
Lecture Notes in Mathematics

Arkansas Tech University
Department of Mathematics

A First Course in Quasi-Linear Partial
Differential Equations
for Physical Sciences and Engineering

Marcel B. Finan
Arkansas Tech University
©All Rights Reserved

October 3, 2019

Preface

Partial differential equations are often used to construct models of the most basic theories underlying physics and engineering. The goal of this book is to develop the most basic ideas from the theory of partial differential equations, and apply them to the simplest models arising from the above mentioned fields.

It is not easy to master the theory of partial differential equations. Unlike the theory of ordinary differential equations, which relies on the fundamental existence and uniqueness theorem, there is no single theorem which is central to the subject. Instead, there are separate theories used for each of the major types of partial differential equations that commonly arise.

It is worth pointing out that the preponderance of differential equations arising in applications, in science, in engineering, and within mathematics itself, are of either first or second order, with the latter being by far the most prevalent. We will mainly cover these two classes of PDEs.

This book is intended for a first course in partial differential equations at the advanced undergraduate level for students in engineering and physical sciences. It is assumed that the student has had the standard three semester calculus sequence, and a course in ordinary differential equations.

Marcel B Finan

August 2009

Contents

Preface	i
The Basics of the Theory of Partial Differential Equation	3
1 Basic Concepts	4
2 Solutions to PDEs/PDEs with constraints	13
Review of Some ODE Results	25
3 The Method of Integrating Factor	26
4 The Method of Separation of Variables for ODEs	31
First Order Partial Differential Equations	35
5 Classification of First Order PDEs	36
6 A Review of Multivariable Calculus	41
6.1 Multiplication of Vectors: The Scalar or Dot Product	41
6.2 Directional Derivatives and the Gradient Vector	50
7 Solvability of Semi-linear First Order PDEs	59
8 Linear First Order PDE: The One Dimensional Spatial Transport Equations	65
9 Solving Quasi-Linear First Order PDE via the Method of Characteristics	71
10 The Cauchy Problem for First Order Quasilinear Equations	75
Second Order Linear Partial Differential Equations	87
11 Second Order PDEs in Two Variables	88
12 Hyperbolic Type: The Wave equation	93
13 Parabolic Type: The Heat Equation in One-Dimensional Space	100
14 Sequences of Functions: Pointwise and Uniform Convergence	109
15 An Introduction to Fourier Series	120

16	Fourier Sines Series and Fourier Cosines Series	133
17	Separation of Variables for PDEs	140
17.1	Second Order Linear Homogenous ODE with Constant Coefficients	140
17.2	The Method of Separation of Variables for PDEs	141
18	Solutions of the Heat Equation by the Separation of Variables Method	147
19	Elliptic Type: Laplace's Equations in Rectangular Domains	154
20	Laplace's Equations in Circular Regions	165
The Laplace Transform Solutions for PDEs		177
21	Essentials of the Laplace Transform	178
22	Solving PDEs Using Laplace Transform	191
The Fourier Transform Solutions for PDEs		199
23	Complex Version of Fourier Series	200
24	An introduction to Fourier Transforms	206
25	Applications of Fourier Transforms to PDEs	214
Appendix		221
Appendix A: The Method of Undetermined Coefficients		222
Appendix B: The Method of Variation of Parameters		229
Answers and Solutions		233
Index		289

The Basics of the Theory of Partial Differential Equation

Many fields in engineering and the physical sciences require the study of ODEs and PDEs. Some of these fields include acoustics, aerodynamics, elasticity, electrodynamics, fluid dynamics, geophysics (seismic wave propagation), heat transfer, meteorology, oceanography, optics, petroleum engineering, plasma physics (ionized liquids and gases), quantum mechanics.

So the study of partial differential equations is of great importance to the above mentioned fields. The purpose of this chapter is to introduce the reader to the basic terms of partial differential equations.

1 Basic Concepts

The goal of this section is to introduce the reader to the basic concepts and notations that will be used in the remainder of this book.

We start this section by reviewing the concept of partial derivatives and the chain rule of functions in two variables.

Let $u(x, y)$ be a function of the independent variables x and y . The **first derivative of u with respect to x** is defined by

$$u_x(x, y) = \frac{\partial u}{\partial x}(x, y) = \lim_{h \rightarrow 0} \frac{u(x+h, y) - u(x, y)}{h}$$

provided that the limit exists.

Likewise, the **first derivative of u with respect to y** is defined by

$$u_y(x, y) = \frac{\partial u}{\partial y}(x, y) = \lim_{h \rightarrow 0} \frac{u(x, y+h) - u(x, y)}{h}$$

provided that the limit exists.

We can define higher order derivatives such as

$$u_{xx}(x, y) = \frac{\partial^2 u}{\partial x^2}(x, y) = \lim_{h \rightarrow 0} \frac{u_x(x+h, y) - u_x(x, y)}{h}$$

$$u_{yy}(x, y) = \frac{\partial^2 u}{\partial y^2}(x, y) = \lim_{h \rightarrow 0} \frac{u_y(x, y+h) - u_y(x, y)}{h}$$

$$u_{xy}(x, y) = \frac{\partial^2 u}{\partial x \partial y}(x, y) = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y} \right) = \lim_{h \rightarrow 0} \frac{u_y(x+h, y) - u_y(x, y)}{h}$$

and

$$u_{yx}(x, y) = \frac{\partial^2 u}{\partial y \partial x}(x, y) = \lim_{h \rightarrow 0} \frac{u_x(x, y+h) - u_x(x, y)}{h}$$

provided that the limits exist.¹

An important formula of differentiation is the so-called **chain rule**. If $u = u(x, y)$, where $x = x(s, t)$ and $y = y(s, t)$, then

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s}.$$

Likewise,

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t}.$$

¹If u_{xy} and u_{yx} are continuous then $u_{xy}(x, y) = u_{yx}(x, y)$.

Example 1.1

Compute the partial derivatives indicated:

(a) $\frac{\partial}{\partial y}(y^2 \sin xy)$

(b) $\frac{\partial^2}{\partial x^2}[e^{x+y}]^2$

Solution.

(a) We have $\frac{\partial}{\partial y}(y^2 \sin xy) = \sin xy \frac{\partial}{\partial y}(y^2) + y^2 \frac{\partial}{\partial y}(\sin xy) = 2y \sin xy + xy^2 \cos xy$.

(b) We have $\frac{\partial}{\partial x}[e^{x+y}]^2 = \frac{\partial}{\partial x}e^{2(x+y)} = 2e^{2(x+y)}$. Thus, $\frac{\partial^2}{\partial x^2}[e^{x+y}]^2 = \frac{\partial}{\partial x}2e^{2(x+y)} = 4e^{2(x+y)}$ ■

Example 1.2

Suppose $u(x, y) = \sin(x^2 + y^2)$, where $x = te^s$ and $y = s + t$. Find $u_s(s, t)$ and $u_t(s, t)$.

Solution.

We have

$$\begin{aligned} u_s(s, t) &= u_x x_s + u_y y_s = 2x \cos(x^2 + y^2) te^s + 2y \cos(x^2 + y^2) \\ &= [2t^2 e^{2s} + 2(s + t)] \cos[t^2 e^{2s} + (s + t)^2]. \end{aligned}$$

Likewise,

$$\begin{aligned} u_t(s, t) &= u_x x_t + u_y y_t = 2x \cos(x^2 + y^2) e^s + 2y \cos(x^2 + y^2) \\ &= [2te^{2s} + 2(s + t)] \cos[t^2 e^{2s} + (s + t)^2] \blacksquare \end{aligned}$$

A **differential equation** is an equation that involves an unknown scalar function (the dependent variable) and one or more of its derivatives. For example,

$$\frac{d^2 y}{dx^2} - 5 \frac{dy}{dx} + 3y = -3 \quad (1.1)$$

or

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + u = 0. \quad (1.2)$$

If the unknown function is a function in one single variable then the differential equation is called an **ordinary differential equation**, abbreviated by ODE. An example of an ordinary differential equation is Equation (1.1). In contrast, when the unknown function is a function of two or more independent variables then the differential equation is called a **partial differential equation**, in short PDE. Equation (1.2) is an example of a partial differential equation. In this book we will be focusing on partial differential equations.

Example 1.3

Identify which variables are dependent and which are independent for the following differential equations.

- (a) $\frac{d^4y}{dx^4} - x^2 + y = 0$.
- (b) $u_{tt} + xu_{tx} = 0$.
- (c) $x\frac{dx}{dt} = 4$.
- (d) $\frac{\partial y}{\partial u} - 4\frac{\partial y}{\partial v} = u + 3y$.

Solution.

- (a) Independent variable is x and the dependent variable is y .
- (b) Independent variables are x and t and the dependent variable is u .
- (c) Independent variable is t and the dependent variable is x .
- (d) Independent variables are u and v and the dependent variable is y ■

Example 1.4

Classify the following as either ODE or PDE.

- (a) $u_t = c^2u_{xx}$.
- (b) $y'' - 4y' + 5y = 0$.
- (c) $z_t + cz_x = 5$.

Solution.

- (a) A PDE with dependent variable u and independent variables t and x .
- (b) An ODE with dependent variable y and independent variable x .
- (c) A PDE with dependent variable z and independent variables t and x ■

The **order** of a differential equation is the highest order derivative occurring in the equation. Thus, (1.1) and (1.2) are second order differential equations.

Example 1.5

Find the order of each of the following partial differential equations:

- (a) $xu_x + yu_y = x^2 + y^2$
- (b) $uu_x + u_y = 2$
- (c) $u_{tt} - c^2u_{xx} = f(x, t)$
- (d) $u_t + uu_x + u_{xxx} = 0$
- (e) $u_{tt} + u_{xxxx} = 0$.

Solution.

- (a) First order (b) First order (c) Second order (d) Third order (e) Fourth

order ■

A first order partial differential equation is called **quasi-linear** if it can be written in the form

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u). \quad (1.3)$$

If $a(x, y, u) = \alpha(x, y)$ and $b(x, y, u) = \beta(x, y)$ then (1.3) is called **semi-linear**. If furthermore, $c(x, y, u) = \gamma(x, y)u + \delta(x, y)$ then (1.3) is called **linear**.

In a similar way, a second order quasi-linear pde has the form

$$a(x, y, u, u_x, u_y)u_{xx} + b(x, y, u, u_x, u_y)u_{xy} + c(x, y, u, u_x, u_y)u_{yy} = d(x, y, u, u_x, u_y).$$

The semi-linear case has the form

$$a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy} = d(x, y, u, u_x, u_y).$$

and the linear case is

$$a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy} + d(x, y)u_x + e(x, y)u_y + f(x, y)u = g(x, y).$$

Note that linear and semi-linear partial differential equations are special cases of quasi-linear equations. However, a quasi-linear PDE needs not be linear: A partial differential equation that is not linear is called **non-linear**. For example, $u_x^2 + 2u_{xy} = 0$ is non-linear. Note that this equation is quasi-linear and semi-linear.

As for ODEs, linear PDEs are usually simpler to analyze/solve than non-linear PDEs.

Example 1.6

Determine whether the given PDE is linear, quasi-linear, semi-linear, or non-linear:

(a) $xu_x + yu_y = x^2 + y^2$.

(b) $uu_x + u_y = 2$.

(c) $u_{tt} - c^2u_{xx} = f(x, t)$.

(d) $u_t + uu_x + u_{xxx} = 0$.

(e) $u_{tt}^2 + u_{xx} = 0$.

Solution.

(a) Linear, quasi-linear, semi-linear.

(b) Quasi-linear, non-linear.

(c) Linear, quasi-linear, semi-linear.

- (d) Quasi-linear, semi-linear, non-linear.
- (e) Non-linear ■

A more precise definition of a linear differential equation begins with the concept of a **linear differential operator** L . The operator L is assembled by summing the basic partial derivative operators, with coefficients depending on the independent variables only. The operator acts on sufficiently smooth functions² are depending on the relevant independent variables. **Linearity** imposes two key requirements:

$$L[u + v] = L[u] + L[v] \quad \text{and} \quad L[\alpha u] = \alpha L[u],$$

for any two (sufficiently smooth) functions u, v and any constant (a scalar) α .

Example 1.7

Define a linear differential operator for the PDE

$$u_t = c^2 u_{xx}.$$

Solution.

Let $L[u] = u_t - c^2 u_{xx}$. Then one can easily check that $L[u + v] = L[u] + L[v]$ and $L[\alpha u] = \alpha L[u]$ ■

A linear partial differential equation is called **homogeneous** if the dependent variable and/or its derivatives appear in terms with degree exactly one. A linear partial differential equation that is not homogeneous is called **non-homogeneous**. In this case, there is a term in the equation that involves only the independent variables.

A homogeneous linear partial differential equation has the form

$$L[u] = 0$$

where L is a linear differential operator and the non-homogeneous case has the form

$$L[u] = f(x, y, \dots).$$

²**Smooth functions** are functions that are continuously differentiable up to a certain order.

Example 1.8

Determine whether the equation is homogeneous or non-homogeneous:

(a) $xu_x + yu_y = x^2 + y^2$.

(b) $u_{tt} = c^2u_{xx}$.

(c) $y^2u_{xx} + xu_{yy} = 0$.

Solution.

(a) Non-homogeneous because of $x^2 + y^2$.

(b) Homogeneous.

(c) Homogeneous ■

Practice Problems

Problem 1.1

Classify the following equations as either ODE or PDE.

(a) $(y''')^4 + \frac{t^2}{(y')^{2+4}} = 0.$

(b) $\frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{y-x}{y+x}.$

(c) $y'' - 4y = 0.$

Problem 1.2

Write the equation

$$u_{xx} + 2u_{xy} + u_{yy} = 0$$

in the coordinates $s = x$, $t = x - y$.

Problem 1.3

Write the equation

$$u_{xx} - 2u_{xy} + 5u_{yy} = 0$$

in the coordinates $s = x + y$, $t = 2x$.

Problem 1.4

For each of the following PDEs, state its order and whether it is linear or non-linear. If it is linear, also state whether it is homogeneous or non-homogeneous:

(a) $uu_x + x^2u_{yyy} + \sin x = 0.$

(b) $u_x + e^{x^2}u_y = 0.$

(c) $u_{tt} + (\sin y)u_{yy} - e^t \cos y = 0.$

Problem 1.5

For each of the following PDEs, determine its order and whether it is linear or not. For linear PDEs, state also whether the equation is homogeneous or not. For non-linear PDEs, circle all term(s) that are not linear.

(a) $x^2u_{xx} + e^xu = xu_{xyy}.$

(b) $e^yu_{xxx} + e^xu = -\sin y + 10xu_y.$

(c) $y^2u_{xx} + e^xuu_x = 2xu_y + u.$

(d) $u_xu_{xxy} + e^xuu_y = 5x^2u_x.$

(e) $u_t = k^2(u_{xx} + u_{yy}) + f(x, y, t).$

Problem 1.6

Which of the following PDEs are linear?

- (a) **Laplace's equation:** $u_{xx} + u_{yy} = 0$.
 (b) **Convection (transport) equation:** $u_t + cu_x = 0$.
 (c) **Minimal surface equation:** $(1 + Z_y^2)Z_{xx} - 2Z_xZ_yZ_{xy} + (1 + Z_x^2)Z_{yy} = 0$.
 (d) **Korteweg-Vries equation:** $u_t + 6uu_x = u_{xxx}$.

Problem 1.7

Classify the following differential equations as ODEs or PDEs, linear or non-linear, and determine their order. For the linear equations, determine whether or not they are homogeneous.

- (a) The **diffusion equation** for $u(x, t)$:

$$u_t = ku_{xx}.$$

- (b) The **wave equation** for $w(x, t)$:

$$w_{tt} = c^2w_{xx}.$$

- (c) The **thin film equation** for $h(x, t)$:

$$h_t = -(hh_{xxx})_x.$$

- (d) The **forced harmonic oscillator** for $y(t)$:

$$y_{tt} + \omega^2y = F \cos(\omega t).$$

- (e) The **Poisson Equation** for the electric potential $\Phi(x, y, z)$:

$$\Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 4\pi\rho(x, y, z)$$

where $\rho(x, y, z)$ is a known charge density.

- (f) **Burger's equation** for $h(x, t)$:

$$h_t + hh_x = \nu h_{xx}.$$

Problem 1.8

Write down the general form of a linear second order differential equation of a function in three variables.

Problem 1.9

Give the orders of the following PDEs, and classify them as linear or non-linear. If the PDE is linear, specify whether it is homogeneous or non-homogeneous.

(a) $x^2 u_{xxy} + y^2 u_{yy} - \log(1 + y^2)u = 0.$

(b) $u_x + u^3 = 1.$

(c) $u_{xxyy} + e^x u_x = y.$

(d) $u u_{xx} + u_{yy} - u = 0.$

(e) $u_{xx} + u_t = 3u.$

Problem 1.10

Consider the second-order PDE

$$u_{xx} + 4u_{xy} + 4u_{yy} = 0.$$

Use the change of variables $v(x, y) = y - 2x$ and $w(x, y) = x$ to show that $u_{ww} = 0.$

Problem 1.11

Write the one dimensional wave equation $u_{tt} = c^2 u_{xx}$ in the coordinates $v = x + ct$ and $w = x - ct.$

Problem 1.12

Write the PDE

$$u_{xx} + 2u_{xy} - 3u_{yy} = 0$$

in the coordinates $v(x, y) = y - 3x$ and $w(x, y) = x + y.$

Problem 1.13

Write the PDE

$$au_x + bu_y = 0, \quad a \neq 0$$

in the coordinates $s(x, y) = bx - ay$ and $t(x, y) = x.$

Problem 1.14

Write the PDE

$$u_x + u_y = 1$$

in the coordinates $s = x - y$ and $t = x.$

Problem 1.15

Write the PDE

$$au_t + bu_x = u, \quad b \neq 0$$

in the coordinates $v = ax - bt$ and $w = x.$

2 Solutions to PDEs/PDEs with constraints

By a **classical solution** or **strong solution** to a partial differential equation we mean a function that satisfies the equation. A PDE might have many classical solutions. To **solve** a PDE is to find all its classical solutions. In the case of only two independent variables x and y , a classical solution $u(x, y)$ is visualized geometrically as a surface, called a **solution surface** or an **integral surface**³ of the PDE in the (x, y, u) space.

A formula that expresses all the solutions of a PDE is called the **general solution** of the equation.

Example 2.1

Show that $u(x, t) = e^{-\lambda^2 \alpha^2 t}(\cos \lambda x - \sin \lambda x)$ is a solution to the equation $u_t - \alpha^2 u_{xx} = 0$.

Solution.

Since

$$\begin{aligned} u_t - \alpha^2 u_{xx} &= -\lambda^2 \alpha^2 e^{-\lambda^2 \alpha^2 t}(\cos \lambda x - \sin \lambda x) \\ &\quad - \alpha^2 e^{-\lambda^2 \alpha^2 t}(-\lambda^2 \cos \lambda x + \lambda^2 \sin \lambda x) = 0, \end{aligned}$$

the given function is a classical solution to the given equation ■

Example 2.2

The function $u(x, y) = x^2 - y^2$ is a solution to Laplace's equation

$$u_{xx} + u_{yy} = 0.$$

Represent this solution graphically.

Solution.

The given integral surface is the hyperbolic paraboloid shown in Figure 2.1.

³The idea behind the name is due to the fact that integration is being used to finding the solution.

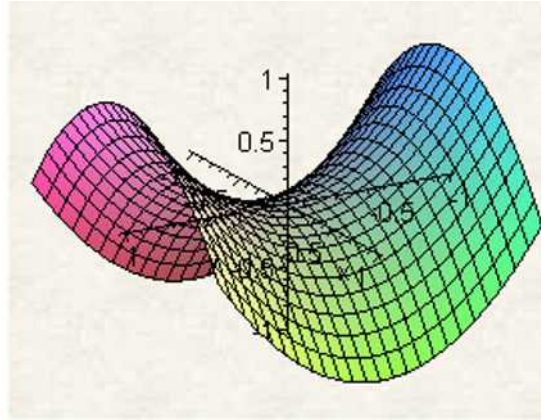


Figure 2.1

Example 2.3

Find the general solution of $u_{xy} = 0$.

Solution.

Integrating first we respect to y we find $u_x(x, y) = f(x)$, where f is an arbitrary differentiable function. Integrating u_x with respect to x we find $u(x, y) = \int f(x)dx + g(y)$, where g is an arbitrary differentiable function ■

Note that the general solution in the previous example involves two arbitrary functions. In general, the general solution of a partial differential equation is an expression that involves arbitrary functions. This is in contrast to the general solution of an ordinary differential equation which involves arbitrary constants.

Usually, a classical solution enjoys properties such as **smoothness** (i.e. differentiability) and continuity. However, in the theory of non-linear pdes, there are solutions that do not require the smoothness property. Such solutions are called **weak solutions** or **generalized solutions**. For example, $u(x) = x$ is a classical solution to the differential equation $uu' = x$. In contrast, $u(x) = |x|$ is a generalized solution since it is not differentiable at 0. In this book, the word solution will refer to a classical solution.

Example 2.4

Show that $u(x, t) = t + \frac{1}{2}x^2$ is a classical solution to the PDE

$$u_t = u_{xx}. \quad (2.1)$$

Solution.

Assume that the domain of definition of u is $D \subset \mathbb{R}^2$. Since $u, u_t, u_x, u_{tx}, u_{xx}$ exist and are continuous in D (i.e., u is smooth in D) and u satisfies equation (2.1), we conclude that u is a classical solution to the given PDE ■

We next consider the structure of solutions to linear partial differential equations. To this end, consider the linear differential operator L as defined in the previous section. The defining properties of linearity immediately imply the key facts concerning homogeneous linear differential equations.

Theorem 2.1

The sum of two solutions to a homogeneous linear differential equation is again a solution, as is the product of a solution by any constant.

Proof.

Let u_1, u_2 be solutions, meaning that $L[u_1] = 0$ and $L[u_2] = 0$. Then, thanks to linearity,

$$L[u_1 + u_2] = L[u_1] + L[u_2] = 0,$$

and hence their sum $u_1 + u_2$ is a solution. Similarly, if α is any constant, and u any solution, then

$$L[\alpha u] = \alpha L[u] = \alpha 0 = 0,$$

and so the scalar multiple αu is also a solution ■

The following result is known as the **superposition principle** for homogeneous linear equations. It states that from given solutions to the equation one can create many more solutions.

Theorem 2.2

If u_1, \dots, u_n are solutions to a common homogeneous linear partial differential equation $L[u] = 0$, then the linear combination $u = c_1 u_1 + \dots + c_n u_n$ is a solution for any choice of constants c_1, \dots, c_n .

Proof.

The key fact is that, thanks to the linearity of L , for any sufficiently smooth functions u_1, \dots, u_n and any constants c_1, \dots, c_n ,

$$\begin{aligned} L[u] &= L[c_1 u_1 + \dots + c_n u_n] = L[c_1 u_1 + \dots + c_{n-1} u_{n-1}] + L[c_n u_n] \\ &= \dots = L[c_1 u_1] + \dots + L[c_n u_n] = c_1 L[u_1] + \dots + c_n L[u_n]. \end{aligned}$$

In particular, if the functions are solutions, so $L[u_1] = 0, \dots, L[u_n] = 0$, then the right hand side of the above equation vanishes, proving that u is also a solution to the homogeneous equation $L[u] = 0$ ■

In physical applications, homogeneous linear equations model unforced systems that are subject to their own internal constraints. External forcing is represented by an additional term that does not involve the dependent variable. This results in the non-homogeneous equation

$$L[u] = f$$

where L is a linear partial differential operator, u is the dependent variable, and f is a given non-zero function of the independent variables alone.

You already learned the basic philosophy for solving of nonhomogeneous linear equations in your study of elementary ordinary differential equations. Step one is to determine the general solution to the homogeneous equation. Step two is to find a particular solution to the non-homogeneous version. The general solution to the non-homogeneous equation is then obtained by adding the two together. Here is the general version of this procedure:

Theorem 2.3

Let u_p be a particular solution to the non-homogeneous linear equation $L[u] = f$. Then the general solution to $L[u] = f$ is given by $u = u_p + u_h$, where u_h is the general solution to the corresponding homogeneous equation $L[u] = 0$.

Proof.

Let us first show that $u = u_p + u_h$ is also a solution to $L[u] = f$. By linearity,

$$L[u] = L[u_p + u_h] = L[u_p] + L[u_h] = f + 0 = f.$$

To show that every solution to the non-homogeneous equation can be expressed in this manner, suppose u satisfies $L[u] = f$. Set $w = u - u_p$. Then, by linearity,

$$L[w] = L[u - u_p] = L[u] - L[u_p] = 0,$$

and hence w is a solution to the homogeneous differential equation. Thus, $u = u_p + w$ ■

As you have noticed by the above discussion, one solution of a linear PDE

leads to the creation of lots of solutions. In contrast, non-linear equations are much tougher to deal with, for example, knowledge of several solutions does not necessarily help in constructing others. Indeed, even finding one solution to a non-linear partial differential equation can be quite a challenge.

PDEs with Constraints

Also, as observed above, a linear partial differential equation has infinitely many solutions described by the general solution. In most applications, this general solution is of little use since it has to satisfy other supplementary conditions, usually called initial or boundary conditions. These conditions determine the unique solution of interest.

A **boundary value problem** is a partial differential equation where either the unknown function or its derivatives have values assigned on the physical boundary of the domain in which the problem is specified. These conditions are called **boundary conditions**. For example, the domain of the following problem is the square $[0, 1] \times [0, 1]$ with boundaries defined by $x = 0$, $x = 1$ for all $0 \leq y \leq 1$ and $y = 0$, $y = 1$ for all $0 \leq x \leq 1$.

$$\begin{aligned} u_{xx} + u_{yy} &= 0 && \text{if } 0 < x, y < 1 \\ u(x, 0) = u(x, 1) &= 0 && \text{if } 0 < x < 1 \\ u_x(0, y) = u_x(1, y) &= 0 && \text{if } 0 < y < 1. \end{aligned}$$

There are three types of boundary conditions which arise frequently in formulating physical problems:

1. **Dirichlet Boundary Conditions:** In this case, the dependent function u is prescribed on the boundary of the bounded domain. For example, if the bounded domain is the rectangular plate $0 < x < L_1$ and $0 < y < L_2$, the boundary conditions $u(0, y)$, $u(L_1, y)$, $u(x, 0)$, and $u(x, L_2)$ are prescribed.

The boundary conditions are called **homogeneous** if the dependent variable is zero at any point on the boundary, otherwise the boundary conditions are called **nonhomogeneous**.

2. **Neumann Boundary Conditions:** In this case, first partial derivatives are prescribed on the boundary of the bounded domain. For example, the Neumann boundary conditions for a rod of length L , where $0 < x < L$, are of the form $u_x(0, t) = \alpha$ and $u_x(L, t) = \beta$, where α and β are constants.

3. Robin or mixed Boundary Conditions: This occurs when the dependent variable and its first partial derivatives are prescribed on the boundary of the bounded domain.

An **initial value problem** (or **Cauchy problem**) is a partial differential equation together with a set of additional conditions on the unknown function or its derivatives at a point in the given domain of the solution. These conditions are called **initial value conditions**. For example, the **transport equation**

$$\begin{aligned}u_t(x, t) + cu_x(x, t) &= 0 \\ u(x, 0) &= f(x).\end{aligned}$$

It can be shown that initial conditions for a linear PDE are necessary and sufficient for the existence of a unique solution.

We say that an initial and/or boundary value problem associated with a PDE is **well-posed** if it has a solution which is *unique* and depends continuously on the data given in the problem. The last condition, namely the continuous dependence is important in physical problems. This condition means that the solution changes by a small amount when the conditions change a little. Such solutions are said to be **stable**.

Example 2.5

For $x \in \mathbb{R}$ and $t > 0$ we consider the initial value problem

$$\begin{aligned}u_{tt} - u_{xx} &= 0 \\ u(x, 0) = u_t(x, 0) &= 0.\end{aligned}$$

Clearly, $u(x, t) = 0$ is a solution to this problem.

(a) Let $0 < \epsilon \ll 1$ be a very small number. Show that the function $u_\epsilon(x, t) = \epsilon^2 \sin\left(\frac{x}{\epsilon}\right) \sin\left(\frac{t}{\epsilon}\right)$ is a solution to the initial value problem

$$\begin{aligned}u_{tt} - u_{xx} &= 0 \\ u(x, 0) &= 0 \\ u_t(x, 0) &= \epsilon \sin\left(\frac{x}{\epsilon}\right).\end{aligned}$$

(b) Show that $\sup\{|u_\epsilon(x, t) - u(x, t)| : x \in \mathbb{R}, t > 0\} = \epsilon^2$. Thus, a small change in the initial data leads to a small change in the solution. Hence, the initial value problem is well-posed.

Solution.

(a) We have

$$\begin{aligned}\frac{\partial u_\epsilon}{\partial t} &= \epsilon \sin\left(\frac{x}{\epsilon}\right) \cos\left(\frac{t}{\epsilon}\right) \\ \frac{\partial^2 u_\epsilon}{\partial t^2} &= -\sin\left(\frac{x}{\epsilon}\right) \sin\left(\frac{t}{\epsilon}\right) \\ \frac{\partial u_\epsilon}{\partial x} &= \epsilon \cos\left(\frac{x}{\epsilon}\right) \sin\left(\frac{t}{\epsilon}\right) \\ \frac{\partial^2 u_\epsilon}{\partial x^2} &= -\sin\left(\frac{x}{\epsilon}\right) \sin\left(\frac{t}{\epsilon}\right).\end{aligned}$$

Thus, $\frac{\partial^2 u_\epsilon}{\partial t^2} - \frac{\partial^2 u_\epsilon}{\partial x^2} = 0$. Moreover, $u_\epsilon(x, 0) = 0$ and $\frac{\partial}{\partial t} u_\epsilon(x, 0) = \epsilon \sin\left(\frac{x}{\epsilon}\right)$.

(b) We have

$$\begin{aligned}\sup\{|u_\epsilon(x, t) - u(x, t)| : x \in \mathbb{R}, t > 0\} &= \epsilon^2 \sup\left\{\left|\sin\left(\frac{x}{\epsilon}\right) \sin\left(\frac{t}{\epsilon}\right)\right| : x \in \mathbb{R}, t > 0\right\} \\ &= \epsilon^2 \blacksquare\end{aligned}$$

A problem that is not well-posed is referred to as an **ill-posed** problem. We illustrate this concept in the next example.

Example 2.6

For $x \in \mathbb{R}$ and $t > 0$ we consider the initial value problem

$$\begin{aligned}u_{tt} + u_{xx} &= 0 \\ u(x, 0) = u_t(x, 0) &= 0.\end{aligned}$$

Clearly, $u(x, t) = 0$ is a solution to this problem.

(a) Let $0 < \epsilon \ll 1$ be a very small number. Show that the function $u_\epsilon(x, t) = \epsilon^2 \sin\left(\frac{x}{\epsilon}\right) \sinh\left(\frac{t}{\epsilon}\right)$, where

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

is a solution to the problem

$$\begin{aligned}u_{tt} + u_{xx} &= 0 \\ u(x, 0) &= 0 \\ u_t(x, 0) &= \epsilon \sin\left(\frac{x}{\epsilon}\right).\end{aligned}$$

- (b) Show that $\sup\{|u_\epsilon(x, t) - u(x, t)| : x \in \mathbb{R}\} = \epsilon^2 \left| \sinh\left(\frac{t}{\epsilon}\right) \right|$.
 (c) Find $\lim_{t \rightarrow \infty} \sup\{|u_\epsilon(x, t) - u(x, t)| : x \in \mathbb{R}\}$.

Solution.

(a) We have

$$\begin{aligned}\frac{\partial u_\epsilon}{\partial t} &= \epsilon \sin\left(\frac{x}{\epsilon}\right) \cosh\left(\frac{t}{\epsilon}\right) \\ \frac{\partial^2 u_\epsilon}{\partial t^2} &= \sin\left(\frac{x}{\epsilon}\right) \sinh\left(\frac{t}{\epsilon}\right) \\ \frac{\partial u_\epsilon}{\partial x} &= \epsilon \cos\left(\frac{x}{\epsilon}\right) \sinh\left(\frac{t}{\epsilon}\right) \\ \frac{\partial^2 u_\epsilon}{\partial x^2} &= -\sin\left(\frac{x}{\epsilon}\right) \sinh\left(\frac{t}{\epsilon}\right).\end{aligned}$$

Thus, $\frac{\partial^2 u_\epsilon}{\partial t^2} + \frac{\partial^2 u_\epsilon}{\partial x^2} = 0$. Moreover, $u_\epsilon(x, 0) = 0$ and $\frac{\partial}{\partial t} u_\epsilon(x, 0) = \epsilon \sin\left(\frac{x}{\epsilon}\right)$.

(b) We have

$$\begin{aligned}\sup\{|u_\epsilon(x, t) - u(x, t)| : x \in \mathbb{R}\} &= \epsilon^2 \sup\left\{\left| \sinh\left(\frac{t}{\epsilon}\right) \sin\left(\frac{x}{\epsilon}\right) \right| : x \in \mathbb{R}\right\} \\ &= \epsilon^2 \left| \sinh\left(\frac{t}{\epsilon}\right) \right|.\end{aligned}$$

(c) We have

$$\lim_{t \rightarrow \infty} \sup\{|u_\epsilon(x, t) - u(x, t)| : x \in \mathbb{R}\} = \lim_{t \rightarrow \infty} \epsilon^2 \left| \sinh\left(\frac{t}{\epsilon}\right) \right| = \infty.$$

Thus, a small change in the initial data leads to a catastrophically change in the solution. Hence, the given problem is ill-posed ■

Practice Problems

Problem 2.1

Determine a and b so that $u(x, y) = e^{ax+by}$ is a solution to the equation

$$u_{xxxx} + u_{yyyy} + 2u_{xxyy} = 0.$$

Problem 2.2

Consider the following differential equation

$$tu_{xx} - u_t = 0.$$

Suppose $u(t, x) = X(x)T(t)$. Show that there is a constant λ such that $X'' = \lambda X$ and $T' = \lambda tT$.

Problem 2.3

Consider the initial value problem

$$xu_x + (x + 1)yu_y = 0, \quad x, y > 1$$

$$u(1, 1) = e.$$

Show that $u(x, y) = \frac{xe^x}{y}$ is the solution to this problem.

Problem 2.4

Show that $u(x, y) = e^{-2y} \sin(x - y)$ is the solution to the initial value problem

$$\begin{cases} u_x + u_y + 2u = 0 & \text{for } x, y > 1 \\ u(x, 0) = \sin x. \end{cases}$$

Problem 2.5

Solve each of the following differential equations:

- (a) $\frac{du}{dx} = 0$ where $u = u(x)$.
- (b) $\frac{\partial u}{\partial x} = 0$ where $u = u(x, y)$.

Problem 2.6

Solve each of the following differential equations:

- (a) $\frac{d^2 u}{dx^2} = 0$ where $u = u(x)$.
- (b) $\frac{\partial^2 u}{\partial x \partial y} = 0$ where $u = u(x, y)$.

Problem 2.7

Show that $u(x, y) = f(y + 2x) + xg(y + 2x)$, where f and g are two arbitrary twice differentiable functions, satisfy the equation

$$u_{xx} - 4u_{xy} + 4u_{yy} = 0.$$

Problem 2.8

Find the differential equation whose general solution is given by $u(x, t) = f(x - ct) + g(x + ct)$, where f and g are arbitrary twice differentiable functions in one variable.

Problem 2.9

Let $p : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function in one variable. Prove that

$$u_t = p(u)u_x$$

has a solution satisfying $u(x, t) = f(x + p(u)t)$, where f is an arbitrary differentiable function. Then find the general solution to $u_t = (\sin u)u_x$.

Problem 2.10

Find the general solution to the pde

$$u_{xx} + 2u_{xy} + u_{yy} = 0.$$

Hint: See Problem 1.2.

Problem 2.11

Let $u(x, t)$ be a function such that u_{xx} exists and $u(0, t) = u(L, t) = 0$ for all $t \in \mathbb{R}$. Prove that

$$\int_0^L u_{xx}(x, t)u(x, t)dx \leq 0.$$

Problem 2.12

Consider the initial value problem

$$u_t + u_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0$$

$$u(x, 0) = 1.$$

(a) Show that $u(x, t) \equiv 1$ is a solution to this problem.

(b) Show that $u_n(x, t) = 1 + \frac{e^{n^2 t}}{n} \sin nx$ is a solution to the initial value problem

$$u_t + u_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0$$

$$u(x, 0) = 1 + \frac{\sin nx}{n}.$$

- (c) Find $\sup\{|u_n(x, 0) - 1| : x \in \mathbb{R}\}$.
 (d) Find $\sup\{|u_n(x, t) - 1| : x \in \mathbb{R}\}$.
 (e) Show that the problem is ill-posed.

Problem 2.13

Find the general solution of each of the following PDEs by means of direct integration.

- (a) $u_x = 3x^2 + y^2$, $u = u(x, y)$.
 (b) $u_{xy} = x^2y$, $u = u(x, y)$.
 (c) $u_{xyz} = 0$, $u = u(x, y, z)$.
 (d) $u_{xtt} = e^{2x+3t}$, $u = u(x, t)$.

Problem 2.14

Consider the second-order PDE

$$u_{xx} + 4u_{xy} + 4u_{yy} = 0.$$

- (a) Use the change of variables $v(x, y) = y - 2x$ and $w(x, y) = x$ to show that $u_{ww} = 0$.
 (b) Find the general solution to the given PDE.

Problem 2.15

Derive the general solution to the PDE

$$u_{tt} = c^2 u_{xx}$$

by using the change of variables $v = x + ct$ and $w = x - ct$.

Review of Some ODE Results

Later on in this book, we will encounter problems where a given partial differential equation is reduced to an ordinary differential equation by means of a given change of variables. Then techniques from the theory of ODE are required in solving the transformed ODE. In this chapter, we include some of the results from ODE theory that will be needed in our future discussions.

3 The Method of Integrating Factor

In this section, we discuss a technique for solving the first order linear non-homogeneous equation

$$y' + p(t)y = g(t) \quad (3.1)$$

where $p(t)$ and $g(t)$ are continuous on the open interval $a < t < b$. Since $p(t)$ is continuous, it has an antiderivative namely $\int p(t)dt$. Let $\mu(t) = e^{\int p(t)dt}$. Multiply Equation (3.1) by $\mu(t)$ and notice that the left hand side of the resulting equation is the derivative of a product. Indeed,

$$\frac{d}{dt}(\mu(t)y) = \mu(t)g(t).$$

Integrate both sides of the last equation with respect to t to obtain

$$\mu(t)y = \int \mu(t)g(t)dt + C$$

Hence,

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)g(t)dt + \frac{C}{\mu(t)}$$

or

$$y(t) = e^{-\int p(t)dt} \int e^{\int p(t)dt} g(t)dt + Ce^{-\int p(t)dt}$$

Notice that the second term of the previous expression is just the general solution for the homogeneous equation

$$y' + p(t)y = 0$$

whereas the first term is a solution to the nonhomogeneous equation. That is, the general solution to Equation (3.1) is the sum of a particular solution of the nonhomogeneous equation and the general solution of the homogeneous equation.

Example 3.1

Solve the initial value problem

$$y' - \frac{y}{t} = 4t, \quad y(1) = 5.$$

Solution.

We have $p(t) = -\frac{1}{t}$ so that $\mu(t) = \frac{1}{t}$. Multiplying the given equation by the integrating factor and using the product rule we notice that

$$\left(\frac{1}{t}y\right)' = 4.$$

Integrating with respect to t and then solving for y we find that the general solution is given by

$$y(t) = t \int 4dt + Ct = 4t^2 + Ct.$$

Since $y(1) = 5$, we find $C = 1$ and hence the unique solution to the IVP is $y(t) = 4t^2 + t$, $0 < t < \infty$ ■

Example 3.2

Find the general solution to the equation

$$y' + \frac{2}{t}y = \ln t, \quad t > 0.$$

Solution.

The integrating factor is $\mu(t) = e^{\int \frac{2}{t} dt} = t^2$. Multiplying the given equation by t^2 to obtain

$$(t^2y)' = t^2 \ln t.$$

Integrating with respect to t we find

$$t^2y = \int t^2 \ln t dt + C.$$

The integral on the right-hand side is evaluated using integration by parts with $u = \ln t$, $dv = t^2 dt$, $du = \frac{dt}{t}$, $v = \frac{t^3}{3}$ obtaining

$$t^2y = \frac{t^3}{3} \ln t - \frac{t^3}{9} + C$$

Thus,

$$y = \frac{t}{3} \ln t - \frac{t}{9} + \frac{C}{t^2} \quad \blacksquare$$

Example 3.3

Solve

$$au_x + bu_y + cu = 0$$

by using the change of variables $s = ax + by$ and $t = bx - ay$.**Solution.**

By the Chain rule for functions of two variables, we have

$$\begin{aligned}u_x &= u_s s_x + u_t t_x = au_s + bu_t \\u_y &= u_s s_y + u_t t_y = bu_s - au_t.\end{aligned}$$

Substituting into the given equation, we find

$$u_s + \frac{c}{a^2 + b^2}u = 0.$$

Solving this equation using the integrating factor method we find

$$u(s, t) = f(t)e^{-\frac{cs}{a^2+b^2}}$$

where f is an arbitrary differentiable function of f . Switching back to x and y we obtain

$$u(x, y) = f(bx - ay)e^{-\frac{c}{a^2+b^2}(ax+by)} \blacksquare$$

Practice Problems

Problem 3.1

Solve the IVP: $y' + 2ty = t$, $y(0) = 0$.

Problem 3.2

Find the general solution: $y' + 3y = t + e^{-2t}$.

Problem 3.3

Find the general solution: $y' + \frac{1}{t}y = 3 \cos t$, $t > 0$.

Problem 3.4

Find the general solution: $y' + 2y = \cos(3t)$.

Problem 3.5

Find the general solution: $y' + (\cos t)y = -3 \cos t$.

Problem 3.6

Given that the solution to the IVP $ty' + 4y = \alpha t^2$, $y(1) = -\frac{1}{3}$ exists on the interval $-\infty < t < \infty$. What is the value of the constant α ?

Problem 3.7

Suppose that $y(t) = Ce^{-2t} + t + 1$ is the general solution to the equation $y' + p(t)y = g(t)$. Determine the functions $p(t)$ and $g(t)$.

Problem 3.8

Suppose that $y(t) = -2e^{-t} + e^t + \sin t$ is the unique solution to the IVP $y' + y = g(t)$, $y(0) = y_0$. Determine the constant y_0 and the function $g(t)$.

Problem 3.9

Find the value (if any) of the unique solution to the IVP $y' + (1 + \cos t)y = 1 + \cos t$, $y(0) = 3$ in the long run?

Problem 3.10

Solve the initial value problem $ty' = y + t$, $y(1) = 7$.

Problem 3.11

Show that if a and λ are positive constants, and b is any real number, then every solution of the equation

$$y' + ay = be^{-\lambda t}$$

has the property that $y \rightarrow 0$ as $t \rightarrow \infty$. Hint: Consider the cases $a = \lambda$ and $a \neq \lambda$ separately.

Problem 3.12

Solve the initial-value problem $y' + y = e^t y^2$, $y(0) = 1$ using the substitution $u(t) = \frac{1}{y(t)}$

Problem 3.13

Solve the initial-value problem $ty' + 2y = t^2 - t + 1$, $y(1) = \frac{1}{2}$

Problem 3.14

Solve $y' - \frac{1}{t}y = \sin t$, $y(1) = 3$. Express your answer in terms of the **sine integral**, $Si(t) = \int_0^t \frac{\sin s}{s} ds$.

4 The Method of Separation of Variables for ODEs

The method of separation of variables that you have seen in the theory of ordinary differential equations has an analogue in the theory of partial differential equations (Section 17). In this section, we review the method for ordinary differentiable equations.

A first order differential equation is **separable** if it can be written with one variable only on the left and the other variable only on the right:

$$f(y)y' = g(t)$$

To solve this equation, we proceed as follows. Let $F(t)$ be an antiderivative of $f(t)$ and $G(t)$ be an antiderivative of $g(t)$. Then by the Chain Rule

$$\frac{d}{dt}F(y) = \frac{dF}{dy} \frac{dy}{dt} = f(y)y'.$$

Thus,

$$f(y)y' - g(t) = \frac{d}{dt}F(y) - \frac{d}{dt}G(t) = \frac{d}{dt}[F(y) - G(t)] = 0.$$

It follows that

$$F(y) - G(t) = C$$

which is equivalent to

$$\int f(y)y' dt = \int g(t) dt + C.$$

As you can see, the result is generally an implicit equation involving a function of y and a function of t . It may or may not be possible to solve this to get y explicitly as a function of t . For an initial value problem, substitute the values of t and y by t_0 and y_0 to get the value of C .

Remark 4.1

If F is a differentiable function of y and y is a differentiable function of t and both F and y are given then the chain rule allows us to find $\frac{dF}{dt}$ given by

$$\frac{dF}{dt} = \frac{dF}{dy} \cdot \frac{dy}{dt}$$

For separable equations, we are given $f(y)y' = \frac{dF}{dt}$ and we are asked to find $F(y)$. This process is referred to as “reversing the chain rule.”

Example 4.1

Solve the initial value problem $y' = 6ty^2$, $y(1) = \frac{1}{25}$.

Solution.

Separating the variables and integrating both sides we obtain

$$\int \frac{y'}{y^2} dt = \int 6t dt$$

or

$$-\int \frac{d}{dt} \left(\frac{1}{y} \right) dt = \int 6t dt.$$

Thus,

$$-\frac{1}{y(t)} = 3t^2 + C.$$

Since $y(1) = \frac{1}{25}$, we find $C = -28$. The unique solution to the IVP is then given explicitly by

$$y(t) = \frac{1}{28 - 3t^2} \blacksquare$$

Example 4.2

Solve the IVP $yy' = 4 \sin(2t)$, $y(0) = 1$.

Solution.

This is a separable differential equation. Integrating both sides we find

$$\int \frac{d}{dt} \left(\frac{y^2}{2} \right) dt = 4 \int \sin(2t) dt.$$

Thus,

$$y^2 = -4 \cos(2t) + C.$$

Since $y(0) = 1$, we find $C = 5$. Now, solving explicitly for $y(t)$ we find

$$y(t) = \pm \sqrt{-4 \cos t + 5}.$$

Since $y(0) = 1$, we have $y(t) = \sqrt{-4 \cos t + 5}$. The interval of existence of the solution is the interval $-\infty < t < \infty$ ■

Practice Problems

Problem 4.1

Solve the (separable) differential equation

$$y' = te^{t^2 - \ln y^2}.$$

Problem 4.2

Solve the (separable) differential equation

$$y' = \frac{t^2 y - 4y}{t + 2}.$$

Problem 4.3

Solve the (separable) differential equation

$$ty' = 2(y - 4).$$

Problem 4.4

Solve the (separable) differential equation

$$y' = 2y(2 - y).$$

Problem 4.5

Solve the IVP

$$y' = \frac{4 \sin(2t)}{y}, \quad y(0) = 1.$$

Problem 4.6

Solve the IVP:

$$yy' = \sin t, \quad y\left(\frac{\pi}{2}\right) = -2.$$

Problem 4.7

Solve the IVP:

$$y' + y + 1 = 0, \quad y(1) = 0.$$

Problem 4.8

Solve the IVP:

$$y' - ty^3 = 0, \quad y(0) = 2.$$

Problem 4.9

Solve the IVP:

$$y' = 1 + y^2, \quad y\left(\frac{\pi}{4}\right) = -1.$$

Problem 4.10

Solve the IVP:

$$y' = t - ty^2, \quad y(0) = \frac{1}{2}.$$

Problem 4.11Solve the equation $3u_y + u_{xy} = 0$ by using the substitution $v = u_y$.**Problem 4.12**

Solve the IVP

$$(2y - \sin y)y' = \sin t - t, \quad y(0) = 0.$$

Problem 4.13State an initial value problem, with initial condition imposed at $t_0 = 2$, having implicit solution $y^3 + t^2 + \sin y = 4$.**Problem 4.14**

Can the differential equation

$$\frac{dy}{dx} = x^2 - xy$$

be solved by the method of separation of variables? Explain.

First Order Partial Differential Equations

Many problems in the mathematical, physical, and engineering sciences deal with the formulation and the solution of first order partial differential equations. Our first task is to understand simple first order equations. In applications, first order partial differential equations are most commonly used to describe dynamical processes, and so time, t , is one of the independent variables. Most of our discussion will focus on dynamical models in a single space dimension, bearing in mind that most of the methods can be readily extended to higher dimensional situations. First order partial differential equations and systems model a wide variety of wave phenomena, including transport of solvents in fluids, flood waves, acoustics, gas dynamics, glacier motion, traffic flow, and also a variety of biological and ecological systems. From a mathematical point of view, first order partial differential equations have the advantage of providing conceptual basis that can be utilized in the study of higher order partial differential equations.

In this chapter we introduce the basic definitions of first order partial differential equations. We then derive the one dimensional spatial transport equation and discuss some methods of solutions. One general method of solvability for quasilinear first order partial differential equation, known as the method of characteristics, is analyzed.

5 Classification of First Order PDEs

In this section, we present the basic definitions pertained to first order PDE. By a **first order partial differential equation** in two variables x and y we mean any equation of the form

$$F(x, y, u, u_x, u_y) = 0. \quad (5.1)$$

In what follows the functions a, b , and c are assumed to be continuously differentiable functions. If Equation (5.1) can be written in the form

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \quad (5.2)$$

then we say that the equation is **quasi-linear**. The following are examples of quasi-linear equations:

$$\begin{aligned} uu_x + u_y + cu^2 &= 0 \\ x(y^2 + u)u_x - y(x^2 + u)u_y &= (x^2 - y^2)u. \end{aligned}$$

If Equation (5.1) can be written in the form

$$a(x, y)u_x + b(x, y)u_y = c(x, y, u) \quad (5.3)$$

then we say that the equation is **semi-linear**. The following are examples of semi-linear equations:

$$\begin{aligned} xu_x + yu_y &= u^2 + x^2 \\ (x + 1)^2u_x + (y - 1)^2u_y &= (x + y)u^2. \end{aligned}$$

If Equation (5.1) can be written in the form

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = d(x, y) \quad (5.4)$$

then we say that the equation is **linear**. Examples of linear equations are:

$$\begin{aligned} xu_x + yu_y &= cu \\ (y - z)u_x + (z - x)u_y + (x - y)u_z &= 0. \end{aligned}$$

A first order pde that is not linear is said to be **non-linear**. Examples of non-linear equations are:

$$u_x + cu_y^2 = xy$$

$$u_x^2 + u_y^2 = c.$$

First order partial differential equations are classified as either linear or non-linear. Clearly, linear equations are a special kind of quasi-linear equation (5.2) if a and b are functions of x and y only and c is a linear function of u . Likewise, semi-linear equations are quasilinear equations if a and b are functions of x and y only. Also, semi-linear equations (5.3) reduces to a linear equation if c is linear in u .

A linear first order partial differential equation is called **homogeneous** if $d(x, y) \equiv 0$ and **non-homogeneous** if $d(x, y) \neq 0$. Examples of linear homogeneous equations are:

$$\begin{aligned} xu_x + yu_y &= cu \\ (y - z)u_x + (z - x)u_y + (x - y)u_z &= 0. \end{aligned}$$

Examples of non-homogeneous equations are:

$$\begin{aligned} u_x + (x + y)u_y - u &= e^x \\ yu_x + xu_y &= xy. \end{aligned}$$

Recall that for an ordinary linear differential equation, the general solution depends mainly on arbitrary constants. Unlike ODEs, in linear partial differential equations, the general solution depends on arbitrary functions.

Example 5.1

Solve the equation $u_t(x, t) = 0$.

Solution.

The general solution is given by $u(x, t) = f(x)$ where f is an arbitrary differentiable function of x ■

Example 5.2

Consider the transport equation

$$au_t(x, t) + bu_x(x, t) = 0$$

where a and b are constants. Show that $u(x, t) = f(bt - ax)$ is a solution to the given equation, where f is an arbitrary differentiable function in one variable.

Solution.

Let $v(x, t) = bt - ax$. Using the chain rule we see that $u_t(x, t) = bf_v(v)$ and $u_x(x, t) = -af_v(v)$. Hence, $au_t(x, t) + bu_x(x, t) = abf_v(v) - abf_v(v) = 0$ ■

Practice Problems

Problem 5.1

Classify each of the following PDE as linear, quasi-linear, semi-linear, or non-linear.

(a) $xu_x + yu_y = \sin(xy)$.

(b) $u_t + uu_x = 0$

(c) $u_x^2 + u^3u_y^4 = 0$.

(d) $(x + 3)u_x + xy^2u_y = u^3$.

Problem 5.2

Show that $u(x, y) = e^x f(2x - y)$, where f is a differentiable function of one variable, is a solution to the equation

$$u_x + 2u_y - u = 0.$$

Problem 5.3

Show that $u(x, y) = x\sqrt{xy}$ satisfies the equation

$$xu_x - yu_y = u$$

subject to

$$u(y, y) = y^2, \quad y \geq 0.$$

Problem 5.4

Show that $u(x, y) = \cos(x^2 + y^2)$ satisfies the equation

$$-yu_x + xu_y = 0$$

subject to

$$u(0, y) = \cos y^2.$$

Problem 5.5

Show that $u(x, y) = y - \frac{1}{2}(x^2 - y^2)$ satisfies the equation

$$\frac{1}{x}u_x + \frac{1}{y}u_y = \frac{1}{y}$$

subject to $u(x, 1) = \frac{1}{2}(3 - x^2)$.

Problem 5.6

Find a relationship between a and b if $u(x, y) = f(ax + by)$ is a solution to the equation $3u_x - 7u_y = 0$ for any differentiable function f such that $f'(x) \neq 0$ for all x .

Problem 5.7

Reduce the partial differential equation

$$au_x + bu_y + cu = 0$$

to a first order ODE by introducing the change of variables $s = bx - ay$ and $t = x$.

Problem 5.8

Solve the partial differential equation

$$u_x + u_y = 1$$

by introducing the change of variables $s = x - y$ and $t = x$.

Problem 5.9

Show that $u(x, y) = e^{-4x}f(2x - 3y)$ is a solution to the first-order PDE

$$3u_x + 2u_y + 12u = 0.$$

Problem 5.10

Derive the general solution of the PDE

$$au_t + bu_x = u, \quad b \neq 0$$

by using the change of variables $v = ax - bt$ and $w = x$.

Problem 5.11

Derive the general solution of the PDE

$$au_x + bu_y = 0, \quad a \neq 0$$

by using the change of variables $s = bx - ay$ and $t = x$.

Problem 5.12

Write the equation

$$u_t + cu_x + \lambda u = f(x, t), \quad c \neq 0$$

in the coordinates $v = x - ct$, $w = x$.

Problem 5.13

Suppose that $u(x, t) = w(x - ct)$ is a solution to the PDE

$$xu_x + tu_t = Au$$

where A and c are constants. Let $v = x - ct$. Write the differential equation with unknown function $w(v)$.

6 A Review of Multivariable Calculus

In this section, we recall some concepts from vector calculus that we encounter later in the book.

6.1 Multiplication of Vectors: The Scalar or Dot Product

Is there such thing as multiplying a vector by another vector? The answer is yes. As a matter of fact there are two types of vector multiplication. The first one is known as **scalar** or **dot product**⁴ and produces a scalar; the second is known as the **vector** or **cross product** and produces a vector. In this section we will discuss the former one leaving the latter one for the next section.

One of the motivation for using the dot product is the physical situation to which it applies, namely that of computing the work done on an object by a given force over a given distance, as shown in Figure 6.1.1.

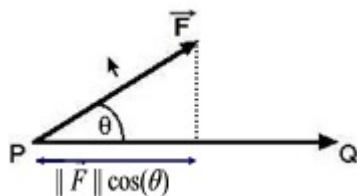


Figure 6.1.1

Indeed, the work W is given by the expression

$$W = \|\vec{F}\| \|\vec{PQ}\| \cos \theta$$

where $\|\vec{F}\| \cos \theta$ is the component of \vec{F} in the direction of \vec{PQ} .

Thus, we define the **dot product** of two vectors \vec{u} and \vec{v} to be the number

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta, \quad 0 \leq \theta \leq \pi$$

⁴Also called **inner product**.

where θ is the angle between the two vectors as shown in Figure 6.1.2.

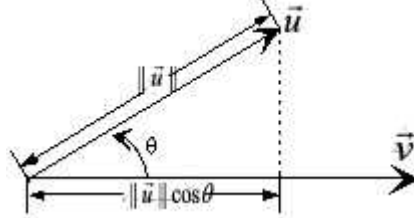


Figure 6.1.2

The above definition is the geometric definition of the dot product. We next provide an algebraic way for computing the dot product. Indeed, let $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ and $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$. Then $\vec{v} - \vec{u} = (v_1 - u_1)\vec{i} + (v_2 - u_2)\vec{j} + (v_3 - u_3)\vec{k}$. Moreover, we have $\|\vec{u}\|^2 = u_1^2 + u_2^2 + u_3^2$, $\|\vec{v}\|^2 = v_1^2 + v_2^2 + v_3^2$ and

$$\begin{aligned}\|\vec{v} - \vec{u}\|^2 &= (v_1 - u_1)^2 + (v_2 - u_2)^2 + (v_3 - u_3)^2 \\ &= v_1^2 - 2v_1u_1 + u_1^2 + v_2^2 - 2v_2u_2 + u_2^2 + v_3^2 - 2v_3u_3 + u_3^2.\end{aligned}$$

Now, applying the Law of Cosines to Figure 6.1.3 we can write

$$\|\vec{v} - \vec{u}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\| \|\vec{v}\| \cos \theta.$$

Thus, by substitution we obtain

$$v_1^2 - 2v_1u_1 + u_1^2 + v_2^2 - 2v_2u_2 + u_2^2 + v_3^2 - 2v_3u_3 + u_3^2 = u_1^2 + u_2^2 + u_3^2 + v_1^2 + v_2^2 + v_3^2 - 2\|\vec{u}\| \|\vec{v}\| \cos \theta$$

or

$$\|\vec{u}\| \|\vec{v}\| \cos \theta = u_1v_1 + u_2v_2 + u_3v_3$$

so that we can define the dot product algebraically by

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

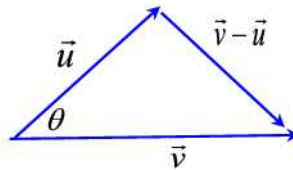


Figure 6.1.3

Example 6.1.1

Compute the dot product of $\vec{u} = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j} + \frac{1}{\sqrt{2}}\vec{k}$ and $\vec{v} = \frac{1}{2}\vec{i} + \frac{1}{2}\vec{j} + \vec{k}$ and the angle between these vectors.

Solution.

