Chapter 6

The Mobius transformation

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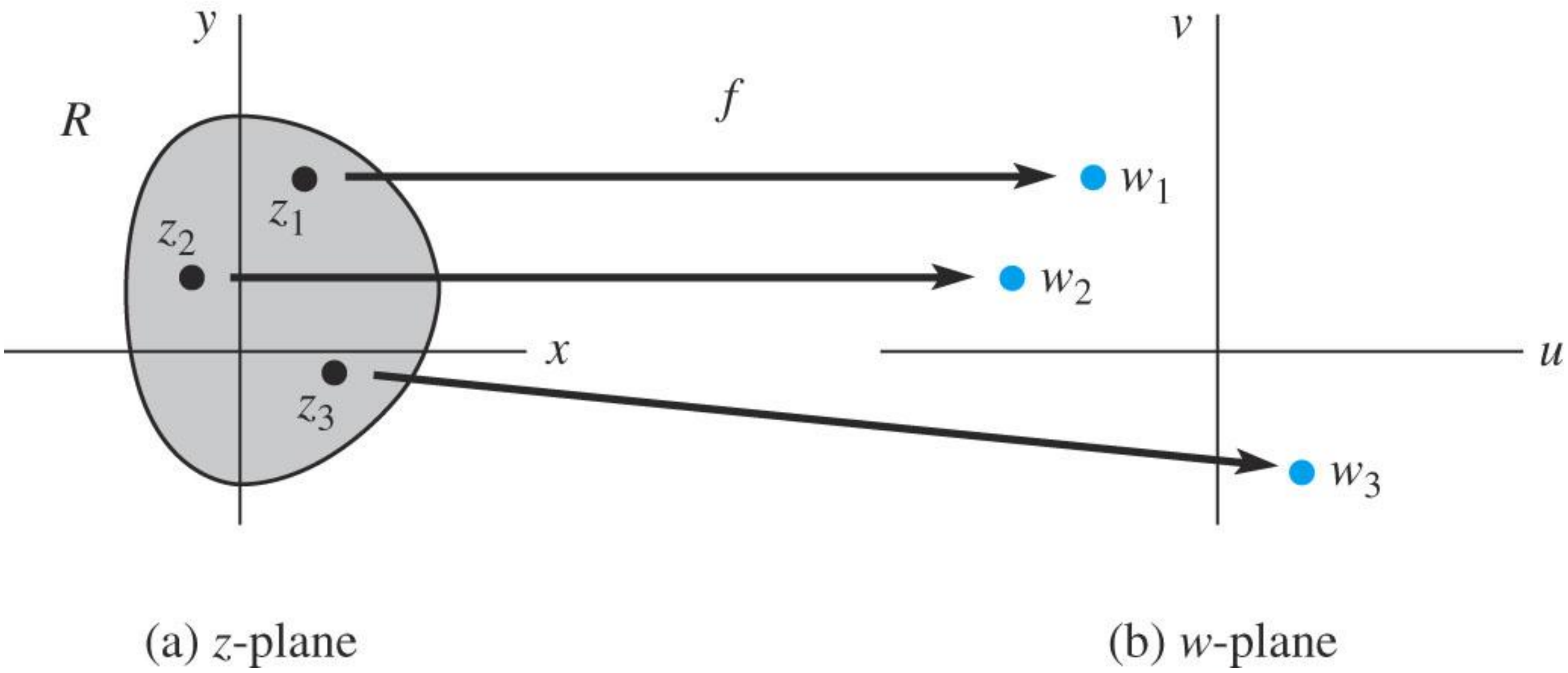
Contents

6.1 Complex Functions as Mappings

Introduction

The complex function $w = f(z) = u(x, y) + iv(x, y)$ may be considered as the planar transformation. We also call $w = f(z)$ is the image of z under f. See Fig 6.1.

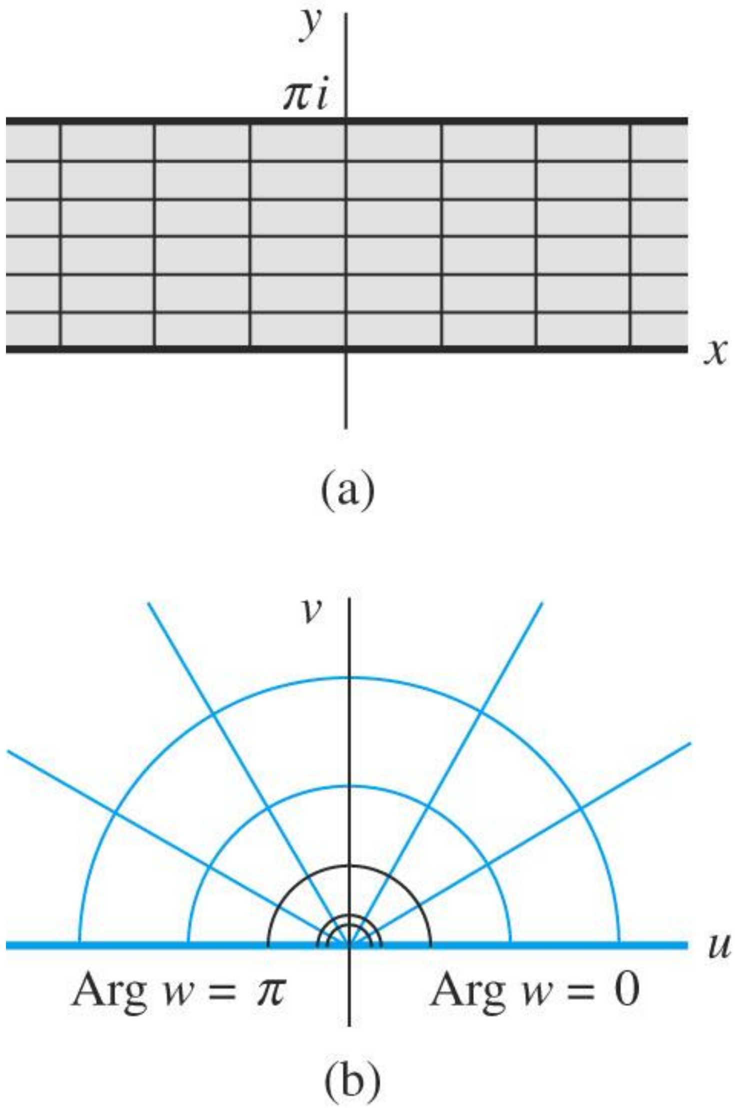
Fig 6.1



Example 1

- Consider the function $f(z) = e^z$. If $z = a + it$, $0 \le t \le \pi$, $w = f(z) = e^a e^{it}$. Thus this is a semicircle with center $w = 0$ and radius $r = e^a$. If $z = t + ib$, $-\infty \le t \le \infty$, $w = f(z) = e^t e^{ib}$. Thus this is a ray with Arg $w = b$, $|w| = e^t$. See Fig 20.2.

