PART-I COMBINATORICS

CHAPTER 1

ELEMENTARY COUNTING PRINCIPLES

Objectives:-On completion of the chapter the students will be able to:

- Identify the elementary counting principle.
- Solve a counting problems using a counting method
- Identify the difference between the permutation and combination of a counting method
- Understand and differentiate the method of permutation with and without repetition.
- Understand and differentiate the method of combination with and without repetition.
- Understand the binomial theorem.

Introduction

Combinatorics is a fascinating branch of discrete Mathematics, which deals with the art of counting. Very often we ask the question, In how many ways can a certain task be done? Usually combinatorics comes to our rescue. In most cases, listing the possibilities and counting them is the least desirable way of finding the answer to such a problem. Enumeration, the counting of objects with certain properties is an important part of combinatorics. We must count objects to solve many different type of problems.

Counting is also required to determine whether there are enough telephone numbers or internet protocol addresses to meet demand. Furthermore, counting techniques are used extensively when probabilities of events are computed.

The elementary counting principle, which we will study section 1.1 to 1.3 can solve a tremendous variety of problems. We can phrase many counting problems in terms of ordered or unordered arrangements of the objects and combinations are used in many counting problems.

1.1 Addition principle

Objective: After completing this section the student should be able to:

- ✓ Understand the addition principle
- ✓ Solve some problems using addition principle.

Theorem1.1 (Addition principle)

Let A and B be two mutually exclusive tasks. Suppose task A can be done in m ways and task B in n ways.Then there are $n + m$ ways to do one of these tasks.

Example 1.1 Suppose that either a member of the mathematics faculty or a student who is mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors?

Solution*:* Let A be the task, choosing a member of the mathematics faculty, can be done in 37 ways. Let B be the task, choosing a mathematics major, can be done in 83 ways. From the sum (addition) rule, it follows that there are $37 + 83 = 120$ possible ways to pick this representative.

The addition principle can be extended to any finite number of pair wise mutually exclusive tasks, using induction, for instance, let $T_1, T_2, T_3, \dots, T_n$ be n pair wise mutually exclusive tasks. Suppose task T_i can be done in m_i ways, where$1 \leq \text{i} \leq \text{n}$.

Then task$T_1, T_2, T_3, \dots, T_n$ can be done to $m_1 + m_2 + \cdots + m_n$ ways.

Example 1.2: A freshman has selected four courses and needs one more course for the next term. There are 15 courses in English, 10 in French and 6 in German. She is eligible to task. In how many ways can she choose the fifth course?

Solution: Let E be the task of selecting a course in English, F be the task of selecting a course in French and G that of selecting a course in German.

These tasks can be done in 15,10 and 6 ways, respectively, and are mutually exclusive, so by addition principle, the fifth course can be selected in $|E| + |F| + |G| = 15 + 10 + 6 = 31$ way.

Example 1.3: A student can choose a computer project from one of three lists. The three lists contain 23,15and 19 possible projects, respectively. How many possible projects are there to choose from?

Solution: Let $A, B\ and\ C$ be the first, second and third lists respectively. And the list can be done 23, 15 and 19 ways respectively. Hence there are 23+15+19=57 projects to choose from.

The sum can be phrased in terms of sets as: if $A_1, A_2, \dots, A_n$ are disjoint sets, then the number of elements in the union of these sets is the sum of the number of elements in them. To relate this to our statement of the sum rule, let T_i be the task of choosing an element from A_i for $i = 1,2, \dots, n$. There are $|A_i|$ ($|A_i|$ is the notation for the cardinality of A_i) ways to do T_i. From the sum rule, since no two of the tasks can be done, at the same time the number of ways to choose an element from one of the sets, which is the number of elements in the union is

$$|A_1 \cup A_2 \cup \dots A_n| = |A_1| + |A_2| + \dots + |A_n|$$

This equality applies only when the sets in question are disjoint.

Exercise 1.1

1. *There are 18 mathematics majors and 325 computer science major at a college. How many ways are there to pick one representative who is either mathematics major or a computer science major?*
2. *Let A and B be finite disjoint sets, where $|A| = a$, and $|B| = b$. find $|A \cup B|$*
3. *Let U be a universal set contain the disjoint set A and B such that $|A| = 2a + b$, $|U| = 2a + 3b$. find $|A \cup B|$*

1.2. The inclusion- exclusion principle

Objective**:** After completing this section the student should be able to:

- ✓ Understand the method of inclusion-exclusion principle
- ✓ Understand the difference between the method of sum rule and inclusion-exclusion principle.

✓ Solve some problems using inclusion-exclusion principle.

When two tasks can be done at the same time, we can't use the sum rule to count the number of ways to do one of the two tasks. Adding the number of ways to do

one of the two tasks. Adding the number of ways to do each task leads to an over count, since the ways to do both tasks are counted twice.

Question: Let $A\ and\ B$ be any two finite sets. How is $|A \cup B|$related $|A|\ and |B|$?

Theorem 1.2 (Inclusion-exclusion principle)

Suppose a task A can be done in m ways, task B in n ways and both can be accomplished in k different ways. Then task $A\ or\ B$ can be done in $m + n - k$

We can phrase this counting principle in terms of sets. Let $A\ and\ B$ be two finite sets. Then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example *1.4*: Find the number of positive integers ≤ 300 and divisible by 2 or 3.

Solution: Let $A = \{x \in \mathbb{N}: x \leq 300\ and\ divisible\ by\ 2\}$

$$B = \{x \in \mathbb{N}: x \leq 300\ and\ divisible\ by\ 3\}$$

Then $A \cap B$ consists of positive integers ≤ 300 that are divisible by 2 and 3. That is, divisible by 6. Thus,

$$A = \{2,4, \dots ,300\}$$

$$B = \{3,6, \dots ,300\} \quad \text{and} \quad A \cap B = \{6,12, \dots ,300\}.$$

Clearly,$|A| = 150$, $|B| = 100\ and\ |A \cap B| = 50$. By theorem 1.2,

$$|A \cup B| = |A| + |B| - |A \cap B| = 150 + 100 - 50 = 200$$

Thus, there are 200 positive integers ≤ 300 and divisible by 2or 3.

Example 1.5 Find the number of positive integers ≤ 3000 and not divisible by 7 or 8.

Solution: Let $A = \{x \in \mathbb{N}: x \leq 3000\ and\ divisible\ by\ 7\}$

$$B = \{x \in \mathbb{N}: x \leq 3000\ and\ divisible\ by\ 8\}$$

We need to find $|A' \cap B'|$

$|A' \cap B'| = |(A \cup B)'| = |U| - |A \cup B|$, Where U be the universal set.

$$= |U| - |A| - |B| + |A \cap B|$$

$$= 3000 - \frac{3000}{7} - \frac{3000}{8} + \frac{3000}{56}$$

$$= 3000 - 428 - 375 + 53 = 2250$$

Corollary 1.3: Let $A, B\ and\ C$ be three finite sets.

Then $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

Proof:

$|A \cup B \cup C| = |A \cup (B \cup C)|$

$= |A| + |B \cup C| - |A \cap (B \cup C)|$ (By theorem 1.2)

$= |A| + [|B| + |C| - |B \cap C|] - [|A \cap B| \cup |A \cap C|]$

$= |A| + |B| + |C| - |B \cap C| - [|A \cap B| + |A \cap C| - |(A \cap B) \cap (A \cap C)|]$

$= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$

Example 1.6: Find the number of positive integers ≤ 2076 and divisible by 3, 5 and 7.

Solution: Let A, B and C denote the sets of positive integers ≤ 2076 and divisible by 3, 5 and 7, respectively, by the corollary 1.3

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= \frac{2076}{3} + \frac{2076}{5} + \frac{2076}{7} - \frac{2076}{15} - \frac{2076}{21} - \frac{2076}{35} + \frac{2076}{105}$$

$$= 692 + 415 + 296 - 138 - 59 - 98 + 19 = 1127$$

Example 1.7:A survey among 100 students show that of the three ice-cream flavors vanilla, chocolate and strawberry,50 students like vanilla, 43 like chocolate, 28 like strawberry ,13 like vanilla and chocolate , 11 like chocolate and strawberry , 12 like strawberry and vanilla and 5 like all of them. Find the number of students surveyed who like each of the following flavors

a) Chocolate but not strawberry

b) Chocolate and strawberry but not vanilla

c) Vanilla or chocolate but not strawberry.

Solution: Let V, C and S symbolize the set of students who like vanilla, chocolate and strawberry flavors, respectively, draw three intersecting circles to represent them in the most general case as in figure 1.1

- 5 students like all flavors', $|V \cap C \cap S| = 5$

- 12 students like both strawberry and vanilla, $|S \cap V| = 12$ but 5 of them like chocolate also, therefore, $|(S \cap V) - C| = 5$

- 11 students like chocolate and strawberry so, $|C \cap S| = 11$ but 5 of them like vanilla , therefore, $|(S \cap C) - V| = 6$

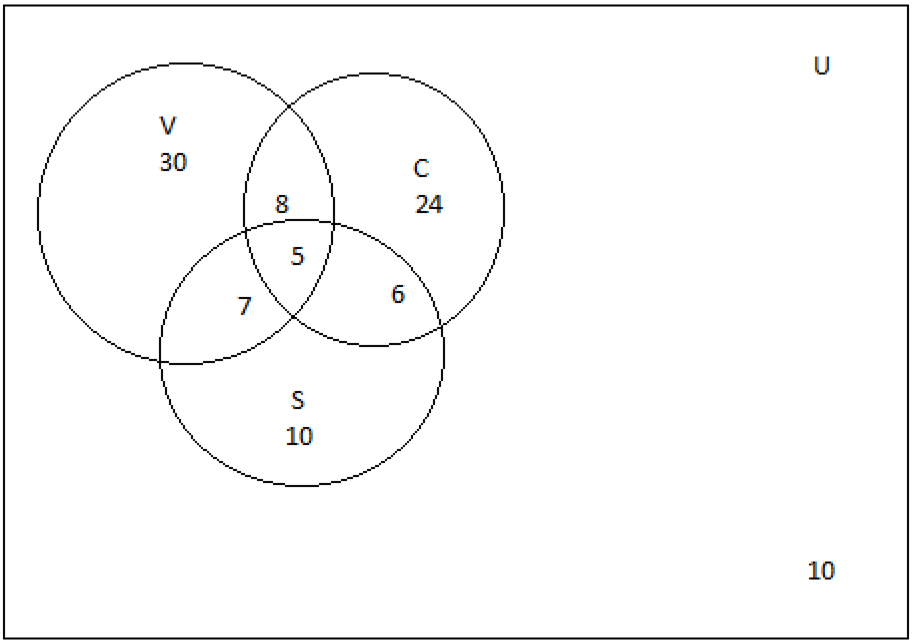


Figure 1.1

- 13 students like vanilla and chocolate, so $|V \cap C| = 13$ but 5 of them like strawberry also, therefore, $|(C \cap V) - S| = 8$

Of the 28 students who like strawberry, we have already a accounted for 7+5+6=18. So, the remaining 10 students belongs to the set $(S - (V \cup C))$

Similarly, $|V - (C \cup S)| = 30$ *and* $|C - (S \cup V)| = 24$

Thus, we have accounted for 90 of the 100 students.

The remaining 10 students lie outside the region $V \cup S \cup C$ as in figure 1.Now,

a) $|C - S| = 8 + 24 = 32$

So, 32 students like chocolate but not strawberry

b) $|(C \cap S) - V| = 6$ Therefore, 6 students like both chocolate and strawberry but not vanilla.

c) $30 + 8 + 24 = 62$ students like vanilla or chocolate but not strawberry. They are presented by the region $(V \ \cap C) - S$.

Exercise 1.2

1. *Let A and B be two sets such that* $|A| = 2a - b$, $|B| = 2a$, $|A \cap B| = a - b$ *and* $|U| = 3a + 2b$. *Find the cardinality of each set.*

a) $A \cup B$ *b)* $A - B$ *c)* B' *d)* $A' - A'$

2. Let A and B be finite disjoint set, where $|A| = a$, $|B| = b$. Find the cardinality of each set

a) $A \cup B$ *b)* $A - B$ *c)* $B - A$

3. *Find the cardinality of each set in (2) where $A \subseteq B$, B is finite*

4. *According to a survey among 160 college students, 95 students takes a course in English , 72 takes a course in French, 67 takes a course in German , 35 take a course in English and in French , 37 takes a course in French and in German, 40 takes a course in*

German and in English and 25 take a course in all three language. Find the number of students in the survey who take a course in

a) English but not German b) English, French or German

1.3. Multiplication principle

Objective: After completing this section the student should be able to:

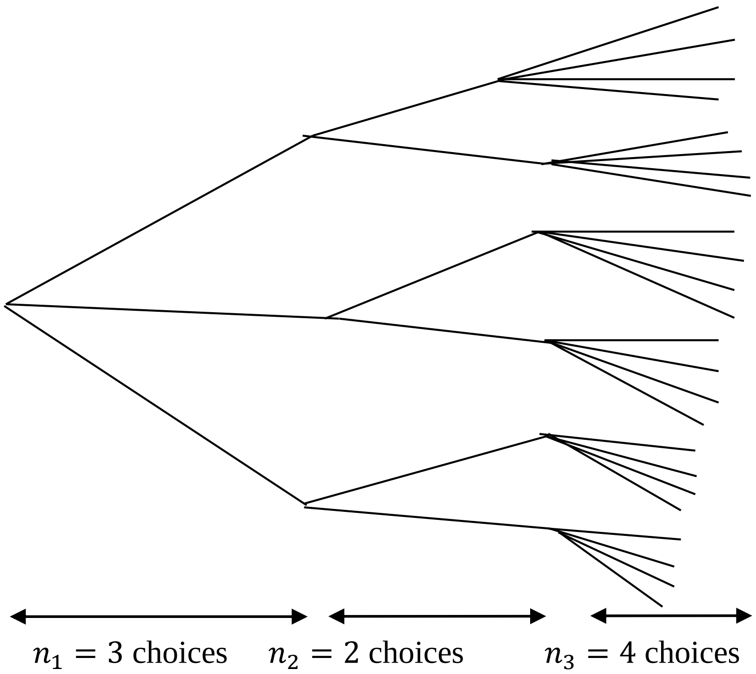
- ✓ Understand the multiplication principle
- ✓ Solve some counting problems using multiplication principle

The most important counting principle is the multiplication principle. It allows for counting (like, example the experiment consisting of both rolling a dice and tossing a coin), and this principle apply when a procedure is made up of separate task. Multiplication principle: if an experiment consisting of k independent steps, in such a way that:

- The first step has n_1 possible out come
- Any outcome of the first can be followed by n_2 outcome of the 2^{nd} step,
- Any one of the first and the second step can be followed n_3 outcome of the 3^{rd} step

.
.
.

Then the total number of outcomes $n_1, n_1, \ldots, n_k$



In total $n_1 n_2 n_3 = 3 \times 2 \times 4 = 24$ possible out coomes.

Figure 1. 2 Illustrate multiplication principles

Example 1.8 How many distinct phone numbers are there if we assume that a phone number is made of 6 digits with the first digit begin different from 0 and 1?

Solution: Assume that a_1 be the first digit, a_2 be the second digits, $a_3, a_{4,} a_{5,} a_6$ be the 3rd , 4th ,5th and 6th digit respectively. But $a_1 \neq 0 \; and \; 1$. So we have 8 possible choice of a_1 and we have 10 possible choices for the digit a_2 to a_6.

Therefore, $8 \times 10^5 = 800{,}000$ distinct phone number.

Example 1.9 In how many ways can the letters of the word 'CAR' be reordered to produce distinct 'words'.

Solution: We have 3 possibilities for the first letter, 2 possibilities for the 2nd letter and have to use the remaining letter. So, there are $3 \times 2 \times 1 = 6$ distinct 'words'.

Theorem 1.4 (Multiplication principle)

Suppose a task T is made up of two subtasks. Subtask T_1 followed by subtask T_2. If subtask T_1 can be done in $m_{\,1}$ways and subtask T_2 in $m_{\,2}$ different way for each way subtask T_1 can be done, then task T can be done in $m_1 m_2$ ways.

Example 1.10 Find the number of two letter words that being with a vowel a,e,i,o or u.

Solution :The task of forming a two-letter word consists of two subtasks $T_1 \; and \; T_2$, T_1 consisting of the first letter and T_2 selecting the second letter; as figure 1.3 shows

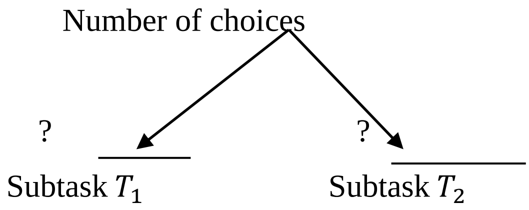


Figure 1.3

Since each word must begin with a vowel, T_1 can be accomplished in five ways. There is no restriction on the choice of the 2nd letter, so T_2 can be done in 26 ways (figure 1.4).

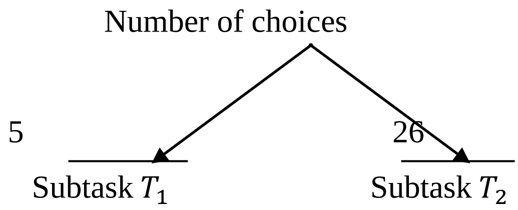


Figure1. 4

Therefore, by the multiplication principle the task can be performed in $5 \times 26 = 130$ different ways. In other words, 130 two letter words begin with a vowel.

The multiplication principle can also be extended to any finite number of subtasks. Suppose a task T can be done by n successive subtasks, $T_1, T_2, \ldots, T_n$. If subtask T_i can be done in m_i different ways afterT_{i-1} has been completed, where $1 \leq i \leq n$, then task T can be done in $m_1 \times m_2 \times m_3 \times \ldots \times m_n$ ways.

The multiplication principle can be applied to prove that a set with size n has 2^n subsets, as shown below.

Example 1.11: Show that a set S with n elements has 2^n subset.

Solution: Every subset of S can be uniquely identified by an $n-$ bit words (see figure1.5). The task of forming an n –bit word can be broken down to n subtasks. Selecting a bit for each of the n- positions. Each position in the word

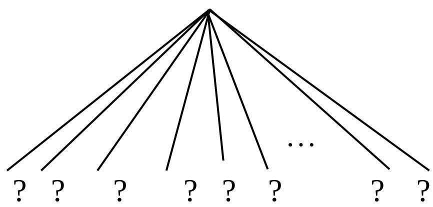


Figure 1.5

has two choices 0 or 1: so by the multiplication principle, the total number of n-bit words that can be formed is 2.2 .2…2=2n(see figure1. 6)

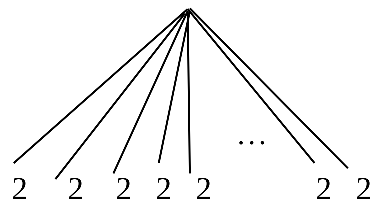


Figure 1. 6

Example 1.12: How many one to one functions are there from a set with m elements to one with n elements?

Solution: First note when $m > n$ there is no one to one functions from a set with m elements to a set with n elements. Now let $m \leq n$. Suppose the elements in the domain are $a_1, a_2, \ldots, a_m$. There are n ways to choose the value of the function at a_1. Since the function is one to one, the value of the function at a_2 can be picked in $(n-1)$ ways (since the value used for a_1 can't be used again). In general, the value of the function at a_k can be choosen in $n-k+1$ ways. By the multiplication principle, there are

$n(n-1)(n-2)\ldots(n-m+1)$ one to one functions from aset with m elements to one with n element.

For instance, there are $5 \times 4 \times 3 = 60$ one to one functions from a set with three elements to a set with five elements.

Exercise 1.3

1. *How many bit strings are there of length eight.*
2. *How many bit strings of length ten begin and end with a 1?*
3. *How many different functions are there from a set with 10 elements to a set with the following numbers of elements*

   a) 2 b) 3 c) 4 d) 5

4. *How many one to one function are there from a set with five elements to a set with the following number of element*

   a) 4 b) 5 c) 6 d) 7

5. *A multi-choice test contains ten questions. There are four possible answer for each question*

   a) How many ways can a student answer the questions on the test if every question is answered?

   b) How many ways can a student answer the question on the test if the student can leave answers blank?

6. A particular brand of shirt comes in 12 colors, has a male version and a female version and comes in three sizes for each sex. How many different types of this shirt are made?

1.4 permutation and combination

Objective: After completing this section the student should be able to:

- ✓ Understand the method of permutation and combination
- ✓ Explain the definition of permutation with and without repetition
- ✓ Explain the definition of combination with and without repetition
- ✓ Solve the counting problems using these methods.

Most counting problems we will be dealing with can be classified into one of four categories. We explain such categories by means of an example.

Example1.13: Consider the set $\{a, b, c, d\}$. Suppose we "select" two letters from these four. Depending on our interpretation, we may obtain the following answers.

i. **Permutations with repetitions.** The order of listing the letters is important, and repetition is allowed. In this case there are $4 \cdot 4 = 16$ possible selections:

aa	ab	ac	ad
ba	bb	bc	bd
ca	cb	cc	cd
da	db	dc	dd

ii. **Permutations without repetitions.** The order of listing the letters is important, and repetition is not allowed. In this case there are $4 \cdot 3 = 12$ possible selections:

	ab	ac	ad
ba		bc	bd
ca	cb		cd
da	db	dc	

iii. **Combinations with repetitions.** The order of listing the letters is **not** important, and repetition is allowed. In this case there are

$$\frac{4 \cdot 3}{2} + 4 = 10 \quad \text{possible selections:}$$

aa	ab	ac	ad
	bb	bc	bd
		cc	cd
			dd

iv. **Combinations without repetitions.** The order of listing the letters is **not** important, and repetition is not allowed. In this case there are.

$\frac{4\cdot 3}{2} = 6$ Possible selections:

	ab	ac	ad
		bc	bd
			cd

Permutations without Repetitions

Definition 1.1. A permutation of a set of distinct objects is an ordered arrangement of these objects, we also are interested in ordered arrangements of some of the elements of a set.

A permutation of a set of n (distinict) elements taken r $(0 \leq r \leq n)$ at a time is an arrangement of r elements of the set. For convenience, it is called a $r-$ permutation. If $r = n$ then the $r-$ permutation is called a $r-$ permutation. The number of $r-$ permutation of a set of size n is denoted by $p(n,r)$, n distinct objects can be reordered in

$$n! = n(n-1)(n-2)\ldots 2 \times 1 \text{ different ways of doing so.}$$

Note: we will use the convention that $o! = 1$.

Example 1.14 :Find the number of permutation, that is, $3-$permutations of the elements of the set $\{a,b,c\}$

Solution: By the multiplication principle, the number of $3-$permutation of three elements is $3 \times 2 \times 1 = 6$. Or can be obtained systematically using a tree diagram.

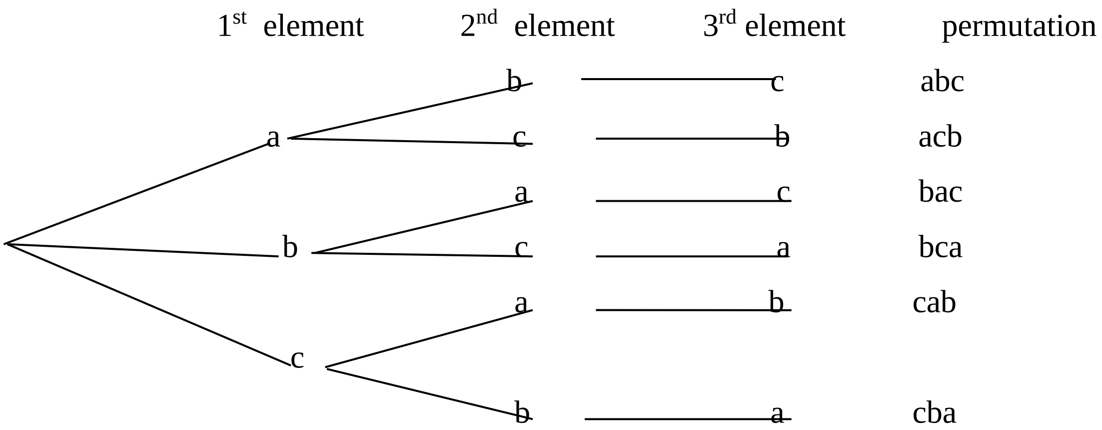


Figure 1.7

Example 1.15 .Eight runners take part in a race. How many different of ways of allocating medals (gold, silver and bronze) are there?

Solution: We choose $r = 3$ medalists from the $n = 8$ runners (the order dosen't matter). The number of $3-$permutation of 8 runners is $8 \times 7 \times 6 = 336$ ways the medals can be handed out, thus, $p(8,3) = 336$.

If we went to choose only $r \leq n$ of the n objects and retain the order in which we choose the object the there are $p(n,r) = n(n-1)(n-2) \ldots (n-r+1)$ different ways of doing so.

Theorem 1.5: The number of $r-$permutation of a set of n (distinict) elements is given by

$$p(n,r) = \frac{\boldsymbol{n}!}{(\boldsymbol{n}-\boldsymbol{r})!}$$

Proof: The first elements of the permutation can be chosen in n ways, since there are n elements in the set. There are $(n-1)$ ways to choose the second elements of the permutation. Since there are $(n-1)$ elements left in the set after using the element picked for the first position.Similarlly, there are $(n-2)$ ways to choose the third element, as so on until there are exactly $n-(r-1) = n-r+1$ ways to choose the n^th^ element. Thus, by the multiplication principle,

$$p(n,r) = n(n-1)(n-2) \ldots (n-r+1)$$

$$= n(n-1)(n-2) \ldots (n-r+1) \frac{(n-r)(n-r-1)\ldots 2\times 1}{(n-r)(n-r-1)\ldots 2\times 1}$$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)(n-r-1)\ldots 2\times 1}{(n-r)(n-r-1)\ldots 2\times 1} = \frac{n!}{(n-r)!}$$

$$p(n,r) = \frac{n!}{(n-r)!}$$

In particular, suppose $r = n$ then $p(n,r) = p(n,n) = \frac{n!}{0!} = n!$. So, $p(n,n) = n!$

Example 1.16 How many ways are there to select a first-prize winner, a second-prize winner and a third-prize winner, from 100 different people who have entered contest?

Solution: The number of ways to pick the three prize winner (1^{st} ,2^{nd} and 3^{rd}) is the number of ordered selections of three elements from a set of 100 elements, that is the 3 −permutations of a set of 100 elements.

$$p(100,3) = \frac{(100)!}{(100-3)!} = \frac{(100)!}{(97)!} = \frac{100\times 99\times 98\times 97!}{(97)!} = 100 \times 99 \times 98 = 970{,}200$$

Example 1.17: Find the number of words that can be formed by scrambling the letter of the word SCRAMBLE (remember, a word is just an arrangement of symbols, it need not make sense)?

Solution: The word SCRAMBLE contains eight distinct letters. Therefore, the number of words that can be formed equals. The number of arrangement of the letters in the word, namely

$$p(8,8) = 8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40{,}320$$

Combinations without Repetitions

Consider again the case that we want to choose $r \le n$ from n objects, but this time we do not want to retain the order. If we retained the order, there would be $p(n,r) = \frac{n!}{(n-r)!}$ possibilities. But $r!$ of these ways result in the same set of r objects. Since the ordering is not important, only their membership is important. We will investigate such unordered arrangements in this section.

Definition 1.2: An r −combination of elements of a set, where $0 \le r \le n$, is an unordered selection of r elements from the set. Thus, an r −combination is simply a subset of the set with r −elements. The number of r −combinations of a set with n elements is denoted by $C(n,r) or \binom{n}{r}$. Both notations frequently appear in combinatorics. The number of combination is also called the binomial coefficient.

Example 1.18: Find the number of r −combinations of the set $\{a,b,c\}$ when $r = 0,1,2\ or\ 3$

Solution:

- Exactly one subset contains zero element, the null set

  Number of $0-c$ombinations: $\square(3,0)=1$
- Three subsets contain one elements each: $\{a\},\{b\}$ and $\{c\}$.

  Number of 1 $-$combinations: $C(3,1)=3$
- Three subset contains two elements each: $\{a,b\},\{b,c\}$, and $\{c,a\}$

  Number of 2 $-$combinations: $C(3,2)=3$
- Finally, exactly one subset contains three elements: the set itself

  Number of 3 $-$compinations: $C(3,3)=1$

Theorem 1.6: The number of $r-$combinatios of a set with n elements, where n is a nonnegative integer and r is an integer with $0\le r\le n$ equals

$$C(n,r)=\frac{n!}{(n-r)!r!}$$

Proof: By definition, there are $C(n,r)$, $r-$ combinations of a set of $n-$elements, Each combination contains $r-$elements and contributes $p(r,r)=r!$;
$r-$permutation, so, the total number of $r-$permutation is $r!\,C(n,r)$. But by definition, there are $p(n,r)=\frac{n!}{(n-r)!}$ $r-$permutations. Therefore,

$$r!\,C(n,r)=\frac{n!}{(n-r)!}$$

That is, $C(n,r)=\frac{n!}{(n-r)!r!}$

Note:

1. $C(n,0)=\frac{n!}{0!(n-0)!}=1$, that is, the number of 0 $-$combinations of a set with n elements is one.
2. $C(n,n)=\frac{n!}{n!(n-n)!}=1$, that is, the number of $n-$combinations of a set with n elements is also one.

Corollary 1.7 Let n and r be nonnegative integer with $r\le n$. Then

$$C(n,r)=C(n,n-r)$$

Proof: From theorem 1.6, it follows that

$$C(n,r)=\frac{n}{r!(n-r)!}\quad \text{And}$$

$$C(n,n-r)=\frac{n}{(n-r)!(n-(n-r))!}=\frac{n!}{(n-r)!(n-n+r)!}$$
$$=\frac{n!}{(n-r)!r!}=\frac{n!}{r!(n-r)!}$$

Hence, $C(n,r) = C(n,n-r)$

Example 1.19

John wants to go to the pub with 3 of his 5 best friends. How many options does John have?

Solution: 3-combination of 5 is

$$C(5,3) = \frac{5!}{3!(5-3)!} = \frac{5!}{3!2!} = \frac{5\times4\times3!}{2\times1\times3!} = \frac{5\times4}{2} = 10$$

Permutations with Repetitions

The permutation and computation examined so far involved unrepeated items. For instance, a letter or digit may be used more than once on a licenses plate. When a dozen dounts are selected, each variety can be chosen repeatedly. This contrasts with the counting problems discussed earlier in the chapter where we only be used at most once. In this section we will show how to solve counting problems where elements may use more than one. Counting permutation when repeated is allowed can easily be done using the product rule.

Example 1.20: How many strings of n length can be formed from the English alphabet?

Solution: By the product rule, since there are 26 letters and since each letter can be used repeatedly, we see that there are 26^n strings of length n.

Theorem 1.8: The number of $r-$permutation of a set of n objects with repetition allowed is n^r

Proof: There are n ways to select an element of the set of each of the r position in the $r-$permutation is allowed, since for each choice all n objects are available. Hence by the multiplication principle there are n^r $r-$permutation when a
repetition is allowed.

Theorem 1.9: The number of permutation of n items of which n_1 items are of one type, n_2 are of a second type, and n_k are of a k^{th} type, is

$$\frac{n!}{n_1!\,n_2!\,\ldots n_k}$$

Example 1.21 Find the number of bytes contain exactly three 0's

Solution:

$$\begin{pmatrix} number\ of\ bytes \\ containg\ exactly\ three\ 0's \end{pmatrix} = \begin{pmatrix} number\ of\ bytes\ contaning\ three\ 0's \\ and\ five\ 1's \end{pmatrix}$$

$$=$$

$$\begin{pmatrix} number\ of\ permutations\ of\ eight\ symbols\ of \\ which\ three\ are\ like\ (0's) and\ five\ are\ alike\ (1's) \end{pmatrix}$$

$$= \frac{8!}{3!5!}$$

$$= 56$$

Example 1.22. Find the number of different arrangement of the letter of the word REFERENCE.

Solution: The word REFERENCE contain nine letters; two R's and four E's, the remaining letters are distinct, now by theorem 1.9, the total number of words are $\frac{9!}{2!4!} = 7560$.

Combinations with Repetitions

Just as permutation can deal with repeated elements, so can combinations (called selections).

Example 1.23 Find the number of 3 $-$combination of the set $S = \{a, b\}$

Solution: S contains $n = 2$ elements. Since each combination must contain three elements $r = 3$. Since $r > n$, the elements of each combination must be repeated. Consequently, a combination may contain three a's, two a's and one b's, one a's and two b's or three b's. Using the set notation, the 3-combinations are

$$\{a, a, a\}, \{a, a, b\}, \{a, b, b\}, and\ \ \{b, b, b\}$$

So, there are four 3-combination of a set of two elements.

Theorem 1.10: The number of r $-$combinations with repetition from a set of n elements is

$$C(n + r - 1, r).$$

Proof: Each r $-$combination with repeated elements from a set of n elements can be considered a string of $r\, x's$ and $(n - 1)$ slashes (that means, for instance if $n = 3$ $xx\ /\ xx\ /\ xx$ indicates that two elements select 1st task, two select 2nd and two select 3rd task) each strings contains $n + r - 1$ symboles,of which r are alike $(x's)$

and $n-1$ are alike (slashes).Therefore by theorem 1.9,the number of such strings, that is r-combination equals

$$\frac{(n+r-1)!}{r!(n-1)!} = C(n+r-1,r) = \binom{n+r-1}{r}$$

Example 1.24. Suppose that a cookie shop has four different kinds of cooking. How many different ways can six cookies be chosen? Assume that only the types of cookie are not the individual cookies or the order in which they are chosen matter

Solution: The number of ways to chose six cookies is the number of 6-combinations of a set with four elements. From theorem 1.10, this equals

$C(4+6-1,6) = C(9,6)$. Since

$$C(9,6) = C(9,3) = \frac{9!}{(9-3)!3!} = \frac{9!}{6!3!} = \frac{9\times8\times7\times6!}{3\times2\times1\times6!} = 84$$

There are 84 different ways to chosen the six cookies.

Table 1.1 gives a summary of the different ways how r object can be drawn from n object

	With repetition	Without repetition
Retaining order	$p(n,r) = \frac{n!}{(n-r)!}$	n^r
Not retaining order	$C(n,r) = \binom{n}{r}$ $= \frac{n!}{(n-r)!\,r!}$	$\binom{n+r-1}{r}$ $= \frac{(n+r-1)!}{r!\,(n-1)!}$

Table 1.1 different ways of drawing from n objects.

Exerices1.4

1. *Make each sentence as true or false. When n is an arbitrary non negative integer and* $0 \le r \le n$

   a) $0! = 0$
   b) $5 \times 4! = 5!$
   c) $(2+5)! = 2! + 5!$
   d) $(2\times3)! = 2!\,3!$
   e) $n(n-1)! = n!$
   f) $p(n,r) = p(n, n-r)$
   g) $1! = 1$
   h) $(m+n)! = m! + n!$
   i) $(mn)! = m!\,n!$
   j) $p(n,0) = 0$

2. *Find the number of two-digit numerals that can be formed using the digits* 2,3,5,6*and* 9 *and that contain no repeated digits.*

3. *How many permutation are there of the set* $\{a, b, c, d, e, f\}$?

4. *Let* $S = \{1,2,3,4,5\}$

   a) List all the 3 −permutation of S

   b) List of the 3 −combination of S

5 . *How many bit strings of length ten contain*

   a) *Exactly four 1 s*

   b) *At most four 1 s*

   c) *An equal number of 0 s and 1 s*

5. *Solve the equation:*

   a) $p(n, 1) = 6$ c) $p(n, n-1) = 5040$

   b) $C(n, 1) = 10$ d) $C(n, n-2) = 55$

6. *In how many different ways can five elements be selected order from a set with three elements when repetition is allowed?*

7. *How many ways are there to select five unordered elements from a set with three elements when repetition is allowed.*

8. *How many different ways are there to choose a dozen dounts from the 21 varieties at a dount shpo?*

9. *A bagel shop has onion bagels, poppy seed bagels, egg bagels, salty bagels and plain bagels. How many ways are there to choose.*

   a. *Six bagels*

   b. *A dozen bagels*

   c. *Two dozen bagels*

   d. *A dozen bagels with at least one of each kind*

   e. *A dozen bagels with at least three egg bagels and no more than two slaty bagels*

10. *How many different strings can be made from the letters in MISSISSIPPI, using all the letters.*

11. *How many different strings can be made from the letter in ORONO, using some or all of the letters*

12. *How many strings with five or more characters can be formed from the letters in SEERESS?*

1.5 The Binomial Theorem

Objective: After completing this section the student should be able to:

- ✓ Explain the binomial theorem
- ✓ Explain the Pascal's triangle and Pascal identity
- ✓ Understand the binomial theorem.
- ✓ Solve same problems using binomial theorem.

The binomial theorem gives the coefficients of the expansion of powers of binomial expression. A binomial expression is simply the sum of terms, such as $x + y$.

Theorem 1.11(The Binomial Theorem) If n is a nonnegative integer and x and y be a real variable, then

$$(x+y)^n \quad = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^n$$

The notation $\sum$ means that the sum extends over all integers.

Proof: Since $(x+y)^n = (x+y)(x+y)\ldots(x+y)$ to n factors. $(x+y)^n$ is expanded by multiplying an x from some of the factors on the RHS and a y from the remaning factors. That is, every term is obtained by selecting an x from any of the $n-r$ factors and a y from the remaining r factor. Thus, every term in the expansion is of the form $c\,x^{n-r}y^r$, where c denotes the coefficient and $0 \le r \le n$.

Notice that the coefficient of $x^{n-r}y^r$ is the number of ways of selecting an x from any $n-r$ of the n factors (and hence a y from the remaining r factors). Therefore

Coefficient of

$$x^{n-r}y^r = \binom{n}{n-r} = \binom{n}{r}$$

So, every term in the expansion is of the form $\binom{n}{r} x^{n-r} y^r$ where $0 \le r \le n$. Thus,

$$(x+y)^n = \sum_{r=0}^{4} \binom{n}{r} x^{n-r} y^n$$

Example 1.25. Find the binomial expansion of $(2a - 3b)^4$

Solution: Here $x = 2a \; and \; y =$ 3b and $n = 4$ .using the binomial theorem

$$(2a-3b)^4 = \big(2a + (-3b)\big) = \sum_{r=0}^{n} \binom{4}{r} x^{4-r} y^r$$

$$= \binom{4}{0}(2a)^4 + \binom{4}{1}(2a)^3(-3b) + \binom{4}{2}(2a)^2(-3b)^2 +$$

$\binom{4}{3}(2a)(-3b)^3 + \binom{4}{4}(-3b)^4$

$= (2a)^4 + 4(2a)^3(-3b) + 6(2a)^2(-3b)^2 + 4(2a)(-3b)^3 + (-3b)^4$

$= 16a^4 - 96\,a^3b + 216\,a^2b^2 - 216ab^3 + 81b^4$

Example 1.26: What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$

Solution : $(2x - 3y)^{25} = (2x + (-3y))^{25}$. By the binomial theorem, we have

$$(2x - 3y)^{25} = \sum_{r=0}^{n} \binom{n}{r} x^{n-r}y^r = \sum_{r=0}^{n} \binom{n}{r} (2x)^{n-r}(-3y)^r.$$

Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $r = 13$,namely,

$$\binom{25}{12}(2x)^{25-13}(-3y)^{13} = \binom{25}{12}(2x)^{12}(-3y)^{13}$$

$$= \binom{25}{12}(2)^{12}x^{12}(-3)^{13}y^{13}$$

$$= \binom{25}{12}(2)^{12}(-3)^{13}x^{12}y^{13}$$

$$= \frac{25!}{13!12!}2^{12}(-13)^{13}$$

We can prove some useful identities using the Binomial Theorem.

Corollary 1.12 Let n be a non negative integer. Then

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n$$

That is, the sum of the binomial coefficients is 2^n , in other words, a set with n elements has 2^n subsets.

Proof: By the binomial theorem

$$(x + y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r}y^n$$

Let $x = y = 1$, then $(x + y)^n = (1 + 1)^n = \sum_{r=0}^{n} \binom{n}{r} 1^{n-r}1^n = \sum_{r=0}^{n} \binom{n}{r}$

Thus, $\sum_{r=0}^{n} \binom{n}{r} = 2^n$

Corollary 1.13 Let n be a positive integer. Then

$$\sum_{r=0}^{n}(-1)^r \binom{n}{r} = 0$$

Proof: By the binomial theorem it follows that

$$0 = 0^n = (1 + (-1))^n = \sum_{r=0}^{n} \binom{n}{r}(1)^{n-r}(-1)^r$$

$$= \sum_{r=0}^{n} \binom{n}{r}(-1)^r$$

$$= \sum_{r=0}^{n}(-1)^n \binom{n}{r}$$

Corollary 1.14

$$\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\cdots+=\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots$$

Where $n \geq 1$,that is, the sum of the “even” binomial coefficient equals that of the “odd” binomial coefficients.

Proof: By using the corollary 1.13, we have

$$0=\sum_{r=0}^{n}(-1)^r\binom{n}{r}$$

$$=\binom{n}{0}(-1)^0+\binom{n}{1}(-1)^1+\binom{n}{2}(-1)^2+\cdots+\binom{n}{n}(-1)^n$$

$$=\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\binom{n}{4}-\binom{n}{5}+\cdots$$

$$=\left[\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\cdots\right]-\left[\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots\right]$$

$$\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\cdots=\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots$$

Corollary 1.15 Let n be a non negative integer. Then

$$\sum_{r=0}^{n}2^r\binom{n}{r}=3^n$$

Proof: Again by the binomial theorem, put $x=1\ and\ y=2$, then

$$(1+2)^n=\sum_{r=0}^{n}\binom{n}{r}(1)^{n-r}(2)^r=\sum_{r=0}^{n}\binom{n}{r}2^r$$

Hence , $\sum_{r=0}^{n}2^r\binom{n}{r}=3^n$

Pascal’s identity and triangle

Definition 1.3 The various binomial coefficients $\binom{n}{r}$,where $0 \leq r \leq n$, can be arranged in the form of a triangle, called Pascal’s triangle [Although Pascal’s triangle is named after pascal, it appeared in 1330 work by the Chinese Mathematician Chushi-kie], as shown in figure 1.8 and figure 1.9

$$\begin{array}{ccccccccccl}
 & & & & \binom{0}{0} & & & & & \longleftarrow & \text{row } 0 \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & & \longleftarrow & \text{row } 1 \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{1} & & & \longleftarrow & \text{row } 2 \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \longleftarrow & \text{row } 3 \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \longleftarrow & \text{row } 4
\end{array}$$

Figure 1.8

1 ← row 0

1 1 ← row 1

1 2 1 ← row 2

1 3 3 1 ← row 3

1 4 6 4 1 ← row 4

Figure1.9

Theorem 1.16 (Pascal identity) Let n and r be positive integers with $n \geq r$. Then

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

Pascal's triangle has the following property

- Pascal's identity, together with the initial condition $\binom{n}{0} = \binom{n}{n} = 1$ for all integers n, can be used to recursive define binomial coefficients.
- Pascal's triangle is symmetric about a vertical line through the middle. This is so since $C(n,r) = C(n,n-r)$

Theorem 1.17 (Vandermonde's identity)

Let m, n and r be non negative integers with r not exceeding either m or n. Then

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

Remark: This identity was discovered by Mathematician Alexander-Theophile Van demonde in the eighteenth century

Proof: Suppose that there are m items in one set and n items in a second set. Then the total number of ways to pick r elements from the union of these sets is $\binom{m+n}{r}$. Another way to pick r elements from the union is to pick k elements from the first set and then $r-k$ elements from the second set, where k is an integer with $0 \leq k \leq r$,this can be done $\binom{m}{k}\binom{n}{r-k}$ ways, using the r elements from the union also equal.

