CHAPTER 13

Rigid Body Motion: II

13.1 INTRODUCTION

We continue our discussion of rigid body motion started in Chapter 9. We briefly summarize the properties of rigid bodies as discussed there. A *rigid body* may be defined as a collection of discrete point particles for which the distance between any pair of particles is constrained to remain constant with time. Actually, these point particles are atoms and molecules that are always in constant vibrational motion. But these vibrations are on a microscopic scale and may be neglected. A perfectly rigid body will have no elastic deformation, and a mechanical pulse signal (a blow) will travel with infinite velocity. In actual practice, all rigid bodies have elastic properties and transmission velocities are $\simeq 10^3$ m/s. In most situations we shall ignore elastic deformation.

The motion of a rigid body can be described by using two coordinate systems, an inertial coordinate system and a body coordinate system, that is, a coordinate system fixed with respect to the body. Furthermore, to specify the position of the body, six coordinate must be specified. Three of these are usually taken to be the coordinates of the center of mass of the rigid body (usually the origin of the body coordinate system is taken to coincide with the center of mass), and the other three coordinates are taken to be the angles that describe the orientation of the body coordinate axes with respect to the inertial (or fixed) coordinate axes. One set of three commonly used independent angles are the Eulerian angles, as will be described in this chapter.

13.2 ANGULAR MOMENTUM AND KINETIC ENERGY

Let us consider a rigid body B as shown in Fig. 13.1. The body is rotating about an axis passing through a single fixed point O, while the coordinate system $OXYZ$ is fixed in the body with its

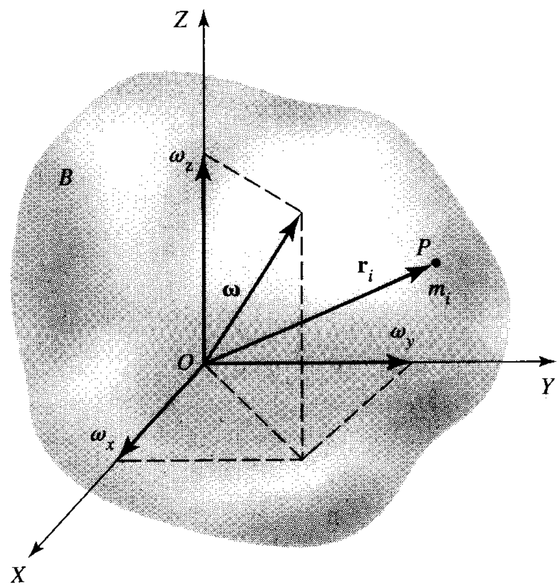


Figure 13.1 Rigid body B rotating with angular velocity $\boldsymbol{\omega}(\omega_x, \omega_y, \omega_z)$ about an axis passing through a single fixed point O.

origin at O. The instantaneous translational velocity $\mathbf{v}_i$ of particle P of mass m_i, which is at a distance $\mathbf{r}_i$ from the origin O, is

$$\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i \tag{13.1}$$

where $\boldsymbol{\omega}$ is the angular velocity of the body with its components $(\omega_x, \omega_y, \omega_z)$ as shown. The angular momentum $\mathbf{L}$ relative to origin O, for a system of particles m_i may be defined as

$$\mathbf{L} = \sum_{i=1}^{n} m_i \mathbf{r}_i \times \mathbf{v}_i \tag{13.2}$$

Substituting for $\mathbf{v}_i$ from Eq. (13.1),

$$\mathbf{L} = \sum_{i=1}^{n} m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) \tag{13.3}$$

Using the identity for a triple cross product,

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{A}) = A^2\mathbf{B} - \mathbf{A}(\mathbf{A} \cdot \mathbf{B}) \tag{13.4}$$

we may write

$$\begin{aligned} \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) &= r_i^2 \boldsymbol{\omega} - \mathbf{r}_i(\mathbf{r}_i \cdot \boldsymbol{\omega}) \\ &= (x_i^2 + y_i^2 + z_i^2)(\hat{\mathbf{i}}\omega_x + \hat{\mathbf{j}}\,\omega_y + \hat{\mathbf{k}}\,\omega_z) \\ &\quad - (\hat{\mathbf{i}}x_i + \hat{\mathbf{j}}\,y_i + \hat{\mathbf{k}}z_i)(x_i\omega_x + y_i\omega_y + z_i\omega_z) \end{aligned} \tag{13.5}$$

Combining this result with Eq. (13.3) and rearranging,

$$\mathbf{L} = \hat{\mathbf{i}}L_x + \hat{\mathbf{j}}L_y + \hat{\mathbf{k}}L_z$$

$$= \hat{\mathbf{i}}\left[\omega_x \sum_{i=1}^{n} m_i(y_i^2 + z_i^2) - \omega_y \sum_{i=1}^{n} m_i x_i y_i - \omega_z \sum_{i=1}^{n} m_i x_i z_i\right]$$

$$+ \hat{\mathbf{j}}\left[-\omega_x \sum_{i=1}^{n} m_i x_i y_i + \omega_y \sum_{i=1}^{n} m_i(x_i^2 + z_i^2) - \omega_z \sum_{i=1}^{n} m_i y_i z_i\right]$$

$$+ \hat{\mathbf{k}}\left[-\omega_x \sum_{i=1}^{n} m_i x_i z_i - \omega_y \sum_{i=1}^{n} m_i y_i z_i + \omega_z \sum_{i=1}^{n} m_i(x_i^2 + y_i^2)\right] \tag{13.6}$$

We may obtain the same result by using a matrix expansion

$$\mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x_i & y_i & z_i \\ (\omega_y z_i - \omega_z y_i) & (\omega_z x_i - \omega_x z_i) & (\omega_x y_i - \omega_y x_i) \end{vmatrix} \tag{13.7}$$

which on simplification and combining with Eq. (13.3) gives the same result as Eq. (13.6).

In short, we may write Eq. (13.6) as

$$\mathbf{L} = \hat{\mathbf{i}}L_x + \hat{\mathbf{j}}L_y + \hat{\mathbf{k}}L_z$$

$$= \hat{\mathbf{i}}[\omega_x I_{xx} - \omega_y I_{xy} - \omega_z I_{xz}] + \hat{\mathbf{j}}[-\omega_x I_{yx} + \omega_y I_{yy} - \omega_z I_{yz}]$$

$$+ \hat{\mathbf{k}}[-\omega_x I_{zx} - \omega_y I_{zy} + \omega_z I_{zz}] \tag{13.8}$$

where the quantities I_{xx}, I_{yy}, and I_{zz} involve the sums of the squares of the coordinates and are called the *moments of inertia* of the body about the coordinate axes; that is (the summation is taken from $i = 1$ to n),

$$I_{xx} = \sum m_i(y_i^2 + z_i^2) = \sum m_i(r_i^2 - x_i^2) = \text{moment of inertia about the } X\text{-axis}$$

$$I_{yy} = \sum m_i(x_i^2 + z_i^2) = \sum m_i(r_i^2 - y_i^2) = \text{moment of inertia about the } Y\text{-axis}$$

$$I_{zz} = \sum m_i(x_i^2 + y_i^2) = \sum m_i(r_i^2 - z_i^2) = \text{moment of inertia about the } X\text{-axis} \tag{13.9}$$

The quantities $I_{xy}, I_{xz}, \ldots,$ involve the sums of the products of the coordinates and are called the *products of inertia*; that is,

$$I_{xy} = I_{yx} = \sum m_i x_i y_i, \qquad xy \text{ product of inertia} \tag{13.10a}$$

$$I_{yz} = I_{zy} = \sum m_i y_i z_i, \qquad yz \text{ product of inertia} \tag{13.10b}$$

$$I_{zx} = I_{xz} = \sum m_i z_i x_i, \qquad zx \text{ product of inertia} \tag{13.10c}$$

It is clear from Eq. (13.8) that **L** is not necessarily always in the same direction as the instantaneous axis of rotation; that is **L** is not always in the same direction as $\boldsymbol{\omega}$. For example, if the Z-axis is the direction of rotation, $\boldsymbol{\omega} = (0, 0, \omega)$; that is, $\omega_x = \omega_y = 0$ and $\omega_z = \omega$, then from Eq. (13.8)

$$L_x = -I_{xz}\omega, \quad L_y = -I_{yz}\omega, \quad L_z = +I_{zz}\omega$$

That is, **L** has a component $L_z = I_{zz}\omega$ in the direction of rotation, but also has two other components in the directions at right angles to the direction of rotation. Thus **L** and $\boldsymbol{\omega}$ are not in the same direction. This point is further illustrated in Example 13.1.

The components of **L** given by Eq. (13.8) may be written in a compact form as

$$\mathbf{L}_k = \sum_{i=1}^{3} \omega_l I_{kl} \tag{13.11}$$

where $k = 1, 2, 3$ and $l = 1, 2, 3$; that is x, y, and z have been replaced by 1, 2, and 3.

Now we are in a position to derive a general expression for the rotational kinetic energy of a body. In simple cases, the axis of rotation always remains normal to a fixed plane. This need not be the case, as we demonstrate now. Let us calculate the kinetic energy of a rigid body that is rotating about an axis passing through a fixed point with an angular velocity $\boldsymbol{\omega}$. A particle of mass m_i at a distance $\mathbf{r}_i$ has a velocity $\mathbf{v}_i$.

$$\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i \tag{13.1}$$

Thus the kinetic energy of the whole body is given by

$$T = \sum_{i=1}^{n} \frac{1}{2} m_i v_i^2 = \sum_{i=1}^{n} \frac{1}{2} m_i \mathbf{v}_i \cdot \mathbf{v}_i = \frac{1}{2} \sum_{i=1}^{n} [(\boldsymbol{\omega} \times \mathbf{r}_i) \cdot (m_i \mathbf{v}_i)] \tag{13.12}$$

But in a triple scalar product, the dot and cross may be interchanged; that is,

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) \tag{13.13}$$

or

$$(\boldsymbol{\omega} \times \mathbf{r}_i) \cdot m_i \mathbf{v}_i = \boldsymbol{\omega} \cdot (\mathbf{r}_i \times m_i \mathbf{v}_i) \tag{13.14}$$

For kinetic energy T, we may write Eq. (13.12) as

$$T = \frac{1}{2} \sum_{i=1}^{n} \boldsymbol{\omega} \cdot (\mathbf{r}_i \times m_i \mathbf{v}_i) \tag{13.15a}$$

Since $\boldsymbol{\omega}$ is the same for all particles, and from the definition of angular momentum given by Eq. (13.2), we may write

$$T = \frac{1}{2} \boldsymbol{\omega} \cdot \left[\sum_{i=1}^{n} (\mathbf{r}_i \times m_i \mathbf{v}_i) \right] \tag{13.15b}$$

or

$$T = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L} \tag{13.16}$$

It may be pointed out that unlike $\mathbf{L}$, which is a vector and has three components, the rotational kinetic energy T is a scalar (a dot product of $\frac{1}{2}\boldsymbol{\omega}$ and $\mathbf{L}$). Also, this expression for T is analogous to the expression for the translational kinetic energy T_{tran} given by

$$T_{\text{tran}} = \tfrac{1}{2}\mathbf{v}_c \cdot \mathbf{p}_c \tag{13.17}$$

where $\mathbf{v}_c$ is the velocity of the center of mass and $\mathbf{p}_c$ is the linear momentum of the system. Using the expression

$$\boldsymbol{\omega} = \hat{\mathbf{i}}\,\omega_x + \hat{\mathbf{j}}\,\omega_y + \hat{\mathbf{k}}\,\omega_z \tag{13.18}$$

and Eq. (13.8) for $\mathbf{L}$ in Eq. (13.16), we may write

$$\begin{aligned} T &= \tfrac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L} = \tfrac{1}{2}\omega_x L_x + \tfrac{1}{2}\omega_y L_y + \tfrac{1}{2}\omega_z L_z \\ &= \tfrac{1}{2}\omega_x^2 I_{xx} + \tfrac{1}{2}\omega_y^2 I_{yy} + \tfrac{1}{2}\omega_z^2 I_{zz} - \omega_x\omega_y I_{xy} - \omega_y\omega_z I_{yz} - \omega_z\omega_x I_{zx} \end{aligned} \tag{13.19}$$

Instead of using (x, y, z), we may use $k = 1, 2, 3$ and $l = 1, 2, 3$ and write T in a compact form as

$$T = \frac{1}{2}\sum_{\substack{k=1 \\ l=1}}^{3} \omega_k \omega_l I_{kl} = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L} \tag{13.20}$$

In many practical situations, a rigid body consists of continuous mass with density ρ, which may not be constant. In such cases, summation must be replaced by volume integration. Thus the moment of inertia and the product of inertia may be written as

$$\begin{aligned} I_{xx} &= \int_V \rho(y^2 + z^2)\,dx\,dy\,dz \\ I_{yy} &= \int_V \rho(x^2 + z^2)\,dx\,dy\,dz \\ I_{zz} &= \int_V \rho(x^2 + y^2)\,dx\,dy\,dz \end{aligned} \tag{13.21a}$$

$$\begin{aligned} I_{xy} &= \int_V \rho xy\,dx\,dy\,dz \\ I_{yz} &= \int_V \rho yz\,dx\,dy\,dz \\ I_{zx} &= \int_V \rho zx\,dx\,dy\,dz \end{aligned} \tag{13.21b}$$

Example 13.1

Two point masses of equal mass m are connected by a massless rigid rod of length $2a$ forming a dumbbell. The dumbbell is constrained to rotate with a constant angular velocity $\boldsymbol{\omega}$ about an axis that makes an angle ϕ with the rod. Calculate the magnitudes and the directions of the angular momentum and the torque that is applied to the system.

Solution

As shown in Fig. Ex. 13.1(a), let the dumbbell rotate with an angular velocity $\boldsymbol{\omega}$ about an axis AOA' passing through O and lying in the inertial coordinate system. (AOA' is also the direction of the axle and the bearings are at O.) The point O is the origin of the coordinate system. The angular momentum of the system due to the two masses is

$$\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 = m\mathbf{r}_1 \times (\boldsymbol{\omega} \times \mathbf{r}_1) + m\mathbf{r}_2 \times (\boldsymbol{\omega} \times \mathbf{r}_2) \tag{i}$$

Note that both $\mathbf{L}_1$ and $\mathbf{L}_2$ point in the same direction as does $\mathbf{L}$, as shown in Fig. Ex. 13.1(a). It is quite clear that $\mathbf{L}$ is not in the same direction as $\boldsymbol{\omega}$. As shown in part (b), if $\mathbf{L}$ is resolved into two components, only $\mathbf{L}_{\parallel}$ is in the direction of $\boldsymbol{\omega}$, while $\mathbf{L}_{\perp}$, although in a plane at right angles to $\boldsymbol{\omega}$, is not zero. The magnitude of the angular momentum is

$$L = ma^2\omega \sin\phi + ma^2\omega \sin\phi = 2ma^2\omega \sin\phi = I\omega \sin\phi \tag{ii}$$

where I is the moment of inertia of the dumbbell about an axis perpendicular to the length of the connecting rod.

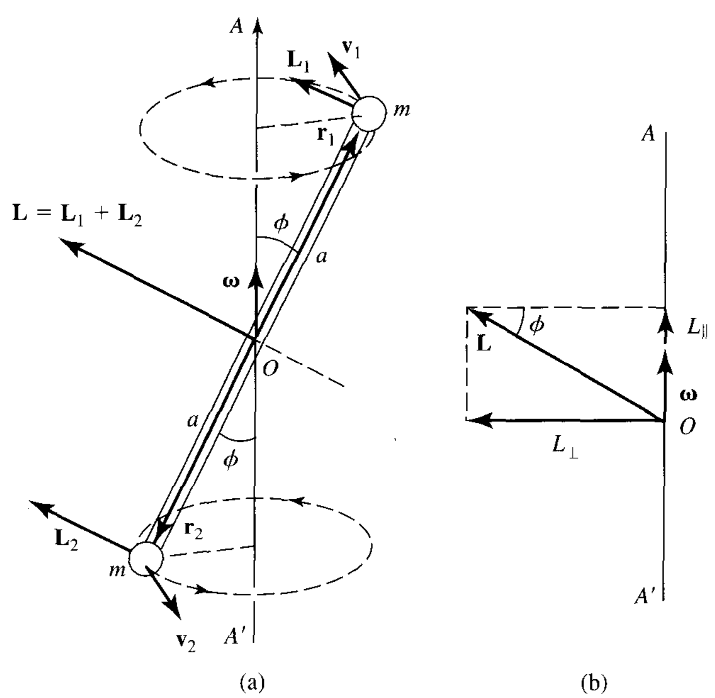


Figure Ex. 13.1

Furthermore, the angular momentum vector **L** is continuously changing direction as it rotates about **ω**. Thus **L** is not constant, and it is necessary to apply a torque **τ** to maintain this motion. By definition

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} = \dot{\mathbf{L}} \tag{iii}$$

where $\dot{\mathbf{L}}$ is a vector in the direction in which the tip (or head) of vector **L** is moving. In analogy to the relation $\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}$, we may write

$$\dot{\mathbf{L}} = \boldsymbol{\omega} \times \mathbf{L} \tag{iv}$$

Thus the magnitude of the applied torque is [substituting for **L** from Eq. (ii)]

$$|\boldsymbol{\tau}| = |\dot{\mathbf{L}}| = \omega L \sin(90^\circ - \phi) = 2ma^2\omega^2 \sin\phi \cos\phi \tag{v}$$

and the direction of the torque, from Eq. (iv), is perpendicular to the plane containing **ω** and **L** at any instant. If—rather than having one dumbbell as in Fig. Ex. 13.1(a)—there are two dumbbells moving symmetrically, by drawing a simple diagram we can show that **L** and **ω** will be in the same direction.

EXERCISE 13.1 Discuss the motion of the double dumbbells if the masses of the one dumbbell are different from those of the other (say twice).

