Operational Research (OR) Chapter One

CHAPTER ONE: Introduction to Operations Research

1. INTRODUCTION

Operations Research (OR) is a science which deals with problem, formulation, solutions and finally appropriate decision making. This subject is new and started after World War II, when the failures of missions were very high. Scientists and technocrats formed team to study the problem arising out of difficult situations and at the later stage solutions to these problems. It is research designed to determine most efficient way to do something new. OR is the use of mathematical models, statistics and algorithm to aid in decision-making. It is most often used to analyze complex real life problems typically with the goal of improving or optimizing performance. Decision making is the main activity of an engineer/manager.

Some decisions can be taken by common sense, sound judgment and experience without using mathematics, and some cases this may not be possible and use of other techniques is inevitable.

With the growth of technology, the World has seen a remarkable change in the size and complexity of organizations. An integral part of this had been the division of labor and segmentation of management responsibilities in these organizations. The results have been remarkable but with this, increasing specialization has created a new problem to meet out organizational challenges. The allocation of limited resources to various activities has gained significant importance in the competitive market. These types of problems need immediate attention which is made possible by the application of OR techniques.

The tools of operations research are not from any one discipline, rather Mathematics, Statistics, Economics, Engineering, Psychology, etc. have contributed to this newer discipline of knowledge. Today, it has become a professional discipline that deals with the application of scientific methods for decision-making, and especially to the allocation of scare resources.

In India first unit of OR started in the year 1957 with its base at RRL Hyderabad. The other group was set up in Defense Science Laboratory which was followed by similar units at different parts of the country. The popular journal of OPSEARCH was established in 1963, to promote research in this field.

Keeping in view the critical economic situation which required drastic increase in production efficiency, OR activities were directed, in all areas of business activities. In the late 50‘s OR was introduced at university level. With the development of PC‘s the use of OR techniques became prominent and effective tool as large amount of computation is required to handle complex problems. In recent years application of OR techniques have achieved significance in all walk of life, may it be industry or office work for making strategically decisions more scientifically.

2. BACKGROUND OF OPERATIONS RESEARCH

The effectiveness of operations research in military spread interests in its other governmental departments and industry. In the U.S.A. the National Research Council formed a committee on operations research in 1951, and the first book on the subject ―Methods of Operations Research‖, by Morse and Kimball, was published. In 1952 the Operations Research Society of America came into being.

Today, almost all organizations make use of OR techniques for decision-making at all levels. This general acceptance to OR has come as managers have learned the advantage of the scientific approach to all industrial problems. Some of the Indian organizations using operations research techniques to solve their varied complex problems are: Railways, Defense, Indian Airlines, Fertilizer Corporation of India, Delhi Cloth Mills, Tata Iron and Steel Co. etc.

A purpose of OR is to provide a rational basis for making decisions in the absence of complete information. OR can also be treated as science devoted to describing, understanding and predicting the behavior of systems, particularly man-machine systems.

3. MEANING OF OR

Defining OR is difficult task as its boundaries and content are not yet fixed. It can be regarded as use of mathematical and quantitative techniques to substantiate the decision being taken. Further, it is multidisciplinary which takes tools from subjects like mathematics, statistics, engineering, economics, psychology etc. and uses them to score the consequences of possible alternative actions. Today it has become professional discipline that deals with the application of scientific methods to decision-making. Salient aspects related to definition stressed by various experts on the subject are as follows:

(*a*) Pocock stresses that OR is an applied science; he states OR is scientific methodology-analytical, experimental, quantitative—which by assessing the overall implication of various alternative courses of action in a management system, provides an improved basis for management decisions‘‘.

(*b*) Morse and Kimball have stressed the quantitative approach of OR and have described it as a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control‖.

(*c*) Miller and Starr see OR as applied decision theory. They state OR is applied decision theory.
It uses any scientific, mathematical or logical means to attempt to cope with the problems that confront the executive, when he tries to achieve a thorough going rationality in dealing with his decision problem‖.

(*d*) Saaty considers OR as tool of improving the quality of answers to problems. He say, OR is the art of giving bad answers to problems which otherwise have worse answers.
Few other definitions of OR are as follows:

• OR is concerned with scientifically deciding how to best design and operate man-machine system usually requiring the allocation of scare resources: *Operations Research Society, America*

• OR is essentially a collection of mathematical techniques and tools which in conjunction with system approach, are applied to solve practical decision problems of an economic or engineering nature‘‘. : *Daellenbach and George*

• OR utilizes the planned approach (updated scientific method) and an interdisciplinary team in order to represent complex functional relationships as mathematical models for the purpose of providing a quantitative analysis‘‘: *Thieraub and Klekamp*

• OR is a scientific knowledge through interdisciplinary team effort for the purpose of determining the best utilization of limited resources: *H.A. Taha*

• OR is a scientific approach to problem solving for executive management‖.

4. FEATURES OF OR

The significant features of operations research include the followings:

(i) **Decision-making.** Every industrial organization faces multi fact problems to identify best possible solution to their problems. OR aims to help the executives to obtain optimal solution with the use of OR techniques. It also helps the decision maker to improve his creative and judicious capabilities, analyses and understand the problem situation leading to better control, better co-ordination, better systems and finally better decisions.

(ii) **Scientific Approach.** OR apply scientific methods, techniques and tools for the purpose of analysis and solutions of the complex problems. In this approach there is no place for guess work and the person bias of the decision maker.

(iii) **Inter-disciplinary Team Approach.** Basically the industrial problems are of complex nature and therefore require a team effort to handle it. This team comprises of scientist/mathematician and technocrats. Who jointly use the OR tools to obtain a optimal solution of the problem. The tries to analyze the cause and effect relationship between various parameters of the problem and evaluates the outcome of various alternative strategies.

(iv) **System Approach.** The main aim of the system approach is to trace for each proposal all significant and indirect effects on all sub-system on a system and to evaluate each action in terms of effects for the system as a whole.

The interrelationship and interaction of each sub-system can be handled with the help of mathematical/analytical models of OR to obtain acceptable solution.

(v) **Use of Computers.** The models of OR need lot of computation and therefore, the use of computers becomes necessary. With the use of computers it is possible to handle complex problems requiring large amount of calculations.

(vi) **The objective** of the operations research models is to attempt and to locate best or optimal solution under the specified conditions. For the above purpose, it is necessary that a measure of effectiveness has to be defined which must be based on the goals of the organization. These measures can be used to compare the alternative courses of action taken during the analysis.

5. PHASES OF OR STUDY

OR is a logical and systematic approach to provide a rational basis for decision-making. The phases of OR must be logical and systematic. The various steps required for the analysis of a problem under OR are as follows:

Step I. Observe the Problem Environment

The first step of OR study is the observation of the environment in which the problem exists. The activities that constitute this step are visits, conferences, observations, research etc. with the help of such activities, the OR analyst gets sufficient information and support to proceed and is better prepared to formulate the problem.

Step II. Analyze and Define the Problem

In this step not only the problem is defined but also uses objectives and limitations of the study that are stressed in the light of the problem. The end results of this step are clear grasp of need for a solution and understanding of its nature.

Step III. Develop a Model

The next step is to develop model, which is representation of same real or abstract situation. OR models are basically mathematical models representing systems, process or environment in form of equations, relationships or formulae. The activities in this step is to defining interrelationships among variables, formulating equations, using known OR models or searching suitable alternate models. The proposed model may be field tested and modified in order to work under stated environmental constraints. A model may also be modified if the management is not satisfied with the answer that it gives.

Step IV. Selection of Data Input

It is an established fact that without authentic and appropriate data the results of the OR models cannot be trusted. Hence, taping right kind of data is a vital step in OR process. Important activities in this step are analyzing internal-external data and facts, collecting opinions and using computer data banks. The purpose of this step is to have sufficient input to operate and test the model.

Step V. Solution and Testing

In this step the solution of the problems is obtained with the help of model and data input. Such a solution is not implemented immediately and this solution is used to test the model and to find its limitations if any. If the solution is not reasonable or if the model is not behaving properly, updating and modification of the model is considered at this stage. The end result of this step is solution that is desirable and supports current organizational objectives.

Step VI. Implementation of the Solution

This is the last phase of the OR study. In OR the decision-making is scientific but implementation of decision involves many behavioral issues. Therefore, implementation authority has to resolve the behavioral issues, involving the workers and supervisors to avoid further conflicts. The gap between management and OR scientist may offer some resistance but must be eliminated before solution is accepted in totality. Both the parties should play positive role, since the implementation will help the organization as a whole. A properly implemented solution obtained through OR techniques results in improved working conditions and wins management support.

6. OUTLINES OF OR MODELS

In OR the problem is expressed in the form of a model. Where, a model is a theoretical abstraction (approximation) of a real-life problem. It can be defined as a simplified representation of an operation or a process in which only the basic aspects or the most important features of a typical problem under investigation are considered. OR analysts have given special impetus to the development and use of techniques like, linear programming, waiting line theory, game theory, inventory controls and simulation. In addition, some other common tools are non-linear programming, integer programming, dynamic programming, sequencing theory, Markov process, network scheduling PERT and CPM, symbolic logic, information theory and utility/value theory. The list, of course, is not exhaustive. The detailed discussion on above will be presented in appropriate chapters, however, brief explanation of these is given below:

(i) Linear Programming (L.P.)

Linear programming is basically a constrained optimization technique which tries to optimize some criterion within some constraints. It consists of an objective function which is some measure of effectiveness like profit, loss or return on investment and several boundary conditions putting restriction on the use of resources. Objective function and boundary conditions are linear in nature. There are methods available to solve a linear programming problem.

(ii) Waiting Line or Queuing Theory

This deal with the situation in which queue is formed or the customers has to wait for service or machines wait for repairmen and therefore concept of a queue is involved. If we assume that there are costs associated with waiting in line, and if there are costs of adding more service facilities, we want to minimize the sum of costs of waiting and the costs of providing service facilities. Waiting line theory helps to make calculations like number of expected member of people in queue, expected waiting time in the queue, expected idle time for the server, etc. These calculations then can be used to determine the desirable number of service facilities or number of servers.

(iii) Game Theory

It is used for decision-making under conflicting situations where there are one or more opponents. The opponents, in game theory, are called players. The motives of the players are dictomized. The success of one player tends to be at the cost of others and hence they are in conflict. Game theory models, a conflict situation arises and helps to improve the decision process by formulating appropriate strategy.

(IV) Inventory Control Models

These models deal with the quantities which are either to be purchased or stocked since each factor involves cost. The purchase and material managers are normally encounter such situations. Therefore, inventory models provide rational answer to these questions in different situations of supply and demand for different kind of materials. Inventory control models help managers to decide ordering time, reordering level and optimal ordering quantity. The approach is to prepare a mathematical model of the situation that expressed total inventory costs in terms of demand, size of order, possible over or under stocking and other relevant factors and then to determine optimal order size, optimum order level etc. using calculus or some other technique.

(v) Simulation

It is basically data generating technique, where sometimes it is risky, cumbersome, or time consuming to conduct real study or experiment to know more about situation or problem. The available analytical methods cannot be used in all situations due to large number of variables or large number of interrelationships among the variables and the complexity of relationship; it is not possible to develop an analytical model representing the real situation. Sometimes, even building of model is possible but its solution may not be possible. Under such situations simulation is used. It should be noted that simulation does not solve the problem by itself, but it only generates the required information or data needed for decision problem or decision-making.

(VI) Non-Linear Programming

These models may be used when either the objective function or some of the constraints are not linear in nature. Non-linearity may be introduced by such factors as discount on price of purchase of large quantities and graduated income tax etc. Linear programming may be employed to approximate the non-linear conditions, but the approximation becomes poorer as the range is extended. Non-linear methods may be used to determine the approximate area in which a solution lies and linear methods may be used to obtain a more exact solution.

(vii) Integer Programming

This method can be used when one or more of the variables can only take integer values. Examples are the number of trucks in a fleet, the number of generators in a power house and so on. Approximate solutions can be obtained without using integer programming methods, but the approximation generally becomes poorer as the number becomes smaller. There are techniques to obtain solution of integer programming problems.

(viii) Dynamic Programming

This is a method of analyzing multistage decision processes, in which each elementary decision is dependent upon those preceding it as well as upon external factors. It drastically reduces the computational efforts otherwise necessary to analyze results of all possible combinations of elementary decisions.

(ix) Sequencing Theory

This is related to waiting line theory and is applicable when the facilities are fixed, but the order of servicing may be controlled. The scheduling of service or the sequencing of jobs is done to minimize the relevant costs and time.

(x) Markov Process

It is used for decision-making in situations where various states are defined. The probability of going from one state to another is known and depends on the present state and is independent of how we have arrived at that state. Theory of Markov process helps us to calculate long run probability of being in a particular state (steady state probability), which is used for decision-making.

(xi) Network Scheduling PERT and CPM

These techniques are used to plan, schedule and monitor large projects such as building construction, maintenance of computer system installation, research and development design etc. The technique aims at minimizing trouble spots, such as, delays, interruptions and production bottlenecks, by identifying critical factors and coordinating various parts of overall job/project. The project/job is diagrammatically represented with the help of network made of arrows representing different activities and interrelationships among them. Such a representation is used for identifying critical activities and critical path. Two basic techniques in network scheduling are Program Evaluation and Review Technique (PERT) and Critical Path Method (CPM). CPM is used when time taken by activities in a project are known for sure and PERT is used when activities time is not known for sure—only probabilistic estimate of time is available to the users.

(xii) Symbolic Logic

It deals with substituting symbols for words, classes of things or functional systems. Symbolic logic involves rules, algebra of logic and propositions. There have been only limited attempts to apply this technique to business problems; however has had extensive application in the design of computing machinery.

(xiii) Information Theory

Information theory is an analytical process transferred from the electrical communications field to operations research. It seeks to evaluate the effectiveness of information flow within a given system. Despite its application mainly to communication networks, it has had a indirect influence in simulating the examination of business organizational structures with a view to improving information or communication flow.

(xiv) Utility/Value Theory

It deals with assigning numerical significance to the worth of alternative choices. To date, this has been only a concept and is in the stage of elementary model formulation and experimentation and can be useful in decision-making process.

7. SCOPE OF OPERATIONS RESEARCH

As presented in the earlier paragraphs, the scope of OR is not only confined to any specific agency like defense services but today it is widely used in all industrial organizations. It can be used to find the best solution to any problem be it simple or complex. It is useful in every field of human activities, where optimization of resources is required in the best way. Thus, it attempts to resolve the conflicts of interest among the components of organization in a way that is best for the organization as a whole. The main fields where OR is extensively used are given below, however, this list is not exhaustive but only illustrative.

(i) National Planning and Budgeting

OR is used for the preparation of Five Year Plans, annual budgets, forecasting of income and expenditures, scheduling of major projects of national importance, estimation of GNP, GDP, population, employment and generation of agriculture yields etc.

(ii) Defense Services

Basically formulation of OR started from USA army, so it has wide application in the areas such as: development of new technology, optimization of cost and time, tender evaluation, setting and layouts of defense projects, assessment of threat analysis‖, strategy of battle, effective maintenance and replacement of equipment, inventory control, transportation and supply depots etc.

(iii) Industrial Establishment and Private Sector Units

OR can be effectively used in plant location and setting finance planning, product and process planning, facility planning and construction, production planning and control, purchasing, maintenance management and personnel management etc. to name a few.

(iv) R & D and Engineering

Research and development being the heart of technological growth, OR has wide scope for and can be applied in technology forecasting and evaluation, technology and project management, preparation of tender and negotiation, value engineering, work/method study and so on.

(v) Business Management and Competition

OR can help in taking business decisions under risk and uncertainty, capital investment and returns, business strategy formation, optimum advertisement outlay, optimum sales force and their distribution, market survey and analysis and market research techniques etc.

(VI) Agriculture and Irrigation

In the area of agriculture and irrigation also OR can be useful for project management, construction of major dams at minimum cost, optimum allocation of supply and collection points for fertilizer/seeds and agriculture outputs and optimum mix of fertilizers for better yield.

(vii) Education and Training

OR can be used for obtaining optimum number of schools with their locations, optimum mix of students/teacher student ratio, optimum financial outlay and other relevant information in training of graduates to meet out the national requirements.

(viii) Transportation

Transportation models of OR can be applied to real life problems to forecast public transport requirements, optimum routing, forecasting of income and expenses, project management for railways, railway network distribution, etc. In the same way it can be useful in the field of communication.

(ix) Home Management and Budgeting

OR can be effectively used for control of expenses to maximize savings, time management, work study methods for all related works. Investment of surplus budget, appropriate insurance of life and properties and estimate of depreciation and optimum premium of insurance etc.

8. ROLE OF OR IN ENGINEERING AND SCIENCE

The information available in science has been used to develop engineering. Whatever available in the engineering is based on basic fundamentals of science? With the growth of technology the practitioners faced many challenges to improve the product and market it efficiently. OR has emerged to help everyone to improve their performance and produce items at optimum cost. Since the modern problems are complex in nature and modern technology is knowledge based‖ and skill intensive‖. The knowledge possessed by single individual or group is limited and OR is a team effort; specialists from all relevant disciplines participate in it, to find out best possible solution of the problem under the given environmental conditions.

It has been described that OR is a systematic method of stating the problem in clear terms, collecting facts and data, analyzing them and then reaching certain conclusions in the form of solutions to the problem, which further can be tested and verified for its optimality in most of the cases. OR is team efforts have been used with the existence of mankind. But OR is a systematic approach using only scientific methods/techniques to find solution which distinguishes OR from team efforts past or present. OR is being effectively used in areas such as: production planning and inventory control, transportation, military operations and weapon system development, personnel management, social services, health services, communication systems, computer networks and information system to name but a few the problems they pose with ever increasing rate are similarly formulated, can be identified by several features they have in common and, last but not least, can be solved by similar methods. These problems are therefore conveniently grouped under the common heading of OR problems.

Under OR study an objective is defined which may have alternative solutions. A decision has to be made based upon choosing from a set of possible alternatives. Each choice offers its own advantages and disadvantages, so that in complex situation the decision maker might not be able to make a preferable option at once and quickly decide why he should prefer one alternative over the other one. To clarify the situation and compare the alternatives in several aspects, OR suggests a series of mathematical operations. Their aim is to analyze the situation critically and thus prepare a decision for those bearing the responsibility for a final choice.

During implementation the operations research team should prepare a detailed instruction for those who have to implement the solution. Translating solution into operating procedure should indicate who should do, what and when? The facilities and information required should also be clearly specified and persons concerned should be taken into confidence for obtaining their cooperation for the success of the program.

9. SIMPLIFICATION OF OR MODELS

The more complex, expensive and large in scale the designed system is, the less allowable in it are willful decisions and the more gain in importance scientific methods which, implemented, provide an estimate of each decision‘s consequences, help discard the unallowable versions and recommend the most successful ones. They help in assessing whether the available information is adequate to prepare a correct decision and, if not, and then indicate what data should be additionally collected. It would be extremely risky to be guided solely by intuition *i.e*., experience and common sense. Modern science and technology evolves so fast that the experience may simply not have been acquired. The calculations that make the process of decision-making easier are the subject matter of operations research.

Under the complex situations some of the models need a lot of computational efforts particularly in case of linear programming. Efforts should be made to simplify the situation and development of model so as to generate the optimal solution with minimum effort. The selection of model for a particular problem has its own bearings and availability of solution. Assumptions to be imposed on the model should be such that, it makes it possible to achieve desirable solution without affecting the constraints of the problem.

In most intricate cases, when scanning an operation and its outcome depend on a large number of intimately interrelated random factors, the analytical techniques fail altogether and the analyst has to employ Monto Carlo methods of statistical modeling. In this case a computer simulates the process of an operation development with all the random variables involved. This manipulation of the process yields an observation of one random operation run. One such realization gives no grounds for decision making, but, once a manifold of them is collected after several runs, it may be handled statistically to find the process means and make inferences about the real system and how in the mean, it is influenced by initial conditions and controllable variables.

Both analytical and statistical models are widely implemented in OR. Each of the models possesses its own advantages and disadvantages. The analytical models are rougher, but they yield more meaningful results. However, statistical models are more accurate, but are bulky, poorly analyzable, need more computational time and do not yield optimal results. Therefore, the analyst should make correct judgment to select either model depending upon requirements and situation of the problem.

10. DEVELOPMENT OF OR IN INDIA

OR being a new discipline started a bit late in India with its inception at Regional Research Laboratory at Hyderabad and at the same time a group was established in Defense Science Laboratory to solve the problems of stores, purchase and planning. OR society was formed in 1953. Today OR subject is very popular and is being taught at graduation and post-graduation level in all the university of the country.

It is also being used in industrial establishment extensively to improve decision-making process.

11. COMPUTERS IN OR

As has been presented earlier that OR tries to find optimal solutions with multiple variables. In most of the cases a large number of iterations are required to reach optimal solution. Manually this task becomes time consuming and single mistake at any point can generate erroneous results. With the development of computers and P.C‘s this has reduced manual efforts considerably and solutions can be obtained in a short period of time and possibility of errors is also minimized considerably.

Storage of information/data is easy and faster with the use of computers because of its memory. The computational time requirements are also less and no paper work is required. Transfer of data from one place to another is also possible through net/computers. The reliability of solutions is also high. For the large size problems, where simulation was to be used, it was not possible to carry it out manually, which is now possible with the use of computers. To handle linear programming problem with multiple variables use to be cumbersome and time taking, can be done at wink of moment without any manual efforts.

12. LIMITATIONS OF OPERATIONS RESEARCH

OR has some limitations however, these are related to the problem of model building and the time and money factors involved in application rather than its practical utility. Some of them are as follows:

(*i*) **Magnitude of Computation.** Operations research models try to find out optimal solution taking into account all the factors. These factors are enormous and expressing them in quantity and establishing relationships among these require voluminous calculations which can be handled by computers.

(*ii*) **Non-Quantifiable Factors.** OR provide solutions only when all elements related to a problem can be quantified. All relevant variables do not lend themselves to quantification. Factors which cannot be quantified find no place in OR study. Models in OR do not take into account qualitative factors or emotional factors which may be quite important.

(*iii*) **Distance between User and Analyst.** OR being specialist‘s job requires a mathematician or statistician, who might not be aware of the business problems. Similarly, a manager fails to understand the complex working of OR. Thus there is a gap between the two. Management itself may offer a lot of resistance due to conventional thinking.

IV) Time and Money Costs. When basic data are subjected to frequent changes, incorporating them into the OR models is a costly proposition. Moreover, a fairly good solution at present may be more desirable than a perfect OR solution available after sometime. The computational time increases depending upon the size of the problem and accuracy of results desired.

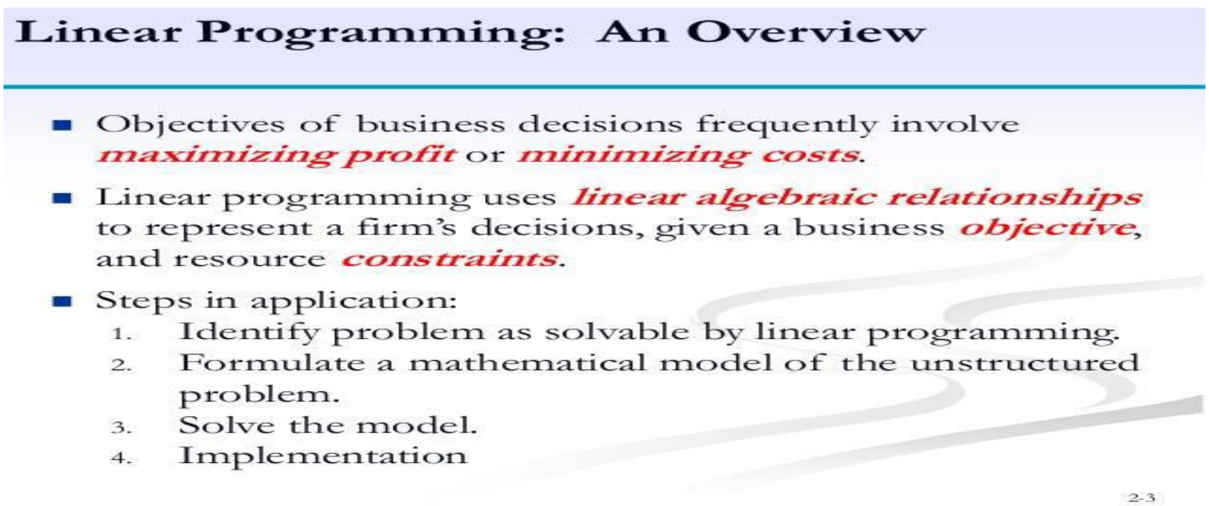
(***v***) **Implementation.** Implementation of any decision is a delicate task. It must take into account the complexities of human relations and behavior. Sometimes, resistance is offered due to psychological factors which may not have any bearing on the problem as well as its solution.

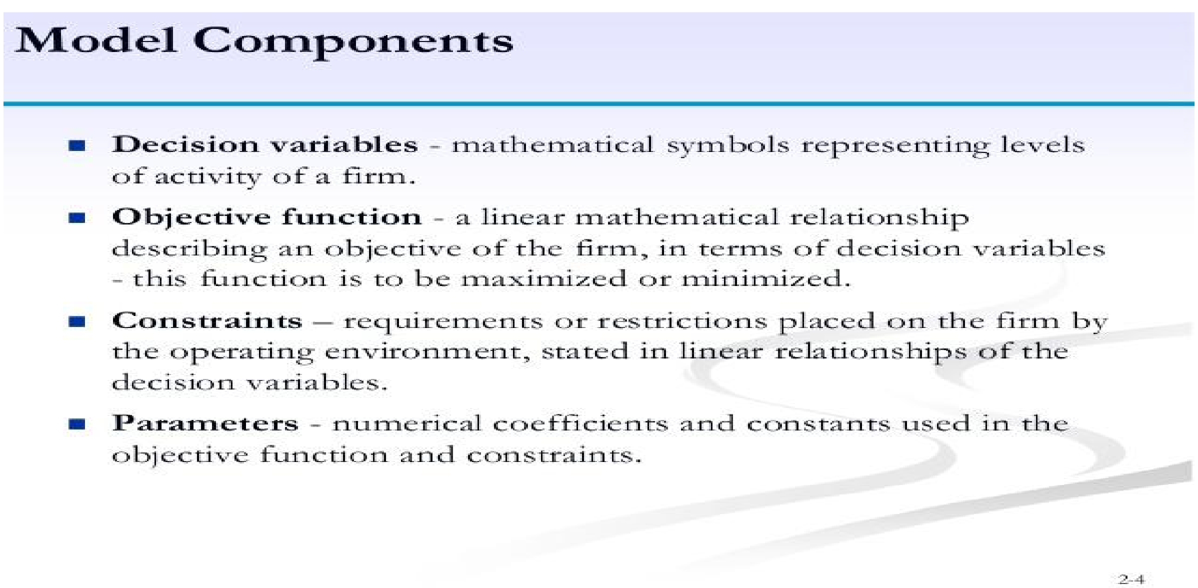
EXERCISES

1. What is operations research and explain briefly its applications in industrial organizations?
2. What are the 3 characteristics of operations research? Discuss.
3. Discuss the importance of OR in decision-making process.
4. Enumerate, with brief description, some of the important techniques used in OR.
5. Discuss the limitations of operation research.
6. Describe the various steps involved in OR study.
7. Discuss significance and scope of operation research.
8. Describe briefly the different phases of operation research.
9. Explain steps involved in the solution of OR problems.
10. Illustrate the importance of features in OR.

Chapter 2: Linear programming

LP is a technique for finding the best alternative from a set of feasible alternatives. It is a optimization method which shows how to allocate scarce resources & how to do such allocation in the best possible way subject to more than one limiting factor/conditions. It is a mathematical technique which involves the allocation of scarce resources in an optimal manner. LP technique is designed to help managers in planning, decision making & allocation of resources. Determination of product mix, transportation schedule, plant location machine, assignment, portfolio selection & allocation of labour are few types of problems that can be solved by LP. In LPP objective function as well as restrictions or constraints can be expressed as linear mathematical function.





Summary of Model Formulation Steps

Step 1 : Clearly define the decision variables

Step 2 : Construct the objective function

Step 3 : Formulate the constraints

2-5

Assumptions of Linear Programming Model

- **Proportionality** - The rate of change (slope) of the objective function and constraint equations is constant.
- **Additivity** - Terms in the objective function and constraint equations must be additive.
- **Divisibility** -Decision variables can take on any fractional value and are therefore continuous as opposed to integer in nature.
- **Certainty** - Values of all the model parameters are assumed to be known with certainty (non-probabilistic).

2-7

Characteristics of Linear Programming Problems

- A decision amongst alternative courses of action is required.
- The decision is represented in the model by **decision variables**.
- The problem encompasses a goal, expressed as an **objective function**, that the decision maker wants to achieve.
- Restrictions (represented by **constraints**) exist that limit the extent of achievement of the objective.
- The objective and constraints must be definable by **linear** mathematical functional relationships.

2-6

Advantages of Linear Programming Model

- It helps decision - makers to use their productive resource effectively.
- The decision-making approach of the user becomes more objective and less subjective.
- In a production process, bottle necks may occur. For example, in a factory some machines may be in great demand while others may lie idle for some time. A significant advantage of linear programming is highlighting of such bottle necks.

2-8

LP Model Formulation
A Maximization Example (1 of 4)

- Product mix problem - Beaver Creek Pottery Company
- How many bowls and mugs should be produced to maximize profits given labor and materials constraints?
- Product resource requirements and unit profit:

	Resource Requirements		
Product	Labor (Hr./Unit)	Clay (Lb./Unit)	Profit ($/Unit)
Bowl	1	4	40
Mug	2	3	50

2-10

LP Model Formulation
A Maximization Example (3 of 4)

Resource Availability: 40 hrs of labor per day
120 lbs of clay

Decision Variables: x_1 = number of bowls to produce per day
x_2 = number of mugs to produce per day

Objective Function: Maximize $Z = \$40x_1 + \$50x_2$
Where Z = profit per day

Resource Constraints: $1x_1 + 2x_2 \leq 40$ hours of labor
$4x_1 + 3x_2 \leq 120$ pounds of clay

Non-Negativity Constraints: $x_1 \geq 0$; $x_2 \geq 0$

LP Model Formulation
A Maximization Example (4 of 4)

Complete Linear Programming Model:

Maximize $Z = \$40x_1 + \$50x_2$

subject to:
$$1x_1 + 2x_2 \leq 40$$
$$4x_2 + 3x_2 \leq 120$$
$$x_1, x_2 \geq 0$$

Feasible Solutions

A ***feasible solution*** does not violate ***any*** of the constraints:

Example: $x_1 = 5$ bowls
$x_2 = 10$ mugs
$Z = \$40x_1 + \$50x_2 = \$700$

Labor constraint check: $1(5) + 2(10) = 25 < 40$ hours
Clay constraint check: $4(5) + 3(10) = 70 < 120$ pounds

2-14

Infeasible Solutions

An ***infeasible solution*** violates ***at least one*** of the constraints:

Example: $x_1 = 10$ bowls
$x_2 = 20$ mugs
$Z = \$40x_1 + \$50x_2 = \$1400$

Labor constraint check: $1(10) + 2(20) = 50 > 40$ hours

2-15

Graphical Solution of LP Models

- Graphical solution is limited to linear programming models containing ***only two decision variables*** (can be used with three variables but only with great difficulty).
- Graphical methods provide ***visualization of how*** a solution for a linear programming problem is obtained.
- Graphical methods can be classified under two categories:
  1. Iso-Profit(Cost) Line Method
  2. Extreme-point evaluation Method.