Fig 6.2



Example 2

- The complex function $f = 1/z$ has domain $z \neq 0$ and

  real part : $u(x, y) = \dfrac{x}{x^2 + y^2}$

  imaginary part : $v(x, y) = \dfrac{-y}{x^2 + y^2}$

  When $a \neq 0$, $u(x, y) = a$ can be written as

  $$x^2 - \frac{1}{a}x + y^2 = 0,\ (x - \frac{1}{2a})^2 + y^2 = (\frac{1}{2a})^2$$

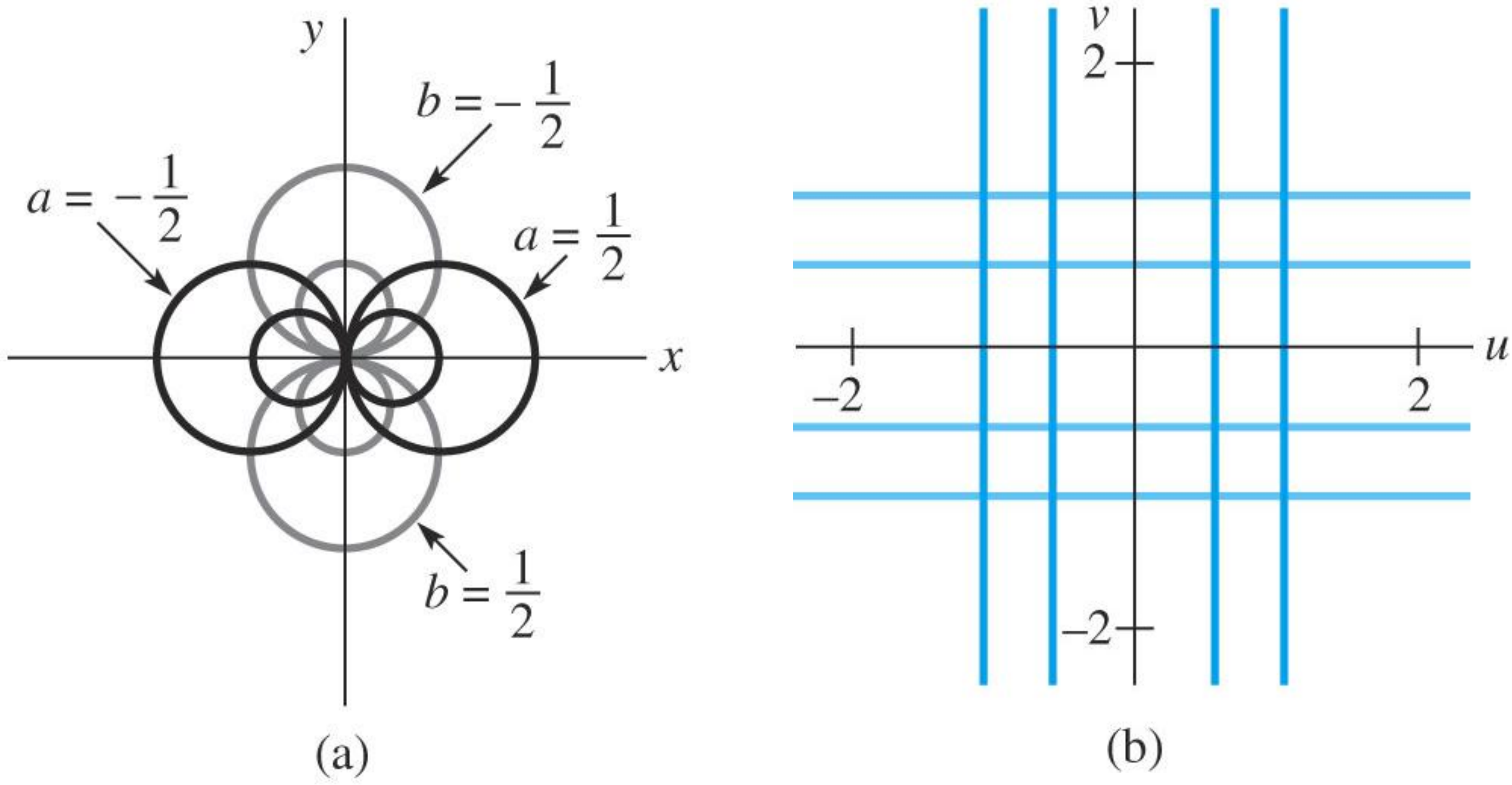
Cont'd

- Likewise $v(x, y) = b$, $b \neq 0$ can be written as

$$x^2 + (y + \frac{1}{2b})^2 = (\frac{1}{2b})^2$$

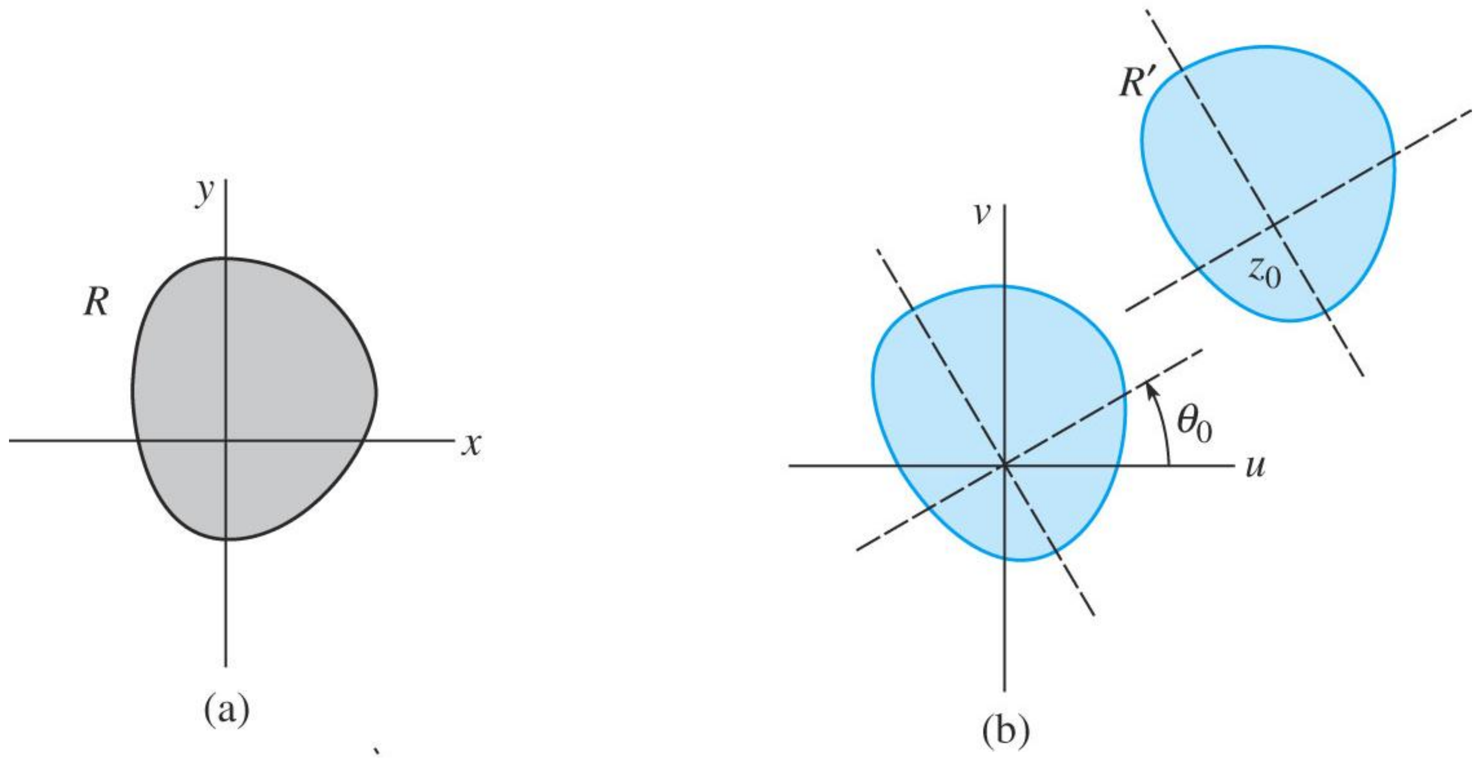
See Fig 6.3.

Fig 6.3



Translation and Rotation

- The function $f(z) = z + z_0$ is interpreted as a translation. The function $g(z) = e^{i\theta_0}z$ is interpreted as a rotation. See Fig 6.4.



Example 3

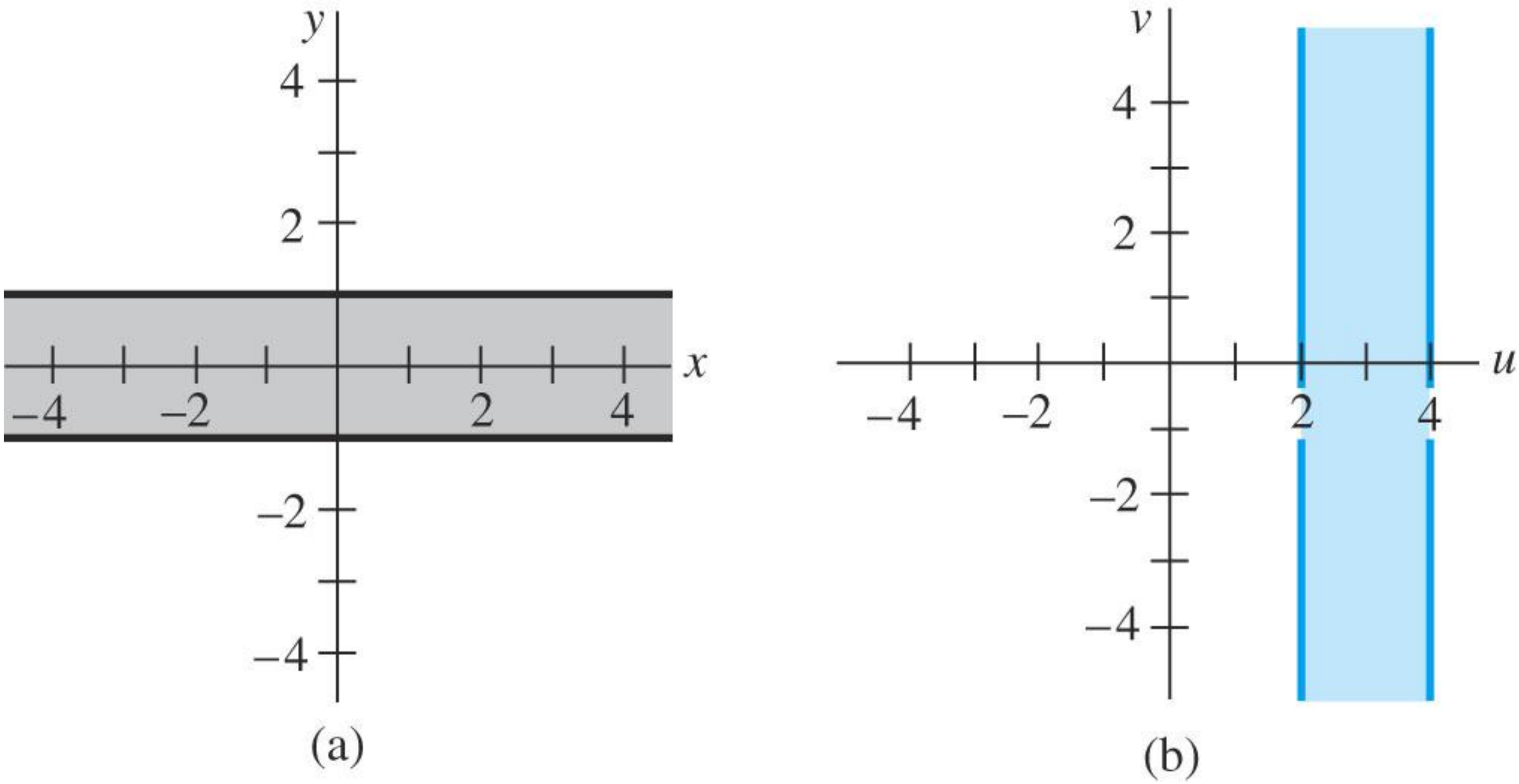
Find a complex function that maps $-1 \leq y \leq 1$ onto $2 \leq x \leq 4$.

