lg)$ is small compared to the tension T and may be neglected.

To obtain a differential equation that describes the motion of the string, consider a small portion AB of length ds and of horizontal length dx between x and $x + dx$, as shown in Fig. 15.1(b). Since for small displacements the tension remains the same, we may write the X and Y components of the tensions acting on this small element to be

$$\sum F_x = T\cos\theta_2 - T\cos\theta_1 \quad \textbf{(15.1a)}$$

$$\sum F_y = T\sin\theta_2 - T\sin\theta_1 \quad \textbf{(15.1b)}$$

If θ_1 and θ_2 are small, $\cos\theta_1 \simeq \cos\theta_2$; hence there is no net horizontal force. This means there is no longitudinal displacement of the string. That is, for small displacements of the string, we are concerned only with the lateral or transverse motion (motion perpendicular to the length of the string). That is, the string vibrates in the XY plane. Also, for small displacements or small angles, we may replace the sine by the tangent; that is,

$$\sin\theta_1 \simeq \tan\theta_1 \quad \text{and} \quad \sin\theta_2 \simeq \tan\theta_2 \quad \textbf{(15.2)}$$

Thus the resultant force in the Y direction is

$$\sum F_y \simeq T\tan\theta_1 - T\tan\theta_2 \quad \textbf{(15.3)}$$

The motion of the string is described by a displacement function $u(x, t)$ of each point x and at an instant of time t.

Let the vertical displacement of the string be u at x and $u + du$ at $x + dx$. According to Newton's second law,

$$\sum F_y = ma_y = m\frac{\partial^2 u}{\partial t^2} \tag{15.4}$$

where m is the mass of a string of length AB, $m = \mu\, dx$, and $u = u(x, t)$, is the lateral displacement of the string at position x and instant of time t. (Partial derivatives are used because u is a function of both x and t.) Thus combining the preceding equations and assuming $ds \simeq dx$,

$$\mu\, dx \frac{\partial^2 u}{\partial t^2} = T \tan\theta_2 - T\tan\theta_1 \tag{15.5}$$

Using

$$T\tan\theta = T\frac{\partial u}{\partial x} \tag{15.6}$$

we may write the net vertical force as

$$T\tan\theta_2 - T\tan\theta_1 = T\left(\frac{\partial u}{\partial x}\right)_B - T\left(\frac{\partial u}{\partial x}\right)_A \tag{15.7}$$

The slope of the string at B may be expanded by using a Taylor series; that is,

$$\left(\frac{\partial u}{\partial x}\right)_B = \left(\frac{\partial u}{\partial x}\right)_A + \left(\frac{\partial^2 u}{\partial x^2}\right)_A dx + \cdots \tag{15.8}$$

which, on substituting in Eq. (15.7) and combining with Eq. (15.5), yields

$$\mu\, dx\frac{\partial^2 u}{\partial t^2} = T\frac{\partial^2 u}{\partial x^2}\, dx \tag{15.9}$$

or

$$\mu\frac{\partial^2 u}{\partial t^2} = T\frac{\partial^2 u}{\partial x^2} \tag{15.10}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 u}{\partial t^2} \tag{15.11}$$

Since the dimensions of μ are $[ML^{-1}]$ and the dimensions of T are of force $[MLT^{-2}]$, the dimension of μ/T are $[L^{-2}T^2]$, that is, the reciprocal of velocity squared. Hence the *wave equation* of the vibrating string is

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0 \tag{15.12}$$

where

$$v = \sqrt{\frac{T}{\mu}} \tag{15.13}$$

v is not simply a velocity of propagation; it has a much deeper physical interpretation, which we shall seek later. Here v may be identified as the *wave velocity* with which the wave propagates along the string.

If there were an external vertical force F_e per unit length acting on the string, Eq. (15.9) would take the form

$$\mu\, dx \frac{\partial^2 u}{\partial t^2} = T \frac{\partial^2 u}{\partial x^2}\, dx + F_e\, dx$$

or

$$\mu \frac{\partial^2 u}{\partial t^2} = T \frac{\partial^2 u}{\partial x^2} + F_e \tag{15.14}$$

We shall not deal with these situations and shall return to Eq. (15.12) for further discussion.

General Solution: Normal Modes of Vibration

Equation (15.12) is a partial differential equation for the function $u(x, t)$ that describes the motion of a vibrating string. To evaluate the function $u(x, t)$, we make use of initial and boundary conditions. Suppose at $t = 0$ the function $u(x, t)$ satisfies the following initial conditions:

$$u(x, 0) = u_0(x) \tag{15.15}$$

$$\left[\frac{\partial u}{\partial t}\right]_{t=0} = \dot{u}_0(x) \tag{15.16}$$

where $u_0(x)$ is the displacement and $\dot{u}_0(x)$ is the velocity of the string at $t = 0$, and both are functions of position x. Since the string is tied at the ends, it must satisfy the following boundary conditions:

$$u(0, t) = u(L, t) = 0 \tag{15.17}$$

That is, the displacement at the ends is zero at all times.

We now proceed to find the solution $u(x, t)$ of the differential equation (15.12). We make use of the method of separation of variables. Let

$$u(x, t) = X(x)\Theta(t) \tag{15.18}$$

where X is a function of x alone and Θ is a function of t alone. From Eq. (15.18),

$$\frac{\partial^2 u}{\partial x^2} = \Theta \frac{d^2X}{dx^2} \quad \text{and} \quad \frac{\partial^2 u}{\partial t^2} = X \frac{d^2\Theta}{dt^2} \tag{15.19}$$

Substituting these in Eq. (15.12) and rearranging, we obtain

$$\frac{v^2}{X} \frac{d^2X}{dx^2} = \frac{1}{\Theta} \frac{d^2\Theta}{dt^2} \tag{15.20}$$

The left side of this equation is a function of x only, while the right side is a function of t only. This is possible for all values of x and t only if each side is equal to a constant. Let this constant be $-\omega^2$. The minus sign indicates that the acceleration of the element of the string is always directed toward the equilibrium position (position of the string when it is along the X-axis), that is, the acceleration is opposite to the displacement. Thus, from Eq. (15.20),

$$\frac{v^2}{X} \frac{d^2X}{dx^2} = -\omega^2 \quad \text{or} \quad \frac{d^2X}{dx^2} + \frac{\omega^2}{v^2} X = 0 \tag{15.21}$$

and

$$\frac{1}{\Theta}\frac{d^2\Theta}{dt^2} = -\omega^2 \quad \text{or} \quad \frac{d^2\Theta}{dt^2} + \omega^2\Theta = 0 \tag{15.22}$$

where ω may be interpreted as the angular frequency. The solution of Eq. (15.21) is

$$X(x) = C\cos\frac{\omega}{v}x + D\sin\frac{\omega}{v}x \tag{15.23}$$

and that of Eq. (15.22) is

$$\Theta(t) = E\cos\omega t + F\sin\omega t \tag{15.24}$$

where C, D, E, and F are the four constants of integration to be evaluated by using the initial and boundary conditions given by Eqs. (15.15) to (15.17).

Thus, by substituting for $X(x)$ and $\Theta(t)$ from Eqs. (15.23) and (15.24) into Eq. (15.18), we get the general solution:

$$u(x, t) = \left(C\cos\frac{\omega}{v}x + D\sin\frac{\omega}{v}x\right)(E\cos\omega t + F\sin\omega t) \tag{15.25}$$

We may now apply the boundary conditions to evaluate the constants C and D. At $x = 0$, $u(0, t) = 0$ for all values of t; that is $X(0) = 0$ in Eq. (15.23):

$$0 = C\cos\left(\frac{\omega}{v}0\right) + D\sin\left(\frac{\omega}{v}0\right)$$

which is possible only if $C = 0$; thus

$$X(x) = D\sin\frac{\omega}{v}x \tag{15.26}$$

At $x = L$, $u(L, t) = 0$ for all values of t; that is, $X(L) = 0$ in Eq.(15.26):

$$0 = D\sin\frac{\omega}{v}L \tag{15.27}$$

Since $C = 0$ and D cannot be zero because that would give a trivial solution, to satisfy Eq. (15.27), we must have

$$\sin\frac{\omega}{v}L = 0 \quad \text{or} \quad \frac{\omega}{v}L = n\pi \tag{15.28}$$

where $n = 1, 2, 3, \ldots$, or, replacing ω by ω_n, and $v = \sqrt{T/\mu}$,

$$\omega_n = \frac{n\pi v}{L} = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}} \tag{15.29}$$

Thus, with $C = 0$ and letting $DE = A_n$ and $DF = B_n$, we may write Eq. (15.25) to be

$$u(x, t) = (A_n\cos\omega_n t + B_n\sin\omega_n t)\sin\frac{n\pi}{L}x \tag{15.30}$$

where $\omega_n = 2\pi\nu_n$, ν_n being the normal frequencies of vibrations. For a given value of n, we may write

$$u(x, t) = A_n \sin \frac{n\pi x}{L} \cos \frac{n\pi v}{L} t + B_n \sin \frac{n\pi x}{L} \sin \frac{n\pi v}{L} t \tag{15.31}$$

Equation (15.30) or (15.31) represents *normal mode of vibration* of the string in particular, the nth mode. The velocity of the normal mode can be obtained by differentiating Eq. (15.31):

$$\begin{aligned} \dot{u}(x, t) &= \frac{d}{dt} u(x, t) \\ &= A_n \sin\left(\frac{n\pi x}{L}\right)\left(-\frac{n\pi v}{L}\right) \sin \frac{n\pi v}{L} t + B_n \sin\left(\frac{n\pi x}{L}\right)\left(\frac{n\pi v}{L}\right) \cos \frac{n\pi v}{L} t \end{aligned} \tag{15.32}$$

We can now evaluate the constants A_n and B_n of the nth mode of vibration by using the initial conditions that, at $t = 0$,

$$u(x, 0) = u_0(x) \quad \text{and} \quad \dot{u}(x,0) = \dot{u}_0(x) \tag{15.33}$$

Using these conditions in Eqs. (15.31) and (15.32), respectively, we obtain

$$u_0(x) = A_n \sin \frac{n\pi x}{L} \tag{15.34}$$

$$\dot{u}_0(x) = \frac{n\pi v}{L} B_n \sin \frac{n\pi x}{L} \tag{15.35}$$

We know from the theory of differential equations that if $u_1(x, t)$ and $u_2(x, t)$ are any two solutions that satisfy the boundary conditions given by Eq. (15.17), then $u(x)$, which is a linear combination of $u_1(x, t)$ and $u_2(x, t)$, that is

$$u(x, t) = u_1(x, t) + u_2(x, t) \tag{15.36}$$

is also a solution. A more general solution is obtained by adding together all the n particular solutions using different constants A_n and B_n corresponding to different frequencies ω_n. Thus the general solution of motion of a vibrating string is a linear combination of a large number of normal modes [from Eq. (15.30)] and is given by

$$u(x, t) = \sum_{n=1}^{\infty} \left(A_n \sin \frac{n\pi x}{L} \cos \omega_n t + B_n \sin \frac{n\pi x}{L} \sin \omega_n t\right) \tag{15.37}$$

where $$\omega_n = \frac{n\pi v}{L}, \quad n = 1, 2, 3, \ldots$$

which is a solution containing an infinite number of arbitrary constants. In initial conditions corresponding to different modes are given, that is, at $t = 0$

$$u(x, 0) = u_0(x) \quad \text{and} \quad \dot{u}(x, 0) = \dot{u}_0(x) \tag{15.38}$$

then from Eq. (15.37) we obtain

$$u_0(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \tag{15.39}$$

$$\dot{u}_0(x) = \sum_{n=1}^{\infty} \frac{n\pi v}{L} B_n \sin \frac{n\pi x}{L} \tag{15.40}$$

Before we get involved in evaluating the constants, we show in Fig. 15.2 plots of $u(x, t)$ versus x for $n = 1, 2, 3, 4$. The mode of vibration in which $n = 1$ is called the *fundamental* or *first harmonic*. The mode of vibration for $n = 2$ is called the *first overtone* or *second harmonic*; similarly, $n = 3$ corresponds to the *second overtone* or *third harmonic*. The frequency of the nth harmonic is n times that of the fundamental frequency. In general, a string vibrates with several modes simultaneously.

The general solution given by Eq. (15.37) consisting of sums of sines and/or cosines is called a *Fourier series*. The general solution is completely known if the coefficients A_n and B_n are known. These coefficients can be evaluated if the initial conditions, that is, the values of $u_0(x)$ and $\dot{u}_0(x)$, are known. We use the Fourier technique to evaluate these constants. Multiply both sides of Eq.(15.39) by $\sin(m\pi x/L)$, where m is an integer; and integrate from $x = 0$ to $x = L$.

$$\int_0^L u_0(x) \sin \frac{m\pi x}{L}\, dx = \int_0^L \sum_{n=1}^{\infty} A_n\left(\sin \frac{n\pi x}{L}\right)\left(\sin \frac{m\pi x}{L}\right) dx \tag{15.41}$$

But all the terms on the right will vanish unless $m = n$. Thus integration yields

$$\int_0^L u_0(x) \sin \frac{n\pi x}{L}\, dx = A_n \int_0^L \sin^2 \frac{n\pi x}{L}\, dx = A_n \frac{L}{2}$$

or

$$A_n = \frac{2}{L} \int_0^L u_0(x) \sin \frac{n\pi x}{L}\, dx \tag{15.42}$$

Similarly, multiplying both sides of Eq. (15.40) by $\sin(m\pi x/L)$ and integrating from $x = 0$ to $x = L$, that is,

$$\int_0^L \dot{u}_0(x) \sin \frac{m\pi x}{L}\, dx = \int_0^L \sum_{n=1}^{\infty} \frac{n\pi v}{L} B_n \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L}\, dx$$

yields, as before,

$$B_n = \frac{2}{n\pi v} \int_0^L \dot{u}(x) \sin \frac{n\pi x}{L}\, dx \tag{15.43}$$

Thus Eqs. (15.42) and (15.43) state that, if displacements $u_0(x)$ and velocity $\dot{u}_0(x)$ are given for all points of the string at one time, A_n and B_n can be evaluated. Once these constants are known, the motion of the string at all subsequent times is known.

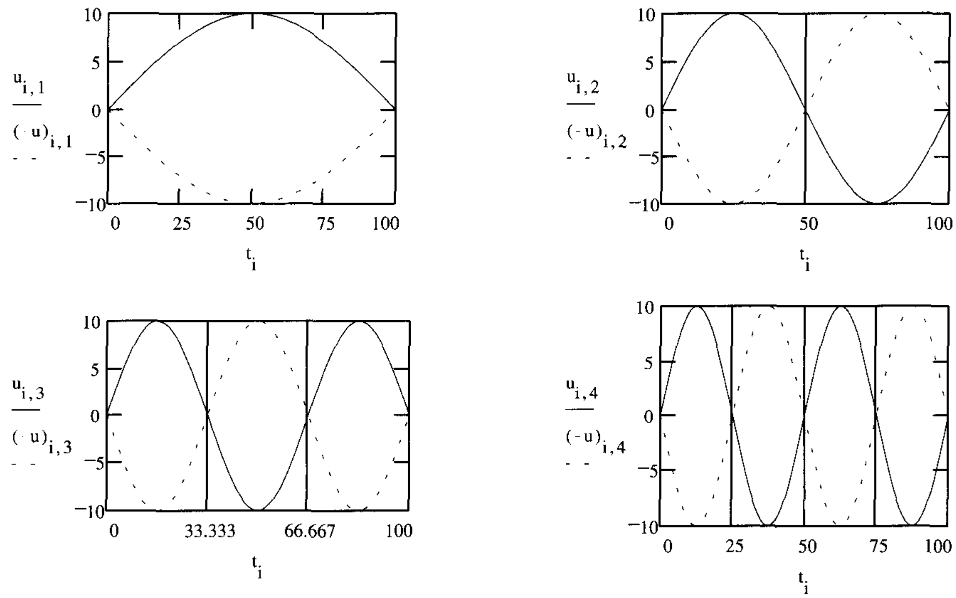
Figure 15.2

Below some possible modes of vibration of a string are shown. In general, a string vibrates in a combination of several modes simultaneously.

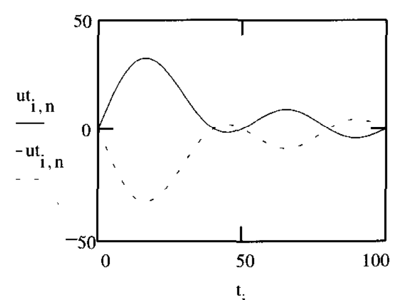
$N := 9 \quad n := 1..N \quad I := 100 \quad i := 0..I \quad x_i := i \quad t_i := i \quad L := 100 \quad v := 1$

N = number of modes $\quad A_n := 10 \quad B_n := 10 \quad \omega_n := n\cdot\frac{\pi}{2}\cdot\frac{v}{L}$

$$u_{i,n} := \left(A_n\cdot\sin\left(\frac{n\cdot\pi\cdot x_i}{L\cdot 2}\right)\cdot\cos\left(\omega_n\cdot t_i\right) + B_n\cdot\sin\left(\frac{n\cdot\pi\cdot x_i}{L\cdot 2}\right)\cdot\sin\left(\omega_n\cdot t_i + \frac{\pi}{2}\right)\right) \qquad ut_{i,n} := u_{i,1} + u_{i,2} + u_{i,3} + u_{i,4}$$



The first four graphs show four different modes, n = 1, 2, 3, and 4. The fifth graph shows the combination of all four modes excited simultaneously.



Example 15.1

A string of length L of mass per unit length μ is fixed at two ends and is under tension T. The string is initially displaced a distance h ($h \ll L$) at the middle of the string and then released. Evaluate the Fourier coefficients for the subsequent motion of the string.