We have

$$\vec{u} \cdot \vec{v} = \frac{1}{\sqrt{2}} \cdot \frac{1}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} + \frac{1}{\sqrt{2}} \cdot 1 = \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}.$$

We also have

$$\begin{aligned} \|\vec{u}\|^2 &= \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{3}{2} \\ \|\vec{v}\|^2 &= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + 1 = \frac{3}{2}. \end{aligned}$$

Thus,

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = \frac{2\sqrt{2}}{3}.$$

Hence,

$$\theta = \cos^{-1} \left(\frac{2\sqrt{2}}{3} \right) \approx 0.34 \text{ rad} \approx 19.5^\circ \blacksquare$$

Remark 6.1.1

The algebraic definition of the dot product extends to vectors with any number of components.

Next, we discuss few properties of the dot product.

Theorem 6.1.1

For any vectors \vec{u}, \vec{v} , and \vec{w} and any scalar λ we have

- (i) Commutative law: $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.
- (ii) Distributive law: $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$.
- (iii) $\vec{u} \cdot (\lambda\vec{v}) = (\lambda\vec{u}) \cdot \vec{v} = \lambda(\vec{u} \cdot \vec{v})$.
- (iv) Magnitude: $\|\vec{u}\|^2 = \vec{u} \cdot \vec{u}$.
- (v) Two nonzero vectors \vec{u} and \vec{v} are **orthogonal** or **perpendicular** if and only if $\vec{u} \cdot \vec{v} = 0$.
- (vi) Two nonzero vectors \vec{u} and \vec{v} are parallel if and only if $\vec{u} \cdot \vec{v} = \pm \|\vec{u}\| \|\vec{v}\|$.
- (vii) $\vec{0} \cdot \vec{v} = \vec{0}$.

Proof.

Write $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$, $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$, and $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$.

Then

(i) $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3 = v_1u_1 + v_2u_2 + v_3u_3 = \vec{v} \cdot \vec{u}$ since product of numbers is commutative.

(ii) $(\vec{u} + \vec{v}) \cdot \vec{w} = ((u_1 + v_1)\vec{i} + (u_2 + v_2)\vec{j} + (u_3 + v_3)\vec{k}) \cdot (w_1\vec{i} + w_2\vec{j} + w_3\vec{k}) = (u_1 + v_1)w_1 + (u_2 + v_2)w_2 + (u_3 + v_3)w_3 = u_1w_1 + u_2w_2 + u_3w_3 + v_1w_1 + v_2w_2 + v_3w_3 = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$.

(iii) $\vec{u} \cdot (\lambda\vec{v}) = (u_1\vec{i} + u_2\vec{j} + u_3\vec{k}) \cdot (\lambda v_1\vec{i} + \lambda v_2\vec{j} + \lambda v_3\vec{k}) = \lambda u_1v_1 + \lambda u_2v_2 + \lambda u_3v_3 = \lambda(u_1v_1 + u_2v_2 + u_3v_3) = \lambda(\vec{u} \cdot \vec{v})$.

(iv) $\|\vec{u}\|^2 = \vec{u} \cdot \vec{u} \cos 0 = \vec{u} \cdot \vec{u}$.

(v) If \vec{u} and \vec{v} are perpendicular then the cosine of their angle is zero and so the dot product is zero. Conversely, if the dot product of the two vectors is zero then the cosine of their angle is zero and this happens only when the two vectors are perpendicular.

(vi) If \vec{u} and \vec{v} are parallel then the cosine of their angle is either 1 or -1 . That is, $\vec{u} \cdot \vec{v} = \pm\|\vec{u}\| \|\vec{v}\|$. Conversely, if $\vec{u} \cdot \vec{v} = \pm\|\vec{u}\| \|\vec{v}\|$ then $\cos \theta = \pm 1$ and this implies that either $\theta = 0$ or $\theta = \pi$. In either case, the two vectors are parallel.

(vii) In 3-D, $\vec{0} = \langle 0, 0, 0 \rangle$ and $\vec{v} = \langle a, b, c \rangle$ so that $\vec{0} \cdot \vec{v} = (0 \times a)\vec{i} + (0 \times b)\vec{j} + (0 \times c)\vec{k} = \vec{0}$ ■

Remark 6.1.2

Note that the unit vectors $\vec{i}, \vec{j}, \vec{k}$ associated with the coordinate axes satisfy the equalities

$$\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1 \text{ and } \vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{i} \cdot \vec{k} = 0.$$

Example 6.1.2

- (a) Show that the vectors $\vec{u} = 3\vec{i} - 2\vec{j}$ and $\vec{v} = 2\vec{i} + 3\vec{j}$ are perpendicular.
 (b) Show that the vectors $\vec{u} = 2\vec{i} + 6\vec{j} - 4\vec{k}$ and $\vec{v} = -3\vec{i} - 9\vec{j} + 6\vec{k}$ are parallel.

Solution.

- (a) We have: $\vec{u} \cdot \vec{v} = 3(2) - 2(3) = 0$. Hence \vec{u} is perpendicular to \vec{v} .
 (b) We have:

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = \frac{2(-3) + (6)(-9) - 4(6)}{[\sqrt{2^2 + (6)^2 + (-4)^2}][\sqrt{(-3)^2 + (-9)^2 + 6^2}]} = -1.$$

Hence, $\theta = \pi$ so that the two vectors are parallel. Another way to see that the vectors are parallel is to notice that $\vec{u} = -\frac{2}{3}\vec{v}$ ■

Projection of a vector onto a line

The **orthogonal projection** of a vector along a line is obtained by taking a vector with same length and direction as the given vector but with its tail on the line and then dropping a perpendicular onto the line from the tip of the vector. The resulting vector on the line is the vector's orthogonal projection or simply its projection. See Figure 6.1.4.

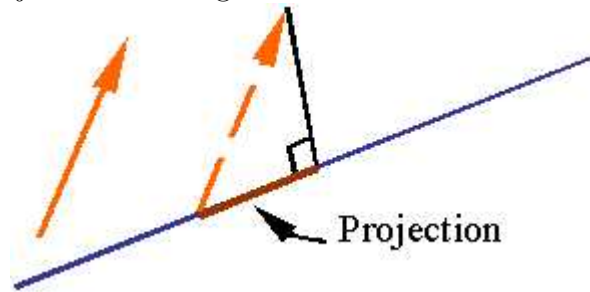


Figure 6.1.4

Now, if \vec{u} is a unit vector along the line of projection and if $\vec{v}_{parallel}$ is the vector projection of \vec{v} onto \vec{u} then

$$\vec{v}_{parallel} = (||\vec{v}|| \cos \theta) \vec{u} = (\vec{v} \cdot \vec{u}) \vec{u}.$$

See Figure 6.1.5. Also, the component perpendicular to \vec{u} is given by

$$\vec{v}_{perpendicular} = \vec{v} - \vec{v}_{parallel}.$$

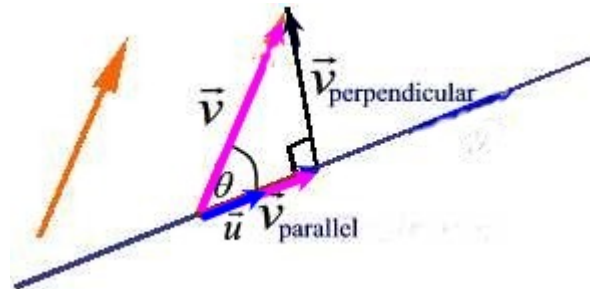


Figure 6.1.5

We call $\text{Comp}_{\vec{u}}\vec{v} = \vec{v} \cdot \vec{u}$ the the **scalar projection** of \vec{v} onto \vec{u} . We call the vector $\text{Proj}_{\vec{u}}\vec{v} = \vec{v}_{parallel}$ the **vector projection** of \vec{v} onto \vec{u} .

It follows that the vector \vec{v} can be written in terms of $\vec{v}_{parallel}$ and $\vec{v}_{perpendicular}$

$$\vec{v} = \vec{v}_{parallel} + \vec{v}_{perpendicular}.$$

Example 6.1.3

Write the vector $\vec{v} = 3\vec{i} + 2\vec{j} - 6\vec{k}$ as the sum of two vectors, one parallel, and one perpendicular to $\vec{w} = 2\vec{i} - 4\vec{j} + \vec{k}$.

Solution.

Let $\vec{u} = \frac{\vec{w}}{\|\vec{w}\|} = \frac{2}{\sqrt{21}}\vec{i} - \frac{4}{\sqrt{21}}\vec{j} + \frac{1}{\sqrt{21}}\vec{k}$. Then,

$$\vec{v}_{parallel} = (\vec{v} \cdot \vec{u})\vec{u} = \left(\frac{6}{\sqrt{21}} - \frac{8}{\sqrt{21}} - \frac{6}{\sqrt{21}} \right) \vec{u} = -\frac{16}{21}\vec{i} + \frac{32}{21}\vec{j} - \frac{8}{21}\vec{k}.$$

Also,

$$\begin{aligned} \vec{v}_{perpendicular} &= \vec{v} - \vec{v}_{parallel} = \left(3 + \frac{16}{21} \right) \vec{i} + \left(2 - \frac{32}{21} \right) \vec{j} + \left(-6 + \frac{8}{21} \right) \vec{k} \\ &= \frac{79}{21}\vec{i} + \frac{10}{21}\vec{j} - \frac{118}{21}\vec{k}. \end{aligned}$$

Hence,

$$\vec{v} = \vec{v}_{parallel} + \vec{v}_{perpendicular} \blacksquare$$

Example 6.1.4

Find the scalar projection and vector projection of $\vec{u} = \langle 1, 1, 2 \rangle$ onto $\vec{v} = \langle -2, 3, 1 \rangle$.

Solution.

We have

$$\begin{aligned} \text{comp}_{\vec{v}}\vec{u} &= \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|} = \frac{1(-2) + (1)(3) + 2(1)}{\sqrt{(-2)^2 + 3^2 + 1^2}} = \frac{3}{\sqrt{14}} \\ \text{Proj}_{\vec{v}}\vec{u} &= \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|} \frac{\vec{v}}{\|\vec{v}\|} \\ &= \frac{3}{14}\vec{v} = -\frac{2}{7}\vec{i} + \frac{9}{14}\vec{j} + \frac{3}{14}\vec{k} \blacksquare \end{aligned}$$

Applications

As pointed out earlier in the section, scalar products are used in Physics. For instance, in finding the work done by a force applied on an object.

Example 6.1.5

A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 70 N. The handle of the wagon is held at an angle of 35° above the horizontal. Find the work done by the force.

Solution.

The work done is

$$W = F \cdot d \cos 35^\circ = 70(100) \cos 35^\circ \approx 5734 \text{ J} \blacksquare$$

Practice Problems

Problem 6.1.1

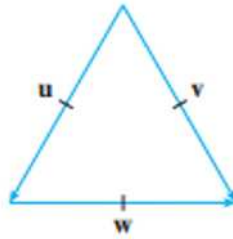
Find $\vec{a} \cdot \vec{b}$ where $\vec{a} = \langle 4, 1, \frac{1}{4} \rangle$ and $\vec{b} = \langle 6, -3, -8 \rangle$.

Problem 6.1.2

Find $\vec{a} \cdot \vec{b}$ where $\|\vec{a}\| = 6$, $\|\vec{b}\| = 5$ and the angle between the two vectors is 120° .

Problem 6.1.3

If \vec{u} is a unit vector, find $\vec{u} \cdot \vec{v}$ and $\vec{u} \cdot \vec{w}$ using the figure below.



Problem 6.1.4

Find the angle between the vectors $\vec{a} = \langle 4, 3 \rangle$ and $\vec{b} = \langle 2, -1 \rangle$.

Problem 6.1.5

Find the angle between the vectors $\vec{a} = \langle 4, -3, 1 \rangle$ and $\vec{b} = \langle 2, 0, -1 \rangle$.

Problem 6.1.6

Determine whether the given vectors are orthogonal, parallel, or neither.

(a) $\vec{a} = \langle -5, 3, 7 \rangle$ and $\vec{b} = \langle 6, -8, 2 \rangle$.

(b) $\vec{a} = \langle 4, 6 \rangle$ and $\vec{b} = \langle -3, 2 \rangle$.

(c) $\vec{a} = -\vec{i} + 2\vec{j} + \vec{k}$ and $\vec{b} = 3\vec{i} + 4\vec{j} - \vec{k}$.

(d) $\vec{a} = 2\vec{i} + 6\vec{j} - 4\vec{k}$ and $\vec{b} = -3\vec{i} - 9\vec{j} + 6\vec{k}$.

Problem 6.1.7

Use vectors to decide whether the triangle with vertices $P(1, -3, -2)$, $Q(2, 0, -4)$, and $R(6, -2, -5)$ is right-angled.

Problem 6.1.8

Find a unit vector that is orthogonal to both $\vec{i} + \vec{j}$ and $\vec{i} + \vec{k}$.

Problem 6.1.9

Find the acute angle between the lines $2x - y = 3$ and $3x + y = 7$.

Problem 6.1.10

Find the scalar and vector projections of the vector $\vec{b} = \langle 1, 2, 3 \rangle$ onto $\vec{a} = \langle 3, 6, -2 \rangle$.

Problem 6.1.11

If $\vec{a} = \langle 3, 0, -1 \rangle$, find a vector \vec{b} such that $\text{comp}_{\vec{a}}\vec{b} = 2$.

Problem 6.1.12

Find the work done by a force $\vec{F} = 8\vec{i} - 6\vec{j} + 9\vec{k}$ that moves an object from the point $(0, 10, 8)$ to the point $(6, 12, 20)$ along a straight line. The distance is measured in meters and the force in newtons.

Problem 6.1.13

A sled is pulled along a level path through snow by a rope. A 30-lb force acting at an angle of 40° above the horizontal moves the sled 80 ft. Find the work done by the force.

6.2 Directional Derivatives and the Gradient Vector

Given a function $z = f(x, y)$ and let (x_0, y_0) be in the domain of f . We wish to find the rate of change of f at (x_0, y_0) in the direction of a unit vector $\vec{u} = \langle a, b \rangle$. To do this, we consider the vertical plane to the graph S of f that passes through the point $P(x_0, y_0, z_0)$ in the direction of \vec{u} . This plane intersects the graph S in a curve C . (See Figure 6.2.1.)

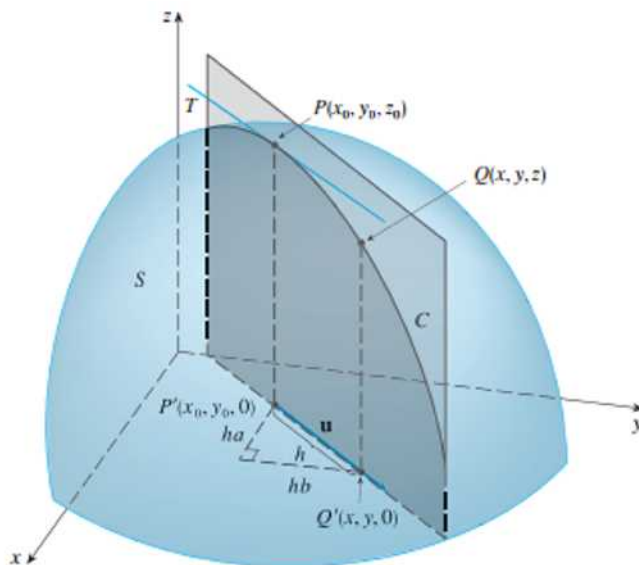


Figure 6.2.1

The slope of the tangent line T to C at the point P is the rate of change of z in the direction of \vec{u} . Let $Q(x, y, z)$ be an arbitrary point on C and let $P'(x_0, y_0, 0)$ and $Q'(x, y, 0)$ be the orthogonal projection of P and Q respectively onto the xy -plane. Then the vectors $\overrightarrow{P'Q'} = \langle x - x_0, y - y_0, 0 \rangle$ is parallel to \vec{u} so that $\overrightarrow{P'Q'} = h\vec{u}$ for some scalar h . Hence, $x = x_0 + ha$, $y = y_0 + hb$ and

$$\frac{f(x, y) - f(x_0, y_0)}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}.$$

If we take the limit of the above average rate as $h \rightarrow 0$, we obtain the rate of change of z (with respect to distance) in the direction of \vec{u} , which is called the **directional derivative** of f at (x_0, y_0) in the direction of \vec{u} . We write

$$f_{\vec{u}}(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}.$$

Notice that if $\vec{u} = \vec{i}$ then $a = 1$ and $b = 0$ so that $f_{\vec{u}}(x_0, y_0) = f_x(x_0, y_0)$. That is, f_x is the rate of change of f in the x -direction. Likewise, if $\vec{u} = \vec{j}$ then $a = 0$ and $b = 1$ so that $f_{\vec{u}}(x_0, y_0) = f_y(x_0, y_0)$.

The following theorem provides a formula for computing the directional derivative.

Theorem 6.2.1

If f is a differentiable function of x and y , then f has a directional derivative in the direction of any unit vector $\vec{u} = \langle a, b \rangle$ and

$$f_{\vec{u}}(x, y) = f_x(x, y)a + f_y(x, y)b.$$

Proof.

Fix a point (x_0, y_0) in the domain of f and consider the single variable function $g(h) = f(x_0 + ha, y_0 + hb)$. Then

$$g'(0) = \lim_{h \rightarrow 0} \frac{g(h) - g(0)}{h} = \lim_{h \rightarrow 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h} = f_{\vec{u}}(x_0, y_0).$$

Let $x = x_0 + ah$ and $y = y_0 + bh$. Using the Chain Rule, we find

$$g'(h) = \frac{\partial f}{\partial x} \frac{dx}{dh} + \frac{\partial f}{\partial y} \frac{dy}{dh} = f_x(x, y)a + f_y(x, y)b.$$

Letting $h = 0$ in the above expression, we find

$$f_{\vec{u}}(x_0, y_0) = g'(0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b \blacksquare \quad (6.2.1)$$

Example 6.2.1

Find $u_{\vec{v}}(4, 0)$ if $u(x, y) = x + y^2$ and $\vec{v} = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$.

Solution.

We have

$$u_{\vec{v}}(4, 0) = u_x(4, 0) \left(\frac{1}{2} \right) + u_y(4, 0) \left(\frac{\sqrt{3}}{2} \right) = \frac{1}{2} \blacksquare$$

The Gradient Vector

The **gradient** is a generalization of the usual concept of derivative of a function of one variable to functions of several variables. For a function $u(x, y)$ or $u(x, y, z)$, the gradient are, respectively,

$$\nabla u(x, y) = u_x \vec{i} + u_y \vec{j} \quad \text{and} \quad \nabla u(x, y, z) = u_x \vec{i} + u_y \vec{j} + u_z \vec{k}.$$

Example 6.2.2

Let $F(x, y, z) = u(x, y) - z$. Find $\nabla F(x, y, z)$.

Solution.

We have

$$\nabla F(x, y, z) = u_x \vec{i} + u_y \vec{j} - \vec{k} \blacksquare$$

Example 6.2.3

Find the gradient vector of $f(x, y, z) = (2x - 3y + 5z)^5$.

Solution.

We have

$$\begin{aligned} f_x(x, y, z) &= 10(2x - 3y + 5z)^4 \\ f_y(x, y, z) &= -15(2x - 3y + 5z)^4 \\ f_z(x, y, z) &= 25(2x - 3y + 5z)^4. \end{aligned}$$

Thus,

$$\nabla f(x, y, z) = 5(2x - 3y + 5z)^4 [2\vec{i} - 3\vec{j} + 5\vec{k}] \blacksquare$$

With the notation for the gradient vector, we can rewrite the expression (6.2.1) for the directional derivative as

$$f_{\vec{u}}(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}.$$

This expresses the directional derivative in the direction of \vec{u} as the scalar projection of the gradient vector onto \vec{u} .

Maximizing the Directional Derivative

Suppose we have a function of two or three variables and we consider all possible directional derivatives of f at a given point. These give the rates of change of f in all possible directions. We can then ask the questions: In which of these directions does f change fastest and what is the maximum rate of change? The answers are provided by the following theorem.

Theorem 6.2.2

The maximum value of the directional derivative of a function $f(x, y)$ or $f(x, y, z)$ at a point (x, y) or (x, y, z) is $\|\nabla f\|$ and it occurs in the direction of the gradient of f at that point.

Proof.

We have

$$f_{\vec{u}}(x, y) = \nabla f \cdot \vec{u} = \|\nabla f\| \|\vec{u}\| \cos \theta = \|\nabla f\| \cos \theta,$$

where θ is the angle between ∇f and \vec{u} . The maximum value of $\cos \theta$ is 1 and this occurs when $\theta = 0$. Therefore the maximum value of $f_{\vec{u}}$ is $\|\nabla f\|$ and it occurs when $\theta = 0$, that is, when \vec{u} has the same direction as ∇f ■

Example 6.2.4

Find the maximum rate of change of the function $u(x, y) = 50 - x^2 - 2y^2$ at the point $(1, -1)$.

Solution.

The maximum rate of change occurs in the direction of the gradient vector:

$$\nabla u(1, -1) = u_x(1, -1)\vec{i} + u_y(1, -1)\vec{j} = -2\vec{i} + 4\vec{j}.$$

The maximum rate of change at $(1, -1)$ is

$$\|\nabla u(1, -1)\| = \sqrt{(-2)^2 + 4^2} = 2\sqrt{5} \blacksquare$$

Significance of the Gradient Vector

Suppose that a curve in 3-D is defined parametrically by the equations $x = x(t)$, $y = y(t)$, $z = z(t)$, where t is a parameter. This curve can be described by the **vector function**

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}.$$

Its derivative is the tangent vector to the curve (See Figure 6.2.2) and is given by

$$\frac{d}{dt}(\vec{r}(t)) = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}.$$

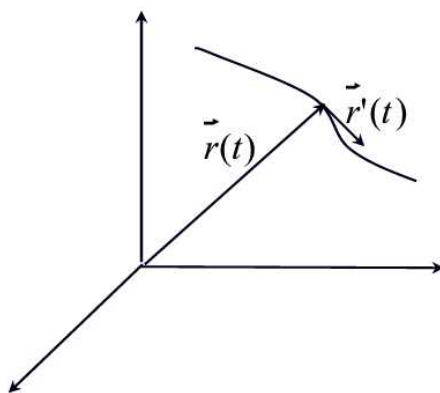


Figure 6.2.2

Now, for a function in two variables $u(x, y)$, the equation $u(x, y) = C$ is called a **level curve** of u (a **level surface** of $u(x, y, z)$). The level curves $u(x, y) = C$ are just the traces of the graph of $u(x, y)$ in the horizontal plane $z = C$ projected down to the xy -plane.

An important property of the gradient of u is that it is normal to a level surface of u at every point. To see this, let S be the level surface $f(x, y, z) = k$ and $P_0(x_0, y_0, z_0)$ be a point on S . Let C be any curve on S that passes through P_0 . We can describe C in parametric form $x = x(t)$, $y = y(t)$, and $z = z(t)$. Any point on C satisfies $f(x(t), y(t), z(t)) = k$. Differentiating both sides of this equation with respect to t we find by means of the Chain Rule

$$f_x(x, y, z)x'(t) + f_y(x, y, z)y'(t) + f_z(x, y, z)z'(t) = 0$$

which can be written as $\nabla f \cdot \vec{r}'(t) = 0$. This means that the gradient is normal to a level surface (respectively a level curve). See Figure 6.2.3.

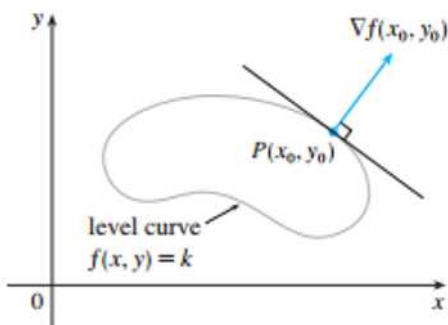


Figure 6.2.3

If we consider a topographical map of a hill and let $f(x, y)$ represent the height above sea level at a point with coordinates (x, y) , then a curve of steepest ascent can be drawn as in Figure 6.2.4 by making it perpendicular to all of the contour lines.

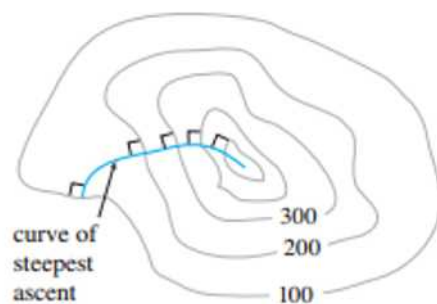


Figure 6.2.4

Vector Fields and Integral Curves

In vector calculus, a **vector field** is a function $\vec{F}(x, y)$ (or $\vec{F}(x, y, z)$ in 3-D space) that assigns a vector to each point of its domain as shown in Figure 6.2.5.

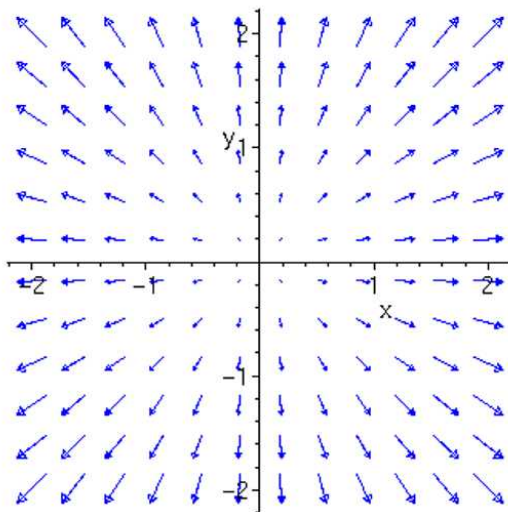


Figure 6.2.5

Creating vector fields manually is very tedious. Thus, vector fields are generally generated using computer softwares such as Mathematica, Maple, or Matlab.

Example 6.2.5

The gradient vector of a function is an example of a vector field called the **gradient vector field**. Sketch the gradient vector field of the function

$$u(x, y) = x^2 + y^2.$$

Describe the level curves of $u(x, y)$.

Solution.

The gradient vector field of the given function is

$$\nabla u(x, y) = 2x\vec{i} + 2y\vec{j}.$$

A level curve is defined by the equation

$$x^2 + y^2 = C, \quad C \geq 0.$$

Thus, level curves are circles centered at the origin. Figure 6.2.6 shows the gradient vector field as well as some of the level curves.

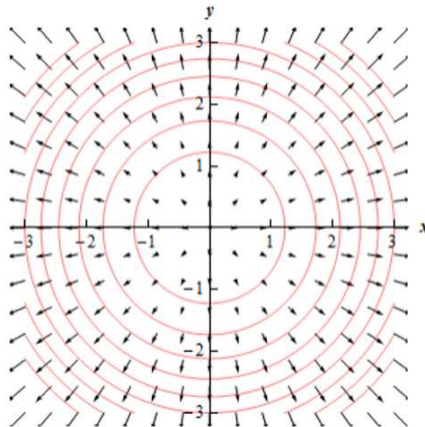


Figure 6.2.6

For example, at the point $(1, 2)$, the corresponding vector in the vector field is the vector with tail $(1, 2)$ and tip $(2, 4)$ ■

An **integral curve** of a vector field is a smooth curve⁵ Γ such that $\vec{F}(x, y)$ assigns a tangent vector at each point of Γ . For example, the integral curves of the vector field $\vec{F}(x, y) = y\vec{i} - x\vec{j}$ are circles centered at the origin. See Figure 6.2.7.

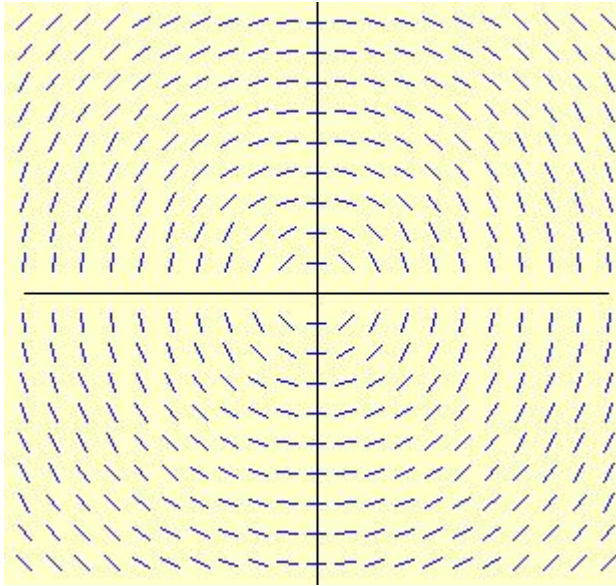


Figure 6.2.7

⁵If $\vec{r}(t)$ is a parametrization of Γ then $\vec{r}'(t)$ is continuous and $\vec{r}'(t) \neq \vec{0}$.

Practice Problems

Problem 6.2.1

Find the gradient of the function

$$F(x, y, z) = e^{xyz} + \sin(xy).$$

Problem 6.2.2

Find the gradient of the function

$$F(x, y, z) = x \cos\left(\frac{y}{z}\right).$$

Problem 6.2.3

Describe the level surfaces of the function $f(x, y, z) = (x - 2)^2 + (y - 3)^2 + (z + 5)^2$.

Problem 6.2.4

Find the directional derivative of $u(x, y) = 4x^2 + y^2$ in the direction of $\vec{a} = \vec{i} + 2\vec{j}$ at the point $(1, 1)$.

Problem 6.2.5

Find the directional derivative of $u(x, y, z) = x^2z + y^3z^2 - xyz$ in the direction of $\vec{a} = -\vec{i} + 3\vec{k}$ at the point (x, y, z) .

Problem 6.2.6

Find the maximum rate of change of the function $u(x, y) = ye^{xy}$ at the point $(0, 2)$ and the direction in which this maximum occurs.

Problem 6.2.7

Find the gradient vector field for the function $u(x, y, z) = e^z - \ln(x^2 + y^2)$.

7 Solvability of Semi-linear First Order PDEs

In this section we discuss the solvability of the semi-linear first order PDE

$$a(x, y)u_x + b(x, y)u_y = f(x, y, u) \quad (7.1)$$

via the **method of characteristics**.

To solve (7.1), we proceed as follows. Suppose we have found a solution $u(x, y)$ to (7.1). This solution may be interpreted geometrically as a surface in (x, y, z) space called the **integral surface** where $z = u(x, y)$. This integral surface can be viewed as the level surface of the function

$$F(x, y, z) = u(x, y) - z = 0.$$

Then equation (7.1) can be written as the dot product

$$\vec{v} \cdot \vec{n} = 0 \quad (7.2)$$

where $\vec{v} = \langle a, b, f \rangle$ is the **characteristic direction** and $\vec{n} = \nabla F(x, y, z) = \langle u_x, u_y, -1 \rangle$. Note that \vec{n} is normal to the surface $F(x, y, z) = 0$ and is pointing downward. Hence, \vec{n} is normal to \vec{v} and this implies that \vec{v} is tangent to the surface $F = 0$ at (x, y, z) . So our task to finding a solution to (7.1) is equivalent to finding a surface \mathcal{S} such that at every point on the surface the vector

$$\vec{v} = a\vec{i} + b\vec{j} + f(x, y, u)\vec{k}.$$

is tangent to the surface. How do we construct such a surface? The idea is to find the integral curves of the vector field \vec{v} (see Section 6.2) and then patch all these curves together to obtain the desired surface.

To this end, we start first by constructing a curve Γ parametrized by t such that at each point of Γ the vector \vec{v} is tangent to Γ . A parametrization of this curve is given by the vector function

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + u(t)\vec{k}.$$

Then the tangent vector is

$$\vec{r}'(t) = \frac{d}{dt}(\vec{r}(t)) = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{du}{dt}\vec{k}.$$

Hence, the vectors $\vec{r}'(t)$ and \vec{v} are parallel so these two vectors are proportional and this leads to the ODE system

$$\frac{dx}{dt} = \frac{dy}{dt} = \frac{du}{dt} = f(x, y, u) \quad (7.3)$$

or in differential form

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{f(x, y, u)}. \quad (7.4)$$

By solving the system (7.3) or (7.4), we are assured that the vector \vec{v} is tangent to the curve Γ which in turn lies in the solution surface \mathcal{S} . In our context, integral curves are called **characteristic curves** or simply **characteristics** of the PDE (7.1). We call (7.3) the **characteristic equations**. The projection of Γ into the xy -plane is called the **projected characteristic curve**.

Once we have found the characteristic curves, the surface \mathcal{S} is the union of these characteristic curves. In summary, by introducing these characteristic equations, we have reduced our partial differential equation to a system of ordinary differential equations. We can use ODE theory to solve the characteristic equations, then piece together these characteristic curves to form a surface. Such a surface will provide us with a solution to our PDE.

Remark 7.1

Solving $\frac{dy}{dx} = \frac{b}{a}$ one obtains the general solution $h(x, y) = k_1$ where k_1 is constant. Likewise, solving $\frac{du}{dx} = \frac{f}{a}$ one obtains the general solution $j(x, y, u) = k_2$ where k_2 is a constant. The constant k_2 is a function of k_1 . For the sake of discussion, suppose that $h(x, y) = k_1$ can be expressed as $y = g(x, k_1)$. Then, the y in $\frac{du}{dx} = \frac{f}{a}$ is being replaced by $g(x, k_1)$ so that the constant in $j(x, y, u) = k_2$ will depend on k_1 .

Example 7.1

Find the general solution to $au_x + bu_y = 0$ where a and b are constants with $a \neq 0$.

Solution.

From (7.3) we can write $\frac{dy}{dx} = \frac{b}{a}$ which yields $bx - ay = k_1$ for some arbitrary constant k_1 . From $\frac{du}{dx} = 0$ we find $u(x, y) = k_2$ where k_2 is a constant. That is, $u(x, y)$ is constant on Γ . Since $(0, -\frac{k_1}{a}, k_2)$ is on Γ , we have

$$u(x, y) = u(0, -\frac{k_1}{a}) = k_2$$

which shows that k_2 is a function of k_1 . Hence,

$$u(x, y) = f(k_1) = f(bx - ay)$$

where f is a differentiable function in one variable ■

In the next example, we show how the initial value problem for the PDE determines the function f .

Example 7.2

Find the unique solution to $au_x + bu_y = 0$, where a and b are constants with $a \neq 0$, with the initial condition $u(x, 0) = g(x)$.

Solution.

From the previous example, we found $u(x, y) = f(bx - ay)$ for some differentiable function f . Since $u(x, 0) = g(x)$, we find $g(x) = f(bx)$ or $f(x) = g\left(\frac{x}{b}\right)$ assuming that $b \neq 0$. Thus,

$$u(x, y) = g\left(x - \frac{a}{b}y\right) \quad \blacksquare$$

Example 7.3

Find the solution to $-3u_x + u_y = 0$, $u(x, 0) = e^{-x^2}$.

Solution.

We have $a = -3$, $b = 1$ and $g(x) = e^{-x^2}$. The unique solution is given by

$$u(x, y) = e^{-(x+3y)^2} \quad \blacksquare$$

Example 7.4

Find the general solution of the equation

$$xu_x + yu_y = xe^{-u}, \quad x > 0.$$

Solution.

We have $a(x, y) = x$, $b(x, y) = y$, and $f(x, y, u) = xe^{-u}$. So we have to solve the system

$$\frac{dy}{dx} = \frac{y}{x}, \quad \frac{du}{dx} = e^{-u}.$$

From the first equation, we can use the separation of variables method to find $y = k_1x$ for some constant k_1 . Solving the second equation by the method of separation of variables, we find

$$e^u - x = k_2.$$

But $k_2 = g(k_1)$ so that

$$e^u - x = g(k_1) = g\left(\frac{y}{x}\right)$$

where g is a differentiable function of one variable ■

Example 7.5

Find the general solution of the equation

$$u_x + u_y - u = y.$$

Solution.

The characteristic equations are

$$\frac{dx}{1} = \frac{dy}{1} = \frac{du}{u+y} = \frac{d(u+y+1)}{u+y+1}.$$

Solving the equation $\frac{dy}{dx} = 1$ we find $y - x = k_1$. Solving the equation $dx = \frac{d(u+y+1)}{u+y+1}$, we find $u + y + 1 = k_2e^x = f(y - x)e^x$, where f is a differentiable function of one variable. Hence,

$$u = -(1 + y) + f(y - x)e^x \quad \blacksquare$$

Example 7.6

Find the general solution to $x^2u_x + y^2u_y = (x + y)u$.

Solution.

Using properties of proportions⁶ we have

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{du}{(x+y)u} = \frac{dx - dy}{x^2 - y^2}.$$

⁶If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a \pm b}{b} = \frac{c \pm d}{d}$. Also, $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{\alpha a + \beta c + \gamma e}{\alpha b + \beta d + \gamma f}$.

Solving $\frac{dy}{dx} = \frac{y^2}{x^2}$ by the method of separation of variables we find $\frac{1}{x} - \frac{1}{y} = k_1$.

From the equation $\frac{du}{(x+y)u} = \frac{d(x-y)}{x^2-y^2}$ we find

$$\frac{du}{u} = \frac{d(x-y)}{x-y}$$

which implies

$$u = k_2(x-y) = f\left(\frac{1}{x} - \frac{1}{y}\right)(x-y) \blacksquare$$

Example 7.7

Find the solution satisfying $yu_x + xu_y = x^2 + y^2$ subject to the conditions $u(x, 0) = 1 + x^2$ and $u(0, y) = 1 + y^2$.

Solution.

Solving the equation $\frac{dy}{dx} = \frac{x}{y}$ we find $x^2 - y^2 = k_1$. On the other hand, we have

$$\begin{aligned} du &= y^{-1}(x^2 + y^2)dx \\ &= ydx + x^2y^{-1}dx \\ &= ydx + x^2y^{-1}\left(\frac{y}{x}dy\right) \\ &= ydx + xdy = d(xy). \end{aligned}$$

Hence,

$$u(x, y) = xy + f(x^2 - y^2).$$

From $u(x, 0) = 1 + x^2$ we find $f(x) = 1 + x$, $x \geq 0$. From $u(0, y) = 1 + y^2$ we find $f(y) = 1 - y$, $y \leq 0$. Hence, $f(x) = 1 + |x|$ and

$$u(x, y) = xy + |x^2 - y^2| \blacksquare$$

Remark 7.2

The method of characteristics discussed in this section applies as well to any quasi-linear first order PDE. See Chapter 9.

Practice Problems**Problem 7.1**

Solve $u_x + yu_y = y^2$ with the initial condition $u(0, y) = \sin y$.

Problem 7.2

Solve $u_x + yu_y = u^2$ with the initial condition $u(0, y) = \sin y$.

Problem 7.3

Find the general solution of $yu_x - xu_y = 2xyu$.

Problem 7.4

Find the integral surface of the IVP: $xu_x + yu_y = u$, $u(x, 1) = 2 + e^{-|x|}$.

Problem 7.5

Find the unique solution to $4u_x + u_y = u^2$, $u(x, 0) = \frac{1}{1+x^2}$.

Problem 7.6

Find the unique solution to $e^{2y}u_x + xu_y = xu^2$, $u(x, 0) = e^{x^2}$.

Problem 7.7

Find the unique solution to $xu_x + u_y = 3x - u$, $u(x, 0) = \tan^{-1} x$.

Problem 7.8

Solve: $xu_x - yu_y = 0$, $u(x, x) = x^4$.

Problem 7.9

Find the general solution of $yu_x - 3x^2yu_y = 3x^2u$.

Problem 7.10

Find $u(x, y)$ that satisfies $yu_x + xu_y = 4xy^3$ subject to the boundary conditions $u(x, 0) = -x^4$ and $u(0, y) = 0$.

8 Linear First Order PDE: The One Dimensional Spatial Transport Equations

Modeling is the process of writing a differential equation to describe a physical situation. In this section we derive the one-dimensional spatial transport equation and use the method of characteristics to solve it.

Linear Transport Equation for Fluid Flows

We shall describe the transport of a dissolved chemical by water that is traveling with uniform velocity c through a long thin tube G with uniform cross section A . (The very same discussion applies to the description of the transport of gas by air moving through a pipe.) We assume the velocity $c > 0$ is in the (rightward) positive direction of the x -axis. We will also assume that the concentration of the chemical is constant across the cross section A located at x so that the chemical changes only in the x -direction and thus the term one-dimensional spatial equation. This condition says that the quantity of the chemical in a portion of the tube is the same but it is traveling with time.

Let $u(x, t)$ be a continuously differentiable function denoting the concentration of the chemical (i.e. amount of chemical per unit volume) at position x and time t . Then at time t_0 , the amount of chemical stored in a section of the tube between positions a and x_0 (see Figure 8.1) is given by the definite integral

$$\int_a^{x_0} Au(s, t_0) ds.$$

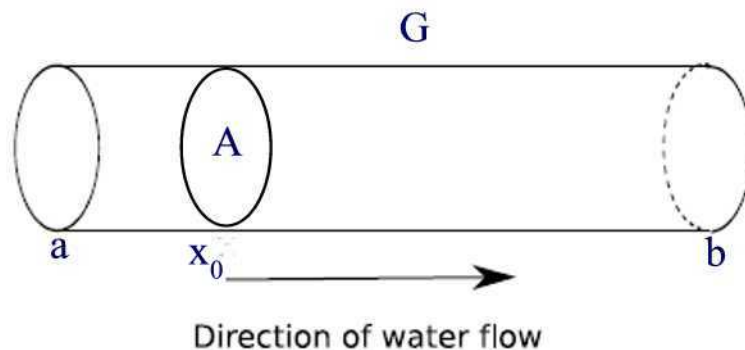


Figure 8.1

Since the water is flowing at a constant speed c , so at time $t_0 + h$ the same quantity of chemical will exist in the portion of the tube between $a + ch$ and $x_0 + ch$. That is,

$$\int_a^{x_0} Au(s, t_0) ds = \int_{a+ch}^{x_0+ch} Au(s, t_0 + h) ds.$$

Taking the derivative of both sides with respect to x_0 and using the Fundamental Theorem of Calculus, we find

$$u(x_0, t_0) = u(x_0 + ch, t_0 + h).$$

Now, taking the derivative of this last equation with respect to h and using the chain rule, with $x = x_0 + ch$, $t = t_0 + h$, we find

$$0 = u_t(x_0 + ch, t_0 + h) + cu_x(x_0 + ch, t_0 + h).$$

Taking the limit of this last equation as h approaches 0 and using the fact that u_t and u_x are continuous, we find

$$u_t(x_0, t_0) + cu_x(x_0, t_0) = 0. \quad (8.1)$$

Since x_0 and t_0 are arbitrary, Equation (8.1) is true for all (x, t) . This equation is called the **transport equation** in one-dimensional space. It is a linear, homogeneous first order partial differential equation.

Note that (8.1) can be written in the form

$$\langle 1, c \rangle \cdot \langle u_t, u_x \rangle = 0$$

so that the left-hand side of (8.1) is the directional derivative of $u(t, x)$ at (t, x) in the direction of the vector $\langle 1, c \rangle$.

Solvability via the method of characteristics

We will use the method of characteristics discussed in Chapter 7 to solve (8.1). The characteristic equations are

$$dt = \frac{dx}{c} = \frac{du}{0}.$$

Thus, to solve (8.1), we solve the system of ODEs

$$\frac{dt}{dx} = \frac{1}{c}, \quad \frac{du}{dx} = 0.$$

Solving the first equation, we find $x - ct = k_1$. Solving the second equation we find

$$u(x, t) = k_2 = f(k_1) = f(x - ct).$$

One can check that this is indeed a solution to (8.1). Indeed, by using the chain rule one finds

$$u_t = -cf'(x - ct) \quad \text{and} \quad u_x = f'(x - ct).$$

Hence, by substituting these results into the equation one finds

$$u_t + cu_x = -cf'(x - ct) + cf'(x - ct) = 0.$$

The solution $u(x, t) = f(x - ct)$ is called the **right traveling wave**, since the graph of the function $f(x - ct)$ at a given time t is the graph of $f(x)$ shifted to the right by the value ct . Thus, with growing time, the function $f(x)$ is moving without changes to the right at the speed c .

An initial value condition determines a unique solution to the transport equation as shown in the next example.

Example 8.1

Find the solution to $u_t - 3u_x = 0$, $u(x, 0) = e^{-x^2}$.

Solution.

The characteristic equations lead to the ODEs

$$\frac{dt}{dx} = -\frac{1}{3}, \quad \frac{du}{dx} = 0.$$

Solving the first equation, we find $3t + x = k_1$. From the second equation, we find $u(x, t) = k_2 = f(k_1) = f(3t + x)$. From the initial condition, $u(x, 0) = f(x) = e^{-x^2}$. Hence,

$$u(x, t) = e^{-(3t+x)^2} \blacksquare$$

Transport Equation with Decay

Recall from ODE that a function u is an exponential decay function if it satisfies the equation

$$\frac{du}{dt} = \lambda u, \quad \lambda < 0.$$

A **transport equation with decay** is an equation given by

$$u_t + cu_x + \lambda u = f(x, t) \quad (8.2)$$

where $\lambda > 0$ and c are constants and f is a given function representing external resources. Note that the decay is characterized by the term λu .

Note that (8.2) is a first order linear partial differential equation that can be solved by the method of characteristics by solving the characteristic equations

$$\frac{dx}{c} = \frac{dt}{1} = \frac{du}{f(x, t) - \lambda u}.$$

Example 8.2

Find the general solution of the transport equation

$$u_t + u_x + u = t.$$

Solution.

The characteristic equations are

$$\frac{dx}{1} = \frac{dt}{1} = \frac{du}{t - u}.$$

From the equation $dx = dt$ we find $x - t = k_1$. Using a property of proportions we can write

$$\frac{dt}{1} = \frac{du}{t - u} = \frac{dt - du}{1 - t + u} = -\frac{d(1 - t + u)}{1 - t + u}.$$

Thus, $1 - t + u = k_2 e^{-t} = f(x - t)e^{-t}$ or $u(x, t) = t - 1 + f(x - t)e^{-t}$ where f is a differentiable function of one variable ■

Practice Problems

Problem 8.1

Find the solution to $u_t + 3u_x = 0$, $u(x, 0) = \sin x$.

Problem 8.2

Solve the equation $au_x + bu_y + cu = 0$.

Problem 8.3

Solve the equation $u_x + 2u_y = \cos(y - 2x)$ with the initial condition $u(0, y) = f(y)$, where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a given function.

Problem 8.4

Show that the initial value problem $u_t + u_x = x$, $u(x, x) = 1$ has no solution.

Problem 8.5

Solve the transport equation $u_t + 2u_x = -3u$ with initial condition $u(x, 0) = \frac{1}{1+x^2}$.

Problem 8.6

Solve $u_t + u_x - 3u = t$ with initial condition $u(x, 0) = x^2$.

Problem 8.7

Show that the decay term λu in the transport equation with decay

$$u_t + cu_x + \lambda u = 0$$

can be eliminated by the substitution $w = ue^{\lambda t}$.

Problem 8.8 (*Well-Posed*)

Let u be the unique solution to the IVP

$$u_t + cu_x = 0$$

$$u(x, 0) = f(x)$$

and v be the unique solution to the IVP

$$u_t + cu_x = 0$$

$$u(x, 0) = g(x)$$

where f and g are continuously differentiable functions.

(a) Show that $w(x, t) = u(x, t) - v(x, t)$ is the unique solution to the IVP

$$u_t + cu_x = 0$$

$$u(x, 0) = f(x) - g(x)$$

(b) Write an explicit formula for w in terms of f and g .

(c) Use (b) to conclude that the transport problem is well-posed. That is, a small change in the initial data leads to a small change in the solution.

Problem 8.9

Solve the initial boundary value problem

$$u_t + cu_x = -\lambda u, \quad x > 0, t > 0$$

$$u(x, 0) = 0, \quad u(0, t) = g(t), \quad t > 0.$$

Problem 8.10

Solve the first-order equation $2u_t + 3u_x = 0$ with the initial condition $u(x, 0) = \sin x$.

Problem 8.11

Solve the PDE $u_x + u_y = 1$.

9 Solving Quasi-Linear First Order PDE via the Method of Characteristics

In this section we develop a method for finding the general solution of a quasi-linear first order partial differential equation

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u). \quad (9.1)$$

This method is called the **method of characteristics** or **Lagrange's method**. This method consists of transforming the PDE to a system of ODEs which can be solved and the found solution is transformed into a solution for the original PDE.

The method of characteristics relies on a geometrical argument. A visualization of a solution is an integral surface with equation $z = u(x, y)$. An alternative representation of this integral surface is

$$F(x, y, z) = u(x, y) - z = 0.$$

That is, an integral surface is a level surface of the function $F(x, y, z)$.

Now, recall from vector calculus that the gradient vector to a level surface at the point (x, y, z) is a normal vector to the surface at that point. That is, the gradient is a vector normal to the tangent plane to the surface at the point (x, y, z) . Thus, the normal vector to the surface $F(x, y, z) = 0$ is given by

$$\vec{n} = \nabla F = F_x \vec{i} + F_y \vec{j} + F_z \vec{k} = u_x \vec{i} + u_y \vec{j} - \vec{k}.$$

Because of the negative z - component, the vector \vec{n} is pointing downward. Now, equation (9.1) can be written as the dot product

$$(a(x, y, u), b(x, y, u), c(x, y, u)) \cdot (u_x, u_y, -1) = 0$$

or

$$\vec{v} \cdot \vec{n} = 0$$

where $\vec{v} = a(x, y, u)\vec{i} + b(x, y, u)\vec{j} + c(x, y, u)\vec{k}$. Thus, \vec{n} is normal to \vec{v} . Since \vec{n} is normal to the surface $F(x, y, z) = 0$, the vector \vec{v} must be tangential to the surface $F(x, y, z) = 0$ and hence must lie in the tangent plane to the surface at every point. Thus, to find a solution to (9.1) we need to find an integral surface such that the surface is tangent to the vector \vec{v} at each of its point.

The required surface can be found as the union of integral curves, that is, curves that are tangent to \vec{v} at every point on the curve. If an integral curve has a parametrization

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + u(t)\vec{k}$$

then the integral curve (i.e. the characteristic) is a solution to the ODE system

$$\frac{dx}{dt} = a(x, y, u), \quad \frac{dy}{dt} = b(x, y, u), \quad \frac{du}{dt} = c(x, y, u) \quad (9.2)$$

or in differential form

$$\frac{dx}{a(x, y, u)} = \frac{dy}{b(x, y, u)} = \frac{du}{c(x, y, u)}. \quad (9.3)$$

Equations (9.2) or (9.3) are called **characteristic equations**. Note that $u(t) = u(x(t), y(t))$ gives the values of u along a characteristic. Thus, along a characteristic, the PDE (9.1) degenerates to an ODE.

Example 9.1

Find the general solution of the PDE $yu u_x + xu u_y = xy$.

Solution.

The characteristic equations are $\frac{dx}{yu} = \frac{dy}{xu} = \frac{du}{xy}$. Using the first two fractions we find $x^2 - y^2 = k_1$. Using the last two fractions we find $u^2 - y^2 = f(x^2 - y^2)$. Hence, the general solution can be written as $u^2 = y^2 + f(x^2 - y^2)$, where f is an arbitrary differentiable single variable function ■

Example 9.2

Find the general solution of the PDE $x(y^2 - u^2)u_x - y(u^2 + x^2)y_y = (x^2 + y^2)u$.

Solution.

The characteristic equations are $\frac{dx}{x(y^2 - u^2)} = \frac{dy}{-y(u^2 + x^2)} = \frac{du}{(x^2 + y^2)u}$. Using a property of proportions we can write

$$\frac{xdx + ydy + udu}{x^2(y^2 - u^2) - y^2(u^2 + x^2) + u^2(x^2 + y^2)} = \frac{du}{(x^2 + y^2)u}.$$

That is

$$\frac{xdx + ydy + udu}{0} = \frac{du}{(x^2 + y^2)u}$$

or

$$xdx + ydy + udu = 0.$$

Hence, we find $x^2 + y^2 + u^2 = k_1$. Also,

$$\frac{\frac{dx}{x} - \frac{dy}{y}}{y^2 - u^2 + u^2 + x^2} = \frac{du}{(x^2 + y^2)u}$$

or

$$\frac{dx}{x} - \frac{dy}{y} = \frac{du}{u}.$$

Hence, we find $\frac{yu}{x} = k_2$. The general solution is given by

$$u(x, y) = \frac{x}{y} f(x^2 + y^2 + u^2)$$

where f is an arbitrary differentiable single variable function ■

Practice Problem

Problem 9.1

Find the general solution of the PDE $(y + u)u_x + u_y = -1$.

Problem 9.2

Find the general solution of the PDE $x(y - u)u_x + y(u - x)u_y = u(x - y)$.

Problem 9.3

Find the general solution of the PDE $u(u^2 + xy)(xu_x - yu_y) = x^4$.

Problem 9.4

Find the general solution of the PDE $(y + xu)u_x - (x + yu)u_y = x^2 - y^2$.

Problem 9.5

Find the general solution of the PDE $(y^2 + u^2)u_x - xyu_y + xu = 0$.

Problem 9.6

Find the general solution of the PDE $u_t + uu_x = x$.

Problem 9.7

Find the general solution of the PDE $(y - u)u_x + (u - x)u_y = x - y$.

Problem 9.8

Solve

$$x(y^2 + u)u_x - y(x^2 + u)u_y = (x^2 - y^2)u.$$

Problem 9.9

Solve

$$\sqrt{1 - x^2}u_x + u_y = 0.$$

Problem 9.10

Solve

$$u(x + y)u_x + u(x - y)u_y = x^2 + y^2.$$

10 The Cauchy Problem for First Order Quasilinear Equations

When solving a partial differential equation, it is seldom the case that one tries to study the properties of the general solution of such equations. In general, one deals with those partial differential equations whose solutions satisfy certain supplementary conditions. In the case of a first order partial differential equation, we determine the particular solution by formulating an initial value problem also known as a **Cauchy problem**.

In this section, we discuss the Cauchy problem for the first order quasilinear partial differential equation

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u). \quad (10.1)$$

Recall that the initial value problem of a first order ordinary differential equation asks for a solution of the equation which has a given value at a given point in \mathbb{R} . The Cauchy problem for the PDE (10.1) asks for a solution of (10.1) which has given values on a given curve in \mathbb{R}^2 . A precise statement of the problem is given next.

Initial Value Problem or Cauchy Problem

Let C be a given curve in \mathbb{R}^2 defined parametrically by the equations

$$x = x_0(t), \quad y = y_0(t)$$

where x_0, y_0 are continuously differentiable functions on some interval I . Let $u_0(t)$ be a given continuously differentiable function on I . The Cauchy problem for (10.1) asks for a continuously differentiable function $u = u(x, y)$ defined in a domain $\Omega \subset \mathbb{R}^2$ containing the curve C and such that:

- (1) $u = u(x, y)$ is a solution of (10.1) in Ω .
- (2) On the curve C , u equals the given function $u_0(t)$, i.e.

$$u(x_0(t), y_0(t)) = u_0(t), \quad t \in I. \quad (10.2)$$

We call C the **initial curve** of the problem, $u_0(t)$ the **initial data**, and (10.2) the **initial condition** or **Cauchy data** of the problem. See Figure 10.1.

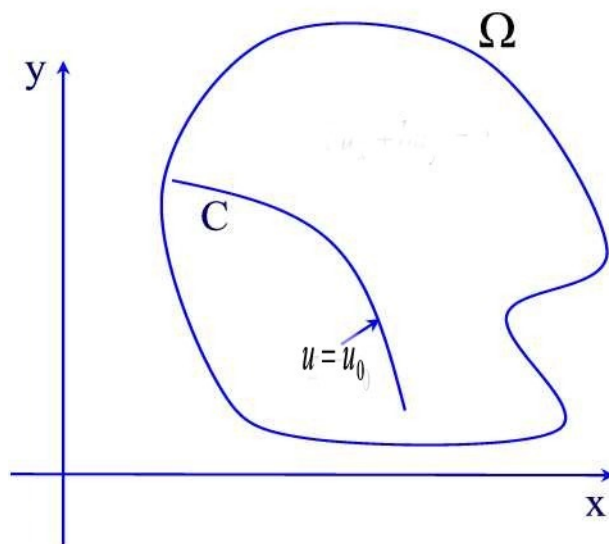


Figure 10.1

If we view a solution $u = u(x, y)$ of (10.1) as an integral surface of (10.1), we can give a simple geometrical statement of the problem: Find a solution surface of (10.1) containing the curve Γ described parametrically by the equations

$$\Gamma : x = x_0(t), \quad y = y_0(t), \quad u = u_0(t), \quad t \in I.$$

Note that the projection of this curve in the xy -plane is just the curve C . The following theorem asserts that under certain conditions the Cauchy problem (10.1) - (10.2) has a unique solution.

Theorem 10.1

Suppose that $x_0(t), y_0(t)$, and $u_0(t)$ are continuously differentiable functions of t in an interval I , and that a, b , and c are functions of x, y , and u with continuous first order partial derivatives with respect to their argument in some domain D of (x, y, u) -space containing the initial curve

$$\Gamma : x = x_0(t), \quad y = y_0(t), \quad u = u_0(t)$$

where $t \in I$. If $(x_0(t), y_0(t), u_0(t))$ is a point on Γ that satisfies the condition

$$a(x_0(t), y_0(t), u_0(t)) \frac{dy_0}{dt}(t) - b(x_0(t), y_0(t), u_0(t)) \frac{dx_0}{dt}(t) \neq 0 \quad (10.3)$$

then by continuity this relation holds in a neighborhood U of $(x_0(t), y_0(t), u_0(t))$ so that Γ is nowhere characteristic in U . In this case, there exists a unique solution $u = u(x, y)$ of (10.1) in U such that the initial condition (10.2) is satisfied for every point on C contained in U . See Figure 10.2. That is, there is a unique integral surface of (10.1) that contains Γ in a neighborhood of $(x_0(t), y_0(t), u_0(t))$.

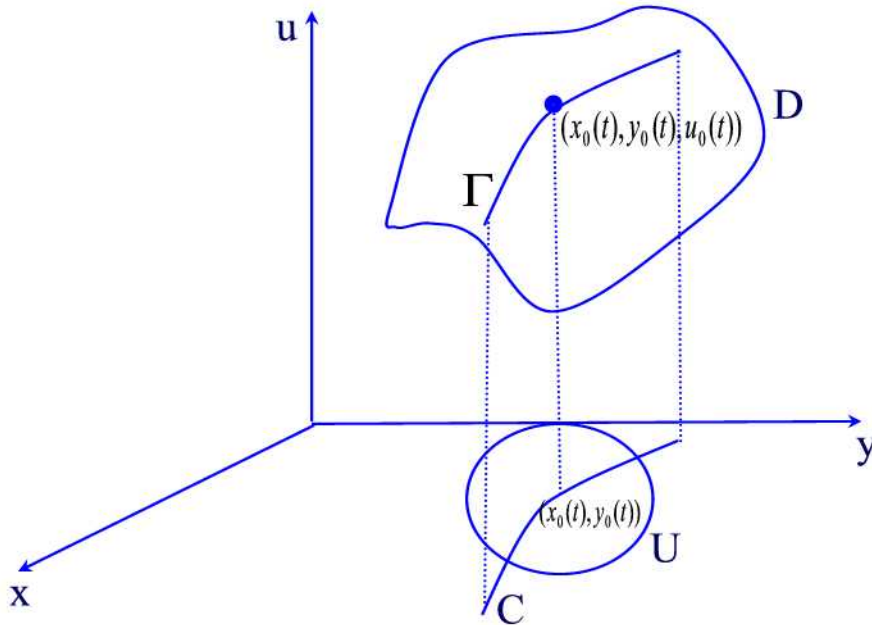


Figure 10.2

It follows that the Cauchy problem has a unique solution if C is **nowhere characteristic**.

