Chapter Four

Series representation of analytic functions

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Chapter 4:Contents

4.1 Sequences

- **Sequence**

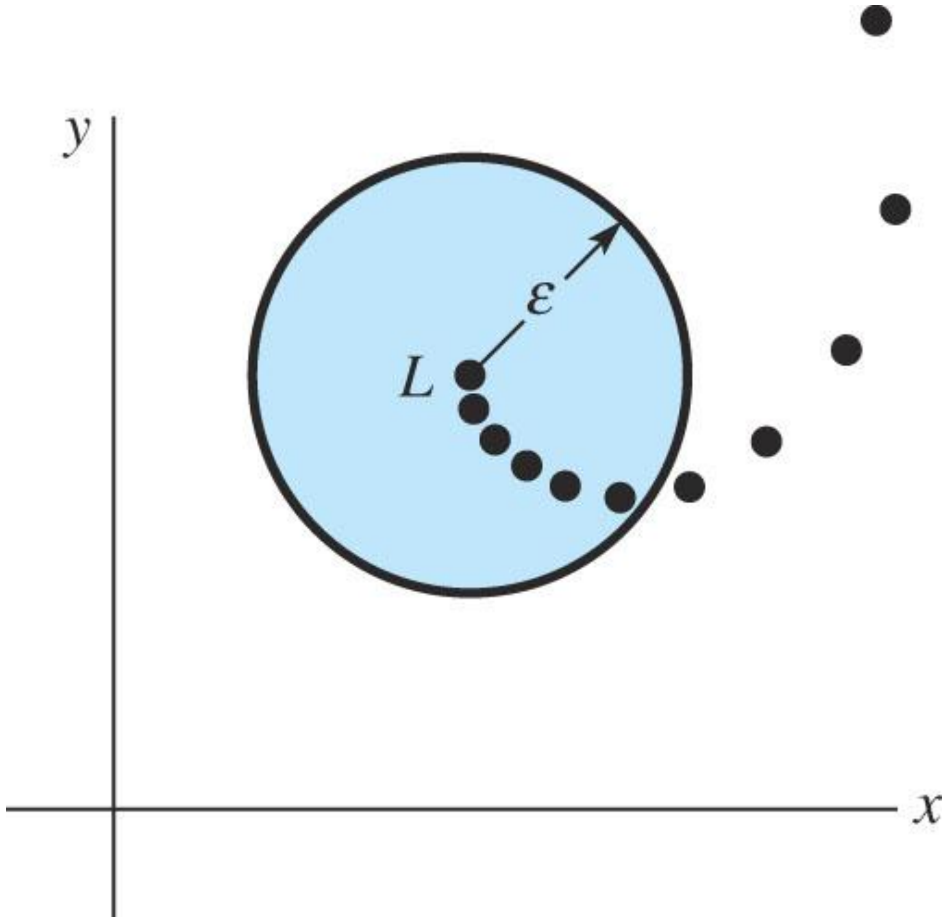
  For example, the sequence $\{1 + i^n\}$ is

$$\begin{array}{cccccc} 1+i\,, & 0\,, & 1-i\,, & 2\,, & 1+i\,, & \cdots \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\ n=1\,, & n=2\,, & n=3\,, & n=4\,, & n=5\,, & \cdots \end{array} \qquad (1)$$

- If $\lim_{n\to\infty} z_n = L$, we say this sequence is convergent. See Fig 4.1.
- Definition of the existence of the limit:

$$\forall \varepsilon > 0,\ \exists N > 0,\ \ni |z_n - L| < \varepsilon,\ \forall n > N$$

Fig 14.1: Illustration

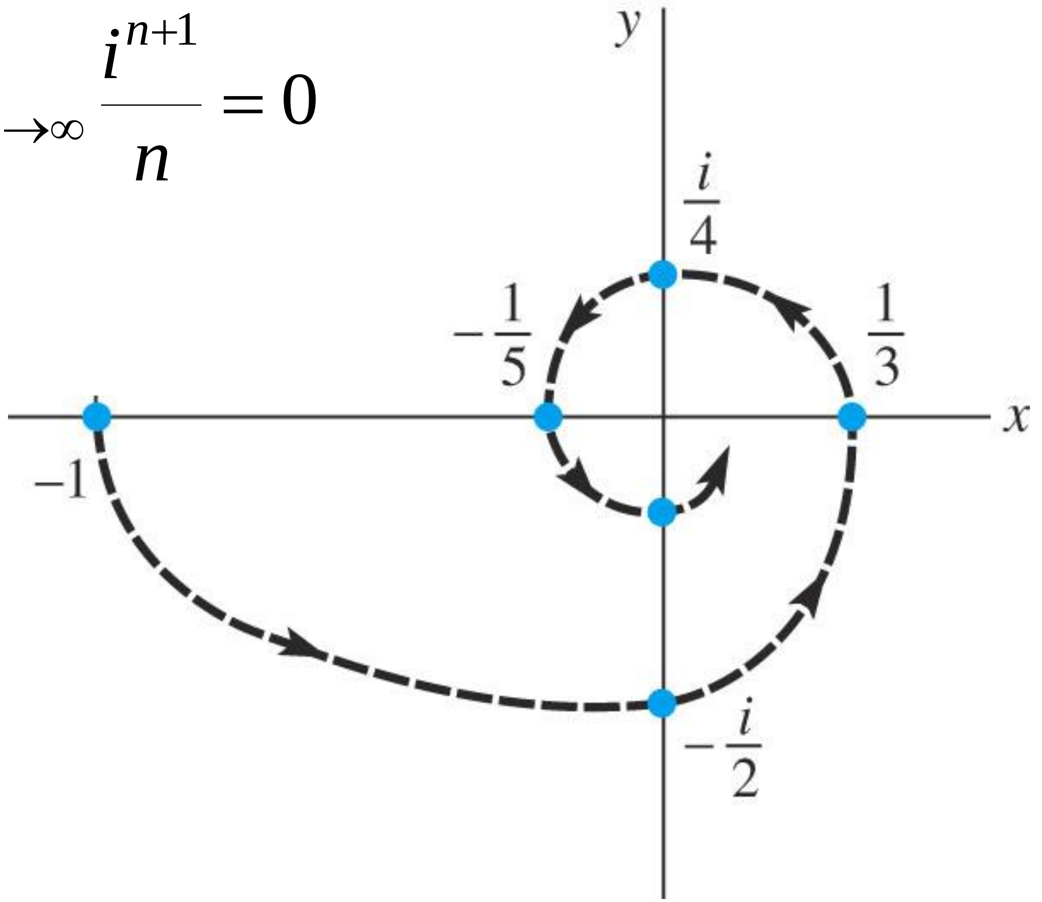


Example 1

The sequence $\left\{ \frac{i^{n+1}}{n} \right\}$ converges, since

$$\lim_{n\to\infty} \frac{i^{n+1}}{n} = 0$$

See Fig 4.2.



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THEOREM 4.1 **Criterion for Convergence**

A sequence $\{z_n\}$ converge to a complex number L if and only if $\text{Re}(z_n)$ converges to $\text{Re}(L)$ and $\text{Im}(z_n)$ converges to $\text{Im}(L)$.

Example 2

The sequence $\left\{\frac{ni}{n+2i}\right\}$ converges to i. Note that

Re(i) = 0 and Im(i) = 1. Then

$$z_n = \frac{ni}{n+2i} = \frac{2n}{n^2+4} + i\frac{n^2}{n^2+4}$$

$$\operatorname{Re}(z_n) = \frac{2n}{n^2+4} \to 0,\ \operatorname{Im}(z_n) = \frac{n^2}{n^2+4} \to 1$$

as $n \to \infty$.

Series

- An infinite series of complex numbers

$$\sum_{k=1}^{\infty} z_k = z_1 + z_2 + \ldots + z_n + \ldots$$

is convergent if the sequence of partial sum $\{S_n\}$, where

$$S_n = z_1 + z_2 + \ldots + z_n$$

converges.

Geometric Series

- For the geometric Series

$$\sum_{k=1}^{\infty} az^{k-1} = a + az + az^2 + \cdots + az^{n-1} + \cdots \qquad (2)$$

the nth term of the sequence of partial sums is

$$S_n = a + az + az^2 + \ldots + az^{n-1} \qquad (3)$$

and

$$S_n = \frac{a(1-z^n)}{1-z} \qquad (4)$$

Cont'd

Since $z^n \to 0$ as $n \to \infty$ whenever $|z| < 1$, we conclude that (2) converges to $a/(1 - z)$ when $|z| < 1$; the series diverges when $|z| \geq 1$.