Solution

See Fig 6.5. We find that $-1 \leq y \leq 1$ is first rotated through 90° and shifted 3 units to the right. Thus the mapping is

$$h(z) = e^{i\pi/2}z + 3 = iz + 3$$

Fig 6.5



Magnification

- A magnification is the function $f(z) = \alpha z$, where α is a fixed positive real number. Note that $|w| = |\alpha z| = \alpha|z|$. If $g(z) = az + b$ and $a = r_0 e^{i\theta_0}$ then the vector is rotated through θ_0, magnified by a factor r_0, and then translated using b.

Example 4

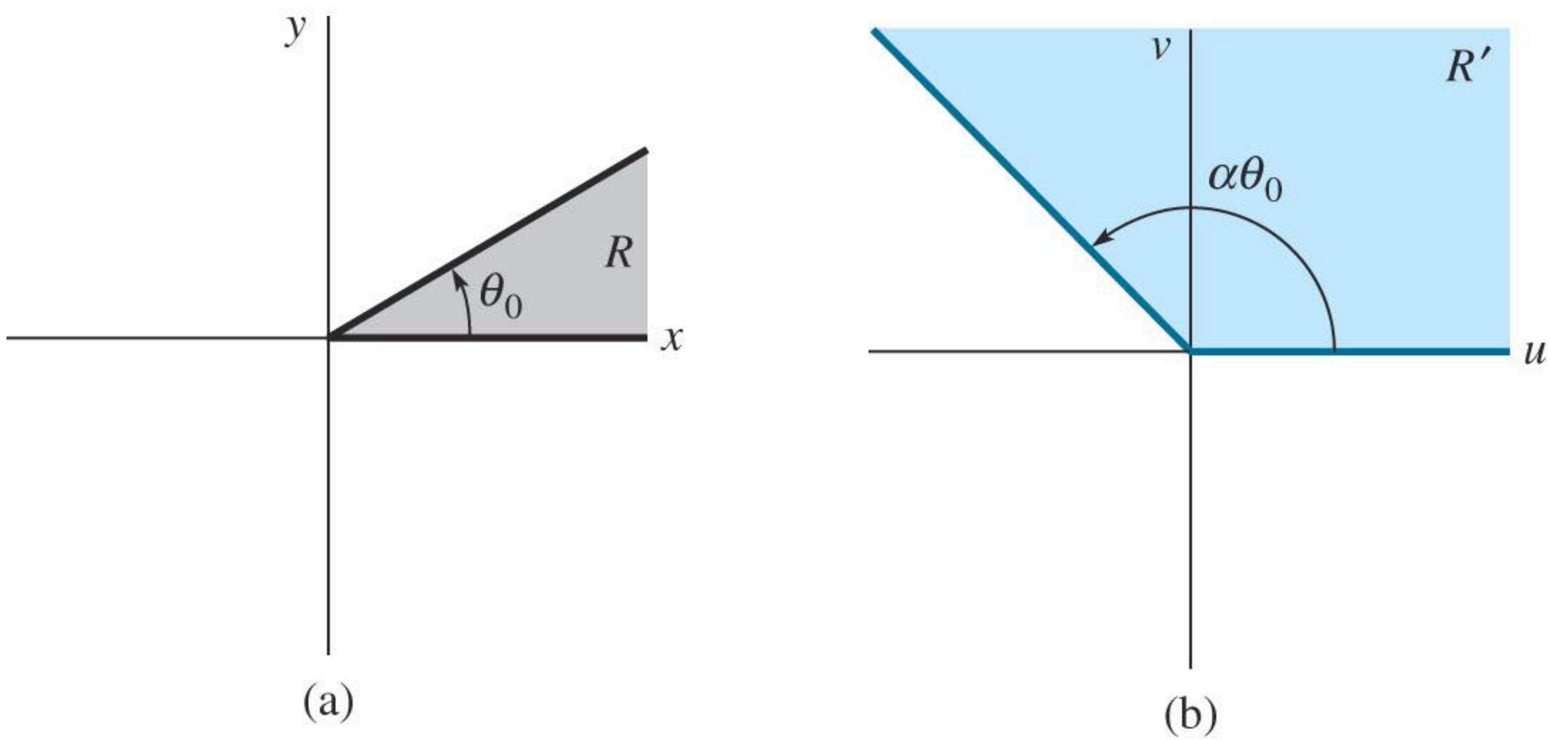
Find a complex function that maps the disk $|z| \leq 1$ onto the disk $|w - (1 + i)| \leq ½$.

Solution

Magnified by ½ and translated to $1 + i$, we can have the desired function as $w = f(z) = ½z + (1 + i)$.

Power Functions

- A complex function $f(z) = z^{\alpha}$ where α is a fixed positive number, is called a real power function. See Fig 6.6. If $z = re^{i\theta}$, then $w = f(z) = r^{\alpha}e^{i\alpha\theta}$.



Example 5

Find a complex function that maps the upper half-plane $y \geq 0$ onto the wedge $0 \leq \text{Arg } w \leq \pi/4$.