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$$

This proved Vander -monde's identity.

Corollary 1.18 If n is a non negative integer then $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$

Proof: We use Vander-mode's identity with $m = r = n$ to obtain:

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{n-k}\binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k}^2$$

The last equality was obtained using the identity $\binom{n}{k} = \binom{n}{n-k}$

Theorem 1.19: Let n and r be non negative integers with $r \leq n$. Then

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$$

Exercise 1.5

1. *Using the binomial theorem expand each.*

   a) $(x+y)^6$ b) $(x+y)^4$ c) $(2x-1)^5$

2. *Find the coefficient of* x^5y^8 *in* $(x+y)^{13}$
3. *Find the coefficient of* x^4y^5 *in* $(2x-3y)^9$
4. *Find the middle in the binomial expansion of each*

   a) $(x+\frac{1}{x})^4$ b) $(2x+\frac{2}{x})^6$ c) $(x-\frac{1}{x})^6$

5. *Prove that if* n *and* k *are integers with* $1 \leq k \leq n$ *. then* $k\binom{n}{k} = n\binom{n-1}{k-1}$
6. $\sum_{r=0}^{n} \binom{2n}{2r} = \sum_{r=1}^{n} \binom{2n}{2r-1}$ *(hint use corollary 1.14)*
7. *find a formula for*

   a) $\sum_{i=2}^{n} \binom{i}{2}$ b) $\sum_{i=3}^{n} \binom{i}{3}$

Chapter summery

- **Addition principle**

  If A and B are two mutually exclusive tasks and can be done in m and n ways, respectively, then task a A or B can be done in $m+n$ ways .

- **Inclusion-exclusion principle**

  Suppose task A can be done in m ways and task B in n ways. If both can be done in k ways, then task A or B can be done in $m+n-k$ ways.

- **Multiplication principle**

  If task T_1 can be done in m_1 ways and task T_2 in m_2 ways corresponding to each way T_1 can occur, these two tasks can be done in that order in m_1m_2 ways.

- An r −permutation of a set of n distinct element is an arrangement of r elements of the set.
- The number of r −permutations of a set of size n is denoted by $p(n,r)$,

$$p(n,r) = \frac{n!}{(n-r)!}$$

- The permutation of n distinct objects are $n!$, $p(n,n) = n!$
- The number of r −permutation of a set of n objects with repetition allowed is n^r.
- An r-combination of a set of n elements is a subset with size r, where $0 \leq r \leq n$.
- The number of r-combination is denoted by $C(n,r)$ or $\binom{n}{r}$.
- $C(n,r) = \frac{n!}{r!(n-r)!}$, $C(n,r) = C(n,n-r$
- The number of r −combinations with repetition from a set of n elements is $C(n+r-1,r)$.
- Let x and y be a real variable and n is a nonnegative integer, then

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^n$$

- $\sum_{r=0}^{n} C(n,r) = 2^n$.

Self Test Exercise 1

1. *Find the number of positive integer* ≤ 2076 *and divisible by*

   a) 3 *or* 4 c) 3 *or* 4 *bit not* 12

   b) $2,3$*or* 5 *but not* 30 d) $3, 4$ *or* 5 *but not* 60

2. *Find the number of positive integers* ≤ 1976 *and divisible by*

   a) *2 or 3*   c) *2,3 0r 5*

   b) *3 or 5*   d) *3,5 or 7*

3. *In how many ways can the letters of the word **AFFECTION** be arranged, keeping the vowels in their natural order and not letting the two **F**'s come together?*
4. *In one version of BASIC a variable name consists of letters or a letter followed by a digit. Find the total number of possible names.*
5. *Of 40 people, 28 smoke and 16 chew tobacco. It is also known that 10 both smoke and chew. How many among the 40 neither smoke nor chew?*
6. *Consider the set*

A = {2,4,6, . . . ,114}.

a) *How many elements are there in A?*

b) *How many are divisible by 3?*

c) *How many are divisible by 5?*

d) *How many are divisible by 15*

e) *How many are divisible by either 3, 5 or both?*

f) *How many are neither divisible by 3 nor 5*

g) *How many are divisible by exactly one of 3 or 5?*

7. *Solve each of the following, where* $n \geq 0$
   a) $c(n, o) = 1$ c) $c(n, 1) = 10$
   b) $c(n, 2) = 28$ d) $c(n, n-2) = 55$
8. *Evaluate*
   a) $\frac{10!}{3!7!}$ c) $p(6,6)$
   b) $p(5,3)$ d) $\frac{5!}{4!}$
9. *Find the number of ways of dividing a set of size* n *into two disjoint subsets of sizes* r *and* $n - r$
10. *Find the number of three digit numerals that can be formed using the digit 2,3,5,6 and 9 if repetitions are not allowed*
11. *Find the number of ways seven boys and three girls can be seated in a row if*
    a) *A boy sit at each end of the row*
    b) *A girl sits at each end of the row*
    c) *The girl sit together at one end of the row*
12. *Find the coefficient of each:*
    a) x^3y^5 *in the expansion of* $(x + y)^8$
    b) x^4y^6 *in the expansion of* $(x - y)^{10}$
    c) x^2y^6 *in the expansion of* $(2x + y)^8$
13. *Use the binomial theorem, expand each*
    a) $(x + y)^6$ c) $(2x - 1)^6$
    b) $(x - y)^5$ d) $(x + 2y)^5$
14. *Find the largest binomial coefficient in the expansion of each*
    a) $(x + y)^5$ c) $(x + y)^6$
    b) $(x + y)^4$ d) $(x + y)^8$

Reference:-

1. A. W.F.Edwards,Pascal's Arithmeticical Triangle. The Johns Hopkins University Press, Baltimore,MD, 2002.
2. Branislav Kisacanin, Mathematical problems and proof, combinatorics, number theory and geometry
3. C.Oliver, " The Twelve Days of Christmas," Mathematics Teacher, Vol. 70
4. D .P.Acharjya Sreekumar, Fundamental Approach to Discrete Mathematics, 3rd eedition.
5. Discrete Mathematics, Handbook of discrete and Compinatorial Mathematics
6. Hewel Hus, Schaum's outline, probability Random Variable
7. J.M.Harris.Jeffry L.Hirst.Michael J.Mossinghoff, Combinatorics and Graph Theory, 2nd edition
8. Kenneth H.Rosen, Discrete Mathematics and its application, 3rd edition.
9. KOSHY, Discrete Mathematics with application
10. M.Eng and J.Casey, "Pascal's Arithmetical Triangle-ASerendipitious Source for programming Activities,'' Mathematics Teacher, Vol 76
11. Mott, Knadel and Baker, Discreate Mathematics
12. P.Z.Chinn,''inductive patterns, finite Difference, and a Missing Region,'' Mathematics Teacher, Vol.81(Sept.1988)
13. R.P.GRIMALDI and B.V.RAMANA pearson, Discreate combinatorial mathematics
14. Seymour Lipschutz,Schaum's outline, Discrete Mathematics, 3rd edition
15. Steven G. Krantz, Discrete Mathematics

CHAPTER 2

ELEMENTARY PROBABILITY THEORY

Objective: At the end of this chapter the student should be able to:

- Explain the elementary property of a probability of an event.
- Calculate the probability of an event.
- Explain the conditional probability
- Calculate the conditional probability of an event
- Identify whether the given two events are dependent or independent event.
- Calculate the expected value

Introduction

Probability theory is a mathematical modeling of the phenomenon of chance or randomness. If a coin is tossed in a random manner, it can land heads (H) or tails (T),with equally likely. Each of them, H or T is an outcome of the experiment to tossing the coin. The set $\{H, T\}$ of the possible outcomes of the experiment is the sample space of the experiment. A probabilistic mathematical model of random phenomena is defined by assigning "probabilities" to all the possible outcomes of an experiment. The reliability of our mathematical model for a given experiment depends upon the closeness of the assigned probabilities to the actual limiting relative frequencies.

In this chapter the basic concepts of probability theory are presented. We see the definition of important terms that are used to solve the probability of a given experiments. In section 2.1 we will study experiments with finitely many out comes that are not necessary equally likely. From section 2.2 to 2.4 we introduce some probability concepts in probability theory, including conditional probability and independence of events. Finally we discussed the concept of the random variable and the expectation and variance of a random variable.

2.1 Sample space and events

Objective: After study this section the student should be able to:

- ✓ Define the terms; random experiment, sample space and events

- ✓ Find the sample space of an experiment
- ✓ Find the events of an experiment

Definition 2.1:

A) Random experiments:

An **experiment** is a procedure that yields one of a given set of possible outcomes. An experiment is called a **random experiment** if its outcome cannot be predicted. Typical experiments of a random experiment are the roll of a die, the toss of a coin or drawing a card from a deck.

B) Sample space

The set S of all possible outcomes of a given experiment is called the ***sample space*** or universal set (S is the notation for a sample space and assumes that S is non- empty). A particular outcome, i.e., an element in S, is called a ***sample point***. Each outcome of a random experiment corresponding to a sample point.

Example 2.1 Find the sample space for the experiment of tossing a coin (a) one and (b) twice

Solution: a) There are two possible outcomes, head or tails, thus

$$S = \{H, T\}$$

b) There are four possible outcomes. They are pairs of heads and tails.

Thus,

$$S = \{HH, HT, TH, TT\}$$

Example 2.2: Find the sample space for the experiment of tossing a coin repeatedly and of counting the number of tosses required until the first head appears.

Solution: Clearly all possible outcomes for this experiment are the terms of the sequence 1,2,3 .... Thus

$$S = \{1,2,3, \ldots\}$$

Note that there are infinite number of outcomes (i.e. the sample space is infinite).

Example 2.3*:* Find the sample space for the experiment of toss a (six sided) die.

The sample consists of the six possible numbers, that is

$$S = \{1,2,3,4,5,6\} .$$

Note that: Any particular experiment can be often having many different sample spaces depending on the observation of interest.

A sample space S is said to be discrete if it consists of a finite number of sample point (as in example 2.1 and 2.3) or countable infinite sample point (as in example 2.2)

C) Event

Since we have identified a sample space S as the set of all possible outcomes of random experiments, we will review some set notation in the following:

- If a is an element of S (or belongs to S), then we write $a \in S$.
- If b is not an element of S (or does not belongs to S) then we write $b \notin S$.
- A set A is called a subset of B, denoted by $A \subset B$, if every element of A is also an element of B.

An event E is a set of outcomes or, in other words, a subset of the sample space S. In particular, the set $\{a\}$ consisting of a single sample point $a \in S$ is called an ***elementary event***. Furthermore, the empty set $\emptyset$ and S itself are subset of S and so $\emptyset$ and S are also events. Since S is the set of all possible outcomes, it is often called the certain event and $\emptyset$ is sometimes called the **impossible event** or the **null event.** An outcome in E is **favorable** outcome (or success), an outcome not in E is an **unfavorable** outcomes (or failure). Since an event is a set, we can combine events to form new events using the various set operations:

I. $A \cup B$ is the event that occurs if and only if A occurs *or* B occurs (or both).

II. $A \cap B$ is the event that occurs if and only if A occurs *and* B occurs.

III. A^c, the complement of A, is the event that occurs if and only if A does *not* occur.

Two events A and B are called ***mutually exclusive*** if they are disjoint, that is, if $A \cap B = \emptyset$. In other words, A and B are mutually exclusive if and only if they cannot occur simultaneously. Three or more events are mutually exclusive if every two of them are mutually exclusive.

Example 2.4: Consider the experiment of example 2.2. Let A be the event that the number of tosses required until the first head appears is even. Let B be the event that the number of tosses required until the first head appears is odd. Let C be the event that the number of tosses required until the first head appears is less than 5. Express events A, B *and* C

A={2,4,6, ... } , B={1,3,5, ... } , C={1,2,3,4}

EXAMPLE 2.5: Consider the experiment of toss a coin three times. Let A be the event of obtaining exactly two heads, let B be that of obtaining at least two heads and C that of obtaining four heads. Express events A, B and C

Solution: By the multiplication principle, the sample space S consists of $2 \times 2 \times 2 = 8$ possible outcome: see figure 2.1

The sample space consists of the following eight elements:

$$S = \{HHH, HHT, HTH, HT\,T, T\,HH, THT\,, T\,TH, T\,T\,T\,\}$$

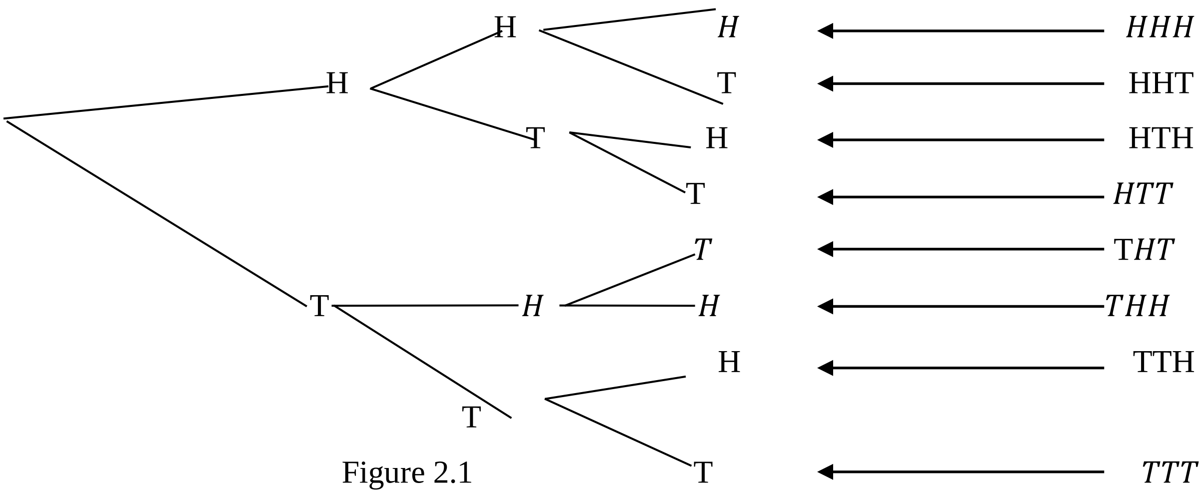


Figure 2.1

$A = \{HHT, THH, HTH\}\,, B = \{HHH, HHT, HTH, THH\,\}$and $\mathrm{C} = \emptyset$ an impossible event.

Then $A \cap B = \{HHT, HTH, THH\} = A$

Example 2.6: Tossing a (six-sided) die. The sample space S consists of the six possible numbers, that is, $S = \{1, 2, 3, 4, 5, 6\}$. Let A be the event that an even number appears, B that an odd number appears, and C that a prime number appears. That is, let

$$A = \{2, 4, 6\}, B = \{1, 3, 5\}, C = \{2, 3, 5\}$$

Then: $A \cup C = \{2, 3, 4, 5, 6\}$ is the event that an even or a prime number occurs.

$B \cap C = \{3, 5\}$ is the event that an odd prime number occurs.

$\mathrm{C}^{\mathrm{c}} = \{1, 4, 6\}$ is the event that a prime number does not occur.

Note that A and B are mutually exclusive: $A \cap B = \emptyset$. In other words, an even number and an odd number cannot occur simultaneously.

EXAMPLE 2.7 Consider the experiment toss a pair of dice. Let A be the event that the sum of two numbers is 6, and let B be the event that the largest of the two number is 4. Express the event A and B

Solution: There are six possible numbers, $1, 2, \dots, 6$, on each die. Thus S consists of the pairs of numbers from 1 to 6, and hence $n(S) = 36$. Figure 2.2 shows these 36 pairs of numbers arranged in an array where the rows are labeled by the first die and the columns by the second die.

$$A = \{(1,5), (2,4), (3,3), (4,2), (5,1)\},$$

$$B = \{(1,4), (2,4), (3,4), (4,4), (4,3), (4,2), (4,1)\}.$$

Then the event "A and B" consists of those pairs of integers whose sum is 6 and whose largest number is 4 or,in other words, the intersection of A and B.

Thus

$$A \cap B = \{(2,4), (4,2)\}$$

Similarly, "A or B," the sum is 6 or the largest is 4, shaded in Fig. 2.2, is the union A

First die

Second die

	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Figure 2.2

Exercise 2.1

1. *Consider a random experiment of tossing a coin three times.*
   a. *Find the sample space S_1, if we wish to observe the number of heads and tails obtained.*
   b. *Find the sample space S_2, if we wish to observe the number of heads in the three tosses.*

2. *Consider an experiment of drawing two cards at random from a bag containing four cards marked with the integer 1 through 4.*

a) *Find the sample space* S_1 *of the experiment if the first card is replaced before the second is drawn.*

b) *Find the sample space* S_2 *of the experiment if the first card is not replaced.*

4. *Consider the experiment of selecting items from a group consisting of three* $\{a, b, c\}$.

a) *Find the sample space* S_1 *of the experiment in which two items are selected without replacement.*

b) *Find the sample space* S_2 *of the experiment in which two items are selected with replacement.*

5. *Find the sample space for experiment consisting of measurement of the voltage V from a transducer, the maximum and minimum of which are* $+5$ *and* -5 *volts, respectively.*

6. *An experiment consists of tossing two dice.*

a) *Find the sample space S*

b) *Find the event A that the sum of the dotes on the dice is greater than* 10

c) *Find the event* B *that the sum of the dotes on the dice is greater than* 12

d) *Find the event* C *that the sum of the dotes on the dice is greater than* 7

2.2 Probability of an event

Objective: After study this section the student should be able to:

- ✓ Define the probability of an event
- ✓ Explain the property of probability
- ✓ Calculate the probability of events

Definition 2.2: Let S be a finite sample space, say $S = \{a_1, a_2, a_3, \dots, a_n\}$. A finite probability space, or probability model, is obtained by assigning to each point a_i in S a real number $\boldsymbol{p_i}$ called the ***probability*** of $\boldsymbol{a}_i$ satisfying the following properties:

I. Each p_i is nonnegative, that is, $p_i \geq 0$.

II. The sum of the p_i is 1, that is, is $p_1 + p_2 + \cdot \cdot \cdot + p_n = 1$

The *probability* of an event A written $P(A)$, is then defined to be the sum of the probabilities of the points in A. The singleton set $\{a_i\}$ is called an ***elementary* event** and, for notational convenience, we write $P(a_i)$ for $P(\{a_i\})$.

Definition 2.3: Let E be an event of a finite sample space S consisting of equally likely out comes. Then the probability of the event, defined as

$$p(E) = \frac{|E|}{|S|} = \frac{number\ of\ ways\ E\ can\ occure}{total\ number\ of\ possible\ out\ comes}$$

EXAMPLE 2.8: Suppose three coins are tossed, and the number of heads is recorded. Let A be the event that at least one head appears, and let B be the event that all heads or tails appears. Find the probability of event A and B.

Solution: The sample space is $S = \{0, 1, 2, 3\}$. The probability of the elements of S.

$P(0) = \frac{1}{8}$, $P(1) = \frac{3}{8}$, $P(2) = \frac{3}{8}$, $P(3) = \frac{1}{8}$

That is, each probability is nonnegative, and the sum of the probabilities is 1.

$A = \{1, 2, 3\}$ *and* $B = \{0, 3\}$.

Then, by definition, $P(A) = P(1) + P(2) + P(3) = \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = \frac{7}{8}$ and

$$P(B) = P(0) + P(3) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

Example 2.9: Let a card be selected from an ordinary deck of 52 playing cards. Let

A = {the card is a spade} and B = {the card is a face card}.

What is *P(A), P(B)*, and *P(A ∩ B)*.

Solution: Using definition 2.3, $\mathrm{P(A)} = \frac{\text{number of spades}}{\text{number of card}} = \frac{13}{52} = \frac{1}{4}$

$$\mathrm{P(B)} = \frac{\text{number of face cards}}{\text{number of card}} = \frac{12}{52} = \frac{3}{12}$$

$$\mathrm{P(A \cap B)} = \frac{\text{number of spade face cards}}{\text{number of card}} = \frac{3}{52}$$

Example 2.10 An urn contains four blue ball and five red ball. What is the probability that a ball chosen from the urn is blue?

Solution: Calculate the probability, note that there are nine possible outcomes, and four of these possible outcomes produce a blue ball, hence, the probability that a blue is chosen is $\frac{4}{9}$.

Elementary properties of probability:

Let S be the sample space and A be an event in S. Then the probability function P defined on the class of all events in a finite probability space has the following properties:

1. For every event A, $0 \leq P(A) \leq 1$.
2. $P(S) = 1$.
3. $p(A^c) = 1 - p(A)$

4. $P(\emptyset) = 0$

5. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Example 2.11: Find the probability of obtaining at least one head when three coins are tossed.

Solution: Let A be the event of obtaining at least one head. Then A^c denotes the event of obtaining no heads occur, $p(A^c) = \frac{1}{8}$, therefore ,

$$p(A) = 1 - p(A^c) = 1 - \frac{1}{8} = \frac{7}{8}$$

Let $E = \{a_1, a_2, a_3, \dots, a_n\}$ be an event of a finite sample space consisting of not necessary equally likely out comes. Let $p(a_i)$ denote the probability that the outcome a_i will occur. Then the probability of E is defined by $p(E) = \sum_{i=1}^{n} p(a_i)$. Thus $p(E)$ is the sum of the probabilities of the outcomes in E.

Example 2.12: Suppose the probability of obtaining a prime number is twice that of obtaining a non- prime number, when a certain loaded die is rolled. Find the probability of obtaining an odd number when it is rolled.

Solution: There are six possible out comes when a die is rolled, of which there are primes:2,3 and 5. The probability of obtaining a prime is twice that of a non-prime; that is $p(prime) = 2p(non - prime)$. Since the sum of the probabilities of the various possible outcome is 1. $3p(prime) + 3p(non - prime) = 1$

$$6p(non - prime) + 3p(non - prime) = 1$$

$$9p(non - prime) = 1$$

$$p(non - prime) = \frac{1}{9}$$

Thus, $p(prime) = 2p(non - prime) = \frac{2}{9}$.

Then, $p(odd\ number) = p(1) + p(3) + p(5) = \frac{1}{9} + \frac{2}{9} + \frac{2}{9} = \frac{5}{9}$.

Definition 2.4: Suppose that S is a set with n elements. The uniform distribution assigns the probability $\frac{1}{n}$ to each elements of S. The experiment of selecting an element from a sample space with a uniform distribution is called selecting an element of S at random.

We now define the probability of an event as the sum of the probabilities of the outcomes in this event.

Definition 2.5: The probability of the event E is the sum of the probabilities of the outcomes in E. that is, $p(E) = \sum_{s \in E} P(s)$

Example 2.13: Suppose that a die is biased (or loaded) so that 3 appears twice as often as each other number but that the other five outcomes are equally likely. What is the probability that an odd number appears when we roll is this die?

Solution: We want to find the probability of the event,$\{1,2,3,4,5,6\}$, $A = \{1,3,5\}$ and given that

$$p(3) = 2p(1) = 2p(2) = 2p(4) = 2p(5) = 2p(6)$$

$$P(E) = \sum_{I=1}^{6} p(a_i) = 1$$

$$p(a_1) + p(a_2) + p(a_3) + p(a_4) + p(a_5) + p(a_6) = 1$$

$$p(1) + p(2) + p(3) + p(4) + p(5) + p(6) = 1$$

$$\frac{1}{2}p(3) + \frac{1}{2}p(3) + p(3) + \frac{1}{2}p(3) + \frac{1}{2}p(3) + \frac{1}{2}p(3) = 1$$

$$\frac{7}{2}\, p(3) = 1 => p(3) = \frac{1}{7}$$

$P(1) = P(2) = P(4) = P(5) = P(6) = \frac{1}{7}, p(3) = \frac{1}{7}.$

It follows that $p(A) = p(1) + p(3) + p(5) = \frac{1}{7} + \frac{2}{7} + \frac{1}{7} = \frac{4}{7}.$

Suppose that there are n equally likely outcomes; each possible outcome has probability $\frac{1}{n}$, since the sum of their probability is 1. Suppose the event E contains m out comes. According to definition 2.5

$p(E) = \sum_{i=1}^{m} \frac{1}{n} = \frac{m}{n}$. Since $|E| = m\ and\ |S| = n$, it follows that

$$p(E) = \frac{m}{n} = \frac{|E|}{|S|}$$

Theorem 2.1 (**Inclusion- exclusion principle**). If A and B are any two events of a finite sample space, S. The probability that at least one of them will occurs is given by

$$p(A \cup B) = p(A) + p(B) - p(A \cap B)$$

Proof: By the inclusion –exclusion principle on a set,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Then $\frac{|A \cup B|}{|S|} = \frac{|A|}{|S|} + \frac{|B|}{|S|} - \frac{|A \cap B|}{|S|}.$

That

$$p(A \cup B) = p(A) + p(B) - p(A \cap B).$$

In particular, if A and B are mutually exclusive $A \cap B = \emptyset$ and hence $p(A \cap B) = 0.$

Therefore,

$$p(A \cup B) = p(A) + p(B)$$

Theorem 2.2: **(Addition Principle):** If A and B are mutually exclusive events of a finite sample space. Then $P(A \cup B) = P(A) + P(B)$.

EXAMPLE 2.14: Suppose a student is selected at random from 100 students where 30 are taking mathematics,20 are taking chemistry, and 10 are taking mathematics and chemistry. Find the probability p that the student is taking mathematics or chemistry.

Solution: Let M = {students taking mathematics} and C = {students taking chemistry}.

$$P(M) = \frac{30}{100} = \frac{3}{10},$$

$$P(C) = \frac{20}{100} = \frac{1}{5}$$

$$\mathrm{P(Mand\ C)} = \mathrm{P(M \cap C)} = \frac{10}{100} = \frac{1}{10}$$

Thus, by the Addition Principle (Theorem 2.2),

$$p = P(M\, or\, C) = P(M \cup C) = P(M) + P(C) - P(M \cap C)$$

$$= \frac{3}{10} + \frac{1}{5} - \frac{1}{10} = \frac{2}{5}$$

Example 2.15 A survey among 50 house wives about the two laundry detergent Lex(L) and Rex(R), shows that 25 like Lex, 30 like Rex, 10 like both, and 5 like neither. A house wives is selected at random from the group surveyed. Find the probability that she likes neither Lex nor Rex.

Solution: Using the Venn diagram in figure 2.2, we have $p(L) = \frac{25}{50}, p(R) = \frac{30}{50}$

$$p(L \cap R) = \frac{10}{50},$$

$$p(L \cup R) = P(L) + P(R) - P(L \cap R) = \frac{1}{2} + \frac{3}{5} - \frac{1}{5} = \frac{9}{10}$$

So, $p(L^c \cap R^c) = p((L \cup R)^c) = 1 - p(L \cup R)$

$$= 1 - \frac{9}{10} = \frac{1}{10}$$

Exercise 2.2

1. *A card is drawn at random from a standard deck of cards. Find the probability of obtaining*

    a) *A king* c) *A king or a queen*

    b) *A club* d) *A club or a diamond*

2. *What probability should be assigned to the out come of heads when a biased coin is tossed, if heads is three times as likely to come up as tail. What probability should be assigned to the outcome of tails?*
3. *Two dice are rolled. Find the probability of obtaining*
   a) *Two five* c) *A five and a six*
   b) *A sum of four* d) *A sum less than five*
4. *Two cards are drawn at random from a standard deck of card. Find the probability that:*
   a) *Both are king* d) *One is a king and the other a queen*
   b) *Both are club*
   c) *One is a club and the other a diamond.*
5. *Let* $U = \{a, b, c, d, e\}$ *be the sample space of an experiment, where the outcomes are equally likely. Find the probability of each event*
   a) $\{a\}$ c) $\{a, c, d\}$
   b) $\{a, b\}$ d) $\emptyset$
6. *Let* $p(A) = 0.9$ *and* $p(B) = 0.8$. *show that* $p(A \cap B) \geq 0.7$
7. *A random experiment has sample space* $S = \{a, b, c\}$. *Suppose that* $p(a, c) = 0.75$ *and* $p(b, c) = 0.6$.*Find the probability of the elementary events.*
8. *Let* A, B *and* C *be three events in* S. *If* $p(A) = p(B) = \frac{1}{3}, p(A \cap B) = \frac{1}{8}, p(A \cap C) = \frac{1}{6}$ *and* $p(B \cap C) = 0$ *find* $P(A \cup B \cup C)$
9. *Consider a telegraph source generating two symbols, dots and dashes. We observe that the dots were twice a likely to occur as the dashes. Find the probability of dot's occurring and the dash's occurring.*
10. *The sample space* S *of a random experiment is given by* $S = \{a, b, c, d\}$ *with probability* $p(a) = 0.2, p(b) = 0.3, p(c) = 0.4,$ *and* $p(d) = 0.1$. *let* A *denote the event* $\{a, b\}$, *and* $B = \{b, c, d\}$.*determine the following probabilities: (a)* $p(B)$;*(b)* $p(A)$;*(c)* $p(A^c)$;*(d)* $p(A \cup B)$ *and (e)* $p(A \cap B)$

2.3. Conditional probability

objective: After study this section the student should be able to:

- ✓ Define conditional probability
- ✓ Calculate the conditional probability of an event

Suppose E be the event of rolling a sum of seven with two dice. Then $p(E) = \frac{6}{36} = \frac{1}{6}$. Suppose a 3 comes up on one of the dice. This reduces the sample space to

$$\{(1,3),(2,3),(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(4,3),(5,3),(6,3)\}$$

consequently, a sum of 7 can be obtained in two ways: $\{(3,4),(4,3)\}$. Therefore, the probability of getting a sum of seven, knowing that a three has been rolled is $\frac{2}{11}$. Thus the additional information has indeed affected the probability of E.

Accordingly, we make the following definition

Definition2.6: The probability that an event A will occur, knowing that a certain other event $B\ (\neq \emptyset)$ has already occurred, is the conditional probability of A, given B, denoted by $p(A\backslash B)$. And defined as

$$p(A\backslash B) = \frac{p(A\cap B)}{p(B)}$$

Suppose S is a sample space, and $n(A)$ denotes the number of elements in A. Then $P(A \cap B) = \frac{n(A\cap B)}{n(S)}, p(B) = \frac{n(B)}{n(S)}$, and so on $p(A\backslash B) = \frac{p(A\cap B)}{p(B)} = \frac{n(A\cap B)}{n(B)}$

EXAMPLE 2.16; (a) A pair of fair dice is tossed. The sample space S consists of the 36 ordered pairs (a, b), where a and b can be any of the integers from 1 to 6.(see example 2.6)Thus the probability of any point is $\frac{1}{36}$. Find the probability that one of the dice is 2 if the sum is 6.

Solution: Let $E = \{sum\ is\ 6\}$ and $A = \{2\ appears\ on\ at\ least\ one\ die\}$

Now E consists of 5 elements and $A \cap E$ consists of two elements; namely

$$E = \{(1,5),(2,4),(3,3),(4,2),(5,1)\} \text{ and } A \cap E = \{(2,4),(4,2)\}$$

By definition 2.6 $P(A\backslash E) = \frac{2}{5}$.

On the other hand A itself consists of 11 elements, that is,

$A = \{(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),(1,2),(3,2),(4,2),(5,2),(6,2)\}$

Since S consists of 36 elements, $P(A) = 11/36$.

(b) A couple has two children; the sample space is $S = \{bb, bg, gb, gg\}$ with probability $\frac{1}{4}$ for each point. Find the probability p that both children are boys if it is known that:

$i)$ At least one of the children is a boy;

(ii) the older child is a boy.

Solution: Let E be the set contain both children are boy, $E = \{bb\}$

i) Here the reduced space consists of three elements, $A = \{\text{bb}, \text{bg}, \text{gb}\}$; then

$p = p(E\backslash A) = \frac{P(A\cap E)}{P(A)} = \frac{1}{3}$

ii) Here the reduced space consists of only two elements $B = \{\text{bb}, \text{bg}\}$;

Then $p = p(E\backslash B) = \frac{P(B\cap E)}{p(B)} = \frac{1}{2}$

Example 2.17: A bit strength of length four is generated at random, so that each of the 16 bit strings of length four is equally likely. What is the probability that it contains at least two consecutive 0's, given that its bit is a 0? (We assume that 0 bit and 1 bit are equally likely)

Solution: Let A be the event that a bit strings of length four contains at least two consecutive $0s$, and B be the event that the first bit of a bit string of length four is a 0. The probability that a bit string of length four has at least two consecutive 0s, given that its first bit is a 0. Equals $p(A\backslash B) = \frac{p(A\cap B)}{p(B)}$.

Since $A \cap B = \{0000, 0010, 0011, 0100\}$ (See figure 2.3) .We see that $p(A \cap B) = \frac{5}{16}$. Since there are eight bit strings of length four that start with 0, we have $p(B) = \frac{8}{16} = \frac{1}{2}$

Consequently, $p(A\backslash B) = \frac{P(A\cap B)}{P(B)} = \frac{5/16}{1/2} = \frac{5}{8}$

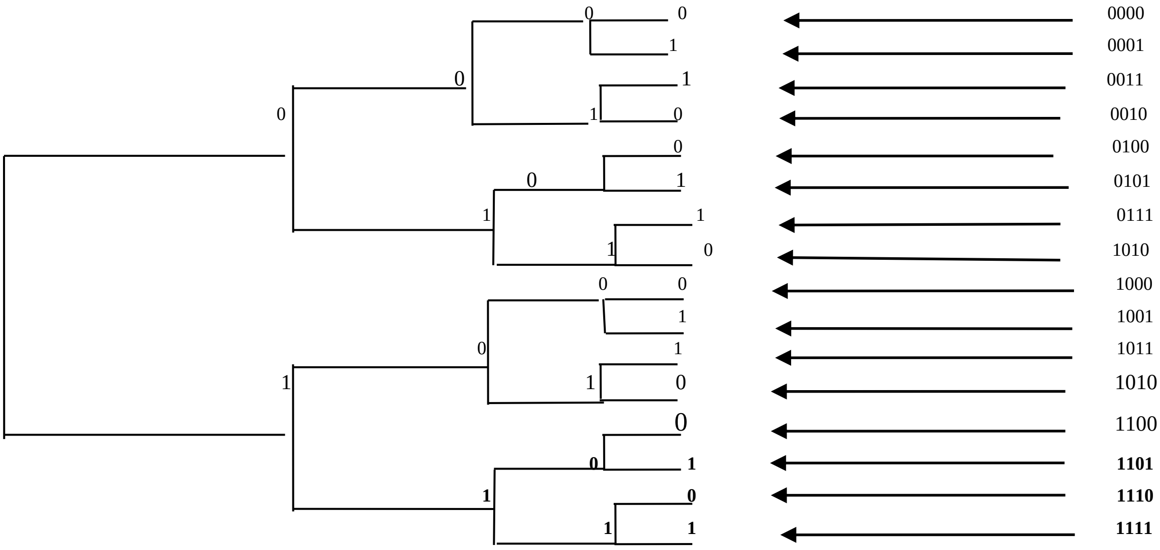


FIGURE 2.3

Theorem 2.3 (Multiplication Theorem for Conditional Probability):

Let A and B be any two events of a finite sample space. Then the probability that both A and B will occure is given by **:**

$$\mathrm{P}(\mathrm{A} \cap \mathrm{B}) = \mathrm{P}(\mathrm{A})\mathrm{P}(\mathrm{B}\backslash\mathrm{A}) = P(B)p(A\backslash B)$$

From theorem 2.3 we can obtain the following Baye's rule:

$$p(A\backslash B) = \frac{p(B\backslash A)p(A)}{p(B)}$$

The multiplication theorem gives us a formula for the probability that events A and B both occur. It can easily be extended to three or more events$\mathrm{A}_1, \mathrm{A}_2, \ldots \mathrm{A}_\mathrm{m}$; that is,

$$P(\mathrm{A}_1 \cap \mathrm{A}_2 \cap \ldots \cap \mathrm{A}_\mathrm{m}) = P(\mathrm{A}_1) \cdot P(\mathrm{A}_2\backslash\mathrm{A}_1) \cdot \cdot \cdot P(\mathrm{A}_\mathrm{m}\backslash\mathrm{A}_1, \mathrm{A}_2, \ldots \mathrm{A}_{\mathrm{m}-1})$$

EXAMPLE 2.17 A lot contains 12 items of which 4 are defective. Three items are drawn at random from the lot one after the other. Find the probability p that all three are non-defective.

Solution: The probability that the first item is non-defective is $\frac{8}{12}$ since 8 of 12 items are non-defective. If the first item is non-defective, then the probability that the next item is non-defective is $\frac{7}{11}$ since only 7 of the remaining 11 items are non-defective. If the first 2 items are non-defective, then the probability that the last item is non-defective is $\frac{6}{10}$ since only 6 of the remaining 10 items are now non-defective.

Thus by the multiplication theorem,

$$p = \frac{8}{12} \cdot \frac{7}{11} \cdot \frac{6}{10} = \frac{14}{55} \approx 0.25$$

Example 2.18: Two marbles are drawn successively from a box of three black and four white marbles. Find the probability that both are black if the first marble is not replaced before the second drawing.

Solution: Let B_1 be the event drawing the first black marble, then $p(B_1) = \frac{3}{7}$ and let B_2 be the event of drawing a second black marble. Since the first marble is not replaced before the second is drawn, there are only two black balls left in the box at the second drawing. Therefore, $p(B_2\backslash B_1) = \frac{2}{6}$. Consequently, the probability of drawing two black balls successively without replacement is given by

$$p(B_1 \cap B_2) = p(B_1) \cdot p(B_2\backslash B_1) = \frac{3}{7} \cdot \frac{2}{6} = \frac{1}{7}$$

Exercise 2.3

1. What is the conditional probability that a family with two children has two boys, given they have at least one boy.

2. Find $p(A\backslash B)$ if a) $A \cap B = \emptyset$, b) $A \subset B$ and c) $B \subset A$

3. Two manufacturing plants produce similar parts. Plant 1 produces 1000 parts, 100 of which are defective. Plant 2 produce 2000 parts, 150 of which are defective. A part is selected at random and found be defective. What is the probability that it came from plant 1?

4. Two cards are drawn at random from a deck. Find the probability that both are aces.

2.4 Independent events

Objective: After study this section the student should be able to:

- ✓ Define the independent event
- ✓ Differentiate the dependant and independent events
- ✓ Identify whether the given events are dependant or not
- ✓ Calculate the probability of dependant events.

Definition 2. 6: Two events are dependent if the occurrence of one event affects the probability of the other event occurring; otherwise, they are independent which means , events A and B in a probability space S are said to be independent if the occurrence of one of them does not influence the occurrence of the other. More specifically, B is independent of A if $P(B)$ is the same as $P(B|A)$.

Now substituting $P(B)$ for $P(B|A)$ in the Multiplication Theorem,

$P(A \cap B) = P(A)P(B|A)$ yields $P(A \cap B) = P(A)P(B)$.

We formally use the above equation as our definition of independence.

Definition 2.7: The event A and B are independent if and only if

$$p(A \cap B) = P(A)P(B)$$

Example 2.19: Suppose A is the event that a randomly generated bit string of length four begins with 1 and B is the event that this bit string contains an even number of ones. Are A and B independent, if the 16 bit strings of length four are equally likely?

Solution: There are eight bit strings of length four that begin with a one:

$$1000, 1001, 1010, 1011, 1101, 1100, 1110 \text{ and } 1111$$

There are also eight bit strings of length four that contain an even number of ones:

$$0000, 0011, 0101, 0110, 1001, 1010, 1100 \text{ and } 1111.$$

Since there are 16 bit strings of length four, it follows that $p(A) = p(B) = \frac{8}{16} = \frac{1}{2}$.

Because $A \cap B = \{1111, 1100, 1010, 1001\}$, we see that $p(A \cap B) = \frac{4}{16} = \frac{1}{4}$. Since $p(A \cap B) = \frac{1}{4} = \frac{1}{2} \times \frac{1}{2} = p(A)p(B)$. We conclude that A and B are independent.

Example 2.20: A fire coin is tossed three times. Consider the events:

$$\mathrm{A} = \{first\ toss\ is\ heads\}$$

$$\mathrm{B} = \{second\ toss\ is\ heads\}$$

$$\mathrm{C} = \{exactly\ two\ heads\ in\ row\}$$

Show that A and C are independent.

Solution: The sample space is

$\mathrm{S} = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$ and the events are:

$\mathrm{A} = \{HHH, HHT, HTH, HTT\}, \mathrm{B} = \{HHH, HHT, THH, THT\} and\ \mathrm{C} = \{HHT, THH\}$

And $\mathrm{A} \cap \mathrm{B} = \{HHH, HHT\}, \mathrm{A} \cap \mathrm{C} = \{HHT\}$ *and* $\mathrm{B} \cap \mathrm{C} = \{HHT, THH\}$

We have:

$$p(A) = \frac{4}{8} = \frac{1}{2}, p(B) = \frac{4}{8} = \frac{1}{2} \quad , p(C) = \frac{2}{8} = \frac{1}{4}$$

$$p(A \cap B) = \frac{1}{4} \ , p(A \cap C) = \frac{1}{8} \ , p(B \cap C) = \frac{1}{4}$$

Accordingly,

$p(A)p(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = p(A \cap B)$, and so A and B are independent

$p(A)p(C) = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8} = p(A \cap C)$, and so A and C are independent

$p(C)p(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \neq p(C \cap B)$, and so C and B are dependent

Example 2.21: Are the event A, that a family with three children has children of both sexes, and B that a family with three children has at most one boy, independent? Assume that the eight ways a family can have three children are equally likely.