The special series

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots \qquad (5)$$

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots \qquad (6)$$

valid for $|z| < 1$.

Example 3

❖ The series

$$\sum_{k=1}^{\infty}\frac{(1+2i)^k}{5^k}=\frac{(1+2i)}{5}+\frac{(1+2i)^2}{5^2}+\frac{(1+2i)^3}{5^3}+\dots$$

is a geometric series with $a = (1 + 2i)/5$ and $z = (1 + 2i)/5$. Since $|z| < 1$, we have

$$\sum_{k=1}^{\infty}\frac{(1+2i)^k}{5^k}=\frac{\frac{1+2i}{5}}{1-\frac{1+2i}{5}}=\frac{i}{2}$$

THEOREM 4.2 **Necessary Condition for Convergence**

If $\sum_{k=1}^{\infty} z_k$ converges, then $\lim_{n\to\infty} z_n = 0$.

THEOREM 4.3 **The nth Term Test for Divergence**

If $\lim_{n\to\infty} z_n \neq 0$, then the series $\sum_{k=1}^{\infty} z_k$ diverges.

DEFINITION 4.1

Absolute Convergence

An infinite series $\sum_{k=1}^{\infty} z_k$ is said be absolutely convergent if $\sum_{k=1}^{\infty} |z_k|$ converges.

Example 4

The series $\sum_{k=1}^{\infty} \frac{i^k}{k^2}$ is absolutely convergent since $|i^k/k^2| = 1/k^2$ and the real series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges.

As in real calculus,

Absolute convergence implies convergence.

Thus the series in Example 4 converges.

THEOREM 4.4 **Ratio Test**

Suppose $\sum_{k=1}^{\infty} z_k$ is a series of nonzero complex terms such that

$$\lim_{n\to\infty} \left| \frac{z_{n+1}}{z_n} \right| = L \qquad (9)$$

(i) If $L < 1$, then the series converges absolutely.

(ii) If $L > 1$ or $L = \infty$, then the series diverges.

(iii) If $L = 1$, the test is inconclusive.

THEOREM 4.5

Root Test

Suppose $\sum_{k=1}^{\infty} z_k$ is a series of complex terms such that

$$\lim_{n\to\infty} \sqrt[n]{|z_n|} = L \tag{10}$$

(i) If $L < 1$, then the series converges absolutely.

(ii) If $L > 1$ or $L = \infty$, then the series diverges.

(iii) If $L = 1$, the test is inconclusive.

Power Series

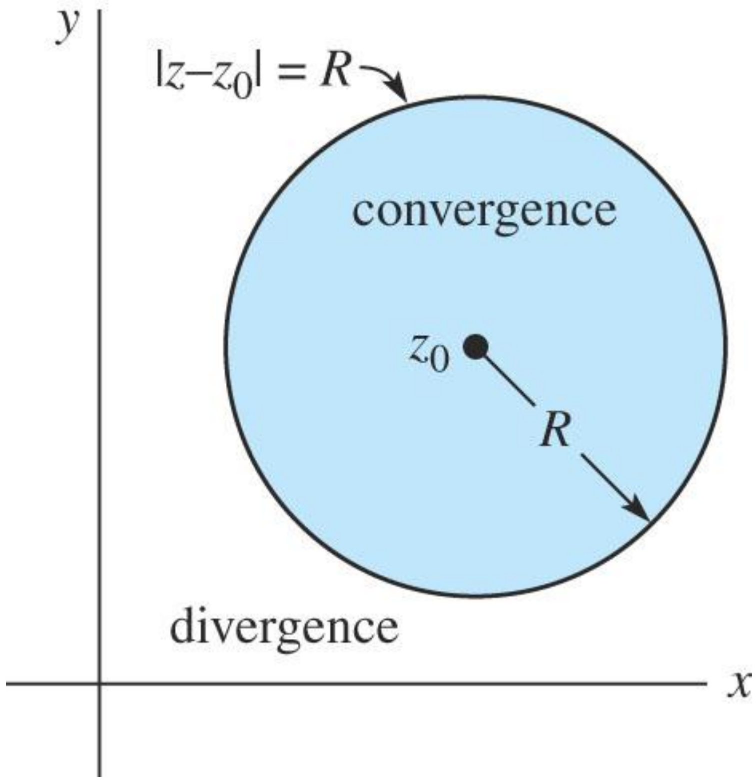
- An infinite series of the form

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots, \quad (11)$$

where a_k are complex constants is called a power series in $z - z_0$. (11) is said to be centered at z_0 and z_0 is referred to the center of the series.

Circle of Convergence

- Every complex power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ has radius of convergence R and has a circle of convergence defined by $|z - z_0| = R$, $0 < R < \infty$. See Fig 19.3.



- The radius R can be

  (i) zero (converges at only $z = z_0$).

  (ii) a finite number (converges at all interior points of the circle $|z - z_0| = R$).

  (iii) ∞ (converges for all z).

A power series may converge at some, all, or none of the points on the circle of convergence.

Example 5

- Consider the series $\sum_{k=1}^{\infty} \frac{z^{k+1}}{k}$, by ratio test

$$\lim_{n\to\infty}\left|\frac{\frac{z^{n+2}}{n+1}}{\frac{z^{n+1}}{n}}\right| = \lim_{n\to\infty}\frac{n+1}{n}|z| = |z|$$

Thus the series converges absolutely for $|z| < 1$ and the radius of convergence $R = 1$.

Summary: R.O.C. using ratio test

- (i) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \neq 0,$ the R.O.C. is $R = 1/L$.

  (ii) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$ the R.O.C. is ∞.

  (iii) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$ the R.O.C. is $R = 0$.

For the power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$

Example 6: R.O.C. using ratio test

- Consider the power series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(z-1-i)^k}{k!}$ with

$$a_n = \frac{(-1)^{n+1}}{n!}, \quad \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+2}}{(n+1)!}}{\frac{(-1)^{n+1}}{n!}}\right| = \lim_{n\to\infty}\left|\frac{1}{n+1}\right| = 0$$

The R.O.C. is ∞.

Example 7: R.O.C. using root test

Consider the power series $\sum_{k=1}^{\infty}\left(\frac{6k+1}{2k+5}\right)^k (z-2i)^k$

$$a_n = \left(\frac{6n+1}{2n+5}\right)^n, \lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{6n+1}{2n+5} = 3$$

This root test shows the R.O.C. is 1/3. The circle of convergence is $|z-2i| = 1/3$; the series converges absolutely for $|z-2i| < 1/3$.

4.2 Taylor Series

THEOREM 4.6 **Continuity**

A power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ represents a continuous function f within its circle of convergence $|z - z_0| = R, R \neq 0.$

THEOREM 4.7 **Term-by-Term Integration**

A power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ can be integrated term by term within its circle of convergence $|z - z_0| = R, R \neq 0,$ for every contour C lying entirely within the circle of convergence.

THEOREM 4.8 **Term-by-Term Differentiation**

A power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ can be differentiated term by term within its circle of convergence $|z - z_0| = R$, $R \neq 0$.

Taylor Series

- Suppose a power series represents a function f for $|z - z_0| < R$, $R \neq 0$, that is

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$
$$= a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \cdots \qquad (1)$$

It follows that

$$f'(z) = \sum_{k=1}^{\infty} k a_k (z - z_0)^{k-1} \qquad (2)$$
$$= a_1 + 2a_2(z - z_0) + 3a_3(z - z_0)^2 + \cdots$$

Cont'd

$$f''(z) = \sum_{k=2}^{\infty} k(k-1)a_k(z-z_0)^{k-2} \qquad (3)$$

$$= 2\cdot 1a_2 + 3\cdot 2a_3(z-z_0) + \cdots$$

$$f^{(3)}(z) = \sum_{k=3}^{\infty} k(k-1)(k-2)a_k(z-z_0)^{k-3} \qquad (4)$$

$$= 3\cdot 2\cdot 1a_3 + \cdots$$

From the above, at $z = z_0$ we have

$$f(z_0) = a_0,\ f'(z_0) = 1!a_1,\ f''(z_0) = 2!a_2, \ldots$$

$$f^{(n)}(z_0) = n!a_n$$

Cont'd

or

$$a_n = \frac{f^{(n)}(z_0)}{n!}, \qquad n \geq 0 \tag{5}$$

When $n = 0$, we interpret the zeroth derivative as $f(z_0)$ and $0! = 1$. Now we have

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k \tag{6}$$

Cont'd

This series is called the Taylor series for f centered at z_0. A Taylor series with center $z_0 = 0$,

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k , \qquad (7)$$

is referred to as a Maclaurin series.

