CHAPTER 10

Gravitational Force and Potential

10.1 INTRODUCTION

In this chapter we shall investigate Newton's universal law of gravitation and its application. We introduce the concepts of gravitational field intensity (or simply gravitational field) **g** and gravitational potential V. We calculate these quantities for different mass distributions by applying Newton's law of gravitation. Gauss's laws will be applied to calculate **g** and V for simple symmetrical mass distributions. Finally, gravitational field equations will be introduced, which are differential equations satisfied by functions such as **g** and V. These equations provide a more general procedure of interest.

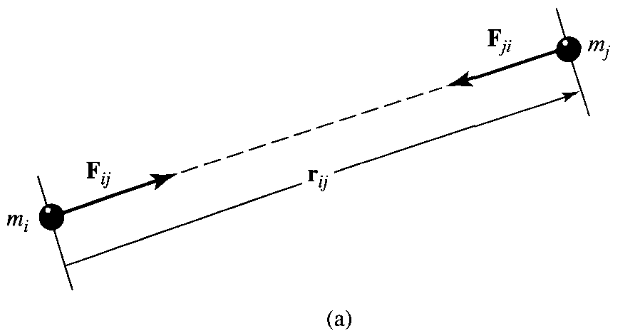
10.2 NEWTON'S UNIVERSAL LAW OF GRAVITATION

Newton's universal law of gravitation (which is a law of force), together with Newton's laws of motion, has been applied by physicists to predict and calculate very precisely the motion of the planets, moons, satellites, and other objects in the universe. In 1666, 23-year-old Isaac Newton stated the universal law of gravitation in the following form:

> ***Newton's Universal Law of Gravitation.*** *The gravitational force (or interaction) of attraction between any two objects in the universe is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.*

Thus the magnitude of the force F between any two objects of masses m_i and m_j separated by a distance r_{ij} is given by

$$F = G\frac{m_i m_j}{r_{ij}^2} \tag{10.1}$$



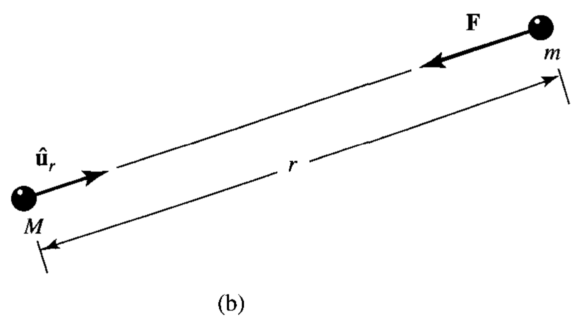


Figure 10.1 (a) Gravitational forces between two masses m_i and m_j. (b) Gravitational force on mass m due to mass M.

where G is the *gravitational constant;* its presently accepted value is

$$G = (6.673 \pm 0.003) \times 10^{-11} \text{ N-m}^2/\text{kg}^2 \tag{10.2}$$

Referring to Fig. 10.1(a), we may write the law in vector form as

$$\mathbf{F}_{ij} = G\frac{m_i m_j}{r_{ij}^2}\frac{\mathbf{r}_{ij}}{r_{ij}} = G\frac{m_i m_j}{r_{ij}^3}\mathbf{r}_{ij} \tag{10.3}$$

where $\mathbf{F}_{ij}$ is the gravitational force by which mass m_i is attracted by mass m_j, $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$ is the distance between the two masses m_i and m_j, and $\mathbf{F}_{ji}$ is the force by which m_j is attracted by mass m_i. According to Newton's third law, we have

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji}$$

$$|\mathbf{F}_{ij}| = |\mathbf{F}_{ji}| = F = G\frac{m_i m_j}{r_{ij}^2} \tag{10.4}$$

From Fig. 10.1(b), mass m is attracted by mass M with a force $\mathbf{F}$; we may write

$$\mathbf{F} = -G\frac{Mm}{r^2}\hat{\mathbf{u}}_r \tag{10.5}$$

where the unit vector $\hat{\mathbf{u}}_r$ is in the direction from M to m. The minus sign indicates that $\mathbf{F}$ is the force of attraction with its line of action passing through a fixed point on the line joining the two masses. Thus the force is directed toward the center of mass M, and the *gravitational force is a* *central force.* The preceding equations are applicable to the situation in which the masses may be considered point masses. This is possible only if the dimensions of the masses are negligible compared to the distances between them.

Let us consider a point mass m at P attracted by an extended body of mass M, as shown in Fig. 10.2. To calculate the force on m at P, we must assume that the gravitational field is a *linear field.* That is, the force at P may be calculated by the vector addition of the individual forces produced by the interactions between the point particle m and the large number of particles in the extended body. The force $d\mathbf{F}$ between m and a small element of volume dV' of mass dm is

$$d\mathbf{F} = -G\frac{m\,dm}{r^2}\,\hat{\mathbf{u}}_r \tag{10.6}$$

where $dm = \rho(\mathbf{r})\,dV'$, $\rho(\mathbf{r})$ being the density. The force $\mathbf{F}$ acting on m due to the extended body of mass M may be obtained by integrating Eq. (10.6); that is,

$$\mathbf{F} = -\int_{V'} G\frac{m\rho(\mathbf{r})}{r^2}\,\hat{\mathbf{u}}_r\,dV' \tag{10.7}$$

where V' indicates integration over the whole volume. If the extended body is a thin shell that has a surface density or area density σ so that $dm = \sigma\,dA$, we may write

$$\mathbf{F} = -\int_{A} G\frac{m\sigma(\mathbf{r})}{r^2}\,\hat{\mathbf{u}}_r\,dA \tag{10.8}$$

where A indicates the integration over the whole area. If the extended body is a line source with a linear mass density λ so that $dm = \lambda\,dL$, we may write

$$\mathbf{F} = -\int_{L} G\frac{m\lambda(\mathbf{r})}{r^2}\,\hat{\mathbf{u}}_r\,dL \tag{10.9}$$

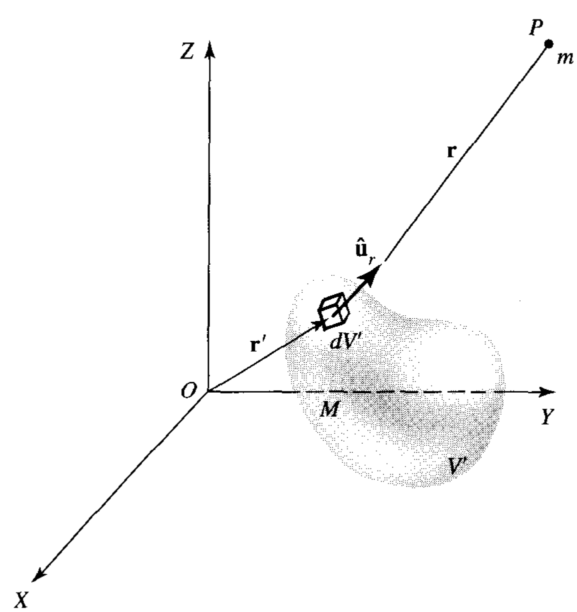


Figure 10.2 Gravitational force on mass m at P due to an extended body of mass M and volume V'.

If the extended body is replaced by a large number of discrete masses $m_1, m_2, m_3, \ldots, m_i$, the force on mass m may be written as

$$\mathbf{F} = -\sum_i G\frac{mm_i}{r_i^2}\hat{\mathbf{u}}_r \tag{10.10}$$

where $\hat{\mathbf{u}}_r$ is the unit vector in the direction along the line joining m_i and m.

According to Eq. (10.7), the system of forces acting on different portions of the extended body due to mass m at P has a resultant force $\mathbf{F}$ acting along a line through the mass m. According to Newton's third law, the force acting on m is $-\mathbf{F}$, as shown in Fig. 10.3. On this line of action of $\mathbf{F}$, we locate a point CG at a distance r from m at P such that

$$F = G\frac{mM}{r^2} \tag{10.11}$$

Under these conditions, the gravitational force between the body of mass M and the particle of mass m is equivalent to a single resultant force $\mathbf{F}$ acting on M at CG and $-\mathbf{F}$ acting on m at P. The extended body behaves as if all its mass is concentrated at CG. The point CG is called the *center of gravity* of the body of mass M relative to the point mass m at P. If the position of m at P changes, so will the position of CG. In general, CG does not coincide with the center of mass of M; it may not even be on the line joining the center of mass of M with P. The center of gravity will coincide with the center of mass under the following conditions: (1) If the mass m is far away from M, the gravitational field will be uniform, different parts of the body will be acted on by the same force, and the center of gravity will coincide with the center of mass; (2) for a symmetrical body, such as a uniform sphere, its center of gravity coincides with its center of mass.

We will encounter another complication if the mass m is also an extended body. In such cases Eqs. (10.6) and (10.7) must be rearranged, which will involve integrals of both m and dm.

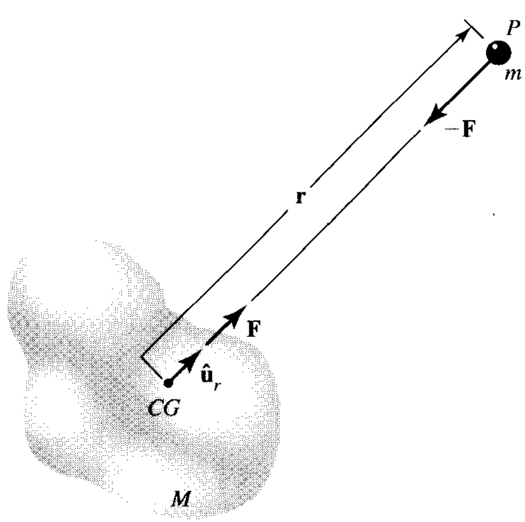


Figure 10.3 Center of gravity CG of an extended body of mass M relative to mass m at point P.

10.3 GRAVITATIONAL FIELD AND GRAVITATIONAL POTENTIAL

As stated before, a gravitational force is a *central force;* that is, it is a purely radial force passing through a given point, the center of force. Furthermore, the gravitational force is *spherically symmetric;* that is, the magnitude of the force depends only on the radial distance from the center of the force and not on its direction. We shall show that *spherically symmetric central forces are conservative;* hence the sum of the kinetic energy and the potential energy is constant. Conversely, *if a central force field is conservative, it must also be spherically symmetric.* (Note of caution: A force that is conservative may or may not be both central and spherically symmetric.)

Suppose a particle of mass m is under the action of a spherically symmetric central force $\mathbf{F}$ with its center of force at O, as shown in Fig. 10.4. In this situation, the force $\mathbf{F}$ has only a radial component F_r, which is a function of $\mathbf{r}$ only and may be written as

$$F_r = F(\mathbf{r}) \tag{10.12}$$

The work dW done by the central force $\mathbf{F}$ when m undergoes a small displacement ds, as shown, is

$$dW = \mathbf{F} \cdot d\mathbf{s} = F\, ds \cos\theta \tag{10.13}$$

But

$$ds \cos\theta = dr$$

where dr is the change of the radial distance from O when mass m undergoes a displacement $d\mathbf{s}$. Thus

$$dW = F\, dr \tag{10.14}$$

Since the magnitude of the force $\mathbf{F}$ depends only on r, the total work done in going from A to B, as shown in Fig. 10.4, will be

$$W_{AB} = \int_{r_A}^{r_B} F(r)\, dr \tag{10.15}$$

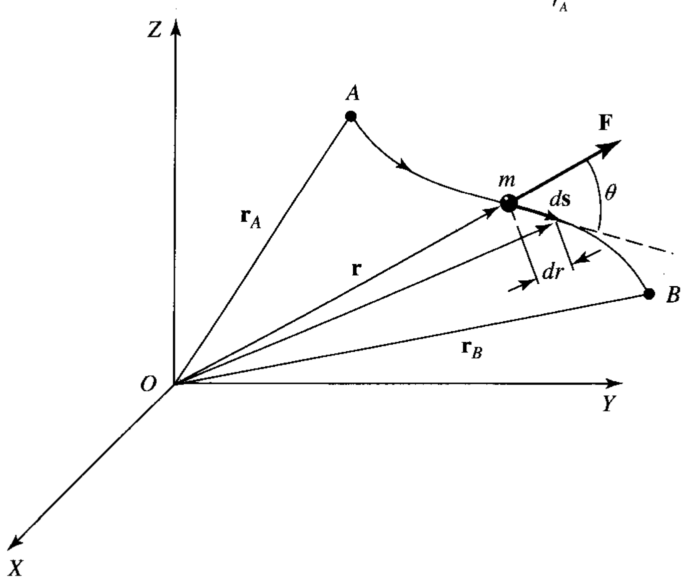


Figure 10.4 Work done by a central force **F** when a mass m is displaced from point A to point B.