2-16

Coordinate Axes
Graphical Solution of Maximization Model (1 of 12)

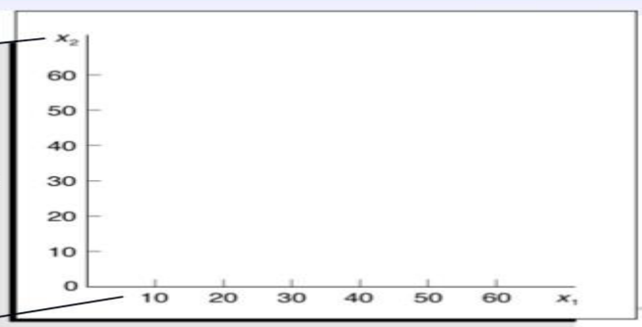
X2 is mugs

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_2 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$



X1 is bowls

Figure 2.2 Coordinates for Graphical Analysis

2-17

Labor Constraint
Graphical Solution of Maximization Model (2 of 12)

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_2 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

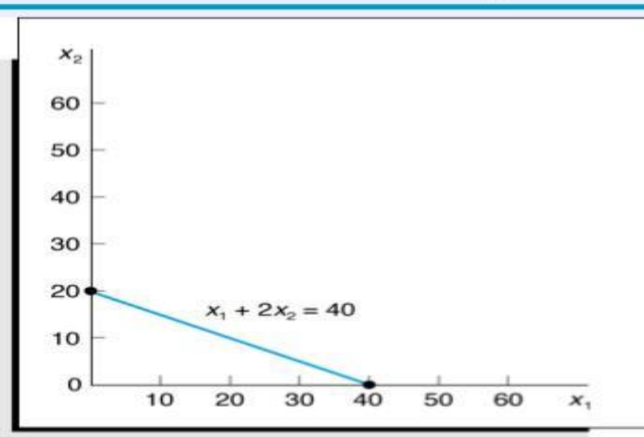


Figure 2.3 Graph of Labor Constraint

2-18

Labor Constraint Area
Graphical Solution of Maximization Model (3 of 12)

Maximize $Z = \$40x_1 + \$50x_2$
subject to: $1x_1 + 2x_2 \leq 40$
$4x_2 + 3x_2 \leq 120$
$x_1, x_2 \geq 0$

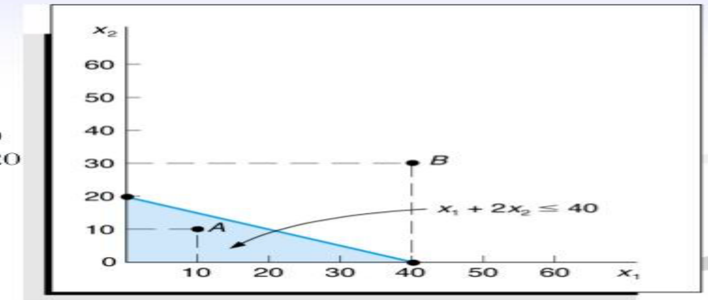


Figure 2.4 Labor Constraint Area

2-19

Clay Constraint Area
Graphical Solution of Maximization Model (4 of 12)

Maximize $Z = \$40x_1 + \$50x_2$
subject to: $1x_1 + 2x_2 \leq 40$
$4x_2 + 3x_2 \leq 120$
$x_1, x_2 \geq 0$

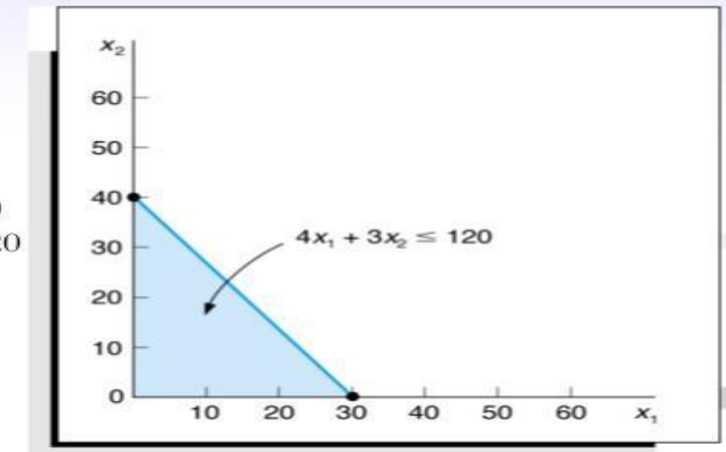


Figure 2.5 Clay Constraint Area

2-20

Both Constraints
Graphical Solution of Maximization Model (5 of 12)

Maximize $Z = \$40x_1 + \$50x_2$
subject to: $1x_1 + 2x_2 \leq 40$
$4x_2 + 3x_2 \leq 120$
$x_1, x_2 \geq 0$

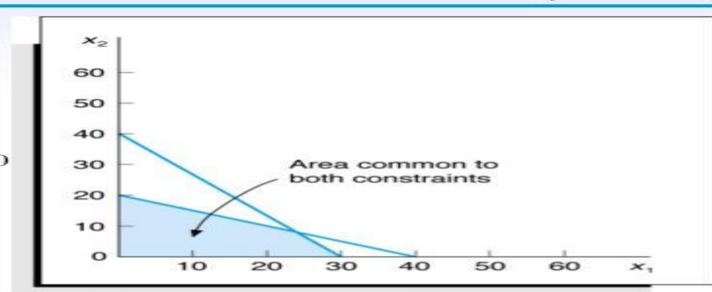


Figure 2.6 Graph of Both Model Constraints

2-21

Feasible Solution Area
Graphical Solution of Maximization Model (6 of 12)

Maximize $Z = \$40x_1 + \$50x_2$
subject to: $1x_1 + 2x_2 \leq 40$
$4x_2 + 3x_2 \leq 120$
$x_1, x_2 \geq 0$

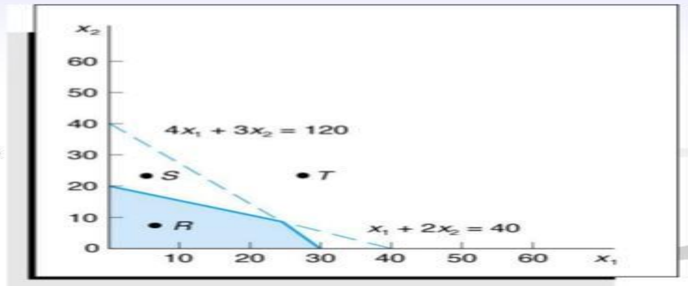


Figure 2.7 Feasible Solution Area

2-22

Objective Function Solution = \$800
Graphical Solution of Maximization Model (7 of 12)

Maximize $Z = \$40x_1 + \$50x_2$
subject to: $1x_1 + 2x_2 \leq 40$
$4x_2 + 3x_2 \leq 120$
$x_1, x_2 \geq 0$

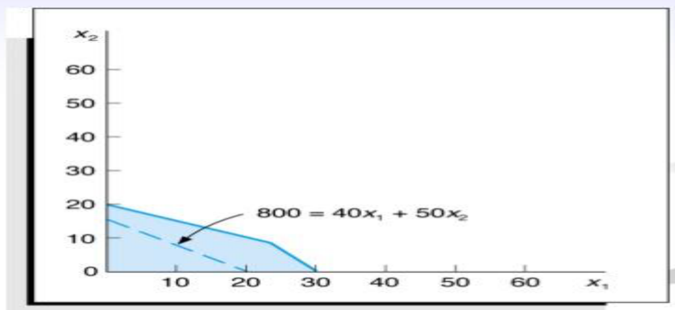


Figure 2.8 Objection Function Line for Z = \$800

2-23

Alternative Objective Function Solution Lines
Graphical Solution of Maximization Model (8 of 12)

Maximize $Z = \$40x_1 + \$50x_2$
subject to: $1x_1 + 2x_2 \leq 40$
$4x_2 + 3x_2 \leq 120$
$x_1, x_2 \geq 0$

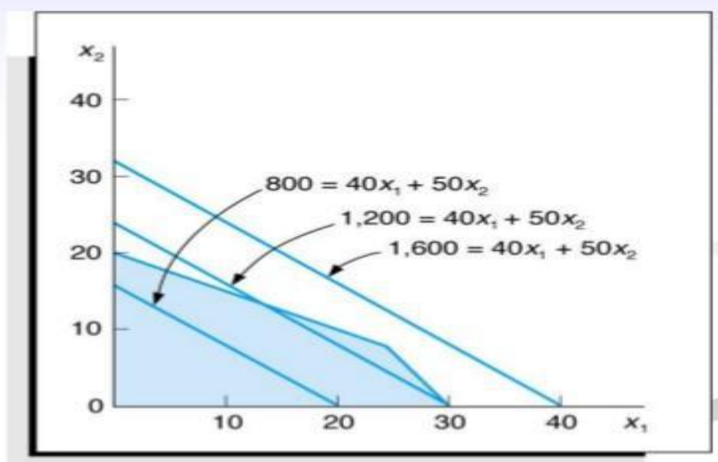


Figure 2.9 Alternative Objective Function Lines

2-24

Optimal Solution
Graphical Solution of Maximization Model (9 of 12)

Maximize $Z = \$40x_1 + \$50x_2$
subject to: $1x_1 + 2x_2 \leq 40$
$4x_2 + 3x_2 \leq 120$
$x_1, x_2 \geq 0$

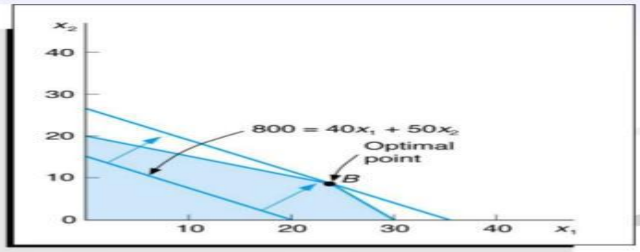


Figure 2.10 Identification of Optimal Solution Point

2-25

Optimal Solution Coordinates
Graphical Solution of Maximization Model (10 of 12)

Maximize $Z = \$40x_1 + \$50x_2$
subject to: $1x_1 + 2x_2 \leq 40$
$4x_2 + 3x_2 \leq 120$
$x_1, x_2 \geq 0$

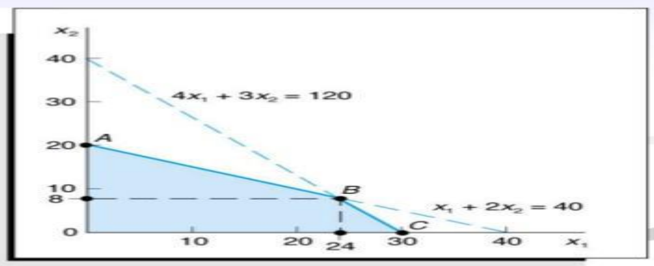


Figure 2.11 Optimal Solution Coordinates

2-26

Extreme (Corner) Point Solutions
Graphical Solution of Maximization Model (11 of 12)

Maximize $Z = \$40x_1 + \$50x_2$
subject to: $1x_1 + 2x_2 \leq 40$
$4x_2 + 3x_2 \leq 120$
$x_1, x_2 \geq 0$

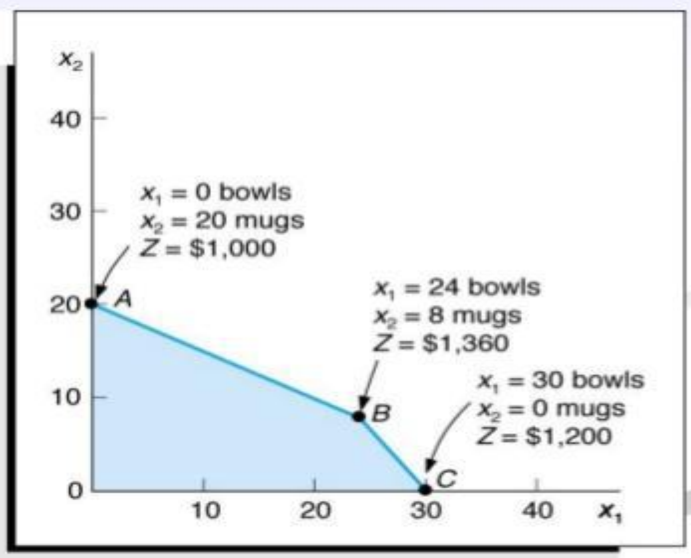


Figure 2.12 Solutions at All Corner Points

2-27

Optimal Solution for New Objective Function Graphical Solution of Maximization Model (12 of 12)

Maximize $Z = \$70x_1 + \$20x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_2 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

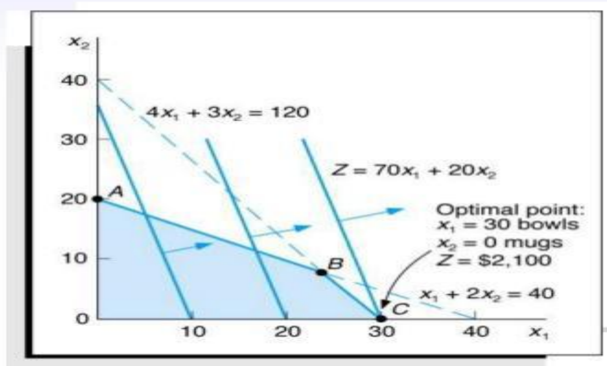


Figure 2.13 Optimal Solution with $Z = 70x_1 + 20x_2$

2-28

Slack Variables

- Standard form requires that all constraints be in the form of equations (equalities).
- A slack variable is ***added to a ≤ constraint*** (weak inequality) to convert it to an equation (=).
- A slack variable typically represents an ***unused resource***.
- A slack variable ***contributes nothing*** to the objective function value.

2-29

Linear Programming Model: Standard Form

Max $Z = 40x_1 + 50x_2 + s_1 + s_2$

subject to: $1x_1 + 2x_2 + s_1 = 40$

$4x_2 + 3x_2 + s_2 = 120$

$x_1, x_2, s_1, s_2 \geq 0$

Where:

x_1 = number of bowls

x_2 = number of mugs

s_1, s_2 are slack variables

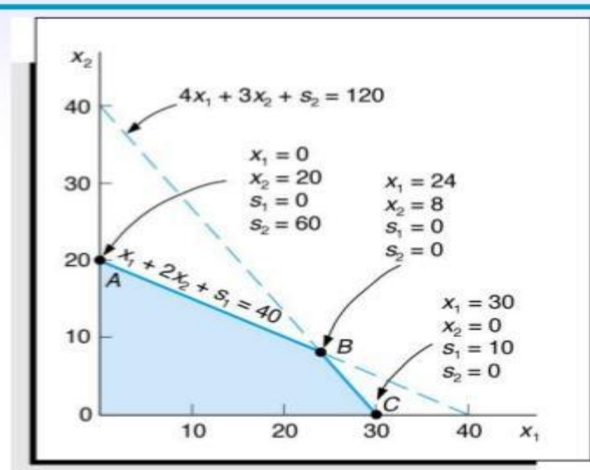


Figure 2.14 Solution Points A, B, and C with Slack

2-30

LP Model Formulation – Minimization (1 of 8)

- Two brands of fertilizer available - Super-gro, Crop-quick.
- Field requires at least 16 pounds of nitrogen and 24 pounds of phosphate.
- Super-gro costs \$6 per bag, Crop-quick \$3 per bag.
- Problem: How much of each brand to purchase to minimize total cost of fertilizer given following data ?

Brand	Chemical Contribution	
	Nitrogen (lb/bag)	Phosphate (lb/bag)
Super-gro	2	4
Crop-quick	4	3

2-31

LP Model Formulation – Minimization (3 of 8)

Decision Variables:

x_1 = bags of Super-gro
x_2 = bags of Crop-quick

The Objective Function:

Minimize $Z = \$6x_1 + 3x_2$
Where: $\$6x_1$ = cost of bags of Super-Gro
$\$3x_2$ = cost of bags of Crop-Quick

Model Constraints:

$2x_1 + 4x_2 \geq 16$ lb (nitrogen constraint)
$4x_1 + 3x_2 \geq 24$ lb (phosphate constraint)
$x_1, x_2 \geq 0$ (non-negativity constraint)

2-33

Constraint Graph – Minimization (4 of 8)

Minimize $Z = \$6x_1 + \$3x_2$
subject to: $2x_1 + 4x_2 \geq 16$
$4x_2 + 3x_2 \geq 24$
$x_1, x_2 \geq 0$

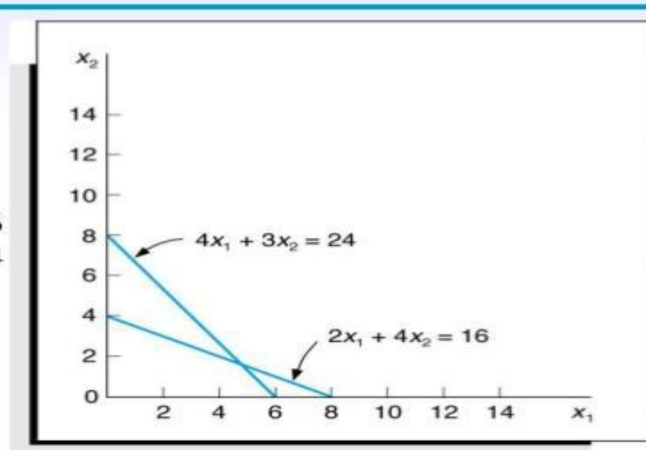


Figure 2.16 Graph of Both Model Constraints

2-34

Feasible Region– Minimization (5 of 8)

Minimize $Z = \$6x_1 + \$3x_2$
subject to: $2x_1 + 4x_2 \geq 16$
$4x_2 + 3x_2 \geq 24$
$x_1, x_2 \geq 0$

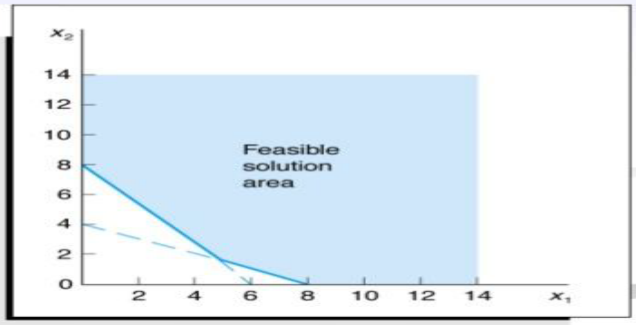


Figure 2.17 Feasible Solution Area

2-35

Optimal Solution Point – Minimization (6 of 8)

Minimize $Z = \$6x_1 + \$3x_2$
subject to: $2x_1 + 4x_2 \geq 16$
$4x_2 + 3x_2 \geq 24$
$x_1, x_2 \geq 0$

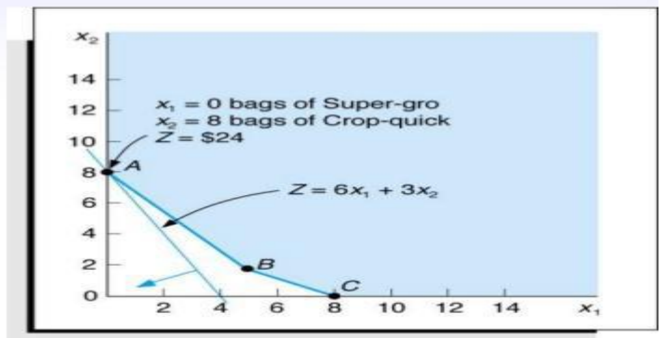


Figure 2.18 Optimum Solution Point

2-36

Surplus Variables – Minimization (7 of 8)

- A surplus variable is ***subtracted from a ≥ constraint*** to convert it to an equation (=).
- A surplus variable ***represents an excess*** above a constraint requirement level.
- A surplus variable ***contributes nothing*** to the calculated value of the objective function.
- Subtracting surplus variables in the farmer problem constraints:

$$2x_1 + 4x_2 - s_1 = 16 \text{ (nitrogen)}$$
$$4x_1 + 3x_2 - s_2 = 24 \text{ (phosphate)}$$

2-37

Graphical Solutions – Minimization (8 of 8)

Minimize $Z = \$6x_1 + \$3x_2 + 0s_1 + 0s_2$

subject to: $2x_1 + 4x_2 - s_1 = 16$

$4x_2 + 3x_2 - s_2 = 24$

$x_1, x_2, s_1, s_2 \geq 0$

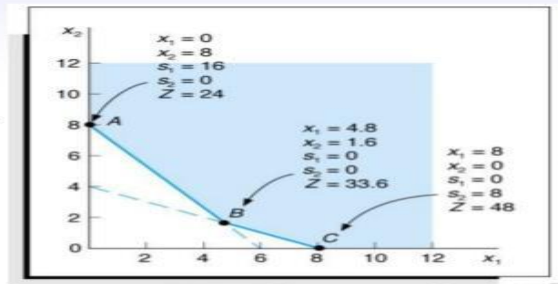


Figure 2.19 Graph of Fertilizer Example

2-38

Irregular Types of Linear Programming Problems

For some linear programming models, the general rules do not apply.

- Special types of problems include those with:
  - Multiple optimal solutions
  - Infeasible solutions
  - Unbounded solutions

2-39

Multiple Optimal Solutions Beaver Creek Pottery

The objective function is **parallel** to a constraint line.

Maximize $Z=\$40x_1 + 30x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_2 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

Where:

x_1 = number of bowls

x_2 = number of mugs

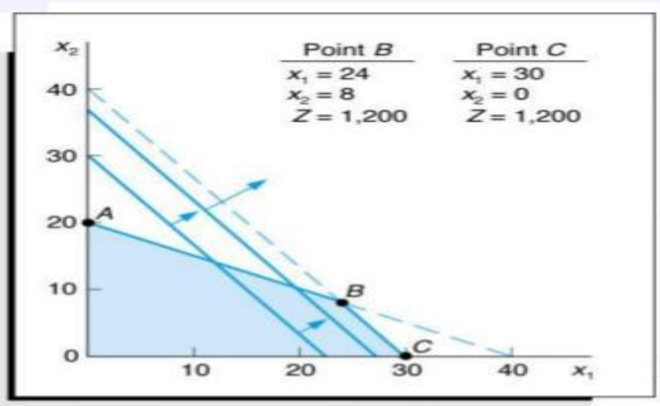


Figure 2.20 Example with Multiple Optimal Solutions

2-40

An Infeasible Problem

Every possible solution **violates** at least one constraint:

Maximize $Z = 5x_1 + 3x_2$

subject to: $4x_1 + 2x_2 \le 8$

$x_1 \ge 4$

$x_2 \ge 6$

$x_1, x_2 \ge 0$

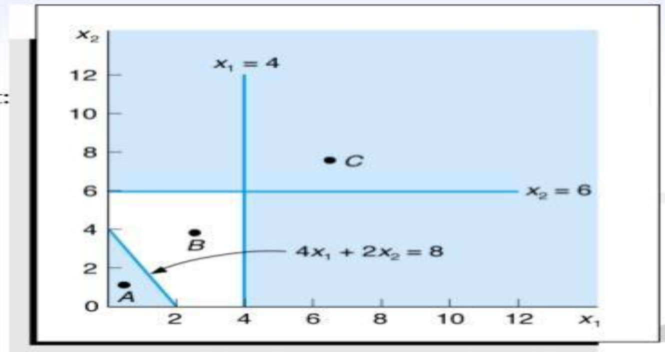


Figure 2.21 Graph of an Infeasible Problem

2-41

An Unbounded Problem

Value of the objective function increases indefinitely:

Maximize $Z = 4x_1 + 2x_2$

subject to: $x_1 \ge 4$

$x_2 \le 2$

$x_1, x_2 \ge 0$

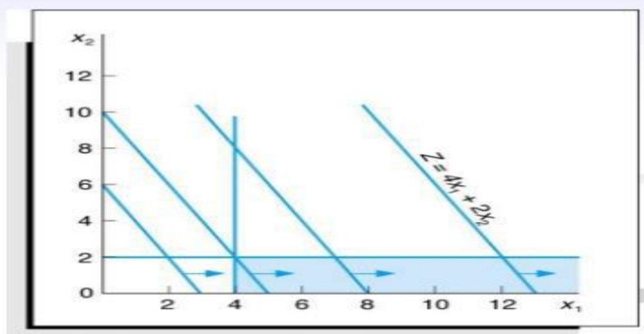


Figure 2.22 Graph of an Unbounded Problem

2-42

Problem Statement Example Problem No. 1 (1 of 3)

- Hot dog mixture in 1000-pound batches.
- Two ingredients, chicken (\$3/lb) and beef (\$5/lb).
- Recipe requirements:

  at least 500 pounds of "chicken"

  at least 200 pounds of "beef"
- Ratio of chicken to beef must be at least 2 to 1.
- Determine optimal mixture of ingredients that will minimize costs.

2-43

Solution
Example Problem No. 1 (2 of 3)

Step 1:

Identify decision variables.

x_1 = lb of chicken in mixture

x_2 = lb of beef in mixture

Step 2:

Formulate the objective function.

Minimize $Z = \$3x_1 + \$5x_2$

where Z = cost per 1,000-lb batch

$\$3x_1$ = cost of chicken

$\$5x_2$ = cost of beef

2-44

Solution
Example Problem No. 1 (3 of 3)

Step 3:

Establish Model Constraints

$x_1 + x_2 = 1{,}000$ lb

$x_1 \geq 500$ lb of chicken

$x_2 \geq 200$ lb of beef

$x_1/x_2 \geq 2/1$ or $x_1 - 2x_2 \geq 0$

$x_1, x_2 \geq 0$

The Model: Minimize $Z = \$3x_1 + 5x_2$

subject to: $x_1 + x_2 = 1{,}000$ lb

$x_1 \geq 50$

$x_2 \geq 200$

$x_1 - 2x_2 \geq 0$

$x_1, x_2 \geq 0$

2-45

Example Problem No. 2 (1 of 3)

Solve the following model graphically:

Maximize $Z = 4x_1 + 5x_2$

subject to: $x_1 + 2x_2 \leq 10$

$6x_1 + 6x_2 \leq 36$

$x_1 \leq 4$

$x_1, x_2 \geq 0$

Step 1: Plot the constraints as equations

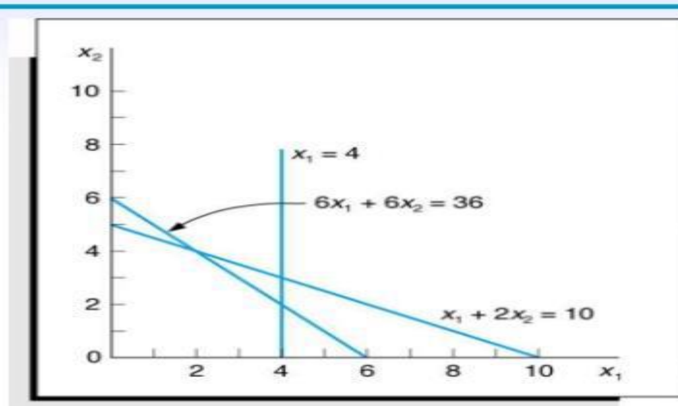


Figure 2.23 Constraint Equations

2-46

Example Problem No. 2 (2 of 3)

Maximize $Z = 4x_1 + 5x_2$

subject to: $x_1 + 2x_2 \leq 10$

$6x_1 + 6x_2 \leq 36$

$x_1 \leq 4$

$x_1, x_2 \geq 0$

Step 2: Determine the feasible solution space

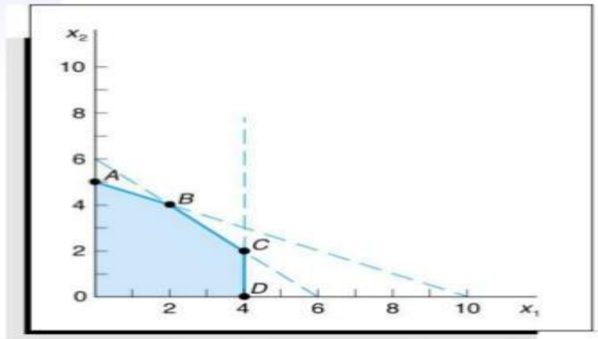


Figure 2.24 Feasible Solution Space and Extreme Points

2-47

Example Problem No. 2 (3 of 3)

Maximize $Z = 4x_1 + 5x_2$

subject to: $x_1 + 2x_2 \leq 10$

$6x_1 + 6x_2 \leq 36$

$x_1 \leq 4$

$x_1, x_2 \geq 0$

Step 3 and 4: Determine the solution points and optimal solution

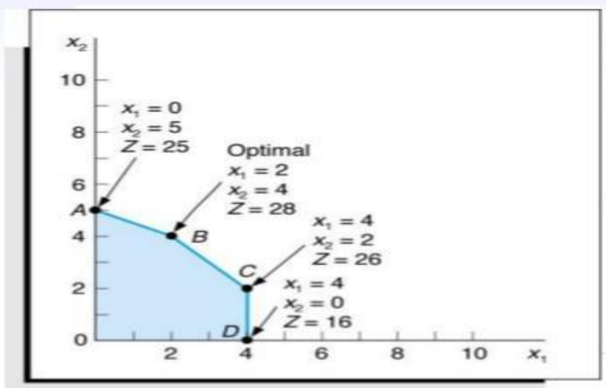


Figure 2.25 Optimal Solution Point

2-48

Simplex Method

Linear programming models could be solved algebraically. The most widely used algebraic procedure for solving linear programming problem is called the Simplex Method. The simplex method is a general-purpose linear-programming algorithm widely used to solve large scale problems. Although it lacks the intuitive appeal of the graphical approach, its ability to handle problems with more than two decision variables makes it extremely valuable for solving problems often encountered in production/operations management. Thus simplex method offers an efficient means of solving more complex linear programming problems.

Characteristics of Simplex Method

In the simplex method, the computational routine is an iterative process. To *iterate* means to repeat; hence, in working toward the optimum solution, the computational routine is repeated over and over, following a standard pattern.

Successive solutions are developed in a systematic pattern until the best solution is reached. Each new solution will yield a value of the objective function as large as or larger than the previous solution. This important feature assures us that we are always moving closer to the optimum answer. Finally, the method indicates when the optimum solution has been reached.

Most real-life linear programming problems have more than two variables, so a procedure called the simplex method is used to solve such problems. This procedure solves the problem in an iterative manner, that is, repeating the same set of procedures time after time until an optimal solution is reached. Each iteration brings a higher value for the objective function so that we are always moving closer to the optimal solution.

The simplex method requires simple mathematical operations (addition, subtraction, multiplication, and division), but the computations are lengthy and tedious, and the slightest error can lead to a good deal of frustration. For these reasons, most users of the technique rely on computers to handle the computations while they concentrate on the solutions. Still, some familiarity with manual computations is helpful in understanding the simplex process. The student will discover that it is better not to use his/her calculator in working through these problems because rounding can easily distort the results. Instead, it is better to work with numbers in fractional form.

Why we should study the Simplex Method?

It is important to understand the ideas used to produce solution. The simplex approach yields not only the optimal solution to the xi variables, and the maximum profit (or minimum cost) but valuable economic information as well.

To be able to use computers successfully and to interpret LP computer print outs, we need to know what the simplex method is doing and why.

We begin by solving a maximization problem using the simplex method. We then tackle a minimization problem.

STEPS OF THE SIMPLEX METHOD FOR MAX. PROBLEMS

Step1. Formulate a LP model of the problem.

Step2. Add slack variables to each constraint to obtain standard form.

Step3. Set up the initial simplex tableau.

Step4. Choose the non-basic variable with the largest entry in the net evaluation row

(Cj – Zj) to bring into the basis. This identifies the pivot (key) column; the column associated with the incoming variable.

Step5. Choose as the pivot row that row with the smallest ratio of ―bi/ aij‖, for aij >0 where j is the pivot column. This identifies the pivot row, the row of the variable leaving the basis when variable j enters.

Step6. A). Divide each element of the pivot row by the pivot element.

B). According to the entering variable, find the new values for remaining variables.

Step7. Test for optimality. If Cj – Zj ≤ 0 for all columns, we have the optimal solution. If not, return to step 4.

STEPS IN SIMPLEX METHOD FOR MIN. PROBLEMS

1. Formulate a LP model of the problem/ Set up the inequalities & equalities describing the problem constraints/.

2. Convert any inequalities to equalities by introducing surplus variable.

3. Add artificial variable to all equalities involving surplus variable.

4. Enter the resulting equalities in the simplex table.

5. Calculate Zj & Cj-Zj value for this solution.

6. Determining the interring variable selecting the one with the highest negative Cj-Zj value.

7. Determining the departing/removing row by choosing the smallest non-negative ratio ignoring infinity & negative.

8. Compute the value for the new key/pivot/ row.

9. Calculate the values for the remaining row.

10. Calculate the Cj-Zj e value for this solution.

11. If there is negative element in the Cj-Zj row then return to step 6. If there is no negative Cj-Zj then the final solution has been obtained.

Simplex method with mixed constraints

The following point may be noted in this regard.

1. A situation may arise when the constraints are ≥ type or ≤ type or = type i.e mixed constraints.

2. If any constraint has negative constant at the RHS, then multiply both sides with -1 & it will reverse the direction of inequalities.

3. Introduce slack variable in case of ≤ type constraint.

4. Introduce surplus & artificial variables in case of ≥ type constraints.

5. Introduce artificial variable in case of = type constraints.

6. Assign zero coefficients to the slack variable & surplus variable in the objective function.

7. Assign +M in case of min. & -M in case of max. Problems to the artificial variables in the objective function.

8. In case of max. Stop where Cj-Zj row contains zero or negative elements.

9. In case of min. stop where Cj-Zj row contains zero or positive elements.

Where, Zj –be contribution loss per unit

Cj- be contribution per unit to the objective function

Cj-Zj be increase/decrease per unit

Zj=Σ (coefficient of Cj column × corresponding coefficients of the constraints)

Example 1

A Furniture Ltd. wants to determine the most profitable combination of products to manufacture given that its resources are limited. The Furniture Ltd., Makes two products, *tables* and *chairs*, which must be processed through *assembly* and *finishing departments*. Assembly has 60 hours available; Finishing can handle up to 48 hours of work. Manufacturing one table requires 4 hours in assembly and 2 hours in finishing. Each chair requires 2 hours in assembly and 4 hours in finishing. Profit is $8 per table and $6 per chair.