THEOREM 4.9 **Taylor's Theorem**

Let f be analytic within a domain D and let z_0 be a point in D. Then f has the series representation

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k \qquad (8)$$

valid for the largest circle C with center at z_0 and radius R that lies entirely within D.

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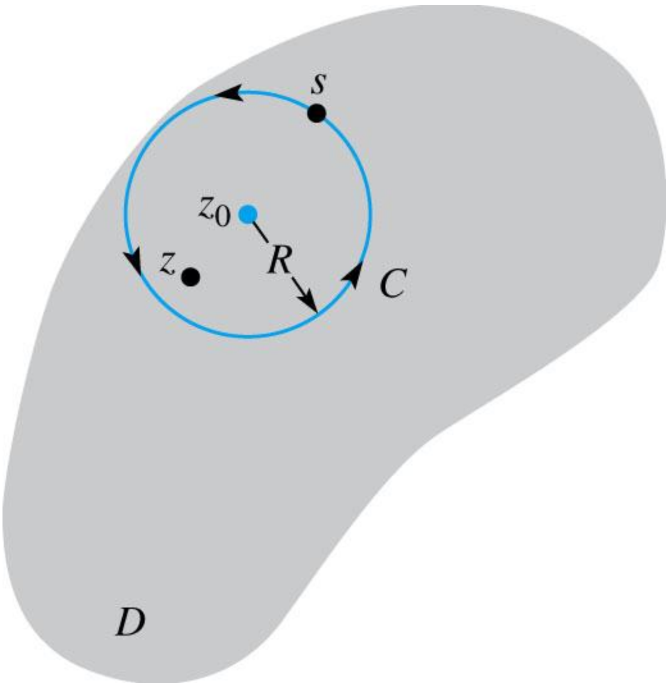

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- Some important Maclurin series

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{z^k}{k!} \qquad (12)$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} \qquad (13)$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} \qquad (14)$$

Example 1

Find the Maclurin series of $f(z) = 1/(1 - z)^2$.

Solution

For $|z| < 1$,

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots \qquad (15)$$

Differentiating both sides of (15)

$$\frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + \ldots = \sum_{k=1}^{\infty} kz^{k-1}$$

4.3 Laurent Series

- **Isolated Singularities**

  Suppose $z = z_0$ is a singularity of a complex function f. For example, 2i and -2i are sigularities of $f(z) = \frac{z}{z^2+4}$. The point z_0 is said to be an *isolated singularity*, if there exists some deleted neighborhood or punctured open disk $0 < |z - z_0| < R$ throughout which is analytic.

A New Kind of Series

- About an isolated singularity, it is possible to represent f by a new kind of series involving both negative and nonnegative integer powers of $z - z_0$; that is

$$f(z) = \ldots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots$$

Cont'd

Using summation notation, we have

$$f(z)=\sum_{k=1}^{\infty}a_{-k}(z-z_0)^{-k}+\sum_{k=0}^{\infty}a_k(z-z_0)^k \qquad (1)$$

The part with negative powers is called the principal part of (1) and will converge for $|1/(z-z_0)| < r^*$ or $|z-z_0| > 1/r^* = r$. The part with nonnegative powers is called the analytic part of (1) and will converge for $|z-z_0| < R$. Hence the sum of these parts converges when $r < |z-z_0| < R$.

Example 1

The function $f(z) = (\sin z)/z^3$ is not analytic at $z = 0$ and hence can not be expanded in a Maclaurin series. We find that

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots$$

converges for all z. Thus

$$f(z) = \frac{\sin z}{z^3} = \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \cdots \qquad (2)$$

This series converges for all z except $z = 0$, $0 < |z|$.

THEOREM 4.10 Laurent's Theorem

Let f be analytic within the annular domain D defined by $r < |z - z^0| < R$. Then f has the series representation

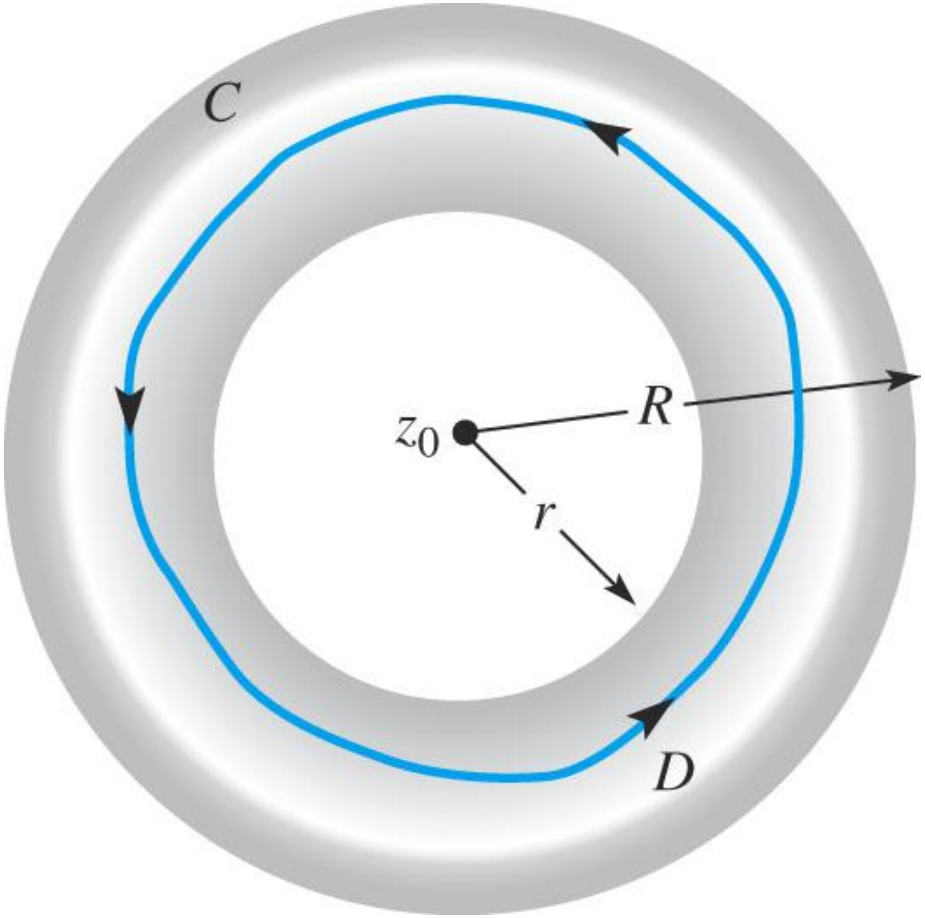
$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k \tag{3}$$

valid for $r < |z - z_0| < R$. The coefficients a_k are given by

$$a_k = \frac{1}{2\pi i}\int_C \frac{f(s)}{(s - z_0)^{k+1}}\,ds, \qquad k = 0, \pm 1, \pm 2, \cdots, \tag{4}$$

where C is a simple closed curve that lies entirely within D and has z_0 in its interior. (see Figure 4.6)

Fig 4.6



Example 4

Expand $f(z) = \dfrac{8z+1}{z(1-z)}$ in a Laurent series valid

for $0 < |z| < 1$.

Solution

(a) We can write

$$f(z) = \frac{8z+1}{z(1-z)} = \frac{8z+1}{z}\frac{1}{1-z} = (8+\frac{1}{z})\left[1+z+z^2+\ldots\right]$$