Solution

Figure Ex. 15.1(a) shows the initial configuration of the string. Hence the initial conditions are

$$\text{For } 0 < x < \frac{L}{2}, \quad \frac{u}{x} = \frac{h}{L/2} \quad \text{or} \quad u = \frac{2h}{L}x \tag{i}$$

$$\text{For } \frac{L}{2} < x < L, \quad \frac{u}{L-x} = \frac{h}{L/2} \quad \text{or} \quad u = \frac{2h}{L}(L-x) \tag{ii}$$

$$\text{At } t = 0, \quad \frac{du_0(x)}{dt} = \dot{u}_0(x) = 0 \tag{iii}$$

Substituting the value of $\dot{u}_0(x)$ from Eq. (iii) in Eq. (15.43) reveals that $B_n = 0$ for all n. The values of A_n can be determined by using the initial conditions given by Eqs. (i) and (ii). Substituting these in Eq.(15.42),

$$A_n = \frac{2}{L}\left[\frac{2h}{L}\int_0^{L/2} x \sin\frac{n\pi x}{L}\,dx + \frac{2h}{L}\int_{L/2}^{L}(L-x)\sin\frac{n\pi x}{L}\,dx\right] \tag{iv}$$

Evaluating integrals for different values of n, we obtain

$$A_n = 0, \qquad \text{if } n \text{ is even} \tag{v}$$

$$A_n = \frac{8h}{n^2\pi^2}\sin\frac{n\pi}{L}, \qquad \text{if } n \text{ is odd} \tag{vi}$$

Substituting these in Eq. (15.37), we obtain the general solution of the form

$$u(x,t) = \frac{8h}{\pi^2}\sin\frac{\pi x}{L}\cos\frac{nvt}{L} + \frac{1}{3^2}\sin\frac{3\pi x}{L}\cos\frac{3\pi vt}{L} + \frac{1}{5^2}\frac{8h}{\pi^2}\sin\frac{5\pi x}{L}\cos\frac{5\pi vt}{L} + \cdots \tag{vii}$$

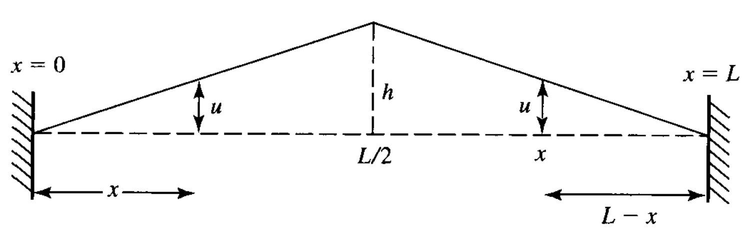


Figure Ex. 15.1(a)

Note that only the odd harmonics have been excited. The plots of the first three odd harmonics are shown in Fig. Ex. 15.1(b), and it is clear that none of the harmonics have been excited that would have a node at the midpoint. the modes shown and others are usually excited simultaneously.

The integration of expression of Eq. (iv) yields the value of A_n. Substituting different values of n, we can calculate the values of different A_n.

$$A_n = \frac{2}{L}\cdot\left[\frac{2\cdot h}{L}\cdot\int_0^{\frac{L}{2}} x\cdot\sin\left(\frac{n\cdot\pi\cdot x}{L}\right)dx + \frac{2\cdot h}{L}\cdot\int_{\frac{L}{2}}^{L}(L-x)\cdot\sin\left(\frac{n\cdot\pi\cdot x}{L}\right)dx\right]$$

$$A_n = 4\cdot h\cdot\frac{\left(2\cdot\sin\left(\frac{1}{2}\cdot n\cdot\pi\right)-\sin(n\cdot\pi)\right)}{\left(n^2\cdot\pi^2\right)} \qquad A_1 = 8\cdot\frac{h}{\pi^2}$$

$n := 1..5 \quad i := 6..19 \quad t_i := i$

$L := 50 \quad h := 2 \quad Ae := 0$

$v := 4\cdot\pi \quad x_i := \frac{2\cdot h}{L}\cdot(L-i)$

$$A_n := 4\cdot h\cdot\frac{\left(2\cdot\sin\left(\frac{1}{2}\cdot n\cdot\pi\right)-\sin(n\cdot\pi)\right)}{\left(n^2\cdot\pi^2\right)} \qquad A_3 = \frac{8}{9}\cdot\frac{h}{\pi^2}$$

$$A_5 = \frac{8}{25}\cdot\frac{h}{\pi^2}$$

$$U1_i := \frac{8\cdot h}{\pi^2}\cdot\sin\left(\frac{\pi\cdot x_i}{L}\right)\cdot\cos\left(\frac{v\cdot t_i}{L}\right)$$

$$U3_i := \frac{8\cdot h}{3^2\cdot\pi^2}\cdot\sin\left(\frac{3\cdot\pi\cdot x_i}{L}\right)\cdot\cos\left(\frac{3\cdot v\cdot t_i}{L}\right)$$

$$U5_i := \frac{8\cdot h}{(5\cdot\pi)^2}\cdot\sin\left(\frac{5\cdot\pi\cdot x_i}{L}\right)\cdot\cos\left(5\cdot v\cdot\frac{t_i}{L}\right)$$

n	A_n
1	1.621
2	0
3	-0.18
4	0
5	0.065

max(U1) = 0.305

max(U3) = 0.091

max(U5) = 0.046

Explain why there are only odd numbers of segments.

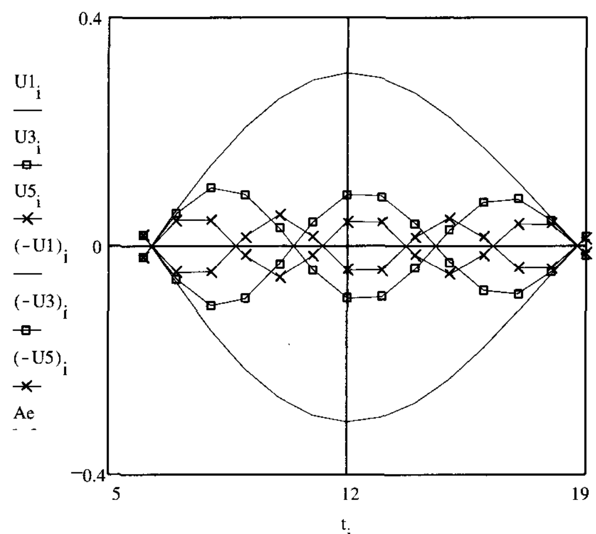


Fig. Ex 15.1(b)

15.3 WAVE PROPAGATION IN GENERAL

Wave motion is not limited to the vibration of strings. It is a phenomenon that occurs in many different branches of physics and involves such cases as sound waves, waves on a liquid surface, and electromagnetic waves. With the advent of quantum mechanics, in which we deal with such abstract ideas as probability waves, the study of wave motion has assumed a much more important and fundamental role. One may be tempted to say that wave motion deals with those phenomena that exhibit periodicity or oscillations. But this is not always true in general. For example, a pulse that travels on a rope or a tidal wave does not exhibit periodicity.

A better definition of wave motion is discussed in terms of energy transport; when a wave reaches a portion of a medium, it sets the particles of the medium into motion. After the wave has passed the particles come to rest, while neighboring particles are set in motion. From this we may conclude that one of the common characteristics of all wave motion is the following:

> *Wave motion provides a mechanism for transfer of energy from one point to another without physical transfer of any material between the points.*

It may be pointed out that wave motions in solids, liquids, and gases do need a medium to transfer energy, while electromagnetic waves can transport energy without requiring a medium to carry them. Thus it is essential that we adopt a more basic viewpoint of wave motion (a kinematical viewpoint instead of the dynamical one stated previously).

Let us discuss the propagation of a single pulse in one dimension. Consider a stretched rope that has been shaken at one end, resulting in a pulse traveling along its length and taking the form shown in Fig. 15.3. This pulse, wave, or disturbance travels along the rope, say along the X-axis, without distortion in form; that is, it has the same shape at t_1 as at any other later time, t_2. We have assumed an ideal case in which the form of the pulse does not change. In actual practice, because of damping, there will be some change in form. The pulse form travels with a constant velocity. The same remarks can be made about any wave disturbance or wave

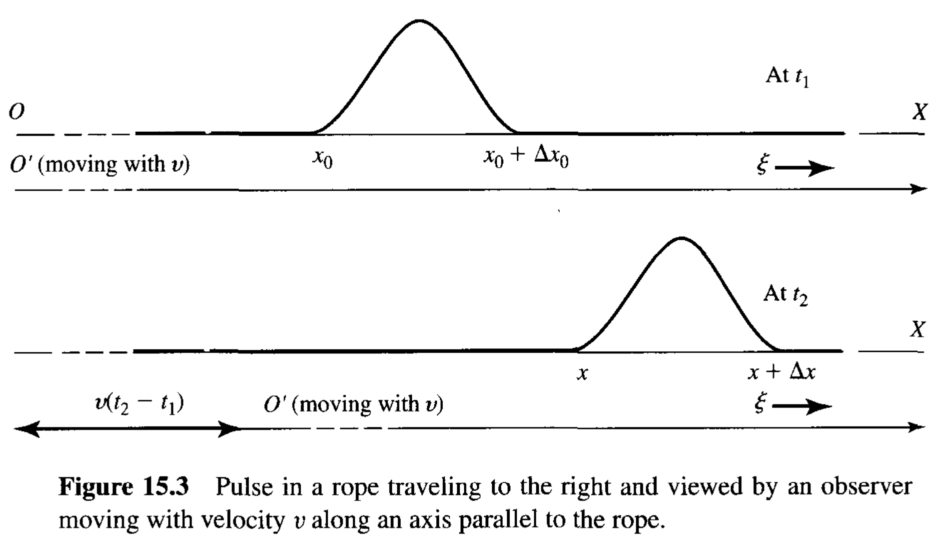


Figure 15.3 Pulse in a rope traveling to the right and viewed by an observer moving with velocity v along an axis parallel to the rope.

motion. Thus we may define it as follows:

> ***Wave motion*** *is a disturbance that propagates itself with constant velocity without changing its form or pattern.*

Suppose a pulse or disturbance is traveling along the $+X$-axis with constant velocity v. Now we view this pulse from the ξ-axis, which is moving with velocity v along and parallel to the X-axis. Furthermore, if the origins of the X-axis and ξ-axis coincide at $t = 0$, we may write

$$\xi = x - vt \tag{15.44}$$

Thus, to an observer in the ξ system, the form and position of the disturbance remains unchanged; that is, the disturbance has such a time dependence that it is a function of ξ alone. Thus the wave propagating to the right is

$$u(x, t) = f(\xi) \equiv f(x - vt) \tag{15.45}$$

where $f(\xi)$ is a completely arbitrary function. Equation (15.45) guarantees that it is a wave traveling to the right. Thus, as t increases, x must increase so that ξ remains constant; hence $f(\xi)$ represents a wave traveling to the right. Similarly, we define

$$\eta = x + vt \tag{15.46}$$

and a wave propagating to the left is given by

$$u(x, t) = g(\eta) \equiv g(x + vt) \tag{15.47}$$

where $g(\eta)$ is another arbitrary function. Once again, as t increases, x must decrease so that η is constant, and then $g(\eta)$ represents a wave propagating to the left. f and g given by Eqs. (15.45) and (15.47) are referred to as *wave forms* and represent the most general type of one-dimensional motion.

By direct substitution of f and g from Eqs. (15.45) and (15.47) into Eq. (15.12), we can show that these satisfy the wave equation.

The general expression for u is a combination of two functions, one of which depends only on ξ and the other only on η; that is, the sum of the two linear functions of ξ and η [individual functions $f(\xi)$ and $g(\eta)$ are also solutions as long as they are linear] is

$$u(x, t) = f(\xi) + g(\eta) = f(x - vt) + g(x + vt) \tag{15.48}$$

Thus the most general solution of the wave equation, Eq. (15.12), is given by Eq. (15.48) or any other *linear* combination of $f(\xi)$ and $g(\eta)$. That Eq. (15.48) is a general solution is consistent with the fact *that the general solution of a second-order differential equation contains two arbitrary functions.*

Let us now proceed to evaluate these functions using initial conditions; that is, at $t = 0$,

$$u = u_0(x) \quad \text{and} \quad \dot{u} = \dot{u}_0(x) \tag{15.49}$$

gives

$$u(x, 0) = f(x) + g(x) = u_0(x) \tag{15.50}$$

and

$$\left[\frac{\partial u}{\partial t}\right]_{t=0} = \left[-v\frac{df}{d\xi} + v\frac{dg}{d\eta}\right]_{t=0} \dot{u}_0(x) \tag{15.51}$$

At $t = 0$, $\xi = \eta = x$, and Eq. (15.51) takes the form

$$v \frac{d}{dx}[-f(x) + g(x)] = \dot{u}_0(x) \tag{15.52}$$

which on integration gives

$$-f(x) + g(x) = \frac{1}{v}\int_0^x \dot{u}_0(x)\,dx + C \tag{15.53}$$

where C is a constant of integration. Adding and subtracting Eqs. (15.50) and (15.53), we obtain

$$f(x) = \frac{1}{2}\left[u_0(x) - \frac{1}{v}\int_0^x \dot{u}_0(x)\,dx - C\right] \tag{15.54}$$

$$g(x) = \frac{1}{2}\left[u_0(x) + \frac{1}{v}\int_0^x \dot{u}_0(x)\,dx + C\right] \tag{15.55}$$

Since these solutions hold for any value of x, we may replace x by ξ or η. Also, the constant C may be dropped because it may be eliminated in linear combinations of solutions. Thus

$$f(\xi) = \frac{1}{2}\left[u_0(\xi) - \frac{1}{v}\int_0^\xi \dot{u}_0(\xi)\,d\xi\right] \tag{15.56}$$

$$g(\eta) = \frac{1}{2}\left[u_0(\eta) + \frac{1}{v}\int_0^\eta \dot{u}_0(\eta)\,d\eta\right] \tag{15.57}$$

Our next step is to see the connection between the general solution obtained in this section and those in the previous section concerning vibrations of strings. The partial differential equation Eq. (15.12) was separated into two differential equations, Eqs. (15.21) and (15.22); that is,

$$\frac{d^2X}{dx^2} + \frac{\omega^2}{v^2}X = 0 \tag{15.21}$$

$$\frac{d^2\Theta}{dt^2} + \omega^2\Theta = 0 \tag{15.22}$$

Instead of writing the solutions in the form of sines and cosines as given by Eqs. (15.23) and (15.24), we may write the solutions of these equations in the following alternative form:

$$X(x) = Ce^{i(\omega/v)x} + De^{-i(\omega/v)x} \tag{15.58}$$

$$\Theta(t) = Ee^{i\omega t} + Fe^{-i\omega t} \tag{15.59}$$

where C, D, E, and F are the constants to be determined from the boundary conditions. Thus the general solution will be of the form

$$\begin{aligned} u(x, t) &= X(x)\Theta(t) = Ae^{\pm i(\omega/v)x}e^{\pm i\omega t} \\ &= Ae^{\pm i(\omega/v)(x\pm vt)} \end{aligned} \tag{15.60}$$

where A is a constant. This states that the general solution $u(x, t)$ is a linear combination of the following terms:

$$\pm i(\omega/v)(x \pm vt) \tag{15.61}$$

Note that these solutions already contain the quantities that are functions of $x + vt$ and $x - vt$. By taking the real part or by adding the complex conjugate and dividing by 2, we obtain the solutions

$$u(x, t) = A \cos \frac{\omega}{v}(x - vt) \tag{15.62}$$

$$u(x, t) = A \cos \frac{\omega}{v}(x + vt) \tag{15.63}$$

and by adding the imaginary parts or subtracting the complex conjugate and dividing by $2i$, we obtain

$$u(x, t) = A \sin \frac{\omega}{v}(x - vt) \tag{15.64}$$

$$u(x, t) = A \sin \frac{\omega}{v}(x + vt) \tag{15.65}$$

The solutions containing $x - vt$ represent waves traveling to the right. While those containing $x + vt$ represent waves traveling to the left. These solutions do not satisfy the boundary conditions because they represent *traveling waves* down the string or medium.

Furthermore, these equations are not satisfied by only one particular value of $-\omega^2$; many more are possible. Thus the general solution is not only a linear combination of harmonic terms given by Eq. (15.60), but also must be summed over all possible frequencies. Thus the most general solution is

$$u(x, t) = \sum_n A_n e^{\pm i(\omega_n/v)(x \pm vt)} \tag{15.66}$$

Once the boundary conditions are known, the constants can be evaluated in a manner similar to the case of evaluation of coefficients in infinite Fourier series. For our discussion, we shall write the solution in the following form, it being understood that the complete solution is summed over all frequencies:

$$u(x, t) = Ae^{i(\omega/v)(x - vt)} \tag{15.67}$$

The quantity k, called the *propagation constant* or *angular wave number* or simply *wave number* (number of waves per unit length) has dimensions of reciprocal length and is defined as

$$k^2 \equiv \frac{\omega^2}{v^2} \quad \text{or} \quad |k| = \frac{\omega}{v} \tag{15.68}$$

Thus the wave equation for X, Eq. (15.21) and its general solution Eq. (15.67) take the forms

$$\frac{d^2X}{dx^2} + k^2X = 0 \tag{15.69}$$

and

$$u(x, t) = Ae^{ik(x-vt)} = Ae^{i(kx-\omega t)} \tag{15.70}$$

If ν is the frequency of vibration so that $\omega = 2\pi\nu$, then the *wavelength* λ is defined as the distance for one complete vibration of the wave

$$\lambda = \frac{v}{\nu} = \frac{2\pi v}{2\pi\nu} = \frac{2\pi v}{\omega} \tag{15.71}$$

Combining this with Eq. (15.68), we get

$$k = \frac{2\pi}{\lambda} \tag{15.72}$$

Let us see what happens if we superimpose two waves, both of the same frequency and amplitude, but one traveling to the right and the other to the left. Thus

$$u = u_1 + u_2 = Ae^{i(kx-\omega t)} + Ae^{i(kx+\omega t)} \tag{15.73}$$

$$u = 2Ae^{-i\omega t}\cos kx \tag{15.74}$$

The real part of this equation yields

$$u(x, t) = 2A\cos kx \cos \omega t \tag{15.75}$$

This wave has the property that it does not propagate forward with time. This superposition of waves leads to the formation of *standing waves*. There are certain points where there is no motion at all because of the cancellation of one wave by the other. Such points are called *nodes*. Since at nodes no motion is possible, no energy is transmitted from one side to the other; hence the pattern is named standing waves. From Eq. (15.75), we can obtain the condition for the position of the nodes to be

$$x = (2n + 1)\frac{\lambda}{4} = (2n + 1)\frac{\pi}{2k} \tag{15.76}$$

Before concluding this section, let us talk about phase velocity and dispersion. To start, let us say that we have a wave motion of a single wave (or frequency) given by Eq. (15.70):

$$u(x, t) = Ae^{i(kx-\omega t)} \tag{15.70}$$

The quantity $kx - \omega t$ is defined as the *phase* ϕ of the wave represented by $u(x, t)$; that is,

$$\phi \equiv kx - \omega t \tag{15.77}$$

The wave pattern or form will remain unchanged in time if ϕ remains constant. For ϕ to remain constant, we must have

$$d\phi = 0 \quad \text{or} \quad k\,dx - \omega\,dt = 0 \tag{15.78}$$

That is, we define the *phase velocity* v_p to be the velocity with which the wave pattern travels; it is given by

$$v_p = \frac{dx}{dt} = \frac{\omega}{k} = v \tag{15.79}$$

That is, for a simple wave the phase velocity v_p is equal to the wave velocity v. This is not true in general. The phase velocity is usually a function of frequency; that is, in a given medium the phase velocity is frequency dependent, $v_p = v_p(k)$. Such a medium is called a *dispersive medium.* In a dispersive medium the phase velocity is not equal to the wave velocity. (As an example, for electromagnetic waves in a given refractive medium, the velocity of the waves is a function of the wavelength.) Thus, in such cases, the wave pattern is modified; it does not remain constant. But even such a pattern will appear unchanged to an observer who is moving with a velocity v_g given by (ω being a function of k)

$$v_g = \frac{d\omega(k)}{dk} \tag{15.80}$$

where v_g is called the *group velocity.*

15.4 LAGRANGE FORMULATION OF A VIBRATING STRING: ENERGY AND POWER

If we calculate the kinetic energy and potential energy of a vibrating string, we can set up the Lagrangian L and the Lagrange equations; hence we can calculate the normal modes of a vibrating string. Furthermore, we know the total energy stored in the string and also the rate at which the energy is being transferred from one portion of the string to the other.