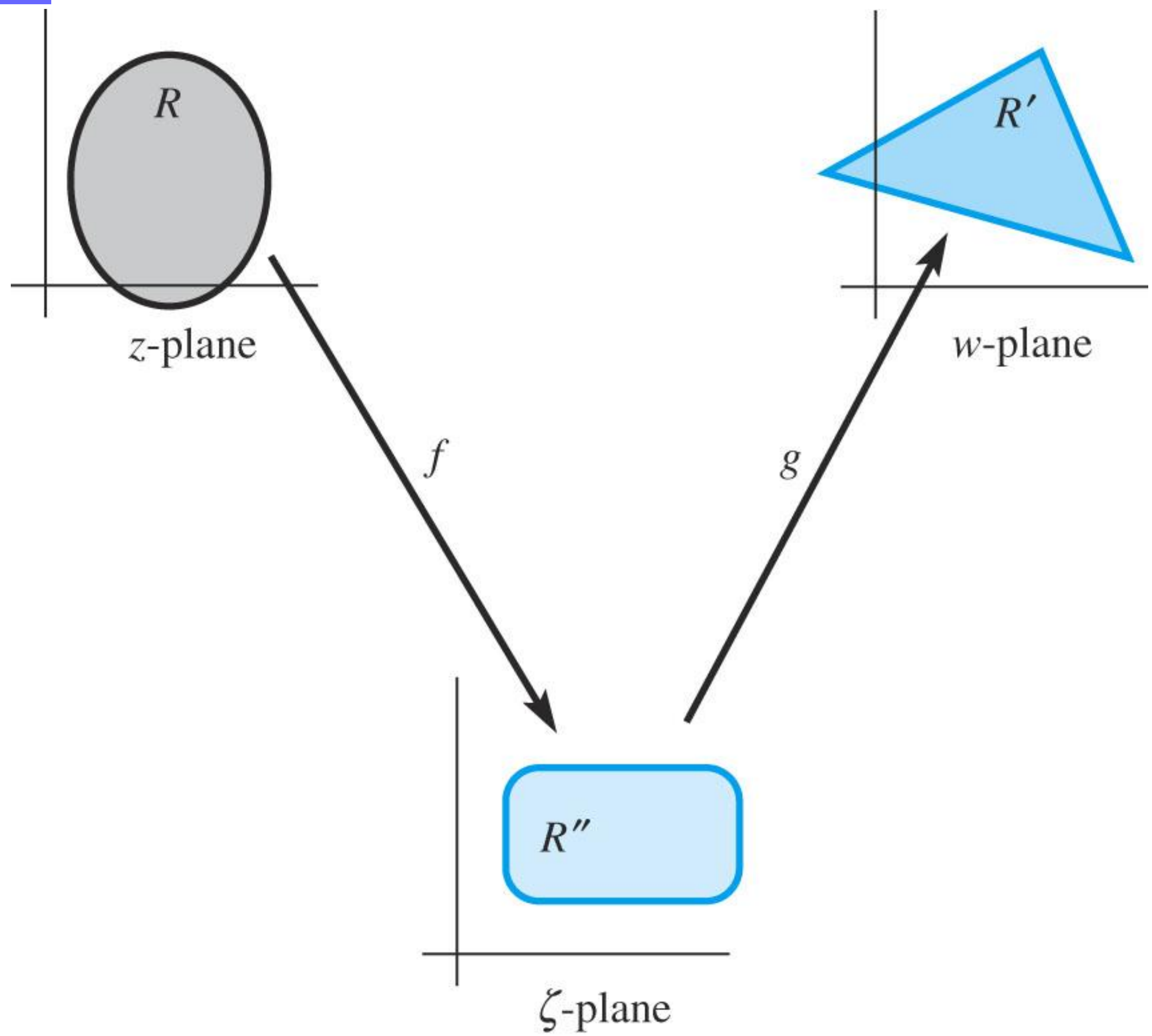
Solution

The upper half-plane can also be described by $0 \leq \text{Arg } w \leq \pi$. Thus $f(z) = z^{1/4}$ will map the upper half-plane onto the wedge $0 \leq \text{Arg } w \leq \pi/4$.

Successive Mapping

- See Fig 6.7. If $\zeta = f(z)$ maps R onto R'', and $w = g(\zeta)$ maps R'' onto R', $w = g(f(z))$ maps R onto R'.

Fig 6.7



Example 6

Find a complex function that maps $0 \le y \le \pi$ onto the wedge $0 \le \text{Arg } w \le \pi/4$.

Solution

We have shown that $f(z) = e^z$ maps $0 \le y \le \pi$ onto to $0 \le \text{Arg } \zeta \le \pi$ and $g(\zeta) = \zeta^{1/4}$ maps $0 \le \text{Arg } \zeta \le \pi$ onto $0 \le \text{Arg } w \le \pi/4$. Thus the desired mapping is $w = g(f(z)) = g(e^z) = e^{z/4}$.

Example 7

Find a complex function that maps $\pi/4 \le \text{Arg } z \le 3\pi/4$ onto the upper half-plane $v \ge 0$.

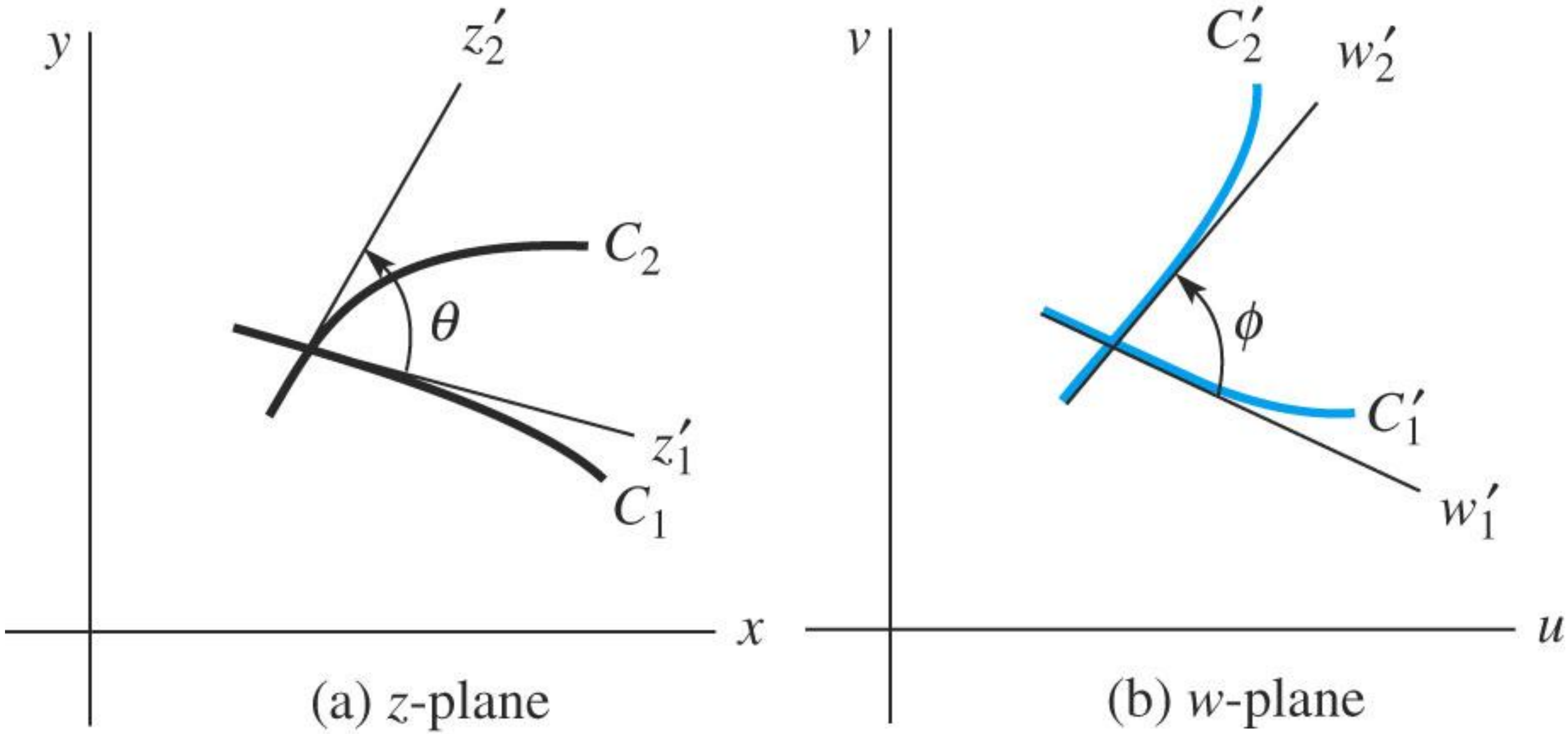
Solution

First rotate $\pi/4 \le \text{Arg } z \le 3\pi/4$ by $\zeta = f(z) = e^{-i\pi/4}z$. Then magnify it by 2, $w = g(\zeta) = \zeta^2$. Thus the desired mapping is $w = g(f(z)) = (e^{-i\pi/4}z)^2 = -iz^2$.

6.2 Conformal Mappings

- **Angle –Preserving Mappings**
  A complex mapping $w = f(z)$ defined on a domain D is called conformal at $z = z_0$ in D when f preserves that angle between two curves in D that intersect at z_0. See Fig 6.10.

Fig 6.10



- Referring to Fig 6.10, we have

$$|z_1' - z_2'|^2 = |z_1'|^2 + |z_2'|^2 - 2|z_1'||z_2'|\cos\theta$$

$$\text{or} \quad \theta = \cos^{-1}\left(\frac{|z_1'|^2 + |z_2'|^2 - |z_1' - z_2'|^2}{2|z_1'||z_2'|}\right) \qquad (1)$$

Likewise

$$\phi = \cos^{-1}\left(\frac{|w_1'|^2 + |w_2'|^2 - |w_1' - w_2'|^2}{2|w_1'||w_2'|}\right) \qquad (2)$$

THEOREM 20.1 **Conformal Mapping**

If $f(z)$ is analytic in the domain D and $f'(z) \neq 0$, then f is conformal at $z = z_0$.

Proof

If a curve C in D is defined by $z = z(t)$, then $w = f(z(t))$ is the image curve in the w-plane. We have

$$w = f(z(t)), w' = f'(z(t))z'(t)$$

If C_1 and C_2 intersect at $z = z_0$, then

$$w_1' = f'(z_0)z_1', w_2' = f'(z_0)z_2'$$

Proof   Cont'd

Since $f'(z_0) \neq 0$, we can use (2) to obtain

$$\phi = \cos^{-1}\left(\frac{\left|f'(z_0)z_1'\right|^2 + \left|f'(z_0)z_2'\right|^2 - \left|f'(z_0)z_1' - f'(z_0)z_2'\right|^2}{2\left|f'(z_0)z_1'\right|\left|f'(z_0)z_2'\right|} \right)$$

$$= \cos^{-1}\left(\frac{\left|z_1'\right|^2 + \left|z_2'\right|^2 - \left|z_1' - z_2'\right|^2}{2\left|z_1'\right|\left|z_2'\right|} \right) = \theta$$

Example 1

(a) The analytic function $f(z) = e^z$ is conformal at all points, since $f'(z) = e^z$ is never zero.