Solution: By assumption, each of the eight ways a family can have three children. $BBB, BBG, BGB, BGG, GBB, GBG, GGB, and\ GGG$, Where G represents girl and B represents boy, has a probability $\frac{1}{8}$. Since

$A = \{BBG, BGB, BGG, GBG, GGB, GBB\}, B = \{BGG, GBG, GGB, GGG\}$ and

$A \cap B = \{BGG, GBG, GGB\}$

It follows that $p(A) = \frac{6}{8} = \frac{3}{4}, p(B) = \frac{4}{8} = \frac{1}{2}$ and $p(A \cap B) = \frac{3}{8}$. Since

$p(A \cap B) = \frac{3}{8} = \frac{3}{4} \times \frac{1}{2} = p(A)p(B)$. We conclude that A and B are independent.

Exercise 2.4

1. Consider the experiment of throwing two fair dice. Let A be the event that the sum of the dice is 7, let B be the event that the sum of the dice is 6, and C be the event that the first die is 4. Show that events A and C are independent, but event B and C are not independent.

2. Suppose we draw a card from a standard deck of 52 cards, replace it, draw another card, and continue for a total of ten draws. Is this an independent trials process?

3. Suppose we draw a card from a standard deck of 52 cards, discard it (i.e. we do not replace it), draw another card and continue for a total of ten draws. Is this an independent trials process?

4. In three flips of a coin, is the event that we have at most one tail independent of the event that not all flips are identical?

2.5 Random variable and expectation

Objective: After study this section the student should be able to:

- ✓ Define a random variable
- ✓ Calculate the expected value

A **random variable** for an experiment with a sample space S is a function that assigns a number to each element of S. Typically instead of using f to stand for such a function we use X (at first, a random variable was conceived of as a variable related to an experiment, explaining the use of X, but it is very helpful in understanding the mathematics to realize it actually is a function on the sample space).

For example, if we consider the process of tossing a coin n times, we have the set of all sequences of n Hs and Ts as our sample space. The "number of heads" random variable takes a sequence and tells us how many heads are in that sequence. Somebody might say "Let X be the number of heads in tossing a coin 5 times." In that case $X(HTHHT) = 3$ while $\mathrm{X(THTHT)} = 2$. It may be rather jarring to see X used to stand for a function, but it is the notation most people use.

Definition 2.7

Consider a random experiment with sample space S. A random variable $X(\tau)$ is a single-valued real function that assigns a real number called the value of $X(\tau)$ to each sample point τ of S.

Note: A random variable is not a variable at all in the usual sense, and it is a function. The sample space S is termed the domain of the random variable $X(\tau)$, and the collection of all numbers (values of $X(\tau)$) is termed the range of the random variable. Thus the range of $X(\tau)$ is a certain subset of the set of all real numbers (figure 2.4)

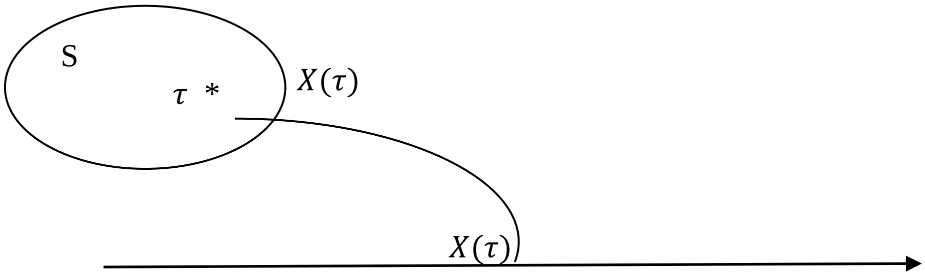


Figure 2.4 Random variable $X(\tau)$ as a function.

Example 2.22 : In the experiment of tossing a coin once, we might define the random variable $X(\tau)$ as (figure 2.5)

$X(H) = 1 \;\; X(T) = 0$

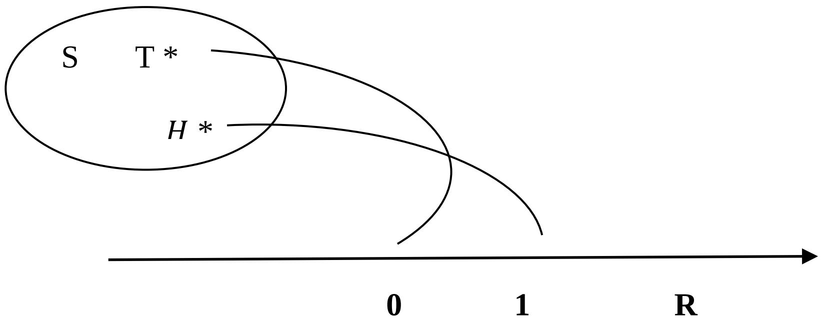


Fig.2.5 One random variable associated with coin tossed

Note that we could also define another random variable, say Y or Z, with

$$Y(H) = 0, Y(T) = 1 \quad or \; Z(H) = 0, Z(T) = 1$$

These events have probabilities that are denoted by

$$p(X = x) = p(\tau: X(\tau) = x)$$

$$p(X \leq x) = p(\tau: X(\tau) \leq x)$$

$$p(X > x) = p(\tau: X(\tau) > x)$$

$$p(x_1 < X \leq x_2) = p(\tau: x_1 < X \leq x_2)$$

EXAMPLE 2.23 In the experiment of tossing a fair coin three times, the sample space S_1 consists of eight equally likely sample points $S_1 = (HHH, \dots, TTT)$. If X is the random variable giving the number of heads obtained, find

(a) $P(X = 2)$;

(b) $P(X < 2)$

Solution:

(a) Let $A \subset S_1$ be the event defined by $X = 2$. Then, we have

$$\mathrm{A = (X = 2) = \{\tau: X(\tau) = 2) = \{HHT, HTH, THH)}$$

Since the sample points are equally likely, we have

$$P(X = 2) = P(A) = \frac{3}{8}$$

(b) Let $B \subset S_1$ be the event defined by $X < 2$. Then

$$B = (X < 2) = \{\tau: X((\tau) < 2) = (HTT, THT, TTH, TTT)$$

and $P(X < 2) = P(B) = \frac{4}{8} = \frac{1}{2}$

Probability Distribution of a Random Variable

Let X be a random variable on a finite sample space S with range space $R_x = \{x_1, x_2, \ldots, x_t\}$ Then X induces a function f which assigns probabilities p_k to the points x_k in R_x as follows:

$$\begin{aligned} f\,(\mathrm{x_k}) &= \mathrm{p_k} = \mathrm{P(X = x_k} \\ &= \text{sum of probabilities of points in S whose image is } \mathrm{x_k}. \end{aligned}$$

The set of ordered pairs $(x_1, f\,(x_1)), (x_2, f\,(x_2)), \ldots, (x_t, f\,(x_t))$ is called the distribution of the random variable X; it is usually given by a table as in Fig. 2.6. This function f has the following two properties:

(i) $f\,(x_k) \geq 0$ *and* $(ii) \sum_k f(x_k) = 1$

Thus R_X with the above assignments of probabilities is a probability space. (Sometimes we will use the pair notation $[x_k, p_k]$ to denote the distribution of X instead of the functional notation [x, f (x)]).

Out come x	x_1	x_2	x_3	· · ·	x_t
Probability $f(x)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	· · ·	$f(x_t)$

Fig. 2.6 Distribution f of a random variable X

In the case that S is an a equiprobable (uniform distribution) space, we can easily obtain the distribution of a random variable from the following result.

Theorem 2.4: Let S be an equiprobable space, and let f be the distribution of a random variable X on S with the range space $R_x = \{x_1, x_2, \ldots, x_t\}$.

Then $p_I = f(x_I) = \frac{number\ of\ points\ in\ S\ whose\ image\ is\ x_i}{number\ of\ points\ in\ S}$

EXAMPLE 2.24: A pair of dice is tossed. Let X be assign to each point in S the sum of the numbers: X is a random variable with range space

$$R_x = \{2,3,4, \ldots 12\}$$

Note $n(S) = 36$, and $R_x = \{2, 3, \ldots, 12\}$. Using Theorem 2.4, we obtain the distribution f of X as follows:

$f(2) = 1/36$, since there is one outcome, $(1, 1)$ whose sum is 2.

$f(3) = 2/36$, Since there are two outcomes, $(1, 2)$ and $(2,1)$, whose sum is 3.

$f(4) = 3/36$, since there are three outcomes, $(1, 3), (2, 2)$ and $(3, 1)$,whose sum is 4.

Similarly,$f(5) = 4/36, f(6) = 5/36, \ldots, f(12) = 1/36$.Thus the distribution of X as follows.

x	2	3	4	5	6	7	8	9	10	11	12
$f(x)$	$^1/_{36}$	$^2/_{36}$	$^3/_{36}$	$^4/_{36}$	$^5/_{36}$	$^6/_{36}$	$^5/_{36}$	$^4/_{36}$	$^3/_{36}$	$^2/_{36}$	$^1/_{36}$

Expected value:

The expected value of a random variable is the sum over all elements in a sample space of the product of the probability of the element and the value of the random variable at this element.

Definition 2.8 The expected value (or expectation) of the random variable $X(s)$ on the sample space Sis equal to

$$E(X) = \sum_{s\epsilon S} p(s)X(s).$$

Note that when the sample space S has n elements

$S = \{x_1, x_2, \ldots, x_n\},\ E(X) = \sum_{i=1}^{n} p(x_i)X(x_i)$

Remark: We are concerned only with random variables with finite expected values here.

Example 2.25 A fair coin is tossed six times. Let S be the sample space of the 64 possible outcomes, and let X be the random variable that assigns to an outcome the number of heads in this outcome. What is the expected value of X?

Solution: The number of heads which can occur with their respective probabilities follows:

x_i	0	1	2	3	4	5	6
p_i	$1/64$	$6/64$	$15/64$	$20/64$	$15/64$	$6/64$	$1/64$

Then the mean or expectation (or expected number of heads) is:

$$E(X) = 0\left(\frac{1}{64}\right) + 1\left(\frac{6}{64}\right) + 2\left(\frac{15}{64}\right) + 3\left(\frac{20}{64}\right) + 4\left(\frac{15}{64}\right) + 5\left(\frac{6}{64}\right) + 6\left(\frac{1}{64}\right) = 3$$

Example 2.26 Three horses a, b and c are in a race, suppose their respective probabilities of winning are $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{6}$. Let X denote the pay of a function for the winning horse, and suppose X pay $\$2, \$6$ or $\$9$ according as a, b or c wins the race. What is the expected value of ?

Solution: Expected value of X is

$$E(X) = X(a)p(a) + X(b)p(b) + X(c)P(c)$$

$$= 2\left(\frac{1}{2}\right) + 6\left(\frac{1}{3}\right) + 9\left(\frac{1}{6}\right) = 4.5$$

Theorem 2.5: If X is a random variable and $p(X = r)$ is the probability that $X = r$, so that $p(X = r) = \sum_{s\in S, X(s)=r} p(s)$, then $E(X) = \sum_{r\in X(S)} p(X = r)$

Proof: suppose that X is a random variable with range $X(S)$, and let $p(X = r)$ be the probability that the random variable X takes the value r. consequently,

$p(X = r)$ is the sum of the probabilities of the outcomes s such that $X(s) = r$. it follows that

$$E(X) = \sum_{r\in X((S)} p(X = r)r$$

Example 2.27 What is the expected value of the sum of the numbers that appear when a pair of a fair dice?

Solution: Let X be the random variable equal to the sum of the numbers that appear when a pair of dice is rolled. In example 2.25 we listed the value of X for the 36 possible outcomes. The range of X is {2,3,4,5,6,7,8,9,10,11,12}

Substitute the value in example 2.25 in the formula, we have

$E(X) = 2p(X = 2) + 3p(X = 3) + 4p(X = 4) + 5p(X = 5) + 6p(X = 6)7p(X = 7) + 8p(X = 8) + 9p(X = 9) + 10p(X = 10) + 11p(X = 11) + 12p(X = 12)$

$= 2\left(\frac{1}{36}\right) + 3\left(\frac{1}{18}\right) + 4\left(\frac{1}{12}\right) + 5\left(\frac{1}{9}\right) + 6\left(\frac{5}{36}\right) + 7\left(\frac{1}{6}\right) + 8\left(\frac{5}{36}\right) + 9\left(\frac{1}{9}\right) + 10\left(\frac{1}{12}\right) + 11\left(\frac{1}{18}\right) + 12\left(\frac{1}{36}\right) = 7$

Theorem 2.5: The expected number of successes when n Bernoulli trials are performed, where $p$the probability of success on each trial is np.

Exercise 2.5

1. *A random sample with replacement of size $n = 2$ is drawn from the set $\{1,2,3\}$, yielding the following 9-element equiprobable sample space*

$S = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

a) Let X denote the sum of the two numbers. Find the distribution f of X, and find the expected value $E(X)$.

b) Let Y denote the minimum of the two numbers. Find the distribution g of Y ,and find the expected value $E(Y)$

2. *A coin is weighted so that $p(H) = 3/4$, and $P(T) = 1/4$. The coin is tossed three times. Let X denote the number of heads that appear*

a) Find the distribution of f of X

b) Find the expected $E(X)$

3. *A player tossed three fair coins. He wins \$5 if three heads occur, \$3 if two heads occur, and \$1 if only one head occurs. On the other hand, he losses \$15 if three tails occur. Find the value of the game to the player.*

4. *What is the expected sum of the tops of n dice when we roll them?*

Chapter summery:

- An experiment is a procedure that yields one of a given set of possible outcome.
- A sample space is the set of all possible outcomes of a given experiment,denoted by *s*.
- An event E is a set of outcome.
- Two events A and B are called mutually exclusive if they are disjoint, i.e. $A \cap B = \emptyset$.
- Let E be an event of a finite sample space S consisting of equally likely outcomes. Then $p(E) = \frac{p(E)}{P(S)}$.
- Suppose that S is a set with n elements. The uniform distribution assigns the probability $\frac{1}{n}$ to each elements of S.
- If A and B are any two events, then $p(A \cup B) = P(A) + P(B) - P(A \cap B)$ (Inclusion exclusion principle)
- If A and B are mutually exclusive events, then $p(A \cup B) = P(A) + P(B)$ (addition principle).

- The conditional probability of A given B, denoted by $p(A\backslash B)$, defined as $p(A\backslash B) = \frac{p(A\cap B)}{p(B)}$.
- The conditional probability of B given A, denoted by $p(A\backslash B)$,defined as $p(B\backslash A) = \frac{p(A\cap B)}{p(A)}$.
- Two events are dependent if the occurrence of one event affects the probability of the event occurring.
- The event A and B are independent if and only if $p(A \cap B) = p(A)p(B)$
- A random variable for an experiment with a sample space S is a function that assigns a number to each element of S.
- Consider a random experiment with sample space S. A random variable $X(\tau)$ is a single-valued real function that assigns a real number called the value of $X(\tau)$ to each sample point τ of S.
- The expected value (or expectation) of the random variable $X(s)$ on the sample space Sis equal to $E(X) = \sum_{s\epsilon S} p(s)X(s)$.

Self Test Exercise 2

1. *A sample space S consists of four elements: that is $S = \{a_1, a_2, a_3, a_4\}$. under which of the following function does S become a probability space*

   a) $p(a_1) = \frac{1}{3}$, $p(a_2) = -\frac{1}{2}$, $p(a_3) = \frac{1}{4}$, $p(a_4) = \frac{1}{5}$

   b) $p(a_1) = \frac{1}{3}$, $p(a_2) = -\frac{1}{3}$, $p(a_3) = \frac{1}{2}$, $p(a_4) = \frac{1}{2}$

   c) $p(a_1) = \frac{1}{2}$, $p(a_2) = \frac{1}{4}$, $p(a_3) = \frac{1}{8}$, $p(a_4) = 0$

   d) $p(a_1) = \frac{3}{6}$, $p(a_2) = \frac{3}{12}$, $p(a_3) = \frac{3}{24}$, $p(a_4) = \frac{3}{24}$

2. *A coin is weighted so that head is three times as likely to appear as tails. Find $p(H)$ and $p(T)$*

3. *Suppose A and B are events with $p(A) = 3/5$, $P(B) = 3/10$ and $p(A\ B) = 1/5$. Find the probability that*

   a) *A does not occur*

   b) *B does not occur*

   c) *A or B occur*

   d) $(A \cap B)^c$

4. *A fair die is tossed. Consider events* $A = \{2,4,6\}$ $B = \{1,2\}$ $C = \{1,2,3,4\}$. *Find*

   a) $P(A \text{ and } B)$ *and* $p(A \text{ or } C)$

   b) $P(A \backslash B)$ *and* $p(B \backslash A)$

   c) $P(A \backslash C)$ *and* $p(C \backslash A)$

   d) $P(B \backslash C)$ *and* $p(C \backslash A)$

5. *A pair of fair dice is tossed. If the numbers appearing are different. Find the probability that*

   a) *The sum is even*

   b) *The sum exceeds nine*

6. *Two dice are rolled. Find the probability of obtaining each event*

   a) A sum of 11, knowing that a six has occurred on one dies.

   b) A sum of 11, knowing that one die shows an odd number.

7. *The Sealords have three children. Assuming that the outcomes are equally likely and independent, find the probability that they have three boys, knowing that:*

   a) The first child is a boy

   b) The first two children are boys.

   c) at least one child is a boy

   d) the second child is a boy

Reference:

1. A. W.F.Edwards,Pascal's Arithmeticical Triangle. The Johns Hopkins University Press, Baltimore,MD, 2002.
2. C.Oliver, " The Twelve Days of Christmas," Mathematics Teacher, Vol. 70
3. David A.Santos, Discrete Mathematics Note
4. Discrete Mathematics, Handbook of discrete and Compinatorial Mathematics
5. Hewel Hus, Schaum's outline, probability Random Variable
6. J.M.Harris.Jeffry L.Hirst.Michael J.Mossinghoff, Combinatorics and Graph Theory, 2nd edition
7. J.Varnadore, " Pascal's Triangle and Fibonacci Numbers," Mathematics Teacher, Vol,81.
8. Ken Bogart and Cliffstein, Discrete math in computer science
9. Kenneth H.Rosen, Discrete Mathematics and its application, 3rd edition.
10. M.Eng and J.Casey, "Pascal's Arithmetical Triangle-ASerendipitious Source for programming Activities,'' Mathematics Teacher, Vol 76
11. Mott, Knadel and Baker, Discreate Mathematics
12. P.Z.Chinn, " inductive patterns, finite Difference, and a Missing Region,'' Mathematics Teacher, Vol.81(Sept.1988)
13. R.P.GRIMALDI and B.V.RAMANA pearson, Discreate combinatorial mathematics
14. Rose, Discrete Mathematics
15. Seymour Lipschutz, Schaum's outline, Discrete Mathematics, 3rd edition

CHAPTER 3
RECURRENCE RELATION

Objectives:- On completion of the chapter the students will be able to:

- Define recurrence relation.
- Solve linear homogeneous recurrence relation with constant coefficient.
- Solve linear nonhomogeneous recurrence relation with constant coefficient.

3.1 Introduction

Many counting problems cannot be solved easily using the methods discussed in chapter 1. One such problem is: how many bit strings of length $n+1$ do not contain two consecutive zeros? To solve this problem let a_n be the number of such strings of length $n+1$. An argument can be given that shows $a_{n+2} = a_{n+1} + a_n$. This equation, called recurrence relation, and the initial conditions $a_0 = 2$ and $a_1 = 3$ determine the sequence $\{a_n\}_{n=0}^{\infty}$. Moreover the explicate formula can be found for a_n from the equation relating the terms of the sequence. As we will see, a similar technique can be found to solve many different types of counting problems.

Definition 3.1: A recurrence relation for a sequence $\{a_n\}_{n=0}^{\infty}$ is an equation that expresses a_n in terms of one or more preceding terms $a_0, a_1, \dots, a_{n-2}, a_{n-1}$. Moreover, the sequence is called the solution to the recurrence relation if it satisfies the recurrence relation.

The following are examples of recurrence relations :

$$a_n = 2a_{n-1} + 1, \quad n \geq 2 \tag{1}$$

$$a_n = 5a_{n-1} - 6a_{n-2}, \quad n \geq 3 \tag{2}$$

For (1) we would need one initial value to find a particular or single a_n. For example, if $a_0 = 1$ then $a_1 = 3$ and $a_2 = 7$.

For (2) we would need two initial values to find a particular or single a_n. For example, if $a_0 = 1$ and $a_1 = 5$ then $a_2 = 19$ and $a_3 = 65$.

Example 3.1 Verify that the solution of the recurrence relation $a_n = 2a_{n-1} + 1, n \geq 2$ with $a_0 = 0$ is $a_n = 2^n - 1$

Solution: We have to do two things

(a) Check that the given formula gives the correct initial value

(b) Check that the given formula solves the recurrence relation.

Putting $n = 0$ in $a_n = 2^n - 1$ gives $a_0 = 1 - 1$ as required. To do (b) we evaluate $2a_{n-1} + 1$ using the given formula and show that it is equal to a_n.

Now $a_{n-1} = 2^{n-1} - 1$ so

$$2a_{n-1} + 1 = 2\left(2^{n-1} - 1\right) + 1 = 2^n - 1$$

Recurrence relations have many applications. Suppose that you put £100 into a savings account yielding 4% compounded annually. Let a_n be the amount (in pounds) in the account after n years. Then a_n is equal to the amount in the account after $n - 1$ years plus the interest for the nth year. For example, a_1 is equal to 100 plus the interest which is 4. Hence $a_1 = 104$.

In general, $a_n = a_{n-1} + (0.04)a_{n-1}$ so that

$$a_n = (1.04)a_{n-1}, \quad n \geq 1$$

with $a_0 = 100$.

Solving this we obtain

$$a_1 = 100(1.04)$$

$$a_2 = (1.04)a_1 = 100(1.04)^2$$

$$a_3 = (1.04)a_2 = 100(1.04)^3$$

and in general

$$a_n = 100(1.04)^n.$$

Exercises 3.1

1. Let $a_n = 2a_{n-1} + a_{n-2}$ with $a_0 = 1$ and $a_1 = 1$. Find a_2, a_3, a_4 and a_5 .
2. Verify that the solution of the recurrence relation $a_n = 3a_{n-1}$ with $a_0 = 4$ is $a_n = 4(3)^n$.
3. Verify that the solution of $a_n = 5a_{n-1} - 12$ with $a_0 = 13$ is $a_n = 10(5)^n + 3$.

3.2 Linear recurrence relation with constant coefficient

Objectives:- On completion of this section the students will be able to:

- Define a linear recurrence relation with constant coefficient of degree k.

A wide variety of recurrence relations occur in models. Some of those recurrence relations can be solved using iteration or some other technique. However, one important class of recurrence relation can be explicitly solved in a systematic way. These are recurrence relations that express the terms of the sequence as linear combination of previous terms.

Definition 3.2: A linear recurrence relation with constant coefficient of degree (order) k is a recurrence relation of the form

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = f(n) \qquad (1)$$

where $c_1, c_2, \ldots, c_k$ are constants and $c_k \neq 0$.

If $f(n)$ is identically zero ($f(n) = 0$) in the recurrence relation (1) defined above, then the recurrence relation (1) is called homogeneous, otherwise it is called nonhomogeneous.

Exercise 3.2:

a. $a_n + 6a_{n-1} = 0$ is linear homogeneous recurrence relation with constant coefficient of degree (order) 1.
b. $a_n - 5a_{n-1} + 6a_{n-2} = 0$ is linear homogeneous recurrence relation with constant coefficient of degree (order) 2.
c. $a_n + 4a_{n-1} + 4a_{n-2} = -5n^2 + n$ is linear non-homogeneous recurrence relation with constant coefficient of degree (order) 2.
d. $a_n - 5a_{n-1} + 6a_{n-2} - a_{n-3} = 4^n$ is linear non-homogeneous recurrence relation with constant coefficient of degree (order) 3.

3.3 Solution of linear recurrence relation with constant coefficient

3.3.1 Solving linear homogeneous recurrence relation with constant coefficient

Objectives:- On completion of this section the students will be able to:

- Solve linear homogeneous recurrence relation with constant coefficient.

The basic approach for solving linear homogeneous recurrence relations is to look for the solutions of the form $a_n = r^n$, where r is constant. Note that $a_n = r^n$ is the solution of the recurrence relation

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = 0 \qquad (2)$$

if and only if

$$r^n + c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k} = 0.$$

When both side of the equation is divided by r^{n-k} we obtain the equation

$$r^k + c_1 r^{k-1} + c_2 r^{k-2} + \cdots + c_{k-1} r + c_k = 0 \quad . \tag{3}$$

Consequently the sequence $\{a_n\}_{n=0}^{\infty}$ with $a_n = r^n$ is the solution if and only if r is the solution of the last equation (3), which is called the **characteristic equation** of the recurrence relation (2). The solutions of the characteristic equation (3) are called **characteristic roots** of the recurrence relation (2). As we will see, these characteristic roots can be used to give an explicit formula for all the solutions of the recurrence relation (2).

Let us first see the rule to find all the possible solutions (general solution) homogeneous recurrence relation with constant coefficients of degree 1.

All the possible solutions or general solution to the linear homogenous recurrence relation with constant coefficient of degree 1

$$a_n + c a_{n-1} = 0$$

is $a_n = p(-c)^n$, where p is a constant.

Example 3.2. Find the solution of the recurrence relation

$$a_n - 5a_{n-1} = 0$$

with initial condition $a_0 = 7$

Solution. The general solution is $a_n = p(5^n)$.

Since $a_0 = 7$,

$$\Longrightarrow \; a_0 = p(5^0) = 7$$

Thus,

$p = 7$ and hence the solution is $a_n = 7(5^n)$.

Now we will develop rules that deal with linear homogeneous recurrence relation with constant coefficients of degree 2. Then corresponding general rules when the degree may be greater that two will be stated. Because the proofs needed to establish the general rules in general case are more complicated. We now turn our attention to linear recurrence relations of degree two.

Theorem 3.1: Consider a linear homogeneous recurrence relation with constant coefficient of degree 2

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} = 0 \tag{4}$$

for $n \geq 2$, where c_1 and c_2 are constants, and consider its characteristic equation

$$r^2 + c_1 r + c_2 = 0\,. \tag{5}$$

i. If the characteristic equation (5) has two distinct roots r_1 and r_2, then the sequence $\{a_n\}_{n=0}^{\infty}$ is the solution of the recurrence relation (4) if and only if

$$a_n = d_1 {r_1}^n + d_2 {r_2}^n \tag{6}$$

where d_1 and d_1 are constants.

ii. If the characteristic equation (5) has only one root r_0, then the sequence $\{a_n\}_{n=0}^{\infty}$ is the solution of the recurrence relation (4) if and only if

$$a_n = d_1 {r_0}^n + d_2 n\, {r_0}^n \tag{7}$$

where d_1 and d_1 are constants.

Proof:

i. We must do two things to prove the theorem. First it must be shown that if r_1 and r_2 are roots of the characteristic equation, and d_1 and d_1 are constants, then the sequence $\{a_n\}_{n=0}^{\infty}$ with $a_n = d_1 {r_1}^n + d_2 {r_2}^n$ is the solution of the recurrence relation. Second it must be shown that if the sequence $\{a_n\}_{n=0}^{\infty}$ is the solution, then $a_n = d_1 {r_1}^n + d_2 {r_2}^n$ for some constants d_1 and d_1.

Now we will show that if $a_n = d_1 {r_1}^n + d_2 {r_2}^n$, then the sequence $\{a_n\}_{n=0}^{\infty}$ is the solution of the recurrence relation. Since r_1 and r_2 are roots of $r^2 + c_1 r + c_2 = 0$, it follows that $r_1^2 = -c_1 r_1 - c_2$ and $r_2^2 = -c_1 r_2 - c_2$.

From these equations , we see that

$$\begin{aligned}
-c_1 a_{n-1} - c_2 a_{n-2} &= -c_1\big(d_1(r_1^{n-1}) + d_2(r_2^{n-1})\big) - c_2\big(d_1(r_1^{n-2}) + d_2(r_2^{n-2})\big) \\
&= -c_1 d_1(r_1^{n-1}) - c_1 d_2(r_2^{n-1}) - c_2 d_1(r_1^{n-2}) - c_2 d_2(r_2^{n-2}) \\
&= d_1(r_1^{n-2})[-c_1 r_1 - c_2] + d_2(r_1^{n-2})[-c_1 r_2 - c_2] \\
&= d_1(r_1^{n-2}) r_1^2 + d_2(r_1^{n-2}) r_2^2 \\
&= d_1 r_1^n + d_2 r_2^n \\
&= a_n
\end{aligned}$$

This shows that the sequence $\{a_n\}$ with $a_n = d_1 r_1^n + d_2 r_2^n$ is the solution of the recurrence relation. To show every solution $\{a_n\}$ of the recurrence relation $a_n + c_1 a_{n-1} + c_2 a_{n-2} = 0$ has $a_n = d_1 r_1^n + d_2 r_2^n$ for $n = 0,1,2,\ldots$ and d_1 and d_2 are constants, suppose that $\{a_n\}$ is the solution of the recurrence relation, and the initial conditions $a_0 = C_0$ and $a_1 = C_1$ hold. It will be shown that there are constants d_1 and d_2 so that the sequence $\{a_n\}$ with $a_n = d_1 r_1^n + d_2 r_2^n$ satisfies these the same initial conditions. This requires that

$$a_0 = C_0 = d_1 + d_2$$
$$a_1 = C_1 = d_1 r_1 + d_2 r_1$$

We can solve these two equations for d_1 and d_2. From the first equation it follows that $d_2 = C_0 - d_1$. Inserting this equation in to the second equation gives $C_1 = d_1 r_1 + (C_0 - d_1) r_2$.

Hence,

$$C_1 = d_1(r_1 - r_2) + C_0 r_2$$

This shows that

$$d_1 = \frac{(C_1 - C_0 r_2)}{r_1 - r_2}$$

and

$$d_2 = C_0 - d_1 = d_2 = C_0 - \frac{(C_1 - C_0 r_2)}{r_1 - r_2} = \frac{C_0 r_1 - C_1}{r_1 - r_2}$$

where these expressions for d_1 and d_2 depend on the fact that $r_1 \neq r_2$. (When $r_1 = r_2$, this theorem is not true.)

Hence, with those values for d_1 and d_2 , the sequence $\{a_n\}$ with $d_1 r_1^n + d_2 r_2^n$ satisfies the two initial conditions. Since this recurrence relation and these initial conditions uniquely determine the sequence, it follows that $a_n = d_1 r_1^n + d_2 r_2^n$.■

ii. Exercise.

Note that: The solution (6) or (7) are all possible solutions of the recurrence relation (4). And we call the solution general solution of the recurrence relation (4).

Example 3.3. Find the general solution of

$$a_n - a_{n-1} - 2a_{n-2} = 0 \text{ , for } n \geq 2$$

Solution: The characteristic equation of the given recurrence relation is $r^2 - r - 2 = 0$. Then, find the roots of the characteristic equation using quadratic formula (factorization).

$$r^2 - r - 2 = 0$$
$$\Rightarrow \quad (r+1)(r-2) = 0$$

So $r = -1$ and $r = 2$. Thus, the characteristic equation has two distinct roots $r = -1$ and $r = 2$

Hence, the general solution is

$$a_n = d_1(-1)^n + d_2(2)^n$$

where d_1 and d_2 are constants.

Example 3.4. Find the solution of

$$a_n = -7a_{n-1} - 12a_{n-2} \text{ , for } n \geq 2$$

with initial condition $a_1 = 0$ and $a_1 = 5$.

Solution: $a_n = -7a_{n-1} - 12a_{n-2}$

$\Rightarrow \qquad a_n + 7a_{n-1} + 12a_{n-2} = 0$

Thus, the characteristic equation is

$$r^2 + 7r + 12 = 0.$$

And finding the roots of the characteristic equation using quadratic formula or factorization, we have $r = 3$ and $= 4$.

Hence, the general solution is

$$a_n = d_1(3)^n + d_2(4)^n$$

where d_1 and d_2 are constants.

Putting $n = 0$ and $n = 1$ in $a_n = d_1(3)^n + d_2(4)^n$ and using the initial conditions gives, we have

$d_1 + d_2 = 0$ and

$3d_1 + 4d_2 = 5.$

Solving these gives $d_1 = -5$ and $d_2 = 5$.

Hence,

$$a_n = -5(3)^n + 5(4)^n$$

is the solution.

Example 3.5: Find the solution of

$$a_n - 3a_{n-1} + \frac{9}{4}a_{n-2} = 0 \quad \text{, for } n \geq 2$$

with $a_0 = 1$ and $a_1 = 10$.

Solution: The characteristic equation of the given recurrence relation is

$$r^2 - 3r + \frac{9}{4} = 0.$$

Finding the roots of the characteristic equation using quadratic formula

$$r = \frac{3 \pm \sqrt{3^2 - 4(1)\left(\frac{9}{4}\right)}}{2(1)} = \frac{3 \pm \sqrt{9-9}}{2} = \frac{3}{2}.$$

So the characteristic equation has double root $r = \frac{3}{2}$.

Hence, the general solution is

$$a_n = d_1\left(\frac{3}{2}\right)^n + d_2 n\left(\frac{3}{2}\right)^n$$

where d_1 and d_2 are constants.

Putting $n = 0$ and $n = 1$ in $a_n = d_1\left(\frac{3}{2}\right)^n + d_2 n\left(\frac{3}{2}\right)^n$ and using the initial conditions gives, we have

$d_1 = 1$ and

$\frac{3}{2}d_1 + \frac{3}{2}d_2 = 10.$

Solving these gives $d_1 = 1$ and $d_2 = \frac{17}{3}$.

Hence,

$$a_n = \left(\frac{3}{2}\right)^n + \frac{17}{3}n\left(\frac{3}{2}\right)^n$$

is the solution.

We will now state the general result about the solution of linear homogeneous recurrence relation with constant coefficients of degree k, where the degree $k \geq 2$ under the assumption that the characteristic equation has k distinct roots $r_1, r_2, \ldots, r_k$ or the characteristic equation has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicity $m_1, m_2, \ldots, m_t$ respectively, so that $m_i \geq 1$, for $i = 1,2,\ldots,t$ and $m_1 + m_2 + \cdots + m_t = k$.

Theorem 3.2: Consider the linear homogeneous recurrence relation with constant coefficient of degree k

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = 0 \tag{8}$$

and consider its characteristic equation

$$r^k + c_1 r^{k-1} + c_2 r^{k-2} + \cdots + c_{k-1} r + c_k = 0. \tag{9}$$

i. If the characteristic equation (9) has k distinct roots $r_1, r_2, \ldots, r_k$, then the sequence $\{a_n\}_{n=0}^{\infty}$ is the solution of the recurrence relation (8) if and only if

$$a_n = d_1 {r_1}^n + d_2 {r_2}^n + \cdots + d_k {r_k}^n \tag{10}$$

where $d_1, d_2, \ldots, d_k$ are constants.

ii. If the characteristic equation (9) has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicity $m_1, m_2, \ldots, m_t$ respectively, so that $m_i \geq 1$, for $i = 1,2,\ldots,t$ and $m_1 + m_2 + \cdots + m_t = k$, then the sequence $\{a_n\}_{n=0}^{\infty}$ is the solution of the recurrence relation (8) if and only if

$a_n = (\alpha_{10} + \alpha_{11}n + \cdots + \alpha_{1(m_1-1)}n^{m_1-1})r_1{}^n + (\alpha_{20} + \alpha_{21}n + \cdots + \alpha_{2(m_2-1)}n^{m_2-1})r_2{}^n + \cdots + (\alpha_{t0} + \alpha_{t1}n + \cdots + \alpha_{t(m_t-1)}n^{m_t-1})r_t{}^n$

(11) where α_{ij} are constants for $0 \le i \le t$ and $0 \le j \le m_i - 1$.

Note that: The solution (10) or (11) are all possible solutions of the recurrence relation (8). And we call the solution general solution of the recurrence relation (8).

Example 3.6. Find the general solution of the recurrence relation

$$a_n + \frac{3}{2}a_{n-1} - \frac{23}{2}a_{n-2} + 28a_{n-3} - 10a_{n-4} = 0 \ \text{, for } n \ge 4.$$

Solution: The characteristic equation the given recurrence relation is

$$r^4 + \frac{3}{2}r^3 - \frac{23}{2}r^2 + 28r - 10 = 0.$$

$$\Rightarrow \quad (r-2)^2\left(r - \frac{1}{2}\right)(r+5) = 0$$

So that $r = 2$, $r = \frac{1}{2}$ and $r = -5$ with the multiplicity of 2 is 2, the multiplicity of $\frac{1}{2}$ is 2 and the multiplicity of -5 is 1.

Hence the general solution of the recurrence relation

$$a_n + \frac{3}{2}a_{n-1} - \frac{23}{2}a_{n-2} + 28a_{n-3} - 10a_{n-4} = 0 \ \text{, for } n \ge 4$$

is

$$a_n = (d_1 + d_2 n)(2\,)^n + d_3\left(\frac{1}{2}\right)^n + d_3(-5)^n$$

where d_1, d_2 , d_3 and d_4 are constants.

Exercise 3.3.1

1. Solve the recurrence relations together with the initial conditions given.
   a. $a_n - 2a_{n-1} = 0$, for $n \ge 1$, $a_0 = 3$
   b. $a_n - 5a_{n-1} + 6a_{n-2} = 0$, for $n \ge 2$, $a_0 = 1$, $a_1 = 0$.
   c. $a_n + 8a_{n-1} + 16a_{n-2} = 0$, for $n \ge 2$, $a_0 = 1$, $a_1 = 1$.
2. Find the general solution (all solutions) of the recurrence relation
   a. $a_n = 6a_{n-1}$, for $n \ge 1$.
   b. $a_n + 10a_{n-1} + 25a_{n-2} = 0$, for $n \ge 2$
   c. $a_n - 2a_{n-1} - a_{n-2} + 2a_{n-3} = 0$, for $n \ge 3$

3.3.2. Solving linear nonhomogeneous recurrence relation with constant coefficient

Objectives:- On completion of this section the students will be able to:

- Solve linear nonhomogeneous recurrence relation with constant coefficient.

We have seen how to solve linear homogeneous recurrence relation with constant coefficients. Is there a relatively simple technique for solving a linear, but not homogeneous, recurrence relation with constant coefficients, such as $a_n = 3a_{n-1} + 2n$? We will see that the answer is yes for a certain families of such recurrence relations.

The recurrence relation $a_n = 3a_{n-1} + 2n$ is an example of **linear nonhomogeneous recurrence relations with constant coefficients**, that is recurrence relation of the form

$$a_n + c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k} = f(n)$$

where $c_1, c_2, \dots, c_k$ are constants and $c_k \neq 0$ and $f(n)$ is a function not identically zero depending only on n. The recurrence relation

$$a_n + c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k} = 0$$

is called the associated homogenous recurrence relation. It plays an important role in the solution of the nonhomogeneous recurrence relation.

Theorem 3.3: Consider the linear nonhomogeneous recurrence relations with constant coefficients of degree k

$$a_n + c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k} = f(n) \qquad (12)$$

and its associated homogenous recurrence relation

$$a_n + c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k} = 0 \qquad (13)$$

Then every solution of the recurrence relation (12) is of the form $\{a_n^p + a_n^g\}_{n=0}^{\infty}$, where a_n^p is the particular solution of the recurrence relation (12) and a_n^h is the solution of the recurrence relation (13).

Proof: Let a_n^p be the particular solution of the non homogeneous recurrence relation. Thus,

$$a_n^p + c_1a_{n-1}^p + \cdots + c_ka_{n-k}^p = f(n). \qquad (14)$$

Now suppose that a_n^s is any other solution of the nonhomogeneous recurrence relation. Thus,

$$a_n^s + c_1a_{n-1}^s + \cdots + c_ka_{n-k}^s = f(n) . \qquad (15)$$

Subtracting the equation (a) from equation (b)

$$[a_n^s + c_1a_{n-1}^s + \cdots + c_ka_{n-k}^s] - [a_n^p + c_1a_{n-1}^p + \cdots + c_ka_{n-k}^p] = 0 + f(n)$$

$$\Longrightarrow (a_n^s - a_n^p) + c_1(a_{n-1}^s - a_{n-1}^p) + \cdots + c_k(a_{n-k}^s - a_{n-k}^p) = f(n)$$

It follows that $a_n^s - a_n^p$ is the solution of the associated homogeneous recurrence relation.

Let $a_n^h = a_n^s - a_n^p$. Thus, $a_n^s = a_n^h + a_n^p$.

Therefore, every solution of the recurrence relation (12) is of the form $\{a_n^h + a_n^p\}_{n=0}^{\infty}$, where a_n^p is the particular solution of the recurrence relation (12) and a_n^h is the solution of the recurrence relation (13).

Problem: What is the general solution of the linear nonhomogeneous recurrence relation (12)?

- The general solution (all solutions) of the linear nonhomogeneous recurrence relation (12) is the sequence $\{a_n^p + a_n^g\}_{n=0}^{\infty}$, where a_n^p is the particular solution of the linear nonhomogeneous recurrence relation (12) and a_n^g is the general solution of the associated homogenous recurrence relation (13).

Problem: How can we find or choose the particular solution a_n^p for the linear nonhomogeneous recurrence relation (12)?

$f(n)$	Choice of a_n^p
$q_m n^m + q_{m-1}n^{n-1} + \cdots + q_1 n + q_0$	$p_m n^m + p_{m-1}n^{n-1} + \cdots + p_1 n + p_0$
$\alpha(b^n)$	$\beta(b^n)$
$(q_m n^m + q_{m-1}n^{n-1} + \cdots + q_1 n + q_0)b^n$	$(p_m n^m + p_{m-1}n^{n-1} + \cdots + p_1 n + p_0)b^n$

Rules:

- If $f(n)$ is one of the terms on the left side of the table choose the particular solution a_n^p from the right side of the table and determine the underdetermined coefficients by using the original equation in (12).
- If b is the root of the characteristic equation to the recurrence relation (13) with multiplicity m, then multiply a_n^p by n^m.

  That is if b is the root of the characteristic equation to the recurrence relation (13) with multiplicity m, choose the particular solution $a_n^p = \beta n^m(b^n)$ for the case $f(n) = \alpha(b^n)$ and choose the particular solution

$$a_n^p = n^m(p_m n^m + p_{m-1}n^{n-1} + \cdots + p_1 n + p_0)b^n$$

  for the case $f(n) = (q_m n^m + q_{m-1}n^{n-1} + \cdots + q_1 n + q_0)b^n$.

Example 3.7. Find a particular solution of the recurrence relation

$$a_n - 4a_{n-1} = -15 \text{ , for } n \geq 1$$

Solution. The given recurrence relation is of the form $a_n - 4a_{n-1} = f(n)$, where $f(n) = -15$. We choose particular solution $a_n^p = q$, where q is a constant.