Let us reconsider the vibrating string shown in Fig. 15.1, which has length L and is fixed at both ends. As shown in Fig. 15.1(b), the element of length dx when in equilibrium is stretched to length ds when vibrating. The tension in the string is T when it is vibrating. Thus the amount of potential energy stored in this vibrating element of the string is, assuming the potential energy to be zero when the string is unstretched,

$$dV = T(ds - dx) = T\left(\frac{ds}{dx} - 1\right) dx \tag{15.81}$$

where

$$\frac{ds}{dx} = \left[1 + \left(\frac{\partial u}{\partial x}\right)^2\right]^{1/2} \tag{15.82}$$

Substituting this in Eq. (15.81), assuming $\partial u/\partial x \ll 1$, and using the binomial theorem for expansion, we obtain

$$dV \simeq \frac{T}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx \tag{15.83}$$

Thus the total potential energy stored in the string may be obtained by integrating Eq. (15.83); that is,

$$V = \frac{T}{2}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx \tag{15.84}$$

The mass of an element of length dx is $\mu\, dx$; hence its kinetic energy is (in order to avoid confusion we will start using K for kinetic energy instead of T, which we are using for tension)

$$dK = \frac{1}{2}\mu\, dx\left(\frac{\partial u}{\partial t}\right)^2 \tag{15.85}$$

while the total kinetic energy of the string is obtained by integrating Eq. (15.85):

$$K = \frac{\mu}{2}\int_0^L \left(\frac{\partial u}{\partial t}\right)^2 dx \tag{15.86}$$

To evaluate V and K, we make use of the solution given by Eq. (15.30),

$$u(x, t) = \Theta_n(t) \sin \frac{\omega_n x}{v} \tag{15.87}$$

where

$$\Theta_n(t) = A_n \cos \omega_n t + B_n \sin \omega_n t \tag{15.88}$$

and we have used the relation given in Eq. (15.29),

$$\omega_n = \frac{n\pi v}{L} = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}} \tag{15.29}$$

Thus, from Eqs. (15.87) and (15.29), we obtain (for all solutions)

$$\frac{\partial u}{\partial x} = \frac{\pi}{L}\sum_{n=1}^{\infty} n\Theta_n \cos \frac{n\pi x}{L} \tag{15.89}$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \dot{\Theta}_n \sin \frac{n\pi x}{L} \tag{15.90}$$

Substituting Eq. (15.89) into Eq. (15.84), we get

$$V = \frac{\pi^2 T}{2L^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(nm\dot{\Theta}_n\Theta_m \int_0^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L}\, dx\right) \tag{15.91}$$

On integrating, we find that only those terms are nonzero for which $m = n$, and each of these terms on integration yields $L/2$. Thus

$$V = \frac{\pi^2 T}{4L} \sum_{n=1}^{\infty} n^2 \Theta_n^2, \quad n = 1, 2, 3, \ldots \tag{15.92}$$

Similarly, substituting Eq. (15.90) into Eq. (15.86), we obtain

$$K = \frac{\mu}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\dot{\Theta}_n \dot{\Theta}_m \int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L}\, dx \right) \tag{15.93}$$

Once again, on integrating we find that only those terms are nonzero for which $m = n$, and each of those terms on integration yields

$$K = \frac{\mu L}{4} \sum_{n=1}^{\infty} \dot{\theta}_n^2 \tag{15.94}$$

while the Lagrangian of the system may be written as

$$L = K - V = \tfrac{1}{4} \sum_{n=1}^{\infty} \left(\mu L \dot{\Theta}_n^2 - \frac{\pi^2 T}{L} n^2 \theta_n^2 \right) \tag{15.95}$$

Note that the potential energy is the sum of quantities of the form $A_n \theta_n^2$, and the kinetic energy has terms of the form $B_n \dot{\theta}_n^2$. The Lagrangian equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\Theta}_n} \right) - \frac{\partial L}{\partial \Theta_n} = 0 \tag{15.96}$$

take the form

$$\ddot{\Theta}_n + \frac{\pi^2 T}{\mu L^2} n^2 \Theta_n = 0 \tag{15.97}$$

where Θ_n is the dependent variable and t is the independent variable. The solutions of these yield the normal coordinates Θ_n. Since n varies from 1 to ∞, the number of normal coordinates for a vibrating string is infinite.

It is now a simple matter to write the total energy E by using Eqs. (15.92) and (15.94):

$$E = K + V = \frac{1}{4} \sum_{n=1}^{\infty} \left(\mu L \dot{\Theta}_n^2 + \frac{\pi^2 T}{L} n^2 \theta_n^2 \right) \tag{15.98}$$

Since $\mu L = M$ (the mass of the string), and from Eq. (15.29),

$$T = \frac{\mu L^2 \omega_n^2}{n^2 \pi^2} = \frac{ML}{n^2 \pi^2} \omega_n^2 \tag{15.99}$$

and, using Eq. (15.88), we may write Eq. (15.98) in the form

$$E = \frac{M}{4} \sum_{n=1}^{\infty} [\omega_n^2(A_n^2 + B_n^2)] \tag{15.100}$$

where A_n and B_n are constants (see Problem 15.16).

Finally, let us calculate the rate of flow of energy, that is, power P, delivered from the left to right across any point x along the string. To calculate power, we make use of the definition that $P = F\dot{u}$, where F is the magnitude of the driving force **F**. F is equal in magnitude to tension T and must be applied in a direction tangent to the string. Thus the component of **F** in the direction of transverse displacement at point x is

$$F_y = -T\sin\theta \simeq -T\tan\theta = -T\frac{\partial u}{\partial x} \tag{15.101}$$

while the component of velocity $\dot{u}$ at the point x is $\partial u/\partial t$. Therefore,

$$P = F_y\dot{u} = \left(-T\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial t}\right) \tag{15.102}$$

The value of P can be evaluated by using the values of $\partial u/\partial x$ and $\partial u/\partial t$ given by Eqs. (15.89) and (15.90), respectively.

Let us calculate P for a particular case. Consider a wave traveling to the right and given by

$$u = f(x - vt) = f(\xi) \tag{15.103}$$

Suppose f is a sinusoidal function of the form

$$u = f(\xi) = A\cos(kx - \omega t) \tag{15.104}$$

Evaluating $\partial u/\partial x$ and $\partial u/\partial t$ and substituting in Eq. (15.102) yields

$$P = k\omega TA^2\sin^2(kx - \omega t) \tag{15.105}$$

Since the average value of $\sin^2(kx - \omega t)$ is $\frac{1}{2}$, the average power P transmitted from left to right will be

$$\langle P\rangle = \tfrac{1}{2}k\omega TA^2 \tag{15.106}$$

15.5 SYSTEM OF PARTICLES: THE LOADED STRING

In previous discussions we considered an idealized string that is characterized by its linear mass density μ. Actually, a string is made up of a finite number of particles. We can view the situation as a number of identical particles each of mass m placed at regular intervals on an elastic string, as shown in Fig. 15.4(a). There are N particles where the equilibrium distance between adjacent particles is d, and the attractive force between adjacent particles is T. Thus the length

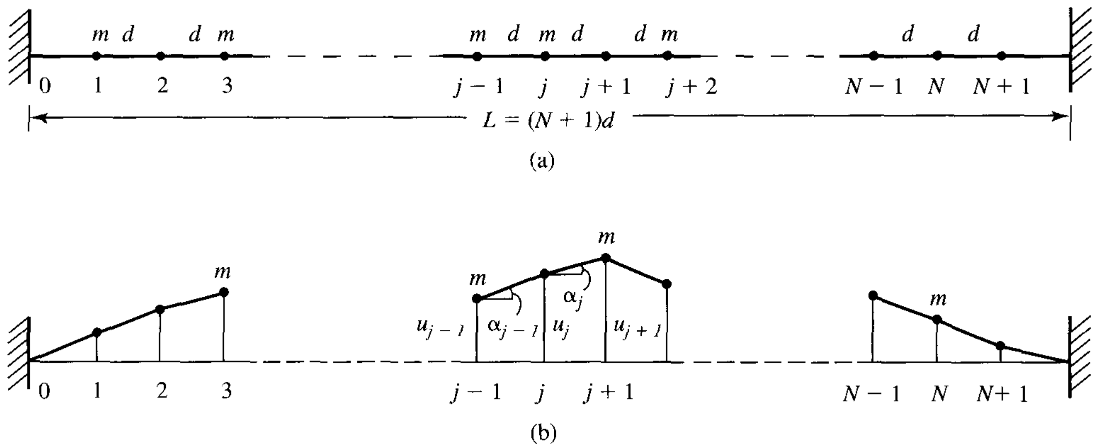


Figure 15.4 **(a)** A large number of identical particles each of mass m placed at regular intervals constitutes an elastic string. **(b)** Transverse displacements of point masses.

of the string, as shown, is $L = (N + 1)d$. Such a string is tied at both ends and is horizontal when in equilibrium. We are interested in investigating a small transverse displacement and hence the oscillations of the particles about equilibrium positions.

Consider small vertical displacements of particles $j - 1, j$, and $j + 1$, each of mass m, and vertical displacements u_{j-1}, u_j, and u_{j+1}. Assuming the displacements to be small means that the angles α_j are small, and the slopes are small; hence we may replace $\sin \alpha_j$ by $\tan \alpha_j$. For a small displacement, the resultant X component of the force on the jth particle is

$$-T\cos \alpha_{j-1} + T\cos \alpha_j \simeq \tfrac{1}{2}\, T(\alpha_{j-1}^2 - \alpha_j^2) \simeq 0$$

The resultant Y component of force on the jth particle for small displacements may be written as

$$\begin{aligned} F_y &= -T\sin \alpha_{j-1} + T\sin \alpha_j \approx -T\tan \alpha_{j-1} + T\tan \alpha_j \\ &= -T\frac{u_j - u_{j-1}}{d} + T\frac{u_{j+1} - u_j}{d} \end{aligned} \tag{15.107}$$

Since

$$F_j = m\ddot{u}_j = m\,\frac{d^2u_j}{dt^2} \tag{15.108}$$

the equation of motion of the jth particle is ($F_j = F_y$)

$$m\,\frac{d^2u_j}{dt^2} = T\left(\frac{u_{j+1} - u_j}{d} - \frac{u_j - u_{j-1}}{d}\right) \tag{15.109}$$

If the number of particles is taken to be very large, we may then assume the string to be smooth and write

$$u(jd, t) = u_j(t) = u(x, t)$$

while the linear mass density is $\mu = m/d$. Thus Eq. (15.109) takes the form

$$\frac{d^2u_j}{dt^2} = \frac{T}{\mu d}\left(\frac{u_{j+1} - u_j}{d} - \frac{u_j - u_{j-1}}{d}\right) \tag{15.110}$$

Before proceeding to solve Eq. (15.110), we shall show that it represents a wave equation. For the right side of Eq. (15.110),

$$\frac{1}{d}\left(\frac{u_{j+1} - u_j}{d} - \frac{u_j - u_{j-1}}{d}\right) = \frac{1}{d}\left[\left(\frac{\partial u}{\partial x}\right)_{(j+1/2)d} - \left(\frac{\partial u}{\partial x}\right)_{(j-1/2)d}\right] = \left(\frac{\partial^2 u}{\partial x^2}\right)_{jd} \tag{15.111}$$

Substituting in Eq. (15.110), we obtain the familiar wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0 \tag{15.12}$$

Let us look at Eq. (15.110), which describes the motion of the jth particle, and try to find a possible solution. Let

$$\frac{T}{md} = \omega_0^2 \tag{15.112}$$

and write Eq. (15.110) as the general wave equation:

$$\ddot{u}_j + 2\omega_0^2 u_j - \omega_0^2(u_{j+1} + u_{j-1}) = 0 \tag{15.113}$$

Since these are N particles, we can write a set of N differential equations, each being similar to Eq. (15.113). Note that we have assumed that $u_0 = 0$ and $u_{j+1} = 0$.

Before solving the general equation, Eq. (15.113), we shall first consider some simple cases. Suppose there is only one particle; that is, $N = 1$. Then Eq. (15.113) takes the form

$$\frac{d^2u_1}{dt^2} + 2\omega_0^2 u_1 = 0 \tag{15.114}$$

which represents transverse harmonic motion of a single particle oscillating with an angular frequency of $\sqrt{2}\omega_0$. This situation is shown in Fig. 15.5(a). If we had two particles, that is, $N = 2$. Eq. (15.113) would yield

$$\frac{d^2u_1}{dt^2} + 2\omega_0^2 u_1 - \omega_0^2 u_2 = 0 \tag{15.115}$$

$$\frac{d^2u_2}{dt^2} + 2\omega_0^2 u_2 - \omega_0^2 u_1 = 0 \tag{15.116}$$

These are coupled equations, similar to those for two coupled oscillators or pendula, having the same natural frequency ω_0. Thus there are two normal modes for $N = 2$. The lower mode has an angular frequency $\omega = \omega_0$, and the higher mode has an angular frequency $\omega = \sqrt{3}\omega_0$, as shown in Fig. 15.5(b)(i) and (ii), respectively.

Let us go back to Eq. (15.113) and try to find the normal modes of oscillation for N particles. Basically, we apply the same technique as used for two particles. For each normal mode,

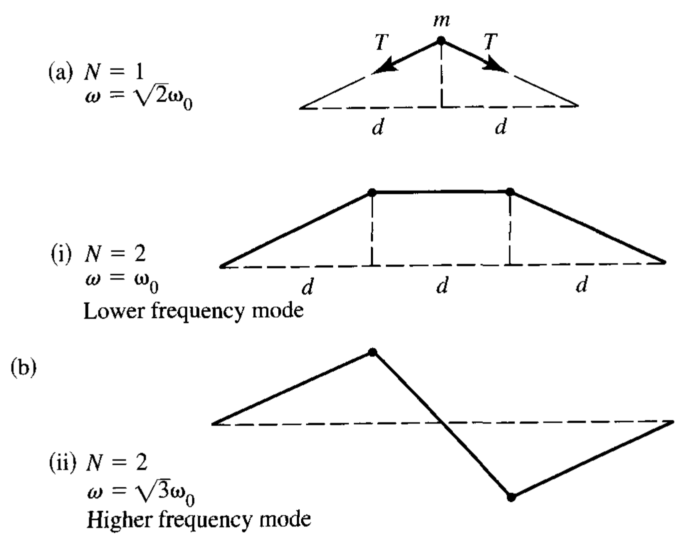


Figure 15.5 Normal modes of vibration for (a) $N = 1$, and (b) $N = 2$ particles.

we seek a sinusoidal solution so that each particle oscillates with the same frequency. Let such a solution be

$$u_j(t) = A_j \cos \omega t, \qquad j = 1, 2, 3, \ldots, N \tag{15.117}$$

where A_j and ω are the amplitude and frequency of the jth particle. We could have equally started with a solution of the form

$$u_j(t) = A_j e^{i\omega t} \tag{15.118}$$

(see Problem 15.23). Thus, if we know A_j and ω, a set of differential equations for Eq. (15.113) can be solved. Furthermore, a solution of the type given by Eq. (15.117) assumes that each particle has zero velocity at $t = 0$. This is obvious if we differentiate Eq. (15.117), which gives

$$\frac{du_j}{dt} = -\omega A_j \sin \omega t, \qquad j = 1, 2, 3, \ldots, N \tag{15.119}$$

Thus, if $t = 0$, $\dot{u}_j = 0$. Substituting the trial solution, Eq. (15.117), in Eq. (15.113), we obtain

$$(-\omega^2 + 2\omega_0^2)A_j - \omega_0^2(A_{j-1} + A_{j+1}) = 0, \qquad j = 1, 2, 3, \ldots, N \tag{15.120}$$

which is equivalent to the following set of equations:

$$\begin{aligned}
(-\omega^2 + 2\omega_0^2)A_1 - \omega_0^2(A_0 + A_2) &= 0 \\
(-\omega^2 + 2\omega_0^2)A_2 - \omega_0^2(A_1 + A_3) &= 0 \\
\vdots \qquad\qquad \vdots & \\
(-\omega^2 + 2\omega_0^2)A_j - \omega_0^2(A_{j-1} + A_{j+1}) &= 0 \\
\vdots \qquad\qquad \vdots & \\
(-\omega^2 + 2\omega_0^2)A_N - \omega_0^2(A_{N-1} + A_{N+1}) &= 0
\end{aligned} \tag{15.121}$$

To have a nontrivial solution, the determinant of the coefficients in Eqs. (15.121) must be zero. That is,

$$\begin{vmatrix} (-\omega^2 + 2\omega_0^2) & -\omega_0^2 & 0 & 0 & 0 & \cdots \\ -\omega_0^2 & (-\omega^2 + 2\omega_0^2) & -\omega_0^2 & 0 & 0 & \cdots \\ 0 & -\omega_0^2 & (-\omega^2 + 2\omega_0^2) & -\omega_0^2 & 0 & \cdots \\ 0 & 0 & -\omega_0^2 & (-\omega^2 + 2\omega_0^2) & -\omega_0^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{vmatrix} = 0 \tag{15.122}$$

For $N = 1$, we get

$$|-\omega^2 + 2\omega^2 0| = 0 \rightarrow \omega = \sqrt{2}\omega_0$$

For $N = 2$, we get

$$\begin{vmatrix} (-\omega^2 + 2\omega_0^2) & -\omega_0^2 \\ -\omega_0^2 & (-\omega^2 + 2\omega_0^2) \end{vmatrix} = 0$$

which gives the frequencies of the two normal modes to be

$$\omega = \omega_0 \quad \text{or} \quad \omega = \sqrt{3}\omega_0$$

These are the results we predicted. This method is simple enough for calculating the frequencies of normal modes as long as we are dealing with a small number of particles. For a very large number of particles, this method is cumbersome. The following alternative approach is desirable.