(b) The analytic function $g(z) = z^2$ is conformal at all points except $z = 0$, since $g'(z) = 2z \neq 0$, for $z \neq 0$.

Example 2

- The vertical strip $-\pi/2 \leq x \leq \pi/2$ is called the fundamental region of the trigonometric function $w = \sin z$. A vertical line $x = a$ in the interior of the region can be described by $z = a + it$, $-\infty \leq t \leq \infty$. We find that

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

and so

$$u + iv = \sin(a + it)$$
$$= \sin a \cosh t + i \cos a \sinh t.$$

Cont'd

Since $\cosh^2 t - \sinh^2 t = 1$, then

$$\frac{u^2}{\sin^2 a} - \frac{v^2}{\cos^2 a} = 1$$

The image of the vertical line $x = a$ is a hyperbola with $\pm \sin a$ as u-intercepts and since $-\pi/2 < a < \pi/2$, the hyperbola crosses the u-axis between $u = -1$ and $u = 1$. Note if $a = -\pi/2$, then $w = -\cosh t$, the line $x = -\pi/2$ is mapped onto the interval $(-\infty, -1]$. Likewise, the line $x = \pi/2$ is mapped onto the interval $[1, \infty)$.

Example 3

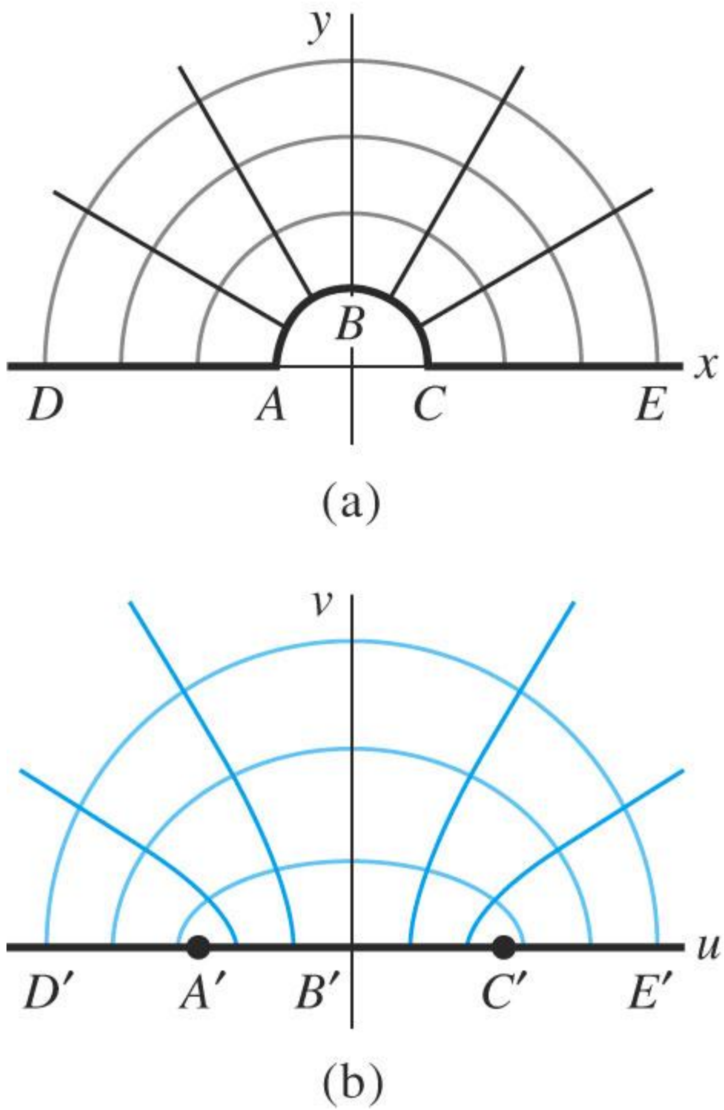
The complex function $f(z) = z + 1/z$ is conformal at all points except $z = \pm 1$ and $z = 0$. In particular, the function is conformal at all points in the upper half-plane satisfying $|z| > 1$. If $z = re^{i\theta}$, then

$w = re^{i\theta} + (1/r)e^{-i\theta}$, and so

$$u = \left(r + \frac{1}{r}\right)\cos\theta, \quad v = \left(r - \frac{1}{r}\right)\sin\theta \qquad (3)$$

Note if $r = 1$, then $v = 0$ and $u = 2\cos\theta$. Thus the semicircle $z = e^{it}$, $0 \le t \le \pi$, is mapped onto $[-2, 2]$ on the u-axis. If $r > 1$, the semicircle $z = re^{it}$, $0 \le t \le \pi$, is mapped onto the upper half of the ellipse $u^2/a^2 + v^2/b^2 = 1$, where $a = r + 1/r$, $b = r - 1/r$. See Fig 6.12.

Fig 6.12



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For a fixed value of θ, the ray $te^{i\theta}$, for $t \geq 1$, is mapped to the point $u^2/\cos^2\theta - v^2/\sin^2\theta = 4$ in the upper half-plane $v \geq 0$. This follows from (3) since

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = \left(t + \frac{1}{t}\right)^2 - \left(t - \frac{1}{t}\right)^2 = 4$$

Since f is conformal for $|z| > 1$ and a ray $\theta = \theta_0$ intersects a circle $|z| = r$ at a right angle, the hyperbolas and ellipses in the w-plane are orthogonal.

THEOREM 6.2 **Transformation Theorem for Harmonic Functions**

If f be an analytic function that maps a domain D onto a domain D'. If U is harmonic in D', then the real-valued function $u(x, y) = U(f(z))$ is harmonic in D.

Cont'd

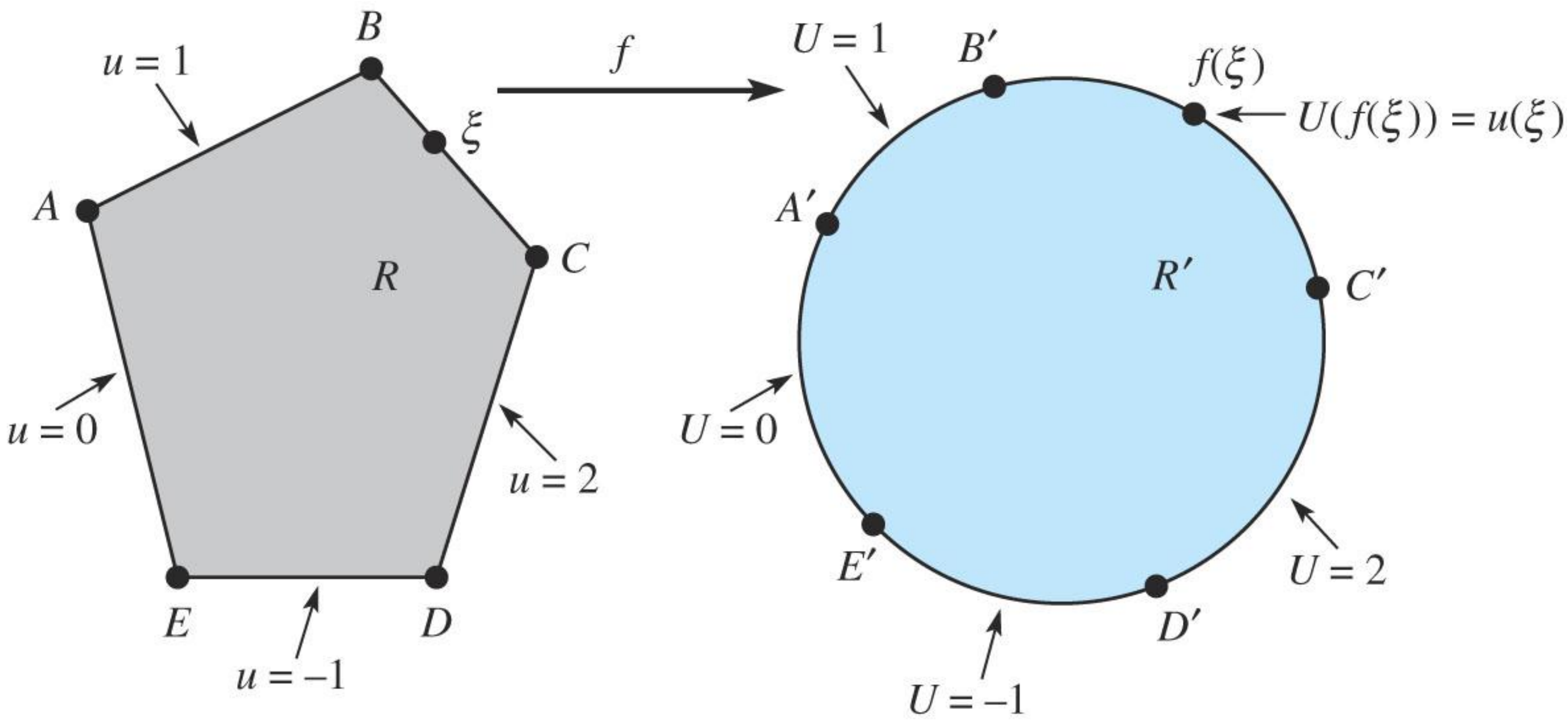
Proof

We will give a special proof for the special case in which D' is simply connected. If U has a harmonic conjugate V in D', then $H = U + iV$ is analytic in D', and so the composite function $H(f(z)) = U(f(z)) + iV(f(z))$ is analytic in D. It follow that the real part $U(f(z))$ is harmonic in D.