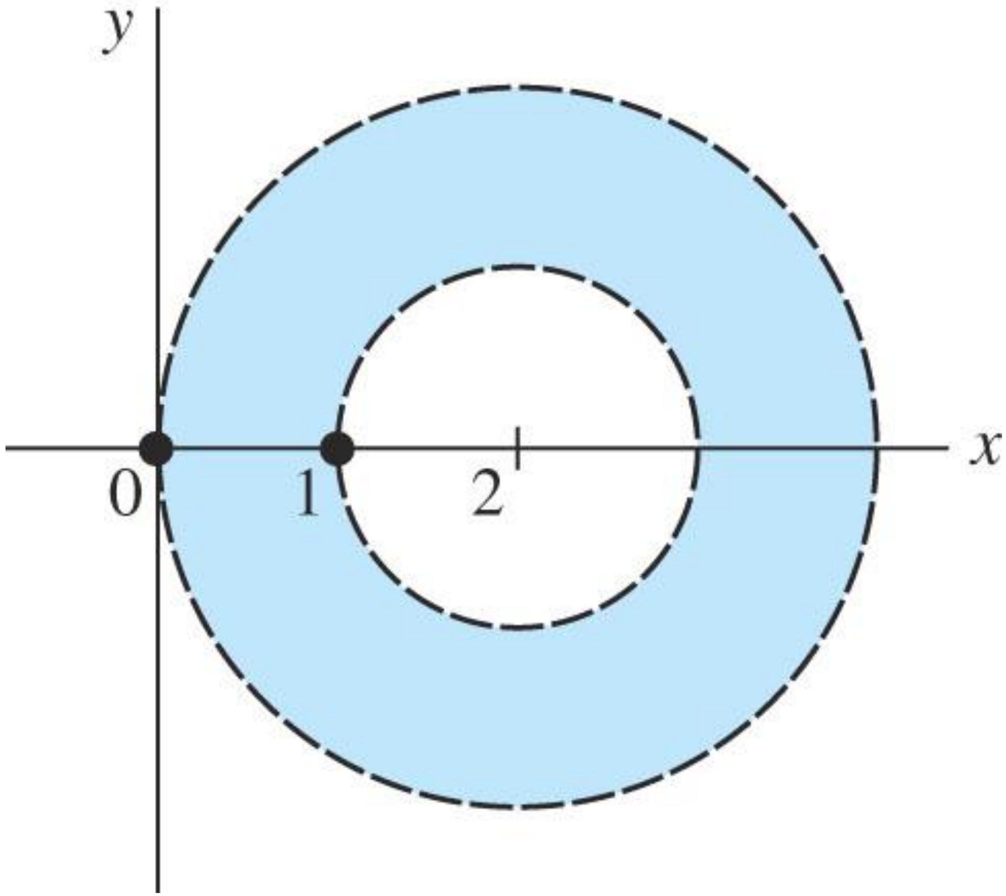
$$= \frac{1}{z} + 9 + 9z + 9z^2 + \ldots$$

Example 5

Expand $f(z) = \dfrac{1}{z(z-1)}$ in a Laurent series valid for $1 < |z - 2| < 2$.

Solution

(a) See Fig 4.9



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The center $z = 2$ is a point of analyticity of f. We want to find two series involving integer powers of $z - 2$; one converging for $1 < |z - 2|$ and the other converging for $|z - 2| < 2$.

$$f(z) = -\frac{1}{z} + \frac{1}{z-1} = f_1(z) + f_2(z)$$

$$f_1(z) = -\frac{1}{z} = -\frac{1}{2+z-2} = -\frac{1}{2}\frac{1}{1+\frac{z-2}{2}}$$

Cont'd

$$= -\frac{1}{2}\left[1 - \frac{z-2}{2} + \left(\frac{z-2}{2}\right)^2 - \left(\frac{z-2}{2}\right)^3 + \dots\right]$$

$$= -\frac{1}{2} + \frac{z-2}{2^2} - \frac{(z-2)^2}{2^3} + \frac{(z-2)^3}{2^4} - \dots$$

This series converges for $|(z-2)/2| < 1$ or $|z-2| < 2$.

Cont'd

$$f_2(z) = \frac{1}{z-1} = \frac{1}{1+z-2} = \frac{1}{z-2}\frac{1}{1+\dfrac{1}{z-2}}$$

$$= \frac{1}{z-2}\left[1 - \frac{1}{z-2} + \left(\frac{1}{z-2}\right)^2 - \left(\frac{1}{z-2}\right)^3 + \dots\right]$$

$$= \frac{1}{z-2} - \frac{1}{(z-2)^2} + \frac{1}{(z-2)^3} - \dots$$

This series converges for $|1/(z-2)| < 1$ or $1 < |z-2|$.

4.4 Zeros and Poles

Introduction

Suppose that $z = z_0$ is an isolated singularity of f and

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k = \sum_{k=1}^{\infty} \frac{a_{-k}}{(z - z_0)^k} + \sum_{k=0}^{\infty} a_k (z - z_0)^k \quad (1)$$

is the Laurent series of f valid for $0 < |z - z_0| < R$. The principal part of (1) is

$$\sum_{k=1}^{\infty} a_{-k} (z - z_0)^{-k} = \sum_{k=1}^{\infty} \frac{a_{-k}}{(z - z_0)^k} \quad (2)$$

Classification

(i) If the principal part is zero, $z = z_0$ is called a *removable singularity*.

(ii) If the principal part contains a finite number of terms, then $z = z_0$ is called a pole. If the last nonzero coefficient is a_{-n}, $n \geq 1$, then we say it is a pole of order n. A pole of order 1 is commonly called a *simple pole*.

(iii) If the principal part contains infinitely many nonzero terms, $z = z_0$ is called an *essential singularity*.

Example 1

We form

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \qquad (2)$$

that $z = 0$ is a removable singularity.

Example 2

(a) From

$$\frac{\sin z}{z^2} = \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - \ldots$$

$0 < |z|$. Thus $z = 0$ is a simple pole.

Moreover, $(\sin z)/z^3$ has a pole of order 2.

A Question

- The Laurent series of $f(z) = 1/z(z-1)$ valid for $1 < |z|$ is (exercise)

$$f(z) = \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \ldots$$

- The point $z = 0$ is an isolated singularity of f and the Laurent series contains an infinite number of terms involving negative integer powers of z. Does it mean that $z = 0$ is an essential singularity?

Cont'd

- The answer is "NO". Since the interested Laurent series is the one with the domain $0 < |z| < 1$, for which we have (exercise)

$$f(z) = -\frac{1}{z} - 1 - z - z^2 - \ldots$$

Thus $z = 0$ is a simple pole for $0 < |z| < 1$.

Zeros

- We say that z_0 is a zero of f if $f(z_0) = 0$. An analytic function f has a zero of order n at $z = z_0$ if

$$f(z_0) = 0\,,\; f'(z_0) = 0\,, f''(z_0) = 0\,, \ldots,\; f^{(n-1)}(z_0) = 0\,,$$

$$\text{but} \quad f^{(n)}(z_0) \neq 0 \qquad (3)$$

Example 3

- The analytic function $f(z) = z \sin z^2$ has a zero at $z = 0$, because f(0)=0.

$$\sin z^2 = z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} - \dots$$

$$f(z) = z \sin z^2 = z^3 \left[1 - \frac{z^4}{3!} + \frac{z^8}{5!} - \dots \right]$$

hence $z = 0$ is a zero of order 3, because $f(0) = 0, f'(0) = 0,$

$f''(0) = 0$ but $f^{(3)}(0) \neq 0$

THEOREM 4.11 **Pole of Order n**

If the functions f and g are analytic at $z = z_0$ and f has a zero of order n at $z = z_0$ and $g(z_0) \neq 0$, then the function $F(z) = g(z)/f(z)$ has a pole of order n at $z = z_0$.

Example 4

(a) Inspection of the rational function

$$F(z) = \frac{2z+5}{(z-1)(z+5)(z-2)^4}$$

shows that the denominator has zeros of order 1 at $z = 1$ and $z = -5$, and a zero of order 4 at $z = 2$. Since the numerator is not zero at these points, $F(z)$ has simple poles at $z = 1$ and $z = -5$ and a pole of order 4 at $z = 2$.

5. Calculus of residues

Contents

5.1. Residues and Residue Theorem

Residue

The coefficient a_{-1} of $1/(z - z_0)$ in the Laurent series is called the residue of the function f at the isolated singularity. We use this notation

$$a_{-1} = \text{Res}(f(z), z_0)$$

Example 1

(a) $z = 1$ is a pole of order 2 of $f(z) = 1/(z-1)^2(z-3)$.
The coefficient of $1/(z-1)$ is $a_{-1} = -\frac{1}{4}$.

THEOREM 5.1 **Residue at a Simple Pole**

If f has a simple pole at $z = z_0$, then

$$\operatorname{Res}(f(z), z_0) = \lim_{z \to z_0} (z - z_0) f(z) \qquad (1)$$

THEOREM 1

Proof

Since $z = z_0$ is a simple pole, we have

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots$$

$$\lim_{z \to z_0}(z - z_0) f(z)$$

$$= \lim_{z \to z_0}[a_{-1} + a_0(z - z_0) + a_1(z - z_0)^2 + \ldots]$$

$$= a_{-1} = \text{Res}(f(z), z_0)$$

THEOREM 5.2 **Residue at a Pole of Order n**

If f has a pole of order n at $z = z_0$, then

$$\operatorname{Res}(f(z), z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) \quad (2)$$

THEOREM 2

Proof

Since $z = z_0$ is a pole of order n, we have the form

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \ldots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \ldots$$

$$(z-z_0)^n f(z) = a_{-n} + \ldots + a_{-1}(z-z_0)^{n-1} + a_0(z-z_0)^n + \ldots$$

Then differentiating $n - 1$ times:

$$\frac{d^{n-1}}{dz^{n-1}}(z-z_0)^n f(z) = (n-1)!a_{-1} + n!a_0(z-z_0) + \cdots \qquad (3)$$

Proof Cont'd

The limit of (3) as $z \to z_0$ is

$$\lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) = (n-1)! a_{-1}$$

$$\text{Res}(f(z), z_0) = a_{-1} = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z)$$

Example 2

The function $f(z) = 1/(z-1)^2(z-3)$ has a pole of order 2 at $z = 1$. Find the residues.