Let us refer back to Eq. (15.120). The requirement that both ends are fixed leads to the boundary conditions

$$A_0 = 0 \quad \text{and} \quad A_{N+1} = 0 \tag{15.123}$$

The existence of normal modes, that for each mode all particles vibrate with the same frequency, imposes certain restrictions on the ratios of the amplitudes. Equation (15.120) may be written as

$$\frac{A_{j-1} + A_{j+1}}{A_j} = \frac{-\omega^2 + 2\omega_0^2}{\omega_0^2}, \qquad j = 1, 2, 3, \ldots, N \tag{15.124}$$

Since for a given mode ω^2 is constant, the right side is constant. Thus, if A_{j-1} and A_j are given, A_{j+1} can be evaluated. For example, if $A_0 = 0$ and A_1 is given, A_2 can be calculated.

Furthermore, Eq. (15.124) implies that since the right side is constant the left side must be constant. Let us not forget that we want to get the value of ω^2. A neat method of doing this is to assume the following form of a solution for A_j:

$$A_j = C \sin j\theta \tag{15.125}$$

where θ is some angle. With similar expressions for A_{j-1}, we may write

$$A_{j-1} + A_{j+1} = C \sin(j - 1)\theta + C \sin(j + 1)\theta = 2C \sin j\theta \cos \theta = 2A_j \cos \theta \tag{15.126}$$

which may be rewritten as

$$\frac{A_{j-1} + A_{j+1}}{A_j} = 2\cos\theta \tag{15.127}$$

The right side is independent of j. Thus, if we could evaluate θ, we would have the value of the constant needed for $2\cos\theta$, which we can substitute in Eq. (15.124) and thus evaluate ω. To do this, we make use of the boundary conditions that $A_j = 0$ for $j = 0$ and $j = N + 1$. From Eq. (15.125), we see that if $j = 0$ and for $j = N + 1$, A_j will be zero only if $(N + 1)\theta$ is an integer multiple of π: that is,

$$(N + 1)\theta = n\pi, \qquad n = 1, 2, 3, \ldots \tag{15.128a}$$

or

$$\theta = \frac{n\pi}{N + 1} \tag{15.128b}$$

Substituting this in Eq. (15.125),

$$A_j = C\sin\left(\frac{nj\pi}{N + 1}\right) \tag{15.129}$$

Using Eqs. (15.124), (15.127), and (15.128), we get the frequencies of the possible normal modes:

$$\frac{A_{j-1} + A_{j+1}}{A_j} = \frac{-\omega^2 + 2\omega_0^2}{\omega_0^2} = 2\cos\left(\frac{n\pi}{N + 1}\right) \tag{15.130}$$

Therefore, the relation for the frequencies of the normal mode (independent of j) is obtained by solving for ω^2:

$$\omega^2 = 2\omega_0^2\left[1 - \cos\left(\frac{n\pi}{N + 1}\right)\right] = 4\omega_0^2\sin^2\left(\frac{n\pi}{2(N + 1)}\right) \tag{15.131}$$

Taking the square root, we have the required frequencies:

$$\omega = 2\omega_0\sin\left(\frac{n\pi}{2(N + 1)}\right) \tag{15.132}$$

From Eq. (15.112), substituting the value of ω_0, and since different values of n correspond to a different normal mode with the corresponding frequency, we may replace ω by ω_n and write the *normal mode frequencies* as

$$\omega_n = 2\sqrt{\frac{T}{md}}\sin\left(\frac{n\pi}{2(N + 1)}\right) \tag{15.133}$$

The same type of procedure can be carried out for the longitudinal oscillations where T/d is replaced by k, the spring constant. After replacing T/d by k. Eq. (15.133) yields the frequencies of the two normal modes of two coupled oscillators after substituting $N = 2$ and $n = 1, 2$.

For all practical purposes, we have solved the problem of N coupled oscillators. We must look closely at the motion these equations describe so that we can obtain a physical interpretation of the situation. To describe the displacement of the jth particle when a collection of N particles is oscillating in the nth mode, Eq. (15.117) must be written in the following form:

$$u_{jn} = A_{jn}\cos\omega_n t \tag{15.134}$$

where A_{jn} is obtained by substituting Eq. (15.128b) in Eq. (15.125) and replacing A_j by A_{jn} and C by C_n, we obtain

$$A_{jn} = C_n \sin\left(\frac{jn\pi}{N+1}\right) \tag{15.135}$$

A_{jn} represents the amplitude of the jth particle in the nth mode of the system, ω_n represents the frequency of the nth mode and is given by Eqs. (15.135) or (15.133); that is,

$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right) \tag{15.136}$$

The solution given in Eq. (15.134) assumes that at time $t = 0$ the particle is at rest. But this difficulty can be overcome and any arbitrary initial conditions can be satisfied by adding a phase factor ϕ_n; that is,

$$u_{jn} = A_{jn} \cos(\omega_n t - \phi_n) \tag{15.137}$$

First, we would like to know the number of possible normal modes. We shall now show that for N oscillators there are only N independent modes; that is, $n = N$ and the corresponding amplitudes and frequencies are A_{jn} and ω_n. For modes beyond $n = N$, that is, for $n = N + 1$, $N + 2, \ldots$, and so on, the preceding equations do not lead to new physical situations. We shall show that for values greater than $n = N$ the amplitudes and frequencies of the normal modes repeat themselves.

Figure 15.6 shows a plot of mode frequency ω_n (always taken to be positive) versus mode number n [for convenience written as $n\pi/2(N + 1)$ instead of n]. If we substitute $n = 0$ or $N + 1$ in Eq. (15.135), the amplitude factors A_{jn} turn out to be zero. These values of n are called *null modes*. For $n = 1$ to $n = N$, we have N different characteristic frequencies as discussed, reaching a maximum value of $\omega_{max} = 2\omega_0$, from Eq. (15.136), for $n = N + 1$, because $\sin(\pi/2)$ becomes 1. But for this maximum value of the characteristic frequency, the corresponding amplitude for $n = N + 1$ from Eq. (15.135) is zero. Let us calculate the characteristic frequency for the mode $n = N + 2$. From Eq. (15.136), we get

$$\omega_{N+2} = 2\omega_0 \sin\left[\frac{(N+2)\pi}{2(N+1)}\right] = 2\omega_0 \sin\left[\pi - \frac{N\pi}{2(N+1)}\right] = 2\omega_0 \sin\left[\frac{N\pi}{2(N+1)}\right] = \omega_N$$

That is,

$$\omega_{N+2} = \omega_N$$

Similarly,

$$\begin{aligned} \omega_{N+3} &= \omega_{N-1} \\ \omega_{N+4} &= \omega_{N-2} \\ \vdots \quad & \quad \vdots \end{aligned} \tag{15.138}$$

Thus there are only $n = N$ number of independent modes; for any further values of n, the modes repeat themselves.

The same is true for the amplitudes as well; that is, the relative amplitudes of the particles in a normal mode repeat themselves. That is, from Eq. (15.135)

$$\begin{aligned} A_{j(N+2)} &= A_{jN} \\ A_{j(N+3)} &= A_{j(N-1)} \\ A_{j(N+4)} &= A_{j(N-2)} \end{aligned}$$

Figure 15.6

Three frequency spectrums are shown below:
(a) the graph of the mode frequency ω versus the mode number n,
(b) the graph of the square of the mode frequency ω versus n, and
(c) the graph (b) for N = 5 (string loaded with 4 masses).

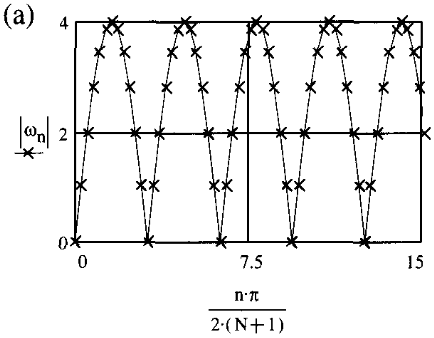
Equation (15.136) gives the frequency of the n modes

$n := 0..100 \quad N := 5 \quad j := 1..N \quad t_j := j \quad \omega0 := 2$

$$\omega_n := 2\cdot\omega0\cdot\sin\left[\frac{n\cdot\pi}{2\cdot(N+1)}\right] \qquad \max(\omega) = 4$$

Explain the difference in graphs (a) and (b).

(a)



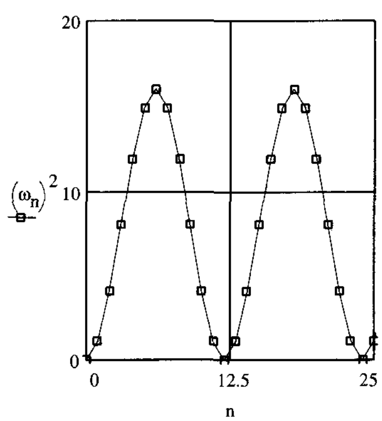
$\omega_1 = 1.035 \qquad (\omega_1)^2 = 1.072$

$\omega_2 = 2 \qquad (\omega_2)^2 = 4$

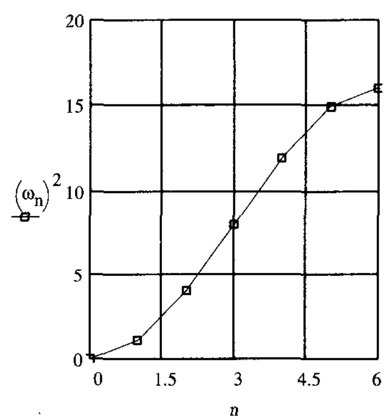
$\omega_3 = 2.828 \qquad (\omega_3)^2 = 8$

$\omega_4 = 3.464 \qquad (\omega_4)^2 = 12$

(b)



(c) Showing only a small region of (b)



Note that $n = 2N + 2$ gives the next null mode. Thus there are only N distinct modes, and if n increases beyond N, it simply duplicates the normal mode for smaller N. The conclusion of this discussion is illustrated in Fig. 15.7, which shows the normal modes of a vibrating string for $N = 12$. Note that $n = 13$ is a null mode, while modes for $n = 14, 15, 16, 17, 18$ repeat the pattern of $n = 12, 11, 10, 9, 8$ with opposite sign. The sinusoidal curves represent the variation in the amplitude A_{jn} for various values of n. One must be careful to note that the frequencies of these sine curves have no relation to the frequencies of the vibrating particles.

Figure 15.7

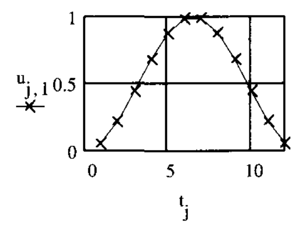
Below the normal modes of a vibrating string for N = 12 particles are shown.

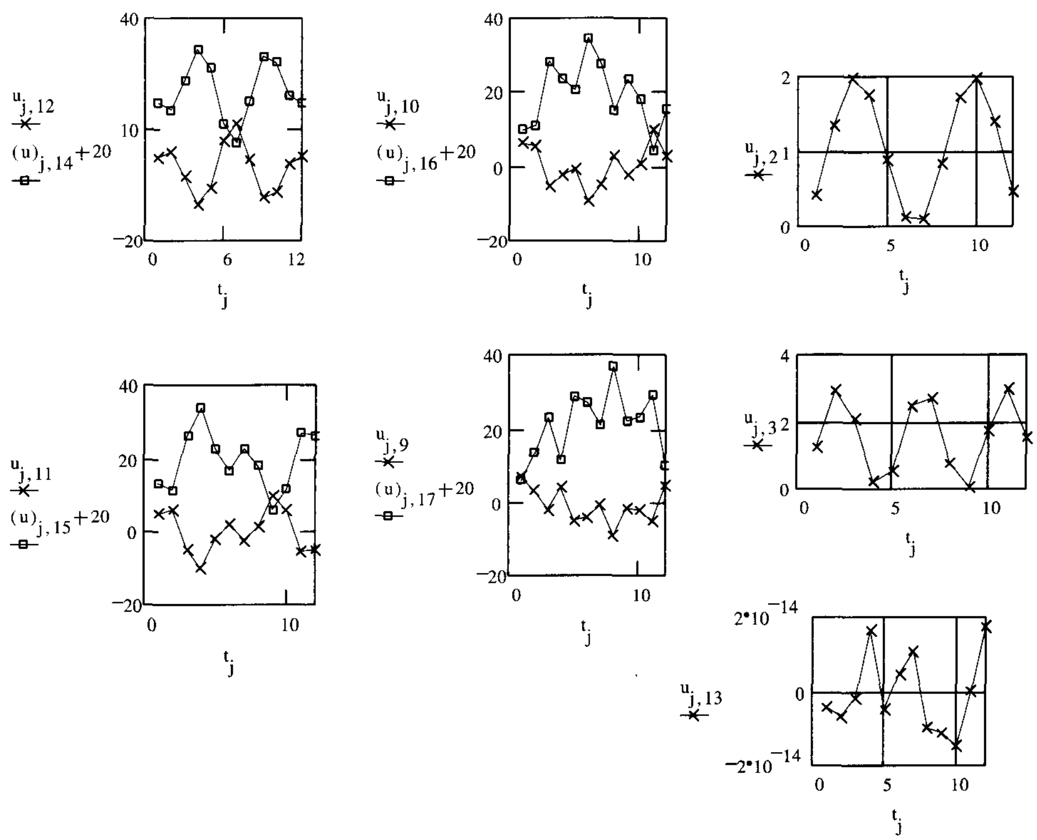
$N := 12 \qquad j := 1..N \qquad t_j := j \qquad n := 0..25 \qquad C_n := n \qquad \omega 0 := 1$

$$A_{j,n} := C_n \cdot \sin\left(\frac{j \cdot n \cdot \pi}{N+1}\right) \qquad \omega_n := 2 \cdot \omega 0 \cdot \sin\left[\frac{n \cdot \pi}{2 \cdot (N+1)}\right] \qquad u_{j,n} := A_{j,n} \cdot \cos\left[\left(\omega_n \cdot t_j\right) - \frac{\pi}{2}\right]$$

The null point is n = N + 1 = 12 + 1 = 13. Hence the modes for:
n = 14, 15, 16, 17, 18, . . . repeat the pattern for
n = 12, 11,10, 9, 8, . . . but with opposite signs.

We have shifted the position of the graphs by adding 20 to one of them.





Let us now discuss specific modes of vibrations, assuming that there is a large number of particles N. The particle displacement corresponding to the mode $n = 1$ is, from Eqs. (15.134) and (15.135),

$$u_{j1} = C_1 \sin\left(\frac{j\pi}{N+1}\right) \cos \omega_1 t, \qquad j = 1, 2, 3, \ldots, N \tag{15.139}$$

This equation implies that at any given time the $C_1 \cos \omega_1 t$ factor is the same for all particles, while the displacements of different particles are given by the factor $\sin[j\pi/(N + 1)]$. The boldface curve in Fig. 15.8(a) is a plot of $\sin[j\pi/(N + 1)]$ versus j for $j = 0$ to $N + 1$ and gives the amplitudes of different particles. As time passes, the particles have different displacements and oscillate with frequency ω_1, as shown in Fig. 15.8(b). The dotted curves give the positions of the particles at different times. For the $n = 2$ mode, the situation is as shown in Fig. 15.9, where the amplitudes are given by the boldface curve, while the dotted curves give the positions of the particles vibrating with frequency ω_2; that is,

$$u_{j2} = C_2 \sin\left(\frac{2j\pi}{N+1}\right) \cos \omega_2 t, \qquad j = 1, 2, 3, \ldots, N \tag{15.140}$$

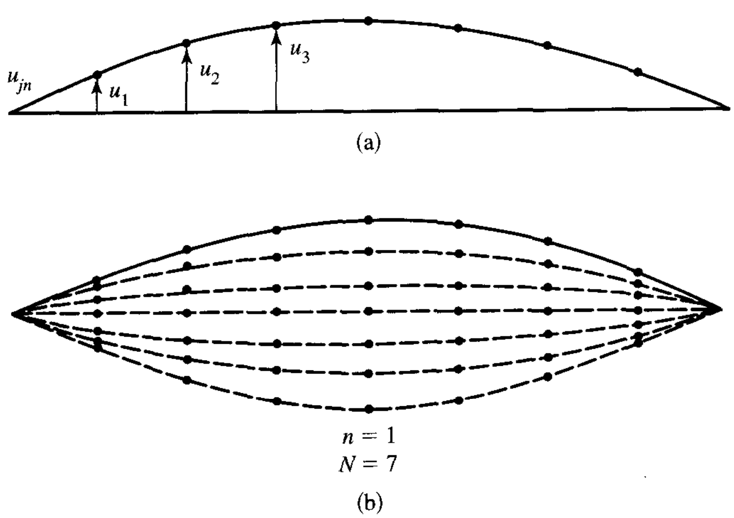


Figure 15.8 (a) Plot of $\sin[j\pi/(N + 1)]$ versus j for $j = 0$ to $N + 1$ shown by the boldface curve (for seven particles). (b) The dotted curves give the positions of the particles at different times vibrating with frequency ω_1.

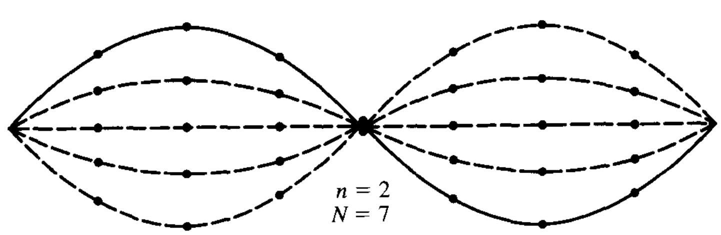


Figure 15.9 For $n = 2$ modes, the amplitudes are shown by a boldface curve, while the dotted curves give the positions of the particles vibrating with frequency ω_2.

15.6 BEHAVIOR OF A WAVE AT DISCONTINUITY: ENERGY FLOW

As an example of discontinuity, consider two semi-infinite strings of different linear mass densities tied together at $x = 0$, as shown in Fig. 15.10. The string that extends over $-\infty \leqslant x \leqslant 0$ has a linear mass density μ_1, and the wave traveling along this string has a velocity v_1, while the string that extends over $0 \leqslant x \leqslant \infty$ has a linear mass density μ_2, and the wave traveling along this string has a velocity v_2. Let the tension in the string be T. We want to investigate the effect of a sudden change in density at $x = 0$ on a continuous harmonic wave.