Solving Dirichlet Problems Using Conformal Mapping

1. Find a conformal mapping $w = f(z)$ that transforms the original region R onto the image R'. The region R' may be a region for which many explicit solutions to Dirichlet problems are known.
2. Transfer the boundary conditions from the R to the boundary conditions of R'. The value of u at a boundary point ξ of R is assigned as the value of U at the corresponding boundary point $f(\xi)$.

Fig 6.13

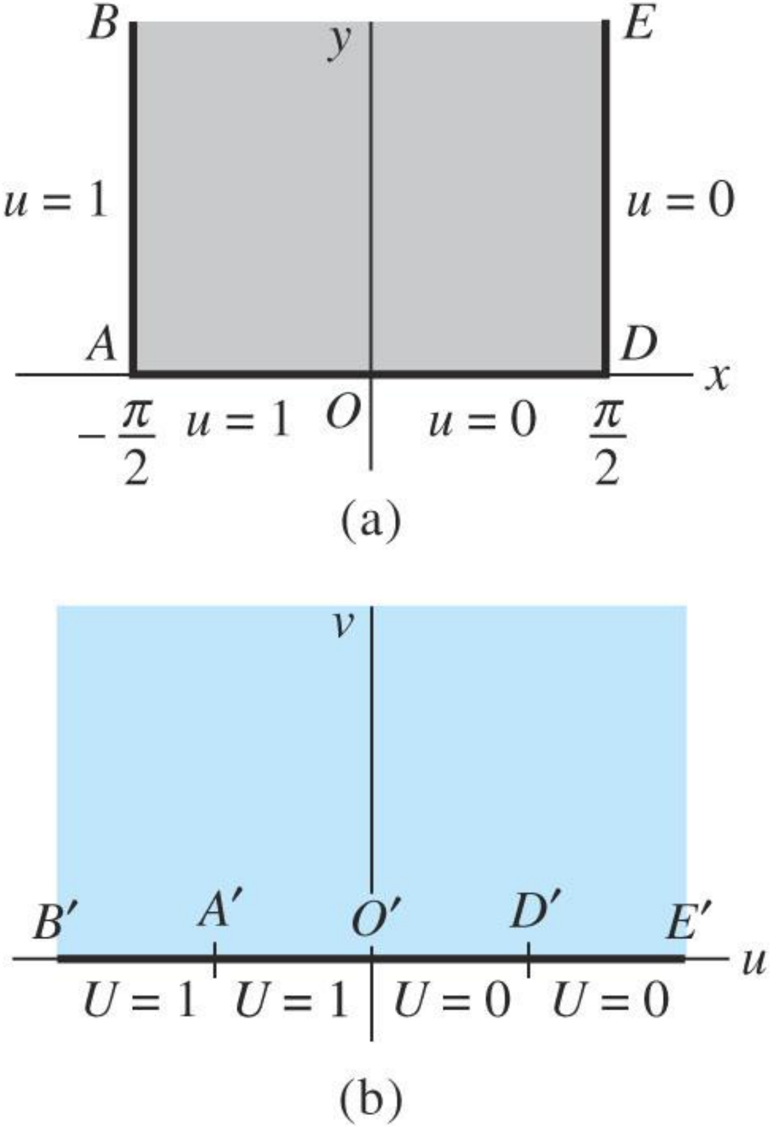


3. Solve the Dirichlet problem in R'. The solution may be apparent from the simplicity of the problem in R' or may be found using Fourier or integral transform methods.
4. The solution to the original Dirichlet problems is $u(x, y) = U(f(z))$.

Example 6

The function $U(u, v) = (1/\pi)$ Arg w is harmonic in the upper half-plane $v > 0$ since it is the imaginary part of the analytic function $g(w) = (1/\pi)$ Ln w. Use this function to solve the Dirichlet problem in Fig 6.14(a).

Fig 6.14



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Solution

The analytic function $f(z)$ = sin z maps the original region to the upper half-plane $v \geq 0$ and maps the boundary segments to the segments shown in Fig 6.14(b). The harmonic function $U(u, v) = (1/\pi)$ Arg w satisfies the transferred boundary conditions $U(u, 0) = 0$ for $u > 0$ and $U(u, 0) = 1$ for $u < 0$.

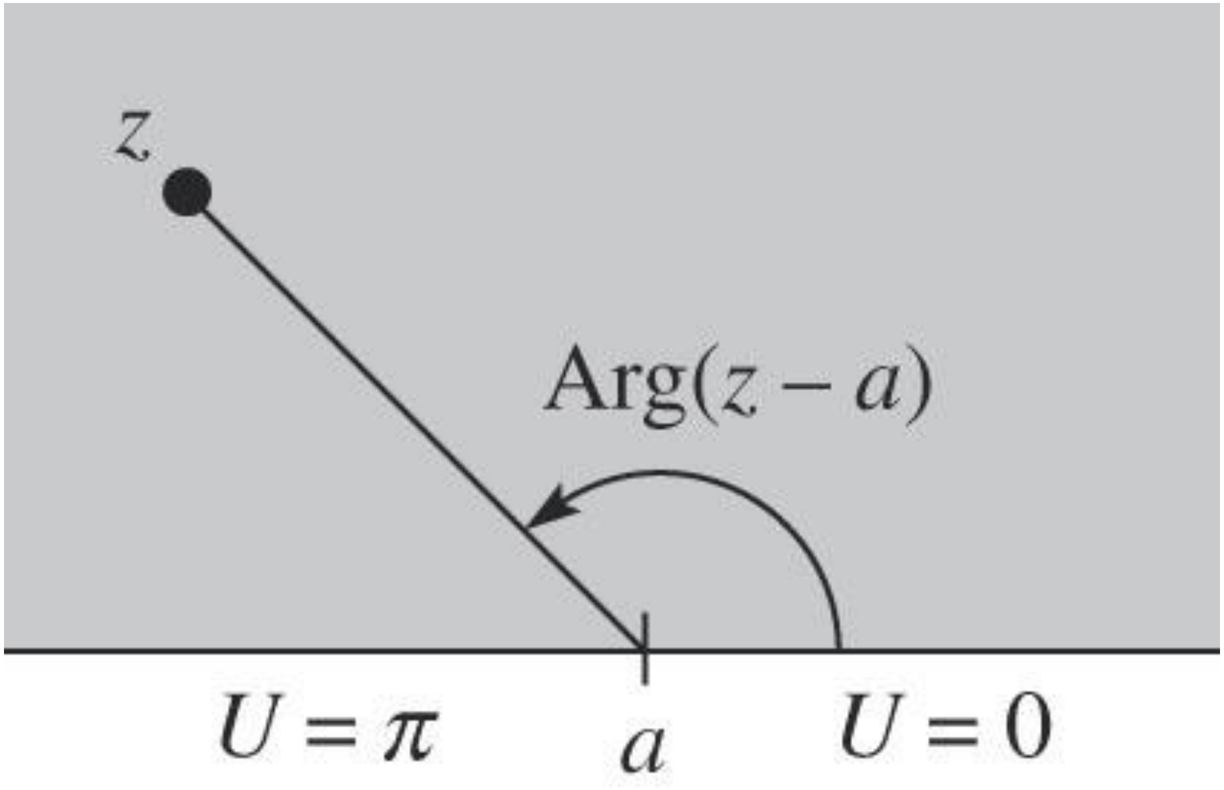
$$u(x, y) = \frac{1}{\pi}\tan^{-1}\left(\frac{\cos x \sinh y}{\sin x \cosh y}\right)$$

- A favorite image region R' for a simply connected region R is the upper half-plane $y \geq 0$. For any real number a, the complex function

$$\text{Ln}(z - a) = \log_e|z - a| + i \text{ Arg }(z - a)$$

is analytic in R' and is a solution to the Dirichlet problem shown in Fig 6.16.