Solution

$$\text{Res}(f(z),\ 1) = \frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}(z-1)^2 f(z) = \lim_{z\to 1}\frac{d}{dz}\frac{1}{z-3}$$

$$= \lim_{z\to 1}\frac{-1}{(z-3)^2} = -\frac{1}{4}$$

Approach for a simple pole

- If f can be written as $f(z) = g(z)/h(z)$ and has a simple pole at z_0 (note that $h(z_0) = 0$), then

$$\operatorname{Res}(f(z), z_0) = \frac{g(z_0)}{h'(z_0)} \tag{4}$$

because

$$\lim_{z \to z_0} (z - z_0)\frac{g(z)}{h(z)} = \lim_{z \to z_0} \frac{g(z)}{\dfrac{h(z) - h(z_0)}{z - z_0}} = \frac{g(z_0)}{h'(z_0)}$$

Example 3

❖ The polynomial $z^4 + 1$ can be factored as $(z - z_1)(z - z_2)(z - z_3)(z - z_4)$. Thus the function $f = 1/(z^4 + 1)$ has four simple poles. We can show that

$$z_1 = e^{\pi i/4},\ z_2 = e^{3\pi i/4},\ z_3 = e^{5\pi i/4},\ z_4 = e^{7\pi i/4}$$

$$\text{Res}(f(z), z_1) = \frac{1}{4z_1^{\ 3}} = \frac{1}{4}e^{-3\pi i/4} = -\frac{1}{4\sqrt{2}} - \frac{1}{4\sqrt{2}}i$$

Cont'd

$$\mathrm{Res}(f(z), z_2) = \frac{1}{4z_2^{\ 3}} = \frac{1}{4}e^{-9\pi i/4} = \frac{1}{4\sqrt{2}} - \frac{1}{4\sqrt{2}}i$$

$$\mathrm{Res}(f(z), z_3) = \frac{1}{4z_3^{\ 3}} = \frac{1}{4}e^{-15\pi i/4} = \frac{1}{4\sqrt{2}} + \frac{1}{4\sqrt{2}}i$$

$$\mathrm{Res}(f(z), z_4) = \frac{1}{4z_4^{\ 3}} = \frac{1}{4}e^{-21\pi i/4} = -\frac{1}{4\sqrt{2}} + \frac{1}{4\sqrt{2}}i$$

THEOREM 5.3

Cauchy's Residue Theorem

Let D be a simply connected domain and C a simply closed contour lying entirely within D. If a function f is analytic on and within C, except at a finite number of singular points $z_1, z_2, \ldots, z_n$ within C, then

$$\oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f(z), z_k) \qquad (5)$$

THEOREM 3

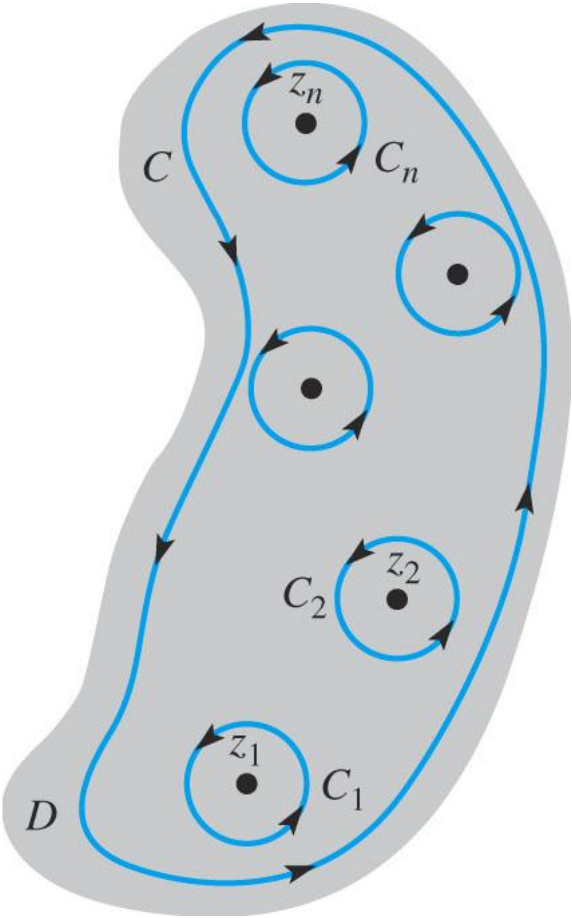
Proof

See Fig 5.1. Recalling from (15) of Sec. 4.3 , we can easily prove this theorem.

$$\oint_{C_k} f(z)dz = 2\pi i \text{Res}(f(z), z_k)$$

$$\oint_C f(z)dz = \sum_{k=1}^{n} \oint_{C_k} f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f(z), z_k)$$

Fig 5.1



Example 4

Evaluate $\oint_C \frac{1}{(z-1)^2(z-3)}\,dz$ where

(b) the contour C is the circle $|z| = 2$

Solution

Cont'd

(b) Since only the pole $z = 1$ lies within the circle, then there is only one singular point z=1 within C, from (5)

$$\oint_C \frac{1}{(z-1)^2(z-3)}\,dz = 2\pi i\ \mathrm{Res}(f(z), 1)$$

$$= 2\pi i\left(-\frac{1}{4}\right) = -\frac{\pi}{2}i$$

5.3 Evaluation of Real Integrals

Introduction

In this section we shall see how the residue theorem can be used to evaluate real integrals of the forms

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta \qquad (1)$$

$$\int_{-\infty}^{\infty} f(x)dx \qquad (2)$$

Integral of the Form $\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$

- Consider the form

$$\int_0^{2\pi} F(\cos\theta, \sin\theta) d\theta$$

The basic idea is to convert an integral form of (1) into a complex integral where the contour C is the unit circle: $z = \cos\theta + i\sin\theta$, $0 \le \theta \le \pi$.

Using

$$dz = ie^{i\theta} d\theta, \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Cont'd

- we can have

$$d\theta = \frac{dz}{iz}\,,\ \cos\theta = \frac{1}{2}(z + z^{-1})\,,\ \sin\theta = \frac{1}{2i}(z - z^{-1}) \qquad (4)$$

The integral in (1) becomes

$$\oint_C F\left(\frac{1}{2}(z + z^{-1})\,,\ \frac{1}{2i}(z - z^{-1})\right)\frac{dz}{iz}$$

where C is $|z| = 1$.

Example 1

Evaluate $\int_0^{2\pi} \frac{1}{(2+\cos\theta)^2}\, d\theta$

Solution

Using (4), we have the form

$$\frac{4}{i}\oint_C \frac{z}{(z^2+4z+1)^2}\, dz$$

We can write

$$f(z) = \frac{z}{(z-z_0)^2(z-z_1)^2}$$

where $z_0 = -2-\sqrt{3}, z_1 = -2+\sqrt{3}.$

Cont'd

Since only z_1 is inside the unit circle, we have

$$\oint_C \frac{z}{(z^2+4z+1)^2}\,dz = 2\pi i \,\mathrm{Res}(f(z),\, z_1)$$

$$\mathrm{Res}(f, z_1) = \lim_{z\to z_1}\frac{d}{dz}(z-z_1)^2 f(z) = \lim_{z\to z_1}\frac{d}{dz}\frac{z}{(z-z_0)^2}$$

$$= \lim_{z\to z_1} -\frac{(z+z_0)}{(z-z_0)^3} = \frac{1}{6\sqrt{3}}$$

Hence

$$\int_0^{2\pi}\frac{1}{(2+\cos\theta)^2}\,d\theta = \frac{4}{i}2\pi i\frac{1}{6\sqrt{3}} = \frac{4\pi}{3\sqrt{3}}$$

Integral of the Form $\int_{-\infty}^{\infty} f(x)dx$

- When f is continuous on $(-\infty, \infty)$, we have

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{r\to\infty}\int_{-r}^{0} f(x)\,dx + \lim_{R\to\infty}\int_{0}^{R} f(x)\,dx \qquad (5)$$

If both limits exist, the integral is said to be convergent; if one or both of them fail to exist, the integral is divergent.