	Hours required for 1 unit of product		Total hours
	Tables	*Chairs*	available
Assembly	4	2	60
Finishing	2	4	48
Profit per unit	$8	$6	

Stated algebraically, the Ltd., problem is

Maximise! Profit $Z = 8X_1 + 6X_2$

Subject to :

Assembly	$4X_1 + 2X_2 \leq 60$
Finishing	$2X_1 + 4X_2 \leq 48$
	All variables ≥ 0

- The first step is to convert the inequalities into equations.

The best combination of tables and chairs may not necessarily use all the time available in each department. We must therefore add to each inequality a variable, which will take up the slack, i.e. the time not used in each department. This variable is called a slack variable.

By adding the slack variables we convert the constraint inequalities in the problem into equations. The slack variable in each department takes on whatever value is required to make the equation relationship hold.

∴ The final form is

Maximize Profit $Z = 8X_1 + 6X_2 + 0S_1 + 0S_2$

Subject to

$$4X_1 + 2X_2 + S_1 = 60$$
$$2X_1 + 4X_2 + S_2 = 60$$

All variables ≥ 0

Tabular solution for Example 1

- The 2nd step is to put the equations into tabular form, called ***tableaus***.

Product mix column

Profit per unit column

constant column (quantities of product in the mix)

Variable columns

Cj			\$8	6	0	0	Cj row
	Product mix	Quantity	X_1	X_2	S_1	S_2	Variable row
\$0	S_1	60	4	2	1	0	
0	S_2	48	2	4	0	1	

Real products slack time

The simplest starting solution is to make no tables or chairs, have all unused time and earn no profit. This solution is technically feasible but not financially attractive. (Because the variables X_1 and X_2 do not appear in the mix, they are equal to zero.)

To find the profit for each solution and to determine whether the solution can be improved upon, we need to add two more rows to the initial simplex tableau: a Z_j row and a $C_j - Z_j$ row.

Column Z_j = Total profit from this particular solution

Tabular solution for Example 1

The four values for Z_j under the variable columns (all 0 $) are the amounts by which profit would be reduced if 1 unit of any of the variables were added to the mix.

Cj			$8	6	0	0	
	Product mix	Quantity	X_1	X_2	S_1	S_2	
$0	S_1	60	4	2	1	0	
0	S_2	48	2	4	0	1	
	Zj	$0	0	0	0	0	
	Cj – Zj		8	6	0	0	

Max

Z_j represents the gross profit given up by adding 1 unit of this variable into the current solution (profit loss per unit). $C_j - Z_j$ is net profit from the introduction 1 unit of each variable into the solution.

By examining the numbers in the $C_j - Z_j$ row we can see that total profit can be increased by 48 for each unit of X_1 (tables). Positive number indicates that profits can be improved for each unit added. We select the largest positive value. Max $C_j - Z_j$ value showing the variable that should be added, replacing one of the variables present in the mix.

- The next step is to determine which variable will be replaced.

  This is done in the following manner:

  Divide quantity column values by their corresponding numbers in the maximum (optimum) column and select the row with the smallest nonnegative ratio as the row to be replaced.

  S_1 row 60/4 = 15 units of Table (X_1) → minimum replaced row.

  S_2 row 48/4 = 24 units of Table (X_1)

1st Simplex Tableau

Cj			$8	6	0	0	
	Product mix	Quantity b_i	X_1	X_2	S_1	S_2	b_i/a_{ij}
0	S_1	60	4	2	1	0	60/4 = 15
0	S_2	48	2	4	0	1	48/2 = 24
	Zj	$0	$0	0	0	0	Intersectional elements
	Cj – Zj		8	6	0	0	(key #)

Max. (optimum entering variable)

a_{ij} = coefficient associated with variable j in the constraint i

For 2nd Simplex Tableau

X_1 = 60/4 = 15, 4/4 = 1, 2/4 = 1/2, 1/4 = 1/4 , 0/4 = 0

Thus new X_1 row should be (15, 1 , 1/2 , 1/4 , 0),

The new values for remaining rows:

[elements in old row] - [key #] x [corresponding elements in replacing row] = new row

Elements in old row	–	key #	x	replacing row	=	new row
48	–	2	x	15	=	18
2	–	2	x	1	=	0
4	–	2	x	1/2	=	3
0	–	2	x	1/4	=	-1/2
1	–	2	x	0	=	1

The computation of Z_j row for 2nd tableau is as follows.

Z_j for X_1 8 x 1 + 0 (0) = 8
Z_j for X_2 8 (1/2) + 0 (3) = 4
Z_j for S_1 8 (1/4) + 0 (-1/2) = 2
Z_j for S_2 8 (0) + 0 (1) = 0

} Profit given up by introducing 1 unit of these variables

Z_j (total profit) = 8 (15) + 0 (18) = $120

2nd Simplex Tableau

Cj			$8	6	0	0	
	Product mix	Quantity b_i	X_1	X_2	S_1	S_2	b_i/a_{ij}
$8	X_1	15	1	1/2	1/4	0	15/ ½ = 30
0	S_2	18	0	3	-1/2	1	18/3 = 6 → min! leaving
	Z_j	$120	$8	4	2	0	
	$C_j - Z_j$		$0	2	-2	0	

Max! entering

X_2 will enter in the product mix and S_2 is leaving.

New X_2 values: 18/3 = 6, 0/3 = 0, 3/3 = 1, -½ /3 = -1/6, 1/3 = 1/3
Thus new X_2 (replacing row) values = 6, 0, 1, -1/6, 1/3 (Assumes same row position as the replaced row)

New Values for X_1 :

Elements in old X_1 row	−	key #	x	replacing row	=	new X_1 row
15	−	1/2	x	6	=	12
1	−	1/2	x	0	=	1
1/2	−	1/2	x	1	=	0
1/4	−	1/2	x	-1/6	=	1/3
0	−	1/2	x	1/3	=	-1/6

New Z_j values :

Z_j (total profit) = 8 (12) + 6 (6) = $132
Z_j for X_1 8 (1) + 6 (0) = 8
Z_j for X_2 8 (0) + 6 (1) = 6
Z_j for S_1 8 (1/3) + 6 (-1/6) = 5/3
Z_j for S_2 8 (- 1/6) + 6 (1/3) = 2/3

3rd Simplex Tableau

Cj			$8	6	0	0	
	Product mix	Quantity b_i	X_1	X_2	S_1	S_2	
$8	X_1	12	1	0	1/3	-1/6	
6	X_2	6	0	1	-1/6	1/3	
	Z_j	$132	$8	6	5/3	2/3	
	$C_j - Z_j$		$0	0	-5/3	-2/3	

There is no positive "$C_j - Z_j$" value, no further profit improvement is possible. Thus the optimum solution is obtained. Profit will be maximized by making 12 tables and 6 chairs and having no unused time in either department (because slack variables do no appear in the product-mix column and are equal to zero). Optimum profit is $132 .

Verification:

Objective function

$Z_j = 8 X_1 + 6 X_2 + 0 \ (S_1 + S_2)$
$Z_j = 8 (12) + 6 (6) + 0 = \$132$

Constraints :

Assembly $4 X_1 + 2 X_2 \leq 60 \rightarrow 4 (12) + 2 (6) \leq 60 \rightarrow 60 \leq 60$
Finishing $2 X_1 + 4 X_2 \leq 48 \rightarrow 2 (12) + 4 (6) \leq 48 \rightarrow 48 \leq 48$

Example 2

PAR Inc. produces golf equipment and decided to move into the market for standard and deluxe golf bags. Each golf bag requires the following operations: Cutting and dyeing the material, Sewing, Finishing (inserting umbrella holder, club separators etc.), Inspection and packaging. Each standard golf-bag will require 7/10 hr. in the cutting and dyeing department, 1/2 hr. in the sewing department, 1 hr. in the finishing department and 1/10 hr. in the inspection & packaging department.

Deluxe model will require 1 hr. in the cutting and dyeing department, 5/6 hr. for sewing, 2/3 hr. for finishing and 1/4 hr. for inspection and packaging

The profit contribution for every standard bag is 10 MU and for every deluxe bag is 9 MU.

In addition the total hours available during the next 3 months are as follows:

Cutting & dyeing dept. 630 hrs.

Sewing dept. 600 hrs.

Finishing 708 hrs.

Inspection & packaging 135 hrs.

The company‘s problem is to determine how many standard and deluxe bags should be produced in the next 3 months?

Let X_1 = number of standard bags

X_2 = number of deluxe bags

Z = the total profit contribution

Objective function:

$$\text{Max!} \quad Z = 10X_1 + 9X_2$$

Subject to constraints

$7/10\,X_1 + 1X_2 \le 630$	cutting and dying
$1/2\,X_1 + 5/6\,X_2 \le 600$	sewing
$1X_1 + 2/3\,X_2 \le 708$	finishing
$1/10\,X_1 + 1/4\,X_2 \le 135$	inspection & packaging
$X_1 \ge 0$, $X_1 \ge 0$	Nonnegative constraints

In linear programming terminology, any unused or idle capacity for a ≤ constraint is referred to as the slack associated with the constraint. Often variables, called slack variables, are added to the formulation of a linear programming problem to represent the slack or idle capacity. Unused capacity makes no contribution to profit; thus slack variables have coefficients of zero in the objective function.

Whenever a linear program is written in a form with all constraints expressed as equalities, it is said to be written in standard form.

After the addition of slack variables to the mathematical statement, the mathematical model becomes

$$\text{Max!} \quad 10X_1 + 9X_2 + 0S_1 + 0S_2 + 0S_3 + 0S_4$$

Subject to

$$\begin{aligned}
7/10\,X_1 + 1X_2 + 1S_1 &= 630\\
1/2\,X_1 + 5/6\,X_2 + 1S_2 &= 600\\
1X_1 + 2/3\,X_2 + 1S_3 &= 708\\
1/10\,X_1 + 1/4\,X_2 + 1S_4 &= 135\\
X_1, X_2, S_1, S_2, S_3, S_4 &\ge 0
\end{aligned}$$

Tabular solution for Example 2

Initial Tableau

C_j			10 MU	9 MU	0 MU	0 MU	0 MU	0 MU	
	Product mix	Quantity b_i	X_1	X_2	S_1	S_2	S_3	S_4	b_i / a_{ij}
0 MU	S_1	630	7/10	1	1	0	0	0	630/ 7/10 =900
0 MU	S_2	600	½	5/6	0	1	0	0	600/ ½ = 1200
0 MU	S_3	708	1	2/3	0	0	1	0	708/1 = 708 (min leaving)
0 MU	S_4	135	1/10	¼	0	0	0	1	135/ 1/10 = 1350
	Z_j	0 MU	0 MU	0 MU	0 MU	0 MU	0 MU	0 MU	
	$C_j - Z_j$		10 MU	9 MU	0 MU	0 MU	0 MU	0 MU	

Max. (Entering)

For the 2nd Simplex Tableau

708/1 = 708, 1/1 = 1, 2/3/1 = 2/3, 0, 0, 1, 0

∴ new X_1 value are : 708, 1, 2/3, 0, 0, 1, 0

Elements in old S_1 row	–	key #	x	new X_1 row	=	new S_1 row
630	–	7/10	x	708	=	134.4
7/10	–	7/10	x	1	=	0
1	–	7/10	x	2/3	=	8/15
1	–	7/10	x	0	=	1
0	–	7/10	x	0	=	0
0	–	7/10	x	1	=	-7/10
0	–	7/10	x	0	=	0

Elements in old S_2 row	–	key #	x	new X_1 row	=	new S_2 row
600	–	1/2	x	708	=	246
1/2	–	1/2	x	1	=	0
5/6	–	1/2	x	2/3	=	1/2
0	–	1/2	x	0	=	0
1	–	1/2	x	0	=	1
0	–	1/2	x	1	=	-1/2
0	–	1/2	x	0	=	0

Elements in old S_4 row	–	key #	x	new X_1row	=	new S_4 row
135	–	1/10	x	708	=	64.2
1/10	–	1/10	x	1	=	0
1/4	–	1/10	x	2/3	=	11/60
0	–	1/10	x	0	=	0
0	–	1/10	x	0	=	0
0	–	1/10	x	1	=	-1/10
1	–	1/10	x	0	=	1

2nd Tableau

C_j			10 MU	9 MU	0 MU	0 MU	0 MU	0 MU	
	Product mix	Quantity b_i	X_1	X_2	S_1	S_2	S_3	S_4	b_i / a_{ij}
0 MU	S_1	134.4	0	8/15	1	0	-7/10	0	252 (min leaving)
0 MU	S_2	246	0	½	0	1	-1/2	0	492
10 MU	X_1	708	1	2/3	0	0	1	0	1062
0 MU	S_4	64.2	0	11/60	0	0	-1/10	1	3852/11
	Z_j	7080 MU	10	20/3	0	0	10	0	
	$C_j - Z_j$		0	7/3	0	0	-10	0	

Max. (Entering)

New X_2 values are: $\frac{134.4}{}$ = 252 , 0, 1, 15/8, 0, -21/16, 0

Elements in old S_2 row	–	key #	x	new X_2row	=	new S_2 row
246	–	1/2	x	252	=	120
0	–	1/2	x	0	=	0
1/2	–	1/2	x	1	=	0
0	–	1/2	x	15/8	=	-15/16
1	–	1/2	x	0	=	1
-1/2	–	1/2	x	-21/16	=	5/32
0	–	1/2	x	0	=	0

Elements in old X_1row	–	key #	x	new X_2row	=	new X_1 row
708	–	2/3	x	252	=	540
1	–	2/3	x	0	=	1
2/3	–	2/3	x	1	=	0
0	–	2/3	x	15/8	=	-5/4
0	–	2/3	x	0	=	0
1	–	2/3	x	-21/16	=	15/8
0	–	2/3	x	0	=	0

Elements in old S_4 row	–	key #	x	new X_2row	=	new S_4 row
64.2	–	11/60	x	252	=	18
0	–	11/60	x	0	=	0
11/60	–	11/60	x	1	=	0
0	–	11/60	x	15/8	=	-11/32
0	–	11/60	x	0	=	0
-1/10	–	11/60	x	-21/10	=	45/320
1	–	11/60	x	0	=	1

3^{rd} Tableau

Cj			10 MU	9 MU	0 MU	0 MU	0 MU	0 MU
	Product mix	Quantity b_i	X_1	X_2	S_1	S_2	S_3	S_4
9 MU	X_2	252	0	1	15/8	0	-21/16	0
0 MU	S_2	120	0	0	-15/16	1	5/32	0
10 MU	X_1	540	1	0	-5/4	0	15/8	0
0 MU	S_4	18	0	0	-11/32	0	45/320	1
	Zj	7668 MU	10	9	135/8+(5/4) = 35/8	0	-189/16+ 300/16=111/16	0
	Cj − Zj		0	0	-35/8	0	-111/16	0

There is no positive Cj − Zj value in the simplex tableau. Therefore no further profit improvement is possible. Thus the optimum solution is obtained.

- Thus: Standard bag production (X_1) = 540 bags .
  Deluxe bag production (X_2) = 252 bags .
- Maximum profit = Z = \$10 (540) + \$9 (252) = \$7668
- Unused hours in Sewing department = 120 hours
  Inspection and packaging department = 18 hours

Example 3

High Tech industries import components for production of two different models of personal computers, called deskpro and portable. High Tech's management is currently interested in developing a weekly production schedule for both products.

The deskpro generates a profit contribution of \$50/unit, and portable generates a profit contribution of \$40/unit. For next week's production, a max of 150 hours of assembly time is available. Each unit of deskpro requires 3 hours of assembly time. And each unit of portable requires 5 hours of assembly time.

High Tech currently has only 20 portable display components in inventory; thus no more than 20 units of portable may be assembled. Only 300 sq. feet of warehouse space can be made available for new production. Assembly of each Deskpro requires 8 sq. ft. of warehouse space, and each Portable requires 5 sq. ft. of warehouse space.

	X_1 - Deskpro	X_2 – Portable	Capacity
Assembly line	3	5	150
Portable Ass	-	1	20
Space	8	5	300
Profit Cont.	\$50/unit	\$40/unit	

Tabular solution for Example 3

	X_1 - Deskpro	X_2 – Portable	Capacity
Assembly line	3	5	150
Portable Ass	-	1	20
Space	8	5	300
Profit Cont.	$50/unit	$40/unit	

X_1 = *number of units of the Deskpro* X_2 = *number of units of the Portable*

Objective Function : Max! $Z = 50 X_1 + 40 X_2$

Subject to :

$3 X_1 + 5 X_2 \leq 150$ Assembly time

$1 X_2 \leq 20$ Portable display

$8 X_1 + 5 X_2 \leq 300$ Warehouse capacity

$X_1, X_2 \geq 0$

Adding a slack variable to each of the constraints permits us to write the problem in standard form:

Objective Function: Max! $Z = 50 X_1 + 40 X_2 + 0 S_1 + 0S_2 + 0S_3$

Subject to:

$$3 X_1 + 5 X_2 + 1S_1 = 150$$
$$1 X_2 + 1S_2 = 20$$
$$8 X_1 + 5 X_2 + 1S_3 = 300$$
$$X_1, X_2, S_1, S_2, S_3 \geq 0$$

Initial Tableau

C_j			$50	$40	$0	$0	$0	
	Product mix	Quantity bi	X_1	X_2	S_1	S_2	S_3	b_i / a_{ij}
$0	S_1	150	3	5	1	0	0	150/3 = 50
$0	S_2	20	0	1	0	1	0	--
$0	S_3	300	8	5	0	0	1	300/8 = 37.5 (min. leaving)
	Z_j	$0	$0	$0	$0	$0	$0	
	$C_j - Z_j$		$50	$40	$0	$0	$0	

Max. (entering)

New X_1 value : 300/8 = 37.5, 8/8 = 1, 5/8, 0, 0, 1/8

Old S_1 row	–	key #	x	new X_1 row	=	new S_1 row
150	–	3	x	75/2	=	37.5
3	–	3	x	1	=	0
5	–	3	x	5/8	=	25/8
1	–	3	x	0	=	1
0	–	3	x	0	=	0
0	–	3	x	1/8	=	-3/8

Old S_2 row	–	key #	x	new X_1row	=	new S_2 row
20	–	0	x	75/2	=	20
0	–	0	x	1	=	0
1	–	0	x	5/8	=	1
0	–	0	x	0	=	0
1	–	0	x	0	=	1
0	–	0	x	1/8	=	0

2^{nd} Tableau

Cj			\$50	\$40	\$0	\$0	\$0	
	Product mix	Quantity bi	X_1	X_2	S_1	S_2	S_3	b_i / a_{ij}
\$0	S_1	75/2	0	25/8	1	0	-3/8	75/2 / 25/8 = 12 (min leaving)
\$0	S_2	20	0	1	0	1	0	20/1=20
\$50	X_1	75/2	1	5/8	0	0	1/8	75/2 / 5/8 = 60
	Zj	\$1875	\$50	\$250/8	\$0	\$0	\$50/8	
	Cj – Zj		\$0	\$70/8	\$0	\$0	\$-50/8	

Max. (Entering)

New X_2 values : 12, 0, 1, 8/25, 0, -3/25

old S_2 row	–	key #	x	new X_2row	=	new S_2 row
20	–	1	x	12	=	8
0	–	1	x	0	=	0
1	–	1	x	1	=	0
0	–	1	x	8/25	=	-8/25
1	–	1	x	0	=	1
0	–	1	x	-3/25	=	3/25

old X_1row	–	key #	x	new X_2row	=	new X_1 row
75/2	–	5/8	x	12	=	30
1	–	5/8	x	0	=	1
5/8	–	5/8	x	1	=	0
0	–	5/8	x	8/25	=	-1/5
0	–	5/8	x	0	=	0
1/8	–	5/8	x	-3/25	=	1/5

3^{rd} Tableau

Cj			\$50	\$40	\$0	\$0	\$0
	Product mix	Quantity Bi	X_1	X_2	S_1	S_2	S_3
\$40	X_2	12	0	1	8/25	0	-3/25
\$0	S_2	8	0	0	-8/25	1	3/25
\$50	X_1	30	1	0	-1/5	0	1/5
	Z_j	\$1980	\$50	\$40	\$14/5	\$0	\$26/5
	$C_j – Z_j$		\$0	\$0	\$-14/5	\$0	\$-26/5

The optimal solution to a linear programming problem has been reached when all of the entries in the net evaluation row $C_j – Z_j$ are zero or negative. In such cases, the optimal solution is the current basic feasible solution.

Thus:

Units of Deskpro production (X_1) = 30 units
Units of Portable production (X_2) = 12 units
S_2 = 8 units

Management should note that there would be eight unused Portable display units.
Maximum profit is \$1980.

Tableau Form: The Special Case

Obtaining tableau form is somewhat more complex if the LP contains ≥ constraints, = constraints, and/or "–ve" right-hand-side values. Here we will explain how to develop tableau form for each of these situations. Example 4, Suppose that in the high-tech industries problem, management wanted to ensure that the combined total production for both models would be at least 25 units. Thus,

Objective Function Max Z = 50X1 + 40X2

Subjective to: 3 X1 + 5 X2 ≤ 150 Assembly time

1X1 ≤ 20 Portable display

8X1 + 5 X2 ≤ 300 Warehouse space

$1X1 + 1X2 \geq 25$ Min. total production

$X_1, X_2 \geq 0$

SOLUTION: First, we use three slack variables and one surplus variable to write the problem in std. Form.

Max $Z = 50X1 + 40X2 + 0S1 + 0S2 + 0S3 + 0S4$

Subject to $3X1 + 5X2 + 1S1 = 150$

$1X2 + 1S2 = 20$

$8X1 + 5X2 + 1S3 = 300$

$1X1 + 1X2 - 1S4 = 25$

All variables ≥ 0

For the initial tableau $X_1 = 0$ $X_2 = 0$

$S_1 = 150$, $S_2 = 20$

$S_3 = 300$, $S_4 = -25$

Tabular solution for Example 4

Clearly this is not a basic feasible solution since S4 = -25 violates the non-negativity requirement.

∴We introduce new variable called ARTIFICIAL VARIABLE.

Artificial variables will be eliminated before the optimal solution is reached. We assign a very large cost to the variable in the objective function.

Objective function

$50X_1 + 40X_2 + 0S_1 + 0S_2 + 0S_3 + 0S_4 - MA_4$

Initial Tableau

C_j			50	40	0	0	0	0	-M	
	Product mix	Quantit bi	X_1	X_2	S_1	S_2	S_3	S_4	A_4	b_i / a_{ij}
0	S_1	150	3	5	1	0	0	0	0	150/3=50
0	S_2	20	0	1	0	1	0	0	0	--
0	S_3	300	8	5	0	0	1	0	0	300/8=37.5
-M	A_4	25	1	1	0	0	0	-1	1	25 Min. leaving
	Z_j	-25M	-M	-M	0	0	0	M	-M	
	$C_j - Z_j$		50+M	40+M	0	0	0	-M	0	

Max entering↑

New X_1 values = 25, 1, 1, 0, 0, 0, -1, 1

Old S_1 row	—	key #	x	new X_1 values	=	new S_1 row
150	—	3	x	25	=	75
3	—	3	x	1	=	0
5	—	3	x	1	=	2
1	—	3	x	0	=	1
0	—	3	x	0	=	0
0	—	3	x	0	=	0
0	—	3	x	-1	=	3
0	—	3	x	1	=	-3
Old S_2 row	—	**key #**	x	**new X_1 values**	=	**new S_2 row**
20	—	0	x	25	=	20
0	—	0	x	1	=	0
1	—	0	x	1	=	1
0	—	0	x	0	=	0
1	—	0	x	0	=	1
0	—	0	x	0	=	0
0	—	0	x	-1	=	0
0	—	0	x	1	=	0
Old S_3 row	—	**key #**	x	**new X_1 values**	=	**new S_3 row**
300	—	8	x	25	=	10
8	—	8	x	1	=	0
5	—	8	x	1	=	-3
0	—	8	x	0	=	0
0	—	8	x	0	=	0
1	—	8	x	0	=	1
0	—	8	x	-1	=	8
0	—	8	x	1	=	-8

2nd Tableau

Cj			\$50	40	0	0	0	0	-M	
	Prodt mix	Quant b_i	X_1	X_2	S_1	S_2	S_3	S_4	A_4	b_i / a_{ij}
\$0	S_1	75	0	2	1	0	0	3	-3	75/3=25
0	S_2	20	0	1	0	1	0	0	0	--
0	S_3	100	0	-3	0	0	1	8	-8	100/8=12.5 Min,leaving
50	X_1	25	1	1	0	0	0	-1	1	--
	Z_j	\$1250	50	50	0	0	0	-50	50	
	$C_j - Z_j$		0	-10	0	0	0	50	-M-50	

Max entering↑

New S4 values: 100/8 = 25/2, 0, -3/8, 0, 0, 1/8, 1

IMPORTANT NOTE!!

Since A4 is an artificial variable that was added simply to obtain an initial basic feasible solution, we can drop its associated column from the simplex tableau. Indeed whenever artificial variables are used, they can be dropped from the simplex tableau as soon as they have been eliminated from the basic feasible solution.

Old S_1 row	−	key #	x	new S_4 values	=	new S_1 row
75	−	3	x	25/2	=	75/2
0	−	3	x	0	=	0
2	−	3	x	-3/8	=	25/8
1	−	3	x	0	=	1
0	−	3	x	0	=	0
0	−	3	x	1/8	=	-3/8
3	−	3	x	1	=	0
Old S_2 row	−	**key #**	x	**new S_4 values**	=	**new S_2 row**
20	−	0	x	25/2	=	20
0	−	0	x	0	=	0
1	−	0	x	-3/8	=	1
0	−	0	x	0	=	0
1	−	0	x	0	=	1
0	−	0	x	1/8	=	0
0	−	0	x	1	=	0
Old X_1 row	−	**key #**	x	**new S_4 values**	=	**new X_1 row**
25	−	-1	x	25/2	=	75/2
1	−	-1	x	0	=	1
1	−	-1	x	-3/8	=	5/8
0	−	-1	x	0	=	0
0	−	-1	x	0	=	0
0	−	-1	x	1/8	=	1/8
-1	−	-1	x	1	=	0

3rd Tableau

C_j			50	40	0	0	0	0	
	Product mix	Quantity b_i	X_1	X_2	S_1	S_2	S_3	S_4	bi / a_{ij}
0	S_1	75/2	0	25/8	1	0	-3/8	0	12 Min. leaving
0	S_2	20	0	1	0	1	0	0	20
0	S_4	25/2	0	-3/8	0	0	1/8	1	---
50	X_1	75/2	1	5/8	0	0	1/8	0	60
	Z_j	1875	50	250/8	0	0	50/8	0	
	$C_j - Z_j$		0	70/8	0	0	-50/8	0	

Max. (Entering)

One more iteration is required. This time X2 comes into the solution and S1 is eliminated. After performing this iteration, the following simplex tableau shows that the optimal solution has been reached.

C_i			50	40	0	0	0	0
	Product mix	Quantity b_i	X_1	X_2	S_1	S_2	S_3	S_4
40	X_2	12	0	1	8/25	0	-3/25	0
0	S_2	8	0	0	-8/25	1	3/25	0
0	S_4	17	0	0	3/25	0	2/25	1
50	X_1	30	1	0	-5/25	0	5/25	0
	Z_i	1980	50	40	14/5	0	26/5	0
	$C_i - Z_i$		0	0	-14/5	0	-26/5	0

It turns out that the optimal solution has been reached. (All $C_i - Z_i \leq 0$ and all artificial variables have been eliminated.)

EQUALITY CONTRAINTS NEGATIVE RIGHT-HAND SIDE VALUES

Simply add an artificial variable A1 to create a basic feasible solution in the initial simplex tableau.

$6X1 + 4X2 - 5X3 = 30 \Rightarrow 6X1 + 4X2 - 5X3 + 1A1 = 30$

One of the properties of the tableau form of a linear program is that the values on the right-hand sides of the constraints have to be nonnegative.

E.g. One units of the portable model (X2) has to be less than or equal to the one units of the deskpro model (X1) after setting aside 5 units of the deskpro for internal company use.

I.e. $X2 \leq X1 - 5$

$- X1 + X2 \leq -5$

(Min)Multiply by –1 => (Max) $X1 - X2 \geq 5$

We now have an acceptable nonnegative right-hand-side value. Tableau form for this constraint can now be obtained by subtracting a surplus variable and adding an artificial variable.

Example 5

• Livestock Nutrition Co. produces specially blended feed supplements. LNC currently has an order for 200 kgs of its mixture.

• This consists of two ingredients

X_1 (a protein source)

X_2 (a carbohydrate source)

The first ingredient, X_1 costs LNC 3MU a kg. The second ingredient, X_2 costs LNC 8MU a kg. The mixture can‘t be more than 40% X1 and it must be at least 30% X2.

• LNC‘s problem is to determine how much of each ingredient to use to minimize cost.

Solution: The cost function can be written as Cost = 3X1 + 8X2 Min!

■LNC must produce 200 kgs of the mixture – no more, no less.

X1 + X2 = 200 kgs

• The mixture can‘t be more than 40% X1, so we may use less than 80 kgs. (40% X 200 = 80). However, we must not exceed 80 kgs.

$X_1 \leq 80$ kgs

• The mixture must be at least 30% X_2. Thus we may use more than 60 kgs, not less than 60 kgs. (30% X 200 = 60)

$X_2 \geq 60$ kgs
Minimize: Cost = 3MU X1 + 8MU X2
Subject to X1 + X2 = 200 kgs
X1 ≤ 80 kgs
X2 ≥ 60 kgs
X1, X2 ≥ 0
• An initial solution: X1 + X2 = 200 kgs

=> X1 + X2 + A1 = 20

↓
Artificial variable: A very expensive substance must not be represented in optimal solution.

• ***An artificial Variable*** is only of value as a computational device; it allows 2 types of restrictions to be treated. These are the equality type (=) & ≥ type

$X_1 \leq 80$ kgs constraint on protein
=> $X_1 + S1 = 80$ kgs
• $X_2 \geq 60$ kgs constraint on carbohydrates

=> X2 - S2 + A2 = 60

X1, X2, S1, S2, A1, A2 ≥ 0

Minimize: Cost = 3X1 + 8X2 + 0S1 + 0S2 + MA1 + MA2

Subject to: X1 + X2 + A1 = 200

X1 + S1 = 80

X2 - S2 + A2 = 60

All variables ≥ 0

Initial Tableau

C_j			3 MU	8 MU	M MU	0 MU	0 MU	M MU	
	Product mix	Quantity b_i	X_1	X_2	A_1	S_1	S_2	A_2	b_i / a_{ij}
M	A_1	200	1	1	1	0	0	0	200/1=200
0	S_1	80	1	0	0	1	0	0	--
M	A_2	60	0	1	0	0	-1	1	60/1=60 Min. replaced row
	Z_j	260M	M	2M	M	0	-M	M	
	$C_j - Z_j$		3-M	8-2M	0	0	M	0	

↑ Optimal column

Computation for 2nd tableau:

Replacing row = new X_2 values : 60/1=60, 0/1=0, 1/1=1, 0/1=0, -1/1=-1, 1/1=1

Old A_1 row	–	key #	x	new X_2 values	=	new A_1 row
200	–	1	x	60	=	140
1	–	1	x	0	=	1
1	–	1	x	1	=	0
1	–	1	x	0	=	1
0	–	1	x	0	=	0
0	–	1	x	-1	=	1
0	–	1	x	1	=	-1

Old S_1 row	–	key #	x	new X_2values	=	new S_1 row
80	–	0	x	60	=	80
1	–	0	x	0	=	1
0	–	0	x	1	=	0
0	–	0	x	0	=	0
1	–	0	x	0	=	1
0	–	0	x	-1	=	0
0	–	0	x	1	=	0

2nd Tableau

C_j			3	8	M	0	0	M	
	Product mix	Quantity b_i	X_1	X_2	A_1	S_1	S_2	A_2	b_i / a_{ij}
M	A_1	140	1	0	1	0	1	-1	140/1=140
0	S_1	80	1	0	0	1	0	0	80/1=80 replaced row, min.
8	X_2	60	0	1	0	0	-1	1	60/0= --
	Z_j	140M + 480	M	8	M	0	M-8	8-M	
	$C_j - Z_j$		3-M	0	0	0	8-M	2M-8	

Optimal column

Computations for 3rd Tableau

Replacing row = new X_1 values: 80/1=80, 1/1=1, 0/1=0, 0/1=0, 1/1=1, 0/1=0, 0/1=0

Old A_1 row	–	key #	x	new X_1 values	=	new A_1 row
140	–	1	x	80	=	60
1	–	1	x	1	=	0
0	–	1	x	0	=	0
1	–	1	x	0	=	1
0	–	1	x	1	=	-1
1	–	1	x	0	=	1
-1	–	1	x	0	=	-1

Old X_2 row	–	key #	x	new X_1 values	=	new X_2 row
60	–	0	x	80	=	60
0	–	0	x	1	=	0
1	–	0	x	0	=	1
0	–	0	x	0	=	0
0	–	0	x	1	=	0
-1	–	0	x	0	=	-1
1	–	0	x	0	=	1

3rd Tableau

C_j			3	8	M	0	0	M	
	Product mix	Quantity bi	X_1	X_2	A_1	S_1	S_2	A_2	b_i / a_{ij}
M	A_1	60	0	0	1	-1	1	-1	60/1=60 replaced row
3	X_1	80	1	0	0	1	0	0	80/0= --
8	X_2	60	0	1	0	0	-1	1	60/-1= - 60 not considered
	Z_j	60M – 720	3	8	M	3-M	M-8	8-M	
	$C_j - Z_j$		0	0	0	M-3	8-M	2M-8	

Optimal column

Computations for the 4^{th} Tableau

Replacing row = new S_2 values: 60/1=60, 0/1=0, 0/1=0, 1/1=1, -1/1=-1, 1/1=1, -1/1=-1

Old X_1 row	–	key #	x	new S_2 values	=	new X_1 row
80	–	0	x	60	=	80
1	–	0	x	0	=	1
0	–	0	x	0	=	0
0	–	0	x	1	=	0
1	–	0	x	-1	=	1
0	–	0	x	1	=	0
0	–	0	x	-1	=	0

Old X_2 row	–	key #	x	new S_2 values	=	new X_2 row
60	–	-1	x	60	=	120
0	–	-1	x	0	=	0
1	–	-1	x	0	=	1
0	–	-1	x	1	=	1
0	–	-1	x	-1	=	-1
-1	–	-1	x	1	=	0
1	–	-1	x	-1	=	0

4^{th} Tableau

Cj			3	8	M	0	0	M
	Product mix	Quantity bi	X_1	X_2	A_1	S_1	S_2	A_2
0	S_2	60	0	0	1	-1	1	-1
3	X_1	80	1	0	0	1	0	0
8	X_2	120	0	1	1	-1	0	0
	Z_j	1200	3	8	8	-5	0	0
	$C_j - Z_j$		0	0	M-8	5	0	M

$\because$ No negative values remain in the $C_j - Z_j$ row, we have reached the OPTIMAL solution.