Then, we have to find q by substituting a_n^p in to the given recurrence relation, that is

$$a_n^p - 4a_{n-1}^p = -15.$$

Observe that $a_n^p = q$ and $a_{n-1}^p = q$. Thus, we have $-4q = -15$.

$$\Rightarrow -3q = -15 \qquad \Rightarrow q = 5$$

Hence, $a_n^p = 5$ is the particular solution of the recurrence relation.

Example 3.8 . Find a particular solution of the recurrence relation

$$a_n = 6a_{n-1} + 7a_{n-2} + 7^n \text{ , for } n \geq 2$$

Solution.

The given recurrence relation is of the form $a_n - 6a_{n-1} - 7a_{n-2} = f(n)$, where $f(n) = 7^n$. The associated homogeneous recurrence relation to the given recurrence relation is

$$a_n - 6a_{n-1} - 7a_{n-2} = 0$$

and its characteristic equation is $r^2 - 6r - 7 = 0$.

Thus,

$$r = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(-7)}}{2(1)} = \frac{6 \pm \sqrt{64}}{2} = \frac{6 \pm 8}{2}$$

So $r = 7$ and $r = -1$.

$f(n) = 7^n$, and $b = 7$ is the root of the characteristic equation with multiplicity 1.

Thus we choose particular solution $a_n^p = qn(7^n)$, where q is a constant.

Then, we have to find q by substituting a_n^p in to the given recurrence relation, that is

$$a_n^p - 6a_{n-1}^p - 7a_{n-2}^p = 7^n.$$

Observe that $a_n^p = qn(7^n)$, $a_{n-1}^p = q(n-1)(7^{n-1})$ and $a_{n-2}^p = q(n-2)(7^{n-2})$.

Thus,

$$[qn(7^n)] - 6[q(n-1)(7^{n-1})] - 7[q(n-2)(7^{n-2})] = 7^n$$

$$\Rightarrow qn(7^n) - 6qn7^{n-1} + 6q7^{n-1} - 7qn7^{n-2} + 14q7^{n-2} = 7^n$$

$$\Rightarrow qn(7^n) - \frac{6}{7}qn(7^n) + \frac{6}{7}q(7^n) - \frac{1}{7}qn(7^n) + \frac{2}{7}q7^{n-2} = 7^n$$

$$\Rightarrow \frac{8}{7}q(7^n) = 7^n \quad \Leftrightarrow \frac{8}{7}q = 1 \;\Leftrightarrow q = \frac{7}{8}$$

Hence, $a_n^p = \frac{7}{8}n(7^n)$ is the particular solution of the recurrence relation.

Example 3.9 . Find the general solution of the recurrence relation

$$a_n = a_{n-1} + a_{n-2} + 5n - 6 \quad \text{, for } n \geq 2$$

Solution. $a_n = a_{n-1} + a_{n-2} + 5n - 6$

$\Longrightarrow \quad a_n - a_{n-1} - a_{n-2} = 5n - 6$

$\Longrightarrow \quad a_n - a_{n-1} - a_{n-2} = f(n)$, where $f(n) = 5n - 6$

The associated homogeneous recurrence relation to the given recurrence relation is

$$a_n - a_{n-1} - a_{n-2} = 0$$

and its characteristic equation is $r^2 - r - 1 = 0$.

Thus,

$$r = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$$

So $r = \frac{1+\sqrt{5}}{2}$ and $r = \frac{1-\sqrt{5}}{2}$.

Hence, the general solution to the associated homogeneous recurrence relation is

$$a_n^g = d_1\left(\frac{1+\sqrt{5}}{2}\right)^n + d_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

where d_1 and d_2 are constants.

$f(n) = 5n - 6$, so we choose the particular solution to the given nonhomogeneous recurrence relation to be $a_n^p = q_1 n + q_2$, where q_1 and q_2 are constants to be determined by substituting a_n^p in to the given recurrence relation.

Thus,

$$a_n^p - a_{n-1}^p - a_{n-2}^p = f(n).$$

And $a_n^p = q_1 n + q_2$, $a_{n-1}^p = q_1(n-1) + q_2$ and $a_{n-2}^p = q_1(n-2) + q_2$.

Thus,

$$a_n^p - a_{n-1}^p - a_{n-2}^p = f(n)$$

$\Longrightarrow \quad (q_1 n + q_2) - (q_1(n-1) + q_2) - (q_1(n-2) + q_2) = f(n)$

$\Longrightarrow \quad q_1 n + q_2 - q_1 n + q_1 - q_2 - q_1 n + 2q_1 - q_2 = 5n - 6$

$\Longrightarrow \quad -q_1 n + (q_2 + 3q_1) = 5n - 6$

$\Longleftrightarrow \quad -q_1 = 5$ and $q_2 + 3q_1 = -6$

Thus, $q_1 = -5$ and $q_2 = 9$.

Hence, the particular solution to the given nonhomogeneous recurrence relation is

$$a_n^p = -5n + 9.$$

Therefore, the general solution to the nonhomogeneous recurrence relation $a_n = a_{n-1} + a_{n-2} + 5n - 6$ is $a_n = a_n^g + a_n^p$.

That is the general solution to the nonhomogeneous recurrence relation

$$a_n = a_{n-1} + a_{n-2} + 5n - 6$$

is

$$a_n = d_1\left(\frac{1+\sqrt{5}}{2}\right)^n + d_2\left(\frac{1-\sqrt{5}}{2}\right)^n - 5n + 9$$

where d_1 and d_2 are constants.

Example 3.10. Find the solution of the recurrence relation

$$a_n - 6a_{n-1} + 9a_{n-2} = 2(3^n) \quad \text{, for } n \geq 2$$

with initial condition $a_0 = -1$ and $a_1 = 5$.

Solution. We are given a recurrence relation

$$a_n - 6a_{n-1} + 9a_{n-2} = f(n), \text{ where } f(n) = 2(3^n).$$

The associated homogeneous recurrence relation to the given recurrence relation is

$$a_n - 6a_{n-1} + 9a_{n-2} = 0$$

and its characteristic equation is $r^2 - 6r + 9 = 0$.

Thus, $\quad r^2 - 6r + 9 = 0$

$\Longrightarrow \; (r-3)^2 = 0$

So the characteristic equation has double root and is $r = 3$.

Hence, the general solution to the associated homogeneous recurrence relation is

$$a_n^g = d_1(3\,)^n + d_2 n(3)^n$$

where d_1 and d_2 are constants.

$f(n) = 2(3^n)$, and $b = 3$ is the root of the characteristic equation and the multiplicity of 3 is 2, so we choose the particular solution to the given nonhomogeneous recurrence relation to be $a_n^p = qn^2(3^n)$, where q is constant to be determined by substituting a_n^p in to the given recurrence relation.

That is $a_n^p - 6a_{n-1}^p + 9a_{n-2}^p = 2(3^n)$. But, $a_n^p = qn^2(3^n)$, $a_{n-1}^p = q(n-1)^2(3^{n-1})$ and $a_{n-2}^p = q(n-2)^2(3^{n-2})$.

Thus,

$a_n^p - 6a_{n-1}^p + 9a_{n-2}^p = 2(3^n)$

$\Longrightarrow [qn^2(3^n)] - 6[q(n-1)^2(3^{n-1})] + 9[q(n-2)^2(3^{n-2})] = 2(3^n)$

$\Longrightarrow qn^2(3^n) - 6[q(n^2 - 2n + 1)(3^{n-1})] + 9[q(n^2 - 4n + 4)(3^{n-2})] = 2(3^n)$

$\Longrightarrow qn^2(3^n) - 6qn^2(3^{n-1}) + 12qn(3^{n-1}) - 6q(3^{n-1}) + 9qn^2(3^{n-2}) - 36qn(3^{n-2}) + 36q(3^{n-2}) = 2(3^n)$

$\Longrightarrow qn^2(3^n) - 2qn^2(3^n) + 4qn(3^n) - 2q(3^n) + qn^2(3^n) - 4qn(3^n) + 4q(3^n) = 2(3^n)$

$\Longrightarrow 2q3^n = 2(3^n) \Leftrightarrow q = 1$

Hence, the particular solution to the given nonhomogeneous recurrence relation is

$$a_n^p = n^2(3^n).$$

Therefore, the general solution to the nonhomogeneous recurrence relation $a_n - 6a_{n-1} + 9a_{n-2} = 2(3^n)$ is $a_n = a_n^g + a_n^p$.

That is the general solution to the nonhomogeneous recurrence relation

$$a_n - 6a_{n-1} + 9a_{n-2} = 2(3^n)$$

is

$$a_n = d_1(3^n) + d_2n(3^n) + n^2(3^n)$$

where d_1 and d_2 are constants.

Then, putting $n = 0$ and $n = 1$ in $a_n = d_1(3^n) + d_2n(3^n) + n^2(3^n)$ and using the initial conditions gives, we have

$d_1 = -1$ and

$3d_1 + 3d_2 + 3 = 5.$

That is $d_1 = -1$ and $3d_1 + 3d_2 = 2.$

Solving these gives $d_1 = -1$ and $d_2 = \frac{5}{3}$.

Hence,

$$a_n = -(3^n) + \frac{5}{3}n(3^n) + n^2(3^n)$$

is the solution.

Example 3.11. Find the general solution of the recurrence relation

$$a_n + 9a_{n-1} + 20a_{n-2} = (n^2 + n - 1)5^n \text{ , for } n \geq 2$$

Solution. We are given a recurrence relation

$$a_n - 10a_{n-1} + 25a_{n-2} = f(n), \text{ where } f(n) = (n^2 + n - 1)5^n.$$

The associated homogeneous recurrence relation to the given recurrence relation is

$$a_n - 10a_{n-1} + 25a_{n-2} = 0$$

and its characteristic equation is $r^2 - 10r + 25 = 0.$

Thus, $r^2 - 10r + 25 = 0$

$\Longrightarrow (r - 5)^2 = 0$

So the characteristic equation has double roots and is $r = 5$.

Hence, the general solution to the associated homogeneous recurrence relation is

$$a_n^g = d_1(5)^n + d_2 n(5)^n$$

where d_1 and d_2 are constants.

$f(n) = (n^2 + n - 1)5^n$, and $b = 5$ is the root of the characteristic equation and the multiplicity of 5 is 2, so we choose the particular solution to the given nonhomogeneous recurrence relation to be $a_n^p = n^2(q_2 n^2 + q_1 n + q_0)5^n$, where q_2 , q_1 and q_0 are constants to be determined by substituting a_n^p in to the given recurrence relation.

That is $a_n^p - 10a_{n-1}^p + 25a_{n-2}^p = (n^2 + n - 1)5^n$.

But, $a_n^p = n^2(q_2 n^2 + q_1 n + q_0)5^n$, $a_{n-1}^p = (n-1)^2(q_2(n-1)^2 + q_1(n-1) + q_0)5^{n-1}$ and $a_{n-2}^p = (n-2)^2(q_2(n-2)^2 + q_1(n-2) + q_0)5^{n-2}$.

Thus,

$$a_n^p - 10a_{n-1}^p + 25a_{n-2}^p = (n^2 + n - 1)5^n$$

$$\Rightarrow [n^2(q_2 n^2 + q_1 n + q_0)5^n] - 10[(n-1)^2(q_2(n-1)^2 + q_1(n-1) + q_0)5^{n-1}] + 25[(n-2)^2(q_2(n-2)^2 + q_1(n-2) + q_0)5^{n-2}] = (n^2 + n - 1)5^n$$

The rest of the steps of the solution are left as an exercise.

Example 3.12. Find the general solution of the recurrence relation

$$a_n - a_{n-1} - 10a_{n-2} + 8a_{n-3} = f(n) \quad \text{, for } n \geq 3$$

where a. $f(n) = 6n$ b. $f(n) = (-1)^n(4)^n$

Solution. The associated homogeneous recurrence relation to the given recurrence relation is $a_n - a_{n-1} - 10a_{n-2} + 8a_{n-3} = 0$, for $n \geq 3$

and its characteristic equation is $r^3 - r^2 - 10r + 8 = 0$.

Thus

$$r^3 - r^2 - 10r + 8 = 0$$

$$\Rightarrow \quad (r-1)(r-2)(r+4) = 0$$

$$\Leftrightarrow r = 1 \ , \ r = 2 \ , \ r = -4$$

So the characteristic equation three distinct root $r = 1$, $r = 2$ and $r = -4$.

Hence, the general solution to the associated homogeneous recurrence relation is

$$a_n^g = d_1(1)^n + d_2(2)^n + d_3(-4)^n$$

where d_1 , d_2 and d_3 are constants.

a. $f(n) = 6n$, so we choose the particular solution to the given nonhomogeneous recurrence relation to be $a_n^p = q_1 n + q_0$, where q_1 and q_0 are constants to be determined by substituting a_n^p in to the given recurrence relation.

Thus,

$$a_n^p - a_{n-1}^p - 10a_{n-2}^p + 8a_{n-3}^p = f(n) \text{ , where } f(n) = 6n$$

$$\Rightarrow [q_1(n-1) + q_0] - [q_1 n + q_0] - 10[q_1(n-2) + q_0] + 8[q_1(n-3) + q_0] = 6n$$

$$\Rightarrow -2q_1 n - 3q_1 - q_0 = 6n$$

$$\Leftrightarrow -2q_1 = 6 \text{ and } -3q_1 - q_0 = 0$$

$$\Leftrightarrow q_1 = -3 \text{ and } q_0 = 9$$

Thus, $a_n^p = -3n + 9$ is the particular solution of the given recurrence relation.

Hence, the general solution is

$$a_n = d_1(1^n) + d_2(2^n) + d_3(-4)^n - 3n + 9$$

where d_1 , d_2 and d_3 are constants.

b. $f(n) = (-1)^n(4)^n = (-4)^n$, and $b = -4$ is the root of the characteristic equation and the multiplicity of -4 is 1, so we choose the particular solution to the given nonhomogeneous recurrence relation to be $a_n^p = qn(-4)^n$, where q is constant to be determined by substituting a_n^p in to the given recurrence relation.

Thus,

$$a_n^p - a_{n-1}^p - 10a_{n-2}^p + 8a_{n-3}^p = f(n) \text{ , where } f(n) = (-4)^n.$$

The rest of the steps are left as an exercise.

Exercise 3.3.2

1. Find the general solution (all the solutions) of the recurrence relation
   a. $a_n = 3a_{n-1} + 2^n$, for $n \geq 1$
   b. $a_n = -4a_{n-1} - 3a_{n-2} + 10n^2 - 2n + 3$, for $n \geq 2$
2. Solve the recurrence relation together with the initial conditions given.
   a. $a_n - 2a_{n-1} = -2n$, for $n \geq 1$, $a_0 = 3$
   b. $a_n + 5a_{n-1} + 6a_{n-2} = 42(4^n)$, for $n \geq 2$, $a_0 = 4$, $a_1 = 1$.
3. Find the general solution (all the solutions) of the recurrence relation
   a. $a_n - 6a_{n-1} + 12a_{n-2} - 8a_{n-3} = n^2$, for $n \geq 3$

Summary

- A recurrence relation for a sequence $\{a_n\}_{n=0}^{\infty}$ is an equation that expresses a_n in terms of one or more preceding terms $a_0, a_1, \dots, a_{n-2}, a_{n-1}$.
- A linear recurrence relation with constant coefficient of degree (order) k is a recurrence relation of the form

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = f(n) \qquad \text{(i)}$$

  where $c_1, c_2, \dots, c_k$ are constants and $c_k \neq 0$. If $f(n)$ is identically zero ($f(n) = 0$), then the recurrence relation (i) is called homogeneous, otherwise it is called nonhomogeneous.
- Consider the linear homogeneous recurrence relation with constant coefficient of degree k

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = 0 \qquad \text{(ii)}$$

  and consider its characteristic equation

$$r^k + c_1 r^{k-1} + c_2 r^{k-2} + \cdots + c_{k-1} r + c_k = 0\,. \qquad \text{(iii)}$$

  iii. If the characteristic equation (iii) has k distinct roots $r_1, r_2, \dots, r_k$, then the sequence $\{a_n\}_{n=0}^{\infty}$ is the solution of the recurrence relation (ii) if and only if

$$a_n = d_1 {r_1}^n + d_2 {r_2}^n + \cdots + d_k {r_k}^n$$

  where $d_1, d_2, \dots, d_k$ are constants.

  iv. If the characteristic equation (iii) has t distinct roots $r_1, r_2, \dots, r_t$ with multiplicity $m_1, m_2, \dots, m_t$ respectively, so that $m_i \geq 1$, for $i = 1,2, \dots, t$ and $m_1 + m_2 + \cdots + m_t = k$, then the sequence $\{a_n\}_{n=0}^{\infty}$ is the solution of the recurrence relation (ii) if and only if

$$a_n = \left(\alpha_{10} + \alpha_{11} n + \cdots + \alpha_{1(m_1-1)} n^{m_1-1}\right) {r_1}^n + \left(\alpha_{20} + \alpha_{21} n + \cdots + \alpha_{2(m_2-1)} n^{m_2-1}\right) {r_2}^n + \cdots + \left(\alpha_{t0} + \alpha_{t1} n + \cdots + \alpha_{t(m_t-1)} n^{m_t-1}\right) {r_t}^n$$

  where α_{ij} are constants for $0 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.
- Consider the linear nonhomogeneous recurrence relations with constant coefficients of degree k

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = f(n) \qquad \text{(iv)}$$

  and its associated homogenous recurrence relation

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = 0 \qquad \text{(v)}$$

The general solution (all solutions) of the linear nonhomogeneous recurrence relation (iv) is the sequence $\{a_n^p + a_n^g\}_{n=0}^{\infty}$, where a_n^p is the particular solution of the linear nonhomogeneous recurrence relation (iv) and a_n^g is the general solution (all possible solutions) of the associated homogenous recurrence relation (v).

Self Test Exercise 3

1. *Solve the recurrence relations together with the initial conditions given.*
   a. $a_n + 6a_{n-1} = 0$, *for* $n \geq 1$, $a_0 = -4$
   b. $a_n - 7a_{n-1} + 12a_{n-2} = 0$, *for* $n \geq 2$, $a_0 = 1$, $a_1 = 0$.
   c. $a_n - \frac{5}{2}a_{n-1} + \frac{25}{16}a_{n-2} = 0$, *for* $n \geq 2$, $a_0 = 2$, $a_1 = 1$.
   d. $a_n = 7a_{n-1} - 10a_{n-2}$, *for* $n \geq 2$, $a_0 = 2$, $a_1 = 1$.
   e. $a_n = 2a_{n-1} - a_{n-2}$, *for* $n \geq 2$, $a_0 = 4$, $a_1 = 1$.
   f. $a_n + 6a_{n-1} + 9a_{n-2} = 0$, *for* $n \geq 2$, $a_0 = 3$, $a_1 = -3$
   g. $a_{n+2} = -4a_{n-1} + 5a_n$, *for* $n \geq 0$, $a_0 = 2$, $a_1 = 8$.
2. *Find the general solution (all solutions) of the recurrence relation*
   a. $a_n + 6a_{n-1} = 0$, *for* $n \geq 1$.
   b. $a_n + \frac{1}{3}a_{n-1} - \frac{2}{9}a_{n-2} = 0$, *for* $n \geq 2$
   c. $a_n - 2a_{n-1} - a_{n-2} + 2a_{n-3} = 0$, *for* $n \geq 3$
3. *Find the solution to the recurrence relation*

$$a_n = 5a_{n-2} - 4a_{n-4} = 0 \quad for \ \ n \geq 4$$

   with $a_0 = 3$, $a_1 = 2$, $a_2 = 6$ *and* $a_3 = 8$.
4. *Find the general solution (all the solutions) of the recurrence relation*
   c. $a_n = 6a_{n-1} + 2^n$, *for* $n \geq 1$
   d. $a_n = -2a_{n-1} - a_{n-2} + 10n^2 - 2n + 3$, *for* $n \geq 2$
   e. $a_n - 5a_{n-1} - 6a_{n-2} = 5^n + 3n$, *for* $n \geq 2$
5. *Solve the recurrence relation together with the initial conditions given.*
   c. $a_n - 4a_{n-1} = -n^2$, *for* $n \geq 1$, $a_0 = 1$
   d. $a_n - 5a_{n-1} + 6a_{n-2} = 42(4^n)$, *for* $n \geq 2$, $a_0 = 4$, $a_1 = 1$.
   e. $a_n + 6a_{n-1} + 9a_{n-2} = n2^n$, *for* $n \geq 2$, $a_0 = 3$, $a_1 = -3$.
   f. $a_n - 2a_{n-1} + a_{n-2} = (n^2 + 3)(5^n)$, *for* $n \geq 2$, $a_0 = 3$, $a_1 = -3$.

6. *Find the general solution (all the solutions) of the recurrence relation*
   a. $a_n - 6a_{n-1} + 12a_{n-2} - 8a_{n-3} = n^2$, *for* $n \geq 3$
   b. $a_n - 6a_{n-1} + 12a_{n-2} - 8a_{n-3} = n^2 2^n$, *for* $n \geq 3$
7. *Solve the recurrence relation*

$$a_n = 8a_{n-2} - 16a_{n-4} + 2 \text{ , for } n \geq 3$$

with initial condition $a_0 = 1$, $a_1 = 2$, $a_2 = 3$ *and* $a_3 = 0$.

Reference:

1. A. W.F.Edwards,Pascal's Arithmeticical Triangle. The Johns Hopkins University Press, Baltimore,MD, 2002.
2. C.Oliver, " The Twelve Days of Christmas," Mathematics Teacher, Vol. 70
3. Hewel Hus, Schaum's outline, probability Random Variable.
4. J.M.Harris.Jeffry L.Hirst.Michael J.Mossinghoff, Combinatorics and Graph Theory, 2nd edition
5. J.Varnadore, " Pascal's Triangle and Fibonacci Numbers," Mathematics Teacher, Vol,81.
6. Kenneth H.Rosen, Discrete Mathematics and its application, 3rd edition.
7. M.Eng and J.Casey, "Pascal's Arithmetical Triangle-ASerendipitious Source for programming Activities,'' Mathematics Teacher, Vol 76
   mathematics
8. Mott, Knadel and Baker, Discreate Mathematics
9. P.Z.Chinn, " inductive patterns, finite Difference, and a Missing Region,'' Mathematics Teacher, Vol.81(Sept.1988)
10. R.P.GRIMALDI and B.V.RAMANA pearson,Discreate combinatorial
11. Rosen , Discrete Mathematics
12. Seymour Lipschutz,Schaum's outline, Discrete Mathematics, 3rd edition
13. Berman G.Fryer K.D, introduction to combinatorics
14. Andweson I, A first course in Discrete Mathematics
15. D.P.Acharjya Sreekumar, Fundamental Approach to Discrete Mathematics.

PART-II GRAPH THEORY

CHAPTER 4

ELEMENTS OF GRAPH THEORY

Objectives: After studying this chapter, you should be able to:

- Introduce some basic definitions and concepts of graph theory
- Present useful result on the degree sequence of a graph.
- Know the terms graph, labelled graph, unlabelled graph, vertex, edge, adjacent, incident, multiple edges, loop, simple graph and subgraph;
- Appreciate the idea of graph theory;
- Determine whether two given graphs are isomorphic;
- Know the terms degree, degree sequence and regular graph;
- State and use the handshaking lemma;
- Realize the terms walk, trail, path, closed walk, closed trail, cycle, connected graph, disconnected graph and component;
- Understand what are meant by complete graphs, null graphs, cycle graphs, the Platonic graphs, cubes and the Petersen graph;
- Distinguish between physical and conceptual tree structures.
- Understand what are meant by bipartite graphs, complete bipartite graphs, trees, path graphs and cubes;
- Apply the concept of graph theory to the real world.

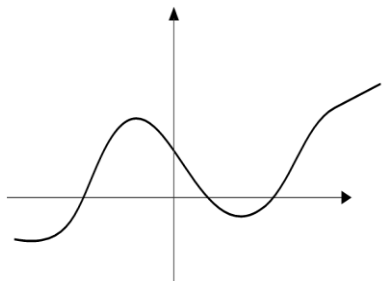
Introduction

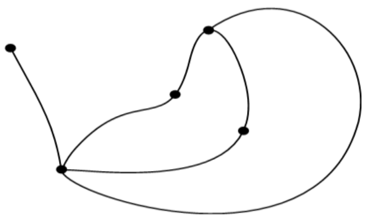
You learn even in high school about graphs of functions. The graph of a function is usually a curve drawn in the $\boldsymbol{xy}-$ plane. See Fig. $(\boldsymbol{a})$. But the word “graph” has other meanings. Infinite or discrete mathematics, a graph is a collection of points and edges or arcs in the plane. Fig. $(\boldsymbol{b})$ illustrates a graph as we are now discussing the concept.

Leonhard Euler (1707–1783) is considered to have been the father of graph theory. His paper in 1736 on the seven bridges of Königsberg is considered to have been the foundational paper in the subject. It is worthwhile now to review that topic.

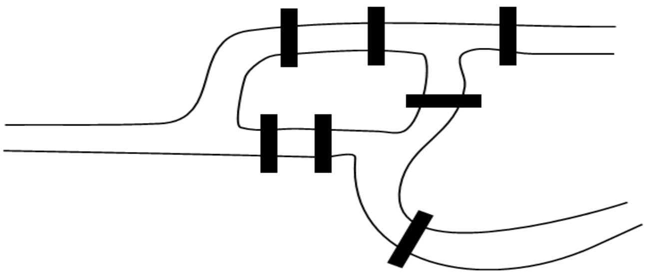
Königsberg is a town, founded in 1256, that was originally in Prussia. Aftera stormy history, the town became part of the Soviet Union and was renamed Kaliningrad in 1946. In any event, during Euler's time the town had seven bridges (named Krämer, Schmiede, Holz, Hohe, Honig, Köttel, and Grünespanning) spanning the Pregel River. Fig. $(\boldsymbol{c})$ gives a simplified picture of how the bridges were originally configured (two of the bridges were later destroyed during World War II, and two others demolished by the Russians). The question that fascinated people in the eighteenth century was whether it was possible to walk a route that never repeats any part of the path and that crosses each bridge exactly once.

Euler in effect invented graph theory and used his ideas to show that it is impossible to devise such a route. We shall, in the subsequent sections, devise a broader version of Euler's ideas and explain his solution of the Königsberg bridge problem.





$(\boldsymbol{a})$A graph of a function in the plane. $(\boldsymbol{b})$ A graph as a combinatorial object.



$(\boldsymbol{c})$ The seven bridges at Königsberg.

4.1. Basic Terminologies

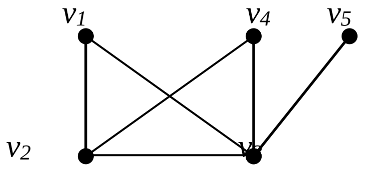
Objectives: After studying this chapter, you should be able to:

- Introduce some basic definitions and concepts of graph theory
- Present useful result on the degree sequence of a graph.
- Explain the terms degree, degree sequence and regular graph;
- Explain the terms graph, labelled graph, unlabelled graph, vertex, edge, adjacent, incident, multiple edges, loop, simple graph and subgraph.
- State and use the handshaking lemma;

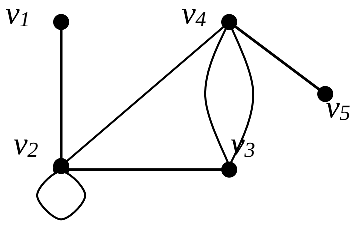
Definition 4.1: A graph G consists of a finite non-empty set $V(G)$ of elements called ***vertices*** together with a finite set $E(G)$ of unordered pairs of (not necessarily distinct) vertices called ***edges***.

Example 4.1: The following are examples of graphs with vertices $V(G)$ and edges $E(G)$.

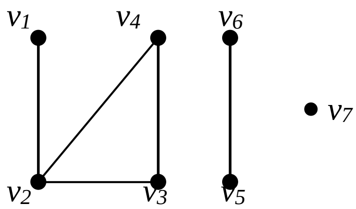
(i) $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$
$E(G) = \{v_1v_2, v_1v_3, v_2v_3, v_2v_4, v_3v_4, v_3v_5\}$



(ii) $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$
$E(G) = \{v_1v_2, v_2v_2, v_2v_3, v_2v_4, v_3v_4 \text{ (twice)}, v_4v_5\}$



(iii) $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$
$E(G) = \{v_1v_2, v_2v_3, v_2v_4, v_3v_4, v_5v_6\}$
v_7 is an isolated vertex.



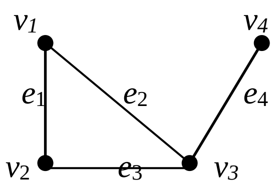
Definition 4.2: In a graph, two or more edges joining the same pair of vertices are ***multiple edges***. An edge joining a vertex to itself is a ***loop***. A graph with no multiple edges or loops is a ***simple graph***.

Thus, the graphs in example4.1 (i) and (iii) are simple, that in (ii) is not, it is a ***non-simple*** graph.

Graphs (such as those in examples (i) and (ii)) which 'come in one piece' are said to be ***connected***. The graph in (iii) is not connected: it is the union of three connected subgraphs, called the ***components*** of the graph.

Definition 4.3: Two vertices u and v of a graph G are ***adjacent*** if there is an edge uv joining them and we then say that u and v are ***incident*** with the edge (or that the edge is ***incident*** with u and v). Similarly, two edges are ***adjacent*** if they have a vertex in common.

Example 4.2: v_1 and v_2 are adjacent in the following graph, each being incident with edge e_1. We call v_1 and v_2 the ***end-vertices*** of e_1. Also edges e_1 and e_2 are adjacent.



The ***degree*** **(*size*)** of a vertex v, denoted by deg(v), is the number of edges incident with v. Thus, in the graph above,

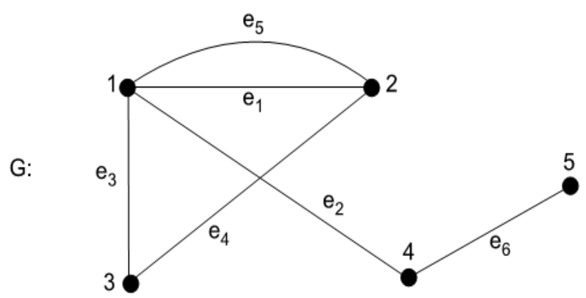
$$\deg(v_1) = 2\ ,\ \deg(v_2) = 2\ ,\ \deg(v_3) = 3\ ,\ \deg(v_4) = 1\ .$$

In example 4.1 (ii), deg (v_2) = 5 (count the loop twice), deg (v_3) = 3.

In example4.1 (iii), deg (v_7) = 0. (An isolated vertex has degree 0.)

The ***order*** of a graph G is the number of vertices, $|V|$, where V is the set of vertices.

If there is more than one edge between the same pair of vertices, then the edges are termed as ***parallel edges***. Consider the graph G as



Here the edges e_1 and e_5 are parallel edges.

A vertex of degree 1 is called a ***leaf***.

By the ***degree sequence*** of a graph, we mean the vertex degrees written in ascending order with repeats where necessary. Thus the degree sequence in example 4.1 (i) is 1,2,2,3,4, that in (iii) is 0,1,1,1,2,2,3.

Theorem 4.1: *Handshaking Lemma (Degree Sum Theorem)*: For a graph with vertex set $\{v_1, v_2, \cdots, v_n\}$ and m edges,

$$\sum_{k=1}^{n} \deg(v_k) = 2m.$$

***Proof*.** Each edge contributes 2 to the sum of vertex degrees.

Note: As a consequence, in any graph the number of vertices of odd degree must be even. And if a sequence $d_1, d_2, \cdots, d_n$ with $d_1 \le d_2 \le \cdots \le d_n$ is the degree sequence of a graph, then $\sum_{j=1}^{n} d_j$ must be even. Also, for a simple graph, $d_n \le n-1$. However, these two conditions are not sufficient for $d_1, d_2, \cdots, d_n$ to be the degree sequence of a simple graph. For example, the sequence 1, 1, 3, 3 is not graphic.

Useful result (Havel-Hakimi)

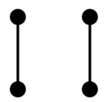
The sequence $d_1, d_2, \cdots, d_{n-1}, j,$
where $d_1 \le d_2 \le \cdots \le d_{n-1} \le j \le n-1,$
is the degree sequence of a simple graph if and only if

$$d_1,\, d_2, \cdots, d_{n-j-1},\, d_{n-j}-1,\, d_{n-j+1}-1, \cdots,\, d_{n-1}-1$$

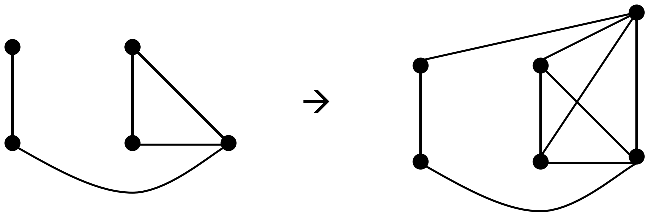
is (when rearranged in ascending order if necessary) the degree sequence of a simple graph.

Example 4.3: From the sequence 2, 2, 3, 3, 4, 4 we derive 1, 2, 2, 2, 3 and 1, 1, 1, 1.

The last sequence is the degree sequence of



Working backwards, we obtain a graph for the original sequence: (non-isomorphic answers are possible).



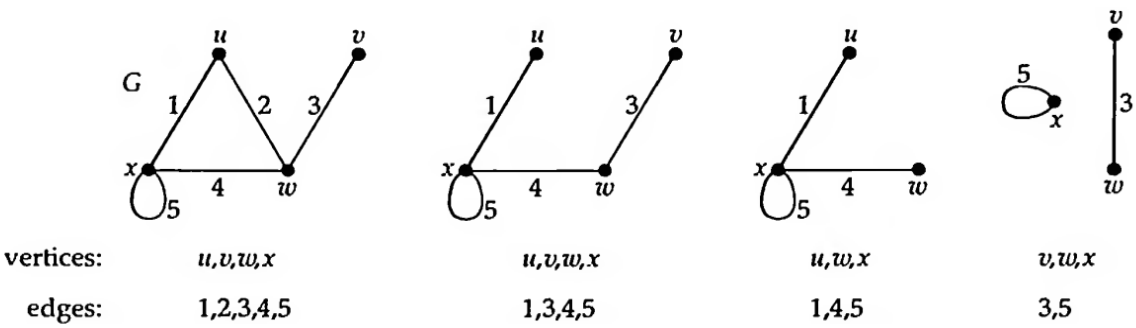
Subgraphs

In mathematics we often study complicated objects by looking at simpler objects of the same type contained in them - subsets of sets, subgroups of groups, and so on. In graph theory we make the following definition.

Definition 4.4: A ***subgraph*** of a graph G is a graph all of whose vertices are vertices of G and all of whose edges are edges of G.

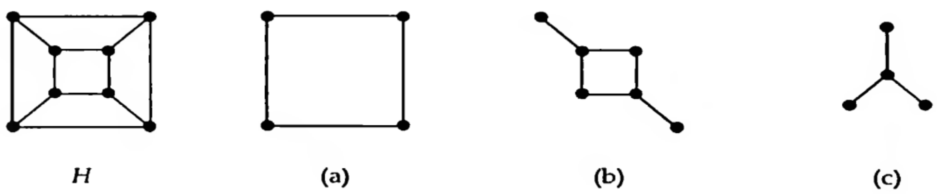
Remark 4.2: Note that G is a sub graph of itself.

Example 4.4: The following graphs are all subgraphs of the graph G on the left, with vertices $\{u, v, w, x\}$ and edges $\{1, 2, 3, 4, 5\}$.



The idea of a subgraph can be extended to unlabelled graphs.

Example 4.5: The following graphs are all subgraphs of the unlabelled graph H on the left; the configuration in graph (c) occurs at each corner of H.



Union of graphs

If G_1 and G_2 be two graphs, then their union $G_1 \cup G_2$is the graph with

$V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

Intersection of graphs

If G_1 and G_2 be two graphs with at least one vertex in common, then their intersection $G_1 \cap G_2$is the graph with

$V(G_1 \cap G_2) = V(G_1) \cap V(G_2)$ and $E(G_1 \cap G_2) = E(G_1) \cap E(G_2)$.

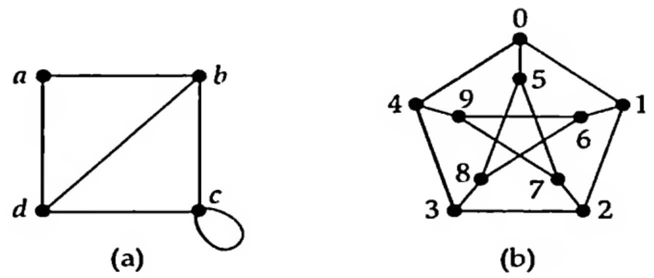
√ Check-List

Put a tick (√) mark if you can perform the task and a cross (x) mark if you can't in the box against the following questions.

1. Can you define simple graph? .... ☐
2. Can you identify the difference between adjacent and incident? .. ☐
3. Can a given graph is simple? .... ☐

Exercise 4.1

1. *Write down the vertices and edges of each of the following graphs. are these graphs simple graphs?*



2. *Draw the graphs whose vertices and edges are as follows. Are these graphs simple graphs?*

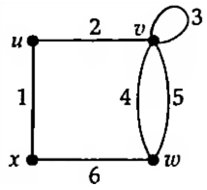
   a. vertices: $\{u, v, w, x\}$ *edges:* $\{uv, vw, vx, wx\}$

   b. vertices: $\{1, 2, 3, 4, 5, 6, 7, 8\}$ *edges:* $\{12, 22, 23, 34, 35, 67, 68, 78\}$

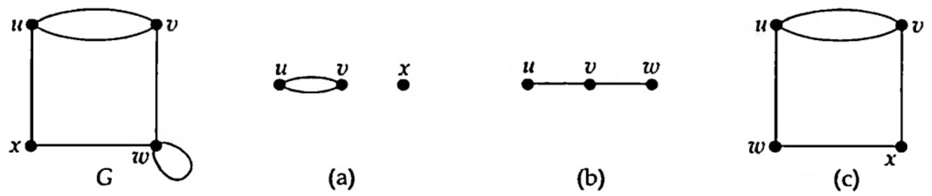
3. *Draw, if possible, simple graphs with the following degree sequences:*

   (i) 2,3,3,4,5,5 (ii) 3,3,5,5,5,5 (iii) 2,2,3,3,4,5,5.

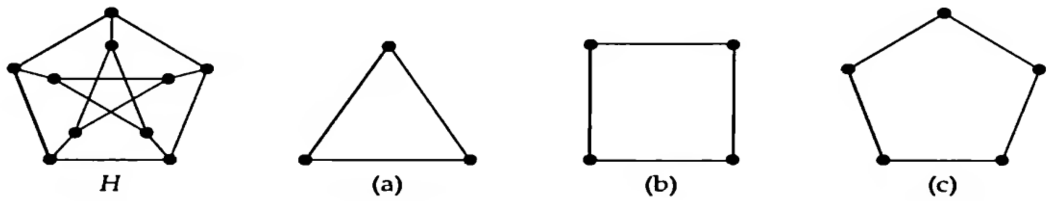
4. *Which of the following statements hold for the graph on the right?*
   a. *vertices v and ware adjacent;*
   b. *vertices v and x are adjacent;*
   c. *vertex u is incident with edge 2;*
   d. *edge 5 is incident with vertex x.*



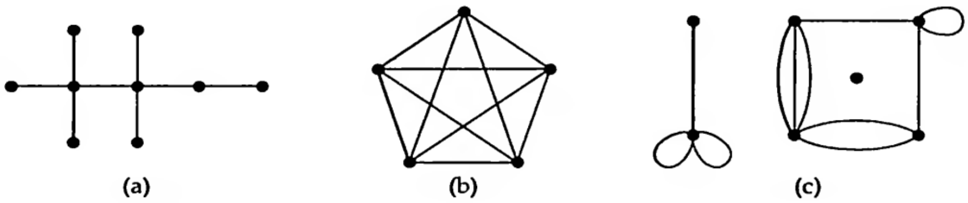
5. *Which of the following graphs are subgraphs of the graph G below?*



6. *Which of the following graphs are subgraphs of the graph H below?*



7. *Write down the degree sequence of each of the following graphs:*



8. *For each of the graphs in Problem 6, write down:the number of edges; the sum of the degrees of all the vertices. What is the connection between your answers? Can you explain why this connection arises?*

9. (a) *Use the handshaking lemma to prove that, in any graph, the number of vertices of odd degree is even.*
   (b) *Verify that the result of part* (a) *holds for each of the graphs in Pro. 6.*

4.2 Isomorphism

Objectives: After studying this topic, you should be able to:

- Introduce some basic concepts of isomorphism
- Determine whether two given graphs are isomorphic

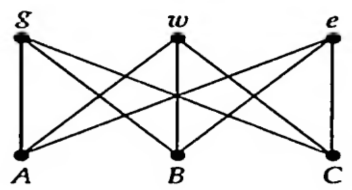
It follows from the definition that a graph is completely determined when we know its vertices and edges, and that two graphs are the same if they have the same vertices and edges. Once we know the vertices and edges, we can draw the graph and, in principle, any picture we draw is as good as any other; the actual way in which the vertices and edges are drawn is irrelevant - although some pictures are easier to use than others!

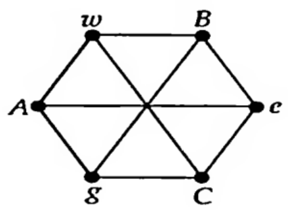
For example, recall the utilities graph, in which three houses A, Band Care joined to the three utilities gas (g), water (w) and electricity (e). This graph is specified completely by the following sets:

vertices: {A, B, C, g, w, e}

edges: {Ag, Aw, Ae, Bg, Bw, Be, Cg, Cw, Ce},

and can be drawn in many ways, such as the following:





Each of these diagrams has six vertices and nine edges, and conveys the same information - each house is joined to each utility, but no two houses are joined, and no two utilities are joined. It follows that these two dissimilar diagrams represent the same graph.

On the other hand, two diagrams may look similar, but represent different graphs. For exan1ple, the diagrams below look similar, but they are not the same graph: for example, AB is an edge of the second graph, but not the first.

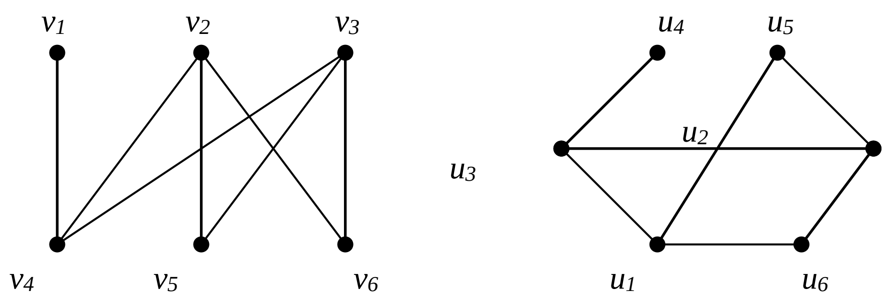
We express this similarity by saying that the graphs represented by these two diagrams are *isomorphic*. This means that the *two graphs have essentially the same structure*: we can relabel the vertices in the first graph to get the second graph - in this case, we simply interchange the labels wand B.