Since this integral and hence the work done depend only on the initial and the final values of r (not on the path itself) the spherically symmetric force must be conservative.

Once we know that the force is conservative, we can proceed to define a potential energy function $U(r)$ of an object in such a spherically symmetric central force field. Thus, in going from A to B, the change in potential energy of an object is

$$\Delta U = U_B - U_A = -\int_{r_A}^{r_B} f(r)\,dr \tag{10.16}$$

From Eqs. (10.15) and (10.16), we obtain

$$W_{AB} = -\Delta U = -(U_B - U_A) \tag{10.17}$$

But the work done is also equal to change in kinetic energy; that is,

$$W_{AB} = K_B - K_A = -(U_B - U_A) \tag{10.18}$$

Thus, if E is the total energy, Eq. (10.18) yields

$$K_A + U_A = K_B + U_B = E \tag{10.19}$$

which is the law of the *conservation of energy*.

Since the gravitational force is an inverse square law force,

$$F(r) = f(r) = \frac{C}{r^2} \tag{10.20}$$

where C is a constant. Substituting this in Eq. (10.16), we get

$$U_B - U_A = -\int_{r_A}^{r_B} \frac{C}{r^2}\,dr$$

which on integration gives

$$U_B - U_A = C\left(\frac{1}{r_B} - \frac{1}{r_A}\right) \tag{10.21}$$

As is usually done , we define $U_A = 0$ when $r_A \to \infty$ and $U_B = U(r)$ where $r_B = r$; thus we get

$$U(r) = \frac{C}{r} \tag{10.22}$$

which states that the potential energy of a particle in a central force field is a function of the distance r from the force center. The constant C is negative for attractive forces and positive for repulsive forces. Since the gravitational force is attractive and has the general form

$$F(r) = -\frac{GMm}{r^2} = \frac{C}{r^2} \quad \text{where } C = GMm \tag{10.23}$$

the *potential energy* of m in the field of M at a distance r from M is

$$U(r) = -\frac{GMm}{r} \tag{10.24}$$

If M is a continuous mass distribution of arbitrary shape, the potential energy of m at a distance r is

$$U(r) = -\int_{V'} \frac{Gm\rho(\mathbf{r})}{r}\, dV' \tag{10.25}$$

To make the preceding three equations independent of m (the test mass), we introduce the concepts of gravitational field and gravitational potential.

The *gravitational field intensity*, or *gravitational field vector*, or simply *gravitational field*, $\mathbf{g}$, is defined as the force per unit mass exerted on a particle in the gravitational field of mass M. That is,

$$\mathbf{g} = \frac{\mathbf{F}}{m} = -\frac{GM}{r^2}\,\hat{\mathbf{u}}_r \tag{10.26}$$

or, for an extended body of mass M, we may write

$$\mathbf{g} = -\int_{V'} \frac{G\rho(\mathbf{r})}{r^2}\,\hat{\mathbf{u}}_r\, dV' \tag{10.27}$$

where $\mathbf{g}$ has the dimensions of force per unit mass, that is, acceleration. The magnitude of this gravitational acceleration on the surface of Earth is approximately 9.8 m/s^2.

Whenever there is a conservative vector field, as is the gravitational force field, we can always introduce a gravitational potential (which is a scalar quantity) to represent this field, provided certain conditions are satisfied. The condition required is that the **curl** of the vector field $\mathbf{g}$ must be zero. Since $\mathbf{g}$ is proportional to $1/r^2$,

$$\textbf{curl}\,\mathbf{g} \equiv \nabla \times \mathbf{g} = 0 \tag{10.28}$$

(as proved in Chapter 6). This condition will also be satisfied if $\mathbf{g}$ is equal to the gradient of a scalar; that is,

$$\mathbf{g} \equiv -\textbf{grad}\,V \equiv -\nabla V \tag{10.29}$$

(remembering $\nabla \times \nabla V = 0$), where V is called the *gravitational potential* and has the dimensions of *energy per unit mass*. Since $\mathbf{g}$ is only r dependent, V will be only r dependent. Substituting for $\mathbf{g}$ from Eq. (10.26) into Eq. (10.29), we get

$$-\frac{GM}{r^2}\,\hat{\mathbf{u}}_r = -\frac{dV}{dr}\,\hat{\mathbf{u}}_r$$

which on integration gives

$$V(r) = -\frac{GM}{r} \tag{10.30}$$

It is not necessary to have a constant of integration in Eq. (10.30) because we assume that $V(r) \to 0$ as $r \to \infty$.

The gravitational potential due to a continuous distribution of mass M may be written as

$$V(r) = -\int_{V'} \frac{G\rho(\mathbf{r})}{r} dV' \tag{10.31}$$

We may summarize this discussion as follows:

Force:

$$\mathbf{F} = -\int_{V'} G \frac{m\rho(\mathbf{r})}{r^2} \hat{\mathbf{u}}_r \, dV' \tag{10.7}$$

Potential energy:

$$U(r) = -\int_{V'} \frac{Gm\rho(\mathbf{r})}{r} dV' \tag{10.25}$$

Gravitational field:

$$\mathbf{g} = -\int_{V'} \frac{G\rho(\mathbf{r})}{r^2} \hat{\mathbf{u}}_r \, dV' \tag{10.27}$$

Gravitational potential:

$$V(r) = -\int_{V'} \frac{G\rho(\mathbf{r})}{r} dV' \tag{10.31}$$

Also,

$$\mathbf{F} = m\mathbf{g} \tag{10.32a}$$

$$U = mV \tag{10.32b}$$

$$\mathbf{g} = -\nabla V = -\mathbf{grad}\ V \tag{10.32c}$$

$$\mathbf{F} = -\nabla U = -\mathbf{grad}\ U \tag{10.32d}$$

Whenever a mass m is placed in the field of M, it is conventional to speak of the potential energy of mass m even though such potential energy resides in the field and not in the mass itself.

10.4 LINES OF FORCE AND EQUIPOTENTIAL SURFACES

The lines of force and equipotential lines in two dimensions and equipotential surfaces in three dimensions are very helpful in visualizing a force field. Let us consider a mass M that produces a gravitational field in the surrounding space and that may be described by the gravitational field

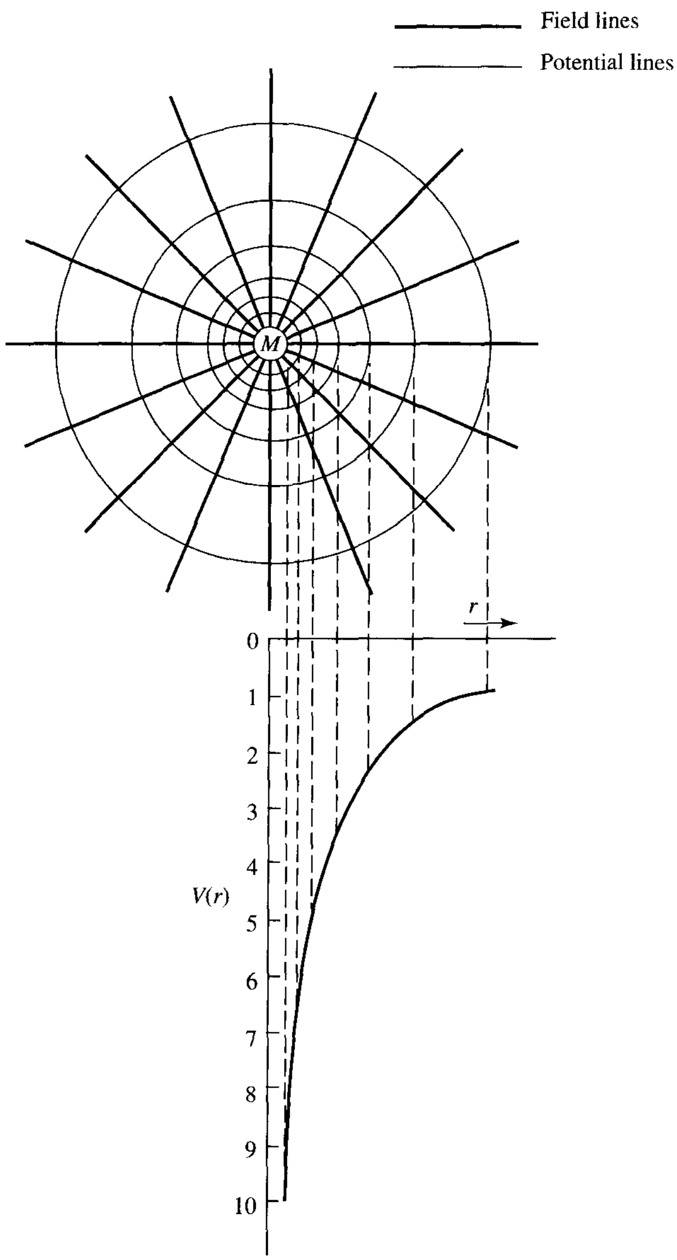


Figure 10.5(a) Gravitational field lines (boldface lines) and equipotential lines due to a sphere of mass M. The graph shows the relative value of $V(r)$ versus r.

vector **g**. We start from an arbitrary point and draw an infinitesimal line element in the direction of the vector **g** at that point. At the end of this line element, we draw another line element in the direction of **g** at this new point. We continue this process, and when we join these small line elements, we obtain a smooth line or curve called the *line of force* or *force field line*. We can draw a large number of such lines in the space surrounding a mass, as shown in Fig. 10.5(a). [See also Fig. 10.5(b).] These lines start from the surface of a mass and extend to infinity. For a single mass point, the force lines are straight lines (or radial) extending to infinity as shown. This is not true in all mass configurations and may be very complicated. For example, Fig. 10.6 will

Figure 10.5(b)

Below is the graph of the gravitational field and gravitational potential versus the distance r from the center of mass M of Earth

$N := 50$ $\quad i := 1..N$ $\quad RE := 6.368 \cdot 10^6$

$G := 6.673 \cdot 10^{-11}$ $\quad M := 5.98 \cdot 10^{24}$ $\quad r_i := i \cdot RE$

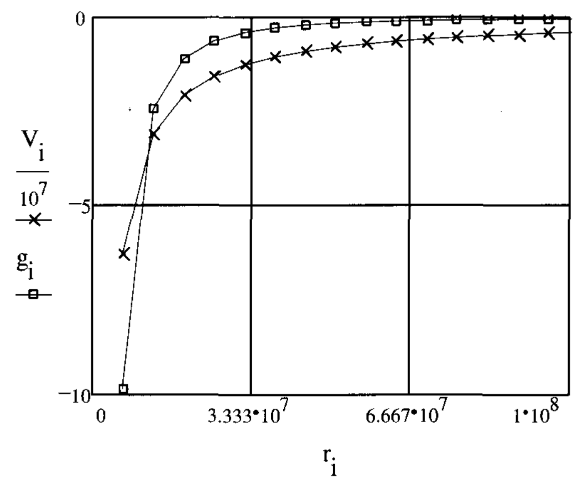
Which value, g or V, decreases faster with a change in distance r and why?

$$g_i := \frac{-G \cdot M}{(r_i)^2} \qquad V_i := \frac{-G \cdot M}{r_i}$$

$g_1 = -9.84$ $\quad V_1 = -6.266 \cdot 10^7$

$g_{20} = -0.025$ $\quad V_{20} = -3.133 \cdot 10^6$

$g_{50} = -0.004$ $\quad V_{50} = -1.253 \cdot 10^6$



show potential curves resulting from two unequal masses. The force field lines will be perpendicular to potential curves at every point.

This picture of the lines of force may be used to describe the direction and magnitude of the field vector **g**. A tangent drawn at any point to the field line gives the direction of the force field (of **F** or **g**) at that point. The density of these lines, the number of lines passing through a unit volume (the volume being small, but including the point), gives the magnitude of the vector field **g** at that point. No two field lines cross each other because **g** is a single-valued function; that is, it has only one value at any given point. It may be pointed out that these field lines have no real existence, but give a vivid picture depicting the properties of the force field.

We now seek to investigate the relation between the force field lines and the gravitational potential lines. Suppose we know the gravitational potential *V* in the space surrounding a mass.