Fig 6.16



- It follows that the solution in R' to the Dirichlet problem with

$$U(x,0) = \begin{cases} c_0, & a < x < b \\ 0, & otherwise \end{cases}$$

is the harmonic function

$$U(x, y) = (c_0/\pi)(\text{Arg}(z - b) - \text{Arg}(z - a))$$

6.3 Linear Fractional Transformations

❖ Linear Fractional Transformation

If a, b, c, d are complex constants with $ad - bc \neq 0$, then the function

$$T(z) = \frac{az + b}{cz + d}$$

is called a llinear fractional transformation. Since

$$T'(z) = \frac{ad - bc}{(cz + d)^2}$$

- T is conformal at z provided

$$\Delta = ad - bc \neq 0 \text{ and } z \neq -d/c.$$

Note when $c \neq 0$, $T(z)$ has a simple zero at $z_0 = -d/c$, and so

$$\lim_{z \to z_0} |T(z)| = \infty,$$

We will write $T(z_0) = \infty$. In addition, if $c \neq 0$, then

$$\lim_{|z| \to \infty} T(z) = \lim_{|z| \to \infty} \frac{a + b/z}{c + d/z} = \frac{a}{c},$$

and we write $T(\infty) = a/c$.

Example 1

If $T(z) = (2z + 1)/(z - i)$, compute $T(0)$, $T(\infty)$, $T(i)$.

Solution

$$T(0) = 1/(-i) = i,\ T(\infty) = \lim_{|z|\to\infty} T(z) = 2,$$

$$T(i) = \lim_{z\to i} |T(z)| = \infty,\ T(i) = \infty$$

Circle Preserving Property

- If $c = 0$, the transformation reduces to a linear function $T(z) = Az + B$. This is a composition of a rotation, magnification, and translation. As such, a linear function will map a circle in the z-plane to a circle in the w-plane. When $c \neq 0$,

$$w = \frac{az+b}{cz+d} = \frac{bc-ad}{c}\frac{1}{cz+d} + \frac{c}{a} \qquad (1)$$

Letting $A = \dfrac{bc - ad}{c}, B = \dfrac{a}{c},\ T(z)$ can be written as

$$z_1 = cz + d, z_2 = \frac{1}{z_1}, w = Az_2 + B \qquad (2)$$

Note that if $|z - z_1| = r, w = \dfrac{1}{z}$, then

$$\left|\frac{1}{w} - \frac{1}{w_1}\right| = \frac{|w - w_1|}{|w||w_1|} = r \ \text{ or } \ |w - w_1| = (r|w_1|)|w - 0| \qquad (3)$$

- It is easy to show that all points w that satisfy

$$|w - w_1| = \lambda |w - w_2| \qquad (4)$$

is a line when $\lambda = 1$ and is a circle when $\lambda > 0$ and $\lambda \neq 1$. It follows from (3) that the image of the circle $|z - z_1| = r$ under the inversion $w = 1/z$ is a circle except when $r = 1/|w_1| = |z_1|$.

THEOREM 20.3 **Circle-Preserving Property**

A linear fractional transformation maps a circle in the z-plane to either a line or a circle in the w-plane. The image is a line if and only if the original circle passes through a pole of the linear fractional transformation.

Example 2

Find the images of the circles $|z| = 1$ and $|z| = 2$ under $T(z) = (z + 2)/(z - 1)$. What are the images of the interiors of these circles?

Solution

The circle $|z| = 1$ passes through the pole $z_0 = 1$ of the linear transformation and so the image is a line. Since $T(-1) = -\frac{1}{2}$ and $T(i) = -(1/2) - (3/2)i$, we conclude that the line is $u = -\frac{1}{2}$.

Cont'd

The image of the interior $|z| = 1$ is either the half-plane $u < -\frac{1}{2}$ or the half-plane $u > -\frac{1}{2}$. Using $z = 0$ as a test point, $T(0) = -2$ and so the image is the half-plane $u < -\frac{1}{2}$.

The circle $|z| = 2$ does not pass through the pole so the image is a circle. For $|z| = 2$,

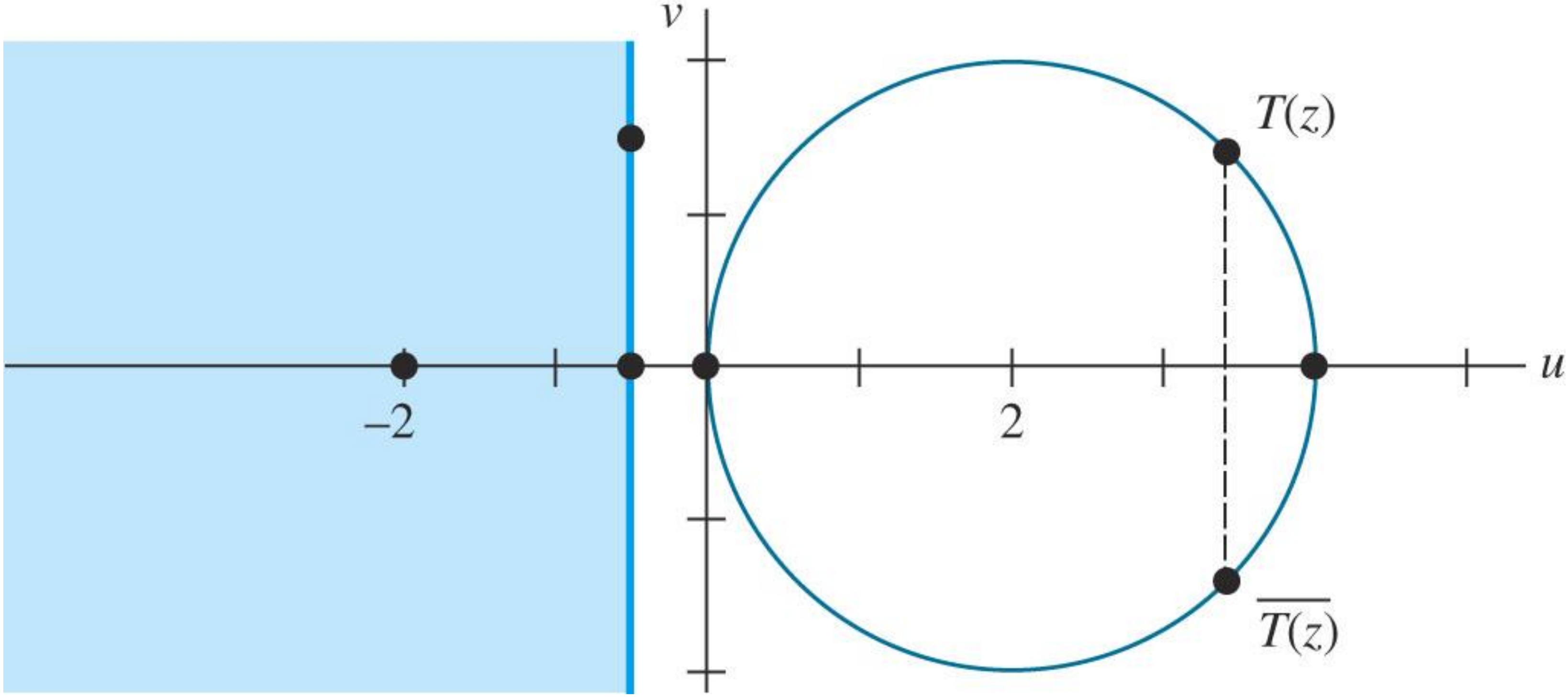
$$|\bar{z}| = 2, \overline{T(z)} = \overline{\frac{z+2}{z-1}} = \frac{\bar{z}+2}{\bar{z}-1} = T(\bar{z})$$

Example 2 (2)

Therefore $\overline{T(z)}$ is a point on the image circle and the image circle is symmetric w.r.t. the u-axis.

Since $T(-2) = 0$ and $T(2) = 4$ the center of the circle is $w = 2$ and the image is the circle $|w - 2| = 2$. The interior of $|z| = 2$ is either the interior or the exterior of the image $|w - 2| = 2$. Since $T(0) = -2$, we conclude that the image is $|w - 2| > 2$. See Fig 6.33.

Fig 6.33



Matrix Methods

- We associate the matrix

$$\mathbf{A} = \begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix} \text{ with } T(z) = \frac{az+b}{cz+d}$$

$$\text{If } T_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1}, \quad T_2(z) = \frac{a_2 z + b_2}{c_2 z + d_2},$$

$$\text{then } T_2(T_1(z)) \text{ is given by } T(z) = \frac{az+b}{cz+d}$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \quad (5)$$

If $w = T(z) = \dfrac{az+b}{cz+d}$, then $z = \dfrac{dw-b}{-cw+a}$

that is, $T^{-1}(w) = \dfrac{dw-b}{-cw+a}$, and the associated

matrix is $\quad \text{adj}\,\mathbf{A} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad (6)$

Example 3

If $T(z)=\dfrac{2z-1}{z+2}$ and $S(z)=\dfrac{z-i}{iz-1}$, find $S^{-1}(T(z))$.

