CHAPTER ONE
ELEMENTARY COUTING PRINCIPLES

1.1 Introduction

This chapter develops some techniques for determining, without direct enumeration, the number of possible outcomes of a particular event or the number of elements in a set. Such sophisticated counting is sometimes called *combinatorial analysis*. It includes the study of permutations and combinations.

1.2 Basic counting principles

Sum Rule Principle:

Suppose some event E can occur in m ways and a second event F can occur in n ways, and suppose both events cannot occur simultaneously. Then E or F can occur in $m + n$ ways.

Product Rule Principle:

Suppose there is an event E which can occur in m ways and, independent of this event, there is a second event F which can occur in n ways. Then combinations of E and F can occur in mn ways.

Permutation

An arrangement of n objects in a specified order is called permutation of the objects.
Permutation Rules:

1. The number of permutations of *n* distinct objects taken all together is n!
   Where $n! = n*(n-1)*(n-2)*.....*3*2*1$
2. The arrangement of n objects in a specified order using r objects at a time is called the permutation of n objects taken r objects at a time. It is written as ${}_nP_r$ and the formula is

$${}_nP_r = \frac{n!}{(n-r)!}$$

3. The number of permutations of n objects in which k_1 are alike k_2 are alike etc is

$$= \frac{n!}{k_1!*k_2*...*k_n}$$

Exercises:

1. Six different statistics books, seven different physics books, and 3 different Economics books are arranged on a shelf. How many different arrangements are possible if;
   i. The books in each particular subject must all stand together
   ii. Only the statistics books must stand together
2. If the permutation of the word WHITE is selected at random, how many of the permutations
   i. Begins with a consonant?
   ii. Ends with a vowel?
   iii. Has a consonant and vowels alternating?

Combination

A selection of objects with out regard to order is called combination.
Example: Given the letters A, B, C, and D list the permutation and combination for selecting two letters.
Solutions:

Permutation

AB	BA	CA	DA
AC	BC	CB	DB
AD	BD	CD	DC

Combination

AB	BC
AC	BD
AD	DC

Note that in permutation AB is different from BA. But in combination AB is the same as BA.

Combination Rule

The number of combinations of r objects selected from n objects is denoted by $_nC_r$ *or* $\binom{n}{r}$ and is given by the formula:

$$\binom{n}{r} = \frac{n!}{(n-r)!*r!}$$

Exercises:

1. Out of 5 Mathematician and 7 Statistician a committee consisting of 2 Mathematician and 3 Statistician is to be formed. In how many ways this can be done if
   a) There is no restriction
   b) One particular Statistician should be included
   c) Two particular Mathematicians can not be included on the committee.
2. If 3 books are picked at random from a shelf containing 5 novels, 3 books of poems, and a dictionary, in how many ways this can be done if
   a) There is no restriction.
   b) The dictionary is selected?
   c) 2 novels and 1 book of poems are selected?

THE INCLUSION–EXCLUSION PRINCIPLE

Let A and B be any finite sets.

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

In other words, to find the number $n(A \cup B)$ of elements in the union of A and B, we add $n(A)$ and $n(B)$ and then we subtract $n(A \cap B)$; that is, we "include" $n(A)$ and $n(B)$, and we "exclude" $n(A \cap B)$. This follows from the fact that, when we add $n(A)$ and $n(B)$, we have counted the elements of $(A \cap B)$ twice.

The above principle holds for any number of sets. We first state it for three sets.

Theorem For any finite sets A, B, C we have

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

That is, we "include" $n(A), n(B), n(C)$, we "exclude" $n(A \cap B), n(A \cap C), n(B \cap C)$, and finally "include" $n(A \cap B \cap C)$.

Pigeonhole Principle: If n pigeonholes are occupied by $n + 1$ or more pigeons, then at least one pigeonhole is occupied by more than one pigeon.

Generalized Pigeonhole Principle: If n pigeonholes are occupied by $kn + 1$ or more pigeons, where k is a positive integer, then at least one pigeonhole is occupied by $k + 1$ or more pigeons.

example

Find the number of mathematics students at a college taking at least one of the languages

French, German, and Russian, given the following data:

65 study French, 20 study French and German,
45 study German, 25 study French and Russian, 8 study all three languages.
42 study Russian, 15 study German and Russian,

We want to find $n(F \cup G \cup R)$ where F, G, and R denote the sets of students studying French, German, and Russian, respectively.

By the Inclusion–Exclusion Principle,

$$n(F \cup G \cup R) = n(F) + n(G) + n(R) - n(F \cap G) - n(F \cap R) - n(G \cap R) + n(F \cap G \cap R)$$
$$= 65 + 45 + 42 - 20 - 25 - 15 + 8 = 100$$

Namely, 100 students study at least one of the three languages.

Now, suppose we have any finite number of finite sets, say, $A_1, A_2, \ldots, A_m$. Let s_k be the sum of the cardinalities

$$n(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k})$$

of all possible k-tuple intersections of the given m sets. Then we have the following general Inclusion–Exclusion Principle.

Theorem $n(A_1 \cup A_2 \cup \cdots \cup A_m) = s_1 - s_2 + s_3 - \cdots + (-1)^{m-1} s_m$.

example

Find the minimum number of students in a class to be sure that three of them are born in the same month.

Here the $n = 12$ months are the pigeonholes, and $k + 1 = 3$ so $k = 2$. Hence among any $kn + 1 = 25$ students (pigeons), three of them are born in the same month.

THE BINOMIAL THEOREM Let x and y be variables, and let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

examples

What is the expansion of $(x + y)^4$?

Solution: From the binomial theorem it follows that

$$\begin{aligned}(x+y)^4 &= \sum_{j=0}^{4} \binom{4}{j} x^{4-j} y^j \\ &= \binom{4}{0} x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4 \\ &= x^4 + 4x^3 y + 6x^2 y^2 + 4x y^3 + y^4.\end{aligned}$$

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?

Solution: From the binomial theorem it follows that this coefficient is

$$\binom{25}{13} = \frac{25!}{13!\,12!} = 5{,}200{,}300.$$

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

Solution: First, note that this expression equals $(2x+(-3y))^{25}$. By the binomial theorem, we have

$$(2x+(-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j}(-3y)^{j}.$$

Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j=13$, namely,

$$\binom{25}{13} 2^{12}(-3)^{13} = -\frac{25!}{13!\,12!} 2^{12}3^{13}.$$

We can prove some useful identities using the binomial theorem, as Corollaries 1, 2, and 3 demonstrate.