Since the gravitational potential V is defined for each point in space and is a single-valued function, we may write

$$V = V(x, y, z) \tag{10.33}$$

Suppose we join all the points having the same value of gravitational potential V_0. The equation representing these points is

$$V = V(x, y, z) = V_0 = \text{constant} \tag{10.34}$$

This is the equation of a surface, called an *equipotential surface*. We can draw a surface for each different value of V_0, hence resulting in a large number, or a whole family, of equipotential surfaces. In a two-dimensional case instead of equipotential surfaces, we get *equipotential lines*. Once again, since $V(x, y, z)$ is a single-valued function, no two equipotential surfaces or lines will cross each other. Suppose we move a mass m from one point to another point on an equipotential line. By definition, no work will be done. This leads us to the conclusion that *the lines of force are everywhere perpendicular (or orthogonal) to the equipotential lines*. This is true because $\mathbf{g} = -\nabla V$; it means that $\mathbf{g}$ cannot have a component along an equipotential surface because V is constant. Thus every line of force must be normal to the equipotential surface, as shown in Fig. 10.5(a). We shall elaborate on this point shortly. Meanwhile, Fig. 10.6 shows equipotential lines resulting from two masses M_1 and M_2. The equipotential surfaces in this case are defined by the equation

$$V = -G\left(\frac{M_1}{r_1} + \frac{M_2}{r_2}\right) = \text{constant} \tag{10.35}$$

Consider a mass at point P and let it be displaced a distance $d\mathbf{s}$. The change in its potential energy, which is equal to the work done, is given by

$$dU = -\mathbf{F} \cdot d\mathbf{s} = -F_s\, ds \tag{10.36}$$

where F_s is the component of the force in the direction of the displacement $d\mathbf{s}$. Equation (10.36) may be written as

$$F_s = -\frac{dU}{ds} \tag{10.37}$$

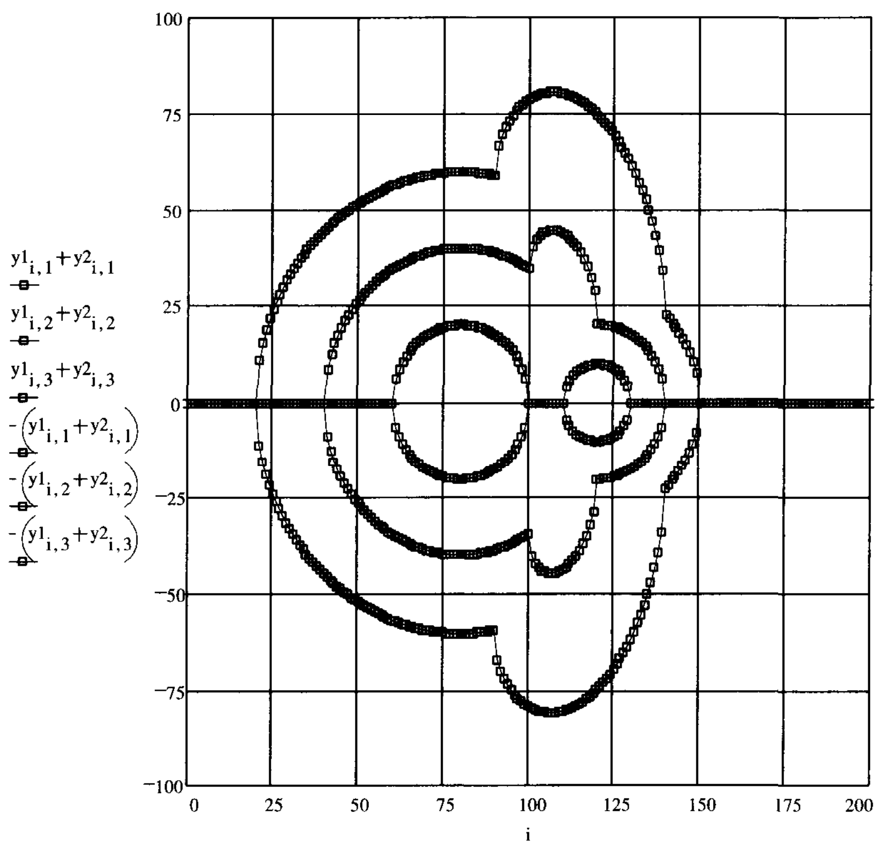
This equation states that the *component of* **F** *in any direction is equal to the negative rate of change of potential energy with distance in that direction*. The right side of Eq. (10.37) is called the *directional derivative* because its value will depend on the direction of $d\mathbf{s}$ relative to **F**. For example, consider two equipotential energy lines U_0 and $U_0 + \Delta U$ or two equipotential lines V_0 and $V_0 + \Delta V$, as shown in Fig. 10.7. If we move form P to Q, which is on the same equipotential line, dU/ds will be zero. But if we move from P to R_1, R_2, or R on a different equipotential line, dU/ds will be different for different paths, such that $dU/ds > dU/ds_1, dU/ds_2, \ldots$. In this case, dU/ds is maximum when ds is the shortest and hence perpendicular to the equipotential line at that point. The particular direction for which dU/ds is maximum in the direction of the

Figure 10.6

Below is a graph of the gravitational potential lines (only three are shown drawn) due to two nearby unequal masses located at the center of the circles (not shown). The force field lines, not shown, are perpendicular to potential lines at every point.

$j := 1..10 \qquad i := 0..200 \qquad r1_j := 20 \cdot j \qquad r2_j := 10 \cdot j \qquad x1_i := i - 80 \qquad x2_i := i - 120$

$$y1_{i,j} := \sqrt{(r1_j)^2 - (x1_i)^2} \qquad y2_{i,j} := \sqrt{(r2_j)^2 - (x2_i)^2}$$



line of force, and the maximum magnitude of dU/ds is the magnitude of the vector force at that point. The maximum value of dU/ds and its direction is called the *gradient* of the potential energy and is equal to the force **F**; that is,

$$\mathbf{F} = -\mathbf{grad}\, U \tag{10.38}$$

Since $\mathbf{F} = m\mathbf{g}$ and $U = mV$, we may write

$$\mathbf{g} = -\mathbf{grad}\, V = -\nabla V \tag{10.39}$$

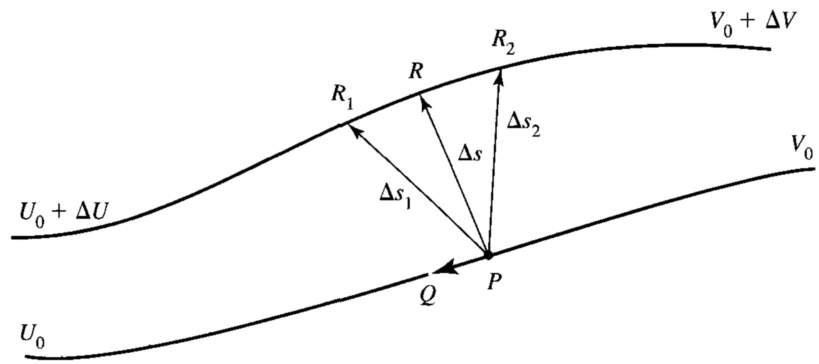


Figure 10.7 Gradient of the potential energy. The magnitude of the gradient is $\Delta U/\Delta s$.

10.5 CALCULATION OF GRAVITATIONAL FORCE AND GRAVITATIONAL POTENTIAL

We shall start by calculating the gravitational force between a uniform spherical shell of mass M and a point mass m. We shall show that any spherical shell may be treated as a point mass located at the center of the shell. Actually, this is true for any uniform spherically symmetric distribution of matter. In any of these situations, instead of calculating the force (which is a vector quantity), it is easier to calculate gravitational potential (which is a scalar quantity). Once the gravitational potential is known, the gravitational force may be calculated from it. We shall elaborate on both these procedures.

Spherical Shell

Consider a thin uniform shell of mass M and radius R, as shown in Fig. 10.8. A particle of mass m is placed outside the shell at point P a distance r ($r > R$) from the center of the shell. We divide the shell into a large number of circular rings like the one shown shaded in the figure. We can calculate the force between one of these rings and mass m and then sum over all the rings. As shown in the figure, the width of the shaded ring is $R\,d\theta$, while the radius of the ring is $R \sin\theta$. The circumference of the ring is $2\pi R \sin\theta$, while the area dA of the circular strip or shaded ring is

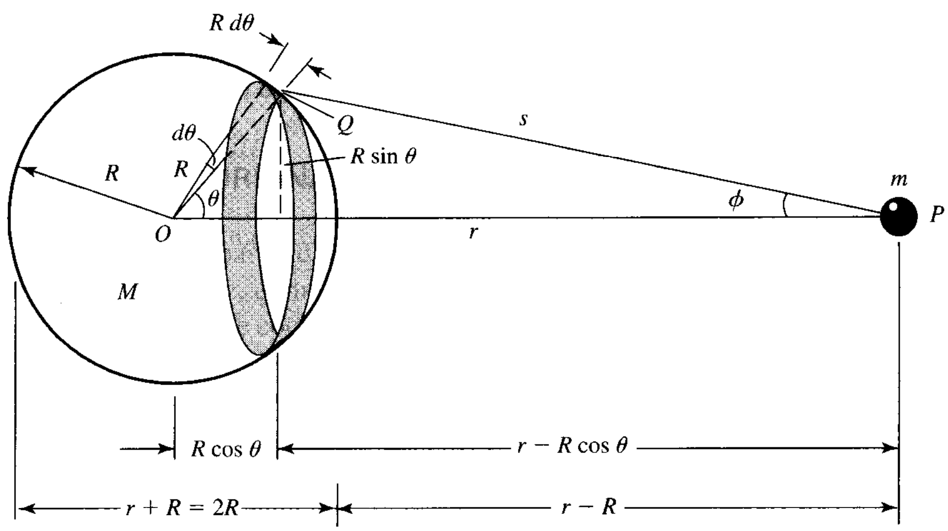
$$dA = (2\pi R \sin\theta)R\,d\theta = 2\pi R^2 \sin\theta\,d\theta \qquad \textbf{(10.40)}$$

If σ is the density per unit area of the material of the shell, then the mass of the whole spherical shell is

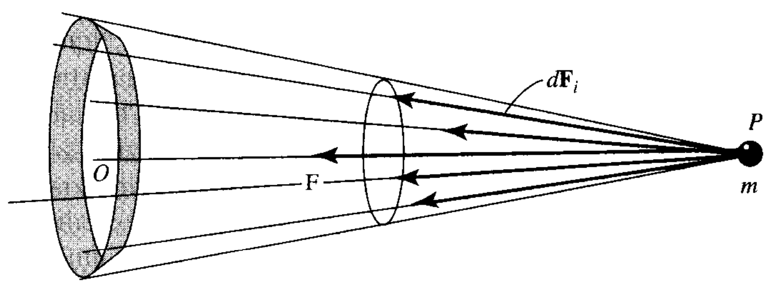
$$M = (4\pi R^2)\sigma, \qquad \sigma = \frac{M}{4\pi R^2} \qquad \textbf{(10.41)}$$

while the mass dM of the shaded ring is

$$dM = \sigma\,dA = \sigma 2\pi R^2 \sin\theta\,d\theta = \frac{M}{2}\sin\theta\,d\theta \qquad \textbf{(10.42)}$$



(a)



(b)

Figure 10.8 Gravitational force between a point mass m and spherical shell of mass M and radius R.