Let an incident wave traveling from the left for $x < 0$ be represented by

$$u_I = A_I \cos(k_1 x - \omega t) \tag{15.141}$$

where A_I is the amplitude of the incident wave, $k_1 = \omega/v_1$, v_1 being the wave velocity. When this wave reaches $x = 0$, the point where the two strings join (the point of discontinuity), part of the wave is reflected back along the first string, while the remaining wave is transmitted. The reflected wave is represented by

$$u_R = A_R \cos(k_1 x + \omega t) \tag{15.142}$$

where A_R is the amplitude of the reflected wave. The transmitted wave is given by

$$u_T = A_T \cos(k_2 x - \omega t) \tag{15.143}$$

where A_T is the amplitude of the transmitted wave and $k_2 = \omega/v_2$, v_2 being the velocity of the wave on the second string to the right of $x = 0$. It may be noted that we could have used solutions of the following form:

$$u_I = \operatorname{Re} A_I e^{i(k_1 x - \omega t)} \tag{15.144}$$

where Re stands for the real part of the expression.

Our aim is to evaluate the reflected and transmitted amplitudes A_R and A_T in terms of the incident amplitude A_I. This can be done by imposing the boundary conditions that at the junction of the two string ($x = 0$) the displacement u and its derivative $\partial u/\partial x$ must be continous. These are the *continuity conditions* and are valid for any other types of wave motion, such as sound waves. The first condition satisfies the requirement that there is no break in the string,

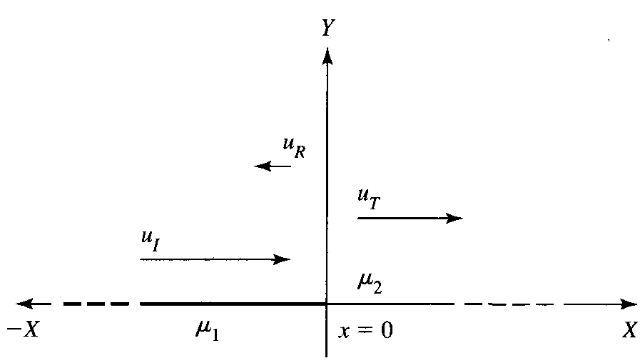


Figure 15.10 Two semi-infinite strings of different linear mass densities tied together at $x = 0$.

while the second condition implies that the restoring force resulting from a displacement y is the same on each side of the junction. If this were not true, then a finite force acting on a vanishing small mass element would produce an infinite acceleration. Thus the boundary conditions may be written as

$$(u_I + u_R)\big|_{x=0} = u_T\big|_{x=0} \tag{15.145}$$

and

$$\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_R}{\partial x}\right)\bigg|_{x=0} = \left(\frac{\partial u_T}{\partial x}\right)\bigg|_{x=0} \tag{15.146}$$

Using Eqs. (15.141), (15.142), and (15.143), the continuity of u, Eq. (15.145), yields

$$A_I + A_R = A_T \tag{15.147}$$

while the continuity of $\partial u/\partial x$, given by Eq. (15.146), yields

$$k_1(A_I - A_R) = k_2 A_T \tag{15.148}$$

Solving these two equations for A_R/A_I and A_T/A_I,

$$\frac{A_R}{A_I} = \frac{k_1 - k_2}{k_1 + k_2} \tag{15.149}$$

$$\frac{A_T}{A_I} = \frac{2k_1}{k_1 + k_2} \tag{15.150}$$

Since $k = \omega/v$ and $v = \sqrt{T/\mu}$, we may write these results as

$$\frac{A_R}{A_I} = \frac{v_2 - v_1}{v_1 + v_2} = \frac{\sqrt{\mu_1} - \sqrt{\mu_2}}{\sqrt{\mu_1} + \sqrt{\mu_2}} \tag{15.151}$$

$$\frac{A_T}{A_I} = \frac{2v_2}{v_1 + v_2} = \frac{2\sqrt{\mu_1}}{\sqrt{\mu_1} + \sqrt{\mu_2}} \tag{15.152}$$

It is clear that the ratio A_T/A_I is always positive; hence the transmitted wave is always in phase with the incident wave. If the second medium is lighter, $v_2 > v_1$ or $\mu_2 < \mu_1$, the ratio A_R/A_I will be positive; hence the reflected wave will be in phase with the incident wave. On the other hand, if the second medium is denser than the first, $v_2 < v_1$ or , $\mu_2 > \mu_1$, A_R/A_I will be negative. This means that the reflected wave is out of phase by π with respect to the incident wave. This type of behavior is typical of many kinds of wave motion.

The intensity, the rate of energy flow, for any type of wave motion is proportional to the square of the amplitude. For this purpose, we define the *reflection coefficient*, R, to be the fraction of the incident energy that is reflected back; that is,

$$R \equiv \left(\frac{A_R}{A_I}\right)^2 \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 = \left(\frac{v_2 - v_1}{v_1 + v_2}\right)^2 \tag{15.153}$$

While the *transmission coefficient,* T defined as the fraction of the incident energy that is transmitted, must satisfy the condition

$$R + T = 1 \tag{15.154}$$

or

$$T \equiv |1 - R| \equiv \frac{4v_1 v_2}{(v_1 + v_2)^2} \tag{15.155}$$

(*Note:* From Eq. (15.153), R becomes larger and larger as the difference between v_1 and v_2 becomes larger, while correspondingly T becomes smaller.]

Finally, let us calculate the rate of energy flow dE/dt across the junction at $x = 0$. This is equal to the work done by the adjacent portion of the string on the particle at $x = 0$ and is equal to the product of the restoring force $-T(\partial u/\partial x)$ and the velocity of the particle $\partial u/\partial t$ both evaluated at $x = 0$. Then

$$\frac{dE}{dt} = \left(-T\frac{\partial u}{\partial x}\right)_{x=0}\left(\frac{\partial u}{\partial t}\right)_{x=0} \tag{15.156}$$

If we want to calculate the energy transmitted to the left of the string at $x = 0$, we let

$$\begin{aligned} u &= u_I + u_R \\ &= A_I \cos(k_1 x - \omega t) + A_R \cos(k_1 x + \omega t) \end{aligned} \tag{15.157}$$

Substituting this in Eq. (15.156), we get

$$\left(\frac{dE}{dt}\right)_- = \omega k_1 T(A_I^2 - A_R^2)\sin^2 \omega t \tag{15.158}$$

Similarly, if we use

$$u = u_T = A_T \cos(k_2 x - \omega t)$$

in Eq. (15.156), we get energy transmitted to the right as

$$\left(\frac{dE}{dt}\right)_+ = \omega k_2 T A_T^2 \sin^2 \omega t \tag{15.159}$$

Since the average value of $\sin^2 \omega t$ over one complete cycle is $\frac{1}{2}$, we may write Eqs. (15.158) as

$$\left[\left(\frac{dE}{dt}\right)_-\right]_{\text{ave}} = \frac{1}{2}\omega k_2 T A_I^2 - \frac{1}{2}\omega k_1 T A_R^2 \tag{15.160}$$

where the first term on the right is the mean rate at which the energy is incident on the junction, while the second term is the mean rate at which the energy is reflected back. Similarly, the mean rate at which the energy is transmitted, from Eq. (15.159), is

$$\left[\left(\frac{dE}{dt}\right)_+\right]_{\text{ave}} = \frac{1}{2}\omega k_2 T A_T^2 \tag{15.161}$$

This is the net rate at which the energy is supplied to the junction from left to right (see Problem 15.28).

15.7 SOUND WAVES: LONGITUDINAL WAVES

So far we have been dealing with transverse waves in solids. These waves consist of crests and troughs. We now start with the discussion of *sound waves,* which are basically longitudinal in nature and consist of compressions and rarefaction. Sound waves can travel in solids and fluids (liquids and gases) and propagate in general in three dimensions. For simplicity, we will deal with sound waves in fluids traveling only in one dimension, say along the X-axis. We use a simple procedure using the results derived already.

We showed that the net upward force acting on a small element of length of a string is, from Eq. (15.101) (replacing F_y by F),

$$F = -T\frac{\partial u}{\partial x} \tag{15.162}$$

while the upward velocity at a point of such an element is

$$\dot{u} = \frac{\partial u}{\partial t} \tag{15.163}$$

Note that $\dot{u}$ is the particle velocity and is not to be confused with the wave velocity v. Also, from Eq. (15.9),

$$\mu\, dx\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(T\frac{\partial u}{\partial x}\right) dx \tag{15.9}$$

Using the preceding three equations, we can show

$$\frac{\partial F}{\partial t} = -T\frac{\partial \dot{u}}{\partial x} \tag{15.164}$$

and

$$\frac{\partial \dot{u}}{\partial t} = -\frac{1}{\mu}\frac{\partial F}{\partial x} \tag{15.165}$$

Equation (15.164) states that the time rate of change of F is proportional to $\partial\dot{u}/\partial x$ (= the difference in the velocities at the ends of the line segment divided by the length of the line segment). Equation (15.165) states that the acceleration of the string is proportional to $\partial F/\partial x$ (= the difference in the forces at the ends of a line segment divided by the length of the line segment).

We can conclude that, for a small amplitude, *for the quantities F and $\dot{u}$, the time rate of change of either is proportional to the space derivative of the other.*

Starting with Eqs. (15.164) and (15.165), and taking derivatives,

$$\frac{\partial F}{\partial t} = -T\frac{\partial \dot{u}}{\partial x}, \qquad \frac{\partial^2 F}{\partial t^2} = -T\frac{\partial^2 \dot{u}}{\partial t\,\partial x}$$

$$\frac{\partial \dot{u}}{\partial t} = -\frac{1}{\mu}\frac{\partial F}{\partial x}, \qquad \frac{\partial^2 \dot{u}}{\partial x\,\partial t} = -\frac{1}{\mu}\frac{\partial^2 F}{\partial x^2}$$

Combining these equations, we get

$$\frac{\partial^2 F}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 F}{\partial t^2}, \qquad v = \sqrt{\frac{T}{\mu}} \tag{15.166}$$

and, similarly

$$\frac{\partial^2 \dot{u}}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \dot{u}}{\partial t^2} \tag{15.167}$$

Thus, instead of writing the usual wave equation, Eq. (15.12), where the displacement $u(x, t)$ is the variable, here we have two wave equations with F and $\dot{u}$ as the two independent variables.

As an application of Eqs. (15.166) and (15.167), let us consider plane sound waves traveling in air in the X direction. This will be equivalent to, as an example, sound waves traveling in an organ pipe. In Eq. (15.166), F is replaced by p, the pressure in excess of atmospheric pressure, while $\dot{u}$ represents the velocity of the volume element of air at any point, μ by ρ, the density of air, and T by B, the bulk modulus. Thus Eqs. (15.164) and (15.165) take the form

$$\frac{\partial p}{\partial t} = -B\frac{\partial \dot{u}}{\partial x} \tag{15.168}$$

and

$$\frac{\partial \dot{u}}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{15.169}$$

Both p and $\dot{u}$ satisfy the equations

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{v}\frac{\partial^2 p}{\partial t^2} \tag{15.170}$$

$$\frac{\partial \dot{u}}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \dot{u}}{\partial t^2} \tag{15.171}$$

where

$$v = \sqrt{\frac{B}{\rho}} \tag{15.172}$$

The power transmitted in the X direction from left to right may be written as (making use of the definition $P = F\dot{u}$)

$$P = p\dot{u} \tag{15.173}$$

Note that p is pressure in excess of atmosphere pressure, while $\dot{u}$ is the particle velocity (and not the wave velocity).

Suppose a pipe of infinite length and cross-sectional area A is placed along the X-axis. When air (or any other fluid) in the pipe is undisturbed, the pressure at any point is p_0 and the mass density is ρ_0. If the pressure changes to $p = p_0 + \Delta p$ while the density changes to $\rho = \rho_0 + \Delta\rho$, then according to the definition of the bulk modulus, B, we may write

$$B = \frac{\Delta p}{\Delta\rho/\rho_0} \quad \text{or} \quad \frac{dp}{d\rho} = \frac{B}{\rho_0} = \frac{1}{K\rho_0} \tag{15.174}$$

where K is the compressibility. Thus the velocity of wave propagation

$$v = \sqrt{\frac{dp}{d\rho}} = \sqrt{\frac{B}{\rho_0}} = \sqrt{\frac{1}{K\rho_0}} \tag{15.175}$$

This relation is good only for propagation of waves in liquids. In gases, the situation is quite different. A small change in pressure will cause a considerable change in temperature. The compressions and rarefactions take place so rapidly that there is no time for heat to flow out or in; hence the process may be assumed to be adiabatic. For such situations, assuming an ideal gas, we have the relation

$$p = B\rho^{\gamma} \tag{15.176}$$

where γ is the ratio of the specific heat of the gas at constant pressure to the specific heat of the gas at constant volume; that is, $\gamma = C_p/C_v$. Combining the above equations and using the approximation $|\partial\eta/\partial x| \ll 1$, we get

$$\frac{dp}{d\rho} = \frac{\gamma p_0}{\rho_0} \tag{15.177}$$

Using the ideal gas equation,

$$p = \frac{RT}{M}\rho \tag{15.178}$$

where R is the gas constant, M the molecular weight, and T the absolute temperature, and using Eq. (15.177), we may write the wave propagation velocity to be, from Eq. (15.175),

$$v = \sqrt{\frac{\gamma p_0}{\rho_0}} = \sqrt{\frac{\gamma RT}{M}} \tag{15.179}$$

which clearly indicates that the temperature T alone determines the velocity of the propagation of sound waves in an ideal gas.

If we extend out discussion of sound waves to their propagation in three dimensions, we get the following equations (as compared to Eqs. (15.170) and (15.171) for one dimension):

$$\frac{\partial^2\rho}{\partial t^2} = \frac{dp}{dt}\nabla^2\rho \tag{15.180}$$

$$\frac{\partial^2 p}{\partial t^2} = \frac{dp}{dt}\nabla^2 p \tag{15.181}$$

where, for example, $p = p(x, y, z, t) = \Psi(x, y, z)\Theta(t)$ and $\Psi(x, y, z) = X(x)\,Y(y)\,Z(z)$.

15.8 FLUID STATICS

A fluid is a substance that does not have a fixed shape. It consists of a continuum of matter and will undergo a finite displacement when an infinitesimal shear stress is applied. A small volume of a fluid can be treated as a continuum if it contains such a large number of molecules that the average distance traveled by molecules between collisions is much smaller than the size of the volume of the fluid. Fluids are characterized by physical and mechanical properties, such as density, pressure, temperature, and velocity. Both liquids and gases are fluids, but there are fundamental differences between the two. Liquids are not easily compressed and hence may be considered to have fixed volumes and densities. This is not so with gases, which can be easily compressed. Gases do not have any fixed shape; they simply fill up any container. Liquids do not have any definite shape, but they do have a distinct surface.

Hydrostatics or fluid statics deals with fluids at rest and *hydrodynamics* or *fluid dynamics* deals with fluids in motion. If the fluid flow is time independent, it is said to be *steady*. The fluid flow is *laminar* or *streamlined* if different layers of fluids move past each other with no mixing. If mixing between layers takes place, the flow is said to be *turbulent*.

In this section, we limit our discussion to fluid statics, while fluid dynamics will be investigated in the following sections. Newton's laws and conservation laws will be applied to fluids, since after all fluids are merely a collection of a large number of particles.

Let us consider a fluid in static equilibrium. Thus each elemental volume of fluid is at rest and the velocity at each point in the fluid is zero. We now discuss two characteristics of fluid statics: (1) A fluid exerts equal pressure in all directions, and (2) pressures at equal depths are the same.

Consider a very small triangular prism, as shown in Fig. 15.11. Let F, F_y, and F_z be the forces acting on the three surfaces of areas A, A_y, and A_z, respectively, as shown. These forces must act normal to the surfaces. If any forces acted tangent to the prism surfaces, the fluid would be set in motion, which is contrary to our assumption of a static fluid.

Thus the only forces acting are the normal forces and the weight W of the fluid. For equilibrium, the forces acting in the Y and Z directions are zero. (We have assumed no forces along the X-direction.)

$$\sum F_y = F\sin\theta - F_y = 0 \tag{15.182}$$

$$\sum F_z = F_z - F\cos\theta - W = 0 \tag{15.183}$$

If P, P_y, and P_z are the pressures (normal force per unit area) acting on the three surfaces, and $W = \rho g[(dx\,dy\,dz)/2]$, we may write Eqs. (15.182) and (15.183) as

$$P\frac{dx\,dz}{\sin\theta}\sin\theta - P_y\,dx\,dz = 0 \tag{15.184}$$

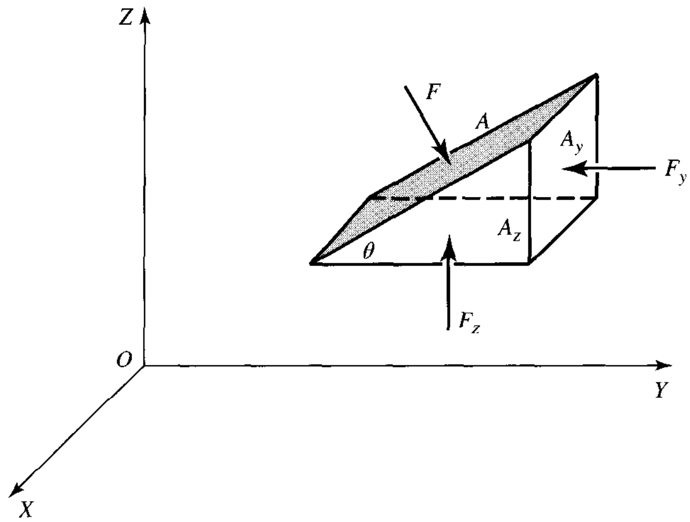


Figure 15.11 Small fluid element (in the shape of a small triangular prism) in static equilibrium.

and
$$P_z\, dx\, dy - P\frac{dx\, dz}{\sin\theta}\cos\theta - \rho g\frac{dx\, dy\, dz}{2} = 0 \tag{15.185}$$

where ρ is the density of the fluid and $dx\, dy\, dz/2$ is the volume of the prism. As dx, dy, and dz go to zero, the last term in Eq. (15.185), the weight term, becomes negligible compared to the pressure term. Hence, from the preceding equations, we conclude

$$P = P_y = P_z \tag{15.186}$$

$$P = \frac{F}{A} = \frac{F_y}{A_y} = \frac{F_z}{A_z} \tag{15.187}$$

which states that the pressure is independent of the direction; that is, *pressure is the same in all directions* and is a scalar quantity. Equation (15.186) is the statement of *Pascal's law.*

Let us now derive an expression for the variation of pressure with vertical position in a static fluid. Consider an infinitesimal volume $dx\, dy\, dz$ of fluid, as shown in Fig. 15.12. Since the

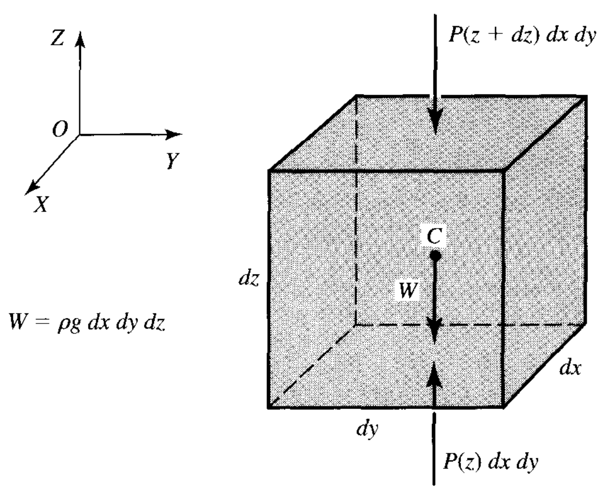


Figure 15.12 Infinitesimal volume *dx dy dz* of a fluid in equilibrium.

fluid is in equilibrium, the sum of the forces in the Z direction must add up to zero; that is,

$$\sum F_z = P(z)\,dx\,dy - P(z + dz)dx\,dy - \rho g\,dx\,dy\,dz = 0 \tag{15.188}$$

The second term, when expanded in a Taylor series about z to the first order, takes the form

$$P(z + dz) = P(z) + \frac{dP}{dz}\,dz + \cdots \tag{15.189}$$

Substituting this in Eq. (15.188) yields

$$\frac{dP(z)}{dz} = -\rho g \tag{15.190}$$

which, on integration, assuming the fluid to be incompressible and $P = P_0$ at $z = 0$, gives

$$P(z) = P_0 - \rho g z \tag{15.191}$$

Since z is taken to be positive upward, this equation states that P increases as z decreases. Also, it states that the pressure at any depth of a column is equal to the sum of the pressure P_0 at the top of the column and the weight of the liquid column.