- If we know (2) is convergent, we can evaluate it by a single limiting process:

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx \qquad (6)$$

It is important to note that (6) may exist even though the improper integral is divergent. For example:

$$\int_{-\infty}^{\infty} x dx \text{ is divergent, since}$$

$$\lim_{R\to\infty}\int_{0}^{R} x dx = \lim_{R\to\infty}\frac{R^2}{2} = \infty$$

- However using (6) we obtain

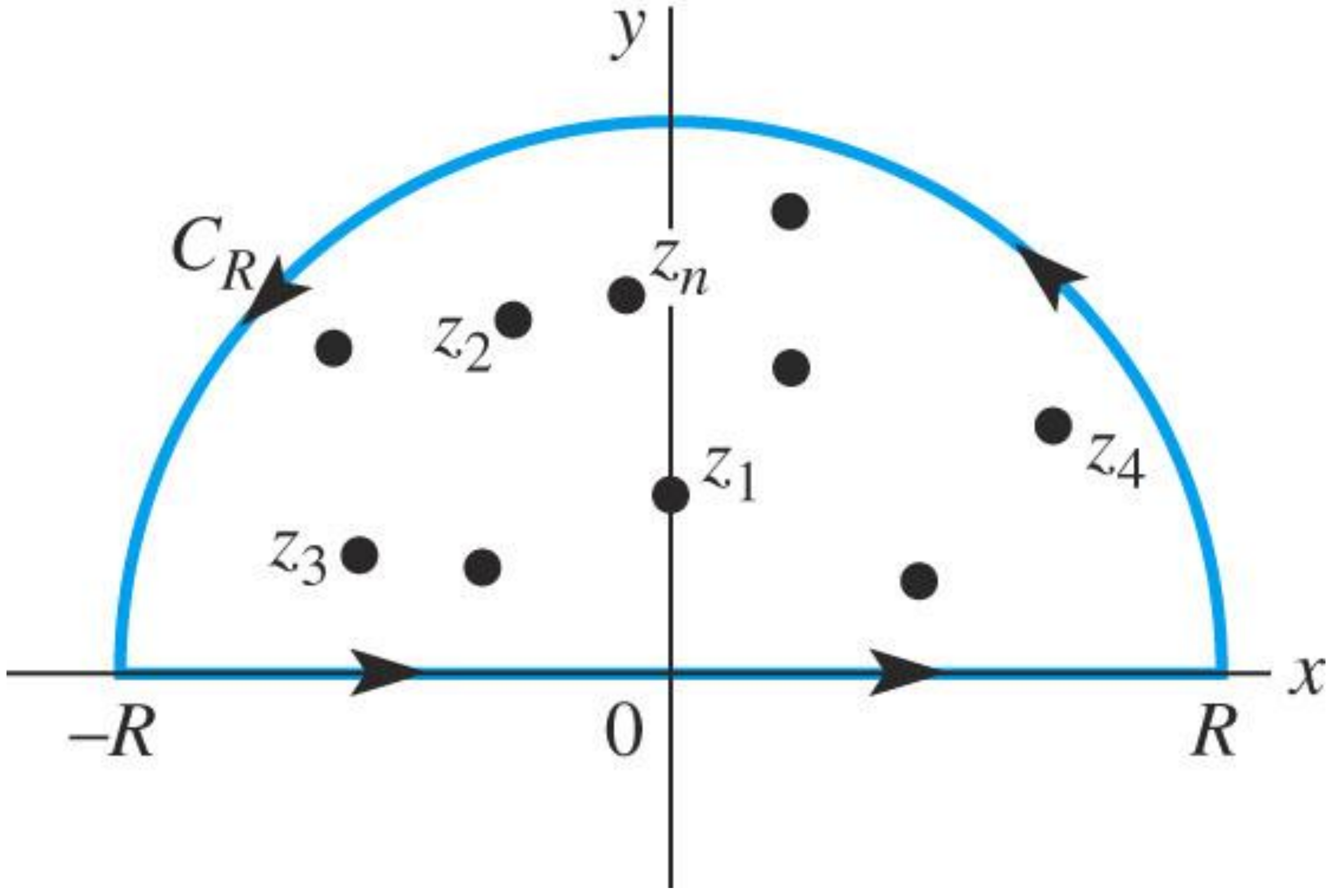
$$\lim_{r\to\infty}\int_{-R}^{R} x\,dx = \lim_{R\to\infty}\left[\frac{R^2}{2}-\frac{(-R)^2}{2}\right]=0 \qquad (7)$$

The limit in (6) is called the Cauchy principal value of the integral and is written

$$\text{P.V.}\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx$$

Fig 5.2

- To evaluate the integral in (2), we first see Fig 5.2.



- By theorem 5.3, we have

$$\oint_C f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx$$

$$= 2\pi i \sum_{k=1}^{n} \text{Res}(f(z), z_k)$$

where z_k, $k = 1, 2, \ldots, n$, denotes poles in the upper half-plane. If we can show the integral $\int_{C_R} f(z)dz \to 0$ as $R \to \infty$, then we have

$$P.\ V. \int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$$

$$= 2\pi i \sum_{k=1}^{n} \text{Res}(f(z), z_k) \qquad (8)$$

Example 2

Evaluate the Cauchy principal value of

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)}\,dx$$

Solution

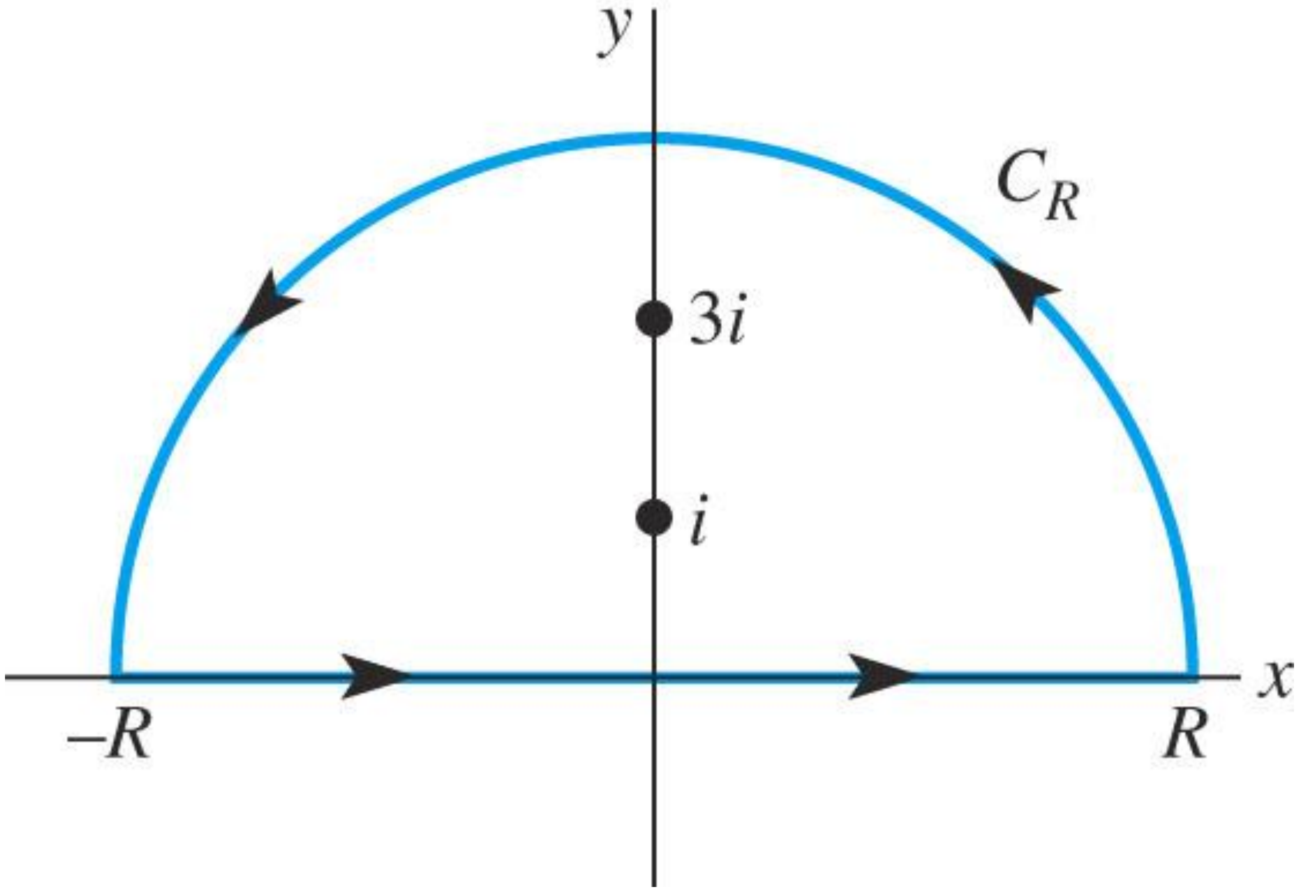
Let $f(z) = 1/(z^2 + 1)(z^2 + 9)$

$= 1/(z + i)(z - i)(z + 3i)(z - 3i)$

Only $z = i$ and $z = 3i$ are in the upper half-plane.