This leads to the following definition.

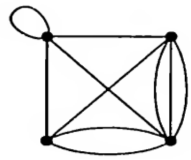
Definition 4.5: Two graphs G_1 and G_2 are ***isomorphic*** if there is a one-one correspondence between their vertex sets $V(G_1)$ and $V(G_2)$ such that for every pair of distinct vertices in G_1 the number of edges joining them is equal to the number of edges joining their corresponding vertices in G_2. Such a one-one correspondence is an ***isomorphism***.

Example 4.6: The following two graphs are isomorphic with the following one-one correspondence: $v_1 \leftrightarrow u_1, v_2 \leftrightarrow u_4, v_3 \leftrightarrow u_2, v_4 \leftrightarrow u_3$.

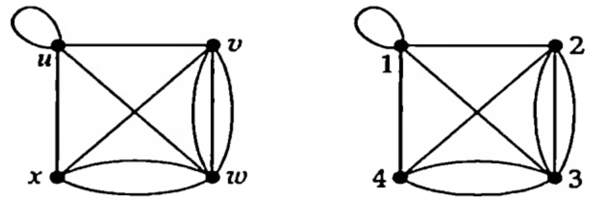
Likewise the following graphs are isomorphic:



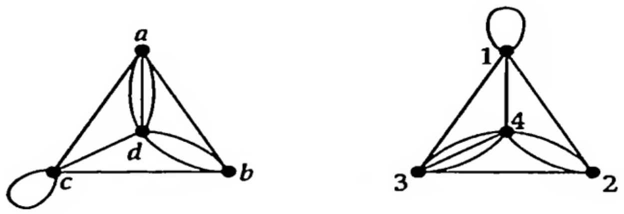
Sometimes it is unnecessary to have labels on the graphs. In such cases, we omit the labels and refer to the resulting object as an ***unlabelled*** graph.



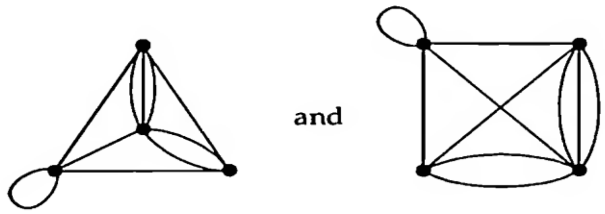
Example 4.7: The unlabelled graphcorresponds to either of the following isomorphic graphs:



Indeed, it also corresponds to either of the following graphs, which are isomorphic to the above two:



We say that two unlabelled graphs such as



are ***isomorphic*** if labels can be attached to their vertices so that they become the same graph.

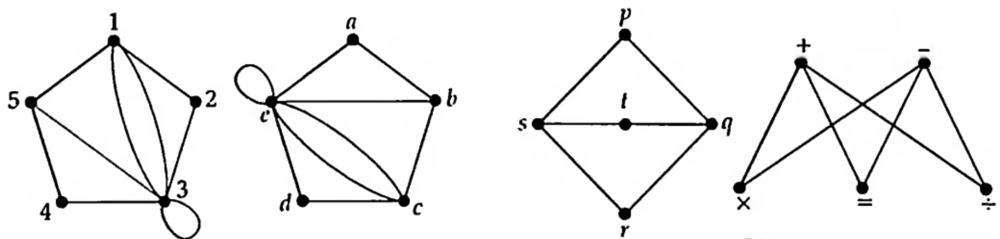
√ Check-List

Put a tick (√) mark if you can perform the task and a cross (x) mark if you can’t in the box against the following questions.

1. Can you define isomorphic graphs?.. ☐
2. Can you determine whether two graphs are isomorphic or not?.......... ☐

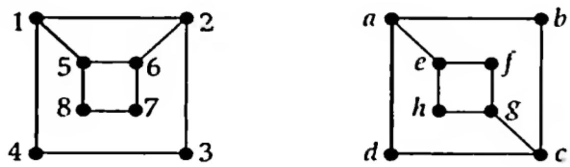
Exercise 4.2

1. *By suitably relabelling the vertices, show that the following pairs of graphs are isomorphic:*

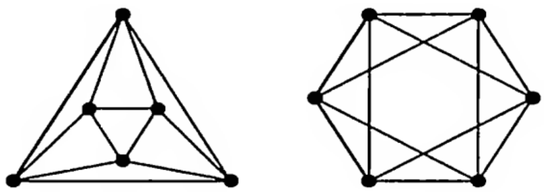


(a) (b)

2. *Are the following two graphs isomorphic? If so, find a suitable one-one correspondence between the vertices of the first and those of the second; if not, explain why no such one-one correspondence exists.*



3. *By suitably labelling the vertices, show that the following unlabelled graphs are isomorphic:*



4.3 Path and Connectivity

Objectives: After studying this topic, you should be able to:

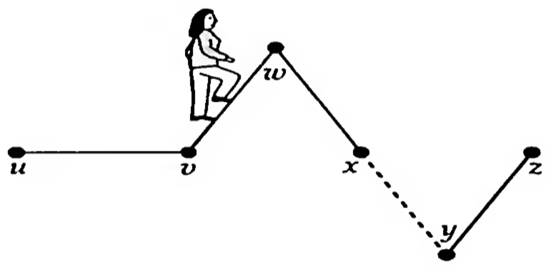
- Explain the terms walk, trail, path, closed walk, closed trail, cycle, connected graph, disconnected graph and component;
- Explain the terms edge connectivity, vertex connectivity, cutset and vertex cutset;
- Introduce concepts relating to how a connected graph can be disconnected.
- Apply the concept of path and connectivity to the real world.

Many applications of graphs involve getting from one vertex to another. For example, you may wish to find the shortest route between one town and another. Other examples include the routeing of a telephone call between one subscriber and another, the flow of current between two terminals of an electrical network, and the tracing of a maze. We now make this idea precise by defining a walk in a graph.

Definition 4.6: A ***walk*** of length k in a graph is a succession of k edges of the form

$$uv, vw, wx, \dots, yz.$$

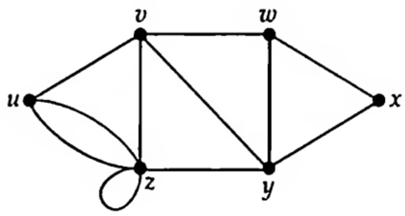
This walk is denoted by $uvwx\dots yz$, and is referred to as a walk between u and z.



We can think of such a walk as going from u to v, then from v to w, then from w to x, and so on, until we arrive eventually at the vertex z. Since the edges are undirected, we can also think of it as a walk from z to y and on, eventually, to x, w, v and u. So we can equally well denote this walk by $zy\dots xwvu$, and refer to it as a walk between z and u.

Note that we do not require all the edges or vertices in a walk to be different.

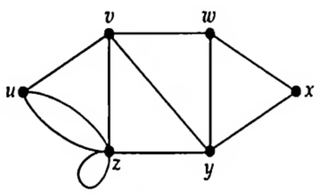
Example 4.8: In the following graph, $uvwxywvzzy$ is a walk of length 9 between the vertices u and y, which includes the edge vw twice and the vertices v, w, y and z twice.



It is sometimes useful to be able to refer to a walk under more restrictive conditions in which we require all the edges, or all the vertices, to be different.

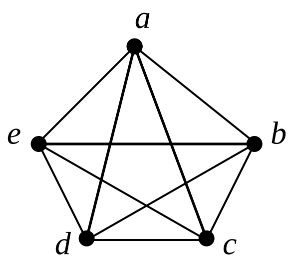
Definition 4.7: A ***trail*** is a walk in which no edge has been traversed more than once (in either direction) but repeated vertices are allowed. It is closed if the last vertex is the same as the first, and open otherwise. A ***path*** is a walk in which all the edges and all the vertices are different.

Example 4.9: In the following graph above, the walk $vzzywxy$ is a trail which is not a path, since the vertices y and z both occur twice, whereas the walk $vwxyz$ has no repeated vertices, and is therefore a path.



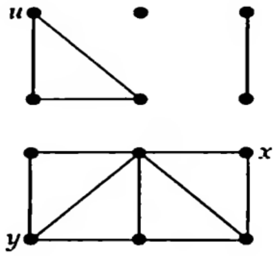
Example 4.10: In the graph below,

- *acbdce* is an open trail
- *acbdcea* is a closed trail
- *acbdcbe* is not a trail, since edge *cb* has been traversed twice, it is a walk.



We can use the concept of a path to define a connected graph. Intuitively, a graph is connected if it is 'in one piece';

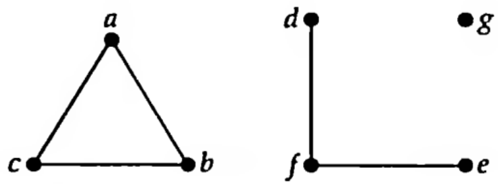
Example 4.11: The following graph is not connected, but can be split into four connected subgraphs.



The observation that there is a path between x and y (which lie in the same subgraph), but not between II and y (which lie in different subgraphs), leads to the following definitions.

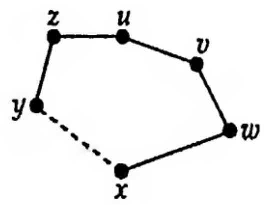
Definition 4.8: A graph is ***connected*** if there is a path between each pair of vertices, and is ***disconnected*** otherwise. An edge in a connected graph is a ***bridge*** if its removal leaves a disconnected graph. Every disconnected graph can be split up into a number of connected subgraphs, called ***components***.

Example 4.12: In the graph in exercise 4.3 (2), the edge tz is a bridge; and the following disconnected graph has three components:

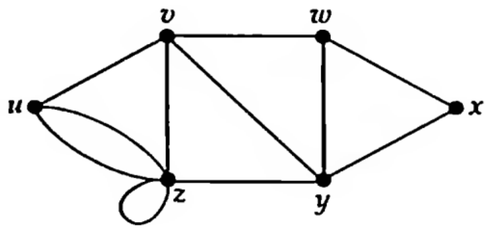


It is also useful to have a special term for those walks or trails that start and finish at the same vertex. We say that they are closed.

Definition 4.9: A ***closed walk*** in a graph is a succession of edges of the form $uv, vw, wx, \dots, yz, zu,$ that starts and ends at the same vertex. A ***closed trail*** is a closed walk in which all the edges are different. A ***cycle*** is a closed walk in which all the edges are different and all the intermediate vertices are different. A walk or trail is ***open*** if it starts and finishes at different vertices.



Example 4.13: Consider the following graph:

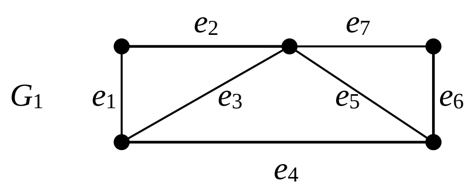


The closed walk $vywxyzv$ is a closed trail which is not a cycle, whereas the closed trails $zz, vwxyv$ and $vwxyzv$ are all cycles. A cycle of length 3, such as $vwyv$ or $wxyw$, is called a mangle. In describing closed walks, we can allow any vertex to be the starting vertex. For example, the triangle $vwyv$ can equally well be written as $wyvw$ or $yvwy$ or (since the direction is immaterial) by $vywv, wvyw$ or $ywvy$.

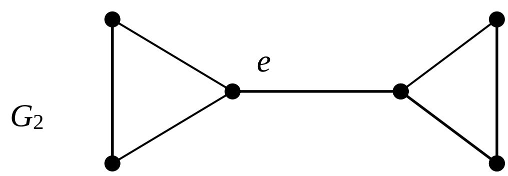
Disconnecting Sets (this involves edges)

Recall that a graph is ***connected*** if there is a path joining each pair of vertices. Let G be a connected graph. By a ***disconnecting*** set we mean a set of edges whose deletion results in a disconnected graph. A ***cutset*** is a disconnecting set, no proper subset of which is a disconnecting set.

Example 4.14: In graph G_1, $\{e_2, e_3, e_4\}$ and $\{e_1, e_2\}$ are disconnecting sets but $\{e_3, e_5\}$ is not.



If a disconnecting set has only one edge e (as in G_2 below), then e is called a ***bridge*** (or ***cut-edge***).



The *edge connectivity* of G is the size of the smallest disconnecting set of *G*, in other words, it is the smallest number of edges whose deletion disconnects *G*. It is denoted by $\lambda(G)$.

In the graph G_1 above since $\{e_1, e_2\}$ is a disconnecting set of size 2, then $\lambda(G_1) \le 2$.

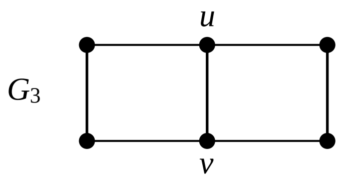
Result

Let $\delta(G)$ denote the minimum degree of *G*. Then $\lambda(G) \le \delta(G)$.

Separating Sets (this involves vertices)

A ***separating set*** is a set of vertices whose removal disconnects *G*. When removing vertices, we must also remove any incident edges.

Example 4.15: In graph G_3, $\{u, v\}$ is a separating set, but $\{u\}$ is not.



If a separating set contains only one vertex *w*, then *w* is called a ***cut-vertex***. The ***vertex connectivity*** of a connected graph *G* is the size of the smallest separating set. In other words, it is the smallest number of vertices whose removal disconnects *G*. It is denoted by $\kappa(G)$.

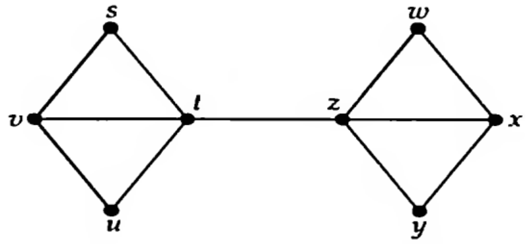
√ Check-List

Put a tick (√) mark if you can perform the task and a cross (x) mark if you can't in the box against the following questions.

1. Can you define walk? .... ☐
2. Do you know the difference between trail and path?.......................... ☐
3. Do you know the difference between closed trail and cycle?.............. ☐
4. Can a simple graph is connected graph? .... ☐

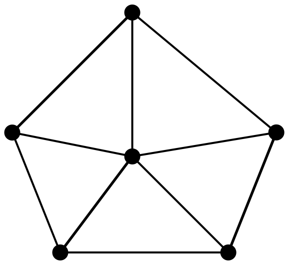
Exercise 4.3

1. *Write down all the paths between s and y in the following graph:*

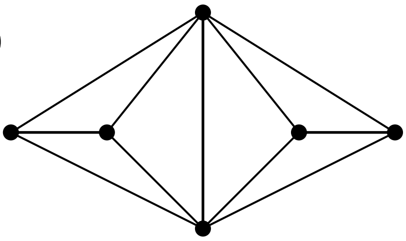


2. *Draw:*
   a. *a connected graph with eight vertices;*
   b. *a disconnected graph with eight vertices and two components;*
   c. *a disconnected graph with eight vertices and three components.*
3. *For the graph on the right, write down:*
   a. *a closed walk that is not a closed trail;*
   b. *a closed trail that is not a cycle;*
   c. *all the cycles of lengths 1,2,3 and 4.*
4. *Find the vertex connectivity and the edge connectivity for each of the following graphs, giving reasons for your answers.*

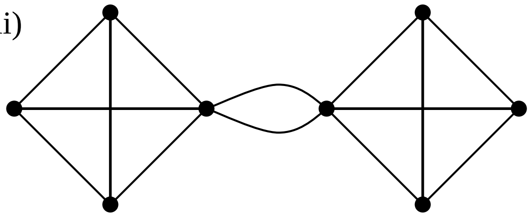
(i)



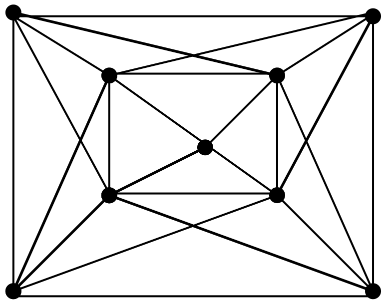
(ii)



(iii)



(iv)



4.4. Complete, regular and bipartite graphs

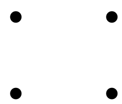
Objectives: After studying this topic, you should be able to:

- Explain what are meant by complete graphs, null graphs, cycle graphs, the Platonic graphs, cubes and the Petersen graph;
- Explain what are meant by bipartite graphs, complete bipartite graphs and path graphs .

Null graphs

These are graphs with no edges and are denoted by N_n.

Example 4.16: the null graph on 4 vertices



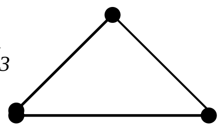
N_4

Complete graphs

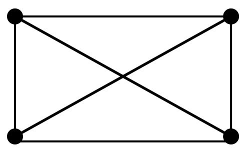
Definition 4.10: The ***complete graph*** on *n* vertices, denoted by K_n, is the simple graph in which every pair of vertices is joined by an edge.

Example 4.17: The following graphs are complete graphs

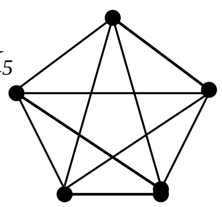
K_3



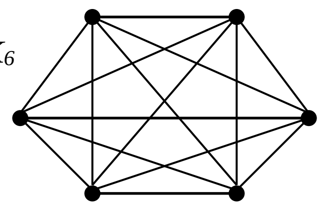
K_4



K_5

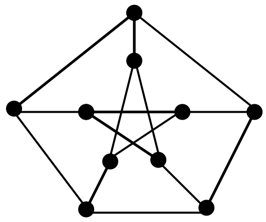


K_6



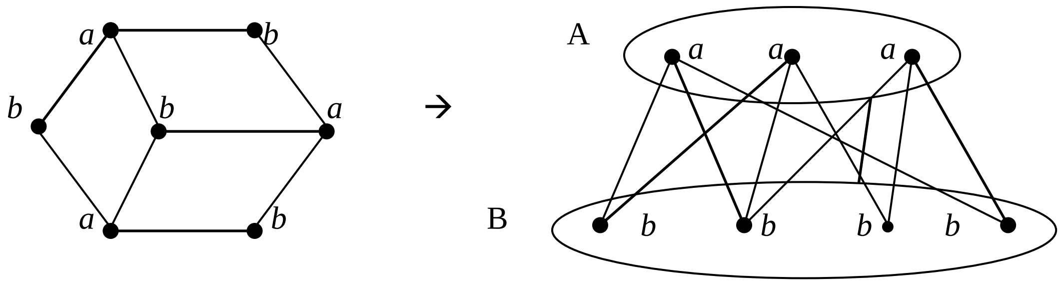
Regular graphs

These are graphs in which every vertex has the same degree. For example, K_n is regular of degree n-1. Regular graphs of degree 3 are called ***cubic*** graphs. The number of edges of a regular graph with n vertices is given by $n(n - 1)/2$. The following regular graph, called the ***Petersen*** graph is an interesting example.



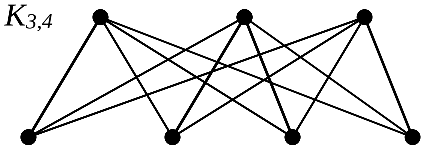
Bipartite graphs

Definition 4.11: A graph G is ***bipartite*** if every vertex can be labelled with either a or b, so that every edge is an ab edge. In these graphs, the vertex set is the union of two non-empty disjoint sets A and B with each edge of the graph joining a vertex in A to a vertex in B.

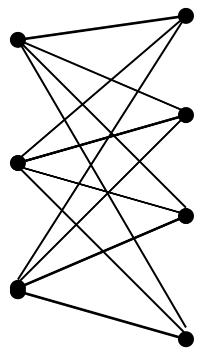


Complete bipartite graphs

$K_{r,s}$ denotes the simple bipartite graph in which the sets A and B (as above) contain r and s vertices and every vertex in A is adjacent to every vertex in B.

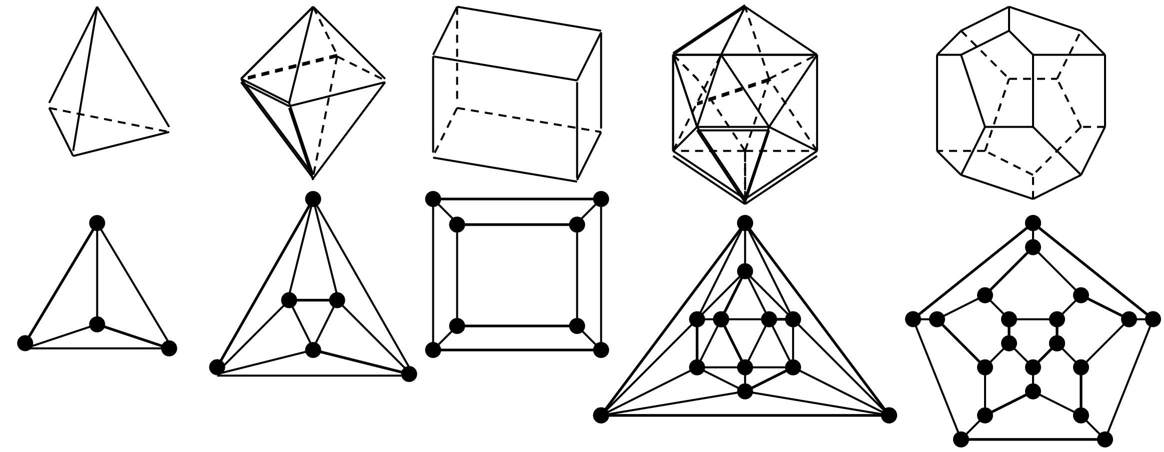


or



The Platonic graphs

These are formed from the vertices and edges of the 5 regular (Platonic) solids:

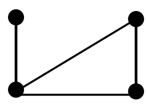


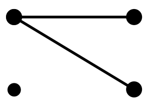
Tetrahedron Octahedron Cube Icosahedron Dodecahedron

The complement of a simple graph

Definition 4.12: The ***complement*** of a simple graph G is the simple graph with the same vertex set as G with two vertices adjacent if and only if they are not adjacent in G. It is denoted by $\overline{G}$.

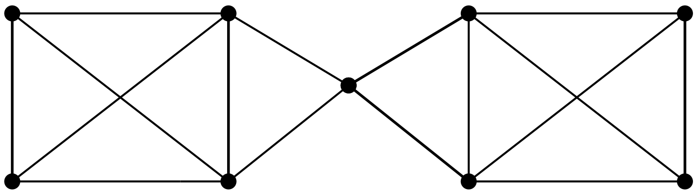
Example 4.18: if G is then is $\overline{G}$



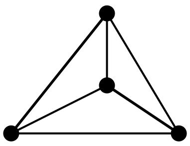


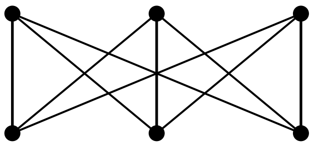
Theorem 4.3: In a connected graph G, we have $\kappa(G) \leq \lambda(G) \leq \delta(G)$, for $G \neq K_n$.

Example 4.19: $\lambda = 2, \quad \kappa = 1$.



$K_4 : \lambda = 3$, κ does not exist for K_n. $K_{3,3} : \lambda = 3$, $\kappa = 3$.





√ Check-List

Put a tick (√) mark if you can perform the task and a cross (x) mark if you can't in the box against the following questions.

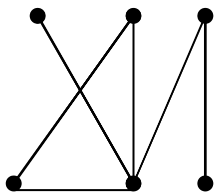
1. Can you define complete graph? ....
2. Can you define regular and bipartite graph? ....
3. Can you explain the relation between a simple graph and its complement?..
4. Can you define platonic graph? ....

Exercises 4.4

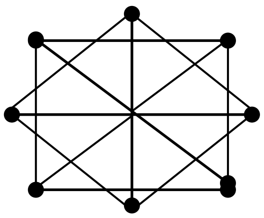
1. *Give (if it exists) an example of each of the following:*
   i. *A Platonic graph that is a complete graph.*
   ii. *A bipartite Platonic graph.*
   iii. *A bipartite graph that is regular of degree 4.*
   iv. *A cubic graph with 7 vertices.*
2. *Determine the number of edges of the following graphs:*

   (i) K_8 *(ii)* $K_{5,7}$ *(iii) the Peterson graph.*
3. *Draw the complements of the following graphs:*

   (i)



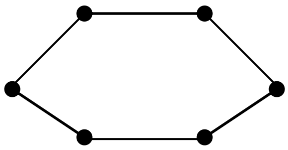
   (ii)



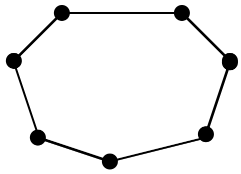
4. *Which of the following graphs are bipartite?*

   (i) Petersen

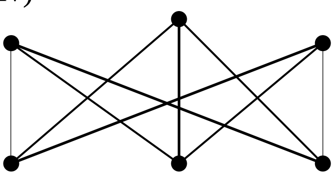
   (ii) C_6 *(6 – cycle):*



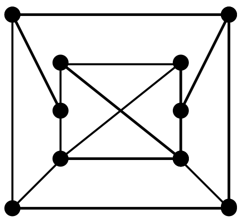
   (iii) C_7 *(7 – cycle):*



   (iv)



   (v)



5. *In the Petersen graph find cutsets with 3, 4 and 5 edges.*

4.5. Eulerian and Hamiltonian Graphs

Objectives: After studying this topic, you should be able to:

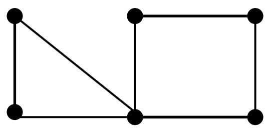
- Define Eulerian and Hamiltonian graphs;
- Discuss Semi-Eulerian graphs;
- introduce the idea of a Semi-Hamiltonian graph;
- Present some simple criteria for determining whether a graph is Hamiltonian.

Eulerian and Semi-Eulerian Graphs

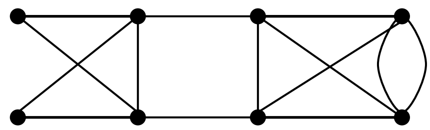
Definition 4.13: A connected graph *G* is ***Eulerian*** if there is a closed trail containing every edge of *G*. We call such a trail an ***Eulerian trail***.

Example 4.20: the following graphs are Eulerian

(i)



(ii)

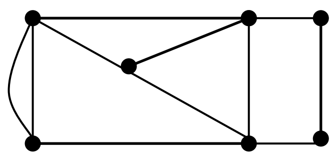


Theorem 4.4: (*Euler 1736*). A connected graph is Eulerian if and only if every vertex has even degree.

To find an Eulerian trail in a given graph, start in an arbitrary vertex and traverse along the edges, ensuring all the edges are traversed before returning to the starting vertex.

If *G* is not Eulerian, but there is an open trail containing every edge of *G*, then *G* is ***semi-Eulerian***.

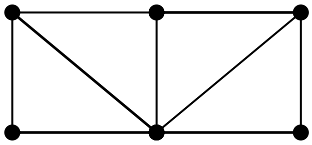
Example 4.21:



Theorem 4.5: A connected graph is semi-Eulerian if and only if precisely two of its vertices have odd degree.

To obtain a semi-Eulerian trail in a given graph, you must start at one of the odd degree vertices and end in the other odd degree vertex.

The following is an example of a graph that is neither Eulerian nor semi-Eulerian.

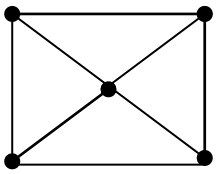


Hamiltonian and semi-Hamiltonian Graphs

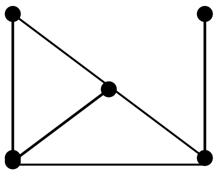
Definition 4.14: A graph G is ***Hamiltonian*** if there is a cycle that passes through every vertex of G. Such a cycle is a ***Hamiltonian cycle***.

If G is not Hamiltonian, but there is an open path which includes every vertex of G, then G is ***semi- Hamiltonian***, and such a path is a ***Hamiltonian path***.

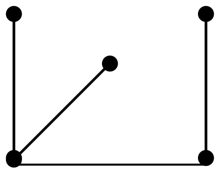
Example 4.22:



Hamiltonian



semi- Hamiltonian



neither Hamiltonian nor semi- Hamiltonian

Theorem 4.6: (*Ore 1960*): Let G be a simple graph with $n \geq 3$ vertices. If $\deg(u) + \deg(v) \geq n$ for each pair of non-adjacent vertices u and v, then G is Hamiltonian.

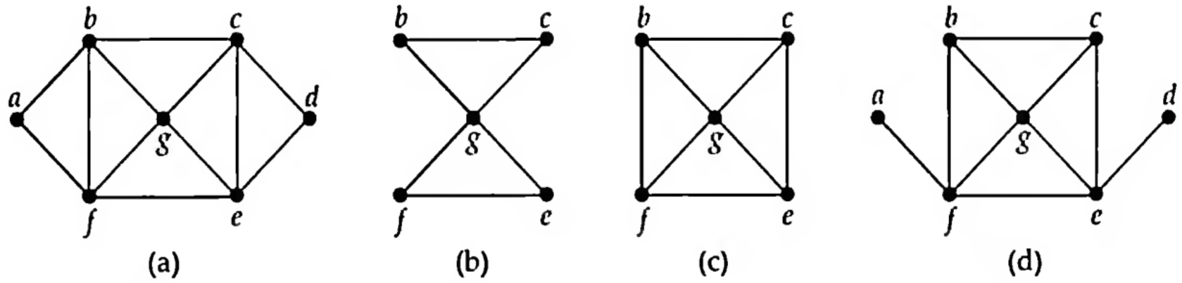
Corollary 4.7: (Dirac 1952). Let G be a simple graph with $n \geq 3$ vertices. If $\deg(v) \geq n/2$ for every vertex v, then G is Hamiltonian.

Note that this is a *necessary* condition, but not *sufficient*.

Example 4.23: The Petersen graph is non-Hamiltonian but K_n, $(n \geq 3)$ graphs are Hamiltonian.

Bipartite graphs can only be Hamiltonian if sets A and B of vertices (as previously defined) have the same number of vertices. It follows that if the total number of vertices in a bipartite graph is odd then it cannot be Hamiltonian. There is no known general criterion for testing whether a graph is semi-Hamiltonian.

Example 4.24: Consider the following four graphs:



graph (a) is both Eulerian and Hamiltonian, as we saw above;

graph (b) is Eulerian - an Eulerian trail is $bcgfegb$; it is not Hamiltonian;

graph (c) is Hamiltonian -a Hamiltonian cycle is $bcgefb$; it is not Eulerian;

graph (d) is neither Eulerian nor Hamiltonian.

√ Check-List

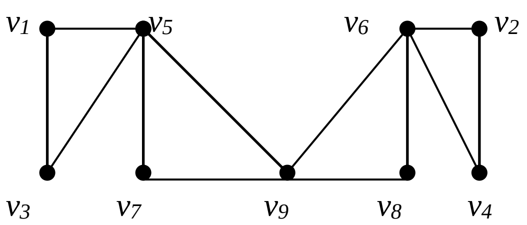
Put a tick (√) mark if you can perform the task and a cross (x) mark if you can't in the box against the following questions.

1. Can you define Eulerian graphs? .... ☐
2. Can you define Hamiltonian graph? .... ☐
3. Can you explain the relation between a Eulerian and Hamiltonian graph? ...... ☐
4. Can you define semi-Eulerian and semi-Hamiltonian graph? .... ☐

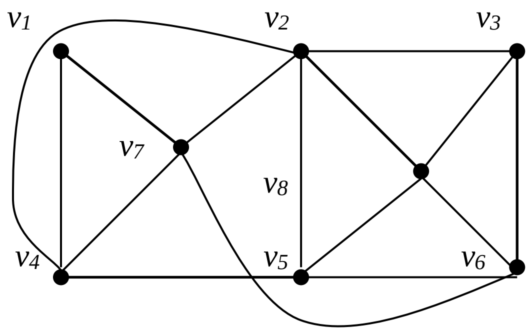
Exercises 4.5

1. *Determine whether the following graphs are Eulerian, semi-Eulerian or neither. For those that are Eulerian or semi-Eulerian, find a suitable trail.*

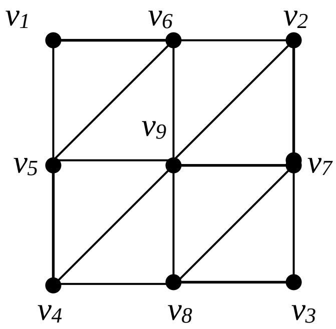
(i)



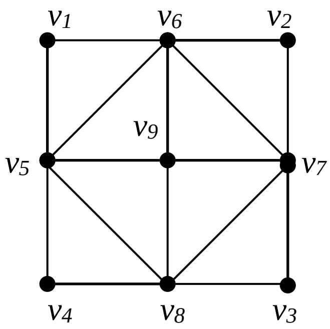
ii)



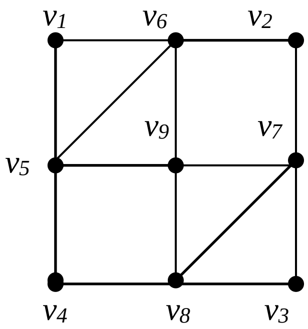
(iii)



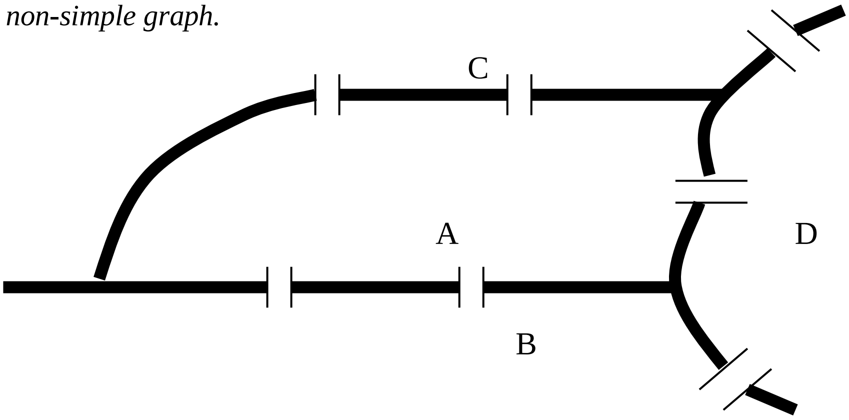
(iv)



(v)

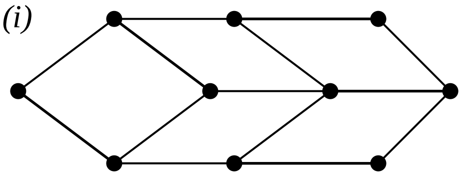


2. *In the Petersen graph,*
   a. *Find walks of lengths 5, 6 and 8.*
   b. *Find cycles of lengths 5, 6, 8 and 9.*
   c. *Can you find a cycle of length 10?*
   d. *What is the size of the smallest cycle?*
3. ***Bridges of Konigsberg.*** *Euler lived in the town of Konigsberg that had rivers passing through it (as shown below) and with bridges connecting four parts of the town. He considered the following problem: Is it possible to cross each of the seven bridges exactly once and return to your starting point? Hint: Let the four parts A, B, C and D be vertices of a suitable non-simple graph.*

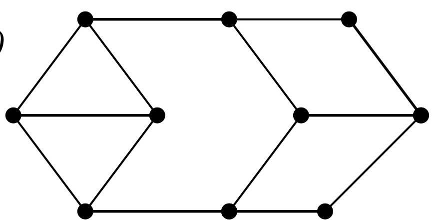


4. *Determine whether the following graphs are Hamiltonian, semi- Hamiltonian or neither. Find a Hamiltonian cycle or path if one exists.*

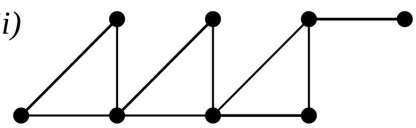
(i)



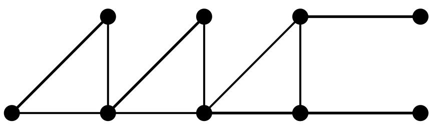
(ii)



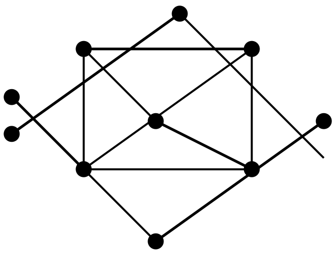
(iii)



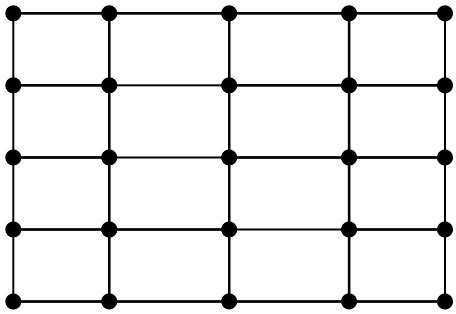
(iv)



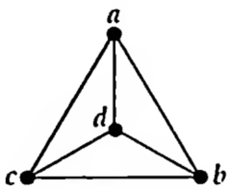
(*v*)



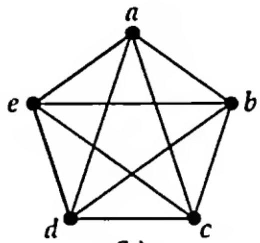
(*vi*)



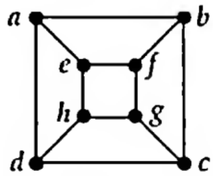
5. *Decide which of the following graphs are Eulerian and/or Hamiltonian, and write down an Eulerian trail or Hamiltonian cycle where possible.*



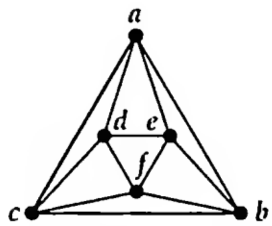
(a)



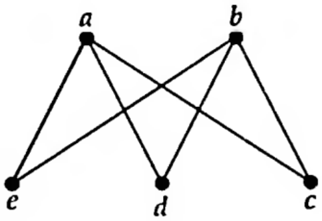
(b)



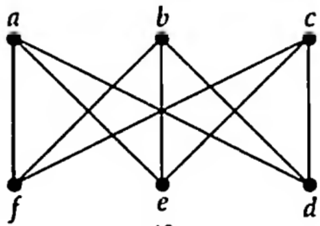
(c)



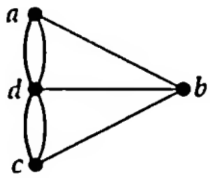
(d)



(e)



(f)



(g)

6. *Determine which of the following graphs are Eulerian:*

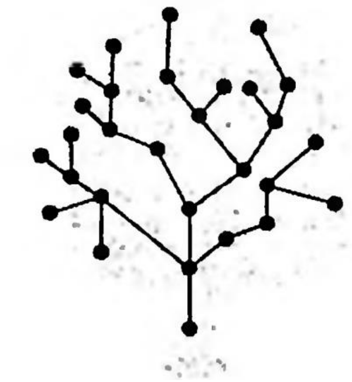
   a. *the complete graph* K_8;
   b. *the complete bipartite graph* $K_{8,8}$;
   c. *the cycle graph* C_8;
   d. *the dodecahedron graph;*
   e. *the cube graph* Q_8.

4.6. Tree Graphs

Objectives: After working through this topic, you should be able to:

- state several properties of a tree, and give several equivalent definitions of a tree;
- distinguish between physical and conceptual tree structures, and give examples of each type;
- appreciate the uss of rooted trees in different areas;
- construct the bipartite graph representation of a given braced rectangular frame- work and use it to determine whether the system is rigid; if so, determine whether the system is minimally braced. define tree.

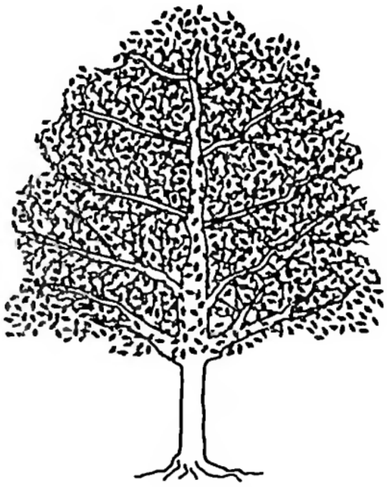
In this topic we focus our attention on one particularly important and useful type of graph - a *tree*. Although trees are relatively simple structures, they form the basis of many of the practical techniques used to model and to design large-scale systems.

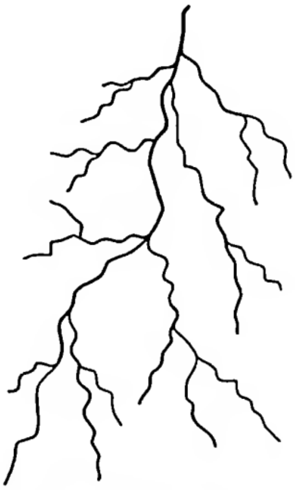


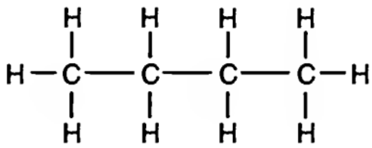
The concept of a tree is one of the most important and commonly used ideas in graph theory, especially in the applications of the subject. It arose in connection with the work of Gustav Kirchhoff on electrical networks in the 1840s, and later with Arthur Cayley's work on the enumeration of molecules in the 1870s. More recently, trees have proved to be of value in such areas as computer science, decision making, linguistics, and the design of gas pipeline systems.

Trees are often used to model situations involving various physical or conceptual tree-like structures. These structures are also commonly referred to as 'trees'. In the following examples, we classify such 'trees' in terms of the type of application in which they occur.

Many trees have a physical structure which may be either natural or artificial and either static or time-dependent. Two examples of natural trees are the biological variety with trunk, branches and leaves, and the drainage system of tributaries forming a river basin. Less obvious examples of tree structures are provided by the chemical structure of certain organic molecules.


a sapling


river tributaries

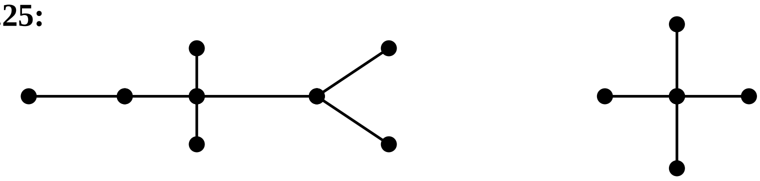


a molecule

One of the most important classes of bipartite graphs is the class of trees. If *G* is a tree then G is bipartite, i.e. all of its vertices can be labelled with either *a* or *b* so that every edge is an ***ab*** edge (no ***aa*** or ***bb*** edges).