The unique solution is found as follows: We solve the PDE by the method of characteristics to obtain the general solution that involves an unknown function. This unknown function is determined using the initial condition. We illustrate this process in the next example.

Example 10.1

Solve the Cauchy problem

$$\begin{aligned} u_x + u_y &= 1 \\ u(x, 0) &= f(x). \end{aligned}$$

Solution.

The initial curve in \mathbb{R}^3 can be given parametrically as

$$\Gamma : x_0(t) = t, \quad y_0(t) = 0, \quad u_0(t) = f(t).$$

We have

$$a(x_0(t), y_0(t), u_0(t)) \frac{dy_0}{dt}(t) - b(x_0(t), y_0(t), u_0(t)) \frac{dx_0}{dt}(t) = -1 \neq 0$$

so by the above theorem the given Cauchy problem has a unique solution. Next we apply the results of the previous section to find the unique solution. If we solve the characteristic equations in non-parametric form

$$\frac{dx}{1} = \frac{dy}{1} = \frac{du}{1}$$

we find $x - y = c_1$ and $u - x = c_2$. Thus, the general solution of the PDE is given by $u = x + F(x - y)$. Using the Cauchy data $u(x, 0) = f(x)$ we find $f(x) = x + F(x)$ which implies that $F(x) = f(x) - x$. Hence, the unique solution is given by

$$u(x, y) = x + f(x - y) - (x - y) = y + f(x - y) \blacksquare$$

Next, if condition (10.3) is not satisfied and Γ is a characteristic curve, i.e.,

$$\begin{aligned} a(x_0, y_0, u_0) \frac{dy_0}{dt} &= b(x_0, y_0, u_0) \frac{dx_0}{dt} \\ a(x_0, y_0, u_0) \frac{du_0}{dt} &= c(x_0, y_0, u_0) \frac{dx_0}{dt} \\ c(x_0, y_0, u_0) \frac{dy_0}{dt} &= b(x_0, y_0, u_0) \frac{du_0}{dt} \end{aligned}$$

for all point on Γ then the problem has infinitely many solutions. To see this, pick an arbitrary point $P_0 = (x_0, y_0, u_0)$ on Γ . Pick a new initial curve Γ' passing through P_0 which is nowhere characteristic in a neighborhood of P_0 . In this case, condition (10.3) is satisfied and the new Cauchy problem has a unique solution. Since there are infinitely many ways of selecting Γ' , we obtain infinitely many solutions. We illustrate this case in the next example.

Example 10.2

Solve the Cauchy problem

$$\begin{aligned} u_x + u_y &= 1 \\ u(x, x) &= x. \end{aligned}$$

Solution.

The initial curve in \mathbb{R}^3 can be given parametrically as

$$\Gamma : x_0(t) = t, \quad y_0(t) = t, \quad u_0(t) = t.$$

We have

$$\begin{aligned} a(x_0, y_0, u_0) \frac{dy_0}{dt} &= b(x_0, y_0, u_0) \frac{dx_0}{dt} = 1 \\ a(x_0, y_0, u_0) \frac{du_0}{dt} &= c(x_0, y_0, u_0) \frac{dx_0}{dt} = 1 \\ c(x_0, y_0, u_0) \frac{dy_0}{dt} &= b(x_0, y_0, u_0) \frac{du_0}{dt} = 1 \end{aligned}$$

so that Γ is a characteristic curve. As in Example 10.1, the general solution of the PDE is $u(x, y) = y + f(x - y)$ where f is an arbitrary differentiable function. Using the Cauchy data $u(x, x) = x$ we find $f(0) = 0$. Thus, the solution is given by

$$u(x, y) = y + f(x - y)$$

where f is an arbitrary function such that $f(0) = 0$. There are infinitely many choices for f . Hence, the problem has infinitely many solutions ■

If condition (10.3) is not satisfied and if

$$c(x_0, y_0, u_0) \frac{dx_0}{dt} \neq a(x_0, y_0, u_0) \frac{du_0}{dt}$$

for some points on Γ then the Cauchy problem has no solutions. We illustrate this case next.

Example 10.3

Solve the Cauchy problem

$$\begin{aligned} u_x + u_y &= 1 \\ u(x, x) &= 1. \end{aligned}$$

Solution.

The initial curve in \mathbb{R}^3 can be given parametrically as

$$\Gamma : x_0(t) = t, \quad y_0(t) = t, \quad u_0(t) = 1.$$

We have

$$a(x_0(t), y_0(t), u_0(t)) \frac{dy_0}{dt}(t) - b(x_0(t), y_0(t), u_0(t)) \frac{dx_0}{dt}(t) = 0.$$

and

$$1 = c(x_0, y_0, u_0) \frac{dx_0}{dt} \neq a(x_0, y_0, u_0) \frac{du_0}{dt} = 0.$$

As in Example 10.1, the general solution to the PDE is given by $u = y + f(x - y)$. Using the Cauchy data $u(x, x) = 1$ we find $f(0) = 1 - x$, which is not possible since the LHS is a fixed number whereas the RHS is a variable expression. Hence, the problem has no solutions ■

Example 10.4

Solve the Cauchy problem

$$\begin{aligned} u_x - u_y &= 1 \\ u(x, 0) &= x^2. \end{aligned}$$

Solution.

The initial curve is given parametrically by

$$\Gamma : x_0(t) = t, \quad y_0(t) = 0, \quad u_0(t) = t^2.$$

We have

$$a(x_0(t), y_0(t), u_0(t)) \frac{dy_0}{dt}(t) - b(x_0(t), y_0(t), u_0(t)) \frac{dx_0}{dt}(t) = 1 \neq 0$$

so the Cauchy problem has a unique solution.

The characteristic equations in non-parametric form are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{du}{1}.$$

Using the first two fractions we find $x + y = c_1$. Using the first and the third fractions we find $u - x = c_2$. Thus, the general solution can be represented by

$$u = x + f(x + y)$$

where f is an arbitrary differentiable function. Using the Cauchy data $u(x, 0) = x^2$ we find $x^2 - x = f(x)$. Hence, the unique solution is given by

$$u = x + (x + y)^2 - (x + y) = (x + y)^2 - y \quad \blacksquare$$

Example 10.5

Solve the initial value problem

$$u_t + uu_x = x, \quad u(x, 0) = 1.$$

Solution.

The initial curve is given parametrically by

$$\Gamma: x_0(t) = t, \quad y_0(t) = 0, \quad u_0(t) = 1.$$

We have

$$a(x_0(t), y_0(t), u_0(t)) \frac{dy_0}{dt}(t) - b(x_0(t), y_0(t), u_0(t)) \frac{dx_0}{dt}(t) = -1 \neq 0$$

so the Cauchy problem has a unique solution.

The characteristic equations in non-parametric form are

$$\frac{dt}{1} = \frac{dx}{u} = \frac{du}{x}.$$

Since

$$\frac{dt}{1} = \frac{d(x+u)}{x+u}$$

we find that $(x+u)e^{-t} = c_1$. Now, using the last two fractions we find $u^2 - x^2 = k_2 = f((x+u)e^{-t})$.

Using the Cauchy data $u(x, 0) = 1$, we find $1 - x^2 = f(1+x)$ or $f(1+x) = (1+x)^2 - 2x(1+x)$. Thus, $f(x) = x^2 - 2x(x-1)$. The unique solution is given by

$$u^2 - x^2 = (x+u)^2 e^{-2t} - 2(x+u)e^{-t}[(x+u)e^{-t} - 1]$$

or

$$u - x = (x+u)e^{-2t} - 2e^{-t}[(x+u)e^{-t} - 1] = 2e^{-t} - (x+u)e^{-2t}.$$

This can be reduced further as follows: $u + ue^{-2t} = x + 2e^{-t} - xe^{-2t} = 2e^{-t} + x(1 - e^{-2t}) \implies u = \frac{2e^{-t}}{1+e^{-2t}} + x \frac{1-e^{-2t}}{1+e^{-2t}} = \operatorname{sech}(t) + x \tanh(t)$ ■

Example 10.6

Solve the initial value problem

$$uu_x + u_y = 1$$

where $u(x, y) = 0$ on the curve $y^2 = 2x$.

Solution.

A parametrization of Γ is

$$\Gamma : x_0(t) = 2t^2, \quad y_0(t) = 2t, \quad u_0(t) = 0, \quad t > 0.$$

We have

$$a(x_0(t), y_0(t), u_0(t)) \frac{dy_0}{dt}(t) - b(x_0(t), y_0(t), u_0(t)) \frac{dx_0}{dt}(t) = -4t \neq 0, \quad t > 0$$

so the Cauchy problem has a unique solution.

The characteristic equations in non-parametric form are

$$\frac{dx}{u} = \frac{dy}{1} = \frac{du}{1}.$$

Using the last two fractions, we find $u - y = k_1$. Using the first and the last fractions, we find $u^2 - 2x = k_2 = f(k_1) = f(u - y)$.

Using the initial condition, we find $f(x) = -x^2$. Hence,

$$u^2 - 2x = -(u - y)^2$$

or equivalently

$$(u - y)^2 + u^2 = 2x.$$

Solving this quadratic equation in u to find

$$2u = y \pm (4x - y^2)^{\frac{1}{2}}.$$

The solution surface satisfying $u = 0$ on $y^2 = 2x$ is given by

$$2u = y - (4x - y^2)^{\frac{1}{2}}.$$

This represents a solution surface only when $y^2 < 4x$. The solution does not exist for $y^2 > 4x$ ■

Practice Problems

Problem 10.1

Solve

$$(y - u)u_x + (u - x)u_y = x - y$$

with the condition $u\left(x, \frac{1}{x}\right) = 0$, $x \neq 0, 1$.

Problem 10.2

Solve the linear equation

$$yu_x + xu_y = u$$

with the Cauchy data $u(x, 0) = x^3$, $x > 0$.

Problem 10.3

Solve

$$x(y^2 + u)u_x - y(x^2 + u)u_y = (x^2 - y^2)u$$

with the Cauchy data $u(x, -x) = 1$, $x > 0$

Problem 10.4

Solve

$$xu_x + yu_y = xe^{-u}$$

with the Cauchy data $u(x, x^2) = 0$, $x > 0$.

Problem 10.5

Solve the initial value problem

$$xu_x + u_y = 0, \quad u(x, 0) = f(x).$$

Problem 10.6

Solve the initial value problem

$$u_t + au_x = 0, \quad u(x, 0) = f(x).$$

Problem 10.7

Solve the initial value problem

$$au_x + u_y = u^2, \quad u(x, 0) = \cos x$$

Problem 10.8

Solve the initial value problem

$$u_x + xu_y = u, \quad u(1, y) = h(y).$$

Problem 10.9

Solve the initial value problem

$$uu_x + u_y = 0, \quad u(x, 0) = f(x)$$

where f is an invertible function.

Problem 10.10

Solve the initial value problem

$$\sqrt{1-x^2}u_x + u_y = 0, \quad u(0, y) = y.$$

Problem 10.11

Consider

$$xu_x + 2yu_y = 0.$$

- (i) Find and sketch the characteristics.
- (ii) Find the solution with $u(1, y) = e^y$.
- (iii) What happens if you try to find the solution satisfying either $u(0, y) = g(y)$ or $u(x, 0) = h(x)$ for given functions g and h ?

Problem 10.12

Solve the equation $u_x + u_y = u$ subject to the condition $u(x, 0) = \cos x$.

Problem 10.13

(a) Find the general solution of the equation

$$u_x + yu_y = u.$$

- (b) Find the solution satisfying the Cauchy data $u(x, 3e^x) = 2$.
- (c) Find the solution satisfying the Cauchy data $u(x, e^x) = e^x$.

Problem 10.14

Solve the Cauchy problem

$$u_x + 4u_y = x(u + 1), \quad u(x, 5x) = 1.$$

Problem 10.15

Solve the Cauchy problem

$$u_x - u_y = u, \quad u(x, -x) = \sin x, \quad x \neq \frac{\pi}{4}.$$

Problem 10.16

(a) Find the characteristics of the equation

$$yu_x + xu_y = 0.$$

(b) Sketch some of the characteristics.

(c) Find the solution satisfying the boundary condition $u(0, y) = e^{-y^2}$.

Problem 10.17

Consider the equation $u_x + yu_y = 0$. Is there a solution satisfying the extra condition

(a) $u(x, 0) = 1$

(b) $u(x, 0) = x$?

If yes, give a formula; if no, explain why.

Second Order Linear Partial Differential Equations

In this chapter we consider the three fundamental second order linear partial differential equations of parabolic, hyperbolic, and elliptic type. These types arise in many applications such as the wave equation, the heat equation and the Laplace's equation. We will study the solvability of each of these equations.

11 Second Order PDEs in Two Variables

In this section we will briefly review second order partial differential equations.

A **second order partial differential equation in the variables** x and y is an equation of the form

$$F(x, y, u, u_x, u_y, u_{xx}, u_{yy}, u_{xy}) = 0. \quad (11.1)$$

If Equation (11.1) can be written in the form

$$A(x, y, u, u_x, u_y)u_{xx} + B(x, y, u, u_x, u_y)u_{xy} + C(x, y, u, u_x, u_y)u_{yy} = D(x, y, u, u_x, u_y) \quad (11.2)$$

then we say that the equation is **quasi-linear**.

If Equation (11.1) can be written in the form

$$A(x, y)u_{xx} + B(x, y)u_{xy} + C(x, y)u_{yy} = D(x, y, u, u_x, u_y) \quad (11.3)$$

then we say that the equation is **semi-linear**.

If Equation (11.1) can be written in the form

$$A(x, y)u_{xx} + B(x, y)u_{xy} + C(x, y)u_{yy} + D(x, y)u_x + E(x, y)u_y + F(x, y)u = G(x, y) \quad (11.4)$$

then we say that the equation is **linear**.

A linear equation is said to be **homogeneous** when $G(x, y) \equiv 0$ and **non-homogeneous** otherwise.

Equation (11.4) resembles the general equation of a conic section

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

which is classified as either parabolic, hyperbolic, or elliptic based on the sign of the **discriminant** $B^2 - 4AC$. We do the same for a second order linear partial differential equation:

- **Hyperbolic:** This occurs if $B^2 - 4AC > 0$ at a given point in the domain of u .
- **Parabolic:** This occurs if $B^2 - 4AC = 0$ at a given point in the domain of u .
- **Elliptic:** This occurs if $B^2 - 4AC < 0$ at a given point in the domain of u .

Example 11.1

Determine whether the equation $u_{xx} + xu_{yy} = 0$ is hyperbolic, parabolic or elliptic.

Solution.

Here we are given $A = 1$, $B = 0$, and $C = x$. Since $B^2 - 4AC = -4x$, the given equation is hyperbolic if $x < 0$, parabolic if $x = 0$ and elliptic if $x > 0$ ■

Second order partial differential equations arise in many areas of scientific applications. In what follows we list some of the well-known models that are of great interest:

1. The *heat equation* in one-dimensional space is given by

$$u_t = ku_{xx}$$

where k is a constant.

2. The *wave equation* in one-dimensional space is given by

$$u_{tt} = c^2u_{xx}$$

where c is a constant.

3. The *Laplace equation* is given by

$$\Delta u = u_{xx} + u_{yy} = 0.$$

Practice Problems

Problem 11.1

Classify each of the following equation as hyperbolic, parabolic, or elliptic:

- (a) Wave propagation: $u_{tt} = c^2 u_{xx}$, $c > 0$.
- (b) Heat conduction: $u_t = c u_{xx}$, $c > 0$.
- (c) Laplace's equation: $\Delta u = u_{xx} + u_{yy} = 0$.

Problem 11.2

Classify the following linear scalar PDE with constant coefficients as hyperbolic, parabolic or elliptic.

- (a) $u_{xx} + 4u_{xy} + 5u_{yy} + u_x + 2u_y = 0$.
- (b) $u_{xx} - 4u_{xy} + 4u_{yy} + 3u_x + 4u_y = 0$.
- (c) $u_{xx} + 2u_{xy} - 3u_{yy} + 2u_x + 6u_y = 0$.

Problem 11.3

Find the region(s) in the xy -plane where the equation

$$(1+x)u_{xx} + 2xyu_{xy} - y^2u_{yy} = 0$$

is elliptic, hyperbolic, or parabolic. Sketch these regions.

Problem 11.4

Show that $u(x, t) = \cos x \sin t$ is a solution to the problem

$$\begin{aligned} u_{tt} &= u_{xx} \\ u(x, 0) &= 0 \\ u_t(x, 0) &= \cos x \\ u_x(0, t) &= 0 \end{aligned}$$

for all $x, t > 0$.

Problem 11.5

Classify each of the following PDE as linear, quasilinear, semi-linear, or non-linear.

- (a) $u_t + uu_x = uu_{xx}$
- (b) $xu_{tt} + tu_{yy} + u^3u_x^2 = t + 1$
- (c) $u_{tt} = c^2u_{xx}$
- (d) $u_{tt}^2 + u_x = 0$.

Problem 11.6

Show that, for all $(x, y) \neq (0, 0)$, $u(x, y) = \ln(x^2 + y^2)$ is a solution of

$$u_{xx} + u_{yy} = 0,$$

and that, for all $(x, y, z) \neq (0, 0, 0)$, $u(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$ is a solution of

$$u_{xx} + u_{yy} + u_{zz} = 0.$$

Problem 11.7

Consider the eigenvalue problem

$$u_{xx} = \lambda u, \quad 0 < x < L$$

$$u_x(0) = k_0 u(0)$$

$$u_x(L) = -k_L u(L)$$

with Robin boundary conditions, where k_0 and k_L are given positive numbers and $u = u(x)$. Can this system have a nontrivial solution $u \not\equiv 0$ for $\lambda > 0$?

Hint: Multiply the first equation by u and integrate over $x \in [0, L]$.

Problem 11.8

Show that $u(x, y) = f(x)g(y)$, where f and g are arbitrary differentiable functions, is a solution to the PDE

$$uu_{xy} = u_x u_y.$$

Problem 11.9

Show that for any $n \in \mathbb{N}$, the function $u_n(x, y) = \sin nx \sinh ny$ is a solution to the Laplace equation

$$\Delta u = u_{xx} + u_{yy} = 0.$$

Problem 11.10

Solve

$$u_{xy} = xy.$$

Problem 11.11

Classify each of the following second-order PDEs according to whether they are hyperbolic, parabolic, or elliptic:

(a) $2u_{xx} - 4u_{xy} + 7u_{yy} - u = 0.$

(b) $u_{xx} - 2 \cos x u_{xy} - \sin^2 x u_{yy} = 0.$

(c) $yu_{xx} + 2(x-1)u_{xy} - (y+2)u_{yy} = 0.$

Problem 11.12

Let $c > 0$. By computing u_x, u_{xx}, u_t , and u_{tt} show that

$$u(x, t) = \frac{1}{2}(f(x + ct) + f(x - ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds$$

is a solution to the PDE

$$u_{tt} = c^2 u_{xx}$$

where f is twice differentiable function and g is a differentiable function. Then compute and simplify $u(x, 0)$ and $u_t(x, 0)$.

Problem 11.13

Consider the second-order PDE

$$y u_{xx} + u_{xy} - x^2 u_{yy} - u_x - u = 0.$$

Determine the region D in \mathbb{R}^2 , if such a region exists, that makes this PDE: (a) hyperbolic, (b) parabolic, (c) elliptic.

Problem 11.14

Consider the second-order hyperbolic PDE

$$u_{xx} + 2u_{xy} - 3u_{yy} = 0.$$

Use the change of variables $v(x, y) = y - 3x$ and $w(x, y) = x + y$ to solve the given equation.

Problem 11.15

Solve the Cauchy problem

$$u_{xx} + 2u_{xy} - 3u_{yy} = 0.$$

$$u(x, 2x) = 1, \quad u_x(x, 2x) = x.$$

12 Hyperbolic Type: The Wave equation

The wave equation has many physical applications from sound waves in air to magnetic waves in the Sun's atmosphere. However, the simplest systems to visualize and describe are waves on a stretched flexible string.

A flexible homogeneous string of length L and constant mass density ρ (i.e., mass per unit length) is stretched horizontally along the x -axis with its left end placed at $x = 0$ and its right end placed at $x = L$. From the left end (and at time $t = 0$) we slightly shake the string and we notice a small vibrations propagate through the string. We make the following physical assumptions:

(a) the string does not furnish any resistance to bending (i.e., perfectly elastic);

(b) the (pulling) tension force on the string is the dominant force and all other forces acting on the string are negligible (no external forces are applied to the string, the damping forces (resistance) and gravitational forces are negligible);

(c) clearly a point on the string moves up and down along a curve but since the horizontal displacement is small compared to the vertical displacement, we will assume that each point of the string moves only vertically. Thus, the horizontal component of the tension force must be constant.

We denote the vertical displacement from the x -axis of the string by $u(x, t)$ which is a function of position x and time t . That is, $u(x, t)$ is the vertical displacement from the equilibrium at position x and time t . Our aim is to find an equation that is satisfied by $u(x, t)$.

A displacement of a tiny piece of the string between points P and Q is shown in Figure 12.1,

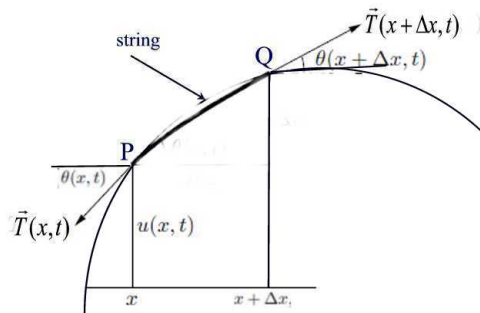


Figure 12.1

where

(i) $\theta(x, t)$ is the angle between $\vec{T}(x, t)$ and \vec{i} at x and time t ; for small vibrations, we have $\theta \approx 0$;

(ii) $\vec{T}(x, t)$ is the (pulling) tension force in the string at position x and time t and $\vec{T}(x + \Delta x, t)$ the tension force at position $x + \Delta x$ and t .

By (c) above, we have

$$\|\vec{T}(x, t)\| \cos [\theta(x, t)] = \|\vec{T}(x + \Delta x, t)\| \cos [\theta(x + \Delta x, t)] = T.$$

Now, at P the vertical component of the tension force is $-\|\vec{T}(x, t)\| \sin [\theta(x, t)]$ (the minus sign occurs due to the component at P is pointing downward) whereas at Q the vertical component is $\|\vec{T}(x + \Delta x, t)\| \sin [\theta(x + \Delta x, t)]$. Then Newton's Law of motion

$$\text{mass} \times \text{acceleration} = \text{net applied forces}$$

gives

$$\rho \Delta x \frac{\partial^2 u}{\partial t^2} = \|\vec{T}(x + \Delta x, t)\| \sin [\theta(x + \Delta x, t)] - \|\vec{T}(x, t)\| \sin [\theta(x, t)].$$

Next, dividing through by T , we obtain

$$\begin{aligned} \frac{\rho}{T} \Delta x \frac{\partial^2 u}{\partial t^2} &= \frac{\|\vec{T}(x + \Delta x, t)\| \sin [\theta(x + \Delta x, t)]}{\|\vec{T}(x + \Delta x, t)\| \cos [\theta(x + \Delta x, t)]} - \frac{\|\vec{T}(x, t)\| \sin [\theta(x, t)]}{\|\vec{T}(x, t)\| \cos [\theta(x, t)]} \\ &= \tan [\theta(x + \Delta x, t)] - \tan [\theta(x, t)] \\ &= u_x(x + \Delta x, t) - u_x(x, t). \end{aligned}$$

Dividing by Δx and letting $\Delta x \rightarrow 0$ we obtain

$$\frac{\rho}{T} \frac{\partial^2 u}{\partial t^2} = u_{xx}(x, t).$$

which can be written as

$$u_{tt}(x, t) = c^2 u_{xx}(x, t) \tag{12.1}$$

where $c^2 = \frac{T}{\rho}$. Note that the units of T are *mass* \times *length*/*time*² and the units of ρ are *mass*/*length*² so that the units of c are *length*/*time*. We call c the **wave speed**.

General Solution of (12.1): D'Alembert Approach

By using the change of variables $v = x + ct$ and $w = x - ct$, we find

$$\begin{aligned}u_t &= cu_v - cu_w \\u_{tt} &= c^2 u_{vv} - 2c^2 u_{vw} + c^2 u_{ww} \\u_x &= u_v + u_w \\u_{xx} &= u_{vv} + 2u_{vw} + u_{ww}\end{aligned}$$

Substituting into Equation (12.1), we find $u_{vw} = 0$ and solving this equation we find $u_v = F(v)$ and $u(v, w) = f(v) + g(w)$ where $f(v) = \int F(v)dv$.

Finally, using the fact that $v = x + ct$ and $w = x - ct$; we get d'Alembert's solution to the one-dimensional wave equation:

$$u(x, t) = f(x + ct) + g(x - ct) \quad (12.2)$$

where f and g are arbitrary differentiable functions.

The function $f(x + ct)$ represents waves that are moving to the left at a constant speed c and the function $g(x - ct)$ represents waves that are moving to the right at the same speed c .

The function $u(x, t)$ in (12.2) involves two arbitrary functions that are determined (normally) by two initial conditions.

Example 12.1

Find the solution to the Cauchy problem

$$\begin{aligned}u_{tt} &= c^2 u_{xx} \\u(x, 0) &= v(x) \\u_t(x, 0) &= w(x).\end{aligned}$$

Solution.

The condition $u(x, 0)$ is the initial position whereas $u_t(x, 0)$ is the initial velocity. We have

$$u(x, 0) = f(x) + g(x) = v(x)$$

and

$$u_t(x, 0) = -cf'(x) + cg'(x) = w(x)$$

which implies that

$$f(x) - g(x) = -\frac{1}{c}W(x) = -\frac{1}{c} \int_0^x w(s)ds.$$

Therefore,

$$f(x) = \frac{1}{2}(v(x) - \frac{1}{c}W(x))$$

and

$$g(x) = \frac{1}{2}(v(x) + \frac{1}{c}W(x)).$$

Finally,

$$\begin{aligned} u(x, t) &= \frac{1}{2}[v(x - ct) + v(x + ct) + \frac{1}{c}(W(x + ct) - W(x - ct))] \\ &= \frac{1}{2}[v(x - ct) + v(x + ct) + \frac{1}{c} \int_{x-ct}^{x+ct} w(s) ds] \blacksquare \end{aligned}$$

Practice Problems

Problem 12.1

Show that if $v(x, t)$ and $w(x, t)$ satisfy equation (12.1) then $\alpha v + \beta w$ is also a solution to (12.1), where α and β are constants.

Problem 12.2

Show that any linear time independent function $u(x, t) = ax + b$ is a solution to equation (12.1).

Problem 12.3

Find a solution to (12.1) that satisfies the homogeneous conditions $u(x, 0) = u(0, t) = u(L, t) = 0$.

Problem 12.4

Solve the initial value problem

$$\begin{aligned}u_{tt} &= 9u_{xx} \\ u(x, 0) &= \cos x \\ u_t(x, 0) &= 0.\end{aligned}$$

Problem 12.5

Solve the initial value problem

$$\begin{aligned}u_{tt} &= u_{xx} \\ u(x, 0) &= \frac{1}{1+x^2} \\ u_t(x, 0) &= 0.\end{aligned}$$

Problem 12.6

Solve the initial value problem

$$\begin{aligned}u_{tt} &= 4u_{xx} \\ u(x, 0) &= 1 \\ u_t(x, 0) &= \cos(2\pi x).\end{aligned}$$

Problem 12.7

Solve the initial value problem

$$\begin{aligned}u_{tt} &= 25u_{xx} \\ u(x, 0) &= v(x) \\ u_t(x, 0) &= 0\end{aligned}$$

where

$$v(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x \geq 0. \end{cases}$$

Problem 12.8

Solve the initial value problem

$$\begin{aligned}u_{tt} &= c^2 u_{xx} \\ u(x, 0) &= e^{-x^2} \\ u_t(x, 0) &= \cos^2 x.\end{aligned}$$

Problem 12.9Prove that the wave equation, $u_{tt} = c^2 u_{xx}$ satisfies the following properties, which are known as invariance properties. If $u(x, t)$ is a solution, then

- (i) Any translate, $u(x - y, t)$ where y is a fixed constant, is also a solution.
- (ii) Any derivative, say $u_x(x, t)$, is also a solution.
- (iii) Any dilation, $u(ax, at)$, is a solution, for any fixed constant a .

Problem 12.10Find $v(r)$ if $u(r, t) = \frac{v(r)}{r} \cos nt$ is a solution to the PDE

$$u_{rr} + \frac{2}{r}u_r = u_{tt}.$$

Problem 12.11Find the solution of the wave equation on the real line ($-\infty < x < +\infty$) with the initial conditions

$$u(x, 0) = e^x, \quad u_t(x, 0) = \sin x.$$

Problem 12.12

The total energy of the string (the sum of the kinetic and potential energies) is defined as

$$E(t) = \frac{1}{2} \int_0^L (u_t^2 + c^2 u_x^2) dx.$$

(a) Using the wave equation derive the equation of conservation of energy

$$\frac{dE(t)}{dt} = c^2(u_t(L, t)u_x(L, t) - u_t(0, t)u_x(0, t)).$$

(b) Assuming fixed ends boundary conditions, that is the ends of the string are fixed so that $u(0, t) = u(L, t) = 0$, for all $t > 0$, show that the energy is constant.

(c) Assuming free ends boundary conditions for both $x = 0$ and $x = L$, that is both $u(0, t)$ and $u(L, t)$ vary with t , show that the energy is constant.

Problem 12.13

For a wave equation with damping

$$u_{tt} - c^2 u_{xx} + du_t = 0, \quad d > 0, \quad 0 < x < L$$

with the fixed ends boundary conditions show that the total energy decreases.

Problem 12.14

(a) Verify that for any twice differentiable $R(x)$ the function

$$u(x, t) = R(x - ct)$$

is a solution of the wave equation $u_{tt} = c^2 u_{xx}$. Such solutions are called traveling waves.

(b) Show that the potential and kinetic energies (see Exercise 12.12) are equal for the traveling wave solution in (a).

Problem 12.15

Find the solution of the Cauchy wave equation

$$u_{tt} = 4u_{xx}$$

$$u(x, 0) = x^2, \quad u_t(x, 0) = \sin 2x.$$

Simplify your answer as much as possible.

13 Parabolic Type: The Heat Equation in One-Dimensional Space

In this section, We will look at a model for describing the distribution of temperature in a solid material as a function of time and space. More specifically, we will derive the heat equation that models the flow of heat in a rod that is insulated everywhere except at the two ends.

Before we begin our discussion of the mathematics of the heat equation, we must first determine what is meant by the term **heat**? Heat is type of energy known as **thermal energy**. Heat travels in waves like other forms of energy, and can change the matter it touches. It can heat it up and cause chemical reactions like burning to occur.

Heat can be released through a chemical reaction (such as the nuclear reactions that make the Sun “burn”) or can be trapped for a limited time by insulators. It is often released along with other kinds of energy such as light waves or sound waves. For example, a burning candle releases light and heat waves. On the other hand, an explosion releases light, heat, and sound waves. The most common units of heat are BTU (British Thermal Unit), Calorie and Joule.

Consider now a thin rod made of homogeneous heat conducting material of uniform density ρ and constant cross section A , wrapped along the x -axis from $x = 0$ to $x = L$ as shown in Figure 13.1.

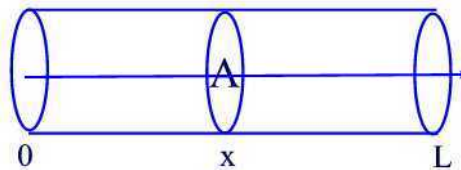


Figure 13.1

Assume the heat flows only in the x -direction, with the lateral sides well insulated, and the only way heat can enter or leave the rod is at either end. Since our rod is thin, the temperature of the rod can be considered constant on any cross section and so depends on the horizontal position along the x -axis and we can hence consider the rod to be a one spatial dimensional rod. We will also assume that heat energy in any piece of the rod is conserved. That is, the heat gained at one end is equal to the heat lost at the other end.

Let $u(x, t)$ be the temperature of the cross section at the point x and the time t . Consider a portion U of the rod from x to $x + \Delta x$ of length Δx as shown in Figure 13.2.

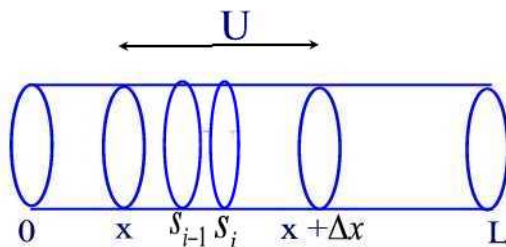


Figure 13.2

Divide the interval $[x, x + \Delta x]$ into n sub-intervals each of length Δs using the partition points $x = s_0 < s_1 < \cdots < s_n = x + \Delta x$. Consider the portion U_i of U of height Δs . The portion U_i is assumed to be thin so that the temperature is constant throughout the volume. From the theory of heat conduction, the quantity of heat Q_i in U_i at time t is given by

$$Q_i = cm_i u(s_{i-1}, t) = c\rho u(s_{i-1}, t)\Delta V_i$$

where m_i is the mass of U_i , ΔV_i is the volume of U_i and c is the **specific heat**, that is, the amount of heat that it takes to raise one unit of mass of the material by one unit of temperature.

But U_i is a cylinder of height Δs and area of base A so that $\Delta V_i = A\Delta s$. Hence,

$$Q_i = c\rho Au(s_{i-1}, t)\Delta s.$$

The quantity of heat in the portion U is given by

$$Q(t) = \lim_{n \rightarrow \infty} \sum_{i=1}^n Q_i = \lim_{n \rightarrow \infty} \sum_{i=1}^n c\rho Au(s_{i-1}, t)\Delta s = \int_x^{x+\Delta x} c\rho Au(s, t)ds.$$

By differentiation, the change in heat with respect to time is

$$\frac{dQ}{dt} = \int_x^{x+\Delta x} c\rho Au_t(s, t)ds.$$

Assuming that u is continuously differentiable, we can apply the mean value theorem for integrals and find $x \leq \xi \leq x + \Delta x$ such that

$$\int_x^{x+\Delta x} u_t(s, t) ds = \Delta x u_t(\xi, t).$$

Thus, the rate of change of heat in U is given by

$$\frac{dQ}{dt} = c\rho A \Delta x u_t(\xi, t).$$

Now, Fourier law of heat transfer says that the rate of heat transfer through any cross section is proportional to the area A and the negative gradient of the temperature normal to the cross section, i.e., $-K A u_x(x, t)$. Note that if the temperature increases as x increases (i.e., the temperature is hotter to the right), $u_x > 0$ so that the heat flows to the left. This explains the minus sign in the formula for Fourier law. Hence, according to this law heat is transferred from areas of high temperature to areas of low temperature. Now, the rate of heat flowing in U through the cross section at x is $-K A u_x(x, t)$ and the rate of heat flowing out of U through the cross section at $x + \Delta x$ is $-K A u_x(x + \Delta x, t)$, where K is the **thermal conductivity**⁷ of the rod. Now, the conservation of energy law states

rate of change of heat in U = rate of heat flowing in – rate of heat flowing
out

or mathematically written as,

$$c\rho A \Delta x u_t(\xi, t) = -K A u_x(x, t) + K A u_x(x + \Delta x, t)$$

or

$$c\rho A \Delta x u_t(\xi, t) = K A [u_x(x + \Delta x, t) - u_x(x, t)].$$

Dividing this last equation by $c\rho A \Delta x$ and letting $\Delta x \rightarrow 0$ we obtain

$$u_t(x, t) = k u_{xx}(x, t) \tag{13.1}$$

where $k = \frac{K}{c\rho}$ is called the **diffusivity constant**.

Equation (13.1) is the one dimensional heat equation which is second order,

⁷It is a property of material to conduct heat. Heat transfer is slow in materials with small thermal conductivity and fast in materials with large thermal conductivity.

linear, homogeneous, and of parabolic type.
The non-homogeneous heat equation

$$u_t = ku_{xx} + f(x)$$

is known as the heat equation with an external **heat source** $f(x)$. An example of an external heat source is the heat generated from a candle placed under the bar.

The function

$$E(t) = \int_0^L u(x, t) dx$$

is called the **total thermal energy**⁸ at time t of the entire rod.

Example 13.1

The two ends of a homogeneous rod of length L are insulated. There is a constant source of thermal energy $q_0 \neq 0$ and the temperature is initially $u(x, 0) = f(x)$.

- (a) Write the equation and the boundary conditions for this model.
- (b) Calculate the total thermal energy of the entire rod.

Solution.

- (a) The model is given by the PDE

$$u_t(x, t) = ku_{xx} + q_0$$

with boundary conditions

$$u_x(0, t) = u_x(L, t) = 0.$$

- (b) First note that

$$\begin{aligned} \frac{d}{dt} \int_0^L u(x, t) dx &= \int_0^L u_t(x, t) dx = \int_0^L ku_{xx} dx + \int_0^L q_0 dx \\ &= ku_x|_0^L + q_0L = q_0L \end{aligned}$$

since $u_x(0, t) = u_x(L, t) = 0$. Integrating with respect to t we find

$$E(t) = q_0Lt + C.$$

⁸The total internal energy in the rod generated by the rod's temperature.

But $C = E(0) = \int_0^L u(x, 0)dx = \int_0^L f(x)dx$. Hence, the total thermal energy is given by

$$E(t) = \int_0^L f(x)dx + q_0Lt \blacksquare$$

Initial Boundary Value Problems

In order to solve the heat equation we must give the problem some initial conditions. If you recall from the theory of ODE, the number of conditions required for solving initial value problems always matched the highest order of the derivative in the equation.

In partial differential equations the same idea holds except now we have to pay attention to the variable we are differentiating with respect to as well. So, for the heat equation we have got a first order time derivative and so we will need one initial condition and a second order spatial derivative and so we will need two boundary conditions.

For the initial condition, we define the temperature of every point along the rod at time $t = 0$ by

$$u(x, 0) = f(x)$$

where f is a given (prescribed) function of x . This function is known as the **initial temperature distribution**.

The boundary conditions will tell us something about what the temperature is doing at the ends of the bar. The conditions are given by

$$u(0, t) = T_0 \text{ and } u(L, t) = T_L.$$

and they are called as the **Dirichlet conditions**. In this case, the general form of the heat equation initial boundary value problem is to find $u(x, t)$ satisfying

$$\begin{aligned} u_t(x, t) &= ku_{xx}(x, t), \quad 0 \leq x \leq L, \quad t > 0 \\ u(x, 0) &= f(x), \quad 0 \leq x \leq L \\ u(0, t) &= T_0, \quad u(L, t) = T_L, \quad t > 0. \end{aligned}$$

In the case of insulated endpoints, i.e., there is no heat flow out of them, we use the boundary conditions

$$u_x(0, t) = u_x(L, t) = 0.$$

These conditions are examples of what is known as **Neumann boundary** conditions. In this case, the general form of the heat equation initial boundary value problem is to find $u(x, t)$ satisfying

$$u_t(x, t) = k u_{xx}(x, t), \quad 0 \leq x \leq L, \quad t > 0$$

$$u(x, 0) = f(x), \quad 0 \leq x \leq L$$

$$u_x(0, t) = u_x(L, t) = 0, \quad t > 0.$$

Practice Problems

Problem 13.1

Show that if $u(x, t)$ and $v(x, t)$ satisfy equation (13.1) then $\alpha u + \beta v$ is also a solution to (13.1), where α and β are constants.

Problem 13.2

Show that any linear time independent function $u(x, t) = ax + b$ is a solution to equation (13.1).

Problem 13.3

Find a linear time independent solution u to (13.1) that satisfies $u(0, t) = T_0$ and $u(L, t) = T_L$.

Problem 13.4

Show that to solve (13.1) with the boundary conditions $u(0, t) = T_0$ and $u(L, t) = T_L$ it suffices to solve (13.1) with the **homogeneous boundary conditions** $u(0, t) = u(L, t) = 0$.

Problem 13.5

Find a solution to (13.1) that satisfies the conditions $u(x, 0) = u(0, t) = u(L, t) = 0$.

Problem 13.6

Let (I) denote equation (13.1) together with initial condition $u(x, 0) = f(x)$, where f is not the zero function, and the homogeneous boundary conditions $u(0, t) = u(L, t) = 0$. Suppose a nontrivial solution to (I) can be written in the form $u(x, t) = X(x)T(t)$. Show that X and T satisfy the ODE

$$X'' - \frac{\lambda}{k}X = 0 \text{ and } T' - \lambda T = 0$$

for some constant λ .

Problem 13.7

Consider again the solution $u(x, t) = X(x)T(t)$. Clearly, $T(t) = T(0)e^{\lambda t}$. Suppose that $\lambda > 0$.

(a) Show that $X(x) = Ae^{x\sqrt{\alpha}} + Be^{-x\sqrt{\alpha}}$, where $\alpha = \frac{\lambda}{k}$ and A and B are arbitrary constants.

(b) Show that A and B satisfy the two equations $A + B = 0$ and $A(e^{L\sqrt{\alpha}} - e^{-L\sqrt{\alpha}}) = 0$.

(c) Show that $A = 0$ leads to a contradiction.

(d) Using (b) and (c) show that $e^{L\sqrt{\alpha}} = e^{-L\sqrt{\alpha}}$. Show that this equality leads to a contradiction. We conclude that $\lambda < 0$.

Problem 13.8

Consider the results of the previous exercise.

- (a) Show that $X(x) = c_1 \cos \beta x + c_2 \sin \beta x$ where $\beta = \sqrt{\frac{-\lambda}{k}}$.
 (b) Show that $\lambda = \lambda_n = -\frac{kn^2\pi^2}{L^2}$, where n is an integer.

Problem 13.9

Show that $u(x, t) = \sum_{i=1}^n u_i(x, t)$, where $u_i(x, t) = c_i e^{-\frac{ki^2\pi^2}{L^2}t} \sin\left(\frac{i\pi}{L}x\right)$ satisfies (13.1) and the homogeneous boundary conditions.

Problem 13.10

Suppose that a wire is stretched between 0 and a . Describe the boundary conditions for the temperature $u(x, t)$ when

- (i) the left end is kept at 0 degrees and the right end is kept at 100 degrees; and
 (ii) when both ends are insulated.

Problem 13.11

Let $u_t = u_{xx}$ for $0 < x < \pi$ and $t > 0$ with boundary conditions $u(0, t) = 0 = u(\pi, t)$ and initial condition $u(x, 0) = \sin x$. Let $E(t) = \int_0^\pi (u_t^2 + u_x^2) dx$. Show that $E'(t) < 0$.

Problem 13.12

Suppose

$$u_t = u_{xx} + 4, \quad u_x(0, t) = 5, \quad u_x(L, t) = 6, \quad u(x, 0) = f(x).$$

Calculate the total thermal energy of the one-dimensional rod (as a function of time).

Problem 13.13

Consider the heat equation

$$u_t = ku_{xx}$$

for $x \in (0, 1)$ and $t > 0$, with boundary conditions $u(0, t) = 2$ and $u(1, t) = 3$ for $t > 0$ and initial condition $u(x, 0) = x$ for $x \in (0, 1)$. A function $v(x)$ that satisfies the equation $v''(x) = 0$, with conditions $v(0) = 2$ and $v(1) = 3$ is called a **steady-state solution**. That is, the steady-state solutions of the heat equation are those solutions that don't depend on time. Find $v(x)$.

Problem 13.14

Consider the equation for the one-dimensional rod of length L with given heat energy source:

$$u_t = u_{xx} + q(x).$$

Assume that the initial temperature distribution is given by $u(x, 0) = f(x)$. Find the equilibrium (steady state) temperature distribution in the following cases.

- (a) $q(x) = 0, u(0) = 0, u(L) = T$.
- (b) $q(x) = 0, u_x(0) = 0, u(L) = T$.
- (c) $q(x) = 0, u(0) = T, u_x(L) = \alpha$.

Problem 13.15

Consider the equation for the one-dimensional rod of length L with insulated ends:

$$u_t = ku_{xx}, \quad u_x(0, t) = u_x(L, t) = 0.$$

- (a) Give the expression for the total thermal energy of the rod.
- (b) Show using the equation and the boundary conditions that the total thermal energy is constant.

Problem 13.16

Suppose

$$u_t = u_{xx} + x, \quad u(x, 0) = f(x), \quad u_x(0, t) = \beta, \quad u_x(L, t) = 7.$$

- (a) Calculate the total thermal energy of the one-dimensional rod (as a function of time).
- (b) From part (a) find the value of β for which a steady-state solution exist.
- (c) For the above value of β find the steady state solution.

14 Sequences of Functions: Pointwise and Uniform Convergence

In the next section, we will be constructing solutions to PDEs involving infinite sums of sines and cosines. These infinite sums or series are called **Fourier series**. Fourier series are examples of series of functions. Convergence of series of functions is defined in terms of convergence of a sequence of functions. In this section we study the two types of convergence of sequences of functions.

Recall that a sequence of numbers $\{a_n\}_{n=1}^{\infty}$ is said to **converge** to a number L if and only if for every given $\epsilon > 0$ there is a positive integer $N = N(\epsilon)$ such that for all $n \geq N$ we have $|a_n - L| < \epsilon$.

What is the analogue concept of convergence when the terms of the sequence are variables? Let $D \subset \mathbb{R}$ and for each $n \in \mathbb{N}$ consider a function $f_n : D \rightarrow \mathbb{R}$. Thus, we obtain a sequence of functions $\{f_n\}_{n=1}^{\infty}$. For such a sequence, there are two types of convergence that we consider in this section: pointwise convergence and uniform convergence.

We say that $\{f_n\}_{n=1}^{\infty}$ **converges pointwise** in D to a function $f : D \rightarrow \mathbb{R}$ if and only if

$$\lim_{n \rightarrow \infty} f_n(x) = f(x)$$

for each $x \in D$. Equivalently, for a given $x \in D$ and $\epsilon > 0$ there is a positive integer $N = N(x, \epsilon)$ such that if $n \geq N$ then $|f_n(x) - f(x)| < \epsilon$. It is important to note that N is a function of both x and ϵ .

Example 14.1

Define $f_n : [0, \infty) \rightarrow \mathbb{R}$ by $f_n(x) = \frac{nx}{1+n^2x^2}$. Show that the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise to the function $f(x) = 0$ for all $x \geq 0$.

Solution.

For all $x \geq 0$,

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \frac{nx}{1+n^2x^2} = 0 = f(x) \blacksquare$$

Example 14.2

For each positive integer n let $f_n : (0, \infty) \rightarrow (0, \infty)$ be given by $f_n(x) = nx$. Show that $\{f_n\}_{n=1}^{\infty}$ does not converge pointwise in $D = (0, \infty)$.

Solution.

This follows from the fact that $\lim_{n \rightarrow \infty} nx = \infty$ for all $x \in D$ ■

One of the weaknesses of this type of convergence is that it does not preserve some of the properties of the base functions $\{f_n\}_{n=1}^{\infty}$. For example, if each f_n is continuous then the pointwise limit function need not be continuous. (See Problem 14.1) A stronger type of convergence which preserves most of the properties of the base functions is the uniform convergence which we define next.

Let D be a subset of \mathbb{R} and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions defined on D . We say that $\{f_n\}_{n=1}^{\infty}$ **converges uniformly** on D to a function $f : D \rightarrow \mathbb{R}$ if and only if for all $\epsilon > 0$ there is a positive integer $N = N(\epsilon)$ such that if $n \geq N$ then $|f_n(x) - f(x)| < \epsilon$ for all $x \in D$.

This definition says that the integer N depends only on the given ϵ (in contrast to pointwise convergence where N depends on both x and ϵ) so that for $n \geq N$, the graph of $f_n(x)$ is bounded above by the graph of $f(x) + \epsilon$ and below by the graph of $f(x) - \epsilon$ as shown in Figure 14.1.

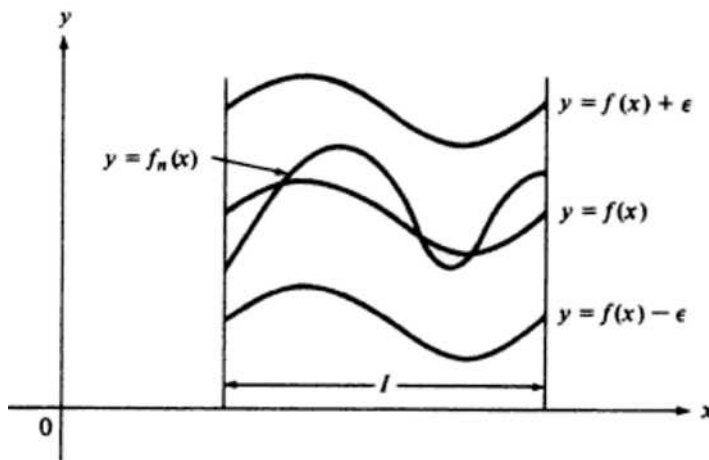


Figure 14.1

Example 14.3

For each positive integer n let $f_n : [0, 1] \rightarrow \mathbb{R}$ be given by $f_n(x) = \frac{x}{n}$. Show that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to the zero function.

Solution.

Let $\epsilon > 0$ be given. Let N be a positive integer such that $N > \frac{1}{\epsilon}$. Then for

$n \geq N$ we have

$$|f_n(x) - f(x)| = \left| \frac{x}{n} - 0 \right| = \frac{|x|}{n} \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon$$

for all $x \in [0, 1]$ ■

Clearly, uniform convergence implies pointwise convergence to the same limit function. However, the converse is not true in general. Thus, one way to show that a sequence of functions does not converge uniformly is to show that it does not converge pointwise.

Example 14.4

Define $f_n : [0, \infty) \rightarrow \mathbb{R}$ by $f_n(x) = \frac{nx}{1+n^2x^2}$. By Example 14.1, this sequence converges pointwise to $f(x) = 0$. Let $\epsilon = \frac{1}{3}$. Show that there is no positive integer N with the property $n \geq N$ implies $|f_n(x) - f(x)| < \epsilon$ for all $x \geq 0$. Hence, the given sequence does not converge uniformly to $f(x)$.

Solution.

For any positive integer N and for $n \geq N$ we have

$$\left| f_n\left(\frac{1}{n}\right) - f\left(\frac{1}{n}\right) \right| = \frac{1}{2} > \epsilon \quad \blacksquare$$

Problem 14.1 shows a sequence of continuous functions converging pointwise to a discontinuous function. That is, pointwise convergence does not preserve the property of continuity. One of the interesting features of uniform convergence is that it preserves continuity as shown in the next example.

Example 14.5

Suppose that for each $n \geq 1$ the function $f_n : D \rightarrow \mathbb{R}$ is continuous in D . Suppose that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f . Let $a \in D$.

(a) Let $\epsilon > 0$ be given. Show that there is a positive integer N such that if $n \geq N$ then $|f_n(x) - f(x)| < \frac{\epsilon}{3}$ for all $x \in D$.

(b) Show that there is a $\delta > 0$ such that for all $|x - a| < \delta$ we have $|f_N(x) - f_N(a)| < \frac{\epsilon}{3}$.

(c) Using (a) and (b) show that for $|x - a| < \delta$ we have $|f(x) - f(a)| < \epsilon$. Hence, f is continuous in D since a was arbitrary.

Solution.

(a) This follows from the definition of uniform convergence.

(b) This follows from the fact that f_N is continuous at $a \in D$.

(c) For $|x - a| < \delta$ we have $|f(x) - f(a)| = |f(a) - f_N(a) + f_N(a) - f_N(x) + f_N(x) - f(x)| \leq |f_N(a) - f(a)| + |f_N(a) - f_N(x)| + |f_N(x) - f(x)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$ ■

From this example, we can write

$$\lim_{x \rightarrow a} \lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \lim_{x \rightarrow a} f_n(x).$$

Indeed,

$$\begin{aligned} \lim_{x \rightarrow a} \lim_{n \rightarrow \infty} f_n(x) &= \lim_{x \rightarrow a} f(x) \\ &= f(a) = \lim_{n \rightarrow \infty} f_n(a) \\ &= \lim_{n \rightarrow \infty} \lim_{x \rightarrow a} f_n(x). \end{aligned}$$

Does pointwise convergence allow the interchange of limits and integration? The answer is no as shown in the next example.

Example 14.6

The sequence of function $f_n : (0, \infty) \rightarrow \mathbb{R}$ defined by $f_n(x) = \frac{x}{n}$ converges pointwise to the zero function. Show that

$$\lim_{n \rightarrow \infty} \int_1^{\infty} f_n(x) dx \neq \int_1^{\infty} \lim_{n \rightarrow \infty} f_n(x) dx.$$

Solution.

We have

$$\int_1^{\infty} \frac{x}{n} dx = \frac{x^2}{2n} \Big|_1^{\infty} = \infty.$$

Hence,

$$\lim_{n \rightarrow \infty} \int_1^{\infty} f_n(x) dx = \infty$$

whereas

$$\int_1^{\infty} \lim_{n \rightarrow \infty} f_n(x) dx = 0 \quad \blacksquare$$

Contrary to pointwise convergence, uniform convergence preserves integration. That is, if $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f on a closed interval $[a, b]$ then

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b \lim_{n \rightarrow \infty} f_n(x) dx.$$

Theorem 14.1

Suppose that $f_n : [a, b] \rightarrow \mathbb{R}$ is a sequence of continuous functions that converges uniformly to $f : [a, b] \rightarrow \mathbb{R}$. Then

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b \lim_{n \rightarrow \infty} f_n(x) dx = \int_a^b f(x) dx.$$

Proof.

From Example 14.5, we have that f is continuous and hence integrable. Let $\epsilon > 0$ be given. By uniform convergence, we can find a positive integer N such that $|f_n(x) - f(x)| < \frac{\epsilon}{b-a}$ for all x in $[a, b]$ and $n \geq N$. Thus, for $n \geq N$, we have

$$\left| \int_a^b f_n(x) dx - \int_a^b f(x) dx \right| \leq \int_a^b |f_n(x) - f(x)| dx < \epsilon.$$

This completes the proof of the theorem ■

Now, what about differentiability? Again, pointwise convergence fails in general to conserve the differentiability property. See Problem 14.1. Does uniform convergence preserve differentiability? The answer is still no as shown in the next example.

Example 14.7

Consider the family of functions $f_n : [-1, 1]$ given by $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$.

(a) Show that f_n is differentiable for each $n \geq 1$.

(b) Show that for all $x \in [-1, 1]$ we have

$$|f_n(x) - f(x)| \leq \frac{1}{\sqrt{n}}$$

where $f(x) = |x|$. Hint: Note that $\sqrt{x^2 + \frac{1}{n}} + \sqrt{x^2} \geq \frac{1}{\sqrt{n}}$.

(c) Let $\epsilon > 0$ be given. Show that there is a positive integer N such that for $n \geq N$ we have

$$|f_n(x) - f(x)| < \epsilon \text{ for all } x \in [-1, 1].$$

Thus, $\{f_n\}_{n=1}^{\infty}$ converges uniformly to the non-differentiable function $f(x) = |x|$.

Solution.

(a) f_n is the composition of two differentiable functions so it is differentiable with derivative

$$f'_n(x) = x \left[x^2 + \frac{1}{n} \right]^{-\frac{1}{2}}.$$

(b) We have

$$\begin{aligned} |f_n(x) - f(x)| &= \left| \sqrt{x^2 + \frac{1}{n}} - \sqrt{x^2} \right| = \left| \frac{(\sqrt{x^2 + \frac{1}{n}} - \sqrt{x^2})(\sqrt{x^2 + \frac{1}{n}} + \sqrt{x^2})}{\sqrt{x^2 + \frac{1}{n}} + \sqrt{x^2}} \right| \\ &= \frac{\frac{1}{n}}{\sqrt{x^2 + \frac{1}{n}} + \sqrt{x^2}} \\ &\leq \frac{\frac{1}{n}}{\frac{1}{\sqrt{n}}} = \frac{1}{\sqrt{n}}. \end{aligned}$$

(c) Let $\epsilon > 0$ be given. Since $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$ we can find a positive integer N such that for all $n \geq N$ we have $\frac{1}{\sqrt{n}} < \epsilon$. Now the answer to the question follows from this and part (b) ■

Even when uniform convergence occurs, the process of interchanging limits and differentiation may fail as shown in the next example.

Example 14.8

Consider the functions $f_n : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f_n(x) = \frac{\sin nx}{n}$.

(a) Show that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to the function $f(x) = 0$.

(b) Note that $\{f_n\}_{n=1}^{\infty}$ and f are differentiable functions. Show that

$$\lim_{n \rightarrow \infty} f'_n(x) \neq f'(x) = \left[\lim_{n \rightarrow \infty} f_n(x) \right]'$$

That is, one cannot, in general, interchange limits and derivatives.

Solution.

(a) Let $\epsilon > 0$ be given. Let N be a positive integer such that $N > \frac{1}{\epsilon}$. Then for $n \geq N$ we have

$$|f_n(x) - f(x)| = \left| \frac{\sin nx}{n} \right| \leq \frac{1}{n} < \epsilon$$

and this is true for all $x \in \mathbb{R}$. Hence, $\{f_n\}_{n=1}^{\infty}$ converges uniformly to the function $f(x) = 0$.

(b) We have $\lim_{n \rightarrow \infty} f'_n(\pi) = \lim_{n \rightarrow \infty} \cos n\pi = \lim_{n \rightarrow \infty} (-1)^n$ which does not converge. However, $f'(\pi) = 0$ ■

Pointwise convergence was not enough to preserve differentiability, and neither was uniform convergence by itself. Even with uniform convergence the process of interchanging limits with derivatives is not true in general. However, if we combine pointwise convergence with uniform convergence we can indeed preserve differentiability and also switch the limit process with the process of differentiation.

Theorem 14.2

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of differentiable functions on $[a, b]$ that converges pointwise to some function f defined on $[a, b]$. If $\{f'_n\}_{n=1}^{\infty}$ converges uniformly on $[a, b]$ to a function g , then the function f is differentiable with derivative equals to g . Thus,

$$\lim_{n \rightarrow \infty} f'_n(x) = g(x) = f'(x) = \left[\lim_{n \rightarrow \infty} f_n(x) \right]'$$

Proof.

First, note that the function g is continuous in $[a, b]$ since uniform convergence preserves continuity. Let c be an arbitrary point in $[a, b]$. Then

$$\int_c^x f'_n(t) dt = f_n(x) - f_n(c), \quad x \in [a, b].$$

Taking the limit of both sides and using the facts that f'_n converges uniformly to g and f_n converges pointwise to f , we can write

$$\int_c^x g(t) dt = f(x) - f(c).$$

Taking the derivative of both sides of the last equation yields $g(x) = f'(x)$ ■

Finally, we conclude this section with the following important result that is useful in testing uniform convergence.

Theorem 14.3

Consider a sequence $f_n : D \rightarrow \mathbb{R}$. Then this sequence converges uniformly to $f : D \rightarrow \mathbb{R}$ if and only if

$$\lim_{n \rightarrow \infty} \sup\{|f_n(x) - f(x)| : x \in D\} = 0.$$

Proof.