An alternative approach to this treatment is the following. Let $\mathbf{w} = \rho\mathbf{g}$ be the weight density, that is, the weight per unit volume acting in the direction of $\mathbf{g}$. Consider two points 1 and 2 in a fluid where the pressures are P_1 and P_2 and separated by an infinitesimal distance $d\mathbf{r}$. Let us construct a right circular cylinder of cross-sectional area dA and length dr, as shown in Fig. 15.13. The only forces acting on the cylinder are due to the liquid pressure and gravity. Since the liquid is in equilibrium, the sum of the components of the forces along $d\mathbf{r}$ must be zero; that is,

$$P_1\,dA - P_2\,dA + \mathbf{w} \cdot d\mathbf{r}\,dA = 0 \tag{15.192}$$

where $dr\,dA = dV$ is the volume of fluid inside the cylinder. Thus the differences in pressure ΔP between the two points is

$$\Delta P = P_2 - P_1 = \mathbf{w} \cdot d\mathbf{r} \tag{15.193}$$

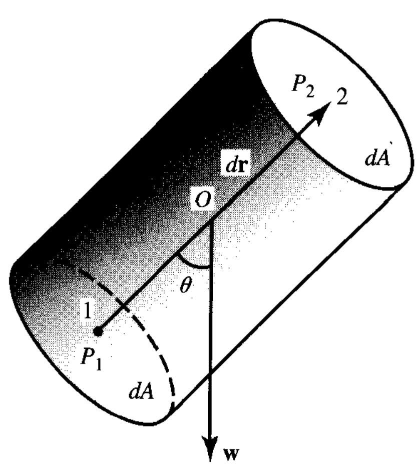


Figure 15.13 In a fluid, point 1 at pressure P_1 is at a distance $d\mathbf{r}$ from point 2 at pressure P_2.

If the two points 1 and 2 are located at distances $\mathbf{r}_1$ and $\mathbf{r}_2$, then the pressure between the two points is obtained by integrating Eq. (15.193); that is,

$$P_2 - P_1 = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{w} \cdot d\mathbf{r} \tag{15.194}$$

This line integral implies that the pressure difference between two points in a fluid depends on gravitational forces and the spatial orientation of the two points. Furthermore, this equation states that "any change in pressure at one point will be transmitted to every point in the fluid," which is the statement of *Pascal's law.*

Integration of Eq. (15.194), assuming $P_1 = P_0$ at $z = 0$ and $P_2 = P(z)$ is the pressure at a distance z above point 1, yields ($\mathbf{w} \cdot d\mathbf{r} = -\rho g\, dz$)

$$P(z) = P_0 - \rho g z \tag{15.191}$$

which is the same as Eq. (15.191). Thus, if at the surface of a lake or pond the atmospheric pressure is $P_a[=1.103 \times 10^5$ Pa (1 Pa = 1 N/m^2)], the pressure at a depth h below will be ($z = -h$)

$$P(h) = P_a + \rho g h \tag{15.195}$$

In Eq. (15.191), we have assumed that the density of the fluid is constant. This is not true, especially in the case of gases. Thus, if there is a change in pressure, it will result in a change of volume. If B is the bulk modulus of the gas,

$$B = -\frac{dP}{dV/V} \quad \text{or} \quad \frac{dV}{V} = -\frac{dP}{B} \tag{15.196}$$

The minus sign is due to the fact that as P increases V decreases. If m is the mass of a gas of volume V, the density is $\rho = m/V$. Hence

$$d\rho = -\frac{m}{V^2}\, dV = \rho\left(-\frac{dV}{V}\right)$$

or

$$\frac{dV}{V} = -\frac{d\rho}{\rho} \tag{15.197}$$

Combining Eq. (15.197) with Eq. (15.196), we have

$$\frac{d\rho}{\rho} = \frac{dP}{B} \tag{15.198}$$

For an ideal gas, the equation of state is

$$PV = nRT \tag{15.199}$$

where n is the number of moles given by $n = m/M$, M being the molecular mass, $R =$ 8.134 J/mol-K, and T is the absolute temperature of the gas. Thus

$$\rho = \frac{m}{V} = \frac{mP}{nRT} = \frac{MP}{RT} \tag{15.200}$$

Let us apply this expression for ρ in calculating the variation of pressure in our atmosphere as a function of altitude z. From Eq. (15.190), substituting the value of ρ from Eq. (15.200) yields

$$\frac{dP}{dz} = -\rho g = -\frac{Mg}{RT}P \tag{15.201}$$

Thus, if P_0 is the pressure at sea level, integrating Eq. (15.201) yields

$$P(z) = P_0 e^{-(Mg/RT)z} \tag{15.202}$$

We may define, providing the temperature remains constant, for an isothermal atmosphere, the *atmosphere scale height* H as

$$H = \frac{RT}{Mg} \tag{15.203}$$

Thus Eq. (15.202), the atmospheric pressure variation with z, takes the form

$$P(z) = P_0 e^{-z/H} \tag{15.204}$$

while the variation in the density takes the form

$$\rho(z) = \rho_0 e^{-z/H} \tag{15.205}$$

Thus H may be defined as the distance in which the density or the pressure decreases by $1/e$ of its initial value. Note that, for a constant density, Eq. (15.204) reduces to the familiar expression for $P(z)$. Assuming z to be small, so that ρ will not change, expanding Eq. (15.204) yields, [using Eq. (15.201)],

$$P(z) = P_0\left(1 - \frac{z}{H} + \cdots\right) \simeq P_0 - P_0\frac{z}{H} + \cdots = P_0 - \rho g H\frac{z}{H} \simeq P_0 - \rho g z$$

Archimedes' Principle

Let us consider the weight of a fluid of volume V, so that

$$\mathbf{W} = \iiint_V \mathbf{w}\, dV = \iiint_V \rho \mathbf{g}\, dV \tag{15.206}$$

Since the fluid is at rest, the weight (or force) is balanced by the forces of pressure exerted by the surrounding fluid on the surface of this volume; that is,

$$\mathbf{F}_b = \iint_S \hat{\mathbf{n}} P\, dA \tag{15.207}$$

For a fluid at rest $\mathbf{F}_b$ must be equal and opposite to $\mathbf{W}$; that is,

$$\mathbf{F}_b = -\mathbf{W} = -\iiint_V \rho \mathbf{g}\, dV = -\rho \mathbf{g} V \qquad \textbf{(15.208)}$$

Thus the buoyant force $\mathbf{F}_b$ on a volume V in a fluid is equal to the weight of the fluid inside the volume V. This is *Archimedes' principle,* which states that *the buoyant force on a body immersed in a fluid is equal to the weight of the volume of the fluid displaced.*

15.9 FLUIDS IN MOTION

The study of fluids in motion can be divided into two parts: fluid kinematics and fluid dynamics. We shall first investigate kinematics. There are two approaches, both suggested by Euler, by which fluids in motion may be investigated. The first approach is the direct application of Newtonian mechanics to a system of particles. Time t is considered to be the only independent coordinate, and the coordinates (x, y, z) are expressed in terms of the initial coordinate (x_0, y_0, z_0) at time t_0 and time t. The resulting equations are called *Lagrangian equations* (this approach is also called Lagrange's method). The resulting equations are so numerous that this method of keeping track of each fluid particle becomes cumbersome. The second approach, also due to Euler, is equally cumbersome, but manageable.

According to the Eulerian system for fluids, we describe such properties of fluids as density $\rho(x, y, z, t)$, velocity $\mathbf{v}(x, y, z, t)$, and pressure P, at different positions (x, y, z) and time t along the path of the fluid. Thus we are focusing our attention on a point in space where the fluid is flowing, instead of the fluid particles themselves. This leads us to define two different time rates of change for any quantity such as ρ, $\mathbf{v}$, or P. The *partial time derivative* $(\partial/\partial t)$ is the time rate of change of a quantity measured at a point fixed in space. The *total time derivative* is the time rate of change of a quantity as measured with respect to a particle moving with the fluid.

As an example, for the velocity vector $\mathbf{v}$,

$$\mathbf{v} = \mathbf{v}(x, y, z) \qquad \textbf{(15.209)}$$

the change in velocity vector is given by

$$\begin{aligned} d\mathbf{v} &= \mathbf{v}(x + dx, y + dy, z + dz, t + dt) - \mathbf{v}(x, y, z, t) \\ &\simeq \frac{\partial \mathbf{v}}{\partial x} dx + \frac{\partial \mathbf{v}}{\partial y} dy + \frac{\partial \mathbf{v}}{\partial z} dz + \frac{\partial \mathbf{v}}{\partial t} dt \end{aligned} \qquad \textbf{(15.210)}$$

In the limit as $dt \to 0$, we may write the total time derivative of $\mathbf{v}$ as

$$\frac{d\mathbf{v}}{dt} = v_x \frac{\partial \mathbf{v}}{\partial x} + v_y \frac{\partial \mathbf{v}}{\partial y} + v_z \frac{\partial \mathbf{v}}{\partial z} + \frac{\partial \mathbf{v}}{\partial t} \qquad \textbf{(15.211)}$$

Similarly, for the total time derivative of pressure P, we may write

$$\frac{dP}{dt} = v_x \frac{\partial P}{\partial x} + v_y \frac{\partial P}{\partial y} + v_z \frac{\partial P}{\partial z} + \frac{\partial P}{\partial t} \qquad \textbf{(15.212)}$$

Quantities v_x, v_y, and v_z ($\equiv dx/dt$, dy/dt, and dz/dt) are the components of the fluid velocity **v** at any point (x, y, z) and time t. Relations of the form of Eqs. (15.211) or (15.212) hold for any quantity describing the fluid. These equations in concise form may be written as

$$\frac{d\mathbf{v}}{dt} = (\mathbf{v} \cdot \nabla)\mathbf{v} + \frac{\partial \mathbf{v}}{\partial t} \tag{15.213}$$

and

$$\frac{dP}{dt} = (\mathbf{v} \cdot \nabla)P + \frac{\partial P}{\partial t} \tag{15.214}$$

From these two equations, we may reduce a common operator

$$\frac{d}{dt} = (\mathbf{v} \cdot \nabla) + \frac{\partial}{\partial t} \tag{15.215}$$

called the *substantial derivative.* This operator is applicable to both vector and scalar quantities.

We now apply these ideas by dividing our discussion into three parts:

1. Continuity equation
2. Equation of motion for an ideal fluid flow
3. Bernoulli's equation

Continuity Equation

We can arrive at the continuity equation by applying the law of conservation of mass to a Eulerian system. Consider a small differential volume element $dx\,dy\,dz$ of fluid surrounding a point (x, y, z), as shown in Fig. 15.14. The velocities of the fluid at different faces are as shown. The mass flowing in across face I (shown shaded) in the time dt is

$$dm_I = \rho(x, y, z, t)v_x(x, y, z, t)\,dy\,dz\,dt \tag{15.216a}$$

where ρ is the mass density, and v_x is the x component of the velocity, which is normal to the area $dy\,dz$. The mass flowing out from face II in time dt is

$$dm_{II} = \rho(x + dx, y, z, t)v_x(x + dx, y, z, t)\,dy\,dz\,dt \tag{15.216b}$$

Thus the net mass of the fluid leaving the volume element in the X direction is

$$dm_{II} - dm_I = [\rho(x + dx, y, z, t)v_x(x + dx, y, z, t) - \rho(x, y, z, t)v_x(x, y, z, t)]\,dy\,dz\,dt \tag{15.217}$$

Using the following expansions,

$$\rho(x + dx, y, z, t) = \rho(x, y, z, t) + \frac{\partial \rho(x, y, z, t)}{\partial x}dx + \cdots$$

$$v_x(x + dx, y, z, t) = v_x(x, y, z, t) + \frac{\partial v_x(x, y, z, t)}{\partial x}dx + \cdots$$

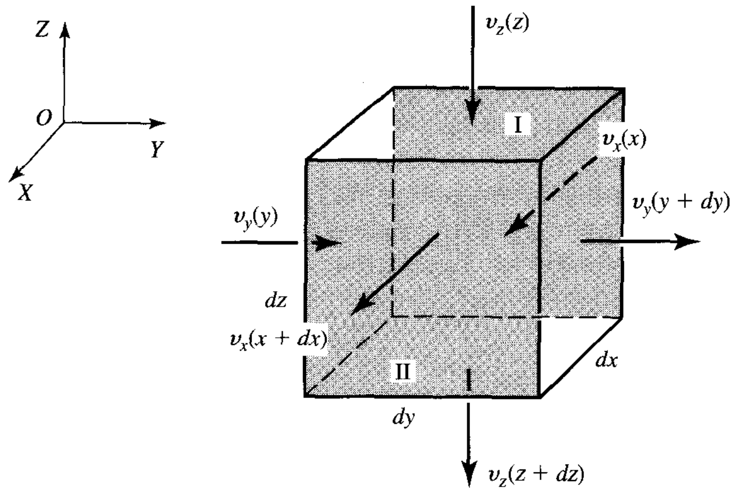


Figure 15.14 Motion of fluid across a small differential volume element $dx\,dy\,dz$ of fluid surrounding a point (x, y, z).

in Eq. (15.217) and neglecting the higher-order terms, we obtain

$$dm_{II} - dm_I = \frac{\partial \rho}{\partial x} v_x \, dx\,dy\,dz\,dt + \rho \frac{\partial v_x}{\partial x} dx\,dy\,dz\,dt \tag{15.218}$$

Applying the same procedure to the remaining faces, the net total mass leaving the volume element $dx\,dy\,dz$ in time dt is

$$dm = \left(\frac{\partial \rho}{\partial x} v_x + \frac{\partial \rho}{\partial y} v_y + \frac{\partial \rho}{\partial z} v_z + \rho \frac{\partial v_x}{\partial x} + \rho \frac{\partial v_y}{\partial y} + \rho \frac{\partial v_z}{\partial z} \right) dx\,dy\,dz\,dt \tag{15.219}$$

This net mass leaving the volume element must be equal to the decrease in mass within the element so as to conserve mass; that is,

$$dm = -\left(\frac{\partial \rho}{\partial t} \right) dx\,dy\,dz\,dt \tag{15.220}$$

Equating Eqs. (15.219) and (15.220), we obtain

$$\frac{\partial \rho}{\partial x} v_x + \frac{\partial \rho}{\partial y} v_y + \frac{\partial \rho}{\partial z} v_z + \rho \frac{\partial v_x}{\partial x} + \rho \frac{\partial v_y}{\partial y} + \rho \frac{\partial v_z}{\partial z} = -\frac{\partial \rho}{\partial t} \tag{15.221}$$

or, in vector notation, we may write this as

$$\nabla \cdot \rho \mathbf{v} + \frac{\partial \rho}{\partial t} = 0 \tag{15.222}$$

Equations (15.221) and (15.222) are the statements of the *continuity equation* and simply represent the law of conservation of mass. Matter is nowhere created or destroyed, and the mass density in any volume element dV ($=dx\, dy\, dz$) moving with the fluid remains constant. The quantity $\rho\mathbf{v}$ is the *mass flux* (also called the momentum density or mass current), defined as the mass of the fluid leaving the volume element in a unit time through a unit area. Thus Eq. (15.222) states that the divergence of the mass flux leaving a volume is equal to the rate at which the mass density decreases.

We now look at a further interpretation of Eq. (15.222). The mass flow can be determined by integrating over a fixed volume V bounded by a surface A with outward normal $\hat{\mathbf{n}}$; that is,

$$\iiint_V \nabla \cdot (\rho\mathbf{v})\, dV + \iiint_V \frac{\partial \rho}{\partial t}\, dV = 0 \qquad \textbf{(15.223)}$$

We rewrite the first term by using Gauss's divergence theorem, Eq. (5.129), and we can take the time differentiation outside the second term because V is a fixed volume. Thus Eq. (15.223) takes the form

$$\iint_A \hat{\mathbf{n}} \cdot (\rho\mathbf{v})\, dA = -\frac{d}{dt}\iiint_V \rho\, dV \qquad \textbf{(15.224)}$$

This equation states that the outward flow of mass across the surface is equal to the rate of decrease of mass inside the volume V.