13.3 INERTIA TENSOR

We proceed to write expressions for kinetic energy and angular momentum in tensor notation. Once again, we are considering a rigid body rotating about an axis passing through a fixed point located inside or outside the body. We shall use i, j for running indexes referring to the particles, while k, l, and s will be used to refer to the coordinate axes. The expression for rotational kinetic energy is

$$T = T_{\text{rot}} = \frac{1}{2}\sum_{i=1}^{n} m_i(\boldsymbol{\omega} \times \mathbf{r}_i)^2 \tag{13.22}$$

Making use of the vector identity

$$(\mathbf{A} \times \mathbf{B})^2 = (\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{B}) = A^2B^2 - (\mathbf{A} \cdot \mathbf{B})^2 \tag{13.23}$$

In Eq. (13.22), we may write

$$T = T_{\text{rot}} = \frac{1}{2}\sum_{i=1}^{n} m_i[\omega^2 r_i^2 - (\boldsymbol{\omega} \cdot \mathbf{r}_i)^2] \tag{13.24}$$

The vector $\mathbf{r}_i$ has components x_{is}, that is (x_{i1}, x_{i2}, x_{i3}), and **ω** has components $\omega_k(\omega_1, \omega_2, \omega_3)$. Thus

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i\left[\left(\sum_{k=1}^{3} \omega_k^2\right)\left(\sum_{s=1}^{3} x_{is}^2\right) - \left(\sum_{k=1}^{3} \omega_k x_{ik}\right)\left(\sum_{l=1}^{3} \omega_l x_{il}\right)\right] \tag{13.25}$$

Making use of the relation

$$\omega_k = \sum_{l} \omega_l \delta_{kl}$$

where [from Eq. (5.176)], we write $\delta_{kl} = 1$ if $k = l$ and $\delta_{kl} = 0$ if $k \neq l$, we may write Eq. (13.25) as

$$T = \frac{1}{2}\sum_i \sum_{k,l} m_i\left[\omega_k\, \omega_l\, \delta_{kl}\left(\sum_s x_{is}^2\right) - \omega_k\, \omega_l x_{ik} x_{il}\right]$$

Since all points in a rigid body have the same angular velocity, we may factor these out and write

$$T = \frac{1}{2}\sum_{k,l} \omega_k \omega_l \sum_i m_i\left[\delta_{kl}\sum_s x_{is}^2 - x_{ik}x_{il}\right] \tag{13.26}$$

If we define I_{kl} to be the klth element of the sum over i, that is,

$$I_{kl} = \sum_{i=1}^{n} m_i\left[\delta_{kl}\sum_s x_{is}^2 - x_{ik}x_{il}\right] \tag{13.27a}$$

or, noting that $x_{i1}^2 + x_{i2}^2 + x_{i3}^2 = r_i^2$, we may write

$$I_{kl} = \sum_{i=1}^{n} m_i\left[\delta_{kl} r_i^2 - x_{ik}x_{il}\right] \tag{13.27b}$$

Then Eq. (13.26) for rotational kinetic energy becomes

$$T = \frac{1}{2}\sum_{k,l} I_{kl}\,\omega_k\,\omega_l \tag{13.28}$$

I_{kl}, given by Eq. (13.27), has *nine* components and constitutes the elements of a quantity **I**, called the *moment of inertia tensor* or simply an *inertia tensor* of a rigid body relative to a body coordinate system. **I** is very similar in form to a 3 × 3 matrix and, as we shall see shortly, it is a factor of proportionality between L and ω and also between T and $\omega\omega$ (a quantity called *dyadic*, discussed in Section 13.7). The dimensions of **I** are (mass) × (length)^2. The elements of **I** can be obtained from Eq. (13.27) and may be written in a 3 × 3 array.

$$\mathbf{I} = \begin{pmatrix} \sum_i m_i(x_{i2}^2 + x_{i3}^2) & -\sum_i m_i x_{i1}x_{i2} & -\sum_i m_i x_{i1}x_{i3} \\ -\sum_i m_i x_{i2}x_{i1} & \sum_i m_i(x_{i1}^2 + x_{i3}^2) & -\sum_i m_i x_{i2}x_{i3} \\ -\sum_i m_i x_{i3}x_{i1} & -\sum_i m_i x_{i3}x_{i2} & \sum_i m_i(x_{i1}^2 + x_{i2}^2) \end{pmatrix} \tag{13.29}$$

which for a single point mass m reduces to

$$\mathbf{I} = m\begin{pmatrix} x_2^2 + x_3^2 & -x_1x_2 & -x_1x_3 \\ -x_2x_1 & x_1^2 + x_3^2 & -x_2x_3 \\ -x_3x_1 & -x_3x_2 & x_1^2 + x_2^2 \end{pmatrix} \tag{13.30}$$

or, in general,

$$\mathbf{I} = I_{kl} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \tag{13.31}$$

The diagonal elements I_{11}, I_{22}, and I_{33}, that is,

$$I_{kk} = \sum_{i=1}^{n} m_i(r_i^2 - x_{ik}^2) \tag{13.32}$$

are called the *moment of inertia* about the k-axis. The off-diagonal elements given by

$$I_{kl} = I_{lk} = -\sum_{i=1}^{n} m_i x_{ik} x_{il} \tag{13.33}$$

are called the *product of inertia.* Since the off-diagonal element are symmetric, $I_{kl} = I_{lk}$, the inertia tensor is a *symmetric tensor.* Hence only six elements of $\mathbf{I}$ are independent. Furthermore, the tensor $\mathbf{I}$ has a positive definite form.

Let us consider a particular element I_{11}; that is,

$$I_{11} = \sum_{i=1}^{n} m_i(r_i^2 - x_{i1}^2) = \sum_{i=1}^{n} m_i(x_{i1}^2 + x_{i2}^2 + x_{i3}^2 - x_{i1}^2) = \sum_{i=1}^{n} m_i(x_{i2}^2 + x_{i3}^2) \tag{13.34}$$

$(x_{i2}^2 + x_{i3}^2)$ is the square of the distance from the ith mass point to the X_1-axis; hence I_{11} is always positive or zero. In general, we may conclude that the diagonal elements I_{kk} are always positive or zero. I_{kk} is zero only if all the masses lie on the kth axis. On the other hand, the off-diagonal elements I_{kl} may be positive, negative, or zero.

Another property of the inertia tensor is the additive property of the elements. That is, the inertia tensor for a body can be considered to be the sum of the tensors for the various portions for the body. Thus, for a continuous distribution, we may write, using Eq. (13.27),

$$I_{kl} = \iiint_V \rho(\mathbf{r})\left[\delta_{kl}\sum_s x_s^2 - x_k x_l\right] dV = \iiint_V \rho(\mathbf{r})[\delta_{kl} r^2 - x_k x_l]\, dV \tag{13.35}$$

where the volume elements $dV = dx_1\, dx_2\, dx_3$, $\rho(\mathbf{r})$ is the density, and the integration is taken over the whole volume. Note that the indexes for the mass of the particles are not needed.

We may arrive at the same expression for the inertia tensor by starting with the expression for angular momentum. That is, by definition

$$\mathbf{L} = \sum_i m_i \mathbf{r}_i \times \mathbf{v}_i = \sum_i m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) \tag{13.36}$$

Using the vector identity

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{A}) = A^2\mathbf{B} - \mathbf{A}(\mathbf{A} \cdot \mathbf{B}) \tag{13.37}$$

we get

$$\mathbf{L} = \sum_{i=1}^{n} m_i[r_i^2 \boldsymbol{\omega} - \mathbf{r}_i(\mathbf{r}_i \cdot \boldsymbol{\omega})] \tag{13.38}$$

Unlike T, the angular momentum is a vector quantity, and hence for the kth component we may write

$$\begin{aligned} L_k &= \sum_i m_i\left[\omega_k \sum_s x_{is}^2 - x_{ik}\sum_l x_{il}\omega_l\right] = \sum_i m_i \sum_l \left[\omega_l \delta_{kl} \sum_s x_{is}^2 - \omega_l x_{ik} x_{il}\right] \\ &= \sum_l \omega_l \sum_i m_i\left[\delta_{kl}\sum_s x_{is}^2 - x_{ik}x_{il}\right] \end{aligned} \tag{13.39}$$

As before, I_{kl} is defined as

$$I_{kl} = \sum_{i=1}^{n} m_i \left[\delta_{kl} \sum_{s} x_{is}^2 - x_{ik} x_{il} \right] \tag{13.27a}$$

and we may write

$$L_k = \sum_{l} I_{kl} \omega_l \tag{13.40}$$

or in tensor notation

$$\mathbf{L} = \mathbf{I} \cdot \boldsymbol{\omega} \tag{13.41}$$

As mentioned earlier and shown in Example 13.1, **L** and **ω** are not in the same direction.

The relation between **L** and T may be arrived at in the following manner. Multiplying both sides of Eq. (13.40) by $\frac{1}{2}\omega_k$ and summing over k,

$$\frac{1}{2}\sum_{k} \omega_k L_k = \frac{1}{2}\sum_{k,l} I_{kl}\omega_k\omega_l = T$$

or

$$T = \frac{1}{2}\sum_{k} \omega_k L_k = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L} \tag{13.42}$$

Substituting for **L** from Eq. (13.41),

$$T = \tfrac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L} = \tfrac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I} \cdot \boldsymbol{\omega} \tag{13.43}$$

From Eq. (13.41), we may conclude that a product of a tensor and a vector is a vector; while from Eq. (13.43) we conclude that the product of two vectors and a tensor is a scalar.

Example 13.2

Calculate the components of a moment of inertia tensor for the following configuration. Point masses of 1,2,3, and 4 units are located at (1,0,0), (1,1,0), (1,1,1), and (1,1,–1).

Solution

n = 4, the number of point masses
m_i = mass of the ith particle
r_i = distance of the ith particle from the origin
x_{i1}, x_{i2}, and x_{i3} are the coordinates for the particles i = 1,2,3,4. All quantities are in arbitrary units. The masses and the coordinates of the particles are shown in the column matrices

$n := 4 \qquad i := 1..n \qquad k := 1..3 \qquad j := 1..3$

$m_i :=$	$x_{i,1} :=$	$x_{i,2} :=$	$x_{i,3} :=$
1	1	0	0
2	1	1	0
3	1	1	1
4	1	1	-1

Calculating distance r_i from the origin and the definition of $\delta_{k,j}$ function,

$$r_i := \sqrt{(x_{i,1})^2 + (x_{i,2})^2 + (x_{i,3})^2} \qquad \delta_{k,j} := \text{if}(k=j,1,0)$$

r_i	$(r_i)^2$
1	1
1.414	2
1.732	3
1.732	3

$$\delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Using Eq. (13.27b), we can calculate the moment of inertia tensor as shown.

$$I_{k,j} := \sum_{i=1}^{n} m_i \cdot \left[\delta_{k,j} \cdot (r_i)^2 - x_{i,k} \cdot x_{i,j}\right]$$

$$I = \begin{pmatrix} 16 & -9 & 1 \\ -9 & 17 & 1 \\ 1 & 1 & 19 \end{pmatrix} \qquad \mathbf{I} = \begin{pmatrix} 16 & -9 & 1 \\ -9 & 17 & 1 \\ 1 & 1 & 19 \end{pmatrix}$$

EXERCISE 13.2 Calculate **I** for point masses 4, 3, 2, and 1 units and located at (1,1,–1), (1,1,1), (1,1,0) and (1,0,0).

Example 13.3

Consider a homogeneous cube of density ρ, mass M, and side L. For origin O at one corner and an axis directed along the edges as shown in Fig. Ex. 13.3, evaluate the elements of the inertia tensor.

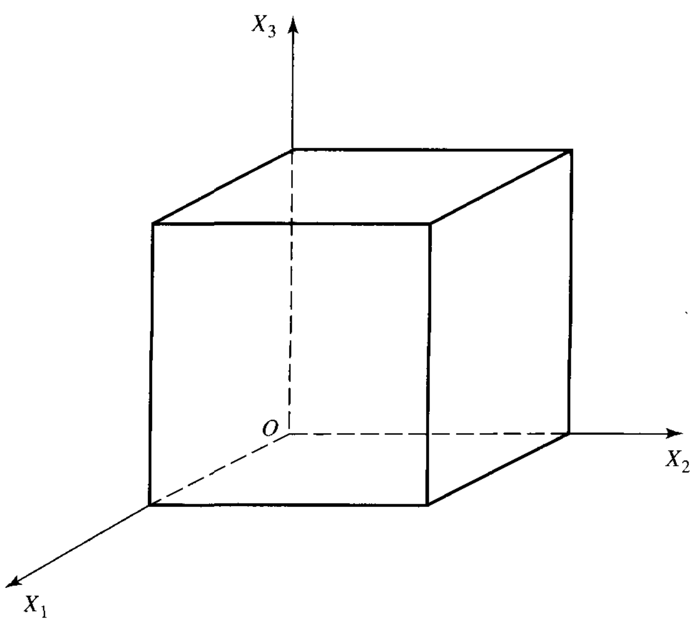


Figure Ex. 13.3

Solution

Calculate the elements of inertia tensor by using Eq. (13.35).

$$I_{k,l} = \int\int\int \rho(r)\cdot\left[\delta_{k,l}\cdot\sum_i (x_i)^2 - x_k\cdot x_l\right] dx\, dy\, dz$$

Since the cube is homogenous, ρ is constant.

$$\delta_{k,l} = \text{if}(k=l,1,0) \qquad M = \rho\cdot L^3$$

The diagonal elements of the inertia tensor are all equal and calculated as shown. (Integrate and then substitute for ρ.)

$$I11 = \rho\cdot\int_0^L\int_0^L\int_0^L (y^2+z^2)\, dz\, dy\, dx$$

$$I11 = \frac{2}{3}\cdot\rho\cdot L^5 = \frac{2}{3}\cdot M\cdot L^2$$

Because of symmetry, all the off-diagonal elements are equal and are calculated as shown.

$$I12 = -\rho\cdot\int_0^L\int_0^L\int_0^L x\cdot y\, dz\, dy\, dx$$

$$I12 = \frac{-1}{4}\cdot\rho\cdot L^5 = \frac{-1}{4}\cdot M\cdot L^2$$

All diagonal elements = I11

All off diagonal elements = I12

$$k := 1..3 \qquad l := 1..3 \qquad M := 1 \qquad L := 1 \qquad \rho := 1$$

$$m := 1..3 \qquad n := 1..3$$

Using different given values, we can calculate the moment of inertia tensor. Note that each element must be multiplied γ.

$$I11 := \frac{2}{3}\cdot M\cdot L^2 \qquad I12 := \frac{-1}{4}\cdot M\cdot L^2$$

$$I_{m,n} := \text{if}(m=n, I11, I12)$$

$$I = \begin{pmatrix} 0.667 & -0.25 & -0.25 \\ -0.25 & 0.667 & -0.25 \\ -0.25 & -0.25 & 0.667 \end{pmatrix} \qquad I = \begin{pmatrix} 0.667 & -0.25 & -0.25 \\ -0.25 & 0.667 & -0.25 \\ -0.25 & -0.25 & 0.667 \end{pmatrix}\cdot\gamma \qquad \gamma = M\cdot L^2$$

EXERCISE 13.3 Repeat the example for a rectangular body of homogeneous density ρ and sides $2a$, a, and a.

13.4 MOMENT OF INERTIA FOR DIFFERENT BODY SYSTEMS (STEINER THEOREM)

We have seen that if we choose a body coordinate system whose origin coincides with the center of mass, it is possible to express the kinetic energy as a sum of the translational and rotational kinetic energy. Hence, it is convenient to know the relationship between inertia tensors

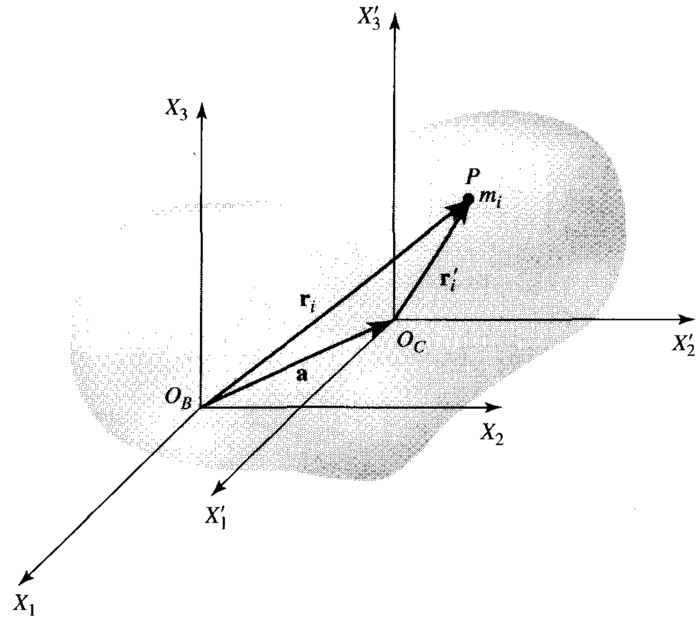


Figure 13.2 Body coordinate system with origin at O_B having its axes oriented parallel to the center-of-mass coordinate axes with origin at O_C.

expressed in different body coordinate systems. Let $\mathbf{I}$ be the inertia tensor defined in a body coordinate system with the origin fixed at O_B and $\mathbf{I}'$ be the inertia tensor defined in a center-of-mass coordinate system with its origin at the center-of-mass O_C, as shown in Fig. 13.2. Furthermore, it is assumed that the Cartesian coordinate axes in the two systems are parallel to each other as shown; that is, they have the same orientation. We wish to find the relation between $\mathbf{I}$ and $\mathbf{I}'$. The components I_{kl} of the inertia tensor $\mathbf{I}$, from Eq. (13.27a) are

$$I_{kl} = \sum_{i=1}^{n} m_i \left[\delta_{kl} \sum_s x_{is}^2 - x_{ik} x_{il} \right] \tag{13.27}$$

while the components I'_{kl} of the inertia tensor $\mathbf{I}'$ are

$$I'_{kl} = \sum_{i=1}^{n} m_i \left[\delta_{kl} \sum_s x_{is}'^2 - x'_{ik} x'_{il} \right] \tag{13.44}$$

Referring to Fig. 13.2, if the center of mass O_C is at a distance $\mathbf{a}$ from the origin O_B, the relation between $\mathbf{r}$ and $\mathbf{r}'$ is

$$\mathbf{r}_i = \mathbf{r}_i' + \mathbf{a} \tag{13.45a}$$

or, in component form,

$$x_{is} = x'_{is} + a_s, \quad s = 1, 2, 3 \tag{13.45b}$$

Substituting this in Eq. (13.27) and rearranging,

$$I_{kl} = \sum_i m_i \left[\delta_{kl} \sum_s (x'_{is} + a_s)^2 - (x'_{ik} + a_k)(x'_{il} + a_l) \right]$$

$$= \sum_i m_i \left[\delta_{kl} \sum_s x'^{\,2}_{is} - x'_{ik} x'_{il} \right] + \sum_i m_i \left[\delta_{ki} \sum_s a_s^2 - a_k a_l \right]$$

$$+ 2 \sum_i m_i [\delta_{kl} x'^{\,2}_{is} a_s] - \sum_i m_i x'_{ik} a_l - \sum_i m_i x'_{il} a_k \tag{13.46}$$

Each of the last three terms on the right side is zero because of the definition of the center of mass with the origin at O_C. That is,

$$\sum_i m_i \mathbf{r}'_i = 0 \quad \text{or} \quad \sum_i m_i x'_{is} = 0$$

Thus Eq. (13.46) with the help of Eq. (13.44) takes the form

$$I_{kl} = I'_{kl} + \sum_i m_i \left[\delta_{kl} \sum_s a_s^2 - a_k a_l \right] \tag{13.47}$$

If, instead of discrete particles, we had an extended rigid body, we would obtain the following relation

$$I_{kl} = I'_{kl} + (\delta_{kl} a_s^2 - a_k a_l) \int\int\int \rho \, dV \tag{13.48}$$

In either situation, the mass M is given by

$$M = \sum_i m_i \quad \text{or} \quad M = \int\int_v\int \rho \, dV$$

and Eq. (13.47) or (13.48) takes the form

$$I_{kl} = I'_{kl} + M(a^2 \delta_{kl} - a_k a_l) \tag{13.49}$$

which is the required relation. It states that the difference in the elements $I_{kl} - I'_{kl}$ is equal to the mass M of the body multiplied by the square of the distance $(a^2 \delta_{kl} - a_k a_l)$.

As a special case, let us find the relation between the diagonal elements; that is,

$$I_{kk} = I'_{kk} + M(a^2 - a_k^2) = I'_{kk} + M d_k^2 \tag{13.50a}$$

where d_k is the shortest distance from the axis of rotation in the body system to the center of mass. The relation of Eq. (13.50a) is the statement of *Steiner's theorem.*

The moment of inertia of a rigid body in a body coordinate system about a given axis is equal to the moment of inertia of the body in the center-of-mass coordinate system about *an axis parallel to the given axis plus the moment of inertia of M located at the center of mass about the given body axis.*

Note that if O_B and O_C coincide, $d_k = 0$, which implies that the body will have a minimum moment of inertia in the center-of-mass coordinate system.

Let us consider a relation between the diagonal elements.

$$I_{11} = I'_{11} + M(a_1^2 + a_2^2 + a_3^2 - a_1^2) = I'_{11} + M(a_2^2 + a_3^2) = I'_{11} + Md_1^2 \qquad \textbf{(13.50b)}$$

where $d_1^2 = a_2^2 + a_3^2$. Equation (13.50b) states that the difference between elements $I_{11} - I'_{11}$ is equal to the mass M of the body multiplied by the square of the distance between the parallel axes. This is the special case of Steiner's theorem and is called the *parallel axes theorem.*

Perpendicular Axis Theorem

As explained earlier, a *plane lamina* is a rigid body whose mass is distributed in a single plane: that is, it has almost zero thickness. Suppose this plane lamina lies in the $X_1 - X_2$ plane; hence $x_3 = 0$. Let σ be the mass per unit area of this body. Let dA be a small area element. In such situations, the diagonal elements of the inertia tensor **I** of a plane lamina are

$$I_{11} = \iint_A \sigma x_2^2 \, dA$$

$$I_{22} = \iint_A \sigma x_1^2 \, dA$$

$$I_{33} = \iint_A \sigma(x_1^2 + x_2^2) \, dA \qquad \textbf{(13.51)}$$

From these relations, we may conclude

$$I_{33} = I_{11} + I_{22} \qquad \textbf{(13.52)}$$

which is the statement of the *perpendicular axis theorem*:

If for a certain rigid body in the form of a plane lamina the moments of inertia about the X_1 and X_2 axes are I_{11} and I_{22}, the moment of inertia about the X_3 axis is equal to $I_{11} + I_{22}$.