Point Q, or any other point on the shaded ring, is at the same distance s from the point mass m at P. The force dF_i on m due to any small section of this ring, such as at Q, points toward that section [see Fig. 10.8(b)]. This force can be resolved into transverse component $dF_i \sin \phi$, which is perpendicular to PO, and another component $dF_i \cos \phi$, which is parallel to PO. Due to the symmetry of the situation, all the transverse components resulting from considering the whole ring add up to zero, while the force components parallel to PO due to the whole ring add up to give

$$dF = \sum dF_i = \sum \frac{Gm\,dM}{s^2} \cos \phi \tag{10.43}$$

or, substituting for dM, we have

$$dF = \frac{GMm}{2s^2} \sin \theta \, d\theta \cos \phi \tag{10.44}$$

The force due to the entire shell is

$$F = \int dF = \int_0^{\pi} \frac{GMm \sin \theta \, d\theta}{2s^2} \cos \phi$$

or

$$F = GMm \int_0^{\pi} \frac{\cos \phi \sin \theta \, d\theta}{2s^2} \tag{10.45}$$

From triangle OPQ, using the law of cosines, we obtain

$$s^2 = r^2 + R^2 - 2rR \cos \theta \tag{10.46}$$

Since r and R are constants, differentiation yields

$$2s \, ds = 2rR \sin \theta \, d\theta \tag{10.47}$$

and, similarly, from the same triangle OPQ, we obtain

$$R^2 = s^2 + r^2 - 2sr \cos \phi$$

or

$$\cos \phi = \frac{s^2 + r^2 - R^2}{2sr} \tag{10.48}$$

Substituting for $\sin \theta \, d\theta$ and $\cos \phi$ from Eqs. (10.47) and (10.48) into Eq. (10.45) and changing the limits by using Eq. (10.46) from $0 \rightarrow \pi$ to $r - R \rightarrow r + R$, we obtain

$$F = \frac{GMm}{4r^2R} \int_{r-R}^{r+R} \left(1 + \frac{r^2 - R^2}{s^2}\right) ds \tag{10.49}$$

which on integration yields

$$F = \frac{GMm}{r^2} \tag{10.50}$$

In vector notation, this may be written as

$$\mathbf{F} = -\frac{GMm}{r^2} \hat{\mathbf{u}}_r, \quad \text{for } r > R \tag{10.51a}$$

and

$$\mathbf{g} = -\frac{GM}{r^2} \hat{\mathbf{u}}_r, \quad \text{for } r > R \tag{10.51b}$$

where $\hat{\mathbf{u}}_r$ is the unit radial vector from the origin O. This result indicates that a uniform spherical shell acts as if the whole mass of the shell were concentrated at the center. A solid uniform spherical body may be assumed to consist of a large number of concentric shells. Each shell may be treated as if its mass is concentrated on the center; hence the mass of the whole sphere may be assumed to be at the center.

To calculate the force on a point mass m placed inside the shell, all we must do is change the lower limit $r - R$ to $R - r$ and the upper limit $r + R$ to $R + r$. Integrating Eq. (10.49) with appropriate limits is

$$F = \frac{GMm}{4r^2R}\left(s + \frac{R^2 - r^2}{s}\right)\Bigg|_{R-r}^{R+r} = 0$$

Thus $$F = 0 \quad \text{and} \quad g = 0, \qquad \text{for } r < R \tag{10.52}$$

It must be kept in mind that this result (for $r < R$) is true only for a spherical shell and not for a solid sphere.

Using the relation given by Eq. (10.16), that is,

$$\Delta U = -\int_{r_A}^{r_B} F(r)\,dr = -\int_{r_A}^{r_B} \mathbf{F}\cdot d\mathbf{r} \tag{10.16}$$

and Eqs. (10.51a) and (10.52), we can calculate the potential energy to be

$$U = -\int\left(-\frac{GMm}{r^2}\right)dr = -\frac{GMm}{r}, \qquad \text{for } r > R \tag{10.53}$$

and $$U = \text{constant} = C_1, \qquad \text{for } r < R$$

We can evaluate the constant by substituting $r = R$ in Eq. (10.53), that is,

$$U = -\frac{GMm}{R} = C_1, \qquad \text{for } r < R \tag{10.54}$$

while the gravitational potential $V(=U/m)$ is

$$V = -\frac{GM}{r}, \qquad \text{for } r > R \tag{10.55}$$

$$V = \text{constant} = C_2 = -\frac{GM}{R}, \qquad \text{for } r < R \tag{10.56}$$

The variation is $\mathbf{g}$ and V for this case is shown in Fig. 10.9.

We can obtain the preceding results by first calculating the potential energy $U(r)$ and then calculating $\mathbf{F}(\mathbf{r})$ from the relation $F = -dU/dr$, as shown next.

The potential energy of mass m at P due to the circular ring of mass dM given by Eq. (10.42) at a distance s (each point of the ring is at the same distance s) is (see Fig. 10.8)

$$dU = \frac{Gm\,dM}{s} = -\frac{GMm}{2}\frac{\sin\theta\,d\theta}{s} \tag{10.57}$$

while the total potential energy of m at P is

$$U(r) = -\frac{GMm}{2}\int_0^{\pi}\frac{\sin\theta\,d\theta}{s} \tag{10.58}$$

Figure 10.9

Below is the graph of the variation in g(r) and V(r) versus r in the case of a spherical shell.

$G = 6.672 \cdot 10^{-11}$ $M = 5.98 \cdot 10^{24}$ $R = 6.37 \cdot 10^{6}$

(Because of the large variation in the values of G, M, and R, divide them with appropriate numbers to make the graphs easier to interpret.)

$G := \frac{6.672 \cdot 10^{-11}}{10^{-11}}$ $M := 5.98 \cdot \frac{10^{24}}{10^{24}}$ $R := \frac{6.37 \cdot 10^{6}}{10^{6}}$

$N := 100$ $n := 1, 5 .. N$ $\theta_n := \pi \cdot \frac{n}{50}$

x and y are used for drawing a spherical surface or a circular surface in two dimensions.

$x_n := R \cdot \cos(\theta_n)$ $y_n := (R \cdot \sin(\theta_n))$ $r_n := \frac{n}{5}$

$V1_n := -\left(\frac{G \cdot M}{r_n}\right)$ $V2_n := -\frac{G \cdot M}{R} \cdot \frac{n}{n}$ $V_n := \mathrm{if}(r_n < R, V2_n, V1_n)$

The expressions give the values of g and V inside and outside the sphere.

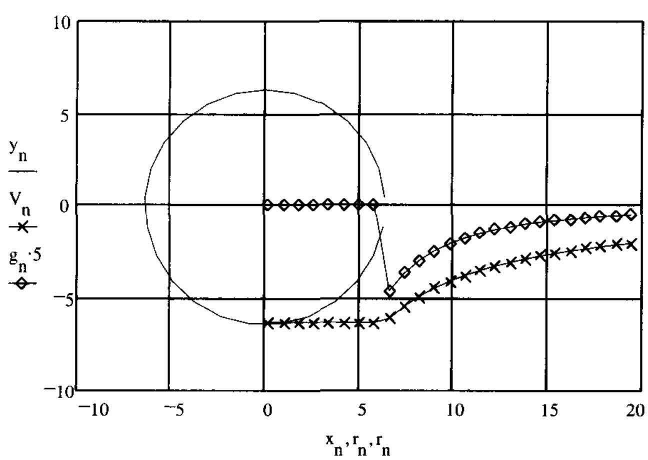
$g1_n := \frac{-G \cdot M}{(r_n)^2}$ $g2_n := 0 \cdot n$ $g_n := \mathrm{if}(r_n < R, g2_n, g1_n)$

min(V) = −6.264

min(g) = −0.916

max(V) = 0

max(g) = 0



(a) Explain the variations in the values of g and V for the values of r given above.

(b) Since max(V) = 0 and max(g) = 0, what do the variations in V and g mean?

From the triangle OPQ in Fig. 10.8, we obtain

$$s^2 = r^2 + R^2 - 2rR\cos\theta \tag{10.46}$$

Differentiating, while keeping in mind that r and R are constants, and rearranging, we get

$$\frac{\sin\theta\, d\theta}{s} = \frac{ds}{Rr}$$

Substituting in Eq. (10.58) yields

$$U(r) = -\frac{GMm}{2Rr}\int ds \tag{10.59}$$

The limits of integration will depend on the position of the point mass *m*, as discussed next.

Case (*i*) $r > R$: That is, the point mass *m* at *P* is outside the shell. As before, the limits $0 \rightarrow \pi$ change to $s_{min} = r - R \rightarrow s_{max} = r + R$. Thus

$$U(r) = -\frac{GMm}{2Rr}\int_{r-R}^{r+R} ds = -\frac{GMm}{2Rr}2R$$

gives

$$U(r) = -\frac{GMm}{r}, \qquad \text{for } r > R \tag{10.60}$$

That is, the potential energy varies as $1/r$, while

$$F = -\frac{dU}{dr} = -\frac{d}{dr}\left(-\frac{GMm}{r}\right)$$

gives

$$F = -\frac{GMm}{r^2}, \qquad \text{for } r > R \tag{10.61}$$

We may also write

$$V(r) = -\frac{GM}{r}, \qquad \text{for } r > R \tag{10.62}$$

and

$$g(r) = -\frac{GM}{r^2}, \qquad \text{for } r > R \tag{10.63}$$

Case (*ii*) $r < R$: That is, the point mass *m* at *P* is inside the shell. Hence the limits of integration $0 \rightarrow \pi$ change to $s_{min} = R - r \rightarrow s_{max} = R + r$. Thus

$$U(r) = -\frac{GMm}{2Rr}\int_{R-r}^{R+r} ds = -\frac{GMm}{2Rr}2r$$

gives

$$U(r) = -\frac{GMm}{R}, \qquad \text{for } r < R \tag{10.64}$$

That is, the potential inside the shell is constant, while

$$F = -\frac{dU}{dr} = -\frac{d}{dr}\left(-\frac{GMm}{R}\right) = 0$$

$$F = 0, \qquad \text{for } r < R \tag{10.65}$$

as expected. We may also write

$$V(r) = -\frac{GM}{R}, \qquad \text{for } r < R \tag{10.66}$$

$$g(r) = 0, \qquad \text{for } r < R \tag{10.67}$$

These results are graphed in Fig. 10.9, as already mentioned.

Solid Sphere

The results derived for a spherical shell may be extended to a solid sphere. The only requirement is that the distribution of matter, that is, the density, be spherically symmetric. Furthermore, the problem becomes simple if the density is uniform.

Case (*i*) $r > R$: That is, mass m is at r outside a solid sphere of mass M and radius R. The sphere may be divided into a large number of shells, each behaving as if the mass of the shell were concentrated at the center. Independent of the variation in density with radial distance (that is, symmetric but not necessarily uniform), as in the case of a shell, we obtain

$$F = -\frac{GMm}{r^2}, \qquad \text{for } r > R \tag{10.68}$$

$$g(r) = -\frac{GM}{r^2}, \qquad \text{for } r > R \tag{10.69}$$

$$U(r) = -\frac{GMm}{r}, \qquad \text{for } r > R \tag{10.70}$$

$$V(r) = -\frac{GM}{r}, \qquad \text{for } r > R \tag{10.71}$$

The graphs of $V(r)$ and $\mathbf{g}(r)$ are shown in Fig. 10.10.

Case (*ii*) $r < R$: That is, the mass m is inside a solid sphere of mass M. Once again we draw spherical shells. All the shells that are outside a sphere of radius r give zero contribution to the force, while the shells inside r contribute to the force. For convenience, let us assume that the density is uniform; that is, the sphere is homogeneous. The fraction of the mass contained within r is

$$\frac{(4\pi/3)r^3\rho}{(4\pi/3)R^3\rho} = \frac{r^3}{R^3}$$

(see Fig. 10.11) where ρ is the density of the material. Thus the mass concentrated at the center is Mr^3/R^3. Hence the force at r is given by

$$F(r) = -\frac{Gm}{r^2}\left(\frac{Mr^3}{r^3}\right) = -\frac{GMm}{r^3}r, \qquad \text{for } r < R \tag{10.72}$$

Figure 10.10

Below is the graph of V(r) and g(r) versus r due to a solid homogeneous sphere of radius R and mass M.

$G=6.672\cdot10^{-11}$ $M=5.98\cdot10^{24}$ $R=6.37\cdot10^{6}$

$G:=\frac{6.672\cdot10^{-11}}{10^{-11}}$ $M:=5.98\cdot\frac{10^{24}}{10^{24}}$ $R:=\frac{6.37\cdot10^{6}}{10^{6}}$

Before graphing, we divided the constants by appropriate powers of 10 to make the graphs easier to interpret.

$G=6.672$ $M=5.98$ $R=6.37$

$N:=100$ $n:=1..N$ $r_n:=\frac{n}{2}$ $\theta_n:=\pi\cdot\frac{n}{50}$

Graphing y versus x gives a circle.