Suppose that f_n converges uniformly to f . Let $\epsilon > 0$ be given. Then there is a positive integer N such that $|f_n(x) - f(x)| < \frac{\epsilon}{2}$ for all $n \geq N$ and all $x \in D$. Thus, for $n \geq N$, we have

$$\sup\{|f_n(x) - f(x)| : x \in D\} \leq \frac{\epsilon}{2} < \epsilon.$$

This shows that

$$\lim_{n \rightarrow \infty} \sup\{|f_n(x) - f(x)| : x \in D\} = 0.$$

Conversely, suppose that

$$\lim_{n \rightarrow \infty} \sup\{|f_n(x) - f(x)| : x \in D\} = 0.$$

Let $\epsilon > 0$ be given. Then there is a positive integer N such that

$$\sup\{|f_n(x) - f(x)| : x \in D\} < \epsilon$$

for all $n \geq N$. But this implies that

$$|f_n(x) - f(x)| < \epsilon$$

for all $x \in D$. Hence, f_n converges uniformly to f in D ■

Example 14.9

Show that the sequence defined by $f_n(x) = \frac{\cos x}{n}$ converges uniformly to the zero function.

Solution.

We have

$$0 \leq \sup\left\{\left|\frac{\cos x}{n}\right| : x \in \mathbb{R}\right\} \leq \frac{1}{n}.$$

Now apply the squeeze rule⁹ for sequences we find that

$$\lim_{n \rightarrow \infty} \sup\left\{\left|\frac{\cos x}{n}\right| : x \in \mathbb{R}\right\} = 0$$

which implies that the given sequence converges uniformly to the zero function on \mathbb{R} ■

⁹If $a_n \leq b_n \leq c_n$ for all $n \geq N$ and if $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$ then $\lim_{n \rightarrow \infty} b_n = L$.

Practice Problems

Problem 14.1

Define $f_n : [0, 1] \rightarrow \mathbb{R}$ by $f_n(x) = x^n$. Define $f : [0, 1] \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1. \end{cases}$$

- (a) Show that the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f .
 (b) Show that the sequence $\{f_n\}_{n=1}^{\infty}$ does not converge uniformly to f . Hint: Suppose otherwise. Let $\epsilon = 0.5$ and get a contradiction by using a point $(0.5)^{\frac{1}{N}} < x < 1$.

Problem 14.2

Consider the sequence of functions

$$f_n(x) = \frac{nx + x^2}{n^2}$$

defined for all x in \mathbb{R} . Show that this sequence converges pointwise to a function f to be determined.

Problem 14.3

Consider the sequence of functions

$$f_n(x) = \frac{\sin(nx + 3)}{\sqrt{n + 1}}$$

defined for all x in \mathbb{R} . Show that this sequence converges pointwise to a function f to be determined.

Problem 14.4

Consider the sequence of functions defined by $f_n(x) = n^2 x^n$ for all $0 \leq x \leq 1$. Show that this sequence does not converge pointwise to any function.

Problem 14.5

Consider the sequence of functions defined by $f_n(x) = (\cos x)^n$ for all $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. Show that this sequence converges pointwise to a noncontinuous function to be determined.

Problem 14.6

Consider the sequence of functions $f_n(x) = x - \frac{x^n}{n}$ defined on $[0, 1]$.

- (a) Does $\{f_n\}_{n=1}^{\infty}$ converge to some limit function? If so, find the limit function and show whether the convergence is pointwise or uniform.
- (b) Does $\{f'_n\}_{n=1}^{\infty}$ converge to some limit function? If so, find the limit function and show whether the convergence is pointwise or uniform.

Problem 14.7

Let $f_n(x) = \frac{x^n}{1+x^n}$ for $x \in [0, 2]$.

- (a) Find the pointwise limit $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ on $[0, 2]$.
- (b) Does $f_n \rightarrow f$ uniformly on $[0, 2]$?

Problem 14.8

For each $n \in \mathbb{N}$ define $f_n : \mathbb{R} \rightarrow \mathbb{R}$ by $f_n(x) = \frac{n + \cos x}{2n + \sin^2 x}$.

- (a) Show that $f_n \rightarrow \frac{1}{2}$ uniformly.
- (b) Find $\lim_{n \rightarrow \infty} \int_2^7 f_n(x) dx$.

Problem 14.9

Show that the sequence defined by $f_n(x) = (\cos x)^n$ does not converge uniformly on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Problem 14.10

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions such that

$$\sup\{|f_n(x)| : 2 \leq x \leq 5\} \leq \frac{2^n}{1 + 4^n}.$$

- (a) Show that this sequence converges uniformly to a function f to be found.
- (b) What is the value of the limit $\lim_{n \rightarrow \infty} \int_2^5 f_n(x) dx$?

15 An Introduction to Fourier Series

In this and the next section we will have a brief look to the subject of Fourier series. The point here is to do just enough to allow us to do some basic solutions to partial differential equations later in the book.

Motivation: In Calculus we have seen that certain functions may be represented as power series by means of the Taylor expansions. These functions must have infinitely many derivatives, and the series provide a good approximation only in some (often small) vicinity of a reference point.

Fourier series constructed of trigonometric rather than power functions, and can be used for functions not only not differentiable, but even discontinuous at some points. The main limitation of Fourier series is that the underlying function should be periodic.

Recall from calculus that a **function series** is a series where the summands are functions. Examples of function series include power series, Laurent series, Fourier series, etc.

Unlike series of numbers, there exist many types of convergence of series of functions, namely, pointwise, uniform, etc. We say that a series of functions $\sum_{n=1}^{\infty} f_n(x)$ **converges pointwise** to a function f if and only if the sequence of partial sums

$$S_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$$

converges pointwise to f . We write

$$\sum_{n=1}^{\infty} f_n(x) = \lim_{n \rightarrow \infty} S_n(x) = f(x).$$

Example 15.1

Show that $\sum_{n=0}^{\infty} x^n$ converges pointwise to a function to be determined for all $-1 < x < 1$.

Solution.

The n^{th} term of the sequence of partial sums is given by

$$S_n(x) = 1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

Since

$$\lim_{n \rightarrow \infty} x^{n+1} = 0, \quad -1 < x < 1,$$

the partial sums converge pointwise to the function $\frac{1}{1-x}$. Thus,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \blacksquare$$

Likewise, we say that a series of functions $\sum_{n=1}^{\infty} f_n(x)$ **converges uniformly** to a function f if and only if the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ converges uniformly to f .

The following theorem provide a tool for uniform convergence of series of functions.

Theorem 15.1 (*Weierstrass M-test*)

Suppose that for each x in an interval I the series $\sum_{n=1}^{\infty} f_n(x)$ is well-defined. Suppose further that

$$|f_n(x)| \leq M_n, \quad \forall x \in I.$$

If $\sum_{n=1}^{\infty} M_n$ (a scalar series) is convergent then the series $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent.

Example 15.2

Show that $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ is uniformly convergent.

Solution.

For all $x \in \mathbb{R}$, we have

$$\left| \frac{\sin(nx)}{n^2} \right| \leq \frac{|\sin(nx)|}{n^2} \leq \frac{1}{n^2}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent being a p -series with $p = 2 > 1$. Thus, by Weierstrass M-test the given series is uniformly convergent \blacksquare

In this section we introduce a type of series of functions known as **Fourier series**. They are given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right], \quad -L \leq x \leq L \quad (15.1)$$

where a_n and b_n are called the **Fourier coefficients**. Note that we begin the series with $\frac{a_0}{2}$ as opposed to simply a_0 to simplify the coefficient formula

for a_n that we will derive later in this section.

The main questions we want to consider next are the questions of determining which functions can be represented by Fourier series and if so how to compute the coefficients a_n and b_n .

Before answering these questions, we look at some of the properties of Fourier series.

Periodicity Property

Recall that a function f is said to be **periodic** with period $T > 0$ if $f(x + T) = f(x)$ for all $x, x + T$ in the domain of f . The smallest value of T for which f is periodic is called the **fundamental period**. A graph of a T -periodic function is shown in Figure 15.1.



Figure 15.1

For a T -periodic function we have

$$f(x) = f(x + T) = f(x + 2T) = \dots$$

Note that the definite integral of a T -periodic function is the same over any interval of length T . By Problem 15.1 below, if f and g are two periodic functions with common period T , then the product fg and an arbitrary linear combination $c_1f + c_2g$ are also periodic with period T . It is an easy exercise to show that the Fourier series (15.1) is periodic with fundamental period $2L$.

Orthogonality Property

Recall from Calculus that for each pair of vectors \vec{u} and \vec{v} we associate a scalar quantity $\vec{u} \cdot \vec{v}$ called the **dot product** of \vec{u} and \vec{v} . We say that \vec{u} and \vec{v} are **orthogonal** if and only if $\vec{u} \cdot \vec{v} = 0$. We want to define a similar concept for functions.

Let f and g be two functions with domain the closed interval $[a, b]$. We define

a function that takes a pair of functions to a scalar. Symbolically, we write

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx.$$

We call $\langle f, g \rangle$ the **inner product** of f and g . We say that f and g are **orthogonal** if and only if $\langle f, g \rangle = 0$. A set of functions is said to be **mutually orthogonal** if each distinct pair of functions in the set is orthogonal. Before we proceed any further into computations, we would like to remind the reader of the following two facts from calculus:

- If $f(x)$ is an odd function defined on the interval $[-L, L]$ then $\int_{-L}^L f(x)dx = 0$.
- If $f(x)$ is an even function defined on the interval $[-L, L]$ then $\int_{-L}^L f(x)dx = 2 \int_0^L f(x)dx$.

Example 15.3

Show that the set $\{1, \cos(\frac{n\pi}{L}x), \sin(\frac{n\pi}{L}x) : n \in \mathbb{N}\}$, where $m \neq n$, is mutually orthogonal in $[-L, L]$.

Solution.

Since the cosine function is even, we have

$$\int_{-L}^L 1 \cdot \cos\left(\frac{n\pi}{L}x\right)dx = 2 \int_0^L \cos\left(\frac{n\pi}{L}x\right)dx = \frac{2L}{n\pi} \left[\sin\left(\frac{n\pi}{L}x\right)\right]_0^L = 0.$$

Since the sine function is odd, we have

$$\int_{-L}^L 1 \cdot \sin\left(\frac{n\pi}{L}x\right)dx = 0.$$

Now, for $n \neq m$ we have

$$\begin{aligned} \int_{-L}^L \cos\left(\frac{m\pi}{L}x\right) \cos\left(\frac{n\pi}{L}x\right)dx &= \frac{1}{2} \int_{-L}^L \left[\cos\left(\frac{(m+n)\pi}{L}x\right) + \cos\left(\frac{(m-n)\pi}{L}x\right) \right] dx \\ &= \frac{1}{2} \left[\frac{L}{(m+n)\pi} \sin\left(\frac{(m+n)\pi}{L}x\right) \right. \\ &\quad \left. + \frac{L}{(m-n)\pi} \sin\left(\frac{(m-n)\pi}{L}x\right) \right]_{-L}^L = 0 \end{aligned}$$

where we used the trigonometric identity

$$\cos a \cos b = \frac{1}{2}[\cos(a + b) + \cos(a - b)].$$

We can also show (see Problem 15.2):

$$\int_{-L}^L \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx = 0$$

and

$$\int_{-L}^L \cos\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx = 0 \blacksquare$$

The reason we care about these functions being orthogonal is because we will exploit this fact to develop a formula for the coefficients in our Fourier series.

Now, in order to answer the first question mentioned earlier, that is, which functions can be expressed as a Fourier series expansion, we need to introduce some mathematical concepts.

A function $f(x)$ is said to be **piecewise continuous** on $[a, b]$ if it is continuous in $[a, b]$ except possibly at finitely many points of discontinuity within the interval $[a, b]$, and at each point of discontinuity, the right- and left-handed limits of f exist. An example of a piecewise continuous function is the function

$$f(x) = \begin{cases} x & 0 \leq x < 1 \\ x^2 - x & 1 \leq x \leq 2. \end{cases}$$

We will say that f is **piecewise smooth** in $[a, b]$ if and only if $f(x)$ as well as its derivatives are piecewise continuous.

The following theorem, proven in more advanced books, ensures that a Fourier decomposition can be found for any function which is piecewise smooth.

Theorem 15.2

Let f be a $2L$ -periodic function. If f is a piecewise smooth on $[-L, L]$ then for all points of discontinuity $x \in [-L, L]$ we have

$$\frac{f(x^-) + f(x^+)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right]$$

where as for points of continuity $x \in [-L, L]$ we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right].$$

Remark 15.1

- (1) Almost all functions occurring in practice are piecewise smooth functions.
- (2) Given a piecewise smooth function f on $[-L, L]$. The above theorem applies to the periodic extension F of f where $F(x + 2nL) = f(x)$ ($n \in \mathbb{Z}$) and $F(x) = f(x)$ on $(-L, L)$. Note that if $f(-L) = f(L)$ then $F(x) = f(x)$ on $[-L, L]$. Otherwise, the end points of $f(x)$ may be jump discontinuities of $F(x)$.

Convergence Results of Fourier Series

We list few of the results regarding the convergence of Fourier series:

- (1) The type of convergence in the above theorem is pointwise convergence.
- (2) The convergence is uniform for a continuous function f on $[-L, L]$ such that $f(-L) = f(L)$.
- (3) The convergence is uniform whenever $\sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2)$ is convergent.
- (4) If $f(x)$ is periodic, continuous, and has a piecewise continuous derivative, then the Fourier Series corresponding to f converges uniformly to $f(x)$ for the entire real line.
- (5) The convergence is uniform on any closed interval that does not contain a point of discontinuity.

Euler-Fourier Formulas

Next, we will answer the second question mentioned earlier, that is, the question of finding formulas for the coefficients a_n and b_n . These formulas for a_n and b_n are called Euler-Fourier formulas which we derive next. We will assume that the RHS in (15.1) converges uniformly to $f(x)$ on the interval $[-L, L]$. Integrating both sides of (15.1) we obtain

$$\int_{-L}^L f(x)dx = \int_{-L}^L \frac{a_0}{2}dx + \int_{-L}^L \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right] dx.$$

Since the trigonometric series is assumed to be uniformly convergent, from Theorem 14.2, we can interchange the order of integration and summation to obtain

$$\int_{-L}^L f(x)dx = \int_{-L}^L \frac{a_0}{2}dx + \sum_{n=1}^{\infty} \int_{-L}^L \left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right] dx.$$

But

$$\int_{-L}^L \cos\left(\frac{n\pi}{L}x\right) dx = \frac{L}{n\pi} \left[\sin\left(\frac{n\pi}{L}x\right) \right]_{-L}^L = 0$$

and likewise

$$\int_{-L}^L \sin\left(\frac{n\pi}{L}x\right) dx = -\frac{L}{n\pi} \left[\cos\left(\frac{n\pi}{L}x\right) \right]_{-L}^L = 0.$$

Thus,

$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx.$$

To find the other Fourier coefficients, we recall the results of Problems 15.2 - 15.3 below.

$$\int_{-L}^L \cos\left(\frac{n\pi}{L}x\right) \cos\left(\frac{m\pi}{L}x\right) dx = \begin{cases} L & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

$$\int_{-L}^L \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx = \begin{cases} L & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

$$\int_{-L}^L \sin\left(\frac{n\pi}{L}x\right) \cos\left(\frac{m\pi}{L}x\right) dx = 0, \quad \forall m, n.$$

Now, to find the formula for the Fourier coefficients a_m for $m > 0$, we multiply both sides of (15.1) by $\cos\left(\frac{m\pi}{L}x\right)$ and integrate from $-L$ to L to obtain

$$\begin{aligned} \int_{-L}^L f(x) \cos\left(\frac{m\pi}{L}x\right) dx &= \int_{-L}^L \frac{a_0}{2} \cos\left(\frac{m\pi}{L}x\right) dx + \sum_{n=1}^{\infty} \left[a_n \int_{-L}^L \cos\left(\frac{n\pi}{L}x\right) \cos\left(\frac{m\pi}{L}x\right) dx \right. \\ &\quad \left. + b_n \int_{-L}^L \sin\left(\frac{n\pi}{L}x\right) \cos\left(\frac{m\pi}{L}x\right) dx \right]. \end{aligned}$$

Hence,

$$\int_{-L}^L f(x) \cos\left(\frac{m\pi}{L}x\right) dx = a_m L$$

and therefore

$$a_m = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{m\pi}{L}x\right) dx.$$

Likewise, we can show that

$$b_m = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{m\pi}{L}x\right) dx.$$

Example 15.4

Find the Fourier series expansion of

$$f(x) = \begin{cases} 0, & x \leq 0 \\ x, & x > 0 \end{cases}$$

on the interval $[-\pi, \pi]$.

Solution.

We have

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_0^{\pi} x dx = \frac{\pi}{2} \\ a_n &= \frac{1}{\pi} \int_0^{\pi} x \cos nx dx = \frac{1}{\pi} \left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2} \right]_0^{\pi} = \frac{(-1)^n - 1}{\pi n^2} \\ b_n &= \frac{1}{\pi} \int_0^{\pi} x \sin nx dx = \frac{1}{\pi} \left[-\frac{x \cos nx}{n} + \frac{\sin nx}{n^2} \right]_0^{\pi} = \frac{(-1)^{n+1}}{n}. \end{aligned}$$

Hence,

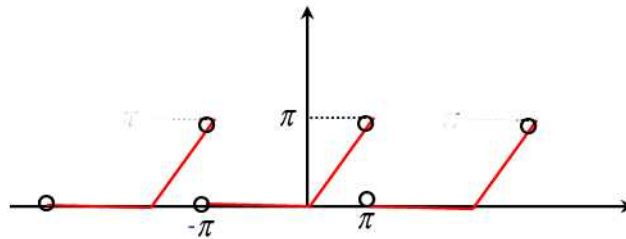
$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{\pi n^2} \cos(nx) + \frac{(-1)^{n+1}}{n} \sin(nx) \right] \quad -\pi < x < \pi \blacksquare$$

Example 15.5

Apply Theorem 15.2 to the function in Example 15.4.

Solution.

Let F be a periodic extension of f of period 2π . See Figure 152.



Graph of $F(x)$

Figure 152

Thus, $f(x) = F(x)$ on the interval $(-\pi, \pi)$. Note that for $x = \pi$, the Fourier series converges to

$$\frac{F(\pi^-) + F(\pi^+)}{2} = \frac{\pi}{2}.$$

Similar result for $x = -\pi$. Clearly, F is a piecewise smooth function so that by the previous theorem we can write

$$\frac{\pi}{4} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{\pi n^2} \cos(nx) + \frac{(-1)^{n+1}}{n} \sin(nx) \right] = \begin{cases} \frac{\pi}{2}, & \text{if } x = -\pi \\ f(x), & \text{if } -\pi < x < \pi \\ \frac{\pi}{2}, & \text{if } x = \pi. \end{cases}$$

Taking $x = \pi$ we have the identity

$$\frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{\pi n^2} (-1)^n = \frac{\pi}{2}$$

which can be simplified to

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

This provides a method for computing an approximate value of π ■

Remark 15.2

An example of a function that does not have a Fourier series representation is the function $f(x) = \frac{1}{x^2}$ on $[-L, L]$. For example, the coefficient a_0 for this function does not exist. Thus, not every function can be written as a Fourier series expansion.

The final topic of discussion here is the topic of differentiation and integration of Fourier series. In particular we want to know if we can differentiate a Fourier series term by term and have the result be the Fourier series of the derivative of the function. Likewise we want to know if we can integrate a Fourier series term by term and arrive at the Fourier series of the integral of the function. Answers to these questions are provided next.

Theorem 15.3

A Fourier series of a *piecewise smooth* function f can always be integrated term by term and the result is a convergent infinite series that always converges to $\int_{-L}^L f(x) dx$ even if the original series has jumps.

Theorem 15.4

A Fourier series of a continuous function $f(x)$ can be differentiated term by term if $f'(x)$ is *piecewise smooth*. The result of the differentiation is the Fourier series of $f'(x)$.

Practice Problems

Problem 15.1

Let f and g be two functions with common domain D and common period T . Show that

- (a) fg is periodic of period T .
- (b) $c_1f + c_2g$ is periodic of period T , where c_1 and c_2 are real numbers.

Problem 15.2

Show that for $m \neq n$ we have

- (a) $\int_{-L}^L \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx = 0$ and
- (b) $\int_{-L}^L \cos\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx = 0$.

Problem 15.3

Compute the following integrals:

- (a) $\int_{-L}^L \cos^2\left(\frac{n\pi}{L}x\right) dx$.
- (b) $\int_{-L}^L \sin^2\left(\frac{n\pi}{L}x\right) dx$.
- (c) $\int_{-L}^L \cos\left(\frac{n\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx$.

Problem 15.4

Find the Fourier coefficients of

$$f(x) = \begin{cases} -\pi, & -\pi \leq x < 0 \\ \pi, & 0 < x < \pi \\ 0, & x = 0, \pi \end{cases}$$

on the interval $[-\pi, \pi]$.

Problem 15.5

Find the Fourier series of $f(x) = x^2 - \frac{1}{2}$ on the interval $[-1, 1]$.

Problem 15.6

Find the Fourier series of the function

$$f(x) = \begin{cases} -1, & -2\pi < x < -\pi \\ 0, & -\pi < x < \pi \\ 1, & \pi < x < 2\pi. \end{cases}$$

Problem 15.7

Find the Fourier series of the function

$$f(x) = \begin{cases} 1 + x, & -2 \leq x \leq 0 \\ 1 - x, & 0 < x \leq 2. \end{cases}$$

Problem 15.8

Show that $f(x) = \frac{1}{x}$ is not piecewise continuous on $[-1, 1]$.

Problem 15.9

Assume that $f(x)$ is continuous and has period $2L$. Prove that

$$\int_{-L}^L f(x) dx = \int_{-L+a}^{L+a} f(x) dx$$

is independent of $a \in \mathbb{R}$. In particular, it does not matter over which interval the Fourier coefficients are computed as long as the interval length is $2L$. [Remark: This result is also true for piecewise continuous functions].

Problem 15.10

Consider the function $f(x)$ defined by

$$f(x) = \begin{cases} 1 & 0 \leq x < 1 \\ 2 & 1 \leq x < 3 \end{cases}$$

and extended periodically with period 3 to \mathbb{R} so that $f(x+3) = f(x)$ for all x .

- (i) Find the Fourier series of $f(x)$.
- (ii) Discuss its limit: In particular, does the Fourier series converge pointwise or uniformly to its limit, and what is this limit?
- (iii) Plot the graphs of $f(x)$ and its extension $F(x)$ on the interval $[0, 3]$.

Problem 15.11

For the following functions $f(x)$ on the interval $-L < x < L$, determine the coefficients a_n , $n = 0, 1, 2, \dots$ and b_n , $n \in \mathbb{N}$ of the Fourier series expansion.

- (a) $f(x) = 1$.
- (b) $f(x) = 2 + \sin\left(\frac{\pi x}{L}\right)$.
- (c) $f(x) = \begin{cases} 1 & x \leq 0 \\ 0 & x > 0. \end{cases}$
- (d) $f(x) = x$.

Problem 15.12

Let $f(t)$ be the function with period 2π defined as

$$f(t) = \begin{cases} 2 & \text{if } 0 \leq x \leq \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} < x \leq 2\pi \end{cases}$$

$f(t)$ has a Fourier series and that series is equal to

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt), \quad 0 < x < 2\pi.$$

Find $\frac{a_3}{b_3}$.

Problem 15.13

Let $f(x) = x^3$ on $[-\pi, \pi]$, extended periodically to all of \mathbb{R} . Find the Fourier coefficients a_n , $n = 1, 2, 3, \dots$.

Problem 15.14

Let $f(x)$ be the square wave function

$$f(x) = \begin{cases} -\pi & -\pi \leq x < 0 \\ \pi & 0 \leq x \leq \pi \end{cases}$$

extended periodically to all of \mathbb{R} . To what value does the Fourier series of $f(x)$ converge when $x = 0$?

Problem 15.15

(a) Find the Fourier series of

$$f(x) = \begin{cases} 1 & -\pi \leq x < 0 \\ 2 & 0 \leq x \leq \pi \end{cases}$$

extended periodically to all of \mathbb{R} . Simplify your coefficients as much as possible.

(b) Use (a) to evaluate the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)}$. Hint: Evaluate the Fourier series at $x = \frac{\pi}{2}$.

16 Fourier Sines Series and Fourier Cosines Series

In this section we discuss some important properties of Fourier series when the underlying function f is either even or odd.

A function f is **odd** if it satisfies $f(-x) = -f(x)$ for all x in the domain of f whereas f is **even** if it satisfies $f(-x) = f(x)$ for all x in the domain of f .

Example 16.1

Show the following

- (a) If f and g are either both even or both odd then fg is even.
- (b) If f is odd and g is even then fg is odd.

Solution.

(a) Suppose that both f and g are even. Then $(fg)(-x) = f(-x)g(-x) = f(x)g(x) = (fg)(x)$. That is, fg is even. Now, suppose that both f and g are odd. Then $(fg)(-x) = f(-x)g(-x) = [-f(x)][-g(x)] = (fg)(x)$. That is, fg is even.

(b) f is odd and g is even. Then $(fg)(-x) = f(-x)g(-x) = -f(x)g(x) = -(fg)(x)$. That is, fg is odd ■

Example 16.2

(a) Show that for any even function $f(x)$ defined on the interval $[-L, L]$, we have

$$\int_{-L}^L f(x)dx = 2 \int_0^L f(x)dx.$$

(b) Show that for any odd function $f(x)$ defined on the interval $[-L, L]$, we have

$$\int_{-L}^L f(x)dx = 0.$$

Solution.

(a) Since $f(x)$ is even, we have $f(-x) = f(x)$ for all x in $[-L, L]$. Thus,

$$\int_{-L}^0 f(x)dx = \int_{-L}^0 f(-x)dx = - \int_L^0 f(u)du = \int_0^L f(x)dx.$$

For this, it follows that

$$\int_{-L}^L f(x)dx = \int_{-L}^0 f(x)dx + \int_0^L f(x)dx = 2 \int_0^L f(x)dx.$$

(b) Since $f(x)$ is odd, we have $f(-x) = -f(x)$ for all x in $[-L, L]$. Thus,

$$\int_{-L}^0 f(x)dx = \int_{-L}^0 [-f(-x)]dx = \int_L^0 f(u)du = -\int_0^L f(x)dx.$$

Hence,

$$0 = \int_{-L}^0 f(x)dx + \int_0^L f(x)dx = \int_{-L}^L f(x)dx \blacksquare$$

Example 16.3

(a) Find the value of the integral $\int_{-L}^L f(x) \sin\left(\frac{n\pi}{L}x\right)dx$ when f is even.

(b) Find the value of the integral $\int_{-L}^L f(x) \cos\left(\frac{n\pi}{L}x\right)dx$ when f is odd.

Solution.

(a) Since the function $\sin\left(\frac{n\pi}{L}x\right)$ is odd and f is even, we have that $f(x) \sin\left(\frac{n\pi}{L}x\right)$ is odd so that

$$\int_{-L}^L f(x) \sin\left(\frac{n\pi}{L}x\right)dx = 0$$

(b) Since the function $\cos\left(\frac{n\pi}{L}x\right)$ is even and f is odd, we have that $f(x) \cos\left(\frac{n\pi}{L}x\right)$ is odd so that

$$\int_{-L}^L f(x) \cos\left(\frac{n\pi}{L}x\right)dx = 0 \blacksquare$$

Even and Odd Extensions

Let $f : (0, L) \rightarrow \mathbb{R}$ be a piecewise smooth function. We define the **odd extension** of this function on the interval $-L < x < L$ by

$$f_{\text{odd}}(x) = \begin{cases} f(x) & 0 < x < L \\ -f(-x) & -L < x < 0. \end{cases}$$

This function will be odd on the interval $(-L, L)$, and will be equal to $f(x)$ on the interval $(0, L)$. We can then further extend this function to the entire real line by defining it to be $2L$ periodic. Let \bar{f}_{odd} denote this extension. We note that \bar{f}_{odd} is an odd function and piecewise smooth so that by Theorem 15.2 it possesses a Fourier series expansion, and from the fact that it is odd all of the a'_n s are zero. Moreover, in the interval $(0, L)$ we have

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right). \quad (16.1)$$

We call (16.1) the **Fourier sine series** of f .

The coefficients b_n are given by the formula

$$\begin{aligned} b_n &= \frac{1}{L} \int_{-L}^L \bar{f}_{odd} \sin\left(\frac{n\pi}{L}x\right) dx = \frac{2}{L} \int_0^L \bar{f}_{odd} \sin\left(\frac{n\pi}{L}x\right) dx \\ &= \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx \end{aligned}$$

since $\bar{f}_{odd} \sin\left(\frac{n\pi}{L}x\right)$ is an even function.

Likewise, we can define the **even extension** of f on the interval $-L < x < L$ by

$$f_{even}(x) = \begin{cases} f(x) & 0 < x < L \\ f(-x) & -L < x < 0. \end{cases}$$

We can then further extend this function to the entire real line by defining it to be $2L$ periodic. Let \bar{f}_{even} denote this extension. Again, we note that \bar{f}_{even} is equal to the original function $f(x)$ on the interval upon which $f(x)$ is defined. Since \bar{f}_{even} is piecewise smooth, by Theorem 15.2 it possesses a Fourier series expansion, and from the fact that it is even all of the b'_n s are zero. Moreover, in the interval $(0, L)$ we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right). \quad (16.2)$$

We call (16.2) the **Fourier cosine series** of f . The coefficients a_n are given by

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi}{L}x\right) dx, \quad n = 0, 1, 2, \dots$$

Example 16.4

Graph the odd and even extensions of the function $f(x) = x$, $0 \leq x \leq 1$.

Solution.

We have $f_{odd}(x) = x$ for $-1 \leq x \leq 1$. The odd extension of f is shown in Figure 16.1(a). Likewise,

$$f_{even}(x) = \begin{cases} x & 0 \leq x \leq 1 \\ -x & -1 \leq x < 0. \end{cases}$$

The even extension is shown in Figure 16.1(b) ■

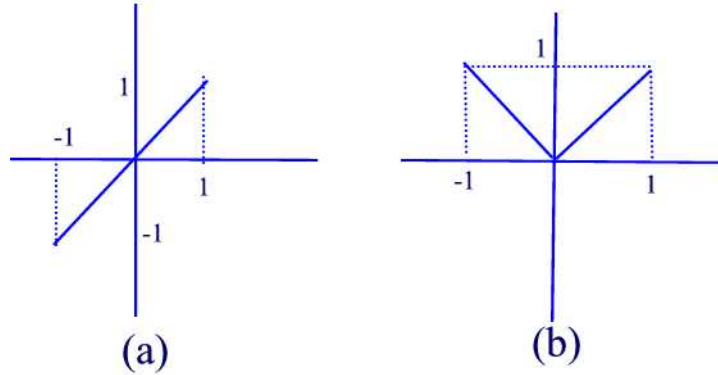


Figure 16.1

Example 16.5

Find the Fourier sine series of the function

$$f(x) = \begin{cases} x, & 0 \leq x \leq \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \leq x \leq \pi. \end{cases}$$

Solution.

We have

$$b_n = \frac{2}{\pi} \left[\int_0^{\frac{\pi}{2}} x \sin nx dx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x) \sin nx dx \right].$$

Using integration by parts we find

$$\begin{aligned} \int_0^{\frac{\pi}{2}} x \sin nx dx &= \left[-\frac{x}{n} \cos nx \right]_0^{\frac{\pi}{2}} + \frac{1}{n} \int_0^{\frac{\pi}{2}} \cos nx dx \\ &= -\frac{\pi \cos(n\pi/2)}{2n} + \frac{1}{n^2} [\sin nx]_0^{\frac{\pi}{2}} \\ &= -\frac{\pi \cos(n\pi/2)}{2n} + \frac{\sin(n\pi/2)}{n^2} \end{aligned}$$

while

$$\begin{aligned} \int_{\frac{\pi}{2}}^{\pi} (\pi - x) \sin nx dx &= \left[-\frac{(\pi - x)}{n} \cos nx \right]_{\frac{\pi}{2}}^{\pi} - \frac{1}{n} \int_{\frac{\pi}{2}}^{\pi} \cos nx dx \\ &= \frac{\pi \cos(n\pi/2)}{2n} - \frac{1}{n^2} [\sin nx]_{\frac{\pi}{2}}^{\pi} \\ &= \frac{\pi \cos(n\pi/2)}{2n} + \frac{\sin(n\pi/2)}{n^2} \end{aligned}$$

Thus,

$$b_n = \frac{4 \sin(n\pi/2)}{\pi n^2},$$

and the Fourier sine series of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} \frac{4 \sin(n\pi/2)}{\pi n^2} \sin nx = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{\pi(2n-1)^2} \sin(2n-1)x \blacksquare$$

Practice Problems

Problem 16.1

Give an example of a function that is both even and odd.

Problem 16.2

Graph the odd and even extensions of the function $f(x) = 1$, $0 \leq x \leq 1$.

Problem 16.3

Graph the odd and even extensions of the function $f(x) = L - x$ for $0 \leq x \leq L$.

Problem 16.4

Graph the odd and even extensions of the function $f(x) = 1 + x^2$ for $0 \leq x \leq L$.

Problem 16.5

Find the Fourier cosine series of the function

$$f(x) = \begin{cases} x, & 0 \leq x \leq \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \leq x \leq \pi. \end{cases}$$

Problem 16.6

Find the Fourier cosine series of $f(x) = x$ on the interval $[0, \pi]$.

Problem 16.7

Find the Fourier sine series of $f(x) = 1$ on the interval $[0, \pi]$.

Problem 16.8

Find the Fourier sine series of $f(x) = \cos x$ on the interval $[0, \pi]$.

Problem 16.9

Find the Fourier cosine series of $f(x) = e^{2x}$ on the interval $[0, 1]$.

Problem 16.10

For the following functions on the interval $[0, L]$, find the coefficients b_n of the Fourier sine expansion.

(a) $f(x) = \sin\left(\frac{2\pi}{L}x\right)$.

(b) $f(x) = 1$

(c) $f(x) = \cos\left(\frac{\pi}{L}x\right)$.

Problem 16.11

For the following functions on the interval $[0, L]$, find the coefficients a_n of the Fourier cosine expansion.

(a) $f(x) = 5 + \cos\left(\frac{\pi}{L}x\right)$.

(b) $f(x) = x$

(c)

$$f(x) = \begin{cases} 1 & 0 < x \leq \frac{L}{2} \\ 0 & \frac{L}{2} < x \leq L. \end{cases}$$

Problem 16.12

Consider a function $f(x)$, defined on $0 \leq x \leq L$, which is even (symmetric) around $x = \frac{L}{2}$. Show that the even coefficients (n even) of the Fourier sine series are zero.

Problem 16.13

Consider a function $f(x)$, defined on $0 \leq x \leq L$, which is odd around $x = \frac{L}{2}$. Show that the even coefficients (n even) of the Fourier cosine series are zero.

Problem 16.14

The Fourier sine series of $f(x) = \cos\left(\frac{\pi x}{L}\right)$ for $0 \leq x \leq L$ is given by

$$\cos\left(\frac{\pi x}{L}\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad n \in \mathbb{N}$$

where

$$b_1 = 0, \quad b_n = \frac{2n}{(n^2 - 1)\pi} [1 + (-1)^n].$$

Using term-by-term integration, find the Fourier cosine series of $\sin\left(\frac{\pi x}{L}\right)$.

Problem 16.15

Consider the function

$$f(x) = \begin{cases} 1 & 0 \leq x < 1 \\ 2 & 1 \leq x < 2. \end{cases}$$

- Sketch the even extension of f .
- Find a_0 in the Fourier series for the even extension of f .
- Find a_n ($n = 1, 2, \dots$) in the Fourier series for the even extension of f .
- Find b_n in the Fourier series for the even extension of f .
- Write the Fourier series for the even extension of f .

17 Separation of Variables for PDEs

Finding analytic solutions to PDEs is essentially impossible. Most of the PDE techniques involve a mixture of analytic, qualitative and numeric approaches. Of course, there are some easy PDEs too. If you are lucky your PDE has a solution with separable variables. In this chapter we discuss the application of the method of separation of variables in the solution of PDEs.

17.1 Second Order Linear Homogenous ODE with Constant Coefficients

In this section, we review the basics of finding the general solution to the ODE

$$ay'' + by' + cy = 0 \quad (17.1)$$

where a, b , and c are constants. The process starts by solving the **characteristic equation**

$$ar^2 + br + c = 0$$

which is a quadratic equation with roots

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We consider the following three cases:

- If $b^2 - 4ac > 0$ then the general solution to (17.1) is given by

$$y(t) = Ae^{\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)t} + Be^{\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)t}.$$

- If $b^2 - 4ac = 0$ then the general solution to (17.1) is given by

$$y(t) = Ae^{-\frac{b}{2a}t} + Bte^{-\frac{b}{2a}t}.$$

- If $b^2 - 4ac < 0$ then

$$r_{1,2} = -\frac{b}{2a} \pm i \frac{\sqrt{4ac - b^2}}{2a}$$

and the general solution to (17.1) is given by

$$y(t) = Ae^{-\frac{b}{2a}t} \cos\left(\frac{\sqrt{4ac - b^2}}{2a}t\right) + Be^{-\frac{b}{2a}t} \sin\left(\frac{\sqrt{4ac - b^2}}{2a}t\right).$$

17.2 The Method of Separation of Variables for PDEs

In developing a solution to a partial differential equation by separation of variables, one assumes that it is possible to separate the contributions of the independent variables into separate functions that each involve only one independent variable. Thus, the method consists of the following steps

1. Factorize the (unknown) dependent variable of the PDE into a product of functions, each of the factors being a function of one independent variable. That is,

$$u(x, y) = X(x)Y(y).$$

2. Substitute into the PDE, and divide the resulting equation by $X(x)Y(y)$.

3. Then the problem turns into a set of separated ODEs (one for $X(x)$ and one for $Y(y)$.)

4. The general solution of the ODEs is found, and boundary initial conditions are imposed.

5. $u(x, y)$ is formed by multiplying together $X(x)$ and $Y(y)$.

We illustrate these steps in the next three examples.

Example 17.1

Find all the solutions of the form $u(x, t) = X(x)T(t)$ of the equation

$$u_{xx} - u_x = u_t.$$

Solution.

It is very easy to find the derivatives of a separable function:

$$u_x = X'(x)T(t), u_t = X(x)T'(t) \text{ and } u_{xx} = X''(x)T(t),$$

which is basically a consequence of the fact that differentiation with respect to x sees t as a constant, and vice versa. Now, the equation $u_{xx} - u_x = u_t$ becomes

$$X''(x)T(t) - X'(x)T(t) = X(x)T'(t).$$

We can separate variables further. Division by $X(x)T(t)$ gives

$$\frac{X''(x) - X'(x)}{X(x)} = \frac{T'(t)}{T(t)}.$$

The expression on the LHS is a function of x whereas the one on the RHS is a function of t only. They both have to be constant. That is,

$$\frac{X''(x) - X'(x)}{X(x)} = \frac{T'(t)}{T(t)} = \lambda.$$

Thus, we have the following ODEs:

$$X'' - X' - \lambda X = 0 \text{ and } T' = \lambda T.$$

The second equation is easy to solve: $T(t) = Ce^{\lambda t}$. The first equation is solved via the characteristic equation $\omega^2 - \omega - \lambda = 0$, whose solutions are

$$\omega = \frac{1 \pm \sqrt{1 + 4\lambda}}{2}.$$

If $\lambda > -\frac{1}{4}$ then

$$X(x) = Ae^{\frac{1+\sqrt{1+4\lambda}}{2}x} + Be^{\frac{1-\sqrt{1+4\lambda}}{2}x}.$$

In this case,

$$u(x, t) = De^{\frac{1+\sqrt{1+4\lambda}}{2}x}e^{\lambda t} + Ee^{\frac{1-\sqrt{1+4\lambda}}{2}x}e^{\lambda t}.$$

If $\lambda = -\frac{1}{4}$ then

$$X(x) = Ae^{\frac{x}{2}} + Bxe^{\frac{x}{2}}$$

and in this case

$$u(x, t) = (D + Ex)e^{\frac{x}{2} - \frac{t}{4}}.$$

If $\lambda < -\frac{1}{4}$ then

$$X(x) = Ae^{\frac{x}{2}} \cos\left(\frac{\sqrt{-(1+4\lambda)}}{2}x\right) + Be^{\frac{x}{2}} \sin\left(\frac{\sqrt{-(1+4\lambda)}}{2}x\right).$$

In this case,

$$u(x, t) = D'e^{\frac{x}{2} + \lambda t} \cos\left(\frac{\sqrt{-(1+4\lambda)}}{2}x\right) + B'e^{\frac{x}{2} + \lambda t} \sin\left(\frac{\sqrt{-(1+4\lambda)}}{2}x\right) \blacksquare$$

Example 17.2

Solve Laplace's equation using the separation of variables method

$$\Delta u = u_{xx} + u_{yy} = 0.$$

Solution.

We look for a solution of the form $u(x, y) = X(x)Y(y)$. Substituting in the Laplace's equation, we obtain

$$X''(x)Y(y) + X(x)Y''(y) = 0.$$

Assuming $X(x)Y(y)$ is nonzero, dividing for $X(x)Y(y)$ and subtracting $\frac{Y''(y)}{Y(y)}$ from both sides, we find:

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}.$$

The left hand side is a function of x while the right hand side is a function of y . This says that they must equal to a constant. That is,

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda$$

where λ is a constant. This results in the following two ODEs

$$X'' - \lambda X = 0 \text{ and } Y'' + \lambda Y = 0.$$

The solutions of these equations depend on the sign of λ .

- If $\lambda > 0$ then the solutions are given

$$\begin{aligned} X(x) &= Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x} \\ Y(y) &= C \cos \sqrt{\lambda}y + D \sin \sqrt{\lambda}y \end{aligned}$$

where A, B, C , and D are constants. In this case,

$$\begin{aligned} u(x, t) &= k_1 e^{\sqrt{\lambda}x} \cos \sqrt{\lambda}y + k_2 e^{\sqrt{\lambda}x} \sin \sqrt{\lambda}y \\ &+ k_3 e^{-\sqrt{\lambda}x} \cos \sqrt{\lambda}y + k_4 e^{-\sqrt{\lambda}x} \sin \sqrt{\lambda}y. \end{aligned}$$

- If $\lambda = 0$ then

$$\begin{aligned} X(x) &= Ax + B \\ Y(y) &= Cy + D \end{aligned}$$

where A, B , and C are arbitrary constants. In this case,

$$u(x, y) = k_1 xy + k_2 x + k_3 y + k_4.$$

- If $\lambda < 0$ then

$$\begin{aligned} X(x) &= A \cos \sqrt{-\lambda}x + B \sin \sqrt{-\lambda}x \\ Y(y) &= C e^{\sqrt{-\lambda}y} + D e^{-\sqrt{-\lambda}y} \end{aligned}$$

where $A, B, C,$ and D are arbitrary constants. In this case,

$$\begin{aligned} u(x, y) &= k_1 \cos \sqrt{-\lambda}x e^{\sqrt{-\lambda}y} + k_2 \cos \sqrt{-\lambda}x e^{-\sqrt{-\lambda}y} \\ &+ k_3 \sin \sqrt{-\lambda}x e^{\sqrt{-\lambda}y} + k_4 \sin \sqrt{-\lambda}x e^{-\sqrt{-\lambda}y} \blacksquare \end{aligned}$$

Example 17.3

Solve using the separation of variables method.

$$y u_x - x u_y = 0.$$

Solution.

Substitute $u(x, y) = X(x)Y(y)$ into the given equation we find

$$yX'Y - xXY' = 0.$$

This can be separated into

$$\frac{X'}{xX} = \frac{Y'}{yY}.$$

The left hand side is a function of x while the right hand side is a function of y . This says that they must equal to a constant. That is,

$$\frac{X'}{xX} = \frac{Y'}{yY} = \lambda$$

where λ is a constant. This results in the following two ODEs

$$X' - \lambda xX = 0 \text{ and } Y' - \lambda yY = 0.$$

Solving these equations using the method of separation of variable for ODEs we find $X(x) = A e^{\frac{\lambda x^2}{2}}$ and $Y(y) = B e^{\frac{\lambda y^2}{2}}$. Thus, the general solution is given by

$$u(x, y) = C e^{\frac{\lambda(x^2+y^2)}{2}} \blacksquare$$

Practice Problems

Problem 17.1

Solve using the separation of variables method

$$\Delta u + \lambda u = 0.$$

Problem 17.2

Solve using the separation of variables method

$$u_t = ku_{xx}.$$

Problem 17.3

Derive the system of ordinary differential equations for $R(r)$ and $\Theta(\theta)$ that is satisfied by solutions to

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

Problem 17.4

Derive the system of ordinary differential equations and boundary conditions for $X(x)$ and $T(t)$ that is satisfied by solutions to

$$u_{tt} = u_{xx} - 2u, \quad 0 < x < 1, \quad t > 0$$

$$u(0, t) = 0 = u_x(1, t) \quad t > 0$$

of the form $u(x, t) = X(x)T(t)$. (Note: you do not need to solve for X and T .)

Problem 17.5

Derive the system of ordinary differential equations and boundary conditions for $X(x)$ and $T(t)$ that is satisfied by solutions to

$$u_t = ku_{xx}, \quad 0 < x < L, \quad t > 0$$

$$u(x, 0) = f(x), \quad u(0, t) = 0 = u_x(L, t) \quad t > 0$$

of the form $u(x, t) = X(x)T(t)$. (Note: you do not need to solve for X and T .)

Problem 17.6

Find all product solutions of the PDE $u_x + u_t = 0$.

Problem 17.7

Derive the system of ordinary differential equations for $X(x)$ and $Y(y)$ that is satisfied by solutions to

$$3u_{yy} - 5u_{xxxxy} + 7u_{xxy} = 0.$$

of the form $u(x, y) = X(x)Y(y)$.

Problem 17.8

Find the general solution by the method of separation of variables.

$$u_{xy} + u = 0.$$

Problem 17.9

Find the general solution by the method of separation of variables.

$$u_x - yu_y = 0.$$

Problem 17.10

Find the general solution by the method of separation of variables.

$$u_{tt} - u_{xx} = 0.$$

Problem 17.11

For the following PDEs find the ODEs implied by the method of separation of variables.

- (a) $u_t = kr(ru_r)_r$
- (b) $u_t = ku_{xx} - \alpha u$
- (c) $u_t = ku_{xx} - au_x$
- (d) $u_{xx} + u_{yy} = 0$
- (e) $u_t = ku_{xxxx}$.

Problem 17.12

Find all solutions to the following partial differential equation that can be obtained via the separation of variables.

$$u_x - u_y = 0.$$

Problem 17.13

Separate the PDE $u_{xx} - u_y + u_{yy} = u$ into two ODEs with a parameter. You do not need to solve the ODEs.

18 Solutions of the Heat Equation by the Separation of Variables Method

In this section we apply the method of separation of variables in solving the one spatial dimension of the heat equation.

The Heat Equation with Dirichlet Boundary Conditions

Consider the problem of finding all nontrivial solutions to the heat equation $u_t = ku_{xx}$ that satisfies the initial time condition $u(x, 0) = f(x)$ and the Dirichlet boundary conditions $u(0, t) = u(L, t) = 0$ (that is, the endpoints are assumed to be at zero temperature) with u not the trivial solution. Let's assume that the solution can be written in the form $u(x, t) = X(x)T(t)$. Substituting into the heat equation we obtain

$$\frac{X''}{X} = \frac{T'}{kT}.$$

Since the LHS only depends on x and the RHS only depends on t , there must be a constant λ such that

$$\frac{X''}{X} = \lambda \text{ and } \frac{T'}{kT} = \lambda.$$

This gives the two ordinary differential equations

$$X'' - \lambda X = 0 \text{ and } T' - k\lambda T = 0.$$

As far as the boundary conditions, we have

$$u(0, t) = 0 = X(0)T(t) \implies X(0) = 0$$

and

$$u(L, t) = 0 = X(L)T(t) \implies X(L) = 0.$$

Note that T is not the zero function for otherwise $u \equiv 0$ and this contradicts our assumption that u is the non-trivial solution.

Next, we consider the three cases of the sign of λ .

Case 1: $\lambda = 0$

In this case, $X'' = 0$. Solving this equation we find $X(x) = ax + b$. Since $X(0) = 0$ we find $b = 0$. Since $X(L) = 0$ we find $a = 0$. Hence, $X \equiv 0$ and $u(x, t) \equiv 0$. That is, u is the trivial solution.

Case 2: $\lambda > 0$

In this case, $X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$. Again, the conditions $X(0) = X(L) = 0$ imply $A = B = 0$ and hence the solution is the trivial solution.

Case 3: $\lambda < 0$

In this case, $X(x) = A \cos \sqrt{-\lambda}x + B \sin \sqrt{-\lambda}x$. The condition $X(0) = 0$ implies $A = 0$. The condition $X(L) = 0$ implies $B \sin \sqrt{-\lambda}L = 0$. We must have $B \neq 0$ otherwise $X(x) = 0$ and this leads to the trivial solution. Since $B \neq 0$, we obtain $\sin \sqrt{-\lambda}L = 0$ or $\sqrt{-\lambda}L = n\pi$ where $n \in \mathbb{N}$. Solving for λ we find $\lambda = -\frac{n^2\pi^2}{L^2}$. Thus, we obtain infinitely many solutions given by

$$X_n(x) = A_n \sin \frac{n\pi}{L}x, \quad n = 1, 2, \dots$$

Now, solving the equation

$$T' - \lambda kT = 0$$

by the method of separation of variables we obtain

$$T_n(t) = B_n e^{-\frac{n^2\pi^2}{L^2}kt}, \quad n = 1, 2, \dots$$

Hence, the functions

$$u_n(x, t) = C_n \sin \left(\frac{n\pi}{L}x \right) e^{-\frac{n^2\pi^2}{L^2}kt}, \quad n = 1, 2, \dots$$

satisfy $u_t = ku_{xx}$ and the boundary conditions $u(0, t) = u(L, t) = 0$.

Now, in order for these solutions to satisfy the initial value condition $u(x, 0) = f(x)$, we invoke the superposition principle of linear PDE to write

$$u(x, t) = \sum_{n=1}^{\infty} C_n \sin \left(\frac{n\pi}{L}x \right) e^{-\frac{n^2\pi^2}{L^2}kt}. \quad (18.1)$$

To determine the unknown constants C_n we use the initial condition $u(x, 0) = f(x)$ in (18.1) to obtain

$$f(x) = \sum_{n=1}^{\infty} C_n \sin \left(\frac{n\pi}{L}x \right).$$

Since the right-hand side is the Fourier sine series of f on the interval $[0, L]$, the coefficients C_n are given by

$$C_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx. \quad (18.2)$$

Thus, the solution to the heat equation is given by (18.1) with the C_n 's calculated from (18.2).

The Heat Equation with Neumann Boundary Conditions

When both ends of the bar are insulated, that is, there is no heat flow out of them, we use the boundary conditions

$$u_x(0, t) = u_x(L, t) = 0.$$

In this case, the general form of the heat equation initial boundary value problem is to find $u(x, t)$ satisfying

$$\begin{aligned} u_t(x, t) &= k u_{xx}(x, t), \quad 0 \leq x \leq L, \quad t > 0 \\ u(x, 0) &= f(x), \quad 0 \leq x \leq L \\ u_x(0, t) &= u_x(L, t) = 0, \quad t > 0. \end{aligned}$$

Since $0 = u_x(0, t) = X'(0)T(t)$ we obtain $X'(0) = 0$. Likewise, $0 = u_x(L, t) = X'(L)T(t)$ implies $X'(L) = 0$. We again consider the following three cases:

- If $\lambda = 0$ then $X(x) = A + Bx$. Since $X'(0) = 0$, we find $B = 0$. Thus, $X(x) = A$ and $T(t) = \text{constant}$ so that $u(x, t) = \text{constant}$ which is impossible if $f(x)$ is not the constant function.
- If $\lambda > 0$ then a simple calculation shows that $u(x, t)$ is the trivial solution. Again, because of the condition $u(x, 0) = f(x)$, this solution is discarded.
- If $\lambda < 0$ then $X(x) = A \cos \sqrt{-\lambda}x + B \sin \sqrt{-\lambda}x$ and upon differentiation with respect to x we find

$$X'(x) = -\sqrt{-\lambda}A \sin \sqrt{-\lambda}x + \sqrt{-\lambda}B \cos \sqrt{-\lambda}x.$$

The conditions $X'(0) = X'(L) = 0$ imply $\sqrt{-\lambda}B = 0$ and $\sqrt{-\lambda}A \sin \sqrt{-\lambda}L = 0$. Hence, $B = 0$ and $\lambda = -\frac{n^2\pi^2}{L^2}$ and

$$X_n(x) = A_n \cos\left(\frac{n\pi}{L}x\right), \quad n = 1, 2, \dots$$

and

$$u_n(x, t) = C_n \cos\left(\frac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2}{L^2}kt}.$$

By the superposition principle, the required solution to the heat equation with Neumann boundary conditions is given by

$$\bar{u}(x, t) = \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2}{L^2}kt}.$$

In order to satisfy the initial condition $u(x, 0) = f(x)$, we let

$$u(x, t) = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2}{L^2}kt}$$

where

$$C_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi}{L}x\right) dx, \quad n = 0, 1, 2, \dots.$$

Practice Problems

Problem 18.1

Find the temperature in a bar of length 2 whose ends are kept at zero and lateral surface insulated if the initial temperature is $f(x) = \sin\left(\frac{\pi}{2}x\right) + 3\sin\left(\frac{5\pi}{2}x\right)$.

Problem 18.2

Find the temperature in a homogeneous bar of heat conducting material of length L with its end points kept at zero and initial temperature distribution given by $f(x) = \frac{x^d}{L^2}(L-x)$, $0 \leq x \leq L$.

Problem 18.3

Find the temperature in a thin metal rod of length L , with both ends insulated (so that there is no passage of heat through the ends) and with initial temperature in the rod $f(x) = \sin\left(\frac{\pi}{L}x\right)$.

Problem 18.4

Solve the following heat equation with Dirichlet boundary conditions

$$\begin{aligned} u_t &= k u_{xx} \\ u(0, t) &= u(L, t) = 0 \\ u(x, 0) &= \begin{cases} 1 & 0 \leq x < \frac{L}{2} \\ 2 & \frac{L}{2} \leq x \leq L. \end{cases} \end{aligned}$$

Problem 18.5

Solve

$$\begin{aligned} u_t &= k u_{xx} \\ u(0, t) &= u(L, t) = 0 \\ u(x, 0) &= 6 \sin\left(\frac{9\pi}{L}x\right). \end{aligned}$$

Problem 18.6

Solve

$$u_t = k u_{xx}$$

subject to

$$\begin{aligned} u_x(0, t) &= u_x(L, t) = 0 \\ u(x, 0) &= \begin{cases} 0 & 0 \leq x < \frac{L}{2} \\ 1 & \frac{L}{2} \leq x \leq L. \end{cases} \end{aligned}$$

Problem 18.7

Solve

$$u_t = ku_{xx}$$

subject to

$$\begin{aligned} u_x(0, t) &= u_x(L, t) = 0 \\ u(x, 0) &= 6 + 4 \cos\left(\frac{3\pi}{L}x\right). \end{aligned}$$

Problem 18.8

Solve

$$u_t = ku_{xx}$$

subject to

$$\begin{aligned} u_x(0, t) &= u_x(L, t) = 0 \\ u(x, 0) &= -3 \cos\left(\frac{8\pi}{L}x\right). \end{aligned}$$

Problem 18.9Find the general solution $u(x, t)$ of

$$\begin{aligned} u_t &= u_{xx} - u, \quad 0 < x < L, \quad t > 0 \\ u_x(0, t) &= 0 = u_x(L, t), \quad t > 0. \end{aligned}$$

Briefly describe its behavior as $t \rightarrow \infty$.**Problem 18.10** (*Energy method*)Let u_1 and u_2 be two solutions to the Neumann boundary value problem

$$\begin{aligned} u_t &= u_{xx} - u, \quad 0 < x < 1, \quad t > 0 \\ u_x(0, t) &= u_x(1, t) = 0, \quad t > 0 \\ u(x, 0) &= g(x), \quad 0 < x < 1 \end{aligned}$$

Define $w(x, t) = u_1(x, t) - u_2(x, t)$.(a) Show that w satisfies the initial value problem

$$w_t = w_{xx} - w, \quad 0 < x < 1, \quad t > 0$$

$$w_x(0, t) = w_x(1, t) = w(x, 0) = 0, \quad 0 < x < 1, \quad t > 0$$

(b) Define $E(t) = \int_0^1 w^2(x, t) dx \geq 0$ for all $t \geq 0$. Show that $E'(t) \leq 0$. Hence, $0 \leq E(t) \leq E(0)$ for all $t > 0$.(c) Show that $E(t) = 0$, $w(x, t) = 0$. Hence, conclude that $u_1 = u_2$.

Problem 18.11

Consider the heat induction in a bar where the left end temperature is maintained at 0, and the right end is perfectly insulated. We assume $k = 1$ and $L = 1$.

- (a) Derive the boundary conditions of the temperature at the endpoints.
 (b) Following the separation of variables approach, derive the ODEs for X and T .
 (c) Consider the equation in $X(x)$. What are the values of $X(0)$ and $X'(1)$? Show that solutions of the form $X(x) = \sin \sqrt{-\lambda}x$ satisfy the ODE and one of the boundary conditions. Can you choose a value of λ so that the other boundary condition is also satisfied?

Problem 18.12

Using the method of separation of variables find the solution of the heat equation

$$u_t = ku_{xx}$$

satisfying the following boundary and initial conditions:

- (a) $u(0, t) = u(L, t) = 0$, $u(x, 0) = 6 \sin\left(\frac{9\pi x}{L}\right)$
 (b) $u(0, t) = u(L, t) = 0$, $u(x, 0) = 3 \sin\left(\frac{\pi x}{L}\right) - \sin\left(\frac{3\pi x}{L}\right)$

Problem 18.13

Using the method of separation of variables find the solution of the heat equation

$$u_t = ku_{xx}$$

satisfying the following boundary and initial conditions:

- (a) $u_x(0, t) = u_x(L, t) = 0$, $u(x, 0) = \cos\left(\frac{\pi x}{L}\right) + 4 \cos\left(\frac{5\pi x}{L}\right)$.
 (b) $u_x(0, t) = u_x(L, t) = 0$, $u(x, 0) = 5$.

Problem 18.14

Find the solution of the following heat conduction partial differential equation

$$u_t = 8u_{xx}, \quad 0 < x < 4\pi, \quad t > 0$$

$$u(0, t) = u(4\pi, t) = 0, \quad t > 0$$

$$u(x, 0) = 6 \sin x, \quad 0 < x < 4\pi.$$

19 Elliptic Type: Laplace's Equations in Rectangular Domains

Boundary value problems are of great importance in physical applications. Mathematically, a boundary-value problem consists of finding a function which satisfies a given partial differential equation and particular boundary conditions. Physically speaking, the problem is independent of time, involving only space coordinates.