Definition 4.15: A ***tree*** is a connected graph without cycles.

Examples 4.25:

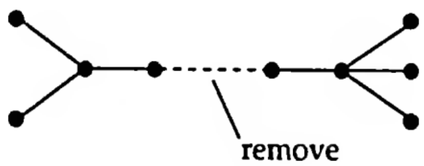


Starting with the tree with just one vertex, we can build up any tree we wish by successively adding a new edge and a new vertex. At each stage, the number of vertices exceeds the number of edges by 1, so every tree with n vertices has exactly n - 1 edges. At no stage is a cycle created, since each added edge joins an old vertex to a new vertex.

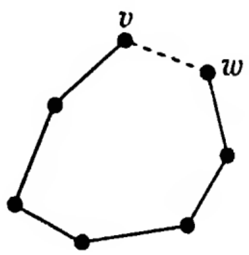
At each stage, the tree remains connected, so any two vertices must be connected by at least one path. However, they cannot be connected by more than one path, since any two such paths would contain a cycle (and possibly other edges as well).

We therefore deduce that any two vertices in a tree are connected by exactly one path.

In particular, any two adjacent vertices are connected by exactly one path - the edge joining them. If this edge is removed, then there is no path between the two vertices.



It follows that the removal of any edge of a tree disconnects the tree. Moreover, any two vertices v and w are connected by a path, and the addition of the edge vw produces a cycle - the cycle consisting of the path and the added edge vw.



So joining any two vertices of a tree by an edge creates a cycle.

Several of the above properties can be used as definitions of a tree. In the following theorem, we state six possible definitions. They are all equivalent: anyone of them can be taken as the definition of a tree, and the other five can then be deduced.

Theorem 4.8: Let G be a graph with n vertices. Then the following statements are equivalent.

i. G is connected and has no cycles;

ii. *G* has *n*-1 edges and has no cycles;

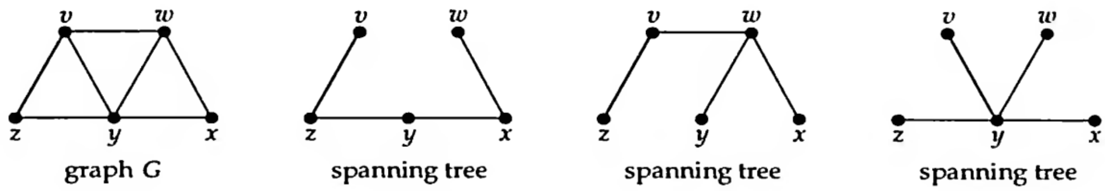
iii. Any two vertices in *G* are connected by exactly one path;

iv. G is connected and the removal of any edge disconnects G;

v. G contains no cycle but the addition of any new edge to *G* creates exactly one cycle;

Definition 4.16: Let G be a connected graph. Then a ***spanning tree*** in G is a subgraph of G that includes every vertex and is also a tree.

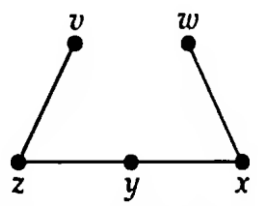
Example 4.26: The following diagram shows a graph and three of its spanning trees.



Given a connected graph, we can construct a spanning tree by using either of the following two methods. We illustrate these by applying them to the graph G above.

***Building-up method*:** Select edges of the graph one at a time, in such a way that no cycles are created; repeat this procedure until all vertices are included.

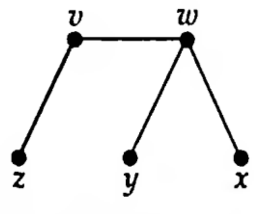
Example 4.27: In the above graph G, we select the edges vz, wx, xy, yz; then no cycles are created. We obtain the following spanning tree.



spanning tree

***Cutting-down method* :** Choose any cycle and remove any one of its edges; repeat this procedure until no cycles remain.

Example 4.28: From the above graph G, we remove the edges vy (destroying the cycle $vwyv$), yz (destroying the cycle $vwyzv$), xy (destroying the cycle $wxyw$). We obtain the following spanning tree.



spanning tree

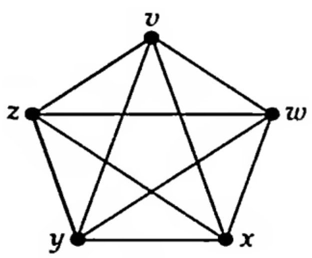
√ Check-List

Put a tick (√) mark if you can perform the task and a cross (x) mark if you can't in the box against the following questions.

1. Can you define a tree? ....
2. Can you construct the spanning tree of a given graph by using both methods?

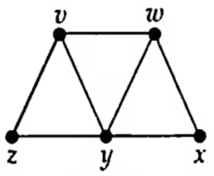
Exercise 4.6

1. *Draw the branching tree representing the outcomes of two throws of a six-sided die.*
2. *Give an example of a tree with seven vertices and*
   a. *exactly two vertices of degree* 1;
   b. *exactly four vertices of degree* 1;
   c. *exactly six vertices of degree* 1.
3. *Use the handshaking lemma to prove that every tree with n vertices, where* $n \geq 2$, *has at least two vertices of degree* 1.
4. *Use a proof by contradiction to show that the removal of an edge cannot disconnect a tree into more than two components.*
5. *Use a proof by contradiction to show that the addition of a new edge to a tree cannot create more than one cycle.*
6. *Use each method to construct a spanning tree in the complete graph* K_5.



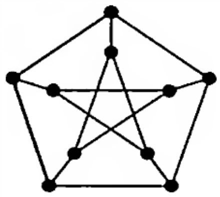
K_5

7. *The graph G below has twenty-one spanning trees. Find as many of them as you can.*



graph G

8. *Find three spanning trees in the Petersen graph.*



4.7. Planar Graphs

Objectives: After studying this topic, you should be able to:

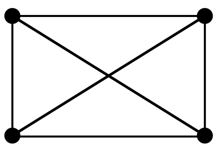
- Introduce the topic of planar graphs, and to present some simple properties of such graphs.
- Explain the terms planar graph, non-planar graph, plane drawing, face, infinite face, degree of a face, subdivision of a graph and contraction of a graph;
- Use the handshaking lemma for planar graphs;
- State and use Euler's formula;
- Explain the term dual graph and describe its properties;
- Present the Kuratowski theorem and discuss its use in determining whether a graph is planar.

In this chapter we investigate the properties of graphs that can be drawn in the plane without any of their edges crossing; such graphs are called planar graphs. In particular, we determine whether the complete bipartite graph K3,3 is planar, thereby solving the utilities problem. We discuss Euler's formula and Kuratowski's theorem; the latter is an important theoretical result which gives a necessary and sufficient condition for a graph to be planar.

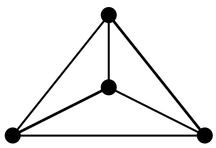
Definition 4.17: A graph *G* is ***planar*** if it can be drawn in the plane without its edges crossing. Such a drawing is called a ***plane drawing*** or a ***plane graph***. A graph G is ***non-planar*** if no plane drawing of G exists.

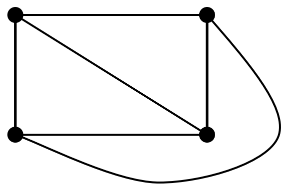
Example 4.29:

(i) The graph K_4 is planar.

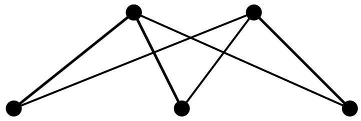


Here are two plane drawings:

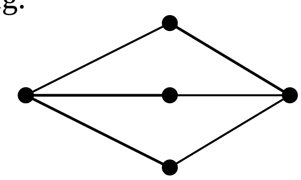




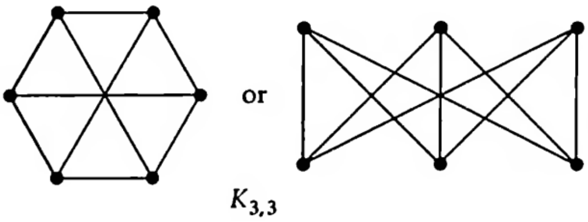
(ii) The graph $K_{2,3}$ is planar.



Here is a plane drawing:



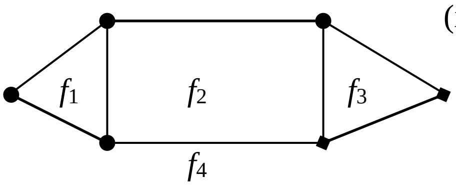
For some graphs, such as $K_{3,3}$, it is impossible to find a drawing that involves no crossings, therefore, $K_{3,3}$ is an example of non planar graph.



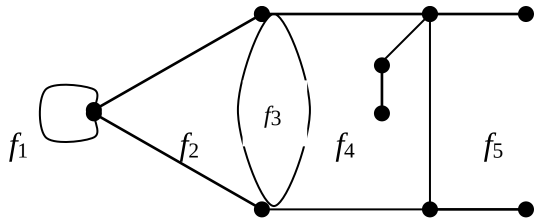
Definition 4.18: A plane graph divides the plane into regions called ***faces***. If we denote the *i*-th face by f_i, then $|f_i|$ represents the number of edges bordering f_i.

Example 4.30:

(i)



(ii)



Graph (i) has 4 faces and graph (ii) has 5.

Theorem 4.9: Let *G* be a connected plane graph, and let *n*, *m* and f be the respective number of vertices, edges and faces of *G*. Then

$$\sum_{i=1}^{f} |f_i| = 2m.$$

Informal proof. Take a walk around every face of the graph. You will traverse every edge exactly twice, because an edge either borders exactly two faces (hence is counted twice as you walk) or borders the same face twice.

In Example 4.30 (i) above, n = 6, m = 8, $f = 4$ and $\sum_{i=1}^{4}|f_i| = 16$.

In (ii), n = 9, m = 12, $f = 5$ and $\sum_{i=1}^{5}|f_i| = 24$.

Theorem 4.10(***Euler's Formula*****)** (Euler 1750): Let G be a connected plane graph, and let n, m and f be the respective number of vertices, edges and faces of G as before. Then

$$n - m + f = 2.$$

Theorem 4.11: Let G be a simple planar graph with $n \geq 3$ vertices and m edges. Then

$$m \leq 3n - 6.$$

Proof. Assume that we have a plane drawing of G with f faces. Since each face is bounded by at least 3 edges, we have $|f_i| \geq 3$ for every i = 1, 2, …, f. Hence, $\sum_{i=1}^{f}|f_i| = 2m \geq 3f$. So $3f \leq 2m$, and by using Euler's formula we have $3(m - n + 2) \leq 2m$ or $m \leq 3n - 6$.

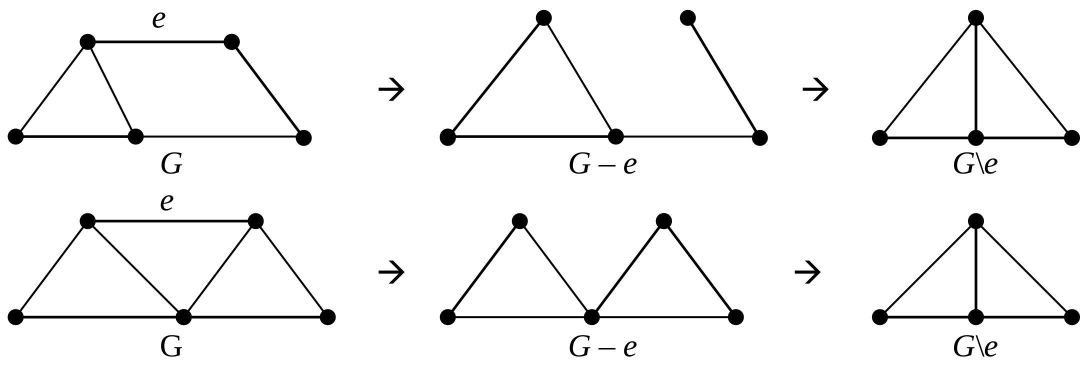
By converse of the above theorem, if $m > 3n - 6$, then we can conclude that a simple graph G is non-planar. Note, however, that if this condition is not satisfied it does not necessarily mean that the graph is planar.

Example 4.31: The graph K_5 has 5 vertices and 10 edges, so we have $10 > 3\times5 - 6 = 9$, and therefore K_5 is non-planar. But in case of the non-planar graph $K_{3,3}$, which has 6 vertices and 9 edges, we have 9 > 3×6 – 6 =12. As 9 is not greater than 12 then the condition is not satisfied, but this does not mean $K_{3,3}$ is planar. The graphs K_5 and $K_{3,3}$ are in fact two important examples of non-planar graphs as we shall see later. These graphs are known as ***Kuratowski graphs***. It is also useful to note that the complete graph K_n has exactly $n(n-1)/2$ edges.

Definition 4.19: A ***minor*** of G is a graph obtained from G by repeatedly deleting and/or contracting edges.

Given an edge e in a graph G, by ***contracting*** the edge e, we mean combining its end vertices by bringing them closer and closer together until they become one and replacing multi-edges by a single edge. We denote the graph obtained in this by G\\e. Deletion of an edge is denoted by G-e.

Example 4.32:



Given a non-planar graph (such as K_5 or $K_{3,3}$) if we 'reverse' a contraction or deletion (i.e. add an edge) then it still remains non-planar. This is the basis for the *Kuratowski theorem***.**

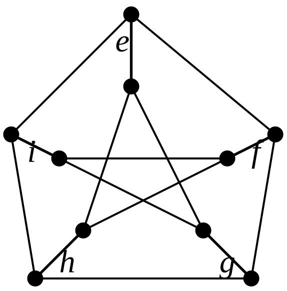
Theorem 4.12 (*Kuratowski theorem*): A graph G is non-planar then it MUST contain either a K_5 or $K_{3,3}$ minor.

G is non-planar $\Leftrightarrow$ G has a K_5 or $K_{3,3}$ minor.

This is a useful result for determining whether a graph is planar or not. (For graphs with not many vertices and edges it may be easier to try finding a plane drawing to show planarity.)

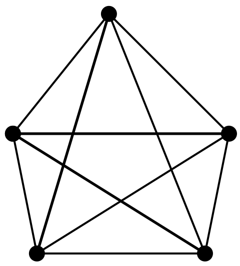
Example 4.33:

1. The Petersen graph is non-planar as it has a K_5 minor. (It also has a $K_{3,3}$ minor.)



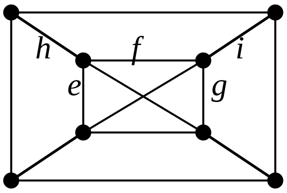
Petersen

Contract e, f, g, h, i.
-->

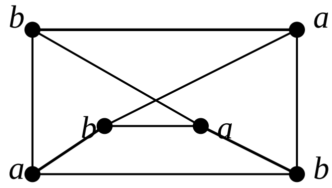


K_5

2. The following graph is non-planar as it has a $K_{3,3}$ minor.



Delete *e, f, g and* contract *h, i.*
-->
$K_{3,3}$



Dual Graphs

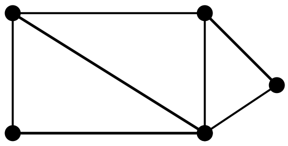
Given a plane drawing of a planar graph *G*, its dual G^* is constructed as follows:

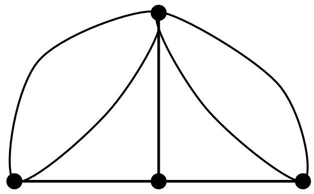
(i) Choose a point v^* inside each face of *G* and make v^* vertices of G^*.

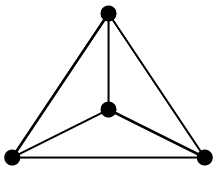
(ii) For each edge *e* of *G* draw a line e^* joining the v^* inside the faces on each side of *e* and make e^* the edges of G^*.

Example 4.34:

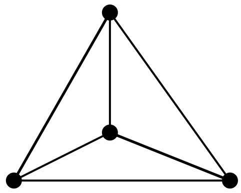
G ← dual → G^*



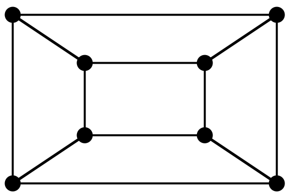




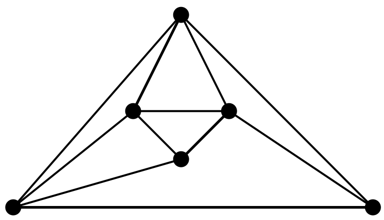
Tetrahedron



Tetrahedron (self dual)



Cube



Octahedron

The following hold for a connected planar graph *G*.

(i) If *G* has *n* vertices, *m* edges and *k* faces, then G^* has *k* vertices, *m* edges and *n* faces.

(ii) G^{**} is isomorphic to *G*.

√ Check-List

Put a tick (√) mark if you can perform the task and a cross (x) mark if you can't in the box against the following questions.

1. Can you define a planar graph? .... ☐
2. Can you state Euler formula and Kuratowski theorem? .... ☐
3. Can you define minor? .... ☐
4. Can you find the minors of a given graph? .... ☐

Exercise 4.7

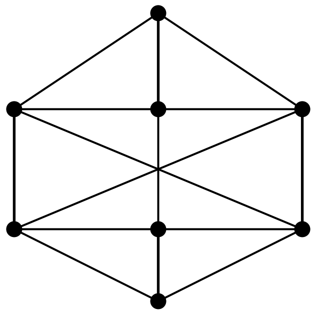
1. *Show that if G is a simple planar graph with no cycles of length 3 or 4 then* $m \le \frac{5}{3}(n-2)$ *, where n is the number of vertices and m the number of edges of G.*

2. *Deduce that the Petersen graph is non-planar.*

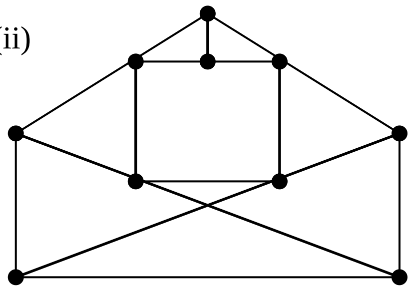
3. *Show that the Petersen graph has a* $K_{3,3}$ *minor.*

4. *Use Kuratowski's theorem to show that the following graphs are non-planar.*

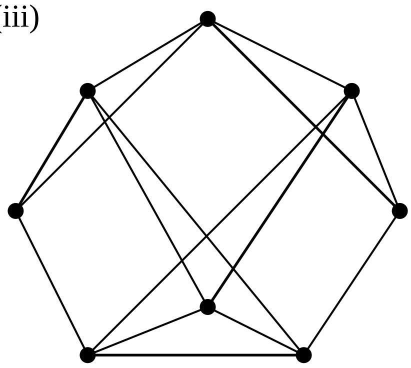
(i)



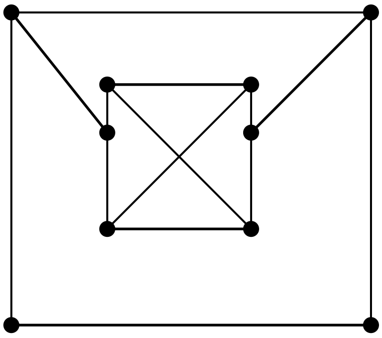
(ii)



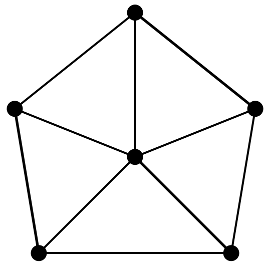
(iii)

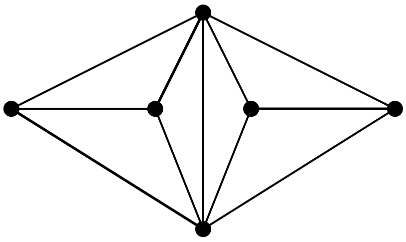


(iv)

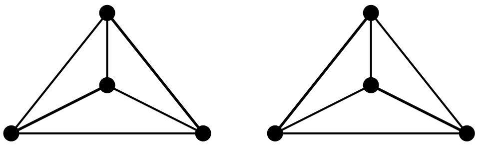


5. *Draw the duals of the following graphs.*





6. *Draw* G^{*} *and* G^{**} *for the following disconnected graph G.*



4.8 Graph Colouring

Objectives: After studying this section, you should be able to:

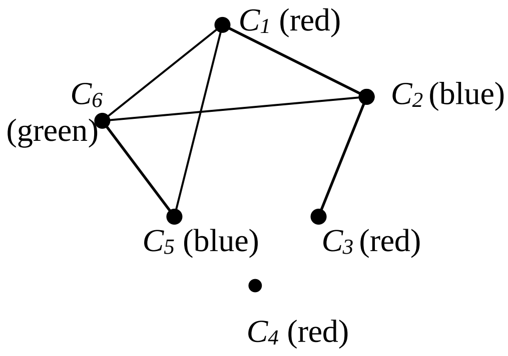
- Introduce the topic of vertex colouring and an application of it to a storage problem.
- Define the term, chromatic number, and present methods of determining it.
- Explain the terms vertex colouring, k-colouring and chromatic number;
- State 4-colour Theorem;
- Apply Brooks' theorem;
- Explain what are meant by colouring problems, the map colouring problem and domination problems, and how they can be represented as vertex decomposition problems.

Introduction

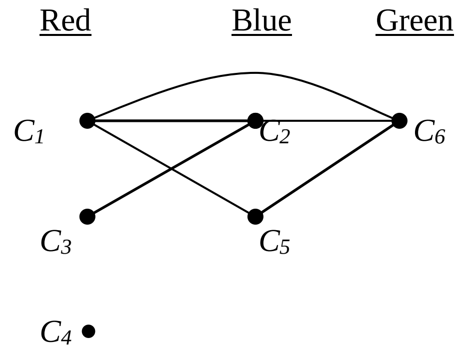
A chemical company wants to ship 6 chemicals, $C_1, \cdots, C_6$, in such a way that those which react violently together are stored in separate containers. The problem is how to store the chemicals using minimum number of containers. Pairs of chemical that react violently together are:

$$C_1C_2,\ C_1C_6,\ C_1C_5,\ C_2C_3,\ C_2C_6, \text{ and } C_5C_6.$$

The problem can be solved by drawing a 'react violently' graph, *F*, whose vertices are C_i and its edges are C_iC_j $(i, j = 1, \cdots, 6)$, where C_i and C_j react together violently. If we now assign colours to the vertices of *F* so that no two adjacent vertices have the same colour, we see that we require at least three colours. This minimum number of colours is called the ***chromatic number*** and is denoted by the Greek letter χ, (chi). For this problem, therefore, $\chi(F) = 3$, as two colours would not be sufficient to colour the adjacent vertices differently. So three containers are required to transport the chemicals safely.



OR:



Definition 4.20: For a simple graph G, we say that G is ***k-colourable*** if we can colour each vertex from a set of k colours in such a way that for every edge its two end vertices have a different colour. The smallest k for which this is possible is called the ***chromatic number*** of G, denoted by $\chi(G)$.

Note that the graph F above is *tripartite*. Its vertices can be split into three sets (red, blue and green) and the edges connect vertices in different sets.

Example 4.35:

$\chi(G) = 1$ iff G has no edges, that is, G is a null graph.

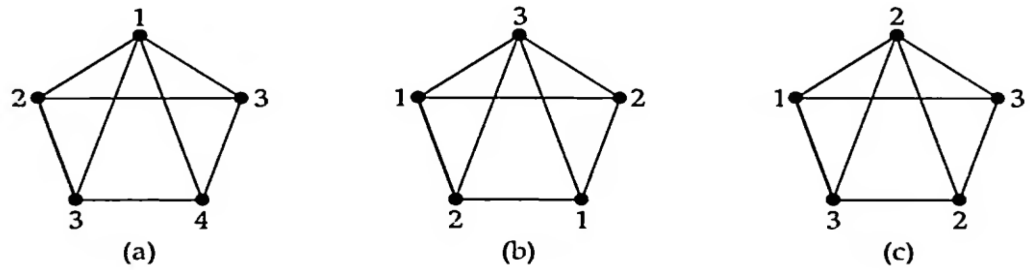
$\chi(G) = 2$ iff G is bipartite and non-null.

$\chi(K_n) = n$, where K_n is a complete graph with n vertices.

$\chi(C_{even}) = 2$ and $\chi(C_{odd}) = 3$, where C is a cycle graph with even or odd number of vertices.

Remark 4.13: The above definitions are given only for simple graphs. Loops must be excluded since, in any k-colouring, the vertices at the ends of each edge must be assigned different colours, so the vertex at both ends of a loop would have to be assigned a different colour from itself. We also exclude multiple edges, since the presence of one edge between two vertices forces them to be coloured differently, and the addition of further edges between them is then irrelevant to the colouring. We therefore restrict our attention to simple graphs.

We usually show a k-colouring by writing the numbers 1,2, ..., k next to the appropriate vertices. For example, diagrams (a) and (b) below illustrate a 4-colouring and a 3-colouring of a graph G with five vertices; note that diagram (c) is 1lot a 3-colouring of G, since the two vertices coloured 2 are adjacent.



Since G has a 3-colouring, χ(G) < 3; thus 3 is an upper bound for χ(G). Also, G contains three mutually adjacent vertices (forming a triangle) that must be assigned different colours, so χ(G) > 3; thus 3 is a lower bound for χ(G). Combining these inequalities, we obtain χ(G) = 3.

If H is a subgraph of G, then $\chi(G) \geq \chi(H)$. For example, the odd cycle C_5 is a subgraph of Petersen graph, P, so $\chi(P) \geq \chi(C_5) \geq 3$. In fact $\chi(P) = 3$.

This is denoted by $\Delta(G)$ and is the **maximum degree** of any vertex in graph G.

The question that we are now interested in is how many k-colourings does a graph have. Let $\rho_G(k)$ be the number of k-colourings of G. Then $\chi(G)$ is the smallest k for which $\rho_G(k) \geq 1$, since this means that there is at least one $\chi(G)$-colouring of G, (and no k-colourings of G for $k < \chi(G)$ because $\rho_G(k) = 0$ for these k). So this gives another method of computing the chromatic number of a graph.

Example 4.36: Let H be the path on 3 vertices. Then, $\rho_H(1) = 0$, $\rho_H(2) = 2$, (so $\chi(H) = 2$), $\rho_H(3) = 12$, $\rho_H(4) = 36$, and so on.

So we have $\rho_H(k) = k(k-1)^2$.

Example 4.37:

1. Null graphs.

   If $G = N_n$ is the null graph with *n* vertices, then $\rho_G(k) = k^n$.

2. Trees.

   If $G = T_n$ is any tree with *n* vertices, then $\rho_G(k) = k(k-1)^{n-1}$.

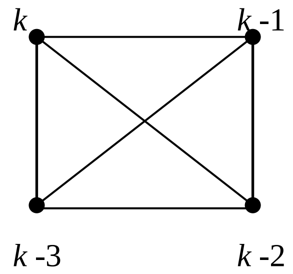
   For example, the graph *H* above is a T_3.

3. Complete graphs.

   If $G = K_n$ is the complete graph with *n* vertices, then

$$\rho_G(k) = k(k-1)(k-2)\cdots(k-n+1).$$

   For example, for the graph K_4 we have



$$\rho_{K_4}(k) = k(k-1)(k-2)(k-3).$$

Theorem 4.14 (*Brooks 1941*): If *G* is a simple connected graph which is not an odd cycle or a K_n, then $\chi(G) \le \Delta(G)$. (That is, *G* is $\Delta(G)$ -colourable.)

Remark 4.15: We have $\chi(C_{odd}) = 3 > 2 = \Delta(C_{odd})$ and we have $\chi(K_n) = n > \Delta(K_n) = n-1$, hence these two classes of graphs are excluded from the statement of Brook's theorem above.

Theorem 4.16: (4-Colour theorem) (*Appel and Haken 1976*).Every simple planar graph is 4-colourable. (i.e. $\chi(G) \le 4$ for any planar graph *G*.)

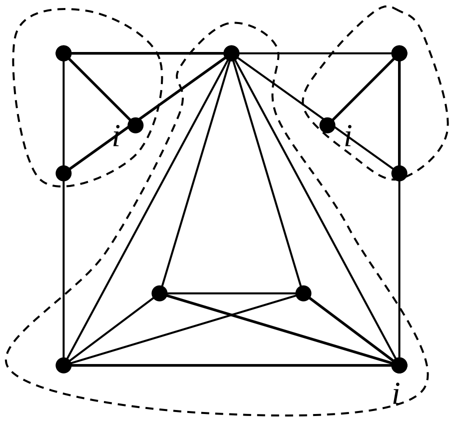
Definition 4.21: An ***independent** set* in a graph G is a set of vertices of G, no two of which are adjacent. The **independence number** of a graph G, denoted by $i(G)$, is the size of the largest independent set. It is useful to know $i(G)$ as it can be used to find the lower bound for $\chi(G)$ as

$$\left\lceil \frac{n}{i(G)} \right\rceil \leq \chi(G) \leq \Delta(G),$$

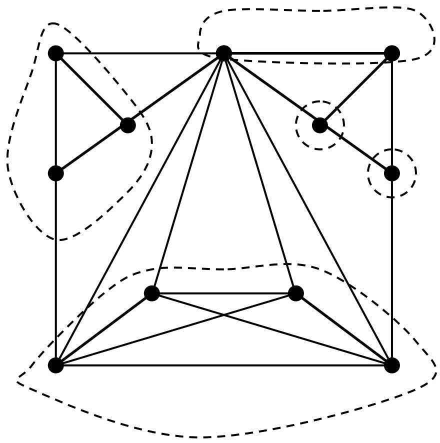
where n is the number of vertices of G and the symbol $\lceil a \rceil$ means the smallest integer greater than or equal to the real number a.

Definition 4.22: If G is an undirected graph, any subgraph of G that is a complete graph is called a ***clique*** in G. A ***clique-partition*** of G is a partition of all vertices of G into cliques. A graph can have many clique-partitions.

Example 4.38: for graph L below



3 parts clique-partition K_5, K_3, K_3



5 parts clique-partition K_4, K_3, K_2, K_1, K_1

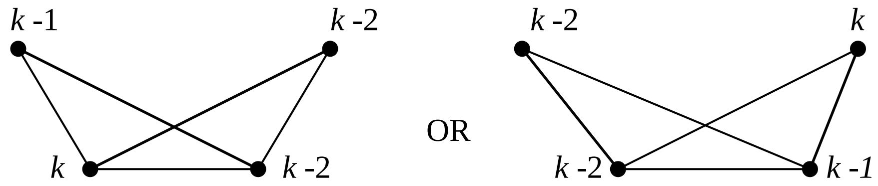
Given a clique-partition of a graph G with k parts, an independent set of G can have at most one vertex from each part, since vertices from the same part are adjacent. So $i(G) \leq k$. For L above, we have found a clique-partition with 3 parts, and one with 5 parts; so $i(L) \leq 3$. Now try to use the clique-partition with 3 parts to find an independent set of size 3. This is possible for L as the 3 vertices labelled i form an independent set of size 3, so $i(L) = 3$.

In the above example, we had equality between $i(L)$ and some k. This need not always be the case. For example the Petersen graph, P, has no K_3's, K_4'$s, \cdots$ as subgraphs; only K_2's (edges) and K_1's (vertices). It has 10 vertices, so every clique-partition with k parts must have $k \geq 5$. But $i(P) = 4 < k$. So clique-partitions cannot always be used to find the independence number M.

Labelling Procedure

Labelling K_4 as above illustrates the general procedure. As you label vertices with $k, k-1, \ldots$ you can label a new vertex with $k-i$ provided its labelled neighbours form an i-clique (a complete subgraph K_i).

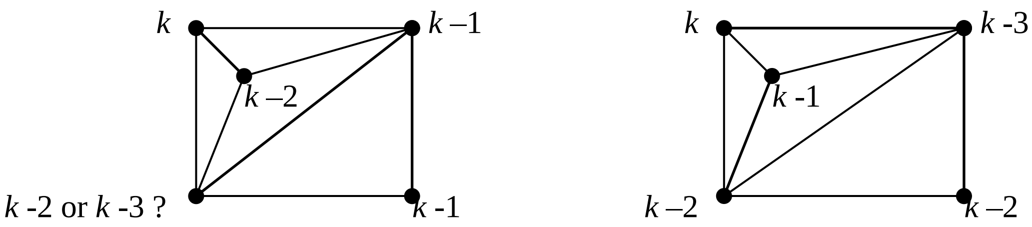
Example 4.39 Consider the graph J blow.



$$\rho_J(k) = k(k-1)(k-2)^2.$$

Sometimes, when labelling, you might reach a conflict, i.e. the labelled neighbours of a vertex you want to label do not form a clique. A different labelling may avoid this.

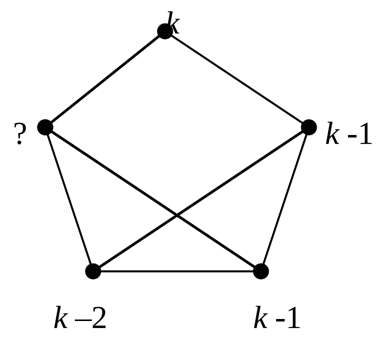
For example, the labelling on the left below reaches a conflict but the one on the right works.



$$\rho_J(k) = k(k-1)(k-2)^2(k-3)$$

However, there are some graphs that CANNOT be labelled in a conflict free manner. We need the *deletion-contraction* theorem to break such graphs into two or more graphs, each of which having a conflict free labelling.

The graph F below, for example, cannot be labelled in a conflict free manner.



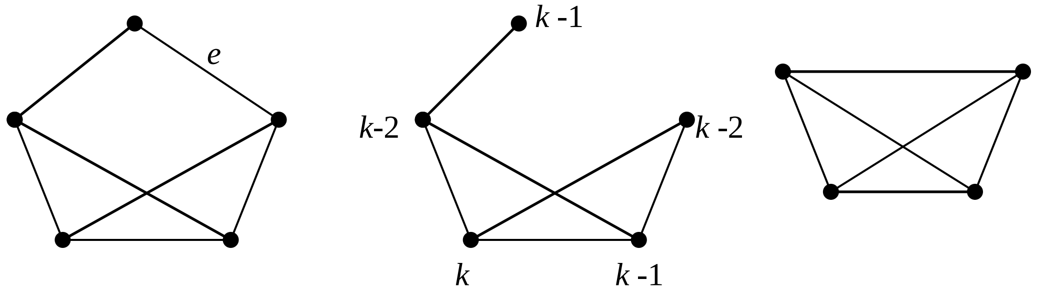
Deletion-Contraction Theorem

Recall that $G - e$ denotes the graph obtained from G by deleting edge e, and $G \setminus e$ denotes the graph obtained by contracting e.

Theorem 4.17: Let e be an edge in a graph G, then $\rho_G(k) = \rho_{G-e}(k) - \rho_{G\setminus e}(k)$.

This is a useful result as the two graphs on the right-hand side of the above equation have fewer edges than G, so we can use the Deletion-Contraction theorem repeatedly until we reach graphs for which we can compute the chromatic polynomial by a conflict free labelling, e.g. for null graphs, complete graphs or trees.

We can now find the chromatic polynomial of F by applying the above result.



F → $F - e$ → $F \setminus e$ (this is K_4)

$$\begin{aligned}
\rho_F(k) &= \rho_{F-e}(k) - \rho_{F\backslash e}(k) \\
&= k(k-1)^2(k-2)^2 - k(k-1)(k-2)(k-3) \\
&= k(k-1)(k-2)[(k-1)(k-2)-(k-3)] \\
&= k(k-1)(k-2)(k^2-4k+5).
\end{aligned}$$

Hence, $\rho_F(1) = \rho_F(2) = 0$, but $\rho_F(3) = 12$. So $\chi_F(k) = 3$ and F has 12 three-colourings. Note that the quadratic expression in $\rho_F(k)$ cannot be factorised (unless complex numbers are used), and that is why F could not be labelled in a conflict free manner.

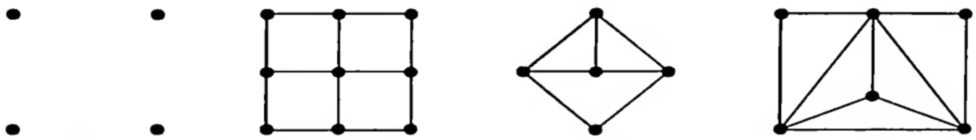
√ Check-List

Put a tick (√) mark if you can perform the task and a cross (x) mark if you can't in the box against the following questions.

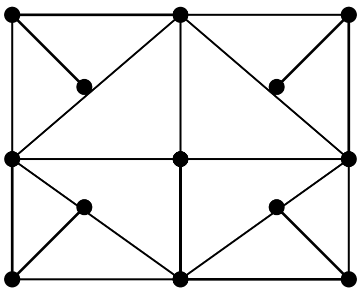
1. Can you define a k-colourable? ....
2. Can you state Brooks theorem? ....
3. Can you 4-colouring theorem? ....
4. Can you find the chromatic number of a given graph? ....

Exercises 4.8

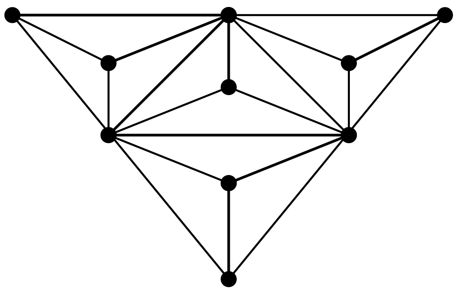
1. *Determine the chromatic number of each of the following graphs.*



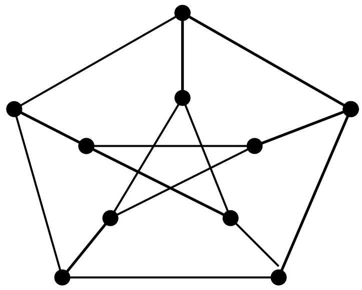
(a) (b) (c) (d)



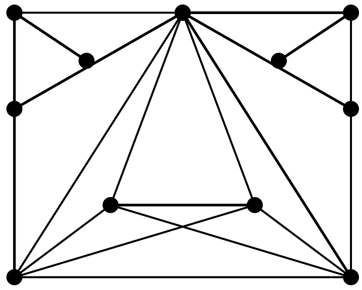
(e)



(f)



Petersen graph
(g)



(h)

2. *For the graphs in 1 above, find the independence number and the Max degree in each case and check that $\chi(G)$ obeys the bounds*

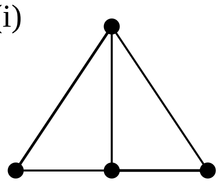
$$\left\lceil \frac{n}{i(G)} \right\rceil \le \chi(G) \le \Delta(G).$$

3. *Write down the chromatic number of each of the following graphs:*

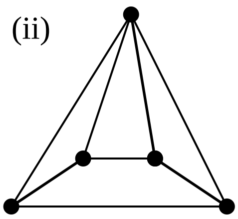
   a. *the complete graph K_n;*
   b. *the complete bipartite graph $K_{r,s}$;*
   c. *the cycle graph C_n $(n > 3)$;*
   d. *a tree.*

4. *Find the chromatic polynomials of the following graphs.*

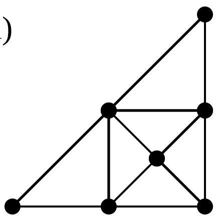
(i)



(ii)



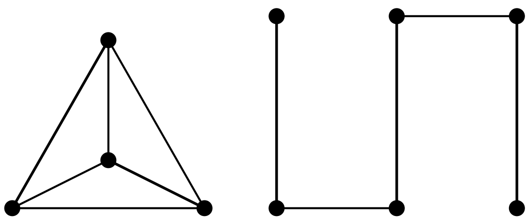
(iii)



5. *In how many ways can the graph in part (iii) above be coloured*

   (a) with 3 colours? (b) with 4 colours?

6. *Find the chromatic polynomial of the disconnected graph.*



(a) (b)

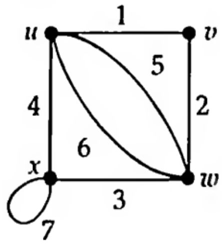
Chapter Summary

- A graph is a diagram consisting of points, called vertices, joined by lines, called edges; each edge joins exactly two vertices.
- In a graph, two or more edges joining the same pair of vertices are *multiple edges*.
- An edge joining a vertex to itself is a *loop*.
- A graph with no multiple edges or loops is a *simple graph*.
- Two vertices *adjacent* if they have an edge in common.
- Two edges are *adjacent* if they have a vertex in common.
- A *subgraph* of a graph G is a graph all of whose vertices are vertices of G and all of whose edges are edges of G.
- Handshaking Lemma states that in any graph, the sum of all the vertex degrees is equal to twice the number of edges.
- Two graphs are *isomorphic* if there is a one-one correspondence between their vertex sets.
- A *walk* in a graph *G* is a finite sequence of edges of the form $v_0v_1, v_1v_2, v_2v_3, \cdots, v_{m-1}v_m$ (also written $v_0v_1v_2v_3\cdots v_m$).
- The number of edges in the walk is called its *length*.
- A *trail* is a walk in which no edge has been traversed more than once (in either direction) but repeated vertices are allowed.
- A *path* is a walk in which all the edges and all the vertices are different.
- A graph is *connected* if there is a path between each pair of vertices, and is *disconnected* otherwise
- A *complete graph*, denoted by K_n, is the simple graph in which every pair of vertices is joined by an edge.
- A graph *G* is *bipartite* if every vertex can be labelled with either *a* or *b*, so that every edge is an *ab* edge.
- A connected graph *G* is *Eulerian* if there is a closed trail containing every edge of *G*. We call such a trail an *Eulerian trail*.
- A graph *G* is *Hamiltonian* if there is a cycle that passes through every vertex of *G*. Such a cycle is a *Hamiltonian cycle*.
- A *tree* is a connected graph without cycles.

- A *spanning tree* in a connected graph G is a subgraph of G that includes every vertex and is also a tree.
- A graph *G* is *planar* if it can be drawn in the plane without its edges crossing.
- A *minor* of *G* is a graph obtained from *G* by repeatedly deleting and/or contracting edges.
- *G* is *k-colourable* if we can colour each vertex from a set of *k* colours in such a way that for every edge its two end vertices have a different colour.The smallest *k* for which this is possible is called the *chromatic number* of *G*, denoted by $\chi(G)$.