Example 13.4

Consider the homogenous cube of density ρ, mass M, and side L discussed in Example 13.3. For a coordinate system with its origin at the center of mass of the cube as shown in Fig. Ex. 13.4, evaluate the elements of the inertia tensor.

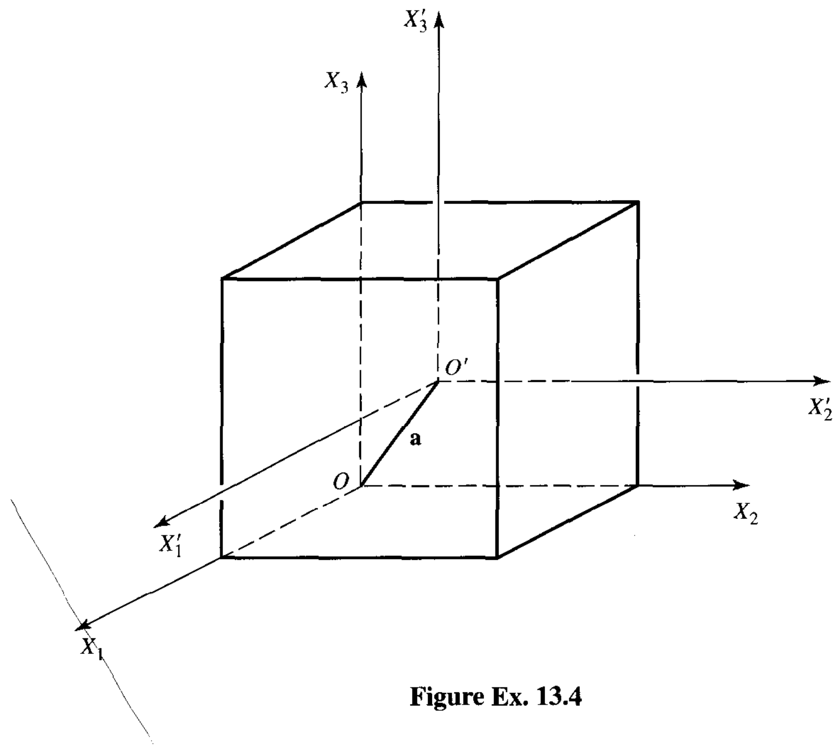


Figure Ex. 13.4

Solution

To transform the elements of an inertia tensor from one coordinate system to another, we make use of the relation given by Eq. (13.49).

$$I_{k,l} \equiv (I')_{k,l} + M\left(a^2 \cdot \delta_{k,l} - a_k \cdot a_l\right)$$

$I_{k,l}$ are the elements of the inertia tensor I evaluated in Example 13.3. Thus

$$I_{1,1} \equiv I_{2,2} \equiv I_{3,3} \equiv \frac{2}{3} \cdot M \cdot L^2$$

$$I_{1,2} \equiv I_{1,3} \equiv I_{2,3} \equiv I_{3,1} \equiv I_{2,1} \equiv I_{3,2} \equiv \frac{-1}{4} \cdot M \cdot L^2$$

$$I := \begin{bmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{bmatrix} \qquad I_{k,l} \equiv I \cdot \gamma \qquad \gamma \equiv M \cdot L^2$$

The center of mass of the cube is at (L/2, L/2, L/2) in the X-system and the components of the vector **a** are

$$a_1 \equiv a_2 \equiv a_3 \equiv \frac{L}{2}$$

$$k := 0..2 \qquad l := 0..2 \qquad n := 0..2 \qquad M := 1$$

Assuming L = 1 and M =1, we can assign values to I and a as shown.

$$I := \begin{bmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{bmatrix} \qquad a_k := \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \qquad a_l = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} \qquad a_l := a_k$$

$$|a| = 0.866025$$

Using the values of the inertia tensor I, we can calculate the components of It.

$\delta_{k,l} := \text{if}(k=l, 1, 0)$

$It_{k,l} := M \cdot \left[\left| (a_k)^2 + (a_l)^2 \right| \cdot \delta_{k,l} - a_k \cdot a_l \right]$

$$It = \begin{pmatrix} 0.25 & -0.25 & -0.25 \\ -0.25 & 0.25 & -0.25 \\ -0.25 & -0.25 & 0.25 \end{pmatrix}$$

The transfer matrix It' components may be calculated as shown. Note that the diagonal elements are the only elements that are not zero.

$It' := I - It$

$$It' = \begin{pmatrix} 0.416667 & 0 & 0 \\ 0 & 0.416667 & 0 \\ 0 & 0 & 0.416667 \end{pmatrix}$$

Thus the transfer matrix It' may be written by using the matrix Iu as shown.

$It' = \frac{1}{6} \cdot M \cdot L^2 \cdot Iu \qquad Iu := \frac{It'}{0.417} \qquad Iu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

EXERCISE 13.4 Repeat the calculations for the case discussed in Exercise 13.3.

13.5 PRINCIPAL MOMENT OF INERTIA AND PRINCIPAL AXES

We have described the inertia tensor of a rigid body with respect to a set of coordinate systems with the origin fixed in the body. A particular set of coordinate axes can be chosen such that the product of inertia elements will be zero in such a set. A set of axes possessing this property is called the *principal axes*. We shall find such a set of axes very useful in many situations in order to understand the description of motion of a rigid body.

Three mutually orthogonal coordinate axes meeting at a point O are said to form a set of *principal axes* provided all the product of inertia elements I_{xy}, I_{yz}, and I_{zx} of the rigid body are zero as expressed in terms of these axes. The point O, the origin of these principal axes, is called the *principal point.* Three coordinate planes, each of which passes through two principal axes, are called *principal planes* at point O.

If the product of inertia elements is zero, then the inertia tensor consists only of diagonal elements; that is,

$$\mathbf{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \tag{13.53}$$

or in compact form we may write

$$I_{kl} = I_k \delta_{kl} \tag{13.54}$$

Furthermore, the use of the principal axes leads to a considerable simplification in the expression for L and T. Thus

$$L_k = \sum_l I_{kl}\omega_l = \sum_l I_k \delta_{kl}\omega_l = I_k \omega_k \tag{13.55}$$

$$T = \frac{1}{2}\sum_{k,l} I_{kl}\,\omega_k\,\omega_l = \frac{1}{2}\sum_{k,l} I_k\delta_{kl}\,\omega_k\,\omega_l = \frac{1}{2}\sum_{k} I_k\,\omega_k^2 \tag{13.56}$$

Before proceeding to understand the mathematical procedure for finding the principal axes so that the resulting moment of inertia will be diagonal, we shall present a physical description of the process and some particular situations of common interest.

For some situations in rigid body dynamics, the principal axes may be determined by examining the symmetry of the body. For example, consider a plane laminar body in the XY plane so that $z = 0$ for every particle. Thus

$$I_{yz} = I_{zx} = 0$$

Furthermore, suppose the lamina has an axis of symmetry, say the X-axis as shown in Fig. 13.3, such that the xy product in $\iiint \rho xy\, dV$ consists of two parts of equal magnitudes but of opposite sign. This results in $I_{xy} = 0$. Thus the inertia tensor is diagonal and the three coordinate axes in this case are the principal axes for a laminar rigid body. This also leads to the fact (using the definition of the product of inertia) that the coordinate axes will be the principal axes if the coordinate planes are planes of symmetry. (A note of caution: A body does not have to be symmetrical for its product of inertia elements to be zero.) But symmetry of a rigid body is helpful in determining the principal axes by inspection. For example, a cylindrical rod (which is a solid of revolution) has one principal axis along the symmetry axis, say the Z-axis through the center line of the cylindrical rod, and the two other principal axes are in a plane perpendicular to the symmetry axis, as shown in Fig. 13.4. The placement of these two principal axes in the XY plane is arbitrary.

Let us consider the relation between $\mathbf{L}$ and $\boldsymbol{\omega}$ for a rigid body when the coordinate axes are the principal axes. In such situations, $\mathbf{L}$ takes the form

$$\mathbf{L} = \hat{\mathbf{i}} I_x\omega_x + \hat{\mathbf{j}} I_y\omega_y + \hat{\mathbf{k}} I_z\omega_z \tag{13.57}$$

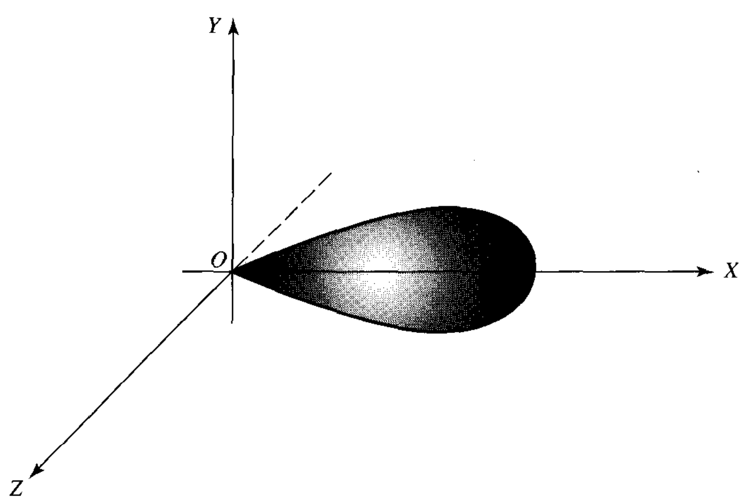


Figure 13.3 Plane laminar body in the XY-plane with the axis of symmetry along the X-axis, which has elements of the product of inertia to be zero.

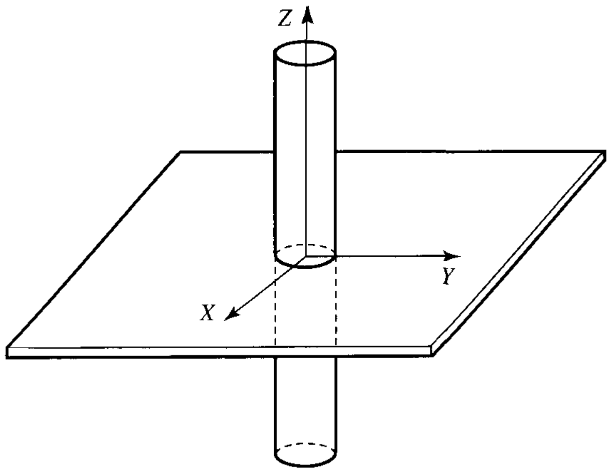


Figure 13.4 Cylindrical rod with one principal axis along the Z-axis and the other two principal axes in a plane perpendicular to the Z-axis.

where I_x, I_y, and I_z are the principal moments of inertia. Let the body rotate about the Z-axis such that $\omega_z = \omega$, while $\omega_x = \omega_y = 0$. Thus Eq. (13.57) takes the form

$$\mathbf{L} = \hat{\mathbf{k}} I_z \omega_z \tag{13.58}$$

which states that the angular momentum is parallel to the axis of rotation; that is, **L** is in the same direction as **ω**. Thus we may conclude: If **L** and **ω** are in the same direction (that is, the direction of rotation), then the axis of rotation is the principal axis. **L** and **ω** will be in different directions if the rotation axis is not the principal axis. An important application of this principal is described next.

Dynamic Balancing

Consider a rotating device such as a fan blade or flywheel. This device will be *statically balanced* if the center of mass lies on the axis of rotation. If the device is *dynamically balanced*, the axis of rotation must be the principal axis; hence **L** and **ω** will lie along this axis of rotation. If the rotational axis is not a principal axis, the angular momentum varies in direction, as shown in Fig. 13.5. Such variations require that there must be a torque acting on the body, that is, $\boldsymbol{\tau} = \dot{\mathbf{L}} = d\mathbf{L}/dt$, and the direction of this torque is at right angles to the direction of rotation. This leads a rigid body, a rotor in this case, to be dynamically unbalanced, resulting in the vibrations and wobbling of the whole system.

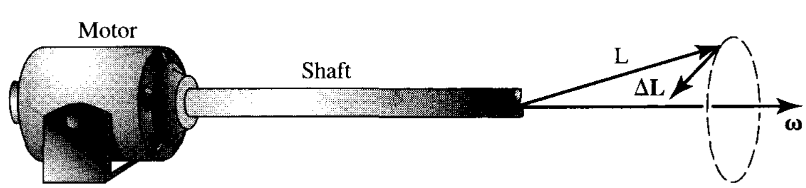


Figure 13.5 **L** not being in the same direction as **ω** results in a rotating device such as a fan blade being dynamically unbalanced.

Determination of Principal Axes

We are given the moment of inertia and the product of inertia elements of a rigid body in terms of arbitrarily chosen coordinate axes through point O. We wish to find the principal axes about the origin at O. The process is called *diagonalizing* the matrix tensor. We make use of the fact that if the rotation axis is the principal axis then both the angular momentum **L** and the angular velocity $\boldsymbol{\omega}$ are directed along this axis and hence are proportional to each other. If I is the moment of inertia about the axis, we may write

$$\mathbf{L} = I\boldsymbol{\omega} = I\omega_x\hat{\mathbf{i}} + I\omega_y\hat{\mathbf{j}} + I\omega_z\hat{\mathbf{k}} \tag{13.59}$$

Thus, using Eq. (13.40), we may write

$$\begin{aligned} L_x &= I\omega_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\ L_y &= I\omega_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\ L_z &= I\omega_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \end{aligned} \tag{13.60}$$

or, after rearranging,

$$\begin{aligned} (I_{xx} - I)\omega_x + I_{xy}\omega_y + I_{xz}\omega_z &= 0 \\ I_{yx}\omega_x + (I_{yy} - I)\omega_y + I_{yz}\omega_z &= 0 \\ I_{zx}\omega_x + I_{zy}\omega_y + (I_{zz} - I)\omega_z &= 0 \end{aligned} \tag{13.61}$$

For these equations to have nontrivial solutions, the determinants of the coefficients must vanish; that is,

$$\begin{vmatrix} I_{xx} - I & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - I & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - I \end{vmatrix} = 0 \tag{13.62}$$

Equation (13.62), called a *secular* or *characteristic equation*, is cubic in I of the form

$$-I^3 + AI^2 + BI + C = 0 \tag{13.63}$$

where A, B, and C are constants and depend on the values of the moment of inertia and product of inertia elements. Each of the three roots I_x, I_y, and I_z (or I_1, I_2, and I_3) corresponds to the moment of inertia about one of the principal axes. These values of I_x, I_y, and I_z are called the *principal moments of inertia.* The direction of any one principal axis is determined by substituting for I equal to one of the three roots I_x, I_y, or I_z, say I_x, in Eq. (13.61) and determining the ratio of the components of the angular velocity $\boldsymbol{\omega}$, that is, to find $\omega_x : \omega_y : \omega_z$. Hence we can determine the directional cosines of the axis about which the moment of inertia is I_x. A similar procedure can be followed for finding the directions of the principal axes corresponding to principal moments of inertia I_y and I_z. This procedure results in the directions of the axes. The magnitude of the angular velocity is arbitrary and we are free to assume any value. The elements of the principal moment of inertia are generally called the *eigenvalues* or *characteristic values* of the inertia tensor. The directions of the principal axes are the *eigenvectors* or the *characteristic vectors.*

In most situations in rigid body dynamics, the body has some regular shape and the principal axes may be determined by determining the symmetry of the body. The axis of symmetry is the principal axis. Furthermore, if the body is a *solid of revolution* and has a moment of inertia I_x along the symmetry axis, then $I_y = I_z$; hence the secular equation has two distinct roots. Similarly, if the secular equation has a triple root, that is, $I_x = I_y = I_z$, it is called a *spherical top*: it is called an *asymmetrical top* if all the roots are distinct, that is, $I_x \neq I_y \neq I_z$. A body is a *rotor* if $I_x = 0$ and $I_y = I_z$, such as a dumbbell and diatomic molecules.

From symmetry properties or otherwise, if one of the principal axes is known, then the other two can be determined by the following procedure. Suppose one of the principal axes is the Z-axis; then the other two principal axes must lie in the XY plane. Since the Z-axis is the principal one, we must have

$$I_{zx} = I_{zy} = 0 \tag{13.64}$$

and the first two equations in (13.61) take the form

$$\begin{aligned}(I_{xx} - I)\omega_x + I_{xy}\omega_y &= 0\\ I_{xy}\omega_x + (I_{yy} - I)\omega_y &= 0\end{aligned} \tag{13.65}$$

Let us define

$$\tan\phi = \frac{\omega_y}{\omega_x} \tag{13.66}$$

where ϕ is the angle between the principal axis and the X-axis. Substituting from Eq. (13.66) into Eq. (13.65) and eliminating I from the resulting two equations,

$$\tan 2\phi = \frac{2I_{xy}}{I_{yy} - I_{xx}} \tag{13.67}$$

This equation gives two values of ϕ between 0° and 180°, and these are directions of the two principal axes in the XY plane.

Example 13.5

Consider a homogeneous cube of density ρ, mass M, and side L, as discussed in Example 13.3. Evaluate the principal axes and their associated moment of inertia.

Solution

The moment of inertia tensor of the cube with the axes directed along the edges, as evaluated in Example 13.3, is

$$\mathbf{I} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{bmatrix}$$

To evaluate the principal moment of inertia, we must solve the secular equation Eq. (13.62),

$$\left|\begin{bmatrix} \frac{2}{3}-I & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3}-I & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3}-I \end{bmatrix}\right| = 0$$

Simplifying this equation gives

$$\frac{121}{864} - \frac{55}{48}\cdot I + 2\cdot I^2 - I^3 = 0$$

Solving for I gives three roots

$$\begin{bmatrix} \frac{1}{6} \\ \frac{11}{12} \\ \frac{11}{12} \end{bmatrix}$$

Now for each of these roots and the value of I given above, we use the secular equation Eq. (13.61), which gives the principal axes corresponding to each root as

I1 = 1/6 I2 = I3 = 11/12

Substitute for the first root I1 = 0.167 = 1/6 in the secular equation, Eq. (13.62). S = 0 gives three equations, which when solved give the three values ω11, ω21, and ω31.

$$S = \left|\begin{bmatrix} \left(\frac{2}{3}-\frac{1}{6}\right)\cdot\omega 11 & -\frac{1}{4}\cdot\omega 21 & -\frac{1}{4}\cdot\omega 31 \\ -\frac{1}{4}\cdot\omega 11 & \left(\frac{2}{3}-\frac{1}{6}\right)\cdot\omega 21 & -\frac{1}{4}\cdot\omega 31 \\ -\frac{1}{4}\cdot\omega 11 & -\frac{1}{4}\cdot\omega 21 & \left(\frac{2}{3}-\frac{1}{6}\right)\cdot\omega 31 \end{bmatrix}\right|$$

Given

$$\left(\frac{2}{3}-\frac{1}{6}\right)\cdot\omega 11 + \left(-\frac{1}{4}\cdot\omega 21 + -\frac{1}{4}\cdot\omega 31\right) = 0$$

$$-\frac{1}{4}\cdot\omega 11 + \left[\left(\frac{2}{3}-\frac{1}{6}\right)\cdot\omega 21 + -\frac{1}{4}\cdot\omega 31\right] = 0$$

$$-\frac{1}{4}\cdot\omega 11 + -\frac{1}{4}\cdot\omega 21 + \left(\frac{2}{3}-\frac{1}{6}\right)\cdot\omega 31 = 0$$

Thus the first eigenvector ω1 will have all equal components. The resulting eigenvector ω1 is as shown.

$$\text{Find}(\omega 11, \omega 21, \omega 31) \rightarrow \begin{pmatrix} \omega 11 \\ \omega 11 \\ \omega 11 \end{pmatrix} \qquad \boldsymbol{\omega 1 = i + j + k}$$

Repeat the above procedure for the second root I2 = 11/12 = 0.917 to obtain the eigenvector. Note that the third root I3 is also 11/12 = 0.917.