It is to use 80 kgs of X_1 and 120 kgs of X_2. This results in a cost of 1200MU.
S_2 represents the amount of X_2 used over the minimum quantity required (60kg)

$$X_2 - S_2 + A_2 = 60 \longrightarrow 120 - 60 + 0 = 60 \longrightarrow 60 = 60 \because A_2 = 0.$$

SPECIAL CASES IN APPLYING SIMPLEX METHOD (COMPLICATED SITUATIONS)

Several complications can occur while solving the LPP. Such problems are

1. Tie for key rows or degeneracy
2. Unbounded problems
3. Multiple optimal solutions
4. Infeasible solutions
5. Tie for the key columns

1. **Tie for the key rows or Degeneracy**: In the simplex method degeneracy occurs when there is tie for the minimum ratio for choosing the departing/ leaving/ variable. The main drawback to degeneracy is the increase in the computation, which reduces the efficiency of simplex method.

As a general rule, the best way to break the tie between the key rows is to select any departing variable arbitrarily, if we are unlucky & cycling does occur we simply go back & select the other.

Alternatively: the following procedure is followed

i) Locate the rows in which smallest non-negative ratio are tied (equal).

ii) Find the coefficient of the slack variables & divide each coefficient by the corresponding positive numbers of the key column in the row, starting from the left to the right in order to break the tie.

iii) If the ratio does not break the tie, find the similar ratios for the coefficient of decision variables.

iv) Compare the resulting ratio, column by column.

v) Select the row which has the smallest ratio. This row becomes the key row.

vi) After resolving of this tie, simplex method is applied to obtain the optimum solution.

NB: If the tie has occurred between artificial variable & other variables, the artificial variable should be selected as departing variable without going for the above procedure.

Example: Solve the following problem by using simplex method.

$Max\ Z=1000x1+4000x2+5000x3$

$St.to\ 3x1+3x3\leq 22$

$x1+2x2+3x3\leq 14$

$3x1+2x2\leq 14$

$x1,\ x2,\ 3\geq 0$

Solution: By introduce slack variable convert inequality constraints into equations. Then the std. form of LPP is: $Max\ Z=1000x1+4000x2+5000x3+0s1+0s2+0s3$

$St.to\ 3x1+3x3+s1=22$

$x1+2x2+3x3+s2=14$

$3x1+2x2+s3=14$

$x1,2,x3,s1,s2,s3\geq 0$

Simplex Table I

Cj	Contribution per unit		1000	4000	5000	0	0	0	
	Basic Variables	Solution Values	X1	X2	X3	S1	S2	S3	Min.Ratio
0	S1	22	3	0	3	1	0	0	22/3
0	S2	14	1	2	3*	0	1	0	14/3← Key row
0	S3	14	3	2	0	0	0	1	∞
	Zj	0	0	0	0	0	0	0	
		Cj-Zj	1000	4000	5000	0	0	0	
					↑Max (entering)				

The entering variable is X3 & the leaving variable is S2 & also the key element is 3.
By introducing the entering variable & removing the leaving variable with the changed values the 2^{nd} simplex table is prepared as follows.

Simplex Table II

Cj	Contribution per unit		1000	4000	5000	0	0	0	
	Basic Variables	Solution Values	X1	X2	X3	S1	S2	S3	Min.Ratio
0	S1	8	2	-2	0	1	-1	0	8/-2=-4
5000	X3	14/3	1/3	2/3	1	0	1/3	0	7
0	S3	14	3	2*	0	0	0	1	7 ← Key row
	Zj	70000/3	5000/3	10000/3	5000	0	0	0	
		Cj-Zj	-2000/3	2000/3	0	0	0	0	
				↑	Max (entering)				

The tie can occur between row X3 & S3. So, in order to break the tie we can use the degeneracy procedures.
Degeneracy is resolved as under:

	S1	S2	S3
X3	0/(2/3)=0	(1/3)(2/3)=1/2	0/(2/3)=0
S3	0/2=0	0/2	1/2

Since the ratio could not break the tie, we have arbitrarily taken S3as a departing variable because the tie can be occurred between decision variable &

Cj	Contribution per unit		1000	4000	5000	0	0	0
	Basic Variables	Solution Values	X1	X2	X3	S1	S2	S3
0	S1	22	5	0	0	1	-1	1
5000	X3	0	-2/3	0	1	0	1/3	-1/3
4000	X2	7	3/2	1	0	0	0	1/2
	Zj	28000	8000/3	4000	5000	0	5000/3	1000/3
		Cj-Zj	-5000/3	0	0	0	-5000/3	-1000/3

slack/surplus variables we take slack or surplus variable as a leaving variable. The entering

variable is X2 & the key element is 2. Therefore, by introducing the entering variable & removing the leaving variable with the changed values the 3^{rd} simplex table is prepared as follows:

Simplex Table III

Since all Cj-Zj≤0, the optimum solution is obtained. X1=0, X2=7, X3=0 &Max Z=28000.

2. **Unbounded Problem**: As a general rule, it can be stated that a key row can't be selected because minimum ratio column contains negative or infinity (∞) the solution is unbounded. The table does not indicate an optimal solution, yet the simplex process is prohibited from continuing.

   Example: Solve the following LPP by using simplex method.

$$Max\ Z = 5x1 + 4x2$$
$$St.to\ x1 \leq 7$$
$$x1 - x2 \leq 8$$
$$x1, x2 \geq 0$$

Solution: By introducing slack variables convert the inequality constraints into equations. Then the std. form of LPP is:

$$Max\ Z = 5x1 + 4x2 + 0s1 + 0s2$$
$$St.to\ x1 + s1 = 7$$
$$x1 - x2 + s2 = 8$$
$$x1, x2, s1, s2 \geq 0$$

From the above LPP formulation the initial simplex table is prepared in the following format.

Simplex Table I

Cj	Contribution per unit		5	4	0	0	
	Basic Variables	Solution Values	X1	X2	S1	S2	Min.Ratio
0	S1	7	1*	0	1	0	7 ← Key row
0	S2	8	1	-1	0	1	8
	Zj	0	0	0	0	0	
		Cj-Zj	5	4	0	0	
			↑	Max (entering)			

From Simplex Table I the entering variable is X1 & the leaving variable is S1 & also the key element is 1.

By introducing the entering variable & removing the leaving variable with the new changed values the 2nd simplex table is prepared as follows.

Simplex Table I

Cj	Contribution per unit		5	4	0	0	
	Basic Variables	Solution Values	X1	X2	S1	S2	Min.Ratio
5	X1	7	1	0	1	0	∞
0	S2	1	0	-1	-1	1	-1=-ve ←Key row
	Zj	35	5	0	5	0	
		Cj-Zj	0	4	-5	0	
				↑	Max (entering)		

On the basis of Cj-Zj row, x2 is the entering variable, but on the basis of minimum ratio it is not possible to decide departing variable, hence the given LPP has unbounded solution.

3. **Multiple Optimal Solutions**: In the final simplex table, if the index row indicates the value of Cj-Zj for a non-basic variable to be zero, there exists an alternative optimum solution. This irrespective of whether the variable is a decision or slack or surplus variable.

Example: Solve the following LPP by using simplex method.

$$Max\ Z = 2000x1 + 3000x2$$
$$St.to\ 6x1 + 9x2 \leq 100$$
$$2x1 + x2 \leq 20$$
$$x1, x2 \geq 0$$

Solution: By introducing slack variables convert the inequality constraints into equations. Then the std. form of LPP is: $Max\ Z = 2000x1 + 3000x2 + 0s1 + 0s2$

$$St.to\ 6x1 + 9x2 + s1 = 100$$
$$2x1 + x2 + s2 = 20$$
$$x1, x2, s1, s2 \geq 0$$

From the above LPP formulation the initial simplex table is prepared in the following format.

Simplex Table I

Cj	Contribution per unit		2000	3000	0	0	
	Basic Variables	Solution Values	X1	X2	S1	S2	Min.Ratio
0	S1	100	6	9*	1	0	100/9←Key row
0	S2	20	2	1	0	1	20
	Zj	0	0	0	0	0	
		Cj-Zj	2000	3000	0	0	
				↑ Max (entering)			

From Simplex Table I the entering variable is X2 & the leaving variable is S1 & also the key element is 9.
By introducing the entering variable & removing the leaving variable with the new changed values the 2nd simplex table is prepared as follows.

Simplex Table II

Cj	Contribution per unit		2000	3000	0	0
	Basic Variables	Solution Values	X1	X2	S1	S2
3000	X2	100/9	2/3	1	1/9	0
0	S2	80/9	4/3	0	-1/9	1
	Zj	100000/3	2000	3000	1000/3	0
		Cj-Zj	0	0	-1000/3	0

Since all the elements in Cj-Zj row are negative or zero, we are having optimum solution. X1=0, X2=100/9 & Max Z=100000/3.
Recalling that the Cj-Zj value indicates the per unit net increase in profit that would be realized from entering a non-basic variable, we can see that the entering variable X1 would neither decrease nor increase profit. It would result in a different solution having the same Zj. In order to compute the value of the alternative optimum solution we introduce X1 as a basic variable replacing S2. The resultant simplex table is given below.

Simplex Table III

Cj	Contribution per unit		2000	3000	0	0
	Basic Variables	Solution Values	X1	X2	S1	S2
3000	X2	20/3	0	1	1/6	-1/2
2000	X1	20/3	1	0	-1/12	3/4
	Zj	100000/3	2000	3000	1000/3	0
		Cj-Zj	0	0	-1000/3	0

Since all Cj-Zj≤0, the optimum solution is obtained. X1=20/3, X2=20/3 &Max Z=100000/3.

4. **Infeasible Problem**: This condition occurs when the problem has incompatible constraints. Final simplex table as shown optimal solution as all Cj-Zj elements positive or zero in case of minimization & negative or zero in case of maximization. However, observing the solution base, we find that an artificial variable as a basic variable. Both of these values are totally meaningless since the artificial variable has no meaning. Hence, in such a situation it is said that LPP has got an infeasible solution. And also an infeasible problem can occur when a negative solution value is appearing in the final simplex table.
   Example: Solve the following LPP by using simplex method.

$$
\begin{aligned}
Max\ Z &= 4x1 + 3x2 \\
St.to\ x1 + x2 &\le 50 \\
x1 + 2x2 &\ge 80 \\
3x1 + 2x2 &\ge 140 \\
x1, x2 &\ge 0
\end{aligned}
$$

Solution: By introducing slack variable in ≤ type constraint & surplus & artificial variables ≥ type constraints & assign zero coefficients to slack & surplus variables & -M to artificial variable in the objective function. The std. form of LPP is:

$$Max\ Z = 4x1 + 3x2 + 0s1 + 0s2 + 0s3 - MA2 - MA3$$

$$
\begin{aligned}
St.to\ x1 + x2 + s1 &= 50 \\
x1 + 2x2 - s2 + A2 &= 80 \\
3x1 + 2x2 - s3 + A3 &= 140 \\
x1, x2, s1, s2, s3, A2, A3 &\ge 0
\end{aligned}
$$

From the above LPP formulation the initial simplex table is prepared in the following format.

Simplex Table I

Cj	Contribution per unit		4	3	0	0	0	-M	-M	
	Basic Variables	Solution Values	X1	X2	S1	S2	S3	A2	A3	Min.Ratio
0	S1	50	1	1	1	0	0	0	0	50
-M	A2	80	1	2	0	-1	0	1	0	80
-M	A3	140	3*	2	0	0	-1	0	1	140/3 ← Key row
	Zj	-220M	-4M	-4M	0	M	M	-M	-M	
		Cj-Zj	4+4M	3+4M	0	-M	-M	0	0	

↑ Max (entering)

From Simplex Table I the entering variable is X1 &the leaving variable is A3 & also the key element is 3.

By introducing the entering variable & removing the leaving variable with the new changed values the 2nd simplex table is prepared as follows.

Simplex Table II

Cj	Contribution per unit		4	3	0	0	0	-M	
	Basic Variables	Solution Values	X1	X2	S1	S2	S3	A2	Min.Ratio
0	S1	10/3	0	1/3	1	0	1/3	0	10 ← Key row
-M	A2	100/3	0	4/3	0	-1	1/3	1	25
4	X1	140/3	1	2/3	0	0	-1/3	0	70
	Zj	(560-100M)/3	4	(8-4M)/3	0	M	(-4-M)/3	-M	
		Cj-Zj	0	(4M+4)/3	0	-M	(4+M)/3	0	

↑ Max (entering)

From Simplex Table II the entering variable is X2 &the leaving variable is S1 &also the key element is 1/3. By introducing the entering variable & removing the leaving variable with the new changed values the 3rd simplex table is prepared as follows.

Simplex Table III

Cj	Contribution per unit		4	3	0	0	0	-M
	Basic Variables	Solution Values	X1	X2	S1	S2	S3	A2
3	X2	10	0	1	3	0	1	0
-M	A2	20	0	0	-4	-1	-1	1
4	X1	40	1	0	-2	0	-1	0
	Zj	190-20M	4	3	1+4M	M	-1+M	-M
		Cj-Zj	0	0	-1-4M	-M	1-M	0

Since all $C_j - Z_j \leq 0$, the optimum solution is obtained. Since artificial variable is present as a basic variable, the given problem has infeasible solution.

5. **Tie for the Key Column:** The non-basic variable that is selected to enter the solution is determined by the largest positive value in case of maximization & the largest negative value in case of minimization. Problem can arise in case of tie between identical Cj-Zj values, i.e. two or more columns have exactly the same positive or negative value in the Cj-Zj row.

CHAPTER 3: DUALITY THEORY AND SENSITIVITY ANALYSIS

3.1. DUALITY: Primal – Dual

Associated with any LP is another LP called the ***dual***. The term dual, in general sense is two or double. In the context of LP, duality implies that each LP can be analyzed in two different ways but having equivalent solution. The first way of stating LPP is called the primal of the problem. The second way of stating the same problem is called dual. In other words each LP maximizing problem has its corresponding dual, minimizing problem & vice versa. Knowledge of the dual provides interesting economic and sensitivity analysis insights. When taking the dual of any LP, the given LP is referred to as the ***primal***. If the primal is a max problem, the dual will be a min. problem and vice versa. The main focus of dual is to find for each resource its best marginal value or shadow price.

The optimal solution for the primal & the dual are equivalent but they are derived through alternative procedures. The dual contains economic information useful to management & it may also be easier to solve than the primal problem. Generally if the LP primal involves maximizing profit function subject to ≤ resource constraints, the dual will involve minimizing total opportunity cost subject to ≥ product profit constraints. Formulating the dual from a given primal is not difficult & once it is formulated the solution procedure is exactly the same as for any LPP.

It may be noted that, sometimes it is computationally easier to solve the dual than the primal.

Finding the Dual of an LP.

The dual of a **max** problem is a **min.** problem

Normal max problem is a problem in which all the variables are required to be nonnegative and all the constraints are ≤ constraints.

Normal min problem is a problem in which all the variables are required to be nonnegative and all the constraints are ≥ constraints.

Similarly, the dual of a normal min problem is a normal max problem.

In order to understand the formulation of the dual we will define the dual problem when its primal is given in the following form:

Finding the Dual of a Max Problem

PRIMAL $\quad \text{Max } z = c1x1 + c2x2 + \ldots + cnxn$

$$\begin{aligned} \text{s.t.} \quad & a11x1 + a12x2 + \ldots + a1nxn \le b1 \\ & a21x1 + a22x2 + \ldots + a2nxn \le b2 \\ & \ldots \ldots \ldots \ldots \\ & am1x1 + am2x2 + \ldots + amnxn \le bm \\ & xj \ge 0 \ (j = 1, 2, \ldots, n) \end{aligned}$$

DUAL $\quad \text{Min } z^* = b1y1 + b2y2 + \ldots + bmym$

s.t. $\quad a11y1 + a21y2 + \ldots + am1ym \geq c1$

$a12y1 + a22y2 + \ldots + am2ym \geq c2$

$\ldots \ldots \ldots \ldots$

$a1ny1 + a2ny2 + \ldots + amnym \geq cn$

$yi \geq 0 \ (i = 1, 2, \ldots, m)$

Finding the Dual of a Min Problem

PRIMAL $\quad \text{Min } z = b1y1 + b2y2 + \ldots + bmym$

s.t. $\quad a11y1 + a21y2 + \ldots + am1ym \geq c1$

$a12y1 + a22y2 + \ldots + am2ym \geq c2$

$\ldots \ldots \ldots \ldots$

$a1ny1 + a2ny2 + \ldots + amnym \geq cn$

$yi \geq 0 \ (i = 1, 2, \ldots, m)$

DUAL $\quad \text{Max } z^* = c1x1 + c2x2 + \ldots + cnxn$

s.t. $\quad a11x1 + a12x2 + \ldots + a1nxn \leq b1$

$a21x1 + a22x2 + \ldots + a2nxn \leq b2$

$\ldots \ldots \ldots \ldots$

$am1x1 + am2x2 + \ldots + amnxn \leq bm$

$xj \geq 0 \ (j = 1, 2, \ldots, n$

Rules for constructing the Dual problem

a) If the objective of the one problem is to maximize, the objective of the other is to minimize.

b) The maximization problem should have all $\leq$ constraints & minimization problem should have $\geq$ constraints if in the maximization case any constraint is $\geq$ type, it can be multiplied by -1 & sings can be got reversed. Similarly if in minimization case any problem has got $\leq$ type, it should have multiplied by -1 & signs can be reversed.

c) The number of primal problem decision variables equals the number of dual constraints & the number of primal constraints equals the number of the dual decision variables i.e. each dual variable corresponds to one constraint in the primal & vice versa. Therefore if the primal problem has "m" constraints & "n" variables the dual problem will have "n" constraints & "m" variables. Such condition is violated in case of = type constraints & should be resolved before proceeding to the solution.

d) If the inequalities are $\leq$ type then in the dual problem they are $\geq$ type & vice versa.

e) The Cj of the primal in the objective function appears as RHS constant of the dual constraints, & the primal RHS constraints appears as unit contribution rate i.e. Cj of the dual decision variable in the objective function. The matrix of the constraint coefficient for one problem is the transpose of the matrix of constraint coefficient of the other problem i.e. rows in the primal becomes columns in the dual.

Primal dual r/ship

The various primal-dual r/ship is summarized below.

Primal **Dual**

- Maximization
- No of variable
- Number of constraint
- $\leq$type constraint
- RHS constant for the ith constraint
- The objective function coefficient for the jth variable
- Variable xj unrestricted in sign
- Coefficient(aij) for the ith variable in the jth constraint
- Constraint ith is = type

- Minimization
- Number of constraint
- Number of variables
- $\geq$type constraint
- Objective function coefficient for the ith variable
- RHS constant for the jth constraint.
- Constraint jth is = type
- Coefficients (aij) for the jth variable in the ith constraint
- Variable yi unrestricted in sing

Economic Interpretation

When the primal is a normal max problem, the dual variables are related to the value of resources available to the decision maker. For this reason, dual variables are often referred to as ***resource shadow prices***. Shadow price reflects the scarcity of the resources. If the resource is not completely used i.e. there is slack, then its shadow price is zero. Shadow price is the rate of change in the optimal objective function value with respects to the unit change in the availability of a resource.

Example

PRIMAL

Let $x1$, $x2$, $x3$ be the number of desks, tables and chairs produced. Let the weekly profit be \$$z$. Then, we must

Max $z = 60x1 + 30x2 + 20x3$

s.t. $8x1 + 6x2 + x3 \leq 48$ (Lumber constraint)

$4x1 + 2x2 + 1.5x3 \leq 20$ (Finishing hour constraint)

$2x1 + 1.5x2 + 0.5x3 \leq 8$ (Carpentry hour constraint)

$x1, x2, x3 \geq 0$

DUAL

Suppose an entrepreneur wants to purchase all of Dakota's resources.

In the dual problem $y1$, $y2$, $y3$ are the resource prices (price paid for one board ft of lumber, one finishing hour, and one carpentry hour).

\$w is the cost of purchasing the resources.

Resource prices must be set high enough to induce Dakota to sell. i.e. total purchasing cost equals total profit.

Min $w = 48y1 + 20y2 + 8y3$

s.t. $8y1 + 4y2 + 2y3 \geq 60$ (Desk constraint)

$6y1 + 2y2 + 1.5y3 \geq 30$ (Table constraint)

$y1 + 1.5y2 + 0.5y3 \geq 20$ (Chair constraint)

$y1, y2, y3 \geq 0$

Example2 write the dual of the following problem

Max $Z = 3x_1+x_2+2x_3-x_4$

St. to $2x_1-x_2+3x_3+x_4=1$

$x_1+x_2-x_3+x_4=3$

$x_1, x_2\geq 0$ & x_3, x_4 unrestricted in sing

Solution: the dual of the above problem is

Min. $Z=y_1+3y_2$

St. to $2y_1+y_2\geq 3$

$-y_1+y_2\geq 1$

$3y_1-y_2=2$

$y_1+y_2=-1$

y_1, y_2 is unrestricted in sing.

Dual of the Dual is Primal

The statement the dual of dual is primal is illustrated below.

Max Z=50x1+20x2

St.to 2x1+4x2≤80

3x1+x2≤60

x1, x2≥0

The dual of the above problem is given below

Min Z=80y1+60y2

$$\text{St.to}\quad \left.\begin{array}{l} 2y1+3y2\geq 50 \\ 4y1+y2\geq 20 \end{array}\right\} \text{ primal objective function coefficients}$$

y1, y2≥0

We now again obtain the dual of the above dual

Max Z=50x1+20x2

St. to 2x1+4x2≤80

3x1+x2≤60

x1, x2≥0. The dual so obtained is, compared with the primal is exactly the same. Hence it is right to say that dual of the dual is primal.

Interpreting the Primal-Dual R/ship

For interpreting the optimal solution to the primal/dual its solution values can be read directly from the optimal solution table of the dual/primal. The method can be summarized in the following steps:-

Step 1: locate the slack/surplus variables in the dual/or primal problem. These variables correspond to the primal/dual basic variable in the optimal solution.

Step2: the values in the index row corresponding to the columns of the slack/surplus variables with change in the sign gives directly the optimal values of the primal basic variables.

Step3: values for slack/surplus variables of the primal are given by the index row under the non basic variables of the dual solution with change in sign.

Step4: the value of the objective function is the same for primal & dual problem.

Example1 min $Z=40x_1+200x_2$

St. to $4x_1+40x_2\geq 160$

$3x_1+10x_2\geq 60$

$8x_1+10x_2\geq 80$

x1, x2≥0 solve this LPP by simplex method.

Solution: Introducing the surplus variables S1, S2 &S3 & artificial variables A1, A2 &A3 in the objective function. We convert the LPP in to the following form:

Minimize Z=40x1+200x2+0S1+0S2+0S3+MA1+MA2+MA3

St. to 4x1+40x2-S1+A1=160

3x1+10x2-S2+A2=60

8x1+10x2-S3+A3=80

x1, x2, x3, s1, s2, s3≥0

The above problem can be solved by simplex method as below

Simplex table I

Cj	Contribution per unit		40	200	0	0	0	M	M	M	
↓	Basic variable	Solution value	X1	X2	S1	S2	S3	A1	A2	A3	Minimum ratio
M	A1	160	4	40*	-1	0	0	1	0	0	160/40=4 → key row
M	A2	60	3	10	0	-1	0	0	1	0	60/10=6
M	A3	80	8	10	0	0	-1	0	0	1	80/10=8
	Zj	300M	15M	60M	-M	-M	-M	M	M	M	
		Cj-Zj	40-15M	200-60M	M	M	M	0	0	0	

Key column

Simplex table II

Cj	Contribution per unit		40	200	0	0	0	M	M	
↓	Basic variable	Solution value	X1	X2	S1	S2	S3	A2	A3	Minimum ratio
200	X2	4	1/10	1	-1/40	0	0	0	0	4(1/10)=40
M	A2	20	2	0	¼	-1	0	1	0	20/2=10
M	A3	40	7*	0	¼	0	-1	0	1	40/7 → key row
	Zj	800+60M	20+9M	200	-5+M/2	-M	-M	M	M	
		Cj-Zj	20-9M	0	5-M/2	M	M	0	0	

Key column

Simplex table III

Cj	Contribution per unit		40	200	0	0	0	M	
↓	Basic variable	Solution value	X1	X2	S1	S2	S3	A2	Minimum ratio
200	X2	24/7	0	1	-1/35	0	1/70	0	(24/7)/(1/70)=240
M	A2	60/7	0	0	5/28	-1	2/7*	1	(60/7)/(2/7)=30 → key row
40	X1	40/7	1	0	1/28	0	-1/7	0	(40/7)/(-1/7)
	Zj	(6400+60M)/7	40	200	(5M-120)/28	-M	(2M-20)/7	M	
		Cj-Zj	0	0	(120-5M)/28	M	(20-2M)/7	0	

Key column

Simplex table IV

Cj	Contribution per unit		40	200	0	0	0
↓	Basic variable	Solution value	X1	X2	S1	S2	S3
200	X2	3	0	1	-21/560	1/20	0
0	S3	30	0	0	5/8	-7/2	1
40	X1	10	1	0	1/8	-1/2	0
	Zj	1000	40	200	-5/2	-10	0
		Cj-Zj	0	0	5/2	10	0

Since all Cj-Zj in the index row are ≥0&the current solution is the optimal solution & is given by x1 =10,x2=3&the minimum value of Z=40x10+200x3 =400+600=1000

Dual problem:

The dual of the above primal is written as follows

Max $Z^*=160y1+60y2+80y3$

Subjected to $4y1+3y2+8y3 \leq 40$

$40y1+10y2+10y3 \leq 200$

$y1, y2 \ \& y3 \geq 0$

Solution introducing the slack variables s_1&$s_2$in the objective function, we convert the LPP in to the following form.

Max $Z^*=160y1+60y2+80y3+0s1+0s2$

Subject to $4y1+3y2+8y3+s1+0s2=40$

$40y1+10y2+10y3+0s1+s2 =200$

$y1, y2, y3, s1 \ \& s2 \geq 0$

The above dual problem can be solved by the simplex method as below

Simplex table I

Cj ↓	Contribution per unit		160	60	80	0	0	
	Basic variable	Solution value	Y 1	Y 2	Y3	S1	S2	Min ratio
0	S1	40	4	3	8	1	0	40/4=10
0	S2	200	40*	10	10	0	1	200/40=5 → Key row
	Z j	0	0	0	0	0	0	
		Cj-Zj	160	60	80	0	0	

↑Key column

Simplex table II

Cj ↓	Contribution per unit		160	60	80	0	0	
	Basic variable	Solution value	Y 1	Y 2	Y3	S1	S2	Min ratio
0	S1	20	0	2	7*	1	-1/10	20/7 → Key row
160	Y 1	5	1	¼	1/4	0	1/40	20
	Z j	800	160	40	40	0	4	
		Cj-Zj	0	20	40	0	-4	

↑Key column

Simplex table III

Cj ↓	Contribution per unit		160	60	80	0	0	
	Basic variable	Solution value	Y 1	Y 2	Y3	S1	S2	Min ratio
80	Y 3	20/7	0	2/7*	1	1/7	-1/70	(20/7)/(2/7)=10 → Key row
160	Y 1	30/7	1	5/28	0	-1/28	2/70	(30/7)/(5/28)=24
	Z j	6400/7	160	360/7	80	40/7	24/7	
		Cj-Zj	0	60/7	0	-40/7	-24/7	

↑Key column

Simplex table IV

Cj	Contribution per unit		160	60	80	0	0
	Basic variable	Solution value	Y 1	Y 2	Y3	S1	S2
60	Y 2	10	0	1	7/2	1/2	-1/20
160	Y 1	5/2	1	0	-5/8	-1/8	3/80
	Z j	1000	160	60	110	10	3
		Cj-Zj	0	0	-110	-10	-3

Since all the Cj-Zj elements in the index row are ≤0, the optimum solution of the dual problem is reached & is given by Y1=5/2&Y2 =10 and the max value of Z*=160x5/2+60x10+800x0 =1000.
The respective coefficients of S1& S2 in last Cj –Zj row (ignoring sign) of the simplex table IV are 10 &3. Therefore x1 =10, x2 =3

Max Z* =min. Z = 1000

Now it is clear that the primal &dual lead to the same solution, even though they are formulated differently. It is also clear that in the final simplex table of primal problem, the absolute value of the numbers in Cj –Zj row under the slack variable represents the solutions to the dual problem. In another words it also happens that the absolute value of the Cj –Zj value of the slack variable in the optimal dual solution represent the optimal value of the primal x1&x2 variables. The minimum opportunity cost derived in the dual must always be equal the maximum profit derived in the primal.

Consequences

• Any feasible solution to the dual can be used to develop a bound on the optimal value of the primal objective function.

• If the primal is unbounded, then the dual problem is infeasible.
If the dual is unbounded, then the primal is infeasible

- The primal and dual have equal optimal objective function values (if the problems have optimal solutions).
- If the primal problems have multiple optimal solutions, then the dual problems have also multiple optimal solutions.

Advantage of duality

- ➢ The dual variables provides the decision maker a basis for deciding how much to pay for additional units of resources.
- ➢ The maximum amount that should be paid for one additional unit of resource is called shadow price.
- ➢ It is quite useful when investigating changes in the parameter of an LPP (the technique known as sensitivity analysis).
- ➢ Fully understanding the interpretation of the shadow price.

3.2. SENSITIVITY ANALYSIS

In an LP model, the input data (also known as parameters) such as,

I. Profit/cost/ contribution(Cj) per unit of decision variables,
II. Availability of resources(bi) &
III. Consumption of resources per unit of decision variables (aij), are assumed constant & known

with certainty during a planning period. However, in real world situations some data may change over time because of the dynamic nature of the business. Such changes in any of these parameters may raise doubt on the validity of the optimal solution of the given LP model. Thus, a decision maker in such situations would like to know how sensitive the optimal solution is to the changes in the original input data values.