Self Test Exercise 4

1. *Consider the graph G shown on the right. Which of the following statements hold for G?*



   a. *vertices v and x are adjacent;*
   b. *edge 6 is incident with vertex w;*
   c. *vertex x is incident with edge 4;*
   d. *vertex wand edges 5 and 6 form a subgraph of G.*

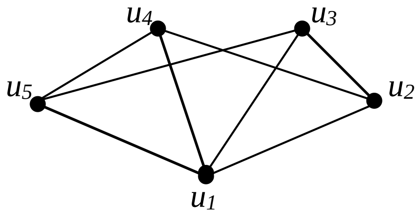
2. *Draw an example of each of the following, each with 5 vertices and 8 edges:*
   i. *a simple graph,*
   ii. *a non-simple graph with no loops,*
   iii. *a non-simple graph with no multiple edges.*

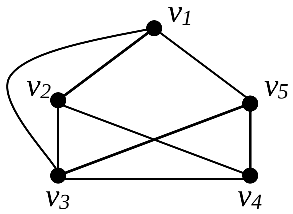
3. *Which of the following are possible degree sequences of simple graphs. Draw graphs for those that are.*

   (i) 1,2,2,3 *(ii) 1,2,3,4* *(iii) 0,0,1,1,2,2*

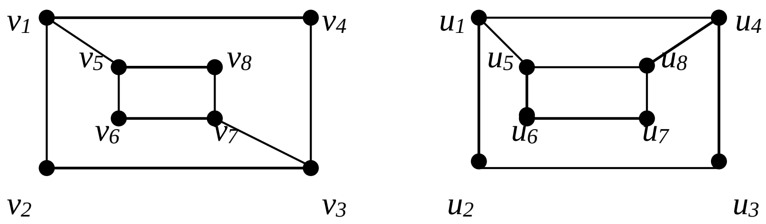
   (iv)1,2,3,3,4 *(v) 2,2,3,3,4,4* *(vi) 2,3,3,4,5,5.*

4. (a) *Show that the following two graphs are isomorphic:*

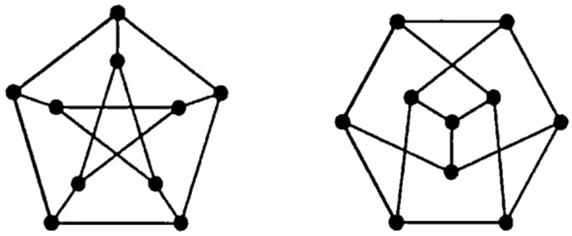




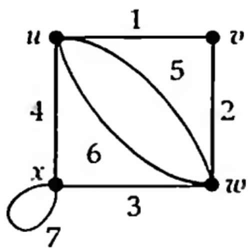
(b) *Give a reason why the following two graphs cannot be isomorphic:*



5. *By suitably labelling the vertices, show that the following graphs are isomorphic:*

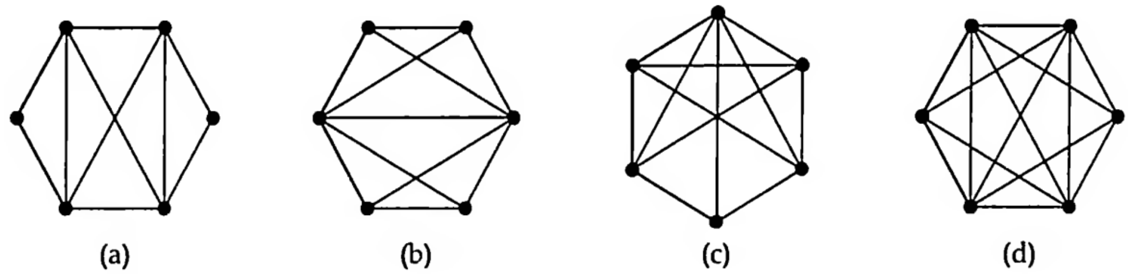


6. *Draw the eleven unlabelled simple graphs with four vertices.*
7. *(a) If two graphs have the same degree sequence, must they be isomorphic?*
   (b) If two graphs are isomorphic, must they have the same degree sequence?
8. *Let G be a graph with degree sequence (1, 2, 3, 4). Write down the number of vertices and number of edges of G, and construct such a graph. Are there any simple graphs with degree sequence (1, 2, 3, 4)?*
9. *Prove that, if G is a simple graph with at least two vertices, then G has two or more vertices of the same degree.*
10. *For the graph shown on the right, write down:*
    a. *a walk of length 7 between u and w;*
    b. *all the cycles of lengths 1,2,3 and 4;*
    c. *a path of maximum length.*



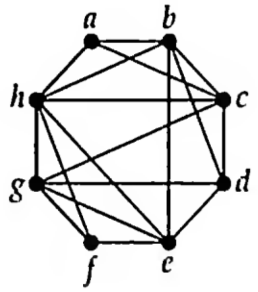
11. *Draw:*
    a. *two non-isomorphic regular graphs with 8 vertices and 12 edges;*
    b. *two non-isomorphic regular graphs with 10 vertices and 20 edges.*
12. *For which values of n, rand s are the following graphs Eulerian? For which values are they semi-Eulerian?*
    a. *the complete graph K_n;*
    b. *the complete bipartite graph $K_{r,s}$;*
    c. *the n-cube Q_n.*

13. *For which values of n, r and s are the graphs in Exercise 9 Hamiltonian? For which values are they semi-Hamiltonian?*
14. *Draw two graphs each with 10 vertices and 13 edges: one that is Eulerian but not Hamiltonian and one that is Hamiltonian but not Eulerian.*
15. *Decide which of the following graphs are planar.*



For each planar graph, give a plane drawing.

16. *By finding a plane drawing, show that the following graph is planar.*



17. *Let G be a planar graph with k components, and let n, m and f denote, respectively, the numbers of vertices, edges and faces in a plane drawing of G.*
    a. *Show that if each component has at least three vertices, then Euler's formula has the form*

$$n - m + f = k + 1.$$

    b. *Deduce that if G is simple and each vertex has degree at least 2, then*

$$m < 311 - 3(k + 1).$$

18. *Give an example of a connected planar graph G with 7 vertices such that its complement is also planar.*

REFERENCE

[1]. I. Anderson, *A First Course in Discrete Mathematics,*2002.

[2]. C. Berge, *Graphs and Hypergraphs*, North-Hoi Land Publishing Company-Amsterdam, 1976.

[3]. A. Bezdek, *Discrete Geometry*, Marcel Dekker, 2003.

[4]. N. L. Biggs, *Discrete mathematics*, Oxford University Press, 2002.

[5]. Eric W., "Discrete mathematics" from MathWorld.

[6]. S. Finch, *Bipartite, k-Colorable and k-Colored Graphs*, 2003.

[7]. Ralph P. Grimaldi, *Discrete and Combinatorial Mathematics: An Applied Introduction* 5th ed. Addison Wesley.

[8]. T. R. Hodkinson and John A. N. Parnell, *Reconstructing the Tree of Life: Taxonomy and systematics of species rich taxa*, CRC Press, 2007.

[9]. B. Hopkins, *Resources for Teaching Discrete Mathematics*, Mathematical Association of America, 2008.

[10]. R. Johnsonbaugh, *Discrete Mathematics*, New Jersey, 2005.

[11]. R. Johnsonbaugh, *Discrete Mathematics*, Prentice Hall, 2008.

[12]. S. G. Krantz, *Discrete Mathematics Demystified*, McGraw-Hill, 2009.

[13]. B. Mohar and C. Thomassen, *Graphs on Surfaces*, Johns Hopkins University press, 2001.

[14]. K.H. Rosen &J.G. Michaels, *Handbook of Discrete and Combinatorial Mathematics*, CRC Press, 2000.

[15]. K. H. Rosen, *Discrete Mathematics and Its Applications* 6th ed. McGraw Hill.

[16]. R. J. Wilson, *Introduction to Graph Theory*, Longman, 1996.

[17]. D. B. West, *Introduction to Graph Theory*, 2005.

CHAPTER 5

DIRECTED GRAPHS

Objectives: After studying this chapter, you should be able to:

- Understand the terms digraph, labelled digraph, unlabelled digraph, vertex, arc, adjacent, incident, multiple arcs, loop, simple digraph, underlying graph and subdigraph;
- Appreciate the uses of rooted trees in different areas;
- determine whether two given digraphs are isomorphic;
- Understand the terms in-degree, out-degree, in-degree sequence and out-degree sequence.
- state and use the handshaking dilemma;
- realize walk, trail, path, closed walk, closed trail, cycle, connected, disconn-ected and strongly connected in the context of digraphs;
- Know the terms Eulerian digraph and Eulerian trail;
- State a necessary and sufficient condition for a connected digraph to be Eulerian;
- Know the terms Hamiltonian digraph and Hamiltonian Cycle;
- describe the use of digraphs in ecology, social networks, the rotating drum problem, and ranking in tournaments.

Introduction

In this chapter we discuss digraphs and their properties. Our treatment of the subject is similar to that of Chapters 4 for graphs, except that we need to take account of the directions of the arcs.

5.1. Basic Terminologies

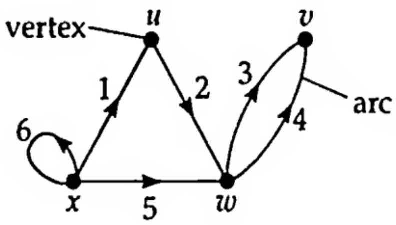
Objectives: After studying this section, you should be able to:

- define the terms digraph, labelled digraph, unlabelled digraph, vertex, arc, adjacent, incident, multiple arcs, loop, simple digraph, underlying graph and subdigraph;
- determine whether two given digraphs are isomorphic;
- explain the terms in-degree, out-degree, in-degree sequence and out-degree sequence
- state and use the handshaking dilemma;

Definition 5.1: A ***directed graph (digraph)*** consists of a set of elements called vertices and a set of elements called arcs. Each arc joins two vertices in a specified direction.

Example 5.1: the digraph shown below has four vertices $\{u, v, w, x\}$ and six arcs $\{1, 2, 3, 4, 5, 6\}$. Arc 1 joins x to u, arc 2 joins u to w, arcs 3 and 4 join w to v,

arc 5 joins x to w, and arc 6 joins the vertex x to itself.

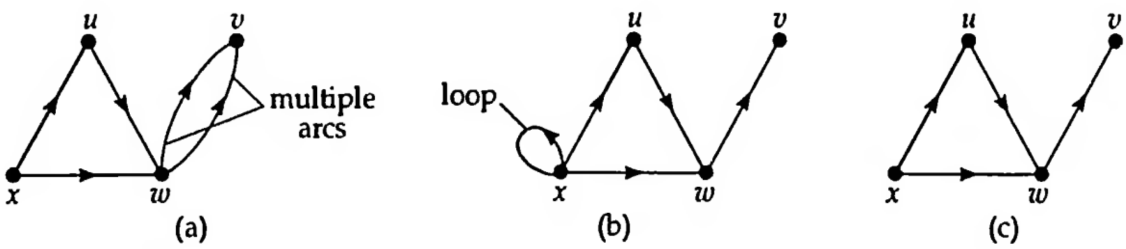


We often denote an arc by specifying its two vertices in order; for example, arc 1 is denoted by xu, arcs 3 and 4 are denoted by wv, and arc 6 is denoted by xx. Note that xu is not the same as ux.

The above digraph contains more than one arc joining w to v, and an arc joining the vertex x to itself. The following terminology is useful when discussing such digraphs.

Definition 5.2: In a digraph, two or more arcs joining the same pair of vertices in the same direction are ***multiple arcs***. An arc joining a vertex to itself is a ***loop***. A digraph with no multiple arcs or loops is a ***simple digraph***.

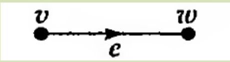
Example 5.2: digraph (a) below has multiple arcs and digraph (b) has a loop, so neither is a simple digraph. Digraph (c) has no multiple arcs or loops, and is therefore a simple digraph.



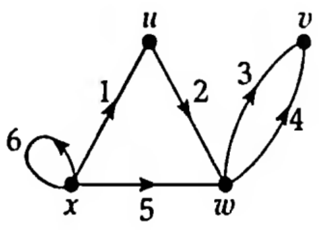
Adjacency and Incidence

The digraph analogues of adjacency and incidence are similar to the corresponding definitions for graphs, except that we take account of the directions of the arcs.

Definition 5.3: The vertices v and w of a digraph are ***adjacent*** vertices if they are joined (in either direction) by an arc e. An arc e that joins v to w is ***incident from v*** and ***incident to*** w; v is ***incident to e***, and w is ***incident from*** e.



Example 5.3: in the digraph below, the vertices u and x are adjacent, vertex w is incident from arcs 2 and 5 and incident to arcs 3 and 4, and arc 6 is incident to (and from) the vertex x.



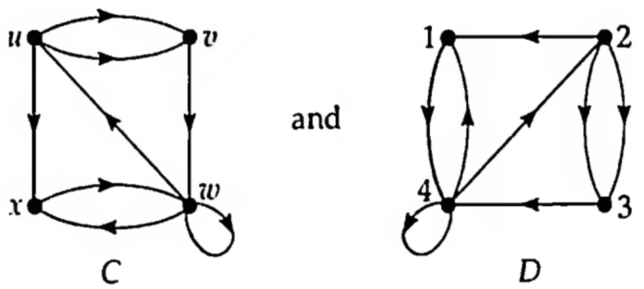
Isomorphism

It follows from the definition that a digraph is completely determined when we know its vertices and arcs, and that two digraphs are the same if they have the same vertices and arcs. Once we know the vertices and arcs, we can draw the digraph and, in principle, any picture we draw is as good as any other; the actual way in which the vertices and arcs are drawn is irrelevant - although some pictures are easier to use than others!

We extend the concept of isomorphism to digraphs, as follows.

Definition 5.4: Two digraphs C and D are ***isomorphic*** if D can be obtained by relabeling the vertices of C - that is, if there is a one-one correspondence between the vertices of C and those of D, such that the arcs joining each pair of vertices in C agree in both number and direction with the arcs joining the corresponding pair of vertices in D.

Example 5.4: the digraphs C and D represented by the diagrams



are not the same, but they are isomorphic, since we can relabel the vertices in the digraph C to get the digraph D, using the following one-one correspondence:

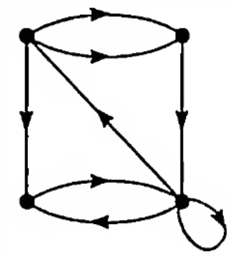
C	$\leftrightarrow$	D
u	$\leftrightarrow$	2
v	$\leftrightarrow$	3
w	$\leftrightarrow$	4
x	$\leftrightarrow$	1

Note that arcs in C correspond to arcs in D - for example:

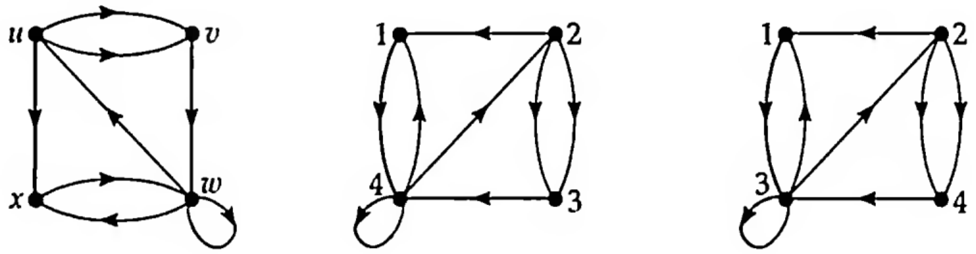
the two arcs from u to v in C correspond to the two arcs from 2 to 3 in D; the arcs $wx$and xw in C correspond to the arcs 41 and 14 in D; the loop ww in C corresponds to the loop 44 in D.

Sometimes it is unnecessary to have labels on the digraphs. In such cases, we omit the labels, and refer to the resulting object as an ll1l1abelled digraph.

Example 5.5: The unlabelled digraph



corresponds to any of the following isomorphic digraphs:



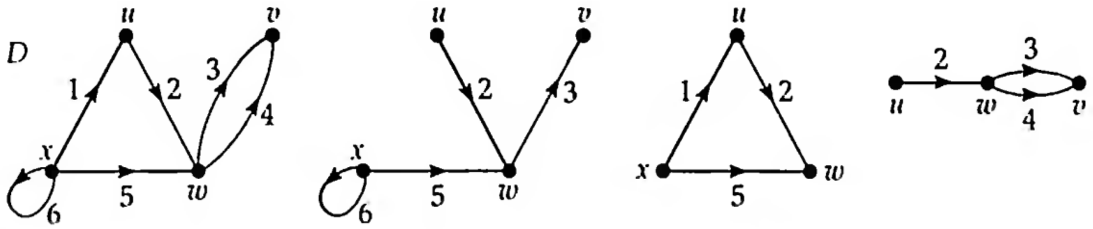
We say that two unlabelled digraphs are ***isomorphic*** if labels can be attached to their vertices so that they become the same digraph.

It is convenient to define a concept analogous to that of a subgraph of a graph.

Definition 5.5: A ***subdigraph*** of a digraph D is a digraph all of whose vertices are vertices of D and all of whose arcs are arcs of D.

Remark 5.1: Note that D is a subdigraph of itself.

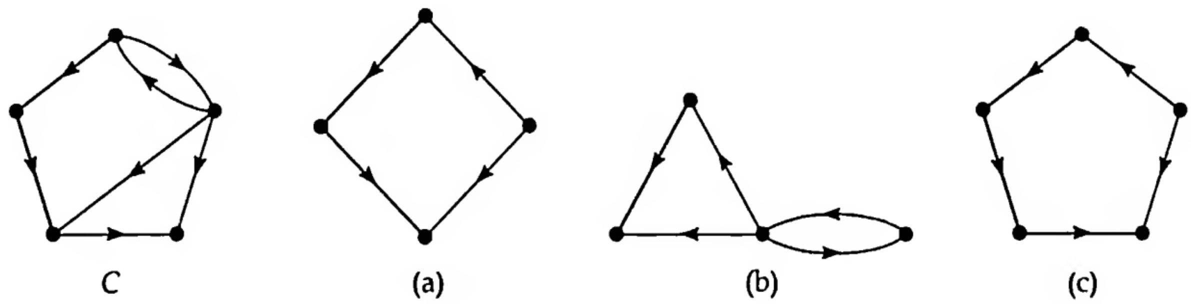
Example 5.6: The following digraphs are all subdigraphs of the digraph D on the left, with vertices $\{u, v, w, x\}$ and arcs $\{1, 2, 3, 4, 5, 6\}$.



vertices:	u,v,w,x	u,v,w,x	u,w,x	u,v,w
edges:	1,2,3,4,5,6	2,3,5,6	1,2,5	2,3,4

The idea of a subdigraph can be extended to unlabelled digraphs.

Example 5.7: The following digraphs are all subdigraphs of the unlabelled digraph C on the left:

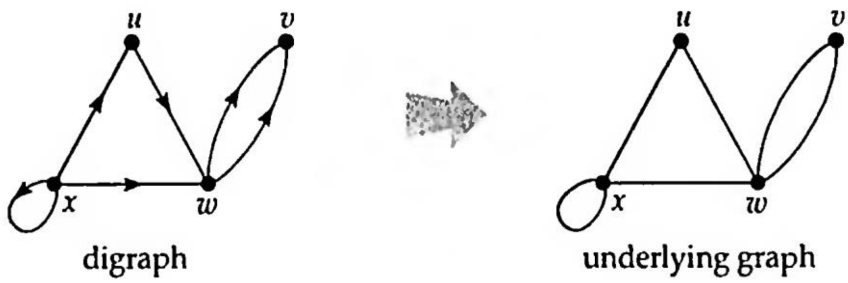


It is also convenient to introduce the idea of the *underlying graph* of a digraph.

Definition 5.6: The ***underlying graph*** of a digraph D is the graph obtained by replacing each arc of D by the corresponding undirected edge.

To obtain the underlying graph, we simply remove the arrows from the arcs.

Example 5.8:

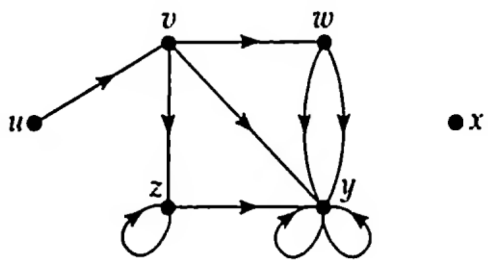


We now give analogues of the degree of a vertex in a graph.

Definition 5.7: In a digraph, the ***out-degree*** of a vertex v is the number of arcs incident from v, and is denoted by *outdeg* $\boldsymbol{v}$; the ***in-degree*** of v is the number of arcs incident to v, and is denoted by *indeg* v.

Remark 5.2: Each loop contributes 1 to both the in-degree and the out-degree of the corresponding vertex.

Example 5.9: The digraph below has the following out-degrees and in-degrees:



outdeg u = 1	outdeg v = 3	outdeg w = 2
indeg u = 0	indeg v = 1	indeg w = 1
outdeg x = 0	outdeg y = 2	outdeg z = 2
indeg x = 0	indeg y = 6	indeg z = 2

There are also analogues of the degree sequence of a graph, corresponding to the out-degree and in-degree of a vertex.

Definition 5.8: The ***out-degree sequence*** of a digraph D is the sequence obtained by listing the out-degrees of D in increasing order, with repeats as necessary. The ***in-degree sequence*** of D is defined analogously.

Example 5.10: The above digraph has out-degree sequence (0, 1,2,2,2,3) and in-degree sequence (0, 0, 1, 1, 2, 6).

Handshaking Dilemma

In the solution to Problem 4.10, you should have noticed that the sum of the out-degrees and the SUI11 of the in-degrees of each digraph are both equal to the number of arcs. A corresponding result holds for any digraph; we call it the handshaking dilemma!

Theorem 5.3(*Handshaking Dilemma*): In any digraph, the sum of all the out-degrees and the sum of all the in-degrees are both equal to the number of arcs.

***Proof*:** In any digraph, each arc has two ends, so it contributes exactly 1 to the sum of the out-degrees and exactly 1 to the sum of the in-degrees. The result follows immediately.

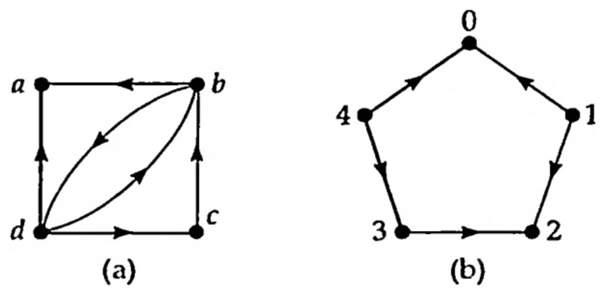
√ Check-List

Put a tick (√) mark if you can perform the task and a cross (x) mark if you can't in the box against the following questions.

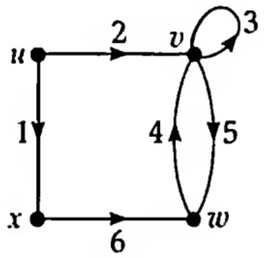
1. Can you define a digraph? .... ☐
2. Can you state Handshaking dilemma? .... ☐
3. Can you define isomorphic digraphs? .... ☐
4. Can you distinguish the difference between in-degree and out-degree? .... ☐

Exercise 5.1

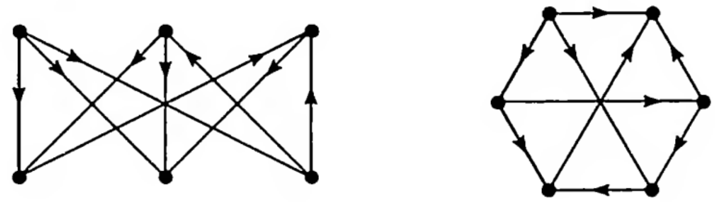
1. *Write down the vertices and arcs of each of the following digraphs. Are these digraphs simple digraphs?*



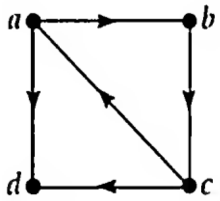
2. *Which of the following statements hold for the digraph on the right?*
   a. *vertices v and w are adjacent;*
   b. *vertices v and x are adjacent;*
   c. *vertex u is incident to arc 2;*
   d. *arc 5 is incident from vertex v.*



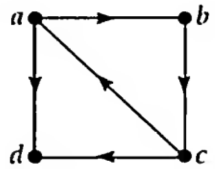
3. *Draw the digraphs whose vertices and arcs are as follows. Are these digraphs simple digraphs?*
   a. *vertices:* $\{u, v, w, x\}$ *arcs:* $\{vw, wu, wv, wx, xu\}$
   b. *vertices:* $\{l, 2, 3, 4, 5,6,7, 8\}$ *arcs:* $\{l2, 22,23,34,35,67,68, 78\}$
4. *By suitably labelling the vertices, show that the following unlabelled digraphs are isomorphic:*



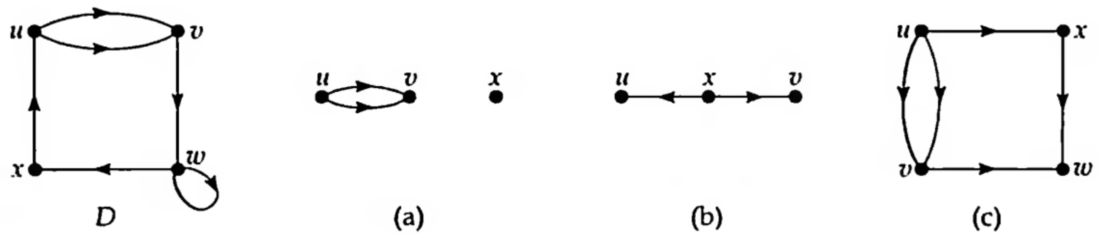
5. *By suitably relabelling the vertices, show that the following digraphs are isomorphic:*



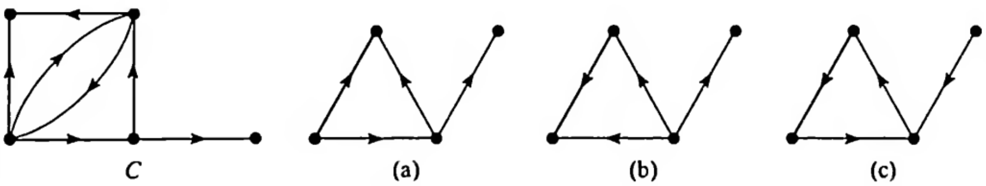
6. *Are the following two digraphs isomorphic? If so, find a suitable one-one correspondence between the vertices of the first and those of the second; if not, explain why no such one-one correspondence exists.*



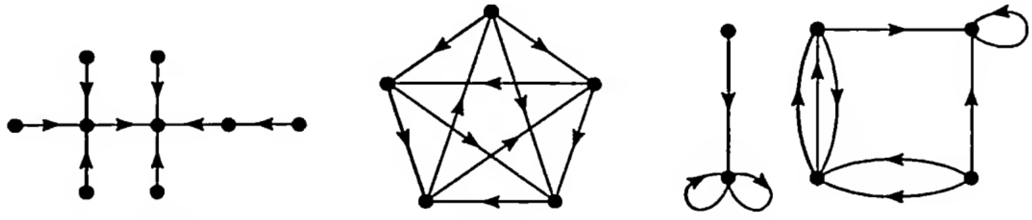
7. *Which of the following digraphs are subdigraphs of the digraph D below?*



8. *Which of the following digraphs are subdigraphs of the digraph C below?*



9. *Write down the out-degree and in-degree sequences of each of the following digraphs:*



(a) (b) (c)

10. *For each of the digraphs in Problem 9, write down:*
    a. *the number of arcs;*
    b. *the sum of the out-degrees of all the vertices;*
    c. *the sum of the in-degrees of all the vertices.*
    d. *What is the connection between your answers? Can you explain why this connection arises?*
11. *Use the handshaking dilemma to prove that, in any digraph, if the number of vertices with odd out-degree is odd, then the number of vertices with odd in-degree is odd.*

5.2. Paths and Connectivity

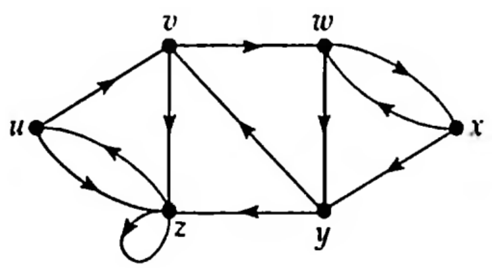
Objectives: After studying this section, you should be able to:

- Explain the terms walk, trail, path, closed walk, closed trail, cycle, connected, disconnected and strongly connected in the context of digraphs;
- explain the terms Eulerian digraph and Eulerian trail;
- state a necessary and sufficient condition for a connected digraph to be Eulerian;
- explain the terms Hamiltonian digraph and Hamiltonian Cycle;

Just as you may be able to get from one vertex of a graph to another by tracing the edges of a walk, trail or path, so you may be able to get from one vertex of a digraph to another by tracing the arcs of a 'directed' walk, trail or path. This means that you have to follow the directions of the arcs as you go, just as if you were driving around a one-way street system in a town. We make this idea precise, as follows.

Definition 5.9: A ***walk*** of length k in a digraph is a succession of k arcs of the form uv, vw, wx, ..., yz. This walk is denoted by $uvwx \dots yz$, and is referred to as a walk from u to z. A ***trail*** is a walk in which all the arcs, but not necessarily all the vertices, are different. A ***path*** is a walk in which all the arcs and all the vertices are different.

Example 5.11: In the following diagram, the walk $vwxyvwyzzu$ is a walk of length 9 from v to u, which includes the arc vw twice and the vertices v, w, y and z twice. The walk $uvwyvz$ is a trail which is not a path, since the vertex v occurs twice, whereas the walk $vwxyz$ has no repeated vertices and is therefore a path.



The terms closed walk, closed trail and cycle also apply to digraphs.

Definition 5.10: A closed walk in a digraph is a succession of arcs of the form

$$uv, vw, wx, \dots, yz, zu.$$

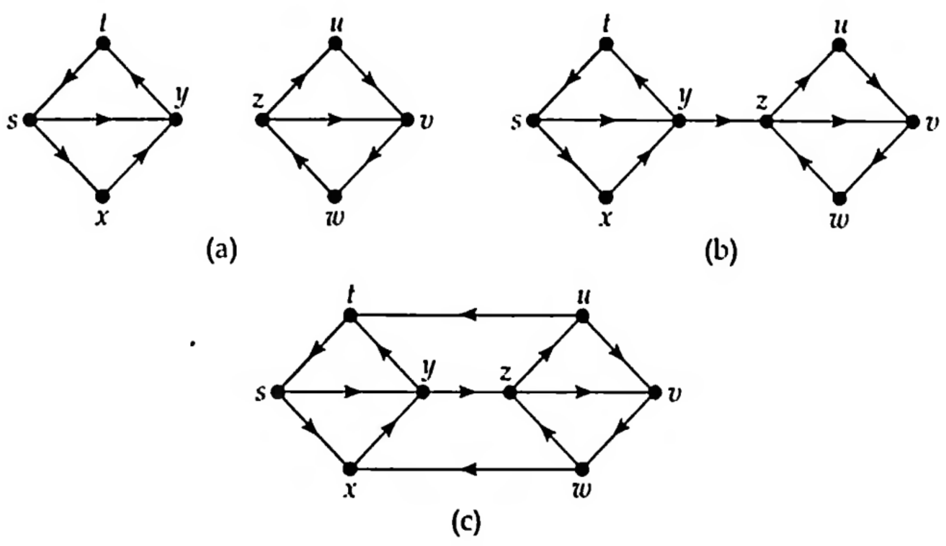
A ***closed*** trail is a closed walk in which all the arcs are different. A ***cycle*** is a closed trail in which all the intermediate vertices are different.

In the digraph above, the closed walk $uvwyvzu$ is a closed trail which is not a cycle (since the vertex v occurs twice), whereas the closed trails zz, wxw, $vwxyv$ and $uvwxyzu$ are all cycles. In describing closed walks, we can allow any vertex to be the starting vertex. For example, the triangle $vwyv$ can also be written as $wyvw$ or $yvwy$.

As with graphs, we can use the concept of a path tell us whether or not a digraph is connected. Recall that a graph is connected if it is 'in one piece', and this means that there is a path between each pair of vertices. For digraphs these two ideas are not the same, and this leads to two different definitions of the word connected for digraphs.

Definition 5.11: A digraph is ***connected*** if its underlying graph is a connected graph, and is ***disconnected*** otherwise. A digraph is ***strongly connected*** if there is a path between each pair of vertices.

Example 5.12: These three types of digraph are illustrated below:



Digraph (a) is disconnected, since its underlying graph is a disconnected graph. Digraph (b) is connected but is not strongly connected since, for example, there is no path from z to y. Digraph (c) is strongly connected, since there are paths joining all pairs of vertices.

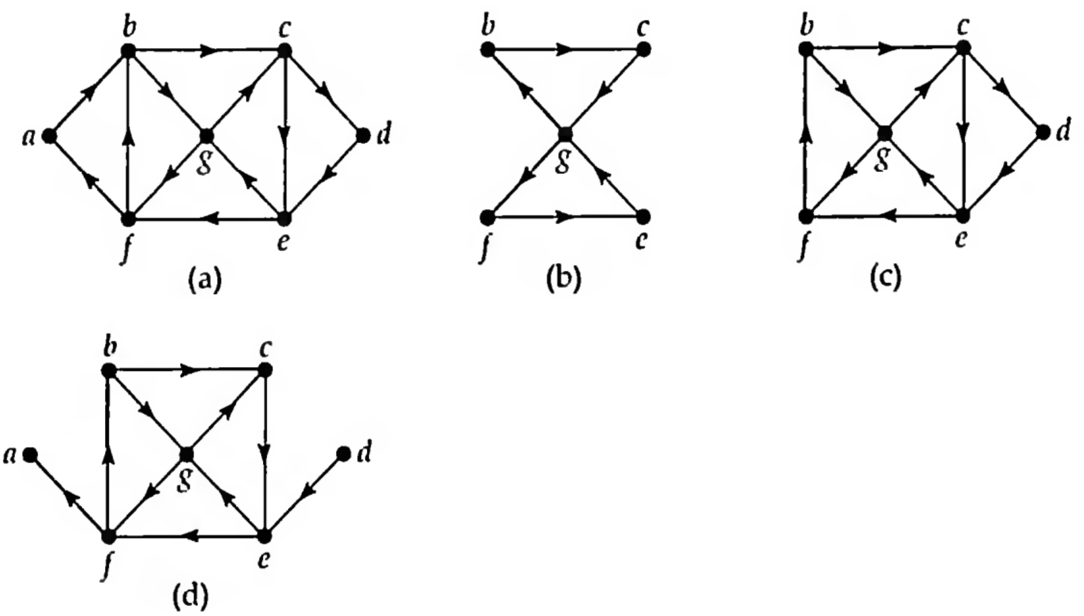
Alternatively, you can think of driving around a one-way street system in a town. If the town is strongly connected, then you can drive from any part of the town to any other, following the directions of the one-way streets as you go; if the town is merely connected, then you can still drive from any part of the town to any other, but you may have to ignore the directions of the one-way streets!

Eulerian and Hamiltonian Digraphs

In Chapter 4, we discussed the problem of finding a route that includes every edge or every vertex of a graph exactly once, and it is natural to consider the corresponding problem for digraphs. This leads to the following definitions.

Definition 5.12: A connected digraph is ***Eulerian*** if it contains a closed trail that includes every arc; such a trail is an ***Eulerian trail***. A connected digraph is ***Hamiltonian*** if it contains a cycle that includes every vertex; such a cycle is a ***Hamiltonian cycle***.

Example 5.13: Consider the following four digraphs:



(a) (b) (c) (d)

digraph (a) is Eulerian - an Eulerian trail is $abcdefgcegfa$ and Hamiltonian - a Hamiltonian cycle is $abcdegfa$;

digraph (b) is Eulerian - an Eulerian trail is $bcgfegb$ it is not Han1iltonian;

digraph (c) is Hamiltonian - a Hamiltonian cycle is $bcd\ e\ g\ f\ b$ it is not Eulerian;

digraph (d) is neither Eulerian nor Hamiltonian.

Much of the earlier discussion of Eulerian and Hamiltonian graphs can be adapted to Eulerian and Hamiltonian digraphs. In particular, there is an analogue of Theorem 3.2. We ask you to discover this analogue in the following problem.

Activity 5.1:

a. Guess a necessary and sufficient condition for a digraph to be Eulerian, involving the in-degree and out-degree of each vertex.

b. Use the condition obtained in part (a) to check which of the digraphs in are Eulerian.

Theorem 5.4: A connected digraph is Eulerian if and only if, for each vertex, the out-degree equals the in-degree.

Theorem 5.5: An Eulerian digraph can be split into cycles, no two of which have an arc in common.

The proofs of these theoren1s are similar to those of Theorems 4.4 and 4.5. In the sufficiency part of the proof of Theorem 5.4, the basic idea is to show that the digraph contains a (directed) cycle, and then to build up the required Eulerian trail from cycles step by step, as in the proof of Theorem 4.4. We omit the details. There is an analogue of Ore's theoren1 for Hamiltonian graphs, but it is harder to state and prove than the theorem for graphs, so we omit it.

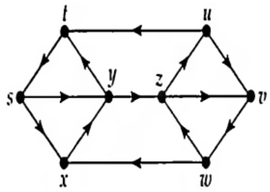
√ Check-List

Put a tick (√) mark if you can perform the task and a cross (x) mark if you can’t in the box against the following questions.

1. Can you define a walk in a digraph? .... ☐
2. Can you define Eulerian digraph and Eulerian trial? .... ☐
3. Can you define Hamiltonian and Hamiltonian cycle? .... ☐

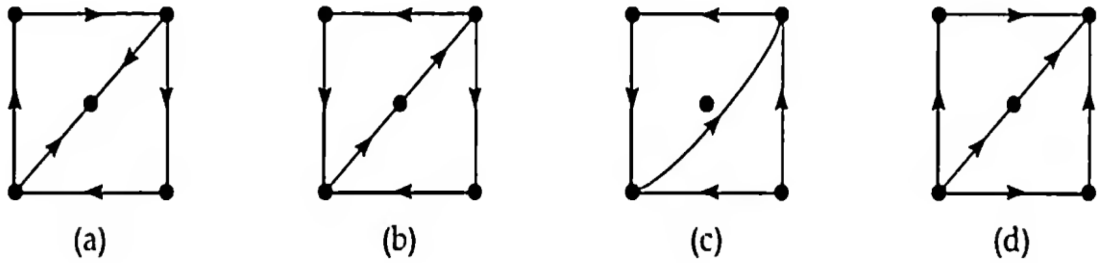
Exercise 5.2

19. *For the digraph on the right, write down:*

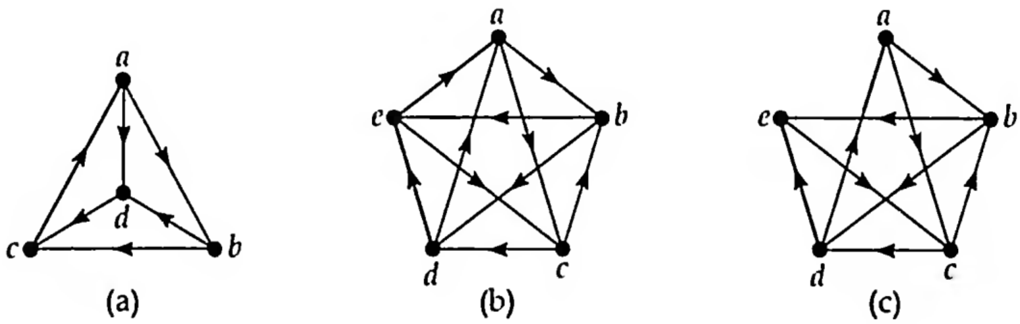


a. *all the paths from t to w;*
b. *all the paths from w to t;*
c. *a closed trail of length 8 containing t and z.*
d. *all the cycles containing both t and w.*

20. *Classify each of the following digraphs as disconnected, connected but not strongly connected, or strongly connected:*



21. *Decide which of the following digraphs are Eulerian and/or Hamiltonian, and write down an Eulerian trail or Hamiltonian cycle where possible.*

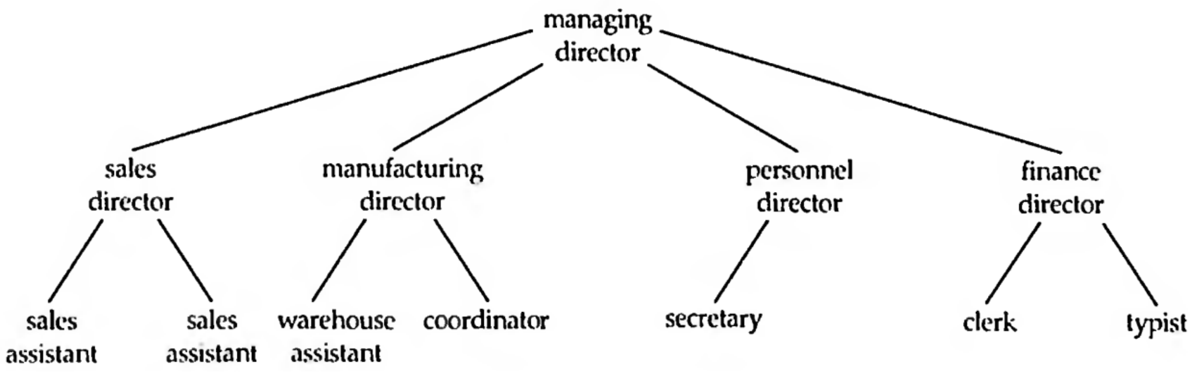


5.3. Rooted Trees

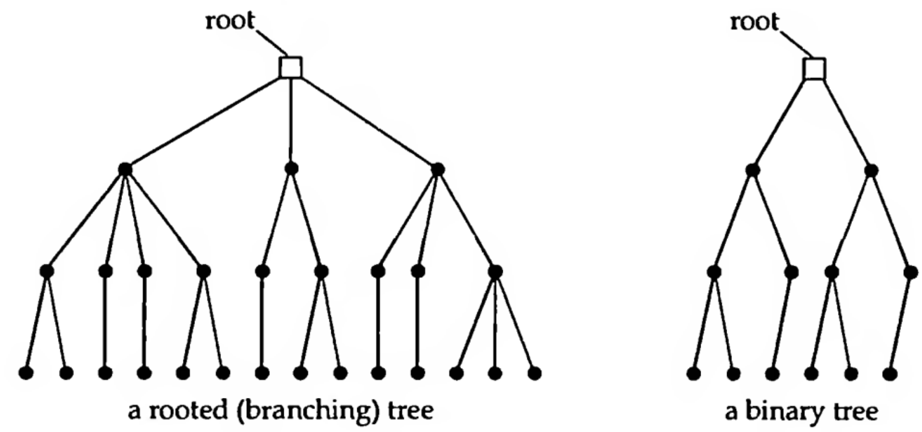
Objectives: After studying this section, you should be able to:

- distinguish between physical and conceptual tree structures, and give examples of each type;
- appreciate the uses of rooted trees in different areas;
- construct the bipartite graph representation of a given braced rectangular framework and use it to determine whether the system is rigid; if so, determine whether the system is minimally braced.