Just as initial-value problems are associated with hyperbolic PDE, boundary value problems are associated with PDE of elliptic type. In contrast to initial-value problems, boundary-value problems are considerably more difficult to solve.

The main model example of an elliptic type PDE is the **Laplace equation**

$$\Delta u = u_{xx} + u_{yy} = 0 \quad (19.1)$$

where the symbol Δ is referred to as the **Laplacian**. Solutions of this equation are called **harmonic functions**.

Example 19.1

Show that, for all $(x, y) \neq (0, 0)$, $u(x, y) = ax^2 - ay^2 + cx + dy + e$ is a harmonic function, where a, b, c, d , and e are constants.

Solution.

We have

$$\begin{aligned} u_x &= 2ax + c \\ u_{xx} &= 2a \\ u_y &= -2ay + d \\ u_{yy} &= -2a. \end{aligned}$$

Plugging these expressions into the equation we find $u_{xx} + u_{yy} = 0$. Hence, $u(x, y)$ is harmonic ■

The Laplace equation is arguably the most important differential equation in all of applied mathematics. It arises in an astonishing variety of mathematical and physical systems, ranging through fluid mechanics, electromagnetism, potential theory, solid mechanics, heat conduction, geometry, probability,

number theory, and on and on.

There are two main modifications of the Laplace equation: the **Poisson equation** (a non-homogeneous Laplace equation):

$$\Delta u = f(x, y)$$

and the **eigenvalue problem** (the **Helmholtz equation**):

$$\Delta u = \lambda u, \quad \lambda \in \mathbb{R}.$$

Solving Laplace's Equation (19.1)

Note first that both independent variables are spatial variables and each variable occurs in a 2nd order derivative and so we will need two boundary conditions for each variable a total of four boundary conditions.

Consider (19.1) in the rectangle

$$\Omega = \{(x, y) : 0 \leq x \leq a, 0 \leq y \leq b\}$$

with the Dirichlet boundary conditions

$$u(0, y) = f_1(y), \quad u(a, y) = f_2(y), \quad u(x, 0) = g_1(x), \quad u(x, b) = g_2(x)$$

where $0 \leq x \leq a$ and $0 \leq y \leq b$.

The separation of variables method is most successful when the boundary conditions are homogeneous. Thus, solving the Laplace's equation in Ω requires solving four initial boundary conditions problems, where in each problem three of the four conditions are homogeneous. The four problems to be solved are

$$(I) \begin{cases} u_{xx} + u_{yy} = 0 \\ u(0, y) = f_1(y), \\ u(a, y) = u(x, 0) = u(x, b) = 0 \end{cases} \quad (II) \begin{cases} u_{xx} + u_{yy} = 0 \\ u(a, y) = f_2(y), \\ u(0, y) = u(x, 0) = u(x, b) = 0 \end{cases}$$

$$(III) \begin{cases} u_{xx} + u_{yy} = 0 \\ u(x, 0) = g_1(x), \\ u(0, y) = u(a, y) = u(x, b) = 0 \end{cases} \quad (IV) \begin{cases} u_{xx} + u_{yy} = 0 \\ u(x, b) = g_2(x), \\ u(0, y) = u(a, y) = u(x, 0) = 0. \end{cases}$$

If we let $u_i(x, y)$, $i = 1, 2, 3, 4$, denote the solution of each of the above problems, then the solution to our original system will be

$$u(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + u_4(x, y).$$

In each of the above problems, we will apply separation of variables to (19.1) and find a product solution that will satisfy the differential equation and the three homogeneous boundary conditions. Using the Principle of Superposition we will find a solution to the problem and then apply the final boundary condition to determine the value of the constant(s) that are left in the problem. The process is nearly identical in many ways to what we did when we were solving the heat equation.

We will illustrate how to find $u(x, y) = u_4(x, y)$. So let's assume that the solution can be written in the form $u(x, y) = X(x)Y(y)$. Substituting in (19.1), we obtain

$$X''(x)Y(y) + X(x)Y''(y) = 0.$$

Assuming $X(x)Y(y)$ is nonzero, that is u is the non-trivial solution. Dividing for $X(x)Y(y)$ and subtracting $\frac{Y''(y)}{Y(y)}$ from both sides, we find:

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}.$$

The left hand side is a function of x while the right hand side is a function of y . This says that they must equal to a constant. That is,

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda$$

where λ is a constant. This results in the following two ODEs

$$X'' - \lambda X = 0 \text{ and } Y'' + \lambda Y = 0.$$

As far as the boundary conditions, we have for all $0 \leq x \leq a$ and $0 \leq y \leq b$

$$u(0, y) = 0 = X(0)Y(y) \implies X(0) = 0$$

$$u(a, y) = 0 = X(a)Y(y) \implies X(a) = 0$$

$$u(x, 0) = 0 = X(x)Y(0) \implies Y(0) = 0$$

$$u(x, b) = g_2(x) = X(x)Y(b).$$

Note that X and Y are not the zero functions for otherwise $u \equiv 0$ and this contradicts our assumption that u is the non-trivial solution.

Consider the first equation: since $X'' - \lambda X = 0$ the solution depends on the sign of λ . If $\lambda = 0$ then $X(x) = Ax + B$. Now, the conditions $X(0) = X(a) = 0$

imply $A = B = 0$ and so $u \equiv 0$. So assume that $\lambda \neq 0$. If $\lambda > 0$ then $X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$. Now, the conditions $X(0) = X(a) = 0$, $\lambda \neq 0$ imply $A = B = 0$ and hence the solution is the trivial solution. Hence, in order to have a nontrivial solution we must have $\lambda < 0$. In this case,

$$X(x) = A \cos \sqrt{-\lambda}x + B \sin \sqrt{-\lambda}x.$$

The condition $X(0) = 0$ implies $A = 0$. The condition $X(a) = 0$ implies $B \sin \sqrt{-\lambda}a = 0$. We must have $B \neq 0$ otherwise $X(x) = 0$ and this leads to the trivial solution. Since $B \neq 0$, we obtain $\sin \sqrt{-\lambda}a = 0$ or $\sqrt{-\lambda}a = n\pi$ where $n \in \mathbb{N}$. Solving for λ we find $\lambda_n = -\frac{n^2\pi^2}{a^2}$. Thus, we obtain infinitely many solutions given by

$$X_n(x) = \sin \frac{n\pi}{a}x, \quad n \in \mathbb{N}.$$

Now, solving the equation

$$Y'' + \lambda Y = 0$$

we obtain

$$Y_n(y) = a_n e^{\sqrt{-\lambda_n}y} + b_n e^{-\sqrt{-\lambda_n}y} = A_n \cosh \sqrt{-\lambda_n}y + B_n \sinh \sqrt{-\lambda_n}y, \quad n \in \mathbb{N}.$$

Using the boundary condition $Y(0) = 0$ we obtain $A_n = 0$ for all $n \in \mathbb{N}$. Hence, the functions

$$u_n(x, y) = B_n \sin \left(\frac{n\pi}{a}x \right) \sinh \left(\frac{n\pi}{a}y \right), \quad n \in \mathbb{N}$$

satisfy (19.1) and the boundary conditions $u(0, y) = u(a, y) = u(x, 0) = 0$.

Now, in order for these solutions to satisfy the boundary value condition $u(x, b) = g_2(x)$, we invoke the superposition principle of linear PDE to write

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin \left(\frac{n\pi}{a}x \right) \sinh \left(\frac{n\pi}{a}y \right). \quad (19.2)$$

To determine the unknown constants B_n we use the boundary condition $u(x, b) = g_2(x)$ in (19.2) to obtain

$$g_2(x) = \sum_{n=1}^{\infty} \left(B_n \sinh \left(\frac{n\pi}{a}b \right) \right) \sin \left(\frac{n\pi}{a}x \right).$$

Since the right-hand side is the Fourier sine series of f on the interval $[0, a]$, the coefficients B_n are given by

$$B_n = \left[\frac{2}{a} \int_0^a g_2(x) \sin\left(\frac{n\pi}{a}x\right) dx \right] \left[\sinh\left(\frac{n\pi}{a}b\right) \right]^{-1}. \quad (19.3)$$

Thus, the solution to the Laplace's equation is given by (19.2) with the B_n 's calculated from (19.3).

Example 19.2

Solve

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(0, y) = f_1(y), \\ u(a, y) = u(x, 0) = u(x, b) = 0. \end{cases}$$

Solution.

Assume that the solution can be written in the form $u(x, y) = X(x)Y(y)$. Substituting in (19.1), we obtain

$$X''(x)Y(y) + X(x)Y''(y) = 0.$$

Assuming $X(x)Y(y)$ is nonzero, dividing for $X(x)Y(y)$ and subtracting $\frac{Y''(y)}{Y(y)}$ from both sides, we find:

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}.$$

The left hand side is a function of x while the right hand side is a function of y . This says that they must equal to a constant. That is,

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda$$

where λ is a constant. This results in the following two ODEs

$$X'' - \lambda X = 0 \text{ and } Y'' + \lambda Y = 0.$$

As far as the boundary conditions, we have for all $0 \leq x \leq a$ and $0 \leq y \leq b$

$$u(0, y) = f_1(y) = X(0)Y(y)$$

$$u(a, y) = 0 = X(a)Y(y) \implies X(a) = 0$$

$$u(x, 0) = 0 = X(x)Y(0) \implies Y(0) = 0$$

$$u(x, b) = 0 = X(x)Y(b) \implies Y(b) = 0.$$

Note that X and Y are not the zero functions for otherwise $u \equiv 0$ and this contradicts our assumption that u is the non-trivial solution.

Consider the second equation: since $Y'' + \lambda Y = 0$ the solution depends on the sign of λ . If $\lambda = 0$ then $Y(y) = Ay + B$. Now, the conditions $Y(0) = Y(b) = 0$ imply $A = B = 0$ and so $u \equiv 0$. So assume that $\lambda \neq 0$. If $\lambda < 0$ then $Y(y) = Ae^{\sqrt{-\lambda}y} + Be^{-\sqrt{-\lambda}y}$. Now, the condition $Y(0) = Y(b) = 0$ imply $A = B = 0$ and hence the solution is the trivial solution. Hence, in order to have a nontrivial solution we must have $\lambda > 0$. In this case,

$$Y(y) = A \cos \sqrt{\lambda}y + B \sin \sqrt{\lambda}y.$$

The condition $Y(0) = 0$ implies $A = 0$. The condition $Y(b) = 0$ implies $B \sin \sqrt{\lambda}b = 0$. We must have $B \neq 0$ otherwise $Y(y) = 0$ and this leads to the trivial solution. Since $B \neq 0$, we obtain $\sin \sqrt{\lambda}b = 0$ or $\sqrt{\lambda}b = n\pi$ where $n \in \mathbb{N}$. Solving for λ we find $\lambda_n = \frac{n^2\pi^2}{b^2}$. Thus, we obtain infinitely many solutions given by

$$Y_n(y) = \sin \left(\frac{n\pi}{b}y \right), \quad n \in \mathbb{N}.$$

Now, solving the equation

$$X'' - \lambda X = 0, \quad \lambda > 0$$

we obtain

$$X_n(x) = a_n e^{\sqrt{\lambda_n}x} + b_n e^{-\sqrt{\lambda_n}x} = A_n \cosh \left(\frac{n\pi}{b}x \right) + B_n \sinh \left(\frac{n\pi}{b}x \right), \quad n \in \mathbb{N}.$$

However, this is not really suited for dealing with the boundary condition $X(a) = 0$. So, let's also notice that the following is also a solution.

$$X_n(x) = A_n \cosh \left(\frac{n\pi}{b}(x - a) \right) + B_n \sinh \left(\frac{n\pi}{b}(x - a) \right), \quad n \in \mathbb{N}.$$

Now, using the boundary condition $X(a) = 0$ we obtain $A_n = 0$ for all $n \in \mathbb{N}$. Hence, the functions

$$u_n(x, y) = B_n \sin \left(\frac{n\pi}{b}y \right) \sinh \left(\frac{n\pi}{b}(x - a) \right), \quad n \in \mathbb{N}$$

satisfy (19.1) and the boundary conditions $u(a, y) = u(x, 0) = u(x, b) = 0$. Now, in order for these solutions to satisfy the boundary value condition $u(0, y) = f_1(y)$, we invoke the superposition principle of linear PDE to write

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{b}y\right) \sinh\left(\frac{n\pi}{b}(x-a)\right). \quad (19.4)$$

To determine the unknown constants B_n we use the boundary condition $u(0, y) = f_1(y)$ in (19.4) to obtain

$$f_1(y) = \sum_{n=1}^{\infty} \left(B_n \sinh\left(-\frac{n\pi}{b}a\right) \right) \sin\left(\frac{n\pi}{b}y\right).$$

Since the right-hand side is the Fourier sine series of f_1 on the interval $[0, b]$, the coefficients B_n are given by

$$B_n = \left[\frac{2}{b} \int_0^b f_1(y) \sin\left(\frac{n\pi}{b}y\right) dy \right] \left[\sinh\left(-\frac{n\pi}{b}a\right) \right]^{-1}. \quad (19.5)$$

Thus, the solution to the Laplace's equation is given by (19.4) with the B_n 's calculated from (19.5) ■

Example 19.3

Solve

$$\begin{aligned} u_{xx} + u_{yy} &= 0, & 0 < x < L, & 0 < y < H \\ u(0, y) &= u(L, y) = 0, & 0 < y < H \\ u(x, 0) &= u_y(x, 0), & u(x, H) &= f(x), & 0 < x < L. \end{aligned}$$

Solution.

Using separation of variables we find

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda.$$

We first solve

$$\begin{cases} X'' - \lambda X = 0 & 0 < x < L \\ X(0) = X(L) = 0. \end{cases}$$

We find $\lambda_n = -\frac{n^2\pi^2}{L^2}$ and

$$X_n(x) = \sin \frac{n\pi}{L}x, \quad n \in \mathbb{N}.$$

Next we need to solve

$$\begin{cases} Y'' + \lambda Y = 0 & 0 < y < H \\ Y(0) - Y'(0) = 0. \end{cases}$$

The solution of the ODE is

$$Y_n(y) = A_n \cosh\left(\frac{n\pi}{L}y\right) + B_n \sinh\left(\frac{n\pi}{L}y\right), \quad n \in \mathbb{N}.$$

The boundary condition $Y(0) - Y'(0) = 0$ implies

$$A_n - B_n \frac{n\pi}{L} = 0.$$

Hence,

$$Y_n = B_n \frac{n\pi}{L} \cosh\left(\frac{n\pi}{L}y\right) + B_n \sinh\left(\frac{n\pi}{L}y\right), \quad n \in \mathbb{N}.$$

Using the superposition principle and the results above we have

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L}x \left[\frac{n\pi}{L} \cosh\left(\frac{n\pi}{L}y\right) + \sinh\left(\frac{n\pi}{L}y\right) \right].$$

Substituting in the condition $u(x, H) = f(x)$ we find

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L}x \left[\frac{n\pi}{L} \cosh\left(\frac{n\pi}{L}H\right) + \sinh\left(\frac{n\pi}{L}H\right) \right].$$

Recall the Fourier sine series of f on $[0, L]$ given by

$$f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L}x$$

where

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx.$$

Thus, the general solution is given by

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L}x \left[\frac{n\pi}{L} \cosh\left(\frac{n\pi}{L}y\right) + \sinh\left(\frac{n\pi}{L}y\right) \right].$$

with the B_n satisfying

$$B_n \left[\frac{n\pi}{L} \cosh\left(\frac{n\pi}{L}H\right) + \sinh\left(\frac{n\pi}{L}H\right) \right] = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx \blacksquare$$

Practice Problems

Problem 19.1

Solve

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(a, y) = f_2(y), \\ u(0, y) = u(x, 0) = u(x, b) = 0. \end{cases}$$

Problem 19.2

Solve

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(x, 0) = g_1(x), \\ u(0, y) = u(a, y) = u(x, b) = 0. \end{cases}$$

Problem 19.3

Solve

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(x, 0) = u(0, y) = 0, \\ u(1, y) = 2y, u(x, 1) = 3 \sin \pi x + 2x \end{cases}$$

where $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Hint: Define $U(x, y) = u(x, y) - 2xy$.

Problem 19.4

Show that $u(x, y) = x^2 - y^2$ and $u(x, y) = 2xy$ are harmonic functions.

Problem 19.5

Solve

$$u_{xx} + u_{yy} = 0, \quad 0 \leq x \leq L, \quad -\frac{H}{2} \leq y \leq \frac{H}{2}$$

subject to

$$\begin{aligned} u(0, y) = u(L, y) = 0, \quad -\frac{H}{2} < y < \frac{H}{2} \\ u(x, -\frac{H}{2}) = f_1(x), \quad u(x, \frac{H}{2}) = f_2(x), \quad 0 \leq x \leq L. \end{aligned}$$

Problem 19.6

Consider a complex valued function $f(z) = u(x, y) + iv(x, y)$ where $i = \sqrt{-1}$. We say that f is **holomorphic** or **analytic** if and only if f can be expressed as a power series in z , i.e.

$$u(x, y) + iv(x, y) = \sum_{n=0}^{\infty} a_n z^n.$$

(a) By differentiating with respect to x and y show that

$$u_x = v_y \text{ and } u_y = -v_x$$

These are known as the Cauchy-Riemann equations.

(b) Show that $\Delta u = 0$ and $\Delta v = 0$.

Problem 19.7

Show that Laplace's equation in polar coordinates is given by

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

Problem 19.8

Solve

$$u_{xx} + u_{yy} = 0, \quad 0 \leq x \leq 2, \quad 0 \leq y \leq 3$$

subject to

$$u(x, 0) = 0, \quad u(x, 3) = \frac{x}{2}$$

$$u(0, y) = \sin\left(\frac{4\pi}{3}y\right), \quad u(2, y) = 7.$$

Problem 19.9

Solve

$$u_{xx} + u_{yy} = 0, \quad 0 \leq x \leq L, \quad 0 \leq y \leq H$$

subject to

$$u_y(x, 0) = 0, \quad u(x, H) = 0$$

$$u(0, y) = u(L, y) = 4 \cos\left(\frac{\pi y}{2H}\right).$$

Problem 19.10

Solve

$$u_{xx} + u_{yy} = 0, \quad x > 0, \quad 0 \leq y \leq H$$

subject to

$$u(0, y) = f(y), \quad |u(x, 0)| < \infty$$

$$u_y(x, 0) = u_y(x, H) = 0.$$

Problem 19.11

Consider Laplace's equation inside a rectangle

$$u_{xx} + u_{yy} = 0, \quad 0 \leq x \leq L, \quad 0 \leq y \leq H$$

subject to the boundary conditions

$$u(0, y) = 0, \quad u(L, y) = 0, \quad u(x, 0) - u_y(x, 0) = 0, \quad u(x, H) = 20 \sin\left(\frac{\pi x}{L}\right) - 5 \sin\left(\frac{3\pi x}{L}\right).$$

Find the solution $u(x, y)$.

Problem 19.12

Solve Laplace's equation $u_{xx} + u_{yy} = 0$ in the rectangle $0 < x, y < 1$ subject to the conditions

$$\begin{aligned} u(0, y) = u(1, y) = 0, \quad 0 < y < 1 \\ u(x, 0) = \sin(2\pi x), \quad u_y(x, 0) = -2\pi \sin(2\pi x), \quad 0 < x < 1. \end{aligned}$$

Problem 19.13

Find the solution to Laplace's equation on the rectangle $0 < x < 1, 0 < y < 1$ with boundary conditions

$$\begin{aligned} u(x, 0) = 0, \quad u(x, 1) = 1 \\ u_x(0, y) = u_x(1, y) = 0. \end{aligned}$$

Problem 19.14

Solve Laplace's equation on the rectangle $0 < x < a, 0 < y < b$ with the boundary conditions

$$\begin{aligned} u_x(0, y) = -a, \quad u_x(a, y) = 0 \\ u_y(x, 0) = b, \quad u_y(x, b) = 0. \end{aligned}$$

Problem 19.15

Solve Laplace's equation on the rectangle $0 < x < \pi, 0 < y < 2$ with the boundary conditions

$$\begin{aligned} u(0, y) = u(\pi, y) = 0 \\ u_y(x, 0) = 0, \quad u_y(x, 2) = 2 \sin 3x - 5 \sin 10x. \end{aligned}$$

20 Laplace's Equations in Circular Regions

In the previous section we solved the Dirichlet problem for Laplace's equation on a rectangular region. However, if the domain of the solution is a disc, an annulus, or a circular wedge, it is useful to study the two-dimensional Laplace's equation in polar coordinates.

It is well known in calculus that the cartesian coordinates (x, y) and the polar coordinates (r, θ) of a point are related by the formulas

$$x = r \cos \theta \text{ and } y = r \sin \theta$$

where $r = (x^2 + y^2)^{\frac{1}{2}}$ and $\tan \theta = \frac{y}{x}$. Using the chain rule we obtain

$$\begin{aligned} u_x &= u_r r_x + u_\theta \theta_x = \cos \theta u_r - \frac{\sin \theta}{r} u_\theta \\ u_{xx} &= u_{xr} r_x + u_{x\theta} \theta_x \\ &= \left(\cos \theta u_{rr} + \frac{\sin \theta}{r^2} u_\theta - \frac{\sin \theta}{r} u_{r\theta} \right) \cos \theta \\ &\quad + \left(-\sin \theta u_r + \cos \theta u_{r\theta} - \frac{\cos \theta}{r} u_\theta - \frac{\sin \theta}{r} u_{\theta\theta} \right) \left(-\frac{\sin \theta}{r} \right) \\ u_y &= u_r r_y + u_\theta \theta_y = \sin \theta u_r + \frac{\cos \theta}{r} u_\theta \\ u_{yy} &= u_{yr} r_y + u_{y\theta} \theta_y \\ &= \left(\sin \theta u_{rr} - \frac{\cos \theta}{r^2} u_\theta + \frac{\cos \theta}{r} u_{r\theta} \right) \sin \theta \\ &\quad + \left(\cos \theta u_r + \sin \theta u_{r\theta} - \frac{\sin \theta}{r} u_\theta + \frac{\cos \theta}{r} u_{\theta\theta} \right) \left(\frac{\cos \theta}{r} \right). \end{aligned}$$

Substituting these equations into $\Delta u = 0$ we obtain

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0. \quad (20.1)$$

Example 20.1

Find the solution to

$$\Delta u = 0, \quad x^2 + y^2 < a^2$$

subject to

- (i) Boundary condition: $u(a, \theta) = f(\theta)$, $0 \leq \theta \leq 2\pi$.
- (ii) Boundedness at the origin: $|u(0, \theta)| < \infty$.
- (iii) Periodicity: $u(r, \theta + 2\pi) = u(r, \theta)$, $0 \leq \theta \leq 2\pi$.

Solution.

First, note that (iii) implies that $u(r, 0) = u(r, 2\pi)$ and $u_\theta(r, 0) = u_\theta(r, 2\pi)$. Next, we will apply the method of separation of variables to (20.1). Suppose that a solution $u(r, \theta)$ of (20.1) can be written in the form $u(r, \theta) = R(r)\Theta(\theta)$. Substituting in (20.1) we obtain

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0.$$

Dividing by $R\Theta$ (under the assumption that $R\Theta \neq 0$) we obtain

$$\frac{\Theta''(\theta)}{\Theta(\theta)} = -r^2 \frac{R''(r)}{R(r)} - r \frac{R'(r)}{R(r)}.$$

The left-hand side is independent of r whereas the right-hand side is independent of θ so that there is a constant λ such that

$$-\frac{\Theta''(\theta)}{\Theta(\theta)} = r^2 \frac{R''(r)}{R(r)} + r \frac{R'(r)}{R(r)} = \lambda.$$

This results in the following ODEs

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0 \tag{20.2}$$

and

$$r^2 R''(r) + rR'(r) - \lambda R(r) = 0. \tag{20.3}$$

The second equation is known as **Euler's equation**. Both of these equations are easily solvable. To solve (20.2), we only have to add the appropriate boundary conditions. We have $\Theta(0) = \Theta(2\pi)$ and $\Theta'(0) = \Theta'(2\pi)$. The periodicity of Θ implies that $\lambda = n^2$ and Θ must be of the form

$$\Theta_n(\theta) = A'_n \cos n\theta + B'_n \sin n\theta, n = 0, 1, 2 \dots$$

The equation in R is of Euler type and its solution must be of the form $R(r) = r^\alpha$. Since $\lambda = n^2$, the corresponding characteristic equation is

$$\alpha(\alpha - 1)r^\alpha + \alpha r^\alpha - n^2 r^\alpha = 0.$$

Solving this equation we find $\alpha = \pm n$. Hence, we let

$$R_n(r) = C_n r^n + D_n r^{-n}, n \in \mathbb{N}.$$

For $n = 0$, $R = 1$ is a solution. To find a second solution, we solve the equation

$$r^2 R'' + rR' = 0.$$

This can be done by dividing through by r and using the substitution $S = R'$ to obtain $rS' + S = 0$. Solving this by noting that the left-hand side is just $(rS)'$ we find $S = \frac{c}{r}$. Hence, $R' = \frac{c}{r}$ and this implies $R(r) = C \ln r$. Thus, $R = 1$ and $R = \ln r$ form a fundamental set of solutions of (20.3) and so a general solution is given by

$$R_0(r) = C_0 + D_0 \ln r.$$

By assumption (ii), $u(r, \theta)$ must be bounded at $r = 0$, and so does R_n . Since r^{-n} and $\ln r$ are unbounded at $r = 0$, we must set $D_0 = D_n = 0$. In this case, the solutions to Euler's equation are given by

$$R_n(r) = C_n r^n, \quad n = 0, 1, 2, \dots$$

Using the superposition principle, and combining the results obtained above, we find

$$u(r, \theta) = C_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta).$$

Now, using the boundary condition $u(a, \theta) = f(\theta)$ we can write

$$f(\theta) = C_0 + \sum_{n=1}^{\infty} (a^n A_n \cos n\theta + a^n B_n \sin n\theta)$$

which is usually written in a more convenient equivalent form by

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta).$$

It is obvious that a_n and b_n are the Fourier coefficients, and therefore can be determined by the formulas

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos n\theta d\theta, \quad n = 0, 1, \dots$$

and

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin n\theta d\theta, \quad n = 1, 2, \dots$$

Finally, the general solution to our problem is given by

$$u(r, \theta) = C_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

where

$$\begin{aligned} C_0 &= \frac{a_0}{2} = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta \\ A_n &= \frac{a_n}{a^n} = \frac{1}{a^n \pi} \int_0^{2\pi} f(\theta) \cos n\theta d\theta, \quad n = 1, 2, \dots \\ B_n &= \frac{b_n}{a^n} = \frac{1}{a^n \pi} \int_0^{2\pi} f(\theta) \sin n\theta d\theta, \quad n = 1, 2, \dots \blacksquare \end{aligned}$$

Example 20.2

Solve

$$\Delta u = 0, \quad 0 \leq \theta < 2\pi, \quad 1 \leq r \leq 2$$

subject to

$$u(1, \theta) = u(2, \theta) = \sin \theta, \quad 0 \leq \theta < 2\pi.$$

Solution.

Use separation of variables. First, solving for $\Theta(\theta)$, we see that in order to ensure that the solution is 2π -periodic in θ , the eigenvalues are $\lambda = n^2$. When solving the equation for $R(r)$, we do NOT need to throw out solutions which are not bounded as $r \rightarrow 0$. This is because we are working in the annulus where r is bounded away from 0 and ∞ . Therefore, we obtain the general solution

$$u(r, \theta) = (C_0 + C_1 \ln r) + \sum_{n=1}^{\infty} [(C_n r^n + D_n r^{-n}) \cos n\theta + (A_n r^n + B_n r^{-n}) \sin n\theta].$$

But

$$C_0 + \sum_{n=1}^{\infty} [(C_n + D_n) \cos n\theta + (A_n + B_n) \sin n\theta] = \sin \theta$$

and

$$C_0 + \sum_{n=1}^{\infty} [(C_n 2^n + D_n 2^{-n}) \cos n\theta + (A_n 2^n + B_n 2^{-n}) \sin n\theta] = \sin \theta$$

Hence, comparing coefficients we must have

$$\begin{aligned} C_0 &= 0 \\ C_n + D_n &= 0 \\ A_n + B_n &= 0 \quad (n \neq 1) \\ A_1 + B_1 &= 1 \\ C_n 2^n + D_n 2^{-n} &= 0 \\ A_n 2^n + B_n 2^{-n} &= 0 \quad (n \neq 1) \\ 2A_1 + 2^{-1}B_1 &= 1. \end{aligned}$$

Solving these equations we find $C_0 = C_n = D_n = 0$, $A_1 = \frac{1}{3}$, $B_1 = \frac{2}{3}$, and $A_n = B_n = 0$ for $n \neq 1$. Hence, the solution to the problem is

$$u(r, \theta) = \frac{1}{3} \left(r + \frac{2}{r} \right) \sin \theta \blacksquare$$

Example 20.3

Solve Laplace's equation inside a 60° wedge of radius a subject to the boundary conditions

$$u_\theta(r, \theta) = 0, \quad u_\theta(r, \frac{\pi}{3}) = 0, \quad u(a, \theta) = \frac{1}{3} \cos 9\theta - \frac{1}{9} \cos 3\theta.$$

You may assume that the solution remains bounded as $r \rightarrow 0$.

Solution.

Separating the variables we obtain the eigenvalue problem

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0$$

$$\Theta'(0) = \Theta'\left(\frac{\pi}{3}\right) = 0.$$

As above, because of periodicity we expect the solution to be of the form

$$\Theta(\theta) = A \cos \sqrt{\lambda}\theta + B \sin \sqrt{\lambda}\theta.$$

The condition $\Theta'(0) = 0$ implies $B = 0$. The condition $\Theta'\left(\frac{\pi}{3}\right) = 0$ implies $\lambda_n = (3n)^2$, $n = 0, 1, 2, \dots$. Thus, the angular solution is

$$\Theta_n(\theta) = A_n \cos 3n\theta, \quad n = 0, 1, 2, \dots$$

The corresponding solutions of the radial problem are

$$R_n(r) = B_n r^{3n} + C_n r^{-3n}, \quad n = 0, 1, \dots$$

To obtain a solution that remains bounded as $r \rightarrow 0$ we take $C_n = 0$. Hence,

$$u(r, \theta) = \sum_{n=0}^{\infty} D_n r^{3n} \cos 3n\theta, \quad n = 0, 1, 2, \dots$$

Using the boundary condition

$$u(a, \theta) = \frac{1}{3} \cos 9\theta - \frac{1}{9} \cos 3\theta$$

we obtain $D_1 a^3 = -\frac{1}{9}$ and $D_3 a^9 = \frac{1}{3}$ and 0 otherwise. Thus,

$$u(r, \theta) = \frac{1}{3} \left(\frac{r}{a}\right)^9 \cos 9\theta - \frac{1}{9} \left(\frac{r}{a}\right)^3 \cos 3\theta \blacksquare$$

Practice Problems

Problem 20.1

Solve the Laplace's equation as in Example 20.1 in the unit disk with $u(1, \theta) = 3 \sin 5\theta$.

Problem 20.2

Solve the Laplace's equation in the upper half of the unit disk with $u(1, \theta) = \pi - \theta$.

Problem 20.3

Solve the Laplace's equation in the unit disk with $u_r(1, \theta) = 2 \cos 2\theta$.

Problem 20.4

Consider

$$u(r, \theta) = C_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

with

$$\begin{aligned} C_0 &= \frac{a_0}{2} = \frac{1}{2\pi} \int_0^{2\pi} f(\phi) d\phi \\ A_n &= \frac{a_n}{a^n} = \frac{1}{a^n \pi} \int_0^{2\pi} f(\phi) \cos n\phi d\phi, \quad n = 1, 2, \dots \\ B_n &= \frac{b_n}{a^n} = \frac{1}{a^n \pi} \int_0^{2\pi} f(\phi) \sin n\phi d\phi, \quad n = 1, 2, \dots \end{aligned}$$

Using the trigonometric identity

$$\cos a \cos b + \sin a \sin b = \cos(a - b)$$

show that

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} f(\phi) \left[1 + 2 \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\theta - \phi) \right] d\phi.$$

Problem 20.5

(a) Using Euler's formula from complex analysis $e^{it} = \cos t + i \sin t$ show that

$$\cos t = \frac{1}{2}(e^{it} + e^{-it}),$$

where $i = \sqrt{-1}$.

(b) Show that

$$1 + 2 \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\theta - \phi) = 1 + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n e^{in(\theta-\phi)} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n e^{-in(\theta-\phi)}.$$

(c) Let $q_1 = \frac{r}{a}e^{i(\theta-\phi)}$ and $q_2 = \frac{r}{a}e^{-i(\theta-\phi)}$. It is defined in complex analysis that the absolute value of a complex number $z = x + iy$ is given by $|z| = (x^2 + y^2)^{\frac{1}{2}}$. Using these concepts, show that $|q_1| < 1$ and $|q_2| < 1$.

Problem 20.6

(a) Show that

$$\sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n e^{in(\theta-\phi)} = \frac{re^{i(\theta-\phi)}}{a - re^{i(\theta-\phi)}}$$

and

$$\sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n e^{-in(\theta-\phi)} = \frac{re^{-i(\theta-\phi)}}{a - re^{-i(\theta-\phi)}}$$

Hint: Each sum is a geometric series with a ratio less than 1 in absolute value so that these series converges.

(b) Show that

$$1 + 2 \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\theta - \phi) = \frac{a^2 - r^2}{a^2 - 2ar \cos(\theta - \phi) + r^2}.$$

Problem 20.7

Show that

$$u(r, \theta) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{f(\phi)}{a^2 - 2ar \cos(\theta - \phi) + r^2} d\phi.$$

This is known as the **Poisson formula** in polar coordinates.

Problem 20.8

Solve

$$u_{xx} + u_{yy} = 0, \quad x^2 + y^2 < 1$$

subject to

$$u(1, \theta) = \theta, \quad -\pi \leq \theta \leq \pi.$$

Problem 20.9

The vibrations of a symmetric circular membrane where the displacement $u(r, t)$ depends on r and t only can be describe by the one-dimensional wave equation in polar coordinates

$$u_{tt} = c^2 \left(u_{rr} + \frac{1}{r} u_r \right), \quad 0 < r < a, \quad t > 0$$

with initial condition

$$u(a, t) = 0, \quad t > 0$$

and boundary conditions

$$u(r, 0) = f(r), \quad u_t(r, 0) = g(r), \quad 0 < r < a.$$

(a) Show that the assumption $u(r, t) = R(r)T(t)$ leads to the equation

$$\frac{1}{c^2} \frac{T''}{T} = \frac{1}{R} R'' + \frac{1}{r} \frac{R'}{R} = \lambda.$$

(b) Show that $\lambda < 0$.

Problem 20.10

Cartesian coordinates and cylindrical coordinates are shown in Figure 20.1 below.

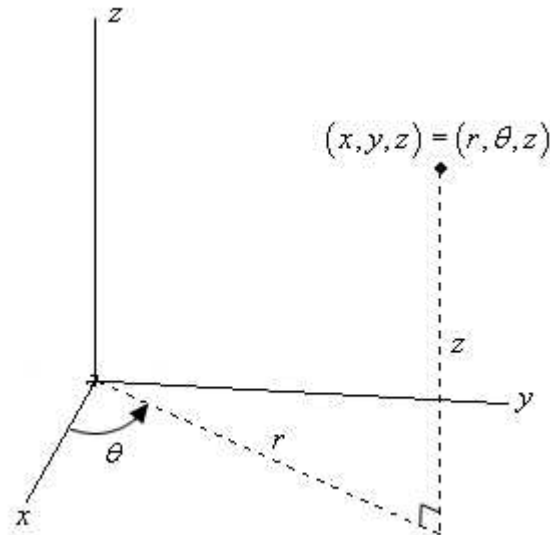


Figure 20.1

- (a) Show that $x = r \cos \theta$, $y = r \sin \theta$, $z = z$.
 (b) Show that

$$u_{xx} + u_{yy} + u_{zz} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz}.$$

Problem 20.11

An important result about harmonic functions is the so-called the **maximum principle** which states: Any harmonic function $u(x, y)$ defined in a domain Ω satisfies the inequality

$$\min_{(x,y) \in \partial\Omega} u(x, y) \leq u(x, y) \leq \max_{(x,y) \in \partial\Omega} u(x, y), \quad \forall (x, y) \in \Omega \cup \partial\Omega$$

where $\partial\Omega$ denotes the boundary of Ω .

Let u be harmonic in $\Omega = \{(x, y) : x^2 + y^2 < 1\}$ and satisfies $u(x, y) = 2 - x$ for all $(x, y) \in \partial\Omega$. Show that $u(x, y) > 0$ for all $(x, y) \in \Omega$.

Problem 20.12

Let u be harmonic in $\Omega = \{(x, y) : x^2 + y^2 < 1\}$ and satisfies $u(x, y) = 1 + 3x$ for all $(x, y) \in \partial\Omega$. Determine

(i) $\max_{(x,y) \in \Omega} u(x, y)$

(ii) $\min_{(x,y) \in \Omega} u(x, y)$

without solving $\Delta u = 0$.

Problem 20.13

Let $u_1(x, y)$ and $u_2(x, y)$ be harmonic functions on a smooth domain Ω such that

$$u_1|_{\partial\Omega} = g_1(x, y) \text{ and } u_2|_{\partial\Omega} = g_2(x, y)$$

where g_1 and g_2 are continuous functions satisfying

$$\max_{(x,y) \in \partial\Omega} g_1(x, y) < \min_{(x,y) \in \partial\Omega} g_2(x, y).$$

Prove that $u_1(x, y) < u_2(x, y)$ for all $(x, y) \in \Omega \cup \partial\Omega$.

Problem 20.14

Show that $r^n \cos(n\theta)$ and $r^n \sin(n\theta)$ satisfy Laplace's equation in polar coordinates.

Problem 20.15

Solve the Dirichlet problem

$$\Delta u = 0, \quad 0 \leq r < a, \quad -\pi \leq \theta \leq \pi$$

$$u(a, \theta) = \sin^2 \theta.$$

Problem 20.16

Solve Laplace's equation

$$u_{xx} + u_{yy} = 0$$

outside a circular disk ($r \geq a$) subject to the boundary condition

$$u(a, \theta) = \ln 2 + 4 \cos 3\theta.$$

You may assume that the solution remains bounded as $r \rightarrow \infty$.

The Laplace Transform

Solutions for PDEs

If in a partial differential equation the time t is one of the independent variables of the searched-for function, we say that the PDE is an **evolution equation**. Examples of evolution equations are the heat equation and the wave equation. In contrast, when the equation involves only spatial independent variables then the equation is called a **stationary equation**. Examples of stationary equations are the Laplace's equations and Poisson equations. There are classes of methods that can be used for solving the initial value or initial boundary problems for evolution equations. We refer to these methods as the methods of **integral transforms**. The fundamental ones are the Laplace and the Fourier transforms. In this chapter we will just consider the Laplace transform.

21 Essentials of the Laplace Transform

Laplace transform has been introduced in an ODE course, and is used especially to solve linear ODEs with constant coefficients, where the equations are transformed to algebraic equations. This idea can be easily extended to PDEs, where the transformation leads to the decrease of the number of independent variables. PDEs in two variables are thus reduced to ODEs. In this section we review the Laplace transform and its properties.

Laplace transform is yet another operational tool for solving constant coefficients linear differential equations. The process of solution consists of three main steps:

- The given “hard” problem is transformed into a “simple” equation.
- This simple equation is solved by purely algebraic manipulations.
- The solution of the simple equation is transformed back to obtain the solution of the given problem.

In this way the Laplace transformation reduces the problem of solving a differential equation to an algebraic problem. The third step is made easier by tables, whose role is similar to that of integral tables in integration.

The above procedure can be summarized by Figure 21.1

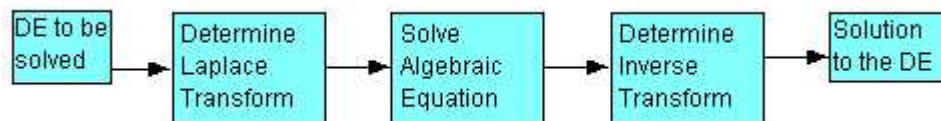


Figure 21.1

In this section we introduce the concept of Laplace transform and discuss some of its properties.

The Laplace transform is defined in the following way. Let $f(t)$ be defined for $t \geq 0$. Then the **Laplace transform** of f , which is denoted by $\mathcal{L}[f(t)]$ or by $F(s)$, is defined by the following equation

$$\mathcal{L}[f(t)] = F(s) = \lim_{T \rightarrow \infty} \int_0^T f(t)e^{-st} dt = \int_0^{\infty} f(t)e^{-st} dt$$

The integral which defines a Laplace transform is an improper integral. An improper integral may **converge** or **diverge**, depending on the integrand. When the improper integral is convergent then we say that the function $f(t)$ possesses a Laplace transform. So what types of functions possess Laplace

transforms, that is, what type of functions guarantees a convergent improper integral.

Example 21.1

Find the Laplace transform, if it exists, of each of the following functions

$$(a) f(t) = e^{at} \quad (b) f(t) = 1 \quad (c) f(t) = t \quad (d) f(t) = e^{t^2}$$

Solution.

(a) Using the definition of Laplace transform we see that

$$\mathcal{L}[e^{at}] = \int_0^{\infty} e^{-(s-a)t} dt = \lim_{T \rightarrow \infty} \int_0^T e^{-(s-a)t} dt.$$

But

$$\int_0^T e^{-(s-a)t} dt = \begin{cases} T & \text{if } s = a \\ \frac{1 - e^{-(s-a)T}}{s-a} & \text{if } s \neq a. \end{cases}$$

For the improper integral to converge we need $s > a$. In this case,

$$\mathcal{L}[e^{at}] = F(s) = \frac{1}{s-a}, \quad s > a.$$

(b) In a similar way to what was done in part (a), we find

$$\mathcal{L}[1] = \int_0^{\infty} e^{-st} dt = \lim_{T \rightarrow \infty} \int_0^T e^{-st} dt = \frac{1}{s}, \quad s > 0.$$

(c) We have

$$\mathcal{L}[t] = \int_0^{\infty} te^{-st} dt = \left[-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_0^{\infty} = \frac{1}{s^2}, \quad s > 0.$$

(d) Again using the definition of Laplace transform we find

$$\mathcal{L}[e^{t^2}] = \int_0^{\infty} e^{t^2-st} dt.$$

If $s \leq 0$ then $t^2 - st \geq 0$ so that $e^{t^2-st} \geq 1$ and this implies that $\int_0^{\infty} e^{t^2-st} dt \geq \int_0^{\infty} dt$. Since the integral on the right is divergent, by the comparison theorem of improper integrals, the integral on the left is also divergent. Now, if $s > 0$ then $\int_0^{\infty} e^{t(t-s)} dt \geq \int_s^{\infty} dt$. By the same reasoning the integral on the

left is divergent. This shows that the function $f(t) = e^{t^2}$ does not possess a Laplace transform ■

The above example raises the question of what class or classes of functions possess a Laplace transform. To answer this question we introduce few mathematical concepts.

A function f that satisfies

$$|f(t)| \leq Me^{at}, \quad t \geq C \quad (21.1)$$

is said to be a function with an **exponential order at infinity**. A function f is called **piecewise continuous** on an interval if the interval can be broken into a finite number of subintervals on which the function is continuous on each open subinterval (i.e. the subinterval without its endpoints) and has a finite limit at the endpoints (**jump discontinuities** and no vertical asymptotes) of each subinterval. Below is a sketch of a piecewise continuous function.



Note that a piecewise continuous function is a function that has a finite number of breaks in it and doesn't blow up to infinity anywhere. A function defined for $t \geq 0$ is said to be **piecewise continuous on the infinite interval** if it is piecewise continuous on $0 \leq t \leq T$ for all $T > 0$.

Example 21.2

Show that the following functions are piecewise continuous and of exponential order at infinity for $t \geq 0$

$$(a) f(t) = t^n \quad (b) f(t) = t^n \sin at$$

Solution.

(a) Since $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \geq \frac{t^n}{n!}$, $t^n \leq n!e^t$. Hence, t^n is piecewise continuous and of exponential order at infinity.

(b) Since $|t^n \sin at| \leq n!e^t$, $t^n \sin at$ is piecewise continuous and of exponential order at infinity ■

The following is an existence result of Laplace transform.

Theorem 21.1

Suppose that $f(t)$ is piecewise continuous on $t \geq 0$ and has an exponential order at infinity with $|f(t)| \leq Me^{at}$ for $t \geq C$. Then the Laplace transform

$$F(s) = \int_0^{\infty} f(t)e^{-st} dt$$

exists as long as $s > a$. Note that the two conditions above are sufficient, but not necessary, for $F(s)$ to exist.

In what follows, we will denote the class of all piecewise continuous functions with exponential order at infinity by \mathcal{PE} . The next theorem shows that any linear combination of functions in \mathcal{PE} is also in \mathcal{PE} . The same is true for the product of two functions in \mathcal{PE} .

Theorem 21.2

Suppose that $f(t)$ and $g(t)$ are two elements of \mathcal{PE} with

$$|f(t)| \leq M_1 e^{a_1 t}, \quad t \geq C_1 \quad \text{and} \quad |g(t)| \leq M_2 e^{a_2 t}, \quad t \geq C_2.$$

(i) For any constants α and β the function $\alpha f(t) + \beta g(t)$ is also a member of \mathcal{PE} . Moreover

$$\mathcal{L}[\alpha f(t) + \beta g(t)] = \alpha \mathcal{L}[f(t)] + \beta \mathcal{L}[g(t)].$$

(ii) The function $h(t) = f(t)g(t)$ is an element of \mathcal{PE} .

We next discuss the problem of how to determine the function $f(t)$ if $F(s)$ is given. That is, how do we invert the transform. The following result on uniqueness provides a possible answer. This result establishes a one-to-one correspondence between the set \mathcal{PE} and its Laplace transforms. Alternatively, the following theorem asserts that the Laplace transform of a member in \mathcal{PE} is unique.

Theorem 21.3

Let $f(t)$ and $g(t)$ be two elements in \mathcal{PE} with Laplace transforms $F(s)$ and $G(s)$ such that $F(s) = G(s)$ for some $s > a$. Then $f(t) = g(t)$ for all $t \geq 0$ where both functions are continuous.

With the above theorem, we can now officially define the inverse Laplace transform as follows: For a piecewise continuous function f of exponential order at infinity whose Laplace transform is F , we call f the **inverse Laplace transform** of F and write $f = \mathcal{L}^{-1}[F(s)]$. Symbolically

$$f(t) = \mathcal{L}^{-1}[F(s)] \iff F(s) = \mathcal{L}[f(t)].$$

Example 21.3

Find $\mathcal{L}^{-1}\left(\frac{1}{s-1}\right)$, $s > 1$.

Solution.

From Example 21.1(a), we have that $\mathcal{L}[e^{at}] = \frac{1}{s-a}$, $s > a$. In particular, for $a = 1$ we find that $\mathcal{L}[e^t] = \frac{1}{s-1}$, $s > 1$. Hence, $\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = e^t$, $t \geq 0$ ■.

The above theorem states that if $f(t)$ is continuous and has a Laplace transform $F(s)$, then there is no other function that has the same Laplace transform. To find $\mathcal{L}^{-1}[F(s)]$, we can inspect tables of Laplace transforms of known functions to find a particular $f(t)$ that yields the given $F(s)$.

When the function $f(t)$ is not continuous, the uniqueness of the inverse Laplace transform is not assured. The following example addresses the uniqueness issue.

Example 21.4

Consider the two functions $f(t) = H(t)H(3-t)$ and $g(t) = H(t) - H(t-3)$, where H is the Heaviside function defined by

$$H(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

- (a) Are the two functions identical?
 (b) Show that $\mathcal{L}[f(t)] = \mathcal{L}[g(t)]$.

Solution.

(a) We have

$$f(t) = \begin{cases} 1, & 0 \leq t \leq 3 \\ 0, & t > 3 \end{cases}$$

and

$$g(t) = \begin{cases} 1, & 0 \leq t < 3 \\ 0, & t \geq 3 \end{cases}$$

Since $f(3) = 1$ and $g(3) = 0$, f and g are not identical.

(b) We have

$$\mathcal{L}[f(t)] = \mathcal{L}[g(t)] = \int_0^3 e^{-st} dt = \frac{1 - e^{-3s}}{s}, s > 0.$$

Thus, both functions $f(t)$ and $g(t)$ have the same Laplace transform even though they are not identical. However, they are equal on the interval(s) where they are both continuous ■

The inverse Laplace transform possesses a linear property as indicated in the following result.

Theorem 21.4

Given two Laplace transforms $F(s)$ and $G(s)$ then

$$\mathcal{L}^{-1}[aF(s) + bG(s)] = a\mathcal{L}^{-1}[F(s)] + b\mathcal{L}^{-1}[G(s)]$$

for any constants a and b .

Convolution integrals are useful when finding the inverse Laplace transform of products. They are defined as follows: The **convolution** of two scalar piecewise continuous functions $f(t)$ and $g(t)$ defined for $t \geq 0$ is the integral

$$(f * g)(t) = \int_0^t f(t-s)g(s)ds.$$

Example 21.5

Find $f * g$ where $f(t) = e^{-t}$ and $g(t) = \sin t$.

Solution.

Using integration by parts twice we arrive at

$$\begin{aligned} (f * g)(t) &= \int_0^t e^{-(t-s)} \sin s ds \\ &= \frac{1}{2} [e^{-(t-s)}(\sin s - \cos s)]_0^t \\ &= \frac{e^{-t}}{2} + \frac{1}{2}(\sin t - \cos t) \quad \blacksquare \end{aligned} \tag{21.2}$$

Next, we state several properties of convolution product, which resemble those of ordinary product.

Theorem 21.5

Let $f(t)$, $g(t)$, and $k(t)$ be three piecewise continuous scalar functions defined for $t \geq 0$ and c_1 and c_2 are arbitrary constants. Then

- (i) $f * g = g * f$ (Commutative Law)
- (ii) $(f * g) * k = f * (g * k)$ (Associative Law)
- (iii) $f * (c_1g + c_2k) = c_1f * g + c_2f * k$ (Distributive Law)

Example 21.6

Express the solution to the initial value problem $y' + \alpha y = g(t)$, $y(0) = y_0$ in terms of a convolution integral.

Solution.

Solving this initial value problem by the method of integrating factor we find

$$y(t) = e^{-\alpha t}y_0 + \int_0^t e^{-\alpha(t-s)}g(s)ds = e^{-\alpha t}y_0 + (e^{-\alpha t} * g)(t) \blacksquare$$

The following theorem, known as the Convolution Theorem, provides a way for finding the Laplace transform of a convolution integral and also finding the inverse Laplace transform of a product.

Theorem 21.6

If $f(t)$ and $g(t)$ are piecewise continuous for $t \geq 0$, and of exponential order at infinity then

$$\mathcal{L}[(f * g)(t)] = \mathcal{L}[f(t)]\mathcal{L}[g(t)] = F(s)G(s).$$

Thus, $(f * g)(t) = \mathcal{L}^{-1}[F(s)G(s)]$.

Example 21.7

Use the convolution theorem to find the inverse Laplace transform of

$$P(s) = \frac{1}{(s^2 + a^2)^2}.$$

Solution.

Note that

$$P(s) = \left(\frac{1}{s^2 + a^2} \right) \left(\frac{1}{s^2 + a^2} \right).$$

So, in this case we have, $F(s) = G(s) = \frac{1}{s^2 + a^2}$ so that $f(t) = g(t) = \frac{1}{a} \sin(at)$.

Thus,

$$(f * g)(t) = \frac{1}{a^2} \int_0^t \sin(at - as) \sin(as) ds = \frac{1}{2a^3} (\sin(at) - at \cos(at)) \blacksquare$$

Example 21.8

Solve the initial value problem

$$4y'' + y = g(t), \quad y(0) = 3, \quad y'(0) = -7$$

Solution.

Take the Laplace transform of all the terms and plug in the initial conditions to obtain

$$4(s^2Y(s) - 3s + 7) + Y(s) = G(s)$$

or

$$(4s^2 + 1)Y(s) - 12s + 28 = G(s).$$

Solving for $Y(s)$ we find

$$\begin{aligned} Y(s) &= \frac{12s - 28}{4\left(s^2 + \frac{1}{4}\right)} + \frac{G(s)}{4\left(s^2 + \frac{1}{4}\right)} \\ &= \frac{3s}{s^2 + \left(\frac{1}{2}\right)^2} - 7\frac{\left(\frac{1}{2}\right)^2}{s^2 + \left(\frac{1}{2}\right)^2} + \frac{1}{4}G(s)\frac{\left(\frac{1}{2}\right)^2}{s^2 + \left(\frac{1}{2}\right)^2} \end{aligned}$$

Hence,

$$y(t) = 3 \cos\left(\frac{t}{2}\right) - 7 \sin\left(\frac{t}{2}\right) + \frac{1}{2} \int_0^t \sin\left(\frac{s}{2}\right) g(t-s) ds.$$

So, once we decide on a $g(t)$ all we need to do is to evaluate the integral and we'll have the solution ■

We conclude this section with the following table of Laplace transform pairs where H is the Heaviside function defined by $H(t) = 1$ for $t \geq 0$ and 0 otherwise.

$f(t)$	$F(s)$
$H(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$	$\frac{1}{s}, s > 0$
$t^n, n = 1, 2, \dots$	$\frac{n!}{s^{n+1}}, s > 0$
$e^{\alpha t}$	$\frac{1}{s-\alpha}, s > \alpha$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}, s > 0$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}, s > 0$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}, s > \omega $
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}, s > \omega $
$e^{\alpha t} f(t), \text{ with } f(t) \leq M e^{at}$	$F(s-\alpha), s > \alpha + a$
$e^{\alpha t} H(t)$	$\frac{1}{s-\alpha}, s > \alpha$
$e^{\alpha t} t^n, n = 1, 2, \dots$	$\frac{n!}{(s-\alpha)^{n+1}}, s > \alpha$
$e^{\alpha t} \sin(\omega t)$	$\frac{\omega}{(s-\alpha)^2+\omega^2}, s > \alpha$
$e^{\alpha t} \cos(\omega t)$	$\frac{s-\alpha}{(s-\alpha)^2+\omega^2}, s > \alpha$
$f(t-\alpha)H(t-\alpha), \alpha \geq 0$ with $ f(t) \leq M e^{at}$	$e^{-\alpha s} F(s), s > a$
$H(t-\alpha), \alpha \geq 0$	$\frac{e^{-\alpha s}}{s}, s > 0$
$t f(t)$	$-F'(s)$
$\frac{t}{2\omega} \sin \omega t$	$\frac{s}{(s^2+\omega^2)^2}, s > 0$
$\frac{1}{2\omega^3} [\sin \omega t - \omega t \cos \omega t]$	$\frac{1}{(s^2+\omega^2)^2}, s > 0$
$f'(t), \text{ with } f(t) \text{ continuous}$ and $ f'(t) \leq M e^{at}$	$sF(s) - f(0)$ $s > \max\{a, 0\} + 1$
$f''(t), \text{ with } f'(t) \text{ continuous}$ and $ f''(t) \leq M e^{at}$	$s^2 F(s) - s f(0) - f'(0)$ $s > \max\{a, 0\} + 1$
$f^{(n)}(t), \text{ with } f^{(n-1)}(t) \text{ continuous}$ and $ f^{(n)}(t) \leq M e^{at}$	$s^n F(s) - s^{n-1} f(0) - \dots$ $-s f^{(n-2)}(0) - f^{(n-1)}(0)$ $s > \max\{a, 0\} + 1$
$\frac{2}{2\sqrt{\pi}} \int_{\frac{\alpha}{2\sqrt{t}}}^{\infty} e^{-u^2} du$	$\frac{e^{-\alpha\sqrt{s}}}{s}$
$\int_0^t f(u) du, \text{ with } f(t) \leq M e^{at}$	$\frac{F(s)}{s}, s > \max\{a, 0\} + 1$

Table \mathcal{L}

Practice Problems

Problem 21.1

Determine whether the integral $\int_0^\infty \frac{1}{1+t^2} dt$ converges. If the integral converges, give its value.

Problem 21.2

Determine whether the integral $\int_0^\infty \frac{t}{1+t^2} dt$ converges. If the integral converges, give its value.

Problem 21.3

Determine whether the integral $\int_0^\infty e^{-t} \cos(e^{-t}) dt$ converges. If the integral converges, give its value.

Problem 21.4

Using the definition, find $\mathcal{L}[e^{3t}]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

Problem 21.5

Using the definition, find $\mathcal{L}[t - 5]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

Problem 21.6

Using the definition, find $\mathcal{L}[e^{(t-1)^2}]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

Problem 21.7

Using the definition, find $\mathcal{L}[(t - 2)^2]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

Problem 21.8

Using the definition, find $\mathcal{L}[f(t)]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

$$f(t) = \begin{cases} 0, & 0 \leq t < 1 \\ t - 1, & t \geq 1 \end{cases}$$

Problem 21.9

Using the definition, find $\mathcal{L}[f(t)]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

$$f(t) = \begin{cases} 0, & 0 \leq t < 1 \\ t - 1, & 1 \leq t < 2 \\ 0, & t \geq 2. \end{cases}$$

Problem 21.10

Let n be a positive integer. Using integration by parts establish the reduction formula

$$\int t^n e^{-st} dt = -\frac{t^n e^{-st}}{s} + \frac{n}{s} \int t^{n-1} e^{-st} dt, \quad s > 0.$$

Problem 21.11

For $s > 0$ and n a positive integer evaluate the limits

$$(a) \lim_{t \rightarrow 0} t^n e^{-st} \quad (b) \lim_{t \rightarrow \infty} t^n e^{-st}$$

Problem 21.12

Use the linearity property of Laplace transform to find $\mathcal{L}[5e^{-7t} + t + 2e^{2t}]$. Find the domain of $F(s)$.

Problem 21.13

Find $\mathcal{L}^{-1}\left(\frac{3}{s-2}\right)$.

Problem 21.14

Find $\mathcal{L}^{-1}\left(-\frac{2}{s^2} + \frac{1}{s+1}\right)$.

Problem 21.15

Find $\mathcal{L}^{-1}\left(\frac{2}{s+2} + \frac{2}{s-2}\right)$.

Problem 21.16

Use Table \mathcal{L} to find $\mathcal{L}[2e^t + 5]$.

Problem 21.17

Use Table \mathcal{L} to find $\mathcal{L}[e^{3t-3}H(t-1)]$.

Problem 21.18

Use Table \mathcal{L} to find $\mathcal{L}[\sin^2 \omega t]$.

Problem 21.19

Use Table \mathcal{L} to find $\mathcal{L}[\sin 3t \cos 3t]$.

Problem 21.20

Use Table \mathcal{L} to find $\mathcal{L}[e^{2t} \cos 3t]$.

Problem 21.21

Use Table \mathcal{L} to find $\mathcal{L}[e^{4t}(t^2 + 3t + 5)]$.