Given

$$\left(\frac{2}{3}-\frac{11}{12}\right)\cdot\omega 12 + \left(-\frac{1}{4}\cdot\omega 22 + -\frac{1}{4}\cdot\omega 32\right) = 0$$

Thus the eigenvector ω2 has two equal components and a third that is equal to the negative of the sum of the other two.

$$-\frac{1}{4}\cdot\omega12+\left[\left(\frac{2}{3}-\frac{11}{12}\right)\cdot\omega22+-\frac{1}{4}\cdot\omega32\right]=0$$

$$-\frac{1}{4}\cdot\omega12+-\frac{1}{4}\cdot\omega22+\left(\frac{2}{3}-\frac{11}{12}\right)\cdot\omega32=0$$

$$\mathrm{Find}(\omega12,\omega22,\omega32)\rightarrow\begin{pmatrix}\omega12\\ \omega22\\ -\omega12-\omega22\end{pmatrix}\qquad \omega2=\mathbf{i}-2\cdot\mathbf{j}+\mathbf{k}$$

Since the two roots ω2 and ω3 are equal, the corresponding eigenvectors must lie in the same plane. (Note that all the three roots are interchangeable, that is, naming them 1, 2, 3, is arbitrary.)

Alternate Direct Treatment

The moment of inertia tensor of the cube with the origin at the corner and axes directed along the edges as evaluated in Exercise 13.3 is (without a constant γ)

$$I:=\begin{bmatrix}\frac{2}{3} & -\frac{1}{4} & -\frac{1}{4}\\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4}\\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3}\end{bmatrix}\qquad c:=\mathrm{eigenvals}(I)\qquad c=\begin{pmatrix}0.917\\0.917\\0.167\end{pmatrix}$$

c = eigenvalues
v = eigenvector

c has three values, two of which are equal

$$v=I\cdot c$$

$$v1:=\mathrm{eigenvec}(I,c_1)\qquad v2:=\mathrm{eigenvec}(I,c_2)\qquad v3:=\mathrm{eigenvec}(I,c_3)$$

The three eigenvectors are as shown

$$v1=\begin{pmatrix}-0.615\\-0.158\\0.773\end{pmatrix}\qquad v2=\begin{pmatrix}-0.615\\-0.158\\0.773\end{pmatrix}\qquad v3=\begin{pmatrix}0.577\\0.577\\0.577\end{pmatrix}$$

$$\mathbf{v1}=\mathbf{v2}=-0.62\cdot\mathbf{i}+-0.16\cdot\mathbf{j}+0.77\cdot\mathbf{k}\qquad \mathbf{v3}=0.58\cdot\mathbf{i}+0.58\cdot\mathbf{j}+0.58\cdot\mathbf{k}$$

Thus the eigenvectors v1 and v2 are in the same plane, while v3 is perpendicular to them.

This implies that the principal axis corresponding to I_3 must lie along the diagonal of the cube, that is, along *OA* as shown in Fig. Ex. 13.5. Since $I_1 = I_2$, this means that the remaining two principal axes must lie in a plane normal to the axis *OA*. This plane is shown shaded in the figure. Thus the second principal axis can be picked in any direction in this plane, while the third one will be perpendicular to the second but in the same plane.

EXERCISE 13.5 Evaluate the principal axes and their associated moment of inertia for the inertia tensor obtained in Exercise 13.3.

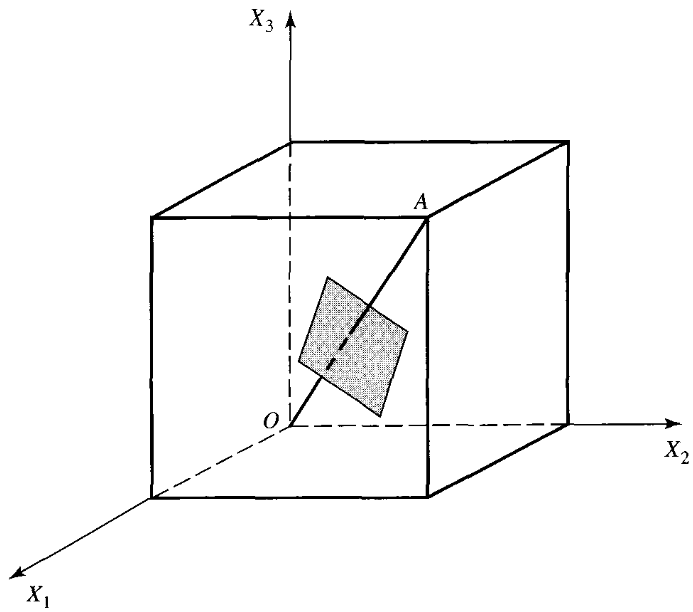


Figure Ex. 13.5

13.6 INERTIA ELLIPSOID

The inertia ellipsoid is helpful in visualizing the inertia tensor geometrically, thereby enabling us to predict some inertial properties of rigid bodies without going deeply into mathematical details. The motion of a rigid body depends on three numbers: I_1, I_2, and I_3, the principal moments of inertia. Bodies that have the same principal moments of inertia will move in exactly the same manner independent of their shape and size (provided we ignore the effects of frictional force and other forces that may be functions of the shape of the body). We show in the following that *the simplest geometrical shape of a body having three given principal moments of inertia is a homogeneous ellipsoid.* Hence, we may conclude that *the motion of any rigid body can be represented by the motion of an equivalent ellipsoid.*

Consider an arbitrary axis of rotation OA passing through a body, as shown in Fig. 13.6. Let P be a point on the axis such that the distance OP is numerically equal to the reciprocal of the square root of the moment of inertia I about OA. That is,

$$OP = \frac{1}{\sqrt{I}} \tag{13.68}$$

If the coordinates of P are x, y, and z and the directional cosines of line OP are l, m, and n, then

$$l = \frac{x}{OP} = x\sqrt{I}, \qquad m = \frac{y}{OP} = y\sqrt{I}, \qquad n = \frac{z}{OP} = z\sqrt{I} \tag{13.69}$$

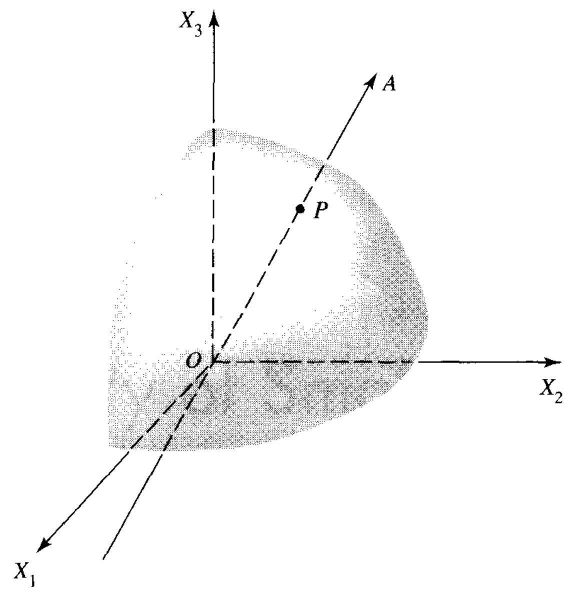


Figure 13.6 Axis of rotation OA through an arbitrarily shaped rigid body and passing through a point P.

Thus the moment of inertia of a rigid body about any line in terms of the directional cosines of that line with its inertia elements for some coordinate system with its origin on the line is

$$I = l^2 I_{xx} + m^2 I_{yy} + n^2 I_{zz} + 2nmI_{yz} + 2lnI_{zx} + 2mlI_{xy} \tag{13.70}$$

Substituting for l, m, and n, Eq. (13.70), after rearranging, takes the form

$$x^2 I_{xx} + y^2 I_{yy} + z^2 I_{zz} + 2yzI_{yz} + 2zxI_{zx} + 2xyI_{xy} = 1 \tag{13.71}$$

This is an equation of a surface (the locus of points P) as the direction of axis OA is varied. It is the equation of a general quardratic surface in three dimensions, and the surface is bounded; hence it must be an *ellipsoid.* If the coordinate axes are the principal axes. Eq. (13.70) takes the form

$$I = l^2 I_{xx} + m^2 I_{yy} + n^2 I_{zz} \tag{13.72}$$

and the inertia ellipsoid, Eq. (13.71), takes the form

$$x^2 I_1 + y^2 I_2 + z^2 I_3 = 1 \tag{13.73}$$

where I_1, I_2, and I_3 (which have replaced I_{xx}, I_{yy}, and I_{zz}) are the principal moments of inertia. The two inertia ellipsoids given by Eqs. (13.71) and (13.73) are shown in Fig. 13.7(a) and (b). Note that the semiaxes of the ellipsoid in Fig. 13.7(a) are

$$\frac{1}{\sqrt{I_{11}}}, \quad \frac{1}{\sqrt{I_{22}}}, \quad \text{and} \quad \frac{1}{\sqrt{I_{33}}}$$

and the semiaxes of the ellipsoid in Fig. 13.7(b) are

$$\frac{1}{\sqrt{I_1}}, \quad \frac{1}{\sqrt{I_2}}, \quad \text{and} \quad \frac{1}{\sqrt{I_3}}$$

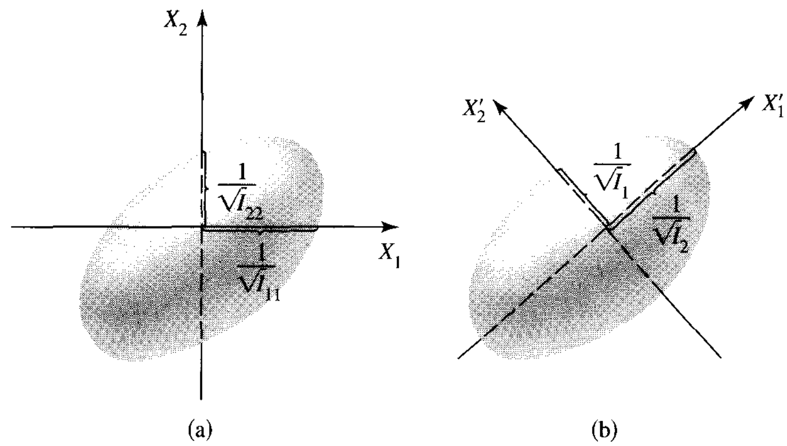


Figure 13.7 (a) Inertia ellipsoid in a nonprincipal coordinate axes system given by Eq. (13.71), and (b) inertia ellipsoid in a principal coordinate axes system given by Eq. (13.73).

That is, the semiaxes are

$$x_k = \frac{1}{\sqrt{I_k}} \tag{13.74}$$

Also, the ellipsoid in Fig. 13.7(b) can be obtained from Fig. 13.7(a) by causing proper rotations.

If two of the I_k are equal, the inertia ellipsoid has rotational symmetry about the third axis. Suppose $I_1 = I_2$, then the intersection of the inertia ellipsoid with the $X_1 - X_2$ planes may be drawn, all having the same moments of inertia. If $I_1 = I_2 = I_3$, the inertia ellipsoid reduces to a sphere, and the moments of inertia about any axis passing through the origin are equal.

13.7 MORE ABOUT THE PROPERTIES OF THE INERTIA TENSOR

We start with the definition of an inertia tensor and tensors in general. Then we introduce a slightly different way of defining an inertia tensor by means of a dyad product. And, finally, we see the similarity in the treatment of tensors as matrices.

The relation between the quantities $\mathbf{L}$ and $\boldsymbol{\omega}$ may be written as

$$\mathsf{I} = \frac{\mathbf{L}}{\boldsymbol{\omega}} \tag{13.75}$$

where I is the quotient of two vector quantities. In general, the quotient of two vector quantities is not necessarily a member of the same class as that of the two dividing factors. Hence, we do not expect the ratio of the two dividing vectors to be a vector. As a matter of fact, it is altogether a different quantity, called a *tensor of the second rank.*

In a Cartesian three-dimensional space, a Cartesian tensor T of the Nth rank may be defined as (1) a quantity that has 3^N components $T_{ijk\ldots N}$, and (2) under orthogonal transformation of coordinates it obeys the following rule:

$$T'_{ijk\ldots}(x') = a_{il}a_{jm}a_{kn}\cdots T_{imn\ldots}(x) \tag{13.76}$$

where a_{il}, a_{jm}, . . . are the elements of transformation. Since we shall not be using any other coordinates except Cartesian, we shall simply use the term tensor **T** instead of Cartesian tensor **T**. Thus from this definition, for $N = 0$, $3^0 = 1$. That is, a tensor of *zero rank* has only *one component*; hence this quantity will be invariant under an orthogonal transformation. We may say that a *scalar is a tensor of zero rank* and has only *one component.* On the other hand, if $N = 1$, $3^1 = 3$; the tensor of *first rank* will have *three components.* These components transfer, according to Eq. (13.76), as

$$T'_i = a_{ij}T_j \tag{13.77}$$

which is similar to the transformation equation for a vector. Thus a *vector is a tensor of the first rank,* and has three components. For $N = 2$, a tensor of *second rank* will have *nine components,* which will transfer as

$$T'_{ij} = a_{il}a_{jm}T_{lm} \tag{13.78}$$

This transformation is similar to a 3 × 3 square matrix, except for one fundamental difference between the two. Unlike a tensor of second rank, a matrix transformation is not limited only to orthogonal transformation. In spite of these differences, we shall make use of the properties of matrices in tensors.

Another way of representing a tensor **I** is as a *dyadic.* We start with the definition of angular momentum, Eq. (13.38):

$$\mathbf{L} = \sum_{i=1}^{n} m_i[r_i^2\boldsymbol{\omega} - \mathbf{r}_i(\mathbf{r}_i \cdot \boldsymbol{\omega})] \tag{13.38}$$

Or we may write this as

$$\mathbf{L} = \left(\sum_{i=1}^{n} m_i r_i^2\right)\boldsymbol{\omega} - \left(\sum_{i=1}^{n} m_i\mathbf{r}_i\mathbf{r}_i\right) \cdot \boldsymbol{\omega} \tag{13.79}$$

The second term on the right has no meaning because we have not yet defined quantities of the form $\mathbf{r}_i\mathbf{r}_i$. We define a *dyad* as a simple pair of two vectors written as **AB**. The quantity **AB** has meaning only when it operates on other quantities. Thus we define the scalar dot product of a dyad with a vector **C** as a vector quantity given by

$$(\mathbf{AB}) \cdot \mathbf{C} = \mathbf{A}(\mathbf{B} \cdot \mathbf{C}) \tag{13.80}$$

or

$$\mathbf{C} \cdot (\mathbf{AB}) = (\mathbf{C} \cdot \mathbf{A})\mathbf{B} \tag{13.81}$$

where $\mathbf{B} \cdot \mathbf{C}$ is a scalar ($= b_1c_1 + b_2c_2 + b_3c_3$); hence $(\mathbf{AB}) \cdot \mathbf{C}$ is a vector. Similarly, $\mathbf{C} \cdot (\mathbf{AB})$ is a vector. But the two vectors given by Eqs. (13.80) and (13.81), in general, will not be equal. That is, *dyad scalar multiplication is not commutative.* If we let

$$\mathbf{T} = \mathbf{AB} \tag{13.82}$$

then we may write

$$\mathbf{T} \cdot \mathbf{C} = \mathbf{A}(\mathbf{B} \cdot \mathbf{C}) \tag{13.83}$$

$$\mathbf{C} \cdot \mathbf{T} = (\mathbf{C} \cdot \mathbf{A})\mathbf{B} \tag{13.84}$$

Also,
$$\mathbf{T} \cdot (\mathbf{C} + \mathbf{D}) = \mathbf{T} \cdot \mathbf{C} + \mathbf{T} \cdot \mathbf{D} \tag{13.85}$$

$$\mathbf{T} \cdot (c\mathbf{C}) = c(\mathbf{T} \cdot \mathbf{C}) \tag{13.86}$$

where c is a constant.

A linear polynomial of dyads is called a *dyadic*, such as $\mathbf{AB} + \mathbf{CD} + \cdots$. Actually, any dyad may be expressed as a dyadic if we express the vectors $\mathbf{A}$ and $\mathbf{B}$ in terms of unit vectors. Thus, if

$$\mathbf{C} = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}}$$

$$\mathbf{A} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$$

$$\mathbf{B} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$$

then the dyad $\mathbf{AB}$ may be written as a dyadic:

$$\begin{aligned}\mathbf{T} = \mathbf{AB} = a_1b_i\hat{\mathbf{i}}\hat{\mathbf{i}} + a_1b_2\hat{\mathbf{i}}\hat{\mathbf{j}} + a_1b_3\hat{\mathbf{i}}\hat{\mathbf{k}} + a_2b_1\hat{\mathbf{j}}\hat{\mathbf{i}} + a_2b_3\hat{\mathbf{j}}\hat{\mathbf{k}} \\ + a_3b_1\hat{\mathbf{k}}\hat{\mathbf{i}} + a_3b_2\hat{\mathbf{k}}\hat{\mathbf{j}} + a_3b_3\hat{\mathbf{k}}\hat{\mathbf{k}}\end{aligned} \tag{13.87}$$

Thus, in matrix notation, we may write

$$\mathbf{T} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} = \begin{pmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{pmatrix} \tag{13.88}$$

Any given component of $\mathbf{T}$ is written as T_{ij}.

In component form, we may write

$$\mathbf{C} = \sum_{j=1}^{3} c_j\hat{\mathbf{u}}_j \tag{13.89}$$

where $\hat{\mathbf{u}}_j = (\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3)$ are the unit vectors; hence

$$(\mathbf{T} \cdot \mathbf{C})_i = \sum_{j=1}^{3} T_{ij}c_j \tag{13.90}$$

$$(\mathbf{C} \cdot \mathbf{T})_i = \sum_{j=1}^{3} c_jT_{ji} \tag{13.91}$$

$$T_{ij} = \hat{\mathbf{u}}_i \cdot (\mathbf{T} \cdot \hat{\mathbf{u}}_j) = (\hat{\mathbf{u}}_i \cdot \mathbf{T}) \cdot \hat{\mathbf{u}}_j \tag{13.92}$$

while
$$\mathbf{T} = \sum_{i,j=1}^{3} T_{ij}\hat{\mathbf{u}}_i\hat{\mathbf{u}}_j \tag{13.93}$$

Now we may define a *unit dyadic* **1** as

$$\mathbf{1} = \hat{\mathbf{i}}\hat{\mathbf{i}} + \hat{\mathbf{j}}\hat{\mathbf{j}} + \hat{\mathbf{k}}\hat{\mathbf{k}} \tag{13.94}$$

and **1** behaves exactly like a unit matrix, giving the results

$$\mathbf{1} \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{1} = \mathbf{A} \tag{13.95}$$

We may also write **1** as a unit tensor such that

$$\mathbf{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{13.96}$$

Finally, we take full advantage of the fact that a tensor of second rank is very similar to a 3 × 3 square matrix in its representation. Hence transformation properties in orthogonal Cartesian coordinates may be directly utilized here. Let us start with a vector **L** in space or fixed in an inertial coordinate system so that

$$L_k = \sum_l I_{kl}\omega_l \tag{13.97}$$

In a body coordinate system that is simply rotated with respect to space coordinates, the angular momentum $\mathbf{L}'$ must have an analogous form:

$$L'_i = \sum I'_{ij}\omega'_j \tag{13.98}$$

Using the transformation properties of vectors, we may write the transformation of **L** and $\boldsymbol{\omega}$ as [note that from Eq. (5.166)

$$x_i = \sum_j \lambda_{ji}x'_j \tag{5.166}$$

where λ_{ji} is an element of the transformation matrix $\boldsymbol{\lambda}$]

$$L_k = \sum_m \lambda_{mk}L'_m \tag{13.99}$$

and

$$\omega_l = \sum_j \lambda_{jl}\omega'_j \tag{13.100}$$

Substituting these in Eq. (13.97), we obtain

$$\sum_m \lambda_{mk}L'_m = \sum_l I_{kl}\sum_j \lambda_{jl}\omega'_j \tag{13.101}$$

Multiplying both sides by λ_{ik} and summing over k,

$$\sum_m \left(\sum_k \lambda_{ik}\lambda_{mk} \right) L'_m = \sum_j \left(\sum_{k,l} \lambda_{ik}\lambda_{kl} I_{kl} \right) \omega'_j \tag{13.102}$$

This left side may be written as

$$\sum_m \left(\sum_k \lambda_{ik}\lambda_{mk} \right) L'_m = \sum_m \delta_{im} L'_m = L'_i$$

That is,

$$L'_i = \sum_j \left(\sum_{k,l} \lambda_{ik}\lambda_{jl} I_{kl} \right) \omega'_j \tag{13.103}$$

But this must be identical to Eq. (13.98). Comparing the two yields

$$I'_{ij} = \sum_{k,l} \lambda_{ik}\lambda_{jl} I_{kl} \tag{13.104}$$

Thus each element I_{kl} of inertia tensor $\mathbf{I}$ in a fixed coordinate system can be transformed into rotated (body) coordinates resulting in elements I'_{ij} of inertia tensor $\mathbf{I}'$. The preceding result may be written as

$$I'_{ij} = \sum_{k,l} \lambda_{ik} I_{kl} \lambda^t_{lj} \tag{13.105}$$

where λ^t_{lj} are the elements of a transposed matrix $\boldsymbol{\lambda}^t$. Just as in matrix notation, we may write

$$\mathbf{I}' = \boldsymbol{\lambda}\mathbf{I}\boldsymbol{\lambda}^t \tag{13.106}$$

Since for orthogonal transformations, $\boldsymbol{\lambda}^t = \boldsymbol{\lambda}^{-1}$, where $\boldsymbol{\lambda}^{-1}$ is the inverse matrix, we may write

$$\mathbf{I}' = \boldsymbol{\lambda}\mathbf{I}\boldsymbol{\lambda}^{-1} \tag{13.107}$$

which is the *similarity transformation* ($\mathbf{I}'$ is similar to $\mathbf{I}$).