Sensitivity analysis is used to determine effects on the optimal solution within specified ranges for the objective function coefficients, constraint coefficients, and right hand side values.
Sensitivity analysis provides answers to certain what-if questions.
Sensitivity analysis is a technique that evaluates the relationship between the optimal solution & changes in the LP model parameters. Sensitivity analysis considered the effects of variations in the input coefficients (also called parameters) when these coefficients are changed one at a time.
Sensitivity analysis provides the sensitive ranges (both lower & upper limits) within which the LP model parameters can vary without changing the optimality of the current optimal solution.
Sensitivity analysis is the study of sensitivity of the optimal solution of an LP problem due to discrete variations (changes) in its parameters. The degree of sensitivity of the solution due to these variations can range from no change at all to a substantial change in the optimal solution of the given LPP. Thus, in sensitivity analysis we determine the range over which the LP model parameters can change without

affecting the current optimal solution. For this, instead of resolving the entire problem as a new problem with new parameters, we may consider the original optimal solution as an initial solution for the purpose of knowing the ranges, both the lower & upper, within which a parameter may assume a value.
Sensitivity analysis can be used to deal not only with errors in estimating input parameters to the LP model but also with management's experiments with possible future changes in the firm that may affect profits. Sensitivity analysis often involves a series of what-if? Questions concerning constraints, variable coefficients, and the objective function

The process of studying the sensitivity of the optimal solution of an LPP is called post-optimality analysis because it is done after an optimal solution, assuming a given set of parameters has been obtained for the model.

Different categories of parameter changes in the original LP model include:

1) Coefficients in the objective function
2) Availability of the resources(RHS of constraints)

1) Changes in the objective function coefficient(cj)

Suppose the coefficient cj in the objective function of an LP model represents either profit or cost per unit of an activity (variable) xj. Then the question that may arise is: what happens to the optimal solution & the objective function value when this coefficient is changed? The test of sensitivity of the objective function value with respect to this coefficient (cj) determines the range (both lower & upper) of values within which each cj can lie without changing the current optimal solution.
Such an analysis can help the decision maker in deciding whether resources from other activities (variables) should be diverted to (diverted away from) a more profitable (less profitable) activity.
Changes in the profit or cost coefficients (contributions) in the objective function can occur for a basic variable (the variable in the solution mix) or non-basic variable (the variable not in the solution mix). The sensitivity ranges for these variables are determined differently.

Case I: change in the coefficient of a non-basic variable

The current optimal solution for a max LPP will remain optimal as long as all Cj-Zj ≤ 0, for all variables. Let C_k be the coefficient of non basic variable X_k in the objective function. Since C_k is the coefficient of non basic variable X_k, it does not affect any value of the Cj values listed in the solution mix column of simplex table associated with basic variables. Since the calculation of $Z_j = C_B B^{-1} a_j$ values do not involve Cj, therefore changes in Cj doesn't alter Zj values & hence Cj-Zj values remain unchanged except $C_k - Z_k$ value due to change in C_k. in other words, any change in this coefficient does not affect feasibility of the optimal solution. This means that unit profit of Xk can be lowered to any level without causing the optimal solution to change. But any increase in its unit profit beyond a certain level (i.e. upper limit) should make this variable eligible to be a basic variable in the new solution mix. Obviously, the Ck-Zk will no longer be negative.

The sensitivity limits (range of optimality) for the contribution per unit of a non- basic variable are calculated as under:

Lower limit= negative infinity ($-\infty$)

Upper limit C_k=original value + absolute value of unit improvement value

Case II: change in the coefficient of basic variable

In the max LPP the change in the coefficient, say Ck of a basic variable Xk affects the Cj-Zj values corresponding to all non-basic variables in the simplex table. It is because the coefficient Ck is listed in the C_B (coefficient of basic variable) column of the simplex table & affects the calculation of Zj values. The sensitivity limits (range of optimality) for the contribution per unit of a basic variable are calculated as under:

Lower limit = original value Ck – (lowest absolute value of improvement ratio or $-\infty$ (if no ratio is negative)

Upper limit = original value Ck + (lowest positive value of improvement ratio or ∞ (if no ratio is positive)

$$\text{Improvement ratio} = \frac{\text{per unit improvement value}}{\text{Input-output coefficient in the variable row}} = \frac{C_j - Z_j}{a_{kj}}$$

The **range of optimality** of the change in the coefficient of basic variable for **min.** LPP is the same with that of **max** LPP.

While performing the sensitivity analysis, the artificial variable columns in the simplex table are ignored. Any change of quantities between basic variables & an artificial variable make no sense, because an artificial variable has no economic interpretation. Thus improvement ratios using coefficients corresponding to artificial variables should not be considered.

Case III: **change in the coefficient of non-basic variable in min. problem**

The procedure for calculating sensitivity limits to a cost min. LPP, where the objective function coefficients are unit costs is identical to case I. in this case, the unit cost coefficient can be increased to any arbitrary level but it can't be decreased by more than per unit improvement value without making it eligible so that a non- basic variable can be entered into the new solution mix.

The sensitivity limits can be calculated as:

Lower limit = original value – unit improvement value

Upper limit = positive infinity (∞)

Ranges of Optimality

The value of the objective function will change if the coefficient multiplies a variable whose value is nonzero.

The optimal solution will remain unchanged as long as:

- an objective function coefficient lies within its range of optimality
- there are no changes in any other input parameters.
- The optimality range for an objective coefficient is the range of values over which the current optimal solution point will remain optimal
- For two variable LP problems the optimality ranges of objective function coefficients can be found by setting the slope of the objective function equal to the slopes of each of the binding constraints
- The range of optimality is valid only when a single objective function coefficients changes.
- A range of optimality of an objective function coefficient is found by determining an interval for the objective function coefficient in which the original optimal solution remains optimal while keeping all other data of the problem constant. The value of the objective function may change in this range.
- If the coefficient is changed by more than the *Allowable Increase* or *Allowable Decrease*, another extreme point of the feasible region will be identified as the optimal solution.
- *Range of optimality*: the range of values for which the solution quantities of the decision variables remains the same.

Example: a company wants to produce three products: A, B & C. the unit profit on these products is 4, 6 & 2 respectively. These products require two types of resources, manpower & raw material. The LP model formulated for determining the optimal product mix is as follows:

Max $Z=4x_1+6x_2+2x_3$

St. to $x_1+x_2+x_3\leq 3$ *(manpower constraint)*

$x_1+4x_2+7x_3\leq 9$ *(raw material constraint)*

$x_1, x_2, x_3\geq 0$

Where x_1, x_2, x_3 = *number of units of product A, B & C respectively to be produced.*

a) Find the optimal product mix & the corresponding profit of the company.
b) Find the range of the profit contribution of product C (i.e. coefficient c3 of the variable x3) in the objective function such that current optimal product mix remains unchanged.
c) What shall be the new optimal product mix when profit per unit for product C is increased from 2 to 10?

d) Find the range of the profit contribution of product A (i.e. coefficient of c1 of variable x1) in the objective function such that the current optimal solution (product mix) remains unchanged.

Solution: a) convert the given LP model into standard form by introducing the slack variables s1 & s2

Max $Z=4x_1+6x_2+2x_3+0s_1+0s_2$

St. to $x_1+x_2+x_3+s_1=3$

$x_1+4x_2+7x_3+s_2=9$

$x_1, x_2, x_3, s_1, s_2\geq 0$

The optimal solution obtained by applying simplex method is shown in table 3.1

Table 3.1 optimal solution

C_j → / ↓C_B	Contribution per unit		4	6	2	0	0
	Basic variables	Solution values	x_1	x_2	x_3	s_1	s_2
4	x_1	1	1	0	-1	4/3	-1/3
6	x_2	2	0	1	2	-1/3	1/3
	Zj	16	4	6	8	10/3	2/3
		Cj-Zj	0	0	-6	-10/3	-2/3

The optimal solution is $x_1 = 1$, $x_2=2$, $x_3 = 0$ & max Z=16

b) effect of change in the coefficient of c_3 of non-basic variable x_3 for product C.

In optimal simplex table 3.1 the variable x3 is non-basic variable & its coefficient c3=2 is not listed in the CB column of the table. This means that a further decrease in its profit contribution per unit weill have no effect on the current optimal product mix. But if c3 is increased beyond a certain value, the product may become profitable to produce. Hence, there is only an upper limit on c3 for which the current optimal product mix will be affected.

Lower limit= negative infinity($-\infty$)

Upper limit = original value (c3) – improvement value

Therefore, the range of optimality = $(-\infty, 8)$ Or

As we know, a change $\triangle 3$ in c3 will cause a change in c3-z3 values in the column corresponding to x3. The change in c3 results in the modified simplex table 3.2

Table 3.2 table with a change in c3

C_j → / ↓C_B	Contribution per unit		4	6	$2+\triangle c3$	0	0
	Basic variables(B)	Solution values(x_B)	x_1	x_2	x_3	s_1	s_2
4	x_1	1	1	0	-1	4/3	-1/3
6	x_2	2	0	1	2	-1/3	1/3
	Zj	16	4	6	8	10/3	2/3
		Cj-Zj	0	0	$\triangle c3 -6$	-10/3	-2/3

For an optimal solution shown in table 3.2 to remain unchanged, we must have

$\triangle c3-6 \leq 0 \Rightarrow \triangle c3 \leq 6$

Recalling that $c3= 2+\triangle 3 \Rightarrow \triangle 3=c3-2$, we substitute this amount in the above inequality

$\Rightarrow \triangle c3 \leq 6$

$\Rightarrow c3-2 \leq 6$

$\Rightarrow c3 \leq 8$ upper limit

This implies that as long as the profit contribution per unit of product C is less than 8 (i.e. the change should not be more than 8) it is not profitable to produce it & the current optimal solution will remain unchanged.

c) If the value of c3 is increased from 2 to 1, the new value of

$c_3-z_3= (c_3- c_B a_3)$ will be, where c_3= new coefficient of x3

$=10-[4 \;\; 6] \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ c_B= coefficient of basic variable

$=10-((4\times-1) + (6 \times 2)$ a_3=constraint coefficient of x3

$=10-(-4+12)$

$=10-8=2>2$

Thus, if the coefficient of variable x3 is increased from 2 to 10, the value of c3-z3 will become positive shown in table 3.3. The variable x3 becomes eligible to enter into the basis. Hence, the solution can't remain optimal any more.

Cj → ↓C_B	Contribution per unit		4	6	10	0	0	
	Basic variables(B)	Solution values(x_B)	x_1	x_2	x_3	s_1	s_2	Min. ratio
4	x_1	1	1	0	-1	4/3	-1/3	1/-1=-1(-ve)
6	x_2	2	0	1	2	-1/3	1/3	2/2=1 → key row
	Zj	16	4	6	8	10/3	2/3	
		Cj-Zj	0	0	2	-10/3	-2/3	
					↑ Max (entering)			

Then, by applying the simplex method, enter x3 into the new solution mix.
The new solution mix after entering the non-basic variable x3 into the solution mix is as shown below.

Table 3.4

Cj → ↓C_B	Contribution per unit		4	6	10	0	0
	Basic variables(B)	Solution values(x_B)	x_1	x_2	x_3	s_1	s_2
4	x_1	2	1	½	0	7/6	-1/6
10	x_3	1	0	½	1	-1/6	1/6
	Zj	18	4	7	10	3	1
		Cj-Zj	0	-1	0	-3	-1

Since all Cj-Zj ≤ 0 in the above table the new optimal solution obtained is $x_1 = 2$, $x_2 = 0$, $x_3 = 1$ & max Z= 18

d) Effects of change in the coefficient c1 of basic variable x1 for product A.

In the simplex table 3.1, the variable x1 appears in column B (in the basic variable column). This means a further decrease in its profit contribution c1 will make it less profitable to produce & therefore the current optimal product mix (solution) will be affected. Also, an increase in the value of c1 beyond a limit will make product A much more profitable & may force the decision maker to decide to produce product A only. Thus, in either case, the current optimal solution (product mix) will be affected & hence we need to know both the lower as well as the upper limit on the value of c1 with in which the optimal solution will not be affected. i.e. range of optimality

Lower limit = original value (c1) – (smallest absolute value of unit improvement ratio)
Upper limit = original value (c1) + (smallest positive value of unit improvement ratio)

Improvement ratio $= \frac{\text{per unit improvement value}}{\text{Input-output coefficient in the variable (xi) row}} = \frac{Cj-Zj}{a_{kj}}$

$= \frac{Cj-Zj}{a_{kj}}$

$= (c_1-z_1)/a_{11}=0/1=0$

$= (c_2-z_2)/a_{12}=0/0$ undefind

$= (c_3-z_3)/a_{13}=-6/-1=6$

$= (c_4-z_4)/a_{14}= (-10/3)/(4/3)=-5/2$ → lowest negative ratio

$= (c_5-z_5)/a_{15}= (-2/3)/(-1/3)=2$ → lowest positive ratio

Therefore,Lower limit=original value (c_1)-(smallest negative absolute value of unit improvement ratio)
$=4- \left|-5/2\right| =4-5/2=3/2$

Upper limit = original value (c1) + (smallest positive value of unit improvement ratio)
=4+2=6

Therefore the range of optimality is (3/2, 6), that means between this range the current optimal solution remain unchanged.

2) Changes in Right-Hand-Side Values of Constraints (the availability of resources)

Case I when the slack variable is not in the solution mix

We know that in the optimal simplex table Cj-Zj numbers (ignoring negative sing) corresponding to the slack variable columns represent shadow prices of the available resources.
Shadow price provides important information in terms of:

- Change in the values of objective function from an increase at one unit of a scarce resource.
- Resource value that should be increased inorder to realise the best marginal increase in the value of the objective function.

From simplex table 3.1, s_1- represents slack variable of manpower(unused manpower resource)
s_2-represents slack variable for raw material(unused raw material)
in table 3.1 there is no slack variable in the solution mix column B (in the column of basic variable). The procedure for finding the range for resource values(range of feasibility) within which current shadow price remain unchanged is summarized below.

1. Treat the slack variable corresponding to resource value if it was an intering variable in solution mix. For this calculate minimium ratio (echange ratio) for every row.

$$\text{Minimum ratio} = \frac{\text{solution value, } x_B}{\text{Input-output coefficients in slack variable column}}$$

2. Find both lower & upper sensitivity limits (range of feasibility)
   Lower limit = original value- smallest positive ratio or-∞ (if no ratio is positive)
   Upper limit = original value + smallest absolute negative ratio or ∞ (if no ratio is negative)

To illustrate the methods of finding the range of variations in availability of resources, we treat s1,s2 column & the solution values column from table 3.1 & calculate ratio

Variables in the base (B)	Solution values (x_B)	Exchange/ Input-output coefficient in s_1 column	Exchange/Min. Ratio(x_B / s_1)	Exchange/ Input-output coefficient in s_2 column	Exchange/Min. Ratio(x_B / s_2)
X_1	1	4/3	1/(4/3)=3/4	-1/3	1/(-1/3)=-3
X_2	2	-1/3	2/(-1/3)=-6	1/3	2/(1/3)=6

From the above table in the 4th column, the smallest positive ratio (3/4) indicates as to by how many hrs.can manpower time resource (availability of resource) be decreased or reduced while smallest absolute nagative ratio(-6) indicates as to by how many hrs.can this resource be increased (added) without changing the current shadow price. Thus the shadow price for manpower resource (10/3) & raw material resource (2/3) are valid over the range as given below.
Range of feasibility:

- For manpower :-
  - ✓ lower limit = original value-smallest positive ratio

    =3-3/4=9/4
  - ✓ upper limit= original value + smallest absolute negative ratio

    $= 3+(|-6|)=3+6=9$

Range of feasiblity for manpower availability is(9/4, 9) that means between this range the current shadow price remain unchanged.

- For raw material:-
  - ✓ Lower limit = original value-smallest positive ratio
    =9-6=3
  - ✓ upper limit= original value + smallest absolute negative ratio
    = 9+(|-3 |)=9+3=12

Range of feasiblity for raw material availability is(3, 12) that means between this range the current shadow price remain unchanged.

Case II: when the slack variable in the basic variable column(column B)
When a slack variable is present in the solution mix column of optimal simplex table, the procedure for finding the range of variation for corresponding RHS of the constriant is as follows:
Lower limit = original value – solution of slack variable
Upper limit = positive infinity (∞)

Case III: change in RHS when constraints are mixed type

i) when surplus variable is not in the basic variable column

Lower limit = original value - smallest absolute negative ratio or -∞ (if no ratio is negative)
Upper limit = original value+smallest positive ratio or-∞ (if no ratio is positive)

ii) When surplus variable is in the basic variable column

Lower limit = negative infinity (-∞)
Upper limit = =original value + solution of surplus variable

- If the new values of RHS in the constraint is changed into "b", then the new values of the basic variable is: $X_B=B^{-1}b$
  Where B^{-1} =matrix coefficients corresponding to slack variables inthe optimal simplex table.
  b= amount of change in the RHS value.
  X_B=basic variables appearing in the B-column of simplex table.

NB:

1) If one or more entries in the solution values column of the simplex table are negative, the dual simplex method can be used to get an optimal solution to the new problem by maintaining feasibility.
2) A resource whose shadow price is bigger in comparision to others , should be increased first to ensure the best mariginal increase in the objective function value.

The sensitivity range for a RHS value is the range of values over which the quantity (RHS) values

can change without changing the solution variable mix, including slack variables.

- Any change in the right hand side of a binding constraint will change the optimal solution.
- Any change in the right-hand side of a nonbinding constraint that is less than its slack or surplus, will cause no change in the optimal solution.

- The amount of change (increase or decrease) in the objective function value that results from a unit change in one of the resources available is called the *dual price* or *dual value*
- If the dual value of a constraint is zero
  - The slack is positive, indicating unused resource
  - Additional amount of resource will simply increase the amount of slack.

- *Range of feasibility*: the range of values for the fight-hand side of a constraint over which the shadow price remains the same
- *Shadow prices*: negative values indicating how much a one-unit decrease in the original amount of a constraint would decrease the final value of the objective function

RHS of Binding Constraint -

- If RHS of non-redundant constraint changes, size of feasible region changes.
  - If size of region increases, optimal objective function improves.
  - If size of region decreases, optimal objective function worsens.

Range of Feasibility

- The set of right - hand side values for which same set of constraints determines the optimal point.
- Within the range of feasibility, shadow prices remain constant; however, the optimal solution will change.

- The range of feasibility for a change in the right hand side value is the *Allowable Increase* and *Allowable Decrease* for the coefficient in which the original shadow price remains constant.

Limitation of sensitivity analysis

No doubt the uncertainty element is considered in this concept; however, it has certain limitations such as:

1. Only one variable is considered at a time for the analysis. Thus, the overall effects of other variables changing all together cannot be considered.
2. Only linear relationship of changing variables is considered so it also suffers from all the limitations of linearity.
3. The extent of uncertainty is not measured in this analysis.

DUAL SIMPLEX METHOD

In the simplex method the optimality conditions are independent of basic variables. This imples that one or more of the values of solution variable may be negative. In such cases, it is possible to find a starting basic but not feasible solution that is dual feasible. A variant of simplex method by which we get an optimal basic feasible solution in a finite number of steps maintaining dual feasibility & complementary slackness is known as **Dual Simplex**.
In dual simplex the solution starts infeasible & optimal (as compared with primal simplex method which starts feasible but non-optimal).

In the dual simplex method we first determine the leaving variable (the variable to leave) in the basic variable column & then the variable to enter in the basic variable column (key column).

Steps in dual simplex

Step1: convert the problem into maximization, if it is initially in the minimization form.

Step2: convert all the constraints into $\leq$ type constraints.

Step3: convert the inequality constraints into equality constraints by adding slack variable & obtain initial basic solution.

Step4: compute Cj-Zj for each column.

a) If all values in Cj-Zj row are negative or zero & all solution value are negative or zero & all solutions are non negative the solution found is the optimum basic feasible solution.the process ends.
b) If all values in Cj-Zj row are negative or zero & at least one values of solution value is negative then proceed to step 5.
c) If all values in Cj-Zj row are positive, the method fails.

Step5: select the row that contains the most negative solution values. This row is called the key/pivot row. The corresponding basic variable leaves the current solution.

Step6: look at the elements of key row.

a) If all elements are non- negative the problem does not have a feasible solution.
b) If at least one element is negative, find the ratios of the corresponding elements of Cj-Zj row to these elements.

   Ignore the ratios associated with positive or zero elements of the key row. Choose the smallest of these ratios corresponding column is the key column &positive associated variable is the entering variable. Mark key element or pivot element.

Step7. Make this key element unity. We carry out the row operations as in the regular simplex method & repeat relations until either an optimal feasible solution is obtained in a finite number of steps.

Example 1: solve the following problem by dual simplex

$$\text{Max } Z=5x_1+3x_2$$

$$\text{Subject to } x_1+x_2\leq 2$$

$$5x_1+2x_2\leq 10$$

$$-2x_1-8x_2\geq -12$$

$$x_1, x_2 \geq 0$$

<u>**Solution**</u>: Multiply constraint three by -1 on both sides in order to have less than or equal to constraint, then we get Max $Z=5x_1+3x_2$

$$\text{Subject to } x_1+x_2\leq 2$$

$$5x_1+2x_2 \leq 10$$

$$2x_1+8x_2\leq 12$$

$$x_1, x_2 \geq 0$$

Introduce slack variable &assign zero coefficient in the objective function &convert the inequality constraints in to equality constraints (standard form of LPP).

$$\text{Max } Z=5x_1+3x_2+0S_1+0S_2+0S_3$$
$$\text{Subject to } x_1+x_2+S_1=2$$
$$5x_1+2x_2+S_2+S_2=10$$
$$2x_1+8x_2+S_3=12$$
$$x_1, x_2, S_1, S_2, S_3 \geq 0$$

Simplex Table I

Cj → Contribution per unit		5	3	0	0	0	
↓ Basic Variables	Solution Values	X1	X2	S1	S2	S3	M.R
0 S1	2	1	1	1	0	0	2/1=2
0 S2	10	5*	2	0	1	0	10/5=2 → key row
0 S3	12	2	8	0	0	1	12/2=6
Zj	0	0	0	0	0	0	
	Cj-zj	5	3	0	0	0	
		↑ Max (entering)					

From the above table degeneracy is occurred, since the minimum ratios are tied b/n row1&row2. Therefore, such degeneracy is resolved as follows:

	S1	S2	S3
S1	1/1=1	0/1=0	0/1=0
S2	0/5=0	1/5*	0/5=0

This is least non-negative ratio. So this is the key row. That means S_2 is the leaving variable & x_1 is the entering variable & key element is 5. By introducing the entering variable & removing the leaving variable & with the changed values the second simplex table is prepared as follows.

Simplex table II

Cj → Contribution per unit		5	3	0	0	0	
↓ Basic variables	Solution values	X1	X2	S1	S2	S3	Min. ratio (SV/Key Column)
0 S1	0	0	3/5	0	-1/5	0	0/3/5=0
5 X1	2	1	2/5	0	1/5	0	2/(2/5)=5
0 S3	8	0	36/5*	0	-2/5	1	8/(36/5)=40/36 → key row
Zj	10	5	2	0	1	0	
	Cj-Zj	0	1	0	-1	0	
			↑Max (entering)				

the entering variable is X_2 & the leaving variable is S_3 & the key element is 36/5.
By introducing the entering variable & removing the leaving variable & with the changed values the third simplex table is prepared as follows.

Simplex table III

Cj → Contribution per unit		5	3	0	0	0	
↓ Basic variables	Solution values	X1	X2	S1	S2	S3	
0 S1	-2/3	0	0	1	-1/6	-1/2*	→ Key row
5 X1	14/9	1	0	0	2/9	-1/18	
3 X3	10/9	0	1	0	-1/18	5/36	
Zj	100/9	5	3	0	17/18	5/36	
	Cj-Zj	0	0	0	-17/18	-5/36	
Ratio	(Cj-Zj)/S1	-	-	-	17/3	5/3	
						↑ Min (entering)	

Since S_1 variable have negative values in the solution value column it is selected as a leaving

variable & S_3 is the entering variable because it has least ratio (least non-negative ratio) in the (Cj-Zj)/S_1 row.
By introducing the entering variable & removing the leaving variable & with the changed values the fourth simplex table is prepared as follows.

Simplex table IV

Cj →	Contribution per unit		5	3	0	0	0
↓	Basic variables	Solution values	X1	X2	S1	S2	S3
0	S3	8	0	0	-12	2	1
5	X1	2	1	0	-2/3	1/3	0
3	X2	0	0	1	5/3	-1/3	0
	Zj	10	5	3	5/3	2/3	0
		Cj-Zj	0	0	-5/3	-2/3	0

Since all elements in Cj-Zj≤0, the optimum solution is obtained & in the solution value columns all elements are non-negative.

CHAPTER 4: INTEGER PROGRAMMING

INTRODUCTION

One of the assumption of LP is that the decision variables are continuous i.e. a variable can have fractional as well as integer values. In many real situations solution to LPPs make sense only if they have integer values.
So, to overcome this drawback of LP the new technique i.e. Integer programming was developed which requires decision variables to have integer values only.
Integer linear programming problem (ILPP) is an LPP, in which all or some of the decision variables are to be assume non-negative integer value. This type of problem is particular importance in business & in industry, where quite often the fractional value may be unrealistic because of some variables does not take fractional values.

• Some business problems can be solved only if variables have ***integer*** values.

E.g. Numbers of men that should be work in a company

– Airline decides on the number of flights to operate in a given sector must be an integer or whole number amount.

– The number of aircraft purchased this year

– The number of machines needed for production

– The number of trips made by a sales person

– The number of police officers assigned to the night shift etc.

Method of obtaining solution to an ILPP

There are different methods to find out the solution with integer values to an ILPP. The most commonly used are listed as follows.

1. Cutting plane method or Gomory fractional method
2. Branch & Bound method

1. Gomory Fractional Method

This method was developed by the scientist Gomory in 1958. Under this method the solution is obtained by the following steps:

Step I: if the problem is minimization type, convert it into that of maximization.

Step II: solve the given problem by simplex method, ignoring the integrality conditions.

Step III: if in the optimal solution all the variables assume integer values that is the desired optimal solution.

If in the optimal solution all the variables do not assume integer values then proceed to

Step IV: test the integrality of the optimum solution

Locate the values in the solution value(SV) column which obtain the highest fraction (among the decision variables which are restricted to be integer) choose the row which containing largest fractions say K^{th} row & express each of the negative fraction as the sum of the negative integer & a non-negative fraction. Then the K^{th} row is written as the form of equation. In this equation, all coefficients which are whole numbers are ignored & are replaced by zeros & integral part is also ignored & equations containing fractional part of all coefficients are obtained.

Step V: the part of this equation corresponding to the technical coefficients is equal to (=) fractional parts of resource availability plus (+) some integer. Thus it may be equal to (=) or greater than (>) the fractional part of resource availability.

Therefore, the inequality can be written as greater than or equal to (≥) type with fractional part on RHS. To convert it into constraint of less than or equal to (≤) type, we multiplied it by -1 & to convert it into equality type constraint introduce another slack variable represented by F1.

Step VI: add this new constraint to the bottom of the optimum simplex table obtained in step II& find the optimal solution by using Dual Simplex Method.

Step VII: If by using the dual simplex method the decision variables assume the integer values the optimum solution is obtained. If still the decision variables does not acquire integer values go to step III again & repeat the procedure until the optimum solution is obtained & all the required variables which are to acquire integer values obtained full fill the condition of integrality.

Example 1: solve the following ILPP

Max $Z=5x1+7x2$

St. to $-2x1+3x2\leq 6$

$6x1+x2\leq 30$

Where x1, x2 are all non- negative integers

Solution: introduce slack variable & assign 0 coefficients in the objective function & convert the inequality constraints into equations as follows:

Max $Z= 5x1+7x2+0s1+0s2$

St. to $-2x1+3x2+s1=6$

$6x1+x2+s2=30$

Where x1, x2, s1 $s2\geq 0$ & x1, x2 are positive integers

Simplex table I

Cj	Contribution per unit		5	7	0	0	
	Basic variables	Solution values	X_1	X_2	S_1	S_2	Min. ratio
0	S_1	6	-2	3*	1	0	6/3=2 → key row
0	S_2	30	6	1	0	1	30/1=30
	Zj	0	0	0	0	0	
		Cj-Zj	5	7	0	0	

↑ Max (entering)

X_2 is the entering variable & s_1 the leaving variable & 3 is the key element

By introducing the entering variable &removing the leaving variable & with the changed values the 2nd simplex table is prepared as follows:

Simplex table II

Cj	Contribution per unit		5	7	0	0	
	Basic variables	Solution values	X1	X2	S1	S2	Min. ratio
7	X2	2	-2/3	1	1/3	0	2/(-2/3)= -3(-ve)
0	S2	28	20/3*	0	-1/3	1	28/(20/3)=21/5 → key row
	zj	14	-14/3	7	7/3	0	
		Cj-Zj	29/3	0	-7/3	0	

↑Max (entering)

X_1 is the entering variable & s_2 is the leaving variable & 20/3 is the key element.

By introducing the entering variable & removing the leaving variable & with the changed values the 3rd simplex table is prepared as follows:

Simplex table III

Cj	Contribution per unit		5	7	0	0
	Basic variables	Solution values	X1	X2	S1	S2
7	X2	24/5	0	1	3/10	1/10
5	X1	21/5	1	0	-1/20	3/20
	Zj	273/5	5	7	37/20	29/20
	Cj-Zj		0	0	-37/20	-29/20

Since all Cj-Zj$\geq$0, the optimum solution is obtained $x_1=21/5$, $x_2=24/5$ &max Z=273/5

Using simplex method an optimum non-integer solution is displayed in simplex table III. Since the optimum solution is not integer value we consider only the fractional part & we will proceed as below to have a fractional cut. $X_1=4+1/5$ $x_2=4+4/5$

Since the fractional part of x_2 is greater than x_1 i.e. 4/5>1/5. We select x_2 as a source row & rewrite the non- integer value of the row corresponding to this variable (x_2). Then the constraint equation:

$24/5=0x_1+1x_2+3/10s_1+1/10s_2$, it is rewrite as $0x_1+1x_2+3/10s_1+1/10s_2=4+4/5$

Thus, we can write the above constraint as: $3/10s_1+1/10s_2=4/5$+some integer (RHS)

The LHS must be greater than or equal to the non- integer part of the RHS i.e.LHS$\geq$4/5.

We can write as $3/10s_1+1/10s_2\geq 4/5$. This inequality called GOMORY's constraint& to convert it into less than or equal to ($\leq$) type, we will multiply it by -1. The resultant constraint is $-3/10s_1-1/10s_2\leq -4/5$. By introducing slack variable F_1, the constraint will be $-3/10S_1-1/10S_2+F_1=-4/5$. This is the required fractional cut. Where, F1is slack variable.

In order to obtain an optimum integer solution this new equality constraint is included in simplex table III as shown below & the problem is then solved by Dual Simplex Method.

Simplex table IV

Cj	Contribution per unit		5	7	0	0	0	
	Basic variables	Solution values	X1	X2	S1	S2	F1	Min. ratio
7	X2	24/5	0	1	3/10	1/10	0	(24/5)/(3/10)=16
5	X1	21/5	1	0	-1/20	3/20	0	(21/5)/(-1/20)=-84(-ve)
0	F1	-4/5	0	0	-3/10*	-1/10	1	(-4/5)/(-3/10)=8/3 ⟶ key row
	Zj	273/5	5	7	37/20	29/20	0	
	Cj-Zj		0	0	-37/20	-29/20	0	
	Ratio (Cj-Zj)/F1		-	-	37/6	29/2	-	
					↑ Min. (entering)			

The entering variable is s_1 & the leaving variable is F_1& the key element is -3/10.
By introducing the entering variable & removing the leaving variable & with the changed values the 5[th] simplex table is prepared as follows:

Simplex table V

Cj	Contribution per unit		5	7	0	0	0
	Basic variable	Solution values	X1	X2	S1	S2	F1
7	X2	4	0	1	0	0	1
5	X1	13/3	1	0	0	1/6	-1/6
0	S1	8/3	0	0	1	1/3	-10/3
	Zj	149/3	5	7	0	5/6	37/6
	Cj-Zj		0	0	0	-5/6	-37/6

Since x_1 was non- integer value second Gomory's cut is now introduced (non-integer values of s_1 is not considered since it is slack variable). The constraint equation is: $x_1+0x_2+0s_1+1/6s_2-1/6F_1=13/3$, this is rewrite as $(1+0)\ x_1+\ (0+1/6)s_2+(-1+5/6)F_1=4+1/3$
Thus, we can write the above constraint as: $1/6s_2+5/6F_1=1/3+$some integer
Now, since the RHS is 1/3+some integer, so the LHS must be greater than or equal to 1/3 i.e.LHS$\geq$1/3
Therefore we can write as: $1/6s_2+5/6F_1\geq 1/3$. This inequality is called Gomory's constraint &to convert it into less than or equal to ($\leq$) type, we will multiply it by -1, the resultant constraint is $-1/6s_2-5/6F_1\leq -1/3$.
By introducing slack variable F2 the constraint will be $-1/6s_2-5/6F_1+F_2=-1/3$. This is the required fractional cut. Introducing second fractional cut in the simplex table, we get

Simplex table VI

Cj	Contribution per unit		5	7	0	0	0		
	Basic variables	Solution values	X1	X2	S1	S2	F1	F2	Min. ratio
7	X2	4	0	1	0	0	1	0	4/0=∞
5	X1	13/3	1	0	0	1/6	-1/6	0	(13/3)/(1/6)=26
0	S1	8/3	0	0	1	1/3	-10/3	0	(8/3)/(1/3)=8
0	F2	-1/3	0	0	0	-1/6*	-5/6	1	(-1/3)/(-1/6)=2 ⟶ key row
	Zj	149/3	5	7	0	5/6	37/6	0	
	Cj-Zj		0	0	0	-5/6	-37/6	0	
	Ratio (Cj-Zj)/F2		-	-	-	5	37/5	-	
						↑ Min.(entering)			

The entering variable is S_2 & the leaving variable is F_2 & the key element is -1/6.
By introducing the entering variable & removing the leaving variable& with the changed values the 7[th] simplex table is prepared as follows:

Simplex table VII

Cj	Contribution per unit		5	7	0	0	0	0
	Basic variables	Solution values	X1	X2	S1	S2	F1	F2
7	X2	4	0	1	0	0	1	0
5	X1	4	1	0	0	0	-1	1
0	S1	2	0	0	1	0	-5	2
0	S2	2	0	0	0	1	5	-6
	Zj	48	5	7	0	0	2	5
	Cj-Zj		0	0	0	0	-2	-5

Since all the values in the solution value column are integers, the solution is optimal. X_1=4, x_2=4 & max Z=48

Example 2: solve the following ILPP by using Gomory fractional cut.