Among the examples of tree structures, one particular type of tree occurs repeatedly. This is the hierarchical structure in which one vertex is singled out as the starting point, and the branches fan out from this vertex. We call such trees ***rooted trees***, and refer to the starting vertex as the ***root***. For example, the tree representing the lines of responsibility of a company is a rooted tree, with the managing director as the root.



A rooted tree is often drawn as follows, with the root indicated by a small square at the top, and the various branches descending from it. When a path from the top reaches a vertex, it may split into several new branches. Although a top-to-bottom direction is often implied, we usually draw a rooted tree as a graph with undirected edges, rather than as a digraph with arcs directed downwards. A rooted tree in which there are at most *two* descending branches at any vertex is a ***binary tree***.

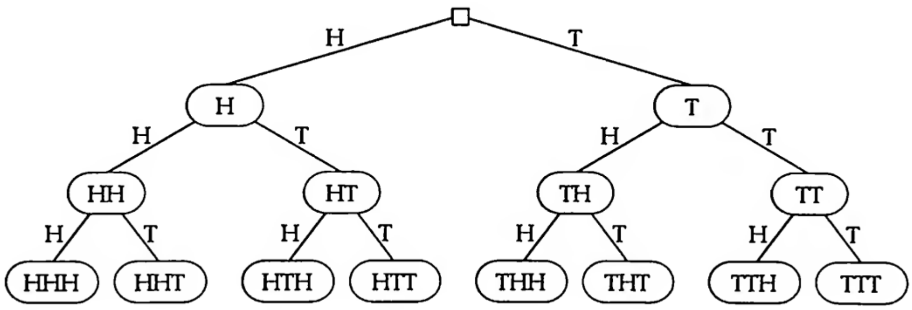


Such trees are often called branching trees. We have already seen two instances of branching trees - the family tree and the hierarchical tree. There are many further examples, as we now show.

Outcomes of Experiments

If we toss a coin or throw a die several times, then the possible outcomes can be represented by a branching tree. In the case of tossing a coin, each possible outcome has two edges leading from it, since the next toss may be a head (H) or a tail (T), and we obtain a binary tree.

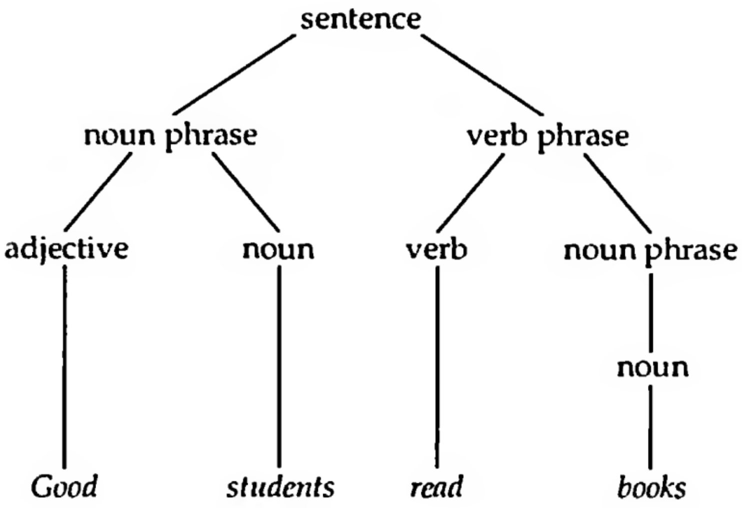
Example 5.13: if we toss a coin three times, then there are eight possible outcomes, and we obtain the following branching tree.



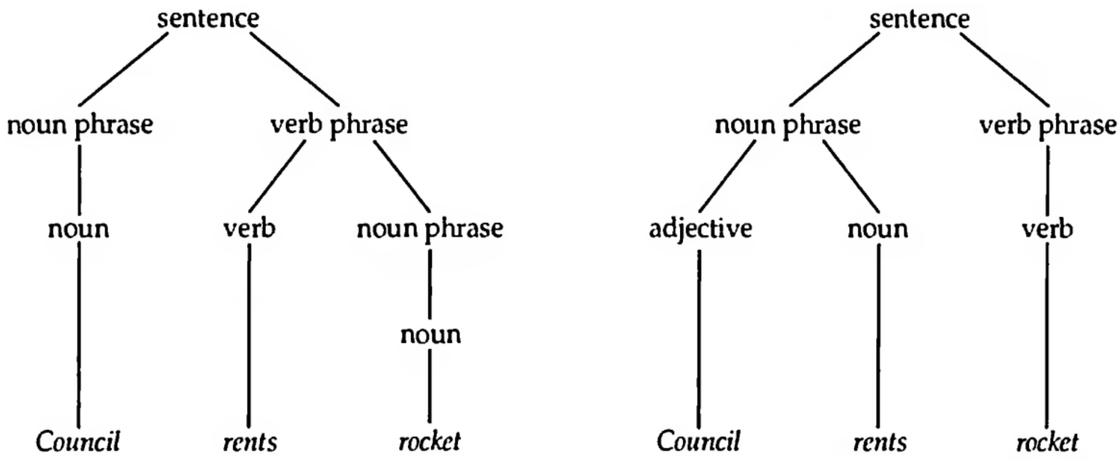
Grammatical Trees

Branching trees occur in the parsing of a sentence in a natural language, such as English. The tree represents the interrelationships between the words and phrases of the sentence, and hence the underlying syntactic structure. Such a branching tree is obtained by splitting the sentence into noun phrases and verb phrases, then splitting these phrases into nouns, verbs, adjectives, and so on.

Example 5.14: The structure of the sentence Good students read books can be represented by the following tree.



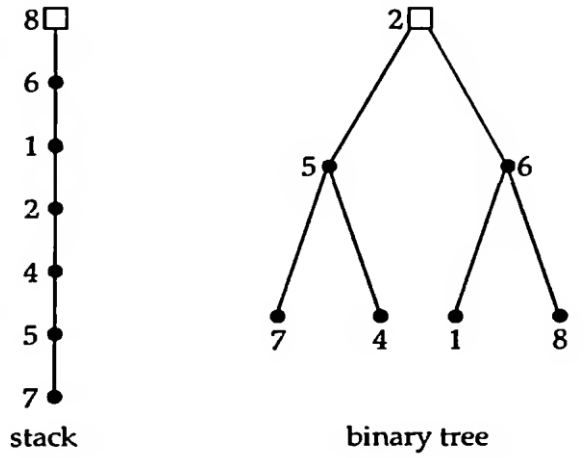
If a sentence is ambiguous, we can use branching trees to distinguish between the different sentences constructions. For example, the newspaper headline Council rents rocket can be interpreted in two ways, as illustrated by the following trees.



Computer Science

Rooted tree structures arise in computer science, where they are used to model and describe branching procedures in programming languages (the languages used to write algorithms to be interpreted by computers). In particular, they are used to store data in a computer's memory in many different ways.

Example 5.15: Consider the list of seven numbers 7, 5, 4, 2, I, 6, 8. The following trees represent ways of storing this list in the memory - as a stack and as a binary tree. Each representation has its advantages, depending on how the data is to be manipulated, but in both representations it is important to distinguish where the data starts, so the trees are rooted trees.



We obtain the tree by writing the numbers in a string 7542168, 'promoting' every second number (5, 2, 6) and then 'promoting' the new second number (2).

√ Check-List

Put a tick (√) mark if you can perform the task and a cross (x) mark if you can't in the box against the following questions.

1. Can you define a rooted tree? .... ☐

Exercise 5.3

1. *Draw the branching tree representing the outcomes of two throws of a six-sided die.*
2. *The ambiguous sentence **Help rape victims** appeared as a newspaper headline l and can be interpreted in two ways. Draw two tree structures that correspond to this sentence.*

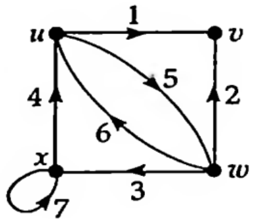
Chapter Summary

- A *digraph* consists of a set of elements called vertices and a set of elements called arcs.
- In a digraph, two or more arcs joining the same pair of vertices in the same direction are *multiple arcs*.

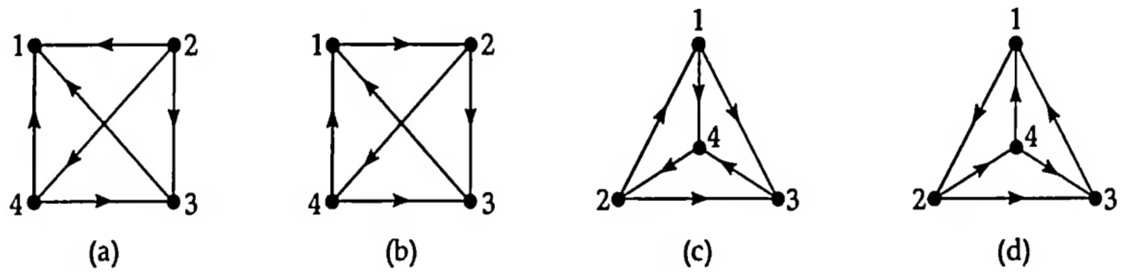
- An arc joining a vertex to itself is a *loop*.
- A digraph with no multiple arcs or loops is a *simple digraph.*
- The vertices v and w of a digraph are *adjacent* vertices if they are joined (in either direction) by an arc e.
- An arc e that joins v to w is *incident from* v and *incident to* w; v is *incident to* e, and w is *incident from* e.
- Two digraphs C and D are *isomorphic* if D can be obtained by relabeling the vertices of C - that is, if there is a one-one correspondence between the vertices of C and those of D.
- A *subdigraph* of a digraph D is a digraph all of whose vertices are vertices of D and all of whose arcs are arcs of D.
- In a digraph, the *out-degree* of a vertex v is the number of arcs incident from v, and is denoted by *outdeg* v;
- The *in-degree* of v is the number of arcs incident to v, and is denoted by *indeg* v.
- *Handshaking Dilemma* states that in any digraph, the sum of all the out-degrees and the sum of all the in-degrees are both equal to the number of arcs.
- A *walk* of length k in a digraph is a succession of k arcs of the form uv, vw, wx, ..., yz. This walk is denoted by $uvwx \dots yz$, and is referred to as a walk from u to z.
- A *trail* is a walk in which all the arcs, but not necessarily all the vertices, are different.
- A *path* is a walk in which all the arcs and all the vertices are different.
- A *closed* trail is a closed walk in which all the arcs are different.
- A *cycle* is a closed trail in which all the intermediate vertices are different.
- A connected digraph is *Eulerian* if it contains a closed trail that includes every arc; such a trail is an *Eulerian trail*.
- A connected digraph is *Hamiltonian* if it contains a cycle that includes every vertex; such a cycle is a *Hamiltonian cycle*.
- The hierarchical structure in which one vertex is singled out as the starting point, and the branches fan out from this vertex is called *rooted trees*.

Self Test Exercise 5

1. *Consider the digraph D shown on the right. Which of the following statements hold for D?*
   a. *vertices u and x are adjacent;*
   b. *arc 2 is incident to vertex w;*
   c. *vertex x is incident from arc 3;*
   d. *vertex x and arc 7 form a subdigraph of D.*

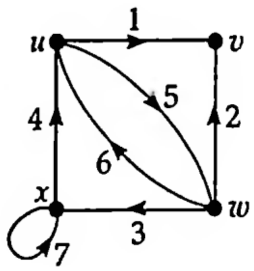


2. *Of the following four digraphs, which two are the same, which one is isomorphic to these two, and which is not isomorphic to any of the others?*

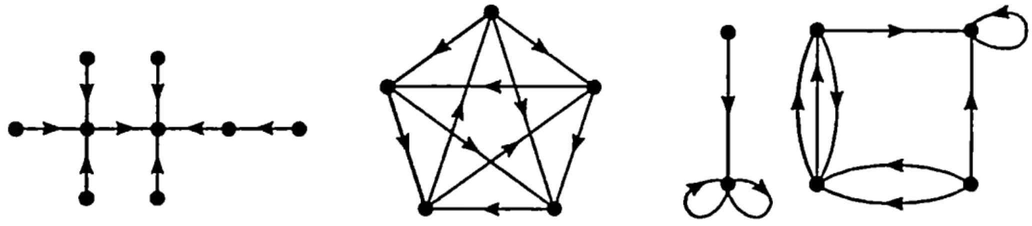


3. *Draw two non-isomorphic non-simple digraphs, each with 4 vertices and 7 arcs. Explain why your digraphs are not isomorphic.*
4. *Write down the out-degree sequence and the in-degree sequence for each of the digraphs in Exercise 2.*
5. *(a) If two digraphs have the same out-degree sequence and the same in- degree sequence, must they be isomorphic?*

   (b) If two digraphs are isomorphic, must they have the same out-degree sequence and the same in-degree sequence?
6. *Draw a digraph with 4 vertices and 7 arcs such that the number of vertices with odd out-degree is odd and the number of vertices with odd in-degree is odd.*
7. *For the digraph shown on the right, write down (if possible):*
   a. *a walk of length 7 from u to w;*
   b. *cycles of lengths 1, 2, 3 and 4;*
   c. *a path of maximum length.*

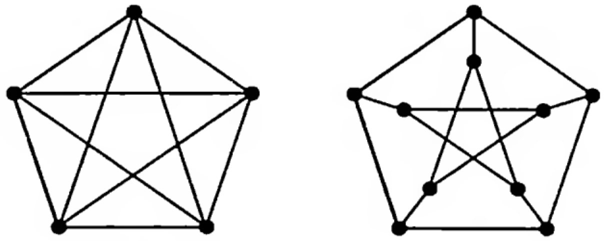


8. *Draw four connected digraphs,* D_1, D_2, D_3 *and* D_4, *each with* 5 *vertices and* 8 *arcs, satisfying the following conditions:*
   - D_1 *is a simple digraph;*
   - D_2 *is a non-simple digraph with no loops;*
   - D_3 *is a digraph with both loops and multiple arcs;*
   - D_4 *is strongly connected.*
9. *Classify each of the following digraphs as disconnected, connected but not strongly connected, or strongly connected:*



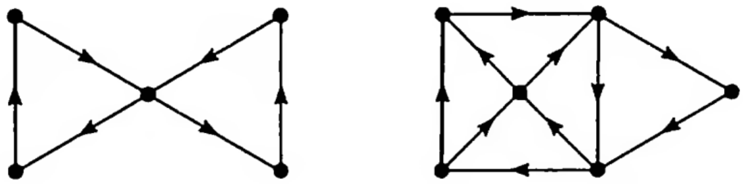
(a) (b) (c)

10. *A graph is orientable if a direction can be assigned to each edge in such a way that the resulting digraph is strongly connected. Show that* K_5 *and the Petersen graph are orientable, and find a graph that is not.*



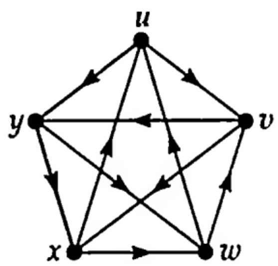
K_5 Petersen

11. *Are the following digraphs Eulerian? Hamiltonian?*



(a) (b)

12. *In the digraph on the right, find:*
    a. *all cycles of lengths 3, 4 and 5;*
    b. *an Eulerian trail;*
    c. *a Hamiltonian cycle.*



REFERENCE

[1]. Anderson, *A First Course in Discrete Mathematics,*2002.

[2]. C. Berge, *Graphs and Hypergraphs*, North-Hoi Land Publishing Company-Amsterdam, 1976.

[3]. A. Bezdek, *Discrete Geometry*, Marcel Dekker, 2003.

[4]. N. L. Biggs, *Discrete mathematics*, Oxford University Press, 2002.

[5]. Eric W., "Discrete mathematics" from MathWorld.

[6]. Ralph P. Grimaldi, *Discrete and Combinatorial Mathematics: An Applied Introduction* 5th ed. Addison Wesley.

[7]. T. R. Hodkinson and J. A. N. Parnell, *Reconstructing the Tree of Life: Taxonomy and systematics of species rich taxa*, CRC Press, 2007.

[8]. B. Hopkins, *Resources for Teaching Discrete Mathematics*, Mathematical Association of America, 2008.

[9]. R. Johnsonbaugh, *Discrete Mathematics*, New Jersey, 2005.

[10]. R. Johnsonbaugh, *Discrete Mathematics*, Prentice Hall, 2008.

[11]. S. G. Krantz, *Discrete Mathematics Demystified*, McGraw-Hill, 2009.

[12]. B. Mohar and C. Thomassen, *Graphs on Surfaces*, Johns Hopkins University press, 2001.

[13]. K.H. Rosen &J.G. Michaels, *Handbook of Discrete and Combinatorial Mathematics*, CRC Press, 2000.

[14]. K. H. Rosen, *Discrete Mathematics and Its Applications* 6th ed. McGraw Hill.

[15]. R. J. Wilson, *Introduction to Graph Theory*, Longman, 1996.

[16]. D. B. West, *Introduction to Graph Theory*, 2005.

CHAPTER 6

MATRICES AND GRAPHS

Objectives: After studying this chapter, you should be able to:

- write down the adjacency matrix and incidence matrix of a given labelled graph or digraph;
- draw the graph or digraph with a given adjacency or incidence matrix;
- use an adjacency matrix to determine the number of walks between two given vertices in a graph or digraph;
- Utilize an adjacency matrix to determine whether a given graph/digraph is connected/strongly connected;
- Illustrate the connections between adjacency matrices and problems in archaeology and genetics;
- Illustrate the connections between adjacency matrices and Markov chains.

Introduction

Up to now, you have seen two ways of representing a graph or digraph - as a diagram of points joined by lines, and as a set of vertices and a set of edges or arcs. The pictorial representation is useful in many situations, especially when we wish to examine the structure of the graph or digraph as a whole, but its value diminishes as soon as we need to describe large or complicated graphs and digraphs. For example, if we need to store a large graph in a computer, then a pictorial representation is unsuitable and some other representation is necessary.

One possibility is to store the set of vertices and the set of edges or arcs. This method is often used, especially when the graph or digraph is 'sparse', with many vertices but relatively few edges or arcs. Another method is to take each vertex in turn and list those vertices adjacent to it; by joining each vertex to its neighbours, we can reconstruct the graph or digraph. Yet another method is to give a table indicating which pairs of vertices are adjacent, or indicating which vertices are incident to which edges or arcs.

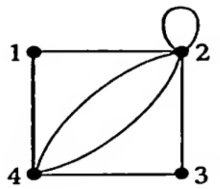
Each of these methods has its advantages, but the last one is particularly useful. Using this method, we represent each graph or digraph by a rectangular array of numbers, called a matrix. Such matrices lend themselves to computational techniques, and are often the most natural way of formulating a problem. There are various types of matrix that we can use to specify a given graph or digraph. Here we describe the two simplest types - the *adjacency matrix* and the *incidence matrix*.

6.1 Adjacency Matrices

Objectives: After studying this topic, you should be able to:

- write down the adjacency matrix of a given labelled graph or digraph;
- draw the graph or digraph with a given adjacency matrix;
- use an adjacency matrix to determine the number of walks between two given vertices in a graph or digraph;
- use an adjacency matrix to determine whether a given graph/digraph is connected/strongly connected;
- describe the connections between adjacency matrices and problems in archaeology and genetics;
- describe the connections between adjacency matrices and Markov chains.

Consider the following example:



$$\begin{array}{cc} & \begin{array}{cccc} \text{col } 1 & \text{col } 2 & \text{col } 3 & \text{col } 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\ \begin{array}{c} \text{row } 1 \rightarrow \\ \text{row } 2 \rightarrow \\ \text{row } 3 \rightarrow \\ \text{row } 4 \rightarrow \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix} \end{array}$$

On the left we have a graph with four labelled vertices, and on the right we have a matrix with four rows and four columns - that is, a $4 \, x \, 4$ matrix. The numbers appearing in the matrix refer to the number of edges joining the corresponding vertices in the graph. For example,

vertices 1 and 2 are joined by 1 edge, so 1 appears in row 1 column 2, and in row 2 column 1;

vertices 2 and 4 are joined by 2 edges, so 2 appears in row 2 column 4, and in row 4 column 2;

vertices 1 and 3 are joined by 0 edges, so 0 appears in row 1 column 3, and in row 3 column 1;

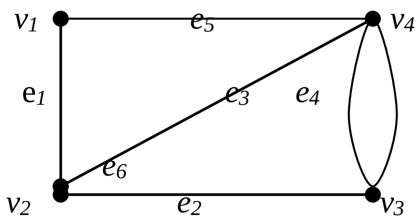
vertex 2 is joined to itself by 1 edge, so 1 appears in row 2 column 2.

We generalize this idea, as follows.

Definition 6.1: Let G be a graph with n vertices labelled $1,2,3,\ldots,n$. The ***adjacency matrix*** $A(G)$ of G is the $n\ x\ n$ matrix in which the entry in row i and column j is the number of edges joining the vertices i and j.

The adjacency matrix of a graph is ***symmetrical*** about the main diagonal (top-left to bottom-right). Also, for a graph without loops, each entry on the main diagonal is 0, and the sum of the entries in any row or column is the degree of the vertex corresponding to that row or column.

Example 6.1: For the graph



the adjacency matrix is

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{pmatrix} \end{array}$$

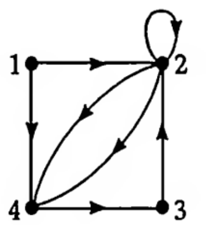
and the incidence matrix is

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array}.$$

The representation of a graph by an adjacency matrix has a digraph analogue that is frequently used when storing large digraphs in a computer. When defining the adjacency matrix of a digraph, we have to take into account the directions of the arcs.

Consider the following example:

Example 6.2:



$$\begin{array}{cc} & \begin{array}{cccc} \text{col } 1 & \text{col } 2 & \text{col } 3 & \text{col } 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\ \begin{array}{c} \text{row } 1 \rightarrow \\ \text{row } 2 \rightarrow \\ \text{row } 3 \rightarrow \\ \text{row } 4 \rightarrow \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

On the left we have a digraph with four labelled vertices, and on the right we have a matrix with four rows and four columns. The numbers appearing in the matrix refer to the number of arcs joining the corresponding vertices in the digraph. For example,

vertices 1 and 2 are joined (in that order) by 1 arc, so 1 appears in row 1 column 2;
vertices 2 and 4 are joined (in that order) by 2 arcs, so 2 appears in row 2 column 4;
vertices 4 and 1 are joined (in that order) by 0 arcs, so 0 appears in row 4 column 1;
vertex 2 is joined to itself by 1 arc, so 1 appears in row 2 column 2.

We generalize this idea, as follows.

Definition 6.2: Let D be a digraph with n vertices labelled. $1, 2, 3, \dots, n$. The ***adjacency matrix*** $\boldsymbol{A(D)}$ of D is the $n \ x \ n$ matrix in which the entry in row i and column j is the number of arcs from vertex i to vertex j.

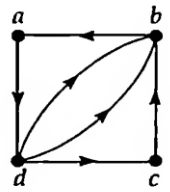
The adjacency matrix of a digraph is not usually symmetrical about the main diagonal. Also, if the digraph has no loops, then each entry on the main diagonal is 0, the sum of the entries in any row is the out-degree of the vertex corresponding to that row, and the sum of the numbers in any column is the in-degree of the vertex corresponding to that column.

Walks in Graphs and Digraphs

We can establish the existence of walks in a graph or digraph by using the adjacency matrix. In the following, we restrict our attention to digraphs: similar results can be derived for graphs.

Consider the following digraph and table:

Example 6.3:



	a	b	c	d
a	0	0	0	1
b	1	0	0	0
c	0	1	0	0
d	0	2	1	0

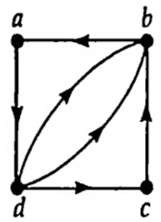
The table shows the number of walks of length 1 between each pair of vertices. For example,

the number of walks of length 1 from a to c is 0, so 0 appears in row 1 column 3;

the number of walks of length 1 from b to a is 1, so 1 appears in row 2 column 1;

the number of walks of length 1 from d to b is 2, so 2 appears in row 4 column 2.

Now a walk of length 1 is an arc, so the table above is the adjacency matrix A of the digraph:



$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

adjacency matrix A

Next, we consider walks of lengths 2 and 3. For example, there are two different walks of length 2 from a to b, because there is one arc from a to d and two arcs from d to b. Similarly, there are two different walks of length 3 from d to d, since there are two arcs from d to b, and one walk of length 2 from b to d, namely, bad.

The solution to the above problem illustrates the following theorem;

Theorem 6.1: Let D be a digraph with n vertices labelled $1,2,\dots,n$, let A be its adjacency matrix with respect to this listing of the vertices, and let k be any positive integer. Then the number of walks of length k from vertex i to vertex j is equal to the entry in row i and column j of the matrix A^k (the kth power of the matrix A).

Write down the adjacency matrix A, calculate the matrices A^2 , A^3 and A^4 , and hence find the numbers of walks of lengths 1, 2,3 and 4 from b to d. Are there walks of lengths 1, 2, 3 or 4 from d to b?

Theorem 6.1 also gives a method of determining whether a digraph is strongly connected, by working directly from its adjacency matrix.

Recall that a digraph is strongly connected if there is a path from vertex i to vertex j, for each pair of distinct vertices i and j, and that a path is a walk in which all the vertices are different. For example, in the digraph considered earlier, there are four vertices, so a path has length 1, 2 or 3. We have seen that the numbers of walks (including the paths) of lengths 1, 2 and 3 between pairs of distinct vertices are given by the non-diagonal entries in the matrices

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix}, \quad \mathbf{A}^2 = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}^3 = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

By examining these matrices, we can see that each pair of distinct vertices is indeed joined by at least one path of length 1, 2 or 3, so the digraph is strongly connected. However, we can check this more easily if we consider the matrix B obtained by adding the three matrices together:

$$\mathbf{B} = \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 = \begin{bmatrix} 2 & 3 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 3 & 3 & 1 & 2 \end{bmatrix}.$$

Let b_{ij} denote the entry in row i and column j in the matrix B. Then each entry b_{ij} is the total number of walks of lengths 1, 2 and 3 from vertex i to vertex j. Since all the non-diagonal entries are positive, each pair of distinct vertices is connected by a path, so the digraph is strongly connected.

We generalize this result in the following theorem; the proof is given at the end of this section.

Theorem 6.2: let D be a digraph with n vertices labelled $1,2,\ldots,n$, let A be its adjacency matrix with respect to this listing of the vertices, and let B be the matrix

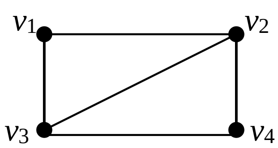
$$B = A + A^2 + \cdots + A^{n-1}$$

Then D is strongly connected if and only if each non-diagonal entry in B is positive - that is, $b_{ij} > 0$ whenever $i \neq j$.

Counting walks

Let $A = (a_{ij})$ be the adjacency matrix of a graph G with vertex set $\{v_1, v_2, \cdots, v_n\}$. The (i, j)th element of A^2 is $\sum_{k=1}^{n} a_{ik}a_{kj}$ and this is the number of walks of length 2 from v_i to v_j.

Example 6.4: If G is



then

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad A^2 = \begin{pmatrix} 2 & 1 & 1 & 2 \\ 1 & 3 & 2 & 1 \\ 1 & 2 & 3 & 1 \\ 2 & 1 & 1 & 2 \end{pmatrix}.$$

Hence, for example, the number of walks of length 2 from v_2 to v_3 is 2, and the number of walks of length 2 from v_2 to v_2 is 3.

Generally, for any positive integer r, the number of walks of length r from v_i to v_j is given by the (i, j)th element of A^r.

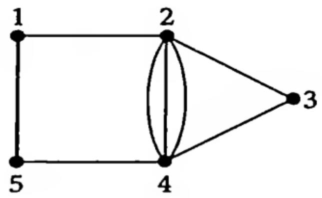
√ Check-List

Put a tick (√) mark if you can perform the task and a cross (x) mark if you can't in the box against the following questions.

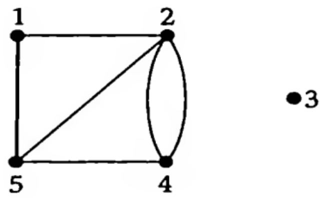
1. Can you define a adjacency matrix of a graph? .... ☐
2. Can you define a adjacency matrix of a digraph? .... ☐

Exercise 6.1

1. *Write down the adjacency matrix of each of the following graphs:*



(a)



(b)

2. *Draw the graph represented by each of the following adjacency matrices:*

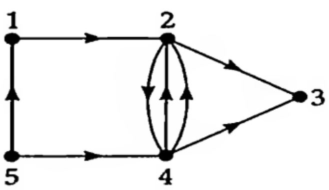
$$\begin{array}{c} \\ 1\\2\\3\\4\\5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 2 & 0 & 1 & 1 \\ 2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 2 & 0 \end{bmatrix} \end{array}$$

(a)

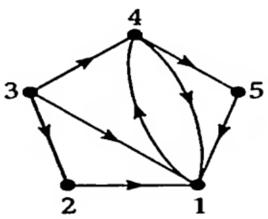
$$\begin{array}{c} \\ 1\\2\\3\\4\\5\\6 \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

(b)

3. *Write down the adjacency matrix of each of the following digraphs:*



(a)



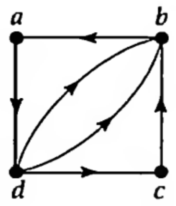
(b)

4. *Draw the digraph represented by each of the following adjacency matrices:*

$$\begin{array}{c} \\ 1\\2\\3\\4\\5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ 1\\2\\3\\4\\5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

(a) (b)

5. *Complete the following tables for the numbers of walks of lengths 2 and 3 in the above digraph.*



	a	*b*	*c*	*d*
a		2		
b				1
c				
d				

numbers of walks of length 2

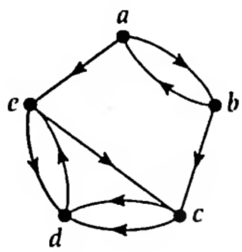
	a	*b*	*c*	*d*
a				
b				
c				
d				2

numbers of walks of length 3

b. *Find the matrix products A^2 and A^3, where A is the adjacency matrix of the above digraph.*

c. *Comment on your results.*

6. *Consider the following digraph:*



7. *Find B for the digraph in Problem 5.6, and hence determine whether the digraph is strongly connected.*

8. *Determine whether the digraph with the following adjacency matrix is strongly connected:*

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

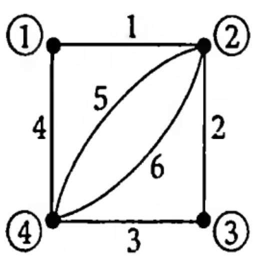
6.2. Incidence Matrices

Objectives: After studying this topic, you should be able to:

- write down the incidence matrix of a given labelled graph or digraph;
- draw the graph or digraph with a given incidence matrix;
- use an incidence matrix to determine the number of walks between two given vertices in a graph or digraph;

For convenience, in this section we restrict our attention to graphs all digraphs with out loops.

Whereas the adjacency matrix of a graph or digraph involves the adjacency of vertices, the incidence matrix involves the incidence of vertices and edges or arcs. To see what is involved, consider the following example:



$$\begin{array}{cc} & \begin{matrix} \text{col } 1 & \text{col } 2 & \text{col } 3 & \text{col } 4 & \text{col } 5 & \text{col } 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{matrix} \\ \begin{matrix} \text{row}(1) \rightarrow \\ \text{row}(2) \rightarrow \\ \text{row}(3) \rightarrow \\ \text{row}(4) \rightarrow \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

On the left we have a graph with four labelled vertices and six labelled edges, and on the righ t we have a matrix with four rows and six columns. Each of the numbers appearing in the matrix is 1 or 0, depending on whether the corresponding vertex and edge are incident with each other. For example,

vertex (1) is incident with edge 4, so 1 appears in row 1 column 4;
vertex (2)is not incident with edge 4, so 0 appears in row 2 column 4.

We generalize this idea, as follows.

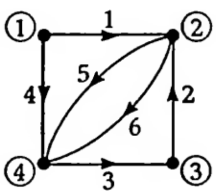
Definition 6.3: Let G be a graph without loops, with n vertices labelled 1, 2, ..., n and m edges labelled $1,2,3,\ldots,m$. The ***incidence matrix I(G)*** of G is the n x m matrix in which the entry in row i and column j is

$$\begin{cases} 1 & \text{if the vertex } i \text{ is incident with the edge } j \\ 0 & \text{otherwise} \end{cases}$$

In the incidence matrix of a graph without loops, each column contains exactly two 1s, as each edge is incident with just two vertices; the sum of the numbers in a row is the degree of the vertex corresponding to that row.

Whereas the adjacency matrix of a digraph involves the adjacency of vertices, the incidence matrix of a digraph involves the incidence of vertices and arcs. Since an arc can be incident from, incident to, or not incident with a vertex, we have to take account of this when defining the matrix. To see what is involved, consider the following example:

Example 6.5:



$$\begin{array}{cc} & \begin{array}{cccccc} \text{col } 1 & \text{col } 2 & \text{col } 3 & \text{col } 4 & \text{col } 5 & \text{col } 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\ \begin{array}{c} \text{row } 1 \rightarrow \\ \text{row } 2 \rightarrow \\ \text{row } 3 \rightarrow \\ \text{row } 4 \rightarrow \end{array} & \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{bmatrix} \end{array}$$

On the left we have a digraph with four labelled vertices and six labelled arcs, and on the right we have a matrix with four rows and six columns. Each of the numbers appearing in the matrix is 1, -1 or 0, depending on whether the corresponding arc is incident from, incident to, or not incident with, the corresponding vertex. For example,

arc 4 is incident from vertex 1, so 1 appears in row 1 column 4;
arc 5 is incident to vertex 4, so -1 appears in row 4 column 5;
arc 4 is not incident with vertex 2, so 0 appears in row 2 column 4.

We generalize this idea, as follows.

Definition 6.4: Let D be a digraph without loops, with n vertices labelled ①, ②, ..., ⓝ and m arcs labelled $1, 2, 3, \dots, m$. The ***incidence matrix*** I(D) of D is the $n \; x \; m$ matrix in which the entry in row i and column j is

$$\begin{cases} 1 & \text{if arc j is incident from vertex i,} \\ -1 & \text{if arc j is incident to vertex,} \\ o & \text{otherwise.} \end{cases}$$

In the incidence matrix of a digraph without loops, each column has exactly one 1 and one -1, since each arc is incident from one vertex and incident to one vertex; the number of 1s in any row is the out-degree of the vertex corre- sponding to that row, and the number of -1s in any row is the in-degree of the vertex corresponding to that row.

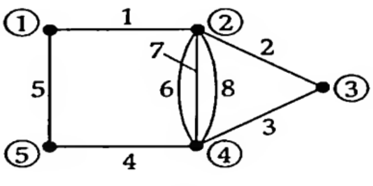
√ Check-List

Put a tick (√) mark if you can perform the task and a cross (x) mark if you can't in the box against the following questions.

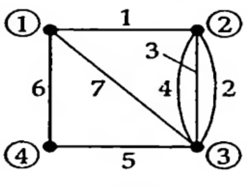
3. Can you define a incidence matrix of a graph without loops? .... ☐
4. Can you define a incidence matrix of a digraph without loops? .... ☐

Exercise 6.2

1. *Write down the adjacency matrix A for K_5. By working out A^3 determine*
   a. *the number of walks of length 3 from a vertex u to a distinct vertex v;*
   b. *the number of triangles (i.e. cycles of length 3).*
2. *Write down the incidence matrix of each of the following graphs:*



(a)



(b)

3. *Draw the graph represented by each of the following incidence matrices:*

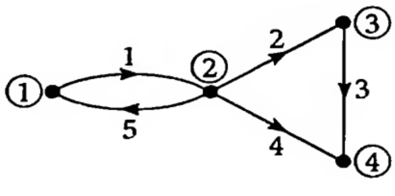
$$\begin{array}{c|c} & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{array}{c} ① \\ ② \\ ③ \\ ④ \\ ⑤ \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

(a)

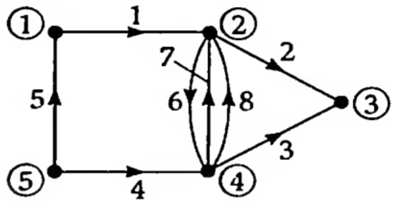
$$\begin{array}{c|c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} ① \\ ② \\ ③ \\ ④ \\ ⑤ \end{array} & \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

(b)

4. *Write down the incidence matrix of each of the following digraphs:*



(a)



(b)

5. *Draw the digraph represented by each of the following incidence matrices:*

$$\begin{array}{c|c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} ① \\ ② \\ ③ \\ ④ \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & -1 & 1 & -1 & 0 \end{bmatrix} \end{array}$$

(a)

$$\begin{array}{c|c} & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{array}{c} ① \\ ② \\ ③ \\ ④ \\ ⑤ \end{array} & \begin{bmatrix} 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

(b)

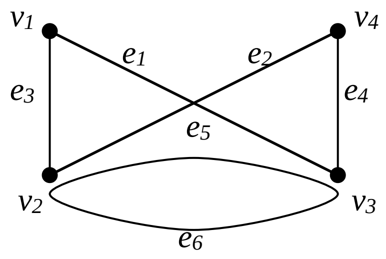
Chapter Summary

- The *adjacency matrix* for a loopless graph G is the $n \times n$ matrix whose ij^{th} element is the number of edges joining v_i to v_j.
- The *adjacency matrix* $A(D)$ of a directed graph D is the $n\ x\ n$ matrix in which the entry in row i and column j is the number of arcs from vertex i to vertex j.
- The *incidence* matrix of a loopless graph G is the $n \times m$ matrix whose ij^{th} element is 1 if v_i is incident with e_j and 0 otherwise.
- The *incidence matrix* I(D) of a digraph D is the $n\ x\ m$ matrix in which the entry in row i and column j is

$$\begin{cases} 1 & \text{if arc j is incident from vertex i,} \\ -1 & \text{if arc j is incident to vertex,} \\ o & \text{otherwise.} \end{cases}$$

Self Test Exercise 6

1. (a) *Write down the adjacency and incidence matrices for the following graph:*



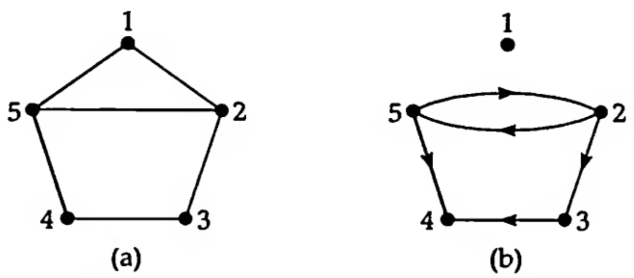
(b) *Draw the graph with the adjacency matrix*

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

(c) *Draw the graph with the incidence matrix*

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

2. *Write down the adjacency matrices of the following graph and digraph.*



3. *Draw the graph corresponding to adjacency matrix (a) and the digraph corresponding to adjacency matrix (b).*

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \end{bmatrix}$$

(a) (b)

4. *The following matrix is the adjacency matrix of a graph G.*

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

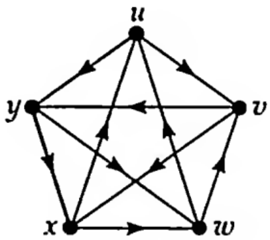
Which three of the following statements are TRUE?

(a) G is connected; *(b) G is regular;*

(c) G is bipartite; *(d) G is a tree;*

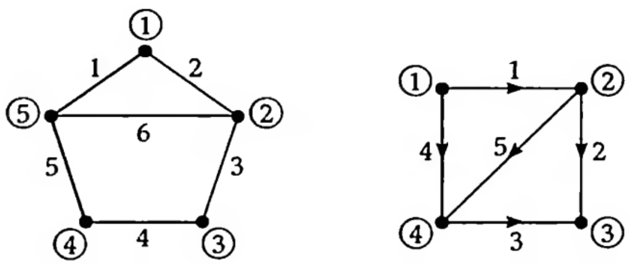
(e) G is Eulerian; *(f) G is Hamiltonian.*

5. *Consider the following digraph:*



Write down the adjacency matrix A, calculate the matrices A^2, A^3 and A^4, and hence find the numbers of walks of lengths 1, 2, 3 and 4 from w to u. Is there a walk of length 1, 2, 3 or 4 from u to w?

6. *Write down the incidence matrices of the following graph and digraph.*



(a) (b)

7. *Draw the digraph whose incidence matrix is*

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

8. *The following matrix is the incidence matrix of a graph G. What is the adjacency matrix of G with the same labelling?*

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

REFERENCE

[1]. C. Berge, *Graphs and Hypergraphs*, North-Hoi Land Publishing Company-Amsterdam, 1976.

[2]. A. Bezdek, *Discrete Geometry*, Marcel Dekker, 2003.

[3]. N. L. Biggs, *Discrete mathematics*, Oxford University Press, 2002.

[4]. Eric W., "Discrete mathematics" from MathWorld.

[5]. Ralph P. Grimaldi, *Discrete and Combinatorial Mathematics: An Applied Introduction* 5th ed. Addison Wesley.

[6]. T. R. Hodkinson and J. A. N. Parnell, *Reconstructing the Tree of Life: Taxonomy and systematics of species rich taxa*, CRC Press, 2007.

[7]. B. Hopkins, *Resources for Teaching Discrete Mathematics*, Mathematical Association of America, 2008.

[8]. R. Johnsonbaugh, *Discrete Mathematics*, New Jersey, 2005.

[9]. R. Johnsonbaugh, *Discrete Mathematics*, Prentice Hall, 2008.

[10]. S. G. Krantz, *Discrete Mathematics Demystified*, McGraw-Hill, 2009.

[11]. B. Mohar and C. Thomassen, *Graphs on Surfaces*, Johns Hopkins University press, 2001.

[12]. K.H. Rosen &J.G. Michaels, *Handbook of Discrete and Combinatorial Mathematics*, CRC Press, 2000.

[13]. K. H. Rosen, *Discrete Mathematics and Its Applications* 6th ed. McGraw Hill.

[14]. R. J. Wilson, *Introduction to Graph Theory*, Longman, 1996.

[15]. D. B. West, *Introduction to Graph Theory*, 2005.