These results indicate the method for transferring an inertia tensor from one system to another rotated system by using the rotation matrix. Furthermore, we may utilize this method to find the principal axes by determining the eigenvalues from the secular equation

$$|I_{ml} - I\delta_{ml}| = 0 \tag{3.108}$$

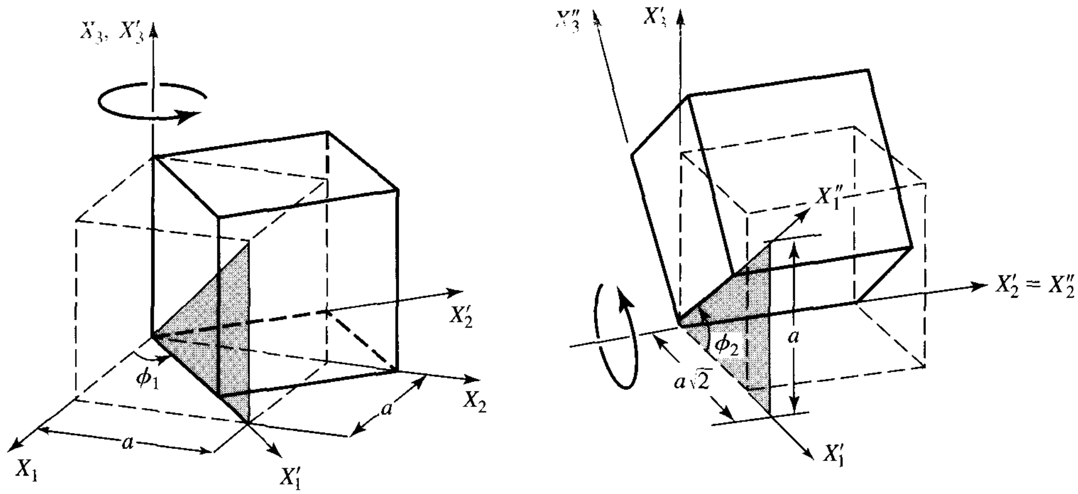
That is,

$$\begin{vmatrix} I_{11} - I & I_{12} & I_{13} \\ I_{21} & I_{22} - I & I_{23} \\ I_{31} & I_{32} & I_{33} - I \end{vmatrix} = 0 \tag{13.109}$$

which is the same as Eq. (13.62). These points are illustrated in the following example.

Example 13.6

Diagonalize the inertia tensor of a cube by rotating the coordinate axes.



$\phi_1 = 45°$
(a)

$\cos\phi_2 = \sqrt{\frac{2}{3}}$, $\sin\phi_2 = \sqrt{\frac{1}{3}}$
(b)

Figure Ex. 13.6

Solution

As before, let the origin be at one corner of the cube. We have to perform a rotation in such a way that the X_1-axis will coincide with the diagonal of the cube. This can be achieved by means of two rotations: (a) perform a rotation through an angle $\phi_1 = 45°$ about the X_3-axis, and (b) perform a rotation through an angle $\phi_2 = \cos^{-1}(\sqrt{2/3})$ about the X_2'-axis, as shown in Fig. Ex. 13.6.

(a)

ϕ1 = first angle of rotation about X3-axis

$\phi1 := 45 \cdot \mathrm{deg} \qquad \phi1 = 0.785 \cdot \mathrm{rad}$

λ1 = the matrix of first rotation

$$\lambda1 := \begin{bmatrix} \cos(\phi1) & \sin(\phi1) & 0 \\ -\sin(\phi1) & \cos(\phi1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \lambda1 = \begin{pmatrix} 0.707 & 0.707 & 0 \\ -0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b)

ϕ2 = second angle of rotation about X2'-axis

$$\phi2 := \mathrm{acos}\left(\sqrt{\frac{2}{3}}\right) \qquad \phi2 = 0.615$$

λ2 = matrix of second rotation

$$\lambda2 := \begin{bmatrix} \cos(\phi2) & 0 & \sin(\phi2) \\ 0 & 1 & 0 \\ -\sin(\phi2) & 0 & \cos(\phi2) \end{bmatrix} \qquad \lambda2 = \begin{pmatrix} 0.816 & 0 & 0.577 \\ 0 & 1 & 0 \\ -0.577 & 0 & 0.816 \end{pmatrix}$$

λ = total matrix rotation

$$\lambda := \lambda2 \cdot \lambda1$$

λ^T = the inverse or transform matrix

$$\lambda = \begin{pmatrix} 0.577 & 0.577 & 0.577 \\ -0.707 & 0.707 & 0 \\ -0.408 & -0.408 & 0.816 \end{pmatrix} \qquad \lambda^T = \begin{pmatrix} 0.577 & -0.707 & -0.408 \\ 0.577 & 0.707 & -0.408 \\ 0.577 & 0 & 0.816 \end{pmatrix}$$

I is from Example 13.3.

$$I = \begin{bmatrix} \frac{2}{3}\cdot\gamma & \frac{-1}{4}\cdot\gamma & \frac{-1}{4}\cdot\gamma \\ \frac{-1}{4}\cdot\gamma & \frac{2}{3}\cdot\gamma & \frac{-1}{4}\cdot\gamma \\ \frac{-1}{4}\cdot\gamma & \frac{-1}{4}\cdot\gamma & \frac{2}{3}\cdot\gamma \end{bmatrix} \qquad I' = \lambda\cdot I\cdot\lambda^T$$

We can now calculate the inverse trasform matrix by substituting the values of λ and λ^T and I.

$$I' = \begin{bmatrix} .5761773 & .5756886 & .5802 \\ -.7068 & .7074 & 0 \\ -.41043348 & -.41008536 & .8145 \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{3}\cdot\gamma & \frac{-1}{4}\cdot\gamma & \frac{-1}{4}\cdot\gamma \\ \frac{-1}{4}\cdot\gamma & \frac{2}{3}\cdot\gamma & \frac{-1}{4}\cdot\gamma \\ \frac{-1}{4}\cdot\gamma & \frac{-1}{4}\cdot\gamma & \frac{2}{3}\cdot\gamma \end{bmatrix} \cdot \begin{bmatrix} 0.576 & -0.707 & -0.41 \\ 0.576 & 0.707 & -0.41 \\ 0.58 & 0 & 0.815 \end{bmatrix}$$

It is clear that we get the same matrix as in Example 13.5.

$$I' = \begin{bmatrix} .1666\cdot\gamma & 0 & 0 \\ 0 & .9165\cdot\gamma & 0 \\ 0 & 0 & .9168\cdot\gamma \end{bmatrix} \qquad \frac{1}{6} = 0.167 \qquad \frac{11}{12} = 0.917$$

The alternate treatment

The procedure is self-explanatory.

$$A := \begin{bmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{bmatrix} \qquad E := \text{eigenvals}(A) \qquad E = \begin{pmatrix} 0.917 \\ 0.917 \\ 0.167 \end{pmatrix}$$

$$\text{diag}(E) = \begin{pmatrix} 0.917 & 0 & 0 \\ 0 & 0.917 & 0 \\ 0 & 0 & 0.167 \end{pmatrix}$$

The three columns are the three eigenvectors.

$$V := \text{eigenvecs}(A) \qquad V = \begin{pmatrix} 0.711 & -0.401 & 0.577 \\ -0.009 & 0.816 & 0.577 \\ -0.703 & -0.416 & 0.577 \end{pmatrix}$$

$$V = \begin{bmatrix} 0.711 & -0.401 & 0.577 \\ -0.009 & 0.816 & 0.577 \\ -0.703 & -0.416 & 0.577 \end{bmatrix}$$

which gives

$$\omega 1 = 0.577\cdot\mathbf{i} + 0.577\cdot\mathbf{j} + 0.577\cdot\mathbf{k}$$

$$\omega 2 = -0.401\cdot\mathbf{i} + 0.816\cdot\mathbf{j} + -0.46\cdot\mathbf{k}$$

$$\omega 3 = -0.71\cdot\mathbf{i} - 0.01\cdot\mathbf{j} + 0.71\cdot\mathbf{k}$$

EXERCISE 13.6 Diagonalize the inertia tensor of the rectangular body discussed in Exercise 13.3 by rotating the coordinate axes.

13.8 EULERIAN ANGLES

We are interested in evaluating a matrix that will enable us to cause transformation from one coordinate system to another. Let us say we want to transfer from coordinates $\mathbf{X}'$ of a fixed or inertial coordinate system to coordinates $\mathbf{X}$ of a body coordinate system. The transformation may be represented by a matrix equation

$$\mathbf{X} = \boldsymbol{\lambda}\mathbf{X}' \tag{13.110}$$

where the rotation matrix $\boldsymbol{\lambda}$ completely specifies the relative orientation of the two systems. Such a rotation matrix should contain three independent angles. Of the several possible choices for these angles, the most common and convenient ones to use are the *Eulerian angles* represented as ϕ, θ, and ψ.

To go from an X' system to an X system, the following sequence of rotations of three angles is followed, as demonstrated in Fig. 13.8:

1. The first rotation is counterclockwise through an angle ϕ about the X_3'-axis and in the X_1'-X_2' plane, transforming the axes $X_i' \rightarrow X_i''$, as shown in Fig. 13.8(a). The transformation matrix for this rotation in the X_1'-X_2' plane is

$$\mathbf{R}_\phi = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{13.111}$$

   The angle ϕ is called the *precession angle.*

2. The second rotation is counterclockwise through an angle θ about the X_1''-axis and in the X_2''-X_3'' plane, transforming the axes $X_i'' \rightarrow X_i'''$, as shown in Fig. 13.8(b). The transformation matrix for this rotation in the X_2''-X_3'' plane is

$$\mathbf{R}_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{13.112}$$

   The angle θ is called the *nutation angle.*

3. The third rotation is counterclockwise through an angle ψ about the X_3'''-axis and in the X_1'''-X_2''' plane, transforming the axes $X_i''' \rightarrow X_i$, as shown in Fig. 13.8(c). The transformation matrix for this rotation is

$$\mathbf{R}_\psi = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{13.113}$$

   The angle ψ is called the *body angle.*

The line NN' formed by the intersection of the planes containing the X_1-X_2 axes of the body system and the X_1'-X_2' axes of the fixed system is called the *line of nodes*. The transforma-

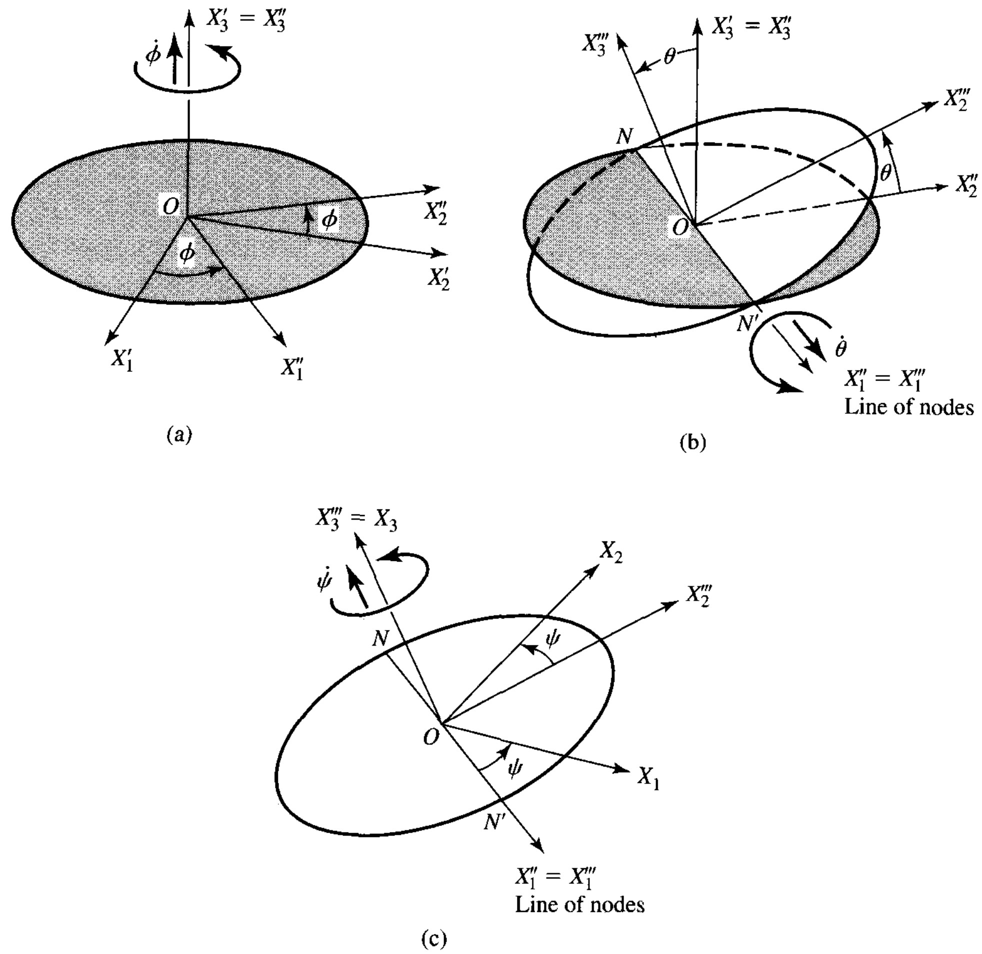


Figure 13.8 Sequence of three angular rotations employed in going from an X' system to an X system.

tion from the fixed coordinate system X_i' to the body coordinate system X_i is given by the rotation matrix $\boldsymbol{\lambda}$ obtained by the product of the three individual matrices $\mathbf{R}_\phi$, $\mathbf{R}_\theta$, and $\mathbf{R}_\psi$ given previously. That is,

$$\boldsymbol{\lambda} = \mathbf{R}_\psi \mathbf{R}_\theta \mathbf{R}_\phi \tag{13.114}$$

$$\boldsymbol{\lambda} = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix}$$

$$\boldsymbol{\lambda} = \begin{pmatrix} \cos\psi\cos\phi & \cos\psi\cos\phi & \sin\psi\sin\theta \\ \quad -\cos\theta\sin\phi\sin\psi & \quad +\cos\theta\cos\phi\sin\psi & \\ -\sin\psi\cos\phi & -\sin\psi\sin\phi & \cos\psi\sin\theta \\ \quad -\cos\theta\sin\phi\cos\psi & \quad +\cos\theta\cos\phi\cos\psi & \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix} \tag{13.115}$$

All infinitesimal rotations can be represented by vector notation. This enables us to represent the three time derivatives of rotation, that is, $\dot{\phi}$, $\dot{\theta}$, and $\dot{\psi}$, as the components of an angular velocity vector $\boldsymbol{\omega}(\omega_\phi, \omega_\theta, \omega_\psi)$. The three components of $\boldsymbol{\omega}$ are not all either along the fixed axes or the body axes. Actually,

$$\omega_\phi = \dot{\phi} \quad \text{is directed along the } X_3' \text{ (fixed) axis}$$

$$\omega_\theta = \dot{\theta} \quad \text{is along the line of nodes}$$

$$\omega_\psi = \dot{\psi} \quad \text{is directed along the } X_3 \text{ (body) axis} \tag{13.116}$$

It is not very convenient to use these components to describe the motion of a rigid body. Rigid body equations of motion are described in terms of a body coordinate system. Thus we must calculate the angular velocity vector $\boldsymbol{\omega}(\omega_1, \omega_2, \omega_3)$ in the body coordinate system. To do this, we must first resolve $\dot{\phi}$, $\dot{\theta}$, and $\dot{\psi}$ along the body axes; that is,

$$\dot{\phi}_1 = \dot{\phi} \sin\theta \sin\psi \qquad \text{along } X_1\text{-axis}$$

$$\dot{\phi}_2 = \dot{\phi} \sin\theta \cos\psi \qquad \text{along } X_2\text{-axis}$$

$$\dot{\phi}_3 = \dot{\phi} \cos\theta \qquad \text{along } X_3\text{-axis} \tag{13.117}$$

$$\dot{\theta}_1 = \dot{\theta} \cos\psi \qquad \text{along } X_1\text{-axis}$$

$$\dot{\theta}_2 = -\dot{\theta} \sin\psi \qquad \text{along } X_2\text{-axis}$$

$$\dot{\theta}_3 = 0 \qquad \text{along } X_3\text{-axis} \tag{13.118}$$

$$\dot{\psi}_1 = 0 \qquad \text{along } X_1\text{-axis}$$

$$\dot{\psi}_2 = 0 \qquad \text{along } X_2\text{-axis}$$

$$\dot{\psi}_3 = \dot{\psi} \qquad \text{along } X_3\text{-axis} \tag{13.119}$$

Using these results, we get the components of $\boldsymbol{\omega}$ to be

$$\omega_1 = \dot{\phi}_1 + \dot{\theta}_1 + \dot{\psi}_2 = \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi$$

$$\omega_2 = \dot{\phi}_2 + \dot{\theta}_2 + \dot{\psi}_2 = \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi$$

$$\omega_3 = \dot{\phi}_3 + \dot{\theta}_3 + \dot{\psi}_3 = \dot{\phi} \cos\theta + \dot{\psi} \tag{13.120}$$

These equations are called *Euler's geometrical equations*. We can make use of these to describe rigid body motion using body axes.

It is important to emphasize that the angular displacements and other rotational quantities may be represented as vectors only if these quantities are infinitesimally small; then they obey the law of vector addition. The exception is the case in which the rotations are in the same plane.

13.9 EULER'S EQUATIONS OF MOTION FOR A RIGID BODY

The translational motion of a rigid body is described by the equations

$$\mathbf{F} = \frac{d\mathbf{P}}{dt}, \qquad \text{where } \mathbf{P} = M\mathbf{V} \tag{13.121}$$

F is the resultant force acting on the body, **P** is its linear momentum, M is its mass, and **V** is the velocity of its center of mass. The rotational motion of a body is described by

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}, \qquad \text{where } \mathbf{L} = \mathbf{I}\boldsymbol{\omega} \tag{13.122}$$

$\boldsymbol{\tau}$ is the net torque acting on the rigid body, **L** is its angular momentum, $\boldsymbol{\omega}$ is its angular velocity, and **I** is the inertia tensor. The methods used for solving equations of translational motion can be directly extended to those for rotational motion only for the special case in which the rotational motion is restricted about a *fixed* axis. For a general case, this is not true.