Problem 21.22

Use Table \mathcal{L} to find $\mathcal{L}^{-1}\left[\frac{10}{s^2+25} + \frac{4}{s-3}\right]$.

Problem 21.23

Use Table \mathcal{L} to find $\mathcal{L}^{-1}\left[\frac{5}{(s-3)^4}\right]$.

Problem 21.24

Use Table \mathcal{L} to find $\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s-9}\right]$.

Problem 21.25

Using the partial fraction decomposition find $\mathcal{L}^{-1}\left[\frac{12}{(s-3)(s+1)}\right]$.

Problem 21.26

Using the partial fraction decomposition find $\mathcal{L}^{-1}\left[\frac{24e^{-5s}}{s^2-9}\right]$.

Problem 21.27

Use Laplace transform technique to solve the initial value problem

$$y' + 4y = g(t), \quad y(0) = 2$$

where

$$g(t) = \begin{cases} 0, & 0 \leq t < 1 \\ 12, & 1 \leq t < 3 \\ 0, & t \geq 3 \end{cases}$$

Problem 21.28

Use Laplace transform technique to solve the initial value problem

$$y'' - 4y = e^{3t}, \quad y(0) = 0, \quad y'(0) = 0.$$

Problem 21.29

Consider the functions $f(t) = e^t$ and $g(t) = e^{-2t}$, $t \geq 0$. Compute $f * g$ in two different ways.

(a) By directly evaluating the integral.

(b) By computing $\mathcal{L}^{-1}[F(s)G(s)]$ where $F(s) = \mathcal{L}[f(t)]$ and $G(s) = \mathcal{L}[g(t)]$.

Problem 21.30

Consider the functions $f(t) = \sin t$ and $g(t) = \cos t$, $t \geq 0$. Compute $f * g$ in two different ways.

(a) By directly evaluating the integral.

(b) By computing $\mathcal{L}^{-1}[F(s)G(s)]$ where $F(s) = \mathcal{L}[f(t)]$ and $G(s) = \mathcal{L}[g(t)]$.

Problem 21.31

Compute $t * t * t$.

Problem 21.32

Compute $H(t) * e^{-t} * e^{-2t}$.

Problem 21.33

Compute $t * e^{-t} * e^t$.

22 Solving PDEs Using Laplace Transform

The same idea for solving linear ODEs using Laplace transform can be exploited when solving PDEs for functions in two variables $u = u(x, t)$. The transformation will be done with respect to the time variable $t \geq 0$, the spatial variable x will be treated as a parameter unaffected by this transform. In particular we define the Laplace transform of $u(x, t)$ by the formula

$$\mathcal{L}(u(x, t)) = U(x, s) = \int_0^{\infty} u(x, t)e^{-st} dt.$$

The time derivatives are transformed in the same way as in the case of functions in one variable, that is, for example

$$\mathcal{L}(u_t)(x, t) = sU(x, s) - u(x, 0)$$

and

$$\mathcal{L}(u_{tt})(x, s) = s^2U(x, s) - su(x, 0) - u_t(x, 0).$$

The spatial derivatives remain unchanged, for example,

$$\mathcal{L}u_x(x, t) = \int_0^{\infty} u_x(x, \tau)e^{-s\tau} d\tau = \frac{\partial}{\partial x} \int_0^{\infty} u(x, \tau)e^{-s\tau} d\tau = U_x(x, s).$$

Likewise, we have

$$\mathcal{L}u_{xx}(x, t) = U_{xx}(x, s).$$

Thus, applying the Laplace transform to a PDE in two variables x and t we obtain an ODE in the variable x and with the parameter s .

Example 22.1

Let $u(x, t)$ be the concentration of a chemical contaminant dissolved in a liquid on a half-infinite domain $x > 0$. Let us assume that at time $t = 0$ the concentration is 0 and on the boundary $x = 0$, constant unit concentration of the contaminant is kept for $t > 0$. The behaviour of this problem is described by the following mathematical model

$$\begin{cases} u_t - u_{xx} = 0 & , x > 0, t > 0 \\ u(x, 0) = 0, \\ u(0, t) = 1, \\ |u(x, t)| < \infty. \end{cases}$$

Find $u(x, t)$.

Solution.

Applying Laplace transform to both sides of the equation we obtain

$$sU(x, s) - u(x, 0) - U_{xx}(x, s) = 0$$

or

$$U_{xx}(x, s) - sU(x, s) = 0.$$

This is a second order linear ODE in the variable x and positive parameter s . Its general solution is

$$U(x, s) = A(s)e^{-\sqrt{s}x} + B(s)e^{\sqrt{s}x}.$$

Since $U(x, s)$ is bounded in both variables, we must have $B(s) = 0$ and in this case we obtain

$$U(x, s) = A(s)e^{-\sqrt{s}x}.$$

Next, we apply Laplace transform to the boundary condition obtaining

$$U(0, s) = \mathcal{L}(1) = \frac{1}{s}.$$

This leads to $A(s) = \frac{1}{s}$ and the transformed solution becomes

$$U(x, s) = \frac{1}{s}e^{-\sqrt{s}x}.$$

Thus,

$$u(x, t) = \mathcal{L}^{-1}\left(\frac{1}{s}e^{-\sqrt{s}x}\right) = \frac{2}{\sqrt{\pi}} \int_{\frac{x}{2\sqrt{t}}}^{\infty} e^{-w^2} dw \blacksquare$$

Example 22.2

Solve the following initial boundary value problem

$$\begin{cases} u_t - u_{xx} = 0 & , x > 0, t > 0 \\ u(x, 0) = 0, \\ u(0, t) = f(t), \\ |u(x, t)| < \infty. \end{cases}$$

Solution.

Following the argument of the previous example we find

$$U(x, s) = F(s)e^{-\sqrt{s}x}, \quad F(s) = \mathcal{L}f(t).$$

Thus, using Theorem 21.6 we can write

$$u(x, t) = \mathcal{L}^{-1} \left(F(s) e^{-\sqrt{s}x} \right) = f * \mathcal{L}^{-1}(e^{-\sqrt{s}x}).$$

It can be shown that

$$\mathcal{L}^{-1}(e^{-\sqrt{s}x}) = \frac{x}{\sqrt{4\pi t^3}} e^{-\frac{x^2}{4t}}.$$

Hence,

$$u(x, t) = \int_0^t \frac{x}{\sqrt{4\pi(t-s)^3}} e^{-\frac{x^2}{4(t-s)}} f(s) ds \blacksquare$$

Example 22.3

Solve the wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & , x > 0, t > 0 \\ u(x, 0) = u_t(x, 0) = 0, \\ u(0, t) = f(t), \\ |u(x, t)| < \infty. \end{cases}$$

Solution.

Applying Laplace transform to both sides of the equation we obtain

$$s^2 U(x, s) - su(x, 0) - u_t(x, 0) - c^2 U_{xx}(x, s) = 0$$

or

$$c^2 U_{xx}(x, s) - s^2 U(x, s) = 0.$$

This is a second order linear ODE in the variable x and positive parameter s . Its general solution is

$$U(x, s) = A(s) e^{-\frac{s}{c}x} + B(s) e^{\frac{s}{c}x}.$$

Since $U(x, s)$ is bounded, we must have $B(s) = 0$ and in this case we obtain

$$U(x, s) = A(s) e^{-\frac{s}{c}x}.$$

Next, we apply Laplace transform to the boundary condition obtaining

$$U(0, s) = \mathcal{L}(f(t)) = F(s).$$

This leads to $A(s) = F(s)$ and the transformed solution becomes

$$U(x, s) = F(s)e^{-\frac{s}{c}x}.$$

Thus,

$$u(x, t) = \mathcal{L}^{-1} \left(F(s)e^{-\frac{x}{c}s} \right) = H \left(t - \frac{x}{c} \right) f \left(t - \frac{x}{c} \right) \blacksquare$$

Remark 22.1

Laplace transforms are useful in solving parabolic and some hyperbolic PDEs. They are not in general useful in solving elliptic PDEs.

Practice Problems

Problem 22.1

Solve by Laplace transform

$$\begin{cases} u_t + u_x = 0 & , x > 0, t > 0 \\ u(x, 0) = \sin x, \\ u(0, t) = 0 \end{cases}$$

Hint: Method of integrating factor of ODEs.

Problem 22.2

Solve by Laplace transform

$$\begin{cases} u_t + u_x = -u & , x > 0, t > 0 \\ u(x, 0) = \sin x, \\ u(0, t) = 0 \end{cases}$$

Problem 22.3

Solve

$$\begin{aligned} u_t &= 4u_{xx} \\ u(0, t) &= u(1, t) = 0 \\ u(x, 0) &= 2 \sin \pi x + 6 \sin 2\pi x. \end{aligned}$$

Hint: A particular solution of a second order ODE must be found using the method of variation of parameters.

Problem 22.4

Solve by Laplace transform

$$\begin{cases} u_t - u_x = u & , x > 0, t > 0 \\ u(x, 0) = e^{-5x}, \\ |u(x, t)| < \infty \end{cases}$$

Problem 22.5

Solve by Laplace transform

$$\begin{cases} u_t + u_x = t & , x > 0, t > 0 \\ u(x, 0) = 0, \\ u(0, t) = t^2 \end{cases}$$

Problem 22.6

Solve by Laplace transform

$$\begin{cases} xu_t + u_x = 0 & , x > 0, t > 0 \\ u(x, 0) = 0, \\ u(0, t) = t \end{cases}$$

Problem 22.7

Solve by Laplace transform

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & , x > 0, t > 0 \\ u(x, 0) = u_t(x, 0) = 0, \\ u(0, t) = \sin t, \\ |u(x, t)| < \infty \end{cases}$$

Problem 22.8

Solve by Laplace transform

$$u_{tt} - 9u_{xx} = 0, \quad 0 \leq x \leq \pi, \quad t > 0$$

$$u(0, t) = u(\pi, t) = 0,$$

$$u_t(x, 0) = 0, \quad u(x, 0) = 2 \sin x.$$

Problem 22.9

Solve by Laplace transform

$$\begin{cases} u_{xy} = 1 & , x > 0, y > 0 \\ u(x, 0) = 1, \\ u(0, y) = y + 1. \end{cases}$$

Problem 22.10

Solve by Laplace transform

$$\begin{cases} u_{tt} = c^2 u_{xx} & , x > 0, t > 0 \\ u(x, 0) = u_t(x, 0) = 0, \\ u_x(0, t) = f(t), \\ |u(x, t)| < \infty. \end{cases}$$

Problem 22.11

Solve by Laplace transform

$$\begin{cases} u_t + u_x = u & , x > 0, t > 0 \\ u(x, 0) = \sin x, \\ u(0, t) = 0 \end{cases}$$

Problem 22.12

Solve by Laplace transform

$$\begin{cases} u_t - c^2 u_{xx} = 0 & , x > 0, t > 0 \\ u(x, 0) = T, \\ u(0, t) = 0, \\ |u(x, t)| < \infty \end{cases}$$

Problem 22.13

Solve by Laplace transform

$$\begin{aligned} u_t - 3u_{xx} &= 0, \quad 0 \leq x \leq 2, \quad t > 0 \\ u(0, t) &= u(2, t) = 0, \\ u(x, 0) &= 5 \sin(\pi x) \end{aligned}$$

Problem 22.14

Solve by Laplace transform

$$\begin{aligned} u_t - 4u_{xx} &= 0, \quad 0 \leq x \leq \pi, \quad t > 0 \\ u_x(0, t) &= u(\pi, t) = 0, \\ u(x, 0) &= 40 \cos \frac{x}{2} \end{aligned}$$

Problem 22.15

Solve by Laplace transform

$$\begin{aligned} u_{tt} - 4u_{xx} &= 0, \quad 0 \leq x \leq 2, \quad t > 0 \\ u(0, t) &= u(2, t) = 0, \\ u_t(x, 0) &= 0, \quad u(x, 0) = 3 \sin \pi x. \end{aligned}$$

The Fourier Transform

Solutions for PDEs

In the previous chapter we discussed one class of integral transform methods, the Laplace transform. In this chapter, we consider a second fundamental class of integral transform methods, the so-called **Fourier transform**.

Fourier series are designed to solve boundary value problems on bounded intervals. The extension of Fourier methods to the entire real line leads naturally to the Fourier transform, an extremely powerful mathematical tool for the analysis of non-periodic functions. The Fourier transform is of fundamental importance in a broad range of applications, including both ordinary and partial differential equations, quantum mechanics, signal processing, control theory, and probability, to name but a few.

23 Complex Version of Fourier Series

We have seen in Section 15 that a $2L$ -periodic function $f : \mathbb{R} \rightarrow \mathbb{R}$ that is piecewise smooth on $[-L, L]$ can be expanded in a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right)$$

at all points of continuity of f . In the context of Fourier analysis, this is referred to as the **real form** of the Fourier series. It is often convenient to recast this series in **complex form** by means of Euler formula

$$e^{ix} = \cos x + i \sin x.$$

It follows from this formula that

$$e^{ix} + e^{-ix} = 2 \cos x \quad \text{and} \quad e^{ix} - e^{-ix} = 2i \sin x$$

or

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \text{and} \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

Hence the Fourier expansion of f can be rewritten as

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \left(\frac{e^{\frac{in\pi x}{L}} + e^{-\frac{in\pi x}{L}}}{2} \right) \right. \\ &\quad \left. + b_n \left(\frac{e^{\frac{in\pi x}{L}} - e^{-\frac{in\pi x}{L}}}{2i} \right) \right] \\ f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}} \end{aligned} \tag{23.1}$$

where $c_0 = \frac{a_0}{2}$ and for $n \in \mathbb{N}$ we have

$$c_n = \frac{a_n - ib_n}{2}$$

$$c_{-n} = \frac{a_n + ib_n}{2}.$$

It follows that if $n \in \mathbb{N}$ then

$$a_n = c_n + c_{-n} \quad \text{and} \quad b_n = i(c_n - c_{-n}). \quad (23.2)$$

That is, a_n and b_n can be easily found once we have formulas for c_n . In order to find these formulas, we need to evaluate the following integral

$$\begin{aligned} \int_{-L}^L e^{\frac{in\pi x}{L}} e^{-\frac{im\pi x}{L}} dx &= \int_{-L}^L e^{\frac{i(n-m)\pi x}{L}} dx \\ &= \frac{L}{i(n-m)\pi} \left[e^{\frac{i(n-m)\pi x}{L}} \right]_{-L}^L \\ &= -\frac{iL}{(n-m)\pi} [\cos[(n-m)\pi] + i \sin[(n-m)\pi] \\ &\quad - \cos[-(n-m)\pi] - i \sin[-(n-m)\pi]] \\ &= 0 \end{aligned}$$

if $n \neq m$. If $n = m$ then

$$\int_{-L}^L e^{\frac{in\pi x}{L}} e^{-\frac{in\pi x}{L}} dx = 2L.$$

Now, if we multiply (23.1) by $e^{-\frac{in\pi x}{L}}$ and integrate from $-L$ to L and apply the last result we find

$$\int_{-L}^L f(x) e^{-\frac{in\pi x}{L}} dx = 2Lc_n$$

which yields the formula for coefficients of the complex form of the Fourier series:

$$c_n = \frac{1}{2L} \int_{-L}^L f(x) e^{-\frac{in\pi x}{L}} dx, \quad n = 0, \pm 1, \pm 2, \dots$$

Example 23.1

Find the complex Fourier coefficients of the function

$$f(x) = x, \quad -\pi \leq x \leq \pi$$

extended to be periodic of period 2π .

Solution.

Using integration by parts and the fact that $e^{i\pi} = e^{-i\pi} = -1$ we find

$$\begin{aligned}
 c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x e^{-inx} dx \\
 &= \frac{1}{2\pi} \left[\left(\frac{ix}{n} \right) e^{-inx} \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \left(\frac{i}{n} \right) e^{-inx} dx \right] \\
 &= \frac{1}{2\pi} \left[\left(\frac{i\pi}{n} \right) e^{-in\pi} + \left(\frac{i\pi}{n} \right) e^{in\pi} \right] \\
 &\quad + \frac{1}{2\pi} \left[\frac{1}{n^2} e^{-in\pi} - \frac{1}{n^2} e^{in\pi} \right] \\
 &= \frac{1}{2\pi} \left[2i \frac{\pi}{n} (-1)^n \right] + \frac{1}{2\pi} (0) = \frac{(-1)^n i}{n}
 \end{aligned}$$

for $n \in \mathbb{N}$ and for $n = 0$, we have

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x dx = 0 \blacksquare$$

Remark 23.1

It is often the case that the complex form of the Fourier series is far simpler to calculate than the real form. One can then use (23.2) to find the real form of the Fourier series. For example, the Fourier coefficients of the real form of the previous function are given by

$$a_n = (c_n + c_{-n}) = 0 \text{ and } b_n = i(c_n - c_{-n}) = \frac{2}{n}(-1)^{n+1}, \quad n \in \mathbb{N}$$

Practice Problems

Problem 23.1

Find the complex Fourier coefficients of the function

$$f(x) = x, \quad -1 \leq x \leq 1$$

extended to be periodic of period 2.

Problem 23.2

Let

$$f(x) = \begin{cases} 0 & -\pi < x < \frac{-\pi}{2} \\ 1 & \frac{-\pi}{2} < x < \frac{\pi}{2} \\ 0 & \pi < x < \pi \end{cases}$$

be 2π -periodic. Find its complex series representation.

Problem 23.3

Find the complex Fourier series of the 2π -periodic function $f(x) = e^{ax}$ over the interval $(-\pi, \pi)$.

Problem 23.4

Find the complex Fourier series of the 2π -periodic function $f(x) = \sin x$ over the interval $(-\pi, \pi)$.

Problem 23.5

Find the complex Fourier series of the 2π -periodic function defined

$$f(x) = \begin{cases} 1 & 0 < x < T \\ 0 & T < x < 2\pi \end{cases}$$

Problem 23.6

Let $f(x) = x^2$, $-\pi < x < \pi$, be 2π -periodic.

- Calculate the complex Fourier series representation of f .
- Using the complex Fourier series found in (a), recover the real Fourier series representation of f .

Problem 23.7

Let $f(x) = \sin \pi x$, $-\frac{1}{2} < x < \frac{1}{2}$, be of period 1.

- Calculate the coefficients a_n , b_n and c_n .
- Find the complex Fourier series representation of f .

Problem 23.8

Let $f(x) = 2 - x$, $-2 < x < 2$, be of period 4.

(a) Calculate the coefficients a_n , b_n and c_n .

(b) Find the complex Fourier series representation of f .

Problem 23.9

Suppose that the coefficients c_n of the complex Fourier series are given by

$$c_n = \begin{cases} \frac{2}{i\pi n} & \text{if } |n| \text{ is odd} \\ 0 & \text{if } |n| \text{ is even.} \end{cases}$$

Find a_n , $n = 0, 1, 2, \dots$ and b_n , $n = 1, 2, \dots$.

Problem 23.10

Recall that any complex number z can be written as $z = Re(z) + iIm(z)$ where $Re(z)$ is called the real part of z and $Im(z)$ is called the **imaginary part**. The **complex conjugate** of z is the complex number $\bar{z} = Re(z) - iIm(z)$. Using these definitions show that $a_n = 2Re(c_n)$ and $b_n = -2Im(c_n)$.

Problem 23.11

Suppose that

$$c_n = \begin{cases} \frac{i}{2\pi n} [e^{-inT} - 1] & \text{if } n \neq 0 \\ \frac{T}{2\pi} & \text{if } n = 0. \end{cases}$$

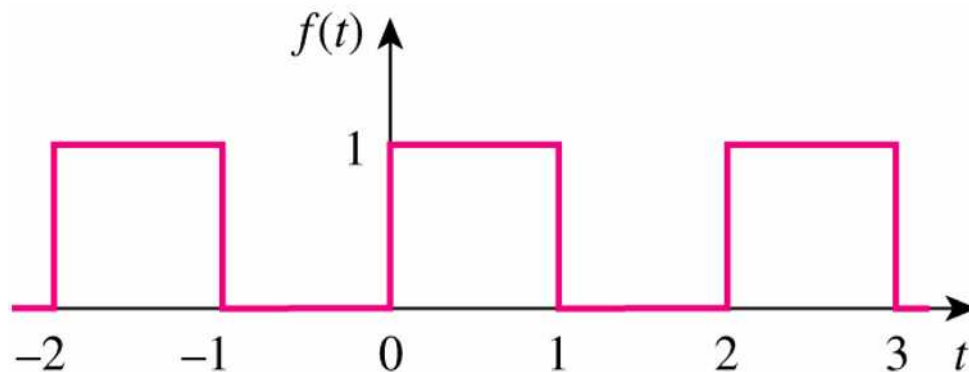
Find a_n and b_n .

Problem 23.12

Find the complex Fourier series of the function $f(x) = e^x$ on $[-2, 2]$.

Problem 23.13

Consider the wave form



- (a) Write $f(x)$ explicitly. What is the period of f .
- (b) Determine a_0 and a_n for $n \in \mathbb{N}$.
- (c) Determine b_n for $n \in \mathbb{N}$.
- (d) Determine c_0 and c_n for $n \in \mathbb{N}$.

Problem 23.14

If z is a complex number we define $\sin z = \frac{1}{2}(e^{iz} - e^{-iz})$. Find the complex form of the Fourier series for $\sin 3x$ without evaluating any integrals.

24 An introduction to Fourier Transforms

One of the problems with the theory of Fourier series discussed so far is that it applies only to periodic functions. There are many times when one would like to divide a function which is not periodic into a superposition of sines and cosines. The Fourier transform is the tool often used for this purpose. Like the Laplace transform, the Fourier transform is often an effective tool in finding explicit solutions to partial differential equations.

We will introduce the Fourier transform of $f(x)$ as a limiting case of a Fourier series. This requires a tedious discussion which we omit and rather explain the underlying ideas. More specifically, the approach we introduce is to construct Fourier series of $f(x)$ on progressively longer and longer intervals, and then take the limit as their lengths go to infinity. This limiting process converts the Fourier sums into integrals, and the resulting representation of a function is renamed the Fourier transform.

To start with, let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a piecewise continuous function with the properties $\lim_{x \rightarrow \pm\infty} f(x) = 0$ and $\int_0^\infty |f(x)| dx < \infty$. Define the function f_L which is equal to f in an interval of the form $[-\pi L, \pi L]$ and vanishes outside this interval. Note that $f(x) = \lim_{L \rightarrow \infty} f_L(x)$. This function can be extended to a periodic function, denoted by f_e , of period $2\pi T$ with $T > L$ and where $f_e(x) = f(x)$ for $|x| \leq \pi L$ and 0 for $-\pi T \leq x \leq -\pi L$ and $\pi L \leq x \leq \pi T$. Note that $f(x) = \lim_{L \rightarrow \infty} f_L(x) = \lim_{L \rightarrow \infty} f_e(x)$. From the previous section we can find the complex Fourier series of f_e to be

$$f_e(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{inx}{T}} \quad (24.1)$$

where

$$c_n = \frac{1}{2\pi T} \int_{-\pi T}^{\pi T} f_e(x) e^{-\frac{inx}{T}} dx.$$

Let $\xi \in \mathbb{R}$. Multiply both sides of (24.1) by $e^{-i\xi x}$ and then integrate both sides from $-\pi T$ to πT . Assuming integration and summation can be interchanged we find

$$\int_{-\pi T}^{\pi T} f_e(x) e^{-i\xi x} dx = \sum_{n=-\infty}^{\infty} c_n \int_{-\pi T}^{\pi T} e^{-i\xi x} e^{\frac{inx}{T}} dx.$$

It can be shown that the RHS converges, say to $\hat{f}(\xi)$, as $L \rightarrow \infty$ (and $T \rightarrow \infty$) Hence, we find

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x} dx. \quad (24.2)$$

The function \hat{f} is called the **Fourier transform** of f . We will use the notation $\mathcal{F}[f(x)] = \hat{f}(\xi)$.

Next, it can be shown that

$$\hat{f}\left(\frac{n}{T}\right) = 2\pi T c_n$$

so that

$$f_e(x) = \frac{1}{2\pi T} \sum_{n=-\infty}^{\infty} \hat{f}\left(\frac{n}{T}\right) e^{\frac{inx}{T}}.$$

It can be shown that as $L \rightarrow \infty$, we have

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{n=-\infty}^{\infty} \hat{f}\left(\frac{n}{T}\right) e^{\frac{inx}{T}} = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi x} d\xi$$

so that

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi x} d\xi \quad (24.3)$$

Equation (24.3) is called the **Fourier inversion formula** and we use the notation $\mathcal{F}^{-1}[\hat{f}(\xi)]$. Now, if we make use of Euler's formula, we can write the Fourier inversion formula in terms of sines and cosines,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) \cos \xi x d\xi + \frac{i}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) \sin \xi x d\xi$$

a superposition of sines and cosines of various frequencies.

Example 24.1

Find the Fourier transform of the function $f(x)$ defined by

$$f(x) = \begin{cases} e^{-ax} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

for some $a > 0$.

Solution.

We have

$$\begin{aligned}\hat{f}(\xi) &= \int_{-\infty}^{\infty} f(x)e^{-i\xi x} dx = \int_0^{\infty} e^{-ax} e^{-i\xi x} dx \\ &= \int_0^{\infty} e^{-ax-i\xi x} dx = \frac{e^{-x(a+i\xi)}}{-(a+i\xi)} \Big|_0^{\infty} \\ &= \frac{1}{a+i\xi} \blacksquare\end{aligned}$$

The following theorem lists the basic properties of the Fourier transform

Theorem 24.1

Let f, g , be piecewise continuous functions. Then we have the following properties:

- (1) **Linearity:** $\mathcal{F}[\alpha f(x) + \beta g(x)] = \alpha \mathcal{F}[f(x)] + \beta \mathcal{F}[g(x)]$, where α and β are arbitrary numbers.
- (2) **Shifting:** $\mathcal{F}[f(x - \alpha)] = e^{-i\alpha\xi} \mathcal{F}[f(x)]$.
- (3) **Scaling:** $\mathcal{F}[f(\frac{x}{\alpha})] = \alpha \mathcal{F}[f(\alpha x)]$.
- (4) **Continuity:** If $\int_{-\infty}^{\infty} |f(x)| dx < \infty$ then \hat{f} is continuous in ξ .
- (5) **Differentiation:** $\mathcal{F}[f^{(n)}(x)] = (i\xi)^n \mathcal{F}[f(x)]$.
- (6) **Integration:** $\mathcal{F}[\int_0^x f(s) ds] = -\frac{1}{i\xi} \mathcal{F}[f(x)]$.
- (7) **Parseval's Relation:** $\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi$.
- (8) **Duality:** $\mathcal{F}[\mathcal{F}[f(x)]] = 2\pi f(-x)$.
- (9) **Multiplication by x^n :** $\mathcal{F}[x^n f(x)] = i^n \hat{f}^{(n)}(\xi)$.
- (10) **Gaussians:** $\mathcal{F}[e^{-\alpha x^2}] = \sqrt{\frac{\pi}{\alpha}} e^{-\frac{\xi^2}{4\alpha}}$.
- (11) **Product:** $\mathcal{F}[(f(x)g(x))] = \frac{1}{2\pi} \mathcal{F}[f(x)] * \mathcal{F}[g(x)]$.
- (12) **Convolution:** $\mathcal{F}[(f * g)(x)] = \mathcal{F}[f(x)] \cdot \mathcal{F}[g(x)]$.

Example 24.2

Determine the Fourier transform of the Gaussian $u(x) = e^{-\alpha x^2}$, $\alpha > 0$.

Solution.

We have

$$\hat{u}(\xi) = \int_{-\infty}^{\infty} e^{-\alpha x^2} e^{-i\xi x} dx.$$

If we differentiate this relation with respect to the variable ξ and then integrate by parts we obtain

$$\begin{aligned}
\hat{u}'(\xi) &= -i \int_{-\infty}^{\infty} x e^{-\alpha x^2} e^{-i\xi x} dx \\
&= \frac{i}{2\alpha} \int_{-\infty}^{\infty} \frac{d}{dx} (e^{-\alpha x^2}) e^{-i\xi x} dx \\
&= \frac{i}{2\alpha} \left[e^{-\alpha x^2 - i\xi x} \Big|_{-\infty}^{\infty} + i\xi \int_{-\infty}^{\infty} (e^{-\alpha x^2}) e^{-i\xi x} dx \right] \\
&= \frac{i^2 \xi}{2\alpha} \int_{-\infty}^{\infty} (e^{-\alpha x^2}) e^{-i\xi x} dx = -\frac{\xi}{2\alpha} \hat{u}(\xi)
\end{aligned}$$

Thus, we have arrived at the ODE $\hat{u}'(\xi) = -\frac{\xi}{2\alpha} \hat{u}(\xi)$ whose general solution has the form

$$\hat{u}(\xi) = C e^{-\frac{\xi^2}{4\alpha}}.$$

Using a result from real analysis which states that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi},$$

we can write

$$\hat{u}(0) = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}} = C,$$

and therefore

$$\hat{u}(\xi) = \sqrt{\frac{\pi}{\alpha}} e^{-\frac{\xi^2}{4\alpha}} \blacksquare$$

Example 24.3

Prove

$$\mathcal{F}[f(-x)] = \hat{f}(-\xi).$$

Solution.

Using a change of variables we find

$$\mathcal{F}[f(-x)] = \int_{-\infty}^{\infty} f(-x) e^{-i\xi x} dx = \int_{-\infty}^{\infty} f(x) e^{i\xi x} dx = \hat{f}(-\xi) \blacksquare$$

Example 24.4

Prove

$$\mathcal{F}[\mathcal{F}[f(x)]] = 2\pi f(-x).$$

Solution.

We have

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi x} d\xi$$

Thus,

$$2\pi f(-x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-i\xi x} d\xi = \mathcal{F}[\hat{f}(\xi)] = \mathcal{F}[\mathcal{F}[f(x)]] \blacksquare$$

The following theorem lists the properties of inverse Fourier transform

Theorem 24.2

Let f and g be piecewise continuous functions.

- (1') **Linearity:** $\mathcal{F}^{-1}[\alpha \hat{f}(\xi) + \beta \hat{g}(\xi)] = \alpha \mathcal{F}^{-1}[\hat{f}(\xi)] + \beta \mathcal{F}^{-1}[\hat{g}(\xi)]$.
- (2') **Derivatives:** $\mathcal{F}^{-1}[f^{(n)}(\xi)] = (-ix)^n f(x)$.
- (3') **Multiplication by ξ^n :** $\mathcal{F}^{-1}[\xi^n \hat{f}(\xi)] = (-i)^n f^{(n)}(x)$.
- (4') **Multiplication by $e^{-i\xi\alpha}$:** $\mathcal{F}^{-1}[e^{-i\xi\alpha} \hat{f}(\xi)] = f(x - \alpha)$.
- (5') **Gaussians:** $\mathcal{F}^{-1}[e^{-\alpha\xi^2}] = \frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{x^2}{4\alpha}}$.
- (6') **Product:** $\mathcal{F}^{-1}[\hat{f}(\xi)\hat{g}(\xi)] = f(x) * g(x)$.
- (7') **Convolution:** $\mathcal{F}^{-1}[\hat{f} * \hat{g}(\xi)] = 2\pi(fg)(x)$.

Remark 24.1

It is important to mention that there exists no established convention of how to define the Fourier transform. In the literature, we can meet an equivalent definition of (24.3) with the constant $\frac{1}{\sqrt{2\pi}}$ or $\frac{1}{2\pi}$ in front of the integral. There also exist definitions with positive sign in the exponent. The reader should keep this fact in mind while working with various sources or using the transformation tables.

Practice Problems

Problem 24.1

Find the Fourier transform of the function

$$f(x) = \begin{cases} 1 & \text{if } -1 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Problem 24.2

Obtain the transformed problem when applying the Fourier transform with respect to the spatial variable to the equation and initial condition

$$u_t + cu_x = 0$$

$$u(x, 0) = f(x).$$

Problem 24.3

Obtain the transformed problem when applying the Fourier transform with respect to the spatial variable to the equation and both initial conditions

$$u_{tt} = c^2 u_{xx}, \quad x \in \mathbb{R}, \quad t > 0$$

$$u(x, 0) = f(x)$$

$$u_t(x, 0) = g(x).$$

Problem 24.4

Obtain the transformed problem when applying the Fourier transform with respect to the spatial variable to the equation and both initial conditions

$$\Delta u = u_{xx} + u_{yy} = 0, \quad x \in \mathbb{R}, \quad 0 < y < L$$

$$u(x, 0) = 0$$

$$u(x, L) = \begin{cases} 1 & \text{if } -a < x < a \\ 0 & \text{otherwise} \end{cases}$$

Problem 24.5

Find the Fourier transform of $f(x) = e^{-|x|^\alpha}$, where $\alpha > 0$.

Problem 24.6

Prove that

$$\mathcal{F}[e^{-x}H(x)] = \frac{1}{1+i\xi}$$

where

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Problem 24.7

Prove that

$$\mathcal{F}\left[\frac{1}{1+ix}\right] = 2\pi e^{\xi}H(-\xi).$$

Problem 24.8

Prove

$$\mathcal{F}[f(x-\alpha)] = e^{-i\xi\alpha}\hat{f}(\xi).$$

Problem 24.9

Prove

$$\mathcal{F}[e^{i\alpha x}f(x)] = \hat{f}(x-\alpha).$$

Problem 24.10

Prove the following

$$\mathcal{F}[\cos(\alpha x)f(x)] = \frac{1}{2}[\hat{f}(\xi+\alpha) + \hat{f}(\xi-\alpha)]$$

$$\mathcal{F}[\sin(\alpha x)f(x)] = \frac{1}{2}[\hat{f}(\xi+\alpha) - \hat{f}(\xi-\alpha)]$$

Problem 24.11

Prove

$$\mathcal{F}[f'(x)] = (i\xi)\hat{f}(\xi).$$

Problem 24.12Find the Fourier transform of $f(x) = 1 - |x|$ for $-1 \leq x \leq 1$ and 0 otherwise.**Problem 24.13**

Find, using the definition, the Fourier transform of

$$f(x) = \begin{cases} -1 & -a < x < 0 \\ 1 & 0 < x < a \\ 0 & \text{otherwise} \end{cases}$$

Problem 24.14

Find the inverse Fourier transform of $\hat{f}(\xi) = e^{-\frac{\xi^2}{2}}$.

Problem 24.15

Find $\mathcal{F}^{-1}\left(\frac{1}{a+i\xi}\right)$.

25 Applications of Fourier Transforms to PDEs

Fourier transform is a useful tool for solving differential equations. In this section, we apply Fourier transforms in solving various PDE problems. Contrary to Laplace transform, which usually uses the time variable, the Fourier transform is applied to the spatial variable on the whole real line.

The Fourier transform will be applied to the spatial variable x while the variable t remains fixed. The PDE in the two variables x and t passes under the Fourier transform to an ODE in the t -variable. We solve this ODE to obtain the transformed solution \hat{u} which can be converted to the original solution u by means of the inverse Fourier transform. We illustrate these ideas in the examples below.

First Order Transport Equation

Consider the initial value problem

$$u_t + cu_x = 0$$

$$u(x, 0) = f(x).$$

Let $\hat{u}(\xi, t)$ be the Fourier transform of u in x . Performing the Fourier transform on both the PDE and the initial condition, we reduce the PDE into an ODE in t

$$\frac{\partial \hat{u}}{\partial t} + i\xi c \hat{u} = 0$$

$$\hat{u}(\xi, 0) = \hat{f}(\xi).$$

Solution of the ODE gives

$$\hat{u}(\xi, t) = \hat{f}(\xi)e^{-i\xi ct}.$$

Thus,

$$u(x, t) = \mathcal{F}^{-1}[\hat{u}(\xi, t)] = f(x - ct)$$

which is exactly the same as we obtained by using the method of characteristics. (Section 8)

Second Order Wave Equation

Consider the two dimensional wave equation

$$u_{tt} = c^2 u_{xx}, \quad x \in \mathbb{R}, \quad t > 0$$

$$\begin{aligned}u(x, 0) &= f(x) \\u_t(x, 0) &= g(x).\end{aligned}$$

Again, by performing the Fourier transform of u in x , we reduce the PDE problem into an ODE problem in the variable t :

$$\begin{aligned}\frac{\partial^2 \hat{u}}{\partial t^2} &= -c^2 \xi^2 \hat{u} \\ \hat{u}(\xi, 0) &= \hat{f}(\xi) \\ \hat{u}_t(\xi, 0) &= \hat{g}(\xi).\end{aligned}$$

General solution to the ODE is

$$\hat{u}(\xi, t) = \Phi(\xi)e^{-i\xi ct} + \Psi(\xi)e^{i\xi ct}$$

where Φ and Ψ are two arbitrary functions of ξ . Performing the inverse transformation and making use of the translation theorem, we get the general solution

$$u(x, t) = \phi(x - ct) + \psi(x + ct)$$

where $\mathcal{F}(\phi) = \Phi$ and $\mathcal{F}(\psi) = \Psi$. But

$$\begin{aligned}\Phi(\xi) &= \frac{1}{2} \left[\hat{f}(\xi) - \frac{1}{i\xi c} \hat{g}(\xi) \right] \\ \Psi(\xi) &= \frac{1}{2} \left[\hat{f}(\xi) + \frac{1}{i\xi c} \hat{g}(\xi) \right].\end{aligned}$$

By using the integration property, we find the inverse transforms of Φ and Ψ

$$\begin{aligned}\phi(x) &= \frac{1}{2} \left[f(x) + \frac{1}{c} \int_0^x g(s) ds \right] \\ \psi(x) &= \frac{1}{2} \left[f(x) - \frac{1}{c} \int_0^x g(s) ds \right].\end{aligned}$$

Application of the translation property then yields directly the D'Alembert solution

$$u(x, t) = \frac{1}{2} [f(x - ct) + f(x + ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds.$$

Second Order Heat Equation

Next, we consider the heat equation

$$u_t = ku_{xx}, \quad x \in \mathbb{R}, \quad t > 0$$

$$u(x, 0) = f(x).$$

Performing Fourier Transform in x for the PDE and the initial condition, we obtain

$$\frac{\partial \hat{u}}{\partial t} = -k\xi^2 \hat{u}$$

$$\hat{u}(\xi, 0) = \hat{f}(\xi).$$

Treating ξ as a parameter, we obtain the solution to the above ODE problem

$$\hat{u}(\xi, t) = \hat{f}(\xi)e^{-k\xi^2 t}.$$

Application of the convolution theorem yields

$$\begin{aligned} u(x, t) &= f(x) * \mathcal{F}^{-1}[e^{-k\xi^2 t}] \\ &= f(x) * \left[\frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}} \right] \\ &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} f(s) e^{-\frac{(x-s)^2}{4kt}} ds. \end{aligned}$$

Laplace's Equation in 2D

Consider the problem

$$\Delta u = u_{xx} + u_{yy} = 0, \quad x \in \mathbb{R}, \quad 0 < y < L$$

$$u(x, 0) = 0$$

$$u(x, L) = \begin{cases} 1 & \text{if } -a < x < a \\ 0 & \text{otherwise.} \end{cases}$$

Performing Fourier Transform in x for the PDE we obtain the second order ODE in y

$$\hat{u}_{yy} = \xi^2 \hat{u}.$$

The general solution is given by

$$\hat{u}(\xi, y) = A(\xi) \sinh(\xi y) + B(\xi) \cosh(\xi y).$$

Using the boundary condition $\hat{u}(\xi, 0) = 0$ we find $B(\xi) = 0$. Using the second boundary condition we find

$$\begin{aligned}\hat{u}(\xi, L) &= \int_{-\infty}^{\infty} u(x, L)e^{-i\xi x} dx \\ &= \int_{-a}^a e^{-i\xi x} dx = \int_{-a}^a \cos(\xi x) dx \\ &= \frac{2 \sin(\xi a)}{\xi}.\end{aligned}$$

Hence,

$$A(\xi) \sinh(\xi L) = \frac{2 \sin(\xi a)}{\xi}$$

and this implies

$$A(\xi) = \frac{2 \sin(\xi a)}{\xi \sinh(\xi L)}.$$

Thus,

$$\hat{u}(\xi, y) = \frac{2 \sin(\xi a)}{\xi \sinh(\xi L)} \sinh(\xi y).$$

Taking inverse Fourier transform we find

$$u(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2 \sin(\xi a)}{\xi \sinh(\xi L)} \sinh(\xi y) e^{i\xi x} d\xi.$$

Using Euler's formula, and the fact that

$$\frac{2 \sin(\xi a)}{\xi \sinh(\xi L)} \sinh(\xi y) \sin(\xi x)$$

is odd in ξ , we arrive at

$$u(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2 \sin(\xi a)}{\xi \sinh(\xi L)} \sinh(\xi y) \cos(\xi x) d\xi \blacksquare$$

Practice Problems

Problem 25.1

Solve, by using Fourier transform

$$\begin{aligned}u_t + cu_x &= 0 \\ u(x, 0) &= e^{-\frac{x^2}{4}}.\end{aligned}$$

Problem 25.2

Solve, by using Fourier transform

$$\begin{aligned}u_t &= ku_{xx} - \alpha u, \quad x \in \mathbb{R} \\ u(x, 0) &= e^{-\frac{x^2}{\gamma}}.\end{aligned}$$

Problem 25.3

Solve the heat equation

$$u_t = ku_{xx}$$

subject to the initial condition

$$u(x, 0) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Problem 25.4

Use Fourier transform to solve the heat equation

$$\begin{aligned}u_t &= u_{xx} + u, \quad -\infty < x < \infty < t > 0 \\ u(x, 0) &= f(x).\end{aligned}$$

Problem 25.5

Prove that

$$\int_{-\infty}^{\infty} e^{-|\xi|y} e^{i\xi x} d\xi = \frac{2y}{x^2 + y^2}.$$

Problem 25.6

Solve the Laplace's equation in the half plane

$$u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \quad 0 < y < \infty$$

subject to the boundary condition

$$u(x, 0) = f(x), \quad |u(x, y)| < \infty.$$

Problem 25.7

Use Fourier transform to find the transformed equation of

$$u_{tt} + (\alpha + \beta)u_t + \alpha\beta u = c^2 u_{xx}$$

where $\alpha, \beta > 0$.

Problem 25.8

Solve the initial value problem

$$u_t + 3u_x = 0$$

$$u(x, 0) = e^{-x}$$

using the Fourier transform.

Problem 25.9

Solve the initial value problem

$$u_t = k u_{xx}$$

$$u(x, 0) = e^{-x}$$

using the Fourier transform.

Problem 25.10

Solve the initial value problem

$$u_t = k u_{xx}$$

$$u(x, 0) = e^{-x^2}$$

using the Fourier transform.

Problem 25.11

Solve the initial value problem

$$u_t + cu_x = 0$$

$$u(x, 0) = x^2$$

using the Fourier transform.

Problem 25.12

Solve, by using Fourier transform

$$\Delta u = 0$$

$$u_y(x, 0) = f(x)$$

$$\lim_{x^2+y^2 \rightarrow \infty} u(x, y) = 0.$$

Appendix

Appendix A: The Method of Undetermined Coefficients

The general solution to the nonhomogeneous differential equation

$$y'' + p(t)y' + q(t)y = g(t), \quad a < t < b \quad (26.1)$$

has the structure

$$y(t) = c_1y_1(t) + c_2y_2(t) + y_p(t)$$

where $y_p(t)$ is a particular solution to the nonhomogeneous equation. We will write $y(t) = y_h(t) + y_p(t)$ where $y_h(t) = c_1y_1(t) + c_2y_2(t)$.

In this and the next section we discuss methods for determining $y_p(t)$. The technique we discuss in this section is known as the **method of undetermined coefficients**.

This method requires that we make an initial assumption about the form of the particular solution $y_p(t)$, but with the coefficients left unspecified, thus the name of the method. We then substitute the assumed expression into equation (26.1) and attempt to determine the coefficients as to satisfy that equation.

The main advantage of this method is that it is straightforward to execute once the assumption is made as to the form of $y_p(t)$. Its major limitation is that it is useful only for equations with constant coefficients and the nonhomogeneous term $g(t)$ is restricted to a very small class of functions, namely functions of the form $e^{\alpha t}P_n(t)\cos\beta t$ or $e^{\alpha t}P_n(t)\sin\beta t$ where $P_n(t)$ is a polynomial of degree n .

We next illustrate the method of undetermined coefficients by several simple examples.

Example 26.1

Find the general solution of the nonhomogeneous equation

$$y'' - 2y' - 3y = 36e^{5t}.$$

Solution.

We seek a function where the combination $y_p'' - 2y_p' - 3y_p$ is equal to $36e^{5t}$. Since the exponential function reproduces itself through differentiation, the most plausible guessing function will be $y_p(t) = Ae^{5t}$ where A is a constant to be determined. Inserting this into the given equation we arrive at

$$25Ae^{5t} - 10Ae^{5t} - 3Ae^{5t} = 36e^{5t}.$$

Simplifying this last equation we find $12Ae^{5t} = 36e^{5t}$. Solving for A we find $A = 3$. Thus, $y_p(t) = 3e^{5t}$ is a particular solution to the differential equation. Next, the characteristic equation $r^2 - 2r - 3 = 0$ has the roots $r_1 = -1$ and $r_2 = 3$. Hence, the general solution to the differential equation is $y(t) = c_1e^{-t} + c_2e^{3t} + 3e^{5t}$ ■

Example 26.2

Find the general solution of

$$y'' - y' + y = 2 \sin 3t.$$

Solution.

The combination $y_p'' - y_p' + y_p$ must be equal to $2 \sin 3t$. Let's try with the guess $y_p(t) = A \sin 3t$. Inserting this into the given differential equation leads to

$$(2 - 16A) \sin 3t - 6A \cos 3t = 0 \quad (26.2)$$

and this is valid for all t . Letting $t = 0$ we find $-6A = 0$ or $A = 0$. Letting $t = \frac{\pi}{6}$ we find $2 - 16A = 0$ or $2 = 0$ which is impossible. This means there is no choice of A that makes equation (26.2) true. Hence, our choice is inadequate. The appearance of sine and cosine in equation (26.2) suggests a guessing of the form $y_p(t) = A \cos 3t + B \sin 3t$. Inserting this into the given differential equation leads to

$$(-8A - 3B) \cos 3t + (3A - 8B) \sin 3t = 2 \sin 3t.$$

Setting $-8A - 3B = 0$ and $3A - 8B = 2$ and solving for A and B we find $A = \frac{6}{73}$ and $B = -\frac{16}{73}$. Thus, a particular solution is

$$y_p(t) = \frac{6}{73} \cos 3t - \frac{16}{73} \sin 3t.$$

Next, the characteristic equation $r^2 - r + 1 = 0$ has roots $r_1 = \frac{1}{2} - i\frac{\sqrt{3}}{2}$ and $r_2 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. Thus, the general solution to the homogeneous equation is

$$y_h(t) = e^{\frac{1}{2}t} \left(c_1 \cos \frac{\sqrt{3}}{2}t + c_2 \sin \frac{\sqrt{3}}{2}t \right).$$

The general solution to the differential equation is

$$y(t) = y_h(t) + y_p(t) = e^{\frac{1}{2}t} \left(c_1 \cos \frac{\sqrt{3}}{2}t + c_2 \sin \frac{\sqrt{3}}{2}t \right) + \frac{6}{73} \cos 3t - \frac{16}{73} \sin 3t \quad \blacksquare$$

Example 26.3

Find the general solution of

$$y'' + 4y' - 2y = 2t^2 - 3t + 6.$$

Solution.

We see from the previous two examples that the trial function has usually the appearance of the nonhomogeneous term $g(t)$. Since $g(t)$ is a quadratic function, we are going to try $y_p(t) = At^2 + Bt + C$. Inserting this into the differential equation leads to

$$-2At^2 + (8A - 2b)t + (2A + 4B - 2C) = 2t^2 - 3t + 6.$$

Equating coefficients of like powers of t we find $A = -1$, $B = -\frac{5}{2}$, and $C = -9$. Thus, a particular solution is

$$y_p(t) = -t^2 - \frac{5}{2}t - 9.$$

We next solve the homogeneous equation. The characteristic equation $r^2 + 4r - 2 = 0$ has the roots $r_1 = -2 - \sqrt{6}$ and $r_2 = -2 + \sqrt{6}$. Thus,

$$y_h(t) = c_1 e^{(-2-\sqrt{6})t} + c_2 e^{(-2+\sqrt{6})t}.$$

The general solution of the given equation is

$$y(t) = y_h(t) + y_p(t) = c_1 e^{(-2-\sqrt{6})t} + c_2 e^{(-2+\sqrt{6})t} - t^2 - \frac{5}{2}t - 9 \blacksquare$$

Remark 26.1

The same principle used in the previous three examples extends to the case where $g(t)$ is a product of any two or all three of the three types of functions discussed above, as the next example illustrates.

Example 26.4

Find a particular solution of

$$y'' - 3y' - 4y = -8e^t \cos 2t.$$

Solution.

We are going to try $y_p(t) = Ae^t \cos 2t + Be^t \sin 2t$. Inserting into the differential equation we find

$$(-10A - 2B)e^t \cos 2t + (2A - 10B)e^t \sin 2t = -8e^t \cos 2t.$$

Thus, A and B satisfy the equations $10A + 2B = 8$ and $2A - 10B = 0$. Solving we find $A = \frac{10}{13}$ and $B = \frac{2}{13}$. Therefore, a particular solution is given by

$$y(t) = \frac{10}{13}e^t \cos 2t + \frac{2}{13}e^t \sin 2t \blacksquare$$

The following example illustrates the use of Theorem 15.2.

Example 26.5

Find the general solution of

$$y'' - 2y' - 3y = 4t - 5 + 6te^{2t}.$$

Solution.

The characteristic equation of the homogeneous equation is $r^2 - 2r - 3 = 0$ with roots $r_1 = -1$ and $r_2 = 3$. Thus,

$$y_h(t) = c_1e^{-t} + c_2e^{3t}.$$

By Theorem 15.2, a guess for the particular solution is $y_p(t) = At + B + Cte^{2t} + De^{2t}$. Inserting this into the differential equation leads to

$$-3At - 2A - 3B - 3Ce^{2t} + (2C - 3D)e^{2t} = 4t - 5 + 6te^{2t}.$$

From this identity we obtain $-3A = 4$ so that $A = -\frac{4}{3}$. Also, $-2A - 3B = -5$ so that $B = \frac{23}{9}$. Since $-3C = 6$ we find $C = -2$. From $2C - 3D = 0$ we find $D = -\frac{4}{3}$. It follows that

$$y(t) = c_1e^{-t} + c_2e^{3t} - \frac{4}{3}t + \frac{23}{9} - \left(2t + \frac{4}{3}\right)e^{2t} \blacksquare$$

Although the method of undetermined coefficients provides a nice general method for finding a particular solution, some difficulty arise as illustrated in the following example.

Example 26.6

Find the general solution of the nonhomogeneous equation

$$y'' - y' - 2y = 4e^{-t}.$$

Solution.

Let's try with $y_p(t) = Ae^{-t}$. Substituting this into the differential equation leads to $0Ae^{-t} = 4e^{-t}$. Thus, A does not exist. Why did the procedure of the previous examples fail here? The reason is that the function e^{-t} that appears in y_p is a solution to the homogeneous equation and so cannot possibly be a solution to the nonhomogeneous equation at the same time. Then comes the question of how to find a correct form for the particular solution.

We will try to solve a simpler equation with the same difficulty and to use its general solution to suggest how to proceed with our given equation. The simpler equation we consider is $y' + y = 4e^{-t}$. By the method of integrating factor we find the general solution $y(t) = 4te^{-t} + ce^{-t}$. The second term is the solution to the homogeneous equation whereas the first one is the solution to the nonhomogeneous equation. We conclude from this discussion that a good guess for the original equation would be $y_p(t) = Ate^{-t}$. If we insert this into the differential equation we end up with $-3Ae^{-t} = 4e^{-t}$. Solving for A we find $A = -\frac{4}{3}$. Thus, $y_p(t) = -\frac{4}{3}te^{-t}$ and the general solution to the differential equation is $y(t) = c_1e^{-t} + c_2e^{2t} - \frac{4}{3}te^{-t}$ ■

Example 26.7

Find the general solution of the nonhomogeneous equation

$$y'' + 2y' + y = 2e^{-t}.$$

Solution.

The characteristic equation is $r^2 + 2r + 1 = 0$ with double roots $r_1 = r_2 = -1$. Thus, $y_h(t) = c_1e^{-t} + c_2te^{-t}$. Our trial function can not contain either e^{-t} or te^{-t} since both are solutions to the homogeneous equation. Thus, a proper guess is $y_p(t) = At^2e^{-t}$. Finding derivatives up to order 2 we find $y_p'(t) = 2Ate^{-t} - At^2e^{-t}$ and $y_p''(t) = 2Ae^{-t} - 4Ate^{-t} + At^2e^{-t}$. Substituting this in the original equation and collecting like terms we find

$$2Ae^{-t} = 2e^{-t}.$$

Solving for A we find $A = 1$ so that $y_p(t) = t^2e^{-t}$. Hence, the general solution is given by

$$y(t) = c_1e^{-t} + c_2te^{-t} + t^2e^{-t} \blacksquare$$

APPENDIX A: THE METHOD OF UNDETERMINED COEFFICIENTS 227

In the following table we list examples of $g(t)$ along with the corresponding form of the particular solution.

Form of $g(t)$	Form of $y_p(t)$
$P_n(t) = a_n t^n + a_{n-1} t^{n-1} + \dots + a_0$	$t^r [A_n t^n + A_{n-1} t^{n-1} + \dots + A_1 t + A_0]$
$P_n(t)e^{\alpha t}$	$t^r [A_n t^n + A_{n-1} t^{n-1} + \dots + A_1 t + A_0] e^{\alpha t}$
$P_n(t)e^{\alpha t} \cos \beta t$ or $P_n(t)e^{\alpha t} \sin \beta t$	$t^r e^{\alpha t} [(A_n t^n + A_{n-1} t^{n-1} + \dots + A_1 t + A_0) \cos \beta t + (B_n t^n + B_{n-1} t^{n-1} + \dots + B_1 t + B_0) \sin \beta t]$

The number r is chosen to be the smallest nonnegative integer such that no term in the assumed form is a solution of the homogeneous equation $ay'' + by' + cy = 0$. The value of r will be 0, 1, or 2.

Example 26.8

Find the general solution of $y'' - y = t + te^t$.

Solution.

The characteristic equation $r^2 - 1 = 0$ has roots $r = \pm 1$. Thus, the homogeneous solution is $y_h(t) = c_1 e^{-t} + c_2 e^t$. A trial function for the particular solution is $A_0 + A_1 t + t(B_0 + B_1 t)e^t$ since e^t is a solution of the homogeneous equation. Inserting into the differential equation we find

$$2B_1 e^t + 2(B_0 + 2B_1 t)e^t + (tB_0 + t^2 B_1)e^t - A_0 - A_1 t - t(B_0 + B_1 t)e^t = t + te^t$$

or

$$-A_0 - A_1 t + (2B_1 + 2B_0 + 4B_1 t)e^t = t + te^t$$

From this we obtain, $A_0 = 0, A_1 = -1, B_1 + B_0 = 0, 4B_1 = 1$. Hence, $A_0 = 0, A_1 = -1, B_0 = -\frac{1}{4}, B_1 = \frac{1}{4}$. So

$$y_p(t) = -t + \frac{1}{4}t(t - 1)e^t$$

and the general solution is

$$y(t) = c_1 e^{-t} + c_2 e^t - t + \frac{1}{4}t(t - 1)e^t \blacksquare$$

Example 26.9

Solve using the method of undetermined coefficients:

$$y'' + y = e^t + t^3, y(0) = 2, y'(0) = 0.$$

Solution.

First, the characteristic equation is $r^2 + 1 = 0$, with roots $r = \pm i$, so the homogeneous solution is $y_h(t) = c_1 \sin t + c_2 \cos t$. The trial function for the particular solution is $y_p(t) = Ae^t + Bt^3 + Ct^2 + Dt + E$. Plugging into the differential equation, we see

$$Ae^t + 6Bt + 2C + Ae^t + Bt^3 + Ct^2 + Dt + E = e^t + t^3.$$

Matching coefficients, we see:

$$2A = 1, \quad B = 1, \quad C = 0, \quad 6B + D = 0, \quad E = 0$$

The particular solution is

$$y_p(t) = \frac{1}{2}e^t + t^3 - 6t,$$

and so the general solution is

$$y(t) = c_1 \sin t + c_2 \cos t + \frac{1}{2}e^t + t^3 - 6t$$

When $t = 0$, this is $y(0) = c_2 + \frac{1}{2} = 2$, so $c_2 = \frac{3}{2}$. The first derivative of the general solution is $y'(t) = c_1 \cos t - \frac{3}{2} \sin t + \frac{1}{2}e^t + 3t^2 - 6$. At $t = 0$, $y'(0) = c_1 + \frac{1}{2} - 6 = 0$, so $c_1 = \frac{11}{2}$. We thus have solution $y(t) = \frac{11}{2} \sin t + \frac{3}{2} \cos t + \frac{1}{2}e^t + t^3 - 6t$ ■

Appendix B: The Method of Variation of Parameters

In this section, we discuss a second method for finding a particular solution to a nonhomogeneous differential equation

$$y'' + p(t)y' + q(t)y = g(t), \quad a < t < b. \quad (27.1)$$

This method has no prior conditions to be satisfied by either $p(t)$, $q(t)$, or $g(t)$. Therefore, it may sound more general than the method of undetermined coefficients. We will see that this method depends on integration while the previous one is purely algebraic which, for some at least, is an advantage.

To use this method, we first find the general solution to the homogeneous equation

$$y(t) = c_1y_1(t) + c_2y_2(t).$$

Then we replace the parameters c_1 and c_2 by two functions $u_1(t)$ and $u_2(t)$ to be determined. From this the method got its name. Thus, obtaining

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t).$$

Observe that if u_1 and u_2 are constant functions then the above y is just the homogeneous solution to the differential equation.

In order to determine the two functions one has to impose two constraints. Finding the derivative of y_p we obtain

$$y_p' = (y_1'u_1 + y_2'u_2) + (y_1u_1' + y_2u_2').$$

Finding the second derivative to obtain

$$y_p'' = y_1''u_1 + y_1'u_1' + y_2''u_2 + y_2'u_2' + (y_1u_1' + y_2u_2)'.$$

Since it is up to us to choose u_1 and u_2 we decide to do that in such a way to make our computation simple. One way to achieving that is to impose the condition

$$y_1u_1' + y_2u_2' = 0. \quad (27.2)$$

Under such a constraint y_p' and y_p'' are simplified to

$$y_p' = y_1'u_1 + y_2'u_2$$

and

$$y_p'' = y_1''u_1 + y_1'u_1' + y_2''u_2 + y_2'u_2'.$$

In particular, y_p'' does not involve u_1'' and u_2'' .