Max $Z=3x_1+x_2+3x_3$

St. to $-x_1+2x_2+x_3\leq4$

$2x_2-3/2x_3\leq1$

$x_1-3x_2+2x_3\leq3$

Where x_1, x_2 & x_3 are non-negative integers.

Solution: for the application of cutting plane method, all coefficients & constraints must be whole numbers. Hence the second constraint must be transformed into $4x_2-3x_3\leq2$

Introduce slack variable & assign 0 coefficients in the objective function & convert the inequality constraints into equations as follows:

Max $Z=3x_1+x_2+3x_3+0s_1+0s_2$

St. to $-x_1+2x_2+x_3+s_1=4$

$4x_2-3x_3+s_2=2$

$x_1-3x_2+2x_3+s_3=3$

Where x_1, x_2, x_3, s_1, s_2, $s_3\geq0$ & x_1, x_2, x_3 are integers.

Initial feasible solution is obtained by setting $x_1 = x_2= x_3=0$, then we get s_1 =4 s_2=2 s_3=3

Simplex table I

Cj	Contribution per unit		3	1	3	0	0	0	
	Basic variables	Solution values	X1	X2	X3	S1	S2	S3	Min. ratio
0	S1	4	-1	2	1	1	0	0	4/-1=-4
0	S2	2	0	4	-3	0	1	0	2/0=∞
0	S3	3	1*	-3	2	0	0	1	3/1=3 ⟶ key row
	Zj	0	0	0	0	0	0	0	
		Cj-Zj	3	1	3	0	0	0	

↑Max (entering)

By selecting arbitrary x_1 is the entering variable & the leaving variable is s_3 & the key element is 1. By introducing the entering variable &removing the leaving variable & with the changed values the 2nd simplex table is prepared as follows:

Simplex table II

Cj	Contribution per unit		3	1	3	0	0	0	
	Basic variables	Solution values	X1	X2	X3	S1	S2	S3	Min. ratio
0	S1	7	0	-1	3	1	0	1	7/-1=-7
0	S2	2	0	4*	-3	0	1	0	2/4=1/2 ⟶ key row
3	X1	3	1	-3	2	0	0	1	3/-3=-1
	Zj	9	3	-9	6	0	0	3	
		Cj-Zj	0	10	-3	0	0	-3	

↑Max (entering)

X_2 is the entering variable & s_2 is the leaving variable & 4 is the key element.

By introducing the entering variable & removing the leaving variable & with the changed values the 3rd simplex table is prepared as follows:

Simplex table III

Cj	Contribution per unit		3	1	3	0	0	0	
	Basic variables	Solution values	X1	X2	X3	S1	S2	S3	Min. ratio
0	S1	15/2	0	0	9/4*	1	¼	1	(15/2)/(9/4)=10 /3 → key row
1	X2	½	0	1	-3/4	0	¼	0	(1/2)/(-3/4)=-2/3
3	X1	9/2	1	0	-1/4	0	¾	1	(9/2)/(-1/4)=-18
	Zj	14	3	1	-3/2	0	5/2	3	
		Cj-Zj	0	0	9/2	0	-5/2	-3	

↑Max (entering)

X_3 is the entering variable & s_1 is the leaving variable & 9/4 is the key element.
By introducing the entering variable & removing the leaving variable & with the changed values the 4^{th} simplex table is prepared as follows:

Simplex table IV

Cj	Contribution per unit		3	1	3	0	0	0
	Basic variables	Solution values	X1	X2	X3	S1	S2	S3
3	X3	10/3	0	0	1	4/9	1/9	4/9
1	X2	3	0	1	0	1/3	1/3	1/3
3	X1	16/3	1	0	0	1/9	7/9	10/9
	Zj	29	3	1	3	2	3	5
		Cj-Zj	0	0	0	-2	-3	-5

Since all Cj-Zj≥0, the optimum solution is obtained, we are having the optimum non-integer solution x_1=16/3, x_2=3, x_3=10/3 &max Z=29

Using simplex method an optimum non-integer solution is displayed in simplex table IV. Since the optimum solution is not integer value we consider only the fractional part & we will proceed as below to have a fractional cut. $X_1=5+1/3$ $x_3=3+1/3$

To construct Gomory constraint the consonant equation with the largest fractional part is selected but here row x_1 & row x_3 have the same fractional part 1/3. So the tie is broken by choosing any row arbitrary. We select x_3 as a source row & rewrite the non- integer value of the row corresponding to this variable (x_3). Then the constraint equation:

$10/3=0x_1+0x_2+1x_3+4/9s_1+1/9s_2+4/9s_3$, it is rewrite as

$(1+0)x_3+(0+4/9)s_1+(0+1/9)s_2+(0+4/9)s_3=3+1/3$

Thus, we can write the above constraint as: $4/9s_1+1/9s_2+4/9s_3=1/3$+some integer (RHS)
The LHS must be greater than or equal to the non- integer part of the RHS i.e.LHS≥1/3.
We can write as $4/9s_1+1/9s_2+4/9s_3\geq 1/3$. This inequality called GOMORY's constraint& to convert it into less than or equal to (≤) type, we will multiply it by -1. The resultant constraint is: $-4/9s_1-1/9s_2-4/9s_3\leq -1/3$. By introducing slack variable F_1, the constraint will be $-4/9S_1-1/9S_2-4/9S_3+F_1=-1/3$. This is the required fractional cut. Where, $F_1$is slack variable.
In order to obtain an optimum integer solution this new equality constraint is included in simplex table IV as shown below & the problem is then solved by Dual Simplex Method.

Simplex table V

Cj	Contribution per unit		3	1	3	0	0	0	0	
	Basic variables	Solution values	X1	X2	X3	S1	S2	S3	F1	Min ratio
3	X3	10/3	0	0	1	4/9	1/9	4/9	0	15/2
1	X2	3	0	1	0	1/3	1/3	1/3	0	1
3	X1	16/3	1	0	0	1/9	7/9	10/9	0	48
0	F1	-1/3	0	0	0	-4/9*	-1/9	-4/9	1	¾ → key row
	Zj	29	3	1	3	2	3	5	0	
		Cj-Zj	0	0	0	-2	-3	-5	0	
	Ratio	(Cj-Zj)/F1	-	-	-	9/2	27	45/4	-	

↑Min (entering)

S_1 is the entering variable &F_1 is the leaving variable & -4/9 is the key element.
By introducing the entering variable & removing the leaving variable & with the changed values the 6[th] simplex table is prepared as follows:

Simplex table VI

Cj	Contribution per unit		3	1	3	0	0	0	0
	Basic variables	Solution values	X1	X2	X3	S1	S2	S3	F1
3	X3	3	0	0	1	0	0	0	1
1	X2	11/4	0	1	0	0	¼	0	¾
3	X1	21/4	1	0	0	0	¾	1	¼
0	S1	3/4	0	0	0	1	¼	1	-9/4
	Zj	55/2	3	1	3	0	5/2	3	9/2
		Cj-Zj	0	0	0	0	-5/2	-3	-9/2

Since all Cj-Zj≤0, the optimal solution is obtained. $X_1=21/4$, $x_2=11/4$, $x_3=3$ & max z=55/2. But the value of x_1& x_2 are non-integer i.e $x_1=5+1/4$, $x_2=2+3/4$.
Since the solution is still non-integer, second Gomory's constraint must be added. Now x_2 has the largest fractional part (3/4) & is chosen as a source row. Now integer value of s_1 is not considered since it is a slack variable.
To construct Gomory's second constraint x_2 row is written as: $x_1+1/4s_2+3/4F_1=11/4$ or

$$(1+0)x_1+(0+1/4)s_2+(0+3/4)F_1=2+3/4$$

Thus, we can write the above constraint as: $1/4s_2+3/4F_1=3/4$+some integer (RHS)
The LHS must be greater than or equal to the non- integer part of the RHS i.e.LHS≥3/4.
We can write as $1/4s_2+3/4F_1\geq3/4$. This inequality called GOMORY's constraint& to convert it into less than or equal to (≤) type, we will multiply it by -1. The resultant constraint is: $-1/4s_2-3/4F_1\leq-3/4$.
By introducing slack variable F_2, the constraint will be $-1/4S_2-3/4F_1+F_2=-3/4$. This is the required fractional cut. Where, F_2 is slack variable.
In order to obtain an optimum integer solution this new equality constraint is included in simplex table VI as shown below & the problem is then solved by Dual Simplex Method.

Simplex table VII

Cj	Contribution per unit		3	1	3	0	0	0	0	0	
	Basic variables	Solution values	X1	X2	X3	S1	S2	S3	F1	F2	Min ratio
3	X3	3	0	0	1	0	0	0	1	0	3
1	X2	11/4	0	1	0	0	¼	0	¾	0	11/3
3	X1	21/4	1	0	0	0	¾	1	¼	0	21
0	S1	3/4	0	0	0	1	¼	1	-9/4	0	-1/3
0	F2	-3/4	0	0	0	0	-1/4	0	-3/4	1	1→ key row
	Zj	55/2	3	1	3	0	5/2	3	9/2	0	
	Cj-Zj		**0**	**0**	**0**	**0**	**-5/2**	**-3**	**-9/2**	**0**	
	Ratio	(Cj-Zj)/F2	-	-	-	-	10	6	6	-	
									↑Min. (entering)		

By selecting arbitrary the entering variable is F_1 & the leaving variable is F_2, -3/4 is the key element.
By introducing the entering variable & removing the leaving variable& with the changed values the 8[th] simplex table is prepared as follows:

Simplex table VIII

Cj	Contribution per unit		3	1	3	0	0	0	0	0
	Basic variables	Solution values	X1	X2	X3	S1	S2	S3	F1	F2
3	X3	2	0	0	1	0	-1/3	0	0	4/3
1	X2	2	0	1	0	0	0	0	0	1
3	X1	5	1	0	0	0	2/3	1	0	1/3
0	S1	3	0	0	0	1	1	1	0	-3
0	F1	1	0	0	0	0	1/3	0	1	-4/3
	Zj	23	3	1	3	0	1	3	0	0
	Cj-Zj		**0**	**0**	**0**	**0**	**-1**	**-3**	**0**	**0**

Since all the values of **Cj-Zj≤0, the integer optimal solution is obtained. $x_1=5$, $x_2=2$, $x_3=2$ & max z=23**

SOME FACTS

- Integer variables may be required when the model represents a one time decision (not an ongoing operation).

- Integer Linear Programming (ILP) models are much more difficult to solve than Linear Programming (LP) models.

- Algorithms that solve integer linear models do not provide valuable sensitivity analysis results

2. Branch & Bound Method

Under this method the solution of ILPP is obtained by the following steps.

Step1: Solve the original problem by using LP. If the answer satisfies the integer constraints we will stop. If not, this value provides an interval upper bound.

Step 2: find any feasible solution that meets the integer constraints for use as a lower bound. Usually rounding down each variable will give the required answer.

Step 3: branch on one variable from step 1 that does not have an integer value. Split the problem into two sub-problems based on integer values that are immediately above &below the integer value.

For example, if $X_1=8.75$ was in the final LP solution, introduce the constraints $X_1 \geq 9$ in the first sub problem &$X_1 \leq 8$ in the second sub-problem.

Step 4: create nodes at the top of these new branches by solving the new problem.

Step 5:

a. If a branch yields a solution to the LP problem that is not feasible, terminate the branch.
b. If a branch yields a solution to the LPP that is feasible but not an integer solution, go to step 6.
c. If the branch yields a feasible integer solution, examine the value of the objective function solution has been reached. If it is not equal to the upper bound, but exceeds the lower bound, set it as the new lower bound &go to step 6. Finally if it is less than the lower bound terminate this branch.

Step 6: Examine both branches again & set the upper bound equal to the maximum value of the objective function at all find nodes. If the upper bound equal the lower bound stop. If not, go back to step 3.

Example1: solve the following ILPP by branch &bound method

Max Z = $6X_1+8X_2$

Subject to $X_1+4X_2 \leq 8$

$7X_1+2X_2 \leq 14$

$X_1\ X_2 \geq 0$&are integers.

Solution: now, in order to plot the constraints on the graph temporarily, we will convert the in equality in to equation.

$X_1 + X_2 = 8$-- (i)

$7X_1 + 2X_2 = 14$ --- (ii)

- ✓ In equation (i) $X_1 + 4X_2 = 8$

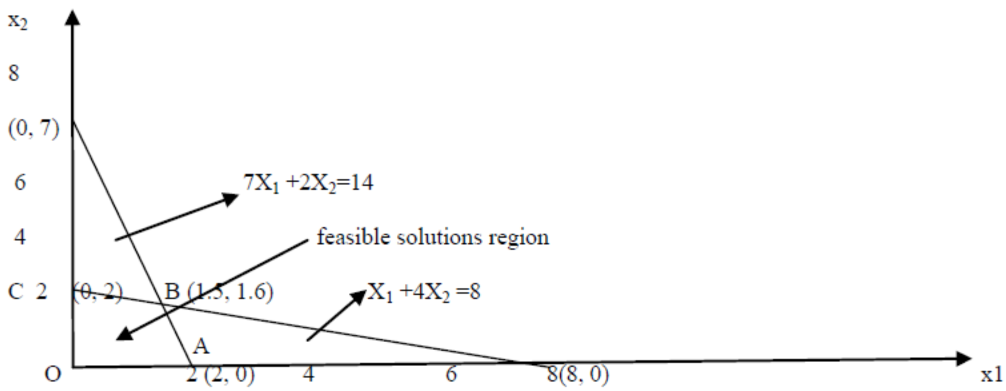
If $X_1 = 0$, then $X_2 = 2$------ (0, 2)

If $X_2 = 0$, then $X_1 = 8$------- (8, 0)

- ✓ In equation (ii) $7X_1 + 2X_2 = 14$

If $X_1 = 0$, then $X_2 = 7$---------- (0, 7)

If $X_2 = 0$, then $X_1 = 2$ --------- (2, 0)



Any combination of value of X_1 &X_2 which satisfies the given constraints is called feasible solution. Area OABC in the figure satisfied by the constraint is shown by shaded area and is called feasible solution region.

Now, we solve the inter-sectional equation at point B, simultaneously.

$$\left.\begin{matrix} X_1 + 4X_2 = 8 \\ 7X_1 + 2X_2 = 14 \end{matrix}\right\}$$ multiplying equation (ii) by 2 &subtracting equation (i) from equation(ii)

$$\begin{array}{l} 14X_1 + 4X_2 = 28 \\ -\ X_1 + 4X_2 = 8 \\ \hline 13X_1 = 20 \end{array}$$

$X_1 = 20/13$ then substitute this value into equation (ii)

$X_1 + 4X_2 = 8$

$20/13 + 4X_2 = 8$

$4X_2 = 8 - 20/13$, $X_2 = 21/13$

Corner points	Coordinates	$Z=6X_1 +8X_2$
A	(2,0)	6x2 +8(0)=12
B	(1.5 ,1.6)	6x1.5 +8x1.6 =21.8
C	(0,2)	6x0 +8x2 =16
O	(0,0)	6x0 +8x0 =0

By solving graphically, we get $X_1 = 1.5$, $X_2 = 1.6$ Max Z = 21.8Since x_1 & x_2 are not integers, this solution is not valid. The max value of 21.8 will serve as an initial upper bound. We note that rounding down gives $x_1=1$, $x_2=1$ & z=14 which is feasible & can be used as lower bound.

Sub-problem A& B: The problem is now divided into two sub-problems A& B. we can consider branching out either variable that does not have integer solution; let us pick x_1 in this time.

Sub-problem A

Max $Z=6x_1+8x_2$

St. to $x_1+4x_2 \le 8$

$7x_1+2x_2 \le 14$

$x_1 \ge 2$

$x_1, x_2 \ge 0$

Sub-problem B

Max $Z=6x_1+8x_2$

St. to $x_1+4x_2 \le 8$

$7x_1+2x_2 \le 14$

$x_1 \le 1$

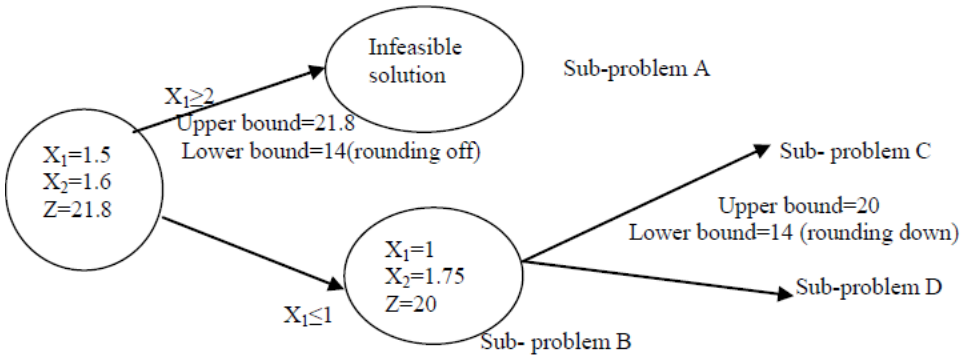
$x_1, x_2 \ge 0$

By solving graphically/simplex method we get

Sub-problem A has infeasible solution

Sub-problem B has optimal solution ($x_1=1$, $x_2=1.75$ & max z=20)

This information is presented in branch from in figure given below:



Sub- problem B's is searched further since it has a non-integer solution. The second upper bound takes on the value 20 replacing 21.8 from the first node. Sub-problem B is now branched into two new sub-problems: C & D. Sub-problem C has the additional constraint of $x_2 \geq 2$ & sub-problem D has also additional constraint $x_2 \leq 1$. The logic for developing these sub-problems is that, since Sub- problem B's optimal solution $x_2=1.75$ is not feasible; the integer answers must lie either in the region $x_2 \geq 2$ or in the region $x_2 \leq 1$.

Sub-problem C	Sub-problem D
Max $Z=6x_1+8x_2$	Max $Z=6x_1+8x_2$
St. to $x_1+4x_2 \leq 8$	St. to $x_1+4x_2 \leq 8$
$7x_1+2x_2 \leq 14$	$7x_1+2x_2 \leq 14$
$x_1 \leq 1$	$x_1 \leq 1$
$x_1 \geq 2$	$x_1 \leq 1$
$x_1, x_2 \geq 0$	$x_1, x_2 \geq 0$

By solving graphically/simplex method we get

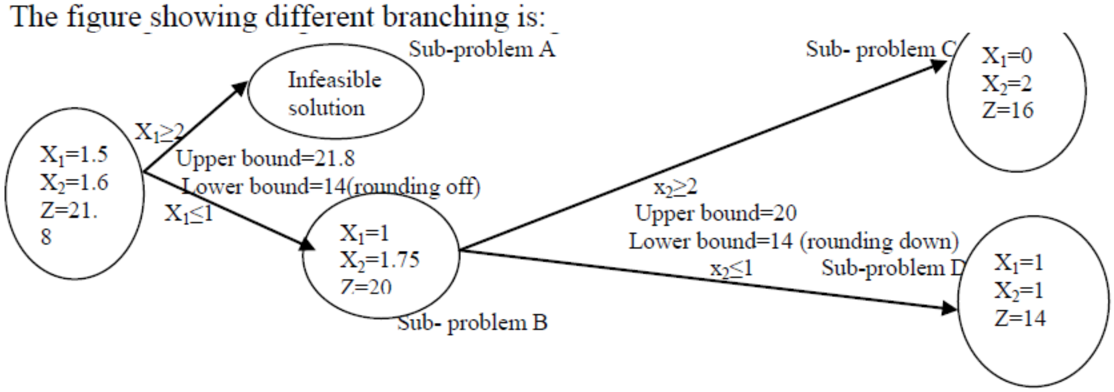
Optimal solution to C:
X1=0, X2=2 & max Z=16

Optimal solution to D:
X1=1, X2=1 & max Z=14

Now, branching process is stopped since x1, x2 both are integers.

Therefore, the optimal solution is at Sub-problem C i.e. x1=0, x2=2 & max Z=16

The figure showing different branching is:



CHAPTER5: SPECIAL TYPES OF LINEAR PROGRAMMING

5.1. TRANSPORTATION PROBLEM

Transportation problems are one types of the LPP, in which objective is to transport various quantities of a single homogeneous commodity, to different destinations in such a way that the total transportation cost is minimized. TPs gave direct relevance to decisions in the area of distribution policy making, where the objective is minimization of transportation cost. The various features of linear programming can be observed in these problems. Here the availability as well as requirements of the various centers are finite & constitute the limited resources. It is also assumed that cost of shipping is linear. Thus these problems could also be solved by simplex method.

Basic Assumption of the TP model

1. Availability of the quantity: Quantity available for distribution at different sources is equal to total requirement of different consumption centers.
   $\sum_{i=1}^{m} ai = \sum_{j=1}^{n} bj$ i.e. quantity of available/supply=quantity of requirement/demand
2. Transportation of items: Items can be conveniently transported from every production center to every consumption center.
3. Cost per unit: The per unit transportation cost of items from one production center to another consumption center is certain.
4. Independent of cost: Per unit cost of transportation is independent of the quantity dispatched.
5. Objective: The objective of TP is to minimize the total cost of transportation for the organization as a whole.

Definitions

The following terms may be defined with reference to the transportation problem.

1. Feasible solution (FS): A set of non-negative individual allocation ($x_{ij} \geq 0$) which simultaneously removes deficiencies is called feasible solution.
2. Basic Feasible Solution (BFS): A feasible solution to a m-origin, n-destination problem is said to be basic if the number of positive allocations are m+n-1 i.e. one less than the sum of rows &columns.
3. Optimal solution: a feasible solution is said to be optimal if it minimizes the total transportation cost. The optimal solution itself may or may not be a basic solution. This is one through successive improvement to the initial basic feasible solution until no further increase in transportation cost is possible.

The Transportation Model Characteristics

- A product is transported from a number of sources to a number of destinations at the minimum possible cost.
- Each source is able to supply a fixed number of units of the product, and each destination has a fixed demand for the product.
- The linear programming model has constraints for supply at each source and demand at each destination.
- All constraints are equalities in a balanced transportation model where supply equals demand.

Constraints contain inequalities in unbalanced models where supply does not equal demand.

Mathematical formulation of the TP

Let us assume 'm' as origins (plant location), i^{th} origins possessing ai units of a certain product, whereas there will be 'n' destinations with destination j^{th} requiring b_j units. Cost of transporting an item is known either directly or indirectly in terms of shipping house etc. let c_{ij} be the cost of shipping one unit from i^{th} source to j^{th} destination. Let "x_{ij}" be the amount to be shipped from i^{th} origin to j^{th} destination. The problem is now to determine non-negative values of "x_{ij}" satisfying both the availability constraint.

$\sum_{j=1}^{n} xij = a_i$ for i=1, 2,……..,m

As well as the requirement

$\sum_{i=1}^{m} xij = b_j b$ for j= 1,2, ……….n

And minimizing the total cost of shipping is.$Z = \sum_{i=1}^{m} \sum_{j=1}^{n} xij\ cij$

That means: min $Z = \sum_{i=1}^{m} \sum_{j=1}^{n} xij\ cij$

St. to $\sum_{j=1}^{n} xij = a_i$ for i=1, 2............... m (Supply constraint)

$\sum_{i=1}^{m} xij = b_j$ for j= 1,2, ……….n (Demand constraint)

$Xij \geq 0$ for all i & j

Example1: A company has three production facilities S1, S2 &S3 with production capacity of 7,9 & 18 units per week of a product respectively. These units are to be shipped to four warehouses D1, D2, D3 &D4 with requirement of 5, 8, 7 & 14 per week respectively. The transportation costs per between factories to allocations are m+n-1i.e. one less than the sum of rows & columns.

Warehouse → Factory ↓	D1	D2	D3	D4	Capacity
S1	19	30	50	10	7
S2	70	30	40	60	9
S3	40	8	70	20	18
Demand	5	8	7	14	34

Formulate this transportation problem as an LP model to minimize the total transportation cost.

Solution: Model formulation

- Let xij=number of units of the product to be transported from factory i (i=1, 2, 3) to warehouse j (j=1, 2, 3, 4)
- Let cij= cost of shipping one unit of the commodity from source i(i=1, 2, 3) to destination j (j=1, 2, 3, 4) for each square (route).

Min Z = $\sum_{i=1}^{3} \sum_{j=1}^{4} x_{ij} c_{ij}$

St. to $\sum_{j=1}^{4} x_{ij} = a_i$ for i=1, 2, 3 (Supply constraint)

$\sum_{i=1}^{3} x_{ij} = b_j$ for j= 1,2, ...n (Demand constraint)

$X_{ij} \geq 0$ for all i & j

$$\text{Min } Z = 19x11 + 30x12 + 50x13 + 10x14 + 70x21 + 30x22 + 40x23 + 60x24 + 40x31 + 8x32 + 70x33 + 20x34$$

St. to

$$\left.\begin{aligned} x11 + x12 + x13 + x14 &= 7 \\ x21 + x22 + x23 + x24 &= 9 \\ x31 + x32 + x33 + x34 &= 18 \end{aligned}\right\} \text{Supply constraints}$$

$$\left.\begin{aligned} x11 + x21 + x31 &= 5 \\ x12 + x22 + x32 &= 8 \\ x13 + x23 + x33 &= 7 \\ x14 + x24 + x34 &= 14 \end{aligned}\right\} \text{Demand constraints}$$

$$xij \geq 0 \; for \; all \; i \; \& \; j$$

Example2: How many tons of wheat to transport from each grain elevator to each mill on a monthly basis in order to minimize the total cost of transportation?

- Data:

Grain Elevator	Supply	Mill	Demand
1. Kansas City	150	A. Chicago	200
2. Omaha	175	B. St.Louis	100
3. Des Moines	275	C. Cincinnati	300
Total	600 tons	total	600 tons

Grain Elevator	Transport cost from Grain Elevator to Mill ($/ton) A. Chicago	B. St. Louis	C. Cincinnati
1. Kansas City	$6	8	10
2. Omaha	7	11	11
3. Des Moines	4	5	12

Subject to $x_{1A} + x_{1B} + x_{1C} = 150$

$x_{2A} + x_{2B} + x_{2C} = 175$

$x_{3A} + x_{3B} + x_{3C} = 275$

$x_{1A} + x_{2A} + x_{3A} = 200$

$x_{1B} + x_{2B} + x_{3B} = 100$

$x_{1C} + x_{2C} + x_{3C} = 300$

$x_{ij} \geq 0$

Where x_{ij} = tons of wheat from each grain elevator, i, i = 1, 2, 3, to each mill j, j = A, B, C
The Transportation Model Tableau Format

- Transportation problems are solved manually within a *tableau* format.
- Each cell in a transportation tableau is analogous to a decision variable that indicates the amount allocated from a source to a destination.
- The supply and demand values along the outside rim of a tableau are called *rim values.*

From \ To	A	B	C	Supply
1	6	8	10	150
2	7	11	11	175
3	4	5	12	275
Demand	200	100	300	600

The Transportation Tableau

The Transportation Model Solution Methods

- Transportation models do not start at the origin where all decision values are zero; they must instead be given an *initial feasible solution*.
- There are two steps to find the optimal solution of TP.

A) Find an initial basic feasible solution.
B) An optimal solution by making successive improvements to initial basic solution until no further decrease in the transportation cost is possible.

A) Methods for Finding Initial Basic Feasible Solutions

The following are the important methods of developing an initial basic feasible solution.

1. North West Corner Method (NWCM)
2. Lowest Cost Entry Method (LCEM) or Matrix Minima Method (M^3)
3. Vogel's Approximation Method (VAM)

1. North West Corner Method (NWCM)

This method is the most systematic & easiest method for obtaining initial feasible solution. Steps involved in this method are stated as follows:
Step1: Construct an empty m×n matrix completed with rows & columns.
Step2: Indicate the row totals & column totals at the end.

Step3: Starting with (1, 1) cell at the North West Corner of the matrix allocate maximum possible quantity keeping in view that allocation can neither be more than the quantity required by the respective warehouses nor more than the quantity available at each supply center.
Step4: Adjust the supply & demand numbers in the respective rows & columns allocation.
Step5: If the supply for the first row is exhausted then moves down to the first cell in the second row & first column & go to step4.

Step6: If the demand for the first column is satisfied, then move to the first cell in the second column & the first row & go to step4.
Step7: If for any cell, supply equal to demand the next allocation can be made in cell either in the next row or column.
Step8: Compute the procedure until the total available quantity is fully allocated to the cells as required.

Generally, in the northwest corner method the largest possible allocation is made to the cell in the upper left-hand corner of the tableau, followed by allocations to adjacent feasible cells.
Example1: Solve the following TP by using NWCM.

Plant \ Warehouse		Warehouse			
Plant		W1	W2	W3	Supply (Si)
plant	P1	7	6	9	20
	P2	5	7	3	28
	P3	4	5	8	17
Demand (Di)		21	25	19	65

Solution:

Plant \ Warehouse		Warehouse			
Plant		W1	W2	W3	Supply (Si)
plant	P1	7 20	6	9	20
	P2	5 1	7 25	3 2	28
	P3	4	5	8 17	17
Demand (Di)		21	25	19	65

As stated in this method, we start with the cell (P1, W1) & allocate the min (S1, D1) =min (20, 21) =20. Therefore we allocate 20 units to this cell which completely exhausts the supply of plant P1 & leaves a balance of (21-20) =1unit of demand at warehouse W1. Now, we move vertically downward to the cell (P2, W1). At this stage the largest possible allocation is the min

(S2, D1-20) = min (28, 1) =1unit.this allocation of 1 unit of the cell (P2, W1) completely satisfies the demand of warehouse W1. However, this leaves a balance of (28-1) =27 units of supply at plant P2. Now, we move again horizontally to the cell (P2, W2). Since the demand of warehouse W2 is 25 units while supply available at plant P2 is 27 units. Therefore the min (27, 25) =25 units are allocated to the cell (P2, W2). The demand of warehouse W2 is now satisfied & a balance of (27-25) =2 units of supply remain at plant P2. Moving again horizontally, we allocate two units the cell ((P2, W3) which completely exhaust the supply at plant P2 & leaves a balance of 17 units demand at warehouse W3. Now, we move vertically downward to the cell (P3, W3). At this cell 17 units are available at plant P3 & 17 units are required at warehouse W3. So we allocate 17 units to this cell (P3, W3).

Hence we have made all the allocations. It may be noted here that there are 5 (3+3-1) allocations which are necessary to proceed further for obtaining the optimal solution.
Therefore, the total transportation cost for this initial feasible solution is:

Total cost = $20\times7+1\times5+2\times57+2\times3+17\times8=462$

Example 2: solve the following TP by using NWCM.

To / from	A	B	C	Supply
1	6	8	10	150
2	7	11	11	175
3	4	5	12	275
Demand	200	100	300	600

Solution:

From / To	A	B	C	Supply
1	6 150	8	10	150
2	7 50	11 100	11 25	175
3	4	5	12 275	275
Demand	200	100	300	600

The Initial NW Corner Solution

- The initial solution is complete when all rim requirements are satisfied.
- Transportation cost is computed by evaluating the objective function:

$$Z = \$6x_{1A} + 8x_{1B} + 10x_{1C} + 7x_{2A} + 11x_{2B} + 11x_{2C} + 4x_{3A} + 5x_{3B} + 12x_{3C}$$
$$= 6(150) + 8(0) + 10(0) + 7(50) + 11(100) + 11(25) + 4(0) + 5(0) + 12(275)$$
$$= \$5,925$$

2. Lowest Cost Entry Method (LCEM)

This method takes into consideration the lowest cost & therefore takes less time to solve the problem.
In the minimum cell cost method as much as possible is allocated to the cell with the minimum cost followed by allocation to the feasible cell with minimum cost. Steps involved in this method are stated as follows:

Step1: Select the cell with the lowest transportation cost among all the rows or columns of the transportation table. If the minimum cost is not unique, then select arbitrary any cell with the lowest cost.
Step2: Allocate as any units as possible to the cell determined in step1 & eliminate that row/ column in which either capacity or requirement is exhausted.
Step3: Adjust the capacity & requirement for the next allocations.