Let us now proceed to obtain Euler's equations of motion for a rigid body in a force field. Equation (13.22), which describes the motion of a rigid body as viewed from a fixed, inertial, or laboratory coordinate system (LCS), may be written as

$$\frac{d\mathbf{L}}{dt} = \dot{\mathbf{L}} = \frac{d}{dt}(\mathbf{I}\cdot\boldsymbol{\omega}) = \boldsymbol{\tau} \tag{13.123}$$

Note that **I** changes as the body rotates. To overcome this difficulty, we refer Eq. (13.123) to a set of axes that are fixed with the rotating body. Let d'/dt be the time derivative with respect to the coordinate axes fixed in the body. Using the results given in Chapter 11 [Eq. (11.26)], Eq. (13.123) takes the form

$$\frac{d\mathbf{L}}{dt} = \frac{d'\mathbf{L}}{dt} + \boldsymbol{\omega}\times\mathbf{L} = \boldsymbol{\tau} \tag{13.124}$$

Since $\mathbf{L} = \mathbf{I}\cdot\boldsymbol{\omega}$, where **I** is constant relative to the body axes, we may substitute in Eq. (13.124) to obtain

$$\frac{d'(\mathbf{I}\cdot\boldsymbol{\omega})}{dt} + \boldsymbol{\omega}\times(\mathbf{I}\cdot\boldsymbol{\omega}) = \boldsymbol{\tau}$$

$$\mathbf{I}\cdot\frac{d'\boldsymbol{\omega}}{dt} + \frac{d'\mathbf{I}}{dt}\cdot\boldsymbol{\omega} + \boldsymbol{\omega}\times(\mathbf{I}\cdot\boldsymbol{\omega}) = \boldsymbol{\tau} \tag{13.125}$$

But

$$\frac{d'\mathbf{I}}{dt} = 0 \quad \text{and} \quad \frac{d'\boldsymbol{\omega}}{dt} = \frac{d\boldsymbol{\omega}}{dt} \tag{13.126}$$

which when substituted in Eq. (13.125) yields

$$\mathbf{I}\cdot\frac{d\boldsymbol{\omega}}{dt} + \boldsymbol{\omega}\times(\mathbf{I}\cdot\boldsymbol{\omega}) = \boldsymbol{\tau} \tag{13.127}$$

For convenience, choose the body axes to be the principal axes so that

$$\mathbf{L} = \mathbf{I} \cdot \boldsymbol{\omega} = \hat{\mathbf{i}}L_1 + \hat{\mathbf{j}}L_2 + \hat{\mathbf{k}}L_3$$

Also $$L_i = I_i\omega_i$$

That is, $$L_1 = I_1\omega_1, \qquad L_2 = I_2\omega_2, \quad \text{and} \quad L_3 = I_3\omega_3$$

Using these relations, Eq. (13.127) may be written in component form as

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_3\omega_2 = \tau_1 \tag{13.128a}$$

$$I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 = \tau_2 \tag{13.128b}$$

$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 = \tau_3 \tag{13.128c}$$

These are known as *Euler's dynamical equations* or simply *Euler's equations* for the motion of a rigid body in a force field. In the absence of a torque, Eqs. (13.128), take the form

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_3\omega_2 = 0 \qquad \tau_1 = 0 \tag{13.129a}$$

$$I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 = 0 \qquad \tau_2 = 0 \tag{13.129b}$$

$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 = 0 \qquad \tau_3 = 0 \tag{13.129c}$$

Furthermore, for the net zero external torque, the angular momentum must remain constant in both magnitude and direction. For this, Euler's equations require that if $\omega_1 \neq 0$, then $\omega_2 = \omega_3 = 0$, and if $\omega_2 \neq 0$, then $\omega_1 = \omega_3 = 0$. These results imply that for no net external torque, only rotations about the body's principal axes are possible.

As is clear from our discussion, the three principal moment of inertia elements I_1, I_2, and I_3 determine the motion of a rigid body. Any two rigid bodies that have the same principal moment of inertia will have the same behavior regardless of their structure and shape. Motions of such bodies are described by means of an equivalent ellipsoid constructed with principal moment of inertia elements, as discussed in Section 13.6.

13.10 FORCE FREE MOTION OF A SYMMETRICAL TOP

We can solve Euler's equations, Eqs. (13.128), for the special case in which $\boldsymbol{\tau} = 0$. Furthermore, we limit our discussion to the case in which the body is symmetrical, that is, a symmetrical top in which two of the principal moments of inertia are the same. We choose the X_3-axis to be the symmetry axis so that $I_1 = I_2 = I_{12} \neq I_3$. Thus Eqs. (13.128) reduce to

$$I_{12}\dot{\omega}_1 + (I_3 - I_{12})\omega_2\omega_3 = 0 \tag{13.130}$$

$$I_{12}\dot{\omega}_2 + (I_{12} - I_3)\omega_1\omega_3 = 0 \tag{13.131}$$

$$I_3\dot{\omega}_3 = 0 \tag{13.132}$$

Before solving these equations, two points must be made clear. First, since the motion is force free, the center of mass of the body is at rest or moving with uniform velocity. There will be no loss of generality if we assume that the center of mass is at rest and is located at the origin of a fixed or laboratory coordinate system. Second, assume that the angular velocity $\boldsymbol{\omega}$ does *not* lie along one of the principal axes of the body coordinate system, because if it does the problem will be a trivial one.

From Eq. (13.132), since $I_3 \neq 0$,

$$\dot{\omega}_3 = 0$$

which on integration gives

$$\omega_3(t) = \text{constant} \tag{13.133}$$

This equation states that for any rigid body rotating with angular velocity $\boldsymbol{\omega}$ the component of angular velocity along the symmetry axis, ω_3, remains constant. (If $\boldsymbol{\omega}$ were along the X_3-axis, the principal axis, the entire angular velocity $\boldsymbol{\omega}$ would remain constant.)

Equations (13.130) and (13.131) may be written as

$$\dot{\omega}_1 + \frac{I_3 - I_{12}}{I_{12}}\,\omega_3\omega_2 = 0 \tag{13.134}$$

and

$$\dot{\omega}_2 - \frac{I_3 - I_{12}}{I_{12}}\,\omega_3\omega_1 = 0 \tag{13.135}$$

Let us define Ω (or Ω_B to be more specific),

$$\Omega_B \equiv \frac{I_3 - I_{12}}{I_{12}}\,\omega_3 = \gamma\omega_3 \tag{13.136}$$

and rewrite Eqs. (13.134) and (13.135) as

$$\dot{\omega}_1 + \Omega_B\omega_2 = 0 \tag{13.137}$$

$$\dot{\omega}_2 - \Omega_B\omega_1 = 0 \tag{13.138}$$

These are two first-order coupled equations and can be solved by the usual procedure for such equations. Multiply the second equation by i and add to the first, that is,

$$(\dot{\omega}_1 + i\dot{\omega}_2) - i\Omega_B(\omega_1 + i\omega_2) = 0 \tag{13.139}$$

Substitute

$$\eta \equiv \omega_1 + i\omega_2 \tag{13.140}$$

and

$$\dot{\eta} = \dot{\omega}_1 + i\dot{\omega}_2 \tag{13.141}$$

into Eq. (13.139), resulting in

$$\dot{\eta} - i\Omega_B\eta = 0 \tag{13.142}$$

Assuming that the phase angle $\delta = 0$ when $t = 0$, the solution of Eq. (13.142) is

$$\eta(t) = Ae^{i\Omega_B t} \tag{13.143}$$

or

$$\omega_1 + i\omega_2 = A \cos \Omega_B t + iA \sin \Omega_B t \tag{13.144}$$

where A is an arbitrary constant. Comparing the two sides,

$$\omega_1(t) = A \cos \Omega_B t \tag{13.145}$$

$$\omega_2(t) = A \sin \Omega_B t \tag{13.146}$$

Squaring the two equations and adding,

$$\omega_1^2 + \omega_2^2 = A^2 \tag{13.147}$$

That is, the sum of the squares of the angular velocity components ω_1 and ω_2 is constant and is equal to A^2. Furthermore, according to Eq. (13.133), ω_3 is constant; therefore, the magnitude of ω is also constant; that is,

$$\omega = |\boldsymbol{\omega}| = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2} = \sqrt{A^2 + \omega_3^2} = \text{constant} \tag{13.148}$$

Equations (13.145) and (13.146) are parametric equations of a circle, and ω_1 and ω_2 are the components of $\boldsymbol{\omega}$ in the X_1X_2 body plane. Thus the components ω_1 and ω_2 of $\boldsymbol{\omega}$ trace out a circle with time in the X_1X_2 plane, which implies that the angular velocity vector $\boldsymbol{\omega}$ precesses in a cone about the X_3-axis (the body symmetry axis) with a constant angular frequencey Ω_B, as shown in Fig. 13.9, while ω_3 remains constant around the symmetry axis. The net result is *to an*

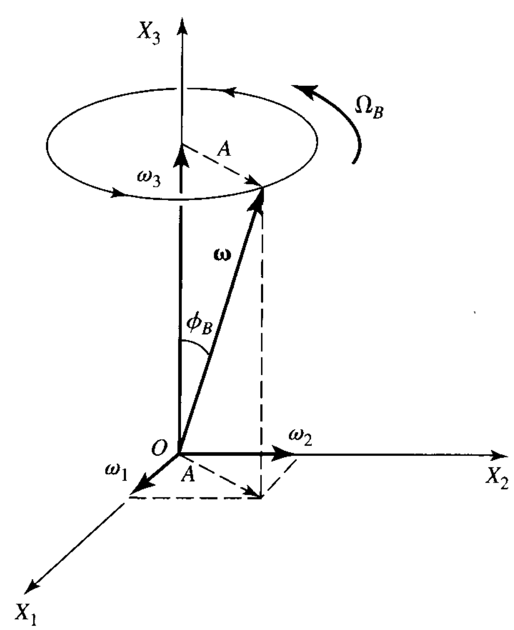


Figure 13.9(a) The angular velocity vector $\boldsymbol{\omega}$ precesses in a cone about the X_3-axis, the body symmetry axis, with a constant angular frequency Ω_B. ϕ_B is the half-angle of the body cone.

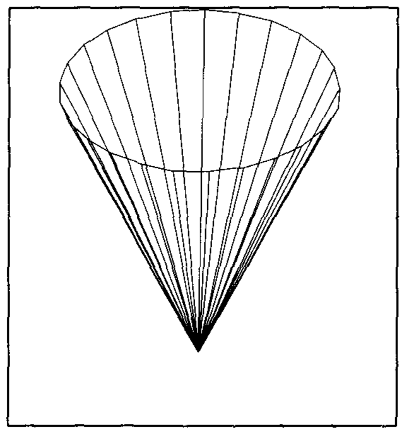
Figure 13.9(b)

Assuming arbitrary values, we can show the angular velocity precessing in a cone.

$n := 1..26 \quad m := 1 \quad I1 := 2 \quad I2 := 2 \quad I12 := 2 \quad I3 := 3 \quad \omega := 3$

$\omega z_n := 1 \quad \Omega B_m := \frac{I3 - I12}{I12} \cdot \omega z_n \cdot m \quad \Omega B_m = 0.5 \quad A_n := \sqrt{\omega^2 - (\omega z_n)^2}$

$\omega x_{n,m} := A_n \cdot \cos\left(\Omega B_m \cdot \frac{n}{2}\right) \quad \omega y_{n,m} := A_n \cdot \sin\left(\Omega B_m \cdot \frac{n}{2}\right) \quad Z_{n,m} := \omega z_n \cdot \pi \cdot 2$



ωx, ωy, Z

Precessing cone

observer in the body coordinate system, $\boldsymbol{\omega}$ *traces out a cone around the body symmetric axis.* This is called the *body cone*, and in the body reference frame its half angle, ϕ_B, is

$$\tan \phi_B = \frac{(\omega_1^2 + \omega_2^2)^{1/2}}{\omega_3} = \frac{A}{\omega_3} \quad \textbf{(13.149)}$$

as shown in Fig. 13.9(a) and (b).

Remember, we have been considering the force free motion of a rigid body. As viewed from the inertial system, there should be two constants of motion, the angular momentum and kinetic energy. Thus, as viewed from the fixed, LCS, or inertial coordinate system,

$$\mathbf{L}(t) = \text{constant} \quad \textbf{(13.150)}$$

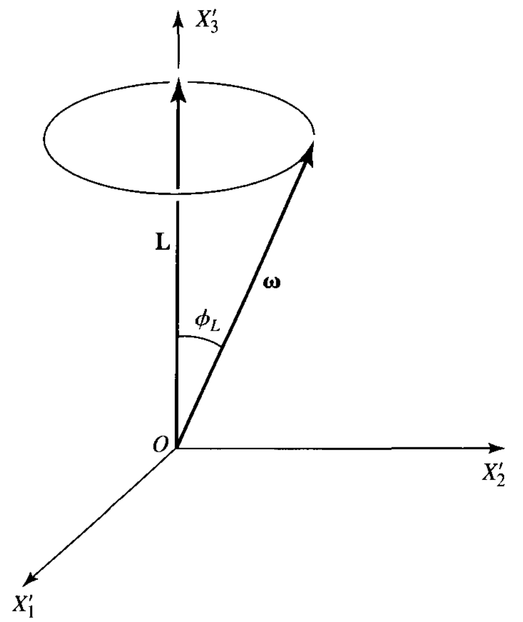


Figure 13.10 As viewed from a fixed, LCS, or inertial coordinate system, the angular vector $\boldsymbol{\omega}$ moves in such a way that its projection on the angular momentum vector $\mathbf{L}$ or the X_3'-axis is constant. ϕ_L is the half-angle of the space cone.

and is fixed about the X_3'-axis, as shown in Fig. 13.10. Since the center of mass is fixed, the kinetic energy is all rotational and constant; that is,

$$T_{\text{rot}} = \tfrac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L} = \text{constant} \tag{13.151}$$

We know that $\mathbf{L}$ is constant; T_{rot} will be constant only if $\boldsymbol{\omega}$ moves in such a way that its projection on the angular momentum vector $\mathbf{L}$ or the X_3'-axis is constant. As shown in Fig. 13.10, the angle ϕ_L between $\boldsymbol{\omega}$ and $\mathbf{L}$ is given by [using the definition of the dot product and Eq. (13.151)]

$$\cos \phi_L = \frac{\boldsymbol{\omega} \cdot \mathbf{L}}{\omega L} = \frac{2T_{\text{rot}}}{\omega L} = \text{constant} \tag{13.152}$$

Angle ϕ_L remains constant and is the half-angle of the *laboratory* or *space cone.* This cone is the result of precession of $\boldsymbol{\omega}$ about the constant angular momentum $\mathbf{L}$ as viewed from the inertial or LCS reference frame, $\mathbf{L}$, $\boldsymbol{\omega}$, and the X_3(body)-axis all lie in one plane, and since $\mathbf{L}$ has been designated to be along the X_3'(LCS)-axis, it has resulted in $\boldsymbol{\omega}$ precessing around the X_3'-axis when viewed in the LCS or inertial coordinate system. On the other hand, when viewed from the body coordinate system, $\boldsymbol{\omega}$ precesses around the X_3(body or symmetry)-axis. The situation is shown in Fig. 13.11(a) and (b) and may be described as one cone rolling on another; that is, the body cone is rolling without slipping around the LCS cone and the line of contact is the direction of the angular velocity $\boldsymbol{\omega}$, which precesses around the X_3-axis when viewed from the body reference frame and around the X_3'-axis when viewed from the LCS frame. The angular frequency of precession of $\boldsymbol{\omega}$ about the X_3-axis (the symmetry axis), as stated earlier, is Eq. (13.136)

$$\Omega_B \equiv \frac{I_3 - I_{12}}{I_{12}}\,\omega_3 = \gamma\omega_3 \tag{13.136}$$

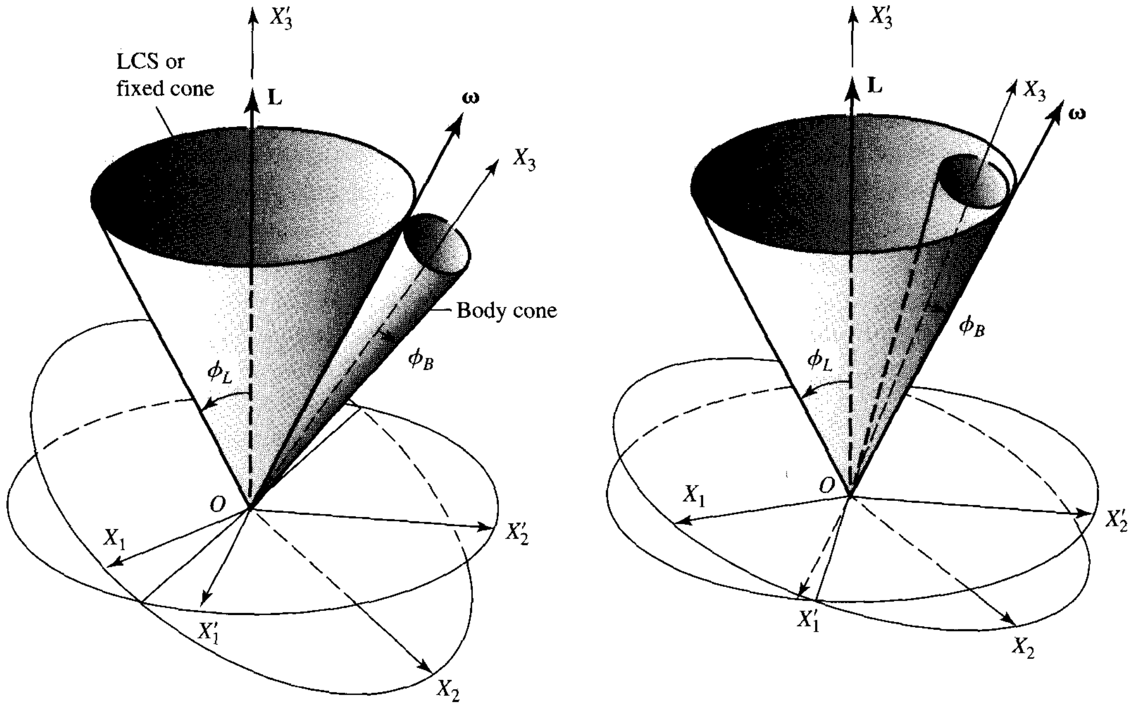


Figure 13.11 Body cone rolling around a LCS cone without slipping. Depending on the values of I_{12} and I_3, the body cone may roll (a) outside or (b) inside the LCS cone.

and the angular frequency of precession of $\boldsymbol{\omega}$ about the X_3'-axis (or $\mathbf{L}$) is

$$\Omega_L = \gamma\omega_3 \frac{\sin \phi_B}{\sin \phi_L} \tag{13.153}$$

Depending on the values of I_{12} and I_3, the body cone may roll outside or inside the LCS cone, as shown in Fig. 13.11.

One striking example is the application of the above theory to the rotating Earth. Earth is known to be slightly flattened near the poles, resulting in an oblate spheroid shape. This gives $I_3 \simeq I_{12}$ and $I_3 > I_{12}$, resulting in

$$\Omega = \frac{I_3 - I_{12}}{I_{12}} \omega_3$$

being very small as compared to ω_3, such that $\Omega \simeq \omega_3/300$. Since the period of Earth's rotation is $(1/\omega) = 1$ day and $\omega_3 \simeq \omega$, we get $(1/\Omega \simeq 300$ days)

$$T_P = \frac{2\pi}{\Omega} = \frac{2\pi I_{12}}{\omega_3(I_3 - I_{12})} = \frac{1 \text{ day}}{0.00327} = 305 \text{ days} \tag{13.154}$$

The measured value is $\simeq$ 440 days. The disagreement is not due to lack of knowledge of I_3 or I_{12}, but to the fact that Earth is not a perfect rigid body nor an oblate spheroid in shape. Actually, the shape of Earth resembles a flattened pear. Thus Earth's rotation axis precesses about the North Pole in a circle with a radius of $\simeq$10 m and with a period of about 430 days. Since latitude is dependent on the rotation axis, a measurable change in latitude results. Such changes in latitude are called the *Chandler wobble* and were discovered by S. C. Chandler in 1891.