$x_n:=R\cdot\cos(\theta_n)$ $y_n:=(R\cdot\sin(\theta_n))$

$V1_n:=-\left(\frac{G\cdot M}{r_n}\right)$ $V2_n:=-\frac{G\cdot M}{2\cdot R^3}\cdot\left[3\cdot R^2-(r_n)^2\right]$ $V_n:=\text{if}(r_n<R, V2_n, V1_n)$

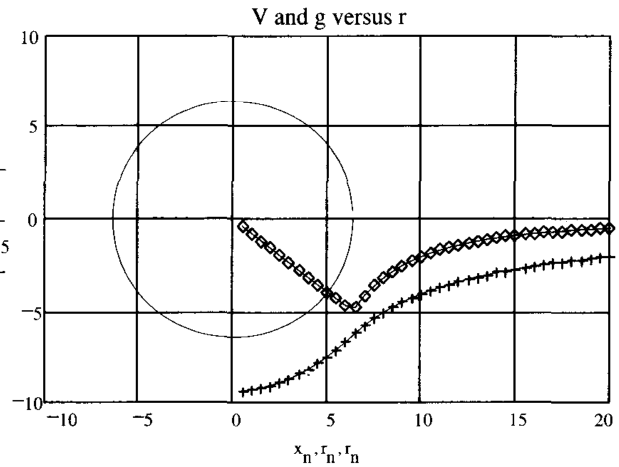
$g1_n:=\frac{-G\cdot M}{(r_n)^2}$ $g2_n:=\frac{-G\cdot M}{R^3}\cdot r_n$ $g_n:=\text{if}(r_n<R, g2_n, g1_n)$

min(V) = −9.376

min(g) = −0.944

max(V) = 0

max(g) = 0



(a) How do you think the plot for F will differ from these? Explain.

(b) Explain the variations in the values of g and V.

The potential energy $U(r)$ of the mass inside the sphere may be calculated by using Eq. (10.72). For $r < R$, we obtain

$$U(r) - U(R) = -\int_r^R F\,dr = -\int_r^R \frac{GMm}{R^3}\,r\,dr$$

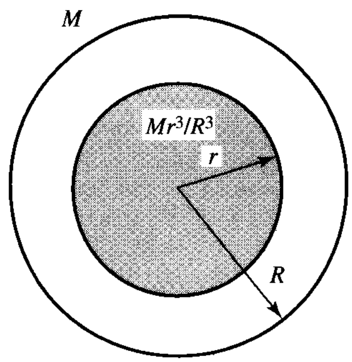


Figure 10.11 Fraction of mass of a sphere of radius r inside a homogeneous sphere of radius R and mass M.

that is,

$$U(r) - U(R) = -\frac{GMm}{2R^3}(R^2 - r^2) \tag{10.73}$$

But at $r = R$, from Eq. (10.70), we obtain

$$U(R) = -\frac{GMm}{R}$$

Substituting this in Eq. (10.73), we get

$$U(r) = -\frac{GMm}{2R^3}(3R^2 - r^2), \qquad \text{for } r < R \tag{10.74}$$

or

$$V(r) = -\frac{GM}{2R^3}(3R^2 - r^2), \qquad \text{for } r < R \tag{10.75}$$

We can calculate $U(r)$ and $V(r)$ at $r = 0$, that is, at the center

$$U(0) = -\frac{3GMm}{2R}, \qquad \text{at } r = 0 \tag{10.76}$$

and

$$V(0) = -\frac{3GM}{2R}, \qquad \text{at } r = 0 \tag{10.77}$$

The graphs of $g(r)$ and $V(r)$ for $r > R$ and $r < R$ are shown in Fig. 10.10.

Shell of Finite Thickness

Consider a shell of finite thickness of inner radius R_1 and outer radius R_2, as shown in Fig. 10.12. We want to calculate the potential at point P at a distance r from the center of the shell. By definition, the potential $V(r)$ at P is

$$V(r) = -G\int_{V'} \frac{\rho(R)}{s}\, dV' \tag{10.78}$$

where dV' is a small volume element at R:

$$dV' = (R \sin\theta\, d\phi)(R\, d\theta)\, dR = R^2 \sin\theta\, dR\, d\theta\, d\phi \tag{10.79}$$

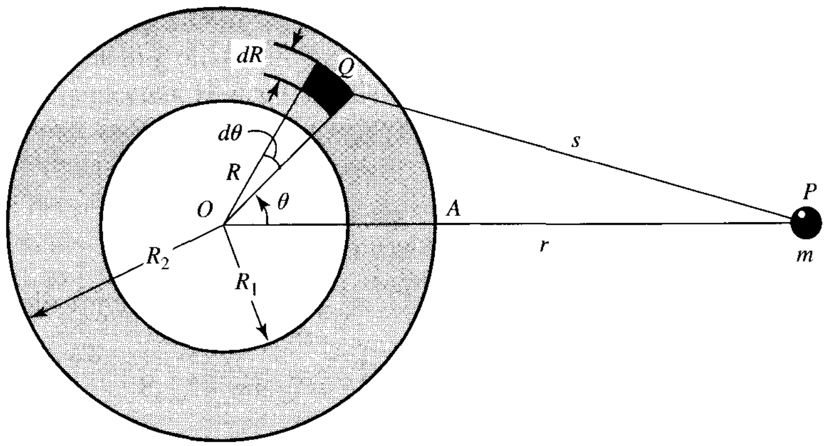


Figure 10.12 Potential and force on a point mass m at P due to a shell of finite thickness.

Because of the symmetry about the line connecting O with P, the azimuthal angle ϕ may be eliminated by integrating over $d\phi$; that is, $\int d\phi = 2\pi$. Also, $\rho(R) = \rho =$ constant for a homogeneous sphere. Thus

$$V(r) = -2\pi G\rho \int_{R_1}^{R_2} R^2\, dR \int_0^{\pi} \frac{\sin\theta}{s}\, d\theta \tag{10.80}$$

From triangle OPQ (Fig. 10.12), we obtain

$$s^2 = R^2 + r^2 - 2Rr\cos\theta \tag{10.81}$$

where R and r are constants; hence differentiating gives

$$2s\, ds = 2Rr \sin\theta\, d\theta$$

or

$$\frac{\sin\theta\, d\theta}{s} = \frac{ds}{Rr}$$

Substituting this in Eq. (10.80) yields

$$V(r) = -2\pi G\rho \int_{R_1}^{R_2} R^2\, dR \int_{s_{\min}}^{s_{\max}} \frac{ds}{Rr} \tag{10.82}$$

From Eq. (10.81), if $\theta = 0$, $s_{\min} = r - R$, and if $\theta = \pi$, $s_{\max} = r + R$. Therefore, for $r > R_2$,

$$V(r) = -\frac{2\pi G\rho}{r} \int_{R_1}^{R_2} R\, dR \int_{r-R}^{r+R} ds \tag{10.83}$$

$$= -\frac{4\pi G\rho}{r} \int_{R_1}^{R_2} R^2\, dR$$

That is,

$$V(r) = -\frac{4\pi}{3} \frac{G\rho}{r} (R_2^3 - R_1^3) \tag{10.84}$$

Since

$$M = \frac{4\pi}{3}\rho(R_2^3 - R_1^3) \tag{10.85}$$

we get the following expression for the gravitational potential outside the shell:

$$V(r) = -\frac{GM}{r}, \qquad \text{for } r > R_2 \tag{10.86}$$

Thus the gravitational potential at any point outside a shell or sphere with a spherically symmetric mass distribution is independent of the distribution. It behaves as if the whole mass were located at the center.

For a point inside the shell, changing the limits in Eq. (10.83) (as we did in a previous case), we get, for $r < R_1$,

$$V(r) = -\frac{2\pi G\rho}{r}\int_{R_1}^{R_2} R\,dR \int_{R-r}^{R+r} ds \tag{10.87}$$

$$= -4\pi G\rho \int_{R_1}^{R_2} R\,dR$$

which gives

$$V(r) = -2\pi G\rho(R_2^2 - R_1^2) = \text{constant}, \qquad \text{for } r < R_1 \tag{10.88}$$

Thus the potential inside ($r < R_1$) the shell is constant and is independent of the position.

The potential inside ($R_1 < r < R_2$) the shell is a little bit tricky to calculate. But an easy approach to this problem is to change the lower limit by replacing R_1 by r in Eq. (10.87) for $V(r)$ for $r < R_1$, and to change the upper limit by replacing R_2 by r in Eq. (10.83) for $V(r)$ for $r > R_2$. Thus, combining the two gives the potential inside the shell:

$$V(R_1 < r < R_2) = -2\pi G\rho(R_2^2 - r^2) - \frac{4\pi G\rho}{3r}(r^3 - R_1^3)$$

That is,

$$V(R_1 < r < R_2) = -4\pi G\rho\left(\frac{R_2^2}{2} - \frac{R_1^3}{3r} - \frac{r^2}{6}\right) \qquad \text{for } R_1 < r < R_2 \tag{10.89}$$

The field intensity vector **g** can be calculated from the relation $g = -dV/dr$ for each of the three regions by using Eqs. (10.86), (10.89), and (10.88). That is,

$$g(r) = -\frac{GM}{r^2}, r > R_2 \tag{10.90}$$

$$g(r) = \frac{4\pi G\rho}{3}\left(\frac{R_1^3}{r^2} - r\right) \qquad R_1 < r < R_2 \tag{10.91}$$

$$g(r) = 0, \qquad r < R_1 \tag{10.92}$$

The plots of $V(r)$ using Eqs. (10.86), (10.88), and (10.89) and of $g(r)$ using Eqs. (10.90), (10.91), and (10.92) are shown in Fig. 10.13. Let us make some important observations. The potential function $V(r)$ plotted in Fig. 10.13 is continuous across the points $r = R_1$ and $r = R_2$, and its gradient $dV(r)/dr$, which is force $g(r)$, is also continuous, as shown. If the potential function $V(r)$

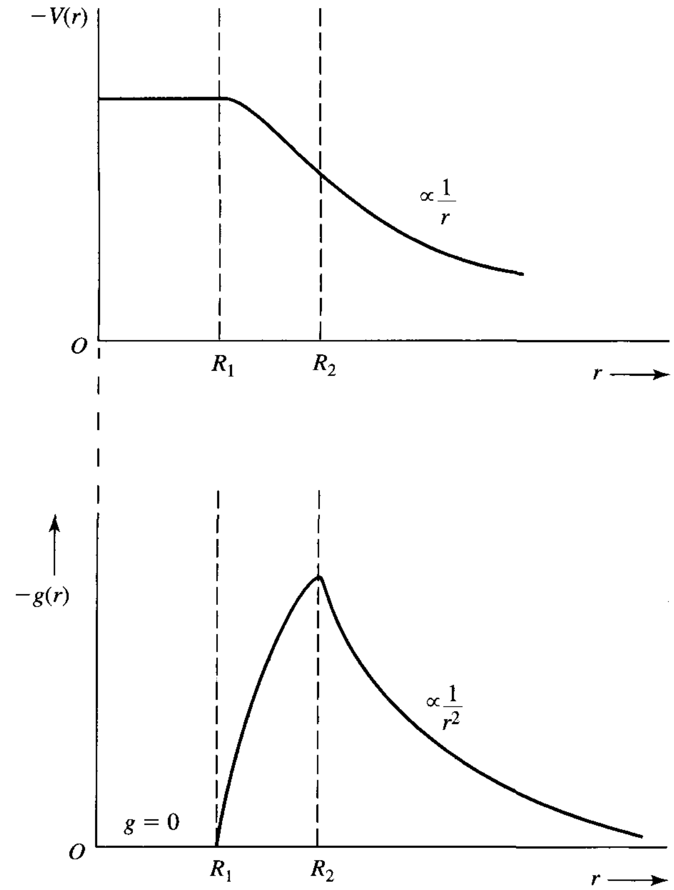


Figure 10.13 Variation of $V(r)$ and $g(r)$ versus r for a shell of finite thickness.

were not continuous, its derivative would be infinite. That is, the force would be infinite, which has no meaning. Thus the potential function must be continuous for the force to have any physical meaning. (Note that the derivative of the force function is not continuous.) That the potential function is continuous may be seen mathematically as follows. In Eq. (10.89), if we substitute $r = R_2$, we get the same result as from Eq. (10.84). Similarly in Eq. (10.89), if we substitute $r = R_1$, we get the same result as from Eq. (10.88).

Example 10.1

Consider a homogeneous circular disk of radius R, thickness t, and average density ρ (mass per unit volume). Calculate the gravitational potential and gravitational field intensity at a point outside the disk and on the axis of symmetry.