Step4: Repeat step 1 to 3 for the reduced table until the entire capacities are exhausted to fill the requirement at different destinations.
Example1: Solve the following TP by using LCEM.

To / From		Warehouse A	B	C	Supply (Si)
plant	X	50	30	220	41
	Y	90	45	170	13
	Z	250	200	60	4
Demand (Di)		13	23	22	58

Solution:

To / From		Warehouse A	B	C	Supply (Si)
plant	X	50 (13)	30 (23)	220 (5)	41
	Y	90	45	170 (13)	13
	Z	250	20	60 (4)	4
Demand (Di)		13	23	22	58

Total cost = $50\times13+30\times23+220\times5+170\times13+60\times4 = 650+690+1100+2210+240=4890$

In this method, we start with the cell which has the minimum cost i.e.(X, B) in which the cost is 30 birr (which is the lowest cost), we allocate minimum (S1, D1) = min (41, 23) = 23 units to fulfill the complete requirement of warehouse B. Since the demand of warehouse B is satisfied, therefore column B will not be considered any more. In the reduced table again allocate the minimum cost cell i.e. (X, A) then we allocate min (S1, D1) = min (18, 13) = 13 units to this cell which fulfill the requirement of warehouse A. Since the demand of warehouse A is satisfied, therefore column A will not be considered any more. Proceeding in the same way the search for minimum cost cell continue until all supply & demand conditions are satisfied.

Example2: solve the following TP by using LCEM.

To / from	A	B	C	Supply
1	6	8	10	150
2	7	11	11	175
3	4	5	12	275
Demand	200	100	300	600

Solution:

To / From	A	B	C	Supply
1	6	8	10	150
2	7	11	11	175
3	4 / 200	5	12	275
Demand	200	100	300	600

The Initial Minimum Cell Cost Allocation

To / From	A	B	C	Supply
1	6	8	10	150
2	7	11	11	175
3	4 / 200	5 / 75	12	275
Demand	200	100	300	600

The Second Minimum Cell Cost Allocation

- The complete initial minimum cell cost solution; total cost = $4,550.
- The minimum cell cost method will provide a solution with a lower cost than the northwest corner solution because it considers cost in the allocation process.

To / From	A	B	C	Supply
1	6	8 / 25	10 / 125	150
2	7	11	11 / 175	175
3	4 / 200	5 / 75	12	275
Demand	200	100	300	600

The Initial Solution

3. Vogel's Approximation Method (VAM)

This method is preferred over other two methods because the initial feasible solution obtained with VAM is either optimal or close to the optimal solution. Therefore, the amount of time required to calculate the optimal solution is reduced. In VAM the basis of allocation is unit cost penalty (the difference between the lowest & the next highest cost) is selected first for allocation & the subsequent allocations are also done keeping in view the highest unit cost penalty.

This Method is based on the concept of *penalty cost* or *regret*.
- A penalty cost is the difference between the largest and the next largest cell cost in a row (or column).
- In VAM the first step is to develop a penalty cost for each source and destination. Various steps in the iteration process are:

Step1: Construct the cost, requirement & availability matrix i.e. cost matrix with column & row information.
Step2: Compute a penalty for each row & column in the transportation table. The penalty for a given column is merely the difference between the smallest cost & the next smallest cost element in that particular row or column.

Step3: Identify the row & column with the largest penalty. In this identified row or column choose the cell which has the smallest cost & allocate the maximum possible quantity to this cell. Delete the row or column in which the capacity or requirement is exhausted.
Whenever the largest penalty among rows or columns is not unique, make an arbitrary choice.
Step4: Repeat step 1 to 3 for the reduced table until the entire capacities are used to fill the requirement at different warehouses.
Step5: From step 4 we will get initial feasible solution. Now for initial feasible solution find the total transportation cost by multiplying the cell allocation by unit cost.
Though this method takes more time as compared to other two methods, but still it gives better solutions & saves more time in reaching the optimum solution.

Example1: A manufacturer wants to 8 loads of his product from production centers X, Y & Z to distribution centers A, B & C. the unit transportation cost from origin O to destination D is given in the following matrix.

O / D	Distribution center			
	A	B	C	Availability
Production center X	50	30	220	1
Y	90	45	170	3
Z	250	200	50	4
requirement	3	3	2	8

Find the IFS by using VAM for this problem.

Solution:

O / D	Distribution center				
	A	B	C	Availability	Unit penalty
Production center X	50	30	220	1	20
Y	90	45	170	3	45
Z	250	20	50 2	4	150 ←
requirement	3	3	2	8	
Unit penalty	40	15	120		

Allocate 2 to ZC cell & eliminate column C. then the reduced matrix is obtained as below:

O / D	A	B	Availability	Unit penalty
Production center X	50	30	1	20
Y	90	45	3	45
Z	250	20 2	2	50 ←
requirement	3	3	6	
Unit penalty	40	15		

Allocate 2 to ZB cell & eliminate row Z. Then the reduced matrix is obtained as below:

O / D	A	B	Availability	Unit penalty
Production center X	50	30	1	20
Y	90	45 1	3	45 ←
requirement	3	1		
Unit penalty	40	15		

Allocate 1 to BY cell & eliminate row B. Then the reduced matrix is obtained as below:

O / D	A	Availability
Production center X	50 1	1
Y	90 2	2
requirement	3	3

Allocate 1 to XA cell & 2 to YA

Therefore, the initial feasible solution will be

D \ O		Distribution center			
		A	B	C	Availability
Production center	X	50 1	30	220	1
	Y	90 2	45 1	170	3
	Z	250	20 2	50 2	4
requirement		3	3	2	8

The initial feasible solution of transportation cost= 1×50+2×90+1×45+2×200+2×50=775

Example2: solve the following TP by using VAM.

To from	A	B	C	Supply
1	6	8	10	150
2	7	11	11	175
3	4	5	12	275
Demand	200	100	300	600

Solution:

To From	A	B	C	Supply	
1	6	8	10	150	2
2	7	11	11	175	4
3	4	5	12	275	1
Demand	200	100	300	600	
	2	3	1		

The VAM Penalty Costs

VAM allocates as much as possible to the minimum cost cell in the row or column with the largest penalty cost. So allocate 175 to cell 2A. Therefore, row 2 will not considered anymore because the supply of plant 2 is completely exhausted or used.

To From	A	B	C	Supply	
1	6	8	10	150	2
2	7 175	11	11	175	
3	4	5	12	275	1
Demand	200	100	300	600	
	2	3	2		

The Initial VAM Allocation

- After each VAM cell allocation, all row and column penalty costs are recomputed.

To From	A	B	C	Supply	
1	6	8	10	150	4
2	7 175	11	11	175	
3	4	5 100	12	275	8
Demand	200	100	300	600	
	2		2		

The Second VAM Allocation

- Recomputed penalty costs after the third allocation.

From \ To	A	B	C	Supply
1	6	8	10 150	150
2	7 175	11	11	175
3	4 25	5 100	12 150	275
Demand	200	100	300	600

The Third VAM Allocation

The initial VAM solution; total cost = $5,125
- VAM and minimum cell cost methods both provide better initial solutions than does the northwest corner method.

From \ To	A	B	C	Supply
1	6	8	10 150	150
2	7 175	11	11	175
3	4 25	5 100	12 150	275
Demand	200	100	300	600

The Initial VAM Solution

Vogel's Approximation Method (VAM) Summary of Steps

1. Determine the penalty cost for each row and column.
2. Select the row or column with the highest penalty cost.
3. Allocate as much as possible to the feasible cell with the lowest transportation cost in the row or column with the highest penalty cost.
4. Repeat steps 1, 2, and 3 until all rim requirements have been met.

Optimality Test

After computation of an initial basic feasible solution, we can proceed to know, whether the solution so obtained is optimum or not. Or once an initial solution is derived, the problem must be solved using either the stepping-stone method or the modified distribution method (MODI). For this purpose two methods are generally followed

A. Stepping stone method

B. Modified distribution method(MODI)

A. Stepping stone method

The stepping stone method is an interactive technique from moving an initial feasible solution to an optimal solution. In order to apply the stepping stone method to transportation problem one rule about the number of shipping routes being used must first be observed. In this rule, the number of occupied routes (square) must always be equal to one less than the sum of the number of rows plus the number of columns. The stepping stone method evaluates the cost effectiveness of shipping goods via transportation routes not currently in the solution.

The stepping stone method for testing the optimality can be summarized in the following

Step 1: prepare transportation table with a given unit cost of transportation along with the Rim requirement

Step 2: determine an initial basic feasible solution using any method (preferably VAM).

Step 3: evaluate an unoccupied cell forte effect of transferring one unit from an occupied cell to the unoccupied cell. This transfer is made by forming a closed path (loop) that retains the supply and demand condition of the problem.

The evaluation is conducted as follow:

a) Select an unused square to be evaluated.

b) Beginning with the selected unused square trace a close path (loop) through at least three occupied cans and finally returning back to the same occupied cell. It is called Home Square. The direction of the movement taken is immaterial because the result will be same in either case. In the closed path formulation of only right angle turn is allowed and therefore skips all other cells which are not at the turning points.

c) At each corner of the closed path assign plus (+) and minus (-) sign alternatively, beginning with plus sign for the unoccupied square to be evaluated. The +ve and –ve signs can be assigned either in a clockwise or counter clockwise direction.

d) Compute the net change in cost along the closed path by assign together the unit transportation costs associated with each of the cell traced in closed path. Comparing the addition to cost with the decreases, will be the improvement index.

e) Repeat steps 3(a) to 3(d) until net change in cost has been calculated for all unoccupied cells.

Step 4: check the sign of each of the net change in the unit transportation costs. If all net change are plus (+) or zero, then we have obtained an optimal solution, otherwise go to step5.

Step 5: select unoccupied cell with most negative net change among all unoccupied cells, if two minus values are equal select that one which will result in moving as many units as possible into the selected unoccupied cell with the minimum cost.

Step 6: assign as many units as possible to unoccupied cell satisfying rim conditions. The maximum numbers of units to be assigned are equal to the smaller circled number ignoring sign among the occupied cells with minus value in the closed path

Step 7: go to step 3 and repeat the procedure until all unoccupied cells are evaluated and the net change is positive or zero values.

Example1: solve the following by using NWCM & test its optimality by stepping stone method. Shipping costs are given in the table below.

To / From	Project A	Project B	Project C	Plant Capacity
Plant W	4	8	8	56
Plant X	16	24	16	82
Plant Y	8	16	24	77
Project Req.	72	102	41	215

SOLUTION: Initial feasible solution by NWCM.

To / From	Project A	B	C	Plant Capacity
plant W	4 56	8	8	56
X	16 16	2 66	16	82
Y	8	1 36	24 41	77
Project Req.	72	102	41	215

Total cost of the initial solution.

Source & destination combination	Quantity shipped unit cost	Result (TC)
WA	56×4	224
XA	16×16	256
XB	66×24	1584
YB	36×16	576
YC	41×24	984
		Total Transportation cost=3624

Since the stone square are 5 as against the required rim requirements m+n-1=5.the solution is non-degenerate. Now we proceed to test its optimality by stepping stone method.
We have traced a closed loop or path for each of unused square & opportunity cost for each of the unused square is determined as follows.
Now, tracing closed loop for cell WB.

To / From	Project A	B	C	Plant Capacity
plant W	4 56 -	8 +	8	56
X	16 16 +	? 66 -	1	82
Y	8	1 36	2 41	77
Project Req.	72	102	41	215

On the basis of closed loop shown as above the opportunity cost of cell WB

WB= WB-WA+XA-XB=8-4+16-24=-4

While tracing a closed path, we must move horizontally or vertically from unused square via stone square back to the original unused square. The path may skip over stone square as well as water square. The corner of the close path may occur only at the unused square & only the most direct route is used.

Assign (+) & (-) signs alternatively at each corner square of the closed path. Beginning with (+) sign at the unused square movement may be either clockwise or anti-clockwise directions. The (+) & (-) signs represent the additions or the subtractions of one unit to a square. The next result is improvement index. An important restriction is that there must be exactly one cell with (+) sign & one cell with a (-) sign in any row or column in which the loop takes a turn. The restrictions ensure that rim requirement would not be violated when units shifted among cells. If there are m+n-1 stone squares & there would be one &only one closed loop for each cell. And even number (at least four cell) must participate in a closed loop & an occupied cell can be considered only once & no more. Closed loop may or may not be square or rectangular in shape in large transportation table, the closed loop may have peculiar configurations & the loop may cross over itself.

Now tracing closed loop for cell WC.

To / From		Project A	B	C	Plant Capacity
plant	W	4 56 -	8	8 +	56
	X	16 16 +	2 - 66	1	82
	Y	8	1 + 36	2 - 41	77
Project Req.		72	102	41	215

Improvement index for unused square (WC). WC=WC-WA+XA-XB+YB-YC=8-4+16-24+16-24=-12.

Now tracing closed loop for cell XC.

To / From		Project A	B	C	Plant Capacity
plant	W	4 56	8	8	56
	X	16 16	2 - 66	1 +	82
	Y	8	1 + 36	2 - 4	77
Project Req.		72	102	41	215

Improvement index for unused square (XC). XC= XC-XB+YB-YC=16-24+16-24=-16.

Finally tracing closed loop for cell YA.

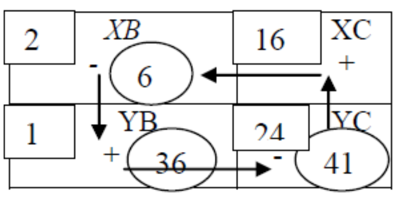
To / From		Project A	B	C	Plant Capacity
plant	W	4 56	8	8	56
	X	16 16 -	2 + 66	1	82
	Y	8 +	1 - 36	2 4	77
Project Req.		72	102	41	215

Improvement index for unused square (YA); YA=YA-XA+XB-YB=8-16+24-16=0.

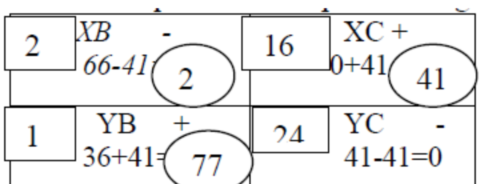
Therefore we have now completed the evaluation of unused squares, each of which represents an alternative square (route) that might be taken. The improvement index for each unused square is shown as follows.

To / From	Project A	B	C	Plant Capacity
plant W	4 (56)	8 (-4)	8 (-12)	56
X	16 (16)	2 (66)	16 (-16)	82
Y	8 (0)	1 (36)	24 (41)	77
Project Req.	72	102	41	215

Select the cell which has the largest negative improvement index, in order to obtain improved solution. So cell XC has the largest negative improvement index; then select it



The closed path for square XC has negative corners at square XB & YC. Select the cell which has smaller quantity from these two squares. So the smaller quantity of these two squares is 41. In order to obtain the new improved solution we add 41 to all squares on the closed path with (+) signs & subtract this quantity from all squares on the path having (-) signs. Then we get:



The movement in various cells is as shown below.

Now the 2nd improved table is reproduced as follows:

To / From	Project A	B	C	Plant Capacity
plant W	4 (56)	8	8	56
X	16 (16)	2 (25)	16 (41)	82
Y	8	1 (77)	24	77
Project Req.	72	102	41	215

Total cost for the 2nd improved solution

Shipping Assignments	Quantity shipped × unit cost	Result (TC)
WA	56× 4	224
XA	16× 16	256
XB	25× 24	600
XC	41× 16	656
YB	77× 16	1232
	Total Transportation cost=2968	

Closed loop paths & improvement index for unused square from the 2nd improved table.

Unused square	Closed path	Improvement index
WB	WB-WA+XA-XB	8-4+16-24=-4
WC	WC-WA+XA-XC	8-4+16-16=4
YA	YA-XA+XB-YB	8-16+14-16=0
YC	YC-XC+XB-YB	24-16+24-16=16

Select the cell which has the most negative opportunity cost. So select cell WB, because it has a negative opportunity cost which is -4. Then construct a closed path for cell WB.

4 *WA* - 5	8 WB +
1 XA + 16	24 - XB 25

The closed path for square (cell) WB has negative corners at square (cell) WA & XB. Then select the smaller quantity of these cells. So the smaller quantity of these corners is 25. Then add this quantity in the cell having (+) sign & subtract this quantity from the cell having (-) sign in order to obtain a new improved solution. Then we get:

4 *WA* - *56-25* 3	8 WB + 0+25 25
1 XA + 16+25= 41	24 XB - 25-25=0

Now the third improved table is

To / From	Project A	B	C	Plant Capacity
plant W	4 31	8 25	8 4	56
X	16 41	2 4	16 41	82
Y	8 -4	1 77	24 12	77
Project Req.	72	102	41	215

The total cost of the third improved solution is

Shipping Assignments	Quantity shipped × unit cost	Result (TC)
WA	31× 4	124
WB	25× 8	200
XA	41× 16	656
XC	41× 16	656
YB	77× 16	1232
Total Transportation cost=2868		

Closed path & improvement index

Unused square	Closed path	improvement index
WC	WC-WA+XA-XC	8-4+16-16 =4
XB	XB-WB+WA-XA	24-8+4-16=4
YA	YA-WA+WB-YB	8-4+8-16=-4
YC	YC-YB+WB-WA+XA-XC	24-16+8-4+16-16 =12

Since the opportunity cost of cell YA is-4 the closed path is traced from cell YA &the next improved table is reproduced below.

To / From	Project A	B	C	Plant Capacity
plant W	4 4	8 56	8 8	56
X	16 41	2 0	16 41	82
Y	8 31	1 46	24 16	77
Project Req.	72	102	41	215

Closed path &total cost

Unused square	Closed path	improvement index
WA	WA-YA +YB-WB	4-8+16-8 =4
XC	WC-XC+XA-YA +YB-WB	8-16+16-8+16-8=8
XB	XB-YB+YA-XA	24-16+8-16=0
YC	YC-XC+XA-YA	24-16+16-8=16

Total cost of optimal solution is as under

Shipping assignment	Quantity shipped× unit cost	Total cost
WB	56×8	448
XA	41 ×16	656
XC	41 ×16	656
YA	31 ×8	248
YB	46 ×16	736
		Total transportation cost =2744

Where the solution is not unique since the opportunity cost of XB cell is zero, it means there exists an alternative solution where the shipping assignment would change. Yet the total transportation cost would be the same. From a practical view point, the existence of alternative optimal solution gives valuable flexibility to the management. The alternative solution is shown as below.

Alternative solution

From \ To		Project A	B	C	Plant Capacity
plant	W	4 4	8 56	8 8	56
	X	16 0	2 41	16 41	82
	Y	8 72	1 5	24 16	77
Project Req.		72	102	41	215

Shipping assignment	Quantity shipped × unit cost	Total cost
WB	56×8	448
XB	41×24	984
XC	41×16	656
YA	72 ×8	576
YB	5 ×16	80
		Total transportation cost =2744

Example2: solve the following by using LCEM & test its optimality by stepping stone method. Shipping costs are given in the table below.

To / from	A	B	C	Supply
1	6	8	10	150
2	7	11	11	175
3	4	5	12	275
Demand	200	100	300	600

SOLUTION: Initial feasible solution by LCEM. The initial solution used as a starting point in this problem is the minimum cell cost method solution because it had the minimum total cost of the three methods used.

From \ To	A	B	C	Supply
1	[6]	[8] 25	[10] 125	150
2	[7]	[11]	[11] 175	175
3	[4] 200	[5] 75	[12]	275
Demand	200	100	300	600

The Minimum Cell Cost Solution

The stepping-stone method determines if there is a cell with no allocation that would reduce cost if used.

From \ To	A	B	C	Supply	
1	[6]	[8] 25	[10] 125	150	151
2	[7]	[11]	[11] 175	175	
3	[4] 200	[5] 75	[12]	275	
Demand	200	100	300	600	

The Allocation of One Ton to Cell 1A

- Must subtract one ton from another allocation along that row.

From \ To	A	B	C	Supply
1	+1 [6]	−1 [8] 25	[10] 125	150
2	[7]	[11]	[11] 175	175
3	[4] 200	[5] 75	[12]	275
Demand	200	100 99	300	600

The Subtraction of One Ton from Cell 1B

- A requirement of this solution method is that units can only be added to and subtracted from cells that already have allocations, thus one ton must be added to a cell as shown.

From \ To	A	B	C	Supply
1	+1 [6]	−1 [8] 25	[10] 125	150
2	[7]	[11]	[11] 175	175
3	−1 [4] 200	+1 [5] 75	[12]	275
Demand	200	100	300	600

The Addition of One Ton to Cell 3B and the Subtraction of One Ton from Cell 3A

- An empty cell that will reduce cost is a potential entering variable.
- To evaluate the cost reduction potential of an empty cell, a closed path connecting used cells to the empty cells is identified.

From \ To	A	B	C	Supply
1	[6]	[8] 25	[10] 125	150
2	[7]	[11]	[11] 175	175
3	[4] 200	[5] 75	[12]	275
Demand	200	100	300	600

2A → 2C → 1C → 1B → 3B → 3A
+ $7 − 11 + 10 − 8 + 5 − 4 = −$1

The Stepping-Stone Path for Cell 2A

- The remaining stepping-stone paths and resulting computations for cells 2B and 3C.

From \ To	A	B	C	Supply
1	[6]	[8] 25	[10] 125	150
2	[7]	[11]	[11] 175	175
3	[4] 200	[5] 75	[12]	275
Demand	200	100	300	600

2B → 2C → 1C → 1B
+ $11 − 11 + 10 − 8 = +$2

The Stepping-Stone Path for Cell 2B

From \ To	A	B	C	Supply
1	[6]	[8] 25	[10] 125	150
2	[7]	[11]	[11] 175	175
3	[4] 200	[5] 75	[12]	275
Demand	200	100	300	600

3C → 1C → 1B → 3B
+ $12 − 10 + 8 − 5 = +$5

The Stepping-Stone Path for Cell 3C

- After all empty cells are evaluated, the one with the greatest cost reduction potential is the entering variable.
- A tie can be broken arbitrarily.

From \ To	A	B	C	Supply
1	[6]	[8] 25	[10] 125	150
2	[7]	[11]	[11] 175	175
3	[4] 200	[5] 75	[12]	275
Demand	200	100	300	600

The Stepping-Stone Path for Cell 1A

When reallocating units to the entering variable (cell), the amount is the minimum amount subtracted on the stepping-stone path.

- At each iteration one variable enters and one leaves (just as in the simplex method).

From \ To	A	B	C	Supply
1	[6] 25	[8]	[10] 125	150
2	[7]	[11]	[11] 175	175
3	[4] 175	[5] 100	[12]	275
Demand	200	100	300	600

The Second Iteration of the Stepping-Stone Method

- Check to see if the solution is optimal.

From \ To	A	B	C	Supply
1	− 6 25	8	+ 10 125	150
2	+ 7	11	− 11 175	175
3	4 175	5 100	12	275
Demand	200	100	300	600

2A → 2C → 1C → 1A
+ $7 − 11 + 10 − 6 = $0

The Stepping-Stone Path for Cell 2A

From \ To	A	B	C	Supply
1	− 6 25	+ 8	10 125	150
2	7	11	11 175	175
3	+ 4 175	− 5 100	12	275
Demand	200	100	300	600

1B → 3B → 3A → 1A
+ $8 − 5 + 4 − 6 = +$1

The Stepping-Stone Path for Cell 1B

- Continuing check for optimality.

From \ To	A	B	C	Supply
1	− 6 25	8	+ 10 125	150
2	7	+ 11	− 11 175	175
3	+ 4 175	− 5 100	12	275
Demand	200	100	300	600

2B → 3B → 3A → 1A → 1C → 2C
+ $11 − 5 + 4 − 6 + 10 − 11 = +$3

The Stepping-Stone Path for Cell 2B

From \ To	A	B	C	Supply
1	+ 6 25	8	− 10 125	150
2	7	11	11 175	175
3	− 4 175	5 100	+ 12	275
Demand	200	100	300	600

3C → 3A → 1A → 1C
+ $12 − 4 + 6 − 10 = +$4

The Stepping-Stone Path for Cell 3C

- The stepping-stone process is repeated until none of the empty cells will reduce costs (i.e. an optimal solution).
- In example, evaluation of four paths indicates no cost reductions; therefore Table 19 solution is optimal.
- Solution and total minimum cost:

x_{1A} = 25 tons, x_{2C} = 175 tons, x_{3A} = 175 tons, x_{1C} = 125 tons, x_{3B} = 100 tons

$$Z = \$6(25) + 8(0) + 10(125) + 7(0) + 11(0) + 11(175) + 4(175) + 5(100) + 12(0) = \$4{,}525$$

- A multiple optimal solution occurs when an empty cell has a cost change of zero and all other empty cells are positive.
- An alternate optimal solution is determined by allocating to the empty cell with a zero cost change.
- Alternate optimal total minimum cost also equals $4,525.

From \ To	A	B	C	Supply
1	6	8	10 150	150
2	7 25	11	11 150	175
3	4 175	5 100	12	275
Demand	200	100	300	600

The Alternative Optimal Solution

The Stepping-Stone Solution Method Summary of Steps

1. Determine the stepping-stone paths and cost changes for each empty cell in the tableau.
2. Allocate as much as possible to the empty cell with the greatest net decrease in cost.
3. Repeat steps 1 and 2 until all empty cells have positive cost changes that indicate an optimal solution.

2. Modified Distribution Method (MODI)

- MODI is a modified version of the stepping-stone method in which math equations replace the stepping-stone paths.

The MODI method allows us to compute improvement indices quickly for each unbiased square without drawing all of the closed paths or loops. Because of this is can often provide considerable time savings over the stepping stone for solving transportation problems.

MODI provides a new means of finding the unused route or square with the largest negative improvement index. Once the largest index is identified, we are required to trace only one closed path just as with the stepping stone method, this path helps to determine the maximum number of units that can be shipped via the best unused route. The following steps are followed to determine the optimality.

Step1: From the given data construct a transportation table with the given cost of transportation & rim conditions (total supply = total demand)

Step2: Determine the initial basic feasible solution using suitable conditions (i.e. NWCM, LCEM or VAM).

Step3: For the current basic feasible solution with m+n-1 occupied cells, calculate index numbers (dual variables) ui= (i= 1, 2, 3 …m) & vj= (j= 1, 2, 3…n) for rows & columns respectively.

For calculating the values of ui & vj the following relationship (formula) for occupied cells is used,

$$cij = ui + vj \; for \; all \; i \& j$$

Step4: For occupied cells, the opportunity cost by using the formula: $dij = cij - (ui + vj) for \; all \; i \; \& \; j$

Step5: Now, the opportunity cost for unoccupied cell is determined by using the formula:

Opportunity cost = actual cost-Implied cost

$$dij = cij - (ui + vj)$$

Step6: Examine unoccupied cells evaluation for dij

a) If dij>0, then the cost of transportation will increase & optimal solution has been arrived at/reached.
b) If dij= 0, then the cost of transportation will remain unchanged. But there exists an alternative solution.
c) If dij<0, then an improved solution can be obtained by introducing cell (i, j) in the basis & go to step 7.

Step7: Select an occupied cell with the largest negative opportunity cost among all unoccupied cells.
Step8: Construct a closed path for the unoccupied cells determined in step 7 & assign plus (+) & minus (-) sign alternatively beginning with plus sign for the selected unoccupied in clockwise or other direction.
Step9: Assign as many units as possible to the unoccupied cell satisfying rim condition. The smallest allocation in a cell with negative sign on the closed path indicated the number of units that can be transported to the unoccupied cells. This quantity is added to the occupied cells on the path marked with plus sign & subtracted from these occupied cells on the path marked with minus signs.
Step10: Go to step 4 & repeat the procedure until all dij≥0, i.e. an optimal solution is reached. Calculate the associated total transportation cost.

Example1: the following is the initial feasible solutions calculated by NWCM& apply MODI method to test its optimality.

To / From		Project A	B	C	Supply (Si)
plant	W	4 (56)	8	8	56
	X	16 (16)	24 (66)	16	82
	Y	8	16 (36)	24 (41)	77
Demand (Di)		72	102	41	215

Total associated cost for the initial feasible solution=4×56+16×16+24×66+16×36+24×41=3624
Solution: To apply the MODI method, we have made minor modification in the transportation table.

vj / ui			V1=4	V2=12	V3=20	
	From / To		Project A	Project B	Project C	Supply (Si)
u1=0	plant	W	4 (56)	8	8	56
u2=12		X	16 (16)	24 (66)	16	82
u3=4		Y	8	16 (36)	24 (41)	77
	Demand (Di)		72	102	41	215

In the above figure we get u & v represented row & column values we have a subscript to denote the specific rows or column values. In our case we have u1, u2 & u3 to represent the rows & v1, v2 & v3 to represent the columns.
In general we can write:
ui= value assigned to row i
vj= value assigned to column j.
cij= cost in square ij (the square of intersection of row I & column j).
For each stone square we have used the following formula to find its cost.

$$ui + vj = cij$$

Since we have the five stone squares, then we have the following five equations
$u1 + v1 = c11;\;\; u2 + v1 = c21; u2 + v2 = c22; u3 + v2 = c32;\;\; u3 + v3 = c33$
Since we are given the cost figure for each stone square, then we substitute the cost value in the above equations. The results are:

$$u1 + v1 = 4;\;\; u2 + v1 = 16;\;\; u2 + v2 = 24;\;\; u3 + u2 = 16;\;\; u3 + v3 = 24$$
$$Let\ u1 = 0, then\ u1 + v1 = 4; 0 + v1 = 4; v1 = 4$$
$$since\ v1 = 4; then\ u2 + v1 = 16; u2 + 4 = 16; u2 = 12$$
$$since\ u2 = 12, then\ u2 + v2 = 24; 12 + v2 = 24; v2 = 12$$
$$since\ v2 = 12, then\ u3 + v2 = 16; u3 + 12 = 16; u3 = 4$$
$$sinceu3 = 4, then\ u3 + v3 = 24; 4 + v3 = 24; v3 = 20$$

Now the initial with ui & vj values

vj / ui		V1=4	V2=12	V3=20	
	From / To	Project A	Project B	Project C	Supply (Si)
u1=0	plant W	4 56	8	8	56
u2=12	X	16 16	24 66	16	82
u3=4	Y	8	16 36	24 41	77
	Demand (Di)	72	102	41	215

Now we proceed to compute the opportunity cost of each of the empty (unused) cell. We have used the following general formula.

$$improvement\ index\ (dij) = cij - (ui + vj)$$

Unused square	$cij - (ui + vj)$	$improvement\ index$
W→B or 1→ 2	$c12 - u1 - v2 = 8 - 0 - 12$	=-4
W→ C or 1→3	$c13 - u1 - v3 = 8 - 0 - 20$	=-12
X→C or 2→3	$c23 - u2 - v3 = 16 - 12 - 20$	-16
Y→ W or3→1	$c31 - u3 - v1 = 8 - 4 - 4$	=0

Since the value of empty cell (water square) $c23$ is the most negative. We draw closed loop through this cell & the new table is obtained as given below:

vj ui	From \ To	v1=4 Project A	v2=12 Project B	v3=20 Project C	Supply (Si)
u1=0	plant W	4 56	8 -4	8 -	56
u2=12	X	16 16	24 66 -	16 +	82
u3=4	Y	8 0	16 36 +	24 - 41	77
	Demand (Di)	72	102	41	215

Non-optimal second solution

vj ui	From \ To	v1=4 Project A	v2=12 Project B	v3=4 Project C	Supply (Si)
u1=0	plant W	4 56	8 -4	8 4	56
u2=12	X	16 16	24 25	16 41	82
u3=4	Y	8 0	16 77	24 16	77
	Demand (Di)	72	102	41	215

Since $ui + vj = cij \; let \; u1 = 0 \; then$

$$stone\ square\ c11 = u1 + v1 = 4; 0 + v1 = 4; v1 = 4$$
$$stone\ square\ c21 = u2 + v1 = 16; u2 + 4 = 16; u2 = 12$$
$$stone\ square\ c22 = u2 + v2 = 24; 12 + v2 = 24; v2 = 12$$
$$stone\ square\ c23 = u2 + v3 = 16; 12 + v3 = 16; v3 = 4$$
$$stone\ square\ c32 = u3 + v2 = 16; u3 + 12 = 16; u3 = 4$$

The calculation of opportunity cost of empty cell (water square) is as below:

Unused square	$cij - (ui + vj)$	$improvement\ index$
W→B or 1→ 2	$c12 - u1 - v2 = 8 - 0 - 12$	=-4
W→ C or 1→3	$c13 - u1 - v3 = 8 - 0 - 4$	=4
Y→A or 3→ 1	$c31 - u3 - v1 = 8 - 4 - 4$	=0
Y→ C or 3→3	$c33 - u3 - v3 = 24 - 4 - 4$	=16

Since the opportunity cost of cell $c12$ is negative, we have improved solution by adopting the procedure stated above.