Another and more familiar precession, Earth's axis about a cone with a half-angle 23.5°, is the result of the external gravitational torques due to the Sun and Moon. (That is, the rotational axis is inclined at 23.5° to the plane of Earth's orbit around the Sun.) This results in a slow precession of Earth'a axis. The period of such precessional motion is 26,000 years. This means that, as time passes, different stars become the polar star. Today the North Star (Polaris) is the polar star; in 3000 B.C., Thuban was the polar star; in 14,000 A.D., Vega will be the polar star. This is due to the precession of the rotational axis of Earth resulting from the gravitational forces of the Sun and Moon.

13.11 MOTION OF A SYMMETRICAL TOP WITH ONE POINT FIXED (THE HEAVY TOP)

A rigid body rotating about some fixed point O under the influence of a torque produced by its weight (in the gravitational force field) is called a *heavy top*. We shall limit our discussion to a special case of a symmetrical top in which $I_3 > I_1 = I_2$ $(=I_{12})$. Furthermore, the fixed point O does not coincide with the center of mass, but still lies on the symmetry axis. Such a situation is shown in Fig. 13.12. The fixed point is O, which coincides with the origins of the fixed and body coordinate systems. Because of the coincidence of the origins of the body and fixed sys-

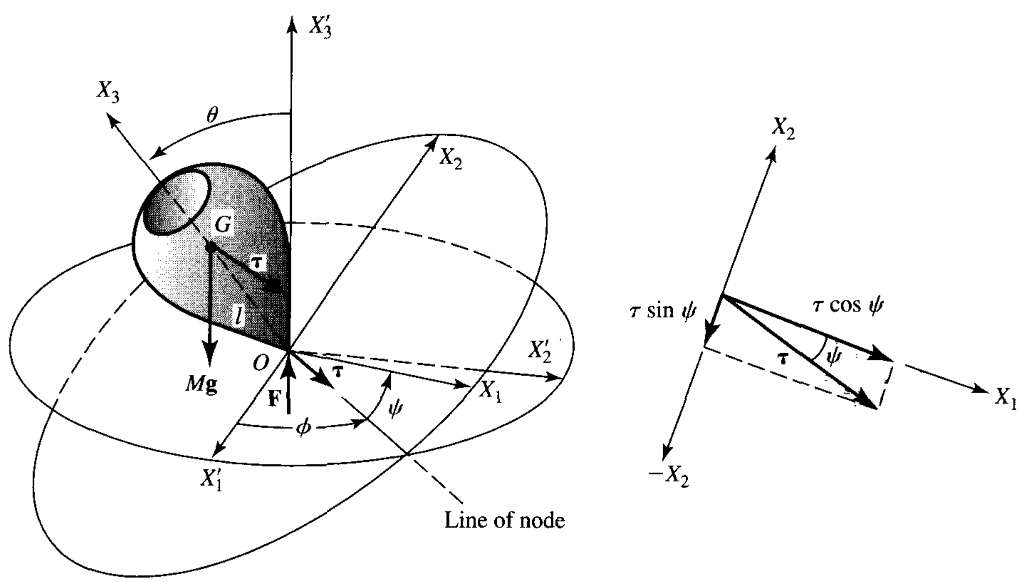


Figure 13.12 Heavy symmetrical top with one point fixed.

tems, the translational kinetic energy will be zero because $v = \dot{r} = 0$. The X_3'(fixed)-axis corresponds to the vertical, and the X_3(body)-axis is the symmetry axis of the top. The tip of the top is stationary at O. The only two forces acting are the reaction force $\mathbf{F}$ passing through point O, which does not produce any torque, while the gravitational force $M\mathbf{g}$ produces a torque $\boldsymbol{\tau}$ that is parallel to the line of nodes, as shown. We can use the Euler angles to describe the motion of the symmetrical top. The torque $\boldsymbol{\tau}$ on the symmetrical top is

$$\boldsymbol{\tau} = \mathbf{r} \times M\mathbf{g} \tag{13.155}$$

thus

$$\tau \simeq Mgl \sin\theta \tag{13.156}$$

The torque $\boldsymbol{\tau}$ that is along the lines of node may be resolved along the body axes, as shown in the insert in Fig. 13.12, resulting in

$$\tau_1 = Mgl \sin\theta \cos\psi \tag{13.157}$$

$$\tau_2 = Mgl \sin\theta \sin\psi \tag{13.158}$$

$$\tau_3 = 0 \tag{13.159}$$

Using the values of ω_1, ω_2, and ω_3 from Eqs. (13.120) and the Euler equations given by Eqs. (13.128), we obtain the following Euler's equations for a symmetrical top:

$$I_{12}\frac{d}{dt}(\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi) + (I_3 - I_{12})(\dot{\phi}\cos\theta + \dot{\psi})$$
$$\times (\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi) = Mgl\sin\theta\cos\psi \tag{13.160}$$

$$I_{12}\frac{d}{dt}(\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi) - (I_3 - I_{12})(\dot{\phi}\cos\theta + \dot{\psi})$$
$$\times (\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi) = -Mgl\sin\theta\sin\psi \tag{13.161}$$

$$I_3\frac{d}{dt}(\dot{\psi} + \dot{\phi}\cos\theta) = 0 \tag{13.162}$$

In principle, Euler's equations can be solved to obtain three first integrals (two angular momenta and one energy); hence the three Euler angles. Since this is quite cumbersome, we will simply summarize the results.

The kinetic energy of the symmetrical top is

$$T = \frac{1}{2}\sum_i I_i\omega_i^2 = \frac{1}{2}I_{12}(\omega_1^2 + \omega_2^2) + \frac{1}{2}I_3\omega_3^2 \tag{13.163}$$

or

$$T = \tfrac{1}{2}I_{12}(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \tfrac{1}{2}I_3(\dot{\phi}\cos\theta + \dot{\psi})^2 \tag{13.164}$$

while the potential energy is

$$V = Mgl\cos\theta \tag{13.165}$$

Thus the Lagrangian L is

$$L = L(\theta, \dot{\phi}, \dot{\theta}, \dot{\psi}) = T - V \tag{13.166}$$

We notice that ϕ and ψ are ignorable or cyclic coordinates. Therefore, the momenta conjugate to these coordinates are constants of motion. The cyclic coordinates are angles; the conjugate momenta are angular momenta. Thus

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = I_{12}\dot{\phi}\sin^2\theta + I_3\cos\theta(\dot{\phi}\cos\theta + \dot{\psi}) = \text{constant} \tag{13.167}$$

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3(\dot{\phi}\cos\theta + \dot{\psi}) = \text{constant} \tag{13.168}$$

These are the two first integrals of motion. Another first integral is the total energy E. Since the symmetrical top is in the gravitational force field, which is conservative, the total energy is a constant of motion and may be written as

$$\begin{aligned} E = T + V &= \tfrac{1}{2} I_{12}(\omega_1^2 + \omega_2^2) + \tfrac{1}{2} I_3\omega_3^2 + Mgl\cos\theta \\ &= \tfrac{1}{2} I_{12}(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \tfrac{1}{2} I_3(\dot{\phi}\cos\theta + \dot{\psi})^2 + Mgl\cos\theta = \text{constant} \end{aligned} \tag{13.169}$$

From Eq. (13.168),

$$p_\psi = I_3(\dot{\phi}\cos\theta + \dot{\psi}) = I_3\omega_3 = \text{constant} \tag{13.170}$$

and

$$\frac{1}{2} I_3\omega_3^2 = \frac{p_\psi^2}{2I_3} = \text{constant} \tag{13.171}$$

Thus not only E, but $E' = E - \frac{1}{2}I_3\omega_3^2$ is also a constant of motion. Substituting the values of $\dot{\phi}$ from Eq. (13.75) in Eq. (13.168) and after rearranging, we get

$$E' = \tfrac{1}{2} I_{12}\dot{\theta}^2 + V(\theta) \tag{13.172}$$

where

$$V(\theta) = \frac{p_\phi - p_\psi\cos\theta}{2I_{12}\sin^2\theta} + Mgl\cos\theta \tag{13.173}$$

and $V(\theta)$ is called the *effective potential*. From Eq. (13.172),

$$\dot{\theta} = \left(\frac{2}{I_{12}}[E' - V(\theta)]\right)^{1/2} \tag{13.174}$$

which on integration gives

$$t(\theta) = \int \frac{d\theta}{\sqrt{(2/I_{12})[E' - V(\theta)]}} \tag{13.175}$$

This equation, in principle, can be solved to obtain $\theta(t)$. These values of $\theta(t)$ can be used to yield the values of $\phi(t)$ and $\psi(t)$. Thus we have all three Eulerian angles that specify the orientation of a rigid body. Hence, the problem at hand is completely solved. Unfortunately, the integration of these equations involves an elliptic integral and the procedure becomes complicated. Hence, it becomes essential to limit ourselves to a qualitative discussion, similar to the one used in describing the motion of a particle in a central force field.

Steady Precession

Figure 13.13 shows the plot of effective potential $V(\theta)$ [given by Eq. (13.173)] versus θ between the physically acceptable range of $0 \leq \theta \leq \pi$. This energy diagram with a minimum effective potential is similar to the diagram for the central force field. For any energy value $E' = E_1'$, the motion is limited between two extreme values, which are similar to the turning points, that is, between $\theta = \theta_1$ and $\theta = \theta_2$, as shown. This implies that the symmetrical axis OX_3 of the rotating top can vary its inclination θ to the vertical between $\theta_1 \leq \theta \leq \theta_2$. If the energy of the top is such that $E' = E_0' = V_{min}$, the value of θ is limited to a single value of $\theta = \theta_0$, as shown. The resulting motion is a steady precession at a fixed angle of inclination θ_0. This is an interesting special case of steady precession in which the axis of the gyroscope or heavy top describes a right

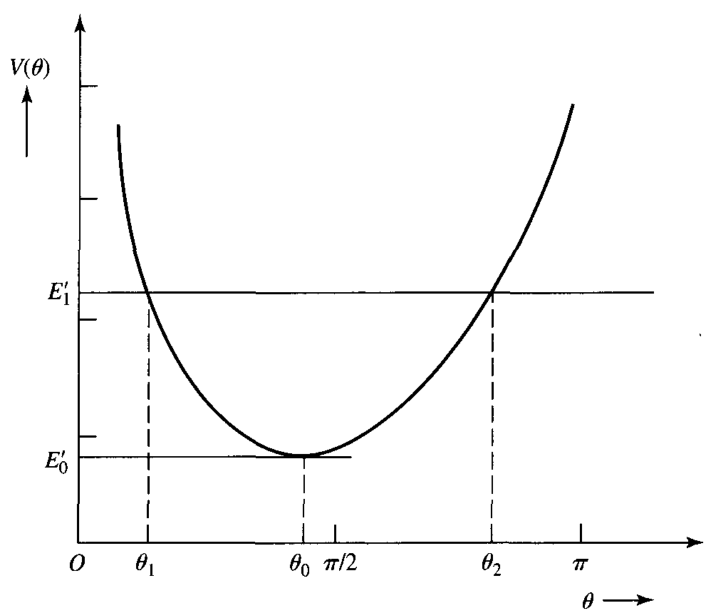


Figure 13.13 Energy diagram: plot of effective potential $V(\theta)$ versus θ ($0 \leq \theta \leq \pi$) for a heavy symmetrical top.

circular cone about the vertical (X_3'-axis). Before discussing the general situation, we shall discuss this case in some detail. We can evaluate the value θ_0 by setting the derivative of the effective potential $V(\theta)$ equal to zero at θ_0. (Note that, in general, $\tau(\theta) = dV(\theta)/d\theta$.) Hence, from Eq. (13.173),

$$\tau(\theta_0) = \left| -\frac{\partial V}{\partial \theta} \right|_{\theta=\theta_0}$$

$$= \frac{(p_\phi - p_\psi \cos\theta_0)^2 \cos\theta_0 - (p_\phi - p_\psi \cos\theta_0)\, p_\psi \sin^2\theta_0}{I_{12}\sin^3\theta_0} + Mgl\sin\theta_0 = 0 \qquad \textbf{(13.176)}$$

Let us define

$$\gamma \equiv p_\phi - p_\psi \cos\theta_0 \qquad \textbf{(13.177)}$$

and rewrite Eq. (13.176) as

$$(\cos\theta_0)\gamma^2 - (p_\psi \sin^2\theta_0)\gamma + (MglI_{12}\sin^4\theta_0) = 0 \qquad \textbf{(13.178)}$$

This is a quadratic in γ, which has two values. For the given value of $\theta = \theta_0$, let us discuss the value of $\dot{\phi}$. The precessional angular velocity $\dot{\phi}_0$ has two possible values, one for each value of γ given by solving Eq. (13.178). A large value of γ results in fast precession and a small value in slow precession; that is,

$$\dot{\phi}_0(+) = \dot{\phi}_{0f} \rightarrow \text{fast precession}$$
$$\dot{\phi}_0(-) = \dot{\phi}_{0s} \rightarrow \text{slow precession} \qquad \textbf{(13.179)}$$

It is this slow precessional angular velocity, that is usually observed in gyroscopes.

Thus, for the symmetry axis at $\theta = \theta_0$ and less than $\pi/2$, the top is rotating about the symmetry axis at frequency ω_3 and the symmetry axis can precess about the fixed axis with two possible frequencies $\dot{\phi}$. A special case is in order. If the top is spinning sufficiently fast and is in the vertical position, the axis of the top will remain fixed in the vertical direction. This condition is called *sleeping,* and the top is a *sleeping top.* If the top slows down due to friction or other causes, the top starts undergoing a nutation (as discussed later) and eventually will topple over.

Let us now discuss the case in which $\theta_0 > \pi/2$. In such a case, the fixed tip of the top is at a position above the center of mass. The symmetrical top is hanging with its axis below the horizontal. Furthermore, the values of $\dot{\phi}_{0f}$ and $\dot{\phi}_{0s}$ have opposite signs. That is, for $\theta_0 > \pi/2$, the fast precession $\dot{\phi}_{0f}$ is in the same direction as that for $\theta_0 < \pi/2$, while the slow precession $\dot{\phi}_{0s}$ takes place in the opposite sense.

θ Motion: Nutation

As discussed earlier in connection with the effective potential $V(\theta)$ versus θ plot in Fig. 13.13, the motion of the symmetrical axis is limited between $\theta_1 < \theta < \theta_2$ for any given energy E' of the top. As θ varies between these limits, the value of $\dot{\phi}$ may or may not change sign. If there is no change in the sign of $\dot{\phi}$, the top precesses monotonically around the fixed, inertial, or LCS X_3'-axis, while the X_3(symmetry)-axis of the body oscillates between $\theta = \theta_1$ and $\theta = \theta_2$. This motion of the top is called *nutation*. The path desribed by the body symmetry axis when projected on a unit sphere in the fixed system is shown in Fig. 13.14(a). On the other hand, if $\dot{\phi}$ does change sign between the limiting values of θ, the precessional angular velocity must have opposite signs at $\theta = \theta_1$ and $\theta = \theta_2$. In this situation the nutational-precessional motion results in a looping motion of the symmetry axis, as shown in Fig. 13.14(b), which is a projection of the symmetry axis on a unit sphere. Note that the changes in $\dot{\phi}$ are not only due to the values of p_ϕ and p_ψ. If these values are such that at $\theta = \theta_1$,

$$(p_\phi - p_\psi \cos\theta)\big|_{\theta=\theta_1} = 0 \tag{13.180}$$

then

$$\dot{\phi}\big|_{\theta=\theta_1} = 0, \qquad \dot{\theta}\big|_{\theta=\theta_1} = 0, \quad \text{and} \quad \dot{\psi} = \omega_3 \tag{13.181}$$

The resulting motion of the projection of the symmetry axis on a sphere is cusplike, as shown in Fig. 13.14(c). (a) and (c) are redrawn for arbitrary values in Fig. 13.14(d).

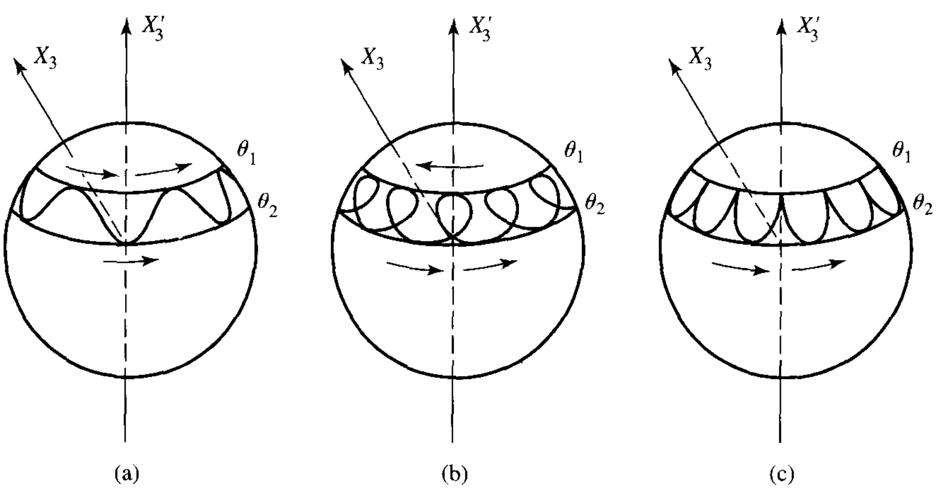


Figure 13.14 θ motion (nutation), the motion of the symmetrical axis limited between $\theta_1 < \theta < \theta_2$. The diagrams show the path of a body symmetry axis as projected on a unit sphere in a fixed system. (a) monotonic precession, (b) looping motion, and (c) cusplike motion of the symmetry axis about a fixed axis.

Figure 13.14(d)

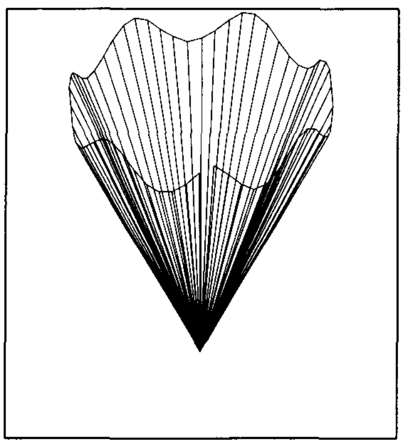
Assuming arbitrary values, we can show the angular velocity precessing in a cone. The two cases (a) and (c) above are redrawn here.

$n := 1..60 \quad m := 1 \quad I1 := 2 \quad I2 := 2 \quad I12 := 3.5 \quad I3 := 4 \quad \omega := 40$

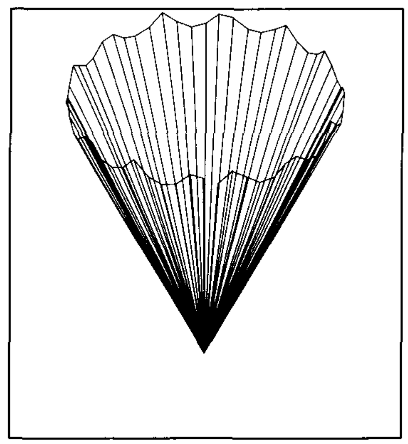
$\omega z := 5 \quad \Omega B_m := \frac{I3 - I12}{I12} \cdot \omega z \quad \Omega B_m = 0.714 \quad A := \sqrt{\omega^2 - (\omega z)^2}$

$\omega x_{n,m} := A \cdot \sin(\Omega B_m \cdot m) \cdot \sin\left(n \cdot \frac{2 \cdot \pi}{60}\right) \qquad \omega y_{n,m} := A \cdot \sin(\Omega B_m \cdot m) \cdot \cos\left(n \cdot \frac{2 \cdot \pi}{60}\right)$

$Za_{n,m} := 4 \cdot \cos(\Omega B_m \cdot n) + 60 \qquad Zc_{n,m} := -4 \cdot \left|\cos(\Omega B_m \cdot n)\right| + 60$



ωx, ωy, Za
Precessing cone



ωx, ωy, Zc
Precessing cone

Compare these with (a) and (c) and explain the difference.