See Fig 5.3.

Fig 5.3



Cont'd

$$\oint_C \frac{1}{(z^2+1)(z^2+9)}\,dz$$

$$= \int_{-R}^{R} \frac{1}{(z^2+1)(z^2+9)}\,dx + \int_{C_R} \frac{1}{(z^2+1)(z^2+9)}\,dz$$

$$= I_1 + I_2$$

$$= 2\pi i\left[\operatorname{Res}(f(z), i) + \operatorname{Res}(f(z), 3i)\right]$$

$$= 2\pi i\left[\frac{1}{16i} - \frac{1}{48i}\right] = \frac{\pi}{12} \qquad (9)$$

Cont'd

Now let $R \to \infty$ and note that on C_R:

$$\left|(z^2+1)(z^2+9)\right| = \left|(z^2+1)\right|\left|(z^2+9)\right|$$

$$\geq \left||z|^2-1\right|\left||z|^2-9\right| = (R^2-1)(R^2-9)$$

From the ML-inequality

$$|I_2| = \left|\int_{C_R} \frac{1}{(z^2+1)(z^2+9)}dz\right| \leq \frac{\pi R}{(R^2-1)(R^2-9)}$$

$$|I_2| \to 0 \text{ as } R \to \infty$$

Cont'd

Thus

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{1}{(x^2+1)(x^2+9)}dx=\frac{\pi}{12}$$

THEOREM 5.4 **Behavior of Integral as R → ∞**

Suppose $f(z) = P(z)/Q(z)$, where the degree of $P(z)$ is n and the degree of $Q(z)$ is $m \geq n + 2$. If C_R is a Semicircular contour $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$, then $\int_{C_R} f(z)\,dz \to 0$ as $R \to \infty$.

Example 3

Evaluate the Cauchy principal value of

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx$$

Solution

We first check that the condition in Theorem 19.15 is satisfied. The poles in the upper half-plane are $z_1 = e^{\pi i/4}$ and $z_2 = e^{3\pi i/4}$. We also knew that

$$\text{Res}(f, z_1) = -\frac{1}{4\sqrt{2}} - \frac{1}{4\sqrt{2}}i,$$

$$\text{Res}(f, z_2) = \frac{1}{4\sqrt{2}} - \frac{1}{4\sqrt{2}}i$$

Cont'd

Thus by (8)

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx$$

$$= 2\pi i[\text{Res}(f, z_1) + \text{Res}(f, z_2)]$$

$$= \frac{\pi}{\sqrt{2}}$$

Integrals of the Form $\int_{-\infty}^{\infty} f(x)\cos\alpha x\,dx$ or $\int_{-\infty}^{\infty} f(x)\sin\alpha x\,dx$

- Using Euler's formula, we have

$$\int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx = \int_{-\infty}^{\infty} f(x)\cos\alpha x\,dx + i\int_{-\infty}^{\infty} f(x)\sin\alpha x\,dx \tag{10}$$

Before proceeding, we give the following sufficient conditions without proof.

THEOREM 5.5 **Behavior of Integral as R → ∞**

Suppose $f(z) = P(z)/Q(z)$, where the degree of $P(z)$ is n and the degree of $Q(z)$ is $m \geq n + 1$. If C_R is a Semicircular contour $z = \text{Re}^{i\theta}$, $0 \leq \theta \leq \pi$, and $\alpha > 0$, then $\int_{C_R} (P(z)/Q(z))e^{i\alpha z}dz \to 0$ as $R \to \infty$.

Example 4

Evaluate the Cauchy principal value of

$$\int_0^\infty \frac{x\sin x}{x^2+9}\,dx$$

Solution

Note that the lower limit of this integral is not $-\infty$. We first check that the integrand is even, so we have

$$\int_0^\infty \frac{x\sin x}{x^2+9}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{x\sin x}{x^2+9}\,dx \qquad (11)$$

Cont'd

With α =1, we now for the integral

$$\oint_C \frac{z}{z^2+9} e^{iz} dz$$

where *C* is the same as in Fig 19.12. Thus

$$\int_{C_R} \frac{z}{z^2+9} e^{iz} dz + \int_{-R}^{R} \frac{x}{x^2+9} e^{ix} dx$$

$$= 2\pi i \, \mathrm{Res}(f(z)e^{iz}, 3i)$$

Cont'd

where $f(z) = z/(z^2 + 9)$. From (4) of Sec 5.1,

$$\text{Res}(f(z)e^{iz}, 3i) = \frac{ze^{iz}}{2z}\Big|_{z=3i} = \frac{e^{-3}}{2}$$

In addition, this problem satisfies the condition of Theorem 5.5, so

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{x}{x^2+9} e^{ix} dx = 2\pi i \frac{e^{-3}}{2} = \frac{\pi}{e^3} i$$

Cont’d

Then

$$\int_{-\infty}^{\infty} \frac{x}{x^2+9} e^{ix} dx$$

$$= \int_{-\infty}^{\infty} \frac{x\cos x}{x^2+9} dx + i\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+9} dx = \frac{\pi}{e^3} i$$

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{x\cos x}{x^2+9} dx = 0, \text{P.V.}\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+9} dx = \frac{\pi}{e^3}$$

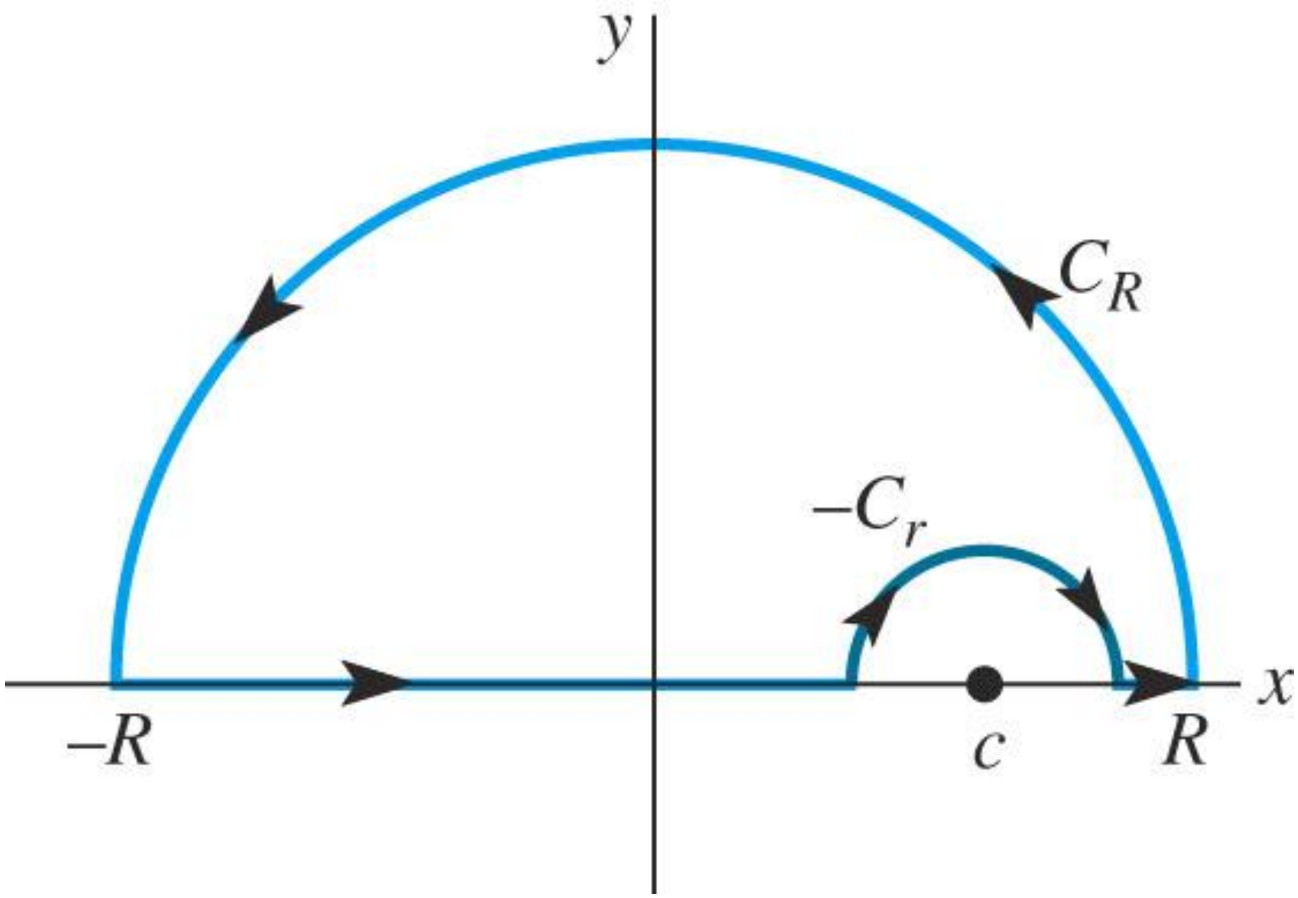
Finally,

$$\int_{0}^{\infty} \frac{x\sin x}{x^2+9} dx = \frac{1}{2}\text{P.V.}\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+9} dx = \frac{\pi}{2e^3}$$