Inserting y_p , y_p' , and y_p'' into equation (27.1) to obtain

$$[y_1''u_1 + y_1'u_1' + y_2''u_2 + y_2'u_2'] + p(t)(y_1'u_1 + y_2'u_2) + q(t)(u_1y_1 + u_2y_2) = g(t).$$

Rearranging terms,

$$[y_1'' + p(t)y_1' + q(t)y_1]u_1 + [y_2'' + p(t)y_2' + q(t)y_2]u_2 + [u_1'y_1' + u_2'y_2'] = g(t).$$

Since y_1 and y_2 are solutions to the homogeneous equation, the previous equation yields our second constraint

$$u_1'y_1' + u_2'y_2' = g(t). \quad (27.3)$$

Combining equation (27.2) and (27.3) we find the system of two equations in the unknowns u_1' and u_2'

$$\begin{aligned} y_1u_1' + y_2u_2' &= 0 \\ u_1'y_1' + u_2'y_2' &= g(t). \end{aligned}$$

Since $\{y_1, y_2\}$ is a fundamental set, the expression $W(t) = y_1y_2' - y_1'y_2$ is nonzero so that one can find unique u_1' and u_2' . Using the method of elimination, these functions are given by

$$u_1'(t) = -\frac{y_2(t)g(t)}{W(t)} \text{ and } u_2'(t) = \frac{y_1(t)g(t)}{W(t)}.$$

Computing antiderivatives to obtain

$$u_1(t) = \int -\frac{y_2(t)g(t)}{W(t)} dt \text{ and } u_2(t) = \int \frac{y_1(t)g(t)}{W(t)} dt.$$

Example 27.1

Find the general solution of

$$y'' - y' - 2y = 2e^{-t}$$

using the method of variation of parameters.

Solution.

The characteristic equation $r^2 - r - 2 = 0$ has roots $r_1 = -1$ and $r_2 = 2$. Thus, $y_1(t) = e^{-t}$, $y_2(t) = e^{2t}$ and $W(t) = 3e^t$. Hence,

$$u_1(t) = - \int \frac{e^{2t} \cdot 2e^{-t}}{3e^t} dt = -\frac{2}{3}t$$

and

$$u_2(t) = \int \frac{e^{-t} \cdot 2e^{-t}}{3e^t} dt = -\frac{2}{9}e^{-3t}.$$

The particular solution is

$$y_p(t) = -\frac{2}{3}te^{-t} - \frac{2}{9}e^{-t}.$$

The general solution is then given by

$$y(t) = c_1e^{-t} + c_2e^{2t} - \frac{2}{3}te^{-t} - \frac{2}{9}e^{-t} \blacksquare$$

Example 27.2

Find the general solution to $(2t - 1)y'' - 4ty' + 4y = (2t - 1)^2e^{-t}$ if $y_1(t) = t$ and $y_2(t) = e^{2t}$ form a fundamental set of solutions to the equation.

Solution.

First we rewrite the equation in standard form

$$y'' - \frac{4t}{2t - 1}y' + \frac{4}{2t - 1}y = (2t - 1)e^{-t}.$$

Since $W(t) = (2t - 1)e^{2t}$ we find

$$u_1(t) = - \int \frac{e^{2t} \cdot (2t - 1)e^{-t}}{(2t - 1)e^{2t}} dt = e^{-t}$$

and

$$u_2(t) = \int \frac{t \cdot (2t - 1)e^{-t}}{(2t - 1)e^{2t}} dt = -\frac{1}{3}te^{-3t} - \frac{1}{9}e^{-3t}.$$

Thus,

$$y_p(t) = te^{-t} - \frac{1}{3}te^{-t} - \frac{1}{9}e^{-t} = \frac{2}{3}te^{-t} - \frac{1}{9}e^{-t}.$$

The general solution is

$$y(t) = c_1t + c_2e^{2t} + \frac{2}{3}te^{-t} - \frac{1}{9}e^{-t} \blacksquare$$

Example 27.3

Find the general solution to the differential equation $y'' + y' = \ln t$, $t > 0$.

Solution.

The characteristic equation $r^2 + r = 0$ has roots $r_1 = 0$ and $r_2 = -1$ so that $y_1(t) = 1$, $y_2(t) = e^{-t}$, and $W(t) = -e^{-t}$. Hence,

$$u_1(t) = - \int \frac{e^{-t} \ln t}{-e^{-t}} dt = \int \ln t dt = t \ln t - t$$

$$u_2(t) = \int \frac{\ln t}{-e^{-t}} dt = - \int e^t \ln t dt = -e^t \ln t + \int \frac{e^t}{t} dt$$

Thus,

$$y_p(t) = t \ln t - t - \ln t + e^{-t} \int \frac{e^t}{t} dt$$

and

$$y(t) = c_1 + c_2 e^{-t} + t \ln t - t - \ln t + e^{-t} \int \frac{e^t}{t} dt \blacksquare$$

Example 27.4

Find the general solution of

$$y'' + y = \frac{1}{2 + \sin t}.$$

Solution.

Since the characteristic equation $r^2 + 1 = 0$ has roots $r = \pm i$, the general solution of the corresponding homogeneous equation $y'' + y = 0$ is given by

$$y_h(t) = c_1 \cos t + c_2 \sin t$$

Since $W(t) = 1$ we find

$$u_1(t) = - \int \frac{\sin t}{2 + \sin t} dt = -t + \int \frac{2}{2 + \sin t} dt$$

$$u_2(t) = \int \frac{\cos t}{2 + \sin t} dt = \ln(2 + \sin t)$$

Hence, the particular solution is

$$y_p(t) = \sin t \ln(2 + \sin t) + \cos t \left(\int \frac{2}{2 + \sin t} dt - t \right)$$

and the general solution is

$$y(t) = c_1 \cos t + c_2 \sin t + y_p(t) \blacksquare$$

Answers and Solutions

Section 1

1.1 (a) ODE (b) PDE (c) ODE.

1.2 $u_{ss} = 0$.

1.3 $u_{ss} + u_{tt} = 0$.

1.4 (a) Order 3, nonlinear (b) Order 1, linear, homogeneous (c) Order 2, linear, nonhomogeneous.

1.5 (a) Linear, homogeneous, order 3.

(b) Linear, non-homogeneous, order 3. The inhomogeneity is $-\sin y$.

(c) Nonlinear, order 2. The non-linear term is $e^x uu_x$.

(d) Nonlinear, order 3. The non-linear terms are $u_x u_{xxy}$ and $e^x uu_y$.

(e) Linear, non-homogeneous, order 2. The inhomogeneity is $f(x, y, t)$.

1.6 (a) Linear. (b) Linear. (c) Nonlinear. (d) Nonlinear.

1.7 (a) PDE, linear, second order, homogeneous.

(b) PDE, linear, second order, homogeneous.

(c) PDE, nonlinear, fourth order.

(d) ODE, linear, second order, nonhomogeneous.

(e) PDE, linear, second order, nonhomogeneous.

(f) PDE, quasilinear, second order.

1.8 $A(x, y, z)u_{xx} + B(x, y, z)u_{xy} + C(x, y, z)u_{yy} + E(x, y, z)u_{xz} + F(x, y, z)u_{yz} + G(x, y, z)u_{zz} + H(x, y, z)u_x + I(x, y, z)u_y + J(x, y, z)u_z + K(x, y, z)u = L(x, y, z)$.

- 1.9** (a) Order 3, linear, homogeneous.
 (b) Order 1, nonlinear.
 (c) Order 4, linear, nonhomogeneous
 (d) Order 2, nonlinear.
 (e) Order 2, linear, homogeneous.

1.10 $u_{ww} = 0$.

1.11 $u_{vw} = 0$.

1.12 $u_{vw} = 0$.

1.13 $u_t = 0$.

1.14 $u_t = 1$.

1.15 $u_w = \frac{1}{b}u$.

Section 2

2.1 $a = b = 0$.

2.2 Substituting into the differential equation we find

$$tX''T - XT' = 0$$

or

$$\frac{X''}{X} = \frac{T'}{tT}.$$

The LHS is a function of x only whereas the RHS is a function of t only. This is true only when both sides are constant. That is, there is λ such that

$$\frac{X''}{X} = \frac{T'}{tT} = \lambda$$

and this leads to the two ODEs $X'' = \lambda X$ and $T' = \lambda tT$.

2.3 We have $xu_x + (x+1)yu_y = \frac{x}{y}(e^x + xe^x) + (x+1)y\left(-\frac{xe^x}{y^2}\right) = 0$ and

$$u(1, 1) = e.$$

2.4 We have $u_x + u_y + 2u = e^{-2y} \cos(x - y) - 2e^{-2y} \sin(x - y) - e^{-2y} \cos(x - y) + 2e^{-2y} \sin(x - y) = 0$ and $u(x, 0) = \sin x$.

2.5 (a) The general solution to this equation is $u(x) = C$ where C is an arbitrary constant.

(b) The general solution is $u(x, y) = f(y)$ where f is an arbitrary differentiable function of y .

2.6 (a) The general solution to this equation is $u(x) = C_1x + C_2$ where C_1 and C_2 are arbitrary constants.

(b) We have $u_y = f(y)$ where f is an arbitrary function of y . Hence, $u(x, y) = \int f(y)dy + g(x)$.

2.7 Let $v(x, y) = y + 2x$. Then

$$\begin{aligned} u_x &= 2f_v(v) + g(v) + 2xg_v(v) \\ u_{xx} &= 4f_{vv}(v) + 4g_v(v) + 4xg_{vv}(v) \\ u_y &= f_v(v) + xg_v(v) \\ u_{yy} &= f_{vv}(v) + xg_{vv}(v) \\ u_{xy} &= 2f_{vv}(v) + g_v(v) + 2xg_{vv}(v) \end{aligned}$$

Hence,

$$\begin{aligned} u_{xx} - 4u_{xy} + 4u_{yy} &= 4f_{vv}(v) + 4g_v(v) + 4xg_{vv}(v) \\ &\quad - 8f_{vv}(v) - 4g_v(v) - 8xg_{vv}(v) \\ &\quad + 4f_{vv}(v) + 4xg_{vv}(v) = 0. \end{aligned}$$

2.8 $u_{tt} = c^2u_{xx}$.

2.9 Let $v = x + p(u)t$. Using the chain rule we find

$$u_t = f_v \cdot v_t = f_v \cdot (p(u) + p_u u_t t).$$

Thus

$$(1 - tf_v p_u)u_t = f_v p.$$

If $1 - tf_v p_u \equiv 0$ on any t -interval I then $f_v p \equiv 0$ on I which implies that $f_v \equiv 0$ or $p \equiv 0$ on I . But either condition will imply that $tf_v p_u \equiv 0$ and

this will imply that $1 = 1 - tf_v p_u = 0$, a contradiction. Hence, we must have $1 - tf_v p_u \neq 0$. In this case,

$$u_t = \frac{f_v p}{1 - tf_v p_u}.$$

Likewise,

$$u_x = f_v \cdot (1 + p_u u_x t)$$

or

$$u_x = \frac{f_v}{1 - tf_v p_u}.$$

It follows that $u_t = p(u)u_x$.

If $u_t = (\sin u)u_x$ then $p(u) = \sin u$ so that the general solution is given by

$$u(x, t) = f(x + t \sin u)$$

where f is an arbitrary differentiable function in one variable.

2.10 $u(x, y) = xf(x - y) + g(x - y)$.

2.11 Using integration by parts, we compute

$$\begin{aligned} \int_0^L u_{xx}(x, t)u(x, t)dx &= u_x(x, t)u(x, t)|_{x=0}^L - \int_0^L u_x^2(x, t)dx \\ &= u_x(L, t)u(L, t) - u_x(0, t)u(0, t) - \int_0^L u_x^2(x, t)dx \\ &= - \int_0^L u_x^2(x, t)dx \leq 0 \end{aligned}$$

Note that we have used the boundary conditions $u(0, t) = u(L, t) = 0$ and the fact that $u_x^2(x, t) \geq 0$ for all $x \in [0, L]$.

2.12 (a) This can be done by plugging in the equations.

(b) Plug in.

(c) We have $\sup\{|u_n(x, 0) - 1| : x \in \mathbb{R}\} = \frac{1}{n} \sup\{|\sin nx| : x \in \mathbb{R}\} = \frac{1}{n}$.

(d) We have $\sup\{|u_n(x, t) - 1| : x \in \mathbb{R}\} = \frac{e^{n^2 t}}{n}$.

(e) We have $\lim_{t \rightarrow \infty} \sup\{|u_n(x, t) - 1| : x \in \mathbb{R}, t > 0\} = \lim_{t \rightarrow \infty} \frac{e^{n^2 t}}{n} = \infty$. Hence, the solution is unstable and thus the problem is ill-posed.

2.13 (a) $u(x, y) = x^3 + xy^2 + f(y)$, where f is an arbitrary function.

(b) $u(x, y) = \frac{x^3y^2}{6} + F(x) + g(y)$, where $F(x) = \int f(x)dx$.

(c) $u(x, t) = \frac{1}{18}e^{2x+3t} + t \int f_1(x)dx + \int f_2(x)dx + g(t)$.

2.14 (b) $u(x, y) = xf(y - 2x) + g(y - 2x)$.

2.15 We have

$$\begin{aligned}u_t &= cu_v - cu_w \\u_{tt} &= c^2u_{vv} - 2c^2u_{vw} + c^2u_{ww} \\u_x &= u_v + u_w \\u_{xx} &= u_{vv} + 2u_{vw} + u_{ww}\end{aligned}$$

Substituting we find $u_{vw} = 0$ and solving this equation we find $u_v = f(v)$ and $u(v, w) = F(v) + G(w)$ where $F(v) = \int f(v)dv$.

Finally, using the fact that $v = x + ct$ and $w = x - ct$; we get d'Alembert's solution to the one-dimensional wave equation:

$$u(x, t) = F(x + ct) + G(x - ct)$$

where F and G are arbitrary differentiable functions.

Section 3

3.1 $y = \frac{1}{2}(1 - e^{-t^2})$.

3.2 $y(t) = \frac{3t-1}{9} + e^{-2t} + Ce^{-3t}$.

3.3 $y(t) = 3 \sin t + \frac{3 \cos t}{t} + \frac{C}{t}$.

3.4 $y(t) = \frac{1}{13}(3 \sin(3t) + 2 \cos(3t)) + Ce^{-2t}$.

3.5 $y(t) = Ce^{-\sin t} - 3$.

3.6 $\alpha = -2$.

3.7 $p(t) = 2$ and $g(t) = 2t + 3$.

3.8 $y_0 = y(0) = -1$ and $g(t) = 2e^t + \cos t + \sin t$.

3.9 1.

3.10 $y(t) = t \ln |t| + 7t$.

3.11 Since $p(t) = a$ we find $\mu(t) = e^{at}$. Suppose first that $a = \lambda$. Then

$$y' + ay = be^{-at}$$

and the corresponding general solution is

$$y(t) = bte^{-at} + Ce^{-at}$$

Thus,

$$\begin{aligned} \lim_{t \rightarrow \infty} y(t) &= \lim_{t \rightarrow \infty} \left(\frac{bt}{e^{at}} + \frac{C}{e^{at}} \right) \\ &= \lim_{t \rightarrow \infty} \frac{b}{ae^{at}} = 0 \end{aligned}$$

Now, suppose that $a \neq \lambda$ then

$$y(t) = \frac{b}{a - \lambda} e^{-\lambda t} + Ce^{-at}$$

Thus,

$$\lim_{t \rightarrow \infty} y(t) = 0.$$

3.12 $y(t) = (-te^t + e^t)^{-1}$.

3.13 $y(t) = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{1}{12t^2}$.

3.14 $y(t) = tSi(t) + (3 - Si(1))t$.

Section 4

4.1 $y(t) = \left(\frac{3}{2}e^{t^2} + C \right)^{\frac{1}{3}}$.

4.2 $y(t) = Ce^{\frac{t^2}{2} - 2t}$.

4.3 $y(t) = Ct^2 + 4$.

$$4.4 \quad y(t) = \frac{2Ce^{4t}}{1+Ce^{4t}}.$$

$$4.5 \quad y(t) = \sqrt{5 - 4 \cos(2t)}.$$

$$4.6 \quad y(t) = -\sqrt{(-2 \cos t + 4)}.$$

$$4.7 \quad y(t) = e^{1-t} - 1.$$

$$4.8 \quad y(t) = \frac{2}{\sqrt{-4t^2+1}}.$$

$$4.9 \quad y(t) = \tan\left(t + \frac{\pi}{2}\right) = -\cot t.$$

$$4.10 \quad y(t) = \frac{3-e^{-t^2}}{3+e^{-t^2}}.$$

$$4.11 \quad u(x, y) = F(y)e^{-3x} + G(x) \text{ where } F(y) = \int f(y)dy.$$

$$4.12 \quad y^2 + \cos y + \cos t + \frac{t^2}{2} = 2.$$

$$4.13 \quad 3y^2y' + \cos y + 2t = 0, \quad y(2) = 0.$$

4.14 The ODE is not separable.

Section 5

5.1 (a) Linear (b) Quasi-linear, nonlinear (c) Nonlinear (d) Semi-linear, non-linear.

5.2 Let $w = 2x - y$. Then $u_x + 2u_y - u = e^x f(w) + 2e^x f_w(w) - 2e^x f_w(w) - e^x f(w) = 0$.

5.3 We have $xu_x - yu_y = x\left(\frac{3}{2}x^{\frac{1}{2}}y^{\frac{1}{2}}\right) - y\left(\frac{1}{2}x^{\frac{3}{2}}y^{-\frac{1}{2}}\right) = x\sqrt{xy} = u$. Also, $u(y, y) = y^2$.

5.4 We have $-yu_x + xu_y = -2xy \sin(x^2 + y^2) + 2xy \sin(x^2 + y^2) = 0$. Moreover, $u(0, y) = \cos y^2$.

5.5 We have $\frac{1}{x}u_x + \frac{1}{y}u_y = \frac{1}{x}(-x) + \frac{1}{y}(1+y) = \frac{1}{y}$. Moreover, $u(x, 1) = \frac{1}{2}(3-x^2)$.

5.6 $3a - 7b = 0$.

5.7 $au_t + cu = 0$.

5.8 $u(x, y) = x + f(x - y)$.

5.9 We have

$$\begin{aligned}u_x &= -4e^{-4x}f(2x - 3y) + 2e^{-4x}f'(2x - 3y) \\u_y &= -3e^{-4x}f'(2x - 3y)\end{aligned}$$

Thus,

$$\begin{aligned}3u_x + 2u_y + 12u &= -12e^{-4x}f(2x - 3y) + 6e^{-4x}f'(2x - 3y) \\&\quad -6e^{-4x}f'(2x - 3y) + 12e^{-4x}f(2x - 3y) = 0.\end{aligned}$$

5.10 $u(x, t) = f(ax - bt)e^{\frac{x}{b}}$.

5.11 $u(x, y) = f(bx - ay)$.

5.12 $cu_w + \lambda u(v, w) = f\left(w, \frac{w-v}{c}\right)$.

5.13 $vw_v(v) = Aw(v)$.

Section 6.1

6.1.1 19.

6.1.2 -15.

6.1.3 $\vec{u} \cdot \vec{v} = \frac{1}{2}$ and $\vec{u} \cdot \vec{w} = -\frac{1}{2}$.

6.1.4 63° .

6.1.5 52° .

6.1.6 (a) Neither (b) Orthogonal (c) Orthogonal (d) Parallel.

6.1.7 PQR is a right triangle at Q .

6.1.8 $\vec{u} = \langle -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle$.

6.1.9 45° .

6.1.10 $\text{Comp}_{\vec{a}}\vec{b} = \frac{9}{7}$ and $\text{Proj}_{\vec{a}}\vec{b} = \langle \frac{27}{49}, \frac{54}{49}, -\frac{18}{49} \rangle$.

6.1.11 $\vec{b} = \langle x, y, 3x - 2\sqrt{10} \rangle$ where x and y are arbitrary numbers.

6.1.12 144 J.

6.1.13 1839 ft - lb = 1839 slug.

Section 6.2

6.2.1 $\nabla F(x, y, z) = (yze^{xyz} + y \cos(xy))\vec{i} + (xze^{xyz} + x \cos(xy))\vec{j} + xy e^{xyz}\vec{k}$.

6.2.2 $\nabla F(x, y, z) = \cos\left(\frac{y}{z}\right)\vec{i} - \frac{x}{z} \sin\left(\frac{y}{z}\right)\vec{j} + \frac{xy}{z^2} \sin\left(\frac{y}{z}\right)\vec{k}$.

6.2.3 The level surfaces are spheres centered at $(2, 3, -5)$ and with radius \sqrt{C} , $C \geq 0$.

6.2.4 $\frac{12}{\sqrt{5}}$.

6.2.5 $\frac{1}{\sqrt{10}}(3x^2 + 6y^3z - 3xy - 2xz + yz)$.

6.2.6 The maximum rate of change is $\sqrt{17}$ and the maximum occurs in the direction of

$$\frac{\nabla u(0, 2)}{\|\nabla u(0, 2)\|} = \frac{4}{\sqrt{17}}\vec{i} + \frac{1}{\sqrt{17}}\vec{j}.$$

6.2.7 $\nabla u(x, y, z) = -\frac{2x}{x^2+y^2}\vec{i} - \frac{2y}{x^2+y^2}\vec{j} + e^z\vec{k}$.

Section 7

$$7.1 \quad u(x, y) = \frac{1}{2}y^2 - \frac{1}{2}y^2 e^{-2x} + \sin(ye^{-x}).$$

$$7.2 \quad u(x, y) = \frac{1}{\csc(ye^{-x}) - x}.$$

$$7.3 \quad u(x, y) = e^{x^2} f(x^2 + y^2).$$

$$7.4 \quad u(x, y) = y(2 + e^{-|\frac{x}{y}|}).$$

$$7.5 \quad u(x, y) = \frac{1}{(x-4y)^2 + 1 - y}.$$

$$7.6 \quad u(x, y) = \frac{1}{e^{-x^2 + e^{2y} - 1} - y}.$$

$$7.7 \quad u(x, y) = \frac{3}{2}x - \frac{3}{2}xe^{-2y} + e^{-y} \tan^{-1}(xe^{-y}).$$

$$7.8 \quad u(x, y) = x^2 y^2, \quad xy \geq 0.$$

$$7.9 \quad uy = k_2 = f(k_1) = f(y + x^3).$$

$$7.10 \quad u(x, y) = y^4 - (y^2 - x^2)^2 = 2x^2 y^2 - x^4.$$

Section 8

$$8.1 \quad u(x, t) = \sin(x - 3t).$$

$$8.2 \quad u(x, y) = k_2 e^{-\frac{c}{a}x} = f(bx - ay) e^{-\frac{c}{a}x}.$$

$$8.3 \quad u(x, y) = x \cos(y - 2x) + f(y - 2x).$$

8.4 Solving the equation $\frac{dy}{dx} = 1$ we find $x - y = k_1$. Solving the equation $\frac{du}{dx} = x$ we find $u(x, y) = \frac{1}{2}x^2 + f(x - y)$ where f is a differentiable function of one variable. Since $u(x, x) = 1$ we find $1 = \frac{1}{2}x^2 + f(0)$ or $f(0) = 1 - \frac{x^2}{2}$ which is impossible since $f(0)$ is a constant. Hence, the given initial value problem has no solution.

$$8.5 \quad u(x, t) = \frac{e^{-3t}}{1 + (x-2t)^2}.$$

$$8.6 \quad u(x, t) = e^{3t} \left[(x - t)^2 + \frac{1}{9} \right] - \frac{1}{3}t - \frac{1}{9}.$$

8.7 Using the chain rule we find $w_t = u_t e^{\lambda t} + \lambda u e^{\lambda t}$ and $w_x = u_x e^{\lambda t}$. Substituting these equations into the original equation we find

$$w_t e^{-\lambda t} - \lambda u + c w_x e^{-\lambda t} + \lambda u = 0$$

or

$$w_t + c w_x = 0$$

8.8 (a) $w(x, t)$ is a solution to the equation follows from the principle of superposition. Moreover, $w(x, 0) = u(x, 0) - v(x, 0) = f(x) - g(x)$.

(b) $w(x, t) = f(x - ct) - g(x - ct)$.

(c) From (b) we see that

$$\sup_{x,t} \{|u(x, t) - v(x, t)|\} = \sup_x \{|f(x) - g(x)|\}.$$

Thus, small changes in the initial data produces small changes in the solution. Hence, the problem is a well-posed problem.

8.9

$$u(x, t) = \begin{cases} g\left(t - \frac{x}{c}\right) e^{-\frac{\lambda}{c}x} & \text{if } x < ct \\ 0 & \text{if } x \geq ct. \end{cases}$$

8.10 $u(x, t) = \sin\left(\frac{2x-3t}{2}\right)$.

8.11 $u(x, y) = x + f(x - y)$.

Section 9

9.1 $u = -y + f(y \ln(y + u) - x)$ where f is an arbitrary differentiable function.

9.2 $u = \frac{f(x+y+u)}{xy}$ where f is an arbitrary differentiable function.

9.3 $x^4 - u^4 - 2xyu^2 = f(xy)$ where f is an arbitrary differentiable function.

9.4 $xy + u = f(x^2 + y^2 - u^2)$ where f is an arbitrary differentiable function.

9.5 $x^2 + y^2 + u^2 = f\left(\frac{y}{u}\right)$ where f is an arbitrary differentiable function.

9.6 $x+u = k_2 e^t = e^t f(u^2 - x^2)$ where f is an arbitrary differentiable function.

9.7 $x^2 + y^2 + u^2 = f(x+y+u)$ where f is an arbitrary differentiable function.

9.8 $x^2 + y^2 - 2u = f(xyu)$ where f is an arbitrary differentiable function.

9.9 $y = \sin^{-1} x + f(u)$ where f is a differentiable function.

9.10 $\frac{1}{2}(x^2 - y^2 - u^2) = f(xy - \frac{u^2}{2})$ where f is a differentiable function.

Section 10

10.1 $u(x, y) = \frac{1-xy}{x+y}$, $x + y \neq 0$.

10.2 $u(x, y) = (x + y)(x^2 - y^2)$.

10.3 $2xyu + x^2 + y^2 - 2u + 2 = 0$.

10.4 $u(x, y) = \ln\left(x + 1 - \frac{y}{x}\right)$.

10.5 $u(x, y) = f(xe^{-y})$.

10.6 $u(t, x) = f(x - at)$.

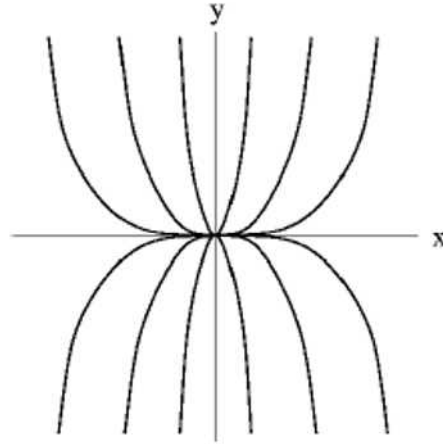
10.7 $u(x, y) = \frac{1}{\sec(x-ay)-y}$.

10.8 $u(x, y) = h\left(y - \frac{x^2}{2} + \frac{1}{2}\right) e^{x-1}$.

10.9 $u(x, y) = f(x - uy)$.

10.10 $u(x, y) = y - \sin^{-1} x$.

10.11 (i) $y = Cx^2$. The characteristics are parabolas in the plane centered at the origin. See figure below.



(ii) $u(x, y) = e^{yx^{-2}}$.

(iii) In the first case, we cannot substitute $x = 0$ into yx^{-2} (the argument of the function f , above) because x^{-2} is not defined at 0. Similarly, in the second case, we'd need to find a function f so that $f(0) = h(x)$. If h is not constant, it is not possible to satisfy this condition for all $x \in \mathbb{R}$.

10.12 $u(x, y) = e^y \cos(x - y)$.

10.13 (a) $u = e^x f(ye^{-x})$ where f is an arbitrary differential function.

(b) We want $2 = u(x, 3x) = e^x f(3e^x e^{-x}) = e^x f(3)$. This equation is impossible so this Cauchy problem has no solutions.

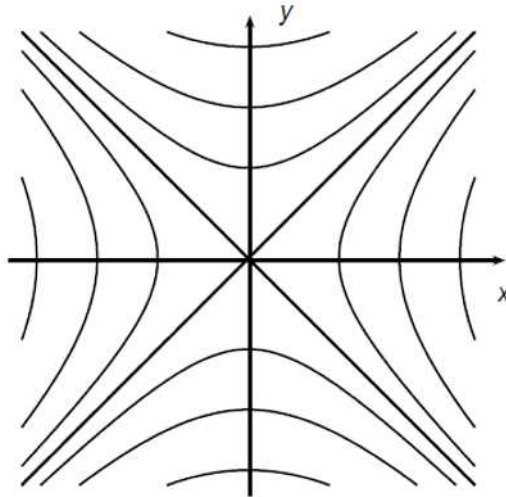
(c) We want $e^x = e^x f(e^x e^{-x}) \implies f(1) = 1$. In this case, there are infinitely many solutions to this Cauchy problem, namely, $u(x, y) = e^x f(ye^{-x})$ where f is an arbitrary function satisfying $f(1) = 1$.

10.14 $u(x, y) = -1 + 2e^{\frac{x^2}{2}} e^{-\frac{(4x-y)^2}{2}}$.

10.15 The Cauchy problem has no solutions.

10.16 (a) The characteristics satisfy the ODE $\frac{dy}{dx} = \frac{x}{y}$. Solving this equation we find $x^2 - y^2 = C$. Thus, the characteristics are hyperbolas.

(b)



(c) The general solution to the PDE is $u(x, y) = f(x^2 - y^2)$ where f is an arbitrary differentiable function. Since $u(0, y) = e^{-y^2}$ we find $f(y) = e^y$. Hence, $u(x, y) = e^{x^2 - y^2}$.

10.17 (a) infinitely many (b) no solutions.

Section 11

11.1 (a) Hyperbolic (b) Parabolic (c) Elliptic.

11.2 (a) Elliptic (b) Parabolic (c) Hyperbolic.

11.3 • The PDE is of hyperbolic type if $4y^2(x^2 + x + 1) > 0$. This is true for all $y \neq 0$. Graphically, this is the xy -plane with the x -axis removed,

• The PDE is of parabolic type if $4y^2(x^2 + x + 1) = 0$. Since $x^2 + x + 1 > 0$ for all $x \in \mathbb{R}$, we must have $y = 0$. Graphically, this is x -axis.

• The PDE is of elliptic type if $4y^2(x^2 + x + 1) < 0$ which can not happen.

11.4 We have

$$\begin{aligned}u_x(x, t) &= -\sin x \sin t, \\u_{xx}(x, t) &= -\cos x \sin t, \\u_t(x, t) &= \cos x \cos t, \\u_{tt}(x, t) &= -\cos x \sin t.\end{aligned}$$

Thus,

$$\begin{aligned}u_{xx}(x, t) &= -\cos x \sin t = u_{tt}(x, t), \\u(x, 0) &= \cos x \sin 0 = 0, \\u_t(x, 0) &= \cos x \cos 0 = \cos x, \\u_x(0, t) &= -\sin 0 \sin t = 0.\end{aligned}$$

11.5 (a) Quasi-linear (b) Semi-linear (c) Linear (d) Nonlinear.

11.6 We have

$$\begin{aligned}u_x &= \frac{2x}{x^2 + y^2} \\u_{xx} &= \frac{2y^2 - 2x^2}{(x^2 + y^2)^2} \\u_y &= \frac{2y}{x^2 + y^2} \\u_{yy} &= \frac{2x^2 - 2y^2}{(x^2 + y^2)^2}\end{aligned}$$

Plugging these expressions into the equation we find $u_{xx} + u_{yy} = 0$. Similar argument holds for the second part of the problem.

11.7 Multiplying the equation by u and integrating, we obtain

$$\begin{aligned}\lambda \int_0^L u^2(x) dx &= \int_0^L uu_{xx}(x) dx \\&= [u(L)u_x(L) - u(0)u_x(0)] - \int_0^L u_x^2(x) dx \\&= - \left[k_L u(L)^2 + k_0 u(0)^2 + \int_0^L u_x^2(x) dx \right]\end{aligned}$$

For $\lambda > 0$, because $k_0, k_L > 0$, the right-hand side is nonpositive and the left-hand side is nonnegative. Therefore, both sides must be zero, and there can be no solution other than $u \equiv 0$, which is the trivial solution.

11.8 Substitute $u(x, y) = f(x)g(y)$ into the left side of the equation to obtain $f(x)g(y)(f(x)g(y))_{xy} = f(x)g(y)f'(x)g'(y)$. Now, substitute the same thing into the right side to obtain $(f(x)g(y))_x(f(x)g(y))_y = f'(x)g(y)f(x)g'(y) = f(x)g(y)f'(x)g'(y)$. So the sides are equal, which means $f(x)g(y)$ is a solution.

11.9 We have

$$(u_n)_{xx} = -n^2 \sin nx \sinh ny \text{ and } (u_n)_{yy} = n^2 \sin nx \sinh ny$$

Hence, $\Delta u_n = 0$.

11.10 $u(x, y) = \frac{x^2y^2}{4} + F(x) + G(y)$, where $F(x) = \int f(x)dx$.

11.11 (a) We have $A = 2$, $B = -4$, $C = 7$ so $B^2 - 4AC = 16 - 56 = -40 < 0$. So this equation is elliptic everywhere in \mathbb{R}^2 .

(b) We have $A = 1$, $B = -2 \cos x$, $C = -\sin^2 x$ so $B^2 - 4AC = 4 \cos^2 x + 4 \sin^2 x = 4 > 0$. So this equation is hyperbolic everywhere in \mathbb{R}^2 .

(c) We have $A = y$, $B = 2(x - 1)$, $C = -(y + 2)$ so $B^2 - 4AC = 4(x - 1)^2 + 4y(y + 2) = 4[(x - 1)^2 + (y + 1)^2 - 1]$. The equation is parabolic if $(x - 1)^2 + (y + 1)^2 = 1$. It is hyperbolic if $(x - 1)^2 + (y + 1)^2 > 1$ and elliptic if $(x - 1)^2 + (y + 1)^2 < 1$.

11.12 Using the chain rule we find

$$\begin{aligned} u_t(x, t) &= \frac{1}{2}(cf'(x + ct) - cf'(x - ct)) + \frac{1}{2c}[g(x + ct)(c) - g(x - ct)(-c)] \\ &= \frac{c}{2}(f'(x + ct) - f'(x - ct)) + \frac{1}{2}(g(x + ct) + g(x - ct)) \\ u_{tt} &= \frac{c^2}{2}(f''(x + ct) + f''(x - ct)) + \frac{c}{2}(g'(x + ct) - g'(x - ct)) \\ u_x(x, t) &= \frac{1}{2}(f'(x + ct) + f'(x - ct)) + \frac{1}{2c}[g(x + ct) - g(x - ct)] \\ u_{xx}(x, t) &= \frac{1}{2}(f''(x + ct) + f''(x - ct)) + \frac{1}{2c}[g'(x + ct) - g'(x - ct)] \end{aligned}$$

By substitution we see that $c^2 u_{xx} = u_{tt}$. Moreover,

$$u(x, 0) = \frac{1}{2}(f(x) + f(x)) + \frac{1}{2c} \int_x^x g(s) ds = f(x)$$

and

$$u_t(x, 0) = g(x).$$

11.13 (a) $1 + 4x^2y > 0$, (b) $1 + 4x^2y = 0$, (c) $1 + 4x^2y < 0$.

11.14 $u(x, y) = f(y - 3x) + g(x + y)$.

11.15 $u(x, y) = f(y - 3x) + g(x + y) = \frac{10x^2 + y^2 - 7xy + 6}{6}$.

Section 12

12.1 Let $z(x, t) = \alpha v(x, t) + \beta w(x, t)$. Then we have

$$\begin{aligned} c^2 z_{xx} &= c^2 \alpha v_{xx} + c^2 \beta w_{xx} \\ &= \alpha v_{tt} + \beta w_{tt} \\ &= z_{tt}. \end{aligned}$$

12.2 Indeed we have $c^2 u_{xx}(x, t) = 0 = u_{tt}(x, t)$.

12.3 $u(x, t) = 0$.

12.4 $u(x, t) = \frac{1}{2}(\cos(x - 3t) + \cos(x + 3t))$.

12.5 $u(x, t) = \frac{1}{2} \left[\frac{1}{1+(x+t)^2} + \frac{1}{1+(x-t)^2} \right]$.

12.6 $u(x, t) = 1 + \frac{1}{8\pi} [\sin(2\pi x + 4\pi t) - \sin(2\pi x - 4\pi t)]$.

12.7

$$u(x, t) = \begin{cases} 1 & \text{if } x - 5t < 0 \text{ and } x + 5t < 0 \\ \frac{1}{2} & \text{if } x - 5t < 0 \text{ and } x + 5t > 0 \\ \frac{1}{2} & \text{if } x - 5t > 0 \text{ and } x + 5t < 0 \\ 0 & \text{if } x - 5t > 0 \text{ and } x + 5t > 0 \blacksquare \end{cases}$$

12.8 $u(x, t) = \frac{1}{2}[e^{-(x+ct)^2} + e^{-(x-ct)^2}] + \frac{t}{2} + \frac{1}{4c} \cos(2x) \sin(2ct)$.

12.9 Just plug the translated/differentiated/dilated solution into the wave equation and check that it is a solution.

12.10 $v(r) = A \cos(nr) + B \sin(nr)$.

12.11 $u(x, t) = \frac{1}{2}[e^{x-ct} + e^{x+ct} + \frac{1}{c}(\cos(x-ct) - \cos(x+ct))]$.

12.12 (a) We have

$$\begin{aligned} \frac{dE}{dt}(t) &= \int_0^L u_t u_{tt} dx + \int_0^L c^2 u_x u_{xt} dx \\ &= \int_0^L u_t u_{tt} dx + c^2 u_t(L, t) u_x(L, t) - c^2 u_t(0, t) u_x(0, t) - c^2 \int_0^L u_t u_{xx} dx \\ &= c^2 u_t(L, t) u_x(L, t) - c^2 u_t(0, t) u_x(0, t) + \int_0^L u_t (u_{tt} - c^2 u_{xx}) dx \\ &= c^2 (u_t(L, t) u_x(L, t) - u_t(0, t) u_x(0, t)) \end{aligned}$$

since $u_{tt} - c^2 u_{xx} = 0$.

(b) Since the ends are fixed, we have $u_t(0, t) = u_t(L, t) = 0$. From (a) we have

$$\frac{dE}{dt}(t) = c^2 (u_t(L, t) u_x(L, t) - u_t(0, t) u_x(0, t)) = 0.$$

(c) Assuming free ends boundary conditions, that is $u_x(0, t) = u_x(L, t) = 0$, we find $\frac{dE}{dt}(t) = 0$.

12.13 Using the previous exercise, we find

$$\frac{dE}{dt}(t) = -d \int_0^L (u_t)^2 dx.$$

The right-hand side is nonpositive, so the energy either decreases or is constant. The latter case can occur only if $u_t(x, t)$ is identically zero, which means that the string is at rest.

12.14 (a) By the chain rule we have $u_t(x, t) = -cR'(x-ct)$ and $u_{tt}(x, t) = c^2 R''(x-ct)$. Likewise, $u_x(x, t) = R'(x-ct)$ and $u_{xx} = R''(x-ct)$. Thus, $u_{tt} = c^2 u_{xx}$.

(b) We have

$$\frac{1}{2} \int_0^L (u_t)^2 dx = \int_0^L \frac{c^2}{2} [R'(x - ct)]^2 dx = \int_0^L \frac{c^2}{2} (u_x)^2 dx.$$

12.15 $u(x, t) = x^2 + 4t^2 + \frac{1}{4} \sin 2x \sin 4t.$

Section 13

13.1 Let $z(x, t) = \alpha u(x, t) + \beta v(x, t)$. Then we have

$$\begin{aligned} kz_{xx} &= k\alpha u_{xx} + k\beta v_{xx} \\ &= \alpha u_t + \beta v_t \\ &= z_t. \end{aligned}$$

13.2 Indeed we have $ku_{xx}(x, t) = 0 = u_t(x, t)$.

13.3 $u(x, t) = T_0 + \frac{T_L - T_0}{L}x.$

13.4 Let \bar{u} be the solution to (13.1) that satisfies $\bar{u}(0, t) = \bar{u}(L, t) = 0$. Let $w(x, t)$ be the time independent solution to (13.1) that satisfies $w(0, t) = T_0$ and $w(L, t) = T_L$. That is, $w(x, t) = T_0 + \frac{T_L - T_0}{L}x$. From Exercise 13.1, the function $u(x, t) = \bar{u}(x, t) + w(x, t)$ is a solution to (13.1) that satisfies $u(0, t) = T_0$ and $u(L, t) = T_L$.

13.5 $u(x, t) = 0.$

13.6 Substituting $u(x, t) = X(x)T(t)$ into (13.1) we obtain

$$k \frac{X''}{X} = \frac{T'}{T}.$$

Since X only depends on x and T only depends on t , we must have that there is a constant λ such that

$$k \frac{X''}{X} = \lambda \text{ and } \frac{T'}{T} = \lambda.$$

This gives the two ordinary differential equations

$$X'' - \frac{\lambda}{k}X = 0 \text{ and } T' - \lambda T = 0.$$

13.7 (a) Letting $\alpha = \frac{\lambda}{k} > 0$ we obtain the ODE $X'' - \alpha X = 0$ whose general solution is given by $X(x) = Ae^{x\sqrt{\alpha}} + Be^{-x\sqrt{\alpha}}$ for some constants A and B .

(b) The condition $u(0, t) = 0$ implies that $X(0) = 0$ which in turn implies $A + B = 0$. Likewise, the condition $u(L, t) = 0$ implies $Ae^{L\sqrt{\alpha}} + Be^{-L\sqrt{\alpha}} = 0$. Hence, $A(e^{L\sqrt{\alpha}} - e^{-L\sqrt{\alpha}}) = 0$.

(c) If $A = 0$ then $B = 0$ and $u(x, t)$ is the trivial solution which contradicts the assumption that u is non-trivial. Hence, we must have $A \neq 0$.

(d) Using (b) and (c) we obtain $e^{L\sqrt{\alpha}} = e^{-L\sqrt{\alpha}}$ or $e^{2L\sqrt{\alpha}} = 1$. This equation is impossible since $2L\sqrt{\alpha} > 0$. Hence, we must have $\lambda < 0$ so that $X(x) = A \cos(x\sqrt{-\alpha}) + B \sin(x\sqrt{-\alpha})$.

13.8 (a) Now, write $\beta = \sqrt{-\frac{\lambda}{k}}$. Then we obtain the equation $X'' + \beta^2 X = 0$ whose general solution is given by

$$X(x) = c_1 \cos \beta x + c_2 \sin \beta x.$$

(b) Using $X(0) = 0$ we obtain $c_1 = 0$. Since $c_2 \neq 0$ we must have $\sin \beta L = 0$ which implies $\beta L = n\pi$ where n is an integer. Thus, $\lambda = -\frac{kn^2\pi^2}{L^2}$, where n is an integer.

13.9 For each integer $i \geq 0$ we have $u_i(x, t) = c_i e^{-\frac{ki^2\pi^2}{L^2}t} \sin\left(\frac{i\pi}{L}x\right)$ is a solution to (13.1). By superposition, $u(x, t)$ is also a solution to (13.1). Moreover, $u(0, t) = u(L, t) = 0$ since $u_i(0, t) = u_i(L, t) = 0$ for $i = 1, \dots, n$.

13.10 (i) $u(0, t) = 0$ and $u(a, t) = 100$ for $t > 0$.

(ii) $u_x(0, t) = u_x(a, t) = 0$ for $t > 0$.

13.11 Solving this problem we find $u(x, t) = e^{-t} \sin x$. We have

$$E(t) = \int_0^\pi [e^{-2t} \sin^2 x + e^{-2t} \cos^2 x] dx = \int_0^\pi e^{-2t} dx = \pi e^{-2t}.$$

Thus, $E'(t) = -2\pi e^{-2t} < 0$ for all $t > 0$.

13.12 $E(t) = \int_0^L f(x) dx + (1 + 4L)t$.

13.13 $v(x) = x + 2$.

13.14 (a) $v(x) = \frac{T}{L}x$.

(b) $v(x) = T$.

(c) $v(x) = \alpha x + T$.

13.15 (a) $E(t) = \int_0^L u(x, t)dx$.

(b) We integrate the equation in x from 0 to L :

$$\int_0^L u_t(x, t)dx = \int_0^L ku_{xx}dx = ku_x(x, t)|_0^L = 0,$$

since $u_x(0, t) = u_x(L, t) = 0$. The left-hand side can also be written as

$$\frac{d}{dt} \int_0^L u(x, t)dx = E'(t).$$

Thus, we have shown that $E'(t) = 0$ so that $E(t)$ is constant.

13.16 (a) The total thermal energy is

$$E(t) = \int_0^L u(x, t)dx.$$

We have

$$\frac{dE}{dt} = \int_0^L u_t(x, t)dx = u_x|_0^L + \int_0^L xdx = (7 - \beta) + \frac{L^2}{2}.$$

Hence,

$$E(t) = \int_0^L f(x)dx + \left[(7 - \beta) + \frac{L^2}{2} \right] t.$$

(b) The steady solution (equilibrium) is possible if the right-hand side vanishes:

$$(7 - \beta) + \frac{L^2}{2} = 0$$

Solving this equation for β we find $\beta = 7 + \frac{L^2}{2}$.

(c) By integrating the equation $u_{xx} + x = 0$ we find the steady solution

$$u(x) = -\frac{x^3}{6} + C_1x + C_2$$

From the condition $u_x(0) = \beta$ we find $C_1 = \beta$. The steady solution should also have the same value of the total energy as the initial condition. This means

$$\int_0^L \left(-\frac{x^3}{6} + \beta x + C_2 \right) dx = \int_0^L f(x) dx = E(0).$$

Performing the integration and then solving for C_2 we find

$$C_2 = \frac{1}{L} \int_0^L f(x) dx + \frac{L^3}{24} - \beta \frac{L}{2}.$$

Therefore, the steady-state solution is

$$u(x) = \frac{1}{L} \int_0^L f(x) dx + \frac{L^3}{24} - \beta \frac{L}{2} + \beta x - \frac{x^3}{6}.$$

Section 14

14.1 (a) For all $0 \leq x < 1$ we have $\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} x^n = 0$. Also, $\lim_{n \rightarrow \infty} f_n(1) = 1$. Hence, the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f .

(b) Suppose the contrary. Let $\epsilon = \frac{1}{2}$. Then there exists a positive integer N such that for all $n \geq N$ we have

$$|f_n(x) - f(x)| < \frac{1}{2}$$

for all $x \in [0, 1]$. In particular, we have

$$|f_N(x) - f(x)| < \frac{1}{2}$$

for all $x \in [0, 1]$. Choose $(0.5)^{\frac{1}{N}} < x < 1$. Then $|f_N(x) - f(x)| = x^N > 0.5 = \epsilon$ which is a contradiction. Hence, the given sequence does not converge uniformly.

14.2 For every real number x , we have

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \frac{nx + x^2}{n^2} = \lim_{n \rightarrow \infty} \frac{x}{n} + \lim_{n \rightarrow \infty} \frac{x^2}{n^2} = 0$$

Thus, $\{f_n\}_{n=1}^{\infty}$ converges pointwise to the zero function on \mathbb{R} .

14.3 For every real number x , we have

$$-\frac{1}{\sqrt{n+1}} \leq f_n(x) \leq \frac{1}{\sqrt{n+1}}.$$

Moreover,

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+1}} = 0.$$

Applying the squeeze rule for sequences, we obtain

$$\lim_{n \rightarrow \infty} f_n(x) = 0$$

for all x in \mathbb{R} . Thus, $\{f_n\}_{n=1}^{\infty}$ converges pointwise to the zero function on \mathbb{R} .

14.4 First of all, observe that $f_n(0) = 0$ for every n in \mathbb{N} . So the sequence $\{f_n(0)\}_{n=1}^{\infty}$ is constant and converges to zero. Now suppose $0 < x < 1$ then $n^2 x_n = n^2 e^{n \ln x}$. But $\ln x < 0$ when $0 < x < 1$, it follows that

$$\lim_{n \rightarrow \infty} f_n(x) = 0 \text{ for } 0 < x < 1$$

Finally, $f_n(1) = n^2$ for all n . So,

$$\lim_{n \rightarrow \infty} f_n(1) = \infty.$$

Therefore, $\{f_n\}_{n=1}^{\infty}$ is not pointwise convergent on $[0, 1]$.

14.5 For $-\frac{\pi}{2} \leq x < 0$ and $0 < x \leq \frac{\pi}{2}$ we have

$$\lim_{n \rightarrow \infty} (\cos x)^n = 0.$$

For $x = 0$ we have $f_n(0) = 1$ for all n in \mathbb{N} . Therefore, $\{f_n\}_{n=1}^{\infty}$ converges pointwise to

$$f(x) = \begin{cases} 0 & \text{if } -\frac{\pi}{2} \leq x < 0 \text{ and } 0 < x \leq \frac{\pi}{2} \\ 1 & \text{if } x = 0. \end{cases}$$

14.6 (a) Let $\epsilon > 0$ be given. Let N be a positive integer such that $N > \frac{1}{\epsilon}$. Then for $n \geq N$

$$\left| x - \frac{x^n}{n} - x \right| = \frac{|x|^n}{n} < \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

Thus, the given sequence converges uniformly (and pointwise) to the function $f(x) = x$.

(b) Since $\lim_{n \rightarrow \infty} f'_n(x) = 1$ for all $x \in [0, 1)$, the sequence $\{f'_n\}_{n=1}^{\infty}$ converges pointwise to $f'(x) = 1$. However, the convergence is not uniform. To see this, let $\epsilon = \frac{1}{2}$ and suppose that the convergence is uniform. Then there is a positive integer N such that for $n \geq N$ we have

$$|1 - x^{n-1} - 1| = |x|^{n-1} < \frac{1}{2}.$$

In particular, if we let $n = N + 1$ we must have $x^N < \frac{1}{2}$ for all $x \in [0, 1)$. But $x = (\frac{1}{2})^{\frac{1}{N}} \in [0, 1)$ and $x^N = \frac{1}{2}$ which contradicts $x^N < \frac{1}{2}$. Hence, the convergence is not uniform.

14.7 (a) The pointwise limit is

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ \frac{1}{2} & \text{if } x = 1 \\ 1 & \text{if } 1 < x \leq 2 \end{cases}$$

(b) The convergence cannot be uniform because if it were f would have to be continuous.

14.8 (a) Let $\epsilon > 0$ be given. Note that

$$|f_n(x) - \frac{1}{2}| = \left| \frac{2 \cos x - \sin^2 x}{2(2n + \sin^2 x)} \right| \leq \frac{3}{4n}.$$

Since $\lim_{n \rightarrow \infty} \frac{3}{4n} = 0$ we can find a positive integer N such that if $n \geq N$ then $\frac{3}{4n} < \epsilon$. Thus, for $n \geq N$ and all $x \in \mathbb{R}$ we have

$$|f_n(x) - \frac{1}{2}| \leq \frac{3}{4n} < \epsilon.$$

This shows that $f_n \rightarrow \frac{1}{2}$ uniformly on \mathbb{R} and also on $[2, 7]$.

(b) We have

$$\lim_{n \rightarrow \infty} \int_2^7 f_n x dx = \int_2^7 \lim_{n \rightarrow \infty} f_n x dx = \int_2^7 \frac{1}{2} dx = \frac{5}{2}.$$

14.9 We have proved earlier that this sequence converges pointwise to the discontinuous function

$$f(x) = \begin{cases} 0 & \text{if } -\frac{\pi}{2} \leq x < 0 \text{ and } 0 < x \leq \frac{\pi}{2} \\ 1 & \text{if } x = 0 \end{cases}$$

Therefore, uniform convergence cannot occur for this given sequence.

14.10 (a) Using the squeeze rule we find

$$\lim_{n \rightarrow \infty} \sup\{|f_n(x)| : 2 \leq x \leq 5\} = 0.$$

Thus, $\{f_n\}_{n=1}^{\infty}$ converges uniformly to the zero function.

(b) We have

$$\lim_{n \rightarrow \infty} \int_2^5 f_n(x) dx = \int_2^5 0 dx = 0.$$

Section 15.

15.1 (a) We have $(fg)(x+T) = f(x+T)g(x+T) = f(x)g(x) = (fg)(x)$.

(b) We have $(c_1f + c_2g)(x+T) = c_1f(x+T) + c_2g(x+T) = c_1f(x) + c_2g(x) = (c_1f + c_2g)(x)$.

15.2 (a) For $n \neq m$ we have

$$\begin{aligned} \int_{-L}^L \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx &= -\frac{1}{2} \int_{-L}^L \left[\cos\left(\frac{(m+n)\pi}{L}x\right) - \cos\left(\frac{(m-n)\pi}{L}x\right) \right] dx \\ &= -\frac{1}{2} \left[\frac{L}{(m+n)\pi} \sin\left(\frac{(m+n)\pi}{L}x\right) \right. \\ &\quad \left. - \frac{L}{(m-n)\pi} \sin\left(\frac{(m-n)\pi}{L}x\right) \right]_{-L}^L \\ &= 0 \end{aligned}$$

where we used the trigonometric identity

$$\sin a \sin b = \frac{1}{2}[-\cos(a+b) + \cos(a-b)].$$

(b) For $n \neq m$ we have

$$\begin{aligned} \int_{-L}^L \cos\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx &= \frac{1}{2} \int_{-L}^L \left[\sin\left(\frac{(m+n)\pi}{L}x\right) - \sin\left(\frac{(m-n)\pi}{L}x\right) \right] dx \\ &= \frac{1}{2} \left[-\frac{L}{(m+n)\pi} \cos\left(\frac{(m+n)\pi}{L}x\right) \right. \\ &\quad \left. + \frac{L}{(m-n)\pi} \cos\left(\frac{(m-n)\pi}{L}x\right) \right]_{-L}^L \\ &= 0 \end{aligned}$$

where we used the trigonometric identity

$$\cos a \sin b = \frac{1}{2} [\sin(a+b) - \sin(a-b)].$$

15.3 (a) L (b) L (c) 0.

15.4

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = 0 \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \\ &= - \int_{-\pi}^0 \cos nx dx + \int_0^{\pi} \cos nx dx = 0 \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \\ &= - \int_{-\pi}^0 \sin nx dx + \int_0^{\pi} \sin nx dx \\ &= \frac{2}{n} [1 - (-1)^n] \end{aligned}$$

15.5 $f(x) = -\frac{1}{6} + \sum_{n=1}^{\infty} \frac{4}{(n\pi)^2} (-1)^n \cos(n\pi x).$

15.6 $f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \left[\cos\left(\frac{n\pi}{2}\right) - (-1)^n \right] \sin\left(\frac{nx}{2}\right).$

15.7 $f(x) = \sum_{n=1}^{\infty} \frac{4}{(n\pi)^2} [1 - (-1)^n] \cos\left(\frac{n\pi}{2}x\right).$

15.8 Since the sided limits at the point of discontinuity $x = 0$ do not exist, the function is not piecewise continuous in $[-1, 1]$.

15.9 Define the function

$$g(a) = \int_{-L+a}^{L+a} f(x) dx.$$

Using the fundamental theorem of calculus, we have

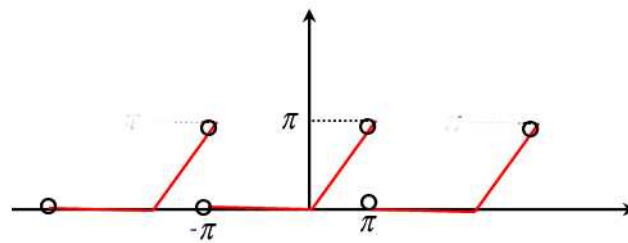
$$\begin{aligned} \frac{dg}{da} &= \frac{d}{da} \int_{-L+a}^{L+a} f(x) dx \\ &= f(L+a) - f(-L+a) = f(-L+a+2L) - f(-L+a) \\ &= f(-L+a) - f(-L+a) = 0 \end{aligned}$$

Hence, g is a constant function, and in particular we can write $g(a) = g(0)$ for all $a \in \mathbb{R}$ which gives the desired result.

15.10 (i) $f(x) = \frac{10}{3} + \sum_{n=1}^{\infty} \left[-\frac{1}{n\pi} \sin\left(\frac{2n\pi}{3}\right) \cos\left(\frac{2n\pi x}{3}\right) - \frac{1}{n\pi} \left(-\cos\left(\frac{2n\pi}{3}\right) + 1\right) \sin\left(\frac{2n\pi x}{3}\right) \right]$.

(ii) Using the theorem discussed in class, because this function and its derivative are piecewise continuous, the Fourier series will converge to the function at each point of continuity. At any point of discontinuity, the Fourier series will converge to the average of the left and right limits.

(iii)



Graph of $F(x)$

15.11 (a) $a_0 = 2$, $a_n = b_n = 0$ for $n \in \mathbb{N}$.

(b) $a_0 = 4$, $a_n = 0$, $b_1 = 1$, and $b_n = 0$.

(c) $a_0 = 1$, $a_n = 0$, $b_n = \frac{1}{\pi n} [(-1)^n - 1]$, $n \in \mathbb{N}$.

(d) $a_0 = a_n = 0$, $b_n = \frac{2L}{\pi n} (-1)^{n+1}$, $n \in \mathbb{N}$.

15.12 -1

15.13 $a_n = 0$ for all $n \in \mathbb{N}$.

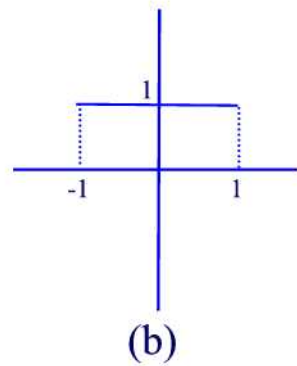
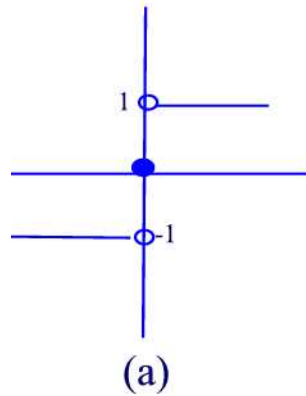
15.14 $\frac{f(0^-)+f(0^+)}{2} = \frac{-\pi+\pi}{2} = 0$.

15.15 (a) $f(x) = \frac{3}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}$.
 (b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}$.

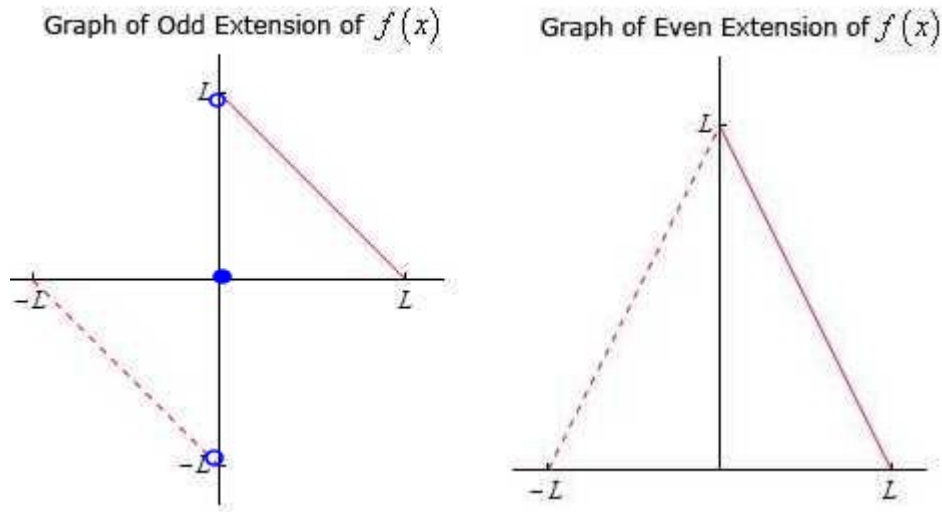
Section 16

16.1 $f(x) = 0$.