For a steady flow, $\partial\rho/\partial t = 0$; hence the mass entering is exactly equal to the mass leaving. Thus Eq. (15.224) takes the form

$$\iint_A \hat{\mathbf{n}} \cdot (\rho\mathbf{v})\, dA = 0 \qquad \textbf{(15.225)}$$

Furthermore, if in addition to a constant fluid density the velocity is constant at the flow areas and is perpendicular to such areas, Eq. (15.225) yields

$$\rho_1 v_1 A_1 = \rho_2 v_2 A_2 \qquad \textbf{(15.226)}$$

and if the fluid is incompressible so that $\rho_1 = \rho_2$,

$$v_1 A_1 = v_2 A_2 = \text{constant} \qquad \textbf{(15.227)}$$

That is, the volume flux vA is constant for incompressible fluid and steady flow.

Let us consider again the case for an incompressible fluid flow, that is, ρ = constant; Eq. (15.222) takes the form

$$\nabla \cdot \mathbf{v} = 0 \qquad \textbf{(15.228)}$$

We know that the divergence of the **curl** of the vector is zero. Therefore, **v** is derivable from a vector potential $\mathbf{\Phi}$. That is, if

$$\mathbf{v} = \nabla \times \mathbf{\Phi} \qquad \textbf{(15.229)}$$

then

$$\nabla \cdot (\nabla \times \mathbf{\Phi}) = 0 \qquad \textbf{(15.230)}$$

These equations are similar to the equations for vector potentials associated with magnetic fields.

In describing fluid flow, the **curl** of the velocity, $\nabla \times \mathbf{v}$, is useful, as we explain now. Consider the relation

$$\iint_A \hat{\mathbf{n}} \cdot (\nabla \times \mathbf{v})\, dA = \oint \mathbf{v} \cdot d\mathbf{r} \qquad \textbf{(15.231)}$$

The expression on the left represents the integral over the surface area A of the normal component of **curl v**, while the right side is obtained by using Stokes' theorem [see Eq. (5.224)]. Figure 15.15 shows two examples of fluid flow, a vortex and a transverse velocity gradient. In both examples, the line integral $\oint \mathbf{v} \cdot d\mathbf{r}$ is nonzero. Hence the **curl** of the velocity ($= \nabla \times \mathbf{v}$) must be nonzero. The quantity $(\nabla \times \mathbf{v})$ may be considered to be a measure of the rate of rotation of the fluid per unit area. In Fig. 15.15(a), the **curl v** has a nonzero value around a vortex. In Fig. 15.15(b), even though there is no vortex and the fluid does not actually circle a point, but because of the transverse velocity gradient, the **curl v** is nonzero. The fluid motion is said to have rotational properties.

If the **curl v** is zero everywhere in the fluid, the motion is said to be irrotational. That is, if about a given point

$$\nabla \times \mathbf{v} = 0 \qquad \textbf{(15.232)}$$

the particles of the fluid will have no angular velocity about that point. Furthermore, if the **curl v** is zero, then **v** is derivable from a scalar potential ϕ. Since the **curl** of a gradient of a scalar is zero, we must have

$$\mathbf{v} = -\nabla \Phi \qquad \textbf{(15.233)}$$

Substituting in Eq. (15.232) gives

$$\nabla \times \nabla \Phi = 0 \qquad \textbf{(15.234)}$$

The equation represents the irrotational flow.

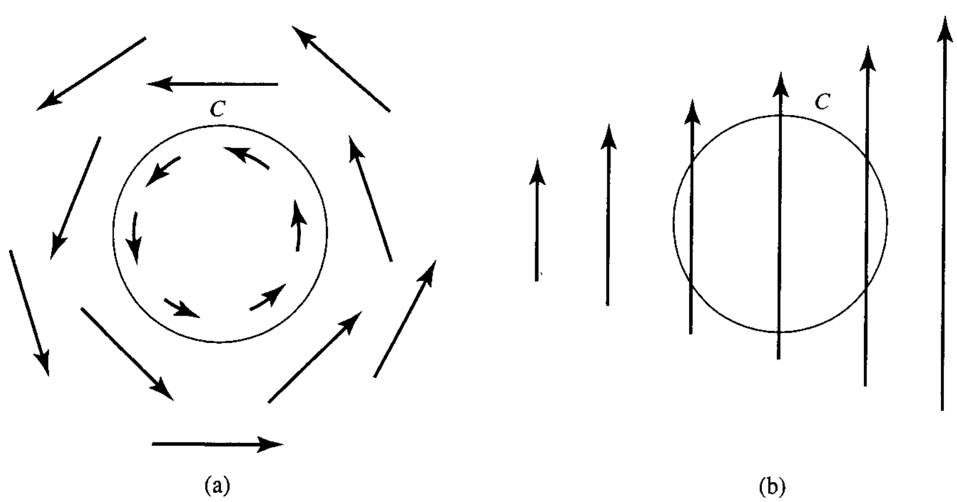


Figure 15.15 Two examples of fluid flow: (a) a vortex, and (b) a transverse velocity gradient. In both cases, $\nabla \times \mathbf{v}$ is nonzero; hence both have rotational flow.

Equation of Motion for an Ideal Fluid Flow

Once again we assume that we are dealing with an ideal fluid; that is, the fluid does not support any shear stress when in equilibrium. But any flowing fluid has viscosity, no matter how small, and hence will have some shearing stress. Thus we assume that an ideal fluid will have no viscosity.

A fluid in motion must not only satisfy the continuity equation, but must satisfy Newton's laws as well. Consider a fluid of volume $dx\,dy\,dz$, as shown in Fig. 15.16, for which the net force acting on the body is not zero. Let us assume that, in addition to pressure, the fluid is acted on by a general body force of **f** per unit volume. Thus the total body force acting on the volume element is $\mathbf{f}\,dx\,dy\,dz$. The force due to the pressure on face I is $p(x, y, z, t)\,dy\,dz$, and that due to face II is $p(x + dx, y, z, t)\,dy\,dz$. Thus, applying Newton's second law in the X direction,

$$dF_x = f_x\,dx\,dy\,dz + p(x, y, z, t)\,dy\,dz - p(x + dx, y, z, t)\,dy\,dz$$

Expanding $p(x + dx, y, z, t)$ to the first order in $dx\,dy\,dz$, we obtain

$$dF_x = \left(f_x - \frac{\partial p}{\partial x}\right) dx\,dy\,dz \tag{15.235}$$

Also, from Newton's second law,

$$dF_x = \frac{d(mv_x)}{dt} = \frac{d}{dt}(\rho v_x)\,dx\,dy\,dz \tag{15.236}$$

Equating these two equations, we get

$$f_x - \frac{\partial p}{\partial x} = \frac{d}{dt}(\rho v_x) \tag{15.237}$$

with similar expressions for the other two directions.

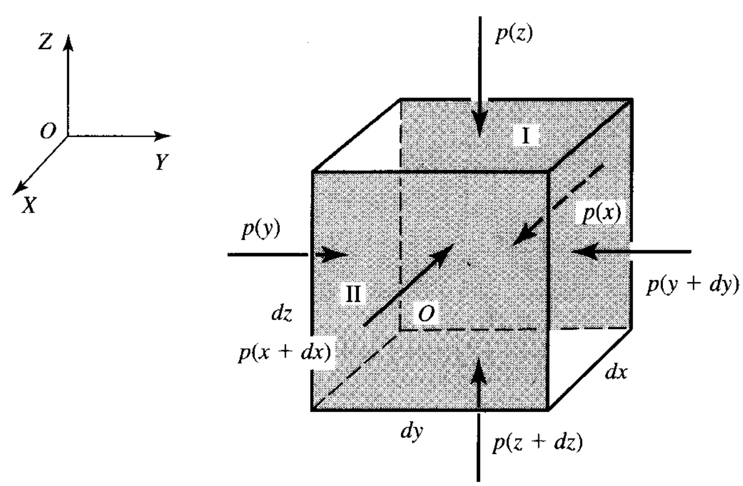


Figure 15.16 A fluid of volume $dx\,dy\,dz$ on which the net force acting is not zero. Besides pressure, the body force per unit is **f**.

In vector notation, these three equations can be combined into one as

$$\mathbf{f} - \nabla p = \frac{d}{dt}(\rho \mathbf{v}) \tag{15.238}$$

Making use of the relation in Eq. (15.215), that is,

$$\frac{d}{dt} = (\mathbf{v} \cdot \nabla) + \frac{\partial}{\partial t} \tag{15.215}$$

Eq. (15.238) may take the form

$$\mathbf{f} - \nabla p = (\mathbf{v} \cdot \nabla)(\rho \mathbf{v}) + \frac{\partial}{\partial t}(\rho \mathbf{v}) \tag{15.239}$$

or

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \frac{1}{\rho} \nabla p = \frac{\mathbf{f}}{\rho} \tag{15.240}$$

Equation (15.238), (15.239), or (15.240) is *Euler's equation of motion* for a fluid. The quantity $\mathbf{f}/\rho$ is the body force per unit mass. If the density ρ depends only on pressure p, the fluid is said to be homogeneous.

If the body force $\mathbf{f}$ is given, we still have five unknowns: density, pressure, and the three components of velocity. The continuity equation and Euler equation provide us with four scalar equations only. If the density (or one other unknown quantity) is known, the problem can be solved.

Bernoulli's Equation

The law of conservation of energy when applied to the motion of the fluid as given by the Euler equation results in Bernoulli's equation. The scalar product of Euler's equation, Eq. (15.238), with velocity vector $\mathbf{v}$ gives

$$\mathbf{f} \cdot \mathbf{v} - \nabla p \cdot \mathbf{v} = \rho \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} \tag{15.241}$$

The product $\mathbf{f} \cdot \mathbf{v}$ (force per unit volume times velocity) is the power per unit volume supplied by the body force $\mathbf{f}$. The second term may be written as

$$-\nabla p \cdot \mathbf{v} = -\frac{\partial p}{\partial x}\frac{dx}{dt} - \frac{\partial p}{\partial y}\frac{dy}{dt} - \frac{\partial p}{\partial z}\frac{dz}{dt} = -\frac{dp}{dt} + \frac{\partial p}{\partial t} \tag{15.242}$$

and the last term in Eq. (15.241) may be written as

$$\rho \frac{dv}{dt} \cdot \mathbf{v} = \rho \frac{d}{dt}\left(\frac{1}{2}v^2\right) = \frac{d}{dt}\left(\frac{1}{2}\rho v^2\right) - \frac{1}{2}v^2\frac{d\rho}{dt} \tag{15.243}$$

Thus Eq. (15.241) takes the form

$$\mathbf{f} \cdot \mathbf{v} - \frac{dp}{dt} + \frac{\partial p}{\partial t} = \frac{d}{dt}\left(\frac{1}{2}\rho v^2\right) - \frac{1}{2}v^2\frac{d\rho}{dt} \tag{15.244}$$

Since the fluid is incompressible ($d\rho/dt = 0$) and the flow is steady ($\partial p/\partial t = 0$), Eq. (15.244) takes the form

$$\mathbf{f} \cdot \mathbf{v} - \frac{dp}{dt} - \frac{d}{dt}\left(\frac{1}{2}\rho v^2\right) = 0 \tag{15.245}$$

Now Euler's equation is in a form that can be integrated. Multiplying both sides by dt and integrating, we get

$$\int (\mathbf{f} \cdot \mathbf{v})\, dt - p - \frac{1}{2}\rho v^2 = \text{constant} \tag{15.246}$$

The first term on the left is the work done by the body force per unit volume. If the body force **f** is derivable from a *scalar potential* Φ so that

$$\mathbf{f} = -\nabla\Phi \tag{15.247}$$

where Φ is the *potential energy per unit volume,* we may write the first term in Eq. (15.246) as

$$W = \int (\mathbf{f} \cdot \mathbf{v})\, dt = \int \mathbf{f} \cdot d\mathbf{r} = \int (-\nabla\Phi) \cdot d\mathbf{r} = -\Phi \tag{15.248}$$

Then Eq. (15.246) takes the form

$$p + \tfrac{1}{2}\rho v^2 + \Phi = \text{constant} \tag{15.249}$$

which is the general form of Bernoulli's equation.

If the body force is the gravitational force, $\Phi = \rho g z$, Eq. (15.249) takes the form

$$p + \tfrac{1}{2}\rho v^2 + \rho g z = \text{constant} \tag{15.250}$$

This equation, which is a statement of the conservation of energy, is known as *Bernoulli's equation* and is applicable to steady flow of incompressible fluid in a gravitational field. The first term, pressure p, represents the work done per unit volume by the fluid, the second term $\frac{1}{2}\rho v^2$ represents the kinetic energy per unit volume of the fluid, and the last term $\rho g z$ is the potential energy per unit volume of the fluid.

15.10 VISCOSITY AND VISCOUS FLOW

In previous discussions, we have assumed that the fluid was nonviscous; hence there was no friction between different layers of fluid when in motion. When adjacent layers of fluids are moving, the shearing force tends to reduce their relative motion. The existence of frictional force is illustrated as follows.

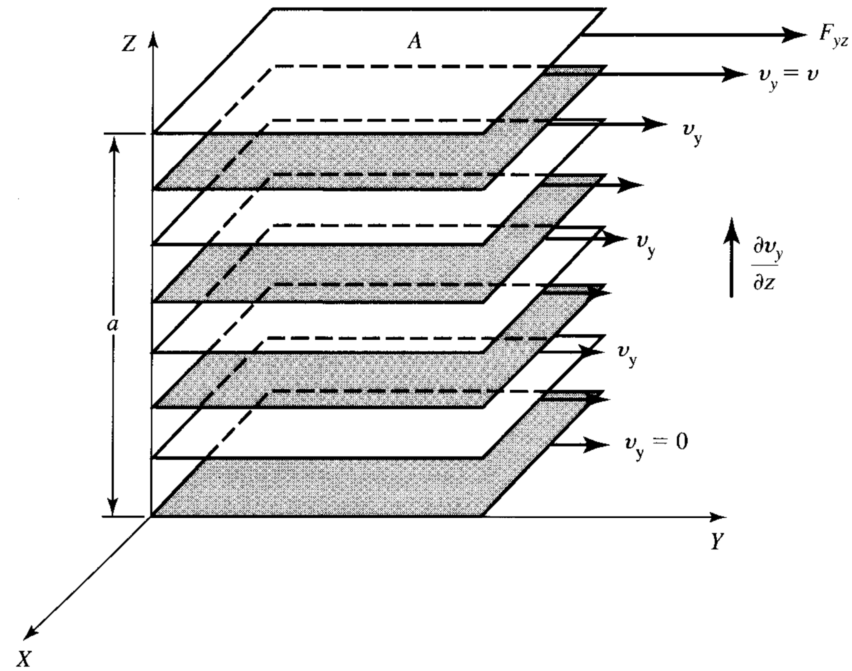


Figure 15.17 Velocity distribution in the case of viscous fluid flow.

Let us assume that the velocity of the fluid is in the Y direction. The fluid is flowing in layers that are parallel to the XY plane, as shown in Fig. 15.17, while the velocity v_y is a function of z only; that is, $v_y = f(z)$. Suppose plate A is in contact with the upper layer of a fluid and is moving with velocity v in the Y direction. A constant force **F** is needed to maintain a constant velocity, indicating the presence of a frictional force within the fluid. A layer that is in contact with the moving plate moves with the velocity of the plate so that there is no relative velocity between them. Similarly, a fluid layer next to a stationary layer will be at rest. That is, there is zero relative velocity between the solid-fluid interface, leading to zero slip at these surfaces.

As shown in Fig. 15.17, the velocity gradient is $\partial v_y/\partial z$ and is positive to the right. The viscous friction produces a positive shearing stress F_{yz} acting from left to right across an area A and parallel to the XY plane such that the normal to this plane is parallel to the Z-axis. The *coefficient of viscosity* η is defined as the ratio of the shearing stress to the velocity gradient; that is,

$$\eta = \frac{F_{yz}/A}{\partial v_y/\partial z} \tag{15.251}$$

Actually, the presence of a velocity gradient implies the existence of a shearing force acting on different layers of the fluid. Equation (15.251) takes a simple form if $F_{yz} = F$ and $\partial v_y/\partial z = v/a$; thus

$$\eta = \frac{F/A}{v/a} \tag{15.252}$$

This definition implies a simple type of distribution in which the shear stress is proportional to the first power of the velocity gradient. This is *Newtonian flow*. In most situations, flow is non-Newtonian and viscosity is a much more complicated function, resulting in a complicated shearing stress. We shall limit our discussion to Newtonian flow and illustrate the preceding definition by applying it to a laminar flow (fluid flows in layers) in circular pipes.

Consider a steady flow of fluid through a circular pipe of cross-sectional area $A = \pi r_0^2$, where r_0 is the radius of the pipe. The velocity everywhere is parallel to the axis of the pipe. As shown in Fig. 15.18, the axis of the pipe is taken along the Y-axis, and the velocity v_y is a function only of the distance r from the axis of the pipe; that is, the velocity gradient is dv_y/dr. Consider a fluid cylinder of radius r and length L so that $A = (2\pi r)L$. Thus the force exerted on this cylinder from the fluid outside this cylinder is

$$F = \eta(2\pi rL)\frac{dv_y}{dr} \tag{15.253}$$

The only forces acting on these fluids are the viscous force and the pressure difference ΔP between the two ends that are a distance L apart. In the absence of a body force and no acceleration, the sum of these two forces must be zero; that is,

$$\Delta P(\pi r^2) + F = 0 \tag{15.254}$$

Substituting for F from Eq. (15.253) and rearranging, we get

$$\frac{dv_y}{dr} = -\frac{\Delta P}{2\eta L}r \tag{15.255}$$

We integrate this outward from the cylinder axis, assuming $v = v_0$ at $r = 0$, and $v = v_y$ at $z = r$:

$$\int_{v_0}^{v_y} dv_y = -\frac{\Delta P}{2\eta L}\int_0^r r\,dr \tag{15.256}$$

$$v_y - v_0 = -\frac{\Delta P}{4\eta L}r^2 \tag{15.257}$$

If we assume that the fluid is at rest at the walls, that is, $v_y = 0$ at $r = r_0$, we get the maximum velocity:

$$v_0 = \frac{\Delta P}{4\eta L}r_0^2 \tag{15.258}$$

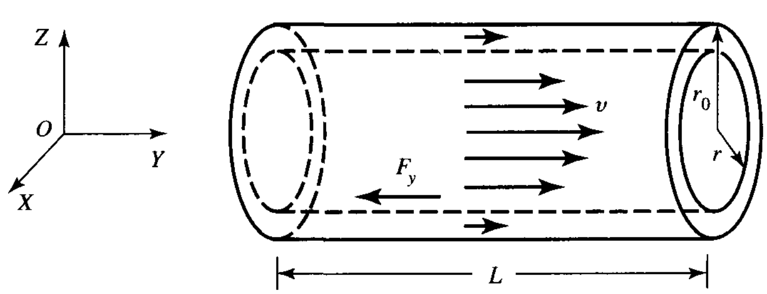


Figure 15.18 Laminar flow in a cylindrical pipe.