Indented Contour

- The complex functions $f(z) = P(z)/Q(z)$ of the improper integrals (2) and (3) did not have poles on the real axis. When $f(z)$ has a pole at $z = c$, where c is a real number, we must use the indented contour as in Fig 5.4.

Fig 5.4



THEOREM 5.6

Behavior of Integral as r → ∞

Suppose f has a simple pole z = c on the real axis. If Cr is the contour defined by $z = c + re^{i\theta}$, $0 \leq \theta \leq \pi$, then

$$\lim_{r \to 0} \int_{C_r} f(z)\, dz = \pi i \,\mathrm{Res}(f(z), c)$$

proof Cont'd

Proof

Since f has a simple pole at $z = c$, its Laurent series is

$$f(z) = a_{-1}/(z - c) + g(z)$$

where $a_{-1} = \text{Res}(f(z), c)$ and g is analytic at c. Using the Laurent series and the parameterization of C_r, we have

$$\int_{C_r} f(z)\,dz$$

$$= a_{-1}\int_0^{\pi} \frac{ire^{i\theta}}{re^{i\theta}}\,d\theta + ir\int_0^{\pi} g(c + re^{i\theta})\,e^{i\theta}d\theta \qquad (12)$$

$$= I_1 + I_2$$

Cont'd

First we see $I_1 = a_{-1}\int_0^{\pi} \frac{ire^{i\theta}}{re^{i\theta}+9} d\theta = a_{-1}\int_0^{\pi} i d\theta$

$$= \pi i a_{-1} = \pi i \mathrm{Res}(f(z), c)$$

Next, g is analytic at c and so it is continuous at c and is bounded in a neighborhood of the point; that is, there exists an $M > 0$ for which $|g(c + re^{i\theta})| \leq M$.

Hence

$$I_2 = \left| ir\int_0^{\pi} g(c+re^{i\theta})e^{i\theta} d\theta \right| =\leq r\int_0^{\pi} M d\theta = \pi r M$$

It follows that $\lim_{r\to 0}|I_2| = 0$ and $\lim_{r\to 0} I_2 = 0$.

We complete the proof.

Example 5

Evaluate the Cauchy principal value of

$$\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2 - 2x + 2)}\, dx$$

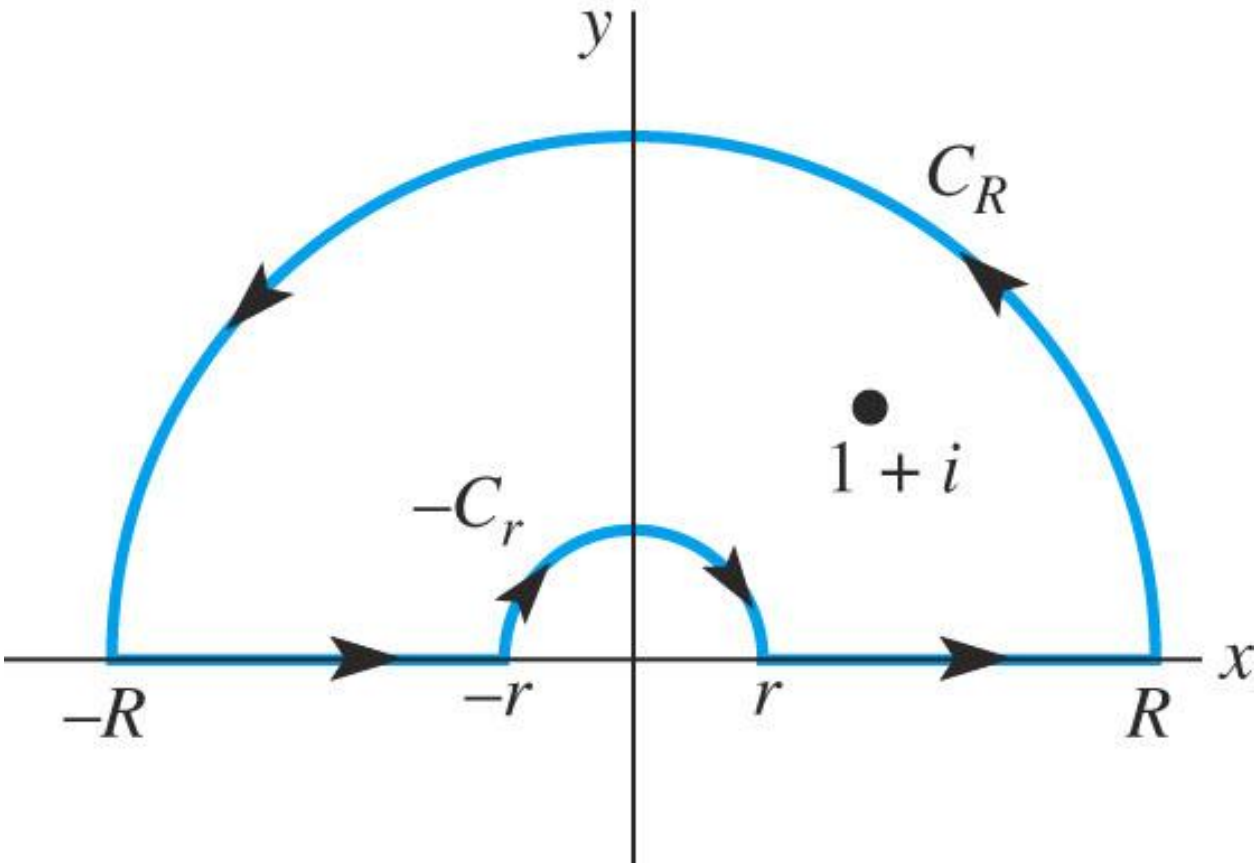
Solution

Since the integral is of form (3), we consider the contour integral

$$\oint_C \frac{e^{iz}dz}{z(z^2 - 2z + 2)}, \quad f(z) = \frac{1}{z(z^2 - 2z + 2)}$$

Fig 5.5

$f(z)$ has simple poles at $z = 0$ and $z = 1 + i$ in the upper half-plane. See Fig 5.5.



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- Now we have

$$\oint_C = \int_{C_R} + \int_{-R}^{-r} + \int_{-C_r} + \int_r^R = 2\pi i \text{Res}(f(z)e^{iz}, 1+i) \quad (13)$$

Taking the limits in (13) as $R \to \infty$ and $r \to 0$, from Theorem 5.5 and 5.6, we have

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2 - 2x + 2)} dx - \pi i \text{Res}(f(z)e^{iz}, 0)$$

$$= 2\pi i \text{Res}(f(z)e^{iz}, 1+i)$$

Cont'd

Now

$$\text{Res}(f(z)e^{iz}, 0) = \frac{1}{2}$$

$$\text{Res}(f(z)e^{iz}, 1+i) = \frac{e^{-1+i}}{4}(1+i)$$

Therefore,

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2-2x+2)}dx = \pi i\frac{1}{2} + 2\pi i\left(-\frac{e^{-1+i}}{4}(1+i)\right)$$

Cont'd

Using $e^{-1+i} = e^{-1}(\cos 1 + i \sin 1)$, then

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{\cos x}{x(x^2-2x+2)}\,dx = \frac{\pi}{2}e^{-1}(\sin 1 + \cos 1)$$

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2-2x+2)}\,dx = \frac{\pi}{2}[1 + e^{-1}(\sin 1 - \cos 1)]$$

Thank You !