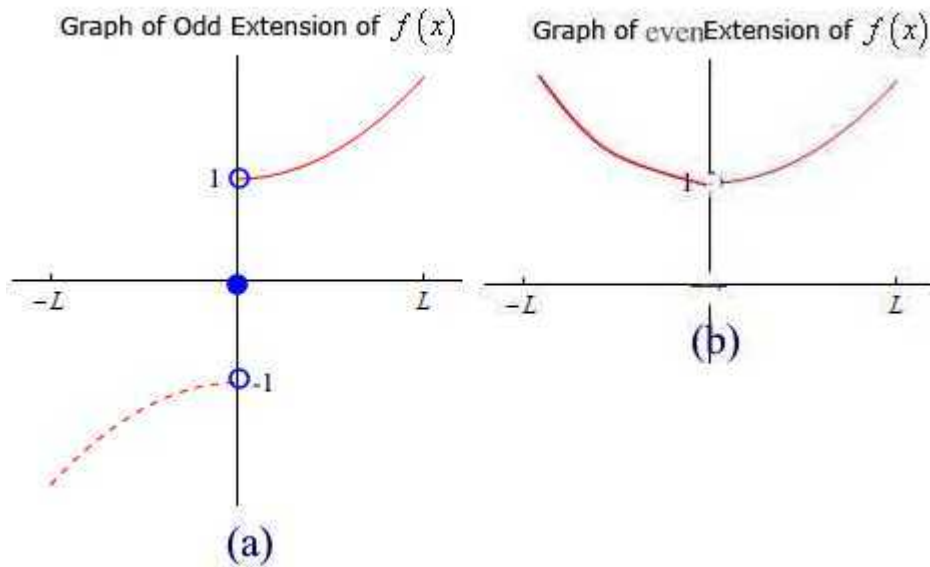
16.2



16.3



16.4



$$16.5 \quad f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2} [2 \cos(n\pi/2) - 1 - (-1)^n] \cos nx.$$

$$16.6 \quad f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi} [(-1)^n - 1] \cos nx.$$

$$16.7 \quad f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} [1 - (-1)^n] \sin nx.$$

$$\mathbf{16.8} \quad f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} n \left(\frac{1+(-1)^n}{n^2-1} \right) \sin nx.$$

$$\mathbf{16.9} \quad f(x) = \frac{1}{2}(e^2 - 1) + \sum_{n=1}^{\infty} \frac{4[(-1)^n e^2 - 1]}{4+n^2\pi^2} \cos(n\pi x).$$

16.10 (a) If $f(x) = \sin\left(\frac{2\pi}{L}x\right)$ then $b_n = 0$ if $n \neq 2$ and $b_2 = 1$.
 (b) If $f(x) = 1$ then

$$b_n = \frac{2}{L} \int_0^L \sin\left(\frac{n\pi}{L}x\right) dx = \frac{2}{n\pi} [1 - (-1)^n].$$

(c) If $f(x) = \cos\left(\frac{\pi}{L}x\right)$ then

$$b_1 = \frac{2}{L} \int_0^L \cos\left(\frac{\pi}{L}x\right) \sin\left(\frac{\pi}{L}x\right) dx = 0$$

and for $n \neq 1$ we have

$$\begin{aligned} b_n &= \frac{2}{L} \int_0^L \cos\left(\frac{\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx \\ &= \frac{1}{2} \frac{2}{L} \int_0^L \left[\sin\left(\frac{\pi x}{L}\right) (1+n) - \sin\left(\frac{\pi x}{L}\right) (1-n) \right] dx \\ &= \frac{1}{L} \left[-\frac{L}{(1+n)\pi} \cos\left(\frac{\pi x}{L}\right) (1+n) + \frac{L}{(1-n)\pi} \cos\left(\frac{\pi x}{L}\right) (1-n) \right]_0^L \\ &= \frac{2n}{(n^2-1)\pi} [1 + (-1)^n]. \end{aligned}$$

16.11 (a) $a_0 = 10$ and $a_1 = 1$, and $a_n = 0$ for $n \neq 1$.

(b) $a_0 = L$ and $a_n = \frac{2L}{(\pi n)^2} [(-1)^n - 1]$, $n \in \mathbb{N}$.

(c) $a_0 = 1$ and $a_n = \frac{2}{\pi n} \sin\left(\frac{\pi n}{2}\right)$, $n \in \mathbb{N}$.

16.12 By definition of Fourier sine coefficients,

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

The symmetry around $x = \frac{L}{2}$ can be written as

$$f\left(\frac{L}{2} + x\right) = f\left(\frac{L}{2} - x\right)$$

for all $x \in \mathbb{R}$. To use this symmetry it is convenient to make the change of variable $x - \frac{L}{2} = u$ in the above integral to obtain

$$b_n = \int_{-\frac{L}{2}}^{\frac{L}{2}} f\left(\frac{L}{2} + u\right) \sin\left[\frac{n\pi}{L}\left(\frac{L}{2} + u\right)\right] du.$$

Since $f\left(\frac{L}{2} + u\right)$ is even in u and for n even $\sin\left[\frac{n\pi}{L}\left(\frac{L}{2} + u\right)\right] = \sin\left(\frac{n\pi u}{L}\right)$ is odd in u , the integrand of the above integral is odd in u for n even. Since the integral is from $-\frac{L}{2}$ to $\frac{L}{2}$ we must have $b_{2n} = 0$ for $n = 0, 1, 2, \dots$ ■

16.13 By definition of Fourier cosine coefficients,

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi}{L}x\right) dx$$

The anti-symmetry around $x = \frac{L}{2}$ can be written as

$$f\left(\frac{L}{2} - y\right) = -f\left(\frac{L}{2} + y\right)$$

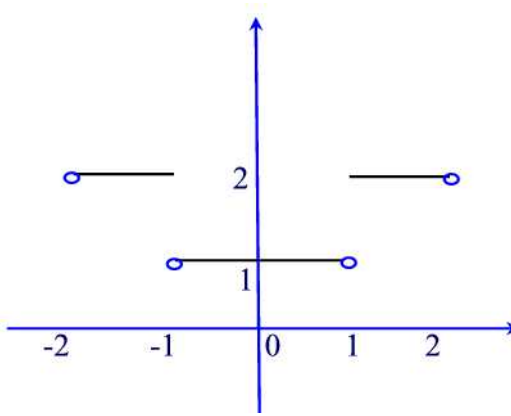
for all $y \in \mathbb{R}$. To use this symmetry it is convenient to make the change of variable $x = \frac{L}{2} + y$ in the above integral to obtain

$$a_n = \int_{-\frac{L}{2}}^{\frac{L}{2}} f\left(\frac{L}{2} + y\right) \cos\left[\frac{n\pi}{L}\left(\frac{L}{2} + y\right)\right] dy.$$

Since $f\left(\frac{L}{2} + y\right)$ is odd in y and for n even $\cos\left[\frac{n\pi}{L}\left(\frac{L}{2} + y\right)\right] = \pm \cos\left(\frac{n\pi y}{L}\right)$ is even in y , the integrand of the above integral is odd in y for n even. Since the integral is from $-\frac{L}{2}$ to $\frac{L}{2}$ we must have $a_{2n} = 0$ for all $n = 0, 1, 2, \dots$.

16.14 $\sin\left(\frac{\pi x}{L}\right) = \frac{2}{\pi} - \frac{2}{\pi} \sum_{n=2}^{\infty} \frac{1+(-1)^n}{n^2-1} \cos\left(\frac{n\pi x}{L}\right).$

16.15 (a)



(b) $a_0 = \frac{2}{2} \int_0^2 f(x) dx = 3.$

(c) We have

$$\begin{aligned} a_n &= \frac{2}{2} \int_0^2 f(x) \cos\left(\frac{n\pi x}{2}\right) dx \\ &= \int_0^1 \cos\left(\frac{n\pi x}{2}\right) dx + \int_1^2 2 \cos\left(\frac{n\pi x}{2}\right) dx \\ &= \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right) \Big|_0^1 + 2 \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right) \Big|_1^2 \\ &= -\frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right). \end{aligned}$$

(d) $b_n = 0$ since $f(x) \sin\left(\frac{n\pi x}{2}\right)$ is odd in $-2 \leq x \leq 2.$

(e)

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \left(-\frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) \right) \cos\left(\frac{n\pi x}{2}\right).$$

Section 17

17.1 We look for a solution of the form $u(x, y) = X(x)Y(y).$ Substituting in the given equation, we obtain

$$X''Y + XY'' + \lambda XY = 0.$$

Assuming $X(x)Y(y)$ is nonzero, dividing for $X(x)Y(y)$ and subtract both sides for $\frac{X''(x)}{X(x)}$, we find:

$$-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} + \lambda.$$

The left hand side is a function of x while the right hand side is a function of y . This says that they must equal to a constant. That is,

$$-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} + \lambda = \delta.$$

where δ is a constant. This results in the following two ODEs

$$X'' + \delta X = 0 \text{ and } Y'' + (\lambda - \delta)Y = 0.$$

- If $\delta > 0$ and $\lambda - \delta > 0$ then

$$\begin{aligned} X(x) &= A \cos \sqrt{\delta}x + B \sin \sqrt{\delta}x \\ Y(y) &= C \cos \sqrt{(\lambda - \delta)}y + D \sin \sqrt{(\lambda - \delta)}y \end{aligned}$$

- If $\delta > 0$ and $\lambda - \delta < 0$ then

$$\begin{aligned} X(x) &= A \cos \sqrt{\delta}x + B \sin \sqrt{\delta}x \\ Y(y) &= C e^{-\sqrt{-(\lambda - \delta)}y} + D e^{\sqrt{-(\lambda - \delta)}y} \end{aligned}$$

- If $\delta = \lambda > 0$ then

$$\begin{aligned} X(x) &= A \cos \sqrt{\delta}x + B \sin \sqrt{\delta}x \\ Y(y) &= Cy + D \end{aligned}$$

- If $\delta = \lambda < 0$ then

$$\begin{aligned} X(x) &= A e^{-\sqrt{-\delta}x} + B e^{\sqrt{-\delta}x} \\ Y(y) &= Cy + D \end{aligned}$$

- If $\delta < 0$ and $\lambda - \delta > 0$ then

$$\begin{aligned} X(x) &= A e^{-\sqrt{-\delta}x} + B e^{\sqrt{-\delta}x} \\ Y(y) &= C \cos \sqrt{(\lambda - \delta)}y + D \sin \sqrt{(\lambda - \delta)}y \end{aligned}$$

- If $\delta < 0$ and $\lambda - \delta < 0$ then

$$X(x) = Ae^{-\sqrt{-\delta}x} + Be^{\sqrt{-\delta}x}$$

$$Y(y) = Ce^{-\sqrt{-(\lambda-\delta)y}} + De^{\sqrt{-(\lambda-\delta)y}}.$$

17.2 Let's assume that the solution can be written in the form $u(x, t) = X(x)T(t)$. Substituting into the heat equation we obtain

$$\frac{X''}{X} = \frac{T'}{kT}.$$

Since X only depends on x and T only depends on t , we must have that there is a constant λ such that

$$\frac{X''}{X} = \lambda \text{ and } \frac{T'}{kT} = \lambda.$$

This gives the two ordinary differential equations

$$X'' - \lambda X = 0 \text{ and } T' - k\lambda T = 0.$$

Next, we consider the three cases of the sign of λ .

Case 1: $\lambda = 0$

In this case, $X'' = 0$ and $T' = 0$. Solving these equations we find $X(x) = ax + b$ and $T(t) = c$.

Case 2: $\lambda > 0$

In this case, $X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$ and $T(t) = Ce^{k\lambda t}$.

Case 3: $\lambda < 0$

In this case, $X(x) = A \cos \sqrt{-\lambda}x + B \sin \sqrt{-\lambda}x$ and $T(t) = Ce^{k\lambda t}$.

17.3 $r^2 R''(r) + rR'(r) - \lambda R(r) = 0$ and $\Theta''(\theta) + \lambda\Theta(\theta) = 0$.

17.4 $X'' = (2 + \lambda)X$, $T'' = \lambda T$, $X(0) = 0$, $X'(1) = 0$.

17.5 $X'' - \lambda X = 0$, $T' = k\lambda T$, $X(0) = 0 = X'(L)$.

17.6 $u(x, t) = Ce^{\lambda(x-t)}$.

17.7 $5X''' - 7X'' - \lambda X = 0$ and $3Y'' - \lambda Y' = 0$.

17.8 $u(x, y) = Ce^{\lambda x - \frac{y}{\lambda}}$.

17.9 $u(x, y) = Ce^{\lambda x} y^\lambda$.

17.10 We look for a solution of the form $u(x, y) = X(x)T(t)$. Substituting in the wave equation, we obtain

$$X''(x)T(t) - X(x)T''(t) = 0.$$

Assuming $X(x)T(t)$ is nonzero, dividing for $X(x)T(t)$ we find:

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)}.$$

The left hand side is a function of x while the right hand side is a function of t . This says that they must equal to a constant. That is,

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = \lambda$$

where λ is a constant. This results in the following two ODEs

$$X'' - \lambda X = 0 \text{ and } T'' - \lambda T = 0.$$

The solutions of these equations depend on the sign of λ .

- If $\lambda > 0$ then the solutions are given

$$\begin{aligned} X(x) &= Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x} \\ T(t) &= Ce^{\sqrt{\lambda}t} + De^{-\sqrt{\lambda}t} \end{aligned}$$

where $A, B, C,$ and D are constants. In this case,

$$u(x, t) = k_1 e^{\sqrt{\lambda}(x+t)} + k_2 e^{\sqrt{\lambda}(x-t)} + k_3 e^{-\sqrt{\lambda}(x+t)} + k_4 e^{-\sqrt{\lambda}(x-t)}.$$

- If $\lambda = 0$ then

$$\begin{aligned} X(x) &= Ax + B \\ T(t) &= Ct + D \end{aligned}$$

where A , B , and C are arbitrary constants. In this case,

$$u(x, t) = k_1xt + k_2x + k_3t + k_4.$$

• If $\lambda < 0$ then

$$X(x) = A \cos \sqrt{-\lambda}x + B \sin \sqrt{-\lambda}x$$

$$T(t) = A \cos \sqrt{-\lambda}t + B \sin \sqrt{-\lambda}t$$

where A , B , C , and D are arbitrary constants. In this case,

$$u(x, t) = k_1 \cos \sqrt{-\lambda}x \cos \sqrt{-\lambda}t + k_2 \cos \sqrt{-\lambda}x \sin \sqrt{-\lambda}t \\ + k_3 \sin \sqrt{-\lambda}x \cos \sqrt{-\lambda}t + k_4 \sin \sqrt{-\lambda}x \sin \sqrt{-\lambda}t.$$

17.11 (a) $u(r, t) = R(r)T(t)$, $T'(t) = k\lambda T$, $r(rR')' = \lambda R$.

(b) $u(x, t) = X(x)T(t)$, $T' = \lambda T$, $kX'' - (\alpha + \lambda)X = 0$.

(c) $u(x, t) = X(x)T(t)$, $T' = \lambda T$, $kX'' - aX' = \lambda X$.

(d) $u(x, t) = X(x)Y(y)$, $X'' = \lambda X$, $Y'' = -\lambda Y$.

(e) $u(x, t) = X(x)T(t)$, $T' = k\lambda T$, $X'''' = \lambda X$.

17.12 $u(x, y) = Ce^{\lambda(x+y)}$.

17.13 $X'' = \lambda X$, $Y' - Y'' + Y = \lambda Y$.

Section 18

18.1 $u(x, t) = \sin\left(\frac{\pi}{2}x\right)e^{-\frac{\pi^2 k}{4}t} + 3 \sin\left(\frac{5\pi}{2}x\right)e^{-\frac{25\pi^2 k}{4}t}$.

18.2 $u(x, t) = \frac{8d}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin\left(\frac{(2n-1)\pi}{L}x\right)e^{-\frac{k(2n-1)^2\pi^2}{L^2}t}$.

18.3 $u(x, t) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(4n^2-1)} \cos\left(\frac{2n\pi}{L}x\right)e^{-k\frac{4n^2\pi^2}{L^2}t}$.

18.4 $u(x, t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right)e^{-\frac{kn^2\pi^2}{L^2}t}$ where

$$C_n = \begin{cases} -\frac{4}{n\pi} & n = 2, 6, 10, \dots \\ 0 & n = 4, 8, 12, \dots \\ \frac{6}{n\pi} & n \text{ is odd.} \end{cases}$$

$$18.5 \quad u(x, t) = 6 \sin\left(\frac{9\pi}{L}x\right) e^{-\frac{81k\pi^2}{L^2}t}.$$

$$18.6 \quad u(x, t) = \frac{1}{2} + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi}{L}x\right) e^{-\frac{kn^2\pi^2}{L^2}t} \text{ where}$$

$$C_n = \begin{cases} -\frac{2}{n\pi} & n = 1, 5, 9, \dots \\ \frac{2}{n\pi} & n = 3, 7, 11, \dots \\ 0 & n \text{ is even} \end{cases}$$

$$18.7 \quad u(x, t) = 6 + 4 \cos\left(\frac{3\pi}{L}x\right) e^{-\frac{9k\pi^2}{L^2}t}.$$

$$18.8 \quad u(x, t) = -3 \cos\left(\frac{8\pi}{L}x\right) e^{-\frac{64k\pi^2}{L^2}t}.$$

18.9

$$u(x, t) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) e^{-\left(1+\frac{n^2\pi^2}{L^2}\right)t}.$$

As $t \rightarrow \infty$, $e^{-\left(1+\frac{n^2\pi^2}{L^2}\right)t} \rightarrow 0$ for each $n \in \mathbb{N}$. Hence, $u(x, t) \rightarrow 0$.

18.10 (b) We have

$$\begin{aligned} E'(t) &= 2 \int_0^1 w(x, t) w_t(x, t) dx \\ &= 2 \int_0^1 w(x, t) [w_{xx}(x, t) - w(x, t)] dx \\ &= 2w(x, t) w_x(x, t) \Big|_0^1 - 2 \left[\int_0^1 w_x^2(x, t) dx + \int_0^1 w^2(x, t) dx \right] \\ &= -2 \left[\int_0^1 w_x^2(x, t) dx + \int_0^1 w^2(x, t) dx \right] \leq 0 \end{aligned}$$

Hence, E is decreasing, and $0 \leq E(t) \leq E(0)$ for all $t > 0$.

(c) Since $w(x, 0) = 0$, we must have $E(0) = 0$. Hence, $E(t) = 0$ for all $t \geq 0$. This implies that $w(x, t) = 0$ for all $t > 0$ and all $0 < x < 1$. Therefore $u_1(x, t) = u_2(x, t)$. This means that the given problem has a unique solution.

$$18.11 \quad (a) \quad u(0, t) = 0 \text{ and } u_x(1, t) = 0.$$

(b) Let's assume that the solution can be written in the form $u(x, t) = X(x)T(t)$. Substituting into the heat equation we obtain

$$\frac{X''}{X} = \frac{T'}{T}.$$

Since X only depends on x and T only depends on t , we must have that there is a constant λ such that

$$\frac{X''}{X} = \lambda \text{ and } \frac{T'}{T} = \lambda.$$

This gives the two ordinary differential equations

$$X'' - \lambda X = 0 \text{ and } T' - \lambda T = 0.$$

As far as the boundary conditions, we have

$$u(0, t) = 0 = X(0)T(t) \implies X(0) = 0$$

and

$$u_x(1, t) = 0 = X'(1)T(t) \implies X'(1) = 0.$$

Note that T is not the zero function for otherwise $u \equiv 0$ and this contradicts our assumption that u is the non-trivial solution.

(c) We have $X' = \sqrt{-\lambda} \cos \sqrt{-\lambda}x$ and $X'' = \lambda \sin \sqrt{-\lambda}x$. Thus, $X'' - \lambda X = 0$. Moreover $X(0) = 0$. Now, $X'(1) = 0$ implies $\cos \sqrt{-\lambda} = 0$ or $\sqrt{-\lambda} = (n - \frac{1}{2})\pi$, $n \in \mathbb{N}$. Hence, $\lambda = - (n - \frac{1}{2})^2 \pi^2$.

18.12 (a) Let's assume that the solution can be written in the form $u(x, t) = X(x)T(t)$. Substituting into the heat equation we obtain

$$\frac{X''}{X} = \frac{T'}{kT}.$$

Since the LHS only depends on x and the RHS only depends on t , there must be a constant λ such that

$$\frac{X''}{X} = \lambda \text{ and } \frac{T'}{kT} = \lambda.$$

This gives the two ordinary differential equations

$$X'' - \lambda X = 0 \text{ and } T' - k\lambda T = 0.$$

As far as the boundary conditions, we have

$$u(0, t) = 0 = X(0)T(t) \implies X(0) = 0$$

and

$$u(L, t) = 0 = X(L)T(t) \implies X(L) = 0.$$

Note that T is not the zero function for otherwise $u \equiv 0$ and this contradicts our assumption that u is the non-trivial solution.

Next, we consider the three cases of the sign of λ .

Case 1: $\lambda = 0$

In this case, $X'' = 0$. Solving this equation we find $X(x) = ax + b$. Since $X(0) = 0$ we find $b = 0$. Since $X(L) = 0$ we find $a = 0$. Hence, $X \equiv 0$ and $u(x, t) \equiv 0$. That is, u is the trivial solution.

Case 2: $\lambda > 0$

In this case, $X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$. Again, the conditions $X(0) = X(L) = 0$ imply $A = B = 0$ and hence the solution is the trivial solution.

Case 3: $\lambda < 0$

In this case, $X(x) = A \cos \sqrt{-\lambda}x + B \sin \sqrt{-\lambda}x$. The condition $X(0) = 0$ implies $A = 0$. The condition $X(L) = 0$ implies $B \sin \sqrt{-\lambda}L = 0$. We must have $B \neq 0$ otherwise $X(x) = 0$ and this leads to the trivial solution. Since $B \neq 0$, we obtain $\sin \sqrt{-\lambda}L = 0$ or $\sqrt{-\lambda}L = n\pi$ where $n \in \mathbb{N}$. Solving for λ we find $\lambda = -\frac{n^2\pi^2}{L^2}$. Thus, we obtain infinitely many solutions given by

$$X_n(x) = A_n \sin \frac{n\pi}{L}x, \quad n \in \mathbb{N}.$$

Now, solving the equation

$$T' - \lambda kT = 0$$

by the method of separation of variables we obtain

$$T_n(t) = B_n e^{-\frac{n^2\pi^2}{L^2}kt}, \quad n \in \mathbb{N}.$$

Hence, the functions

$$u_n(x, t) = C_n \sin \left(\frac{n\pi}{L}x \right) e^{-\frac{n^2\pi^2}{L^2}kt}, \quad n \in \mathbb{N}$$

satisfy $u_t = ku_{xx}$ and the boundary conditions $u(0, t) = u(L, t) = 0$. Now, in order for these solutions to satisfy the initial value condition $u(x, 0) = 6 \sin\left(\frac{9\pi x}{L}\right)$, we invoke the superposition principle of linear PDE to write

$$u(x, t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2}{L^2}kt}. \quad (13.4)$$

To determine the unknown constants C_n we use the initial condition $u(x, 0) = 6 \sin\left(\frac{9\pi x}{L}\right)$ in (13.4) to obtain

$$6 \sin\left(\frac{9\pi x}{L}\right) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right).$$

By equating coefficients we find $C_9 = 6$ and $C_n = 0$ if $n \neq 9$. Hence, the solution to the problem is given by

$$u(x, t) = 6 \sin\left(\frac{9\pi x}{L}\right) e^{-\frac{81\pi^2}{L^2}kt}.$$

(b) Similar to (a), we find

$$u(x, t) = 3 \sin\left(\frac{\pi}{L}x\right) e^{-\frac{\pi^2}{L^2}kt} - \sin\left(\frac{3\pi}{L}x\right) e^{-\frac{9\pi^2}{L^2}kt}$$

18.13 $u(x, t) = \cos\left(\frac{\pi x}{L}\right) e^{-\frac{\pi^2}{L^2}kt} + 4 \cos\left(\frac{5\pi x}{L}\right) e^{-\frac{25\pi^2}{L^2}kt}.$

(b) $u(x, t) = 5.$

18.14 $u(x, t) = 6 \sin xe^{-8t}.$

Section 19

19.1 $u(x, y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{b}y\right) \sinh\left(\frac{n\pi}{b}x\right)$ where

$$B_n = \left[\frac{2}{b} \int_0^b f_2(y) \sin\left(\frac{n\pi}{b}y\right) dy \right] \left[\sinh\left(\frac{n\pi}{b}a\right) \right]^{-1}.$$

19.2 $u(x, y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{a}x\right) \sinh\left(\frac{n\pi}{a}(y-b)\right)$ where

$$B_n = \left[\frac{2}{a} \int_0^a g_1(x) \sin\left(\frac{n\pi}{a}x\right) dx \right] \left[\sinh\left(-\frac{n\pi}{a}b\right) \right]^{-1}.$$

19.3 $u(x, y) = 2xy + \frac{3}{\sinh \pi} \sin \pi x \sinh \pi y.$

19.4 If $u(x, y) = x^2 - y^2$ then $u_{xx} = 2$ and $u_{yy} = -2$ so that $\Delta u = 0$.
If $u(x, y) = 2xy$ then $u_{xx} = u_{yy} = 0$ so that $\Delta u = 0$.

19.5

$$u(x, y) = \sum_{n=1}^{\infty} \left[A_n \cosh \left(\frac{n\pi}{L} y \right) + B_n \sinh \left(\frac{n\pi}{L} y \right) \right] \sin \frac{n\pi}{L} x.$$

where

$$A_n = \left[\frac{2}{L} \int_0^L (f_1(x) + f_2(x)) \sin \frac{n\pi}{L} x dx \right] \left[\cosh \left(\frac{n\pi H}{2L} \right) \right]^{-1}$$

and

$$B_n = \left[\frac{2}{L} \int_0^L (f_2(x) - f_1(x)) \sin \frac{n\pi}{L} x dx \right] \left[\sinh \left(\frac{n\pi H}{2L} \right) \right]^{-1}$$

19.6 (a) Differentiating term by term with respect to x we find

$$u_x + iv_x = \sum_{n=0}^{\infty} n a_n (x + iy)^{n-1}.$$

Likewise, differentiating term by term with respect to y we find

$$u_y + iv_y = \sum_{n=0}^{\infty} n a_n i (x + iy)^{n-1}.$$

Multiply this equation by i we find

$$-iu_y + v_y = \sum_{n=0}^{\infty} n a_n (x + iy)^{n-1}.$$

Hence, $u_x + iv_x = v_y - iu_y$ which implies $u_x = v_y$ and $v_x = -u_y$.

(b) We have $u_{xx} = (v_y)_x = (v_x)_y = -u_{yy}$ so that $\Delta u = 0$. Similar argument for $\Delta v = 0$.

19.7 Polar and Cartesian coordinates are related by the expressions $x =$

$r \cos \theta$ and $y = r \sin \theta$ where $r = (x^2 + y^2)^{\frac{1}{2}}$ and $\tan \theta = \frac{y}{x}$. Using the chain rule we obtain

$$\begin{aligned} u_x &= u_r r_x + u_\theta \theta_x = \cos \theta u_r - \frac{\sin \theta}{r} u_\theta \\ u_{xx} &= u_{xr} r_x + u_{x\theta} \theta_x \\ &= \left(\cos \theta u_{rr} + \frac{\sin \theta}{r^2} u_\theta - \frac{\sin \theta}{r} u_{r\theta} \right) \cos \theta \\ &\quad + \left(-\sin \theta u_r + \cos \theta u_{r\theta} - \frac{\cos \theta}{r} u_\theta - \frac{\sin \theta}{r} u_{\theta\theta} \right) \left(-\frac{\sin \theta}{r} \right) \\ u_y &= u_r r_y + u_\theta \theta_y = \sin \theta u_r + \frac{\cos \theta}{r} u_\theta \\ u_{yy} &= u_{yr} r_y + u_{y\theta} \theta_y \\ &= \left(\sin \theta u_{rr} - \frac{\cos \theta}{r^2} u_\theta + \frac{\cos \theta}{r} u_{r\theta} \right) \sin \theta \\ &\quad + \left(\cos \theta u_r + \sin \theta u_{r\theta} - \frac{\sin \theta}{r} u_\theta + \frac{\cos \theta}{r} u_{\theta\theta} \right) \left(\frac{\cos \theta}{r} \right) \end{aligned}$$

Substituting these equations into (21.1) we obtain the desired equation.

19.8 $u(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + u_4(x, y)$ where

$$u_1(x, y) = 0$$

$$u_2(x, y) = \sum_{n=1}^{\infty} \left[-\frac{2}{n\pi} \cdot \frac{(-1)^n}{\sinh\left(\frac{3n\pi}{2}\right)} \right] \sin \frac{n\pi}{2} x \sinh \left(\frac{n\pi}{2} y \right)$$

$$u_3(x, y) = \frac{1}{\sinh\left(\frac{8\pi}{3}\right)} \sinh \left(\frac{4\pi(x-2)}{3} \right) \sin \left(\frac{4\pi}{3} y \right)$$

$$u_4(x, y) = \sum_{n=1}^{\infty} \frac{14(1 - (-1)^n)}{n\pi \sinh\left(\frac{2n\pi}{3}\right)} \sin \left(\frac{n\pi}{3} y \right) \sinh \left(\frac{n\pi}{3} x \right).$$

19.9

$$u(x, y) = \frac{4}{\sinh\left(\frac{\pi L}{2H}\right)} \left\{ \sinh \left(\frac{\pi x}{2H} \right) - \sinh \left(\frac{\pi(x-L)}{2H} \right) \right\} \cos \frac{\pi y}{2H}.$$

19.10 $u(x, t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-\sqrt{\lambda_n} x} \cos \sqrt{\lambda_n} y$ where

$$A_0 = \frac{2}{H} \int_0^H f(y) dy$$

$$A_n = \frac{2}{H} \int_0^H f(y) \cos \frac{n\pi}{H} y dy.$$

19.11

$$u(x, y) = \frac{20}{\left(\frac{\pi}{L} \cosh\left(\frac{\pi}{L}\right) H + \sinh\left(\frac{\pi}{L}\right) H\right)} \sin\left(\frac{\pi x}{L}\right) - \frac{5}{\left(\frac{3\pi}{L} \cosh\left(\frac{3\pi}{L}\right) H + \sinh\left(\frac{3\pi}{L}\right) H\right)} \sin\left(\frac{3\pi x}{L}\right) \blacksquare$$

19.12 $u(x, y) = \sin(2\pi x) e^{-2\pi y}$.

19.13 $u(x, y) = y$.

19.14 $u(x, y) = \frac{1}{2}x^2 - \frac{1}{2}y^2 - ax + by + C$ where C is an arbitrary constant.

19.15 $u(x, y) = \frac{2 \cosh 3y \sin 3x}{3 \sinh 6} - \frac{5 \cosh 10y \sin 10x}{10 \sinh 20}$.

Section 20

20.1 $u(r, \theta) = 3r^5 \sin 5\theta$.

20.2 $u(r, \theta) = \frac{\pi}{4} + \sum_{n=1}^{\infty} r^n \left[\frac{1 - (-1)^n}{n^2 \pi} \cos n\theta + \frac{\sin n\theta}{n} \right]$.

20.3 $u(r, \theta) = C_0 + r^2 \cos 2\theta$.

20.4 Substituting C_0 , A_n , and B_n into the right-hand side of $u(r, \theta)$ we find

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} f(\phi) d\phi + \sum_{n=1}^{\infty} \frac{r^n}{\pi a^n} \int_0^{2\pi} f(\phi) [\cos n\phi \cos n\theta + \sin n\phi \sin n\theta] d\phi$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(\phi) \left[1 + 2 \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\theta - \phi) \right] d\phi.$$

20.5 (a) We have $e^{it} = \cos t + i \sin t$ and $e^{-it} = \cos t - i \sin t$. The result follows by adding these two equalities and dividing by 2.

(b) This follows from the fact that

$$\cos n(\theta - \phi) = \frac{1}{2}(e^{in(\theta-\phi)} + e^{-in(\theta-\phi)}).$$

(c) We have $|q_1| = \frac{r}{a} \sqrt{\cos^2(\theta - \phi) + \sin^2(\theta - \phi)} = \frac{r}{a} < 1$ since $0 < r < a$. A similar argument shows that $|q_2| < 1$.

20.6 (a) The first sum is a convergent geometric series with ratio q_1 and sum

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n e^{in(\theta-\phi)} &= \frac{\frac{r}{a} e^{i(\theta-\phi)}}{1 - q_1} \\ &= \frac{r e^{i(\theta-\phi)}}{a - r e^{i(\theta-\phi)}} \end{aligned}$$

Similar argument for the second sum.

(b) We have

$$\begin{aligned} 1 + 2 \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\theta - \phi) &= 1 + \frac{r e^{i(\theta-\phi)}}{a - r e^{i(\theta-\phi)}} \\ &\quad + \frac{r e^{-i(\theta-\phi)}}{a - r e^{-i(\theta-\phi)}} \\ &= 1 + \frac{r}{a e^{-i(\theta-\phi)} - r} + \frac{r}{a e^{-i(\theta-\phi)} - r} \\ &= 1 + \frac{r}{a \cos(\theta - \phi) - r - ai \sin(\theta - \phi)} \\ &\quad + \frac{r}{a \cos(\theta - \phi) - r + ai \sin(\theta - \phi)} \\ &= 1 + \frac{r[a \cos(\theta - \phi) - r + ai \sin(\theta - \phi)]}{a^2 + 2ar \cos(\theta - \phi) + r^2} \\ &\quad + \frac{r[a \cos(\theta - \phi) - r - ai \sin(\theta - \phi)]}{a^2 - 2ar \cos(\theta - \phi) + r^2} \\ &= \frac{a^2 - r^2}{a^2 - 2ar \cos(\theta - \phi) + r^2}. \end{aligned}$$

20.7 We have

$$\begin{aligned} u(r, \theta) &= \frac{1}{2\pi} \int_0^{2\pi} f(\phi) \left[1 + 2 \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\theta - \phi) \right] d\phi \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(\phi) \frac{a^2 - r^2}{a^2 - 2ar \cos(\theta - \phi) + r^2} d\phi \\ &= \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{f(\phi)}{a^2 - 2ar \cos(\theta - \phi) + r^2} d\phi. \end{aligned}$$

20.8 $u(r, \theta) = 2 \sum_{n=1}^{\infty} (-1)^{n+1} r^n \frac{\sin n\theta}{n}.$

20.9 (a) Differentiating $u(r, t) = R(r)T(t)$ with respect to r and t we find

$$u_{tt} = RT'' \text{ and } u_r = R'T \text{ and } u_{rr} = R''T.$$

Substituting these into the given PDE we find

$$RT'' = c^2 \left(R''T + \frac{1}{r}R'T \right)$$

Dividing both sides by c^2RT we find

$$\frac{1}{c^2} \frac{T''}{T} = \frac{R''}{R} + \frac{1}{r} \frac{R'}{R}.$$

Since the RHS of the above equation depends on r only, and the LHS depends on t only, they must equal to a constant λ .

(b) The given boundary conditions imply

$$u(a, t) = 0 = R(a)T(t) \implies R(a) = 0$$

$$u(r, 0) = f(r) = R(r)T(0)$$

$$u_t(r, 0) = g(r) = R(r)T'(0).$$

If $\lambda = 0$ then $R'' + \frac{1}{r}R' = 0$ and this implies $R(r) = C \ln r$. Using the condition $R(a) = 0$ we find $C = 0$ so that $R(r) = 0$ and hence $u \equiv 0$. If $\lambda > 0$ then $T'' - \lambda c^2 T = 0$. This equation has the solution

$$T(t) = A \cos(c\sqrt{\lambda}t) + B \sin(c\sqrt{\lambda}t).$$

The condition $u(r, 0) = f(r)$ implies that $A = f(r)$ which is not possible. Hence, $\lambda < 0$

20.10 (a) Follows from the figure and the definitions of trigonometric functions in a right triangle.

(b) The result follows from equation (20.1).

20.11 By the maximum principle we have

$$\min_{(x,y) \in \partial\Omega} u(x, y) \leq u(x, y) \leq \max_{(x,y) \in \partial\Omega} u(x, y), \quad \forall (x, y) \in \Omega$$

But $\min_{(x,y) \in \partial\Omega} u(x, y) = u(1, 0) = 1$ and $\max_{(x,y) \in \partial\Omega} u(x, y) = u(-1, 0) = 3$. Hence,

$$1 \leq u(x, y) \leq 3$$

and this implies that $u(x, y) > 0$ for all $(x, y) \in \Omega$.

20.12 (i) $u(1, 0) = 4$ (ii) $u(-1, 0) = -2$.

20.13 Using the maximum principle and the hypothesis on g_1 and g_2 , for all $(x, y) \in \Omega \cup \partial\Omega$ we have

$$\begin{aligned} \min_{(x,y) \in \partial\Omega} u_1(x, y) &= \min_{(x,y) \in \partial\Omega} g_1(x, y) \\ &\leq u_1(x, y) \leq \max_{(x,y) \in \partial\Omega} u_1(x, y) \\ &= \max_{(x,y) \in \partial\Omega} g_1(x, y) < \min_{(x,y) \in \partial\Omega} g_2(x, y) \\ &= \min_{(x,y) \in \partial\Omega} g_2(x, y) = \min_{(x,y) \in \partial\Omega} u_2(x, y) \\ &\leq u_2(x, y) \leq \max_{(x,y) \in \partial\Omega} u_2(x, y) = \max_{(x,y) \in \partial\Omega} g_2(x, y) \blacksquare \end{aligned}$$

20.14 We have

$$\begin{aligned} \Delta(r^n \cos(n\theta)) &= \frac{\partial^2}{\partial r^2}(r^n \cos(n\theta)) + \frac{1}{r} \frac{\partial}{\partial r}(r^n \cos(n\theta)) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}(r^n \cos(n\theta)) \\ &= n(n-1)r^{n-2} \cos(n\theta) + nr^{n-2} \cos(n\theta) - r^{n-2}n^2 \cos(n\theta) = 0 \end{aligned}$$

Likewise, $\Delta(r^n \sin(n\theta)) = 0$.

20.15 $u(r, \theta) = \frac{1}{2} - \frac{r^2}{2a^2} \cos 2\theta.$

20.16 $u(r, \theta) = \ln 2 + 4 \left(\frac{a}{r}\right)^3 \cos 3\theta.$

Section 21

21.1 Convergent.

21.2 Divergent.

21.3 Convergent.

21.4 $\frac{1}{s-3}, s > 3.$

21.5 $\frac{1}{s^2} - \frac{5}{s}, s > 0.$

21.6 $f(t) = e^{(t-1)^2}$ does not have a Laplace transform.

21.7 $\frac{4}{s} - \frac{4}{s^2} + \frac{2}{s^3}, s > 0.$

21.8 $\frac{e^{-s}}{s^2}, s > 0.$

21.9 $-\frac{e^{-2s}}{s} + \frac{1}{s^2}(e^{-s} - e^{-2s}), s \neq 0.$

21.10 $-\frac{t^n e^{-st}}{s} + \frac{n}{s} \int t^{n-1} e^{-st} dt, s > 0.$

21.11 (a) 0 (b) 0.

21.12 $\frac{5}{s+7} + \frac{1}{s^2} + \frac{2}{s-2}, s > 2.$

21.13 $3e^{2t}, t \geq 0.$

21.14 $-2t + e^{-t}, t \geq 0.$

21.15 $2(e^{-2t} + e^{2t}), t \geq 0.$

21.16 $\frac{2}{s-1} + \frac{5}{s}, s > 1.$

$$21.17 \frac{e^{-s}}{s-3}, \quad s > 3.$$

$$21.18 \frac{1}{2} \left(\frac{1}{s} - \frac{s}{s^2+4\omega^2} \right), \quad s > 0.$$

$$21.19 \frac{3}{s^2+36}, \quad s > 0.$$

$$21.20 \frac{s-2}{(s-2)^2+9}, \quad s > 3.$$

$$21.21 \frac{2}{(s-4)^3} + \frac{3}{(s-4)^2} + \frac{5}{s-4}, \quad s > 4.$$

$$21.22 2 \sin 5t + 4e^{3t}, \quad t \geq 0.$$

$$21.23 \frac{5}{6}e^{3t}t^3, \quad t \geq 0.$$

$$21.24$$

$$\begin{cases} 0, & 0 \leq t < 2 \\ e^{9(t-2)}, & t \geq 2. \end{cases}$$

$$21.25 3e^{3t} - 3e^{-t}, \quad t \geq 0.$$

$$21.26 4[e^{3(t-5)} - e^{-3(t-5)}]H(t-5), \quad t \geq 0.$$

$$21.27 y(t) = 2e^{-4t} + 3[H(t-1) - H(t-3)] - 3[e^{-4(t-1)}H(t-1) - e^{-4(t-3)}H(t-3)], \quad t \geq 0.$$

$$21.28 \frac{1}{5}e^{3t} + \frac{1}{20}e^{-2t} - \frac{1}{4}e^{2t}, \quad t \geq 0.$$

$$21.29 \frac{e^t - e^{-2t}}{3}.$$

$$21.30 \frac{t}{2} \sin t.$$

$$21.31 \frac{t^5}{120}.$$

$$21.32 \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}.$$

$$21.33 -t + \frac{e^t}{2} - \frac{e^{-t}}{2}.$$

Section 22

$$22.1 \quad u(x, t) = \sin(x - t) - H(t - x) \sin(x - t).$$

$$22.2 \quad u(x, t) = [\sin(x - t) - H(t - x) \sin(x - t)]e^{-t}.$$

$$22.3 \quad u(x, t) = 2e^{-4\pi^2 t} \sin \pi x + 6e^{-16\pi^2 t} \sin 2\pi x.$$

$$22.4 \quad u(x, t) = [\sin(x - t) - H(t - x) \sin(x - t)]e^t.$$

$$22.5 \quad u(x, t) = \frac{1}{2}t^2 + \frac{1}{2}H(t - x)(t - x)^2.$$

$$22.6 \quad u(x, t) = \left(t - \frac{1}{2}x^2\right) H\left(t - \frac{1}{2}x^2\right).$$

$$22.7 \quad u(x, t) = \mathcal{L}^{-1}\left(\frac{e^{-\frac{s}{c}x}}{s^2+1}\right) = H\left(t - \frac{x}{c}\right) \sin\left(t - \frac{x}{c}\right).$$

$$22.8 \quad u(x, t) = 2 \sin x \cos 3t.$$

$$22.9 \quad u(x, y) = y(x + 1) + 1.$$

$$22.10 \quad u(x, t) = -c \int_0^t f(t - \tau) H\left(\tau - \frac{x}{c}\right) d\tau.$$

$$22.11 \quad u(x, t) = e^{-5x} e^{-4t}.$$

$$22.12 \quad u(x, t) = \mathcal{L}^{-1}\left(-\frac{T}{s} e^{-\frac{\sqrt{s}}{c}x} + \frac{T}{s}\right).$$

$$22.13 \quad u(x, t) = 5e^{-3\pi^2 t} \sin(\pi x).$$

$$22.14 \quad u(x, t) = 40e^{-t} \cos \frac{x}{2}.$$

$$22.15 \quad u(x, t) = 3 \sin \pi x \cos 2\pi t.$$

Section 23

$$23.1 \quad \frac{(-1)^n i}{n\pi}.$$

$$23.2 \quad f(x) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{1}{n\pi} \sin\left(\frac{n\pi}{2}\right) (e^{inx} + e^{-inx}).$$

$$23.3 \quad f(x) = \frac{\sinh a\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n (a+in)}{(a^2+n^2)} e^{inx}.$$

$$23.4 \quad f(x) = \frac{e^{ix} - e^{-ix}}{2i}.$$

$$23.5 \quad f(x) = \frac{1}{2\pi} \left\{ T + \sum_{n=-\infty}^{-1} \frac{i}{n} [e^{-inT} - 1] e^{inx} + \sum_{n=1}^{\infty} \frac{i}{n} [e^{-inT} - 1] e^{inx} \right\}.$$

$$23.6 \quad (a) \quad f(x) = \frac{\pi^2}{3} + \sum_{n=-\infty}^{-1} \frac{2}{n^2} (-1)^n e^{inx} + \sum_{n=1}^{\infty} \frac{2}{n^2} (-1)^n e^{inx}.$$

$$(b) \quad f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos nx.$$

23.7 (a)

$$a_0 = 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \sin \pi x dx = -\frac{2}{\pi} [\cos \frac{\pi}{2} - \cos -\frac{\pi}{2}] = 0$$

$$a_n = 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \sin \pi x \cos 2n\pi x dx = 0$$

$$b_n = 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \sin \pi x \sin 2n\pi x dx = \frac{8n}{\pi - 4n^2\pi}$$

$$c_0 = 0$$

$$c_n = \frac{4(-1)^n n}{i(\pi - 4n^2\pi)}$$

$$c_n = \frac{4i(-1)^n n}{\pi - 4n^2\pi}.$$

$$(b) \quad f(x) = \frac{4}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n n}{i(1-4n^2)} e^{2n\pi ix}.$$

23.8 (a)

$$a_0 = \frac{1}{2} \int_{-2}^2 (2-x) dx = 4$$

$$a_n = \frac{1}{2} \int_{-2}^2 (2-x) \cos \left(\frac{n\pi}{2} x \right) dx = 0$$

$$b_n = \frac{1}{2} \int_{-2}^2 (2-x) \sin \left(\frac{n\pi}{2} x \right) dx = \frac{4(-1)^n}{n\pi}$$

$$(b) \quad f(x) = 2 - \sum_{n=1}^{\infty} \frac{2(-1)^{n+1} i}{n\pi} e^{-\left(\frac{in\pi}{2} x\right)} + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1} i}{n\pi} e^{\left(\frac{in\pi}{2} x\right)}.$$

23.9 $a_n = c_n + c_{-n} = 0$. We have for $|n|$ odd $b_n = i\frac{4}{in\pi} = \frac{4}{n\pi}$ and for $|n|$ even $b_n = 0$.

23.10 Note that for any complex number z we have $z + \bar{z} = 2\operatorname{Re}(z)$ and $z - \bar{z} = -2i\operatorname{Im}(z)$. Thus,

$$c_n + \bar{c}_n = a_n$$

which means that $a_n = 2\operatorname{Re}(c_n)$. Likewise, we have

$$c_n - \bar{c}_n = ib_n$$

That is $ib_n = -2i\operatorname{Im}(c_n)$. Hence, $b_n = -2\operatorname{Im}(c_n)$.

23.11 $a_n = 2\operatorname{Re}(c_n) = \frac{1}{\pi n} \sin(nT)$ and $b_n = \frac{1 - \cos(nT)}{n\pi}$.

23.12 $f(x) = i \sum_{n=-\infty}^{\infty} \frac{i \sin(2-in\pi)}{2-in\pi} e^{\frac{in\pi}{2}x}$.

23.13 (a) We have

$$f(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & 1 < t < 2 \end{cases}$$

and $f(t+2) = f(t)$ for all $t \in \mathbb{R}$.

(b) We have

$$a_0 = \frac{2}{L} \int_0^L f(x) dx = \int_0^2 dx = \int_0^1 dx = 1$$

$$a_n = \int_0^1 \cos n\pi x dx = \frac{\sin n\pi}{n\pi} = 0.$$

(c) We have

$$b_n = \int_0^1 \sin n\pi x dx = \frac{1 - \cos n\pi}{n\pi} = \frac{1 - (-1)^n}{n\pi}.$$

Hence,

$$b_n = \begin{cases} \frac{2}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

(d) We have $c_0 = \frac{a_0}{2} = \frac{1}{2}$ and for $n \in \mathbb{N}$ we have

$$c_n = \frac{a_n - ib_n}{2} = \begin{cases} -\frac{i}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$23.14 \sin 3x = \frac{1}{2}(e^{3ix} - e^{-3ix}).$$

Section 24

24.1

$$\hat{f}(\xi) = \begin{cases} 2\frac{\sin \xi}{\xi} & \text{if } \xi \neq 0 \\ 2 & \text{if } \xi = 0. \end{cases}$$

24.2

$$\begin{aligned} \frac{\partial \hat{u}}{\partial t} + i\xi c \hat{u} &= 0 \\ \hat{u}(\xi, 0) &= \hat{f}(\xi). \end{aligned}$$

24.3

$$\begin{aligned} \frac{\partial^2 \hat{u}}{\partial t^2} &= -c^2 \xi^2 \hat{u} \\ \hat{u}(\xi, 0) &= \hat{f}(\xi) \\ \hat{u}_t(\xi, 0) &= \hat{g}(\xi). \end{aligned}$$

24.4

$$\begin{aligned} \hat{u}_{yy} &= \xi^2 \hat{u} \\ \hat{u}(\xi, 0) = 0, \hat{u}(\xi, L) &= \frac{2 \sin \xi a}{\xi}. \end{aligned}$$

$$24.5 \frac{1}{\alpha - i\xi} + \frac{1}{\alpha + i\xi} = \frac{2\alpha}{\alpha^2 + \xi^2}.$$

24.6 We have

$$\begin{aligned} \mathcal{F}[e^{-x}H(x)] &= \int_{-\infty}^{\infty} e^{-x}H(x)e^{-i\xi x} dx \\ &= \int_0^{\infty} e^{-x(1+i\xi)} dx = -\frac{e^{-x(1+i\xi)}}{1+i\xi} \Big|_0^{\infty} = \frac{1}{1+i\xi}. \end{aligned}$$

24.7 Using the duality property, we have

$$\mathcal{F}\left[\frac{1}{1+ix}\right] = \mathcal{F}[\mathcal{F}[e^{-\xi}H(\xi)]] = 2\pi e^{\xi}H(-\xi).$$

24.8 We have

$$\begin{aligned}\mathcal{F}[f(x - \alpha)] &= \int_{-\infty}^{\infty} f(x - \alpha)e^{-i\xi x} dx \\ &= e^{-i\xi\alpha} \int_{-\infty}^{\infty} f(u)e^{-i\xi u} du \\ &= e^{-i\xi\alpha} \hat{f}(\xi)\end{aligned}$$

where $u = x - \alpha$.

24.9 We have

$$\mathcal{F}[e^{i\alpha x} f(x)] = \int_{-\infty}^{\infty} e^{i\alpha x} f(x)e^{-i\xi x} dx = \int_{-\infty}^{\infty} f(x)e^{-i(\xi - \alpha)x} dx = \hat{f}(\xi - \alpha).$$

24.10 We will just prove the first one. We have

$$\begin{aligned}\mathcal{F}[\cos(\alpha x)f(x)] &= \mathcal{F}\left[\frac{f(x)e^{i\alpha x}}{2} + f(x)\frac{e^{-i\alpha x}}{2}\right] \\ &= \frac{1}{2}[\mathcal{F}[f(x)e^{i\alpha x}] + \mathcal{F}[f(x)e^{-i\alpha x}]] \\ &= \frac{1}{2}[\hat{f}(\xi - \alpha) + \hat{f}(\xi + \alpha)].\end{aligned}$$

24.11 Using the definition and integration by parts we find

$$\begin{aligned}\mathcal{F}[f'(x)] &= \int_{-\infty}^{\infty} f'(x)e^{-i\xi x} dx \\ &= f(x)e^{-i\xi x}\Big|_{-\infty}^{\infty} + (i\xi) \int_{-\infty}^{\infty} f(x)e^{-i\xi x} dx \\ &= (i\xi)\hat{f}(\xi)\end{aligned}$$

where we used the fact that $\lim_{x \rightarrow \pm\infty} f(x) = 0$.

24.12 $\frac{2}{\xi^2}(1 - \cos \xi)$.

24.13 $\frac{2}{i\xi}(1 - \cos \xi a)$.

$$24.14 \mathcal{F}^{-1}[\hat{f}(\xi)] = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

$$24.15 \mathcal{F}^{-1}\left(\frac{1}{a+i\xi}\right) = e^{-ax}, \quad x \geq 0.$$

Section 25

$$25.1 u(x, t) = \mathcal{F}^{-1}[u(\xi, t)] = e^{-\frac{(x-ct)^2}{4}}.$$

25.2

$$\begin{aligned} u(x, t) &= \sqrt{\frac{\gamma}{4\pi}} e^{-\alpha t} \mathcal{F}^{-1}[e^{-\xi^2(kt+\frac{\gamma}{4})}] \\ &= \sqrt{\frac{\gamma}{4\pi}} e^{-\alpha t} \cdot \sqrt{\frac{\pi}{kt+\frac{\gamma}{4}}} \cdot e^{-\frac{x^2}{4(kt+\frac{\gamma}{4})}} \\ &= \sqrt{\frac{\gamma}{4kt+\gamma}} e^{-\frac{x^2}{4kt+\gamma}} e^{-\alpha t}. \end{aligned}$$

$$25.3 u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty e^{-\frac{(x-s)^2}{4kt}} ds.$$

25.4

$$\begin{aligned} u(x, t) &= e^t \mathcal{F}^{-1}[e^{-\xi^2 t}] \\ &= e^{-\alpha t} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}. \end{aligned}$$

25.5 We have

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-|\xi|y} e^{i\xi x} d\xi &= \int_{-\infty}^0 e^{\xi y} e^{i\xi x} d\xi + \int_0^{\infty} e^{-\xi y} e^{i\xi x} d\xi \\ &= \frac{1}{y+ix} e^{\xi(y+ix)} \Big|_{-\infty}^0 - \frac{1}{y-ix} e^{\xi(-y+ix)} \Big|_0^{\infty} \\ &= \frac{1}{y+ix} + \frac{1}{y-ix} = \frac{2y}{x^2+y^2}. \end{aligned}$$

25.6

$$\begin{aligned}
 u(x, y) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-|\xi|y} e^{i\xi x} d\xi \\
 &= \frac{1}{2\pi} f(x) * \left[\frac{2y}{x^2 + y^2} \right] \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \frac{2y}{(x - \xi)^2 + y^2} d\xi.
 \end{aligned}$$

$$\mathbf{25.7} \quad \hat{u}_{tt} + (\alpha + \beta)\hat{u}_t + \alpha\beta\hat{u} = -c^2\xi^2\hat{u}.$$

$$\mathbf{25.8} \quad u(x, t) = e^{-(x-3t)}.$$

$$\mathbf{25.9} \quad u(x, t) = e^{-(x-kt)}.$$

$$\mathbf{25.10} \quad u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-s^2 - \frac{(x-s)^2}{4kt}} ds.$$

$$\mathbf{25.11} \quad u(x, t) = (x - ct)^2.$$

$$\mathbf{25.12} \quad u(x, t) = f(x) * \mathcal{F}^{-1}\left[-\frac{1}{|\xi|}e^{-|\xi|y}\right].$$

Index

- Boundary value problem, 17
- Burger's equation, 11

- Cauchy data, 75
- Cauchy problem, 75
- Characteristic curve, 60
- Characteristic direction, 59
- Characteristic equation, 140
- Characteristic equations, 60, 72
- Characteristics, 60
- Classical solution, 13
- Convection (transport) equation, 11
- Convolution, 183

- Discriminant, 88
- Differential equation, 5
- Diffusion equation, 11
- Diffusivity constant, 102
- Directional derivative, 50
- Dirichlet boundary conditions, 17
- Dirichlet conditions, 104
- Dot product, 41

- Eigenvalue problem, 155
- Elliptic, 88
- Euler equation, 166
- Euler-Fourier Formulas, 125
- Even extension, 135
- Even function, 133
- Evolution equation, 177
- Exponential order at infinity, 180

- First derivative, 4
- First order PDE, 36
- Forced harmonic oscillator, 11
- Fourier coefficients, 121
- Fourier cosine series, 135
- Fourier inversion formula, 207
- Fourier law, 102
- Fourier series, 109, 121
- Fourier sine series, 135
- Fourier transform, 207
- Function series, 120
- Fundamental period, 122

- General solution, 13
- Generalized solution, 14
- Gradient, 51
- Gradient vector field, 56

- Harmonic function, 154
- Heat equation, 89, 100
- Heat source, 103
- Helmholtz equation, 155
- Homogeneous, 37, 88
- Homogeneous linear PDE, 8
- Hyperbolic, 88

- Ill-posed, 19
- Initial curve, 75
- Initial data, 75
- Initial temperature distribution, 104
- Initial value conditions, 18

- Initial value problem, 18
- Inner product, 123
- integral curve, 57
- Integral surface, 13, 59
- integral transforms, 177
- Integrating factor, 26
- Inverse Laplace transform, 182

- Korteweg-Vries equation, 11

- Lagrange's method, 71
- Laplace equation, 89, 154
- Laplace transform, 178
- Laplace's equation, 11
- Laplacian, 154
- Level curve, 54
- Level surface, 54
- Linear, 7, 36
- linear, 88
- Linear differential operator, 8
- Linear operator, 8

- Method of characteristics, 59, 66, 71
- Method of undetermined coefficients, 222
- Method of Variation of Parameters, 229
- Minimal surface equation, 11
- Mixed boundary condition, 18
- Mutually orthogonal, 123

- Neumann boundary conditions, 17, 105
- Non-homogeneous, 37, 88
- Non-homogeneous PDE, 8
- Non-linear, 7, 36
- Nowhere characteristic, 77

- Odd extension, 134
- Odd function, 133

- Order, 6
- Ordinary differential equation, 5
- Orthogonal, 123
- Orthogonal projection, 45

- Parabolic, 88
- Partial differential equation, 5
- Piecewise continuous, 124, 180
- Piecewise smooth, 124
- Pointwise convergence, 109, 120
- Poisson Equation, 11
- Poisson equation, 155
- Projected characteristic curve, 60

- Quasi-linear, 7, 36, 88

- Right traveling wave, 67

- Scalar projection, 46
- Semi-linear, 7, 36, 88
- Separable, 31
- Separation of variables, 31
- Smooth functions, 8
- Solution surface, 13
- Specific heat, 101
- Squeeze rule, 117
- Stable solution, 18
- stationary equation, 177
- Strong solution, 13
- Superposition principle, 15

- Thermal conductivity, 102
- Thermal energy, 100
- Thin film equation, 11
- Total thermal energy, 103
- Transport equation, 18
- Transport equation with decay, 68
- Transport equation in 1-D space, 66

- Uniform convergence, 110, 121

Vector field, [55](#)

Vector function, [53](#)

Wave equation, [89](#), [93](#)

wave equation, [11](#)

Weak solution, [14](#)

Weierstrass M-test, [121](#)

Well-posed, [18](#)