Solution

Let $S^{-1}(T(z))=\dfrac{az+b}{cz+d}$, where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \text{adj}\begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -2+i & -1+2i \\ 1-2i & 2+i \end{pmatrix}, \text{ then}$$

$$S^{-1}(T(z))=\frac{(-2+i)z+1+2i}{(1-2i)z+2+i}$$

Triples to Triples

- The linear fractional transformation

$$T(z) = \frac{z - z_1}{z - z_3}\frac{z_2 - z_3}{z_2 - z_1}$$

has a zero at $z = z_1$, a pole at $z = z_3$ and $T(z_2) = 1$. Thus $T(z)$ maps three distinct complex numbers z_1, z_2, z_3 to 0, 1, and ∞, respectively. The term

The term $\frac{z - z_1}{z - z_3}\frac{z_2 - z_3}{z_2 - z_1}$ is called the cross-ratio of

z, z_1, z_2, z_3.

- Likewise, the linear fractional transformation

$$S(w) = \frac{w - w_1}{w - w_3} \frac{w_2 - w_3}{w_2 - w_1}$$

sends w_1, w_2, w_3 to 0, 1, and ∞, and so S^{-1}maps 0, 1, and ∞ to w_1, w_2, w_3. It follows that $w = S^{-1}(T(z))$ maps the triple z_1, z_2, z_3 to the triples w_1, w_2, w_3. From $w = S^{-1}(T(z))$, we have $S(w) = T(z)$ and

$$\frac{w - w_1}{w - w_3} \frac{w_2 - w_3}{w_2 - w_1} = \frac{z - z_1}{z - z_3} \frac{z_2 - z_3}{z_2 - z_1} \qquad (7)$$

Example 4

Construct a linear fractional transformation that maps the points 1, i, −1 on the circle $|z| = 1$ to the points −1, 0 and 1 on the real x-axis.

Solution

From (7) we get

$$\frac{w+1}{w-1}\frac{0-1}{0-(-1)} = \frac{z-1}{z+1}\frac{i+1}{i-1} \quad \text{or} \quad -\frac{w+1}{w-1} = -i\frac{z-1}{z+1}$$

Solving for w, we get $w = -i(z - i)/(z + i)$.

Example 5

Construct a linear fractional transformation that maps the points ∞, 0, 1 on the real *x*-axis to the points 1, *i*, −1 on the circle $|w| = 1$.

Solution

Since $z_1 = \infty$, the terms $z - z_1$ and $z_2 - z_1$ in the cross-product are replaced by 1. Then

$$\frac{w-1}{w+1}\frac{i+1}{i-1} = \frac{1}{z-1}\frac{0-1}{1} \quad \text{or} \quad S(w) = -i\frac{w+1}{w-1} = \frac{-1}{z-1} = T(z)$$

Cont'd

If we use the matrix method to find $w = S^{-1}(T(z))$,

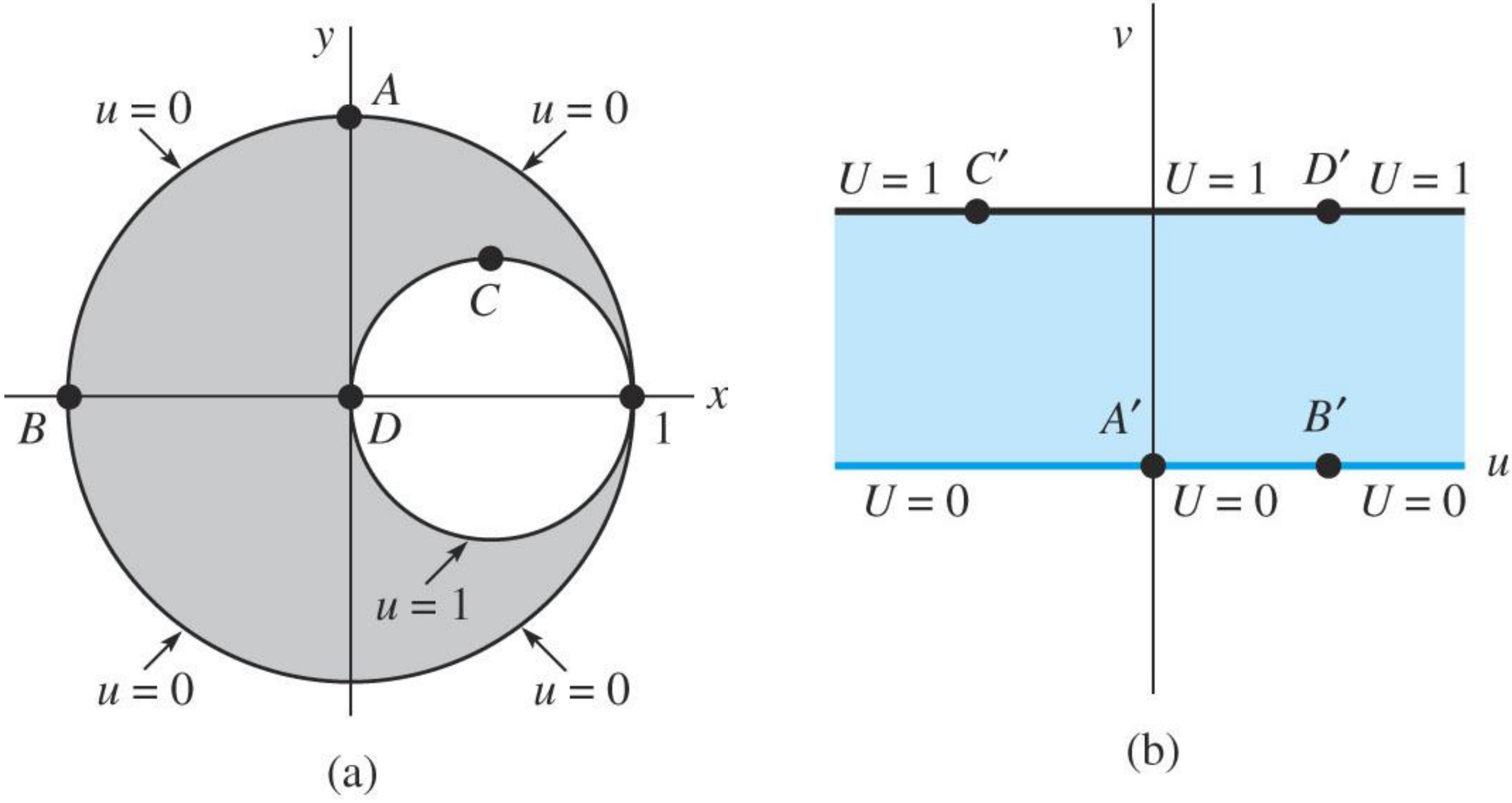
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \operatorname{adj}\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -i & -1+i \\ -i & 1+i \end{pmatrix}$$

and so $w = \dfrac{-iz-1+i}{-iz+1+i} = \dfrac{z-1-i}{z-1+i}$.

Example 6

Solve the Dirichlet problem in Fig 6.35(a) using conformal mapping by constructing a linear fractional transformation that maps the given region into the upper half-plane.

Fig 6.35(a)



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Solution

The boundary circles $|z| = 1$ and $|z - ½| = ½$ each pass through $z = 1$. We can map each boundary circle to a line by selecting a linear fractional transformation that has a pole at $z = 1$. If we require $T(i) = 0$ and $T(-1) = 1$, then

$$T(z) = \frac{z-i}{z-1}\frac{-1-1}{-1-i} = (1-i)\frac{z-i}{z-1}$$

Since $T(0) = 1+i, T(\frac{1}{2}+\frac{1}{2}i) = -1+i$, T maps the interior of $|z| = 1$ onto the upper half-plane and maps $|z - ½| = ½$ onto the line $v = 1$. See Fig 6.35(b).

Example 6 (3)

The harmonic function $U(u, v) = v$ is the solution to the simplified Dirichlet problem in the w-plane, and so $u(x, y) = U(T(z))$ is the solution to the original Dirichlet problem in the z-plane.

Since the imaginary part of $T(z) = (1-i)\dfrac{z-i}{z-1}$ is

$\dfrac{1-x^2-y^2}{(x-1)^2+y^2}$, the solution is $u(x, y) = \dfrac{1-x^2-y^2}{(x-1)^2+y^2}$

Cont'd

The level curves $u(x, y) = c$ can be written as

$$\left(x - \frac{c}{1+c}\right)^2 + y^2 = \left(\frac{1}{1+c}\right)^2$$

and are therefore circles that pass through $z = 1$. See Fig 6.36.

Fig 6.36