Solution

A circular disk of radius R is shown in Fig. Ex. 10.1(a). Its mass is given by

$$M = \rho V' = \rho(\pi R^2)t \tag{i}$$

where t is the thickness of the disk. Let us consider a ring of radius r and width dr, as shown in Fig. Ex. 10.1(a). As shown in Fig. Ex. 10.1(b), any small element of this ring is at the same distance s from point P. We shall calculate potential at point P due to this ring. The point P is at the same distance from all parts of the ring and lies on the axis of the disk. The mass of the ring is

$$dm = \rho\, dV' = \rho(2\pi r\, dr\, t) \tag{ii}$$

Hence the potential at point P due this ring is

$$dV = -\frac{G\, dm}{s} = -\frac{G2\pi\rho\, tr\, dr}{(z^2 + r^2)^{1/2}} \tag{iii}$$

Refer to two small elements of the ring at A and B in Fig. Ex. 10.1(b). The intensity at P due to these is given by $\mathbf{g}_1$ and $\mathbf{g}_2$ pointing along the lines PA and PB, respectively. When $\mathbf{g}_1$ and $\mathbf{g}_2$ are resolved, the horizontal components cancel. Because of the symmetry of the situation, all the horizontal components cancel and the vertical components add.

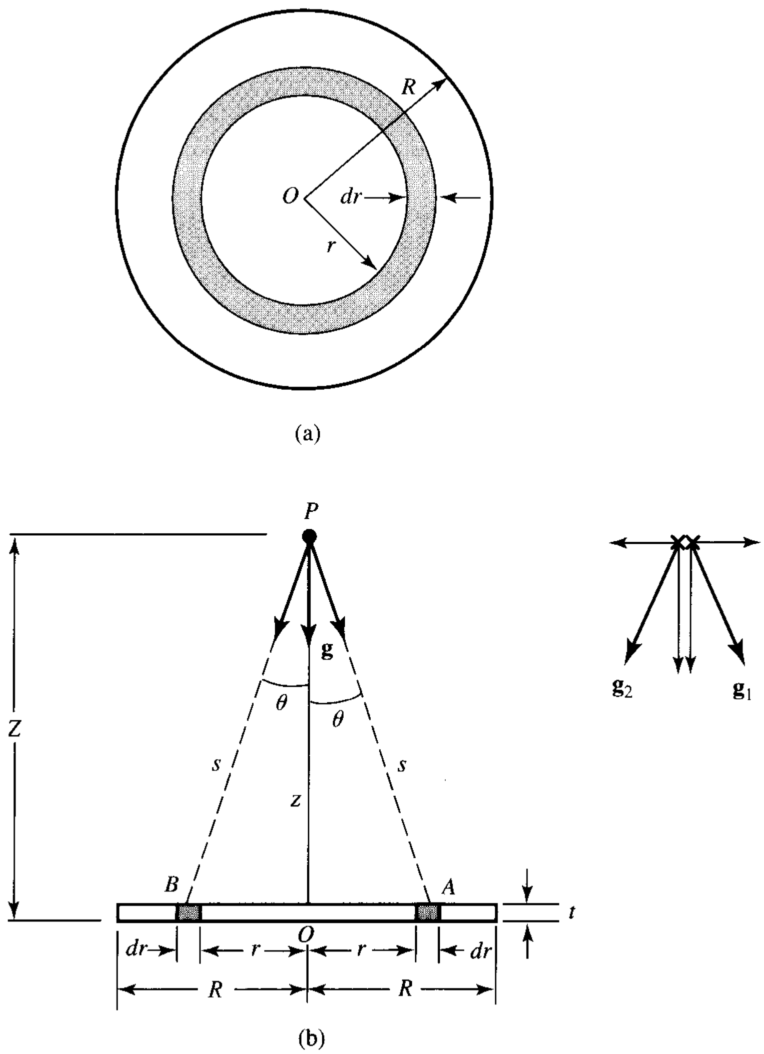


Figure Ex. 10.1

Using the value of dV from Eq. (iii), we can calculate V by integrating it as shown.

$$V = \int 1\, dV \qquad V = -2\cdot\pi\cdot\rho\cdot G\cdot t\cdot\int_0^R \frac{r}{\sqrt{z^2+r^2}}\,dr$$

$$V = 2\cdot\pi\cdot\rho\cdot G\cdot t\cdot\left(-\sqrt{z^2+R^2}+z\right)$$

To calculate g, we have two alternatives: direct integration or the definition g = dV/dz.

$$g = \frac{-G\cdot dm}{s^2} \qquad g = \int_0^R -\frac{2\cdot\pi\cdot\rho\cdot G\cdot t\cdot z\cdot r}{\left(z^2+r^2\right)^{\frac{3}{2}}}\,dr \qquad g = \frac{d}{dz}V \qquad g = -\left[\frac{d}{dz}2\cdot\pi\cdot\rho\cdot G\cdot t\cdot\left(-\sqrt{z^2+R^2}+z\right)\right]$$

$$g = 2\cdot\pi\cdot\rho\cdot G\cdot t\cdot\frac{\left(-\sqrt{z^2+R^2}+z\right)}{\sqrt{z^2+R^2}}$$

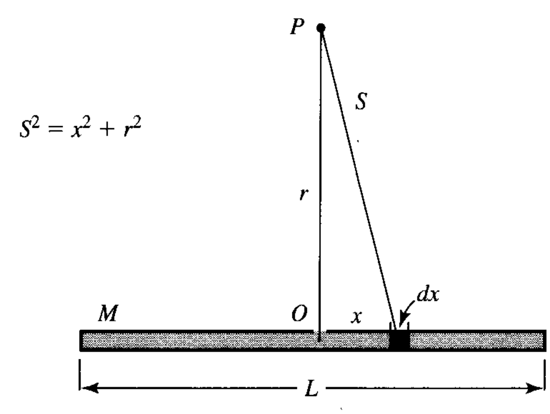
$$g = 2\cdot\pi\cdot\rho\cdot G\cdot t\cdot\frac{\left(-\sqrt{z^2+R^2}+z\right)}{\sqrt{z^2+R^2}} \qquad g = 2\cdot\pi\cdot\rho\cdot G\cdot t\cdot\left(-1+\frac{1}{\sqrt{z^2+R^2}}\cdot z\right)$$

How will the graphs of V and g versus z look?

EXERCISE 10.1 Repeat the example for the case of a circular ring of radius R and a linear mass density λ, but with the same mass as the disk. Compare the two results.

Example 10.2

Consider a thin rod of length L and mass M. Calculate the gravitational potential and gravitational field intensity at a point P that is at a distance r from the center of the rod and perpendicular to the rod as shown. Make a plot of F versus r.



Solution

We can apply Eq. (10.31) using M/L as linear density, and replacing the volume element dV by the length element dx and r by the perpendicular distance from the rod. The gravitational potential energy V is

$$V = \frac{-G\cdot M}{L}\cdot\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{1}{\sqrt{x^2+r^2}}\,dx$$

Given the value of V, we can calculate the value of g by differentiating V. Simplifying the expression for g gives the value of F (= mg).
The graph of F versus r is as shown.

$$V = G\cdot M\cdot\frac{\left(-\ln\left(L+\sqrt{L^2+4\cdot r^2}\right)+\ln\left(-L+\sqrt{L^2+4\cdot r^2}\right)\right)}{L}$$

$$g = -\frac{d}{dr}G\cdot M\cdot\frac{\left(-\ln\left(L+\sqrt{L^2+4\cdot r^2}\right)+\ln\left(-L+\sqrt{L^2+4\cdot r^2}\right)\right)}{L}$$

$$g = 8\cdot G\cdot M\cdot\frac{r}{\left[\sqrt{L^2+4\cdot r^2}\cdot\left[\left(L+\sqrt{L^2+4\cdot r^2}\right)\cdot\left(L-\sqrt{L^2+4\cdot r^2}\right)\right]\right]}$$

$i := 0 .. 10 \qquad r_i := i$

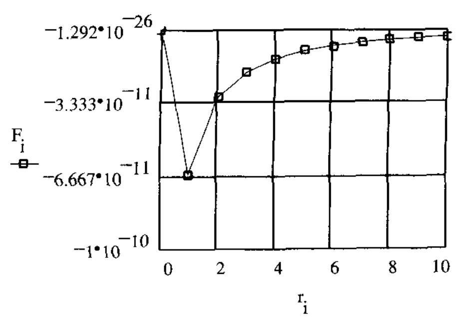
$G := 6.672\cdot 10^{-11} \qquad M := 5 \qquad m := 1 \qquad L := 10$

$$F_i := 8\cdot G\cdot M\cdot m\cdot\frac{r_i}{\left[-4\cdot\sqrt{L^2+4\cdot(r_i)^2}\cdot(r_i)^2\right]}$$

How do you explain the variation in the value of F for
(a) r = 0,
(b) r very small as compared to L and
(c) r very large as compared to L?

Graph V and g versus r.

F_i
0
$-6.542\cdot 10^{-11}$
$-3.097\cdot 10^{-11}$
$-1.907\cdot 10^{-11}$
$-1.302\cdot 10^{-11}$
$-9.436\cdot 10^{-12}$
$-7.119\cdot 10^{-12}$
$-5.54\cdot 10^{-12}$
$-4.42\cdot 10^{-12}$
$-3.6\cdot 10^{-12}$
$-2.984\cdot 10^{-12}$



Gravitational force field versus r

$\min(F) = -6.542\cdot 10^{-11} \qquad \max(F) = 0$

EXERCISE 10.2 Repeat the example for the case of a cylindrical rod of radius a, length L, and mass M.

10.6 GAUSS'S LAW

Gauss's law is used extensively in connection with electric fields in electrostatics. Actually, Gauss's law is applicable to any situation that involves the inverse-square force law; it is fair to say that Gauss's law is a compact form of the statement of the inverse-square law. Since gravitational force is an inverse-square force, let us apply Gauss's law and see its usefulness in calculating gravitational field intensity **g** in simple situations.

Let us consider a point mass M. The gravitational field **g** at a distance r from M is given by

$$\mathbf{g}(\mathbf{r}) = -\frac{GM}{r^2}\hat{\mathbf{u}}_r \quad \textbf{(10.93)}$$

Draw a sphere of radius r with point mass M at the center. We define the radially outward direction as positive. A quantity *flux* ϕ of the gravitational field **g** through the surface of the sphere is defined as, using Eq. (10.93),

$$\phi = 4\pi r^2 g_r = -4\pi GM \quad \textbf{(10.94)}$$

where g_r is the radial component of **g** and $4\pi r^2$ is the surface area of a sphere of radius r. We shall show that the total flux due to any mass is independent of the distance r.

Let us consider a mass M that is completely enclosed by an arbitrarily shaped surface, as shown in Fig. 10.14. Such an arbitrary surface is called a *Gaussian surface* (GS). Let us consider a point P on this surface where the outward normal to the surface makes an angle θ with **r** from M to P. Resolve **g** into two components, a radial (or normal) component and a transverse component (component parallel to the surface). It is only the normal component that contributes to the flux ϕ. Since the normal component of **g** is $-(GM/r^2)\cos\theta$, the flux $d\phi$ through an element of area dA is

$$d\phi = -\frac{GM}{r^2}\cos\theta\, dA = \mathbf{g}\cdot d\mathbf{A} = \hat{\mathbf{n}}\cdot\mathbf{g}\, dA \quad \textbf{(10.95)}$$

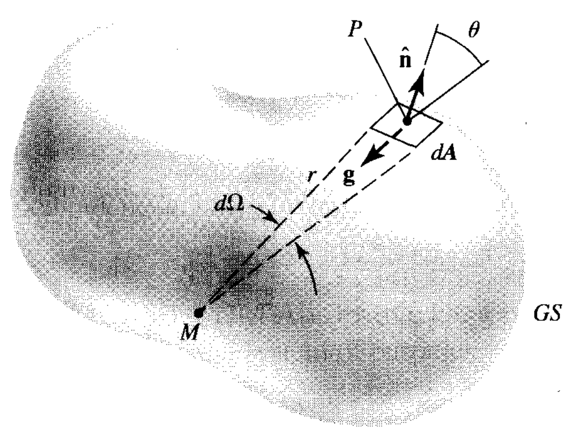


Figure 10.14 Mass M enclosed by a Gaussian surface (GS).

The projection of dA perpendicular to $\mathbf{r}$ is $dA \cos\theta = \hat{\mathbf{n}} \cdot d\mathbf{A}$, while $dA \cos\theta/r^2 = d\Omega$, where $d\Omega$ is the solid angle subtended by dA at M, as shown. Thus Eq. (10.95) may be written as

$$d\phi = -GM\, d\Omega \tag{10.96}$$

The total flux ϕ due to $\mathbf{g}$ is obtained by integrating Eq. (10.96) in which contributions from all the solid angle elements are taken into consideration. Remembering that the complete solid angle is 4π, we get

$$\phi = \int d\phi = -GM \int d\Omega = -4\pi GM \tag{10.97}$$

This is the same result we obtained using a spherical surface, as in Eq. (10.94).