The third improved non- optimal solution is given below

vj / ui		v1=4	v2=8	v3=4	
	From / To	Project A	Project B	Project C	Supply (Si)
u1=0	plant W	4 - 31	8 + 25	8 4	56
u2=12	X	16 41	24 4	16 41	82
u3=8	Y	8 + -4	~~16~~ - 77	24	77
	Demand (Di)	72	102	41 12	215

Since the value of cell $c31$ is negative we proceed to get improved solution.

To / From	Project A	Project B	Project C	Supply (Si)
plant W	4 4	8 56	8 8	56
X	16 41	24 0	16 41	82
Y	8 31	16 46	24 16	77
Demand (Di)	72	102	41	215

Since the opportunity cost of all unoccupied cells or water squares is positive, therefore the optimal solution is obtained.
The total cost of the optimal solution is:

Shipping Assignment	Quantity shipped	Unit cost	Total Cost
WB	56	8	448
XA	41	16	656
XC	41	16	656
YA	31	8	248
YB	46	16	736
	Total Transportation cost		=2744

Example2: solve the following by using LCEM & test its optimality by MODI method.
Shipping costs are given in the table below.

To / from	A	B	C	Supply
1	6	8	10	150
2	7	11	11	175
3	4	5	12	275
Demand	200	100	300	600

Solution:- In the table, the extra left-hand column with the u_i symbols and the extra top row with the v_j symbols represent values that must be computed.
- Computed for all cells with allocations:

$u_i + v_j = c_{ij}$ = unit transportation cost for cell $_{ij}$.

	v_j	$v_A =$	$v_B =$	$v_C =$	
u_i	From \ To	A	B	C	Supply
$u_1 =$	1	(6)	25 (8)	125 (10)	150
$u_2 =$	2	(7)	(11)	175 (11)	175
$u_3 =$	3	200 (4)	75 (5)	(12)	275
	Demand	200	100	300	600

The Minimum Cell Cost Initial Solution

- Formulas for cells containing allocations:

x_{1B}: $u_1 + v_B = 8$
x_{1C}: $u_1 + v_C = 10$
x_{2C}: $u_2 + v_C = 11$
x_{3A}: $u_3 + v_A = 4$
x_{3B}: $u_3 + v_B = 5$

	v_j	$v_A = 7$	$v_B = 8$	$v_C = 10$	
u_i	From \ To	A	B	C	Supply
$u_1 = 0$	1	(6)	25 (8)	125 (10)	150
$u_2 = 1$	2	(7)	(11)	175 (11)	175
$u_3 = -3$	3	200 (4)	75 (5)	(12)	275
	Demand	200	100	300	600

The Initial Solution with All ui and vj Values

- Five equations with 6 unknowns therefore let $u_1 = 0$ and solve to obtain:

$v_B = 8$, $v_C = 10$, $u_2 = 1$, $u_3 = -3$, $v_A = 7$

- Each MODI allocation replicates the stepping-stone allocation.
- Use following to evaluate all empty cells:

$$c_{ij} - u_i - v_j = d_{ij}$$

Where d_{ij} equals the cost increase or decrease that would occur by allocating to a cell.

- For the empty cells in the table above:

x_{1A}: $k_{1A} = c_{1A} - u_1 - v_A = 6 - 0 - 7 = -1$
x_{2A}: $k_{2A} = c_{2A} - u_2 - v_A = 7 - 1 - 7 = -1$
x_{2B}: $k_{2B} = c_{2B} - u_2 - v_B = 11 - 1 - 8 = +2$
x_{3C}: $k_{3C} = c_{3C} - u_3 - v_C = 12 - (-3) - 10 = +5$

- After each allocation to an empty cell, the u_i and v_j values must be recomputed.

	v_j	$v_A =$	$v_B =$	$v_C =$	
u_i	From \ To	A	B	C	Supply
$u_1 =$	1	25 (6)	(8)	125 (10)	150
$u_2 =$	2	(7)	(11)	175 (11)	175
$u_3 =$	3	175 (4)	100 (5)	(12)	275
	Demand	200	100	300	600

The Second Iteration of the MODI Solution Method

- Recomputing u_i and v_j values:

x_{1A}: $u_1 + v_A = 6$, $v_A = 6$ x_{1C}: $u_1 + v_C = 10$, $v_C = 10$ x_{2C}: $u_2 + v_C = 11$, $u_2 = 1$
x_{3A}: $u_3 + v_A = 4$, $u_3 = -2$ x_{3B}: $u_3 + v_B = 5$, $v_B = 7$

u_i	v_j / From \ To	$v_A = 6$ A	$v_B = 7$ B	$v_C = 10$ C	Supply
$u_1 = 0$	1	6 / 25	8	10 / 125	150
$u_2 = 1$	2	7	11	11 / 175	175
$u_3 = -2$	3	4 / 175	5 / 100	12	275
	Demand	200	100	300	600

The New ui and vj Values for the Second Iteration

- Cost changes for the empty cells, $c_{ij} - u_i - v_j = d_{ij}$;

$$x_{1B}: d_{1B} = c_{1B} - u_1 - v_B = 8 - 0 - 7 = +1$$
$$x_{2A}: d_{2A} = c_{2A} - u_2 - v_A = 7 - 1 - 6 = 0$$
$$x_{2B}: d_{2B} = c_{2B} - u_2 - v_B = 11 - 1 - 7 = +3$$
$$x_{3C}: d_{2B} = c_{2B} - u_3 - v_C = 12 - (-2) - 10 = +4$$

- Since none of the values are negative, solution obtained is optimal.
- Cell 2A with a zero cost change indicates a multiple optimal solution.

The Modified Distribution Method (MODI) Summary of Steps

1. Develop an initial solution.
2. Compute the u_i and v_j values for each row and column.
3. Compute the cost change, d_{ij}, for each empty cell.
4. Allocate as much as possible to the empty cell that will result in the greatest net decrease in cost (most negative d_{ij})
5. Repeat steps 2 through 4 until all d_{ij} values are positive or zero.

ASSINGMENT PROBLEM (AP)

Introduction

Assignment problem is special types of LPP. It deals with in the allocating the various resources or items to various activities on one to one basis in such a way that the time or cost involved is minimized & sale or profit is maximized. Such types of a problem can also solved with the help of simplex method or by transportation method but simpler & more efficient method for getting the solution are through assignment problem.

Several problems of management have a structure identical with the assignment problem. A departmental head may have six people available for assignment & six jobs to assign. He may like to know which job should be assigned to which person so that all these jobs can be completed in the shortest possible time. Similarly, in marketing set up by making an estimate of sales performance for different territories one could assign a particular salesman to a particular territory with a view to maximize overall sales. It may be noted that with ‘n’ facilitation with ‘n’ jobs there are n possible assignments. Now in the assumption that each one of the persons can perform each one of the jobs one at a time, then the problem is to find assignment that is which job should be assigned to which person so that total cost of performing all jobs is minimized. For this purpose the assignment problem is constructed as follows:

Table of an assignment problem

	Jobs					
Persons	1	2	3	j^{th}	n	Total
1	C11	C12	C13	----	C1n	1
2	C21	C22	C23	----	C2n	1
3	C31	C32	C33	-----	C3n	1
i^{th}	----	----	----	-----	Cij	1
n	Cn1	Cn2	Cn3	----	Cnn	1
Total	1	1	1	1	1	n

There are finite numbers of persons & jobs and all the persons are capable of taking up all the jobs with different time or costs. Here cij is the cost or time or effectiveness, when ith person is assigned to jth job. This is a special type of transportation problem where m=n, that is number of rows is equal to the number of columns & ai=bj=1, that is total requirements of rows & columns are constant. A separate computational device is required to solve assignment problem.

Formulation of Assignment Problem

Let xij be a variable defined by

$$Xij=\begin{cases} 0 \text{ if the } i^{th} \text{ job is not assigned to } j^{th} \text{ machine.} \\ 1 \text{if the } i^{th} \text{ job is assigned to } j^{th} \text{ machine.} \end{cases}$$

Then clearly, since only one job is to be assigned to each machine. We have

$\sum_{i=1}^{n} xij = 1$ & $\sum_{j=1}^{n} xij = 1$ also the total assigned cost is given by $Z = \sum_{i=1}^{n}\sum_{j=1}^{n} xij\,cij$

Thus the assignment problem takes the following mathematical form:

Minimize $Z = \sum_{i=1}^{n}\sum_{j=1}^{n} xij\,cij$

St. to $\sum_{i=1}^{n} xij = 1\,, j = 1, 2, \ldots\ldots\ldots . n$

$\sum_{j=1}^{n} xij = 1\,, i = 1, 2, \ldots\ldots\ldots\ldots n$

With $xij = 0 \; or \; 1$

Example: Consider problem of assigning five operators to five machines. The assignment costs are given below:

Operations

		I	II	III	IV	V
Machines	A	10	5	12	15	16
	B	3	9	18	3	6
	C	10	7	2	2	2
	D	5	11	9	7	12
	E	7	9	10	4	12

Formulate an LP model to determine an optimal assignment. Write the objective function & constraints in detail.

Solution: A key decision is to determine which operator should be assigned to which machine. Let us designate the assignment of jth operator to the ith machine by the decision variable xij (i= A, B, C, D, E & j= I, II, III, IV, V) where xij = 0 or 1.

$$
\begin{aligned}
minimize\ Z = {} & (10x11 + 5x12 + 12x13 + 15x14 + 16x15) \\
& + (3x21 + 9x22 + 18x23 + 3x24 + 6x25) \\
& + (10x31 + 7x32 + 2x33 + 2x34 + 2x35) \\
& + (5x41 + 11x42 + 9x43 + 7x44 + 12x45) \\
& + (7x51 + 9x52 + 10x53 + 4x54 + 12x55)
\end{aligned}
$$

$$
\begin{aligned}
st.to\ x11 + x12 + x13 + x14 + x15 = 1\ for\ machine\ A \\
x21 + x22 + x23 + x24 + x25 = 1\ for\ machine\ B \\
x31 + x32 + x33 + x34 + x35 = 1\ for\ machine\ C \\
x41 + x42 + x43 + x44 + x45 = 1\ for\ macine\ D \\
x51 + x52 + x53 + x54 + x55 = 1\ for\ macine\ E
\end{aligned}
$$

Each machine must be assigned to one & only one operator.

$$
\begin{aligned}
x11 + x21 + x31 + x41 + x51 = 1\ for\ operator\ I \\
x12 + x22 + x32 + x42 + x52 = 1\ for\ operator\ II \\
x13 + x23 + x33 + x43x + x53 = 1\ for\ operator\ III \\
x14 + x24 + x34 + x44 + x54 = 1\ for\ oprator\ IV \\
x15 + x25 + x35 + x45x + x55 = 1\ for\ operator\ V
\end{aligned}
$$

Solution Method of Assignment Problem

An assignment problem can be solved by using four methods. These are:

1. Hungarian Method
2. Enumeration Method
3. Simplex Method
4. Transportation Method

But the most commonly used for solving assignment problem is the Hungarian Method.

1. Hungarian Method

The Hungarian mathematician D.koning developed simpler and more efficient method for solving assignment which is known as Hungarian techniques or methods. The Hungarian methods of assignment provides as with an efficient method of finding the optimal solution without having to make a direct comparison of every solution. The Hungarian method is based up on the following principles.

i. If a constant is added to every element of a row and /or a column of the cost matrix of an assignment problem the resulting assignment problem has the same optimum solution as the original problem and vice versa.

ii. The solution having zero total cost is considered as optimum solution.

Steps for solving AP by using Hungarian method.

When the objective function is that of the minimization type, the Hungarian method of solving assignment problem can be summarized in the following steps.

Step1: Starting with the first row, locate the smallest cost element in each row of the cost table. Now subtract this smallest element from each element in that row .as as a result there shall be al least one zero in each row of this new table.

Step2: In the reduced cost table obtained in step1, consider each column &locate the smallest element in it. Subtract the smallest element in each column from every element of that column. As a result of this step there would be at least one zero in each of the rows &columns of the second reduced cost table.

Step3: make the assignment for the reduced matrix obtained from step1&2 in the following way.

i. Examine the row successively, until a row with exactly one zero is found make assignment to this single zero by putting square (□) around it and cross out (×) all other zeros appearing in the corresponding column as they will not be used to make any other assignment in that column. Proceed in this manner until all rows have been examined.

ii. Examine the columns successively until a column with exactly one zero is found. Make an assignment to this single zero by putting square around it &cross out (×) all other zeros appearing in the corresponding row. Proceed in this manner until all columns have been examined.

iii. Repeat step 3 (i) &3 (ii) until all zeros in rows and columns are either marked or crossed out. If the number of assignment (marked) made is equal to number of rows /columns, then it is an optimum solution &there is exactly one assignment in each row &in each column. There may be some row or columns without assignment, in such a case, we proceed to step 4.

Step 4: draw the minimum number of horizontal and vertical lines necessary too cover all the zeros in the reduced matrix obtained from step 3 in the following way:

a) Mark (✓) all rows that do not have any assignment.
b) Mark (✓) all columns that have zero in marked rows(step 4(a))
c) Mark (✓) all rows (not always marked) that have assignment in marked columns (step 4(b))
d) Repeat step 4(a) to 4(c) until no more rows or columns can be marked.
e) Draw straight lines through all unmarked rows and marked columns.

It may be pointed out here that you may also draw the minimum number of lines to cover all zeros by inspection. it should however be observed that in all n×n matrices less than 'n' lines will cover the zeros only when there is no solution among them. Conversely, it the minimum number of lines required for covering all the zeros is 'n' then the solution stage has reached.

Step 5: It the number of lines drawn are equal to 'n' that is equal to the number of rows or columns then it is an optimum solution, otherwise go to step 6.

Step 6: select the smallest element among all the uncovered elements. Subtract this smallest element from all the uncovered elements and add it to the element which lies at the intersection of two lines. Then we obtain another reduced matrix for fresh assignments.

Step 7: Go to step 3 and repeat the produce until the number of assignments become equal to the number of rows or columns. In such a case we shall observe that every row and columns has an assignment. Then the current solution is the optimal solution.

Example 1: A company is faced with the problem of assigning five jobs to five machines each job must be done an only one machines the cost of preparing each job an each machine is given below (in birr).

		Machines				
		M1	M2	M3	M4	M5
Jobs	J1	7	5	9	8	11
	J2	9	12	7	11	10
	J3	8	5	4	6	9
	J4	7	3	6	9	5
	J5	4	6	7	5	11

The problem is to determine the assignment of jobs to machines so that it will result in minimum cost.

Solution: The step by step producer is as follows.

Step 1: select the minimum element in each row and subtract this element from every element in that row. The resultant reduced matrix is shown below in table 1.

Table 1

Jobs		Machines M1	M2	M3	M4	M5
	J1	2	0	4	3	6
	J2	2	5	0	4	3
	J3	4	1	0	2	5
	J4	4	0	3	6	2
	J5	0	2	3	1	7

Minimum element is 5 in the first row, 7 in the second row, 4 in the third row, 3 in the fourth row & 4 in the fifth row. This is called row reduction.

Step2: Next, select the minimum element in each column & subtract this element from every element in that column. The columns having zero elements will not change. While in column 4 & column 5 the minimum element is 1 & 2 respectively. This is called column reduction. The resultant reduced matrix is shown in table2.

Table 2

Jobs		Machines M1	M2	M3	M4	M5
	J1	2	0	4	2	4
	J2	2	5	0	3	1
	J3	4	1	0	1	3
	J4	4	0	3	5	0
	J5	0	2	3	0	5

Step3: Zero assignments. Now we attempt to make a complete set of assignments using only a single zero element in each row or column. Since row J1 contains only single zero, therefore assignment is made in cell (J1, M2) i.e. c12 & zeros appearing in the corresponding column M2 is crossed out. Similarly we go to the next row & find that single zero appearing in the 2nd row & assignment is made at (J2, M3) i.e. c23 & zeros appearing in the corresponding column M3 is crossed out.

Now row J4 has single zero & assignment is made in cell (J4, M5). Since there are two zeros in row J5, we cannot make assignment in that row J5. Looking column wise, we find that column M1 has single zero therefore we make an assignment in cell (J5, M1) i.e. c51 & crossed out the zeros appearing in the corresponding row J5. The assignments so made are shown in table 3.

Table 3

Jobs		Machines M1	M2	M3	M4	M5
	J1	2	$\boxed{0}$	4	2	4
	J2	2	5	$\boxed{0}$	3	1
	J3	4	1	$\cancel{0}$	1	3
	J4	4	$\cancel{0}$	3	5	$\boxed{0}$
	J5	$\boxed{0}$	2	3	$\cancel{0}$	5

This it is possible to make four of the five necessary assignments using the zero element position. So the optimum solution is not reached, so go to the next step 4.

Step4: Draw minimum number of lines horizontal & vertical to cover all possible zeros usually all of the zeros can be obtained by inspection. However, we shall use the method given earlier in explaining the various steps. The various steps in drawing the minimum number of lines are:

i) Mark the row which has no assignment, Row J3

ii) Mark column which has zero in the marked row, column M3

iii) Mark the row which has assignment in marked column, row J2

iv) Repeat step (i) to step (iii) until no more rows or columns can be marked.

v) Draw lines through unmarked rows & marked columns.

The minimum numbers of lines drawn are shown in table 4.

Table 4

Jobs	Machines →	M1	M2	M3	M4	M5
	J1	~~2~~	~~0~~	~~4~~	~~2~~	~~4~~
	J2	2	5	~~0~~	3	1✓
	J3	4	1	~~0~~	1	3✓
	J4	~~4~~	~~0~~	~~3~~	~~5~~	~~0~~
	J5	~~0~~	~~2~~	~~3~~	~~0~~	~~5~~
				✓		

The number of lines drawn is equal to the number of assignments made.

Step5: To create one more zero, we examine the elements not covered by these lines & select the smallest element; viz. 1 is the smallest element not covered by these lines. Subtract this element from all uncovered elements & add it to the element lying at the intersection of the two lines. Therefore, the reduced matrix so obtained is shown below in table 5.

Table 5

Jobs	Machines →	M1	M2	M3	M4	M5
	J1	2	0	5	2	4
	J2	1	4	0	2	0
	J3	3	0	0	0	2
	J4	4	0	4	5	0
	J5	0	2	4	0	5

Now we make fresh assignments, proceeding in the usual way the set of assignments made are shown in table 6.

Table 6

Jobs	Machines →	M1	M2	M3	M4	M5
	J1	2	$\boxed{0}$	5	2	4
	J2	1	4	$\boxed{0}$	2	~~0~~
	J3	3	~~0~~	~~0~~	$\boxed{0}$	2
	J4	4	~~0~~	4	5	$\boxed{0}$
	J5	$\boxed{0}$	2	4	~~0~~	5

The optimum solution is reached. Because the number of assignments is equal to the number of rows or columns or every row & column has an assignment.

Assign Job	To Machine	Cost (in birr)
J1	M2	5
J2	M3	7
J3	M4	6
J4	M5	5
J5	M1	4
	Minimum total cost= birr 27	

Maximization case in assignment problem

The Hungarian method can also be used for maximization case. The problem of maximization can be converted into minimization case by selecting the largest element among all elements of the profit matrix or sale matrix & subtracting it from all other elements in the matrix including itself. We can then proceed as usual & obtain the optimum solution by adding the original values of these cells to which the assignments have been made.

Example1: Five jobs are to be processed & five machines are available. Any machine can process any job which resulting profit as follows.

		Machines				
		A	B	C	D	E
Jobs	1	32	38	40	28	40
	2	40	24	28	21	36
	3	41	27	33	30	37
	4	22	38	41	36	36
	5	29	33	40	35	39

Find the assignment pattern that maximizes profit.

Solution: Convert the profit matrix into opportunity cost matrix by subtracting the highest element (41) from all elements of the given matrix. Then we get:

		Machines				
		A	B	C	D	E
Jobs	1	9	3	1	13	1
	2	1	17	13	20	5
	3	0	14	8	11	4
	4	19	3	0	5	5
	5	12	8	1	6	2

Step1: **Row reduction**: select the minimum element in each row & subtracts this element from every elements of that row. The reduced matrix is shown below:

		Machines				
		A	B	C	D	E
Jobs	1	8	2	0	12	0
	2	0	16	12	19	4
	3	0	14	8	11	4
	4	19	3	0	5	5
	5	11	7	0	5	1

Step2: **Column reduction**: select the minimum element in each column & subtracts this element from every elements of that column. Columns having zero elements will not change. The reduced matrix is shown below:

		Machines				
		A	B	C	D	E
Jobs	1	8	0	0	7	0
	2	0	14	12	14	4
	3	0	12	8	6	4
	4	19	1	0	0	5
	5	11	5	0	0	1

Step3: Zero assignments,; now we attempt to make a complete set of assignments using only a single zero element in each row or column.

		Machines				
		A	B	C	D	E
Jobs	1	8	$\boxed{0}$	$\cancel{0}$	7	$\cancel{0}$
	2	$\boxed{0}$	14	12	14	4
	3	$\cancel{0}$	12	8	6	4
	4	19	1	$\cancel{0}$	$\boxed{0}$	5
	5	11	5	$\boxed{0}$	$\cancel{0}$	1

OR

		Machines				
		A	B	C	D	E
Jobs	1	8	$\boxed{0}$	$\cancel{0}$	7	$\cancel{0}$
	2	$\boxed{0}$	14	12	14	4
	3	$\cancel{0}$	12	8	6	4
	4	19	1	$\boxed{0}$	$\cancel{0}$	5
	5	11	5	$\cancel{0}$	$\boxed{0}$	1

Step4: draw minimum numbers of lines horizontally & vertically to cover all possible zeros.

i) Mark the row which has no assignment, Row 3

ii) Mark column which has zero in the marked row, column A

iii) Mark the row which has assignment in marked column, row 2

iv) Draw lines through unmarked rows & marked columns. Then we get:

		Machines				
		A	B	C	D	E
Jobs	1	8	$\boxed{0}$	$\cancel{0}$	7	$\cancel{0}$
	2	$\boxed{0}$	14	12	14	4✓
	3	$\cancel{0}$	12	8	6	4✓
	4	19	1	$\boxed{0}$	$\cancel{0}$	5
	5	11	5	$\cancel{0}$	$\boxed{0}$	1
		✓				

Step5: modify the above table by subtracting the smallest uncovered element (4) from all elements not covered by the lines & adding this element at the intersection of the two lines. Then we get.

		Machines				
		A	B	C	D	E
	1	12	0	0	7	0
Jobs	2	0	10	8	10	0
	3	0	8	4	2	0
	4	23	1	0	0	5
	5	15	5	0	0	1

Now we make fresh assignments, proceeding in the usual way we get

		Machines				
		A	B	C	D	E
Jobs	1	12	$\boxed{0}$	$\cancel{0}$	7	$\cancel{0}$
	2	$\boxed{0}$	10	8	10	$\cancel{0}$
	3	$\cancel{0}$	8	4	2	$\boxed{0}$
	4	23	1	$\boxed{0}$	$\cancel{0}$	5
	5	15	5	$\cancel{0}$	$\boxed{0}$	1

OR

		A	B	C	D	E
Jobs	1	12	$\boxed{0}$	$\cancel{0}$	7	$\cancel{0}$
	2	$\cancel{0}$	10	8	10	$\boxed{0}$
	3	$\boxed{0}$	8	4	2	$\cancel{0}$
	4	23	1	$\cancel{0}$	$\cancel{0}$	5
	5	15	5	$\boxed{0}$	$\boxed{0}$	1

Step6: The optimal solution is reached, because every row & column has an assignment.

Assign Jobs	To Machine	profit
1	B	38
2	A	37
3	E	40
4	C	35
5	D	41
	Maximum Total Profit	=191

OR

Assign Jobs	To Machine	profit
1	B	38
2	E	37
3	A	40
4	D	35
5	C	41
	Maximum Total Profit	=191

Therefore, the given assignment problem has alternative optimal solution.

Chapter 6: Network Analysis with CPM and PERT

6.1 Basic concepts and definitions

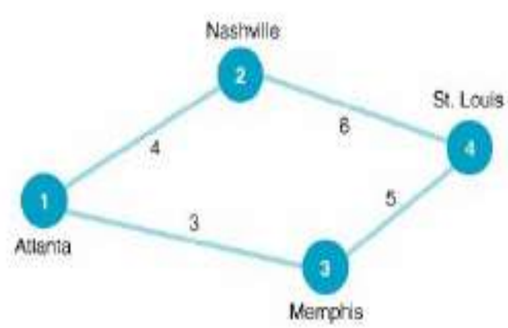
Network: A network is an arrangement of paths connected at various points, through which one or more items move from one point to another. Everyone is familiar with such networks as highway systems, telephone networks, railroad systems, and television networks. For example, a railroad network consists of a number of fixed rail routes (paths) connected by terminals at various junctions of the rail routes.

A network is an arrangement of paths connected at various points, through which items move.

In recent years using network models has become a very popular management science technique for a couple of very important reasons. First, a network is drawn as a diagram, which literally provides a picture of the system under analysis. This enables a manager to visually interpret the system and thus enhances the manager's understanding. Second, a large number of real-life systems can be modeled as networks, which are relatively easy to conceive and construct. Networks are popular because they provide a picture of a system and because a large number of systems can be easily modeled as networks.

Network Components:

Networks are illustrated as diagrams consisting of two main components: nodes and branches. **Nodes** represent junction points for example, an intersection of several streets. **Branches** connect the nodes and reflect the flow from one point in the network to another. Nodes are denoted in the network diagram by circles, and branches are represented by lines connecting the nodes. Nodes typically represent localities, such as cities, intersections, or air or railroad terminals; branches are the paths connecting the nodes, such as roads connecting cities and intersections or railroad tracks or air routes connecting terminals. For example, the different railroad routes between Atlanta, Georgia, and St. Louis, Missouri, and the intermediate terminals are shown in Figure1



***Nodes**, denoted by circles, represent junction points connecting branches.*
Branches, represented as lines, connect nodes and show flow from one point to another.

The network shown in Figure .1 has four nodes and four branches. The node representing Atlanta is referred to as the origin, and any of the three remaining nodes could be the destination, depending on what we are trying to determine from the network. Notice that a number has been assigned to each node. Numbers provide a more convenient means of identifying the nodes and branches than do names. For example, we can refer to the origin (Atlanta) as node 1 and the branch from Atlanta to Nashville as branch 12.

Typically, a value that represents a distance, length of time, or cost is assigned to each branch. Thus, the purpose of the network is to determine the shortest distance, shortest length of time, or lowest cost between points in the network. In **Figure .1**, the values 4, 6, 3, and 5, corresponding to the four branches, represent the lengths of time, in hours, between the attached nodes. Thus, a traveler can see that the route to St. Louis through Nashville requires 10 hours and the route to St. Louis through Memphis requires 8 hours. The values assigned to branches typically represent distance, time, or cost.

Definition of PERT

PERT is an acronym for Program (Project) Evaluation and Review Technique, in which planning, scheduling, organizing, coordinating and controlling uncertain activities take place. The technique studies and represents the tasks undertaken to complete a project, to identify the least time for completing a task and the minimum time required to complete the whole project. It was developed in the late 1950s. It is aimed to reduce the time and cost of the project.

PERT uses time as a variable which represents the planned resource application along with performance specification. In this technique, first of all, the project is divided into activities and events. After that proper sequence is ascertained, and a network is constructed. After that time needed in each activity is calculated and the critical path (longest path connecting all the events) is determined.

Definition of CPM

Developed in the late 1950s, Critical Path Method or CPM is an algorithm used for planning, scheduling, coordination and control of activities in a project. Here, it is assumed that the activity duration is fixed and certain. CPM is used to compute the earliest and latest possible start time for each activity.

The process differentiates the critical and non-critical activities to reduce the time and avoid the queue generation in the process. The reason for the identification of critical activities is that, if any activity is delayed, it will cause the whole process to suffer. That is why it is named as Critical Path Method. In this method, first of all, a list is prepared consisting of all the activities needed to complete a project, followed by the computation of time required to complete each activity. After that, the dependency between the activities is determined. Here, ‘path’ is defined as a sequence of activities in a network. The critical path is the path with the highest length.

6.2 The critical path method (CPM):

One of the most popular uses of networks is for project analysis. Such projects as the construction of a building, the development of a drug, or the installation of a computer system can be represented as networks. These networks illustrate the way in which the parts of the project are organized, and they can be used to determine the time duration of the projects. The network techniques that are used for project analysis are CPM and PERT. CPM stands for critical path method, and PERT is an acronym for project evaluation and review technique. These two techniques are very similar.

There were originally two primary differences between CPM and PERT. With CPM, a single, or deterministic, estimate for activity time was used, whereas with PERT probabilistic time estimates were employed. The other difference was related to the mechanics of drawing the project network. In PERT, activities were represented as arcs, or arrowed lines, between two nodes, or circles, whereas in CPM, activities were represented as the nodes or circles. However, these were minor differences, and over time CPM and PERT have been effectively merged into a single technique, conventionally referred to as simply CPM/PERT.

CPM and PERT were developed at approximately the same time (although independently) during the late 1950s. The fact that they have already been so frequently and widely applied attests to their value as management science techniques.

The three time estimates for each activity are the most likely time, the optimistic time, and the pessimistic time in PERT while CPM contains only one time estimate. The **most likely time** is the time that would most frequently occur if the activity were repeated many times. The **optimistic time** is the shortest possible time within which the activity could be completed if everything went right. The **pessimistic time** is the longest possible time the activity would require to be completed, assuming that everything went wrong. In general, the person most familiar with an activity makes these estimates to the best of his or her knowledge and ability. In other words, the estimate is subjective.

6.3 Comparison of CPM and PERT:

BASIS FOR COMPARISON	PERT	CPM
Meaning	PERT is a project management technique, used to manage uncertain activities of a project.	CPM is a statistical technique of project management that manages well defined activities of a project.
What is it?	A technique of planning and control of time.	A method to control cost and time.
Orientation	Event-oriented	Activity-oriented
Evolution	Evolved as Research & Development project	Evolved as Construction project
Model	Probabilistic Model	Deterministic Model
Focuses on	Time	Time-cost trade-off
Estimates	Three time estimates	One time estimate
Appropriate for	High precision time estimate	Reasonable time estimate
Management of	Unpredictable Activities	Predictable activities
Nature of jobs	Non-repetitive nature	Repetitive nature
Critical and Non-critical activities	No differentiation	Differentiated
Suitable for	Research and Development	Non-research projects like civil

BASIS FOR COMPARISON	PERT	CPM
	Project	construction, ship building etc.
Crashing concept	Not Applicable	Applicable

6.4 Key Differences Between PERT and CPM

The most important differences between PERT and CPM are provided below:

1. PERT is a project management technique, whereby planning, scheduling, organizing, coordinating and controlling uncertain activities are done. CPM is a statistical technique of project management in which planning, scheduling, organizing, coordination and control of well-defined activities take place.
2. PERT is a technique of planning and control of time. Unlike CPM, which is a method to control costs and time.
3. While PERT is evolved as a research and development project, CPM evolved as a construction project.
4. PERT is set according to events while CPM is aligned towards activities.
5. A deterministic model is used in CPM. Conversely, PERT uses a probabilistic model.
6. There are three times estimates in PERT, i.e. optimistic time (to), most likely time ™, pessimistic time (tp). On the other hand, there is only one estimate in CPM.
7. PERT technique is best suited for a high precision time estimate, whereas CPM is appropriate for a reasonable time estimate.
8. PERT deals with unpredictable activities, but CPM deals with predictable activities.
9. PERT is used where the nature of the job is non-repetitive. In contrast to, CPM involves the job of repetitive nature.
10. There is a demarcation between critical and non-critical activities in CPM, which is not in the case of PERT.
11. PERT is best for research and development projects, but CPM is for non-research projects like construction projects.
12. Crashing is a compression technique applied to CPM, to shorten the project duration, along with the least additional cost. The crashing concept is not applicable to PERT.

Conclusion

The difference between these two project management tools is getting unclear as the techniques are merged with the path of time. That is why, in most projects, they are being used as a single project. The primary point that distinguishes PERT from CPM is that the former gives the extreme importance of time, i.e. if the time is minimized, consequently the cost will also be reduced. However, cost optimization is the basic element, in the latter.