PROBLEMS

13.1. Combine Eq. (13.7) with Eq. (13.3) to obtain the results in Eq. (13.6).

13.2. Prove the following identities.

(a) $\mathbf{A} \times (\mathbf{B} \times \mathbf{A}) = A^2B - \mathbf{A}(\mathbf{A} \cdot \mathbf{B})$

(b) $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{C}) - (\mathbf{B} \cdot \mathbf{C})(\mathbf{A} \cdot \mathbf{D})$

(c) $(\mathbf{A} \times \mathbf{B})^2 = (\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{B}) = (\mathbf{A} \cdot \mathbf{A})(\mathbf{B} \cdot \mathbf{B}) - (\mathbf{B} \cdot \mathbf{A})(\mathbf{A} \cdot \mathbf{B}) = A^2B^2 - (\mathbf{A} \cdot \mathbf{B})^2$

13.3. Find the angular momentum and kinetic energy for the rotation of a uniform square lamina of side L and mass M about a diagonal with an angular velocity $\boldsymbol{\omega}$.

13.4. Consider a uniform rectangular lamina of sides a and b and surface density (mass per unit area) σ. It is rotating about a diagonal with constant angular velocity ω. Find the magnitude and the direction of the angular momentum about an axis passing through **(a)** its center, and **(b)** one of its corners. Also calculate the rotational kinetic energy.

13.5. A uniform disk of radius R and mass M is rotating with uniform angular velocity ω about an axis that makes an angle θ with the axis of the disk. Calculate **(a)** the angular momentum (magnitude as well as direction), and **(b)** the total rotational kinetic energy.

13.6. A particle of mass m is rotating in a vertical plane with angular velocity $\boldsymbol{\omega}$, which lies in the XY plane and makes an angle of 45° with the X-axis, as shown in Fig. P13.6. Calculate velocity $\mathbf{v}$, angular momentum $\mathbf{L}$, and the rotational kinetic energy. Are $\boldsymbol{\omega}$ and $\mathbf{L}$ in the same direction? What does this imply?

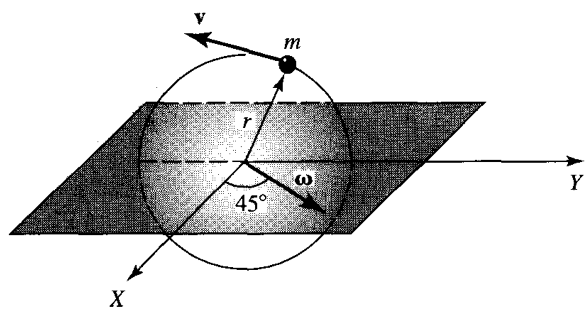


Figure P13.6

13.7. A thin uniform disk of mass m, radius r, and thickness h rolls without slipping about the Z-axis. It is supported by an axle of length R through its center (as shown in Fig. P13.7) and circles around the Z-axis with angular velocity Ω. Calculate the instantaneous angular velocity $\boldsymbol{\omega}$ of the disk and its angular momentum $\mathbf{L}$. Are $\boldsymbol{\omega}$ and $\mathbf{L}$ parallel to each other?

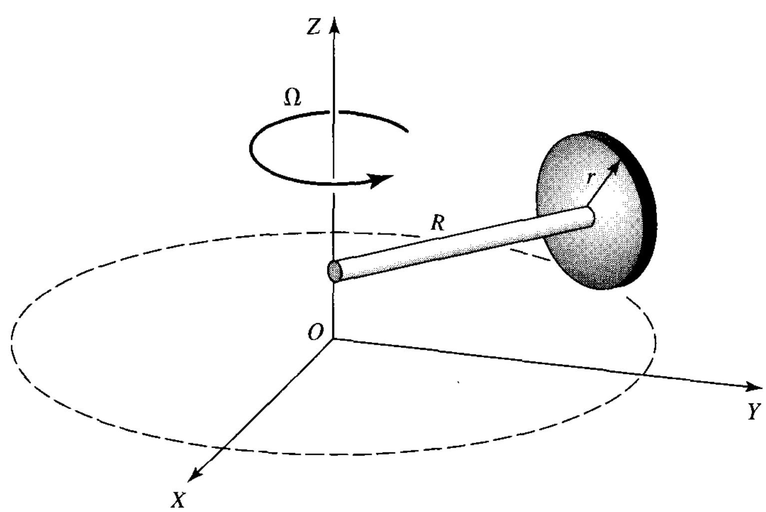


Figure P13.7

13.8. The axis of symmetry of a misaligned armature makes an angle ϕ with the rotation axis, as shown in Fig. P13.8. If the rotational angular velocity of the armature is $\dot{\theta}$, what are the reactions at the bearings at A and B? The disk has a radius of R, thickness h, and mass M.

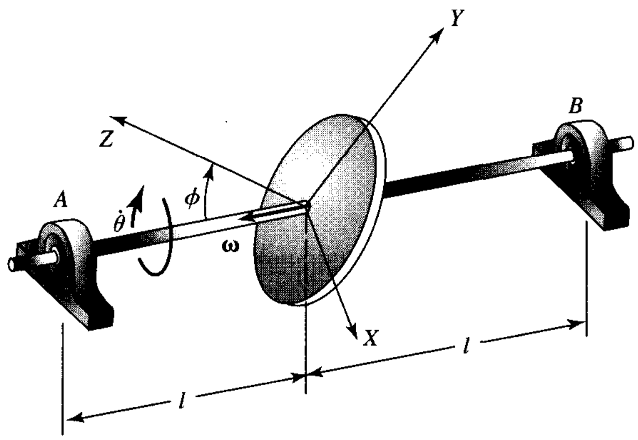


Figure P13.8

13.9. Calculate the components of a moment of inertia tensor for the following configuration. Point masses 2, 3, 1, and 5 units located at (0, 0, 1), (0, 1, 1), (1, −1, 1), and (1, −1, −1).

13.10. Calculate the components of a moment of inertia tensor for the following configuration. Point masses 2, 3, 6, and 8 units located at (0, 1, 2), (0, 2, 2), (2, 2, 2), and (2, −2, −2).

13.11. Find the elements of the inertia tensor of a rod of mass M and length l. The origin of the coordinate system is at its center, the X-axis is along the length of the rod, and the Z-axis is perpendicular to the rod.

13.12. Find the inertia tensor for a square lamina of length L and mass M for a coordinate system whose origin is located **(a)** at one corner, and **(b)** at the center of the lamina.

13.13. Find the inertia tensor of a rectangular lamina of sides L and W and mass M for a coordinate system whose origin is **(a)** at one corner, and **(b)** at the center of the lamina.

13.14. Consider a homogeneous sphere of mass M and radius R. Find a coordinate system whose origin is at the center of the sphere, calculate the moments of inertia I_1, I_2, and I_3.

13.15. Show that for any homogeneous regular polyhedron the principal moments of inertia will all be equal for a coordinate system whose origin is at the center of the polyhedron. Find the radius of a homogeneous solid sphere of the same mass that has the same moment of inertia elements.

13.16. Find the inertia tensor of a rectangular block of mass M and dimension $a \times b \times c$. The origin of the axes coincides with the center of mass, the Z-axis is parallel to the thickness c, and the Y-axis is parallel to a diagonal of rectangular $a \times b$, as shown in Fig. P13.16. Find the relation between the coordinates axes and the principal coordinate axes.

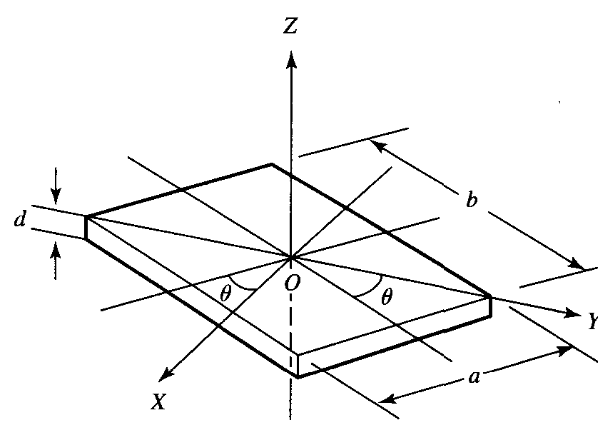


Figure P13.16

13.17. Consider a right triangular solid of mass M and length L along the X_1-, X_2-, and X_3-axes, as shown in Fig. P13.17. Calculate the elements of the inertia tensor for these axes.

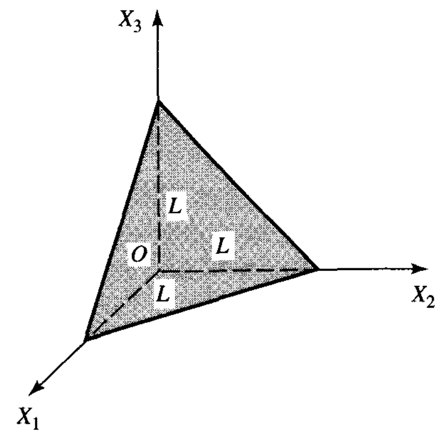


Figure P13.17

13.18. Six particles, each of mass m, are fixed at the end of massless rods of length $2l$. The rods are perpendicular to each other, as shown in Fig. P13.18. For the axis along the three rods, calculate the elements of the inertia tensor for this configuration. Show that these axes are the principal axes.

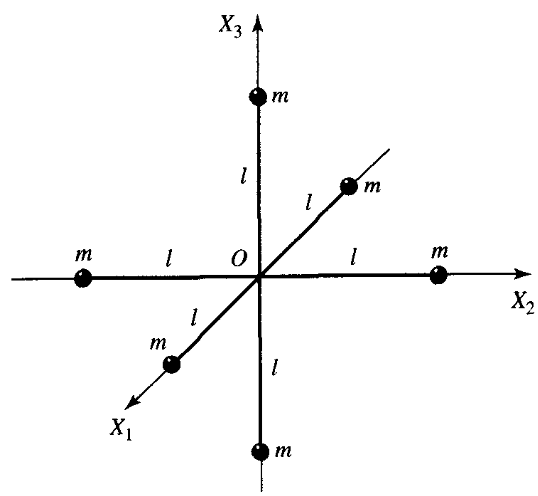


Figure P13.18

13.19. Show that any one of the three principal moments of inertia cannot be greater than the sum of the other two.

13.20. Consider a uniform density object of mass M in the shape of an ellipsoid whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

where the axes $2a > 2b > 2c$ are the dimensions of the solid. Find the principal moments of inertia I_1, I_2, and I_3 and the principal axes.

13.21. Consider a homogeneous cone of mass M, height h, and radius R. Let the origin be at the apex of the cone and the X_3-axis be along the axis of symmetry of the cone. Calculate the elements of the inertia tensor. Now make the transformation so that the center of mass of the cone is also the origin of the coordinate axes. Find the principal moment of inertia about this new coordinate system.

13.22. Show that in a plane rectangular lamina the direction of the principal axis at a corner is given by

$$\tan 2\phi = \frac{2(Mab/4)}{(Ma^2/3) - (Mb^2/3)} = \frac{3ab}{2(a^2 - b^2)}$$

13.23. For the following two cases, draw a sketch of the ellipsoid of inertia: **(a)** a uniform circular disk of radius R and mass M; **(b)** a solid rectangular parallelepiped of mass M and sides a, $2a$, and $4a$.

13.24. Draw a sketch of the ellipsoid of inertia of a solid right circular cylinder of radius R and length L ($= 8R$). What should be the ratio R/L so that the ellipsoid of inertia at the center is a sphere?

13.25. Find the principal moment of inertia and principal axes for the right triangular solid discussed in Problem 13.17 and shown in Fig. P13.17. What types of rotation and rotation matrices are needed to go from the given axes to the principal axes?

13.26. Consider a right elliptical cylinder of mass M. The cylinder is bounded by plane ends with $Z = -c$ and $Z = +c$, while its wall is an elliptical surface defined by $(x/a)^2 + (y/b)^2 = 1$. Calculate the moment of inertia for rotation of the cylinder about the Z-axis.

13.27. In Problem 13.12, transfer from **(a)** to **(b)** by a proper rotation matrix.

13.28. In Problem 13.13, transfer from **(a)** to **(b)** by a proper rotation matrix.

13.29. The trace of a tensor $\mathbf{I}$ is defined as the sum of the diagonal elements; that is, trace $\mathbf{I} \equiv \Sigma_k I_{kk}$. By performing a similarity transformation, show that the trace is invariant under a coordinate transformation; that is, trace $\mathbf{I} \equiv$ trace $\mathbf{I}'$, where $\mathbf{I}'$ is the tensor in a coordinate system rotated with respect to the coordinate system of $\mathbf{I}$.

13.30. Show that the determinant of the elements of a tensor is invariant under different coordinate systems rotated with respect to each other.

13.31. Consider a tensor $\mathbf{T} = \mathbf{AB} + \mathbf{BA}$, where $\mathbf{A} = 10\hat{\mathbf{u}}_1 - 6\hat{\mathbf{u}}_2 + 4\hat{\mathbf{u}}_3$ and $\mathbf{B} = \hat{\mathbf{u}}_2 + 2\hat{\mathbf{u}}_3$. Transform this tensor into a coordinate system rotated 45° about the X_3-axis. Diagonalize the resulting tensor.

13.32. Diagonalize the following tensor and find the principal axes:

$$\mathbf{T} = \begin{pmatrix} 7 & \sqrt{6} & -\sqrt{3} \\ \sqrt{6} & 2 & -5\sqrt{2} \\ -\sqrt{3} & -5\sqrt{2} & -3 \end{pmatrix}$$

13.33. Consider a thin homogeneous plate of mass M and of dimensions l and w that lies in the X_1-X_2 plane. Show that its inertia tensor has the following form:

$$\mathbf{T} = \begin{pmatrix} A & -C & 0 \\ -C & B & 0 \\ 0 & 0 & A + B \end{pmatrix}$$

Calculate the value of A, B, and C in terms of M, l, and w.

Now rotate the coordinate axis through angle θ about the X_3-axis. Show that the new inertia tensor takes the form

$$\mathbf{T}' = \begin{pmatrix} A' & -C' & 0 \\ -C' & B' & 0 \\ 0 & 0 & A' + B' \end{pmatrix}$$

Calculate the values of A', B', and C' in terms of A, B, and C and θ.

Show that if the angle of rotation θ is given by the expression

$$\theta = \frac{1}{2}\tan^{-1}\frac{2C}{B - A}$$

then the X_2- and X_3-axes will be the principal axes.

13.35. Obtain the components of angular velocity $\boldsymbol{\omega}$ directly from the transformation matrix $\boldsymbol{\lambda}$ given by Eq. (13.115).

13.36. Obtain the inverse transformation matrix of $\boldsymbol{\lambda}$ given in Eq. (13.115), and then obtain the components of $\boldsymbol{\omega}'$.

13.37. By using Fig. 13.8, obtain the components of $\boldsymbol{\omega}$ along the fixed X_i'-axes; that is, calculate ω_1', ω_2', and ω_3'.

13.38. Derive Eq. (13.153).

13.39. Find the componets of a tensor that corresponds to a rotation by an angle θ about the Z-axis and followed by a rotation by an angle ϕ about the Y-axis.

13.40. Consider a homogeneous sphere with moments of inertia $I_1 = I_2 = I_3$. Find the equations of motion of the sphere by using Euler's equations.

13.41. A uniform rod of length l and mass m is mounted on a horizontal frictionless axle through its center. The axle is mounted on a platform that is rotating with angular velocity Ω, as shown in Fig. P13.41. The axis of the platform passes through the center of the rod. Using Euler's equations, calculate the angle θ that the rod makes with the horizontal as a function of time. Show that for a small θ the motion of the rod is simple harmonic with angular frequency $[(I_3 - I_2)/I_1]^{1/2}\Omega$.

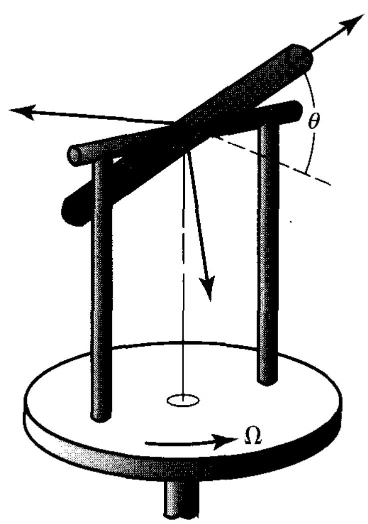


Figure P13.41

13.42. Derive Eqs. (13.198) and (13.199).

13.43. Consider the force free motion of a symmetrical top and show that the angular velocity $\boldsymbol{\omega}$, the angular momentum $\mathbf{L}$ about the fixed (space) X_3'-axis, and the body X_3-axis are coplanar.

13.44. A circular disk of mass M and radius R is rotating freely under no external torque. The angle between the axis of symmetry of the disk and the angular velocity $\boldsymbol{\omega}$ is ϕ. Calculate the time in which the axis of symmetry describes a cone about the direction of $\mathbf{L}$, that is, about the invariable line.

13.45. Consider the force free rotation of a plane lamina. By using Euler's equations, show that the component of the angular velocity in the plane of the lamina is constant in magnitude. Under what conditions will the component of the angular velocity normal to the plane of the lamina be constant?

13.46. Consider a symmetrical rigid body moving freely in space and powered by two jet engines that are symmetrically placed with respect to the symmetry body axis (that is, X_3-axis) and supply a constant torque $\boldsymbol{\tau}$ about the symmetry axis. Find the general expression for the angular velocity $\boldsymbol{\omega}$ as a function of time. Show that $\boldsymbol{\omega}$ increases in magnitude with time, and its components perpendicular to the X_3-axis describe a constant ellipse.

13.47. Consider a rigid body with three different principal moments of inertia, $I_1 > I_2 > I_3$, rotating freely about its center of mass. Show by using Euler's equations that the rotational motion of the body is stable about either the axis of greatest moments of inertia or the axis of least moment of inertia.

13.48. A symmetrical rigid body rotates freely about a fixed point free of any external torque. Let θ be the angle between the axis of rotation and the axis of the system. The moment of inertia I_s about the symmetry axis is greater than the moment of inertia I_n about an axis normal to the symmetry axis. Show that the angle between the axis of rotation and the invariable line (the $\mathbf{L}$ vector) is

$$\tan^{-1}\left[\frac{(I_s - I_n)\tan\theta}{I_s + I_n\tan^2\theta}\right]$$

What is the maximum possible value for this angle?

13.49. A flywheel (a disk of mass M and radius R) is mounted with its axis vertical in a truck and works as a stabilizing gyroscope. Suppose the disk is rotating at full speed ω. Show that the torque needed to make it precess in a vertical plane is $\tau \simeq \frac{1}{2} MR^2\omega\Omega$, where Ω is the frequency of precession.

13.50. Suppose a heavy top is spinning in a stable configuration. What is the effect of friction on the motion as friction gradually reduces the value of ω_3?

13.51. A simple gyroscope consists of a disk of 0.2-kg mass and has a radius of 0.06 m; it is mounted at the center of a light rod of length 0.12 m. It is set spinning such that precessional frequency is 0.2 revolution per second. Calculate approximately the spinning frequency.

13.52. A symmetrical rigid body rotates with an angular velocity $\boldsymbol{\omega}$ in three-dimensional motion about its center of mass. If there is a frictional torque $-b\boldsymbol{\omega}$ due to air drag, show that the component of $\boldsymbol{\omega}$ in the direction of the symmetry axis decreases exponentially with time.

13.53. To investigate the turning points of the nutational motion of a symmetrical top, we substitute $\dot{\theta} = 0$. Show that the resulting equation is a cubic in $\cos\theta$ with two real roots and one imaginary root for θ.

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