From Fig. 10.14 and Eqs. (10.95) and (10.96), we may conclude that the flux $d\phi$ is a scalar product of $\mathbf{g}$ and $d\mathbf{A}$; that is,

$$d\phi = \mathbf{g} \cdot d\mathbf{A} = \hat{\mathbf{n}} \cdot \mathbf{g}\, dA \tag{10.98}$$

where $\hat{\mathbf{n}} \cdot d\mathbf{A} = dA \cos\theta =$ projection of area dA perpendicular to $\mathbf{g}$ or $\mathbf{r}$.

Let us extend our discussion to a large number of masses $M_1, M_2, M_3, \ldots$, inside an arbitrary closed surface. At any point P on this surface, the total gravitational field is

$$\mathbf{g} = \mathbf{g}_1 + \mathbf{g}_2 + \mathbf{g}_3 + \cdots \tag{10.99}$$

while the total gravitational flux through the enclosed surface is

$$\phi = \int \mathbf{g} \cdot d\mathbf{A} = \int \hat{\mathbf{n}} \cdot \mathbf{g}\, dA \tag{10.100}$$

Combining Eqs. (10.99) and (10.100),

$$\phi = \int \mathbf{g}_1 \cdot d\mathbf{A} + \int \mathbf{g}_2 \cdot d\mathbf{A} + \cdots = -4\pi G(M_1 + M_2 + \cdots)$$

If $M_{\text{total}} = M_1 + M_2 + \cdots$, then

$$\phi = -4\pi G M_{\text{total}} \tag{10.101}$$

Equation (10.101) is a statement of Gauss's law, and its validity is based on the fact that the force is an inverse-square law. Once the flux is calculated by using Eq. (10.101), we can use Eq. (10.100) in simple symmetrical situations to calculate $\mathbf{g}$, as illustrated in Example10.3.

Example 10.3

By using Gauss's law, calculate the gravitational field intensity at a distance x from an infinite plane sheet having a surface mass density σ.

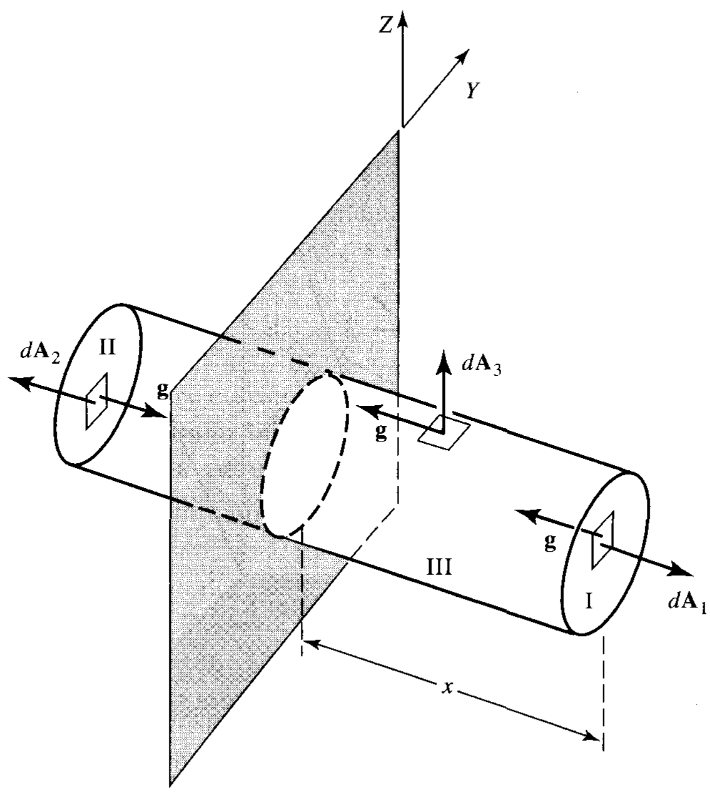


Figure Ex. 10.3

Solution

Draw a Gaussian surface in the form of a right circular cylinder, as shown in Fig. Ex. 10.3. The two end caps of the cylinder each have a surface area A. The total flux through the cylinder is due to the three surface areas (right, left, and curved) and is given by

$$\int \hat{\mathbf{n}} \cdot \mathbf{g}\, dA = \int \mathbf{g} \cdot d\mathbf{A} = \int_{\text{right}} \mathbf{g} \cdot d\mathbf{A} + \int_{\text{left}} \mathbf{g} \cdot d\mathbf{A} + \int_{\text{curved}} \mathbf{g} \cdot d\mathbf{A} \tag{i}$$

g will always be perpendicular to the sheet. One the curved surface of the cylinder, an element like $d\mathbf{A}_3$ (as shown in Fig. Ex. 10.3) will be perpendicular to **g**; hence there is no flux through the curved surface; that is,

$$\int_{\text{curved}} \mathbf{g} \cdot d\mathbf{A} = \int g\, dA \cos 90^\circ = 0 \tag{ii}$$

At the end caps, **g** is always antiparallel to the areas as shown; hence

$$\int_{\text{right}} \mathbf{g} \cdot d\mathbf{A} = \int g\, dA \cos 180^\circ = -gA \tag{iii}$$

$$\int_{\text{left}} \mathbf{g} \cdot d\mathbf{A} = \int g\, dA \cos 180^\circ = -gA \tag{iv}$$

Thus the total flux is

$$\Phi = \int_{\text{total}} \mathbf{g} \cdot d\mathbf{A} = -2gA \tag{v}$$

where A is the area of each cap, which is also equal to the area of the sheet enclosed.

The mass of the enclosed sheet is $M_{\text{total}} = \sigma A$. Hence the total flux according to Gauss's law is

$$\Phi = -4\pi G M_{\text{total}} = -4\pi G(\sigma A) \tag{vi}$$

Combining Eqs. (v) and (vi) gives

$$-2gA = -4\pi G(\sigma A)$$

or

$$g = 2\pi G\sigma \tag{vii}$$

Thus g is independent of the distance from the plane sheet; that is, it is the same everywhere and is directed toward and perpendicular to the sheet.

EXERCISE 10.3 By using Gauss's law, calculate the gravitational field intensity just outside a spherical shell having a mass M, radius R, and surface density σ.

10.7 GRAVITATIONAL FIELD EQUATIONS

We have briefly outlined the procedure for calculating **g** and ϕ by using Gauss's law for symmetrical mass distributions and also by the direct application of the inverse-square gravitational force law. A more general procedure of interest will be to find differential equations that are satisfied by the gravitational field intensity $\mathbf{g}(\mathbf{r})$ and the gravitational potential $V(\mathbf{r})$. Thus, from Eq. (10.29), we know the relation between **g** and V to be

$$\mathbf{g}(\mathbf{r}) = -\nabla V(\mathbf{r}) \tag{10.102}$$

Taking the curl on both sides, and noting that the curl of a gradient is zero, we get

$$\nabla \times \mathbf{g} = -\nabla \times \nabla V = 0 \tag{10.103}$$

That is,

$$\nabla \times \mathbf{g}(\mathbf{r}) = 0 \tag{10.104}$$

This vector equation is a set of three differential equations giving relations between the components (g_x, g_y, g_z) of **g**; that is,

$$\frac{\partial g_z}{\partial y} - \frac{\partial g_y}{\partial z} = 0, \qquad \frac{\partial g_x}{\partial z} - \frac{\partial g_z}{\partial x} = 0, \qquad \frac{\partial g_y}{\partial x} - \frac{\partial g_x}{\partial y} = 0 \tag{10.105}$$

These equations are satisfied by any gravitational field. For a unique determination of **g**, we need a relation between **g** and the distribution of mass. This can be achieved as follows.

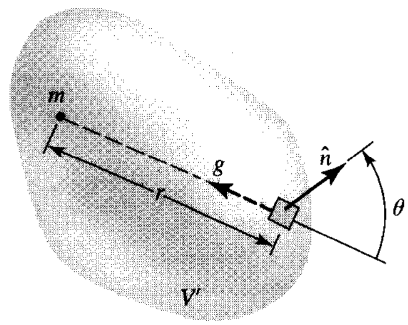


Figure 10.15 Mass m enclosed by a volume V' of surface area A.

As shown in Fig. 10.15, let us consider a mass m enclosed by a volume V' having a surface area A. Thus the flux ϕ through this area is

$$\phi = \iint_A \hat{\mathbf{n}} \cdot \mathbf{g}\, dA \tag{10.106}$$

But ϕ is also given by Eq. (10.97); that is,

$$\phi = -4\pi GM$$

or for a continuous mass distribution

$$\phi = -\iiint_{V'} 4\pi G\, dm = -\iiint_{V'} 4\pi G\rho\, dV' \tag{10.107}$$

where $dm = \rho\, dV'$, ρ being the mass density. Equating the preceding two equations gives

$$\iint_A \hat{\mathbf{n}} \cdot \mathbf{g}\, dA = -\iiint_{V'} 4\pi G\rho\, dV' \tag{10.108}$$

Gauss's divergence theorem applied to any vector $\mathbf{B}$ is (see Chapter 5)

$$\iint_A \hat{\mathbf{n}} \cdot \mathbf{B}\, dA = -\iiint_{V'} \nabla \cdot \mathbf{B}\, dV' \tag{10.109}$$

Applying this to the left side of Eq. (10.108), we obtain

$$\iint_A \hat{\mathbf{n}} \cdot \mathbf{g}\, dA = -\iiint_{V'} \nabla \cdot \mathbf{g}\, dV' \tag{10.110}$$

Substituting this in Eq. (10.108) gives

$$\iiint_{V'} \nabla \cdot \mathbf{g}\, dV' = -\iiint_{V'} 4\pi G\rho\, dV'$$

or

$$\iiint_{V'} (\nabla \cdot \mathbf{g} + 4\pi G\rho)\, dV' = 0 \tag{10.111}$$

Since this holds for any arbitrary volume V', we may write

$$\nabla \cdot \mathbf{g} + 4\pi G\rho = 0$$

or

$$\nabla \cdot \mathbf{g} = -4\pi G\rho \tag{10.112}$$

which is the required relation between $\mathbf{g}$ and the mass distribution described by the density $\rho(x, y, z)$. In Cartesian coordinates, Eq. (10.112) may be written as

$$\frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z} = -4\pi G\rho \tag{10.113}$$

Knowledge of $\rho(x, y, z)$, using Eqs. (10.105) and (10.113) and the boundary condition that $g \to 0$ as $r \to \infty$, will uniquely determine $\mathbf{g}$.

Substituting $\mathbf{g} = -\nabla V$ in Eq. (10.112) yields

$$\nabla \cdot \mathbf{g} = \nabla \cdot (-\nabla V) = -4\pi G\rho$$

or

$$\nabla^2 V = 4\pi G\rho \tag{10.114}$$

Rewriting in Cartesian coordinates gives

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 4\pi G\rho \tag{10.115}$$

This equation is called *Poisson's equation* and uniquely determines the value of $V(r)$ with the boundary conditions that $V(r) \to 0$ as $r \to \infty$. The general solution of Eq. (10.114) is

$$V(r) = \iiint \frac{G\rho(r)}{r}\, dV' \tag{10.116}$$

where $\rho(r)$ is the density of the volume element dV'. This is in agreement with the value of $V(r)$ given earlier by Eq. (10.31).

In short, Newton's theory of gravitation may be completely summarized by a set of three equations [(10.26), (10.29), (10.114)]; that is,

$$\mathbf{g}(\mathbf{r}) = \frac{\mathbf{F}}{m}, \qquad \mathbf{g} = -\nabla V, \quad \text{and} \quad \nabla^2 V = -4\pi G\rho \tag{10.117}$$

or, alternatively, Eqs. (10.26), (10.104), and (10.112); that is,

$$\mathbf{g}(r) = \frac{F}{m}, \qquad \nabla \times \mathbf{g} = 0, \quad \text{and} \quad \nabla \cdot \mathbf{g} = -4\pi G\rho \tag{10.118}$$

PROBLEMS

10.1. Starting with Kepler's laws of planetary motion and Newton's laws of motion, derive Newton's universal law of gravitation.