Substituting this in Eq. (15.257) yields

$$v_y = \frac{\Delta P}{4\eta L}(r_0^2 - r^2) \tag{15.259}$$

Since $A = \pi r^2$ and $dA = 2\pi r\, dr$, the total fluid current I or mass flow through the pipe is given by

$$I = \iint_A \rho v_y\, dA = 2\pi\rho \int_0^{r_0} v_y r\, dr \tag{15.260}$$

Substituting for v_y from Eq. (15.259) and integrating, we obtain

$$\frac{I}{\rho} = \frac{\pi r_0^4}{8\eta L}\Delta P \tag{15.261}$$

which is the statement of *Poiseuille's law*. Equation (15.261) contains measurable quantities; hence η can be calculated from it.

We can find the average velocity $\bar{v}$ of the fluid by using the definition of mass flow. Consider the expression

$$\rho\bar{v}A = \rho \int v_y\, dA = \text{mass flow} \tag{15.262}$$

Substituting for v_y from Eq. (15.259), $dA = 2\pi r\, dr$, and integrating from $r = 0$ to $r = r_0$, we obtain

$$\bar{v} = \frac{\Delta P}{2\eta L r_0^2}\int_0^{r_0}(r_0^2 - r^2)r\, dr = \frac{r_0^2}{8\eta L}\Delta P \tag{15.263}$$

which gives the relation between the pressure drop and the average velocity.

Laminar (Streamline) and Turbulent Motions

Let us now investigate the motion of an object in a fluid and its relation to frictional forces. Suppose a sphere of radius r is moving with a small constant velocity v in a liquid of viscosity η. It is assumed that the velocity is small enough so that we can have a streamlined motion. Since the sphere is moving with uniform velocity, the applied forces must be equal to the frictional force F. We can evaluate F by means of dimensional analysis. Let us assume that the frictional force F is a function of r, v, and η. Thus we may write

$$F = Kr^a v^b \eta^c \tag{15.264}$$

where K is a dimensionless constant that cannot be evaluated from dimensional analysis. Substituting the dimensions of various quantities, we get

$$[MLT^{-2}] = [L]^a[LT^{-1}]^b[ML^{-1}T^{-1}]^c \tag{14.265}$$

which gives

$$a = b = c = 1 \qquad \textbf{(15.266)}$$

Hence

$$F = Krv\eta \qquad \textbf{(15.267)}$$

The value of K can be determined experimentally. This is done by measuring the force required to pull a sphere of known radius through a liquid of known viscosity. K is found to be 6π. Thus Eq. (15.267) takes the form

$$F = 6\pi rv\eta \qquad \textbf{(15.268)}$$

which is known as *Stokes' law*.

We can now discuss the motion of a small sphere falling through a viscous fluid at constant velocity. According to Archimedes' principle, the net weight of the sphere is

$$F_{\text{net}} = \frac{4\pi}{3} r^3(\rho_s - \rho_l)g \qquad \textbf{(12.269)}$$

where ρ_s and ρ_l are the densities of the material of the sphere and that of the liquid, respectively. This force must be equal to the frictional force given by Eq. (15.268). That is,

$$6\pi rv\eta = \frac{4\pi}{3} r^3(\rho_s - \rho_l)g$$

That is,

$$v = \frac{2g}{9\eta}(\rho_s - \rho_l)r^2 \qquad \textbf{(15.270)}$$

Thus, by measuring v, since all the other quantities are known, we can calculate η. It is important to remember that the preceding results are applicable only if the motion is laminar or streamlined. For example, a stone falling through glycerine may have streamlined motion, but not if falling through water.

Sir Osborne Reynolds found that, as the velocity of an object through any liquid increases, there is a critical velocity when a sudden change from laminar motion to turbulent motion occurs. This critical velocity v_c depends on the density ρ of the liquid, its viscosity η, and diameter d of the cylindrical tube in which the liquid is flowing. Thus we may once again make use of dimensional analysis and write

$$v_c = R_e\rho^a\eta^b d^c \qquad \textbf{(15.271)}$$

where R_e is a dimensionless quantity called the *Reynolds number*. Substituting dimensions for different quantities, we obtain

$$[LT^{-1}] = [ML^{-3}]^a[ML^{-1}T^{-1}]^b[L]^c$$

which yields

$$a = -1, \quad b = 1, \quad \text{and} \quad c = -1$$

Therefore,

$$v_c = R_e \frac{\eta}{\rho d} \tag{15.272}$$

and

$$R_e = \frac{\rho d v_c}{\eta} \tag{15.273}$$

Thus, using cylindrical tubes, knowing d, ρ, and η, we can measure v_c and hence calculate R_e. Since the velocity of a liquid in a tube varies from a maximum along the axis to zero at the edges, we must use average velocities over the whole cross section in order to calculate critical velocity. From his experimental work using the flow of liquids through glass tubes, Reynolds concluded that the flow of liquids is laminar if $R_e < 2000$, whereas the flow of liquids is turbulent if $R_e > 4000$. For a liquid where the predominantly viscous forces damp out any fluctuations, Reynolds numbers have low values. On the contrary, if the viscous forces are significant, Reynolds numbers will be large, indicating the existence of turbulent flow.

When an object is moving below the critical velocity v_c, the motion is laminar, and the frictional force is caused by viscosity. As soon as the velocity is greater than the critical velocity, the motion is turbulent; eddies are set up in front of the moving object. The frictional force now mainly depends on the pressure difference between the front and back of the object and very slightly on the viscosity. Since the pressure difference depends on the cross-sectional area of the object, we may write the frictional force as

$$F = K v^a \rho^b A^c \tag{15.274}$$

Once again dimensional analysis yields

$$F = K \rho A v^2 \tag{15.275}$$

where K depends on the shape of the body and may have a value varying from 0.9 to 0.01.

PROBLEMS

15.1. Derive Eqs. (15.34) and (15.35).

15.2. A string of length L and mass m is tied at both ends. The midpoint of the string is pulled a distance $h = L/10$ in the vertical direction and released. Find an expression that describes the motion of the string.

15.3. A uniform string of length L and linear mass density μ under tension T, is displaced initially ($h \ll L$), as shown in Fig. P15.3. Find the general solution of the equation that describes the motion of the vibrating string and evaluate the coefficients by using initial conditions.

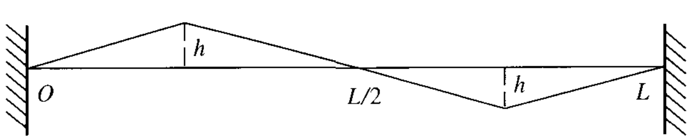


Figure P15.3

15.4. A uniform string of length L and linear mass density μ, under tension T, is initially in an equilibrium position but has a velocity given by

$$v = ax, \qquad 0 < x < \frac{L}{2}$$

$$v = a(x - L), \qquad \frac{L}{2} < x < L$$

where a is a constant. Find the general solution of the equation that describes the motion of the vibrating string and evaluate the coefficients by using initial conditions. Make graphs to describe the nature of the vibrating string.

15.5. A string of length L and mass m is tied at $x = 0$ and the end $x = L$ is tied to a ring that slides without friction on a vertical rod. Show that the boundary condition at end $x = L$ is $(\partial u/\partial x)_{x=L} = 0$, and find the normal frequencies and normal modes of vibrations. Make the graphs to describe the nature of the vibrating string.

15.6. Find the general solution for the equation of motion and normal modes of vibrations of a string with the following initial conditions:

$$u(x, 0) = A \sin \frac{3\pi x}{L} \quad \text{and} \quad \dot{u}(x, 0) = 0$$

15.7. A stretched string of length L and mass m is set into vibration by striking it over a length $2a$ at the center. The situation is described by the following initial conditions:

$$u(x, 0) = 0$$

$$\dot{u}(x, 0) = 0, \qquad \text{for } x < \frac{L}{2} - a \quad \text{and} \quad x > \frac{L}{2} + a$$

Describe the motion of the string.

15.8. Calculate the characteristic frequencies and its amplitudes for different modes for a vibrating string under the following initial conditions:

$$u(x, 0) = \frac{4x(L - x)}{L^2} \quad \text{and} \quad \dot{u}(x, 0) = 0$$

15.9. A string of length L and mass m, under tension T, is fixed at both ends. If the string is pulled in such a way that it has a parabolic shape given by $y = a(L - x)x$ and then released, investigate the motion of the string by graphing different vibrating modes.

15.10. Consider a string of length L and mass m. The end $x = L$ is tied, while at the end $x = 0$ a sinusoidal force of $a \sin \omega_0 t$ (a and ω_0 are constants) is applied. Find the solution in which all portions of the string vibrate with the same frequency ω_0; that is, find the solution of the equation for steady-state motion. Discuss the nature of vibrations by graphing.

15.11. Consider a stretched string of L and mass m tied at both ends. A force proportional to the position of the string, that is,

$$F(x, t) = F_0 \sin \frac{n\pi x}{L} \cos \omega t$$

where n is an integer, is applied along the length of the string. Investigate the steady-state motion of the string by assuming a similar dependence for $u(x, t)$.

15.12. Solve Problem 15.11 for a more general situation in which the applied force has the form $F(x, t) = F_0(x) \cos \omega t$, where $F_0(x)$ is zero at both ends of the string.

15.13. If the wave function $u(x, t) = Ae^{i(\omega t - kx)}$ is such that the quantities ω and v are complex, while k is real, then such a wave is damped in time. (Assume $\omega = \alpha + i\beta$ and $v = u + iw$, where α, β, u, and w are real.)

15.14. Solve Problem 15.2 (pulled $L/10$ at the midpoint) by superimposing waves $f(x - ct)$ and $g(x + ct)$. Draw wave forms at different time intervals.

15.15. Derive Eqs. (15.92) and (15.94).

15.16. Derive Eq. (15.100).

15.17. Evaluate P given in Eq. (15.102) by using Eqs. (15.89) and (15.90).

15.18. Derive Eq. (15.109) by first evaluating K, V, and L; then use the Lagrange equation.

15.19. In a vibrating string fixed at both ends, if p_n is the generalized momentum conjugate to θ_n, what is the Hamiltonian function H?

15.20. Consider a uniform string of length L and linear mass density μ, tied at both ends and under tension T. A force of $a \cos \omega t$ (a and ω being constants) is applied at $x = L/2$. Initially, the string is at rest with its middle point having a displacement of h $(=L/20)$. The retarding frictional force is proportional to the velocity $(= -b\dot{x})$ and acts all along the string. Find the solution of the motion of the vibrating string for the underdamped case.

15.21. In Problem 15.20, suppose the string is vibrating in the nth mode. Calculate **(a)** the rate at which the driving force is doing work, and **(b)** the average rate of doing this work.

15.22. Consider a string fixed at both ends and vibrating in a viscous medium. The damping force on any portion of the string is directly proportional to the velocity $(= -b\dot{x})$. Show that the general solution of the motion of the vibrating string is satisfied by

$$u(x, t) = e^{-\alpha x} \sin \omega\left(\frac{x}{v} - t\right)$$

where α is a constant.

15.23. Obtain the solution of Eq. (15.113), starting with Eq. (15.128) instead of Eq. (15.127).

15.24. Show that the equations of motion for longitudinal vibrations of a loaded string are exactly of the same form as transverse vibrations, provided we replace T/d by k, the force constant of the string.

15.25. Discuss the wave propagation along a string loaded with two different types of particle masses that alternate in their positions:

$$m_j = \begin{cases} m_1, & \text{for } j \text{ even} \\ m_2, & \text{for } j \text{ odd} \end{cases}$$

Show that the $\omega - k$ curve has two branches in this case.

15.26. Consider the situation discussed in Problem 15.5, where the right end of the string is attached to a ring around a vertical rod. Once the string is set into vibration, discuss the reflection of waves from the right end for **(a)** a massless and frictionless ring, **(b)** a ring of small mass and little friction, **(c)** a ring of small mass and large friction, and **(d)** a very heavy ring.

15.27. Suppose, in Section 15.6, the incident wave in coming from the right and meets a junction between two strings of different densities. Calculate the relative amplitudes and intensities for the reflected and transmitted waves. Also calculate the energy being transmitted from one string to the other, and vice versa.

15.28. In connection with Eq. (15.161), show that this is the rate at which the energy is supplied to the junction from left to right.

15.29. Consider a infinitely long string, as shown in Fig. P15.29. For $x < 0$ and $x > L$, the linear mass density of the string is μ_1, and for $0 < x < L$, the linear mass density is $\mu_2\,(>\mu_1)$. A wave of amplitude A_0 and frequency ω is incident from the left side. Find the reflected and transmitted intensities at A and B. How do these values change for a relative change in the values of μ_1 and μ_2?

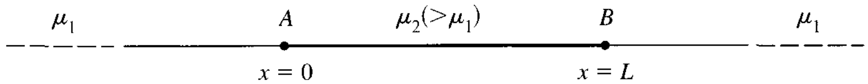


Figure P15.29

15.30. A stretched string of infinite length is under tension T. A wave of frequency ω and velocity ω/k is incident from the left. Calculate the reflected and transmitted amplitudes when mass M is attached **(a)** at $x = 0$. and **(b)** at $x = L$.

15.31. An electrical transmission line has a uniform inductance L per unit length and a uniform capacitance C per unit length. Show that the alternating current i in such a transmission line satisfies the wave equation

$$\frac{\partial^2 i}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 i}{\partial t^2}, \qquad \text{where } v = 1/\sqrt{LC}$$

15.32. Show that the spherical wave $p = f(r - vt)/r$ satisfies the longitudinal wave equation for sound waves.

15.33. Consider a right circular cone of half-angle ϕ, height h, and mass density ρ_l. The cone is floating in a liquid of density ρ_l. Show that the cone will be in stable equilibrium only if the vertex points vertically upward. Determine the frequency of small oscillations for this system.

15.34. We know that the density of air in the atmosphere varies with the altitude. Let us assume that the density is constant and equal to 1.3 kg/m^3, that is, the density at standard temperature and pressure at sea level. What would be the total thickness of the atmosphere?

15.35. For an incompressible fluid, if two components of velocity are given, how would you determine the third? If $v_x = 3x^2y^2zt^3$ and $v_y = x^2y^2z^2t^3$, calculate v_z.

15.36. Rewrite Euler's equations (15.238) and (15.240) for the motion of the fluids in cylindrical polar coordinates.

15.37. Rewrite Euler's equations (15.238) and (15.240) for the motion of the fluids in spherical polar coordinates.

15.38. Show that the velocity **v** for an ideal incompressible fluid that experiences irrotational flow may be derived from a scalar potential satisfying Laplace's equation; that is, $\nabla^2\phi = 0$, where $\mathbf{v} = -\nabla\phi$.

15.39. Using Gauss's divergence theorem where appropriate, write **(a)** the continuity equation in integral form, and **(b)** Euler's equation in integral form.

15.40. Consider a sealed cubical container half filled with water and half with air. A small hole is made in the base of the container. Determine the velocity of efflux of the water as a function of the water level. How does this compare with the results obtained for an open container? You may assume that the water is incompressible and that the entire process is isothermal.

15.41. A container of cross-sectional area A and height H is filled with an ideal incompressible fluid. The fluid is drained through a small hole of cross-sectional area a. Calculate the time required to drain half of the fluid.

15.42. The velocity distribution v for an incompressible fluid in a turbulent flow through a circular pipe of radius r_0 is given by

$$v = v_0\left(1 - \frac{r}{r_0}\right)^{1/7}$$

where v_0 is the velocity at the axis. Calculate the volumetric flow rate.

15.43. Using Euler's equation, Eq. (15.240), derive an expression for the conservation of angular momentum.

15.44. The function $\phi = c/r$, where c is a constant and r is a distance from a fixed point, satisfies Laplace's equation $\nabla^2\phi = 0$ except at $r = 0$. Discuss the nature of the fluid flow if ϕ represents the velocity potential.

15.45. A circular pipe of length l and radius r_0, open at both ends, is held vertically. Air is blown across the top open end. What is the differential equation for the sound waves set up in the pipe? Using the appropriate boundary conditions, determine the normal modes of vibrations along the axis of the pipe.

15.46. Repeat Problem 15.45 if the lower end of the pipe is closed.

15.47. Show that the force F resulting from the fluid viscosity may be written as

$$\mathbf{F} = \iint_A \eta\nabla\mathbf{v} \cdot d\mathbf{A}$$

Calculate the body force **f**, that is, the force per unit volume.

15.48. Find the increase in the density of water 30 m below the surface of a lake. The bulk modulus of water is 2×10^4 atm and its density is 1000 kg/m^3. For each 10-m depth, the pressure increases by 1 atm.

15.49. Let P_0 be the pressure at sea level and P at the top of a column of air H meter in height. Assuming a uniform temperature of T K, show that $\log_{10} P_0 - \log_{10} P = C[H/T]$, where C is a constant. Assuming pressure at sea level to be 1.013×10^5 N/m^2 and the density of air at 0°C to be 1.29 kg/m^3.

15.50. Consider a streamlined flow of water. At some fixed point, the velocity of the water is 60 cm/s and the rate of change of velocity with distance is 12 cm/s/cm. Calculate the acceleration of the water at the fixed point.

15.51. A horizontal tube 20-cm long and 1.0 cm in diameter is connected at one end to a water tank whose constant-level height is 2 m. Calculate the coefficient of viscosity if 500 cm^3 flows through the tube in 5 min.

15.52. Consider a horizontal capillary tube of length L and radius R connected to an airtight vessel of volume V. Air escapes from the vessel through the capillary tube, and in time t the air pressure reduces from P_1 to P_2. If the atmospheric pressure is P_0, show that the coefficient of viscosity is given by

$$\eta = \frac{\pi R^4 P_0 t}{8LV \log_e \dfrac{(P_1 - P_0)(P_2 + P_0)}{(P_2 - P_0)(P_1 + P_0)}}$$

(*Hint:* Consider a small section of a capillary tube of length dx where pressure is P, and then integrate over the whole length and use Boyle's law.)

15.53. Using the expression derived in Problem 15.52, calculate the coefficient of viscosity for the following data: $L = 1$ m, $R = 0.05$ m, $V = 0.5$ m^3, the original pressure is 81 cm of mercury, the final pressure after 30 s is 79.5 cm of mercury, the atmospheric pressure is 76 cm of mercury, and the temperature is assumed to be constant throughout.

15.54. Assuming a Reynolds number of 1200 for a cylindrical pipe of 2.5-cm radius, calculate the critical velocity of (**a**) water and (**b**) glycerin, both at 20°C.

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*The asterisk indicates works of an advanced nature.