10.2. Consider a planet of mass M and radius R and with a uniform density ρ. A tunnel is bored through this planet to connect any two points on its surface. An object of mass m is thrown in this tunnel.

(a) Show that the motion of this object is simple harmonic and calculate the time period of this motion.

(b) Calculate the time period if the planet were Earth.

(c) Calculate the time period if the hole were drilled through the Moon.

10.3. In Fig. 10.3, if there are a large number of forces acting on M, the result of combining them should be a single force and a torque. Why is there no torque? (*Hint:* Take an arbitrary point P and show that the forces pass through it.)

10.4. An object of mass m released at a very large distance from Earth falls toward Earth's center. Calculate the time it will take **(a)** to reach halfway to Earth's center, and **(b)** to reach from halfway to Earth's center. Compare the two time intervals. Assume Earth to be a point mass.

10.5. Using the expression $U(r) = -GMm/r$ for $r > R$, show that as a mass m is moved from the surface of Earth to a height h the change in the potential energy is $\simeq mgh$.

10.6. A particle of mass m in a certain force field given by $F = -k/x^3$ is moving toward the center of the force. Calculate the time it will take the particle to move from a point at a distance D from the center to the center of the force.

10.7. Suppose an object is dropped from a height h ($h \ll R_E$, where R_E is the radius of Earth). Show that the speed with which it will hit the ground is

$$v = \sqrt{2gh}\left(1 - \frac{1}{2}\frac{h}{R_E}\right)$$

10.8. An object has a free fall in the gravitational field of Earth from infinity to Earth's surface, while another object falls from a height $h = R_E$ with constant acceleration g. Show that they both arrive at Earth's surface with the same speed.

10.9. Draw gravitational field lines and equipotential lines for a thin rod of finite length. What can you say about the equipotential surfaces of the rod?

10.10. The gravitational potential at any point P due to two masses M_1 and M_2 is given by (see Fig. P10.10)

$$V(r) = -\frac{GM_1}{r_1} - \frac{GM_2}{r_2}$$

Suppose $M_1 = nM_2$, where $n = 2$ or 3. Outline a method for drawing equipotential lines, and draw them for these two cases.

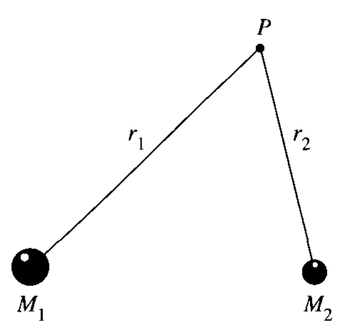


Figure P10.10

10.11. In Problem 10.10, if the masses are not point masses but are spheres of finite sizes, what changes will take place in the equipotential lines?

10.12. Draw lines of force and equipotential lines due to two masses M_1 and M_2 when **(a)** $M_1 = M_2$, and **(b)** $M_1 \gg M_2$.

10.13. Explain the steps necessary to arrive at Eq. (10.89).

10.14. Consider a uniform hemispherical shell of radius R and mass M with its center at $z = 0$. Let the Z-axis be its symmetry axis. Calculate the gravitational potential and field intensity at any point on the Z-axis. Graph $V(r)$ and $g(r)$. How do these compare with those due to a full (solid) shell?

10.15. Consider a uniform solid hemisphere of radius R and mass M with its center at $z = 0$. Let the Z-axis be its symmetry axis. Calculate the gravitational potential and field intensity at any point on the Z-axis and graph the results.

10.16. Consider a planet of radius R_1 and mass M that is surrounded by a cloud of mixed gases with an average density $\bar{\rho}$. Calculate the gravitational potential and the gravitational field intensity in regions I, II, and III (see Fig. P10.16). Graph $V(r)$ and $g(r)$ for different regions.

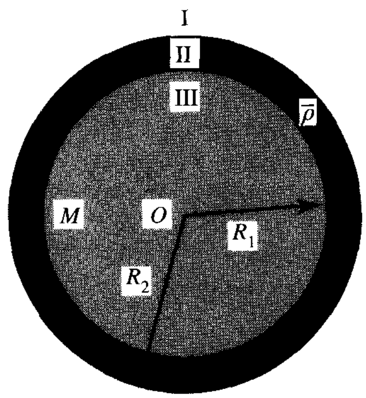


Figure P10.16

10.17. Consider a sphere of radius R having a variable density given by $\rho = \rho_0 e^{-ar}$ for $r < R$, where a is a constant. Calculate the gravitational potential and intensity at some point r inside and outside the sphere.

10.18. In a tunnel at a distance h below Earth's surface, the density of Earth's material is ρ_h. What will be the change in the time period of a clock pendulum, that is, $\Delta T/T$, at this depth? Will the clock run fast or slow? Calculate your answer in terms of h, M, R_E, and ρ_h. If we measure $\Delta T/T$ experimentally, can we determine Earth's mass, knowing the other variables?

10.19. Consider a thin cylindrical rod of length L, radius a, and mass M. Calculate gravitational potential and the gravitational field intensity at a distance r from the center of the rod and in a direction perpendicular to the rod. Graph the results.

10.20. Consider a thin rod of length L and mass M. Calculate the gravitational potential and gravitational field intensity at a point P that is at a distance r ($\gg L$) from the center of the rod and making an angle θ with the rod, as shown in Fig. P10.20. Calculate only to the second order in L/r. Graph the results and compare them with the results derived in the text.

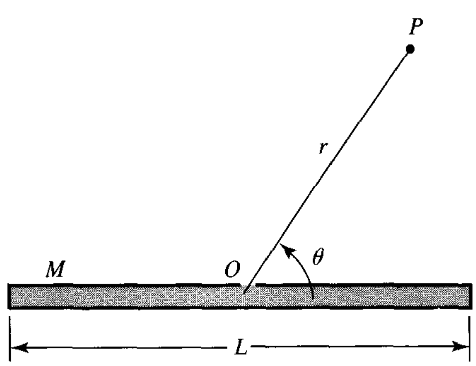


Figure P10.20

10.21. Calculate the gravitational potential and the gravitational field intensity due to a thin circular ring of mass M and radius R at a point in the plane of the ring. For large distances from the ring, expand the expression for the potential and find the first-order correction term. Draw the corresponding graphs.

10.22. Calculate the gravitation potential and the gravitational field intensity due to a thin circular ring of mass M and radius R for a point P on the Z-axis at right angles to the plane of the disk, as shown in Fig. P10.22. Assume $z \gg R$, and expand the expression for the potential, keeping terms only in the second order in R/z. Graph $V(z)$ and $g(z)$.

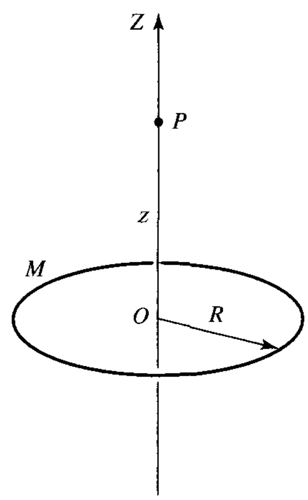


Figure P10.22

10.23. Calculate the gravitational potential and the gravitational field intensity due to a thin circular ring of mass M and radius R at a point P at a distance r from the center of the disk and making an angle θ with the Z-axis, as shown in Fig. P10.23. Assume that $r \gg R$, and expand the expression for the potential, keeping only the second-order terms in R/r. Graph $V(r)$ and $g(r)$.

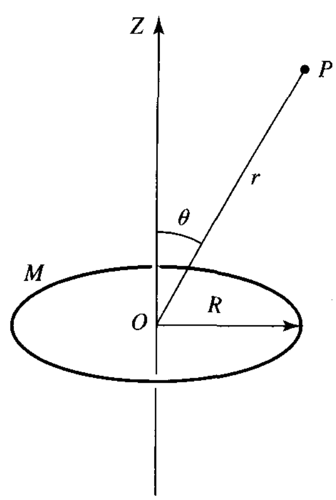


Figure P10.23

10.24. Calculate the gravitational potential and the gravitational field intensity due to a thin circular disk of mass M and radius R for a point in the plane of the ring. For large distances from the ring, expand the expression for the potential and find the first-order correction term. Graph $V(r)$ and $g(r)$.

10.25. Calculate the gravitational potential and the gravitational field intensity due to a thin circular disk of mass M and radius R for point P on the Z-axis at right angles to the plane of the disk. Assume $z \gg R$, and expand the expression for the potential, keeping terms only of the second order in R/z. Graph $V(z)$ and $g(z)$.

10.26. Calculate the gravitational potential and the gravitational field intensity due to a thin circular disk of mass M and radius R at a point P at a distance r from the center of the disk and making an angle θ with the Z-axis, similar to Fig. P10.23. Assume that $r \gg R$, and expand the expression for the potential, keeping only the second-order terms in R/r. Graph $V(r)$ and $g(r)$.

10.27. Consider a body that has a cylindrical symmetry with density $\rho(r, \theta)$ for $r < R$ and $\rho = 0$ for $r > R$. Calculate the gravitational potential at a point (r, θ) far away from the body. (Expand in powers of R/r.)

10.28. Consider a system of binary stars, each of mass M and separated by a distance $2r$. These stars orbit about their common center of mass. A mass m is located at a point P, as shown in Fig. P10.28.

(a) Calculate the gravitational potential and gravitational field intensity and force at point P.

(b) Repeat part (a) if $z \gg r$, and $z \ll r$.

(c) Suppose the mass m is at point O and then is slightly displaced. Show that it executes simple harmonic motion.

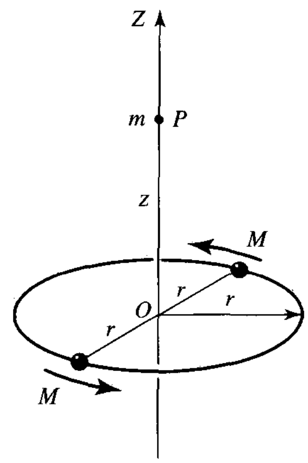


Figure P10.28

10.29. A and B are two thin concentric shells of radii R_1 and R_2 and masses M_1 and M_2, respectively. A point mass m is located at a distance r from O. Calculate the gravitational potential and field intensity in the three regions shown in Fig. P10.29. If a mass m is released from infinity, what will be its speed when it reaches O?

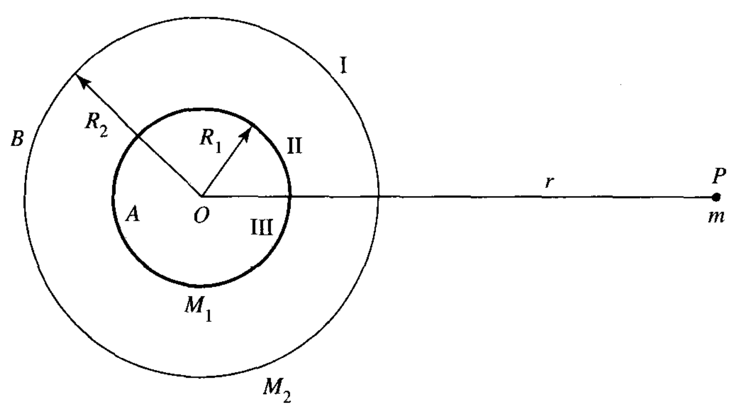


Figure P10.29

10.30. By applying Gauss's law, calculate the gravitational field and gravitational potential due to a homogenous sphere of radius R.

10.31. Calculate the gravitational field intensity and gravitational potential at a distance x from an infinite sheet of surface density σ in the XY plane.

10.32. A mass m is placed at a depth h in a tunnel in Earth. Show by using Gauss's law that the force exerted on this mass is due to the mass of the spherical portion below the tunnel. What will be the force on m at one-half the radius of Earth?

10.33. By using Gauss's law, calculate the gravitational field intensity inside and outside an infinitely long cylindrical shell of radius R and mass M.

10.34. By using Gauss's law, calculate the gravitational field intensity at a distance x from an infinitely long cylindrical rod of mass density σ per unit length.

10.35. Show that if we consider a mass M outside an enclosed surface, then the net flux through this surface is zero.

10.36. Show that the equations $\nabla \times \mathbf{g} = 0$, $\nabla \cdot \mathbf{g} = -4\pi G\rho$, and $\nabla^2 V = 4\pi G\rho$ are all satisfied by the gravitational field intensity and gravitational potential in Problems 10.30 and 10.31.

10.37. Show that **curl g** $= 0$.

SUGGESTIONS FOR FURTHER READING

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